Chin. Ann. Math.
31B(6), 2010, 879–920
DOI: 10.1007/s11401-010-0613-4

Chinese Annals of Mathematics, Series B

Boundary Shape Control of the Navier-Stokes Equations and Applications***

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(*Dedicated to Professor Roger Temam on the Occasion of his* 70*th Birthday*)

Abstract In this paper, the geometrical design for the blade's surface $\Im$ in an impeller or for the profile of an aircraft, is modeled from the mathematical point of view by a boundary shape control problem for the Navier-Stokes equations. The objective function is the sum of a global dissipative function and the power of the fluid. The control variables are the geometry of the boundary and the state equations are the Navier-Stokes equations. The Euler-Lagrange equations of the optimal control problem are derived, which are an elliptic boundary value system of fourth order, coupled with the Navier-Stokes equations. The authors also prove the existence of the solution of the optimal control problem, the existence of the solution of the Navier-Stokes equations with mixed boundary conditions, the weak continuity of the solution of the Navier-Stokes equations with respect to the geometry shape of the blade's surface and the existence of solutions of the equations for the Gâteaux derivative of the solution of the Navier-Stokes equations with respect to the geometry of the boundary.

Keywords Blade, Boundary shape control, General minimal surface, Navier-Stokes equations, Euler-Lagrange equations
2000 MR Subject Classification 65N30, 76U05, 76M05

1 Introduction

Blade's shape design for impellers is driven by the need of improving performances and reliability. We are interested in the geometric design entirely from the mathematical point of view. As it is well-known that the blade's surface is a part of the boundary of the flow's channel in the impeller, the mathematical theory and methods of the boundary shape control problem for the Navier-Stokes equations can be used to design the blades and profile of, for example, airfoil etc. This idea is motivated by the classical minimal surface problem which accounts to find a surface $\Im$ spanning on a closed Jordan curve C and such that $J(\Im)=\inf\limits_{S\in\mathcal{F}}J(S)$, where $J(S)=\iint_S \mathrm{d}S$ is the area of S.

Manuscript received April 12, 2010. Published online October 22, 2010.
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***Project supported by the National High-Tech Research and Development Program of China (No. 2009AA01A135), the National Natural Science Foundation of China (Nos. 10926080, 10971165, 10871156) and Xian Jiaotong University (No. XJJ2008033).

In this paper, we attempt to set the principle for a fully mathematical design of the surface for the blade in an impeller. This principle models a general minimal surface by minimizing a functional proposed in this paper. A key point in this modeling process is the theoretical rationality and the realizability of our design procedure. Using tensor analysis we realize this procedure and obtain the Euler-Lagrange equations for the blade's surface which is an elliptic boundary value system coupled with the Navier-Stokes equations and with the linearized Navier-Stokes equations. We prove the existence of solutions of the control problem, the existence of solutions for the N-S equations with mixed boundary condition, and prove uniform weak continuity of the solution with respect to the surface $\Im$ of the blade.

This paper is organized as follows. In Section 2, we give some preliminary results. In Section 3, we derive the rotating Navier-Stokes equations in the channel inside the impeller with mixed boundary conditions under a new coordinate system, prove the uniform positiveness of the bilinear form and uniform continuity of the trilinear form. In Section 4, we prove the existence of solution of the Navier-stokes equations with mixed boundary conditions. In Section 5, we derive the equations for the Gâteaux derivative of the solution of the Navier-Stokes equation and prove the existence of its solution. In Section 6, we present the objective functional and derive the Euler-Lagrange equations. In Section 7, we prove the existence of the solution of the optimal control problem.

2 Preliminary Results — The Geometry of the Blade's Surface

Let $(x^1, x^2) \in D \subset E^2$ ($2D$-Euclidian Space), and let (r, θ, z) denote a polar cylindrical coordinate system rotating with the impeller's angular velocity ω.

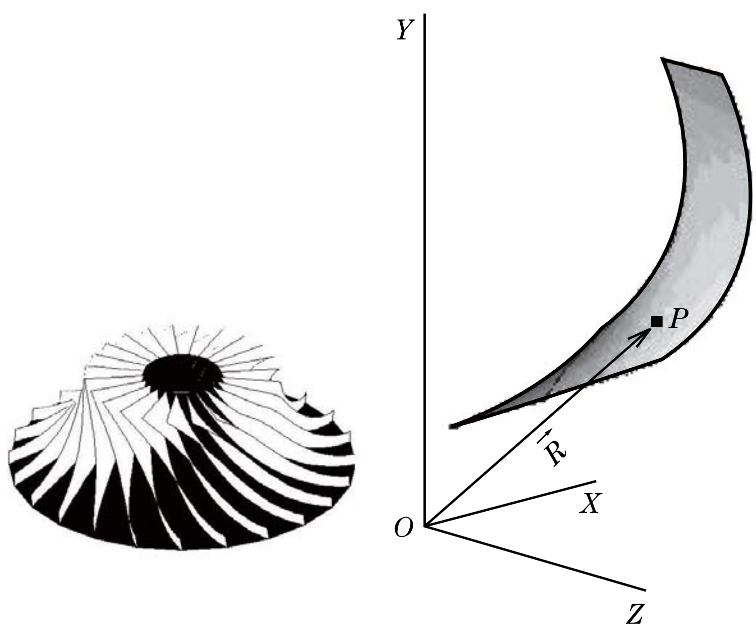

Figure 1 Impeller and blade

$(\vec{e}_r, \vec{e}_\theta, \vec{k})$ is the corresponding base vectors, here z-axis being the rotating axis of the impeller, N the number of blade and $\varepsilon = \frac{\pi}{N}$. The angle between two successive blades is $\frac{2\pi}{N}$. The flow passage of the impeller is bounded by $\partial\Omega_\varepsilon = \Gamma_{\text{in}} \cup \Gamma_{\text{out}} \cup \Gamma_t \cup \Gamma_b \cup \Im_+ \cup \Im_-$. The middle surface $\Im$ of the blade is defined as the immersion $\overrightarrow{\Re}$ of the closure of a domain $D \subset R^2$, where $\overrightarrow{\Re} : D \to \Re^3$ is a smooth injective mapping which can be expressed by that for any point $\overrightarrow{\Re}(x) \in \Im$ by

$$\overrightarrow{\Re}(x) = x^2 \overrightarrow{e}_r + x^2 \Theta(x^1, x^2) \overrightarrow{e}_\theta + x^1 \overrightarrow{k}, \quad \forall x = (x^1, x^2) \in \overline{D}, \tag{2.1}$$

where $\Theta \in C^3(D, R)$ is a smooth function; $x = (x^1, x^2)$ is called a Gaussian coordinate system on $\Im$. It is easy to prove that there exists a family $\Im_\xi$ of surfaces with a single parameter ξ to cover the domain Ω_ε defined by the mapping $D \to \Im_\xi = \{\overrightarrow{R}(x^1, x^2; \xi) : \forall (x^1, x^2) \in D\}$, $-1 < \xi < 1$:

$$\overrightarrow{R}(x^1, x^2; \xi) = x^2 \overrightarrow{e}_r + x^2(\varepsilon\xi + \Theta(x^1, x^2)) \overrightarrow{e}_\theta + x^1 \overrightarrow{k}. \tag{2.2}$$

It is clear that the metric tensor $a_{\alpha\beta}$ of $\Im_\xi$ is homogenous and nonsingular independent of ξ, and is given as follows:

$$\begin{aligned} &a_{\alpha\beta} = \frac{\partial \overrightarrow{R}}{\partial x^\alpha} \frac{\partial \overrightarrow{R}}{\partial x^\beta} = \delta_{\alpha\beta} + r^2 \Theta_\alpha \Theta_\beta, \quad a = \det(a_{\alpha\beta}) = 1 + r^2(\Theta_1^2 + \Theta_2^2) > 0, \\ &a^{\alpha\beta} a_{\beta\lambda} = \delta^\alpha_\lambda. \end{aligned} \tag{2.3}$$

From this, a curvilinear coordinate system (x^1, x^2, ξ) in $\Re^3$ is established,

$$(r, \theta, z) \to (x^1, x^2, \xi) : \quad x^1 = z, \quad x^2 = r, \quad \xi = \varepsilon^{-1}(\theta - \Theta(x^1, x^2)). \tag{2.4}$$

Under this special coordinate system, the flow passage domain

$$\Omega_\varepsilon = \{\overrightarrow{R}(x^1, x^2, \xi) = x^2 \overrightarrow{e}_r + x^2(\varepsilon\xi + \Theta(x^1, x^2)) \overrightarrow{e}_\theta + x^1 \overrightarrow{k}, \ \forall (x^1, x^2, \xi) \in \Omega\}$$

is mapped into a fixed domain in E^3 (3D Euclidian Space):

$$\Omega = \{(x^1, x^2) \in D, -1 \le \xi \le 1\}, \quad \text{in } \Re^3,$$

which is independent of the surface $\Im$ of the blade. The Jacobian of the coordinate transformation is $J\big(\frac{\partial(r,\theta,z)}{\partial(x^1,x^2,\xi)}\big) = \varepsilon$. This shows that the transformation $\{r, \theta, z\} \to \{x^1, x^2, \xi\}$ is nonsingular.

Let $(x^{1'}, x^{2'}, x^{3'}) = (r, \theta, z)$. Corresponding metric tensor of E^3 in the cylindrical coordinate (r, θ, z) is denoted by $g_{1'1'} = 1,\ g_{2'2'} = r^2,\ g_{3'3'} = 1,\ g_{i'j'} = 0\ (\forall\, i' \ne j')$. According to the rules of tensor transformation under coordinate transformation, we have the following transformation formulae

$$g_{ij} = g_{i'j'} \frac{\partial x^{i'}}{\partial x^i} \frac{\partial x^{j'}}{\partial x^j}.$$

Substituting (2.3) into the above formula, the covariant and contra-variant components of the metric tensor of E^3 in the new curvilinear coordinate system are given by

$$\begin{aligned} &g_{\alpha\beta} = a_{\alpha\beta}, \quad g_{3\beta} = g_{\beta 3} = \varepsilon r^2 \Theta_\beta, \quad g_{33} = \varepsilon^2 r^2, \quad g = \det(g_{ij}) = \varepsilon^2 r^2, \\ &g^{\alpha\beta} = \delta^{\alpha\beta}, \quad g^{3\beta} = g^{\beta 3} = -\varepsilon^{-1}\Theta_\beta, \quad g^{33} = \varepsilon^{-2} r^{-2}(1 + r^2|\nabla\Theta|^2) = (r\varepsilon)^{-2} a, \end{aligned} \tag{2.5}$$

where the notations $|\nabla\Theta|^2 = \Theta_1^2 + \Theta_2^2$, and $\Theta_\alpha = \frac{\partial\Theta}{\partial x^\alpha}$ will be frequently used through out this paper.

Tensor calculations show then the following result (see [20]).

Proposition 2.1 *In the new coordinate system* (x^α, ξ), *let* $(\overrightarrow{e}_i,\ i = 1, 2, 3)$ *denote the base vectors, and a vector* $\overrightarrow{v}$ *in* $\Re^3$ *is expressed as* $\overrightarrow{v} = v^i \overrightarrow{e}_i$, *where* $v^i, v_i = g_{ij} v^j$ *are respectively called the contra-variant and covariant components of the vector* $\overrightarrow{v}$. *In the new coordinate system* (x^α, ξ), *the following formulae are valid:*

(1) *Angular velocity vector* $\overrightarrow{\omega}$

$$\begin{aligned} &\overrightarrow{\omega} = \omega \overrightarrow{e}_1 - \omega\varepsilon^{-1}\Theta_1 \overrightarrow{e}_3, \\ &\omega^1 = \omega, \quad \omega^2 = 0, \quad \omega^3 = -\omega\varepsilon^{-1}\Theta_1; \end{aligned} \tag{2.6}$$

(2) *Coriolis Force*

$$\begin{aligned} &\boldsymbol{C} = 2\overrightarrow{\omega} \times \overrightarrow{w} = -2\omega r \Pi(w,\Theta)\overrightarrow{e}_2 + 2\omega\varepsilon^{-1}\Big(r\Theta_2\Pi(w,\Theta) + \frac{w^2}{r}\Big)\overrightarrow{e}_3, \\ &C^1 = 2(\overrightarrow{\omega} \times \overrightarrow{w})^1 = 0, \quad C^2 = 2(\overrightarrow{\omega} \times \overrightarrow{w})^2 = -2\omega r \Pi(w,\Theta), \\ &C^3 = 2(\overrightarrow{\omega} \times \overrightarrow{w})^3 = 2\omega\varepsilon^{-1}\Big(r\Theta_2\Pi(w,\Theta) + \frac{w^2}{r}\Big), \\ &\Pi(w,\Theta) := \varepsilon w^3 + w^\lambda \Theta_\lambda; \end{aligned} \tag{2.7}$$

(3) *Unit normal vector to* $\Im$

$$\begin{aligned} &\overrightarrow{n} = \frac{\overrightarrow{e}_1 \times \overrightarrow{e}_2}{|\overrightarrow{e}_1 \times \overrightarrow{e}_2|} = -\frac{x^2\Theta_\alpha}{\sqrt{a}}\overrightarrow{e}_\alpha + (\varepsilon r)^{-1}\sqrt{a}\,\overrightarrow{e}_3, \\ &n^\lambda = -r\frac{\Theta_\lambda}{\sqrt{a}}, \quad n^3 = (\varepsilon r)^{-1}\sqrt{a}; \end{aligned} \tag{2.8}$$

(4) *Second fundamental form* (*curvature tensors for a* 2*D manifold*)

$$b_{\alpha\beta} = \frac{1}{2}r^2(2\Theta_2\Theta_\alpha\Theta_\beta + \varepsilon^{-1}\Theta_\sigma(\Theta_\alpha\Theta_{\sigma\beta} + \Theta_\beta\Theta_{\sigma\alpha}))\frac{1}{\sqrt{a}}; \tag{2.9}$$

(5) *Mean Curvature* H *and Gaussian Curvature* K.
Let

$$|\nabla\Theta|^2 = \Theta_1^2 + \Theta_2^2, \quad \|\nabla\Theta\|^2 = a^{\alpha\beta}\Theta_\alpha\Theta_\beta.$$

Then

$$\begin{aligned} &2H = \frac{1}{2}r^2(\Theta_2\|\nabla\Theta\|^2 + \varepsilon^{-1}a^{\alpha\beta}\Theta_\sigma\Theta_\alpha\Theta_{\sigma\beta})\frac{1}{\sqrt{a}}, \\ &K = \frac{b}{a} = \frac{\det(b_{\alpha\beta})}{a}. \end{aligned} \tag{2.10}$$

It is obvious that $\xi =$ constant corresponds to a surface $\Im_\xi$ which has the same geometry as $\Im$. This is based on the fundamental theorem in differential geometry, as it is well-known that the geometry of $\Im$ is completely determined by $(a_{\alpha\beta})$, $(b_{\alpha\beta})$ in the following sense. We recall that $\mathcal{O}^3$ denotes the set of all orthogonal matrices Q of order three; and that $\mathcal{O}^3_+ = \{Q \in \mathcal{O}^3; \det(Q) = 1\}$ denotes the set of all proper orthogonal matrices of order three. $\mathbf{J}_+(x) = \mathbf{c} + Q \circ \mathbf{x}$ is a proper isometry of $\mathbf{E}^3 : \mathbf{E}^3 \to \mathbf{E}^3$ with $\mathbf{c} \in \mathbf{E}^3$, $Q \in \mathcal{O}^3_+$.

Theorem 2.1 (see [3]) *Two immersions $\overrightarrow{R} \in C^1(D;\boldsymbol{E}^3)$ and $\widetilde{\overrightarrow{R}} \in C^1(D;\boldsymbol{E}^3)$ share the same fundamental forms $(a_{\alpha\beta})$ and $(b_{\alpha\beta})$ over an open connected subset D of $\Re^3$ if and only if*

$$\widetilde{\overrightarrow{R}} = \boldsymbol{J}_+ \circ \overrightarrow{R}, \quad \textit{where } \boldsymbol{J}_+ \textit{ is a proper isometry of } \boldsymbol{E}^3. \tag{2.11}$$

Furthermore, If two matrice fields $(a_{\alpha\beta}) \in C^2(D;\mathcal{S}^2_>)$ and $(b_{\alpha\beta}) \in C^2(D;\mathcal{S}^2)$ satisfy the Gauss and Codazzi equations in D

$$\begin{aligned}\partial_\beta\Gamma_{\alpha\sigma,\tau} - \partial_\sigma\Gamma_{\alpha\beta,\tau} + \Gamma^\mu_{\alpha\beta}\Gamma_{\sigma\tau,\mu} - \Gamma^\mu_{\alpha\sigma}\Gamma_{\beta\tau,\mu} &= b_{\alpha\sigma}b_{\beta\tau} - b_{\alpha\beta}b_{\sigma\tau},\\ \partial_\beta b_{\alpha\sigma} - \partial_\sigma b_{\alpha\beta} + \Gamma^\mu_{\alpha\sigma}b_{\beta\mu} - \Gamma^\mu_{\alpha\beta}b_{\sigma\mu} &= 0,\end{aligned}$$

where

$$\begin{aligned}\Gamma_{\alpha\beta,\tau} &= \frac{1}{2}(\partial_\alpha a_{\alpha\tau} + \partial_\alpha a_{\beta\tau} - \partial_\tau a_{\alpha\beta}),\\ \Gamma^\sigma_{\alpha\beta} &= a^{\sigma\tau}\Gamma_{\alpha\beta,\tau}, \quad \text{where } (a^{\alpha\beta}) = (a_{\alpha\beta})^{-1},\end{aligned}$$

then there exists an immersion $\overrightarrow{R} \in C^3(D;\mathbf{E}^3)$ such that

$$a_{\alpha\beta} = \partial_\alpha\overrightarrow{R}\partial_\beta\overrightarrow{R}, \quad b_{\alpha\beta} = \partial^2_{\alpha\beta}\overrightarrow{R} \cdot \Big\{\frac{\partial_1\overrightarrow{R} \times \partial_2\overrightarrow{R}}{|\partial_1\overrightarrow{R} \times \partial_2\overrightarrow{R}|}\Big\}.$$

Because $\Im_\xi$ results from a rotation of angle $\xi\varepsilon$ of $\Im$. Theorem 2.1 can be applied to $\Im_\xi$, which means that $\forall\,\xi \in [-1,1]$, it has the same geometrical characteristics $a_{\alpha\beta}, b_{\alpha\beta}, K, H, \cdots$.

Subsequently, we will frequently employ the third fundamental tensor of $\Im$

$$c_{\alpha\beta} = a^{\lambda\sigma}b_{\alpha\lambda}b_{\beta\sigma}, \tag{2.12}$$

and its inverse matrix $(\widehat{c}^{\;\alpha\beta}) = (c_{\alpha\beta})^{-1}$, $(\widehat{b}^{\alpha\beta}) = (b_{\alpha\beta})^{-1}$ defined by

$$\widehat{b}^{\alpha\beta}b_{\beta\lambda} = \delta^\alpha_\lambda, \quad \widehat{c}^{\;\alpha\beta}c_{\beta\lambda} = \delta^\alpha_\lambda. \tag{2.13}$$

Furthermore, let us introduce the permutation tensors in Euclidean space E^3 and on the 2D manifold $\Im$,

$$\varepsilon_{ijk} = \begin{cases}\sqrt{g},\\ -\sqrt{g},\\ 0,\end{cases} \qquad \varepsilon_{ijk} = \begin{cases}\dfrac{1}{\sqrt{g}}, & (i,j,k): \text{even permutation of (1,2,3)},\\ -\dfrac{1}{\sqrt{g}}, & (i,j,k): \text{odd permutation of (1,2,3)},\\ 0, & \text{otherwise},\end{cases} \tag{2.14}$$

where $g = \det(g_{ij})$, and g_{ij} is metric tensor of $\Re^3$,

$$\varepsilon_{\alpha\beta} = \begin{cases}\sqrt{a},\\ -\sqrt{a},\\ 0,\end{cases} \qquad \varepsilon_{\alpha\beta} = \begin{cases}\dfrac{1}{\sqrt{a}}, & (\alpha,\beta): \text{even permutation of (1,2)},\\ -\dfrac{1}{\sqrt{a}}, & (\alpha,\beta): \text{odd permutation of (1,2)},\\ 0, & \text{otherwise}.\end{cases} \tag{2.15}$$

3 Rotating Navier-Stokes Equations with Mixed Boundary Conditions in Turbo-Machinery

At first, we consider the three-dimensional rotating Navier-Stokes equations in a frame rotating around the axis of a rotating impeller with an angular velocity ω:

$$\begin{aligned}&\frac{\partial\rho}{\partial t}+\mathrm{div}(\rho w)=0,\\&\rho a=\nabla\sigma+f,\\&\rho c_v\Big(\frac{\partial T}{\partial t}+w^j\nabla_jT\Big)-\mathrm{div}(\kappa\,\mathrm{grad}\,T)+p\mathrm{div}\,w-\Phi=h,\\&p=p(\rho,T),\end{aligned}\tag{3.1}$$

where ρ is the density of the fluid, w the velocity of the fluid, h the heat source, T the temperature, k the coefficient of heat conductivity, C_v the specific heat at constant volume, and μ the viscosity. Furthermore, the strain rate tensor, stress tensor,the dissipation function and viscous tensor are respectively given by:

$$\begin{aligned}e_{ij}(w)&=\frac{1}{2}(\nabla_iw_j+\nabla_jw_i),\quad i,j=1,2,3,\\e^{ij}(w)&=g^{ik}g^{jm}e_{km}(w)=\frac{1}{2}(\nabla^iw^j+\nabla^jw^i),\\\sigma^{ij}(w,p)&=A^{ijkm}e_{km}(w)-g^{ij}p,\quad\Phi=A^{ijkm}e_{ij}(w)e_{km}(w),\\A^{ijkm}&=\lambda g^{ij}g^{km}+\mu(g^{ik}g^{jm}+g^{im}g^{jk}),\quad\lambda=-\frac{2}{3}\mu,\end{aligned}\tag{3.2}$$

where g_{ij} and g^{ij} are the covariant and contra-variant components of the metric tensor of the three-dimensional Euclidean space in the curvilinear coordinates (x^1,x^2,ξ) defined by (2.4), respectively. Then the covariant derivatives of the velocity vector and the Christoffel symbols are

$$\nabla_iw^j=\frac{\partial w^j}{\partial x^i}+\Gamma^j_{ik}w^k;\quad\nabla_iw_j=\frac{\partial w_j}{\partial x^i}-\Gamma^k_{ij}w_k,\quad\Gamma^i_{jk}=g^{il}\Big(\frac{\partial g_{kl}}{\partial x^j}+\frac{\partial g_{jl}}{\partial x^k}-\frac{\partial g_{jk}}{\partial x^l}\Big).\tag{3.3}$$

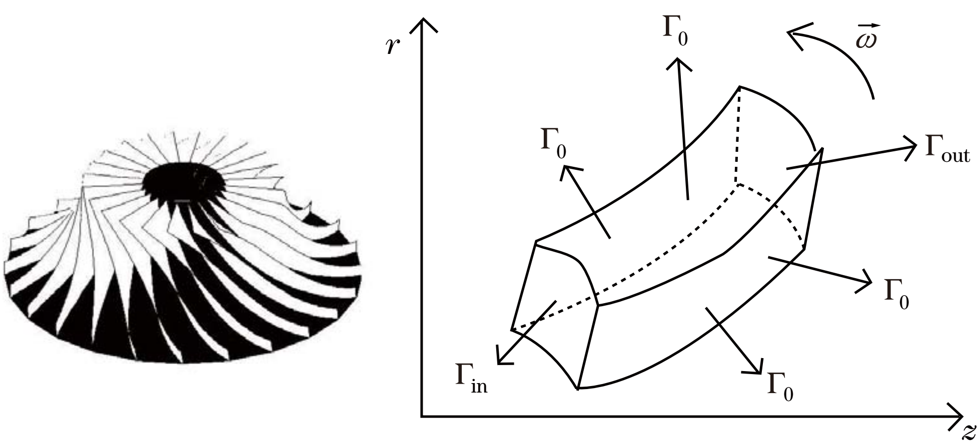

Figure 2 Impeller and passage of flow

The absolute acceleration of the fluid is given by

$$\begin{aligned} a^i &= \frac{\partial w^i}{\partial t} + w^j \nabla_j w^i + 2\varepsilon^{ijk}\omega_j w_k - \omega^2 r^i, \\ a &= \frac{\partial w}{\partial t} + (w\nabla)w + 2\vec{\omega} \times \vec{w} + \vec{\omega} \times (\vec{\omega} \times \vec{R}), \end{aligned} \tag{3.4}$$

where $\vec{\omega} = \omega \vec{k}$ is the vector of angular velocity, $\vec{k}$ the unit vector along the z axis, and $\vec{R}$ the radial vector of the fluid particle. The flow domain occupied by the fluid in the channel in the impeller is denoted by Ω_ε. The boundary $\partial\Omega_\varepsilon$ of the flow domain Ω_ε consists the inflow boundary $\Gamma_{\rm in}$, the outflow boundary $\Gamma_{\rm out}$, the positive blade's surface $\Im_+$, the negative blade's surface $\Im_-$ and the top wall Γ_t and the bottom wall Γ_b:

$$\partial\Omega_\varepsilon = \Gamma = \Gamma_{\rm in} \cup \Gamma_{\rm out} \cup \Im_- \cup \Im_+ \cup \Gamma_t \cup \Gamma_b \tag{3.5}$$

(see Figure 2). The boundary conditions are given by

$$\begin{cases} w|_{\Im_- \cup \Im_+} = 0, \quad w|_{\Gamma_b} = 0, \quad w|_{\Gamma_t} = 0, \\ \sigma^{ij}(w,p)n_j|_{\Gamma_{\rm in}} = g^i_{\rm in}, \quad \sigma^{ij}(w,p)n_j|_{\Gamma_{\rm out}} = g^i_{\rm out} \quad \text{(Natural conditions)}, \\ \dfrac{\partial T}{\partial n} + \kappa(T - T_0) = 0, \quad \text{where } \kappa \geq 0 \text{ is constant.} \end{cases} \tag{3.6}$$

If the fluid is incompressible and the flow is stationary, then

$$\begin{cases} {\rm div} w = 0, \\ (w\nabla)w + 2\vec{\omega} \times \vec{w} + \nabla p - \nu {\rm div}(e(w)) = -\vec{\omega} \times (\vec{\omega} \times \vec{R}) + f, \\ w|_{\Gamma_0} = 0, \quad \Gamma_0 = \Im_+ \cup \Im_- \cup \Gamma_t \cup \Gamma_b, \\ (-pn + 2\nu e(w))|_{\Gamma_{\rm in}} = g_{\rm in}, \quad \Gamma_1 = \Gamma_{\rm in} \cup \Gamma_{\rm out}, \\ (-pn + 2\nu e(w))|_{\Gamma_{\rm out}} = g_{\rm out}, \\ w|_{t=0} = w_0(x), \quad \Omega_\varepsilon. \end{cases} \tag{3.7}$$

For the polytropic ideal gas and the stationary flow, system (3.1) turns to the conservative form

$$\begin{cases} {\rm div}(\rho w) = 0, \\ {\rm div}(\rho w \otimes w) + 2\rho\omega \times w + R\nabla(\rho T) = \mu \Delta w + (\lambda + \mu)\nabla {\rm div} w - \rho\omega \times (\omega \times \vec{R}), \\ {\rm div}\Big[\rho\Big(\dfrac{|w|^2}{2} + c_v T + RT\Big)w\Big] = \kappa \Delta T + \lambda {\rm div}(w {\rm div} w) + \mu {\rm div}[w\nabla w] + \dfrac{\mu}{2}\Delta |w|^2, \end{cases} \tag{3.8}$$

while for isentropic ideal gases, it turns to

$$\begin{cases} {\rm div}(\rho w) = 0, \\ {\rm div}(\rho w \otimes w) + 2\rho\omega \times w + \alpha \nabla(\rho^\gamma) = 2\mu {\rm div}(e) + \lambda \nabla {\rm div} w - \rho\omega \times (\omega \times \vec{R}), \end{cases} \tag{3.9}$$

where $\gamma > 1$ is the specific heat radio and α is a positive constant.

The rate of work done by the impeller and the global dissipative energy are respectively

$$I(\Im, w(\Im)) = \iint_{\Im_- \cup \Im_+} \sigma \cdot n \cdot e_\theta \omega r {\rm d}\Im, \quad J(\Im, w(\Im)) = \iiint_{\Omega_\varepsilon} \Phi(w){\rm d}V, \tag{3.10}$$

where e_θ is the base vector along the angular direction in the cylindrical coordinate system.

Let us employ the new coordinate system defined by (2.2). The flow domain Ω_ε is mapped into $\Omega = D\times[-1,1]$, where D is a domain in $(x^1, x^2) \in \Re^2$ limited by four arcs $\widehat{AB}, \widehat{CD}, \widehat{CB}, \widehat{DA}$ such that

$$\partial D = \gamma_0 \cup \gamma_1, \quad \gamma_0 = \widehat{AB} \cup \widehat{CD}, \quad \gamma_1 = \widehat{CB} \cup \widehat{DA},$$

and there exist four positive functions $\gamma_0(z), \widetilde{\gamma}_0(z), \gamma_1(z), \widetilde{\gamma}_1(z)$ such that

$$\begin{aligned}
&r := x^2 = \gamma_0(x^1) = \gamma_0(z), \quad \text{on } \widehat{AB}, \quad x^2 = \widetilde{\gamma}_0(x^1), \quad \text{on } \widehat{CD},\\
&r := x^2 = \gamma_1(x^1) = \gamma_1(z), \quad \text{on } \widehat{DA}, \quad x^2 = \widetilde{\gamma}_1(x^1), \quad \text{on } \widehat{BC},\\
&r_0 \le \gamma_0(z) \le r_1, \quad \text{on } \widehat{AB}, \quad r_0 \le \widetilde{\gamma}_0(z) \le r_1, \quad \text{on } \widehat{CD},\\
&r_0 \le \gamma_1(z) \le r_1, \quad \text{on } \widehat{DA}, \quad r_0 \le \widetilde{\gamma}_1(z) \le r_1, \quad \text{on } \widehat{BC}.
\end{aligned} \tag{3.11}$$

We have

$$\begin{aligned}
&\partial\Omega = \widetilde{\Gamma}_0 \cup \widetilde{\Gamma}_1, \quad \widetilde{\Gamma}_1 = \widetilde{\Gamma}_{\text{out}} \cup \widetilde{\Gamma}_{\text{in}}, \quad \widetilde{\Gamma}_0 = \widetilde{\Gamma}_b \cup \widetilde{\Gamma}_t \cup \{\xi = 1\} \cup \{\xi = -1\},\\
&\widetilde{\Gamma}_{\text{in}} = \overrightarrow{\Re}(\Gamma_{\text{in}}), \quad \widetilde{\Gamma}_{\text{out}} = \overrightarrow{\Re}(\Gamma_{\text{out}}), \quad \widetilde{\Gamma}_b = \overrightarrow{\Re}(\Gamma_b), \quad \widetilde{\Gamma}_t = \overrightarrow{\Re}(\Gamma_t)
\end{aligned} \tag{3.12}$$

and

$$\partial D = \gamma_0 \cup \gamma_1, \quad \gamma_0 = (D \cap \widetilde{\Gamma}_b) \cup (D \cap \widetilde{\Gamma}_t), \quad \gamma_1 = (D \cup \widetilde{\Gamma}_{\text{out}}) \cup (D \cup \widetilde{\Gamma}_{\text{in}}), \tag{3.13}$$

where $\overrightarrow{\Re}$ is defined by (2.1).

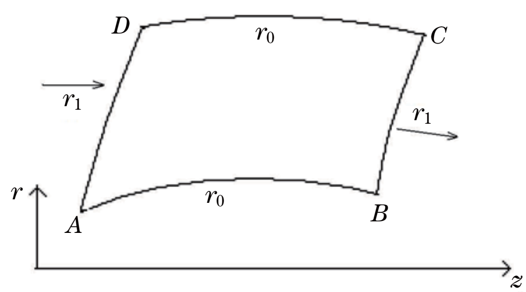

Figure 3 Sectional graph D of meridian plane in channel flow Ω_ε

Let the Sobolev spaces be

$$V(\Omega) := \{v \mid v \in H^1(\Omega)^3,\ v|_{\widetilde{\Gamma}_0} = 0\}, \quad H^1_\Gamma(\Omega) = \{q \mid q \in H^1(\Omega),\ q|_{\widetilde{\Gamma}_0} = 0\}, \tag{3.14}$$

which are equipped with the usual Sobolev norm $\|\cdot\|_{1,\Omega}$. The relation $v = 0$ on the boundary is to be understood in the sense of traces. Then variational formulation of the Navier-Stokes problems (3.7) and (3.9) are respectively given by

$$\begin{cases}
\text{Find } (w,p),\ w \in V(\Omega),\ p \in L^2(\Omega), \text{ such that}\\
a(w,v) + 2(\omega \times w, v) + b(w,w,v) - (p, \text{div} v) = \langle F, v\rangle, \quad \forall\, v \in V(\Omega),\\
(q, \text{div} w) = 0, \quad \forall\, q \in L^2(\Omega)
\end{cases} \tag{3.15}$$

and

$$\begin{cases}
\text{Find } (w,\rho),\ w \in V(\Omega),\ \rho \in L^\gamma(\Omega), \text{ such that}\\
a(w,v) + 2(\omega \times w, v) + b(\rho w,w,v) + (-p + \lambda \text{div} w, \text{div} v) = \langle F, v\rangle, \quad \forall\, v \in V(\Omega),\\
(\nabla q, \rho w)) = \langle \rho w n, q\rangle|_{\Gamma_1}, \quad \forall\, q \in H^1_\Gamma(\Omega),
\end{cases} \tag{3.16}$$

where

$$
\begin{aligned}
&\langle F, v\rangle := \langle f, v\rangle + \langle \widetilde{g}, v\rangle_{\widetilde{\Gamma}_1}, \quad \langle \widetilde{g}, v\rangle = \langle g_{\rm in}, v\rangle|_{\widetilde{\Gamma}_{\rm in}} + \langle g_{\rm out}, v\rangle|_{\widetilde{\Gamma}_{\rm out}},\\
&a(w,v) = \int_\Omega A^{ijkm} e_{ij}(w) e_{km}(v)\sqrt{g}\,\mathrm{d}x\mathrm{d}\xi,\\
&b(w,w,v) = \int_\Omega g_{km} w^j \nabla_j w^k v^m \sqrt{g}\,\mathrm{d}x\mathrm{d}\xi.
\end{aligned}
\tag{3.17}
$$

Next we rewrite (3.7) and (3.9) in the new coordinate system. Because the second kind of Christoffel symbols in the new coordinate system can be explicitly expressed in terms of Θ

$$
\begin{cases}
\Gamma^\alpha_{\beta\gamma} = -r\delta_{2\alpha}\Theta_\beta\Theta_\gamma, \quad \Gamma^\alpha_{3\beta} = -\varepsilon r\delta_{2\alpha}\Theta_\beta,\\
\Gamma^3_{\alpha\beta} = \varepsilon^{-1} r^{-1}(\delta_{2\alpha}\delta^\lambda_\beta + \delta_{2\beta}\delta^\lambda_\alpha)\Theta_\lambda + \varepsilon^{-1}\Theta_{\alpha\beta} + \varepsilon^{-1} r\Theta_2\Theta_\alpha\Theta_\beta,\\
\Gamma^3_{3\alpha} = \Gamma^3_{\alpha 3} = r^{-1}\delta_{2\alpha} + r\Theta_2\Theta_\alpha, \quad \Gamma^\alpha_{33} = -\varepsilon^2 r\delta_{2\alpha}, \quad \Gamma^3_{33} = \varepsilon r\Theta_2,
\end{cases}
\tag{3.18}
$$

the covariant derivatives of the velocity field $\nabla_i w^j = \frac{\partial w^j}{\partial x^i} + \Gamma^j_{ik} w^k$ possess the following forms.

Lemma 3.1 *Under the curvilinear coordinate system (x^1, x^2, ξ) defined by (2.4), the covariant derivatives of the velocity field can be expressed as*

$$
\begin{cases}
\nabla_\alpha w^\beta = \dfrac{\partial w^\beta}{\partial x^\alpha} - r\delta^\beta_2 \Theta_\alpha \Pi(w,\Theta),\\
\nabla_\alpha w^3 = \dfrac{\partial w^3}{\partial x^\alpha} + \varepsilon^{-1}(x^2)^{-1} w^2\Theta_\alpha + \varepsilon^{-1} w^\beta\Theta_{\alpha\beta} + (\varepsilon x^2)^{-1} a_{2\alpha}\Pi(w,\Theta),\\
\nabla_3 w^\alpha = \dfrac{\partial w^\alpha}{\partial \xi} - x^2\varepsilon\delta_{2\alpha}\Pi(w,\Theta), \quad \nabla_3 w^3 = \dfrac{\partial w^3}{\partial\xi} + \dfrac{w^2}{x^2} + x^2\Theta_2\Pi(w,\Theta),\\
\mathrm{div} w = \dfrac{1}{r}\dfrac{\partial(rw^\alpha)}{\partial x^\alpha} + \dfrac{\partial w^3}{\partial \xi}, \quad \Pi(w,\Theta) = \varepsilon w^3 + w^\beta\Theta_\beta,
\end{cases}
\tag{3.19}
$$

while the strain velocity tensors can be split in the form

$$
\begin{aligned}
e_{ij}(w) &= \varphi_{ij}(w) + \psi_{ij}(w,\Theta),\\
\psi_{ij}(w,\Theta) &= \psi^\lambda_{ij}(w)\Theta_\lambda + \psi^{\lambda\sigma}_{ij}(w)\Theta_\lambda\Theta_\sigma + e^*_{ij}(w,\Theta),
\end{aligned}
\tag{3.20}
$$

where the first terms without Θ are

$$
\begin{aligned}
&\varphi_{\alpha\beta}(w) = \frac{1}{2}\Big(\frac{\partial w^\alpha}{\partial x^\beta} + \frac{\partial w^\beta}{\partial x^\alpha}\Big),\\
&\varphi_{3\alpha}(w) = \frac{1}{2}\Big(\frac{\partial w^\alpha}{\partial \xi} + \varepsilon^2 r^2\frac{\partial w^3}{\partial x^\alpha}\Big), \quad \varphi_{33}(w) = \varepsilon^2 r^2\Big(\frac{\partial w^3}{\partial\xi} + \frac{w^2}{r}\Big)
\end{aligned}
\tag{3.21}
$$

and the second terms containing Θ are

$$
\begin{cases}
\psi^\lambda_{\alpha\beta}(w) = \dfrac{1}{2}\varepsilon r^2\Big(\dfrac{\partial w^3}{\partial x^\alpha}\delta^\lambda_\beta + \dfrac{\partial w^3}{\partial x^\beta}\delta^\lambda_\alpha\Big),\\
\psi^\lambda_{3\alpha}(w) = \dfrac{1}{2}\varepsilon r^2\Big(\dfrac{\partial w^\lambda}{\partial x^\alpha} + \delta^\lambda_\alpha\Big(\dfrac{\partial w^3}{\partial\xi} + \dfrac{2}{r}w^2\Big)\Big), \quad \psi^\lambda_{33}(w) = \varepsilon r^2\dfrac{\partial w^\lambda}{\partial\xi},\\
\psi^{\lambda\sigma}_{\alpha\beta}(w) = \dfrac{1}{2} r^2\Big(\dfrac{\partial w^\lambda}{\partial x^\alpha}\delta_{\beta\sigma} + \dfrac{\partial w^\lambda}{\partial x^\beta}\delta_{\sigma\alpha} + \dfrac{2}{r}w^2\delta_{\alpha\lambda}\delta_{\sigma\beta}\Big),\\
\psi^{\lambda\sigma}_{3\alpha}(w) = \frac{1}{2} r^2\dfrac{\partial w^\lambda}{\partial\xi}\delta_{\alpha\sigma}, \quad \psi^{\lambda\sigma}_{33}(w) = 0.
\end{cases}
\tag{3.22}
$$

$$e^*_{\alpha\beta}(w,\Theta)=\frac{1}{2}r^2w^\sigma\partial_\sigma(\Theta_\alpha\Theta_\beta),\quad e^*_{3\alpha}(w)=\frac{1}{2}\varepsilon r^2w^\sigma\Theta_{\sigma\alpha},\quad e^*_{33}(w)=0. \tag{3.23}$$

The proof is omitted here.

Throughout this paper, Latin indices and exponents $i,j,k\cdots$ vary in the set $\{1,2,3\}$, while Greek indices and exponents $\alpha,\beta,\gamma\cdots$ vary in the set $\{1,2\}$. Furthermore, the summation convention with respect to repeated indices or exponents is systematically used in conjunction with this rule.

From now on, we consider incompressible flows only. Let the viscosity tensor be

$$A^{ijkl}=\mu(g^{ik}g^{jl}+g^{il}g^{jk}). \tag{3.24}$$

Then in the new coordinate system the dissipative function is as follows

$$\begin{aligned}\Phi(w,v)&=A^{ijkl}e_{kl}(w)e_{ij}(v)=2\mu g^{ik}g^{jl}e_{kl}(w)e_{ij}(v)\\&=2\mu[e_{\alpha\beta}(w)e_{\alpha\beta}(v)+g^{33}g^{33}e_{33}(w)e_{33}(v)\\&\quad+g^{3\alpha}g^{3\beta}(e_{33}(w)e_{\alpha\beta}(v)+e_{\alpha\beta}(w)e_{33}(v))+2(g^{3\alpha}g^{3\beta}+g^{\alpha\beta}g^{33})e_{3\alpha}(w)e_{3\beta}(v)\\&\quad+2g^{\alpha\beta}g^{3\lambda}(e_{\beta\lambda}(w)e_{3\alpha}(v)+e_{3\alpha}(w)e_{\beta\lambda}(v))+2g^{3\alpha}g^{33}(e_{33}(w)e_{3\alpha}(v)+e_{3\alpha}(w)e_{33}(v))\\&=2\mu[e_{\alpha\beta}(w)e_{\alpha\beta}(v)+g^{33}g^{33}e_{33}(w)e_{33}(v)+2(\varepsilon^{-2}\Theta_\alpha\Theta_\beta+\delta^{\alpha\beta}g^{33})e_{3\alpha}(w)e_{3\beta}(v)\\&\quad+\varepsilon^{-2}\Theta_\alpha\Theta_\beta(e_{33}(w)e_{\alpha\beta}(v)+e_{\alpha\beta}(w)e_{33}(v))\\&\quad-2\varepsilon^{-1}\Theta_\beta(e_{\alpha\beta}(w)e_{3\alpha}(v)+e_{3\alpha}(w)e_{\alpha\beta}(v))\\&\quad-2\varepsilon^{-1}\Theta_\alpha g^{33}(e_{33}(w)e_{3\alpha}(v)+e_{3\alpha}(w)e_{33}(v)).\end{aligned}$$

Taking (2.5) into account, simple calculations show that

$$\begin{aligned}\Phi(w,v)=2\mu[&(e_{\alpha\beta}(w)+\varepsilon^{-2}\Theta_\alpha\Theta_\beta e_{33}(w)-2\varepsilon^{-1}\Theta_\alpha e_{3\beta}(w))\\&\cdot(e_{\alpha\beta}(v)+\varepsilon^{-2}\Theta_\alpha\Theta_\beta e_{33}(v)-2\varepsilon^{-1}\Theta_\beta e_{3\alpha}(v))\\&+(\varepsilon^{-2}r^{-2}e_{33}(w)-2\varepsilon^{-1}\Theta_\alpha e_{3\alpha}(w))(\varepsilon^{-2}r^{-2}e_{33}(v)-2\varepsilon^{-1}\Theta_\beta e_{3\beta}(v))\\&+2\varepsilon^{-4}r^{-2}|\nabla\Theta|^2e_{33}(w)e_{33}(v)+2g^{33}e_{3\alpha}(w)e_{3\alpha}(v)\\&-6\varepsilon^{-2}\Theta_\alpha\Theta_\beta e_{3\alpha}(w)e_{3\beta}(v)].\end{aligned} \tag{3.25}$$

Lemma 3.2 *Assume that the mapping Θ is smooth and satisfies*

$$\Theta\in\mathcal{F}_1=\Big\{\phi\in C^2(\Omega),\ \inf_D\{|\nabla\phi|\}\le\frac{1}{2}r_0^{-1}\Big\}. \tag{3.26}$$

Then the three dimensional viscosity tensor $A^{ijkl}=\mu(g^{ik}g^{jl}+g^{il}g^{jk})$ is uniformly positive definite in D, i.e., for any symmetric matrices of order three t_{ij}, it holds

$$\begin{aligned}&A^{ijkl}t_{kl}t_{ij}\ge\mu|t|^2,\\&|t|^2:=t_{\alpha\beta}t_{\alpha\beta}+(r\varepsilon)^{-2}t_{3\alpha}t_{3\alpha}+\frac{1}{2}(r\varepsilon)^{-4}t_{33}t_{33}.\end{aligned} \tag{3.27}$$

Proof Indeed by (3.25),

$$A^{ijkl}t_{kl}t_{ij}=2\mu[t_{\alpha\beta}t_{\alpha\beta}+g^{33}g^{33}t_{33}t_{33}+2(g^{3\alpha}g^{3\beta}+g^{\alpha\beta}g^{33})t_{3\alpha}(w)t_{3\beta}(v)$$

$$
\begin{aligned}
&+ 2g^{3\alpha}g^{3\beta}t_{33}t_{\alpha\beta} + 4g^{\alpha\beta}g^{3\lambda}t_{\beta\lambda}t_{3\alpha} + 4g^{3\alpha}g^{33}t_{33}t_{3\alpha}] \\
= {}& 2\mu[(t_{\alpha\beta} + \varepsilon^{-2}\Theta_\alpha\Theta_\beta t_{33} - 2\varepsilon^{-1}\Theta_\alpha t_{3\beta})(t_{\alpha\beta} + \varepsilon^{-2}\Theta_\alpha\Theta_\beta t_{33} - 2\varepsilon^{-1}\Theta_\beta t_{3\alpha}) \\
&+ (\varepsilon^{-2}r^{-2}t_{33} - 2\varepsilon^{-1}\Theta_\alpha t_{3\alpha})(\varepsilon^{-2}r^{-2}t_{33} - 2\varepsilon^{-1}\Theta_\beta t_{3\beta}) \\
&+ 2\varepsilon^{-4}r^{-2}|\nabla\Theta|^2 t_{33}t_{33} + 2g^{33}t_{3\alpha}t_{3\alpha} - 6\varepsilon^{-2}(\Theta_\alpha t_{3\alpha})^2].
\end{aligned}
$$

For a positive constant p_0, using the Young's inequality

$$
2ab \le p_0 a^2 + \frac{1}{p_0}b^2, \quad (a+b)^2 \ge (1-p_0)a^2 + \Big(1 - \frac{1}{p_0}\Big)b^2,
$$

we assert

$$
\begin{aligned}
A^{ijkl}t_{kl}t_{ij} \ge 2\mu\Big[&(1-p_0)t_{\alpha\beta}t_{\alpha\beta} + (1-p_0)\varepsilon^{-4}r^{-4}t_{33}t_{33} + 2g^{33}t_{3\alpha}t_{3\alpha} \\
&+ \Big(1 - \frac{1}{p_0}\Big)(\varepsilon^{-2}\Theta_\alpha\Theta_\beta t_{33} - 2\varepsilon^{-1}\Theta_\alpha t_{3\beta})(\varepsilon^{-2}\Theta_\alpha\Theta_\beta t_{33} - 2\varepsilon^{-1}\Theta_\beta t_{3\alpha}) \\
&+ \Big(4\Big(1 - \frac{1}{p_0}\Big) - 6\Big)(\varepsilon^{-1}\Theta_\alpha t_{3\alpha})^2 + 2\varepsilon^{-4}r^{-2}|\nabla\Theta|^2 t_{33}t_{33}\Big].
\end{aligned}
$$

Since

$$
\begin{aligned}
&\Big(1 - \frac{1}{p_0}\Big)(\varepsilon^{-2}\Theta_\alpha\Theta_\beta t_{33} - 2\varepsilon^{-1}\Theta_\alpha t_{3\beta})(\varepsilon^{-2}\Theta_\alpha\Theta_\beta t_{33} - 2\varepsilon^{-1}\Theta_\beta t_{3\alpha}) \\
= {}& \Big(1 - \frac{1}{p_0}\Big)(\varepsilon^{-2}|\nabla\Theta|^2 t_{33} - 2\varepsilon^{-1}\Theta_\alpha t_{3\alpha})^2
\end{aligned}
$$

and $g^{33} = \varepsilon^{-2}r^{-2}(1 + r^2|\nabla\Theta|^2)$, it yields

$$
\begin{aligned}
A^{ijkl}t_{kl}t_{ij} \ge 2\mu\Big[&(1-p_0)t_{\alpha\beta}t_{\alpha\beta} + (1-p_0)\varepsilon^{-4}r^{-4}t_{33}t_{33} + 2\varepsilon^{-2}r^{-2}t_{3\alpha}t_{3\alpha} \\
&+ \Big(1 - \frac{1}{p_0}\Big)(\varepsilon^{-2}|\nabla\Theta|^2 t_{33} - 2\varepsilon^{-1}\Theta_\alpha t_{3\alpha})^2 + \Big(4\Big(1 - \frac{1}{p_0}\Big) - 6\Big)(\varepsilon^{-1}\Theta_\alpha t_{3\alpha})^2 \\
&+ 2\varepsilon^{-2}|\nabla\Theta|^2 t_{3\alpha}t_{3\alpha} + 2\varepsilon^{-4}|\nabla\Theta|^4 t_{33}t_{33}\Big].
\end{aligned}
$$

Let $p_0 = \frac{1}{2},\ 1 - p_0 = \frac{1}{2}$. Then $1 - \frac{1}{p_0} = -1$. In addition,

$$
\begin{aligned}
\Big|\Big(1 - \frac{1}{p_0}\Big)(\varepsilon^{-2}|\nabla\Theta|^2 t_{33} - 2\varepsilon^{-1}\Theta_\alpha t_{3\alpha})^2\Big| &\le 2\varepsilon^{-4}|\nabla\Theta|^4(t_{33})^2 + 2\times 4\varepsilon^{-2}|\nabla\Theta|^2 t_{3\alpha}t_{3\alpha}, \\
\Big|\Big(4\Big(1 - \frac{1}{p_0}\Big) - 6\Big)(\varepsilon^{-1}\Theta_\alpha t_{3\alpha})^2\Big| &\le 10\varepsilon^{-2}|\nabla\Theta|^2 t_{3\alpha}t_{3\alpha}.
\end{aligned}
$$

To sum up, we find

$$
\begin{aligned}
A^{ijkl}t_{kl}t_{ij} \ge {}& 2\mu\Big[(1-p_0)t_{\alpha\beta}t_{\alpha\beta} + (1-p_0)\varepsilon^{-4}r^{-4}t_{33}t_{33} + 2\varepsilon^{-2}r^{-2}t_{3\alpha}t_{3\alpha} \\
&+ \Big(1 - \frac{1}{p_0}\Big)(\varepsilon^{-2}|\nabla\Theta|^2 t_{33} - 2\varepsilon^{-1}\Theta_\alpha t_{3\alpha})^2 + \Big(4\Big(1 - \frac{1}{p_0}\Big) - 6\Big)(\varepsilon^{-1}\Theta_\alpha t_{3\alpha})^2 \\
&+ 2\varepsilon^{-2}|\nabla\Theta|^2 t_{3\alpha}t_{3\alpha} + 2\varepsilon^{-4}|\nabla\Theta|^4 t_{33}t_{33}\Big] \\
\ge {}& 2\mu\Big[\frac{1}{2}t_{\alpha\beta}t_{\alpha\beta} + \frac{1}{2}\varepsilon^{-4}r^{-4}t_{33}t_{33} + 2\varepsilon^{-2}r^{-2}t_{3\alpha}t_{3\alpha} \\
&- 4\varepsilon^{-4}|\nabla\Theta|^4(t_{33})^2 - 20\varepsilon^{-2}|\nabla\Theta|^2 t_{3\alpha}t_{3\alpha}\Big]
\end{aligned}
$$

$$= 2\mu\Big[\frac{1}{2}t_{\alpha\beta}t_{\alpha\beta} + \frac{1}{2}\varepsilon^{-4}r^{-4}(1-8r^4|\nabla\Theta|^4)t_{33}t_{33} + 2\varepsilon^{-2}r^{-2}(1-10r^2|\nabla\Theta|^2)t_{3\alpha}t_{3\alpha}\Big].$$

By assumptions, we have

$$1-8r^4|\nabla\Theta|^4 \geq \frac{1}{2} \Rightarrow |\nabla\Theta| \leq \frac{1}{2}r^{-1} \leq \frac{1}{2}r_0^{-1},$$
$$1-10r^2|\nabla\Theta|^2 \geq \frac{1}{2} \Rightarrow |\nabla\Theta| \leq \frac{1}{\sqrt{20}}r^{-1} \leq \frac{1}{\sqrt{20}}r_0^{-1},$$

and we obtain

$$A^{ijkl}t_{kl}t_{ij} \geq \mu\Big[t_{\alpha\beta}t_{\alpha\beta} + \frac{1}{2}\varepsilon^{-4}r^{-4}t_{33}t_{33} + \varepsilon^{-2}r^{-2}t_{3\alpha}t_{3\alpha}\Big].$$

The proof is completed.

Let us introduce a scalar product $((u,v))_\Omega$ on the Hilbert space $V(\Omega) = \{v \in H^1(\Omega)^3, v|_{\tilde{\Gamma}_1} = 0\}$, the dissipation functional and the associated bilinear form $a(\cdot\,,\cdot)$

$$\begin{cases} ((w,v)) := \|\varphi(w,v)\|^2 \\ \qquad\qquad = \mu\Big(\varphi_{\alpha\beta}(w)\varphi_{\alpha\beta}(v) + (r\varepsilon)^{-2}\varphi_{3\alpha}(w)\varphi_{3\alpha}(v) + \frac{1}{2}(r\varepsilon)^{-4}\varphi_{33}(w)\varphi_{33}(v)\Big), \\ \|w\|^2 = \|\varphi(w)\|^2 = ((w,w)), \\ ((w,v))_\Omega := \int_\Omega ((w,v))r\varepsilon \mathrm{d}\xi \mathrm{d}x, \quad \|w\|_\Omega^2 := ((w,w))_\Omega, \\ \Phi(w,v) := A^{ijkl}e_{kl}(w)e_{ij}(v), \\ a(u,v) := \int_\Omega \Phi(u,v)r\varepsilon \mathrm{d}x\mathrm{d}\xi, \quad J(u) := \frac{1}{2}a(u,u). \end{cases} \tag{3.28}$$

In particular, we will use the notation

$$\begin{aligned} (e(w),e(v)) &:= \mu\Big(e_{\alpha\beta}(w)e_{\alpha\beta}(v) + (r\varepsilon)^{-2}e_{3\alpha}(w)e_{3\alpha}(v) + \frac{1}{2}(r\varepsilon)^{-4}e_{33}(w)e_{33}(v)\Big), \\ (\varphi(w),\varphi(v)) &:= \mu\Big(\varphi_{\alpha\beta}(w)\varphi_{\alpha\beta}(v) + (r\varepsilon)^{-2}\varphi_{3\alpha}(w)\varphi_{3\alpha}(v) + \frac{1}{2}(r\varepsilon)^{-4}\varphi_{33}(w)\varphi_{33}(v)\Big), \\ (\psi(w,\Theta),\psi(v,\Theta)) &:= \mu\Big(\psi_{\alpha\beta}(w,\Theta)\psi_{\alpha\beta}(v,\Theta) + (r\varepsilon)^{-2}\psi_{3\alpha}(w,\Theta)\psi_{3\alpha}(v,\Theta) \\ &\quad + \frac{1}{2}(r\varepsilon)^{-4}\psi_{33}(w,\Theta)\psi_{33}(v,\Theta)\Big), \\ (\varphi(w),\psi(v,\Theta)) &:= \mu\Big(\varphi_{\alpha\beta}(w)\psi_{\alpha\beta}(v,\Theta) + (r\varepsilon)^{-2}\varphi_{3\alpha}(w)\psi_{3\alpha}(v,\Theta) \\ &\quad + \frac{1}{2}(r\varepsilon)^{-4}\varphi_{33}(w)\psi_{33}(v,\Theta)\Big), \\ \|e(w)\|^2 &= (e(w),e(w)), \quad \|\varphi(w)\|^2 = (\varphi(w),\varphi(w)), \\ \|\psi(w,\Theta)\|^2 &= (\psi(w,\Theta),\psi(w,\Theta)). \end{aligned} \tag{3.29}$$

Lemma 3.3 *Assume that the determinant a of the metric tensor of the surface $\Im$ defined by (2.3) such that $\Im \in \mathcal{F}_1$ where $\mathcal{F}_1$ is a manifold in a Banach space $C^2(D)$ defined by (3.26). Then following estimates of the dissipative function $\Phi(w,v)$, $\forall\, w,v \in V(\Omega)$ hold*

$$\begin{cases} \Phi(w,w) \geq \kappa_0\|\varphi(w)\|^2 = \kappa_0\|w\|^2, \\ |\Phi(w,v)| \leq \kappa_3\|\varphi(w)\|\|\varphi(v)\|, \end{cases} \tag{3.30}$$

where $C(\Omega)$ *is a constant independent of* Θ *and*

$$\begin{cases}\kappa_0 := \dfrac{1}{2} - \kappa_1 C_2(\Omega) > 0, \ \textit{if } \kappa_1 \textit{ small enough},\\ \kappa_1 := \sup\limits_D \{3(1+r_1^2)|\nabla\Theta|^2, 3(1+r_1^4)|\nabla\Theta|^4\},\\ \kappa_2 = \sup\limits_D \{3|\nabla\Theta|^2, 3|\nabla\Theta|^4, 3r_1^2(1+r_1^2|\nabla\Theta|^2)|\nabla^2\Theta|^2\},\\ \kappa_3 = 12\mu\kappa_2(1+\kappa_1 C_2(\Omega)),\end{cases} \tag{3.31}$$

where $C_1(\Omega)$*,* $C_2(\Omega)$ *are defined by*

$$\begin{cases} C_1(\Omega)\|\varphi(w)\|^2 \le \||\psi(w)\||^2 \le C_2(\Omega)\|\varphi(w)\|^2,\\ \||\psi(w)\||^2 = \sum\limits_\lambda \|\psi^\lambda(w)\|^2 + \sum\limits_{\lambda,\sigma}\|\psi^{\lambda,\sigma}(w)\|^2 + w^\lambda w^\lambda,\end{cases} \tag{3.32}$$

and r_1 *is defined by* (3.11)*. Later on, the constant* $C(\Omega)$ *appearing in different places may have different meanings at different occasions.*

Proof By virtue of (3.20), (3.27) and (3.33), we claim

$$\begin{aligned}\Phi(w,w) \ge (e(w),e(w)) &= (\varphi(w)+\psi(w,\Theta), \varphi(w)+\psi(w,\Theta))\\ &= (\varphi(w),\varphi(w)) + (\psi(w,\Theta),\psi(w,\Theta)) + 2(\varphi(w),\psi(w,\Theta)).\end{aligned} \tag{3.33}$$

By the symmetry of the indices and by the Cauchy and Young inequalities, we infer that

$$\begin{aligned}2((\varphi(w),\psi(w))) &= 2\mu\Big[\varphi_{\alpha\beta}(w)\psi_{\alpha\beta}(w,\Theta) + (r\varepsilon)^{-2}\varphi_{3\alpha}(w)\psi_{3\alpha}(w,\Theta)\\ &\quad + \frac{1}{2}(r\varepsilon)^{-4}\varphi_{33}(w)\psi_{33}(w,\Theta)\Big]\\ &\le 2\mu\Big[\sqrt{\varphi_{\alpha\beta}(w)\varphi_{\alpha\beta}(w)}\sqrt{\psi_{\alpha\beta}(w,\Theta)\psi_{\alpha\beta}(w,\Theta)}\\ &\quad + (r\varepsilon)^{-2}\sqrt{\varphi_{3\alpha}(w)\varphi_{3\alpha}(w)}\sqrt{\psi_{3\alpha}(w,\Theta)\psi_{3\alpha}(w,\Theta)}\\ &\quad + \frac{1}{2}(r\varepsilon)^{-4}|\varphi_{33}(w)||\psi_{33}(w,\Theta)|\Big]\\ &\le 2\mu\Big[\frac{1}{2}\|\varphi(w)\|^2 + 2\|\psi(w,\Theta)\|^2\Big].\end{aligned} \tag{3.34}$$

Hence

$$\Phi(w,w) \ge \frac{1}{2}\|\varphi(w)\|^2 - \|\psi(w,\Theta)\|^2. \tag{3.35}$$

Due to (3.22) and (3.23),

$$\begin{aligned}\|\psi(w,\Theta)\|^2 &= \|\psi^\lambda(w)\Theta_\lambda + \psi^{\lambda\sigma}(w)\Theta_\lambda\Theta_\sigma + e^*(w,\Theta)\|^2\\ &= ((\psi^\lambda,\psi^\sigma))\Theta_\lambda\Theta_\sigma + ((\psi^{\lambda\sigma},\psi^{\nu\mu}))\Theta_\lambda\Theta_\sigma\Theta_\nu\Theta_\mu + ((e^*(w,\Theta),e^*(w,\Theta)))\\ &\quad + 2((\psi^\lambda(w),\psi^{\nu\mu}(w)))\Theta_\lambda\Theta_\nu\Theta_\mu + 2((e^*(w,\Theta),\psi^\lambda(w)))\Theta_\lambda\\ &\quad + 2((e^*(w,\Theta),\psi^{\lambda\sigma}(w)))\Theta_\lambda\Theta_\sigma.\end{aligned}$$

Set

$$\begin{cases}|\nabla\Theta|^2 = \sum\limits_\lambda |\Theta_\lambda|^2, \quad |\nabla\nabla\Theta|^2 = \sum\limits_{\alpha,\beta}(\Theta_{\alpha\beta})^2 = |\nabla^2\Theta|^2,\\ \sum\limits_\sigma \nabla_\sigma\Big(\sum\limits_{\alpha,\beta}(\Theta_\alpha\Theta_\beta)\Big) = |\nabla((\nabla\Theta)(\nabla\Theta))| = 2|\nabla^2\Theta||\nabla\Theta|.\end{cases} \tag{3.36}$$

Taking (3.20), (3.22) and (3.23) into account and using the Cauchy's inequality, we claim that

$$
\begin{aligned}
|((\psi^\lambda(w,\Theta),\psi^\sigma(w,\Theta)))\Theta_\lambda\Theta_\sigma| &\le \sum_\lambda \|\psi^\lambda\|^2|\nabla\Theta|^2,\\
((\psi^{\lambda\sigma}(w,\Theta),\psi^{\nu\mu}(w,\Theta)))\Theta_\lambda\Theta_\sigma\Theta_\nu\Theta_\mu &\le \sum_{\lambda,\sigma}\|\psi^{\lambda\sigma}\|^2|\nabla\Theta|^4,\\
2((\psi^{\lambda\sigma}(w,\Theta),\psi^\nu(w,\Theta)))\Theta_\lambda\Theta_\sigma\Theta_\nu &\le 2\sqrt{\sum_{\lambda,\sigma}\|\psi^{\lambda\sigma}\|^2}\sqrt{\sum_\nu\|\psi^\nu\|^2}\,|\nabla\Theta|^3\\
&\le \sum_\lambda\|\psi^\lambda\|^2|\nabla\Theta|^2+\sum_{\lambda,\sigma}\|\psi^{\lambda\sigma}\|^2|\nabla\Theta|^4,\\
((e^*(w,\Theta),e^*(w,\Theta))) &= \|e^*(w,\Theta)\|^2 \le r^2|w|_*^2|\nabla^2\Theta|^2(1+r^2|\nabla\Theta|^2)\\
&= r^2a|w|_*^2|\nabla^2\Theta|^2,\\
2((e^*(w,\Theta),\psi^\lambda(w)))\Theta_\lambda &\le 2\|e^*(w,\Theta)\|\|\psi^\lambda(w)\||\Theta_\lambda|\\
&\le \|e^*(w,\Theta)\|^2+\sum_\lambda\|\psi^\lambda(w)\|^2|\nabla\Theta|^2,\\
2((e^*(w,\Theta),\psi^{\lambda\sigma}(w)))\Theta_\lambda\Theta_\sigma &\le \|e^*(w,\Theta)\|^2+\sum_{\lambda,\sigma}\|\psi^{\lambda\sigma}(w)\|^2|\nabla\Theta|^4,
\end{aligned}
$$

where

$$
\begin{aligned}
\|\psi^\lambda(w)\|^2 &= ((\psi^\lambda_{\alpha\beta}(w),\psi^\lambda_{\alpha\beta}))+(r\varepsilon)^{-2}((\psi^\lambda_{3\alpha},\psi^\lambda_{3\alpha}))+\frac{1}{2}(r\varepsilon)^{-4}((\psi^\lambda_{33}(w),\psi^\lambda_{33})),\\
|w|_*^2 &= w^1w^1+w^2w^2=w^\lambda w^\lambda.
\end{aligned}
\tag{3.37}
$$

To sum up, we have

$$
\|\psi(w,\Theta)\|^2 \le 3\Big(|\nabla\Theta|^2\sum_\lambda\|\psi^\lambda(w)\|^2+|\nabla\Theta|^4\sum_{\lambda,\sigma}\|\psi^{\lambda,\sigma}(w)\|^2+r^2a|\nabla^2\Theta|^2|w|_*^2\Big). \tag{3.38}
$$

Set

$$
\begin{cases}
k_1=\sup\limits_D\{3|\nabla\Theta|^2,3|\nabla\Theta|^4,3r_1^2(1+r_1^2|\nabla\Theta|^2)|\nabla^2\Theta|^2\},\\
\||\psi(w)\||^2:=\sum\limits_\lambda\|\psi^\lambda(w)\|^2+\sum\limits_{\lambda,\sigma}\|\psi^{\lambda,\sigma}(w)\|^2+|w|_*^2.
\end{cases}
\tag{3.39}
$$

We conclude that

$$
\|\psi(w,\Theta)\|^2 \le k_1\||\psi(w)\||^2. \tag{3.40}
$$

On the other hand, (3.21) and (3.22) show that there exists constants $C_1(\Omega)$ and $C_2(\Omega)$ independent of w such that

$$
C_1(\Omega)\|\varphi(w)\|^2 \le \||\psi(w)\||^2 \le C_2(\Omega)\|\varphi(w)\|^2. \tag{3.41}
$$

Therefore,

$$
\|\psi(w,\Theta)\|^2 \le k_1C_2(\Omega)\|\varphi(w)\|^2. \tag{3.42}
$$

Let us return to (3.39). We infer that

$$
\Phi(w,\Theta) \ge \frac{1}{2}\|\varphi(w)\|^2-m_1C_2(\Omega)\|\varphi(w)\|^2 \ge \kappa_0\|\varphi(w)\|^2, \tag{3.43}
$$

where

$$\kappa_0 := \frac{1}{2} - k_1 C_2(\Omega) > 0, \quad \text{if } m_1 \text{ is small enough.} \tag{3.44}$$

Therefore (4.30) holds.

Next, we consider the continuity of the dissipation function Φ. Note that the Cauchy's inequality shows that

$$A_{\alpha\beta}B_{\alpha\beta} \le \sqrt{A_{\alpha\beta}A_{\alpha\beta}}\sqrt{B_{\alpha\beta}B_{\alpha\beta}},$$

and then

$$\begin{aligned}
l_{\alpha\beta}(w,\Theta) &:= (e_{\alpha\beta}(w) + \varepsilon^{-2}\Theta_\alpha\Theta_\beta e_{33}(w) - 2\varepsilon^{-1}\Theta_\beta e_{3\alpha}(w)),\\
l_{\alpha\beta}(w,\Theta)l_{\alpha\beta}(v,\Theta) &\le \sqrt{l_{\alpha\beta}(w,\Theta)l_{\alpha\beta}(w,\Theta)}\sqrt{l_{\alpha\beta}(v,\Theta)l_{\alpha\beta}(v,\Theta)},\\
l_{\alpha\beta}(w,\Theta)l_{\alpha\beta}(w,\Theta) &\le 3[e_{\alpha\beta}(w)e_{\alpha\beta}(w) + \varepsilon^{-4}|\nabla\Theta|^4 e_{33}(w)e_{33}(w) + 4\varepsilon^{-2}|\nabla\Theta|^2 e_{3\alpha}(w)e_{3\alpha}(w)]\\
&\le k_2(e(w),e(w)),\\
l_{\alpha\beta}(w,\Theta)l_{\alpha\beta}(v,\Theta) &\le k_2\sqrt{(e(w),e(w))}\sqrt{(e(v),e(v))},\\
k_2 &:= 3\max_D\{1, r_1^4|\nabla\Theta|^4, 8r_1^2|\nabla\Theta|^2\}.
\end{aligned}$$

From (3.25) we claim that

$$\Phi(w,v) \le 2\mu 3k_2\sqrt{(e(w),e(w))}\sqrt{(e(v),e(v))}.$$

From the triangle inequality and $e(w) = \varphi(w) + \psi(w)$, we get

$$(e(w),e(w)) \le 2[(\varphi(w),\varphi(w)) + (\psi(w),\psi(w))] \le 2(1 + k_1C_2(\Omega))(\varphi(w),\varphi(w)).$$

From (3.25), we assert $\Phi(w,v) \le 12\mu k_2(1+k_1C_2(\Omega))\|\varphi(w)\|\|\varphi(v)\|$. The proof is completed.

Next, we consider the bilinear form. To do this, at first we have

Lemma 3.4 *The function* $\|\cdot\|_\Omega$ *defined by* (3.28) *is a norm on the Hilbert space* $V(\Omega)$,

$$V(\Omega) := \{v \in H^1(D)^3, v|_{\tilde\Gamma_1} = 0\}. \tag{3.45}$$

Proof Indeed it is enough to prove that $\|w\|_\Omega = 0$, $w \in V(\Omega) \Rightarrow w = 0$. Indeed, this means that

$$\|w\|_\Omega = 0, \quad \text{i.e., } \varphi_{ij}(w) = 0.$$

We have to prove $w = 0$. Firstly, the following identity holds:

$$\partial_\gamma(\partial_\alpha w^\beta) = \partial_\gamma\varphi_{\alpha\beta}(w) + \partial_\alpha\varphi_{\gamma\beta}(w) - \partial_\beta\varphi_{\alpha\gamma}(w).$$

This shows that

$$\varphi_{\alpha\beta}(w) = 0, \text{ in } D \Rightarrow \partial_\gamma\partial_\alpha w^\beta = 0, \text{ in } \mathcal{D}'(D).$$

By a classical result from distribution theory, each function w is therefore a polynomial of degree at most 1 (recall that the set D is connected). In other words, there exist constants c_α and $d_{\alpha\beta}$ such that

$$w^\alpha(x) = c_\alpha + d_{\alpha\beta}x^\beta, \quad \forall\, x = (x^1,x^2) \in D.$$

But $\varphi_{\alpha\beta}(w)=0$ also implies that $d_{\alpha\beta}=-d_{\beta\alpha}$. Hence there exist two vectors $\overrightarrow{c}, \overrightarrow{d} \in \Re^2$ such that $w=\overrightarrow{c}+\overrightarrow{d}\times\overrightarrow{x}$, $\forall x \in D$. Since $w|_{\widetilde{\Gamma}}=0$ and the set where such a vector field w^α vanishes is always of zero area unless $\overrightarrow{c}=\overrightarrow{d}=0$, it follows that $w^\alpha=0$ when the area $\widetilde{\Gamma}_0>0$. On the other hand, in view of the boundary condition (3.13)

$$\varphi_{33}(w)=\varepsilon^2 r^2\Big(\frac{\partial w^3}{\partial\xi}+\frac{w^2}{r}\Big)=0\Rightarrow\frac{\partial w^3}{\partial\xi}=0\Rightarrow w^3=0.$$

The proof is completed.

Lemma 3.5 *The norm $\|\cdot\|_\Omega$ and the semi-norm*

$$|w|^2_{1,\Omega}=\int_\Omega\Big[\sum_{i=1}^3\Big(\sum_{\alpha=1}^2\Big(\frac{\partial w^i}{\partial x^\alpha}\Big)^2+\Big(\frac{\partial w^i}{\partial\xi}\Big)^2\Big)\Big]r\varepsilon\mathrm{d}\xi\mathrm{d}x,\quad\forall\,w\in V(\Omega)$$

are equivalent on $V(\Omega)$, i.e., there exist constants $C_i(\Omega)>0$ $(i=3,4)$ only depending upon Ω such that

$$C_3(\Omega)|w|_{1,\Omega}\le\|w\|_\Omega\le C_4(\Omega)|w|_{1,\Omega},\quad\forall\,w\in V(\Omega).\tag{3.46}$$

Proof Firstly, we indicate that in view of (3.11), (3.12), there exist constants $C_i(\Omega)>0$ $(i=3,4)$ depending upon Ω only such that

$$C_3(\Omega)\Big(\sum_{i,j=1}^3\|\varphi_{ij}(w)\|^2_{0,\Omega}\Big)^{\frac12}\le\|w\|_\Omega\le C_4(\Omega)\Big(\sum_{i,j=1}^3\|\varphi_{ij}(w)\|^2_{0,\Omega}\Big)^{\frac12},\quad\forall\,w\in V(\Omega),\tag{3.47}$$

and $\varphi_{ij}(w)$ can be viewed as the strain tensor in Cartesian coordinates in $\Re^3$. Then according to the Korn's inequality (see [14, 15]), $\Big(\sum\limits_{i,j=1}^3\|\varphi_{ij}(w)\|^2_{0,\Omega}\Big)^{\frac12}$ is a norm equivalent to $\|w\|_{1,\Omega}$, therefore this yields (3.46). The proof is completed.

Lemma 3.6 *The bilinear form $a(\cdot,\cdot)=\int_\Omega\Phi(\cdot,\cdot)\sqrt{g}\,\mathrm{d}\xi\mathrm{d}x$ defined by* (3.25) *is a symmetric, continuous and uniformly coercive mapping from $V(\Omega)\times V(\Omega)$ into $\Re$:*

(i) *Symmetry: $a(w,v)=a(v,w)$, $\forall\,w,v\in V(\Omega)$;*

(ii) *Continuity: $|a(w,v)|\le\kappa_1(\Omega)\|w\|_\Omega\|v\|_\Omega$, $\forall\,w,v\in V(\Omega)$;*

(iii) *If the function $\Theta\in\mathcal{S}$, then $a(w,v)$ is coercive uniformly with respect to Θ:*

$$a(w,w)\ge\kappa_0\|w\|^2_\Omega,\tag{3.48}$$

where κ_0,κ_1 are defined by (3.31).

Proof The conclusions follow immediately from Lemma 3.3.

Next, we consider the trilinear form and Coriolis force form

$$b(w,u,v)=\int_D\int_{-1}^1 g_{km}w^j\nabla_j u^k v^m\sqrt{g}\,\mathrm{d}\xi\mathrm{d}x,\tag{3.49}$$

$$C(w,v):=\int_D\int_{-1}^1 2g_{ij}(\overrightarrow{\omega}\times w)^i v^j\sqrt{g}\,\mathrm{d}\xi\mathrm{d}x$$

$$= \int_D \int_{-1}^{1} 2r\omega[(w^2\Theta_\beta - \delta_{2\beta}\Pi(w,\Theta))v^\beta + \varepsilon w^2 v^3] r\varepsilon \mathrm{d}\xi \mathrm{d}x. \tag{3.50}$$

By virtue of (2.5) and (3.19), we have

$$\begin{aligned}
B(w,u,v) : &= g_{km} w^j \nabla_j u^k v^m \\
&= \Big(w^\lambda \frac{\partial u^\alpha}{\partial x^\lambda} + w^3 \frac{\partial u^\alpha}{\partial \xi}\Big)(a_{\alpha\beta}v^\beta + r^2\varepsilon\Theta_\alpha v^3) \\
&\quad + \Big(w^\lambda \frac{\partial u^3}{\partial x^\lambda} + w^3 \frac{\partial u^3}{\partial \xi}\Big)(\varepsilon r^2\Theta_\beta v^\beta + r^2\varepsilon^2 v^3) \\
&\quad - r\delta_{2\alpha}\Pi(w,\Theta)\Pi(u,\Theta)(a_{\alpha\beta}v^\beta + r^2\varepsilon\Theta_\alpha v^3) \\
&\quad + [\varepsilon^{-1}w^\lambda u^\beta\Theta_{\lambda\beta} + (r\varepsilon)^{-1}(u^2\Pi(w,\Theta) + w^2\Pi(u,\Theta)) \\
&\quad + \varepsilon^{-1}r\Theta_2\Pi(w,\Theta)\Pi(u,\Theta)](\varepsilon r^2\Theta_\beta v^\beta + r^2\varepsilon^2 v^3), \\
B(w,u,v) : &= \Big(w^\lambda \frac{\partial u^\alpha}{\partial x^\lambda} + w^3 \frac{\partial u^\alpha}{\partial \xi}\Big)(a_{\alpha\beta}v^\beta + r^2\varepsilon\Theta_\alpha v^3) \\
&\quad + \Big(w^\lambda \frac{\partial u^3}{\partial x^\lambda} + w^3 \frac{\partial u^3}{\partial \xi}\Big)(\varepsilon r^2\Theta_\beta v^\beta + r^2\varepsilon^2 v^3) + \pi_{ijk}w^i u^j v^k,
\end{aligned} \tag{3.51}$$

where

$$\begin{aligned}
&\pi_{\alpha\beta,\lambda} = r^2\Theta_\lambda\Theta_{\alpha\beta} + r\Theta_\lambda(\delta_{2\alpha}\Theta_\beta + \delta_{2\beta}\Theta_\alpha) - r\delta_{2\lambda}\Theta_\alpha\Theta_\beta, \\
&\pi_{\alpha 3,\lambda} = r\varepsilon(\delta_{2\alpha}\Theta_\lambda - \delta_{2\lambda}\Theta_\alpha), \quad \pi_{3\beta,\lambda} = r\varepsilon(\delta_{2\beta}\Theta_\lambda + r\Theta_\lambda\Theta_\beta - a_{2\lambda}\Theta_\beta), \\
&\pi_{33,\lambda} = -r\varepsilon^2\delta_{2\lambda}, \\
&\pi_{\alpha\beta,3} = r\varepsilon(\Theta_\alpha\delta_{2\beta} + \Theta_\beta\delta_{2\alpha}), \quad \pi_{\alpha 3,3} = 0, \\
&\pi_{3\beta,3} = r\varepsilon^2(\delta_{2\beta} + r\Theta_\beta - r^2\Theta_2\Theta_\beta), \quad \pi_{33,3} = r\varepsilon^3.
\end{aligned} \tag{3.52}$$

Lemma 3.7 *The trilinear form $b(\,\cdot\,,\,\cdot\,,\,\cdot\,)$ is uniformly continuous*

$$|b(w,u,v)| \le C(\Omega)(1+\kappa_3)\|w\|_\Omega\|u\|_\Omega\|v\|_\Omega, \tag{3.53}$$

if the mapping Θ is smooth enough and satisfies

$$\sup_\Omega(|\nabla\Theta|, |\nabla\Theta|^2, |\nabla^2\Theta|) \le \kappa_3. \tag{3.54}$$

The from $C(\,\cdot\,,\,\cdot\,)$ is uniformly continuous

$$|C(w,v)| \le C(\Omega)\omega(1+k_3)\|w\|_{0,2,\Omega}\|v\|_{0,2,\Omega}, \tag{3.55}$$

and

$$C(w,w) = 0. \tag{3.56}$$

Proof Indeed, from (3.51) and (3.52), by virtue of a standard process as in [13, 16], we assert that (3.52) is valid. Similarly, from (3.50) it yields directly (3.55). In addition, the Coriolis form is

$$C(w,w) := \int_D \int_{-1}^{1} 2g_{ij}(\overrightarrow{\omega} \times w)^i w^j \sqrt{g}\, \mathrm{d}\xi \mathrm{d}x = \int_D \int_{-1}^{1} [2(\overrightarrow{\omega} \times \mathbf{w}) \cdot \mathbf{w}]\varepsilon x^2 \mathrm{d}\xi \mathrm{d}x = 0, \tag{3.57}$$

i.e.,

$$C(w,w) = 0.$$

The proof is completed.

4 The Existence of the Solution of Rotating Navier-Stokes Equations with Mixed Boundary Conditions

Next, we consider the existence of the solutions for the Navier-Stokes equations. Indeed the flow's domain is an unbounded domain. In Section 3, we introduce an artificial boundary, with the inflow boundary $\Gamma_{\rm in}$ and the outflow boundary $\Gamma_{\rm out}$, and impose the natural boundary conditions (3.6). We can also impose the pressures $p|_{\Gamma_{\rm in}} = p_{\rm in}, \quad p|_{\Gamma_{\rm out}} = p_{\rm out}$, or fluxes

$$\int_{\Gamma_{\rm in}} \rho w \cdot n {\rm d}\Gamma = Q, \quad \int_{\Gamma_{\rm out}} \rho w \cdot n {\rm d}\Gamma = Q.$$

Let us consider the energy inequality. Since

$$(2\omega \times w, w) = 0, \tag{4.1}$$

the moment equations (3.11) show that $a(w,w) + b(w,w,w) = (f,w)$. However,

$$\begin{aligned}
b(w,w,w) &= \int_\Omega w^j \nabla_j w^i g_{ik} w^k \sqrt{g} {\rm d}x{\rm d}\xi \\
&= \int_\Omega (\nabla_j (w^j w^i) - w^i {\rm div} w) g_{ik} w^k \sqrt{g} {\rm d}x{\rm d}\xi \\
&= \int_\Omega ({\rm div}(|w|^2 w) - g_{ik} w^i w^j \nabla_j w^k) \sqrt{g} {\rm d}x{\rm d}\xi \\
&= \int_{\Gamma_1} |w|^2 w \cdot n {\rm d}\Gamma - b(w,w,w), \\
b(w,w,w) &= \frac{1}{2} \int_{\Gamma_1} |w|^2 w \cdot n {\rm d}\Gamma.
\end{aligned} \tag{4.2}$$

Here we denote

$$|w|^2 = g_{ik} w^i w^k, \quad \Gamma_1 = \Gamma_{\rm in} \cup \Gamma_{\rm out}. \tag{4.3}$$

The inflow and outflow fluxes of kinetic energy are respectively given by

$$K_{\rm in}(w) = \int_{\Gamma_{\rm in}} |w|^2 w \cdot n {\rm d}\Gamma, \quad K_{\rm out}(w) = \int_{\Gamma_{\rm out}} |w|^2 w \cdot n {\rm d}\Gamma,$$

where $w \cdot n = g_{ij} w^i n^j$ and n is the outward normal unit vector of the inflow or the outflow boundaries. Therefore (4.1) shows that

$$b(w,w,w) = K_{\rm out}(w) + K_{\rm in}(w). \tag{4.4}$$

Let us come back to (3.7), (3.15) and (3.17),

$$\begin{cases}
a(w,w) + b(w,w,w) - \langle g_{\rm in}, w \rangle - \langle g_{\rm out}, w \rangle = \langle f, w \rangle, \\
b(w,w,w) - \langle g_{\rm in}, w \rangle - \langle g_{\rm out}, w \rangle = \displaystyle\int_{\Gamma_{\rm in} \cup \Gamma_{\rm out}} [pwn - 2\nu e(w) n \cdot w] {\rm d}S, \\
2\nu e(w) \cdot n \cdot w = \nu (g_{ik} \nabla_j w^k + g_{jk} \nabla_i w^k) n^j w^i = \nu \Big(\dfrac{\partial w}{\partial s} n + \dfrac{\partial |w|^2}{\partial n} \Big) \\
\text{is the flux of dissipative energy,}
\end{cases}$$

where $P = \frac{1}{2}|w|^2 + p$ is a total pressure, for a non-viscous flow it is conservative by the Bernoulli theorem, and s is the direction along the steam line. According to the conservation law of energy, we assert that $b(w,w,w) - \langle g_{\rm in}, w\rangle - \langle g_{\rm out}, w\rangle = 0$. Therefore $a(w,w) = \langle f, w\rangle,\ |w|_{1,\Omega} \le \frac{1}{\nu}|f|_{-1,\Omega}$. By a standard method, it is easy to prove that there exists at least one solution for the Navier-Stokes equations. If the energy from outside is thus that

$$b(w,w,w) - \langle g_{\rm in}, w\rangle - \langle g_{\rm out}, w\rangle \ne 0,$$

for example, for a hydroelectric and compressor, then we have the following theorem.

Theorem 4.1 *Suppose that the exterior force f and normal stress g at the inflow and the outflow boundaries $\Gamma_1 = \Gamma_{\rm in} \cup \Gamma_{\rm out}$ satisfy*

$$\|F\|_* := \|f\|_{0,\Omega} + \|g_{\rm in}\|_{-\frac{1}{2},\Gamma_{\rm in}} + \|g_{\rm out}\|_{-\frac{1}{2},\Gamma_{\rm out}} \le \frac{\mu^2}{C^2(\Omega)(1+\kappa_3^2)}, \tag{4.5}$$

and the mapping Θ defined in (2.1) *is a $C^2(D)$-function satisfying* (3.59) *and* (3.53)*. Then there exists a smooth solution of the variational problem* (3.15)

$$\begin{cases} \textit{Find } (w,p),\ \ w \in V(\Omega),\ \ p \in L^2(\Omega),\ \ \textit{such that} \\ a(w,v) + 2(\omega \times w, v) + b(w,w,v) - (p, {\rm div} v) = \langle F, v\rangle, \quad \forall\, v \in V(\Omega), \\ (q, {\rm div} w) = 0, \quad \forall\, q \in L^2(\Omega), \end{cases}$$

satisfying

$$C(\Omega)|w|_{1,\Omega} \le \|w\|_\Omega \le \frac{\kappa_0}{2C(\Omega)(1+\kappa_3)}\Big[1 - \sqrt{1 - \frac{4C^2(\Omega)(1+\kappa_3^2)\|F\|_*}{\kappa_0^2}}\,\Big], \tag{4.6}$$

where $\|w\|_\Omega$ is defined in (3.37)*, $C(\Omega)$ is a constant depending on Ω which has different meaning at different place.*

Proof To prove the theorem for a steady Navier-Stokes problem, it is convenient to construct the solution as the limit of Galerkin approximations in terms of the eigenfunctions of the corresponding stationary Stokes problem. Galerkin equations are a system of algebraic equations and the Galerkin approximation solution w is a solution of the finite dimensional problem

$$a(w,v) + 2(\omega \times w, v) + b(w,w,v) = \langle F, v\rangle, \quad \forall\, v \in V_m := {\rm span}\{\phi_1, \phi_2 \cdots, \phi_m\}, \tag{4.7}$$

where ϕ_i $(i = 1, 2, \cdots m)$ are the eigenfunctions of the corresponding Stokes operator. Let S_ρ denote the spheres in V_m satisfying inequality (4.6). Assume that $w_* \in S_\rho$. We find w such that

$$a(w,v) + 2(\omega \times w, v) + b(w_*, w, v) = \langle F, v\rangle, \quad \forall\, v \in V_m, \tag{4.8}$$

(4.8) is uniquely solvable. To do this it is enough to prove that for any $w_* \in S_\rho$, $w = 0$ is the only one solution of (4.8) with $(F = 0)$. Owing to Lemma 3.6,

$$a(w,w) \ge \kappa_0\|w\|_\Omega^2$$

and by virtue of (3.59) and (4.2), we assert that

$$\kappa_0\|w\|_\Omega^2 \le |b(w_*, w, w)| \le C_6(\Omega)(1+\delta_3)\|w_*\|_\Omega\|w\|_\Omega^2$$
$$< C(\Omega)(1+\kappa_3)\frac{c(\Omega,\Theta)\mu}{(1+\delta_3)C(\Omega)}\|w\|_\Omega^2. \tag{4.9}$$

This implies that $w = 0$. In order to apply Brouwer's fixed point theorem, we have to show that the mapping $w_* \Rightarrow w$ takes the ball S_ρ defined by (4.7) into itself. Since w_* satisfies (4.7) and

$$Fw = g_{ij}F^iw^j = (\delta_{\alpha\beta} + r^2\Theta_\alpha\Theta_\beta)F^\alpha w^\beta + \varepsilon\Theta_\alpha(F^\alpha w^3 + F^3w^\alpha) + \varepsilon^2r^2F^3w^3,$$

we claim that

$$\begin{aligned}\kappa_0\|\nabla w\|_\Omega^2 &\le |b(w_*, w, w)| + |\langle F, w\rangle| \\ &\le (1+\kappa_3)[C(\Omega)\|w_*\|_\Omega\|w\|_\Omega^2 + C(\Omega)\|F\|_*\|w\|_\Omega], \\ \kappa_0\|w\|_\Omega &\le (1+\kappa_3)[C(\Omega)\|w_*\|_\Omega\|w\|_\Omega + C(\Omega)\|F\|_*],\end{aligned} \tag{4.10}$$

For simplicity, let

$$X := 1 - \frac{C^2(\Omega)(1+\kappa_3)^2\|F\|_*}{\kappa_0^2}.$$

Therefore

$$\begin{aligned}\|w\|_\Omega &\le \frac{C(\Omega)(1+\kappa_3)\|F\|_*}{\kappa_0 - C(\Omega)(1+\kappa_3)\|w^*\|_\Omega} \le \frac{8C(\Omega)(1+\kappa_3)\|F\|_*}{\frac{\kappa_0}{2}(1+\sqrt{X})} = \frac{\kappa_0}{2C^2(\Omega)}\frac{1-X}{1+\sqrt{X}} \\ &= \frac{\kappa_0}{2C(\Omega)(1+\kappa_3)}[1-\sqrt{X}].\end{aligned}$$

This is (4.6). Thus Brouwer's fixed point theorem can be applied and it gives the existence of the Galerkin approximations satisfying (6.8). Hence, by a standard compactness argument there exists at least a subsequence of the Galerkin approximation converging to a weak solution $w \in V(\Omega)$ of the steady problem (3.11):

$$\begin{cases}\text{Find } (w,p),\ w \in V(\Omega),\ p \in L^2(\Omega),\ \text{such that} \\ a(w,v) + 2(\omega \times w, v) + b(w,w,v) + -(p, \operatorname{div} v) = \langle F, v\rangle, \quad \forall\, v \in V(\Omega), \\ (q, \operatorname{div} w) = 0, \quad \forall\, q \in L^2(\Omega).\end{cases}$$

Its smoothness is easily proven if one obtains a further estimate from the Galerkin approximations by setting $v = -Aw$ in (6.9). This gives

$$\mu\|Aw\|_0^2 = -2(\omega \times w, Aw) - b(w,w,Aw) + \langle F, Aw\rangle. \tag{4.11}$$

Because Aw is solenoidal, one has the rather unusual trace estimate

$$|2(\omega \times w, Aw)| \le C(\Omega)\|w\|_0\|Aw\|_0, \quad |\langle F, Aw\rangle| \le c_3\|F\|_*\|Aw\|_0, \tag{4.12}$$

which we combine with (4.12) and the Agmon's inequality

$$\|w\|_\infty \le C(\Omega)\|\nabla w\|_0^{\frac{1}{2}}\|Aw\|_0^{\frac{1}{2}}, \quad \forall w \in D(A) \tag{4.13}$$

to get

$$\mu\|Aw\|_0^2 \le C(\Omega)\|\nabla w\|_0^{\frac{3}{2}}\|Aw\|_0^{\frac{3}{2}} + C(\Omega)\|w\|_0\|Aw\|_0 + c_3\|F\|_*\|Aw\|_0. \tag{4.14}$$

Then, by using the Young's inequality, we obtain

$$\mu\|Aw\|_0 \le \frac{2C^2(\Omega)}{\mu^2}\|\nabla w\|_0^3 + \frac{8C^2(\Omega)}{\mu}\|w\|_0^2 + \frac{8C^2(\Omega)}{\mu}\|F\|_*^2, \tag{4.15}$$

which is then inherited by the solution. The full classical smoothness of the solution can now be obtained by using the L^2-regularity theory for the steady Stokes equations. This completes the proof of Theorem 4.1.

5 Gâteaux Derivatives and Their Equations

In order to obtain the Gâteaux derivative of the solution of the Navier-Stokes equations with respect to the boundary shape Θ, we first consider the Navier-Stokes equations in the new coordinate system (x^α, ξ) defined by (2.4). Indeed, we refer to [20].

Theorem 5.1 *Under the new coordinate system, the incompressible rotating stationary Navier-Stokes equations* (3.7) *can be explicitly expressed via* Θ :

$$\begin{cases} \dfrac{\partial w^\alpha}{\partial x^\alpha} + \dfrac{\partial w^3}{\partial \xi} + \dfrac{w^2}{r} = \dfrac{1}{r}\dfrac{\partial (rw^\alpha)}{\partial x^\alpha} + \dfrac{\partial w^3}{\partial \xi} = \widetilde{\mathrm{div}}_2 w + \dfrac{\partial w^3}{\partial \xi} = 0, \\ \mathcal{N}^k(w,p,\Theta) := \mathcal{L}^k(w,p,\Theta) + N^k(w,w) = f^k, \quad \forall\, k = 1,2,3, \\ \mathcal{L}^k(w,p,\Theta) := -\nu\widetilde{\Delta} w^k - \nu(r\varepsilon)^{-2} a\dfrac{\partial^2 w^k}{\partial \xi^2} - \nu P_j^{k3}(\Theta)\dfrac{\partial w^j}{\partial \xi} - 2\nu\varepsilon^{-1}\Theta_\beta\dfrac{\partial^2 w^k}{\partial\xi\partial x^\beta} \\ \qquad\qquad -\nu P_j^{k\beta}(\Theta)\dfrac{\partial w^j}{\partial x^\beta} - \nu q_j^k(\Theta) w^j + g^{k\beta}(\Theta)\nabla_\beta p + g^{k3}(\Theta)\partial_\xi p + C^k(w,\omega), \end{cases} \tag{5.1}$$

where $\boldsymbol{C}(w,\omega)$ *is the Coriolis forces defined in* (2.7)*,*

$$\begin{cases} N^k(w,w) = \dfrac{\partial (w^3 w^k)}{\partial \xi} + \partial_\beta(w^k w^\beta) + \pi_{ij}^k w^i w^j = \dfrac{\partial (w^3 w^k)}{\partial \xi} + B^k(w,w), \\ B^k(w,w) := \partial_\beta(w^k w^\beta) + \pi_{ij}^k w^i w^j, \end{cases} \tag{5.2}$$

$$\begin{cases} P_\alpha^{\lambda\beta}(\Theta) = \dfrac{1}{r}\delta_{\beta 2}\delta_\alpha^\lambda, \quad P_3^{\lambda\beta}(\Theta) = 0, \\ P_\alpha^{3\beta}(\Theta) = 2(r\varepsilon)^{-1}(\delta_{2\beta}\Theta_\alpha + r\Theta_{\alpha\beta}), \quad P_3^{3\beta} = \dfrac{3}{r}\delta_{\beta 2}, \\ P_\lambda^{\alpha 3}(\Theta) = -[(r\varepsilon)^{-1}(\delta_{\alpha\lambda}\Theta_2 + 2\delta_{2\alpha}\Theta_\lambda) + \varepsilon^{-1}\delta_{\alpha\lambda}\Delta\Theta], \quad P_3^{\alpha 3} = -2r^{-1}\delta_{2\alpha}, \\ P_\sigma^{33}(\Theta) = 2\varepsilon^{-2}(r^{-3}\delta_{2\sigma} - \Theta_\beta\Theta_{\beta\sigma}), \\ P_3^{33}(\Theta) = -(r\varepsilon)^{-1}(\Theta_2 + r\Delta\Theta), \end{cases} \tag{5.3}$$

$$\begin{cases} q_\sigma^\alpha(\Theta) = -r^{-2}\delta_{2\alpha}\delta_{2\sigma}, \quad q_3^\alpha(\Theta) = 0, \quad q_3^3(\Theta) = 0, \\ q_\sigma^3(\Theta) := (r\varepsilon)^{-1}[r^{-1}\delta_{2\sigma}\Theta_2 + 3\Theta_{2\sigma}] + \varepsilon^{-1}\partial_\sigma\Delta\Theta, \end{cases} \tag{5.4}$$

$$\begin{cases} \Theta_\alpha := \dfrac{\partial\Theta}{\partial x^\alpha}, \quad \Theta_{\alpha\beta} := \dfrac{\partial^2\Theta}{\partial x^\alpha\partial x^\beta}, \quad \Pi(w,\Theta) := \varepsilon w^3 + w^\lambda\Theta_\lambda, \\ \Delta\Theta := \Theta_{\alpha\alpha} = \Theta_{11} + \Theta_{22}, \quad |\nabla\Theta|^2 = \Theta_1^2 + \Theta_2^2 \end{cases} \tag{5.5}$$

and

$$\pi_{ij}^k = \Gamma_{ij}^k + r^{-1}\delta_{2i}\delta_{jk}. \tag{5.6}$$

By using the above formula, we claim that there exists the Gâteaux derivative of the solution of the Navier-Stokes equations exists and satisfies the following linearized Navier-Stokes equation.

Theorem 5.2 *Assume that a solution* $(w(\Theta), p(\Theta))$ *of the Navier-Stokes problem* (3.7) *such that we can define a mapping* $\Theta \Rightarrow (w(\Theta), p(\Theta))$ *from* $H_0^1(D) \cap H^2(D)$ *to* $H^{1,q}(\Omega) \times L^{2,q}(\Omega)$. *Then the Gâteaux derivative of* (w, p) *at a point* $\Theta \in H_0^1(D) \cap H^2(D)$ *with respect to any direction* $\eta \in H_0^1(D) \cap H^2(D)$ *exists,* $\widehat{w}\eta \doteq \frac{\mathcal{D}w}{\mathcal{D}\Theta}\eta$, $\widehat{p}\eta \doteq \frac{\mathcal{D}p}{\mathcal{D}\Theta}\eta$ *and it satisfies the following linearized equations*:

$$\begin{cases} \widetilde{\operatorname{div}} w := \dfrac{1}{r}\dfrac{\partial(r\widehat{w}^\alpha)}{\partial x^\alpha} + \dfrac{\partial \widehat{w}^3}{\partial \xi} = 0, \\ -\nu\Delta\widehat{w}^k - \nu(r\varepsilon)^{-2}a\dfrac{\partial^2\widehat{w}^k}{\partial\xi^2} - \nu P_j^{k3}(\Theta)\dfrac{\partial\widehat{w}^k}{\partial\xi} - 2\nu\varepsilon^{-1}\Theta_\beta\dfrac{\partial^2\widehat{w}^k}{\partial\xi\partial x^\beta} \\ -\nu P_j^{k\beta}(\Theta)\frac{\partial\widehat{w}^j}{\partial x^\beta} - \nu q_j^k(\Theta)\widehat{w}^j + g^{k\beta}\partial_\beta\widehat{p} + g^{k3}\partial_\xi\widehat{p} \\ +C^k(\widehat{w}, \omega) + N^k(w, \widehat{w}) + N^k(\widehat{w}, w) + R^k(w, p, \Theta) = 0, \end{cases} \tag{5.7}$$

$$\begin{cases} \widehat{w} = 0, \quad \text{on } \Gamma_s \cap \{\xi = \xi_k\}, \\ \nu\dfrac{\partial\widehat{w}}{\partial n} - \widehat{p}n = 0, \quad \text{on } \Gamma_{\text{in}} \cap \Gamma_{\text{out}}, \end{cases} \tag{5.8}$$

where

$$\begin{cases} R^\alpha(w,p,\Theta) := -\dfrac{\partial}{\partial x^\beta}\Big\{ - 2\nu\varepsilon^{-1}r\Theta_\beta\dfrac{\partial^2 w^\alpha}{\partial\xi^2} + \nu(r\varepsilon)^{-1}\dfrac{\partial w^\lambda}{\partial\xi}(\delta_{\alpha\lambda}\delta_{\beta 2} + 2\delta_{2\alpha}\delta_{\lambda\beta}) \\ \qquad -3\nu\varepsilon^{-1}\dfrac{\partial^2 w^\alpha}{\partial\xi\partial x^\beta} - \varepsilon^{-1}\partial_\xi p\delta_{\alpha\beta} - 2r\delta_{2\alpha}\Pi(w,\Theta)w^\beta\Big\}, \\ R^3(w,p,\Theta) \quad = -\dfrac{\partial}{\partial x^\beta}\Big\{\nu\varepsilon^{-1}\dfrac{\partial^2 w^\sigma}{\partial x^\beta\partial x^\sigma} + \nu\varepsilon^{-1}\dfrac{\partial}{\partial\xi}\Big(\dfrac{\partial w^3}{\partial x^\beta} - 2\dfrac{\partial w^\sigma}{\partial x^\sigma}\Theta_\beta\Big) \\ \qquad -2\nu\varepsilon^{-1}r\Theta_\beta\dfrac{\partial^2 w^3}{\partial\xi^2} - 2\nu(r\varepsilon)^{-1}\Big(\Big(\delta_{2\beta} + r\dfrac{\partial}{\partial x^\beta}\Big)\dfrac{\partial w^3}{\partial\xi} + \dfrac{\partial w^\beta}{\partial r} \\ \qquad +\dfrac{w^2}{r}\delta_{2\beta}\Big) - \varepsilon^{-1}\dfrac{\partial p}{\partial x^\beta} + \varepsilon^{-2}\Theta_\beta\dfrac{\partial p}{\partial\xi} - \dfrac{\partial}{\partial x^\alpha}(\varepsilon^{-1}w^\alpha w^\beta) \\ \qquad +(r\varepsilon)^{-1}(a_{2\alpha}\delta_\lambda^\beta + a_{2\lambda}\delta_\alpha^\beta + a_{\alpha\lambda}\delta_2^\beta - \delta_{\alpha\lambda}\delta_2^\beta)w^\lambda w^\alpha \\ \qquad +2r(\Theta_2\delta_{\alpha\beta} + \Theta_\alpha\delta_{2\beta})w^3 w^\alpha + r\varepsilon\delta_{2\beta}w^3 w^3\Big\}. \end{cases} \tag{5.9}$$

The variational formulation associated with (5.7) *and* (5.8) *is given by*

$$\begin{cases} \text{Find } \widehat{w} \in V(\Omega),\ \widehat{p} \in L_0^2(\Omega) \text{ such that } \forall\, v \in V(\Omega), \\ a_0(\widehat{\boldsymbol{w}}, \boldsymbol{v}) + (\boldsymbol{C}(\widehat{w}, \omega), \boldsymbol{v}) + b(\widehat{w}, w, v) + b(w, \widehat{w}, v) - (\widehat{p}, \partial_\alpha v^\alpha + \partial_\xi v^3) + (l(\widehat{w}, \Theta), v) \\ \quad = (\boldsymbol{R}(w,p,\Theta), \boldsymbol{v}), \\ \Big(\dfrac{1}{r}\dfrac{\partial(r\widehat{w}^\alpha)}{\partial x^\alpha} + \dfrac{\widehat{w}^2}{r} + \dfrac{\partial\widehat{w}^3}{\partial\xi}, q\Big) = 0, \quad \forall\, q \in L^2(\Omega), \end{cases} \tag{5.10}$$

where

$$\begin{aligned} a_0(\widehat{w}, v) &= \int_\Omega \nu g_{ij}\Big[\frac{\partial w^i}{\partial x^\alpha}\frac{\partial w^j}{\partial x^\alpha} + (r\varepsilon)^{-2}a\frac{\partial w^i}{\partial\xi}\frac{\partial w^j}{\partial\xi}\Big]\mathrm{d}x\mathrm{d}\xi, \\ (l(\widehat{w}, \Theta), v) &= \nu\int_\Omega\Big[- \varepsilon^{-1}\Theta_\beta g_{ij}\frac{\partial\widehat{w}^i}{\partial x^\beta}\frac{\partial v^j}{\partial\xi} + d_{ij}^k(\Theta)\frac{\partial\widehat{w}^i}{\partial x^k}v^j + g_{ij}q_m^i\widehat{w}^m v^j\Big]\mathrm{d}x\mathrm{d}\xi, \\ d_{ij}^k(\Theta) &:= g_{mi}P_j^{km}(\Theta) - \delta_\beta^k\partial_\beta g_{ij} \end{aligned} \tag{5.11}$$

with

$$\begin{aligned}(\boldsymbol{R}(w,p,\Theta),\boldsymbol{v}) = \int_\Omega \Big[\Big(& -2\nu\varepsilon^{-1}r\Theta_\beta\frac{\partial^2 w^\alpha}{\partial\xi^2} + \nu(r\varepsilon)^{-1}\frac{\partial w^\lambda}{\partial\xi}(\delta_{\alpha\lambda}\delta_{\beta 2} + 2\delta_{2\alpha}\delta_{\lambda\beta}) \\ & - 3\nu\varepsilon^{-1}\frac{\partial^2 w^\alpha}{\partial\xi\partial x^\beta} - \varepsilon^{-1}\partial_\xi p\delta_{\alpha\beta} - 2r\delta_{2\alpha}\Pi(w,\Theta)w^\beta\Big)\partial_\beta(a_{\alpha\lambda}v^\lambda + \varepsilon r^2\Theta_\alpha v^3) \\ & + \Big(\nu\varepsilon^{-1}\frac{\partial^2 w^\sigma}{\partial x^\beta\partial x^\sigma} + \nu\varepsilon^{-1}\frac{\partial}{\partial\xi}\Big(\frac{\partial w^3}{\partial x^\beta} - 2\frac{\partial w^\sigma}{\partial x^\sigma}\Theta_\beta\Big) - 2\nu\varepsilon^{-1}r\Theta_\beta\frac{\partial^2 w^3}{\partial\xi^2} \\ & - 2\nu(r\varepsilon)^{-1}\Big(\Big(\delta_{2\beta} + r\frac{\partial}{\partial x^\beta}\Big)\frac{\partial w^3}{\partial\xi} + \frac{\partial w^\beta}{\partial r} + \frac{w^2}{r}\delta_{2\beta}\Big) - \varepsilon^{-1}\frac{\partial p}{\partial x^\beta} + \varepsilon^{-2}\Theta_\beta\frac{\partial p}{\partial\xi} \\ & - \frac{\partial}{\partial x^\alpha}(\varepsilon^{-1}w^\alpha w^\beta) + (r\varepsilon)^{-1}(a_{2\alpha}\delta^\beta_\lambda + a_{2\lambda}\delta^\beta_\alpha + a_{\alpha\lambda}\delta^\beta_2 - \delta_{\alpha\lambda}\delta^\beta_2)w^\lambda w^\alpha \\ & + 2r(\Theta_2\delta_{\alpha\beta} + \Theta_\alpha\delta_{2\beta})w^3 w^\alpha + r\varepsilon\delta_{2\beta}w^3 w^3\Big) \\ & \cdot\frac{\partial}{\partial x^\beta}(\varepsilon r^2\Theta_\lambda v^\lambda + \varepsilon^2 r^2 v^3)\Big]\sqrt{g}\,\mathrm{d}x\mathrm{d}\xi. \end{aligned} \tag{5.11'}$$

Proof The Navier-Stokes equations (5.1) read

$$\begin{aligned} & \frac{\partial w^\alpha}{\partial x^\alpha} + \frac{w^2}{r} + \frac{\partial w^3}{\partial x^3} = 0, \\ & \mathcal{N}^\alpha(w,p,\Theta)\vec{e}_\alpha + \mathcal{N}^3(w,p,\Theta)\vec{e}_3 = f^\alpha\vec{e}_\alpha + f^3\vec{e}_3. \end{aligned} \tag{5.12}$$

The Gâteaux derivative with respect to Θ along any director $\eta \in \mathcal{W} := H^2(D)\cap H^1_0(D)$ is denoted by $\frac{\mathcal{D}}{\mathcal{D}\Theta}\eta$. Then from (5.12), we obtain

$$\begin{aligned} & \frac{\mathcal{D}}{\mathcal{D}\Theta}\mathcal{N}^\alpha(w,p,\Theta)\vec{e}_\alpha\eta + \frac{\mathcal{D}}{\mathcal{D}\Theta}\mathcal{N}^3(w,p,\Theta)\vec{e}_3\eta + \mathcal{N}^\alpha(w,p,\Theta)\frac{\mathcal{D}\vec{e}_\alpha}{\mathcal{D}\Theta}\eta \\ & + \mathcal{N}^3(w,p,\Theta)\frac{\mathcal{D}\vec{e}_3}{\mathcal{D}\Theta}\eta = f^\alpha\frac{\mathcal{D}\vec{e}_\alpha}{\mathcal{D}\Theta}\eta + f^3\frac{\mathcal{D}\vec{e}_3}{\mathcal{D}\Theta}\eta, \\ & \frac{\mathcal{D}}{\mathcal{D}\Theta}\mathcal{N}^\alpha(w,p,\Theta)\vec{e}_\alpha + \frac{\mathcal{D}}{\mathcal{D}\Theta}\mathcal{N}^3(w,p,\Theta)\vec{e}_3 + [\mathcal{N}^\alpha(w,p,\Theta) - f^\alpha]\frac{\mathcal{D}\vec{e}_\alpha}{\mathcal{D}\Theta} \\ & + [\mathcal{N}^3(w,p,\Theta) - f^3]\frac{\mathcal{D}\vec{e}_3}{\mathcal{D}\Theta}\vec{e}_3 = 0. \end{aligned}$$

Substituting (5.1) into the above equations, we claim that

$$\frac{\mathcal{D}}{\mathcal{D}\Theta}\mathcal{N}^k(w,p,\Theta) \doteq \frac{\mathcal{D}}{\mathcal{D}\Theta}\mathcal{L}^k(w,p,\Theta) + \frac{\mathcal{D}}{\mathcal{D}\Theta}N^k(w,p,\Theta) = 0.$$

However

$$\begin{cases} \dfrac{\mathcal{D}}{\mathcal{D}\Theta}\mathcal{L}^k(w,p,\Theta)\eta = \dfrac{\partial}{\partial w}\mathcal{L}^k(w,p,\Theta)\widehat{w}\eta + \dfrac{\partial}{\partial p}\mathcal{L}^k(w,p,\Theta)\widehat{p}\eta + \dfrac{\partial}{\partial\Theta}\mathcal{L}^k(w,p,\Theta)\eta, \\ \dfrac{\mathcal{D}}{\mathcal{D}\Theta}N^k(w,w,\Theta)\eta = \dfrac{\partial}{\partial w}N^k(w,w,\Theta)\widehat{w}\eta + \dfrac{\partial}{\partial\Theta}N^k(w,w,\Theta)\eta. \end{cases} \tag{5.13}$$

Since $\mathcal{L}$ are the linear operators and N are the bilinear operators defined by (5.2), we assert

$$\begin{cases} \dfrac{\partial}{\partial w}\mathcal{L}^k(w,p,\Theta)\widehat{w}\eta + \dfrac{\partial}{\partial p}\mathcal{L}^k(w,p,\Theta)\widehat{p}\eta = \mathcal{L}^k(\widehat{w},\widehat{p},\Theta)\eta, \\ \dfrac{\partial}{\partial w}N^k(w,w,\Theta)\widehat{w}\eta = (N^k(\widehat{w},w,\Theta) + N^k(w,\widehat{w},\Theta))\eta. \end{cases}$$

Therefore

$$
\begin{cases}
\dfrac{\mathcal{D}}{\mathcal{D}\Theta}\mathcal{N}^k(w,p,\Theta)\eta = \mathcal{L}^k(\widehat{w},\widehat{p},\Theta)\eta + N^k(w,\widehat{w})\eta + N^k(\widehat{w},w)\eta + R^k(w,p,\Theta)\eta = 0,\\
R^k(w,p,\Theta) = \dfrac{\partial}{\partial\Theta}\mathcal{L}^k(w,p,\Theta)\eta + \dfrac{\partial}{\partial\Theta}N^k(w,p,\Theta)\eta\\
\quad = -2\nu(r\varepsilon)^{-2}\Theta_\alpha\eta_\alpha\dfrac{\partial^2 w^k}{\partial\xi^2} - \nu\dfrac{\partial}{\partial\Theta}P_j^{k3}(\Theta)\eta\dfrac{\partial w^j}{\partial\xi} - 2\nu\varepsilon^{-1}\eta_\beta\dfrac{\partial^2 w^k}{\partial\xi\partial x^\beta}\\
\qquad -\nu\dfrac{\partial}{\partial\Theta}P_j^{k\beta}(\Theta)\eta\dfrac{\partial w^j}{\partial x^\beta} - \nu\dfrac{\partial}{\partial\Theta}q_j^k(\Theta)\eta w^j + \dfrac{\partial}{\partial\Theta}g^{k\beta}(\Theta)\eta\partial_\beta p + \dfrac{\partial}{\partial\Theta}g^{k3}(\Theta)\eta\partial_\xi p\\
\qquad +2\omega r[-w^\lambda\delta_2^k + \varepsilon^{-1}(\delta_{\lambda 2}\Pi(w,\Theta) + \Theta_2 w^\lambda)\delta_3^k]\eta_\lambda + \dfrac{\partial}{\partial\Theta}\pi_{ij}^k(\Theta)\eta w^i w^j.
\end{cases}
\tag{5.14}
$$

In order to obtain the expressions for the $R^k(w,p,\Theta)$, at the first, by (2.3), (2.7), (3.18) and (5.6), it is clear that

$$
\begin{aligned}
&\frac{\partial a}{\partial\Theta}\eta = 2r^2\Theta_\beta\eta_\beta, \quad \frac{\partial C^1}{\partial\Theta}\eta = 0, \quad \frac{\partial C^2}{\partial\Theta}\eta = -2r\omega w^\beta\eta_\beta,\\
&\frac{\partial C^3}{\partial\Theta}\eta = 2\omega\varepsilon^{-1}(r\Pi(w,\Theta)\delta_{2\beta} + r\Theta_2 w^\beta)\eta_\beta,\\
&\frac{\partial\pi^\alpha_{\beta\gamma}}{\partial\Theta}\eta = -r\delta_{2\alpha}(\Theta_\beta\delta_\gamma^\lambda + \Theta_\gamma\delta_\beta^\lambda)\eta_\lambda, \quad \frac{\partial\pi^\alpha_{3\beta}}{\partial\Theta}\eta = \frac{\partial\pi^\alpha_{\beta 3}}{\partial\Theta}\eta = -r\varepsilon\delta_{2\alpha}\eta_\beta, \quad \frac{\partial\pi^\alpha_{33}}{\partial\Theta}\eta = 0,\\
&\frac{\partial\pi^3_{\alpha\beta}}{\partial\Theta}\eta = (r\varepsilon)^{-1}(a_{2\alpha}\delta_\beta^\lambda + a_{2\beta}\delta_\alpha^\lambda + a_{\alpha\beta}\delta_2^\lambda - \delta_{\alpha\beta}\delta_2^\lambda)\eta_\lambda + \varepsilon^{-1}\eta_{\alpha\beta},\\
&\frac{\partial\pi^3_{3\alpha}}{\partial\Theta}\eta == \frac{\partial\pi^3_{\alpha 3}}{\partial\Theta}\eta = r(\Theta_2\delta_{\alpha\beta} + \Theta_\alpha\delta_{2\beta})\eta_\beta, \quad \frac{\partial\pi^3_{33}}{\partial\Theta}\eta = r\varepsilon\eta_2.
\end{aligned}
$$

By using the above formula and (5.1)–(5.4) and (5.6), we assert that

$$
\begin{aligned}
\frac{\partial}{\partial\Theta}\mathcal{L}^\alpha(w,p,\Theta)\eta &= \nu(\varepsilon)^{-1}\frac{\partial w^\alpha}{\partial\xi}\widetilde{\Delta}\eta + \Big[-2\nu\varepsilon^{-1}r\Theta_\beta\frac{\partial^2 w^\alpha}{\partial\xi^2} + \nu(r\varepsilon)^{-1}\frac{\partial w^\lambda}{\partial\xi}(\delta_{\alpha\lambda}\delta_{\beta 2} + 2\delta_{2\alpha}\delta_{\lambda\beta})\\
&\quad - 2\nu\varepsilon^{-1}\frac{\partial^2 w^\alpha}{\partial\xi\partial x^\beta} - \varepsilon^{-1}\partial_\xi p\delta_{\alpha\beta}\Big]\eta_\beta,\\
\frac{\partial}{\partial\Theta}\mathcal{L}^3(w,p,\Theta)\eta &= -\nu(\varepsilon)^{-1}\frac{\partial w^3}{\partial\xi}\widetilde{\Delta}\eta - \nu\varepsilon^{-1}w^\sigma\partial_\sigma\widetilde{\Delta}\eta + 2\nu\varepsilon^{-1}\Big(\frac{\partial w^\sigma}{\partial\xi}\Theta_\beta - \frac{\partial w^\sigma}{\partial x^\beta}\Big)\eta_{\beta\sigma}\\
&\quad + \Big[-2\nu\varepsilon^{-1}r\Theta_\beta\frac{\partial^2 w^3}{\partial\xi^2} + 2\nu(\varepsilon)^{-2}\frac{\partial w^\sigma}{\partial\xi}\Theta_{\beta\sigma} - (r\varepsilon)^{-1}\delta_{2\beta}\frac{\partial w^3}{\partial\xi} - 2\nu\varepsilon^{-1}\frac{\partial^2 w^3}{\partial\xi\partial x^\beta}\\
&\quad - 2\nu(r\varepsilon)^{-1}\frac{\partial w^\beta}{\partial r} - \varepsilon^{-1}rw^2\delta_{2\beta} - \varepsilon^{-1}\frac{\partial p}{\partial x^\beta} + \varepsilon^{-2}\Theta_\beta\frac{\partial p}{\partial\xi}\Big]\eta_\beta,\\
\frac{\partial}{\partial\Theta}N^\alpha(w,w,\Theta)\eta &= -[r\delta_{2\alpha}(\Theta_\beta\delta_\gamma^\lambda + \Theta_\gamma\delta_\beta^\lambda)w^\beta w^\gamma + 2r\varepsilon\delta_{2\alpha}w^3 w^\lambda]\eta_\lambda,\\
\frac{\partial}{\partial\Theta}N^3(w,w,\Theta)\eta &= \varepsilon^{-1}w^\alpha w^\beta\eta_{\alpha\beta} + [(r\varepsilon)^{-1}(a_{2\alpha}\delta_\beta^\lambda + a_{2\beta}\delta_\alpha^\lambda + a_{\alpha\beta}\delta_2^\lambda - \delta_{\alpha\beta}\delta_2^\lambda)w^\beta w^\alpha\\
&\quad + 2r(\Theta_2\delta_{\alpha\lambda} + \Theta_\alpha\delta_{2\lambda})w^3 w^\alpha + r\varepsilon\delta_{2\lambda}w^3 w^3]\eta_\lambda.
\end{aligned}
$$

Hence, from (5.14) it is yields that

$$
\begin{aligned}
R^\alpha(w,p,\Theta)\eta :&= \nu(\varepsilon)^{-1}\frac{\partial w^\alpha}{\partial\xi}\widetilde{\Delta}\eta + \Big[-2\nu\varepsilon^{-1}r\Theta_\beta\frac{\partial^2 w^\alpha}{\partial\xi^2}\\
&\quad + \nu(r\varepsilon)^{-1}\frac{\partial w^\lambda}{\partial\xi}(\delta_{\alpha\lambda}\delta_{\beta 2} + 2\delta_{2\alpha}\delta_{\lambda\beta}) - 2\nu\varepsilon^{-1}\frac{\partial^2 w^\alpha}{\partial\xi\partial x^\beta} - \varepsilon^{-1}\partial_\xi p\delta_{\alpha\beta}\Big]\eta_\beta
\end{aligned}
$$

$$
\begin{aligned}
&\qquad - [2r\delta_{2\alpha}\Theta_\lambda w^\lambda w^\beta + 2r\varepsilon\delta_{2\alpha}w^3w^\beta]\eta_\beta,\\
R^3(w,p,\Theta)\eta = {}& -\nu\varepsilon^{-1}w^\sigma\partial_\sigma\widetilde{\Delta}\eta + \nu\varepsilon^{-1}\Big(2\frac{\partial w^\sigma}{\partial\xi}\Theta_\beta - 2\frac{\partial w^\sigma}{\partial x^\beta} - \frac{\partial w^3}{\partial\xi}\delta_{\beta\sigma}\Big)\eta_{\beta\sigma}\\
&+ \Big[- 2\nu\varepsilon^{-1}r\Theta_\beta\frac{\partial^2 w^3}{\partial\xi^2} + 2\nu(\varepsilon)^{-2}\frac{\partial w^\sigma}{\partial\xi}\Theta_{\beta\sigma} - 2\nu(r\varepsilon)^{-1}\Big(\Big(\delta_{2\beta} + r\frac{\partial}{\partial x^\beta}\Big)\frac{\partial w^3}{\partial\xi}\\
&+ \frac{\partial w^\beta}{\partial r} + \frac{w^2}{r}\delta_{2\beta}\Big) - \varepsilon^{-1}\frac{\partial p}{\partial x^\beta} + \varepsilon^{-2}\Theta_\beta\frac{\partial p}{\partial\xi}\Big]\eta_\beta\\
&+ \varepsilon^{-1}w^\alpha w^\beta\eta_{\alpha\beta} + [(r\varepsilon)^{-1}(a_{2\alpha}\delta^\beta_\lambda + a_{2\lambda}\delta^\beta_\alpha + a_{\alpha\lambda}\delta^\beta_2 - \delta_{\alpha\lambda}\delta^\beta_2)w^\lambda w^\alpha\\
&+ 2r(\Theta_2\delta_{\alpha\beta} + \Theta_\alpha\delta_{2\beta})w^3w^\alpha + r\varepsilon\delta_{2\beta}w^3w^3]\eta_\beta.
\end{aligned}
$$

(5.9) is derived. It is obvious that since the arbitrary of the direction η with homogenous boundary conditions in $V(\Omega)$, (5.9) can be expressed as follows

$$
\begin{cases}
R^\alpha(w,p,\Theta) := -\dfrac{\partial}{\partial x^\beta}\Big\{ - 2\nu\varepsilon^{-1}r\Theta_\beta\dfrac{\partial^2 w^\alpha}{\partial\xi^2} + \nu(r\varepsilon)^{-1}\dfrac{\partial w^\lambda}{\partial\xi}(\delta_{\alpha\lambda}\delta_{\beta 2} + 2\delta_{2\alpha}\delta_{\lambda\beta})\\
\qquad -3\nu\varepsilon^{-1}\dfrac{\partial^2 w^\alpha}{\partial\xi\partial x^\beta} - \varepsilon^{-1}\partial_\xi p\delta_{\alpha\beta} - 2r\delta_{2\alpha}\Pi(w,\Theta)w^\beta\Big\},\\
R^3(w,p,\Theta) \quad = -\dfrac{\partial}{\partial x^\beta}\Big\{\nu\varepsilon^{-1}\dfrac{\partial^2 w^\sigma}{\partial x^\beta\partial x^\sigma} + \nu\varepsilon^{-1}\dfrac{\partial}{\partial\xi}\Big(\dfrac{\partial w^3}{\partial x^\beta} - 2\dfrac{\partial w^\sigma}{\partial x^\sigma}\Theta_\beta\Big) - 2\nu\varepsilon^{-1}r\Theta_\beta\dfrac{\partial^2 w^3}{\partial\xi^2}\\
\qquad -2\nu(r\varepsilon)^{-1}\Big(\Big(\delta_{2\beta} + r\dfrac{\partial}{\partial x^\beta}\Big)\dfrac{\partial w^3}{\partial\xi} + \dfrac{\partial w^\beta}{\partial r} + \dfrac{w^2}{r}\delta_{2\beta}\Big) - \varepsilon^{-1}\dfrac{\partial p}{\partial x^\beta} + \varepsilon^{-2}\Theta_\beta\dfrac{\partial p}{\partial\xi}\\
\qquad -\dfrac{\partial}{\partial x^\alpha}(\varepsilon^{-1}w^\alpha w^\beta) + (r\varepsilon)^{-1}(a_{2\alpha}\delta^\beta_\lambda + a_{2\lambda}\delta^\beta_\alpha + a_{\alpha\lambda}\delta^\beta_2 - \delta_{\alpha\lambda}\delta^\beta_2)w^\lambda w^\alpha\\
\qquad +2r(\Theta_2\delta_{\alpha\beta} + \Theta_\alpha\delta_{2\beta})w^3w^\alpha + r\varepsilon\delta_{2\beta}w^3w^3\Big\}.
\end{cases}
$$

In order to consider the variational formulation for the Gâteaux derivatives $(\widehat{w},\widehat{p})$, let compute $(\mathbf{R}(w,p,\Theta),\mathbf{v})$. Indeed,

$$
\begin{aligned}
(\mathbf{R}(w,p,\Theta),\mathbf{v}) &= \int_\Omega g_{ij}R^iv^j\sqrt{g}\mathrm{d}x\mathrm{d}\xi\\
&= \int_\Omega [a_{\alpha\beta}R^\alpha v^\beta + \varepsilon r^2\Theta_\alpha(R^\alpha v^3 + R^3v^\alpha) + \varepsilon^2r^2R^3v^3]\sqrt{g}\mathrm{d}x\mathrm{d}\xi\\
&= \int_\Omega [R^\alpha(w,p,\Theta)(a_{\alpha\lambda}v^\lambda + \varepsilon r^2\Theta_\alpha v^3) + R^3(w,p,\Theta)(\varepsilon r^2\Theta_\lambda v^\lambda + \varepsilon^2r^2v^3)]\sqrt{g}\mathrm{d}x\mathrm{d}\xi,
\end{aligned}
$$

from which it follows

$$
\begin{aligned}
(\mathbf{R}(w,p,\Theta),\mathbf{v}) = \int_\Omega \Big[&\Big(- 2\nu\varepsilon^{-1}r\Theta_\beta\frac{\partial^2 w^\alpha}{\partial\xi^2} + \nu(r\varepsilon)^{-1}\frac{\partial w^\lambda}{\partial\xi}(\delta_{\alpha\lambda}\delta_{\beta 2} + 2\delta_{2\alpha}\delta_{\lambda\beta}) - 3\nu\varepsilon^{-1}\frac{\partial^2 w^\alpha}{\partial\xi\partial x^\beta}\\
&- \varepsilon^{-1}\partial_\xi p\delta_{\alpha\beta} - 2r\delta_{2\alpha}\Pi(w,\Theta)w^\beta\Big)\partial_\beta(a_{\alpha\lambda}v^\lambda + \varepsilon r^2\Theta_\alpha v^3)\\
&+ \Big(\nu\varepsilon^{-1}\frac{\partial^2 w^\sigma}{\partial x^\beta\partial x^\sigma} + \nu\varepsilon^{-1}\frac{\partial}{\partial\xi}\Big(\frac{\partial w^3}{\partial x^\beta} - 2\frac{\partial w^\sigma}{\partial x^\sigma}\Theta_\beta\Big) - 2\nu\varepsilon^{-1}r\Theta_\beta\frac{\partial^2 w^3}{\partial\xi^2}\\
&- 2\nu(r\varepsilon)^{-1}\Big(\Big(\delta_{2\beta} + r\frac{\partial}{\partial x^\beta}\Big)\frac{\partial w^3}{\partial\xi} + \frac{\partial w^\beta}{\partial r} + \frac{w^2}{r}\delta_{2\beta}\Big) - \varepsilon^{-1}\frac{\partial p}{\partial x^\beta} + \varepsilon^{-2}\Theta_\beta\frac{\partial p}{\partial\xi}
\end{aligned}
$$

$$-\frac{\partial}{\partial x^\alpha}(\varepsilon^{-1}w^\alpha w^\beta)+(r\varepsilon)^{-1}(a_{2\alpha}\delta^\beta_\lambda+a_{2\lambda}\delta^\beta_\alpha+a_{\alpha\lambda}\delta^\beta_2-\delta_{\alpha\lambda}\delta^\beta_2)w^\lambda w^\alpha$$
$$+2r(\Theta_2\delta_{\alpha\beta}+\Theta_\alpha\delta_{2\beta})w^3w^\alpha+r\varepsilon\delta_{2\beta}w^3w^3\Big)\frac{\partial}{\partial x^\beta}(\varepsilon r^2\Theta_\lambda v^\lambda+\varepsilon^2r^2v^3)\Big]\sqrt{g}\,\mathrm{d}x\mathrm{d}\xi.$$

This is (5.11′). The proof is completed.

Corollary 5.1 *Assume that Θ is small enough and such that* (3.38) *holds and that the solution $(w,p)\in H^2(\Omega)\times H^1(\Omega)$. Then there exists a constant C_9 independent of w and Θ, which makes the following estimate to be valid:*

$$\|R(w,\Theta)\|_*\le C(\Omega)(1+\kappa_3)(\|w\|^2_{2.\Omega}+\|p\|_{1,\Omega}). \tag{5.15}$$

Theorem 5.3 *Assume that the assumptions in Theorem* 5.1 *are satisfied. Furthermore, $(w,p)\in V(\Omega)\cap H^3(\Omega)\times H^2(\Omega)$ is a solution of* (3.7) *and satisfies*

$$\|w\|_\Omega\le\frac{1}{2}\frac{\kappa_0}{C(\Omega)(1+\kappa_3)},\quad \|w\|_{2,\Omega}+\|p\|_{1,\Omega}\le\frac{\kappa_0}{2C^3(\Omega)(1+\kappa_3)(1+\kappa_3)^2}. \tag{5.16}$$

Then there exists a pair $(\widehat{w},\widehat{p})\in V(\Omega)\times L^2(\Omega)$ solutions of (5.10)*, i.e., the G-derivatives of the solution of the Navier-Stokes equation* (3.7)*, and it satisfies*

$$\|\widehat{w}\|_\Omega\le\frac{\kappa_0}{4C(\Omega)(1+\kappa_3)}\Big[1-\sqrt{1-\frac{4C^2(\Omega)(1+\kappa_3^2)\|R(w,\Theta)\|_*}{\kappa_0^2}}\Big]. \tag{5.17}$$

Proof First, the bilinear form

$$\widetilde{a}(\widehat{w},v):=a(\widehat{w},v)+b(w;\widehat{w},v)+b(\widehat{w};w,v)+2(\omega\times\widehat{w},v)$$

is continuous form $V(\Omega)\times L^2(\Omega)$ to R; in particular, it is coercive. Indeed, by Lemma 3.6 and Lemma 3.7, we claim that

$$\widetilde{a}(\widehat{w},\widehat{w})\ge\kappa_0\|\widehat{w}\|^2_\Omega-2C(\Omega)(1+\kappa_3)\|w\|_\Omega\|\widehat{w}\|^2_\Omega=(\kappa_0-2C(\Omega)(1+\kappa_3)\|w\|_\Omega)\|\widehat{w}\|^2_\Omega.$$

In view of (4.6),

$$\kappa_0-2C(\Omega)(1+\kappa_3)\|w\|_\Omega\ge\kappa_0-2C(\Omega)\kappa_3\frac{\kappa_0}{2C(\Omega)(1+\kappa_3)}\ge\frac{1}{2}\kappa_0.$$

Therefore, $\widetilde{a}(\widehat{w},\widehat{w})\ge\frac{\kappa_0}{2}\|\widehat{w}\|^2_\Omega$. Furthermore, combining (5.13) and (5.14), we assert that

$$\|R(w,\Theta)\|_*\le\frac{\kappa_0}{2C^2(\Omega)(1+\kappa_3)^2}.$$

By an analog argument as in the proof of Theorem 4.1, we claim that there exists a smooth solution $(\widehat{w},\widehat{p})$ of the variational problem (5.10) which satisfies

$$\|\widehat{w}\|_\Omega\le\frac{\kappa_0}{4C(\Omega)(1+\kappa_3)}\Big[1-\sqrt{1-\frac{4C^2(\Omega)(1+\kappa_3)^2\|R(w,\Theta)\|_*}{\kappa_0^2}}\Big].$$

This is (5.15) and this completes our proof.

For the compressible case, we resolve

$$\begin{aligned}
&\operatorname{div}(\widehat{w}\rho + w\widehat{\rho}) = 0,\\
&\operatorname{div}(\rho\widehat{w}^i w + \rho w^i\widehat{w} + \widehat{\rho} w w^i) + 2\widehat{\rho}(\omega\times w)^i + 2\rho(\omega\omega\times\widehat{w})^i\\
&\quad + a g^{ij}\nabla_j(\gamma\rho^{\gamma-1}\widehat{\rho}) - \nabla_j(A^{ijkm}e_{km}(\widehat{w})) = S^i(w,\rho),
\end{aligned} \tag{5.18}$$

where

$$S^i(w,p,\Theta) = -\partial_\beta S^{i;\beta}(w,p;\Theta) + \partial^2_{\lambda\sigma} S^{i;(\lambda,\sigma)}(w,p;\Theta) \tag{5.19}$$

with

$$\begin{aligned}
S^{\alpha;\beta}(w,p,\Theta) &= r\delta_2^\alpha[(\delta_\lambda^\beta\Theta_\sigma + \delta_\sigma^\beta\Theta_\lambda)w^\lambda w^\sigma + 2\varepsilon w^3 w^\beta] + 2\rho r\omega\delta_{\alpha 2}w^\beta\\
&\quad - \varepsilon^{-1}\delta_{\alpha\beta}\frac{\partial(a\rho^\gamma)}{\partial\xi} + 2\nu\varepsilon^{-1}[-\Theta_\alpha g^{jm}\nabla_j e^\beta_{3m}(w) - (\varepsilon^{-2}\Theta_\alpha\Theta_\beta\\
&\quad + \delta_{\alpha\beta}g^{33})\nabla_3 e_{33}(w) + \varepsilon^{-1}(\Theta_\alpha\delta_{\beta\lambda} + \Theta_\lambda\delta_{\alpha\beta})\nabla_\lambda e_{33}(w)\\
&\quad + \varepsilon^{-1}(\Theta_\alpha\delta_{\beta\lambda} + 2\Theta_\beta\delta_{\alpha\lambda} + \Theta_\lambda\delta_{\alpha\beta})\nabla_3 e_{3\lambda}(w)\\
&\quad - \delta_{\alpha\beta}\nabla_\lambda e_{3\lambda}(w) - \nabla_\beta e_{3\alpha}(w) - \nabla_3 e_{\alpha\beta}(w)\\
&\quad - r^{-1}(\delta_{\alpha 2}\delta_{\beta\lambda} + 3\delta_{\alpha\lambda}\delta_{\beta 2})(e_{3\lambda}(w) - \varepsilon^{-1}\Theta_\lambda e_{33}(w))],\\
S^{\alpha;(\lambda,\sigma)}(w,p,;\Theta) &= -2\nu\varepsilon^{-1}[(\delta_{\alpha\beta}\delta_{\gamma\lambda} + \delta_{\alpha\gamma}\delta_{\beta\lambda})(e_{3\gamma}(w) - \varepsilon^{-1}\Theta_\gamma e_{33}(w))\\
&\quad + \Theta_\alpha g^{jm}(\nabla_j e^{\beta\lambda}_{3m}(w,\Theta)];
\end{aligned} \tag{5.20}$$

$$\begin{aligned}
S^{3;\beta}(w,p,\Theta) &= -[(r\varepsilon)^{-1}(\delta_{2\lambda}\delta_\sigma^\beta + \delta_{2\sigma}\delta_\lambda^\beta + r^2(\delta_2^\beta\Theta_\lambda\Theta_\sigma + \Theta_2\delta_\sigma^\beta\Theta_\lambda\\
&\quad + \Theta_2\Theta_\sigma\delta_\lambda^\beta))w^\lambda w^\sigma + 2r(\delta_\lambda^\beta\Theta_2 + \delta_2^\beta\Theta_\lambda)w^3 w^\lambda + r\varepsilon\delta_2^\beta w^3 w^3]\\
&\quad + 2r\omega((w^3 + \varepsilon^{-1}w^\lambda\Theta_\lambda)\delta_{\beta 2} + \varepsilon^{-1}w^\beta\Theta_2) - \varepsilon^{-1}\nabla_\beta p\\
&\quad + 2\varepsilon^{-2}\Theta_\beta\frac{\partial p}{\partial\xi} + 2\nu g^{3k}g^{jm}\nabla_j e^\beta_{km}(w)\\
&\quad + 2\nu\varepsilon^{-1}[g^{33}(4\varepsilon^{-1}\nabla_3 e_{33}(w) - \nabla_\beta e_{33}(w)) + \nabla_\lambda e_{\beta\lambda}(w)\\
&\quad + \varepsilon^{-1}(2\delta_{\beta\lambda}\nabla_\sigma e_{3\sigma}(w) - \nabla_\lambda e_{3\beta}(w) + \nabla_\beta e_{3\lambda}(w))\Theta_\lambda\\
&\quad - 2\varepsilon^{-2}(2\nabla_3 e_{3\lambda}(w) + \nabla_\lambda e_{33}(w))] - 2\nu[-(r\varepsilon)^{-1}e_{2\beta}(w)\\
&\quad + e_{3\gamma}(w)[\varepsilon^{-2}r^{-1}(\delta_{\gamma 2}\Theta_\beta - 3\delta_{\beta 2}\Theta_\gamma) + r\varepsilon^{-2}(\delta_{\beta\gamma}\Theta_2 - \delta_{2\beta}\Theta_\gamma)|\nabla\Theta|^2]\\
&\quad + e_{33}(w)(r\varepsilon)^{-3}\delta_{2\beta}(4 + 3r^2|\nabla\Theta|^2)],\\
S^{3;(\lambda,\sigma)}(w,p,;\Theta) &= 2\nu[\varepsilon^{-1}(\varepsilon^{-2}\Theta_\lambda\Theta_\sigma + \delta_{\lambda\sigma}g^{33})e_{33}(w) - \varepsilon^{-2}(\Theta_\lambda\delta_{\gamma\sigma} + \Theta_\gamma\delta_{\lambda\sigma})e_{3\gamma}(w)]\\
&\quad - \varepsilon^{-2}w^\lambda w^\sigma + 2\nu g^{3k}g^{jm}\nabla_j e^{\lambda\sigma}_{km}(w,\Theta).
\end{aligned} \tag{5.21}$$

6 Control Problem of the Boundary Shape

The notations used in this paper are usual, for example, the norms in the space $L^2(D)$ and $H^m(D), m \geq 1$ are denoted as usually by $|\cdot|_{0,D}$ and $\|\cdot\|_{m,D}$ and those in the spaces $L^\infty(D)$ and $W^{1,\infty}(D)$ are denoted by $|\cdot|_{0,\infty,D}$ and $\|\cdot\|_{1,\infty,D}$. The same notations are used for the norms in the corresponding spaces of vector fields, such spaces being then denoted by boldface letters. Strong and weak convergences are denoted by $\to$ and $\rightharpoonup$, respectively (a review of all the properties relevant here about weak convergence and lower semi-continuity is found, e.g., in [19]). Naturally, we choose global dissipative energy to be the objective functional for the shape control of the blade surface. The global dissipative energy functional is given by

$$\begin{aligned}
&\Phi(w,v)=A^{ijkl}e_{kl}(w)e_{ij}(v),\\
&J(S)=\frac{1}{2}\iiint_{\Omega_\varepsilon}\Phi(w(S),w(S))\mathrm{d}V=\iiint_{\Omega_\varepsilon}A^{ijkl}e_{kl}(w)e_{ij}(w)\sqrt{g}\,\mathrm{d}x\mathrm{d}\xi,\\
&A^{ijkl}=\mu(g^{ik}g^{jl}+g^{il}g^{jk})\quad\text{for the incompressible viscous fluid,}
\end{aligned}\tag{6.1}$$

where $\Omega=D\times[-1,1]$ and Ω_ε is the flow passage in the Turbo-machinery bounded by $\Gamma_t\cup\Gamma_b\cup\Gamma_{\text{in}}\cup\Gamma_{\text{out}}\cup S_+\cup S_-$. We propose the following variational principle for the geometric design of the blade:

$$\begin{cases}\text{Find a surface } \Im\in\mathcal{F} \text{ of the blade such that}\\ J(\Im)=\inf\limits_{S\in\mathcal{F}}J(S),\\ \mathcal{F}=\{\zeta\in H^2(D),\ \zeta=\Theta_0,\ \zeta=\Theta_*,\ \text{on }\partial D,\ \|\zeta\|_{2,D}\le\kappa_0\},\end{cases}\tag{6.2}$$

where Θ_0 and Θ_* are functions in $H^2(D)$. The $\Im$ which achieves minimum of the object dissipative energy functional, is called a general minimal surface. In other words, from the mathematical point of view, this minimum problem of geometric sharp of the surface of the blade is a general minimal surface problem.

Note that (6.2) is also an optimal control problem with distributed parameters, where the control variable is the surface of the blade and the Navier-Stokes equations are the state equations of this control problem.

Subsequently, we establish the Euler-Lagrange equations of the optimization problem (6.2) with the Navier-Stokes equations being its state equations.

In order to investigate the optimal control problem (6.2), we should consider the object functional J in a fixed domain in the new coordinate system. In this case, we rewrite with (3.25)

$$\begin{aligned}
J(\Im):&=\frac{1}{2}a(w,w)=\frac{1}{2}\int_\Omega\Phi(w)\sqrt{g}\,\mathrm{d}\xi\mathrm{d}x,\\
\Phi(w):&=A^{ijkl}e_{kl}(w)e_{ij}(w)\\
&=A^{ijkl}[\varphi_{kl}(w)\varphi_{ij}(w)+2\varphi_{kl}(w)\psi_{ij}(w,\Theta)+\psi_{kl}(w,\Theta)\psi_{ij}(w,\Theta)].
\end{aligned}\tag{6.3}$$

Lemma 6.1 *Assume that (w,p) is the solution of the Navier-Stokes equations (3.7) associated with $\Theta\in H^1(D)$ which defines a mapping $(w(\Theta),p(\Theta))$:*

$$\Theta\in C^2(D)\Rightarrow(w(\Theta),p(\Theta))\in\boldsymbol{H}^1(\Omega)\times L^2(D).$$

Then the strain rate tensor $e_{ij}(w)$ of the velocity w defined by (3.2) possesses a Gâteaux derivative $\frac{\mathcal{D}}{\mathcal{D}\Theta}e_{ij}(w)\eta$ at any point $\Theta\in C^2(D)$ along every direction $\eta\in W(D):=H_0^1(D)\cap H^2(D)$, and

$$\frac{\mathcal{D}}{\mathcal{D}\Theta}e_{ij}(w)\eta=e_{ij}(\widehat{w})\eta+e_{ij}^{\lambda}(w)\eta_\lambda+e_{ij}^{\lambda\sigma}(w)\eta_{\lambda\sigma},\tag{6.4}$$

where

$$\begin{cases} \widehat{w} = \dfrac{\mathcal{D}}{\mathcal{D}\Theta} w, \quad \eta_\lambda = \dfrac{\partial \eta}{\partial x^\lambda}, \quad \eta_{\lambda\sigma} = \dfrac{\partial^2 \eta}{\partial x^\lambda \partial x^\sigma}, \\ e^\lambda_{\alpha\beta}(w) = \psi^\lambda_{\alpha\beta}(w) + (\psi^{\lambda\sigma}_{\alpha\beta}(w) + \psi^{\sigma\lambda}_{\alpha\beta}(w))\Theta_\sigma + \dfrac{1}{2} r^2 w^\sigma (\delta_{\alpha\lambda}\Theta_{\sigma\beta} + \delta_{\beta\lambda}\Theta_{\alpha\sigma}), \\ e^{\lambda\sigma}_{\alpha\beta}(w) = \dfrac{1}{2} r^2 w^\sigma (\delta_{\alpha\lambda}\Theta_\beta + \delta_{\beta\lambda}\Theta_\alpha), \\ e^\lambda_{3\alpha}(w) = \psi^\lambda_{3\alpha}(w) + (\psi^{\lambda\sigma}_{3\alpha}(w) + \psi^{\sigma\lambda}_{3\alpha}(w))\Theta_\sigma, \quad e^{\lambda\sigma}_{3\alpha}(w) = \dfrac{1}{2}\varepsilon r^2 w^\sigma \delta_{\alpha\lambda}, \\ e^\lambda_{33}(w) = \psi^\lambda_{33}(w), \quad e^{\lambda\sigma}_{33}(w) = 0, \end{cases} \tag{6.5}$$

where $\psi, \psi^\lambda, \psi^{\lambda\sigma}$ *are defined by* (3.22).

Proof By similar manner in the proof of Theorem 5.2 and using (3.20)–(3.22),

$$\frac{\mathcal{D}}{\mathcal{D}\Theta} e_{ij}(w)\eta = \frac{\partial e_{ij}}{\partial w}\widehat{w}\eta + \frac{\partial e_{ij}}{\partial \Theta}\eta = e_{ij}(\widehat{w})\eta + \frac{\partial e_{ij}}{\partial \Theta}\eta,$$
$$\frac{\partial e_{ij}}{\partial \Theta}\eta = \psi^\gamma_{ij}(w)\eta_\gamma + \psi^{\lambda\sigma}_{ij}(w)(\delta_{\lambda\gamma}\Theta_\sigma\delta_{\sigma\gamma}\Theta_\lambda)\eta_\gamma + \frac{\partial e^*_{ij}}{\partial \Theta}\eta,$$

where

$$\frac{\partial e^*_{\alpha\beta}}{\partial \Theta}\eta = \frac{1}{2} r^2 w^\sigma \partial_\sigma(\Theta_\alpha\delta_{\beta\gamma} + \Theta_\beta\delta_{\alpha\gamma})\eta_\gamma, \quad \frac{\partial e^*_{3\alpha}}{\partial \Theta}\eta = \frac{1}{2}\varepsilon r^2 w^\sigma \eta_{\alpha\sigma}, \quad \frac{\partial e^*_{33}}{\partial \Theta}\eta = 0.$$

From this it is easy to obtain (6.4) and (6.5) . The proof is completed.

Lemma 6.2 *The dissipative functions* $\Phi(w)$ *defined by* (4.1) *is Gâteaux differentiable at* $\Theta \in C^2(D)$ *along any direction* $\eta \in W$*. The Gâteaux derivative is a polynomial of degree* 5 :

$$\frac{\mathcal{D}\Phi(w)}{\mathcal{D}\Theta}\eta = \Phi^0(\widehat{w}, w)\eta + \Phi^\lambda(w,\Theta)\eta_\lambda + \Phi^{\lambda\sigma}(w,\Theta)\eta_{\lambda\sigma}, \tag{6.6}$$

where

$$\begin{aligned} \Phi^0(\widehat{w}, w) &= 2A^{ijkl} e_{ij}(\widehat{w}) e_{kl}(w) \\ &= 4\mu[e_{\alpha\beta}(\widehat{w})e_{\alpha\beta}(w) + g^{33}g^{33}e_{33}(\widehat{w})e_{33}(w) \\ &\quad + 2(\varepsilon^{-2}\Theta_\alpha\Theta_\beta + \delta^{\alpha\beta}g^{33})e_{3\alpha}(\widehat{w})e_{3\beta}(w) \\ &\quad + \varepsilon^{-2}\Theta_\alpha\Theta_\beta(e_{33}(\widehat{w})e_{\alpha\beta}(w) + e_{\alpha\beta}(w)e_{33}(\widehat{w})) \\ &\quad - 2\varepsilon^{-1}\Theta_\beta(e_{\alpha\beta}(\widehat{w})e_{3\alpha}(w) + e_{3\alpha}(\widehat{w})e_{\alpha\beta}(w)) \\ &\quad - 2\varepsilon^{-1}\Theta_\alpha g^{33}(e_{33}(\widehat{w})e_{3\alpha}(v) + e_{3\alpha}(\widehat{w})e_{33}(v))], \\ \Phi^\lambda(w,\Theta) &= 2A^{ijkl}e_{kl}(w)e^\lambda_{ij}(w) + \frac{\mathcal{D}A^{ijkl}}{\mathcal{D}\Theta}\eta e_{kl}(w)e_{ij}(w) \\ &= 4\mu[(e_{\alpha\beta}(w) + \varepsilon^{-2}\Theta_\alpha\Theta_\beta e_{33}(w) - 2\varepsilon^{-1}\Theta_\beta e_{3\alpha}(w))e^\lambda_{\alpha\beta}(w) \\ &\quad + (2(r\varepsilon)^{-2}(r^2\Theta_\alpha\Theta_\beta + a\delta_{\alpha\beta})e_{3\beta}(w) - 2\varepsilon^{-1}\Theta_\beta e_{\alpha\beta}(w) \\ &\quad - 2\varepsilon^{-1}\Theta_\alpha g^{33}e_{33}(w))e^\lambda_{3\alpha}(w) \\ &\quad + (g^{33}g^{33}e_{33}(w) + \varepsilon^{-2}\Theta_\alpha\Theta_\beta e_{\alpha\beta}(w) - 2\varepsilon^{-1}\Theta_\alpha g^{33}e_{3\alpha}(w))e^\lambda_{33}(w)] \\ &\quad + 2\mu[4r^{-4}\varepsilon^{-4}a\Theta_\lambda e_{33}(w)e_{33}(w) + 4\varepsilon^{-2}(\Theta_\alpha\delta_{\beta\lambda} + \Theta_\lambda\delta_{\alpha\beta})e_{3\alpha}(w)e_{3\beta}(w) \qquad (6.7) \\ &\quad + 4\varepsilon^{-2}\Theta_\alpha e_{33}(w)e_{\alpha\lambda}(w) - 4\varepsilon^{-1}e_{3\alpha}(w)e_{\alpha\lambda}(w) \end{aligned}$$

$$
\begin{aligned}
&\quad - 4\varepsilon^{-3}r^{-2}(a\delta_{\alpha\lambda} + 2r^2\Theta_\alpha\Theta_\lambda)e_{33}(w)e_{3\alpha}(w)],\\
\Phi^{\lambda\sigma}(w,\Theta) &= 2A^{ijkl}e^{\lambda\sigma}_{ij}(w)e_{kl}(w)\\
&= 4\mu[e_{\alpha\beta}(w)e^{\lambda\sigma}_{\alpha\beta}(w) + g^{33}g^{33}e_{33}(w)e^{\lambda\sigma}_{33}(w)\\
&\quad + 2(\varepsilon^{-2}\Theta_\alpha\Theta_\beta + \delta^{\alpha\beta}g^{33})e_{3\alpha}(w)e^{\lambda\sigma}_{3\beta}(w)\\
&\quad + \varepsilon^{-2}\Theta_\alpha\Theta_\beta(e_{33}(w)e^{\lambda\sigma}_{\alpha\beta}(w) + e_{\alpha\beta}(w)e^{\lambda\sigma}_{33}(w))\\
&\quad - 2\varepsilon^{-1}\Theta_\beta(e_{\alpha\beta}(w)e^{\lambda\sigma}_{3\alpha}(w) + e_{3\alpha}(w)e^{\lambda\sigma}_{\alpha\beta}(w))\\
&\quad - 2\varepsilon^{-1}\Theta_\alpha g^{33}(e_{33}(w)e^{\lambda\sigma}_{3\alpha}(w) + e_{3\alpha}(w)e^{\lambda\sigma}_{33}(w))]\\
&= 2\mu w^\sigma\Big[\varepsilon^{-1}\Big(\frac{\partial w^\lambda}{\partial\xi} + r^2\frac{\partial w^3}{\partial x^\lambda}\Big) + r^2\Big(\frac{\partial w^\nu}{\partial x^\lambda} - \delta_{\lambda\nu}\frac{\partial w^3}{\partial\xi}\Big)\Theta_\nu - \varepsilon^{-1}r^2\Theta_\lambda\Theta_\nu\frac{\partial w^\nu}{\partial\xi}\Big]\\
&\quad + 2\mu r^2 w^\nu w^\sigma\Theta_{\nu\lambda},
\end{aligned}
$$

where $e^{\lambda}_{ij}(w)$, $e^{\lambda\sigma}_{ij}(w)$ *are defined by* (5.12).

Proof At first,

$$
\frac{\mathcal{D}\Phi(w,\Theta)}{\mathcal{D}\Theta}\eta = 2A^{ijkl}\frac{\mathcal{D}e_{kl}(w)}{\mathcal{D}\Theta}\eta e_{ij}(w) + \frac{\mathcal{D}A^{ijkl}}{\mathcal{D}\Theta}\eta e_{kl}(w)e_{ij}(w). \tag{6.8}
$$

Due to (5.11) and (5.12), we assert that

$$
\begin{aligned}
\frac{\mathcal{D}\Phi(w,\Theta)}{\mathcal{D}\Theta} &= \Big[2A^{ijkl}(e_{kl}(\widehat{w})\eta + e^{\lambda}_{kl}(w)\eta_\lambda + e^{\lambda\sigma}_{kl}(w)\eta_{\lambda\sigma}) + \frac{\mathcal{D}A^{ijkl}}{\mathcal{D}\Theta}\eta e_{kl}(w)\Big]e_{ij}(w)\\
&= \Phi^0(\widehat{w},w,\Theta)\eta + \Phi^\lambda(w,\Theta)\eta_\lambda + \Phi^{\lambda\sigma}(w,\Theta)\eta_{\lambda\sigma}.
\end{aligned} \tag{6.9}
$$

Thanks to (2.5) and (3.24), simple calculations show that

$$
\begin{aligned}
\frac{\mathcal{D}A^{ijkl}}{\mathcal{D}\Theta}\eta e_{kl}(w)e_{ij}(w) &= 2\mu[4r^{-4}\varepsilon^{-4}a\Theta_\lambda e_{33}(w)e_{33}(w)\\
&\quad + 4\varepsilon^{-2}(\Theta_\alpha\delta_{\beta\lambda} + \Theta_\lambda\delta_{\alpha\beta})e_{3\alpha}(w)e_{3\beta}(w)\\
&\quad + 4\varepsilon^{-2}\Theta_\alpha e_{33}(w)e_{\alpha\lambda}(w) - 4\varepsilon^{-1}e_{3\alpha}(w)e_{\alpha\lambda}(w)\\
&\quad - 4\varepsilon^{-3}r^{-2}(a\delta_{\alpha\lambda} + 2r^2\Theta_\alpha\Theta_\lambda)e_{33}(w)e_{3\alpha}(w)]\eta_\lambda, \qquad (6.10)\\
2A^{ijkl}e^{\lambda}_{kl}(w)e_{ij}(w) &= 4\mu[(e_{\alpha\beta}(w) + \varepsilon^{-2}\Theta_\alpha\Theta_\beta e_{33}(w) - 2\varepsilon^{-1}\Theta_\beta e_{2\alpha}(w))e^{\lambda}_{\alpha\beta}(w)\\
&\quad + (2(r\varepsilon)^{-2}(r^2\Theta_\alpha\Theta_\beta + a\delta_{\alpha\beta})e_{3\beta}(w) - 2\varepsilon^{-1}\Theta_\beta e_{\alpha\beta}(w)\\
&\quad - 2\varepsilon^{-1}\Theta_\alpha g^{33}e_{33}(w))e^{\lambda}_{3\alpha}(w) + (g^{33}g^{33}e_{33}(w)\\
&\quad + \varepsilon^{-2}\Theta_\alpha\Theta_\beta e_{\alpha\beta}(w) - 2\varepsilon^{-1}\Theta_\alpha g^{33}e_{3\alpha}(w))e^{\lambda}_{33}(w)]. \qquad (6.11)
\end{aligned}
$$

In particular,

$$
\begin{aligned}
\Phi^{\lambda\sigma}(w,\Theta) &= 2A^{ijkl}e^{\lambda\sigma}_{ij}(w)e_{kl}(w)\\
&= 4\mu[e_{\alpha\beta}(w)e^{\lambda\sigma}_{\alpha\beta}(w) + g^{33}g^{33}e_{33}(w)e^{\lambda\sigma}_{33}(w)\\
&\quad + 2(\varepsilon^{-2}\Theta_\alpha\Theta_\beta + \delta^{\alpha\beta}g^{33})e_{3\alpha}(w)e^{\lambda\sigma}_{3\beta}(w)\\
&\quad + \varepsilon^{-2}\Theta_\alpha\Theta_\beta(e_{33}(w)e^{\lambda\sigma}_{\alpha\beta}(w) + e_{\alpha\beta}(w)e^{\lambda\sigma}_{33}(w))\\
&\quad - 2\varepsilon^{-1}\Theta_\beta(e_{\alpha\beta}(w)e^{\lambda\sigma}_{3\alpha}(w) + e_{3\alpha}(w)e^{\lambda\sigma}_{\alpha\beta}(w))\\
&\quad - 2\varepsilon^{-1}\Theta_\alpha g^{33}(e_{33}(w)e^{\lambda\sigma}_{3\alpha}(w) + e_{3\alpha}(w)e^{\lambda\sigma}_{33}(w))].
\end{aligned}
$$

By virtue of (6.5), $e^{\lambda\sigma}_{33}(w) = 0$, we obtain

$$
\begin{aligned}
\Phi^{\lambda\sigma}(w,\Theta) &= 4\mu[(e_{\alpha\beta}(w) + \varepsilon^{-2}\Theta_\alpha\Theta_\beta e_{33}(w) - 2\varepsilon^{-1}\Theta_\beta e_{3\alpha}(w))e^{\lambda\sigma}_{\alpha\beta}(w) \\
&\quad + (2(r\varepsilon)^{-2}(r^2\Theta_\alpha\Theta_\beta + a\delta_{\alpha\beta})e_{3\beta}(w) - 2\varepsilon^{-1}\Theta_\beta e_{\alpha\beta}(w) - 2\varepsilon^{-1}g^{33}\Theta_\alpha e_{33}(w))e^{\lambda\sigma}_{3\alpha}(w)] \\
&= 4\mu\Big[(e_{\alpha\beta}(w) + \varepsilon^{-2}\Theta_\alpha\Theta_\beta e_{33}(w) - 2\varepsilon^{-1}\Theta_\beta e_{3\alpha}(w))\frac{1}{2}r^2 w^\sigma(\delta_{\alpha\lambda}\Theta_\beta + \delta_{\beta\lambda}\Theta_\alpha) \\
&\quad + (2(r\varepsilon)^{-2}(r^2\Theta_\alpha\Theta_\beta + a\delta_{\alpha\beta})e_{3\beta}(w) - 2\varepsilon^{-1}\Theta_\beta e_{\alpha\beta}(w) \\
&\quad - 2\varepsilon^{-1}g^{33}\Theta_\alpha e_{33}(w))\frac{1}{2}\varepsilon r^2 w^\sigma\delta_{\alpha\lambda}\Big] \\
&= 4\mu\frac{1}{2}r^2 w^\sigma[(2\varepsilon^{-2}\Theta_\lambda|\nabla\Theta|^2 - 2g^{33}\Theta_\lambda)e_{33}(w) + 2\varepsilon^{-1}r^{-2}\delta_{\alpha\lambda}(a - r^2|\nabla\Theta|^2)e_{3\alpha}(w)] \\
&= 4\mu\frac{1}{2}r^2 w^\sigma[-2(r\varepsilon)^{-2}\Theta_\lambda e_{33}(w) + 2\varepsilon^{-1}r^{-2}e_{3\lambda}(w)] \\
&= \varepsilon^{-2}w^\sigma[-\Theta_\lambda e_{33}(w) + \varepsilon e_{3\lambda}(w)].
\end{aligned}
$$

Taking (3.20)–(3.23) into account yields

$$
\begin{aligned}
\Phi^{\lambda\sigma}(w,\Theta) &= 4\mu\varepsilon^{-2}w^\sigma[-\Theta_\lambda e_{33}(w) + \varepsilon e_{3\lambda}(w)] \\
&= 2\mu w^\sigma\Big[\varepsilon^{-1}\Big(\frac{\partial w^\lambda}{\partial\xi} + r^2\frac{\partial w^3}{\partial x^\lambda}\Big) + r^2\Big(\frac{\partial w^\nu}{\partial x^\lambda} - \delta_{\lambda\nu}\frac{\partial w^3}{\partial\xi}\Big)\Theta_\nu - \varepsilon^{-1}r^2\Theta_\lambda\Theta_\nu\frac{\partial w^\nu}{\partial\xi}\Big] \\
&\quad + 2\mu r^2 w^\nu w^\sigma\Theta_{\nu\lambda}.
\end{aligned} \tag{6.12}
$$

Summing up (6.7) is derived. The proof is completed.

Theorem 6.1 *Assume that $\Theta \in C^2(D,R)$ is an injective mapping. Then the object functional J defined by* (6.1) *has a Gâteaux derivative* $\mathrm{grad}_\Theta J \equiv \dfrac{\mathcal{D}J}{\mathcal{D}\Theta}$ *in every direction $\eta \in W := H^2(D)\cap H^1_0(D)$ that is determined by*

$$
\begin{aligned}
\langle \mathrm{grad}_\Theta(J(\Theta)),\eta\rangle = \iint_D[&\widehat{\Phi}^0(w;\widehat{w})\eta + \widehat{\Phi}^\lambda(w,\Theta)\eta_\lambda + \widehat{\Phi}^{\lambda\sigma}(w,\Theta)\eta_{\lambda\sigma} \\
&+ 2\mu r^2 W^{\nu\sigma}\Theta_{\lambda\nu}\eta_{\lambda\sigma}]\varepsilon r\mathrm{d}x^1\mathrm{d}x^2,
\end{aligned} \tag{6.13}
$$

where

$$
\begin{cases}
\widehat{\Phi}^0(w;\widehat{w}) = \displaystyle\int_{-1}^1 \Phi^0(w;\widehat{w})\mathrm{d}\xi, \quad \widehat{\Phi}^\lambda(w,\Theta) = \int_{-1}^1 \Phi^\lambda(w,\Theta)\mathrm{d}\xi, \\
\widehat{\Phi}^{\lambda\sigma}(w,\Theta) = \displaystyle\int_{-1}^1 2\mu w^\sigma\Big[\varepsilon^{-1}\Big(\frac{\partial w^\lambda}{\partial\xi} + r^2\frac{\partial w^3}{\partial x^\lambda}\Big) + r^2\Big(\frac{\partial w^\nu}{\partial x^\lambda} - \delta_{\lambda\nu}\frac{\partial w^3}{\partial\xi}\Big)\Theta_\nu \\
\qquad\qquad -\varepsilon^{-1}r^2\Theta_\lambda\Theta_\nu\dfrac{\partial w^\nu}{\partial\xi}\Big]\mathrm{d}\xi, \\
W^{\alpha\beta} = \displaystyle\int_{-1}^1 w^\alpha w^\beta\mathrm{d}\xi,
\end{cases} \tag{6.14}
$$

and where $\widehat{w} = \frac{\mathcal{D}w}{\mathcal{D}\Theta}$ is the Gâteaux derivative of the velocity w of the fluid at the point Θ, and Φ^0, Φ^λ are defined by (6.5).

Proof Indeed, taking into account of (6.5) and (6.6), and $\sqrt{g} = \varepsilon r$ we assert that

$$
\langle \mathrm{grad}_\Theta(J(\Theta)),\eta\rangle = \int_D\int_{-1}^1 \frac{\mathcal{D}\Phi(w,\Theta)}{\mathcal{D}\Theta}\eta\varepsilon r\mathrm{d}\xi\mathrm{d}x
$$

$$= \int_D \int_{-1}^{1} 2\mu[\Phi^0(\widehat{w}, w)\eta + \Phi^\lambda(w,\Theta)\eta_\lambda + \Phi^{\lambda\sigma}(w,\Theta)\eta_{\lambda\sigma}]r\varepsilon \mathrm{d}\xi \mathrm{d}x. \tag{6.15}$$

By using (6.6) and (6.7) it is easy to obtain (6.13). The proof is completed.

Taking integration by part of (6.13) and considering homogenous boundary conditions for $\eta \in W(D)$, this implies

$$\langle \mathrm{grad}_\Theta(J(\Theta)), \eta\rangle = \int_D \varepsilon[\partial_{\lambda\sigma}(2\mu r^3 W^{\sigma\nu}\Theta_{\nu\lambda}) + r\widehat{\Phi}^{\lambda\sigma}(w,\Theta)) - \partial_\lambda(r\widehat{\Phi}^\lambda(w,\Theta)) + r\widehat{\Phi}^0(w,\widehat{w})]\eta \mathrm{d}x.$$

From the above discussion we obtain directly the following result.

Theorem 6.2 *The Euler-Lagrange equation for the extremum Θ of J is given by:*

$$\begin{aligned} &\frac{\partial^2}{\partial x^\lambda \partial x^\sigma}\Big(2\mu r^3 W^{\nu\sigma}\frac{\partial^2\Theta}{\partial x^\nu \partial x^\lambda}\Big) + \frac{\partial^2}{\partial x^\lambda \partial x^\sigma}(r\widehat{\Phi}^{\lambda\sigma}(w,\Theta)) \\ &- \frac{\partial}{\partial x^\lambda}(r\widehat{\Phi}^\lambda(w,\Theta)) + \widehat{\Phi}^0(w,\widehat{w})r = 0, \\ &\Theta|_\gamma = \Theta_0, \quad \frac{\partial\Theta}{\partial n}\Big|_\gamma = \Theta_*, \end{aligned} \tag{6.16}$$

and the variational formulation associated with (6.16) *reads*

$$\begin{cases} \text{Find } \Theta \in V(D) = \Big\{q \,\Big|\, q \in H^2(D),\ q|_\gamma = \Theta_0,\ \dfrac{\partial q}{\partial n}\Big|_\gamma = \Theta_*\Big\} \text{ such that} \\ \displaystyle\int_D \{(2\mu r^2 W^{\lambda\alpha}\Theta_{\alpha\lambda} + \widehat{\Phi}^{\lambda\sigma}(w,\Theta))\eta_{\lambda\sigma} + \widehat{\Phi}^\lambda(w,\Theta)\eta_\lambda + \widehat{\Phi}^0(w,\widehat{w})\eta\}\varepsilon r \mathrm{d}x, \\ \hfill \forall \eta \in H_0^2(D). \end{cases} \tag{6.17}$$

7 The Controllability

In this section, we discuss the existence of solutions of the optimal control problem (4.1) and (4.2) for the incompressible case. As well-known, the object functional

$$J(\Theta) = \frac{1}{2}\int_\Omega A^{ijkl}(\Theta)e_{ij}(w(\Theta))e_{kl}(w(\Theta))\sqrt{g}\, \mathrm{d}x\mathrm{d}\xi = \frac{1}{2}a(w(\Theta), w(\Theta)) \tag{7.1}$$

depends upon the existence of a solution w to the Navier-Stokes equations. Since the solution w of the Navier-Stokes equations and A^{ijkl} are functions of Θ, J itself is a function of Θ and the general minimum problem is

$$\begin{cases} \text{Find the surface } \Im \text{ of the blade such that} \\ J(\Im) = \inf\limits_{S\in\mathcal{F}} J(S), \\ \mathcal{F} = \{\zeta \in H^2(D),\ \zeta = \Theta_0,\ \zeta = \Theta_*, \text{ on } \partial D,\ \|\zeta\|_{2,D} \le \kappa_0\}. \end{cases} \tag{7.2}$$

However, J can be read as a function of $w:\ J(\Theta) = \widetilde{J}(w(\Theta))$. As well-known, if there exists a Gâteaux derivative $\frac{DJ}{D\Theta}$ of $J(\Theta)$ with respect to Θ at Θ^*, then the minimum point Θ of (4.2) must satisfy

$$\mathrm{grad}_\Theta J(\Theta) = 0, \tag{7.3}$$

and from this, if Θ is smooth enough and it must satisfy Euler-Lagrange equations.

At first, it is well-known that the following theorem, which is a theorem analogous to the Generalized Weierstrass Theorem (see [7, 19]) in the calculus of variation, gives a sufficient condition for the existence.

Theorem 7.1 *Let X be a reflexive Banach space, and U a bounded and weakly closed subset of X. If the functional J is weakly lower semi-continuous on U, then J is bounded from below and J achieves its infimum on U.*

We consider the functional defined in a closed set of the Sobolev space

$$V(\Omega) := \{u \mid u \in H^{1,p}(\Omega), u|_{\Gamma_0} = 0, \partial\Omega = \Gamma_0 \cup \Gamma_1, \mathrm{meas}(\Gamma_0) \neq 0\}.$$

Lemma 7.1

$$\widetilde{J}(w) = \frac{1}{2}a(w, w)$$

is weakly lower semi-continuous with respect to w in $\boldsymbol{H}^1(\Omega)$.

Proof Indeed, assume

$$w_k \rightharpoonup w_0 \text{ (weakly)}, \quad \text{in}\, H^1(D).$$

Owing to

$$\begin{aligned} &0 \le a(w_k - w_0, w_k - w_0) = a(w_k, w_k) - 2(w_k, w_0) + a(w_0, w_0) \\ &\Rightarrow a(w_k, w_k) \ge 2a(w_k, w_0) - a(w_0, w_0), \end{aligned}$$

we have

$$\lim_{k\to\infty} \inf \widetilde{J}(w_k) \ge a(w_0, w_0) - \frac{1}{2}a(w_0, w_0) = \frac{1}{2}a(w_0, w_0) = \widetilde{J}(w_0).$$

By virtue of Lemma 7.1, we directly obtain the following result.

Lemma 7.2 *If the solution $w(\Theta)$ of the Navier-Stokes equations satisfies:*

$$\textbf{\textit{Assumption P}}: \quad \Theta_n \rightharpoonup \Theta_0 \ (weakly) \Rightarrow w_n = w(\Theta_n) \rightharpoonup w_0 = w(\Theta_0) \ (weakly),$$

then the functional $J(\Theta)$ defined by (7.1) *is weakly lower semi-continuous with respect to Θ.*

Finally, we have the next theorem.

Theorem 7.2 *Assume that (w, p) is a solution of the Navier-Stokes equations with mixed boundary conditions such that*

$$\inf_D \Big\{ \int_{-1}^1 w^1 w^1 \mathrm{d}\xi, \int_{-1}^1 w^2 w^2 \mathrm{d}\xi, \int_{-1}^1 \Big[\Big(\frac{\partial w^3}{\partial x^\alpha}\Big)\Big(\frac{\partial w^3}{\partial x^\alpha}\Big) + \Big(\frac{\partial w^3}{\partial \xi} + \frac{w^2}{r}\Big)^2\Big]\mathrm{d}\xi\Big\} > 0. \tag{7.4}$$

Then at least there exists a two dimensional surface $\Im$ defined by a smooth mapping

$$\Theta : D \to \mathcal{F} = \{\zeta \in H^2(D), \|\zeta\|_{2,D} \le \kappa_0, \ \zeta|_{\partial D} = \Theta_0, \ \partial_\nu \zeta|_{\partial D} = \Theta_*\},$$

such that $J(\Theta)$ achieves its minimum at $\{\Theta\}$,

$$\Theta \in \mathcal{F}, \quad J(\Theta) = \inf_{\zeta \in \mathcal{F}} J(\zeta). \tag{7.5}$$

Furthermore, Θ also is a stationary point:

$$\langle \mathrm{grad} J(\Theta), \eta\rangle = \int_\Omega \mathrm{grad}_\Theta \Phi(w,\Theta)\mathrm{d}V = 0.$$

Proof Indeed, according to Theorem 7.1 it is enough to prove that

(i) The manifold $\mathcal{F}(D)$ is sequentially weakly closed, i.e.,

$$\forall \overrightarrow{\zeta}_l \in \mathcal{F}(D),\ l \geq 1 \text{ and } \zeta_l \rightharpoonup \zeta, \text{ in } H^2(D) \Rightarrow \zeta \in \mathcal{F}(D).$$

(ii) The functional J is sequentially weakly lower semi-continuous over the manifold $\mathcal{W}(D)$, i.e.,

$$\zeta_l \in \mathcal{W}(D),\ l \geq 1 \text{ and } \zeta_l \rightharpoonup \zeta \in \mathcal{F}(D) \text{ in } H^2(D) \Rightarrow J(\eta) \leq \lim\inf_{l\to\infty} J(\eta_l).$$

(iii) The functional J is bounded from below, i.e., there exist constants C_1 and C_2 such that

$$C_1 > 0, \quad J(\zeta) \geq C_1\|\zeta\|_{2,D} + C_2, \quad \forall \zeta \in \mathcal{F}. \tag{7.6}$$

In fact, the Hilbert space $H^2(D)$ is a reflexive Banach Space. Furthermore,

(i) Let $\zeta_l \in \mathcal{F}(D)$ $(l \geq 1)$ be such that $\zeta_l \rightharpoonup \zeta$ in $H^2(D)$. We have to prove that $\zeta \in \mathcal{F}(D)$.

Since the trace operator $\mathrm{tr}I$ and $\mathrm{tr}\frac{\partial}{\partial n}$ from $H^2(D)$ into $L^2(D)$ are continuous with respect to the strong topologies of both spaces, it remains so with respect to the weak topologies of both spaces. Hence $\zeta_l|_{\partial D} \rightharpoonup \zeta|_{\partial D}$ and $\partial_n\zeta_l|_{\partial D} \rightharpoonup \partial_n\zeta|_{\partial D}$ into $L^2(D)$ and thus $\zeta|_{\partial D} = \Theta_0,\ \partial_n\zeta|_{\partial D} = \Theta_*$ since $\mathrm{tr}\,\zeta_l = \Theta_0,\ \mathrm{tr}_n\zeta_l = \Theta_*$, for all $l \geq 1$. Moreover, the weakly convergence sequence in $H^2(D)$ is a bounded sequence in $H^2(D)$, hence $\|\zeta\|_{2,D} \leq \kappa_0$ and $\zeta \in \mathcal{F}(D)$.

(ii) According to Lemma 7.1 and Lemma 7.2, it is enough to prove that the solution $(w(\Theta), p(\Theta))$ of the rotating Navier-Stokes equations (3.15) is weakly continuous with respect to Θ. That means that for any weakly continuous sequence Θ^k $(k = 1, 2, \cdots)$ in $\mathcal{W}$, the corresponding sequence of the solutions $(w(\Theta^k), p(\Theta^k))$ is weakly continuous. By virtue of Theorem 4.1, (4.6) shows that there exists a subsequence of $(w(\Theta^k))$ (for simplicity, we still denote it $(w(\Theta^k))$), which is weakly convergent, i.e., there exists a $w_* \in V(D)$ such that

$$w(\Theta^k) \rightharpoonup w_*, \quad \text{in } V(D).$$

(iii) It remains to prove that the functional J is coercive on the manifold $\mathcal{F}$, i.e., (7.6) holds. By proceeding as for the proof of (3.35) it is easy to derive that

$$J(\Theta) = \int_\Omega \Phi(w,w) r\varepsilon \mathrm{d}\xi \mathrm{d}x \geq \int_\Omega \mu\Big[\frac{1}{2}\|\psi(w,\Theta)\|^2 - \|\varphi(w)\|^2\Big] r\varepsilon\mathrm{d}\xi\mathrm{d}x. \tag{7.7}$$

By virtue of (3.20), (3.23) and using the Young's inequality, we have

$$\begin{aligned}\|\psi(w,\Theta)\|^2 &= \|\psi^\lambda\Theta_\lambda + \psi^{\lambda\sigma}\Theta_\lambda\Theta_\sigma + e^*(w,\Theta)\|^2\\ &= \|e^*(w,\Theta)\|^2 + \|\psi^\lambda(w)\Theta_\lambda + \psi^{\lambda\sigma}(w)\Theta_\lambda\Theta_\sigma\|^2\\ &\quad + 2(e^*(w), \psi^\lambda(w)\Theta_\lambda + \psi^{\lambda\sigma}(w)\Theta_\lambda\Theta_\sigma).\end{aligned} \tag{7.8}$$

On the other hand, from (3.29) and (3.23), we get

$$
\begin{aligned}
\|e^*(w)\|^2 &= e^*_{\alpha\beta}(w)e^*_{\alpha\beta}(w) + (r\varepsilon)^{-2}e^*_{3\alpha}(w)e^*_{3\alpha}(w) + \frac{1}{2}(r\varepsilon)^{-4}e^*_{33}(w)e^*_{33}(w)\\
&= \frac{1}{4}r^4 w^\lambda w^\sigma \partial_\lambda(\Theta_\alpha\Theta_\beta)\partial_\sigma(\Theta_\alpha\Theta_\beta) + \frac{1}{4}r^2 w^\lambda w^\sigma \Theta_{\lambda\alpha}\Theta_{\sigma\beta}\\
&= \frac{1}{4}r^2 w^\lambda w^\sigma(2r^2(\Theta_{\lambda\alpha}\Theta_{\sigma\alpha}|\nabla\Theta|^2 + \Theta_{\lambda\alpha}\Theta_{\sigma\beta}\Theta_\alpha\Theta_\beta) + \Theta_{\lambda\alpha}\Theta_{\sigma\alpha})\\
&= \frac{1}{2}r^4\sum_\alpha (w^\lambda\Theta_{\alpha\lambda})^2 + \frac{1}{4}r^2(w^\lambda\Theta_\alpha\Theta_{\lambda\alpha})^2 + \frac{1}{4}r^2 w^\lambda w^\sigma\Theta_{\lambda\alpha}\Theta_{\sigma\alpha} \qquad (7.9)\\
&\geq \frac{1}{4}r^2 w^\lambda w^\sigma \Theta_{\lambda\alpha}\Theta_{\sigma\alpha}\\
&= \frac{1}{4}r^2(w^{11}\Theta^2_{11} + w^{22}\Theta^2_{22} + (w^{11}+w^{22})\Theta^2_{12}) + \frac{1}{2}r^2 w^{12}(\Theta_{11}+\Theta_{22})\Theta_{12},\\
w^{\lambda\sigma} &= w^\lambda w^\sigma,
\end{aligned}
$$

where $a_{\alpha\beta}$ and a are defined by (2.3). Taking (3.22) and (3.23) into account, simple calculations show that

$$
\begin{aligned}
&\|\psi^\lambda(w)\Theta_\lambda + \psi^{\lambda\sigma}(w)\Theta_\lambda\Theta_\sigma\|^2\\
&= (\psi^\lambda(w),\psi^\sigma(w))\Theta_\lambda\Theta_\sigma + (\psi^{\lambda\sigma}(w),\psi^{\mu\nu}(w))\Theta_\lambda\Theta_\sigma\Theta_\nu\Theta_\mu + 2((\psi^\lambda(w),\psi^{\nu\mu}(w))\Theta_\lambda\Theta_\nu\Theta_\mu,\\
&(\psi^\lambda(w),\psi^\sigma(w))\Theta_\lambda\Theta_\sigma\\
&= \Big[\frac{1}{2}\varepsilon^2 r^4\Big(\frac{\partial w^3}{\partial x^\alpha}\frac{\partial w^3}{\partial x^\alpha}\delta_{\lambda\sigma} + \frac{\partial w^3}{\partial x^\lambda}\frac{\partial w^3}{\partial x^\sigma}\Big) + \frac{1}{4}r^2\Big(\frac{\partial w^\lambda}{\partial x^\alpha}\frac{\partial w^\sigma}{\partial x^\alpha} + 2\frac{\partial w^\lambda}{\partial x^\sigma}\Big(\frac{\partial w^3}{\partial \xi} + \frac{2}{r}w^2\Big)\\
&\quad + \Big(\frac{\partial w^3}{\partial \xi} + \frac{2}{r}w^2\Big)^2\delta_{\lambda\sigma}\Big) + \varepsilon^{-2}\Big(\frac{\partial w^\lambda}{\partial \xi}\frac{\partial w^\sigma}{\partial \xi}\Big)\Big]\Theta_\lambda\Theta_\sigma\\
&= \frac{1}{2}r^2\Big[r^2\frac{\partial w^3}{\partial x^\alpha}\frac{\partial w^3}{\partial x^\alpha} + \frac{1}{2}\Big(\frac{\partial w^3}{\partial \xi} + \frac{2}{r}w^2\Big)^2\Big]|\nabla\Theta|^2 + \frac{1}{2}\varepsilon^2 r^4\Big(\frac{\partial w^3}{\partial x^\lambda}\Theta_\lambda\Big)^2 + \varepsilon^{-2}\Big(\frac{\partial w^\lambda}{\partial \xi}\Theta_\lambda\Big)^2\\
&\quad + \frac{1}{4}r^2\Big(\frac{\partial w^\lambda}{\partial x^\alpha}\Theta_\lambda\Big)\Big(\frac{\partial w^\sigma}{\partial x^\alpha}\Theta_\sigma\Big) + \frac{1}{2}r^2\frac{\partial w^\lambda}{\partial x^\sigma}\Big(\frac{\partial w^3}{\partial \xi} + \frac{2}{r}w^2\Big)\Theta_\lambda\Theta_\sigma\\
&\geq \frac{1}{4}r^2\Big[2r^2\frac{\partial w^3}{\partial x^\alpha}\frac{\partial w^3}{\partial x^\alpha} + \frac{1}{2}\Big(\frac{\partial w^3}{\partial \xi} + \frac{w^2}{r}\Big)^2\Big]|\nabla\Theta|^2 + \frac{1}{2}r^2\frac{\partial w^\lambda}{\partial x^\sigma}\Big(\frac{\partial w^3}{\partial \xi} + \frac{2}{r}w^2\Big)\Theta_\lambda\Theta_\sigma,\\
&(\psi^{\lambda\sigma}(w),\psi^{\nu\mu}(w))\Theta_\lambda\Theta_\sigma\Theta_\nu\Theta_\mu\\
&= \Big[\frac{1}{2}r^4\Big(\Big(\frac{\partial w^\lambda}{\partial x^\alpha}\Theta_\lambda\Big)\Big(\frac{\partial w^\sigma}{\partial x^\alpha}\Theta_\sigma\Big) + \frac{1}{r}w^2\Big(\frac{\partial w^\sigma}{\partial x^\lambda} + \frac{\partial w^\lambda}{\partial x^\sigma}\Big)\Theta_\lambda\Theta_\sigma\Big)\\
&\quad + \frac{1}{4}\varepsilon^{-2}r^2\Big(\frac{\partial w^\lambda}{\partial \xi}\Theta_\lambda\Big)^2\Big]|\nabla\Theta|^2 + r^2 w^2 w^2|\nabla\Theta|^4\\
&= \Big[\frac{1}{2}r^4\Big(\Big(\frac{\partial w^\lambda}{\partial x^\alpha}\Theta_\lambda\Big)\Big(\frac{\partial w^\sigma}{\partial x^\alpha}\Theta_\sigma\Big) + 2\frac{1}{r}w^2\Big(\frac{\partial w^\lambda}{\partial x^\sigma}\Big)\Theta_\lambda\Theta_\sigma\Big)\\
&\quad + \frac{1}{4}\varepsilon^{-2}r^2\Big(\frac{\partial w^\lambda}{\partial \xi}\Theta_\lambda\Big)^2\Big]|\nabla\Theta|^2 + r^2 w^2 w^2|\nabla\Theta|^4,\\
&(\psi^{\mu\nu}(w),\psi^\lambda(w))\Theta_\lambda\Theta_\nu\Theta_\mu\\
&= \frac{1}{4}\varepsilon r^4\Big[2\frac{\partial w^\lambda}{\partial x^\alpha}\frac{\partial w^3}{\partial x^\alpha} + \frac{2}{r}w^2\frac{\partial w^3}{\partial x^\lambda} + \varepsilon^{-2}\frac{\partial w^\lambda}{\partial \xi}\Big(\frac{\partial w^3}{\partial \xi} + \frac{2}{r}w^2\Big)\Big]\Theta_\lambda|\nabla\Theta|^2
\end{aligned}
$$

$$+\Big[\frac{1}{2}\varepsilon r^4\frac{\partial w^\lambda}{\partial x^\nu}\frac{\partial w^3}{\partial x^\mu}+\frac{1}{4}\varepsilon^{-1}r^2\frac{\partial w^\lambda}{\partial \xi}\frac{\partial w^\nu}{\partial x^\mu}\Big]\Theta_\lambda\Theta_\mu\Theta_\nu.$$

Hence, using $(\frac{\partial w^\sigma}{\partial x^\lambda}+\frac{\partial w^\lambda}{\partial x^\sigma})\Theta_\lambda\Theta_\sigma=2\frac{\partial w^\sigma}{\partial x^\lambda}\Theta_\lambda\Theta_\sigma$, we have

$$\begin{aligned}
&\|\psi^\lambda(w)\Theta_\lambda+\psi^{\lambda\sigma}(w)\Theta_\lambda\Theta_\sigma\|^2\\
&\geq\frac{1}{4}r^2\Big[2r^2\frac{\partial w^3}{\partial x^\alpha}\frac{\partial w^3}{\partial x^\alpha}+\frac{1}{2}\Big(\frac{\partial w^3}{\partial \xi}\Big)^2\Big]|\nabla\Theta|^2\\
&\quad+\frac{1}{2}r^2\Big[\frac{\partial w^\lambda}{\partial x^\sigma}\Big(\frac{\partial w^3}{\partial \xi}+\frac{4}{r}w^2\Big)-\frac{1}{2}w^2w^2\delta_{\lambda\sigma}\Big]\Theta_\lambda\Theta_\sigma\\
&\quad+\frac{1}{4}\varepsilon r^4\Big[2\frac{\partial w^\lambda}{\partial x^\alpha}\frac{\partial w^3}{\partial x^\alpha}+\frac{2}{r}w^2\frac{\partial w^3}{\partial x^\lambda}+(r\varepsilon)^{-2}\frac{\partial w^\lambda}{\partial \xi}\Big(\frac{\partial w^3}{\partial \xi}+\frac{2}{r}w^2\Big)\Big]\Theta_\lambda|\nabla\Theta|^2\\
&\quad+\Big[\frac{1}{2}\varepsilon r^4\frac{\partial w^\lambda}{\partial x^\nu}\frac{\partial w^3}{\partial x^\mu}+\frac{1}{4}\varepsilon^{-1}r^2\frac{\partial w^\lambda}{\partial \xi}\frac{\partial w^\nu}{\partial x^\mu}\Big]\Theta_\lambda\Theta_\mu\Theta_\nu.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
&2(e^*(w),\psi^\lambda(w)\Theta_\lambda+\psi^{\lambda\sigma}(w)\Theta_\lambda\Theta_\sigma)\\
&=2(e^*(w),\psi^\lambda(w)\Theta_\lambda)+2(e^*(w),\psi^{\lambda\sigma}(w)\Theta_\lambda\Theta_\sigma)\\
&=\frac{1}{2}r^2w^\nu\Theta_{\alpha\nu}\Big[2r^2\Big(\varepsilon\frac{\partial w^3}{\partial x^\alpha}+\frac{\partial w^\lambda}{\partial x^\alpha}\Theta_\lambda+\frac{2}{r}w^2\Theta_\alpha\Big)|\nabla\Theta|^2+2r^2\Big(\frac{\partial w^\lambda}{\partial x^\beta}\Theta_\lambda+\varepsilon\frac{\partial w^3}{\partial x^\beta}\Big)\Theta_\beta\Theta_\alpha\\
&\quad+\frac{\partial w^\lambda}{\partial x^\alpha}\Theta_\lambda+\Big(\frac{\partial w^3}{\partial \xi}+\frac{2}{r}w^2\Big)\Theta_\alpha+\varepsilon^{-1}\frac{\partial w^\lambda}{\partial \xi}\Theta_\lambda\Theta_\alpha\Big].
\end{aligned}$$

To sum up,

$$\begin{aligned}
\|\psi(w,\Theta)\|^2\geq{}&\frac{1}{4}r^2(w^{11}\Theta_{11}^2+w^{22}\Theta_{22}^2+(w^{11}+w^{22})\Theta_{12}^2)\\
&+\frac{1}{4}r^2\Big[2r^2\frac{\partial w^3}{\partial x^\alpha}\frac{\partial w^3}{\partial x^\alpha}+\frac{1}{2}\Big(\frac{\partial w^3}{\partial \xi}+\frac{w^2}{r}\Big)^2\Big]|\nabla\Theta|^2+T(w,\Theta),\\
T(w,\Theta)={}&\frac{1}{2}r^2w^{12}(\Theta_{11}+\Theta_{22})\Theta_{12}+\frac{1}{2}r^2\Big[\frac{\partial w^\lambda}{\partial x^\sigma}\Big(\frac{\partial w^3}{\partial \xi}+\frac{4}{r}w^2\Big)-\frac{1}{2}w^2w^2\delta_{\lambda\sigma}\Big]\Theta_\lambda\Theta_\sigma\\
&+\frac{1}{4}\varepsilon r^4\Big[2\frac{\partial w^\lambda}{\partial x^\alpha}\frac{\partial w^3}{\partial x^\alpha}+\frac{2}{r}w^2\frac{\partial w^3}{\partial x^\lambda}+(r\varepsilon)^{-2}\frac{\partial w^\lambda}{\partial \xi}\Big(\frac{\partial w^3}{\partial \xi}+\frac{2}{r}w^2\Big)\Big]\Theta_\lambda|\nabla\Theta|^2\\
&+\Big[\frac{1}{2}\varepsilon r^4\frac{\partial w^\lambda}{\partial x^\nu}\frac{\partial w^3}{\partial x^\mu}+\frac{1}{4}\varepsilon^{-1}r^2\frac{\partial w^\lambda}{\partial \xi}\frac{\partial w^\nu}{\partial x^\mu}\Big]\Theta_\lambda\Theta_\mu\Theta_\nu\\
&+\frac{1}{2}r^2w^\nu\Theta_{\alpha\nu}\Big[2r^2\Big(\varepsilon\frac{\partial w^3}{\partial x^\alpha}+\frac{\partial w^\lambda}{\partial x^\alpha}\Theta_\lambda+\frac{2}{r}w^2\Theta_\alpha\Big)|\nabla\Theta|^2\\
&+2r^2\Big(\frac{\partial w^\lambda}{\partial x^\beta}\Theta_\lambda+\varepsilon\frac{\partial w^3}{\partial x^\beta}\Big)\Theta_\beta\Theta_\alpha\\
&+\frac{\partial w^\lambda}{\partial x^\alpha}\Theta_\lambda+\Big(\frac{\partial w^3}{\partial \xi}+\frac{2}{r}w^2\Big)\Theta_\alpha+\varepsilon^{-1}\frac{\partial w^\lambda}{\partial \xi}\Theta_\lambda\Theta_\alpha\Big].
\end{aligned}\tag{7.10}$$

Since w is bounded in $\mathbf{H}^1(\Omega)$, by Theorem 5.3 and $\Theta\in\mathcal{F}$, we claim that

$$\|T\|_{0,\Omega}+\|\varphi(w)\|_{0,\Omega}^2\leq C,\tag{7.11}$$

where C is a constant. Therefore, from (9.7), (9.10) and (9.11),

$$J(\Theta)=\int_\Omega\Phi(w,w)r\varepsilon\mathrm{d}\xi\mathrm{d}x\geq\int_\Omega\Big\{\frac{1}{4}\mu r^2\Big[(w^{11}\Theta_{11}^2+w^{22}\Theta_{22}^2+(w^{11}+w^{22})\Theta_{12}^2)$$

$$
\begin{aligned}
&+\Big(2r^2\frac{\partial w^3}{\partial x^\alpha}\frac{\partial w^3}{\partial x^\alpha}+\frac{1}{2}\Big(\frac{\partial w^3}{\partial \xi}+\frac{w^2}{r}\Big)^2\Big)|\nabla\Theta|^2\Big]+\mu T(w,\Theta)-\mu\|\varphi(w)\|^2\Big\}\varepsilon r\mathrm{d}\xi\mathrm{d}x\\
\geq&\int_D\frac{1}{4}\mu r^2\Big[\varepsilon\Big((W^{11}\Theta_{11}^2+W^{22}\Theta_{22}^2+(W^{11}+W^{22})\Theta_{12}^2)\\
&+\int_{-1}^1\Big(2r^2\frac{\partial w^3}{\partial x^\alpha}\frac{\partial w^3}{\partial x^\alpha}+\frac{1}{2}\Big(\frac{\partial w^3}{\partial \xi}+\frac{w^2}{r}\Big)^2\Big)\mathrm{d}\xi|\nabla\Theta|^2\Big)\Big]r\mathrm{d}x-C\\
\geq&\frac{\mu}{8}\varepsilon r_0^2\mu\inf_D\Big\{W^{11},W^{22},\int_{-1}^1\Big(\frac{\partial w^3}{\partial x^\alpha}\frac{\partial w^3}{\partial x^\alpha}+\Big(\frac{\partial w^3}{\partial \xi}+\frac{w^2}{r}\Big)^2\Big)\mathrm{d}\xi\Big\}\|\Theta\|_{2,D}^2-C\\
\geq&C_1\|\Theta\|_{2,D}^2+C_2,\quad\forall\Theta\in\mathcal{F},
\end{aligned}\tag{7.12}
$$

where

$$
\begin{aligned}
&W^{\lambda\sigma}=\int_{-1}^1 w^\lambda w^\sigma\mathrm{d}\xi,\\
&C_1=\frac{\mu}{8}\varepsilon r_0^2\inf_D\Big\{W^{11},W^{22},\int_{-1}^1\Big(\frac{\partial w^3}{\partial x^\alpha}\frac{\partial w^3}{\partial x^\alpha}+\Big(\frac{\partial w^3}{\partial \xi}+\frac{w^2}{r}\Big)^2\Big)\mathrm{d}\xi\Big\}.
\end{aligned}\tag{7.13}
$$

Thanks to the first equation (5.1), we have

$$
\frac{\partial w^\alpha}{\partial x^\alpha}=-\Big(\frac{\partial w^3}{\partial \xi}+\frac{1}{r}w^2\Big),
$$

and therefore,

$$
C_1=\frac{\nu}{8}\varepsilon r_0^2\inf_D\Big\{W^{11},W^{22},\int_{-1}^1\Big[\frac{\partial w^3}{\partial x^\alpha}\frac{\partial w^3}{\partial x^\alpha}+\Big(\frac{\partial w^\alpha}{\partial x^\alpha}\Big)\Big(\frac{\partial w^\beta}{\partial x^\beta}\Big)\Big]\mathrm{d}\xi\Big\}>0.\tag{7.14}
$$

(7.6) is valid. The proof is completed.

8 Second Model

Let us consider second minimization function, the power done by the impeller, or the left force of the aircraft, for example, the airfoil:

$$
I(\Im)=\int_{\Im_-\cup\Im_+}\sigma(\mathbf{w},p)\cdot\mathbf{n}\cdot\mathbf{e}_\theta r\omega\mathrm{d}S,\tag{8.1}
$$

where $\mathbf{n}$ is the unite normal vector to the surface $\Im$, ω the angular velocity of the impeller, $\mathrm{d}S=\sqrt{a}$ is the element on the surface $\Im$,

$$
\sigma(\mathbf{w},p)=(-p+\lambda\operatorname{div}w)g_{ij}+2\mu e_{ij}(\mathbf{w})
$$

the stress tensor ($\lambda=0$ corresponds the impressible fluid), and (e_r,e_θ,k) are the bases vectors of rotating cylindrical coordinate system. Our purpose is that find a surface $\Im$ of the blade such that

$$
I(\Im)=\inf_{\mathcal{S}\in\mathcal{F}}I(\mathcal{S}),\tag{8.2}
$$

where $\mathcal{F}$ denotes a set of the smooth surface spanning on a given Jordan's curve $C\in E^3$. Under the new coordinate system, (8.1) can be rewritten as

$$
I(\Im)=\int_D\{((-p+\lambda\operatorname{div}\mathbf{w})g_{ij}+2\mu e_{ij}(\mathbf{w}))n^i(\mathbf{e}_\theta)^j r\omega\sqrt{a}|_{\xi=+1}
$$

$$-(-p+\lambda\mathrm{div}\,\mathbf{w})g_{ij}+2\mu e_{ij}(\mathbf{w}))n^i(\mathbf{e}_\theta)^j r\omega\sqrt{a}|_{\xi=-1}\}\mathrm{d}x, \tag{8.3}$$

where $\mathbf{n}$ and $\mathbf{e}_\theta$ can be found

$$\begin{cases}\mathbf{n}=n^i\mathbf{e}_i, \quad n^\alpha=-r\dfrac{\Theta_\alpha}{\sqrt{a}}, \quad n^3=(r\varepsilon)^{-1}\dfrac{1+r^2\Theta_2^2}{\sqrt{a}},\\ \mathbf{e}_\theta=e_\theta^i\mathbf{e}_i, \quad e_\theta^\alpha=0, \quad e_\theta^3=(r\varepsilon)^{-1}.\end{cases} \tag{8.4}$$

In view of (2.5), (3.20), (3.21) and (3.22), we have

$$\begin{aligned}
g_{ij}n^i(e_\theta)^j &= (r\varepsilon)^{-1}(g_{\alpha 3}n^\alpha+g_{33}n^3)=\frac{1-r^2\Theta_1^2}{\sqrt{a}},\\
e_{3\alpha}(\mathbf{w}) &= \frac{1}{2}\Big(a_{\alpha\beta}\frac{\partial w^\beta}{\partial\xi}+\varepsilon r^2\Theta_\alpha\frac{\partial w^3}{\partial\xi}\Big)+\frac{1}{2}\Big(\Theta_\beta\frac{\partial w^\beta}{\partial x^\alpha}+\varepsilon\frac{\partial w^3}{\partial x^\alpha}\Big)\\
&\quad+\Big(\frac{1}{2}\varepsilon r^2\Theta_{\alpha\sigma}+r\varepsilon\delta_{2\sigma}\Theta_\alpha\Big)w^\sigma,\\
e_{33}(\mathbf{w}) &= r^2\varepsilon^2\frac{\partial w^3}{\partial\xi}+r\varepsilon^2w^2+\varepsilon r^2\Theta_\alpha\frac{\partial w^\alpha}{\partial\xi},\\
e_{ij}(\mathbf{w}))n^i(\mathbf{e}_\theta)^j &= \frac{1}{r^2\varepsilon^2\sqrt{a}}[r^2\Theta_\alpha e_{3\alpha}(\mathbf{w})+(1+r^2\Theta_2^2)e_{33}(\mathbf{w})]\\
&= \frac{1}{r\varepsilon\sqrt{a}}\Big[\frac{1}{2}r\varepsilon\Big(1+\frac{1}{2}r^2(\Theta_2^2-\Theta_1^2)\Big)\frac{\partial w^3}{\partial\xi}+\frac{1}{2}r\Theta_\alpha(1+r^2(\Theta_2^2-\Theta_1^2))\frac{\partial w^\alpha}{\partial\xi}\\
&\quad-\frac{1}{2}r\Theta_\alpha\Big(\frac{\partial w^\beta}{\partial x^\alpha}+\varepsilon\frac{\partial w^3}{\partial x^\alpha}\Big)+\varepsilon\Big((1-r^2\Theta_1^2)\delta_{2\sigma}-\frac{1}{2}r^3\Theta_\alpha\Theta_{\alpha\sigma}\Big)w^\sigma\Big].
\end{aligned} \tag{8.5}$$

Therefore, the integrand in (8.3) can be expressed by

$$\begin{aligned}
A_D(w,p,\Theta) :&= ((-p+\lambda\mathrm{div}\,\mathbf{w})g_{ij}+2\mu e_{ij}(\mathbf{w}))n^i(\mathbf{e}_\theta)^j r\omega\sqrt{a}\\
&= \Big[\frac{1-r^2\Theta_1^2}{\sqrt{a}}(-p+\mathrm{div}\mathbf{w})+2\mu\Big(\frac{1}{r\varepsilon\sqrt{a}}\Big[\frac{1}{2}r\varepsilon\Big(1+\frac{1}{2}r^2(\Theta_2^2-\Theta_1^2)\Big)\frac{\partial w^3}{\partial\xi}\\
&\quad+\frac{1}{2}r\Theta_\alpha(1+r^2(\Theta_2^2-\Theta_1^2))\frac{\partial w^\alpha}{\partial\xi}-\frac{1}{2}r\Theta_\alpha\Big(\frac{\partial w^\beta}{\partial x^\alpha}+\varepsilon\frac{\partial w^3}{\partial x^\alpha}\Big)\\
&\quad+\varepsilon\Big((1-r^2\Theta_1^2)\delta_{2\sigma}-\frac{1}{2}r^3\Theta_\alpha\Theta_{\alpha\sigma}\Big)w^\sigma\Big]\Big)\Big]r\omega\sqrt{a}.
\end{aligned} \tag{8.6}$$

Taking the boundary conditions into account yields that

$$\begin{cases}\mathbf{w}|_{\xi=\pm1}=0, \quad \Big(\dfrac{\partial w^\beta}{\partial x^\alpha}+\varepsilon\dfrac{\partial w^3}{\partial x^\alpha}\Big)\Big|_{\xi=\pm1}=0,\\ \mathrm{div}\,\mathbf{w}=\dfrac{\partial w^3}{\partial\xi}.\end{cases} \tag{8.7}$$

In particular, if the fluid is incompressible then

$$\begin{cases}\mathrm{div}\,\mathbf{w}=\dfrac{\partial w^\alpha}{\partial x^\alpha}+\dfrac{\partial w^3}{\partial\xi}+\dfrac{w^2}{r}=0,\\ \dfrac{\partial w^\alpha}{\partial x^\alpha}\Big|_{\xi=\pm1}=0, \quad \Big(\dfrac{\partial w^3}{\partial\xi}\Big)\Big|_{\xi=\pm1}=0.\end{cases} \tag{8.8}$$

Substituting (8.7) into (8.6) leads to

$$\begin{cases} A_D(w,p,\Theta) = \Big[(1-r^2\Theta_1^2)\Big(-p+\lambda\dfrac{\partial w^3}{\partial\xi}\Big)+\mu\Big(\Big(1+\dfrac{1}{2}r^2(\Theta_2^2-\Theta_1^2)\Big)\dfrac{\partial w^3}{\partial\xi} \\ \qquad +\varepsilon^{-1}\Theta_\alpha(1+r^2(\Theta_2^2-\Theta_1^2))\dfrac{\partial w^\alpha}{\partial\xi}\Big)\Big]r\omega \quad \text{for the compressible flow,} \\ A_D(w,p,\Theta) = \Big[-(1-r^2\Theta_1^2)p+\mu\varepsilon^{-1}\Theta_\alpha(1+r^2(\Theta_2^2-\Theta_1^2))\dfrac{\partial w^\alpha}{\partial\xi}\Big]r\omega \\ \qquad\qquad \text{for the incompressible flow.} \end{cases} \tag{8.9}$$

Hence, we conclude that

$$I(\Im) = \int_D [A_D(w,p,\Theta)|_{\xi=1} - A_D(w,p,\Theta)|_{\xi=-1}]r\omega \mathrm{d}x. \tag{8.10}$$

Theorem 8.1 *Assume that $\Theta\in C^2(D,R)$ is an injective mapping. Then the object functional I defined by (8.1) has a Gâteaux derivative $\mathrm{grad}_\Theta I \equiv \frac{\mathcal{D}I}{\mathcal{D}\Theta}$ in every direction $\eta\in W := H^2(D)\cap H_0^1(D)$ that is determined by*

$$\begin{aligned}\langle \mathrm{grad}_\Theta(I(\Theta)),\eta\rangle = \iint_D &[(E^\alpha(w,p,\Theta)\eta_\alpha + E_0(w,p,\Theta))|_{\xi=1} \\ &- (E^\alpha(w,p,\Theta)\eta_\alpha + E_0(w,p,\Theta))|_{\xi=-1}\eta]\omega r \mathrm{d}x,\end{aligned} \tag{8.11}$$

where

$$\begin{aligned} E^\alpha(\mathbf{w};p) &= 2r^2p\Theta_1\delta_{1\alpha} + r^2(\mu\Theta_2\delta_{2\alpha} - (2\lambda+\mu)\Theta_1\delta_{1\alpha})\frac{\partial w^3}{\partial\xi} \\ &\quad + \mu\varepsilon^{-1}(2r^2\Theta_\beta(\Theta_2\delta_{2\alpha}-\Theta_1\delta_{1\alpha}) + (1+r^2(\Theta_2^2-\Theta_1^2))\delta_{\alpha\beta})\frac{\partial w^\beta}{\partial\xi}, \\ E_0(\mathbf{w},p,\Theta) &= \Big(\lambda(1-r^2\Theta_1^2)+\mu\Big(1+\frac{1}{2}r^2(\Theta_2^2-\Theta_1^2)\Big)\Big)\frac{\partial\widehat{w}^3}{\partial\xi} \\ &\quad + \varepsilon^{-1}\Theta_\alpha(1+r^2(\Theta_2^2-\Theta_1^2))\frac{\partial\widehat{w}^\alpha}{\partial\xi} + (r^2\Theta_1^2-1)\widehat{p}, \end{aligned} \tag{8.12}$$

and where $\widehat{w} = \dfrac{\mathcal{D}w}{\mathcal{D}\Theta}$ is the Gâteaux derivative of the velocity (w) of the fluid at the point Θ, and $\widehat{p} = \frac{\mathcal{D}p}{\mathcal{D}\Theta}$ is the G$\widehat{a}$teaux derivative of the pressure at the point Θ.

Proof Indeed, taking into account of (8.10), and $\sqrt{g}=\varepsilon r$, we assert that

$$\begin{aligned} \langle \mathrm{grad}_\Theta(I(\Theta)),\eta\rangle &= \iint_D \Big[\frac{\mathcal{D}A_D(\mathbf{w},p,\Theta)}{\mathcal{D}\Theta}\Big|_{\xi=1}\eta - \frac{\mathcal{D}A_D(\mathbf{w},p,\Theta)}{\mathcal{D}\Theta}\Big|_{\xi=1}\eta\Big]\omega r\mathrm{d}x, \\ \frac{\mathcal{D}A_D(\mathbf{w},p,\Theta)}{\mathcal{D}\Theta}\eta &= \frac{\partial A_D(\mathbf{w},p,\Theta)}{\partial\Theta}\eta + \frac{\partial A_D(\mathbf{w},p,\Theta)}{\partial w}\widehat{w}\eta + \frac{\partial A_D(\mathbf{w},p,\Theta)}{\partial p}\widehat{p}\eta. \end{aligned} \tag{8.13}$$

By virtue of (8.9), it yields

$$\begin{aligned} \frac{\partial A_D(\mathbf{w},p,\Theta)}{\partial\Theta}\eta &= E^\alpha(\mathbf{w},p,\Theta)\eta_\alpha, \\ \frac{\partial A_D(\mathbf{w},p,\Theta)}{\partial\mathbf{w}}\widehat{\mathbf{w}}\eta &= \Big[\lambda(1-r^2\Theta_1^2)\frac{\partial\widehat{w}^3}{\partial\xi} + \mu\Big(1+\frac{1}{2}r^2(\Theta_2^2-\Theta_1^2)\Big)\frac{\partial\widehat{w}^3}{\partial\xi} \end{aligned}$$

$$+\mu\varepsilon^{-1}\Theta_\beta(1+r^2(\Theta_2^2-\Theta_1^2))\frac{\partial\widehat{w}^\beta}{\partial\xi}\Big]\eta,$$

$$\frac{\partial A_D(\mathbf{w},p,\Theta)}{\partial p}\widehat{p}\eta=(1-r^2\Theta_1^2)\widehat{p}\eta,$$

where

$$\begin{aligned}E^\alpha(\mathbf{w},p,\Theta):=&\Big[2r^2\Big(p-\lambda\frac{\partial\mathbf{w}^3}{\partial\xi}\Big)\Theta_1\delta_\alpha+\mu r^2(\Theta_2\delta_{2\alpha}-\Theta_1\delta_{1\alpha})\frac{\partial\mathbf{w}^3}{\partial\xi}\\&+\mu\varepsilon^{-1}(2r^2\Theta_\beta(\Theta_2\delta_{2\alpha}-\Theta_1\delta_{1\alpha})+(1+r^2(\Theta_2^2-\Theta_1^2))\delta_{\alpha\beta})\frac{\partial\mathbf{w}^\beta}{\partial\xi}\Big].\end{aligned}\tag{8.14}$$

Substituting above equalities into (8.13) leads to (8.11) and (8.12). The proof is completed.

Taking integration by part of (6.13) and considering homogenous boundary conditions for $\eta\in W(D)$, this implies that

$$\begin{aligned}\langle\mathrm{grad}_\Theta(I(\Theta)),\eta\rangle=\int_D&\{[-\partial_\alpha E^\alpha(\mathbf{w},p,\Theta)+rE_0(\widehat{\mathbf{w}},\widehat{p},\Theta)]|_{\xi=1}\\&-[-\partial_\alpha E^\alpha(\mathbf{w},p,\Theta)+rE_0(\widehat{\mathbf{w}},\widehat{p},\Theta)]|_{\xi=-1}\}\omega\eta\mathrm{d}x.\end{aligned}\tag{8.15}$$

Owing to (8.14) we assert that

$$\begin{aligned}\partial_\alpha E^\alpha(\mathbf{w},p,\Theta)&=A^{\alpha\beta}(\mathbf{w},p,\Theta)\Theta_{\alpha\beta}+\Pi(\mathbf{w},p,\Theta),\\A^{22}(\mathbf{w},p,\Theta)&=2\mu\varepsilon^{-1}r^3\Big(2\Theta_2\frac{\partial\mathbf{w}^2}{\partial\xi}+\Theta_1\frac{\partial\mathbf{w}^1}{\partial\xi}\Big),\\A^{12}(\mathbf{w},p,\Theta)&=2\mu\varepsilon^{-1}r^3\Big(\Theta_2\frac{\partial\mathbf{w}^1}{\partial\xi}-\Theta_1\frac{\partial\mathbf{w}^2}{\partial\xi}\Big),\\A^{11}(w,p,\Theta)&=r^3\Big(2p-(2\lambda+\mu)\frac{\partial\mathbf{w}^3}{\partial\xi}-2\mu\varepsilon^{-1}\Theta_\beta\frac{\partial\mathbf{w}^\beta}{\partial\xi}-4\mu\varepsilon^{-1}\Theta_1\frac{\partial\mathbf{w}^1}{\partial\xi}\Big),\\\Pi(w,p,\Theta)&=r^2\Big(2r\Theta_1\frac{\partial p}{\partial x^2}+3\Theta_2\frac{\partial\mathbf{w}^3}{\partial\xi}\Big)+6\mu\varepsilon^{-1}r^2\Theta_2\Theta_\beta\frac{\partial\mathbf{w}^\beta}{\partial\xi}\\&\quad+\mu\varepsilon^{-1}(1+3r^2(\Theta_2^2-\Theta_1^2))\frac{\partial\mathbf{w}^2}{\partial\xi}.\end{aligned}\tag{8.16}$$

Let us introduce the notation

$$[A]^-=A|_{\xi=1}-A|_{\xi=-1}.$$

Then (8.15) becomes

$$\langle\mathrm{grad}_\Theta(I(\Theta)),\eta\rangle=\int_D([A^{\alpha\beta}]^-\Theta_{\alpha\beta}+[\Pi]^-+r[E_0(\widehat{\mathbf{w}},\widehat{p},\Theta)]^-)\omega\eta\mathrm{d}x.\tag{8.17}$$

We get

(1) $[A^{22}(\mathbf{w},p,\Theta)]^-=2\mu\varepsilon^{-1}r^3\Big(2\Theta_2\Big[\frac{\partial\mathbf{w}^2}{\partial\xi}\Big]^-+\Theta_1\Big[\frac{\partial\mathbf{w}^1}{\partial\xi}\Big]^-\Big)$,

(2) $[A^{12}(\mathbf{w},p,\Theta)]^-=2\mu\varepsilon^{-1}r^3\Big(\Theta_2\Big[\frac{\partial\mathbf{w}^1}{\partial\xi}\Big]^--\Theta_1\Big[\frac{\partial\mathbf{w}^2}{\partial\xi}\Big]^-\Big)$,

(3) $[A^{11}(w,p,\Theta)]^-=r^3\Big(2p-(2\lambda+\mu)\Big[\frac{\partial\mathbf{w}^3}{\partial\xi}\Big]^-$

$$- 2\mu\varepsilon^{-1}\Theta_\beta\Big[\frac{\partial \mathbf{w}^\beta}{\partial\xi}\Big]^- - 4\mu\varepsilon^{-1}\Theta_1\Big[\frac{\partial \mathbf{w}^1}{\partial\xi}\Big]^-\Big), \tag{8.18}$$

(4) $$[\Pi(w,p,\Theta)]^- = r^2\Big(2r\Theta_1\Big[\frac{\partial p}{\partial r}\Big]^- + 3\Theta_2\Big[\frac{\partial \mathbf{w}^3}{\partial\xi}\Big]^-\Big) + 6\mu\varepsilon^{-1}r^2\Theta_2\Theta_\beta\Big[\frac{\partial \mathbf{w}^\beta}{\partial\xi}\Big]^- + \mu\varepsilon^{-1}(1+3r^2(\Theta_2^2-\Theta_1^2))\Big[\frac{\partial \mathbf{w}^2}{\partial\xi}\Big]^-,$$

(5) $$[E_0(\mathbf{w},p,\Theta)]^- = \Big(\lambda(1-r^2\Theta_1^2) + \mu\Big(1+\frac{1}{2}r^2(\Theta_2^2-\Theta_1^2)\Big)\Big)\Big[\frac{\partial \widehat{w}^3}{\partial\xi}\Big]^- + \varepsilon^{-1}\Theta_\alpha((1+r^2(\Theta_2^2-\Theta_1^2))\Big[\frac{\partial \widehat{w}^\alpha}{\partial\xi}\Big]^- + (r^2\Theta_1^2-1)[\widehat{p}]^-.$$

If the flow is incompressible, then

$$\frac{\partial w^3}{\partial\xi}\Big|_{\xi=\pm1} = 0, \quad \Big[\frac{\partial w^3}{\partial\xi}\Big]^- = 0.$$

From the above discussion we obtain directly the following theorem.

Theorem 8.2 *The Euler-Lagrange equation for the extremum Θ of I is given by*

$$\begin{cases}[A^{\alpha\beta}(\mathbf{w},p,\Theta)]^-\Theta_{\alpha\beta} + [\Pi(\mathbf{w},p,\Theta)]^- + r[E_0(\widehat{\mathbf{w}},\widehat{p},\Theta)]^- = 0,\\ \Theta|_\gamma = \Theta_0,\end{cases} \tag{8.19}$$

and the variational formulation associated with (6.16) *reads*

$$\begin{cases}\textit{Find } \Theta\in V(D) = \Big\{q\,\Big|\, q\in H^2(D),\ q|_\gamma = \Theta_0,\ \dfrac{\partial q}{\partial n}\Big|_\gamma = \Theta_*\Big\} \textit{ such that}\\ \displaystyle\int_D\Big\{\iint_D[[E^\alpha(w,p,\Theta)]^-\eta_\alpha + [E_0(w,p,\Theta)]^-\eta]\Big\}\omega r\mathrm{d}x = 0, \quad \forall\,\eta\in H_0^2(D),\end{cases} \tag{8.20}$$

where

$$[E^\alpha(\mathbf{w},p,\Theta)]^- := \Big[2r^2\Big([p]^- - \lambda\Big[\frac{\partial \boldsymbol{w}^3}{\partial\xi}\Big]^-\Big)\Theta_1\delta_\alpha + \mu r^2(\Theta_2\delta_{2\alpha} - \Theta_1\delta_{1\alpha})\Big[\frac{\partial \boldsymbol{w}^3}{\partial\xi}\Big]^- + \mu\varepsilon^{-1}(2r^2\Theta_\beta(\Theta_2\delta_{2\alpha}-\Theta_1\delta_{1\alpha}) + (1+r^2(\Theta_2^2-\Theta_1^2))\delta_{\alpha\beta})\Big[\frac{\partial \boldsymbol{w}^\beta}{\partial\xi}\Big]^-\Big]. \tag{8.21}$$

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