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Lipschitz Properties in Variable Exponent Problems via Relative Rearrangement

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(*Dedicated to Professor Roger Temam on the Occasion of his* 70*th Birthday*)

Abstract The author first studies the Lipschitz properties of the monotone and relative rearrangement mappings in variable exponent Lebesgue spaces completing the result given in [9]. This paper is ended by establishing the Lipschitz properties for quasilinear problems with variable exponent when the right-hand side is in some dual spaces of a suitable Sobolev space associated to variable exponent.

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1 Introduction

The notion of relative rearrangement introduced by J. Mossino and R. Temam turns out to be an important tool for studying problems involving monotone rearrangement. Here, again, we shall use it to show the Lipschitz property of some maps. The first one is based on the following lemma proved in [23, 19].

Lemma 1.1 *Let Ω be a bounded open set of $\mathbb{R}^N$, $u \in L^1(\Omega)$, $v \in L^\infty(\Omega)$. Then, for a.e. $s \in \Omega_* =]0, |\Omega|[$,*

$$(u+v)_*(s) - u_*(s) = \int_0^1 v_{*(u+tv)}(s)\mathrm{d}t.$$

Here, $v_{*(u+tv)}$ is the relative rearrangement of v with respect to $u+tv$, and u_* (resp. $(u+v)_*$) is the decreasing rearrangement of u (resp. $u+v$).

From which, we shall derive that

$$\|\,|u_* - v_*|_*\|_{p^*(\cdot)} \le c_L \|u-v\|_{p(\cdot)}, \quad \forall\, u \in L^{p(\cdot)}(\Omega),\ \forall\, v \in L^{p(\cdot)}(\Omega).$$

See below for the definitions of $L^{p(\cdot)}(\Omega)$, u_* (resp v_* and p^*). We recall that, such inequality is not true if we replace the increasing rearrangement of p, $p^*(\cdot)$ by its decreasing rearrangement $p_*(\cdot)$. The counterexample is given in [9].

The second application of the relative rearrangement concerns the pointwise estimates in PDE. We recall that the recent development of the study on variable exponent is partly due

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to the fact that it has a connection with some model in fluids mechanics (see [2]) where the operator given by

$$\widehat{a}(x, \nabla u) = (1 + |\nabla u|^2)^{\frac{p(x)-2}{2}} \nabla u$$

is considered. Many results (see [4, 5, 7, 9, 11–13, 16]) have been given for the Lebesgue space with variable exponent define by

$$L^{p(\cdot)}(\Omega) = \Big\{ u : \Omega \to \mathbb{R},\ \Phi_p(u) = \int_\Omega |u(x)|^{p(x)} \mathrm{d}x < +\infty \Big\},$$

where $p : \Omega \to [1, +\infty[$ is a bounded measurable function.

The space $L^{p(\cdot)}(\Omega)$ is endowed with the modular norm:

$$\|u\|_{p(\cdot)} = \inf \Big\{ \lambda > 0 : \Phi_p\Big(\frac{u}{\lambda}\Big) \le 1 \Big\}.$$

We shall summarize the properties that we shall use in the next section.

We shall consider two types of model of the form

$$-\mathrm{div}\, \big(\widehat{a}(x, \nabla u)\big) = -\mathrm{div}\, (F). \tag{1.1}$$

The first one shall be considered under the following growth condition on $\widehat{a}$:

$$[\widehat{a}(x, \xi) - \widehat{a}(x, \xi')] \cdot [\xi - \xi'] \ge \alpha_0 (1 + |\xi| + |\xi'|)^{p(x)-2} |\xi - \xi'|^2$$

for a.e. $x \in \Omega$, $\forall\, \xi$, $\xi' \in \mathbb{R}^N$ if $p(x) \ge 2$, some $\alpha_0 > 0$. In that case the function

$$u \in W_0^{1,p(\cdot)}(\Omega), \quad F = (f_1, \cdots, f_n), \quad f_i \in L^{q(\cdot)}(\Omega), \quad \frac{1}{p(x)} + \frac{1}{q(x)} = 1, \quad \text{a.e.}$$

The second model concerns the equation of the form

$$\widehat{a}(x, \nabla u) = \Big(a_1\Big(x, \frac{\partial u}{\partial x_1}\Big); \cdots ; a_N\Big(x, \frac{\partial u}{\partial x_N}\Big)\Big).$$

Each a_i has its one growth as for instance, $\forall\, t \in \mathbb{R}$, $\forall\, \sigma \in \mathbb{R}$,

$$(a_i(x, t) - a_i(x, \sigma))(t - \sigma) \ge \alpha_0 (1 + |t| + |\sigma|)^{p_i(x)-2} |t - \sigma|^2$$

for a.e. $x \in \Omega$, for some $\alpha_0 > 0$, $p(x) \ge 2$.

We shall consider the solution $u \in W^{1,p_1(\cdot),\cdots,p_N(\cdot)}(\Omega) \cap W_0^{1,1}(\Omega)$, and F is in an adequate space.

We shall prove some pointwise inequalities related to the difference of two solutions u_1, u_2 of (1.1) say, if $w = |u_1 - u_2|$, then we shall show for instance

$$-\frac{\mathrm{d}w_*}{\mathrm{d}s}(s) \le c_N(\alpha_0) s^{\frac{1}{N}-1} ([|\delta F|^2]_{*w}(s))^{\frac{1}{2}} \quad \text{ for a.e. } s,\ \delta F = F_1 - F_2,$$

from which we shall derive the Lipschitz property for equation (1.1).

2 Notation and Preliminary Results

For our purpose, we consider (for simplicity) Ω an open bounded set and $p : \Omega \to [1, +\infty[$ a measurable function. We shall denote by u_* (resp. u^*) the decreasing (resp. increasing) rearrangement of a measurable function $u : \Omega \to \mathbb{R}$ that is the generalized inverse of the distribution function given by

$$t \to |\{u > t\}| = \text{ measure}\{u \in \Omega : u(x) > t\} \quad (u^*(s) = -(-u)_*(s),\ \forall\, s \in]0, |\Omega|[= \Omega_*).$$

As usual, we set $|E|$ the Lebesgue measure of a measurable set E, and χ_E its characteristic function.

The scalar product of two vectors $X,\ Y$ in $\mathbb{R}^N$ shall be denoted by (X, Y) or $X \cdot Y$ and the associated norm $|X| = \sqrt{X \cdot X}$.

Setting

$$\Phi_p(u) = \int_\Omega |u(x)|^{p(x)} \mathrm{d}x \le +\infty,$$

we consider the norm

$$\|u\|_{p(\cdot)} = \inf \Big\{ \lambda > 0 : \Phi_p\Big(\frac{u}{\lambda}\Big) \le 1 \Big\} \tag{2.1}$$

and

$$L^{p(\cdot)}(\Omega) = \{u : \Omega \to \mathbb{R} \text{ measurable such that } |u|_{p(\cdot)} < +\infty\}.$$

The space $(L^{p(\cdot)}(\Omega); |\cdot|_{p(\cdot)})$ is a Banach function space and an equivalent norm for u is the following Amemiya norm:

$$\| \, |u| \|_{p(\cdot)} = \inf_{\lambda > 0} \lambda \Big(1 + \Phi_p\Big(\frac{u}{\lambda}\Big)\Big). \tag{2.2}$$

More precisely, one has

$$\|u\|_{p(\cdot)} \le \| \, |u| \|_{p(\cdot)} \le 2\|u\|_{p(\cdot)}. \tag{2.3}$$

We set

$$L^1_+(\Omega) = \{v \in L^1(\Omega) : v \ge 0\} \quad \text{and} \quad L^{p(\cdot)}_+(\Omega) = L^{p(\cdot)}(\Omega) \cap L^1_+(\Omega).$$

We recall also that if $v \in L^1(\Omega),\ u \in L^1(\Omega)$ then $\lim\limits_{\lambda \searrow 0} \frac{(u+\lambda v)_* - u_*}{\lambda}$ exists in a weak sense. This limit is called the relative rearrangement of v with respect to $u : v_{*u}$.

More precisely, we have (see [6, 15, 19, 21, 22])

Theorem 2.1 *Let Ω be a bounded measurable set in $\mathbb{R}^N$, u, v be two functions in $L^1(\Omega)$ and $\omega : \overline{\Omega}_* \to \mathbb{R}$ be defined by*

$$\omega(s) = \int_{\{u > u_*(s)\}} v(x) \mathrm{d}x + \int_0^{s - |u > u_*(s)|} (v|_{\{u = u_*(s)\}})_*(\sigma) \mathrm{d}\sigma,$$

where $v|_{\{u=u_*(s)\}}$ is the restriction of v to $\{u = u_*(s)\}$. Then one has $\frac{(u+\lambda v)_* - u_*}{\lambda} \underset{\lambda \to 0}{\rightharpoonup} \frac{\mathrm{d}\omega}{\mathrm{d}s}$ weakly in $L^p(\Omega_*)$ if $v \in L^p(\Omega)$, p is a constant with $1 \le p < +\infty$ and in $L^\infty(\Omega_*)$-weak-star if $p = +\infty$.

Moreover, $|\frac{\mathrm{d}\omega}{\mathrm{d}s}|_{L^p(\Omega_*)} \le |v|_{L^p(\Omega)}$ and $\int_{\Omega_*} \frac{\mathrm{d}\omega}{\mathrm{d}s} \mathrm{d}s = \int_\Omega v(x) \mathrm{d}x$.

See [1, 8, 9] for other aspects and properties.

Definition 2.1 *Under the same notations as Theorem* 2.1 *the relative rearrangement of* v *with respect to* u *is* $\frac{\mathrm{d}\omega}{\mathrm{d}s}$ *and is denoted by* v_{*u}. *In particular, one has*

$$\text{if } v_1 \le v_2 \text{ then } v_{1*u} \le v_{2*u}, \quad v_i \in L^1(\Omega).$$

Set $\Omega(u) = \{x \in \Omega : |\{u = u(x)\}| = 0\}$. *Then for a.e.* $s \in \Omega_*$,

$$[\big(v_1 v_2 \chi_{\Omega(u)}\big)_{*u}(s)]^2 \le \big(v_1^2 \chi_{\Omega(u)}\big)_{*u}(s)\big(v_2^2 \chi_{\Omega(u)}\big)_{*u}(s), \quad \text{if } v_i \in L^2(\Omega),\ i = 1, 2.$$

One property that we shall use for the relative rearrangement is

Proposition 2.1 *Let* $v \ge 0,\ u$ *be two functions in* $L^1(\Omega)$. *Then*

$$(v_{*u})_{**} \le v_{**}.$$

Here

$$v_{**}(s) = \frac{1}{s}\int_0^s v_*(\sigma)\mathrm{d}\sigma, \quad s \in \Omega_*.$$

There is a link between the derivative of u_* and relative rearrangement of the gradient of u as it was proved in [17–19, 21, 22]. We will use only the following results.

Theorem 2.2 (PSR Inequality: Poincaré-Sobolev Inequality for the Relative Rearrangement)

(a) *Let* $u \in W_0^{1,1}(\Omega),\ u \ge 0$. *Then* $u_* \in W^{1,1}_{\mathrm{loc}}(]0, |\Omega|[\,)$,

$$-u_*'(s) \le \frac{s^{\frac{1}{N}-1}}{N\alpha_N^{\frac{1}{N}}}|\nabla u|_{*u}(s) \quad \text{a.e. in } \Omega_*$$

and

$$-u_{**}'(s) \le \frac{s^{\frac{1}{N}-1}}{N\alpha_N^{\frac{1}{N}}}(|\nabla u|_{*u})_{**}(s) \quad \text{a.e. in } \Omega_*,$$

where α_N *is the measure of the unit ball in* $\mathbb{R}^N$.

(b) *Let* $u \in W^{1,1}(\Omega)$. *Then* $u_* \in W^{1,1}_{\mathrm{loc}}(]0, |\Omega|[\,)$,

$$-u_*'(s) \le \frac{\min(s, |\Omega| - s)^{\frac{1}{N}-1}}{Q(\Omega)}|\nabla u|_{*u}(s) \quad \text{a.e. in } \Omega_*,$$

provided that Ω *is a Lipschitz connected open set of* $\mathbb{R}^N$. *Here,* $Q(\Omega)$ *is a positive constant depending only on* Ω.

The following results are proved in [9].

Theorem 2.3 *Let* $u : \Omega \to \mathbb{R}_+$ *and* $p : \Omega \to [1, +\infty[$ *be two measurable functions. Then*

$$\frac{1}{2(1 + |\Omega|)}\|u_*\|_{p^*(\cdot)} \le \|u\|_{p(\cdot)} \le 2(1 + |\Omega|)\|u_*\|_{p_*(\cdot)},$$

where u_* *(resp.* p_**) is the decreasing rearrangement of* u *(resp.* p*) and* p^* *the increasing rearrangement of* p.

Theorem 2.4 *Let $p : \Omega \to [1, +\infty[$ be a bounded measurable function. Assume that the increasing rearrangement of p, p^* satisfies: $1 < p^*(0)$ and that in a neighborhood of the origin 0, we have $|p^*(s) - p^*(t)| \le \frac{A}{|\operatorname{Ln}|s-t||}$ for some $A > 0$. Thus, for all $v \ge 0$ and u in $L^1(\Omega)$, if $v \in L^{p(\cdot)}(\Omega)$, then $(v_{*u})_{**} \in L^{p^*(\cdot)}(\Omega_*)$ and $(v_{*u})_* \in L^{p^*(\cdot)}(\Omega_*)$.*

Moreover, there exist two constants $c_1 > 0$, $c_2 > 0$ such that

$$\|(v_{*u})_*\|_{p^*(\cdot)} \le \|(v_{*u})_{**}\|_{p^*(\cdot)} \le c_1 \|v_*\|_{p^*(\cdot)} \le c_2 \|v\|_{p(\cdot)}.$$

Lemma 2.1 *Under the same assumptions of Theorem 3.1, one has for all $\lambda > 0$,*

$$\int_{\{u_*>\lambda\}} \Big(\frac{u_*}{\lambda}\Big)^{p^*(s)}(s)\mathrm{d}s \le \int_{\{u>\lambda\}} \Big(\frac{u}{\lambda}\Big)^{p(x)}(x)\mathrm{d}x \le \int_{\{u_*>\lambda\}} \Big(\frac{u_*}{\lambda}\Big)^{p_*(s)}(s)\mathrm{d}s.$$

Finally, we have the following theorem (see [19]).

Theorem 2.5 *Let $u \in W^{1,1}(\Omega)$ with Ω being an open bounded connected Lipschitz set if $\gamma_0 u \not\equiv 0$ (the trace of u on the boundary) and Ω is an arbitrary open set otherwise. We assume that the measure $\{x \in \Omega : \nabla u(x) = 0\} = 0$. Then for any sequence u_n converging to $u \in W^{1,1}(\Omega)$ with $\gamma_0 u_n = 0$ if $\gamma_0 u = 0$, and for any $b \in L^p(\Omega)$, $1 \le p < +\infty$, we have $b_{*u_n} \xrightarrow[n\to+\infty]{} b_{*u}$ strongly in $L^p(\Omega_*)$.*

3 Main Results

3.1 On the Lipschitz property of the mappings $u \in L^{p(\cdot)}(\Omega) \to u_* \in L^{p^*(\cdot)}(\Omega_*)$ and $v \in L^{p(\cdot)}(\Omega) \to (v_{*u})_* \in L^{p^*(\cdot)}(\Omega_*)$

Theorem 3.1 *Let $p : \Omega \to [1, +\infty[$ be a measurable bounded function such that the increasing rearrangement p^* of p satisfies $p^*(0) > 1$ and $|p^*(t) - p^*(\sigma)|\,|\operatorname{Ln}|t - \sigma|| \le A$ near zero for some constant $A > 0$. Then there exists a constant $c_L > 0$ such that*

$$\|[(u + v)_* - u_*]_{**}\|_{p^*(\cdot)} \le c_L \|v\|_{p(\cdot)},$$

$\forall\, u \in L^1(\Omega)$, $\forall\, v \in L^{p(\cdot)}_+(\Omega)$, where we denote by $g_{**}(s) = \frac{1}{s}\int_0^s g_*(\sigma)\mathrm{d}s$ for $g \in L^1(\Omega)$.

Corollary 3.1 *Under the same assumptions as for Theorem 3.1, one has*

$$\|\,|u_* - v_*|_*\|_{p^*(\cdot)} \le \|\,|u_* - v_*|_{**}\|_{p^*(\cdot)} \le c_L \|u - v\|_{p(\cdot)},$$

where c_L is the same constant as in Theorem 3.1, whenever u and v are in $L^{p(\cdot)}(\Omega)$.

Proof of Theorem 3.1 We first assume that $u \in W^1_0(\Omega) \cap C^\infty(\Omega)$ with measure $\{x \in \Omega : \nabla u(x) = 0\} = 0$ and $v \in C^1_0(\overline{\Omega})$, $v \ge 0$.

Thus $u + tv$ satisfies conditions of Theorem 2.5 for all t and thus the map

$$\begin{array}{rcl} [0,1] & \to & L^1(\Omega_*) \\ t & \mapsto & v_{*(u+tv)} \end{array} \quad \text{is uniformly continuous.}$$

For $\sigma \in \overline{\Omega}_*$, we set $g(\sigma) = \int_0^1 v_{*(u+tv)}(\sigma)\mathrm{d}t$. We know from Lemma 1.1 that

$$(u + v)_*(\sigma) - u_*(\sigma) = g(\sigma), \quad \forall\, \sigma \in \overline{\Omega}_*. \tag{3.1}$$

Therefore, one has

$$\frac{1}{s}\int_0^s [(u+v)_*(\sigma)-u_*(\sigma)]_*\mathrm{d}s = g_{**}(s). \tag{3.2}$$

Let us consider $m \in \mathbb{N}-\{0\}$ and $t_i = \frac{i}{m},\ i=0,\cdots,m$. We set

$$g_m(\sigma) = \frac{1}{m}\sum_{i=0}^{m-1} v_{*(u+t_iv)}(\sigma). \tag{3.3}$$

Lemma 3.1 *One has a constant $c_L > 0$ such that*

$$\|g_{m**}\|_{p^*(\cdot)} \le c_L\|v\|_{p(\cdot)}, \quad \forall\, m.$$

Proof By convexity property of the mapping $h \to h_{**}(s)$, we have

$$g_{m**}(s) \le \frac{1}{m}\sum_{i=0}^{m-1}(v_{*(u+t_iv)})_{**}(s), \tag{3.4}$$

therefore,

$$\|g_{m**}\|_{p^*(\cdot)} \le \frac{1}{m}\sum_{i=0}^{m-1}\|(v_{*(u+t_iv)})_{**}\|_{p^*(\cdot)}. \tag{3.5}$$

Using Theorem 2.4 of the preliminary result section

$$\|(v_{*(u+t_iv)})_{**}\|_{p^*(\cdot)} \le c_L\|v\|_{p(\cdot)}. \tag{3.6}$$

From relations (3.5) and (3.6), we get the result.

Lemma 3.2 *The sequence g_m converges strongly to g in $L^1(\Omega_*)$.*
In particular, for all $s>0$, we have

$$g_{m**}(s) \to g_{**}(s) \quad \textit{as } m \to +\infty.$$

Proof Since the mapping $t \in [0,1] \to v_{*(u+tv)} \in L^1(\Omega_*)$ is uniformly continuous, letting $\varepsilon > 0$, there exists $\delta > 0$: if $m \ge \frac{1}{\delta}$, then

$$\int_{\Omega_*} |v_{*(u+tv)} - v_{*(u+t_iv)}|(\sigma)\mathrm{d}\sigma \le \varepsilon \tag{3.7}$$

for $|t-t_i| \le \frac{1}{m}$.

Therefore

$$\int_{\Omega_*}|g_m(\sigma)-g(\sigma)|\mathrm{d}\sigma \le \sum_{i=0}^{m-1}\int_{t_i}^{t_{i+1}}\int_{\Omega_*}|v_{*(u+tv)}-v_{*(u+t_iv)}|(\sigma)\mathrm{d}\sigma \le \varepsilon. \tag{3.8}$$

This shows the first statement, while for the second one, the result follows from the first statement and the fact that, $\forall\, s>0$,

$$|g_{m**}(s)-g_{**}(s)| \le \frac{1}{s}\int_{\Omega_*}|g_m(\sigma)-g(\sigma)|\mathrm{d}\sigma. \tag{3.9}$$

End of the proof of Theorem 3.1 One has, from the Fatou's property applied to the Banach function norm on $L^{p^*(\cdot)}(\Omega_*)$, that

$$\|g_{**}\|_{p^*(\cdot)} \leq \liminf_{m\to+\infty} \|g_{m**}\|_{p^*(\cdot)}. \tag{3.10}$$

We conclude with relation (2.2) and Lemma 3.1 to derive

$$\|[(u+v)_* - u_*]_{**}\|_{p^*(\cdot)} \leq c_L \|v\|_{p(\cdot)}. \tag{3.11}$$

Let $u \in L^1(\Omega)$. Then there exists a sequence $u_n \in W_0^{1,1}(\Omega) \cap C^\infty(\Omega)$ with measure $\{x \in \Omega : \nabla u_n(x) = 0\} = 0$ such that

$$u_n(x) \to u(x) \quad \text{a.e. and strongly in } L^1(\Omega).$$

There exists also a sequence of $v_n \in C_c^\infty(\Omega)$ such that $v_n \to v$ in $L^{p(\cdot)}(\Omega)$ strongly. Those convergences imply that

$$(u_n + v_n)_* - u_{n*} \to (u+v)_* - u_* \quad \text{strongly in } L^1(\Omega_*).$$

Therefore, for all $s > 0$,

$$[(u_n + v_n)_* - u_{n*}]_{**}(s) \to [(u+v)_* - u_*]_{**}(s). \tag{3.12}$$

Since

$$\|[(u_n + v_n)_* - u_{n*}]_{**}\|_{p^*(\cdot)} \leq c_L \|v_n\|_{p(\cdot)}, \tag{3.13}$$

one derives

$$\|[(u+v)_* - u_*]_{**}\|_{p^*(\cdot)} \leq c_L \|v\|_{p(\cdot)}. \tag{3.14}$$

Proof of Corollary 3.1 We replace v by $v - u$ and notice that for all $s \in \Omega_*$,

$$|u_* - v_*|_*(s) \leq |u_* - v_*|_{**}(s),$$

thus we derive the result if u and v are in $L_+^{p(\cdot)}(\Omega)$.

Otherwise, we shall consider $T_k(\sigma) = \min\big(|\sigma|, k\big)\ \mathrm{sign}\,(\sigma),\ \sigma \in \mathbb{R}$ and u_k (resp. v_k) defined by $u_k = T_k(u) + k \geq 0$ (resp. $v_k = T_k(v) + k$). Therefore

$$\|\,|T_k(u_*) - T_k(v_*)|_{**}\|_{p^*(\cdot)} \leq c_L \|T_k(u) - T_k(v)\|_{p(\cdot)}, \quad \text{letting } k \to +\infty,$$

we have the result.

Corollary 3.2 *Under the same assumptions as for Theorem* 3.1, *one has for all* $v_1,\ v_2$ *and* u *in* $L^{p(\cdot)}(\Omega)$,

$$\|\,|v_{1*u} - v_{2*u}|_*\|_{p^*(\cdot)} \leq \|\,|v_{1*u} - v_{2*u}|_{**}\|_{p^*(\cdot)} \leq c_L \|v_1 - v_2\|_{p(\cdot)}.$$

In particular, $v_{1*u} \in L^{p^*(\cdot)}$ *if* v_1 *and* u *are in* $L^{p(\cdot)}(\Omega)$.

Proof From Corollary 3.1 of Theorem 3.1, we know that for all $\lambda > 0$,

$$\Big\| \Big| \frac{(u+\lambda v_1)_* - (u+\lambda v_2)_*}{\lambda} \Big|_{**} \Big\|_{p^*(\cdot)} \le c_L \|v_1 - v_2\|_{p(\cdot)}. \tag{3.15}$$

Let us set $g_\lambda(s) = \frac{(u+\lambda v_1)_* - (u+\lambda v_2)_*}{\lambda}(s)$. We know that g_λ converges weakly to $v_{1*u} - v_{2*u}$ in $L^1(\Omega_*)$ (see Theorem 2.1). Let us choose $\lambda \doteq \frac{1}{m}$, $m \ge 1$. Then, from the Mazur's lemma, there exist $(\alpha_{jn})_{j\ge n}$, and $\sum\limits_{j=n}^{m_n} \alpha_{jn} = 1,\ \alpha_{jn} \ge 0$, $h_n \doteq \sum\limits_{j=n}^{m_n} \alpha_{jn} g_{\frac{1}{j}}$ converges strongly to $v_{1*u} - v_{2*u}$ in $L^1(\Omega_*)$.

Thus $|h_n|_* \to |v_{1*u} - v_{2*u}|_*$ strongly in $L^1(\Omega_*)$ as $n \to \infty$ and

$$|h|_{**}(s) \xrightarrow[n\to+\infty]{} |v_{1*u} - v_{2*u}|_{**}(s) \quad \text{for all } s > 0. \tag{3.16}$$

But, we have also

$$|h_n|_{**}(s) \le \sum_{j=n}^{m_n} \alpha_{jn} |g_{\frac{1}{j}}|_{**}(s), \quad \forall\, s \in \Omega_*. \tag{3.17}$$

From the relations (3.15) and (3.17), we derive

$$\|\, |h_n|_{**} \|_{p^*(\cdot)} \le \sum_{j=n}^{m_n} \alpha_{jn} \|\, |g_{\frac{1}{j}}|_{**} \|_{p^*(\cdot)} \le c_L \|v_1 - v_2\|_{p(\cdot)}. \tag{3.18}$$

From relation (3.16) and Fatou's lemma, we have

$$\|\, |v_{1*u} - v_{2*u}|_{**} \|_{p^*(\cdot)} \le c_L \|v_1 - v_2\|_{p(\cdot)}. \tag{3.19}$$

We always have

$$\|\, |v_{1*u} - v_{2*u}|_* \|_{p^*(\cdot)} \le \|\, |v_{1*u} - v_{2*u}|_{**} \|_{p^*(\cdot)}. \tag{3.20}$$

So, from (3.19) and (3.20), we get the result.

3.2 Pointwise estimates for quasilinear equation with variable exponents

The purpose of this section is not to give existence result but only to prove some qualitative properties of the quasilinear equations,

$$-\mathrm{div}\,(\widehat{a}(x, \nabla u)) + b(x, \nabla u) = -\mathrm{div}\,(\overrightarrow{F}),$$

when u is a Sobolev spaces with variable exponents. We shall distinguish two types of operators, the first one will contain the Acerbi-Mingione equation.

3.2.1 Acerbi-Mingione type operators

Let $p \in L^\infty(\Omega),\ p : \Omega \to]1, +\infty[$. We shall consider a mapping $\widehat{a} : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$, where Ω is an open bounded set of $\mathbb{R}^N$, satisfying at least the following condition:

(C1) *There exist two constants $\alpha_0 > 0,\ a_0 > 0$ such that for a.e. $x \in \Omega$, for all $\xi \in \mathbb{R}^N,\ \xi' \in \mathbb{R}^N$,*

$$(\widehat{a}(x,\xi) - \widehat{a}(x,\xi'), \xi - \xi') \ge \alpha_0 |\xi - \xi'|^2 (a_0 + |\xi| + |\xi'|)^{p(x)-2}.$$

Since for all $\delta > 0$, we have

$$|\xi - \xi'|^\delta \le (|\xi| + |\xi'|)^\delta \le (a_0 + |\xi| + |\xi'|)^\delta,$$

we deduce as in [14] the following proposition.

Proposition 3.1 *If $\widehat{a}$ satisfies condition* (C1)*, then, for all $\delta \ge 0$,*

$$(\widehat{a}(x,\xi) - \widehat{a}(x,\xi'), \xi - \xi') \ge \alpha_0 |\xi - \xi'|^{2+\delta} (a_0 + |\xi| + |\xi'|)^{p(x)-2-\delta}.$$

Following the proof of [14], we have the results below.

Proposition 3.2 *Let us consider $\widehat{a}(x,\xi) = (1 + |\xi|^2)^{\frac{p(x)-2}{2}} \xi,\ \xi \in \mathbb{R}^N,\ x \in \Omega$. Then, $\widehat{a}$ satisfies condition* (C1)*. Moreover, one can choose $a_0 = 2$ if $\operatorname*{ess\,inf}_{\Omega} p(x) \ge 2$, otherwise $a_0 = 1$.*

Theorem 3.2 *Let $p \in L^\infty(\Omega),\ p : \Omega \to [2, +\infty[,\ F_1, F_2$ be two functions such that $F_i \in L^{q(\cdot)}(\Omega)^N,\ \frac{1}{q(x)} + \frac{1}{p(x)} = 1$ a.e. Consider u_1 and u_2 two elements in $W_0^{1,p(\cdot)}(\Omega) = W_0^{1,1}(\Omega) \cap W^{1,p(\cdot)}(\Omega)$ satisfying, $\forall \varphi \in W_0^{1,p(\cdot)}(\Omega)$,*

$$\int_\Omega \widehat{a}(x, \nabla u_i) \cdot \nabla \varphi \mathrm{d}x = \int_\Omega F_i \cdot \nabla \varphi \mathrm{d}x.$$

Let $w = |u_1 - u_2|,\ \delta F = F_1 - F_2$. Then, there exists a constant $c_N(\alpha_0, a_0, p) > 0$ such that

$$-\frac{\mathrm{d}w_*}{\mathrm{d}s}(s) \le c_N(\alpha_0, a_0, p) s^{\frac{1}{N}-1} [|\delta F|^2]_{*w}^{\frac{1}{2}}(s),$$

provided that $|\delta F| \in L^2(\Omega)$. Here $c_N(\alpha_0,\ a_0,\ p) = \frac{1}{\alpha_0 N \alpha_N^{\frac{1}{N}}} \left| \frac{1}{a_0^{p(x)-2}} \right|_\infty$.

Proof We set $\delta\widehat{a}(x) = \widehat{a}(x, \nabla u_1) - \widehat{a}(x, \nabla u_2)$ and $u_{12} = u_1 - u_2$. For a fixed $s \in \Omega_*$, we consider the test function, $\varphi_s(x) = \big(w(x) - w_*(s)\big)_+ \operatorname{sign}(u_{12}(x)),\ x \in \Omega$. Then we have as in [18, 19]

$$[\delta\widehat{a} \cdot \nabla u_{12}]_{*w}(s) = [\delta F \cdot \nabla u_{12}]_{*w}(s). \tag{3.21}$$

Let

$$h(x) = a_0 + |\nabla u_1(x)| + |\nabla u_2(x)|. \tag{3.22}$$

Then, from Proposition 3.1, we have almost everywhere in Ω,

$$(\delta\widehat{a} \cdot \nabla u_{12})(x) \ge \alpha_0 |\nabla w(x)|^2 h^{p(x)-2}. \tag{3.23}$$

From relations (3.21) and (3.23), one has, using again the relative rearrangement properties,

$$\alpha_0 [|\nabla w|^2 h^{p(x)-2}]_{*w}(s) \le [|\delta F|\, |\nabla w|]_{*w}(s) \tag{3.24}$$

and

$$[|\delta F|\, |\nabla w|]_{*w}(s) \le [|\delta F|^2]_{*w}^{\frac{1}{2}}(s) [|\nabla w|^2]_{*w}^{\frac{1}{2}}. \tag{3.25}$$

Since $h \ge a_0$, we derive from (3.24) and (3.25) that

$$\alpha_0 [|\nabla w|^2 h^{p(x)-2}]_{*w}(s) \le \left| \frac{1}{a_0^{p(\cdot)-2}} \right|_\infty^{\frac{1}{2}} [|\nabla w|^2 h^{p(x)-2}]_{*w}^{\frac{1}{2}}(s) [|\delta F|_{*w}^2]^{\frac{1}{2}}. \tag{3.26}$$

We deduce

$$[|\nabla w|^2 h^{p(x)-2}]_{*w}(s) \le c(\alpha_0, a_0, p)^2 (|\delta F|^2)_{*w}(s) \quad \text{with } c(\alpha_0, a_0, p) = \frac{1}{\alpha_0}\Big|\frac{1}{a_0^{p(\cdot)-2}}\Big|_\infty^{\frac12}. \tag{3.27}$$

From the PSR (Poincaré-Sobolev inequality for the relative rearrangement) (see Theorem 2.2) one has

$$\begin{aligned} -\frac{\mathrm{d}w_*}{\mathrm{d}s}(s) &\le \frac{s^{\frac1N - 1}}{N\alpha_N^{\frac1N}} \cdot |\nabla w|_{*w} \le \frac{s^{\frac1N - 1}}{N\alpha_N^{\frac1N}} \cdot [|\nabla w|^2]_{*w}^{\frac12}(s) \\ &\le \frac{s^{\frac1N - 1}}{N\alpha_N^{\frac1N}} \Big|\frac{1}{a_0^{p(\cdot)-2}}\Big|_\infty^{\frac12} [h^{p(\cdot)-2}|\nabla w|^2]_{*w}^{\frac12}. \end{aligned} \tag{3.28}$$

By (3.27), we deduce

$$-\frac{\mathrm{d}w_*}{\mathrm{d}s}(s) \le \frac{s^{\frac1N - 1}}{N\alpha_N^{\frac1N}} \Big|\frac{1}{a_0^{p(\cdot)-2}}\Big|_\infty^{\frac12} \cdot c(\alpha_0, a_0, p)(|\delta F|^2)_{*w}^{\frac12}(s). \tag{3.29}$$

Setting $c_N(\alpha_0,\ a_0,\ p) = \frac{c(\alpha_0,a_0,p)}{N\alpha_N^{\frac1N}}|\frac{1}{a_0^{p(\cdot)-2}}|_\infty^{\frac12}$, we derive from (3.29) the pointwise estimate of Theorem 3.2.

Corollary 3.3 *Under the same conditions as for Theorem* 3.2, *if* $r : \Omega \to [2, +\infty[$ *is a bounded measurable function, then we have, for all* $s \in \Omega_*$,

$$w_*(s) \le c_N(\alpha_0, a_0, p) b(s) \|f_*\|_{r^*(\cdot)}$$

with $f(t) = (|\delta F|^2_{*w})^{\frac12}(t)$ *and* $b_1(s) = \|\big(\chi_{[s,|\Omega|]}(t) \cdot t^{\frac1N - 1}\big)_*\|_{\overline{r}^*(\cdot)}$, $\overline{r}_*(s) = \frac{r^*(s)}{r^*(s)-1}$.

Proof We integrate the relation (3.29) from s to $|\Omega|$,

$$w_*(s) \le c_N(\alpha_0, a_0, p) \int_0^{|\Omega|} \chi_{[s,|\Omega|]}(t) t^{\frac1N - 1} f(t) \mathrm{d}t. \tag{3.30}$$

By the Hardy-Littlewood inequality, we have

$$w_*(s) \le c_N(\alpha_0, a_0, p) \int_{\Omega_*} (\chi_{[s,|\Omega|]}(t) t^{\frac1N - 1})_*(t) f_*(t) \mathrm{d}t.$$

By the Hölder inequality, we deduce

$$w_*(s) \le c_N(\alpha_0, a_0) b_1(s) \|f_*\|_{r^*(\cdot)}.$$

Remark 3.1 $\forall\, \sigma \in \Omega_*,\ \forall\, s \in \Omega_*,\ (\chi_{[s,\Omega|]}(t) t^{\frac1N - 1})_*(\sigma) = (\sigma + s)^{\frac1N - 1} \chi_{[0,|\Omega|-s]}(\sigma)$.

Corollary 3.4 *Under the same assumption as for Theorem* 3.2, *if* $\delta F \in L^{r^*(\cdot)}(\Omega)^N$, $r^*(0) > 2$ *bounded, then we have*

$$w_*(s) \le \widetilde{c}_N(\alpha_0, a_0) b_1(s) \|\delta F\|_{r^*(\cdot)},$$

provided that r^* *satisfies* $|r^*(t) - r^*(\sigma)|\,|\,\mathrm{Ln}\,(t - \sigma)| \le A$ *near zero.*

Proof One has (see Proposition 2.1)

$$\big((|\delta F|^2\big)_{*w})_*(t) \leq (|\delta F|^2)_{**}(t) \quad \text{for all } t \in \Omega_*.$$

Then

$$\|f_*\|_{r^*(\cdot)} \leq c\| \, |\delta F|^2_{**}\|_{\frac{r^*(\cdot)}{2}}.$$

By [13], the Hardy inequality is true so that

$$\| \, |\delta F|^2_{**}\|_{\frac{r^*(\cdot)}{2}} \leq c\| \, |\delta F|^2_{*}\|_{\frac{r^*(\cdot)}{2}} \leq c\| \, |\delta F|_*\|_{r^*(\cdot)} \leq c\|\delta F\|_{r^*(\cdot)} \quad \text{(by Theorem 2.3)}.$$

Similar result as Corollary 3.4 can be found in [3] if r is a constant function.

Corollary 3.5 *Under the same assumptions as for Corollary* 3.4 *of Theorem* 3.2*, let* ρ *be a Banach function norm rearrangement invariant and* $\|w\|_{L(\Omega,\rho)} = \rho(|w|_*)$*. If* $\rho(b) < +\infty$*, then*

$$\big\|u_1 - u_2\big\|_{L(\Omega,\rho)} = \rho(|u_1 - u_2|_*) \leq \widetilde{c}_N(\alpha_0, a_0)\rho(b)\|F_1 - F_2\|_{r^*(\cdot)}.$$

3.2.2 Pointwise inequality for anisotropic variable exponent equations

We want to derive similar results as for Theorem 3.2 but for operator of the type

$$\widehat{a}(x,\xi) = \big(a_1(x,\xi_1), \cdots, a_N(x,\xi_N)\big)$$

for $\xi = (\xi_1, \cdots, \xi_N) \in \mathbb{R}^N$, $x \in \Omega$ and each a_i satisfying condition (C1), say

(C2) *There exist* $\alpha_{0i} > 1,\ a_{0i} > 0$ *such that*

$$\big(a_i(x,t) - a_i(x,\sigma)\big)(t-\sigma) \geq \alpha_{0i}\big(a_{0i} + |t| + |\sigma|\big)^{p_i(x)-2}|t-\sigma|^2$$

for a.e. $x \in \Omega,\ \forall\, (t,\sigma) \in \mathbb{R} \times \mathbb{R}$*.*

Here, $p_i : \Omega \to]1, +\infty[$ *is a bounded measurable function.*

We start with the cases when $\operatorname*{essinf}_{\Omega} p_i(x) \geq 2$ for all $i \in \{1, \cdots, N\}$. For this, we shall consider the following Sobolev space, for $\overrightarrow{p} = (p_1, \cdots, p_N)$,

$$\mathbf{V}_{\overrightarrow{p}} = W_0^{1,p_1(\cdot),\cdots,p_N(\cdot)}(\Omega) : \Big\{v \in W_0^{1,1}(\Omega) : \int_\Omega \Big|\frac{\partial v}{\partial x_i}(x)\Big|^{p_i(x)} \mathrm{d}x < +\infty \text{ for } i = 1, \cdots, N\Big\}.$$

If necessary, we can endow this space with Banach function norm

$$\|v\|_{\mathbf{V}_{\overrightarrow{p}}} = |v|_{L^1(\Omega)} + \sum_{i=1}^N \Big\|\frac{\partial v}{\partial x_i}\Big\|_{p_i(\cdot)}$$

or the equivalent norm

$$\sum_{i=1}^N \Big\|\frac{\partial v}{\partial x_i}\Big\|_{p_i(\cdot)}.$$

We denote by $q_i(x) = \frac{p_i(x)}{p_i(x)-1}$ the conjugate of $p(x)$.

Theorem 3.3 *Let* F_1 *and* F_2 *be in* $\prod_{i=1}^N L^{q_i(\cdot)}(\Omega)$ *such that* $\delta F = F_1 - F_2$ *satisfies* $|\delta F| \in L^2(\Omega)$*. Let* u_1 *and* u_2 *be two elements of* $\mathbf{V}_{\overrightarrow{p}}$ *satisfying*

$$\int_\Omega \widehat{a}(x, \nabla u_j) \cdot \nabla \varphi \mathrm{d}x = \int_\Omega F_j \cdot \nabla \varphi \mathrm{d}x, \quad j = 1, 2$$

for all $\varphi \in \mathbf{V}_{\overrightarrow{p}}$. *We asume that* $p_i(x) \geq 2$ *a.e.,* $i = 1, \cdots, N$.

Then there exists a constant $c_N(\widehat{a}) > 0$ *depending only on* N *and* α_{0i}, a_{0i}, p_i *for almost every* $s \in \Omega_*$,

$$-\frac{\mathrm{d}w_*}{\mathrm{d}s}(s) \leq c_N(\widehat{a}) s^{\frac{1}{N}-1} [(|\delta F|^2)_{*w}(s)]^{\frac{1}{2}} \qquad \textit{with } w = |u_1 - u_2|.$$

One can choose $c_N(\widehat{a}) = \frac{1}{N\alpha_N^{\frac{1}{N}} \alpha_0 a_0}$, $\alpha_0 = \min\limits_{1\leq i\leq N} \alpha_{01}$, $a_0 = \min\limits_{1\leq i\leq N} \left[\operatorname*{essinf}\limits_{\Omega} a_{0i}^{p(x)-2} \right]$.

Proof The idea is similar to Theorem 3.2. We shall introduce

$$\begin{aligned} &\delta\widehat{a} = \widehat{a}(x, \nabla u_1) - \widehat{a}(x, \nabla u_2), \\ &\delta F = F_1 - F_2, \quad w = |u_1 - u_2|, \quad u_{12} = u_1 - u_2. \end{aligned}$$

We choose for s (fixed) $\in \Omega_*$ as a test function

$$\varphi_s(x) = (w(x) - w_*(s))_+ \ \mathrm{sign}\,(u_{12}(x)).$$

Then

$$\left[\delta\widehat{a} \cdot \nabla u_{12}\right]_{*w}(s) = [\delta F \cdot \nabla u_{12}]_{*w}(s). \tag{3.31}$$

We shall consider $\alpha_0 = \min\limits_{1\leq i\leq N} \alpha_{0i}$ and $a_0 = \min\limits_{1\leq i\leq N} \left[\operatorname*{essinf}\limits_{\Omega} a_{0i}^{p_i(x)-2} \right]$ and we define the function, $k : \Omega \to \mathbb{R}$ measurable,

$$k(x) = \begin{cases} \dfrac{\sum\limits_{i=1}^{N} \alpha_{0i}\big(a_{0i} + |\partial_i u_1| + |\partial_i u_2|\big)^{p_i(x)-2} |\partial_i w|^2}{|\nabla w|^2}(x), & \text{if } \nabla w(x) \neq 0, \\ \alpha_0 a_0, & \text{if } \nabla w(x) = 0. \end{cases}$$

Here $\partial_i u_j = \frac{\partial u_j}{\partial x_i}$, $\partial_i w = \frac{\partial w}{\partial x_i}$.

Then, $k(x) \geq \alpha_0 a_0 > 0$ for a.e. x, and for a.e. $x \in \Omega$,

$$[\widehat{a}(x, \nabla u_1) - \widehat{a}(x, \nabla u_2)] \cdot \nabla u_{12} \geq k(x) |\nabla w(x)|^2. \tag{3.32}$$

From relations (3.31) and (3.32), one has

$$\begin{aligned} [k|\nabla w|^2]_{*w}(s) &\leq |\delta F \cdot \nabla u_{12}]_{*w}(s) \leq [(|\delta F|^2)_{*w}(s)]^{\frac{1}{2}} [(|\nabla w|^2)_{*w}(s)]^{\frac{1}{2}} \\ &\leq \frac{1}{(\alpha_0 a_0)^{\frac{1}{2}}} [(|\delta F|^2)_{*w}(s)]^{\frac{1}{2}} [(k|\nabla w|^2)_{*w}(s)]^{\frac{1}{2}}. \end{aligned} \tag{3.33}$$

Therefore, we have, for a.e. $s \in \Omega_*$,

$$[k|\nabla w|^2]_{*w}(s) \leq \frac{1}{(\alpha_0 a_0)} (|\delta F|^2)_{*w}(s). \tag{3.34}$$

Next, we use the PSR inequality

$$\begin{aligned} -\frac{\mathrm{d}w_*}{\mathrm{d}s}(s) &\leq \frac{s^{\frac{1}{N}-1}}{N\alpha_N^{\frac{1}{N}}} \cdot |\nabla w|_{*w}(s) \leq \frac{s^{\frac{1}{N}-1}}{N\alpha_N^{\frac{1}{N}}} [(|\nabla w|^2)_{*w}(s)]^{\frac{1}{2}} \\ &\leq \frac{s^{\frac{1}{N}-1}}{N\alpha_N^{\frac{1}{N}}} \frac{1}{\sqrt{\alpha_0 a_0}} [(k|\nabla u|^2)_{*w}(s)]^{\frac{1}{2}}. \end{aligned} \tag{3.35}$$

Combining relations (3.34) and (3.35), we obtain, for a.e. $s \in \Omega_*$,

$$-\frac{\mathrm{d}w_*}{\mathrm{d}s}(s) \leq \frac{s^{\frac{1}{N}-1}}{N\alpha_N^{\frac{1}{N}}} \frac{1}{(\alpha_0 a_0)} [(|\delta F|^2)_{*w}(s)]^{\frac{1}{2}}. \tag{3.36}$$

We then obtain the same corollaries as in Theorem 3.2. In particular, we have

Corollary 3.6 *Assume that $|\delta F| \in L^{r(\cdot)}(\Omega)$ for some bounded measurable function with $r^* > 2$ such that r^* satisfies near zero, $|r^*(t) - r^*(\sigma)|\,|\mathrm{Ln}\,|t-\sigma|| \leq A$ for some constant A. Then, there exists a constant $c_\Omega > 0$ depending only on Ω, $\widehat{a}$, $\overrightarrow{p}$, r, such that for a.e. $s \in \Omega_*$,*

$$w_*(s) \leq c_\Omega \cdot b_1(s) \| \, |F_1 - F_2| \, \|_{r(\cdot)},$$

where b_1 is as in Corollary 3.3 *of Theorem* 3.2.

3.2.3 Operator Acerbi-Mingione with a perturbation term

We can generalize the above results by adding a nonlinear term. We shall illustrate this through an example.

(B1) *Let $b : \Omega \times \mathbb{R}^N \to \mathbb{R}$ a nonlinear function satisfying the following growth.*
There exist constants $\beta_0 \geq 0$, $b_0 \geq 0$ such that for all $\xi \in \mathbb{R}^N$, $\xi' \in \mathbb{R}^N$, for a.e., $x \in \Omega$,

$$|b(x,\xi) - b(x,\xi')| \leq \beta_0 (b_0 + |\xi| + |\xi'|)^{p(x)-2} |\xi - \xi'|.$$

Theorem 3.4 *Let b (resp. $\widehat{a}$) be a nonlinear function satisfying* (B1) (*resp.* (C1)). *We assume that $b_0 \leq a_0$ and $p(x) \geq 2$ a.e. $x \in \Omega$. Let F_1, F_2 be two elements of $L^{q(\cdot)}(\Omega)^N$, $\frac{1}{q(x)} + \frac{1}{p(x)} =$ 1 a.e. Let u_1, u_2 be two functions in $W_0^{1,p(\cdot)}(\Omega)$ satisfying $\forall\, \varphi \in W_0^{1,p(\cdot)}(\Omega)$, $j = 1, 2$,*

$$\int_\Omega \widehat{a}(x, \nabla u_j) \cdot \nabla\varphi \mathrm{d}x + \int_\Omega b(x, \nabla u_j)\varphi \mathrm{d}x = \int_\Omega F_j \cdot \nabla\varphi \mathrm{d}x. \tag{3.37}$$

Then, there exist two constants $c_1 > 0$, $c_2 > 0$ depending on the data $\widehat{a}, b, \Omega, N, p$ such that for a.e. $s \in \Omega_*$,

$$\int_{w > w_*(s)} \big(a_0 + |\nabla u_1| + |\nabla u_2|\big)^{p(x)-2} |\nabla w|^2(x) \mathrm{d}x \leq c_1 \int_0^s \mathrm{e}^{c_2 \int_s^\tau a_{12}(t)\mathrm{d}t} \big(|\delta F|^2\big)_{*w}(\tau) \mathrm{d}\tau,$$

where $w = |u_1 - u_2|$, $\delta F = F_1 - F_2$, provided that $|\delta F| \in L^2(\Omega)$, and

$$a_{12}(t) = c_2 t^{\frac{2}{N}-2} \int_{w > w_*(t)} (a_0 + |\nabla u_1| + |\nabla u_2|)^{p(x)-2} \mathrm{d}x$$

belongs to $L^1(0, |\Omega|)$.

One can choose $c_2 = \frac{2}{\alpha_0^2 a_{0m}}$, $c_1 = \frac{2\beta_0^2}{a_{0m}(N(\alpha_N^{\frac{1}{N}}))^2}$, $a_{0m} = \operatorname*{essinf}_\Omega a_0^{p(x)-2}$.

Proof The idea is similar to the above proofs of Theorem 3.2 and Theorem 3.3 and uses the properties of monotone rearrangement and relative rearrangement as in [17, 19].

Let us set $\delta\widehat{a} = \widehat{a}(x, \nabla u_1) - \widehat{a}(x, \nabla u_2)$, $u_{12} = u_1 - u_2$, $\delta b = b(x, \nabla u_1) - b(x, \nabla u_2)$. We recall that $w_* \in W^{1,1}_{\mathrm{loc}}(\Omega_*)$ and for $s \in \Omega_*$, the function $\varphi(x) = (w(x) - w_*(s))_+ \operatorname{sign}(u_{12}(x))$.

Then, one has

$$[\delta\widehat{a}\cdot\nabla u_{12}]_{*w}(s) - w_*'(s)\int_{w>w_*(s)}\delta b\,\mathrm{sign}\,(u_{12})\mathrm{d}x = [\delta F\cdot\nabla u_{12}]_{*w}(s). \tag{3.38}$$

We set $k_0(x) = \big(a_0 + |\nabla u_1(x)| + |\nabla u_2(x)|\big)^{p(x)-2}$. By the growth conditions (C1) and (B1) on $\widehat{a}$ and b, we have

$$\alpha_0(k_0|\nabla w|^2)_{*w}(s) \le |w_*'(s)|\Big(\beta_0\int_{w>w_*(s)}k_0|\nabla w|\mathrm{d}x\Big) + [(|\delta F|^2)_{*w}(s)]^{\frac12}[(|\nabla w|^2)_{*w}(s)]^{\frac12}. \tag{3.39}$$

From the PSR inequality (see Theorem 2.2), we have

$$|w_*'(s)| = -w_*'(s) \le \frac{s^{\frac1N-1}}{N\alpha_N^{\frac1N}}[\big(|\nabla w|^2\big)_{*w}(s)]^{\frac12} \tag{3.40}$$

and

$$[\big(|\nabla w|^2\big)_{*w}(s)]^{\frac12} \le \frac{1}{\sqrt{a_{0m}}}[\big(k_0|\nabla w|^2\big)_{*w}(s)]^{\frac12} \tag{3.41}$$

with $a_{0m} = \operatorname*{essinf}_{\Omega} a_0^{p(x)-2}$. Therefore, we obtain from (3.39) to (3.41) that

$$\alpha_0[\big(k_0|\nabla w|^2\big)_{*w}(s)]^{\frac12} \le \frac{s^{\frac1N-1}}{N\alpha_N^{\frac1N}}\frac{\beta_0}{\sqrt{a_{0m}}}\int_{w>w_*(s)}k_0|\nabla w|\mathrm{d}x + \frac{1}{\sqrt{a_{0m}}}[\big(|\delta F|^2\big)_{*w}(s)]^{\frac12}. \tag{3.42}$$

By the Cauchy-Schwarz's inequality, we have

$$\int_{w>w_*(s)}k_0|\nabla w|\mathrm{d}x \le \Big(\int_{w>w_*(s)}k_0|\nabla w|^2\mathrm{d}x\Big)^{\frac12}\Big(\int_{w>w_*(s)}k_0\mathrm{d}x\Big)^{\frac12}. \tag{3.43}$$

Let us set $y(s) = \int_{w>w_*(s)}k_0|\nabla w|^2\mathrm{d}x$. Then by the definition of relative rearrangement, we have

$$y'(s) = (k_0|\nabla w|^2)_{*w}(s) \quad \text{ for a.e. } s\in\Omega_*.$$

Therefore, relations (3.42) and (3.43) infer

$$y'(s) \le c_2 s^{\frac2N-2}y(s)\int_{w>w_*(s)}k_0\mathrm{d}x + c_1(|\delta F|^2)_{*w}(s) \tag{3.44}$$

with $c_2 = \frac{2}{\alpha_0^2 a_{0m}}$, $c_1 = \frac{2\beta_0^2}{a_{0m}(N\alpha_N^{\frac1N})^2}$.

From the above Gronwall inequality, we deduce

$$y(s) \le c_1\int_0^s \mathrm{e}^{c_2\int_\sigma^\tau a_{12}(t)\mathrm{d}t}\big(|\delta F|^2\big)_{*w}(\tau)\mathrm{d}\tau, \tag{3.45}$$

provided that $a_{12}(t) \equiv c_2 t^{\frac2N-2}\int_{w>w_*(t)}k_0\mathrm{d}x$ is in $L^1(\Omega_*)$.

Remark 3.2 The condition that $a_{12}\in L^1(\Omega_*)$ depends on p may be detail according to each situation.

For example, if $p(x) = p =$constant and $c_2 \neq 0$ then if $2 \leq p < \frac{2N}{N-2}$ if $N \geq 3$ or $p < +\infty$ for $N = 2$, we have $\int_{\Omega_*} a_{12}(t)\mathrm{d}t < +\infty$.

Corollary 3.7 *Under the same assumptions as for Theorem* 3.4*, if $c_2 a_{12} \in L^1(\Omega_*)$, then*

$$\int_\Omega |\nabla(u_1 - u_2)|^{p(x)}\mathrm{d}x \leq c_1 \mathrm{e}^{c_2 \int_{\Omega_*} a_{12}(t)\mathrm{d}t} \|F_1 - F_2\|^2_{L^2(\Omega)^N}$$

with c_1 and c_2 given as in the proof of Theorem 3.4.

Proof One has

$$|\nabla(u_1 - u_2)|^{p(x)}(x) \leq k_0(x)|\nabla w|^2(x) \quad \text{for a.e. } x, \tag{3.46}$$

since

$$y(|\Omega|) = \int_\Omega k_0(x)|\nabla(u_1 - u_2)|^2\mathrm{d}x,$$

and from Theorem 3.4 that

$$\int_\Omega k_0(x)|\nabla(u_1 - u_2)|^2(x)\mathrm{d}x = y(|\Omega|) \leq c_1 \mathrm{e}^{c_2 \int_{\Omega_*} a_{12}(t)\mathrm{d}t} \|F_1 - F_2\|^2_{L^2(\Omega)^N}. \tag{3.47}$$

From relations (3.46) and (3.47) we derive the result.

Corollary 3.8 *Under the same assumption as for Theorem* 3.4*, we have for all $s > 0$,*

$$w_*(s) \leq \frac{1}{N\alpha_N^{\frac{1}{N}} a_{0m}} \Big(\frac{N}{N-2}\Big)^{\frac{1}{2}} (s^{\frac{2}{N}-1} - |\Omega|^{\frac{2}{N}-1})^{\frac{1}{2}} \Big(\int_\Omega k_0|\nabla w|^2\mathrm{d}x\Big)^{\frac{1}{2}}, \quad \textit{if } N \geq 3$$

and

$$w_*(s) \leq \frac{1}{N\alpha_N^{\frac{1}{N}} a_{0m}} \Big[\mathrm{Ln}\Big(\frac{|\Omega|}{s}\Big)\Big]^{\frac{1}{2}} \Big(\int_\Omega k_0|\nabla w|^2\mathrm{d}x\Big)^{\frac{1}{2}}, \quad \textit{if } N = 2$$

with $k_0(x) = (a_0 + |\nabla u_1(x)| + |\nabla u_2(x)|)^{p(x)-2}$, $a_{0m} = \operatorname*{ess\,inf}_\Omega a_0^{p(x)-2}$, $w = |u_1 - u_2|$.

Proof From the PSR inequality (Theorem 2.2), we derive as before

$$-w_*'(s) \leq \frac{s^{\frac{1}{N}-1}}{N\alpha_N^{\frac{1}{N}}} \frac{1}{a_{0m}} [k_0|\nabla w|^2]^{\frac{1}{2}}_{*w}(s). \tag{3.48}$$

Integrating this last relation and applying the Cauchy-Schwarz's inequality, one has

$$w_*(s) \leq \frac{1}{N\alpha_N^{\frac{1}{N}} a_{0m}} \Big(\int_s^{|\Omega|} t^{\frac{2}{N}-2}\mathrm{d}t\Big)^{\frac{1}{2}} \Big(\int_\Omega k_0|\nabla w|^2\mathrm{d}x\Big)^{\frac{1}{2}}. \tag{3.49}$$

From the above, we have the result.

Remark 3.3 From the proof of Corollary 3.7, we have an estimate of

$$y(|\Omega|) = \int_\Omega k_0|\nabla w|^2\mathrm{d}x \leq c_1 \mathrm{e}^{c_2 \int_{\Omega_*} a_{12}(t)\mathrm{d}t} \|F_1 - F_2\|^2_{L^2(\Omega)^N}.$$

More results and cases shall be given in [20].

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