

Pseudo-Anosov Mapping Classes and Their Representations by Products of Two Dehn Twists

Chaohui ZHANG*

Abstract Let \tilde{S} be a Riemann surface of analytically finite type (p, n) with $3p - 3 + n > 0$. Let $a \in \tilde{S}$ and $S = \tilde{S} - \{a\}$. In this article, the author studies those pseudo-Anosov maps on S that are isotopic to the identity on \tilde{S} and can be represented by products of Dehn twists. It is also proved that for any pseudo-Anosov map f of S isotopic to the identity on \tilde{S} , there are infinitely many pseudo-Anosov maps F on $S - \{b\} = \tilde{S} - \{a, b\}$, where b is a point on S , such that F is isotopic to f on S as b is filled in.

Keywords Riemann surface, Pseudo-Anosov map, Dehn twist, Teichmüller space, Bers fiber space

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1 Introduction

According to Thurston [14], an orientation-preserving homeomorphism f of a Riemann surface is called pseudo-Anosov if there exists a pair $(\mathcal{F}_+, \mathcal{F}_-)$ of transverse measured foliations on the surface with $f(\mathcal{F}_+) = \lambda \mathcal{F}_+$ and $f(\mathcal{F}_-) = \frac{1}{\lambda} \mathcal{F}_-$ for some $\lambda > 1$ (see also [5, 12–13]).

Let \tilde{S} be a Riemann surface of type (p, n) with $3p - 3 + n > 0$. Fix a point $a \in \tilde{S}$. Then $S = \tilde{S} - \{a\}$ is of type $(p, n+1)$. Let \mathcal{F} be the set of maps on S that fix a and are isotopic to the identity on \tilde{S} . By [2, Theorem 10] (see also [4, Theorems 4.2 and 4.3]), \mathcal{F} is isomorphic to the Fuchsian group G that uniformizes \tilde{S} under a universal covering $\varrho: \mathbb{H} = \{z: \text{Im}z > 0\} \rightarrow \tilde{S}$. Throughout this paper, we write $f = g^*$ if $g \in G$ corresponds to $f \in \mathcal{F}$ under the isomorphism. In [7, Theorem 2], Kra showed that $g^* \in \mathcal{F}$ is pseudo-Anosov if and only if g is an essential hyperbolic element of G ; that is, its axis c projects to a filling geodesic \tilde{c} in the sense that the complement $\tilde{S} - \{\tilde{c}\}$ consists of disks and once punctured disks. Let $\mathcal{F}_0 \subset \mathcal{F}$ denote the subset consisting of elements g^* for essential hyperbolic elements of $g \in G$.

It is obvious that if α_1, α_2 are simple closed geodesics on S that are trivial on \tilde{S} , then any products

$$\prod_i t_1^{r_i} \circ t_2^{-s_i}, \quad r_i, s_i \in \mathbb{Z}^+ - \{0\}, \quad (1.1)$$

where t_i is the Dehn twist along α_i , are in \mathcal{F} . That is, (1.1) is of form g^* for some $g \in G$. On the other hand, certain elements $g^* \in \mathcal{F}$ are isotopic to products of Dehn twists along two filling simple closed geodesics α_1 and α_2 on S that are also nontrivial on \tilde{S} .

The main purpose of this paper is to investigate elements in \mathcal{F}_0 that are isotopic to a map of form (1.1).

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*Department of Mathematics, Morehouse College, Atlanta, GA 30314, USA. E-mail: czhang@morehouse.edu

Theorem 1.1 *There exist infinitely many pseudo-Anosov maps $f = g^* \in \mathcal{F}_0$ that can not be isotopic to any products of Dehn twists along two simple curves that are trivial on \tilde{S} . Furthermore, if \tilde{S} contains at least one puncture, there exist infinitely many pseudo-Anosov maps $f = g^* \in \mathcal{F}_0$ that are isotopic to products of Dehn twists along two simple curves that are trivial on \tilde{S} .*

Now we assume that $f = g^* \in \mathcal{F}_0$ is isotopic to a product (1.1) for α_i being nontrivial on \tilde{S} . Let $\tilde{\alpha}_i$ denote the geodesic homotopic to α_i on \tilde{S} . The Dehn twist $t_{\tilde{\alpha}_i}$ can be lifted to a mapping $\tau_i : \mathbb{H} \rightarrow \mathbb{H}$ so that $\tau_i^* = t_i$. The map τ_i determines a collection \mathcal{U}_i of disjoint maximal half-planes D_i each of which is invariant under τ_i . Let H_i be the complement of \mathcal{U}_i in \mathbb{H} . Then $\tau_i|_{H_i}$ is the identity. See Section 4 for an illustration.

Theorem 1.2 *There exist infinitely many elements $f = g^* \in \mathcal{F}_0$ that are isotopic to products (1.1) with $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ being nontrivial. Furthermore, if $f = g^* \in \mathcal{F}_0$ is isotopic to a product (1.1), then either α_1 and α_2 are trivial on \tilde{S} , or $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ are nontrivial. In later case, we let τ_i denote the lift of $t_{\tilde{\alpha}_i}$ so that $\tau_i^* = t_i$. Then the following two conditions hold:*

- (1) *The pair $\{\alpha_1, \alpha_2\}$ fills S , $\tilde{\alpha}_1 = \tilde{\alpha}_2$, and thus $t_{\tilde{\alpha}_1} = t_{\tilde{\alpha}_2}$ and $\sum_i (r_i - s_i) = 0$;*
- (2) *There are maximal elements $D_1 \in \mathcal{U}_1$ and $D_2 \in \mathcal{U}_2$ such that $D_1 \cap D_2 \neq \emptyset$, $\partial D_1 \cap \partial D_2 = \emptyset$, and the axis of g lies in $D_1 \cap D_2$.*

Denote by \mathcal{L} the set of pseudo-Anosov maps on S obtained from products of Dehn twists along two filling simple closed geodesics. Let $a' \in S$ and $\dot{S} = S - \{a'\}$. In [16, Theorem 1.2], we showed that for any element $f \in \mathcal{L}$, there exist infinitely many pseudo-Anosov maps F on $\dot{S} = S - \{a'\}$ isotopic to f on S as a' is filled in.

Unfortunately, by [6, Corollary 1.3], we know that not every pseudo-Anosov map on S is in \mathcal{L} . Also, it is not clear whether every element of \mathcal{F}_0 is in \mathcal{L} . In contrast, $\mathcal{F}_0 \cap \mathcal{L}$ contains infinitely many elements. A question arises as to whether there exist pseudo-Anosov maps F on \dot{S} isotopic to a given map f in $\mathcal{F}_0 - \mathcal{L}$ on S (if the set is not empty). Our last result is the following:

Theorem 1.3 *Let \tilde{S} be a Riemann surface of type (p, n) with $3p - 3 + n > 0$. For any $f \in \mathcal{F}_0$, there exist infinitely many pseudo-Anosov maps F on \dot{S} that are isotopic to f on S as a' is filled in.*

Here we recall the main theorem of Imayoshi, Ito and Yamamoto [8]. Denote $M = \tilde{S} \times \tilde{S}$, $\vec{a} = \{a, a'\}$, and $\Delta = \{(x, y) \in M : x \neq y\}$. Since F is isotopic to the identity on \tilde{S} , there is an isotopy $H : \tilde{S} \times I \rightarrow \tilde{S}$ such that $H(\cdot, 0) = F$ and $H(\cdot, 1) = \text{id}$. Then $s_1 = F(a, t)$ and $s_2 = F(a', t)$, where $1 \leq t \leq 1$, are closed curve on \tilde{S} , which define a pure braids $[b_F]$ represented by $b_F = (s_1, s_2)$ in the fundamental group $\pi_1(M - \Delta, \vec{a})$. By Theorem 1.3 and the main theorem of [8], we obtain infinitely many essential pure braids $[b_F]$ so that s_1 and s_2 are nontrivial and nonparallel.

2 Notation and Background

In this section, we review some basic facts on Teichmüller theory (see [1–3, 7] for more details). Denote by $L_\infty(\mathbb{H}, G)$ the space of measurable functions on the hyperbolic plane \mathbb{H} satisfying

$$(\mu \circ g)(z) \cdot \frac{\overline{g'(z)}}{g'(z)} = \mu(z) \quad \text{for all } g \in G.$$

Let $M(G) \subset L_\infty(\mathbb{H}, G)$ denote the unit ball. For each element $\mu \in M(G)$, there exists a quasi-conformal mapping $w^\mu : \mathbb{C} \rightarrow \mathbb{C}$ that fixes 0, 1 and also satisfies these properties: (i) $w^\mu G (w^\mu)^{-1}$ is a group of Möbius transformations, (ii) w^μ is conformal on $\overline{\mathbb{H}} = \{z \in \mathbb{C} : \text{Im} z \leq 0\}$, (iii) the Beltrami coefficient $\frac{\partial_{\bar{z}} w^\mu(z)}{\partial_z w^\mu(z)}$ of w^μ on \mathbb{H} is $\mu(z)$, (iv) for any fixed $z \in \mathbb{C}$, the function

$$M(G) \ni \mu \mapsto w^\mu(z) \in \mathbb{C}$$

is holomorphic (see [1]). For μ, ν in $M(G)$, we set $\mu \sim \nu$ if and only if

$$w^\mu \circ g \circ (w^\mu)^{-1} = w^\nu \circ g \circ (w^\nu)^{-1} \quad \text{for each } g \in G.$$

The Teichmüller space $T(\tilde{S})$ is defined as the quotient $M(G)/\sim$ equipped with the quotient structure. The equivalence class of $\mu \in M(G)$ is denoted by $[\mu]$. $T(\tilde{S})$ is a complex manifold with dimension $3p - 3 + n$. The Bers fiber space $F(\tilde{S})$ over $T(\tilde{S})$ is defined by the total space

$$F(\tilde{S}) = \{([\mu], z) : [\mu] \in T(\tilde{S}), z \in w^\mu(\mathbb{H})\}.$$

The projection $\pi : F(\tilde{S}) \rightarrow T(\tilde{S})$ that sends a point $([\mu], z)$ to $[\mu]$ is holomorphic. Bers [2, Theorem 9] states that there is an isomorphism $\varphi : F(\tilde{S}) \rightarrow T(S)$ making the following diagram commutative:

$$\begin{array}{ccc} F(\tilde{S}) & \xrightarrow{\varphi} & T(S) \\ \pi \downarrow & & \eta_a \downarrow \\ T(\tilde{S}) & \xrightarrow{\text{id}} & T(\tilde{S}) \end{array} \quad (2.1)$$

where $\eta_a : T(S) \rightarrow T(\tilde{S})$ is defined by forgetting the puncture a . The group of isotopy classes of selfmaps of \tilde{S} is called the mapping class group of \tilde{S} and is denoted by $\text{Mod}_{\tilde{S}}$. Let $\theta \in \text{Mod}_{\tilde{S}}$ and w be a representative of θ . Then w can be lifted to an automorphism $\hat{w} : \mathbb{H} \rightarrow \mathbb{H}$ under the universal covering $\varrho : \mathbb{H} \rightarrow \tilde{S}$. Let $\text{mod } \tilde{S}$ denote the group that consists of equivalence classes $[\hat{w}]$ of \hat{w} , where two lifts \hat{w} and $\hat{w}' : \mathbb{H} \rightarrow \mathbb{H}$ of w are considered equivalent (we write $\hat{w} \sim \hat{w}'$) if they induce the same automorphism by conjugation on G . \hat{w} naturally extends to $\partial\mathbb{H}$, and $\hat{w}|_{\partial\mathbb{H}} = \hat{w}'|_{\partial\mathbb{H}}$ if and only if $\hat{w} \sim \hat{w}'$.

The group $\text{mod } \tilde{S}$ acts on $F(\tilde{S})$ in a fiber preserving way, and the group G , which is isomorphic to the fundamental group $\pi_1(\tilde{S}, a)$, can be regarded as a normal subgroup of $\text{mod } \tilde{S}$ so that $\text{mod } \tilde{S}/G \cong \text{Mod}_{\tilde{S}}$. Let Mod_S^a be the subgroup of Mod_S that consists of mapping classes on S fixing a . From [2, Theorem 10], the group $\text{mod } \tilde{S}$ is isomorphic to Mod_S^a under the isomorphism $\varphi^* : \text{mod } \tilde{S} \rightarrow \text{Mod}_S^a$, defined as

$$\text{mod } \tilde{S} \ni [\hat{w}] \xrightarrow{\varphi^*} \hat{w}^* = \varphi \circ [\hat{w}] \circ \varphi^{-1} \in \text{Mod}_S^a. \quad (2.2)$$

An element $\theta \in \text{Mod}_S^a$ is called a reducible mapping class if there is a curve system $\mathcal{C} = \{c_1, \dots, c_s\}$ of independent simple closed geodesics on S with $f(\{c_1, \dots, c_s\}) = \{c_1, \dots, c_s\}$ for certain representative f in θ . There is a smallest positive integer K such that f^K maps each loop in \mathcal{C} to itself and the restriction of f^K to each component of $S - \{c_1, \dots, c_s\}$ is either the identity or a pseudo-Anosov map.

We assume that θ is reducible and projects to a pseudo-Anosov mapping class $\tilde{\theta}$ on \tilde{S} that is induced by a map w . By [15, Lemmas 3.1 and 3.2], the curve system \mathcal{C} consists of only one curve c_1 that bounds a twice punctured disk enclosing a and another puncture of \tilde{S} , which is equivalent to that c_1 , becomes a trivial loop on \tilde{S} . If we denote by $[\hat{w}]$ the element of $\text{mod } \tilde{S}$

corresponding to θ , then $\widehat{w} : \overline{\mathbb{H}} \rightarrow \overline{\mathbb{H}}$ fixes a parabolic fixed point of G . Conversely, every element $[\widehat{w}]$ fixing the fixed point of a parabolic element of G corresponds to a reducible mapping class in Mod_S^a which is reduced by a single closed geodesic that is trivial on \widetilde{S} .

The natural projection $\eta_a^* : \text{Mod}_S^a \rightarrow \text{Mod}_{\widetilde{S}}$ induced by (2.1) makes the diagram commutative

$$\begin{array}{ccc} \text{mod } \widetilde{S} & \xrightarrow{\varphi^*} & \text{Mod}_S^a \\ \pi^* \downarrow & & \eta_a^* \downarrow \\ \text{Mod}_{\widetilde{S}} & \xrightarrow{\text{id}} & \text{Mod}_{\widetilde{S}} \end{array} \quad (2.3)$$

Thus the kernel $\ker(\eta_a^*)$ of $\eta_a^* : \text{Mod}_S^a \rightarrow \text{Mod}_{\widetilde{S}}$ is $G^* = \varphi^*(G)$. For every element $g \in G$, $g^* = \varphi^*(g)$ defines a mapping class on S that projects to the trivial mapping class on \widetilde{S} . Conversely, any mapping class on S that projects to the trivial mapping class is of the form g^* for some $g \in G$.

Let g be a simple hyperbolic element with $c \subset \mathbb{H}$ its axis, which means that $\widetilde{c} = \varrho(c)$ is a simple closed geodesic on \widetilde{S} . [7, Theorem 2] and [11] show that g^* is induced by a spin $t_2^{-1} \circ t_1$, where t_1 and t_2 are the Dehn twists along boundary geodesics $\{\alpha_1, \alpha_2\}$ of an a -punctured cylinder P so that they are both homotopic to \widetilde{c} on \widetilde{S} . If $g \in G$ is parabolic, g^* is represented by a Dehn twist along a loop α that bounds a twice punctured disk $\Delta \subset S$ enclosing a and another puncture of \widetilde{S} corresponding to the conjugacy class of g in G . Since every essential hyperbolic element g is written as a word generated by simple hyperbolic and parabolic elements of G , the pseudo-Anosov class g^* can be represented as a word generated by spins and Dehn twists.

3 Proof of Theorem 1.1

It suffices to show that there are (infinitely many) essential hyperbolic elements g of G , so that g^* can not be isotopic to a finite product (1.1), where α_i bounds a twice punctured disk Δ_i that encloses a for $i = 1, 2$.

Let $a = x_1, x_2, \dots, x_n$ ($n \geq 2$) denote the punctures of S . Thus $\widetilde{S} = S \cup \{a\}$ has punctures x_2, \dots, x_n . For every $i = 2, \dots, n$, let

$$S_i = S \cup \{x_i\} \quad \text{and} \quad \widetilde{S}_i = \widetilde{S} \cup \{x_i\} = S_i \cup \{a\}.$$

Let G_i denote the Fuchsian group that uniformizes \widetilde{S}_i . G_i acts on the Bers fiber space $F(\widetilde{S}_i)$ fiber wise, and is regarded as a normal subgroup of $\text{mod } \widetilde{S}_i$. Let $\varphi_i : F(\widetilde{S}_i) \rightarrow T(S_i)$ denote the Bers isomorphism. φ_i induces a group isomorphism φ_i^* of $\text{mod } \widetilde{S}_i$ onto $\text{Mod}_{S_i}^a$ by conjugation.

Let Mod_S^{a, x_i} be the subgroup of Mod_S consisting of mapping classes fixing both a and x_i , and $\eta_i^* : \text{Mod}_S^{a, x_i} \rightarrow \text{Mod}_{S_i}^a$ the natural projection defined by forgetting the puncture x_i . We fix an isomorphism $\pi_1(\widetilde{S}_i, a) \cong G_i$ as well as the isomorphism $\pi_1(\widetilde{S}, a) \cong G$. Clearly, there exists a naturally defined projection ξ_i of $\pi_1(\widetilde{S}, a)$ onto $\pi_1(\widetilde{S}_i, a)$ by forgetting the puncture x_i . Then we obtain a projection $\zeta_i : G \rightarrow G_i$ making the diagram

$$\begin{array}{ccc} G & \xrightarrow{\zeta_i} & G_i \\ \cong \downarrow & & \cong \downarrow \\ \pi_1(\widetilde{S}, a) & \xrightarrow{\xi_i} & \pi_1(\widetilde{S}_i, a) \end{array} \quad (3.1)$$

commutative.

Lemma 3.1 *Let G and G_i be regarded as normal subgroups of $\text{mod } \tilde{S}$ and $\text{mod } \tilde{S}_i$, respectively. With notations above, for every $i = 2, \dots, n$, the diagram*

$$\begin{array}{ccc} G & \xrightarrow{\varphi^*} & \text{Mod}_{\tilde{S}}^{a, x_i} \\ \zeta_i \downarrow & & \eta_i^* \downarrow \\ G_i & \xrightarrow{\varphi_i^*} & \text{Mod}_{\tilde{S}_i}^a \end{array} \quad (3.2)$$

commutes.

Proof Fix a set of generators of G . Let $g \in G$. Without loss of generality, we assume that g is one of the generators of G . The general case is handled by the properties of group homomorphisms. Then g is either parabolic or simple hyperbolic. Let $\tilde{c} \in \pi_1(\tilde{S}, a)$ be a loop that corresponds to g under the fixed isomorphism.

If g is parabolic, then \tilde{c} goes around x_i , $\zeta_i(g)$ is trivial, and hence $\varphi_i^* \circ \zeta_i(g)$ is trivial. On the other hand, by [7, Theorem 2] and [11], g^* is the Dehn twist along the loop c bounding a twice punctured disk Δ that encloses a and x_i . As x_i is filled in, c shrinks and thus g^* becomes a trivial loop, which means that $\eta_i^*(g^*)$ is trivial. This shows that

$$\eta_i^*(g^*) = \varphi_i^* \circ \zeta_i(g)$$

if g is a parabolic element corresponding to x_i .

If g is a parabolic element that corresponds to x_j ($i \neq j$) $\zeta_i(g) \in G_i$ corresponds to x_j (as a puncture of \tilde{S}_i). So $\varphi_i^* \circ \zeta_i(g)$ is the Dehn twist along the loop c that bounds a twice punctured disk Δ enclosing a and x_j . On the other hand, g^* is the Dehn twist along $\partial\Delta$. Since x_i lies outside of Δ , as x_i is filled in, Δ does not vanish. So $\eta_i^*(g^*)$ is also the Dehn twist along $\partial\Delta$. So in this case, we again have

$$\eta_i^*(g^*) = \varphi_i^* \circ \zeta_i(g).$$

If g is simple hyperbolic, the argument is similar to the above. Instead of having a twice punctured disk Δ , g^* is a spin defined by an a -punctured cylinder $P \subset S$ that does not contain any other punctures of \tilde{S} . Thus $\eta_i^*(g^*)$ is the spin defined by $P \subset S_i$. If we follow the other path, we see that $\zeta_i(g) \in G_i$ is also simple hyperbolic, and it is easy to check that $\varphi_i^* \circ \zeta_i(g)$ is the spin determined by P .

Now we claim that there are infinitely many essential hyperbolic elements g of G such that $\zeta_i(g) \in G_i$ are also essential elements. One example is demonstrated below, from which one can generate infinitely many essential elements by taking powers of generators or permuting generators.

Let \overline{S} denote the compactification of S , \tilde{c}_1, \tilde{c}_2 be two simple closed geodesics on \overline{S} so that \tilde{c}_1 and \tilde{c}_2 go through a and $\{\tilde{c}_1, \tilde{c}_2\}$ fills \overline{S} , and Q_1, \dots, Q_k be the disk components of $\overline{S} - \{\tilde{c}_1, \tilde{c}_2\}$. They are all polygons whose boundaries are geodesic segments (some of which may be identical). We assume that all the points x_2, \dots, x_n lie in Q_1 , say, and $a = x_1$ is a vertex of Q_1 . In Q_1 , we can take a parabolic basis $\tilde{e}_1, \dots, \tilde{e}_n \in \pi_1(\tilde{S}, a)$. That is, \tilde{e}_i is a loop representative that starts from a , goes around x_i exactly once in the clockwise direction, and then return to a .

Note that \tilde{c}_1 and \tilde{c}_2 also represent two nontrivial elements in $\pi_1(\tilde{S}, a)$. For $1 \leq i \leq n$, let $T_i \in G$ be the elements that correspond to \tilde{e}_i , and h_1, h_2 the elements corresponding to \tilde{c}_1, \tilde{c}_2 , respectively, under the isomorphism $\pi_1(\tilde{S}, a) \cong G$. Note that $T_i \in G$ are parabolic, while $h_1, h_2 \in G$ are hyperbolic.

We define

$$g = T_2 \circ (T_2 \circ T_3) \cdots \circ (T_2 \circ \cdots \circ T_n) \circ h_2 \circ h_1. \quad (3.3)$$

Consider the curve

$$\tilde{\lambda} = \tilde{c}_1 \cdot \tilde{c}_2 \cdot (\tilde{e}_n \cdots \tilde{e}_2) \cdots (\tilde{e}_3 \cdot \tilde{e}_2) \cdot \tilde{e}_2.$$

See Figure 1. We notice that the complement of $\tilde{\lambda}$ on Q_1 consists of one disk and $n - 1$ once punctured disks (Figure 1 above shows the portion of the curve $\tilde{\lambda}$ in Q_1), and the complement of $\tilde{\lambda}$ in $\tilde{S} - Q_1$ is determined by $\overline{S} - \{\tilde{c}_1, \tilde{c}_2\}$ that consists of disks only. It follows that $\tilde{\lambda}$ is a filling curve. Now the axis of g projects to a geodesic homotopic to the filling curve $\tilde{\lambda}$ on \tilde{S} .

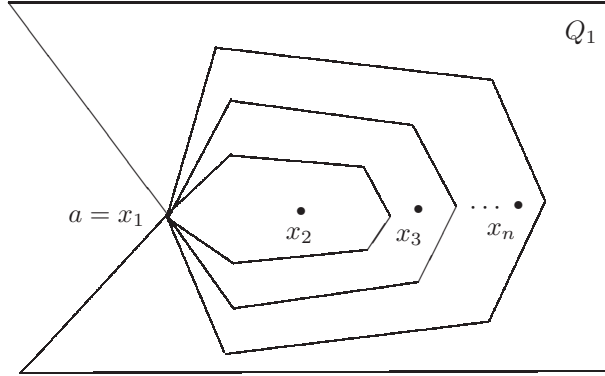


Figure 1 The portion of the curve $\tilde{\lambda}$ in Q_1

It follows that the element g defined in (3.3) is essential hyperbolic. Moreover, as x_i is filled in, the homomorphism $\pi_1(\tilde{S}, a) \rightarrow \pi_1(\tilde{S}_i, a)$ only kills the loop \tilde{e}_i . The image loop still fills \tilde{S}_i , which says that $\zeta_i(g)$ corresponds to the element that fills \tilde{S}_i . Therefore, $\zeta_i(g)$ is an essential hyperbolic element of G_i . Finally, we need

Lemma 3.2 *Let g be defined in (3.3). Then $\eta_i^*(g^*) \in \text{Mod}_{S_i}^a$ represents a pseudo-Anosov mapping class.*

Proof We know that $\zeta_i(g)$ is an essential hyperbolic element of G_i . Hence, by [7, Theorem 2], $\varphi_i^* \circ \zeta_i(g)$ is pseudo-Anosov. By Lemma 3.1, $\eta_i^*(g^*) = \varphi_i^* \circ \zeta_i(g)$. We conclude that $\eta_i^*(g^*) \in \text{Mod}_{S_i}^a$ is a pseudo-Anosov class.

Proof of Theorem 1.1 Let g be defined in (3.3). Assume that g^* is represented by (1.1) for α_1 and α_2 being boundaries of twice punctured disks Δ_1 and Δ_2 enclosing a . Write $\alpha_1 = \partial\Delta_1$ and $\alpha_2 = \partial\Delta_2$. Let $\{a, x_i\}$ be the punctures included in Δ_1 , and $\{a, x_j\}$ be the punctures included in Δ_2 . If $x_i = x_j$, then on the surface $S_i = S \cup \{x_i\}$, both $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ shrink to the puncture a . It follows that the mapping class $\eta_i^*(g^*)$ is trivial, which contradicts Lemma 3.2.

We assume that $x_i \neq x_j$. Observe that as x_i is filled in, the loop α_1 shrinks to the puncture a while α_2 remains noncontractible on S_i . This means that

$$\eta_i^* \left(\prod_i t_1^{r_i} \circ t_2^{-s_i} \right) = \hat{t}_2^{-\sum_i s_i}, \quad (3.4)$$

where \hat{t}_2 denotes the Dehn twist along the loop $\tilde{\alpha}_2$ regarded as a loop on S_i . But (3.4) is a power of Dehn twist that is a special kind of reducible mapping class. This again contradicts Lemma 3.2.

Finally, by the argument of [18, Section 4], we can conclude that there exist infinitely many pseudo-Anosov maps $f = g^* \in \mathcal{F}_0$ that are isotopic to products of Dehn twists along two simple curves that are trivial on \tilde{S} , if \tilde{S} contains at least one puncture. This completes the proof of Theorem 1.1.

4 Proof of Theorem 1.2

Proof of Theorem 1.2(1) We need to construct a pair of geodesics $\{\alpha_1, \alpha_2\}$ on S so that it fills S and $\tilde{\alpha}_1 = \tilde{\alpha}_2$. Then any products of form (1.1) with $\sum_i (r_i - s_i) = 0$ have the required properties.

Since $3p - 3 + n > 0$, we can take a simple closed geodesic $\tilde{\alpha}_0 \subset \tilde{S}$. Note that $\tilde{\alpha}_0$ can also be viewed as a curve on S whose geodesic representative is denoted by α_1 . Since \mathcal{F}_0 contains infinitely elements (see [7, Theorem 2]), we can pick an element $f \in \mathcal{F}_0$. By definition, f is pseudo-Anosov and is isotopic to the identity on \tilde{S} . By a theorem of Masur and Minsky [10], for sufficiently large integer k , the geodesic representative α_2 of $f^k(\alpha_1)$ together with α_1 itself fills S . We must have $\tilde{\alpha}_1 = \tilde{\alpha}_2 = \tilde{\alpha}_0$.

To prove the rest of the results, we let \mathcal{I} denote the subset of $\mathcal{F} \cap \mathcal{L}$ that consists of elements of form (1.1) for α_1 and α_2 being boundaries of twice punctured disks $\Delta_1, \Delta_2 \subset S$ both of which enclose a . Let $\chi \in \mathcal{F} \cap (\mathcal{L} - \mathcal{I})$. There is $g \in G$ such that $\chi = g^*$. Recall that $\eta_a^* : \text{Mod}_S^a \rightarrow \text{Mod}_{\tilde{S}}$ denotes the natural projection induced by (2.1). If $\tilde{\alpha}_1$ is contractible while $\tilde{\alpha}_2$ is not contractible on \tilde{S} , then

$$\eta_a^* \left(\prod_i t_1^{r_i} \circ t_2^{-s_i} \right) = t_{\tilde{\alpha}_2}^{\sum -s_i}. \quad (4.1)$$

On the other hand, $\eta_a^*(g^*)$ is a trivial mapping class. This is a contradiction.

Let us now consider the case that both $\tilde{\alpha}_i$, $i = 1, 2$, are not contractible on \tilde{S} . Clearly, if $\tilde{\alpha}_1$ is disjoint from $\tilde{\alpha}_2$, (1.1) projects to a multi-twist that is nontrivial. This is a contradiction.

If $\tilde{\alpha}_1$ intersects $\tilde{\alpha}_2$, then (1.1) projects to

$$\Theta = \prod_i t_{\tilde{\alpha}_1}^{r_i} \circ t_{\tilde{\alpha}_2}^{-s_i}.$$

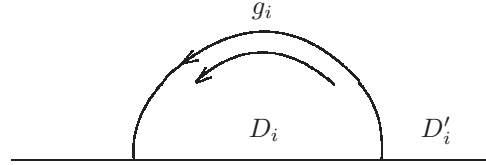
Let Σ be a system of geodesics such that one component \tilde{R} of $\tilde{S} - \Sigma$ contains $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$. We assume that $\tilde{R} - \Sigma$ is a union of disks, once punctured disks, and annuli with boundary loops in Σ . Then from [9], we know that $\Theta|_{\tilde{R}}$ is pseudo-Anosov. In particular, this implies that Θ is nontrivial.

We conclude that $\tilde{\alpha}_1 = \tilde{\alpha}_2$, i.e., $t_{\tilde{\alpha}_1} = t_{\tilde{\alpha}_2}$. Since $\chi \in \mathcal{F}$, it projects to the trivial mapping class. Hence, in order for the maps with form (1.1) to project to the identity, we must have

$$\sum_i (r_i - s_i) = 0.$$

Since g is essential, by [7, Theorem 2], g^* is pseudo-Anosov, which means that $\{\alpha_1, \alpha_2\}$ fills S . This proves Theorem 1.2(1).

Proof of Theorem 1.2(2) To prove Theorem 1.2(2), we first need to describe a lift τ_i of $t_{\tilde{\alpha}_i}$ to the hyperbolic plane \mathbb{H} . Let $\hat{\alpha}_i \subset \mathbb{H}$ be a geodesic with $\varrho(\hat{\alpha}_i) = \tilde{\alpha}_i$, and D_i, D'_i be the components of $\mathbb{H} - \hat{\alpha}_i$. A lift $\tau_i : \mathbb{H} \rightarrow \mathbb{H}$ with respect to D_i can be constructed as follows. Let $g_i \in G$ be the primitive simple hyperbolic element such that $g_i(D_i) = D_i$. We assume that g_i is oriented as shown in Figure 2.

Figure 2 A maximal element for the lift τ_i of $t_{\tilde{\alpha}_i}$

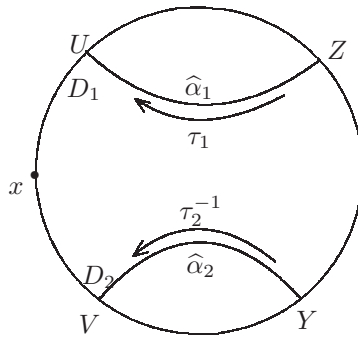
We then take an earthquake g_i -shift on D_i and leave D'_i fixed. Then we define $\tau_i : \mathbb{H} \rightarrow \mathbb{H}$ via G -invariance, which gives rise to a collection \mathcal{U}_i of half planes in \mathbb{H} in a partial order. In Figure 2, the arrow underneath $\hat{\alpha}_i$ indicates the direction of the motion of τ_i on D_i . There are infinitely many disjoint maximal elements $D_i(j)$ of \mathcal{U}_i , each of which is invariant under τ_i (D_i is just one of them). The restriction of τ_i to the complement H_i of the union of these disjoint maximal elements is the identity. It was shown in [17] that among the preimages $\{\varrho^{-1}(\tilde{\alpha}_i)\}$, one may choose a geodesic $\hat{\alpha}_i$ and hence a component $D_i \in \mathcal{U}_i$, so that the lifts τ_i with respect to D_i satisfy the conditions $\tau_i^* = t_i$.

Since $\tilde{\alpha}_1 = \tilde{\alpha}_2$, $\{\varrho^{-1}(\tilde{\alpha}_1)\} = \{\varrho^{-1}(\tilde{\alpha}_2)\}$. We see that, for any $D_1 \in \mathcal{U}_1$ and any $D_2 \in \mathcal{U}_2$, $\partial D_1 \cap \partial D_2 = \emptyset$. Suppose that there does not exist any $D_i \in \mathcal{U}_i$ such that $D_1 \cap D_2 \neq \emptyset$. Then for any $D_i \in \mathcal{U}_i$, either D_1 and D_2 are disjoint, or $D_1 \subset D_2$, or $D_2 \subset D_1$. All of these cases imply that τ_1 commutes with τ_2 , which is equivalent to that τ_1^* commutes with τ_2^* . But $\tau_i^* = t_i$. We assert that t_1 commutes with t_2 , which further implies that α_1 is disjoint from α_2 . This contradicts the fact that $\{\alpha_1, \alpha_2\}$ fills S .

We conclude that there exist $D_i \in \mathcal{U}_i$ such that $D_1 \cap D_2 \neq \emptyset$. The pair $\{D_1, D_2\}$ is drawn in Figure 3. Clearly, $D_1 \cup D_2 = \mathbb{H}$. Denote by (U, V) and $[U, V]$ the open and the closed circular arc on $\partial\mathbb{H}$ connecting the two labeled points U and V on $\partial\mathbb{H}$ without passing through any other labeled points. Let $x \in (U, V)$. Then $\tau_1^{r_1} \tau_2^{-s_1}(x) \in (U, V)$. By induction, one shows that for any finite product

$$\zeta = \prod_i \tau_1^{r_i} \tau_2^{-s_i},$$

$\zeta(x) \in (U, V)$, and its m -th iteration $\zeta^m(x) \in (U, V)$. But we know that $\zeta^* = g^*$ for some $g \in G$. In particular, the iterations of ζ and g on the boundary circle are the same. Hence $g^m(x)$ tends to a point $x_0 \in [U, V]$ as $m \rightarrow +\infty$. By definition, x_0 is the attracting fixed point of g . If $x_0 = U$ or V , then g would share a fixed point with a simple hyperbolic element of G , which is impossible. Therefore $x_0 \in (U, V)$.

Figure 3 Stable region (U, V) and unstable region (Y, Z) for the iteration of ζ

Similarly, as $m \rightarrow -\infty$, $g^m(x)$ tends to a point $y_0 \in (Y, Z)$ that is the repelling fixed point of g . It follows that the geodesic c connecting x_0 and y_0 , which is the axis of g , is completely in the interior of the region $D_1 \cap D_2$. This completes the proof of Theorem 1.2.

5 Proof of Theorem 1.3

Under the universal covering $\varrho : \mathbb{H} \rightarrow \tilde{S}$, the point a determines a set $A = \{\varrho^{-1}(a)\}$ that is a discrete subset of \mathbb{H} invariant under the action of G . Consider the complement $\mathbb{H} - A$. Then G keeps $\mathbb{H} - A$ invariant and ϱ restricts to a covering $\varrho : \mathbb{H} - A \rightarrow \tilde{S}$ with the covering group G . On the other hand, $\mathbb{H} - A$ can be thought of as a Riemann surface with infinite type. Let $\varrho_1 : \mathbb{H} \rightarrow \mathbb{H} - A$ be the universal covering with a covering group G_1 , which is isomorphic to the fundamental group $\pi_1(\mathbb{H} - A, \hat{a}_0)$ for a fixed point $\hat{a}_0 \in \mathbb{H} - A$ with $\varrho(\hat{a}_0) = a_0$. Observe that $\pi_1(\mathbb{H} - A, \hat{a}_0)$ is generated by infinitely many simple small loops δ_j each of which goes around a point $\hat{a}_j \in A$ once and zero times around any other \hat{a}_k for $j \neq k$ and $\hat{a}_k \in A$. Under the isomorphism of $\pi_1(\mathbb{H} - A, \hat{a}_0)$ onto G_1 , δ_j corresponds to a primitive parabolic element γ_j of G_1 . Let $z_j \in \mathbb{R}$ be the fixed point of γ_j .

Lemma 5.1 *Let $\{u_{ij}\} \in \mathbb{H} - A$ ($i = 1, 2, \dots$) be a sequence such that $u_{ij} \rightarrow z_j$ nontangentially as $i \rightarrow \infty$. Then $\varrho_1(u_{ij})$ tends to the puncture a_j .*

Proof Draw a horodisk D_j at z_j that is invariant under γ_j . Then $D_j / \langle \gamma_j \rangle$ is an a_j -punctured disk conformally embedded in $\mathbb{H} - A$. Since $u_{ij} \rightarrow z_j$ nontangentially, we may assume that u_{ij} are not γ_j -equivalent. It follows that $\varrho_1(u_{ij})$ are all distinct and tend to the puncture a_j , as asserted.

By construction, there is an exact sequence of covering groups

$$1 \longrightarrow G_1 \hookrightarrow \dot{G} \longrightarrow G \longrightarrow 1. \quad (5.1)$$

This is equivalent to that \dot{G} is a semi-product of G_1 and G . We need to examine the representatives f of g^* and its lifts to $\mathbb{H} - A$ under ϱ .

Lemma 5.2 *Fix a point $\hat{a} \in A$. Then there exists a quasiconformal map $\omega : \mathbb{H} \rightarrow \mathbb{H}$ such that the following conditions hold:*

- (1) *the map ω leaves $A = \{\varrho^{-1}(a)\} \subset \mathbb{H}$ invariant,*
- (2) *if ω is regarded as a selfmap of $\mathbb{H} - A$ onto itself, then $\varrho \circ \omega = f \circ \varrho$,*
- (3) *the map ω commutes with every element of G ,*
- (4) $\omega^{-1} \circ g(\hat{a}) = g \circ \omega^{-1}(\hat{a}) = \hat{a}$.

Proof By using topological arguments (see, for example, [7, Proposition 1]), we know that for $\hat{a} \in A$, we can construct a quasiconformal map ω of \mathbb{H} that satisfies (3) and (4). For convenience, we outline the construction as follows. Connect \hat{a} and $g(\hat{a})$ by a geodesic segment Γ . By fattening Γ , we obtain a flat ellipse E containing \hat{a} and $g(\hat{a})$. There is a quasiconformal map in E which sends \hat{a} to $g(\hat{a})$ and is the identity outside of E (see [7, Lemma 1] for the construction). We then define ω via G -invariance. Evidently, ω possesses properties (3) and (4).

To see that (1) is satisfied, we choose a point $\hat{a}' \in A$. There is an element $h \in G$ such that $h(\hat{a}) = \hat{a}'$. Then $\omega(\hat{a}') = \omega \circ h(\hat{a}) = h \circ \omega(\hat{a}) = h \circ g^{-1}(\hat{a}) \in A$ since G keeps A invariant. Hence (1) holds. Finally, from (4) and the construction of the Bers isomorphism (see [2, Theorem 9] or [7, Theorem 2]), we know that the map ω , if regarded as a map of $\mathbb{H} - A$ onto itself, descends to $f : S \rightarrow S$ under the restricted covering $\varrho : \mathbb{H} - A \rightarrow \tilde{S} - \{a\} \cong S$. So (2) is satisfied.

Therefore, we can lift the map $\omega : \mathbb{H} - A \rightarrow \mathbb{H} - A$ to $\widehat{\omega} : \mathbb{H} \rightarrow \mathbb{H}$ through the covering map ϱ_1 that satisfies

$$\varrho_1 \circ \widehat{\omega} = \omega \circ \varrho_1. \quad (5.2)$$

Clearly, the composition $\varrho_0 = \varrho \circ \varrho_1 : \mathbb{H} \rightarrow \widetilde{S}$ is a universal covering with the covering group \dot{G} . Combining with Lemma 5.2(4) and (5.1), we compute

$$\varrho_0 \circ \widehat{\omega} = (\varrho \circ \varrho_1) \circ \widehat{\omega} = \varrho \circ \omega \circ \varrho_1 = f \circ (\varrho \circ \varrho_1) = f \circ \varrho_0,$$

which says that $\widehat{\omega}$ is a lift of f through ϱ_0 . Hence, $\widehat{\omega}$ is of form $\dot{h}_1 \circ \widehat{f} \circ \dot{h}_2$ for $\dot{h}_1, \dot{h}_2 \in \dot{G}$, where \widehat{f} is one of the lifts of f .

More careful investigation on the map ω yields

Lemma 5.3 *As a map of $\mathbb{H} - A$ onto itself, ω has the following properties:*

- (1) *the restriction $\omega|_{\partial\mathbb{H}}$ is the identity,*
- (2) *the action of ω on A is fixed-point free,*
- (3) *for every simple hyperbolic or parabolic element h of G , the action of $h \circ \omega$ on A is also fixed-point free.*

Proof (1) is obvious since ω commutes with every element of G . Suppose that for some $\widehat{a}' \in A$ we have $\omega(\widehat{a}') = \widehat{a}'$. Choose $h \in G$ so that $h(\widehat{a}) = \widehat{a}'$. That is $\omega \circ h(\widehat{a}) = h(\widehat{a})$. Since ω commutes with each element of G , we get $h \circ \omega(\widehat{a}) = h(\widehat{a})$. By Lemma 5.2(4), $h \circ g(\widehat{a}) = h(\widehat{a})$. It follows that $g(\widehat{a}) = \widehat{a}$, contradicting the fact that g has no fixed point inside of \mathbb{H} . This proves (2).

To prove (3), we assume that for $\widehat{a}' \in A$, we have

$$h \circ \omega(\widehat{a}') = \widehat{a}'. \quad (5.3)$$

Choose $g_0 \in G$ so that $\widehat{a}' = g_0(\widehat{a})$. Then (5.3) becomes $h \circ \omega \circ (g_0(\widehat{a})) = g_0(\widehat{a})$. Since ω commutes with h , $h \circ g_0 \circ \omega(\widehat{a}) = g_0(\widehat{a})$, or $g_0^{-1} \circ h \circ g_0 \circ \omega(\widehat{a}) = \widehat{a}$. Set $g_0^{-1} \circ h \circ g_0 = h_0$. Then h_0 is also parabolic or simple hyperbolic, depending on whether h is parabolic or simple hyperbolic. Thus we obtain

$$h_0 \circ \omega(\widehat{a}) = \widehat{a}. \quad (5.4)$$

Now from Lemma 5.2(4), $\omega(\widehat{a}) = g(\widehat{a})$. It follows from (5.4) that

$$h_0 \circ g(\widehat{a}) = \widehat{a}. \quad (5.5)$$

Notice that $h_0 \in G$ is either parabolic or simple hyperbolic, while g is essential. We see that $h_0 \circ g \neq \text{id}$. From (5.5), we conclude that $h_0 \circ g$ fixes a point inside of \mathbb{H} and thus it is an elliptic Möbius transformation. This contradicts that G is a torsion free Fuchsian group. This proves (3).

Proof of Theorem 1.3 It suffices to show that there are infinitely many pseudo-Anosov mapping classes on \dot{S} that are isotopic to f on S as a' is filled in.

Let $h \in G$ be any simple hyperbolic element. Consider the map $h \circ \omega : \mathbb{H} - A \rightarrow \mathbb{H} - A$. By Lemma 5.3(1), $\omega|_{\partial\mathbb{H}} = \text{id}$. Hence $h \circ \omega|_{\partial\mathbb{H}}$ fixes no parabolic fixed point of G . By Lemma 5.3(3), $h \circ \omega|_A$ is fixed point free. Let $\widehat{\omega}_0 : \mathbb{H} \rightarrow \mathbb{H}$ be a lift of $h \circ \omega|_{\mathbb{R}}$ which satisfies

$$\varrho_1 \circ \widehat{\omega}_0 = h \circ \omega \circ \varrho_1. \quad (5.6)$$

Suppose that $\widehat{\omega}_0$ fixes some fixed point z_j of γ_j . Choose a sequence $\{u_{ij}\} \in \mathbb{H}$ that tends to the fixed point z_j of γ_j non-tangentially. By (5.6), for all u_{ij} , $i = 1, 2, \dots$, we have

$$\varrho_1 \circ \widehat{\omega}_0(u_{ij}) = h \circ \omega \circ \varrho_1(u_{ij}). \quad (5.7)$$

Let $i \rightarrow \infty$. Then $u_{ij} \rightarrow z_j$. By continuity, we obtain

$$\varrho_1 \circ \widehat{\omega}_0(z_j) = h \circ \omega \circ \varrho_1(z_j).$$

By assumption, we have that $\widehat{\omega}_0$ fixes z_j . So $\varrho_1(z_j) = h \circ \omega \circ \varrho_1(z_j)$. By Lemma 5.1, we get $\lim_{i \rightarrow \infty} \varrho_1(u_{ij}) = a_j$; that is, $h \circ \omega$ fixes a_j . This contradicts Lemma 5.3(3).

We conclude that $\widehat{\omega}_0$ cannot fix the fixed point of any parabolic element γ_j of \dot{G} that emerges from a point in the set A .

We also need to prove that $\widehat{\omega}_0$ does not fix any parabolic fixed point of \dot{G} other than z_j . Suppose for the contrary, we assume that $\widehat{\omega}_0$ fixes a parabolic fixed point x of \dot{G} . Let $\dot{\gamma} \in \dot{G}$ be the parabolic element fixing x . From (5.1), there is a nontrivial element $\gamma \in G$ such that

$$\varrho_1 \circ \dot{\gamma}^m = \gamma^m \circ \varrho_1 \quad (5.8)$$

for any integer m . Since $\dot{\gamma} \in \dot{G}$ is parabolic, for any $u \in \mathbb{H}$, both $\dot{\gamma}^m(u)$ and $\dot{\gamma}^{-m}(u)$ tend to the fixed point x of $\dot{\gamma}$ in \mathbb{R} . From (5.8), we get that both $\gamma^m \varrho_1(u)$ and $\gamma^{-m} \varrho_1(u)$ tend to $\varrho_1(x)$. This implies that $\gamma \in G$ is parabolic and its fixed point is $\varrho_1(x)$. It follows that x projects (under ϱ_1) to a parabolic fixed point of G . By hypothesis, $\widehat{\omega}_0(x) = x$. We thus obtain

$$\varrho_1(x) = \varrho_1 \circ \widehat{\omega}_0(x) = h \circ \omega \circ \varrho_1(x),$$

which tells us that $h \circ \omega$ fixes $\varrho_1(x)$, a parabolic fixed point of G . By Lemma 5.3(1), we have $\omega|_{\partial \mathbb{H}} = \text{id}$. We conclude that $h(\varrho_1(x)) = \varrho_1(x)$. But h is simple hyperbolic; it can not fix a parabolic fixed point of G . This contradiction proves that $\widehat{\omega}_0$ does not fix any parabolic fixed point of \dot{G} other than z_j , and hence $\widehat{\omega}_0$ does not fix any parabolic fixed point of \dot{G} .

Now from (5.1) we know that $\varrho_0 : \mathbb{H} \rightarrow \tilde{S}$ is a covering map with the group \dot{G} . To see that $\widehat{\omega}_0$ projects via ϱ_0 to the map f that represents $g^* \in \mathcal{F}_0$, we notice that $\varrho_0 = \varrho \circ \varrho_1$. From (5.1), (5.6), (5.7) and Lemma 5.2(2), one computes

$$\varrho_0 \circ \widehat{\omega}_0 = \varrho \circ \varrho_1 \circ \widehat{\omega}_0 = \varrho \circ h \circ \omega \circ \varrho_1 = \varrho \circ \omega \circ \varrho_1 = f \circ (\varrho \circ \varrho_1) = f \circ \varrho_0.$$

It follows that $\widehat{\omega}_0 \dot{G} \widehat{\omega}_0^{-1} = \dot{G}$ and $\widehat{\omega}_0$ projects to f . Moreover, its equivalence class $[\widehat{\omega}_0]$ is an element of $\text{mod } S$. Let $\psi : F(S) \rightarrow T(\dot{S})$ denote a Bers isomorphism. Then ψ induces an isomorphism ψ^* of $\text{mod } S$ onto $\text{Mod}_S^{a'}$. By the above argument, we see that $\widehat{\omega}_0^* = \psi^*([\widehat{\omega}_0]) \in \text{Mod}_S^{a'}$ projects to the mapping class g^* .

Now suppose that $\widehat{\omega}_0^*$ is a reducible mapping class on \dot{S} that is reduced by a curve system $\{c_1, c_2, \dots, c_s\}$ for $s \geq 1$. By taking a suitable power, we may assume that $\widehat{\omega}_0^*$ leaves each curve in the system invariant. If $s \geq 2$, then at least one curve in the system, c_1 , say, is also noncontractible on S . Let c'_1 denote the corresponding curve on S . This implies that g^* leaves c'_1 invariant. This contradicts the fact that g^* is pseudo-Anosov. So the only possibility is that $s = 1$ and c'_1 is a trivial curve on S . That is, c_1 is a curve that is the boundary of a twice punctured disk enclosing two punctures, one of which is a' . But in this case, by [15, Lemmas 3.1 and 3.2], we have that $\widehat{\omega}_0$ fixes a parabolic fixed point of \dot{G} . This also contradicts the above argument.

We conclude that $\widehat{\omega}_0^* = \psi^*([\widehat{\omega}_0]) \in \text{Mod}_S^{a'}$ is a pseudo-Anosov element projecting to g^* . Since there are infinitely many simple hyperbolic elements in G , there are infinitely many pseudo-Anosov elements $\widehat{\omega}_0^*$ in $\text{Mod}_S^{a'}$. This completes the proof of Theorem 1.3.

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References

- [1] Ahlfors, L. V. and Bers, L., Riemann's mapping theorem for variable metrics, *Ann. Math.*, **72**(2), 1960, 385–404.
- [2] Bers, L., Fiber spaces over Teichmüller spaces, *Acta. Math.*, **130**, 1973, 89–126.
- [3] Bers, L., An extremal problem for quasiconformal mappings and a theorem by Thurston, *Acta. Math.*, **141**, 1978, 73–98.
- [4] Birman, J. S., Braids, Links and Mapping Class Groups, *Annals of Mathematics Studies*, **82**, Princeton University Press, Princeton, 1974.
- [5] Fathi, A., Laudenbach, F. and Poenaru, V., Travaux de Thurston sur les surfaces, *Seminaire Orsay, Asterisque*, Vol. 66–67, Soc. Math. France, Paris, 1979.
- [6] Hubert, P. and Lanneau E., Veech group without parabolic elements, *Duke Math. J.*, **133**(2), 2006, 335–346.
- [7] Kra, I., On the Nielsen-Thurston-Bers type of some self-maps of Riemann surfaces, *Acta. Math.*, **146**, 1981, 231–270.
- [8] Imai, Y., Ito, M. and Yamamoto, H., On the Nielsen-Thurston-Bers type of some self-maps of Riemann surfaces with two specified points, *Osaka J. Math.*, **40**, 2003, 669–685.
- [9] Long, D. D., Constructing pseudo-Anosov maps, *Knot Theory and Manifolds, Lecture Notes in Math.*, **1144**, Springer-Verlag, New York, 1985, 108–114.
- [10] Masur, H. and Minsky, Y., Geometry of the complex of curves I: Hyperbolicity, *Invent. Math.*, **138**, 1999, 103–149.
- [11] Nag, S., Non-geodesic discs embedded in Teichmüller spaces, *Amer. J. Math.*, **104**, 1982, 339–408.
- [12] Penner, R. C., The action of the mapping class group on isotopy classes of curves and arcs in surfaces, Thesis, Massachusetts Institute of Technology, 1982.
- [13] Penner, R. C., A construction of pseudo-Anosov homeomorphisms, *Proc. Amer. Math. Soc.*, **104**, 1988, 1–19.
- [14] Thurston, W. P., On the geometry and dynamics of diffeomorphisms of surfaces, *Bull. Amer. Math. Soc.*, **19**, 1988, 417–431.
- [15] Zhang, C. H., Hyperbolic mapping classes and their lifts on Bers fiber space, *Chin. Ann. Math.*, **28B**(1), 2007, 55–66.
- [16] Zhang, C. H., Remarks on Thurston's construction of pseudo-Anosov maps of Riemann surfaces, *Chin. Ann. Math.*, **29B**(1), 2007, 85–94.
- [17] Zhang, C. H., Pseudo-Anosov maps and fixed points of boundary homeomorphisms compatible with a Fuchsian group, *Osaka J. Math.*, to appear, 2009.
- [18] Zhang, C. H., Hyperbolic lengths of filling geodesics of Riemann surfaces with punctures, *Osaka J. Math.*, **45**, 2008, 773–787.