Chin. Ann. Math.
30B(4), 2009, 433–446
DOI: 10.1007/s11401-007-0474-7

Chinese Annals of Mathematics, Series B

Blow-Up for a Semi-linear Advection-Diffusion System with Energy Conservation***

Dapeng DU*   Jing LÜ**

Abstract The authors study radial solutions to a model equation for the Navier-Stokes equations. It is shown that the model equation has self-similar singular solution if $5 \leq n \leq 9$. It is also shown that the solution will blow up if the initial data is radial, large enough and $n \geq 5$.

Keywords Navier-Stokes equations, Self-similar singular solutions, Blow-up
2000 MR Subject Classification 35K55, 35B05, 76A02

1 Introduction

In this paper, we study singular solutions to the following system:

$$\frac{\partial \mathbf{u}}{\partial t} - \Delta \mathbf{u} + 2a\mathbf{u}\nabla\mathbf{u} + (1-a)\nabla|\mathbf{u}|^2 + (\operatorname{div}\mathbf{u})\mathbf{u} = 0, \tag{1.1}$$

where $a \in (0,1)$, $\mathbf{u}(\mathbf{x},t)$ is a time-dependent vector field on $\mathbb{R}^n \times (0,T)$. (1.1) was first introduced by Plecháč and Šverák [7] as a model equation for Navier-Stokes equation. In [7], they proposed several conjectures and verified them by both formal calculations and numerical simulations. In the present paper, we justify some of the conjectures in [7] and give a blow-up result.

Instead of working in the general case, we only consider the so-called radial vector field solution

$$\mathbf{u}(\mathbf{x},t) = -v(r,t)\mathbf{x}, \tag{1.2}$$

where $r = |\mathbf{x}|$ and v is a scalar function. A direct calculation gives

$$v_t = v_{rr} + \frac{n+1}{r}v_r + 3rvv_r + (n+2)v^2, \tag{1.3}$$

where subscripts denote corresponding partial derivatives.

Our primary interest is to study whether (1.3) admits a self-similar singular solution which is of the form

$$v(r,t) = \frac{1}{2(T-t)}u\Big(\frac{r}{\sqrt{2(T-t)}}\Big). \tag{1.4}$$

Substituting (1.4) into (1.3) gives

$$u'' + \frac{n+1}{r}u' + 3ruu' + (n+2)u^2 - ru' - 2u = 0. \tag{1.5}$$

Manuscript received November 15, 2007. Revised December 31, 2008. Published online June 8, 2009.
*School of Mathematical Sciences, Fudan University, Shanghai 200433, China.
E-mail: dpdu@fudan.edu.cn
**Department of Mathematics, Shanghai Maritime University, Shanghai 200135, China.
E-mail: jinglv@dbc.shmtu.edu.cn
***Project supported by the China Postdoctoral Science Foundation (No. 20070410683).

Notice that (1.5) has two trivial solutions: $u_\infty = \frac{2n-4}{n-4} r^{-2}$ and $u = \frac{2}{n+2}$. Here and in the sequel, we denote $\frac{2}{n+2}$ by β.

The appropriate initial condition is

$$u(0) = \alpha, \quad u'(0) = 0. \tag{1.6}$$

The condition $u'(0) = 0$ is to ensure that the constructed singular solution to (1.1) is smooth before blow-up time. Here and in the sequel, we denote by u_α the solution to (1.5)–(1.6).

We also require u to have some decay at infinity:

$$u(r) \approx r^{-2}, \quad r \to \infty. \tag{1.7}$$

Roughly speaking, (1.7) is the only reasonable decay we can expect. The reason will become clear from the discussion later.

Now the question becomes how to study (1.5)–(1.7). Most of our methods are borrowed from the literature for a semi-linear heat equation

$$u_t - \Delta u = |u|^{p-1} u, \quad p > 1. \tag{1.8}$$

The reader may refer to [3–5] and the references therein for more information.

Our first result is on the existence of solution to (1.5)–(1.7).

Theorem 1.1 *Assume* $5 \le n \le 9$*. Define* $k(n) = \inf\{k \in N \mid k \ge \frac{n+2}{n-4}\}$*. Then* (1.5)–(1.7) *has at least* $k(n) - 2$ *solutions.*

Remark 1.1 The assumption $5 \le n \le 9$ is necessary. In the case $n \le 4$, Plecháč and Šverák [7] showed that (1.2) has global smooth solutions if the initial data is smooth and decay faster than $r^{-\frac{n+2}{3}}$ $(n \le 3)$ or $r^{-2} \ln r$ $(n = 4)$. Therefore, we see that (1.5)–(1.7) has no non-trivial solution if $n \le 4$. In the case $n \ge 10$, people tend to believe that (1.5)–(1.7) has no solution either. There are some numerical evidence in [7].

The proof of Theorem 1.1 consists of two steps. The first step is to show that (1.5)–(1.6) has $k(n) - 2$ solutions on $[0, \infty)$ by a shooting argument. This essentially has been done in [7]. The next step is to show that such solutions satisfy (1.7) by using some typical ODE arguments and PDE analysis. The precise statement is the following theorem.

Theorem 1.2 *Assume that* $n > 4$ *and* $u \in C^2[0, \infty)$ *is a nonconstant solution to* (1.5)*. Then* u *satisfies* (1.7)*. Moreover,* $\lim\limits_{r\to\infty} r^2 u(r) > 0$*.*

Roughly speaking, the proof of Theorem 1.2 goes as follows. First we use some delicate qualitative analysis to show that u has some decay at infinity

$$\lim_{r\to\infty} [|u(r)| + r|u'(r)|] = 0. \tag{1.9}$$

Then we view u as the solution to

$$f'' + \Big(- r + \frac{n+1}{r} + 3ru \Big) f' + [-2 + (n+2)u] f = 0, \quad \text{in } (1, \infty). \tag{1.10}$$

Clearly, the general solution to (1.10) has the form $c_1 f_1 + c_2 f_2$. By using some weighted spaces and (1.9), we can prove that f_1 and f_2 can be chosen so that f_1 is unbounded and f_2 has the decay like (1.7). Now (1.9) tells us that u satisfies (1.7). Finally some PDE analysis implies $\lim\limits_{r\to\infty} r^2 u(r) > 0$.

Remark 1.2 From the first part of Theorem 1.2, we see that if the solution to (1.5) exists globally, then it decays at least as fast as (1.7). At the same time, the second part implies that the solution can not decay faster than (1.7). Therefore, (1.7) is the only reasonable decay we can expect.

Remark 1.3 Theorem 1.2 also tells us that at the blow-up time, the constructed singular solution to (1.1) equals $c\frac{\mathbf{x}}{|\mathbf{x}|^2}$, where $c = \lim\limits_{r\to 0} r^2 u(r)$.

Both Theorems 1.1 and 1.2 are conjectures in [7]. There is also one interesting conjecture regarding the asymptotic behavior of u_α when $\alpha \to \infty$. Here recall that u_α is the solution to (1.5) and (1.6). It will be used when we carry out shooting argument. Regarding this conjecture, we get a slightly weaker statement, which is sufficient for our purpose.

Theorem 1.3 *Assume $n > 4$. Then for any given $M > 1$ and $0 > \lambda > \lambda_1 = \frac{1}{2}\Big[4 - n - 3\cdot \frac{2n-4}{n-4} + \sqrt{\left(n-4+3\cdot\frac{2n-4}{n-4}\right)^2 - 4(2n-4)}\ \Big]$, there exists an $\alpha_0 = \alpha_0(\lambda, n)$ such that $\forall\, \alpha > \alpha_0$, (1.5)–(1.6) has a solution u_α on $[0, M]$, and*

$$\|u_\alpha - u_\infty\|_{C^2[\frac{1}{M}, M]} \le C\alpha^{\frac{\lambda}{2}}, \quad C = C(\lambda, n, M). \tag{1.11}$$

Here recall that $u_\infty = \frac{2n-4}{n-4} r^{-2}$ is a solution to (1.5).

First we note that the complex number λ_1 comes from a characteristic polynomial for a second order ODE. The reader may refer to Section 2 for details.

We also get a blow-up result. We impose on (1.1) the following initial condition:

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^n. \tag{1.12}$$

Theorem 1.4 *Assume that $n > 4$, $\mathbf{u}_0$ is of the form $\mathbf{u}_0(\mathbf{x}) = -v_0(|\mathbf{x}|)\cdot \mathbf{x}$, $\mathbf{u}_0 \in L^\infty(\mathbb{R}^n)$ and $\mathbf{u}$ solves* (1.1) *and* (1.12)*. Then there exists a constant $C = C(n)$ such that if*

$$\int_{\mathbb{R}^n} \mathbf{u}_0(x)\cdot |\mathbf{x}|^{3-n+\frac{n-4}{6}} \mathrm{e}^{-|\mathbf{x}|} \mathrm{d}\mathbf{x} \ge C, \tag{1.13}$$

then

$$\|\mathbf{u}(\,\cdot\,, t)\|_{L^\infty(\mathbb{R}^n)} \ge \frac{c}{\sqrt{T-t}}, \quad t \in (0, T) \tag{1.14}$$

for some positive constants c and T, $T \in (0, 1)$.

We prove this theorem by working with (1.3). First we choose an appropriate weight. Then we derive a differential inequality which is satisfied by the weighted average of v, where v is the solution to (1.3). Now by analyzing this differential inequality and some standard PDE analysis, we get Theorem 1.4.

Remark 1.4 To some degree, Theorem 1.4 is optimal in the sense that (1.1) and (1.12) has full regularity if $n \le 4$. See Remark 1.1 in the present paper or [7, Theorem 1.2] for details.

This paper is organized as follows. In Section 2, we prove Theorem 1.3. In Section 3, we give the proof of Theorems 1.1 and 1.2. Finally, we prove Theorem 1.4 in Section 4.

2 Local Existence and Asymptotic Behavior

In this section we prove Theorem 1.3. The key is a change of variable, which reduces Theorem 1.3 to a slightly unusual stability problem. We begin with the change of variables

$$t = \ln(\varepsilon^{-\frac{1}{2}} r), \quad w_\varepsilon(t) = \varepsilon \mathrm{e}^{2t} u_\alpha(\varepsilon^{\frac{1}{2}}\mathrm{e}^t), \quad \varepsilon = \alpha^{-1}. \tag{2.1}$$

Then question (1.5) becomes

$$w_\varepsilon'' + (n-4+3w_\varepsilon)w_\varepsilon' + (n-4)w_\varepsilon^2 - (2n-4)w_\varepsilon - \varepsilon e^{2t} w_\varepsilon' = 0. \tag{2.2}$$

The initial condition becomes

$$w_\varepsilon(t) \to 0, \quad w_\varepsilon'(t) \to 0, \quad \text{as } t \to -\infty.$$

Notice that $A_0 = \frac{2n-4}{n-4}$ solves (2.2). The linearization operator about A_0 is

$$L_\varepsilon \psi = \psi'' + (n-4+3A_0)\psi' + (2n-4)\psi - \varepsilon e^{2t}\psi'.$$

The key estimate is Lemma 2.1, which is a refined version of stability theorem for linear ODEs. Now the problem is reduced to showing that w_ε converges to A_0, where w_ε is in some weighted function spaces. We begin with the definition of such spaces.

For all $f \in C^k[a,b]$, $-\infty < a < b < \infty$, $\lambda < 0$, define

$$\|f\|_{C_\lambda^k[a,b]} = \sup_{t\in[a,b]} \Big[\sum_{i=0}^{k} |f^{(i)}(t) e^{-\lambda(t-a)}|\Big],$$
$$C_\lambda^k[a,b] = \{f \in C^k[a,b],\ \|f\|_{C_\lambda^k[a,b]} < \infty\}.$$

Lemma 2.1 *Consider the following initial problem:*

$$\begin{cases} u'' + (A+d_1(t))u' + (B+d_2(t))u = f(t), & \text{in } (a,b), \\ u(a) = u_0,\ u'(a) = u_1, \end{cases} \tag{2.3}$$

where $A, B \in (0,\infty)$, $A^2 > 4B$, and $d_1(t), d_2(t) \in C^0[a,b]$. For all $\lambda > \frac{-A+\sqrt{A^2-4B}}{2}$, there exists a $c_0 = c_0(A,B,\lambda)$ such that if

$$|d_1|_{L^\infty} + |d_2|_{L^\infty} < c_0, \tag{2.4}$$

then for all $f \in C_\lambda^0[a,b]$, (2.3) has a unique solution $u \in C_\lambda^2[a,b]$ and u satisfies the inequality

$$\|u\|_{C_\lambda^k[a,b]} \le C_0(\|f\|_{C_\lambda^0[a,b]} + |u_0| + |u_1|), \quad C_0 = C_0(A,B,\lambda). \tag{2.5}$$

Proof The only nontrivial part is (2.5). It also suffices to show (2.5) in the special case of $u_0 = u_1 = 0$, $d_1 = d_2 = 0$ and $a = 0$. The general case directly follows from $u'' + Au' + Bu = f(t) - d_1(t)u' - d_2(t)u$.

Define $\lambda_{1,2} = \frac{-A\pm\sqrt{A^2-4B}}{2}$. Then $0 > \lambda_1 > \lambda_2$. By the method of variational constants, we know

$$u(t) = \int_0^t K(t-s)f(s)\mathrm{d}s, \quad K(t) = \frac{e^{\lambda_2 t} - e^{\lambda_1 t}}{\lambda_2 - \lambda_1}.$$

Hence $\forall\, \lambda > \lambda_1$, we have

$$\begin{aligned}
\|u\|_{C_\lambda^2} &= \sup_{t\in[a,b]} e^{-\lambda t}\Big[\Big|\int_0^t K(t-s)f(s)\mathrm{d}s\Big| + \Big|\int_0^t K'(t-s)f(s)\mathrm{d}s\Big| + \Big|\int_0^t K''(t-s)f(s)\mathrm{d}s\Big|\Big] \\
&\le \sup_{t\in[a,b]} C\cdot\Big[e^{(\lambda_1-\lambda)t}\int_0^t e^{-\lambda_1 s}\cdot e^{\lambda s}\cdot\|f\|_{C_\lambda^0}\mathrm{d}s + 1\Big] \\
&= \sup_{t\in[a,b]} C\Big\{\frac{1}{\lambda-\lambda_1}[1-e^{(\lambda_1-\lambda_2)t}] + \frac{1}{\lambda-\lambda_2}[1-e^{\lambda-\lambda_2}] + 1\Big\}\cdot\|f\|_{C_\lambda^0} \\
&\le C_0\cdot\|f\|_{C_\lambda^0}.
\end{aligned}$$

The lemma is proved.

We define $\lambda_1 = \frac{1}{2}[4 - n - 3A_0 + \sqrt{(n-4+3A_0)^2 - 4(2n-4)}\,]$, the larger root for the characteristic polynomial for L_ε. Note $\lambda_1 < 0$. We also define

$$c_1(\lambda, n) = \frac{1}{2}\min\{c_0(\lambda, n-4+3A_0, 2n-4), 1\}, \tag{2.6}$$

$$C_1(\lambda, n) = 4C_0(\lambda, n-4+3A_0, 2n-4) + 4. \tag{2.7}$$

Here c_0 and C_0 are the ones defined in Lemma 2.1. Next we give a lemma, which is a key ingredient in the proof of Theorem 1.3. Roughly, it says that system (2.2) is exponentially stable at the equilibriums A_0, in certain region when ε is small enough.

Lemma 2.2 *Impose for* (2.2) *the following initial condition:*

$$w_\varepsilon(a) = A_0 + p_1, \quad w_\varepsilon'(a) = p_2 \tag{2.8}$$

for some $a \in (-\infty, \frac{1}{2}\ln\frac{c_1}{\varepsilon})$. *Then for all* $\lambda < \lambda_1$, $|(p_1, p_2)| < \frac{c_0}{16}$, *there exists a unique smooth solution* w_ε *to* (2.2) *and* (2.8) *in* $(a, \frac{1}{2}\ln\frac{c_1}{\varepsilon})$, *and* w_ε *satisfies the following inequality:*

$$\|w_\varepsilon - A_0\|_{C^2_\lambda[a, \frac{1}{2}\ln\frac{c_1}{\varepsilon}]} \le C_1|p|, \quad p = (p_1, p_2). \tag{2.9}$$

Proof We will use Lemma 2.1 and Banach fixed point theorem to prove this lemma. Clearly we can assume $p \neq 0$. We look for the solution of the form $w_\varepsilon(t) = A_0 + |p|\psi(t;p)$. The equation for ψ is

$$\begin{cases} L_\varepsilon\psi = -|p|[3\psi\psi' + (n-4)\psi^2], \\ \psi(a) = \dfrac{p_1}{|p|}, \quad \psi'(a) = \dfrac{p_2}{|p|}. \end{cases} \tag{2.10}$$

From Lemma 2.1, we know that there exists a solution operator $G_{\varepsilon,p}$ such that $\forall f \in C^0_\lambda[a, \frac{1}{2}\ln\frac{c_1}{\varepsilon}]$,

$$\begin{cases} L_\varepsilon G_{\varepsilon,p} f = f, \\ (G_{\varepsilon,p} f)(a) = \dfrac{p_1}{|p|}, \quad (G_{\varepsilon,p} f)'(a) = \dfrac{p_2}{|p|}. \end{cases}$$

Now (2.10) can be written as the following integral equation:

$$\psi = T_\varepsilon\psi, \quad T_\varepsilon\psi \triangleq G_{\varepsilon,p}\{-p[3t\psi\psi' + (n-4)\psi^2]\}.$$

Using (2.5), we see that T_ε is from $C^0_\lambda[a, \frac{1}{2}\ln\frac{c_1}{\varepsilon}]$ to itself under the assumption that $\lambda < \lambda_1$ and $|p| < \frac{c_0}{16}$. Therefore Lemma 2.2 follows from Banach fixed point theorem.

Before proving Theorem 1.3, we still need one technical claim and a small lemma. Here we note that although a stronger statement holds, we choose only to state what we need for the sake of convenience. First we give the claim, which describes what happens when ε equals zero.

Claim 2.1 $w_0(t) \to A_0$, $w_0'(t) \to 0$, as $t \to \infty$.

Proof We note that the steady-state equation to (2.2) has two equilibria points in the phase space. One is the saddle point $(0,0)$, and the other is the stable point $(A_0, 0)$. We obtain the claim from the global property of the steady-state equation. For details, see [7].

Next we state the lemma. Roughly speaking, it means that w_ε converges to w_0 pointwise.

Lemma 2.3 *For* $t \in \mathbb{R}$, *there holds*

$$w_\varepsilon(t) \to w_0(t), \quad w_\varepsilon'(t) \to w_0'(t), \quad \textit{as } \varepsilon \to 0.$$

Proof First, we do change of variables to (1.5). Let

$$\rho=\varepsilon^{-\frac{1}{2}}r,\quad \varphi_\varepsilon(\rho)=\varepsilon u_\varepsilon(\varepsilon^{\frac{1}{2}}\rho),\quad \varepsilon=\frac{1}{\alpha}.$$

Then we have

$$\varphi_\varepsilon''+3\rho\varphi_\varepsilon\varphi_\varepsilon'+\frac{n+1}{\rho}\varphi_\varepsilon'+(n+2)\varphi_\varepsilon^2-\varepsilon(\rho\varphi_\varepsilon'+2\varphi_\varepsilon)=0 \tag{2.11}$$

and the initial condition

$$\varphi_\varepsilon(0)=1,\quad \varphi_\varepsilon'(0)=0. \tag{2.12}$$

By the continuity of ODE solution with respect to parameters, for $\rho\in\mathbb{R}$, we have

$$\varphi_\varepsilon(\rho)\to\varphi_0(\rho),\quad \varphi_\varepsilon'(\rho)\to\varphi_0'(\rho),\quad \text{as } \varepsilon\to 0.$$

We do change of variables again. Let

$$t=\ln\rho,\quad w_\varepsilon(t)=\mathrm{e}^{2t}\varphi_\varepsilon(\mathrm{e}^t).$$

Then the conclusion follows.

Now we are able to prove Theorem 1.3.

Proof of Theorem 1.3 By Claim 2.1, for any λ $(\lambda_1<\lambda<0)$, we pick a fixed $K>1$ such that

$$(|w_0(\ln K)-A_0|+|w_0'(\ln K)|)<\frac{c_1(\lambda,n)}{40C_1(\lambda,n)}.$$

Here c_1 and C_1 are the ones defined in (2.6) and (2.7). By Lemma 2.3, there exists an ε_1 such that $K<(\frac{c_1}{\varepsilon_1})^{\frac{1}{3}}$. For any $\varepsilon\in(-\varepsilon_1,\varepsilon_1)$, equations (2.11)–(2.12) have a unique solution φ_ε such that

$$\|\varphi_\varepsilon-\varphi_0\|_{C^2[0,K]}\le\frac{1}{2}\min\Big\{\inf_{\rho\in[0,K]}\varphi_0(\rho),\frac{c_1}{40K^3C_1}\Big\}.$$

This implies $\varphi_\varepsilon(\rho)>0$ on $\rho\in[0,K]$, and $(|w_\varepsilon(K)-A_0|+|w_\varepsilon'(K)|)\le\frac{c_1}{20C_1}$.

Now we can use Lemma 2.2 to extend our solution w_ε from $(-\infty,\ln K)$ to $(-\infty,\frac{1}{2}\ln\frac{c_1}{\varepsilon})$. Moreover, w_ε satisfies

$$\|w_\varepsilon-A_0\|_{C^2_\lambda[\ln K,\frac{1}{2}\ln\frac{c_1}{\varepsilon}]}\le c_1.$$

Going back to (u,r) coordinate, we get

$$|u_\alpha(\sqrt{c_1})-u_\infty(\sqrt{c_1})|+|u_\alpha'(\sqrt{c_1})-u_\infty'(\sqrt{c_1})|\le C\cdot\alpha^{\frac{\lambda}{2}},\quad C=C(\lambda,n). \tag{2.13}$$

By Gronwall's inequity, we get our expected conclusion.

3 Self-similar Singular Solutions

In this section, we study the solutions to equation (1.5). We first show Theorem 1.2. The proof mainly consists of three steps. The first step is to show that the solution has some decay at infinity. Then we prove that the solution u decays at the optimum rate. Finally we derive a lower bound on the decay rate of u.

We start with the first step, which is to show the following lemma.

Lemma 3.1 *Suppose that* $n>4$ *and* $u\in C^2[a,\infty)$ *is a nonconstant solution to* (1.5). *Then we have* $\lim[|u(r)|+r|u'(r)|]=0$.

Before presenting the detailed proof, first let us see why we could expect such a result. We rewrite equation (1.5) as

$$u'' + \nu u' = -\frac{\mathrm{d}}{\mathrm{d}u}V(u) \tag{3.1}$$

with $\nu(r) = \frac{r+1}{n} + 3ru(r) - r$ and $V(u) = \frac{n+2}{3}u^3 - u^2$. Now we view the variable r as time and the solution u as the position of a particle with unit mass which moves along a vertical line. Assume that there was no gravitational force. Then (3.1) implies that the motion of our particle is determined by the friction coefficient ν and the potential V. We will explain why Lemma 3.1 is possible by describing the long time behavior of the motion. Notice that V has one local maximum 0 and one local minimum β. Here recall that $\beta = \frac{2}{n+2}$ is a constant solution to (1.5).

We will assume that after a long period of time, the particle will be above β and move downward, and the other cases can be explained in a similar way. We describe the motion of our particle. Because the particle is above β and moves downward, the force induced by potential will keep the particle moving downward which never returns before reaching β. Notice that the friction coefficient ν is negative when the particle is near β. This means that the friction force points the same direction as the particle moves. These two forces will push the particle to pass β. After the particle passes β, the force induced by potential becomes pointing upward. However, at the same time the friction is negative and large. Therefore, one may think that the friction force will beat the force induced by potential and push the particle moving downward without returning. The particle also will not reach the origin, otherwise both the force induced by the potential and the friction force point downward. Therefore, these two forces together will push the particle to reach infinity in finite time, which contradicts the assumptions in Lemma 3.1.

In summary, we can expect that the particle moves toward the origin and goes to infinity. This also implies that the friction force also goes to zero as the particle approaches the origin, otherwise the friction force will push the particle across origin, which leads to a contradiction. Going back to mathematics, we see that the statement above is exactly what Lemma 3.1 says. Finally, we emphasize that it is very useful to think about u in terms of the motion of the particle. Next we give details of the proof. We first state two claims. We will use them to control the monotonicity of u.

Claim 3.1 Suppose that u is a nonconstant solution to (1.5) and r_0 is a local minimum of u. Then $0 < u(r_0) < \beta$.

Proof Because r_0 is a local minimum, we have $u(r_0) = 0$, $u'(r_0) = 0$ and $u''(r_0) > 0$. Going back to (1.5), we get $(n+2)u^2(r_0) - 2u(r_0) < 0$. This proves our claim.

The similar argument give the following claim.

Claim 3.2 Suppose that u is a nonconstant solution to (1.5) and r_0 is a local maxima of u. Then $u(r_0) > \beta$ or $u(r_0) < 0$.

Next we give a claim, which basically says that once u becomes negative, it will blow up. The important consequence is that if u exists all the time, then u must be a positive solution. We prove this claim by deriving a first order differential inequality from (1.5). First notice that u is decreasing, otherwise there will exist a negative local minimum, which contradicts Claim 3.1.

Claim 3.3 Assume that $r_0 > 100n$, $u(r_0) \le 0$, $u'(r_0) < 0$ and u solves (1.5). Then u blows up at finite time.

Proof We have $u'(r) < 0$, $\forall\, r \geq r_0$. Therefore $u(r) < 0$, $\forall\, r \geq r_0$. Now (1.5) and the assumption $r_0 > 100n$ give that $[u'' + (n+2)u^2] = -\frac{n+1}{r}u' - 3ruu_r + ru' + 2u \leq 0$, $\forall\, r \geq r_0$. Let $V(r) = -u(r)$. Then we have $V'' \geq (n+2)V^2$. Multiplying this inequality by V' and integrating from r_0 to r, we have

$$V'(r) \geq \sqrt{\frac{2(n+2)}{3}[V^3(r) - V^3(r_0)] + [V'(r_0)]^2} \triangleq f(V(r)).$$

Because $\int_{r_0}^{\infty} \frac{1}{f(V)} \mathrm{d}V < +\infty$, we know that V blows up at finite time. So is u. The claim is proved.

Next we give a lemma. In the language of particle's motion, it basically says that once the particle is below β, it will remain below all the time. This is mainly because the friction ν is negative and large.

Lemma 3.2 *Suppose that $n > 4$, u solves equation* (1.5), $u(r_0) = \beta$ *and* $u'(r_0) < 0$ *for some* $r_0 > 100n$. *Then* $u(r) < \beta$ *for all* $r > r_0$.

Proof We prove this lemma by contradiction argument. The proof roughly goes as follows. After assuming the contrary, we can get an interval on which there are some estimates on u and u'. Then we derive contradiction based on these estimates and some contradiction arguments.

Suppose that there exists an r_1 such that $u(r_1) = \beta$, $u'(r_1) > 0$ and $u(r) < \beta$ for any $r \in (r_0, r_1)$. Continuity implies that there exists an r_2 such that $u(r_2) = \min\limits_{r \in [r_0, r_1]} u(r)$. From Claim 3.1, we see $u(r_2) > 0$, i.e.,

$$0 < u(r) \leq \beta, \quad r \in [r_0, r_1]. \tag{3.2}$$

Next we derive a bound on u'. (1.5) implies $u''(r_0) < 0$. Therefore, there exists an r_3 such that $r_3 \in (r_0, r_1)$ and $u'(r_3) = \min\limits_{r \in [r_0, r_1]} u'(r)$. So $u''(r_3) = 0$. Hence (1.5) and (3.2) give that $u'(r_3) > -\frac{1}{n}$, i.e.,

$$u'(r) > -\frac{1}{n}, \quad r \in [r_0, r_1]. \tag{3.3}$$

Set $\Phi = \beta - u$. The equation for Φ is

$$\Phi'' + \Big(\frac{n+1}{r} + 3u - r\Big)\Phi' + (n+2)u\Phi = 0.$$

Now let $\Psi = \exp\{\int_{r_0}^{r} \frac{1}{2}(\frac{n+1}{r} + 3u - r)\mathrm{d}r\}\Phi$. Then Ψ satisfies

$$\Psi'' + V_1\Psi = 0, \quad \Psi(r_0) = 0, \quad \Psi'(r_0) = -u'(r_0), \tag{3.4}$$

where

$$V_1 = (n+2)u - \frac{1}{4}\Big(\frac{n+1}{r} + 3u - r\Big)^2 - \frac{1}{2}\Big(-\frac{n+1}{r^2} + 3u' - 1\Big).$$

From (3.2) and (3.3), we see that $V_1 < 0$ on $[r_0, r_1]$. Therefore, (3.4) implies $\Psi > 0$ on $(r_0, r_1]$. As a result, $u(r_1) < \beta$, a contradiction. This lemma is proved.

Before the two most important lemmas, we still need the following two technical claims.

Claim 3.4 Suppose that $f(x)$ is a continuous differentiable function on $[a, \infty)$ and satisfies the following condition:

(C) If x_0 is a local extremum of f, then $|f(x_0)| \leq M$.

Moreover, assume $\sup\limits_{b} \int_a^b f(x)\mathrm{d}x < \infty$. Then

$$\|f(x)\|_{L^\infty[a,\infty)} \leq \max\{|f(a)|, M\}.$$

Proof Assume the contrary, then there exists an x_1 such that

$$|f(x_1)| > \max\{|f(a)|, M\}.$$

Without loss of generality, we can assume $f(x_1) > 0$. Then condition (C) implies $f(x) \geq f(x_1)$, $\forall\, x \geq x_1$, which contradicts the assumption that $\int_a^\infty f(x)\mathrm{d}x$ is finite. The claim is proved.

Claim 3.5 Suppose that $u \in C^2$ is a nonconstant solution to (1.5). Then any local extremum to u or u' must be nonzero.

Proof It directly follows from the uniqueness of solutions to the ODEs.

Now we are able to prove Lemma 3.1.

Proof of Lemma 3.1 The proof consists of three steps. First we show that u has a limit at infinity. Then we prove that this limit is finite. Finally we show that $u(r)$ and $ru(r)$ go to zero as r approaches infinity.

Step 1 $u(r)$ monotonically converges to some $\beta_1 \in [0, \infty]$ as $r \to \infty$.

Claim 3.3 implies that u is a positive solution. Lemma 3.2 and the fact that u is not a constant tell us that there are only two possibilities. One is $u > \beta$ on $(r_1, +\infty)$ for some large r_1. The other is $u < \beta$ on $(r_2, +\infty)$ for some large r_2. We first deal with the former case. Claim 3.1 implies that u has no local minimum in $r_1 f$.

Therefore u has at most one local maxima in $(r_1, +\infty)$. Hence u has neither local minima nor maxima near infinity. This implies Step 1.

The latter can be analyzed in the similar way with the use of Claim 3.2. Step 1 is proved.

Step 2 $\beta_1 < \infty$.

From the point of view of the motion of the particle, this step is more or less obvious because both the friction force and the force induced by potential will push down the particle if its location is high enough. Next we prove this step by contradiction argument. Assume the contrary. Then there is an r_3 such that $u(r) \geq 1$ and $u'(r) \geq 0$, $\forall\, r \geq r_3$. Going back to (1.5), we get $u''(r) \leq 2u(r) - (n+2)u^2(r) \leq -n$, $\forall\, r \geq r_3$. Let $a = u'(r_3)$. We have $u'(r_3 + 1 + a) = u'(r_3) + \int_{r_3}^{r_3+1+a} u''(r)\mathrm{d}r \leq -n$, which is a contradiction. Hence $\beta_1 < \infty$. Step 2 is proved.

Step 3 $\lim\limits_{r\to\infty}[|u(r)| + r|u'(r)|] = 0$.

The proof roughly goes as follows. First we use Claim 3.4 to show u and u' converge to zero at infinity. Going back to (1.5), we get $ru'(r)$ converges as r approaches infinity. Because u is bounded, $ru'(r)$ must converge to zero as r goes to infinity. Then we can use (1.5) to show that u also converges to zero at infinity.

First we derive some controls on the behavior of u' and u'' at infinity. Define $E_1 = \{r_0 \mid r_0$ is a local extremum of $u'\}$. There are two possibilities:

(1) $\exists\, M$ such that $\forall\, r_0 \in E_1$, $r_0 \leq M$;

(2) $\forall\, M < \infty$, $\exists\, r_0 \in E_1$ such that $r_0 > M$.

In Case (1), $u'(r)$ monotonically converges to some β_2 as $r \to \infty$. Because $\lim\limits_{r\to\infty} u(r) < \infty$, we have $\beta_2 = 0$. In Case (2), there exists a sequence of $\{r_k\} \subset E_1$ such that $\lim\limits_{k\to\infty} r_k = \infty$. From (1.5) we know $|u'(r_k)| \leq \frac{C}{r_k}$, $C = C(u, n)$. By applying Claim 3.4 to u' on $[r_k, \infty)$, we get $\lim\limits_{r\to\infty} u'(r) = 0$. Similarly, we have $\lim\limits_{r\to\infty} u''(r) = 0$. From Step 1, we know that $\lim\limits_{r\to\infty}[(n+2)u^2 - 2u]$ exists. From (1.5) and controls on u, u' and u'', we know that $\lim\limits_{r\to\infty}[r(3u-1)u'(r)]$ also exists. We denote this limit by β_3. Next we show $\beta_3 = 0$. Assume the contrary. Then

$|u'(r)| \geq \frac{c}{r}$ for some $c > 0$ and all large r. This contradicts the boundary of u. So $\beta_3 = 0$. Consequently, we have $\lim\limits_{r\to\infty}[(n+2)u^2 - 2u] = 0$, which means $\lim\limits_{r\to\infty} u(r) = 0$ or β. This property and the fact $\beta_3 = 0$ imply that $ru'(r)$ equals zero at infinity.

Next we rule out the case $\lim\limits_{r\to\infty} u(r) = \beta$. There are three possibilities:

(1) $u'(r) \geq 0$, $u(r) < \beta$, $r \in [r_6, \infty)$, for some $r_6 > 0$;

(2) $u'(r) \leq 0$, $u(r) > \beta$, $r \in [r_7, \infty)$, for some $r_7 > 0$;

(3) $u(r) \equiv \beta$, $r \in [r_8, \infty)$, for some $r_8 > 0$.

Case (3) is impossible because u is not a constant. The point of view of the motion of the particle implies that Case (1) is also impossible, because in this case both the friction force and the force induced by potential will push the particle to move upward. Next we use contradiction argument to rule out Case (1) rigorously. Assume that Case (1) holds true. Then (1.5) yields $u''(r) \geq 0$, after r_6. This means $u'(r) \geq u'(r_6) > 0$. Clearly this lower bound on u' implies that u will become larger than β for sufficiently large r, a contradiction. So Case (1) is impossible. Case (2) can be ruled out in a similar way. Hence we have that u converges to zero at infinity.

Case (3) can be easily ruled out. Next we deal with case (1). Notice that there is an $r_9 \in [r_6, \infty)$ such that $r_9 > \frac{100n^2}{n-4}$ and $u''(r_9) < 0$, otherwise $\liminf\limits_{r\to\infty} u'(r) > 0$, a contradiction. But then we have

$$u''(r_9) + \Big[\frac{n+1}{r_9} + 3r_9 u(r_9) - r_9\Big]u'(r_9) + (n+2)u^2(r_9) - 2u(r_9) < 0,$$

which contradicts (1.5). Therefore Case (1) cannot happen. Case (2) can be ruled out in a similar way. The only new thing is that one may need a much larger r_a to make $\frac{n+1}{r_9} + 3r_9 u(r_9) - r_9 < 0$. Hence we have $\lim\limits_{r\to\infty} u(r) = 0$. Going back to (1.5), we get

$$\lim_{r\to\infty}[3ru(r) - r]u'(r) = 0.$$

Now we need the following little claim, whose proof is pretty standard and is omitted.

In summary, we have

$$\lim_{r\to\infty}[|ru'(r)| + |u(r)|] = 0. \tag{3.5}$$

Thus Lemma 3.1 is proven.

Next we prove the decay of u at infinity. First we introduce some function spaces, which quantitatively describe how u decays at infinity. Define

$$\begin{aligned} X_{\mu,r_0} &= \Big\{f \in C^0[r_0,\infty) \,\Big|\, \sup_{r\in[r_0,\infty)} |r^\mu f(r)| < \infty\Big\}, \\ \|f\|_{X_{\mu,r_0}} &= \sup_{r\in[r_0,\infty)} |r^\mu f(r)|, \\ Y_{\mu,r_0} &= \{f \in C^2[r_0,\infty) \mid f, f'' \in X_{\mu,r_0},\ f' \in X_{1+\mu,r_0}\}, \\ \|f\|_{Y_{\mu,r_0}} &= \|f\|_{X_{\mu,r_0}} + \|f'\|_{X_{1+\mu,r_0}} + \|f''\|_{X_{\mu,r_0}}. \end{aligned}$$

The equation under consideration is

$$u'' + \frac{3}{r}u' - (ru' + 2u) + d_1 u' + d_2 u = f. \tag{3.6}$$

Notice that (1.5) can be written as the form above. That is why we consider (3.6). In general, (3.6) will only have one parameter family of solutions which decays at infinity. All the rest solutions grow pretty fast at infinity. To pick up the right solution, we impose on (3.6) the following initial condition

$$u(r_0) = u_0. \tag{3.7}$$

The following lemma is the precise description of the picture above.

Lemma 3.3 *For all $\mu \in (0,2)$, $r_0 \in (1,\infty)$, there exists a constant $c_2 = c_2(\mu) > 0$ such that if $\|d_1\|_{C^0} + \|d_2\|_{C^0} \le c_2$, then $\forall\, f \in X_{\mu,r_0}$, (3.6)–(3.7) has a solution u in Y_{μ,r_0} and u is unique in the class of bounded solutions: $\{f \in C^2[r_0,\infty) \mid \|f\| < \infty\}$.*

Proof We first deal with the existence. The procedure is standard. First we build up a priori estimates. Then we rewrite the equation as an integral equation and apply Banach fixed point theorem to get the existence result. It suffices to consider the case $u_0 = 0$. The procedure is standard: find the representation formula, get the a priori estimates and apply Banach fixed point theorem. If $d_1 = d_2 = 0$, then change of variables $v = r^2 u$ of variation implies the following explicit solution to (3.6)–(3.7):

$$Sf(r) = \frac{-1}{r^2}\int_{r_0}^{r}\Big[r\mathrm{e}^{\frac{r^2}{2}}\int_r^\infty s\mathrm{e}^{-\frac{s^2}{2}} f(s)\mathrm{d}s\Big]\mathrm{d}r.$$

$\forall f \in X_{\mu,r_0}$, we have

$$\begin{aligned}|r^\mu Sf(r)| &= \Big|r^{\mu-2}\int_{r_0}^{r} 2\Big[r\mathrm{e}^{\frac{r^2}{2}}\int_r^\infty \mathrm{e}^{-\frac{s^2}{2}} f(s)\mathrm{d}\Big(\frac{1}{2}s^2\Big)\Big]\mathrm{d}r\Big| \\ &\le r^{\mu-2}\int_{r_0}^{r} r^{1-\mu}\mathrm{d}r \cdot \|f\|_{X_{\mu,r_0}} \\ &\le \frac{2}{2-\mu}\|f\|_{X_{\mu,r_0}}.\end{aligned}$$

In a similar way, we obtain

$$\|Sf\|_{Y_{\mu,r_0}} \le C_2\|f\|_{X_{\mu,r_0}}, \quad C_2 = C_2(\mu).$$

(3.6)–(3.7) can be written as the following integral equation

$$u = Tu \triangleq S(f - d_1u' - d_2u).$$

Taking $c_2 = \frac{1}{2C_2+1}$, we see that T is a contraction from the set $\{u : \|u\|_{Y_N,r_0} \le 4M\}$ to itself, where $M = \|Sf\|_{Y_N,r_0}$. Therefore the existence follows from Banach fixed point theorem.

Next we consider the uniqueness. It is equivalent to showing that except for the solutions we construct in the existence part, all the other solutions to the homogeneous equation go to infinity at infinity. We only need to show that (3.6) has an unbounded solution on $[r_0,\infty)$ when $f = 0$. Because (3.6) is linear and $r_0 > 0$, we know that (3.6) has a solution V such that $V(10+r_0) = 1$ and $V'(10+r_0) = 1$. From (3.6) and the bound on d_1 and d_2, we see that $V(r) > (r-10-r_0)^2$, $\forall\, r > 10+r_0$. The lemma is proved.

Remark 3.1 From the proof, we see that if $f \in X_{2+\delta,r_0}$, $\delta > 0$, then the solution $u \in Y_{2,r_0}$.

Proof of Theorem 1.2 From Lemma 3.2, Lemma 3.3 and Remark 3.1, we know that u satisfies (1.7). This means that the singular solution v constructed from u by (1.4) is bounded in the region $[\delta,\infty)\times[0,T]$, $\delta > 0$. Standard bootstrap argument implies that v is smooth in this region. Therefore $\lim\limits_{s\to\infty} s^2u(s)$ exists. Now it remains to prove that this limit is nonzero. Assume the contrary. Then we know $v = 0$ on $[\delta,\infty)\times\{T\}$. But this means $v = 0$ in $[\delta,\infty)\times[0,T]$ by backward uniqueness for (1.2) (see [2, Theorem 1.1]), a contradiction. Therefore, $\lim_{s\to\infty} s^2u(s) > 0$. The theorem is proved.

Next we prove Theorem 1.1.

For $\alpha > 0$, $\alpha \neq \beta$, we define an index:

$$i(\alpha) = \#\{r \in (0,\infty) \mid u_\alpha(r) = \beta,\ u_\alpha > 0 \text{ on } [0,r)\}.$$

After defining the index above, we explain how to use shooting argument to get the global solution to (1.5). In short, the change of index gives solution. Below are the details. Suppose that the index changes at some initial data α_0. The key observation is that u' is not nonzero at the intersection points between u_α and β. This property, implicit function theorem and the change of index at α_0 imply the following scenario for the intersection points between u_α and β. As α approaches α_0, some of these points will converge to the intersection points between u_{α_0} and β and the rest will go to infinity. This property will enable us to show that u_{α_0} is a global solution to (1.5).

The following lemma is the key step in employing shooting argument.

Lemma 3.4 *Assume $n > 4$. Then there exists $\delta(n) > 0$ such that if $|\alpha - \beta| < \delta(n)$, then $i(\alpha) \geq k(n)$. Here $k(n)$ is the one defined in Theorem* 1.1.

Proof Set $V_\varepsilon = \frac{u_{\beta+\varepsilon}-\beta}{\varepsilon}$. Then V_ε satisfies

$$\begin{cases} V_\varepsilon'' + \Big[\dfrac{n+1}{r} - \dfrac{n-4}{n+2}r\Big]V_\varepsilon' + 2V_\varepsilon + \varepsilon[3rV_\varepsilon V_\varepsilon' + (n+2)V_\varepsilon^2] = 0, \\ V_\varepsilon(0) = 1,\ V_\varepsilon'(0) = 0. \end{cases} \tag{3.8}$$

Define $V_0 = \lim_{\varepsilon\to 0} V_\varepsilon$. Then from [7] we know that V_0 has $k(n)$ zeros in $(0,\infty)$. Take a large K such that all zeros belong to $(0,K)$. Now implicit function theorem and well-posedness for (3.8) imply that there exists a $\delta_1 = \delta_1(n) > 0$ such that if $|\varepsilon| \leq \delta_1(n)$, then V_ε has $k(n)$ zeros in $(0,K)$ and $\|V_\varepsilon\|_{L^\infty[0,K]} \leq C$, $C = C(n)$. Going back to u_α, we see that this lemma is proved.

Now we still need a technical lemma, which may be viewed as the a priori estimate for u_α.

Lemma 3.5 *Suppose $u_\alpha \in C^2[0,M]$, $\alpha \leq K$ and $u_\alpha > 0$ on $[0,M]$. Then*

$$\|u_\alpha\|_{C^3[0,K]} \leq C, \quad C = C(K,M,n).$$

Proof Multiplying (1.5) by u_α' and integrating from 0 to r gives $\|u_\alpha\|_{C^1[0,K]} \leq C$. Now Lemma 3.5 comes directly from the standard bootstrap argument.

Lemma 3.6 *If α is large enough, then the index $i(\alpha) \leq 3$.*

Proof Pick a sufficiently large number M, say, $200n$. Lemma 3.2 implies that u_α intersects β at most twice after M. Using Theorem 1.3, we get that u_α intersects β exactly once on $[\frac{1}{M}, M]$ and $u(\frac{1}{M}) > M$ if α is large enough. Now it remains to prove $u_\alpha > \beta$ on $[0, \frac{1}{M}]$.

The situation is similar to the second part in the proof of Lemma 3.1, but a little more complicated. We prove it by contradiction argument. Assume the contrary. Then there exist two points r_1 and r_2 such that $u(r_1) = u(r_2) = \beta$, $u \leq \beta$ on $[r_1, r_2]$ and $u \geq \beta$ on $[r_2, \frac{1}{M}]$. Now Claim 3.1 implies that $u > 0$ on $[r_1, r_2]$. The bound on u also yields $u'(r_1) \leq 0$ and $u'(r_2) \geq 0$. Using (1.5), we get $u''(r_2) \leq 0$. Therefore the absolute maximum of u' on $[r_1, r_2]$ must be achieved at one local maximum of u', say, r_3. Using (1.5) and the bound u on $[r_1, r_2]$, we get $u'(r_3) < \frac{1}{M}$. Therefore $u'(r_2) \leq u'(r_3) < \frac{1}{M}$. Now notice that Claim 3.1 implies that u has at most one local maximum on $[r_2, \frac{1}{M}]$. Let r_4 be the local maximum if it exists or be $\frac{1}{M}$ if it does not exist. Then we know that $u(r_4) \geq u(\frac{1}{M}) \geq M$ and there is no local maximum in (r_2, r_4). Hence $u' \geq 0$ on $[r_2, r_4]$. Going back to (1.5), we see $u'' \leq 0$ on $[r_2, r_4]$. This yields $u'(r) \leq u'(r_2) \leq \frac{1}{M}$ for $r \in [r_2, r_4]$. This means $u(r_4) < 1$, a contradiction. Therefore $u > \beta$ on $[0, \frac{1}{M}]$. This lemma is proved.

With these three lemmas in hand, we are able to employ shooting argument to prove Theorem 1.1. The idea is that the jump of index gives solution. We present the details below.

Proof of Theorem 1.1 First we shoot from above. $\forall\, 3 < j \leq k(n)$, we define

$$\alpha_j = \sup\{\alpha \mid i(\alpha) \geq j\}. \tag{3.9}$$

From Theorem 1.3 and Lemma 3.1, we know $i(\alpha) \leq 3$ for sufficiently large α. Therefore $\alpha_j < \infty$. Next we prove that u_{α_j} solves (1.5)–(1.7). From (3.9) and well-posedness for (1.5)–(1.6), we know $i(\alpha_j) \leq j-1$. Now Lemma 3.5 implies that there exist two sequences $\{\alpha_{j,n}\}$ and $\{r_n\}$ such that $u_{\alpha_{j,n}}(r_n) = \beta$, $\alpha_{j,n} \to \alpha_j$ and $r_n \to \infty$, otherwise $i(\alpha_j) \geq j$, a contradiction. From Lemma 3.5 and the uniqueness for (1.5)–(1.6), we know that u_{α_j} is defined globally in $[0, \infty)$, i.e., $u_{\alpha_j} \in C^2[0, \infty)$. Now using Theorem 1.2, we obtain that u_{α_j} solves (1.5)–(1.7).

Now we introduce the standard ceiling function

$$\mathrm{cl}(x) = \inf\{k \in Z \mid x \leq k\}.$$

From Lemma 3.1, we know that the index $i(\alpha)$ at most jumps by 2. Hence there are at least $\mathrm{cl}(\frac{k(n)-3}{2})$ different α_j's. In other words, there are at least $\mathrm{cl}(\frac{k(n)-3}{2})$ solutions to (1.5)–(1.7) with $\alpha > \beta$.

Similarly, by shooting form below, we see that (1.5)–(1.7) admits $\mathrm{cl}(\frac{k(n)-2}{2})$ solutions. Notice $\mathrm{cl}(\frac{k(n)-3}{2}) + \mathrm{cl}(\frac{k(n)-2}{2}) = k(n) - 2$. The theorem is proved.

4 Blow-Up

In this section, we will prove Theorem 1.4. The idea is to look at (1.3) as the perturbation of the following equation

$$u_t = (n+2)u^2.$$

To fulfill this idea, we use the so-called function. Roughly speaking, first we choose an appropriate weight. Then the weighted average of v satisfies a differential inequality, which implies Theorem 1.4. We give the details below.

Proof of Theorem 1.4 Assume the contrary. Then this assumption and standard argument in PDEs imply that $u \in C^\infty(\mathbb{R}^n \times \mathbb{R}_+) \cap C(\overline{\mathbb{R}^n \times \mathbb{R}_+})$. It is also straightforward to show that $\mathbf{u}$ is of the form (1.2), i.e.,

$$\mathbf{u}(\mathbf{x}, t) = -v(r, t)\mathbf{x}, \quad r = |\mathbf{x}|,$$

where $v \in C^\infty(\mathbb{R}_+ \times \mathbb{R}_+) \cap C(\overline{\mathbb{R}_+ \times \mathbb{R}_+})$.

For all $n > 4$, define $\delta = \frac{n-4}{6}$ and a measure μ on $\mathbb{R}_+$:

$$\mathrm{d}\mu = c_n r^{3+\delta} \mathrm{e}^{-r} \mathrm{d}r, \tag{4.1}$$

where c_n is chosen such that $\int_{\mathbb{R}^+} \mathrm{d}\mu = 1$. We also define

$$J_v(t) = \int_{\mathbb{R}^+} v(r, t)\mathrm{d}\mu. \tag{4.2}$$

Next we calculate the differential inequality for $J_v(t)$. During the calculation, we will move all the spatial derivatives to the weight $r^{3+\delta}\mathrm{e}^{-r}$.

Then we have

$$\begin{aligned}
\frac{\mathrm{d}J_v(t)}{\mathrm{d}t} &= c_n\int_{\mathbb{R}^+} v_t(r,t)r^{3+\delta}\mathrm{e}^{-r}\mathrm{d}r \\
&= c_n\int_{\mathbb{R}^+}\Big[v_{rr}+\frac{n+1}{r}v_r+3rvv_r+(n+2)v^2\Big]\cdot r^{3+\delta}\mathrm{e}^{-r}\mathrm{d}r \\
&= \int_{\mathbb{R}^+}\Big[v+(n-5-2\delta)\cdot r^{-1}\cdot v+(2-n-\delta)(2+\delta)r^{-2}v \\
&\quad + \Big(n-4-\frac{3}{2}\delta\Big)v^2+\Big(6+\frac{3}{2}\delta\Big)\cdot rv^2\Big]\mathrm{d}\mu \\
&\geq \frac{n-4}{2}\int_{\mathbb{R}^+} v^2\mathrm{d}\mu - C \\
&\geq \frac{n-4}{2}J_v^2(t)-C.
\end{aligned}\tag{4.3}$$

Now Theorem 1.4 follows from the differential inequality above in a standard way.

Acknowledgement The first author would like to express his deep gratitude to Professor Vladimir Šverák for his valuable guidance and advices.

References

[1] Budd, C. J. and Qi, Y. W., The existence of bounded solutions of a semilinear elliptic equation, *J. Diff. Eqs.*, **82**(2), 1989, 207–218.

[2] Escauriaza, L., Seregin, G. and Šverák, V., Backward uniqueness for the heat operator in half-space, *Algebra i Analiz*, **15**(1), 2003, 201–214.

[3] Giga, Y. and Kohn, R. V., Asymptotically self-similar blow-up of semilinear heat equations, *Comm. Pure Appl. Math.*, **38**(3), 1985, 297–319.

[4] Lepin, L. A., Self-similar solutions of a semilinear heat equation (in Russian), *Mat. Modle.*, **2**(3), 1990, 63–74.

[5] Lepin, L. A., Spectra of eigenfunctions for a semilinear heat equation, *Dokl. Akad. Nauk SSSR*, **311**(5), 1990, 1049–1051.

[6] Plecháč, P. and Šverák, V., On self-similar singular solutions of the complex Ginzburg-Landau equation, *Comm. Pure Appl. Math.*, **54**(10), 2001, 1215–1242.

[7] Plecháč, P. and Šverák, V., Singular and regular solutions of a non-linear parabolic system, *Nonlinearty*, **16**(6), 2003, 2083–2097.

[8] Troy, W. C., The existence of bounded solutions of a semilinear heat equation, *SIAM J. Math. Anal.*, **18**(2), 1987, 332–336.

[9] Zhang, P., Remark on the regularities of Kato's solutions to Navier-Stokes equations with initial data in $L^d(\mathbb{R}^d)$, *Chin. Ann. Math.*, **29B**(3), 2008, 265–272.