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Detection of Some Elements in the Stable Homotopy Groups of Spheres**

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(*Dedicated to Professor Jinkun Lin on His* 65*th Birthday*)

Abstract Let A be the mod p Steenrod algebra and S be the sphere spectrum localized at an odd prime p. To determine the stable homotopy groups of spheres π_*S is one of the central problems in homotopy theory. This paper constructs a new nontrivial family of homotopy elements in the stable homotopy groups of spheres $\pi_{p^nq+2pq+q-3}S$ which is of order p and is represented by $k_0h_n \in \mathrm{Ext}_A^{3,p^nq+2pq+q}(\mathbb{Z}_p,\mathbb{Z}_p)$ in the Adams spectral sequence, where $p \geq 5$ is an odd prime, $n \geq 3$ and $q = 2(p-1)$. In the course of the proof, a new family of homotopy elements in $\pi_{p^nq+(p+1)q-1}V(1)$ which is represented by $\beta_*i'_*i_*(h_n) \in \mathrm{Ext}_A^{2,p^nq+(p+1)q+1}(H^*V(1),\mathbb{Z}_p)$ in the Adams sequence is detected.

Keywords Stable homotopy groups of spheres, Adams spectral sequence, May spectral sequence, Steenrod algebra
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1 Introduction and the Main Results

Let A be the mod p Steenrod algebra and S be the sphere spectrum localized at an odd prime p. To determine the stable homotopy groups of spheres π_*S is one of the central problems in homotopy theory. One of the main tools to reach it is the Adams spectral sequence:

$$E_2^{s,t} = \mathrm{Ext}_A^{s,t}(\mathbb{Z}_p,\mathbb{Z}_p) \Rightarrow \pi_{t-s}S,$$

where the $E_2^{s,t}$-term is the cohomology of A.

Throughout this paper, we fix $q = 2(p-1)$.

From [1], $\mathrm{Ext}_A^{1,*}(\mathbb{Z}_p,\mathbb{Z}_p)$ has $\mathbb{Z}_p$-basis consisting of $a_0 \in \mathrm{Ext}_A^{1,1}(\mathbb{Z}_p,\mathbb{Z}_p)$, $h_i \in \mathrm{Ext}_A^{1,p^iq}(\mathbb{Z}_p,\mathbb{Z}_p)$ for all $i \geq 0$ and $\mathrm{Ext}_A^{2,*}(\mathbb{Z}_p,\mathbb{Z}_p)$ has $\mathbb{Z}_p$-basis consisting of α_2, a_0^2, a_0h_i $(i > 0)$, g_i $(i \geq 0)$, k_i $(i \geq 0)$, e_i $(i \geq 0)$, and h_ih_j $(j \geq i+2, i \geq 0)$ whose internal degrees are $2q+1$, 2, p^iq+1, $p^{i+1}q+2p^iq$, $2p^{i+1}+p^iq$, $p^{i+1}q$ and p^iq+p^jq respectively.

Let M be the Moore spectrum modulo a prime $p \geq 5$ given by the cofibration

$$S \xrightarrow{p} S \xrightarrow{i} M \xrightarrow{j} \Sigma S. \tag{1.1}$$

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Let $\alpha : \Sigma^q M \longrightarrow M$ be the Adams map and K be its cofibre given by the cofibration

$$\Sigma^q M \xrightarrow{\alpha} M \xrightarrow{i'} K \xrightarrow{j'} \Sigma^{q+1} M, \tag{1.2}$$

where $q = 2(p-1)$. This spectrum which we briefly write as K is known to be the Toda-Smith spectrum $V(1)$. Let $V(2)$ be the cofibre of $\beta : \Sigma^{(p+1)q} K \longrightarrow K$ given by the cofibration

$$\Sigma^{(p+1)q} K \xrightarrow{\beta} K \xrightarrow{\bar{i}} V(2) \xrightarrow{\bar{j}} \Sigma^{(p+1)q+1} K. \tag{1.3}$$

If a family of homotopy generators x_i in $E_2^{s,*}$ converges nontrivially in the Adams spectral sequence, then we get a family of homotopy elements f_i in $\pi_* S$ and we say that f_i is represented by $x_i \in E_2^{s,*}$ and has filtration s in the Adams spectral sequence. So far, not so many families of homotopy elements in $\pi_* S$ have been detected.

In 1981, a family $\varsigma_{n-1} \in \pi_{p^n q+q-3} S$ for $n \geq 2$ which has filtration 3 in the Adams spectral sequence and is represented by

$$h_0 b_{n-1} \in \mathrm{Ext}_A^{3,p^n q+q}(\mathbb{Z}_p, \mathbb{Z}_p) \tag{1.4}$$

was detected in [2].

Recently, Lin Jinkun got a series of results and detected some new families of lower filtration in the stable homotopy groups of spheres $\pi_* S$.

In [3], Lin Jinkun and Zheng Qibing obtained the following theorem and detected a new family of filtration 7 in the stable homotopy groups of spheres.

Theorem 1.1 (cf. [3]) *Let $p \geq 7$, $n \geq 4$. Then the product*

$$b_{n-1} g_0 \widetilde{\gamma}_3 \neq 0 \in \mathrm{Ext}_A^{7,p^n q+3(p^2+p+1)q}(\mathbb{Z}_p, \mathbb{Z}_p)$$

is a permanent cycle in the Adams spectral sequence and it converges to a nontrivial element of order p in $\pi_{p^n q+3(p^2+p+1)q-7} S$..

Lin [4] detected a new family of filtration 6 in the stable homotopy groups of spheres and proved the following theorem.

Theorem 1.2 (cf. [4]) *Let $p \geq 7$, $n \geq 4$. Then the product*

$$h_n g_0 \widetilde{\gamma}_3 \neq 0 \in \mathrm{Ext}_A^{6,p^n q+3(p^2+p+1)q}(\mathbb{Z}_p, \mathbb{Z}_p)$$

is a permanent cycle in the Adams spectral sequence and it converges to a nontrivial element of order p in $\pi_{p^n q+3(p^2+p+1)q-6} S$.

In [5], Lin detected a new family of filtration 3 in the stable homotopy groups of spheres. Lin's family was constructed by using the Cohen family ς_n and he obtained the following theorem.

Theorem 1.3 (cf. [5]) *Let $p \geq 5$, $n \geq 3$. Then*

(1)

$$i_*(h_1 h_n) \neq 0 \in \mathrm{Ext}_A^{2,p^n q+pq}(H^* M, \mathbb{Z}_p)$$

is a permanent cycle in the Adams spectral sequence and it converges to a nontrivial element $\xi_n \in \pi_{p^n q+pq-2}M$.

(2) *For* $\xi_n \in \pi_{p^n q+pq-2}M$ *obtained in* (1),

$$j\xi_n \in \pi_{p^n q+pq-3}S$$

is a nontrivial element of order p *which is represented* (*up to nonzero scalar*) *by*

$$(b_0h_n + h_1b_{n-1}) \in \mathrm{Ext}_A^{3,p^n q+pq}(\mathbb{Z}_p, \mathbb{Z}_p)$$

in the Adams spectral sequence.

With those known results, the author made use of the May spectral sequence and the Adams spectral sequence to detect some new nontrivial families of higher filtration in the stable homotopy groups of spheres π_*S (cf. [6–12]).

In this paper, we make a further research on the basis of [5] and also detect a family of homotopy elements in $\pi_{p^n q+pq-3}S$ which has filtration 3 and is represented by $k_0h_n \in \mathrm{Ext}_A^{3,p^n q+2pq+q}(\mathbb{Z}_p, \mathbb{Z}_p)$ in the Adams spectral sequence. Our result is the following theorem.

Theorem 1.4 *Let* $p \geq 5$, $n \geq 3$. *Then*

$$k_0h_n \neq 0 \in \mathrm{Ext}_A^{3,p^n q+2pq+q}(\mathbb{Z}_p, \mathbb{Z}_p)$$

is a permanent cycle in the Adams spectral sequence and it converges to a nontrivial element of order p *in* $\pi_{p^n q+2pq+q-3}S$.

The construction of the above k_0h_n-element is parallel to that of $(b_0h_n + h_1b_{n-1})$-element given in [5]. Theorem 1.4 will be proved on the basis of the following theorem.

Theorem 1.5 *Let* $p \geq 5, n \geq 3$. *Then*

$$\beta_* i'_* i_*(h_n) \neq 0 \in \mathrm{Ext}_A^{2,p^n q+(p+1)q+1}(H^*K, \mathbb{Z}_p)$$

is a permanent cycle in the Adams spectral sequence and it converges to a nontrivial element $\zeta_n \in \pi_{p^n q+(p+1)q-1}K$.

Remark 1.1 The k_0h_n-element obtained in Theorem 1.4 is an indecomposable element in π_*S, i.e., it is not a composition of elements of lower filtration in π_*S, because h_n $(n > 0)$ is known to die in the Adams spectral sequence.

This paper is organized as follows. After giving some useful propositions in Section 2, the proofs of the main theorems will be given in Section 3.

2 Some Preliminaries on Low-Dimensional Ext Groups

In this section, we will prove some results on Ext groups of low dimension which will be used in the proofs of the main theorems.

Proposition 2.1 *Let $p \geq 5$, $n \geq 3$, $a_0 \in \mathrm{Ext}_A^{1,1}(\mathbb{Z}_p, \mathbb{Z}_p)$, $h_n \in \mathrm{Ext}_A^{1,p^n q}(\mathbb{Z}_p, \mathbb{Z}_p)$, and $b_n \in \mathrm{Ext}_A^{2,p^{n+1}q}(\mathbb{Z}_p, \mathbb{Z}_p)$ respectively. Then we have the following:*

(1) $\mathrm{Ext}_A^{4,p^n q+pq+2}(\mathbb{Z}_p, \mathbb{Z}_p) \cong \mathbb{Z}_p\{a_0 a_0 h_1 h_n\}$.

(2) $\mathrm{Ext}_A^{5,p^n q+(p+2)q}(\mathbb{Z}_p, \mathbb{Z}_p) = 0$.

Proof (1) See [13, Theorem 4.1].

(2) The proof is similar to that given in the proof of [6, Proposition 1.2]. We can show that in the May spectral sequence $E_1^{5,p^n q+(p+2)q,*} = 0$. Then

$$\mathrm{Ext}_A^{5,p^n q+(p+2)q}(\mathbb{Z}_p, \mathbb{Z}_p) = 0.$$

Here the proof is omitted.

The following lemma is used in the proofs of many propositions in this section.

First recall spectra $V(k) = \{V(k)_n\}$ for $n \geq -1$ which are so-called Toda-Smith spectra. The spectrum $V(n)$ is given in [14] such that the Z_p-cohomology

$$H^*(V(n), \mathbb{Z}_p) \cong E(n) = E(\mathcal{Q}_0, \mathcal{Q}_1, \cdots, \mathcal{Q}_n),$$

the exterior algebra generator by Milnor basis elements $\mathcal{Q}_0$, $\mathcal{Q}_1$, $\cdots$, $\mathcal{Q}_n$ in A. The spectra $V(n)$ for $n \geq -1$ are defined inductively by $V(-1) = S$ and the cofibration

$$\Sigma^{2(p^n-1)} V(n-1) \xrightarrow{\alpha^{(n)}} V(n-1) \xrightarrow{i_n} V(n) \xrightarrow{j_n} \Sigma^{2p^n-1} V(n-1). \tag{2.1}$$

When $n = 0$, 1, 2, the above cofibration sequences just are the cofibration sequences (1.1), (1.2) and (1.3) respectively. $\alpha^{(n)}$ stand for the maps p, α, β in (1.1), (1.2) and (1.3) respectively. Here $V(-1) = S,\ V(0) = M,\ V(1) = K,\ i_0 = i,\ i_1 = i',\ i_2 = \bar{i},\ j_0 = j,\ j_1 = j',\ j_2 = \bar{j}$. The existence of $V(n)$ is assured (cf. [14, Theorem 1.1]) for $n = 1$, $p \geq 3$ and for $n = 2$, $p \geq 5$.

By the definition of Ext groups, from (2.1) we can easily have the following lemma.

Lemma 2.1 *With notation as above, we have the following two long exact sequences:*

(1)
$$\begin{aligned} \cdots \longrightarrow \mathrm{Ext}_A^{s-1,t-(2p^n-1)}(H^*V(n-1), \square) &\xrightarrow{\alpha_*^{(n)}} \mathrm{Ext}_A^{s,t}(H^*V(n-1), \square) \\ &\xrightarrow{(i_n)_*} \mathrm{Ext}_A^{s,t}(H^*V(n), \square) \xrightarrow{(j_n)_*} \mathrm{Ext}_A^{s,t-(2p^n-1)}(H^*V(n), \square) \longrightarrow \cdots. \end{aligned}$$

(2)
$$\begin{aligned} \cdots \longrightarrow \mathrm{Ext}_A^{s-1,t-(2p^n-1)}(\square, H^*V(n-1)) &\xrightarrow{\alpha^{(n)*}} \mathrm{Ext}_A^{s,t}(\square, H^*V(n-1)) \\ &\xrightarrow{(j_n)^*} \mathrm{Ext}_A^{s,t}(\square, H^*V(n)) \xrightarrow{(i_n)^*} \mathrm{Ext}_A^{s,t}(\square, H^*V(n)) \longrightarrow \cdots. \end{aligned}$$

Here $\square$ is an arbitrary A-comodule.

Proposition 2.2 *Let $p \geq 5$, $n \geq 3$. Then $\mathrm{Ext}_A^{3,p^n q+(p+2)q+1}(H^*M, H^*M)$ has a unique generator*

$$\overline{\overline{h_n g_0}}\,,$$

where $\overline{\overline{h_n g_0}}$ satisfies

$$i^* j_* \overline{\overline{h_n g_0}} = h_n g_0,$$

the generator of $\mathrm{Ext}_A^{3,p^n q+(p+2)q}(\mathbb{Z}_p, \mathbb{Z}_p)$ stated in [15, *Table* 8.1].

Proof First consider the exact sequence

$$\begin{aligned}&\mathrm{Ext}_A^{3,p^nq+(p+2)q+1}(\mathbb{Z}_p,\mathbb{Z}_p)\xrightarrow{j^*}\mathrm{Ext}_A^{3,p^nq+(p+2)q}(\mathbb{Z}_p,H^*M)\\ \xrightarrow{i^*}\ &\mathrm{Ext}_A^{3,p^nq+(p+2)q}(\mathbb{Z}_p,\mathbb{Z}_p)\xrightarrow{p^*}\mathrm{Ext}_A^{4,p^nq+(p+2)q+1}(\mathbb{Z}_p,\mathbb{Z}_p)\end{aligned}$$

induced by (1.1). Since we know that $\mathrm{Ext}_A^{3,p^nq+(p+2)q+1}(\mathbb{Z}_p,\mathbb{Z}_p)$ is zero (cf. [15, Table 8.1]) and $\mathrm{Ext}_A^{4,p^nq+(p+2)q+1}(\mathbb{Z}_p,\mathbb{Z}_p)$ is zero (cf. [5, Proposition 2.1]), the above i^* is an isomorphism. Then we see that $\mathrm{Ext}_A^{3,p^nq+(p+2)q}(\mathbb{Z}_p,H^*M)$ has a unique generator

$$\overline{h_ng_0}\,,$$

where $\overline{h_ng_0}$ satisfies

$$i^*\overline{h_ng_0}=h_ng_0,$$

the unique generator of $\mathrm{Ext}_A^{3,p^nq+(p+2)q}(\mathbb{Z}_p,\mathbb{Z}_p)$ stated in [15, Table 8.1].

At last, look at the following exact sequence induced by (1.1)

$$\begin{aligned}&\mathrm{Ext}_A^{3,p^nq+(p+2)q+1}(\mathbb{Z}_p,H^*M)\xrightarrow{i_*}\mathrm{Ext}_A^{3,p^nq+(p+2)q+1}(H^*M,H^*M)\\ \xrightarrow{j_*}\ &\mathrm{Ext}_A^{3,p^nq+(p+2)q}(\mathbb{Z}_p,H^*M)\xrightarrow{p_*}\mathrm{Ext}_A^{4,p^nq+(p+2)q+1}(\mathbb{Z}_p,H^*M).\end{aligned}$$

Since the first group is zero by virtue of $\mathrm{Ext}_A^{3,p^nq+(p+2)q+r}(\mathbb{Z}_p,\mathbb{Z}_p)=0$ for $r=1,2$ (cf. [15, Table 8.1]) and the fourth group is zero by virtue of the fact that $\mathrm{Ext}_A^{4,p^nq+(p+2)q+t}(\mathbb{Z}_p,\mathbb{Z}_p)=0$ for $t=1,2$ (cf. [5, Proposition 2.1]), the above j_* is an isomorphism. Thus we can know that $\mathrm{Ext}_A^{3,p^nq+(p+2)q+1}(H^*M,H^*M)$ has a unique generator

$$\overline{\overline{h_ng_0}}\,,$$

where $\overline{\overline{h_ng_0}}$ satisfies

$$j_*\overline{\overline{h_ng_0}}=\overline{h_ng_0}\,.$$

This finishes the proof of Proposition 2.2.

Proposition 2.3 *Let $p\geq 5, n\geq 3$. Then*

$$\mathrm{Ext}_A^{3,p^nq+(p+2)q}(H^*M,H^*M)\cong\mathbb{Z}_p\{i_*j_*\overline{\overline{h_ng_0}}\,,j^*i^*\overline{\overline{h_ng_0}}\}.$$

Proof Consider the exact sequence

$$\begin{aligned}&\mathrm{Ext}_A^{2,p^nq+(p+2)q-1}(\mathbb{Z}_p,\mathbb{Z}_p)\xrightarrow{p^*}\mathrm{Ext}_A^{3,p^nq+(p+2)q}(\mathbb{Z}_p,\mathbb{Z}_p)\\ \xrightarrow{j^*}\ &\mathrm{Ext}_A^{3,p^nq+(p+2)q-1}(\mathbb{Z}_p,H^*M)\xrightarrow{i^*}\mathrm{Ext}_A^{3,p^nq+(p+2)q-1}(\mathbb{Z}_p,\mathbb{Z}_p)\end{aligned}$$

induced by (1.1). Since $\mathrm{Ext}_A^{2,p^nq+(p+2)q-1}(\mathbb{Z}_p,\mathbb{Z}_p)=0=\mathrm{Ext}_A^{3,p^nq+(p+2)q-1}(\mathbb{Z}_p,\mathbb{Z}_p)$ (cf. [15, Table 8.1]), the above j^* is an isomorphism. Moreover, we also know $\mathrm{Ext}_A^{3,p^nq+(p+2)q}(\mathbb{Z}_p,\mathbb{Z}_p)=\mathbb{Z}_p\{h_ng_0\}$ (cf. [15, Table 8.1]). Thus we can have

$$\mathrm{Ext}_A^{3,p^nq+(p+2)q-1}(\mathbb{Z}_p,H^*M)=\mathbb{Z}_p\{j^*(h_ng_0)\}.$$

Now observe the following exact sequence

$$\begin{aligned}&\mathrm{Ext}_A^{2,p^nq+(p+2)q-1}(\mathbb{Z}_p,H^*M)\xrightarrow{p_*}\mathrm{Ext}_A^{3,p^nq+(p+2)q}(\mathbb{Z}_p,H^*M)\\ \xrightarrow{i_*}\;&\mathrm{Ext}_A^{3,p^nq+(p+2)q}(H^*M,H^*M)\xrightarrow{j_*}\mathrm{Ext}_A^{3,p^nq+(p+2)q-1}(\mathbb{Z}_p,H^*M)\xrightarrow{p_*}\end{aligned}$$

induced by (1.1). Since $\mathrm{Ext}_A^{2,p^nq+(p+2)q+r}(\mathbb{Z}_p,\mathbb{Z}_p)=0$ for $r=-1,0$ (cf. [1]), we can easily get

$$\mathrm{Ext}_A^{2,p^nq+(p+2)q-1}(\mathbb{Z}_p,H^*M)=0.$$

By virtue of the fact

$$\mathrm{Ext}_A^{3,p^nq+(p+2)q-1}(\mathbb{Z}_p,H^*M)=\mathbb{Z}_p\{j^*(h_ng_0)\},$$

we have that the image of the second p_* is zero since

$$p_*j^*(h_ng_0)=j^*p_*(h_ng_0)=j^*p^*(g_0h_n)=0.$$

From the fact that

$$\begin{aligned}&\mathrm{Ext}_A^{3,p^nq+(p+2)q}(\mathbb{Z}_p,H^*M)\cong\mathbb{Z}_p\{\overline{h_ng_0}\}\cong\mathbb{Z}_p\{j_*\overline{\overline{h_ng_0}}\},\\ &\mathrm{Ext}_A^{3,p^nq+(p+2)q-1}(\mathbb{Z}_p,H^*M)\cong\mathbb{Z}_p\{j^*(h_ng_0)\}\cong\mathbb{Z}_p\{j^*i^*j_*\overline{\overline{h_ng_0}}\}\cong\mathbb{Z}_p\{j_*j^*i^*\overline{\overline{h_ng_0}}\},\end{aligned}$$

we can easily get

$$\mathrm{Ext}_A^{3,p^nq+(p+2)q}(H^*M,H^*M)\cong\mathbb{Z}_p\{i_*j_*\overline{\overline{h_ng_0}},j^*i^*\overline{\overline{h_ng_0}}\}.$$

This shows Proposition 2.3.

Proposition 2.4 *Let $p\ge 5, n\ge 3$. Then we have*
(1) $i^*d_2(i_*j_*(\overline{\overline{h_ng_0}}))\neq 0$;
(2) $d_2(j^*i^*(\overline{\overline{h_ng_0}}))\neq 0$, *where*

$$d_2:\mathrm{Ext}_A^{3,p^nq+(p+2)q}(H^*M,H^*M)\longrightarrow\mathrm{Ext}_A^{5,p^nq+(p+2)q+1}(H^*M,H^*M)$$

is the differential of the Adams spectral sequence.

Proof (1) From [5, p. 488] we know that

$$d_2(i_*(h_ng_0))\neq 0.$$

By Proposition 2.2,

$$d_2(i_*(h_ng_0))=d_2(i_*i^*j_*(\overline{\overline{h_ng_0}}))=d_2(i^*i_*j_*(\overline{\overline{h_ng_0}}))=i^*d_2(i_*j_*(\overline{\overline{h_ng_0}})).$$

The desired result follows.

(2) Consider the exact sequence

$$\begin{aligned}&\mathrm{Ext}_A^{4,p^nq+(p+2)q}(\mathbb{Z}_p,\mathbb{Z}_p)\xrightarrow{p^*}\mathrm{Ext}_A^{5,p^nq+(p+2)q+1}(\mathbb{Z}_p,\mathbb{Z}_p)\\ \xrightarrow{j^*}\;&\mathrm{Ext}_A^{5,p^nq+(p+2)q}(\mathbb{Z}_p,H^*M)\xrightarrow{i^*}\mathrm{Ext}_A^{5,p^nq+(p+2)q}(\mathbb{Z}_p,\mathbb{Z}_p)\end{aligned}$$

induced by (1.1). We claim that the above j^* is an isomorphism. By Proposition 2.1, we have

$$\mathrm{Ext}_A^{5,p^n q+(p+2)q}(\mathbb{Z}_p,\mathbb{Z}_p)=0.$$

It follows that the above j^* is an epimorphism. From [5, Proposition 2.1], we know that

$$\mathrm{Ext}_A^{4,p^n q+(p+2)q}(\mathbb{Z}_p,\mathbb{Z}_p)\cong\mathbb{Z}_p\{g_0b_{n-1}\}.$$

Since

$$p^*(g_0b_{n-1})=a_0g_0b_{n-1}=0\quad(a_0g_0=0\text{ by [15, Table 8.2]}),$$

we have

$$\text{kernel } j^*=\text{image } p^*=0.$$

It follows that the above j^* also is a monomorphism. The proof of the claim is finished. From [5, Proposition 2.1], we have

$$\alpha_2b_0h_n\neq 0\in\mathrm{Ext}_A^{5,p^n q+(p+2)q+1}(\mathbb{Z}_p,\mathbb{Z}_p).$$

By the claim we can get

$$j^*(\alpha_2b_0h_n)\neq 0\in\mathrm{Ext}_A^{5,p^n q+(p+2)q}(\mathbb{Z}_p,H^*M).$$

At the same time, from [5, Lemma 3.2], we have that up to nonzero scalar

$$d_2(h_ng_0)=\alpha_2b_0h_n.$$

Note that

$$i^*j_*\overline{\overline{h_ng_0}}=h_ng_0.$$

It follows that

$$j^*d_2(j_*i^*\overline{\overline{h_ng_0}})\neq 0.$$

By

$$j^*d_2(j_*i^*\overline{\overline{h_ng_0}})=j_*d_2(j^*i^*\overline{\overline{h_ng_0}}),$$

we can have

$$d_2(j^*i^*(\overline{\overline{h_ng_0}}))\neq 0.$$

This finishes the proof of the second part of Proposition 2.4.

Proposition 2.5 *Let $p\geq 5, n\geq 3$. Then*

$$\mathrm{Ext}_A^{3,p^n q+(p+1)q+2}(H^*K,H^*M)=0.$$

Proof Consider the exact sequence

$$\begin{aligned}\mathrm{Ext}_A^{3,p^nq+(p+1)q+3}(H^*M,\mathbb{Z}_p) &\xrightarrow{j^*} \mathrm{Ext}_A^{3,p^nq+(p+1)q+2}(H^*M,H^*M)\\ &\xrightarrow{i^*} \mathrm{Ext}_A^{3,p^nq+(p+1)q+2}(H^*M,\mathbb{Z}_p).\end{aligned}$$

Since the first group and the third group are zero by the fact that $\mathrm{Ext}_A^{3,p^nq+(p+1)q+r}(\mathbb{Z}_p,\mathbb{Z}_p)=0$ for $r=1,2,3$ (cf. [15, Table 8.1]), the second group is zero.

Look at the exact sequence

$$\begin{aligned}&\mathrm{Ext}_A^{3,p^nq+pq+2}(\mathbb{Z}_p,\mathbb{Z}_p) \xrightarrow{i_*} \mathrm{Ext}_A^{3,p^nq+pq+2}(H^*M,\mathbb{Z}_p)\\ \xrightarrow{j_*}\ &\mathrm{Ext}_A^{3,p^nq+pq+1}(\mathbb{Z}_p,\mathbb{Z}_p) \xrightarrow{p_*} \mathrm{Ext}_A^{4,p^nq+pq+2}(\mathbb{Z}_p,\mathbb{Z}_p)\end{aligned}$$

induced by (1.1). Since we know that $\mathrm{Ext}_A^{3,p^nq+pq+1}(\mathbb{Z}_p,\mathbb{Z}_p)\cong\mathbb{Z}_p\{a_0h_1h_n\}$ (cf. [15, Table 8.1]) and $\mathrm{Ext}_A^{4,p^nq+pq+2}(\mathbb{Z}_p,\mathbb{Z}_p)\cong\mathbb{Z}_p\{a_0a_0h_1h_n\}$ by Proposition 2.1, the above p_* is an isomorphism. image $j_*=0$ since p_* is an isomorphism. image $i_*=0$ by the fact that $\mathrm{Ext}_A^{3,p^nq+pq+2}(\mathbb{Z}_p,\mathbb{Z}_p)=0$ (cf. [15, Table 8.1]). Thus we can have

$$\mathrm{Ext}_A^{3,p^nq+pq+2}(H^*M,\mathbb{Z}_p)=0.$$

Observe the following exact sequence induced by (1.1):

$$\begin{aligned}\mathrm{Ext}_A^{2,p^nq+pq}(\mathbb{Z}_p,\mathbb{Z}_p) &\xrightarrow{p_*} \mathrm{Ext}_A^{3,p^nq+pq+1}(\mathbb{Z}_p,\mathbb{Z}_p) \xrightarrow{i_*} \mathrm{Ext}_A^{3,p^nq+pq+1}(H^*M,\mathbb{Z}_p)\\ &\xrightarrow{j_*} \mathrm{Ext}_A^{3,p^nq+pq}(\mathbb{Z}_p,\mathbb{Z}_p) \xrightarrow{p_*} \mathrm{Ext}_A^{4,p^nq+pq+1}(\mathbb{Z}_p,\mathbb{Z}_p).\end{aligned}$$

Since $\mathrm{Ext}_A^{2,p^nq+pq}(\mathbb{Z}_p,\mathbb{Z}_p)\cong\mathbb{Z}_p\{h_1h_n\}$ and $\mathrm{Ext}_A^{3,p^nq+pq+1}(\mathbb{Z}_p,\mathbb{Z}_p)\cong\mathbb{Z}_p\{a_0h_1h_n\}$ (cf. [15, Table 8.1]), we know that the first p_* is an isomorphism. Similarly by virtue of the facts that $\mathrm{Ext}_A^{3,p^nq+pq}(\mathbb{Z}_p,\mathbb{Z}_p)\cong\mathbb{Z}_p\{b_0h_n,h_1b_{n-1}\}$ (cf. [15, Table 8.1]) and $\mathrm{Ext}_A^{4,p^nq+pq+1}(\mathbb{Z}_p,\mathbb{Z}_p)\cong\mathbb{Z}_p\{a_0b_0h_n,a_0h_1b_{n-1}\}$ (cf. [5, Proposition 2.1]), we get that the second p_* is also an isomorphism. Thus it follows that

$$\mathrm{Ext}_A^{3,p^nq+pq+1}(H^*M,\mathbb{Z}_p)=0.$$

Look at the exact sequence

$$\begin{aligned}0=\mathrm{Ext}_A^{3,p^nq+pq+2}(H^*M,\mathbb{Z}_p) &\xrightarrow{j^*} \mathrm{Ext}_A^{3,p^nq+pq+1}(H^*M,H^*M)\\ &\xrightarrow{i^*} \mathrm{Ext}_A^{3,p^nq+pq+1}(H^*M,\mathbb{Z}_p)=0\end{aligned}$$

induced by (1.1). It is easy to get that the second group is zero.

At last consider the following exact sequence

$$\begin{aligned}0=\mathrm{Ext}_A^{3,p^nq+(p+1)q+2}(H^*M,H^*M) &\xrightarrow{i'_*} \mathrm{Ext}_A^{3,p^nq+(p+1)q+2}(H^*K,H^*M)\\ &\xrightarrow{j'_*} \mathrm{Ext}_A^{3,p^nq+pq+1}(H^*M,H^*M)=0\end{aligned}$$

induced by (1.2). The desired result follows.

Proposition 2.6 *Let $p \geq 5, n \geq 3$. Then*

$$\mathrm{Ext}_A^{2,p^nq+(p+1)q+1}(H^*K,\mathbb{Z}_p) \cong \mathbb{Z}_p\{\beta_* i'_* i_*(h_n)\},$$

where

$$\beta_* : \mathrm{Ext}_A^{1,p^nq}(H^*K,\mathbb{Z}_p) \longrightarrow \mathrm{Ext}_A^{2,p^nq+(p+1)q+1}(H^*K,\mathbb{Z}_p)$$

is the connecting homomorphism induced by $\beta : \Sigma^{(p+1)q}K \longrightarrow K$.

Proof Look at the exact sequence

$$\begin{aligned} &\mathrm{Ext}_A^{1,p^nq+pq-1}(\mathbb{Z}_p,\mathbb{Z}_p) \xrightarrow{p_*} \mathrm{Ext}_A^{2,p^nq+pq}(\mathbb{Z}_p,\mathbb{Z}_p) \\ &\xrightarrow{i_*} \mathrm{Ext}_A^{2,p^nq+pq}(H^*M,\mathbb{Z}_p) \xrightarrow{j_*} \mathrm{Ext}_A^{2,p^nq+pq-1}(\mathbb{Z}_p,\mathbb{Z}_p) \end{aligned}$$

induced by (1.1). Since the first group and the fourth group are zero, the above i_* is an isomorphism. By the fact that

$$\mathrm{Ext}_A^{2,p^nq+pq}(\mathbb{Z}_p,\mathbb{Z}_p) \cong \mathbb{Z}_p\{h_1h_n\},$$

we can have

$$\mathrm{Ext}_A^{2,p^nq+pq}(H^*M,\mathbb{Z}_p) \cong \mathbb{Z}_p\{i_*(h_1h_n)\}.$$

At last observe the following exact sequence

$$\begin{aligned} &\mathrm{Ext}_A^{2,p^nq+(p+1)q+1}(H^*M,\mathbb{Z}_p) \xrightarrow{i'_*} \mathrm{Ext}_A^{2,p^nq+(p+1)q+1}(H^*K,\mathbb{Z}_p) \\ &\xrightarrow{j'_*} \mathrm{Ext}_A^{2,p^nq+pq}(H^*M,\mathbb{Z}_p) \xrightarrow{\alpha_*} \mathrm{Ext}_A^{3,p^nq+(p+1)q+1}(H^*M,\mathbb{Z}_p) \end{aligned}$$

induced by (1.2). Since the first group is zero by $\mathrm{Ext}_A^{2,p^nq+(p+1)q+r}(\mathbb{Z}_p,\mathbb{Z}_p) = 0$ for $r = 0,1$ (cf. [1]) and the fourth group is zero by $\mathrm{Ext}_A^{3,p^nq+(p+1)q+t}(\mathbb{Z}_p,\mathbb{Z}_p) = 0$ for $t = 0,1$ (cf. [15, Table 8.1]), the above j'_* is an isomorphism. Thus we can have

$$\mathrm{Ext}_A^{2,p^nq+(p+1)q+1}(H^*K,\mathbb{Z}_p) \cong \mathbb{Z}_p\{\Delta\}.$$

Here Δ is the unique generator of $\mathrm{Ext}_A^{2,p^nq+(p+1)q+1}(H^*K,\mathbb{Z}_p)$ and satisfies

$$j'_*(\Delta) = i_*(h_1h_n).$$

From [14, (5.4)], we have

$$j'\beta i' i \in \left[\sum\nolimits^{pq-1} S, M\right]$$

is represented by

$$i_*(h_1) \in \mathrm{Ext}_A^{1,pq}(H^*M,\mathbb{Z}_p)$$

in the Adams spectral sequence. It follows that

$$(j'\beta i' i)_*(h_n) = i_*(h_1h_n) = j'_*(\Delta).$$

Note the fact that j'_* is an isomorphism. It is easy to get that

$$\beta_* i'_* i_*(h_n) = \Delta.$$

Therefore this completes the proof of the proposition.

Proposition 2.7 *Let $p \geq 5, n \geq 3$. Then*

$$\mathrm{Ext}_A^{2,p^n q+(p+1)q+1}(H^*K, H^*M) \cong \mathbb{Z}_p\{\beta_* i'_*(\widetilde{h}_n)\},$$

where $\widetilde{h}_n \in \mathrm{Ext}_A^{1,p^n q}(H^*M, H^*M)$ is the unique generator of $\mathrm{Ext}_A^{1,p^n q}(H^*M, H^*M)$ and satisfies

$$i^*(\widetilde{h}_n) = i_*(h_n).$$

Proof Consider the exact sequence induced by (1.1):

$$\mathrm{Ext}_A^{2,p^n q+pq+1}(\mathbb{Z}_p, \mathbb{Z}_p) \xrightarrow{i_*} \mathrm{Ext}_A^{2,p^n q+pq+1}(H^*M, \mathbb{Z}_p)$$
$$\xrightarrow{j_*} \mathrm{Ext}_A^{2,p^n q+pq}(\mathbb{Z}_p, \mathbb{Z}_p) \xrightarrow{p_*} \mathrm{Ext}_A^{3,p^n q+pq+1}(\mathbb{Z}_p, \mathbb{Z}_p).$$

Since $\mathrm{Ext}_A^{2,p^n q+pq+1}(\mathbb{Z}_p, \mathbb{Z}_p) = 0$, image $i_* = 0$. Since $\mathrm{Ext}_A^{2,p^n q+pq}(\mathbb{Z}_p, \mathbb{Z}_p) = \mathbb{Z}_p\{h_1 h_n\}$ and $\mathrm{Ext}_A^{3,p^n q+pq+1}(\mathbb{Z}_p, \mathbb{Z}_p) = \mathbb{Z}_p\{a_0 h_1 h_n\}$ (cf. [15, Table 8.1]), the above p_* is an isomorphism, and then image $j_* = 0$. Thus it follows that

$$\mathrm{Ext}_A^{2,p^n q+pq+1}(H^*M, \mathbb{Z}_p) = 0.$$

Look at the exact sequence

$$\mathrm{Ext}_A^{2,p^n q+(p+1)q+2}(H^*M, \mathbb{Z}_p) \xrightarrow{i'_*} \mathrm{Ext}_A^{2,p^n q+(p+1)q+2}(H^*K, \mathbb{Z}_p) \xrightarrow{j'_*} \mathrm{Ext}_A^{2,p^n q+pq+1}(H^*M, \mathbb{Z}_p) = 0$$

induced by (1.2). Since we know that $\mathrm{Ext}_A^{2,p^n q+(p+1)q+2}(H^*M.\mathbb{Z}_p) = 0$ by the facts that $\mathrm{Ext}_A^{2,p^n q+(p+1)q+r}(\mathbb{Z}_p, \mathbb{Z}_p) = 0$ for $r = 1, 2$ and $\mathrm{Ext}_A^{2,p^n q+pq+1}(H^*M, \mathbb{Z}_p) = 0$, we can have

$$\mathrm{Ext}_A^{2,p^n q+(p+1)q+2}(H^*K, \mathbb{Z}_p) = 0.$$

Observe the following exact sequence

$$\mathrm{Ext}_A^{3,p^n q+(p+1)q+2}(H^*M, \mathbb{Z}_p) \xrightarrow{i'_*} \mathrm{Ext}_A^{3,p^n q+(p+1)q+2}(H^*K, \mathbb{Z}_p) \xrightarrow{j'_*} \mathrm{Ext}_A^{3,p^n q+pq+1}(H^*M, \mathbb{Z}_p)$$

induced by (1.2). From the proof of Proposition 2.5 we know that the first group and the third group are zero. Thus the middle group

$$\mathrm{Ext}_A^{3,p^n q+(p+1)q+2}(H^*K, \mathbb{Z}_p) = 0.$$

At last, look at the following exact sequence

$$\mathrm{Ext}_A^{2,p^n q+(p+1)q+2}(H^*K, \mathbb{Z}_p) \xrightarrow{j^*} \mathrm{Ext}_A^{2,p^n q+(p+1)q+1}(H^*K, H^*M)$$
$$\xrightarrow{i^*} \mathrm{Ext}_A^{2,p^n q+(p+1)q+1}(H^*K, \mathbb{Z}_p) \xrightarrow{p^*} \mathrm{Ext}_A^{3,p^n q+(p+1)q+2}(H^*K, \mathbb{Z}_p)$$

induced by (1.1). Since the first group and the fourth group are zero, the above i^* is an isomorphism. From Proposition 2.6, we have

$$\mathrm{Ext}_A^{2,p^nq+(p+1)q+1}(H^*K,\mathbb{Z}_p)\cong\mathbb{Z}_p\{\beta_*i'_*i_*(h_n)\}.$$

Thus we can easily have that there exists an element

$$\overline{\overline{\Delta}}\in\mathrm{Ext}_A^{2,p^nq+(p+1)q+1}(H^*K,H^*M)$$

such that

$$\mathrm{Ext}_A^{2,p^nq+(p+1)q+1}(H^*K,H^*M)\cong\mathbb{Z}_p\{\overline{\overline{\Delta}}\}$$

and

$$i^*(\overline{\overline{\Delta}})=\beta_*i'_*i_*(h_n).$$

Since $i^*(\widetilde{h}_n)=i_*(h_n)$, we have

$$i^*(\overline{\overline{\Delta}})=\beta_*i'_*i_*(h_n)=\beta_*i'_*i^*(\widetilde{h}_n)=i^*\beta_*i'_*(\widetilde{h}_n).$$

By the fact that i^* is an isomorphism, it follows that

$$\overline{\overline{\Delta}}=\beta_*i'_*(\widetilde{h}_n).$$

Thus this completes the proof of the proposition.

3 Proofs of Theorems 1.4 and 1.5

Let

$$\begin{array}{ccccccc}\cdots\xrightarrow{\overline{a}_2} & \Sigma^{-2}E_2 & \xrightarrow{\overline{a}_1} & \Sigma^{-1}E_1 & \xrightarrow{\overline{a}_0} & S \\ & \downarrow\overline{b}_2 & & \downarrow\overline{b}_1 & & \downarrow\overline{b}_0 \\ & \Sigma^{-2}KG_2 & & \Sigma^{-1}KG_1 & & KG_0=K\mathbb{Z}_p\end{array}\tag{3.1}$$

be the minimal Adams resolution of S satisfying the following.

(1) $E_s\xrightarrow{\overline{b}_s}KG_s\xrightarrow{\overline{c}_s}E_{s+1}\xrightarrow{\overline{a}_s}\Sigma E_s$ are cofibrations for all $s\geq 0$ which induce short exact sequences in $\mathbb{Z}_p$-cohomology;

(2) KG_s is a wedge sum of suspensions of Eilenberg-Maclane spectra of type $K\mathbb{Z}_p$;

(3) π_tKG_s are the $E_1^{s,t}$-terms,

$$(\overline{b}_s\overline{c}_{s-1})_*:\pi_tKG_{s-1}\longrightarrow\pi_tKG_s$$

are the $d_1^{s-1,t}$-differentials of the Adams spectral sequence and

$$\pi_tKG_s\cong\mathrm{Ext}_A^{s,t}(\mathbb{Z}_p,\mathbb{Z}_p)\quad\text{(cf. [16, p. 180]).}$$

Then

$$
\begin{array}{ccccccc}
\cdots \xrightarrow{\overline{a}_2\wedge 1_W} & \Sigma^{-2}E_2\wedge W & \xrightarrow{\overline{a}_1\wedge 1_W} & \Sigma^{-1}E_1\wedge W & \xrightarrow{\overline{a}_0\wedge 1_W} & W & \\
& \downarrow \overline{b}_2\wedge 1_W & & \downarrow \overline{b}_1\wedge 1_W & & \downarrow \overline{b}_0\wedge 1_W & \quad (3.2)\\
& \Sigma^{-2}KG_2\wedge W & & \Sigma^{-1}KG_1\wedge W & & KG_0\wedge W &
\end{array}
$$

is an Adams resolution of arbitrary finite spectrum W.

From [17, pp. 204–206], the Moore spectrum M is a commutative ring spectrum with multiplication $m_M : M\wedge M \longrightarrow M$ and there is $\overline{m}_M : \Sigma M \longrightarrow M\wedge M$ such that

$$
\begin{aligned}
&m_M(i\wedge 1_M) = 1_M, && (j\wedge 1_M)\overline{m}_M = 1_M,\\
&m_M\overline{m}_M = 0, && (i\wedge 1_M)m_M + \overline{m}_M(j\wedge 1_M) = 1_{M\wedge M},\\
&m_M T = -m_M, && T\overline{m}_M = \overline{m}_M,\\
&m_M(1_M\wedge i) = -1_M, && (1_M\wedge j)\overline{m}_M = 1_M,
\end{aligned}
$$

where $T : M\wedge M \longrightarrow M\wedge M$ is the switching map.

Definition 3.1 (cf. [17]) *A spectrum X is called an M-module spectrum if $p\wedge 1_X = 0$.*

Consequently, the cofibration

$$X \xrightarrow{p\wedge 1_X} X \xrightarrow{i\wedge 1_X} M\wedge X \xrightarrow{j\wedge 1_X} \Sigma X$$

split, i.e., there is a homotopy equivalence

$$M\wedge X = X\vee \Sigma X$$

and there are maps

$$m_X : M\wedge X \longrightarrow X, \quad \overline{m}_X : \Sigma X \longrightarrow M\wedge X$$

satisfying

$$
\begin{aligned}
&m_X(i\wedge 1_X) = 1_X, \quad (j\wedge 1_X)\overline{m}_X = 1_X, \quad m_X\overline{m}_X = 0,\\
&\overline{m}_X(j\wedge 1_X) + (i\wedge 1_X)m_X = 1_{M\wedge X}.
\end{aligned}
$$

Definition 3.2 (cf. [17]) *The M-module actions m_X, $\overline{m}_X$ are called associative if*

$$m_X(1_M\wedge m_X) = -m_X(m_X\wedge 1_X) \quad \textit{and} \quad (1_M\wedge \overline{m}_M)\overline{m}_X = (\overline{m}_M\wedge 1_X)\overline{m}_X.$$

Let X and X' be M-module spectra. Then we define a homomorphism

$$d : [\Sigma^s X', X] \longrightarrow [\Sigma^{s+1}X', X]$$

by

$$d(f) = m_X(1_M\wedge f)\overline{m}_{X'}$$

for $f\in[\Sigma^s X', X]$. This operation d is called a derivation (of maps between M-module spectra) which has the following properties:

Lemma 3.1 (cf. [17, Theorem 2.2]) (1) *d is a derivative and*

$$d(fg) = fd(g) + (-1)^{|g|}d(f)g$$

for $f \in [\Sigma^s X', X]$, $g \in [\Sigma^t X'', X']$, where X, X', X'' are M-module spectra.

(2) *Let W', W be arbitrary spectra and $h \in [\Sigma^r W', W]$. Then*

$$d(h \wedge f) = (-1)^{|h|} h \wedge d(f)$$

for $f \in [\Sigma^s X', X]$.

(3) *$d^2 = 0 : [\Sigma^s X', X] \longrightarrow [\Sigma^{s+2} X', X]$ for associative spectra X', X.*

From [17, (3.4)], K is an M-module spectrum, i.e., there are M-module actions $m_K : K \wedge M \longrightarrow K$, $\overline{m}_K : \Sigma K \longrightarrow K \wedge M$ satisfying

$$\begin{aligned} &m_K(1_K \wedge i) = 1_K, \quad && (1_K \wedge j)\overline{m}_K = 1_K, \\ &m_K \overline{m}_K = 0, && (1_K \wedge i)m_K + (1_K \wedge j)\overline{m}_K = 1_{K \wedge M}. \end{aligned}$$

Moreover, from [17, (2.6)] and [17, (3.7)] we have

$$\begin{aligned} d(ij) &= -1_M, \quad & d(\alpha) &= 0, \\ d(i') &= 0, & d(j') &= 0, \\ d(\beta) &= 0. \end{aligned}$$

Remark 3.1 In this paper, all the notations are the same as those of [5].

Let L be the cofiber of $\alpha_1 = j\alpha i : \Sigma^{q-1} S \longrightarrow S$ and K' be the cofiber of $\alpha i : \Sigma^q S \longrightarrow M$ given by the following two cofibrations:

$$\Sigma^{q-1} S \xrightarrow{\alpha_1} S \xrightarrow{i''} L \xrightarrow{j''} \Sigma^q S \qquad \text{(cf. [5, (2.3)])}, \tag{3.3}$$

$$\Sigma^q S \xrightarrow{\alpha i} M \xrightarrow{\upsilon} K' \xrightarrow{y} \Sigma^{q+1} S \qquad \text{(cf. [5, (2.4)])}. \tag{3.4}$$

Let $\alpha' = \alpha_1 \wedge 1_K$. Consider the following two commutative diagrams of 3×3-Lemma in the stable homotopy category:

$$\begin{array}{ccccccccc}
\Sigma M & & \xrightarrow{\upsilon} & & \Sigma K' & & \xrightarrow{1_{K'} \wedge p} & & \Sigma K' \\
& \searrow (\upsilon \wedge 1_M)\overline{m}_M & & \nearrow 1_{K'} \wedge j & & \searrow y & & \nearrow z & \\
& & K' \wedge M & & & & \Sigma^{q+2} S & & \\
& \nearrow 1_{K'} \wedge i & & \searrow \pi & & \nearrow jj' & & \searrow \alpha i & \\
K' & & \xrightarrow{x} & & K & & \xrightarrow{j'\alpha'} & & \Sigma^2 M
\end{array}$$

and

$$\begin{array}{ccccccccc}
M & & \xrightarrow{(i'' \wedge 1_K)i'} & & L \wedge K & & \xrightarrow{j'' \wedge 1_K} & & \Sigma^q K \\
& \searrow i' & & \nearrow i'' \wedge 1_K & & \searrow \overline{r} & & \nearrow \pi & \\
& & K & & & & \Sigma^q K' \wedge M & & \\
& \nearrow \alpha' & & \searrow j' & & \nearrow (\upsilon \wedge 1_M)\overline{m}_M & & \searrow \varepsilon & \\
\Sigma^{q-1} K & & \xrightarrow{j'\alpha'} & & \Sigma^{q+1} M & & \xrightarrow{\alpha} & & \Sigma M
\end{array}$$

By the above two commutative diagrams of 3×3-Lemma in the stable homotopy category, we easily have the following two lemmas.

Lemma 3.2 *There exist three cofibrations*

$$K' \xrightarrow{x} K \xrightarrow{jj'} \Sigma^{q+2} S \xrightarrow{z} \Sigma K', \tag{3.5}$$

$$\Sigma^{-1} K \xrightarrow{j'\alpha'} \Sigma M \xrightarrow{(\upsilon \wedge 1_M)\overline{m}_M} K' \wedge M \xrightarrow{\pi} K, \tag{3.6}$$

$$M \xrightarrow{(i'' \wedge 1_K)i'} L \wedge K \xrightarrow{\overline{r}} \Sigma^q K' \wedge M \xrightarrow{\varepsilon} \Sigma M. \tag{3.7}$$

Lemma 3.3 *There exist the following relations:*

$$\varepsilon(\upsilon \wedge 1_M)\overline{m}_M = \alpha, \quad \overline{r}(i'' \wedge 1_K) = (\upsilon \wedge 1_M)\overline{m}_M j',$$
$$\pi\overline{r} = j'' \wedge 1_K, \qquad \varepsilon(1_{K'} \wedge i)\upsilon j' = -2j'\alpha'.$$

From [18, p. 434], there are $\overline{\triangle} \in [\Sigma^{-1-1} L \wedge K, K]$ and $\widetilde{\Delta} \in [\Sigma^{-1} K, L \wedge K]$ satisfying

$$\overline{\Delta}(i'' \wedge 1_K) = (j'' \wedge 1_K)\widetilde{\Delta} = i'j' \in [\Sigma^{-q-1} K, K], \quad jj'\overline{\Delta} = 0.$$

From [15, p. 484], there is $\overline{\Delta}_{K'} \in [\Sigma^{-q-1} L \wedge K, K']$ satisfying

$$\overline{\Delta}_{K'}(i'' \wedge 1_K) = \upsilon j' \in [\Sigma^{-q-1} K, K'], \quad \overline{\Delta}(i'' \wedge 1_K) = (j'' \wedge 1_K)\widetilde{\Delta} = i'j'.$$

Lemma 3.4 $\overline{\Delta}_{K'} = (1_{K'} \wedge j)\overline{r}$.

Proof From Lemma 3.3 we have

$$(1_{K'} \wedge j)\overline{r}(i'' \wedge 1_K) = (1'_K \wedge j)(\upsilon \wedge 1_M)\overline{m}_M j' = (\upsilon \wedge 1_{S^0})(1_M \wedge j)\overline{m}_M j' = \upsilon j' = \overline{\Delta}_{K'}(i'' \wedge 1_K),$$

which shows that

$$(1_{K'} \wedge j)\overline{r} = \overline{\Delta}_{K'} + \overline{g}(j'' \wedge 1_K)$$

for some $\overline{g} \in [K, \Sigma K']$.

Consider the exact sequence induced by (3.4)

$$[K, \Sigma^{q+1} S] \xrightarrow{(\alpha i)_*} [K, \Sigma M] \xrightarrow{\upsilon_*} [K, \Sigma K'] \xrightarrow{y_*} [K, \Sigma^{q+2} S] \xrightarrow{(\alpha i)_*} [K, \Sigma^2 M].$$

From the proof of [5, Proposition 2.18], we know that

$$[K, \Sigma M] = 0.$$

It follows that image $\upsilon_* = 0$. By

$$[K, \Sigma^{q+2} S] \cong \mathbb{Z}_p\{jj'\},$$

we have

$$(\alpha i)_*(jj') = \alpha i j j' \neq 0.$$

Thus we have image $y_* = 0$. By image $\upsilon_* = 0$ and image $y_* = 0$, we obtain

$$[K, \Sigma K'] = 0.$$

Then we have

$$(1_{K'} \wedge j)\overline{r} = \overline{\Delta}_{K'}.$$

Lemma 3.5 (cf. [5, Lemma 3.3 and (3.4)]) *Let $p \geq 5, n \geq 3$. Then there exists an element*

$$\eta'_{n,2} \in [\Sigma^{p^n q+q} K, E_2 \wedge K]$$

such that

$$(\overline{b}_2 \wedge 1_K)\eta'_{n,2} = h_0 h_n \wedge 1_K \in [\Sigma^{p^n q+q} K, KG_2 \wedge K], \quad (1_{E_2} \wedge \alpha')\eta'_{n,2} = 0,$$

where $h_0 h_n \in \pi_{p^n q+q} KG_2 \cong \mathrm{Ext}_A^{2,p^n q+q}(\mathbb{Z}_p, \mathbb{Z}_p)$ and $\alpha' = j\alpha i \wedge 1_K \in [\Sigma^{q-1} K, K]$. There also exists an element

$$f_2 \in [\Sigma^{p^n q+(p+2)q+3} M, E_5 \wedge L \wedge K]$$

such that

$$(1_{E_2} \wedge (i'' \wedge 1_K)\beta)\eta'{}_{n,2} i' = (\overline{a}_2 \overline{a}_3 \overline{a}_4 \wedge 1_{L\wedge K}) f_2.$$

Corollary 3.1 *For $f_2 \in [\Sigma^{p^n q+(p+2)q+3} M, E_5 \wedge L \wedge K]$ which is given in Lemma 3.5, we have*

$$(1_{E_4} \wedge \varepsilon(1_{K'} \wedge ij))(\overline{a}_4 \wedge 1_{K'\wedge M})(1_{E_5} \wedge \overline{r}) d(f_2 ij) = 0. \tag{3.8}$$

Proof From [5], we have (cf. [5, (3.6)])

$$(\overline{a}_4 \wedge 1_M)(1_{E_5} \wedge \varepsilon(1_{K'} \wedge i)\overline{\Delta}_{K'}) d(f_2 ij) = 0.$$

Here f_2 is given in [5, (3.4)].

By [17, (1.7)], we have

$$(\overline{a}_4 \wedge 1_M)(1_{E_5} \wedge \varepsilon(1_{K'} \wedge i))(1_{E_5} \wedge \overline{\Delta}_{K'}) d(f_2 ij) = 0.$$

By Lemma 3.4, we have

$$(\overline{a}_4 \wedge 1_M)(1_{E_5} \wedge \varepsilon(1_{K'} \wedge i))(1_{E_5} \wedge (1_{K'} \wedge j)\overline{r}) d(f_2 ij) = 0.$$

By [17, (1.7)], it follows that

$$(\overline{a}_4 \wedge 1_M)(1_{E_5} \wedge \varepsilon(1_{K'} \wedge i))(1_{E_5} \wedge (1_{K'} \wedge j))(1_{E_5} \wedge \overline{r}) d(f_2 ij) = 0.$$

Thus

$$(\overline{a}_4 \wedge 1_M)(1_{E_5} \wedge \varepsilon(1_{K'} \wedge ij))(1_{E_5} \wedge \overline{r}) d(f_2 ij) = 0.$$

By [17, (1.7)], the corollary follows.

Let W be the cofibre of $\varepsilon(1_{K'} \wedge ij) : \Sigma^{q-2} K' \wedge M \longrightarrow M$ given by the cofibration

$$\Sigma^{q-2} K' \wedge M \xrightarrow{\varepsilon(1_{K'} \wedge ij)} M \xrightarrow{w_4} W \xrightarrow{u_4} \Sigma^{q-1} K' \wedge M. \tag{3.9}$$

Lemma 3.6 *There exists an element*

$$f' \in [\Sigma^{p^n q+(p+2)q+1} M, E_4 \wedge W]$$

which satisfies

$$(\overline{a}_2\overline{a}_3 \wedge 1_K)(1_{E_4} \wedge \pi u_4)f' = 0.$$

Proof By (3.8) and (3.9), we have

$$(\overline{a}_4 \wedge 1_{K'\wedge M})(1_{E_5} \wedge \overline{r})d(f_2 ij) = (1_{E_4} \wedge u_4)f' \tag{3.10}$$

with $f' \in [\Sigma^{p^n q+(p+2)q+1} M, E_4 \wedge W]$ and by composing $(\overline{a}_2\overline{a}_3 \wedge 1_K)(1_{E_4} \wedge \pi)$ on (3.10) we have

$$(\overline{a}_2\overline{a}_3 \wedge 1_K)(1_{E_4} \wedge \pi u_4)f' = (\overline{a}_2\overline{a}_3\overline{a}_4 \wedge 1_K)(1_{E_5} \wedge \pi\overline{r})d(f_2 ij). \tag{3.11}$$

By composing ij on [5, (3.4)], we have

$$(1_{E_2} \wedge (i'' \wedge 1_K)\beta)\eta'_{n,2} i' ij = (\overline{a}_2\overline{a}_3\overline{a}_4 \wedge 1_{L\wedge K}) f_2 ij \tag{3.12}$$

with $\eta'_{n,2} \in [\Sigma^{p^n q+q} K, E_2 \wedge K]$.

Note that $d(1_K) = 0$ and $d(\beta) = 0$. Then by applying the derivation d on (3.12) we have

$$(1_{E_2} \wedge (i'' \wedge 1_K)\beta) d(\eta'_{n,2} i' ij) = (\overline{a}_2\overline{a}_3\overline{a}_4 \wedge 1_{L\wedge K}) d(f_2 ij). \tag{3.13}$$

Note that $\pi\overline{r} = j'' \wedge 1_K$. By composing $(1_{E_2} \wedge \pi\overline{r})$ on (3.13) we have

$$(\overline{a}_2\overline{a}_3\overline{a}_4 \wedge 1_K)(1_{E_5} \wedge \pi\overline{r})d(f_2 ij) = 0 \tag{3.14}$$

and by (3.11), (3.14) we get

$$(\overline{a}_2\overline{a}_3 \wedge 1_K)(1_{E_4} \wedge \pi u_4)f' = 0. \tag{3.15}$$

Thus the lemma is proved.

Let U be the cofibre of $\pi u_4 : W \longrightarrow \Sigma^{q-1} K$ given by the cofibration

$$W \xrightarrow{\pi u_4} \Sigma^{q-1} K \xrightarrow{w_5} U \xrightarrow{u_5} \Sigma W. \tag{3.16}$$

Lemma 3.7 *w_5 induces zero homomorphism in $\mathbb{Z}_p$-cohomology.*

Proof Consider the following homomorphism induced by w_5:

$$w_5^* : H^* U \longrightarrow H^{*+q-1} K.$$

From the cellar structures of U and K, we can have that

$$H^t K = \begin{cases} \mathbb{Z}_p, & t = 0, 1, q+1, q+2, \\ 0, & \text{others}, \end{cases}$$

and the top cell of U has degree $2q+1$. It easily follows that w_5^* must be a zero homomorphism in $\mathbb{Z}_p$-cohomology.

Lemma 3.8 *There exist three homotopy elements*

$$f_2' \in [\Sigma^{p^n q+(p+2)q} M, E_2 \wedge U],$$
$$f_3' \in [\Sigma^{p^n q+(p+2)q+1} M, E_3 \wedge U],$$
$$g_2 \in [\Sigma^{p^n q+(p+2)q} M, KG_2 \wedge W]$$

such that

$$(\overline{a}_2\overline{a}_3 \wedge 1_W)f' = (1_{E_2} \wedge u_5)f_2', \quad f_2' = (\overline{a}_2 \wedge 1_U)f_3',$$
$$(1_{E_3} \wedge u_4)(\overline{a}_3 \wedge 1_W)f' = -(1_{E_3} \wedge u_4u_5)f_3' + (1_{E_3} \wedge u_4)(\overline{c}_2 \wedge 1_W)g_2.$$

Proof From (3.15) and (3.16), we have

$$(\overline{a}_2\overline{a}_3 \wedge 1_W)f' = (1_{E_2} \wedge u_5)f_2' \tag{3.17}$$

with $f_2' \in [\Sigma^{p^n q+(p+2)q} M, E_2 \wedge U]$.

By (3.17) and (3.2) we have

$$(\overline{b}_2 \wedge 1_W)(1_{E_2} \wedge u_5)f_2' = (\overline{b}_2 \wedge 1_W)(\overline{a}_2\overline{a}_3 \wedge 1_W)f' = 0.$$

Thus it follows that

$$(1_{KG_2} \wedge u_5)(\overline{b}_2 \wedge 1_U)f_2' = 0. \tag{3.18}$$

By (3.18), (3.16) and the fact that w_5 induces zero homomorphism in $\mathbb{Z}_p$-cohomology (cf. Lemma 3.7), we have

$$(\overline{b}_2 \wedge 1_U)f_2' = (1_{KG_2} \wedge w_5)g = 0 \tag{3.19}$$

with $g \in [\Sigma^{p^n q+(p+1)q+1} M, KG_2 \wedge K]$, so by (3.2) we obtain

$$f_2' = (\overline{a}_2 \wedge 1_U)f_3' \tag{3.20}$$

with $f_3' \in [\Sigma^{p^n q+(p+2)q+1} M, E_3 \wedge U]$. By [17, (1.7)], from (3.20) and (3.17) we have

$$(\overline{a}_2\overline{a}_3 \wedge 1_W)f' = -(\overline{a}_2 \wedge 1_W)(1_{E_3} \wedge u_5)f_3'.$$

Then we have

$$(\overline{a}_3 \wedge 1_W)f' = -(1_{E_3} \wedge u_5)f_3' + (\overline{c}_2 \wedge 1_W)g_2 \tag{3.21}$$

with $g_2 \in [\Sigma^{p^n q+(p+2)q} M, KG_2 \wedge W]$. By composing $(1_{E_3} \wedge u_4)$ on (3.21), we have

$$(1_{E_3} \wedge u_4)(\overline{a}_3 \wedge 1_W)f' = -(1_{E_3} \wedge u_4u_5)f_3' + (1_{E_3} \wedge u_4)(\overline{c}_2 \wedge 1_W)g_2. \tag{3.22}$$

We finish the proof of the lemma.

Lemma 3.9 *The cofibre of $\varepsilon(1_{K'}\wedge i)\upsilon:\Sigma^q M\longrightarrow \Sigma M$ is U given by the cofibration*

$$\Sigma^q M\xrightarrow{\varepsilon(1_{K'}\wedge i)\upsilon}\Sigma M\xrightarrow{w_6}U\xrightarrow{u_6}\Sigma^{q+1}M. \tag{3.23}$$

There exist two relations that

$$u_4u_5=(\upsilon\wedge 1_M)\overline{m}_M u_6,\quad \varepsilon(1_{K'}\wedge ij)(\upsilon\wedge 1_M)\overline{m}_M=\varepsilon(1_{K'}\wedge i)\upsilon.$$

Proof By the three cofibrations (3.6), (3.9), and (3.16), we can get the following commutative diagram (3.24) of 3×3-Lemma in stable homotopy category (cf. [19, pp. 292–293]).

$$\begin{array}{ccccccccc}
W & & \xrightarrow{\pi u_4} & & \Sigma^{q-1}K & & \xrightarrow{j'\alpha'} & & \Sigma^{q+1}M \\
 & u_4\searrow & & \pi\nearrow & & \searrow w_5 & & \nearrow u_6 & & \searrow(\upsilon\wedge 1_M)\overline{m}_M \\
 & & \Sigma^{q-1}K'\wedge M & & & & U & & & & \Sigma^q K'\wedge M \\
 & (\upsilon\wedge 1_M)\overline{m}_M\nearrow & & \searrow\varepsilon(1_{K'}\wedge ij) & & \nearrow w_6 & & \searrow u_5 & & \nearrow u_4 \\
\Sigma^q M & & \xrightarrow{\varepsilon(1_{K'}\wedge i)\upsilon} & & \Sigma M & & \xrightarrow{w_4} & & \Sigma W
\end{array} \tag{3.24}$$

By the commutative diagram (3.24), Lemma 3.8 follows.

Lemma 3.10 *With notations as above, we have*

$$(\overline{a}_3\overline{a}_4\wedge 1_{K'\wedge M})(1_{E_5}\wedge\overline{r})d(f_2ij)=(1_{E_3}\wedge(\upsilon\wedge 1_M)\overline{m}_M u_6)f_3'-(\overline{c}_2\wedge 1_{K'\wedge M})(1_{KG_2}\wedge u_4)g_2.$$

Proof By (3.22), [17, (1.7)] and the relation $u_4u_5=(\upsilon\wedge 1_M)\overline{m}_M u_6$ (cf. Lemma 3.9), we have

$$(\overline{a}_3\wedge 1_{K'\wedge M})(1_{E_4}\wedge u_4)f'=(1_{E_3}\wedge(\upsilon\wedge 1_M)\overline{m}_M u_6)f_3'-(\overline{c}_2\wedge 1_{K'\wedge M})(1_{KG_2}\wedge u_4)g_2. \tag{3.25}$$

By composing $(\overline{a}_3\wedge 1_{K'\wedge M})$ on (3.10), we have

$$(\overline{a}_3\wedge 1_{K'\wedge M})(1_{E_4}\wedge u_4)f'=(\overline{a}_3\overline{a}_4\wedge 1_{K'\wedge M})(1_{E_5}\wedge\overline{r})d(f_2ij). \tag{3.26}$$

Combining (3.25) and (3.26) yields

$$\begin{aligned}&(\overline{a}_3\overline{a}_4\wedge 1_{K'\wedge M})(1_{E_5}\wedge\overline{r})d(f_2ij)\\ &=(1_{E_3}\wedge(\upsilon\wedge 1_M)\overline{m}_M u_6)f_3'-(\overline{c}_2\wedge 1_{K'\wedge M})(1_{KG_2}\wedge u_4)g_2.\end{aligned} \tag{3.27}$$

Thus we complete the proof of this lemma.

Lemma 3.11 *There exist two elements*

$$f_4'\in[\Sigma^{p^nq+(p+2)q+1}M,E_3\wedge K],\quad f_5'\in[\Sigma^{p^nq+(p+2)q+1}M,E_3\wedge K'\wedge M]$$

such that

$$(1_{E_3}\wedge u_6)f_3'=(1_{E_3}\wedge j')f_4',\quad f_4'=(1_{E_3}\wedge\pi)f_5'.$$

Proof By Lemma 3.3, we have

$$\alpha = \varepsilon(\upsilon \wedge 1_M)\overline{m}_M.$$

Then

$$\begin{aligned}(1_{E_3} \wedge \alpha u_6)f_3' &= (1_{E_3} \wedge \varepsilon(\upsilon \wedge 1_M)\overline{m}_M u_6)f_3' \quad (\text{by } u_4u_5 = (\upsilon \wedge 1_M)\overline{m}_M u_6)\\ &= (1_{E_3} \wedge \varepsilon u_4u_5)f_3'\\ &= (1_{E_3} \wedge \varepsilon)(1_{E_3} \wedge u_4u_5)f_3' \quad (\text{by } (3.22))\\ &= (1_{E_3} \wedge \varepsilon)[(1_{E_3} \wedge u_4)(\overline{c}_2 \wedge 1_W)g_2 - (1_{E_3} \wedge u_4)(\overline{a}_3 \wedge 1_W)f']\\ &= (1_{E_3} \wedge \varepsilon u_4)(\overline{c}_2 \wedge 1_W)g_2 - (1_{E_3} \wedge \varepsilon u_4)(\overline{a}_3 \wedge 1_W)f'\\ &= (\overline{c}_2 \wedge 1_M)(1_{KG_2} \wedge \varepsilon u_4)g_2 - (1_{E_3} \wedge \varepsilon u_4)(\overline{a}_3 \wedge 1_W)f' \quad (\text{by } 1_{KG_2} \wedge \varepsilon \simeq 0)\\ &= -(1_{E_3} \wedge \varepsilon u_4)(\overline{a}_3 \wedge 1_W)f'\\ &= (1_{E_3} \wedge \varepsilon)(\overline{a}_3 \wedge 1_{K' \wedge M})(1_{E_4} \wedge u_4)f' \quad (\text{by } (3.26))\\ &= (1_{E_3} \wedge \varepsilon)(\overline{a}_3 \wedge 1_{K' \wedge M})(\overline{a}_4 \wedge 1_{K' \wedge M})(1_{E_5} \wedge \overline{r})d(f_2 ij)\\ &= (\overline{a}_3\overline{a}_4 \wedge 1_M)(1_{E_5} \wedge \varepsilon\overline{r})d(f_2 ij) \quad (\text{by } (3.7))\\ &= 0.\end{aligned}$$

Hence, by (1.2) we have

$$(1_{E_3} \wedge u_6)f_3' = (1_{E_3} \wedge j')f_4' \tag{3.28}$$

with $f_4' \in [\Sigma^{p^n q+(p+2)q+1}M, E_3 \wedge K]$.

Similarly, by Lemma 3.3 we have

$$\varepsilon(1_{K'} \wedge i)\upsilon j' = -2j'\alpha'.$$

Then we have

$$\begin{aligned}-2(1_{E_3} \wedge j'\alpha')f_4' &= (1_{E_3} \wedge \varepsilon(1_{K'} \wedge i)\upsilon j')f_4'\\ &= (1_{E_3} \wedge \varepsilon(1_{K'} \wedge i)\upsilon)(1_{E_3} \wedge j')f_4' \quad (\text{by } (3.28))\\ &= (1_{E_3} \wedge \varepsilon(1_{K'} \wedge i)\upsilon)(1_{E_3} \wedge u_6)f_3' \quad (\text{by } (3.24))\\ &= (1_{E_3} \wedge \varepsilon(1_{K'} \wedge ij)(\upsilon \wedge 1_M)\overline{m}_M)(1_{E_3} \wedge u_6)f_3'\\ &= (1_{E_3} \wedge \varepsilon(1_{K'} \wedge ij))(1_{E_3} \wedge (\upsilon \wedge 1_M)\overline{m}_M u_6)f_3' \quad (\text{by } (3.24))\\ &= (1_{E_3} \wedge \varepsilon(1_{K'} \wedge ij))(1_{E_3} \wedge u_4u_5)f_3' \quad (\text{by } (3.9))\\ &= (1_{E_3} \wedge \varepsilon(1_{K'} \wedge ij)u_4u_5)f_3'\\ &= 0.\end{aligned}$$

Thus, by (3.6) we have

$$f_4' = (1_{E_3} \wedge \pi)f_5' \tag{3.29}$$

with $f_5' \in [\Sigma^{p^n q+(p+2)q+1}M, E_3 \wedge K' \wedge M]$. This completes the proof of Lemma 3.11.

Lemma 3.12 *For the above* $f_5' \in [\Sigma^{p^n q+(p+2)q+1}M, E_3 \wedge K' \wedge M]$, *we have*

$$(\overline{b}_3 \wedge 1_{K' \wedge M})f_5' = 0.$$

Proof The proof will be given later.

Now we give the proof of Theorem 1.5.

Proof of Theorem 1.5 From Lemma 3.12, we have

$$(\overline{b}_3 \wedge 1_{K' \wedge M})f_5' = 0. \tag{3.30}$$

By virtue of (3.2), we have

$$f_5' = (\overline{a}_3 \wedge 1_{K' \wedge M})f_6' \tag{3.31}$$

with $f_6' \in [\Sigma^{p^n q+(p+2)q+2}M, E_4 \wedge K' \wedge M]$. By (3.27) and (3.2), we have

$$\begin{aligned}
&(\overline{a}_2\overline{a}_3\overline{a}_4 \wedge 1_{K' \wedge M})(1_{E_5} \wedge \overline{r})d(f_2 ij) \\
&= (\overline{a}_2 \wedge 1_{K' \wedge M})(1_{E_3} \wedge (\upsilon \wedge 1_M)\overline{m}_M u_6)f_3' \\
&= (\overline{a}_2 \wedge 1_{K' \wedge M})(1_{E_3} \wedge (\upsilon \wedge 1_M)\overline{m}_M)(1_{E_3} \wedge u_6)f_3' \quad \text{(by (3.28))} \\
&= (\overline{a}_2 \wedge 1_{K' \wedge M})(1_{E_3} \wedge (\upsilon \wedge 1_M)\overline{m}_M)(1_{E_3} \wedge j')f_4' \quad \text{(by (3.29))} \\
&= (\overline{a}_2 \wedge 1_{K' \wedge M})(1_{E_3} \wedge (\upsilon \wedge 1_M)\overline{m}_M)(1_{E_3} \wedge j')(1_{E_3} \wedge \pi)f_5' \quad \text{(by (3.31))} \\
&= (\overline{a}_2 \wedge 1_{K' \wedge M})(1_{E_3} \wedge (\upsilon \wedge 1_M)\overline{m}_M)(1_{E_3} \wedge j')(1_{E_3} \wedge \pi)(1_{E_3} \wedge \pi)(\overline{a}_3 \wedge 1_{K' \wedge M})f_6' \\
&= (\overline{a}_2\,\overline{a}_3 \wedge 1_{K' \wedge M})(1_{E_4} \wedge (\upsilon \wedge 1_M)\overline{m}_M j')(1_{E_4} \wedge \pi)f_6'.
\end{aligned}$$

That is,

$$\begin{aligned}
&(\overline{a}_2\overline{a}_3\overline{a}_4 \wedge 1_{K' \wedge M})(1_{E_5} \wedge \overline{r})d(f_2 ij) \\
&= (\overline{a}_2\overline{a}_3 \wedge 1_{K' \wedge M})(1_{E_4} \wedge (\upsilon \wedge 1_M)\overline{m}_M j')(1_{E_4} \wedge \pi)f_6'.
\end{aligned} \tag{3.32}$$

From [5, Proposition 2.2], we have

$$[(\overline{b}_4 \wedge 1_K)(1_{E_4} \wedge \pi)f_6'] \in \mathrm{Ext}_A^{4,p^n q+(p+2)q+2}(H^*K, H^*M) = 0.$$

By (3.1), we know that the d_1-cycle $(\overline{b}_4 \wedge 1_K)(1_{E_4} \wedge \pi)f_6'$ is a d_1-boundary. It follows that

$$(\overline{b}_4 \wedge 1_K)(1_{E_4} \wedge \pi)f_6' = (\overline{b}_4 \wedge 1_K)(\overline{c}_3 \wedge 1_K)f_7'$$

for some $f_7' \in [\Sigma^{p^n q+(p+2)q+2}M, KG_3 \wedge K]$. Thus we have

$$(1_{E_4} \wedge \pi)f_6' = (\overline{c}_3 \wedge 1_K)f_7' + (\overline{a}_4 \wedge 1_K)f_8' \tag{3.33}$$

with $f_8' \in [\Sigma^{p^n q+(p+2)q+3}M, E_5 \wedge K]$. Then by (3.32), (3.33) and (3.2), we have

$$(\overline{a}_2\overline{a}_3\overline{a}_4 \wedge 1_{K' \wedge M})(1_{E_5} \wedge \overline{r})d(f_2 ij) = (\overline{a}_2\overline{a}_3\overline{a}_4 \wedge 1_{K' \wedge M})(1_{E_5} \wedge (\upsilon \wedge 1_M)\overline{m}_M j')f_8'. \tag{3.34}$$

Moreover, by composing $(1_{E_2} \wedge \overline{r})$ on (3.13), it is easy to get that

$$(\overline{a}_2\overline{a}_3\overline{a}_4 \wedge 1_{K' \wedge M})(1_{E_5} \wedge \overline{r})d(f_2 ij) = (1_{E_2} \wedge \overline{r}(i'' \wedge 1_K)\beta)d(\eta'_{n,2}i'ij). \tag{3.35}$$

Combining (3.34) and (3.35) yields

$$(1_{E_2} \wedge \overline{r}(i'' \wedge 1_K)\beta)d(\eta'_{n,2}i'ij) = (\overline{a}_2\overline{a}_3\overline{a}_4 \wedge 1_{K' \wedge M})(1_{E_5} \wedge (\upsilon \wedge 1_M)\overline{m}_M j')f'_8. \tag{3.36}$$

From Lemma 3.3, we have

$$\overline{r}(i'' \wedge 1_K) = (\upsilon \wedge 1_M)\overline{m}_M j'.$$

Then (3.36) can turn into

$$(1_{E_2} \wedge (\upsilon \wedge 1_M)\overline{m}_M j'\beta)d(\eta'_{n,2}i'ij) = (1_{E_2} \wedge (\upsilon \wedge 1_M)\overline{m}_M j')(\overline{a}_2\overline{a}_3\overline{a}_4 \wedge 1_K)f'_8. \tag{3.37}$$

By (3.37) and (3.6), we have

$$(1_{E_2} \wedge j'\beta)d(\eta'_{n,2}i'ij) = (1_{E_2} \wedge j')(\overline{a}_2\overline{a}_3\overline{a}_4 \wedge 1_K)f'_8 + (1_{E_2} \wedge j'\alpha')f'_9 \tag{3.38}$$

with $f'_9 \in [\Sigma^{p^nq+(p+1)q+1}M, E_2 \wedge K]$. From [5, p. 489], we know that the left-hand side of (3.38) has filtration 4. However, since the first term of the right-hand side of (3.38) has filtration ≥ 5, the second term of (3.38) must be of filtration 4. So f'_9 has filtration ≤ 3. Notice the facts that $\mathrm{Ext}_A^{3,p^nq+(p+1)q+2}(H^*K, H^*M) = 0$ (cf. Proposition 2.5) and $\mathrm{Ext}_A^{2,p^nq+(p+1)q+1}(H^*K, H^*M) \cong \mathbb{Z}_p\{\beta_* i'_*(\widetilde{h}_n)\}$ (cf. Proposition 2.7). Then we have

$$(\overline{b}_2 \wedge 1_K)f'_9 = (1_{KG_2} \wedge \beta)(1_{KG_2} \wedge i')(\widetilde{h}_n).$$

Let

$$\varrho_n = (\overline{a}_0\overline{a}_1 \wedge 1_K)f'_9.$$

Then ϱ_n is represented by

$$\beta_* i'_*(\widetilde{h}_n)$$

in the Adams spectral sequence. It follows that

$$\zeta_n = \varrho_n i$$

is represented by

$$i^*\beta_* i'_*(\widetilde{h}_n) = \beta_* i'_* i^*(\widetilde{h}_n) = \beta_* i'_* i_*(h_n) \neq 0 \in \mathrm{Ext}_A^{2,p^nq+(p+1)q+1}(H^*K, \mathbb{Z}_p)$$

in the Adams spectral sequence (cf. Proposition 2.6). Thus Theorem 1.5 is proved.

Proof of Lemma 3.12 We first recall three cofibrations given in [5]:

$$\Sigma^{-1}K \xrightarrow{\upsilon j'} \Sigma^q K' \xrightarrow{\overline{\psi}} K'_2 \xrightarrow{\overline{\rho}} K \qquad \text{(cf. [5, (2.5)])}, \tag{3.39}$$

$$\Sigma^{q-1}K'_1 \xrightarrow{\varepsilon(1_{K'_1} \wedge i)} M \xrightarrow{w_2} X \xrightarrow{u_2} \Sigma^q K' \quad \text{(cf. [5, (3.7)])}, \tag{3.40}$$

$$X \xrightarrow{\overline{\psi}u_2} K'_2 \xrightarrow{w_3} K' \wedge W \xrightarrow{u_3} \Sigma X \qquad \text{(cf. [5, (3.10)])} \tag{3.41}$$

with the relation

$$u_2u_3 = -\upsilon j'\pi.$$

By composing $(\overline{a}_2 \wedge 1_{K'\wedge M})$ on (3.27), we have

$$\begin{aligned}
&(\overline{a}_2\overline{a}_3\overline{a}_4 \wedge 1_{K'\wedge M})(1_{E_5} \wedge \overline{r})d(f_2ij)\\
&= (\overline{a}_2 \wedge 1_{K'\wedge M})(1_{E_3} \wedge (\upsilon \wedge 1_M)\overline{m}_M u_6)f_3' \quad \text{(by (3.28))}\\
&= (\overline{a}_2 \wedge 1_{K'\wedge M})(1_{E_3} \wedge (\upsilon \wedge 1_M)\overline{m}_M)(1_{E_3} \wedge j')f_4' \quad \text{(by (3.29))}\\
&= (\overline{a}_2 \wedge 1_{K'\wedge M})(1_{E_3} \wedge (\upsilon \wedge 1_M)\overline{m}_M)(1_{E_3} \wedge j')(1_{E_3} \wedge \pi)f_5'.
\end{aligned}$$

That is,

$$\begin{aligned}
&(\overline{a}_2\overline{a}_3\overline{a}_4 \wedge 1_{K'\wedge M})(1_{E_5} \wedge \overline{r})d(f_2ij)\\
&= (\overline{a}_2 \wedge 1_{K'\wedge M})(1_{E_3} \wedge (\upsilon \wedge 1_M)\overline{m}_M)(1_{E_3} \wedge j')(1_{E_3} \wedge \pi)f_5'.
\end{aligned} \tag{3.42}$$

By composing $(1_{E_2} \wedge (1_{K'} \wedge j))$ on (3.42), we have

$$\begin{aligned}
&(1_{E_2} \wedge (1_{K'} \wedge j))(\overline{a}_2\overline{a}_3\overline{a}_4 \wedge 1_{K'\wedge M})(1_{E_5} \wedge \overline{r})d(f_2ij)\\
&= (1_{E_2} \wedge (1_{K'} \wedge j))(\overline{a}_2 \wedge 1_{K'\wedge M})(1_{E_3} \wedge (\upsilon \wedge 1_M)\overline{m}_M)(1_{E_3} \wedge j')(1_{E_3} \wedge \pi)f_5'.
\end{aligned} \tag{3.43}$$

On the one hand, for the left-hand side of (3.43), we have

$$\begin{aligned}
&(1_{E_2} \wedge (1_{K'} \wedge j))(\overline{a}_2\overline{a}_3\overline{a}_4 \wedge 1_{K'\wedge M})(1_{E_5} \wedge \overline{r})d(f_2ij)\\
&= -(\overline{a}_2\overline{a}_3\overline{a}_4 \wedge 1_{K'})(1_{E_5} \wedge (1_{K'} \wedge j)\overline{r})d(f_2ij) \quad \text{(by Lemma 3.4)}\\
&= -(\overline{a}_2\overline{a}_3\overline{a}_4 \wedge 1_{K'})(1_{E_5} \wedge \overline{\Delta}_{K'})d(f_2ij).
\end{aligned}$$

On the other hand, for the right-hand side of (3.43) we have

$$\begin{aligned}
&(1_{E_2} \wedge (1_{K'} \wedge j))(\overline{a}_2 \wedge 1_{K'\wedge M})(1_{E_3} \wedge (\upsilon \wedge 1_M)\overline{m}_M)(1_{E_3} \wedge j')(1_{E_3} \wedge \pi)f_5'.\\
&= -(\overline{a}_2 \wedge 1_{K'})(1_{E_3} \wedge (1_{K'} \wedge j)(\upsilon \wedge 1_M)\overline{m}_M j'\pi)f_5'\\
&= -(\overline{a}_2 \wedge 1_{K'})(1_{E_3} \wedge (\upsilon \wedge 1_{S^0})(1_M \wedge j)\overline{m}_M j'\pi)f_5' \quad \text{(by } (1_M \wedge j)\overline{m}_M = 1_M)\\
&= -(\overline{a}_2 \wedge 1_{K'})(1_{E_3} \wedge \upsilon j'\pi)f_5' \quad \text{(by } u_2u_3 = -\upsilon j'\pi)\\
&= (\overline{a}_2 \wedge 1_{K'})(1_{E_3} \wedge u_2u_3)f_5'.
\end{aligned}$$

Thus we have

$$(\overline{a}_2\overline{a}_3\overline{a}_4 \wedge 1_{K'})(1_{E_5} \wedge \overline{\Delta}_{K'})d(f_2ij) = -(\overline{a}_2 \wedge 1_{K'})(1_{E_3} \wedge u_2u_3)f_5'. \tag{3.44}$$

Let X be the cofibre of $\varepsilon(1_{K'} \wedge i) : \Sigma^{q-1}K' \longrightarrow M$ given by the cofibration

$$\Sigma^{q-1}K' \xrightarrow{\varepsilon(1_{K'}\wedge i)} M \xrightarrow{w_2} X \xrightarrow{u_2} \Sigma^q K' \quad \text{(cf. [5, (3.7)])}.$$

It follows from [5, (3.7)] and [5, (3.6)] that

$$(\overline{a}_4 \wedge 1_{K'})(1_{E_5} \wedge \overline{\Delta}_{K'})d(f_2ij) = (1_{E_4} \wedge u_2)f_3$$

for some $f_3 \in [\Sigma^{p^n q+(p+2)q+1} M, E_4 \wedge X]$. By composing $(\overline{a}_2\overline{a}_3 \wedge 1_{K'})$ on [5, (3.8)], we have

$$(\overline{a}_2\overline{a}_3\overline{a}_4 \wedge 1_{K'})(1_{E_5} \wedge \overline{\Delta}_{K'})d(f_2 ij) = (\overline{a}_2\overline{a}_3 \wedge 1_{K'})(1_{E_4} \wedge u_2)f_3. \tag{3.45}$$

Combining (3.44) and (3.45) yields

$$(\overline{a}_2 \wedge 1_{K'})(1_{E_3} \wedge u_2 u_3)f_5' = -(\overline{a}_2\overline{a}_3 \wedge 1_{K'})(1_{E_4} \wedge u_2)f_3. \tag{3.46}$$

By [17, (1.7)], (3.46) can turn into

$$(1_{E_2} \wedge u_2)(\overline{a}_2 \wedge 1_X)(1_{E_3} \wedge u_3)f_5' = -(1_{E_2} \wedge u_2)(\overline{a}_2\overline{a}_3 \wedge 1_X)f_3. \tag{3.47}$$

From (3.47) and (3.40) we have

$$(\overline{a}_2 \wedge 1_X)(1_{E_3} \wedge u_3)f_5' = -(\overline{a}_2\overline{a}_3 \wedge 1_X)f_3 + (1_{E_2} \wedge w_2)\overline{f}_4 \tag{3.48}$$

with $\overline{f}_4 \in [\Sigma^{p^n q+(p+2)q-1} M, E_2 \wedge M]$. Note that

$$(\overline{b}_2 \wedge 1_M)\overline{f}_4 \in [\Sigma^{p^n q+(p+2)q-1} M, KG_2 \wedge M] = 0$$

by the exact sequence

$$[\Sigma^{p^n q+(p+2)q-1} M, KG_2] \xrightarrow{(1\wedge i)_*} [\Sigma^{p^n q+(p+2)q-1} M, KG_2 \wedge M] \xrightarrow{(1\wedge j)_*} [\Sigma^{p^n q+(p+2)q-2} M, KG_2]$$

induced by (1.1), where the first group and the last group are zero by the fact that

$$\pi_{p^n q+(p+2)q+r} KG_2 \cong \mathrm{Ext}_A^{2,p^n q+(p+2)q+r}(\mathbb{Z}_p, \mathbb{Z}_p) = 0$$

for $r = 0, -1, -2$ (cf. [1]). Hence, we can have

$$\overline{f}_4 = (\overline{a}_2 \wedge 1_M)\overline{f}_5$$

for some $\overline{f}_5 \in [\Sigma^{p^n q+(p+2)q} M, E_3 \wedge M]$. By (3.2) and (3.48), we have

$$(1_{E_3} \wedge u_3)f_5' = -(\overline{a}_3 \wedge 1_X)f_3 + (1_{E_3} \wedge w_2)\overline{f}_5' + (\overline{c}_2 \wedge 1_X)g_6 \tag{3.49}$$

with $g_6 \in [\Sigma^{p^n q+(p+2)q} M, KG_2 \wedge X]$. So we have

$$(\overline{b}_3 \wedge 1_X)(1_{E_3} \wedge u_3)f_5' = (\overline{b}_3 \wedge 1_X)(1_{E_3} \wedge w_2)\overline{f}_5 + (\overline{b}_3\overline{c}_2 \wedge 1_X)g_6. \tag{3.50}$$

From Proposition 2.3, we have

$$\mathrm{Ext}_A^{3,p^n q+(p+2)q}(H^*M, H^*M) \cong \mathbb{Z}_p\{i_* j_* \overline{\overline{h_n g_0}}, j^* i^* \overline{\overline{h_n g_0}}\,\}.$$

Thus it follows that

$$(\overline{b}_3 \wedge 1_M)\overline{f}_5 = \lambda_1 \overline{\overline{h_n g_0}} ij + \lambda_2 (1_{KG_3} \wedge ij)\overline{\overline{h_n g_0}}$$

for some $\lambda_1, \lambda_2 \in \mathbb{Z}_p$, where $\overline{\overline{h_n g_0}} \in [\Sigma^{p^n q+(p+2)q+1} M, KG_3 \wedge M]$. And so

$$0 = \lambda_1 (\overline{c}_3 \wedge 1_M)\overline{\overline{h_n g_0}} ij + \lambda_2 (\overline{c}_3 \wedge 1_M)(1_{KG_3} \wedge ij)\overline{\overline{h_n g_0}}\,.$$

By composing i on the above equality, we get

$$\lambda_2(\overline{c}_3 \wedge 1_M)(1_{KG_3} \wedge ij)\overline{\overline{\overline{h_n g_0}}}\, i = 0.$$

From Proposition 2.4, we see that

$$d_2(i^*(ij)_*\overline{\overline{\overline{h_n g_0}}}\,) = i^* d_2(i_* j_* \overline{\overline{\overline{h_n g_0}}}\,) \neq 0.$$

Then we get

$$(\overline{c}_3 \wedge 1_M)(1_{KG_3} \wedge ij)\overline{\overline{\overline{h_n g_0}}}\, i \neq 0.$$

Thus, we have

$$\lambda_2 = 0, \quad \lambda_1(\overline{c}_3 \wedge 1_M)\overline{\overline{\overline{h_n g_0}}}\, ij = 0.$$

Note that

$$d_2(j^* i^* \overline{\overline{\overline{h_n g_0}}}\,) \neq 0$$

by Proposition 2.4(2). It follows that

$$(\overline{c}_3 \wedge 1_M)\overline{\overline{\overline{h_n g_0}}}\, ij \neq 0.$$

Thus we have

$$\lambda_1 = 0.$$

From the above discussion, we know that

$$(\overline{b}_3 \wedge 1_M)\overline{f}_5 = 0.$$

Then (3.50) can turn into

$$(\overline{b}_3 \wedge 1_X)(1_{E_3} \wedge u_3) f_5' = (\overline{b}_3 \overline{c}_2 \wedge 1_X) g_6. \tag{3.51}$$

The argument of the proof from [5, (3.16)] to [5, p. 491] shows that $(\overline{b}_3 \wedge 1_X)(1_{E_3} \wedge u_3) f_6 = -(\overline{b}_3 \overline{c}_2 \wedge 1_X)\widetilde{l}_0$ in [5, (3.16)] implies $(\overline{b}_3 \wedge 1_{K' \wedge M}) f_6 = 0$. By a similar argument as in [5], we can also show that (3.51) implies that (3.30) holds.

Proof of Theorem 1.4 By Theorem 1.5, we get that

$$\beta_* {i'}_* i_*(h_n) \neq 0 \in \mathrm{Ext}_A^{2, p^n q + (p+1)q + 1}(H^* K, \mathbb{Z}_p)$$

is a permanent cycle in the Adams spectral sequence and converges to a nontrivial element $\zeta_n \in \pi_{p^n q + (p+1)q - 1} K$.

Consider the following composition of maps

$$\overline{f} : \Sigma^{p^n q + (p+1)q - 1} S \xrightarrow{\zeta_n} K \xrightarrow{jj'\beta} \Sigma^{-pq+2} S.$$

Since ζ_n is represented up to nonzero scalar by $\beta_* i'_* i_*(h_n) \in \mathrm{Ext}_A^{2,p^n q+(p+1)q+1}(H^*K, \mathbb{Z}_p)$ in the Adams spectral sequence, the above $\overline{f}$ is represented up to nonzero scalar by

$$\overline{c} = (jj'\beta)_* \beta_* i'_* i_*(h_n)$$

in the Adams spectral sequence.

Meanwhile, it is well-known that the β-element

$$\beta_2 = jj'\beta^2 i'i$$

is represented by

$$k_0 \in \mathrm{Ext}_A^{2,2pq+q}(\mathbb{Z}_p, \mathbb{Z}_p)$$

in the Adams spectral sequence. By the knowledge of Yoneda products we can see that $\overline{f}$ is represented (up to nonzero scalar) by

$$\overline{c} = k_0 h_n \neq 0 \in \mathrm{Ext}_A^{3,q(p^n+2p+1)}(\mathbb{Z}_p, \mathbb{Z}_p)$$

in the Adams spectral sequence (cf. [15, Table 8.1]).

Moreover, from [1] we know that

$$\mathrm{Ext}_A^{3-r,q(p^n+2p+1)-r+1}(\mathbb{Z}_p, \mathbb{Z}_p) = 0$$

for $r \geq 2$. Then we see that $k_0 h_n$ cannot be hit by any differential in the Adams spectral sequence, and so the corresponding homotopy element $\overline{f} \in \pi_* S$ is nontrivial and of order p. This finishes the proof of Theorem 1.4.

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References

[1] Liulevicius, A., The factorizations of cyclic reduced powers by secondary cohomology operations, *Mem. Amer. Math. Soc.*, **42**, 1962, 112.

[2] Cohen, R., Odd primary families in stable homotopy theory, *Mem. Amer. Math. Soc.*, **242**, 1981, viii+92.

[3] Lin, J. and Zheng, Q., A new family of filtration seven in the stable homotopy of spheres, *Hiroshima Math. J.*, **28**(2), 1998, 183–205.

[4] Lin, J., New families in the stable homotopy of spheres revisited, *Acta Math. Sin. Engl. Ser.*, **18**(1), 2002, 95–106.

[5] Lin, J., A new family of filtration three in the stable homotopy of spheres, *Hiroshima Math. J.*, **31**(3), 2001, 477–492.

[6] Liu, X., A nontrivial product in the stable homotopy groups of spheres, *Sci. China Ser. A*, **47**(6), 2004, 831–841.

[7] Liu, X., A new family of filtration $s+6$ in the stable homotopy groups of spheres, *Acta Math. Sci. Ser. B* (*Engl. Ed.*), **26**(2), 2006, 193–201.

[8] Liu, X., Non-triviality of two homotopy elements in $\pi_* S$, *J. Korean Math. Soc.*, **43**(4), 2006, 783–801.

[9] Liu, X. and Wang, X., The convergence of $\widetilde{\gamma}_s(b_0 h_n - h_1 b_{n-1})$, *Chin. Ann. Math.*, **27B**(3), 2006, 329–340.

[10] Liu, X., A nontrivial product of filtration $s+5$ in the stable homotopy of spheres, *Acta Math. Sin. Engl. Ser.*, **23**(3), 2007, 385–392.

[11] Liu, X., On the convergnce of products $\widetilde{\gamma}_s h_1 h_n$ in the Adams spectral sequence, *Acta Math. Sin. Engl. Ser.*, **23**(6), 2007, 1025–1032.

[12] Liu, X. and Wang, X., A four-filtrated May spectral sequence and its applications, *Acta Math. Sin. Engl. Ser.*, in press.

[13] Wang, X., Ordered cochain complex and $H^{4,*}(A_p)$ for $p \geq 5$, *Beijing Math.*, **1**, 1995, 80–99.

[14] Toda, H., On spectra realizing exterior part of the Steenrod algebra, *Topology*, **10**, 1971, 53–65.

[15] Aikawa, T., 3-dimensional cohomology of the mod p Steenrod algebra, *Math. Scand.*, **47**(1), 1980, 91–115.

[16] Cohen, R. and Goerss, P., Secondary cohomology operations that detect homotopy classes, *Topology*, **23**, 1984, 177–194.

[17] Toda, H., Algebra of stable homotopy of $\mathbb{Z}_p$-spaces and applications, *J. Math. Kyoto Univ.*, **11**, 1971, 197–251.

[18] Oka, S., Multiplicative structure of finite ring spectra and stable homotopy of spheres, Algebraic Topology, Aarhus 1982 (Aarhus, 1982), 418–441, *Lecture Notes in Math.*, **1051**, Springer, Berlin, 1984.

[19] Thomas, E. and Zahler, R., Generalized higher order cohomology operations and stable homotopy groups of spheres, *Advances in Math.*, **20**(3), 1976, 287–328.

[20] Liu, X., A new infinite family $\alpha_1\beta_2\gamma_s$ in π_*S, *JP J. Geom. Topol.*, **7**(1), 2007, 51–63.

[21] Oka, S., A new family in the stable homotopy groups of spheres, *Hiroshima Math. J.*, **5**, 1975, 87–114.