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New Monotonicity Formulae for Semi-linear Elliptic and Parabolic Systems***

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Abstract The authors establish a general monotonicity formula for the following elliptic system

$$\Delta u_i + f_i(x, u_1, \cdots, u_m) = 0 \quad \text{in } \Omega,$$

where $\Omega \subset\subset \mathbb{R}^n$ is a regular domain, $(f_i(x, u_1, \cdots, u_m)) = \nabla_{\vec{u}} F(x, \vec{u})$, $F(x, \vec{u})$ is a given smooth function of $x \in \mathbb{R}^n$ and $\vec{u} = (u_1, \cdots, u_m) \in \mathbb{R}^m$. The system comes from understanding the stationary case of Ginzburg-Landau model. A new monotonicity formula is also set up for the following parabolic system

$$\partial_t u_i - \Delta u_i - f_i(x, u_1, \cdots, u_m) = 0 \quad \text{in } (t_1, t_2) \times \mathbb{R}^n,$$

where $t_1 < t_2$ are two constants, $(f_i(x, \vec{u}))$ is given as above. The new monotonicity formulae are focused more attention on the monotonicity of nonlinear terms. The new point of the results is that an index β is introduced to measure the monotonicity of the nonlinear terms in the problems. The index β in the study of monotonicity formulae is useful in understanding the behavior of blow-up sequences of solutions. Another new feature is that the previous monotonicity formulae are extended to nonhomogeneous nonlinearities. As applications, the Ginzburg-Landau model and some different generalizations to the free boundary problems are studied.

Keywords Elliptic systems, Parabolic system, Monotonicity formula, Ginzburg-Landau model
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1 Introduction

In this paper, we will establish a general monotonicity formula for the following elliptic system:

$$\Delta u_i + f_i(x, u_1, \cdots, u_m) = 0 \quad \text{in } \Omega, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is a regular domain, $(f_i(x, u_1, \cdots, u_m)) = \nabla_{\vec{u}} F(x, \vec{u})$, $F(x, \vec{u})$ is a given smooth function of $x \in \mathbb{R}^n$ and $\vec{u} = (u_1, \cdots, u_m)$ (the precise smoothness will be given in theorems). Here we assume that the solution $\vec{u} \in H^1_{\rm loc}(\Omega)$ satisfies (1.1) in the variational sense to be defined in Section 2. We remark that smooth solutions to (1.1) satisfy (1.1) in the variational sense

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naturally. Our motivation for studying the system (1.1) comes from understanding the stationary case of Ginzburg-Landau model (see [3, 20, 21]). We shall also establish a monotonicity formula for regular solutions of the following parabolic system:

$$\partial_t u_i - \Delta u_i - f_i(x, u_1, \cdots, u_m) = 0 \quad \text{in } (t_1, t_2) \times \mathbb{R}^n, \tag{1.2}$$

where $t_1 < t_2$ are two constants, $(f_i(x, \vec{u}))$ is given as above. One new point in our monotonicity formula is that we introduce an index β, which measures the monotonicity of the nonlinear term $\vec{f} = (f_1, \cdots, f_m)$. This index β also gives us the rate of scaled sequence of the blow-up process for implied solutions. Another new feature is that we extend the previous monotonicity formulae to nonhomogeneous nonlinearities. Our main results are Theorems 2.1, 2.2, 3.1 and 3.2 below. As applications of our new monotonicity formulae, we study the Ginzburg-Landau model and some different generalizations to the free boundary problems. The applications are in Propositions 4.1–4.5 below. Our work is motivated from the monotonicity formulae given by G. S. Weiss [29–32] and the monotonicity formula given by Alt, Caffarelli and Friedman [2] for free boundary problems. For more background related to our work, one may see the appendix in Section 5.

Before we state the monotonicity formulae, we introduce some notations and concepts. As in [32], we denote by $x \cdot y$ the Euclidean inner product in $\mathbb{R}^n \times \mathbb{R}^n$, by $|x|$ the Euclidean norm in $\mathbb{R}^n$, by $B_r(x_0) = \{x \in \mathbb{R}^n \mid |x - x_0| < r\}$ the ball of center x_0 and radius r, by $Q_r(x_0, t_0) = (t_0 - r^2, t_0 + r^2) \times B_r(x_0)$ the cylinder of radius r and height $2r^2$, by $T_r^-(t_0) = (t_0 - 4r^2, t_0 - r^2) \times \mathbb{R}^n$ the horizontal layer from $t_0 - 4r^2$ to $t_0 - r^2$, and by $T_r^+(t_0) = (t_0 + r^2, t_0 + 4r^2) \times \mathbb{R}^n$ the horizontal layer from $t_0 + r^2$ to $t_0 + 4r^2$. We write $T_r^-(T)$ and $T_r^+(T)$ as T_r^- and T_r^+ for notation convenience. We use

$$G_{(t_0,x_0)}(t, x) = 4\pi(t_0 - t)|4\pi(t_0 - t)|^{-\frac{n}{2}-1} \exp\Big(-\frac{|x - x_0|^2}{4(t_0 - t)}\Big)$$

to denote the backward heat kernel, defined in $((-\infty, t_0) \cup (t_0, +\infty)) \times \mathbb{R}^n$. Furthermore, by ν we will always refer to the outer unit normal on a given surface. We mean by $H^1_{\mathrm{loc}}(\Omega)$ and $H^1(Q_T)$ the usual local Sobolev space and parabolic Sobolev spaces respectively as defined in [18].

Roughly speaking, our new monotonicity formulae for (1.1) and (1.2) are as follows. We will show that for the variational solution $\vec{u}$ to (1.1), the function

$$\Phi_{x_0}(r) := r^{-n-2\beta+2} \int_{B_r(x_0)} (|\nabla \vec{u}|^2 - 2F(x, \vec{u})) - \beta r^{-n-2\beta+1} \int_{\partial B_r(x_0)} \vec{u}^2$$

is increasing in $r \in (0, \delta)$ if $\forall\, r \in (0, \delta)$,

$$\int_{B_r(x_0)} (2(\beta - 1)F(x, \vec{u}) - \beta \vec{u}\vec{f}(x, \vec{u}) - \nabla_x F(x, \vec{u}) \cdot (x - x_0)) \geq 0, \tag{1.3}$$

and for the variational solution $\vec{u}$ to (1.2), the functions

$$\Psi^-(r) := r^{-2\beta} \int_{T_r^-} (|\nabla \vec{u}|^2 - 2F(x, \vec{u}))G_{(T,x_0)} - \frac{\beta}{2} r^{-2\beta} \int_{T_r^-} \frac{1}{T - t} \vec{u}^2 G_{(T,x_0)}$$

and

$$\Psi^+(r) := r^{-2\beta} \int_{T_r^+} (|\nabla \vec{u}|^2 - 2F(x, \vec{u}))G_{(T,x_0)} - \frac{\beta}{2} r^{-2\beta} \int_{T_r^+} \frac{1}{T - t} \vec{u}^2 G_{(T,x_0)}$$

are increasing in $r \in (0, \delta)$ provided for the index $\beta \in \mathbb{R}$ and the radial variable $\forall\, r \in (0, \delta)$ there hold

$$\int_{T_r^-} (2(\beta - 1)F(x, \vec{u}) - \beta\vec{u}\vec{f}(x, \vec{u}) - \nabla_x F(x, \vec{u}) \cdot (x - x_0))G_{(T,x_0)} \geq 0 \tag{1.4}$$

and

$$\int_{T_r^+} (2(\beta - 1)F(x, \vec{u}) - \beta\vec{u}\vec{f}(x, \vec{u}) - \nabla_x F(x, \vec{u}) \cdot (x - x_0))G_{(T,x_0)} \geq 0. \tag{1.5}$$

We remark that the conditions (1.3)–(1.5) are automatically true if the weaker point-wise condition is satisfied:

$$2(\beta - 1)F(x, \vec{u}) - \beta\vec{u}\vec{f}(x, \vec{u}) - \nabla_x F(x, \vec{u}) \cdot (x - x_0) \geq 0, \quad \forall\, x \in \Omega. \tag{1.6}$$

We will give illustration by examples in Section 4. From the expression (1.6) above, it is clear that the number β measures the monotonicity of the nonlinear terms, and our new monotonicity formulae are focused more attention on the monotonicity of nonlinear nonhomogeneous terms. Our method can also be used to study elliptic and parabolic systems with variable coefficients.

The remaining part of the paper is organized as follows. In Section 2, we establish the monotonicity formula for (1.1) and characterize the scaled blow-up sequences. In Section 3, we establish the monotonicity formula for (1.2) and characterize the scaled blow-up sequences. In Section 4, we state the interesting applications of our results in Section 2 and Section 3 to the Ginzburg-Landau model and various extensions of the free-boundary problems considered by other authors (see, for examples, [29, 32]).

2 Monotonicity Formula for the Elliptic System

Consider the elliptic system

$$\Delta u_i + f_i(x, u_1, \cdots, u_m) = 0, \quad i = 1, \cdots, m, \quad \text{in } \Omega, \tag{2.1}$$

where $\Omega \subset\subset \mathbb{R}^n$ and $(f_i(x, \vec{u}))$ is the gradient with respect to the $\vec{u} = (u_1, \cdots, u_m)$ variables of a given smooth function $F(x, \vec{u})$. In order to define the variational solution of (2.1), we need some notations. We denote

$$D\phi = \begin{pmatrix} \partial_1\phi_1 & \cdots & \partial_n\phi_1 \\ \vdots & & \vdots \\ \partial_1\phi_n & \cdots & \partial_n\phi_n \end{pmatrix}$$

for $\phi = (\phi_1, \cdots, \phi_n) \in H^1_{\text{loc}}(\mathbb{R}^n; \mathbb{R}^n)$. Denote $\vec{u}^2 = \sum\limits_{i=1}^m u_i^2$, $\vec{u}\vec{f}(\vec{u}) = \sum\limits_{i=1}^m u_i f_i(\vec{u})$, $|\nabla\vec{u}|^2 = \sum\limits_{i=1}^m |\nabla u_i|^2$, $\nabla\vec{u} \cdot x = (\nabla u_1 \cdot x, \cdots, \nabla u_m \cdot x)$, $(\nabla\vec{u} \cdot v)^2 = \sum\limits_{i=1}^m (\nabla u_i \cdot v)^2$, $\vec{u}\nabla\vec{u} \cdot v = \sum\limits_{i=1}^m u_i \nabla u_i \cdot v$ for any vector v, and $\nabla\vec{u}D\phi\nabla\vec{u} = \sum\limits_{i=1}^m \nabla u_i D\phi \nabla u_i$. We say that $\vec{u} \in H^1_{\text{loc}}(\Omega)$ if every component $u_i \in H^1_{\text{loc}}(\Omega)$, $i = 1, \cdots, m$.

Definition 2.1 *$\vec{u}$ is called a solution to* (2.1) *in the sense of variations, or simply a variational solution, if the following three conditions are satisfied simultaneously:*

(1) $u_i \in H^1_{\text{loc}}(\Omega)$, $u_i f_i(x, \vec{u}), F(x, \vec{u}), \nabla_x F(x, \vec{u}) \in L^1_{\text{loc}}(\Omega)$;

(2) $\vec{u}$ *satisfies* (2.1) *in the sense of distributions;*

(3) *the first variation with respect to domain variables of the functional*

$$G(\vec{v}) = \int_\Omega (|\nabla \vec{v}|^2 - 2F(x, \vec{v}))$$

vanishes at $\vec{v} = \vec{u}$, *i.e.,*

$$\begin{aligned} 0 &= -\frac{\mathrm{d}}{\mathrm{d}\varepsilon} G(\vec{u}(x + \varepsilon\phi(x)))\Big|_{\varepsilon=0} \\ &= \int_\Omega ((|\nabla \vec{u}|^2 - 2F(x, \vec{u})) \operatorname{div} \phi - 2\nabla \vec{u} D\phi \nabla \vec{u} - 2\nabla_x F(x, \vec{u}) \cdot \phi) \end{aligned}$$

for any $\phi \in C_0^1(\Omega; \mathbb{R}^n)$.

The main result in this section reads as follows.

Theorem 2.1 (Monotonicity Formula) *Assume that* $B_\delta(x_0) \subset\subset \Omega$ *and* $\vec{u}$ *is a solution to* (2.1) *in the sense of variations. If there exists a real number* $\beta \in \mathbb{R}$ *such that* $\forall\, r \in (0, \delta)$,

$$\int_{B_r(x_0)} (2(\beta - 1)F(x, \vec{u}) - \beta \vec{u} \vec{f}(x, \vec{u}) - \nabla_x F(x, \vec{u}) \cdot (x - x_0)) \geq 0,$$

then the function

$$\Phi_{x_0}(r) := r^{-n-2\beta+2} \int_{B_r(x_0)} (|\nabla \vec{u}|^2 - 2F(x, \vec{u})) - \beta r^{-n-2\beta+1} \int_{\partial B_r(x_0)} \vec{u}^2,$$

defined in $(0, \delta)$, *satisfies the monotonicity formula*

$$\begin{aligned} \Phi_{x_0}(\sigma) - \Phi_{x_0}(\rho) &= 2 \int_\rho^\sigma r^{-n-2\beta+1} \int_{B_r(x_0)} (2(\beta - 1)F(x, \vec{u}) - \beta \vec{u} \vec{f}(x, \vec{u}) - \nabla_x F(x, \vec{u}) \cdot (x - x_0)) \\ &\quad + 2 \int_\rho^\sigma r^{-n-2\beta+2} \int_{\partial B_r(x_0)} \Big(\nabla \vec{u} \cdot \nu - \beta \frac{\vec{u}}{r}\Big)^2 \\ &\geq 0 \end{aligned}$$

for all $0 < \rho < \sigma < \delta$, *where*

$$\Big(\nabla \vec{u} \cdot \nu - \beta \frac{\vec{u}}{r}\Big)^2 = \sum_{i=1}^m \Big(\nabla u_i \cdot \nu - \beta \frac{u_i}{r}\Big)^2.$$

Remark 2.1 The condition involving the integral term that

$$\int_{B_r(x_0)} (2(\beta - 1)F(x, \vec{u}) - \beta \vec{u} \vec{f}(x, \vec{u}) - \nabla_x F(x, \vec{u}) \cdot (x - x_0)) \geq 0, \quad \forall\, r \in (0, \delta)$$

is not convenient to verify sometimes. We therefore prefer to state a stronger point-wise condition

$$2(\beta - 1)F(x, \vec{u}) - \beta \vec{u} \vec{f}(x, \vec{u}) - \nabla_x F(x, \vec{u}) \cdot (x - x_0) \geq 0, \quad \forall\, x \in B_\delta(x_0).$$

Proof of Theorem 2.1 We may assume that $x_0 = 0$ by a translation. We take after approximation $\phi_\varepsilon(x) = \eta_\varepsilon(x)x$ as test function in Definition 2.1(3) for small positive ε with

$\eta_\varepsilon(x) = \max(0, \min(1, \frac{r-|x|}{\varepsilon}))$, and obtain

$$\begin{aligned}
0 &= \int (n(|\nabla\vec{u}|^2 - 2F(x,\vec{u}))\eta_\varepsilon - 2|\nabla\vec{u}|^2\eta_\varepsilon - 2\nabla_x F(x,\vec{u}) \cdot \eta_\epsilon(x)x) \\
&\quad + \int ((|\nabla\vec{u}|^2 - 2F(x,\vec{u}))\nabla\eta_\varepsilon \cdot x - 2\nabla\vec{u} \cdot x\nabla\vec{u} \cdot \nabla\eta_\varepsilon) \\
&\to \int_{B_r(0)} (n(|\nabla\vec{u}|^2 - 2F(x,\vec{u})) - 2|\nabla\vec{u}|^2 - 2\nabla_x F(x,\vec{u}) \cdot x) \\
&\quad - r\int_{\partial B_r(0)} (|\nabla\vec{u}|^2 - 2F(x,\vec{u}) - 2(\nabla\vec{u} \cdot \nu)^2)
\end{aligned}$$

for a.e. $r \in (0,\delta)$ as $\varepsilon \to 0$, i.e.,

$$\begin{aligned}
0 &= (n-2)\int_{B_r(0)} (|\nabla\vec{u}|^2 - 2F(x,\vec{u})) - 4\int_{B_r(0)} F(x,\vec{u}) \\
&\quad - 2\int_{B_r(0)} \nabla_x F(x,\vec{u}) \cdot x - r\int_{\partial B_r(0)} (|\nabla\vec{u}|^2 - 2F(x,\vec{u}) - 2(\nabla\vec{u} \cdot \nu)^2).
\end{aligned} \tag{2.2}$$

By Definition 2.1(2), we can apply mollifier $u_{i,\rho}$ to (2.1) for every u_i $(i = 1, \cdots, m)$, where $\rho > 0$, and get

$$-\Delta u_{i,\rho} = (f_i(x, u_1, \cdots, u_m))_\rho.$$

Multiplying this equation by u_i and integrating over $B_r(0)$, then sending $\rho \to 0+$, we can easily derive the identity

$$\int_{B_r(0)} |\nabla\vec{u}|^2 = \int_{\partial B_r(0)} \vec{u}\nabla\vec{u} \cdot \nu + \int_{B_r(0)} \vec{u}\vec{f}(x,\vec{u}) \tag{2.3}$$

for a.e. $r \in (0,\delta)$. Next, multiplying (2.2) by $-r^{-n-2\beta+1}$ and using (2.3), we obtain

$$\begin{aligned}
0 &= -(n-2)r^{-n-2\beta+1}\int_{B_r(0)} (|\nabla\vec{u}|^2 - 2F(x,\vec{u})) + 4r^{-n-2\beta+1}\int_{B_r(0)} F(x,\vec{u}) \\
&\quad + 2r^{-n-2\beta+1}\int_{B_r(0)} \nabla_x F(x,\vec{u}) \cdot x + r^{-n-2\beta+2}\int_{\partial B_r(0)} (|\nabla\vec{u}|^2 - 2F(x,\vec{u}) - 2(\nabla\vec{u} \cdot \nu)^2) \\
&= (-n-2\beta+2)r^{-n-2\beta+1}\int_{B_r(0)} (|\nabla\vec{u}|^2 - 2F(x,\vec{u})) - 4(\beta-1)r^{-n-2\beta+1}\int_{B_r(0)} F(x,\vec{u}) \\
&\quad + 2r^{-n-2\beta+1}\int_{B_r(0)} \nabla_x F(x,\vec{u}) \cdot x + 2\beta r^{-n-2\beta+1}\Big(\int_{\partial B_r(0)} \vec{u}\nabla\vec{u} \cdot \nu + \int_{B_r(0)} \vec{u}\vec{f}(x,\vec{u})\Big) \\
&\quad + r^{-n-2\beta+2}\int_{\partial B_r(0)} (|\nabla\vec{u}|^2 - 2F(x,\vec{u})) - 2r^{-n-2\beta+2}\int_{\partial B_r(0)} (\nabla\vec{u} \cdot \nu)^2.
\end{aligned}$$

That is,

$$\begin{aligned}
&(-n-2\beta+2)r^{-n-2\beta+1}\int_{B_r(0)} (|\nabla\vec{u}|^2 - 2F(x,\vec{u})) + r^{-n-2\beta+2}\int_{\partial B_r(0)} (|\nabla\vec{u}|^2 - 2F(x,\vec{u})) \\
&= \Big(2r^{-n-2\beta+2}\int_{\partial B_r(0)} (\nabla\vec{u} \cdot \nu)^2 - 2\beta r^{-n-2\beta+1}\int_{\partial B_r(0)} \vec{u}\nabla\vec{u} \cdot \nu\Big) \\
&\quad + 2r^{-n-2\beta+1}\int_{B_r(0)} (2(\beta-1)F(x,\vec{u}) - \beta\vec{u}\vec{f}(x,\vec{u}) - \nabla_x F(x,\vec{u}) \cdot x) \\
&:= \mathrm{I}_1 + \mathrm{I}_2.
\end{aligned} \tag{2.4}$$

We point out that I_1 in the right-hand side of the above equality is just equal to

$$\frac{\partial}{\partial r}\Big(\beta r^{-n-2\beta+1}\int_{\partial B_r(0)}\vec{u}^2\Big)+2r^{-n-2\beta+2}\int_{\partial B_r(0)}\Big(\nabla\vec{u}\cdot\nu-\beta\frac{\vec{u}}{r}\Big)^2. \tag{2.5}$$

This can be easily seen from the next computation

$$\begin{aligned}\frac{\partial}{\partial r}\Big(\beta r^{-n-2\beta+1}\int_{\partial B_r(0)}\vec{u}^2\Big)&=\frac{\partial}{\partial r}\Big(\beta r^{-2\beta}\int_{\partial B_1(0)}\vec{u}^2(ry)\Big)\\&=-2\beta^2r^{-2\beta-1}\int_{\partial B_1(0)}\vec{u}^2(ry)+2\beta r^{-2\beta}\int_{\partial B_1(0)}\vec{u}(ry)\nabla\vec{u}(ry)\cdot y\\&=-2\beta^2r^{-n-2\beta}\int_{\partial B_r(0)}\vec{u}^2+2\beta r^{-n-2\beta+1}\int_{\partial B_r(0)}\vec{u}\nabla\vec{u}\cdot\nu.\end{aligned}$$

Inserting (2.5) into (2.4), we then achieve

$$\begin{aligned}&(-n-2\beta+2)r^{-n-2\beta+1}\int_{B_r(0)}(|\nabla\vec{u}|^2-2F(x,\vec{u}))\\&\quad+r^{-n-2\beta+2}\int_{\partial B_r(0)}(|\nabla\vec{u}|^2-2F(x,\vec{u}))-\frac{\partial}{\partial r}\Big(\beta r^{-n-2\beta+1}\int_{\partial B_r(0)}\vec{u}^2\Big)\\&=2r^{-n-2\beta+2}\int_{\partial B_r(0)}\Big(\nabla\vec{u}\cdot\nu-\beta\frac{\vec{u}}{r}\Big)^2\\&\quad+2r^{-n-2\beta+1}\int_{B_r(0)}(2(\beta-1)F(x,\vec{u})-\beta\vec{u}\vec{f}(x,\vec{u})-\nabla_xF(x,\vec{u})\cdot x),\end{aligned}$$

namely,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}r}\Phi_0(r)&=2r^{-n-2\beta+2}\int_{\partial B_r(0)}\Big(\nabla\vec{u}\cdot\nu-\beta\frac{\vec{u}}{r}\Big)^2\\&\quad+2r^{-n-2\beta+1}\int_{B_r(0)}(2(\beta-1)F(x,\vec{u})-\beta\vec{u}\vec{f}(x,\vec{u})-\nabla_xF(x,\vec{u})\cdot x)\\&\geq 0\end{aligned}\tag{2.6}$$

for a.e. $r\in(0,\delta)$. Integrating (2.6) from ρ to σ, we can establish the monotonicity formula in the theorem.

We now consider the blow-up analysis for variational solutions to (2.1). Let $\vec{u}$ be a function in $B_\delta(x_0)$. For a given sequence $0<\rho_k\to 0$, we define the scaled sequence as

$$\vec{u}_k(x):=\rho_k^{-\beta}\vec{u}(x_0+\rho_kx).$$

We want to acquire some information on the limit's behavior when $\vec{u}$ is a variational solution to the nonlinear elliptic system (2.1). In fact, we have the following theorem.

Theorem 2.2 (Blow-up) *Suppose that $0<\rho_k\to 0$ as $k\to\infty$, and $\vec{u}$ is a variational solution to* (2.1) *defined in $B_\delta(x_0)$ such that the conclusions in Theorem* 2.1 *hold true. Suppose in addition that $\vec{u}$ satisfies at x_0 the growth estimate*

$$\sup_{r\in(0,\delta)}\max\Big\{\beta r^{-n-2\beta+1}\int_{\partial B_r(x_0)}\vec{u}^2,\ r^{-n-2\beta+2}\int_{B_r(x_0)}F(x,\vec{u})\Big\}<+\infty.$$

Then for any open domain $D \subset\subset \mathbb{R}^n$ and $k \geq k(D)$, the scaled sequence

$$\vec{u}_k(x) = \rho_k^{-\beta}\vec{u}(x_0 + \rho_k x)$$

is bounded in $H^1(D)$ and any weak H^1-limit with respect to a subsequence $k \to \infty$ is homogeneous of degree β.

Proof First, we can get for $0 < R < \infty$ that

$$\begin{aligned}
\Phi_{x_0}(\rho_k R) = & (\rho_k R)^{-n-2\beta+2}\int_{B_{\rho_k R}(x_0)} |\nabla \vec{u}|^2 - 2(\rho_k R)^{-n-2\beta+2}\int_{B_{\rho_k R}(x_0)} F(x,\vec{u}) \\
& - \beta(\rho_k R)^{-n-2\beta+1}\int_{\partial B_{\rho_k R}(x_0)} \vec{u}^2 \\
= & R^{-n-2\beta+2}\int_{B_R(0)} |\nabla \vec{u}_k|^2 - 2(\rho_k R)^{-n-2\beta+2}\int_{B_{\rho_k R}(x_0)} F(x,\vec{u}) \\
& - \beta(\rho_k R)^{-n-2\beta+1}\int_{\partial B_{\rho_k R}(x_0)} \vec{u}^2,
\end{aligned}$$

and we know that $\vec{u}_k$ is bounded in $H^1(D)$ for $k \geq k(D)$ by the monotonicity formula and the assumed growth estimate.

Then by the results of Theorem 2.1, we know that Φ is nondecreasing and bounded in $(0, \delta)$, which means that Φ has a right limit at 0, and for $0 < R < S$,

$$\begin{aligned}
0 \leftarrow & \Phi_{x_0}(\rho_k S) - \Phi_{x_0}(\rho_k R) \\
= & 2\int_{\rho_k R}^{\rho_k S} r^{-n-2\beta+1}\int_{B_r(x_0)} (2(\beta-1)F(x,\vec{u}) - \beta\vec{u}\vec{f}(x,\vec{u}) - \nabla_x F(x,\vec{u})\cdot(x-x_0)) \\
& + 2\int_R^S r^{-n-2\beta+2}\int_{\partial B_r(0)} \Big(\nabla\vec{u_k}\cdot\nu - \beta\frac{\vec{u_k}}{r}\Big)^2 \\
\geq & 0,
\end{aligned}$$

namely,

$$\begin{aligned}
0 \leftarrow & \Phi_{x_0}(\rho_k S) - \Phi_{x_0}(\rho_k R) \\
\geq & 2\int_R^S r^{-n-2\beta+2}\int_{\partial B_r(0)} \Big(\nabla\vec{u_k}\cdot\nu - \beta\frac{\vec{u_k}}{r}\Big)^2 \\
= & 2\int_{B_S(0)\setminus B_R(0)} |x|^{-n-2\beta}(\nabla\vec{u_k}(x)\cdot x - \beta\vec{u_k}(x))^2, \quad \text{as } k \to \infty.
\end{aligned}$$

For any subsequence $k \to \infty$ such that $\vec{u}_k \rightharpoonup \vec{u}_0$ weakly in $H^1_{\mathrm{loc}}(\mathbb{R}^n)$, using the lower semicontinuity of the L^2-norm with respect to weak convergence, we obtain

$$\nabla\vec{u}_0(x)\cdot x - \beta\vec{u}_0(x) = 0$$

a.e. in $\mathbb{R}^n$, which implies easily that $\vec{u}_0$ is homogeneous of degree β.

Remark 2.2 One new point is that the index β may be arbitrarily chosen adapted to different situations. In the previous papers [29–32], the sequence studied in Theorem 2.2 with $\beta > 0$ is called the blow-up sequence. Another extension to the former work in Theorem 2.2 is that the nonlinearity $F(x,\vec{u})$ could be nonhomogeneous, which could also lead to a blow-up limit which is homogeneous of β.

3 Monotonicity Formula for a Parabolic System

Parallel to Section 2, we consider the parabolic system

$$\frac{\partial u_i}{\partial t} - \Delta u_i = f_i(x, u_1, \cdots, u_m), \quad i = 1, \cdots, m, \quad \text{in } (t_1, t_2) \times \mathbb{R}^n, \tag{3.1}$$

where t_1, t_2 are two constants. For convenience, we need some notations (see [32]). Considering vector functions $\vec{u} \in H^1_{\rm loc}((0,T) \times \mathbb{R}^n; \mathbb{R}^m)$ and $\psi \in H^1_{\rm loc}((0,T) \times \mathbb{R}^n; \mathbb{R}^{n+1})$, we denote by $\partial_t \vec{u} = \partial_0 \vec{u}$ the time derivative, by $\nabla \vec{u} = (\partial_1 \vec{u}, \cdots, \partial_n \vec{u})$ the space gradient, by $\nabla_{t,x} \vec{u} = (\partial_0 \vec{u}, \partial_1 \vec{u}, \cdots, \partial_n \vec{u})$ the time-space gradient, by ${\rm div}_{t,x} \psi = \sum\limits_{k=0}^{n} \partial_k \psi_k$ the time-space divergence, and by

$$D\psi = \begin{pmatrix} \partial_1 \psi_0 & \cdots & \partial_n \psi_0 \\ \partial_1 \psi_1 & \cdots & \partial_n \psi_1 \\ \vdots & & \vdots \\ \partial_1 \psi_n & \cdots & \partial_n \psi_n \end{pmatrix}$$

the space Jacobian. Moreover, $\vec{u}\partial_t \vec{u} = \sum\limits_{i=1}^{m} u_i \partial_t u_i$, $(\nabla \vec{u} \cdot x + 2t\partial_t \vec{u} - \beta \vec{u})^2 = \sum\limits_{i=1}^{m} (\nabla u_i \cdot x + 2t\partial_t u_i - \beta u_i)^2$. We give the definition of a variational solution to (3.1).

Definition 3.1 *We call $\vec{u}$ a variational solution to* (3.1) *if $\vec{u}$ satisfies:*

(1) $\vec{u} \in H^1_{\rm loc}((t_1,t_2) \times \mathbb{R}^n)$, *and* $u_i f_i(x, \vec{u}), F(x, \vec{u}), \nabla_x F(x, \vec{u}) \in L^1_{\rm loc}((t_1, t_2) \times \mathbb{R}^n)$ *for* $i = 1, \cdots, m$;

(2) $\vec{u}$ *satisfied* (3.1) *in the sense of distributions;*

(3) *the first variation with respect to the time-space domain variations of the following functional*

$$\mathbb{G}(\vec{u}, \vec{v}) = \int_{t_1+\delta}^{t_2-\delta} \int_{\mathbb{R}^n} (|\nabla \vec{v}|^2 - 2F(\vec{v})) + 2\int_{t_1+\delta}^{t_2-\delta} \int_{\mathbb{R}^n} \vec{v}\partial_t \vec{u}$$

vanishes at $\vec{v} = \vec{u}$, i.e.,

$$\begin{aligned} 0 &= -\frac{\rm d}{{\rm d}\varepsilon} \mathbb{G}(\vec{u}, \vec{u}((t,x) + \varepsilon\psi(t,x)))\Big|_{\varepsilon=0} \\ &= \int_{t_1+\delta}^{t_2-\delta} \int_{\mathbb{R}^n} ((|\nabla \vec{u}|^2 - 2F(x,\vec{u}))\, {\rm div}_{t,x} \psi - 2\nabla_{t,x} \vec{u} D\psi \nabla \vec{u} - 2\partial_t \vec{u} \nabla_{t,x} \vec{u} \cdot \psi \\ &\quad - 2\nabla_x F(x, \vec{u}) \cdot (\psi_1, \cdots, \psi_n)) - \Big[\int_{\mathbb{R}^n} (|\nabla \vec{u}|^2 - 2F(\vec{u}))\psi_0\Big]\Big|_{t=t_1+\delta}^{t=t_2-\delta} \end{aligned}$$

for almost every small and positive δ and any $\psi = (\psi_0, \psi_1, \cdots, \psi_n) \in C^1((0,T) \times \mathbb{R}^n; \mathbb{R}^{n+1})$ such that

$${\rm supp}\, \psi(t) \subset\subset \mathbb{R}^n, \quad \forall\, t \in (t_1, t_2).$$

We now state the monotonicity formulae for variational solutions to (3.1). Recall that $T_r^- = (T - 4r^2, T - r^2) \times \mathbb{R}^n$ and $T_r^+ = (T + r^2, T + 4r^2) \times \mathbb{R}^n$.

Theorem 3.1 (Monotonicity Formulae) *Let $x_0 \in \mathbb{R}^n$ and $\vec{u}$ be a variational solution to* (3.1) *in $((t_1, T) \cup (T, t_2)) \times \mathbb{R}^n$, where $t_1 \le T \le t_2$. If there exists a real number $\beta \in \mathbb{R}$ such that $\forall\, r \in (0, \delta)$, there hold*

$$\int_{T_r^-} (2(\beta - 1)F(x, \vec{u}) - \beta \vec{u}\vec{f}(x, \vec{u}) - \nabla_x F(x, \vec{u})(x - x_0)) G_{(T,x_0)} \ge 0$$

and

$$\int_{T_r^+} (2(\beta-1)F(x,\vec{u}) - \beta\vec{u}\vec{f}(x,\vec{u}) - \nabla_x F(x,\vec{u})(x-x_0))G_{(T,x_0)} \geq 0,$$

respectively, then the functions

$$\Psi^-(r) := r^{-2\beta}\int_{T_r^-}(|\nabla\vec{u}|^2 - 2F(x,\vec{u}))G_{(T,x_0)} - \frac{\beta}{2}r^{-2\beta}\int_{T_r^-}\frac{1}{T-t}\vec{u}^2 G_{(T,x_0)}$$

and

$$\Psi^+(r) := r^{-2\beta}\int_{T_r^+}(|\nabla\vec{u}|^2 - 2F(x,\vec{u}))G_{(T,x_0)} - \frac{\beta}{2}r^{-2\beta}\int_{T_r^+}\frac{1}{T-t}\vec{u}^2 G_{(T,x_0)}$$

are well-defined in the interval $(0, \frac{\sqrt{T-t_1}}{2})$ *and* $(0, \frac{\sqrt{t_2-T}}{2})$, *and for any* $0 < \rho < \sigma < \frac{\sqrt{T-t_1}}{2}$ *and* $0 < \rho < \sigma < \frac{\sqrt{t_2-T}}{2}$ *they satisfy the monotonicity formulae*

$$\begin{aligned}
&\Psi^-(\sigma) - \Psi^-(\rho) \\
&= 2\int_\rho^\sigma r^{-2\beta-1}\int_{T_r^-}(2(\beta-1)F(x,\vec{u}) - \beta\vec{u}\vec{f}(x,\vec{u}) - \nabla_x F(x,\vec{u})\cdot(x-x_0))G_{(T,x_0)} \\
&\quad + \int_\rho^\sigma r^{-2\beta-1}\int_{T_r^-}\frac{1}{T-t}(\nabla\vec{u}\cdot(x-x_0) - 2(T-t)\partial_t\vec{u} - \beta\vec{u})^2 G_{(T,x_0)} \\
&\geq 0
\end{aligned}$$

and

$$\begin{aligned}
&\Psi^+(\sigma) - \Psi^+(\rho) \\
&= 2\int_\rho^\sigma r^{-2\beta-1}\int_{T_r^+}(2(\beta-1)F(x,\vec{u}) - \beta\vec{u}\vec{f}(x,\vec{u}) - \nabla_x F(x,\vec{u})\cdot(x-x_0))G_{(T,x_0)} \\
&\quad + \int_\rho^\sigma r^{-2\beta-1}\int_{T_r^+}\frac{1}{T-t}(\nabla\vec{u}\cdot(x-x_0) - 2(T-t)\partial_t\vec{u} - \beta\vec{u})^2 G_{(T,x_0)} \\
&\geq 0,
\end{aligned}$$

respectively.

Proof We only give a proof for the monotonicity of Ψ^- because we can replace in what follows the interval $(-4r^2, -r^2)$ by $(r^2, 4r^2)$ to obtain a proof for Ψ^+. Without loss of generality, we can assume that $x_0 = 0$ and $T = 0$. We omit the index $(0,0)$ in $G_{(0,0)}$ and simply denote it by G. We also denote $T_r^-(0)$ by T_r^-. We will use frequently the facts that $\nabla G = \frac{xG}{2t}$ and $\partial_t G + \Delta G = 0$ in $\{t<0\}\cup\{t>0\}$ in the following calculations without explicitly mentioned again.

We take after approximation $\psi(t,x) = (2t,x)G(t,x)\eta_\epsilon(x)$ as test function in Definition 3.1(3) with $\eta_\epsilon \in H_0^{1,\infty}(\mathbb{R}^n)$ to be chosen later. We first have to calculate $\mathrm{div}_{t,x}\psi$, $\nabla_{t,x}\vec{u}D\psi\nabla\vec{u}$, and $\partial_t\vec{u}\nabla_{t,x}\vec{u}\cdot\psi$. In fact,

$$\begin{aligned}
\mathrm{div}_{t,x}\psi &= (2G + 2t\partial_t G + \mathrm{div}(xG))\eta_\varepsilon + \nabla\eta_\varepsilon\cdot xG \\
&= 2G\eta_\varepsilon + \nabla\eta_\varepsilon\cdot xG, \qquad (3.2)
\end{aligned}$$

$$\begin{aligned}\nabla_{t,x}\vec{u}D\psi\nabla\vec{u} &= \sum_{j=1}^{n}\sum_{k=1}^{n}\partial_j\vec{u}\partial_j\psi_k\partial_k\vec{u} + \sum_{j=1}^{n}\partial_j\vec{u}\partial_j\psi_0\partial_t\vec{u} \\ &= \sum_{j=1}^{n}\sum_{k=1}^{n}\partial_j\vec{u}(\delta_{jk}G\eta_\varepsilon + x_k\partial_j G\eta_\varepsilon + x_k G\partial_j\eta_\varepsilon)\partial_k\vec{u} \\ &\quad + \sum_{j=1}^{n}\partial_j\vec{u}(2t\partial_j G\eta_\varepsilon + 2tG\partial_j\eta_\varepsilon)\partial_t\vec{u} \\ &= G\eta_\epsilon|\nabla\vec{u}|^2 + \frac{G\eta_\epsilon}{2t}(\nabla\vec{u}\cdot x)^2 + G(\nabla\vec{u}\cdot x)(\nabla\vec{u}\cdot\nabla\eta_\epsilon) \\ &\quad + (G\eta_\epsilon\nabla\vec{u}\cdot x + 2tG\nabla\vec{u}\cdot\nabla\eta_\epsilon)\partial_t\vec{u} \qquad (3.3)\end{aligned}$$

and

$$\partial_t\vec{u}\nabla_{t,x}\vec{u}\cdot\psi = \partial_t\vec{u}\sum_{k=0}^{n}\partial_k\vec{u}\psi_k = 2t(\partial_t\vec{u})^2 G\eta_\varepsilon + G\eta_\varepsilon\partial_t\vec{u}\nabla\vec{u}\cdot x. \qquad (3.4)$$

Taking $t_1 = -4r^2$, $t_2 = -r^2$, and inserting (3.2), (3.3) and (3.4) into Definition 3.1(3), we obtain for a.e. $r \in (0, \frac{\sqrt{T-t_1}}{2})$ that

$$\begin{aligned}0 &= \int_{-4r^2}^{-r^2}\int_{\mathbb{R}^n}((|\nabla\vec{u}|^2 - 2F(x,\vec{u}))\,\mathrm{div}_{t,x}\psi - 2\nabla_{t,x}\vec{u}D\psi\nabla\vec{u} - 2\partial_t\vec{u}\nabla_{t,x}\vec{u}\cdot\psi \\ &\quad - 2\nabla_x F(x,\vec{u})\cdot(\psi_1,\cdots,\psi_n)) - \Big[\int_{\mathbb{R}^n}(|\nabla\vec{u}|^2 - 2F(x,\vec{u}))\psi_0\Big]\Big|_{t=-4r^2}^{t=-r^2} \\ &= \int_{T_r^-}\Big((|\nabla\vec{u}|^2 - 2F(x,\vec{u}))(2G\eta_\epsilon + \nabla\eta_\epsilon\cdot xG) - 2G\eta_\epsilon|\nabla\vec{u}|^2 - \frac{G\eta_\epsilon}{t}(\nabla\vec{u}\cdot x)^2 \\ &\quad - 2G(\nabla\vec{u}\cdot x)(\nabla\vec{u}\cdot\nabla\eta_\epsilon) - 2G\eta_\epsilon\partial_t\vec{u}\nabla\vec{u}\cdot x - 4tG\partial_t\vec{u}\nabla\vec{u}\cdot\nabla\eta_\epsilon - 4t(\partial_t\vec{u})^2 G\eta_\epsilon \\ &\quad - 2G\eta_\epsilon\partial_t\vec{u}\nabla\vec{u}\cdot x - 2\nabla_x F(x,\vec{u})\cdot(\psi_1,\cdots,\psi_n)\Big) - \Big[\int_{\mathbb{R}^n}(|\nabla\vec{u}|^2 - 2F(\vec{u}))\psi_0\Big]\Big|_{t=-4r^2}^{t=-r^2}.\end{aligned}$$

We combine all the terms containing $G\eta_\epsilon$ together and all the terms containing $\nabla\eta_\epsilon$ together to rewrite the above identity as

$$\begin{aligned}0 &= \int_{T_r^-} G\eta_\epsilon\Big(-4F(x,\vec{u}) - \frac{1}{t}(\nabla\vec{u}\cdot x + 2t\partial_t\vec{u})^2 - 2\nabla_x F(x,\vec{u})\cdot x\Big) \\ &\quad - \Big[\int_{\mathbb{R}^n}(|\nabla\vec{u}|^2 - 2F(x,\vec{u}))\psi_0\Big]\Big|_{t=-4r^2}^{t=-r^2} \\ &\quad - \int_{T_r^-}((|\nabla\vec{u}|^2 - 2F(x,\vec{u}))\nabla\eta_\epsilon\cdot xG - 2G(\nabla\vec{u}\cdot x)(\nabla\vec{u}\cdot\nabla\eta_\varepsilon) - 4tG\partial_t\vec{u}\nabla\vec{u}\cdot\nabla\eta_\varepsilon) \\ &:= \mathrm{I}_1 - \mathrm{I}_2 - \mathrm{I}_3. \qquad (3.5)\end{aligned}$$

As in the proof of (2.3), we have

$$\begin{aligned}\int_{T_r^-}|\nabla\vec{u}|^2 G\eta_\epsilon &= -\int_{T_r^-}(\vec{u}\eta_\epsilon\nabla\vec{u}\cdot\nabla G + G\eta_\epsilon\vec{u}(\partial_t\vec{u} - \vec{f}(x,\vec{u})) + \vec{u}G\nabla\vec{u}\cdot\nabla\eta_\varepsilon) \\ &= -\int_{T_r^-}\Big(G\eta_\epsilon\Big(\frac{1}{2t}\vec{u}\nabla\vec{u}\cdot x + \vec{u}(\partial_t\vec{u} - \vec{f}(x,\vec{u}))\Big) + \vec{u}G\nabla\vec{u}\cdot\nabla\eta_\varepsilon\Big). \qquad (3.6)\end{aligned}$$

Inserting (3.6) into (3.5) we get

$$\begin{aligned}
\mathrm{I}_1 &:= \int_{T_r^-} G\eta_\epsilon\Big(-4F(x,\vec{u}) - 2\nabla_x F(x,\vec{u})\cdot x - \frac{1}{t}(\nabla\vec{u}\cdot x + 2t\partial_t\vec{u})^2\Big)\\
&= \int_{T_r^-} G\eta_\epsilon\Big(2\beta(|\nabla\vec{u}|^2 - 2F(x,\vec{u})) + 4(\beta-1)F(x,\vec{u}) - 2\nabla_x F(x,\vec{u})\cdot x\\
&\quad - \frac{1}{t}(\nabla\vec{u}\cdot x + 2t\partial_t\vec{u})^2 - 2\beta|\nabla\vec{u}|^2\Big)\\
&= \int_{T_r^-} \Big(G\eta_\epsilon\Big(2\beta(|\nabla\vec{u}|^2 - 2F(x,\vec{u})) + 4(\beta-1)F(x,\vec{u}) - 2\nabla_x F(x,\vec{u})\cdot x\\
&\quad - \frac{1}{t}(\nabla\vec{u}\cdot x + 2t\partial_t\vec{u})^2\Big)\\
&\quad + G\eta_\epsilon\Big(\frac{\beta\vec{u}}{t}\nabla\vec{u}\cdot x + 2\beta\vec{u}(\partial_t\vec{u} - \vec{f}(x,\vec{u}))\Big) + 2\beta\vec{u}G\nabla\vec{u}\cdot\nabla\eta_\epsilon\Big)\\
&= \int_{T_r^-} \Big(G\eta_\epsilon\Big(2\beta(|\nabla\vec{u}|^2 - 2F(x,\vec{u})) + 4(\beta-1)F(x,\vec{u}) - 2\nabla_x F(x,\vec{u})\cdot x\\
&\quad - \frac{1}{t}(\nabla\vec{u}\cdot x + 2t\partial_t\vec{u} - \beta\vec{u})^2\Big)\\
&\quad - G\eta_\epsilon\Big(\frac{\beta\vec{u}}{t}(\nabla\vec{u}\cdot x + 2t\partial_t\vec{u}) - \frac{\beta^2\vec{u}^2}{t} + 2\beta\vec{u}\vec{f}(x,\vec{u})\Big) + 2\beta\vec{u}G\nabla\vec{u}\cdot\nabla\eta_\epsilon\Big).
\end{aligned}$$

From the above identity we can write $\mathrm{I}_1 - \mathrm{I}_2$ as

$$\begin{aligned}
\mathrm{I}_1 - \mathrm{I}_2 &= \Big(2\beta\int_{T_r^-} G\eta_\epsilon(|\nabla\vec{u}|^2 - 2F(x,\vec{u})) - \Big[\int_{\mathbb{R}^n}(|\nabla\vec{u}|^2 - 2F(x,\vec{u}))\psi_0\Big]\Big|_{t=-4r^2}^{t=-r^2}\Big)\\
&\quad - \int_{T_r^-} G\eta_\epsilon\Big(\frac{\beta\vec{u}}{t}(\nabla\vec{u}\cdot x + 2t\partial_t\vec{u}) - \frac{\beta^2\vec{u}^2}{t}\Big)\\
&\quad + \int_{T_r^-} G\eta_\epsilon\Big(2(2(\beta-1)F(x,\vec{u}) - \beta\vec{u}\vec{f}(x,\vec{u}) - \nabla_x F(x,\vec{u})\cdot x)\\
&\quad - \frac{1}{t}(\nabla\vec{u}\cdot x + 2t\partial_t\vec{u} - \beta\vec{u})^2\Big)\\
&\quad + 2\beta\int_{T_r^-} \vec{u}G\nabla\eta_\epsilon\cdot\nabla\vec{u}\\
&:= \mathrm{J}_1 - \mathrm{J}_2 + \mathrm{J}_3 + \mathrm{J}_4.
\end{aligned} \tag{3.7}$$

Then it follows from (3.5) and (3.7) that

$$0 = \mathrm{J}_1 - \mathrm{J}_2 + \mathrm{J}_3 + \mathrm{J}_4 - \mathrm{I}_3. \tag{3.8}$$

Notice that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}r}\Big(r^{-2\beta}\int_{T_r^-} G\eta_\epsilon(|\nabla\vec{u}|^2 - 2F(x,\vec{u}))\Big)\\
&= -2\beta r^{-2\beta-1}\int_{T_r^-} G\eta_\epsilon(|\nabla\vec{u}|^2 - 2F(x,\vec{u})) + r^{-2\beta}\frac{\mathrm{d}}{\mathrm{d}r}\Big(\int_{T_r^-} G\eta_\epsilon(|\nabla\vec{u}|^2 - 2F(x,\vec{u}))\Big)\\
&= -2\beta r^{-2\beta-1}\int_{T_r^-} G\eta_\epsilon(|\nabla\vec{u}|^2 - 2F(x,\vec{u})) + 2r^{-2\beta-1}\Big[t\int_{\mathbb{R}^n} G\eta_\epsilon(|\nabla\vec{u}|^2 - 2F(x,\vec{u}))\Big]\Big|_{t=-4r^2}^{t=-r^2}
\end{aligned}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}r}\Big(\frac{\beta}{2}r^{-2\beta}\int_{T_r^-}\frac{G(t,x)}{t}\eta_\epsilon(x)\vec{u}^2(t,x)\Big)=\frac{\mathrm{d}}{\mathrm{d}r}\Big(\frac{\beta}{2}\int_{T_1^-}\frac{G(t,x)}{t}\eta_\epsilon(rx)\frac{\vec{u}^2(r^2t,rx)}{r^{2\beta}}\Big)$$
$$=\frac{\beta}{2}\int_{T_1^-}\frac{G(t,x)}{t}\eta_\epsilon(rx)\partial_r\Big(\frac{\vec{u}^2(r^2t,rx)}{r^{2\beta}}\Big)+\frac{\beta}{2}\int_{T_1^-}\frac{G(t,x)}{t}\frac{\vec{u}^2(r^2t,rx)}{r^{2\beta}}\partial_r(\eta_\epsilon(rx))$$
$$=r^{-2\beta-1}\int_{T_r^-}G\eta_\epsilon\Big(\frac{\beta}{t}\vec{u}\nabla\vec{u}\cdot x+2\beta\vec{u}\partial_t\vec{u}-\frac{\beta^2}{t}\vec{u}^2\Big)+\frac{\beta}{2}r^{-2\beta-1}\int_{T_r^-}\frac{G(t,x)}{t}\vec{u}^2(t,x)\nabla\eta_\epsilon(x)\cdot x,$$

that is,

$$r^{-2\beta-1}\,\mathrm{J}_1=-\frac{\mathrm{d}}{\mathrm{d}r}\Big(r^{-2\beta}\int_{T_r^-}G\eta_\epsilon(|\nabla\vec{u}|^2-2F(x,\vec{u}))\Big),$$
$$r^{-2\beta-1}\,\mathrm{J}_2=\frac{\mathrm{d}}{\mathrm{d}r}\Big(\frac{\beta}{2}r^{-2\beta}\int_{T_r^-}\frac{1}{t}G\eta_\epsilon\vec{u}^2\Big)-\frac{\beta}{2}r^{-2\beta-1}\int_{T_r^-}\frac{1}{t}\vec{u}^2G\nabla\eta_\epsilon\cdot x.$$

Using these two facts, we can rewrite (3.8) as

$$\begin{aligned}0=&-\frac{\mathrm{d}}{\mathrm{d}r}\Big(r^{-2\beta}\int_{T_r^-}G\eta_\epsilon(|\nabla\vec{u}|^2-2F(x,\vec{u}))\Big)-\frac{\mathrm{d}}{\mathrm{d}r}\Big(\frac{\beta}{2}r^{-2\beta}\int_{T_r^-}\frac{1}{t}G\eta_\epsilon\vec{u}^2\Big)+r^{-2\beta-1}\,\mathrm{J}_3\\&+\frac{\beta}{2}r^{-2\beta-1}\int_{T_r^-}\frac{1}{t}\vec{u}^2G\nabla\eta_\epsilon\cdot x+r^{-2\beta-1}\,\mathrm{J}_4-r^{-2\beta-1}\,\mathrm{I}_3.\end{aligned}\tag{3.9}$$

Choosing $\eta_\epsilon(x)=\min(1,\max(0,2-\epsilon|x|))$ for small $\epsilon>0$, we have

$$\frac{\beta}{2}r^{-2\beta-1}\int_{T_r^-}\frac{1}{t}\vec{u}^2G\nabla\eta_\epsilon\cdot x+r^{-2\beta-1}\,\mathrm{J}_4-r^{-2\beta-1}\,\mathrm{I}_3=O(\epsilon).$$

For any $0<\rho<\sigma<\frac{\sqrt{T-t_1}}{2}$, integrating (3.9) from ρ to σ and then letting $\epsilon\to0$, we conclude that

$$\begin{aligned}\Psi^-(\sigma)-\Psi^-(\rho)=&\,2\int_\rho^\sigma r^{-2\beta-1}\int_{T_r^-}(2(\beta-1)F(x,\vec{u})-\beta\vec{u}\vec{f}(x,\vec{u})-\nabla_xF(x,\vec{u})\cdot x)G_{(T,x_0)}\\&+\int_\rho^\sigma r^{-2\beta-1}\int_{T_r^-}\frac{1}{T-t}(\nabla\vec{u}\cdot(x-x_0)+2(T-t)\partial_t\vec{u}-\beta\vec{u})^2G_{(T,x_0)}\\\geq&\,0.\end{aligned}\tag{3.10}$$

The proof of the theorem is completed.

We now study the blow-up of scaled solutions. For a given point (T,x_0) and a given sequence $0>\rho_k\to0$, we define the scaled sequence as

$$\vec{u}_k(t,x):=\rho_k^{-\beta}\vec{u}(T+\rho_k^2t,x_0+\rho_kx),$$

and want to obtain more information on the solution's behavior. In fact, we have the following result.

Theorem 3.2 (Blow-up) *Let* $\vec{u}\in H^1_{\mathrm{loc}}(((t_1,T)\cup(T,t_2))\times\mathbb{R}^n)$ *be a variational solution to* (3.1) *such that the conclusions in Theorem* 3.1 *hold true, where* $t_1\leq T\leq t_2$. *Suppose that* $x_0\in\mathbb{R}^n$ *and the growth estimates*

$$\sup_{r\in(0,\frac{\sqrt{T-t_1}}{4})}\max\Big\{\beta r^{-2\beta}\int_{T_r^-}\frac{1}{T-t}\vec{u}^2G_{(T,x_0)},r^{-2\beta}\int_{T_r^-}F(x,\vec{u})G_{(T,x_0)}\Big\}<+\infty$$

and

$$\sup_{r\in(0,\frac{\sqrt{t_2-T}}{4})} \max\left\{\beta r^{-2\beta}\int_{T_r^+}\frac{1}{T-t}\vec{u}^2 G_{(T,x_0)}, r^{-2\beta}\int_{T_r^+}F(x,\vec{u})G_{(T,x_0)}\right\} < +\infty$$

are satisfied. Then for any open set $D \subset\subset ((-\sqrt{T-t_1}\,,0)\cup(0,\sqrt{t_2-T}\,))\times\mathbb{R}^n$ *and* $k \geq k(D)$ *the sequence*

$$\vec{u}_k(t,x) = \rho_k^{-\beta}\vec{u}(T+\rho_k^2 t, x_0+\rho_k x)$$

is bounded in $H^1(D)$ *and any weak* H^1*-limit* $\vec{u}_0$ *with respect to a subsequence is a function homogeneous of degree* β *on paths* $\theta \to (\theta^2 t, \theta x)$ *for* $\theta > 0$ *and* $(t,x) \in ((-\sqrt{T-t_1}\,,0)\cup(0,\sqrt{t_2-T}\,))\times\mathbb{R}^n$*, i.e.,*

$$\vec{u}_0(\lambda^2 t, \lambda x) = \lambda^\beta \vec{u}_0(t,x)$$

for any $\lambda > 0$ *and for any* $(t,x) \in ((-\sqrt{T-t_1}\,,0)\cup(0,\sqrt{t_2-T}\,))\times\mathbb{R}^n$*.*

Proof We give the proof only for the case $t_2 = T$ to avoid clumsy notation. As in the proof of Theorem 3.1, we assume that $x_0 = 0$ and $T = 0$. Using the notations introduced there, we calculate for $0 < R < \infty$ that

$$\begin{aligned}\Psi^-(\rho_k R) &= (\rho_k R)^{-2\beta}\int_{T^-_{\rho_k R}}|\nabla\vec{u}|^2 G - 2(\rho_k R)^{-2\beta}\int_{T^-_{\rho_k R}}F(x,\vec{u})G - \frac{\beta}{2}(\rho_k R)^{-2\beta}\int_{T^-_{\rho_k R}}\frac{1}{(-t)}\vec{u}^2 G\\ &= R^{-2\beta}\int_{T^-_R}|\nabla\vec{u}_k|^2 G - 2(\rho_k R)^{-2\beta}\int_{T^-_{\rho_k R}}F(x,\vec{u})G - \frac{\beta}{2}R^{-2\beta}\int_{T^-_R}\frac{1}{(-t)}\vec{u}_k^2 G.\end{aligned}$$

We see that the sequence $\vec{u}_k$ and $\nabla\vec{u}_k$ are bounded in $L^2(D)$ for $k \geq k(D)$ by the assumed growth estimate and the monotonicity formula Theorem 3.1. Thus for $k \geq k(D)$ the sequence $\vec{u}_k$ is bounded in $H^1(D)$.

In view of Theorem 3.1, we know that Ψ^- is nondecreasing and bounded in $(0,\delta)$, which means that Ψ^- has a real right limit at 0 and for $0 < R < S < \infty$,

$$\begin{aligned}0 &\leftarrow \Psi^-(\rho_k S) - \Psi^-(\rho_k R)\\ &= 2\int_{\rho_k R}^{\rho_k S} r^{-2\beta-1}\int_{T_r^-}(2(\beta-1)F(x,\vec{u}) - \beta\vec{u}\vec{f}(x,\vec{u}))G\\ &\quad + \int_R^S r^{-2\beta-1}\int_{T_r^-}\frac{1}{(-t)}(\nabla\vec{u}_k\cdot x + 2t\partial_t\vec{u}_k - \beta\vec{u}_k)^2 G.\end{aligned}$$

Consequently,

$$\int_R^S r^{-2\beta-1}\int_{T_r^-}\frac{1}{(-t)}(\nabla\vec{u}_k\cdot x + 2t\partial_t\vec{u}_k - \beta\vec{u}_k)^2 G \to 0, \quad \text{as } k\to\infty.$$

We take a subsequence $k \to \infty$ such that $\vec{u}_k \rightharpoonup \vec{u}_0$ weakly in $H^1_{\mathrm{loc}}((-\infty,0)\times\mathbb{R}^n)$. By the lower semi-continuity of the L^2-norm with respect to weak convergence, we can obtain

$$\nabla\vec{u}_0(t,x)\cdot x + 2t\partial_t\vec{u}_0(t,x) - \beta\vec{u}_0(t,x) = 0$$

a.e. in $(-\infty,0)\times\mathbb{R}^n$, which implies easily that $\vec{u}_0$ is homogeneous of degree β on paths $\theta \to (\theta^2 t, \theta x)$ for $\theta > 0$ and $(t,x) \in (-\infty,0)\times\mathbb{R}^n$.

4 Some Applications to the Ginzburg-Landau Model and Free Boundary Problems

This section is devoted to various applications of our previous results to different physical interesting models. Since our conditions on the nonlinearities are very general, we can invoke these theorems in different situations to achieve some partial results.

4.1 On the Ginzburg-Landau model

We consider the famous Ginzburg-Landau model

$$\Delta \vec{u} + \frac{1}{\epsilon^2}\vec{u}(1-\vec{u}^2) = 0 \quad \text{in } \Omega.$$

Set $F(\vec{u}) = -\frac{1}{4\epsilon^2}(1-\vec{u}^2)^2$. We derive

$$2(\beta-1)F(\vec{u}) - \beta\vec{u}f(\vec{u}) = \frac{1}{2\epsilon^2}(\vec{u}^2-1)((\beta+1)\vec{u}^2+(\beta-1)) \geq 0,$$

provided

$$\begin{cases} \vec{u}^2 \leq \dfrac{1-\beta}{1+\beta} \text{ or } \vec{u}^2 \geq 1, & \text{if } \beta > 0, \\ \vec{u}^2 \leq 1 \text{ or } \vec{u}^2 \geq \dfrac{1-\beta}{1+\beta}, & \text{if } -1 < \beta \leq 0, \\ \vec{u}^2 \leq 1, & \text{if } \beta \leq -1. \end{cases} \tag{4.1}$$

It then follows from Theorem 2.1 directly that the function

$$\Phi_{x_0}(r) = r^{-n-2\beta+2}\int_{B_r(x_0)} \Big(|\nabla\vec{u}|^2 + \frac{1}{2\epsilon^2}(1-\vec{u}^2)^2\Big) - \beta r^{-n-2\beta+1}\int_{\partial B_r(x_0)} \vec{u}^2$$

is nondecreasing in r under the assumption (4.1). For (4.1) to be satisfied, $\vec{u}^2$ has to be bounded away from 1 in the case $\beta > 0$, which seems somewhat unrealistic. Nevertheless, a noteworthy fact occurs at $\beta = 0$, in which case no matter what the value of $\vec{u}^2$ ranges, we always have $2(\beta-1)F(\vec{u}) - \beta\vec{u}f(\vec{u}) \geq 0$. This observation leads us to the next proposition.

Proposition 4.1 *Let $\vec{u} \in H^1_{\mathrm{loc}}(\Omega) \cap L^4_{\mathrm{loc}}(\Omega)$ be a variational solution to the Ginzburg-Landau model*

$$\Delta \vec{u} + \frac{1}{\epsilon^2}\vec{u}(1-\vec{u}^2) = 0 \quad \textit{in } \Omega.$$

Then for any ball $B_\delta(x_0) \subset\subset \Omega$, the function

$$\Phi_{x_0}(r) = r^{-n+2}\int_{B_r(x_0)} \Big(|\nabla\vec{u}|^2 + \frac{1}{2\epsilon^2}(1-\vec{u}^2)^2\Big)$$

is nondecreasing in $r \in (0,\delta)$ and satisfies the monotonicity formula

$$\frac{\mathrm{d}}{\mathrm{d}r}\Phi_{x_0}(r) = r^{-n+1}\int_{B_r(x_0)} \frac{1}{\epsilon^2}(\vec{u}^2-1)^2 + 2r^{-n+2}\int_{\partial B_r(x_0)} (\nabla\vec{u}\cdot\nu)^2 \geq 0.$$

Notice that we have chosen $\beta = 0$, resulting in no boundary term in the monotonicity formula. To apply Theorem 2.2, we only have to check

$$\sup_{r\in(0,\delta)} r^{-n+2}\int_{B_r(x_0)} F(\vec{u}) < +\infty$$

with $F(\vec{u}) = -\frac{1}{4\epsilon^2}(1-\vec{u}^2)^2$, which holds true trivially. We then obtain the following proposition.

Proposition 4.2 *Let $\vec{u} \in H^1_{\rm loc}(\Omega) \cap L^4_{\rm loc}(\Omega)$ be a variational solution to the Ginzburg-Landau model*

$$\Delta\vec{u} + \frac{1}{\epsilon^2}\vec{u}(1-\vec{u}^2) = 0 \quad \textit{in } \Omega.$$

For any open domain $D \subset\subset \mathbb{R}^n$ and $k \geq k(D)$, the scaled sequence $\vec{u}_k(x) = \vec{u}(x_0 + \rho_k x)$ is bounded in $H^1(D)$ and any weak H^1-limit with respect to a subsequence $k \to \infty$ is homogeneous of degree 0.

For the Ginzburg-Landau equation of parabolic type

$$\frac{\partial\vec{u}}{\partial t} - \Delta\vec{u} - \frac{1}{\epsilon^2}\vec{u}(1-\vec{u}^2) = 0 \quad \text{in } (t_1, t_2) \times \mathbb{R}^n,$$

similar results can be obtained by using Theorems 3.1 and 3.2.

Proposition 4.3 *Let $\vec{u} \in H^1_{\rm loc}(((t_1,T) \cup (T,t_2)) \times \mathbb{R}^n) \cap L^4_{\rm loc}(((t_1,T) \cup (T,t_2)) \times \mathbb{R}^n)$ be a variational solution to*

$$\frac{\partial\vec{u}}{\partial t} - \Delta\vec{u} - \frac{1}{\epsilon^2}\vec{u}(1-\vec{u}^2) = 0 \quad \textit{in } (t_1, t_2) \times \mathbb{R}^n,$$

where $t_1 \leq T \leq t_2$. Then the functions

$$\Psi^-(r) = \int_{T_r^-} \Big(|\nabla\vec{u}|^2 + \frac{1}{2\epsilon^2}(1-\vec{u}^2)^2\Big) G_{(T,x_0)}$$

and

$$\Psi^+(r) = \int_{T_r^+} \Big(|\nabla\vec{u}|^2 + \frac{1}{2\epsilon^2}(1-\vec{u}^2)^2\Big) G_{(T,x_0)}$$

are well-defined in the interval $(0, \frac{\sqrt{T-t_1}}{2})$ and $(0, \frac{\sqrt{t_2-T}}{2})$, and for any $0 < \rho < \sigma < \frac{\sqrt{T-t_1}}{2}$ and $0 < \rho < \sigma < \frac{\sqrt{t_2-T}}{2}$ they satisfy the monotonicity formulae

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}r}\Psi^-(r) &= r^{-1}\int_{T_r^-} \frac{1}{\epsilon^2}(\vec{u}^2-1)^2 G_{(T,x_0)} + r^{-1}\int_{T_r^-} \frac{1}{T-t}(\nabla\vec{u}\cdot(x-x_0) - 2(T-t)\partial_t\vec{u})^2 G_{(T,x_0)} \\ &\geq 0\end{aligned}$$

and

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}r}\Psi^+(r) &= r^{-1}\int_{T_r^+} \frac{1}{\epsilon^2}(\vec{u}^2-1)^2 G_{(T,x_0)} + r^{-1}\int_{T_r^+} \frac{1}{T-t}(\nabla\vec{u}\cdot(x-x_0) - 2(T-t)\partial_t\vec{u})^2 G_{(T,x_0)} \\ &\geq 0,\end{aligned}$$

respectively. Moreover, for any open set $D \subset\subset ((-\sqrt{T-t_1}\,, 0) \cup (0, \sqrt{t_2-T}\,)) \times \mathbb{R}^n$ and $k \geq k(D)$ the sequence $\vec{u}_k(t,x) = \vec{u}(T + \rho_k^2 t, x_0 + \rho_k x)$ is bounded in $H^1(D)$ and any weak H^1-limit $\vec{u}_0$ with respect to a subsequence is a function homogeneous of degree 0 on paths $\theta \to (\theta^2 t, \theta x)$ for $\theta > 0$ and $(t,x) \in ((-\sqrt{T-t_1}\, 0) \cup (0, \sqrt{t_2-T}\,)) \times \mathbb{R}^n$.

4.2 On the free-boundary problem

We now give an application of Theorems 2.1 and 2.2 to the following free-boundary problem with nonhomogeneous nonlinearity:

$$2\Delta u = \lambda_+(pu^{p-1} + f(x))\chi_{\{u>0\}} - \lambda_-(p(-u)^{p-1} + g(x))\chi_{\{u<0\}} \tag{4.2}$$

in a bounded domain $\Omega \subset \mathbb{R}^n$, where λ_+ and λ_- are real numbers, $p < 2$ and $p \neq 0$. We point out that when $f(x) \equiv g(x) \equiv 0$, (4.2) reduces to the equation

$$2\Delta u = \lambda_+ p u^{p-1} \chi_{\{u>0\}} - \lambda_- p(-u)^{p-1} \chi_{\{u<0\}} \tag{4.3}$$

considered by Weiss and others (see [30]). For the single equation (4.2), the associated functional $F(x,u)$ is given by

$$F(x,u) = -\frac{\lambda_+}{2}(u^p + f(x)u)\chi_{\{u>0\}} - \frac{\lambda_-}{2}((-u)^p + g(x)(-u))\chi_{\{u<0\}}.$$

Suppose that $B_\delta(x_0) \subset\subset \Omega$ and u is a variational solution to (4.2). We define for $r \in (0,\delta)$,

$$\begin{aligned}\Phi_{x_0}(r) = {}& r^{-n-2\beta+2} \int_{B_r(x_0)} (|\nabla u|^2 + \lambda_+ u^p \chi_{\{u>0\}} + \lambda_-(-u)^p \chi_{\{u<0\}}) \\ &+ r^{-n-2\beta+2} \int_{B_r(x_0)} (\lambda_+ f(x) u \chi_{\{u>0\}} + \lambda_- g(x)(-u)\chi_{\{u<0\}}) \\ &- \beta r^{-n-2\beta+1} \int_{\partial B_r(x_0)} u^2,\end{aligned} \tag{4.4}$$

and verify that

$$\begin{aligned}&2(\beta-1)F(x,\vec{u}) - \beta \vec{u}\vec{f}(x,\vec{u}) - \nabla_x F(x,\vec{u}) \cdot (x-x_0) \\ = {}& -(\beta-1)(\lambda_+(u^p + f(x)u)\chi_{\{u>0\}} + \lambda_-((-u)^p + g(x)(-u))\chi_{\{u<0\}}) \\ &+ \frac{\beta}{2}(\lambda_+(pu^p + f(x)u)\chi_{\{u>0\}} + \lambda_-(p(-u)^p + g(x)(-u))\chi_{\{u<0\}}) \\ &+ \frac{1}{2}(\lambda_+ u \chi_{\{u>0\}} \nabla f(x) \cdot (x-x_0) + \lambda_-(-u)\chi_{\{u<0\}} \nabla g(x) \cdot (x-x_0)) \\ = {}& \Big(1 + \frac{\beta}{2}p - \beta\Big)(\lambda_+ u^p \chi_{u>0} + \lambda_-(-u)^p \chi_{\{u<0\}}) \\ &+ \Big(1 - \frac{\beta}{2}\Big)(\lambda_+ f(x) u \chi_{\{u>0\}} + \lambda_- g(x)(-u)\chi_{\{u<0\}}) \\ &+ \frac{1}{2}(\lambda_+ u \chi_{\{u>0\}} \nabla f(x) \cdot (x-x_0) + \lambda_-(-u)\chi_{\{u<0\}} \nabla g(x) \cdot (x-x_0)).\end{aligned}$$

We assume that λ_+ and λ_- are nonnegative. To ensure that the last three terms in the expression above are nonnegative point-wise, we need

$$\begin{aligned}&\beta \le \frac{2}{2-p}, \\ &\beta \le 2 \quad \text{if } f(x), g(x) \ge 0 \quad \text{or} \quad \beta \ge 2 \quad \text{if } f(x), g(x) \le 0, \\ &\nabla f(x) \cdot (x-x_0) \ge 0 \quad \text{and} \quad \nabla g(x) \cdot (x-x_0) \ge 0.\end{aligned}$$

Two typical choices for $f(x)$ and $g(x)$ valid in our analysis are the potentials $|x-x_0|^2$ and $-\frac{1}{|x-x_0|}$. Applying Theorems 2.1 and 2.2 to (4.2), we obtain the proposition below.

Proposition 4.4 *Let u be a variational solution to the free boundary problem*

$$2\Delta u = \lambda_+(pu^{p-1} + f(x))\chi_{\{u>0\}} - \lambda_-(p(-u)^{p-1} + g(x))\chi_{\{u<0\}}$$

in a bounded domain $\Omega \subset \mathbb{R}^n$, where $p < 2$ and $p \neq 0$. Assume that $B_\delta(x_0) \subset\subset \Omega$, $\lambda_+, \lambda_- \ge 0$, $\nabla f(x) \cdot (x-x_0) \ge 0$ and $\nabla g(x) \cdot (x-x_0) \ge 0$.

(1) *If $f(x), g(x) \geq 0$, then for $\beta \leq \min\{\frac{2}{2-p}, 2\}$, the function $\Phi_{x_0}(r)$ defined in (4.4) is nondecreasing in $r \in (0, \delta)$. Furthermore, if*

$$\sup_{r\in(0,\delta)} \beta r^{-n-2\beta+1} \int_{\partial B_r(x_0)} \vec{u}^2 < +\infty,$$

then for any open domain $D \subset\subset \mathbb{R}^n$ and $k \geq k(D)$, the scaled sequence $\vec{u}_k(x) = \rho_k^{-\beta} u(x_0 + \rho_k x)$ is bounded in $H^1(D)$ and any weak H^1-limit with respect to a subsequence $k \to \infty$ is homogeneous of degree β.

(2) *If $f(x), g(x) \leq 0$ and $1 \leq p < 2$, then for $2 \leq \beta \leq \frac{2}{2-p}$, the function $\Phi_{x_0}(r)$ defined in (4.4) is nondecreasing in $r \in (0, \delta)$. Furthermore, if*

$$\begin{aligned}\sup_{r\in(0,\delta)} \Big\{ &\beta r^{-n-2\beta+1} \int_{\partial B_r(x_0)} \vec{u}^2,\ -\lambda_+ r^{-n-2\beta+2} \int_{B_r(x_0)} f(x) u \chi_{\{u>0\}}, \\ &-\lambda_- r^{-n-2\beta+2} \int_{B_r(x_0)} g(x)(-u)\chi_{\{u<0\}} \Big\} < +\infty,\end{aligned}$$

then for any open domain $D \subset\subset \mathbb{R}^n$ and $k \geq k(D)$, the scaled sequence $\vec{u}_k(x) = \rho_k^{-\beta} u(x_0 + \rho_k x)$ is bounded in $H^1(D)$ and any weak H^1-limit with respect to a subsequence $k \to \infty$ is homogeneous of degree β.

Another very interesting extension to the model (4.3) is to replace the classical operator Δ by the Schrödinger operator $\Delta - |x|^2$, which appears in many branches of physics. Indeed, we at first use the model modified a little from (4.2) that

$$2\Delta u = \lambda_+(pu^{p-1} + f(x)u)\chi_{\{u>0\}} - \lambda_-(p(-u)^{p-1} - g(x)u)\chi_{\{u<0\}} \tag{4.5}$$

in a bounded domain $\Omega \subset \mathbb{R}^n$, where $p < 2$ and $p \neq 0$. Notice that in this case

$$\begin{aligned} F(x,u) &= -\frac{\lambda_+}{2}\Big(u^p + \frac{1}{2} f(x) u^2\Big)\chi_{\{u>0\}} - \frac{\lambda_-}{2}\Big((-u)^p + \frac{1}{2} g(x)u^2\Big)\chi_{\{u<0\}}, \\ uf(x,u) &= -\frac{\lambda_+}{2}(pu^p + f(x)u^2)\chi_{\{u>0\}} - \frac{\lambda_-}{2}(p(-u)^p + g(x)u^2)\chi_{\{u<0\}} \end{aligned}$$

and

$$\nabla_x F(x,u)\cdot(x-x_0) = -\frac{\lambda_+}{4} u^2 \chi_{\{u>0\}} \nabla f(x)\cdot(x-x_0) - \frac{\lambda_-}{4} u^2\chi_{\{u<0\}}\nabla g(x)\cdot(x-x_0).$$

Suppose that $B_\delta(x_0) \subset\subset \Omega$ and u is a variational solution to (4.5). We define for $r \in (0, \delta)$,

$$\begin{aligned} \Phi_{x_0}(r) = r^{-n-2\beta+2} &\int_{B_r(x_0)} (|\nabla u|^2 + \lambda_+ u^p \chi_{\{u>0\}} + \lambda_-(-u)^p\chi_{\{u<0\}}) \\ &+ r^{-n-2\beta+2} \int_{B_r(x_0)} \Big(\frac{\lambda_+}{2} f(x)u^2\chi_{\{u>0\}} + \frac{\lambda_-}{2} g(x)u^2\chi_{\{u<0\}}\Big) \\ &- \beta r^{-n-2\beta+1} \int_{\partial B_r(x_0)} u^2, \end{aligned} \tag{4.6}$$

and verify that

$$\begin{aligned} &2(\beta-1)F(x,\vec{u}) - \beta\vec{u}\vec{f}(x,\vec{u}) - \nabla_x F(x,\vec{u})\cdot(x-x_0) \\ =&\Big(1+\frac{\beta}{2}p-\beta\Big)(\lambda_+u^p\chi_{\{u>0\}} + \lambda_-(-u)^p\chi_{\{u<0\}}) + \frac{1}{2}(\lambda_+ f(x)u^2\chi_{\{u>0\}} + \lambda_- g(x)u^2\chi_{\{u<0\}}) \\ &+\frac{\lambda_+}{4}u^2\chi_{\{u>0\}}\nabla f(x)\cdot(x-x_0) + \frac{\lambda_-}{4}u^2\chi_{\{u<0\}}\nabla g(x)\cdot(x-x_0). \end{aligned}$$

We assume $\lambda_+, \lambda_- > 0$, and choose $f(x) = \frac{2}{\lambda_+}|x-x_0|^2$, $g(x) = \frac{2}{\lambda_-}|x-x_0|^2$. Then the expression above is nonnegative provided $\beta \le \frac{2}{2-p}$ and we can apply the conclusions of Theorems 2.1 and 2.2.

Proposition 4.5 *Let u be a variational solution to the free boundary problem*

$$2(\Delta - |x-x_0|^2)u = \lambda_+ p u^{p-1}\chi_{\{u>0\}} - \lambda_- p(-u)^{p-1}\chi_{\{u<0\}}$$

in a bounded domain $\Omega \subset \mathbb{R}^n$, where $p < 2$ and $p \ne 0$. Assume that $B_\delta(x_0) \subset\subset \Omega$, $\lambda_+, \lambda_- > 0$. Then for $\beta \le \frac{2}{2-p}$, the function $\Phi_{x_0}(r)$ defined in (4.6) is nondecreasing in $r \in (0, \delta)$. Furthermore, if

$$\sup_{r\in(0,\delta)} \beta r^{-n-2\beta+1} \int_{\partial B_r(x_0)} \vec{u}^2 < +\infty,$$

then for any open domain $D \subset\subset \mathbb{R}^n$ and $k \ge k(D)$, the scaled sequence $\vec{u}_k(x) = \rho_k^{-\beta} u(x_0 + \rho_k x)$ is bounded in $H^1(D)$ and any weak H^1-limit with respect to a subsequence $k \to \infty$ is homogeneous of degree β.

The analogies of Propositions 4.4 and 4.5 for the parabolic case (see [32]) can be obtained exactly in the same way, and we leave the details to the interested reader.

5 Appendix

As a comparison of our new monotonicity inequalities to previous works, we now give a brief review about the previous monotonicity formulae related. We point out that the monotonicity formulae of Perelman [23] and Hamilton [16] on Ricci flow will not be included here. As we said before, our work is motivated from the monotonicity formulae given by Weiss [29–32] and the monotonicity formula given by Alt, Caffarelli and Friedman [2]. In [29–32], Weiss introduced the "boundary-adjusted energy", and obtained some monotonicity formulae. In [30], he studied the critical points with respect to the energy

$$w \mapsto F(w) = \int_\Omega (|\nabla w|^2 + \lambda_+\chi_{\{w>0\}} w^p + \lambda_-\chi_{\{w<0\}}(-w)^p)$$

with $p \in [0, 2)$ and found that: assuming that u is a solution and $B_\delta(x_0) \subset \Omega$, then for $\beta = \frac{2}{2-p}$ and for any $0 < \rho < \sigma < \delta$, the function

$$\Phi(r) = r^{-n-2\beta+2} \int_{B_r(x_0)} (|\nabla u|^2 + \lambda_+\chi_{\{u>0\}} u^p + \lambda_-\chi_{\{u<0\}}(-u)^p) - \beta r^{-n-2\beta+1} \int_{\partial B_r(x_0)} u^2$$

defined in $(0, \delta)$ satisfies the monotonicity formula

$$\Phi(\sigma) - \Phi(\rho) = 2\int_\rho^\sigma r^{-n-2\beta+2} \Big(\int_{\partial B_r(x_0)} \Big(\nabla u \cdot \nu - \beta \frac{u}{r}\Big)^2 \Big) \mathrm{d}r \ge 0.$$

In [31], the monotonicity formula for $\Delta u = \chi_{\{u>0\}}$ has the same form of $\Phi(r)$ with $p = 1$. In [32], Weiss studied the gradient flow in $L^2(\mathbb{R}^n)$ with respect to the energy

$$w \mapsto F(w) = \int_{\mathbb{R}^n} (|\nabla w|^2 + \lambda_+\chi_{\{w>0\}} w^p + \lambda_-\chi_{\{w<0\}}(-w)^p)$$

with $p \in [0,2)$ and found that: assuming that $t_1 \le T \le t_2$, $x_0 \in \mathbb{R}^n$ and u is a solution with some smooth conditions, then for $\beta = \frac{2}{2-p}$ and for any $0 < \rho < \sigma < \delta$, the functions

$$\Psi^-(r) = r^{-2\beta} \int_{T_r^-} (|\nabla u|^2 + \lambda_+ \chi_{\{u>0\}} u^p + \lambda_- \chi_{\{u<0\}} (-u)^p) G_{(T,x_0)} - \frac{\beta}{2} r^{-2\beta} \int_{T_r^-} \frac{1}{T-t} u^2 G_{(T,x_0)}$$

and

$$\Psi^+(r) = r^{-2\beta} \int_{T_r^+} (|\nabla u|^2 + \lambda_+ \chi_{\{u>0\}} u^p + \lambda_- \chi_{\{u<0\}} (-u)^p) G_{(T,x_0)} - \frac{\beta}{2} r^{-2\beta} \int_{T_r^+} \frac{1}{T-t} u^2 G_{(T,x_0)}$$

are well-defined in the interval $(0, \frac{\sqrt{T-t_1}}{2})$ and $(0, \frac{\sqrt{t_2-T}}{2})$ respectively, and satisfy for any $0 < \rho < \sigma < \frac{\sqrt{T-t_1}}{2}$ and $0 < \rho < \sigma < \frac{\sqrt{t_2-T}}{2}$ respectively, the monotonicity formulae

$$\begin{aligned}\Psi^-(\sigma) - \Psi^-(\rho) &= \int_\rho^\sigma r^{-2\beta-1} \int_{T_r^-} \frac{1}{T-t} (\nabla u \cdot (x-x_0) - 2(T-t)\partial_t u - \beta u)^2 G_{(T,x_0)} \\ &\ge 0\end{aligned}$$

and

$$\begin{aligned}\Psi^+(\sigma) - \Psi^+(\rho) &= \int_\rho^\sigma r^{-2\beta-1} \int_{T_r^+} \frac{1}{T-t} (\nabla u \cdot (x-x_0) - 2(T-t)\partial_t u - \beta u)^2 G_{(T,x_0)} \\ &\ge 0.\end{aligned}$$

In [2], Alt, Caffarelli and Friedman established a monotonicity formula for variational problems with two phases and their free boundaries. The monotonicity formula of Alt-Caffarelli-Friedman plays an important role as a fundamental and powerful tool in free boundary problems. Roughly speaking, they found that

$$\Phi(r) = \Big(\frac{1}{r^2} \int_{B_r(x_0)} \frac{|\nabla h_1|^2}{|x-x_0|^{N-2}}\Big)\Big(\frac{1}{r^2} \int_{B_r(x_0)} \frac{|\nabla h_2|^2}{|x-x_0|^{N-2}}\Big)$$

is increasing in r $(0 < r < R)$ for the sub-solutions h_1, h_2 to $\Delta u = 0$ in $B_R(x_0)$ $(R > 0)$ with $h_1 h_2 = 0$ and $h_1(x_0) = h_2(x_0) = 0$. We can consult also [7]. In [5], Caffarelli, Jerison and Kenig found that there is a dimensional constant C such that

$$\begin{aligned}\Phi(r) &= \Big(\frac{1}{r^2} \int_{B_r} \frac{|\nabla u_+|^2}{|X|^{n-2}} \,\mathrm{d}X\Big)\Big(\frac{1}{r^2} \int_{B_r} \frac{|\nabla u_-|^2}{|X|^{n-2}} \,\mathrm{d}X\Big) \\ &\le C\Big(1 + \int_{B_1} \frac{|\nabla u_+(X)|^2}{|X|^{n-2}} \,\mathrm{d}X + \int_{B_1} \frac{|\nabla u_-(X)|^2}{|X|^{n-2}} \,\mathrm{d}X\Big)^2\end{aligned}$$

with $0 < r \le 1$ for $\Delta u_\pm \ge -1$ in the sense of distributions, where u_+ and u_- satisfy $u_+(X)u_-(X) = 0$ for all $X \in B_1$. Various monotonicity formulae of other types have caught many authors' attentions in the past several years. Let us briefly review some progress in them. The well-known monotonicity formula, for minimal hyper-surfaces in [27],

$$\frac{\mathrm{d}}{\mathrm{d}r}\Big(\frac{\mathcal{H}^n(M \cap B_r)}{r^n}\Big) = \frac{\mathrm{d}}{\mathrm{d}r} \int_{M \cap B_r} \frac{|x^\perp|^2}{|x|^{n+2}} \,\mathrm{d}\mathcal{H}^n,$$

where $\mathcal{H}^n$ is the n-dimensional Hausdorff measure on $\mathbb{R}^{n+1}$, is a local statement in balls $B_r \subset \mathbb{R}^{n+1}$, which plays an important role in analyzing singularity set. There are many references about the topic. Fleming obtained the monotonicity formula for area minimizing currents in [12]. Allard proved the monotonicity formula for stationary rectifiable n-varifolds in [1]. Schoen and Uhlenbeck established the monotonicity formula for harmonic maps in [26]. Price proved the monotonicity for weakly stationary harmonic maps and Yang-Mills equations in [24]. Giga and Kohn obtained in [14] the monotonicity formula for the solutions to semi-linear heat equations $\partial_t u - \Delta u - |u|^{p-1}u = 0$ with blow-up analysis, where $p > 1$, and Pacard established its localization for weakly stationary solutions to the corresponding elliptic equation in [22]. Struwe derived the monotonicity formula involving the associated energy densities for the equation $\partial_t u - \Delta u \in T^{\perp}N$ in [28]. Riviere [25], Lin and Riviere [20], Bourgain, Brezis, and Mironescu [3] set up some monotonicity formulae for Ginzburg-Landau model. The famous monotonicity formula for mean curvature flow, which was found by Huisken [17], says that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{M_t} G\,\mathrm{d}\mu_t = -\int_{M_t}\Big|\vec{H} - \frac{x^{\perp}}{2t}\Big|^2 G\,\mathrm{d}\mu_t,$$

which involves the backward heat kernel function

$$G(x,t) = \frac{1}{(-4\pi t)^{\frac{n}{2}}}\exp\Big(\frac{|x|^2}{4t}\Big)$$

for $t < 0$ and $x \in \mathbb{R}^{n+k}$. Monotonicity formulae for geometric evolution equations on more general domains were also derived by Hamilton in [15]. In [10, 11], the local monotonicity formula had been given by Ecker in the "heat-ball"

$$E_r^{\gamma} = \Big\{(x,t) \in \mathbb{R}^n \times \mathbb{R},\ t < 0,\ \Phi^{\gamma} > \frac{1}{r^{n-\gamma}}\Big\} = \bigcup_{-\frac{r^2}{4\pi} < t < 0} B_{R_r^{\gamma}(t)} \times \{t\},$$

where

$$\Phi^{\gamma}(x,t) = \frac{1}{(-4\pi t)^{\frac{n-\gamma}{2}}}\mathrm{e}^{\frac{|x|^2}{4t}}, \quad R_r^{\gamma}(t) = \sqrt{2(n-\gamma)\log\Big(\frac{-4\pi t}{r^2}\Big)}.$$

It can be written as follows:

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}r}\Big[\frac{1}{R^{n-\gamma}}\int_{E_r^{\gamma}}\frac{n-\gamma}{-2t}\Big(e(u) - \frac{\beta}{2t}u^2\Big) - \frac{x}{2t}\cdot Du\Big(\frac{\partial u}{\partial t} + \frac{x}{2t}\cdot Du + \frac{\beta}{t}u\Big)\mathrm{d}x\mathrm{d}t\Big]\\ &= \frac{n-\gamma}{r^{n-\gamma+1}}\int_{E_r^{\gamma}}\Big(\frac{\partial u}{\partial t} + \frac{x}{2t}\cdot Du + \frac{\beta}{t}u\Big)^2\mathrm{d}x\mathrm{d}t,\end{aligned}$$

where u is a solution to $u_t - \Delta u - |u|^{p-1}u = 0$, $x \in \mathbb{R}^n$, $t < 0$ and $p > 1$. The monotonicity formula also appears in the parabolic potential theory (see [9]). For a function v and any $t > 0$, define

$$I(t; v) = \int_{-t}^{0}\int_{\mathbb{R}^n}|\nabla v(s,x)|^2 G(-s,x)\,\mathrm{d}x\mathrm{d}s.$$

In [4], Caffarelli found that

$$\Phi(t) = \Phi(t; h_1, h_2) = \frac{1}{t^2}I(t; h_1)I(t; h_2)$$

is monotone nondecreasing in t $(0 < t < 1)$ for nonnegative sub-caloric functions h_1, h_2 in the strip $[-1,0] \times \mathbb{R}^n$, $h_1(0,0) = h_2(0,0) = 0$ and $h_1 \cdot h_2 = 0$ with a polynomial growth at infinity. Its localization can be stated as follows: there exists a constant $C = C(n,\psi) > 0$ such that

$$\Phi(t; w_1, w_2) \le C\|h_1\|^2_{L^2(Q_1^-)}\|h_2\|^2_{L^2(Q_1^-)}$$

for any $0 < t < \frac{1}{2}$. Here $\psi(x) \geq 0$ is a C^∞ cut-off function with $\operatorname{supp} \psi \subset B_{\frac{3}{4}}$ and $\psi|_{B_{\frac{1}{2}}} = 1$ and $w_i = h_i \psi$ (see [9]). In [6], this formula was generalized for parabolic equations with variable coefficients, and was written as

$$\begin{aligned}&\frac{1}{t}\int_{-t}^{0}\int_{\mathbb{R}^n} |\nabla(u_1\psi)|^2 G(x,-s)\,\mathrm{d}x\mathrm{d}s \cdot \frac{1}{t}\int_{-t}^{0}\int_{\mathbb{R}^n} |\nabla(u_2\psi)|^2 G(x,-s)\,\mathrm{d}x\mathrm{d}s\\ &\leq C(\|u_1\|^4_{L^2(Q_2)} + \|u_2\|^4_{L^2(Q_2)}).\end{aligned}$$

One may also see the recent works [19] of Fanghua Lin and [8] of Caffarelli and Lin for new applications of monotonicity inequalities.

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