Chin. Ann. Math.
31B(4), 2010, 529–540
DOI: 10.1007/s11401-010-0587-2

Chinese Annals of Mathematics, Series B

Observability Inequality for the Kirchhoff-Rayleigh Plate Like Equation in a Short Time**

Dongli LI*

Abstract In this paper, for any given observation time and suitable moving observation domains, the author establishes a sharp observability inequality for the Kirchhoff-Rayleigh plate like equation with a suitable potential in any space dimension. The approach is based on a delicate energy estimate. Moreover, the observability constant is estimated by means of an explicit function of the norm of the coefficient involved in the equation.

Keywords Kirchhoff-Rayleigh plate like equation, Finite speed of propagation, Observability inequality in a short time
2000 MR Subject Classification 93B07, 93B05

1 Introduction

Given arbitrarily a (short) time $T > 0$ and a bounded domain Ω of $\mathbb{R}^n$ $(n \geq 1)$ with C^4 boundary Γ, we put $Q = (0,T) \times \Omega$ and $\Sigma = (0,T) \times \Gamma$. For any $t \in [0,\infty)$, assume that $G(t)$ is a suitable (proper) subdomain of Ω. Throughout this paper, we will use C to denote a generic positive constant which may vary from line to line.

We consider the following Kirchhoff-Rayleigh plate like equation with a potential $q \in L^\infty(0,T;L^p(\Omega))$ $(p \geq n)$:

$$\begin{cases} w_{tt} + \Delta^2 w - \Delta w_{tt} + qw = 0, & \text{in } Q, \\ w = \Delta w = 0, & \text{on } \Sigma, \\ w(0) = w_0,\ w_t(0) = w_1, & \text{in } \Omega, \end{cases} \tag{1.1}$$

where the initial datum (w_0, w_1) is supposed to belong to $(H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$, the state space of system (1.1). Thanks to the standard operator semigroup theory, it is easy to show that system (1.1) admits a mild solution $w \in C([0,T]; H^2(\Omega) \cap H_0^1(\Omega)) \cap C^1([0,T]; H_0^1(\Omega))$.

The main purpose of this paper is, for any given $T > 0$, to find a constant $\mathcal{C}(q) > 0$ and a class of subdomains $\{G(t) \mid t \in [0,T]\}$ such that all weak solutions w of (1.1) satisfy

$$\|\Delta w_0\|^2_{L^2(\Omega)} + \|w_1\|^2_{H_0^1(\Omega)} \leq \mathcal{C}(q) \int_0^T \int_{G(t)} (|w|^2 + |\Delta w|^2) \mathrm{d}x\mathrm{d}t,$$
$$\forall\, (w_0, w_1) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega). \tag{1.2}$$

Manuscript received May 14, 2009. Rivised January 21, 2010. Published online June 21, 2010.
*Key Laboratory of Systems and Control, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China. E-mail: lidongli@amss.ac.cn
**Project supported by the National Natural Science Foundation of China (Nos. 10831007, 60821091, 60974035).

In particular, we shall analyze the explicit and sharp dependence of $C(q)$ on the potential q.

The above inequality, the so-called rapid observability inequality for system (1.1), allows estimating the total energy of solutions in terms of the energy localized in the moving observation subdomain $G(\cdot)$ in an arbitrarily short time. It is relevant to rapid controllability problems. Especially, in this linear setting this inequality is equivalent to the so-called rapid exact controllability property, i.e., for any given T, that of driving solutions to the rest by means of the control forces localized in $(0,T)\times G(\cdot)$ (see [10]). This type of inequality, with explicit estimates on the observability observability constant, is also relevant to the controllability of semilinear problems (see [7, 11, 13]). We refer to [5, 8, 15, 18] for more related background.

When the propagation speed of solutions to the system under consideration is infinite, the (rapid) observability/controllability problems are studied by many authors (see [2, 3, 5, 6, 15, 17] and the references therein). However, when this speed is finite, things become more delicate (see [1]), and the desired observability/controllability holds true only when the time T is large enough or other conditions on the observer/controller are imposed. For the above Kirchhoff-Rayleigh plate like equation, we shall show the finite speed of propagation for its solutions (see Proposition 3.1 in Section 3). Because of this, in order to establish the desired rapid observability estimate (1.2), we need some complicated conditions on the time-variant observer $G(\cdot)$ (see Section 2 for more details).

Similar rapid (internal) observability problems were considered for wave equations in [7, 11, 13], and for Maxwell equations in [14]. Note however that, as far as I know, there is no result published on the rapid observability estimate for the fourth order equations with finite speed of propagation.

The rest of this paper is organized as follows. In Section 2, we state the main result of this work. A proof of the finite speed of propagation on system (1.1) is given in Section 3. In Section 4, we collect some preliminary results that will be used later. Finally, Section 5 is devoted to prove our main result.

2 Statement of the Main Result

To begin with, without loss of generality, we assume that

$$\begin{aligned}&\inf\{x_1\in\mathbb{R}\mid \exists\, x'\in\mathbb{R}^{n-1},\ \text{such that } (x_1,x')\in\Omega\}=0,\\ &\sup\{x_1\in\mathbb{R}\mid \exists\, x'\in\mathbb{R}^{n-1},\ \text{such that } (x_1,x')\in\Omega\}\stackrel{\triangle}{=}\beta>0.\end{aligned}$$

Also, for any $T>0$ and any $\sigma\in(0,T)$, put

$$\begin{aligned}&a=\frac{T-\sigma}{\beta},\quad K_\sigma=\{(t,x_1)\in(0,T)\times[0,\beta]\mid ax_1<t<ax_1+\sigma\},\\ &D_\sigma=(K_\sigma\times\mathbb{R}^{n-1})\cap Q.\end{aligned}\tag{2.1}$$

Denote the set of all subsets in Ω by 2^Ω. As in [7, 13], we introduce the following assumption on the class $\mathcal{G}$ of observers:

(A1) *Class $\mathcal{G}$ is a family of set-valued functions $G:[0,\infty)\to 2^\Omega$ with the properties:*

(i) *Any $G(\cdot) \in \mathcal{G}$ is continuous with respect to the Hausdorff metric*;

(ii) *For any $T > 0$, there exists a $G(\cdot) \in \mathcal{G}$ and a $\sigma \in (0, T)$, such that*

$$\widetilde{G}_T \triangleq \{(t, x) \in Q \mid x \in G(t),\ t \in (0, T)\} \supseteq D_\sigma. \tag{2.2}$$

As an example, we consider the special case that $\Omega = (0, 1)$ and $Q = (0, T) \times (0, 1)$. For any $\mu > 0$, set

$$\begin{aligned} \alpha(t) &= \frac{1+\mu}{T} t - \frac{\mu}{2}, \quad t \geq 0, \\ G(\mu, T, t) &= \Big\{ x \in (0, 1) \, \Big| \, |x - \alpha(t)| < \frac{\mu}{2} \Big\}. \end{aligned} \tag{2.3}$$

We see that $G(\cdot) = G(\mu, T, \cdot)$ satisfies (2.2) provided that $\frac{\sigma}{T-\sigma} < \mu$. Hence the class

$$\mathcal{G} = \bigsqcup_{\substack{\mu > 0 \\ T \in (0, \infty)}} G(\mu, T, \cdot)$$

satisfies (A1) (see Figure 1). In this case, the sets D_σ and $\widetilde{G}_T$ are given by the parallelograms $ABOC$ and $ADOE$ in I

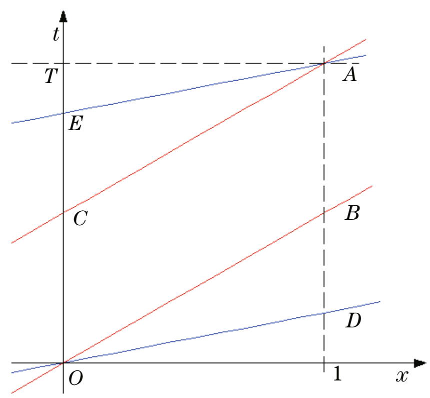

Figure 1 Sets D_σ and $\widetilde{G}_T$

The main result of this paper is stated as follows.

Theorem 2.1 *Assume that* (A1) *holds and $q \in L^\infty(0, T; L^p(\Omega))$ with some $p \in [n, \infty]$. Then, for any $T > 0$, there exists an observer $G(\cdot) \in \mathcal{G}$ such that all weak solutions w of* (1.1) *satisfy estimate* (1.2) *with some observability constant $\mathcal{C}(q) > 0$ to be of the form*

$$\mathcal{C}(q) = C \exp(C r^{\frac{1}{2 - \frac{n}{p}}}), \tag{2.4}$$

where

$$r = \|q\|_{L^\infty(0, T; L^p(\Omega))}. \tag{2.5}$$

Several remarks are in order.

Remark 2.1 In this paper, we establish the desired observability estimate (1.2) by means of a careful energy estimate. Here we choose to work in the state space $(H^2(\Omega)\cap H_0^1(\Omega))\times H_0^1(\Omega)$. But some other choices of the state spaces are possible. For example, one could consider similar problems in state spaces of the form $H_0^1(\Omega)\times L^2(\Omega)$ or $\{w\in H^3(\Omega)\mid w|_\Gamma=\Delta w|_\Gamma=0\}\times(H^2(\Omega)\cap H_0^1(\Omega))$, where the Kirchhoff-Rayleigh plate like system is also well-posed. But the corresponding analysis will be technically more involved.

Remark 2.2 One of the key points to derive inequality (1.2) for system (1.1) is the possibility to decompose the Kirchhoff-Rayleigh plate like operator $\partial_t^2+\Delta^2-\partial_t^2\Delta$ as follows:

$$\partial_t^2+\Delta^2-\partial_t^2\Delta=(\partial_{tt}-\Delta)(I-\Delta)+\Delta,$$

where I is the identity operator; another point is that we impose the boundary conditions $w=\Delta w=0$ on Σ. We set

$$z=w-\Delta w, \tag{2.6}$$

where w is the solution to system (1.1). Then, the Kirchhoff-Rayleigh plate like system (1.1) can be written equivalently as the following coupled elliptic-wave system:

$$\begin{cases} w-\Delta w=z, & \text{in } Q,\\ z_{tt}-\Delta z+w-z+qw=0, & \text{in } Q,\\ w=z=0, & \text{on } \Sigma,\\ z(0)=w_0-\Delta w_0,\ z_t(0)=w_1-\Delta w_1, & \text{in } \Omega. \end{cases} \tag{2.7}$$

Note however that a similar equivalence does not hold any more when the boundary conditions in system (1.1) are replaced by $w=\frac{\partial w}{\partial\nu}=0$ on Σ, where ν is the unit outward normal vector of Ω at boundary Γ. It would be quite interesting to study the same rapid observability problem for the Kirchhoff-Rayleigh plate like system with the later boundary conditions, but this is an unsolved problem.

Remark 2.3 By means of the well-known duality argument (see [9, Lemma 2.4, p. 282] and [15, Theorem 3.2, p. 19], for examples), it is easy to deduce a rapid controllability result for the Kirchhoff-Rayleigh plate like equation from our observability estimate in Theorem 2.1. Since this method is standard, we omit the details.

Remark 2.4 As mentioned before, condition (A1) is complicated. However, as remarked in [7, 13], this condition is necessary for the rapid observation for the wave equations in some sense (by considering the main result in [1]). Indeed, for any fixed observer, the rapid observation for the wave equations is impossible except for some trivial case. By adopting the approaches developed in [4], one can show that similar phenomenon occurs for the Kirchhoff-Rayleigh plate like equation. Nevertheless, the detailed analysis is complicated and we shall present this result in a forthcoming paper.

3 Finite Propagation Speed for the Kirchhoff-Rayleigh Plate Like Equation

In this section, we prove the property of finite speed of propagation for solutions to the Kirchhoff-Rayleigh plate like equation. We believe that this result should be known in the previous literature but we have not found an exact reference.

We consider the following equation (without boundary condition):

$$\begin{cases} w_{tt} + \Delta^2 w - \Delta w_{tt} + qw = 0, & \text{in } (0,T)\times\Omega, \\ w(0) = w_0,\ w_t(0) = w_1, & \text{in } \Omega. \end{cases} \tag{3.1}$$

Similarly to [12], fix any $(L, x_0) \in (0,\infty)\times\Omega$ such that

$$\overline{\{x\in\mathbb{R}^n \mid |x-x_0|\le L\}} \subset \Omega. \tag{3.2}$$

Choose $\ell > 1$ arbitrarily and denote $t_0 = \frac{L}{\ell}$. Set

$$\Omega_t = \{x\in\mathbb{R}^n \mid |x-x_0| \le \ell(t_0-t)\}, \quad \forall\, t\in[0,t_0]. \tag{3.3}$$

Define an (modified) energy of system (3.1) in the subdomain Ω_t by

$$\begin{aligned} E(t) = \frac{1}{2}\int_{\Omega_t} &[|\nabla\Delta w(t,x)|^2 + |\Delta w(t,x)|^2 + |\Delta w_t(t,x) - w_t(t,x)|^2 \\ &+ |\nabla w_t(t,x)|^2 + |\nabla w(t,x)|^2 + |w(t,x)|^2]\mathrm{d}x. \end{aligned} \tag{3.4}$$

We have the following a prior estimate for solutions to (3.1).

Proposition 3.1 *Let $T > 0$, $w_0 \in H^3_{\rm loc}(\Omega)$ and $w_1 \in H^2_{\rm loc}(\Omega)$ be given, $q \in L^\infty(0,T;$ $L^p_{\rm loc}(\Omega))$ with some $p\in[n,\infty]$. Let $w \in C([0,T];\, H^3_{\rm loc}(\Omega)) \cap C^1([0,T]; H^2_{\rm loc}(\Omega))$ be a weak solution to equation* (3.1)*. Then, for any $(L,x_0)\in(0,\infty)\times\Omega$ satisfying* (3.2)*, there exists an $\ell > 0$, such that the energy E(t) of* (3.1) *satisfies*

$$E(t) \le CE(0), \quad \forall\, t\in[0,t_0\wedge T], \tag{3.5}$$

where $C = C(t_0,T,x_0,\ell) > 0$.

Remark 3.1 It is easy to see that the finite speed of propagation for equation (1.1) is a direct consequence of the above a prior estimate. Indeed, in the case that $t_0 \le T$, by (3.5), we see that $E(t)\equiv 0$ whenever $E(0) = 0$. This means that the cone

$$\mathcal{C} \stackrel{\triangle}{=} \{(s,x)\in[0,t_0]\times\mathbb{R}^n \mid |x-x_0|\le\ell(t_0-s)\}$$

is the domain of determination for (t_0,x_0). Therefore, the data outside $\mathcal{C}$ does not affect the value of the (smooth) solution to equation (1.1) at (t_0,x_0). Consequently, the propagation speed is finite.

Proof of Proposition 3.1 We borrow some idea from [12]. Recalling the definition of Ω_t in (3.3), and noting (3.2), we obtain

$$E(t) = \frac{1}{2}\int_0^{\ell(t_0-t)}\mathrm{d}r\int_{|x-x_0|=r}[|\nabla\Delta w(t,x)|^2 + |\Delta w(t,x)|^2$$

$$+|\Delta w_t(t,x)-w_t(t,x)|^2+|\nabla w_t(t,x)|^2+|\nabla w(t,x)|^2+|w(t,x)|^2]\mathrm{d}S_x, \tag{3.6}$$

where $\mathrm{d}S_x$ stands for the area element on $|x-x_0|=r$. Therefore,

$$\begin{aligned}\frac{\mathrm{d}E(t)}{\mathrm{d}t} &= \frac{1}{2}\int_{\Omega_t}\frac{\partial}{\partial t}[|\nabla\Delta w|^2+|\Delta w|^2+|\Delta w_t-w_t|^2+|\nabla w_t|^2+|\nabla w|^2+|w|^2]\mathrm{d}x\\ &\quad -\frac{\ell}{2}\int_{\partial\Omega_t}[|\nabla\Delta w|^2+|\Delta w|^2+|\Delta w_t-w_t|^2+|\nabla w_t|^2+|\nabla w|^2+|w|^2]\mathrm{d}(\partial\Omega_t).\end{aligned} \tag{3.7}$$

On the other hand, denote by $\nu=\nu(x)$ the unit out normal vector at $x\in\partial\Omega_t$. Multiplying the first equation in (3.1) by Δw_t and w_t, respectively, and integrating it in Ω_t, using integration by parts, we get

$$\begin{aligned}\frac{1}{2}\int_{\Omega_t}\frac{\partial}{\partial t}[|\nabla\Delta w|^2+|\Delta w_t|^2+|\nabla w_t|^2]\mathrm{d}x &= \int_{\partial\Omega_t}w_{tt}\nu\cdot\nabla w_t\mathrm{d}(\partial\Omega_t)+\int_{\partial\Omega_t}\Delta w_t\nu\cdot\nabla\Delta w\mathrm{d}(\partial\Omega_t)\\ &\quad+\int_{\Omega_t}qw\Delta w_t\mathrm{d}x, \end{aligned}\tag{3.8}$$

$$\begin{aligned}\frac{1}{2}\int_{\Omega_t}\frac{\partial}{\partial t}[|\Delta w|^2+|\nabla w_t|^2+|w_t|^2]\mathrm{d}x &= \int_{\partial\Omega_t}w_t\nu\cdot\nabla w_{tt}\mathrm{d}(\partial\Omega_t)+\int_{\partial\Omega_t}\Delta w\nu\cdot\nabla w_t\mathrm{d}(\partial\Omega_t)\\ &\quad-\int_{\partial\Omega_t}w_t\nu\cdot\nabla\Delta w\mathrm{d}(\partial\Omega_t)-\int_{\Omega_t}qww_t\mathrm{d}x, \end{aligned}\tag{3.9}$$

and also,

$$\begin{aligned}\int_{\Omega_t}\frac{\partial}{\partial t}(w_t\Delta w_t)\mathrm{d}x &= \int_{\Omega_t}(w_{tt}\Delta w_t+w_t\Delta w_{tt})\mathrm{d}x\\ &= \int_{\partial\Omega_t}[w_{tt}\nu\cdot\nabla w_t+w_t\nu\cdot\nabla w_{tt}]\mathrm{d}(\partial\Omega_t)-\frac{1}{2}\int_{\Omega_t}\frac{\partial}{\partial t}|\nabla w_t|^2\mathrm{d}x, \end{aligned}\tag{3.10}$$

$$\begin{aligned}\int_{\Omega_t}\frac{\partial}{\partial t}\left(|\nabla w|^2+|w|^2\right)\mathrm{d}x &= 2\int_{\Omega_t}\left(\nabla w\cdot\nabla w_t+ww_t\right)\mathrm{d}x\\ &= 2\int_{\partial\Omega_t}w\nu\cdot\nabla w_t\mathrm{d}(\partial\Omega_t)-2\int_{\Omega_t}w(\Delta w_t-w_t)\mathrm{d}x. \end{aligned}\tag{3.11}$$

Hence, by (3.8)–(3.11), it follows that

$$\begin{aligned}&\frac{1}{2}\int_{\Omega_t}\frac{\partial}{\partial t}[|\nabla\Delta w|^2+|\Delta w|^2+|\Delta w_t-w_t|^2+|\nabla w_t|^2+|\nabla w|^2+|w|^2]\mathrm{d}x\\ &=\int_{\partial\Omega_t}[(\Delta w_t-w_t)\nu\cdot\nabla\Delta w+(\Delta w+w)\nu\cdot\nabla w_t]\mathrm{d}(\partial\Omega_t)+\int_{\Omega_t}(q-1)w(\Delta w_t-w_t)\mathrm{d}x. \end{aligned}\tag{3.12}$$

Choosing ℓ such that $\ell\geq 4$, and combining (3.7) and (3.12), we arrive at

$$\frac{\mathrm{d}E(t)}{\mathrm{d}t}\leq\int_{\Omega_t}(q-1)w(\Delta w_t-w_t)\mathrm{d}x. \tag{3.13}$$

In view of (3.13), by noting $q\in L^\infty(0,T;L^p_{\mathrm{loc}}(\Omega))$ with some $p\in[n,\infty]$, $\frac{1}{p}+\frac{1}{2}+\frac{p-2}{2p}=1$, using Hölder's inequality and Sobolev embedding theorem, and recalling the definition of $E(t)$ in (3.4), it is easy to deduce that

$$\frac{\mathrm{d}E(t)}{\mathrm{d}t}\leq\int_{\Omega_t}|(q-1)||w||(\Delta w_t-w_t)|\mathrm{d}x$$

$$
\begin{aligned}
&\le C\|\Delta w_t(t,\cdot)-w_t(t,\cdot)\|_{L^2(\Omega_t)}\|w(t,\cdot)\|_{L^{\frac{2p}{p-2}}(\Omega_t)}\\
&\le C(\|\Delta w_t(t,\cdot)-w_t(t,\cdot)\|^2_{L^2(\Omega_t)}+\|w(t,\cdot)\|^2_{L^2(\Omega_t)})\\
&\le CE(t). \qquad (3.14)
\end{aligned}
$$

Now, the desired estimate (3.5) follows from (3.14), immediately. This completes the proof of Proposition 3.1.

4 Some Preliminaries

In this section, we show some preliminary results, which will play a key role in our proof of the main result, i.e., Theorem 2.1.

Denote the energy of system (1.1) by

$$
\widehat{E}(t) \stackrel{\triangle}{=} \frac{1}{2}(\|\Delta w(t,\cdot)\|^2_{L^2(\Omega)}+\|w_t(t,\cdot)\|^2_{H^1_0(\Omega)}). \tag{4.1}
$$

Consider also the modified energy function

$$
\widehat{E}^*(t) \stackrel{\triangle}{=} \widehat{E}(t)+\frac{1}{2}r^{\frac{2}{2-\frac{n}{p}}}\|w(t,\cdot)\|^2_{L^2(\Omega)}. \tag{4.2}
$$

It is clear that both energies are equivalent.

To begin with, we show the following result.

Lemma 4.1 *Let $q\in L^\infty(0,T;L^p(\Omega))$ with $p\in[n,\infty]$. Then, for any $(w(0),w_t(0))\in (H^2(\Omega)\cap H^1_0(\Omega))\times H^1_0(\Omega)$, the corresponding weak solution $w(\cdot)$ to* (1.1) *satisfies* (*recall* (2.5) *for r*)

$$
\widehat{E}^*(t)\le C\widehat{E}^*(s)\mathrm{e}^{Cr^{\frac{1}{2-\frac{n}{p}}}}, \quad \forall\, t,s\in[0,T]. \tag{4.3}
$$

Proof We proceed as in the proof of Lemma 1 in [16, p. 358] and [2]. Noting system (1.1), it is easy to get

$$
\frac{\partial \widehat{E}^*(t)}{\partial t} = -\int_\Omega qww_t\mathrm{d}x + r^{\frac{2}{2-\frac{n}{p}}}\int_\Omega ww_t\mathrm{d}x \le \Big|\int_\Omega qww_t\mathrm{d}x\Big| + r^{\frac{2}{2-\frac{n}{p}}}\Big|\int_\Omega ww_t\mathrm{d}x\Big|. \tag{4.4}
$$

Put $p_1=\frac{2p}{n-2}$ and $p_2=\frac{2p}{p-n}$. Noting that $\frac{1}{p}+\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{2}=1$ and $\frac{1}{2(\frac{n}{p})^{-1}}+\frac{1}{2(1-\frac{n}{p})^{-1}}+\frac{1}{2}=1$, by Hölder's inequality and Sobolev embedding theorem, we get

$$
\begin{aligned}
\Big|\int_\Omega qww_t\mathrm{d}x\Big| &\le \int_\Omega |q||w||w_t|\mathrm{d}x\\
&\le r\||w(t,\cdot)|^{\frac{n}{p}}\|_{L^{p_1}(\Omega)}\||w(t,\cdot)|^{1-\frac{n}{p}}\|_{L^{p_2}(\Omega)}\|w_t(t,\cdot)\|_{L^2(\Omega)}\\
&= r^{\frac{1}{2-\frac{n}{p}}}\|w(t,\cdot)\|^{\frac{n}{p}}_{L^{\frac{2n}{n-2}}(\Omega)}\big(r^{\frac{1-\frac{n}{p}}{2-\frac{n}{p}}}\|w(t,\cdot)\|^{1-\frac{n}{p}}_{L^2(\Omega)}\big)\|w_t(t,\cdot)\|_{L^2(\Omega)}\\
&\le Cr^{\frac{1}{2-\frac{n}{p}}}\widehat{E}^*(t). \qquad (4.5)
\end{aligned}
$$

Similarly,

$$r^{\frac{2}{2-\frac{n}{p}}}\Big|\int_\Omega ww_t\mathrm{d}x\Big| \le \frac{r^{\frac{1}{2-\frac{n}{p}}}}{2}\int_\Omega (r^{\frac{2}{2-\frac{n}{p}}}|w|^2+|w_t|^2)\mathrm{d}x \le Cr^{\frac{1}{2-\frac{n}{p}}}\widehat{E}^*(t). \tag{4.6}$$

Now, combining (4.4)–(4.6), and applying Gronwall's inequality, we conclude the desired estimate (4.3). This completes the proof of Lemma 4.1.

Further, by using multiplier techniques similar to those in [10], the following lemma holds.

Lemma 4.2 *Let* $0 \le S_1 < S_2 < T_2 < T_1 \le T$ *and* $q \in L^\infty(0,T;L^p(\Omega))$ *with* $p \in [n,\infty]$. *Then the weak solution* $w(\,\cdot\,)$ *of* (1.1) *satisfies*

$$\int_{S_2}^{T_2}\|w_t(t,\,\cdot\,)\|^2_{H_0^1(\Omega)}\mathrm{d}t \le C(1+r)\int_{S_1}^{T_1}\|\Delta w(t,\,\cdot\,)\|^2_{L^2(\Omega)}\mathrm{d}t. \tag{4.7}$$

Proof Put $g(t)=(t-S_1)^2(T_1-t)^2$. Multiplying the first equation of system (1.1) by $g(t)w(t)$ and integrating it in $(S_1,T_1)\times\Omega$, using integration by parts, and by Hölder's inequality and Sobolev embedding theorem, it is easy to see that

$$\begin{aligned}
&\int_{S_1}^{T_1} g(t)\|w_t(t,\,\cdot\,)\|^2_{H_0^1(\Omega)}\mathrm{d}t\\
={}&\int_{S_1}^{T_1}\int_\Omega g_t(t)w_t\Delta w\mathrm{d}x\mathrm{d}t - \int_{S_1}^{T_1}\int_\Omega g_t(t)ww_t\mathrm{d}x\mathrm{d}t\\
&+\int_{S_1}^{T_1}\int_\Omega g(t)|\Delta w|^2\mathrm{d}x\mathrm{d}t + \int_{S_1}^{T_1}\int_\Omega g(t)q|w|^2\mathrm{d}x\mathrm{d}t\\
\le{}&\frac{1}{2}\int_{S_1}^{T_1} g(t)\|w_t(t,\,\cdot\,)\|^2_{H_0^1(\Omega)}\mathrm{d}t + C(1+r)\int_{S_1}^{T_1}\|\Delta w(t,\,\cdot\,)\|^2_{L^2(\Omega)}\mathrm{d}t.
\end{aligned} \tag{4.8}$$

Thus, we get (4.7), which completes the proof of Lemma 4.2.

5 Proof of Theorem 2.1

We are now in a position to prove Theorem 2.1. Without loss of generality, we may assume $|a|<1$ (recall that a is defined in (2.1)). The proof is divided into several steps.

Step 1 To begin with, put

$$q_1(t,x)=\begin{cases} q(t,x), & \text{if } (t,x)\in Q,\\ 0, & \text{if } (t,x)\in((-\infty,0)\cup(T,\infty))\times\Omega.\end{cases} \tag{5.1}$$

Assume that (W,Z) satisfy the following system:

$$\begin{cases} W-\Delta W = Z, & \text{in } \mathbb{R}\times\Omega,\\ Z_{tt}-\Delta Z+W-Z+q_1W=0, & \text{in } \mathbb{R}\times\Omega,\\ W=Z=0, & \text{on } \mathbb{R}\times\Gamma,\\ Z(0)=w_0-\Delta w_0,\ Z_t(0)=w_1-\Delta w_1, & \text{in } \Omega.\end{cases} \tag{5.2}$$

By (5.1), it is clear that $q_1 \equiv q$ in Q. Hence, comparing system (5.2) with system (2.7), we find

$$(W, Z) \equiv (w, z), \quad \text{in } Q. \tag{5.3}$$

We introduce the following coordinate transformation:

$$\begin{cases} t = \overline{t} + a\overline{x}_1, \\ x = \overline{x}. \end{cases} \tag{5.4}$$

Set

$$(\widetilde{w}, \widetilde{z})(\overline{t}, \overline{x}) = (W, Z)(t, x)(= (W, Z)(\overline{t} + a\overline{x}_1, \overline{x})), \quad (\overline{t}, \overline{x}) \in \mathbb{R} \times \Omega. \tag{5.5}$$

Then, by (5.2), $(\widetilde{w}, \widetilde{z})$ solve the following equation:

$$\begin{cases} \widetilde{w} - a^2 \widetilde{w}_{\overline{t}\overline{t}} + 2a\widetilde{w}_{\overline{t}\overline{x}_1} - \sum\limits_i \widetilde{w}_{\overline{x}_i\overline{x}_i} = \widetilde{z}, & \text{in } \mathbb{R} \times \Omega, \\ (1 - a^2)\widetilde{z}_{\overline{t}\overline{t}} + 2a\widetilde{z}_{\overline{t}\overline{x}_1} - \sum\limits_i \widetilde{z}_{\overline{x}_i\overline{x}_i} + \widetilde{w} - \widetilde{z} + q_2\widetilde{w} = 0, & \text{in } \mathbb{R} \times \Omega, \\ \widetilde{w} = \widetilde{z} = 0, & \text{on } \mathbb{R} \times \Gamma, \end{cases} \tag{5.6}$$

where $q_2(\overline{t}, \overline{x}) = q_1(\overline{t} + a\overline{x}_1, \overline{x})$, $\sum\limits_i = \sum\limits_i^n$.

Next, for any given $s \in \mathbb{R}$, denote

$$v(\overline{t}, \overline{x}) = \int_s^{\overline{t}} \widetilde{z}(\tau, \overline{x})\mathrm{d}\tau + \chi(\overline{x}), \tag{5.7}$$

with χ satisfying

$$\begin{cases} \chi(\overline{x}) + \sum\limits_i \chi_{\overline{x}_i\overline{x}_i} = (1 - a^2)\widetilde{z}_{\overline{t}}(s) + 2a\widetilde{z}_{\overline{x}_1}(s), & \text{in } \Omega, \\ \chi = 0, & \text{on } \Gamma. \end{cases} \tag{5.8}$$

It is easy to check that

$$\begin{cases} v_{\overline{t}} = \widetilde{w} - a^2\widetilde{w}_{\overline{t}\overline{t}} + 2a\widetilde{w}_{\overline{t}\overline{x}_1} - \sum\limits_i \widetilde{w}_{\overline{x}_i\overline{x}_i}, & \text{in } \mathbb{R} \times \Omega, \\ (1 - a^2)v_{\overline{t}\overline{t}} + 2av_{\overline{t}\overline{x}_1} - \sum\limits_i v_{\overline{x}_i\overline{x}_i} + \displaystyle\int_s^{\overline{t}} \widetilde{w}(\tau, \overline{x})\mathrm{d}\tau & \\ \quad -v + \displaystyle\int_s^{\overline{t}} q_2(\tau, \overline{x})\widetilde{w}(\tau, \overline{x})\mathrm{d}\tau = 0, & \text{in } \mathbb{R} \times \Omega, \\ \widetilde{w} = v = 0, & \text{on } \mathbb{R} \times \Gamma, \\ v(s) = \chi(\overline{x}), \ v_{\overline{t}}(s) = \widetilde{z}(s), & \text{in } \Omega. \end{cases} \tag{5.9}$$

Step 2 Next, we introduce some energy estimate. First, denote the energy function of the system (5.9) by

$$\mathcal{E}(\overline{t}) \triangleq \frac{1}{2}[\|v(\overline{t}, \cdot)\|^2_{H_0^1(\Omega)} + (1 - a^2)\|v_{\overline{t}}(\overline{t}, \cdot)\|^2_{L^2(\Omega)}]. \tag{5.10}$$

Multiplying the second equation of system (5.9) by $v_{\overline{t}}$ and integrating it in Ω, using integration by parts, we have

$$\frac{\mathrm{d}\,\mathcal{E}(\overline{t})}{\mathrm{d}t} = -\int_\Omega \Big(\int_s^{\overline{t}} (1 + q_2(\tau, \overline{x}))\widetilde{w}(\tau, \overline{x})\mathrm{d}\tau\Big) v_{\overline{t}}(\overline{t}, \overline{x})\mathrm{d}\overline{x} + \int_\Omega vv_{\overline{t}}\mathrm{d}\overline{x}. \tag{5.11}$$

Using the first equation in (5.2) and integrating by part, one obtains

$$\|W(t,\cdot)\|^2_{H_0^1(\Omega)} \le C\|Z(t,\cdot)\|^2_{L^2(\Omega)}, \quad \forall\, t \in \mathbb{R}. \tag{5.12}$$

Then, by (5.5), (5.7) and (5.12), we could also get

$$\|\widetilde{w}(\overline{t},\cdot)\|^2_{L^2(\Omega)} \le C\|v_{\overline{t}}(\overline{t},\cdot)\|^2_{L^2(\Omega)}, \quad \forall\, \overline{t} \in \mathbb{R}. \tag{5.13}$$

Similarly to the proof of Lemma 4.1, by Hölder's inequality and Sobolev embedding theorem, and noting (5.13), we conclude that

$$\mathcal{E}(\overline{t}) \le Cr\mathcal{E}(s), \quad \forall\, s, \overline{t} \in [-2T, 2T]. \tag{5.14}$$

Define

$$\begin{cases} \mathcal{D}(A) = H^2(\Omega) \cap H_0^1(\Omega), \\ Az = -\Delta z, \quad \forall\, z \in \mathcal{D}(A). \end{cases} \tag{5.15}$$

Fix s_0 and s_0' such that

$$0 < s_0 < s_0' < \sigma, \tag{5.16}$$

and put $\psi(\overline{t}) = \overline{t}^2(\sigma - \overline{t})^2$. By (5.5), noting (5.6) and (5.13), and using integration by parts, we get

$$\begin{aligned}
&\int_0^\sigma \int_\Omega (1+q_2)\widetilde{w}\psi(A^{-1}\widetilde{z})\mathrm{d}\overline{x}\mathrm{d}\overline{t} \\
&= \int_0^\sigma \int_\Omega \Big(-(1-a^2)\widetilde{z}_{\overline{t}\overline{t}} - 2a\widetilde{z}_{\overline{t}\overline{x}} + \sum_i \widetilde{z}_{\overline{x}_i\overline{x}_i} + \widetilde{z}\Big)\psi(A^{-1}\widetilde{z})\mathrm{d}\overline{x}\mathrm{d}\overline{t} \\
&= (1-a^2)\int_0^\sigma \psi\|\widetilde{z}_{\overline{t}}(\overline{t},\cdot)\|^2_{H^{-1}(\Omega)}\mathrm{d}\overline{t} - \frac{1-a^2}{2}\int_0^\sigma\int_\Omega \psi_{\overline{t}\overline{t}}(A^{-\frac12}\widetilde{z})^2\mathrm{d}\overline{x}\mathrm{d}\overline{t} \\
&\quad + 2a\int_0^\sigma\int_\Omega [\psi(A^{-\frac12}\widetilde{z}_{\overline{x}_1})(A^{-\frac12}\widetilde{z}_{\overline{t}}) + (A^{-\frac12}\widetilde{z}_{\overline{x}_1})\psi_{\overline{t}}(A^{-\frac12}\widetilde{z})]\mathrm{d}\overline{x}\mathrm{d}\overline{t} \\
&\quad - \int_0^\sigma\int_\Omega \psi\widetilde{z}^2\mathrm{d}\overline{x}\mathrm{d}\overline{t} + \int_0^\sigma\int_\Omega \psi(A^{-\frac12}\widetilde{z})^2\mathrm{d}\overline{x}\mathrm{d}\overline{t}.
\end{aligned} \tag{5.17}$$

Further, it is obvious that

$$\int_\Omega |A^{-\frac12}\widetilde{z}_{\overline{x}_1}|^2\mathrm{d}\overline{x} \le C\|\widetilde{z}_{\overline{x}_1}(\overline{t},\cdot)\|^2_{H^{-1}(\Omega)} \le C\|\widetilde{z}(\overline{t},\cdot)\|^2_{L^2(\Omega)}. \tag{5.18}$$

Combining (5.17) and (5.18), we deduce

$$\int_0^\sigma \|\widetilde{z}_{\overline{t}}(\overline{t},\cdot)\|^2_{H^{-1}(\Omega)}\mathrm{d}\overline{t} \le C(1+r)\int_0^\sigma \|\widetilde{z}(\overline{t},\cdot)\|^2_{L^2(\Omega)}\mathrm{d}\overline{t}. \tag{5.19}$$

Step 3 We finish the proof of Theorem 2.1 in this step.

By (4.1)–(4.3), (4.7), (5.2)–(5.7) and (5.10), we arrive at

$$\|\Delta w_0\|^2_{L^2(\Omega)} + \|w_1\|^2_{H_0^1(\Omega)}$$

$$
\begin{aligned}
&\le C(1+r)\mathrm{e}^{Cr^{\frac{1}{2-\frac{n}{p}}}}\int_{\varepsilon T}^{(1-\varepsilon)T}(\|\Delta w(t,\cdot)\|^2_{L^2(\Omega)}+\|w_t(t,\cdot)\|^2_{H_0^1(\Omega)})\mathrm{d}t\\
&\le C(1+r)\mathrm{e}^{Cr^{\frac{1}{2-\frac{n}{p}}}}\int_0^T\|\Delta w(t,\cdot)\|^2_{L^2(\Omega)}\mathrm{d}t\\
&\le C\mathrm{e}^{Cr^{\frac{1}{2-\frac{n}{p}}}}\int_0^T\|z(t,\cdot)\|^2_{L^2(\Omega)}\mathrm{d}t\\
&\le C\mathrm{e}^{Cr^{\frac{1}{2-\frac{n}{p}}}}\int_{-2T}^{2T}\|\widetilde{z}(\overline{t},\cdot)\|^2_{L^2(\Omega)}\mathrm{d}\overline{t}\\
&\le C\mathrm{e}^{Cr^{\frac{1}{2-\frac{n}{p}}}}\int_{-2T}^{2T}\|v_{\overline{t}}(\overline{t},\cdot)\|^2_{L^2(\Omega)}\mathrm{d}\overline{t}\\
&\le C\mathrm{e}^{Cr^{\frac{1}{2-\frac{n}{p}}}}\int_{-2T}^{2T}\varepsilon(\overline{t})\mathrm{d}\overline{t}. \qquad (5.20)
\end{aligned}
$$

Combining (5.5), (5.10), (5.12), (5.14) and (5.20), for any $s\in[-2T,2T]$, one finds

$$
\|\Delta w_0\|^2_{L^2(\Omega)}+\|w_1\|^2_{H_0^1(\Omega)}\le C\mathrm{e}^{Cr^{\frac{1}{2-\frac{n}{p}}}}(\|\widetilde{z}(s,\cdot)\|^2_{L^2(\Omega)}+\|\widetilde{z}_{\overline{t}}(s,\cdot)\|^2_{H^{-1}(\Omega)}). \qquad (5.21)
$$

Integrating the above inequality with respect to s from s_0 to s_0' (with s_0 and s_0' satisfying (5.16)), and by (5.19), we conclude that

$$
\|\Delta w_0\|^2_{L^2(\Omega)}+\|w_1\|^2_{H_0^1(\Omega)}\le C\mathrm{e}^{Cr^{\frac{1}{2-\frac{n}{p}}}}\int_0^{\sigma}\|\widetilde{z}(\overline{t},\cdot)\|^2_{L^2(\Omega)}\mathrm{d}\overline{t}. \qquad (5.22)
$$

Hence, recalling (2.6), (5.3) and (5.5), and using assumption (A1), we obtain

$$
\begin{aligned}
\|\Delta w_0\|^2_{L^2(\Omega)}+\|w_1\|^2_{H_0^1(\Omega)}&\le C\mathrm{e}^{Cr^{\frac{1}{2-\frac{n}{p}}}}\int_{D_\sigma}|z(t,x)|^2\mathrm{d}x\mathrm{d}t\\
&\le C\mathrm{e}^{Cr^{\frac{1}{2-\frac{n}{p}}}}\int_0^T\int_{G(t)}(|w|^2+|\Delta w|^2)\mathrm{d}x\mathrm{d}t, \qquad (5.23)
\end{aligned}
$$

which yields the desired estimate (1.2). This completes the proof of Theorem 2.1.

Acknowledgement The author thanks Professor Xu Zhang for his help.

References

[1] Bardos, C., Lebeau, G. and Rauch, J., Sharp sufficient conditions for the observation, control and stabilization of waves from the boundary, *SIAM J. Control Optim.*, **30**(5), 1992, 1024–1065.

[2] Duyckaerts, T., Zhang, X. and Zuazua, E., On the optimality of the observability inequalities for parabolic systems with potentials, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **25**(1), 2008, 1–41.

[3] Fu, X., A weighted identity for partial differential operators of second order and its applications, *C. R. Math. Acad. Sci. Paris*, **342**(8), 2006, 579–584.

[4] Fu, X., Yong, J. and Zhang, X., Controllability and observability of the heat equations with hyperbolic memory kernel, *J. Diff. Eqns.*, **247**(8), 2009, 2395–2439.

[5] Fursikov, A. V. and Imanuvilov, O. Y., Controllability of Evolution Equations, Lecture Notes Series, **34**, Research Institute of Mathematics, Seoul National University, Seoul, Korea, 1994.

[6] Lasiecka, I., Triggiani, R. and Zhang, X., Global uniqueness, observability and stabilization of nonconservative Schrödinger equations via pointwise Carleman estimates, Part I: $H^1(\Omega)$-estimates, *J. Inverse and Ill-Posed Problems*, **12**(1), 2004, 43–123.

[7] Li, D., Rapid exact controllability of semilinear hyperbolic equations, *J. Sichuan Univ.* (*Nat. Sci. Ed.*), to appear.

[8] Li, T. T., Controllability and Observability for Quasilinear Hyperbolic Systems, Springer-Verlag, New York, 2009.

[9] Li, X. and Yong, J., Optimal Control Theory for Infinite-Dimensional Systems, Systems & Control: Foundations & Applications, Birkhäuser, Boston, 1995.

[10] Lions, J.-L., Contrôlabilité exacte, perturbations et stabilisation de systèmes distribués, Tome 1, Recherches en Mathématiques Appliquées, **8**, Masson, Paris, 1988.

[11] Liu, K. and Yong, J., Rapid exact controllability of the wave equation by controls distrubuted on a time-variant subdomain, *Chin. Ann. Math.*, **20B**(1), 1999, 65–76.

[12] Yong, J. and Zhang, X., Heat equation with memory in anisotropic and non-homogeneous media, preprint.

[13] Zhang, X., Rapid exact controllability of the semilinear wave equation, *Chin. Ann. Math.*, **20B**(3), 1999, 377–384.

[14] Zhang, X., Exact internal controllability of Maxwell equations, *Appl. Math. Optim.*, **41**(2), 2000, 155–170.

[15] Zhang, X., Exact Controllability of Semi-linear Distributed Parameter Systems (in Chinese), Higher Education Press, Beijing, 2004.

[16] Zhang, X. and Zuazua, E., A sharp observability inequality for Kirchhoff plate systems with potentials, *Comput. Appl. Math.*, **25**(2–3), 2006, 353–373.

[17] Zuazua, E., Contrôlabilité exacte d'un modèle de plaques vibrantes en un temps arbitrairement petit, *C. R. Math. Acad. Sci. Paris*, **304**(7), 1987, 173–176.

[18] Zuazua, E., Controllability and observability of partial differential equations: some results and open problems, Handbook of Differential Equations: Evolutionary Equations, Vol. 3, Elsevier Science, 2006, 527–621.