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A Rigidity Phenomenon on Riemannian Manifolds with Reverse Excess Pinching***

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Abstract The authors introduce the Hausdorff convergence to discuss the differentiable sphere theorem with excess pinching. Finally, a type of rigidity phenomenon on Riemannian manifolds is derived.

Keywords Volume comparison theorem, Hausdorff convergence, Differentiable sphere theorem, Harmonic coordinate, Harmonic radius
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1 Introduction

For a Riemannian manifold, its curvature and topology have a very close relation and form the core of differential geometry. On one hand, the curvature of a Riemannian manifold will reflect its topology. Many famous theorems in differential geometry, like Myer's theorem, the Volume Comparison Theorem, the $\frac{1}{4}$-pinching sphere theorem, etc., have fully reflected this point. On the other hand, the topology of a Riemannian manifold can also control the curvature of the manifold as shown by the Gauss-Bonnet-Chern formula, the Hamilton's theorem and so on. Among all these results, the sphere theorems have always been an interesting subject in differential geometry and lots of references of this respect appear during recent years (see [2–5, 9, 10, 14]). As a celebrated result to reflect the geometry and topology of a manifold, Klingenberg's $\frac{1}{4}$-pinching sphere theorem attracted much attention and many related similar results gave the generalization from various aspects. In 1977, for instance, K. Grove and K. Shiohama [5] concluded that M would be homeomorphic to n-sphere S^n if the sectional curvature of M satisfies that $K_M \geq 1$ while its diameter $\operatorname{diam}(M) > \frac{\pi}{2}$. Another theorem belongs to L. Coghlan and Y. Itokawa [2] which says that a compact, simply connected Riemannian manifold without boundary M^{2n} must be homeomorphic to S^{2n} if its sectional curvature K_M varies in $(0, 1]$ and the volume $V(M)$ is less than $\frac{3}{2}V(S^{2n})$. Recently, J. Y. Wu [3] and Y. Wen [4] respectively gave a little step forward towards its original conjecture in which the volume condition was $0 < V(M) < 3V(S^{2n})$.

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All the topological sphere theorems above can be understood roughly as follows: when the geometry conditions perturbed in a controlled manner based on those of the standard sphere S^n, one can also derive that the topology is the same as S^n.

For the topological sphere theorems, one of the important but difficult questions is to generalize them up to the differentiable sphere theorems (see [22–28]). As is well known, if the dimension $n \le 6$, a manifold being homeomorphic to S^n must be diffeomorphic to S^n, but it is not true when the dimension $n \ge 7$. The famous Milnor's exotic sphere, for example, tells this point. Joint with the discussions above, we may conjecture that if we perturb the geometry conditions in a more slight manner than in the topological sphere theorem, we can expect that the differential structure of the manifolds concerned is the same as the standard sphere S^n. Actually, a lot of proved results about differentiable sphere theorem (see [22–28]) have shown this point.

As one of the generalizations of the $\frac{1}{4}$-pinching sphere theorem, L. Coghlan and Y. Itokawa's theorem then faces the problem whether it is sufficient to conclude the diffeomorphisim. In their original paper, L. Coghlan and Y. Itokawa have also submitted this problem. This problem is still open now, and it causes the great interest to study the topology and geometry of the manifolds with $0 < K_M \le 1$. And we have some related results about this subject (see [34–36]).

Here, we introduce the Hausdorff convergence to discuss a differentiable sphere theorem with perturbed geometry conditions based on the ones on the standard S^n and finally derive a rigidity phenomena on this kind of manifolds.

2 Preliminaries

According to Gromov's idea, if we have a sequence of manifolds satisfying some geometric conditions, we can think about the limit of them in the sense of Hausdorff convergence. In this section, we list some relevant definitions and preliminaries.

The first concept we need is the Hausdorff convergence of manifolds. Here we only give a rough introduction to this point and just list some important definitions and lemmas as the tools. One can refer to [14–18, 21–27] for their detailed explanations.

Definition 2.1 *A sequence of Riemannian n-manifolds $\{(M_i, g_i)\}$ is said to converge in the $C^{1,\alpha}$ topology to a $C^{1,\alpha}$ Riemannian manifold (M, g), denoted by $\lim\limits_{i\to\infty}(M_i, g_i) = (M, g)$, if M is a smooth manifold with a $C^{1,\alpha}$ metric tensor g, and there is a sequence of diffeomorphisms $F_i : M \to M_i$, for i sufficiently large, such that the pull back metrics $F_i^* g_i$ converge to g in the $C^{1,\alpha}$ topology on M.*

When one talks about the Hausdorff convergence and the properties of the limit manifolds, an essential tool — harmonic coordinate — is usually proposed. Here, we also list a rough sketch of the related definitions and theorems about the harmonic coordinate for the proceeding of our results. Let $0 < \alpha < 1$. For a function f on M, the scaling invariant $C^{1,\alpha}$-norm of f on $\mathbb{B}_r(x)$ is defined by

$$\|f\|^*_{C^{1,\alpha}} = \sup_{y\in\mathbb{B}_r(x)} |f(y)| + r \sup_{y\in\mathbb{B}_r(x)} \{|D^1 f(x)|\} + r^{1+\alpha} \sup_{y\in\mathbb{B}_r(x)} \Big\{\frac{|D^1 f(y_1) - D^1 f(y_2)|}{d(y_1, y_2)^\alpha}\Big\}.$$

Definition 2.2 *Let M be a manifold, $x \in M$. Given $Q \geq 1$, the $C^{1,\alpha}$ harmonic radius at x with respect to Q is the largest number $r_h(x) > 0$, such that the metric ball $\mathbb{B}_r(x)$ has a harmonic coordinate satisfying the following two conditions:*

(1) $Q^{-1}\delta_{ij} \leq (g_{ij}) \leq Q\delta_{ij}$ *as tensors;*

(2) $\|g_{ij}\|^*_{C^{1,\alpha}} \leq Q$.

The harmonic radius of M, denoted by $r_h(M)$, is the infimum of the harmonic radii at all points.

Obviously, the estimate of the lower bound of the harmonic radius of M is an important matter if one wants to use this special coordinate. Here, we introduce the following theorem without the proof, and one can refer to [21].

Lemma 2.1 *Let $\alpha \in (0,1)$, $Q > 1$ and (M,g) be a smooth n-dimensional Riemannian manifold. Suppose that for some positive numbers Λ and i, we have*

$$|\mathrm{Ric}| \leq \Lambda \quad \textit{and} \quad \mathrm{Inj}(M) \geq i.$$

Then there exists a positive constant $C = C(n, Q, \alpha, i, \Lambda)$ such that the harmonic radius of M satisfies $r_h(M) \geq C$.

We also point out that the harmonic coordinate is in some sense almost the best one among all kinds of coordinates, and it simplifies some expressions on the manifold such as the curvature, Laplacian, etc. For example, under the harmonic coordinate, the Ricci curvature and the Laplacian can take respectively the following forms:

$$\Delta = g^{ij}\frac{\partial^2}{\partial u^i \partial u^j}, \quad -\mathrm{Ric}_{ij} = \frac{1}{2}\Delta g_{ij} + Q(g, \partial g),$$

where Q is some universal analytic expression that is a polynomial in the matrix g, quadratic in ∂g and has a denominator term depending on $\sqrt{\det g_{ij}}$.

The following lemma indicates that the $C^{1,\alpha}$ harmonic radius is continuous in the $C^{1,\alpha}$ topology.

Lemma 2.2 *Let $\{(M_i, g_i)\}$ be a sequence of Riemannian manifolds which converge strongly in the $C^{1,\alpha}$ topology to a limit $C^{1,\alpha}$ manifold (M, g). Then*

$$r_h(M) = \lim_{i\to\infty} r_h(M_i).$$

Proof One can refer to [26, Chapter 10] or proceed just as [23, Proposition 1.1], which gives the continuous property of the $L^{1,p}$ harmonic radius in the $L^{1,p}$ topology.

In the sequel, we cite an essential convergence theorem of the manifolds (see [22, Theorem 1.1]).

Lemma 2.3 *Given three positive numbers λ, i, D. Let $\{(M_i, g_i)\}$ be a sequence of manifolds satisfying*

$$|\mathrm{Ric}| \leq \lambda, \quad \mathrm{Inj} \geq i, \quad \mathrm{diam} \leq D.$$

Then there is a subsequence of $\{(M_i, g_i)\}$, also denoted by $\{(M_i, g_i)\}$, converging in the $C^{1,\alpha}$ topology to a $C^{1,\alpha}$ Riemannian manifold (M, g).

Proposition 2.1 *A sequence of closed Riemannian n-manifolds $\{(M_i, g_i)\}$ converges in the $C^{1,\alpha}$ topology to a $C^{1,\alpha}$ Riemannian manifold (M, g). We identify M_i with M via diffeomorphism F_i. Denote the distance function to $p \in M$ by ρ_i and ρ respectively under the metric g_i and g. Then for any smooth functions $f \in C^\infty(\mathbb{R})$, $\phi \in C^\infty(M)$, we have*

$$\lim_{i\to\infty}\int_M f(\rho_i)\Delta_i\phi \mathrm{d}\,\mathrm{vol}_{g_i} = \int_M f(\rho)\Delta\phi \mathrm{d}\,\mathrm{vol}_g,$$

where Δ_i and Δ respectively denote the Laplacian of the metric g_i and g.

Proof It is just similar to [24, Proposition 2.5]. But we here list the outline. Firstly, following the discussions in Lemmas 2.1 and 2.2 above about the harmonic radius and its continuity, we know that on the limit manifold (M, g) one can also talk about the Laplacian with the simple form. Now, the following two points then ensure the proof of our proposition.

Fact 1 Under the assumptions of the proposition, ρ_i will converge to ρ uniformly.

Fact 2 $\Delta_i\phi$ will converge to $\Delta\phi$ in the C^α topology (see [26]) because of the existence of harmonic coordinate on (M, g) (see [25]) and the simple expression of the Laplacian $\Delta = g^{ij}\frac{\partial^2}{\partial u^i \partial u^j}$ under the harmonic coordinate.

3 The Rigidity Theorem

Before coming to our differentiable sphere theorem, we first list the following preliminaries.

Definition 3.1 *A smooth manifold M^n is called a twisted sphere if there are smooth embeddings $h_1, h_2 : \mathbb{B}_{1+\epsilon} \to M^n$ such that*

$$h_1(\overline{\mathbb{B}_1}) \cup h_2(\overline{\mathbb{B}_1}) = M^n, \quad h_1(\mathbb{B}_1) \cap h_2(\mathbb{B}_1) = \emptyset,$$

where $\mathbb{B}_r \subset \mathbb{R}^n$ denotes the open ball of radius r and $\overline{\mathbb{B}_1}$ denotes the closure of $\mathbb{B}_1$.

Definition 3.2 *Notations as above, denote the diffeomorphism $h_2^{-1} \circ h_1 : \partial\mathbb{B}_1 = S^{n-1} \to S^{n-1}$ by $f : S^{n-1} \to S^{n-1}$. f is said to be isotopic to a diffeomorphism $g : S^{n-1} \to S^{n-1}$, if there exists a smooth 1-parameter family of diffeomorphism F_t such that $F_1 = f$ and $F_0 = g$.*

With the above definitions, we have the following proposition to conclude the diffeomorphism between the manifold and the standard sphere S^n. The proof will be omitted and one can check it from [1].

Lemma 3.1 *Notations as above, let M^n be a twisted sphere. If f is isotopic to the identity* Id*, then M^n is diffeomorphic to the standard sphere S^n. Especially, M^n will be diffeomorphic to the standard sphere S^n if $f =$* Id.

We in the sequel give a geometrical element which is zero on the unit standard sphere S^n.

Definition 3.3 *We define the π-excess of a manifold M, denoted by $e_\pi(M)$, to be $e_\pi(M) = \sup\limits_{x\in M}\sup\limits_{d(p,q)=\pi}(d(p,x) + d(q,x) - d(p,q))$.*

Now we state the main rigidity theorem.

Theorem 3.1 *For any positive integer n, there exists a positive number η depending only on n such that if (M^{2n}, g) is a $2n$-dimensional compact, simply connected Riemannian manifold without boundary, with the metric g and satisfies*

$$0 < K_M \leq 1, \quad e_\pi(M) \leq \eta,$$

then M^{2n} is diffeomorphic to S^{2n}.

Proof We prove this theorem by contradiction. If not, there exists a decreasing positive sequence $\{\eta_i\}$ satisfying that $\eta_i \to 0$ as $i \to \infty$ and a sequence of $2n$-dimensional Riemannian manifolds $\{(M_i, g_i)\}$ exists which satisfies, for each i,

$$0 < K_{M_i} \leq 1, \quad e_\pi(M_i) \leq \eta_i,$$

and M_i is not diffeomorphic to S^{2n}.

Firstly, we conclude that the sequence $\{(M_i, g_i)\}$ satisfies the conditions in Lemma 2.3.

As we all know, the injectivity radius of each M_i satisfies that $\mathrm{Inj}(M_i) \geq \pi$. Now what we need to do is to give a uniform upper bound of the diameter. We get this by showing that

$$\limsup_{i\to\infty} \mathrm{diam}(M_i) = \pi, \tag{3.1}$$

where $\mathrm{diam}(M_i)$ is the diameter of M_i with respect to the metric g_i. For that, let p_i, q_i be some points of M_i with $d(p_i, q_i) = \mathrm{diam}(M_i)$, where $d(p_i, q_i)$ is the distance between points p_i and q_i with respect to g_i of M_i. If $\limsup\limits_{i\to\infty} \mathrm{diam}(M_i) > \pi + \epsilon$, without loss of generality, we assume that $\lim\limits_{i\to\infty} \mathrm{diam}(M_i) > \pi + \epsilon$. We then easily choose a point w_i between p_i and q_i for large i such that $d(p_i, w_i) = \pi$. Then $\eta_i \geq e_\pi(M_i) \geq d(p_i, q_i) + d(w_i, q_i) - d(w_i, p_i) \geq 2\epsilon$. Let $i \to \infty$. We then get the contradiction and (3.1) is then valid for us to conclude the existence of the uniform upper bound of the diameter of M_i.

From the discussion above, taking advantage of Lemma 2.3, a subsequence of $\{(M_i, g_i)\}$ can be chosen, also denoted by $\{(M_i, g_i)\}$, such that $\lim\limits_{j\to\infty} (M_j, g_j) = (M, g)$ in the $C^{1,\alpha}$ topology and the limit smooth manifold M carries the properties described in the lemmas in Section 2. Especially, the limit metric satisfies that $g \in C^{1,\alpha}$. In the following, we come to draw out the limit manifold M with the limit metric g.

It is easy to know from the convergence of the metric that (M, g) has diameter $\mathrm{diam}(M) = \pi$, $e_\pi(M) = 0$. Also, according to [29], we know that $\mathrm{Inj}(M) \geq \limsup\limits_{j\to\infty} \mathrm{Inj}(M_j)$ if the convergence is in the C^1 topology. So the injectivity radius of M satisfies that $\mathrm{Inj}(M) = \pi$.

In the following, we set out to prove that the limit metric g is not only in $C^{1,\alpha}$, but actually smooth. We describe this in three steps.

Step 1 The equation that the distance function on M satisfies.

Fixed a point p on M, it is obviously from the geometry of M that one can find a point q such that $d(p, q) = \pi$. Let $x \in M$ be any point and denote the distance function from p and q respectively by $\rho_p(x)$ and $\rho_q(x)$. We conclude from $e_\pi(M) = 0$ that

$$\rho_p(x) + \rho_q(x) = \pi$$

for any $x \in M$.

We now set out to give out the equation which the function $\rho_p(x)$ satisfies.

Let $\phi \geq 0$ be a smooth function on M. From the discussion about harmonic coordinates and harmonic radius on M, the simple expression of Laplacian and Proposition 2.1, identifying M_i with M via diffeomorphism F_i, we can deduce together with the curvature conditions that

$$\begin{aligned}\int_M \rho_p \Delta\phi \mathrm{d}\,\mathrm{vol}_g &= \lim_{j\to\infty}\int_M (\rho_j)_p \Delta_j \phi \mathrm{d}\,\mathrm{vol}_{g_j} = \lim_{j\to\infty}\int_M \phi\Delta_j(\rho_j)_p \mathrm{d}\,\mathrm{vol}_{g_j} \\ &\geq \lim_{j\to\infty}(2n-1)\int_M \phi\frac{\cos(\rho_j)_p}{\sin(\rho_j)_p}\mathrm{d}\,\mathrm{vol}_{g_j} = (2n-1)\int_M \phi\frac{\cos\rho_p}{\sin\rho_p}\mathrm{d}\,\mathrm{vol}_g.\end{aligned}$$

Similarly,

$$\int_M \rho_q \Delta\phi \mathrm{d}\,\mathrm{vol}_g \geq (2n-1)\int_M \phi\frac{\cos\rho_q}{\sin\rho_q}\mathrm{d}\,\mathrm{vol}_g.$$

However, we already have $\rho_p(x) + \rho_q(x) = \pi$. Hence,

$$\begin{aligned}\int_M \rho_p \Delta\phi \mathrm{d}\,\mathrm{vol}_g &= -\int_M \rho_q \Delta\phi \mathrm{d}\,\mathrm{vol}_g \leq -(2n-1)\int_M \phi\frac{\cos\rho_q}{\sin\rho_q}\mathrm{d}\,\mathrm{vol}_g \\ &= -(2n-1)\int_M \phi\frac{\cos(\pi-\rho_p)}{\sin(\pi-\rho_p)}\mathrm{d}\,\mathrm{vol}_g = (2n-1)\int_M \phi\frac{\cos\rho_p}{\sin\rho_p}\mathrm{d}\,\mathrm{vol}_g.\end{aligned}$$

Therefore,

$$\int_M \rho_p \Delta\phi \mathrm{d}\,\mathrm{vol}_g = (2n-1)\int_M \phi\frac{\cos\rho_p}{\sin\rho_p}\mathrm{d}\,\mathrm{vol}_g.$$

This means that

$$\Delta\rho_p = (2n-1)\frac{\cos\rho_p}{\sin\rho_p}$$

weakly. This is just the equation we want to derive in this step.

Step 2 The expression of g in terms of the distance function ρ_p.

Following M. Anderson and J. Cheeger [23], for any $x \in M$, choose an orthonormal basis e_i of T_xM and consider the following $n(2n+1)$ unit vectors

$$e_i,\ \frac{\sqrt{2}}{2}(e_i + e_j), \quad i, j = 1, 2, \cdots, 2n \text{ and } i < j.$$

Name these vectors as $v_1, \cdots, v_{n(2n+1)}$. Let γ_m, $m = 1, \cdots, n(2n+1)$ be the minimal geodesics from some points nearby x, denoted by p^m, passing through x with velocity v_m. Denote $\rho^m(x)$ to be the distance functions from p^m. Since $|\nabla\rho^m(x)| = 1$, we have the following system of $n(2n+1)$ equations

$$g^{kl}\frac{\partial\rho^m}{\partial u^k}\frac{\partial\rho^m}{\partial u^l} = 1, \quad m = 1, \cdots, n(2n+1).$$

In order to continue, we need a simple observation of the knowledge in higher algebra stated in the following proposition.

Proposition 3.1 *e_i $(i = 1, \cdots, n)$ is an orthonormal basis of an n-dimensional Hilbert space. We construct $\frac{n(n+1)}{2}$ unit vectors*

$$e_i,\ \frac{\sqrt{2}}{2}(e_i + e_j), \quad i, j = 1, 2, \cdots, n \text{ and } i < j$$

and denote them by $v_1, \cdots, v_{\frac{n(n+1)}{2}}$*. Let* v_i *be coordinated under some orthonormal basis by* $v_i = \{b^i_1, b^i_2, \cdots, b^i_n\}$*. Then the matrix*

$$A = (b^i_k b^i_l)_{\frac{n(n+1)}{2} \times n^2}$$

has rank $\frac{n(n+1)}{2}$*.*

Proof For simplicity, we denote

$$e_i = e_{ii} = \{b^1_{ii}, b^2_{ii}, \cdots, b^n_{ii}\}, \quad \frac{\sqrt{2}}{2}(e_i + e_j) = e_{ij} = \{b^1_{ij}, b^2_{ij}, \cdots, b^n_{ij}\}, \quad i < j$$

and denote by A the following block matrix

$$A = \begin{pmatrix} b^1_{11}e_{11} & b^2_{11}e_{11} & \cdots & b^n_{11}e_{11} \\ b^1_{12}e_{12} & b^2_{12}e_{12} & \cdots & b^n_{12}e_{12} \\ \vdots & \vdots & & \vdots \\ b^1_{ij}e_{ij} & b^2_{ij}e_{ij} & \cdots & b^n_{ij}e_{ij} \\ \vdots & \vdots & & \vdots \\ b^1_{nn}e_{nn} & b^2_{nn}e_{nn} & \cdots & b^n_{nn}e_{nn} \end{pmatrix}.$$

What we need to do is to show that the row vectors of A are linearly independent. For this, denote the row with subscript ij by β_{ij} and assume that

$$k_{11}\beta_{11} + k_{12}\beta_{12} + \cdots + k_{ij}\beta_{11} + \cdots + k_{nn}\beta_{nn} = 0.$$

This equality immediately means that

$$\begin{cases} k_{11}b^1_{11}e_{11} + k_{12}b^1_{12}e_{12} + \cdots + k_{nn}b^1_{nn}e_{nn} = 0, \\ k_{11}b^2_{11}e_{11} + k_{12}b^2_{12}e_{12} + \cdots + k_{nn}b^2_{nn}e_{nn} = 0, \\ \cdots \\ k_{11}b^l_{11}e_{11} + k_{12}b^l_{12}e_{12} + \cdots + k_{nn}b^l_{nn}e_{nn} = 0, \\ \cdots \\ k_{11}b^n_{11}e_{11} + k_{12}b^n_{12}e_{12} + \cdots + k_{nn}b^n_{nn}e_{nn} = 0. \end{cases}$$

Taking the inner product with e_1 for each equation above, we then derive from the orthonormal property that

$$\begin{cases} k_{11}b^1_{11} + \frac{\sqrt{2}}{2}k_{12}b^1_{12} + \frac{\sqrt{2}}{2}k_{13}b^1_{13} + \cdots + \frac{\sqrt{2}}{2}k_{1n}b^1_{1n} = 0, \\ k_{11}b^2_{11} + \frac{\sqrt{2}}{2}k_{12}b^2_{12} + \frac{\sqrt{2}}{2}k_{13}b^2_{13} + \cdots + \frac{\sqrt{2}}{2}k_{1n}b^2_{1n} = 0, \\ \cdots \\ k_{11}b^l_{11} + \frac{\sqrt{2}}{2}k_{12}b^l_{12} + \frac{\sqrt{2}}{2}k_{13}b^l_{13} + \cdots + \frac{\sqrt{2}}{2}k_{1n}b^l_{1n} = 0, \\ \cdots \\ k_{11}b^n_{11} + \frac{\sqrt{2}}{2}k_{12}b^n_{12} + \frac{\sqrt{2}}{2}k_{13}b^n_{13} + \cdots + \frac{\sqrt{2}}{2}k_{1n}b^n_{1n} = 0, \end{cases}$$

which means that

$$k_{11}e_{11} + \frac{\sqrt{2}}{2}k_{12}e_{12} + \cdots + \frac{\sqrt{2}}{2}k_{1n}e_{1n} = 0$$

and then shows that

$$k_{11}e_{11} + \frac{1}{2}k_{12}(e_{11}+e_{22}) + \cdots + \frac{1}{2}k_{1n}(e_{11}+e_{nn}) = 0.$$

For e_{ii} $(i = 1, 2, \cdots, n)$ is a basis, one can easily deduce that $k_{1j} = 0$. The same procedure then shows that $k_{ij} = 0$ and the proposition now is proved.

Going on with the equations $g^{kl}\frac{\partial\rho^m}{\partial u^k}\frac{\partial\rho^m}{\partial u^l} = 1$, $m = 1, \cdots, n(2n+1)$, $k, l = 1, \cdots, 2n$, one views this as a system of linear equation with g^{kl} as unknown and $(\frac{\partial\rho^m}{\partial u^k}\frac{\partial\rho^m}{\partial u^l})$ as coefficient. According to Proposition 3.1, at the point x, the coefficient matrix has rank $n(2n+1)$. Then after elementary operations, we will derive that the following system

$$\sum_{k\le l} g^{kl}\omega_{kl}\frac{\partial\rho^m}{\partial u^k}\frac{\partial\rho^m}{\partial u^l} = 1, \quad m = 1, \cdots, n(2n+1),\ k, l = 1, \cdots, 2n$$

has nonsingular coefficient matrix, where $\omega_{kl} = 1$ for $k = l$ and $\omega_{kl} = 2$ for $k < l$. By continuity, the coefficient matrix is nonsingular in a neighborhood of x. Hence one can solve g^{kl} in terms of $\frac{\partial\rho^m}{\partial u^k}$. This gives the expression of g in terms of the distance function ρ_p.

Step 3 The smoothness of the metric g.

Together with Step 1, we have the weak equation $g^{kl}\frac{\partial^2\rho_p}{\partial u^k\partial u^l} - (2n-1)\frac{\cos\rho_p}{\sin\rho_p} = 0$, and the regularity theory (see [30]) immediately shows that $\rho_p \in C^{3,\alpha}$ since $g^{kl} \in C^{1,\alpha}$. Turn to Step 2, we then get that $g^{kl} \in C^{2,\alpha}$ since $\rho_p \in C^{3,\alpha}$. Continuing this bootstrap argument shows that g is actually smooth.

After the three steps above, we conclude that the limit manifold (M, g) is actually a smooth Riemannian manifold with injectivity radius $\mathrm{Inj}(M) = \pi = \mathrm{diam}(M)$ and $e_\pi(M) = 0$. In the following, we come to prove that (M, g) is diffeomorphic to S^{2n}.

Fix two points $p, q \in M$ with $d(p, q) = \pi$. From the fact that the injectivity radius of M is π, we know that the exponential maps $\exp_p, \exp_q : \mathbb{B}_{\frac{2\pi}{3}} \to M$ are both smooth embeddings. Here, $\mathbb{B}_{\frac{2\pi}{3}} \subset \mathbb{R}^{2n}$ is the standard ball with radius $\frac{2\pi}{3}$. Also, one can easily deduce that $\exp_p(\mathbb{B}_{\frac{\pi}{2}}) \cap \exp_q(\mathbb{B}_{\frac{\pi}{2}}) = \emptyset$ and $\exp_p(\overline{\mathbb{B}_{\frac{\pi}{2}}}) \cup \exp_q(\overline{\mathbb{B}_{\frac{\pi}{2}}}) = M$, which states that M is actually a twisted sphere. Also, from $e_\pi(M) = 0$ and $\mathrm{Inj}(M) = \pi$, one can easily derive that $\partial B_\pi(p) = \{q\}$. $\forall\, x \in \partial B_{\frac{\pi}{2}}(q)$, from the fact above, there exists one geodesic $\gamma_{p,x}$ joining p with x and one $\gamma_{q,x}$ joining q with x which both have length $\frac{\pi}{2}$. However, $d(p, q) = \pi$ means that the two geodesics together form a geodesic, denoted by $\gamma_{p,x,q}$, joining p with q. And Toponogov theorem then assures that this is the unique geodesic joining p and q passing through x. So we can adapt ahead the coordinates at T_pM and T_qM such that

$$\exp_p\Big(\frac{\pi}{2}, \theta\Big) = \exp_q\Big(\frac{\pi}{2}, \theta\Big), \quad \forall\, \theta \in S^{2n-1}.$$

Then it is easy to show that

$$\exp_p^{-1} \circ \exp_q|_{\partial\mathbb{B}_{\frac{\pi}{2}}} = \mathrm{Id} : \partial\mathbb{B}_{\frac{\pi}{2}} \to \partial\mathbb{B}_{\frac{\pi}{2}}$$

Then we claim that M is diffeomorphic to S^{2n} according to Lemma 3.1. Therefore, the elements of a subsequence of $\{M_i\}$ are all diffeomorphic to S^{2n}, which contradicts our construction of $\{M_i\}$. Thus we complete the proof of the rigidity theorem.

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