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Mean Curvature Flow with Convex Gauss Image**

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Abstract In this paper, the mean curvature flow of complete submanifolds in Euclidean space with convex Gauss image and bounded curvature is studied. The confinable property of the Gauss image under the mean curvature flow is proved, which in turn helps one to obtain the curvature estimates. Then the author proves a long time existence result. The asymptotic behavior of these solutions when $t \to \infty$ is also studied.

Keywords Mean curvature flow, Convex Gauss image, Curvature estimates, Long time existence
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1 Introduction

There are many works on the mean curvature flow of hypersurfaces in Riemannian manifolds (see [6, 7, 9, 10] for example). The impressive features of mean curvature flow for codimension one are as follows.

(1) If the initial hypersurface $M_0 \subset \mathbb{R}^{m+1}$ is uniformally convex, then the hypersurfaces under the mean curvature contract smoothly to a single point in finite time and the shapes of the hypersurfaces become spherical at the end of the contraction. If the ambient manifold is a general Riemannian manifold, such a contraction is still working.

(2) If the initial hypersurface $M_0 \subset \mathbb{R}^{m+1}$ is an entire graph with linear growth, then there is long time existence for the mean curvature flow and the shapes of the hypersurfaces become flat.

We know that J. Moser [13] proved that an entire minimal graph in $\mathbb{R}^{m+1}$ given by $x^{m+1} = f(x^1, \cdots, x^m)$ with bounded gradient $|\nabla f| < c < \infty$ has to be hyperplane. This is closely related to the result of Ecker-Huisken [6], which reveals the second feature of the mean curvature flow of hypersurfaces mentioned above. On the other hand, Moser's result [13] has been generalized to higher codimension in [5, 8], and in author's joint work with J. Jost [11]. This viewpoint is the underline motivation of the present work.

It is natural to study the mean curvature flow of higher codimension. In recent years some interesting works have been done in [1–3, 15–19]. In the present paper, we show the second feature in higher codimension. The terminology of linear growth in [6] can be interpreted as the image under the Gauss map of the hypersurface lies in an open hemisphere. We investigate the mean curvature flow of submanifolds with convex Gauss image naturally.

Due to the curved normal bundle, the evolution equation of the squared norm of the second fundamental form (3.5) is more difficult to deal with than the hypersurface case.

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Consider the image of the Gauss map under the mean curvature flow. If the image under the Gauss map of the initial submanifold lies in a geodesic ball B_{R_0} of radius R_0 in the Grassmanian manifold, we can prove that the deforming submanifolds under the mean curvature flow still lie in the same geodesic ball, provided $R_0 < \frac{\sqrt{2}}{4}\pi$. This is an adequate generalization of "linear growth preserving property" in [6] for the codimension one case. This is Theorem 4.1 of this paper. We call it the "confinable property", with whose help we obtain the curvature estimates (see Theorem 4.2).

By using Huisken's monotonicity formula for the backward heat kernel, Ecker-Huisken derived a maximum principle for parabolic equations on certain complete manifolds. Since the Gauss image assumption and the curvature assumption, we see that the mean curvature flow equations are uniformly parabolic. The resulting manifolds under the mean curvature flow have Euclidean volume growth up to a constant. On those manifolds, the curvature and its covariant derivatives have at most polynomial growth. Hence, Ecker-Huisken's maximum principle is applicable in our consideration. The confinable property of the Gauss image under the mean curvature flow and curvature estimates are proved by this maximum principle.

Combining those properties, we are able to prove the following main theorem in this paper.

Theorem 1.1 *Let $F : M \to \mathbb{R}^{m+n}$ be a complete m-submanifold which has bounded curvature. Suppose that the image under the Gauss map from M into $\mathbf{G}_{m,n}$ lies in a geodesic ball of radius $R_0 < \frac{\sqrt{2}}{12}\pi$. Then the evolution equations of mean curvature flow have a long time smooth solution.*

Remark 1.1 Here we need not assume the initial manifold is an entire graph, which is a conclusion of the Gauss image assumption.

We also study the asymptotic behavior of these solutions when $t \to \infty$, namely we study the rescaled mean curvature flow in Section 5. The corresponding results as in [6] can be obtained similarly.

2 A Bochner Type Formula

Let $F : M \to \mathbb{R}^{m+n}$ be an m-submanifold in $(m+n)$-dimensional Euclidean space with the second fundamental form B which can be viewed as a cross-section of the vector bundle $\mathrm{Hom}(\odot^2 TM, NM)$ over M, where TM and NM denote the tangent bundle and the normal bundle along M, respectively. A connection on $\mathrm{Hom}(\odot^2 TM, NM)$ is induced from those of TM and NM naturally. We investigate the higher codimension $n \geq 2$ situation in this paper.

For $\nu \in \Gamma(NM)$, the shape operator $A^\nu : TM \to TM$ satisfies

$$\langle B_{XY}, \nu\rangle = \langle A^\nu(X), Y\rangle.$$

The second fundamental form, curvature tensor of the submanifold, curvature tensor of the normal bundle and that of the ambient manifold satisfy the Gauss equations, the Codazzi equations and the Ricci equations.

Taking the trace of B gives the mean curvature vector H of M in $\mathbb{R}^{m+n}$, a cross-section of the normal bundle.

Choose a local orthonormal frame field $\{e_i, e_\alpha\}$ along M with dual frame field $\{\omega_i, \omega_\alpha\}$, such that e_i are tangent vectors to M. The induced Riemannian metric of M is given by $\mathrm{d}s_M^2 = \sum_i \omega_i^2$ and the induced structure equations of M are

$$\begin{aligned}
&\mathrm{d}\omega_i = \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0,\\
&\mathrm{d}\omega_{ij} = \omega_{ik} \wedge \omega_{kj} + \omega_{i\alpha} \wedge \omega_{\alpha j},\\
&\Omega_{ij} = d\omega_{ij} - \omega_{ik} \wedge \omega_{kj} = -\frac{1}{2} R_{ijkl}\omega_k \wedge \omega_l.
\end{aligned}$$

By Cartan's lemma, we have

$$\omega_{\alpha i} = h_{\alpha ij}\omega_j.$$

To have the curvature estimates, we need the Bochner type formula for the squared norm of the second fundamental form. It is done in [14] for minimal submanifolds in an arbitrary ambient Riemannian manifold. Now, for any submanifold in Euclidean space, by same calculation as in the paper [14] we have the following formula.

Proposition 2.1

$$\begin{aligned}
(\nabla^2 B)_{XY} = {}& \nabla_X \nabla_Y H + \langle B_{Xe_i}, H\rangle B_{Ye_i} - \langle B_{XY}, B_{e_ie_j}\rangle B_{e_ie_j}\\
&+ 2\langle B_{Xe_j}, B_{Ye_i}\rangle B_{e_ie_j} - \langle B_{Ye_i}, B_{e_ie_j}\rangle B_{Xe_j} - \langle B_{Xe_i}, B_{e_ie_j}\rangle B_{Ye_j},
\end{aligned} \tag{2.1}$$

where ∇^2 stands for the trace Laplacian operator.

Denote

$$B_{ij} = B_{e_ie_j} = (\overline{\nabla}_{e_i} e_j)^N = h_{\alpha ij} e_\alpha,$$

where $\{e_\alpha\}$ is a local orthonormal frame field of the normal bundle near $x \in M$. Let $S_{\alpha\beta} = h_{\alpha ij}h_{\beta ij}$. Then $|B|^2 = \sum\limits_\alpha S_{\alpha\alpha}$.

Noting

$$\begin{aligned}
&-\langle B_{kl}, B_{ij}\rangle\langle B_{ij}, B_{kl}\rangle = -h_{\alpha kl}h_{\alpha ij}h_{\beta ij}h_{\beta kl} = -\sum_{\alpha,\beta} S_{\alpha\beta}^2,\\
&2\,\langle B_{il}, B_{jk}\rangle\langle B_{kl}, B_{ij}\rangle - 2\,\langle B_{jk}, B_{kl}\rangle\langle B_{il}, B_{ij}\rangle\\
={}& 2\sum_{\alpha\neq\beta}(\langle A^{e_\beta}A^{e_\alpha}, A^{e_\alpha}A^{e_\beta}\rangle - 2\langle A^{e_\beta}A^{e_\alpha}, A^{e_\beta}A^{e_\alpha}\rangle) = -\sum_{\alpha\neq\beta}|[A^{e_\alpha}, A^{e_\beta}]|^2,
\end{aligned}$$

we then have

$$\langle \nabla^2 B, B\rangle = \langle \nabla_i\nabla_j H, B_{ij}\rangle + \langle B_{ik}, H\rangle\langle B_{il}, B_{kl}\rangle - \sum_{\alpha\neq\beta}|[A^{e_\alpha}, A^{e_\beta}]|^2 - \sum_{\alpha,\beta} S_{\alpha\beta}^2.$$

The following expression follows immediately.

Proposition 2.2

$$\Delta|B|^2 = 2|\nabla B|^2 + 2\langle\nabla_i\nabla_j H, B_{ij}\rangle + 2\langle B_{ij}, H\rangle\langle B_{ik}, B_{jk}\rangle - 2\sum_{\alpha\neq\beta}|[A^{e_\alpha}, A^{e_\beta}]|^2 - 2\sum_{\alpha,\beta} S_{\alpha\beta}^2. \tag{2.2}$$

3 Evolution Equations

We now consider the MCF for a submanifold in $\mathbb{R}^{m+n}$. Namely, consider a one-parameter family $F_t = F(\,\cdot\,, t)$ of immersions $F_t : M \to \mathbb{R}^{m+n}$ with corresponding images $M_t = F_t(M)$ such that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}F(x,t) = H(x,t), \quad x \in M,\\
&F(x,0) = F(x)
\end{aligned} \tag{3.1}$$

are satisfied, where $H(x,t)$ is the mean curvature vector of M_t at $F(x,t)$ in $\mathbb{R}^{m+n}$. We also have

$$\frac{\mathrm{d}g_{ij}}{\mathrm{d}t} = -2\langle H, B_{ij}\rangle, \tag{3.2}$$

$$\frac{\mathrm{d}g^{ij}}{\mathrm{d}t} = 2\, g^{ik} g^{jl}\langle H, B_{kl}\rangle, \tag{3.3}$$

$$\frac{\mathrm{d}g}{\mathrm{d}t} = -2\,|H|^2\, g, \tag{3.4}$$

where $g = \det(g_{ij})$. We now derive the evolution equation for the squared norm of the second fundamental form.

Lemma 3.1 *The second fundamental form satisfies*

$$\Big(\frac{\mathrm{d}}{\mathrm{d}t} - \Delta\Big)|B|^2 \le -\,2\,|\nabla|B||^2 + 3|B|^4. \tag{3.5}$$

Remark 3.1 Compare (3.5) with the corresponding formula for the hypersurfaces, we see that now the curvature estimates are more delicate.

Proof For fixed x_0 and t_0, choose a local orthonormal frame $\{e_i\}$ of M_{t_0} near x_0 which is normal at x_0. By the immersion F_{t_0}, we have $\{e_i\}$ on M, which is not orthonormal in general. Then by F_t we obtain $\{F_{t*}e_i\}$ which is denoted by $\{e_i\}$ for simplicity. We also choose a local orthonormal frame field $\{e_\alpha\}$ of the normal bundle of M_t near x_0. Then at (x_0, t_0),

$$\begin{aligned}\frac{\mathrm{d}h_{\alpha ij}}{\mathrm{d}t} &= \overline{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}\langle \overline{\nabla}_{e_i}e_j, e_\alpha\rangle = \langle \overline{\nabla}_H\overline{\nabla}_{e_i}e_j, e_\alpha\rangle + \langle \overline{\nabla}_{e_i}e_j, \overline{\nabla}_H e_\alpha\rangle \\ &= \langle \overline{\nabla}_{e_i}\overline{\nabla}_{e_j}H, e_\alpha\rangle + \langle B_{ij}, \overline{\nabla}_H e_\alpha\rangle = \langle \overline{\nabla}_{e_i}(\nabla_{e_j}H + (\overline{\nabla}_{e_j}H)^T), e_\alpha\rangle + \langle B_{ij}, \overline{\nabla}_H e_\alpha\rangle \\ &= \langle \nabla_{e_i}\nabla_{e_j}H, e_\alpha\rangle - h_{\alpha ik}h_{\beta jk}H_\beta + h_{\beta ij}\langle \overline{\nabla}_H e_\alpha, e_\beta\rangle. \end{aligned}\tag{3.6}$$

Since in a non-orthonormal frame field $g_{ij} = \langle F_*e_i . F_*e_j\rangle$ (except at t_0) is not a unit matrix,

$$|B|^2 = g^{ik}g^{jl}h_{\alpha ij}\, h_{\alpha kl}.$$

We have, at (x_0, t_0),

$$\frac{\mathrm{d}|B|^2}{\mathrm{d}t} = 2\,\frac{\mathrm{d}g^{ik}}{\mathrm{d}t}h_{\alpha ij}h_{\alpha kj} + 2\,\frac{\mathrm{d}h_{\alpha ij}}{\mathrm{d}t}h_{\alpha ij}. \tag{3.7}$$

From (3.6) we have

$$\frac{\mathrm{d}h_{\alpha ij}}{\mathrm{d}t}h_{\alpha ij} = h_{\alpha ij}\langle \nabla_{e_i}\nabla_{e_j}H, e_\alpha\rangle - h_{\alpha ij}h_{\alpha ik}h_{\beta jk}H_\beta. \tag{3.8}$$

Noting (3.3), we have

$$\frac{\mathrm{d}g^{ik}}{\mathrm{d}t}h_{\alpha ij}h_{\alpha kj} = 2\,h_{\alpha ij}h_{\alpha kj}\langle H, B_{ik}\rangle = 2\,h_{\alpha ij}h_{\alpha kj}h_{\beta ik}H_\beta. \tag{3.9}$$

Substituting (3.8) and (3.9) into (3.7) gives

$$\frac{1}{2}\frac{\mathrm{d}|B|^2}{\mathrm{d}t} = \langle \nabla_i\nabla_j H, B_{ij}\rangle + \langle B_{ij}, H\rangle\langle B_{ik}, B_{jk}\rangle. \tag{3.10}$$

From (2.2) and (3.10), we obtain the evolution equation for the squared norm of the second fundamental form

$$\frac{1}{2}\Big(\frac{\mathrm{d}}{\mathrm{d}t}-\Delta\Big)|B|^2=-|\nabla B|^2+\sum_{\alpha\neq\beta}|[A^{e_\alpha},A^{e_\beta}]|^2+\sum_{\alpha,\beta}S^2_{\alpha\beta}. \tag{3.11}$$

We know from [14] in general that

$$\sum_{\alpha\neq\beta}|[A^{e_\alpha},A^{e_\beta}]|^2+\sum_{\alpha,\beta}S^2_{\alpha\beta}\le\Big(2-\frac{1}{n}\Big)|B|^4.$$

When the codimension $n\ge 2$, the above estimate was refined (see [4, 12])

$$\sum_{\alpha\neq\beta}|[A^{e_\alpha},A^{e_\beta}]|^2+\sum_{\alpha,\beta}S^2_{\alpha\beta}\le\frac{3}{2}|B|^4.$$

On the other hand, by the Schwartz inequality,

$$\nabla|B|\le|\nabla B|.$$

Therefore, the inequality (3.5) is obtained.

For any $p\in M$, let $\{e_1,\cdots,e_m\}$ be a local orthonormal frame field near p. Define the Gauss map $\gamma:p\to\gamma(p)$ which is obtained by parallel translation of T_pM to the origin in the ambient space $\mathbb{R}^{m+n}$. The image of the Gauss map lies in a Grassmannian $\mathbf{G}_{m,n}$. It is a symmetric space of compact type.

For any $P\in\mathbf{G}_{m,n}$, there are m vectors $v_1,\cdots,v_m$ spanning P. Then we have Plücker coordinates $v_1\wedge\cdots\wedge v_m$ for P up to a constant. The Gauss map γ can be described by $p\to e_1\wedge\cdots\wedge e_m$. Since

$$\begin{aligned}\mathrm{d}(e_1\wedge\cdots\wedge e_m)&=\mathrm{d}e_1\wedge\cdots\wedge e_m+\cdots+e_1\wedge\cdots\wedge\mathrm{d}e_m\\&=\omega_{\alpha 1}e_\alpha\wedge e_2\wedge\cdots\wedge e_m+\cdots+e_1\wedge\cdots\wedge e_{m-1}\wedge\omega_{\alpha m}e_\alpha\\&=\omega_{\alpha i}e_{\alpha i}\end{aligned}$$

and the canonical metric on $\mathbf{G}_{m,n}$ is defined by

$$\mathrm{d}s^2=\sum_{\alpha,i}\omega^2_{\alpha i},$$

where $\{e_{\alpha i}=e_1\wedge\cdots\wedge e_{i-1}\wedge e_\alpha\wedge e_{i+1}\wedge\cdots\wedge e_m\}$ is an orthonomal basis for $T\mathbf{G}_{m,n}$ (see [22, pp. 188–194]), it follows that

$$\gamma^*\omega_{\alpha i}=h_{\alpha ij}\omega_j \tag{3.12}$$

and the tension field of the Gauss map

$$\begin{aligned}\tau(\gamma)&=h_{\alpha ijj}e_{\alpha i}=h_{\alpha jji}e_{\alpha i}=h_{\alpha jji}e_1\wedge\cdots\wedge e_{i-1}\wedge e_\alpha\wedge e_{i+1}\wedge\cdots\wedge e_m\\&=\sum_i e_1\wedge\cdots\wedge e_{i-1}\wedge\nabla_{e_i}H\wedge e_{i+1}\wedge\cdots\wedge e_m,\end{aligned} \tag{3.13}$$

where we use the Codazzi equation. In [19], there is the following relation.

Proposition 3.1

$$\frac{\mathrm{d}\gamma}{\mathrm{d}t}=\tau(\gamma(t)). \tag{3.14}$$

4 Main Estimates

We consider the mean curvature flow of a complete manifold. We will assume that integration by parts is permitted and all integrals are finite for the submanifolds and functions we will consider in the sequel. We have the following maximum principle for parabolic equations on complete manifolds.

Define the backward heat kernel $\rho = \rho(x,t)$ by

$$\rho(x,t) = \frac{1}{(4\pi(t_0-t))^{\frac{n}{2}}} \exp\Big(-\frac{|x|^2}{4(t_0-t)}\Big), \quad t_0 > t,\ x \in \mathbb{R}^{m+n}.$$

We have the following formula. It is derived for the mean curvature flow in Euclidean space. By (3.4), the formula is unchanged in higher codimension.

Proposition 4.1 (See [10]) *For a function $f(x,t)$ on M, we have*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_M f\rho\,\mathrm{d}\mu_t = \int_M \Big(\frac{\mathrm{d}}{\mathrm{d}t}f - \Delta f\Big)\rho\,\mathrm{d}\mu_t - \int_M f\rho\Big|H + \frac{F^{\perp}}{2(t_0-t)}\Big|^2 \mathrm{d}\mu_t, \tag{4.1}$$

where $\mathrm{d}\mu_t$ is the volume form of M_t.

Corollary 4.1 (See [6]) *Suppose that the function $f = f(x,t)$ satisfies the inequality*

$$\Big(\frac{\mathrm{d}}{\mathrm{d}t} - \Delta\Big) f \le \langle \mathbf{a}, \nabla f\rangle$$

for some vector field $\mathbf{a}$ with uniformly bounded norm on $M \times [0,t_1]$ for some $t_1 > 0$. Then

$$\sup_{M_t} f \le \sup_{M_0} f \quad \text{for all } t \in [0,t_1].$$

Now, we consider the convex Gauss image situation which is preserved under the flow, as shown in the following theorem.

Theorem 4.1 (Confinable Property) *If the Gauss image of the initial submanifold M is contained in a geodesic ball of the radius $\rho_0 < \frac{\sqrt{2}}{4}\pi$ in $\mathbf{G}_{m,n}$, then the Gauss images of all the submanifolds under the mean curvature flow are also contained in the same geodesic ball.*

Proof We consider a smooth bounded function on $\mathbf{G}_{m,n}$

$$h = 1 + \varepsilon - \cos(\sqrt{2}\,\rho),$$

where ρ is the distance function from a point in $\mathbf{G}_{m,n}$, $\varepsilon > 0$ is a fixed constant. When $\rho < \frac{\sqrt{2}}{4}\pi$, h is convex. By the Hessian comparison theorem, we have

$$\mathrm{Hess}(h) \ge 2\cos(\sqrt{2}\rho)\, g, \tag{4.2}$$

where g is the metric tensor on $\mathbf{G}_{m,n}$. Hence, from (3.12) and (4.2) we have

$$\mathrm{Hess}(h)(\gamma_* e_i, \gamma_* e_i) \ge 2\cos(\sqrt{2}\,\rho)\,|B|^2.$$

The composition function $h \circ \gamma$ of h with the Gauss map γ defines a function on $M_t = F(M,t)$. Using Proposition 3.1, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(h\circ\gamma) = \mathrm{d}h\Big(\frac{\mathrm{d}\gamma}{\mathrm{d}t}\Big) = \mathrm{d}h(\tau(\gamma)).$$

By the composition formula (see [21, p. 28]),

$$\Delta(h\circ\gamma)=\mathrm{Hess}(h)(\gamma_* e_i,\gamma_* e_i)+\mathrm{d}h(\tau(\gamma)),$$

where $\{e_i\}$ is a local orthonormal frame field on M_t.

It follows that

$$\Big(\frac{\mathrm{d}}{\mathrm{d}t}-\Delta\Big)h\circ\gamma\le -2\cos(\sqrt{2}\rho\circ\gamma)\,|B|^2. \tag{4.3}$$

Thus, we can use Corollary 4.1 to get conclusion.

For simplicity $h\circ\gamma$ is denoted by h_1 in the sequel. On the other hand,

$$|\nabla h_1|^2=|\langle\nabla h,\gamma_* e_i\rangle\langle\nabla h,\gamma_* e_i\rangle|\le 2\sin^2(\sqrt{2}\,\rho\circ\gamma)|B|^2. \tag{4.4}$$

From (4.3) and (4.4), we have

$$\Big(\frac{\mathrm{d}}{\mathrm{d}t}-\Delta\Big)h_1\le -\cos(\sqrt{2}\,\rho\circ\gamma)\,|B|^2-\frac{\cos(\sqrt{2}\,\rho\circ\gamma)}{2\sin^2(\sqrt{2}\,\rho\circ\gamma)}|\nabla h_1|^2. \tag{4.5}$$

For any $q>0$,

$$\begin{aligned}\Big(\frac{\mathrm{d}}{\mathrm{d}t}-\Delta\Big)h_1^q&=qh_1^{q-1}\Big(\frac{\mathrm{d}}{\mathrm{d}t}-\Delta\Big)h_1-q(q-1)\,h_1^{q-2}|\nabla h_1|^2\\&\le -qh_1^{q-1}\cos(\sqrt{2}\rho\circ\gamma)|B|^2-\Big(q(q-1)h_1^{q-2}+qh_1^{q-1}\frac{\cos(\sqrt{2}\rho\circ\gamma)}{2\sin^2(\sqrt{2}\rho\circ\gamma)}\Big)|\nabla h_1|^2.\end{aligned} \tag{4.6}$$

From (3.5) and (4.6), we have

$$\begin{aligned}\Big(\frac{\mathrm{d}}{\mathrm{d}t}-\Delta\Big)(|B|^2h_1^q)&=|B|^2\Big(\frac{\mathrm{d}}{\mathrm{d}t}-\Delta\Big)h_1^q+h_1^q\Big(\frac{\mathrm{d}}{\mathrm{d}t}-\Delta\Big)|B|^2-2\,\nabla|B|^2\cdot\nabla h_1^q\\&\le(-q\cos(\sqrt{2}\,\rho\circ\gamma)+3h_1)|B|^4h_1^{q-1}\\&\quad-\Big[q(q-1)h_1^{q-2}+q\frac{\cos(\sqrt{2}\,\rho\circ\gamma)}{2\sin^2(\sqrt{2}\,\rho\circ\gamma)}h_1^{q-1}\Big]|B|^2|\nabla h_1|^2\\&\quad-2\,h_1^q|\nabla|B||^2-2\,\nabla|B|^2\cdot\nabla h_1^q\\&=[3(1+\varepsilon)-(3+q)\cos(\sqrt{2}\rho\circ\gamma)]h_1^{q-1}|B|^4\\&\quad-\Big[q(q-1)+q\frac{\cos(\sqrt{2}\rho\circ\gamma)}{2\sin^2(\sqrt{2}\rho\circ\gamma)}h_1\Big]h_1^{q-2}|B|^2|\nabla h_1|^2\\&\quad-2\,h_1^q|\nabla|B||^2-2\,\nabla|B|^2\cdot\nabla h_1^q.\end{aligned} \tag{4.7}$$

By using the Young inequality, we have

$$\begin{aligned}-2\nabla|B|^2\cdot\nabla h_1^q&=-(h_1^{-q}\nabla h_1^q)\cdot\nabla(|B|^2h_1^q)+|B|^2h_1^{-q}|\nabla h_1^q|^2-\nabla|B|^2\cdot\nabla h_1^q\\&\le -q(h_1^{-1}\nabla h_1)\cdot\nabla(|B|^2h_1^q)+q^2|B|^2h_1^{q-2}|\nabla h_1|^2\\&\quad+\frac{1}{2}q^2h_1^{q-2}|B|^2|\nabla h_1|^2+2\,h_1^q|\nabla|B||^2\\&\le -q(h_1^{-1}\nabla h_1)\cdot\nabla(|B|^2h_1^q)+\frac{3}{2}q^2h_1^{q-2}|B|^2|\nabla h_1|^2+2\,h_1^q|\nabla|B||^2.\end{aligned} \tag{4.8}$$

Thus, (4.7) becomes

$$\begin{aligned}\Big(\frac{\mathrm{d}}{\mathrm{d}t}-\Delta\Big)(|B|^2h_1^q)\le{}&[3(1+\varepsilon)-(3+q)\cos(\sqrt{2}\,\rho\circ\gamma)]|B|^4h_1^{q-1}\\&+\Big(\frac{1}{2}q+1-\frac{\cos(\sqrt{2}\,\rho\circ\gamma)}{2\sin^2(\sqrt{2}\,\rho\circ\gamma)}h_1\Big)q\,h_1^{q-2}|B|^2|\nabla h_1|^2\\&-q(h_1^{-1}\nabla h_1)\cdot\nabla(|B|^2h_1^q).\end{aligned}\tag{4.9}$$

We now give the following result.

Theorem 4.2 *Let M be a complete m-submanifold in $\mathbb{R}^{m+n}$ with bounded curvature. Suppose that the image under the Gauss map from M into $\mathbf{G}_{m,n}$ lies in a geodesic ball of radius $R_0<\frac{\sqrt{2}}{12}\pi$. If M_t is a smooth solution of (3.1), then there is the following estimate*

$$\sup_{M_t}|B|^2h_1^q\le\sup_{M_0}|B|^2h_1^q,\tag{4.10}$$

where q is a fixed constant depending on R_0.

Proof Let $r_0=\cos(\sqrt{2}\,R_0)$. Then $r_0>\frac{\sqrt{3}}{2}$. It follows that

$$\frac{3}{2r_0}-\frac{r_0}{2(1-r_0^2)}<0.$$

It is possible to choose $\varepsilon>0$ satisfying

$$\Big(\frac{3}{2r_0}-\frac{r_0}{2(1-r_0^2)}\Big)\varepsilon+\frac{3}{2r_0}-\frac{1}{2}-\frac{r_0}{2(1+r_0)}\le 0.\tag{4.11}$$

Set

$$q=3\Big(\frac{1+\varepsilon}{r_0}-1\Big).$$

Then for $r=\cos(\sqrt{2}\,\rho\circ\gamma)\ge r_0$,

$$3(1+\varepsilon)-(3+q)r=3(1+\varepsilon)-3(1+\varepsilon)\frac{r}{r_0}\le 0,$$

which implies that the first term of the right-hand side of (4.9) is non-positive. Note

$$\begin{aligned}\frac{1}{2}q+1-\frac{r}{2(1-r^2)}(1+\varepsilon-r)&=\frac{3}{2}\Big(\frac{1+\varepsilon}{r_0}-1\Big)+1-\frac{r}{2(1-r^2)}(1+\varepsilon-r)\\&=\Big(\frac{3}{2r_0}-\frac{r}{2(1-r^2)}\Big)\varepsilon+\frac{3}{2r_0}-\frac{1}{2}-\frac{r}{2(1+r)},\end{aligned}\tag{4.12}$$

which is non-increasing in r. By (4.11), (4.12) is non-positive when $r\ge r_0$. It follows that under the conditions of the theorem, (4.9) becomes

$$\Big(\frac{\mathrm{d}}{\mathrm{d}t}-\Delta\Big)(|B|^2h_1^q)\le -q(h_1^{-1}\nabla h_1)\cdot\nabla(|B|^2h_1^q).\tag{4.13}$$

From (4.4), we have

$$|h_1^{-1}\nabla h_1|\le\frac{\sqrt{2}\sin(\sqrt{2}\,\rho\circ\gamma)}{1+\varepsilon-\cos(\sqrt{2}\,\rho\circ\gamma)}|B|.$$

Let

$$f(\theta)=\frac{\sin\theta}{1+\varepsilon-\cos\theta}.$$

Since $f''(\theta)|_{f'(\theta)=0}\le 0$, we have

$$f(\theta)\le f(\theta)|_{f'(\theta)=0}=\frac{\sqrt{1-\frac{1}{(1+\varepsilon)^2}}}{1+\varepsilon-\frac{1}{1+\varepsilon}}=\frac{\sqrt{(1+\varepsilon)^2-1}}{(1+\varepsilon)^2-1}=\frac{\sqrt{\varepsilon(\varepsilon+2)}}{\varepsilon(\varepsilon+2)}. \tag{4.14}$$

It follows that

$$|h_1^{-1}\nabla h_1|\le\frac{\sqrt{2\,\varepsilon(\varepsilon+2)}}{\varepsilon(\varepsilon+2)}|B|.$$

Thus, we can use Corollary 4.1 and the estimate (4.10) is obtained.

Corollary 4.2 *Suppose that the image under the Gauss map from M into $\mathbf{G}_{m,n}$ lies in a geodesic ball of radius $R_0<\frac{\sqrt{2}}{12}\pi$. If M_t is a smooth solution of (3.1), then there is the following estimate*

$$\sup_{M_t}|B|^2\le\frac{c}{t}, \tag{4.15}$$

where c depends only on the bound of the Gauss image of its initial manifold.

Proof From (4.3),

$$\begin{aligned}\Big(\frac{\mathrm{d}}{\mathrm{d}t}-\Delta\Big)h_1^q&=qh_1^{q-1}\Big(\frac{\mathrm{d}}{\mathrm{d}t}-\Delta\Big)h_1-q(q-1)h_1^{q-2}|\nabla h_1|^2\\&\le-2qh_1^{q-1}\cos(\sqrt{2}\,\rho\circ\gamma)|B|^2-q(q-1)h_1^{q-2}|\nabla h_1|^2.\end{aligned}$$

Noting (4.13), we have

$$\begin{aligned}\Big(\frac{\mathrm{d}}{\mathrm{d}t}-\Delta\Big)(t|B|^2h_1^q+h_1^q)\le&-q(h_1^{-1}\nabla h_1)\cdot\nabla(t|B|^2h_1^q)+|B|^2h_1^q\\&-2qh_1^{q-1}\cos(\sqrt{2}\,\rho\circ\gamma)|B|^2-q^2h_1^{q-2}|\nabla h_1|^2+qh_1^{q-2}|\nabla h_1|^2.\end{aligned} \tag{4.16}$$

Since $q(h_1^{-1}\nabla h_1)\cdot\nabla h_1^q=q^2h_1^{q-2}|\nabla h_1|^2$, (4.16) becomes

$$\begin{aligned}\Big(\frac{\mathrm{d}}{\mathrm{d}t}-\Delta\Big)(t|B|^2h_1^q+h_1^q)\le&-q(h_1^{-1}\nabla h_1)\cdot\nabla(t|B|^2h_1^q+h_1^q)+|B|^2h_1^q\\&-2qh_1^{q-1}\cos(\sqrt{2}\rho\circ\gamma)|B|^2+qh_1^{q-2}|\nabla h_1|^2.\end{aligned}$$

Noting (4.4), the above inequality becomes

$$\begin{aligned}\Big(\frac{\mathrm{d}}{\mathrm{d}t}-\Delta\Big)(t|B|^2h_1^q+h_1^q)\le&-q(h_1^{-1}\nabla h_1)\cdot\nabla(t|B|^2h_1^q+h_1^q)\\&+|B|^2h_1^{q-2}(h_1^2-2qh_1\cos(\sqrt{2}\,\rho\circ\gamma)+2q\sin^2(\sqrt{2}\,\rho\circ\gamma)).\end{aligned} \tag{4.17}$$

Let

$$\begin{aligned}A(r)&=(h_1^2-2qh_1\cos(\sqrt{2}\,\rho\circ\gamma)+2q\sin^2(\sqrt{2}\,\rho\circ\gamma))\\&=(1+\varepsilon-r)^2-2q(1+\varepsilon-r)r+2q(1-r^2)\\&=(1+\varepsilon-r)^2-2q(r+\varepsilon r-1),\end{aligned} \tag{4.18}$$

where $r=\cos(\sqrt{2}\,\rho\circ\gamma)$. Since $A'(r)<0$ and $q=3\big(\frac{1+\varepsilon}{r_0}-1\big)$, for $r\ge r_0$ we have

$$A\le(1+\varepsilon-r_0)^2-2q(r_0+\varepsilon r_0-1)=(1+\varepsilon-r_0)\Big(\frac{6}{r_0}-r_0-5-5\varepsilon\Big).$$

We know that ε is chosen by (4.11). If necessary we choose ε larger such that $A\le 0$. Therefore, from (4.17) we have

$$\Big(\frac{\mathrm{d}}{\mathrm{d}t}-\Delta\Big)(t|B|^2h_1^q+h_1^q)\le -q(h_1^{-1}\nabla h_1)\cdot\nabla(t|B|^2h_1^q+h_1^q)$$

and by Corollary 4.1 again we have the desired estimate.

5 Proof of the Main Theorem

We are now in a position to prove the following theorem.

Theorem 5.1 *Let $F:M\to\mathbb{R}^{m+n}$ be a complete m-submanifold which has bounded curvature. Suppose that the image under the Gauss map from M into $\mathbf{G}_{m,n}$ lies in a geodesic ball of radius $R_0<\frac{\sqrt{2}}{12}\pi$. Then the mean curvature flow equations* (3.1) *have a long time smooth solution.*

Proof Let $P_0\in\mathbf{G}_{m,n}$ be a fixed point which is described by

$$P_0=\varepsilon_1\wedge\cdots\wedge\varepsilon_m,$$

where $\varepsilon_1,\cdots,\varepsilon_m$ are orthonormal vectors in $\mathbb{R}^{m+n}$. Choose complementary orthonormal vectors $\varepsilon_{m+1},\cdots,\varepsilon_{m+n}$, such that $\{\varepsilon_1,\cdots,\varepsilon_m,\varepsilon_{m+1},\cdots,\varepsilon_{m+n}\}$ is an orhtonormal base in $\mathbb{R}^{m+n}$.

Let $p:\mathbb{R}^{m+n}\to\mathbb{R}^m$ be the natural projection defined by

$$p(x^1,\cdots,x^m;x^{m+1},\cdots,x^{m+n})=(x^1,\cdots,x^m),$$

which induces a map from M to $\mathbb{R}^m$. It is a smooth map from a complete manifold to $\mathbb{R}^m$.

For any point $x\in M$ choose a local orhtonormal tangent frame field $\{e_1,\cdots,e_m\}$ near x. Let $v=v_ie_i\in TM$. Its projection

$$p_*v=\langle v_ie_i,\varepsilon_j\rangle\varepsilon_j=v_i\langle e_i,\varepsilon_j\rangle\varepsilon_j.$$

Now, we consider the case of the image under the Gauss map γ containing in a geodesic ball of radius $R_0<\frac{\sqrt{2}}{12}\pi$ and centered at P_0. For any $P\in\gamma(M)$,

$$w\overset{\text{def.}}{=}\langle P,P_0\rangle=\langle e_1\wedge\cdots\wedge e_m,\varepsilon_1\wedge\cdots\wedge\varepsilon_m\rangle=\det W,$$

where $W=(\langle e_i,\varepsilon_j\rangle)$. The Jordan angles between P and P_0 are

$$\theta_i=\cos^{-1}(\lambda_i),$$

where λ_i^2 are eigenvalues of the symmetric matrix W^TW (see [20]). It is well known that

$$W^TW=O^T\Lambda O,$$

where O is an orthogonal matrix and

$$\Lambda=\begin{pmatrix}\lambda_1^2 & & 0\\ & \ddots & \\ 0 & & \lambda_r^2\end{pmatrix},\qquad r=\min(m,n),$$

where each $0 \le \lambda_i^2 < 1$. We know that

$$w = \prod \cos\theta_i.$$

On the other hand, the distance between P_0 and P (see [22, pp. 188–194])

$$\mathrm{d}(P_0, P) = \sqrt{\sum \theta_i^2},$$

which is less than $\frac{\sqrt{2}}{12}\pi$ by the assumption. It follows that

$$w > w_0 = \Big(\cos\frac{\sqrt{2}}{12}\,\pi\Big)^r.$$

We now compare the length of any tangent vector v to M with its projection $p_* v$.

$$|p_* v|^2 = \sum_{j=1}^{m} (v_i \langle e_i, \varepsilon_j\rangle)^2 = (WV)^T WV,$$

where $V = (v^1, \cdots, v^m)^T$. Hence,

$$|p_* v|^2 \ge (\lambda')^2 |v|^2 > w^2 |v|^2 > w_0^2 |v|^2, \tag{5.1}$$

where $\lambda' = \min_i\{\lambda_i\}$. The induced metric $\mathrm{d}s^2$ on M from $\mathbb{R}^{m+n}$ is complete, so is the homothetic metric $\widetilde{\mathrm{d}}s^2 = w_0^2\,\mathrm{d}s^2$. (5.1) implies

$$p : (M, \widetilde{\mathrm{d}}s^2) \to (\mathbb{R}^m, \text{canonical metric})$$

increases the distance. It follows that p is a covering map from a complete manifold into $\mathbb{R}^m$, and a deffeomorphism, since $\mathbb{R}^m$ is simply connected. Hence, the induced Riemannian metric on M can be expressed as $(\mathbb{R}^m, \mathrm{d}s^2)$ with

$$\mathrm{d}s^2 = g_{ij}\,\mathrm{d}x^i \mathrm{d}x^j.$$

Furthermore, the immersion $F : M \to \mathbb{R}^{m+n}$ is realized by a graph $(x, f(x))$ with $f : \mathbb{R}^m \to \mathbb{R}^n$ and

$$g_{ij} = \delta_{ij} + \frac{\partial f^\alpha}{\partial x^i}\frac{\partial f^\alpha}{\partial x^j}.$$

It follows that any eigenvalue of (g_{ij}) is not less than 1.

At each point in M, its image m-plane P under the Gauss map is spanned by

$$f_i = \varepsilon_i + \frac{\partial f^\alpha}{\partial x^i}\varepsilon_\alpha.$$

It follows that

$$|f_1 \wedge \cdots \wedge f_m|^2 = \det\Big(\delta_{ij} + \sum_\alpha \frac{\partial f^\alpha}{\partial x^i}\frac{\partial f^\alpha}{\partial x^j}\Big), \quad \sqrt{g} = |f_1 \wedge \cdots \wedge f_m|.$$

The m-plane P is also spanned by

$$p_i = g^{-\frac{1}{2m}} f_i.$$

Furthermore,

$$|p_1 \wedge \cdots \wedge p_m| = 1.$$

We then have

$$\langle P, P_0\rangle = \det(\langle \varepsilon_i, p_j\rangle) = \begin{pmatrix} g^{-\frac{1}{2m}} & & 0 \\ & \ddots & \\ 0 & & g^{-\frac{1}{2m}} \end{pmatrix} = \frac{1}{\sqrt{g}} > w_0$$

and

$$\sqrt{g} \le \frac{1}{w_0}.$$

Thus, we prove that any eigenvalue of $(g_{ij}) \le \frac{1}{w_0^2}$. Noting Theorem 4.1, we know that the equation (3.1) is uniformly parabolic and has a unique smooth solution on some short time interval. By the curvature estimate (see Theorems 4.1 and 4.2), we have uniform estimate on $|B|$. Then we can proceed as in [10, Proposition 2.3]) to estimate all derivatives of B in terms of their initial data

$$\sup_{M_t} |\nabla^q B| \le C(m),$$

where $C(m)$ only depends on q, m and $\sup\limits_{M_0} |\nabla^j B|$ for $0 \le j \le q$. It follows that this solution can be extended to all $t > 0$.

We assume $0 \in M$ and define coordinate functions

$$x^i = \langle F, \varepsilon_i\rangle, \quad y^\alpha = \langle F, \varepsilon_\alpha\rangle.$$

Denote

$$X = \sqrt{\sum_{i=1}^m (x^i)^2}\,, \quad Y = \sqrt{\sum_{\alpha=m+1}^{m+n} (y^\alpha)^2}\,.$$

It is easy to verify that

$$\Big(\frac{\mathrm{d}}{\mathrm{d}t} - \Delta\Big) y^\alpha = 0, \quad \Big(\frac{\mathrm{d}}{\mathrm{d}t} - \Delta\Big) Y^2 = -2\sum |\nabla y^\alpha|^2 \le 0.$$

Corollary 4.1 implies that if the height function of M_0 is finite, then the height function of M_t is also finite under the evolution.

If the height function is going to infinity, we can consider rescaled mean curvature flow as done in [6]. Define

$$\widetilde{F}(\widetilde{t}) = \frac{1}{\sqrt{2t+1}} F(t),$$

where $\widetilde{t} = \log(2t+1)$. Hence

$$\frac{\partial}{\partial \widetilde{t}} \widetilde{F} = \widetilde{H} - \widetilde{F}.$$

It is not hard to verify that the Gauss map $\widetilde{\gamma}$ of the rescaled mean curvature flow is as same as the original γ. Furthermore, the previous estimates (4.15) translate to

$$|\widetilde{A}|^2 \le (2t+1)|A|^2 \le C$$

which is dependent on the initial bound on M.

Choose an orthonormal frame field near $p \in M$ along M in $\mathbb{R}^{m+n}$, such that $e_i \in TM$ and $e_\alpha \in NM$ with $\nabla_{e_j} e_i|_p = \nabla_{e_i} e_\alpha|_p = 0$. We have

Lemma 5.1

$$\Big(\frac{\mathrm{d}}{\mathrm{d}t}-\Delta\Big)\langle F,e_\alpha\rangle = 2\,\langle H,e_\alpha\rangle + S_{\alpha\beta}\langle F,e_\beta\rangle + C_{\alpha\beta}\langle F,e_\beta\rangle$$

with anti-symmetric $C_{\alpha\beta}$ in α and β, and

$$\begin{aligned}\Big(\frac{\mathrm{d}}{\mathrm{d}t}-\Delta\Big)\sum_\alpha\langle F,e_\alpha\rangle^2 &= 4\,\langle H,e_\alpha\rangle\langle F,e_\alpha\rangle - 2\sum_\alpha|\nabla\langle F,e_\alpha\rangle|^2 + 2\,S_{\alpha\beta}\langle F,e_\alpha\rangle\langle F,e_\beta\rangle\\ &\le C\Big(\sum_\alpha\langle F,e_\alpha\rangle^2+1\Big) - 2\sum_\alpha|\nabla\langle F,e_\alpha\rangle|^2. \end{aligned}\tag{5.2}$$

Proof Since at the point p

$$\begin{aligned}\nabla_{e_i}A^{e_\alpha}(e_i) &= \nabla_{e_i}\langle B_{ij},e_\alpha\rangle e_j = \nabla_{e_i}(\langle\overline{\nabla}_{e_j}e_i,e_\alpha\rangle e_j) = \langle\overline{\nabla}_{e_j}\overline{\nabla}_{e_i}e_i,e_\alpha\rangle e_j\\ &= \langle\overline{\nabla}_{e_j}(\nabla_{e_i}e_i+B_{ii}),e_\alpha\rangle e_j = \langle B_{e_j\nabla_{e_i}e_i},e_\alpha\rangle + \langle\overline{\nabla}_{e_j}H,e_\alpha\rangle e_j = \langle\nabla_{e_j}H,e_\alpha\rangle e_j,\end{aligned}\tag{5.3}$$

we have

$$\begin{aligned}\Delta\langle F,e_\alpha\rangle &= e_ie_i\langle F,e_\alpha\rangle = -e_i\langle F,A^{e_\alpha}(e_i)\rangle = -\langle e_i,A^{e_\alpha}(e_i)\rangle - \langle F,\overline{\nabla}_{e_i}A^{e_\alpha}(e_i)\rangle\\ &= -\langle H,e_\alpha\rangle - \langle F,\nabla_{e_i}A^{e_\alpha}(e_i)\rangle - \langle F,B_{e_iA^{e_\alpha}(e_i)}\rangle\\ &= -\langle H,e_\alpha\rangle - \langle\nabla_{e_i}H,e_\alpha\rangle\langle F,e_i\rangle - \langle F,B_{e_iA^{e_\alpha}(e_i)}\rangle\\ &= -\langle H,e_\alpha\rangle - \langle\nabla_{e_i}H,e_\alpha\rangle\langle F,e_i\rangle - \langle F,B_{ij}\rangle\langle B_{ij},e_\alpha\rangle\\ &= -\langle H,e_\alpha\rangle - \langle\nabla_{e_i}H,e_\alpha\rangle\langle F,e_i\rangle - S_{\alpha\beta}\langle F,e_\beta\rangle.\end{aligned}\tag{5.4}$$

On the other hand,

$$\Big\langle\frac{\mathrm{d}e_\alpha}{\mathrm{d}t},e_i\Big\rangle = -\Big\langle e_\alpha,\frac{\mathrm{d}e_i}{\mathrm{d}t}\Big\rangle = -\langle e_\alpha,\nabla_{e_i}H\rangle,\quad \frac{\mathrm{d}e_\alpha}{\mathrm{d}t} = -\langle e_\alpha,\nabla_{e_i}H\rangle e_i + C_{\alpha\beta}e_\beta$$

with anti-symmetric $C_{\alpha\beta}$ in α, β. It follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle F,e_\alpha\rangle = \langle H,e_\alpha\rangle + \Big\langle F,\frac{\mathrm{d}e_\alpha}{\mathrm{d}t}\Big\rangle = \langle H,e_\alpha\rangle - \langle e_\alpha,\nabla_{e_i}H\rangle\langle F,e_i\rangle + C_{\alpha\beta}\langle F,e_\beta\rangle.\tag{5.5}$$

Furthermore, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\sum_\alpha\langle F,e_\alpha\rangle^2 &= 2\sum_\alpha\langle F,e_\alpha\rangle\frac{\mathrm{d}}{\mathrm{d}t}\langle F,e_\alpha\rangle\\ &= 2\sum_\alpha\langle H,e_\alpha\rangle\langle F,e_\alpha\rangle - 2\sum_\alpha\langle\nabla_{e_i}H,e_\alpha\rangle\langle F,e_i\rangle\langle F,e_\alpha\rangle + 2C_{\alpha\beta}\langle F,e_\beta\rangle\langle F,e_\alpha\rangle\\ &= 2\sum_\alpha\langle H,e_\alpha\rangle\langle F,e_\alpha\rangle - 2\sum_\alpha\langle\nabla_{e_i}H,e_\alpha\rangle\langle F,e_i\rangle\langle F,e_\alpha\rangle,\end{aligned}\tag{5.6}$$

and

$$\begin{aligned}\Delta\langle F,e_\alpha\rangle^2 &= 2\,|\nabla\langle F,e_\alpha\rangle|^2 + 2\,\langle F,e_\alpha\rangle\Delta\langle F,e_\alpha\rangle\\ &= 2\,|\nabla\langle F,e_\alpha\rangle|^2 + 2\,\langle F,e_\alpha\rangle(-\langle H,e_\alpha\rangle - \langle\nabla_{e_i}H,e_\alpha\rangle\langle F,e_i\rangle - S_{\alpha\beta}\langle F,e_\beta\rangle).\end{aligned}\tag{5.7}$$

Hence,

$$\Big(\frac{\mathrm{d}}{\mathrm{d}t}-\Delta\Big)\sum_\alpha\langle F,e_\alpha\rangle^2 = 4\,\langle H,e_\alpha\rangle\langle F,e_\alpha\rangle - 2\sum_\alpha|\nabla\langle F,e_\alpha\rangle|^2 + 2\,S_{\alpha\beta}\langle F,e_\alpha\rangle\langle F,e_\beta\rangle.\tag{5.8}$$

Noting the estimates of $|H|^2$, $|B|^2$ and $S_{\alpha\beta}$, we obtain the desired estimate.

Then, we can proceed in the same way as in [6] to derive

Theorem 5.2 *Let $F : M \to \mathbb{R}^{m+n}$ be a complete m-submanifold with bounded curvature. Suppose that the image under the Gauss map from M into $\mathbf{G}_{m,n}$ lies in a geodesic ball of radius $R_0 < \frac{\sqrt{2}}{12}\pi$. If in addition assume that*

$$\sum_\alpha \langle F, e_\alpha\rangle^2 \le C'(1+|F|^2)^{1-\delta}$$

is valid on M for some constants $C' < \infty, \delta > 0$, then the solution $\widetilde{M}_{\widetilde{t}}$ of the rescaled equation converges for $\widetilde{t} \to \infty$ to a limiting submanifold $\widetilde{M}_\infty$ satisfying the equation

$$F^\perp = H.$$

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