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On the Distribution of Integral Ideals and Hecke Grössencharacters**

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Abstract The author uses analytic methods to study the distribution of integral ideals and Hecke Grössencharacters in algebraic number fields. Nowak's results on the distribution of integral ideals, and Chandrasekharan and Good's results on the distribution of Hecke Grössencharacters are improved.

Keywords Dedekind zeta-function, Integral ideal, Grössencharacters
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1 Introduction and Main Results

Let K be an algebraic number field of finite degree n over the field $\mathbb{Q}$ of rational numbers. The Dedekind zeta-function $\zeta_K(s)$ of the field K is defined by, for $\sigma > 1$,

$$\zeta_K(s) = \sum_{\mathfrak{a}} \frac{1}{N(\mathfrak{a})^s}, \quad s = \sigma + \mathrm{i}t,$$

where $\mathfrak{a}$ varies over the integral ideals of K, and $N(\mathfrak{a})$ denotes its norm.

For a large real variable x, denote $A(x)$ the number of integral ideals $\mathfrak{a}$ with norm $N(\mathfrak{a}) \leq x$, i.e.,

$$A(x) = \sum_{N(\mathfrak{a}) \leq x} 1.$$

It was already known to Weber [1] that

$$A(x) = cx + O(x^{1-\frac{1}{n}}), \tag{1.1}$$

where c is a constant depending only on K, namely

$$c = h\frac{2^{r_1+r_2}\pi^{r_2}R}{\omega\sqrt{|\Delta|}}.$$

Here r_1 is the number of real conjugates and $2r_2$ is the number of non-real conjugates of K (thus $r_1 + 2r_2 = n$), ω is the number of roots of unity in K, Δ is the discriminant, R is the

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so-called regulator, and h is the class number. The estimate (1.1) was improved by Landau [2] to

$$A(x) = cx + O(x^{1-\frac{2}{n+1}+\varepsilon}), \tag{1.2}$$

where and throughout this paper ε is a sufficiently small positive constant, and may not be the same at each occurrence. In the opposite direction, Landau also showed that the Ω-result of this error term is $\Omega(x^{\frac{n-1}{2n}})$.

No improvement of the upper bound (1.2) had been established until early 1990's. For quadratic fields, the problem is essentially a planar lattice point problem. Based on Huxley's deep works on planar lattice point problems, Huxley and Watt [3] proved

$$A(x) = cx + O(x^{\frac{23}{73}}(\log x)^{\frac{315}{146}}).$$

For cubic fields, Müller [4] proved

$$A(x) = cx + O(x^{\frac{43}{96}+\varepsilon}).$$

For any algebraic number field of degree $n \geq 3$, Nowak [5] successfully combined Landau's classical method with Titchmarch's two-dimensional exponential sum method to obtain the best result hitherto

$$A(x) = cx + \begin{cases} O(x^{1-\frac{2}{n}+\frac{8}{n(5n+2)}}(\log x)^{\frac{10}{5n+2}}) & \text{for } \ 3 \leq n \leq 6, \\ O(x^{1-\frac{2}{n}+\frac{3}{2n^2}}(\log x)^{\frac{2}{n}}) & \text{for } \ n \geq 7. \end{cases} \tag{1.3}$$

In this paper, we are able to improve Nowak's result when n is large.

Theorem 1.1 *For any number field of degree n over $\mathbb{Q}$, we have*

$$A(x) = cx + O(x^{1-\frac{3}{n+6}+\varepsilon}).$$

Remark 1.1 When n is large, our result $1 - \frac{3}{n+6}$ is much better than Nowak's result $1 - \frac{2}{n} + \frac{3}{2n^2}$. In fact, it is easy to check that our result is better than Nowak's result when n is greater than 9.

Our arguments can also be used to study the distribution of Hecke Grössen characters on ideals in algebraic number field K of finite degree n over $\mathbb{Q}$. More precisely, we want to study the sum

$$B(x) = \sum_{N(\mathfrak{a}) \leq x} \lambda(\mathfrak{a}),$$

where λ is the Hecke Grössencharacter on ideals. The detailed definition of the Hecke Grössencharacter on ideals will be given in Section 4 because of its complex description.

Although, Landau never studied this problem, his method essentially gives

$$B(x) = \sum_{N(\mathfrak{a}) \leq x} \lambda(\mathfrak{a}) \ll x^{1-\frac{2}{n+1}+\varepsilon}.$$

In [6], Chandraseknaran and Narasimhan were able to remove the factor x^{ε} of the above result, i.e.,

$$B(x) \ll x^{1-\frac{2}{n+1}}. \tag{1.4}$$

In this paper, we will improve Chandraseknaran and Narasimhan's result when n is large.

Theorem 1.2 *For any number field of degree n over $\mathbb{Q}$, we have*

$$B(x) \ll x^{1-\frac{3}{n+6}+\varepsilon}.$$

Remark 1.2 Our result is better than the result of Chandraseknaran and Narasimhan when n is greater than 9.

2 Preliminaries

To prove our results, we need the following lemmas.

Lemma 2.1 (cf. [7]) *Let $h(x)$ be the function given by*

$$h(x) = \begin{cases} 1, & \text{if } x > 1, \\ \frac{1}{2}, & \text{if } x = 1, \\ 0, & \text{if } x < 1. \end{cases}$$

Then

$$\frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}T}^{c+\mathrm{i}T} x^s \frac{\mathrm{d}s}{s} = h(x) + O\Big(\frac{x^c}{T|\log x|}\Big)$$

for any $x > 0$, $x \neq 1$, $T > 0$ and $0 < c \leq 2$, with an absolute implied constant. If $x = 1$, the factor $|\log x|$ is omitted.

Lemma 2.2 (cf. [8]) *Let K be an algebraic number field of degree n and $\zeta_K(s)$ the Dedekind zeta-function. Then*

$$\zeta_K\Big(\frac{1}{2} + \mathrm{i}t\Big) \ll t^{\frac{n}{6}+\varepsilon}, \quad t \geq 1$$

for any fixed $\varepsilon > 0$.

Lemma 2.3 (cf. [9]) *Let K be an algebraic number field of degree n and $\mathfrak{f}$ an integral ideal. Let $\zeta_K(\frac{1}{2} + \mathrm{i}t, \lambda)$ be the Hecke zeta-function with Grössencharacter $\lambda \bmod \mathfrak{f}$ (see Section 4 for detailed definition). Then we have that for any fixed $\varepsilon > 0$,*

$$\zeta_K\Big(\frac{1}{2} + \mathrm{i}t, \lambda\Big) \ll t^{\frac{n}{6}+\varepsilon}, \quad t \geq 1.$$

Lemma 2.4 (cf. [7]) *Let f be a function holomorphic on an open neighborhood of a strip $a \leq \sigma \leq b$, for some real numbers $a < b$, such that $|f(s)| \ll \exp(|s|^A)$ for some $A \geq 0$ and $a \leq \sigma \leq b$.*

Assume that for $t \in \mathbb{R}$,

$$|f(a + \mathrm{i}t)| \leq M_a(1 + |t|)^{\alpha}, \quad |f(b + \mathrm{i}t)| \leq M_b(1 + |t|)^{\beta}.$$

Then

$$|f(\sigma + \mathrm{i}t)| \leq \{M_a(1 + |t|)^{\alpha}\}^{\frac{b-\sigma}{b-a}} \{M_b(1 + |t|)^{\beta}\}^{\frac{\sigma-a}{b-a}}.$$

3 Proof of Theorem 1.1

Let K be an algebraic number field of finite degree n over $\mathbb{Q}$ and $\mathfrak{C}$ an ideal class of K. We define the Dedekind zeta-function of class $\mathfrak{C}$ by the relation

$$\zeta_K(s,\mathfrak{C}) = \sum_{\mathfrak{a}\in\mathfrak{C}} \frac{1}{N(\mathfrak{a})^s},$$

where the sum is taken over the non-zero integral ideals in $\mathfrak{C}$. Recall that the Dedekind zeta-function is defined by

$$\zeta_K(s) = \sum_{\mathfrak{a}} \frac{1}{N(\mathfrak{a})^s},$$

where $\mathfrak{a}$ runs over all non-zero integral ideals of K. Then clearly

$$\zeta_K(s) = \sum_{\mathfrak{C}} \zeta_K(s,\mathfrak{C}).$$

It was already known to Hecke [2] that $\zeta_K(s,\mathfrak{C})$ is a meromorphic function, with a simple pole at $s=1$, with the residue

$$c' = \frac{2^{r_1+r_2}\pi^{r_2}R}{\omega\sqrt{|\Delta|}}.$$

It also satisfies the functional equation

$$\xi(s,\mathfrak{C}) = \xi(1-s,\widetilde{\mathfrak{C}}),$$

where

$$\xi(s,\mathfrak{C}) = B^s\Gamma^{r_1}\Big(\frac{s}{2}\Big)\Gamma^{r_2}(s)\zeta_K(s,\mathfrak{C}), \quad B = 2^{-r_2}\pi^{-\frac{n}{2}}\sqrt{|\Delta|}.$$

Then the Dedekind zeta-function $\zeta_K(s)$ has a simple pole at $s=1$ with residue hc', where h is the class number of K, and satisfies a similar functional equation. This shows that the Dedekind zeta-function is a meromorphic function on $\mathbb{C}$ of order 1. We can also write the Dedekind zeta-function as

$$\zeta_K(s) = \sum_{k=1}^{\infty}\Big(\sum_{N(\mathfrak{a})=k} 1\Big)k^{-s} := \sum_{k=1}^{\infty} a_k k^{-s}. \tag{3.1}$$

Chandraseknaran and Good [10] proved that a_k is a multiplicative function, and satisfies

$$a_k \le d(k)^n, \tag{3.2}$$

where $d(k)$ is the divisor function, and n is the degree of $K/\mathbb{Q}$.

By (3.1) and Lemma 2.1, we have

$$A(x) = \sum_{N(\mathfrak{a})\le x} 1 = \sum_{k\le x} a_k = \frac{1}{2\pi\mathrm{i}}\int_{b-\mathrm{i}T}^{b+\mathrm{i}T} \zeta_K(s)\frac{x^s}{s}\mathrm{d}s + O\Big(\frac{x^{1+\varepsilon}}{T}\Big), \tag{3.3}$$

where $b = 1+\varepsilon$ and $1 \le T \le x$ is a parameter to be chosen later. Here we have used (3.2).

Next we move the integration to the parallel segment with $\mathrm{Re}s = \frac{1}{2}+\varepsilon$. By Cauchy's residue theorem, we have

$$\begin{aligned} A(x) = {} & \frac{1}{2\pi\mathrm{i}}\Big\{ \int_{\frac{1}{2}+\varepsilon-\mathrm{i}T}^{\frac{1}{2}+\varepsilon+\mathrm{i}T} + \int_{\frac{1}{2}+\varepsilon+\mathrm{i}T}^{b+\mathrm{i}T} + \int_{b-\mathrm{i}T}^{\frac{1}{2}+\varepsilon-\mathrm{i}T} \Big\}\zeta_K(s)\frac{x^s}{s}\mathrm{d}s \\ & + \mathrm{Res}_{s=1}\zeta_K(s)x + O\Big(\frac{x^{1+\varepsilon}}{T}\Big) \\ := {} & cx + \mathrm{J}_1 + \mathrm{J}_2 + \mathrm{J}_3 + O\Big(\frac{x^{1+\varepsilon}}{T}\Big), \end{aligned} \tag{3.4}$$

where

$$c = h\frac{2^{r_1+r_2}\pi^{r_2}R}{\omega\sqrt{|\Delta|}}.$$

For J_1, we have

$$\mathrm{J}_1 \ll x^{\frac{1}{2}+\varepsilon} + x^{\frac{1}{2}+\varepsilon}\int_1^T \Big|\zeta_K\Big(\frac{1}{2}+\varepsilon+\mathrm{i}t\Big)\Big|t^{-1}\mathrm{d}t.$$

Then by Lemma 2.2, we have

$$\begin{aligned} \mathrm{J}_1 & \ll x^{\frac{1}{2}+\varepsilon} + x^{\frac{1}{2}+\varepsilon}\log T \max_{T_1\le T}\Big\{T_1^{-1}\int_{\frac{T_1}{2}}^{T_1} \Big|\zeta_K\Big(\frac{1}{2}+\varepsilon+\mathrm{i}t\Big)\Big|\mathrm{d}t\Big\} \\ & \ll x^{\frac{1}{2}+\varepsilon} + x^{\frac{1}{2}+\varepsilon}\log T \max_{T_1\le T}\Big\{T_1^{-1}\int_{\frac{T_1}{2}}^{T_1} t^{\frac{n}{6}+\varepsilon}\mathrm{d}t\Big\} \\ & \ll x^{\frac{1}{2}+\varepsilon} + x^{\frac{1}{2}+\varepsilon}T^{\frac{n}{6}+\varepsilon}. \end{aligned} \tag{3.5}$$

For $\sigma > 1$, we have

$$\zeta_K(\sigma+\mathrm{i}t) \ll 1.$$

Then by Lemmas 2.2 and 2.4, we have, for $\frac{1}{2} \le \sigma \le 1$,

$$\zeta_K(\sigma+\mathrm{i}t) \ll (1+|t|)^{\frac{n}{6}\times\frac{1-\sigma}{1-\frac{1}{2}}+\varepsilon} \ll (1+|t|)^{\frac{n}{3}(1-\sigma)+\varepsilon}. \tag{3.6}$$

Therefore for the integrals over the horizontal segments, we have

$$\begin{aligned} \mathrm{J}_2 + \mathrm{J}_3 & \ll \int_{\frac{1}{2}+\varepsilon}^{b} x^{\sigma}|\zeta_K(\sigma+\mathrm{i}T)|T^{-1}\mathrm{d}\sigma \\ & \ll \max_{\frac{1}{2}+\varepsilon\le\sigma\le b} x^{\sigma}T^{\frac{n}{3}(1-\sigma)+\varepsilon}T^{-1} \\ & = \max_{\frac{1}{2}+\varepsilon\le\sigma\le b}\Big(\frac{x}{T^{\frac{n}{3}}}\Big)^{\sigma}T^{\frac{n}{3}-1+\varepsilon} \\ & \ll \frac{x^{1+\varepsilon}}{T} + x^{\frac{1}{2}+\varepsilon}T^{\frac{n}{6}-1+\varepsilon}. \end{aligned} \tag{3.7}$$

From (3.4), (3.5) and (3.7), we have

$$A(x) = cx + O(x^{1+\varepsilon}T^{-1+\varepsilon}) + O(x^{\frac{1}{2}+\varepsilon}T^{\frac{n}{6}+\varepsilon}). \tag{3.8}$$

Taking $T = x^{\frac{3}{n+6}}$ in (3.8), we have

$$A(x) = cx + O(x^{1-\frac{3}{n+6}+\varepsilon}).$$

This completes the proof of Theorem 1.1.

4 Proof of Theorem 1.2

Let K be an algebraic number field of finite degree n over $\mathbb{Q}$, and α be any element in K. For the conjugates of α, let

$$\alpha^{(q)} \in \mathbb{R} \ \ (1 \le q \le r_1), \quad \alpha^{(q+r_2)} = \overline{\alpha^{(q)}} \ \ (r_1 + 1 \le q \le r_1 + r_2 = r + 1).$$

Then a Grössencharacter for numbers to the modulus $\mathfrak{f}$, $\mathfrak{f}$ an integral ideal of K, is given by

$$\mu(\alpha) := \prod_{q=1}^{r_1} |\alpha^{(q)}|^{-\mathrm{i}b_q} \prod_{q=r_1+1}^{r+1} \Big(\frac{\alpha^{(q)}}{|\alpha^{(q)}|}\Big)^{2a_q} |\alpha^{(q)}|^{-2\mathrm{i}b_q},$$

where the exponents $2a_q \in \mathbb{Z}$ $(r_1 + 1 \le q \le r + 1)$ and $b_q \in \mathbb{R}$ $(1 \le q \le r + 1)$ satisfy some relations, which ensure $\mu(\eta) = 1$ for any unit $\eta \equiv 1 \ (\mathrm{mod}\, \mathfrak{f})$ with $\eta^{(q)} > 0$ $(1 \le q \le r_1)$.

A totally multiplicative function λ defined on the semigroup of ideals of K is called Grössencharacter for ideals modulo $\mathfrak{f}$ belonging to μ, if its value on principal ideals can be expressed in the form

$$\lambda(\alpha) = \chi(\alpha)v(\alpha)\mu(\alpha),$$

where χ is a multiplicative character to the modulus $\mathfrak{f}$, and v is a sign character, i.e., a function defined by

$$v(\alpha) = \prod_{q=1}^{r_1} \Big(\frac{\alpha^{(q)}}{|\alpha^{(q)}|}\Big)^{a_q} \qquad \text{for some } a_q \in \{0, 1\}.$$

A Hecke zeta-function with Grössencharacter λ is given by analytic continuation of

$$\zeta_K(s, \lambda) = \sum_{\mathfrak{a}} \frac{\lambda(\mathfrak{a})}{N(\mathfrak{a})^s}, \quad \sigma = \mathrm{Re}\, s > 1,$$

where the sum is taken over all ideals of K.

If λ is not the principal character, then $\xi_K(s, \lambda) = A^s \Gamma(s, \lambda)\zeta_K(s, \lambda)$ is an entire function and obeys the functional equation

$$\xi_K(s, \lambda) = W\overline{\xi_K(1 - \overline{s}, \lambda)}$$

with some complex number W of modulus 1, where

$$A = (|\Delta| N(\mathfrak{f}) \pi^{-n} 2^{-2r_2})^{\frac{1}{2}},$$
$$\Gamma(s, \lambda) = \prod_{q=1}^{r+1} \Gamma\Big(\frac{e_q}{2}(s + |a_q| + \mathrm{i}b_q)\Big),$$

with e_q equaling 1 if $1 \le q \le r_1$, and 2 if $r_1 + 1 \le q \le r + 1$.

We also write the Hecke zeta-function with Grössencharacter λ as

$$\zeta_K(s, \lambda) = \sum_{k=1}^{\infty} \Big(\sum_{N(\mathfrak{a})=k} \lambda(\mathfrak{a})\Big) k^{-s} := \sum_{k=1}^{\infty} c_k k^{-s}. \tag{4.1}$$

Chandraseknaran and Narasimhan [6] proved that c_k are multiplicative, and satisfy

$$c_k \ll k^{\varepsilon} \tag{4.2}$$

for any $\varepsilon > 0$.

By (4.1) and Lemma 2.1, we have

$$B(x) = \sum_{N(\mathfrak{a})\le x} \lambda(\mathfrak{a}) = \sum_{k\le x} c_k = \frac{1}{2\pi \mathrm{i}} \int_{b-\mathrm{i}T}^{b+\mathrm{i}T} \zeta_K(s,\lambda)\frac{x^s}{s}\mathrm{d}s + O\Big(\frac{x^{1+\varepsilon}}{T}\Big), \tag{4.3}$$

where $b = 1+\varepsilon$ and $1 \le T \le x$ is a parameter to be chosen later. Here we have used (4.2).

Next we move the integration to the parallel segment with $\mathrm{Re}\, s = \frac{1}{2}+\varepsilon$. By Cauchy's residue theorem, we have

$$\begin{aligned} B(x) &= \frac{1}{2\pi\mathrm{i}}\Big\{ \int_{\frac{1}{2}+\varepsilon-\mathrm{i}T}^{\frac{1}{2}+\varepsilon+\mathrm{i}T} + \int_{\frac{1}{2}+\varepsilon+\mathrm{i}T}^{b+\mathrm{i}T} + \int_{b-\mathrm{i}T}^{\frac{1}{2}+\varepsilon-\mathrm{i}T} \Big\}\zeta_K(s,\lambda)\frac{x^s}{s}\mathrm{d}s + O\Big(\frac{x^{1+\varepsilon}}{T}\Big) \\ &:= \mathrm{J}_1 + \mathrm{J}_2 + \mathrm{J}_3 + O\Big(\frac{x^{1+\varepsilon}}{T}\Big). \end{aligned} \tag{4.4}$$

Here we have used that $\zeta_K(s,\lambda)$ is an entire function.

For J_1, we have

$$\mathrm{J}_1 \ll x^{\frac{1}{2}+\varepsilon} + x^{\frac{1}{2}+\varepsilon}\int_1^T \Big|\zeta_K\Big(\frac{1}{2}+\varepsilon+\mathrm{i}t,\lambda\Big)\Big| t^{-1}\mathrm{d}t.$$

Then by Lemma 2.3, we have

$$\begin{aligned} \mathrm{J}_1 &\ll x^{\frac{1}{2}+\varepsilon} + x^{\frac{1}{2}+\varepsilon}\log T \max_{T_1\le T}\Big\{T_1^{-1}\int_{\frac{T_1}{2}}^{T_1}\Big|\zeta_K\Big(\frac{1}{2}+\varepsilon+\mathrm{i}t,\lambda\Big)\Big|\mathrm{d}t\Big\} \\ &\ll x^{\frac{1}{2}+\varepsilon} + x^{\frac{1}{2}+\varepsilon}\log T \max_{T_1\le T}\Big\{T_1^{-1}\int_{\frac{T_1}{2}}^{T_1} t^{\frac{n}{6}+\varepsilon}\mathrm{d}t\Big\} \\ &\ll x^{\frac{1}{2}+\varepsilon} + x^{\frac{1}{2}+\varepsilon}T^{\frac{n}{6}+\varepsilon}. \end{aligned} \tag{4.5}$$

For $\sigma > 1$, we have

$$\zeta_K(\sigma+\mathrm{i}t,\lambda) \ll 1.$$

By Lemmas 2.3 and 2.4, we have that for $\frac{1}{2}\le\sigma\le 1$,

$$\zeta_K(\sigma+\mathrm{i}t,\lambda) \ll (1+|t|)^{\frac{n}{6}\times\frac{1-\sigma}{1-\frac{1}{2}}+\varepsilon} \ll (1+|t|)^{\frac{n}{3}(1-\sigma)+\varepsilon}. \tag{4.6}$$

For the integrals over the horizontal segments, we have

$$\begin{aligned} \mathrm{J}_2 + \mathrm{J}_3 &\ll \int_{\frac{1}{2}+\varepsilon}^{b} x^{\sigma}|\zeta_K(\sigma+\mathrm{i}T,l)|T^{-1}\mathrm{d}\sigma \\ &\ll \max_{\frac{1}{2}+\varepsilon\le\sigma\le b} x^{\sigma}T^{\frac{n}{3}(1-\sigma)+\varepsilon}T^{-1} \\ &= \max_{\frac{1}{2}+\varepsilon\le\sigma\le b}\Big(\frac{x}{T^{\frac{n}{3}}}\Big)^{\sigma}T^{\frac{n}{3}-1+\varepsilon} \\ &\ll \frac{x^{1+\varepsilon}}{T} + x^{\frac{1}{2}+\varepsilon}T^{\frac{n}{6}-1+\varepsilon}. \end{aligned} \tag{4.7}$$

From (4.4), (4.5) and (4.7), we have

$$B(x) \ll x^{1+\varepsilon}T^{-1+\varepsilon} + x^{\frac{1}{2}+\varepsilon}T^{\frac{n}{6}+\varepsilon}. \tag{4.8}$$

Taking $T = x^{\frac{3}{n+6}}$ in (4.8), we have

$$B(x) \ll x^{1-\frac{3}{n+6}+\varepsilon}.$$

This completes the proof of Theorem 1.2.

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