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Strongly Gorenstein Flat Modules and Dimensions

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Abstract There is a variety of nice results about strongly Gorenstein flat modules over coherent rings. These results are done by Ding, Lie and Mao. The aim of this paper is to generalize some of these results, and to give homological descriptions of the strongly Gorenstein flat dimension (of modules and rings) over arbitrary associative rings.

Keywords Strongly Gorenstein flat modules, Gorenstein projective modules, Gorenstein global (resp., weak) dimension
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1 Introduction

Throughout the paper, all rings are associative with identity, and an R-module will mean a right R-module unless explicitly stated otherwise.

Let R be a ring, and M be an R-module. As usual, we use $\mathrm{pd}_R(M)$, $\mathrm{id}_R(M)$ and $\mathrm{fd}_R(M)$ to denote the classical projective dimension, injective dimension and flat dimension of M, respectively.

For a two-sided Noetherian ring R, Auslander and Bridger [1] introduced the G-dimension, $\mathrm{Gdim}_R(M)$ for every finitely generated R-module M. They showed that $\mathrm{Gdim}_R(M) \leq \mathrm{pd}_R(M)$ for all finitely generated R-modules M, and the equality holds if $\mathrm{pd}_R(M)$ is finite.

Several decades later, Enochs and Jenda [11, 12] introduced the notion of Gorenstein projective dimension (G-projective dimension for short) as an extension of G-dimension to modules that are not necessarily finitely generated, and the Gorenstein injective dimension (G-injective dimension for short) as a dual notion of Gorenstein projective dimension. To complete the analogy with the classical homological dimension, Enochs, Jenda and Torrecillas [10] introduced the Gorenstein flat dimension. Some references concerning the Gorenstein projective, injective and flat dimensions are [3, 5 6, 10–12, 15].

Recall that an R-module M is called Gorenstein projective, if there exists an exact sequence of projective R-modules: $\mathbf{P}: \cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$, such that $M \cong \mathrm{Im}\,(P_0 \to P^0)$ holds, and the functor $\mathrm{Hom}_R(-, Q)$ leaves $\mathbf{P}$ exact whenever Q is projective. The complex $\mathbf{P}$ is called a complete projective resolution.

A Gorenstein injective R-module is defined dually.

An R-module M is called Gorenstein flat, if there exists an exact sequence of flat R-modules: $\mathbf{F}: \cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \cdots$, such that $M \cong \mathrm{Im}\,(F_0 \to F^0)$ holds, and the functor

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$- \otimes_R I$ leaves $\mathbf{F}$ exact whenever I is a left injective R-module. The complex $\mathbf{F}$ is called a complete flat resolution.

The Gorenstein projective, injective and flat dimensions are defined in terms of Gorenstein projective, injective and flat resolutions, respectively, and denoted by $\mathrm{Gpd}\,(-)$, $\mathrm{Gid}\,(-)$ and $\mathrm{Gfd}\,(-)$, respectively (see [5, 12, 15]).

In [3], the authors proved the equality

$$\sup\{\mathrm{Gpd}_R(M) \mid M \text{ is a (left) } R\text{-module}\} = \sup\{\mathrm{Gid}_R(M) \mid M \text{ is a (left) } R\text{-module}\}.$$

They called the common value of the above quantities the left Gorenstein global dimension of R, and denoted it by $\mathrm{l.Ggldim}\,(R)$. Similarly, they set

$$\mathrm{l.wGgldim}\,(R) = \sup\{\mathrm{Gfd}_R(M) \mid M \text{ is a (left) } R\text{-module}\},$$

which they called the left Gorenstein weak dimension of R. Similarly, one can define the right (resp. weak) Gorenstein global dimension, and by analogy it shall be denoted by (resp. $\mathrm{r.wGgldim}\,(R)$) $\mathrm{r.Ggldim}\,(R)$.

An R-module M is called strongly Gorenstein flat[1] (see [9]), if there exists an exact sequence of projective R-modules: $\mathbf{P} : \cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$, such that $M \cong \mathrm{Im}\,(P_0 \to P^0)$ holds, and the functor $\mathrm{Hom}_R(-, F)$ leaves $\mathbf{P}$ exact whenever F is flat. The strongly Gorenstein flat dimension is defined in term of strongly Gorenstein flat resolution, and denoted by $\mathrm{SGfd}\,(-)$. The strongly Gorenstein flat dimension of a ring R, $\mathrm{SGFD}\,(R)$, is defined to be the supremum of the set $\mathrm{SGfd}_R(M)$, such that M is an R-module (see [9]).

In [9], the authors gave examples of the strongly Gorenstein flat module which is not projective (flat), and (Gorenstein) flat module which is not strongly Gorenstein flat. In addition, they proved that, if R is (left) coherent, strongly Gorenstein flat modules lie between projective modules and Gorenstein flat modules, and these inclusions can be strict (see [9, Proposition 2.3, Examples 2.18–2.20]). By definition, every strongly Gorenstein flat module is Gorenstein projective. Unfortunately, as in [9], in this paper, we are not able to find examples of the Gorenstein projective module which is not strongly Gorenstein flat. If one can find a Gorenstein projective module M over a coherent ring which is not Gorenstein flat, then M is not strongly Gorenstein flat (by [9, Proposition 2.3]). However, whether Gorenstein projective modules and strongly Gorenstein flat modules coincide or not, we think that this paper gives new things. If the class of strongly Gorenstein flat modules happens to be the class of Gorenstein projective modules, then Theorem 2.4 and Propositions 2.5 and 3.1 give new results about Gorenstein projective modules.

Section 2 deals with the establishment of the fundamentals Theorem 2.1. This result generalizes [9, Proposition 2.10(1)–(2)] to any associative ring. The result is also the main ingredient in the functorial description of the strongly Gorenstein flat dimension (see Theorem 2.4). Next,

[1] This definition has not any relation with the notion of strongly Gorenstein flat modules defined in [2]. In our viewpoint, it was better to call the notion of strongly Gorenstein flat modules defined in [9], strongly (or ultra) Gorenstein projective modules. Indeed, there is not yet a guarantee that a strongly Gorenstein flat module in [9] is Gorenstein flat (except under some conditions like coherence (see [9, Proposition 2.3])), while it is clear that every strongly Gorenstein flat module (in [9]) is Gorenstein projective. However, in spite of our disagreement with this name, we will support it. It is left to the reader to see the difference between the notions of strongly Gorenstein flat modules in [2] and [9].

we get Proposition 2.3. In Section 2, we also investigate strongly Gorenstein flat precovers. Recall that a strongly Gorenstein flat precover of a module M is a homomorphism of modules, $G \to M$, where G is strongly Gorenstein flat, such that the sequence

$$\mathrm{Hom}_R(G', G) \to \mathrm{Hom}_R(G', M) \to 0$$

is exact for every strongly Gorenstein flat module G'. We show that every module M with a finite and strongly Gorenstein flat dimension admits a Gorenstein projective precover (see Theorem 2.2). Note that this result generalizes [9, Theorem 4.1], which is only proved for coherent rings with a finite SGFD $(-)$.

In Section 3, it is proved that for any ring, a (right) strongly Gorenstein flat dimension is equal to a (right) Gorenstein global dimension (see Theorem 3.1). Several characterizations of these dimensions are also given in this section.

In this paper, by $\mathcal{P}(R)$ and $\mathcal{F}(R)$ we denote the classes of all projective and flat R-modules, respectively, and by $\mathcal{SGF}(R)$ and $\mathcal{GP}(R)$ we denote the classes of all strongly Gorenstein flat and Gorenstein projective R-modules, respectively.

Given a class $\mathfrak{X}$ of R-modules, we set

$$\mathfrak{X}^{\perp} = \mathrm{Ker}\,\mathrm{Ext}_R^1(\mathfrak{X}, -) = \{M \mid \mathrm{Ext}_R^1(X, M) = 0 \text{ for all } X \in \mathfrak{X}\},$$
$${}^{\perp}\mathfrak{X} = \mathrm{Ker}\,\mathrm{Ext}_R^1(-, \mathfrak{X}) = \{M \mid \mathrm{Ext}_R^1(M, X) = 0 \text{ for all } X \in \mathfrak{X}\}.$$

The class $\mathfrak{X}^{\perp}$ (resp., ${}^{\perp}\mathfrak{X}$) is usually called the right (resp., left) orthogonal complement relative to the functor $\mathrm{Ext}_R^1(-,-)$ of the class $\mathfrak{X}$.

Moreover, we set ${}^{\perp\infty}\mathfrak{X} = \{M \mid \mathrm{Ext}_R^i(M, X) = 0 \text{ for all } X \in \mathfrak{X} \text{ and all } i > 0\}$.

2 Strongly Gorenstein Flat Modules

In this section, we give a detailed treatment of strongly Gorenstein flat modules. The main purpose is to give functorial descriptions of the strongly Gorenstein flat dimension. We start with the following result which is an immediate consequence of the definition of strongly Gorenstein flat modules.

Proposition 2.1 *An R-module M is strongly Gorenstein flat if and only if*

(1) $\mathrm{Ext}_R^i(M, F) = 0$ *for all flat R-modules F and all $i > 0$ (i.e., $M \in {}^{\perp\infty}\mathcal{F}(R)$),*

(2) *there exists an exact sequence $0 \to M \to P^0 \to P^1 \to \cdots$ where all P^i are projectives and* $\mathrm{Hom}_R(-, F)$ *leaves this sequence exact whenever F is flat.*

Let $\mathfrak{X}$ be a class of R-modules. Recall that $\mathfrak{X}$ is projectively resolving (see [15]), if $\mathcal{P}(R) \subseteq \mathfrak{X}$ and for any short exact sequence $0 \to M \to M' \to M'' \to 0$ where $M'' \in \mathfrak{X}$, $M \in \mathfrak{X}$ if and only if $M' \in \mathfrak{X}$.

Theorem 2.1 *The class of all strongly Gorenstein flat R-modules is resolved, in the sense that if $0 \to M \to M' \to M'' \to 0$ is a short exact sequence of R-modules, where M'' is strongly Gorenstein flat, then M is strongly Gorenstein flat if and only if M' is strongly Gorenstein flat.*

Furthermore, the class of all strongly Gorenstein flat R-modules is closed under arbitrary direct sums and under direct summands.

Proof It is clear that $\mathcal{P}(R) \subseteq \mathcal{SGF}(R)$ (by [9, Remark 2.2(1)]). So, we consider any short exact sequence of R-modules $0 \to M' \to M \to M'' \to 0$ where M'' is strongly Gorenstein flat.

First, suppose that M' is strongly Gorenstein flat. We claim that M is also strongly Gorenstein flat. Since ${}^{\perp\infty}\mathcal{F}(R)$ is projectively resolving (by [14, Lemma 2.2.9]), by Proposition 2.1, we get that M belongs to ${}^{\perp\infty}\mathcal{F}(R)$. Thus, to show that M is strongly Gorenstein flat, we only have to prove the existence of an exact sequence $0 \to M \to P^0 \to P^1 \to \cdots$ where all P^i are projectives and $\mathrm{Hom}_R(-, F)$ leaves this sequence exact whenever F is flat (by Proposition 2.1). By assumption, there exist exact projective resolutions

$$\mathbf{M}' = 0 \to M' \to P'_0 \to P'_1 \to \cdots \quad \text{and} \quad \mathbf{M}'' = 0 \to M'' \to P''_0 \to P''_1 \to \cdots,$$

where $\mathrm{Hom}(-, F)$ keeps the exactness of these sequences whenever F is flat, and all the cokernels of $\mathbf{M}'$ and $\mathbf{M}''$ are strongly Gorenstein flats (such sequences exist by the definition of strongly Gorenstein flat modules). Consider the following diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & & 0 \arrow[d] & \\
0 \arrow[r] & M' \arrow[r, "\alpha"] \arrow[d, "f'"] & M \arrow[r, "\beta"] & M'' \arrow[r] \arrow[d, "f''"] & 0 \\
0 \arrow[r] & P'_0 \arrow[r] & P'_0 \oplus P''_0 \arrow[r] & P''_0 \arrow[r] & 0
\end{tikzcd}$$

Since M'' is strongly Gorenstein flat, we have $\mathrm{Ext}^1_R(M'', P'_0) = 0$. Hence, the following sequence is exact:

$$0 \longrightarrow \mathrm{Hom}_R(M'', P'_0) \xrightarrow{\mathrm{Hom}_R(\beta, P'_0)} \mathrm{Hom}_R(M, P'_0) \xrightarrow{\mathrm{Hom}_R(\alpha, P'_0)} \mathrm{Hom}_R(M', P'_0) \longrightarrow 0.$$

Thus, there exists an R-morphism $\gamma : M \to P'_0$, such that $f' = \gamma \circ \alpha$. It is easy to check that the morphism $f : M \to P'_0 \oplus P''_0$ defined by setting $f(m) = (\gamma(m), f'' \circ \beta(m))$ for each $m \in M$ completes the above diagram and makes it commutative. Then, using the Snake Lemma, we get the following commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & M' \arrow[r, "\alpha"] \arrow[d, "f'"] & M \arrow[r, "\beta"] \arrow[d, "f"] & M'' \arrow[r] \arrow[d, "f''"] & 0 \\
0 \arrow[r] & P'_0 \arrow[r] \arrow[d] & P'_0 \oplus P''_0 \arrow[r] \arrow[d] & P''_0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{coker}(f') \arrow[r] \arrow[d] & \mathrm{coker}(f) \arrow[r] \arrow[d] & \mathrm{coker}(f'') \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Since $\mathrm{coker}(f')$ and $\mathrm{coker}(f'')$ are strongly Gorenstein flat, they belong to ${}^{\perp\infty}\mathcal{F}(R)$ which is projectively resolving. Then, $\mathrm{coker}(f)$ belongs also to ${}^{\perp\infty}\mathcal{F}(R)$. Accordingly, $\mathrm{Hom}_R(-, F)$

keeps the exactness of the short exact sequence $0 \to M \to P_0' \oplus P_0'' \to \mathrm{coker}\,(f) \to 0$ whenever F is flat. By induction, we can construct a commutative diagram with the form

$$\begin{tikzcd}
0 \arrow[r] & \mathbf{M}' \arrow[r] \arrow[d, equal] & \mathbf{M} \arrow[r] \arrow[d, equal] & \mathbf{M}'' \arrow[r] \arrow[d, equal] & 0 \\
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & M' \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & M'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P_0' \arrow[r] \arrow[d] & P_0' \oplus P_0'' \arrow[r] \arrow[d] & P_0'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P_1' \arrow[r] \arrow[d] & P_1' \oplus P_1'' \arrow[r] \arrow[d] & P_1'' \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

such that $\mathrm{Hom}_R(,F)$ leaves $\mathbf{M}$ exact whenever F is flat. Consequently, M is strongly Gorenstein flat.

Now suppose that M is strongly Gorenstein flat. We claim that M' is strongly Gorenstein flat. As above, M belongs to ${}^{\perp\infty}\mathcal{F}(R)$. Hence, we have to prove that M satisfies the condition (2) of Proposition 2.1. To do it, we pick a short exact sequence $0 \to M \to P \to X \to 0$ where P is projective and X is strongly Gorenstein flat (such a sequence exists by [9, Remark 2.2(3)]), and consider the following push-out diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & M' \arrow[r, equal] \arrow[d] & M' \arrow[d] & & \\
0 \arrow[r] & M \arrow[r] \arrow[d] & P \arrow[r] \arrow[d, dotted] & X \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M'' \arrow[r, dotted] \arrow[d] & Y \arrow[r] \arrow[d] & X \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

From the first part of this proof, Y is strongly Gorenstein flat. Hence, it admits a right projective resolution $\mathbf{Y} : 0 \to Y \to F^0 \to F^1 \to \cdots$ which is still exact by $\mathrm{Hom}_R(-,F)$ whenever F is flat. In addition, the short exact sequence $0 \to M' \to P \to Y \to 0$ is still exact by $\mathrm{Hom}_R(-,F)$

whenever F is flat since Y is strongly Gorenstein flat. Finally, it is easy to check that

$$\begin{tikzcd}[column sep=small]
\mathbf{M}': & 0 \arrow[r] & M' \arrow[r] & P \arrow[rr] \arrow[dr] & & F^0 \arrow[r] & F^1 \arrow[r] & \cdots \\
& & & & Y \arrow[ur] \arrow[dr] & & & \\
& & & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

is also exact by $\mathrm{Hom}_R(-,F)$ whenever F is flat.

The closing of $\mathcal{SGF}(R)$ under direct sums is due to [9, Remark 2.2(2)], while its closing under direct summands is deduced from [15, Proposition 1.4].

Remark 2.1 Note that Theorem 2.1 generalizes [9, Proposition 2.10], except the third statement of this theorem which we will generalize later to not necessarily coherent rings.

Lemma 2.1 *Let M be any R-module. Consider two exact sequences*

$$\begin{array}{lllllll}
0 \to & K_n & \to & G_{n-1} & \to & \cdots & \to G_0 \to M \to 0, \\
0 \to & K'_n & \to & G'_{n-1} & \to & \cdots & \to G'_0 \to M \to 0,
\end{array}$$

where all G_i and G'_i are strongly Gorenstein flat modules. Then, K_n is strongly Gorenstein flat if and only if K'_n is strongly Gorenstein flat.

Proof Since the class of strongly Gorenstein flat modules is projectively resolving and closed under arbitrary sums and under direct summands, by Theorem 2.1, the stated result is a direct consequence of [1, Lemma 3.12].

Hereafter, we immediately deal with strongly Gorenstein flat precovers. We begin with a definition of precovers.

Definition 2.1 (Precovers) *Let $\mathfrak{X}$ be any class of R-modules, and M be an R-module. An $\mathfrak{X}$-precover of M is an R-homomorphism $\varphi : X \to M$ where $X \in \mathfrak{X}$, such that the sequence*

$$\mathrm{Hom}_R(X', X) \xrightarrow{\mathrm{Hom}_R(X',\varphi)} \mathrm{Hom}_R(X', M) \longrightarrow 0$$

is exact for every $X' \in \mathfrak{X}$.

For more details about precovers, the reader may consult [13, Chapters 5 and 6]. Instead of saying $\mathcal{SGF}(R)$-precover, we shall use the term "strongly Gorenstein flat precover".

Theorem 2.2 *Let M be an R-module with a finite and strongly Gorenstein flat dimension n. Then, M admits a surjective and strongly Gorenstein flat precover $\varphi : G \twoheadrightarrow M$, where $K = \ker(\varphi)$ satisfies $\mathrm{pd}_R(K) = n-1$ (if $n = 0$, this should be interpreted as $K = 0$). Moreover, if $\mathrm{pd}_R(M) < \infty$, then G is projective.*

Proof Pick an exact sequence $0 \to G' \to P_{n-1} \to \cdots \to P_0 \to M \to 0$ where $P_0, \cdots, P_{n-1}$ are projectives. By Proposition 2.1, G' is clearly strongly Gorenstein flat. Hence, by the definition of "strongly Gorenstein flat", there is an exact sequence $(*)$ $0 \to G' \to Q_0 \to \cdots \to Q_{n-1} \to G \to 0$ where G is strongly Gorenstein flat and all Q_i are projectives, such that $\mathrm{Hom}_R(-,F)$ leaves this sequence exact whenever F is flat, in particular, whenever F is projective. Then, $(*)$ is the beginning of a co-proper right $\mathcal{P}(R)$-resolution of G' (see [15, Defintion 1.5]). Thus, using [15, Proposition 1.8], there exists homomorphisms $Q_i \to P_{n-1-i}$ for $i = 0, \cdots, n-1$, and $G \to M$, such that the following diagram is commutative:

$$\begin{tikzcd}
0 \arrow[r] & G' \arrow[r] \arrow[d, equal] & Q_0 \arrow[r] \arrow[d, dotted] & \cdots \arrow[r] & Q_{n-1} \arrow[r] \arrow[d, dotted] & G \arrow[r] \arrow[d, dotted] & 0 \\
0 \arrow[r] & G' \arrow[r] & P_{n-1} \arrow[r] & \cdots \arrow[r] & P_0 \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

This diagram gives a chain map between complexes

$$\begin{tikzcd}
0 \arrow[r] & Q_0 \arrow[r] \arrow[d] & \cdots \arrow[r] & Q_{n-1} \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P_{n-1} \arrow[r] & \cdots \arrow[r] & P_0 \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

which induces an isomorphism in homology. Its mapping cone is exact, and all the modules in it, except for $P_0 \oplus G$ (which is strongly Gorenstein flat), are projectives. Hence, the kernel K of $\varphi : P_0 \oplus G \twoheadrightarrow M$ satisfies $\mathrm{pd}_R(K) \leq n-1$ (and then equal to n, otherwise $\mathrm{SGfd}_R(M) \leq n-1$).

Since K has a finite projective dimension, we have $\mathrm{Ext}^1_R(A, K) = 0$ for any strongly Gorenstein flat module A (by [9, Lemma 2.4 (1)]). Thus the homomorphism

$$\mathrm{Hom}_R(A, \varphi) : \mathrm{Hom}_R(A, P_0 \oplus G) \to \mathrm{Hom}_R(A, M)$$

is surjective. Hence, φ is the desired precover.

If $\mathrm{pd}_R(M) < \infty$, then $\mathrm{pd}_R(G) < \infty$. Hence, it is projective (by [9, Lemma 2.4(2)]).

Remark 2.2 Note that Theorem 2.2 generalizes [9, Theorem 4.1] in two senses. Firstly, the condition $\mathrm{r.SGFD}(R) < \infty$ (in [9, Theorem 4.1]) is large, which suffices to assume $\mathrm{SGfd}_R(M) < \infty$. Secondly, the coherence condition is not necessary.

As mentioned in Remark 2.1, the next result generalizes the third statement of [9, Proposition 2.10].

Corollary 2.1 *Let $0 \to G' \to G \to M \to 0$ be a short exact sequence, where G and G' are strongly Gorenstein flat modules, and $\mathrm{Ext}^1_R(M, F) = 0$ holds for every flat module F. Then, M is strongly Gorenstein flat.*

Proof Since $\mathrm{SGfd}_R(M) \leq 1$, by Theorem 2.2, there is an exact sequence $0 \to P \to G \to M \to 0$ where P is projective and G is strongly Gorenstein flat. By our assumption $\mathrm{Ext}^1_R(M, P) = 0$, this sequence splits, and hence M is strongly Gorenstein flat (by Theorem 2.1).

Corollary 2.2 *Let M be an R-module with a finite and strongly Gorenstein flat dimension n. Then, there exists an exact sequence $0 \to M \to H \to G \to 0$, where $\mathrm{pd}_R(H) = n$ and G is strongly Gorenstein flat.*

Proof If $n = 0$, the result is obvious by the definition of strongly Gorenstein flat modules. So, we may assume $n > 0$.

From Proposition 2.2, there exists a short exact sequence $0 \to K \to G' \to M \to$ where G' is strongly Gorenstein flat, such that $\mathrm{pd}_R(K) = n-1$. Pick a short exact sequence $0 \to G' \to$

$P \to G \to 0$, where P is projective and G is strongly Gorenstein flat. Consider the following push-out diagram:

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & & \\ & K \arrow[r, equal] \arrow[d] & K \arrow[d] & & \\ 0 \arrow[r] & G' \arrow[r] \arrow[d] & P \arrow[r] \arrow[d, dotted] & G \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & M \arrow[r, dotted] \arrow[d] & H \arrow[r] \arrow[d] & G \arrow[r] & 0 \\ & 0 & 0 & & \end{tikzcd}$$

Clearly, $\mathrm{pd}_R(H) \leq n$. If $n = 1$, then we must have $\mathrm{pd}_R(H) = 1$, otherwise M becomes strongly Gorenstein flat (by Theorem 2.1 and from the short exact sequence $0 \to M \to H \to G \to 0$). If $n > 1$, from the short exact sequence $0 \to K \to P \to H \to 0$, we get

$$\mathrm{pd}_R(H) = \mathrm{pd}_R(K) + 1 = n$$

since $\mathrm{pd}_R(K) \neq \mathrm{pd}_R(P)$. Then, in all cases, we have $\mathrm{pd}_R(H) = n$ as desired.

From [9, Lemma 2.4 (1)] and by using a standard argument, we get the following lemma.

Lemma 2.2 *Consider an exact sequence $0 \to K_n \to G_{n-1} \to \cdots \to G_0 \to M \to 0$, where all G_i are strongly Gorenstein flat modules. Then,*

$$\mathrm{Ext}^i_R(K_n, F) = \mathrm{Ext}^{n+i}_R(M, F)$$

for all integers $i > 0$ and all modules F with $\mathrm{fd}_R(F) < \infty$.

Using Theorem 2.1, Proposition 2.1 and Lemma 2.2 together with standard arguments, we immediately obtain the next result.

Proposition 2.2 *Let $0 \to K \to G \to M \to 0$ be an exact sequence of R-modules, where G is strongly Gorenstein flat. If M is strongly Gorenstein flat, so is K. Otherwise, we get*

$$\mathrm{SGfd}_R(K) = \mathrm{SGfd}_R(M) - 1.$$

Theorem 2.3 *For any R-modules M and M', we have*

$$\mathrm{SGfd}_R(M \oplus M') = \max\{\mathrm{SGfd}_R(M), \mathrm{SGfd}_R(M')\}.$$

Proof The inequality $\mathrm{SGfd}_R(M \oplus M') \leq \max\{\mathrm{SGfd}_R(M), \mathrm{SGfd}_R(M')\}$ follows from the fact that $\mathcal{SGF}(R)$ is closed under direct sums. For the converse inequality, we may assume that $\mathrm{SGfd}_R(M \oplus M') = n$ is finite, and then proceed by induction on n.

The induction start is clear, because the class $\mathcal{SGF}(R)$ is closed under direct summands. If $n > 0$, pick exact sequences $0 \to K \to P \to M \to 0$ and $0 \to K' \to P' \to M' \to 0$ where P and

P' are projectives. We get a commutative diagram with split-exact rows

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & M \arrow[r] \arrow[u] & M \oplus M' \arrow[r] \arrow[u] & M' \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & P \arrow[r] \arrow[u] & P \oplus P' \arrow[r] \arrow[u] & P' \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & K \arrow[r] \arrow[u] & K \oplus K' \arrow[r] \arrow[u] & K' \arrow[r] \arrow[u] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

Applying Proposition 2.2 to the middle column in this diagram, we get $\mathrm{SGfd}_R(K \oplus K') = n-1$. Hence, the induction hypothesis yields that

$$\max\{\mathrm{SGfd}_R(K), \mathrm{SGfd}_R(K)\} \leq \mathrm{SGfd}_R(K \oplus K').$$

Thus,

$$\begin{aligned}
\max\{\mathrm{SGfd}_R(M), \mathrm{SGfd}_R(M)\} &\leq \max\{\mathrm{SGfd}_R(K)+1, \mathrm{SGfd}_R(K)+1\} \\
&\leq \mathrm{SGfd}_R(K \oplus K') + 1 \\
&= \mathrm{SGfd}_R(M \oplus M').
\end{aligned}$$

The next result gives a functorial description of the strongly Gorenstein flat dimension of modules.

Theorem 2.4 *Let M be an R-module with a finite and strongly Gorenstein flat dimension, and let n be an integer. Then, the following conditions are equivalent:*

(1) $\mathrm{SGfd}_R(M) \leq n$;

(2) $\mathrm{Ext}_R^i(M, L) = 0$ *for all $i > n$ and all R-modules L with* $\mathrm{fd}_R(L) < \infty$;

(3) $\mathrm{Ext}_R^i(M, F) = 0$ *for all $i > n$ and all flat R-modules F*;

(4) *For every exact sequence* $0 \to K_n \to G_{n-1} \to \cdots \to G_0 \to M \to 0$ *where* $G_0, \cdots, G_{n-1}$ *are strongly Gorenstein flats, K_n is also strongly Gorenstein flat.*

Proof Obviously, we have (2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (1). So we only have to prove the last two implications to complete a cycle.

(1) $\Rightarrow$ (2) Assume that $\mathrm{SGfd}_R(M) = m \leq n$. By definition, there exists an exact sequence $0 \to G_m \to G_{m-1} \to \cdots \to G_0 \to M \to 0$, where all G_i are strongly Gorenstein flats. By Lemma 2.2 and [9, Lemma 2.4(1)], we conclude that the equalities $\mathrm{Ext}_R^i(M, L) \cong \mathrm{Ext}_R^{i-m}(G_n, L)$ whenever L has a finite flat dimension and $i > m$ (in particular when $i > n$).

(3) $\Rightarrow$ (4) Set $\mathrm{SGfd}_R(M) = m < \infty$ and suppose $n < m$. Consider an exact sequence $0 \to G_m \to \cdots \to G_0 \to M \to 0$, where all G_i are strongly Gorenstein flats. Set $K_{m-1} =$

coker $(G_m \to G_{m-1})$. Thus, by the hypothesis conditions and Lemma 2.2, from the exact sequence

$$0 \to K_{m-1} \to G_{m-2} \to \cdots \to G_0 \to M \to 0,$$

we get $\mathrm{Ext}_R^1(K_{m-1}, F) = \mathrm{Ext}_R^m(M, F) = 0$ for all flat modules F. Hence, applying Corollary 2.1 to the short exact sequence $0 \to G_m \to G_{m-1} \to K_{m-1} \to 0$, we conclude that K_{m-1} is strongly Gorenstein flat. Then, $\mathrm{SGfd}_R(M) \leq m-1$, which is impossible. This contradiction completes the proof.

Recall that a right (resp., left) R-module M is called FP-injective (or absolutely pure), if $\mathrm{Ext}_R^1(N, M) = 0$ (or equivalently $\mathrm{Ext}_R^i(N, M) = 0$ for all $i > 0$) for every finitely presented right (resp., left) R-module N. The FP-injective dimension of the right (resp., left) R-module M, denoted by $\text{FP-id}_R(M)$, is defined to be the smallest non-negative integer n, such that $\mathrm{Ext}_R^{n+1}(N, M) = 0$ for every finitely presented right (resp., left) R-module (see [7, 19]).

Remark 2.3 Note that in Theorem 2.4, only the implication (3) $\Rightarrow$ (4) needs the condition $\mathrm{SGfd}(-) < \infty$. However, in [9, Lemma 3.4], the authors proved this implication without assuming $\mathrm{SGfd}(-) < \infty$, but they forced the ring R to be left coherent and they assumed $\text{FP-id}_R(R) < \infty$.

Proposition 2.3 *Let $0 \to A \to B \to C \to 0$ be an exact sequence of R-modules. If two of $\mathrm{SGfd}_R(A)$, $\mathrm{SGfd}_R(B)$ and $\mathrm{SGfd}_R(C)$ are finite, so is the third. Moreover,*

(1) $\mathrm{SGfd}_R(B) \leq \max\{\mathrm{SGfd}_R(A), \mathrm{SGfd}_R(C)\}$ *with equality, if* $\mathrm{SGfd}_R(A)+1 \neq \mathrm{SGfd}_R(C)$,

(2) $\mathrm{SGfd}_R(A) \leq \max\{\mathrm{SGfd}_R(B), \mathrm{SGfd}_R(C)-1\}$ *with equality, if* $\mathrm{SGfd}_R(B) \neq \mathrm{SGfd}_R(C)$,

(3) $\mathrm{SGfd}_R(C) \leq \max\{\mathrm{SGfd}_R(B), \mathrm{SGfd}_R(A)+1\}$ *with equality, if* $\mathrm{SGfd}_R(B) \neq \mathrm{SGfd}_R(A)$.

Proof It suffices to prove that if two of $\mathrm{SGfd}_R(A)$, $\mathrm{SGfd}_R(B)$ and $\mathrm{SGfd}_R(C)$ are finite, so is the third. While, by using Theorem 2.4, the proof of the other assertions is standard homological algebra.

Suppose $\mathrm{SGfd}_R(A)$ and $\mathrm{SGfd}_R(C)$ are finite, and set $n = \max\{\mathrm{SGfd}_R(A), \mathrm{SGfd}_R(C)\}$. Pick exact sequences

$$0 \to G \to P_{n-1} \to \cdots \to P_0 \to A \to 0 \quad \text{and} \quad 0 \to G' \to Q_{n-1} \to \cdots \to Q_0 \to A \to 0,$$

where all P_i and Q_i are projective modules. From Proposition 2.1 and Theorem 2.1, G and G' are strongly Gorenstein flat modules. Consider the following commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & G \arrow[r] \arrow[d] & P_{n-1} \arrow[r] \arrow[d] & \cdots \arrow[r] & P_0 \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & X \arrow[r] \arrow[d] & P_{n-1} \oplus Q_{n-1} \arrow[r] \arrow[d] & \cdots \arrow[r] & P_0 \oplus Q_0 \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & G' \arrow[r] \arrow[d] & Q_{n-1} \arrow[r] \arrow[d] & \cdots \arrow[r] & Q_0 \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & & 0 & 0 &
\end{tikzcd}$$

From Theorem 2.1, X is strongly Gorenstein flat, and so $\mathrm{SGfd}_R(B)$ is finite.

Now, suppose that $\mathrm{SGfd}_R(A)$ and $\mathrm{SGfd}_R(B)$ are finite and pick a short exact sequence $0 \to X \to P \to B \to 0$ where P is projective. From Proposition 2.2, $\mathrm{SGfd}_R(X)$ is also finite. Consider the following push-out diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & X \arrow[r] \arrow[d, equal] & Z \arrow[r, dotted] \arrow[d, dotted] & A \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & X \arrow[r] & P \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & 0 \\
 & & C \arrow[r, equal] \arrow[d] & C \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

From the first part of this proof, $\mathrm{SGfd}_R(Z)$ is finite, and so, by Proposition 2.2, $\mathrm{SGfd}_R(C)$ is finite.

Now, suppose that $\mathrm{SGfd}_R(B)$ and $\mathrm{SGfd}_R(C)$ are finite and pick a short exact sequence $0 \to X \to P \to C \to 0$ where P is projective. It is clear that $\mathrm{SGfd}_R(X)$ is finite (by Proposition 2.2). Consider the following push-out diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & A \arrow[r, equal] \arrow[d] & A \arrow[d] & \\
0 \arrow[r] & X \arrow[r] \arrow[d, equal] & Z \arrow[r, dotted] \arrow[d, dotted] & B \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & X \arrow[r] & P \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

From the first part of this proof, we have that $\mathrm{SGfd}_R(Z)$ is finite, since $\mathrm{SGfd}_R(X)$ and $\mathrm{SGfd}_R(B)$ are finite. On the other hand, since P is projective, the middle vertical sequence splits, and then $Z \cong A \oplus P$. Hence, by Theorem 2.3, $\mathrm{SGfd}_R(A)$ is finite. This completes our proof.

Proposition 2.4 *Every finitely presented Gorenstein projective module is strongly Gorenstein flat.*

Proof Let M be a finitely presented Gorenstein projective module. From Proposition 2.1 and by induction, to prove that M is strongly Gorenstein projective, it suffices to construct a short exact sequence of modules $0 \to M \to P \to M' \to 0$ where P is finitely generated projective such that M is (finitely presented) Gorenstein projective, and to show that $\mathrm{Ext}^i(M, F) = 0$ for every flat module F and all $i > 0$.

We start with the desired short exact sequence. From [15, Proposition 2.4], M embeds in a free module L such that L/M is Gorenstein projective. Since M is finitely generated, we can find a finitely generated free module L_0 containing in L as a direct summand, which contains M (i.e., $M \subseteq L_0 \subseteq L$, L_0 finitely generated free, and L/L_0 is free). Hence, consider the following diagram with the exact square:

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
0 \arrow[r] & M \arrow[r, hook] \arrow[d, equal] & L_0 \arrow[r] \arrow[d, hook] & L_0/M \arrow[r] & 0 \\
0 \arrow[r] & M \arrow[r, hook] & L \arrow[r] \arrow[d] & L/M \arrow[r] & 0 \\
 & & L/L_0 \arrow[d] & & \\
 & & 0 & &
\end{tikzcd}$$

From [18, Exercise 2.7, p. 29] and the Snake Lemma, the above diagram can be completed as

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & M \arrow[r, hook] \arrow[d, equal] & L_0 \arrow[r] \arrow[d, hook] & L_0/M \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M \arrow[r, hook] & L \arrow[r] \arrow[d] & L/M \arrow[r] \arrow[d] & 0 \\
 & & L/L_0 \arrow[r, equal] \arrow[d] & L/L_0 \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

From the right vertical exact sequence, we conclude that $L/M \cong L_0/M \oplus L/L_0$ (since L/L_0 is free). Hence, by [15, Theorem 2.5], L_0/M is Gorenstein projective which is clearly finitely presented.

Now, let F be an arbitrary flat module. By Lazards Theorem in [4, Section 1], there is a direct system $(L_i)_{i \in I}$ of finitely generated free R-modules such that $\varinjlim L_i \cong F$. From [4, Exercise 3, p. 187], we have $\mathrm{Ext}^1(M, F) \cong \varinjlim \mathrm{Ext}^1(M, L_i) = 0$.

Remark 2.4 If R is a two-sided coherent ring, then by Proposition 2.4, $\mathrm{SGfd}_R(M) =$

$\mathrm{Gpd}_R(M)$ for every (right and left) finitely presented module M. Thus, [9, Proposition 2.17 and Theorem 3.6(1) $\Leftrightarrow$ (2)] are direct consequences of [8, Theorems 6 and 7].

Let R be a ring, and M an R-module. The cotorsion dimension of a module M, i.e., $\mathrm{cd}_R(M)$, is defined to be the smallest integer $n \geq 0$ such that $\mathrm{Ext}_R^{n+1}(F, M) = 0$ for all flat R-modules F. The right global cotorsion dimension of R, i.e., $\mathrm{r.cot.D}(R)$, is defined as the supremum of the cotorsion dimensions of R-modules (see [17]).

Proposition 2.5 *For any R-module M, we have*

(1) $\mathrm{SGfd}_R(M) \leq \mathrm{pd}_R(M)$ *with equality, if* $\mathrm{fd}_R(M) < \infty$,

(2) $\mathrm{Gpd}_R(M) \leq \mathrm{SGfd}_R(M)$ *with equality, if* $\mathrm{SGfd}_R(M) < \infty$ *or* $\mathrm{r.cot.D}(R) < \infty$.

Consequently, if M is a Gorenstein projective module, then either M is strongly Gorenstein flat or $\mathrm{SGfd}_R(M) = \infty$.

Proof (1) The first inequality follows from the fact that every projective module is strongly Gorenstein flat, whereas if $\mathrm{fd}_R(M) < \infty$, the equality holds by [9, Lemma 2.4 (2)].

(2) The desired inequality follows from the fact that every strongly Gorenstein flat is Gorenstein projective.

If $\mathrm{SGfd}_R(M) = n < \infty$, from [15, Theorem 2.20], to show that $\mathrm{Gpd}_R(M) = n$, it suffices to find a projective module P such that $\mathrm{Ext}_R^n(M, P) \neq 0$. Since $\mathrm{SGfd}_R(M) = n$, there is some flat module F such that $\mathrm{Ext}_R^n(M, F) \neq 0$. Consider any short exact sequence $0 \to F' \to P \to F \to 0$ where P is projective (and certainly F' will be flat). We get the long exact sequence of homology

$$\cdots \to \mathrm{Ext}_R^m(M, P) \to \mathrm{Ext}_R^m(M, F) \to \mathrm{Ext}_R^{m+1}(M, F') = 0 \to \cdots .$$

It now follows that also $\mathrm{Ext}_R^n(M, P) \neq 0$, as desired.

If $\mathrm{r.cot.D}(R) < \infty$, every flat module has a finite projective dimension (see [17, Theorem 7.2.5(1)]). Thus, clearly the notions of Gorenstein projective modules and strongly Gorenstein flat modules coincide (by [15, Proposition 2.3]). Thus, we have the desired result.

The last statement holds by (2) above.

3 Strongly Gorenstein Flat Dimension of Rings

In this section, we investigate the strongly Gorenstein flat dimension of rings. Our first result in this section shows that it coincides with the Gorenstein global dimension defined in [3].

Theorem 3.1 *For any ring R, we have* $\mathrm{r.SGFD}(R) = \mathrm{r.Ggldim}(R)$.

Proof ($\geq$) The inequality follows from Proposition 2.5.

($\leq$) To prove this inequality, we may assume $\mathrm{r.Ggldim}(R) < \infty$. From [3, Corollary 2.7] and [17, Theorem 7.2.5(1)], we have $\mathrm{r.cot.D}(R) \leq \mathrm{l.Ggldim}(R) < \infty$. Thus, the desired result is a direct consequence of Proposition 2.5(2).

Remark 3.1 Note that Theorem 3.1 generalizes [9, Proposition 2.16] in the sense that $\mathrm{r.SGFD}(R) = 0$ (i.e., every R-module is strongly Gorenstein flat) if and only if $\mathrm{r.Ggldim}(R) = 0$, which means by [3, Proposition 2.6], that R is quasi-Frobenius.

Proposition 3.1 *The following statements are equivalent for any ring R:*

(1) $\mathrm{r.cot.D}\,(R) \leq n$;

(2) $\mathrm{SGfd}_R(F) \leq n$ *for every flat module F.*

Proof (1) $\Rightarrow$ (2) It is obvious, because for any flat module F, $\mathrm{SGfd}_R(M) \leq \mathrm{pd}_R(F) \leq n$, since $\mathrm{r.cot.D}\,(R) \leq n$.

(2) $\Rightarrow$ (1) Let F be an arbitrary flat module. By hypothesis and from Proposition 2.5(1), $\mathrm{pd}_R(F) = \mathrm{SGfd}_R(F) \leq n$. Accordingly, $\mathrm{r.cot.D}\,(R) \leq n$.

Remark 3.2 By specializing the above proposition in the case $n = 0$, we obtain [9, Proposition 2.15].

Recall that a pair $(\mathfrak{X}, \mathfrak{Y})$ of classes of R-modules is called a cotorsion theory (see [13]), if $\mathfrak{X}^\perp = \mathfrak{Y}$ and ${}^\perp\mathfrak{Y} = \mathfrak{X}$. By $\mathcal{I}_n$ we denote the class of all R-modules with a injective dimension less than or equal to n.

Theorem 3.2 *Let R be a ring, and n a positive integer. The following are equivalent:*

(1) $\mathrm{r.SGFD}\,(R)(= \mathrm{r.Ggldim}\,(R)) \leq n$;

(2) $(\mathcal{SGF}\,(R), \mathcal{I}_n)$ *is a cotorsion theory;*

(3) $(\mathcal{GP}\,(R), \mathcal{I}_n)$ *is a cotorsion theory;*

(4) $\mathrm{id}_R(P) \leq n$ *for every projective module P, and* $\mathrm{pd}_R(I) \leq n$ *for every injective module I;*

(5) $\mathrm{id}_R(P) \leq n$ *for every projective module P, and* $\mathrm{pd}_R(I) < \infty$ *for every injective module I;*

(6) $\mathrm{id}_R(P) < \infty$ *for every projective module P, and* $\mathrm{pd}_R(I) \leq n$ *for every injective module I;*

(7) $\mathrm{id}_R(F) \leq n$ *for every flat module F, and* $\mathrm{fd}_R(I) \leq n$ *for every injective I;*

(8) $\mathrm{id}_R(F) \leq n$ *for every flat module F, and* $\mathrm{fd}_R(I) < \infty$ *for every injective I.*

Proof (1) $\Rightarrow$ (2) Assume that $\mathrm{r.SGFD}\,(R) \leq n$. First, we claim that $\mathcal{I}_n = \mathcal{SGF}\,(R)^\perp$. Consider $I \in \mathcal{I}_n$ and $G \in \mathcal{SGF}\,(R)$. By the definition of strongly Gorenstein flat modules, there exists an exact sequence $\rightarrow G \rightarrow P_0 \rightarrow P_1 \rightarrow \cdots \rightarrow P_{n-1} \rightarrow G' \rightarrow 0$ where all P_i are projective. Then, $\mathrm{Ext}^1_R(G, I) = \mathrm{Ext}^{n+1}_R(G', I) = 0$. Thus, $I \in \mathcal{SGF}\,(R)^\perp$. Accordingly, $\mathcal{I}_n \subseteq \mathcal{SGF}\,(R)$.

Now, consider $J \in \mathcal{SGF}\,(R)^\perp$. Since $\mathcal{SGF}\,(R)$ is projectively resolving, we get that $\mathrm{Ext}^i_R(G, J) = 0$ for all $G \in \mathcal{SGF}\,(R)$ and all $i > 0$ (by [14, Corollary 2.2.11(a)]). For an arbitrary R-module M, consider an exact sequence $0 \rightarrow G \rightarrow P_{n-1} \rightarrow \cdots \rightarrow P_0 \rightarrow M \rightarrow 0$ where all P_i are projective. Since $\mathrm{SGFD}\,(R) \leq n$, it is clear that G is strongly Gorenstein flat. Then, for all $i > n$, $\mathrm{Ext}^i_R(M, J) = \mathrm{Ext}^{i-n}_R(G, J) = 0$. Thus, $J \in \mathcal{I}_n$. Consequently, $\mathcal{SGF}\,(R)^\perp \supseteq \mathcal{I}_n$. So, we have the desired equality.

Then, we claim that ${}^\perp\mathcal{I}_n = \mathcal{SGF}\,(R)$. It is clear that $\mathcal{F}\,(R) \subseteq \mathcal{I}_n$ (by [3, Corollary 2.7]), since $\mathrm{r.Ggldim}\,(R) \leq n$. Consider $G \in {}^\perp\mathcal{I}_n$. We have $\mathrm{SGfd}_R(G) \leq n < \infty$. In addition, since $\mathcal{I}_n$ is injectively resolving, by [14, Corollary 2.2.11(b)], we get $\mathrm{Ext}^i_R(G, F) = 0$ for all $F \in I_n$, in particular, for all flat modules F (recall that $\mathcal{F}\,(R) \subseteq \mathcal{I}_n$). Thus, by Theorem 2.4, G is strongly Gorenstein flat. Hence, ${}^\perp\mathcal{I}_n \subseteq \mathcal{SGF}\,(R)$. In addition, it is clear that $\mathcal{SGF}\,(R) \subseteq^\perp (\mathcal{SGF}\,(R)^\perp) = {}^\perp\mathcal{I}_n$. Thus, ${}^\perp\mathcal{I}_n = \mathcal{SGF}\,(R)$. Accordingly, $(\mathcal{SGF}\,(R), \mathcal{I}_n)$ is a cotorsion theory.

(2) $\Rightarrow$ (1) Assume that $(\mathcal{SGF}(R), \mathcal{I}_n)$ is a cotorsion theory. Let M be an arbitrary R-module. Pick an exact sequence $0 \to G \to P_{n-1} \to \cdots \to P_0 \to M \to 0$ where all P_i are projectives. For each $I \in \mathcal{I}_n$, we have $\mathrm{Ext}_R^1(G, I) = \mathrm{Ext}_R^{n+1}(M, I) = 0$. Thus, $G \in {}^{\perp}\mathcal{I}_n = \mathcal{SGF}(R)$. Then, $\mathrm{SGfd}_R(M) \leq n$. Consequently, $\mathrm{SGFD}(R) \leq n$.

(1) $\Leftrightarrow$ (3) Replacing Theorem 2.4 by [15, Theorem 2.20] in the proof of (1) $\Rightarrow$ (2) and remembering that the class of Gorenstein projective modules is projectively resolving (by [15, Theorem 2.5]), this equivalence is proved in an analogous fashion to (1) $\Leftrightarrow$ (2).

(1) $\Leftrightarrow$ (4) This is the right version of [16, Theorem 2.1].

The implications (4) $\Rightarrow$ (5) and (4) $\Rightarrow$ (6) are obvious, while if (5) or (6) is satisfied, then by the left version of [16, Theorem 2.1], $\mathrm{r.Ggldim}(R) < \infty$. Hence, by [16, Proposition 2.3], the statements (5) and (6) are equivalent, and mean that $\mathrm{r.Ggldim}(R) \leq n$. So, (5) $\Rightarrow$ (1) and (6) $\Rightarrow$ (1) are clear.

(1) $\Rightarrow$ (7) This is a direct consequence of [3, Corollary 2.7].

(7) $\Rightarrow$ (8) It is obvious.

(8) $\Rightarrow$ (5) We have only to prove that $\mathrm{pd}_R(I) < \infty$ for every injective module I. By [17, Theorem 7.2.5(2)], we have

$$\mathrm{r.cot.D}(R) \leq \sup\{\mathrm{id}_R(P) \mid P \text{ projective}\} \leq \sup\{\mathrm{id}_R(F) \mid F \text{ flat}\} \leq n.$$

Thus, for any flat module F, we have $\mathrm{pd}_R(F) \leq n < \infty$. Then, given any injective module I, since $\mathrm{fd}_R(I) < \infty$, we get $\mathrm{pd}_R(I) < \infty$. This completes the proof.

Remark 3.3 If R is left coherent, by [7, Theorem 3.8], $\text{FP-id}_R(R) = \sup\{\mathrm{fd}_R(I) \mid I \text{ is injective}\}$. Thus, the theorem above, especially the equivalence of (1), (2) and (7), generalizes [9, Theorem 4.2] to a not necessarily coherent ring, and note that the condition $\text{FP-id}_R(R) \leq n$ in [9, Theorem 4.2] can be replaced by $\text{FP-id}_R(R) < \infty$ (by (8) of our theorem). Moreover, we have just proved that the equivalence [9, Theorem 4.2(1)$\Leftrightarrow$(3)] does not need the coherence condition.

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