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On Holomorphic Automorphisms of a Class of Non-homogeneous Rigid Hypersurfaces in $\mathbb{C}^{N+1}$**

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Abstract The author determines the real-analytic infinitesimal CR automorphisms of a class of non-homogeneous rigid hypersurfaces in $\mathbb{C}^{N+1}$ near the origin, and the connected component containing the identity transformation of all locally holomorphic automorphisms of these hypersurfaces near the origin.

Keywords Real-analytic infinitesimal CR automorphisms, Rigid hypersurfaces, Holomorphic automorphisms
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1 Introduction

It is well-known that the set of all locally holomorphic automorphisms of the hyperquadric $S_{n+1} = \{(z, w) \in \mathbb{C}^n \times \mathbb{C} \mid \operatorname{Im} w = |z|^2\}$ is the group of $\mathrm{SU}(n+1, 1)$ given by fractional linear transformations. This group plays an important role in the study of spherical CR manifold (cf. [19]). It is interesting and important to determine all locally holomorphic automorphisms of a real submanifold in $\mathbb{C}^n$ (cf. [7]). A criterion for the finite dimensionality of the automorphism group of a hypersurface was given by Stanton [17, 18]. Baouendi, Ebenfelt and Rothschild [3, 4] also studied the condition under which the Lie algebra of locally defined infinitesimal CR automorphisms of a real submanifold is finite dimensional. On the other hand, Beloshapka [5] obtained a description of the Lie algebra of infinitesimal automorphisms of any quadric Q. Shevchenko [16] constructed canonical forms for nondegenerate CR-quadrics of codimension two in a complex space and gave a complete description of the algebra of infinitesimal holomorphic automorphisms. Ežov and Schmalz [9] realized arbitrary automorphism of a non-degenerate $(n, 2)$-quadric by a rational map of degree not more than two. For higher codimension, it is known that each $(3, 3)$-quadric possessing non-linear automorphisms is equivalent to one of eight quadrics (cf. [13, 14]), whose automorphism groups are determined in [1].

For higher degree model surface, Beloshapka considered the surface Q_3 in the space $\mathbb{C}^n \oplus \mathbb{C}^{n^2} \oplus \mathbb{C}^k$ with coordinates $(z \in \mathbb{C}^n,\ w_2 \in \mathbb{C}^{n^2},\ w_3 \in \mathbb{C}^k)$, given by the equations $\operatorname{Im} w_2 = \langle z, \overline{z}\rangle$, $\operatorname{Im} w_3 = 2 \operatorname{Re} \Phi(z, \overline{z})$, where $\langle z, \overline{z}\rangle$ is an n^2 scalar linearly independent Hermitian form, and $\Phi(z, \overline{z})$ is a homogeneous $\mathbb{C}^k$-valued form of degree three, and gave the structure of the automorphism algebra of the cubic (cf. [7] and references therein). See [6, 15] for results for the polynomial models of even higher codimension and degree. It is also interesting to consider

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the case when the domain and the image of a holomorphic mapping are different, e.g. from B^n to B^N, $n \neq N$. Huang, Ji and Xu classified the proper holomorphic mappings $f : B^n \to B^N$ between the unit balls in $\mathbb{C}^n$ and $\mathbb{C}^N$ with $N \geq n \geq 1$ (cf. [10, 11]).

All models above are homogeneous, i.e., the holomorphic automorphisms act on them transitively. Moreover, these models can be given the structure of nilpotent groups. Kolář [12] gave a complete description of local automorphism groups for Levi degenerate hypersurfaces of finite type in $\mathbb{C}^2$. Here we consider a class of non-homogeneous rigid hypersurfaces in $\mathbb{C}^{N+1}$.

Let M be a real rigid hypersurface through the origin in $\mathbb{C}^{N+1}$, i.e., there are coordinates $(z_1, \cdots, z_N, w)$ such that M is given by an equation of the following form:

$$\operatorname{Im} w = F(z, \overline{z}). \tag{1.1}$$

Consider the non-homogenous rigid hypersurfaces of the form

$$\Gamma = \Big\{(z_1, \cdots, z_N, w) \in \mathbb{C}^{N+1} \,\Big|\, \operatorname{Im} w = \sum_{j=1}^{N} |z_j|^{2n_j}\Big\}, \tag{1.2}$$

where $n_j \in \mathbb{Z}_+$ and $n_j > 1$.

By a germ at the origin of holomorphic automorphism of Γ, we mean a local biholomorphism of $\mathbb{C}^{N+1}$ defined in a neighborhood U of the origin that maps $U \cap \Gamma$ into Γ. We denote by $\operatorname{Aut}(\Gamma, 0)$ the set of germs at the origin of holomorphic automorphisms of Γ. Also denote by $\operatorname{hol}(\Gamma, 0)$ the set of real-analytic infinitesimal CR automorphisms of Γ at the origin, i.e., $\operatorname{hol}(\Gamma, 0)$ consists of all germs at the origin of vector fields X tangent to Γ such that the local 1-parameter group of transformations generated by X are biholomorphic transformations of $\mathbb{C}^{N+1}$ preserving Γ. From [2, Proposition 12.4.22], $\operatorname{hol}(\Gamma, 0)$ can be written in the following form:

$$\operatorname{hol}(\Gamma, 0) = \Big\{X(z, w) = 2\operatorname{Re}\Big(\sum_{\mu=1}^{N} f_\mu(z, w)\frac{\partial}{\partial z_\mu} + g(z, w)\frac{\partial}{\partial w}\Big)\Big\}, \tag{1.3}$$

where X is tangent to Γ, $f_\mu(z, w)$ and $g(z, w)$ are holomorphic functions near the origin and $z = (z_1, z_2, \cdots, z_N)$. By $\operatorname{Aut}_0 \Gamma$ we denote the set of germs in $\operatorname{Aut}(\Gamma, 0)$ preserving the origin. Denote by $\operatorname{hol}_0 \Gamma$ the set of vector fields in $\operatorname{hol}(\Gamma, 0)$ vanishing at the origin.

In this paper, we obtain an explicit formula of $\operatorname{hol}(\Gamma, 0)$.

Theorem 1.1 *Suppose* $X = 2\operatorname{Re}\Big[\sum_{\mu=1}^{N} f_\mu(z, w)\frac{\partial}{\partial z_\mu} + g(z, w)\frac{\partial}{\partial w}\Big] \in \operatorname{hol}(\Gamma, 0)$. *Then locally in a neighborhood of the origin, the functions* f_μ, g *can be written in the following form:*

$$\begin{cases} f_\mu(z, w) = \Big(\dfrac{n_1}{n_\mu}\alpha_1 + \mathrm{i}\beta_\mu\Big) z_\mu + \dfrac{n_1}{n_\mu}\alpha_2 z_\mu w, \\ g(z, w) = n_1\alpha_2 w^2 + 2n_1\alpha_1 w + \alpha_3, \end{cases} \tag{1.4}$$

where $\alpha_k,\ \beta_\mu \in \mathbb{R},\ k = 1, 2, 3,\ \mu = 1, \cdots, N.$

We also get the connected component of the identity transformation of $\operatorname{Aut}(\Gamma, 0)$, which is denoted by $\operatorname{Aut}_{\mathrm{id}}(\Gamma, 0)$.

Theorem 1.2 $(F_1,\cdots,F_N,G)\in \mathrm{Aut}_{\mathrm{id}}(\Gamma,0)$ *if and only if functions* F_μ *and* G *can be written in the following form:*

$$F_\mu(z,w)=\frac{\lambda^{\frac{n_1}{n_\mu}}\mathrm{e}^{\mathrm{i}\theta_\mu}z_\mu}{(1+\gamma_1 w)^{\frac{1}{n_\mu}}},\quad G(z,w)=\frac{\lambda^{2n_1}w}{1+\gamma_1 w}+\gamma_2, \tag{1.5}$$

where $\lambda\in\mathbb{R}_+,\ \gamma_1,\gamma_2,\theta_\mu\in\mathbb{R},\ \mu=1,\cdots,N.$

We will prove Theorem 1.1 in Section 2. In Section 3, we obtain a representation of $\mathrm{Aut}(\Gamma,0)$, and get the connected component of the identity transformation of $\mathrm{Aut}(\Gamma,0)$ by using this representation.

2 Real-Analytic Infinitesimal CR Automorphisms of Γ

By definition (1.2), Γ is defined by the equation

$$\rho(z,w,\overline{z},\overline{w})=\sum_{j=1}^{N}z_j^{n_j}\overline{z}_j^{n_j}-\frac{w-\overline{w}}{2\mathrm{i}}=0. \tag{2.1}$$

Since any $X=2\,\mathrm{Re}\Big[\sum\limits_{\mu=1}^{N}f_\mu(z,w)\frac{\partial}{\partial z_\mu}+g(z,w)\frac{\partial}{\partial w}\Big]\in \mathrm{hol}(\Gamma,0)$ is tangent to Γ, we have

$$\mathrm{Re}\Big[\mathrm{i}g\Big(z,u+\mathrm{i}\sum_{j=1}^{N}|z_j|^{2n_j}\Big)+2\sum_{\mu=1}^{N}n_\mu z_\mu^{n_\mu-1}\overline{z}_\mu^{n_\mu}f_\mu\Big(z,u+\mathrm{i}\sum_{j=1}^{N}|z_j|^{2n_j}\Big)\Big]=0 \tag{2.2}$$

for $w=u+\mathrm{i}\sum\limits_{j=1}^{N}|z_j|^{2n_j}$ and $(z,u)\in U$, where U is a small neighborhood of the origin in $\mathbb{C}^N\times\mathbb{R}$.

Proof of Theorem 1.1 The theorem is proved by solving equation (2.2) in the class of formal power series. This method was originally used by Beloshapka [7] for homogeneous models.

Let $X=2\,\mathrm{Re}\Big[\sum\limits_{\mu=1}^{N}f_\mu(z,w)\frac{\partial}{\partial z_\mu}+g(z,w)\frac{\partial}{\partial w}\Big]\in\mathrm{hol}(\Gamma,0)$. By Taylor's expansion with respect to variable w at the point $w=u$, we have

$$\begin{aligned} f_\mu\Big(z,u+\mathrm{i}\sum_{j=1}^{N}|z_j|^{2n_j}\Big)&=\sum_{m=0}^{\infty}f_\mu^{(m)}(z,u)\frac{\mathrm{i}^m\Big(\sum\limits_{j=1}^{N}|z_j|^{2n_j}\Big)^m}{m!},\\ g\Big(z,u+\mathrm{i}\sum_{j=1}^{N}|z_j|^{2n_j}\Big)&=\sum_{m=0}^{\infty}g^{(m)}(z,u)\frac{\mathrm{i}^m\Big(\sum\limits_{j=1}^{N}|z_j|^{2n_j}\Big)^m}{m!}, \end{aligned} \tag{2.3}$$

where $f_\mu^{(m)}(z,u),\ g^{(m)}(z,u)$ indicate differentiation with respect to w. Since $f_\mu(z,u)$ and $g(z,u)$ are holomorphic in z, we can write

$$f_\mu(z,u)=\sum_{k=0}^{\infty}f_{\mu k}(z,u),\quad g(z,u)=\sum_{k=0}^{\infty}g_k(z,u), \tag{2.4}$$

where

$$f_{\mu k}(tz,u)=t^k f_{\mu k}(z,u),\quad g_k(tz,u)=t^k g_k(z,u).$$

Now substitute (2.3) and (2.4) into (2.2), we get

$$\begin{aligned}
0=&\frac{\mathrm{i}}{2}\sum_{k=0}^{\infty}\Big[g_k(z,u)+\mathrm{i}g'_k(z,u)\Delta-\frac{1}{2}g''_k(z,u)\Delta^2-\frac{\mathrm{i}}{6}g'''_k(z,u)\Delta^3+\cdots\Big]\\
&-\frac{\mathrm{i}}{2}\sum_{k=0}^{\infty}\Big[\overline{g_k(z,u)}-\mathrm{i}\,\overline{g'_k(z,u)}\Delta-\frac{1}{2}\overline{g''_k(z,u)}\Delta^2+\frac{\mathrm{i}}{6}\overline{g'''_k(z,u)}\Delta^3+\cdots\Big]\\
&+n_1 z_1^{n_1-1}\overline{z}_1^{n_1}\sum_{k=0}^{\infty}\Big[f_{1k}(z,u)+\mathrm{i}f'_{1k}(z,u)\Delta-\frac{1}{2}f''_{1k}(z,u)\Delta^2+\cdots\Big]\\
&+n_1 z_1^{n_1}\overline{z}_1^{n_1-1}\sum_{k=0}^{\infty}\Big[\overline{f_{1k}(z,u)}-\mathrm{i}\,\overline{f'_{1k}(z,u)}\Delta-\frac{1}{2}\overline{f''_{1k}(z,u)}\Delta^2+\cdots\Big]\\
&\qquad\vdots\\
&+n_N z_N^{n_N-1}\overline{z}_N^{n_N}\sum_{k=0}^{\infty}\Big[f_{Nk}(z,u)+\mathrm{i}f'_{Nk}(z,u)\Delta-\frac{1}{2}f''_{Nk}(z,u)\Delta^2+\cdots\Big]\\
&+n_N z_N^{n_N}\overline{z}_N^{n_N-1}\sum_{k=0}^{\infty}\Big[\overline{f_{Nk}(z,u)}-\mathrm{i}\,\overline{f'_{Nk}(z,u)}\Delta-\frac{1}{2}\overline{f''_{Nk}(z,u)}\Delta^2+\cdots\Big],
\end{aligned}\tag{2.5}$$

where $\Delta=\sum\limits_{j=1}^{N}|z_j|^{2n_j}$, "$\,'\,$" indicates differentiation with respect to w, and "$\cdots$" denotes the other terms of $z^\alpha\overline{z}^\beta$ with $|\beta|\geq 3$. Here $\alpha=(\alpha_1,\cdots,\alpha_N)$, $\beta=(\beta_1,\cdots,\beta_N)$ are multi-indices with $\alpha_j,\beta_j\geq 0$ and $z^\alpha:=z_1^{\alpha_1}z_2^{\alpha_2}\cdots z_N^{\alpha_N}$, $\overline{z}^\beta:=\overline{z}_1^{\beta_1}\overline{z}_2^{\beta_2}\cdots\overline{z}_N^{\beta_N}$, $|\beta|:=\sum\limits_{j=1}^{N}\beta_j$. In the following, we call a term is of type (k,l) if it has the form $\sum\limits_{\substack{|\alpha|=k\\|\beta|=l}}h_\alpha^\beta(u)z^\alpha\overline{z}^\beta$ for some function $h_\alpha^\beta(u)$.

Let us collect terms of type (k,l). Firstly, we consider the terms of type $(k,0)$ in (2.5). Note that $g_k(z,u)$ and $f_{\mu k}(z,u)$ are terms of type $(k,0)$ and $\overline{g_k(z,u)}$, $\overline{f_{\mu k}(z,u)}$ are terms of type $(0,k)$. Consider terms of type $(m,0)$, $m>0$ for example. Since $n_j>1$, $j=1,\cdots,N$, terms in the third to the last rows in (2.5) contain the factors $\overline{z}^\beta$ with $|\beta|\geq 1$, and in the second row all terms but $(0,0)$ include the factors $\overline{z}^\beta$ ($|\beta|\geq 1$), therefore terms of type $(m,0)$ only appear in the first row. Furthermore, in this row, the terms concerning $g_k^{(\delta)}(z,u)$ ($\delta\geq 1$) also contain the factors $\overline{z}^\beta$ with $|\beta|\geq 1$. So they only exist in the first summation in this row, i.e., $\frac{\mathrm{i}}{2}g_m(z,u)$. Therefore, on the right-hand side in equation (2.5),

$$\begin{aligned}
(0,0)\text{ term}:&\ -\operatorname{Im}g_0(z,u),\\
(m,0)\text{ term}:&\ \frac{\mathrm{i}}{2}g_m(z,u),\quad m>0.
\end{aligned}$$

So we have

$$\operatorname{Im}g_0(z,u)=0,\quad g_m(z,u)=0,\quad m>0.\tag{2.6}$$

To determine $f_{\mu k}(z,u)$ ($\mu=1,\cdots,N$), let us consider all terms of type (k_μ,n_μ) ($k_\mu\geq n_\mu-1$) which contain $z_\mu^{n_\mu-1}\overline{z}_\mu^{n_\mu-1}$ on the right-hand side of (2.5),

$$(n_\mu-1,n_\mu)\text{ term}:\ n_\mu z_\mu^{n_\mu-1}\overline{z}_\mu^{n_\mu}f_{\mu 0}(z,u),$$

$$
\begin{aligned}
(n_\mu, n_\mu) \text{ term}: \quad & -\frac{1}{2}(g_0'(z,u)+\overline{g_0'(z,u)})|z_\mu|^{2n_\mu} \\
& + n_\mu f_{\mu 1}(z,u) z_\mu^{n_\mu-1}\overline{z}_\mu^{n_\mu} + n_\mu \overline{f_{\mu 1}(z,u)} z_\mu^{n_\mu}\overline{z}_\mu^{n_\mu-1} \\
& = -g_0'(z,u)|z_\mu|^{2n_\mu} + n_\mu f_{\mu 1}(z,u) z_\mu^{n_\mu-1}\overline{z}_\mu^{n_\mu} \\
& + n_\mu \overline{f_{\mu 1}(z,u)} z_\mu^{n_\mu}\overline{z}_\mu^{n_\mu-1}, \\
(k+n_\mu-1, n_\mu) \text{ term}: \quad & -\frac{1}{2} g_{k-1}'(z,u)|z_\mu|^{2n_\mu} + n_\mu f_{\mu k}(z,u) z_\mu^{n_\mu-1}\overline{z}_\mu^{n_\mu} \\
& = n_\mu f_{\mu k}(z,u) z_\mu^{n_\mu-1}\overline{z}_\mu^{n_\mu}, \quad k \geq 2.
\end{aligned} \tag{2.7}
$$

We have used (2.6) in (2.7). Therefore,

$$
f_{\mu 0}(z,u) = 0, \quad f_{\mu k}(z,u) = 0, \quad k \geq 2, \tag{2.8}
$$

$$
g_0'(z,u)|z_\mu|^2 - n_\mu f_{\mu 1}(z,u)\overline{z}_\mu - n_\mu \overline{f_{\mu 1}(z,u)} z_\mu = 0, \quad \mu = 1, \cdots, N. \tag{2.9}
$$

To determine $f_{\mu 1}(z,u)$, we consider all terms of type $(2n_\mu, 2n_\mu)$ which contain $z_\mu^{2n_\mu-1}\overline{z}_\mu^{2n_\mu-1}$ in (2.5), i.e.,

$$
\begin{aligned}
0 = & -\frac{\mathrm{i}}{4}(g_0''(z,u) - \overline{g_0''(z,u)})|z_\mu|^{4n_\mu} + \mathrm{i} n_\mu f_{\mu 1}'(z,u) z_\mu^{n_\mu-1}\overline{z}_\mu^{n_\mu}|z_\mu|^{2n_\mu} \\
& - \mathrm{i} n_\mu \overline{f_{\mu 1}'(z,u)} z_\mu^{n_\mu}\overline{z}_\mu^{n_\mu-1}|z_\mu|^{2n_\mu}.
\end{aligned} \tag{2.10}
$$

Then by (2.6), we get

$$
f_{\mu 1}'(z,u)\overline{z}_\mu - \overline{f_{\mu 1}'(z,u)} z_\mu = 0. \tag{2.11}
$$

Now let us collect all terms of type $(3n_\mu, 3n_\mu)$ which contain $z_\mu^{3n_\mu-1}\overline{z}_\mu^{3n_\mu-1}$ in (2.5). We have

$$
\begin{aligned}
0 = & \frac{1}{12}(g_0'''(z,u) + \overline{g_0'''(z,u)})|z_\mu|^{6n_\mu} - \frac{1}{2} n_\mu f_{\mu 1}''(z,u) z_\mu^{n_\mu-1}\overline{z}_\mu^{n_\mu}|z_\mu|^{4n_\mu} \\
& - \frac{1}{2} n_\mu \overline{f_{\mu 1}''(z,u)} z_\mu^{n_\mu}\overline{z}_\mu^{n_\mu-1}|z_\mu|^{4n_\mu}.
\end{aligned} \tag{2.12}
$$

Therefore,

$$
\frac{1}{3} g_0'''(z,u)|z_\mu|^2 - n_\mu f_{\mu 1}''(z,u)\overline{z}_\mu - n_\mu \overline{f_{\mu 1}''(z,u)} z_\mu = 0, \quad \mu = 1, \cdots, N. \tag{2.13}
$$

From (2.9), (2.11) and (2.13), we get

$$
g_0'''(z,u) = 0, \quad f_{\mu 1}''(z,u) = 0, \quad \mu = 1, \cdots, N. \tag{2.14}
$$

It is easy to see from (2.9) that each $f_{\mu 1}(z,u)$, $\mu = 1, \cdots, N$, can not contain the factor z_k with $k \neq \mu$. Now we conclude that

$$
\begin{cases}
f_{\mu 1}''(z,u) = 0, & \mu = 1, \cdots, N, \\
f_{\mu k}(z,u) = 0, & k \neq 1, \\
\operatorname{Im} g_0(z,u) = 0, & g_0'''(z,u) = 0, \\
g_m(z,u) = 0, & m \neq 0.
\end{cases} \tag{2.15}
$$

Since $f_\mu(z,w)$ and $g(z,w)$ are holomorphic functions near the origin, and $f_{\mu 1}(z,u)$ ($\mu = 1, \cdots, N$) cannot contain the factor z_k with $k \neq \mu$, together with (2.15), we find that f_μ and g must be written in the following form:

$$
\begin{aligned}
f_\mu(z,w) &= f_{\mu 1}(z,w) = a_\mu z_\mu + b_\mu z_\mu w, \quad \mu = 1, \cdots, N, \\
g(z,w) &= g_0(z,w) = \gamma + \beta w + \alpha w^2
\end{aligned} \tag{2.16}
$$

for some $\alpha,\beta,\gamma\in\mathbb{R},\ a_\mu,b_\mu\in\mathbb{C}$.

Since $X=2\,\mathrm{Re}\Big[\sum\limits_{\mu=1}^{N}f_\mu(z,w)\frac{\partial}{\partial z_\mu}+g(z,w)\frac{\partial}{\partial w}\Big]\in\mathrm{hol}(\Gamma,0)$ is tangent to Γ, f_μ and g given by (2.16) satisfy (2.2), i.e.,

$$\mathrm{Re}\Big[\mathrm{i}(\gamma+\beta w+\alpha w^2)+2\sum_{\mu=1}^{N}n_\mu z_\mu^{n_\mu-1}\overline{z}_\mu^{n_\mu}(a_\mu z_\mu+b_\mu z_\mu w)\Big]=0 \tag{2.17}$$

for $w=u+\mathrm{i}\sum\limits_{j=1}^{N}|z_j|^{2n_j}$ and $(z,u)\in U$. Then

$$\begin{aligned}0=&\,\mathrm{Re}\Big[\mathrm{i}\gamma+\mathrm{i}\beta\Big(u+\mathrm{i}\sum_{j=1}^{N}|z_j|^{2n_j}\Big)+\mathrm{i}\alpha\Big(u+\mathrm{i}\sum_{j=1}^{N}|z_j|^{2n_j}\Big)^2\\&+2\sum_{\mu=1}^{N}n_\mu\Big(a_\mu+b_\mu u+\mathrm{i}b_\mu\sum_{j=1}^{N}|z_j|^{2n_j}\Big)|z_\mu|^{2n_\mu}\Big]\\=&\sum_{\mu=1}^{N}\Big[(2n_\mu\,\mathrm{Re}\,a_\mu-\beta)|z_\mu|^{2n_\mu}+(2n_\mu\,\mathrm{Re}\,b_\mu-2\alpha)u|z_\mu|^{2n_\mu}\\&-2n_\mu\,\mathrm{Im}\,b_\mu|z_\mu|^{2n_\mu}\Big(\sum_{j=1}^{N}|z_j|^{2n_j}\Big)\Big].\end{aligned} \tag{2.18}$$

Thus,

$$\beta=2n_\mu\,\mathrm{Re}\,a_\mu,\quad \alpha=n_\mu\,\mathrm{Re}\,b_\mu,\quad \mathrm{Im}\,b_\mu=0,\quad \mu=1,\cdots,N. \tag{2.19}$$

Let $\alpha_1,\alpha_2,\ \alpha_3$ and $\beta_\mu\in\mathbb{R}$ denote $\mathrm{Re}\,a_1,b_1,\gamma$ and $\mathrm{Im}\,a_\mu$, $\mu=1,\cdots,N$, respectively. Then

$$a_\mu=\frac{n_1}{n_\mu}\alpha_1+\mathrm{i}\beta_\mu,\quad b_\mu=\frac{n_1}{n_\mu}\alpha_2,\quad \mu=1,\cdots,N. \tag{2.20}$$

Thus, (1.4) follows from (2.16) and (2.20). This proves Theorem 1.1.

3 Locally Holomorphic Automorphisms of Γ

Let $(z,w)\mapsto(F_{1t}(z,w),\cdots,F_{Nt}(z,w),G_t(z,w))$ be a one-parameter group generated by $X=2\,\mathrm{Re}\Big[\sum\limits_{\mu=1}^{N}f_\mu(z,w)\frac{\partial}{\partial z_\mu}+g(z,w)\frac{\partial}{\partial w}\Big]\in\mathrm{hol}(\Gamma,0)$ with $F_{\mu0}(z,w)=z_\mu,\ G_0(z,w)=w$, i.e., $F_{\mu t}$ and G_t are solutions to the initial problem of the following ordinary differential equation,

$$\begin{cases}\dfrac{\mathrm{d}F_{\mu t}}{\mathrm{d}t}=f_\mu(F_{1t},\cdots,F_{Nt},G_t)=\Big(\dfrac{n_1}{n_\mu}\alpha_1+\mathrm{i}\beta_\mu\Big)F_{\mu t}+\dfrac{n_1}{n_\mu}\alpha_2F_{\mu t}G_t,\\ \dfrac{\mathrm{d}G_t}{\mathrm{d}t}=g(F_{1t},\cdots,F_{Nt},G_t)=n_1\alpha_2G_t^2+2n_1\alpha_1G_t+\alpha_3,\\ F_{\mu0}(z,w)=z_\mu,\\ G_0(z,w)=w,\end{cases} \tag{3.1}$$

where $\alpha_k,\beta_\mu\in\mathbb{R},\ k=1,2,3,\ \mu=1,\cdots,N$. Recall that $\mathrm{hol}_0\,\Gamma$ is the set of vector fields in $\mathrm{hol}(\Gamma,0)$ vanishing at the origin.

Proposition 3.1 *The transformation* $(F_1, \cdots, F_N, G) : \Gamma \mapsto \Gamma$ *generated by any* $X = 2\,\mathrm{Re}\Big[\sum\limits_{\mu=1}^{N} f_\mu(z,w)\frac{\partial}{\partial z_\mu} + g(z,w)\frac{\partial}{\partial w}\Big] \in \mathrm{hol}_0\,\Gamma$ *can be written in the following form:*

$$F_\mu(z,w) = \frac{\lambda^{\frac{n_1}{n_\mu}} \mathrm{e}^{\mathrm{i}\xi_\mu} z_\mu}{(1+\gamma w)^{\frac{1}{n_\mu}}}, \quad G(z,w) = \frac{\lambda^{2n_1} w}{1+\gamma w}, \tag{3.2}$$

where $\lambda \in \mathbb{R}_+,\ \gamma, \xi_\mu \in \mathbb{R},\ \mu = 1, \cdots, N.$

Proof Since $X \in \mathrm{hol}_0\,\Gamma$ vanishes at the origin, $f_\mu,\ g$ can be written as (1.4) with $\alpha_3 = 0$. Now let us solve the ordinary equation (3.1) with $\alpha_3 = 0$.

(I) When $\alpha_1 = 0$, the second equation in (3.1) can be written as

$$\frac{\mathrm{d}G_t}{\mathrm{d}t} = n_1 \alpha_2 G_t^2 \tag{3.3}$$

with $G_0(z,w) = w$. So we have

$$G_t(z,w) = \frac{w}{1 - n_1\alpha_2 t w}. \tag{3.4}$$

Then substitute (3.4) into the first equation in (3.1) to get

$$\frac{\mathrm{d}F_{\mu t}}{\mathrm{d}t} = \Big(\mathrm{i}\beta_\mu + \frac{n_1}{n_\mu}\alpha_2 G_t\Big)F_{\mu t} = \Big(\mathrm{i}\beta_\mu + \frac{n_1\alpha_2 w}{n_\mu(1 - n_1\alpha_2 t w)}\Big)F_{\mu t}. \tag{3.5}$$

It is easy to see that

$$F_{\mu t}(z,w) = \frac{\mathrm{e}^{\mathrm{i}\beta_\mu t} z_\mu}{(1 - n_1\alpha_2 t w)^{\frac{1}{n_\mu}}}. \tag{3.6}$$

Denote $\xi_\mu = \beta_\mu t,\ \delta_1 = -n_1\alpha_2 t.$ Then

$$F_{\mu t}(z,w) = \frac{\mathrm{e}^{\mathrm{i}\xi_\mu} z_\mu}{(1+\delta_1 w)^{\frac{1}{n_\mu}}}, \quad G_t(z,w) = \frac{w}{1+\delta_1 w}, \tag{3.7}$$

where $\xi_\mu, \delta_1 \in \mathbb{R}$.

(II) When $\alpha_1 \neq 0$, the second equation in (3.1) can be written as

$$\frac{\mathrm{d}G_t}{\mathrm{d}t} = n_1\alpha_2 G_t^2 + 2n_1\alpha_1 G_t. \tag{3.8}$$

By multiplying G_t^{-2} on both sides in (3.8) and setting $X = G_t^{-1}$, we have

$$\frac{\mathrm{d}X}{\mathrm{d}t} = -2n_1\alpha_1 X - n_1\alpha_2 \tag{3.9}$$

with initial data $X(0) = w^{-1}$. Now solving this linear ordinary equation of first order, we obtain

$$X = \Big(\frac{1}{w} + \frac{\alpha_2}{2\alpha_1}\Big)\mathrm{e}^{-2n_1\alpha_1 t} - \frac{\alpha_2}{2\alpha_1}. \tag{3.10}$$

Therefore,

$$G_t = X^{-1} = \frac{2\alpha_1 \mathrm{e}^{2n_1\alpha_1 t} w}{2\alpha_1 + \alpha_2(1 - \mathrm{e}^{2n_1\alpha_1 t})w}. \tag{3.11}$$

Then substitute (3.11) into the first equation in (3.1) to get

$$\begin{aligned}\frac{\mathrm{d}F_{\mu t}}{\mathrm{d}t}&=\Big(\frac{n_1}{n_\mu}\alpha_1+\mathrm{i}\beta_\mu+\frac{n_1}{n_\mu}\alpha_2G_t\Big)F_{\mu t}\\&=\Big[\frac{n_1}{n_\mu}\alpha_1+\mathrm{i}\beta_\mu+\frac{n_1}{n_\mu}\frac{2\alpha_1\alpha_2\mathrm{e}^{2n_1\alpha_1t}w}{2\alpha_1+\alpha_2(1-\mathrm{e}^{2n_1\alpha_1t})w}\Big]F_{\mu t}\end{aligned}\tag{3.12}$$

with $F_{\mu 0}(z,w)=z_\mu$. So we have

$$F_{\mu t}(z,w)=\frac{(2\alpha_1)^{\frac{1}{n_\mu}}\mathrm{e}^{(\frac{n_1}{n_\mu}\alpha_1+\mathrm{i}\beta_\mu)t}z_\mu}{[2\alpha_1+\alpha_2(1-\mathrm{e}^{2n_1\alpha_1t})w]^{\frac{1}{n_\mu}}}.\tag{3.13}$$

Since $\alpha_1\neq 0$, we get

$$F_{\mu t}(z,w)=\frac{\mathrm{e}^{(\frac{n_1}{n_\mu}\alpha_1+\mathrm{i}\beta_\mu)t}z_\mu}{[1+\frac{\alpha_2}{2\alpha_1}(1-\mathrm{e}^{2n_1\alpha_1t})w]^{\frac{1}{n_\mu}}},\quad G_t(z,w)=\frac{\mathrm{e}^{2n_1\alpha_1t}w}{1+\frac{\alpha_2}{2\alpha_1}(1-\mathrm{e}^{2n_1\alpha_1t})w}.\tag{3.14}$$

Denote $\lambda=\mathrm{e}^{\alpha_1t}\in\mathbb{R}_+$ and $\delta_2=\frac{\alpha_2}{2\alpha_1}(1-\mathrm{e}^{2n\alpha_1t})\in\mathbb{R}$. Then

$$F_{\mu t}(z,w)=\frac{\lambda^{\frac{n_1}{n_\mu}}\mathrm{e}^{\mathrm{i}\xi_\mu}z_\mu}{(1+\delta_2w)^{\frac{1}{n_\mu}}},\quad G_t(z,w)=\frac{\lambda^{2n_1}w}{1+\delta_2w}.\tag{3.15}$$

From (3.7) and (3.15) we have (3.2). The proposition is proved.

Let $M\subset\mathbb{C}^n$ be a CR submanifold and $p_0\in M$. Then M is said to be of finite type m at p_0 if the tangent space of M at point p_0 is spanned by commutators of length m of sections of $T^{1,0}M\oplus T^{0,1}M$ and is not spanned by commutators of length up to $m-1$. By of finite type we mean that the type at each point $p\in M$ is less than a fixed positive integer.

The complex tangential subbundles $T^{1,0}\Gamma$ of the CR manifold Γ is spanned by

$$Z_j=\frac{\partial}{\partial z_j}+2in_jz_j^{n_j-1}\overline{z}_j^{n_j}\frac{\partial}{\partial w},\quad j=1,\cdots,N,\tag{3.16}$$

and $T^{0,1}\Gamma=\overline{T^{1,0}\Gamma}$, which is spanned by $\overline{Z}_j=\frac{\partial}{\partial\overline{z}_j}-2in_jz_j^{n_j}\overline{z}_j^{n_j-1}\frac{\partial}{\partial\overline{w}},\ j=1,\cdots,N$. We have

$$\begin{aligned}&[Z_j,\overline{Z}_j]=-2\mathrm{i}n_j^2|z_j|^{2n_j-2}T,\\&[Z_j,[Z_j,\overline{Z}_j]]=-2\mathrm{i}n_j^2(n_j-1)z_j^{n_j-2}\overline{z}_j^{n_j-1}T,\\&[[Z_j,[Z_j,\overline{Z}_j]],\overline{Z}_j]=2\mathrm{i}n_j^2(n_j-1)^2|z_j|^{2n_j-4}T,\\&\qquad\vdots\\&[[\underbrace{Z_j,\cdots,[Z_j,[Z_j,}_{n_j}\underbrace{\overline{Z}_j]],\cdots,],\overline{Z}_j}_{n_j}]=(-1)^{n_j}2\mathrm{i}(n_j!)^2T,\end{aligned}\tag{3.17}$$

where $T=\frac{\partial}{\partial w}+\frac{\partial}{\partial\overline{w}}$.

By taking $\alpha_1=\alpha_2=0$ in (3.1), it is easily to see that $(F_1(z,w),\cdots,F_N(z,w),G(z,w))$ with F_μ and G satisfying

$$F_\mu(z,w)=z_\mu\mathrm{e}^{\mathrm{i}\xi_\mu},\quad G(z,w)=w+t_1,\tag{3.18}$$

where $\xi_\mu,t_1\in\mathbb{R},\ \mu=1,\cdots,N$, is a biholomorphic transformation from Γ to Γ. Let $\mathcal{T}$ denote the group of such transformations.

Proposition 3.2 $\mathrm{Aut}(\Gamma,0)=\mathcal{T}\circ\mathrm{Aut}_0\,\Gamma$.

Proof Suppose that H is an arbitrarily chosen element in $\mathrm{Aut}(\Gamma,0)$. We claim that H maps $(0,\cdots,0)$ to $(0,\cdots,0,t_1)$ with $t_1\in\mathbb{R}$. Let

$$P:=\{(0,\cdots,0,t)\mid t\in\mathbb{R}\}. \tag{3.19}$$

Clearly, $P\subset\Gamma$. By (3.17), we see that Γ is of type 2 at the points $(z_1,\cdots,z_N,w)$ with $\sum\limits_{\mu=1}^{N}|z_\mu|\neq 0$, and is of type $n=2\min\{n_1,\cdots,n_N\}>2$ at the points in P. Since the type is preserved under locally biholomorphic transformations, there is no biholomorphic transformation mapping $(0,\cdots,0)$ to $(z_1,\cdots,z_N,w)$ with $\sum\limits_{\mu=1}^{N}|z_\mu|\neq 0$. Consequently, for $H\in\mathrm{Aut}(\Gamma,0)$,

$$H(0,\cdots,0)=(0,\cdots,0,t_1)$$

for some $t_1\in\mathbb{R}$. Set $H_1=(F_1,\cdots,F_N,G)$ with $F_\mu,\ G$ given by (3.18). Then H_1 is a biholomorphic automorphism of Γ mapping $(0,\cdots,0)$ to $(0,\cdots,0,t_1)$. So we have

$$H=H_1\circ H_2, \tag{3.20}$$

with $H_2:=H_1^{-1}\circ H\in\mathrm{Aut}(\Gamma,0)$. Since

$$H_2(0)=H_1^{-1}\circ H(0)=H_1^{-1}\circ H_1(0)=0,$$

therefore, $H_2\in\mathrm{Aut}_0\,\Gamma$. Hence,

$$\mathrm{Aut}(\Gamma,0)\subset\mathcal{T}\circ\mathrm{Aut}_0\,\Gamma. \tag{3.21}$$

Obviously, $\mathcal{T}\circ\mathrm{Aut}_0\,\Gamma\subset\mathrm{Aut}(\Gamma,0)$. The proposition is proved.

Proof of Theorem 1.2 From (3.17) we can see that the real-analytic hypersurface Γ is of finite type. Hence, by [8, Corollary 1.6], $\mathrm{Aut}_0\,\Gamma$ is a Lie group. It is obvious that $\mathrm{hol}_0\,\Gamma$ is its Lie algebra. Therefore, $\mathrm{hol}_0\,\Gamma$ can generate a connected component of $\mathrm{Aut}_0\,\Gamma$. From Proposition 3.1, the transformation generated by any $X=2\,\mathrm{Re}\Big[\sum\limits_{\mu=1}^{N}f_\mu(z,w)\frac{\partial}{\partial z_\mu}+g(z,w)\frac{\partial}{\partial w}\Big]\in\mathrm{hol}_0\,\Gamma$ can be written as

$$\begin{aligned}T_{\lambda\xi\gamma}(z,w)&=(F_1(z,w),F_2(z,w),\cdots,F_N(z,w),G(z,w))\\&=\Big(\frac{\lambda\mathrm{e}^{\mathrm{i}\xi_1}z_1}{(1+\gamma w)^{\frac{1}{n_1}}},\frac{\lambda^{\frac{n_1}{n_2}}\mathrm{e}^{\mathrm{i}\xi_2}z_2}{(1+\gamma w)^{\frac{1}{n_2}}},\cdots,\frac{\lambda^{\frac{n_1}{n_N}}\mathrm{e}^{\mathrm{i}\xi_N}z_N}{(1+\gamma w)^{\frac{1}{n_N}}},\frac{\lambda^{2n_1}w}{1+\gamma w}\Big)\end{aligned} \tag{3.22}$$

for some $\lambda\in\mathbb{R}_+,\ \xi_\mu,\gamma\in\mathbb{R},\ \mu=1,\cdots,N$. Consequently,

$$T=\{T_{\lambda\xi\gamma}\mid\lambda\in\mathbb{R}_+,\ \xi=(\xi_1,\cdots,\xi_N)\in\mathbb{R}^N,\ \gamma\in\mathbb{R}\}$$

is a connected component of $\mathrm{Aut}_0\,\Gamma$. Then by Proposition 3.2, $\mathcal{T}\circ T$ is a connected component of $\mathrm{Aut}(\Gamma,0)$, whose elements can be written as (1.5). Clearly, the identity transformation is in this component. This proves Theorem 1.2.

Remark 3.1 We use $\Gamma^\star$ to denote (1.2) with $n_{i_1} = \cdots = n_{i_m} = 1$, where $1 \le i_l \le N$. We can also determine the real analytic infinitesimal automorphism of $\Gamma^\star$ near the origin and the connected component of the unit of $\mathrm{Aut}_0\,\Gamma^\star$, which is more complicated and will appear elsewhere.

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