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Inequalities of Eigenvalues for the Dirac Operator on Compact Complex Spin Submanifolds in Complex Projective Spaces***

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Abstract For a compact complex spin manifold M with a holomorphic isometric embedding into the complex projective space, the authors obtain the extrinsic estimates from above and below for eigenvalues of the Dirac operator, which depend on the data of an isometric embedding of M. Further, from the inequalities of eigenvalues, the gaps of the eigenvalues and the ratio of the eigenvalues are obtained.

Keywords Eigenvalue, Dirac operator, Yang-type inequality, Test spinor
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1 Introduction

The Laplace operator and the Dirac operator are fundamental differential operators in Riemannian manifold. The estimates of their eigenvalues are important in geometry, analysis and physics.

Let $\Omega \subset \mathbb{R}^n$ be the bounded domain in n-dimentional Euclidean space $\mathbb{R}^n$. Consider the Dirichlet eigenvalue problem of the Laplacian

$$\begin{cases} \Delta u = \xi u, & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{1.1}$$

where Δ is the positive Laplacian in Ω. It is well-known that this problem has a real and purely discrete spectrum

$$0 < \xi_1 < \xi_2 \le \xi_3 \cdots \to \infty.$$

Many mathematicians studied the eigenvalue inequalities for the problem (1.1). The work of Payne, Pólya and Weinberger [16], Hile and Protter [13], Yang [18] are the important contributions to this aspect. Furthermore, many mathematicians [2, 3, 8, 10, 13, 14] investigated the other cases, such as n-dimensional compact homogeneous Riemannian manifold, the connected bounded domain in an n-dimensional unit sphere, n-dimensional compact minimal submanifold

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in a unit sphere and so on. As we all known, complex projective space is a kind of important complex manifold. It is also quite interesting to study its eigenvalues. Some results have been obtained in [9, 17] for the eigenvalues of the Laplacian on the connected bounded domain of complex projective space.

It is interesting to study the analogues of the eigenvalues between the Laplace operator and the Dirac operator. Estimates from above for the eigenvalues of the Dirac operator on n-dimensional compact Riemannain spin manifold can be obtained by various ways (see e.g. [1, 4–6, 11]). When M is a compact connected n-dimensional Riemannian spin manifold isometrically immersed in Euclidean space $\mathbb{R}^N$ for some N, N. Anghel [1] obtained the eigenvalue inequality for the Dirac operator, i.e.,

Theorem 1.1 *If M is an n-dimensional spin manifold isometrically immersed in some $\mathbb{R}^N$, and $0 \leq \lambda_1 \leq \lambda_k \leq \cdots \to \infty$ are the eigenvalues of the square of the classical Dirac operator, counted with multiplicities, then*

$$\lambda_{k+1} - \lambda_k \leq \frac{4}{nk}\sum_{i=1}^{k}\lambda_i + \frac{1}{n}\sup_M h^2, \tag{1.2}$$

where $h^2 = |\boldsymbol{h}|^2$, $\boldsymbol{h}$ is the second fundamental form of the immersion.

Recently, D. G. Chen [7] improved the inequality of (1.2) under the same conditions, i.e., he obtained

$$\sum_{i=1}^{k}(\mu_{k+1} - \mu_i)\Big(\mu_{k+1} - \Big(1 + \frac{4}{n}\Big)\mu_i\Big) \leq 0, \tag{1.3}$$

and the weak inequality

$$\mu_{k+1} \leq \Big(1 + \frac{4}{n}\Big)\frac{1}{k}\sum_{i=1}^{k}\mu_i, \tag{1.4}$$

where $\mu_i = \lambda_i + \frac{1}{4}\sup h^2$. Since $\frac{1}{k}\sum\limits_{i=1}^{k}\lambda_i \leq \lambda_k$, one can infer that (1.4) is sharper than (1.2).

In this paper, we study the eigenvalues of the Dirac operator of the complex spin manifold with a holomorphic isometric embedding into the complex projective space. In Section 3, we obtain the analogous eigenvalue inequalities of the Dirac operator on a compact complex spin manifold M with a holomorphic isometric embedding into the complex projective space $\mathbb{CP}^{n+m}$ (see Theorem 3.2). From the Theorem 3.2, we obtain the gaps of the eigenvalues. Moreover, the lower order eigenvalue inequality for the Dirac operator is obtained in Section 4 (see Theorem 4.2). As a straightforward application of [10, Theorem 3.1], we deduce the bounds of μ_{k+1}/μ_1 (see Theorem 4.4), where $\mu_i = \lambda_i + 2n(n+1) - \frac{1}{4}\inf R$, R is the scalar curvature of M, λ_i are the eigenvalues of the square of the Dirac operator.

2 Preliminaries

Let M be an n-dimensional compact Riemannian spin manifold. Denote by $\mathbb{S}$ the spinor bundle over M. Let ∇ be the Levi-Cività connection of M and denote by the same symbol its corresponding lift to the spinor bundle $\mathbb{S}$. It is known (see [15, 11]) that there exists a positive definite Hermitian metric $\langle\,\cdot\,,\,\cdot\,\rangle$ on $\mathbb{S}$. Moreover, the Hermitian metric $\langle\,\cdot\,,\,\cdot\,\rangle$, Riemannian metric, the Levi-Cività connection ∇ and Clifford multiplication "$\cdot$" satisfy the following compatible conditions

$$X\langle\varphi,\psi\rangle = \langle\nabla_X\varphi,\psi\rangle + \langle\varphi,\nabla_X\psi\rangle,$$
$$\nabla_X(Y\cdot\varphi) = \nabla_X Y\cdot\varphi + Y\cdot\nabla_X\varphi,$$
$$\langle X\cdot\varphi, X\cdot\psi\rangle = |X|^2\langle\varphi,\psi\rangle$$

for any tangent vector fields $X, Y \in \Gamma(TM)$ and any spinor fields $\varphi,\psi \in \Gamma(\mathbb{S})$. The Dirac operator is a first order elliptic differential operator $D : \Gamma(\mathbb{S}) \to \Gamma(\mathbb{S})$, which is locally given by

$$D = \sum_{i=1}^{n} e_i\cdot\nabla_i, \tag{2.1}$$

where $\{e_1,\cdots,e_n\}$ is the local orthonormal frame of TM. For $f \in C^\infty(M)$ and $\varphi \in \Gamma(\mathbb{S})$, one gets

$$D(f\varphi) = \mathrm{grad}(f)\cdot\varphi + fD\varphi, \tag{2.2}$$
$$D^2(f\varphi) = \Delta(f)\varphi - 2\nabla_{\mathrm{grad}(f)}\varphi + fD^2\varphi, \tag{2.3}$$

where Δ is the positive scalar Laplacian. For the Dirac operator D, one has the Schrödinger-Lichnerowicz formula

$$D^2 = \nabla^*\nabla + \frac{1}{4}R, \tag{2.4}$$

where R is the scalar curvature of M. In addition, the Dirac operator D of spinor bundle is self-adjoint and elliptic on the compact manifold without boundary. Therefore, D^2 has a discrete spectrum contained in $\mathbb{R}$, numbered like

$$0 \le \lambda_1 \le \lambda_2 \le \cdots \nearrow \infty,$$

and one can find an orthonormal basis $\{\varphi_j\}_{j\in\mathbb{N}}$ of $L^2(\mathbb{S})$ consisting of eigenspinors of D^2 (i.e., $D^2\varphi_j = \lambda_j\varphi_j,\ j\in\mathbb{N}$). Such a system $\{\lambda_j;\varphi_j\}_{j\in\mathbb{N}}$ is called a spectral decomposition of $L^2(\mathbb{S})$ generated by D^2, or, in short, a spectral resolution of D^2. Throughout the paper, we denote $(\,\cdot\,,\,\cdot\,) = \Re\int_M\langle\,\cdot\,,\,\cdot\,\rangle$.

3 Upper Bounds

In this section, taking the similar arguments as in [9], we obtain the eigenvalue estimates for Dirac operator over compact complex spin manifold M with a holomorphic isometric embedding into the complex projective space. The following result provides a general inequality for eigenvalues of the Dirac operator on compact Riemannian spin manifold.

Lemma 3.1 *Let M be an n-dimensional compact Riemannian spin manifold. Let D be the Dirac operator of the spinor bundle $\mathbb{S}$ over M, and $\{\lambda_j;\varphi_j\}_{j\in\mathbb{N}}$ be a spectral resolution of D^2. For any real function $g \in C^\infty(M)$, we have*

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\int_M|\mathrm{grad}(g)|^2\langle\varphi_i,\varphi_i\rangle \le \sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\|\Delta g\varphi_i - 2\nabla_{\mathrm{grad}(g)}\varphi_i\|^2. \tag{3.1}$$

Proof Define a_{ij}, b_{ij} and ψ_i, for $i, j = 1, \cdots, k$, by

$$\begin{cases} a_{ij} = (g\varphi_i, \varphi_j), \\ \psi_i = g\varphi_i - \displaystyle\sum_{j=1}^{k} a_{ij}\varphi_j, \\ b_{ij} = \Big(\varphi_i, \dfrac{1}{2}\Delta g\varphi_j - \nabla_{\mathrm{grad}(g)}\varphi_j\Big). \end{cases} \tag{3.2}$$

A simple observation yields

$$\begin{aligned} &a_{ij} = a_{ji}, \quad b_{ij} = -b_{ji}, \quad 2b_{ij} = (\lambda_i - \lambda_j)a_{ij}, \\ &(\psi_i, \varphi_j) = 0 \quad \text{for } 1 \le j \le k. \end{aligned} \tag{3.3}$$

From the Rayleigh inequality, it is easy to get

$$\lambda_{k+1}\|\psi_i\|^2 \le (D^2\psi_i, \psi_i). \tag{3.4}$$

By (2.3), (3.4) can be written as

$$(\lambda_{k+1} - \lambda_i)\|\psi_i\|^2 \le (\Delta g\varphi_i - 2\nabla_{\mathrm{grad}(g)}\varphi_i, \psi_i). \tag{3.5}$$

From (3.2) and the second line of (3.3), we have

$$(\Delta g\varphi_i - 2\nabla_{\mathrm{grad}(g)}\varphi_i, \psi_i) = (\Delta g\varphi_i - 2\nabla_{\mathrm{grad}(g)}\varphi_i, g\varphi_i) + \sum_{j=1}^{k}(\lambda_i - \lambda_j)a_{ij}^2. \tag{3.6}$$

Using the Schwarz inequality and (3.5), we obtain

$$\begin{aligned} &(\lambda_{k+1} - \lambda_i)[(\Delta g\varphi_i - 2\nabla_{\mathrm{grad}(g)}\varphi_i, \psi_i)]^2 \\ &= (\lambda_{k+1} - \lambda_i)\Big[\Big(\Delta g\varphi_i - 2\nabla_{\mathrm{grad}(g)}\varphi_i + 2\sum_{j=1}^{k} b_{ij}\varphi_j, \psi_i\Big)\Big]^2 \\ &\le (\lambda_{k+1} - \lambda_i)\|\psi_i\|^2\Big\|\Delta g\varphi_i - 2\nabla_{\mathrm{grad}(g)}\varphi_i + 2\sum_{j=1}^{k} b_{ij}\varphi_j\Big\|^2 \\ &\le (\Delta g\varphi_i - 2\nabla_{\mathrm{grad}(g)}\varphi_i, \psi_i)\Big\|\Delta g\varphi_i - 2\nabla_{\mathrm{grad}(g)}\varphi_i + 2\sum_{j=1}^{k} b_{ij}\varphi_j\Big\|^2, \end{aligned}$$

i.e.

$$(\lambda_{k+1} - \lambda_i)(\Delta g\varphi_i - 2\nabla_{\mathrm{grad}(g)}\varphi_i, \psi_i) \le \Big\|\Delta g\varphi_i - 2\nabla_{\mathrm{grad}(g)}\varphi_i + 2\sum_{j=1}^{k} b_{ij}\varphi_j\Big\|^2. \tag{3.7}$$

Multiplying (3.7) by $(\lambda_{k+1} - \lambda_i)$ and taking sum on i from 1 to k, we have

$$\begin{aligned} &\sum_{i=1}^{k}(\lambda_{k+1} - \lambda_i)^2(\Delta g\varphi_i - 2\nabla_{\mathrm{grad}(g)}\varphi_i, g\varphi_i) + \sum_{i,j=1}^{k}(\lambda_i - \lambda_j)(\lambda_{k+1} - \lambda_i)^2 a_{ij}^2 \\ &\le \sum_{i=1}^{k}(\lambda_{k+1} - \lambda_i)\Big\|\Delta g\varphi_i - 2\nabla_{\mathrm{grad}(g)}\varphi_i + 2\sum_{j=1}^{k} b_{ij}\varphi_j\Big\|^2. \end{aligned} \tag{3.8}$$

From the equation (2.4) in [1], we have

$$(\Delta g\varphi_i - 2\nabla_{\mathrm{grad}(g)}\varphi_i, g\varphi_i) = \int_M |\mathrm{grad} g|^2|\varphi_i|^2. \tag{3.9}$$

By (2.3), (3.3) and integration by parts, one obtains

$$\begin{aligned}
&\sum_{i=1}^k(\lambda_{k+1}-\lambda_i)\|\Delta g\varphi_i - 2\nabla_{\mathrm{grad}(g)}\varphi_i + 2\sum_{j=1}^k b_{ij}\varphi_j\|^2\\
&= \sum_{i=1}^k(\lambda_{k+1}-\lambda_i)\Big(\|\Delta g\varphi_i - 2\nabla_{\mathrm{grad}(g)}\varphi_i\|^2 + 4\sum_{i=1}^k b_{ij}(\Delta g\varphi_i - 2\nabla_{\mathrm{grad}(g)}\varphi_i, g\varphi_i) + 4\sum_{j=1}^k b_{ij}^2\Big)\\
&= \sum_{i=1}^k(\lambda_{k+1}-\lambda_i)\Big[\|\Delta g\varphi_i - 2\nabla_{\mathrm{grad}(g)}\varphi_i\|^2 + 4\sum_{i,j=1}^k((\lambda_j-\lambda_i)a_{ij}+b_{ij})b_{ij}\Big]\\
&= \sum_{i=1}^k(\lambda_{k+1}-\lambda_i)\|\Delta g\varphi_i - 2\nabla_{\mathrm{grad}(g)}\varphi_i\|^2 - 4\sum_{i,j=1}^k(\lambda_{k+1}-\lambda_i)b_{ij}^2.
\end{aligned} \tag{3.10}$$

Inserting (3.9) and (3.10) into (3.8), and using (3.3), it is easy to derive (3.1).

In the following, we will prove the main theorem by using Lemma 3.1.

Theorem 3.1 *Let M be an n-dimentional compact complex spin manifold admitting a holomorphic isometric embedding into the complex projective space $\mathbb{CP}^{n+m}$, and λ_i be the eigenvalues of the square of the Dirac operator D. Then we have*

$$\sum_{i=1}^k(\lambda_{k+1}-\lambda_i)^2 \le \frac{2}{n}\sum_{i=1}^k(\lambda_{k+1}-\lambda_i)\Big(\lambda_i + 2n(n+1) - \frac{1}{4}R_0\Big), \tag{3.11}$$

where $R_0 = \inf\limits_M R$, R is the scalar curvature of M. Take $\mu_i = \lambda_i + 2n(n+1) - \frac{1}{4}R_0$. Then (3.11) can be written as

$$\sum_{i=1}^k(\mu_{k+1}-\mu_i)\Big(\mu_{k+1} - \Big(1+\frac{2}{n}\Big)\mu_i\Big) \le 0. \tag{3.12}$$

Proof Let $Z = (Z^0, Z^1, \cdots, Z^{n+m})$ be the homogeneous coordinate system of $\mathbb{CP}^{n+m}$, where $Z^p \in \mathbb{C}$. Denote $f_{p\overline{q}}$, for $p, q = 0, 1, \cdots, n+m$, by

$$f_{p\overline{q}} = \frac{Z^p\overline{Z^q}}{\sum\limits_{r=0}^{n+m} Z^r\overline{Z^r}}. \tag{3.13}$$

Then

$$f_{p\overline{q}} = \overline{f_{q\overline{p}}}, \quad \sum_{p,q=0}^{n+m} f_{p\overline{q}}\,\overline{f_{p\overline{q}}} = 1.$$

For any point $P \in M$, we can choose a new homogeneous coordinate of $\mathbb{CP}^{n+m}$ such that, at P,

$$\widetilde{Z}^0 \neq 0, \quad \widetilde{Z}^1 = \cdots = \widetilde{Z}^{n+m} = 0 \quad \text{and} \quad Z^p = \sum_{r=0}^{n+m} C_{pr}\widetilde{Z}^r,$$

where the matrix $C=(C_{pr})\in U(n+m+1)$, i.e., C_{pr} satisfies

$$\sum_{s=0}^{n+m} C_{pr}\overline{C}_{qr}=\delta_{pq},\quad CC^{\dagger}=I,$$

where I denote the $(n+m+1)\times(n+m+1)$ identity matrix and $C^{\dagger}$ denote the complex conjugate and transpose matrix of the matrix C.

Denote

$$z^p=\widetilde{Z}^p/\widetilde{Z}^0\quad \text{for } p=0,\cdots,n+m.$$

Then one knows that $z=(z^1,\cdots,z^{n+m})$ is the local holomorphic coordinate system of M in a neighborhood U of the point $P\in M$ and $z^{n+\alpha}=h^{\alpha}(z^1,\cdots,z^n)\in\mathcal{O}(U)$ satisfies

$$\frac{\partial h^{\alpha}}{\partial z^p}=0\quad \text{for } 1\le p\le n,\ 1\le\alpha\le m. \tag{3.14}$$

Moreover, at P,

$$z^1=\cdots=z^{n+m}=0. \tag{3.15}$$

Then one can infer

$$\widetilde{f}_{p\overline{q}}=\frac{\widetilde{Z}^p\overline{\widetilde{Z}^q}}{\sum\limits_{r=0}^{n+m}\widetilde{Z}^r\overline{\widetilde{Z}^r}}=\begin{cases}\dfrac{1}{1+\sum\limits_{r=0}^{n+m}z^r\overline{z}^r}, & p=q=0;\\ \dfrac{z^p}{1+\sum\limits_{r=0}^{n+m}z^r\overline{z}^r}, & p\ge 1,\ q=0;\\ \dfrac{\overline{z}^q}{1+\sum\limits_{r=0}^{n+m}z^r\overline{z}^r}, & q\ge 1,\ p=0;\\ \dfrac{z^p\overline{z}^q}{1+\sum\limits_{r=0}^{n+m}z^r\overline{z}^r}, & 1\le p,\ q\le n+m.\end{cases} \tag{3.16}$$

Now define the real functions by

$$\begin{aligned}&G_{p\overline{q}}=\Re(f_{p\overline{q}}),\quad F_{p\overline{q}}=\Im(f_{p\overline{q}});\\ &\widetilde{G}_{p\overline{q}}=\Re(\widetilde{f}_{p\overline{q}}),\quad \widetilde{F}_{p\overline{q}}=\Im(\widetilde{f}_{p\overline{q}}),\quad \text{for } p,q=0,1,\cdots,n+m.\end{aligned} \tag{3.17}$$

The relation between $f_{p\overline{q}}$ and $\widetilde{f}_{p\overline{q}}$ is

$$f_{p\overline{q}}=\sum_{p,q=0}^{n+m}C_{ps}C_{qt}\widetilde{f}_{s\overline{t}}. \tag{3.18}$$

Obviously, one gets

$$G_{p\overline{q}}=\widetilde{G}_{q\overline{p}},\quad F_{p\overline{q}}=-\widetilde{F}_{q\overline{p}}.$$

Under the local coordinate system, when $z\in U$, we have $\mathrm{d}\,s_M^2=\sum\limits_{p=1}^{n}\mathrm{d}\,z^p\,\overline{\mathrm{d}\,z^p}+O(z^2)$. For an n-dimensional complex submanifold M in $\mathbb{CP}^{n+m}$, we have $\Delta_{\mathbb{CP}^{n+m}}f=\Delta_M f+\sum\limits_{i=1}^{m}f_{n+i,n+i}$.

From (3.14)–(3.17), at P, we have

$$\begin{cases} \nabla \widetilde{G}_{p\overline{q}} = 0,\ \nabla \widetilde{F}_{p\overline{q}} = 0, \quad \text{when } pq \neq 0 \text{ or } p = q = 0, \\ \nabla_p \widetilde{G}_{q\overline{0}} = \delta_{pq}, \quad \nabla_p \widetilde{F}_{q\overline{0}} = \delta_{pq}, \\ \nabla_p \widetilde{G}_{0\overline{q}} = \delta_{pq}, \quad \nabla_p \widetilde{F}_{0\overline{q}} = -\delta_{pq}. \end{cases} \tag{3.19}$$

Making use of the same arguments as in [9] or [17], we can obtain, at P,

$$\begin{cases} \displaystyle\sum_{p,q=0}^{n+m} (G_{p\overline{q}}^2 + F_{p\overline{q}}^2) = \sum_{p,q=0}^{n+m} f_{p\overline{q}}\, \overline{f_{p\overline{q}}} = \sum_{p,q=0}^{n+m} \widetilde{f}_{p\overline{q}}\, \overline{\widetilde{f}_{p\overline{q}}} = 1, \\ \displaystyle\sum_{p,q=0}^{n+m} G_{p\overline{q}} \nabla G_{p\overline{q}} + F_{p\overline{q}} \nabla F_{p\overline{q}} = 0, \\ \displaystyle\sum_{p,q=0}^{n+m} (\nabla G_{p\overline{q}} \cdot \nabla G_{p\overline{q}} + \nabla F_{p\overline{q}} \cdot \nabla F_{p\overline{q}}) = 4n, \\ \displaystyle\sum_{p,q=0}^{n+m} (\Delta G_{p\overline{q}} \cdot \Delta G_{p\overline{q}} + \Delta F_{p\overline{q}} \cdot \Delta F_{p\overline{q}}) = 16n(n+1), \\ \displaystyle\sum_{p,q=0}^{n+m} (\Delta G_{p\overline{q}} \cdot \nabla F_{p\overline{q}} + \Delta F_{p\overline{q}} \cdot \nabla F_{p\overline{q}}) = 0. \end{cases} \tag{3.20}$$

Applying Lemma 3.1 to the functions $G_{p\overline{q}}$ and $F_{p\overline{q}}$ and taking sum on p and q from 0 to $n+m$, we get

$$\begin{aligned} 4n \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \leq{} & \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i) \sum_{p,q=0}^{n+m} \int_M ((\Delta G_{p\overline{q}} \cdot \Delta G_{p\overline{q}} + \Delta F_{p\overline{q}} \cdot \Delta F_{p\overline{q}}) \langle \varphi_i, \varphi_i \rangle \\ & - 2 \langle \nabla_{(\Delta G_{p\overline{q}} \nabla F_{p\overline{q}} + \Delta F_{p\overline{q}} \nabla F_{p\overline{q}})} \varphi_i, \varphi_i \rangle + 4 (\langle \nabla_{\mathrm{grad}(G_{p\overline{q}})} \varphi_i, \nabla_{\mathrm{grad}(G_{p\overline{q}})} \varphi_i \rangle \\ & + \langle \nabla_{\mathrm{grad}(F_{p\overline{q}})} \varphi_i, \nabla_{\mathrm{grad}(F_{p\overline{q}})} \varphi_i \rangle)) \\ ={} & 16n(n+1) \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i) + 4 \sum_{p,q=0}^{n+m} \int_M (\langle \nabla_{\mathrm{grad}(G_{p\overline{q}})} \varphi_i, \nabla_{\mathrm{grad}(G_{p\overline{q}})} \varphi_i \rangle \\ & + \langle \nabla_{\mathrm{grad}(F_{p\overline{q}})} \varphi_i, \nabla_{\mathrm{grad}(F_{p\overline{q}})} \varphi_i \rangle). \end{aligned} \tag{3.21}$$

For the rest of the proof, we need the following lemma.

Lemma 3.2 *Under the same assumptions of Theorem* 3.1*, we have*

$$\sum_{p,q=0}^{n+m} (\langle \nabla_{\mathrm{grad}(G_{p\overline{q}})} \varphi_i, \nabla_{\mathrm{grad}(G_{p\overline{q}})} \varphi_i \rangle + \langle \nabla_{\mathrm{grad}(F_{p\overline{q}})} \varphi_i, \nabla_{\mathrm{grad}(F_{p\overline{q}})} \varphi_i \rangle) = 2 \langle \nabla \varphi_i, \nabla \varphi_i \rangle. \tag{3.22}$$

Proof It is sufficient to calculate the above quality at any point $P \in M$. $Z = (Z^1, \cdots, Z^{n+m})$, $\widetilde{Z} = (\widetilde{Z}^1, \cdots, \widetilde{Z}^{n+m})$, $G_{p\overline{q}}$, $F_{p\overline{q}}$, $\widetilde{G}_{p\overline{q}}$, $\widetilde{G}_{p\overline{q}}$ are defined as above. Since the manifold M isometrically embeds into $\mathbb{CP}^{n+m}$, by (3.14), (3.18) and (3.19), a straightforward calculation yields, at $P \in M$,

$$\sum_{p,q=0}^{n+m} \langle \nabla_{\mathrm{grad}(G_{p\overline{q}})} \varphi_i, \nabla_{\mathrm{grad}(G_{p\overline{q}})} \varphi_i \rangle + \langle \nabla_{\mathrm{grad}(F_{p\overline{q}})} \varphi_i, \nabla_{\mathrm{grad}(F_{p\overline{q}})} \varphi_i \rangle$$

$$
\begin{aligned}
&= \sum_{j,k=1}^{2n} \sum_{p,q=0}^{n+m} ((\nabla_j(G_{p\overline{q}})\nabla_k(G_{p\overline{q}}) + \nabla_j(F_{p\overline{q}})\nabla_k(F_{p\overline{q}}))\langle \nabla_j\varphi_i, \nabla_k\varphi_i\rangle) \\
&= \sum_{j,k=1}^{2n} \sum_{p,q=0}^{n+m} \Re(\nabla_j(f_{p\overline{q}})\nabla_k(\overline{f}_{p\overline{q}}))\langle \nabla_j\varphi_i, \nabla_k\varphi_i\rangle \\
&= \sum_{j,k=1}^{2n} \sum_{p,q=0}^{n+m} \Re\Big(\sum_{s,r,u,v}^{n+m} C_{ps}\overline{C}_{qr}\overline{C}_{pu}C_{qv}\nabla_j(\widetilde{f}_{s\overline{r}})\nabla_k(\overline{\widetilde{f}_{u\overline{v}}})\Big)\langle \nabla_j\varphi_i, \nabla_k\varphi_i\rangle \\
&= \sum_{j,k=1}^{2n} \Re\Big(\sum_{s,r,u,v}^{n+m} \sum_{p,q=0}^{n+m} (C_{ps}\overline{C}_{pu})(\overline{C}_{qr}C_{qv})\nabla_j(\widetilde{f}_{s\overline{r}})\nabla_k(\overline{\widetilde{f}_{u\overline{v}}})\Big)\langle \nabla_j\varphi_i, \nabla_k\varphi_i\rangle \\
&= \sum_{j,k=1}^{2n} \sum_{p,q=0}^{n+m} \Re(\nabla_j(\widetilde{f}_{p\overline{q}})\nabla_k(\overline{\widetilde{f}_{p\overline{q}}}))\langle \nabla_j\varphi_i, \nabla_k\varphi_i\rangle \\
&= \sum_{j,k=1}^{2n} \sum_{p,q=0}^{n+m} ((\nabla_j(\widetilde{G}_{p\overline{q}})\nabla_k(\widetilde{G}_{p\overline{q}}) + \nabla_j(\widetilde{F}_{p\overline{q}})\nabla_k(\widetilde{F}_{p\overline{q}}))\langle \nabla_j\varphi_i, \nabla_k\varphi_i\rangle) \\
&= 2\sum_{j,k=1}^{2n} \delta_{jk}\langle \nabla_j\varphi_i, \nabla_k\varphi_i\rangle \\
&= 2\langle \nabla\varphi_i, \nabla\varphi_i\rangle.
\end{aligned}
$$

Inserting (3.22) into (3.21), it is easy to get

$$
n\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \le 4n(n+1)\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i) + 2\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)(\nabla\varphi_i, \nabla\varphi_i). \tag{3.23}
$$

From the Shrödinger-Lichnerowicz formula (2.4), one gets

$$
n\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \le 4n(n+1)\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i) + 2\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\Big(\lambda_i - \frac{1}{4}R_0\Big).
$$

This completes the proof of Theorem 3.1.

From Theorem 3.1, we can obtain the gaps of the consecutive eigenvalues.

Corollary 3.1 *Under the same assumptions of Theorem* 3.1*, we have*

$$
\lambda_{k+1}-\lambda_k \le 2\sqrt{\Big[\Big(\frac{1}{nk}\sum_{i=1}^{k}\lambda_i + 2(n+1) - \frac{1}{4n}R_0\Big)^2 - \Big(1+\frac{2}{n}\Big)\sum_{i=1}^{k}\Big(\lambda_j - \frac{1}{k}\sum_{i=1}^{k}\lambda_i\Big)^2\Big]}\ . \tag{3.24}
$$

Corollary 3.2 *Under the same assumptions of Theorem* 3.1*, we have the following second Yang-type inequality*

$$
\mu_{k+1} \le \Big(1+\frac{2}{n}\Big)\frac{1}{k}\sum_{i=1}^{k}\mu_i. \tag{3.25}
$$

In fact, by the same arguments as Theorem 3.1, we can obtain more general results.

Theorem 3.2 *Let M be an n-dimensional compact complex manifold with a holomorphic isometric embedding into complex projective space $\mathbb{CP}^{n+m}$. Let D be the Dirac operator of any Dirac bundle $\not{S}$ over M, and $\{\lambda_j;\varphi_j\}_{j\in\mathbb{N}}$ be a spectral resolution of D^2. Then*

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \le \frac{2}{n}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)(\lambda_i+2n(n+1)-(\Re\varphi_i,\varphi_i)), \tag{3.26}$$

where

$$\Re=\sum_{i<j}e_i\cdot e_j\cdot \mathcal{R}_{e_i,e_j}, \quad \mathcal{R}_{e_i,e_j}=[\nabla_i,\nabla_j]-\nabla_{[e_i,e_j]}, \tag{3.27}$$

is the curvature endomorphism on the Dirac bundle over M. If $(\Re\varphi_i,\varphi_i)\ge C$, where C is a constant, then one has

$$\sum_{i=1}^{k}(\nu_{k+1}-\nu_i)\Big(\nu_{k+1}-\Big(1+\frac{2}{n}\Big)\nu_i\Big)\le 0, \tag{3.28}$$

where $\nu_i=\lambda_i+2n(n+1)-C$.

4 Lower Bounds

In this section, lower bounds of the eigenvalues for the Dirac operator are derived. First of all, we prepare the following lemma, which is used in the proof of the lower bounds of the eigenvalues.

Lemma 4.1 *Let M be an n-dimensional compact Riemannian spin manifold. Let D be the Dirac operator of the spinor bundle $\mathbb{S}$ over M and $\{\lambda_j;\varphi_j\}_{j\in\mathbb{N}}$ be a spectral resolution of D^2. For any real function $g_i\in C^\infty(M)$ satisfying $(g_i\varphi_1,\varphi_j)=0$ for $j=2,\cdots,i$ on M, we have*

$$(\lambda_{i+1}-\lambda_1)\int_M|\mathrm{grad}(g_i)|^2\langle\varphi_1,\varphi_1\rangle\le\|\Delta g_i\varphi_1-2\nabla_{\mathrm{grad}(g_i)}\varphi_1\|^2. \tag{4.1}$$

Proof Define ψ_i by

$$\psi_i=g_i\varphi_1-(g_i\varphi_1,\varphi_1)\varphi_1.$$

Obviously, one can find

$$(\psi_i,\varphi_j)=0 \quad \text{for } 1\le j\le i.$$

From the Rayleigh inequality, one obtains

$$\lambda_{i+1}\|\psi_i\|^2\le(\psi_i,D^2\psi_i),$$

where $\|\psi_i\|^2=\int_M\langle\psi_i,\psi_i\rangle$. From (2.3) and $D^2\varphi_1=\lambda_1\varphi_1$, one infers

$$(\lambda_{i+1}-\lambda_1)\|\psi_i\|^2\le(\Delta g_i\varphi_1-2\nabla_{\mathrm{grad}(g_i)}\varphi_1,\psi_i). \tag{4.2}$$

From the Schwarz inequality and (4.2), one obtains

$$\begin{aligned}&(\lambda_{i+1}-\lambda_1)(\Delta g_i\varphi_1-2\nabla_{\mathrm{grad}(g_i)}\varphi_1,\psi_i)^2\\ \le\ &(\lambda_{i+1}-\lambda_1)\|\psi_i\|^2\|\Delta g_i\varphi_1-2\nabla_{\mathrm{grad}(g_i)}\varphi_1\|^2\\ \le\ &(\Delta g_i\varphi_1-2\nabla_{\mathrm{grad}(g_i)}\varphi_1,\psi_i)\|\Delta g_i\varphi_1-2\nabla_{\mathrm{grad}(g_i)}\varphi_1\|^2,\end{aligned}$$

i.e.

$$(\lambda_{i+1}-\lambda_1)(\Delta g_i\varphi_1-2\nabla_{\mathrm{grad}(g_i)}\varphi_1,\psi_i)\le \|\Delta g_i\varphi_1-2\nabla_{\mathrm{grad}(g_i)}\varphi_1\|^2. \tag{4.3}$$

In fact, from the definition of ψ_i and the Green formula, a straightforward calculation yields

$$(\lambda_{i+1}-\lambda_1)(\Delta g_i\varphi_1-2\nabla_{\mathrm{grad}(g_i)}\varphi_1,\psi_i)=\int_M |\mathrm{grad}(g_i)|^2\langle\varphi_1,\varphi_1\rangle. \tag{4.4}$$

Inserting (4.4) into (4.3) yields the inequality (4.1).

The similar arguments of the following results can be found in [17]. But for the completeness, here we give a short proof of the estimate.

Theorem 4.1 *If M is an n-dimensional compact complex spin manifold admitting a holomorphic isometric embedding into the complex projective space $\mathbb{CP}^{n+m}$, one get*

$$\sum_{i=1}^{2n}(\lambda_{i+1}-\lambda_1)\le 4\Big(\lambda_1+2n(n+1)-\frac{1}{4}R_0\Big), \tag{4.5}$$

where λ_i is the i-th eigenvalue of the square of the Dirac operator D, $R_0=\inf\limits_M R$, R is the scalar curvature of M. Take $\mu_i=\lambda_i+2n(n+1)-\frac{1}{4}R_0$. Then (4.5) can be written as

$$\sum_{i=1}^{2n}(\mu_{i+1}-\mu_1)\le 4\mu_1. \tag{4.6}$$

Proof We continue to use the same notations as above section. Since $C=(C_{pq})\in U(n+m+1)$, a straightforward calculation yields

$$A=(A_{\beta\alpha})=(C_{ps}\overline{C}_{qt})\in U((n+m+1)^2),\quad \alpha=(p,q),\ \beta=(t,s).$$

The matrix A can be written as $A=A_1+\sqrt{-1}\,A_2$, where A_1,A_2 are the real matrices. Since the matrix A is the unitary matrix, we get

$$D=(D_{\alpha\beta}):=\begin{pmatrix}A_1 & -A_2\\ A_2 & A_1\end{pmatrix}\in O(2(n+m+1)^2).$$

Denote the functions $G_{p\overline{q}}$ and $F_{p\overline{q}}$ by g_α, $\widetilde{G}_{p\overline{q}}$ and $\widetilde{F}_{p\overline{q}}$ by $\widetilde{g}_\alpha$, $p,q=0,1,\cdots,n+m$. The real relations (3.18) becomes

$$g_\alpha=D_{\alpha\beta}\widetilde{g}_\beta,\quad 1\le\alpha,\beta\le 2(n+m+1)^2. \tag{4.7}$$

Without loss of generality, we rearrange the $2(n+m+1)^2$ functions $\widetilde{g}_\alpha$ such that the first $4n$ functions are

$$\widetilde{G}_{1\overline{0}},\cdots,\widetilde{G}_{n\overline{0}},\widetilde{F}_{1\overline{0}},\cdots,\widetilde{F}_{n\overline{0}},\widetilde{G}_{0\overline{1}},\cdots,\widetilde{G}_{0\overline{n}},\widetilde{F}_{0\overline{1}},\cdots,\widetilde{F}_{0\overline{n}},$$

denoted by $\widetilde{g}_{s0}$ and $\widetilde{g}_{0t}$, where $s,t=1,\cdots,n$. And we still denote the other $2(n+m+1)^2-4n$ functions by $\widetilde{g}_\alpha$. Therefore, from (3.19), we have

$$\begin{cases}\nabla_p\widetilde{g}_{p0}=1, & p=1,\cdots,2n,\\ \nabla_p\widetilde{g}_{0p}=1, & p=1,\cdots,n,\\ \nabla_p\widetilde{g}_{0p}=-1, & p=n+1,\cdots,2n,\\ \nabla_p\widetilde{g}_\alpha=0, & \alpha=4n+1,\cdots,2(n+m+1)^2.\end{cases} \tag{4.8}$$

From (4.7) and (4.8), we have

$$\begin{cases} \displaystyle\sum_{\alpha=1}^{2(n+m+1)^2} |\nabla g_\alpha|^2 = 4n, \\ \displaystyle\sum_{\alpha=1}^{2(n+m+1)^2} \nabla g_\alpha \Delta g_\alpha = 0, \\ \displaystyle\sum_{\alpha=1}^{2(n+m+1)^2} |\Delta g_\alpha|^2 = 16n(n+1). \end{cases} \tag{4.9}$$

Denote the matrix H by

$$H = (H_{\alpha\beta}), \quad H_{\alpha\beta} = (g_\alpha \varphi_1, \varphi_{\beta+1}) \quad \text{for } 1 \le \alpha, \beta \le 2(n+m+1)^2,$$

where $\{(\lambda_\beta, \varphi_\beta)\}_{\beta \ge 1}$ is the spectral resolution of D^2. From the **QR**-factorization theorem, there exists an orthogonal matrix $K \in O(2(n+m+1)^2)$ such that KH is a real upper triangular matrix, i.e.

$$K_{\alpha\beta} H_{\beta\gamma} = (K_{\alpha\beta} g_\beta \varphi_1, \varphi_{\gamma+1}) = 0 \quad \text{for } \alpha > \gamma.$$

Considering the relation (4.7) and denoting the real $2(n+m+1)^2 \times 2(n+m+1)^2$ matrix $O_{\alpha\beta}$ by $O = KD$, one infers

$$O \in O(2(n+m+1)^2), \quad (O_{\alpha\beta} \widetilde{g}_\beta \varphi_1, \varphi_{\gamma+1}) = 0 \quad \text{for } \alpha > \gamma. \tag{4.10}$$

Defining the real functions h_α by $h_\alpha = O_\alpha^\beta \widetilde{g}_\beta$, for $1 \le \alpha \le 2(n+m+1)^2$, from (4.9), we obtain

$$\begin{cases} \displaystyle\sum_{\alpha=1}^{2(n+m+1)^2} |\nabla h_\alpha|^2 = 4n, \\ \displaystyle\sum_{\alpha=1}^{2(n+m+1)^2} |\Delta h_\alpha|^2 = 16n(n+1), \\ \displaystyle\sum_{\alpha=1}^{2(n+m+1)^2} \nabla h_\alpha \Delta h_\alpha = 0. \end{cases} \tag{4.11}$$

We claim that

$$|\nabla h_\alpha|^2 \le 2 \quad \text{for } 1 \le \alpha \le 2(n+m+1)^2. \tag{4.12}$$

In fact, from (4.8) and $O \in O(2(n+m+1)^2)$, we have

$$\begin{aligned} |\nabla h_\alpha|^2 &= \sum_{p=1}^{2n} \sum_{\beta=1}^{2(n+m+1)^2} O_{\alpha\beta} \nabla_p \widetilde{g}_\beta \sum_{\gamma=1}^{2(n+m+1)^2} O_{\alpha\gamma} \nabla_p \widetilde{g}_\gamma \\ &= \sum_{p=1}^{2n} (O_{\alpha(p,0)} \nabla_p \widetilde{g}_{p0} + O_{\alpha(0,p)} \nabla_p \widetilde{g}_{0p})^2 \\ &= \sum_{p=1}^{n} (O_{\alpha(p,0)} + O_{\alpha(0,p)})^2 + \sum_{p=n+1}^{2n} (O_{\alpha(p,0)} - O_{\alpha(0,p)})^2 \end{aligned}$$

$$\leq \sum_{p=1}^{2n}[(O_{\alpha(p,0)})^2 + (O_{\alpha(0,p)})^2 + 2|O_{\alpha(p,0)}O_{\alpha(0,p)}|]$$
$$\leq 2\sum_{p=1}^{2n}[(O_{\alpha(p,0)})^2 + (O_{\alpha(0,p)})^2]$$
$$\leq 2\sum_{\beta=1}^{2(n+m+1)^2}(O_{\alpha\beta})^2 = 2.$$

Applying Lemma 4.1 to the functions h_α, taking sum on α from 1 to $2(n+m+1)^2$, and using (4.11) and the Schrödinger-Lichnerowicz formula (2.4), we get

$$\sum_{\alpha=1}^{2(n+m+1)^2}\lambda_{\alpha+1}(|\nabla h_\alpha|^2\varphi_1,\varphi_1) \leq 4n\lambda_1 + 16n(n+1) + 8\lambda_1 - 2R_0. \tag{4.13}$$

From (4.12), we infer

$$\begin{aligned}\sum_{\alpha=1}^{2(n+m+1)^2}\lambda_{\alpha+1}|\nabla h_\alpha|^2 &\geq \sum_{\alpha=1}^{2n}\lambda_{\alpha+1}|\nabla h_\alpha|^2 + \lambda_{2n+1}\sum_{\alpha=2n+1}^{2(n+m+1)^2}|\nabla h_\alpha|^2\\ &= \sum_{\alpha=1}^{2n}\lambda_{\alpha+1}|\nabla h_\alpha|^2 + \lambda_{2n+1}\Big(4n - \sum_{\alpha=1}^{2n}|\nabla h_\alpha|^2\Big)\\ &= \sum_{\alpha=1}^{2n}\lambda_{\alpha+1}|\nabla h_\alpha|^2 + \lambda_{2n+1}\Big(\sum_{\alpha=1}^{2n}(2 - |\nabla h_\alpha|^2)\Big)\\ &\geq \sum_{\alpha=1}^{2n}\lambda_{\alpha+1}|\nabla h_\alpha|^2 + \sum_{\alpha=1}^{2n}(2 - |\nabla h_\alpha|^2)\lambda_{\alpha+1}\\ &= 2\sum_{\alpha=1}^{2n}\lambda_{\alpha+1},\end{aligned}$$

i.e.

$$\sum_{\alpha=1}^{2(n+m+1)^2}\lambda_{\alpha+1}|\nabla h_\alpha|^2 \geq 2\sum_{\alpha=1}^{2n}\lambda_{\alpha+1}. \tag{4.14}$$

Inserting (4.14) into (4.13), we obtain (4.5).

Let $m = 2n$ be the real dimension of the manifold M. The inequalities of eigenvalues (3.12) and (4.6) can be written as, respectively,

$$\sum_{i=1}^{k}(\mu_{k+1} - \mu_i)\Big(\mu_{k+1} - \Big(1 + \frac{4}{m}\Big)\mu_i\Big) \leq 0, \quad \sum_{i=1}^{m}(\mu_{i+1} - \mu_1) \leq 4\mu_1,$$

where

$$\mu_i = \lambda_i + \frac{1}{2}m(m+2) - \frac{1}{4}R_0 = \lambda_i + 2n(n+1) - \frac{1}{4}R_0.$$

Since the above inequalities satisfy the assumptions of [10, Theorem 3.1], we have

Theorem 4.2 *Under the same assumptions as Theorem* 3.2, *we have*

(1) *if* $m \geq 41$ *and* $k \geq 41$, *then*

$$\mu_{k+1} \leq k^{2/m}\mu_1;$$

(2) *for any* k,

$$\mu_{k+1} \leq \Big(1 + \frac{a(\min\{m, k-1\})}{m}\Big)k^{2/m}\mu_1,$$

where the bound of $a(s)$ *can be formulated as*

$$\begin{cases} a(1) \leq 2.64, \\ a(s) \leq 2.2 - 4\log\Big(1 + \dfrac{1}{50}(s-3)\Big) \quad \text{for } s \geq 2. \end{cases}$$

From [10], we can deduce the simple and clear inequality

$$\mu_{k+1} \leq \Big(1 + \frac{4}{m}\Big)k^{2/m}\mu_1. \tag{4.15}$$

From [12, Lemma 1.12.6], (4.15) is a best possible estimate in the sense of order.

By the same arguments as Theorem 4.2, a general result can be derived as follows.

Theorem 4.3 *Let* M *be an* n*-dimensional compact complex manifold admitting a holomorphic isometric embedding into complex projective space* $\mathbb{CP}^{n+m}$. *Let* D *be the Dirac operator of any Dirac bundle* $\not{S}$ *over* M, *and* $\{\lambda_j; \varphi_j\}_{j\in\mathbb{N}}$ *be a spectral resolution of* D^2. *Then*

$$\sum_{i=1}^{2n}(\lambda_{i+1} - \lambda_1) \leq 4(\lambda_1 + 2n(n+1) - (\Re\varphi_1, \varphi_1)),$$

where $\Re$ *is given in* (3.27). *Moreover, if* $(\Re\varphi_1, \varphi_1) \geq C$, *where* C *is a constant, then we have*

$$\sum_{i=1}^{2n}(\nu_{i+1} - \nu_1) \leq 4\nu_1,$$

where $\nu_i = \lambda_i + 2n(n+1) - C$.

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