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# The Sharp Estimates of all Homogeneous Expansions for a Class of Quasi-convex Mappings on the Unit Polydisk in $\mathbb{C}^{n*}$

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**Abstract** In this paper, the sharp estimates of all homogeneous expansions for $f$ are established, where $f(z) = (f_1(z), f_2(z), \cdots, f_n(z))'$ is a $k$-fold symmetric quasi-convex mapping defined on the unit polydisk in $\mathbb{C}^n$ and

$$\frac{D^{tk+1}f_p(0)(z^{tk+1})}{(tk+1)!} = \sum_{l_1,l_2,\cdots,l_{tk+1}=1}^{n} |a_{p\,l_1l_2\cdots l_{tk+1}}|\mathrm{e}^{\mathrm{i}\frac{\theta_{p\,l_1}+\theta_{pl_2}+\cdots+\theta_{pl_{tk+1}}}{tk+1}} z_{l_1}z_{l_2}\cdots z_{l_{tk+1}},$$
$$p = 1, 2, \cdots, n.$$

Here $\mathrm{i} = \sqrt{-1}$, $\theta_{pl_q} \in (-\pi, \pi]$ $(q = 1, 2, \cdots, tk+1)$, $l_1, l_2, \cdots, l_{tk+1} = 1, 2, \cdots, n$, $t = 1, 2, \cdots$. Moreover, as corollaries, the sharp upper bounds of growth theorem and distortion theorem for a $k$-fold symmetric quasi-convex mapping are established as well. These results show that in the case of quasi-convex mappings, Bieberbach conjecture in several complex variables is partly proved, and many known results are generalized.

**Keywords** Estimates of all homogeneous expansions, Quasi-convex mapping, Quasi-convex mapping of type A, Quasi-convex mapping of type B
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## 1 Introduction

In the case of one complex variable, the following Bieberbach conjecture (i.e., de Branges theorem) is well-known.

**Theorem A** (see [1]) *If $f(z) = z + \sum\limits_{n=2}^{\infty} a_n z^n$ is a biholomorphic function on the unit disk $U$, then*

$$|a_n| \leqslant n, \quad n = 2, 3, \cdots.$$

However, in the case of several complex variables, Cartan [2] pointed out that the above theorem does not hold. So people mainly investigated the case of the subclasses of biholomorphic mappings Bieberbach conjecture in several complex variables. In 1992, Zhang, Dong and

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Wang [3] first established the sharp estimates of all homogeneous expansions for normalized biholomorphic convex mappings on the unit ball in a complex Banach space with a brief proof. But with respect to the estimates of all homogeneous expansions for normalized biholomorphic starlike mappings, quasi-convex mappings of type A and quasi-convex mappings of type B on the Euclidean unit ball $B^n$ in $\mathbb{C}^n$, Roper and Suffridge [4] stated that the corresponding Bieberbach conjecture does not hold about the second homogeneous expansions with concrete counterexamples. Taking into account the above reason, people chiefly show interest in studying the estimates of homogeneous expansions for the subclasses of biholomorphic mappings on the unit polydisk $U^n$ in $\mathbb{C}^n$. In 1999, Gong [1] posed the following conjecture.

**Conjecture A** *If $f : U^n \to \mathbb{C}^n$ is a normalized biholomorphic starlike mapping on the unit polydisk $U^n$ in $\mathbb{C}^n$, then*

$$\frac{\|D^m f(0)(z^m)\|}{m!} \leqslant m\|z\|^m, \quad z \in U^n,\ m = 2, 3, \cdots .$$

It is obvious that the above conjecture is quite similar to the famous Bieberbach conjecture in one complex variable. Recently, Liu [5] investigated the estimates of all homogeneous expansions for a class of quasi-convex mappings (including quasi-convex mappings of type A and quasi-convex mappings of type B), Liu and Liu [6] established the estimates of all homogeneous expansions for a class of $k$-fold symmetric quasi-convex mappings of type $B$ and order $\alpha$.

On the other hand, at present, the sharp growth, covering and distortion theorem for quasi-convex mappings of type B on $U^n$ is not given, the sharp distortion theorem for quasi-convex mappings (including quasi-convex mappings of type A) on $U^n$ is not given as well (see [7]).

In view of the additional condition for the above mappings (see [5–6]) are somewhat special, a natural question arises to how to weaken the additional condition in order to obtain the generalization of known results. Our paper is to answer this question.

Let $X$ be a complex Banach space with norm $\|\cdot\|$, $X^*$ be the dual space of $X$ and let $T(x) = \{T_x \in X^* : \|T_x\| = 1, T_x(x) = \|x\|\}$. Let $E$ be the unit ball in $X$, let $\partial E$ be the boundary of $E$ and let $\overline{E}$ be the closure of $E$. Let $U$ stand for the Euclidean unit disk in $\mathbb{C}$, let $U^n$ be the unit polydisk in $\mathbb{C}^n$, and let $\partial_0 U^n$ denote the characteristic boundary (i.e., the boundary on which the maxium modulus of the holomorphic function can be attained) of $U^n$. Let the symbol $'$ mean transpose. Let $\mathbb{N}$ be the set of all positive integers.

We now recall some definitions and notations as follows.

**Definition 1.1** (see [8]) *Suppose that $f : E \to X$ is a normalized locally biholomorphic mapping. Denote*

$$G_f(\alpha, \beta) = \frac{2\alpha}{T_u[(Df(\alpha u))^{-1}(f(\alpha u) - f(\beta u))]} - \frac{\alpha + \beta}{\alpha - \beta}.$$

*If*

$$\operatorname{Re} G_f(\alpha, \beta) \geqslant 0, \quad u \in \partial E,\ \alpha, \beta \in U,$$

*then $f$ is said to be a quasi-convex mapping of type $A$ on $E$.*

**Definition 1.2** (see [8]) *Suppose that $f : E \to X$ is a normalized locally biholomorphic mapping. If*

$$\operatorname{Re}\{T_x[(Df(x))^{-1}(D^2f(x)(x^2)+Df(x)x)]\} \geqslant 0, \quad x \in E,$$

*then $f$ is said to be a quasi-convex mapping of type $B$ on $E$.*

When $X=\mathbb{C}^n$, Definitions 1.1 and 1.2 were originally introduced by Roper and Suffridge [4].

**Definition 1.3** (see [9]) *Suppose that $f : E \to X$ is a normalized locally biholomorphic mapping. If*

$$\operatorname{Re}\{T_x[(Df(x))^{-1}(f(x)-f(\xi x))]\} \geqslant 0, \quad x \in E, \ \xi \in \overline{U},$$

*then $f$ is said to be a quasi-convex mapping on $E$.*

When $X=\mathbb{C}$, Definitions 1.1–1.3 are the same; this implies that a quasi-convex function is equivalent to a normalized biholomorphic convex function in one complex variable.

**Definition 1.4** (see [10]) *Suppose $f \in H(E)$. It is said that $f$ is $k$-fold symmetric if $\mathrm{e}^{-\frac{2\pi \mathrm{i}}{k}} f\big(\mathrm{e}^{\frac{2\pi \mathrm{i}}{k}}x\big)=f(x)$ for all $x \in E$, where $k \in \mathbb{N}$ and $\mathrm{i}=\sqrt{-1}$.*

**Definition 1.5** (see [11]) *Suppose that $\Omega$ is a domain (connected open set) in $X$ which contains $0$. It is said that $x=0$ is a zero of order $k$ of $f(x)$ if $f(0)=0,\cdots,D^{k-1}f(0)=0$, but $D^kf(0)\neq 0$, where $k\in\mathbb{N}$.*

**Definition 1.6** (see [6]) *Suppose that $\alpha \in [0,1)$ and $f : E \to X$ is a normalized locally biholomorphic mapping. If*

$$\operatorname{Re}\{T_x[(Df(x))^{-1}(D^2f(x)(x^2)+Df(x)x)]\} \geqslant \alpha\|x\|, \quad x \in E,$$

*then $f$ is said to be quasi-convex of type $B$ and order $\alpha$ on $E$.*

Let $K(E)$ denote the set of all normalized biholomorphic convex mappings on $E$. Let $Q_A(E)$ (resp. $Q_B(E)$) be the set of all quasi-convex mappings of type $A$ (resp. type $B$) on $E$ and let $Q(E)$ be the set of all quasi-convex mappings on $E$.

## 2 The Sharp Estimates of all Homogeneous Expansions for a Class of Quasi-convex Mappings

In order to establish our main results in this section, we shall first give the following lemmas. It is easy to prove the following results.

**Lemma 2.1** *Suppose that $f$ is a normalized locally biholomorphic mapping on $U^n$. Then $f \in Q_B(U^n)$ if and only if*

$$\operatorname{Re}\frac{g_j(z)}{z_j} \geqslant 0, \quad z=(z_1,\cdots,z_n)' \in U^n,$$

*where* $g(z)=(g_1(z),\cdots,g_n(z))'=(Df(z))^{-1}(D^2f(z)(z^2)+Df(z)z)$ *is a column vector in* $\mathbb{C}^n$, $j$ *satisfies* $|z_j|=\|z\|=\max\limits_{1\leqslant k\leqslant n}\{|z_k|\}$.

**Lemma 2.2** (see [12]) *Suppose* $g(z)=(g_1(z),g_2(z),\cdots,g_n(z))'\in H(U^n)$, $g(0)=0$, $Dg(0)=I$. *If* $\operatorname{Re}\frac{g_j(z)}{z_j}\geqslant 0$ $(z\in U^n)$, *where* $|z_j|=\|z\|=\max\limits_{1\leqslant k\leqslant n}\{|z_k|\}$, *then*

$$\frac{\|D^mg(0)(z^m)\|}{m!}\leqslant 2\|z\|^m,\quad z\in U^n,\ m=2,3,\cdots.$$

**Lemma 2.3** (see [5]) *If* $f(z)$ *is a normalized locally biholomorphic mapping on* $U^n$, *and* $g(z)=(Df(z))^{-1}(D^2f(z)(z^2)+Df(z)z)\in H(U^n)$, *then*

$$\frac{D^2f(0)(z^2)}{2!}=\frac{1}{2}\cdot\frac{D^2g(0)(z^2)}{2!},$$

$$m(m-1)\frac{D^mf(0)(z^m)}{m!}=\frac{D^mg(0)(z^m)}{m!}+\frac{2D^2f(0)(z,\frac{D^{m-1}g(0)(z^{m-1})}{(m-1)!})}{2!}+\cdots+\frac{(m-1)D^{m-1}f(0)(z^{m-2},\frac{D^2g(0)(z^2)}{2!})}{(m-1)!},\quad z\in U^n,\ m=3,4,\cdots.$$

**Lemma 2.4** (see [8]) $K(E)\subset Q(E)=Q_A(E)\subset Q_B(E)$. *In some concrete complex Banach spaces, we even have* $K(E)\subsetneq Q(E)$.

**Lemma 2.5** *Suppose* $f(z)\in H(U^n)$, *and*

$$\frac{D^mf_p(0)(z^m)}{m!}=\sum_{l_1,l_2,\cdots,l_m=1}^{n}a_{p\,l_1l_2\cdots l_m}z_{l_1}z_{l_2}\cdots z_{l_m},\quad p=1,2,\cdots,n,$$

*where* $a_{p\,l_1l_2\cdots l_m}=\frac{1}{m!}\frac{\partial^mf_p(0)}{\partial z_{l_1}\partial z_{l_2}\cdots\partial z_{l_m}}$, $l_1,l_2,\cdots,l_m=1,2,\cdots,n$, $m=2,3,\cdots$. *Then*

$$\begin{aligned}\frac{1}{m!}D^mf_p(0)(z^{m-1},w)=\frac{1}{m}\Big(&\sum_{l_1,l_2,\cdots,l_m=1}^{n}a_{p\,l_1l_2\cdots l_m}w_{l_1}z_{l_2}\cdots z_{l_m}\\&+\sum_{l_1,l_2,\cdots,l_m=1}^{n}a_{p\,l_1l_2\cdots l_m}z_{l_1}w_{l_2}z_{l_3}\cdots z_{l_m}+\cdots\\&+\sum_{l_1,l_2,\cdots,l_m=1}^{n}a_{p\,l_1l_2\cdots l_m}z_{l_1}z_{l_2}\cdots z_{l_{m-1}}w_{l_m}\Big),\\&z\in U^n,\ p=1,2,\cdots,n,m=2,3,\cdots,\end{aligned}$$

*where* $w=(w_1,w_2,\cdots,w_n)'\in\mathbb{C}^n$ *which satisfies* $\|w\|=\max\limits_{1\leqslant p\leqslant n}\{|w_p|\}<2$.

**Proof** $\forall\,\lambda\in\mathbb{C}$ with $|\lambda|\leqslant\frac{1}{2}$, by a straightforward computation, it yields that

$$\begin{aligned}&D^mf_p(0)\Big(\underbrace{\frac{z+\lambda w}{2},\frac{z+\lambda w}{2},\cdots,\frac{z+\lambda w}{2}}_{m}\Big)\\&=\frac{D^mf_p(0)(z^m)}{2^m}+\frac{mD^mf_p(0)(z^{m-1},w)}{2^m}\lambda+\cdots+\frac{D^mf_p(0)(w^m)}{2^m}\lambda^m.\end{aligned}\tag{2.1}$$

Note that $\frac{D^m f_p(0)(z^m)}{m!} = \sum\limits_{l_1,l_2,\cdots,l_m=1}^{n} a_{p\,l_1l_2\cdots l_m} z_{l_1}z_{l_2}\cdots z_{l_m}$. Therefore,

$$
\begin{aligned}
&D^m f_p(0)\Big(\underbrace{\frac{z+\lambda w}{2},\frac{z+\lambda w}{2},\cdots,\frac{z+\lambda w}{2}}_{m}\Big)\\
&=\frac{m!}{2^m}\Big(\sum_{l_1,l_2,\cdots,l_m=1}^{n} a_{p\,l_1l_2\cdots l_m}(z_{l_1}+\lambda w_{l_1})(z_{l_2}+\lambda w_{l_2})\cdots(z_{l_m}+\lambda w_{l_m})\Big)\\
&=\frac{m!}{2^m}\Big(\sum_{l_1,l_2,\cdots,l_m=1}^{n} a_{p\,l_1l_2\cdots l_m} z_{l_1}z_{l_2}\cdots z_{l_m}\Big)+\frac{m!}{2^m}\Big(\sum_{l_1,l_2,\cdots,l_m=1}^{n} a_{p\,l_1l_2\cdots l_m} w_{l_1}z_{l_2}\cdots z_{l_m}\\
&\quad+\sum_{l_1,l_2,\cdots,l_m=1}^{n} a_{p\,l_1l_2\cdots l_m} z_{l_1}w_{l_2}z_{l_3}\cdots z_{l_m}+\cdots+\sum_{l_1,l_2,\cdots,l_m=1}^{n} a_{p\,l_1l_2\cdots l_m} z_{l_1}z_{l_2}\cdots z_{l_{m-1}}w_{l_m}\Big)\lambda\\
&\quad+\cdots+\frac{m!}{2^m}\Big(\sum_{l_1,l_2,\cdots,l_m=1}^{n} a_{p\,l_1l_2\cdots l_m} w_{l_1}w_{l_2}\cdots w_{l_m}\Big)\lambda^m. \qquad (2.2)
\end{aligned}
$$

Comparing with the coefficient of the right-hand sides of (2.1) and (2.2) with respect to $\lambda$, we obtain

$$
\begin{aligned}
&\frac{1}{m!}D^m f_p(0)(z^{m-1},w)\\
&=\frac{1}{m}\Big(\sum_{l_1,l_2,\cdots,l_m=1}^{n} a_{p\,l_1l_2\cdots l_m} w_{l_1}z_{l_2}\cdots z_{l_m}+\sum_{l_1,l_2,\cdots,l_m=1}^{n} a_{p\,l_1l_2\cdots l_m} z_{l_1}w_{l_2}z_{l_3}\cdots z_{l_m}+\cdots\\
&\quad+\sum_{l_1,l_2,\cdots,l_m=1}^{n} a_{p\,l_1l_2\cdots l_m} z_{l_1}z_{l_2}\cdots z_{l_{m-1}}w_{l_m}\Big),\quad z\in U^n,\ p=1,2,\cdots,n,\ m=2,3,\cdots,
\end{aligned}
$$

where $w=(w_1,w_2,\cdots,w_n)'\in\mathbb{C}^n$ which satisfies $\|w\|=\max\limits_{1\leqslant p\leqslant n}\{|w_p|\}<2$. This completes the proof.

**Lemma 2.6** *Let*

$$
\left\|\begin{pmatrix}
\sum\limits_{l_1,l_2,\cdots,l_m=1}^{n}|a_{1l_1l_2\cdots l_m}|\mathrm{e}^{\mathrm{i}\frac{\theta_{1l_1}+\theta_{1l_2}+\cdots+\theta_{1l_m}}{m}}z_{l_1}z_{l_2}\cdots z_{l_m}\\
\sum\limits_{l_1,l_2,\cdots,l_m=1}^{n}|a_{2l_1l_2\cdots l_m}|\mathrm{e}^{\mathrm{i}\frac{\theta_{2l_1}+\theta_{2l_2}+\cdots+\theta_{2l_m}}{m}}z_{l_1}z_{l_2}\cdots z_{l_m}\\
\vdots\\
\sum\limits_{l_1,l_2,\cdots,l_m=1}^{n}|a_{nl_1l_2\cdots l_m}|\mathrm{e}^{\mathrm{i}\frac{\theta_{nl_1}+\theta_{nl_2}+\cdots+\theta_{nl_m}}{m}}z_{l_1}z_{l_2}\cdots z_{l_m}
\end{pmatrix}\right\|\leqslant C_m\|z\|^m,\quad z=\begin{pmatrix}z_1\\z_2\\\vdots\\z_n\end{pmatrix}\in U^n,
$$

*where $m=2,3,\cdots$, each $a_{p\,l_1l_2\cdots l_m}(p,l_1,l_2,\cdots,l_m=1,2,\cdots,n)$ is a complex number which is independent of $z_p$ $(p=1,2,\cdots,n)$, $\mathrm{i}=\sqrt{-1}$, each $\theta_{pl_q}\in(-\pi,\pi]$ $(q=1,2,\cdots,m;p,l_1,l_2,\cdots,l_m=1,2,\cdots,n)$ which is independent of $z_p$ $(p=1,2,\cdots,n)$, $\|z\|=\max\limits_{1\leqslant p\leqslant n}\{|z_p|\}$, and each $C_m$ $(m=2,3,\cdots)$ is a nonnegative real constant which is only dependent on $m$. Then*

$$
A_m=\max_{1\leqslant p\leqslant n}\Big\{\sum_{l_1,l_2,\cdots,l_m=1}^{n}|a_{p\,l_1l_2\cdots l_m}|\Big\}\leqslant C_m,\quad m=2,3,\cdots.
$$

**Proof** $\forall z \in U^n \setminus \{0\}$, according to the hypothesis of Lemma 2.6, we have

$$\Big| \sum_{l_1,l_2,\cdots,l_m=1}^{n} |a_{p\,l_1 l_2 \cdots l_m}| \mathrm{e}^{\mathrm{i}\frac{\theta_{p\,l_1}+\theta_{pl_2}+\cdots+\theta_{pl_m}}{m}} \frac{z_{l_1}}{\|z\|}\frac{z_{l_2}}{\|z\|}\cdots\frac{z_{l_m}}{\|z\|}\Big| \leqslant C_m, \quad p=1,2,\cdots,n.$$

In particular, taking $z_{l_q} = \mathrm{e}^{-\mathrm{i}\frac{\theta_{p\,l_q}}{m}}\|z\|$, $q=1,2,\cdots,m$, we conclude that

$$\sum_{l_1,l_2,\cdots,l_m=1}^{n} |a_{p\,l_1 l_2 \cdots l_m}| \leqslant C_m, \quad p=1,2,\cdots,n.$$

That is

$$A_m = \max_{1\leqslant p\leqslant n}\Big\{\sum_{l_1,l_2,\cdots,l_m=1}^{n} |a_{p\,l_1 l_2 \cdots l_m}|\Big\} \leqslant C_m, \quad m=2,3,\cdots.$$

This completes the proof.

Now we shall prove the following theorem.

**Theorem 2.1** *If $f \in Q_B(U^n)$ $(Q_A(U^n)$ or $Q(U^n))$, and*

$$\frac{D^s f_p(0)(z^s)}{s!} = \sum_{l_1,l_2,\cdots,l_s=1}^{n} |a_{p\,l_1 l_2\cdots l_s}| \mathrm{e}^{\mathrm{i}\frac{\theta_{p\,l_1}+\theta_{pl_2}+\cdots+\theta_{pl_s}}{s}} z_{l_1} z_{l_2}\cdots z_{l_s}, \quad p=1,2,\cdots,n,$$

*where* $|a_{p\,l_1 l_2\cdots l_s}| \mathrm{e}^{\mathrm{i}\frac{\theta_{p\,l_1}+\theta_{pl_2}+\cdots+\theta_{pl_s}}{s}} = \frac{1}{s!}\frac{\partial^s f_p(0)}{\partial z_{l_1}\partial z_{l_2}\cdots\partial z_{l_s}}$, $\mathrm{i}=\sqrt{-1}$, $\theta_{pl_q} \in (-\pi,\pi]$ $(q=1,2,\cdots,s)$, $l_1,l_2,\cdots,l_s = 1,2,\cdots,n$, $s=2,3,\cdots,m-1$, *then*

$$\frac{\|D^m f(0)(z^m)\|}{m!} \leqslant \frac{2}{m(m-1)}\Big(1+\sum_{s=2}^{m-1} sA_s\Big)\|z\|^m, \quad z\in U^n,\ m=3,4,\cdots,$$

*where* $A_s = \max\limits_{1\leqslant p\leqslant n}\Big\{\sum\limits_{l_1,l_2,\cdots,l_s=1}^{n} |a_{p\,l_1 l_2\cdots l_s}|\Big\}$, $s=2,3,\cdots,m-1$.

**Proof** Assume $f \in Q_B(U^n)$, $\forall\, z\in U^n\setminus\{0\}$. Denote $z_0 = \frac{z}{\|z\|}$. Let $g(z) = (Df(z))^{-1} \cdot(D^2 f(z)(z^2) + Df(z)z)$, $w = \frac{D^{m-s+1}g(0)(z^{m-s+1})}{(m-s+1)!}$, $s=2,3,\cdots,m-1$, and $j$ satisfies $|z_j| = \|z\| = \max\limits_{1\leqslant k\leqslant n}\{|z_k|\}$. In view of the hypothesis of Theorem 2.1, Lemmas 2.1, 2.2 and 2.5, we conclude that

$$\begin{aligned}
&\Big|\frac{1}{s!}D^s f_j(0)\Big(z_0^{s-1}, \frac{D^{m-s+1}g(0)(z_0^{m-s+1})}{(m-s+1)!}\Big)\Big| \\
={}& \frac{1}{s}\Big|\sum_{l_1,l_2,\cdots,l_s=1}^{n} |a_{jl_1 l_2\cdots l_s}| \mathrm{e}^{\mathrm{i}\frac{\theta_{jl_1}+\theta_{jl_2}+\cdots+\theta_{jl_s}}{s}} \cdot \frac{D^{m-s+1}g_{l_1}(0)(z_0^{m-s+1})}{(m-s+1)!}\frac{z_{l_2}}{\|z\|}\cdots\frac{z_{l_s}}{\|z\|} \\
&+ \sum_{l_1,l_2,\cdots,l_s=1}^{n} |a_{jl_1 l_2\cdots l_s}| \mathrm{e}^{\mathrm{i}\frac{\theta_{jl_1}+\theta_{jl_2}+\cdots+\theta_{jl_s}}{s}} \cdot \frac{z_{l_1}}{\|z\|}\frac{D^{m-s+1}g_{l_2}(0)(z_0^{m-s+1})}{(m-s+1)!}\frac{z_{l_3}}{\|z\|}\cdots\frac{z_{l_s}}{\|z\|} + \cdots \\
&+ \sum_{l_1,l_2,\cdots,l_s=1}^{n} |a_{jl_1 l_2\cdots l_s}| \mathrm{e}^{\mathrm{i}\frac{\theta_{jl_1}+\theta_{jl_2}+\cdots+\theta_{jl_s}}{s}} \cdot \frac{z_{l_1}}{\|z\|}\cdots\frac{z_{l_{s-1}}}{\|z\|}\frac{D^{m-s+1}g_{l_s}(0)(z_0^{m-s+1})}{(m-s+1)!}\Big|
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \frac{1}{s}\Big(\Big|\sum_{l_1,l_2,\cdots,l_s=1}^{n}|a_{jl_1l_2\cdots l_s}|\mathrm{e}^{\mathrm{i}\frac{\theta_{jl_1}+\theta_{jl_2}+\cdots+\theta_{jl_s}}{s}}\cdot\frac{D^{m-s+1}g_{l_1}(0)(z_0^{m-s+1})}{(m-s+1)!}\frac{z_{l_2}}{\|z\|}\cdots\frac{z_{l_s}}{\|z\|}\Big|\\
&\quad+\Big|\sum_{l_1,l_2,\cdots,l_s=1}^{n}|a_{jl_1l_2\cdots l_s}|\mathrm{e}^{\mathrm{i}\frac{\theta_{jl_1}+\theta_{jl_2}+\cdots+\theta_{jl_s}}{s}}\cdot\frac{z_{l_1}}{\|z\|}\frac{D^{m-s+1}g_{l_2}(0)(z_0^{m-s+1})}{(m-s+1)!}\frac{z_{l_3}}{\|z\|}\cdots\frac{z_{l_s}}{\|z\|}\Big|+\cdots\\
&\quad+\Big|\sum_{l_1,l_2,\cdots,l_s=1}^{n}|a_{jl_1l_2\cdots l_s}|\mathrm{e}^{\mathrm{i}\frac{\theta_{jl_1}+\theta_{jl_2}+\cdots+\theta_{jl_s}}{s}}\cdot\frac{z_{l_1}}{\|z\|}\cdots\frac{z_{l_{s-1}}}{\|z\|}\frac{D^{m-s+1}g_{l_s}(0)(z_0^{m-s+1})}{(m-s+1)!}\Big|\Big)\\
&\leqslant \frac{1}{s}\Big(\sum_{l_1,l_2,\cdots,l_s=1}^{n}|a_{jl_1l_2\cdots l_s}|\frac{|D^{m-s+1}g_{l_1}(0)(z_0^{m-s+1})|}{(m-s+1)!}\frac{|z_{l_2}|}{\|z\|}\cdots\frac{|z_{l_s}|}{\|z\|}\\
&\quad+\sum_{l_1,l_2,\cdots,l_s=1}^{n}|a_{jl_1l_2\cdots l_s}|\frac{|z_{l_1}|}{\|z\|}\frac{|D^{m-s+1}g_{l_2}(0)(z_0^{m-s+1})|}{(m-s+1)!}\frac{|z_{l_3}|}{\|z\|}\cdots\frac{|z_{l_s}|}{\|z\|}+\cdots\\
&\quad+\sum_{l_1,l_2,\cdots,l_s=1}^{n}|a_{jl_1l_2\cdots l_s}|\frac{|z_{l_1}|}{\|z\|}\cdots\frac{|z_{l_{s-1}}|}{\|z\|}\frac{|D^{m-s+1}g_{l_s}(0)(z_0^{m-s+1})|}{(m-s+1)!}\Big)\\
&\leqslant \frac{1}{s}(\underbrace{2A_s+2A_s+\cdots+2A_s}_{s})=2A_s.
\end{aligned}
$$

This implies that

$$
\frac{1}{s!}\Big|D^s f_j(0)\Big(z_0^{s-1},\frac{D^{m-s+1}g(0)(z_0^{m-s+1})}{(m-s+1)!}\Big)\Big|\leqslant 2A_s,\quad z_0\in\partial U^n\,. \tag{2.3}
$$

In particular, when $z_0\in\partial_0 U^n$, by (2.3), we deduce that

$$
\frac{1}{s!}\Big|D^s f_p(0)\Big(z_0^{s-1},\frac{D^{m-s+1}g(0)(z_0^{m-s+1})}{(m-s+1)!}\Big)\Big|\leqslant 2A_s,\quad p=1,2,\cdots,n\,. \tag{2.4}
$$

Taking into account

$$
D^s f_p(0)\Big(z^{s-1},\frac{D^{m-s+1}g(0)(z^{m-s+1})}{(m-s+1)!}\Big)\in H(\overline{U^n}),\quad p=1,2,\cdots,n,
$$

by the maximum modulus theorem of holomorphic functions on the unit polydisk and (2.4), it yields that

$$
\frac{1}{s!}\Big|D^s f_p(0)\Big(z_0^{s-1},\frac{D^{m-s+1}g(0)(z_0^{m-s+1})}{(m-s+1)!}\Big)\Big|\leqslant 2A_s,\quad z_0\in\partial U^n,\ p=1,2,\cdots,n.
$$

We have

$$
\frac{1}{s!}\Big\|D^s f(0)\Big(z_0^{s-1},\frac{D^{m-s+1}g(0)(z_0^{m-s+1})}{(m-s+1)!}\Big)\Big\|\leqslant 2A_s\,.
$$

That is,

$$
\frac{1}{s!}\Big\|D^s f(0)\Big(z^{s-1},\frac{D^{m-s+1}g(0)(z^{m-s+1})}{(m-s+1)!}\Big)\Big\|\leqslant 2A_s\|z\|^m,\quad z\in U^n,\ s=2,3,\cdots,m-1. \tag{2.5}
$$

By Lemma 2.3 and (2.5), we obtain

$$\begin{aligned}\frac{m(m-1)\|D^m f(0)(z^m)\|}{m!} &\leqslant \frac{\|D^m g(0)(z^m)\|}{m!}+\frac{2\|D^2 f(0)(z,\frac{D^{m-1}g(0)(z^{m-1})}{(m-1)!})\|}{2!}\\ &+\cdots+\frac{(m-1)\|D^{m-1}f(0)(z^{m-2},\frac{D^2 g(0)(z^2)}{2!})\|}{(m-1)!}\\ &\leqslant 2\Big(1+\sum_{s=2}^{m-1} sA_s\Big)\|z\|^m.\end{aligned}$$

This implies that

$$\frac{\|D^m f(0)(z^m)\|}{m!}\leqslant \frac{2}{m(m-1)}\Big(1+\sum_{s=2}^{m-1} sA_s\Big)\|z\|^m,\quad z\in U^n,\ m=3,4,\cdots,$$

where $A_s=\max\limits_{1\leqslant p\leqslant n}\Big\{\sum\limits_{l_1,l_2,\cdots,l_s=1}^{n}|a_{p\,l_1l_2\cdots l_s}|\Big\}$, $s=2,3,\cdots,m-1$. Consequently, the desired result holds. By Lemma 2.4, the desired result for $f\in Q_A(U^n)$ or $Q(U^n)$ also holds. This completes the proof.

**Remark 2.1** In general, if $f\in H(U^n)$, then $\frac{D^s f_p(0)(z^s)}{s!}=\sum\limits_{l_1,l_2,\cdots,l_s=1}^{n} a_{p\,l_1l_2\cdots l_s}z_{l_1}z_{l_2}\cdots z_{l_s}$ $(p=1,2,\cdots,n)$, therefore $a_{p\,l_1l_2\cdots l_s}$ in Theorem 2.1 except $a_{p\,\underbrace{l_ql_q\cdots l_q}_{s}}$ $(q=1,2,\cdots,s)$ satisfy a certain condition.

**Corollary 2.1** *Suppose $k\in\mathbb{N}$. If $f\in Q_B(U^n)$ ($Q_A(U^n)$ or $Q(U^n)$), and $z=0$ is a zero of order $k+1$ of $f(z)-z$, then*

$$\frac{\|D^{k+1}f(0)(z^{k+1})\|}{(k+1)!}\leqslant\frac{2}{(k+1)k}\|z\|^{k+1},\quad z\in U^n.$$

*The above estimate is sharp.*

**Proof** When $k=1$, in view of the hypothesis of Corollary 2.1 and Lemma 2.3 (the case of $m=2$), the result follows. When $k\geqslant 2$, $m=k+1$, according to the hypothesis of Corollary 2.1, it is known that $A_s=0$, $s=2,3,\cdots,k$. From Theorem 2.1, we deduce that

$$\frac{\|D^{k+1}f(0)(z^{k+1})\|}{(k+1)!}\leqslant\frac{2}{(k+1)k}\|z\|^{k+1},\quad z\in U^n.$$

This completes the proof.

It is not difficult to verify that

$$f(z)=\Big(\int_0^{z_1}\frac{\mathrm{d}t}{(1-t^k)^{\frac{2}{k}}},\frac{z_2}{z_1}\int_0^{z_1}\frac{\mathrm{d}t}{(1-t^k)^{\frac{2}{k}}},\cdots,\frac{z_n}{z_1}\int_0^{z_1}\frac{\mathrm{d}t}{(1-t^k)^{\frac{2}{k}}}\Big)',\quad z\in U^n$$

satisfies the hypothesis of Corollary 2.1. Taking $z=(r,0,\cdots,0)'$ $(0\leqslant r<1)$, we have

$$\frac{\|D^{k+1}f(0)(z^{k+1})\|}{(k+1)!}=\frac{2}{(k+1)k}r^{k+1}.$$

Hence, the estimate of Corollary 2.1 is sharp.

**Corollary 2.2** *Suppose $k \in \mathbb{N}$. If $f$ is a $k$-fold symmetric quasi-convex mapping of type $B$ (quasi-convex mapping of type $A$ or quasi-convex mapping) defined on $U^n$, and*

$$\begin{aligned}&\frac{D^{tk+1}f_p(0)(z^{tk+1})}{(tk+1)!}\\ =&\sum_{l_1,l_2,\cdots,l_{tk+1}=1}^{n}|a_{p\,l_1l_2\cdots l_{tk+1}}|\mathrm{e}^{\mathrm{i}\frac{\theta_{p\,l_1}+\theta_{pl_2}+\cdots+\theta_{pl_{tk+1}}}{tk+1}}z_{l_1}z_{l_2}\cdots z_{l_{tk+1}},\quad p=1,2,\cdots,n,\end{aligned}$$

*where* $|a_{p\,l_1l_2\cdots l_{tk+1}}|\mathrm{e}^{\mathrm{i}\frac{\theta_{p\,l_1}+\theta_{pl_2}+\cdots+\theta_{pl_{tk+1}}}{tk+1}} = \frac{1}{(tk+1)!}\frac{\partial^{tk+1}f_p(0)}{\partial z_{l_1}\partial z_{l_2}\cdots\partial z_{l_{tk+1}}}$, $\theta_{pl_q} \in (-\pi,\pi]$ $(q=1,2,\cdots,tk+1)$, $l_1,l_2,\cdots,l_{tk+1}=1,2,\cdots,n$, $t=1,2,\cdots$, *then*

$$\frac{\|D^{tk+1}f(0)(z^{tk+1})\|}{(tk+1)!}\leqslant\frac{\prod\limits_{r=1}^{t}((r-1)k+2)}{(tk+1)\cdot t!k^t}\|z\|^{tk+1},\quad z\in U^n,\ t=1,2,\cdots. \tag{2.6}$$

*The above estimates are sharp.*

**Proof** It is known that $z=0$ is a zero of order $k+1$ $(k\in\mathbb{N})$ of $f(z)-z$ if $f$ is a $k$-fold symmetric normalized holomorphic mapping $f(z)$ $(f(z)\not\equiv z)$ defined on $U^n$. In view of the hypothesis of Corollaries 2.2 and 2.1, we conclude that

$$\frac{\|D^{k+1}f(0)(z^{k+1})\|}{(k+1)!}\leqslant\frac{2}{(k+1)k}\|z\|^{k+1},\quad z\in U^n.$$

That is, (2.6) holds for $t=1$. Assume now that (2.6) holds for $t=1,2,\cdots,j$ for some integer $j\geqslant 2$. This implies

$$\frac{\|D^{tk+1}f(0)(z^{tk+1})\|}{(tk+1)!}\leqslant\frac{\prod\limits_{r=1}^{t}((r-1)k+2)}{(tk+1)\cdot t!k^t}\|z\|^{tk+1},\quad z\in U^n,\ t=1,2,\cdots,j. \tag{2.7}$$

In view of (2.7), we take

$$C_{tk+1}=\frac{\prod\limits_{r=1}^{t}((r-1)k+2)}{(tk+1)\cdot t!k^t},\quad t=1,2,\cdots,j.$$

Notice that $A_m=0,\ 2\leqslant m\neq tk+1$ $(t=1,2,\cdots)$ from the hypothesis of Corollary 2.2. Again according to Lemma 2.6 and Theorem 2.1, we deduce that

$$\begin{aligned}&\frac{\|D^{(j+1)k+1}f(0)(z^{(j+1)k+1})\|}{[(j+1)k+1]!}\\ \leqslant&\frac{2}{[(j+1)k+1](j+1)k}\Big[1+\sum_{t=1}^{j}(tk+1)A_{tk+1}\Big]\|z\|^{(j+1)k+1}\\ \leqslant&\frac{2}{[(j+1)k+1](j+1)k}\Big[1+\sum_{t=1}^{j}(tk+1)\frac{\prod\limits_{r=1}^{t}((r-1)k+2)}{(tk+1)\cdot t!k^t}\Big]\|z\|^{(j+1)k+1}\end{aligned}$$

$$= \frac{2}{[(j+1)k+1](j+1)k}\Big[1+\sum_{t=1}^{j}\frac{\prod\limits_{r=1}^{t}((r-1)k+2)}{t!k^t}\Big]\|z\|^{(j+1)k+1}$$
$$= \frac{2}{[(j+1)k+1](j+1)k}\cdot\frac{\prod\limits_{r=2}^{j+1}((r-1)k+2)}{j!k^j}\|z\|^{(j+1)k+1}$$
$$= \frac{\prod\limits_{r=1}^{j+1}((r-1)k+2)}{((j+1)k+1)\cdot(j+1)!k^{j+1}}\|z\|^{(j+1)k+1}.$$

That is, (2.6) holds for $t=j+1$. This completes the proof.

It is easy to verify that

$$f(z)=\Big(\int_0^{z_1}\frac{\mathrm{d}t}{(1-t^k)^{\frac{2}{k}}},\frac{z_2}{z_1}\int_0^{z_1}\frac{\mathrm{d}t}{(1-t^k)^{\frac{2}{k}}},\cdots,\frac{z_n}{z_1}\int_0^{z_1}\frac{\mathrm{d}t}{(1-t^k)^{\frac{2}{k}}}\Big)',\quad z\in U^n$$

satisfies the hypothesis of Corollary 2.2. Taking $z=(r,0,\cdots,0)'$ $(0\leqslant r<1)$, we have

$$\frac{\|D^{tk+1}f(0)(z^{tk+1})\|}{(tk+1)!}\leqslant\frac{\prod\limits_{r=1}^{t}((r-1)k+2)}{(tk+1)\cdot t!k^t}r^{tk+1},\quad t=1,2,\cdots.$$

Hence, the estimate of Corollary 2.2 is sharp.

When $l_1=p,\ l_2=\cdots=l_{tk+1}=l\ (l=1,2,\cdots,n)$, we notice that

$$\arg a_{pp\underbrace{l\cdots l}_{tk}}=\frac{\theta_{pp}+tk\theta_{pl}}{tk+1}$$

for $a_{pp\underbrace{l\cdots l}_{tk}}\neq0$. It is obvious that Corollary 2.2 (the case of $f\in Q_B(U^n)$) is the corresponding result of [6] (the case of $\alpha=0$), and the methods of their proofs are different.

Setting $k=1$ in Corollary 2.2, we can deduce the following result.

**Corollary 2.3** *If $f\in Q_B(U^n)$ $(Q_A(U^n)$ or $Q(U^n))$, and*

$$\frac{D^mf_p(0)(z^m)}{m!}=\sum_{l_1,l_2,\cdots,l_m=1}^{n}|a_{p\,l_1l_2\cdots l_m}|\mathrm{e}^{\mathrm{i}\frac{\theta_{p\,l_1}+\theta_{pl_2}+\cdots+\theta_{pl_m}}{m}}z_{l_1}z_{l_2}\cdots z_{l_m},\quad p=1,2,\cdots,n,$$

*where* $|a_{p\,l_1l_2\cdots l_m}|\mathrm{e}^{\mathrm{i}\frac{\theta_{p\,l_1}+\theta_{pl_2}+\cdots+\theta_{pl_m}}{m}}=\frac{1}{m!}\frac{\partial^mf_p(0)}{\partial z_{l_1}\partial z_{l_2}\cdots\partial z_{l_m}}$, $\mathrm{i}=\sqrt{-1}$, $\theta_{pl_q}\in(-\pi,\pi]$ $(q=1,2,\cdots,m)$, $l_1,l_2,\cdots,l_m=1,2,\cdots,n$, $m=2,3,\cdots$, *then*

$$\frac{\|D^mf(0)(z^m)\|}{m!}\leqslant\|z\|^m,\quad z\in U^n,\ m=2,3,\cdots.$$

*The above estimates are sharp.*

**Proof** Take $k=1$ in Corollary 2.2. Denote $m=t+1$. It follows the result. This completes the proof. The example which shows that the estimates of Corollary 2.3 are sharp is the same as the example in [5].

When $l_1 = p,\ l_2 = \cdots = l_m = l\ (l = 1, 2, \cdots, n)$, note that $\arg a_{pp\underbrace{l\cdots l}_{m-1}} = \frac{\theta_{pp}+(m-1)\theta_{pl}}{m}$ for $a_{pp\underbrace{l\cdots l}_{m-1}} \neq 0$. It is clear that Corollary 2.3 is Theorem 3.1 of [5].

## 3 The Sharp Upper Bounds of Growth Theorem and Distortion Theorem for a *k*-fold Symmetric Quasi-convex Mapping

**Corollary 3.1** *With the same assumptions of Corollary* 2.2*, we have*

$$\|f(z)\| \leqslant \int_0^{\|z\|} \frac{\mathrm{d}t}{(1-t^k)^{\frac{2}{k}}}, \quad z \in U^n.$$

*The above estimate is sharp.*

**Proof** By Corollary 2.2, applying a method similar to that in [5], Corollary 3.1 can be proved (the details of the proof are omitted here).

**Corollary 3.2** *With the same assumptions of Corollary* 2.2*, then we have*

$$\|Df(z)z\| \leqslant \frac{\|z\|}{(1-\|z\|^k)^{\frac{2}{k}}}, \quad z \in U^n.$$

*The above estimate is sharp.*

**Proof** According to Corollary 2.2, with an analogous method in [5], Corollary 3.2 can be proved (the details of the proof are omitted here).

It is easy to verify that

$$f(z) = \Big( \int_0^{z_1} \frac{\mathrm{d}t}{(1-t^k)^{\frac{2}{k}}}, \frac{z_2}{z_1} \int_0^{z_1} \frac{\mathrm{d}t}{(1-t^k)^{\frac{2}{k}}}, \cdots, \frac{z_n}{z_1} \int_0^{z_1} \frac{\mathrm{d}t}{(1-t^k)^{\frac{2}{k}}} \Big)', \quad z \in U^n$$

satisfies the condition of Corollaries 3.1 and 3.2.

Taking $z = (r, 0, \cdots, 0)'$ $(0 \leqslant r < 1)$, we have

$$\|f(z)\| = \int_0^r \frac{\mathrm{d}t}{(1-t^k)^{\frac{2}{k}}} \quad \text{and} \quad \|Df(z)z\| = \frac{r}{(1-r^k)^{\frac{2}{k}}}.$$

Therefore the estimates of Corollaries 3.1 and 3.2 are both sharp.

When $k = 1,\ l_1 = p,\ l_2 = \cdots = l_m = l\ (l = 1, 2, \cdots, n)$, Corollaries 3.2 and 3.3 are Corollaries 3.1 and 3.2 in [5], respectively.

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