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Two-Dimensional Riemann Problems for the Compressible Euler System**

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(*Dedicated to Professor Andrew Majda for his* 60*th Birthday*)

Abstract Riemann problems for the compressible Euler system in two space dimensions are complicated and difficult, but a viable alternative remains missing. The author lists merits of one-dimensional Riemann problems and compares them with those for the current two-dimensional Riemann problems, to illustrate their worthiness. Two-dimensional Riemann problems are approached via the methodology promoted by Andy Majda in the spirits of modern applied mathematics; that is, simplified model is built via asymptotic analysis, numerical simulation and theoretical analysis. A simplified model called the pressure gradient system is derived from the full Euler system via an asymptotic process. State-of-the-art numerical methods in numerical simulations are used to discern small-scale structures of the solutions, e.g., semi-hyperbolic patches. Analytical methods are used to establish the validity of the structure revealed in the numerical simulation. The entire process, used in many of Majda's programs, is shown here for the two-dimensional Riemann problems for the compressible Euler systems of conservation laws.

Keywords Characteristic decomposition, Guderley reflection, Hodograph transform, Pressure gradient system, Self-similar, Semi-hyperbolic wave, Triple point paradox, Riemann problem, Riemann variable
2000 MR Subject Classification 35L65, 35J70, 35R35, 35J65

1 Systems

We consider the two-dimensional (2-D) compressible Euler system

$$\begin{cases}\rho_t + \nabla\cdot(\rho\mathbf{u}) = 0,\\ (\rho\mathbf{u})_t + \nabla\cdot(\rho\mathbf{u}\otimes\mathbf{u} + pI) = 0,\\ (\rho E)_t + \nabla\cdot(\rho E\mathbf{u} + p\mathbf{u}) = 0,\end{cases} \tag{1.1}$$

where ρ is density, $\mathbf{u}$ is velocity vector, p is pressure, $E = \frac{|\mathbf{u}|^2}{2} + \frac{p}{(\gamma-1)\rho}$ is the total energy density, and $\gamma > 1$ is the gas constant. We also consider the so-called pressure gradient system

$$\begin{cases}u_t + p_x = 0,\\ v_t + p_y = 0,\\ E_t + (up)_x + (vp)_y = 0,\end{cases} \tag{1.2}$$

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where $E = p + \frac{u^2+v^2}{2}$. Cauchy problems for both systems are open.

2 Search for Model

The full Euler system is known to be hard. Simplified models are appreciated. We propose to asymptotically replace the Euler by

$$\begin{cases} \rho \longrightarrow 1 + \dfrac{1}{\gamma}\rho, \\ \mathbf{u} \longrightarrow \dfrac{1}{\gamma}\mathbf{u}, \\ p \longrightarrow \dfrac{1}{\gamma}p, \end{cases} \tag{2.1}$$

in the limit $\gamma \to \infty$. To leading orders of the equations we obtain the pressure gradient system.

In the asymptotic process we note that the sound speed

$$c = \sqrt{\frac{\gamma p}{\rho}} \sim \sqrt{p}$$

remains at order one, thus our asymptotic system catches acoustic waves.

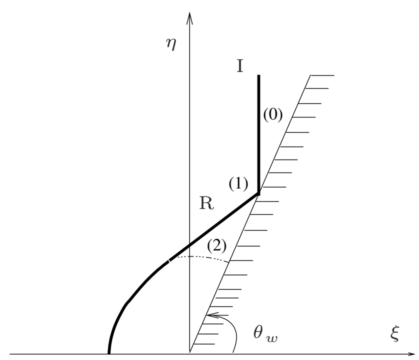

Figure 1 Regular reflection

The pressure decouples from the velocity field to form

$$\left(\frac{p_t}{p}\right)_t - \Delta p = 0,$$

which is indeed simpler than the Euler system. The pressure gradient system is the first two-dimensional system of two or more equations to have the existence established of a global regular reflection on a wedge (see Figure 1 and [42]). Other progresses are in [1, 6, 11, 14, 31, 32, 40, 41].

3 Riemann Problems: 2-D

We propose Riemann problems as initial-value problems in which the initial values are independent of the spatial radius

$$r = \sqrt{x^2 + y^2}\,, \quad (x, y) \in \mathbb{R}^2.$$

A typical realization is to have four constants instead of an arbitrary function of the polar angle θ (see Figure 2).

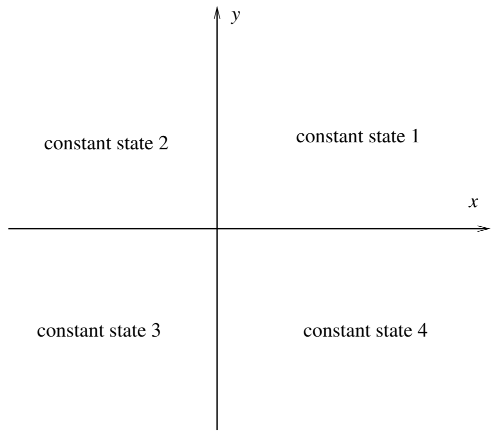

Figure 2 Four-constant Riemann problem

The most important feature of the Riemann problems is that we can propose to look for the so-called self-similar solutions that depend only on the variables $\xi = \frac{x}{t}$, $\eta = \frac{y}{t}$. In this regard, we include some initial-boundary value problems, such as a planar shock hitting a straight wedge, Riemann problems, as long as the solutions are self-similar.

There are other possibilities for Riemann problems. For example, one may search for elementary waves (rather than working with given initial data) as Riemann problems, as James Glimm and associates once attempted (see Figure 3). By elementary waves, we mean that waves will make up general solutions.

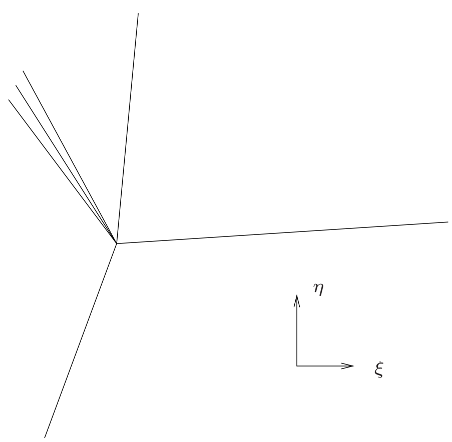

Figure 3 Elementary waves

To judge 2-D Riemann problems, let us list the merits of 1-D Riemann problems:

(1) Building blocks (for solutions to Cauchy problems);

(2) Asymptotic states (of solutions as time approaches infinity);

(3) Elementary waves;

(4) Simple and easy.

And here are features of our 2-D Riemann problems:

(1) Complicated and hard;

(2) Their solutions reveal a lot of internal structures of solutions;

(3) Their solutions depend only on two independent variables;

(4) They can be used to build general approximate solutions.

So it is clear that an ideal generalization of the one-dimensional Riemann problem is not available, the current 2-D Riemann problem is below expectation, but it is still valuable to use to get into two-dimensional problems.

4 Early Results

There are long-standing interests in multi-dimensional piecewise smooth solutions (see, e.g., [8, 25, 27]). In 1986, the four-constant two-dimensional Riemann problem for a typical scalar conservation law was solved (see [34, 37]). For the full Euler, a set of educated guesses of solutions to the four-wave two-dimensional Riemann problems was proposed in 1990 (see [38]). (The four-wave two-dimensional Riemann problems are special cases of the four-constant two-dimensional Riemann problems.) Assuming axial symmetry, the two-dimensional Riemann problems were solved for the Euler in 1996 (see [39, 44]). The axial case catches solutions that exhibit structures of an eye with an eye-wall that are important features of a hurricane.

We would like to mention an interesting structure of a solution to the two-dimensional Riemann problem for the scalar conservation law

$$u_t + \left(\frac{u^2}{2}\right)_x + \left(\frac{u^3}{3}\right)_y = 0$$

from Guckenheimer [9], sketched in Figure 4. The initial data is a triple discontinuity meeting at

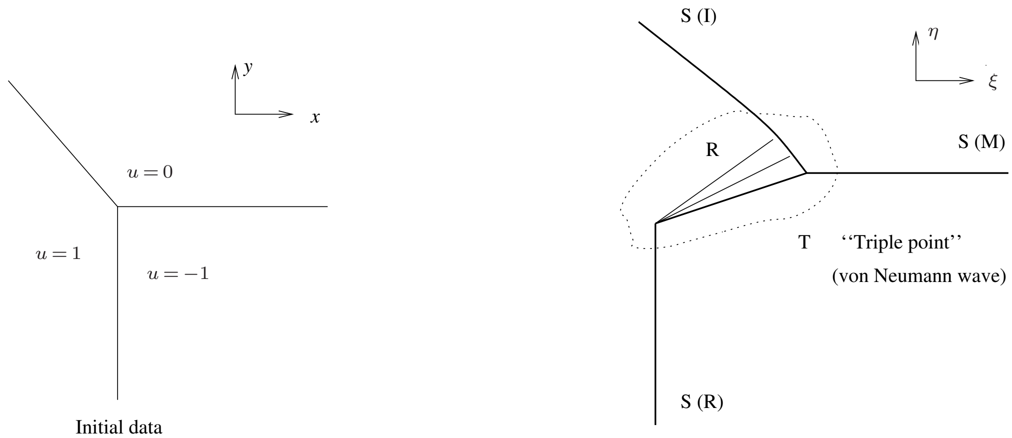

Figure 4 Guckenheimer solution

one point, but the solution does not keep the simple structure; instead, the triple point exhibits the so-called von Neumann structure with a small rarefaction wave (R). In Figure 4, S(I) is interpreted as incident shock, S(M) as Mach stem, while S(R) as reflected shock in analogy to the Mach reflection in a planar shock hitting a wedge.

This brings up Euler system's von Neumann triple point paradox, for which John Hunter and collaborators have produced numerical evidence of Guderley reflection (see [10, 35, 36]). See [30] for physical experimental evidence. An illustration is shown in Figure 5.

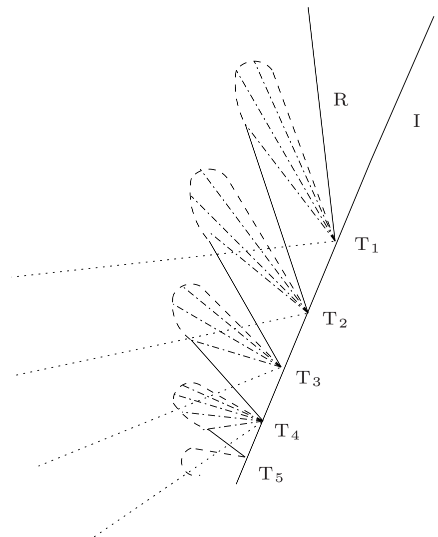

Figure 5 Guderley reflection

In addition, we find that the 4R interaction (two forward, two backward) of the two-dimensional Riemann problems contains patches of solutions adjacent to sonic curves, that are like the small waves in the Guderley reflection. See Figure 6, which is from [7]. These small patches are hyperbolic, but one family of characteristics fails to connect to infinity. So they will be called semi-hyperbolic. Other similar numerical results are in [2, 12, 13, 29].

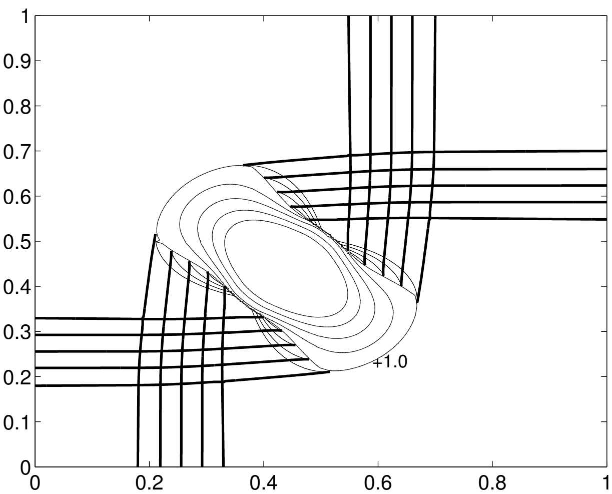

Figure 6 Four semi-hyperbolic patches (netted by thin and heavy curves)

We are interested in constructing the pieces of semi-hyperbolic waves adjacent to the sonic curves. To place the semi-hyperbolic waves in perspective, we notice three wave types:

(1) Simple waves;

(2) Interactions of binary planar waves;

(3) Semi-hyperbolic waves.

A simple wave occupies a region in which one family of characteristics are all straight lines. A planar wave is a simple wave in which one family of characteristics are straight and parallel. Interaction of two planar waves is typically not a simple wave anymore. Semi-hyperbolic waves are locally hyperbolic, but one family of characteristics all start on and end on the sonic curve (or a transonic shock wave). The wave fans in the Guderley reflection of Figure 5 and the four patches in the 4R interaction of Figure 6 are semi-hyperbolic.

5 Self-similar Euler: Simple Waves

We develop analytic methods. First we write the self-similar form of the Euler system:

$$\begin{cases}(u-\xi)i_\xi+(v-\eta)i_\eta+2\kappa i(u_\xi+v_\eta)=0,\\ (u-\xi)u_\xi+(v-\eta)u_\eta+i_\xi=0,\\ (u-\xi)v_\xi+(v-\eta)v_\eta+i_\eta=0,\end{cases} \tag{5.1}$$

where $i=\frac{c^2}{\gamma-1}$, $\kappa=\frac{\gamma-1}{2}$. The two nonlinear eigenvalues are

$$\Lambda_\pm=\frac{(u-\xi)(v-\eta)\pm c\sqrt{(u-\xi)^2+(v-\eta)^2-c^2}}{(u-\xi)^2-c^2}. \tag{5.2}$$

For catching the simple waves, we formulate the identities (see [21]):

$$\begin{cases}\partial^+(\partial^-u)+\dfrac{\partial^+\Lambda_--\partial^-\Lambda_+}{\Lambda_+-\Lambda_-}\partial^-u=\dfrac{\Lambda_+\Lambda_-}{\Lambda_+-\Lambda_-}\Big[\dfrac{\partial^-\Lambda_-}{\Lambda_-^2}\partial^+u-\dfrac{\partial^+\Lambda_+}{\Lambda_+^2}\partial^-u\Big],\\ \partial^-(\partial^+u)+\dfrac{\partial^+\Lambda_--\partial^-\Lambda_+}{\Lambda_+-\Lambda_-}\partial^+u=\dfrac{\Lambda_+\Lambda_-}{\Lambda_+-\Lambda_-}\Big[\dfrac{\partial^-\Lambda_-}{\Lambda_-^2}\partial^+u-\dfrac{\partial^+\Lambda_+}{\Lambda_+^2}\partial^-u\Big],\end{cases}$$

where

$$\partial^\pm\Lambda_\pm=[\partial_U\Lambda_\pm-\Lambda_\mp^{-1}\partial_V\Lambda_\pm-(\gamma-1)\partial_{c^2}\Lambda_\pm(U-\Lambda_\mp^{-1}V)]\partial^\pm u.$$

The notations are $\partial^\pm=\partial_\xi+\Lambda_\pm\partial_\eta$, $U=u-\xi$, $V=v-\eta$, and $\Lambda_\pm$ are regarded as functions of three independent variables (U,V,c^2) in $\partial_U\Lambda_\pm$, $\partial_V\Lambda_\pm$, and $\partial_{c^2}\Lambda_\pm$.

We then see that $\partial^-u=0$ along an entire curve of plus characteristics, provided that it is zero at any point at all. This way, we conclude that a wave that is adjacent to a constant state must be a simple wave.

6 Main Approaches

We note that we can approach the theoretical construction of solutions via two directions:

(1) Hodograph transform;

(2) Direct characteristics decomposition.

We recall the classical hodograph transform $(x, y) \to (u, v)$ is for a homogeneous system

$$\begin{cases} u_x + a(u,v)u_y + b(u,v)v_y = 0, \\ v_x + c(u,v)u_y + d(u,v)v_y = 0, \end{cases} \tag{6.1}$$

and the system in the (u, v) plane is linear.

For possible hodograph transform for the self-similar Euler (5.1), we again use $(\xi, \eta) \to (u, v)$ and regard i as a function of (u, v): $i = i(u, v)$. It brings the system to a single equation

$$(c^2 - i_u^2)i_{vv} + 2i_u i_v i_{uv} + (c^2 - i_v^2)i_{uu} = i_u^2 + i_v^2 - 2c^2 \tag{6.2}$$

in the hodograph plane. This equation is not linear, but it is linearly degenerate.

To show it, we introduce the inclination angles of characteristics (α, β):

$$\tan\alpha = \Lambda_+, \quad \tan\beta = \Lambda_-.$$

Note that $\omega = \frac{\alpha-\beta}{2}$ is the so-called (pseudo-)Mach angle (see [5]). The system in the hodograph plane is

$$\begin{cases} \overline{\partial}_+\alpha = \dfrac{1+\gamma}{4c} \cdot \sin(\alpha - \beta) \cdot (m - \tan^2\omega), \\ \overline{\partial}_-\beta = \dfrac{1+\gamma}{4c} \cdot \sin(\alpha - \beta) \cdot (m - \tan^2\omega), \\ \partial_0 c = \kappa \dfrac{\cos\frac{\alpha+\beta}{2}}{\sin\omega}, \end{cases} \tag{6.3}$$

with

$$\overline{\partial}_+ c = -\kappa, \quad \overline{\partial}_- c = \kappa,$$

where

$$\overline{\partial}_+ = \sin\beta\partial_u - \cos\beta\partial_v, \quad \overline{\partial}_- = \sin\alpha\partial_u - \cos\alpha\partial_v, \quad \partial_0 = \partial_u,$$

and

$$m = \frac{3-\gamma}{1+\gamma}.$$

It is linearly degenerate since the variable α is differentiated along a direction determined by β.

We have obtained other identities to handle high-order estimates and the one-to-one correspondence between the self-similar plane and the hodograph plane (see [22]).

7 Application 1: Two Rarefaction Wave Interaction

It is also called a wedge of gas expanding into vacuum (see Figure 7).

It was considered by Suchkov [33] in 1963 and by Levine and Mackie [15, 26] in 1968. The interaction zone is illustrated in Figure 8.

Wave interaction region in the hodograph plane is illustrated in Figure 9.

We obtain the solutions, one typical solution is shown in Figure 10, where θ_s is defined by $\tan^2\theta_s = m$.

These results are from the paper with Jiequan Li (see [22]).

Why does it work well here, but not well in the steady case? The relation

$$\xi = u + i_u, \quad \eta = v + i_v$$

is different from the steady case

$$0 = u + i_u, \quad 0 = v + i_v.$$

The hodograph transform for (5.1) was carried out a while ago in [28], but it seems that it has not been used well until recently (see [16, 17]).

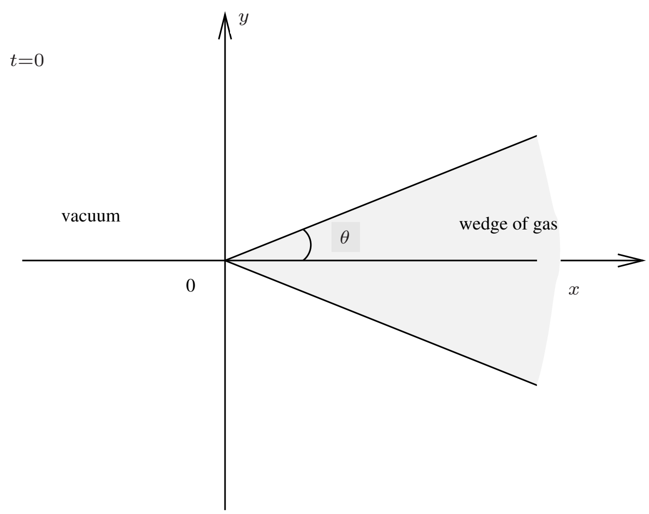

Figure 7 Initial set-up of a wedge of gas expanding into vacuum

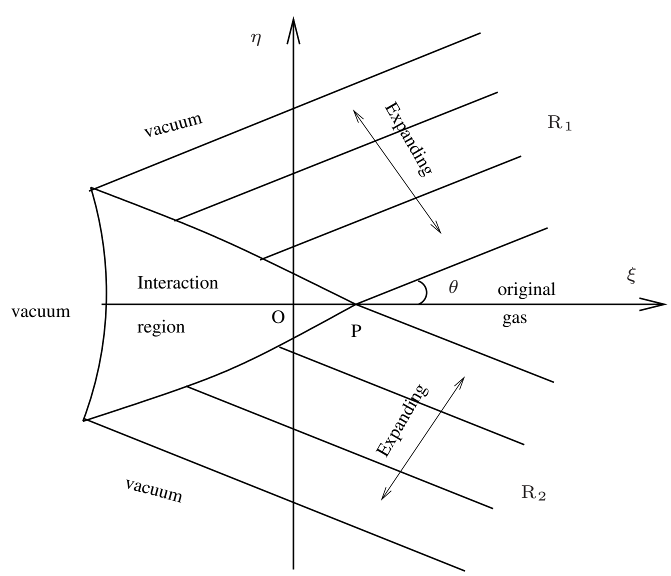

Figure 8 Interaction position of a wedge of gas expanding into vacuum

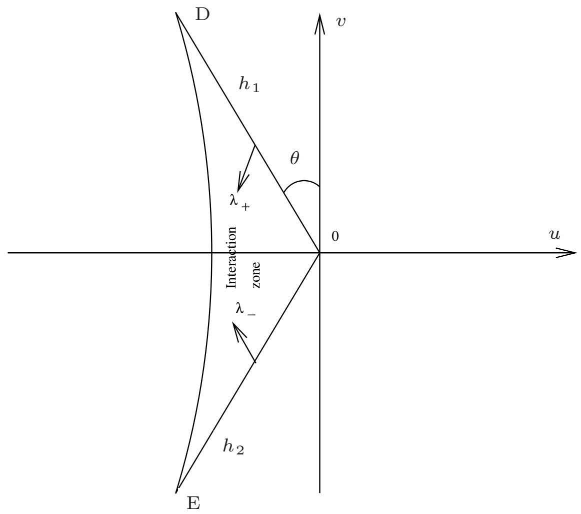

Figure 9 Interaction zone in the hodograph plane

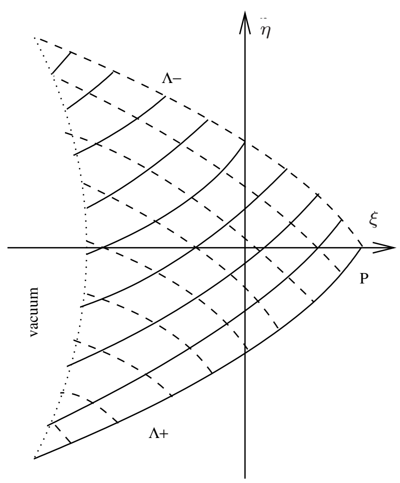

Figure 10 $\theta_s < \theta < 2\theta_s$

8 Direct Method

We want to by-pass the hodograph transform. We obtain (see [4, 19])

$$\begin{cases} \overline{\partial}^{\,+}(-\beta + \psi(\omega)) = \dfrac{\sin^2\omega[\cos(2\omega) - \kappa]}{c(\kappa + \sin^2\omega)}, \\ \overline{\partial}^{\,-}(\alpha + \psi(\omega)) = \dfrac{\sin^2\omega[\cos(2\omega) - \kappa]}{c(\kappa + \sin^2\omega)}, \\ \overline{\partial}^{\,0}[c^2(1 + \kappa M^2)] = 2c\kappa M, \end{cases} \tag{8.1}$$

where

$$M^2 = \frac{u^2 + v^2}{c^2}, \quad \overline{\partial}^{\,+} = \cos\alpha\partial_\xi + \sin\alpha\partial_\eta, \quad \overline{\partial}^{\,-} = \cos\beta\partial_\xi + \sin\beta\partial_\eta,$$
$$\overline{\partial}^{\,0} := -\cos\zeta\partial_\xi - \sin\zeta\partial_\eta, \quad \zeta = \frac{\alpha + \beta}{2},$$

and

$$\psi(\omega) := \sqrt{\frac{\gamma+1}{\gamma-1}}\,\arctan\Big(\sqrt{\frac{\gamma-1}{\gamma+1}}\,\cot\omega\Big). \tag{8.2}$$

The Riemann variables $\psi - \beta$ and $\psi + \alpha$ correspond to the classical Riemann invariants for homogeneous systems.

Applying the direct method to the expansion of a wedge of gas, we obtain the same conclusion as before. The computations are not easier than the hodograph method, though.

9 Application 2: Semi-hyperbolic Patches

Next in line is the semi-hyperbolic patch. A semi-hyperbolic patch has one family of characteristics that start and end on sonic curves or transonic shocks. They are abound (see Figure 11).

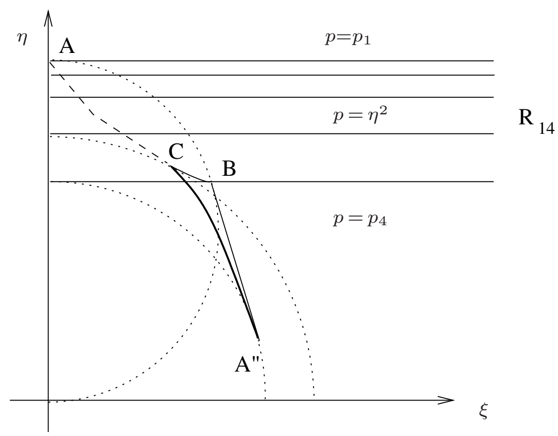

Figure 11 Set-up for construction of a semi-hyperbolic patch

In [32], we consider the pressure gradient system

$$p_{\theta\theta} - \frac{r^2(r^2-p)}{p}p_{rr} + \frac{rp_r}{p}\Big(p + \frac{r^3p_r}{p} - 2r^2\Big) = 0$$

in self-similar polar coordinates (r, θ). It decomposes to

$$\begin{aligned} \partial^+\partial^- p &= q\cdot(\partial^+ p - \partial^- p)\cdot\partial^- p,\\ \partial^-\partial^+ p &= q\cdot(\partial^- p - \partial^+ p)\cdot\partial^+ p, \end{aligned} \tag{9.1}$$

where $q := \frac{r^2}{4p(r^2-p)}$. Here $\partial^\pm$ are again derivatives along characteristics. We consider the set-up as in Figure 11. The horizontal planar wave $p = \eta^2$ is given up to the boundary AB, and the curve BC is given as a convex characteristic curve of the minus family, with point C being sonic. The characteristics in the domain ABC are drawn in Figure 12.

Maximum principle holds for $\partial^\pm p$ in the semi-hyperbolic region. A cute proof using Figure 13 is in [32].

Envelope forms in the simple wave region, thus shock is present (see Figure 14). Thus the patch ABA$''$CA is semi-hyperbolic. We have the same conclusion for the Euler (see [24]). Note the new decomposition

$$\begin{cases} c\overline{\partial}^+\Big(\dfrac{\overline{\partial}^-c}{\sin^2\omega}\Big) + \Big[\dfrac{8\sin^2\omega}{\gamma-1}+2\Big]\overline{\partial}^+c\Big(\dfrac{\overline{\partial}^-c}{\sin^2\omega}\Big) = \dfrac{\nu}{\cos^2\omega}\Big(\dfrac{\overline{\partial}^-c}{\sin^2\omega}\Big)[\overline{\partial}^+c+\overline{\partial}^-c], \\ c\overline{\partial}^-\Big(\dfrac{\overline{\partial}^+c}{\sin^2\omega}\Big) + \Big[\dfrac{8\sin^2\omega}{\gamma-1}+2\Big]\overline{\partial}^-c\Big(\dfrac{\overline{\partial}^+c}{\sin^2\omega}\Big) = \dfrac{\nu}{\cos^2\omega}\Big(\dfrac{\overline{\partial}^+c}{\sin^2\omega}\Big)[\overline{\partial}^-c+\overline{\partial}^+c]. \end{cases}$$

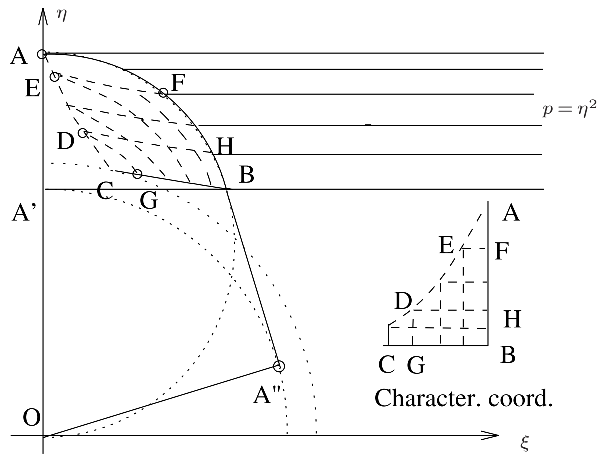

Figure 12 Solution for a semi-hyperbolic patch AFHBGCDEA

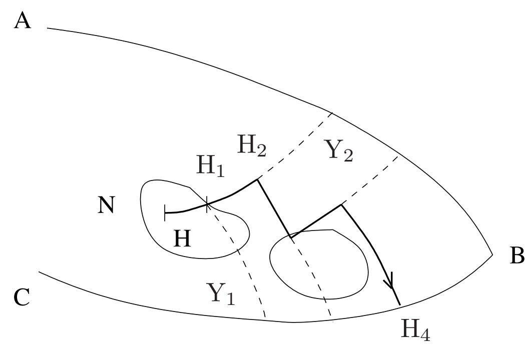

Figure 13 Establishing the maximum principle

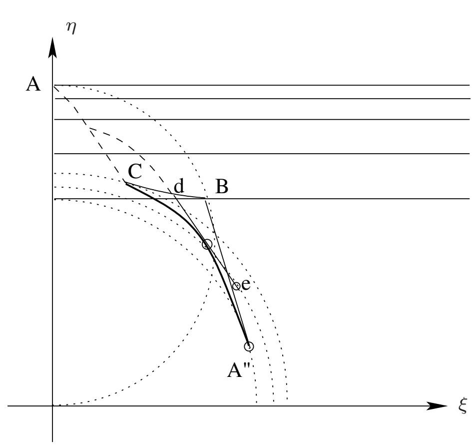

Figure 14 Envelope forms along CA$''$

In the interaction of 4R case, we note that one simple case occurs when the central subsonic region degenerates to vacuum so that no semi-hyperbolic wave is reflected, and a global continuous solution is obtained (see [23]).

10 Summary

We bring new life to the hodograph method, so that it also works for the Euler system of 3×3. We find Riemann variables

$$\{-\beta + \psi(\omega), \alpha + \psi(\omega)\},$$

where

$$\psi(\omega) := \sqrt{\frac{\gamma+1}{\gamma-1}}\,\arctan\Big(\sqrt{\frac{\gamma-1}{\gamma+1}}\,\cot\omega\Big)$$

for the 3×3 Euler system. And we build semi-hyperbolic patches of solutions.

We may apply these methods and ideas to applications in Mach (Guderley) reflection, channel flow, flow around airfoil, de Laval nozzle, etc. (see, e.g., [3]).

With regards to numerics: Theoretical work and numerics need this mutual movement — challenge and promote each other.

See the survey paper [18] for more details. See the books [20, 43] for more background. We apologize for not able to cover work of Canic, Keyfitz, Kim, Tesdall, Hunter, Guiqiang Chen, Feldman, T. P. Liu, V. Elling, Shuxing Chen, Huicheng Yin, Zhouping Xin, Denis Serre, et al in more details.

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