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Regularity of Keldys-Fichera Boundary Value Problem for Degenerate Elliptic Equations***

Limei LI* Tian MA**

(*Dedicated to Professor Roger Temam on the Occasion of his* 70*th Birthday*)

Abstract The authors discuss the $W^{1,p}$-solutions and the interior regularity of weak solutions for the Keldys-Fichera boundary value problem using the acute angle principle, the reversed Hölder inequality and the generalized poincaré inequalities.

Keywords Keldys-Fichera boundary value problem, $W^{1,p}$-regularity, Interior regularity
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1 Introduction

The Keldys-Fichera boundary value problem for linear equations with nonnegative characteristic form of second order is well-known. Oleinik and Radkevich made a detailed discussion on this subject (see [8]). Ma and Yu [7] investigated the existence of the Keldys-Fichera boundary value problem of degenerate quasilinear elliptic equations of second order and discussed the maximum principle, the comparison principle and the modular estimate by using the acute angle principle for weakly continuous mappings. Chen and Xuan discussed this boundary value problem for degenerate elliptic equations, and obtained the existence and uniqueness of the solutions by using the pseudo-monotone operator method in [2, 3]. Li [6] studied the Keldys-Fichera boundary value problem for a class of quasilinear elliptic equations with double degenerate and proved the existence of solution by means of the weighted Sobolev space and the pseudo-monotone operator method. Wang [10] investigated the regularity of a type of elliptic equation by using the compactness method and obtained an optimal Hölder estimates. In this paper, we study the regularity of the Keldys-Fichera boundary value problem by using the reversed Hölder inequality, the generalized poincaré inequalities to discuss the $W^{1,p}$-solutions and the interior regularity of weak solutions for the following equations:

$$\begin{cases} Lu = D_i[a_{ij}(x,u)D_ju + b_i(x)u] - c(x,u) = f(x), & x \in \Omega, \\ u(x) = 0, & x \in \Sigma_2 \cup \Sigma_3, \end{cases} \tag{1.1}$$

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*Corresponding author. Mathematical College, Sichuan University, Chengdu 610064, China.
E-mail: matlilm@yahoo.cn
**Mathematical College, Sichuan University, Chengdu 610064, China. E-mail: matian56@sina.com
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where $\Omega \subset R^n$ is an open set, and Σ_i $(i = 1, 2, 3)$ are defined by

$$\Sigma_3 = \{x \in \partial\Omega \mid a_{ij}(x,0)N_iN_j > 0\},$$
$$\Sigma_2 = \{x \in \partial\Omega\backslash\Sigma_3 \mid b_i(x) \cdot N_i > 0\},$$
$$\Sigma_1 = \partial\Omega\backslash(\Sigma_2 \cup \Sigma_3),$$

where $\overrightarrow{N} = (N_1, \cdots, N_n)$ is the unit outward normal vector on $\partial\Omega$.

An important example relating to degenerate elliptic equations is the following well-known Tricomi equation, which is of especially interest in the aerodynamics,

$$y\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad (x, y) \in R^2. \tag{1.2}$$

The Tricomi equation is a mixed equation of elliptic-hyperbolic type. As $y > 0$, equation (1.2) is elliptic and when $y < 0$ it is hyperbolic. Equation (1.2) can be divided into two equations to be considered respectively as follows

$$y\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{for } (x, y) \in R^2_+. \tag{1.3}$$

where $R^2_+ = \{(x, y) \in R^2 \mid y > 0\}$, and

$$\frac{\partial^2 u}{\partial y^2} - y\frac{\partial^2 u}{\partial x^2} = 0 \quad \text{for } (x, y) \in R^2_+. \tag{1.4}$$

It is easy to see that equation (1.3) is a degenerate elliptic equation and (1.4) is a hyperbolic equation in R^2_+. If $u_i(x, y)$ $(i = 1, 2)$ are respectively the solutions of (1.3) and (1.4), with

$$u_1(x, 0) = u_2(x, 0), \qquad \forall x \in R^1,$$

then the function

$$u(x, y) = \begin{cases} u_1(x, y), & \text{as } y \geq 0, \\ u_2(x, -y), & \text{as } y \leq 0 \end{cases}$$

is a weak solution of Tricomi equation (1.2).

Generally, most of the mixed equations of elliptic-hyperbolic type can be divided into the degenerate elliptic and Hyperbolic equations to be discussed respectively. In general, for degenerate elliptic boundary value problem with Dirichlet boundary condition, the set of degenerate points on boundary $\partial\Omega$ is of nozero measure. It implies that the Dirchlet boundary value problem for degenerate elliptic is not well-posed anymore, and instead of it the Keldys-Fichera boundary value problem works. On the well-posedness of Keldys-Fichera boundary value problem for degenerate elliptic equations, the readers are referred to [8].

2 Recapitulation on Known Results

In this section, we present some known results in [7] for Keldys-Fichera boundary value problem for the degenerate quasilinear elliptic equation. Let Ω be a bounded domain in R^n, $n \geq 2$ with piecewise C^1-boundary $\partial\Omega$. Suppose that the coefficients of equation (1.1) satisfy Carathéodory conditions and

(H_1) Symmetry: $a_{ij}(x, z) = a_{ji}(x, z)$ for all $x, z \in \Omega$;

(H_2) There exist a constant $\beta > 0$ and a nonnegative continuous function $\lambda(x)$ on $\overline{\Omega}$ such that

$$\beta^{-1}a_{ij}(x,0)\xi_i\xi_j \le a_{ij}(x,z)\xi_i\xi_j \le \beta a_{ij}(x,0)\xi_i\xi_j, \tag{2.1}$$

$$\lambda(x)|\xi|^2 \le a_{ij}(x,0)\xi_i\xi_j; \tag{2.2}$$

(H_3) $\Omega' = \{x \in \Omega \mid \lambda(x) = 0\}$ is a set of zero measure in R^n, and there are bounded subdomains with the cone property $\Omega_n \subset\subset \Omega\backslash\Omega'$, such that $\Omega_n \subset \Omega_{n+1}$ and $\bigcup\limits_{n=1}^{\infty} \Omega_n = \Omega\backslash\Omega'$;

(H_4) $b_i(x) \in C^1(\overline{\Omega})$ $(1 \le i \le n)$, and

$$|a_{ij}(x,z)| \le C, \tag{2.3}$$

$$C[|z|^k + |z|^2] - g_1(x) \le c(x,z)z - \frac{1}{2}D_ib_i(x)z^2, \tag{2.4}$$

$$|c(x,z)| \le C|z|^{k-1} + g_2(x), \tag{2.5}$$

where $k > 1,\ C > 0$ are constants, $g_1 \in L^1(\Omega),\ g_2 \in L^{k'}(\Omega),\ \frac{1}{k} + \frac{1}{k'} = 1$.

Let X be a linear space, X_1, X_2 be the completion of X respectively with the norms $\|\cdot\|_1$ and $\|\cdot\|_2$. Denote by

$$X = \{v \in C^1(\overline{\Omega}),\ v|_{\Sigma_3} = 0 \text{ and } \|v\|_2 < \infty\}$$

endowed with the norm

$$\|v\|_2 = \Big[\int_\Omega (|\nabla v|^2 + |v|^2)\mathrm{d}x + \int_{\partial\Omega} |v|^2 \mathrm{d}s\Big]^{\frac{1}{2}} + \Big[\int_\Omega |v|^k \mathrm{d}x\Big]^{\frac{1}{k}}.$$

Let X_1 be the completion of X under the norm

$$\|v\|_1 = \Big[\int_\Omega (a_{ij}(x,0)D_ivD_jv + |v|^2)\mathrm{d}x\Big]^{\frac{1}{2}} + \Big[\int_{\Sigma_1\cup\Sigma_2} |\overrightarrow{b}\cdot\overrightarrow{N}|v^2\mathrm{d}s\Big]^{\frac{1}{2}} + \Big[\int_\Omega |v|^k\mathrm{d}x\Big]^{\frac{1}{k}},$$

where $\overrightarrow{b} = \{b_1(x), b_2(x), \cdots, b_n(x)\}$. Obviously, X is a separable normed space, X_1 is a reflexive Banach space, and X_2 is a separable Banach space.

A weak solution of (1.1) is defined to be an element $u \in X_1$ such that

$$\int_\Omega [a_{ij}(x,u)D_juD_iv + b_i(x)uD_iv + c(x,u)v + fv]\,\mathrm{d}x - \int_{\Sigma_1} \overrightarrow{b}\cdot\overrightarrow{N}uv\mathrm{d}s = 0, \quad \forall v \in X_2. \tag{2.6}$$

Firstly, we introduce the acute angle principle for weakly continuous operator.

Definition 2.1 (see [7]) *Let X_1, X_2 be two Banach spaces. A mapping $G : X_1 \to X_2^*$ is called to be weakly continuous, if for any $x_n, x_0 \in X_1, x_n \rightharpoonup x_0$ in X_1, there exists a subsequence $\{x_{n_k}\}$ such that*

$$\lim_{x\to\infty} \langle Gx_{n_k}, y\rangle = \langle Gx_0, y\rangle, \quad \forall y \in X_2.$$

Theorem 2.1 *Suppose that $G : X_1 \to X_2^*$ is weakly continuous. If there exists a bounded open set $\Omega \subset X_1$, such that*

$$\langle Gu, u\rangle \ge 0, \quad \forall u \in \partial\Omega \cap X,$$

then the equation $Gu = 0$ has a solution in X_1.

Theorem 2.2 *Under the conditions* (H$_1$)–(H$_4$)*, if* $f \in L^{k'}(\Omega)$*, then problem* (1.1) *has a weak solution in* X_1*.*

Now, we present the known results in [7] about the maximum principle, L^∞-modular estimates and the comparison principle for weak solutions of degenerate elliptic Keldys-Fichera boundary value problem. Let $\Omega \subset R^n$ be a bounded domain. We firstly consider the linear case. Give the following operator:

$$L_1 u = D_i(a_{ij}(x) D_j u + b_i(x) u) - c(x) u,$$

$b_i \in C^1(\overline{\Omega})$ and $a_{ij}(x) = a_{ji}(x)$, furthermore,

$$0 \le a_{ij}(x)\xi_i\xi_j, \quad \forall x \in \overline{\Omega},\ \xi \in R^n.$$

Let $\widetilde{X}_1$ be the completion of $C^1(\overline{\Omega})$ with the norm

$$\|u\|_{\widetilde{X}_1} = \Big[\int_\Omega (a_{ij}(x) D_i u D_j u + u^2)\mathrm{d}x + \int_{\Sigma_1 \cup \Sigma_2} |\overrightarrow{b} \cdot \overrightarrow{N}| u^2 \mathrm{d}s\Big]^{\frac{1}{2}}.$$

We say that $u \in \widetilde{X}_1 \cap W^{1,p}(\Omega) \cap L^q(\Omega)$ $(1 < p \le 2, \frac{1}{p} + \frac{1}{q} = 1)$ satisfies $L_1 u \ge 0$ (or $L_1 u \le 0$) in weak sense, if $\forall v \in \widetilde{X}_1 \cap W^{1,p}(\Omega) \cap L^q(\Omega)$ with $v|_{\Sigma_2 \cup \Sigma_3} = 0$, $v \ge 0$ in Ω, the following inequality holds:

$$\int_\Omega [a_{ij}(x) D_j u D_i v + b_i(x) u D_i v + c(x) uv]\mathrm{d}x - \int_{\Sigma_1} \overrightarrow{b} \cdot \overrightarrow{N} uv \mathrm{d}s \le 0 \ (\text{or } \ \ge 0). \tag{2.7}$$

Theorem 2.3 *Let* $\Sigma_2 \cup \Sigma_3 \ne \varnothing$*, and* $b^*(x) < c(x)$*,* $\forall x \in \Omega$*, where*

$$b^*(x) = \max\Big\{D_i b_i(x), \frac{1}{2} D_i b_i(x)\Big\}. \tag{2.8}$$

If $u \in \widetilde{X}_1 \cap W^{1,p}(\Omega) \cap L^q(\Omega)$ $(\frac{1}{p} + \frac{1}{q} = 1)$ *satisfies* $L_1 u \ge 0$ (*or* $L_1 u \ge 0$) *in weak sense, then the nonnegative maximum* (*nonpositive minimum*) *of* u *must be achieved in* $\overline{\Sigma_2 \cup \Sigma_3}$*.*

Here we present the modular estimate theorem for weak solutions of equations (1.1). The condition (H$_2$) is changed to read

$$0 \le a_{ij}(x, 0)\xi_i\xi_j, \quad \forall x \in \Omega,\ \xi \in R^n. \tag{2.9}$$

Theorem 2.4 *Assume that* $\Sigma_2 \cup \Sigma_3 \ne \varnothing$ *and* L *satisfies* (H$_1$), (H$_3$), (2.9) *and*

$$b^*(x) < c(x, z) z^{-1} \quad \textit{for } (x, z) \in \overline{\Omega} \times R. \tag{2.10}$$

If $u \in \widetilde{X}_1 \cap W^{1,p}(\Omega) \cap L^k(\Omega)$ $(\frac{p}{p-1} \le k)$ *satisfies* (2.6)*,* $\forall v \in \widetilde{X}_1 \cap W^{1,p}(\Omega) \cap L^k(\Omega)$ *with* $v|_{\Sigma_2 \cup \Sigma_3} = 0$*, then*

$$|u| = \max\Big\{\sup_{\Omega}\Big|\frac{f}{C^*}\Big|,\ \sup_{\Sigma_2 \cup \Sigma_3} |u|\Big\} = M,$$

where $C^*(x) = \inf\limits_{z \in R^1} [c(x, z) z^{-1} - D_i b_i(x)]$*.*

Finally, we present the comparison principle.

Theorem 2.5 *Let* $b^*(x) \le c(x, z) z^{-1}$*,* $\forall (x, z) \in \Omega \times R^1$*. If* $f(x) \le 0$ *and* u *is a weak solution of* (1.1) *in* $X_1 \cap W^{1,p}(\Omega)$ $\big(1 < p, \frac{p}{p-1} \le k\big)$*, then* $u(x) \ge 0$ *on* Ω*.*

Remark 2.1 In the degenerate elliptic equations, if the terms $D_i(b_i(x) u) \equiv 0$, $1 \le i \le n$, hold then in all the theorems of this section, the condition $u \in \widetilde{X}_1 \cap W^{1,p}(\Omega) \cap L^k(\Omega)$ $\big(\frac{1}{p} + \frac{1}{k} = 1\big)$ can be relaxed as that $u \in \widetilde{X}_1$.

3 $W^{1,p}$-Solutions of the Quasilinear Equations

We start with an abstract regularity result which is useful for the existence problem of $W^{m,p}(\Omega)$-solutions of degenerate quasilinear elliptic equations of order $2m$. Let X, X_1, X_2 be the spaces defined in Definition 2.1, and Y be a reflective Banach space, at the same time $Y \hookrightarrow X_1$.

Lemma 3.1 (see [7]) *Under the hypotheses of Theorem* 2.1, *there exists a sequence of* $\{u_n\} \subset X$, $u_n \rightharpoonup u_0$ *in* X_1 *such that* $\langle Gu_n, u_n\rangle = 0$. *Furthermore, if we can derive that* $\|u_n\|_Y < C$, *where* C *is a constant, then the solution* u_0 *of* $Gu = 0$ *belongs to* Y.

The proof of Lemma 3.1 is obvious.

Now we return to discuss the existence of $W^{1,p}$-solutions of equation (1.1). Let Ω be a bounded domain of R^n and C^∞.

Theorem 3.1 (see [7]) *Under the conditions* (H_1)–(H_4) *and* $f \in L^{k'}$, *if there is a real number* $\beta > 1$ *such that*

$$\int_\Omega |\lambda(x)|^{-\beta} \mathrm{d}x < \infty, \quad \lambda(x) \text{ defined as in (2.2)},$$

then (1.1) *has a weak solution* $u \in X_1 \cap W^{1,p}(\Omega)$, $p = \frac{2\beta}{1+\beta} > 1$. *Moreover, if* $\Sigma_2 \cap \Sigma_3 \neq \varnothing$, *and when* $b_i \not\equiv 0$ *for some* $1 \le i \le n$, $k \ge \frac{2\beta}{\beta-1}$, *and* $c(x,z)z^{-1} - D_i b_i \ge \alpha > 0$, $\forall (x,z) \in \Omega \times R^1$, *otherwise,* $c(x,z)z^{-1} \ge \alpha > 0$, *then the solution* $u \in L^\infty(\Omega)$ *provided* $f \in L^\infty(\Omega)$.

Proof According to Lemma 3.1, it suffices to prove that there is a constant $C > 0$ such that for any $u \in X$ (X is as that in Section 2) with $\langle Lu, u\rangle = 0$, we have

$$\|u\|_{W^{1,p}} \le C, \quad p = \frac{2\beta}{1+\beta}. \tag{3.1}$$

From (2.6), we know

$$\begin{aligned}\langle Lu, u\rangle = &\int_\Omega [a_{ij}(x,u) D_i u D_j u + b_i(x) u D_i u + c(x,u)u + fu] \mathrm{d}x \\ &- \int_{\Sigma_1} \overrightarrow{b} \cdot \overrightarrow{N} u^2 \mathrm{d}s = 0, \quad u \in X.\end{aligned}$$

Due to (H_2) and (2.4), we have

$$\begin{aligned}\langle Lu, u\rangle &= \int_\Omega \Big[a_{ij}(x,u) D_i u D_j u + c(x,u)u - \frac{1}{2} D_i b_i u^2 + fu\Big] \mathrm{d}x \\ &\quad + \frac{1}{2}\int_{\Sigma_2} \overrightarrow{b} \cdot \overrightarrow{N} u^2 \mathrm{d}s - \frac{1}{2}\int_{\Sigma_1} \overrightarrow{b} \cdot \overrightarrow{N} u^2 \mathrm{d}s \\ &\ge \int_\Omega [\beta^{-1}\lambda(x)|\nabla u|^2 + C|u|^k - fu - g_1] \mathrm{d}x + \frac{1}{2}\int_{\Sigma_1 \cup \Sigma_2} |\overrightarrow{b} \cdot \overrightarrow{N}| u^2 \mathrm{d}s.\end{aligned}$$

Consequently, we have

$$\int_\Omega [g_1 + C_1|f|^{k'}] \mathrm{d}x \ge \int_\Omega \Big[\beta^{-1}\lambda(x)|\nabla u|^2 + \frac{C}{2}|u|^k\Big] \mathrm{d}x. \tag{3.2}$$

By the reversed Hölder inequality (see [1]), we get

$$\int_\Omega \lambda(x)|\nabla u|^2 \mathrm{d}x \geq \Big[\int_\Omega |\lambda|^{-\beta}\mathrm{d}x\Big]^{-\frac{1}{\beta}}\Big[\int_\Omega |\nabla u|^{\frac{2\beta}{1+\beta}}\mathrm{d}x\Big]^{\frac{1+\beta}{\beta}}. \tag{3.3}$$

From (3.2) and (3.3), the estimate (3.1) follows. The second conclusion follows from Theorem 2.4 and Remark 2.1. The proof is completed.

Example 3.1 We consider the $W^{1,p}$-solutions of the following Keldys equation:

$$\begin{cases} \dfrac{\partial}{\partial x}\Big(x^{\alpha_1} f_1(u)\dfrac{\partial u}{\partial x}\Big) + \dfrac{\partial}{\partial y}\Big(y^{\alpha_2} f_2(u)\dfrac{\partial u}{\partial y}\Big) - u = f(x), \\ u|_{\Sigma_3} = 0, \end{cases} \tag{3.4}$$

where $\Sigma_2 = \varnothing$, and $\Omega = (0,1)\times(0,1)$, $\Sigma_3 = \{(x,y) \mid x = 1,\ 0 < y \leq 1 \text{ and } y = 1,\ 0 < x \leq 1\}$, $f_1, f_2 \in C(R)$ satisfy the condition

$$0 < C_1 \leq f_1(z), f_2(z) \leq C_2 < \infty.$$

It is easy to see that

$$\lambda(x,y) = \min\{C_1 x^{\alpha_1}, C_2 y^{\alpha_2}\},$$

where $C_1, C_2 > 0$ are constants. If $0 < \alpha_1, \alpha_2 < \frac{1}{2}$, then for $\beta = 2$, we have

$$\int_0^1\int_0^1 |\lambda|^{-2}\mathrm{d}x\mathrm{d}y \leq C_1^2 \int_0^1 x^{-2\alpha_1}\mathrm{d}x \int_0^1 y^{-2\alpha_2}\mathrm{d}y < +\infty.$$

Furthermore, we have $c(x,z)z^{-1} = 1 > 0$. Therefore, by Theorem 3.1, equation (3.4) has a weak solution $u \in W^{1,\frac{4}{3}}(\Omega) \cap L^\infty(\Omega)$ provided $f \in L^\infty(\Omega)$.

Next we investigate the $W^{1,p}$-solutions of the degenerate quasilinear elliptic equations as follows

$$\begin{cases} -D_i(a_{ij}(x,u)D_j u + b_i(x)u) + c(x,u,\nabla u) = f(x), \quad x \in \Omega, \\ u|_{\Sigma_2\cup\Sigma_3} = 0. \end{cases} \tag{3.5}$$

Suppose that

(A_1) The conditions (H_1) and (H_2) in Theorem 2.2 hold, Σ_i $(i = 1,2,3)$ are the same as in (1.1), and the measure of $\Sigma_2 \cup \Sigma_3$ is nonzero on $\partial\Omega$;

(A_2) For the function $\lambda(x)$ in (H_2), there exists a real number $\beta_0 > 1$, such that

$$\int_\Omega |\lambda(x)|^{-\beta_0}\mathrm{d}x < \infty;$$

(A_3) $b_i \in C^1(\overline{\Omega})$ and there is a function $g(x) \in L^1(\Omega)$ such that

$$g(x) \leq c(x,z,\xi)z - \frac{1}{2}D_i b_i z^2, \tag{3.6}$$

$$\begin{cases} |a_{ij}(x,z)| \leq C, \\ |c(x,z,\xi)| \leq C[|z|^{\alpha_1} + |\xi|^{\alpha_2} + 1], \end{cases} \tag{3.7}$$

$$0 \leq \alpha_1 < \frac{n(\beta_0 - 1) + 2\beta_0}{n(1+\beta_0) - 2\beta_0}, \quad 0 \leq \alpha_2 < \frac{n(\beta_0 - 1) + 2\beta_0}{n(1+\beta_0)}.$$

Let $X = \{u \in C^1(\overline{\Omega}) \mid u|_{\Sigma_3} = 0\}$, and Y be the completion of X with the norm

$$\|u\|_Y \quad = \quad \Big[\int_\Omega a_{ij}(x,0)D_iuD_ju\mathrm{d}x\Big]^{\frac{1}{2}} + \Big[\int_\Omega |\nabla u|^p\mathrm{d}x\Big]^{\frac{1}{p}} + \Big[\int_{\Sigma_1\cup\Sigma_2} |\overrightarrow{b}\cdot\overrightarrow{N}|u^2\mathrm{d}s\Big]^{\frac{1}{2}}.$$

Since $\mathrm{mes}\,\Sigma_2 \cup \Sigma_3 \neq 0$, by the generalized Poincaré inequalities (see [9]), we know that $\|u\|_Y \geq C\|u\|_{W^{1,p}}$, i.e., $Y \hookrightarrow W^{1,p}(\Omega)$. For equation (3.5), we always take $p = \frac{2\beta_0}{1+\beta_0}$ and

$$\beta_0 \geq \begin{cases} 1, & \text{if } b_i \equiv 0, \ \forall 1 \leq i \leq n, \\ n, & \text{if } b_i \not\equiv 0, \ \text{for some } 1 \leq i \leq n. \end{cases} \tag{3.8}$$

$u \in Y$ is called a weak solution of (3.5). If $\forall v \in Y$,

$$\int_\Omega [a_{ij}(x,u)D_juD_iv + b_iuD_iv + c(x,u,\nabla u)v - fv]\mathrm{d}x - \int_{\Sigma_1} \overrightarrow{b}\cdot\overrightarrow{N}uv\mathrm{d}s = 0. \tag{3.9}$$

By applying Theorems 2.1, 2.3 and Remark 2.1, we can obtain the following theorem.

Theorem 3.2 *Let conditions* (A_1)–(A_3) *be satisfied and* $f \in L^{p'}(\Omega)$ $(\frac{1}{p'} + \frac{1}{p} = 1)$. *Then problem* (3.5) *has a weak solution* $u \in Y$. *Moreover, if* $f \in L^\infty(\Omega)$, *and*

$$\inf_{\substack{z\in R^1\\ \xi\in R^n}} [c(x,z,\xi)z^{-1} - D_ib_i(x)] \geq \alpha > 0, \tag{3.10}$$

then the solution $u \in L^\infty(\Omega)$.

Proof Denote by $\langle Gu, v\rangle$ the left part of equality (3.9). It is easy to show that the inner product $\langle Gu, v\rangle$ defines a bounded continuous mapping $G : Y \to Y^*$ owing to (3.7) and (3.8). Firstly, we check the acute angle condition. Let $u \in Y$, we have

$$\begin{aligned}
\langle Gu, u\rangle &= \int_\Omega \Big[a_{ij}(x,u)D_iuD_ju - \frac{1}{2}D_ib_iu^2 + c(x,u,\nabla u)u - fu\Big]\mathrm{d}x \\
&\quad + \frac{1}{2}\int_{\Sigma_2} \overrightarrow{b}\cdot\overrightarrow{N}u^2\mathrm{d}s - \frac{1}{2}\int_{\Sigma_1} \overrightarrow{b}\cdot\overrightarrow{N}u^2\mathrm{d}s \\
&\geq \int_\Omega [\beta^{-1}a_{ij}(x,0)D_iuD_ju + g - fu]\mathrm{d}x + \frac{1}{2}\int_{\Sigma_1\cup\Sigma_2} |\overrightarrow{b}\cdot\overrightarrow{N}|u^2\mathrm{d}s \\
&\quad \text{(due to (2.1) and (3.6))} \\
&\geq \frac{\beta^{-1}}{2}\int_\Omega \lambda(x)|\nabla u|^2\mathrm{d}x + \frac{\beta^{-1}}{2}\int_\Omega [a_{ij}(x,0)D_iuD_ju + g - fu]\mathrm{d}x \\
&\quad + \frac{1}{2}\int_{\Sigma_1\cup\Sigma_2} |\overrightarrow{b}\cdot\overrightarrow{N}|u^2\mathrm{d}s \quad \text{(by (2.2))} \\
&\geq \frac{\beta^{-1}}{2}\Big[\int_\Omega |\lambda|^{-\beta_0}\mathrm{d}x\Big]^{-\frac{1}{\beta_0}}\Big[\int_\Omega |\nabla u|^p\mathrm{d}x\Big]^{\frac{2}{p}} + \frac{\beta^{-1}}{2}\int_\Omega a_{ij}(x,0)D_iuD_ju\mathrm{d}x \\
&\quad + \frac{1}{2}\int_{\Sigma_1\cup\Sigma_2} |\overrightarrow{b}\cdot\overrightarrow{N}|u^2\mathrm{d}s - C\int_\Omega |u|^p\mathrm{d}x - C\int_\Omega [|g| + |f|^{p'}]\mathrm{d}x \\
&\quad \text{(thanks to the reversed Hölder inequality).}
\end{aligned}$$

According to (A_2) and $p < 2$, from the above inequality, we can derive

$$\langle Gu, u\rangle \geq 0, \quad \forall u \in Y, \ \|u\|_Y = R \text{ great enough}.$$

Next we need to verify the continuous condition in Theorem 2.1.

Let $u_n \rightharpoonup u_0$ in Y (when $\beta_0 = 1$, let $u_n \underset{\rightarrow}{*} u_0$ in Y), and

$$\lim_{n\to\infty} \langle Gu_n - Gu_0, u_n - u_0\rangle = 0.$$

One can easily show that

$$\lim_{n\to\infty} \langle Gu_n, v\rangle = \langle Gu_0, v\rangle, \quad \forall v \in Y.$$

Here we omit the details of proof. Therefore the first conclusion of the theorem follows from Theorem 2.1.

Finally, by (3.10), we can obtain the second conclusion from Theorem 2.3 and Remark 2.1 by using the same method as the proof of Theorem 2.4 (see [7]). The proof is completed.

4 Interior Regularity

In this section, we concern the interior regularity of weak solutions of equation (3.5). Here a weak solution u of (3.5) means that u satisfies (3.9) for any $v \in C^1(\overline{\Omega})$ with $v|_{\Sigma_3} = 0$. We always assume that

$$0 \le a_{ij}(x,z)\xi_i\xi_j, \quad \forall (x,z) \in \Omega \times R,\ \xi \in R^n,$$

and the set $\Omega' = \{x \in \Omega \mid a_{ij}(x,z)\xi_i\xi_j = 0 \text{ for some } \xi \in R^n \text{ and } |\xi| \neq 0\}$ is independent of z, $\operatorname{mes}\Omega' = 0$ in R^n.

Suppose that $c(x,z,\xi) \in C^1(\overline{\Omega} \times R \times R^n)$ and

$$|c(x,z,\xi)| \le g(x,z), \quad g \in C(\Omega \times R^1). \tag{4.1}$$

Theorem 4.1 *Let* (4.1) *hold and* $f \in C^1(\overline{\Omega})$*. If* $u \in \widetilde{X} \cap L^\infty(\Omega)$ *is a weak solution of problem* (3.5)*,* $\widetilde{X}$ *defined as that in Section* 2*, then* $u \in C^\alpha(\Omega\backslash\Omega') \cap H^2_{\mathrm{loc}}(\Omega\backslash\Omega')$*,* $0 < \alpha < 1$*.*

Proof Since $\Omega\backslash\Omega'$ is open, for any $x_0 \in \Omega\backslash\Omega'$, there exists a close ball $\overline{B}_{2\delta}(x_0) = \{x \in \Omega \mid |x - x_0| \le 2\delta\} \subset \Omega\backslash\Omega'$ for some $\delta > 0$. It suffices to verify that $u \in C^\alpha(B_\delta(x_0)) \cap H^2_{\mathrm{loc}}(B_\delta(x_0))$.

Take $\eta \in C_0^\infty(\Omega)$ such that $\operatorname{supp}\eta \subset B_{2\delta}(x_0)$ and

$$0 \le \eta(x) \le 1, \quad \eta(x) = 1, \quad \text{as } x \in B_\delta(x_0).$$

Let $w = \eta u$. Then

$$\int_\Omega a_{ij}(x,u)D_jwD_iv\mathrm{d}x = \int_\Omega \eta(x)a_{ij}(x,u)D_juD_iv\mathrm{d}x + \int_\Omega a_{ij}(x,u)uD_j\eta D_iv\mathrm{d}x. \tag{4.2}$$

Putting $v = \eta v$ in (3.9), we have

$$\begin{aligned}&\int_\Omega \eta a_{ij}(x,u)D_iuD_jv\mathrm{d}x\\ &= -\int_\Omega [a_{ij}(x,u)vD_iuD_j\eta + b_iuvD_i\eta + b_iu\eta D_iv + c(x,u,\nabla u)\eta v - f\eta v]\mathrm{d}x.\end{aligned} \tag{4.3}$$

On the other hand,

$$-\int_\Omega a_{ij}(x,u)vD_iuD_j\eta\mathrm{d}x = \int_\Omega A_{ij}(x,u)D_j\eta D_iv\mathrm{d}x + \int_\Omega \Big[\frac{\partial A_{ij}}{\partial x_i}D_i\eta + A_{ij}(x,u)D_{ij}\eta\Big]v\mathrm{d}x,$$

where

$$A_{ij}(x,z)=\int_0^z a_{ij}(x,y)\mathrm{d}y.$$

Since $\operatorname{supp}\eta\subset B_{2\delta}(x_0)$, from (4.2) and (4.3), we have

$$\begin{aligned}&\int_{B_{2\delta}} a_{ij}(x,u)D_jwD_iv\mathrm{d}x\\ =&\int_{B_{2\delta}}[A_{ij}(x,u)D_j\eta+b_i(x)u\eta+a_{ij}(x,u)uD_j\eta]D_iv\mathrm{d}x\\ &+\int_{B_{2\delta}}\Big[\frac{\partial A_{ij}(x,u)}{\partial x_i}D_j\eta+A_{ij}(x,u)D_{ij}\eta+f\eta-b_i(x)uD_i\eta\Big]v\mathrm{d}x. \end{aligned}\tag{4.4}$$

Denote

$$\begin{cases} g_i(x)=A_{ij}(x,u)D_j\eta+a_{ij}(x,u)uD_j\eta+b_i(x)u\eta,\\ g(x)=\dfrac{\partial A_{ij}(x,u)}{\partial x_i}D_j\eta+A_{ij}(x,u)D_{ij}\eta+f\eta-b_i(x)uD_i\eta-c(x,u,\nabla u)\eta.\end{cases}$$

Because $\overline{B}_{2\delta}\subset\Omega\backslash\Omega'$, there exists a constant $\varepsilon>0$ such that

$$\varepsilon|\xi|^2\le a_{ij}(x,z)\xi_i\xi_j,\quad \forall (x,z)\in\overline{B}_{2\delta}(x_0)\times R.$$

Hence $w\in W^{1,2}(B_{2\delta})\cap L^\infty(B_{2\delta})$ is a weak solution of the following equation:

$$\begin{cases}-D_i(a_{ij}(x,u)D_jw)=g-D_ig_i,\quad x\in B_{2\delta}(x_0),\\ w|_{\partial B_{2\delta}}=0.\end{cases}$$

Owing to $u\in L^\infty(B_{2\delta})$ and (4.1), $g(x),g_i(x)\in L^\infty(B_{2\delta})$, and thanks to the de Giorgi estimates (see [4]), we get that $w\in C^\alpha(\overline{B}_{2\delta})$, which implies $u\in C^\alpha(\overline{B}_\delta)$ for some $0<\alpha<1$.

Noticing that (4.4) holds true for any $v\in H_0^1(B_\delta)$, and $Dw=Du$ in B_δ, therefore we obtain

$$\begin{aligned}&\int_{B_\delta}[a_{ij}(x,u)D_ju-A_{ij}(x,u)D_j\eta-a_{ij}(x,u)uD_j\eta-b_i(x)u\eta]D_iv\mathrm{d}x\\ &-\int_{B_\delta}\Big[\frac{\partial A_{ij}}{\partial x_i}D_j\eta+A_{ij}(x,u)D_{ij}\eta+f\eta-b_i(x)uD_i\eta-c(x,u,\nabla u)\eta\Big]v\mathrm{d}x\\ =&\,0,\quad \forall v\in H_0^1(B_\delta).\end{aligned}$$

Thus, u restricting on B_δ is a weak solution of the equation

$$D_iA_i(x,u,\nabla u)+B(x,u,\nabla u)=0,\quad x\in B_\delta(x_0),$$

where

$$\begin{cases} A_i(x,u,\nabla u)=a_{ij}(x,u)D_ju-A_{ij}(x,u)D_j\eta-a_{ij}(x,u)uD_j\eta-b_i(x)u\eta,\\ B(x,u,\nabla u)=\dfrac{\partial A_{ij}}{\partial x_i}D_j\eta+A_{ij}(x,u)D_{ij}\eta+f\eta-b_i(x)uD_i\eta-c(x,u,\nabla u)\eta.\end{cases}$$

According to the assumptions, it is easy to see that $A_i,B\in C^1(\overline{B}_\delta\times R\times R^n)$ and $u\in W^{1,2}(B_\delta)\cap C^\alpha(\overline{B}_\delta)$. By means of the H^2-regularity of quasilinear elliptic equation (see [5]), we derive that $u\in H^2_{\mathrm{loc}}(B_\delta)$. Thus the theorem is proven.

Next, we consider the interior $W^{2,p}$-regularity of (3.5). Assume that $a_{ij}(x,z) \in C^1(\overline{\Omega})$ are independent of z, and

$$|c(x,z,\xi)| \leq C(|z|^k + |\xi|^q + 1), \tag{4.5}$$

where $0 \leq k,\ 0 \leq q < 2$.

Theorem 4.2 *Let* (4.5) *be satisfied, and* $b_i(x) \in C^1(\overline{\Omega})$, $f \in L^{k^*}(\Omega)$, $k^* = \frac{k+1}{k}$. *If* $u \in \widetilde{X} \cap L^{k+1}(\Omega)$ *is a weak solution of* (3.5), *then* $u \in W^{2,p}_{\rm loc}(\Omega \backslash \Omega')$, $p = \min\{2, \frac{k+1}{k}, \frac{2}{q}\}$. *Furthermore, if* $a_{ij}, b, c \in C^\infty(\overline{\Omega} \times R \times R^n)$, *and* $\frac{np}{n-2p} > k+1$, $\frac{np}{n-p} > 2$, *then* $u \in C^\infty(\Omega \backslash \Omega')$.

Proof As the proof of Theorem 4.1, we can get that $w = \eta u \in W^{1,2}(B_{2\delta}) \cap L^{k+1}(B_{2\delta})$ is a weak solution of the following equation:

$$\begin{cases} -D_i(a_{ij}(x)D_j w) = g - D_i g_i, \quad x \in B_{2\delta}, \\ w|_{\partial B_{2\delta}} = 0, \end{cases}$$

where

$$\begin{cases} g_i = 2a_{ij}(x)uD_j\eta + b_i\eta u, \\ g = D_i a_{ij} u D_j \eta + a_{ij} u D_{ij}\eta + f\eta - b_i(x) u D_i \eta - c(x,u,\nabla u)\eta. \end{cases}$$

By (4.5) and $\widetilde{X} \cap L^{k+1}(\Omega) \hookrightarrow w^{1,2}(B_{2\delta}) \cap L^{k+1}(B_{2\delta})$, we can see that $g \in L^{\widetilde{k}}(B_{2\delta})$, $D_i g_i \in L^2(B_{2\delta})$, $\widetilde{k} = \min\{\frac{k+1}{k}, \frac{2}{q}\}$. According to the L^p-estimates, one can obtain $w \in W^{2,p}(B_{2\delta})$, i.e., $u \in W^{2,p}(B_\delta)$, $p = \min\{2, \frac{k+1}{k}, \frac{2}{q}\}$. The first conclusion is proven. By iteration, one can derive the second conclusion of this theorem. The proof is completed.

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