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# Explicit Traveling Wave Solutions to Nonlinear Evolution Equations

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**Abstract** First of all, some technical tools are developed. Then the author studies explicit traveling wave solutions to nonlinear dispersive wave equations, nonlinear dissipative dispersive wave equations, nonlinear convection equations, nonlinear reaction diffusion equations and nonlinear hyperbolic equations, respectively.

**Keywords** Explicit traveling wave solutions, Nonlinear partial differential equations, Reduction of order
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## 1 Introduction

Some nonlinear partial differential equations have explicit traveling wave solutions. Many mathematicians found explicit traveling wave solutions to differential equations (see [2, 3, 5, 8, 9, 11–15, 17, 21, 23, 27, 28, 30, 32, 33, 35, 39, 40, 43]). We investigate several classes of nonlinear partial differential equations in one-dimensional or $n$-dimensional space and establish the explicit traveling wave solutions to these equations. We provide a systematic treatment of these solutions.

For nonlinear dispersive wave equations (e.g., generalized Korteweg-de Vries equations, generalized $n$-dimensional Schrödinger equation), nonlinear dissipative dispersive wave equations (e.g., Korteweg-de Vries-Burgers equations, $n$-dimensional Ginzburg-Landau equation), nonlinear convection equations (e.g., one-dimensional Burgers equation, $n$-dimensional Burgers equation), nonlinear reaction diffusion equations (e.g., $n$-dimensional generalized Fisher's equation, $n$-dimensional Belousov-Zhabotinskii system of reaction-diffusion equations) and nonlinear hyperbolic equations (e.g., $n$-dimensional Klein-Gordon equation, $n$-dimensional Sine-Gordon equation), we derive the explicit traveling wave solutions. Many of the ideas and results are new. These differential equations have strong backgrounds in physics, chemistry, biology and fluid mechanics. The basic idea to find the explicit traveling wave solutions is to reduce higher order differential equations to lower order differential equations. The method we develop can be applied to solving the explicit traveling wave solutions to many other differential equations. We will not study the stability or instability of these waves.

We introduce some technical lemmas as follows.

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**Lemma 1.1** *Let $a \neq 0$, $A > 0$, $B > 0$ and $p > 0$ be constants. Consider the following initial value problem:*

$$\phi'(z) = a\phi(z)\{A - B[\phi(z)]^p\}, \quad \phi(0) = \Big(\frac{A}{2B}\Big)^{\frac{1}{p}}.$$

*It has the explicit bounded solution*

$$\phi(z) = \Big[\frac{A}{B}\frac{1}{1+\exp\{-aApz\}}\Big]^{\frac{1}{p}}.$$

**Proof** The given differential equation may be written as

$$\Big\{\frac{1}{p[\phi(z)]^p}\Big\}' + \frac{aA}{[\phi(z)]^p} = -\frac{\phi'(z)}{[\phi(z)]^{p+1}} + \frac{aA}{[\phi(z)]^p} = aB.$$

Multiplying this first order linear differential equation by its integrating factor $\exp(aApz)$, we get

$$\Big\{\frac{1}{[\phi(z)]^p}\exp(aApz)\Big\}' = \Big\{\frac{B}{A} + \frac{B}{A}\exp(aApz)\Big\}'.$$

Integrating it with respect to $z$, we find that

$$\frac{1}{[\phi(z)]^p}\exp(aApz) = \frac{B}{A} + \frac{B}{A}\exp(aApz).$$

The rest of the proof of Lemma 1.1 is simple and is omitted.

**Lemma 1.2** *Let $a \neq 0$, $A > 0$, $B > 0$ and $p > 0$ be constants. Consider the initial value problem*

$$\phi'(z) = a\phi(z)\{A + B[\phi(z)]^p\}, \quad \phi(0) = \Big(-\frac{A}{2B}\Big)^{\frac{1}{p}},$$

*where $(-1)^{\frac{1}{p}}$ makes sense — it is a real number. It has the explicit bounded solution*

$$\phi(z) = \Big[-\frac{A}{B}\frac{1}{1+\exp\{-aApz\}}\Big]^{\frac{1}{p}}.$$

**Proof** The proof is very simple and is omitted.

**Lemma 1.3** *Consider the second order nonlinear differential equation*

$$u''(z) = u(z)\{\alpha^2 - \beta^2[u(z)]^{2m}\},$$

*where $m > 0$, $\alpha > 0$ and $\beta > 0$ are constants. It has the explicit bounded solution*

$$u(z) = \Big[\frac{(m+1)\alpha^2}{\beta^2}\mathrm{sech}^2(\alpha mz)\Big]^{\frac{1}{2m}}.$$

*In particular, if $m = 1$, then the differential equation is $u''(z) = u(z)\{\alpha^2 - \beta^2[u(z)]^2\}$ and the solutions are*

$$u(z) = \pm\sqrt{2}\frac{\alpha}{\beta}\,\mathrm{sech}(\alpha z).$$

**Proof** If we multiply the differential equation $u''(z) = u(z)\{\alpha^2 - \beta^2[u(z)]^{2m}\}$ by $2u'(z)$ and integrate with respect to $z$, taking the integration constant to be equal to zero, we have

$$[u'(z)]^2 = \alpha^2[u(z)]^2 - \frac{1}{m+1}\beta^2[u(z)]^{2m+2}.$$

By solving it, we get

$$u'(z) = \pm\sqrt{\alpha^2 - \frac{1}{m+1}\beta^2[u(z)]^{2m}}\,u(z).$$

If we multiply this equation by $2m[u(z)]^{2m-1}$, we find

$$\{[u(z)]^{2m}\}' = \pm 2m\sqrt{\alpha^2 - \frac{1}{m+1}\beta^2[u(z)]^{2m}}\,[u(z)]^{2m}.$$

Let

$$\phi(z) = \sqrt{\alpha^2 - \frac{1}{m+1}\beta^2[u(z)]^{2m}}.$$

Then

$$[u(z)]^{2m} = \frac{m+1}{\beta^2}\{\alpha^2 - [\phi(z)]^2\}.$$

Now

$$\Big\{\frac{m+1}{\beta^2}[\alpha^2 - [\phi(z)]^2]\Big\}' = \pm 2m\phi(z)\Big\{\frac{m+1}{\beta^2}[\alpha^2 - [\phi(z)]^2]\Big\}.$$

By simplifying it, we have

$$\phi'(z) = \pm m\{\alpha^2 - [\phi(z)]^2\}.$$

Solving it, we get

$$\phi(z) = \pm\alpha\tanh(\alpha m z).$$

Hence

$$[u(z)]^{2m} = \frac{m+1}{\beta^2}\{\alpha^2 - \alpha^2[\tanh(\alpha m z)]^2\} = \frac{(m+1)\alpha^2}{\beta^2}[\mathrm{sech}(\alpha m z)]^2.$$

Finally, we obtain the explicit bounded solution.

**Lemma 1.4** *Let $m > 0$, $\alpha > 0$ and $\beta > 0$ be constants. Consider the differential equation*

$$u''(z) = u(z)\Big\{\alpha^2 + \beta^2[u(z)]^{2m} - \frac{m+2}{\sqrt{m+1}}\alpha\beta[u(z)]^m\Big\}.$$

*It has the explicit bounded solutions*

$$u(z) = \Big[\sqrt{m+1}\frac{\alpha}{\beta}\frac{1}{1+\exp\{\pm m\alpha z\}}\Big]^{\frac{1}{m}}.$$

*In particular, if $m = 1$, then the differential equation is*

$$u''(z) = u(z)\Big\{\alpha^2 + \beta^2[u(z)]^2 - \frac{3}{\sqrt{2}}\alpha\beta u(z)\Big\}$$

*and the solutions are*

$$u(z) = \sqrt{2}\frac{\alpha}{\beta}\frac{1}{1+\exp\{\pm\alpha z\}}.$$

**Proof** Let us reduce the order of the differential equation. Let

$$u'(z) = \pm u(z)\Big\{\alpha - \frac{\beta}{\sqrt{m+1}}[u(z)]^m\Big\}.$$

Then

$$\begin{aligned} u''(z) &= u(z)\Big\{\alpha - \frac{\beta}{\sqrt{m+1}}[u(z)]^m\Big\}\{\alpha - \sqrt{m+1}\beta[u(z)]^m\} \\ &= u(z)\Big\{\alpha^2 - \frac{m+2}{\sqrt{m+1}}\alpha\beta[u(z)]^m + \beta^2[u(z)]^{2m}\Big\}. \end{aligned}$$

Therefore, we obtain the explicit bounded solutions by using Lemma 1.1.

## 2 Nonlinear Dispersive Wave Equations

In this section, we are going to establish the explicit traveling wave solutions to $n$-dimensional Boussinesq equation, nonlinear Korteweg-de Vries equations, nonlinear system of Korteweg-de Vries equations, general two-dimensional and three-dimensional Kadomtsev-Petviashvili equations and $n$-dimensional cubic nonlinear Schrödinger equation. This section is primarily motivated by [4–6, 25–27].

**Motivation equation I** Consider the nonlinear cubic Schrödinger equation

$$\mathrm{i}\frac{\partial u}{\partial t} + \alpha\frac{\partial^2 u}{\partial x^2} + \beta|u|^2 u = 0,$$

where $\alpha > 0$ and $\beta > 0$ are constants. Let $a$, $c$ and $\omega \in \mathbb{R}$ be real constants, such that $c^2 > a^2$. Then the nonlinear cubic Schrödinger equation has the explicit solitary wave solutions

$$u(x,t) = \pm\sqrt{\frac{2\alpha c^2}{\beta}}\exp\{\mathrm{i}(ax + \alpha(c^2 - a^2)t + \omega)\}\mathrm{sech}(c(x - 2\alpha a t)).$$

**Motivation equation II** Consider the nonlinear Schrödinger equation

$$\mathrm{i}\frac{\partial u}{\partial t} + \alpha\frac{\partial^2 u}{\partial x^2} + \beta|u|^2 u = \gamma u,$$

where $\alpha > 0$, $\beta > 0$ and $\gamma > 0$ are constants. The nonlinear Schrödinger equation has the explicit standing wave solutions

$$u(x) = \pm\sqrt{\frac{2\gamma}{\beta}}\,\mathrm{sech}\Big(\sqrt{\frac{\gamma}{\alpha}}x\Big).$$

Motivated by these results, we investigate the explicit traveling wave solutions to nonlinear dispersive wave equations.

**Theorem 2.1** *Consider the generalized $n$-dimensional Boussinesq equation*

$$\frac{\partial^2 u}{\partial t^2} - \triangle u + \beta\triangle(r^2u^{p+1} - s^2u^{2p+1}) + \alpha\triangle^2 u = 0, \tag{2.1}$$

*where $\alpha > 0$, $\beta > 0$, $p > 0$ and $r > 0$ are positive constants, $s \geq 0$ is also a constant, $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ is a spatial variable and $t > 0$ is a temporal variable. Let $\mathbf{a} = (a_1, a_2, \cdots, a_n)$ be any nonzero real constant vector and let $c$ be any real constant, such that $|c| < |\mathbf{a}|$. If $s = 0$, then the generalized $n$-dimensional Boussinesq equation has the explicit traveling wave solution*

$$u(\mathbf{x}, t) = \Big[(p+2)\frac{|\mathbf{a}|^2 - c^2}{2|\mathbf{a}|^2 r^2\beta}\mathrm{sech}^2\Big(\sqrt{\frac{|\mathbf{a}|^2 - c^2}{|\mathbf{a}|^4\alpha}}\frac{p(\mathbf{a}\cdot\mathbf{x} + ct)}{2}\Big)\Big]^{\frac{1}{p}}. \tag{2.2}$$

*If $s > 0$ and*

$$c = |\mathbf{a}|\Big\{1 - \frac{(p+1)r^4\beta}{(p+2)^2s^2}\Big\}^{\frac{1}{2}},$$

*then*

$$u(\mathbf{x}, t) = \left\{\frac{\frac{(p+1)r^2|\mathbf{a}|}{(p+2)s^2}}{1 + \exp\Big\{\pm\frac{p\sqrt{p+1}}{p+2}\frac{r^2}{s}\sqrt{\frac{\beta}{\alpha}}\Big(|\mathbf{a}|\cdot|\mathbf{x}| + |\mathbf{a}|\Big(\mathbf{1} - \frac{(\mathbf{p+1})\mathbf{r^4}\beta}{(\mathbf{p+2})^\mathbf{2}\mathbf{s^2}}\Big)^{\frac{1}{2}}\mathbf{t}\Big)\Big\}}\right\}^{\frac{1}{p}}.$$

**Proof** Let $u(\mathbf{x}, t) = \phi(\mathbf{a}\cdot\mathbf{x} + ct)$ be a traveling wave solution, where $c$ is a real constant such that $c^2 < |\mathbf{a}|^2$, and set $z = \mathbf{a}\cdot\mathbf{x} + ct$. Then

$$c^2\phi''(z) - |\mathbf{a}|^2\phi''(z) + |\mathbf{a}|^2\beta\{r^2[\phi(z)]^{p+1} - s^2[\phi(z)]^{2p+1}\}'' + |\mathbf{a}|^4\alpha\phi^{(4)}(z) = 0.$$

Integrating this equation twice with respect to $z$ and letting the integration constants be equal to zero, we have

$$c^2\phi(z) - |\mathbf{a}|^2\phi(z) + |\mathbf{a}|^2\beta\{r^2[\phi(z)]^{p+1} - s^2[\phi(z)]^{2p+1}\} + |\mathbf{a}|^4\alpha\phi''(z) = 0.$$

Therefore, we have

$$\phi''(z) = \Big\{\frac{|\mathbf{a}|^2 - c^2}{|\mathbf{a}|^4\alpha} - \frac{r^2\beta}{|\mathbf{a}|^2\alpha}[\phi(z)]^p + \frac{s^2\beta}{|\mathbf{a}|^2\alpha}[\phi(z)]^{2p}\Big\}\phi(z).$$

Now by using Lemma 1.3 (for $s = 0$) and Lemma 1.4 (for $s > 0$), where $c$ is determined by the equation

$$\frac{p+2}{\sqrt{p+1}}\frac{s}{|\mathbf{a}|}\sqrt{\frac{\beta}{\alpha}}\sqrt{\frac{|\mathbf{a}|^2 - c^2}{|\mathbf{a}|^4\alpha}} = \frac{r^2\beta}{|\mathbf{a}|\alpha},$$

we can finish the proof.

**Theorem 2.2** *Consider the nonlinear system of Boussinesq equations*

$$\frac{\partial u}{\partial t} + \frac{\partial w}{\partial x} + \frac{\partial}{\partial x}(uw) + \frac{\partial^3 w}{\partial x^3} = 0, \tag{2.3}$$

$$\frac{\partial w}{\partial t} + \frac{\partial u}{\partial x} + w\frac{\partial w}{\partial x} = 0. \tag{2.4}$$

*Let $c$ be any real constant. Then the nonlinear system of Boussinesq equations has the explicit traveling wave solution*

$$u(x,t) = \frac{2c^2 \exp\{c(x+ct)\}}{[1+\exp\{c(x+ct)\}]^2} - 1, \tag{2.5}$$

$$w(x,t) = -\frac{2c}{1+\exp\{c(x+ct)\}}. \tag{2.6}$$

**Proof** Let $z = x + ct$, $v(x,t) = 1 + u(x,t)$. And let

$$v(x,t) = \phi(x+ct), \quad w(x,t) = \psi(x+ct).$$

Then

$$c\phi'(z) + [\phi(z)\psi(z)]' + \psi'''(z) = 0,$$
$$c\psi'(z) + \phi'(z) + \psi(z)\psi'(z) = 0.$$

Integrating this system of differential equations with respect to $z$ and letting the integration constant be equal to zero, we have

$$[c+\psi(z)]\phi(z) + \psi''(z) = 0,$$
$$\phi(z) + \Big[c + \frac{1}{2}\psi(z)\Big]\psi(z) = 0.$$

By canceling out $\phi$, we get

$$\psi''(z) = \psi(z)[c+\psi(z)]\Big[c + \frac{1}{2}\psi(z)\Big].$$

Next, we reduce the order of the differential equation. Let

$$\psi'(z) = -\psi\Big[c + \frac{1}{2}\psi(z)\Big].$$

Then

$$\psi''(z) = -[c+\psi(z)]\psi'(z) = \psi(z)[c+\psi(z)]\Big[c + \frac{1}{2}\psi(z)\Big].$$

Therefore, by using Lemma 1.2, we find

$$\psi(z) = -\frac{2c}{1+\exp\{cz\}}, \quad \phi(z) = \frac{2c^2\exp\{cz\}}{[1+\exp\{cz\}]^2}.$$

The proof is finished.

**Theorem 2.3** *Consider the nonlinear Korteweg-de Vries equation*

$$\frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} + u^p\frac{\partial u}{\partial x} = 0, \tag{2.7}$$

*where $p > 0$ is a constant. The nonlinear Korteweg-de Vries equation has the explicit traveling wave solution*

$$u(x,t) = \Big\{\frac{1}{2}(p+1)(p+2)\Big[c\ \mathrm{sech}\Big(\frac{1}{2}cpz\Big)\Big]^2\Big\}^{\frac{1}{p}}, \tag{2.8}$$

*where $z = x - c^2 t$, $c$ is a real constant and $c^2$ is the wave speed. For $p = 1$, $p = 2$, $p = 3$, the traveling wave solutions are given by, respectively,*

$$\phi_1(z) = 3c^2 \operatorname{sech}^2\Big(\frac{1}{2}cz\Big), \quad \phi_2(z) = \sqrt{6}c \operatorname{sech}(cz), \quad \phi_3(z) = \Big\{10c^2 \operatorname{sech}^2\Big(\frac{3}{2}cz\Big)\Big\}^{\frac{1}{3}}.$$

**Proof** A traveling wave solution takes the form $u(x,t) = \phi(z) = \phi(x - c^2 t)$, where $z = x - c^2 t$ for some number $c > 0$. Therefore,

$$-c^2\phi'(z) + \phi'''(z) + [\phi(z)]^p\phi'(z) = 0.$$

Integrating this equation with respect to $z$ and letting the constant of integration be equal to zero, we have

$$-c^2\phi(z) + \phi''(z) + \frac{1}{p+1}[\phi(z)]^{p+1} = 0.$$

We get

$$\phi''(z) = \phi(z)\Big\{c^2 - \frac{1}{p+1}[\phi(z)]^p\Big\}.$$

The proof is finished by using Lemma 1.3.

**Theorem 2.4** *Consider the generalized nonlinear Korteweg-de Vries equation*

$$\frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} + \frac{\partial}{\partial x}(r^2 u^{p+1} - s^2 u^{2p+1}) = 0, \tag{2.9}$$

*where $r > 0$, $s > 0$ and $p > 0$ are constants. Then the generalized nonlinear Korteweg-de Vries equation has the explicit traveling wave solutions*

$$u(x,t) = \left\{\frac{\frac{(p+1)r^2}{(p+2)s^2}}{1 + \exp\Big\{\pm\frac{p\sqrt{p+1}r^2}{(p+2)s}\Big(x - \frac{(p+1)r^4}{(p+2)^2 s^2}t\Big)\Big\}}\right\}^{\frac{1}{p}}. \tag{2.10}$$

**Proof** Let $u(x,t) = \phi(x - c^2 t)$ be a traveling wave solution, where $c$ is a constant, and set $z = x - c^2 t$. Then

$$-c^2\phi'(z) + \phi'''(z) + \{r^2[\phi(z)]^{p+1} - s^2[\phi(z)]^{2p+1}\}' = 0.$$

Integrating this equation with respect to $z$ and letting the integration constant be equal to zero, we have

$$-c^2\phi(z) + \phi''(z) + r^2[\phi(z)]^{p+1} - s^2[\phi(z)]^{2p+1} = 0.$$

In other words, we get

$$\phi''(z) = \phi(z)\{c^2 - r^2[\phi(z)]^p + s^2[\phi(z)]^{2p}\}.$$

Let

$$c = \frac{\sqrt{p+1}\,r^2}{(p+2)s}.$$

The proof is finished by using Lemma 1.4.

**Theorem 2.5** *Consider the Korteweg-de Vries equation with strong nonlinear functions*

$$\frac{\partial u}{\partial t}+3u^2\frac{\partial u}{\partial x}+3\Big(\frac{\partial u}{\partial x}\Big)^2+3u\frac{\partial^2 u}{\partial x^2}+\frac{\partial^3 u}{\partial x^3}=0. \tag{2.11}$$

*Let $c$ be any real constant. Then the Korteweg-de Vries equation with strong nonlinear functions has the explicit traveling wave solution*

$$u(x,t)=\frac{c}{1+\exp\{-cz\}},\quad z=x-c^2t. \tag{2.12}$$

**Proof** Let $u(x,t)=\phi(x-c^2t)$ be a traveling wave solution, where $c$ is a real constant, and set $z=x-c^2t$ . Then

$$-c^2\phi'(z)+3[\phi(z)]^2\phi'(z)+3\{\phi'(z)\}^2+3\phi(z)\phi''(z)+\phi'''(z)=0.$$

Integrating this equation with respect to $z$ and letting the integration constant be equal to zero, we have

$$-c^2\phi(z)+[\phi(z)]^3+3\phi(z)\phi'(z)+\phi''(z)=0.$$

Let us reduce the order of the differential equation. Let $D\neq 0$ be a constant and

$$\phi'(z)=D\phi(z)[c-\phi(z)].$$

Then

$$\phi''(z)=D^2\phi(z)[c-\phi(z)][c-2\phi(z)].$$

Plugging the derivatives back into the differential equation, we find

$$-c^2\phi(z)+[\phi(z)]^3+3D[\phi(z)]^2[c-\phi(z)]+D^2\phi[c-\phi(z)][c-2\phi(z)]=0.$$

By canceling out $\phi$, we find

$$-c^2+[\phi(z)]^2+3D\phi[c-\phi(z)]+D^2[c-\phi(z)][c-2\phi(z)]=0.$$

By comparing the coefficients, we have

$$c^2+c^2D^2=0,\quad 3cD-3cD^2=0,\quad 1-3D+2D^2=0.$$

It is easy to see that $D=1$. Therefore, by using Lemma 1.1, Theorem 2.5 is proved.

**Theorem 2.6** *Consider the Korteweg-de Vries equation with nonlinear dispersion*

$$\frac{\partial u}{\partial t}+\frac{\partial}{\partial x}(u^2)+\frac{\partial^3}{\partial x^3}(u^2)=0. \tag{2.13}$$

*Let $c$ be any real constant. Then the Korteweg-de Vries equation with nonlinear dispersion has the explicit traveling wave solutions*

$$u(x,t)=-\frac{2}{3}c\pm\frac{2}{3}c\sin\Big(\frac{1}{2}(x+ct)\Big). \tag{2.14}$$

**Proof** Let $u(x,t) = \phi(x+ct)$ be a traveling wave solution, where $c$ is a real constant, and set $z = x + ct$. Then

$$c\phi' + \{[\phi(z)]^2\}' + \{[\phi(z)]^2\}''' = 0.$$

Integrating this equation with respect to $z$ and letting the integration constant be equal to zero, we have

$$c\phi(z) + [\phi(z)]^2 + \{[\phi(z)]^2\}'' = 0.$$

Multiplying this equation by $2\{[\phi(z)]^2\}'$, integrating it with respect to $z$ and letting the integration constant be equal to zero, we have

$$\frac{4}{3}c[\phi(z)]^3 + [\phi(z)]^4 + (\{[\phi(z)]^2\}')^2 = 0.$$

Thus

$$\{[\phi(z)]^2\}' = \pm\sqrt{-\frac{4}{3}c[\phi(z)]^3 - [\phi(z)]^4}.$$

Equivalently,

$$\phi' = \pm\frac{1}{2}\sqrt{-\frac{4}{3}c\phi(z) - [\phi(z)]^2}.$$

Thus

$$\Big(\phi(z) + \frac{2}{3}c\Big)' = \pm\frac{1}{2}\sqrt{\Big[\frac{2}{3}c\Big]^2 - \Big[\phi(z) + \frac{2}{3}c\Big]^2}.$$

Therefore, we have obtained the explicit traveling wave solution.

**Theorem 2.7** *Consider the nonlinear system of Korteweg-de Vries equations*

$$\frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} + 3\frac{\partial}{\partial x}(u^2) = 6\beta^2\frac{\partial}{\partial x}(v^2), \tag{2.15}$$

$$\frac{\partial v}{\partial t} + \frac{\partial^3 v}{\partial x^3} + 3u\frac{\partial v}{\partial x} = 0, \tag{2.16}$$

*where $\beta > 0$ is a constant. Let $c > 0$ be any real constant. Then the nonlinear system of Korteweg-de Vries equations has the solitary traveling wave solutions*

$$u(x,t) = c^2\Big\{\mathrm{sech}\Big(\frac{c}{2}(x - c^2 t)\Big)\Big\}^2, \tag{2.17}$$

$$v(x,t) = \frac{c^2}{2\beta}\Big\{\mathrm{sech}\Big(\frac{c}{2}(x - c^2 t)\Big)\Big\}^2. \tag{2.18}$$

**Proof** Let $u(x,t) = \phi(x - c^2 t)$ and $v(x,t) = \psi(x - c^2 t)$ be traveling wave solutions, and set $z = x - c^2 t$. Then

$$-c^2\phi'(z) + \phi'''(z) + 3\{[\phi(z)]^2\}' = 6\beta^2\{[\psi(z)]^2\}',$$
$$-c^2\psi'(z) + \psi'''(z) + 3\phi(z)\psi'(z) = 0.$$

Integrating this system of differential equations with respect to $z$, letting the integration constant be equal to zero, and assuming that $\phi(z) = d\psi(z)$ with a constant $d$, we have

$$\begin{aligned}-c^2\phi(z) + \phi''(z) + 3[\phi(z)]^2 &= 6\beta^2[\psi(z)]^2,\\ -c^2\psi(z) + \psi''(z) + \frac{3d}{2}[\psi(z)]^2 &= 0.\end{aligned}$$

By comparing these two equations, we may let $\frac{3}{2} = 3 - \frac{6\beta^2}{d^2}$, i.e. $d = 2\beta$. The differential equation reduces to

$$-c^2\phi(z) + \phi''(z) + \frac{3}{2}[\phi(z)]^2 = 0,$$

i.e.,

$$\phi''(z) = \phi(z)\Big[c^2 - \frac{3}{2}\phi(z)\Big].$$

The proof is finished by using Lemma 1.3.

**Theorem 2.8** *Consider the generalized two-dimensional nonlinear Korteweg-de Vries equation*

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(r^2u^{p+1} - s^2u^{2p+1}) + \beta\frac{\partial^3 u}{\partial x^3} + \alpha\frac{\partial^3 u}{\partial x\partial y^2} = 0, \tag{2.19}$$

*where $\alpha \neq 0$ and $\beta \neq 0$ are real constants, $p > 0$ and $r > 0$ are positive constants, $s \geq 0$ is also a constant. Let $z = ax + by + ct$, where $a$, $b$ and $c$ are real constants, such that*

$$\frac{1}{a^2\beta + b^2\alpha} > 0, \quad \frac{c}{a^3\beta + ab^2\alpha} < 0.$$

*If $s = 0$, then the generalized two-dimensional nonlinear Korteweg-de Vries equation has the explicit traveling wave solution*

$$u(x,y,t) = \Big\{ -\frac{(p+2)c}{2ar^2}\operatorname{sech}^2\Big(\frac{p}{2}\sqrt{-\frac{c}{a^3\beta + ab^2\alpha}}(ax + by + ct)\Big)\Big\}^{\frac{1}{p}}. \tag{2.20}$$

*If $s > 0$ and*

$$c = -\frac{p+1}{(p+2)^2}\frac{ar^4}{s^2},$$

*then*

$$u(x,t) = \Big\{\frac{\frac{(p+1)\,r^2}{(p+2)s^2}}{1 + \exp\Big\{\pm\sqrt{\frac{1}{a^2\beta+b^2\alpha}}\frac{p\sqrt{p+1}\,r^2}{(p+2)s}\Big[ax + by - \frac{p+1}{(p+2)^2}\frac{ar^4}{s^2}t\Big]\Big\}}\Big\}^{\frac{1}{p}}.$$

**Proof** Let $u(x,y,t) = \phi(ax + by + ct)$ be a traveling wave solution. Then

$$c\phi'(z) + a\{r^2[\phi(z)]^{p+1} - s^2[\phi(z)]^{2p+1}\}' + a^3\beta\phi'''(z) + ab^2\alpha\phi'''(z) = 0.$$

Integrating this equation with respect to $z$ and letting the integration constant be equal to zero, we have

$$c\phi(z) + a\{r^2[\phi(z)]^{p+1} - s^2[\phi(z)]^{2p+1}\} + a^3\beta\phi''(z) + ab^2\alpha\phi''(z) = 0.$$

Now we have

$$\phi''(z) = \Big\{ - \frac{c}{a^3\beta + ab^2\alpha} - \frac{ar^2}{a^3\beta + ab^2\alpha}[\phi(z)]^p + \frac{as^2}{a^3\beta + ab^2\alpha}[\phi(z)]^{2p} \Big\}\phi(z).$$

Therefore, by using Lemma 1.3 (for $s = 0$) and Lemma 1.4 (for $s > 0$), we find the traveling wave solutions, where $c$ is determined by the equation

$$\frac{p+2}{\sqrt{p+1}}\sqrt{-\frac{c}{a^3\beta + ab^2\alpha}}\sqrt{\frac{as^2}{a^3\beta + ab^2\alpha}} = \frac{ar^2}{a^3\beta + ab^2\alpha}.$$

The proof is finished.

**Theorem 2.9** *Consider the generalized two-dimensional Kadomtsev-Petviashvili equation*

$$\frac{\partial}{\partial x}\Big[\frac{\partial u}{\partial t} + r^2(p+1)u^p\frac{\partial u}{\partial x} - s^2(2p+1)u^{2p}\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3}\Big] + \varepsilon^2\frac{\partial^2 u}{\partial y^2} = 0, \tag{2.21}$$

*where $p > 0$, $r > 0$ and $\varepsilon > 0$ are positive constants, $s \geq 0$ is also a constant. Let $\xi = ax+by+ct$, where $a$, $b$ and $c$ are constants, such that*

$$ac + b^2\varepsilon^2 < 0.$$

*If $s = 0$, the generalized two-dimensional Kadomtsev-Petviashvili equation has the explicit traveling wave solution*

$$\phi(z) = \Big\{ - \frac{(ac+b^2\varepsilon^2)(p+2)}{2a^2r^2}\mathrm{sech}^2\Big(\frac{p}{2}\sqrt{-\frac{ac+b^2\varepsilon^2}{a^4}}(ax+by+ct)\Big)\Big\}^{\frac{1}{p}}. \tag{2.22}$$

*If $s > 0$ and*

$$c = -\frac{1}{a}\Big[b^2\varepsilon^2 + \frac{p+1}{(p+2)^2}\frac{a^2r^4}{s^2}\Big],$$

*then*

$$u(x,t) = \Big\{ \frac{\frac{(p+1)r^2}{(p+2)s^2}}{1 + \exp\Big\{ \pm \frac{p\sqrt{p+1}}{p+2}\frac{r^2}{|a|s}\Big[ax + by - \frac{1}{a}\Big(b^2\varepsilon^2 + \frac{p+1}{(p+2)^2}\frac{a^2r^4}{s^2}\Big)t\Big]\Big\}} \Big\}^{\frac{1}{p}}.$$

**Proof** Let

$$u(x,y,t) = \phi(ax+by+ct)$$

be a traveling wave solution. Then

$$a\{c\phi'(z) + ar^2(p+1)[\phi(z)]^p\phi'(z) - as^2(2p+1)[\phi(z)]^{2p}\phi'(z) + a^3\phi'''(z)\}' + b^2\varepsilon^2\phi''(z) = 0.$$

Integrating this equation twice with respect to $\xi$ and letting the integration constants be equal to zero, we have

$$a\{c\phi(z) + ar^2[\phi(z)]^{p+1} - as^2[\phi(z)]^{2p+1} + a^3\phi''(z)\} + b^2\varepsilon^2\phi(z) = 0.$$

Therefore, we have

$$a^4\phi''(z) + (ac + b^2\varepsilon^2)\phi(z) + a^2r^2[\phi(z)]^{p+1} - a^2s^2[\phi(z)]^{2p+1} = 0.$$

Thus

$$\phi''(z) = \phi(z)\Big\{-\frac{ac + b^2\varepsilon^2}{a^4} - \frac{r^2}{a^2}[\phi(z)]^p + \frac{s^2}{a^2}[\phi(z)]^{2p}\Big\}.$$

Therefore, by using Lemma 1.3 (for $s = 0$) and Lemma 1.4 (for $s > 0$), we find the traveling wave solutions, where $c$ is determined by the equation

$$\frac{p+2}{\sqrt{p+1}}\frac{s}{|a|}\sqrt{-\frac{ac + b^2\varepsilon^2}{a^4}} = \frac{r^2}{a^2}.$$

The proof is finished.

**Corollary 2.1** (I) *Consider the two-dimensional Kadomtsev-Petviashvili equation*

$$\frac{\partial}{\partial x}\Big(\frac{\partial u}{\partial t} + 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3}\Big) + \varepsilon^2\frac{\partial^2 u}{\partial y^2} = 0, \tag{2.23}$$

*where $\varepsilon > 0$ is a positive constant. The two-dimensional Kadomtsev-Petviashvili equation has a traveling wave solution*

$$u(x, y, t) = -\frac{ac + b^2\varepsilon^2}{2a^2}\mathrm{sech}^2\Big(\frac{1}{2}\sqrt{-\frac{ac + b^2\varepsilon^2}{a^4}}(ax + by + ct)\Big). \tag{2.24}$$

(II) *Consider the two-dimensional Kadomtsev-Petviashvili equation*

$$\frac{\partial}{\partial x}\Big(\frac{\partial u}{\partial t} + 12u^2\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3}\Big) + \varepsilon^2\frac{\partial^2 u}{\partial y^2} = 0. \tag{2.25}$$

*The two-dimensional Kadomtsev-Petviashvili equation has the explicit traveling wave solution*

$$u(x, y, t) = \Big\{-\frac{ac + b^2\varepsilon^2}{2a^2}\mathrm{sech}^2\Big(\sqrt{-\frac{ac + b^2\varepsilon^2}{a^4}}(ax + by + ct)\Big)\Big\}^{\frac{1}{2}}. \tag{2.26}$$

**Proof** By using Theorem 2.9, we may finish the proof immediately.

**Theorem 2.10** *Consider the generalized three-dimensional Kadomtsev-Petviashvili equation*

$$\frac{\partial}{\partial x}\Big[\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(r^2u^{p+1} - s^2u^{2p+1}) + \alpha^2\frac{\partial^3 u}{\partial x^3}\Big] + \beta^2\frac{\partial^2 u}{\partial y^2} + \gamma^2\frac{\partial^2 u}{\partial z^2} = 0, \tag{2.27}$$

*where $\alpha > 0$, $\beta > 0$, $\gamma > 0$, $p > 0$ and $r > 0$ are positive constants, $s \geq 0$ is also a constant. Let*

$$\xi = ax + by + cz + \mathrm{d}t,$$

*where $a$, $b$, $c$ and $d$ are real constants, such that*

$$\frac{ad + b^2\beta^2 + c^2\gamma^2}{a^4\alpha^2} < 0.$$

*If $s = 0$, then the generalized three-dimensional Kadomtsev-Petviashvili equation has the explicit traveling wave solution*

$$u(x, y, z, t) = \Big\{ - \frac{p+2}{2} \frac{ad + b^2\beta^2 + c^2\gamma^2}{a^2r^2} \cdot \operatorname{sech}^2\Big(\sqrt{-\frac{ad + b^2\beta^2 + c^2\gamma^2}{a^4\alpha^2}} \frac{p(ax + by + cz + \mathrm{d}t)}{2}\Big)\Big\}^{\frac{1}{p}}. \tag{2.28}$$

*If $s > 0$ and*

$$d = -\frac{1}{a}\Big[b^2\beta^2 + c^2\gamma^2 + \frac{p+1}{(p+2)^2}\frac{a^2r^4}{s^2}\Big],$$

*then*

$$u(x, y, z, t) = \Big\{\frac{\frac{(p+1)r^2}{(p+2)s^2}}{1 + \exp\Big\{\pm \frac{p\sqrt{p+1}r^2}{|a|(p+2)s\alpha}\Big[ax + by + cz - \frac{1}{a}\Big(b^2\beta^2 + c^2\gamma^2 + \frac{p+1}{(p+2)^2}\frac{a^2r^4}{s^2}\Big)t\Big]\Big\}}\Big\}^{\frac{1}{p}}. \tag{2.29}$$

**Proof** Let

$$u(x, y, z, t) = \phi(ax + by + cz + \mathrm{d}t)$$

be a traveling wave solution. Then

$$a\{d\phi'(\xi) + a\{r^2[\phi(\xi)]^{p+1} - s^2[\phi(\xi)]^{2p+1}\}' + a^3\alpha^2\phi'''(\xi)\}' + b^2\beta^2\phi''(\xi) + c^2\gamma^2\phi''(\xi) = 0.$$

Integrating this equation with respect to $\xi$ and letting the integration constants be equal to zero, we have

$$a\{d\phi(\xi) + ar^2[\phi(\xi)]^{p+1} - as^2[\phi(\xi)]^{2p+1} + a^3\alpha^2\phi''(\xi)\} + b^2\beta^2\phi(\xi) + c^2\gamma^2\phi(\xi) = 0.$$

Hence

$$a^4\alpha^2\phi''(\xi) = -[ad + b^2\beta^2 + c^2\gamma^2]\phi(\xi) - a^2r^2[\phi(\xi)]^{p+1} + a^2s^2[\phi(\xi)]^{2p+1}.$$

Therefore, we have

$$\phi''(\xi) = \Big\{ - \frac{ad + b^2\beta^2 + c^2\gamma^2}{a^4\alpha^2} - \frac{r^2}{a^2\alpha^2}[\phi(\xi)]^p + \frac{s^2}{a^2\alpha^2}[\phi(\xi)]^{2p}\Big\}\phi(\xi).$$

Finally, we obtain the explicit traveling wave solution by using Lemma 1.3 (for $s = 0$) and Lemma 1.4 (for $s > 0$), where $d$ is determined by the equation

$$\frac{p+2}{\sqrt{p+1}}\frac{s}{|a|\alpha}\sqrt{-\frac{ad + b^2\beta^2 + c^2\gamma^2}{a^4\alpha^2}} = \frac{r^2}{a^2\alpha^2}.$$

The proof is finished.

**Theorem 2.11** *Consider the $n$-dimensional nonlinear Schrödinger equation*

$$\mathrm{i}\frac{\partial u}{\partial t} + \alpha\triangle u + \beta|u|^2u + \gamma|u|^4u = 0,$$

*where $\alpha > 0$ and $\beta > 0$ are positive constants, $\gamma \le 0$ is also a constant. Let $A$ and $B$ be real parameters and let $\mathbf{a} = (a_1, a_2, \cdots, a_n)$ be a nonzero real constant vector. Then the nonlinear Schrödinger equation has the explicit solitary wave solutions*

$$\begin{aligned} u(\mathbf{x},t) &= \pm\sqrt{\frac{2\alpha}{\beta}}|\mathbf{a}|B\exp\{\mathrm{i}A(\mathbf{a}\cdot\mathbf{x}-2A|\mathbf{a}|^2\alpha t)+\mathrm{i}|\mathbf{a}|^2\alpha(A^2+B^2)t\} \\ &\quad\cdot\ \mathrm{sech}(B(\mathbf{a}\cdot\mathbf{x}-2A|\mathbf{a}|^2\alpha t)), \quad \text{if } \gamma = 0, \qquad (2.30) \\ u(\mathbf{x},t) &= \pm\exp\{\mathrm{i}A(\mathbf{a}\cdot\mathbf{x}-2A|\mathbf{a}|^2\alpha t)+\mathrm{i}|\mathbf{a}|^2\alpha(A^2+B^2)t\} \\ &\quad\cdot\Big\{\frac{4|\mathbf{a}|^2\alpha B^2}{\beta}\frac{1}{1+\exp\{\pm 2|B|(\mathbf{a}\cdot\mathbf{x}-2A|\mathbf{a}|^2\alpha t)\}}\Big\}^{\frac{1}{2}}, \quad \text{if } \gamma = -\frac{3\beta^3}{16|\mathbf{a}|^2\alpha B^2}. \qquad (2.31) \end{aligned}$$

**Proof** Let $\mathbf{a} = (a_1, a_2, \cdots, a_n)$ be any nonzero real constant vector. Let

$$u(\mathbf{x},t) = \exp\{\mathrm{i}\omega(\mathbf{a}\cdot\mathbf{x}+ct)+\mathrm{i}bt\}\phi(\mathbf{a}\cdot\mathbf{x}+ct)$$

be a solitary wave solution of the nonlinear Schrödinger equation, and set $z = \mathbf{a}\cdot\mathbf{x}+ct$. Then

$$\begin{aligned} \frac{\partial u}{\partial t} &= \mathrm{i}[c\omega'(z)+b]\exp\{\mathrm{i}\omega(z)+\mathrm{i}bt\}\phi(z)+c\exp\{\mathrm{i}\omega(z)+\mathrm{i}bt\}\phi'(z), \\ \frac{\partial u}{\partial x_k} &= \mathrm{i}a_k\omega'(z)\exp\{\mathrm{i}\omega(z)+\mathrm{i}bt\}\phi(z)+a_k\exp\{\mathrm{i}\omega(z)+\mathrm{i}bt\}\phi'(z), \\ \frac{\partial^2 u}{\partial x_k^2} &= {a_k}^2\exp\{\mathrm{i}\omega(z)+\mathrm{i}bt\}\cdot\{\mathrm{i}\omega''(z)\phi(z)-[\omega'(z)]^2\phi(z)+2\mathrm{i}\omega'(z)\phi'(z)+\phi''(z)\}, \\ \triangle u &= |\mathbf{a}|^2\exp\{\mathrm{i}\omega(z)+\mathrm{i}bt\}\cdot\{\mathrm{i}\omega''(z)\phi(z)-[\omega'(z)]^2\phi(z)+2\mathrm{i}\omega'(z)\phi'(z)+\phi''(z)\}. \end{aligned}$$

Therefore, we get

$$\begin{aligned} &\mathrm{i}\frac{\partial u}{\partial t}+\alpha\triangle u+\beta|u|^2u+\gamma|u|^4u \\ &= \exp\{\mathrm{i}\omega(z)+\mathrm{i}bt\}\cdot\{-[c\omega'(z)+b]\phi(z)+\mathrm{i}c\phi'(z)+|\mathbf{a}|^2\alpha[\mathrm{i}\omega''(z)\phi(z) \\ &\quad -(\omega'(z))^2\phi(z)+2\mathrm{i}\omega'(z)\phi'(z)+\phi''(z)]+\beta[\phi(z)]^3+\gamma[\phi(z)]^5\} \\ &= \exp\{\mathrm{i}\omega(z)+\mathrm{i}bt\}\cdot\{-[c\omega'(z)+b]\phi(z)-|\mathbf{a}|^2\alpha(\omega'(z))^2\phi(z)+|\mathbf{a}|^2\alpha\phi''(z) \\ &\quad +\beta[\phi(z)]^3+\gamma[\phi(z)]^5+\mathrm{i}[c\phi'(z)+|\mathbf{a}|^2\alpha\omega''(z)\phi(z)+2|\mathbf{a}|^2\alpha\omega'(z)\phi'(z)]\} \\ &= 0. \end{aligned}$$

The real part and the imaginary part should be equal to zero, that is,

$$\begin{aligned} &-[c\omega'(z)+b]\phi(z)-|\mathbf{a}|^2\alpha[\omega'(z)]^2\phi(z)+|\mathbf{a}|^2\alpha\phi''(z)+\beta[\phi(z)]^3+\gamma[\phi(z)]^5 = 0, \\ &c\phi'(z)+|\mathbf{a}|^2\alpha\omega''(z)\phi(z)+2|\mathbf{a}|^2\alpha\omega'(z)\phi'(z) = 0. \end{aligned}$$

Let $\omega(z) = Az$ for some real parameter $A$. Then

$$\omega'(z) = A, \quad \omega''(z) = 0.$$

It is easy to see that

$$c+2A|\mathbf{a}|^2\alpha = 0.$$

Therefore, we get

$$\begin{aligned}\phi''(z) &= \phi(z)\Big\{A^2 + \frac{Ac+b}{|\mathbf{a}|^2\alpha} - \frac{\beta}{|\mathbf{a}|^2\alpha}[\phi(z)]^2 - \frac{\gamma}{|\mathbf{a}|^2\alpha}[\phi(z)]^4\Big\} \\ &= \phi(z)\Big\{\Big(\frac{b}{|\mathbf{a}|^2\alpha} - A^2\Big) - \frac{\beta}{|\mathbf{a}|^2\alpha}[\phi(z)]^2 - \frac{\gamma}{|\mathbf{a}|^2\alpha}[\phi(z)]^4\Big\}.\end{aligned}$$

Let

$$b = |\mathbf{a}|^2\alpha(A^2 + B^2)$$

for two real parameters $A$ and $B$. Then

$$\phi''(z) = \phi(z)\Big\{B^2 - \frac{\beta}{|\mathbf{a}|^2\alpha}[\phi(z)]^2 - \frac{\gamma}{|\mathbf{a}|^2\alpha}[\phi(z)]^4\Big\}.$$

Finally, by Lemma 1.3 and Lemma 1.4, we obtain the explicit traveling wave solutions

$$\begin{aligned}&\phi(z) = \pm\sqrt{\frac{2\alpha}{\beta}}|\mathbf{a}|B \operatorname{sech}(Bz), \quad \text{if } \gamma = 0, \\ &\phi(z) = \pm\Big\{\frac{4|\mathbf{a}|^2\alpha B^2}{\beta}\frac{1}{1+\exp\{\pm 2|B|z\}}\Big\}^{\frac{1}{2}}, \quad \text{if } \gamma = -\frac{3\beta^2}{16|\mathbf{a}|^2\alpha B^2}.\end{aligned}$$

The proof is finished.

**Theorem 2.12** *Consider the $n$-dimensional Landau-Lifschitz system*

$$\frac{\partial \mathbf{Z}}{\partial t} = \mathbf{Z} \times \triangle \mathbf{Z}, \qquad \text{in } \mathbb{R}^n \times \mathbb{R}^+,$$

*where $\mathbf{Z} = (Z_1, Z_2, Z_3)$ and $|\mathbf{Z}(\mathbf{x}, t)| = 1$ for all $(\mathbf{x}, t) \in \mathbb{R}^n \times \mathbb{R}^+$. It has the explicit traveling wave solutions*

$$\mathbf{Z}(\mathbf{x}, t) = \mathbf{Z}_1 \cos\alpha + \{\mathbf{Z}_2 \cos(\mathbf{a}\cdot\mathbf{x} - (|\mathbf{a}|^2\cos\alpha)t) + \mathbf{Z}_3 \sin(\mathbf{a}\cdot\mathbf{x} - (|\mathbf{a}|^2\cos\alpha)t)\}\sin\alpha,$$

*where $\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3 \in \mathbb{R}^3$ are real constant vectors, such that $\mathbf{Z}_1 \times \mathbf{Z}_2 = \mathbf{Z}_3$, $\mathbf{Z}_3 \times \mathbf{Z}_1 = \mathbf{Z}_2$ and $|\mathbf{Z}_1| = |\mathbf{Z}_2| = |\mathbf{Z}_3| = 1$, $\alpha$ and $c$ are real constants.*

**Proof** The proof is simple and is omitted.

## 3 Nonlinear Dissipative Dispersive Wave Equations

In this section, we are going to establish the explicit traveling wave solutions of general Korteweg-de Vries-Burgers equation and the $n$-dimensional Ginzburg-Landau equation. This section is primarily motivated by [2, 3, 7–9, 11–14, 16, 17, 19, 21, 23, 28, 30, 32, 33, 41–43, 45].

**Theorem 3.1** *Consider the general Korteweg-de Vries-Burgers equation*

$$\frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} + \frac{\partial}{\partial x}f(u) = \alpha\frac{\partial^2 u}{\partial x^2}, \tag{3.1}$$

*where $f(u) = Au^2 - B^2u^3$ is a nonlinear smooth function of $u$, $\alpha > 0$, $A$ and $B \geq 0$ are real constants, $|A| + |B| > 0$. Then the general Korteweg-de Vries-Burgers equation has the explicit traveling wave solution*

$$u(x, t) = -\frac{3\alpha^2}{50A}\Big\{1 + \tanh\Big(\frac{\alpha}{10}\Big(x + \frac{6\alpha^2}{25}t\Big)\Big)\Big\}^2, \tag{3.2}$$

*if* $f(u) = Au^2$ *for any nonzero real constant* $A \neq 0$*;*

$$u(x,t) = \frac{\alpha B + \sqrt{2}A}{3\sqrt{2}B^2}\Big\{1 + \tanh\Big(\frac{\alpha B + \sqrt{2}A}{6B}\Big(x + \frac{2\alpha^2 B^2 - 2A^2 + \sqrt{2}AB\alpha}{9B^2}t\Big)\Big)\Big\}, \tag{3.3}$$

*if* $f(u) = Au^2 - B^2u^3$*, where* $B > 0$ *and* $A > -\frac{\alpha B}{\sqrt{2}}$ *are constants;*

$$u(x,t) = \frac{\sqrt{2}A - \alpha B}{3\sqrt{2}B^2}\Big\{1 + \tanh\Big(\frac{\alpha B - \sqrt{2}A}{6B}\Big(x + \frac{2\alpha^2 B^2 - 2A^2 - \sqrt{2}AB\alpha}{9B^2}t\Big)\Big)\Big\}, \tag{3.4}$$

*if* $f(u) = Au^2 - B^2u^3$*, where* $B > 0$ *and* $A > \frac{\alpha B}{\sqrt{2}}$ *are constants.*

**Proof** Let $u(x,t) = \phi(x + ct)$ be a traveling wave solution to the general Korteweg-de Vries-Burgers equation, where $c$ is a real constant to be determined. Set $z = x + ct$. Then

$$c\phi'(z) + \phi'''(z) + [f(\phi(z))]' = \alpha\phi''(z).$$

Integrating this equation with respect to $z$ and letting the integration constant be equal to zero, we get

$$c\phi(z) + \phi''(z) + f(\phi(z)) = \alpha\phi'(z).$$

First of all, let us consider the case $f(u) = -a^2u^2$ for some constant $a > 0$. Let $\phi(z) = [\psi(z)]^2$. Then

$$c[\psi(z)]^2 + 2\psi(z)\psi''(z) + 2[\psi'(z)]^2 - a^2[\psi(z)]^4 = 2\alpha\psi(z)\psi'(z).$$

Let us reduce the order of the differential equation. Suppose that

$$\psi'(z) = \psi(z)[p + q\psi(z)]$$

for two real constants $p$ and $q$. Then

$$\psi''(z) = \psi(z)[p + q\psi(z)][p + 2q\psi(z)].$$

Substituting the derivatives back into the differential equation

$$c[\psi(z)]^2 + 2\psi(z)\psi''(z) + 2[\psi'(z)]^2 - a^2[\psi(z)]^4 = 2\alpha\psi(z)\psi'(z),$$

we get

$$\begin{aligned} &c[\psi(z)]^2 + 2[\psi(z)]^2[p + q\psi(z)][p + 2q\psi(z)] + 2[\psi(z)]^2[p + q\psi(z)]^2 - a^2[\psi(z)]^4 \\ &= 2\alpha[\psi(z)]^2[p + q\psi(z)]. \end{aligned}$$

By canceling out $[\psi(z)]^2$, we find

$$c + 2[p + q\psi(z)][p + 2q\psi(z)] + 2[p + q\psi(z)]^2 - a^2[\psi(z)]^2 = 2\alpha[p + q\psi(z)].$$

We compare the coefficients and find out

$$c + 4p^2 = 2\alpha p, \quad 6q^2 = a^2, \quad 10pq = 2\alpha q.$$

Solving the system, we obtain

$$c = \frac{6\alpha^2}{25}, \quad p = \frac{\alpha}{5}, \quad q = -\sqrt{\frac{1}{6}}a.$$

Therefore, we obtain the solution

$$\begin{aligned}\psi(z) &= \frac{\sqrt{6}\alpha}{5a}\frac{1}{1+\exp\left\{-\frac{\alpha}{5}\left(x+\frac{6\alpha^2}{25}t\right)\right\}}\\ &= \frac{\sqrt{6}\alpha}{10a}\left\{1+\tanh\left(\frac{\alpha}{10}\left(x+\frac{6\alpha^2}{25}t\right)\right)\right\}.\end{aligned}$$

Finally, we have the first explicit traveling wave solution

$$\phi(z) = [\psi(z)]^2 = \frac{3\alpha^2}{50a^2}\left\{1+\tanh\left(\frac{\alpha}{10}\left(x+\frac{6\alpha^2}{25}t\right)\right)\right\}^2.$$

For the differential equation

$$\frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} + a^2\frac{\partial}{\partial x}(u^2) = \alpha\frac{\partial^2 u}{\partial x^2},$$

letting $u(x,t) = -v(x,t)$, we find that $v$ solves the differential equation

$$\frac{\partial v}{\partial t} + \frac{\partial^3 v}{\partial x^3} - a^2\frac{\partial}{\partial x}(v^2) = \alpha\frac{\partial^2 v}{\partial x^2}.$$

Thus

$$u(x,t) = -\frac{3\alpha^2}{50a^2}\left\{1+\tanh\left(\frac{\alpha}{10}\left(x+\frac{6\alpha^2}{25}t\right)\right)\right\}^2$$

is an explicit traveling wave solution.

Next, let us consider the cubic case $f(u) = Au^2 - B^2u^3$ for some real constants $A$ and $B > 0$. Again, let us reduce the order of the differential equation. Let

$$\phi'(z) = \phi(z)[p + q\phi(z)]$$

for two constants $p$ and $q$. Then

$$\phi''(z) = \phi(z)[p + q\phi(z)][p + 2q\phi(z)].$$

Substituting the derivatives back into the differential equation

$$c\phi(z) + \phi''(z) + A[\phi(z)]^2 - B^2[\phi(z)]^3 = \alpha\phi'(z),$$

we have

$$c\phi(z) + \phi(z)[p + q\phi(z)][p + 2q\phi(z)] + A[\phi(z)]^2 - B^2[\phi(z)]^3 = \alpha\phi(z)[p + q\phi(z)].$$

Let us cancel out $\phi(z)$. We find

$$c + [p + q\phi(z)][p + 2q\phi(z)] + A\phi(z) - B^2[\phi(z)]^2 = \alpha[p + q\phi(z)].$$

By comparing the coefficients, we get

$$c+p^2=\alpha p,\quad A+3pq=\alpha q,\quad 2q^2=B^2.$$

Solving the system, we get two sets of solutions

$$c=\frac{2\alpha^2B^2-2A^2\mp\sqrt{2}AB\alpha}{9B^2},\quad p=\frac{\alpha B\mp\sqrt{2}A}{3B},\quad q=\pm\sqrt{\frac{1}{2}}b.$$

Now we obtain the second explicit traveling wave solution

$$\begin{aligned}\phi(z)&=\sqrt{2}\frac{\alpha B+\sqrt{2}A}{3B^2}\frac{1}{1+\exp\Big\{-\frac{\alpha B+\sqrt{2}A}{3B}\Big(x+\frac{2\alpha^2B^2-2A^2+\sqrt{2}AB\alpha}{9B^2}t\Big)\Big\}}\\&=\frac{\alpha B+\sqrt{2}A}{3\sqrt{2}B^2}\Big\{1+\tanh\Big(\frac{\alpha B+\sqrt{2}A}{6B}\Big(x+\frac{2\alpha^2B^2-2A^2+\sqrt{2}AB\alpha}{9B^2}t\Big)\Big)\Big\},\end{aligned}$$

if we choose

$$c=\frac{2\alpha^2B^2-2A^2+\sqrt{2}AB\alpha}{9B^2},\quad p=\frac{\alpha B+\sqrt{2}A}{3B},\quad q=-\sqrt{\frac{1}{2}}B,$$

and let

$$A>-\frac{\alpha B}{\sqrt{2}}.$$

We also get the third explicit traveling wave solution

$$\begin{aligned}\phi(z)&=\sqrt{2}\frac{\sqrt{2}A-\alpha B}{3B^2}\frac{1}{1+\exp\Big\{-\frac{\alpha B-\sqrt{2}A}{3B}\Big(x+\frac{2\alpha^2B^2-2A^2-\sqrt{2}AB\alpha}{9B^2}t\Big)\Big\}}\\&=\frac{\sqrt{2}A-\alpha B}{3\sqrt{2}B^2}\Big\{1+\tanh\Big(\frac{\alpha B-\sqrt{2}A}{6B}\Big(x+\frac{2\alpha^2B^2-2A^2-\sqrt{2}AB\alpha}{9B^2}t\Big)\Big)\Big\},\end{aligned}$$

if we choose

$$c=\frac{2\alpha^2B^2-2A^2-\sqrt{2}AB\alpha}{9B^2},\quad p=\frac{\alpha B-\sqrt{2}A}{3B},\quad q=\sqrt{\frac{1}{2}}B,$$

and let

$$A>\frac{\alpha B}{\sqrt{2}}.$$

By coupling the two cases $B=0$ and $B>0$ together, we finish the proof.

**Corollary 3.1** (I) *Consider the following nonlinear Korteweg-de Vries-Burgers equation*

$$\frac{\partial u}{\partial t}+\frac{\partial^3u}{\partial x^3}+\frac{\partial}{\partial x}(u^2)=\alpha\frac{\partial^2u}{\partial x^2},\tag{3.5}$$

*where $\alpha>0$ is a positive constant. Then the nonlinear Korteweg-de Vries-Burgers equation has the explicit traveling wave solution*

$$u(x,t)=-\frac{3\alpha^2}{50}\Big\{1+\tanh\Big(\frac{\alpha}{10}\Big(x+\frac{6\alpha^2}{25}t\Big)\Big)\Big\}^2.\tag{3.6}$$

(II) *Consider the modified Korteweg-de Vries-Burgers equation*

$$\frac{\partial u}{\partial t}+\frac{\partial^3 u}{\partial x^3}-\frac{\partial}{\partial x}(u^3)=\alpha\frac{\partial^2 u}{\partial x^2}, \tag{3.7}$$

*where $\alpha>0$ is a positive constant. Then the modified Korteweg-de Vries-Burgers equation has the explicit traveling wave solution*

$$u(x,t)=\frac{\alpha}{3\sqrt{2}}\Big\{1+\tanh\Big(\frac{\alpha}{6}\Big(x+\frac{2\alpha^2}{9}t\Big)\Big)\Big\}. \tag{3.8}$$

(III) *Consider the general Korteweg-de Vries-Burgers equation*

$$\frac{\partial u}{\partial t}-\frac{\partial^3 u}{\partial x^3}+\beta^2\frac{\partial}{\partial x}(u^3)=\alpha\frac{\partial^2 u}{\partial x^2}, \tag{3.9}$$

*where $\alpha>0$ and $\beta>0$ are positive constants. Then the general Korteweg-de Vries-Burgers equation has the explicit traveling wave solution*

$$u(x,t)=\frac{\alpha}{3\sqrt{2}\beta}\Big\{1+\tanh\Big(\frac{\alpha}{6}\Big(-x+\frac{2\alpha^2}{9}t\Big)\Big)\Big\}. \tag{3.10}$$

**Proof** (I) and (II) are straightforward to prove. In (III), letting $A=0$, $b=\beta$, $y=-x$ and $v(y,t)=u(x,t)$, we find that $v$ satisfies the following differential equation:

$$\frac{\partial v}{\partial t}+\frac{\partial^3 v}{\partial y^3}-\beta^2\frac{\partial}{\partial y}(v^3)=\alpha\frac{\partial^2 v}{\partial y^2}.$$

Therefore,

$$v(y,t)=\frac{\alpha}{3\sqrt{2}\beta}\Big\{1+\tanh\Big(\frac{\alpha}{6}\Big(y+\frac{2\alpha^2}{9}t\Big)\Big)\Big\}.$$

The rest of the proof follows right away.

**Theorem 3.2** *Consider the generalized $n$-dimensional Ginzburg-Landau equation*

$$\frac{\partial u}{\partial t}-(1+\alpha\mathrm{i})\triangle u+(1+\beta\mathrm{i})|u|^2u+(\gamma+\mathrm{i}\delta)|u|^4u-\varepsilon u=0, \tag{3.11}$$

*where $\alpha\neq0$, $\beta\neq0$, $\gamma\geq0$, $\delta\geq0$ and $\varepsilon>0$ are real constants. Let $\mathbf{a}=(a_1,a_2,\cdots,a_n)$ be any nonzero real constant vector. Suppose that $\alpha=\beta$. If $\gamma=\delta=0$, then the $n$-dimensional Ginzburg-Landau equation has four explicit traveling wave solutions*

$$\begin{aligned}u_{1\pm}(\mathbf{x},t)=&\sqrt{\frac{8\varepsilon}{8+9\alpha^2}}\exp\Big\{\pm\mathrm{i}\frac{3\alpha\sqrt{\varepsilon}}{|\mathbf{a}|\sqrt{8+9\alpha^2}}\Big[\mathbf{a}\cdot\mathbf{x}\mp\frac{6|\mathbf{a}|\sqrt{\alpha\varepsilon}}{\sqrt{8+9\alpha^2}}\Big(\alpha+\frac{1}{\alpha}\Big)t\Big]+\mathrm{i}\alpha\varepsilon\frac{10+9\alpha^2}{8+9\alpha^2}t\Big\}\\&\cdot\frac{1}{1+\exp\Big\{\pm\frac{2\sqrt{\varepsilon}}{|\mathbf{a}|\sqrt{8+9\alpha^2}}\Big[\mathbf{a}\cdot\mathbf{x}\mp\frac{6|\mathbf{a}|\sqrt{\alpha\varepsilon}}{\sqrt{8+9\alpha^2}}\Big(\alpha+\frac{1}{\alpha}\Big)t\Big]\Big\}}\end{aligned}$$

*and*

$$u_{2\pm}(\mathbf{x},t)=-\sqrt{\frac{8\varepsilon}{8+9\alpha^2}}\exp\Big\{\pm\mathrm{i}\frac{3\alpha\sqrt{\varepsilon}}{|\mathbf{a}|\sqrt{8+9\alpha^2}}\Big[\mathbf{a}\cdot\mathbf{x}\mp\frac{6|\mathbf{a}|\sqrt{\alpha\varepsilon}}{\sqrt{8+9\alpha^2}}\Big(\alpha+\frac{1}{\alpha}\Big)t\Big]+\mathrm{i}\alpha\varepsilon\frac{10+9\alpha^2}{8+9\alpha^2}t\Big\}$$

$$\cdot \frac{1}{1+\exp\Big\{\pm \frac{2\sqrt{\varepsilon}}{|\mathbf{a}|\sqrt{8+9\alpha^2}}\Big[\mathbf{a}\cdot\mathbf{x} \mp \frac{6|\mathbf{a}|\sqrt{\alpha\varepsilon}}{\sqrt{8+9\alpha^2}}\Big(\alpha+\frac{1}{\alpha}\Big)t\Big]\Big\}}.$$

*Suppose that $\gamma = \alpha\delta$ and $3\alpha^2 < 4\gamma\varepsilon$. Then the $n$-dimensional Ginzburg-Landau equation has four explicit traveling wave solutions*

$$\begin{aligned}
u_{1\pm}(\mathbf{x},t) &= \pm\exp[\mathbf{i}A_1(\mathbf{a}\cdot\mathbf{x}+c_1t)+\mathbf{i}b_1t]\Big(\frac{-3\alpha+2A_1\sqrt{3|\mathbf{a}|^2\gamma}}{8\alpha\gamma}\Big)^{\frac{1}{2}}\\
&\quad\cdot\Big\{1+\tanh\Big[\Big(\frac{1}{4}\sqrt{\frac{3}{|\mathbf{a}|^2\gamma}}-\frac{A_1}{2\alpha}\Big)(\mathbf{a}\cdot\mathbf{x}+c_1t)\Big]\Big\}^{\frac{1}{2}},\\
u_{2\pm}(\mathbf{x},t) &= \pm\exp[\mathbf{i}A_2(\mathbf{a}\cdot\mathbf{x}+c_2t)+\mathbf{i}b_2t]\Big(\frac{-3\alpha-2A_2\sqrt{3|\mathbf{a}|^2\gamma}}{8\alpha\gamma}\Big)^{\frac{1}{2}}\\
&\quad\cdot\Big\{1+\tanh\Big[\Big(-\frac{1}{4}\sqrt{\frac{3}{|\mathbf{a}|^2\gamma}}-\frac{A_2}{2\alpha}\Big)(\mathbf{a}\cdot\mathbf{x}+c_2t)\Big]\Big\}^{\frac{1}{2}},
\end{aligned}$$

*where*

$$\begin{aligned}
A_1 &= \frac{\sqrt{3\alpha^2}+\sqrt{3\alpha^2+\alpha^2(3+4\alpha^2)(3+16\gamma\varepsilon)}}{2(3+4\alpha^2)\sqrt{|\mathbf{a}|^2\gamma}} > 0,\\
b_1 &= 2{A_1}^2|\mathbf{a}|^2\Big(\alpha+\frac{1}{\alpha}\Big)-\alpha\varepsilon, \quad c_1 = -2A_1|\mathbf{a}|^2\Big(\alpha+\frac{1}{\alpha}\Big) < 0,\\
A_2 &= \frac{-\sqrt{3\alpha^2}-\sqrt{3\alpha^2+\alpha^2(3+4\alpha^2)(3+16\gamma\varepsilon)}}{2(3+4\alpha^2)\sqrt{|\mathbf{a}|^2\gamma}} < 0,\\
b_2 &= 2{A_2}^2|\mathbf{a}|^2\Big(\alpha+\frac{1}{\alpha}\Big)-\alpha\varepsilon, \quad c_2 = -2A_2|\mathbf{a}|^2\Big(\alpha+\frac{1}{\alpha}\Big) > 0.
\end{aligned}$$

**Proof** Let

$$u(\mathbf{x},t) = \exp\{\mathrm{i}\omega(\mathbf{a}\cdot\mathbf{x}+ct)+\mathrm{i}bt\}\phi(\mathbf{a}\cdot\mathbf{x}+ct)$$

be a traveling wave solution of the $n$-dimensional Ginzburg-Landau equation, where $\mathbf{a} = (a_1, a_2, \cdots, a_n)$ is a nonzero real constant vector, $b$ and $c$ are real constants, $\phi$ and $\omega$ are real functions of $z$. Let $z = \mathbf{a}\cdot\mathbf{x}+ct$. It is easy to find the following partial derivatives of $u$:

$$\begin{aligned}
\frac{\partial u}{\partial t} &= [\mathrm{i}c\omega'(z)+\mathrm{i}b]\exp\{\mathrm{i}\omega(z)+\mathrm{i}bt\}\phi(z)+c\exp\{\mathrm{i}\omega(z)+\mathrm{i}bt\}\phi'(z),\\
\frac{\partial u}{\partial x_k} &= \mathrm{i}a_k\omega'(z)\exp\{\mathrm{i}\omega(z)+\mathrm{i}bt\}\phi(z)+a_k\exp\{\mathrm{i}\omega(z)+\mathrm{i}bt\}\phi'(z),\\
\frac{\partial^2 u}{\partial x_k^2} &= \mathrm{i}{a_k}^2\omega''(z)\exp\{\mathrm{i}\omega(z)+\mathrm{i}bt\}\phi(z)-{a_k}^2[\omega'(z)]^2\exp\{\mathrm{i}\omega(z)+\mathrm{i}bt\}\phi(z)\\
&\quad+2\mathrm{i}{a_k}^2\omega'(z)\exp\{\mathrm{i}\omega(z)+\mathrm{i}bt\}\phi'(z)+{a_k}^2\exp\{\mathrm{i}\omega(z)+\mathrm{i}bt\}\phi''(z),\\
\triangle u &= \mathrm{i}|\mathbf{a}|^2\omega''(z)\exp\{\mathrm{i}\omega(z)+\mathrm{i}bt\}\phi(z)-|\mathbf{a}|^2[\omega'(z)]^2\exp\{\mathrm{i}\omega(z)+\mathrm{i}bt\}\phi(z)\\
&\quad+2\mathrm{i}|\mathbf{a}|^2\omega'(z)\exp\{\mathrm{i}\omega(z)+\mathrm{i}bt\}\phi'(z)+|\mathbf{a}|^2\exp\{\mathrm{i}\omega(z)+\mathrm{i}bt\}\phi''(z).
\end{aligned}$$

Now we find that

$$\frac{\partial u}{\partial t}-(1+\alpha\mathrm{i})\triangle u+(1+\beta\mathrm{i})|u|^2u+(\gamma+\mathrm{i}\delta)|u|^4u-\varepsilon u$$

$$
\begin{aligned}
&= \exp\{\mathrm{i}\omega(z) + \mathrm{i}bt\} \cdot \{[\mathrm{i}c\omega'(z) + \mathrm{i}b]\phi(z) + c\phi'(z) \\
&\quad - (1+\alpha\mathrm{i})[\mathrm{i}|\mathbf{a}|^2\omega''(z)\phi(z) - |\mathbf{a}|^2[\omega'(z)]^2\phi(z) \\
&\quad + 2\mathrm{i}|\mathbf{a}|^2\omega'(z)\phi'(z) + |\mathbf{a}|^2\phi''(z)] + (1+\beta\mathrm{i})[\phi(z)]^3 + (\gamma + \mathrm{i}\delta)[\phi(z)]^5 - \varepsilon\phi(z)\} \\
&= \{c\phi'(z) + |\mathbf{a}|^2[\omega'(z)]^2\phi(z) - |\mathbf{a}|^2\phi''(z) + |\mathbf{a}|^2\alpha\omega''(z)\phi(z) \\
&\quad + 2|\mathbf{a}|^2\alpha\omega'(z)\phi'(z) + [\phi(z)]^3 + \gamma[\phi(z)]^5 - \varepsilon\phi(z)\}\exp\{\mathrm{i}\omega(z) + \mathrm{i}bt\} \\
&\quad + \mathrm{i}\{[c\omega'(z) + b]\phi(z) - |\mathbf{a}|^2\omega''(z)\phi(z) - 2|\mathbf{a}|^2\omega'(z)\phi'(z) \\
&\quad + |\mathbf{a}|^2\alpha[\omega'(z)]^2\phi(z) - |\mathbf{a}|^2\alpha\phi''(z) + \beta[\phi(z)]^3 + \delta[\phi(z)]^5\}\exp\{\mathrm{i}\omega(z) + \mathrm{i}bt\} = 0.
\end{aligned}
$$

Both the real part and the imaginary part are equal to zero, i.e.,

$$
\begin{aligned}
&c\phi'(z) + |\mathbf{a}|^2[\omega'(z)]^2\phi(z) - |\mathbf{a}|^2\phi''(z) + |\mathbf{a}|^2\alpha\omega''(z)\phi(z) \\
&+ 2|\mathbf{a}|^2\alpha\omega'(z)\phi'(z) + [\phi(z)]^3 + \gamma[\phi(z)]^5 - \varepsilon\phi(z) = 0
\end{aligned}
$$

and

$$
\begin{aligned}
&[c\omega'(z) + b]\phi(z) - |\mathbf{a}|^2\omega''(z)\phi(z) - 2|\mathbf{a}|^2\omega'(z)\phi'(z) \\
&+ |\mathbf{a}|^2\alpha[\omega'(z)]^2\phi(z) - |\mathbf{a}|^2\alpha\phi''(z) + \beta[\phi(z)]^3 + \delta[\phi(z)]^5 = 0.
\end{aligned}
$$

Equivalently, we get

$$
\frac{\phi''(z)}{\phi(z)} = [\omega'(z)]^2 + \alpha\omega''(z) - \frac{\varepsilon}{|\mathbf{a}|^2} + \frac{\phi'(z)}{\phi(z)}\Big[\frac{c}{|\mathbf{a}|^2} + 2\alpha\omega'(z)\Big] + \frac{1}{|\mathbf{a}|^2}[\phi(z)]^2 + \frac{\gamma}{|\mathbf{a}|^2}[\phi(z)]^4
$$

and

$$
\frac{\phi''(z)}{\phi(z)} = [\omega'(z)]^2 + \frac{c\omega'(z) + b}{|\mathbf{a}|^2\alpha} - \frac{1}{\alpha}\omega''(z) - \frac{2}{\alpha}\frac{\phi'(z)}{\phi(z)}\omega'(z) + \frac{\beta}{|\mathbf{a}|^2\alpha}[\phi(z)]^2 + \frac{\delta}{|\mathbf{a}|^2\alpha}[\phi(z)]^4.
$$

Subtracting the second equation from the first equation, we find

$$
\begin{aligned}
&\Big(\alpha + \frac{1}{\alpha}\Big)\omega''(z) + \frac{\phi'(z)}{\phi(z)}\Big[\frac{c}{|\mathbf{a}|^2} + 2\alpha\omega'(z) + \frac{2}{\alpha}\omega'(z)\Big] + \frac{1}{|\mathbf{a}|^2}\Big(1 - \frac{\beta}{\alpha}\Big)[\phi(z)]^2 \\
&- \frac{\alpha\varepsilon + c\omega'(z) + b}{|\mathbf{a}|^2\alpha} + \frac{\alpha\gamma - \delta}{|\mathbf{a}|^2\alpha}[\phi(z)]^4 = 0.
\end{aligned}
$$

Let $A$ be a real constant and set

$$
\begin{aligned}
&\omega(z) = Az, \quad \alpha = \beta, \quad \delta = \alpha\gamma, \\
&b = -(Ac + \alpha\varepsilon) = 2A^2|\mathbf{a}|^2\Big(\alpha + \frac{1}{\alpha}\Big) - \alpha\varepsilon, \\
&c = -2A|\mathbf{a}|^2\Big(\alpha + \frac{1}{\alpha}\Big).
\end{aligned}
$$

Then

$$
\frac{\phi''(z)}{\phi(z)} = A^2 - \frac{\varepsilon}{|\mathbf{a}|^2} + \frac{\phi'(z)}{\phi(z)}\Big[\frac{c}{|\mathbf{a}|^2} + 2A\alpha\Big] + \frac{1}{|\mathbf{a}|^2}[\phi(z)]^2 + \frac{\gamma}{|\mathbf{a}|^2}[\phi(z)]^4.
$$

Let us reduce the order of the differential equation. Let

$$
\phi'(z) = \phi(z)[p + q\phi(z)].
$$

Then

$$\phi''(z) = \phi(z)[p + q\phi(z)][p + 2q\phi(z)].$$

Substituting the derivatives back into the differential equation, we find

$$[p + q\phi(z)][p + 2q\phi(z)] = A^2 - \frac{\varepsilon}{|\mathbf{a}|^2} + [p + q\phi(z)]\Big(\frac{c}{|\mathbf{a}|^2} + 2A\alpha\Big) + \frac{1}{|\mathbf{a}|^2}[\phi(z)]^2 + \frac{\gamma}{|\mathbf{a}|^2}[\phi(z)]^4.$$

By comparing the coefficients, we obtain the system of equations

$$\begin{aligned} p^2 &= A^2 - \frac{\varepsilon}{|\mathbf{a}|^2} + p\Big(\frac{c}{|\mathbf{a}|^2} + 2A\alpha\Big), \\ 3pq &= q\Big(\frac{c}{|\mathbf{a}|^2} + 2A\alpha\Big), \\ 2q^2 &= \frac{1}{|\mathbf{a}|^2}. \end{aligned}$$

It is easy to find that

$$A^2 + 2p^2 = \frac{\varepsilon}{|\mathbf{a}|^2}, \quad p = -\frac{2A}{3\alpha}, \quad q = \pm\frac{1}{\sqrt{2}|\mathbf{a}|}.$$

By solving the system, we have four sets of solutions

$$\begin{aligned} &A = \pm\frac{3\sqrt{\alpha\varepsilon}}{|\mathbf{a}|\sqrt{8 + 9\alpha^2}}, \quad b = \alpha\varepsilon\frac{10 + 9\alpha^2}{8 + 9\alpha^2}, \quad c = \mp\frac{6|\mathbf{a}|\sqrt{\alpha\varepsilon}}{\sqrt{8 + 9\alpha^2}}\Big(\alpha + \frac{1}{\alpha}\Big), \\ &p = \mp\frac{2\sqrt{\varepsilon}}{|\mathbf{a}|\sqrt{8 + 9\alpha^2}}, \quad q = \frac{1}{\sqrt{2}|\mathbf{a}|} \end{aligned}$$

and

$$\begin{aligned} &A = \pm\frac{3\sqrt{\alpha\varepsilon}}{|\mathbf{a}|\sqrt{8 + 9\alpha^2}}, \quad b = \alpha\varepsilon\frac{10 + 9\alpha^2}{8 + 9\alpha^2}, \quad c = \mp\frac{6|\mathbf{a}|\sqrt{\alpha\varepsilon}}{\sqrt{8 + 9\alpha^2}}\Big(\alpha + \frac{1}{\alpha}\Big), \\ &p = \mp\frac{2\sqrt{\varepsilon}}{|\mathbf{a}|\sqrt{8 + 9\alpha^2}}, \quad q = -\frac{1}{\sqrt{2}|\mathbf{a}|}. \end{aligned}$$

Finally, by using Lemma 1.1 and Lemma 1.2, we obtain four explicit traveling wave solutions

(I)

$$\phi(z) = \sqrt{\frac{8\varepsilon}{8 + 9\alpha^2}}\frac{1}{1 + \exp\Big\{\frac{2\sqrt{\varepsilon}}{|\mathbf{a}|\sqrt{8+9\alpha^2}}z\Big\}},$$

if

$$c = -\frac{6|\mathbf{a}|\sqrt{\alpha\varepsilon}}{\sqrt{8 + 9\alpha^2}}\Big(\alpha + \frac{1}{\alpha}\Big), \quad p = -\frac{2\sqrt{\varepsilon}}{|\mathbf{a}|\sqrt{8 + 9\alpha^2}}, \quad q = \frac{1}{\sqrt{2}|\mathbf{a}|};$$

(II)

$$\phi(z) = -\sqrt{\frac{8\varepsilon}{8 + 9\alpha^2}}\frac{1}{1 + \exp\Big\{-\frac{2\sqrt{\varepsilon}}{|\mathbf{a}|\sqrt{8+9\alpha^2}}z\Big\}},$$

if

$$c = \frac{6|\mathbf{a}|\sqrt{\alpha\varepsilon}}{\sqrt{8+9\alpha^2}}\Big(\alpha + \frac{1}{\alpha}\Big), \quad p = \frac{2\sqrt{\varepsilon}}{|\mathbf{a}|\sqrt{8+9\alpha^2}}, \quad q = \frac{1}{\sqrt{2}|\mathbf{a}|};$$

(III)

$$\phi(z) = -\sqrt{\frac{8\varepsilon}{8+9\alpha^2}}\frac{1}{1+\exp\Big\{\frac{2\sqrt{\varepsilon}}{|\mathbf{a}|\sqrt{8+9\alpha^2}}z\Big\}},$$

if

$$c = -\frac{6|\mathbf{a}|\sqrt{\alpha\varepsilon}}{\sqrt{8+9\alpha^2}}\Big(\alpha + \frac{1}{\alpha}\Big), \quad p = -\frac{2\sqrt{\varepsilon}}{|\mathbf{a}|\sqrt{8+9\alpha^2}}, \quad q = -\frac{1}{\sqrt{2}|\mathbf{a}|};$$

(IV)

$$\phi(z) = \sqrt{\frac{8\varepsilon}{8+9\alpha^2}}\frac{1}{1+\exp\Big\{-\frac{2\sqrt{\varepsilon}}{|\mathbf{a}|\sqrt{8+9\alpha^2}}z\Big\}},$$

if

$$c = \frac{6|\mathbf{a}|\sqrt{\alpha\varepsilon}}{\sqrt{8+9\alpha^2}}\Big(\alpha + \frac{1}{\alpha}\Big), \quad p = \frac{2\sqrt{\varepsilon}}{|\mathbf{a}|\sqrt{8+9\alpha^2}}, \quad q = -\frac{1}{\sqrt{2}|\mathbf{a}|}.$$

Now let us consider the more complicated case $\gamma > 0$. Suppose that

$$\phi'(z) = \phi(z)\{p + q[\phi(z)]^2\}$$

for two real constants $p$ and $q$. Then

$$\phi''(z) = \phi(z)\{p + q[\phi(z)]^2\}\{p + 3q[\phi(z)]^2\}.$$

If we plug these derivatives back into the differential equation, we get

$$\begin{aligned}\{p + q[\phi(z)]^2\}\{p + 3q[\phi(z)]^2\} = A^2 - \frac{\varepsilon}{|\mathbf{a}|^2} + \{p + q[\phi(z)]^2\}\Big[\frac{c}{|\mathbf{a}|^2} + 2A\alpha\Big] \\ + \frac{1}{|\mathbf{a}|^2}[\phi(z)]^2 + \frac{\gamma}{|\mathbf{a}|^2}[\phi(z)]^4.\end{aligned}$$

Let us compare the coefficients. We have the system of equations

$$\begin{aligned}p^2 &= A^2 - \frac{\varepsilon}{|\mathbf{a}|^2} + p\Big[\frac{c}{|\mathbf{a}|^2} + 2A\alpha\Big], \\ 4pq &= q\Big[\frac{c}{|\mathbf{a}|^2} + 2A\alpha\Big] + \frac{1}{|\mathbf{a}|^2}, \\ 3q^2 &= \frac{\gamma}{|\mathbf{a}|^2}.\end{aligned}$$

It is not difficult to find that

$$\frac{3+4\alpha^2}{4\alpha^2}A^2 - \frac{3q}{4\alpha\gamma}A - \frac{3+16\gamma\varepsilon}{16|\mathbf{a}|^2\gamma} = 0.$$

Now we have

$$A=\Big\{\frac{3q}{4\alpha\gamma}\pm\sqrt{\frac{1}{16|\mathbf{a}|^2\alpha^2\gamma}\Big[3+(3+4\alpha^2)(3+16\gamma\varepsilon)\Big]}\Big\}\frac{1}{\big\{\frac{3+4\alpha^2}{2\alpha^2}\big\}},$$
$$p=-\frac{A}{2\alpha}+\frac{3q}{4\gamma},\quad q=\pm\sqrt{\frac{\gamma}{3|\mathbf{a}|^2}}.$$

Then, we obtain two sets of solutions

$$A_1=\frac{\sqrt{3\alpha^2}+\sqrt{3\alpha^2+\alpha^2(3+4\alpha^2)(3+16\gamma\varepsilon)}}{2(3+4\alpha^2)\sqrt{|\mathbf{a}|^2\gamma}}>0,$$
$$p_1=\frac{1}{4}\sqrt{\frac{3}{|\mathbf{a}|^2\gamma}}-\frac{A_1}{2\alpha}<0,\quad q_1=\sqrt{\frac{\gamma}{3|\mathbf{a}|^2}}>0,$$
$$b_1=2{A_1}^2|\mathbf{a}|^2\Big(\alpha+\frac{1}{\alpha}\Big)-\alpha\varepsilon,\quad c_1=-2A_1|\mathbf{a}|^2\Big(\alpha+\frac{1}{\alpha}\Big)<0,$$

and

$$A_2=\frac{-\sqrt{3\alpha^2}-\sqrt{3\alpha^2+\alpha^2(3+4\alpha^2)(3+16\gamma\varepsilon)}}{2(3+4\alpha^2)\sqrt{|\mathbf{a}|^2\gamma}}<0,$$
$$p_2=-\frac{1}{4}\sqrt{\frac{3}{|\mathbf{a}|^2\gamma}}-\frac{A_2}{2\alpha}>0,\quad q_2=-\sqrt{\frac{\gamma}{3|\mathbf{a}|^2}}<0,$$
$$b_2=2{A_2}^2|\mathbf{a}|^2\Big(\alpha+\frac{1}{\alpha}\Big)-\alpha\varepsilon,\quad c_2=-2A_2|\mathbf{a}|^2\Big(\alpha+\frac{1}{\alpha}\Big)>0.$$

Finally, we obtain the bounded explicit solutions

$$\phi_1(z)=\pm\Big(\frac{-3\alpha+2A_1\sqrt{3|\mathbf{a}|^2\gamma}}{8\alpha\gamma}\Big)^{\frac{1}{2}}\Big\{1+\tanh\Big[\Big(\frac{1}{4}\sqrt{\frac{3}{|\mathbf{a}|^2\gamma}}-\frac{A_1}{2\alpha}\Big)z\Big]\Big\}^{\frac{1}{2}},$$

and

$$\phi_2(z)=\pm\Big(\frac{-3\alpha-2A_2\sqrt{3|\mathbf{a}|^2\gamma}}{8\alpha\gamma}\Big)^{\frac{1}{2}}\Big\{1+\tanh\Big[\Big(-\frac{1}{4}\sqrt{\frac{3}{|\mathbf{a}|^2\gamma}}-\frac{A_2}{2\alpha}\Big)z\Big]\Big\}^{\frac{1}{2}}.$$

The proof is finished.

## 4 Nonlinear Convection Equations

In this section, we are going to establish the explicit traveling wave solutions to the one-dimensional Burgers equation and $n$-dimensional Burgers equation. This section is primarily motivated by [1] and [34].

**Theorem 4.1** *Consider the one-dimensional Burgers equation*

$$\frac{\partial u}{\partial t}-\alpha\frac{\partial^2 u}{\partial x^2}+\beta\frac{\partial}{\partial x}(u^p)=0,$$

*where $\alpha>0$, $\beta>0$ and $p>1$ are positive constants. Then the Burgers equation has the exact traveling wave solutions*

$$u(x,t)=\Big\{-\frac{c\alpha}{\beta+\beta\exp\{-c(p-1)(x+c\alpha t)\}}\Big\}^{\frac{1}{p-1}},$$

*where $z = x + c\alpha t$, $c < 0$ is any real constant.*

**Proof** Let $u(x,t) = \phi(x + c\alpha t)$ be a traveling wave solution, and set $z = x + c\alpha t$. Then

$$c\alpha\phi'(z) - \alpha\phi''(z) + \beta\{[\phi(z)]^p\}' = 0.$$

Integrating this equation with respect to $z$, and letting the integration constant be equal to zero, we have

$$c\alpha\phi(z) - \alpha\phi'(z) + \beta[\phi(z)]^p = 0.$$

By dividing this equation by $\alpha$, we find

$$\phi'(z) = \phi(z)\Big\{c + \frac{\beta}{\alpha}[\phi(z)]^{p-1}\Big\}.$$

By Lemma 1.1, we have

$$\phi(z) = \Big\{ -\frac{c\alpha}{\beta + \beta\exp\{-c(p-1)z\}}\Big\}^{\frac{1}{p-1}}.$$

The proof is finished.

**Theorem 4.2** *Consider the $n$-dimensional Burgers equation*

$$\frac{\partial \mathbf{u}}{\partial t} = \triangle\mathbf{u} + 2(\mathbf{u}\cdot\nabla)\mathbf{u}, \quad \text{in } \mathbb{R}^n \times \mathbb{R}^+. \tag{4.1}$$

*Let $\mathbf{a} = (a_1, a_2, \cdots, a_n) \in \mathbb{R}^n$ be any nonzero real constant vector. Then the $n$-dimensional Burgers equation has the explicit traveling wave solution*

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{a} + \mathbf{a}\tanh(\mathbf{a}\cdot\mathbf{x} + 2|\mathbf{a}|^2 t). \tag{4.2}$$

**Proof** Let $\mathbf{x} = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$, $\mathbf{a} = (a_1, a_2, \cdots, a_n) \in \mathbb{R}^n$ be a fixed constant vector and $c$ be a real constant. Suppose that $\mathbf{u}(\mathbf{x}, t) = \mathbf{a}\phi(\mathbf{a}\cdot\mathbf{x} + ct)$ is a traveling wave solution, and set $z = \mathbf{a}\cdot\mathbf{x} + ct$. Then

$$\frac{\partial \mathbf{u}}{\partial t} = c\mathbf{a}\phi'(z), \quad \frac{\partial \mathbf{u}}{\partial x_i} = a_i\mathbf{a}\phi'(z), \quad \frac{\partial^2 \mathbf{u}}{\partial x_i^2} = {a_i}^2\mathbf{a}\phi''(z),$$

$$(\mathbf{u}\cdot\nabla)\mathbf{u} = \sum_{j=1}^{n} u_j\frac{\partial \mathbf{u}}{\partial x_j} = \sum_{j=1}^{n}[a_j\phi(z)][a_j\mathbf{a}\phi'(z)] = |\mathbf{a}|^2\mathbf{a}\phi(z)\phi'(z)$$

and

$$\sum_{i=1}^{n}\frac{\partial^2 \mathbf{u}}{\partial x_i^2} = |\mathbf{a}|^2\mathbf{a}\phi''(z).$$

If we plug these partial derivatives back into the differential equation, we get

$$c\mathbf{a}\phi'(z) = |\mathbf{a}|^2\mathbf{a}\phi''(z) + 2|\mathbf{a}|^2\mathbf{a}\phi(z)\phi'(z).$$

If $\mathbf{a} \neq \mathbf{0}$, then

$$c\phi'(z) = |\mathbf{a}|^2\phi''(z) + 2|\mathbf{a}|^2\phi(z)\phi'(z).$$

Integrating this equation with respect to $z$ and letting the integration constant be equal to zero, we find

$$c\phi(z) = |\mathbf{a}|^2\phi'(z) + |\mathbf{a}|^2[\phi(z)]^2.$$

In other words, we have

$$\phi'(z) = \phi(z)\Big[\frac{c}{|\mathbf{a}|^2} - \phi(z)\Big].$$

Solving it by using Lemma 1.1, we obtain

$$\phi(z) = \frac{c}{|\mathbf{a}|^2\big[1 + \exp\big\{-\frac{cz}{|\mathbf{a}|^2}\big\}\big]}.$$

Let

$$c = 2|\mathbf{a}|^2.$$

Therefore, the $n$-dimensional Burgers equation has the explicit traveling wave solution

$$\mathbf{u}(\mathbf{x}, t) = \frac{2\mathbf{a}}{1 + \exp\{-2(\mathbf{a}\cdot\mathbf{x} + ct)\}} = \mathbf{a} + \mathbf{a}\tanh(\mathbf{a}\cdot\mathbf{x} + 2|\mathbf{a}|^2 t).$$

The proof is finished.

## 5 Nonlinear Reaction Diffusion Equations

**Motivation equation** Consider the reaction-diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(1-u)(u-a), \quad 0 < a < \frac{1}{2}.$$

Let $u(x,t) = \phi(x+ct)$ be a traveling wave solution, where $c \neq 0$ is a constant, representing the wave speed, and set $z = x + ct$. Then

$$c\phi'(z) = \phi''(z) + \phi(z)[1-\phi(z)][\phi(z) - a].$$

Let

$$\phi'(z) = D\phi(z)[1-\phi(z)]$$

for some constant $D \neq 0$. Then

$$\phi''(z) = D^2\phi(z)[1-\phi(z)][1-2\phi(z)].$$

Plugging these derivatives back into the differential equation, we get

$$cD\phi(z)[1-\phi(z)] = D^2\phi(z)[1-\phi(z)][1-2\phi(z)] + \phi(z)[1-\phi(z)][\phi(z)-a].$$

By canceling out $\phi(z)[1-\phi(z)]$, we get

$$cD = D^2[1-2\phi(z)] + \phi(z) - a.$$

By comparing the coefficients, we see

$$cD = D^2 - a, \quad 2D^2 = 1.$$

Solving it, we have

$$D = \pm\frac{1}{\sqrt{2}}, \quad c = \pm\frac{1-2a}{\sqrt{2}}.$$

It has two explicit traveling wave fronts

$$\phi(z) = \frac{\exp\{Dz\}}{1+\exp\{Dz\}} = \frac{\exp\Big\{\pm\frac{z}{\sqrt{2}}\Big\}}{1+\exp\Big\{\pm\frac{z}{\sqrt{2}}\Big\}},$$
$$z = x + ct, \quad c = \pm\frac{1-2a}{\sqrt{2}}.$$

Motivated by the explicit traveling wave solutions to this simple equation, we investigate explicit traveling wave solutions to some reaction diffusion equations. In this section, we are going to establish the explicit traveling wave solutions to the $n$-dimensional Fisher's equation, the $n$-dimensional generalized Fisher's equation, the $n$-dimensional Belousov-Zhabotinskii system of reaction-diffusion equations and the $n$-dimensional McKean-Nagumo reaction diffusion equation. This section is primarily motivated by [15, 31, 36, 37, 39].

**Theorem 5.1** *Consider the $n$-dimensional Fisher's equation*

$$\frac{\partial u}{\partial t} = \alpha^2\triangle u + \beta^2 u(1-u), \tag{5.1}$$

*where $\alpha > 0$ and $\beta > 0$ are positive constants. Let $\mathbf{a} = (a_1, a_2, \cdots, a_n)$ be a nonzero real constant vector. Then the $n$-dimensional Fisher's equation has two explicit traveling wave solutions*

$$u(\mathbf{x}, t) = \left\{\frac{\exp\Big\{\pm\frac{\beta}{\sqrt{6}|\mathbf{a}|\alpha}\Big(\mathbf{a}\cdot\mathbf{x} \pm \frac{5|\mathbf{a}|\alpha\beta}{\sqrt{6}}t\Big)\Big\}}{1+\exp\Big\{\pm\frac{\beta}{\sqrt{6}|\mathbf{a}|\alpha}\Big(\mathbf{a}\cdot\mathbf{x} \pm \frac{5|\mathbf{a}|\alpha\beta}{\sqrt{6}}t\Big)\Big\}}\right\}^2. \tag{5.2}$$

**Proof** Let $\mathbf{a} = (a_1, a_2, \cdots, a_n)$ be a nonzero real constant vector and let $c$ be a real constant. Let $z = \mathbf{a}\cdot\mathbf{x} + ct$. A traveling wave solution $u(\mathbf{x}, t) = \phi(\mathbf{a}\cdot\mathbf{x} + ct)$ to the $n$-dimensional Fisher's equation satisfies the equation

$$c\phi'(z) = |\mathbf{a}|^2\alpha^2\phi''(z) + \beta^2\phi(z)[1-\phi(z)].$$

Suppose that $\phi(z) = [\psi(z)]^2$. Then

$$2c\psi(z)\psi'(z) = 2|\mathbf{a}|^2\alpha^2\psi(z)\psi''(z) + 2|\mathbf{a}|^2\alpha^2[\psi'(z)]^2 + \beta^2[\psi(z)]^2\{1-[\psi(z)]^2\}.$$

Let us reduce the order of the differential equation. Let $\psi'(z) = D\psi(z)[1-\psi(z)]$ for some constant $D \neq 0$. Then

$$\psi''(z) = D^2\psi(z)[1-\psi(z)][1-2\psi(z)].$$

Plugging the derivatives back into the differential equation, we get

$$\begin{aligned}2cD[\psi(z)]^2[1-\psi(z)] &= 2|\mathbf{a}|^2D^2\alpha^2[\psi(z)]^2[1-\psi(z)][1-2\psi(z)]\\&\quad+2|\mathbf{a}|^2D^2\alpha^2[\psi(z)]^2[1-\psi(z)]^2+\beta^2[\psi(z)]^2\{1-[\psi(z)]^2\}.\end{aligned}$$

By canceling out $[\psi(z)]^2[1-\psi(z)]$, we have

$$2cD = 2|\mathbf{a}|^2D^2\alpha^2[1-2\psi(z)]+2|\mathbf{a}|^2D^2\alpha^2[1-\psi(z)]+\beta^2[1+\psi(z)].$$

Comparing the coefficients, we find that

$$2cD = 4|\mathbf{a}|^2D^2\alpha^2+\beta^2, \quad 6|\mathbf{a}|^2D^2\alpha^2=\beta^2.$$

Finally, we find two sets of solutions $c=\pm\frac{5|\mathbf{a}|\alpha\beta}{\sqrt{6}}$, $D=\pm\frac{\beta}{\sqrt{6}|\mathbf{a}|\alpha}$. Now, by using Lemma 1.1, we obtain

$$\psi(z)=\frac{\exp\Big\{\pm\frac{\beta}{\sqrt{6}|\mathbf{a}|\alpha}\Big(\mathbf{a}\cdot\mathbf{x}\pm\frac{5|\mathbf{a}|\alpha\beta}{\sqrt{6}}t\Big)\Big\}}{1+\exp\Big\{\pm\frac{\beta}{\sqrt{6}|\mathbf{a}|\alpha}\Big(\mathbf{a}\cdot\mathbf{x}\pm\frac{5|\mathbf{a}|\alpha\beta}{\sqrt{6}}t\Big)\Big\}}.$$

Now we can obtain the explicit traveling wave solutions to the $n$-dimensional Fisher's equation.

**Theorem 5.2** *Consider the $n$-dimensional generalized Fisher's equation*

$$\frac{\partial u}{\partial t}=\alpha^2\triangle u+\beta^2u(1-u^p)(u^p-a), \tag{5.3}$$

*where $p>0$, $\alpha>0$ and $\beta>0$ are positive constants and $a$ is a real number. Let $\mathbf{a}=(a_1,a_2,\cdots,a_n)\in\mathbb{R}^n$ be any nonzero real constant vector. Then the generalized Fisher's equation has two explicit traveling wave solutions*

$$u(\mathbf{x},t)=\left\{\frac{\exp\Big\{\pm\frac{p\beta}{|\mathbf{a}|\alpha\sqrt{p+1}}\Big(\mathbf{a}\cdot\mathbf{x}\pm\frac{1-(p+1)a}{\sqrt{p+1}}|\mathbf{a}|\alpha\beta t\Big)\Big\}}{1+\exp\Big\{\pm\frac{p\beta}{|\mathbf{a}|\alpha\sqrt{p+1}}\Big(\mathbf{a}\cdot\mathbf{x}\pm\frac{1-(p+1)a}{\sqrt{p+1}}|\mathbf{a}|\alpha\beta t\Big)\Big\}}\right\}^{\frac{1}{p}}. \tag{5.4}$$

**Proof** Let $z=\mathbf{a}\cdot\mathbf{x}+ct$ and $u(\mathbf{x},t)=\phi(\mathbf{a}\cdot\mathbf{x}+ct)$ be a traveling wave solution to the generalized Fisher's equation. Then

$$c\phi'(z)=|\mathbf{a}|^2\alpha^2\phi''(z)+\beta^2\phi(z)\{1-[\phi(z)]^p\}\{[\phi(z)]^p-a\}.$$

Let us reduce the order of the differential equation. Let $D\neq 0$ be a constant and

$$\phi'(z)=D\phi(z)\{1-[\phi(z)]^p\}.$$

Then

$$\phi''(z)=D^2\phi(z)\{1-[\phi(z)]^p\}\{1-(p+1)[\phi(z)]^p\}.$$

Plugging the derivatives back into the differential equation, we get

$$\begin{aligned}cD\phi(z)\{1-[\phi(z)]^p\} &= |\mathbf{a}|^2D^2\alpha^2\phi(z)\{1-[\phi(z)]^p\}\{1-(p+1)[\phi(z)]^p\}\\&\quad+\beta^2\phi(z)\{1-[\phi(z)]^p\}\{[\phi(z)]^p-a\}.\end{aligned}$$

By canceling out $\phi(z)\{1-[\phi(z)]^p\}$, we get

$$cD = |\mathbf{a}|^2 D^2 \alpha^2 \{1-(p+1)[\phi(z)]^p\} + \beta^2\{[\phi(z)]^p - a\}.$$

Then, by comparing the coefficients, we find

$$cD = |\mathbf{a}|^2 D^2 \alpha^2 - a\beta^2, \quad (p+1)|\mathbf{a}|^2 D^2 \alpha^2 = \beta^2.$$

Solving it, we have

$$D = \pm \frac{\beta}{|\mathbf{a}|\alpha\sqrt{p+1}}, \quad c = \pm \frac{1-(p+1)a}{\sqrt{p+1}} |\mathbf{a}|\alpha\beta.$$

Therefore, by using Lemma 1.1, we find the explicit traveling wave solutions to the generalized Fisher's equation.

**Theorem 5.3** *Consider the generalized $n$-dimensional Belousov-Zhabotinskii system of reaction-diffusion equations*

$$\frac{\partial u}{\partial t} = D_1 \triangle u + \alpha u(1-u) + \beta uv, \tag{5.5}$$

$$\frac{\partial v}{\partial t} = D_2 \triangle v + \gamma v(1-v) + \delta uv, \tag{5.6}$$

*where $D_1 > 0$ and $D_2 > 0$ are positive constants, $\alpha$, $\beta$, $\gamma$ and $\delta$ are real constants, such that $\alpha + \gamma > 0$. Let $\mathbf{a} = (a_1, a_2, \cdots, a_n) \in \mathbb{R}^n$ be any nonzero real constant vector. If*

$$\beta = \frac{D_2 - 5D_1}{D_1 + D_2}\alpha - \frac{6D_1}{D_1 + D_2}\gamma,$$
$$\delta = -\frac{6D_2}{D_1 + D_2}\alpha - \frac{5D_2 - D_1}{D_1 + D_2}\gamma,$$

*then the generalized Belousov-Zhabotinskii system of reaction-diffusion equations has two explicit traveling wave solutions*

$$u(\mathbf{x}, t) = \left\{ \frac{\exp\left\{ \pm\sqrt{\frac{\alpha+\gamma}{|\mathbf{a}|^2(D_1+D_2)}}\left(\mathbf{a}\cdot\mathbf{x} \pm \frac{|\mathbf{a}|(D_2\alpha - D_1\gamma)}{\sqrt{(D_1+D_2)(\alpha+\gamma)}}t\right)\right\}}{1 + \exp\left\{ \pm\sqrt{\frac{\alpha+\gamma}{|\mathbf{a}|^2(D_1+D_2)}}\left(\mathbf{a}\cdot\mathbf{x} \pm \frac{|\mathbf{a}|(D_2\alpha - D_1\gamma)}{\sqrt{(D_1+D_2)(\alpha+\gamma)}}t\right)\right\}} \right\}^2, \tag{5.7}$$

$$v(\mathbf{x}, t) = \frac{1}{\left\{1 + \exp\left\{ \pm\sqrt{\frac{\alpha+\gamma}{|\mathbf{a}|^2(D_1+D_2)}}\left(\mathbf{a}\cdot\mathbf{x} \pm \frac{|\mathbf{a}|(D_2\alpha - D_1\gamma)}{\sqrt{(D_1+D_2)(\alpha+\gamma)}}t\right)\right\}\right\}^2}. \tag{5.8}$$

**Proof** Let $z = \mathbf{a}\cdot\mathbf{x} + ct$,

$$u(\mathbf{x}, t) = [\phi(z)]^2 \quad \text{and} \quad v(\mathbf{x}, t) = [1-\phi(z)]^2.$$

Then

$$c\{[\phi(z)]^2\}' = |\mathbf{a}|^2 D_1 \{[\phi(z)]^2\}'' + \alpha[\phi(z)]^2\{1-[\phi(z)]^2\} + \beta[\phi(z)]^2[1-\phi(z)]^2,$$
$$c\{[1-\phi(z)]^2\}' = |\mathbf{a}|^2 D_2 \{[1-\phi(z)]^2\}'' + \gamma[1-\phi(z)]^2\{1-[1-\phi(z)]^2\} + \delta[\phi(z)]^2[1-\phi(z)]^2.$$

Equivalently, we have

$$\begin{aligned}2c\phi(z)\phi'(z) &= 2|\mathbf{a}|^2 D_1\phi(z)\phi''(z) + 2|\mathbf{a}|^2 D_1[\phi'(z)]^2\\ &\quad + \alpha[\phi(z)]^2\{1-[\phi(z)]^2\} + \beta[\phi(z)]^2[1-\phi(z)]^2,\\ 2c[1-\phi(z)][1-\phi(z)]' &= 2|\mathbf{a}|^2 D_2[1-\phi(z)][1-\phi(z)]'' + 2|\mathbf{a}|^2 D_2[-\phi'(z)]^2\\ &\quad + \gamma[1-\phi(z)]^2\{1-[1-\phi(z)]^2\} + \delta[\phi(z)]^2[1-\phi(z)]^2.\end{aligned}$$

Let us reduce the order of the differential equation. Suppose that $p \neq 0$ is a constant and

$$\phi'(z) = p\phi(z)[1-\phi(z)].$$

Then

$$\phi''(z) = p^2\phi(z)[1-\phi(z)][1-2\phi(z)].$$

Plugging these derivatives back into the differential equations, we have

$$\begin{aligned}2cp[\phi(z)]^2[1-\phi(z)] &= 2|\mathbf{a}|^2 D_1 p^2[\phi(z)]^2[1-\phi(z)][1-2\phi(z)] + 2|\mathbf{a}|^2 D_1 p^2[\phi(z)]^2[1-\phi(z)]^2\\ &\quad + \alpha[\phi(z)]^2\{1-[\phi(z)]^2\} + \beta[\phi(z)]^2[1-\phi(z)]^2,\\ -2cp[1-\phi(z)]^2\phi(z) &= -2|\mathbf{a}|^2 D_2 p^2\phi(z)[1-\phi(z)]^2[1-2\phi(z)] + 2|\mathbf{a}|^2 D_2 p^2[\phi(z)]^2[1-\phi(z)]^2\\ &\quad + \gamma[1-\phi(z)]^2\{1-[1-\phi(z)]^2\} + \delta[\phi(z)]^2[1-\phi(z)]^2.\end{aligned}$$

Let us cancel out $[\phi(z)]^2[1-\phi(z)]$ in the first equation and cancel out $\phi(z)[1-\phi(z)]^2$ in the second equation. We have a simpler system of equations

$$\begin{aligned}2cp &= 2|\mathbf{a}|^2 D_1 p^2[1-2\phi(z)] + 2|\mathbf{a}|^2 D_1 p^2[1-\phi(z)] + \alpha[1+\phi(z)] + \beta[1-\phi(z)],\\ 2cp &= 2|\mathbf{a}|^2 D_2 p^2[1-2\phi(z)] - 2|\mathbf{a}|^2 D_2 p^2\phi(z) + \gamma[\phi(z)-2] + \delta\phi(z).\end{aligned}$$

By comparing the coefficients, we have

$$\begin{aligned}2cp &= 2|\mathbf{a}|^2 D_1 p^2 + 2|\mathbf{a}|^2 D_1 p^2 + \alpha + \beta,\\ 0 &= -4|\mathbf{a}|^2 D_1 p^2 - 2|\mathbf{a}|^2 D_1 p^2 + \alpha - \beta,\\ 2cp &= 2|\mathbf{a}|^2 D_2 p^2 - 2\gamma,\\ 0 &= -4|\mathbf{a}|^2 D_2 p^2 - 2|\mathbf{a}|^2 D_2 p^2 + \gamma - \delta.\end{aligned}$$

Simplifying the system, we get

$$\begin{aligned}\beta &= cp - 5|\mathbf{a}|^2 D_1 p^2,\\ \delta &= 6|\mathbf{a}|^2 D_2 p^2 - \gamma,\\ cp &= \frac{D_2\alpha - D_1\gamma}{D_1 + D_2},\\ p^2 &= \frac{\alpha+\gamma}{|\mathbf{a}|^2(D_1+D_2)}.\end{aligned}$$

Solving this system, we find two sets of solutions

$$\beta = \frac{D_2 - 5D_1}{D_1+D_2}\alpha - \frac{6D_1}{D_1+D_2}\gamma,$$

$$\delta = -\frac{6D_2}{D_1 + D_2}\alpha - \frac{5D_2 - D_1}{D_1 + D_2}\gamma,$$
$$c = \pm\frac{|\mathbf{a}|(D_2\alpha - D_1\gamma)}{\sqrt{(D_1 + D_2)(\alpha + \gamma)}},$$
$$p = \pm\sqrt{\frac{\alpha + \gamma}{|\mathbf{a}|^2(D_1 + D_2)}}.$$

Therefore, by using Lemma 1.1, we obtain the solutions

$$\phi(z) = \frac{\exp\left\{\pm\sqrt{\frac{\alpha+\gamma}{|\mathbf{a}|^2(D_1+D_2)}}\left(\mathbf{a}\cdot\mathbf{x} \pm \frac{|\mathbf{a}|(D_2\alpha - D_1\gamma)}{\sqrt{(D_1+D_2)(\alpha+\gamma)}}t\right)\right\}}{1 + \exp\left\{\pm\sqrt{\frac{\alpha+\gamma}{|\mathbf{a}|^2(D_1+D_2)}}\left(\mathbf{a}\cdot\mathbf{x} \pm \frac{|\mathbf{a}|(D_2\alpha - D_1\gamma)}{\sqrt{(D_1+D_2)(\alpha+\gamma)}}t\right)\right\}}.$$

The proof is finished.

**Theorem 5.4** (I) *Consider the $n$-dimensional McKean-Nagumo reaction diffusion equation*

$$\frac{\partial u}{\partial t} = \triangle u + \alpha[H(u-\theta) - u], \quad 0 < \theta < \frac{1}{2}, \tag{5.9}$$

*where $\alpha > 0$ and $\theta > 0$ are constants, $H$ stands for the Heaviside step function: $H(u-\theta) = 0$ for all $u < \theta$, $H(0) = \frac{1}{2}$ and $H(u-\theta) = 1$ for all $u > \theta$. Let $\mathbf{a} = (a_1, a_2, \cdots, a_n)$ be a nonzero real constant vector. Then this equation has a traveling wave front*

$$\begin{aligned} u(\mathbf{x}, t) &= U_{\text{front}}(z) \\ &= \begin{cases} \theta\exp\left\{\frac{c+\sqrt{c^2+4\alpha}}{2}(\mathbf{a}\cdot\mathbf{x} + ct)\right\} & \text{for } \; z = \mathbf{a}\cdot\mathbf{x} + ct < 0, \\ 1 + (\theta - 1)\exp\left\{\frac{c-\sqrt{c^2+4\alpha}}{2}(\mathbf{a}\cdot\mathbf{x} + ct)\right\} & \text{for } \; z = \mathbf{a}\cdot\mathbf{x} + ct > 0, \end{cases} \end{aligned} \tag{5.10}$$

*where the wave speed*

$$c = \sqrt{\frac{|\mathbf{a}|^2\alpha(1-2\theta)^2}{\theta(1-\theta)}},$$

*and $z = \mathbf{a}\cdot\mathbf{x} + ct$ is the moving coordinate.*

(II) *Consider the $n$-dimensional reaction diffusion equation*

$$\frac{\partial u}{\partial t} + \alpha(1-2\theta) = \triangle u + \alpha[H(u-\theta) - u]. \tag{5.11}$$

*This equation has a traveling wave back*

$$u(\mathbf{x}, t) \equiv 2\theta - U_{\text{front}}(z) \tag{5.12}$$

*with the same speed as the traveling wave front.*

**Proof** Let $z = \mathbf{a}\cdot\mathbf{x} + ct$ and let $u(\mathbf{x}, t) = \phi(\mathbf{a}\cdot\mathbf{x} + ct)$ be a traveling wave solution of the $n$-dimensional McKean-Nagumo reaction diffusion equation. Then

$$c\phi'(z) = |\mathbf{a}|^2\phi''(z) + \alpha[H(\phi(z) - \theta) - \phi(z)].$$

Suppose that the traveling wave front satisfies the conditions $\phi < \theta$ on $(-\infty, 0)$, $\phi(0) = \theta$ and $\phi > \theta$ on $(0, \infty)$. Then

$$\phi''(z) - \frac{c}{|\mathbf{a}|^2}\phi'(z) - \frac{\alpha}{|\mathbf{a}|^2}\phi(z) = 0, \quad \text{on } (-\infty, 0),$$

$$\phi''(z) - \frac{c}{|\mathbf{a}|^2}\phi'(z) + \frac{\alpha}{|\mathbf{a}|^2}[1-\phi(z)] = 0, \quad \text{on } (0,\infty),$$
$$\lim_{z\to-\infty}\phi(z) = 0, \quad \lim_{z\to\infty}\phi(z) = 1, \quad \phi(0) = \theta.$$

The characteristic equation for these differential equations is

$$\lambda^2 - \frac{c}{|\mathbf{a}|^2}\lambda - \frac{\alpha}{|\mathbf{a}|^2} = 0.$$

The eigenvalues are

$$\lambda = \frac{c \pm \sqrt{c^2+4|\mathbf{a}|^2\alpha}}{2|\mathbf{a}|^2}.$$

Therefore, the traveling wave solution is given by

$$u(\mathbf{x},t) = \theta\exp\Big\{\frac{c+\sqrt{c^2+4|\mathbf{a}|^2\alpha}}{2|\mathbf{a}|^2}(\mathbf{a}\cdot\mathbf{x}+ct)\Big\}, \quad \text{on } (-\infty,0)$$

and

$$u(\mathbf{x},t) = 1+(\theta-1)\exp\Big\{\frac{c-\sqrt{c^2+4|\mathbf{a}|^2\alpha}}{2|\mathbf{a}|^2}(\mathbf{a}\cdot\mathbf{x}+ct)\Big\}, \quad \text{on } (0,\infty).$$

The wave speed is determined by the condition

$$\phi(0^-) = \phi(0^+).$$

That is

$$\theta\frac{c+\sqrt{c^2+4|\mathbf{a}|^2\alpha}}{2|\mathbf{a}|^2} = (\theta-1)\frac{c-\sqrt{c^2+4|\mathbf{a}|^2\alpha}}{2|\mathbf{a}|^2}.$$

Finally, we find the wave speed

$$c = \sqrt{\frac{|\mathbf{a}|^2\alpha(1-2\theta)^2}{\theta(1-\theta)}}.$$

The proof is finished.

## 6 Nonlinear Hyperbolic Equations

In this section, we are going to establish the explicit traveling wave solutions to the $n$-dimensional Klein-Gordon equation and the $n$-dimensional Sine-Gordon equation. This section is primarily motivated by [10, 18, 20, 22, 24, 29, 35, 38, 44].

**Theorem 6.1** *Consider the generalized $n$-dimensional Klein-Gordon equation*

$$\frac{\partial^2 u}{\partial t^2} - \alpha^2\triangle u - \beta^2 u^{2p+1} + \gamma^2 u^{p+1} - \delta^2 u = 0, \tag{6.1}$$

*where $\alpha>0$, $\beta>0$, $\gamma>0$, $\delta>0$ and $p>0$ are positive constants. Let $\mathbf{a}=(a_1,a_2,\cdots,a_n)\in\mathbb{R}^n$ be any nonzero real constant vector. If $\beta=0$, then the generalized Klein-Gordon equation has the explicit traveling wave solution*

$$u(\mathbf{x},t) = \Big\{\frac{p+2}{2}\frac{\delta^2}{\gamma^2}\operatorname{sech}^2\Big(\frac{p}{2}\frac{\delta}{\sqrt{c^2-|\mathbf{a}|^2\alpha^2}}(\mathbf{a}\cdot\mathbf{x}+ct)\Big)\Big\}^{\frac{1}{p}}, \tag{6.2}$$

*where $c$ is a constant such that $|c| > |\mathbf{a}|\alpha$. Set $z = \mathbf{a}\cdot\mathbf{x} + ct$. If $\beta > 0$ and*

$$\frac{(p+2)^2}{p+1}\beta^2\delta^2 = \gamma^4,$$

*then*

$$u(\mathbf{x},t) = \Big\{\sqrt{p+1}\frac{\delta}{\beta}\frac{1}{1+\exp\Big\{\pm p\sqrt{\frac{\delta^2}{c^2-|\mathbf{a}|^2\alpha^2}}(\mathbf{a}\cdot\mathbf{x}+ct)\Big\}}\Big\}^{\frac{1}{p}}.$$

**Proof** Let $u(\mathbf{x},t) = \phi(\mathbf{a}\cdot\mathbf{x}+ct)$ be a traveling wave solution to the generalized $n$-dimensional Klein-Gordon equation, and let $z = \mathbf{a}\cdot\mathbf{x}+ct$, where $c$ is a real constant, such that $c^2 > |\mathbf{a}|^2\alpha^2$. Then

$$c^2\phi''(z) - |\mathbf{a}|^2\alpha^2\phi''(z) - \beta^2[\phi(z)]^{2p+1} + \gamma^2[\phi(z)]^{p+1} - \delta^2\phi(z) = 0.$$

This equation may be rewritten as

$$\phi''(z) = \phi(z)\Big\{\frac{\delta^2}{c^2-|\mathbf{a}|^2\alpha^2} - \frac{\gamma^2}{c^2-|\mathbf{a}|^2\alpha^2}[\phi(z)]^p + \frac{\beta^2}{c^2-|\mathbf{a}|^2\alpha^2}[\phi(z)]^{2p}\Big\}.$$

By using Lemma 1.3 (for $\beta = 0$) and Lemma 1.4 (for $\beta > 0$), we find the traveling wave solutions, where

$$\frac{p+2}{\sqrt{p+1}}\sqrt{\frac{\delta^2}{c^2-|\mathbf{a}|^2\alpha^2}}\sqrt{\frac{\beta^2}{c^2-|\mathbf{a}|^2\alpha^2}} = \frac{\gamma^2}{c^2-|\mathbf{a}|^2\alpha^2}.$$

Now we finish the proof.

**Theorem 6.2** *Consider the generalized $n$-dimensional Sine-Gordon equation*

$$\frac{\partial^2 u}{\partial t^2} - \alpha^2\triangle u + 2\beta^2\gamma\sin(2\gamma u) = 0, \tag{6.3}$$

*where $\alpha > 0$, $\beta > 0$ and $\gamma > 0$ are positive constants. Let $\mathbf{a} = (a_1, a_2, \cdots, a_n) \in \mathbb{R}^n$ be any nonzero real constant vector. Then the generalized Sine-Gordon equation has the explicit traveling wave solution*

$$u(\mathbf{x},t) = \frac{1}{\gamma}\arccos\Big(\tanh\Big(\frac{2\beta\gamma(\mathbf{a}\cdot\mathbf{x}+ct)}{\sqrt{|\mathbf{a}|^2\alpha^2-c^2}}\Big)\Big), \tag{6.4}$$

*where $c$ is a constant, such that $|c| < |\mathbf{a}|\alpha$.*

**Proof** Let $u(\mathbf{x},t) = \phi(\mathbf{a}\cdot\mathbf{x}+ct)$ be a traveling wave solution, where $c$ is a real constant, and let $z = \mathbf{a}\cdot\mathbf{x}+ct$. Then

$$c^2\phi''(z) - |\mathbf{a}|^2\alpha^2\phi''(z) + 2\beta^2\gamma\sin(2\gamma\phi(z)) = 0.$$

It is not difficult to find

$$\phi''(z) = \frac{2\beta^2\gamma}{|\mathbf{a}|^2\alpha^2-c^2}\sin(2\gamma\phi(z)).$$

Multiplying this equation by $2\phi'$, integrating the result with respect to $z$, and letting the integration constant be equal to zero, we have

$$[\phi'(z)]^2 = \frac{2\beta^2}{|\mathbf{a}|^2\alpha^2 - c^2}[1 - \cos(2\gamma\phi(z))] = \frac{4\beta^2}{|\mathbf{a}|^2\alpha^2 - c^2}\sin^2(\gamma\phi(z)).$$

Now

$$\phi'(z) = \pm\frac{2\beta}{\sqrt{|\mathbf{a}|^2\alpha^2 - c^2}}\sin(\gamma\phi(z)).$$

By separating the variables, we get

$$-\frac{1}{\sin(\gamma\phi)}\mathrm{d}\phi = \pm\frac{2\beta}{\sqrt{|\mathbf{a}|^2\alpha^2 - c^2}}\mathrm{d}z.$$

Equivalently, we have

$$-\frac{\gamma\sin(\gamma\phi(z))}{1 - \cos^2(\gamma\phi(z))}\mathrm{d}\phi = \pm\frac{2\beta\gamma}{\sqrt{|\mathbf{a}|^2\alpha^2 - c^2}}\mathrm{d}z.$$

Finally, we obtain

$$\tanh^{-1}(\cos(\gamma\phi)) = \pm\frac{2\beta\gamma z}{\sqrt{|\mathbf{a}|^2\alpha^2 - c^2}}.$$

The proof is finished.

**Corollary 6.1** *Consider the $n$-dimensional Sine-Gordon equation*

$$\frac{\partial^2 u}{\partial t^2} - \triangle u + \sin u = 0.$$

*It has the explicit traveling wave solution*

$$u(\mathbf{x}, t) = 2\arccos\Big(\tanh\Big(\frac{\mathbf{a}\cdot\mathbf{x} + ct}{\sqrt{|\mathbf{a}|^2 - c^2}}\Big)\Big).$$

**Proof** It follows from Theorem 6.2.

## References


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