Chin. Ann. Math.
31B(3), 2010, 347–360
DOI: 10.1007/s11401-009-0037-1

Chinese Annals of Mathematics, Series B

The Riemannian Manifolds with Boundary and Large Symmetry***

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Abstract Seventy years ago, Myers and Steenrod showed that the isometry group of a Riemannian manifold without boundary has a structure of Lie group. In 2007, Bagaev and Zhukova proved the same result for a Riemannian orbifold. In this paper, the authors first show that the isometry group of a Riemannian manifold M with boundary has dimension at most $\frac{1}{2}\dim M(\dim M-1)$. Then such Riemannian manifolds with boundary that their isometry groups attain the preceding maximal dimension are completely classified.

Keywords Riemannian manifold with boundary, Isometry, Rotationally symmetric metric, Principal orbit
2000 MR Subject Classification 53C99, 57S15

1 Introduction

Let M be a connected smooth Riemannian manifold with or without boundary. A priori, there exist two definitions of isometry on M. The first one is defined to be a distance-preserving bijection of M as we think of M as a metric space. The second is defined to be a diffeomorphism of M onto itself which preserves the metric tensor. In the case of Riemannian manifolds without boundary, these two definitions are equivalent according to Myers and Steenrod [9] in 1939 (see also [7, pp. 169–172] for a proof). Moreover, Myers and Steenrod [9] proved the following result on the isometry group of a Riemannian manifold without boundary.

Fact 1.1 *Let M be a connected smooth Riemannian manifold without boundary. Then the isometry group $I(M)$ is a Lie transformation group with respect to the compact-open topology. For each $x\in M$, the isotropy subgroup $I_x(M)$ is compact. If M is compact, $I(M)$ is also compact.*

Kobayashi [6] gave a different proof to Fact 1.1 by the concept of G-structure from the original one by Myers-Steenrod. Furthermore, Kobayashi proved that there exists a natural embedding of the isometry group $I(M)$ into the orthonormal frame bundle $O(M)$ of M such that $I(M)$ becomes a closed submanifold of $O(M)$. It is this submanifold structure that makes $I(M)$ into a Lie transformation group. Following this idea, he further proved the result below.

Manuscript received February 17, 2009. Revised October 30, 2009. Published online April 20, 2010.
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***Project supported by the National Natural Science Foundation of China (Nos. 10601053, 10671096, 10871184, 10971104).

Fact 1.2 (see [6, pp. 46–47]) *Let M be an n-dimensional connected Riemannian manifold without boundary. Then the isometry group $I(M)$ has dimension at most $\frac{(n+1)n}{2}$. If $\dim I(M) = \frac{n(n+1)}{2}$, then M is isometric to one of the following spaces of constant sectional curvature:*

(a) *an n-dimensional Euclidean space $\mathbf{R}^n$,*

(b) *an n-dimensional unit sphere $S^n = \{x \in \mathbf{R}^{n+1} : |x| = 1\}$ in $\mathbf{R}^{n+1}$,*

(c) *an n-dimensional projective space $\mathbf{R}P^n = S^n/\{\pm 1\}$,*

(d) *an n-dimensional, simply connected hyperbolic space $\mathbf{H}^n$ of constant sectional curvature -1.*

As long as the isometry group of a Riemannian orbifold is concerned, quite recently, Bagaev and Zhukova [1] showed the same result as Facts 1.1–1.2. They generalized the idea of Kobayashi to their setting by using the orthonormal frame bundle of a Riemannian orbifold. In this paper, we consider a special class of orbifolds — manifolds with boundary. We firstly observe that the dimension of the isometry group $I(M)$ of a Riemannian manifold M with boundary does not exceed $\frac{1}{2}\dim M(\dim M - 1)$. Then we classify such Riemannian manifolds M with boundary that the isometry groups $I(M)$ attain the preceding maximal dimension. We divide the lengthy classification list into three parts: Theorems 1.1–1.3, due to that their proofs will use different ideas. The notations in Fact 1.2 will be used in the following theorems.

Theorem 1.1 *Let M be an n-dimensional compact, connected smooth Riemannian manifold with boundary and $n \geq 2$. Suppose that the isometry group $I(M)$ is of dimension $\frac{n(n-1)}{2}$. Then M is diffeomorphic to either of the following four manifolds: the closed n-dimensional unit ball $\overline{D^n} = \{x \in \mathbf{R}^n : |x| \leq 1\}$ in $\mathbf{R}^n$, the two cylinder-like manifolds $S^{n-1} \times [0,1]$ and $\mathbf{R}P^{n-1} \times [0,1]$, and the manifold $\mathbf{R}P^n \setminus U$ constructed from $\mathbf{R}P^n$ with an n-dimensional open disk $U \subset \mathbf{R}P^n$ removed, where the closure $\overline{U}$ in $\mathbf{R}P^n$ is diffeomorphic to $\overline{D^n}$. Furthermore, we can characterize the metric tensor g_M of M as follows.*

(1) *If M is diffeomorphic to $\overline{D^n}$, then the metric g_M of M is rotationally symmetric with respect to a unique interior point O of M. That is, g_M can be expressed by $g_M = \mathrm{d}t^2 + \varphi^2(t) g_{S^{n-1}}$, where $g_{S^{n-1}}$ is the standard metric on the unit sphere $S^{n-1} \subset \mathbf{R}^n$ and the function $\varphi : (0, R] \to (0, \infty)$ is smooth, $\varphi(0) = 0$, and*

$$\varphi^{(\text{even})}(0) = 0, \quad \dot{\varphi}(0) = 1.$$

(2) *If M is diffeomorphic to $S^{n-1} \times [0,1]$, then the metric g_M can be expressed by $\mathrm{d}t^2 + f^2(t) g_{S^{n-1}}$, where T is a positive number and f is a positive smooth function on $[-\frac{T}{2}, \frac{T}{2}]$. The similar statement holds for M diffeomorphic to $\mathbf{R}P^{n-1} \times [0,1]$.*

(3) *Suppose that M is diffeomorphic to $\mathbf{R}P^n \setminus U$. Then we can find a Riemannian manifold $M' = S^n \times [-\frac{T}{2}, \frac{T}{2}]$ endowed with the metric $\mathrm{d}t^2 + f^2(t) g_{S^{n-1}}$, where $f : [-\frac{T}{2}, \frac{T}{2}] \to (0, \infty)$ is an even smooth function, and an involutive isometry β of M' defined by $\beta(x,t) = (-x,-t)$ such that M is the quotient space of M' by the group $\{1, \beta\}$.*

Theorem 1.2 *Let M be a noncompact connected Riemannian manifold with boundary ∂M and of dimension $n \geq 2$ such that $\dim I(M) = \frac{n(n-1)}{2}$ and ∂M has at least one compact component. Then M is diffeomorphic to either $S^{n-1} \times [0,1)$ or $\mathbf{R}P^{n-1} \times [0,1)$. In the former case, the metric g_M of M can be expressed by $\mathrm{d}t^2 + f^2(t) g_{S^{n-1}}$, where $f : [0,T) \to (0,\infty)$ is* *a smooth function and T is a positive number or ∞. Moreover, M is complete if and only if $T = \infty$. The similar statement holds for the latter case.*

Theorem 1.3 *Let M be a connected Riemannian manifold with noncompact boundary ∂M and of dimension $n \geq 2$ such that $\dim I(M) = \frac{n(n-1)}{2}$. Denote by $\mathbf{H}^k$, $k \geq 2$, the k-dimensional complete simple connected Riemannian space of constant sectional curvature -1. Then M is diffeomorphic to either $\mathbf{R}^{n-1} \times [0,1]$ or $\mathbf{R}^{n-1} \times [0,1)$. Furthermore, we can characterize the metric tensor g_M of M as follows.*

(1) *If M is diffeomorphic to $\mathbf{R}^{n-1} \times [0,1]$, then there exists a positive number T and a smooth function $f : [0,T] \to (0,\infty)$ such that the metric tensor g_M on M can be expressed by $g_M = \mathrm{d}t^2 + f^2(t) g_{\mathbf{R}^{n-1}}$ or $g_M = \mathrm{d}t^2 + f^2(t) g_{\mathbf{H}^{n-1}}$ with $t \in [0,T]$. Of course, we identify $\mathbf{H}^1$ with $\mathbf{R}^1$. The metric g_M in this case is always complete.*

(2) *If M is diffeomorphic to $\mathbf{R}^{n-1} \times [0,1)$, then there exist a number $T \in (0,\infty]$ and a smooth function $f : [0,T) \to (0,\infty)$ such that the metric tensor g_M on M can be expressed by $g_M = \mathrm{d}t^2 + f^2(t) g_{\mathbf{R}^{n-1}}$ or $g_M = \mathrm{d}t^2 + f^2(t) g_{\mathbf{H}^{n-1}}$ with $t \in [0,T)$. Moreover, the metric g_M is complete if and only if $T = \infty$.*

This paper is organized as follows. In Section 2, we prove the fact that the two definitions of isometry coincide on Riemannian manifolds with boundary (see Proposition 2.1). It seems that Bagaev-Zhukova [1] did not mention this fact in their setting of Riemannian orbifolds. The idea of making reduction to the boundary in the proof of Proposition 2.1 will be used many times afterwards. In this section we also show the above mentioned observation that the isometry group $I(M)$ of a Riemannian manifold M with boundary is a Lie transformation group of dimension at most $\frac{1}{2} \dim M(\dim M - 1)$ (see Theorem 2.1). Although our proof of the observation is based on the idea of Proposition 2.1, to avoid the troublesome argument of point set topology, we also use the result in [1] that $I(M)$ has a Lie group structure. In Sections 3–5, we use the metric geometry and the theory of transformation group to prove Theorems 1.1–1.3.

2 Some Properties of Isometry Group

In the following sections, we always let M be an n-dimensional connected, smooth Riemannian manifold with boundary and $n \geq 2$. With the induced metric from M, the boundary ∂M of M is an $(n-1)$-dimensional Riemannian manifold without boundary. Note that ∂M has at most countable connected components. Consider a diffeomorphism ϕ of M onto itself which preserves the metric tensor. If p is an interior point of M, then ϕ maps p to another interior point, say q, and the differential map $D\phi$ at p induces an orthogonal transform from the tangent space at p to the one at q. If u is a point on the boundary ∂M, then the tangent space $T_u M$ at u should be thought of as the upper half space

$$\{x = (x_1, x_2, \cdots, x_n) \in \mathbf{R}^n : x_n \geq 0\}$$

of the Euclidean space $\mathbf{R}^n$. That is,

$$T_u M = T_u(\partial M) + \{\lambda \mathbf{n}_u : \lambda \geq 0\},$$

where $\mathbf{n}_u$ is the inner unit normal vector at u. Since ϕ maps u to another point v on ∂M, the differential map $D\phi$ at u maps $\mathbf{n}_u$ to $\mathbf{n}_v$, and maps $T_u(\partial M)$ orthogonally onto $T_v(\partial M)$. Hence, in this sense, we may also call that the differential map $D\phi$ at u is an orthogonal transform from T_uM onto T_vM. Hence ϕ leaves the boundary ∂M invariant and induces an isometry of ∂M. Recall the two definitions of isometry on a Riemannian manifold with or without boundary given in the beginning of the introduction section. By [9], they are equivalent on ∂M, i.e., a distance-preserving bijection on ∂M is a diffeomorphism of ∂M which preserves the metric tensor on ∂M, and vice versa.

Let $d(\cdot\,,\cdot)$ be the distance function on M induced by the metric tensor of M and ψ a bijection on M which preserves $d(\cdot\,,\cdot)$. Since ψ is a homeomorphism of M onto itself, its restrictions to the boundary ∂M is a homeomorhism onto itself, so is the restriction to the interior of M. In fact, we have a stronger property about ψ in the following proposition.

Proposition 2.1 *A distance-preserving bijection ψ of M is a diffeomorphism which preserves the metric tensor of M, and vice versa. That is, the two definitions of isometry of M are equivalent.*

Proof We only prove that a distance-preserving bijection ψ of M is a diffeomorphism which preserves the metric tensor of M, since the vice versa part is easy. We first consider the property of ψ near a point p in the interior $\mathrm{Int}(M)$ of M. There exists an open neighborhood $U \subset \mathrm{Int}(M)$ of p such that the restriction $\psi|_U$ of ψ to U is a distance-preserving map onto the open neighborhood $V = \psi(U) \subset \mathrm{Int}(M)$ of $q = \psi(p)$. Since the two definitions of isometry for Riemannian manifolds without boundary are equivalent (see [7, pp. 169–172] for a proof), $\psi|_U : U \to V$ is a diffeomorphism preserving the metric tensor. Hence, $\psi|_{\mathrm{Int}(M)} : \mathrm{Int}(M) \to \mathrm{Int}(M)$ is also a diffeomorphism preserving the metric tensor. It suffices to prove that for each point $p \in \partial M$, ψ is smooth near p and $D\psi$ at p is an orthogonal transform from T_pM onto T_qM, where $q = \psi(p) \in \partial M$. We divide the proof into two steps.

Step 1 Recall that $\mathbf{n}_p$ denotes the inner unit normal vector at p. Choose $\delta > 0$ so small that the geodesic $\gamma(p,t) := \exp_p(t\mathbf{n}_p)$, $t \in [0,\delta]$, satisfies

$$d(\gamma(p,t), \partial M) = d(\gamma(p,t), \gamma(p,0)) = t. \tag{2.1}$$

Since the geodesic emanating from each point with this property is unique, the image $\psi \circ \gamma$ of γ under the distance-preserving bijection ψ is also a geodesic perpendicular to the boundary ∂M at the initial point q. Actually, we will see later that the differential $D\psi$ at p maps $\mathbf{n}_p$ to $\mathbf{n}_q$. In ∂M, we choose a small open neighborhood $V \subset \partial M$ of p such that for each point $p' \in V$ the geodesic $\gamma_(p',t)$, $t \in [0, \frac{\delta}{2}]$, satisfies (2.1). Then the map $\gamma(\cdot\,,t) : p' \mapsto \gamma(p',t)$ gives a diffeomorphism of V onto a hypersurface V_t in $\mathrm{Int}(M)$ for each $t \in (0, \frac{\delta}{2}]$. We can define the similar map from the neighborhood $\psi(V) \subset \partial M$ of q and denote the map also by $\gamma(\cdot\,,t)$. We observe that

$$\psi \circ \gamma(\cdot\,,t) = \gamma(\cdot\,,t) \circ \psi$$

holds for each $p \in V$ and each $t \in [0, \frac{\delta}{2}]$ so that $\gamma(\psi(V),t) = \psi(V_t)$. Hence the map $\psi|_V$ can be thought of as the composition of three diffeomorphisms:

$$\psi|_V(\cdot) = (\gamma(\cdot\,,t))^{-1} \circ \psi \circ \gamma(\cdot\,,t).$$

So $\psi|_V$ is a diffeomorphism of V onto $\psi(V)$. Since $\psi|_{V_t}$ is an isometry of V_t onto $\psi(V_t)$ for each $0 < t \leq \frac{\delta}{2}$, letting $t \to 0$, we see that $\psi|_V$ is a one-to-one distance-preserving map of V onto $\psi(V)$. Since the two definitions of isometry for Riemannian manifolds without boundary are equivalent, $\psi|_V$ is a diffeomorphism onto $\psi(V)$ preserving the metric tensor.

Step 2 By Step 1 and its preceding argument, we take a small open neighborhood $U \subset M$ of p such that the exponential map $\exp_p$ at p is a diffeomorphism from some neighborhood $\widetilde{U}$ of 0 in T_pM onto U. Recall that the partial derivative of ψ exists at the direction of the inner unit normal $\mathbf{n}_p$, and equals $\mathbf{n}_q$. On the other hand, by Step 1, $\psi|_{\partial M}$ preserves the metric tensor on ∂M. Hence we see that the differential map $D\psi$ at each p in $U \cap \partial M$ is an orthogonal transform from T_pM to $T_{\psi(p)}M$. Since ψ is a homeomorphism, we assume that $\psi(U)$ is contained in a normal coordinate neighborhood of q. Since both $\exp_p$ and $\exp_q$ are local diffeomorphisms, the equality

$$\psi \circ \exp_p = \exp_q \circ (D\psi(p)), \quad \text{in } \widetilde{U},$$

gives that $\psi|_U$ is a diffeomorphism onto $\psi(U)$. Moreover, ψ preserves the metric tensor on M.

The above proof essentially follows the idea in [7, pp. 169–172]. We repeat it here because the idea and notations will be used many times later. The following lemma is elementary and useful, but its proof is omitted.

Lemma 2.1 *Let ϕ be an isometry of M. If ϕ has a fixed point $p \in M$ and the differential $D\phi$ at p is the identity map of T_pM, then ϕ is the identity map of M.*

Proposition 2.2 (1) *Let ϕ be an isometry of M. Then the restriction $\phi|_{\partial M}$ to the boundary ∂M is an isometry of ∂M. Moreover, if ϕ leaves each point of a component B of ∂M fixed, then ϕ is the identity map of M.*

(2) *Let ϕ be an element in the identity component $I^0(M)$ of the isometry group of M. The restriction $\phi|_B$ to a connected component B of ∂M is an element in the identity component $I^0(B)$ of the isometry group of B. This map*

$$\iota : I^0(M) \to I^0(B), \quad \phi \mapsto \phi|_B$$

gives a continuous monomorphism with the image closed in $I^0(B)$. That is, ι is a regular embedding of the Lie group $I^0(M)$ into the Lie group $I^0(B)$.

Proof (1) The first statement have been shown in the proof of Proposition 2.1. If each point p of a component B of ∂M is fixed by ϕ, then the differential $D\phi$ at p is the identity map of T_pM. By Lemma 2.1, ϕ is the identity map of M.

(2) For a given point p in a connected component B of ∂M, we claim that each element ϕ in $I^0(M)$ maps p to a point in B. Otherwise, we assume that $\phi(p)$ lies in another component B' distinct from B. Choosing a path $\{\phi_t\}$ in $I^0(M)$ with $\phi_0 = \mathrm{id}_M$ and $\phi_1 = \phi$, we obtain a path $\{\phi_t(p)\}$ connecting p and $\phi(p)$. Since each diffeomorphism of M maps ∂M onto ∂M, we find that the path $\{\phi_t(p)\}$ lies on ∂M, which leads to a contradiction. By the proof of Proposition 2.1, we know that $\phi|_B$ is an isometry of B, which is actually an element of $I^0(B)$. Moreover, if $\phi|_B$ is the identity map of B, then by (1), ϕ is the identity map of M. Thus, the map $\iota : I^0(M) \to I^0(B)$, $\phi \mapsto \phi|_B$ gives a continuous monomorphism of $I^0(M)$ into $I^0(B)$.

Finally, we need to show that the image of $I^0(M)$ under ι is closed in $I^0(B)$ with respect to the compact-open topology. We divide the proof into two steps.

Step 1 We show that for a sequence $\{\phi^n\}$ of isometries in $I^0(M)$ such that

$$\phi^n|_B \to \mathrm{id}_B \quad \text{in } I^0(B),$$

there holds $\phi^n \to \mathrm{id}_M$ in $I^0(M)$. By Fact 1.1, $I^0(B)$ has the structure of a Lie group, whose topology from its smooth structure coincides with the compact-open topology. We may assume that $\phi^n|_B \to \mathrm{id}_B$ in the C^1 topology of $I^0(B)$. That is, $\phi^n|_K$ converges to the identity map id_K in the sense of the C^1 norm in each compact neighborhood K in the topological space B. Then, by using the normal coordinate charts with respect to each point in K (see the last three lines of the proof of Proposition 2.1), we find that there exists a compact neighborhood $\mathcal{K}$ of M such that $K \subset \mathcal{K}$ and $\phi^n(p)$ converges uniformly to p for each point $p \in \mathcal{K}$.

Step 2 We show that if a sequence $\{\phi^n\}$ of isometries in $I^0(M)$ satisfies $\phi^n|_B \to \phi$ in $I^0(B)$, then there exists $\psi \in I^0(M)$ such that $\psi|_B = \phi$. Since, by [1, 9], both $I^0(B)$ and $I^0(M)$ have structures of Lie groups, they can be endowed with a Riemannian metric. By the Cauchy criterion, $\phi^n|_B$ converges in $I^0(B)$ if and only if $\phi^n|_B(\phi^m|_B)^{-1}$ converges to id_B as $m, n \to \infty$. Then, Step 1 tells us that $\phi^n(\phi^m)^{-1}$ converges to id_M as $m, n \to \infty$. That is, ϕ^n converges to some $\psi \in I^0(M)$ such that $\psi|_B = \phi$.

As an immediate corollary of Proposition 2.2 and Fact 1.2, we obtain the following theorem.

Theorem 2.1 *The isometry group $I(M^n)$ has a structure of Lie group of dimension at most $\frac{n(n-1)}{2}$.*

3 Proof of Theorem 1.1

Let M be a Riemannian manifold satisfying the condition of Theorem 1.1 in this section. By Proposition 2.2 and Fact 1.2, the isometry group $I(B)$ of each component B of ∂M attains the maximal dimension $\frac{n(n-1)}{2}$, so B is isometric to either S^{n-1} or $\mathbf{R}P^{n-1}$ with constant sectional curvature 1. If $n = 2$, ∂M consists of circles. But our argument later also goes through in this case.

Suppose that there exists a component B isometric to the sphere S^{n-1}. By the proof of Fact 1.2 (see [6, pp. 46–47]), $G := I^0(M)$ is isomorphic to $\mathrm{SO}(n)$ and its action on B is just the linear action of $\mathrm{SO}(n)$ on S^{n-1}. We may identify G with $\mathrm{SO}(n)$, B with S^{n-1} up to a scaling of metric. Recall that G acts transitively on S^{n-1} and the isotropy group G_x at each point x of S^{n-1} is isomorphic to $H := \mathrm{SO}(n-1)$. Here we use the notation in the proof of Proposition 2.1. Choose a positive number $\delta > 0$ such that the map $\gamma(\,\cdot\,, t)$ is a diffeomorphism of B onto a hypersurface B_t in $\mathrm{Int}(M)$ for each $t \in (0, \delta]$. Since the G-action interchanges with $\gamma(\,\cdot\,, t)$, G leaves each B_t invariant. We claim that the G-action on M has the principal orbit whose type is $\mathrm{SO}(n)/\mathrm{SO}(n-1)$. Recall that the union of the principal orbits forms an open and dense subset of M. For the detail of this, see Theorem 4.27 in [5, pp. 216–220], where only manifolds without boundary are considered. But Theorem 4.27 in [5] also holds for our case by virtue of the map $\gamma(\,\cdot\,, t)$. The claim follows from that the union $M(H)$ of orbits with type

G/H contains the open subset $\bigcup_{t\in[0,\delta)} B_t$ in M. Therefore, every component of ∂M is isometric to S^{n-1}. By Theorems 4.19 and 4.27 in [5], the orbit space $M(H)/G$ is a connected smooth 1-manifold, whose boundary coincides with the orbit space $\partial M/G$. Hence, ∂M has at most two components. We also have the similar argument when ∂M has a component isometric to $\mathbf{R}P^{n-1}$. Summing up, we have proved the lemma below.

Lemma 3.1 *∂M has at most two components, each of which is isometric to either S^{n-1} or $\mathbf{R}P^{n-1}$. Moreover, if ∂M has two components, then the two components are isometric up to a scaling of metric.*

Lemma 3.2 *Let ∂M have two components. Then M is diffeomorphic to either $S^{n-1}\times[0,1]$ or $\mathbf{R}P^{n-1}\times[0,1]$. If M is diffeomorhic to $S^{n-1}\times[0,1]$, then there exist a positive number T and a positive smooth function $f(t)$ on $[0,T]$ such that the Riemannian metric g_M on M can be expressed by*

$$g_M = \mathrm{d}t^2 + f^2(t) g_{S^{n-1}}.$$

The similar statement holds for M diffeomorphic to $\mathbf{R}P^{n-1}\times[0,1]$.

Proof We only prove the case that both the two component of ∂M are isometric to S^{n-1}. By the proof of Lemma 3.1, the orbit space $M(H)/G$ is a closed interval. On the other hand, the orbit space $M(H)/G$ is a dense subset in the total orbit space M/G. Therefore, $M = M(H)$, i.e., all the orbits of G-action on M are of principal type. So M is a smooth fiber bundle with fiber S^{n-1} over a compact interval. Actually, it will be proved that M is diffeomorphic to the product of S^{n-1} and a compact interval.

We use the notation in the proof of Proposition 2.1 in what follows. Let B and B' be the components of ∂M. The two spheres B and B' may have different sizes. Choose a point $p\in B$. Then we claim that the geodesic $\gamma(p,t)$ with initial velocity $\mathbf{n}_p$ terminates at some point $p'\in B'$ and with the ending velocity $\mathbf{n}_{p'}$ at a positive time determined later. Indeed, choosing $p'\in\partial M$ such that $d(p,B') = d(p,p') =: T(p)$, we can find a geodesic between p and p' whose length is $T(p)$. It is clear that this geodesic is perpendicular to the boundary ∂M at p'. It is also perpendicular to ∂M at p. Otherwise, we can find another point q on B and a path ℓ connecting q and p' of length less than $T(p)$. Choosing an element $\alpha\in G$ mapping q to p, we obtain a path $\alpha(\ell)$ connecting p and $\alpha(p')\in B'$ of length less than $T(p)$. This gives a contradiction to the definition of $d(p,B')$. Therefore, this geodesic is exactly the one $\{\gamma(p,t) : t\in[0,T(p)]\}$ in the claim. We claim again that $T(p)$ equals the distance T between B and B'. Actually, there exists a p_0 such that the geodesic $\gamma(p_0,[0,T(p_0)])$ satisfies

$$T(p_0) = d(p_0, \gamma(p_0, T(p_0))) = d(B,B').$$

But G acts transitively on the set of geodesics

$$\{\gamma(p,[0,T(p)]) : p\in B\}$$

connecting B and B', which implies that these geodesics have the same length T. So each point p in B has the same distance T to B'.

We claim that the map $\gamma(\,\cdot\,,t) : B \to M$ is a diffeomorphism of B onto the hypersurface B_t for each $t \in [0,T]$. Similarly as the preceding paragraph, this map is surjective. It is also injective. Otherwise, there exist $0 < T' < T$ and two distinct points p_1 and p_2 in B such that $\gamma(p_1,T') = \gamma(p_2,T') =: q$. Then we get a curve $\gamma(p_1,[0,T']) \cup \gamma(p_2,[T',T])$ irregular at point q with length T connecting B and B', which is a contradiction. Since this map is equivariant with respect to the G actions on B and B_t, every point of B_t is a regular point of the map by the Sard theorem. Combining above, we know that the map $\gamma(\,\cdot\,,t) : B \to B_t$ is one-to-one, onto and its differential does not degenerate, that is, it is a diffeomorphism.

Now we show that any two geodesics in the set $\{\gamma(p,[0,T]) : p \in B\}$ are equal if they intersect. Suppose that there exist distinct points $p_1, p_2 \in B$ and positive times t_1, t_2 in $(0,T)$ such that $\gamma(p_1,t_1) = \gamma(p_2,t_2)$. Then by the argument in the preceding paragraph, we know $t_1 \neq t_2$, say $t_1 < t_2$. Then the piecewise smooth curve $\gamma(p_1,[0,t_1]) \cup \gamma(p_2,[t_2,T])$ connecting B and B' has length $t_1 + (T - t_2) < T$. This is a contradiction.

The statements in the above two paragraphs shows that the map

$$\gamma(\,\cdot\,,\,\cdot\,) : [0,T] \times B \to M, \quad (p,t) \mapsto \gamma(p,t)$$

is a diffeomorphism. Moreover, since G leaves each hypersurface B_t invariant and acts isometrically and effectively on it, B_t is a sphere with constant sectional curvature. Therefore, the Riemannian metric g_M of M can be written as $g_M = \mathrm{d}t^2 + f^2(t) g_{S^{n-1}}$ for some positive smooth function $f(t)$ on $[0,T]$.

Lemma 3.3 *If ∂M is connected, then ∂M must be isometric to the unit sphere S^{n-1}.*

Proof Suppose that ∂M is isometric to the real projective space $\mathbf{R}P^{n-1}$ with the constant sectional curvature and $n \geq 3$. Then n should be even since $\mathbf{R}P^{\text{even}}$ does not bound by the unoriented cobordism theory (see [9, pp. 52–53]). Denote by G the identity component of the isometry group of M. Since $\dim G = \frac{n(n-1)}{2}$, G is isomorphic to $\mathrm{SO}(n)/\{\pm 1\}$ and its action on the boundary $\mathbf{R}P^{n-1}$ is induced by the linear action of $\mathrm{SO}(n)$ on S^{n-1}. Moreover, the isotropy subgroup $H := G_p$ at each point p on ∂M is isomorphic to $\mathrm{SO}(n-1)$. Following the proof of Lemma 3.1, the G action on M has principal orbits of type $G/H = \mathbf{R}P^{n-1}$. Denote by $M(H)$ the union of principal orbits. Then the orbit space $M(H)/G$ is diffeomorphic to the interval $[0,1)$. Since $M(H)/G$ is dense in the total orbit space M/G, for the G action on M, there exists only one orbit G/J other than the principal ones. It is this orbit G/J that corresponds to the endpoint 1 of the orbit space M/G. Since H can be thought of as a proper subgroup of J in the sense of conjugacy, the Lie algebra $\mathfrak{J}$ of J contains a subalgebra isomorphic to $\mathfrak{so}(n-1)$. Simple computation shows that $\mathfrak{J}$ is isomorphic to either $\mathfrak{so}(n)$ or $\mathfrak{so}(n-1)$.

Case 1 If $\mathfrak{J}$ is $\mathfrak{so}(n)$, then J equals G. This means that topologically M is the cone of $\mathbf{R}P^{n-1}$, which does not have the structure of a manifold. Actually, the cone of $\mathbf{R}P^{n-1}$ is homeomorphic to the orbifold $\overline{D^n}/\{\pm 1\}$. This is a contradiction.

Case 2 If $\mathfrak{J}$ is isomorphic to $\mathfrak{so}(n-1)$, then the exception orbit G/J is a submanifold of codimension 1 in M. So the G action on G/J is also effective. Since G attains the largest-possible dimension $\frac{n(n-1)}{2}$, G/J is isometric to $\mathbf{R}P^{n-1}$ by Fact 1.2, which implies that G/J is also a principal orbit. This is a contradiction.

Lemma 3.4 *If ∂M is isometric to the sphere S^{n-1}, then M is homeomorphic to either $\overline{D^n}$ or $\mathbf{R}P^n \setminus U$, where U is an n-dimensional open disk such that its closure $\overline{U}$ in $\mathbf{R}P^n$ is homeomorphic to the closed unit ball $\overline{B^n} \subset \mathbf{R}^n$.*

Proof We use the notation in the proof of Lemmas 3.1–3.3. Recall that $G = \mathrm{SO}(n)$ and $H = \mathrm{SO}(n-1)$. By the similar argument in the proof of Lemma 3.3, there exists only one orbit G/J other than the principal orbits among all the orbits of the G action on M. Also by the argument in the proof of Lemma 3.3, we know that either J is $\mathrm{SO}(n)$ or J contains a subgroup of finite index and isomorphic to $\mathrm{SO}(n-1)$.

Case 1 If J is $\mathrm{SO}(n)$, then M is the cone of S^{n-1} homeomorphic to the closed unit disk $\overline{D^n}$.

Case 2 If $\mathrm{SO}(n-1)$ is a subgroup of J of finite index, then the exceptional orbit G/J is a submanifold of codimension 1 in M. Hence G acts effectively on G/J. Since ∂M is connected, G/J has to be isometric to $\mathbf{R}P^{n-1}$ by Fact 1.2. Therefore, M is homeomorphic to the mapping cone $S^{n-1} \times [0,1]/\sim$, where the equivalent relation $\sim$ on $S^{n-1} \times [0,1]$ is defined by $(x,1) \sim (-x,1)$ (for the concept of mapping cone, see [4, p. 13]). Clearly, M is also homeomorphic to the punctured real projective space $\mathbf{R}P^n \setminus U$.

Example 3.1 Define an effective $\mathrm{SO}(n)$-action on

$$\mathbf{R}P^n = \{[r_0 : r_1 : \cdots : r_n] \mid (r_0, \cdots, r_n) \in \mathbf{R}^{n+1} \setminus \{0\}\}$$

by the linear action on the last n-coordinates. Then this action has the unique fixed point $p = [1 : 0 : \cdots : 0]$, the principal orbits are isomorphic to S^{n-1}, and the unique exceptional orbit is isomorphic to $\mathbf{R}P^{n-1}$. Here the exceptional orbit means an orbit which has the same dimension as the principal orbit, but is not principal (see [2, p. 181]). Removing the equivariant open neighborhood U of p, we obtain the manifold $\mathbf{R}P^n \setminus U$ with an $\mathrm{SO}(n)$-action such that all the orbits are principal and isomorphic to S^{n-1} except one exceptional orbit $\mathbf{R}P^{n-1}$.

Lemma 3.5 *If M is homeomorphic to the n-dimensional closed disk $\overline{D^n}$, then there exist a point O in the interior of M and a positive number $R > 0$ such that the exponential map $\exp_O$ at O is a diffeomorphism of the closed ball centered at the origin $0 \in T_O M$ and of radius R in $T_O M$ onto M. Moreover, the Riemannian metric g_M of M is rotationally symmetric with respect to O so that it can be expressed by*

$$g_M = \mathrm{d}t^2 + \varphi^2(t) g_{S^{n-1}},$$

where the function $\varphi : (0, R] \to (0, \infty)$ is smooth, $\varphi(0) = 0$, and

$$\varphi^{(\mathrm{even})}(0) = 0, \quad \dot{\varphi}(0) = 1.$$

Proof We use the notation in the proof of Theorem 2.1 and Lemma 3.4. Denote by B the boundary ∂M. We know that B is a sphere of constant sectional curvature. By the proof of Lemma 3.4, there exists a unique fixed point $O \in \mathrm{Int}(M)$ of the G action on M. The G action on the tangent space $T_O M$ at O gives an isomorphism of G onto the special orthogonal transformation group $\mathrm{SO}(T_OM)$ of the Euclidean space T_OM. We denote by $B(0,r)$ the set of tangent vectors of length less than r at O, and by $S(0,r)$ the set of tangent vectors of length r at O. We also denote by $S(r)$ the set of points in M with distance r to O, by exp the exponential map at O.

Choose a point $p \in B$ such that $d(O,p) = d(O,B) =: R$. Then there exists a unit tangent vector V_0 at O such that $p = \exp(RV_0)$, and the geodesic $\exp(tV_0)$ ($t \in [0,R]$) is perpendicular to B at p. Choose an arbitrary unit tangent vector V at O. We claim that the geodesic $\exp(tV)$ ($t \geq 0$) meets the boundary B perpendicularly at time R. Actually, choosing an isometry $\alpha \in G$ such that the differential $\mathrm{d}\alpha$ at O maps V_0 to V, we find that the geodesic $\exp_O(tV)$ ($t \in [0,R]$) is the image of the one $\exp(tV_0)$ ($t \in [0,R]$) under the isometry α. Since G acts transitively on B, for each point $p \in B$, there exists a $V \in S(0,1)$ such that $\exp(RV) = p$. By the uniqueness of the geodesic perpendicular to B at a given point, we find that if $\exp(RV) = \exp(RW)$ for any two vectors $V, W \in S(0,1)$, then $V = W$. Hence $\exp : S(0,R) \to S(R)$ is a smooth bijection and there is no cut point of O in the interior of M. So, exp gives a diffeomorphism of $B(0,R)$ onto $\mathrm{Int}(M)$. To prove that this diffeomorphism can extend to the boundary, by the Gauss lemma, we only need to show that the restriction of exp to $S(0,R)$ is a diffeomorphism onto $S(R) = B$.

Recall that G acts on both $S(0,R)$ and $S(R)$. Moreover, the exponential map $\exp : S(0,R) \to S(R)$ is an equivariant smooth bijection map with respect to the G actions. By the Sard theorem, there exists a regular value of $\exp|_{S(0,R)}$. On the other hand, by the equivariant property, all points of $S(R)$ are regular values. That is, the map $\exp : S(0,R) \to S(R)$ is a diffeomorphism.

The similar statement holds for each $(n-1)$-dimensional sphere $S(r)$, $0 < r \leq R$, in M. So the metric g_M of M has rotational symmetry with respect to O. The expression of g_M follows from the argument in [10, pp. 12–13].

Remark 3.1 We can list closed geodesic balls with suitable radii in the three spaces $\mathbf{R}^n$, S^n and H^n as concrete examples of the manifold M in Lemma 3.5. Simultaneously, the geodesic annuli of these three spaces form examples of the manifold in Lemma 3.4. The manifold M in our consideration need not have constant sectional curvature, whose curvature can be computed explicitly in terms of the function f (see [10, pp. 65–68]). Because of the large symmetry on them, this class of manifolds, including geodesic balls in $\mathbf{R}^n$, S^n and H^n, may be thought of as the simplest class of compact Riemannian manifolds with boundary.

Lemma 3.6 *We use the notation in the proof of Lemma* 3.4. *Suppose that M is homeomorphic to $\mathbf{R}P^n \setminus U$. Then we can find a Riemannian manifold $M' = S^n \times [-\frac{T}{2}, \frac{T}{2}]$ endowed with the metric $\mathrm{d}t^2 + f^2(t) g_{S^{n-1}}$, where $f : [-\frac{T}{2}, \frac{T}{2}] \to (0,\infty)$ is an even smooth function, and an involutive isometry β of M' defined by $\beta(x,t) = (-x,-t)$ such that M is the quotient space of M' by the group $\{1,\beta\}$. Here $-x$ means the antipodal point of x in S^{n-1}. Of course, M is diffeomorphic to $\mathbf{R}P^n \setminus U$.*

Proof First of all, let us forget the Riemannian metric on M. Consider a topological model of M — the mapping cone $S^{n-1} \times [0,1]/\sim$. Recall that the equivalent relation $\sim$ means $(x,1) \sim (-x,1)$, where $x \mapsto -x$ is the deck transformation of the 2-fold covering $S^{n-1} \to \mathbf{R}P^{n-1}$. Then

M is the quotient of $M' := S^{n-1} \times [0, 2]$ by the group generated by the involution β of M given by

$$\beta(x, t) = (-x, 2 - t).$$

Then we endow M' with the induced Riemannian metric from M. Since each isometry of M can be lifted to two isometries of M', M' also satisfies the condition of Theorem 1.1. By Lemma 3.2, there exist a positive number T and a smooth function $f : [-\frac{T}{2}, \frac{T}{2}] \to (0, \infty)$ such that M' is diffeomorphic to $S^{n-1} \times [-\frac{T}{2}, \frac{T}{2}]$ and the metric $g_{M'}$ is given by

$$g_{M'} = \mathrm{d}t^2 + f^2(t) g_{S^{n-1}},$$

where $g_{S^{n-1}}$ is the standard metric on the unit sphere S^{n-1}. On the other hand, since the deck transform $\beta : M' \to M'$, $(x, t) \mapsto (-x, -t)$ is an isometry of M', we can see that $-x$ is actually the antipodal point of $x \in S^{n-1}$ and $f(t)$ is an even function.

We finally complete the proof of Theorem 1.1 by combining all the lemmas in this section.

4 Proof of Theorem 1.2

Let M be a Riemannian manifold satisfying the condition of Theorem 1.2 in this section. Let B be a compact component of ∂M. By Proposition 2.2 and Fact 1.2, the isometry group $I(B)$ of B has the largest-possible dimension $\frac{n(n-1)}{2}$, so B is isometric to either S^{n-1} or $\mathbf{R}P^{n-1}$ with constant sectional curvature. Suppose the former case holds. Then $G := I^0(M)$ is isomorphic to $\mathrm{SO}(n)$ and the isotropy subgroup G_x at each point $x \in B$ is isomorphic to $H := \mathrm{SO}(n-1)$. By the same argument in the proof of Lemma 3.1, the G action on M has principal orbit type G/H and ∂M has at most two components. We claim that ∂M is connected, i.e. $\partial M = B$. Actually, since the orbit space $M(H)/G$ is a connected smooth 1-manifold, whose boundary coincides with the orbit space $\partial M/G$, $M(H)/G$ is a compact interval if ∂M has two components. Since the orbit space $M(H)/G$ is dense in the total orbit space M/G, we have that $M(H) = M$ and M is diffeomorphic to $S^{n-1} \times [0, 1]$. This contradicts the noncompactness of M. By the same reason, we also have $M(H) = M$. We have the similar argument for the latter case. Summing up, we obtain the following results.

Lemma 4.1 *M is diffeomorphic to either $S^{n-1} \times [0, 1)$ or $\mathbf{R}P^{n-1} \times [0, 1)$.*

Lemma 4.2 (1) *Let M be complete. If M is diffeomorphic to $S^{n-1} \times [0, 1)$, then the metric g_M of M can be expressed by*

$$g_M = \mathrm{d}t^2 + f^2(t) g_{S^{n-1}},$$

where $f : [0, \infty) \to (0, \infty)$ is a smooth function. The similar statement holds for M diffeomorphic to $\mathbf{R}P^{n-1} \times [0, \infty)$.

(2) *Let M be noncomplete. If M is diffeomorphic to $S^{n-1} \times [0, 1)$, then there exists a finite positive number T such that the metric g_M of M can be expressed by*

$$g_M = \mathrm{d}t^2 + f^2(t) g_{S^{n-1}},$$

where $f : [0, T) \to (0, \infty)$ is a smooth function. The similar statement holds for M diffeomorphic to $\mathbf{R}P^{n-1} \times [0, 1)$.

Proof We only prove the case that M is diffeomorphic to $S^{n-1}\times[0,1)$. We use the notation in the proof of Proposition 2.1 and Lemma 4.1. Choose an arbitrary point q in the interior of M such that the distance of q to the boundary B is D. Cutting M along the orbit through q of the G action, we obtain a compact part M_1 diffeomorphic to $S^{n-1}\times[0,1]$ and a noncompact part diffeomorphic to M, on both of which G acts isometrically. By the proof of Lemma 3.2, the map $\gamma(\,\cdot\,,\,\cdot\,):B\times[0,D]\to M_1$ is a diffeomorphism. Since q is arbitrary, there exists $T\in(0,\infty]$ such that the map $\gamma(\,\cdot\,,\,\cdot\,):B\times[0,T)\to M$ is a diffeomorphism. Moreover, T is ∞ if and only if M is complete.

5 Proof of Theorem 1.3

Since the compact transformation group theory cannot be applied directly to isometry groups of Riemannian manifolds with noncompact boundary, we need new ideas to classify Riemannian manifolds with noncompact boundary whose isometry groups attain the maximal dimension.

Denote by G_k and $\mathcal{G}_k$ the identity components of the isometry groups of $\mathbf{R}^k$ and $\mathbf{H}^k$, respectively. Recall that $\mathcal{G}_k$ is the identity component of $O(1,k)$ and semisimple for each $k\geq 2$. However, G_k is the semidirect product of $\mathrm{SO}(k)$ and $\mathbf{R}^k$, and it is not semisimple for each $k\geq 1$ (see [10, p. 5 and p. 77]). Let M be a Riemannian manifold satisfying the assumption of Theorem 1.5 throughout this section. By Proposition 2.2 and Fact 1.2, every component of ∂M with the induced Riemannian metric from M is isometric to either $\mathbf{R}^{n-1}$ or the $(n-1)$-dimensional complete and simply connected Riemannian manifold $H^{n-1}(c)$ of constant sectional curvature $c<0$. Note that all $H^{n-1}(c)$'s, $c<0$, have the same isometry group isomorphic to the semidirect product of $\mathcal{G}_{n-1}$ and $\mathbf{Z}_2$. Suppose that a component of ∂M is isometric to $\mathbf{R}^{n-1}$. Then $I^0(M)$ is isomorphic to G_{n-1}, which acts effectively and isometrically on each component of ∂M. Hence, we find that each component of ∂M should be isometric to $\mathbf{R}^{n-1}$. The similar argument goes through if ∂M has a component isometric to $H^{n-1}(c)$ for some $c<0$.

Lemma 5.1 *Each component of ∂M is isometric to either $\mathbf{R}^{n-1}$ and $H^{n-1}(c)$ for some $c<0$. Moreover, the components of ∂M are mutually isometric up to a scaling of metric.*

Lemma 5.2 *We use the notation of Proposition 2.1. Let B be a component of the boundary ∂M and p an arbitrary point of B. Let I be the maximal existence interval of the geodesic $\gamma(p,t)=\exp_p(t\mathbf{n}_p)$ perpendicular to B at the initial point p. Then the map*

$$\gamma:B\times I\to M,\quad (q,t)\mapsto\gamma(q,t)$$

is well-defined and gives a diffeomorphism of $B\times I$ onto M. Consequently, if I is a compact interval, then M is diffeomorphic to $B\times[0,1]$; if I is an interval open at the right endpoint, then M is diffeomorphic to $B\times[0,1)$.

Proof Denote by G the identity component of $I(M)$. Then $G=G_{n-1}$ if B is isometric to $\mathbf{R}^{n-1}$, $G=\mathcal{G}_{n-1}$ if B is isometric to $H^{n-1}(c)$ for some $c<0$.

Since G acts transitively on B, we can see that, for every point $q\in B$, the geodesic $\gamma(q,t)$ perpendicular to B at the initial point q also has the maximal existence interval I. We claim that for any two distinct points $p, q \in B$, the two geodesics $\{\gamma(p,t) : t \in I\}$ and $\{\gamma(q,t) : t \in I\}$ do not intersect at a point x such that $x = \gamma(p,s) = \gamma(q,s)$ for some $s \in I$ and $d(x,p) = d(x,q) = s$. Otherwise, there exist two distinct points $p, q \in B$ and $s > 0$ satisfying $x = \gamma(p,s) = \gamma(q,s)$ and $d(x,p) = d(x,q) = s$. Since G acts transitively on B, there exists an $\alpha \in G$ mapping p to q. By the equality $\gamma(\alpha(p), s) = \alpha(\gamma(p,s))$, we can see that x is a fixed point of α. However, we can choose an α having no fixed point on B. Then, we reach an contradiction. Here are the details of choosing such α. If B is $\mathbf{R}^{n-1}$, α may be chosen to be the translation $x \mapsto x + (q - p)$. If B is $H^{n-1}(c)$ for some $c < 0$ and $n \geq 3$, we may assume without loss of generality that $c = -1$ and B is the upper half space

$$\{x = (x_1, \cdots, x_{n-2}, x_{n-1}) : x_1, \cdots, x_{n-2} \in \mathbf{R},\ x_{n-1} > 0\}$$

endowed with the hyperbolic metric

$$\frac{\mathrm{d}x_1^2 + \cdots + \mathrm{d}x_{n-1}^2}{x_{n-1}^2}.$$

Express the two distinct points p and q into coordinate forms: $p = (p_1, \cdots, p_{n-1})$, $q = (q_1, \cdots, q_{n-1})$. If $p_{n-1} = q_{n-1}$, α may also be chosen to be the similar translation as above; if $p_{n-1} \neq q_{n-1}$, we take

$$\alpha : (x_1, \cdots, x_{n-1}) \mapsto \frac{q_{n-1}}{p_{n-1}}(x - (p_1, \cdots p_{n-2}, 0)) + (q_1, \cdots, q_{n-2}, 0).$$

We claim that the subset $B_t = \{\gamma(p,t) : p \in B\}$ is a Riemannian submanifold isometric to B for each $t \in I$. By the equality $\alpha \circ \gamma(\,\cdot\,, t) = \gamma(\,\cdot\,, t) \circ \alpha$ for every $\alpha \in G$, B_t is exactly an orbit of the G action, so it is a submanifold of M. Remember that the map $\gamma(\,\cdot\,, t) : B \to B_t$ is surjective and G-equivariant. By the claim in the preceding paragraph, this map is one-to-one. Hence, it gives a diffeomorphism of B onto B_t. Since G acts effectively and isometrically on B_t, the claim follows from Fact 1.2.

The left part of the proof is similar to that of Lemma 3.2. There also holds that for each $(p,t) \in B \times I$,

$$d(B, B_t) = d(\gamma(p,t), B) = d(\gamma(p,t), p) = t.$$

And the geodesic $\{\gamma(p,t) : t \in I\}$ is perpendicular to B_t at point $\gamma(p,t)$.

The proof of Theorem 1.3 follows from Lemma 5.2.

Acknowledgements The authors would like to express their deep gratitude to the anonymous referee for the valuable and concrete comments, which improve greatly the readability of this manuscript. In particular, Example 3.1 is due to the referee. A lot of thanks from the third author should go to Beijing International Mathematical Research Center for the hospitality and financial support during the course of this work.

References

[1] Bagaev, A. V. and Zhukova, N. I., The isometry groups of Riemannian orbifolds, *Sib. Math. J.*, **48**, 2007, 579–592.

[2] Bredon, G. E., Introduction to Compact Transformation Groups, Pure and Applied Mathematics, Vol. 46, Academic Press, New York, 1972.

[3] van Dantzig, D. and van der Waerden, B. L., Über metrisch homogene Räume, *Abh. Math. Sem. Univ. Hamburg*, **6**, 1928, 374–376.

[4] Hatcher, A., Algebraic Topology, Cambridge University Press, Cambridge, 2002.

[5] Kawakubo, K., The Theory of Transformation Groups, Oxford University Press, New York, 1991.

[6] Kobayashi, S., Transformation Groups in Differential Geometry, Ergebnisse der Mathematik und Ihrer Grenzgebiete, Vol. 70. Springer-Verlag, Heidelberg, 1972.

[7] Kobayshi, S. and Nomizu, K., Foundations of Differential Geometry, Vol. I, Interscience Publishers, New York, 1963.

[8] Milnor, J. W. and Stasheff, J. D., Characteristic Classes, Princeton University Press, Princeton, 1974.

[9] Myers, S. B. and Steenrod, N. E., The group of isometries of a Riemannian manifold, *Ann. of Math.* (2), **40**, 1939, 400–416.

[10] Petersen, P., Riemannian Geometry, 2nd ed., Graduate Texts in Mathematics, Vol. 171, Springer-Verlag, New York, 2006.