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# The Relative Transpose over Cohen-Macaulay Finite Artin Algebras**

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**Abstract** The relative transpose via Gorenstein projective modules is introduced, and some corresponding results on the Auslander-Reiten sequences and the Auslander-Reiten formula to this relative version are generalized.

**Keywords** Gorenstein projective module, Cohen-Macaulay finite algebra, Relative transpose, Auslander-Reiten sequence, Auslander-Reiten formula
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## 1 Introduction and Preliminaries

Auslander-Reiten theory plays a fundamental role in the modern representation theory of Artin algebras (see [1]). It mainly consists of Auslander-Reiten sequences and Auslander-Reiten quivers, which deeply explore the essential relation among indecomposable modules. Hence, we can grasp thoroughly the properties of modules categories of Artin algebras.

As a key ingredient in this theory, the transpose plays a central role. The classical Auslander-Reiten transpose is constructed via projective modules. For modules category of a Cohen-Macaulay finite Artin algebra (i.e., there are only finitely many isomorphism classes of finitely generated indecomposable Gorenstein projective modules), we introduce the relative transpose via Gorenstein projective modules, and generalize some corresponding results on the Auslander-Reiten sequences and the Auslander-Reiten formula to this relative version. Note that there are only finitely many isomorphism classes of indecomposable projective modules for an Artin algebra. Hence, as a generalization, it is natural to consider Cohen-Macaulay finite Artin algebra.

In this section, we will fix the notation and recall some definitions used in this paper. For details, we refer to [2]. Let $R$ be an associative ring. Denote by $R$-Mod the category of $R$-modules and by $R$-mod the full subcategory of all finitely generated $R$-modules.

A complete projective resolution (see [3]) is an exact sequence of projective modules, $\mathcal{P}^\bullet = \cdots \to P^{-1} \to P^0 \to P^1 \to P^2 \to \cdots$, such that $\mathrm{Hom}^\bullet(\mathcal{P}^\bullet, Q)$ is exact for every projective $R$-module $Q$; and an $R$-module $M$ is called Gorenstein projective if there is a complete projective resolution $\mathcal{P}^\bullet$ such that $M \cong \mathrm{Im}(P^{-1} \to P^0)$. It is clear that a projective $R$-module is

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Gorenstein projective and that in a complete projective resolution, all the images and hence all the kernels and cokernels are Gorenstein projective. Denote by $R$-GProj the full subcategory of Gorenstein projective $R$-modules and by $R$-Gproj the full subcategory of finitely generated Gorenstein projective $R$-modules. Note that $R$-GProj is closed under extensions, the kernel of an epimorphism, arbitrary coproducts and direct summands. For more facts, we refer to [2, 4].

For example, if $R$ is a self-injective ring, we can easily see that $R$-GProj= $R$-Mod.

A Gorenstein projective resolution ($R$-GProj-resolution for short) of $R$-module $M$ is a complex $\cdots \to G^{-1} \to G^0 \to 0$, where each $G^{-n}$ is Gorenstein projective, together with a morphism $G^0 \to M$, such that the complex $\mathcal{G}^\bullet = \cdots \to G^{-1} \to G^0 \to M \to 0$ is exact. The resolution is called proper if $\mathrm{Hom}^\bullet(E, \mathcal{G}^\bullet)$ is exact for all Gorenstein projective $R$-modules $E$. The resolution is said to be of length $n$ if $G^{-n} \neq 0$ and $G^i = 0$ for all $i < -n$. Recall that the Gorenstein projective dimension, $\mathrm{Gpd}_R M$, of $R$-module $M$ is defined as follows: if $M = 0$, set $\mathrm{Gpd}_R M = -\infty$; if $M \neq 0$ and $M$ has no $R$-GProj-resolution of finite length, set $\mathrm{Gpd}_R M = \infty$; if $M \neq 0$ and $M$ has an $R$-GProj-resolution of finite length, set $\mathrm{Gpd}_R M$ to be the smallest integer $n \geq 0$ such that $M$ has an $R$-GProj-resolution of length $n$. Denote by $f\mathrm{G}R$-Mod the full subcategory of $R$-modules with finite Gorenstein projective dimensions.

Let $\mathcal{X}$ be a class of $R$-modules. Recall from [5, 6] that a right $\mathcal{X}$-approximation of $R$-module $M$ is a morphism $f : X \to M$ with $X \in \mathcal{X}$ such that the induced sequence $\mathrm{Hom}_R(X', X) \to \mathrm{Hom}_R(X', M) \to 0$ is exact for all $X' \in \mathcal{X}$. Similarly, a left $\mathcal{X}$-approximation of $R$-module $M$ is a morphism $f : M \to X$ with $X \in \mathcal{X}$ such that the induced sequence $\mathrm{Hom}_R(X, X') \to \mathrm{Hom}_R(M, X') \to 0$ is exact for all $X' \in \mathcal{X}$. It is well-known that if $M$ is a finitely generated $R$-module with finite Gorenstein projective dimension, then $M$ admits a right $R$-Gproj-approximation (see [4, Theorem 2.10]).

Recall from [7, 8] that a ring $R$ is called Cohen-Macaulay finite if there are only finitely many isomorphism classes of finitely generated indecomposable Gorenstein projective $R$-modules. Throughout this paper, $A$ is a Cohen-Macaulay finite Artin $k$-algebra over a commutative Artin ring $k$ and all $A$-modules are finitely generated. Denote by $A$-mod and $A$-Gproj the category of finitely generated left $A$-modules and the full subcategory of finitely generated Gorenstein projective $A$-modules, respectively. Let $\{G_i\}_{i=1}^n$ be all nonisomorphic finitely generated Gorenstein projective $A$-modules and $G = \bigoplus_{i=1}^{n} G_i$. It is clear that $A$-Gproj=add $G$. Let $B = \mathrm{End}_A(G)^{\mathrm{op}}$. Denote by mod-$B$ the category of finitely generated right $B$-modules and $\underline{\text{mod-}B}$ the stable category of mod-$B$ modulo Proj-$B$. Then $G$ is an $A$-$B$ bimodule in a natural manner. We fix such a triple $(A_A, G_B, B)$.

It is clear that for any $A$-module $M$, there is a right $A$-Gproj-approximation and minimal right $A$-Gproj-approximation. Hence $M$ admits an $A$-Gproj-presentation; that is, there is an exact sequence $G_1 \xrightarrow{f_1} G_0 \xrightarrow{f_0} M \to 0$ such that $G_1 \to \mathrm{Im}\, f_1$ and $G_0 \to M$ are right $A$-Gproj-approximation.

## 2 The Relative Transpose

Now we shall introduce the relative $G$-transpose which is defined in a way similar to the one defined by Xi Chang-Chang (see [9]).

**Definition 2.1** *Define the category* $\mathrm{Mor}(A\text{-Gproj})$ : *an object is a morphism* $f : G_1 \to G_0$

*in* $A$-Gproj *such that* $G_1 \to \operatorname{Im} f$ *and* $G_0 \to \operatorname{Coker} f$ *are right* $A$-Gproj-*approximation, and a morphism from* $f : G_1 \to G_0$ *to* $f' : G_1' \to G_0'$ *is a pair* $(g_1, g_0)$ *where* $g_i : G_i \to G_i'$ *for* $i = 0, 1$, *such that the diagram*

$$\begin{tikzcd} G_1 \arrow[r, "f"] \arrow[d, "g_1"'] & G_0 \arrow[d, "g_0"'] \\ G_1' \arrow[r, "f'"] & G_0' \end{tikzcd}$$

*commutes. We call* $\mathrm{Mor}(A\text{-Gproj})$ *the morphism category of* $A$-Gproj.

**Remark 2.1** $\mathrm{Mor}(A\text{-Gproj})$ is an additive category.

Denote by $A\text{-Gproj}(M, N)$ the subgroup of $\mathrm{Hom}_A(M, N)$ of $A$-maps from $M$ to $N$ which factors through the finitely generated Gorenstein projective modules, and by $A\text{-}\underline{\mathrm{mod}}/A\text{-Gproj}$ the stable category of $A$-mod modulo $A$-Gproj, i.e., the objects of $A\text{-mod}/A\text{-Gproj}$ are the same as those of $A$-mod, and the morphism space from $M$ to $N$ of $A\text{-mod}/A\text{-Gproj}$ is the quotient group $\mathrm{Hom}_A(M, N)/A\text{-Gproj}(M, N)$. Simply denote the functor $\mathrm{Hom}_A(\cdot\, , \cdot)$ by $(\cdot\, , \cdot)$.

**Definition 2.2** *Define the functor* $F : \mathrm{Mor}(A\text{-Gproj}) \to A\text{-mod}/A\text{-Gproj}$ *by* $F(f) = \operatorname{Coker} f$ *for all* $f : G_1 \to G_0$ *in* $\mathrm{Mor}(A\text{-Gproj})$, *and* $F(g_1, g_0) = \underline{\operatorname{Coker}(g_1, g_0)}$, *where* $\operatorname{Coker}(g_1, g_0) : \operatorname{Coker} f \to \operatorname{Coker} f'$ *is the unique morphism which makes the diagram*

$$\begin{tikzcd} G_1 \arrow[r, "f"] \arrow[d, "g_1"'] & G_0 \arrow[r, "\pi"] \arrow[d, "g_0"'] & \operatorname{Coker} f \arrow[r] \arrow[d] & 0 \\ G_1' \arrow[r, "f'"] & G_0' \arrow[r, "\pi'"] & \operatorname{Coker} f' \arrow[r] & 0 \end{tikzcd}$$

*commutes.*

**Remark 2.2** It is not hard to check that $\underline{\operatorname{Coker}(g_1, g_0)}$ is independent of the choice of the pair $(g_1, g_0)$ and that $F$ is a dense and full functor.

**Definition 2.3** *Define* $\mathcal{P}(f, f')$ *as the class of the morphisms* $(g_1, g_0)$ *with the property that there is some* $h : G_0 \to G_1'$ *such that* $f'hf = g_0 f$.

**Remark 2.3** It is clear that $\mathcal{P}$ is a relation on $\mathrm{Mor}(A\text{-Gproj})$.

**Lemma 2.1** *Use the above notation. Then the functor* $F$ *induces a functor*

$$\widetilde{F} : \mathrm{Mor}(A\text{-Gproj})/\mathcal{P} \to A\text{-mod}/A\text{-Gproj},$$

*which is an equivalence of categories.*

**Proof** We claim that $F(g_1, g_0) = 0$ if and only if $(g_1, g_0) : f \to f'$ is in $\mathcal{P}(f, f')$.

Let $(g_1, g_0) : f \to f'$ be in $\mathcal{P}(f, f')$. Then there is some $h : G_0 \to G_1'$ such that $f'hf = g_0 f$. So we have a morphism $\phi : \operatorname{Coker} f \to G_0'$ such that $\phi\pi = g_0 - f'h$. Hence $\operatorname{Coker}(g_1, g_0)\pi = \pi' g_0 = \pi'(g_0 - f'h) = \pi'\phi\pi$. Since $\pi$ is an epimorphism, we get $\operatorname{Coker}(g_1, g_0) = \pi'\phi$. This means $F(g_1, g_0) = 0$.

Let $F(g_1, g_0) = 0$. Then there is some $\psi : \operatorname{Coker} f \to G_0'$ such that $\pi'\psi = \operatorname{Coker}(g_1, g_0)$. So $\pi'(g_0 - \psi\pi) = 0$. Hence $\operatorname{Im}(g_0 - \psi\pi)$ lies in $\operatorname{Im} f'$. Since $f' : G_1' \to \operatorname{Im} f'$ is a right $A$-Gproj-approximation, it follows that there is a morphism $h' : G_0 \to G_1'$ such that $g_0 - \psi\pi = f'h'$. So $f'h'f = g_0 f$. This means that $(g_1, g_0) : f \to f'$ is in $\mathcal{P}(f, f')$.

By Remark 2.2, $\widetilde{F} : \mathrm{Mor}(A\text{-Gproj})/\mathcal{P} \to A\text{-mod}/A\text{-Gproj}$ is an equivalence of categories. Its quasi-inverse functor is denoted by $\widetilde{F}^{-1}$.

**Definition 2.4** *Define the functor* $J : \mathrm{Mor}(A\text{-Gproj}) \to \underline{\text{mod-}B}$ *by* $J(f) = \mathrm{Coker}(f, G)$ *for all* $f : G_1 \to G_0$ *in* $\mathrm{Mor}(A\text{-Gproj})$*, and* $J(g_1, g_0) = \underline{\mathrm{Coker}((g_0, G), (g_1, G))}$*, where* $\mathrm{Coker}((g_0, G), (g_1, G)) : \mathrm{Coker}(f', G) \to \mathrm{Coker}(f, G)$ *is the unique morphism which makes the diagram*

$$\begin{tikzcd}
(G_0', G) \arrow[r, "{(f',G)}"] \arrow[d, "{(g_0,G)}"'] & (G_1', G) \arrow[r, "\pi'"] \arrow[d, "{(g_1,G)}"'] & \mathrm{Coker}(f', G) \arrow[r] \arrow[d] & 0 \\
(G_0, G) \arrow[r, "{(f,G)}"] & (G_1, G) \arrow[r, "\pi"] & \mathrm{Coker}(f, G) \arrow[r] & 0
\end{tikzcd}$$

*commutes.*

**Lemma 2.2** *The functor* $J$ *induces a faithful functor* $\widetilde{J} : \mathrm{Mor}(A\text{-Gproj})/\mathcal{P} \to \underline{\text{mod-}B}$.

**Proof** We claim that $J(g_1, g_0) = 0$ if and only if $(g_1, g_0) : f \to f'$ is in $\mathcal{P}(f, f')$.

Let $(g_1, g_0) : f \to f'$ be in $\mathcal{P}(f, f')$. Then there is some $h : G_0 \to G_1'$ such that $f'hf = g_0 f$. So $(f, G)(g_0, G) = (f, G)(h, G)(f', G)$. It follows that $((g_1, G) - (f, G)(h, G))(f', G) = 0$. So we have a morphism $\phi : \mathrm{Coker}(f', G) \to (G_1, G)$ such that $\phi\pi' = (g_1, G) - (f, G)(h, G)$. Hence $\mathrm{Coker}((g_0, G), (g_1, G))\pi' = \pi(g_1, G) = \pi((g_1, G) - (f, G)(h, G)) = \pi\phi\pi'$. By $\pi'$ is an epimorphism, we get $\mathrm{Coker}((g_0, G), (g_1, G)) = \pi\phi$. Thus $F(g_1, g_0) = 0$.

Let $F(g_1, g_0) = 0$. Then there is some $\psi : \mathrm{Coker}(f', G) \to (G_1, G)$ such that $\pi\psi = \mathrm{Coker}((g_0, G), (g_1, G))$. So $\pi((g_1, G) - \psi\pi') = 0$. Hence $\mathrm{Im}((g_1, G) - \psi\pi')$ lies in $\mathrm{Im}(f, G)$. Since $(G_1', G)$ is projective, it follows that there is some $(h', G) : (G_1', G) \to (G_0, G)$ such that $(g_1, G) - \psi\pi' = (f, G)(h', G)$. So we have $(f, G)(h', G)(f', G) = (g_1, G)(f', G) = (f, G)(g_0, G)$. Therefore $f'h'f = g_0 f$. This means that $(g_1, g_0) : f \to f'$ is in $\mathcal{P}(f, f')$.

Therefore, $\widetilde{J} : \mathrm{Mor}(A\text{-Gproj})/\mathcal{P} \to \underline{\text{mod-}B}$ is a faithful functor.

**Theorem 2.1** *Let* $\mathrm{Tr}_G = J \circ \widetilde{F}^{-1}$*. Then the functor* $\mathrm{Tr}_G : A\text{-mod}/A\text{-Gproj} \to \underline{\text{mod-}B}$ *is a faithful functor. We call* $\mathrm{Tr}_G(M)$ *the relative transpose* (*or* G*-transpose*) *of arbitrary* $A$*-module* $M$.

**Proof** We can easily see that $\mathrm{Tr}_G$ is a faithful functor by Lemmas 2.1 and 2.2.

**Remark 2.4** Clearly, if $G = A$, then $\mathrm{Tr}_G(M)$ is the usual transpose of $A$-module $M$.

In fact, the functor $\mathrm{Tr}_G : A\text{-mod}/A\text{-Gproj} \to \underline{\text{mod-}B}$ is defined as follows: for any $A$-module $M$, $\mathrm{Tr}_G(M) = \mathrm{Coker}(f_1, G)$, where $G_1 \xrightarrow{f_1} G_0 \xrightarrow{f_0} M \to 0$ is a minimal $A$-Gproj-presentations of $M$. For $A$-module $M$ and $N$, we take minimal $A$-Gproj-presentation of $M$ and $N$ respectively. Let $h : M \to N$. Then there exists the following exact commutative diagram:

$$\begin{tikzcd}
G_1 \arrow[r, "f_1"] \arrow[d, "g_1"'] & G_0 \arrow[r, "f_0"] \arrow[d, "g_0"'] & M \arrow[r] \arrow[d, "h"] & 0 \\
G_1' \arrow[r, "f_1'"] & G_0' \arrow[r, "f_0'"] & N \arrow[r] & 0
\end{tikzcd} \tag{2.1}$$

and consequently $\mathrm{Tr}_G(h)$ becomes $\underline{\tau}$, where $\tau : \mathrm{Tr}_G(M) \to \mathrm{Tr}_G(N)$ is the unique morphism such that the diagram

$$\begin{tikzcd}
(G_0', G) \arrow[r, "{(f_1', G)}"] \arrow[d, "{(g_0,G)}"'] & (G_1', G) \arrow[r] \arrow[d, "{(g_1,G)}"'] & \mathrm{Tr}_G(M) \arrow[r] \arrow[d, "\tau"] & 0 \\
(G_0, G) \arrow[r, "{(f_1, G)}"] & (G_1, G) \arrow[r] & \mathrm{Tr}_G(N) \arrow[r] & 0
\end{tikzcd}$$

is commutative. Note that $\mathrm{Tr}_G(h)$ is independent of the choice of the pair $(g_1, g_0)$.

**Corollary 2.1** *The following hold:*

(1) $\mathrm{Tr}_G M = 0$ *if and only if* $M \in A$-Gproj,

(2) $\mathrm{Tr}_G(M \oplus N) = \mathrm{Tr}_G(M) \oplus \mathrm{Tr}_G(N)$,

(3) *If* $M_1$ *and* $M_2$ *are indecomposable such that* $M_i \notin A$-Gproj *for* $i = 1, 2$*, then* $M_1 \cong M_2$ *if and only if* $\mathrm{Tr}_G(M_1) \cong \mathrm{Tr}_G(M_2)$.

**Proof** By the definition of $\mathrm{Tr}_G$, we can easily deduce that the corollary holds.

**Theorem 2.2** *Let* $M$ *be an indecomposable* $A$*-module such that* $M \notin A$-Gproj*. Then* $\mathrm{Tr}_G(M)$ *is indecomposable and* $\mathrm{End}({}_AM)/A\text{-Gproj} \cong \underline{\mathrm{End}}(\mathrm{Tr}_G(M)_B)$.

**Proof** Since $\mathrm{Tr}_G$ is a faithful functor by Theorem 2.1, it follows that the morphism $\mathrm{End}({}_AM)/A\text{-Gproj} \to \underline{\mathrm{End}}(\mathrm{Tr}_G(M)_B)$ is injective. Let $G_1 \to G_0 \to M \to 0$ be a minimal $A$-Gproj-presentation of $M$. Since $M$ is an indecomposable $A$-module such that $M \notin A$-Gproj, it follows that $\mathrm{Hom}_A(G_0, G) \to \mathrm{Hom}_A(G_1, G) \to \mathrm{Tr}_G(M) \to 0$ is a minimal projective presentation of the right $B$-module $\mathrm{Tr}_G(M)$. Therefore, by (2.1), we can get that the morphism $\mathrm{End}({}_AM)/A\text{-Gproj} \to \underline{\mathrm{End}}(\mathrm{Tr}_G(M)_B)$ is surjective. So $\mathrm{End}({}_AM)/A\text{-Gproj} \cong \underline{\mathrm{End}}(\mathrm{Tr}_G(M)_B)$.

Now we show that $\mathrm{Tr}_G(M)$ is indecomposable. In order to show this, we show that a morphism $h \in \mathrm{End}({}_AM)$ is an isomorphism if and only if so is $\tau$. If $h$ is an isomorphism, then $g_1$ and $g_0$ are automorphisms by the definition of minimal $A$-Gproj-presentation. Thus $\tau$ is an isomorphism. Conversely, if $\tau$ is an isomorphism, since the sequence $(G_0, G) \to (G_1, G) \to \mathrm{Tr}_G M \to 0$ is a minimal $B$-projective presentation of $\mathrm{Tr}_G M$, it follows that $(g_1, G)$ and $(g_0, G)$ are automorphisms. So $g_1$ and $g_0$ are also automorphisms. This completes the proof.

Recall that an $A$-module $M$ is called torsion if $\mathrm{Hom}_A(M, A) = 0$. Let $J$ be the full subcategory of finitely generated torsion right $B$-modules and $A\text{-Gproj}^{\leq 1}$ the full subcategory of finitely generated $A$-modules $M$ with Gorenstein projective dimensions at most 1. Then we have

**Theorem 2.3** *The functor* $\mathrm{Tr}_G$ *induces* $\widetilde{\mathrm{Tr}}_G : A\text{-Gproj}^{\leq 1}/A\text{-Gproj} \to J$*, which is a faithful functor.*

**Proof** Since $\mathrm{Gpd}_A M \leq 1$, it follows that $M$ has a minimal $A$-Gproj-presentation $0 \to G_1 \to G_0 \to M \to 0$. Then $\mathrm{Hom}_A(G_0, G) \to \mathrm{Hom}_A(G_1, G) \to \mathrm{Tr}_G M \to 0$ is exact. Simply denote the functor $\mathrm{Hom}_B(\cdot\,,\cdot)$ by ${}_B(\cdot\,,\cdot)$, we get an exact sequence

$$0 \to_B (\mathrm{Tr}_G M, B) \to_B (\mathrm{Hom}_A(G_1, G), B) \to_B (\mathrm{Hom}_A(G_0, G), B).$$

Since $\mathrm{Hom}_B(\mathrm{Hom}_A(G_i, G), B) \cong \mathrm{Hom}_A(G, G_i)$ for $i = 0, 1$, it follows that

$$0 \to \mathrm{Hom}_B(\mathrm{Tr}_G M, B) \to \mathrm{Hom}_A(G, G_1) \to \mathrm{Hom}_A(G, G_0)$$

is exact. So $\mathrm{Hom}_B(\mathrm{Tr}_G M, B) = 0$. This completes the proof.

## 3 Relative Auslander-Reiten Sequences and Auslander-Reiten Formula

Now we generalize Auslander-Reiten sequences to this relative version.

**Theorem 3.1** *Let $M$ be an indecomposable $A$-module such that $M \notin A$-Gproj. Then there is an Auslander-Reiten sequence in $B$-mod of the form*

$$0 \to \mathrm{DTr}_G(M) \to X \to \mathrm{Hom}_A(G, M) \to 0.$$

**Proof** Let $G_1 \to G_0 \to M \to 0$ be a minimal $A$-Gproj-presentation of $M$. This induces the following three exact sequences:

$$(G, G_1) \to (G, G_0) \to (G, M) \to 0,$$
$$0 \to \mathrm{DTr}_G M \to D(G_1, G) \xrightarrow{g} D(G_0, G), \tag{3.1}$$
$$\mathrm{D}((G, G_1), \cdot\,) \to \mathrm{D}((G, G_0), \cdot\,) \to \mathrm{D}((G, M), \cdot\,) \to 0. \tag{3.2}$$

Since we have the following series of isomorphisms

$$\begin{aligned}
\mathrm{D}((G, G_i), \cdot\,) &\cong \mathrm{D}(((G_i, G), (G, G)), \cdot\,)\\
&\cong \mathrm{D}(((G_i, G), B), \cdot\,)\\
&\cong \mathrm{D}((G_i, G) \otimes_B (B, \cdot\,))\\
&\cong \mathrm{D}((G_i, G) \otimes_B \cdot\,)\\
&\cong (\,\cdot\,, \mathrm{D}(G_i, G)),
\end{aligned}$$

by (3.2), we get an exact sequence

$$(\,\cdot\,, \mathrm{D}(G_1, G)) \to (\,\cdot\,, \mathrm{D}(G_0, G)) \to \mathrm{D}((G, M), \cdot\,) \to 0.$$

By (3.1), we get an exact sequence of functors

$$0 \to (\,\cdot\,, \mathrm{DTr}_G M) \to (\,\cdot\,, D(G_1, G)) \to (\,\cdot\,, D(G_0, G)).$$

Hence we can obtain the following exact sequence of functors

$$0 \to (\,\cdot\,, \mathrm{DTr}_G M) \to (\,\cdot\,, D(G_1, G)) \to (\,\cdot\,, D(G_0, G)) \to \mathrm{D}((G, M), \cdot\,) \to 0.$$

Let $S_{(G,M)} = (\,\cdot\,, (G, M))/\mathrm{rad}(\,\cdot\,, (G, M))$. Since $(\,\cdot\,, (G, M))$ is a projective functor, it follows that the morphism $(\,\cdot\,, (G, M)) \to S_{(G,M)} \to \mathrm{D}((G, M), \cdot\,)$ factors through $(\,\cdot\,, D(G_0, G))$. Clearly, the morphism $(\,\cdot\,, (G, M)) \to (\,\cdot\,, D(G_0, G))$ is induced by a morphism $g' : (G, M) \to D(G_0, G)$. Let $X$ be a pullback of $g$ and $g'$. Then the following sequence

$$0 \to (\,\cdot\,, \mathrm{DTr}_G M) \to (\,\cdot\,, X) \to (\,\cdot\,, (G, M)) \to S_{(G,M)} \to 0 \tag{3.3}$$

is exact in the functor category. Since $\mathrm{DTr}_G M$ is indecomposable by Theorem 2.2, it follows that (3.3) is a minimal projective resolution of $S_{(G,M)}$ in the functor category. Thus we get that the sequence

$$0 \to \mathrm{DTr}_G(M) \to X \to \mathrm{Hom}_A(G, M) \to 0$$

is an Auslander-Reiten sequence in $B$-mod.

There is an exact sequence involving the $G$-transpose as the following shows.

**Proposition 3.1** *Let $G_1 \to G_0 \to M \to 0$ be a minimal $A$-*Gproj*-presentation of $M$. For any $A$-module $Z$, we have the following exact sequence:*

$$0 \to (M, Z) \to (G_0, Z) \to (G_1, Z) \to \mathrm{Tr}_G(M) \otimes_B (G, Z) \to 0.$$

**Proof** By the hypothesis, we have an exact sequence $(G_0, G) \to (G_1, G) \to \mathrm{Tr}_G(M) \to 0$ of right $B$-modules, which yields the following exact sequence:

$$(G_0, G) \otimes_B (G, Z) \to (G_1, G) \otimes_B (G, Z) \to \mathrm{Tr}_G(M) \otimes_B (G, Z) \to 0.$$

Also, we have an exact sequence

$$0 \to (M, Z) \to (G_0, Z) \to (G_1, Z).$$

Because $(G_i, G) \otimes_B (G, Z) \cong (G_i, Z)$ for $i = 0, 1$, we deduce that the proposition holds.

**Corollary 3.1** *Let $G_1 \to G_0 \to M \to 0$ be a minimal $A$-*Gproj*-presentation of $M$. Then for any $A$-module $Z$, we have*

$$l_k(M, Z) - l_k(\mathrm{Hom}_B((G, Z), \mathrm{DTr}_G(M))) = l_k(G_0, Z) - l_k(G_1, Z),$$

*where $l_k(X)$ stands for the length of $k$-module $X$.*

Now we generalize some corresponding results on the Auslander-Reiten formula to this relative version. Before we do this, we first prove a result which might be considered as an analogue of the defect of exact sequences.

**Definition 3.1** *Let $\delta : 0 \to X \to Y \to Z \to 0$ be an exact sequence in $A$-*mod *such that $0 \to (G, X) \to (G, Y) \to (G, Z) \to 0$ is exact for all $G \in A$-*Gproj*. For $A$-module $M$ and $B$-module $N$, define $\delta^*(M)$ and $\delta_G(N)$ as follows:*

$$0 \to (M, X) \to (M, Y) \to (M, Z) \to \delta^*(M) \to 0,$$
$$0 \to ((G, Z), N) \to ((G, Y), N) \to ((G, X), N) \to \delta_G(N) \to 0.$$

**Lemma 3.1** *The $k$-lengths of $\delta^*(M)$ and $\delta_G(\mathrm{DTr}_G(M))$ are equal for all $M$ in $A$-*mod.

**Proof** This theorem follows directly from Corollary 3.1.

Now we have the following generalization of Auslander-Reiten formula.

**Theorem 3.2** *For any $A$-module $M$ and $Z$, we have*

$$l_k(\mathrm{Ext}^1_B((G, Z), \mathrm{DTr}_G(M))) = l_k((M, Z)/A\text{-}\mathrm{Gproj}(M, Z)) = l_k(\mathrm{Tor}^B_1(\mathrm{Tr}_G(M), (G, Z))).$$

**Proof** Let $\delta : 0 \to K \to G_0 \xrightarrow{f} Z \to 0$ be an exact sequence such that $f : G_0 \to Z$ is a right $A$-Gproj-approximation of $Z$. Then $\delta$ induces another exact sequence $\delta' : 0 \to (G, K) \to (G, G_0) \to (G, Z) \to 0$. By Lemma 3.1, the $k$-lengths of $\delta^*(M)$ and $\delta_G(\mathrm{DTr}_G(M))$ are equal. It is clear that the $k$-length of $\delta^*(M)$ is the same as that of $(M, Z)/A\text{-Gproj}(M, Z)$. On the other hand, by tensoring $\mathrm{Tr}_G(M)$ to the sequence $\delta'$ and using the adjunction, we have an exact sequence $0 \to ((G, Z), \mathrm{DTr}_G(M)) \to ((G, G_0), \mathrm{DTr}_G(M)) \to ((G, K), \mathrm{DTr}_G(M)) \to \mathrm{DTor}_1^B(\mathrm{Tr}_G(M), (G, Z)) \to 0$. This shows that the length of $\delta_G(\mathrm{DTr}_G(M))$ is the same as that of $\mathrm{DTor}_1^B(\mathrm{Tr}_G(M), (G, Z))$. But if we apply $(\,\cdot\,, \mathrm{DTr}_G(M))$ to the sequence $\delta'$, we get that this number is also equal to the $k$-length of $\mathrm{Ext}_B^1((G, Z), \mathrm{DTr}_G(M))$. Thus the theorem is proved.

## References


[1] Auslander, M., Reiten, I. and Smalø, S. O., Representation Theory of Artin Algebras, Cambridge Studies in Advanced Mathematics, **36**, Cambridge University Press, Cambridge, 1995.

[2] Enochs, E. E. and Jenda, O. M. G., Relative Homological Algebra, De Gruyter Expositions in Mathematics, **30**, Walter de Gruyter, Berlin, 2000.

[3] Enochs, E. E. and Jenda, O. M. G., Gorenstein injective and projective modules, *Math. Z.*, **220**(4), 1995, 611–633.

[4] Holm, H., Gorenstein homological dimensions, *J. Pure Appl. Algebra*, **189**(1–3), 2004, 167–193.

[5] Auslander, M. and Smalø, S. O., Preprojective modules over Artin algebras, *J. Algebra*, **66**, 1980, 61–122.

[6] Auslander, M. and Reiten, I., Applications of contravariantly finite subcategories, *Adv. Math.*, **86**, 1991, 111–152.

[7] Bruns, W. and Herzog, J., Cohen-Macaulay Rings, Revised Edition, Cambridge Studies in Advanced Mathematics, **39**, Cambridge University Press, Cambridge, 1998.

[8] Beligiannis, A., Cohen-Macaulay modules, (co)torsion pairs and virtually Gorenstein algebras, *J. Algebra*, **288**(1), 2005, 137–211.

[9] Xi, C. C., The reltive Auslander-Reiten theory of modules, preprint.