Chin. Ann. Math.
31B(4), 2010, 481–490
DOI: 10.1007/s11401-010-0592-5

Chinese Annals of Mathematics, Series B

Quasi-*d*-Koszul Modules and Applications***

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Abstract Let R be a Noetherian semiperfect algebra. A necessary and sufficient condition for a finitely generated R-module to be quasi-d-Koszul is given, which improves one of the main results in [1]. The authors also give a necessary and sufficient condition for the Minimal Horseshoe Lemma holding in mod(R). As an application, it is proved that the "Minimal Horseshoe Lemma" is true in the category of quasi-d-Koszul modules under certain conditions.

Keywords Quasi-d-Koszul algebras, Quasi-d-Koszul modules, Minimal Horseshoe Lemma
2000 MR Subject Classification 16W30

1 Introduction

Koszul algebras were first introduced by Priddy in 1970 (see [2]), which are a class of quadratic algebras with lots of nice homological properties. Motivated by cubic AS-regular algebras, Berger [3] first introduced the notion of nonquadratic Koszul algebras, which were usually called d-Koszul algebras later (see [4, 5]). In order to study the d-Koszul property for any finitely generated graded modules, the notion of weakly d-Koszul modules were introduced in 2007 (see [6]). In order to study the d-Koszul property for any Noetherian semiperfect algebras, the notion of quasi-d-Koszul algebras were introduced in 2008 (see [7]). In order to study the d-Koszul property for any finitely generated modules over a Noetherian semiperfect algebra, the notion of quasi-d-Koszul modules were introduced in 2009 (see [1]).

The following statement is well-known for d-Koszul modules (see [5]):

(1) Let A be a d-Koszul algebra and M a finitely 0-generated graded module. Then M is a d-Koszul module if and only if the Koszul dual of M, $\bigoplus\limits_{i\geq 0}\mathrm{Ext}_A^i(M, A_0)$ is generated in degree 0 as a graded $\bigoplus\limits_{i\geq 0}\mathrm{Ext}_A^i(A_0, A_0)$-module.

Motivated by the above result, [1] is an attempt to find a similar characterization for quasi-d-Koszul modules and the following is the main result:

(2) Let R be a quasi-d-Koszul algebra and M be a quasi-d-Koszul R-module. Then $\mathcal{E}^{ev}(M)$ is generated in degree 0 as a graded $\mathbf{E}^{ev}(R)$-module, where

$$\mathcal{E}^{ev}(M) := \bigoplus_{i\geq 0}\mathrm{Ext}_A^{2i}(M, R/J), \quad \mathbf{E}^{ev}(R) := \bigoplus_{i\geq 0}\mathrm{Ext}_A^{2i}(R/J, R/J).$$

One of the aims of this paper is to find equivalent descriptions for quasi-d-Koszul modules similar to that for d-Koszul modules introduced above and Theorem 2.1 is our main result.

Manuscript received September 23, 2009. Published online June 21, 2010.
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***Project supported by the Zhejiang Innovation Project (No. T200905).

The other aim of this paper is to give some applications for quasi-d-Koszul modules. It is well-known that "Horseshoe Lemma" plays an important role and is a basic tool in homological algebra. It is also well-known that one of the key subjects in homological algebra is to compute (co-)homological groups of different algebras. To do this, finding or constructing projective resolution is unavoidable. However, Horseshoe Lemma provides a method to construct new projective resolutions from the known ones. We also know that using minimal projective resolution to compute homology groups is more convenient than the ordinary ones.

Motivated by the above, one can ask the following question: Do we have a "Minimal Horseshoe Lemma"? That is, replacing "projective resolution" in the classic Horseshoe Lemma by "minimal projective resolution". Recall that a projective resolution of M

$$\cdots \longrightarrow P_n \xrightarrow{d_n} \cdots \longrightarrow P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \longrightarrow 0$$

is called "minimal" if for all $n \geq 0$, $\Omega^n(M) := \ker d_{n-1} \ll P_{n-1}$, where $\Omega^n(M)$ is usually called the n^{th} syzygy of M, $\Omega^0(M) := M$ and the symbol "$\ll$" means "superfluous".

It is easy to see that the above question is not true in general. We will also give an easy example to explain this (see Example 3.1). Lemma 3.3 is a necessary and sufficient condition for the Minimal Horseshoe Lemma to be true.

As applications of quasi-d-Koszul modules, we prove that the Minimal Horseshoe Lemma holds in the category of quasi-d-Koszul modules under certain conditions. More precisely, we obtain Theorem 3.1.

Now we give some notations and definitions which will be used later.

In the rest of this paper, unless specially stated, R denotes a Noetherian semiperfect algebra and $d \geq 2$ a fixed integer. Let $\mathrm{mod}(R)$ denote the category of finitely generated R-modules and J denote the Jacobson radical of R.

Definition 1.1 (see [1]) *Let $M \in \mathrm{mod}(R)$ and $\cdots \longrightarrow P_n \xrightarrow{d_n} \cdots \longrightarrow P_0 \xrightarrow{d_0} M \longrightarrow 0$ be a minimal projective resolution of M. Then M is called a quasi-d-Koszul module if and only if $\Omega^i(M) \cap J^2 P_{i-1} = J\Omega^i(M)$ for any odd number $i \geq 0$ and $\Omega^i(M) \subseteq J^{d-1} P_{i-1}$, $\Omega^i(M) \cap J^d P_{i-1} = J\Omega^i(M)$ for any even number $i \geq 0$.*

2 An Equivalent Description of Quasi-d-Koszul Modules

Let $M \in \mathrm{mod}(R)$. Then M possesses a minimal projective resolution

$$\mathcal{Q} : \cdots \longrightarrow Q_n \xrightarrow{f_n} \cdots \longrightarrow Q_1 \xrightarrow{f_1} Q_0 \xrightarrow{f_0} M \longrightarrow 0.$$

For the sake of convenience, we denote $S_i := \ker f_{i-1}$, the i^{th} syzygy of M.

Let $E(R) := \bigoplus_{i \geq 0} \mathrm{Ext}^i_R(R/J, R/J)$ and $\mathcal{E}(M) := \bigoplus_{i \geq 0} \mathrm{Ext}^i_R(M, R/J)$. Similarly to the graded case, we also call $E(R)$ the Ext-algebra of R and $\mathcal{E}(M)$ the Koszul dual of M, respectively. It should be noted that $E(R)$ is a positively graded algebra (not bigraded) and $\mathcal{E}(M)$ is a graded $E(R)$-module (not bigraded) since now R and M are not graded. In order to use the technique of graded case, we try to give a "second grading" on $\mathrm{Ext}^i_R(R/J, R/J)$ and $\mathrm{Ext}^i_R(M, R/J)$.

Lemma 2.1 *Using the above notations, we have $\mathrm{Ext}^i_R(M, R/J) \cong \mathrm{Hom}_{R/J}(S_i/JS_i, R/J)$ for all $i \geq 0$.*

Proof Note that M has a minimal projective resolution and R/J is semisimple as an

R-module, thus

$$\begin{aligned}\mathrm{Ext}^i_R(M,R/J) &= H^i\,\mathrm{Hom}_R(\mathcal{Q},R/J)\cong \mathrm{Hom}_R(Q_i,R/J)\\ &\cong \mathrm{Hom}_R(S_i,R/J)\cong \mathrm{Hom}_{R/J}(S_i/JS_i,R/J).\end{aligned}$$

Under the above assumptions, S_i/JS_i admits the following filtration:

$S_i/JS_i = (S_i\cap JQ_{i-1})/JS_i \supseteq (S_i\cap J^2Q_{i-1})/JS_i \supseteq (S_i\cap J^3Q_{i-1}+JS_i)/JS_i \supseteq \cdots \supseteq (S_i\cap J^nQ_{i-1}+JS_i)/JS_i\supseteq\cdots$. Because R/J is semisimple, thus the above filtration is stable. That is, there exists an integer N_i, such that $(S_i\cap J^{N_i}Q_{i-1}+JS_i)/JS_i = (S_i\cap J^{N_i+1}Q_{i-1}+JS_i)/JS_i=\cdots$.

Therefore, as R/J-modules, we have

$$\begin{aligned}S_i/JS_i &\cong S_i/(S_i\cap J^2Q_{i-1})\oplus(S_i\cap J^2Q_{i-1})/(S_i\cap J^3Q_{i-1}+JS_i)\\ &\quad\oplus(S_i\cap J^3Q_{i-1}+JS_i)/(S_i\cap J^4Q_{i-1}+JS_i)\\ &\quad\oplus\cdots\oplus(S_i\cap J^{N_i}Q_{i-1}+JS_i)/JS_i\\ &\cong S_i/(S_i\cap J^2Q_{i-1})\oplus(S_i\cap J^2Q_{i-1})/(JS_i\cap J^2Q_{i-1}+S_i\cap J^3Q_{i-1})\\ &\quad\oplus(S_i\cap J^3Q_{i-1})/(JS_i\cap J^3Q_{i-1}+S_i\cap J^4Q_{i-1})\\ &\quad\oplus\cdots\oplus(S_i\cap J^{N_i}Q_{i-1}+JS_i)/JS_i.\end{aligned}$$

Set

$$H^i_j := (S_i\cap J^{j+1}Q_{i-1})/(JS_i\cap J^{j+1}Q_{i-1}+S_i\cap J^{j+2}Q_{i-1}),\quad j=0,\ 1,\ \cdots,\ N_i-1.$$

Lemma 2.2 *Using the above notations, we have the following isomorphism:*

$$\mathrm{Ext}^i_R(M,R/J)\cong\Big(\bigoplus_{j\ge0}\mathrm{Hom}_{R/J}(H^i_j,R/J)\Big)\bigoplus\mathrm{Hom}_{R/J}((S_i\cap J^{N_i}Q_{i-1}+JS_i)/JS_i,R/J).$$

Now we can give a second grading on $\mathcal{E}(M)$ as follows:

$$\mathrm{Ext}^i_R(M,R/J)_j=\begin{cases}\mathrm{Ext}^0_R(M,R/J), & \text{if } i=j=0,\\ \mathrm{Hom}_{R/J}(H^i_{j-i},R/J), & \text{if } j=i,\ i+1,\ i+N_i-1,\\ 0, & \text{otherwise.}\end{cases}$$

Let

$$\overline{\mathrm{Ext}}^i_R(M,R/J)=\bigoplus_{j\ge i}\mathrm{Hom}_{R/J}(H^i_{j-i},R/J),\quad i\ge1$$

and

$$\overline{\mathcal{E}}(M)=\mathrm{Ext}^0_R(M,R/J)\bigoplus\Big(\bigoplus_{i\ge1}\overline{\mathrm{Ext}}^i_R(M,R/J)\Big).$$

Proposition 2.1 (see [7]) *The following statements are true.*

(1) $\mathrm{Ext}^s_R(R/J,R/J)_t\cdot\mathrm{Ext}^i_R(M,R/J)\subseteq\mathrm{Ext}^{i+s}_R(M,R/J)_{i+s}\oplus\cdots\oplus\mathrm{Ext}^{i+s}_R(M,R/J)_{i+t}$;

(2) $\overline{E}(R)$ *is a graded subalgebra of* $E(A)$ *and* $\overline{\mathcal{E}}(M)$ *is a graded submodule of* $\mathcal{E}(M)$*, where* $\overline{E}(R)=\overline{\mathcal{E}}(R/J)$;

(3) $\mathrm{Ext}^1_R(R/J,R/J)\cdot\mathrm{Ext}^i_R(M,R/J)=\mathrm{Ext}^{i+1}_R(M,R/J)_{i+1}$ *for all* $i\ge0$.

Lemma 2.3 *Let* $M\in\mathrm{mod}(R)$ *and* $\mathcal{Q}$ *a minimal projective resolution of* M*. Let* $f:\mathbb{N}\to\mathbb{N}$ *be an increasing set function with* $f(i)\ge i\ge1$*. Then*

(1) *If* $S_i\subseteq J^{f(i)}Q_{i-1}$*, then* $\mathrm{Ext}^i_R(M,R/J)_j=0$ *for all* $j<f(i)+i-1$;

(2) $\mathrm{Ext}^i_R(M,R/J)=\overline{\mathrm{Ext}}^i_R(M,R/J)$ *and* $\overline{\mathrm{Ext}}^i_R(M,R/J)=\mathrm{Ext}^i_R(M,R/J)_{f(i)}$ *if and only if* $S_i\subseteq J^{f(i)-i+1}Q_{i-1}$ *and* $S_i\cap J^{f(i)-i+2}Q_{i-1}=JS_i$.

Proof (1) For any natural number $k \leq f(i)$, we have $S_i \cap J^k Q_{i-1} = S_i$ since $S_i \subseteq J^{f(i)} Q_{i-1}$.

$$\begin{aligned}&\mathrm{Ext}^i_R(M, R/J)_j = \mathrm{Hom}_{R/J}(H^i_{j-i}, R/J)\\ \cong\ &\mathrm{Hom}_{R/J}((S_i \cap J^{j-i+1}Q_{i-1})/(JS_i \cap J^{j-i+1}Q_{i-1} + S_i \cap J^{j-i+2}Q_{i-1}), R/J).\end{aligned}$$

Note that if $j < f(i) + i - 1$, then $j - i + 2 \leq f(i)$ and of course $j - i + 1 \leq f(i)$. Therefore,

$$\begin{aligned}&\mathrm{Hom}_{R/J}((S_i \cap J^{j-i+1}Q_{i-1})/(JS_i \cap J^{j-i+1}Q_{i-1} + S_i \cap J^{j-i+2}Q_{i-1}), R/J)\\ =\ &\mathrm{Hom}_{R/J}(S_i/(JS_i \cap J^{j-i+1}Q_{i-1} + S_i), R/J) = 0.\end{aligned}$$

Now we finish the proof of (1).

(2) Since $\overline{\mathrm{Ext}}^i_R(M, R/J) = \mathrm{Ext}^i_R(M, R/J)_{f(i)}$, we have $\mathrm{Ext}^i_R(M, R/J)_j = 0$ if $j \neq f(i)$. That is, $H^i_{j-i} = (S_i \cap J^{j-i+1}Q_{i-1})/(JS_i \cap J^{j-i+1}Q_{i-1} + S_i \cap J^{j-i+2}Q_{i-1}) = 0,\ \forall\, j \neq f(i)$. It is obvious that $i \leq f(i) \leq i + N_i - 1$. We divide it into two cases.

(a) For $i \leq j \leq f(i) - 1$. By assumption, we have $H^i_{j-i} = (S_i \cap J^{j-i+1}Q_{i-1})/(JS_i \cap J^{j-i+1}Q_{i-1} + S_i \cap J^{j-i+2}Q_{i-1}) = 0,\ \forall\, j = i, \cdots, f(i) - 1$.

We give the proof by an induction on $j - i$. Since $H^i_0 = (S_i \cap JQ_{i-1})/(JS_i \cap JQ_{i-1} + S_i \cap J^2Q_{i-1}) = S_i/(S_i \cap J^2Q_{i-1}) = 0$, it is implied that $S_i \subseteq J^2Q_{i-1}$. Since $H^i_1 = (S_i \cap J^2Q_{i-1})/(JS_i \cap J^2Q_{i-1} + S_i \cap J^3Q_{i-1}) = S_i/(JS_i \cap J^2Q_{i-1} + S_i \cap J^3Q_{i-1}) = 0$, it is implied that $S_i = JS_i \cap J^2Q_{i-1} + S_i \cap J^2Q_{i-1} = JS_i + S_i \cap J^3Q_{i-1}$. Thus $S_i = S_i \cap J^3Q_{i-1}$ since $JS_i \ll S_i$. In particular, we have $S_i \subseteq J^3Q_{i-1}$. Following this clue, we can get $S_i \subseteq J^{f(i)-i+1}Q_{i-1}$.

(b) For $f(i) + 1 \leq j \leq i + N_i - 1$. Similarly to (a), we have $H^i_{f(i)-i+1} = H^i_{f(i)-i+2} = \cdots = H^i_{N_i-1} = 0$.

Recall that $H^i_{j-i} = (S_i \cap J^{j-i+1}Q_{i-1})/(JS_i \cap J^{j-i+1}Q_{i-1} + S_i \cap J^{j-i+2}Q_{i-1})$. Therefore, we have the following equations:

$$\begin{aligned}S_i \cap J^{f(i)-i+2}Q_{i-1} &= JS_i \cap J^{f(i)-i+2}Q_{i-1} + S_i \cap J^{f(i)-i+3}Q_{i-1},\\ S_i \cap J^{f(i)-i+3}Q_{i-1} &= JS_i \cap J^{f(i)-i+3}Q_{i-1} + S_i \cap J^{f(i)-i+4}Q_{i-1},\\ &\ \ \vdots\\ S_i \cap J^{N_i}Q_{i-1} &= JS_i \cap J^{N_i}Q_{i-1} + S_i \cap J^{N_i+1}Q_{i-1}.\end{aligned}$$

Thus, we have

$$S_i \cap J^{f(i)-i+2}Q_{i-1} = JS_i \cap J^{f(i)-i+2}Q_{i-1} + \cdots + JS_i \cap J^{N_i}Q_{i-1} + S_i \cap J^{N_i+1}Q_{i-1}.$$

Note that we have obtained $S_i \subseteq J^{f(i)-i+1}Q_{i-1}$ in (a). Now using the method in (a), we can get $S_i \cap J^{f(i)-i+2}Q_{i-1} = JS_i$ which completes the proof of necessity.

Now we give the proof of sufficiency. By assumption, we have $S_i \subseteq J^{f(i)-i+1}Q_{i-1}$. By (1), we have $\mathrm{Ext}^i_R(M, R/J)_j = 0$ for all $j < f(i)$. In order to finish the proof, we have to show $\mathrm{Ext}^i_R(M, R/J)_j = 0$ for all $j > f(i)$. It suffices to prove $H^i_{j-i} = (S_i \cap J^{j-i+1}Q_{i-1})/(JS_i \cap J^{j-i+1}Q_{i-1} + S_i \cap J^{j-i+2}Q_{i-1}) = 0$ for all $j > f(i)$. Note that $J^{j-i+1}Q_{i-1} \subseteq J^{f(i)-i+1}Q_{i-1}$ since $j > f(i)$. Therefore,

$$\begin{aligned}&\mathrm{Ext}^i_R(M, R/J)_j\\ =\ &\mathrm{Hom}_{R/J}(H^i_{j-i}, R/J)\\ \cong\ &\mathrm{Hom}_{R/J}((S_i \cap J^{j-i+1}Q_{i-1})/(JS_i \cap J^{j-i+1}Q_{i-1} + S_i \cap J^{j-i+2}Q_{i-1}), R/J)\\ =\ &\mathrm{Hom}_{R/J}((S_i \cap J^{j-i+1}Q_{i-1} \cap J^{f(i)-i+1}Q_{i-1})/(JS_i \cap J^{j-i+1}Q_{i-1} + S_i \cap J^{j-i+2}Q_{i-1}), R/J)\\ \cong\ &\mathrm{Hom}_{R/J}((JS_i \cap J^{j-i+1}Q_{i-1})/(JS_i \cap J^{j-i+1}Q_{i-1} + S_i \cap J^{j-i+2}Q_{i-1}), R/J)\\ =\ &0.\end{aligned}$$

Lemma 2.4 *Let $M \in \mathrm{mod}(R)$. Then M is a quasi-d-Koszul module if and only if for any $i \geq 1$, $k \in \mathbb{N}$, we have $\mathrm{Ext}_R^i(M, R/J) = \overline{\mathrm{Ext}}_R^i(M, R/J)$ and*

$$\overline{\mathrm{Ext}}_R^i(M, R/J) = \begin{cases} \mathrm{Ext}_R^i(M, R/J)_i, & i = 2k+1, \\ \mathrm{Ext}_R^i(M, R/J)_{d-2+i}, & i = 2k. \end{cases}$$

Proof Let $\cdots \longrightarrow Q_n \xrightarrow{f_n} \cdots \longrightarrow Q_1 \xrightarrow{f_1} Q_0 \xrightarrow{f_0} M \longrightarrow 0$ be a minimal projective resolution. Then M is a quasi-d-Koszul module if and only if for all $i \geq 0$, we have

(a) $S_i \subseteq JQ_{i-1}$ and $JS_i = J^2 Q_{i-1} \cap S_i$ $(i = 2k+1)$,

(b) $S_i \subseteq J^{d-1} Q_{i-1}$ and $JS_i = J^d Q_{i-1} \cap S_i$ $(n = 2k)$, where $k \in \mathbb{Z}$. By Lemma 2.3, conditions (a) and (b) are equivalent to $\mathrm{Ext}_R^i(M, R/J) = \overline{\mathrm{Ext}}_R^i(M, R/J)$ $(\forall\, i \geq 1)$ and

$$\overline{\mathrm{Ext}}_R^i(M, R/J) = \begin{cases} \mathrm{Ext}_R^i(M, R/J)_i, & i = 2k+1, \\ \mathrm{Ext}_R^i(M, R/J)_{d-2+i}, & i = 2k. \end{cases}$$

Lemma 2.5 (see [1, 7]) *The following statements are true.*

(1) *R is a quasi-d-Koszul algebra if and only if for all $i \geq 1$, $k \in \mathbb{N}$, we have $\mathrm{Ext}_R^i(R/J, R/J)$ $= \overline{\mathrm{Ext}}_R^i(R/J, R/J)$ and*

$$\overline{\mathrm{Ext}}_R^i(R/J, R/J) = \begin{cases} \mathrm{Ext}_R^i(R/J, R/J)_i, & i = 2k+1, \\ \mathrm{Ext}_R^i(R/J, R/J)_{d-2+i}, & i = 2k. \end{cases}$$

(2) *Let R be a quasi-d-Koszul algebra and M a quasi-d-Koszul module. Then for all $i \geq 0$, we have $\mathrm{Ext}_R^{2i}(M, R/J) = \mathrm{Ext}_R^{2i}(R/J, R/J) \cdot \mathrm{Ext}_R^0(M, R/J)$.*

Theorem 2.1 *Let R be a quasi-d-Koszul algebra and M be a finitely generated R-module. Then M is a quasi-d-Koszul module if and only if the following conditions hold:*

(1) $\mathrm{Ext}_R^i(M, R/J) = \overline{\mathrm{Ext}}_R^i(M, R/J)$ $(i \geq 1)$;

(2) $\overline{\mathrm{Ext}}_R^i(M, R/J) = \mathrm{Ext}_R^i(M, R/J)_i$ *for any odd number $i \geq 0$;*

(3) $\mathrm{Ext}_A^i(R/J, R/J)_s \cdot \mathrm{Ext}_A^0(M, R/J) = \mathrm{Ext}_A^i(M, R/J)_s$ *for any even number $i \geq 0$;*

(4) $\mathrm{Ext}_A^{2i}(M, R/J) = \mathrm{Ext}_A^{2i}(R/J, R/J) \cdot \mathrm{Ext}_A^0(M, R/J)$ *for any integer $i \geq 0$.*

Proof ($\Rightarrow$) Suppose that M is a quasi-d-Koszul module. By Lemma 2.4, for any $i \geq 1$, $k \in \mathbb{N}$, we have $\mathrm{Ext}_R^i(M, R/J) = \overline{\mathrm{Ext}}_R^i(M, R/J)$ and

$$\overline{\mathrm{Ext}}_R^i(M, R/J) = \begin{cases} \mathrm{Ext}_R^i(M, R/J)_i, & i = 2k+1, \\ \mathrm{Ext}_R^i(M, R/J)_{d-2+i}, & i = 2k. \end{cases}$$

Hence (1) and (2) hold. Note that R is a quasi-d-Koszul algebra, by Lemma 2.5, for all $i = 2k \geq 0$, we have $\mathrm{Ext}_R^{2k}(R/J, R/J)_s = \mathrm{Ext}_R^{2k}(R/J, R/J)_{2k+d-2}$. Since M is a quasi-d-Koszul R-module, by Lemma 2.5 again, we have $\mathrm{Ext}_R^{2k}(M, R/J) = \mathrm{Ext}_R^{2k}(R/J, R/J) \cdot \mathrm{Ext}_R^0(M, R/J)$. Thus (4) holds. But $\mathrm{Ext}_R^{2k}(M, R/J) = \mathrm{Ext}_R^{2k}(M, R/J)_{2k+d-2}$, so for all $i = 2k \geq 0$, we have

$$\begin{aligned} \mathrm{Ext}_R^i(R/J, R/J)_s \cdot \mathrm{Ext}_R^0(M, R/J) &= \mathrm{Ext}_R^{2k}(R/J, R/J)_{2k+d-2} \cdot \mathrm{Ext}_R^0(M, R/J) \\ &= \mathrm{Ext}_R^{2k}(M, R/J)_{2k+d-2} \\ &= \mathrm{Ext}_R^i(M, R/J)_{2k+d-2} \\ &= \mathrm{Ext}_R^i(M, R/J)_s. \end{aligned}$$

Therefore, (3) holds.

($\Leftarrow$) According to (3), for all $i = 2k \geq 0$, we have $\mathrm{Ext}_R^i(R/J, R/J)_s \cdot \mathrm{Ext}_R^0(M, R/J) = \mathrm{Ext}_R^i(M, R/J)_s$. Note that R is a quasi-d-Koszul algebra, by Lemma 2.5, for all $i = 2k \geq 0$, we have $\mathrm{Ext}_R^{2k}(R/J, R/J)_s = \mathrm{Ext}_R^{2k}(R/J, R/J)_{2k+d-2}$. By (4), we get $\mathrm{Ext}_R^{2k}(M, R/J) = \mathrm{Ext}_R^{2k}(R/J, R/J) \cdot \mathrm{Ext}_R^0(M, R/J)$. Therefore,

$$\begin{aligned}\overline{\mathrm{Ext}}_R^{2k}(M, R/J) &= \mathrm{Ext}_R^{2k}(M, R/J) = \mathrm{Ext}_R^{2k}(R/J, R/J) \cdot \mathrm{Ext}_R^0(M, R/J) \\ &= \mathrm{Ext}_R^{2k}(R/J, R/J)_{2k+d-2} \cdot \mathrm{Ext}_R^0(M, R/J) = \mathrm{Ext}_R^{2k}(M, R/J)_{2k+d-2}.\end{aligned}$$

Now combining conditions (1) and (2), by Lemma 2.4, we obtain that M is a quasi-d-Koszul module.

3 Applications of Quasi-*d*-Koszul Modules

In this section, R denotes an augmented Noetherian semiperfect algebra with Jacobson radical J.

Lemma 3.1 *Let* $X \xrightarrow{f} Y \xrightarrow{g} Z$ *be a sequence of R/J-modules. Then*

$$0 \longrightarrow R\bigotimes_{R/J} X \xrightarrow{1\otimes f} R\bigotimes_{R/J} Y \xrightarrow{1\otimes g} R\bigotimes_{R/J} Z \longrightarrow 0$$

is an exact sequence of R-modules if and only if

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

is an exact sequence of R/J-modules.

Proof ($\Rightarrow$) It is a routine check.
($\Leftarrow$) It is obvious since R/J is semisimple.

Lemma 3.2 *Let* $0 \longrightarrow K \longrightarrow M \longrightarrow N \longrightarrow 0$ *be an exact sequence in* $\mathrm{mod}(R)$*. Then* $JK = K \cap JM$ *if and only if we have the following commutative diagram with exact rows and columns*

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \Omega^1(K) \arrow[r] \arrow[d] & \Omega^1(M) \arrow[r] \arrow[d] & \Omega^1(N) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P_0 \arrow[r] \arrow[d] & Q_0 \arrow[r] \arrow[d] & L_0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

where $P_0 \longrightarrow K \longrightarrow 0$, $Q_0 \longrightarrow M \longrightarrow 0$ *and* $L_0 \longrightarrow N \longrightarrow 0$ *are projective covers, respectively.*

Proof ($\Rightarrow$) Clearly, we obtain the exact sequence

$$0 \longrightarrow K/JK \longrightarrow M/JM \longrightarrow N/JN \longrightarrow 0.$$

Note that for any $M \in \mathrm{mod}(R)$, $R\bigotimes_{R/J} M/JM \longrightarrow M \longrightarrow 0$ is a projective cover. Now setting

$$P_0 := R\bigotimes_{R/J} K/JK, \quad Q_0 := R\bigotimes_{R/J} M/JM, \quad L_0 := R\bigotimes_{R/J} N/JN.$$

By Lemma 3.1, we have the following exact sequence

$$0 \longrightarrow P_0 \longrightarrow Q_0 \longrightarrow L_0 \longrightarrow 0$$

since R/J is semisimple. Therefore, we have the desired diagram.

($\Leftarrow$) Suppose that we have the above commutative diagram. Note that the projective cover of a module is unique up to isomorphisms. We may assume that

$$P_0 := R\bigotimes_{R/J} K/JK, \quad Q_0 := R\bigotimes_{R/J} M/JM, \quad L_0 := R\bigotimes_{R/J} N/JN.$$

From the middle row of the diagram, we have the following exact sequence

$$0 \longrightarrow R\bigotimes_{R/J} K/JK \longrightarrow R\bigotimes_{R/J} M/JM \longrightarrow R\bigotimes_{R/J} N/JN \longrightarrow 0.$$

By Lemma 3.1, we have the following short exact sequence as R/J-modules

$$0 \longrightarrow K/JK \longrightarrow M/JM \longrightarrow N/JN \longrightarrow 0,$$

which implies $JK = K \cap JM$.

Lemma 3.3 *Let* $0 \longrightarrow K \longrightarrow M \longrightarrow N \longrightarrow 0$ *be a short exact sequence in* $\mathrm{mod}(R)$*. Then* $J\Omega^i(K) = \Omega^i(K) \cap J\Omega^i(M)$ *for all* $i \geq 0$ *if and only if the Minimal Horseshoe Lemma holds.*

Proof By Lemma 3.2, $J\Omega^i(K) = \Omega^i(K) \cap J\Omega^i(M)$ for all $i \geq 0$ if and only if, for all $i \geq 0$, we have the following commutative diagram with exact rows and columns

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \Omega^{i+1}(K) \arrow[r] \arrow[d] & \Omega^{i+1}(M) \arrow[r] \arrow[d] & \Omega^{i+1}(N) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P_i \arrow[r] \arrow[d] & Q_i \arrow[r] \arrow[d] & L_i \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Omega^{i}(K) \arrow[r] \arrow[d] & \Omega^{i}(M) \arrow[r] \arrow[d] & \Omega^{i}(N) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

where P_i, Q_i and L_i are projective covers of $\Omega^i(K)$, $\Omega^i(M)$ and $\Omega^i(N)$, respectively. Now putting this commutative diagrams together, we finish the proof.

Theorem 3.1 *Let* $0 \longrightarrow K \longrightarrow M \longrightarrow N \longrightarrow 0$ *be an exact sequence in the category of quasi-d-Koszul modules such that* $JK = K \cap JM$*. Then the Minimal Horseshoe Lemma holds.*

Proof By Lemma 3.2, we have the following commutative diagram with exact rows and columns

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \Omega^1(K) \arrow[r] \arrow[d] & \Omega^1(M) \arrow[r] \arrow[d] & \Omega^1(N) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P_0 \arrow[r] \arrow[d] & Q_0 \arrow[r] \arrow[d] & L_0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

since $JK = K \cap JM$, which implies the following commutative diagram with exact rows

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \Omega^1(K) \arrow[r] \arrow[d] & \Omega^1(M) \arrow[r] \arrow[d] & \Omega^1(N) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & JP_0 \arrow[r] \arrow[d] & JQ_0 \arrow[r] \arrow[d] & JL_0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & JK \arrow[r] \arrow[d] & JM \arrow[r] \arrow[d] & JN \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Applying the functor $R/J \bigotimes_R$ to the above diagram, we have the following commutative diagram with exact rows

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & R/J \bigotimes_R \Omega^1(K) \arrow[r, "\beta"] \arrow[d, "\alpha"] & R/J \bigotimes_R \Omega^1(M) \arrow[r] \arrow[d, "\gamma"] & R/J \bigotimes_R \Omega^1(N) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & R/J \bigotimes_R JP_0 \arrow[r] & R/J \bigotimes_R JQ_0 \arrow[r] & R/J \bigotimes_R JL_0 \arrow[r] & 0
\end{tikzcd}$$

where α and γ are monomorphisms since K, M are quasi-d-Koszul modules, β is a monomorphism induced by the commutativity of the left square. Hence $J\Omega^1(K) = \Omega^1(K) \cap J\Omega^1(M)$.

By Lemma 3.2 again, we have the following commutative diagram with exact rows and columns

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \Omega^2(K) \arrow[r] \arrow[d] & \Omega^2(M) \arrow[r] \arrow[d] & \Omega^2(N) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P_1 \arrow[r] \arrow[d] & Q_1 \arrow[r] \arrow[d] & L_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Omega^1(K) \arrow[r] \arrow[d] & \Omega^1(M) \arrow[r] \arrow[d] & \Omega^1(N) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Note that $K,\ M$ and N are quasi-d-Koszul modules, we have the following commutative diagram with exact rows

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \Omega^2(K) \arrow[r] \arrow[d] & \Omega^2(M) \arrow[r] \arrow[d] & \Omega^2(N) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & J^{d-1}P_1 \arrow[r] & J^{d-1}Q_1 \arrow[r] & J^{d-1}L_1 \arrow[r] & 0
\end{tikzcd}$$

Now applying the functor $R/J \bigotimes_R$ to the above diagram and repeating the above procedures, we finish the proof by Lemma 3.3.

Now we end this paper with an example to expound that the Minimal Horseshoe Lemma does not hold in general.

Example 3.1 Let $M \in \text{mod}(R)$ with $\text{Rad}(M) \neq 0$, where $\text{Rad}(M)$ denotes the Jacobson radical of M. Set $K = \text{Rad}(M)$, $N = M/\text{Rad}(M)$. Then we have the following commutative diagram with exact rows and columns

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{P}_* \arrow[r] \arrow[d] & \mathcal{Q}_* \arrow[r] \arrow[d] & \mathcal{L}_* \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

where both sides are minimal projective resolutions and the middle column is a projective resolution. Now we claim that $Q_0 \longrightarrow M \longrightarrow 0$ must not be a projective cover. If not, note that $N = M/\text{Rad}(M)$, we have $Q_0 = L_0$, which forces $P_0 = 0$. It is impossible since $K \neq 0$. Therefore, the "Minimal Horseshoe Lemma" does not hold in this case.

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