# $\eta$-Invariant and Flat Vector Bundles***

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(*Dedicated to the memory of Shiing-Shen Chern*)

**Abstract** We present an alternate definition of the mod **Z** component of the Atiyah-Patodi-Singer $\eta$ invariant associated to (not necessary unitary) flat vector bundles, which identifies explicitly its real and imaginary parts. This is done by combining a deformation of flat connections introduced in a previous paper with the analytic continuation procedure appearing in the original article of Atiyah, Patodi and Singer.

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## 1 Introduction

Let $M$ be an odd dimensional oriented closed spin manifold carrying a Riemannian metric $g^{TM}$. Let $S(TM)$ be the associated Hermitian bundle of spinors. Let $E$ be a Hermitian vector bundle over $M$ carrying a unitary connection $\nabla^E$. Moreover, let $F$ be a Hermitian vector bundle over $M$ carrying a unitary flat connection $\nabla^F$. Let

$$D^{E\otimes F} : \Gamma(S(TM)\otimes E \otimes F) \longrightarrow \Gamma(S(TM)\otimes E \otimes F) \tag{1.1}$$

denote the corresponding (twisted) Dirac operator, which is formally self-adjoint (cf. [4]).

For any $s \in \mathbf{C}$ with $\mathrm{Re}\,(s) \gg 0$, following [1], set

$$\eta(D^{E\otimes F}, s) = \sum_{\lambda \in \mathrm{Spec}(D^{E\otimes F})\setminus\{0\}} \frac{\mathrm{Sgn}(\lambda)}{|\lambda|^s}. \tag{1.2}$$

Then by [1], one knows that $\eta(D^{E\otimes F}, s)$ is a holomorphic function in $s$ when $\mathrm{Re}\,(s) > \frac{\dim M}{2}$. Moreover, it extends to a meromorphic function over $\mathbf{C}$, which is holomorphic at $s = 0$. The $\eta$ invariant of $D^{E\otimes F}$, in the sense of Atiyah-Patodi-Singer [1], is defined by

$$\eta(D^{E\otimes F}) = \eta(D^{E\otimes F}, 0), \tag{1.3}$$

while the corresponding *reduced* $\eta$ invariant is defined and denoted by

$$\bar{\eta}(D^{E\otimes F}) = \frac{\dim(\ker D^{E\otimes F}) + \eta(D^{E\otimes F})}{2}. \tag{1.4}$$

The $\eta$ and reduced $\eta$ invariants play an important role in the Atiyah-Patodi-Singer index theorem for Dirac operators on manifolds with boundary (cf. [1]).

In [2] and [3], it is shown that the following quantity

$$\rho(D^{E\otimes F}) := \bar{\eta}(D^{E\otimes F}) - \operatorname{rk}(F)\,\bar{\eta}(D^E) \quad \mod \mathbf{Z} \tag{1.5}$$

does not depend on the choice of $g^{TM}$ as well as the metrics and (Hermitian) connections on $E$. Also, a Riemann-Roch theorem is proved in [3, (5.3)], which gives a $K$-theoretic interpretation of the analytically defined invariant $\rho(D^{E\otimes F}) \in \mathbf{R}/\mathbf{Z}$. Moreover, it is pointed out in [3, Remark (1), p. 89] that the above mentioned $K$-theoretic interpretation applies also to the case where $F$ is a non-unitary flat vector bundle, while on [3, p. 93] it shows how one can define the reduced $\eta$-invariant in case $F$ is non-unitary, by working on non-self-adjoint elliptic operators, and then extend the Riemann-Roch result [3, (5.3)] to an identity in $\mathbf{C}/\mathbf{Z}$ (instead of $\mathbf{R}/\mathbf{Z}$). The idea of analytic continuation plays a key role in obtaining this Riemann-Roch result, as well as its non-unitary extension.

In this paper, we show that by using the idea of analytic continuation, one can construct the $\mathbf{C}/\mathbf{Z}$ component of $\bar{\eta}(D^{E\otimes F})$ directly, without passing to analysis of non-self-adjoint operators, in the case where $F$ is a non-unitary flat vector bundle. Consequently, this leads to a direct construction of $\rho(D^{E\otimes F})$ in this case. We will use a deformation introduced in [9] for flat connections in our construction.

In the next section, we will first recall the above mentioned deformation from [9] and then give our construction of $\bar{\eta}(D^{E\otimes F}) \mod \mathbf{Z}$ and $\rho(D^{E\otimes F}) \in \mathbf{C}/\mathbf{Z}$ in the case where $F$ is a non-unitary flat vector bundle.

## 2 The $\eta$ and $\rho$ Invariants Associated to Non-unitary Flat Vector Bundles

This section is organized as follows. In Subsection 2.1, we construct certain secondary characteristic forms and classes associated to non-unitary flat vector bundles. In Subsection 2.2, we present our construction of the mod $\mathbf{Z}$ component of the reduced $\eta$-invariant, as well as the $\rho$-invariant, associated to non-unitary flat vector bundles. Finally, we include some further remarks in Subsection 2.3.

### 2.1 Chern-Simons classes and flat vector bundles

We fix a square root of $\sqrt{-1}$ and let $\varphi : \Lambda(T^*M) \to \Lambda(T^*M)$ be the homomorphism defined by $\varphi : \omega \in \Lambda^i(T^*M) \to (2\pi\sqrt{-1})^{-i/2}\omega$. The formulas in what follows will not depend on the choice of the square root of $\sqrt{-1}$.

If $W$ is a complex vector bundles over $M$ and $\nabla_0^W$, $\nabla_1^W$ are two connections on $W$. Let $W_t$, $0 \le t \le 1$, be a smooth path of connections on $W$ connecting $\nabla_0^W$ and $\nabla_1^W$. We define Chern-Simons form $\mathrm{CS}(\nabla_0^W, \nabla_1^W)$ to be the differential form given by

$$\mathrm{CS}(\nabla_0^W, \nabla_1^W) = -\left(\frac{1}{2\pi\sqrt{-1}}\right)^{\frac{1}{2}}\varphi\int_0^1 \mathrm{Tr}\Big[\frac{\partial\nabla_t^W}{\partial t}\exp(-(\nabla_t^W)^2)\Big]dt. \tag{2.1}$$

Then (cf. [10, Chapter 1])

$$d\,\mathrm{CS}(\nabla_0^W, \nabla_1^W) = \mathrm{ch}(W, \nabla_1^W) - \mathrm{ch}(W, \nabla_0^W). \tag{2.2}$$

Moreover, it is well known that up to exact forms, $\mathrm{CS}(\nabla_0^W, \nabla_1^W)$ does not depend on the path of connections on $W$ connecting $\nabla_0^W$ and $\nabla_1^W$.

Let $(F, \nabla^F)$ be a flat vector bundle carrying the flat connection $\nabla^F$. Let $g^F$ be a Hermitian metric on $F$. We do not assume that $\nabla^F$ preserves $g^F$. Let $(\nabla^F)^*$ be the adjoint connection of $\nabla^F$ with respect to $g^F$.

From [8, (4.1), (4.2)] and [7, §1, (g)], one has

$$(\nabla^F)^* = \nabla^F + \omega(F, g^F) \tag{2.3}$$

with

$$\omega(F, g^F) = (g^F)^{-1}(\nabla^F g^F). \tag{2.4}$$

Then

$$\nabla^{F,e} = \nabla^F + \frac{1}{2}\omega(F, g^F) \tag{2.5}$$

is a Hermitian connection on $(F, g^F)$ (cf. [7, (1.33)] and [8, (4.3)]).

Following [9, (2.47)], for any $r \in \mathbf{C}$, set

$$\nabla^{F,e,(r)} = \nabla^{F,e} + \frac{\sqrt{-1}\,r}{2}\omega(F, g^F). \tag{2.6}$$

Then for any $r \in \mathbf{R}$, $\nabla^{F,e,(r)}$ is a Hermitian connection on $(F, g^F)$.

On the other hand, following [7, (0.2)], for any integer $j \geq 0$, let $c_{2j+1}(F, g^F)$ be the Chern form defined by

$$c_{2j+1}(F, g^F) = (2\pi\sqrt{-1})^{-j}2^{-(2j+1)}\mathrm{Tr}[\omega^{2j+1}(F, g^F)]. \tag{2.7}$$

Then $c_{2j+1}(F, g^F)$ is a closed form on $M$. Let $c_{2j+1}(F)$ be the associated cohomology class in $H^{2j+1}(M, \mathbf{R})$, which does not depend on the choice of $g^F$.

For any $j \geq 0$ and $r \in \mathbf{R}$, let $a_j(r) \in \mathbf{R}$ be defined as

$$a_j(r) = \int_0^1 (1 + u^2 r^2)^j du. \tag{2.8}$$

With these notation we can now state the following result first proved in [9, Lemma 2.12].

**Proposition 2.1** *The following identity in $H^{\mathrm{odd}}(M, \mathbf{R})$ holds for any $r \in \mathbf{R}$,*

$$\mathrm{CS}(\nabla^{F,e}, \nabla^{F,e,(r)}) = -\frac{r}{2\pi}\sum_{j=0}^{+\infty}\frac{a_j(r)}{j!}c_{2j+1}(F). \tag{2.9}$$

### 2.2 $\eta$ and $\rho$ invariants associated to flat vector bundles

We now make the same assumptions as in the beginning of Section 1, except that we no longer assume $\nabla^F$ there is unitary.

For any $r \in \mathbf{C}$, let

$$D^{E \otimes F}(r) : \Gamma(S(TM) \otimes E \otimes F) \longrightarrow \Gamma(S(TM) \otimes E \otimes F) \tag{2.10}$$

denote the Dirac operator associated to the connection $\nabla^{F,e,(r)}$ on $F$. Since when $r \in \mathbf{R}$, $\nabla^{F,e,(r)}$ is Hermitian on $(F, g^F)$, $D^{E \otimes F}(r)$ is formally self-adjoint and one can define the associated reduced $\eta$-invariant as in (1.4).

By the variation formula for the reduced $\eta$-invariant (cf. [1, 6]), one gets that for any $r \in \mathbf{R}$,

$$\bar{\eta}(D^{E \otimes F}(r)) - \bar{\eta}(D^{E \otimes F}(0)) \equiv \int_M \widehat{A}(TM) \operatorname{ch}(E) \operatorname{CS}(\nabla^{F,e}, \nabla^{F,e,(r)}) \mod \mathbf{Z}, \tag{2.11}$$

where $\widehat{A}$ and ch are standard notations for the Hirzebruch $\widehat{A}$-class and Chern character respectively (cf. [10, Chapter 1]).

Let $D^{E \otimes F,e}$ denote the Dirac operator $D^{E \otimes F}(0)$.

From (2.9) and (2.11), one gets that for any $r \in \mathbf{R}$,

$$\bar{\eta}(D^{E \otimes F}(r)) \equiv \bar{\eta}(D^{E \otimes F,e}) - \frac{r}{2\pi} \int_M \widehat{A}(TM) \operatorname{ch}(E) \sum_{j=0}^{+\infty} \frac{a_j(r)}{j!} c_{2j+1}(F) \mod \mathbf{Z}. \tag{2.12}$$

Recall that even though when $\operatorname{Im}(r) \neq 0$, $D^{E \otimes F}(r)$ might not be formally self-adjoint, the $\eta$-invariant can still be defined, as outlined in [3, p. 93]. On the other hand, from (2.5) and (2.6), one sees that

$$\nabla^F = \nabla^{F,e,(\sqrt{-1})}. \tag{2.13}$$

We denote the associated Dirac operator $D^{E \otimes F}(\sqrt{-1})$ by $D^{E \otimes F}$.

We also recall that

$$\int_0^1 (1 - u^2)^j du = \frac{2^{2j}(j!)^2}{(2j+1)!}. \tag{2.14}$$

We can now state the main result of this paper as follows.

**Theorem 2.2** *Formula* (2.12) *holds indeed for any* $r \in \mathbf{C}$. *In particular, one has*

$$\bar{\eta}(D^{E \otimes F}) \equiv \bar{\eta}(D^{E \otimes F,e}) - \frac{\sqrt{-1}}{2\pi} \int_M \widehat{A}(TM) \operatorname{ch}(E) \sum_{j=0}^{+\infty} \frac{2^{2j} j!}{(2j+1)!} c_{2j+1}(F) \mod \mathbf{Z}. \tag{2.15}$$

*Equivalently,*

$$\operatorname{Re}(\bar{\eta}(D^{E \otimes F})) \equiv \bar{\eta}(D^{E \otimes F,e}) \mod \mathbf{Z},$$

$$\operatorname{Im}(\bar{\eta}(D^{E \otimes F})) = -\frac{1}{2\pi} \int_M \widehat{A}(TM) \operatorname{ch}(E) \sum_{j=0}^{+\infty} \frac{2^{2j} j!}{(2j+1)!} c_{2j+1}(F). \tag{2.16}$$

**Proof** Clearly, the right-hand side of (2.12) is a holomorphic function in $r \in \mathbf{C}$. On the other hand, by [3, p. 93], $\bar{\eta}(D^{E \otimes F}(r)) \mod \mathbf{Z}$ is also holomorphic in $r \in \mathbf{C}$. By (2.12) and the uniqueness of the analytic continuation, one sees that (2.12) holds indeed for any $r \in \mathbf{C}$. In particular, by putting together (2.12) and (2.13), one gets (2.15).

Recall that when $\nabla^F$ preserves $g^F$, the $\rho$-invariant has been defined in (1.5). Now if we no longer assume that $\nabla^F$ preserves $g^F$, then by Theorem 2.2, one sees that one gets the following formula of the associated (extended) $\rho$-invariant.

**Corollary 2.3** *The following identity holds:*

$$\rho(D^{E \otimes F}) \equiv \bar{\eta}(D^{E \otimes F, e}) - \mathrm{rk}(F)\,\bar{\eta}(D^E)$$

$$- \frac{\sqrt{-1}}{2\pi} \int_M \widehat{A}(TM)\,\mathrm{ch}(E) \sum_{j=0}^{+\infty} \frac{2^{2j}j!}{(2j+1)!} c_{2j+1}(F) \mod \mathbf{Z}. \qquad (2.17)$$

*Equivalently,*

$$\mathrm{Re}\,(\rho(D^{E \otimes F})) \equiv \bar{\eta}(D^{E \otimes F, e}) - \mathrm{rk}(F)\,\bar{\eta}(D^E) \mod \mathbf{Z},$$

$$\mathrm{Im}\,(\rho(D^{E \otimes F})) = -\frac{1}{2\pi} \int_M \widehat{A}(TM)\,\mathrm{ch}(E) \sum_{j=0}^{+\infty} \frac{2^{2j}j!}{(2j+1)!} c_{2j+1}(F). \qquad (2.18)$$

It is pointed out in [3] that the Riemann-Roch formula proved in [3, (5.3)] still holds for $\rho(D^{E \otimes F})$ in the case where $\nabla^F$ does not preserve $g^F$. One way to understand this is that the argument in the proof of [3, (5.3)] given in [3] works line by line to give a $K$-theoretic interpretation of $\bar{\eta}(D^{E \otimes F, e}) - \mathrm{rk}(F)\,\bar{\eta}(D^E)$. By (2.17) it then gives such an interpretation for $\rho(D^{E \otimes F})$.

### 2.3 Further remarks

**Remark 2.4** The argument in proving Theorem 2.2 works indeed for any twisted vector bundles $F$, not necessary a flat vector bundle. This gives a direct formula for the mod $\mathbf{Z}$ part of the $\eta$-invariant for non-self-adjoint Dirac operators.

**Remark 2.5** In [11, Theorem 2.2], a $K$-theoretic formula for $D^{E \otimes F}(r) \mod \mathbf{Z}$ has been given in the $r \in \mathbf{R}$ case. As a consequence, one gets an alternate $K$-theoretic formula for $\rho(D^{E \otimes F})$ in [11, (4.6)] which holds in the case where $\nabla^F$ preserves $g^F$. By combining the arguments in [11] with Theorem 2.2 proved above, one can indeed extend [11, Theorem 2.2] and [11, (4.6)] to the case where $\nabla^F$ might not preserve $g^F$. We leave this to the interested reader. Here we only mention that this will provide an alternate $K$-theoretic interpretation of $\rho$-invariants in the case where $\nabla^F$ does not preserve $g^F$.

**Remark 2.6** We refer to [9] where we have employed the deformation (2.6) to study and generalize certain Riemann-Roch-Grothendieck formulas due to Bismut-Lott [7] and Bismut [5], for flat vector bundles over fibred spaces.

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