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Dehn Twists and Products of Mapping Classes of Riemann Surfaces with One Puncture

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Abstract Let S be a Riemann surface that contains one puncture x. Let $\mathscr{S}$ be the collection of simple closed geodesics on S, and let $\mathscr{F}$ denote the set of mapping classes on S isotopic to the identity on $S \cup \{x\}$. Denote by t_c the positive Dehn twist about a curve $c \in \mathscr{S}$. In this paper, the author studies the products of forms $\left(t_b^{-m} \circ t_a^n\right) \circ f^k$, where $a, b \in \mathscr{S}$ and $f \in \mathscr{F}$. It is easy to see that if $a = b$ or a, b are boundary components of an x-punctured cylinder on S, then one may find an element $f \in \mathscr{F}$ such that the sequence $\left(t_b^{-m} \circ t_a^n\right) \circ f^k$ contains infinitely many powers of Dehn twists. The author shows that the converse statement remains true, that is, if the sequence $\left(t_b^{-m} \circ t_a^n\right) \circ f^k$ contains infinitely many powers of Dehn twists, then a, b must be the boundary components of an x-punctured cylinder on S and f is a power of the spin map $t_b^{-1} \circ t_a$.

Keywords Riemann surfaces, Simple closed geodesics, Dehn twists, Products, Bers isomorphisms
2000 MR Subject Classification 32G15, 30C60, 30F60

1 Statement of the Results

1.1 Introduction

The investigation of products of Dehn twists on compact Riemann surfaces dates back to [5]. It was shown by Dehn–Lickorish [5, 8] that some non-trivial products of Dehn twists about a set of Lickorish generators on a compact Riemann surface S of genus $p \geq 2$ are isotopic to Dehn twists. In [11], Thurston proved that certain products of pseudo-Anosov maps and Dehn twists may also give rise to Dehn twists. Later, Long-Morton [9] and Fathi [6] proved that some products of pseudo-Anosov maps may once again lead to Dehn twists.

More concretely, let $\mathscr{S}$ be the set of simple closed geodesics on S and let $a, b \in \mathscr{S}$ be such that their geometric intersection number $i(a, b) = 1$. Let t_a, t_b denote the positive Dehn twists about the curves a and b. Then the braids relation tells us that $t_a \circ [t_b, t_a] = t_b$ (where $[t_a, t_b]$ denotes the commutator of t_a and t_b), which implies that $(t_a \circ t_b)^6 = t_c$ for some $c \in \mathscr{S}$. For more information on presentations in the mapping class groups, we refer to [4] and the literature therein.

In this paper, we study a similar problem whether combinations by different kinds of mapping classes may lead to simple mapping classes (such as Dehn twists, for example). Throughout the paper, we let S be an analytically finite Riemann surface of genus $p \geq 2$ that contains one

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puncture x. Set $\widetilde{S} = S \cup \{x\}$. Then $\widetilde{S}$ is compact. Let $\lambda(a, b; m, n)$ denote the mapping classes represented by the maps

$$t_b^{-m} \circ t_a^n, \tag{1.1}$$

where $a, b \in \mathscr{S}$ and m, n are positive integers with $m \neq n$. By abuse of languages, we write $\lambda(a, b; m, n) = t_b^{-m} \circ t_a^n$.

Let $\mathscr{F}$ be the set of mapping classes of S projecting to the identity on $\widetilde{S}$. In this paper, we investigate products of elements $\lambda(a, b; m, n)$ and $f \in \mathscr{F}$. Let a_0 denote a geodesic in $\mathscr{S}$ so that $\{a, a_0\}$ forms the boundary of an x-punctured cylinder on S, that is, a and a_0 are disjoint and are homotopic to each other on $\widetilde{S}$ as x is filled in. Throughout the paper, we call a_0 a geodesic parallel to a (there are infinitely many such geodesics a_0 (see [15])). Our goal is to identify those elements $\lambda(a, b; m, n)$ and f for which their products are represented by simple Dehn twists.

1.2 Special cases

(1) $b = a$. In this case, for any positive large integers m, n with $m \neq n$, we have

$$\lambda(a, b; m, n) = t_a^{-m+n} \quad \text{and} \quad \lambda(a_0, b; m, n) = t_a^{-m} \circ t_{a_0}^n.$$

If we take

$$f = t_a \circ t_{a_0}^{-1}, \tag{1.2}$$

then $f \in \mathscr{F}$ and one easily checks that for integers $k_0 = m - n$ and $k_1 = m$, the products $\lambda(a, b; m, n) \circ f^{k_0}$ and $\lambda(a_0, b; m, n) \circ f^{k_1}$ are powers of the Dehn twist along a_0.

(2) $b = a_0$. In this case, we find that

$$\lambda(a, b; m, n) = t_{a_0}^{-m} \circ t_a^n \quad \text{and} \quad \lambda(a_0, b; m, n) = t_{a_0}^{n-m}.$$

Now for the elements f defined as (1.2), there are integers $k_0 = -m$ and $k_1 = n - m$, such that $\lambda(a, b; m, n) \circ f^{k_0}$ and $\lambda(a_0, b; m, n) \circ f^{k_1}$ are also represented by the powers of the Dehn twist along a.

1.3 Statement of the main result

The theorems of this paper essentially show that the above examples (1) and (2) are the only instantiations for which both sequences $\lambda(a, b; m_i, n_i) \circ f^{k_i}$ and $\lambda(a_0, b; M_i, N_i) \circ f^{K_i}$, $f \in \mathscr{F}$ and $a, b \in \mathscr{S}$, could possibly be represented by powers of the known Dehn twists t_a and t_{a_0}.

Theorem 1.1 *Let $a, b \in \mathscr{S}$ and $f \in \mathscr{F}$ be a non-trivial element. If there is a sequence of triples $\{m_i, n_i, k_i\}$ with $m_i, n_i, k_i \to +\infty$ as $i \to +\infty$, such that each mapping class in the set*

$$\mathscr{L}(a) = \{\lambda(a, b; m_i, n_i) \circ f^{k_i},\ i = 1, 2, \cdots\}$$

is represented by a power of t_a (resp. t_{a_0} for a geodesic a_0 parallel to a), then $b = a_0$ (resp. $b = a$), in which cases, f is represented by an appropriate power of the map $t_a \circ t_{a_0}^{-1}$.

Our next result is similar to Theorem 1.1.

Theorem 1.2 *Let a, b, f be as in Theorem 1.1. Let a_0 be a geodesic parallel to a. If there are sequences of triples $\{m_i, n_i, k_i\}$ and $\{M_i, N_i, K_i\}$, where $m_i, n_i, k_i, M_i, N_i, K_i \to +\infty$ as $i \to +\infty$, such that each mapping class in the set*

$$\mathscr{L}(a, a_0) = \{\lambda(a, b; m_i, n_i) \circ f^{k_i},\ \lambda(a_0, b; M_i, N_i) \circ f^{K_i};\ i = 1, 2, \cdots\} = L(a) \cup L(a_n)$$

is represented by a power of t_a or t_{a_0}, then either $b = a$ or $b = a_0$, in which cases, f is represented by an appropriate power of the map $t_a \circ t_{a_0}^{-1}$. More precisely, if $L(a)$ only contains powers of t_a (resp. powers of t_{a_0}), then so does $L(a_0)$, which occurs only when $b = a_0$ (resp. $b = a$).

1.4 Remarks

Because of the symmetry of a and b, we can conclude that if there are sequences $\{m_i, n_i, k_i\}$ and $\{M_i, N_i, K_i\}$ such that every mapping class in $\mathscr{L}(b, b_0) = \{\lambda(a, b; m_i, n_i) \circ f^{k_i},\ \lambda(a, b_0; M_i, N_i) \circ f^{K_i};\ i = 1, 2, \cdots\}$ is represented by a power of t_b or t_{b_0}, then $a = b$ or $a = b_0$.

This paper is organized as follows. In Section 2, we present some background materials including some lifts of Dehn twists and explore some dynamic properties for those lifts. In Section 3, we study the compositions $\lambda(a, b; m_i, n_i) \circ f^{k_i}$, where $a, b \in \mathscr{S}$ and $f \in \mathscr{F}$, that are represented by powers of Dehn twists. We show that if $\mathscr{L}(a)$ consists of powers of t_a, then $i(a, b) = 0$ which implies that $a = b$ or a is disjoint from b. Section 5 is devoted to the proof of the results.

2 Background and Properties of Lifts of Dehn Twists to the Hyperbolic Plane

2.1 The mapping class group Mod_S and its projection

Let

$$\mathbf{H} = \{z \in \mathbf{C} : |z| < 1\}$$

be the hyperbolic disk. Denote by $\mathbf{S}^1$ the boundary of $\mathbf{H}$. Let

$$\varrho : \mathbf{H} \to \widetilde{S}$$

be the universal holomorphic covering map with the covering group G. We know from the hypothesis that G is a Fuchsian group that contains only hyperbolic elements.

Let $\mathrm{QC}(G)$ be the group of quasiconformal self-maps f of $\mathbf{H}$ such that $fGf^{-1} = G$. An equivalent relation "$\sim$" in $\mathrm{QC}(G)$ is defined as follows. Two maps $f, f' \in \mathrm{QC}(G)$ are said to be equivalent if they share the common boundary values on $\mathbf{S}^1$. Denote by $[f]$ the equivalence class of an element $f \in \mathrm{QC}(G)$.

By [2], there is an isomorphism φ^* of the quotient group $\mathrm{QC}(G)/\sim$ onto the mapping class group Mod_S that consists of isotopy classes of self-maps of S. Since S contains only one puncture, each mapping class fixes the puncture x. Under the Bers isomorphism, $\varphi^*(G)$ is a subgroup of Mod_S consisting of elements that project to a trivial mapping class of $\widetilde{S}$. By abuse of language, in what follows we use the symbol $[f]^*$ to denote the mapping class $\varphi^*([f])$ as well as a representative of $\varphi^*([f])$ for an element $f \in \mathrm{QC}(G)$. In particular, for an element $h \in G$, we use the symbol h^* to denote the mapping class $\varphi^*(h)$ as well as a representative of $\varphi^*(h)$.

Note that every element f in $\mathrm{QC}(G)$ descends to a homeomorphism $\widetilde{f}$ of $\widetilde{S}$ under the universal covering map ϱ. Denote by

$$i : \mathrm{Mod}_S \to \mathrm{Mod}_{\widetilde{S}}$$

the natural projection defined by sending a mapping class $[f]^*$ on S to the mapping class of $\widetilde{f}$.

2.2 Dehn twists and their lifts to H

Let $\widehat{c} \subset \mathbf{H}$ be a geodesic so that $\varrho(\widehat{c}) = \widetilde{c}$ is a simple closed geodesic. Let $t_{\widetilde{c}}$ be the positive Dehn twist along $\widetilde{c}$. Let Δ be a component of $\mathbf{H}\backslash\{\widehat{c}\}$. The positive Dehn twist $t_{\widetilde{c}}$ can be lifted to a map $\tau_{\widehat{c}} : \mathbf{H} \to \mathbf{H}$ with respect to Δ that determines a collection $\mathscr{U}_{\widehat{c}}$ of half-planes in a partial order defined by inclusion. All maximal elements of $\mathscr{U}_{\widehat{c}}$ are disjoint and each maximal element is an invariant region under the action of $\tau_{\widehat{c}}$. Moreover, $\tau_{\widehat{c}}$ restricts to the identity on the complement $\Omega_{\widehat{c}}$ of all maximal elements of $\mathscr{U}_{\widehat{c}}$. See [13–15] for more details.

For every maximal element $\Delta \in \mathscr{U}_{\widehat{c}}$, there are infinitely many maximal elements $\Delta' \in \mathscr{U}_{\widehat{c}}$ such that $\Delta' \subset \mathbf{H}\backslash\overline{\Delta}$. More precisely, for any hyperbolic element $g \in G$ whose repelling fixed point is covered by Δ and whose attracting fixed point is outside of Δ, $g(\mathbf{H}\backslash\overline{\Delta})$ is contained in a maximal element Δ' of $\mathscr{U}_{\widehat{c}}$ that is disjoint from Δ. Further, $g(\mathbf{H}\backslash\overline{\Delta})$ is also maximal if the axis c_g of g satisfies the property that $\varrho(c_g)$ intersects $\widetilde{c}$ only once.

Let $c \subset S$ be a simple closed geodesic isotopic to $\widetilde{c}$ on $\widetilde{S}$. From [13, Lemma 3.2], there is a lift $\tau_{\widehat{c}}$ constructed above, such that $[\tau_{\widehat{c}}]^* = t_c$. Note that if $\tau_{\widehat{c}}$ is a lift of $t_{\widetilde{c}}$ with respect to Δ, then $g^{-1}\tau$ is also a lift of $t_{\widetilde{c}}$ (but is with respect to $\Delta^* = \mathbf{H}\backslash\overline{\Delta}$). By [13, Lemma 3.2] again, $[g^{-1}\tau_{\widehat{c}}]^*$ is represented by the Dehn twist t_{c_0} along another simple closed geodesic c_0 parallel to c. Since every maximal element $\Delta \in \mathscr{U}_{\widehat{c}}$ determines the same $\tau_{\widehat{c}}$, we conclude that the set of maximal elements of $\mathscr{U}_{\widehat{c}}$ is one-to-one correspondent with the set of x-punctured cylinders on S all of which share the common boundary component c (see also [15]).

Let $f : S \to S$ be a homeomorphism. Assume that f is isotopic to t_c. Then, of course, f fixes the puncture x. As such, there is an element $[\omega] \in \mathrm{QC}(G)/\sim$, such that $[\omega]^*$ is represented by f.

Lemma 2.1 *With the above conditions, for every maximal element $\Delta \in \mathscr{U}_{\widehat{c}}$, ω leaves the interval $\Delta \cap \mathbf{S}^1$ invariant, i.e., $w(\Delta \cap \mathbf{S}^1) = \Delta \cap \mathbf{S}^1$.*

Proof By hypothesis, f and t_c represent the same isotopy class of maps in Mod_S. Under the Bers isomorphism of mod $\widetilde{S}$ onto Mod_S, we see that $[\omega] = [\tau_{\widehat{c}}]$, which says that $\omega|_{\mathbf{S}^1} = \tau_{\widehat{c}}|_{\mathbf{S}^1}$. By construction, $\tau_{\widehat{c}}$ keeps each maximal element Δ of $\mathscr{U}_{\widehat{c}}$ invariant. This implies that $\tau_{\widehat{c}}|_{\mathbf{S}^1}$ leaves invariant each interval $\Delta \cap \mathbf{S}^1$. So ω leaves invariant each interval $\Delta \cap \mathbf{S}^1$ as well.

2.3 Iterates of half-planes under $\tau_{\widehat{c}}$

Let h be a loxodromic Möbius transformation acting on $\widehat{\mathbf{C}}$, and let X, Y denote its attracting and repelling fixed points, respectively. Beardon [1, Theorem 4.3.10] stated that for any small neighborhoods U_X, U_Y of X, Y, respectively, there is an integer k, which depends only on U_Y and U_X and is independent of the choice of $z \in \widehat{\mathbf{C}}\backslash U_Y$, such that $h^k(z) \in U_X$.

In our application, $h = g \in G$ is a hyperbolic element keeping Δ invariant. In this situation, X, Y are attracting and repelling fixed points of g lying in $\mathbf{S}^1$. For any half-plane $\Delta_1 \subset \Delta$ with

$\partial\Delta_1$, the geodesic boundary of Δ_1 in $\mathbf{H}$, projecting to a simple closed geodesic $\varrho(\partial\Delta_1)$, the half-planes $g^k(\Delta_1)$ are all disjoint and shrink to X as $k \to +\infty$, which means that $g^m(\Delta_1) \subset U_X$ for large m and the Euclidean area of $g^m(\Delta_1)$ is smaller than that of Δ_1.

Now we proceed to examine the iteration of Δ_1 under the action of $\tau_{\widehat{c}}^k$. As mentioned before, we further assume that either $\varrho(\partial\Delta_1) = \widetilde{c}$ or $\varrho(\partial\Delta_1)$ is disjoint from $\widetilde{c}$. First, we observe from the construction that for any integer $k \neq 0$, $\tau_{\widehat{c}}^k(\partial\Delta_1) \cap \partial\Delta_1 = \emptyset$. Second, based upon the result mentioned above and by the construction of $\tau_{\widehat{c}}$, the regions $\tau_{\widehat{c}}^k(\Delta_1)$ are all half-planes and the sequence $\{\tau_{\widehat{c}}^k(\Delta_1)\}$ uniformly shrinks to the attracting fixed point X of g as $k \to +\infty$, as long as Δ_1 stays away from a small neighborhood of the repelling fixed point of g. Thus the Euclidean area of $\tau_{\widehat{c}}^k(\Delta_1)$ shrinks to zero as $k \to +\infty$.

In the sequel, we call the attracting (resp. repelling) fixed point of g the attracting (resp. repelling) endpoint of Δ with respect to $\tau_{\widehat{c}}$.

3 Mapping Classes Represented by Dehn Twists

In this section, we study the composite mapping classes $\lambda(a,b;m,n) \circ f^k$, where $f \in \mathscr{F}$ and $\lambda(a,b;m,n)$ are defined as in (1.1), that are also represented by Dehn twists. For any curve $c \in \mathscr{S}$, we let $\widetilde{c}$ denote the geodesic on $\widetilde{S}$ homotopic to c if c is viewed as a curve on $\widetilde{S}$. We begin with some general cases.

Lemma 3.1 *Suppose that $\lambda(a,b;m,n) \circ f^k = t_c^p$ for some Dehn twist t_c along a simple closed geodesic c on S. Then $\widetilde{a} = \widetilde{b} = \widetilde{c}$.*

Proof By assumption, $\lambda(a,b;m,n) \circ f^k = t_c^p$. Thus they project to $t_{\widetilde{c}}^p$. But since $i(f) = \mathrm{id}$, a calculation shows that

$$i\left(\lambda(a,b;m,n) \circ f^k\right) = i\left(\lambda(a,b;m,n)\right) = t_{\widetilde{b}}^{-m} \circ t_{\widetilde{a}}^n.$$

It follows that

$$t_{\widetilde{b}}^{-m} \circ t_{\widetilde{a}}^n = t_{\widetilde{c}}^p. \tag{3.1}$$

Since $\widetilde{S}$ is compact, $\widetilde{a}$ and $\widetilde{b}$ are both non-trivial. Thus $\widetilde{c}$ is also non-trivial. If $\widetilde{a} \neq \widetilde{b}$. There are two cases: (1) $\widetilde{a}$ is disjoint from $\widetilde{b}$; (2) $\widetilde{a}$ intersects $\widetilde{b}$. In the former case, the left-hand side of (3.1) is a multi-twist, while the right-hand side of (3.1) is a single Dehn twist. This is a contradiction. In the later case, $\widetilde{a}$ and $\widetilde{b}$ span a minimal surface on which $t_{\widetilde{b}}^{-m} \circ t_{\widetilde{a}}^n$ represents a hyperbolic mapping class. So by the classification of mapping classes (see [3]), on $\widetilde{S}$, $t_{\widetilde{b}}^{-m} \circ t_{\widetilde{a}}^n$ is pseudo-hyperbolic. In particular, it cannot be a parabolic mapping class, whereas $t_{\widetilde{c}}$, as a mapping class of $\widetilde{S}$, is parabolic. This contradicts (3.1). It follows that $\widetilde{a} = \widetilde{b}$. Now from (3.1), we get that $p = n - m$ and $\widetilde{a} = \widetilde{c}$. Thus Lemma 3.1 is proved.

Let $\tau_{\widehat{a}}, \tau_{\widehat{b}} : \mathbf{H} \to \mathbf{H}$ denote the lifts of $t_{\widetilde{a}}$ and $t_{\widetilde{b}}$, such that $[\tau_{\widehat{a}}]^* = t_a$ and $[\tau_{\widehat{b}}]^* = t_b$ (see Section 2.2). Let $\mathscr{U}_{\widehat{a}}$ and $\mathscr{U}_{\widehat{b}}$ be the corresponding collections of half-planes determined by $\tau_{\widehat{a}}$ and $\tau_{\widehat{b}}$, respectively. Let $i(a,b)$ denote the geometric intersection numbers between a and b. Then $i(a,b) \geq 0$ and $i(a,b) = 0$ if and only if $a = b$ or a is disjoint from b.

Lemma 3.2 *Let $\lambda(a,b;m,n)\circ f^k$ be represented by t_c^p for some curve $c\in\mathscr{S}$. If $i(a,b)>0$, then there are maximal elements $\Delta_1\in\mathscr{U}_{\widehat{a}}$ and $\Delta_2\in\mathscr{U}_{\widehat{b}}$ such that*

$$\Delta_1\cap\Delta_2\neq\emptyset,\quad \partial\Delta_1\cap\partial\Delta_2=\emptyset\quad \textit{and}\quad \Delta_1\cup\Delta_2=\mathbf{H}. \tag{3.2}$$

Proof By Lemma 3.1, we have $p=n-m\neq 0$ and $\widetilde{c}=\widetilde{a}=\widetilde{b}$. By assumption, $i(a,b)>0$. We claim that $\Omega_{\widehat{a}}\cap\Omega_{\widehat{b}}=\emptyset$. Suppose on the contrary that

$$\Omega_{\widehat{a}}\cap\Omega_{\widehat{b}}\neq\emptyset. \tag{3.3}$$

Let $\{\varrho^{-1}(\widetilde{a})\}$ denote the collection of (disjoint) geodesics $\widehat{c}$ in $\mathbf{H}$ with $\varrho(\widehat{c})=\widetilde{c}$. Since $\widetilde{a}=\widetilde{b}$, we conclude that $\{\varrho^{-1}(\widetilde{b})\}=\{\varrho^{-1}(\widetilde{a})\}$. It follows from (3.3) and the constructions of $\tau_{\widehat{a}}$ and $\tau_{\widehat{b}}$ that $\tau_{\widehat{a}}$ commutes with $\tau_{\widehat{b}}$. Therefore, t_a commutes with t_b, which means that $a=b$ or a is disjoint from b, contradicting that $i(a,b)>0$.

We must have $\Omega_{\widehat{a}}\cap\Omega_{\widehat{b}}=\emptyset$. But then there exist maximal elements $\Delta_1\in\mathscr{U}_{\widehat{a}}$, $\Delta_2\in\mathscr{U}_{\widehat{b}}$, such that (3.2) holds. This proves the lemma.

As a special situation, if $\lambda(a,b;m,n)\circ f^k=t_a^p$ for some integer p, then the conclusion of Lemma 3.2 remains valid.

Figure 1 depicts the relative position of Δ_1 and Δ_2 obtained from Lemma 3.2. Let $g_1\in G$ be a primitive simple hyperbolic element such that $g_1(\Delta_1)=\Delta_1$ and the orientation of g_1 is the same as $t_{\widetilde{a}}$. By [7, Theorem 2] and [8, Theorem 2], g_1^* is represented by $t_a\circ t_{a_0}^{-1}$ for an $a_0\in\mathscr{S}$ parallel to a.

Lemma 3.3 *Let $\tau_{\widehat{a}_0}$ be the lift of $t_{\widetilde{a}}$ such that $[\tau_{\widehat{a}_0}]^*=t_{a_0}$. Then $\mathbf{H}\backslash\overline{\Delta}_1$ is a maximal element of $\mathscr{U}_{\widehat{a}_0}$.*

Proof Since $\tau_{\widehat{a}}$ is the list of $t_{\widetilde{a}}$ with $[\tau_{\widehat{a}}]^*=t_a$, Δ_1 is a maximal element of $\mathscr{U}_{\widehat{a}}$, that is, $\tau_{\widehat{a}}$ keeps Δ_1 invariant and has no fixed point on $\Delta_1\cap\mathbf{S}^1$. Observe that $g_1^{-1}\tau_{\widehat{a}}$ is also a lift of $t_{\widetilde{a}}$ that satisfies the properties: (i) $g_1^{-1}\tau_{\widehat{a}}$ keeps $\mathbf{H}\backslash\overline{\Delta}_1$ invariant; and (ii) the identity region of $g_1^{-1}\tau_{\widehat{a}}$ is contained in Δ_1. It follows that $\mathbf{H}\backslash\overline{\Delta}_1$ is a maximal element determined by $g_1^{-1}\tau_{\widehat{a}}$.

On the other hand, one calculates

$$\left[g_1^{-1}\tau_{\widehat{a}}\right]^*=(g_1^*)^{-1}\left[\tau_{\widehat{a}}\right]^*=t_{a_0}\circ t_a^{-1}\circ t_a=t_{a_0}.$$

So $\tau_{\widehat{a}_0}=g_1^{-1}\tau_{\widehat{a}}$, which shows that $\mathscr{U}_{\widehat{a}_0}$ is the collection of half-planes determined by $g_1^{-1}\tau_{\widehat{a}}$, and so $\mathbf{H}\backslash\overline{\Delta}_1$ is a maximal element of $\mathscr{U}_{\widehat{a}_0}$.

4 Mapping Classes Interpreted as Automorphisms of H

Let Δ_1 and Δ_2 be as in Lemma 3.2, and c_g be the axis of g. There are various possibilities concerning the relative positions among c_g, Δ_1 and Δ_2. The aim of this section is to classify all the possible cases.

Assume first that $c_g\neq\partial\Delta_1$ and $c_g\neq\partial\Delta_2$. Let $\{A,B\}$ denote the fixed points of g, that is, $c_g\cap\mathbf{S}^1=\{A,B\}$. Clearly, the endpoints $\{A,B\}$ cannot be in $\{U,V,X,Y\}$. Otherwise, the group G would not be discrete. In each of the following figures, we denote by

$$\Delta_1^*=\mathbf{H}\backslash\overline{\Delta}_1\quad\text{and}\quad\Delta_2^*=\mathbf{H}\backslash\overline{\Delta}_2.$$

Let U and X be the endpoints of Δ_1 that are attracting and repelling endpoints with respect to $\tau_{\widehat{a}}$ (that is, $\tau_{\widehat{a}}^k(z) \to U$ and $\tau_{\widehat{a}}^{-k}(z) \to X$ for $z \in \Delta_1 \cap \mathbf{S}^1$ as $k \to +\infty$). Likewise, we denote by V and Y the endpoints of Δ_2 that are attracting and repelling with respect to $\tau_{\widehat{b}}^{-1}$.

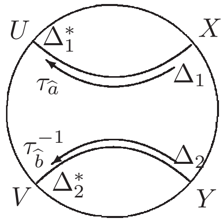

Figure 1 Δ_1 and Δ_2 satisfy the condition (3.2).

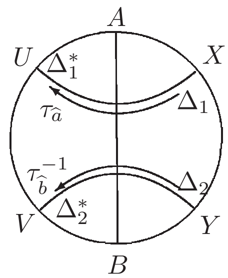

Figure 2 The axis c_g crosses both $\partial\Delta_1$ and $\partial\Delta_2$.

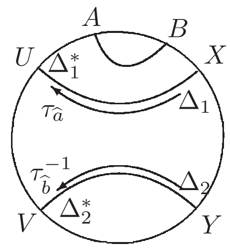

Figure 3 The axis $c_g \subset \Delta_1^* \subset \Delta_2$.

We observe that the arc (UV) (here and below (UV) denotes the arc on $\mathbf{S}^1$ connecting U and V without passing through X or Y) is a stable region for the iteration of $\tau_{\widehat{b}}^{-m}\tau_{\widehat{a}}^{n}$ (in the sense that $\tau_{\widehat{b}}^{-m}\tau_{\widehat{a}}^{n}(z) \in (UV)$ whenever $z \in (UV)$), whereas (XY) is stable for the iteration of the inverse of $\tau_{\widehat{b}}^{-m}\tau_{\widehat{a}}^{n}$. More precisely, we notice that $\widetilde{a} = \widetilde{b}$ is a simple closed geodesic. As a consequence, we obtain

$$\tau_{\widehat{a}}(\Delta_1^*) = \Delta_1^*, \quad \tau_{\widehat{b}}(\Delta_1^*) \cap \Delta_1^* = \emptyset \quad \text{and} \quad \tau_{\widehat{b}}(\Delta_1^*) \subset \Delta_1 \cap \Delta_2. \tag{4.1}$$

Similarly, we have

$$\tau_{\widehat{b}}(\Delta_2^*) = \Delta_2^*, \quad \tau_{\widehat{a}}(\Delta_2^*) \cap \Delta_2^* = \emptyset \quad \text{and} \quad \tau_{\widehat{a}}(\Delta_2^*) \subset \Delta_2 \cap \Delta_1. \tag{4.2}$$

It follows from (4.1) and (4.2) that

$$\tau_{\widehat{b}}^{-m}\tau_{\widehat{a}}^{n}(\Delta_1^*) \cap \mathbf{S}^1 \subset (UV) \quad \text{and} \quad \tau_{\widehat{b}}^{-m}\tau_{\widehat{a}}^{n}(\Delta_2^*) \cap \mathbf{S}^1 \subset (UV).$$

Case 1 The axis c_g is not contained in Δ_1. In this case, either c_g crosses both $\partial\Delta_1$ and $\partial\Delta_2$ (see Figure 2), or c_g lies in Δ_2 (see Figures 3 and 4).

Case 2 The axis c_g is contained in Δ_1. In this case, either $c_g \subset \Delta_1 \cap \Delta_2$ (see Figure 5 and Figure 7), or c_g crosses $\partial\Delta_2$ (see Figure 6), or $c_g \subset \Delta_2^*$ (see Figure 8).

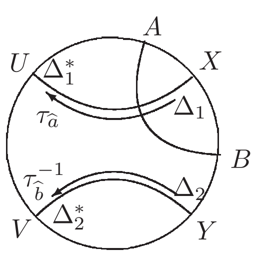

Figure 4 The axis $c_g \subset \Delta_2$ crosses $\partial\Delta_1$.

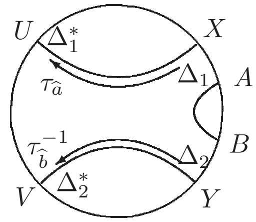

Figure 5 The axis $c_g \subset \Delta_1 \cap \Delta_2$ does not separate $\partial\Delta_1$ and $\partial\Delta_2$.

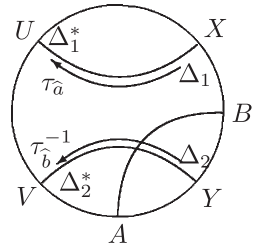

Figure 6 The axis $c_g \subset \Delta_1$ crosses $\partial\Delta_2$.

By considering the inverse of the map $\tau_{\widehat{b}}^{-m}\tau_{\widehat{a}}^{n}g^k$, we see that Figure 8 can be reduced to Figure 3, and Figure 6 can be reduced to Figure 4. As such, it is adequate to discuss the cases depicted in Figures 2–5 and 7.

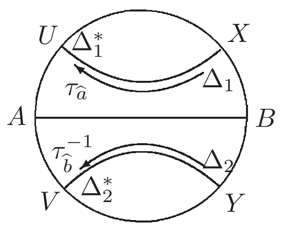

Figure 7 The axis $c_g \subset \Delta_1 \cap \Delta_2$ separates $\partial\Delta_1$ and $\partial\Delta_2$.

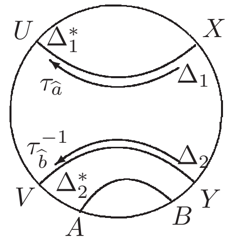

Figure 8 The axis $c_g \subset \Delta_2^* \subset \Delta_1$.

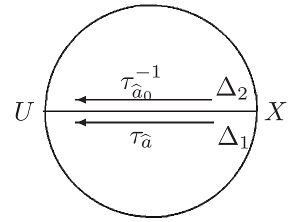

Figure 9 The axis $c_g = \partial\Delta_1 = \partial\Delta_2$.

Now we are ready to prove the following result.

Theorem 4.1 *Under the same conditions as in Theorem* 1.1, *we have* $i(a,b)=0$.

Proof Assume that $i(a,b)>0$. Write $f=g^*\in\mathscr{F}$. By Lemma 3.2, there are elements $\Delta_1\in\mathscr{U}_{\widehat{a}}$ and $\Delta_2\in\mathscr{U}_{\widehat{b}}$, such that (3.2) holds. We also assume that $\mathscr{L}(a)$ in Theorem 1.1 consists of the powers of t_a.

According to the above discussion, we only need to consider Figures 2–5, 7, and the cases $c_g=\partial\Delta_2$ or $\partial\Delta_1$. Since G contains only hyperbolic elements, g is hyperbolic. Denote by

$$F_i=\tau_{\widehat{b}}^{-m_i}\tau_{\widehat{a}}^{n_i}g^{k_i}. \tag{4.3}$$

We know by the assumption that $[F_i]^*$ is represented by powers of t_a. So from Lemma 2.1, we have $F_i(\Delta_1)=\Delta_1$. Note that $\Delta_1^*=\mathbf{H}\backslash\overline{\Delta}_1$ and F_i are homeomorphisms of $\mathbf{H}$ onto itself. We see that $F_i(\Delta_1)=\Delta_1$ if and only if $F_i(\Delta_1^*)=\Delta_1^*$.

By examining Figures 5 and 7 (where c_g is disjoint from Δ_1^*), we find that for large k_i, $g^{k_i}(\Delta_1^*)$ accumulates at the attracting fixed point of g that is away from the repelling endpoint X (with respect to $\tau_{\widehat{a}}$) of Δ_1 and stays in Δ_1. So for large integer n_i, $\tau_{\widehat{a}}^{n_i}g^{k_i}(\Delta_1^*)$ accumulates at the point U that is away from the repelling endpoint Y (with respect to $\tau_{\widehat{b}}^{-1}$) of Δ_2. It follows that $F_i(\Delta_1^*)$ accumulates at V for large integer m_i. In particular, $F_i(\Delta_1^*)\neq\Delta_1^*$. Hence also $F_i(\Delta_1)\neq\Delta_1$.

If Figure 2 or Figure 4 occurs and the point A is the attracting fixed point of g, then $g^{k_i}(\Delta_1^*)\subset\Delta_1^*$. From (4.1) and (4.2), we easily see that $F_i(\Delta_1^*)\neq\Delta_1^*$. Hence also $F_i(\Delta_1)\neq\Delta_1$. If the point A is the repelling fixed point of g, by considering the inverse of F_i, we assert that $F_i^{-1}(\Delta_1^*)\neq\Delta_1^*$ and thus $F_i^{-1}(\Delta_1)\neq\Delta_1$. If Figure 3 occurs, then one easily checks that $F_i(\Delta_1)\subset\Delta_2$ and is near to the point V. This particularly implies that $F_i(\Delta_1)\neq\Delta_1$.

Now we proceed to consider the case $c_g=\partial\Delta_2$. If the attracting fixed point A of g coincides with the point V, then $F_i(\Delta_1^*)$ is disjoint from Δ_1^*. So $F_i(\Delta_1^*)\neq\Delta_1^*$ and $F_i(\Delta_1)\neq\Delta_1$. If the point A coincides with the point Y, then $g^{k_i}(\Delta_1^*)\subset\Delta_2$ shrinks to the point Y, which stays away from the repelling endpoint X (with respect to $\tau_{\widehat{a}}$) and then $\tau_{\widehat{a}}^{n_i}g^{k_i}(\Delta_1^*)=\tau_{\widehat{a}}^{n_i}(g^{k_i}(\Delta_1^*))\subset\Delta_1$ shrinks to the point U, which stays away from the repelling endpoint Y with respect to $\tau_{\widehat{b}}^{-1}$. It turns out that $F_i(\Delta_1^*)\subset\Delta_2$ shrinks to the point V. This particularly implies that $F_i(\Delta_1^*)\neq\Delta_1^*$. Hence also $F_i(\Delta_1)\neq\Delta_1$.

Finally, if $c_g=\partial\Delta_1$, then $g^{k_i}(\Delta_1^*)=\Delta_1^*$. So $\tau_{\widehat{a}}^{n_i}g^{k_i}(\Delta_1^*)=\Delta_1^*$. We conclude that $\tau_{\widehat{b}}^{-m_i}\tau_{\widehat{a}}^{n_i}g^{k_i}(\Delta_1^*)=\tau_{\widehat{b}}^{-m_i}(\Delta_1^*)$ is disjoint from Δ_1^*. It follows that $F_i(\Delta_1)\neq\Delta_1$.

5 Proof of the Theorems

Proof of Theorem 1.1 From Theorem 4.1, we assert that $i(a,b)=0$. If $\widetilde{b}\neq\widetilde{a}$, we claim that the mapping classes $\lambda(a,b;m_i,n_i)\circ f^{k_i}$ would not be powers of Dehn twists. Otherwise, $\lambda(a,b;m_i,n_i)\circ f^{k_i}$ also projects to a Dehn twist. But the projection of $\lambda(a,b;m_i,n_i)\circ f^{k_i}$ is just $t_{\widetilde{a}}^{m_i}\circ t_{\widetilde{b}}^{n_i}$ that cannot be a Dehn twist (since $\widetilde{b}\neq\widetilde{a}$). This leads to a contradiction. We conclude that $\widetilde{b}=\widetilde{a}$, which says that $b=a$ or $b=a_0$ for some a_0 parallel to a. If $b=a$, then $\lambda(a,b;m,n)=t_a^{n-m}$. In this case, one can show (see [12], for example) that $\lambda(a,b;m,n)\circ f^{k_i}$, where $f\in\mathscr{F}$, are never powers of t_a unless f is trivial.

We conclude that $b=a_0$. Now, Figure 1 should be replaced with Figure 9. In the figure, $\Delta_2=\Delta_1^*$, $\Delta_1=\Delta_2^*$ and $\partial\Delta_1=\partial\Delta_2$. We notice that the relative position of c_g, $\partial\Delta_1=\partial\Delta_2$ falls into these cases: (i) c_g crosses $\partial\Delta_1=\partial\Delta_2$; (ii) $c_g\subset\Delta_1$; (iii) $c_g\subset\Delta_2$; or (iv) $c_g=\partial\Delta_1=\partial\Delta_2$. In cases (i)–(iii), one easily checks that for $F_i=\tau_{\hat{a}_0}^{-m_i}\tau_{\hat{a}}^{n_i}g^{k_i}$, F_i does not keep Δ_1 and Δ_2 invariant. This tells us that $[F_i]^*$ can never be powers of t_a. This contradicts the hypothesis. We see that only (iv) can occur. In this case, $f=g^*=t_{a_0}^q\circ t_a^{-q}$ for an appropriate integer q.

This completes the proof of Theorem 1.1.

Proof of Theorem 1.2 The conditions guarantee that both $\mathscr{L}(a)$ and $\mathscr{L}(a_0)$ consist of powers of t_a and t_{a_0}. If $i(a,b)>0$, by Lemma 3.2, there exist maximal elements $\Delta_1\in\mathscr{U}_{\hat{a}}$ and $\Delta_2\in\mathscr{U}_{\hat{b}}$, such that (3.2) holds. Let $g_1\in G$ be a primitive hyperbolic element with $g_1(\Delta_1)=\Delta_1$. By [7, Theorem 2] and [10, Theorem 2], $g_1^*=t_a\circ t_{a_1}^{-1}$, where $a_1\in\mathscr{S}$ is a curve parallel to a. If $a_1=a_0$, then by Lemma 3.3, $\Delta_1^*=\mathbf{H}\backslash\overline{\Delta_1}$ is a maximal element of $\mathscr{U}_{\hat{a}_0}$. Thus for the element $g\in G$ with $g^*=f$, by the similar argument of Theorem 4.1, $\tau_{\hat{b}}^{-m_i}\tau_{\hat{a}_0}^{n_i}g^{k_i}(\Delta_1^*)\neq\Delta_1^*$ and $\tau_{\hat{b}}^{-m_i}\tau_{\hat{a}}^{n_i}g^{k_i}(\Delta_1)\neq\Delta_1$. Hence also $\tau_{\hat{b}}^{-m_i}\tau_{\hat{a}_0}^{n_i}g^{k_i}(\Delta_1)\neq\Delta_1$ and $\tau_{\hat{b}}^{-m_i}\tau_{\hat{a}}^{n_i}g^{k_i}(\Delta_1^*)\neq\Delta_1^*$. This says that $\mathscr{L}(a)$ contains some elements that are not represented by powers of t_a or t_{a_0}, and so does $\mathscr{L}(a_0)$. This leads to a contradiction.

If $a_1\neq a_0$, then there is a maximal element $\Delta_0\in\mathscr{U}_{\hat{a}}$ with $\Delta_0\subset\Delta_1^*$ such that $g_0^*=t_a\circ t_{a_0}^{-1}$ for the hyperbolic element $g_0\in G$ keeping Δ_0 invariant. Then by Lemma 3.3 again, $\Delta_0^*=\mathbf{H}\backslash\overline{\Delta_0}$ is a maximal element of $\mathscr{U}_{\hat{a}_0}$. By the same argument of Theorem 4.1, one easily verifies that for the element $g\in G$ with $g^*=f$, we have

$$\tau_{\hat{b}}^{-m_i}\tau_{\hat{a}_0}^{n_i}g^{k_i}(\Delta_0)\neq\Delta_0\quad\text{and also}\quad\tau_{\hat{b}}^{-m_i}\tau_{\hat{a}_0}^{n_i}g^{k_i}(\Delta_1)\neq\Delta_1.$$

Since $\tau_{\hat{b}}^{-m_i}\tau_{\hat{a}_0}^{n_i}g^{k_i}$ are homeomorphisms of $\mathbf{H}$ onto itself, we obtain

$$\tau_{\hat{b}}^{-m_i}\tau_{\hat{a}_0}^{n_i}g^{k_i}(\Delta_0^*)\neq\Delta_0^*\quad\text{and also}\quad\tau_{\hat{b}}^{-m_i}\tau_{\hat{a}_0}^{n_i}g^{k_i}(\Delta_1^*)\neq\Delta_1^*.\tag{5.1}$$

It follows from (5.1) and Lemma 2.1 that $\tau_{\hat{b}}^{-m_i}\tau_{\hat{a}_0}^{n_i}g^{k_i}$ can never be powers of the Dehn twists t_a or t_{a_0}. This again contradicts the hypothesis, proving that $i(a,b)=0$.

Similarly, we can show that $i(a_0,b)>0$ cannot occur. So we also have $i(a_0,b)=0$.

To prove $b=a$ or $b=a_0$, we use Lemma 3.1 to deduce $\widetilde{b}=\widetilde{a}=\widetilde{a}_0$. This says that b is homotopic to a as x is filled in. If $b\neq a$, then since $i(a,b)=0$, b must be disjoint from a, that is, $\{a,b\}$ forms the boundary of an x-punctured cylinder P. We claim that b cannot be disjoint from a_0; otherwise, P would be disjoint from the x-punctured cylinder bounded by $\{a,a_0\}$, and this is impossible.

We conclude that b cannot be disjoint from a_0. This means that either b intersects a_0 or $b = a_0$. If the later occurs, we are done. If the former occurs, then $i(a_0, b) > 0$. This once again leads to a contradiction.

The second statement of the theorem is the same as in the proof of Theorem 1.1. The remaining part of Theorem 1.2 also follows from Theorem 1.1. This completes the proof of Theorem 1.2.

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