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Porosity of Self-affine Sets**

Lifeng XI*

Abstract In this paper, it is proved that any self-affine set satisfying the strong separation condition is uniformly porous. The author constructs a self-affine set which is not porous, although the open set condition holds. Besides, the author also gives a C^1 iterated function system such that its invariant set is not porous.

Keywords Porosity, Self-affine set, Open set condition
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1 Introduction

Given a metric space X, denote by $B(a,r)$ the open ball centered at $a \in X$ with radius r. Let $M \subset X$, $x \in X$ and $R > 0$. Set $p(x,R,M) = \sup\{r \geq 0 \mid \text{there exists } z \in X \text{ with } B(z,r) \subset B(x,R)\backslash M\}$ and

$$\underline{p}(M,x) := \liminf_{R\to 0+} \frac{p(x,R,M)}{R} \tag{1.1}$$

Definition 1.1 *A subset M $(\subset X)$ is said to be porous if $\underline{p}(M,x) > 0$ for any $x \in M$. Furthermore, M is said to be uniformly porous, if $\inf\limits_{x\in M} \underline{p}(M,x) > 0$.*

Remark 1.1 A porous set is always nowhere dense. In particular, any porous subset of Euclidean space has zero Lebesgue measure. Notice that the porosity and uniform perfectness (see [9, 11]) are invariants under the bi-Lipschitz mapping.

Porosity in $\mathbb{R}$ was used (under another form) already by A. Enjoy [2] in 1915. Probably the theory of σ-porous sets was started in 1967 by Solvendo [3] who applied σ-porous sets in the theory of boundary behavior of functions and who used for the first time the term porous set. In the differentiation theory σ-porous sets were used for the first time in 1978 (see [1]).

[7] proved that any C^{1+a} $(a > 0)$ self-conformal set satisfying the open set condition is uniformly porous. As a result, the self-similar set satisfying the open set condition is uniformly porous.

Remark 1.2 When the open set condition does not hold, the porosity of self-similar set maybe fails. Let $f_1(x) = \frac{x}{3}$, $f_2(x) = \frac{x+1}{3}$, $f_3(x) = \frac{x+u}{3}$, where $u = \sum\limits_{i=1}^{+\infty} \frac{1}{10^{i!}}$. Suppose that S_u is the invariant set of $\{f_i\}_{i=1}^3$. Then $\dim_H S_u = 1$ and $\mathcal{H}^1(S_u) = 0$ (see [11]). By Schief's Theorem (see [10]), S_u does not satisfy the open set condition since $\mathcal{H}^1(S_u) = 0$. Notice that

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*Institute of Mathematics, Zhejiang Wanli University, Ningbo 315100, Zhejiang, China.
E-mail: xilf@zwu.edu.cn
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S_u is not porous, because $\dim_H S_u = 1$ and the Hausdorff dimension of a porous set in $\mathbb{R}$ is strictly smaller than 1 (see [8]).

The motivation of this paper is to study the following questions:
(1) How is the porosity for self-affine sets?
(2) Can we get the uniform porosity for C^1 IFS?

The family $\{A_i\}_{i=1}^m$ of affine mappings from $\mathbb{R}^n$ to $\mathbb{R}^n$ is said to be non-degenerate and contractive, if there is a norm $\|\cdot\|$ of $\mathbb{R}^n$ such that

$$d_i\|x-y\| \le \|A_i(x)-A_i(y)\| \le c_i\|x-y\| \quad \text{for all } x, y \in \mathbb{R}^n,$$

where $0 < d_i < c_i < 1$ for $i = 1, \cdots, m$.

Given contractive non-degenerate affine mappings $\{A_i\}_{i=1}^m$, let $E = \bigcup\limits_{i=1}^{m} A_i(E)$ be the corresponding self-affine set. We say E satisfies the strong separation condition, if $A_i(E) \cap A_j(E) = \emptyset$ for any $i \neq j$. We say E satisfies the open set condition, if there is a non-empty open set U such that $\bigcup\limits_{i=1}^{m} A_i(U) \subset U$ and $A_i(U) \cap A_j(U) = \emptyset$ for any $i \neq j$.

Our main results can be stated as follows.

Theorem 1.1 *Any self-affine set satisfying the strong separation condition is uniformly porous.*

The porosity of self-affine set maybe fail although the open set condition holds as in Theorem 1.2. Given integers a, b with $a < b$, let $C_{a,b} = \{a, a+1, \cdots, b-1, b\}$.

Theorem 1.2 *Suppose that $k_1, k_2 \in \mathbb{N}$ with $k_2 > k_1 \ge 5$, and Γ is a subset of $\mathbb{Z} \times \mathbb{Z}$ with $[C_{0,(k_1-1)} \times C_{0,(k_2-1)}] \backslash [C_{2,(k_1-3)} \times C_{2,(k_2-3)}] \subset \Gamma \subsetneqq [C_{0,(k_1-1)} \times C_{0,(k_2-1)}]$. Let*

$$A_i \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{k_1} & 0 \\ 0 & \frac{1}{k_2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \frac{a_i}{k_1} \\ \frac{b_i}{k_2} \end{pmatrix} \quad \text{for any } (a_i, b_i) \in \Gamma.$$

Then the self-affine set $F = \bigcup\limits_{i=1}^{\#\Gamma} A_i(F)$ is not porous.

For C^1 IFS, we can not get the porosity as in Theorem 1.3.

Theorem 1.3 *There are C^1 injections $g_1, g_2 : [0,1] \to [0,1]$ in $\mathbb{R}$ with*

$$g_1([0,1]) \cap g_2([0,1]) = \emptyset \quad \text{and} \quad \bigcup_{i=1}^{2} g_i([0,1]) \subset [0,1],$$

such that the invariant set $H = g_1(H) \cup g_2(H)$ $(\subset \mathbb{R})$ has positive Lebesgue measure and thus H is not porous.

We organize the paper as follows. Section 2 is on the porosity of self-affine set. In Section 3, an invariant set of C^1 IFS is constructed to prove Theorem 1.3.

2 Self-affine Set

Proof of Theorem 1.1 Given a norm $\|\cdot\|$ of $\mathbb{R}^n$, let $\{A_i : \mathbb{R}^n \to \mathbb{R}^n\}_i$ be contractive non-degenerate affine mappings satisfying $\|A_i(x) - A_i(y)\| \le c_i\|x-y\|$, where $c_i \in (0,1)$,

$i = 1, \cdots, m$. Given i, let $A_i(x) = B_i(x) + b_i$, where B_i is linear and $b_i \in \mathbb{R}^n$. Write $A_{i_1 \cdots i_k} = A_{i_1} \circ \cdots \circ A_{i_k}$ for any $i_1 \cdots i_k \in \bigcup_{t=1}^{\infty} \{1, \cdots, m\}^t$, and $B_{i_1 \cdots i_k} = B_{i_1} \circ \cdots \circ B_{i_k}$. Then

$$A_{i_1 \cdots i_k}(x) = B_{i_1 \cdots i_k}(x) + (b_{i_1} + B_{i_1} b_{i_2} + \cdots + B_{i_1 \cdots i_{k-1}} b_{i_k}).$$

Let $|\cdot|$ be the Euclidean metric on $\mathbb{R}^n$. Then there is a constant $c > 0$ such that $(\sqrt{c})^{-1} |x| \leq \|x\| \leq \sqrt{c}|x|$ for all x. That means for all $x, y \in \mathbb{R}^n$,

$$|A_{i_1 \cdots i_k}(x) - A_{i_1 \cdots i_k}(y)| \leq c\Big(\prod_{t=1}^{k} c_{i_t}\Big)|x - y|. \tag{2.1}$$

Let $E = \bigcup_{i=1}^{m} A_i(E)$ be the self-affine set with $A_i(E) \cap A_j(E) = \emptyset$ for any $i \neq j$. Given subsets $C, D \subset \mathbb{R}^n$, let $d(C, D) = \inf\{|x - y| : x \in C, y \in D\}$. Let $\lambda = \min_{i \neq j} d(A_i(E), A_j(E))c^{-1} > 0$.

Lemma 2.1 *There exists a constant $\eta_0 \in (0, \frac{1}{2}]$ such that for any $x \in E$ and $v \in \mathbb{R}^n$ with $|v| = 1$,*

$$\Big\{t : |t| \leq \frac{\lambda}{2} \text{ and } B(x + tv, \eta_0 \lambda) \subset \mathbb{R}^n \backslash E\Big\} \neq \emptyset.$$

Proof Since E is totally disconnected, the segment $\{x + tv : |t| \leq \frac{\lambda}{2}\}$ has non-empty intersection with the open set $\mathbb{R}^n \backslash E$. Let $\eta(x, v) = \sup\{\eta \in (0, \frac{1}{2}] :$ there exists $t \in [-\frac{\lambda}{2}, \frac{\lambda}{2}]$ such that $B(x + tv, \eta\lambda) \subset \mathbb{R}^n \backslash E\} > 0$. Because $\mathbb{R}^n \backslash E$ is open, the function $\eta(x, v)$ is lower continuous on the compact set $E \times [-\frac{\lambda}{2}, \frac{\lambda}{2}]$. Therefore we let $\eta_0 = \min_{(x,v) \in E \times [-\frac{\lambda}{2}, \frac{\lambda}{2}]} \eta(x, v)$. Then $\eta_0 > 0$.

For any linear mapping $L : \mathbb{R}^n \to \mathbb{R}^n$, write

$$\alpha(L) = \inf_{|x|=1} |L(x)| \quad \text{and} \quad \beta(L) = \sup_{|x|=1} |L(x)|.$$

Then we have the following lemma.

Lemma 2.2 *Given a point $x \in E$, there is an infinite sequence $i_1 \cdots i_k \cdots$ such that $\{x\} = \bigcap_{k=1}^{\infty} A_{i_1 \cdots i_k}(E)$. Let $r_k = \alpha(B_{i_1 \cdots i_k})\lambda$. Then we have*

$$B(x, r_k) \cap [E \backslash A_{i_1 \cdots i_k}(E)] = \emptyset \quad \text{and} \quad \lim_{k \to \infty} r_k = 0, \quad \inf_k \frac{r_{k+1}}{r_k} \geq \min_i \alpha(B_i).$$

Proof We conclude that for any t with $t \leq k$,

$$\alpha(B_{i_1 \cdots i_{t-1}}) \geq c\alpha(B_{i_1 \cdots i_k}). \tag{2.2}$$

In fact, it follows from (2.1) that $\{B_i\}_i$ are linear mappings with $|B_{i_t \cdots i_k}(x)| \leq c(c_{i_t} \cdots c_{i_k})|x| \leq c|x|$, i.e., $|B^{-1}_{i_t \cdots i_k}(y)| \geq c^{-1}|y|$ for all $y \in \mathbb{R}^n$. Take $x_0 \in \mathbb{R}^n$ such that $|x_0| = 1$ and $|B_{i_1 \cdots i_{t-1}}(x_0)| = \alpha(A_{i_1 \cdots i_{t-1}})$. Let $y_0 = B^{-1}_{i_t \cdots i_k}(x_0)$, where $|y_0| \geq c^{-1}|x_0| = c^{-1}$. Then

$$\alpha(A_{i_1 \cdots i_k}) \leq \Big|B_{i_1 \cdots i_{t-1}} B_{i_t \cdots i_k}\Big(\frac{y_0}{|y_0|}\Big)\Big| = \frac{|B_{i_1 \cdots i_{t-1}}(x_0)|}{|y_0|} \leq c^{-1}\alpha(B_{i_1 \cdots i_{t-1}}).$$

On the other hand, for any sequence $i_1 \cdots i_{t-1}$ and $i_t \neq j_t$,

$$d(A_{i_1 \cdots i_{t-1}} A_{i_t}(E), A_{i_1 \cdots i_{t-1}} A_{j_t}(E)) \geq \alpha(B_{i_1 \cdots i_{t-1}}) \min_{i \neq j} d(A_i(E), A_j(E)). \tag{2.3}$$

Therefore, using (2.2), we have

$$\begin{aligned} d(A_{i_1\cdots i_k}(E), E\backslash A_{i_1\cdots i_k}(E)) &\geq \Big[\min_{t\leq k}\alpha(B_{i_1\cdots i_{t-1}})\Big]\min_{i\neq j} d(A_i(E),A_j(E)) \\ &\geq [c^{-1}\alpha(B_{i_1\cdots i_k})]\min_{i\neq j} d(A_i(E),A_j(E)) \\ &\geq \lambda\alpha(B_{i_1\cdots i_k}) = r_k. \end{aligned}$$

That means

$$B(x,r_k)\cap[E\backslash A_{i_1\cdots i_k}(E)] = \emptyset. \tag{2.4}$$

It follows from (2.1) that $\alpha(B_{i_1\cdots i_k}) \leq c\Big(\prod_{t=1}^{k} c_{i_t}\Big)$, which implies $\lim_{k\to\infty} r_k = 0$. We also have

$$\begin{aligned} \alpha(B_{i_1\cdots i_{k+1}}) &= \inf_{|x|=1}|B_{i_1\cdots i_k}B_{i_{k+1}}(x)| \\ &\geq \inf_{|x|=1}\Big|B_{i_1\cdots i_k}\Big(\frac{B_{i_{k+1}}x}{|B_{i_{k+1}}x|}\Big)\Big|\cdot\inf_{|x|=1}|B_{i_{k+1}}x| \\ &\geq \alpha(B_{i_1\cdots i_k})\alpha(B_{i_{k+1}}), \end{aligned}$$

which implies $\frac{r_{k+1}}{r_k} \geq \min_i \alpha(B_i)$.

By Lemma 2.2, we need only to prove that for any $x\in E$,

$$\liminf_{k\to\infty}\frac{p(x,r_k,E)}{r_k} \geq \eta_0.$$

Suppose $E\subset B(0,R_0)$. Let $y_k = A^{-1}_{i_1\cdots i_k}(x)\in E$. There are two orthogonal bases $\{u_1,\cdots,u_n\}$, $\{v_1,\cdots,v_n\}\subset\mathbb{R}^n$ with $|u_i| = |v_i| = 1$ for all i, such that

$$B^{-1}_{i_1\cdots i_k}(u_i) = d_i v_i, \tag{2.5}$$

where

$$d_1 = \alpha^{-1}(B_{i_1\cdots i_k}) \geq d_2 \geq \cdots \geq d_n = \beta^{-1}(B_{i_1\cdots i_k}). \tag{2.6}$$

It follows from Lemma 2.1 that there exists a constant $\eta_0\in(0,\frac{1}{2}]$ such that an open ball

$$B(y_k+tv_1,\eta_0\lambda)\subset\mathbb{R}^n\backslash E \quad \text{with } |t|\leq\frac{\lambda}{2}.$$

Lemma 2.3 *Let* $\Omega_k = \Big\{(y_k+tv_1)+(r_k\eta_0)\sum_{i=1}^n t_i d_i v_i : \sum_{i=1}^n t_i^2 \leq 1\Big\}$ *and* $\Lambda_k = \Big\{y_k + r_k\sum_{i=1}^n t_i d_i v_i : \sum_{i=1}^n t_i^2 \leq 1\Big\}$. *Then we have the following conclusions:*

(1) $\Omega_k\subset\mathbb{R}^n\backslash E$;
(2) $\Lambda_k = A^{-1}_{i_1\cdots i_k}(B(x,r_k))$;
(3) $\Omega_k\subset\Lambda_k$.

Proof To prove (1), we need only to verify $\Omega_k\subset B(y_k+tv_1,\eta_0\lambda)$, and this follows from $r_k\eta_0 d_i \leq (r_k d_1)\eta_0 = \eta_0\lambda$ immediately. By (2.5) and (2.6), we get (2).

To verify (3), we notice that

$$(y_k+tv_1)+(r_k\eta_0)\sum_{i=1}^n t_i d_i v_i = y_k + r_k\Big[\frac{t}{\lambda}+\eta_0 t_1\Big]d_1v_1 + r_k\sum_{i=2}^n[\eta_0 t_i]d_iv_i,$$

where $\left|\frac{t}{\lambda}\right| \le \frac{1}{2}$, $\eta_0 \le \frac{1}{2}$ and $|t_1| \le 1$, and thus

$$\Big[\frac{t}{\lambda}+\eta_0 t_1\Big]^2+\sum_{i=2}^{n}[\eta_0 t_i]^2 \le \Big(\frac{t}{\lambda}\Big)^2+2\Big|\frac{t}{\lambda}\Big|\cdot|\eta_0 t_1|+\sum_{i=1}^{n}[\eta_0 t_i]^2 \le \frac{1}{4}+2\Big(\frac{1}{2}\Big)^2+\eta_0^2=1,$$

which implies $\Omega_k \subset \Lambda_k$.

Notice

$$\begin{aligned} A_{i_1\cdots i_k}(\Omega_k) &= \Big\{A_{i_1\cdots i_k}(y_k+tv_1)+(r_k\eta_0)\sum_{i=1}^{n}t_i u_i : \sum_{i=1}^{n}t_i^2 \le 1\Big\} \\ &= B(A_{i_1\cdots i_k}(y_k+tv_1),\eta_0 r_k). \end{aligned} \tag{2.7}$$

Since $\Omega_k \subset \Lambda_k$, we have

$$A_{i_1\cdots i_k}(\Omega_k) \subset A_{i_1\cdots i_k}(\Lambda_k) = B(x,r_k), \tag{2.8}$$

due to Lemma 2.3. On the other hand, $\Omega_k \subset \mathbb{R}^n\backslash E$ and

$$A_{i_1\cdots i_k}(\Omega_k)\cap[\mathbb{R}^n\backslash A_{i_1\cdots i_k}(E)] \subset B(x,r_k)\cap[\mathbb{R}^n\backslash A_{i_1\cdots i_k}(E)]=\emptyset, \tag{2.9}$$

due to Lemma 2.2 and $r_k=\alpha(B_{i_1\cdots i_k})\lambda$. Therefore,

$$\begin{aligned} A_{i_1\cdots i_k}(\Omega_k)\cap E &\subset \{A_{i_1\cdots i_k}(\Omega_k)\cap[\mathbb{R}^n\backslash A_{i_1\cdots i_k}(E)]\}\cup\{A_{i_1\cdots i_k}(\Omega_k)\cap A_{i_1\cdots i_k}(E)\} \\ &= A_{i_1\cdots i_k}(\Omega_k)\cap A_{i_1\cdots i_k}(E) \subset A_{i_1\cdots i_k}(\Omega_k\cap E)=\emptyset. \end{aligned} \tag{2.10}$$

And thus, by (2.7), (2.8) and (2.10), we have

$$B(A_{i_1\cdots i_k}(y_k+tv_1),\eta_0 r_k) \subset (\mathbb{R}^n\backslash E)\cap B(x,r_k), \tag{2.11}$$

which implies

$$\liminf_{k\to\infty}\frac{p(x,r_k,E)}{r_k} \ge \eta_0.$$

This completes the proof of Theorem 1.1.

Proof of Theorem 1.2 We will prove that the self-affine set $F=\bigcup_{i=1}^{\#\Gamma}A_i(F)$ is not porous. Here $k_2>k_1\ge 5$ and

$$[C_{0,k_1-1}\times C_{0,k_2-1}]\backslash[C_{2,k_1-3}\times C_{2,k_2-3}] \subset \Gamma,$$

which implies the point

$$\Big(\frac{1}{k_1},\frac{1}{k_2}\Big)\in F \tag{2.12}$$

and the boundary $\partial([0,1]^2)$ of $[0,1]^2$ is contained in the self-affine set F, where $\partial([0,1]^2) = ([0,1]\times\{0,1\})\cup(\{0,1\}\times[0,1])$. Furthermore, the boundary of any rectangle $A_{i_1\cdots i_t}([0,1]^2)$ is also contained in F, i.e.,

$$\partial A_{i_1\cdots i_t}([0,1]^2)\subset F. \tag{2.13}$$

For any $t\ge 1$, let $\rho_t=k_2^{-t}$ and

$$I_t=\Big[\frac{1}{k_1}-\rho_t,\frac{1}{k_1}+\rho_t\Big]\times\Big[\frac{1}{k_2}-\rho_t,\frac{1}{k_2}+\rho_t\Big],$$

the square of side $2\rho_t$ centered at $(\frac{1}{k_1},\frac{1}{k_2})$.

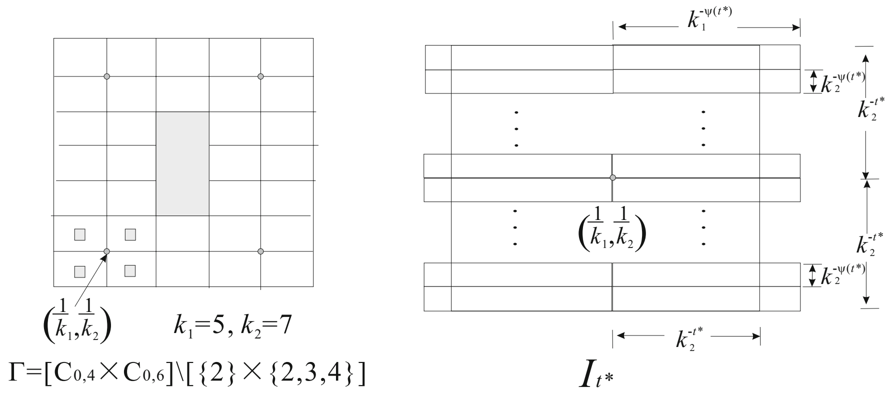

Figure 1

Suppose on the contrary that F is porous. Then at the point $(\frac{1}{k_1},\frac{1}{k_2})\in F$, there exists a constant $\varsigma>0$ such that for each t, there is a square of side $\varsigma\rho_t$ which is contained in $I_t\backslash F$.

Given $t\geq 1$, let $\psi(t)$ be a positive integer satisfying

$$k_1^{-\psi(t)-1}<k_2^{-t}\leq k_1^{-\psi(t)}. \tag{2.14}$$

In fact, since $k_2>k_1$ and $\frac{\psi(t^*)}{t}\to\frac{\log k_2}{\log k_1}>1$ as $t\to\infty$, there exists an integer t^* such that $\varsigma\rho_{t^*}=\varsigma k_2^{-t^*}>k_2^{-\psi(t^*)}$.

Let $T^*=k_2^{\psi(t^*)-t^*}-1$, and $\Theta_{t^*}=\bigcup\limits_{i=0}^{T^*}[0,k_1^{-\psi(t^*)}]\times[i\cdot k_2^{-\psi(t^*)},(i+1)k_2^{-\psi(t^*)}]$ which is a collection of rectangles as in Figure 1. Let

$$\pi_1(x,y)\equiv(x,y),\quad \pi_2(x,y)\equiv(-x,y),\quad \pi_3(x,y)\equiv(-x,-y),\quad \pi_4(x,y)\equiv(x,-y),$$

and $\Pi_i=(\frac{1}{k_1},\frac{1}{k_2})+\pi_i\Theta_{t^*}$ $(i=1,2,3,4)$. Then as in Figure 1,

$$I_{t^*}\subset\Pi_1\cup\Pi_2\cup\Pi_3\cup\Pi_4. \tag{2.15}$$

Notice that each rectangle of width $k_1^{-\psi(t^*)}$ and height $k_2^{-\psi(t^*)}$ appearing in Π_i $(i=1,2,3,4)$ can be written in form of $A_{i_1\cdots i_{\psi(t^*)}}([0,1]^2)$, and thus its boundary is contained in F.

Suppose that S_{t^*} is an open square of side $\varsigma\rho_{t^*}$ such that $S_{t^*}\subset I_{t^*}\backslash F$. Since $\varsigma\rho_{t^*}>k_2^{-\psi(t^*)}$, where $k_2^{-\psi(t^*)}$ is the height of the small rectangle mentioned above, by (2.15), there exists such a rectangle R with its boundary ∂R satisfying $\partial R\cap S_{t^*}\neq\emptyset$. Here $\partial R\cap S_{t^*}\subset\partial R\subset F$ and $\partial R\cap S_{t^*}\subset S_{t^*}\subset\mathbb{R}^2\backslash F$. This is a contradiction.

3 An Example of C^1 IFS

In this section, we will obtain an invariant set H of C^1 IFS in $\mathbb{R}$ such that $\mathcal{H}^1(H)>0$. Thus H is not porous, since the porous set in $\mathbb{R}$ has zero Lebesgue measure (see [8]).

Let $a_n=\frac{1}{2}+\frac{1}{n+3}<1$ and $\delta_{n+1}=a_n-a_{n+1}=\frac{1}{(n+3)(n+4)}$ for $n\geq 1$. Then

$$\lim_{n\to\infty}\frac{a_n}{a_{n+1}}=\lim_{n\to\infty}\frac{\delta_n}{\delta_{n+1}}=1. \tag{3.1}$$

For two intervals J_1,J_2, we denote $J_1<J_2$ if $\sup\limits_{x\in J_1}x\leq\inf\limits_{y\in J_2}y$.

Divide the unit interval $[0,1]$ into three intervals: $[0,1] = I_1 \cup G \cup I_2$, where I_r is closed with length $\frac{a_1}{2}$ for each $r = 1, 2$, G is open and $I_1 < G < I_2$.

By induction, for $i_1 \cdots i_k \in \{1,2\}^k$, we can divide the closed interval $I_{i_1\cdots i_k}$ of length $|I_{i_1\cdots i_k}| = \frac{a_k}{2^k}$ into

$$I_{i_1\cdots i_k} = I_{i_1\cdots i_k 1} \cup G_{i_1\cdots i_k} \cup I_{i_1\cdots i_k 2}, \tag{3.2}$$

where $I_{i_1\cdots i_k 1} < G_{i_1\cdots i_k} < I_{i_1\cdots i_k 2}$ with

$$|I_{i_1\cdots i_k 1}| = |I_{i_1\cdots i_k 2}| = \frac{a_{k+1}}{2^{k+1}} \quad \text{and} \quad |G_{i_1\cdots i_k}| = \frac{\delta_{k+1}}{2^k}.$$

Let $H = \bigcap\limits_{k\geq 1} \bigcup\limits_{i_1\cdots i_k} I_{i_1\cdots i_k}$. For Lebesgue measure $\mathcal{H}^1$, we have

$$\mathcal{H}^1(H) = \lim_{k\to\infty} \sum_{i_1\cdots i_k} \mathcal{H}^1(I_{i_1\cdots i_k}) = \lim_{k\to\infty} 2^k \frac{a_k}{2^k} = \frac{1}{2} > 0. \tag{3.3}$$

That means H is not porous.

We will show that H is the invariant set of certain C^1 IFS $\{g_1, g_2\}$. On H the functions g_1, g_2 can be defined by

$$\{g_{i_0}(x)\} = \bigcap_{k\geq 1} I_{i_0 i_1\cdots i_k} \quad \text{for } \{x\} = \bigcap_{k\geq 1} I_{i_1\cdots i_k}. \tag{3.4}$$

For the definitions of $\{g_1, g_2\}$ on the gaps, we need the following lemma.

Lemma 3.1 *Given sequence $i_0 i_1 \cdots i_k$ ($k \geq 1$), let $G_{i_1\cdots i_k} = (c_{i_1\cdots i_k}, d_{i_1\cdots i_k})$ and $G_{i_0 i_1\cdots i_k} = (c_{i_0 i_1\cdots i_k}, d_{i_0 i_1\cdots i_k})$. Then there is a C^1 increasing and contractive injection $f_{i_0 i_1\cdots i_k} : G_{i_1\cdots i_k} \to G_{i_0 i_1\cdots i_k}$ defined on $G_{i_1\cdots i_k}$ such that*

(1) $f_{i_0 i_1\cdots i_k}(G_{i_1\cdots i_k}) = G_{i_0 i_1\cdots i_k}$,
(2) $f'_{i_0 i_1\cdots i_k}(c_{i_1\cdots i_k}) = f'_{i_0 i_1\cdots i_k}(d_{i_1\cdots i_k}) = \frac{1}{2}$,
(3) $|f'_{i_0 i_1\cdots i_k}(x) - \frac{1}{2}| \leq \frac{2}{k+5}$ *for any* $x \in G_{i_1\cdots i_k}$.

Proof Let

$$\Phi(\zeta) = \begin{cases} 0, & \text{if } \ \zeta \leq 0 \text{ or } \zeta \geq 1, \\ 4\zeta, & \text{if } 0 \leq \zeta \leq \dfrac{1}{2}, \\ 4 - 4\zeta, & \text{if } \dfrac{1}{2} \leq \zeta \leq 1, \end{cases}$$

and $\varphi(x) = \int_{-\infty}^{x} \Phi(\zeta)\mathrm{d}\zeta$. Then $\varphi \in C^1(\mathbb{R}, \mathbb{R})$ satisfies

$$\varphi(0) = \varphi'(0) = \varphi'(1) = 0, \quad \varphi(1) = 1 \quad \text{and} \quad 0 \leq \varphi'(y) \leq 2 \quad \text{for } y \in \mathbb{R}.$$

For any $x \in G_{i_1\cdots i_k}$, let

$$f(x) = c_{i_0 i_1\cdots i_k} + \frac{x - c_{i_1\cdots i_k}}{2} + \Big(\frac{\delta_{k+2}}{2\delta_{k+1}} - \frac{1}{2}\Big)\frac{\delta_{k+1}}{2^k} \cdot \varphi\Big(\frac{x - c_{i_1\cdots i_k}}{\frac{\delta_{k+1}}{2^k}}\Big).$$

Then $f(c_{i_1\cdots i_k}) = c_{i_0 i_1\cdots i_k}$, $f'(c_{i_1\cdots i_k}) = f'(d_{i_1\cdots i_k}) = \frac{1}{2}$, where

$$\frac{d_{i_1\cdots i_k} - c_{i_1\cdots i_k}}{\frac{\delta_{k+1}}{2^k}} = \frac{|G_{i_1\cdots i_k}|}{\frac{\delta_{k+1}}{2^k}} = 1. \tag{3.5}$$

Therefore

$$f(d_{i_1\cdots i_k})-f(c_{i_1\cdots i_k})=\frac{\delta_{k+1}}{2^k}\frac{\delta_{k+2}}{2\delta_{k+1}}\varphi(1)=\frac{\delta_{k+2}}{2^{k+1}}=|G_{i_0i_1\cdots i_k}|.$$

For each $x\in G_{i_1\cdots i_k}$,

$$f'(x)=\frac{1}{2}+\frac{1}{2}\Big(\frac{\delta_{k+2}}{\delta_{k+1}}-1\Big)\varphi'\Big(\frac{x-c_{i_1\cdots i_k}}{\frac{\delta_{k+1}}{2^k}}\Big). \tag{3.6}$$

As $0\le\varphi'(y)\le 2$ and $\frac{\delta_{k+2}}{\delta_{k+1}}=\frac{k+3}{k+5}$, we have

$$\Big|f'(x)-\frac{1}{2}\Big|\le\frac{2}{k+5}. \tag{3.7}$$

Here $0<\frac{2}{k+5}\le\frac{2}{5}<\frac{1}{2}$ for $k\ge 0$. Then $|f'(x)|\in\big[\frac{1}{10},\frac{9}{10}\big]\subset(0,1)$ for each $x\in G_{i_1\cdots i_k}$, and thus f is an increasing contraction satisfying $f(G_{i_1i_2\cdots i_k})=G_{i_0i_1\cdots i_k}$.

Let $f_{i_0i_1\cdots i_k}=f$ and this lemma follows.

Remark 3.1 In the above lemma, we only need $k\ge 0$. In particular, for $k=0$, on G we also get two C^1 mappings $f_1:G\to G_1$ and $f_2:G\to G_2$ satisfying Lemma 3.1(1)–(3).

Then by (3.1), for any $x\in H$,

$$|g_1'(x)|=|g_2'(x)|=\lim_{k\to\infty}\frac{|I_{i_0i_1\cdots i_k}|}{|I_{i_1\cdots i_k}|}=\lim_{k\to\infty}\frac{\frac{a_{k+1}}{2^{k+1}}}{\frac{a_k}{2^k}}=\frac{1}{2}. \tag{3.8}$$

For any sequence $i_1\cdots i_k$ with $k\ge 0$, on the open interval $G_{i_1\cdots i_k}$, let

$$g_{i_0}|_{G_{i_1\cdots i_k}}=f_{i_0i_1\cdots i_k}:G_{i_1\cdots i_k}\to G_{i_0i_1\cdots i_k}$$

as in Lemma 3.1. Then it follows from Lemma 3.1 that g_1, g_2 are C^1 injective contractions with $g_1([0,1])\cap g_2([0,1])=I_1\cap I_2=\emptyset$, and H is their invariant set with $\mathcal{H}^1(E)>0$.

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