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On Smash Products of Transitive Module Algebras***

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Abstract Let H be a semisimple Hopf algebra over a field of characteristic 0, and A a finite-dimensional transitive H-module algebra with a 1-dimensional ideal. It is proved that the smash product $A\#H$ is isomorphic to a full matrix algebra over some right coideal subalgebra N of H. The correspondence between A and such N and the special case $A = k(X)$ of function algebra on a finite set X are considered.

Keywords Semisimple Hopf algebra, Smash product, Transitive module algebra
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1 Introduction

Let k be a field, G a finite group and X a finite transitive G-set. Then the group algebra kG is a finite-dimensional Hopf algebra and the function algebra $k(X) = \mathrm{Hom}(X, k)$ is a commutative kG-module algebra via the module structure induced by the G action. Harrison [3] showed that the smash product of kG and $k(X)$ is isomorphic to a full matrix algebra over kN, where N is the stabilizer of some $x \in X$. The similar question about smash products over finite dimensional Hopf algebras was also discussed by Blattner and Montgomery, Van den Bergh, and Koppinen. In [1], Blattner and Montgomery showed that $H\#H^* \cong \mathrm{End}_k H \cong M_n(k)$ for an n-dimensional Hopf algebra H over k. This result was also proved independently by Van den Bergh [10]. In [5], Koppinen strengthened this result to the case $K\#H^* \cong A \otimes \mathrm{End}K$ and $A\#H \cong K \otimes \mathrm{End}A$, where A is a right coideal Frobenius subalgebra of H^* of the dual Hopf algebra of a finite-dimensional Hopf algebra H, and $K = (H^*/A^+H^*)^* \subseteq H$. An easy fact is that the module algebras in these results are transitive with a 1-dimensional ideal (see [12]).

It is natural to ask whether the smash product $A\#H$ is a full matrix algebra over some right coideal subalgebra of H for a general transitive module algebra. We prove in this paper that the conclusion is also true for H being a semisimple Hopf algebra over a field k of characteristic 0, and A a transitive H-module algebra with a 1-dimensional ideal.

We arrange this paper as follows. Section 2 is devoted to some properties of transitive module algebras. In Section 3, we prove the main theorem (i.e. Theorem 3.1): Let H be a finite-dimensional semisimple Hopf algebra, and A an s-dimensional transitive H-module algebra with a 1-dimensional ideal $k\lambda$. Then the smash product $A\#H$ is isomorphic to $M_s(N)$,

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where $N = \{h \in H \mid h_{(1)} \cdot \lambda \otimes h_{(2)} = \lambda \otimes h\}$. If H is not semisimple, a counterexample for the Theorem 3.1 is also given in this section. In Section 4, we apply our result to the case $A = k(X)$, a function algebra on a finite set X which has a Hopf algebra action.

Throughout this paper, k will be a field of characteristic 0. And we assume that all algebras and Hopf algebras are finite dimensional over k.

2 Transitive Module Algebras

Let H be a Hopf algebra. An H-module algebra A is an associative algebra with a left H-action such that

$$h \cdot 1_A = \varepsilon(h)1_A \quad \text{and} \quad h \cdot (ab) = \sum_{(h)} (h_{(1)} \cdot a)(h_{(2)} \cdot b)$$

for any $h \in H$ and $a, b \in A$. An ideal I of A is called an H-ideal if $HI \subseteq I$, i.e., I is invariant under the H-action. A non-zero H-ideal is called minimal if it does not contain proper non-zero H-ideal.

In [12], Zhu gave the definition of transitive action for Hopf algebras.

Definition 2.1 *Let H be a Hopf algebra. An H-module algebra A is called transitive if it satisfies the following conditions:*

(C1) $A^H = \{a \in A \mid h \cdot a = \varepsilon(h)a,\ \forall\, h \in H\} = k1_A$;

(C2) *A has no non-zero proper H-ideal.*

As explained in [12], none of the conditions (C1) and (C2) is superfluous. But for semisimple Hopf algebras and semisimple module algebras with 1-dimensional ideals, (C1) and (C2) are equivalent.

Lemma 2.1 *Let H be a Hopf algebra with invertible antipode S, and A be an H-module algebra. If A is semisimple, then*

$$A = I_1 \oplus I_2 \oplus \cdots \oplus I_n,$$

where each I_j $(1 \le j \le n)$ is a minimal H-ideal.

For each $1 \le j \le n$,

$$I_j = J_{j,1} \oplus J_{j,2} \oplus \cdots \oplus J_{j,m_j},$$

where $J_{j,k}$ $(1 \le k \le m_j)$ are minimal ideals of A such that $I_j = HJ_{j,k}$ for any $1 \le k \le m_j$.

Proof Let I be any H-ideal of A. Since A is semisimple, there exists an ideal J such that $A = I \oplus J$. We show that J is also an H-ideal. Write $1 = e_1 + e_2$, where e_1 and e_2 are central idempotents of A with $e_1 \in I$ and $e_2 \in J$.

For any $a \in J$ and $h \in H$, we have

$$(h \cdot a)e_1 = \sum_{(h)} h_{(1)} \cdot (a(S(h_{(2)}) \cdot e_1)) = 0.$$

So $h \cdot a \in J$ and thus J is an H-ideal of A. Hence the H-ideals of A are completely reducible, and therefore $A = I_1 \oplus I_2 \oplus \cdots \oplus I_n$.

Let J be an ideal of A. For any $a \in A$, $b \in J$ and $h \in H$,

$$a(h \cdot b) = \sum_{(h)} h_{(2)} \cdot ((S^{-1}(h_{(1)}) \cdot a)b) \in HJ,$$
$$(h \cdot b)a = \sum_{(h)} h_{(1)} \cdot (b(S(h_{(2)}) \cdot a)) \in HJ.$$

So HJ is an H-ideal of A.

If I is a minimal H-ideal of A, then $I = J_1 \oplus J_2 \oplus \cdots \oplus J_m$ is a direct sum of minimal ideals J_i of A. For any $1 \leq k \leq m$, $HJ_k \subseteq HI = I$ is an H-ideal of A. By the minimality of I, $HJ_k = I$.

Proposition 2.1 *Let H be a semisimple Hopf algebra and A be a semisimple H-module algebra with a 1-dimensional ideal. Then the following conditions are equivalent:*

(1) *A is a transitive H-module algebra;*

(2) *$A^H = k1_A$;*

(3) *A has no non-zero proper H-ideal.*

Proof We only need prove that (2) and (3) are equivalent. Assume that $k\lambda$ is the 1-dimensional ideal of A.

(2) $\Rightarrow$ (3) Assume $A^H = k1_A$. Let I be a non-zero H-ideal of A. Suppose $I \neq A$. Then by Lemma 2.1, we get $A = I \oplus J$ for some H-ideal J of A. Write $1_A = e_1 + e_2$, where $e_1 \in I$ and $e_2 \in J$ are central idempotents of A. Then for any $h \in H$,

$$h \cdot 1_A = h \cdot e_1 + h \cdot e_2 = \varepsilon(h)e_1 + \varepsilon(h)e_2.$$

So $h \cdot e_1 = \varepsilon(h)e_1$ and $h \cdot e_2 = \varepsilon(h)e_2$. Thus $e_1, e_2 \in A^H$ which contradicts the assumption that $A^H = k1_A$. Hence A has no non-zero proper H-ideal.

(3) $\Rightarrow$ (2) If A has no non-zero proper H-ideal, then $A = H \cdot \lambda$ by Lemma 2.1. Let t be an integral of H such that $\varepsilon(t) \neq 0$. We have that

$$A^H = t \cdot A = tH \cdot \lambda \subseteq k(t \cdot \lambda)$$

is 1-dimensional.

Lemma 2.2 (see [12, Lemma 2.4]) *Let H be a semisimple Hopf algebra and A a transitive H-module algebra. Assume that t is the integral of H with $\varepsilon(t) = (\dim H)1_k$. Then for any $a \in A$, we have*

$$t \cdot a = (\dim H \cdot \mathrm{tr}(a)/\dim A)1_A,$$

where $\mathrm{tr}(a)$ denotes the trace of left multiplication of a on A. And A is a semisimple algebra.

If H is a semisimple Hopf algebra and A a transitive H-module algebra, by Lemma 2.2, we have that A is semisimple. Then A has a canonical right H^*-comodule structure deduced from the left H-module structure:

$$h \cdot a = \sum_{(a)} a_{\langle 0\rangle}\langle h, a_{\langle 1\rangle}\rangle, \quad \forall\, h \in H,\ a \in A.$$

For any A-module U and H^*-module V, the H^*-comodule structure map gives $U \otimes V$ an A-module structure:

$$a \cdot (u \otimes v) = \sum_{(a)} a_{\langle 0 \rangle} \cdot u \otimes a_{\langle 1 \rangle} \cdot v, \quad \forall\, a \in A,\ u \in U,\ v \in V.$$

Let $M_1, \cdots, M_s$ be the list of isomorphic classes of simple A-modules and $e_1, \cdots, e_s$ the corresponding idempotents of A. Suppose that the A-module $M_i \otimes H^*$ (where H^* acts by left multiplication) has the decomposition:

$$M_i \otimes H^* = \bigoplus_{k=1}^{s} N_i^k M_k, \tag{2.1}$$

where N_i^k is the multiplicity of M_k in $M_i \otimes H^*$. Then $D_k = \mathrm{End}_A M_k$ is a skew field and we have the following result.

Proposition 2.2 *$N_i^k \dim D_k \dim A = \dim H \dim M_i \dim M_k$. Hence*

$$(\dim D_k \dim A) \mid [\dim H (\dim M_k)^2].$$

Proof Suppose $A = \bigoplus_{k=1}^{s} n_k M_k$, where n_k is the multiplicity of M_k in A. By Lemma 2.2, we have

$$t \cdot e_k = (\dim H \cdot \mathrm{tr}(e_k)/\dim A) 1_A = (n_k \dim H \dim M_k / \dim A) 1_A.$$

Now we compute the trace of e_k on A-module $M_i \otimes H^*$:

$$\begin{aligned}\mathrm{tr}(e_k) &= \sum_{(e_k)} \mathrm{tr}|_{M_i}(e_{k\langle 0 \rangle}) \mathrm{tr}|_{H^*}(e_{k\langle 1 \rangle}) = \sum_{(e_k)} \mathrm{tr}|_{M_i}(e_{k\langle 0 \rangle}) \langle t, e_{k\langle 1 \rangle} \rangle \\ &= \mathrm{tr}|_{M_i}(t \cdot e_k) = n_k \dim H \dim M_k \dim M_i / \dim A.\end{aligned}$$

On the other hand, the decomposition (1) implies

$$\mathrm{tr}(e_k) = N_i^k \dim M_k.$$

Compare the two equations above, we get

$$N_i^k \dim D_k \dim A = \dim H \dim M_i \dim M_k.$$

Let $i = k$. We have $\dim H (\dim M_k)^2 / \dim D_k \dim A = N_k^k$ for each k. Since N_k^k is a positive integer, the proof is finished.

By an analogous calculation, the result (see [12, Corollary 2.8]) is still true without the assumption that k is algebraically closed.

Corollary 2.1 (see [12, Corollary 2.8]) *Let A be a transitive module algebra of a semisimple Hopf algebra H, M be a simple A-module and (A', M') the stabilizer of (A, M) defined in* [11]. *Then*

$$(\dim A)(\dim A') = (\dim M)^2 \dim H.$$

3 Main Results

Throughout this section, unless otherwise specified, H is a finite-dimensional semisimple Hopf algebra with antipode S over the field k (thus $S^2 = 1$, see [6]), and A is an s-dimensional transitive H-module algebra with a 1-dimensional ideal $k\lambda$ where λ is a central idempotent in A. Then $A = H\lambda$ by Lemma 2.1 and A is semisimple by Lemma 2.2.

Define

$$N = \Big\{h \in H \,\Big|\, \sum_{(h)} h_{(1)} \cdot \lambda \otimes h_{(2)} = \lambda \otimes h\Big\}.$$

Clearly, N is a right coideal subalgebra of H, $1_H \in N$ and

$$N = \Big\{h \in H \,\Big|\, \sum_{(h)} S(h_{(1)}) \cdot \lambda \otimes h_{(2)} = \lambda \otimes h\Big\}.$$

In this section, we prove the main result of this paper.

Theorem 3.1 *Let H be a finite-dimensional semisimple Hopf algebra and A be an s-dimensional transitive H-module algebra with a 1-dimensional ideal $k\lambda$. Then the smash product $A\#H$ is isomorphic to $M_s(N)$, where $N = \Big\{h \in H \,\Big|\, \sum_{(h)} h_{(1)} \cdot \lambda \otimes h_{(2)} = \lambda \otimes h\Big\}$.*

The theorem will be proved by first showing that H is free as both left and right N-module. Since $k\lambda$ is a 1-dimensional ideal of A, we can define a map θ from H to N via

$$\sum_{(h)} (S(h_{(1)}) \cdot \lambda)\lambda \otimes h_{(2)} = \lambda \otimes \theta(h), \quad \forall\, h \in H.$$

Lemma 3.1 (1) *Let t be the integral of H such that $\varepsilon(t) = 1$. Then $t \cdot \lambda = k_t 1_A \neq 0$ with $k_t \in k$;*

(2) $\sum_{(h)} (h_{(1)} \cdot \lambda)\lambda \otimes h_{(2)} = \sum_{(h)} h_{(1)} \cdot \lambda \otimes \theta(h_{(2)}) = \lambda \otimes \theta(h)$;

(3) $\theta(H) \subseteq N$ *and θ is a right H-comodule projection*;

(4) $(h \cdot \lambda)\lambda = \varepsilon(\theta(h))\lambda = \theta(h) \cdot \lambda$;

(5) *θ is both left and right N-module morphism.*

Proof (1) $k1_A = A^H = t \cdot A = tH \cdot \lambda \subseteq kt \cdot \lambda$, so $0 \neq t \cdot \lambda \in A^H = k1_A$. The conclusion is clear.

(2) For any $h \in H$, by the definition of θ,

$$\sum_{(h)} h_{(1)} \otimes (S(h_{(2)}) \cdot \lambda)\lambda \otimes h_{(3)} = \sum_{(h)} h_{(1)} \otimes \lambda \otimes \theta(h_{(2)}).$$

So

$$\sum_{(h)} h_{(1)} \cdot ((S(h_{(2)}) \cdot \lambda)\lambda) \otimes h_{(3)} = \sum_{(h)} h_{(1)} \cdot \lambda \otimes \theta(h_{(2)}).$$

Since λ is central in A,

$$\sum_{(h)} h_{(1)} \cdot ((S(h_{(2)}) \cdot \lambda)\lambda) = \sum_{(h)} h_{(1)} \cdot (\lambda(S(h_{(2)}) \cdot \lambda)) = (h \cdot \lambda)\lambda$$

for any $h \in H$. Thus $\sum\limits_{(h)}(h_{(1)} \cdot \lambda)\lambda \otimes h_{(2)} = \sum\limits_{(h)} h_{(1)} \cdot \lambda \otimes \theta(h_{(2)})$. Assume $\sum\limits_{(h)} h_{(1)} \cdot \lambda \otimes \theta(h_{(2)}) = \lambda \otimes g$ for some $g \in H$. Then $\sum\limits_{(h)} t \cdot (h_{(1)} \cdot \lambda) \otimes \theta(h_{(2)}) = t \cdot \lambda \otimes \theta(h) = t \cdot \lambda \otimes g$. So $g = \theta(h)$ and (2) is proved.

(3) For any $h \in H$, by definition,

$$\begin{aligned}\sum_{(h)} \theta(h)_{(1)} \cdot \lambda \otimes \theta(h)_{(2)} &= \sum_{(h)} h_{(2)(1)} \cdot ((S(h_{(1)}) \cdot \lambda)\lambda) \otimes h_{(2)(2)} \\ &= \sum_{(h)} h_{(2)} \cdot ((S(h_{(1)}) \cdot \lambda)\lambda) \otimes h_{(3)} \\ &= \sum_{(h)} \lambda(h_{(1)} \cdot \lambda) \otimes h_{(2)} = \lambda \otimes \theta(h).\end{aligned}$$

Hence $\theta(h) \in N$. Then θ is a right H-comodule projection from the definitions of θ and N.

(4) Apply $\mathrm{id} \otimes \varepsilon$ to both sides of the equation $\sum\limits_{(h)}(h_{(1)} \cdot \lambda)\lambda \otimes h_{(2)} = \lambda \otimes \theta(h)$ in (2), we get $(h \cdot \lambda)\lambda = \varepsilon(\theta(h))\lambda$. Since $\theta(h) \in N$, we have $\varepsilon(\theta(h))\lambda = \sum\limits_{(h)}(\theta(h)_{(1)} \cdot \lambda)\varepsilon(\theta(h)_{(2)}) = \theta(h) \cdot \lambda$.

(5) For any $h \in H,\ n \in N$,

$$\lambda \otimes \theta(hn) = \sum_{(h,n)}(h_{(1)}n_{(1)} \cdot \lambda)\lambda \otimes h_{(2)}n_{(2)} = \sum_{(h)}(h_{(1)} \cdot \lambda)\lambda \otimes h_{(2)}n = \lambda \otimes \theta(h)n.$$

So $\theta(hn) = \theta(h)n$ and θ is a right N-module morphism.

Next, we show that θ is a left N-module morphism. One has

$$\sum_{(h,n)} S(n_{(1)}) \otimes (n_{(2)}h_{(1)} \cdot \lambda)\lambda \otimes n_{(3)}h_{(2)} = \sum_{(n)} S(n_{(1)}) \otimes \lambda \otimes \theta(n_{(2)}h).$$

Then

$$\begin{aligned}\sum_{(h,n)} S(n_{(1)}) \cdot ((n_{(2)}h_{(1)} \cdot \lambda)\lambda) \otimes n_{(3)}h_{(2)} &= \sum_{(n)} S(n_{(1)}) \cdot \lambda \otimes \theta(n_{(2)}h), \\ \sum_{(h,n)} (h_{(1)} \cdot \lambda)\lambda(S(n_{(1)}) \cdot \lambda)\lambda \otimes n_{(2)}h_{(2)} &= \sum_{(n)} (S(n_{(1)}) \cdot \lambda)\lambda \otimes \theta(n_{(2)}h).\end{aligned}$$

Thus

$$\lambda \otimes n\theta(h) = \lambda \otimes \theta(\theta(n)h) = \lambda \otimes \theta(nh).$$

So $\theta(nh) = n\theta(h)$ and θ is also a left N-module morphism.

Lemma 3.2 *Let H be a finite-dimensional* (*not necessarily semisimple*) *Hopf algebra, L be a right coideal subalgebra of H and $1 \in L$. If L is a direct summand of H in ${}_LM^H$, then L is a Frobenius algebra.*

Proof Since L is a left L-module direct summand of H, $-\otimes_L H$ is faithful. By [7, Lemma 2.2], L is a simple object in ${}_LM^H$.

Let T be a right integral of H^* and t a left integral of H such that $T(t) = T(S(t)) = 1$. Assume that $H = L \oplus M$ in ${}_LM^H$. Write $t = t_1 + t_2$ for $t_1 \in L,\ t_2 \in M$. By [9, Section

5.1] and [8, Proposition 3], $(H^*, \rightharpoonup)$ is a free left H-module with basis T, and $(H, \leftharpoondown)$ is a free right H^*-module with basis t. Since $S(h \rightharpoonup T) \rightharpoonup t = h$ for any $h \in H$ and L, M are right H-comodules, one has

$$g = S(g \rightharpoonup T) \rightharpoonup t = S(g \rightharpoonup T) \rightharpoonup t_1, \quad \forall\, g \in L.$$

So $S(1 \rightharpoonup T) \rightharpoonup t_1 = 1$ and $1 = \varepsilon(1) = T(S(t_1)) \neq 0$. By [7, Lemma 3.5], L is a Frobenius algebra.

By the two lemmas above, N is a Frobenius algebra. Thus we get the following proposition easily.

Proposition 3.1 *H is free as right and left N-module.*

Next, we calculate the order of H as free N-module. Obviously, $I = k\lambda$ is a simple A-module. Let (A', I') be the stabilizer of (A, I). By calculation, $A' = \left\{h \in H \,\middle|\, \sum\limits_{(h)} (h_{(2)} \cdot a)\lambda \otimes h_{(1)} = a\lambda \otimes h, \forall\, a \in A\right\}$.

Lemma 3.3 $S(A') = \left\{h \in H \,\middle|\, \sum\limits_{(h)} (h_{(1)} \cdot a)\lambda \otimes h_{(2)} = a\lambda \otimes h,\ \forall\, a \in A\right\} = N.$

Proof Firstly, we prove $S(A') \subseteq N$. For any $h \in S(A')$, $\sum\limits_{(h)} (h_{(1)} \cdot \lambda)\lambda \otimes h_{(2)} = \lambda \otimes h$. By Lemma 3.1(2),

$$\lambda \otimes \theta(h) = \sum_{(h)} (h_{(1)} \cdot \lambda)\lambda \otimes h_{(2)}.$$

So $h = \theta(h) \in N$ and $S(A') \subseteq N$.

Conversely, for any $g \in N$ and $a \in A$, we have $a\lambda = k_a\lambda$ for some $k_a \in k$. Thus

$$a\lambda \otimes g = k_a\lambda \otimes g = \sum_{(g)} k_a g_{(1)} \cdot \lambda \otimes g_{(2)} = \sum_{(g)} g_{(1)} \cdot (\lambda a) \otimes g_{(2)} = \sum_{(g)} \lambda(g_{(1)} \cdot a) \otimes g_{(2)}.$$

So $g \in S(A')$ and $N \subseteq S(A')$. Therefore, $S(A') = N$.

Since $\dim A \dim A' = \dim H$ by Corollary 2.1 and S is bijective, $\dim A \dim N = \dim H$. Let $\dim A = s$. Then there exist $h_1, \cdots, h_s \in H$ such that

$$H = h_1 N \oplus \cdots \oplus h_s N.$$

Obviously, $a_i = h_i \cdot \lambda$ $(i = 1, \cdots, s)$ is a k-basis of A.

We are now in a position to prove our main theorem.

Proof of Theorem 3.1 Define a map

$$\varphi : A\#H \to \mathrm{End}_N H$$

by $\varphi(h \cdot \lambda \# h')(l) = \sum\limits_{(h)} h_{(1)} \theta(S(h_{(2)}) h' l)$ for any $h, h', l \in H$.

Since for any $n \in N$,

$$\begin{aligned}\varphi(hn \cdot \lambda \# h')(l) &= \sum_{(h,n)} h_{(1)} n_{(1)} \theta(S(n_{(2)})S(h_{(2)})h'l) \\ &= \sum_{(h,n)} h_{(1)} \theta(n_{(1)} S(n_{(2)})S(h_{(2)})h'l) \\ &= \sum_{(h)} \varepsilon(n) h_{(1)} \theta(S(h_{(2)})h'l) \\ &= \varepsilon(n)\varphi(h \cdot \lambda \# h')(l)\end{aligned}$$

and H is free as right N-module, the definition of φ is reasonable. To prove the theorem, we need only prove that φ is an algebra isomorphism.

We first show that φ is an algebra morphism.

For any $h, h', g, g', l \in H$,

$$\begin{aligned}\varphi(h \cdot \lambda \# h')\varphi(g \cdot \lambda \# g')(l) &= \sum_{(g)} \varphi(h \cdot \lambda \# h')(g_{(1)}\theta(S(g_{(2)})g'l)) \\ &= \sum_{(h,g)} h_{(1)}\theta(S(h_{(2)})h'g_{(1)})\theta(S(g_{(2)})g'l), \\ \varphi((h \cdot \lambda \# h')(g \cdot \lambda \# g'))(l) &= \sum_{(h,h')} \varphi(h_{(1)} \cdot (\lambda(S(h_{(2)})h'_{(1)}g \cdot \lambda))\# h'_{(2)}g')(l) \\ &= \sum_{(h,h')} \varepsilon(\theta(S(h_{(2)})h'_{(1)}g))\varphi(h_{(1)} \cdot \lambda \# h'_{(2)}g')(l) \\ &= \sum_{(h,h')} \varepsilon(\theta(S(h_{(3)})h'_{(1)}g))h_{(1)}\theta(S(h_{(2)})h'_{(2)}g'l) \\ &= \sum_{(h,h',g)} \varepsilon(\theta(S(h_{(3)})h'_{(1)}g_{(1)}))h_{(1)}\theta(S(h_{(2)})h'_{(2)}g_{(2)}S(g_{(3)})g'l) \\ &= \sum_{(h,g)} h_{(1)}\theta(S(h_{(2)})h'g_{(1)})\theta(S(g_{(2)})g'l).\end{aligned}$$

This shows that $\varphi(h \cdot \lambda \# h')\varphi(g \cdot \lambda \# g') = \varphi((h \cdot \lambda \# h')(g \cdot \lambda \# g'))$, as desired.

Next, we prove that φ is bijective. Since $\dim(A\#H) = \dim(\mathrm{End}_N H) < \infty$, it suffices to prove that φ is injective. If $\varphi(\sum h_i \cdot \lambda \# g_i) = 0$, i.e., $\varphi(\sum h_i \cdot \lambda \# g_i)(l) = \sum_{(h_i)} h_{i(1)}\theta(S(h_{i(2)})g_i l) = 0$ for any $l \in H$, we get

$$\begin{aligned}&\sum_{(h_i)} (h_{i(1)}\theta(S(h_{i(2)})g_i l))_{(1)} \cdot \lambda \# (h_{i(1)}\theta(S(h_{i(2)})g_i l))_{(2)} \\ &= \sum_{(h_i,g_i,l)} h_{i(1)}\theta(S(h_{i(4)})g_{i(1)}l_{(1)}) \cdot \lambda \# h_{i(2)}S(h_{i(3)})g_{i(2)}l_{(2)} \\ &= \sum_{(h_i,g_i,l)} h_{i(1)}\theta(S(h_{i(2)})g_{i(1)}l_{(1)}) \cdot \lambda \# g_{i(2)}l_{(2)} \\ &= \sum_{(h_i,g_i,l)} h_{i(1)} \cdot (\lambda(S(h_{i(2)})g_{i(1)}l_{(1)} \cdot \lambda))\# g_{i(2)}l_{(2)} \\ &= \sum_{(g_i,l)} (h_i \cdot \lambda)(g_{i(1)}l_{(1)} \cdot \lambda)\# g_{i(2)}l_{(2)}\end{aligned}$$

$$
\begin{aligned}
&=\sum_{(l)}(h_i\cdot\lambda\#g_i)(l_{(1)}\cdot\lambda\#l_{(2)})\\
&=0.
\end{aligned}
$$

Since $H\cdot\lambda=A$, we may choose $l\in H$ such that $l\cdot\lambda=1_A$. Then

$$\sum h_i\cdot\lambda\#g_i=\sum_{(l)}(h_i\cdot\lambda\#g_i)(l_{(1)}\cdot\lambda\#l_{(2)})(\lambda\#S(l_{(3)}))=0.$$

This means that φ is injective.

Let H be a finite-dimensional Hopf algebra and B a right coideal subalgebra of H. As in the case of Hopf algebras (see [9]), an element $x\in B$ is called a left integral in B if $bx=\varepsilon(b)x$ for all $b\in B$. Similarly, right and two-sided integrals in B are defined. Let H be a semisimple Hopf algebra, and A an s-dimensional Frobenius right coideal subalgebra of H^*. Then A is separable by [7]. By [5], A contains a two-sided integral x^* such that $\varepsilon(x^*)=1$. Hence kx^* is a 1-dimensional ideal of A. We view A as a left H-module algebra in a natural way. One checks that $A^H=k1_A$. So we get the following conclusion.

Corollary 3.1 *Let H be a semisimple Hopf algebra and A an s-dimensional Frobenius right coideal subalgebra of H^*. Then*

(1) *A is a transitive H-module algebra*;

(2) *A has an integral T such that $T(1)=1$. Then the smash product of A and H is isomorphic to $M_s(N)$ as algebras, where $N=\left\{h\in H\,\middle|\,\sum_{(h)}h_{(1)}\rightharpoonup T\otimes h_{(2)}=T\otimes h\right\}$.*

In [5], Koppinen proved that if H is a finite-dimensional Hopf algebra, A a right coideal Frobenius subalgebra of H^*, and $K=(H^*/A^+H^*)^*\subseteq H$, then $K\#H^*\cong A\otimes\mathrm{End}K$ and $A\#H\cong K\otimes\mathrm{End}A$ as algebras. If H is also semisimple, we have $K=N$ with N defined as in Corollary 3.1. However, without the assumption that H is semisimple, for a transitive H-module algebra A with a 1-dimensional ideal even if it is semisimple, this is false in general, as the following example shows.

Example 3.1 Let H_4 be the Sweedler's 4-dimensional Hopf algebra. As an algebra, it is generated by x and g subject to the relations

$$xg=-gx,\quad x^2=0\quad\text{and}\quad g^2=1.$$

Its coalgebra structure is determined by

$$\Delta(g)=g\otimes g\quad\text{and}\quad\Delta(x)=x\otimes g+1\otimes x.$$

Let $A=k\{p_1,p_2\}$. Then A is semisimple obviously. The action of H_4 on A is determined by

$$g\cdot p_1=p_2,\quad g\cdot p_2=p_1;\quad x\cdot p_1=\alpha(p_1+p_2),\quad x\cdot p_2=-\alpha(p_1+p_2),$$

where $\alpha\in k$. One checks that the action is transitive. But when $\alpha\neq 0$, $N=k\{1_H\}$. By considering the dimension $A\#H$ and $M_2(N)$, they are not isomorphic.

Under the hypotheses in Theorem 3.1, for A, we can find a right coideal Frobenius subalgebra $N \subseteq H$ such that $A\#H \cong M_s(N)$. Let $A' = (H/(N^+H))^* \subseteq H^*$. Then A' is a transitive H-module algebra with a 1-dimensional ideal kx^*. By a simple calculation, we get

$$\Big\{h \in H \,\Big|\, \sum_{(h)} h_{(1)} \rightharpoonup x^* \otimes h_{(2)} = x^* \otimes h\Big\} = (H^*/A'^+H^*)^* = N.$$

So $A'\#H \cong M_s(N)$.

Example 3.2 We use [4, Example 15]. Let H be a Hopf algebra such that $H \cong kM$ as algebras, where kM is a group algebra with the group M defined by

$$M = \langle a, b, g \mid a^4 = e,\ b^2 = a^2,\ ba = a^{-1}b,\ ag = ga,\ bg = gb,\ g^2 = e\rangle.$$

The coalgebra structure of H is given as follows:

$$\begin{aligned}
\Delta(a) &= \frac{1}{2}(a \otimes a + ag \otimes a + a \otimes b - ag \otimes b),\\
\Delta(b) &= \frac{1}{2}(b \otimes b + bg \otimes b + b \otimes a - bg \otimes a),\\
\Delta(g) &= g \otimes g, \quad S(g) = g,\\
S(a) &= \frac{1}{2}(a^3 + a^3g + a^2b - a^2bg),\\
S(b) &= \frac{1}{2}(b^3 + b^3g + b^2a - b^2ag),\\
\varepsilon(a) &= \varepsilon(b) = \varepsilon(g) = 1.
\end{aligned}$$

Then H is semisimple and has an integral $t = (e+g)(e+b)(e+a+a^2+a^3)$. Let $A = k\{p_1, \cdots, p_8 \mid p_ip_j = \delta_{ij}p_i\}$. An action of H on A is given by

$$\begin{aligned}
&g \cdot p_i = p_i;\\
&a \cdot p_1 = p_2, \quad a \cdot p_2 = p_3, \quad a \cdot p_3 = p_4, \quad a \cdot p_4 = p_1,\\
&a \cdot p_5 = p_6, \quad a \cdot p_6 = p_7, \quad a \cdot p_7 = p_8, \quad a \cdot p_8 = p_5;\\
&b \cdot p_1 = p_5, \quad b \cdot p_5 = p_3, \quad b \cdot p_3 = p_7, \quad b \cdot p_7 = p_1,\\
&b \cdot p_2 = p_8, \quad b \cdot p_8 = p_4, \quad b \cdot p_4 = p_6, \quad b \cdot p_6 = p_2.
\end{aligned}$$

Then A is a transitive H-module algebra. One can verify that $N = k\{e, g\}$ and

$$\begin{aligned}
A' = k\{&I_e + I_g,\ I_a + I_{ga},\ I_b + I_{gb},\ I_{a^2} + I_{ga^2},\\
&I_{a^3} + I_{ga^3},\ I_{b^3} + I_{gb^3},\ I_{ba} + I_{gba},\ I_{ba^3} + I_{gba^3}\} \subseteq H^*,
\end{aligned}$$

where I_x $(x \in M)$ denotes the k-valued function $I_x(y) = \delta_{x,y}$.

We note that A' is isomorphic to A as algebras in the above example. However, in general, A and A' are not isomorphic as algebras such as the following example.

Example 3.3 Let H be a finite-dimensional Hopf algebra such that $\dim H = n < \infty$ and $k\#_\sigma H^*$ a crossed product. Then $A = k\#_\sigma H^*$ is a transitive H-module algebra via $h \cdot (k\#f) = k\#h \rightharpoonup f$. It is easy to see $N = k$ and $A' = H^*$. By [2], $A\#H \cong M_n(k) \cong A'\#H$, but A and A' are not isomorphic as algebras generally even if H is semisimple.

4 Application to Function Algebra $A = k(X)$

In this section, we apply our result to the function algebra $A = k(X)$. Let X be a finite set. We define $k(X)$ to be the vector space with basis $\{p_x \mid x \in X\}$ and a multiplication on $k(X)$ is defined as

$$p_x p_y = \delta_{x,y} p_x, \quad \forall\, x, y \in X. \tag{4.1}$$

Then $A = k(X)$ is a semisimple algebra and kp_x $(x \in X)$ are all the minimal ideals of A.

For simplicity, we write $X = \{1, 2, \cdots, n\}$.

We apply our main theorem to the case that $A = k(X)$.

Theorem 4.1 *Let H be a semisimple Hopf algebra, and $A = k(X)$ be a transitive H-module algebra. Then the smash product of A and H is isomorphic to $M_n(N_{11})$ as algebras, where $N_{11} = \left\{h \in H \,\middle|\, \sum\limits_{(h)} h_{(1)} \cdot p_1 \otimes h_{(2)} = p_1 \otimes h\right\}$.*

A more detail structure of $A\#H$ is presented as follows.

Lemma 4.1 *Let H be a semisimple Hopf algebra, and $A = k(X)$ be a transitive H-module algebra. Let t be the integral of H such that $\varepsilon(t) = 1$. Then*

$$t \cdot p_i = \alpha_i 1_A, \quad i = 1, \cdots, n$$

for some $0 \neq \alpha_i \in k$ with $\sum\limits_{i=1}^{n} \alpha_i = 1$.

Proof Since the action is transitive, for any p_i, $H \cdot p_i = A$ and $t \cdot p_i \in A^H$, we have $t \cdot p_i = \alpha_i 1_A$ for some $\alpha_i \in k$. Note that $1_A = \sum\limits_{i=1}^{n} p_i$, so $t \cdot 1_A = \sum\limits_{i=1}^{n} t \cdot p_i = \sum\limits_{i=1}^{n} \alpha_i 1_A$, $\sum\limits_{i=1}^{n} \alpha_i = 1$. Because $1_A \in t \cdot A = tH \cdot p_i \subset kt \cdot p_i = k\alpha_i 1_A$, $\alpha_i \neq 0$ for each i.

For non-zero $\alpha_1, \cdots, \alpha_n \in k$ in Lemma 4.1, define $N_{ij} = \left\{h \in H \,\middle|\, \sum\limits_{(h)} h_{(1)} \cdot p_j \otimes h_{(2)} = p_i \otimes \frac{\alpha_j}{\alpha_i} h\right\}$. Then $N_{ij} \in {}_{N_{ii}}M^H \cap M^H_{N_{jj}}$ and $N_{11}, \cdots, N_{nn}$ are subalgebras of H as well. We also get $N_{ij} N_{jk} \subseteq N_{ik}$. By the definition of N_{ij}, the sum $\sum\limits_{j=1}^{n} N_{ij}$ is a direct sum.

Proposition 4.1 *Let H be a semisimple Hopf algebra. If the action of H on $A = k(X)$ is transitive, then*

(1) $H = N_{i1} \oplus \cdots \oplus N_{in}$, $i = 1, \cdots, n$;

(2) $\varepsilon(N_{ij}) \neq 0$ *for any* i, j $(1 \leq i, j \leq n)$;

(3) N_{ii} *is a Frobenius algebra for each* i $(1 \leq i \leq n)$.

Proof (1) For any $h \in H$, we suppose

$$\sum_{(h)} S^{-1}(h_{(1)}) \cdot p_i \otimes h_{(2)} = p_1 \otimes h_1 + \cdots + p_n \otimes h_n \in A \otimes H. \tag{4.2}$$

Since $\alpha_i \neq 0$ for any i, in a similar way as in Lemma 3.1(2), we get $h_j \in N_{ij}$. By equation

(4.2), we have

$$p_i \otimes h = \sum_{(h_1,\cdots,h_n)} h_{1(1)} \cdot p_1 \otimes h_{1(2)} + \cdots + h_{n(1)} \cdot p_n \otimes h_{n(2)}$$
$$= p_i \otimes \frac{\alpha_1}{\alpha_i} h_1 + \cdots + p_i \otimes \frac{\alpha_n}{\alpha_i} h_n.$$

Thus $h = \frac{\alpha_1}{\alpha_i} h_1 + \cdots + \frac{\alpha_n}{\alpha_i} h_n \in N_{i1} \oplus \cdots \oplus N_{in}$. Therefore $H = N_{i1} \oplus \cdots \oplus N_{in}$.

(2) For a fixed i, if $\varepsilon(N_{ij}) = 0$ for some j, act $\mathrm{id} \otimes \varepsilon$ on both sides of equation (4.2). We have

$$S^{-1}(h) \cdot p_i = \varepsilon(h_1)p_1 + \cdots + \varepsilon(h_{j-1})p_{j-1} + \varepsilon(h_{j+1})p_{j+1} + \cdots + \varepsilon(h_n)p_n,$$

which means

$$H \cdot p_i \subseteq kp_1 + \cdots + kp_{j-1} + kp_{j+1} + \cdots + kp_n.$$

But $A = H \cdot p_i$ and this is a contradiction, so $\varepsilon(N_{ij}) \neq 0$.

(3) follows from Lemma 3.2 and part (1).

Suppose that the action of H on A is transitive and $H = N_{i1} \oplus \cdots \oplus N_{in}$. By Proposition 4.1(3), for any i, N_{ii} is a Frobenius algebra. Define two maps

$$N_{ij} \otimes_{N_{jj}} N_{ji} \to N_{ii}, \quad N_{ji} \otimes_{N_{ii}} N_{ij} \to N_{jj} \tag{4.3}$$

by the multiplication of H.

Since $N_{ij} \in {}_{N_{ii}}M^H$ and $N_{ij} \in M^H_{N_{jj}}$, we see that N_{ij} is free left N_{ii}-module and free right N_{jj}-module. $N_{ij}N_{ji} (\subseteq N_{ii})$ is also in ${}_{N_{ii}}M^H$, so it is free left N_{ii}-module. By Proposition 4.1(2), $\varepsilon(N_{ij}) \neq 0$ and $\varepsilon(N_{ji}) \neq 0$, so $N_{ij}N_{ji} \neq 0$ and $N_{ij}N_{ji} = N_{ii}$. Similarly, $N_{ji}N_{ij} = N_{jj}$. So the maps defined by (4.3) are surjective, hence N_{ii} and N_{jj} are Morita equivalent. By Morita equivalent theory, the dimensions of all N_{ij} are the same. Pick $h_{i1} \in N_{i1}$ such that $\varepsilon(h_{i1}) = 1$ and $N_{i1} = h_{i1}N_{11}$. Then

$$N_{i1}H = N_{i1}(N_{11} \oplus \cdots \oplus N_{1n}) = N_{i1} \oplus \cdots \oplus N_{in} = H.$$

On the other hand,

$$N_{i1}H = h_{i1}N_{11}(N_{11} \oplus \cdots \oplus N_{1n}) = h_{i1}H.$$

So $h_{i1}H = H$, then there exists $h_{1i} \in H$ such that $h_{i1}h_{i1} = 1$. Since H is finite dimensional, h_{i1} is the inverse of h_{1i}. We get $h_{1i} \in H_{1i}$, so $N_{1i} = h_{1i}N_{11}$ and $N_{ij} = N_{i1}N_{1i} = h_{i1}N_{11}h_{1j}$.

Let G be the group generated by $h_{i1}, h_{1i},\ i = 1, 2, \cdots n$. Then $G \subseteq H$. But the group G is not contained in the set $G(H)$ of group-like elements of H in general. In Example 3.2, A is an H-module algebra and the action is transitive. But one can verify that $N_{ii} = \{e, g\}$ for any i and $G = \{e, a^3, a^2, a, b^3, ba, b, ba^3\} \subsetneq G(H)$.

Let $M_n(N_{11})$ be the algebra of all n by n matrices over N_{11}. This algebra is the free N_{11}-module with basis $\{e_{ij} \mid 1 \leq i, j \leq n\}$ and the multiplication is given by

$$(h_1 e_{ij})(h_2 e_{kl}) = \delta_{j,k} h_1 h_2 e_{il}$$

for any $h_1, h_2 \in N_{11}$.

Now we prove Theorem 4.1.

Proof of Theorem 4.1 By Proposition 4.1, $H = N_{i1} \oplus \cdots \oplus N_{in}$ for each i. Define a map Φ from $M_n(N_{11})$ to $A\#H$ as

$$\Phi(h \cdot e_{ij}) = \frac{\alpha_i}{\alpha_j} p_i \# h_{i1} h h_{1j}$$

for any $h \in N_{11}$ and $h_{i1}, h_{1j} \in G$ defined above. For any $h, h' \in N_{11}$, we have

$$\begin{aligned}
&(p_i \# h_{i1} h h_{1k})(p_j \# h_{j1} h' h_{1l}) \\
= &\sum_{(h_{i1}, h, h_{1k})} p_i(h_{i1(1)} h_{(1)} h_{1k(1)} \cdot p_j) \# h_{i1(2)} h_{(2)} h_{1k(2)} h_{j1} h' h_{1l} \\
= &\sum_{(h_{i1}, h, h_{1k})} h_{i1(1)} h_{(1)} h_{1k(1)} \cdot (p_k p_j) \# h_{i1(2)} h_{(2)} h_{1k(2)} h_{j1} h' h_{1l} \\
= &\,\delta_{kj} \frac{\alpha_k}{\alpha_i} p_i \# h_{i1} h h' h_{1l}.
\end{aligned}$$

So Φ is an algebra homomorphism. One may check that Φ is surjective. And by dimension considerations, Φ is an algebra isomorphism.

Theorem 4.2 *Let H be a semisimple Hopf algebra and $A = k\{p_x, x \in X \mid p_x p_y = \delta_{x,y} p_x\}$ be an H-module algebra. Then the smash product $A\#H$ is isomorphic to a direct sum of full matrix algebras over some right coideal subalgebras of H.*

Proof By Lemma 2.1,

$$A = I_1 \oplus I_2 \oplus \cdots \oplus I_m$$

is a direct sum of minimal H-ideals of A. For any $1 \le i \le m$, let $X_i = \{x \mid p_x I_i \ne 0\} \subseteq X$. Then $I_i = k\{p_x \mid x \in X_i\}$ and X equals the disjoint union of $X_1, \cdots, X_m$. One may check that

$$A\#H \cong I_1\#H \oplus I_2\#H \oplus \cdots \oplus I_m\#H.$$

The action of H on each I_i is transitive by Proposition 2.1. Let $n_i = |X_i|$ and $N_i = \Big\{h \in H \Big| \sum\limits_{(h)} h_{(1)} \cdot p_x \otimes h_{(2)} = p_x \otimes h\Big\}$ for some $x \in X_i$. Then $I_i\#H \cong M_{n_i}(N_i)$ by Theorem 4.1. Hence the conclusion holds.

When H is not semisimple, the result in Proposition 4.1 is false in general, as the Example 3.1 shows. In Example 3.1, if $\alpha \ne 0$, then

$$N_{11} = k\{1_H\}, \quad N_{12} = k\{g\}.$$

However,

$$H = N_{11} + N_{12} + k\{x, gx\}.$$

But we have the following result.

Proposition 4.2 *Let H be a finite-dimensional Hopf algebra, and $A = k(X)$ be a transitive H-module algebra. If $H = N_{i1} + \cdots + N_{in}$ for some i and $\varepsilon(N_{ji}) \ne 0$, then $H = N_{j1} + \cdots + N_{jn}$.*

Proof Suppose that t is a non-zero left integral of H. Then $t = t_1 + \cdots + t_n$, $t_k \in N_{ik}$ for $H = N_{i1} + \cdots + N_{in}$. Pick $a \in N_{ji}$ such that $\varepsilon(a) = 1$. We get $t = at = at_1 + \cdots + at_n \in N_{j1} + \cdots + N_{jn}$. Since $N_{j1} + \cdots + N_{jn} \subseteq H$ is a right H-comodule, then $H = N_{j1} + \cdots + N_{jn}$.

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