Chin. Ann. Math.
32B(3), 2011, 387–396
DOI: 10.1007/s11401-011-0645-4

Chinese Annals of Mathematics, Series B

Chen's Theorem with Small Primes*

Yingjie LI[1] Yingchun CAI[2]

Abstract Let N be a sufficiently large even integer. Let p denote a prime and P_2 denote an almost prime with at most two prime factors. In this paper, it is proved that the equation $N = p + P_2$ $(p \leq N^{0.945})$ is solvable.

Keywords Chen's Theorem, Sieve method, Mean value theorem
2000 MR Subject Classification 11N36

1 Introduction

In 1966, Jingrun Chen [4] made great progress in the research of the binary Goldbach conjecture. In 1973, Jingrun Chen [5] proved what is now called the Chen's theorem: Let N be a sufficiently large even integer. Let p denote a prime and P_2 denote an almost prime with at most two prime factors. Then the equation

$$N = p + P_2 \tag{1.1}$$

is solvable. In fact, Chen's theorem can be expressed in a more precise form: Let $S(N)$ be the number of solutions to the equation (1.1). Then

$$S(N) \geq \frac{0.67C(N)N}{\log^2 N},$$

where

$$C(N) = \prod_{p>2}\Big(1 - \frac{1}{(p-1)^2}\Big)\prod_{\substack{p|N\\p>2}}\frac{p-1}{p-2}.$$

Chen's constant 0.67 was improved by many authors. The historical record is as follows: 0.689 by Halberstam and Richert [9], 0.754, 0.81 by Chen [7, 8], 0.828 by Cai and Lu [2], 0.836 by Wu [14], and 0.867 by Cai [3].

Chen's theorem with a small prime p was studied in [1]: Let $S(N,\theta)$ be the number of solutions of the equation

$$N = p + P_2, \quad p \leq N^{\theta}. \tag{1.2}$$

For $\theta = 0.95$, we have $S(N,\theta) > \frac{0.01C(N)N^{\theta}}{\log^2 N}$.

The aim of this paper is to propose a better result.

Theorem 1.1 *For $\theta = 0.945$, we have $S(N,\theta) > \frac{0.001C(N)N^{\theta}}{\log^2 N}$.*

Manuscript received August 16, 2010. Published online April 19, 2011.
[1]College of Information Technology, Shanghai Ocean University, Shanghai 201306, China.
E-mail: thelyj@163.com
[2]Corresponding author. Department of Mathematics, Tongji University, Shanghai 200092, China.
E-mail: yingchuncai@mail.tongji.edu.cn
*Project supported by the National Natural Science Foundation of China (No. 11071186), the Science Foundation for the Excellent Youth Scholars of Shanghai (No. ssc08017) and the Doctoral Research Fund of Shanghai Ocean University.

2 Some Lemmas

Let $\mathscr{A}$ denote a finite integral set and $\mathscr{P}$ denote an infinite set of primes. $\overline{\mathscr{P}}$ denotes the set of primes that do not belong to $\mathscr{P}$. Let $z \ge 2$, and put

$$P(z) = \prod_{p<z,\, p\in\mathscr{P}} p, \quad S(\mathscr{A};\mathscr{P},z) = \sum_{a\in\mathscr{A},(a,P(z))=1} 1,$$
$$\mathscr{A}_d = \{a \mid a \in \mathscr{A}, a \equiv 0 \pmod{d}\}, \quad \mathscr{P}(q) = \{p \mid p \in \mathscr{P}, (p,q) = 1\}.$$

Lemma 2.1 (see [10]) *If*
(A$_1$) $|\mathscr{A}_d| = \frac{\omega(d)}{d}X + r_d,\ \mu(d) \ne 0,\ (d, \overline{\mathscr{P}}) = 1;$
(A$_2$) $\sum\limits_{z_1\le p<z_2} \frac{\omega(p)}{p} = \log\frac{\log z_2}{\log z_1} + O\big(\frac{1}{\log z_1}\big), \quad 2 \le z_1 < z_2,$
where $\omega(d)$ *is multiplicative with* $0 \le \omega(p) < p$. $X > 1$ *is independent of* d. *Then*

$$S(\mathscr{A};\mathscr{P},z) \ge XV(z)\Big\{f(s) + O\Big(\frac{1}{\log^{\frac13} D}\Big)\Big\} - R_D,$$
$$S(\mathscr{A};\mathscr{P},z) \le XV(z)\Big\{F(s) + O\Big(\frac{1}{\log^{\frac13} D}\Big)\Big\} + R_D,$$

where

$$C(\omega) = \prod_p \Big(1 - \frac{\omega(p)}{p}\Big)\Big(1 - \frac{1}{p}\Big)^{-1}, \quad R_D = \sum_{\substack{d<D\\ d\mid P(z)}} |r_d|,$$
$$V(z) = C(\omega)\frac{\mathrm{e}^{-\gamma}}{\log z}\Big(1 + O\Big(\frac{1}{\log z}\Big)\Big), \quad s = \frac{\log D}{\log z}.$$

Here γ *denotes Euler constant.* $f(s)$ *and* $F(s)$ *are determined by the following differential-difference equations:*

$$\begin{cases} F(s) = \dfrac{2\mathrm{e}^\gamma}{s}, \quad f(s) = 0, \quad 0 < s \le 2,\\ (sF(s))' = f(s-1), \quad (sf(s))' = F(s-1), \quad s \ge 2. \end{cases}$$

Lemma 2.2 (see [11]) *We have*

$$F(s) = \frac{2\mathrm{e}^\gamma}{s}, \quad 0 < s \le 3,$$
$$F(s) = \frac{2\mathrm{e}^\gamma}{s}\Big(1 + \int_2^{s-1} \frac{\log(t-1)}{t}\mathrm{d}t\Big), \quad 3 \le s \le 5,$$
$$f(s) = \frac{2\mathrm{e}^\gamma \log(s-1)}{s}, \quad 2 \le s \le 4,$$
$$f(s) = \frac{2\mathrm{e}^\gamma}{s}\Big(\log(s-1) + \int_3^{s-1} \frac{\mathrm{d}t}{t} \int_2^{t-1} \frac{\log(u-1)}{u}\mathrm{d}u\Big), \quad 4 \le s \le 6.$$

Lemma 2.3 (see [11]) *For any given constant* $A > 0$, *there exists a constant* $B = B(A) > 0$, *such that*

$$\sum_{d\le D} \max_{(l,d)=1} \max_{y\le x} \Big| \sum_{\substack{p\le y\\ p\equiv l \pmod{d}}} 1 - \frac{\mathrm{Li}y}{\varphi(d)} \Big| \ll \frac{x}{\log^A x},$$

where $\mathrm{Li}x = \int_2^x \frac{\mathrm{d}t}{\log t}$, $D = x^{\frac12}\log^{-B} x$.

Lemma 2.4 (see [13]) *Let $g(n)$ be a number-theoretic function such that $\sum_{n\le x}\frac{g^2(n)}{n}\ll \log^c x$, where $c>0$. For $(al,q)=1$, define*

$$H(z,h,a,q,l)=\sum_{\substack{z\le ap\le z+h\\ ap\equiv l \pmod q}}1-\frac{1}{\varphi(q)}\Big(\mathrm{Li}\Big(\frac{z+h}{a}\Big)-\mathrm{Li}\Big(\frac{z}{a}\Big)\Big).$$

Then for any constant $A>0$, there exists a constant $B=B(A,c)>0$, such that

$$\sum_{d\le D}\max_{(l,d)=1}\max_{h\le y}\max_{\frac{x}{2}\le z\le x}\Big|\sum_{\substack{a\le x^{\beta}\\ (a,d)=1}}g(a)H(z,h,a,d,l)\Big|\ll\frac{y}{\log^A x}$$

for $\frac{3}{5}<\theta\le 1,\ y=x^{\theta},\ 0\le\beta<\frac{5\theta-3}{2},\ \lambda=\theta-\frac{1}{2},\ D=x^{\lambda}\log^{-B}x$.

Lemma 2.5 (see [12]) *Suppose that $\omega(u)$ is the solution to the following equations:*

$$\begin{cases}\omega(u)=\dfrac{1}{u}, & 1\le u\le 2,\\ (u\omega(u))'=\omega(u-1), & u>2.\end{cases}$$

Then we have $\omega(u)<\frac{1}{1.763},\ u\ge 2$.

Lemma 2.6 *Let $\omega(u)$ be defined in Lemma 2.5. Let $x>1,\ x^{\frac{19}{24}+\varepsilon}\le y\le\frac{x}{\log x},\ z=x^{\frac{1}{u}},\ P_1(z)=\prod_{p<z}p$. Then for any $u>1$, we have*

$$\sum_{\substack{x-y\le n\le x\\ (n,P_1(z))=1}}1=\omega(u)\frac{y}{\log z}+O\Big(\frac{y}{\log^2 z}\Big). \tag{2.1}$$

Proof We will prove it by mathematical induction.

Firstly, when $1<u\le 2$, by Huxley's prime number theorem in shorter intervals and the definition of $\omega(u)$ in Lemma 2.5, we have

$$\sum_{\substack{x-y\le n\le x\\ (n,P_1(z))=1}}1=\sum_{x-y\le p\le x}1=\frac{y}{\log x}+O\Big(\frac{y}{\log^2 x}\Big)=\omega(u)\frac{y}{\log z}+O\Big(\frac{y}{\log^2 z}\Big).$$

So (2.1) holds for $1<u\le 2$.

Next, we assume that (2.1) is true for $k<u\le k+1$ (k being a natural number). When $k+1<u\le k+2$, let $\mathscr{P}_1$ be the set of all prime numbers and $\mathscr{N}=\{n:x-y\le n\le x\}$. Then we have

$$\sum_{\substack{x-y\le n\le x\\ (n,P_1(z))=1}}1=S(\mathscr{N};\mathscr{P}_1,z).$$

If $k+1<u\le k+2$, we have

$$\begin{aligned}S(\mathscr{N};\mathscr{P}_1,x^{\frac{1}{u}})&=S(\mathscr{N};\mathscr{P}_1,x^{\frac{1}{k+1}})+\sum_{x^{\frac{1}{u}}\le p<x^{\frac{1}{k+1}}}S(\mathscr{N}_p;\mathscr{P}_1,p)\\ &=\sum_{\substack{x-y\le n\le x\\ (n,P_1(x^{\frac{1}{k+1}}))=1}}1+\sum_{x^{\frac{1}{u}}\le p<x^{\frac{1}{k+1}}}\sum_{\substack{\frac{x-y}{p}\le n_1\le\frac{x}{p}\\ (n_1,P_1(p))=1}}1.\end{aligned}\tag{2.2}$$

Since $p = \left(\frac{x}{p}\right)^{\frac{\frac{1}{\log \frac{x}{p}}}{\log p}}$ and $k < \frac{\log \frac{x}{p}}{\log p} = \frac{\log x}{\log p} - 1 \le k+1$, $\frac{y}{p} \ge \left(\frac{x}{p}\right)^{\frac{7}{12}+\varepsilon}$ for $x^{\frac{1}{u}} \le p < x^{\frac{1}{k+1}}$, by assumption, (2.1)–(2.2), the prime number theorem and the definition of $\omega(u)$, we get

$$\begin{aligned} S(\mathscr{N}; \mathscr{P}_1, x^{\frac{1}{u}}) &= (k+1)\omega(k+1)\frac{y}{\log x} + \int_{x^{\frac{1}{u}}}^{x^{\frac{1}{k+1}}} \omega\Big(\frac{\log x}{\log t} - 1\Big)\frac{y}{t\log^2 t}\mathrm{d}t \\ &\quad + O\Big(\int_{x^{\frac{1}{u}}}^{x^{\frac{1}{k+1}}} \frac{y}{t\log^3 t}\mathrm{d}t\Big) + O\Big(\frac{y}{(\log x^{\frac{1}{k+1}})^2}\Big) \\ &= \omega(u)\frac{y}{\log x^{\frac{1}{u}}} + O\Big(\frac{y}{(\log x^{\frac{1}{u}})^2}\Big). \end{aligned}$$

Hence, (2.1) holds when $k+1 < u \le k+2$.

By the principle of mathematical induction, (2.1) is true for all $u > 1$. Thus the proof of Lemma 2.6 is completed.

3 Weighted Sieve Method

In the following two sections, we suppose that N is a sufficiently large even integer and p, p_1, p_2, p_3, p_4 denote primes. Put

$$\mathscr{A} = \{a \mid a = N - p,\ \ p \le N^{\theta}\}, \quad \theta = 0.945, \quad \mathscr{P} = \{p \mid (p, N) = 1\}.$$

Then

$$\begin{aligned} X &= \mathrm{Li}N^{\theta} \sim \frac{N^{\theta}}{\log N^{\theta}}, \quad (d, N) = 1, \quad D = \frac{N^{\frac{\theta}{2}}}{\log^B N}, \quad B = B(5) > 0, \\ r_d &= \pi(N^{\theta}; d, N) - \frac{\mathrm{Li}N^{\theta}}{\varphi(d)}, \quad \omega(d) = \frac{d}{\varphi(d)}, \quad \mu(d) \ne 0, \quad (d, N) = 1. \end{aligned}$$

Lemma 3.1 (see [5]) *We have*

$$S(N, \theta) > S - \frac{1}{2}S_1 - \frac{1}{2}S_2 - S_3 + O(N^{\frac{9.95}{10.95}}),$$

where

$$\begin{aligned} S &= \sum_{\substack{a \in \mathscr{A}, (a,N)=1 \\ (a, P(N^{\frac{1}{10.95}}))=1}} 1, \qquad S_1 = \sum_{\substack{N^{\frac{1}{10.95}} \le p < N^{\frac{1}{3.3}} \\ (p,N)=1}} S(\mathscr{A}_p; \mathscr{P}, N^{\frac{1}{10.95}}), \\ S_2 &= \sum_{\substack{a \in \mathscr{A}, (a,N)=1 \\ (a, P(N^{\frac{1}{10.95}}))=1}} \rho_2(a), \quad S_3 = \sum_{\substack{a \in \mathscr{A}, (a,N)=1 \\ (a, P(N^{\frac{1}{10.95}}))=1}} \rho_3(a), \\ \rho_2(a) &= \begin{cases} 1, & a = p_1p_2p_3,\ N^{\frac{1}{10.95}} \le p_1 < N^{\frac{1}{3.3}} \le p_2 < p_3,\ (a, N) = 1, \\ 0, & \textit{otherwise}, \end{cases} \\ \rho_3(a) &= \begin{cases} 1, & a = p_1p_2p_3,\ N^{\frac{1}{3.3}} \le p_1 < p_2 < p_3,\ (a, N) = 1, \\ 0, & \textit{otherwise}. \end{cases} \end{aligned}$$

Lemma 3.2 *For S_1, we have*

$$S_1 \le \sum_{\substack{N^{\frac{1}{10.95}} \le p < N^{\frac{\theta}{2}-\frac{2.5}{10.95}} \\ (p,N)=1}} S(\mathscr{A}_p;\mathscr{P},N^{\frac{1}{10.95}}) + \sum_{\substack{N^{\frac{\theta}{2}-\frac{2.5}{10.95}} \le p < N^{\frac{1}{3.3}} \\ (p,N)=1}} S\Big(\mathscr{A}_p;\mathscr{P},\Big(\frac{D}{p}\Big)^{\frac{1}{2.5}}\Big)$$
$$= S_4 + S_5.$$

Proof

$$S_1 = \sum_{\substack{N^{\frac{1}{10.95}} \le p < N^{\frac{\theta}{2}-\frac{2.5}{10.95}} \\ (p,N)=1}} S(\mathscr{A}_p;\mathscr{P},N^{\frac{1}{10.95}}) + \sum_{\substack{N^{\frac{\theta}{2}-\frac{2.5}{10.95}} \le p < N^{\frac{1}{3.3}} \\ (p,N)=1}} S(\mathscr{A}_p;\mathscr{P},N^{\frac{1}{10.95}})$$
$$\le S_4 + S_5.$$

Lemma 3.3 (see [6]) *We have*

$$S_5 \le S_6 - \frac{1}{2}S_7 + \frac{1}{2}S_8 + O(N^{0.9}), \tag{3.1}$$

where

$$S_6 = \sum_{\substack{N^{\frac{\theta}{2}-\frac{2.5}{10.95}} \le p < N^{\frac{1}{3.3}} \\ (p,N)=1}} S\Big(\mathscr{A}_p;\mathscr{P},\Big(\frac{D}{p}\Big)^{\frac{1}{3.67}}\Big),$$

$$S_7 = \sum_{\substack{N^{\frac{\theta}{2}-\frac{2.5}{10.95}} \le p < N^{\frac{1}{3.3}} \\ (p,N)=1}} \sum_{\substack{(\frac{D}{p})^{\frac{1}{3.67}} \le p_1 < (\frac{D}{p})^{\frac{1}{2.5}} \\ (p_1,N)=1}} S\Big(\mathscr{A}_{pp_1};\mathscr{P},\Big(\frac{D}{p}\Big)^{\frac{1}{3.67}}\Big),$$

$$S_8 = \sum_{\substack{N^{\frac{\theta}{2}-\frac{2.5}{10.95}} \le p < N^{\frac{1}{3.3}} \\ (p,N)=1}} \sum_{\substack{(\frac{D}{p})^{\frac{1}{3.67}} \le p_2 < p_3 < p_1 < (\frac{D}{p})^{\frac{1}{2.5}} \\ (p_1p_2p_3,N)=1}} S(\mathscr{A}_{pp_1p_2p_3};\mathscr{P}(p_2),p_3).$$

Proof By Buchstab's identity, we have

$$S\Big(A_p;\mathscr{P},\Big(\frac{D}{p}\Big)^{\frac{1}{2.5}}\Big) = S\Big(A_p;\mathscr{P},\Big(\frac{D}{p}\Big)^{\frac{1}{3.67}}\Big) - \sum_{\substack{(\frac{D}{p})^{\frac{1}{3.67}} \le p_1 < (\frac{D}{p})^{\frac{1}{2.5}} \\ (p_1,N)=1}} S\Big(A_{pp_1};\mathscr{P},\Big(\frac{D}{p}\Big)^{\frac{1}{3.67}}\Big)$$
$$+ \sum_{\substack{(\frac{D}{p})^{\frac{1}{3.67}} \le p_2 < p_1 < (\frac{D}{p})^{\frac{1}{2.5}} \\ (p_1p_2,N)=1}} S(A_{pp_1p_2};\mathscr{P},p_2), \tag{3.2}$$

$$S\Big(A_p;\mathscr{P},\Big(\frac{D}{p}\Big)^{\frac{1}{2.5}}\Big) \le S\Big(A_p;\mathscr{P},\Big(\frac{D}{p}\Big)^{\frac{1}{3.67}}\Big) - \sum_{\substack{(\frac{D}{p})^{\frac{1}{3.67}} \le p_1 < (\frac{D}{p})^{\frac{1}{2.5}} \\ (p_1,N)=1}} S\Big(A_{pp_1};\mathscr{P}(p_1),\Big(\frac{D}{p}\Big)^{\frac{1}{2.5}}\Big)$$
$$- \sum_{\substack{(\frac{D}{p})^{\frac{1}{3.67}} \le p_1 < p_2 < (\frac{D}{p})^{\frac{1}{2.5}} \\ (p_1p_2,N)=1}} S(A_{pp_1p_2};\mathscr{P}(p_1),p_2) \tag{3.3}$$

and

$$\begin{aligned}
&\sum_{\substack{(\frac{D}{p})^{\frac{1}{3.67}} \le p_2 < p_1 < (\frac{D}{p})^{\frac{1}{2.5}} \\ (p_1p_2,N)=1}} S(A_{pp_1p_2}; \mathscr{P}, p_2) - \sum_{\substack{(\frac{D}{p})^{\frac{1}{3.67}} \le p_1 < p_2 < (\frac{D}{p})^{\frac{1}{2.5}} \\ (p_1p_2,N)=1}} S(A_{pp_1p_2}; \mathscr{P}(p_1), p_2) \\
= &\sum_{\substack{(\frac{D}{p})^{\frac{1}{3.67}} \le p_2 < p_3 < p_1 < (\frac{D}{p})^{\frac{1}{2.5}} \\ (p_1p_2p_3,N)=1}} S(A_{pp_1p_2p_3}; \mathscr{P}(p_2), p_3) \\
&+ \sum_{\substack{(\frac{D}{p})^{\frac{1}{3.67}} \le p_2 < p_1 < (\frac{D}{p})^{\frac{1}{2.5}} \\ (p_1p_2,N)=1}} S(A_{pp_1p_2^2}; \mathscr{P}, p_2). \qquad (3.4)
\end{aligned}$$

Now adding (3.2) and (3.3), suming over p in the interval $[N^{\frac{\theta}{2}-\frac{2.5}{10.95}}, N^{\frac{1}{3.3}})$ and by (3.4), we get Lemma 3.3, where the trivial inequality

$$\begin{aligned}
&\sum_{\substack{N^{\frac{\theta}{2}-\frac{2.5}{10.95}} \le p < N^{\frac{1}{3.3}} \\ (p,N)=1}} \sum_{\substack{(\frac{D}{p})^{\frac{1}{3.67}} \le p_2 < p_1 < (\frac{D}{p})^{\frac{1}{2.5}} \\ (p_1p_2,N)=1}} S(A_{pp_1p_2^2}; \mathscr{P}, p_2) \\
\ll &\sum_{\substack{N^{\frac{\theta}{2}-\frac{2.5}{10.95}} \le p < N^{\frac{1}{3.3}} \\ (p,N)=1}} \sum_{\substack{(\frac{D}{p})^{\frac{1}{3.67}} \le p_2 < p_1 < (\frac{D}{p})^{\frac{1}{2.5}} \\ (p_1p_2,N)=1}} \Big(\frac{N^\theta}{pp_1p_2^2} + 1\Big) \ll N^{0.9}
\end{aligned}$$

is used.

Hence, combining Lemmas 3.1–3.3, we get

$$S(N,\theta) > S - \frac{1}{2}S_4 - \frac{1}{2}S_6 + \frac{1}{4}S_7 - \frac{1}{4}S_8 - \frac{1}{2}S_2 - S_3 + O(N^{\frac{9.95}{10.95}}). \qquad (3.5)$$

4 Proof of the Theorem

4.1 Estimation of the lower bound of S

Suppose $D = \frac{N^{\frac{\theta}{2}}}{\log^B N}$ with $B = B(5) > 0$. By Lemma 2.3, we have

$$R_D = \sum_{d\le D} \Big|\pi(N^\theta; d, N) - \frac{\mathrm{Li}N^\theta}{\varphi(d)}\Big| \le \sum_{d\le D} \max_{y\le N^\theta} \max_{(l,d)=1} \Big|\pi(y; d, l) - \frac{\mathrm{Li}y}{\varphi(d)}\Big| \ll \frac{N^\theta}{\log^5 N}. \qquad (4.1)$$

Since

$$C(\omega) = 2\prod_{p>2}\Big(1 - \frac{1}{(p-1)^2}\Big)\prod_{p|N,p>2}\Big(\frac{p-1}{p-2}\Big) = 2C(N), \qquad (4.2)$$

by Lemmas 2.1–2.2, (4.1) and (4.2), we get

$$\begin{aligned}
S &\ge 8(1+o(1))\frac{C(N)N^\theta}{\theta^2\log^2 N}\Big(\log\Big(\frac{10.95\theta}{2} - 1\Big) + \int_2^{\frac{10.95\theta}{2}-2} \frac{\log(s-1)}{s}\log\frac{\frac{10.95\theta}{2}-1}{s+1}\mathrm{d}s\Big) \\
&> 12.9972\frac{C(N)N^\theta}{\log^2 N}. \qquad (4.3)
\end{aligned}$$

4.2 Estimation of the upper bounds of S_4 and S_6

Let $R_D(p) = \sum\limits_{d<\frac{D}{p}, d|P(N^{\frac{1}{10.95}})} |r_{dp}|$. By Lemma 2.3, we get

$$\sum_{\substack{N^{\frac{1}{10.95}} \le p < N^{\frac{\theta}{2}-\frac{2.5}{10.95}} \\ (p,N)=1}} R_D(p) \le \sum_{d\le D} \max_{y\le N^\theta} \max_{(l,d)=1} \left|\pi(y;d,l) - \frac{\mathrm{Li} y}{\varphi(d)}\right| \ll \frac{N^\theta}{\log^5 N}. \tag{4.4}$$

By Lemmas 2.1–2.2, (4.2), (4.4), the prime number theorem and partial integration, we have

$$\begin{aligned} S_4 &\le 21.9(1+o(1))\mathrm{e}^{-\gamma} \frac{C(N)N^\theta}{\theta \log^2 N} \sum_{\substack{N^{\frac{1}{10.95}} \le p < N^{\frac{\theta}{2}-\frac{2.5}{10.95}} \\ (p,N)=1}} \frac{1}{p} F\Big(\frac{10.95}{2}\theta - 10.95\frac{\log p}{\log N}\Big) \\ &\le 21.9(1+o(1))\mathrm{e}^{-\gamma} \frac{C(N)N^\theta}{\theta \log^2 N} \int_{N^{\frac{1}{10.95}}}^{N^{\frac{\theta}{2}-\frac{2.5}{10.95}}} \frac{1}{u\log u} F\Big(\frac{10.95}{2}\theta - 10.95\frac{\log u}{\log N}\Big)\mathrm{d}u \\ &\le 8(1+o(1))\frac{C(N)N^\theta}{\theta^2 \log^2 N}\Big(\log\Big(\frac{(10.95\theta-2)(10.95\theta-5)}{10}\Big) \\ &\quad + \int_2^{\frac{10.95}{2}\theta-2} \frac{\log(s-1)}{s} \log\frac{(\frac{10.95}{2}\theta-1)(\frac{10.95}{2}\theta-1-s)}{s+1}\mathrm{d}s\Big) \\ &\le 14.1914\frac{C(N)N^\theta}{\log^2 N}. \end{aligned} \tag{4.5}$$

Similarly, we have

$$\begin{aligned} S_6 &\le 8(1+o(1))\frac{C(N)N^\theta}{\theta^2 \log^2 N}\Big(\log\Big(\frac{10}{(3.3\theta-2)(10.95\theta-5)}\Big)\Big)\Big(1+\int_2^{2.67}\frac{\log(x-1)}{x}\mathrm{d}x\Big) \\ &< 4.9577\frac{C(N)N^\theta}{\log^2 N}. \end{aligned} \tag{4.6}$$

4.3 Estimation of the upper bounds of S_2 and S_3

Let $D_1 = N^\lambda \log^{-B} N$. Here λ and $B = B(5) > 0$ are determined by Lemma 2.4. By the method in [5] and Huxley's prime number theorem in shorter intervals, we get

$$\begin{aligned} S_2 &\le 4(1+o(1))\frac{C(N)}{\log D_1} \sum_{N^{\frac{1}{10.95}} \le p_1 < N^{\frac{1}{3.3}} \le p_2 < (\frac{N}{p_1})^{\frac{1}{2}}} \sum_{N-N^\theta \le p_1p_2p_3 < N} 1 \\ &\le 8(1+o(1))\frac{C(N)N^\theta}{(2\theta-1)\log^2 N}\int_{2.3}^{9.95}\frac{\log\big(2.3-\frac{3.3}{t+1}\big)}{t}\mathrm{d}t \\ &< 6.9078\frac{C(N)N^\theta}{\log^2 N}. \end{aligned} \tag{4.7}$$

Similarly, we have

$$S_3 \le 8(1+o(1))\frac{C(N)N^\theta}{(2\theta-1)\log^2 N}\int_2^{2.3}\frac{\log(t-1)}{t}\mathrm{d}t < 0.1682\frac{C(N)N^\theta}{\log^2 N}. \tag{4.8}$$

4.4 Estimation of the lower bound of S_7

Let $R_D(pp_1) = \sum\limits_{d<\frac{D}{pp_1}, d|P((\frac{D}{p})^{\frac{1}{3.67}})} |r_{dpp_1}|$. By Lemma 2.3, we have

$$\sum_{\substack{N^{\frac{\theta}{2}-\frac{2.5}{10.95}}\le p<N^{\frac{1}{3.3}}\\(p,N)=1}}\ \sum_{\substack{(\frac{D}{p})^{\frac{1}{3.67}}\le p_1<(\frac{D}{p})^{\frac{1}{2.5}}\\(p_1,N)=1}} R_D(pp_1) \le \sum_{d\le D}\max_{y\le N^\theta}\max_{(l,d)=1}\left|\pi(y;d,l)-\frac{\mathrm{Li}y}{\varphi(d)}\right| \ll \frac{N^\theta}{\log^5 N}. \tag{4.9}$$

By Lemmas 2.1–2.2, (4.2), (4.9), the prime number theorem and partial integration, we obtain

$$\begin{aligned}S_7 &\ge 7.34(1+o(1))\mathrm{e}^{-\gamma}\frac{C(N)N^\theta}{\theta\log N}\\ &\quad\times \sum_{\substack{N^{\frac{\theta}{2}-\frac{2.5}{10.95}}\le p<N^{\frac{1}{3.3}}\\(p,N)=1}}\ \sum_{\substack{(\frac{D}{p})^{\frac{1}{3.67}}\le p_1<(\frac{D}{p})^{\frac{1}{2.5}}\\(p_1,N)=1}} \frac{1}{pp_1\log\frac{D}{p}} f\Big(3.67-3.67\frac{\log p_1}{\log\frac{D}{p}}\Big)\\ &\ge 8(1+o(1))\frac{C(N)N^\theta}{\theta^2\log^2 N}\Big(\log\Big(\frac{10}{(3.3\theta-2)(10.95\theta-5)}\Big)\Big)\int_{1.5}^{2.67}\frac{\log\big(2.67-\frac{3.67}{x+1}\big)}{x}\mathrm{d}x\\ &> 0.9625\frac{C(N)N^\theta}{\log^2 N}.\end{aligned} \tag{4.10}$$

4.5 Estimation of the upper bound of S_8

We set

$$E_1=\max\Big(\frac{N-N^\theta}{\mathrm{e}},\frac{D}{p_2^{3.67}},N^{\frac{\theta}{2}-\frac{2.5}{10.95}}\Big),\quad E_2=\min\Big(\frac{N}{\mathrm{e}},\frac{D}{p_1^{2.5}},N^{\frac{1}{3.3}}\Big),$$
$$E_3=\frac{N-N^\theta}{p_1p_2p_3N^{\frac{1}{3.3}}},\quad E_4=\frac{N}{p_1p_2p_3N^{\frac{\theta}{2}-\frac{2.5}{10.95}}},\quad E_5=\Big(\frac{D}{N^{\frac{1}{3.3}}}\Big)^{\frac{1}{3.67}},\quad E_6=\Big(\frac{D}{N^{\frac{\theta}{2}-\frac{2.5}{10.95}}}\Big)^{\frac{1}{2.5}}.$$

Then

$$\begin{aligned}S_8 &= \sum_{\substack{N^{\frac{\theta}{2}-\frac{2.5}{10.95}}\le p<N^{\frac{1}{3.3}}\\(p,N)=1}}\ \sum_{\substack{(\frac{D}{p})^{\frac{1}{3.67}}\le p_2<p_3<p_1<(\frac{D}{p})^{\frac{1}{2.5}}\\(p_1p_2p_3,N)=1}}\ \sum_{\substack{a\in\mathscr{A},pp_1p_2p_3|a\\(a,\frac{N}{p_2}P(p_3))=1}} 1+O(N^{\frac{9}{10}})\\ &= \sum_{\substack{N^{\frac{\theta}{2}-\frac{2.5}{10.95}}\le p<N^{\frac{1}{3.3}}\\(p,N)=1}}\ \sum_{\substack{(\frac{D}{p})^{\frac{1}{3.67}}\le p_2<p_3<p_1<(\frac{D}{p})^{\frac{1}{2.5}}\\(p_1p_2p_3,N)=1}}\ \sum_{\substack{p_4=N-pp_1p_2p_3n\\(n,\frac{N}{p_2}P(p_3))=1}} 1+O(N^{\frac{9}{10}})\\ &= S_8'+O(N^{\frac{9}{10}}),\end{aligned}$$

where

$$S_8'=\sum_{\substack{E_5\le p_2<p_3<p_1<E_6\\(p_1p_2p_3,N)=1}}\ \sum_{\substack{E_3\le n\le E_4\\(n,\frac{N}{p_2}P(p_3))=1}}\ \sum_{\substack{p_4=N-p(p_1p_2p_3n)\\E_1\le p<E_2\\(p,N)=1}} 1.$$

Now we consider

$$\begin{aligned}\mathscr{E} = \Big\{ & e : e = p_1p_2p_3n, E_5 \le p_2 < p_3 < p_1 < E_6,\ (p_1p_2p_3, N) = 1, \\ & E_3 \le n \le E_4,\ \Big(n, \frac{N}{p_2}P\Big(p_3\Big)\Big) = 1\Big\}, \\ \mathscr{L} = \{ & l : l = N - ep, e \in \mathscr{E}, E_1 \le p < E_2\}.\end{aligned}$$

Obviously, $(\mathscr{E}, N) = 1$. Since

$$N^{\frac{1}{2}} < e < N^{0.76},\ e \in \mathscr{E}; \quad |\mathscr{E}| < \sum_{E_5 \le p_2 < p_3 < p_1 < E_6} \frac{N}{p_1p_2p_3N^{\frac{\theta}{2}-\frac{2.5}{10.95}}} \ll N^{0.76},$$

the number of elements not exceeding $N^{\frac{1}{2}}$ in $\mathscr{L} \ll N^{0.76}$. S_8' does not exceed the number of primes in $\mathscr{L}$, hence

$$S_8 \le S(\mathscr{L}; \mathscr{P}, z) + O(N^{\frac{9}{10}}), \quad z \le N^{\frac{1}{2}}. \tag{4.11}$$

Thus we can choose

$$\begin{aligned}X_1 = \sum_{e\in\mathscr{E}} \sum_{E_1 \le p < E_2} 1 = & \sum_{N^{\frac{\theta}{2}-\frac{2.5}{10.95}} \le p < N^{\frac{1}{3.3}}} \sum_{\substack{(\frac{D}{p})^{\frac{1}{3.67}} \le p_2 \le p_3 < p_1 < (\frac{D}{p})^{\frac{1}{2.5}} \\ (p_1p_2p_3, N)=1}} \sum_{\substack{\frac{N-N^\theta}{pp_1p_2p_3} \le n \le \frac{N}{pp_1p_2p_3} \\ (n, \frac{N}{p_2}P(p_3))=1}} 1 \\ & \le X + O(N^{\frac{9}{10}}),\end{aligned} \tag{4.12}$$

where

$$X = \sum_{N^{\frac{\theta}{2}-\frac{2.5}{10.95}} \le p < N^{\frac{1}{3.3}}} \sum_{(\frac{D}{p})^{\frac{1}{3.67}} \le p_2 \le p_3 < p_1 < (\frac{D}{p})^{\frac{1}{2.5}}} \sum_{\substack{\frac{N-N^\theta}{pp_1p_2p_3} \le n \le \frac{N}{pp_1p_2p_3} \\ (n, NP(p_3))=1}} 1.$$

Let $z^2 = D_1 = N^\lambda \log^{-B} N$. Here λ and $B = B(5) > 0$ are determined by Lemma 2.4. Set $g(a) = \sum_{\substack{e=a \\ e\in\mathscr{E}}} 1$. By Lemma 2.4, we have

$$\begin{aligned}R_{D_1} = & \sum_{\substack{d \le D_1 \\ d|P(D_1^{0.5})}} \Big| \sum_{e\in\mathscr{E}} \Big(\sum_{\substack{E_1 \le p < E_2 \\ ep \equiv N(d)}} 1 - \frac{1}{\varphi(d)} \sum_{E_1 \le p < E_2} 1 \Big) \Big| \\ & \le \sum_{d \le D_1} \max_{(l,d)=1} \max_{h \le N^\theta} \max_{\frac{N}{2} \le z \le N} \Big| \sum_{\substack{a \le N^\beta \\ (a,d)=1}} g(a)H(z,h,a,d,l) \Big| \ll \frac{N^\theta}{\log^5 N}.\end{aligned} \tag{4.13}$$

Hence, by (4.13) and Lemmas 2.1–2.2, we get

$$S(\mathscr{L}; \mathscr{P}, D_1^{0.5}) \le 8(1 + o(1))C(N) \frac{X_1}{(2\theta - 1)\log N} + O\Big(\frac{N^\theta}{\log^5 N}\Big). \tag{4.14}$$

Combining (4.11)–(4.12) and (4.14), we obtain

$$S_8 \le 8(1 + o(1))C(N) \frac{X}{(2\theta - 1)\log N} + O\Big(\frac{N^\theta}{\log^5 N}\Big). \tag{4.15}$$

Since

$$\frac{\log \frac{N}{pp_1p_2p_3}}{\log p_3} > 4, \quad \Big(\frac{N}{pp_1p_2p_3}\Big)^{\frac{19}{24}+\varepsilon} < \frac{N^\theta}{pp_1p_2p_3} < \frac{N}{pp_1p_2p_3},$$

by Lemma 2.6, Lemma 2.5, the prime number theorem and partial integration, we get

$$\begin{aligned} X &\le (1+o(1)) \sum_{N^{\frac{\theta}{2}-\frac{2.5}{10.95}} \le p < N^{\frac{1}{3.3}}} \; \sum_{(\frac{D}{p})^{\frac{1}{3.67}} \le p_2 \le p_3 < p_1 < (\frac{D}{p})^{\frac{1}{2.5}}} \omega\Big(\frac{\log \frac{N}{pp_1p_2p_3}}{\log p_3}\Big) \frac{\frac{N^\theta}{pp_1p_2p_3}}{\log p_3} \\ &< \frac{N^\theta}{1.763}(1+o(1)) \sum_{N^{\frac{\theta}{2}-\frac{2.5}{10.95}} \le p < N^{\frac{1}{3.3}}} \frac{1}{p} \int_{(\frac{D}{p})^{\frac{1}{3.67}}}^{(\frac{D}{p})^{\frac{1}{2.5}}} \frac{\mathrm{d}u}{u \log u} \int_u^{(\frac{D}{p})^{\frac{1}{2.5}}} \frac{\mathrm{d}s}{s \log^2 s} \int_s^{(\frac{D}{p})^{\frac{1}{2.5}}} \frac{\mathrm{d}t}{t \log t} \\ &= \frac{2}{1.763}(1+o(1)) \frac{N^\theta}{\theta \log N}(6.17 \log 1.468 - 2.34) \log \frac{10}{(3.3\theta-2)(10.95\theta-5)}. \end{aligned}$$

This, together with (4.15), gives

$$S_8 < 0.159 \frac{C(N)N^\theta}{\log^2 N}. \tag{4.16}$$

4.6 Proof of Theorem 1.1

By (3.5), (4.3), (4.5)–(4.8), (4.10) and (4.16), we obtain

$$\begin{aligned} S(N,\theta) &> \Big(12.9972 - \frac{14.1914}{2} - \frac{4.9577}{2} + \frac{0.9625}{4} - \frac{0.159}{4} - \frac{6.9078}{2} - 0.1682\Big) \frac{C(N)N^\theta}{\log^2 N} \\ &= 0.001425 \frac{C(N)N^\theta}{\log^2 N}. \end{aligned}$$

This completes the proof of Theorem 1.1.

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