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The Infinite Dimensional Hyperbolic Space $\mathbb{H}^\infty$ Does Not Have Property A**

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Abstract The author constructs a sequence of cubes in the infinitely dimensional hyperbolic space $\mathbb{H}^\infty$ which is equi-coarsely equivalent to $\mathbb{Z}_2^n$. As a corollary, it is proved that the infinitely dimensional hyperbolic space $\mathbb{H}^\infty$ does not have property A.

Keywords Coarse geometry, Property A, Hyperbolic space
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1 Introduction

Yu [10] introduced the concept of property A for the metric spaces. It was proved that this property has important applications in the study of coarse Baum-Connes conjecture for the discrete metric spaces with bounded geometry, Novikov conjecture for the finite generated group (see [10]) and exactness of C^*-algebras (see [3, 7]).

Yu [10] proved that property A for the metric space X implies a coarse embedding of X into Hilbert space. Recently, Nowak [6] constructed a locally finite metric space which can coarsely embedded into Hilbert space but does not have property A.

In this note, we construct a sequence of cubes $\mathbb{H}_2^n$ $(n = 1, 2, \cdots)$ in the infinite dimensional hyperbolic space $\mathbb{H}^\infty$,

$$\mathbb{H}^\infty = \big\{(z, x_1, x_2, \cdots) \colon z^2 - (x_1^2 + x_2^2 + \cdots) = 1,\ z \geq 1\big\},$$

which is equi-coarsely equivalent to the sequence of $\mathbb{Z}_2^n$. By using the result proved by Nowak [6] that $\lim\limits_{n\to\infty} \operatorname{diam}_{\mathbb{Z}_2^n}(1, \varepsilon) = +\infty$, it follows that the hyperbolic space $\mathbb{H}^\infty$ does not have property A.

2 Preliminaries

For discrete metric spaces, we use the definition of property A given by Higson and Roe [4, 9].

Definition 2.1 *A discrete metric space X is said to have property A if for any $R > 0$, $\varepsilon > 0$, there exist a map $\xi \colon X \to l_1(X)_{1,+}$ and a positive number S such that*

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(i) $\|\xi_x - \xi_y\| \le \varepsilon$, *if* $d(x,y) \le R$,

(ii) $\operatorname{supp} \xi_x \subseteq B(x,S)$, $\forall\, x \in X$.

Here $l_1(X)_{1,+} = \{\xi\colon \xi \in l_1(X), \|\xi\| = 1$ *and* $\xi(x) \ge 0$, $\forall\, x \in X\}$, *and* $B(x,S)$ *is the ball centered at* x *with radius* S *in* X.

For the metric spaces with bounded geometry, this definition is equivalent to the original one given in [10]. For general metric spaces, we have the following definitions.

Definition 2.2 (see [10]) *A metric space* X *is said to have property A if there exists a discrete subspace* Γ *of* X *such that* Γ *is* C*-dense in* X *(i.e.,* $X = \{x\colon d(x,\Gamma) \le C\}$*) and* Γ *has property A.*

Definition 2.3 (see [2]) *Let* X *be a metric space,* H *be a separable and infinite-dimensional Hilbert space. A map* $f : X \to H$ *is said to be a coarse embedding if there exist non-decreasing functions* ρ_1 *and* ρ_2 *from* $\mathbb{R}_+ = [0,+\infty)$ *to* $\mathbb{R}_+$ *such that*

(i) $\rho_1(d(x,y)) \le \|f(x) - f(y)\| \le \rho_2(d(x,y))$, $\forall\, x,y \in X$,

(ii) $\lim\limits_{r\to\infty} \rho_i(r) = +\infty$ *for* $i = 1,2$.

For the metric spaces with property A, Nowak [6] introduced the following definition.

Definition 2.4 (see [6]) *Let* X *be a discrete metric space,* $R > 0$, $\varepsilon > 0$. *We define* $\operatorname{diam}_X(R,\varepsilon)$ *to be*

$$\inf\{S\colon \operatorname{supp} \xi_x \subseteq B(x,S),\ \forall\, x \in X\},$$

where ξ *is a map* $\xi\colon X \to l_1(X)_{1,+}$ *and satisfies Definition* 2.1(i) *if it exists, otherwise we set* $\operatorname{diam}_X(R,\varepsilon) = +\infty$.

Remark 2.1 (1) X has property A if and only if $\operatorname{diam}_X(R,\varepsilon) < \infty$ for all $R > 0$, $\varepsilon > 0$.

(2) If $R_1 \le R_2$, then $\operatorname{diam}_X(R_1,\varepsilon) \le \operatorname{diam}_X(R_2,\varepsilon)$.

(3) Let X and Y be discrete metric spaces, $f\colon X \to Y$ be a coarse embedding. Suppose that Y has property A. Then for every $R > 0$, $\varepsilon > 0$, we have

$$\rho_-(\operatorname{diam}_X(R,\varepsilon)) \le 3 \operatorname{diam}_Y(\rho_+(R),\varepsilon).$$

(4) Suppose that (X_n, d_n) is a sequence of metric spaces with property A. If there exist $R > 0$, $\varepsilon > 0$ such that $\lim\limits_{n\to\infty} \operatorname{diam}_{X_n}(R,\varepsilon) = \infty$, we construct the disjoint union $X = \bigsqcup\limits_{n=1}^{\infty} X_n$ with the metric d which satisfies the following conditions:

(a) d restricts to d_n on X_n,

(b) $d(X_n, X_{n+1}) \ge n+1$,

(c) $d(X_n, X_m) = \sum\limits_{k=n}^{m-1} d(X_k, X_{k+1})$ for $n < m$.

Then X does not have property A.

The third statement in the remark can be proved by combining Proposition 3.6 and Theorem 3.11 in [5]. The fourth statement can be proved by the same method of [6, Theorem 5.1].

Let (X, d_1), (Y, d_2) be metric spaces. We will consider the Cartesian product $X \times Y$ with the metric

$$d((x_1,y_1),(x_2,y_2)) = d_1(x_1,x_2) + d_2(y_1,y_2), \quad \forall\, x_1,x_2 \in X,\ y_1,y_2 \in Y.$$

Especially for the case $X = Y$, this definition can be generalized to n-copies of X. For a finitely generated amenable group Γ, Nowak [6] proved the following theorem.

Theorem 2.1 *Let Γ be a finitely generated amenable group. Then for any $0 < \varepsilon < 2$,*

$$\lim_{n\to\infty} \mathrm{diam}_{\Gamma^n}(1, \varepsilon) = +\infty. \tag{2.1}$$

It follows from this theorem that for any non-trivial finite group Γ, the metric space $X = \coprod_{n=1}^{\infty} \Gamma^n$ does not have property A and it can coarsely embedded into Hilbert space (see [6]). For the special case $\Gamma = \mathbb{Z}_2 = \{0, 1\}$, we get the disjoint union $X = \coprod_{n=1}^{\infty} \mathbb{Z}_2^n$ with a metric d which satisfies the following conditions:

(1) $\mathrm{d}(x, y) = \sum_{i=1}^{n} \mathrm{d}(x_i, y_i),\ \forall\, x = (x_1, \cdots, x_n),\ y = (y_1, \cdots, y_n) \in \mathbb{Z}_2^n$,

(2) $d(\mathbb{Z}_2^n, \mathbb{Z}_2^{n+1}) = n + 1$,

(3) For $n \le m,\ d(\mathbb{Z}_2^n, \mathbb{Z}_2^m) = \sum_{k=n}^{m-1} \mathrm{d}(\mathbb{Z}_2^k, \mathbb{Z}_2^{k+1}) = \frac{m(m+1)}{2} - \frac{n(n+1)}{2}$.

Then X does not have property A and it can coarsely embedded into Hilbert space.

3 Cubes in $\mathbb{H}^\infty$

In this section, we will construct a sequence of cubes $\mathbb{H}_2^n$ in $\mathbb{H}^\infty$ such that the sequence of the metric spaces $\mathbb{Z}_2^n$ are equi-coarsely equivalent to the sequence of $\mathbb{H}_2^n$.

Definition 3.1 (see [1]) *Let X_n, Y_n $(n = 1, 2, \cdots)$ be two sequences of discrete metric spaces. A sequence of map $F = \{f_n\}$ where $f_n\colon X_n \to Y_n$ $(n = 1, 2, \cdots)$ is said to be equi-coarse embedding if there exist non-decreasing functions ρ_- and ρ_+ from $\mathbb{R}_+ = [0, +\infty)$ to $\mathbb{R}_+$ such that*

(i) $\rho_-(d(x, y)) \le d(f_n(x), f_n(y)) \le \rho_+(d(x, y))$ *for all* $x, y \in X_n,\ n \in \mathbb{N}$,

(ii) $\lim_{r\to\infty} \rho_\pm(r) = +\infty$.

If there exists a constant $C > 0$ such that $f_n(X_n)$ are C-dense in Y_n, i.e., $Y_n = \{y\colon y \in Y_n,\ d(y, Y_n) \le C\}$, for every n, then we say that the two sequences of metric spaces $\{X_n\}$ and $\{Y_n\}$ are equi-coarsely equivalent.

By Remark 2.1(3), we have the following lemma.

Lemma 3.1 *Let X_n, Y_n $(n = 1, 2, \cdots)$ be two equi-coarsely equivalent sequences of discrete metric spaces. Suppose that Y_n has property A for every n and if there exist $R > 0,\ \varepsilon > 0$ such that $\lim_{n\to\infty} \mathrm{diam}_{X_n}(R, \varepsilon) = \infty$, then we have*

$$\lim_{n\to\infty} \mathrm{diam}_{Y_n}(\rho_+(R), \varepsilon) = +\infty.$$

Let $\mathbb{H}^\infty$ be the infinite dimensional hyperbolic space. For convenience, we represent the coordinates of the points in $\mathbb{H}^\infty$ as the form $x = (z, y, x_1, x_2, \cdots)$, where $z^2 - (y^2 + x_1^2 + x_2^2 + \cdots) = 1$ and $z \ge 1$. Let $O = (1, 0, 0, \cdots)$ be the vertex of $\mathbb{H}^\infty$.

On the z-y plane in $\mathbb{H}^\infty$, we consider the hyperbola $z^2 - y^2 = 1$ $(z \ge 1)$. Suppose that the hyperbolic distance between the point $(z, y, 0, 0, \cdots)$ and the vertex $O = (1, 0, 0, \cdots)$ is t. Then we have the parameter representations of z and y as follows

$$z = \cosh t, \quad y = \sinh t.$$

Theorem 3.1 *There exists a sequence of cubes $\mathbb{H}_2^n$ in the infinite dimensional hyperbolic space $\mathbb{H}^\infty$ which is equi-coarsely equivalent to $\mathbb{Z}_2^n$.*

Proof Let $S(t) = \{x\colon x = (\cosh t, y, x_1, \cdots) \in \mathbb{H}^\infty, d_H(x, O) = t\}$ be the sphere in $\mathbb{H}^\infty$ with radius t $(t \geq 1)$. In the following, we shall construct an n-dimensional cube in the sphere $S(t)$ $(t \geq n)$ based on the point $(\cosh t, \sinh t, 0, 0, \cdots)$.

(a) We first compute the coordinates of the points which are of the form $x = (\cosh t, y_1, 0_N, x_1, 0, \cdots) \in S(t)$ $(x_1 > 0)$, and have hyperbolic distance 1 with the base point $x_t = (\cosh t, \sinh t, 0, \cdots)$. Here, we use the notation $0_N = 0, 0, \cdots, 0$ $(N \geq 0)$ to denote the N-zeros, and $0_\infty = (0, 0, \cdots)$ to denote the infinite many zeros.

By the fact that $x \in S(t)$ and $d_H(x, x_t) = 1$, we have

$$y_1^2 + x_1^2 = \sinh^2 t, \tag{3.1}$$

$$\cosh^2 t - y_1 \sinh t = \cosh 1. \tag{3.2}$$

From these equations, we get

$$y_1 = \sinh t - \frac{\cosh 1 - 1}{\sinh t}, \tag{3.3}$$

$$x_1^2 = 2(\cosh 1 - 1) - \Big(\frac{\cosh 1 - 1}{\sinh t}\Big)^2. \tag{3.4}$$

Hereinafter, we use the following notations:

$a_0 = 2(\cosh 1 - 1)$ and $a^2 = a_0\big(1 - \frac{\cosh 1 - 1}{2\sinh^2 t}\big) = x_1^2$ $(a > 0)$ which depends on t.

We have $a_0 = 2(\cosh 1 - 1) \approx 1.086\,(< 1.087)$ and $a \leq 1.043$ for $t \geq 1$.

(b) Next, we compute the second coordinates y $(y > 0)$ of the points

$$(\cosh t, y, x_1, x_2 \cdots) \in S(t),$$

where $x_n = a$ or 0 and the multiplicity of $x_n = a$ is i. It is obvious that the y-coordinate does not depend on the positions of a, it depends only on the multiplicity of a. So, it is sufficient for us to compute y_i of the points as $(\cosh t, y_i, a, a, \cdots, a, 0_\infty) \in S(t)$ $(y_i > 0)$, where the multiplicity of a is i.

From the equality

$$\cosh^2 t - (y_i^2 + ia^2) = 1,$$

we get

$$y_i^2 = \sinh^2 t - ia^2. \tag{3.5}$$

(c) Now, we construct the n-dimensional cube $\mathbb{H}_2^n$ in $\mathbb{H}^\infty$. We set the vertices of the cube as following

$$\begin{cases} (\cosh t, y_0, 0, 0, \cdots, 0, 0_\infty), \\ (\cosh t, y_1, a, 0, \cdots, 0, 0_\infty), \\ \qquad\qquad \vdots \\ (\cosh t, y_i, a_1, a_2, \cdots, a_n, 0_\infty), \\ \qquad\qquad \vdots \\ (\cosh t, y_n, a, a, \cdots, a, 0_\infty), \end{cases}$$

where $a_n = a$ or 0. The multiplicities of $a_k = a$ $(1 \le k \le n)$ are denoted by the subscripts of y. There are 2^n-many vertices.

(d) We compute the hyperbolic distances between any two vertices in $\mathbb{H}_2^n$.
Let

$$\mathrm{A} = (\cosh t, y_{i+j_1}, a_1, a_2, \cdots, a_n, 0_\infty) \quad \text{and} \quad \mathrm{B} = (\cosh t, y_{i+j_2}, b_1, b_2, \cdots, b_n, 0_\infty)$$

be two vertices in $\mathbb{H}_2^n$, where there exist $i + j_1$ numbers of $a_k = a$ in A and $i + j_2$ numbers of $b_k = a$ in B, and there exist i numbers a in A and B which have the same positions. We have

$$\begin{aligned}
\cosh d(\mathrm{A},\mathrm{B}) &= \cosh^2 t - (y_{i+j_1} y_{i+j_2} + ia^2) \\
&= \cosh^2 t - \Big[\sqrt{\sinh^2 t - (i+j_1)a^2} \cdot \sqrt{\sinh^2 t - (i+j_2)a^2} + ia^2\Big] \\
&= \cosh^2 t - \Big[\sinh^2 t \cdot \sqrt{1 - \frac{(i+j_1)a^2}{\sinh^2 t}} \cdot \sqrt{1 - \frac{(i+j_2)a^2}{\sinh^2 t}} + ia^2\Big] \\
&= \cosh^2 t - \Big\{\sinh^2 t \cdot \Big[1 - \frac{(i+j_1)a^2}{2\sinh^2 t} + r_1\Big]\Big[1 - \frac{(i+j_2)a^2}{2\sinh^2 t} + r_2\Big] + ia^2\Big\} \\
&= \cosh^2 t - \Big\{\sinh^2 t \cdot \Big[1 - \frac{(2i+j_1+j_2)a^2}{2\sinh^2 t} + r_3\Big] + ia^2\Big\} \\
&= 1 + \frac{(j_1+j_2)}{2} a^2 + r_4 \\
&= 1 + \frac{(j_1+j_2)}{2} a_0 \Big\{1 - \frac{\cosh 1 - 1}{2\sinh^2 t}\Big\} + r_4 \\
&= 1 + \frac{(j_1+j_2)}{2} a_0 + r_5,
\end{aligned} \tag{3.6}$$

where r_1, r_2, r_3, r_4, r_5 are the errors and

$$r_3 = \frac{(i+j_1)(i+j_2)a^4}{4\sinh^4 t} - r_1 \frac{(i+j_2)a^2}{2\sinh^2 t} - r_2 \frac{(i+j_1)a^2}{2\sinh^2 t} + r_1 r_2, \tag{3.7}$$

$$r_4 = -r_3 \cdot \sinh^2 t, \tag{3.8}$$

$$r_5 = r_4 - \frac{(j_1+j_2)a_0^2}{8\sinh^2 t}. \tag{3.9}$$

Noticing $\max\{(i+j_1), (i+j_2)\} \le n \le t$, we have $\frac{(i+j_k)a^2}{\sinh^2 t} < 1$ $(k = 1, 2)$. By a simple calculation and the simple inequality $0 \le \big(1 - \frac{1}{2}x\big) - \sqrt{1-x} < \frac{1}{2}x^2$ $(0 \le x < 1)$, we get the following estimates:

$$\begin{aligned}
&|r_1| \le \frac{1}{2}\Big[\frac{(i+j_1)a^2}{\sinh^2 t}\Big]^2, \quad |r_2| \le \frac{1}{2}\Big[\frac{(i+j_2)a^2}{\sinh^2 t}\Big]^2, \quad |r_3| \le \frac{(i+j_1)(i+j_2)a^4}{\sinh^4 t}, \\
&|r_4| \le \frac{(i+j_1)(i+j_2)a^4}{\sinh^2 t}, \quad |r_5| \le \frac{5}{8}\Big(\frac{n}{\sinh t}\Big)^2 a_0^2.
\end{aligned}$$

From the last inequality, we get

$$|r_5| \le \frac{1}{4} a_0 \quad \text{for } t \ge n. \tag{3.10}$$

By the equality (3.6) and the inequality (3.10), we get

$$\max\Big\{1, 1 + \frac{2(j_1+j_2)-1}{4} a_0\Big\} \le \cosh d(\mathrm{A},\mathrm{B}) \le 1 + \frac{2(j_1+j_2)+1}{4} a_0. \tag{3.11}$$

Therefore, we have

$$\max\Big\{0, \ln\Big[1+\frac{2(j_1+j_2)-1}{2}a_0\Big]\Big\} \le d(\mathrm{A},\mathrm{B}) \le \ln\Big[2+\frac{2(j_1+j_2)+1}{2}a_0\Big]. \tag{3.12}$$

(e) Finally, we define the maps f_n from $\mathbb{Z}_2^n$ to $\mathbb{H}_2^n$ by

$$f_n(z_1, z_2, \cdots, z_n) = (\cosh t, y_i, 0_N, a_1, a_2, \cdots, a_n, 0, \cdots),$$

where $(z_1, z_2, \cdots, z_n) \in \mathbb{Z}_2^n$, $z_k = 1$ or 0, $a_k = a$ or 0 according to $z_k = 1$ or 0, and i is the multiplicity of $a_k = a$. It is obvious that f_n are one to one and surjective. By the equation (3.12), we have that the sequence of maps f_n is an equi-coarse embedding from $\mathbb{Z}_2^n$ to $\mathbb{H}_2^n$. This completes the proof of the theorem.

We recall that the cubes $\mathbb{H}_2^n$ are in the spheres $S(t)$ $(t \ge n)$ of $\mathbb{H}^\infty$. Set $t = \frac{n(n+1)}{2}$ for $\mathbb{H}_2^n$. Then we get a discrete subspace $Y = \bigsqcup\limits_{n=1}^{\infty} \mathbb{H}_2^n$ of $\mathbb{H}^\infty$. Similarly to the proof of Theorem 5.1 in [6], we get the following corollary.

Corollary 3.1 *The infinite dimensional hyperbolic space $\mathbb{H}^\infty$ does not have property A.*

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