Chin. Ann. Math.
31B(4), 2010, 447–460
DOI: 10.1007/s11401-010-0595-2

Chinese Annals of Mathematics, Series B

Exact Boundary Observability of Unsteady Supercritical Flows in a Tree-Like Network of Open Canals

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Abstract The author establishes the exact boundary observability of unsteady supercritical flows in a tree-like network of open canals with general topology. An implicit duality between the exact boundary controllability and the exact boundary observability is also given for unsteady supercritical flows.

Keywords Exact boundary observability, Saint-Venant system, Tree-like network of open canals, Quasilinear hyperbolic system
2000 MR Subject Classification 35L65, 93B07

1 Introduction

The one-dimensional mathematical model of unsteady flows in an open canal was given by de Saint-Venant [14]. In [4], the authors gave a corresponding model of Saint-Venant system for a network of open canals, in which the interface conditions at any given joint point of open canals are given.

In recent years, based on the result on the semi-global classical solution in [9], the exact boundary controllability for general first order quasilinear hyperbolic systems has been established (see [10, 11]). Then this result has been applied to get the exact boundary controllability of unsteady subcritical flows in a network of open canals (see [5, 6, 12, 13]). On the other hand, with the interface conditions given in [3], the exact boundary controllability of unsteady supercritical flows in a tree-like network of open canals has been established (see [1]).

Moreover, the exact boundary observability for first order quasilinear hyperbolic systems has been studied in [7, 8], in which an implicit duality between the exact boundary controllability and the exact boundary observability is also given. Based on this result, the exact boundary observability of unsteady subcritical flows in a tree-like network of open canals has been obtained (see [2]).

In this paper, under the assumption that the observed value is accurate, i.e., there is no measuring error in the observation, we will establish the exact boundary observability of supercritical unsteady flows in a tree-like network of open canals with general topology, in which the observed values are physically meaningful and practically handleable. Moreover, we will also show an implicit duality between the exact boundary controllability and the exact boundary observability for unsteady supercritical flows.

Manuscript received December 9, 2009. Published online June 21, 2010.
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This paper is organized as follows. We recall the known results on the exact boundary observability for first order quasilinear hyperbolic systems in Section 2, then the corresponding exact boundary observability of unsteady supercritical flows in a single open canal and in a star-like network of open canals will be presented in Sections 3 and 4. Finally the exact boundary observability of unsteady flows in a tree-like network of open canals will be given in Section 5.

2 Exact Boundary Observability for a Kind of Quasilinear Hyperbolic System

For the purpose of this paper, in this section we recall the result given in [7, 8] only for the following quasilinear hyperbolic system of diagonal form

$$\frac{\partial u_i}{\partial t} + \lambda_i(u)\frac{\partial u_i}{\partial x} = F_i(u), \quad i = 1, \cdots, n, \tag{2.1}$$

where $u = (u_1, \cdots, u_n)^{\mathrm{T}}$ is the unknown vector function of (t, x), $\lambda_i(u)$ and $F_i(u)$ $(i = 1, \cdots, n)$ are C^1 functions of u,

$$F_i(0) = 0, \quad i = 1, \cdots, n \tag{2.2}$$

and on the domain under consideration

$$\begin{aligned} &\lambda_i(u) < 0, \quad i = 1, \cdots, n \\ (\text{resp.} \quad &\lambda_i(u) > 0, \quad i = 1, \cdots, n). \end{aligned} \tag{2.3}$$

The boundary conditions are given as follows:

$$\begin{aligned} &x = L: \quad u_i = h_i(t), \quad i = 1, \cdots, n \\ (\text{resp.} \quad &x = 0: \quad u_i = h_i(t), \quad i = 1, \cdots, n), \end{aligned} \tag{2.4}$$

where h_i $(i = 1, \cdots, n)$ are C^1 functions of t.

By means of [7, 8], we have the following theorem.

Theorem 2.1 *Let*

$$T > \max_{i=1,\cdots,n} \frac{L}{|\lambda_i(0)|}. \tag{2.5}$$

For any given initial condition

$$t = 0: \quad u = \varphi(x), \quad 0 \le x \le L, \tag{2.6}$$

such that $\|\varphi\|_{C^1[0,L]}$ *is suitably small and the conditions of* C^1 *compatibility for the mixed initial-boundary value problem* (2.1), (2.6) *and* (2.4) *are satisfied at the point* $(t, x) = (0, L)$ (resp. $(0, 0)$), *if we have the observed values* $u_i = \overline{u}_i(t)$ $(i = 1, \cdots, n)$ *at* $x = 0$ (resp. $u_i = \overline{\overline{u}}_i(t)$ $(i = 1, \cdots, n)$ *at* $x = L$) *on the interval* $[0, T]$*, then the initial data* $\varphi(x)$ *can be uniquely determined and the following observability inequality holds:*

$$\|\varphi\|_{C^1[0,L]} \le C \sum_{i=1}^{n} \|\overline{u}_i\|_{C^1[0,T]} \quad \Big(\text{resp.} \quad \|\varphi\|_{C^1[0,L]} \le C \sum_{i=1}^{n} \|\overline{\overline{u}}_i\|_{C^1[0,T]}\Big). \tag{2.7}$$

Here and hereafter, C denotes a positive constant.

Proof We give the proof under the assumption that all eigenvalues $\lambda_i(u)$ $(i = 1, \cdots, n)$ are negative (see (2.3)).

Since there is no zero eigenvalue, we may change the status of t and x and solve a rightward Cauchy problem for system (2.1) with the initial condition

$$x = 0: \quad u = \overline{u}(t), \quad 0 \le t \le T, \tag{2.8}$$

where $\overline{u}(t) = (\overline{u}_1(t), \cdots, \overline{u}_n(t))^{\mathrm{T}}$ with small C^1 norm. By the theory of semi-global C^1 solution for quasilinear hyperbolic systems (see [9]), there exists a unique C^1 solution $u = \widetilde{u}(t, x)$ on the whole maximum determinate domain and

$$\|\widetilde{u}\|_{C^1} \le C \sum_{i=1}^{n} \|\overline{u}_i\|_{C^1[0,T]}. \tag{2.9}$$

Obviously, $u = \widetilde{u}(t, x)$ is the restriction of the solution $u = u(t, x)$ to the original mixed problem on the corresponding domain.

By (2.5), the maximum determinate domain must intersect $x = L$ and contains the interval $0 \le x \le L$ on the x-axis. Thus the initial data can be uniquely determined and the observability inequality (2.7) holds.

3 Exact Boundary Observability of Unsteady Supercritical Flows in a Single Open Canal

Now we apply the theory on the exact boundary observability to unsteady supercritical flows. In this section we first consider the case of a single open canal. Let L be the length of the canal. Taking the x-axis along the inverse direction of flow, this canal can be parameterized lengthwise by $x \in [0, L]$. Suppose that there is no friction and the canal is horizontal and cylindrical. The corresponding Saint-Venant system can be written as (see. [4, 6, 14])

$$\begin{cases} \dfrac{\partial A}{\partial t} + \dfrac{\partial (AV)}{\partial x} = 0, \\ \dfrac{\partial V}{\partial t} + \dfrac{\partial S}{\partial x} = 0, \end{cases} \quad t \ge 0,\ 0 \le x \le L, \tag{3.1}$$

where $A = A(t, x)$ stands for the area of the cross section at x occupied by the water at time t, $V = V(t, x)$ is the average velocity over the cross section and

$$S = \frac{1}{2} V^2 + g h(A) + g Y_b, \tag{3.2}$$

where g is the gravity constant, constant Y_b denotes the altitude of the bed of canal and

$$h = h(A) \tag{3.3}$$

is the depth of the water, $h(A)$ being a suitably smooth function of A such that

$$h'(A) > 0. \tag{3.4}$$

Consider an equilibrium state $(A, V) = (A_0, V_0)$ of system (3.1) with $A_0 > 0$, which belongs to the supercritical case, i.e.,

$$|V_0| > \sqrt{gA_0h'(A_0)}\,. \tag{3.5}$$

Without loss of generality, we suppose that

$$V_0 < -\sqrt{gA_0h'(A_0)}\,. \tag{3.6}$$

The boundary condition is then given as follows:

$$x = L: \quad Q \stackrel{\text{def}}{=} AV = q(t), \quad V = v(t). \tag{3.7}$$

By Theorem 2.1, we have the following theorem on the exact boundary observability.

Theorem 3.1 *Under assumption* (3.6)*, let*

$$T > \frac{L}{|\widetilde{\lambda}_2|} = \max\Big(\frac{L}{|\widetilde{\lambda}_1|}, \frac{L}{|\widetilde{\lambda}_2|}\Big), \tag{3.8}$$

where

$$\widetilde{\lambda}_1 \stackrel{\text{def}}{=} V_0 - \sqrt{gA_0h'(A_0)} < \widetilde{\lambda}_2 \stackrel{\text{def}}{=} V_0 + \sqrt{gA_0h'(A_0)} < 0. \tag{3.9}$$

For any given initial condition

$$t = 0: \quad (A, V) = (A_0(x), V_0(x)), \quad 0 \le x \le L, \tag{3.10}$$

such that the norm $\|(A_0(x) - A_0, V_0(x) - V_0)\|_{C^1[0,L]}$ *is suitably small and the conditions of* C^1 *compatibility with* (3.1) *and* (3.7) *are satisfied at the point* $(t, x) = (0, L)$*, if we have the observed values* $A = a(t)$ *and* $V = v(t)$ *at* $x = 0$ *on the interval* $[0, T]$*, then the initial data* $(A_0(x), V_0(x))$ *can be uniquely determined and the following observability inequality holds:*

$$\|(A_0(x) - A_0, V_0(x) - V_0)\|_{C^1[0,L]} \le C(\|a(t) - A_0\|_{C^1[0,T]} + \|v(t) - V_0\|_{C^1[0,T]}). \tag{3.11}$$

Proof In a neighbourhood of the supercritical equilibrium state (A_0, V_0), (3.1) is a strictly hyperbolic system with two distinct real eigenvalues

$$\lambda_1 \stackrel{\text{def}}{=} V - \sqrt{gAh'(A)} < \lambda_2 \stackrel{\text{def}}{=} V + \sqrt{gAh'(A)} < 0. \tag{3.12}$$

Introducing the Riemann invariants r and s as follows:

$$\begin{cases} 2r = V - V_0 - G(A), \\ 2s = V - V_0 + G(A), \end{cases} \tag{3.13}$$

where

$$G(A) = \int_{A_0}^{A} \sqrt{\frac{gh'(A)}{A}}\mathrm{d}A, \tag{3.14}$$

we have

$$\begin{cases} V = r + s + V_0, \\ A = H(s - r) > 0, \end{cases} \tag{3.15}$$

where H is the inverse function of $G(A)$ with

$$H(0) = A_0, \tag{3.16}$$

$$H'(0) = \sqrt{\frac{A_0}{gh'(A_0)}} > 0. \tag{3.17}$$

Taking (r, s) as new unknown variables, the equilibrium state $(A, V) = (A_0, V_0)$ corresponds to $(r, s) = (0, 0)$ and system (3.1) reduces to the following system of diagonal form:

$$\begin{cases} \dfrac{\partial r}{\partial t} + \lambda_1(r, s)\dfrac{\partial r}{\partial x} = 0, \\ \dfrac{\partial s}{\partial t} + \lambda_2(r, s)\dfrac{\partial s}{\partial x} = 0, \end{cases} \tag{3.18}$$

where

$$\begin{cases} \lambda_1(r, s) = r + s + V_0 - \sqrt{gH(s - r)h'(H(s - r))} < 0, \\ \lambda_2(r, s) = r + s + V_0 + \sqrt{gH(s - r)h'(H(s - r))} < 0. \end{cases} \tag{3.19}$$

The boundary condition (3.7) now becomes

$$x = L: \quad P_1(t, r, s) \stackrel{\text{def}}{=} (r + s + V_0)H(s - r) - q(t) = 0, \tag{3.20}$$

$$P_2(t, r, s) \stackrel{\text{def}}{=} (r + s + V_0) - v(t) = 0 \tag{3.21}$$

and the corresponding observed values become

$$x = 0: \quad H(s - r) = a(t), \quad 0 \le t \le T, \tag{3.22}$$

$$r + s + V_0 = v(t), \quad 0 \le t \le T. \tag{3.23}$$

Moreover, the initial condition (3.10) can be written as

$$t = 0: \quad (r, s) = (r_0(x), s_0(x)), \tag{3.24}$$

where

$$r_0(x) = \frac{1}{2}(V_0(x) - V_0 - G(A_0(x))), \quad s_0(x) = \frac{1}{2}(V_0(x) - V_0 + G(A_0(x))). \tag{3.25}$$

When $(r, s) = (0, 0)$, noting (3.6), we have

$$\det\left|\frac{\partial(P_1, P_2)}{\partial(r, s)}\right| = -2V_0\sqrt{\frac{A_0}{gh'(A_0)}} > 0. \tag{3.26}$$

By the implicit function theorem, in a neighbourhood of $(r, s) = (0, 0)$, (3.20), (3.21) can be equivalently rewritten as

$$x = L: \quad r = \alpha(t), \quad s = \beta(t), \tag{3.27}$$

where α and β are C^1 functions of t. Moreover, noting (3.14), by (3.22) and (3.23), at $x = 0$ we have

$$\begin{cases} r = r(t) \stackrel{\text{def}}{=} \dfrac{1}{2}\Big(v(t) - V_0 - \displaystyle\int_{A_0}^{a(t)} \sqrt{\frac{gh'(A)}{A}}\mathrm{d}A\Big), \\ s = s(t) \stackrel{\text{def}}{=} \dfrac{1}{2}\Big(v(t) - V_0 + \displaystyle\int_{A_0}^{a(t)} \sqrt{\frac{gh'(A)}{A}}\mathrm{d}A\Big), \end{cases} \quad 0 \le t \le T \tag{3.28}$$

and

$$\|(r, s)\|_{C^1[0,T]} \le C(\|a(t) - A_0\|_{C^1[0,T]} + \|v(t) - V_0\|_{C^1[0,T]}). \tag{3.29}$$

Thus, noting (3.8), by Theorem 2.1 $(r_0(x), s_0(x))$ can be uniquely determined by the observed values $r(t)$ and $s(t)$ $(0 \le t \le T)$ at $x = 0$ and

$$\|(r_0(x), s_0(x))\|_{C^1[0,L]} \le C(\|a(t) - A_0\|_{C^1[0,T]} + \|v(t) - V_0\|_{C^1[0,T]}). \tag{3.30}$$

Then, noting (3.25), it is easy to see that $(A_0(x), V_0(x))$ can be uniquely determined and (3.11) holds. This proves Theorem 3.1.

The procedure of resolution given by Theorem 3.1 can be illustrated by Figure 1, in which the point E $(x = 0)$ is the end point of the water flow and "$\rightarrow$" stands for the direction of the water flow. Moreover, we need only two observed values taken at E (marked by $\bullet$), but no observation at another end (marked by $\circ$).

Figure 1 The procedure of resolution given by Theorem 3.1.

4 Exact Boundary Observability of Unsteady Supercritical Flows in a Star-Like Network of Open Canals

Now, we consider the exact boundary observability of unsteady supercritical flows in a star-like network composed of N open canals $c_1, \cdots, c_N$. Let the multiple node be the point O. Suppose that the single node of canal c_1 is the end point E and the water flows from other single nodes (through O) to the point E (see Figure 2, in which "$\rightarrow$" stands for the direction of the water flow).

Let L_i be the length of the i-th canal $(i = 1, \cdots, N)$. For $i = 1, \cdots, N$, by taking the joint point O as $x = 0$, the i-th canal can be parameterized lengthwise by $x \in [0, L_i]$ and all the quantities associated with the i-th canal are indexed by i.

Suppose that there is no friction and all the canals are horizontal and cylindrical. The corresponding Saint-Venant system is (see [4, 6])

$$\begin{cases} \dfrac{\partial A_i}{\partial t} + \dfrac{\partial (A_i V_i)}{\partial x} = 0, \\ \dfrac{\partial V_i}{\partial t} + \dfrac{\partial S_i}{\partial x} = 0, \end{cases} \quad t \ge 0,\ 0 \le x \le L_i,\ i = 1, \cdots, N, \tag{4.1}$$

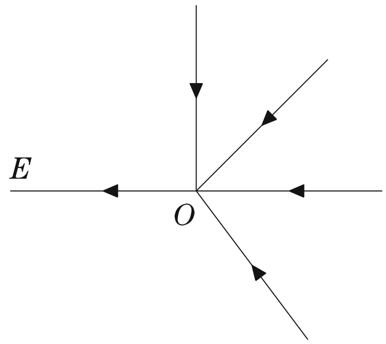

Figure 2 The single node of canal c_1 is the end point E and the water flows from other single nodes (through O) to the point E.

where

$$S_i = \frac{1}{2}V_i^2 + gh_i(A_i) + gY_{bi}, \quad i = 1, \cdots, N, \tag{4.2}$$

Y_{bi} $(i = 1, \cdots, N)$ being constants and

$$h_i'(A_i) > 0, \quad i = 1, \cdots, N. \tag{4.3}$$

The interface conditions at the joint point O are given by the total energy interface condition

$$\sum_{i=1}^{N} A_i V_i S_i = 0 \tag{4.4}$$

and the total flux interface condition

$$\sum_{i=1}^{N} A_i V_i = 0 \tag{4.5}$$

(see [1, 3]), while, at another end of each canal except E we have the boundary conditions

$$x = L_i : \quad Q_i \stackrel{\text{def}}{=} A_i V_i = q_{i1}(t), \quad V_i = v_{i1}(t), \quad i = 2, \cdots, N. \tag{4.6}$$

Consider an equilibrium state $(A_i, V_i) = (A_{i0}, V_{i0})$ of system (4.1) with $A_{i0} > 0$ $(i = 1, \cdots, N)$, which belongs to a supercritical case, i.e.,

$$V_{i0} < -\sqrt{gA_{i0}h_i'(A_{i0})}, \quad i = 1, \cdots, N, \tag{4.7}$$

and, corresponding to (4.4), (4.5), satisfies

$$\sum_{i=1}^{N} A_{i0} V_{i0} S_{i0} = 0, \tag{4.8}$$

$$\sum_{i=1}^{N} A_{i0} V_{i0} = 0, \tag{4.9}$$

where

$$S_{i0}=\frac{1}{2}V_{i0}^2+gh_i(A_{i0})+gY_{bi},\quad i=1,\cdots,N. \tag{4.10}$$

Theorem 4.1 *Let*

$$T>\frac{1}{|\widetilde{\lambda}_{12}|}+\max_{i=2,\cdots,N}\frac{1}{|\widetilde{\lambda}_{i2}|}, \tag{4.11}$$

where

$$\widetilde{\lambda}_{i1}\overset{\text{def}}{=}\frac{1}{L_i}\Big(V_{i0}-\sqrt{gA_{i0}h_i'(A_{i0})}\Big)<\widetilde{\lambda}_{i2}\overset{\text{def}}{=}\frac{1}{L_i}\Big(V_{i0}+\sqrt{gA_{i0}h_i'(A_{i0})}\Big)<0,\quad i=1,\cdots,N. \tag{4.12}$$

For any given initial condition

$$t=0:\quad (A_i,V_i)=(A_{i0}(x),V_{i0}(x)),\quad 0\le x\le L_i,\ i=1,\cdots,N, \tag{4.13}$$

satisfying that $\sum\limits_{i=1}^{N}\|(A_{i0}(x)-A_{i0},V_{i0}(x)-V_{i0})\|_{C^1[0,L_i]}$ *is suitably small and the conditions of piecewise* C^1 *compatibility with* (4.1) *and* (4.4)–(4.6) *are satisfied at the joint point* O *and all other ends except* E*, if we have the observed values* $A_1=a_1(t)$*,* $V_1=v_1(t)$ *at point* E *and* $A_i=a_i(t)$*,* $V_i=v_i(t)$ $(i=2,\cdots,N-1)$ *at point* O *on the interval* $[0,T]$ *(the number of observed values is equal to* $2(N-1)$*), then the initial data* $(A_{i0}(x),V_{i0}(x))$ $(i=1,\cdots,N)$ *can be uniquely determined and the following observability inequality holds:*

$$\begin{aligned}&\sum_{i=1}^{N}\|(A_{i0}(x)-A_{i0},V_{i0}(x)-V_{i0})\|_{C^1[0,L_i]}\\ \le{}& C\Big(\sum_{i=1}^{N-1}\|a_i(t)-A_{i0}\|_{C^1[0,T]}+\sum_{i=1}^{N-1}\|v_i(t)-V_{i0}\|_{C^1[0,T]}\Big).\end{aligned} \tag{4.14}$$

Proof In a neighbourhood of the supercritical equilibrium state (A_{i0},V_{i0}) $(i=1,\cdots,N)$, (4.1) is a hyperbolic system with real eigenvalues

$$\lambda_{i1}\overset{\text{def}}{=}V_i-\sqrt{gA_ih_i'(A_i)}<\lambda_{i2}\overset{\text{def}}{=}V_i+\sqrt{gA_ih_i'(A_i)}<0,\quad i=1,\cdots,N. \tag{4.15}$$

For $i=1,\cdots,N$, introducing the Riemann invariants r_i and s_i as follows:

$$\begin{cases}2r_i=V_i-V_{i0}-G_i(A_i),\\ 2s_i=V_i-V_{i0}+G_i(A_i),\end{cases} \tag{4.16}$$

where

$$G_i(A_i)=\int_{A_{i0}}^{A_i}\sqrt{\frac{gh_i'(A_i)}{A_i}}\,\mathrm{d}A_i, \tag{4.17}$$

we have

$$\begin{cases}V_i=r_i+s_i+V_{i0},\\ A_i=H_i(s_i-r_i)>0,\end{cases} \tag{4.18}$$

where H_i is the inverse function of $G_i(A_i)$, and

$$H_i(0) = A_{i0}, \tag{4.19}$$

$$H_i'(0) = \sqrt{\frac{A_{i0}}{gh_i'(A_{i0})}} > 0. \tag{4.20}$$

Thus, system (4.1) can be equivalently rewritten as

$$\begin{cases} \dfrac{\partial r_i}{\partial t} + \lambda_{i1}(r_i, s_i)\dfrac{\partial r_i}{\partial x} = 0, \\ \dfrac{\partial s_i}{\partial t} + \lambda_{i2}(r_i, s_i)\dfrac{\partial s_i}{\partial x} = 0, \end{cases} \quad t \geq 0,\ 0 \leq x \leq L_i,\ i = 1, \cdots, N, \tag{4.21}$$

where

$$\begin{aligned} &\lambda_{i1}(r_i, s_i) \overset{\text{def}}{=} r_i + s_i + V_{i0} - \sqrt{gH_i(s_i - r_i)h_i'(H_i(s_i - r_i))} \\ &< \lambda_{i2}(r_i, s_i) \overset{\text{def}}{=} r_i + s_i + V_{i0} + \sqrt{gH_i(s_i - r_i)h_i'(H_i(s_i - r_i))} < 0, \quad i = 1, \cdots, N. \end{aligned} \tag{4.22}$$

Moreover, the initial condition (4.13) becomes

$$t = 0: \quad (r_i, s_i) = (r_{i0}(x), s_{i0}(x)), \quad 0 \leq x \leq L_i,\ i = 1, \cdots, N, \tag{4.23}$$

where

$$\begin{cases} r_{i0}(x) = \dfrac{1}{2}(V_{i0}(x) - V_{i0} - G_i(A_{i0}(x))), \\ s_{i0}(x) = \dfrac{1}{2}(V_{i0}(x) - V_{i0} - G_i(A_{i0}(x))), \end{cases} \quad i = 1, \cdots, N. \tag{4.24}$$

As in the proof of Theorem 3.1, (r_1, s_1) at $x = L_1$ and (r_i, s_i) $(i = 2, \cdots, N-1)$ at $x = 0$ can be uniquely determined by the observed values $a_i(t)$ and $v_i(t)$ $(i = 1, \cdots, N-1)$, respectively, as follows:

$$\begin{cases} r_i = \dfrac{1}{2}\Big(v_i(t) - V_{i0} - \displaystyle\int_{A_{i0}}^{a_i(t)} \sqrt{\frac{gh_i'(A_i)}{A_i}}\mathrm{d}A_i\Big), \\ s_i = \dfrac{1}{2}\Big(v_i(t) - V_{i0} + \displaystyle\int_{A_{i0}}^{a_i(t)} \sqrt{\frac{gh_i'(A_i)}{A_i}}\mathrm{d}A_i\Big), \end{cases} \quad i = 1, \cdots, N-1 \tag{4.25}$$

and

$$\|(r_i, s_i)\|_{C^1[0,T]} \leq C(\|a_i(t) - A_{i0}\|_{C^1[0,T]} + \|v_i(t) - V_{i0}\|_{C^1[0,T]}), \quad i = 1, \cdots, N-1. \tag{4.26}$$

Now, for $i = 1$, changing the status of t and x in (4.21) and using (4.25) as the initial data on $x = L_1$, we can solve a leftward Cauchy problem on canal c_1. As in the proof of Theorem 2.1, noting (4.11), we get that $(r_{10}(x), s_{10}(x))$ can be uniquely determined by $a_1(t)$ and $v_1(t)$ and

$$\|(r_{10}(x), s_{10}(x))\|_{C^1[0,L_1]} \leq C(\|a_1(t) - A_{10}\|_{C^1[0,T]} + \|v_1(t) - V_{10}\|_{C^1[0,T]}); \tag{4.27}$$

moreover, there exists T_1,

$$T_1 > \max_{i=2,\cdots,N} \frac{1}{|\widetilde{\lambda}_{i2}|}, \tag{4.28}$$

such that at $x = 0$, on the interval $[0, T_1]$, (r_1, s_1) can be also uniquely determined by $a_1(t)$ and $v_1(t)$ and

$$\|(r_1, s_1)\|_{C^1[0,T_1]} \leq C(\|a_1(t) - A_{10}\|_{C^1[0,T]} + \|v_1(t) - V_{10}\|_{C^1[0,T]}). \tag{4.29}$$

At $x = 0$, the interface conditions (4.4) and (4.5) now become

$$P_1 \stackrel{\text{def}}{=} \sum_{i=1}^{N} (r_i + s_i + V_{i0}) H_i(s_i - r_i) \Big(\frac{1}{2}(r_i + s_i + V_{i0})^2 + g h_i(H_i(s_i - r_i)) + gY_i \Big) = 0 \tag{4.30}$$

and

$$P_2 \stackrel{\text{def}}{=} \sum_{i=1}^{N} (r_i + s_i + V_{i0}) H_i(s_i - r_i) = 0. \tag{4.31}$$

Since when $(r_i, s_i) = (0, 0)$ $(i = 1, \cdots, N)$,

$$\det \left| \frac{\partial(P_1, P_2)}{\partial(r_N, s_N)} \right| = 2A_{N0}V_{N0} \sqrt{\frac{A_{i0}}{g h_i'(A_{i0})}}\, (V_{N0}^2 - gA_{N0}h_N'(A_{N0})) < 0, \tag{4.32}$$

by the implicit function theorem, in a neighbourhood of $(r_i, s_i) = (0, 0)$ $(i = 1, \cdots, N)$, (4.30), (4.31) can be equivalently rewritten as

$$\begin{cases} r_N = g_{N1}(r_1, s_1, \cdots, r_{N-1}, s_{N-1}), \\ s_N = g_{N2}(r_1, s_1, \cdots, r_{N-1}, s_{N-1}), \end{cases} \tag{4.33}$$

where g_{N1} and g_{N2} are C^1 functions with respect to their arguments with

$$g_{N1}(0, \cdots, 0) = g_{N2}(0, \cdots, 0) = 0. \tag{4.34}$$

So at $x = 0$, on the interval $[0, T_1]$, (r_N, s_N) can be uniquely determined by $a_i(t)$ and $v_i(t)$ $(i = 1, \cdots, N - 1)$ and

$$\|(r_N, s_N)\|_{C^1[0,T_1]} \leq C\Big(\sum_{i=1}^{N-1} \|a_i(t) - A_{i0}\|_{C^1[0,T]} + \sum_{i=1}^{N-1} \|v_i(t) - V_{i0}\|_{C^1[0,T]} \Big). \tag{4.35}$$

Now, for $i = 2, \cdots, N$, changing the status of t and x in (4.21) and using (4.25) and (4.33) as the initial data on $x = 0$, we can solve the rightward Cauchy problem on each canal c_i respectively. Noting (4.28), as in the proof of Theorem 2.1, we get that $(r_{i0}(x), s_{i0}(x))$ $(i = 2, \cdots, N)$ can be uniquely determined and

$$\begin{aligned} &\|(r_{i0}(x), s_{i0}(x))\|_{C^1[0,L_i]} \\ &\leq C\Big(\sum_{i=1}^{N-1} \|a_i(t) - A_{i0}\|_{C^1[0,T]} + \sum_{i=1}^{N-1} \|v_i(t) - V_{i0}\|_{C^1[0,T]} \Big), \quad i = 2, \cdots, N. \end{aligned} \tag{4.36}$$

Note (4.24). The combination of (4.27) and (4.36) yields the desired conclusion.

The procedure of resolution given by Theorem 4.1 can be illustrated by Figure 3, in which at the node marked by $\bullet$, all related values should be observed; at the node marked by $\odot$, a part of related values should be observed; and at the node marked by $\circ$, no observation is needed.

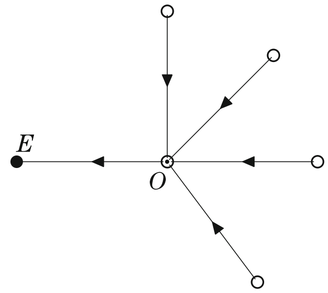

Figure 3 The procedure of resolution given by Theorem 4.1.

5 Exact Boundary Observability of Unsteady Supercritical Flows in a Tree-Like Network of Open Canals

We now consider the exact boundary observability of unsteady flows in a tree-like network composed by N open canals $c_1, \cdots, c_N$. Suppose that a single node is the end point E and the water flows from other single nodes to the point E (see Figure 4, in which "$\rightarrow$" stands for the direction of the water flow).

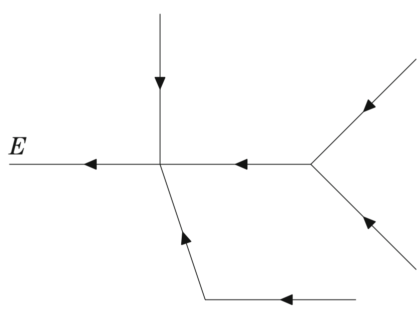

Figure 4 A single node is the end point E and the water flows from other single nodes to the point E.

For $i = 1, \cdots, N$, let d_{i0} and d_{i1} be the x-coordinates of two ends of the i-canal C_i, $d_{i0} < d_{i1}$ and $L_i = d_{i1} - d_{i0}$ be its length. Suppose that the water in the i-canal flows from d_{i1} to d_{i0} $(i = 1, \cdots, N)$. Under the assumption that there is no friction and all the canals are horizontal and cylindrical, the corresponding Saint-Venant system can be written as

$$\begin{cases} \dfrac{\partial A_i}{\partial t} + \dfrac{\partial (A_i V_i)}{\partial x} = 0, \\ \dfrac{\partial V_i}{\partial t} + \dfrac{\partial S_i}{\partial x} = 0, \end{cases} \quad t \geq 0,\ d_{i0} \leq x \leq d_{i1},\ i = 1, \cdots, N, \tag{5.1}$$

where S_i $(i = 1, \cdots, N)$ are given by (4.2).

When d_{i1} is a simple node, we have the flux boundary condition

$$x = d_{i1}: \quad Q_i \stackrel{\text{def}}{=} A_i V_i = q_{i1}(t), \quad V_i = v_{i1}(t). \tag{5.2}$$

While, when d_{i1} is a multiple node, at d_{i1} we have the total energy interface condition

$$\sum_{\substack{j \in J_{i1} \\ j \neq i}} A_j V_j S_j = A_i V_i S_i \tag{5.3}$$

and the total flux interface condition

$$\sum_{\substack{j \in J_{i1} \\ j \neq i}} A_j V_j = A_i V_i, \tag{5.4}$$

where J_{i1} denotes the set of indices corresponding to all the canals jointed at d_{i1}.

Based on Theorem 4.1, we choose a group of observed values as follows:

(i) For simple nodes, we take the observation only on the end point E. Suppose that E is the simple node of canal c_i. We observe A_i and V_i on it.

(ii) For any given multiple node, suppose that it is the joint point of k canals: $c_{i_1}, \cdots, c_{i_k}$, which constitute a star-like subnetwork. Suppose that the end point for this star-like subnetwork belongs to c_{i_1}. We observe $A_{i_2}, V_{i_2}, \cdots, A_{i_{k-1}}, V_{i_{k-1}}$ on this multiple node: there are $2(k-2)$ observed values on it.

Using this principle, we can get the following theorems.

Theorem 5.1 *Consider a supercritical equilibrium state* $(A_i, V_i) = (A_{i0}, V_{i0})$ $(i = 1, \cdots, N)$ *of system* (4.1) *with* $A_{i0} > 0$ $(i = 1, \cdots, N)$, *which satisfies*

$$V_{i0} < -\sqrt{g A_{i0} h_i'(A_{i0})}, \quad i = 1, \cdots, N. \tag{5.5}$$

Let

$$\widetilde{\lambda}_{i1} \stackrel{\text{def}}{=} V_{i0} - \sqrt{g A_{i0} h_i'(A_{i0})} < \widetilde{\lambda}_{i2} \stackrel{\text{def}}{=} V_{i0} + \sqrt{g A_{i0} h_i'(A_{i0})} < 0 \tag{5.6}$$

and

$$T > \max_{d_{i1} \in K} \sum_{j \in D_i} \frac{L_j}{|\widetilde{\lambda}_{j2}|}, \tag{5.7}$$

where K *stands for the set of all simple nodes except point* E, *and* D_i *the set of indices corresponding to all the canals in the string-like subnetwork connecting the points* E *and* d_{i1}.

For any given initial condition

$$t = 0: \quad (A_i, V_i) = (A_{i0}(x), V_{i0}(x)), \quad i = 1, \cdots, N, \tag{5.8}$$

such that the conditions of piecewise C^1 *compatibility are satisfied and* $\sum_{i=1}^{N} \|(A_{i0}(x) - A_{i0}, V_{i0}(x) - V_{i0})\|_{C^1[d_{i0}, d_{i1}]}$ *is suitably small, if we choose the observed values on the interval* $[0, T]$ *according to the principle mentioned above, then the initial data can be uniquely determined and we have the corresponding observability inequality.*

This theorem can be proved similarly to the proof of Theorem 4.1.

More precisely, we have the following theorem.

Theorem 5.2 *For any given tree-like network of open canals, if there are l simple nodes, then we need $2(l-1)$ observed values.*

Proof We prove this theorem by induction on the number m of the multiple nodes. If there is only 1 multiple node in the network, then it is a star-like network and the conclusion comes directly from Theorem 4.1.

Suppose that the conclusion is valid for any given network with m multiple nodes. Consider a network with $m+1$ multiple nodes and l simple nodes. Cutting the network at a multiple node M such that this network can be regarded as a subnetwork with m multiple nodes plus k canals, each of which has one original simple node (see Figure 5).

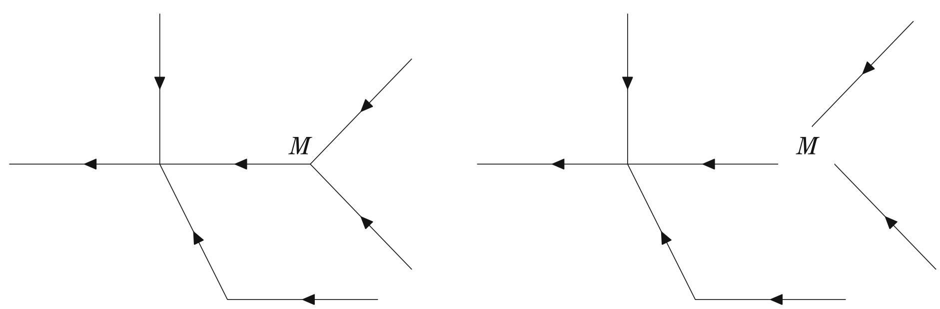

Figure 5 The network can be regarded as a subnetwork with m multiple nodes plus k canals, each of which has one original simple node.

This subnetwork should have $l-(k-1)$ simple nodes. According to the assumption of induction, we need $2[l-(k-1)-1]=2(l-k)$ observed values for the subnetwork and there is no observation at M. Moreover, the star-like network with M as its center node contains $k+1$ canals, then, by Theorem 5.1, for the original network, we need $2(k-1)$ observed values at M and there is no observation at all the original simple nodes in this star-like subnetwork. Therefore, the total number of the observed values is equal to $2(l-k)+2(k-1)=2(l-1)$.

Thus, Theorem 5.2 is obtained by induction.

Remark 5.1 Comparing with the results given in [1], we can find an implicit duality between the exact boundary controllability and the exact boundary observability of unsteady supercritical flows in a tree-like network as follows:

(1) In a tree-like network, the number of the observed values is equal to the number of the boundary controls. If the network contains l simple nodes, then both the number of the observed values and the number of the boundary controls are equal to $2(l-1)$.

(2) The observability time is equal to the controllability time. Both of them satisfy (5.7).

(3) The observed values are given on the ending simple node E and all the multiple nodes, while the controls are acted only on the simple nodes except E (see Figure 6, in which the

observations are taken on bold nodes "•", while the boundary controls are acted on hollow nodes "∘").

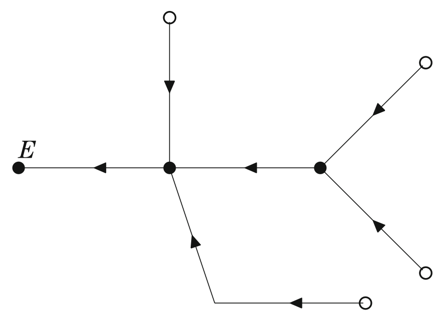

Figure 6 The observations are taken on bold nodes "•", while the boundary controls are acted on hollow nodes "∘".

References

[1] Gu, Q. L., Exact boundary controllability of unsteady supercritical flows in a tree-like network of open canals, *Math. Meth. Appl. Sci.*, **31**, 2008, 1497–1508.

[2] Gu, Q. L. and Li, T. T., Exact boundary observability of unsteady flows in a tree-like network of open canals, *Math. Meth. Appl. Sci.*, **32**, 2009, 395–418.

[3] Gugat, M., Leugering, G. and Schmidt, E. G., Global controllability between steady supercritical flows in channel networks, *Math. Meth. Appl. Sci.*, **27**, 2004, 781–802.

[4] Leugering, G. and Schmidt, E. G., On the modelling and stabilization of flows in networks of open canals, *SIAM J. Control Optim.*, **41**, 2002, 164–180.

[5] Li, T. T., Exact boundary controllability for quasilinear hyperbolic systems and its application to unsteady flows in a network of open canals, *Math. Meth. Appl. Sci.*, **27** , 2004, 1089–1114.

[6] Li, T. T., Exact boundary controllability of unsteady flows in a network of open canals, *Math. Nachr.*, **278**, 2005, 278–289.

[7] Li, T. T., Observabilité exacte frontière pour des systèmes hyperboliques quasi linéaires, *C. R. Acad. Sci. Paris, Ser. I*, **342**, 2006, 937–942.

[8] Li, T. T., Exact boundary observability for quasilinear hyperbolic systems, *ESIAM: Control, Optimisation and Calculus of Variations*, **14**, 2008, 759–766.

[9] Li, T. T. and Jin, Y., Semi-global C^1 solution to the mixed initial-boundary value problem for quasilinear hyperbolic systems, *Chin. Ann. Math.*, **22B**(3), 2001, 325–336.

[10] Li, T. T. and Rao, B. P., Exact boundary controllability for quasilinear hyperbolic systems, *SIAM J. Control Optim.*, **41**, 2003, 1748–1755.

[11] Li, T. T. and Rao, B. P., Local exact boundary controllability for a class of quasilinear hyperbolic systems, *Chin. Ann. Math.*, **23B**(2), 2002, 209–218.

[12] Li, T. T. and Rao, B. P., Contrôlabilité exacte frontière de l'écoulement d'un fluide non-stationnaire dans un réseau du type d'arbre de canaux ouverts, *C. R. Acad. Sci. Paris, Ser. I*, **339**, 2004, 867–872.

[13] Li, T. T. and Rao, B. P., Exact boundary controllability of unsteady flows in a tree-like network of open canals, *Meth. Appl. Anal.*, **11**, 2004, 353–366.

[14] de Saint-Venant, B., Théorie du mouvement non permanent des eaux, avec application aux crues des rivières et l'introduction des marées dans leur lit, *C. R. Acad. Sci.*, **73**, 1871, 147–154, 237–240.