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Strong Convergence for Weighted Sums of Negatively Associated Arrays**

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Abstract Let $\{X_{ni}\}$ be an array of rowwise negatively associated random variables and $T_{nk} = \sum\limits_{i=1}^{k} i^\alpha X_{ni}$ for $\alpha \geq -1$, $S_{nk} = \sum\limits_{|i|\leq k} \phi(\frac{i}{n^\eta})\frac{1}{n^\eta}X_{ni}$ for $\eta \in (0,1]$, where ϕ is some function. The author studies necessary and sufficient conditions of

$$\sum_{n=1}^{\infty} A_n P\Big(\max_{1\leq k\leq n} |T_{nk}| > \epsilon B_n\Big) < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} C_n P\Big(\max_{0\leq k\leq m_n} |S_{nk}| > \epsilon D_n\Big) < \infty$$

for all $\epsilon > 0$, where A_n, B_n, C_n and D_n are some positive constants, $m_n \in \mathbb{N}$ with $\frac{m_n}{n^\eta} \to \infty$. The results of Lanzinger and Stadtmüller in 2003 are extended from the i.i.d. case to the case of the negatively associated, not necessarily identically distributed random variables. Also, the result of Pruss in 2003 on independent variables reduces to a special case of the present paper; furthermore, the necessity part of his result is complemented.

Keywords Tail probability, Negatively associated random variable, Weighted sum
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1 Introduction

Let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed (i.i.d.) random variables (r.v.s). In this case the tail behavior of $\sum\limits_{i=1}^{n} X_i$ has been discussed by many authors. Since many useful linear statistics, e.g., least-squares estimators, nonparametric regression function estimators and jackknife estimates among others (see [13, 20]), are weighted sums of random variables, the limit properties of the weighted sums of the i.i.d. r.v.s have received considerable attention from scientists in probability and statistics. See, for example, [4, 5, 7–9, 16]. In particular, Lanzinger and Stadtmüller [8] considered the asymptotic behavior of the distribution function of $\sum\limits_{i=1}^{n} i^\alpha X_i$ for $\alpha \geq -1$ in the so-called Baum-Katz laws (see [3]). These theorems reflect exactly the moment conditions on X_1. Also, some authors discussed precise asymptotics for complete convergence of sums of random variables (see [15, 21]) in recent years.

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In this paper, our aim is to study again the results on weighted sums of Lanzinger and Stadtmüller [8] under not necessarily identically distributed negatively associated arrays assumptions. First we give the definition of negatively associated r.v.s.

Definition 1.1 *A finite family of r.v.s* $\{X_i, 1 \leq i \leq n\}$ *is said to be negatively associated (NA), if for every pair of disjoint subsets A and B of* $\{1, 2, \cdots, n\}$, *we have*

$$\mathrm{Cov}(f_1(X_i, i \in A), f_2(X_j, j \in B)) \leq 0,$$

whenever f_1 *and* f_2 *are coordinatewise increasing and the covariance exists. An infinite family of r.v.s is NA if every finite subfamily is NA.*

The notion of negative association was first introduced by Alam and Saxena [1]. Joag-Dev and Proschan [6] showed that many well-known multivariate distributions possess the NA property. Some examples include: (a) the multinomial, (b) the convolution of unlike multinomials, (c) the multivariate hypergeometric, (d) the Dirichlet, (e) the Dirichlet compound multinomial, (f) the negatively correlated normal distribution, (g) the permutation distribution, (h) the random sampling without replacement, and (i) the joint distribution of ranks. Because of its wide applications in multivariate statistical analysis and system reliability, the notion of negative association has received considerable attention recently. We refer to [6] for fundamental properties, [14] for the three series theorem, [18] for moment equalities, [19] for the law of the iterated logarithm, [2, 10, 11] for complete convergence, [12] for some strong law, and [17] for the central limit theorem of random fields.

The layout of the paper is as follows. In Section 2, we give the main results. The proofs of the main results will be provided in Section 3, and the proof of a preliminary lemma is put in Appendix.

2 Main Results

In this section, let $\{X_{ni}, 1 \leq i \leq n, n \geq 1\}$ be a triangular array of rowwise NA random variables, and X be some random variable. $\{X_{ni}\} \prec X$ means that for all $x > 0$ and some $K > 0$ such that

$$\frac{1}{n}\sum_{i=1}^{n} P(|X_{ni}| \geq x) \leq KP(|X| \geq x).$$

$\{X_{ni}\} \succ X$ stands for $\frac{1}{n}\sum\limits_{i=1}^{n} P(|X_{ni}| \geq x) \geq CP(|X| \geq x)$ for all $x > 0$ and some positive constant C. Define a weighted sum by $T_{nk} = \sum\limits_{i=1}^{k} i^{\alpha} X_{ni}$ for $\alpha \geq -1$.

In the sequel, let $C > 0$ and $c > 0$ denote generic constants whose value may change from one place to another; $a = O(b)$ means $a \leq Cb$; $\log^+ x = \max(1, \log x)$. Our main results are as follows.

Theorem 2.1 *Let* $\alpha \geq 0$.

(a) *Assume* $\gamma > 0$ *and* $\beta > \max\{0, \gamma^{-1} - 1\}$. *If* $\{X_{ni}\} \prec X$ *and* $E|X|^\gamma < \infty$, *then*

$$\sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2} P\Big(\max_{1\le k\le n} |T_{nk}| > \epsilon n^{\alpha+\beta+1}\Big) < \infty \quad \text{for all } \epsilon > 0. \tag{2.1}$$

Conversely, if $\{X_{ni}\} \succ X$, *and assume* $X_{ni} = X_i$ *for* $1 \le i \le n$ *when* $\gamma(\beta+1) < 2$, *then* (2.1) *implies* $E|X|^\gamma < \infty$.

In particular, assume that $\{X_{ni} = X_i, i \ge 1\}$ *is a sequence of identically distributed NA random variables, then* (2.1) *is equivalent to* $E|X_1|^\gamma < \infty$.

(b) *Assume* $\gamma > 1$ *and* $\max\{-\frac{1}{2}, \gamma^{-1} - 1\} < \beta \le 0$. *If* $EX_{ni} = 0$, $\{X_{ni}\} \prec X$ *and* $E|X|^\gamma < \infty$, *then* (2.1) *remains true.*

Conversely, if $\{X_{ni}\} \succ X$, *and assume* $X_{ni} = X_i$ *for* $1 \le i \le n$ *when* $\gamma(\beta+1) < 2$, *then* (2.1) *implies* $E|X|^\gamma < \infty$ *and* $n^{-(\alpha+\beta+1)} \max_{1\le k\le n} \Big|\sum_{i=1}^{k} i^\alpha EX_{ni}\Big| \to 0$, *further* $EX_{ni} = 0$ *when* $\{X_{ni}\}$ *are identically distributed.*

In particular, assume that $\{X_{ni} = X_i, i \ge 1\}$ *is a sequence of identically distributed NA random variables with* $EX_i = 0$, *then* (2.1) *is equivalent to* $E|X_1|^\gamma < \infty$ *and* $EX_1 = 0$.

Corollary 2.1 (a) *Let* $0 < t < 1$, $r > t$. *If* $\{X_{ni}\} \prec X$ *and* $E|X|^r < \infty$, *then*

$$\sum_{n=1}^{\infty} n^{\frac{r}{t}-2} P\Big(\max_{1\le k\le n} \Big|\sum_{i=1}^{k} X_{ni}\Big| > \epsilon n^{\frac{1}{t}}\Big) < \infty \quad \text{for all } \epsilon > 0. \tag{2.2}$$

Conversely, if $\{X_{ni}\} \succ X$, *and assume* $X_{ni} = X_i$ *for* $1 \le i \le n$ *when* $\frac{r}{t} < 2$, *then* (2.2) *implies* $E|X|^r < \infty$.

(b) *Let* $1 \le t < 2$, $r > t$. *If* $EX_{ni} = 0$, $\{X_{ni}\} \prec X$ *and* $E|X|^r < \infty$, *then* (2.2) *remains true.*

Conversely, if $\{X_{ni}\} \succ X$, *and assume* $X_{ni} = X_i$ *for* $1 \le i \le n$ *when* $\frac{r}{t} < 2$, *then* (2.2) *implies* $E|X|^r < \infty$ *and* $n^{-\frac{1}{t}} \max_{1\le k\le n} \Big|\sum_{i=1}^{k} EX_{ni}\Big| \to 0$, *further* $EX_{ni} = 0$ *when* $\{X_{ni}\}$ *are identically distributed.*

Remark 2.1 If $\{X_{ni}, 1 \le i \le n, n \ge 1\}$ is a triangular array of independent random variables with $\{X_{ni}\} \prec X$, Pruss [16] proved the sufficiency part of Corollary 2.1. Therefore, Corollary 2.1 extends the result of Pruss [16] from independent variables to the NA setting and complements his necessity part.

Theorem 2.2 *Let* $-1 < \alpha < 0$. *Set*

(A1) $0 < \gamma < \frac{1}{|\alpha|}$; (i) $\beta > \max\{0, \gamma^{-1} - 1\}$ *or* (ii) $\gamma > 1$, $\max\{\frac{1}{2}, \gamma^{-1} - \alpha\} - 1 < \beta \le 0$, $EX_{ni} = 0$,

(A2) $\gamma = \frac{1}{|\alpha|}$; (i) $\beta > 0$ *or* (ii) $\gamma > 1$, $\max\{\frac{1}{2}, \gamma^{-1} - \alpha\} - 1 < \beta \le 0$, $EX_{ni} = 0$,

(A3) $\gamma > \frac{1}{|\alpha|}$; (i) $\beta > 0$ *or* (ii) $\gamma > 1$, $\max\{\frac{1}{2}, \gamma^{-1} - \alpha\} - 1 < \beta \le 0$, $EX_{ni} = 0$.

(a) *Assume that* (A1) *is satisfied. If* $\{X_{ni}\} \prec X$ *and* $E|X|^\gamma < \infty$, *then*

$$\sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2} P\Big(\max_{1\le k\le n} |T_{nk}| > \epsilon n^{\alpha+\beta+1}\Big) < \infty \quad \text{for all } \epsilon > 0. \tag{2.3}$$

Conversely, if $\{X_{ni}\} \succ X$, *and assume* $X_{ni} = X_i$ *for* $1 \le i \le n$ *when* $\gamma(\beta+1) < 2$, *then* (2.3) *implies* $E|X|^\gamma < \infty$, *and* $n^{-(\alpha+\beta+1)} \max_{1\le k\le n} \Big|\sum_{i=1}^k i^\alpha EX_{ni}\Big| \to 0$ *under* (ii), *further* $EX_{ni} = 0$ *under* (ii) *and* $\{X_{ni}\}$ *are identically distributed.*

(b) *Assume that* (A2) *is satisfied. If* $\{X_{ni}\} \prec X$ *and* $E|X|^\gamma < \infty$, *then*

$$\sum_{n=1}^{\infty} \frac{n^{\gamma(\alpha+\beta+1)-1}}{\log^+ n} P\Big(\max_{1\le k\le n} |T_{nk}| > \epsilon n^{\alpha+\beta+1}\Big) < \infty \quad \text{for all } \epsilon > 0. \tag{2.4}$$

Conversely, if $\{X_{ni}\} \succ X$, *and assume* $X_{ni} = X_i$ *for* $1 \le i \le n$ *when* $\gamma(\alpha+\beta+1) < 1$, *then* (2.4) *implies* $E|X|^\gamma < \infty$, *and* $n^{-(\alpha+\beta+1)} \max_{1\le k\le n} \Big|\sum_{i=1}^k i^\alpha EX_{ni}\Big| \to 0$ *under* (ii), *further* $EX_{ni} = 0$ *under* (ii) *and* $\{X_{ni}\}$ *are identically distributed.*

(c) *Assume that* (A3) *is satisfied. If* $\{X_{ni}\} \prec X$ *and* $E|X|^\gamma < \infty$, *then*

$$\sum_{n=1}^{\infty} n^{\gamma(\alpha+\beta+1)-1} P\Big(\max_{1\le k\le n} |T_{nk}| > \epsilon n^{\alpha+\beta+1}\Big) < \infty \quad \text{for all } \epsilon > 0. \tag{2.5}$$

Conversely, if $\{X_{ni}\} \succ X$, *and assume* $X_{ni} = X_i$ *for* $1 \le i \le n$ *when* $\gamma(\alpha+\beta+1) < 1$, *then* (2.5) *implies* $E|X|^\gamma < \infty$, *and* $n^{-(\alpha+\beta+1)} \max_{1\le k\le n} \Big|\sum_{i=1}^k i^\alpha EX_{ni}\Big| \to 0$ *under* (ii), *further* $EX_{ni} = 0$ *under* (ii) *and* $\{X_{ni}\}$ *are identically distributed.*

Theorem 2.3 *Let* $\alpha = -1$.

(a) *Assume* $0 < \gamma < 1$ *and* $\beta > \gamma^{-1} - 1$. *If* $\{X_{ni}\} \prec X$ *and* $E\Big[\frac{|X|^\gamma}{(\log^+ |X|)^{2\gamma}}\Big] < \infty$, *then*

$$\sum_{n=1}^{\infty} \frac{n^{\gamma(\beta+1)-2}}{(\log^+ n)^\gamma} P\Big(\max_{1\le k\le n} |T_{nk}| > \epsilon n^{\beta}\log n\Big) < \infty \quad \text{for all } \epsilon > 0. \tag{2.6}$$

Conversely, if $\{X_{ni}\} \succ X$, *and assume* $X_{ni} = X_i$ *for* $1 \le i \le n$ *when* $\gamma(\beta+1) < 2$, *then* (2.6) *implies* $E\Big[\frac{|X|^\gamma}{(\log^+ |X|)^{2\gamma}}\Big] < \infty$.

(b) *Assume* $\beta > 0$. *If* $\{X_{ni}\} \prec X$ *and* $E\Big[\frac{|X|}{(\log^+ |X|)^{2}}\Big] < \infty$, *then*

$$\sum_{n=1}^{\infty} \frac{n^{\beta-1}}{(\log^+ n)^2} P\Big(\max_{1\le k\le n} |T_{nk}| > \epsilon n^{\beta}\log n\Big) < \infty \quad \text{for all } \epsilon > 0. \tag{2.7}$$

Conversely, if $\{X_{ni}\} \succ X$, *and assume* $X_{ni} = X_i$ *for* $1 \le i \le n$ *when* $\beta < 1$, *then* (2.7) *implies* $E\Big[\frac{|X|}{(\log^+ |X|)^{2}}\Big] < \infty$.

(c) *Assume* $\gamma > 1$ *and* $\beta > 0$. *If* $\{X_{ni}\} \prec X$ *and* $E\Big[\frac{|X|^\gamma}{(\log^+ |X|)^{2\gamma}}\Big] < \infty$, *then*

$$\sum_{n=1}^{\infty} \frac{n^{\gamma\beta-1}}{(\log^+ n)^\gamma} P\Big(\max_{1\le k\le n} |T_{nk}| > \epsilon n^{\beta}\log n\Big) < \infty \quad \text{for all } \epsilon > 0. \tag{2.8}$$

Conversely, if $\{X_{ni}\} \succ X$, *and assume* $X_{ni} = X_i$ *for* $1 \le i \le n$ *when* $\gamma\beta < 1$, *then* (2.8) *implies* $E\Big[\frac{|X|^\gamma}{(\log^+ |X|)^{2\gamma}}\Big] < \infty$.

Theorem 2.4 *Let $0 < \eta \le 1$, and let $\phi : \mathcal{R} \to \mathcal{R}$ be positive, continuous, decreasing and integrable on* $[0, \infty)$. *Assume that $\{X_{ni},\ i \in \mathcal{Z},\ n \ge 1\}$ is an array of rowwise NA random variables with $P(|X_{ni}| \ge x) \le CP(|X| \ge x)$ for all $x > 0$, $i \in \mathcal{Z}$ and $n \ge 1$. If one of the following conditions* (a)–(c) *is satisfied:*

(a) *Let $\gamma \ge 1,\ \beta > 0$. Then $E|X|^\gamma < \infty$;*

(b) *Let $\gamma > 1,\ \max\{-\frac{\eta}{2}, \eta(\gamma^{-1} - 1)\} < \beta \le 0$. Then $E|X|^\gamma < \infty$ and $EX_{ni} = 0$;*

(c) *Let $0 < \gamma < 1,\ \beta > \eta(\gamma^{-1} - 1)$ and $\int_0^\infty (\phi(t))^\gamma \mathrm{d}t < \infty$. Then $E|X|^\gamma < \infty$,*

then for $m_n \in \mathbb{N}$ with $\frac{m_n}{n^\eta} \to \infty$,

$$\sum_{n=1}^{\infty} n^{\gamma(\eta+\beta)-1-\eta} P\Big(\max_{0\le k\le m_n}\Big|\sum_{|i|\le k}\phi\Big(\frac{i}{n^\eta}\Big)\frac{1}{n^\eta}X_{ni}\Big| > \epsilon n^\beta\Big) < \infty \quad \text{for all } \epsilon > 0, \tag{2.9}$$

$$\sum_{n=1}^{\infty} n^{\gamma(\eta+\beta)-1-\eta} P\Big(\Big|\sum_{i=-\infty}^{\infty}\phi\Big(\frac{i}{n^\eta}\Big)\frac{1}{n^\eta}X_{ni}\Big| > \epsilon n^\beta\Big) < \infty \quad \text{for all } \epsilon > 0. \tag{2.10}$$

Conversely, assume that $\{X, X_{ni}, |i| \le m_n, n \ge 1\}$ is an array of identically distributed rowwise NA random variables, and further that X_{ni} are independent when $\gamma(\eta + \beta) - 1 - \eta < 0$. Then (2.9) *implies* (a), (b) *and* (c).

Remark 2.2 Assume that $X_{ni} = X_i$ for all $n \ge 1$, and that $\{X_i\}$ is a sequence of i.i.d. random variables. Then Theorems 2.1–2.4 reduce to the results of Lanzinger and Stadtmüller [8]. So, Theorems 2.1–2.4 extend the results of Lanzinger and Stadtmüller [8] from the i.i.d. case to the case of negatively associated, not necessarily identically distributed random variables.

3 Proofs of Main Results

We now give two lemmas, which will be useful later in the section.

Lemma 3.1 *Let $\{X_i,\ i \ge 1\}$ be a sequence of NA random variables and $\{a_{ni}, 1 \le i \le n, n \ge 1\}$ an array of real numbers. Then there exists some constant $A > 0$ such that, for $n \ge 1$,*

$$\frac{1}{2}\sum_{j=1}^{n} P(|a_{nj}X_j| > \epsilon) \le (1 + A)P\Big(\max_{1\le j\le n}|a_{nj}X_j| > \epsilon\Big) + \sum_{j=1}^{n} P(|a_{nj}X_j| > \epsilon)P\Big(\max_{1\le j\le n}|a_{nj}X_j| > \epsilon\Big).$$

Further, if $\lim\limits_{n\to\infty} P\Big(\max\limits_{1\le j\le n}|a_{nj}X_j| > \epsilon\Big) = 0$, then there exists some constant $C > 0$ such that for n large enough we have $\sum\limits_{j=1}^{n} P(|a_{nj}X_j| > \epsilon) \le CP\Big(\max\limits_{1\le j\le n}|a_{nj}X_j| > \epsilon\Big)$.

The proof of Lemma 3.1 can be found in the proof of Theorem 2.1 in [10].

Lemma 3.2 *Let $\{Y_{ni}, 1 \le i \le k_n, n \ge 1\}$ be a triangular array of rowwise NA random variables, where $\{k_n, n \ge 1\}$ is a sequence of positive integer numbers, and let $\{a_n, n \ge 1\}$ be* *a sequence of positive constants. Suppose that for every $\epsilon > 0$ there exists $q \geq 2$ such that for $\delta = \frac{\epsilon}{72q}$,*

(i) $\sum_{n=1}^{\infty} a_n \sum_{i=1}^{k_n} P(|Y_{ni}| > \delta) < \infty$;

(ii) $\sum_{n=1}^{\infty} a_n \Big(\sum_{i=1}^{k_n} EY_{ni}^2 I(|Y_{ni}| \leq \delta)\Big)^q < \infty$;

(iii) *in the case* $\liminf_{n\to\infty} a_n = 0$, $\sum_{i=1}^{k_n} P(|Y_{ni}| > \delta) \to 0$ *and* $\sum_{i=1}^{k_n} EY_{ni}^2 I(|Y_{ni}| \leq \delta) \to 0$, *as* $n \to \infty$.

If $\max_{1\leq k\leq k_n} \Big| \sum_{i=1}^{k} EY_{ni} I(|Y_{ni}| \leq \delta)\Big| \to 0$ *as* $n \to \infty$, *then*

$$\sum_{n=1}^{\infty} a_n P\Big(\max_{1\leq k\leq k_n} \Big| \sum_{i=1}^{k} Y_{ni}\Big| > \epsilon\Big) < \infty \quad \textit{for all } \epsilon > 0.$$

The proof of Lemma 3.2 is given in Appendix.

Remark 3.1 (i) and (ii) in Lemma 3.2 imply that when $\liminf_{n\to\infty} a_n > 0$,

$$\sum_{i=1}^{k_n} P(|Y_{ni}| > \delta) \to 0 \quad \text{and} \quad \sum_{i=1}^{k_n} EY_{ni}^2 I(|Y_{ni}| \leq \delta) \to 0, \quad \text{as } n \to \infty. \tag{3.1}$$

Therefore, (i), (ii) and (iii) in Lemma 3.2 show that (3.1) holds.

Proof of Theorem 2.1 (a) **Sufficiency** We apply Lemma 3.2 here. Set

$$Y_{ni} = n^{-(\alpha+\beta+1)} i^{\alpha} X_{ni}.$$

We now verify that Y_{ni} satisfy the conditions of Lemma 3.2.

(i) We observe for all $\epsilon > 0$,

$$\sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2} \sum_{i=1}^{n} P(|Y_{ni}| > \epsilon) = O(1) \sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2} \sum_{i=1}^{n} P(|X| > \epsilon n^{\alpha+\beta+1} i^{-\alpha}).$$

Note that

$$\sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2} \sum_{i=1}^{n} P(|X| > \epsilon n^{\alpha+\beta+1} i^{-\alpha}) < \infty$$
$$\text{iff } \int_1^{\infty} x^{\gamma(\beta+1)-2} \mathrm{d}x \int_1^x P(|X| > \epsilon x^{\alpha+\beta+1} y^{-\alpha}) \mathrm{d}y < \infty.$$

Let $u = x^{\alpha+\beta+1} y^{-\alpha}$, $v = y$. Then

$$\begin{aligned}
\infty &> \int_1^{\infty} x^{\gamma(\beta+1)-2} \mathrm{d}x \int_1^x P(|X| > \epsilon x^{\alpha+\beta+1} y^{-\alpha}) \mathrm{d}y \\
&= \frac{1}{\alpha+\beta+1} \int_1^{\infty} u^{\frac{\gamma(\beta+1)-(\alpha+\beta+2)}{\alpha+\beta+1}} P(|X| > \epsilon u) \mathrm{d}u \int_1^{u^{\frac{1}{1+\beta}}} v^{\frac{\alpha\gamma(\beta+1)-\alpha}{\alpha+\beta+1}} \mathrm{d}v \\
&= \frac{1}{(\alpha\gamma+1)(\beta+1)} \Big[\int_1^{\infty} u^{\gamma-1} P(|X| > \epsilon u) \mathrm{d}u - \int_1^{\infty} u^{\frac{\gamma(\beta+1)-(\alpha+\beta+2)}{\alpha+\beta+1}} P(|X| > \epsilon u) \mathrm{d}u \Big],
\end{aligned}$$

which is equivalent to $E|X|^\gamma < \infty$. Therefore

$$\sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2} \sum_{i=1}^{n} P(|Y_{ni}| > \epsilon) < \infty. \tag{3.2}$$

(ii) Note $E|X|^\gamma < \infty$ and $\gamma(\beta+1)-1 > 0$, so

$$\begin{aligned}\sum_{i=1}^{n} P(|Y_{ni}| > \epsilon) &= O(1) \sum_{i=1}^{n} P(|X| > \epsilon n^{\alpha+\beta+1} i^{-\alpha}) \\ &\le O(1)\epsilon^{\gamma} n^{-\gamma(\alpha+\beta+1)} E|X|^\gamma \sum_{i=1}^{n} i^{\alpha\gamma} = O(n^{-[\gamma(\beta+1)-1]}) \to 0. \end{aligned}\tag{3.3}$$

(iii) If $\gamma \ge 2$, then $EX^2 < \infty$. Therefore we have

$$\sum_{i=1}^{n} EY_{ni}^2 I(|Y_{ni}| \le \epsilon) \le n^{-2(\alpha+\beta+1)} EX^2 \sum_{i=1}^{n} i^{2\alpha} = O(n^{-(2\beta+1)}).$$

If $0 < \gamma < 2$ and $E|X|^\gamma < \infty$, then

$$\sum_{i=1}^{n} EY_{ni}^2 I(|Y_{ni}| \le \epsilon) \le \epsilon^{2-\gamma} n^{-\gamma(\alpha+\beta+1)} E|X|^\gamma \sum_{i=1}^{n} i^{\alpha\gamma} = O(n^{-[\gamma(\beta+1)-1]}).$$

Hence, according to $\beta > \max\{0, \gamma^{-1}-1\}$ we have $\sum\limits_{i=1}^{n} EY_{ni}^2 I(|Y_{ni}| \le \epsilon) \to 0$ and there exists $q \ge 2$ such that

$$\sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2} \Big(\sum_{i=1}^{n} EY_{ni}^2 I(|Y_{ni}| \le \epsilon)\Big)^q < \infty. \tag{3.4}$$

(iv) Similarly to the arguments as in (iii), one can verify

$$\max_{1\le k\le n} \Big|\sum_{i=1}^{k} EY_{ni} I(|Y_{ni}| \le \epsilon)\Big| \to 0, \quad \text{as } n \to \infty.$$

Therefore, from (i)–(iv), according to Lemma 3.2, we have

$$\sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2} P\Big(\max_{1\le k\le n} |T_{nk}| > \epsilon n^{\alpha+\beta+1}\Big) < \infty \quad \text{for all } \epsilon > 0.$$

Necessity If (2.1) holds, then when $\gamma(\beta+1) \ge 2$,

$$P\Big(\max_{1\le k\le n} |T_{nk}| > \epsilon n^{\alpha+\beta+1}\Big) \to 0 \quad \text{for all } \epsilon > 0 \quad \text{as } n \to \infty. \tag{3.5}$$

When $\gamma(\beta+1) < 2$, noticing $X_{ni} = X_i$, we find

$$\begin{aligned}\infty &> \sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2} P\Big(\max_{1\le k\le n} |T_{nk}| > \epsilon n^{\alpha+\beta+1}\Big) \\ &= \sum_{j=1}^{\infty} \sum_{2^{j-1}\le n<2^j} n^{\gamma(\beta+1)-2} P\Big(\max_{1\le k\le n} \Big|\sum_{i=1}^{k} i^\alpha X_i\Big| > \epsilon n^{\alpha+\beta+1}\Big)\end{aligned}$$

$$\geq C\sum_{j=1}^{\infty}2^{j[\gamma(\beta+1)-1]}P\Big(\max_{1\leq k\leq 2^{j-1}}\Big|\sum_{i=1}^{k}i^{\alpha}X_i\Big|>\epsilon 2^{j(\alpha+\beta+1)}\Big),$$

which follows from $\gamma(\beta+1)-1>0$ that

$$P\Big(\max_{1\leq k\leq 2^{j-1}}\Big|\sum_{i=1}^{k}i^{\alpha}X_i\Big|>\epsilon 2^{j(\alpha+\beta+1)}\Big)\to 0\quad\text{for all }\epsilon>0\quad\text{as }j\to\infty. \tag{3.6}$$

For any $n\geq 1$, there exists some $j\geq 1$ such that $2^{j-1}\leq n<2^j$. Therefore, from (3.6) it yields that

$$\begin{aligned}&P\Big(\max_{1\leq k\leq n}\Big|\sum_{i=1}^{k}i^{\alpha}X_i\Big|>\epsilon n^{\alpha+\beta+1}\Big)\\ &\leq P\Big(\max_{1\leq k\leq 2^{j}}\Big|\sum_{i=1}^{k}i^{\alpha}X_i\Big|>\epsilon\cdot 2^{2(\alpha+\beta+1)}\cdot 2^{(j+1)(\alpha+\beta+1)}\Big)\to 0\quad\text{for all }\epsilon>0\quad\text{as }j\to\infty,\end{aligned}$$

i.e., (3.5) holds for $\gamma(\beta+1)<2$. Hence, according to (3.5) and Lemma 3.1, we have

$$\sum_{i=1}^{n}P(|i^{\alpha}X_{ni}|>\epsilon n^{\alpha+\beta+1})\leq CP\Big(\max_{1\leq i\leq n}|i^{\alpha}X_{ni}|>\epsilon n^{\alpha+\beta+1}\Big). \tag{3.7}$$

Note $\max_{1\leq k\leq n}|k^{\alpha}X_{nk}|\leq 2\max_{1\leq k\leq n}|T_{nk}|$. Therefore, from (2.1), (3.7) and $\{X_{ni}\}\succ X$ we find

$$\sum_{n=1}^{\infty}n^{\gamma(\beta+1)-2}\sum_{i=1}^{n}P(|i^{\alpha}X|>\epsilon n^{\alpha+\beta+1})<\infty\quad\text{for all }\epsilon>0.$$

Hence, similarly to the arguments in (i), we have $E|X|^{\gamma}<\infty$.

(b) **Sufficiency** Since $\max\{-\frac{1}{2},\gamma^{-1}-1\}<\beta\leq 0$ implies $2\beta+1>0$ and $\gamma(\beta+1)-1>0$, (i)–(iii) in (a) are still true. So, according to Lemma 3.2, it suffices to show

$$\max_{1\leq k\leq n}\Big|\sum_{i=1}^{k}EY_{ni}I(|Y_{ni}|\leq\epsilon)\Big|\to 0,\quad\text{as }n\to\infty.$$

By $EX_{ni}=0$ and $E|X|^{\gamma}<\infty$ ($\gamma>1$) we have

$$\begin{aligned}\max_{1\leq k\leq n}\Big|\sum_{i=1}^{k}EY_{ni}I(|Y_{ni}|\leq\epsilon)\Big|&\leq\sum_{i=1}^{n}E|Y_{ni}|I(|Y_{ni}|>\epsilon)\leq\epsilon^{1-\gamma}\sum_{i=1}^{n}E|Y_{ni}|^{\gamma}\\&=O\Big(n^{-[\gamma(\beta+1)-1]}\Big)\to 0,\quad\text{as }n\to\infty.\end{aligned}$$

Necessity Similarly to the arguments in (a), we have $E|X|^{\gamma}<\infty$. Thus, from the sufficiency we know

$$\sum_{n=1}^{\infty}n^{\gamma(\beta+1)-2}P\Big(\max_{1\leq k\leq n}|T_{nk}-ET_{nk}|>\epsilon n^{\alpha+\beta+1}\Big)<\infty\quad\text{for all }\epsilon>0. \tag{3.8}$$

From (2.1) and (3.8) it follows that $\frac{\max_{1\le k\le n}\left|\sum_{i=1}^{k} i^\alpha EX_{ni}\right|}{n^{\alpha+\beta+1}} \to 0$. Further if $\{X, X_{ni}, 1\le i\le n, n\ge 1\}$ is an array of identically distributed random variables, then

$$\frac{\max\limits_{1\le k\le n}\left|\sum\limits_{i=1}^{k} i^\alpha EX_{ni}\right|}{n^{\alpha+\beta+1}} = \frac{|EX|\sum\limits_{i=1}^{n} i^\alpha}{n^{\alpha+\beta+1}} \sim \frac{|EX|}{(1+a)n^\beta} \to 0.$$

Since $\beta\le 0$, $EX_{ni} = EX = 0$.

Proof of Theorem 2.2 We only prove (b). The proofs of (a) and (c) are analogous.

Sufficiency We apply Lemma 3.2. Set $Y_{ni} = n^{-(\alpha+\beta+1)} i^\alpha X_{ni}$. We now verify that Y_{ni} satisfy the conditions of Lemma 3.2.

(1) We observe that for all $\epsilon>0$,

$$\sum_{n=1}^{\infty}\frac{n^{\gamma(\alpha+\beta+1)-1}}{\log^+ n}\sum_{i=1}^{n} P(|Y_{ni}|>\epsilon) = O(1)\sum_{n=1}^{\infty}\frac{n^{\gamma(\alpha+\beta+1)-1}}{\log^+ n}\sum_{i=1}^{n} P(|X|>\epsilon n^{\alpha+\beta+1} i^{-\alpha}).$$

Note that

$$\sum_{n=1}^{\infty}\frac{n^{\gamma(\alpha+\beta+1)-1}}{\log^+ n}\sum_{i=1}^{n} P(|X|>\epsilon n^{\alpha+\beta+1} i^{-\alpha})<\infty$$
$$\text{iff } \int_1^\infty \frac{x^{\gamma(\alpha+\beta+1)-1}}{\log^+ x}\mathrm{d}x\int_1^x P(|X|>\epsilon x^{\alpha+\beta+1}y^{-\alpha})\mathrm{d}y<\infty.$$

Let $u = x^{\alpha+\beta+1}y^{-\alpha},\ v=y$. Then

$$\begin{aligned}
\infty &> \int_1^\infty \frac{x^{\gamma(\alpha+\beta+1)-1}}{\log^+ x}\mathrm{d}x\int_1^x P(|X|>\epsilon x^{\alpha+\beta+1}y^{-\alpha})\mathrm{d}y\\
&= \int_1^\infty u^{\gamma-1}P(|X|>\epsilon u)\mathrm{d}u\int_1^{u^{\frac{1}{1+\beta}}}\frac{1}{v\log^+(uv^\alpha)}\mathrm{d}v\\
&= \frac{1}{\alpha}\int_1^\infty u^{\gamma-1}P(|X|>\epsilon u)\cdot \log(\log^+ u+\alpha\log^+ v)\big|_{v=1}^{v=u^{\frac{1}{1+\beta}}}\mathrm{d}u\\
&= \frac{1}{\alpha}\int_1^\infty u^{\gamma-1}\Big(\log^+\frac{\alpha+\beta+1}{1+\beta}+\log\log^+ u-\log\log^+ u\Big)P(|X|>\epsilon u)\mathrm{d}u\\
&= \frac{1}{\alpha}\log^+\frac{\alpha+\beta+1}{1+\beta}\int_1^\infty u^{\gamma-1}P(|X|>\epsilon u)\mathrm{d}u,
\end{aligned}$$

which is equivalent to $E|X|^\gamma<\infty$. Therefore

$$\sum_{n=1}^{\infty}\frac{n^{\gamma(\alpha+\beta+1)-1}}{\log^+ n}\sum_{i=1}^{n} P(|Y_{ni}|>\epsilon)<\infty.$$

(2) Note $\gamma(\beta+1)-1 = -\frac{\alpha+\beta+1}{\alpha}>0$ and $\beta>0$ under (i), $2\beta+1>0$ and $\gamma(\beta+1)-1>-\gamma\alpha>0$ under (ii). So $\sum_{i=1}^{n} P(|Y_{ni}|\ge\epsilon) = O(n^{-[\gamma(\beta+1)-1]})\to 0$,

$$\sum_{i=1}^{n} EY_{ni}^2 I(|Y_{ni}|\le\epsilon) = \begin{cases} O(n^{-(2\beta+1)}), & \text{if } \gamma\ge 2,\\ O(n^{-[\gamma(\beta+1)-1]}), & \text{if } \gamma<2\end{cases} \to 0$$

and further there exists some $q \ge 2$ such that

$$\sum_{n=1}^{\infty} \frac{n^{\gamma(\alpha+\beta+1)-1}}{\log^+ n} \Big(\sum_{i=1}^{n} EY_{ni}^2 I(|Y_{ni}| \le \epsilon) \Big)^q < \infty.$$

(3) Since $\gamma > 1$,

$$\max_{1\le k\le n} \Big| \sum_{i=1}^{k} EY_{ni} I(|Y_{ni}| \le \epsilon) \Big| = \begin{cases} O(n^{-\beta}), & \text{under (i)}, \\ O(n^{-[\gamma(\beta+1)-1]}), & \text{under (ii)} \end{cases} \to 0.$$

Therefore, according to (1)–(3), (2.4) holds by using Lemma 3.2.

Necessity Following the line used in the proof of Theorem 2.1, the necessity can be proved similarly.

Proof of Theorem 2.3 We only prove (a). The proofs of (b) and (c) are similar.

Sufficiency We apply Lemma 3.2. Set $Z_{ni} = i^{-1} n^{-\beta} (\log n)^{-1} X_{ni}$. We now verify that Z_{ni} satisfy the conditions of Lemma 3.2.

(1) We observe that for all $\epsilon > 0$,

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{n^{\gamma(\beta+1)-2}}{(\log^+ n)^{\gamma}} \sum_{i=1}^{n} P(|Z_{ni}| > \epsilon) &= O(1) \sum_{n=1}^{\infty} \frac{n^{\gamma(\beta+1)-2}}{(\log^+ n)^{\gamma}} \sum_{i=1}^{n} P(|X| > \epsilon i n^{\beta} \log n) \\ &= O(1) \sum_{n=1}^{\infty} \frac{n^{\gamma(\beta+1)-2}}{(\log^+ n)^{\gamma}} \sum_{i=1}^{n} \sum_{k > i n^{\beta} \log n} P(k-1 \le |X| < k) \\ &= O(1) \sum_{n=1}^{\infty} \frac{n^{\gamma(\beta+1)-2-\beta}}{(\log^+ n)^{\gamma+1}} \sum_{k=n^{\beta}\log^+ n+1}^{n^{\beta+1}\log^+ n} kP(k-1 \le |X| < k) \\ &\quad + O(1) \sum_{n=1}^{\infty} \frac{n^{\gamma(\beta+1)-1}}{(\log^+ n)^{\gamma}} \sum_{k=n^{\beta+1}\log^+ n+1}^{\infty} P(k-1 \le |X| < k) \\ &:= O(1)\{\mathrm{I}_{n1} + \mathrm{I}_{n2}\}. \end{aligned}$$

Note that if $f(x) = x^{\beta} \log x$ then $f^{-1}(x) \sim (\frac{\beta x}{\log x})^{\frac{1}{\beta}}$ as $x \to \infty$, where f^{-1} is the inverse function of f. Hence

$$\begin{aligned} \mathrm{I}_{n1} &\le \sum_{n=\mathrm{e}}^{\infty} \frac{n^{\gamma(\beta+1)-2-\beta}}{(\log n)^{\gamma+1}} \sum_{k=n^{\beta}\log n+1}^{n^{\beta+1}\log n} kP(k-1 \le |X| < k) + C \\ &\le C + \sum_{k=\mathrm{e}^{\beta}+1}^{\mathrm{e}^{\beta+1}} kP(k-1 \le |X| < k) \sum_{n=\mathrm{e}}^{(\frac{\beta k}{\log k})^{\frac{1}{\beta}}} \frac{n^{\gamma(\beta+1)-2-\beta}}{(\log n)^{\gamma+1}} \\ &\quad + \sum_{k=\mathrm{e}^{\beta+1}+1}^{\infty} kP(k-1 \le |X| < k) \sum_{n=(\frac{(\beta+1)k}{\log k})^{\frac{1}{\beta+1}}}^{(\frac{\beta k}{\log k})^{\frac{1}{\beta}}} \frac{n^{\gamma(\beta+1)-2-\beta}}{(\log n)^{\gamma+1}} \\ &\le C + C \sum_{k=\mathrm{e}^{\beta+1}+1}^{\infty} kP(k-1 \le |X| < k) \cdot \frac{1}{(\log k)^{\gamma+1}} \cdot \Big(\frac{k}{\log k} \Big)^{\frac{\gamma(\beta+1)-1-\beta}{\beta+1}} \end{aligned}$$

$$\leq C + C \sum_{k=e^{\beta+1}+1}^{\infty} \frac{k^{\gamma}}{(\log k)^{2\gamma}} P(k-1 \leq |X| < k)$$
$$\leq C + CE\Big[\frac{|X|^{\gamma}}{(\log^{+}|X|)^{2\gamma}}\Big] < \infty.$$

Similarly

$$\mathrm{I}_{n2} = C + \sum_{n=e}^{\infty} \frac{n^{\gamma(\beta+1)-1}}{(\log^{+} n)^{\gamma}} \sum_{k=n^{\beta+1}\log^{+} n+1}^{\infty} P(k-1 \leq |X| < k)$$
$$\leq C + CE\Big[\frac{|X|^{\gamma}}{(\log^{+}|X|)^{2\gamma}}\Big] < \infty.$$

(2) Since $E\Big[\frac{|X|^{\gamma}}{(\log^{+}|X|)^{2\gamma}}\Big] < \infty$ and $\gamma(\beta+1) > 1$, and noticing $\frac{|x|^{\gamma}}{(\log|x|)^{2\gamma}}$ is increasing function for $|x| > \mathrm{e}$, we have

$$\sum_{i=1}^{n} P(|Z_{ni}| \geq \epsilon) = O(1) \sum_{i=1}^{n} P\Big(\frac{|X|^{\gamma}}{(\log^{+}|X|)^{2\gamma}} \geq \frac{\epsilon^{\gamma} i^{\gamma} n^{\beta\gamma}(\log^{+} n)^{\gamma}}{(\log^{+}\epsilon + \log^{+} i + \beta\log^{+} n + \log\log^{+} n)^{2\gamma}}\Big)$$
$$= O(1) \frac{(\log^{+} n)^{2\gamma}}{n^{\beta\gamma(\log n)^{\gamma}}} \sum_{i=1}^{n} i^{-\gamma} = O(n^{-[\gamma(\beta+1)-1]}\log n) \to 0.$$

(3) Since $E\Big[\frac{|X|^{\gamma}}{(\log^{+}|X|)^{2\gamma}}\Big] < \infty$ and $\gamma(\beta+1) > 1$, there exists some $\delta > 0$ such that $(\gamma-\delta)(\beta+1) > 1$ and $E|X|^{\gamma-\delta} < \infty$. Hence

$$\sum_{i=1}^{n} EZ_{ni}^{2} I(|Z_{ni}| \leq \epsilon) \leq \epsilon^{2-(\gamma-\delta)} \sum_{i=1}^{n} E|Z_{ni}|^{\gamma-\delta} = O(n^{-[(\gamma-\delta)(\beta+1)-1]}(\log n)^{-(\gamma-\delta)}).$$

Therefore $\sum_{i=1}^{n} EZ_{ni}^{2} I(|Z_{ni}| \leq \epsilon) \to 0$ and there exists $q \geq 2$ such that

$$\sum_{n=1}^{\infty} \frac{n^{\gamma(\beta+1)-1}}{(\log^{+} n)^{\gamma}} \Big(\sum_{i=1}^{n} EZ_{ni}^{2} I(|Z_{ni}| \leq \epsilon)\Big)^{q} < \infty.$$

(4) Since $0 < \gamma < 1$, similarly to the arguments in (3) we find

$$\max_{1\leq k\leq n} \Big|\sum_{i=1}^{k} EZ_{ni} I(|Z_{ni}| \leq \epsilon)\Big| \to 0, \quad \text{as } n \to \infty.$$

Therefore, according to Lemma 3.2, from (1)–(4) we obtain (2.6).

Necessity If (2.6) holds, similarly to the arguments in Theorem 2.1 we get

$$P\Big(\max_{1\leq k\leq n} |T_{nk}| > \epsilon n^{\beta}\log n\Big) \to 0, \quad \text{as } n \to \infty.$$

Thus, by using Lemma 3.2, we have

$$\infty > \sum_{n=1}^{\infty} \frac{n^{\gamma(\beta+1)-2}}{(\log^{+} n)^{\gamma}} \sum_{i=1}^{n} P(|i^{-1}X| > \epsilon n^{\beta}\log n)$$

$$\geq \sum_{k=\mathrm{e}}^{\infty} P\Big(k^{\beta+1}\log k < \frac{|X|}{\epsilon} \leq (k+1)^{\beta+1}\log(k+1)\Big) \sum_{n=\mathrm{e}}^{k} \frac{n^{\gamma(\beta+1)-1}}{(\log n)^{\gamma}}.$$

We find

$$\sum_{n=\mathrm{e}}^{k} \frac{n^{\gamma(\beta+1)-1}}{(\log n)^{\gamma}} \sim \int_{\mathrm{e}}^{k} \frac{x^{\gamma(\beta+1)-1}}{(\log x)^{\gamma}}\mathrm{d}x \sim \frac{1}{\gamma(\beta+1)} \cdot \frac{k^{\gamma(\beta+1)}}{(\log k)^{\gamma}}, \quad \text{as } k \to \infty.$$

Therefore

$$\sum_{k=\mathrm{e}}^{\infty} \frac{k^{\gamma(\beta+1)}}{(\log k)^{\gamma}} P\Big(k^{\beta+1}\log k < \frac{|X|}{\epsilon} \leq (k+1)^{\beta+1}\log(k+1)\Big) < \infty,$$

which is equivalent to $E\Big[\frac{|X|^{\gamma}}{(\log^{+}|X|)^{2\gamma}}\Big] < \infty$.

Proof of Theorem 2.4 We only prove (2.9). The proof of (2.10) is similar.

Sufficiency We apply Lemma 3.2. Set $W_{ni} = \phi\big(\frac{i}{n^{\eta}}\big)\frac{1}{n^{\eta}}X_{ni}$. We now verify that W_{ni} satisfy the conditions of Lemma 3.2.

(1) We observe that for all $\epsilon > 0$,

$$\begin{aligned}\sum_{n=1}^{\infty} n^{\gamma(\eta+\beta)-1-\eta} \sum_{i=-m_n}^{m_n} P(|W_{ni}| > \epsilon) &\leq \sum_{n=1}^{\infty} n^{\gamma(\eta+\beta)-1-\eta} \sum_{i=-\infty}^{\infty} P(|W_{ni}| > \epsilon) \\ &= O(1) \sum_{n=1}^{\infty} n^{\gamma(\eta+\beta)-1-\eta} \sum_{i=-\infty}^{\infty} P\Big(|X| > \epsilon \frac{n^{\eta+\beta}}{\phi(\frac{i}{n^{\eta}})}\Big). \quad (3.9)\end{aligned}$$

Following the line of Theorem 4 in [8, p. 997], we find

$$\sum_{n=1}^{\infty} n^{\gamma(\eta+\beta)-1-\eta} \sum_{i=-\infty}^{\infty} P\Big(|X| > \epsilon \frac{n^{\eta+\beta}}{\phi(\frac{i}{n^{\eta}})}\Big) \leq C \sum_{k=1}^{\infty} k^{\gamma} P\Big(k-1 \leq \frac{|X|}{\epsilon} < k\Big) \leq CE|X|^{\gamma}. \quad (3.10)$$

Hence, (3.9) and (3.10) yield

$$\sum_{n=1}^{\infty} n^{\gamma(\eta+\beta)-1-\eta} \sum_{i=-m_n}^{m_n} P(|W_{ni}| > \epsilon) < \infty.$$

(2) According to $E|X|^{\gamma} < \infty$, the integrability of $\phi(\cdot)$ and $\int_0^{\infty}(\phi(t))^{\gamma}\mathrm{d}t < \infty$ when $0 < \gamma < 1$, we have

$$\begin{aligned}\sum_{i=-m_n}^{m_n} P(|W_{ni}| > \epsilon) &= O(1) \sum_{i=-\infty}^{\infty} P\Big(|X| > \epsilon \frac{n^{\eta+\beta}}{\phi(\frac{i}{n^{\eta}})}\Big) \\ &\leq \sum_{i=-\infty}^{\infty} \epsilon^{-\gamma} n^{-\gamma(\eta+\beta)} E|X|^{\gamma} \Big(\phi\Big(\frac{i}{n^{\eta}}\Big)\Big)^{\gamma} \\ &= O(n^{-\gamma(\eta+\beta)+\eta}) \int_{-\infty}^{\infty} (\phi(t))^{\gamma}\mathrm{d}t = O(n^{-\gamma(\eta+\beta)+\eta}).\end{aligned}$$

Note $\gamma(\eta+\beta) - \eta > 0$ under (a) or (b) or (c). Therefore $\sum\limits_{i=-m_n}^{m_n} P(|W_{ni}| > \epsilon) \to 0$.

(3) When $\gamma \geq 2$, since $E|X_{ni}|^2 \leq CE|X|^2 < \infty$, we have

$$\sum_{i=-m_n}^{m_n} EW_{ni}^2 I(|W_{ni}| \leq \epsilon) \leq C \sum_{i=-\infty}^{\infty} \frac{1}{n^{2(\eta+\beta)}} \phi^2\Big(\frac{i}{n^\eta}\Big) \leq C \frac{1}{n^{\eta+2\beta}} \int_{-\infty}^{\infty} \phi^2(t)\mathrm{d}t \leq Cn^{-(\eta+2\beta)};$$

when $0 < \gamma < 2$, we have

$$\begin{aligned}\sum_{i=-m_n}^{m_n} EW_{ni}^2 I(|W_{ni}| \leq \epsilon) &\leq \epsilon^{2-\gamma} \sum_{i=-\infty}^{\infty} E|W_{ni}|^\gamma = O(n^{-\gamma(\eta+\beta)+\eta}) \int_{-\infty}^{\infty} (\phi(t))^\gamma \mathrm{d}t \\ &= O(n^{-\gamma(\eta+\beta)+\eta}).\end{aligned}$$

Since $\eta + 2\beta > 0$ when $\gamma \geq 2$, and $\gamma(\eta+\beta) - \eta > 0$ when $0 < \gamma < 2$,

$$\sum_{i=-m_n}^{m_n} EW_{ni}^2 I(|W_{ni}| \leq \epsilon) \to 0$$

and there exists $q \geq 2$ such that

$$\sum_{n=1}^{\infty} n^{\gamma(\eta+\beta)-1-\eta} \Big(\sum_{i=-m_n}^{m_n} EW_{ni}^2 I(|W_{ni}| \leq \epsilon)\Big)^q < \infty.$$

(4) Under (a), i.e., $\gamma \geq 1, \beta > 0$ and $E|X|^\gamma < \infty$, we have

$$\begin{aligned}&\max_{0\leq k\leq m_n} \Big| \sum_{|i|\leq k} EW_{ni} I(|W_{ni}| \leq \epsilon)\Big| \\ &\leq \sum_{i=-\infty}^{\infty} E|W_{ni}| \leq Cn^{-\beta} \int_{-\infty}^{\infty} \phi(t)\mathrm{d}t = O(n^{-\beta}) \to 0, \quad \text{as } n \to \infty.\end{aligned}$$

Under (b), by $\gamma > 1, \gamma(\eta+\beta) - \eta > 0$ and $EX_{ni} = 0$, we have

$$\begin{aligned}&\max_{0\leq k\leq m_n} \Big| \sum_{|i|\leq k} EW_{ni} I(|W_{ni}| \leq \epsilon)\Big| \\ &\leq \sum_{i=-\infty}^{\infty} E|W_{ni}| I(|W_{ni}| > \epsilon) = O(n^{-\gamma(\eta+\beta)+\eta}) \to 0, \quad \text{as } n \to \infty.\end{aligned}$$

Under (c), according to $0 < \gamma < 1$, $\gamma(\eta+\beta) - \eta > 0$ and $\int_{-\infty}^{\infty} (\phi(t))^\gamma \mathrm{d}t$, we have

$$\max_{0\leq k\leq m_n} \Big| \sum_{|i|\leq k} EW_{ni} I(|W_{ni}| \leq \epsilon)\Big| = O(n^{-\gamma(\eta+\beta)+\eta}) \int_{-\infty}^{\infty} (\phi(t))^\gamma \mathrm{d}t \to 0, \quad \text{as } n \to \infty.$$

Therefore, according to Lemma 3.2, from (1)–(4) we obtain (2.9).

Necessity Let (2.9) hold. Note that $\frac{m_n}{n^\eta} \to \infty$ implies that there exists some positive integer N_0 such that $m_n \geq n^\eta$ for $n \geq N_0$.

Since $\{X, X_{ni}, |i| \leq m_n, n \geq 1\}$ is independent and identically distributed random variables when $\gamma(\eta+\beta) < 1 + \eta$, following the line of Theorem 4 in [8, p. 998], we find

$$\infty > \sum_{n=1}^{\infty} n^{\gamma(\eta+\beta)-1-\eta} P\Big(\max_{0\leq k\leq m_n} \Big| \sum_{|i|\leq k} \phi\Big(\frac{i}{n^\eta}\Big) \frac{1}{n^\eta} X_{ni}\Big| > \epsilon n^\beta\Big)$$

$$\geq C\sum_{n=1}^{\infty} n^{\gamma(\eta+\beta)-1}P\Big(|X|>c\frac{n^{\eta+\beta}}{\phi(1)}\Big),$$

which is equivalent to $E|X|^{\gamma}<\infty$. When $\gamma(\eta+\beta)\geq 1+\eta$, (2.9) implies

$$P\Big(\max_{0\leq k\leq m_n}\Big|\sum_{|i|\leq k}\phi\Big(\frac{i}{n^{\eta}}\Big)\frac{1}{n^{\eta}}X_{ni}\Big|>\epsilon n^{\beta}\Big)\to 0,\quad \text{as } n\to\infty.$$

Thus, by using Lemma 3.2, we have

$$\sum_{n=1}^{\infty} n^{\gamma(\eta+\beta)-1-\eta}\sum_{i=-m_n}^{m_n}P\Big(\Big|\phi\Big(\frac{i}{n^{\eta}}\Big)\frac{1}{n^{\eta}}X_{ni}\Big|>\epsilon n^{\beta}\Big)<\infty \tag{3.11}$$

and

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\gamma(\eta+\beta)-1-\eta}\sum_{i=-m_n}^{m_n}P\Big(\Big|\phi\Big(\frac{i}{n^{\eta}}\Big)\frac{1}{n^{\eta}}X_{ni}\Big|>\epsilon n^{\beta}\Big)\\
&\geq \sum_{n=N_0}^{\infty} n^{\gamma(\eta+\beta)-1-\eta}\sum_{i=-n^{\eta}}^{n^{\eta}}P\Big(\Big|\phi\Big(\frac{i}{n^{\eta}}\Big)\frac{1}{n^{\eta}}X_{ni}\Big|>\epsilon n^{\beta}\Big)\\
&\geq \sum_{n=1}^{\infty} n^{\gamma(\eta+\beta)-1}P\Big(|X|>c\frac{n^{\eta+\beta}}{\phi(1)}\Big).
\end{aligned}\tag{3.12}$$

(3.11) and (3.12) yield $E|X|^{\gamma}<\infty$.

For the case (b), $E|X|^{\gamma}<\infty$ follows from

$$\sum_{n=1}^{\infty} n^{\gamma(\eta+\beta)-1-\eta}P\Big(\max_{0\leq k\leq m_n}\Big|\sum_{|i|\leq k}\phi\Big(\frac{i}{n^{\eta}}\Big)\frac{1}{n^{\eta}}(X_{ni}-EX_{ni})\Big|>\epsilon n^{\beta}\Big)<\infty. \tag{3.13}$$

(2.9) and (3.13) yield $EX_{ni}=EX=0$.

Appendix

Lemma A.1 (see [18]) *Let $\{X_i, 1\leq i\leq n\}$ be a sequence of NA random variables with zero means and finite second moments. Let $B_n^2=\sum_{i=1}^{n}EX_i^2$. Then for all $x>0$, $\alpha>0$ and $0<\beta<1$,*

$$\begin{aligned}
&P\Big(\max_{1\leq k\leq n}\Big|\sum_{i=1}^{k}X_i\Big|\geq x\Big)\\
&\leq 2P\Big(\max_{1\leq k\leq n}|X_k|>\alpha\Big)+\frac{2}{1-\beta}\exp\Big(-\frac{x^2\beta}{2(\alpha x+B_n^2)}\cdot\Big\{1+\frac{2}{3}\ln\Big(1+\frac{\alpha x}{B_n^2}\Big)\Big\}\Big).
\end{aligned}$$

Proof of Lemma 3.2 Define $Y'_{ni}=-\delta I(Y_{ni}<-\delta)+Y_{ni}I(|Y_{ni}|\leq\delta)+\delta I(Y_{ni}>\delta)$, $Y''_{ni}=Y_{ni}-Y'_{ni}$. Then

$$\sum_{n=1}^{\infty}a_nP\Big(\max_{1\leq k\leq k_n}\Big|\sum_{i=1}^{k}Y_{ni}\Big|>\epsilon\Big)$$

$$\le \sum_{n=1}^{\infty} a_n \Big\{ P\Big(\max_{1\le k\le k_n} \Big| \sum_{i=1}^{k} Y'_{ni} \Big| > \frac{\epsilon}{2} \Big) + P\Big(\max_{1\le k\le k_n} \Big| \sum_{i=1}^{k} Y''_{ni} \Big| > \frac{\epsilon}{2} \Big) \Big\}$$
$$:= \mathrm{I}_{3n} + \mathrm{I}_{4n}.$$

Obviously, from (i) we find $\mathrm{I}_{4n} \le \sum_{n=1}^{\infty} a_n \sum_{i=1}^{k_n} P(|Y_{ni}| > \delta) < \infty$. Note

$$\max_{1\le k\le k_n} \Big| \sum_{i=1}^{k} EY'_{ni} \Big| \le \delta \sum_{i=1}^{k_n} P(|Y_{ni}| > \delta) + \max_{1\le k\le k_n} \Big| \sum_{i=1}^{k} EY_{ni} I(|Y_{ni}| \le \delta) \Big| \to 0.$$

So, to prove $\mathrm{I}_{3n} < \infty$, it suffices to show

$$\mathrm{I}^*_{3n} := \sum_{n=1}^{\infty} a_n P\Big(\max_{1\le k\le k_n} \Big| \sum_{i=1}^{k} (Y'_{ni} - EY'_{ni}) \Big| > \frac{\epsilon}{3} \Big) < \infty.$$

Since $\{Y'_{ni} - EY'_{ni}, 1 \le i \le k_n, n \ge 1\}$ is still a triangular array of rowwise NA random variables from the definition of Y'_{ni} and the NA property, we apply Lemma A.1 here for $x = \frac{\epsilon}{3}$, $\beta = \frac{1}{2}$ and $\alpha = 2\delta = \frac{\epsilon}{36q}$. Note $\max_{1\le k\le k_n} |Y'_{ni} - EY'_{ni}| \le 2\delta = \alpha$, so we have

$$\mathrm{I}^*_{3n} \le 4 \sum_{n=1}^{\infty} a_n \exp\Big\{ - \frac{x^2}{4(\alpha x + B_n^2)} \Big\} \cdot \Big(\frac{B_n^2}{\alpha x + B_n^2} \Big)^{\frac{x^2}{6(\alpha x + B_n^2)}},$$

where $B_n^2 = \sum_{i=1}^{k_n} E(Y'_{ni} - EY'_{ni})^2$. We observe

$$B_n^2 \le \sum_{i=1}^{k_n} E(Y'_{ni})^2 \le \delta^2 \sum_{i=1}^{k_n} P(|Y_{ni}| > \delta) + \sum_{i=1}^{k_n} EY_{ni}^2 I(|Y_{ni}| \le \delta) \to 0.$$

So, $B_n^2 \le \alpha x$ for n large enough, and

$$\exp\Big\{ - \frac{x^2}{4(\alpha x + B_n^2)} \Big\} \cdot \Big(\frac{B_n^2}{\alpha x + B_n^2} \Big)^{\frac{x^2}{6(\alpha x + B_n^2)}}$$
$$\le \exp\Big\{ - \frac{x}{8\alpha} \Big\} \cdot \Big(\frac{B_n^2}{\alpha x} \Big)^{\frac{x}{12\alpha}} = \exp\Big\{ - \frac{3q}{2} \Big\} \Big(\frac{\epsilon^2}{108q} \Big)^{q} (B_n^2)^q.$$

Therefore, there exists a constant $C > 0$ such that

$$\begin{aligned}
\mathrm{I}^*_{3n} &\le 4 \exp\Big\{ - \frac{3q}{2} \Big\} \Big(\frac{\epsilon^2}{108q} \Big)^{q} \sum_{n=1}^{\infty} a_n (B_n^2)^q + C \\
&\le 2^{q+1} \exp\Big\{ - \frac{3q}{2} \Big\} \Big(\frac{\epsilon^2}{108q} \Big)^{q} \Big[\delta^{2q} \sum_{n=1}^{\infty} a_n \Big(\sum_{i=1}^{k_n} P(|Y_{ni}| > \delta) \Big)^q \\
&\quad + \sum_{n=1}^{\infty} a_n \Big(\sum_{i=1}^{k_n} EY_{ni}^2 I(|Y_{ni}| \le \delta) \Big)^q \Big] + C \\
&\le C \sum_{n=1}^{\infty} a_n \sum_{i=1}^{k_n} P(|Y_{ni}| > \delta) + C \sum_{n=1}^{\infty} a_n \Big(\sum_{i=1}^{k_n} EY_{ni}^2 I(|Y_{ni}| \le \delta) \Big)^q + C \\
&< \infty.
\end{aligned}$$

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