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Stable Rank One and Real Rank Zero for Crossed Products by Finite Group Actions with the Tracial Rokhlin Property***

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Abstract The authors prove that the crossed product of an infinite dimensional simple separable unital C^*-algebra with stable rank one by an action of a finite group with the tracial Rokhlin property has again stable rank one. It is also proved that the crossed product of an infinite dimensional simple separable unital C^*-algebra with real rank zero by an action of a finite group with the tracial Rokhlin property has again real rank zero.

Keywords C^*-algebra, Stable rank one, Real rank zero
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1 Introduction

The Rokhlin property in ergodic theory was adopted to the context of von Neumann algebras by A. Connes in [1]. It was adopted by R. Hermann and A. Ocneanu for UHF-algebras in [5]. M. Rordam [16] and A. Kishimoto [7] introduced the Rokhlin property to a much more general context of C^*-algebras. More recently, N. C. Phillips and H. Osaka studied finite group actions which satisfy certain type of Rokhlin property on some C^*-algebras in [12–15]. In [15], N. C. Phillips proved that the crossed product of an infinite dimensional simple separable unital C^*-algebra with tracial rank zero by an action of a finite group with the tracial Rokhlin property again has tracial rank zero. In [12], H. Osaka and N. C. Phillips proved that if A is a simple unital C^*-algebras with real rank zero and stable rank one such that the order on projections of A is determined by traces, and $\alpha \in \mathrm{Aut}(A)$ has the tracial Rokhlin property, then the crossed product algebra $C^*(\mathbb{Z}, A, \alpha)$ has stable rank one and real rank zero. Recently in [13], H. Osaka and N. C. Phillips proved that for a separable unital C^*-algebra A, a finite group G, and an action $\alpha : G \to \mathrm{Aut}(A)$ with the Rokhlin property, if A has stable rank one then the crossed product algebra $C^*(G, A, \alpha)$ has stable rank one, and if A has real rank zero then the the crossed product algebra $C^*(G, A, \alpha)$ has real rank zero.

In this paper, using the method and technique of N. C. Phillips, we could get the same result of [13] under the weaker assumption with the Rokhlin property replaced by the tracial Rokhlin property, i.e., we prove that the crossed product of an infinite dimensional simple separable

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unital C^*-algebra with stable rank one by an action of a finite group with the tracial Rokhlin property has again stable rank one, and that the crossed product of an infinite dimensional simple separable unital C^*-algebra with real rank zero by an action of a finite group with the tracial Rokhlin property has again real rank zero.

2 Preliminaries and Definitions

A unital C^*-algebra A is said to have stable rank one, and written as $\mathrm{tsr}(A) = 1$, if $\mathrm{GL}(A)$ is dense in A, i.e., the set of invertible elements is dense in A.

A unital C^*-algebra A is said to have real rank zero, and written as $\mathrm{RR}(A) = 0$, if the set of invertible self-adjoint elements is dense in A_{sa}.

We say that a C^*-algebra A has the property SP, if every nonzero hereditary C^*-subalgebra of A contains a nonzero projection.

Let a and b be two positive elements in a C^*-algebra A. We write $[a] \le [b]$, if there exists a partial isometry $v \in A^{**}$, such that for every $c \in \mathrm{Her}(a)$, $v^*c, cv^* \in A$, $vv^* = P_{[a]}$, where $P_{[a]}$ is the range projection in A^{**}, and $v^*cv \in \mathrm{Her}(b)$. We write $[a] = [b]$, if $v^*\mathrm{Her}(a)v = \mathrm{Her}(b)$. Let n be a positive integer. We write $n[a] \le [b]$, if there are n mutually orthogonal positive elements $b_1, b_2, \cdots, b_n \in \mathrm{Her}(b)$ such that $[a] \le [b_i]$, $i = 1, 2, \cdots, n$.

Let $0 < \sigma_1 < \sigma_2 \le 1$ be two positive numbers. Define

$$f_{\sigma_1}^{\sigma_2}(t) = \begin{cases} 1, & \text{if } t \ge \sigma_2, \\ \dfrac{t-\sigma_1}{\sigma_2-\sigma_1}, & \text{if } \sigma_1 \le t \le \sigma_2, \\ 0, & \text{if } 0 < t \le \sigma_1. \end{cases}$$

Definition 2.1 (see [2, Definition 1.1]) *A unital C^*-algebra A is said to have tracial stable rank one if for any $\varepsilon > 0$, any finite subset $F \subseteq A$ containing a nonzero element $b \ge 0$, any $0 < \sigma_3 < \sigma_4 < \sigma_1 < \sigma_2 < 1$ and any integer $n > 0$, there exist a nonzero projection $p \in A$ and a C^*-algebra B of A with $1_B = p$ and $\mathrm{tsr}(B) = 1$, such that*

(1) $\|xp - px\| < \varepsilon$ *for all* $x \in F$,

(2) $pxp \in_\varepsilon B$ *for all* $x \in F$,

(3) $n[1-p] \le [p]$, *and* $n[f_{\sigma_1}^{\sigma_2}((1-p)b(1-p))] \le [f_{\sigma_3}^{\sigma_4}(pbp)]$.

If A has tracial stable rank one, we will write $\mathrm{Tsr}(A) = 1$.

Definition 2.2 (see [17, Definition 1.4]) *A unital C^*-algebra A is said to have tracial real rank zero, if for any $\varepsilon > 0$, any finite subset $F \subseteq A$ containing a nonzero element $b \ge 0$, any $0 < \sigma_3 < \sigma_4 < \sigma_1 < \sigma_2 < 1$ and any integer $n > 0$, there exist a nonzero projection $p \in A$ and a C^*-algebra B of A with $1_B = p$ and $\mathrm{RR}(B) = 0$, such that*

(1) $\|xp - px\| < \varepsilon$ *for all* $x \in F$,

(2) $pxp \in_\varepsilon B$ *for all* $x \in F$,

(3) $n[1-p] \le [p]$, *and* $n[f_{\sigma_1}^{\sigma_2}((1-p)b(1-p))] \le [f_{\sigma_3}^{\sigma_4}(pbp)]$.

If A has tracial real rank zero, we will write $\mathrm{TRR}(A) = 0$.

Definition 2.3 (see [15, Definition 1.2]) *Let A be an infinite dimensional simple separable unital C^*-algebra, and $\alpha : G \to \mathrm{Aut}(A)$ be an action of a finite group G on A. We say that α has the tracial Rokhlin property, if for any finite set $F \subseteq A$, any $\varepsilon > 0$ and any positive element $x \in A$ with $\|x\| = 1$, there are mutually orthogonal projections $e_g \in A$ for $g \in G$ such that*

(1) $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ *for all* $g, h \in G$,

(2) $\|e_g a - a e_g\| < \varepsilon$ *for all* $g \in G$ *and* $a \in F$,

(3) *With* $e = \sum_{g\in G} e_g$, *the projection* $1 - e$ *is Murray-von Neumann equivalent to a projection in the hereditary subalgebra of* A *generated by* x,

(4) *With* e *as in* (3), *we have* $\|exe\| > 1 - \varepsilon$.

Definition 2.4 (see [15, Definition 1.1]) *Let* A *be an infinite dimensional simple separable unital* C^**-algebra, and* $\alpha : G \to \mathrm{Aut}(A)$ *be an action of a finite group* G *on* A. *We say that* α *has the strict Rokhlin property* (*also called the Rokhlin property in* [13]), *if for any finite set* $F \subseteq A$ *and any* $\varepsilon > 0$, *there are mutually orthogonal projections* $e_g \in A$ *for* $g \in G$ *such that*

(1) $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ *for all* $g, h \in G$,

(2) $\|e_g a - a e_g\| < \varepsilon$ *for all* $g \in G$ *and* $a \in F$,

(3) $\sum_{g\in G} e_g = 1$.

Generally speaking, the tracial Rokhlin property does not imply the strict Rokhlin property even in a simple case (see [15]).

Theorem 2.1 (see [15, Corollary 1.6 and Lemma 1.13]) *Let* A *be an infinite dimensional simple unital* C^**-algebra, and* $\alpha : G \to \mathrm{Aut}(A)$ *be an action of a finite group* G *on* A *which has the tracial Rokhlin property. Then*

(1) $C^*(G, A, \alpha)$ *is simple,*

(2) *A has the property SP or* α *has the strict Rokhlin property.*

Theorem 2.2 (see [13, Proposition 3.11]) *Let* A *be an infinite dimensional separable unital* C^**-algebra, and let* $\alpha : G \to \mathrm{Aut}(A)$ *be an action of a finite group* G *on* A *which has the strict Rokhlin property. Then*

(1) *If* A *has stable rank one, then so does* $C^*(G, A, \alpha)$,

(2) *If* A *has real rank zero, then so does* $C^*(G, A, \alpha)$.

3 The Main Results

Lemma 3.1 (see [15, Proposition 1.12]) *Let* A *be an infinite dimensional simple separable unital* C^**-algebra with the property SP, and* $\alpha : G \to \mathrm{Aut}(A)$ *be an action of a finite group* G *on* A, *such that* $C^*(G, A, \alpha)$ *is also simple. Let* $B \subseteq C^*(G, A, \alpha)$ *be a nonzero hereditary subalgebra. Then there exists a nonzero projection* $p \in A$ *which is Murray-von Neumann equivalent in* $C^*(G, A, \alpha)$ *to a projection in* B.

Lemma 3.2 (see [15, Lemma 1.17]) *Let* A *be an infinite dimensional finite simple separable unital* C^**-algebra, and* $\alpha : G \to \mathrm{Aut}(A)$ *be an action of a finite group* G *on* A *which has the tracial Rokhlin property. Let* $F \subseteq A$ *be finite,* $\varepsilon > 0$, *and let* $x \in A$ *be a positive element with* $\|x\| = 1$. *Then there are mutually orthogonal projections* $e_g \in A$ *for* $g \in G$ *such that*

(1) $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ *for all* $g, h \in G$,

(2) $\|e_g a - a e_g\| < \varepsilon$ *for all* $g \in G$ *and all* $a \in F$,

(3) *The projection* $e = \sum_{g\in G} e_g$ *is* α *invariant, i.e.,* $\alpha_g(e) = e$ *for all* $g \in G$,

(4) *With e as in (3), the projection $1-e$ is Murray-von Neumann equivalent to a projection in the hereditary subalgebra of A generated by x,*

(5) *With e as in (3), we have $\|exe\| > 1-\varepsilon$.*

Lemma 3.3 (see [15, Lemma 2.1]) *Let $n \in \mathbb{N}$, and $(e_{i,j})_{1\leq j,k\leq n}$ be a system of matrix units for M_n. For every $\varepsilon > 0$, there is $\delta > 0$ such that, whenever B is a unital C^*-algebra, and $w_{j,k}$, for $1 \leq j,k \leq n$, are elements of B,*

(1) $\|w_{j,k}^* - w_{k,j}\| < \delta$ *for* $1 \leq j,k \leq n$,

(2) $\|w_{j_1,k_1}w_{j_2,k_2} - \delta_{j_2,k_1}w_{j_1,k_2}\| < \delta$ *for* $1 \leq j_1, j_2, k_1, k_2 \leq n$,

(3) $w_{j,j}$ *are orthogonal projections with* $\sum\limits_{j=1}^{n} w_{j,j} = 1$.

Then there exists a unital homomorphism $\varphi : M_n \to B$, such that $\varphi(e_{j,j}) = w_{j,j}$ for $1 \leq j \leq n$ and $\|\varphi(e_{j,k}) - w_{j,k}\| < \varepsilon$ for $1 \leq j,k \leq n$.

Lemma 3.4 (see [4, Theorem 3.4]) *Let A be a simple unital C^*-algebra. Then the following are equivalent:*

(1) *For any $\varepsilon > 0$ and any finite subset $F \subseteq A$ containing a nonzero positive element $b \geq 0$, there exist a nonzero projection $p \in A$ and a C^*-algebra B of A with $1_B = p$ and $\mathrm{tsr}(B) = 1$, such that $\|xp - px\| < \varepsilon$ for all $x \in F$, $pxp \in_\varepsilon B$, for all $x \in F$, and $[1-p] \leq [b]$,*

(2) $\mathrm{tsr}(A) = 1$,

(3) $\mathrm{Tsr}(A) = 1$.

Lemma 3.5 (see [17, Theorem 3.3]) *Let A be a simple unital C^*-algebra. Then the following are equivalent:*

(1) *For any $\varepsilon > 0$ and any finite subset $F \subseteq A$ containing a nonzero positive element $b \geq 0$, there exist a nonzero projection $p \in A$ and a C^*-algebra B of A with $1_B = p$ and $\mathrm{RR}(B) = 0$, such that $\|xp - px\| < \varepsilon$ for all $x \in F$, $pxp \in_\varepsilon B$, for all $x \in F$, and $[1-p] \leq [b]$,*

(2) $\mathrm{RR}(A) = 0$,

(3) $\mathrm{TRR}(A) = 0$.

Theorem 3.1 *Let A be an infinite dimensional simple separable unital C^*-algebra with stable rank one. Let $\alpha : G \to \mathrm{Aut}(A)$ be an action of a finite group G on A which has the tracial Rokhlin property. Then the crossed product algebra $C^*(G, A, \alpha)$ has stable rank one.*

Proof By Theorem 2.1(2), A has the property SP or α has the strict Rokhlin property. We prove this theorem by two steps. Firstly, we suppose that A has the property SP. By Theorem 2.1(1), $C^*(G, A, \alpha)$ is a simple C^*-algebra.

Suppose $G = \{g_1, g_2, \cdots, g_m\}$, where g_1 is the unit of G. By Lemma 3.4, we need to show that for any finite subset S of the form $S = F \cup \{u_{g_i} : 1 \leq i \leq m\}$, where F is a finite subset of the unit ball of A and $u_{g_i} \in C^*(G, A, \alpha)$ is the canonical unitary implementing the automorphism α_{g_i}, any $\varepsilon > 0$ and any nonzero positive element $b \in C^*(G, A, \alpha)$, there exist a C^*-subalgebra $D \subseteq C^*(G, A, \alpha)$ and a projection $p \in C^*(G, A, \alpha)$ with $1_D = p$ and $\mathrm{tsr}(D) = 1$, such that

(1) $\|pa - ap\| < \varepsilon$ for any $a \in S$,

(2) $pxp \in_\varepsilon D$ for any $a \in S$,

(3) $[1_A - p] \leq [b]$.

Since C^*-algebra A has the property SP, by Lemma 3.1 there exists a nonzero projection $r \in A$ which is Murray-von Neumann equivalent to a projection in $\overline{bC^*(G,A,\alpha)b}$, i.e., $[r] \le [b]$. By [9, Lemma 3.5.7], there are orthogonal nonzero projections $r_1, r_2 \in A$ such that $r_1, r_2 \le r$. Then we have $[r_1] + [r_2] = [r_1 + r_2] \le [r]$.

Set $\delta = \frac{\varepsilon}{16m}$. Choose $\eta > 0$ according to Lemma 3.3 for m given above and δ in place of ε. Moreover we may require $\eta < \frac{\varepsilon}{8m(m+1)}$. Applying Lemma 3.2 to α with F given above, η in place with ε, and r_1 in place of x, we have projections $e_{g_i} \in A$ for $1 \le i \le m$, such that

(1)′ $\|\alpha_{g_i}(e_{g_j}) - e_{g_i g_j}\| < \eta$ for any $1 \le i, j \le m$,

(2)′ $\|e_{g_i} a - a e_{g_i}\| < \eta$ for any $1 \le i \le m$ and any $a \in F$,

(3)′ $u_{g_i} e u_{g_i}^* = \alpha_{g_i}(e) = e$ for every $1 \le i \le m$, where $e = \sum_{i=1}^{m} e_{g_i}$,

(4)′ $1_A - e \preceq r_1$.

By (1)′ and (2)′, we have $\|ea - ae\| \le \sum_{i=1}^{m} \|e_{g_i} a - a e_{g_i}\| < m\eta$. Define $w_{g_i, g_j} = u_{g_i g_j^{-1}} e_{g_j}$ for every $1 \le i, j \le m$. We claim that the $w_{g_i, g_j} \in eC^*(G,A,\alpha)e$ $(1 \le i, j \le m)$ satisfy the conditions in Lemma 3.3. We prove it as follows:

$$\begin{aligned} \|w_{g_i,g_j}^* - w_{g_j,g_i}\| &= \|e_{g_j}(u_{g_i g_j^{-1}})^* - u_{g_i g_j^{-1}} e_{g_i}\| \\ &\le \|u_{g_i g_j^{-1}} e_{g_j} (u_{g_i g_j^{-1}})^* - e_{g_i}\| \\ &= \|\alpha_{g_i g_j^{-1}}(e_{g_j}) - e_{g_i}\| < \eta. \end{aligned}$$

Moreover, using $e_{g_i} e_{g_j} = \delta_{g_i, g_j} e_{g_j}$, we have

$$\begin{aligned} \|w_{g_i,g_j} w_{g_k,g_l} - \delta_{g_k,g_j} w_{g_i,g_l}\| &= \|u_{g_i g_j^{-1}} e_{g_j} u_{g_k g_l^{-1}} e_{g_l} - \delta_{g_k,g_j} u_{g_i g_l^{-1}} e_{g_l}\| \\ &= \|u_{g_i g_j^{-1}} e_{g_j} u_{g_k g_l^{-1}} e_{g_l} - u_{g_i g_j^{-1} g_k g_l^{-1}} e_{g_l g_k^{-1} g_j} e_{g_l}\| \\ &= \|u_{g_i g_j^{-1} g_k g_l^{-1}} ((u_{g_k g_l^{-1}})^* e_{g_j} u_{g_k g_l^{-1}} - e_{g_k g_l^{-1} g_j}) e_{g_l}\| < \eta. \end{aligned}$$

Finally, we have $\sum_{i=1}^{m} w_{g_i, g_i} = e$. This proves the claim.

Let (f_{ij}) $(1 \le i, j \le m)$ be a system of matrix units for M_m. By Lemma 3.3, there exists a unital homomorphism $\psi_0 : M_m \to eC^*(G,A,\alpha)e$ such that $\|\psi_0(f_{ij}) - w_{g_i,g_j}\| < \delta$ for all $1 \le i, j \le m$, and $\psi_0(f_{ii}) = e_{g_i}$ for all $1 \le i \le m$. Now we define a unital injective homomorphism $\psi : M_m \otimes e_{g_1} A e_{g_1} \to eC^*(G,A,\alpha)e$ by

$$\psi(f_{ij} \otimes a) = \psi_0(f_{i1}) a \psi_0(f_{i1})$$

for all $1 \le i, j \le m$ and $a \in e_{g_1} A e_{g_1}$. Then

$$\psi(f_{ij} \otimes e_{g_1}) = \psi_0(f_{i1}) e_{g_1} \psi_0(f_{1j}) = \psi_0(f_{ij}) = e_{g_i} \psi_0(f_{ij}) e_{g_j},$$

and so $\psi(1_{M_m} \otimes e_{g_1}) = e$. Let $k_{i,j}$ be the integer such that $g_{k_{i,j}} = g_i g_j$. For $1 \le i \le m$, we have

$$\begin{aligned} \Big\| e u_{g_i} e - \psi\Big(\sum_{j=1}^{m} f_{(k_{i,j})j} \otimes e_{g_1}\Big)\Big\| &= \Big\| e u_{g_i} e - \sum_{j=1}^{m} \psi_0(f_{(k_{i,j})j})\Big\| \le \sum_{j=1}^{m} \|u_{g_i} e_{g_j} - \psi_0(f_{(k_{i,j})j})\| \\ &= \sum_{j=1}^{m} \|w_{g_i g_j, g_j} - \psi_0(f_{(k_{i,j})j})\| < m\delta \le \frac{\varepsilon}{4}. \end{aligned}$$

Now let $a \in F$. Set
$$c = \sum_{i=1}^{m} f_{ii} \otimes e_{g_1} \alpha_{g_i}^{-1}(a) e_{g_1} \in M_m \otimes e_{g_1} A e_{g_1}.$$
Using $\|e_{g_i} a e_{g_j}\| \leq \|e_{g_i} a - a e_{g_i}\| + \|a e_{g_i} e_{g_j}\|$, we have
$$\Big\| eae - \sum_{i=1}^{m} e_{g_i} a e_{g_i} \Big\| \leq \sum_{i \neq j} \|e_{g_i} a e_{g_j}\| < m(m-1)\eta.$$
Using the inequity above and the inequalities
$$\begin{aligned} &\|\psi_0(f_{i1}) e_{g_1} - u_{g_i} e_{g_1}\| < \delta, \\ &\|e_{g_1} \alpha_{g_i}^{-1}(a) e_{g_1} - \alpha_{g_i}^{-1}(e_{g_i} a e_{g_i})\| < 2\eta, \end{aligned}$$
we have
$$\begin{aligned} \|eae - \psi(c)\| &= \Big\| eae - \sum_{i=1}^{m} \psi_0(f_{i1}) e_{g_1} \alpha_{g_i}^{-1}(a) e_{g_1} \psi_0(f_{1i}) \Big\| \\ &< 2m\delta + \Big\| eae - \sum_{i=1}^{m} u_{g_i} e_{g_1} \alpha_{g_i}^{-1}(a) e_{g_1} u_{g_i}^* \Big\| \\ &< 2m\delta + 2m\eta + \Big\| eae - \sum_{i=1}^{m} u_{g_i} \alpha_{g_i}^{-1}(e_{g_i} a e_{g_i}) u_{g_i}^* \Big\| \\ &< 2m\delta + 2m\eta + m(m-1)\eta \leq \frac{\varepsilon}{4}. \end{aligned}$$
So there is a finite set $T \subseteq M_m \otimes e_{g_1} A e_{g_1}$ such that for every $a \in S = F \cup \{u_{g_i} : 1 \leq i \leq m\}$, there is a $c \in T$, such that $\|\psi(c) - eae\| < \frac{\varepsilon}{4}$. Furthermore, ψ has the property that if $a \in e_{g_1} A e_{g_1}$, then $\psi(f_{11} \otimes a) = a$. By [9, Lemma 3.5.6], there are equivalent nonzero projections $s_1, s_2 \in A$ such that $s_1 \leq e_{g_1}$ and $s_2 \leq r_2$. Since $M_m \otimes e_{g_1} A e_{g_1}$ is a simple C^*-algebra and $\mathrm{tsr}(M_m \otimes e_{g_1} A e_{g_1}) = 1$, by Lemma 3.4 there exist a projection $q \in M_m \otimes e_{g_1} A e_{g_1}$ and a unital subalgebra $D_0 \subseteq q(M_m \otimes e_{g_1} A e_{g_1})q$ with $\mathrm{tsr}(D_0) = 1$, such that

(1) $\|qc - cq\| < \frac{\varepsilon}{4}$ for all $c \in T$,

(2) For every $c \in T$, there exists a $d \in D_0$ with $\|qcq - d\| < \frac{\varepsilon}{4}$,

(3) $1_{M_m} \otimes e_{g_1} - q \preceq f_{11} \otimes s_1$ in $M_m \otimes e_{g_1} A e_{g_1}$.

Take $p = \psi(q)$, and set $D = \psi(D_0)$, which is a unital subalgebra of $pC^*(G, A, \alpha)p$. Then $e - p = \psi(1_{M_m} \otimes e_{g_1} - q) \preceq \psi(f_{11} \otimes s_1) = s_1 \sim s_2$. Since ψ is injective, we have $\mathrm{tsr}(D) = 1$.

Let $a \in S$. Choose $c \in T$, such that $\|\psi(c) - eae\| < \frac{\varepsilon}{4}$. Then, by using $pe = \psi(q)\psi(1_{M_m} \otimes e_{g_1}) = ep = p$, we have
$$\begin{aligned} \|pa - ap\| &\leq 2\|ea - ae\| + \|peae - eaep\| \\ &\leq 2\|ea - ae\| + 2\|eae - \psi(c)\| + \|qb - bq\| \\ &< 2m\delta + 2\frac{\varepsilon}{4} + \frac{\varepsilon}{4} \leq \varepsilon. \end{aligned}$$
Furthermore, choosing $d \in D_0$ such that $\|qcq - d\| < \frac{\varepsilon}{4}$, we see that the element $\psi(d) \in D$ satisfies
$$\|pap - \psi(d)\| \leq \|eae - \psi(c)\| + \|qcq - d\| < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} \leq \varepsilon.$$

Finally, in $C^*(G, A, \alpha)$, we have

$$[1_A - p] = [(1_A - e)] + [(e - p)] \le [r_1] + [s_1] = [r_1] + [s_2] \le [r_1] + [r_2] \le [r] \le [b].$$

So we have $[1_A - p] \le [b]$.

Secondly, we suppose that α has the strict Rokhlin property. By Theorem 2.2, we have $\mathrm{tsr}(C^*(G, A, \alpha)) = 1$.

Theorem 3.2 *Let A be an infinite dimensional simple separable unital C^*-algebra with real rank zero. Let $\alpha : G \to \mathrm{Aut}(A)$ be an action of a finite group G on A which has the tracial Rokhlin property. Then the crossed product algebra $C^*(G, A, \alpha)$ has real rank zero.*

Proof By Theorem 2.1(1), $C^*(G, A, \alpha)$ is a simple C^*-algebra.

Suppose that $G = \{g_1, g_2, \cdots, g_m\}$, where g_1 is the unit of G. By Lemma 3.5, we need to show that for any finite subset S of the form $S = F \cup \{u_{g_i} : 1 \le i \le m\}$, where F is a finite subset of the unit ball of A and $u_{g_i} \in C^*(G, A, \alpha)$ is the canonical unitary implementing the automorphism α_{g_i}, any $\varepsilon > 0$ and any nonzero positive element $b \in C^*(G, A, \alpha)$, there exist a C^*-subalgebra $D \subseteq C^*(G, A, \alpha)$ and a projection $p \in C^*(G, A, \alpha)$ with $1_D = p$ and $\mathrm{RR}(D) = 0$, such that

(1) $\|pa - ap\| < \varepsilon$ for any $a \in S$,

(2) $pxp \in_\varepsilon D$ for any $a \in S$,

(3) $[1_A - p] \le [b]$.

Since the C^*-algebra A has the property SP, by Lemma 3.1 there exists a nonzero projection $r \in A$ which is Murray-von Neumann equivalent to a projection in $\overline{bC^*(G, A, \alpha)b}$, i.e., $[r] \le [b]$. By [9, Lemma 3.5.7], there are orthogonal nonzero projections $r_1, r_2 \in A$, such that $r_1, r_2 \le r$. Then we have $[r_1] + [r_2] \le [r]$.

Set $\delta = \frac{\varepsilon}{16m}$. Choose $\eta > 0$ according to Lemma 3.3 for m given above and δ in place of ε. Moreover we may require $\eta < \frac{\varepsilon}{8m(m+1)}$. Applying Lemma 3.2 to α with F given above, η in place of ε, and r_1 in place of x, we have projections $e_{g_i} \in A$ for $1 \le i \le m$, such that

$(1)'$ $\|\alpha_{g_i}(e_{g_j}) - e_{g_i g_j}\| < \eta$ for any $1 \le i, j \le m$,

$(2)'$ $\|e_{g_i} a - a e_{g_i}\| < \eta$ for any $1 \le i \le m$ and any $a \in F$,

$(3)'$ $u_{g_i} e u_{g_i}^* = \alpha_{g_i}(e) = e$ for every $1 \le i \le m$, where $e = \sum\limits_{i=1}^{m} e_{g_i}$,

$(4)'$ $1_A - e \preceq r_1$.

By $(1)'$ and $(2)'$, we have $\|ea - ae\| \le \sum\limits_{i=1}^{m} \|e_{g_i} a - a e_{g_i}\| < m\eta$. Define $w_{g_i, g_j} = u_{g_i g_j^{-1}} e_{g_j}$ for every $1 \le i, j \le m$. Using the same estimates as in the proof of Theorem 3.1, we find a unital injective homomorphism $\psi : M_m \otimes e_{g_1} A e_{g_1} \to eC^*(G, A, \alpha)e$ and a finite set $T \subseteq M_m \otimes e_{g_1} A e_{g_1}$, such that for any $a \in S$, there is a $c \in T$, such that $\|\psi(c) - eae\| < \frac{\varepsilon}{4}$. Furthermore, ψ has the property that if $a \in e_{g_1} A e_{g_1}$, then $\varphi(f_{11} \otimes a) = a$, where $f_{11} \in M_n$ denotes the usual $(1, 1)$ matrix unit. By [9, Lemma 3.5.6], there are equivalent nonzero projections $s_1, s_2 \in A$ such that $s_1 \le e_{g_1}$ and $s_2 \le r_2$. Then we have $[s_1] = [s_2] \le [r_2]$. Since $\mathrm{RR}(M_m \otimes e_{g_1} A e_{g_1}) = 0$, by Lemma 3.5 there are projection $q \in M_m \otimes e_{g_1} A e_{g_1}$ and a unital subalgebra $D_0 \subseteq qM_m \otimes e_{g_1} A e_{g_1} q$ with $1_{D_0} = q$ and $\mathrm{RR}(D_0) = 0$, such that

(1) $\|qc - cq\| < \frac{\varepsilon}{4}$ for all $c \in T$,

(2) For every $c \in T$, there exists a $d \in D_0$ with $\|qcq - d\| < \frac{\varepsilon}{4}$,

(3) $1_{M_m} \otimes e_{g_1} - q \preceq f_{11} \otimes s_1$ in $M_m \otimes e_{g_1} A e_{g_1}$.

Take $p = \psi(q)$, and set $D = \psi(D_0)$, which is a unital subalgebra of $pC^*(G, A, \alpha)p$. Then $e - p = \psi(1_{M_m} \otimes e_{g_1} - q) \preceq \psi(f_{11} \otimes s_1) = s_1 \sim s_2$. Since ψ is injective, we have $\mathrm{RR}(D) = 0$.

Let $a \in S$. Choose $c \in T$ such that $\|\psi(c) - eae\| < \frac{\varepsilon}{4}$. Then, by using $pe = \psi(q)\psi(1_{M_m} \otimes e_{g_1}) = ep = p$, we have

$$\begin{aligned}\|pa - ap\| &\leq 2\|ea - ae\| + \|peae - eaep\| \\ &\leq 2\|ea - ae\| + 2\|eae - \psi(c)\| + \|qb - bq\| \\ &< 2m\delta + 2\frac{\varepsilon}{4} + \frac{\varepsilon}{4} \leq \varepsilon.\end{aligned}$$

Furthermore, chosen $d \in D_0$ such that $\|qcq - d\| < \frac{\varepsilon}{4}$, the element $\psi(d) \in D$ satisfies

$$\|pap - \psi(d)\| \leq \|eae - \psi(c)\| + \|qcq - d\| < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} \leq \varepsilon.$$

Finally, in $C^*(G, A, \alpha)$, we have

$$[1_A - p] = [(1_A - e)] + [(e - p)] \leq [r_1] + [s_1] = [r_1] + [s_2] \leq [r_1] + [r_2] \leq [r] \leq [b].$$

So we have $[1_A - p] \leq [b]$.

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