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Energy Decay for the Cauchy Problem of the Linear Wave Equation of Variable Coefficients with Dissipation**

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Abstract Decay of the energy for the Cauchy problem of the wave equation of variable coefficients with a dissipation is considered. It is shown that whether a dissipation can be localized near infinity depends on the curvature properties of a Riemannian metric given by the variable coefficients. In particular, some criteria on curvature of the Riemannian manifold for a dissipation to be localized are given.

Keywords Wave equation, Riemannian metric, Localized dissipation near infinity
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1 Introduction and Main Results

Let $n \geq 2$ be an integer. We consider the energy decay of solutions to the initial value problem

$$\begin{cases} u_{tt} - \operatorname{div} A(x)\nabla u + a(x)u_t = 0, & \text{on } (0,\infty)\times\mathbb{R}^n, \\ u(0,x) = u_0(x),\ u_t(0,x) = u_1(x), & \text{on } \mathbb{R}^n, \end{cases} \tag{1.1}$$

where $A(x) = (a_{ij}(x))$ are symmetric and positively definite matrices for all $x \in \mathbb{R}^n$, $a_{ij}(x)$ are smooth functions on $\mathbb{R}^n$, and $a \in L^\infty(\mathbb{R}^n)$ is a nonnegative function.

We define the energy of the problem (1.1) as

$$E(t) = \frac{1}{2}\int_{\mathbb{R}^n} \Big(u_t^2 + \sum_{i,j=1}^{n} a_{ij}(x) u_{x_i} u_{x_j}\Big)\mathrm{d}x. \tag{1.2}$$

We are interested in whether $E(t)$ decays in some way.

If $A(x) = (\delta_{ij})$, we say that the problem (1.1) is of constant coefficients. In the case of constant coefficients, a wealth of results on this problem are available in the literature. For the Cauchy problem, see [21, 28–30] and many other papers. For exterior domains, see [17, 20] and the references therein. For bounded domains, see [1, 7, 11, 13, 14, 16, 18, 25–27] and many other papers.

In this paper, we consider the problem (1.1) with a general $A(x)$ where $A(x)$ is given by the material in application. We refer to the problem (1.1) as the variable coefficient problem. The main tool here is the geometrical method which is powerful to cope with variable coefficients.

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This method was introduced by [32] for the controllability of the wave equation with variable coefficients and was extended in [2–7, 15, 22, 33, 34] and many others. For a survey on the geometric method, see [10]. Very recently, this method is used to study the problems with quasilinearly principal parts in [35–37].

We shall combine [17] and [32] to use the multiplier technique to derive some estimates on $E(t)$. The key is to use a multiplier of the geometric version. We mention that earlier multipliers for the Klein-Gordon equation were given by Morawetz [24] and for the control problem were given by Ho [12].

For the constant coefficient problem, Nakao [17] established the following decay of $E(t)$ by the multiplier technique in the Euclidean space $\mathbb{R}^n$. Let $L > 0$ be given. If there is an $\varepsilon > 0$ such that

$$a(x) \geq \varepsilon \quad \text{for } |x| \geq L,\ x \in \mathbb{R}^n, \tag{1.3}$$

then

$$E(t) \leq \frac{c}{1+t}(E(0) + \|u_0\|^2), \quad t > 0,\ \|u_0\|^2 = \int_{\mathbb{R}^n} u_0^2 \mathrm{d}x. \tag{1.4}$$

The condition (1.3) is referred to as a localized dissipation near infinity if $a(x)$ is only effective for L large enough. In fact, [17] considered an exterior domain problem. In this paper, we consider whether a similar estimate like (1.4) holds for the variable coefficient problem (1.1) under a similar condition as (1.3). We show that this problem depends closely on the geometric properties of a Riemannian metric, given by (1.5) below.

We define

$$g = A^{-1}(x), \quad x \in \mathbb{R}^n \tag{1.5}$$

as a Riemannian metric on $\mathbb{R}^n$, and consider the couple $(\mathbb{R}^n, g)$ as a Riemannian manifold. For each $x \in \mathbb{R}^n$, the Riemannian metric g induces the inner product and the norm on the tangent space $R^n_x = R^n$ by

$$\langle X, Y\rangle_g = \langle A^{-1}(x)X, Y\rangle, \quad |X|_g^2 = \langle X, X\rangle_g, \quad X, Y \in \mathbb{R}^n, \tag{1.6}$$

where $\langle \cdot\,, \cdot \rangle$ is the standard inner product of the Euclidean space $\mathbb{R}^n$. For $w \in H^1(\mathbb{R}^n)$, we have

$$|\nabla_g w|_g^2 = \sum_{i,j=1}^n a_{ij}(x) w_{x_i} w_{x_j}, \quad x \in \mathbb{R}^n,$$

where ∇_g is the gradient of the Riemannian metric g.

We introduce a space

$$H^1(g, \mathbb{R}^n) = \{w \mid w \in L^2(\mathbb{R}^n),\ |\nabla_g w|_g \in L^2(\mathbb{R}^n)\} \tag{1.7}$$

with a norm

$$\|w\|_g^2 = \int_{\mathbb{R}^n} (|\nabla_g w|_g^2 + w^2) \mathrm{d}x.$$

If there are $c_1 > 0$ and $c_2 > 0$ such that

$$c_1|X|^2 \leq \langle A(x)X, X\rangle \leq c_2|X|^2 \quad \text{for all } x \in \mathbb{R}^n,\ X \in \mathbb{R}^n,$$

then $H^1(g, \mathbb{R}^n) = H^1(\mathbb{R}^n)$.

For decay of the energy, we need the following assumption.

Assumption 1.1 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. There is a vector field H on $\mathbb{R}^n$ such that*

$$D_g H(X, X) \geq \sigma |X|_g^2, \quad X \in \mathbb{R}_x^n,\ x \in \overline{\Omega}, \tag{1.8}$$

where $\sigma > 0$ is a constant and D_g is the Levi-Civita connection of the Riemannian metric g.

Our main results are as follows.

Theorem 1.1 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set such that Assumption 1.1 holds true. Suppose that there is an $\varepsilon > 0$ such that*

$$a(x) \geq \varepsilon \quad \textit{for all } x \in \mathbb{R}^n \setminus \Omega. \tag{1.9}$$

Then, for each $(u_0, u_1) \in H^1(g, \mathbb{R}^n) \times L^2(\mathbb{R}^n)$, there is a unique solution

$$u(t) \in C([0, \infty), H^1(g, \mathbb{R}^n)) \cap C^1([0, \infty), L^2(\mathbb{R}))$$

to the problem (1.1) such that

$$\|u(t)\|^2 \leq c(E(0) + \|u_0\|^2), \tag{1.10}$$

where $\|\cdot\|$ is the usual norm of $L^2(\mathbb{R}^n)$ and $c > 0$ is a constant independent of solutions. Moreover, the estimate (1.4) holds true for a solution of the problem (1.1).

Assumption 1.1 was introduced by Yao [32] for the controllability of the wave equation with variable coefficients, which is also a useful condition for the controllability and the stabilization of the quasilinear wave equation (see [35–37]). Existence of such a vector field depends on the sectional curvature of the Riemannian manifold $(\mathbb{R}^n, g)$. There are a number of methods and examples in [32] to find out a vector field H that satisfies Assumption 1.1.

If there is a vector field H such that

$$D_g H > 0 \quad \text{for all } x \in \mathbb{R}^n, \tag{1.11}$$

then Assumption 1.1 holds for any bounded open set $\Omega \subset \mathbb{R}^n$ and the damping region (1.9) can be a neighborhood near infinity. Therefore, we say that the problem (1.1) has a localized dissipation near infinity if the condition (1.11) holds.

Let h be a strictly convex function of the metric g on $\overline{\Omega}$. Then $H = \nabla_g h$ satisfies Assumption 1.1. One of candidates for strictly convex functions is the distance function of the metric g. Let $x_0 \in \mathbb{R}^n$ be given. Let ρ denote the distance function of the metric g from x_0 to $x \in \mathbb{R}^n$. If $A = (\delta_{ij})$, then g is the standard metric of R^n and $\rho(x) = |x - x_0|$. For a general metric g, like (1.5), the structure of $\rho(x)$ is very complicated. For the properties of this function, see any Riemannian geometry book, for example, [31].

Let

$$\Xi = \{x \mid x \in \mathbb{R}^n,\ D_g^2 h(x) > 0\}, \tag{1.12}$$

where $D_g^2 h$ is the Hessian of a function h in the metric g, given by

$$h(x) = \frac{1}{2}\rho^2(x), \quad x \in \mathbb{R}^n. \tag{1.13}$$

It is well-known that $\Xi \subset \mathbb{R}^n$ is an open set and $x^0 \in \Xi$. If a bounded open set $\Omega \subset \mathbb{R}^n$ is such that

$$\overline{\Omega} \subset \Xi,$$

then Assumption 1.1 holds for Ω with $H = \nabla_g h$.

If $A = (\delta_{ij})$, then $D_g^2 h = \nabla^2 h$ is the unit matrix and $\Xi = \mathbb{R}^n$, where $h = \frac{|x-x^0|^2}{2}$. In general, how large $\Xi \subset \mathbb{R}^n$ is depends on the sectional curvature of the Riemannian manifold $(\mathbb{R}^n, g)$ closely. It is well-known that if $(\mathbb{R}^n, g)$ has non-positive sectional curvature, then $\Xi = \mathbb{R}^n$.

To verify the condition (1.11), there is other choice on curvature. If $(\mathbb{R}^n, g)$ is a noncompact complete Riemannian manifold with everywhere positive sectional curvature, then there exists a strictly convex function h on $(\mathbb{R}^n, g)$ by Green and Wu [9]. Then the vector filed $H = \nabla_g h$ satisfies the condition (1.11).

We have obtained the following result.

Theorem 1.2 *If* $(\mathbb{R}^n, g)$ *satisfies one of the following assumptions:*

(a) $(\mathbb{R}^n, g)$ *has non-positive sectional curvature, or*

(b) $(\mathbb{R}^n, g)$ *is noncompact complete Riemannian manifold and has positive sectional curvature everywhere,*

then the problem (1.1) *has a localized dissipation near infinity.*

In general, the condition (1.11) is not true. We have the following result.

Theorem 1.3 *If the sectional curvature of* $(\mathbb{R}^n, g)$ *has a positive lower bound, then the problem* (1.1) *does not have a localized dissipation near infinity in the sense* (1.11).

Next, let us consider decay of the second order energy. We introduce the space

$$H^2(g, \mathbb{R}^n) = \{w \mid w \in L^2(\mathbb{R}^n),\ \operatorname{div} A\nabla w \in L^2(\mathbb{R}^n)\} \tag{1.14}$$

with the norm

$$\|w\|^2_{H^2(g,\mathbb{R}^n)} = \|\operatorname{div} A\nabla w\|^2 + \|w\|^2. \tag{1.15}$$

Let u be a solution to the problem (1.1). We define

$$E_2(t) = \|u_{tt}(t)\|^2 + \||\nabla_g u_t|_g\|^2, \quad t \geq 0. \tag{1.16}$$

For the second order energy, the decay is much more rapid as in the case of constant coefficients (see [17]).

Theorem 1.4 *For* $(u_0, u_1) \in H^2(g, \mathbb{R}^n) \times H^1(g, \mathbb{R}^n)$, *there exists a unique solution*

$$u(t) \in C([0,\infty), H^2(g, \mathbb{R}^n)) \cap C^1([0,\infty), H^1(g, \mathbb{R}^n))$$

of the problem (1.1) *that satisfies*

$$E_2(t) \leq \frac{c}{1+t^2}(\|u_0\|^2_{H^2(g,\mathbb{R}^n)} + \|u_1\|^2_{H^1(g,\mathbb{R}^n)}), \tag{1.17}$$

$$\|\operatorname{div} A\nabla u\|^2 \leq \frac{c}{1+t}(\|u_0\|^2_{H^2(g,\mathbb{R}^n)} + \|u_1\|^2_{H^1(g,\mathbb{R}^n)}), \tag{1.18}$$

where $c > 0$ *is a constant independent of solutions.*

Finally, as an application, we consider the existence of global solutions of the nonlinear wave equation

$$\begin{cases} u_{tt} - \operatorname{div} A\nabla u + a u_t = f(u), & (t,x) \in [0,T) \times \mathbb{R}^n, \\ u(0,x) = u_0(x),\ u_t(0,x) = u_1(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.19}$$

where $f(u)$ is a nonlinear source term.

We make the following assumptions on f and on $A(x)$, respectively.

Assumption 1.2 *Assume that $f(s)$ is locally Lipschitz continuous in $s \in \mathbb{R}$ such that*

$$|f(s)| \leq c|s|^{1+\alpha}, \quad |f(s) - f(\zeta)| \leq c(|s| + |\zeta|)^\alpha |s - \zeta| \tag{1.20}$$

for some $c > 0$ and $\alpha > 0$.

Assumption 1.3 *There is a $\sigma_0 > 0$ such that*

$$\langle A(x)X, X\rangle \geq \sigma_0 |X|^2 \quad \text{for all } x \in \mathbb{R}^n, \ X \in \mathbb{R}^n. \tag{1.21}$$

Following Nakao [17], we have the following result.

Theorem 1.5 *Let Assumptions* 1.1–1.3 *hold. Let $2 \leq n \leq 3$ and $\frac{4}{n} < \alpha \leq \frac{2}{n-2}$ ($2 < \alpha < \infty$ if $n = 2$). Then when $E(0) + \|u_0\|^2$ is small enough, problem* (1.19) *has a global solution in $C([0,\infty), H^1(g, \mathbb{R}^n)) \cap C^1([0,\infty), L^2(\mathbb{R}^n))$ such that*

$$\sup_{t\geq 0} \|u(t)\| < \infty, \quad E(t) \leq \frac{c(u_0, u_1)}{1+t}, \quad t \geq 0$$

for this solution u.

Of course, similar results are true as in [17, Theorem 2]. We omit them.

2 Proofs of the Main Results

We work on $\mathbb{R}^n$ with two metrics, the standard metric $\langle \cdot, \cdot \rangle$ and the Riemannian metric $g = \langle \cdot, \cdot \rangle_g$ given by (1.5).

If $f \in C^1(\mathbb{R}^n)$, we define the gradient $\nabla_g f$ of f in the Riemannian metric g, via the Riesz representation theorem, by

$$X(f) = \langle \nabla_g f, X\rangle_g, \tag{2.1}$$

where X is any vector field on $(\mathbb{R}^n, g)$. The following lemma provides further relations (see [32, Lemma 2.1]).

Lemma 2.1 *Let $x = (x_1, \cdots, x_n)$ be the natural coordinate system in $\mathbb{R}^n$. Let f, h be functions and H, X be vector fields. Then*

$$\langle H(x), A(x)X(x)\rangle_g = \langle H(x), X(x)\rangle, \quad x \in \mathbb{R}^n, \tag{2.2}$$

$$\nabla_g f = \sum_{i=1}^n \Big(\sum_{j=1}^n a_{ij}(x) f_{x_j}\Big) \frac{\partial}{\partial x_i} = A(x)\nabla f, \quad x \in \mathbb{R}^n, \tag{2.3}$$

where ∇f is the gradient of f in the standard metric, and

$$\nabla_g f(h) = \langle \nabla_g f, \nabla_g h\rangle_g = \langle \nabla f, A(x)\nabla h\rangle, \quad x \in \mathbb{R}^n, \tag{2.4}$$

$$\langle \nabla_g f, \nabla_g (H(f))\rangle_g = D_g H(\nabla_g f, \nabla_g f) + \frac{1}{2}\operatorname{div}(|\nabla_g f|_g^2 H) - \frac{1}{2}|\nabla_g f|_g^2 \operatorname{div} H, \quad x \in \mathbb{R}^n, \tag{2.5}$$

where $\operatorname{div} H$ is the divergence of the vector field H in the standard metric and the matrix $A(x)$ is given in problem (1.1).

We assume that the initial data $u_0, u_1 \in C_0^\infty(\mathbb{R}^n)$. Let u be a solution of problem (1.1). Then $u(t,x)$ has a compact support on $\mathbb{R}^n$ for each $t > 0$. Let H be a vector field on $\mathbb{R}^n$. Using the above formulas in Lemma 2.1, we multiply the equation in (1.1) by $H(u)$, integrate by parts over $\mathbb{R}^n$ with respect to the variable x, and obtain (see [32, Proposition 2.1])

$$\frac{\mathrm{d}}{\mathrm{d}t}(u_t, H(u)) + \int_{\mathbb{R}^n} p(u_t^2 - |\nabla_g u|_g^2)\mathrm{d}x + \int_{\mathbb{R}^n} D_g H(\nabla_g u, \nabla_g u)\mathrm{d}x + \int_{\mathbb{R}^n} a u_t H(u)\mathrm{d}x = 0, \quad (2.6)$$

where $(\cdot\,,\,\cdot)$ is the standard inner product of $L^2(\mathbb{R}^n)$ and $p = \frac{\operatorname{div} H}{2}$.

Let q be a function. We multiply the equation (1.1) by qu, integrate by parts, and have

$$\frac{\mathrm{d}}{\mathrm{d}t}[2(u_t, qu) + (aqu, u)] + \int_{\mathbb{R}^n} q(|\nabla_g u|_g^2 - u_t^2)\mathrm{d}x - \int_{\mathbb{R}^n} u^2 \operatorname{div} A\nabla q \mathrm{d}x = 0. \quad (2.7)$$

In addition, differentiating the energy (1.2) with respect to time t yields

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) + \int_{\mathbb{R}^n} a(x)u_t^2 \mathrm{d}x = 0. \quad (2.8)$$

Proof of Theorem 1.1 Let vector field H be such that the condition (1.8) holds. We take two bounded open sets $\widehat{\Omega}$, $\widehat{\widehat{\Omega}} \subset \mathbb{R}^n$ such that

$$\overline{\Omega} \subset \widehat{\Omega} \subset \overline{\widehat{\Omega}} \subset \widehat{\widehat{\Omega}}. \quad (2.9)$$

Let $\varphi, \phi \in C_0^\infty(\mathbb{R}^n)$ be cut-off functions such that

$$0 \le \varphi \le 1, \quad 0 \le \phi \le 1, \quad \varphi = \begin{cases} 1, & x \in \Omega, \\ 0, & x \notin \widehat{\Omega}, \end{cases} \quad \phi = \begin{cases} 1, & x \in \widehat{\Omega}, \\ 0, & x \notin \widehat{\widehat{\Omega}}. \end{cases} \quad (2.10)$$

Let $q = a\phi$ in the identity (2.7). We have

$$\int_{\mathbb{R}^n} a\phi u_t^2 \mathrm{d}x = \int_{\mathbb{R}^n} a\phi |\nabla_g u|_g^2 \mathrm{d}x + \frac{\mathrm{d}}{\mathrm{d}t}[2(u_t, a\phi u) + (a^2\phi u, u)] - \int_{\mathbb{R}^n} u^2 \operatorname{div} A\nabla(a\phi)\mathrm{d}x. \quad (2.11)$$

We replace H with φH in the identity (2.6) and replace q with

$$q_0 = \frac{1}{2}\operatorname{div}\varphi H - \sigma\varphi \quad (2.12)$$

in the identity (2.7), respectively, where $\sigma > 0$ is given in (1.8). Then we add up the two identities and obtain

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}X(t) + \int_{\mathbb{R}^n} [D_g(\varphi H)(\nabla_g u, \nabla_g u) + \sigma\varphi(u_t^2 - |\nabla_g u|_g^2)]\mathrm{d}x \\ &- \int_{\mathbb{R}^n} u^2 \operatorname{div} A\nabla q_0 \mathrm{d}x + \int_{\mathbb{R}^n} a\varphi u_t H(u)\mathrm{d}x = 0, \end{aligned} \quad (2.13)$$

where

$$X(t) = (u_t, \varphi H(u) + 2q_0 u) + (aq_0 u, u). \quad (2.14)$$

Let $k > 0$ be a constant. We multiply the identity (2.8) by k, then add it to the identity (2.13), and have

$$\frac{\mathrm{d}}{\mathrm{d}t}[X(t) + kE(t)] + \int_{\mathbb{R}^n} Y(k, u)\mathrm{d}x - \int_{\mathbb{R}^n} u^2 \operatorname{div} A\nabla q_0 \mathrm{d}x + \int_{\mathbb{R}^n} a\varphi u_t H(u)\mathrm{d}x = 0, \quad (2.15)$$

where

$$Y(k,u) = D_g(\varphi H)(\nabla_g u, \nabla_g u) + kau_t^2 + \sigma\varphi(u_t^2 - |\nabla_g u|_g^2). \tag{2.16}$$

Since $a \geq a\phi \geq 0$, we obtain via (1.8)

$$\begin{aligned}\int_{\mathbb{R}^n} Y(k,u)\mathrm{d}x &\geq \int_{\mathbb{R}^n\setminus\Omega} \Big[D_g(\varphi H)(\nabla_g u, \nabla_g u) + \frac{ka}{4}u_t^2 + \sigma\varphi(u_t^2 - |\nabla_g u|_g^2)\Big]\mathrm{d}x \\ &\quad + \int_\Omega [D_g H(\nabla_g u, \nabla_g u) + \sigma(u_t^2 - |\nabla_g u|_g^2)]\mathrm{d}x + \frac{k}{2}\int_{\mathbb{R}^n} au_t^2\mathrm{d}x + \frac{k}{4}\int_{\mathbb{R}^n} a\phi u_t^2\mathrm{d}x \\ &\geq \frac{k}{4}\int_{\mathbb{R}^n} a\phi u_t^2\mathrm{d}x - \int_{\widehat{\Omega}\setminus\Omega}(\sigma_1+\sigma)|\nabla_g u|_g^2\mathrm{d}x + \sigma\int_\Omega u_t^2\mathrm{d}x + \int_{\mathbb{R}^n}\frac{ka}{2}u_t^2\mathrm{d}x, \end{aligned} \tag{2.17}$$

where

$$\sigma_1 = \sup_{\substack{X\in\mathbb{R}^n_x \\ |X|_g=1,\, x\in\widehat{\Omega}\setminus\Omega}} |D_g(\varphi H)(X,X)|. \tag{2.18}$$

Using identity (2.11) in inequality (2.17) yields

$$\begin{aligned}\int_{\mathbb{R}^n} Y(k,u)\mathrm{d}x &\geq \int_{\widehat{\Omega}\setminus\Omega}\Big(\frac{ka}{4} - \sigma - \sigma_1\Big)|\nabla_g u|_g^2\mathrm{d}x + \sigma\int_\Omega u_t^2\mathrm{d}x + \int_{\mathbb{R}^n}\frac{ka}{2}u_t^2\mathrm{d}x \\ &\quad + \frac{k}{4}\frac{\mathrm{d}}{\mathrm{d}t}[2(u_t, a\phi u) + (a^2\phi u, u)] - \frac{k}{4}\int_{\mathbb{R}^n} u^2 \mathrm{div}\, A\nabla(a\phi)\mathrm{d}x. \end{aligned} \tag{2.19}$$

If

$$k \geq \frac{4}{\varepsilon}(\sigma + \sigma_1), \tag{2.20}$$

from (2.19) and (1.9), we have

$$\int_{\mathbb{R}^n} Y(k,u)\mathrm{d}x \geq \sigma\int_{\mathbb{R}^n} u_t^2\mathrm{d}x + \frac{k}{4}\frac{\mathrm{d}}{\mathrm{d}t}[2(u_t, a\phi u) + (a^2\phi u, u)] + \frac{k}{4}\int_{\mathbb{R}^n} au_t^2\mathrm{d}x - \frac{k\sigma_2}{4}\int_{\widehat{\widehat{\Omega}}} u^2\mathrm{d}x, \tag{2.21}$$

where

$$\sigma_2 = \sup_{x\in\widehat{\widehat{\Omega}}}|\mathrm{div}\, A\nabla(a\phi)|. \tag{2.22}$$

Letting $q = 1$ in (2.7) yields

$$\int_{\mathbb{R}^n} u_t^2\mathrm{d}x = \int_{\mathbb{R}^n}|\nabla_g u|_g^2\mathrm{d}x + \frac{\mathrm{d}}{\mathrm{d}t}[2(u_t, u) + (au, u)]. \tag{2.23}$$

Moreover,

$$\Big|\int_{\mathbb{R}^n} a\varphi u_t H(u)\mathrm{d}x\Big| \leq \frac{\sigma_3}{k}\int_{\mathbb{R}^n}|\nabla_g u|_g^2\mathrm{d}x + \frac{k}{8}\int_{\mathbb{R}^n} au_t^2\mathrm{d}x, \tag{2.24}$$

where

$$\sigma_3 = 2\sup_{x\in\widehat{\Omega}}(a\varphi^2|H|_g^2). \tag{2.25}$$

Finally, we use (2.24), (2.23) and (2.21) in (2.15) to obtain that if

$$k \geq \max\Big[\frac{3\sigma_3}{\sigma}, \frac{4}{\varepsilon}(\sigma + \sigma_1)\Big], \tag{2.26}$$

then

$$\frac{\mathrm{d}}{\mathrm{d}t}Z(k,u) + \frac{\sigma}{3}E(t) + \frac{k}{8}\int_{\mathbb{R}^n} au_t^2\mathrm{d}x \leq \Big(\frac{k}{4}\sigma_2 + \sigma_4\Big)\int_{\widehat{\widehat{\Omega}}} u^2\mathrm{d}x, \tag{2.27}$$

where

$$Z(k,u)=\Big(u_t,\varphi H(u)+\Big[2q_0+\frac{4}{3}\sigma+\frac{k}{2}a\phi\Big]u\Big)+\Big(u,\Big[\frac{2\sigma}{3}a+aq_0+\frac{k}{4}a^2\phi\Big]u\Big)+kE(t), \quad (2.28)$$

$$\sigma_4=\sup_{x\in\widehat{\widehat{\Omega}}}|\mathrm{div}\,A\nabla q_0|. \quad (2.29)$$

Let $k>0$ be given such that the condition (2.26) holds. By (2.28), we have

$$\begin{aligned}Z(k,u)&\geq kE(t)+\frac{2\sigma}{3}(au,u)-\sup_{x\in\widehat{\Omega}}\varphi|H|_g\|u_t\|\|\nabla_g u\|\\&\quad-\sup_{x\in\widehat{\widehat{\Omega}}}\Big|2q_0+\frac{4}{3}\sigma+\frac{k}{2}a\phi\Big|\|u_t\|\|u\|-\sup_{x\in\widehat{\widehat{\Omega}}}|aq_0|\int_{\widehat{\widehat{\Omega}}}u^2\mathrm{d}x\\&\geq\frac{\sigma\varepsilon}{3}\|u\|^2+\Big(k-\sup_{x\in\widehat{\Omega}}\varphi|H|_g-\frac{3\sigma_5^2}{4\sigma\varepsilon}\Big)E(t)-\Big(\sup_{x\in\widehat{\widehat{\Omega}}}a|q_0|+\frac{2\sigma\varepsilon}{3}\Big)\int_{\widehat{\widehat{\Omega}}}u^2\mathrm{d}x \quad (2.30)\end{aligned}$$

and

$$Z(k,u)\leq\Big(k+\sup_{x\in\widehat{\Omega}}\varphi|H|_g+\sigma_5\Big)E(t)+(\sigma_5+\sigma_6)\|u\|^2, \quad (2.31)$$

where

$$\sigma_5=\sup_{x\in\widehat{\widehat{\Omega}}}\Big|2q_0+\frac{4}{3}\sigma+\frac{k}{2}a\phi\Big|,\quad \sigma_6=\frac{1}{2}\sup_{x\in\widehat{\widehat{\Omega}}}a\Big(\frac{2\sigma}{3}+q_0+\frac{k}{4}a\phi\Big).$$

Now we fix a $k>0$ such that

$$k\geq\max\Big[\frac{3\sigma_3}{\sigma},\frac{4}{\varepsilon}(\sigma+\sigma_1),\sup_{x\in\widehat{\Omega}}\varphi|H|_g+\frac{3\sigma_5^2}{4\sigma\varepsilon}\Big]. \quad (2.32)$$

We integrate (2.27) over $[s,t]$ with respect to time t where $0\leq s\leq t$. Using the estimates (2.30) and (2.31), we obtain constants $c_1>0$ and $c_2>0$ which are independent of $t>0$ and a solution u such that

$$E(t)+\|u(t)\|^2+c_1\int_s^t E(\tau)\mathrm{d}\tau+\frac{k}{8}\int_{\mathbb{R}^n}au_t^2\mathrm{d}x\leq c_2(E(s)+\|u(s)\|^2)+c_2\int_s^t\int_{\widehat{\widehat{\Omega}}}u^2\mathrm{d}x\mathrm{d}\tau \quad (2.33)$$

for k satisfying the inequality (2.32), where the following estimate is used:

$$\|u(t)\|^2\leq\|u(s)\|^2+\varepsilon\int_s^t u_t^2\,\mathrm{d}\tau+C_\varepsilon\int_s^t u^2\,\mathrm{d}\tau.$$

Using the compactness-uniqueness argument in Lemma 2.2 below, the lower order term in (2.33) can be absorbed. We then have constants $c_1>0$ and $c_2>0$ such that

$$E(t)+\|u(t)\|^2+c_1\int_s^t E(\tau)\mathrm{d}\tau\leq c_2(E(s)+\|u(s)\|^2) \quad (2.34)$$

for $0\leq s\leq t$ with $t-s\geq T_0$, where

$$T_0=\frac{2}{\sigma}\sup_{x\in\widehat{\widehat{\Omega}}}|H|_g. \quad (2.35)$$

The above number T_0 is given by Yao [32, Theorem 1.1], which is referred as to the length of wave.

The estimate (2.34) and the identity (2.8) together yield

$$E(t)\leq\frac{c_2+1}{1+t}(E(0)+\|u_0\|^2),\quad t>0. \quad (2.36)$$

Lemma 2.2 *Let $T \geq T_0$ be given, where T_0 is defined by (2.35). Then for $\eta > 0$ given, there exists a $c_\eta > 0$ such that*

$$\int_t^{t+T}\int_{\widehat{\widehat{\Omega}}} u^2 \mathrm{d}x\mathrm{d}\tau \leq \eta \int_t^{t+T} E(\tau)\mathrm{d}\tau + c_\eta \int_t^{t+T}\int_{\mathbb{R}^n} a u_t^2 \mathrm{d}x\mathrm{d}\tau, \quad t > 0. \tag{2.37}$$

Proof As usual, we prove the lemma by contradiction. We assume that for some $\eta_0 > 0$ the number c_{η_0} does not exist. Then for any $m \geq 1$ there are solutions u_m and $t_m > 0$ such that

$$\int_{t_m}^{t_m+T}\int_{\widehat{\widehat{\Omega}}} u_m^2 \mathrm{d}x\mathrm{d}\tau \geq \eta_0 \int_{t_m}^{t_m+T} E_m(\tau)\mathrm{d}\tau + m \int_{t_m}^{t_m+T}\int_{\mathbb{R}^n} a u_{mt}^2 \mathrm{d}x\mathrm{d}\tau, \tag{2.38}$$

where $E_m(t)$ are the $E(t)$ with u replaced by u_m.

Let

$$v_m(t) = \frac{u_m(t_m + t)}{\lambda_m}, \quad \lambda_m^2 = \int_{t_m}^{t_m+T}\int_{\widehat{\widehat{\Omega}}} u_m^2 \mathrm{d}x\mathrm{d}\tau. \tag{2.39}$$

Then the inequality (2.38) means that $v_m \in H^1(g, (0,T) \times \mathbb{R}^n)$ satisfies

$$1 \geq \eta_0 \int_0^T\int_{\mathbb{R}^n} (v_{mt}^2 + |\nabla_g v_m|_g^2)\mathrm{d}x\mathrm{d}\tau + m \int_0^T\int_{\mathbb{R}^n} a v_{mt}^2 \mathrm{d}x\mathrm{d}\tau, \tag{2.40}$$

$$\|v_m\|^2_{L^2((0,T)\times\widehat{\widehat{\Omega}})} = 1, \tag{2.41}$$

where

$$H^1(g, (0,T) \times \mathbb{R}^n) = \{w \mid w, w_t, |\nabla_g w|_g \in L^2((0,T) \times \mathbb{R}^n)\}$$

with the norm

$$\|w\|^2_{H^1(g,(0,T)\times\mathbb{R}^n)} = \int_0^T\int_{\mathbb{R}^n} (w_t^2 + |\nabla_g w|_g^2 + w^2)\mathrm{d}x\mathrm{d}t.$$

We may assume that there is a $v_0 \in H^1(g, (0,T) \times \mathbb{R}^n)$ such that

$$v_m \rightharpoonup v_0, \quad \text{in } H^1(g, (0,T) \times \mathbb{R}^n) \text{ weakly}, \tag{2.42}$$

$$v_m \to v_0, \quad \text{in } L^2((0,T) \times \widehat{\widehat{\Omega}}) \text{ strongly}. \tag{2.43}$$

By (2.40), we have

$$\int_{\mathbb{R}^n} a v_{0t}^2 \mathrm{d}x = 0,$$

that is,

$$a v_{0t} = 0, \ \ \text{in } \mathbb{R}^n; \quad v_{0t} = 0, \ \ \text{in } \mathbb{R}^n \setminus \Omega \text{ via (1.9)}. \tag{2.44}$$

By (2.41) and (2.43), we obtain

$$\|v_0\|^2_{L^2((0,T)\times\widehat{\widehat{\Omega}})} = 1. \tag{2.45}$$

Moreover, by (2.42)–(2.44), $w = v_{0t}$ solves the problem

$$\begin{cases} w_{tt} - \operatorname{div} A\nabla w = 0, & (t,x) \in (0,T) \times \widehat{\Omega}, \\ w = 0, & (t,x) \in (0,T) \times \mathbb{R}^n \setminus \Omega. \end{cases} \tag{2.46}$$

Then an observability estimate, as in [32, Theorem 1.1], implies

$$v_{0t} = 0, \quad (t,x) \in (0,T) \times \mathbb{R}^n,$$

which yields

$$\operatorname{div} A\nabla v_0 = 0, \quad x \in \mathbb{R}^n. \tag{2.47}$$

Then $v_0 \in H^1(g, \mathbb{R}^n)$ implies that v_0 is constant. Furthermore, $v_0 \in L^2(\mathbb{R}^n)$ gives

$$v_0 = 0, \quad \text{in } \mathbb{R}^n,$$

which contradicts the relation (2.45).

Proof of Theorem 1.3 Since $w = u_t$ solves problem (1.1) with the initial data (u_1, $\operatorname{div} A\nabla u_0 - au_1$), replacing u with u_t in the inequality (2.34) yields

$$E_2(t) + \|u_t(t)\|^2 + c_1 \int_s^t E_2(\tau)\mathrm{d}\tau \le c_2(E_2(s) + \|u_t(s)\|^2), \quad t - T_0 \ge s \ge 0. \tag{2.48}$$

We integrate the inequality (2.48) over $[0, t - T_0]$ and obtain via (2.34) and (2.48) that

$$\begin{aligned} c_1 \int_0^{t-T_0} \tau E_2(\tau)\mathrm{d}\tau &\le c_2 \int_0^{t-T_0} (E_2(\tau) + \|u_t(\tau)\|^2)\mathrm{d}\tau \\ &\le c_2^2(E_2(0) + \|u_1\|^2 + E(0) + \|u_0\|^2) \\ &\le c_3(\|u_0\|^2_{H^2(g,\mathbb{R}^n)} + \|u_1\|^2_{H^1(g,\mathbb{R}^n)}), \quad t > T_0, \end{aligned} \tag{2.49}$$

which implies the estimate (1.17) since $E_2'(t) \le 0$.

The estimate (1.18) follows from the inequalities (1.17), (2.36) and the equation in the problem (1.1).

Proof of Theorem 1.4 We prove the theorem by contradiction. We assume that there is a vector field H on $(\mathbb{R}^n, g)$ such that the condition (1.11) holds. By Gallot, Hulin, and Lafontaine [8, Chapter II, Problem 2.98], there is a closed geodesic $r : [0, b] \to \mathbb{R}^n$ with $r(0) = r(b) = x_0 \in \mathbb{R}^n$. Then

$$\dot r(0) = -\dot r(b). \tag{2.50}$$

Let

$$f(t) = \langle H(r(t)), \dot r(t)\rangle_g, \quad t \in [0, b].$$

By (2.50), we get

$$f(0) = -f(b). \tag{2.51}$$

The condition (1.11) implies

$$f'(t) = \langle D_{g_{\dot r(t)}} H, \dot r(t)\rangle_g = D_g H(\dot r(t), \dot r(t)) > 0, \quad t \in [0, b],$$

which implies

$$f(b) > f(0),$$

that is, by (2.51),

$$\langle H(x_0), \dot r(0)\rangle_g < 0. \tag{2.52}$$

Now let

$$p(t) = \langle H(\alpha(t)), \alpha(t)\rangle_g, \quad \alpha(t) = \exp_{x_0} t(-\dot r(0)),$$

where $\exp_{x_0} : \mathbb{R}^n_{x_0} \to (\mathbb{R}^n, g)$ is the exponential map of the metric g. Then the same argument on $p(t)$ as above for $f(t)$ gives

$$-\langle H(x_0), \dot{r}(0)\rangle_g = \langle H(x_0), \dot{\alpha}(0)\rangle_g < 0,$$

which contradicts the relation (2.52).

Proof of Theorem 1.5 This is completed by following the proof of Theorem 2 in [17], where just minor changes are needed. Assumption 1.3 implies

$$|||\nabla w||| \leq \frac{1}{\sqrt{\sigma_0}} |||\nabla_g w|||, \quad w \in H^1(g, \mathbb{R}^n),$$

which is used when one applies the Gagliardo-Nirenberg estimate (see [23]) to the source term $f(u)$.

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