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# On the Equi-nuclearity of Roe Algebras of Metric Spaces****

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**Abstract** The authors define the equi-nuclearity of uniform Roe algebras of a family of metric spaces. For a discrete metric space $X$ with bounded geometry which is covered by a family of subspaces $\{X_i\}_{i=1}^{\infty}$, if $\{C_u^*(X_i)\}_{i=1}^{\infty}$ are equi-nuclear and under some proper gluing conditions, it is proved that $C_u^*(X)$ is nuclear. Furthermore, it is claimed that in general, the coarse Roe algebra $C^*(X)$ is not nuclear.

**Keywords** Nuclear $C^*$-algebra, Uniform Roe algebra, Equi-nuclear uniform Roe algebra
**2000 MR Subject Classification** 46L07, 46L80

## 1 Introduction

Many interesting geometrical properties of spaces and groups are captured by the structure of $C^*$-algebras associated to those objects. For example, a discrete group $G$ is amenable if and only if the full $C^*$-algebra $C^*(G)$ is nuclear (see [2, 7]). The notion of property A for metric spaces, which was introduced by Yu [12] as a weak form of amenability and was studied later in [1 , 3, 10], is equivalent to the nuclearity of the uniform Roe algebras (see [2, 3, 11]). The notion of equi-property A was introduced in [5], where Dadarlat and Guentner proved the following fact. Given a metric space $X$ which is covered by a family of subspaces $\{X_i\}_{i\in I}$, if $\{X_i\}_{i\in I}$ have equi-property A and moreover we have a gluing condition, then $X$ has property A. This theorem is the key to study the permanence of property A under various group operations, such as free product, extensions and relative hyperbolic groups and so on (see [5, 11]).

Naturally, we want to propose a definition of equi-nuclearity for a family of $C^*$-algebras. However, for general $C^*$-algebras, it is difficult to give such a definition. So, in this paper, we introduce the equi-nuclearity for a family of uniform Roe algebras. We use analytic techniques to prove the following theorem.

**Theorem 1.1** *Let $X$ be a metric space with bounded geometry. Suppose that for all $R > 0$ and $\varepsilon > 0$, there is a partition of unit $(\phi_i)_{i=1}^{\infty}$ on $X$ such that*

(1) $\forall\, x, y \in X$ *with* $d(x, y) \leq R$, $\sum\limits_{i=1}^{\infty} |\phi_i(x) - \phi_i(y)| \leq \varepsilon$,

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(2) $\{\phi_i\}_{i=1}^{\infty}$ *subordinate to a family of subspaces* $\{X_i\}_{i=1}^{\infty}$ *and* $\{C_u^*(X_i)\}_{i=1}^{\infty}$ *are equi-nuclear. Then* $C_u^*(X)$ *is nuclear.*

We know that the equi-property A implies the equi-nuclearity, but we do not know whether the reverse is true. Our theorem says that with the same gluing condition (1) in Theorem 1.1, these two definitions imply the same result since the metric space $X$ has property A if and only if the uniform Roe algebra $C_u^*(X)$ is nuclear (see [2, 11]). Moreover, with this definition and this theorem, we can prove the permanence of the nuclearity. For example, if $G$ is a finitely generated group which is hyperbolic relative to subgroups $G_1, G_2, \cdots, G_n$, then $C_u^*(G)$ is nuclear if and only if each subalgebra $C_u^*(G_i)$ is nuclear. With our theorem, the proof is only the routine of [5]. In the last section, we claim that the coarse Roe algebras are not nuclear by using a result of Choi [4].

## 2 Preliminaries

The uniform Roe algebra associated to a discrete metric space with bounded geometry (in particular, a finitely generated group with word length metric) plays an important role in both index theory (see [6]) and exactness problem in $C^*$-algebra theory (see [8]). Let $(X, d)$ be a discrete metric space with bounded geometry. Denote by $\mathcal{B}(l^2(X))$ the $C^*$-algebra of all bounded linear operators on the Hilbert space $l^2(X)$. Any operator in $\mathcal{B}(l^2(X))$ has a natural form of $X \times X$ matrix

$$T = [t(x, y)]_{(x,y)\in X\times X}.$$

$T$ has finite propagation if there exists $R \geq 0$ such that $t(x, y) = 0$ for $d(x, y) \geq R$. The propagation of $T$ is the smallest possible value of $R$. Denote by $\mathrm{Prop}(T)$ the propagation of $T$.

**Definition 2.1** *The collection of all bounded finite propagation operators on* $l^2(X)$ *is a* $*$*-subalgebra of* $\mathcal{B}(l^2(X))$*; its norm completion is called uniform Roe algebra, denoted by* $C_u^*(X)$*.*

If $t_{x,y}$ is an element of a $C^*$-algebra $\mathscr{A}$, we also obtain a $C^*$-algebra by a similar definition, denoted by $C_u^*(X, \mathscr{A})$. Especially, when $\mathscr{A} = \mathbb{M}_n(\mathbb{C})$, we get $C_u^*(X, \mathbb{M}_n(\mathbb{C}))$.

Let $Y$ be a subspace of $X$. The uniform Roe algebra $C_u^*(Y)$ acts on the Hilbert space $l^2(Y)$, a subspace of $l^2(X)$. Thus the correspondence $T \mapsto T \oplus 0$ of $T \in C_u^*(Y)$ gives an inclusion

$$C_u^*(Y) \hookrightarrow C_u^*(X).$$

From this point of view, $C_u^*(Y)$ is a subalgebra of $C_u^*(X)$. Let $P$ denote the projection from $l^2(X)$ to $l^2(Y)$. Then $C_u^*(Y) = PC_u^*(X)P$.

**Definition 2.2** *Let* $Y$ *be a subspace of* $X$*, and let* $Y(R) = \{x \in X \mid d(x, Y) \leq R\}$*. Denote by* $C_u^*(Y, X)$ *the operator norm closure of the set of all finite propagation operators* $T$ *on* $l^2(X)$ *whose support is contained in* $Y(R) \times Y(R)$ *for some* $R > 0$ *(depending on* $T$*).*

We note that $C_u^*(Y, X)$ is a closed two-sided ideal in $C_u^*(X)$ and $C_u^*(Y, X)$ is an inductive limit, i.e.,

$$C_u^*(Y, X) = \lim_{\rightarrow} C_u^*(Y(n)) = \overline{\bigcup_{n=1}^{\infty} C_u^*(Y(n))}.$$

**Definition 2.3** *Let $\mathscr{A}, \mathscr{B}$ be $C^*$-algebras. A linear map $\varphi : \mathscr{A} \to \mathscr{B}$ is said to be completely positive if $\varphi_n : \mathbb{M}_n(\mathscr{A}) \to \mathbb{M}_n(\mathscr{B})$, defined by*

$$\varphi_n([a_{i,j}]_{n\times n}) = [\varphi(a_{i,j})]_{n\times n},$$

*is positive for every $n$ in $\mathbb{N}$ (Here $\mathbb{M}_n(\mathscr{A}) = \{[a_{ij}]_{n\times n} \mid a_{ij} \in \mathscr{A}\}$). Moreover, if*

$$\|\varphi(a)\| \le \|a\|,$$

*we call $\varphi$ contractive completely positive.*

**Definition 2.4** *A $C^*$-algebra $\mathscr{A}$ is called nuclear if for any finite subset $\mathcal{F} \subset \mathscr{A}$ and $\varepsilon > 0$, there exist $n \in \mathbb{N}$ and contractive completely positive maps $\varphi : \mathscr{A} \to \mathbb{M}_n(\mathbb{C}),\ \psi : \mathbb{M}_n(\mathbb{C}) \to \mathscr{A}$ such that $\|\psi \circ \varphi(a) - a\| < \varepsilon$ for any $a \in \mathcal{F}$.*

**Definition 2.5** *A $C^*$-algebra $\mathscr{A}$ is called exact if there exists a faithful representation $\pi : \mathscr{A} \to B(H)$ such that $\pi$ is nuclear. That means for any finite subset $\mathcal{F} \subset \mathscr{A}$ and $\varepsilon > 0$, there exist $n \in \mathbb{N}$ and contractive completely positive maps $\varphi : \mathscr{A} \to \mathbb{M}_n(\mathbb{C}),\ \psi : \mathbb{M}_n(\mathbb{C}) \to B(H)$ such that $\|\psi \circ \varphi(a) - \pi(a)\| < \varepsilon$ for any $a \in \mathcal{F}$.*

From these definitions, we know that the nuclearity of $C^*$-algebras implies exactness.

**Theorem 2.1** (see [1, 11]) *Let $X$ be a discrete metric space with bounded geometry. Then $X$ has property* A *if and only if $C_u^*(X)$ is nuclear.*

Because $C_u^*(X)$ is the completion of the finite propagation operators on $l^2(X)$, the finite subset can be chosen to be the one of operators with finite propagation. And from the proof of Theorem 2.1, one can find that there exists $\lambda(n) \in \mathbb{N}$ such that

$$\sup\{\mathrm{Prop}(\psi(T)) \mid T \in \mathbb{M}_n(\mathbb{C})\} \le \lambda(n).$$

## 3 Equi-nuclearity for Uniform Roe Algebras

Now we give the definition of equi-nuclearity of uniform Roe algebras. From now on, we assume that $X$ is a discrete metric space with bounded geometry, and $\{X_i\}_{i=1}^{\infty}$ is a family of subspaces of $X$ such that $\bigcup\limits_{i=1}^{\infty} X_i = X$ and for any $R \ge 0,\ X_i(R) = \{x \in X : \mathrm{dist}(x, X_i) \le R\}$. For any finite subset $\mathcal{F} \subseteq C_u^*(X)$, $\mathrm{Prop}(\mathcal{F}) = \max\{\mathrm{Prop}(T) : T \in \mathcal{F}\}$, $\sharp\mathcal{F}$ denotes the number of elements in $\mathcal{F}$. Call a family of subsets $\mathcal{F}_i \subseteq C_u^*(X_i)$ uniformly finite if the number of elements in every subset $\mathcal{F}_i$ is uniformly finite, i.e., $\sup\limits_i\{\sharp\mathcal{F}_i\} < \infty$.

**Definition 3.1** *We call the family $\{C_u^*(X_i)\}_{i=1}^{\infty}$ equi-nuclear if for any $\varepsilon > 0$ and any family of uniformly finite subsets $\mathcal{F}_i \subseteq C_u^*(X_i)$ such that $\sup\limits_{i\in\mathbb{N}}\{\mathrm{Prop}(\mathcal{F}_i)\} < \infty$, there exist $n \in \mathbb{N}$, $\lambda(n) \in \mathbb{R}$ and contractive completely positive maps $\varphi_i : C_u^*(X_i) \to \mathbb{M}_n(\mathbb{C}),\ \psi_i : \mathbb{M}_n(\mathbb{C}) \to C_u^*(X_i)$ such that $\|\psi_i \circ \varphi_i(a) - a\| < \varepsilon$ for any $a \in \mathcal{F}_i$ and*

$$\max\{\mathrm{Prop}(\psi_i(E_{s,t})) \mid E_{s,t} \in \mathbb{M}_n(\mathbb{C}) \text{ are matrix units}\} \le \lambda(n), \quad i = 1, 2, \cdots.$$

**Remark 3.1** In fact, the condition

$$\max\{\mathrm{Prop}(\psi_i(E_{s,t})) \mid E_{s,t} \in \mathbb{M}_n(\mathbb{C}) \text{ are matrix units}\} \le \lambda(n), \quad i = 1, 2, \cdots$$

is equivalent to the condition that there exists $\gamma(n) \in \mathbb{N}$ such that

$$\sup\{\mathrm{Prop}(\psi_i(T)) \mid T \in \mathbb{M}_n(\mathbb{C})\} \le \gamma(n), \quad i = 1, 2, \cdots .$$

The proof is obvious.

**Lemma 3.1** *For any $R \ge 0$, $\{C_u^*(X_i)\}_{i=1}^\infty$ are equi-nuclear if and only if $\{C_u^*(X_i(R))\}_{i=1}^\infty$ are equi-nuclear.*

**Proof** **Sufficiency** For any $\varepsilon > 0$ and any uniformly finite subsets $\mathcal{F}_i \subseteq C_u^*(X_i)$ such that $\sup\limits_{i\in\mathbb{N}}\{\mathrm{Prop}(\mathcal{F}_i)\} < \infty$, since $\{C_u^*(X_i(R))\}_{i=1}^\infty$ are equi-nuclear and $\mathcal{F}_i \subseteq C_u^*(X_i) \subseteq C_u^*(X_i(R))$, there exist $n \in \mathbb{N}$, $\lambda(n) \in \mathbb{R}$ and contractive completely positive maps

$$\varphi_i : C_u^*(X_i(R)) \to \mathbb{M}_n(\mathbb{C}), \quad \psi_i : \mathbb{M}_n(\mathbb{C}) \to C_u^*(X_i(R))$$

such that $\|\psi_i \circ \varphi_i(a) - a\| < \varepsilon$ for any $a \in \mathcal{F}_i$ and

$$\max\{\mathrm{Prop}(\psi_i(E_{s,t})) \mid E_{s,t} \in \mathbb{M}_n(\mathbb{C})\} \le \lambda(n), \quad i = 1, 2, \cdots .$$

Consider $C_u^*(X_i)$ as a subalgebra of $C_u^*(X_i(R))$. Let

$$P_i : l^2(X_i(R)) \to l^2(X_i)$$

be the associated projection. Then for the $n$ above and any $i = 1, 2, \cdots$, define

$$\widetilde{\varphi}_i = \varphi_i|_{C_u^*(X_i)} : C_u^*(X_i) \to \mathbb{M}_n(\mathbb{C})$$

and

$$\begin{aligned} \widetilde{\psi}_i : \mathbb{M}_n(\mathbb{C}) &\to C_u^*(X_i) \\ T &\mapsto P_i\psi_i(T)P_i. \end{aligned}$$

It is easy to check that for any $i = 1, 2, \cdots$, $\widetilde{\varphi}_i, \widetilde{\psi}_i$ are all contractive completely positive maps and for any $a \in \mathcal{F}_i$, $P_iaP_i = a$. So

$$\|\widetilde{\psi}_i \circ \widetilde{\varphi}_i(a) - a\| = \|P_i\psi_i(\varphi_i(a))P_i - a\| = \|P_i\psi_i(\varphi_i(a))P_i - P_iaP_i\| \le \varepsilon$$

and

$$\max\{\mathrm{Prop}(\widetilde{\psi}_i(E_{s,t})) \mid E_{s,t} \in \mathbb{M}_n(\mathbb{C})\} \le \lambda(n), \quad i = 1, 2, \cdots .$$

So $\{C_u^*(X_i)\}_{i=1}^\infty$ are equi-nuclear.

**Necessity** Since $X$ has bounded geometry, there exists $N \in \mathbb{N}$ such that for any $x \in X$, $\sharp B(x, R) \le N$. Let $I = \{1, 2, \cdots, N\}$, $X \times I$ be the Cartesian product. Denote by $d_X$ the metric on $X$ and endow a metric on $X \times I$ by $d((x, i), (y, j)) = d_X(x, y) + |i - j|$. Then for every $i \in \mathbb{N}$, $X_i \times I$ is a subspace of $X \times I$ with the induced metric. Since $X$ has bounded geometry and it is easy to find a subspace $Y_i$ of $X_i \times I$ such that there exists a bijection $f_i$ from $X_i(R)$ to $Y_i$. Denote by $\delta_x$ the Dirac function. Let

$$\begin{aligned} U_i : l^2(X_i(R)) &\to l^2(Y_i) \\ \delta_x &\mapsto \delta_{f_i(x)} \end{aligned}$$

be a unitary operator and define

$$\begin{aligned} \mathrm{Ad}\, U_i : C_u^*(X_i(R)) &\to C_u^*(Y_i) \\ T &\mapsto U_i T U_i^*. \end{aligned}$$

Then $C_u^*(X_i(R))$ is isomorphic to $C_u^*(Y_i)$. In fact, it suffices to prove that the image of $\mathrm{Ad}\, U_i$ is contained in $C_u^*(Y_i)$ for each $i \in \mathbb{N}$. Suppose $\mathrm{Prop}(T) \leq S$. If $d(f_i(x), f_i(y)) \geq S + 2R + N$, then $d_X(x, y) \geq S$, so

$$\langle U_i T U_i^* \delta_{f_i(x)}, \delta_{f_i(y)} \rangle = \langle T \delta_x, \delta_y \rangle = 0,$$

which means $\mathrm{Prop}(U_i T U_i^*) \leq S + 2R + N$ and the isomorphism is obvious.

By adjusting the basis of $l^2(X_i \times I)$, we know

$$C_u^*(X_i, \mathbb{M}_N(\mathbb{C})) \cong \mathbb{M}_N(C_u^*(X_i)).$$

Then we have

$$C_u^*(X_i \times I) \cong C_u^*(X_i, \mathbb{M}_N(\mathbb{C})) \cong \mathbb{M}_N(C_u^*(X_i)) \cong C_u^*(X_i) \otimes \mathbb{M}_N(\mathbb{C}).$$

For any $\varepsilon > 0$ and any uniformly finite subsets $\mathcal{F}_i \subseteq C_u^*(X_i(R))$ such that $\sup\limits_{i \in \mathbb{N}} \{\mathrm{Prop}(\mathcal{F}_i)\} < \infty$, we identify $C_u^*(X_i(R))$ with $C_u^*(Y_i) \subseteq C_u^*(X_i) \otimes \mathbb{M}_n(\mathbb{C})$. Then for any $a_r^{(i)} \in \mathcal{F}_i$, we have

$$a_r^{(i)} = \sum_{s,t=1}^{n} b_{s,t}^{(i,r)} \otimes E_{s,t},$$

where $b_{s,t}^{(i,r)} \in C_u^*(X_i)$. Let

$$\mathcal{L}_i = \{ b_{s,t}^{(i,r)} \in C_u^*(X_i) \mid 1 \leq s, t \leq N,\ 1 \leq r \leq \sharp \mathcal{F}_i \}.$$

Then

$$\sharp \mathcal{L}_i \leq N^2 \cdot \sharp \mathcal{F}_i$$

and

$$\sup_{i \in \mathbb{N}} \{\mathrm{Prop}(\mathcal{L}_i)\} \leq \sup_{i \in \mathbb{N}} \{\mathrm{Prop}(\mathcal{F}_i)\} + 2R + N.$$

As $C_u^*(X_i)$ are equi-nuclear, there exist $n \in \mathbb{N}$, $\lambda(n) \in \mathbb{R}$ and contractive completely positive maps

$$\varphi_i : C_u^*(X_i) \to \mathbb{M}_n(\mathbb{C}), \quad \psi_i : \mathbb{M}_n(\mathbb{C}) \to C_u^*(X_i)$$

such that $\|\psi_i \circ \varphi_i(b) - b\| < \varepsilon$ for any $b \in \mathcal{L}_i$ $(i = 1, 2, \cdots)$ and

$$\max\{\mathrm{Prop}(\psi_i(E_{s,t})) \mid E_{s,t} \in \mathbb{M}_n(\mathbb{C})\} \leq \lambda(n), \quad i = 1, 2, \cdots.$$

Define

$$\widetilde{\varphi}_i = \varphi_i \otimes \mathrm{id} : C_u^*(X_i) \otimes \mathbb{M}_N(\mathbb{C}) \to \mathbb{M}_n(\mathbb{C}) \otimes \mathbb{M}_N(\mathbb{C})$$

and

$$\widetilde{\psi}_i = \psi_i \otimes \mathrm{id} : \mathbb{M}_n(\mathbb{C}) \otimes \mathbb{M}_N(\mathbb{C}) \to C_u^*(X_i) \otimes \mathbb{M}_N(\mathbb{C}).$$

By [1, Theorem 3.5.3], we know that $\widetilde{\varphi}_i$ and $\widetilde{\psi}_i$ are contractive completely positive maps. Let $P_i' : l^2(X_i \times I) \to l^2(Y)$ be the projection and identify $C_u^*(X_i(R))$ with $C_u^*(Y)$. Define

$$\widehat{\varphi}_i = \widetilde{\varphi}_i|_{C_u^*(X_i(R))} : C_u^*(X_i(R)) \to \mathbb{M}_n(\mathbb{C}) \otimes \mathbb{M}_N(\mathbb{C})$$

and

$$\widehat{\psi}_i = P_i' \widetilde{\psi}_i P_i' : \mathbb{M}_n(\mathbb{C}) \otimes \mathbb{M}_N(\mathbb{C}) \to C_u^*(X_i(R)).$$

Then $\widehat{\varphi}_i$ and $\widehat{\psi}_i$ are contractive completely positive maps.

For any $a \in \mathcal{F}_i \subseteq C_u^*(X_i(R))$, we know $a = \sum\limits_{s,t=1}^{n} b_{s,t} \otimes E_{s,t}$. Since $P_i' a P_i' = a$, so

$$\begin{aligned}
\|\widehat{\psi}_i \circ \widehat{\varphi}_i(a) - a\| &= \|P_i' \widetilde{\psi}_i(\widetilde{\varphi}_i(a)) P_i' - a\| \\
&\le \sum_{s,t=1}^{N} \|P_i' \psi_i \circ \varphi_i(b_{s,t} \otimes E_{s,t}) P_i' - P_i'(b_{s,t} \otimes E_{s,t}) P_i'\| \\
&\le \sum_{s,t=1}^{N} \|P_i'(\psi_i \circ \varphi_i(b_{s,t}) - b_{s,t}) \otimes E_{s,t} P_i'\| \\
&\le N^2 \varepsilon
\end{aligned}$$

and

$$\max\{\mathrm{Prop}(\widehat{\psi}_i(E_{s,t} \otimes E_{s',t'})) \mid E_{s,t} \in \mathbb{M}_n(\mathbb{C}),\ E_{s',t'} \in \mathbb{M}_N(\mathbb{C})\} \le N \cdot \lambda(n) + 2R$$

for $i = 1, 2, \cdots$. So $\{C_u^*(X_i(R))\}_{i=1}^{\infty}$ are equi-nuclear. We complete the proof.

**Corollary 3.1** *Let $X$ be a metric space with bounded geometry, $Y$ be a subspace of $X$. Then $C_u^*(Y)$ is nuclear if and only if $C_u^*(Y, X)$ is nuclear.*

**Proof** If $C_u^*(Y)$ is nuclear, for any $n \in \mathbb{N}$, by a similar proof as that of the above lemma, $C_u^*(Y(n))$ is nuclear. As $C_u^*(Y, X) = \varinjlim C_u^*(Y(n))$, the nuclearity is preserved by the inductive limits.

Conversely, by using the projection $P : X \to Y$ similarly as the above lemma, it is easy to prove that $C_u^*(Y)$ is nuclear.

**Lemma 3.2** *Let $X$ be a metric space with bounded geometry. Then for any $T \in C_u^*(X)$ with $\mathrm{Prop}(T) \le R$, there exists some $N_{(R)} > 0$ such that*

$$\|T\| \le N_{(R)} \cdot \sup_{(x,y):\, d(x,y) \le R} |T_{x,y}|.$$

**Proof** As $X$ has bounded geometry, we know that there exists some $N_{(R)} > 0$ such that $\sharp B(x, R) \le N_{(R)}$ for any $x \in X$. For any $(\xi_x) \in l^2(X)$, $(T\xi)_x = \sum\limits_y T_{x,y} \xi_y$. By Cauchy-Schwarz inequality, we know

$$|(T\xi)_x|^2 \le N_{(R)} \cdot \Big( \sup_{(x,y)\colon d(x,y)\le R} |T_{x,y}| \Big)^2 \sum_{y\colon d(x,y)\le R} |\xi_y|^2.$$

So by

$$\begin{aligned} \|T\xi\|^2 &\le N_{(R)} \cdot \Big( \sup_{(x,y)\colon d(x,y)\le R} |T_{x,y}| \Big)^2 \sum_x \Big( \sum_{y\colon d(x,y)\le R} |\xi_y|^2 \Big) \\ &\le N_{(R)} \cdot \Big( \sup_{(x,y)\colon d(x,y)\le R} |T_{x,y}| \Big)^2 \sum_y \Big( \sum_{x: d(x,y)\le R} |\xi_y|^2 \Big) \\ &\le N_{(R)}^2 \cdot \Big( \sup_{(x,y)\colon d(x,y)\le R} |T_{x,y}| \Big)^2 \sum_y |\xi_y|^2, \end{aligned}$$

we get the conclusion.

**Proof of Theorem 1.1** For any finite subset $\mathcal{F} \subset C_u^*(X)$, as the finite propagation operators are dense in $C_u^*(X)$, we assume that the elements in $\mathcal{F}$ are all finite propagation operators. Let $R = \max\{\mathrm{Prop}(T) : T \in \mathcal{F}\}$, $M = \max\{\|T\| : T \in \mathcal{F}\}$. For any $\varepsilon > 0$, by assumption, there is a partition of unit $\{\phi_i\}_{i=1}^\infty$ subordinated to a cover $\{X_i\}_{i=1}^\infty$ of $X$ such that $\forall\, x, y \in X$ with $d(x,y) \le R$,

$$\sum_{i=1}^\infty |\phi_i(x) - \phi_i(y)| \le \frac{\varepsilon}{M}$$

and $\{C_u^*(X_i)\}_{i=1}^\infty$ are equi-nuclear. So by Lemma 3.1 we know that $\{C_u^*(X_i(R))\}_{i=1}^\infty$ are equi-nuclear. Let $P_i : l^2(X) \to l^2(X_i(R))$ be the projection. For the uniformly finite subsets $\mathcal{F}_i = P_i \mathcal{F} P_i$, there exist $n \in \mathbb{N}$ and contractive completely positive maps

$$\varphi_i : C_u^*(X_i(R)) \to \mathbb{M}_n(\mathbb{C}), \quad \psi_i : \mathbb{M}_n(\mathbb{C}) \to C_u^*(X_i(R))$$

such that $\|\psi_i \circ \varphi_i(a) - a\| < \varepsilon$ for any $a \in \mathcal{F}_i$ and

$$\max\{\mathrm{Prop}(\psi_i(E_{s,t})) \mid E_{s,t} \in \mathbb{M}_n(\mathbb{C})\} \le \lambda(n), \quad i = 1, 2, \cdots.$$

Let

$$\prod_{i=1}^\infty \mathbb{M}_n(\mathbb{C}) = \{(a_i)_{i=1}^\infty \mid a_i \in \mathbb{M}_n(\mathbb{C})\ (i = 1, 2, \cdots) \text{ such that } \sup_i \|a_i\| \le \infty\}.$$

Define

$$\begin{aligned} \varphi : C_u^*(X) &\to \prod_{i=1}^\infty \mathbb{M}_n(\mathbb{C}) \\ T &\mapsto (\varphi_i(T_i))_{i=1}^\infty, \end{aligned}$$

where $T_i = P_i T P_i$. As all $\varphi_i$ are all contractive completely positive maps, so $\varphi$ is also contractive completely positive.

Define $V_i : l^2(X) \to l^2(X_i)$ by setting $V_i \delta_x = \phi_i(x)^{\frac{1}{2}} \delta_x$. We now define

$$\begin{aligned} \psi : \prod_{i=1}^\infty \mathbb{M}_n(\mathbb{C}) &\to B(l^2(X)) \\ (S_i)_{i=1}^\infty &\mapsto \sum_{i=1}^\infty V_i^* \psi_i(S_i) V_i. \end{aligned}$$

$\psi$ is well-defined and contractive since

$$\Big\|\Big(\sum_{i=1}^{\infty} V_i^*\psi_i(S_i)V_i\Big)\xi\Big\|^2 \le \sum_{i=1}^{\infty}\|V_i^*\psi_i(S_i)V_i\xi\|^2 \le \max_i\|S_i\|^2\sum_{i=1}^{\infty}\|V_i\xi\|^2 \le \max_i\|S_i\|^2\|\xi\|^2.$$

And from the form of $\psi$, it is obviously completely positive. Since

$$\langle\psi((S_i)_{i=1}^{\infty})\delta_x,\delta_y\rangle = \Big\langle\sum_{i=1}^{\infty}V_i^*(\psi_i(S_i))V_i\delta_x,\delta_y\Big\rangle = \sum_{i=1}^{\infty}\phi_i(x)^{\frac12}\phi_i(y)^{\frac12}\langle\psi_i(S_i)\delta_x,\delta_y\rangle$$

and for any $i\in\mathbb{N}$,

$$\mathrm{Prop}\{\psi_i(E_{s,t})\mid E_{s,t}\in\mathbb{M}_n(\mathbb{C})\}\le\lambda(n),$$

we know that $\psi((S_i)_{i=1}^{\infty})$ have finite propagation and thus $\psi((S_i)_{i=1}^{\infty})\in C_u^*(X)$. If $d(x,y)\le R$ and $x\in X_{i_0}$, then $x,y\in X_{i_0}(R)$. Assume

$$\mathcal{O}=\{i\in\mathbb{N}\mid x,y\in X_i(R)\}.$$

Then

$$1=\sum_{i=1}^{\infty}\phi_i(x)=\sum_{i:\,x\in X_i}\phi_i(x)=\sum_{i\in\mathcal{O}}\phi_i(x). \tag{$*$}$$

We have

$$\begin{aligned}
&|\langle\psi\circ\varphi(T)\delta_x,\delta_y\rangle-\langle T\delta_x,\delta_y\rangle|\\
&=\Big|\sum_{i=1}^{\infty}\phi_i(x)^{\frac12}\phi_i(y)^{\frac12}\langle\psi_i\varphi_i(T_i)\delta_x,\delta_y\rangle-\langle T_{i_0}\delta_x,\delta_y\rangle\Big|\\
&\le\Big|\sum_{i=1}^{\infty}\phi_i(x)^{\frac12}\phi_i(y)^{\frac12}\langle(\psi_i\varphi_i(T_i)-T_i)\delta_x,\delta_y\rangle\Big|+\Big|\sum_{i=1}^{\infty}\phi_i(x)^{\frac12}\phi_i(y)^{\frac12}\langle T_i\delta_x,\delta_y\rangle-\langle T_{i_0}\delta_x,\delta_y\rangle\Big|\\
&\le\varepsilon\cdot\sum_{i=1}^{\infty}\phi_i(x)^{\frac12}\phi_i(y)^{\frac12}+\Big|\sum_{i\in\mathcal{O}}\phi_i(x)^{\frac12}\phi_i(y)^{\frac12}\langle T_{i_0}\delta_x,\delta_y\rangle-\langle T_{i_0}\delta_x,\delta_y\rangle\Big|\\
&\le\varepsilon+\Big|\sum_{i\in\mathcal{O}}\phi_i(x)^{\frac12}\phi_i(y)^{\frac12}-1\Big|\cdot|\langle T_{i_0}\delta_x,\delta_y\rangle|\quad\text{(by Cauchy-Schwarz inequality)}\\
&\le\varepsilon+\sum_{i\in\mathcal{O}}|\phi_i(x)^{\frac12}-\phi_i(y)^{\frac12}|^2\cdot|\langle T_{i_0}\delta_x,\delta_y\rangle|\quad\text{(by $(*)$)}\\
&\le\varepsilon+\sum_{i\in\mathcal{O}}|\phi_i(x)-\phi_i(y)|\cdot\|T\|\\
&\le 2\,\varepsilon.
\end{aligned}$$

Since $X$ has bounded geometry, there exists an $N\in\mathbb{N}$ such that for any $x\in X$, $\sharp B(x,R)\le N$. Hence by Lemma 3.2 $\|\psi\circ\varphi(T)-T\|\le 2N\varepsilon$, then $C_u^*(X)$ is nuclear.

**Remark 3.2** With Theorem 1.1, we can give an analytic proof that if a finitely generated group $G$ is hyperbolic relative to the subgroups $G_1,G_2,\cdots,G_n$, then $C_u^*(G)$ is nuclear if and only if $C_u^*(G_i)$ $(i=1,2,\cdots,n)$ are nuclear. But we omit the proof here because it is only a routine of [5].

## 4 Coarse Roe Algebras are not Nuclear

The coarse Roe algebras arise from index theory for general noncompact complete Riemannian manifold (see [9]) and have been associated to coarse geometry of proper metric spaces (see [6, 12]). But unlike the uniform Roe algebra, unless the metric space is finite, its coarse Roe algebra is not nuclear.

**Definition 4.1** *Let $X$ be a discrete metric space, and $H$ be an infinite dimensional Hilbert space. A bounded operator $T : l^2(X) \otimes H \to l^2(X) \otimes H$ is said to have propagation at most $R$ if for all $\varphi, \psi \in l^2(X) \otimes H$ with $d(\mathrm{supp}(\varphi), \mathrm{supp}(\psi)) > R$ such that*

$$\langle T\varphi, \psi\rangle = 0.$$

*Note that if $X$ is discrete, then we can write*

$$l^2(X) \otimes H = \bigoplus_{x\in X} (\delta_x \otimes H),$$

*where $\delta_x$ is the Dirac function at $x$. Every bounded operator acting on $l^2(X) \otimes H$ has a corresponding matrix representation*

$$T = (T_{x,y})_{x,y\in X},$$

*where $T_{x,y} : \delta_y \otimes H \to \delta_x \otimes H$ is a bounded operator. We say $T$ locally compact if $T_{x,y}$ is a compact operator for all $x, y \in X$. Since $T$ has finite propagation $R$, it is equivalent to say that the matrix coefficient $T_{x,y}$ of $T$ vanishes when $d(x, y) > R$.*

**Definition 4.2** *The collection of all locally compact, finite propagation operators on $l^2(X)\otimes H$ is a $*$-subalgebra of $B(l^2(X) \otimes H)$. Its norm completion, denoted by $C^*(X)$, is called the coarse Roe algebra of $X$.*

Let $F_2$ be the free group with two generators, $C^*(F_2)$ be the full $C^*$-algebra of $F_2$. Let

$$\prod_{i=1}^{\infty} \mathbb{M}_i(\mathbb{C}) = \Big\{ (a_i)_{i=1}^{\infty} \,\Big|\, a_i \in \mathbb{M}_i(\mathbb{C}) \ (i = 1, 2, \cdots) \text{ such that } \sup_i \|a_i\| \le \infty \Big\}.$$

**Theorem 4.1** (see [1, 4]) *The full $C^*$-algebra $C^*(F_2)$ can isometrically embed into the $C^*$-algebra $\prod\limits_{i=1}^{\infty} \mathbb{M}_i(\mathbb{C})$.*

**Theorem 4.2** *Let $X$ be a discrete countable metric space. Then the coarse Roe algebra $C^*(X)$ can not be nuclear.*

**Proof** If $C^*(X)$ is nuclear, then it is exact. By Theorem 4.1, we can view $C^*(F_2)$ as a subalgebra of $\prod\limits_{i=1}^{\infty} \mathbb{M}_i(\mathbb{C})$ and it is a subalgebra of $C^*(X)$ too. Because the exactness passes to subalgebras, we get that $C^*(F_2)$ is exact. By [1, Proposition 3.7.11], it is implied that $F_2$ is amenable, which is a contradiction.

**Corollary 4.1** *$l^\infty(X, K(H))$ is not isomorphic to $l^\infty(X, K(H))$.*

**Proof** By Theorem 4.1, we view $C^*(F_2)$ as a subalgebra of $l^\infty(X, K(H))$, then we know that $l^\infty(X, K(H))$ is not nuclear. But $l^\infty(X) \otimes K(H)$ is a nuclear $C^*$-algebra, so they are not isomorphic.

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