Chin. Ann. Math.
32B(4), 2011, 549–568
DOI: 10.1007/s11401-011-0658-z

Chinese Annals of Mathematics, Series B

Global Existence of the Equilibrium Diffusion Model in Radiative Hydrodynamics*

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Abstract This paper is devoted to the analysis of the Cauchy problem for a system of PDEs arising in radiative hydrodynamics. This system, which comes from the so-called equilibrium diffusion regime, is a variant of the usual Euler equations, where the energy and pressure functionals are modified to take into account the effect of radiation and the energy balance containing a nonlinear diffusion term acting on the temperature. The problem is studied in the multi-dimensional framework. The authors identify the existence of a strictly convex entropy and a stability property of the system, and check that the Kawashima-Shizuta condition holds. Then, based on these structure properties, the well-posedness close to a constant state can be proved by using fine energy estimates. The asymptotic decay of the solutions are also investigated.

Keywords Radiative hydrodynamics, Initial value problem, Equilibrium diffusion regime, Energy method
2000 MR Subject Classification 76N10, 35L65, 35L45, 35Q80

1 Introduction and Main Results

We are interested in the following system of nonlinear PDEs:

$$\begin{cases}\partial_t\rho + \nabla_x\cdot(\rho u) = 0,\\ \partial_t(\rho u) + \mathrm{Div}_x(\rho u\otimes u) + \nabla_x p^* = 0,\\ \partial_t(\rho E^*) + \nabla_x\cdot(\rho E^* u + p^* u) = \displaystyle\sum_{1\le j,k\le n}\partial_{x_j}(\kappa\,\partial_{x_k}\theta^4)\end{cases} \tag{1.1}$$

with $\kappa > 0$. The unknowns $\rho > 0$, $u \in \mathbb{R}^n$ and $\theta > 0$, which depend on the time and space variables $t \ge 0$, $x \in \mathbb{R}^n$, are the density, velocity and temperature of a radiating gas, respectively. In the so-called equilibrium diffusion model, the specific total energy E^*, the internal energy e^* and the pressure p^* are defined by the relations

$$p^* = p + \frac{1}{3}\mathcal{P}\theta^4,\quad E^* = e^* + \frac{|u|^2}{2},\quad e^* = e + \frac{\mathcal{P}\theta^4}{\rho}, \tag{1.2}$$

completed with the perfect gas law as an equation of state for the material pressure

$$p = \frac{\rho\theta}{\gamma},\quad e = \frac{\theta}{\gamma(\gamma-1)}, \tag{1.3}$$

Manuscript received May 26, 2010. Revised April 6, 2011.
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*Project supported by the Fundamental Research Funds for the Central Universities (No. 2009B27514) and the National Natural Science Foundation of China (No. 10871059).

where $\gamma > 1$ is the adiabatic constant. The dimensionless constant $\mathcal{P} > 0$ is the ratio of the radiative energy to the fluid energy. As $\mathcal{P} = 0$, we recover the standard equations of gas dynamics with a nonlinear thermal diffusion. Hence, $\mathcal{P}$ modifies the pressure law and the definition of the total energy, and characterizes the influence of radiation on the fluid dynamics. The system (1.1)–(1.3) can be obtained by considering the small optical depth regime from a more complete system coupling gas dynamics to a kinetic equation describing the evolution of the radiative intensity. Here we neglect viscous and heat conduction effects as well as any relativistic effect. The diffusion term in the energy equation still comes from the asymptotics and the coefficient κ depending on scattering and emission/absorption properties of the material. For further details on the physical background, we refer to [19]. Many details on the derivation of the equations can be found in [1, 14, 5, 12]. We also refer to [6] for a review of mathematical topics issued from the radiative transfer theory.

Clearly, the constant state $\overline{\rho} > 0$, $u = 0$ and $\overline{\theta} > 0$ is a particular and physically relevant solution to (1.1)–(1.3). We wish to prove the global well-posedness of the problem at the price of considering the initial data close to such a constant state. More precisely, our main result states as follows.

Theorem 1.1 *Let $\overline{v} = (\overline{\rho}, 0, \overline{\theta})^{\mathrm{T}}$ be a constant state with $\overline{\rho} > 0$ and $\overline{\theta} > 0$. Let $d > \frac{n}{2} + 2$ be an integer. There exists a positive constant ε such that for any initial data $v_0 = (\rho_0, u_0, \theta_0)^{\mathrm{T}}$ satisfying $\|v_0 - \overline{v}\|_{H^d(\mathbb{R}^n)} \le \varepsilon$, there exists a unique global solution to* (1.1)–(1.3)*, such that $v = (\rho, u, \theta)^{\mathrm{T}}$ satisfies $v - \overline{v} \in C([0, +\infty); H^d(\mathbb{R}^n))$. Furthermore, the solution v satisfies the following inequality:*

$$\|v(t) - \overline{v}\|^2_{H^d(\mathbb{R}^n)} + \int_0^t \Big(\|\nabla_x v(\tau)\|^2_{H^{d-1}(\mathbb{R}^n)} + \Big\| \sum_{1 \le j \le n} \partial_{x_j} \theta(\tau) \Big\|^2_{H^d(\mathbb{R}^n)} \Big) \mathrm{d}\tau \le C \|v_0 - \overline{v}\|^2_{H^d(\mathbb{R}^n)}$$

for some constant C which may depend on ε but is uniform with respect to the time variable t.

The analysis is completed with the following result, where the asymptotic trend to the constant state is established.

Theorem 1.2 *Let the assumptions of Theorem* 1.1 *be fulfilled. Assume that the initial data v_0 satisfies $v_0 - \overline{v} \in H^d(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$. There exists positive constants ε and C, such that if $\|v_0 - \overline{v}\|_{H^d(\mathbb{R}^n)} + \|v_0 - \overline{v}\|_{L^1(\mathbb{R}^n)} \le \varepsilon$, then the global solution v obtained in Theorem* 1.1 *satisfies the following decay estimate:*

$$\|v(t) - \overline{v}\|_{H^{d-1}(\mathbb{R}^n)} \le C(\|v_0 - \overline{v}\|_{H^d(\mathbb{R}^n)} + \|v_0 - \overline{v}\|_{L^1(\mathbb{R}^n)})\ (1 + t)^{-\frac{n}{4}}.$$

For the usual gas dynamics system with viscosity and/or heat conduction, the analysis of the well-posedness close to a constant state and convergence to the equilibrium dates back to [17, 18]. Then, general structure properties which imply stability estimates were exhibited in [21]. In particular, these properties are at the basis of the analysis of the non-equilibrium model of radiative hydrodynamics performed for the one-dimensional framework in [9, 10]. For this model, the temperature of radiation does not equilibrate to the material temperature and radiation is coupled with hydrodynamics through an additional elliptic equation. Similarly, the structure properties of the non-equilibrium model can be used to deal with perturbations of smooth shock profiles (see [12, 13]). Recently, the method has been extended for the general hyperbolic system of balance laws in [7] for the one-dimensional setting and in [22, 11] for multi-dimensional problems. We refer to [2, 3] for applications to asymptotic problems. In this work, our purpose is to prove the analogue of the results in [10] for the multi-dimensional equilibrium model (1.1)–(1.3). According to the above mentioned references, the main argument of the analysis is two-fold. First, we need to identify a strictly convex entropy for the system. Second, we shall check that the Kawashima-Shizuta stability condition holds. The paper is then organized as follows. In Section 2, we show that the system (1.1)–(1.3) admits a strictly convex entropy, and establish the structure property. Section 3 is devoted to the energy estimates, which prove that the system is globally well-posed for small enough perturbations of a constant state, i.e. Theorem 1.1. Finally, in Section 4, we show the convergence of the solution to the equilibrium state with an algebraic rate, i.e. Theorem 1.2 (which is compared to [9, Theorem 8.1] for the one-dimension non-equilibrium model).

Remark 1.1 In the one-dimensional case ($n = 1$), we can slightly relax the regularity requirements in Theorem 1.1 by assuming $d \geq 2$ only. We refer to Remark 3.1 for the modification of the proof, which relies on the Gagliardo-Nirenberg inequality.

2 Entropy Function and the Structure of System (1.1)

It is convenient to introduce the vector of conserved quantities $w = (\rho, \rho u, \rho E^*)^{\mathrm{T}}$, which takes values in $\mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^+$. The system (1.1) can be recast as

$$\partial_t w + \sum_{j=1}^{n} \partial_{x_j}\{F_j(w)\} = \begin{pmatrix} 0 \\ 0 \\ \sum\limits_{1\leq j,k\leq n} \partial^2_{x_j x_k} g(w) \end{pmatrix} \tag{2.1}$$

with

$$F_j(w) = \begin{pmatrix} \rho u_j \\ \rho u_j u + p^* l_j \\ (\rho E^* + p^*) u_j \end{pmatrix}, \quad g(w) = \kappa \theta^4,$$

where l_j denotes the jth unit vector in $\mathbb{R}^n$.

Of course, we can work equivalently with the density $\rho > 0$, the velocity $u \in \mathbb{R}^n$ and the temperature $\theta > 0$, and regard them as independent thermodynamical variables. Denoting $v = (\rho, u, \theta)^{\mathrm{T}}$ and using (1.2)–(1.3), we can consider w as a function of v: The mapping $v \mapsto w(v)$ is a diffeomorphism from $\mathcal{D}_v = \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^+$ onto $\mathcal{D}_w = \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^+$, and the Jacobian of this mapping reads

$$D_v w(v) = \begin{pmatrix} 1 & 0 & 0 \\ u & \rho \mathbb{I}_n & 0 \\ E & \rho u^t & \frac{\rho}{\gamma(\gamma-1)} + 4\mathcal{P}\theta^3 \end{pmatrix},$$

where $\mathbb{I}_n$ stands for the $n \times n$ unit matrix. Therefore, as long as the solution is smooth, the system (1.1) can be written equivalently as

$$D_v w(v) \partial_t v + \sum_{j=1}^{n} D_v F_j(w(v)) \partial_{x_j} v = \begin{pmatrix} 0 \\ 0 \\ \sum\limits_{1\leq j,k\leq n} \partial^2_{x_j x_k} g(w) \end{pmatrix}. \tag{2.2}$$

Equation (2.2) is completed by imposing the initial data

$$v(0,x) = v_0(x) := (\rho_0, u_0, \theta_0)^{\mathrm{T}}(x),$$

and $w_0(x) = w(v_0(x))$ is the initial data for the system (2.1).

In the pure hydrodynamic case, $\mathcal{P} = 0$, the Euler equations admit an entropy $w \mapsto \eta(w) \in \mathbb{R}$, which is a strictly convex function of the conserved quantities w, such that there exists an entropy-flux Q verifying $\partial_t \eta + \sum_{j=1}^{n} \partial_{x_j} Q_j = 0$. Since here the definitions of the pressure p^*, the total energy E^* and the internal energy e^* account for the influence of radiation, the definition of the entropy functional is modified too. We set $\eta = -\rho s^*$ and impose the following modified equation of state:

$$\theta \mathrm{d}s^* = \mathrm{d}e^* + p^* \mathrm{d}\frac{1}{\rho}.$$

We define the entropy variable

$$U = D_w \eta(w)^{\mathrm{T}} = \Big(-s^* - \frac{|u|^2}{2\theta} + \frac{4\mathcal{P}\theta^3}{3\rho} + \frac{1}{\gamma-1},\ \frac{u}{\theta}, -\frac{1}{\theta}\Big)^{\mathrm{T}}.$$

We can check that the Hessian matrix $D^2_{ww}\eta(w)$ is indeed positive definite.

The remarkable fact is that the entropy provides a natural symmetrization for the system (2.2). Indeed, let us now consider U as a function of the state variable v: the mapping $v \mapsto U(v)$ is a diffeomorphism, the Jacobian matrix of which reads

$$D_v U(v) = \begin{pmatrix} \frac{1}{\gamma\rho} & -\frac{u^t}{\theta} & -\frac{1}{\gamma(\gamma-1)\theta} + \frac{|u|^2}{2\theta^2} \\ 0 & \frac{1}{\theta}\mathbb{I}_n & -\frac{u}{\theta^2} \\ 0 & 0 & \frac{1}{\theta^2} \end{pmatrix}.$$

Multiply (2.2) from the left by $(D_v U(v))^{\mathrm{T}}$. We obtain

$$A^0(v)\partial_t v + \sum_{j=1}^{n} A_j(v)\partial_{x_j} v - B(\theta) \sum_{1\le j,k\le n} \partial^2_{x_j x_k} v = b \tag{2.3}$$

with

$$A^0(v) = (D_v U(v))^{\mathrm{T}} D_v w(v) = \begin{pmatrix} \frac{1}{\gamma\rho} & 0 & 0 \\ 0 & \frac{\rho}{\theta}\mathbb{I}_n & 0 \\ 0 & 0 & \frac{\rho}{\gamma(\gamma-1)\theta^2} + 4\mathcal{P}\theta \end{pmatrix},$$

$$A_j(v) = (D_v U(v))^{\mathrm{T}} D_v F_j(w(v)) = \begin{pmatrix} \frac{u_j}{\gamma\rho} & \frac{1}{\gamma} l_j^t & 0 \\ \frac{1}{\gamma} l_j & \rho u_j \mathbb{I}_n & (\frac{\rho}{\gamma\theta} + \frac{4}{3}\mathcal{P}\theta^4) l_j \\ 0 & (\frac{\rho}{\gamma\theta} + \frac{4}{3}\mathcal{P}\theta^4) l_j^t & \frac{\rho u_j}{\theta^2\gamma(\gamma-1)} + 2\mathcal{P}\theta u_j \end{pmatrix},$$

$$B(\theta) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 4\kappa\theta \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 0 \\ 12\kappa\Big(\sum_{1\le j\le n} \partial_{x_j}\theta\Big)^2 \end{pmatrix}.$$

We summarize the manipulations as follows.

Proposition 2.1 *We can rewrite the system* (2.1) *in the form* (2.3), *where, for any* $v \in \mathcal{D}_v$, *we have*

(i) $A^0(v)$ *is symmetric and positive definite;*

(ii) $A_j(v)$ *is symmetric for any* $j = 1, 2, \cdots, n$.

The entropy function $\eta(w)$ *is a strictly convex function of* w *in the sense that the Hessian* $D^2_{ww}\eta(w)$ *is positive definite.*

Referring to [4, 15, 16], Proposition 2.1 tells us that (2.3) is a symmetrizable system in the sense of Friedrichs. Furthermore, (2.3) has another important structure property which can be seen as a stability condition: It satisfies the so-called Kawashima-Shizuta condition (see [21]), that we discuss now.

We start by considering the linearized version of the system (2.3): Given a constant state $\overline{v} = (\overline{\rho}, 0, \overline{\theta}) \in \mathcal{D}_v$, we are concerned with

$$A^0(\overline{v})\partial_t v + \sum_{1\le j\le n} A_j(\overline{v})\partial_{x_j} v - B(\overline{\theta}) \sum_{1\le j,k\le n} \partial^2_{x_j x_k} v = 0. \tag{2.4}$$

For $\omega = (\omega_1, \cdots, \omega_n) \in \mathbb{S}^{n-1}$, we set

$$\mathbb{A}(\omega) = \sum_{j=1}^{n} A_j(\overline{v})\omega_j, \quad \mathbb{B}(\omega) = \sum_{1\le j,k} B(\overline{\theta})\omega_j\omega_k. \tag{2.5}$$

Proposition 2.2 *The linear system* (2.4) *satisfies the following stability condition: For any* $\mu \in \mathbb{R}$ *and* $\omega \in \mathbb{S}^{n-1}$, *we have*

$$(\mu A^0(\overline{v}) + \mathbb{A}(\omega))\psi = 0, \quad \mathbb{B}(\omega)\psi = 0$$

iff $\psi = 0$ *in* $\mathbb{R}^{n+2}$.

Proof We recall that $\overline{v} = (\overline{\rho}, 0, \overline{\theta})$. Let

$$\omega = (\omega_1, \cdots, \omega_n)^{\mathrm{T}} \in \mathbb{S}^{n-1} \quad \text{and} \quad \psi = (y_0, y_1, \cdots, y_n, y_{n+1})^{\mathrm{T}} \in \mathbb{R}^{n+2}.$$

The equality $\mathbb{B}(\omega)\psi = 0$ yields $y_{n+1} = 0$. Then we have

$$(\mu A^0(\overline{v}) + \mathbb{A}(\omega))\psi = \begin{pmatrix} \frac{1}{\gamma\overline{\rho}} & \frac{1}{\gamma}\omega^{\mathrm{T}} & 0 \\ \frac{1}{\gamma}\omega & \frac{\overline{\rho}}{\overline{\theta}}\mathbb{I}_n & \big(\frac{\overline{\rho}}{\gamma\overline{\theta}} + \frac{4}{3}\mathcal{P}\overline{\theta}^4\big)\omega \\ 0 & \big(\frac{\overline{\rho}}{\gamma\overline{\theta}} + \frac{4}{3}\mathcal{P}\overline{\theta}^4\big)\omega^{\mathrm{T}} & \frac{\overline{\rho}}{\gamma(\gamma-1)\overline{\theta}^2} + 4\mathcal{P}\overline{\theta} \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \\ 0 \end{pmatrix},$$

which yields $y_n = 0 = y_{n-1} = \cdots = y_1 = y_0$. Thus we have $\psi = 0$ in $\mathbb{R}^{n+2}$. The proof is completed.

Note that the stability condition in Proposition 2.2 can be rephrased by saying that for any $\omega \in \mathbb{S}^{n-1}$, the kernel of $\mathbb{B}(\omega)$ does not contain eigenvectors of the matrix $\{A^0(\overline{v})\}^{-1}\mathbb{A}(\omega)$. This stability condition was first introduced in [21] and further developed in [7, 22]. We propose to use the stability condition to investigate the global existence and the asymptotic decay of the smooth solutions to the symmetric hyperbolic-parabolic coupled system. It was shown in [21, Theorem 1.1] that the system (2.4) satisfies the stability condition stated in Proposition 2.2 iff it admits a compensating matrix in the following sense.

Proposition 2.3 *There exists a matrix-valued function*

$$\omega \in \mathbb{S}^{n-1} \mapsto K(\omega) \in \mathcal{M}_{(n+2)\times(n+2)},$$

such that

(i) $\omega \mapsto K(\omega)$ *is a* C^∞ *function on* $\mathbb{S}^{n-1}$*, and it satisfies* $K(-\omega) = -K(\omega)$ *for any* $\omega \in \mathbb{S}^{n-1}$;

(ii) $K(\omega)A_0(\overline{v})$ *is a skew-symmetric matrix for any* $\omega \in \mathbb{S}^{n-1}$;

(iii) *Denoting* $[X] = \frac{X+X^{\mathrm{T}}}{2}$ *the symmetric part of the matrix* X*, the matrix* $[K(\omega)\mathbb{A}(\omega)] + \mathbb{B}(\omega)$ *is symmetric and positive definite for any* $\omega \in \mathbb{S}^{n-1}$.

The matrix $K(\omega)$ is called the compensating matrix. The proof of the equivalence between Proposition 2.2 and Proposition 2.3 in the one-dimensional case can be found in [9, Definitions 4.1 and 4.2, Theorem 4.1] in connection to the study of the non-equilibrium model of radiative hydrodynamics. Here we omit the proof and only mention that the existence of a compensating matrix is a crucial ingredient in the proofs of Theorems 1.1 and 1.2. More precisely, as we will see below, the compensating matrix will be used to obtain the necessary $L^2([0,t];H^{d-1}(\mathbb{R}^n))$ estimate on the space derivatives of v. We will write the problem as a perturbation of the linearized system (2.4) (see (3.10) below). Then, the compensating matrix is the structure ingredient, which will allow us to derive nonlinear estimates on Sobolev norms of the unknown (see Theorem 3.1, that eventually justifies the global existence statement). Moreover, $K(\omega)$ will also be used to derive the point-wise estimates of the solution to the homogeneous system (2.4), with which we can get the asymptotic decay of the solutions to the equilibrium diffusion model (2.3).

3 Global Well-Posedness

This section is devoted to the global existence result in Theorem 1.1. The local in time existence of smooth solutions may be proved by the standard iterative scheme and a fixed point argument, according to the general method devised in [8, 16, 15]. We skip the proof of such a local existence result, and refer to [9, 10] for details on the non-equilibrium model. To show that the solutions to (2.3) are globally defined, we need further energy estimates.

We fix an integer $d > \frac{n}{2} + 2$. Then for any positive time $T > 0$, we define the functional space

$$X(0,T) = \left\{ v = \begin{pmatrix} \rho \\ u \\ \theta \end{pmatrix} \;\middle|\; \begin{array}{l} v - \overline{v} \in C([0,T];H^d(\mathbb{R}^n)), \\ \nabla_x v \in L^2([0,T];H^{d-1}(\mathbb{R}^n)), \\ \sum\limits_j \partial_{x_j}\theta \in L^2([0,T];H^d(\mathbb{R}^n)) \end{array} \right\}.$$

The solution is expected to be in the space $X(0,T)$. We introduce the following energy functional:

$$N(t)^2 = \sup_{0\le\tau\le t} \|v(\tau) - \overline{v}\|^2_{H^d(\mathbb{R}^n)} + \int_0^t \Big(\|\nabla_x v(\tau)\|^2_{H^{d-1}(\mathbb{R}^n)} + \Big\| \sum_{1\le j\le n} \partial_{x_j}\theta(\tau) \Big\|^2_{H^d(\mathbb{R}^n)} \Big) \mathrm{d}\tau.$$

Remark that, since $d > \frac{n}{2} + 2$, the Sobolev imbedding theorem yields

$$\|v(\tau) - \overline{v}\|_{W^{2,\infty}(\mathbb{R}^n)} \leq CN(t), \quad \forall \tau \in [0, t]$$

for some numerical constant C. The key estimate can be formulated as follows.

Theorem 3.1 *Assume that the hypotheses of Theorem* 1.1 *are fulfilled. Let $T > 0$, and let $v \in X(0, T)$ be a solution to* (2.3). *Then there exists a non-decreasing function $C : \mathbb{R}^+ \to \mathbb{R}^+$, such that the following nonlinear inequality*

$$N^2(t) \leq C(N(t))(N^2(0) + N(t)^3)$$

holds for any $0 \leq t \leq T$.

Let us assume temporarily that Theorem 3.1 holds, and let us explain how the statement can be used to justify that the solution is defined for any non-negative time. To this end, we follow the arguments in [20]. Indeed, let us suppose $N(T) \leq 1$, so that the estimate in Theorem 3.1 can be recast as

$$N(t)^2 \leq C_0(N(0)^2 + N(t)^3)$$

for some positive constant C_0. If we assume furthermore that $N(T) \leq \frac{1}{2C_0}$ holds, then we obtain

$$N(T)^2 \leq 2C_0 N(0)^2 = 2C_0\|v(0) - \overline{v}\|^2_{H^d(\mathbb{R}^n)}. \tag{3.1}$$

Disposing of these preliminary remarks, we consider an initial data satisfying

$$\|v(0) - \overline{v}\|^2_{H^d(\mathbb{R}^n)} \leq \frac{1}{4}\frac{1}{8C_0^3}.$$

We suppose that the solution is not globally defined, which means that $\lim\limits_{t\to T_c} N(t) = \infty$ for some finite time T_c. In particular, there exists a T_0, such that

$$N(t) \geq N(T_0) = \frac{1}{4C_0} \quad \text{for any } T_0 \leq t < T_c.$$

But $N(T_0) < \frac{1}{2C_0}$ implies that we can find $T_0 < T_1 < T_c$ verifying $\frac{1}{4C_0} < N(T_1) \leq \frac{1}{2C_0}$. Therefore, applying (3.1) to $T = T_1$, we have

$$N(T_1)^2 \leq 2C_0\|v(0) - \overline{v}\|^2_{H^d(\mathbb{R}^n)} \leq \frac{1}{16\ C_0^2},$$

which contradicts the definition of T_1.

The proof of Theorem 3.1 consists of two steps. First, we show the $L^\infty(H^d)$ estimates. Second, the $L^2(H^{d-1})$ estimates are obtained by using the Kawashima-Shizuta stability condition.

3.1 $L^\infty(H^d)$ estimates

The first step consists of establishing the following claim.

Proposition 3.1 *Let the assumptions of Theorem* 3.1 *be fulfilled. Then there exists a non-decreasing function* $C:\mathbb{R}^+\to\mathbb{R}^+$*, such that for any* $0\le t\le T$*, we have*

$$\begin{aligned}&\|v(t)-\overline{v}\|^2_{H^d(\mathbb{R}^n)}+\int_0^t\Big\|\sum_{1\le j\le n}\partial_{x_j}\theta(\tau)\Big\|^2_{H^d(\mathbb{R}^n)}\mathrm{d}\tau\\ &\le C(N(t))\Big(N(0)^2+N(t)\int_0^t\|\nabla_x v(\tau)\|^2_{H^{d-1}(\mathbb{R}^n)}\mathrm{d}\tau\Big).\end{aligned}\tag{3.2}$$

We start by proving the L^2 estimate by using the entropy function η. Then we derive the higher order estimates.

Lemma 3.1 *Let the assumptions of Theorem* 3.1 *be fulfilled. We have*

$$\|v(t)-\overline{v}\|^2_{L^2(\mathbb{R}^n)}+\int_0^t\Big\|\sum_{1\le j\le n}\partial_{x_j}\theta(\tau)\Big\|^2_{L^2(\mathbb{R}^n)}\mathrm{d}\tau\le C(N(t))N(0)^2.\tag{3.3}$$

Proof It is convenient to define from the entropy η the following relative entropy:

$$\widetilde{\eta}(w)=\eta(w)-\eta(\overline{w})-D_w\eta(\overline{w})(w-\overline{w}).$$

It is a non-negative quantity, which vanishes iff $w=\overline{w}$. More precisely, we have

$$\widetilde{\eta}(w)\ge 0,\quad \widetilde{\eta}(\overline{w})=0,\quad D_w\widetilde{\eta}(\overline{w})=0.$$

Since $\widetilde{\eta}$ is a strictly convex function of $w=(\rho,\rho u,\rho E^*)^{\mathrm{T}}$, we have that $\widetilde{\eta}(w)$ is equivalent to the quadratic function $|w-\overline{w}|^2$, and then to $|v-\overline{v}|^2$, provided that $|v-\overline{v}|$ remains in a bounded set. Hence we have

$$c|v-\overline{v}|^2\le\widetilde{\eta}(w)\le C|v-\overline{v}|^2\tag{3.4}$$

for some numerical constants c and C depending on $\|v-\overline{v}\|_{L^\infty([0,T]\times\mathbb{R}^n)}$ (and thus on $N(t)$). Accordingly, the entropy flux $q_j(w)=-\rho u_j s^*$ is modified as follows:

$$\widetilde{q}_j(w)=q_j(w)-q_j(\overline{w})-D_w\eta(\overline{w})(F_j(w)-F_j(\overline{w})),$$

and we have

$$\partial_t\widetilde{\eta}(w)+\sum_{j=1}^n\partial_{x_j}\{\widetilde{q}_j(w)\}=-\sum_{1\le k\le n}\partial_{x_k}\Big(\Big(\frac{1}{\theta}-\frac{1}{\overline{\theta}}\Big)\sum_{1\le j\le n}\partial_{x_j}(\kappa\theta^4)\Big)-4\kappa\theta\Big|\sum_{1\le j\le n}\partial_{x_j}\theta\Big|^2.$$

Integrating this relation over $[0,t]\times\mathbb{R}^n$ yields

$$\int_{\mathbb{R}^n}\widetilde{\eta}(w)\mathrm{d}x\Big|_0^t+\int_0^t\int_{\mathbb{R}^n}4\kappa\theta\Big|\sum_{1\le j\le n}\partial_{x_j}\theta\Big|^2\mathrm{d}x\mathrm{d}\tau=0.$$

We complete the proof of (3.3) by using (3.4).

In order to estimate the space derivatives of the unknown, we shall use the following lemma, the proof of which will be detailed later.

Lemma 3.2 *Let $v \in X(0,T)$ be the unique solution to* (2.3). *Then for any $0 \leq t \leq T$, we have*

$$\|\partial_t v(t)\|_{H^{d-1}(\mathbb{R}^n)} \leq C\Big(\|\nabla_x v(t)\|_{H^{d-1}(\mathbb{R}^n)} + \Big\|\sum_{1\leq j\leq n} \partial_{x_j}\theta(t)\Big\|_{H^d(\mathbb{R}^n)}\Big) \tag{3.5}$$

for some C possibly depending on $N(t)$.

As a matter of fact, note that the Sobolev embedding theorem can be used to deduce from (3.5) the following estimate of the L^∞ norm of $\partial_t v$:

$$\|\partial_t v(t)\|_{L^\infty(\mathbb{R}^n)} \leq C\Big(\|\nabla_x v(t)\|_{H^{d-1}(\mathbb{R}^n)} + \Big\|\sum_{1\leq j\leq n} \partial_{x_j}\theta(t)\Big\|_{H^d(\mathbb{R}^n)}\Big). \tag{3.6}$$

The space derivatives of v are then estimated as follows.

Lemma 3.3 *Let $\alpha \in \mathbb{N}^n$ satisfy $1 \leq |\alpha| \leq d$, where we still assume $d > \frac{n}{2} + 2$. Let $v \in X(0,T)$ be the unique solution to* (2.3). *Then for any $0 \leq t \leq T$, we have*

$$\begin{aligned}
&\|\partial_x^\alpha v(t)\|^2_{L^2(\mathbb{R}^n)} + \int_0^t \Big\|\partial_x^\alpha\Big(\sum_{1\leq j\leq n}\partial_{x_j}\theta\Big)(\tau)\Big\|^2_{L^2(\mathbb{R}^n)}\mathrm{d}\tau \\
&\leq C\Big(N(0)^2 + N(t)\int_0^t \Big(\|\nabla_x v(\tau)\|^2_{H^{d-1}(\mathbb{R}^n)} + \Big\|\sum_{1\leq j\leq n}\partial_{x_j}\theta(\tau)\Big\|^2_{H^d(\mathbb{R}^n)}\Big)\mathrm{d}\tau\Big).
\end{aligned} \tag{3.7}$$

Proof Apply ∂_x^α to the system (2.3), then take the scalar product with the vector $\partial_x^\alpha v$, and integrate the resulting equality over $[0,t]\times\mathbb{R}^n$. We obtain

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^n} A^0(v)\partial_x^\alpha v\cdot\partial_x^\alpha v\mathrm{d}x\Big|_0^t + \int_0^t\int_{\mathbb{R}^n} 4\kappa\theta\Big|\partial_x^\alpha\sum_{1\leq j\leq n}\partial_{x_j}\theta\Big|^2\mathrm{d}x\mathrm{d}\tau \\
&= \int_0^t\int_{\mathbb{R}^n}\Big(\frac{\mathrm{I}_1+\mathrm{I}_2}{2} - (\mathrm{I}_3+\mathrm{I}_4) + \mathrm{I}_5 - \mathrm{I}_6 + \mathrm{I}_7\Big)\mathrm{d}x\mathrm{d}\tau
\end{aligned} \tag{3.8}$$

with

$$\begin{aligned}
&\mathrm{I}_1 = \partial_t\{A^0(v)\}\partial_x^\alpha v\cdot\partial_x^\alpha v, && \mathrm{I}_2 = \sum_{j=1}^n \partial_{x_j}\{A_j(v)\}\partial_x^\alpha v\cdot\partial_x^\alpha v, \\
&\mathrm{I}_3 = [\partial_x^\alpha, A^0(v)]\partial_t v\cdot\partial_x^\alpha v, && \mathrm{I}_4 = \sum_{j=1}^n [\partial_x^\alpha, A_j(v)]\partial_{x_j} v\cdot\partial_x^\alpha v, \\
&\mathrm{I}_5 = \sum_{1\leq j,k\leq n}[\partial_x^\alpha, B(\theta)]\partial^2_{x_jx_k}v\cdot\partial_x^\alpha v, && \mathrm{I}_6 = \sum_{1\leq j,k\leq n}\partial_{x_j}\{B(\theta)\}\partial_x^\alpha\{\partial_{x_k}v\}\cdot\partial_x^\alpha v, \\
&\mathrm{I}_7 = \kappa\Big\{\partial_x^\alpha\Big[\sum_{1\leq j\leq n}\partial_{x_j}\theta\Big]\Big\}^2\partial_x^\alpha\theta.
\end{aligned}$$

We first estimate the integrals with I_1 and I_2. By using (2.3), we are led to

$$\begin{aligned}\int_0^t\int_{\mathbb{R}^n}|\mathrm{I}_1|\mathrm{d}x\mathrm{d}\tau &\le C\int_0^t\int_{\mathbb{R}^n}|\partial_x^\alpha v|^2|\partial_t v|\mathrm{d}x\mathrm{d}\tau\\ &\le C\int_0^t\int_{\mathbb{R}^n}|\partial_x^\alpha v|^2\Big(|\nabla_x v|+\Big|\sum_{1\le j,k\le n}\partial^2_{x_jx_k}\theta\Big|+\Big|\sum_{1\le j\le n}\partial_{x_j}\theta\Big|^2\Big)\mathrm{d}x\mathrm{d}\tau\\ &\le CN(t)\int_0^t\|\partial_x^\alpha v(\tau)\|^2_{L^2(\mathbb{R}^n)},\\ \int_0^t\int_{\mathbb{R}^n}|\mathrm{I}_2|\mathrm{d}x\mathrm{d}\tau &\le CN(t)\int_0^t\|\partial_x^\alpha v(\tau)\|^2_{L^2(\mathbb{R}^n)}.\end{aligned}$$

In these estimates we used the Sobolev embedding theorem. Note that the assumption $d>\frac{n}{2}+2$ is needed to dominate the term involving $\theta_{x_jx_k}$ by $N(t)$ in the integral of I_1. In the one-dimensional case $n=1$, this assumption can be slightly relaxed ($d\ge 2$ instead) by using the Gagliardo-Nirenberg inequality (see Remark 3.1 below). To estimate the integral of I_3, we use the Cauchy-Schwarz inequality as follows:

$$\int_0^t\int_{\mathbb{R}^n}|\mathrm{I}_3|\mathrm{d}x\mathrm{d}\tau\le\int_0^t\|\partial_x^\alpha v(\tau)\|_{L^2(\mathbb{R}^n)}\|[\partial_x^\alpha,A^0(v)]v_t(\tau)\|_{L^2(\mathbb{R}^n)}\mathrm{d}\tau.$$

Then, we use classical tame estimates for commutators and composition of functions (see for example [15, Proposition 2.1]). We get

$$\begin{aligned}\|[\partial_x^\alpha,A^0(v)]v_t(\tau)\|_{L^2(\mathbb{R}^n)} &= \|[\partial_x^\alpha,A^0(v)-A^0(\overline{v})]v_t(\tau)\|_{L^2(\mathbb{R}^n)}\\ &\le C(\|\partial_t v(\tau)\|_{L^\infty(\mathbb{R}^n)}\|\nabla_x A^0(v(\tau))\|_{H^{d-1}(\mathbb{R}^n)}\\ &\quad+\|\partial_t v(\tau)\|_{H^{d-1}(\mathbb{R}^n)}\|\nabla_x A^0(v(\tau))\|_{L^\infty(\mathbb{R}^n)}),\end{aligned}$$

which is combined with

$$\|\nabla_x A^0(v(\tau))\|_{H^{d-1}(\mathbb{R}^n)}\le C\|v(\tau)-\overline{v}\|_{H^d(\mathbb{R}^n)}\le CN(t)$$

and

$$\|\nabla_x A^0(v(\tau))\|_{L^\infty(\mathbb{R}^n)}\le C\|\nabla_x v\|_{H^{d-1}(\mathbb{R}^n)}\le CN(t).$$

Coming back to (3.5) and (3.6), we conclude with the following estimate for the integral of I_3:

$$\int_0^t\int_{\mathbb{R}^n}|\mathrm{I}_3|\mathrm{d}x\mathrm{d}\tau\le CN(t)\int_0^t\Big(\|\nabla_x v(\tau)\|^2_{H^{d-1}(\mathbb{R}^n)}+\Big\|\sum_{1\le j\le n}\partial_{x_j}\theta(\tau)\Big\|^2_{H^d(\mathbb{R}^n)}\Big)\mathrm{d}\tau.$$

Similar manipulations lead to the following estimate for I_4:

$$\int_0^t\int_{\mathbb{R}^n}|\mathrm{I}_4|\mathrm{d}x\mathrm{d}\tau\le CN(t)\int_0^t\|\nabla_x v(\tau)\|^2_{H^{d-1}(\mathbb{R}^n)}\mathrm{d}\tau.$$

Estimations of the integrals of I_5 and I_6 rely on the specific form of the matrix $B(\theta)$, which has only one non-zero component, depending linearly on the temperature. Thus, similar to the estimate of the integral of I_3, we first apply the Cauchy-Schwarz inequality, and then use the tame estimate for commutators. We obtain

$$\begin{aligned}
\int_0^t\int_{\mathbb{R}^n} |\mathrm{I}_5|\mathrm{d}x\mathrm{d}\tau &\le CN(t)\int_0^t \Big(\|\nabla v(\tau)\|^2_{H^{d-1}(\mathbb{R}^n)} + \Big\|\sum_{1\le j\le n}\partial_{x_j}\theta(\tau)\Big\|^2_{H^d(\mathbb{R}^n)}\Big)\mathrm{d}\tau,\\
\int_0^t\int_{\mathbb{R}^n} |\mathrm{I}_6|\mathrm{d}x\mathrm{d}\tau &\le C\int_0^t\int_{\mathbb{R}^n}\Big|\sum_{1\le j\le n}\partial_{x_j}\theta\Big|\Big|\partial_x^\alpha\sum_{1\le j\le n}\partial_{x_j}\theta\Big||\partial_x^\alpha v|\mathrm{d}x\mathrm{d}\tau\\
&\le CN(t)\int_0^t\Big(\Big\|\sum_{1\le j\le n}\partial_{x_j}\theta(\tau)\Big\|^2_{H^d(\mathbb{R}^n)} + \|\nabla v(\tau)\|^2_{H^{d-1}(\mathbb{R}^n)}\Big)\mathrm{d}\tau.
\end{aligned}$$

Finally, we have

$$\begin{aligned}
\int_0^t\int_{\mathbb{R}^n} |\mathrm{I}_7|\mathrm{d}x\mathrm{d}\tau &\le C\int_0^t \|\partial_x^\alpha\theta(\tau)\|_{L^2(\mathbb{R}^n)}\Big\|\partial_x^\alpha\Big(\sum_{1\le j\le n}\partial_{x_j}\theta(\tau)\Big)^2\Big\|_{L^2(\mathbb{R}^n)}\mathrm{d}\tau\\
&\le CN(t)\int_0^t\Big\|\sum_{1\le j\le n}\partial_{x_j}\theta(\tau)\Big\|^2_{H^d(\mathbb{R}^n)}\mathrm{d}\tau.
\end{aligned}$$

With the estimates above, we go back to (3.8). Using the fact that the matrix $A^0(v)$ is symmetric positive definite for any $v\in\mathcal{D}_v$, we arrive at (3.7).

Proof of Proposition 3.1 Let us sum up the estimates (3.7) over all the multi-indices α with $1\le|\alpha|\le d$. Combining this information with (3.3) yields (3.2) and completes the proof of Proposition 3.1, which is up to the justification of Lemma 3.2.

Proof of Lemma 3.2 Equation (2.2) can be rewritten as

$$\begin{aligned}
\partial_t v = &-\sum_{1\le j\le n}[\widetilde{A}_j(v)-\widetilde{A}_j(\overline{v})]\partial_{x_j}v - \sum_{1\le j\le n}\widetilde{A}_j(\overline{v})\partial_{x_j}v\\
&+[\widetilde{B}(v)-\widetilde{B}(\overline{v})]\begin{pmatrix}0\\0\\\sum\limits_{1\le j,k\le n}\partial^2_{x_jx_k}\theta\end{pmatrix} + \widetilde{B}(\overline{v})\begin{pmatrix}0\\0\\\sum\limits_{1\le j,k\le n}\partial^2_{x_jx_k}\theta\end{pmatrix}\\
&+[\widetilde{F}(v)-\widetilde{F}(\overline{v})]\begin{pmatrix}0\\0\\\Big|\sum\limits_{1\le j\le n}\partial_{x_j}\theta\Big|^2\end{pmatrix} + \widetilde{F}(\overline{v})\begin{pmatrix}0\\0\\\Big|\sum\limits_{1\le j\le n}\partial_{x_j}\theta\Big|^2\end{pmatrix}
\end{aligned}$$

with $\widetilde{A}(v)=\{A^0(v)\}^{-1}A_j(v)$, $\widetilde{B}(v)=\{A^0(v)\}^{-1}B(\theta)$ and $\widetilde{F}(v)=\kappa\{A^0(v)\}^{-1}$. Since $d-1>\frac{n}{2}+1$, the Sobolev space $H^{d-1}(\mathbb{R}^n)$ is an algebra. Therefore, we have for any $t\in[0,T]$,

$$\begin{aligned}
\|\partial_t v(t)\|_{H^{d-1}(\mathbb{R}^n)} \le C(\|v(t)-\overline{v}\|_{H^{d-1}(\mathbb{R}^n)}+1)\Big(&\|\nabla_x v(t)\|_{H^{d-1}(\mathbb{R}^n)}\\
&+\Big\|\sum_{1\le j,k\le n}\partial^2_{x_jx_k}\theta(t)\Big\|_{H^{d-1}(\mathbb{R}^n)} + \Big\|\sum_{1\le j\le n}\partial_{x_j}\theta(t)\Big\|^2_{H^{d-1}(\mathbb{R}^n)}\Big).
\end{aligned}$$

Remark 3.1 In the one-dimensional case ($n=1$), we can combine the Gagliardo-Nirenberg and Hölder inequalities to estimate the term involving θ_{xx} in the integral of I_1. Indeed, we have

$$\begin{aligned}\int_0^t\int_R |\partial_x^l v|^2|\theta_{xx}|\mathrm{d}x\mathrm{d}\tau &\le \int_0^t \|\partial_x^l v(\tau)\|^2_{L^2(\mathbb{R})}\|\theta_{xx}(\tau)\|_{L^\infty(\mathbb{R})}\mathrm{d}\tau \\ &\le CN(t)\int_0^t \|\partial_x^l v(\tau)\|_{L^2(\mathbb{R})}\|\theta_{xx}(\tau)\|^{\frac12}_{L^2}\|\theta_{xxx}(\tau)\|^{\frac12}_{L^2(\mathbb{R})}\mathrm{d}\tau \\ &\le CN(t)\int_0^t (\|\theta_{xxx}(\tau)\|^2_{L^2(\mathbb{R})}+\|\partial_x^l v(\tau)\|^{\frac43}_{L^2(\mathbb{R})}\|\theta_{xx}(\tau)\|^{\frac23}_{L^2(\mathbb{R})})\mathrm{d}\tau \\ &\le CN(t)\int_0^t (\|\theta_x(\tau)\|^2_{H^d(\mathbb{R})}+\|v_x(\tau)\|^2_{H^{d-1}(\mathbb{R})})\mathrm{d}\tau,\end{aligned}$$

where in the last inequality we used $d\ge 2$.

3.2 $L^2(H^{d-1})$ estimates

The next step consists in deriving an estimate of the right-hand side of (3.2). To this end, we follow the method introduced in [21] and further developed in [7, 22] (see also [2, 3] for further applications). We work on the Fourier transform of the equation and the properties listed in Proposition 2.3, and of the compensating matrix K to show the estimate on the space derivative of v.

Proposition 3.2 *Let the assumptions of Theorem* 3.1 *be fulfilled. Then there exists a non-decreasing function $C:\mathbb{R}^+\to\mathbb{R}^+$, such that for any $0\le t\le T$, we have*

$$\begin{aligned}\int_0^t \|\nabla_x v(\tau)\|^2_{H^{d-1}(\mathbb{R}^n)}\mathrm{d}\tau \le C(N(t))\Big(&\|v(t)-\overline{v}\|^2_{H^d(\mathbb{R}^n)}+\|v_0-\overline{v}\|^2_{H^d(\mathbb{R}^n)} \\ &+\int_0^t\Big\|\sum_{1\le j\le n}\partial_{x_j}\theta(\tau)\Big\|^2_{H^d(\mathbb{R}^n)}\mathrm{d}\tau\Big).\end{aligned}\tag{3.9}$$

Proof We obtain the linearized system by writing (2.3) as follows:

$$A^0(\overline{v})\partial_t(v-\overline{v})+\sum_{1\le j\le n}A_j(\overline{v})\partial_{x_j}(v-\overline{v})=H,\tag{3.10}$$

where we set

$$\begin{aligned}H=&-A^0(\overline{v})\sum_{1\le j\le n}[\{A^0(v)\}^{-1}A_j(v)-\{A^0(\overline{v})\}^{-1}A_j(\overline{v})]\partial_{x_j}v \\ &+4\kappa A^0(\overline{v})[\{A^0(v)\}^{-1}\theta-\{A^0(\overline{v})\}^{-1}\overline{\theta}]\begin{pmatrix}0\\0\\ \Big|\sum_{1\le j,k\le n}\partial^2_{x_jx_k}\theta\Big|\end{pmatrix} \\ &+4\kappa\overline{\theta}\begin{pmatrix}0\\0\\ \Big|\sum_{1\le j,k\le n}\partial^2_{x_jx_k}\theta\Big|\end{pmatrix} \\ &+12\kappa A^0(\overline{v})[\{A^0(v)\}^{-1}-\{A^0(\overline{v})\}^{-1}]\begin{pmatrix}0\\0\\ \Big|\sum_{1\le j\le n}\partial_{x_j}\theta\Big|^2\end{pmatrix}+12\kappa\begin{pmatrix}0\\0\\ \Big|\sum_{1\le j\le n}\partial_{x_j}\theta\Big|^2\end{pmatrix}.\end{aligned}$$

Since $d > \frac{n}{2} + 2$, the Sobolev space $H^{d-1}(\mathbb{R}^n)$ is an algebra. Hence, we have for any $\tau \in [0, t]$,

$$\|H(\tau)\|_{H^{d-1}(\mathbb{R}^n)} \le C\Big(N(t)\|\nabla_x v(\tau)\|_{H^{d-1}(\mathbb{R}^n)} + \Big\|\sum_{1\le j\le n} \partial_{x_j}\theta(\tau)\Big\|_{H^d(\mathbb{R}^n)}\Big). \tag{3.11}$$

We apply the Fourier transform to (3.10), and then multiply it from the left by $-i\widehat{v-\overline{v}}^* K(\frac{\xi}{|\xi|})$, where z^* denotes the transposed conjugate of z and K is the compensating matrix defined in Proposition 2.3. The real part of the resulting equation reads

$$\begin{aligned}
&\mathrm{Im}\Big(\widehat{v-\overline{v}}^* K\Big(\frac{\xi}{|\xi|}\Big)A^0(\overline{v})(\widehat{v-\overline{v}})_t\Big) + |\xi|\widehat{v-\overline{v}}^* K\Big(\frac{\xi}{|\xi|}\Big)\mathbb{A}\Big(\frac{\xi}{|\xi|}\Big)\widehat{v-\overline{v}} \\
&= \mathrm{Im}\Big(\widehat{v-\overline{v}}^* K\Big(\frac{\xi}{|\xi|}\Big)\widehat{H}\Big),
\end{aligned} \tag{3.12}$$

where the matrix $\mathbb{A}(\frac{\xi}{|\xi|})$ is defined by (2.5). However, by property (ii) in Proposition 2.3, KA^0 is skew-symmetric, which allows us to write

$$\mathrm{Im}\Big(\widehat{v-\overline{v}}^* K\Big(\frac{\xi}{|\xi|}\Big)A^0(\overline{v})(\widehat{v-\overline{v}})_t\Big) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Im}\Big(\widehat{v-\overline{v}}^* K\Big(\frac{\xi}{|\xi|}\Big)A^0(\overline{v})\widehat{v-\overline{v}}\Big), \tag{3.13}$$

while we also have (with $\mathbb{B}(\frac{\xi}{|\xi|})$ defined by (2.5))

$$\begin{aligned}
|\xi|\widehat{v-\overline{v}}^* K\Big(\frac{\xi}{|\xi|}\Big)\mathbb{A}\Big(\frac{\xi}{|\xi|}\Big)\widehat{v-\overline{v}} &= |\xi|\widehat{v-\overline{v}}^*\Big(\Big[K\Big(\frac{\xi}{|\xi|}\Big)\mathbb{A}\Big(\frac{\xi}{|\xi|}\Big)\Big] + \mathbb{B}\Big(\frac{\xi}{|\xi|}\Big)\Big)\widehat{v-\overline{v}} \\
&\quad - \frac{1}{|\xi|}\widehat{v-\overline{v}}^* B(\overline{\theta})\sum_{j,k}\xi_j\xi_k\widehat{v-\overline{v}} \\
&\ge \alpha_1|\xi||\widehat{v-\overline{v}}|^2 - \alpha_2\frac{1}{|\xi|}\Big|\sum_{1\le j\le n}\xi_j\widehat{\theta-\overline{\theta}}\Big|^2
\end{aligned} \tag{3.14}$$

for some positive constants $\alpha_1, \alpha_2 > 0$. To obtain (3.14), we used the specific form of the matrix $B(\overline{\theta})$ and property (iii) in Proposition 2.3. Eventually, the Cauchy-Schwarz and Young inequalities yield

$$\Big|\mathrm{Im}\Big(\widehat{v-\overline{v}}^* K\Big(\frac{\xi}{|\xi|}\Big)\widehat{H}\Big)\Big| \le \epsilon|\xi||\widehat{v-\overline{v}}|^2 + C_\epsilon\frac{1}{|\xi|}|\widehat{H}|^2 \tag{3.15}$$

for any $\epsilon > 0$. Choose ϵ to be sufficiently small, insert (3.13)–(3.15) in (3.12), and then multiply the resulting equality by $|\xi|^{2l-1}$ $(1 \le l \le d)$. Proceeding that way, we are led to

$$\begin{aligned}
|\xi|^{2l}|\widehat{v-\overline{v}}|^2 &\le C\Big(|\xi|^{2l-2}\Big|\sum_{1\le j\le n}\xi_j\widehat{\theta-\overline{\theta}}\Big|^2 + |\xi|^{2l-2}|\widehat{H}|^2 \\
&\quad - |\xi|^{2l-1}\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Im}\Big(\widehat{v-\overline{v}}^* K\Big(\frac{\xi}{|\xi|}\Big)A^0(\overline{v})\widehat{v-\overline{v}}\Big)\Big).
\end{aligned}$$

Integrate this inequality over $[0, t] \times \mathbb{R}^n$, and use the Plancherel Theorem to obtain

$$\begin{aligned}
&\int_0^t\int_{\mathbb{R}^n}\sum_{|\alpha|=l-1}|\partial_x^\alpha\nabla_x v(\tau,x)|^2\mathrm{d}x\mathrm{d}\tau \\
&\le C\int_0^t\int_{\mathbb{R}^n}\Big(\sum_{|\alpha|=l-1}\Big|\partial_x^\alpha\sum_{1\le j\le n}\partial_{x_j}\theta(\tau,x)\Big|^2 + \sum_{|\alpha|=l-1}|\partial_x^\alpha H(\tau,x)|^2\Big)\mathrm{d}x\mathrm{d}\tau \\
&\quad + C\ \mathrm{Im}\int_{\mathbb{R}^n}|\xi|^{2l-1}\Big(\widehat{v-\overline{v}}^* K\Big(\frac{\xi}{|\xi|}\Big)A^0(\overline{v})\widehat{v-\overline{v}}\Big)\mathrm{d}\xi\Big|_0^t.
\end{aligned}$$

The matrix $K(\frac{\xi}{|\xi|})$ is uniformly bounded for $\xi \in \mathbb{R}^n$, $|\xi| \neq 0$, so that

$$\begin{aligned}
&\mathrm{Im}\int_{\mathbb{R}^n} |\xi|^{2l-1}\Big(\widehat{v-\overline{v}}^{*} K\Big(\frac{\xi}{|\xi|}\Big) A^0(\overline{v})\widehat{v-\overline{v}}\Big)\mathrm{d}\xi\Big|_0^t \\
&\le C\Big(\int_{\mathbb{R}^n}(1+|\xi|^2)^l|\widehat{v-\overline{v}}|^2\mathrm{d}\xi + \int_{\mathbb{R}^n}(1+|\xi|^2)^l|\widehat{v_0-\overline{v}}|^2\mathrm{d}\xi\Big) \\
&\le C(\|v(t)-\overline{v}\|^2_{H^l(\mathbb{R}^n)} + \|v_0-\overline{v}\|^2_{H^l(\mathbb{R}^n)}).
\end{aligned}$$

Then summing over $1 \le l \le d$ leads to

$$\begin{aligned}
\int_0^t \|\nabla_x v(\tau)\|^2_{H^{d-1}(\mathbb{R}^n)}\mathrm{d}\tau \le C\Big(&\|v(t)-\overline{v}\|^2_{H^d(\mathbb{R}^n)} + \|v_0-\overline{v}\|^2_{H^d(\mathbb{R}^n)} \\
&+ \int_0^t \Big(\Big\|\sum_{1\le j\le n}\partial_{x_j}\theta(\tau)\Big\|^2_{H^{d-1}(\mathbb{R}^n)} + \|H\|^2_{H^{d-1}(\mathbb{R}^n)}\Big)\mathrm{d}\tau\Big).
\end{aligned}$$

We complete the proof of (3.9) by using the H^{d-1} estimate of the perturbation H, i.e. (3.11).

4 Decay Estimate

In this section, we show that the solutions obtained in the previous section tend to reach the equilibrium state $\overline{v}$ as time becomes large, and we are also able to identify an (algebraic) rate of convergence. As in [10], it is convenient to introduce a new unknown $z(t,x)$ by setting

$$w - \overline{w} = D_v w(\overline{v}) z.$$

Since $D_v w(\overline{v})$ is not singular, $|z|$ is equivalent to $|w-\overline{w}|$ or $|v-\overline{v}|$. We now write (2.1) as follows:

$$A^0(\overline{v})\partial_t z + \sum_{1\le j\le n} A_j(\overline{v})\partial_{x_j} z - B(\overline{\theta})\sum_{1\le j,k\le n}\partial^2_{x_j x_k} z = \sum_{1\le j\le n}(\partial_{x_j} H_j + \partial_{x_j} G), \tag{4.1}$$

where in the left-hand side we make the linearized operators appear, with the matrices $A^0(\overline{v})$, $A_j(\overline{v})$ $(1 \le j \le n)$ and $B(\overline{\theta})$ defined in Section 2 and evaluated at the constant state $\overline{v}$. In the right-hand side, the functions H_j and G are defined by

$$\begin{aligned}
H_j &= -D_v U(\overline{v})^{\mathrm{T}}(F_j(w) - F_j(\overline{w}) - D_w F_j(\overline{w})(w-\overline{w})), \\
G &= D_v U(\overline{v})^{\mathrm{T}}\Big(0, 0, \sum_{1\le k\le n}(D_w g(w) - D_w g(\overline{w}))w_{x_k}\Big)^{\mathrm{T}}
\end{aligned}$$

with $g(w) = \kappa\theta^4$.

Let us set

$$\Phi(\xi) = -\{A^0(\overline{v})\}^{-1}\Big[\sum_{1\le j\le n}\mathrm{i}\xi_j A_j(\overline{v}) + \sum_{1\le j,k\le n} B(\overline{\theta})\xi_j\xi_k\Big].$$

We define the following family of operators, parametrized by the time variable,

$$h \longmapsto \mathrm{e}^{t\Phi}h = \mathcal{F}^{-1}\{\mathrm{e}^{t\Phi(\xi)}\widehat{h}(\xi)\} = \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}\mathrm{e}^{t\Phi(\xi)}\widehat{h}(\xi)\mathrm{e}^{\mathrm{i}x\cdot\xi}\mathrm{d}\xi.$$

Actually, by using the Fourier transform, we realize that $\mathrm{e}^{t\Phi}h$ is nothing but the solution to the following homogeneous linear equation:

$$\begin{cases} A^0(\overline{z})\partial_t z + \sum\limits_{1\le j\le n} A_j(\overline{v})\partial_{x_j} z - B(\overline{\theta}) \sum\limits_{1\le j,\, k\le n} \partial^2_{x_j x_k} z = 0, \\ z(0,x) = h(x). \end{cases} \tag{4.2}$$

Then, we obtain the solution to (4.1) as the fixed point of the Duhamel formula

$$z(t,x) = \mathrm{e}^{t\Phi} z_0 + \int_0^t \mathrm{e}^{(t-\tau)\Phi}\{A^0(\overline{v})\}^{-1} \sum_{1\le j\le n} (\partial_{x_j} H_j + \partial_{x_j} G)\mathrm{d}\tau, \tag{4.3}$$

where $z_0(x)$ is the initial data $z_0 = \{D_v w(\overline{v})\}^{-1}(w_0(x) - \overline{w})$. The analysis of the asymptotic trend to equilibrium then relies on the following properties of the operator $\mathrm{e}^{t\Phi}$.

Lemma 4.1 *The following assertions hold:*
(i) *For any $h \in C_c^\infty(\mathbb{R}^n)$, $t \ge 0$ and any multi-index $\alpha \in \mathbb{N}^n$, we have*

$$\partial_x^\alpha(\mathrm{e}^{t\Phi}h) = \mathrm{e}^{t\Phi}(\partial_x^\alpha h), \tag{4.4}$$

(ii) *There exist two positive constants $C, c > 0$, such that for any $h \in C_c^\infty(\mathbb{R}^n)$, $t \ge 0$ and any multi-index $\alpha \in \mathbb{N}^n$, we have*

$$\|\partial_x^\alpha(\mathrm{e}^{t\Phi}h)\|_{L^2(\mathbb{R}^n)} \le C(\mathrm{e}^{-ct}\|\partial_x^\alpha h\|_{L^2(\mathbb{R}^n)} + (1+t)^{-\frac{n}{4}-\frac{|\alpha|}{2}}\|h\|_{L^1(\mathbb{R}^n)}). \tag{4.5}$$

We shall also need the following quite elementary estimates.

Lemma 4.2 *Let $n \in \mathbb{N}$, $n \neq 0$ and $c > 0$. There exists a positive constant L which depends on n and c, such that for any $t > 0$, we have*

$$\int_0^t \mathrm{e}^{-c(t-\tau)}(1+\tau)^{-\frac{n}{4}}\mathrm{d}\tau \le L(1+t)^{-\frac{n}{4}}, \tag{4.6}$$

$$\int_0^t (1+t-\tau)^{-\frac{n}{4}-\frac{1}{2}}(1+\tau)^{-\frac{n}{2}}\mathrm{d}\tau \le L(1+t)^{-\frac{n}{4}}. \tag{4.7}$$

We postpone the proof of Lemmas 4.1 and 4.2 to the Appendix. For the time being, let us show how they can be used to derive the decay estimate in Theorem 1.2.

Proof of Theorem 1.2 Let us set

$$M(t) = \sup_{\tau\in[0,t]} (1+\tau)^{\frac{n}{4}}\|z(\tau)\|_{H^{d-1}(\mathbb{R}^n)}.$$

We wish to show that $M(t)$ remains bounded for all $t \in \mathbb{R}^+$, provided that the quantity

$$E_s = \|v_0 - \overline{v}\|_{H^d(\mathbb{R}^n)} + \|v_0 - \overline{v}\|_{L^1(\mathbb{R}^n)}$$

is sufficiently small. Theorem 1.1 and the definition of the variable z have already told us that

$$\|z(t)\|_{H^d(\mathbb{R}^n)} \le c\|w(t) - \overline{w}\|_{H^d(\mathbb{R}^n)} \le CE_s$$

holds. Next, let $\alpha \in \mathbb{N}^n$ with $|\alpha| = l \leq d-1$. We apply ∂_x^α to (4.3), and then evaluate the L^2-norm:

$$\|\partial_x^\alpha z(t)\|_{L^2(\mathbb{R}^n)} \leq \|\partial_x^\alpha(\mathrm{e}^{t\Phi}z_0)\|_{L^2(\mathbb{R}^n)} + \int_0^t \Big\|\partial_x^\alpha \mathrm{e}^{(t-\tau)\Phi}\{A^0(\overline{v})\}^{-1}\sum_{1\leq j\leq n}\partial_{x_j}(H_j+G)\Big\|_{L^2(\mathbb{R}^n)}\mathrm{d}\tau, \tag{4.8}$$

where we used (4.4). Then we use (4.5) to estimate the terms in the right-hand side of (4.8). The first term is dominated by

$$C(\mathrm{e}^{-ct}\|\partial_x^\alpha z_0\|_{L^2(\mathbb{R}^n)} + (1+t)^{-\frac{n}{4}-\frac{|\alpha|}{2}}\|z_0\|_{L^1(\mathbb{R}^n)}) \leq C(1+t)^{-\frac{n}{4}}E_s,$$

while the second term can be estimated by

$$C\Big(\int_0^t \mathrm{e}^{-c(t-\tau)}\Big\|\partial_x^\alpha\sum_{1\leq j\leq n}\partial_{x_j}(H_j+G)\Big\|_{L^2(\mathbb{R}^n)}\mathrm{d}\tau + \int_0^t(1+t-\tau)^{-\frac{n}{4}-\frac{1}{2}}\sum_{1\leq j\leq n}\|H_j+G\|_{L^1(\mathbb{R}^n)}\mathrm{d}\tau\Big),$$

where (4.4) is used again. Insert this information in (4.8) and sum over $|\alpha| = l$ and $0 \leq l \leq d-1$. We obtain

$$\|z(t)\|_{H^{d-1}(\mathbb{R}^n)} \leq C\Big((1+t)^{-\frac{n}{4}}E_s + \int_0^t \mathrm{e}^{-c(t-\tau)}\Big\|\sum_{1\leq j\leq n}\partial_{x_j}(H_j+G)\Big\|_{H^{d-1}(\mathbb{R}^n)}\mathrm{d}\tau + \int_0^t(1+t-\tau)^{-\frac{n}{4}-\frac{1}{2}}\Big\|\sum_{1\leq j\leq n}H_j+G\Big\|_{L^1(\mathbb{R}^n)}\mathrm{d}\tau\Big).$$

Coming back to the definitions of H_j and G, we have

$$\Big\|\sum_{1\leq j\leq n}\partial_{x_j}H_j\Big\|_{H^{d-1}(\mathbb{R}^n)} \leq C\|w-\overline{w}\|_{H^{d-1}(\mathbb{R}^n)}\|\nabla_x w\|_{H^{d-1}(\mathbb{R}^n)} \leq CE_s\|z(\tau)\|_{H^{d-1}(\mathbb{R}^n)},$$

$$\begin{aligned}\Big\|\sum_{1\leq j\leq n}\partial_{x_j}G\Big\|_{H^{d-1}(\mathbb{R}^n)} &\leq C\|\theta-\overline{\theta}\|_{H^{d-1}(\mathbb{R}^n)}\times\Big(\Big\|\sum_{1\leq j\leq n}\partial_{x_j}\theta\Big\|^2_{H^{d-1}(\mathbb{R}^n)} + \Big\|\sum_{1\leq j\leq n}\partial_{x_j}\theta\Big\|_{H^d(\mathbb{R}^n)}\Big)\\ &\leq CE_s\|z(\tau)\|_{H^{d-1}(\mathbb{R}^n)},\\ \|H_j\|_{L^1(\mathbb{R}^n)} &\leq C\|w-\overline{w}\|^2_{L^2(\mathbb{R}^n)} \leq C\|z(\tau)\|^2_{H^{d-1}(\mathbb{R}^n)},\\ \|G\|_{L^1(\mathbb{R}^n)} &\leq C\|\theta-\overline{\theta}\|_{L^2(\mathbb{R}^n)}\|\nabla_x\theta\|_{L^2(\mathbb{R}^n)} \leq C\|z(\tau)\|^2_{H^{d-1}(\mathbb{R}^n)}.\end{aligned}$$

Thus we arrive at

$$\|z(t)\|_{H^{d-1}(\mathbb{R}^n)} \leq C(1+t)^{-\frac{n}{4}}E_s + CE_s\int_0^t \mathrm{e}^{-c(t-\tau)}\|z(\tau)\|_{H^{d-1}(\mathbb{R}^n)}\mathrm{d}\tau + C\int_0^t(1+t-\tau)^{-\frac{n}{4}-\frac{1}{2}}\|z(\tau)\|^2_{L^2(\mathbb{R}^n)}\mathrm{d}\tau.$$

However, for any $0 \leq \tau \leq t$, we have $\|z(\tau)\|_{H^{d-1}(\mathbb{R}^n)} \leq M(t)(1+\tau)^{-\frac{n}{4}}$. Hence, we make use of Lemma 4.2 to establish that

$$\|z(t)\|_{H^{d-1}(\mathbb{R}^n)} \leq C(1+t)^{-\frac{n}{4}}(E_s + E_sM(t) + M(t)^2)$$

holds. It implies

$$M(t) \leq CE_s + CE_sM(t) + CM(t)^2.$$

If E_s is sufficiently small, the discriminant of the polynomial $P(M) = CM^2+(CE_s-1)M+CE_s$ is positive, and P admits two positive roots. Therefore, $M(t)$ remains bounded by the smallest root. This completes the proof of Theorem 1.2.

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Appendix Proofs of Lemmas 4.1 and 4.2

We start with a point-wise estimate for the Fourier transform of the solution to the homogeneous system (4.2).

Lemma A.1 *Let the hypothesis stated in Lemma* 4.1 *be satisfied. There exist* $C, c > 0$ *such that*

$$|\mathrm{e}^{t\Phi(\xi)}| \le C\mathrm{e}^{-c\lambda(\xi)t} \tag{A.1}$$

for $\xi \in \mathbb{R}^n$, $t > 0$, *with* $\lambda(\xi) = \frac{|\xi|^2}{1+|\xi|^2}$.

Proof The Fourier transform of the solution z to (4.2) is

$$\widehat{z}(t,\xi) = \mathrm{e}^{t\Phi(\xi)}\widehat{h}(\xi).$$

Thus (A.1) means

$$|\widehat{z}(t,\xi)| \le C\mathrm{e}^{-c\lambda(\xi)t}|\widehat{h}(\xi)|.$$

Therefore, we apply the energy method in the Fourier space to the homogeneous system (4.2). Let us apply the Fourier transform to (4.2). We have

$$A^0(\overline{v})\widehat{z}_t + \mathrm{i}\sum_{1\le j\le n} A_j(\overline{v})\xi_j\widehat{z} + B(\overline{\theta})\sum_{1\le j,k\le n}\xi_j\xi_k\widehat{z} = 0. \tag{A.2}$$

Multiply (A.2) from the left by $\widehat{z}^*$, the transposed conjugate of $\widehat{z}$, and then take the real part of each term. We are led to

$$\frac{1}{2}(\widehat{z}^* A^0(\overline{v})\widehat{z})_t + 4\kappa\overline{\theta}\Big|\sum_{1\le j\le n}\xi_j\widehat{z}_{n+2}\Big|^2 = 0, \tag{A.3}$$

where z_{n+2} denotes the last component of the vector variable z. For the derivation of the last equality, we used the fact that the matrices A^0 and A_j $(j = 1, \cdots, n)$ are real symmetric. The next step uses the compensating matrix $K(\frac{\xi}{|\xi|})$ $(|\xi| \ne 0)$. We multiply (A.2) from the left by $-\mathrm{i}\widehat{z}^*K(\frac{\xi}{|\xi|})$. Since $K(\frac{\xi}{|\xi|})A^0(\overline{v})$ is skew-symmetric, the matrix $\mathrm{i}K(\frac{\xi}{|\xi|})A^0(\overline{v})$ is hermitian. Thus, by taking the real part of each term, we have

$$-\frac{1}{2}\Big(\widehat{z}^*\mathrm{i}K\Big(\frac{\xi}{|\xi|}\Big)A^0(\overline{v})\widehat{z}\Big)_t + |\xi|\widehat{z}^*K\Big(\frac{\xi}{|\xi|}\Big)\mathbb{A}\Big(\frac{\xi}{|\xi|}\Big)\widehat{z} + \mathrm{Im}\sum_{j,k}\widehat{z}^*K\Big(\frac{\xi}{|\xi|}\Big)B(\overline{\theta})\xi_j\xi_k\widehat{z} = 0. \tag{A.4}$$

The properties of the compensating matrix allow us to derive the following inequality:

$$\begin{aligned}|\xi|\widehat{z}^*K\Big(\frac{\xi}{|\xi|}\Big)\mathbb{A}\Big(\frac{\xi}{|\xi|}\Big)\widehat{z} &= |\xi|\widehat{z}^*\Big(\Big[K\Big(\frac{\xi}{|\xi|}\Big)\mathbb{A}\Big(\frac{\xi}{|\xi|}\Big)\Big] + \mathbb{B}\Big(\frac{\xi}{|\xi|}\Big)\Big)\widehat{z} - |\xi|\widehat{z}^*\mathbb{B}\Big(\frac{\xi}{|\xi|}\Big)\widehat{z}\\ &\ge 2C_2|\xi||\widehat{z}|^2 - C_3\frac{1}{|\xi|}\Big|\sum_{1\le j\le n}\xi_j\widehat{z}_{n+2}\Big|^2.\end{aligned}$$

Then using the Cauchy-Schwarz inequality yields

$$\mathrm{Im}\sum_{j,k}\widehat{z}^*K\Big(\frac{\xi}{|\xi|}\Big)B(\overline{\theta})\xi_j\xi_k\widehat{z} \ge -\epsilon|\xi||\widehat{z}|^2 - C_\epsilon|\xi|\Big|\sum_{1\le j\le n}\xi_j\widehat{z}_{n+2}\Big|^2,$$

where the small positive constant ϵ can be chosen so that $\epsilon \le C_2$. Using the above two inequalities in (A.4), we obtain

$$-\frac{1}{2}\Big(\widehat{z}^*\mathrm{i}K\Big(\frac{\xi}{|\xi|}\Big)A^0(\overline{v})\widehat{z}\Big)_t + C_2|\xi||\widehat{z}|^2 - C_4\Big(\frac{1}{|\xi|}+|\xi|\Big)\Big|\sum_{1\le j\le n}\xi_j\widehat{z}_{n+2}\Big|^2 \le 0. \tag{A.5}$$

Finally, we choose a small constant $\varsigma > 0$, which is determined by

$$\begin{aligned} &\varsigma C_4 \le 4\kappa\overline{\theta}, \\ &c_0|\varphi|^2 \le E_\varsigma[\varphi] \le C_0|\varphi|^2, \end{aligned} \tag{A.6}$$

where $E_\varsigma[\varphi]$ is defined by

$$E_\varsigma[\varphi] = \varphi^* A^0(\overline{v})\varphi - \varsigma\frac{|\xi|}{1+|\xi|^2}\varphi^*\mathrm{i}K\Big(\frac{\xi}{|\xi|}\Big)A^0(\overline{v})\varphi.$$

Recall that $A^0(\overline{v})$ is symmetric and positive definite, and thus the condition (A.6) can be fulfilled.

With these preparations, let us multiply (A.5) by $\frac{\varsigma|\xi|}{1+|\xi|^2}$, and add the resulting inequality to (A.3). We get

$$\frac{1}{2}\{E_\varsigma[\widehat{z}]\}_t + c\lambda(\xi)E_\varsigma[\widehat{z}] \le 0,$$

where $c = \frac{\varsigma C_2}{C_0}$ and $\lambda(\xi) = \frac{|\xi|^2}{1+|\xi|^2}$. Solving this differential inequality yields

$$E_\varsigma[\widehat{z}] \le E_\varsigma[\widehat{h}]\mathrm{e}^{-2c\lambda(\xi)t}.$$

We conclude by using (A.6).

We are now ready to complete the proof of Lemma 4.1.

Proof of Lemma 4.1 Inequality (4.4) follows directly by applying ∂_x^α to (4.2). Next we show (4.5). The Plancherel Theorem yields

$$\|\partial_x^\alpha(\mathrm{e}^{t\Phi}h)\|_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n}|\mathrm{i}^\alpha\xi^\alpha\mathrm{e}^{t\Phi(\xi)}\widehat{h}(\xi)|^2\mathrm{d}\xi \le C\int_{\mathbb{R}^n}|\mathrm{i}^\alpha\xi^\alpha|^2\mathrm{e}^{-2c\lambda(\xi)t}|\widehat{h}(\xi)|^2\mathrm{d}\xi,$$

where we used (A.1). Next we split the last integral into two parts I_1 and I_2 according to $|\xi| \le 1$ and $|\xi| \ge 1$. When $|\xi| \le 1$, we have $\lambda(\xi) \ge \frac{|\xi|^2}{2}$. Hence we obtain

$$\begin{aligned} \mathrm{I}_1 &\le C\int_{|\xi|\le 1}|\xi|^{2|\alpha|}|\widehat{h}(\xi)|^2\mathrm{e}^{-c|\xi|^2t}\mathrm{d}\xi \\ &\le C\|\widehat{h}\|^2_{L^\infty(\mathbb{R}^n)}\int_{|\xi|\le 1}|\xi|^{2|\alpha|}\mathrm{e}^{-c|\xi|^2t}\mathrm{d}\xi \\ &\le C\|h\|^2_{L^1(\mathbb{R}^n)}(1+t)^{-|\alpha|-\frac{n}{2}}. \end{aligned}$$

If $|\xi| \ge 1$, we use $\lambda(\xi) \ge \frac{1}{2}$, which leads to

$$\mathrm{I}_2 \le C\mathrm{e}^{-ct}\int_{|\xi|\ge 1}|\mathrm{i}^\alpha\xi^\alpha|^2|\widehat{h}(\xi)|^2\mathrm{d}\xi \le C\mathrm{e}^{-ct}\|\partial_x^\alpha h\|^2_{L^2(\mathbb{R}^n)}.$$

We finish the proof of Lemma 4.1 by combining the estimates of I_1 and I_2.

Proof of Lemma 4.2 Let us write

$$\int_0^t \mathrm{e}^{-c(t-\tau)}(1+\tau)^{-\frac{n}{4}}\mathrm{d}\tau = \int_0^{\frac{t}{2}} \mathrm{e}^{-c(t-\tau)}(1+\tau)^{-\frac{n}{4}}\mathrm{d}\tau + \int_{\frac{t}{2}}^t \mathrm{e}^{-c(t-\tau)}(1+\tau)^{-\frac{n}{4}}\mathrm{d}\tau$$
$$\le \mathrm{e}^{-ct}\int_0^{\frac{t}{2}} \mathrm{e}^{c\tau}\mathrm{d}\tau + \Big(1+\frac{t}{2}\Big)^{-\frac{n}{4}}\mathrm{e}^{-ct}\int_{\frac{t}{2}}^t \mathrm{e}^{c\tau}\mathrm{d}\tau$$
$$\le \frac{1}{c}\Big(\mathrm{e}^{-\frac{ct}{2}} + \Big(1+\frac{t}{2}\Big)^{-\frac{n}{4}}\Big),$$

which leads readily to (4.6).

Let us prove (4.7) in the case $n \ge 3$. We introduce the change of variable $\sigma = \sqrt{\frac{1+\tau}{2+t}}$ so that the integral to be investigated reads

$$J(t) = \int_0^t (1+t-\tau)^{-\frac{n}{4}-\frac{1}{2}}(1+\tau)^{-\frac{n}{2}}\mathrm{d}\tau = 2(2+t)^{-\frac{3n}{4}+\frac{1}{2}}\int_{\frac{1}{\sqrt{2+t}}}^{\sqrt{\frac{1+t}{2+t}}} \frac{\sigma^{1-n}}{(1-\sigma^2)^{\frac{n}{4}+\frac{1}{2}}}\mathrm{d}\sigma.$$

Without loss of generality, we can restrict ourselves to the situation where $t \ge 2$. Accordingly, we have $\frac{1}{\sqrt{2+t}} \le \frac{1}{2} \le \sqrt{\frac{1+t}{2+t}}$ and split

$$J(t) = 2(2+t)^{-\frac{3n}{4}+\frac{1}{2}}\Big(\int_{\frac{1}{\sqrt{2+t}}}^{\frac{1}{2}} \cdots \mathrm{d}\sigma + \int_{\frac{1}{2}}^{\sqrt{\frac{1+t}{2+t}}} \cdots \mathrm{d}\sigma\Big)$$
$$\le 2(2+t)^{-\frac{3n}{4}+\frac{1}{2}}\Big(\Big(\frac{4}{3}\Big)^{\frac{n}{4}+\frac{1}{2}}\int_{\frac{1}{\sqrt{2+t}}}^{\frac{1}{2}} \frac{1}{\sigma^{n-1}}\mathrm{d}\sigma + 2^{n-1}\int_{\frac{1}{2}}^{\sqrt{\frac{1+t}{2+t}}} \frac{\mathrm{d}\sigma}{(1-\sigma^2)^{\frac{n}{4}+\frac{1}{2}}}\Big)$$
$$\le 2(2+t)^{-\frac{3n}{4}+\frac{1}{2}}\Big(\frac{(\frac{4}{3})^{\frac{n}{4}+\frac{1}{2}}}{n-2}(2+t)^{\frac{n}{2}-1} + \frac{2^n}{\frac{n}{4}-\frac{1}{2}}\Big(1-\frac{1+t}{2+t}\Big)^{-\frac{n}{4}+\frac{1}{2}}\Big)$$
$$\le 2(2+t)^{-\frac{3n}{4}+\frac{1}{2}}\Big(\frac{(\frac{4}{3})^{\frac{n}{4}+\frac{1}{2}}}{n-2}(2+t)^{\frac{n}{2}-1} + \frac{2^n}{\frac{n}{4}-\frac{1}{2}}(2+t)^{\frac{n}{4}-\frac{1}{2}}\Big)$$
$$\le C(n)\ (2+t)^{-\frac{3n}{4}+\frac{1}{2}}(2+t)^{\frac{n}{2}-1}$$
$$\le C(n)(1+t)^{-\frac{n}{4}-\frac{1}{2}}.$$

The estimates in the cases of $n = 1$ or $n = 2$ can be obtained by direct computations. We skip the details.