

The Global Existence of Small Solutions to the Oldroyd-B Model

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Abstract The Cauchy problem to the Oldroyd-B model is studied. In particular, it is shown that if the smooth solution (u, τ) to this system blows up at a finite time T^* , then $\int_0^{T^*} \|\nabla u(t)\|_{L^\infty} dt = \infty$. Furthermore, the global existence of smooth solution to this system is given with small initial data.

Keywords Cauchy problem, Oldroyd-B model, Global existence

2000 MR Subject Classification 82D60, 35Q35, 35B45

1 Introduction

In this paper, we consider the blow up principle and the global existence of small solutions to the following Oldroyd-B model:

$$\begin{cases} \operatorname{Re} \left(\frac{\partial u}{\partial t} + u \cdot \nabla u \right) = -\nabla p + (1 - \varepsilon) \Delta u + \operatorname{div} \tau, \\ \operatorname{div} u = 0, \\ \frac{\partial \tau}{\partial t} + (u \cdot \nabla) \tau = (\nabla u) \tau + \tau (\nabla^T u) - \frac{1}{\operatorname{We}} \tau + \frac{\varepsilon}{\operatorname{We}} (\nabla u + \nabla^T u), \end{cases} \quad (1.1)$$

where $u : (x, t) \in \mathbb{R}^n \times [0, T] \rightarrow u(x, t) \in \mathbb{R}^n$ is the fluid velocity, $p : (x, t) \in \mathbb{R}^n \times [0, T] \rightarrow p(x, t) \in \mathbb{R}$ represents the hydrodynamic pressure, and $\tau : (x, t) \in \mathbb{R}^n \times [0, T] \rightarrow \tau(x, t) \in \mathbb{R}^{n \times n}$ denotes the extra-stress tensor, $n = 2, 3$ is the space dimension. The following parameters are dimensionless: the Reynolds number $\operatorname{Re} \in \mathbb{R}_+$, the Weissenberg number $\operatorname{We} \in \mathbb{R}_+$ and the elastic viscosity fraction $\varepsilon \in (0, 1)$. And the initial data satisfy

$$u(x, 0) = u_0(x), \quad \tau(x, 0) = \tau_0(x). \quad (1.2)$$

The Oldroyd-B model is a classical model for dilute solutions of polymers suspended in a viscous incompressible solvent (see [2]). The system (1.1) describes the motion of the incompressible fluid satisfying the Oldroyd constitutive law (see [17]). Viscoelastic material can be viewed as the intermediate state between the fluid and solid. This kind of material exhibits elastic behavior, such as memory effects as well as fluid properties. Many complicated hydrodynamical and rheological behaviors of complex fluids and the electro-magnetic behavior of the materials (see [2, 5, 6, 10, 14, 18]) can be regarded as consequence of internal elastic properties.

Manuscript received January 8, 2010. Revised August 28, 2010. Published online January 25, 2011.

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In 2001, Chemin and Masmoudi [3] established a criterion for the system involving both τ and ∇u . [4, 11, 13, 15] dealt with the inertial Oldroyd-B model for the deformation tensor and established global-in-time existence for small initial data.

In this paper, we will prove a similar result like the well-known Beale-Kato-Majda criterion (see [1]) for the two and three dimensional Oldroyd-B model for the extra-stress tensor with the relaxation term (finite Weissenberg number) and global existence with small initial data. The main results of the paper are presented by the following two theorems:

Theorem 1.1 *Let $s > [\frac{n}{2}] + 1$ for $n = 2, 3$ be an integer, $u_0 \in H^s(\mathbb{R}^n)$, $\tau_0 \in H^s(\mathbb{R}^n)$ with $\nabla \cdot u_0 = 0$. Then there exists a positive time T , such that Cauchy problem (1.1) with (1.2) has a unique solution on $[0, T]$, $(u, \tau) \in L^\infty([0, T]; H^s(\mathbb{R}^n))$ with $\nabla u \in L^2([0, T]; H^s(\mathbb{R}^n))$. Furthermore, if T^* is the maximal time for existence, then*

$$\int_0^{T^*} \|\nabla u(t)\|_{L^\infty} dt = \infty. \quad (1.3)$$

Theorem 1.2 *Let $s > [\frac{n}{2}] + 1$ for $n = 2, 3$ be an integer, $u_0 \in H^s(\mathbb{R}^n)$, $\tau_0 \in H^s(\mathbb{R}^n)$ with $\nabla \cdot u_0 = 0$, and $0 < \varepsilon < \frac{4}{5}$. Then Cauchy problem (1.1) with (1.2) has a unique global solution $(u, \tau) \in L^\infty([0, \infty); H^s(\mathbb{R}^n))$ with $\nabla u \in L^2([0, \infty); H^s(\mathbb{R}^n))$ provided that $\|u_0\|_{H^3}$ and $\|\tau_0\|_{H^3}$ are small enough.*

This paper is organized as follows. Some lemmas are given in Section 2, and Theorem 1.1 and Theorem 1.2 are proved respectively in Section 3 and Section 4.

2 Preliminaries

Before proving the main theorems, we first introduce three useful estimates.

Lemma 2.1 *For $u \in H^s(\mathbb{R}^n)$, $\nabla u \in L^\infty(\mathbb{R}^n)$ where $n = 2$ or 3 , and for any positive integer $s > 1$, set $v = D_x^\alpha u$ where $\alpha \in Z_+^n$ with $|\alpha| = s$. Then we have*

$$\int [D_x^\alpha(u \cdot \nabla u) - u \cdot \nabla D_x^\alpha u] v \leq C(n) \|\nabla u\|_{L^\infty} \|u\|_{H^s}^2, \quad (2.1)$$

where $C(n)$ is a constant depending only on n .

Proof First

$$\int [D_x^\alpha(u \cdot \nabla u) - u \cdot \nabla D_x^\alpha u] v \leq \|D_x^\alpha(u \cdot \nabla u) - u \cdot \nabla D_x^\alpha u\|_{L^2} \|v\|_{L^2},$$

then we can easily get (2.1) by the Sobolev inequality

$$\|D_x^\alpha(fg) - fD_x^\alpha g\|_{L^2} \leq C(n)(\|f\|_{H^s}\|g\|_{L^\infty} + \|\nabla f\|_{L^\infty}\|g\|_{H^{s-1}}), \quad (2.2)$$

where $f = u$ and $g = \nabla u$.

Lemma 2.2 *For $(\nabla u, \tau) \in H^s(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ where $n = 2$ or 3 , and for any positive integer $s > 1$, set $v = D_x^\alpha u$, $\eta = D_x^\alpha \tau$ where $\alpha \in Z_+^n$ with $|\alpha| = s$. Then we have*

$$\begin{aligned} & \int [D_x^\alpha(u \cdot \nabla \tau) - u \cdot \nabla D_x^\alpha \tau] \eta \\ & \leq \delta \|\nabla u\|_{H^s}^2 + \frac{C(n)}{\delta} \|\tau\|_{H^s}^2 (\|\nabla u\|_{L^\infty} + \|\tau\|_{L^\infty}^2 + 1) + C(n) \|\nabla u\|_{L^\infty} \|\tau\|_{H^s}^2, \end{aligned} \quad (2.3)$$

where $\delta > 0$ and $C(n)$ is a constant depending only on n .

Proof By the Cauchy inequality, we have

$$\begin{aligned} \int [\mathbf{D}_x^\alpha(u \cdot \nabla \tau) - u \cdot \nabla \mathbf{D}_x^\alpha \tau] \eta &\leq \|\mathbf{D}_x^\alpha(u \cdot \nabla \tau) - u \cdot \nabla \mathbf{D}_x^\alpha \tau\|_{L^2} \|\eta\|_{L^2} \\ &\leq C(n) \left(\sum_{i=1}^{s-1} \|\mathbf{D}_x^{s+1-i} u\|_{L^4} \|\mathbf{D}_x^i \tau\|_{L^4} + \|\nabla u\|_{L^\infty} \|\tau\|_{H^s} \right) \|\eta\|_{L^2}. \end{aligned}$$

Then for any integer $1 \leq j < s$, using the Gagliardo-Nirenberg interpolation inequalities (see [7, 9, 16]),

$$\|\mathbf{D}_x^j f\|_{L^4} \leq C(n) \|\mathbf{D}_x^s f\|_{L^2}^{\frac{n-4j}{2(n-2s)}} \|f\|_{L^\infty}^{\frac{n-4(s-j)}{2(n-2s)}}, \quad (2.4)$$

we obtain

$$\|\mathbf{D}_x^{s+1-i} u\|_{L^4} \leq C(n) \|\nabla v\|_{L^2}^{\frac{n-4(s-i)}{2(n-2s)}} \|\nabla u\|_{L^\infty}^{\frac{n-4i}{2(n-2s)}}, \quad (2.5)$$

$$\|\mathbf{D}_x^i \tau\|_{L^4} \leq C(n) \|\eta\|_{L^2}^{\frac{n-4i}{2(n-2s)}} \|\tau\|_{L^\infty}^{\frac{n-4(s-i)}{2(n-2s)}}. \quad (2.6)$$

So

$$\begin{aligned} &\int [\mathbf{D}_x^\alpha(u \cdot \nabla \tau) - u \cdot \nabla \mathbf{D}_x^\alpha \tau] \eta \\ &\leq C(n) \left(\sum_{i=1}^{s-1} \|\nabla v\|_{L^2}^{\frac{n-4(s-i)}{2(n-2s)}} \|\nabla u\|_{L^\infty}^{\frac{n-4i}{2(n-2s)}} \|\eta\|_{L^2}^{\frac{n-4i}{2(n-2s)}} \|\tau\|_{L^\infty}^{\frac{n-4(s-i)}{2(n-2s)}} + \|\nabla u\|_{L^\infty} \|\eta\|_{L^2} \right) \|\eta\|_{L^2} \\ &\leq C(n) \sum_{i=1}^{s-1} \|\nabla u\|_{H^s}^{\frac{n-4(s-i)}{2(n-2s)}} \|\tau\|_{H^s}^{\frac{3n-4(s+i)}{2(n-2s)}} \|\nabla u\|_{L^\infty}^{\frac{n-4i}{2(n-2s)}} \|\tau\|_{L^\infty}^{\frac{n-4(s-i)}{2(n-2s)}} + C(n) \|\nabla u\|_{L^\infty} \|\tau\|_{H^s}^2. \end{aligned}$$

Finally by means of the Young's inequality we obtain (2.3).

Lemma 2.3 For $(\nabla u, \tau) \in H^s(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ where $n = 2$ or 3 , and for any positive integer $s > 1$, set $v = \mathbf{D}_x^\alpha u$, $\eta = \mathbf{D}_x^\alpha \tau$ where $\alpha \in Z_+^n$ with $|\alpha| = s$. Then we have

$$\begin{aligned} \int \mathbf{D}_x^\alpha(\nabla u \tau) \eta + \int \mathbf{D}_x^\alpha(\tau \nabla^T u) \eta &\leq C(n) \|\nabla u\|_{H^s} \|\tau\|_{L^\infty} \|\tau\|_{H^s} + \delta \|\nabla u\|_{H^s}^2 \\ &\quad + \frac{C(n)}{\delta} \|\tau\|_{H^s}^2 (\|\nabla u\|_{L^\infty} + \|\tau\|_{L^\infty}^2 + 1) \\ &\quad + C(n) \|\nabla u\|_{L^\infty} \|\tau\|_{H^s}^2, \end{aligned} \quad (2.7)$$

where $\delta > 0$ and $C(n)$ is a constant depending only on n .

Proof By the Cauchy inequality, we get

$$\begin{aligned} \int \mathbf{D}_x^\alpha(\nabla u \tau) \eta + \int \mathbf{D}_x^\alpha(\tau \nabla^T u) \eta &\leq (\|\mathbf{D}_x^\alpha(\nabla u \tau)\|_{L^2} + \|\mathbf{D}_x^\alpha(\tau \nabla^T u)\|_{L^2}) \|\eta\|_{L^2} \\ &\leq C(n) \left(\|\nabla v\|_{L^2} \|\tau\|_{L^\infty} + \sum_{i=1}^{s-1} \|\mathbf{D}_x^{s+1-i} u\|_{L^4} \|\mathbf{D}_x^i \tau\|_{L^4} \right. \\ &\quad \left. + \|\nabla u\|_{L^\infty} \|\tau\|_{H^s} \right) \|\eta\|_{L^2}. \end{aligned}$$

By using (2.5) and (2.6), we obtain

$$\begin{aligned} \int D_x^\alpha(\nabla u \tau) \eta + \int D_x^\alpha(\tau \nabla^T u) \eta &\leq C(n) \|\nabla u\|_{H^s} \|\tau\|_{L^\infty} \|\tau\|_{H^s} \\ &+ C(n) \sum_{i=1}^{s-1} \|\nabla u\|_{H^s}^{\frac{n-4(s-i)}{2(n-2s)}} \|\tau\|_{H^s}^{\frac{3n-4(s+i)}{2(n-2s)}} \|\nabla u\|_{L^\infty}^{\frac{n-4i}{2(n-2s)}} \|\tau\|_{L^\infty}^{\frac{n-4(s-i)}{2(n-2s)}} \\ &+ C(n) \|\nabla u\|_{L^\infty} \|\tau\|_{H^s}^2. \end{aligned}$$

Then by the Young's inequality, we get (2.7).

3 Proof of Theorem 1.1

As the local existence of the Oldroyd-B model is well-known, here we only present the H^s -estimates for the solution of (1.1)–(1.2), $s > [\frac{n}{2}] + 1$.

Proof of Theorem 1.1 Actually let (u, τ) be the unique local solution to (1.1)–(1.2). It is sufficient to prove for any integer $s > [\frac{n}{2}] + 1$,

$$\|u\|_{H^s} + \|\tau\|_{H^s} \leq C_0, \quad t \leq T^* \quad (3.1)$$

for some constant C_0 provided that

$$\int_0^{T^*} \|\nabla u(t)\|_{L^\infty} dt \triangleq M_0 < +\infty. \quad (3.2)$$

In what follows, we are going to present the L^2 estimate up to s order derivative of u and τ .

First, a standard energy estimate for system (1.1) gives

$$\begin{aligned} &\frac{\text{Re}}{2} \frac{d}{dt} \int |u|^2 + (1-\varepsilon) \int |\nabla u|^2 + \frac{\text{We}}{2\varepsilon} \frac{d}{dt} \int |\tau|^2 + \frac{1}{\varepsilon} \int |\tau|^2 \\ &= \frac{\text{We}}{\varepsilon} \int \nabla_j u_i \tau_{jl} \tau_{il} + \frac{\text{We}}{\varepsilon} \int \tau_{ij} \nabla_j u_l \tau_{il} + \int \nabla_j u_i \tau_{ij} \\ &\leq C(n) \left(\frac{\text{We}}{\varepsilon} \|\nabla u\|_{L^\infty} \|\tau\|_{L^2}^2 + \|\nabla u\|_{L^2} \|\tau\|_{L^2} \right), \end{aligned} \quad (3.3)$$

where $C(n)$ is a constant depending only on n . Thus by the Cauchy inequality, we have

$$\begin{aligned} &\frac{\text{Re}}{2} \frac{d}{dt} \|u\|_{L^2}^2 + \frac{1-\varepsilon}{2} \|\nabla u\|_{L^2}^2 + \frac{\text{We}}{2\varepsilon} \frac{d}{dt} \|\tau\|_{L^2}^2 \\ &\leq C(n) \left(\|\nabla u\|_{L^\infty} + \frac{1}{2(1-\varepsilon)} \frac{\varepsilon}{\text{We}} \right) \left(\text{Re} \|u\|_{L^2}^2 + \frac{\text{We}}{\varepsilon} \|\tau\|_{L^2}^2 \right). \end{aligned} \quad (3.4)$$

Then

$$\text{Re} \|u\|_{L^2}^2 + \frac{\text{We}}{\varepsilon} \|\tau\|_{L^2}^2 \leq \left(\text{Re} \|u_0\|_{L^2}^2 + \frac{\text{We}}{\varepsilon} \|\tau_0\|_{L^2}^2 \right) e^{C(n) \left(M_0 + \frac{1}{2(1-\varepsilon)} \frac{\varepsilon}{\text{We}} T^* \right)}. \quad (3.5)$$

Next, by differentiating (1.1) we obtain

$$\begin{aligned} \text{Re} \left(\frac{\partial \nabla u}{\partial t} + u \cdot \nabla(\nabla u) \right) &= -\nabla \nabla p + (1-\varepsilon) \Delta(\nabla u) + \text{div} \nabla \tau - \text{Re} \nabla u \cdot \nabla u, \\ \frac{\partial(\nabla \tau)}{\partial t} + u \cdot \nabla(\nabla \tau) + \nabla u \cdot \nabla \tau &= \nabla(\nabla u) \tau + (\nabla u)(\nabla \tau) + \nabla \tau(\nabla^T u) \\ &\quad + \tau \nabla(\nabla^T u) - \frac{1}{\text{We}} \nabla \tau + \frac{\varepsilon}{\text{We}} (\nabla \nabla u + \nabla \nabla^T u). \end{aligned}$$

In view of $\nabla \cdot u = 0$, we get by the standard L^2 estimate

$$\begin{aligned}
& \operatorname{Re} \frac{1}{2} \frac{d}{dt} \int |\nabla u|^2 + (1 - \varepsilon) \int |\nabla^2 u|^2 + \frac{\operatorname{We}}{\varepsilon} \frac{1}{2} \frac{d}{dt} \int |\nabla \tau|^2 + \frac{1}{\varepsilon} \int |\nabla \tau|^2 \\
&= -\operatorname{Re} \int \nabla_i u \cdot \nabla u_j \nabla_i u_j - \frac{\operatorname{We}}{\varepsilon} \int \nabla_i u \cdot \nabla \tau_{jl} \nabla_i \tau_{jl} + \frac{\operatorname{We}}{\varepsilon} \int \nabla_i \nabla_k u_j \tau_{kl} \nabla_i \tau_{jl} \\
&\quad + \frac{\operatorname{We}}{\varepsilon} \int \nabla_k u_j \nabla_i \tau_{kl} \nabla_i \tau_{jl} + \frac{\operatorname{We}}{\varepsilon} \int \nabla_i \tau_{jk} \nabla_k u_l \nabla_i \tau_{jl} \\
&\quad + \frac{\operatorname{We}}{\varepsilon} \int \tau_{jk} \nabla_i \nabla_k u_l \nabla_i \tau_{jl} + \int \nabla_i \nabla_l u_j \nabla_i \tau_{jl} \\
&\leq \operatorname{Re} \|\nabla u\|_{L^\infty} \|\nabla u\|_{L^2}^2 + \frac{\operatorname{We}}{\varepsilon} \|\nabla u\|_{L^\infty} \|\nabla \tau\|_{L^2}^2 + \frac{\operatorname{We}}{\varepsilon} \|\tau\|_{L^\infty} \|\nabla \tau\|_{L^2} \|\nabla^2 u\|_{L^2} \\
&\quad + \frac{\operatorname{We}}{\varepsilon} \|\nabla u\|_{L^\infty} \|\nabla \tau\|_{L^2}^2 + \frac{\operatorname{We}}{\varepsilon} \|\nabla u\|_{L^\infty} \|\nabla \tau\|_{L^2}^2 \\
&\quad + \frac{\operatorname{We}}{\varepsilon} \|\tau\|_{L^\infty} \|\nabla^2 u\|_{L^2} \|\nabla \tau\|_{L^2} + \|\nabla^2 u\|_{L^2} \|\nabla \tau\|_{L^2}.
\end{aligned} \tag{3.6}$$

By the Cauchy inequality, we have

$$\begin{aligned}
& \frac{\operatorname{Re}}{2} \frac{d}{dt} \|\nabla u\|_{L^2}^2 + \frac{1 - \varepsilon}{2} \|\nabla^2 u\|_{L^2}^2 + \frac{\operatorname{We}}{2\varepsilon} \frac{d}{dt} \|\nabla \tau\|_{L^2}^2 \\
&\leq C(n) \left(\|\nabla u\|_{L^\infty} + \frac{1}{2(1 - \varepsilon)} \frac{\operatorname{We}}{\varepsilon} \|\tau\|_{L^\infty}^2 + \frac{1}{2(1 - \varepsilon)} \right) \left(\operatorname{Re} \|\nabla u\|_{L^2}^2 + \frac{\operatorname{We}}{\varepsilon} \|\nabla \tau\|_{L^2}^2 \right).
\end{aligned} \tag{3.7}$$

Using the equation for τ

$$\frac{\partial \tau}{\partial t} + (u \cdot \nabla) \tau = (\nabla u) \tau + \tau (\nabla^T u) - \frac{1}{\operatorname{We}} \tau + \frac{\varepsilon}{\operatorname{We}} (\nabla u + \nabla^T u),$$

by [12] one can get

$$\|\tau\|_{L^\infty} \leq \left(\|\tau_0\|_{L^\infty} + \frac{\varepsilon}{\operatorname{We}} M_0 \right) e^{M_0} \triangleq M_1. \tag{3.8}$$

Hence we obtain

$$\operatorname{Re} \|u\|_{H^1}^2 + \frac{\operatorname{We}}{\varepsilon} \|\tau\|_{H^1}^2 \leq \left(\operatorname{Re} \|u_0\|_{H^1}^2 + \frac{\operatorname{We}}{\varepsilon} \|\tau_0\|_{H^1}^2 \right) e^{C(n) \left(M_0 + \frac{1}{2(1 - \varepsilon)} \frac{\operatorname{We}}{\varepsilon} M_1^2 T^* + \frac{1}{2(1 - \varepsilon)} T^* \right)}. \tag{3.9}$$

Finally, for any positive integer $s > [\frac{n}{2}] + 1$, we have the following equations for $v = D_x^\alpha u$, $\eta = D_x^\alpha \tau$ where $\alpha \in Z_+^n$ with $|\alpha| = s$:

$$\operatorname{Re} \left(\frac{\partial v}{\partial t} + u \cdot \nabla v \right) = -\operatorname{Re} [D_x^\alpha (u \cdot \nabla u) - u \cdot D_x^\alpha u] - \nabla D_x^\alpha p + (1 - \varepsilon) \Delta v + \operatorname{div} \eta, \tag{3.10}$$

$$\begin{aligned}
& \frac{\partial \eta}{\partial t} + (u \cdot \nabla) \eta = -[D_x^\alpha (u \cdot \nabla \tau) - u \cdot \nabla D_x^\alpha \tau] + D_x^\alpha (\nabla u \tau) + D_x^\alpha (\tau \nabla^T u) \\
& \quad - \frac{1}{\operatorname{We}} \eta + \frac{\varepsilon}{\operatorname{We}} (\nabla v + \nabla^T v),
\end{aligned} \tag{3.11}$$

and $\operatorname{div} v = 0$. After multiplying both sides of (3.10) and (3.11) by v and η respectively, and integrating by parts, we obtain

$$\frac{\operatorname{Re}}{2} \frac{d}{dt} \int |v|^2 + (1 - \varepsilon) \int |\nabla v|^2 = \int \nabla_i \eta_{ij} v_j - \operatorname{Re} \int [D_x^\alpha (u \cdot \nabla u) - u \cdot \nabla D_x^\alpha u] v, \tag{3.12}$$

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \int |\eta|^2 + \frac{1}{\operatorname{We}} \int |\eta|^2 = - \int [D_x^\alpha (u \cdot \tau) - u \cdot \nabla D_x^\alpha \tau] \eta + \int D_x^\alpha (\nabla u \tau) \eta \\
& \quad + \int D_x^\alpha (\tau \nabla^T u) \eta + \frac{\varepsilon}{\operatorname{We}} \int (\nabla_i v_j + \nabla_j v_i) \eta_{ij}
\end{aligned} \tag{3.13}$$

with the help of $\operatorname{div} v = 0$. Combining (3.12) and $\frac{\operatorname{We}}{\varepsilon} \cdot (3.13)$ yields

$$\begin{aligned} & \frac{\operatorname{Re}}{2} \frac{d}{dt} \int |v|^2 + (1 - \varepsilon) \int |\nabla v|^2 + \frac{\operatorname{We}}{2\varepsilon} \frac{d}{dt} \int |\eta|^2 + \frac{1}{\varepsilon} \int |\eta|^2 \\ &= -\operatorname{Re} \int [D_x^\alpha (u \cdot \nabla u) - u \cdot \nabla D_x^\alpha u] v - \frac{\operatorname{We}}{\varepsilon} \int [D_x^\alpha (u \cdot \nabla \tau) - u \cdot \nabla D_x^\alpha \tau] \eta \\ &+ \frac{\operatorname{We}}{\varepsilon} \int D_x^\alpha (\nabla u \tau) \eta + \frac{\operatorname{We}}{\varepsilon} \int D_x^\alpha (\tau \nabla^T u) \eta + \int (\nabla_i v_j) \eta_{ij}. \end{aligned} \quad (3.14)$$

Using (2.1), (2.3) and (2.7), we obtain

$$\begin{aligned} & \frac{\operatorname{Re}}{2} \frac{d}{dt} \|u\|_{H^s}^2 + (1 - \varepsilon) \|\nabla u\|_{H^{s-1}}^2 + \frac{\operatorname{We}}{2\varepsilon} \frac{d}{dt} \|\tau\|_{H^s}^2 + \frac{1}{\varepsilon} \|\tau\|_{H^s}^2 \\ & \leq C(n) \operatorname{Re} \|\nabla u\|_{L^\infty} \|u\|_{H^s}^2 + C(n) \frac{\operatorname{We}}{\varepsilon} \|\nabla u\|_{H^s} \|\tau\|_{L^\infty} \|\tau\|_{H^s} \\ & + \frac{\operatorname{We}}{\varepsilon} \delta \|\nabla u\|_{H^s}^2 + \frac{C(n)}{\delta} \frac{\operatorname{We}}{\varepsilon} \|\tau\|_{H^s}^2 (\|\nabla u\|_{L^\infty} + \|\tau\|_{L^\infty}^2 + 1) \\ & + C(n) \frac{\operatorname{We}}{\varepsilon} \|\nabla u\|_{L^\infty} \|\tau\|_{H^s}^2 + \|\nabla u\|_{H^s} \|\tau\|_{H^s}. \end{aligned} \quad (3.15)$$

By the Cauchy inequality and choosing δ small enough, we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left(\operatorname{Re} \|u\|_{H^s}^2 + \frac{\operatorname{We}}{\varepsilon} \|\tau\|_{H^s}^2 \right) + \frac{1 - \varepsilon}{2} \|\nabla u\|_{H^s}^2 \\ & \leq C(n) \left[\|\nabla u\|_{L^\infty} + \frac{\operatorname{We}}{\varepsilon} \frac{1}{2(1 - \varepsilon)} \|\tau\|_{L^\infty}^2 + \frac{1}{2(1 - \varepsilon)} \right] \left(\operatorname{Re} \|u\|_{H^s}^2 + \frac{\operatorname{We}}{\varepsilon} \|\tau\|_{H^s}^2 \right). \end{aligned} \quad (3.16)$$

With the aid of the Gronwall's inequality, we get that $\|u\|_{H^s}$ and $\|\tau\|_{H^s}$ are bounded as $t \leq T^*$ if (3.2) is satisfied. This completes the proof of Theorem 1.1.

4 Proof of Theorem 1.2

Similarly to the proof of Theorem 1.1, we are going to present a global priori estimate of the solutions constructed in Theorem 1.1. Let (u, τ) be the unique solution constructed in Theorem 1.1 with the initial data (u_0, τ_0) which satisfies all the requirements in Theorem 1.1.

Actually, combining (3.3), (3.6) and (3.14), we can get

$$\begin{aligned} & \operatorname{Re} \frac{1}{2} \frac{d}{dt} \|u\|_{H^2}^2 + \frac{\operatorname{We}}{\varepsilon} \frac{1}{2} \frac{d}{dt} \|\tau\|_{H^2}^2 + (1 - \varepsilon) \|\nabla u\|_{H^2}^2 + \frac{1}{\varepsilon} \|\tau\|_{H^2}^2 \\ &= \frac{\operatorname{We}}{\varepsilon} \int \nabla_j u_i \tau_{jl} \tau_{il} + \frac{\operatorname{We}}{\varepsilon} \int \tau_{ij} \nabla_j u_l \tau_{il} + \int \nabla_j u_i \tau_{ij} \\ & - \operatorname{Re} \int \nabla_i u \cdot \nabla u_j \nabla_i u_j - \frac{\operatorname{We}}{\varepsilon} \int (\nabla_i u \cdot \nabla \tau_{jl} - \nabla_i \nabla_k u_j \tau_{kl} \\ & - \nabla_k u_j \nabla_i \tau_{kl} - \nabla_i \tau_{jk} \nabla_k u_l - \tau_{jk} \nabla_i \nabla_k u_l) \nabla_i \tau_{jl} + \int \nabla_i \nabla_l u_j \nabla_i \tau_{jl} \\ & - \operatorname{Re} \int (\nabla_{ij}^2 u \cdot \nabla u_k + \nabla_i u \cdot \nabla \nabla_j u_k + \nabla_j u \cdot \nabla \nabla_i u_k) \nabla_{ij}^2 u_k \\ & - \frac{\operatorname{We}}{\varepsilon} \int (\nabla_i u \cdot \nabla \nabla_j \tau_{kl} + \nabla_j u \cdot \nabla \nabla_i \tau_{kl} + \nabla_{ij}^2 u \cdot \nabla \tau_{kl} - \nabla_{ij}^2 \nabla_r u_k \tau_{rl} \\ & - \nabla_i \nabla_r u_k \nabla_j \tau_{rl} - \nabla_j \nabla_r u_k \nabla_i \tau_{kl} - \nabla_r u_k \nabla_{ij}^2 \tau_{rl} - \nabla_{ij}^2 \tau_{kr} \nabla_r u_l \\ & - \nabla_i \tau_{kr} \nabla_j \nabla_r u_l - \nabla_j \tau_{kr} \nabla_i \nabla_r u_l - \tau_{kr} \nabla_{ij}^2 \nabla_r u_l) \nabla_{ij}^2 \tau_{kl} + \int \nabla_{ij}^2 \nabla_l u_k \nabla_{ij}^2 \tau_{kl}, \end{aligned}$$

so

$$\begin{aligned}
& \operatorname{Re} \frac{1}{2} \frac{d}{dt} \|u\|_{H^2}^2 + \frac{\operatorname{We}}{\varepsilon} \frac{1}{2} \frac{d}{dt} \|\tau\|_{H^2}^2 + (1 - \varepsilon) \|\nabla u\|_{H^2}^2 + \frac{1}{\varepsilon} \|\tau\|_{H^2}^2 \\
& \leq C \left(\frac{\operatorname{We}}{\varepsilon} \|\nabla u\|_{L^\infty} \|\tau\|_{L^2}^2 + \operatorname{Re} \|\nabla u\|_{L^\infty} \|\nabla u\|_{L^2}^2 + \frac{\operatorname{We}}{\varepsilon} \|\nabla u\|_{L^\infty} \|\nabla \tau\|_{L^2}^2 \right. \\
& \quad + \frac{\operatorname{We}}{\varepsilon} \|\nabla^2 u\|_{L^2} \|\tau\|_{L^\infty} \|\nabla \tau\|_{L^2} + \operatorname{Re} \|\nabla u\|_{L^\infty} \|\nabla^2 u\|_{L^2}^2 + \frac{\operatorname{We}}{\varepsilon} \|\nabla u\|_{L^\infty} \|\nabla^2 \tau\|_{L^2}^2 \\
& \quad + \frac{\operatorname{We}}{\varepsilon} \|\nabla^2 u\|_{L^4} \|\nabla \tau\|_{L^4} \|\nabla^2 \tau\|_{L^2} + \frac{\operatorname{We}}{\varepsilon} \|\nabla^3 u\|_{L^2} \|\nabla^2 \tau\|_{L^2} \|\tau\|_{L^\infty} \Big) \\
& \quad + \|\nabla u\|_{L^2} \|\tau\|_{L^2} + \|\nabla^2 u\|_{L^2} \|\nabla \tau\|_{L^2} + \|\nabla^3 u\|_{L^2} \|\nabla^2 \tau\|_{L^2}.
\end{aligned}$$

Using the Gagliardo-Nirenberg interpolation inequalities, we have

$$\|\nabla \tau\|_{L^4} \leq C(n) \|\tau\|_{H^2}, \quad \|\nabla^2 u\|_{L^4} \leq C(n) \|\nabla u\|_{H^2}.$$

Therefore, we get

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \left(\operatorname{Re} \|u\|_{H^2}^2 + \frac{\operatorname{We}}{\varepsilon} \|\tau\|_{H^2}^2 \right) + (1 - \varepsilon) \|\nabla u\|_{H^2}^2 + \frac{1}{\varepsilon} \|\tau\|_{H^2}^2 \\
& \leq C(n) \left(\|\nabla u\|_{H^2} \frac{\operatorname{We}}{\varepsilon} \|\tau\|_{H^2}^2 + \operatorname{Re} \|\nabla u\|_{H^2}^2 \|u\|_{H^2} \right) + \|\nabla u\|_{H^2} \|\tau\|_{H^2}.
\end{aligned} \tag{4.1}$$

Similarly, for general positive integer $s > [\frac{n}{2}] + 1$, we can get the following estimate

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \left(\operatorname{Re} \|u\|_{H^s}^2 + \frac{\operatorname{We}}{\varepsilon} \|\tau\|_{H^s}^2 \right) + (1 - \varepsilon) \|\nabla u\|_{H^s}^2 + \frac{1}{\varepsilon} \|\tau\|_{H^s}^2 \\
& \leq C(n) \left(\|\nabla u\|_{H^s} \frac{\operatorname{We}}{\varepsilon} \|\tau\|_{H^s}^2 + \operatorname{Re} \|\nabla u\|_{H^s}^2 \|u\|_{H^s} \right) + \|\nabla u\|_{H^s} \|\tau\|_{H^s}.
\end{aligned} \tag{4.2}$$

When $0 < \varepsilon < \frac{4}{5}$, we can choose $k_1, k_2 > 0$ satisfying $k_1 + k_2 < 1 - \varepsilon$ and $\frac{1}{4k_2} < \frac{1}{\varepsilon}$ such that

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \left(\operatorname{Re} \|u\|_{H^s}^2 + \frac{\operatorname{We}}{\varepsilon} \|\tau\|_{H^s}^2 \right) + (1 - \varepsilon) \|\nabla u\|_{H^s}^2 + \frac{1}{\varepsilon} \|\tau\|_{H^s}^2 \\
& \leq k_1 \|\nabla u\|_{H^s}^2 + \frac{1}{4k_1} \left(C(n) \frac{\operatorname{We}}{\varepsilon} \|\tau\|_{H^s}^2 \right)^2 + C(n) \operatorname{Re} \|\nabla u\|_{H^s}^2 \|u\|_{H^s} + k_2 \|\nabla u\|_{H^s}^2 + \frac{1}{4k_2} \|\tau\|_{H^s}^2,
\end{aligned}$$

which implies

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \left(\operatorname{Re} \|u\|_{H^s}^2 + \frac{\operatorname{We}}{\varepsilon} \|\tau\|_{H^s}^2 \right) \leq -[(1 - \varepsilon) - k_1 - k_2 - C(n) \operatorname{Re} \|u\|_{H^s}] \|\nabla u\|_{H^s}^2 \\
& \quad - \left[\frac{1}{\varepsilon} - \frac{1}{4k_2} - \frac{1}{4k_1} C(n)^2 \frac{\operatorname{We}^2}{\varepsilon^2} \|\tau\|_{H^s}^2 \right] \|\tau\|_{H^s}^2.
\end{aligned} \tag{4.3}$$

Set

$$A_0 = \max \left\{ \|u_0\|_{H^s}^2 + \frac{\operatorname{We}}{\operatorname{Re} \varepsilon} \|\tau_0\|_{H^s}^2, \frac{\varepsilon}{\operatorname{We}} \operatorname{Re} \|u_0\|_{H^s}^2 + \|\tau_0\|_{H^s}^2 \right\}.$$

If A_0 is so small that

$$(1 - \varepsilon) - k_1 - k_2 - C(n) \operatorname{Re} \sqrt{A_0} > 0 \tag{4.4}$$

and

$$\frac{1}{\varepsilon} - \frac{1}{4k_2} - \frac{1}{4k_1} C(n)^2 \frac{\operatorname{We}^2}{\varepsilon^2} A_0 > 0, \tag{4.5}$$

from (4.3) it turns out that

$$\operatorname{Re}\|u\|_{H^s}^2 + \frac{\operatorname{We}}{\varepsilon}\|\tau\|_{H^s}^2 \leq \operatorname{Re}\|u_0\|_{H^s}^2 + \frac{\operatorname{We}}{\varepsilon}\|\tau_0\|_{H^s}^2.$$

Hence $\|u\|_{H^s}^2 \leq A_0$ and $\|\tau\|_{H^s}^2 \leq A_0$. In view of the choice of A_0 in (4.4) and (4.5), (4.3) tells us $\operatorname{Re}\|u\|_{H^s}^2 + \frac{\operatorname{We}}{\varepsilon}\|\tau\|_{H^s}^2$ is monotone-decreasing. Therefore, if the initial data are small enough, then $\|u\|_{H^s}$ and $\|\tau\|_{H^s}$ will be bounded for all $t < \infty$. Using the blow up principle in Theorem 1.1, we finish the proof of Theorem 1.2.

Acknowledgement The author would like to thank Prof. Jiaxing Hong, Prof. Chun Liu and Prof. Yi Zhou for a lot of suggestions to this paper.

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