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On the Normal Subgroup with Exactly Two G-Conjugacy Class Sizes***

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Abstract Let G be a finite group with a non-central Sylow r-subgroup R, $Z(G)$ the center of G, and N a normal subgroup of G. The purpose of this paper is to determine the structure of N under the hypotheses that N contains R and the G-conjugacy class size of every element of N is either 1 or m. Particularly, it is shown that N is Abelian if $N \cap Z(G) = 1$ and the G-conjugacy class size of every element of N is either 1 or m.

Keywords Normal subgroups, Conjugacy class sizes, Nilpotent groups
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1 Introduction

All groups considered in this paper are finite. Let G be a group, π a set of some primes and x an element of G. x^G denotes the conjugacy class containing x, $|x^G|$ denotes the size of x^G, x_π and $x_{\pi'}$ denote π-component and π'-component of x, respectively. Moreover, we write G_π for a Hall π-subgroup of G, $G_{\pi'}$ for a Hall π'-subgroup of G, and n_π for the π-part of n whenever n is a positive integer.

In 1904, Burnside proved that if a group G has a conjugacy class with prime power size, then G is not simple (see [3, Corollary II, p. 322]). Since then, many authors have investigated the relationship between the structure of a group and its conjugacy class sizes (for example, [1, 2, 4, 5, 8–14]). Among these results, a classic result by Itô [8] asserts that a group G is nilpotent if $|x^G| = 1$ or m for every $x \in G$. Recently, Beltrán and Felipe [2] proved that every Hall p'-subgroup of a p-solvable group is nilpotent if $|x^G| = 1$ or m for every p'-element x of G. On the other hand, the structure of a normal subgroup N of a group G was given if N is the union of some G-conjugacy classes (see [9–12]). Now, we are interested in the following question: Let G be a finite group and let N be a normal subgroup of G. If $|x^G| = 1$ or m for every element $x \in N$, is N nilpotent?

Our main result is the following theorem.

Theorem 1.1 *Let G be a finite group with a non-central Sylow r-subgroup R and N a normal subgroup of G containing R. If $|x^G| = 1$ or m for every element x of N, then N is nilpotent.*

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2 Preliminaries

We first list some lemmas which are useful in the proof of our main result.

Lemma 2.1 (see [5, Lemma 1.1]) *Let N be a normal subgroup of a group G and x an element of G. Then*

(a) $|x^N|$ *divides* $|x^G|$;

(b) $|(Nx)^{G/N}|$ *divides* $|x^G|$.

Corollary 2.1 *Let N be a normal subgroup of a group G and $N \cap Z(G) = 1$. If $|x^G| = 1$ or m for every element x of N, then N is Abelian.*

Proof By the assumption of this corollary, we may assume $|N| = km + 1$, where k is the number of noncentral G-conjugacy classes contained in N. It follows that $(|N|, m) = 1$. By Lemma 2.1, we deduce that N is Abelian.

Lemma 2.2 (see [7, Theorem 33.4]) *Let G be a group. A prime p does not divide any conjugacy class size of G if and only if G has a central Sylow p-subgroup.*

Lemma 2.3 *Let π be a set of some primes and N be a normal subgroup of a group G. If $\overline{x} = xN$ is a π-element, then there exists a π-element x^* of G such that $\overline{x} = \overline{x^*}$.*

Proof Let $o(\overline{x}) = n_0$ and $o(x) = n \cdot m$ such that n is a π-number and $(n, m) = 1$. Then $n_0 \mid n$ and $x^{n_0} \in N$. Since $(n, m) = 1$, there exist integers u and v such that $un + vm = 1$. It follows from $x = x^{un} \cdot x^{vm}$ that $xN = (x^m)^v N$. It is clear that $x^* = (x^m)^v$ is a π-element.

Lemma 2.4 (see [6, Theorem 1]) *Let G be a group acting transitively on a set Ω with $|\Omega| > 1$. Then there exist a prime p and a p-element $x \in G$ such that x acts without fixed point on Ω.*

3 Proof of Theorem 1.1

Now, we are equipped to prove the main result.

Assume that Theorem 1.1 is not true. Let G be a counterexample with minimal order, and $Z(G)$ be the center of the group G. Without loss of generality, we may replace N by $NZ(G)$. Therefore we may assume that $Z(G) \leq N$. We will complete the proof by the following steps.

Step 1 $N_p \nleq Z(G)$ for any prime divisor $p(\neq r)$ of $|N|$.

If not, there exists a prime divisor $q(\neq r)$ of the order of N such that $N_q \leq Z(G)$ and thus $N_q \trianglelefteq G$. Consider the quotient groups G/N_q and N/N_q. For convenience, we use "$\widetilde{\ }$" to work in the factor group mod N_q. Obviously, $\widetilde{R}$ is a non-central Sylow r-subgroup of $\widetilde{G}$, and $\widetilde{N}$ is a normal subgroup of $\widetilde{G}$ containing $\widetilde{R}$. Let $\widetilde{x}$ be an element of $\widetilde{N}$ and y an element of G. We may assume that x is a q'-element of N by Lemma 2.3. If $\widetilde{x}\,\widetilde{y} = \widetilde{y}\,\widetilde{x}$, then $[x, y] \in N_q$. So y normalizes the group $\langle x \rangle \times N_q$, and hence $[x, y] \in \langle x \rangle$. Consequently, $[x, y] = 1$. So $C_{\widetilde{G}}(\widetilde{x}) = \widetilde{C_G(x)}$. Therefore, $|\widetilde{x}^{\widetilde{G}}| = |\widetilde{G} : C_{\widetilde{G}}(\widetilde{x})| = |\widetilde{G} : \widetilde{C_G(x)}| = |G : C_G(x)| = 1$ or m. This means that the hypotheses of the theorem are inherited by factor group $\widetilde{G}$ and $\widetilde{N}$. We conclude that $\widetilde{N}$ is nilpotent by the minimal choice of G. Since $N_q \leq Z(G)$ while $Z(G) \leq Z(N)$, N is nilpotent, a contradiction.

In the following, we will consider the quotient groups $G/Z(G)$ and $N/Z(G)$. For convenience, we use "$\overline{\ }$" to work in the factor group mod $Z(G)$.

Step 2 For any $1 \neq \overline{x} \in \overline{N}$, we have

(i) If $o(\overline{x})$ is a power of a prime p, then $C_N(x)_{p'} \leq Z(C_G(x))$.

(ii) If $o(\overline{x})$ is not a prime power, then $C_N(x) \leq Z(C_G(x))$.

If $o(\overline{x})$ is a power of p, then by Lemma 2.3 we may assume that x is a p-element. For any p'-element $y \in N \cap C_G(x) = C_N(x)$, since $C_G(xy) = C_G(x) \cap C_G(y) \subseteq C_G(x)$, the assumption of the theorem implies $C_G(xy) = C_G(x)$, and hence $C_G(x) \subseteq C_G(y)$. It follows that $y \in Z(C_G(x))$, and therefore $C_N(x)_{p'} \leq Z(C_G(x))$.

If $o(\overline{x})$ is not a prime power, then $o(x)$ is also not a prime power. So we may assume $x = x_1 x_2 \cdots x_s$, where the order of each x_i is a power of a prime p_i and x_i commutes pairwise with $p_i \neq p_j$ $(i, j = 1, 2, \cdots, s$ and $s \geq 2)$. Since $o(\overline{x})$ is not a prime power, there at least exist two elements beyond $Z(G)$ among x_i $(1 \leq i \leq s)$, say, x_1 and x_2. Noticing that x, x_1 and x_2 are non-central elements and $C_G(x) = C_G(x_1) \cap C_G(x_2 \cdots x_s) = C_G(x_2) \cap C_G(x_1 x_3 \cdots x_s)$, we have $C_G(x) = C_G(x_1) = C_G(x_2)$ by the assumption of the theorem. On the other hand, (i) implies that $C_N(x)_{p_1'} \leq Z(C_G(x))$ and $C_N(x)_{p_2'} \leq Z(C_G(x))$, while $C_N(x)_{p_1'} C_N(x)_{p_2'} = C_N(x)$. So $C_N(x) \leq Z(C_G(x))$.

Step 3 Let $1 \neq \overline{x}, \overline{y} \in \overline{N}$ and $o(\overline{x})$ be not a prime power. If $C_G(x) \neq C_G(y)$, then $C_N(x) \cap C_N(y) = Z(G)$.

By (ii), we have $C_N(x) \leq Z(C_G(x))$. If there exists an element a such that $1 \neq a \in C_N(x) \cap C_N(y)$ but $a \notin Z(G)$, then $C_G(x) = C_G(a)$, and therefore $C_N(x) = C_N(a)$. Also, $a \in C_N(y)$ implies $y \in C_N(a) = C_N(x) \leq Z(C_G(x))$. It follows that $C_G(x) \subseteq C_G(y)$. Hence $C_G(x) = C_G(y)$, a contradiction.

Step 4 For any $1 \neq \overline{x} \in \overline{N}$, we have

(iii) If $o(\overline{x})$ is a power of a prime q, then $|\overline{x}^{\overline{G}}|_{q'} = m_{q'}$.

(iv) If $o(\overline{x})$ is not a prime power, then $|\overline{x}^{\overline{G}}| = m$.

Let $C_{\overline{G}}(\overline{x}) = \overline{H}$. For every element $\overline{y} \in \overline{H}$ of order of a power of a prime t where $t \in q'$, we may assume that y is a t-element and $y \in H$ by Lemma 2.3. Therefore we have that $\langle \overline{x}, \overline{y} \rangle$ is a cyclic group and hence $\langle x, y \rangle$ is Abelian. So $\overline{y} \in \overline{C_G(x)}$. It follows that $|\overline{H}|_t = |\overline{C_G(x)}|_t$. So $|\overline{H}|_{q'} = |\overline{C_G(x)}|_{q'}$. Therefore

$$|\overline{x}^{\overline{G}}|_{q'} = \frac{|\overline{G}|_{q'}}{|\overline{H}|_{q'}} = \frac{|\overline{G}|_{q'}}{|\overline{C_G(x)}|_{q'}} = \frac{|G|_{q'}/|Z(G)|_{q'}}{|C_G(x)|_{q'}/|Z(G)|_{q'}} = \frac{|G|_{q'}}{|C_G(x)|_{q'}} = |G : C_G(x)|_{q'} = m_{q'}.$$

Next, if $\overline{x}$ is not a prime power order element, then $o(x)$ is also not a prime power. So we may assume that $x = x_1 x_2 \cdots x_s$, where the order of each x_i is a power of a prime p_i and x_i commute pairwise with $p_i \neq p_j$ $(i, j = 1, 2, \cdots, s$ and $s \geq 2)$. Since $o(\overline{x})$ is not a prime power, there at least exist two elements beyond $Z(G)$ among x_i $(1 \leq i \leq s)$, say, x_1 and x_2. So $\overline{x} = \overline{x}_1 \, \overline{x}_2 \, \overline{x}_3 \cdots \overline{x}_s$. Obviously, $C_{\overline{G}}(\overline{x}) = C_{\overline{G}}(\overline{x}_1) \cap C_{\overline{G}}(\overline{x}_2 \, \overline{x}_3 \cdots \overline{x}_s) = C_{\overline{G}}(\overline{x}_2) \cap C_{\overline{G}}(\overline{x}_1 \, \overline{x}_3 \cdots \overline{x}_s)$, and it is clear that $C_{\overline{G}}(\overline{x}) \leq C_{\overline{G}}(\overline{x}_1)$ and $C_{\overline{G}}(\overline{x}) \leq C_{\overline{G}}(\overline{x}_2)$. Hence $|\overline{x}_1^{\overline{G}}| \mid |\overline{x}^{\overline{G}}|$ and $|\overline{x}_2^{\overline{G}}| \mid |\overline{x}^{\overline{G}}|$. By (iii), we have $m_{p_1'} \mid |\overline{x}^{\overline{G}}|$ and $m_{p_2'} \mid |\overline{x}^{\overline{G}}|$. So $|\overline{x}^{\overline{G}}| = m$.

Step 5 If $1 \neq \overline{x} \in \overline{N}$ is not a prime power order element, then $C_{\overline{G}}(\overline{x}) = \overline{C_G(x)}$, particularly, $C_{\overline{N}}(\overline{x}) = \overline{C_N(x)}$.

Since

$$|\overline{x}^{\overline{G}}| = |\overline{G} : C_{\overline{G}}(\overline{x})| \leq |\overline{G} : \overline{C_G(x)}| = |G : C_G(x)| = m$$

while $|\overline{x}^{\overline{G}}| = m$ by (iv), thus we obtain $C_{\overline{G}}(\overline{x}) = \overline{C_G(x)}$. Particularly, $C_{\overline{N}}(\overline{x}) = C_{\overline{G}}(\overline{x}) \cap \overline{N} = \overline{C_G(x)} \cap \overline{N} = \overline{C_G(x) \cap N} = \overline{C_N(x)}$.

Step 6 Let $g_0 \in N - Z(G)$ and $\overline{g}_0$ be not a prime power order element. Consider the conjugacy class $\overline{g}_0^{\overline{N}}$ of $\overline{g}_0$ in $\overline{N}$. Then there exists some non-central element x of N such that $\overline{g}_0^{\overline{N}} \cap \overline{C_N(x)} = \emptyset$.

Consider the conjugacy class g_0^N of g_0 in N. Noting that N operates transitively on the set g_0^N with $|g_0^N| > 1$, we have that there exists an element x of N such that x operates without fixed point on g_0^N by Lemma 2.4. It follows that $g_0^N \cap C_N(x) = \emptyset$. So $\overline{g}_0^{\overline{N}} \cap \overline{C_N(x)} = \emptyset$.

Step 7 There exists a $\{p, r\}$-element g of N such that $\overline{g}$ is a $\{p, r\}$-element of $\overline{N}$ for any prime divisor $p(\neq r)$ of $|N|$.

According to Step 1, for any prime divisor $p(\neq r)$ of $|N|$, there exists a non-central p-element in N, say, x. By (i), we have $C_N(x) = C_N(x)_p \times C_N(x)_{p'}$. We claim $C_N(x)_r \nleq Z(G)$. Otherwise, $C_N(x)_r \leq Z(G)$. As R is non-central, there exists a non-central r-element z such that $z \in N \setminus C_N(x)_r$. So $C_N(x)_r < \langle C_N(x)_r, z\rangle \leq C_G(z)$, in contradiction to $|z^G| = |x^G| = m$. Take $z \in C_N(x)_r \setminus Z(G)$ and let $g = xz$. Thus $\overline{g} = \overline{x}\,\overline{z}$. It is clear that g and $\overline{g}$ satisfy the requirement of Step 7.

Step 8 If there exist an r-element x of N and a prime divisor $p(\neq r)$ of $|N|$ such that $p \nmid |\overline{C_N(x)}|$, then $|\overline{N}|_p \mid m$.

As

$$|x^N| = |N : C_N(x)| = |\overline{N} : \overline{C_N(x)}|,$$

it leads to $|\overline{N}|_p \mid |x^N|$ since p does not divide $|\overline{C_N(x)}|$. It follows that $|\overline{N}|_p \mid |x^G|$ by Lemma 2.1. The hypotheses of the theorem imply $|\overline{N}|_p \mid m$.

Step 9 $|\overline{C_N(y)}|$ and $|\overline{N}|$ have the same prime divisors for any r-element y of N.

If it is not true, there exist an r-element x_0 of N and a prime divisor p of N such that p does not divide $|\overline{C_N(x_0)}|$. Obviously, $p \neq r$. By Step 8, we have $|\overline{N}|_p \mid m$.

By Step 7, we may take a $\{p, r\}$-element $g \in N$ such that $\overline{g}$ is a $\{p, r\}$-element. Applying Step 6, we have that there exists a non-central element x of N such that $\overline{g}^{\overline{N}} \cap \overline{C_N(x)} = \emptyset$. Consider that $\overline{C_N(x)}$ operates on $\overline{g}^{\overline{N}}$ by conjugation. Notice that no element in $\overline{C_N(x)}$ distinct from 1 centralizes any element in $\overline{g}^{\overline{N}}$ by Steps 3 and 5. So all orbits of $\overline{C_N(x)}$ on $\overline{g}^{\overline{N}}$ have the same length $|\overline{C_N(x)}|$, which implies $|\overline{C_N(x)}| \mid |\overline{g}^{\overline{N}}|$. Also $|\overline{g}^{\overline{N}}| \mid |\overline{g}^{\overline{G}}|$ by Lemma 2.1, so

$$|\overline{C_N(x)}| \mid |\overline{g}^{\overline{G}}|. \tag{3.1}$$

On the other hand, it is obvious that $\overline{C_N(g)}$ operates without fixed points on $\overline{g}^{\overline{G}} - \overline{g}^{\overline{G}} \cap \overline{C_N(g)}$. By Steps 3 and 5 once again, we have

$$|\overline{C_N(g)}| \mid (|\overline{g}^{\overline{G}}| - |\overline{g}^{\overline{G}} \cap \overline{C_N(g)}|). \tag{3.2}$$

Since N contains a Sylow r-subgroup R of G, $|\overline{C_N(g)}|_r = |\overline{C_N(x)}|_r$, from which the relationship between (3.1) and (3.2) becomes

$$|\overline{C_N(g)}|_r \mid |\overline{g}^{\overline{G}} \cap \overline{C_N(g)}|. \tag{3.3}$$

Notice $|\overline{N}|_p \mid m$. Also, Step 4 implies that $|\overline{N}|_p \mid |\overline{g}^{\overline{G}}|$. Obviously, $|\overline{C_N(g)}|_p \leq |\overline{N}|_p$. So $|\overline{C_N(g)}|_p \mid |\overline{g}^{\overline{G}}|$. Noticing (3.2), we have

$$|\overline{C_N(g)}|_p \mid |\overline{g}^{\overline{G}} \cap \overline{C_N(g)}|. \tag{3.4}$$

By (3.3) and (3.4), we have

$$|\overline{C_N(g)}|_{\{p,r\}} \mid |\overline{g}^{\overline{G}} \cap \overline{C_N(g)}|. \tag{3.5}$$

Noticing that $\overline{C_N(g)}$ is Abelian by (ii), we get

$$|\overline{g}^{\overline{G}} \cap \overline{C_N(g)}| = |\overline{C_N(g)}|_{\{p,r\}}, \tag{3.6}$$

a contradiction.

Step 10 The final contradiction.

Let p be a prime divisor of $|\overline{N}|$ with $p \neq r$. Choose a non-central r-element x_0 such that $|\overline{C_N(x_0)}|_p$ is as small as possible. By Step 9 we have $|\overline{C_N(x_0)}|_p > 1$. So, it is clear that we may take a $\{p,r\}$-element $g \in C_N(x_0)$ such that $\overline{g}$ is also a $\{p,r\}$-element. Arguing as in Step 9, we may see that (3.1) and (3.2) still hold, and therefore (3.3) also holds.

Noting $C_N(g) = C_N(x_0)$ by Step 3, we have $|\overline{C_N(g)}|_p = |\overline{C_N(x_0)}|_p$. Noticing $|\overline{C_N(x_0)}|_p \leq |\overline{C_N(x)}|_p$ by the choice of x_0, we have $|\overline{C_N(g)}|_p \leq |\overline{C_N(x)}|_p$. Consequently, $|\overline{C_N(g)}|_p \mid |\overline{g}^{\overline{G}}|$ by (3.1). By using a similar argument as in Step 9, (3.4) is obtained. Arguing as in Step 9 once again, we have (3.5), and thus equation (3.6) holds, a contradiction.

Corollary 3.1 *Let G be a finite group with a non-central Sylow r-subgroup R and N a normal subgroup of G containing R. If $|x^G| = 1$ or m for every element x of N, then either N is Abelian or $N_{r'} \leq Z(G)$.*

Proof By Theorem 1.1, we have that N is nilpotent. So, for any $p(\neq r)$-element x of N, we have $r \nmid |G : C_G(x)|$. If $N_{r'} \not\leq Z(G)$, then $r \nmid m$. So $R \leq Z(N)$. However, since R is non-central, there exists a non-central r-element x_0 such that $N \leq C_N(x_0)$. Thus we have $N_{r'} \leq Z(C_G(x_0))$ by (i) in the proof of Theorem 1.1. So N is Abelian.

Corollary 3.2 (see [8, Theorem 1]) *Let G be a group. If $|x^G| = 1$ or m for every element x in G, then G is nilpotent.*

Proof Let $N = G$. Obviously, N satisfies the hypotheses of Theorem 1.1. So $N = G$ is nilpotent by Theorem 1.1.

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