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REGULAR ARTICLE

Non-self-averaging of a two-person game with only positive spillover: a new formulation of Avatamsaka's dilemma

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Abstract In this game [Aruka in Avatamsaka game structure and experiment on the web. In: Aruka Y (ed) Evolutionary controversies in economics. Springer, Tokyo, pp 115–132, 2001], selfishness may not be determined even if an agent selfishly adopts the strategy of defection. Individual selfishness can only be realized if the other agent cooperates, therefore gain from defection can never be assured by defection alone. The sanction by defection as a reaction of the rival agent cannot necessarily reduce the selfishness of the rival. In this game, explicit direct reciprocity cannot be guaranteed. Now we introduce different spillovers or payoff matrices, so that each agent may then be faced with a different payoff matrix. A ball in the urn is interpreted as the number of cooperators, and the urn as a payoff matrix. We apply Ewens' sampling formula to our urn process in this game theoretic environment. In this case, there is a similar result as in the classic case, because there is "self-averaging" for the variances of the number who cooperate. Applying Pitman's sampling formula to the urn process, the invariance of the random partition vectors under the properties of exchangeability and size-biased permutation does not hold in general. Pitman's sampling formula depends on the two-parameter Poisson–Dirichlet distribution whose special case is just Ewens' formula. In the Ewens setting, only one probability α of a new entry matters. On the other hand, there is an additional probability θ of an unknown entry, as will be argued in the Pitman formula. More concretely, we will investigate the effects

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of different payoff sizes from playing a series of different games for newly emerging agents. As Aoki and Yoshikawa (Non-self-averaging in macroeconomic models: a criticism of modern micro-founded macroeconomics, Economics Discussion Papers 2007-49. http://www.economics-ejournal.org. November 26, 2007) and Aoki (J Econ Interact Coord 3:1–3, 2008) dealt with a product innovation and a process innovation, they criticized Lucas' representative method and the idea that players face micro shocks drawn from the same unchanged probability distribution. In the light of Aoki and Yoshikawa (Non-self-averaging in macroeconomic models: a criticism of modern micro-founded macroeconomics, Economics Discussion Papers 2007-49. http://www.economics-ejournal.org. November 26, 2007), we show the same argument in our Avatamsaka game with different payoffs. In this setting, innovations occurring in urns may be regarded as increases of the number of cooperators in urns whose payoffs are different.

1 Introduction

1.1 Towards a new method to deal with two conspicuous features of modern society

When discussing modern society, there are two conspicuous features: a huge number of players and a set of unexpected results.

First, it is trivial that there are a huge number of players with their intensive and/or extensive interactions. The total population of the world is 6.5×10^9 players, and is gradually approaching $\times 10^{10}$. Including global financial activities, the order of magnitude of interactions will be 10^{15} or so. The interaction of human activities might amount to interactions based on the number of molecules in a box of gas, i.e., 10^{23}.[1] Social sciences must now treat a huge number, just as in statistical physics.

Second, in this secular world, people with good intentions are not always rewarded. This perception could actually be traced back to ancient beliefs such as Tyche or Fortuna of ancient mythology. In the social sciences, this one belief has inspired and promoted applications of probability theory since David Hume.

Briefly look at this story in light of Mainzer's work (Mainzer 2007a,b; Aruka 2009). In the ancient cultures, the goddesses Tyche in Greece and Fortuna in Rome were well known. Fortuna however does not always guarantee fortunes for individuals. According to Mainzer (2007b), the Christian tradition rejected the idea of goddesses of fortune and fate, and Kairos, the opportune moment, instead became highlighted in the new era. Individuals can prepare for the moment that may bring them opportunities. Mainzer (2007a), in connection with non-linear scientific innovations of the last century, insightfully argued Kairos as a "right moment" in order to judge Mandelbrot's parables (Mandelbrot and Hudson 2004): the Joseph effect and the Noah effect. In the modern world with its increasing complexity, it is relevant to describe contingencies in connection with Kairos, a kind of a macroscopically weak control mechanism.

Reconsidering "a free act of human self-determination in a stream of non-linearity and randomness in history", the classical philosophical arguments could be revived.

[1] See Fujiwara (2008).

It is easy to realize that individuals never rely only upon their own a priori preferences. On the other hand, “chances” cannot be ignored independently of “individual abilities” (including learning abilities) and “circumstances”. Things like novelties emerging either from nature or society could never happen without intermediation by chance. Thus, how chances may intermediate emergences of thinking must be explored, so that we are ready to argue Hegel’s methodology positively in the new methodological view. Then it is certain that “Good intentions may lead to bad effects without their subjective intentions. Hegel called it a stratagem of reason” (Mainzer 2007a, p. 408).

In the following discussions, equilibrium does not imply a kind of “self-filling prophecy” game result for participating agents, whose expectations must often be circumvented.

1.2 An interactive chance

Kant distinguished contingencies as being more empirical, more logical, and more intelligible. An empirical contingency that depends on a certain cause is meant as the “event.” Sun irradiation is not a necessary condition for heating a stone, but a stone necessarily falls—on the basis of gravity—to the earth at any moment. Seemingly, these are opposite observations. A concept may fail to catch all the events around it. A property (attribute or type) that does not follow from the definition of a concept (category) is called logically “chance” (Mainzer 2007a, p. 408).

This is the reason why the idea of type distribution with exchangeable agents should be used, because it is difficult to identify the attributes or types by themselves with reference to events at hand. Due to the limitations of the empirical data available, however, a microscopic approach in terms of the given or fixed types must be given up, using some idea of state property instead. An example is income transition in the society. In this example, middle class property is a state property, if the income range of the middle class is pertinently defined. The ingredients of various types in the middle class of income is then ignored. Such a simplification is called a metonymy from types to state properties, according to Hildenbrand (1994). Exchangeable agents are virtually used, instead of types in a precise sense. Types are replaced with state variables. A state variable then becomes a surrogate variable for type. Thus, the focus is on the combinatorial dynamics of the total number of state and the state variables. It may be helpful to reproduce Fig. 1[2] in order to deal with transitional and exchangeable types.[3]

[2] This figure is almost equivalent to Figure 3 in Aruka (2007).

[3] This interpretation is supplemented by citing Bowles (2004), in terms of behavioral dynamics: “Thus individuals are the bearers of behavioral rules. Analytical attention is focused on the success or failure of these behavioral rules themselves as they either diffuse and become pervasive in a population or fail to do so and are confined to minor ecological niches or are eliminated. The dramatis personae of the social dynamic thus are not individuals but behavioral rules: how they fare is the key; what individuals do is important for how this contributes to the success or failure of behavioral rules.”

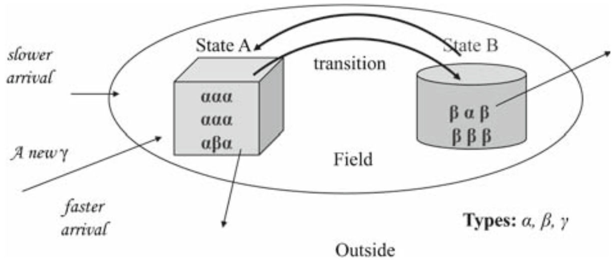

Fig. 1 The idea of transmission mechanism

1.3 A macro-microscopic linkage and our approach

The "Price Equation" (Price 1970, 1972) in biology is well known as a simple statistical statement on the expected change of the frequency of a gene. Recently, the *Journal of Economic Behavior and Organization*[vol. 53(1), 2004] has arranged a special issue on the Price Equation in the context of Henrich's contributions (Henrich 2004). In this gene, x_i denotes an altruistic allele if individual i retains $x_i = 1$, while an egoistic allele if $x_i = 0$. x_i can then express the current frequency of this gene. Let w_i be the number of offspring of i, namely, the absolute fitness of i while $\bar{w}_i$ is the average fitness of the population. In Henrich's contribution (2004), "we ignore any effects arising from the transmission process (e.g., recombination, mutation, etc.)." It then follows:

$$\overline{w}\Delta\overline{x} = \beta_{w_j x_j}\mathrm{Var}(x_j) + E\left(\beta_{w_{ij}x_{ij}}\mathrm{Var}(x_{ij})\right) \tag{1}$$
$$\text{Selection between-groups} + \text{Selection within-groups}$$

Here $\beta_{w_{ij}x_{ij}}$ means the within-group regression coefficients of x_{ij} with respect to w_{ij} in group j. This type of Price Equation "tells us that the change in the frequency of allele created by natural selection acting on individuals can be partitioned into between-group and within-group components" (Henrich 2004).

This is a good practice to measure contributions of interactive factors due to between-groups and within-groups. It is not a better one, however, because this description lacks an overarching mechanism working between within-groups and between-groups. In the above equation, this is just a transition mechanism in the Master Equation approach. Without this overarching mechanism, the signs of between-groups effects as indicative of group selection may not be definitely ascertained (Aruka 2004).

The assumption of a mutant gene to promote altruistic behavior for this group may be rejected because it reduces the fitness of its carrier. In order to make group selection feasible, a transmission mechanism is indispensable, as Henrich stated (2004). Cultural group selection based on such a transmission could be viable if and only if a certain macroscopic order is implemented in the interactive field. In other words, a promising way to take a cooperative system formation into account is to introduce a macroscopic order in the concerned field in which individual agents act bilaterally with groups. If such a binding condition of group selection to form a macroscopic order could be found, an overarching mechanism could be found to illustrate the processes of a dispersive social system.[4]

Thus, even in the case of describing such an interactive process between cooperation and defection, we could be induced to adopt the stochastic approach employed by Masanao Aoki's seminal works (Aoki 1996, 2002, 2008; Aoki and Yoshikawa 2006, 2007), written partly in cooperation with Hiroshi Yoshikawa. It might be hypothetically true that human beings have both the altruistic gene and the egoistic gene. However, individuals never rely only upon their own distinct genes, since chance must be consecutively created in the course of the sequential interactions of variant clusters of exchangeable agents. So, interactive chances matter.

2 A two-person game with positive spillover

2.1 A famous tale from Buddhism Sutra

There is a famous tale of *Avatamsaka Sutra*, mahayana buddhist sutra. Suppose that two men sat down at a table, across from one another. They are tied up with rope except for one arm, then each is given a long spoon. They cannot feed themselves because the spoon is too long, but each can feed the other. There is plenty of food on the table.

- If they cooperate to feed each other, they can both be happy. This defines Paradise.
- One may be so kind as to feed the other but the other may not cooperate. This must give rise to a feeling of hate for the other. This describes a situation of Hell.

The gain structure will not essentially depend on the altruistic willingness to cooperate. This tale often is cited all over the world, recently, in *Chicken Soup for the Soul* (Canfield and Hansen 2001), a best-selling book in the US.

Aruka (2001) formulated this tale as two-person, two-strategy game and called this game Avatamsaka game. The main features of the Avatamsaka game are summarized as:

- Any defector may use his rival's cooperation to get his gain. Without the rival's cooperation, however, his gain cannot be guaranteed.
- The expected value of a gain for any agent could be reinforced, if it should be generated by a higher average rate of cooperation as measured by the frequency of mutual cooperation. Here there is a macroscopically weak control mechanism, which does not mean a personal mutual fate control described in Thibaut and Kelley (1959).
- The sanction by defection as a reaction of the rival agent cannot necessarily reduce the selfishness of the rival. In this game, then, any explicit direct reciprocity cannot be automatically guaranteed.

In the Avatamsaka game (Aruka 2001), selfishness may not be determined even if the agent selfishly adopts the strategy of defection. Individual selfishness can only be realized if the other agent cooperates. Any certain gain from defection can never be

[4] Hildenbrand (1994) called it "macroscopic microeconomic" linkage.

assured by defection alone. The sanction by defection as a reaction of the rival agent cannot necessarily reduce the selfishness of the rival. In this game, any explicit direct reciprocity cannot be guaranteed.

2.2 A historical note

2.2.1 The "mutual fate control" game

The authors recently learned that the same formulation, looking at the payoff structure formally, had already been explored in the field of psychology. In psychology, it is very important to know how one could affect the other's behavior. Our payoff structure can show this. In the field of social psychology, the payoff structure of the Avatamsaka game was argued. The same game appeared earlier, in 1959, in the book titled *The social psychology of groups* (Thibaut and Kelley 1959). The game was called "mutual fate control." Psychologists created experiments based on this game. Recently, Mitropulos revived it (Mitropoulos 2001).

2.2.2 The path dependence of the Avatamsaka game

Our original view focused on an emerging/evolving environment, i.e., path dependency. We further focus on two kinds of averaging.

- Self-averaging: eventually players' behavior could be independent from the other players.
- Non-self-averaging: the invariance of the random partition vectors under the properties of exchangeability and size-biased permutation does not hold in general.

2.3 Geometric structures of two-person games

In order to clarify the characteristics of the Avatamsaka game, we compare our Avatamsaka game with a class of Prisoner's Dilemmas, taking two numerical examples shown in Tables 1 and 2.

$$\text{Avatamsaka} \begin{bmatrix} R & S \\ T & P \end{bmatrix} = \begin{bmatrix} 1 & 0.25 \\ 1 & 0.25 \end{bmatrix}$$

Table 1 Avatamsaka

Player 1	Player 2	
	Cooperation	Defection
Cooperation	(1, 1)	(0.25, 1)
Defection	(1, 0.25)	(0.25, 0.25)

$$\text{Prisoner's Dilemma} \begin{bmatrix} R & S \\ T & P \end{bmatrix} = \begin{bmatrix} 1 & 0.3 \\ 0.9 & 0.1 \end{bmatrix}$$

Table 2 Prisoner's Dilemma

Player 1	Player 2	
	Cooperation	Defection
Cooperation	(1, 1)	(0.1, 0.9)
Defection	(0.9, 0.1)	(0.3, 0.3)

In order to clarify the properties of the two games, we may use the letters *R, S, T, P* and the differences of $D_g = T - R$, $D_r = P - S$.

Here we call

$$D_r = P - S$$

the "Risk Aversion Dilemma" and

$$D_g = T - R$$

the "Risk Seeking Dilemma."

First, the Avatamsaka game is specified as follows:

$$D_g = T - R = 0 \tag{2}$$
$$D_r = P - S = 0 \tag{3}$$

On the other hand, a Prisoner's Dilemma game is specified as follows:

$$D_g = T - R = 0.2 \tag{4}$$
$$D_r = P - S = 0.2 \tag{5}$$

Note that dilemmas may be generated unless "complementarity" for both players holds.

We call

$$R - S = T - P$$

the "spillover". In the Avatamsaka game, spillovers are always positive. So each player can then increase his rewards by the other player's strategy switching (Aruka 2001, p. 118). [Regarding the Avatamsaka original papers, see Aruka (2001, 2002) and Akiyama and Aruka (2006)].

2.4 Tanimoto's geometrical expression of a two-person game

Tanimoto geometry (Tanimoto 2007a,b; Tanimoto and Sagara 2007) is used to describe the characteristics of games.

The Tanimoto geometry is composed by the following equations:

$$P = 1 - 0.5 r_1 \cos\left(\frac{\pi}{4}\right) \tag{6}$$

$$R = 1 + 0.5 r_1 \cos\left(\frac{\pi}{4}\right) \tag{7}$$

$$S = 1 + r r_1 \cos\left(\frac{\pi}{4} + \theta\right) \tag{8}$$

$$T = 1 + r r_1 \cos\left(\frac{\pi}{4} + \theta\right) \tag{9}$$

$$\text{Here } r = \frac{r_2}{r_1}; \quad r_1 = PS, \quad r_2 = SM$$

Using this description, we can illustrate the key points *R, S, T, P* on the two-dimensional plane of Fig. 2. Here the horizontal axis shows Player 1's payoff while the vertical axis shows Player 2's. An Avatamsaka game is shown in Fig. 3 and a Prisoner's Dilemma game is shown in Fig. 4.

We define the contour on the parametric plane (θ, r), depicting the contour of $T + P = R + S$ on this plane. A contour is shown in Fig. 5; $T + P > R + S$ is inside the contour and $T + P < R + S$ is outside the contour. We show these relationships in Fig. 6. A Prisoner's Dilemma game is to be located on the vertical line $\theta = \frac{\pi}{2}$. From this point of view, a Prisoner's Dilemma game may be regarded as a reference standard for any two-person game.

Note that our Avatamsaka game has a property of $T + P = R + S$. This implies that our game could always be defined as the contours, if the various payoffs were allowed to be introduced, as we observe later. Different contours correspond to different payoffs of the Avatamsaka game, so players can walk around between different states of payoffs. In the latter case, our argument may be ubiquitous on the contours plane.

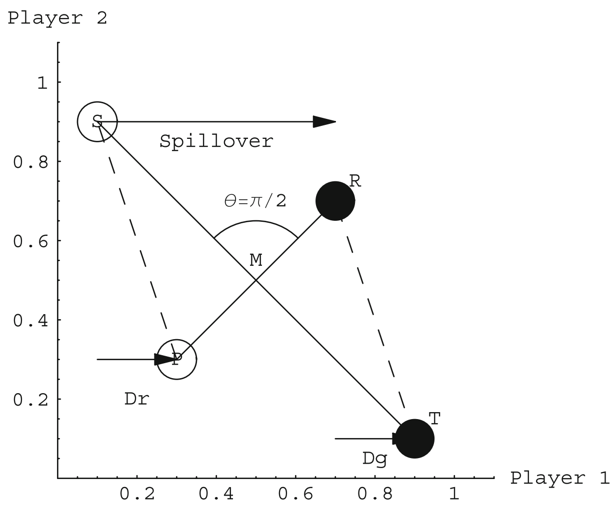

Fig. 2 The Tanimoto geometry

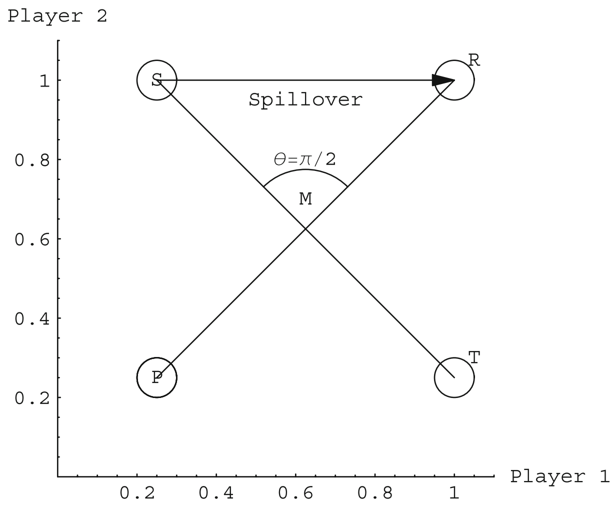

Fig. 3 Avatamsaka

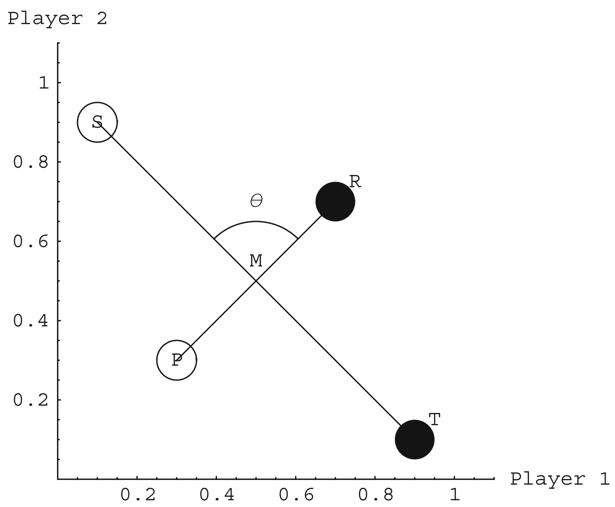

Fig. 4 Prisoner's Dilemma

The Avatamsaka game by definition has infinite equilibria even given a fixed payoff structure. Originally, our game was always allowed to be on any Nash equilibrium. The dynamics of our game are a kind of equilibrium dynamics. In the following stochastic setting, by resorting to Aoki's method, players could never have their reasonable self-filling prophecies even if players should wander on the contours. Also notice that our notion of equilibrium does not necessarily guarantee each agent's prophecy. This is a new finding.

Finally, compare the Avatamsaka game with the donor-recipient type prisoner's dilemma, which can be regarded as an exceptional case ($c = 0$) of ($T = b$,

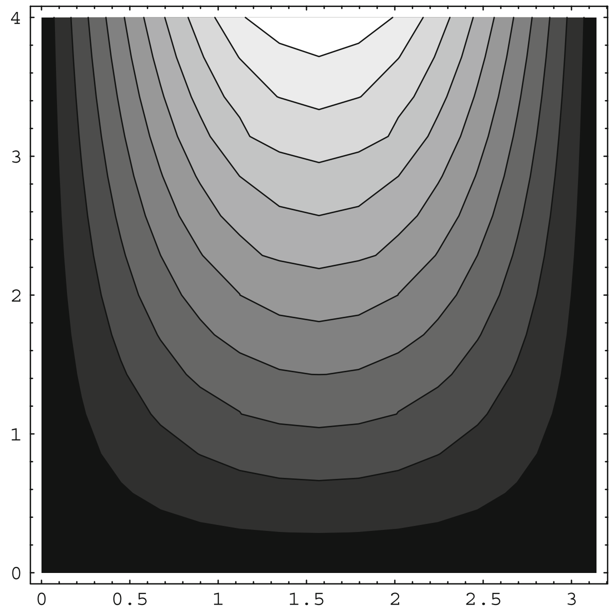

Fig. 5 The contours of $T + P = R + S$

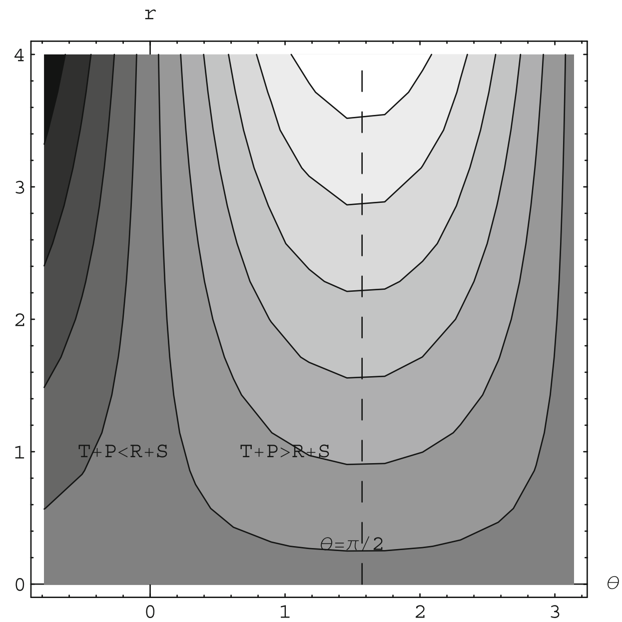

Fig. 6 The relationships on the contours

$R = b - c$, $P = 0$ and $S = -c$).[5] In our game, such a cost is not observed, instead of taking account of transition risks posed by the failure of agent expectations.[6] In this article, the dynamics of expectation—which are not based on the benefit-cost relationship—matter. Our game could be the suggested case by linear transformation if a small perturbation is given on the off-diagonal elements of the payoff structure. An introduction of perturbation suggests that our game becomes out of equilibrium. Such disequilibrium must be important but, for the moment, we confine ourselves to observing only equilibrium dynamics.

2.5 A last attempt to formulate the Avatamsaka game in terms of an evolutionary game

In Akiyama and Aruka (2006), the Avatamsaka game as an evolutionary game of dyadic pairs of interactive agents was discussed. We focused on the so-called memory effects on agent decisions, supposing a repeated game with "action noise" (or matching noise).[7] In reality, interaction with a person does not happen only once. In a repeated game, each agent needs a strategy to decide her action from the memory of the past actions (e.g., "Tit-For-Tat" for repeated Prisoner's Dilemma). Here agents' memory size becomes important. Action noise is defined as the mistakes that players sometimes make when they hand in their actions because we really sometimes make mistakes.

In our setting, this is verified by the following propositions:

- Memory size is 0: There is no evolutionary stable strategy. The population shares of (All D, All C)[8] do not change from the initial generation.
- Memory size is greater than 2: Pavlov is the *ESS* (evolutionary stable strategy) for any memory size.

Here Pavlov's policy (Nowak and Sigmund 1993) is defined in the following way:

- If she feels "good", she would not change her current action.
- If she feels "uneasy", she will change her current action.

In the case of the infinitely repeated game, we have a definite conclusion that Pavlov is the *ESS* for $m \leq 2$. If the probability of another repetition of the Avatamsaka game is less than 1.0, Pavlov is the *ESS* for any memory size.

[5] This was pointed out by an anonymous referee to whom we are indebted for this description.

[6] The risk may be the risk of realized expectation, whether the expectation is cooperation or defection, when agents wander around.

[7] As Bowles (2004, p. 62) argued, this technique is not superficial: "What is called matching noise is another way that chance affects evolutionary dynamics. When small numbers of individuals in a heterogeneous population are randomly paired to interact, the realized distribution of types with whom one is paired over a given period may diverge significantly from the expected distribution. The difference between the realized distribution and the expected distribution reflects matching noise and may have substantial effects. …How, then, are evolutionary models different? First, mutations, behavioral innovations, and matching noise are distinct because these sources of stochastic events are endogenous to evolutionary models."

[8] D denotes "Defection" while C denotes "Cooperation".

This result can easily be applied to any dependent game like the Avatamsaka game (with any number of players and with any number of strategies). Thus, Pavlovian strategy is the *ESS* in a dependent game: *Keep changing my action until the others benefit me*.

Now we will try to formulate the Avatamsaka game in view of stochastic processes where we can deal with a truly emerging mutant. In contrast with our last results, stable equilibrium states may not necessarily be guaranteed.

3 The Avatamsaka stochastic process under a given payoff matrix

3.1 The simplest Polya's urn process

In the original stage, an urn contains a white ball and a red ball only. Draw out one ball, and return that ball with an additional ball of the same color. Repeat this trial again and again. The number of balls increases by one with each trial. So after the completion of two trials,

- The total number of balls after the second trial = 2 + 1 + 1 = 4.
- Hence the expected value of white balls after two successive trials is 2(=4/2) since the number of red ball is equal to the number of white ball at the initial point.

After the completion of n trials, what is the probability that the urn contains just one white ball? This must be equivalent to the probability of n-fold successive draws of red balls.

$$\underbrace{\frac{1}{2}\frac{1+1}{2+1}\frac{1+1+1}{2+1+1}\cdots\frac{n}{n+1}}_{\text{Path-dependent}} = \frac{1}{n+1}.$$

Suppose the total number of white balls can be k at the l-th trial $(1 \leq k \leq n+1)$. We describe this situation as

$$P(l,k) = \frac{1}{l+1}.$$

By induction, it is easy to prove that

$$P(n,k) = \frac{1}{n+1}.$$

This result shows that any event (n, k) at the n-th trial can emerge, i.e., any number of white balls can go everywhere at the ratio of $\frac{1}{n+1}$.

A short demonstration of the classic, simplest case of the Polya urn process:

In this kind of short simulation, it is trivial that the trend of this trial must be seen within a short time like 100 trials (Figs. 7, 8).

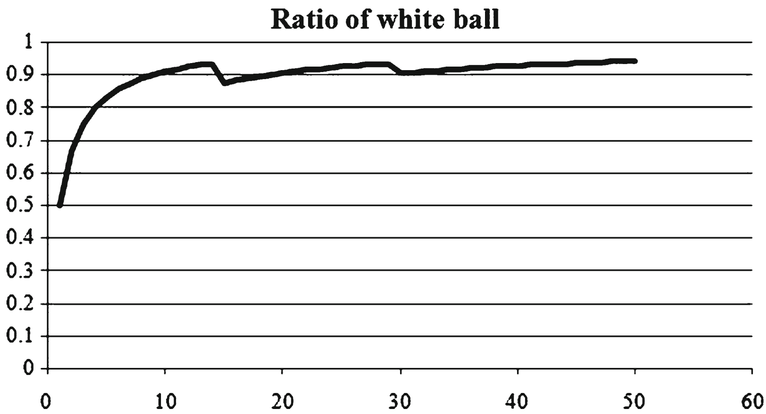

Fig. 7 Simulation 1 (This simulation is originally designed by Excel Macro Programming.)

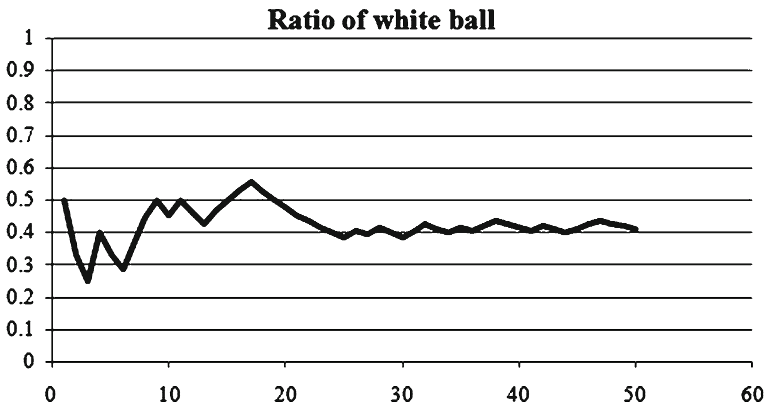

Fig. 8 Simulation 2 (This simulation is originally designed by Excel Macro Programming.)

3.2 Avatamsaka characteristics

Avatamsaka characteristics are summarized as follows:

- The increase of the ratio $\frac{C}{C+D}$ (C-ratio) implies an average increase of C agents who changed their strategies from D to C, which leads to the expectation of mutual cooperation. This reflects the property of positive spillover.
- In the simple case, then, it is trivial that the expectation of mutual cooperation could be reinforced.
- So this game has a macro structure of cooperation that can loosely control a path of interaction.

Now, apply a Polya urn stochastic process to our Avatamsaka game experiment to predict an asymptotic structure of the game, according to Aruka (2002). This stochastic process was employed by Brian Arthur (1994) and in particular in Arthur et al. (1987). In order to apply Arthur's theorem, we impose the assumptions:

Assumption 1 There are a finite number of players who join the Avatamsaka Game by pairs.

Assumption 2 The ratio of cooperation, or C-ratio for each player, is identified with the proportion of the total possible gains for each player.

3.3 New applications of the urn process to Avatamsaka game

A red ball is interpreted as a gain for the player who loves to use the Defect Strategy in our two-person game, while a white ball is a gain for the player who loves to use the Cooperation Strategy. Player 1 can then continue to increase his gain if the proportion of his gain is kept higher than Player 2's. The Polya original restriction never allows a player to change his mind. A non-linear inspection for players may be taken into account to obtain a more realistic result (Aruka 2001).

There are several applications of the urn process to the Avatamsaka game. Here, we apply a classic Polya urn process with a given payoff matrix to the Avatamsaka game. In this case, we may discover a kind of "averaging", in the sense that a player's behavior could be independent from the others. Fortunately, however, we have more candidates for doing this. We have the next two sampling formulas.

- Ewens' sampling formula: a K-dimensional Polya urn process with multiple payoff matrices and possible new agents;
- Pitman's sampling formula: a two-parameter Poisson–Dirichlet distribution.

In the latter case, there may be a "non-averaging" property in the sense that the invariance of the random partition vectors under the properties of exchangeability and size-biased permutation does not hold in general (Pitman 2002).

3.3.1 A comparison with the market

In the stock or commodity exchanges, any trade or sequence of trades must be settled at the end of session. Once started, any trade must have an end within a definite period, even in the futures market. A distribution of trader's types can affect the trading results, but not vice versa. Compared with the stock exchange system, each game round in a repeated game must change its own environment. A distribution of agent's types can affect the results of the game, and vice versa. Agents must inherit their previous results. This situation describes path-dependency in a repeated game.

3.4 Self-averaging in the Avatamsaka game with a special rule

3.4.1 A balanced urn game

According to Aoki and Yoshikawa (2007), there is another urn model to discuss: the idea of a "balanced urn" in the context of Flajolet et al. (2005).

The urn contains two different colored balls: white and red. If a white ball is drawn from the urn, a white balls and $b - a$ red balls are returned into the urn. If a red ball is drawn from the urn, b red balls are returned into the urn. This type of ball replacement is expressed in the replacement matrix, R:

$$R = \begin{pmatrix} a & b-a \\ 0 & b \end{pmatrix}$$

The urn is called balanced, if

- The sum of the first row = $a + b - a = b$.
- The sum of the second row = b.

The matrix R then is a balanced triangular urn.

Proposition 1 *The number of white balls in the balanced triangular urn model is non-self-averaging.* (For the proof, see Aoki and Yoshikawa 2007, pp. 11–13.)

Before discussing the properties of non-self-averaging of strategy arrangements in a more general case, we apply this modeling directly to our particular Avatamsaka game: if a player is faced with the cooperation of a rival, the player can react by C-ratio of $\frac{a}{b-a}$. On the other hand, the player can react merely by defecting when faced with the defection of a rival.[9] The game must be non-self-averaging according to Proposition 1.[10]

In the balanced urn, the total number of balls in the urn is constant regardless of the color of a drawn ball. The mixture of the two colors (types) must be path-dependent. This may create a non-self-averaging property for the game. If a player is faced with the cooperation of a rival, the player can react by the C-ratio of a to $b - a$. On the other hand, the player can react merely by defecting when he is faced with the defection of a rival; b is constant as the number of defections is always constant. This special unchanged attitude of D-agents can rather contribute a stable expectation of non-self-averaging.

3.4.2 A non-linear Polya urn dynamic game: a gain-ratio in the total possible points

Let X_i be a gain-ratio in the total possible gains that Player i gains. The gain-ratio = $\frac{\text{Gain}}{\text{TotalGain}}$. The initial total potential of gains for each player is defined as $2N - 1$. In the next period, 2, the total maximal gains will grow by the number of players $2N$. In the period n, therefore, the gain-ratio for Player i at time n will be

$$X_i^n = \frac{b_i^n}{(2N-1)n}.$$

Suppose there is such a sample space Ω that

$$\begin{cases} X_i : \Omega \to [0, 1] \\ \omega \in \Omega \to X_i(\omega) \in [0, 1]. \end{cases}$$

[9] This statement needs an assumption like Assumption 3, given in Sect. 4.

[10] We can add to a more general proposition: in non-balanced triangular urn models as depending on the values of parameters, no-self-averaging emerges (Aoki and Yoshikawa 2007, p. 14).

There is a probability $x = X_i(\omega)$ for a sample ω. Then, the non-linear evolution of the proportion, i.e., Arthur's dynamics:

$$X_i^{n+1} = X_i^n + \frac{1}{n+1}\left(\beta_i^n(x) - x_i^n\right) \tag{10}$$

Inspecting the expected value of X_i^n leads to the result:

Proposition 2 *Players shall initially be motivated by the behaviors of the other players. Eventually, however, players' behavior could be independent from the others* (For the proof, see Aruka 2002).

3.5 Relaxing the assumption of a given payoff matrix

Suppose there are a number of different urns with various payoff matrices, and each has its own size of spillover. A different spillover in our Avatamsaka game may change the inclinations of players' reactions, but these inclinations are not necessarily symmetrical.

3.5.1 Inclinations of a player's strategy

An urn with a greater spillover might sometimes be more attractive for a cooperative player in a game with a high level of cooperation, because the players could earn greater gains. A cooperative player in a game with a low level of cooperation might be attracted to enter an urn with a smaller spillover than in a game with a much higher level of cooperation. But, he can become a defecting player. There may thus be various players' plans for their strategies. Any player depends on his state, which may change, while an urn, that is, a payoff matrix occurs stochastically.

3.5.2 Nothing to do with any particular transition mechanism

Thus, as we suppose that any urn must be random for players, any player must be random for urns. In our Avatamsaka game, the size effects due to different spillovers are irrelevant, so any specific or particular transition mechanism need not be assumed ex ante.

4 The key ideas for the new economics

4.1 The economics of the master equation and the fluctuations

The stochastic evolution of the state vector can be described in terms of the master equation as equivalent to the Chapman–Kolmogorov differential equation system. The master equation leads to bringing the aggregate dynamics, from which the Fokker–Planck equation could be derived. Thus, we can explicitly argue the fluctuations in a dynamic system. These settings can indispensably be connected with the following key ideas, making the type classification of agents in the system feasible, and tracking the variations in cluster size, as featured by statistical physics and combinatorial stochastic processes (Aoki and Yoshikawa 2006).

In Aoki's new economics, there are exchangeable agents in the combinatorial stochastic process, as in the urn process. The exchangeable agents come out by the use of random partition vectors, as in statistical physics or population genetics. The partition vector provides the state information, allowing discussion of the size-distribution of the components and their cluster dynamics with the exchangeable agents.

Define a maximum countable set, in which the probability density of transition from state i to state j is given, respectively. In this setting, the dynamics of the heterogeneous interacting agents gives the field where an agent can become another agent. It is also important to note that this way of thinking welcomes the unknown agents.

4.1.1 A K-dimensional Polya distribution

A K-dimensional Polya distribution is stated by the use of parameter θ. We then have a transition rate:

$$w(\mathbf{n}, \mathbf{n} - \mathbf{e}_i + \mathbf{e}_j) = \frac{n_i}{n} \frac{n_j + \theta_j}{n - 1 + \theta}$$

where

$$e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$$

1 is located ith position in the above row. It is also noted that

$$n = n_1 + \cdots + n_K; \theta_j > 0$$

and

$$\theta = \sum_{j}^{K} \theta_j.$$

A Jump Markov Process' Stationary State then follows:

$$\pi(\mathbf{n}) w(\mathbf{n}, \mathbf{n} - \mathbf{e}_i + \mathbf{e}_j) = \pi(\mathbf{n} - \mathbf{e}_i + \mathbf{e}_j) w(\mathbf{n} - \mathbf{e}_i + \mathbf{e}_j, \mathbf{n}) \tag{11}$$

$$\pi(\mathbf{n}) = \frac{w(\mathbf{n} - \mathbf{e}_i + \mathbf{e}_j, \mathbf{n})}{w(\mathbf{n}, \mathbf{n} - \mathbf{e}_i + \mathbf{e}_j)} \pi(\mathbf{n} - \mathbf{e}_i + \mathbf{e}_j) \tag{12}$$

Hence we have the stationary distribution:

$$\pi(\mathbf{n}) = \frac{n!}{\theta^{[n]}} \prod_{i=1}^{K} \frac{\theta_i^{[n_i]}}{n_i!} \tag{13}$$

where

$$\theta^{[n]} = \theta(\theta + 1) \cdots (\theta + n - 1). \tag{14}$$

4.2 A general urn process

Suppose that balls (or agents) and boxes (or urns) are both indistinguishable. We then have a partition vector:

$$\mathbf{a} = (a_1, a_2, \ldots, a_n).$$

a_i is the number of boxes containing i balls. The number of balls is:

$$\sum_{i=1}^{n} i a_i = n.$$

The number of categories is:

$$\sum_{i=1}^{n} a_i = K_n.$$

K_n is the number of occupied boxes.

The number of configurations then is

$$N(\mathbf{a}) = \frac{n!}{\prod_{j=1}^{n} (j!)^{a_j} a_j!} = \frac{n!}{(1!)^{a_1} (2!)^{a_2} \cdots (n!)^{a_n} a_1! a_2! \cdots a_n!}.$$

4.2.1 A new type entry in an urn process

Let **a** be a state vector. Suppose that one new type agent enters an empty box. We then have the equation:

$$w(\mathbf{a}, \mathbf{a} + \mathbf{e}_1) = \frac{\theta}{n + \theta}. \tag{15}$$

Suppose also that an agent enters a cluster of size j. Adding one to size $j + 1$ while reducing one from size j:

$$w(\mathbf{a}, \mathbf{a} + \mathbf{e}_{j+1} - \mathbf{e}_j) = \frac{j a_j}{n + \theta}. \tag{16}$$

On the contrary, suppose that an agent leaves a cluster of size j. Adding one to size $j - 1$ while reducing one from size j:

$$w(\mathbf{a}, \mathbf{a} - \mathbf{e}_j + \mathbf{e}_{j-1}) = \frac{j a_j}{n}. \tag{17}$$

We then have Ewens' Sampling Formula

$$\pi(\mathbf{a}) = \frac{n!}{\theta^{[n]}} \prod_{j=1}^{n} \left(\frac{\theta}{j}\right)^{a_j} \frac{1}{a_j!} \tag{18}$$

where

$$\sum_{j=1}^{n} j a_j = n; \quad \sum_{j=1}^{n} a_j = K_n.$$

4.2.2 The probability that the number of clusters is k

Let the probability that the number of clusters is k be the sum of a newcomer who comes in a new cluster with probability $\frac{n}{n+\theta}$ and a newcomer who comes in an existing cluster with probability $1 - \frac{n}{n+\theta}$:

$$q_{n,k} := \Pr(K_n = k|n) \tag{19}$$

$$q_{n+1,k} = \frac{n}{n+\theta} q_{n,k} + \frac{\theta}{n+\theta} q_{n,k-1} \tag{20}$$

In this case, there are the boundary conditions:

$$q_{n,1} = \frac{(n-1)!}{\theta^{[n]}} \tag{21}$$

$$q_{n,n} = \frac{\theta^n}{\theta^{[n]}}. \tag{22}$$

It then follows:

$$q_{n,k} = \frac{\theta^k}{\theta^{[n]}} c(n,k) \tag{23}$$

where

$$\begin{aligned}
c(n+1,k) &= nc(n,k) + c(n,k-1) \\
\theta^{[n+1]} &= \sum_{m=0}^{n} s(n,m)\theta^m = \theta(\theta+1)(\theta+2)\cdots(\theta+n-1)(\theta+n) \\
&= \theta^{[n]}(\theta+n) = \theta \cdot \theta^{[n]} + n \cdot \theta^{[n]} \\
\theta \cdot \theta^{[n]} &= \sum_{m=0}^{n} s(n,m)\theta \cdot \theta^m \\
&= \sum_{m=0}^{n} s(n,m)\theta^{m+1} = \sum_{m-1=0}^{n} s(n,m-1)\theta^m = c(n,k-1)
\end{aligned}$$

$$n\theta^{[n]} = n\sum_{m=0}^{n} s(n,m)\theta^m = nc(n,k).$$

The final equation is called sign-less Stirling Number of the first kind.

4.3 Pitman's Chinese restaurant process

Suppose that there are an infinite number of round tables in the Chinese restaurant that are labeled by an integer from 1 to n. The first customer, numbered 1, takes a seat at the table numbered 1. Suppose that the customers from No.1 to No. k in turn take their seats at their tables from No.1 to No. k. Here the c_j customers take their seats at the jth table (Pitman 1995; Yamato and Sibuya 2000, 2003).

Now the new arrival comes! The next arriving customer has two options: either to take a seat at the kth table by the probability

$$\frac{\theta + k\alpha}{\theta + n}$$

or to take a seat at the table j, one of the remaining tables $j = 1, \ldots, k$ by the probability

$$\frac{c_j - \alpha}{\theta + n}.$$

Here two parameters, θ and α are used. Thus, the solution:

$$\frac{n!\theta^{[k:\alpha]}}{\theta^{[n]}} \prod_{j=1}^{n} \left(\frac{(1-\alpha)^{[j-1]}}{j!}\right)^{c_j} \frac{1}{c_j!} \tag{24}$$

where

$$\theta^{[j]} = (\theta+1)\cdots(\theta+j-1),$$
$$\theta^{[j:\alpha]} = (\theta+\alpha)\cdots(\theta+(j-1)\alpha).$$

Ewens' sampling formula (Ewens 1972) gives the invariance of the random partition vectors under the properties of exchangeability and size-biased permutation. The Ewens sampling formula is the case with one parameter; in a special case of two-parameter Poisson–Dirichlet distributions:

$$\frac{n!}{\theta^{[n]}} \prod_{j=1}^{n} \left(\frac{\theta}{j}\right)^{a_j} \frac{1}{a_j!}$$

In the case of the two-parameter Poisson–Dirichlet model, there will be a non-averaging system in the limit.

5 An application of the two-parameter Poisson–Dirichlet model to Avatamsaka

5.1 Non-self-averaging

Aoki and Yoshikawa (2006) have shown that the two-parameter Poisson–Dirichlet models are qualitatively different from the one-parameter version. An additional parameter could then generate non-self-averaging, even if there were a situation of self-averaging in the one-parameter model (Aoki and Yoshikawa 2007, p. 6). An urn state is self-averaging if the number of balls in each urn could eventually be convergent, on average, in the following sense (Aoki and Yoshikawa 2007, p. 4): "Non-self averaging" means that a size-dependent (i.e., "extensive" in physics) random variable X of the model has the coefficient of variation that does not converge to zero as model size goes to infinity. The coefficient of variation (C.V.) of an extensive random variable, X, defined by

$$\text{C.V.}(X) = \frac{\sqrt{\text{variance}(X)}}{\text{mean}(X)}$$

is normally expected to converge to zero as model size (e.g. the number of economic agents) goes to infinity. In this case, the model is said to be "self-averaging."

5.2 Aoki's application

An application of the two-parameter Poisson–Dirichlet model to an economic system (Aoki and Yoshikawa 2007) shows a growing economy of multiple sectors where the waves of innovations arrive stochastically. Innovation, when it occurs, either raises productivity in one of the existing sectors, or creates a new sector. Thus, the number of sectors is not given, but increases over time.

By the time the nth innovation occurs, the total of K_n sectors are formed in the economy where the i-th sector has experienced n_i innovations ($i = 1, 2, \ldots, k_n$). By definition, the following equality holds:

$$n_1 + n_2 + \cdots + n_k = n$$

when $K_n = k$. If the n-th innovation creates a new sector (sector k), then $n_k = 1$.

5.2.1 Between different spillovers

The Avatamsaka game is characterized by the positive spillovers for both players; larger or smaller spillovers all preserve Avatamsaka characteristics, of course. A typical change of spillover size is illustrated in Fig. 9. As stated in Sect. 1.4, our Avatamsaka game could always be defined as the contours, if the various different payoffs are allowed to be introduced. Different contours correspond to different payoffs of the Avatamsaka game. Given the different payoff states, players can wander among different states. So we can describe a kind of dynamic equilibrium.

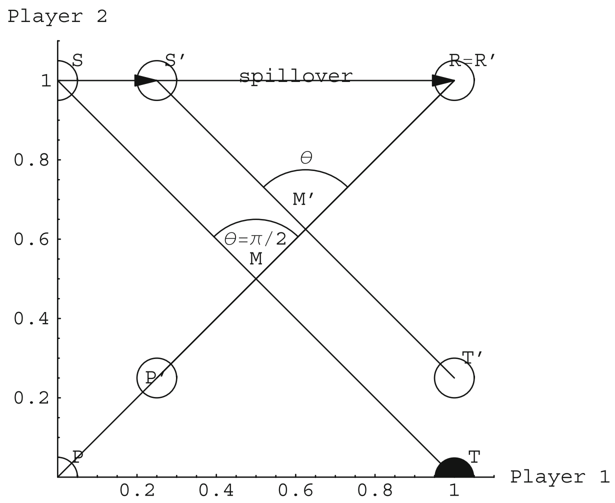

Fig. 9 Spillover

Fig. 10 Various spillovers (These clips are picked up from Microsoft Office Sample Collections.)

We can measure a difference of any two spillovers by

$$\frac{\eta_1}{\eta_2} = \frac{SR}{S'R'}.$$

Incidentally, we note that we can get a greater spillover structure in a Prisoner's Dilemma game by simply extending the *PR* line (Fig. 10).

Given a particular spillover, the expected total gain in the urn will increase, if the level of cooperation in the game increases. It is natural that more cooperative players could bring about a higher average size gain in the given box. This coincides with the previous assumption (Assumption 2).

5.3 An application to Avatamsaka game

Suppose there are many different urns with various different payoff matrices, each of which has its own size spillover. A different spillover in the Avatamsaka game may change the inclinations of players' reactions. However, these inclinations are not necessarily symmetrical. An urn with a greater spillover might sometimes be more attractive for a cooperative player in the context of a high level of cooperation, because the player could earn greater gains. A cooperative player in the context of a low level of cooperation might enter an urn with a smaller spillover than he would with a much higher level of cooperation. He can change his mind to become a defecting player. There may thus be various players' plans for their strategies. Any player depends on his state, which may change, while an urn, that is, a payoff matrix occurs stochastically.

Thus, any urn must be random for players, and any player must be random for urns. In this Avatamsaka game, size effects due to different spillovers may be ignored. A specific or particular transition mechanism need not be assumed ex ante.

Now replace "innovations" with "increases of the level of cooperation" in this game. By the time n-th cooperation occurs, the total of K_n payoff urns are formed in the whole game space where the i-th payoff urn has experienced n_i cooperations $(i = 1, 2, \ldots, k_n)$. By definition, the following equality holds:

$$n_1 + n_2 + \cdots + n_k = n \tag{25}$$

when $K_n = k$. If n-th cooperation creates a new payoff matrix (urn k), then $n_k = 1$. It is noted that

$$n = \sum_j j a_j(n). \tag{26}$$

So there are a finite number of urns into which various types of payoff matrices are embedded: $1, \ldots, K$.

In this environment, we can have n inventions to increase the level of cooperation x_i. In other words, the level of cooperation in urn i may grow due to stochastic multiple inventions occurring in this urn.

Assumption 3 Cooperation accelerates cooperation, i.e., the larger the number of cooperators, the larger the number of cooperators and/or the greater the total gain in the urn will be.

Assumption 4 Due to an Avatamsaka property, a player can compare the situations between the urns given to him by normalizing his own gain.

We then assume a particular type growth:

$$x_i = \eta_i \gamma^{n_i}, \quad \eta_i > 0, \quad \gamma > 1, \quad \text{for } i = 1, \ldots, k. \tag{27}$$

First of all, the part γ^{n_i} reflects Assumption 3.

As shown above, η_i indicates an element of the set of different spillovers:

$$E = (\eta_1, \ldots, \eta_n)\,.$$

Fig. 11 A stochastic interaction (These clips are picked up from Microsoft Office Sample Collections.)

$a_j(n)$ is the number of urns where j inventions have occurred. Here $a_j(n)$ is an element of the partition vector $a(n)$. K_n can then be expressed as

$$K_n = \sum_j^n a_j(n). \tag{28}$$

Expanding the exponential $\exp(n_i \ln \gamma)$ and rounding the remaining terms except for the first two, we obtain the next approximation:

$$\gamma^{n_i} \approx 1 + \ln(\gamma)n_i.$$

Hence it follows:

$$x_i = \eta_i + \eta_i \ln(\gamma)n_i. \tag{29}$$

Due to Assumption 4, we normalize x_i by the use of η_i (Fig. 11).

$$\tilde{x}_i = \frac{x_i}{\eta_i}$$

Here we then define:

$$X_n = \sum_i^{K_n} \tilde{x}_i. \tag{30}$$

This shows the aggregate behavior of cooperation dynamics, i.e., cluster urn dynamics. Thus, from these Eqs. (27)–(30) in the above, we obtain

$$X_n \approx K_n + \beta \sum_j^n j a_j(n). \tag{31}$$

where $\beta = \ln(\gamma) > 0$. It then turns out that X_n depends on how cooperation occurs.

5.4 A result

We have just transformed our Avatamsaka game form into the same Eq. (31) in essentially the same context as (Aoki and Yoshikawa 2007). Thus, by the same reasoning, we could conclude the same result. Hence we have the following proposition:[11]

Proposition 3 *In the two-parameter Poisson–Dirichlet model, the aggregate cooperation behavior X_n is non-self-averaging.* (For the proof, see Aoki and Yoshikawa 2007, pp. 6–10.)

6 Concluding remarks

There is a particular type of self-fulfilling prophecy that is like the Oedipus effect. According to Robert K. Merton, this effect is that a true prophetic statement—a prophecy declared as truth when it is not—may sufficiently influence people, either through fear or logical confusion, so that their reactions ultimately fulfill the false prophecy. This implies that the collective macro-state of various social orders (order parameter) can be averaged over its parts.

We apply this self-fulfilling rhetoric to our rational expectation hypothesis. This hypothesis may be self-fulfilling either through fear or logical confusion, of course, but this prophecy must be faced with some logical failure. It could be verified in the framework of complex dynamics that this hypothesis only referred to a unilateral direction of the whole dynamics, in which a single rational individual had to contribute to the rational macro-state of the economy. Actually, we need the other aspect of the full feedback: "Its order parameters strongly influence the individuals of the society by orienting (enslaving) their activities and by activating or deactivating their attitudes and capabilities." (Mainzer 2007a, p. 395) This is just the slaving principle as a whole elucidated as synergetics by Hermann Haken (Weidlich 2002, 2006, 2007). This dynamic thus is encompassed by critical values, outside of which the system falls into an unstable situation. So the prophecy to be fulfilled might be circumvented. In our new setting of the Avatamsaka game, it has turned out that we often are faced with non-convergent, unstable situations in the face of some particular types of interactive chances.

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