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Hopf Superquivers*

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Abstract In this paper, a super version of the Hopf quiver theory is developed. The notion of Hopf superquivers is introduced. It is shown that only the path supercoalgebras of Hopf superquivers admit graded Hopf superalgebra structures. A complete classification of such graded Hopf superalgebras is given. A superquiver setting for general pointed Hopf superalgebras is also built up. In particular, a super version of the Gabriel type theorem and the Cartier-Gabriel decomposition theorem is given.

Keywords Hopf superalgebra, Superquiver, Path supercoalgebra
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1 Introduction

Hopf superalgebras (see, e.g., [12, 15]) are natural generalizations of supergroups and Lie superalgebras and play important roles in various branches of mathematics and physics. The problem of their classification and representation has been an active research theme for decades.

As well-known, quiver methods are very useful in constructing algebras and studying their representations (see [2] and references therein). Very nice quiver settings for the usual Hopf algebras were built in various works (see [5, 6, 8, 19]) and shown their advantage in studying some interesting classes of Hopf algebras as well as their representations (see for instance [3, 17] among many other related works). It seems of interest to extend the Hopf quiver theory to the super situation.

Our main aim is to provide a super version of the Hopf quiver theory and, by taking advantage of it, to contribute to the classification and representation theory of Hopf superalgebras. As Hopf superalgebras are compatible combination of superalgebras and supercoalgebras, suitable quiver settings for superalgebras and supercoalgebras are necessary for our object. A recent work of Han and Zhao [10] gave a nice superquiver setting for superalgebras. What we are going to use is a superquiver setting for supercoalgebras, which is dual to [10] and can be seen as a super version of [4].

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We start by introducing the notion of Hopf superquivers. We show that the path supercoalgebra of a given superquiver admits graded Hopf superalgebra structures if and only if it is a Hopf superquiver. A complete classification of graded Hopf superalgebra structures on a given Hopf superquiver is given via collections of group representations. Then we give a Gabriel type theorem for coradically graded pointed Hopf superalgebras, as well as a super analogue of the Cartier-Gabriel decomposition theorem. This provides a superquiver setting for general pointed Hopf superalgebras. Finally, we give some examples to elucidate our approaches. In particular, the graded version of the quantum supergroup $U_q(\mathrm{osp}(1|2))$ is presented via a Hopf superquiver.

Throughout the paper, we work over a field k. Vector spaces, superalgebras, supercoalgebras, linear mappings and unadorned $\otimes$ are over k. The readers are referred to [16, 14] for general knowledge of Hopf algebras and Hopf superalgebras, and to [2] for that of quivers and their applications to algebras and representation theory.

2 Hopf Superalgebras and Superquivers

This section is devoted to some preliminaries.

2.1 Hopf superalgebras

Let $\mathbb{Z}_2 = \{\overline{0}, \overline{1}\}$ denote the cyclic group of order 2. A super space is $\mathbb{Z}_2$-graded vector space $V = V_{\overline{0}} \oplus V_{\overline{1}}$. Let $\mathfrak{p}$ denote the parity map, namely, $\mathfrak{p}(v) = \overline{0}$ if $v \in V_{\overline{0}}$ is even and $\mathfrak{p}(v) = \overline{1}$ if $v \in V_{\overline{1}}$ is odd. A superalgebra is a $\mathbb{Z}_2$-graded algebra $A = A_{\overline{0}} \oplus A_{\overline{1}}$, that is, if $a \in A_\mu,\ b \in A_\nu,\ \mu, \nu \in \mathbb{Z}_2$, then $ab \in A_{\mu+\nu}$. The unit element 1_A, written by 1 if no confusion should cause, lies in $A_{\overline{0}}$. A supercoalgebra is, likewise, a $\mathbb{Z}_2$-graded space $C = C_{\overline{0}} \oplus C_{\overline{1}}$ such that $\Delta(c) \in \bigoplus_{\mu+\nu=\sigma} C_\mu \otimes C_\nu$ if $c \in C_\sigma$. The counit ε vanishes on the odd space $C_{\overline{1}}$.

A superbialgebra means a superalgebra and supercoalgebra B such that the comultiplication and the counit are even superalgebra homomorphisms, or equivalently the multiplication and the unit maps are even supercoalgebra homomorphisms. In particular, the comultiplication Δ satisfies the condition

$$\Delta(ab) = (-1)^{\mathfrak{p}(a_2)\mathfrak{p}(b_1)} a_1 b_1 \otimes a_2 b_2. \tag{2.1}$$

Here and hereafter, we use the Sweedler sigma notation $\Delta(a) = a_1 \otimes a_2$ for the coproduct. A Hopf superalgebra is a superbialgebra H having an antipode $\mathcal{S} : H \to H$ which is a degree preserving linear map obeying the usual axioms

$$\mathcal{S}(a_1)a_2 = \varepsilon(a) = a_1\mathcal{S}(a_2), \quad \forall\, a \in H. \tag{2.2}$$

We remark that superalgebras, supercoalgebras, superbialgebras and Hopf superalgebras can be understood as algebras, coalgebras, bialgebras and Hopf algebras in the symmetric category of supervector spaces (see [14]).

2.2 Superquivers

Recall that, a quiver is a quadruple $Q = (Q_0, Q_1, s, t)$, where Q_0 is the set of vertices, Q_1 is the set of arrows, and $s, t :\ Q_1 \to Q_0$ are two maps assigning respectively the source and the target for each arrow. A path of length $l \geq 1$ in the quiver Q is a finitely ordered sequence of l arrows $a_l \cdots a_1$ such that $s(a_{i+1}) = t(a_i)$ for $1 \leq i \leq l-1$. By convention a vertex is said to be a trivial path of length 0.

A superquiver is a pair $(Q, \mathfrak{p})$ where Q is a quiver and $\mathfrak{p}$ is a parity map $\mathfrak{p} : Q_1 \to \mathbb{Z}_2$. Let $Q_{1,\overline{0}}$ (resp. $Q_{1,\overline{1}}$) denote $\mathfrak{p}^{-1}(\overline{0})$ (resp. $\mathfrak{p}^{-1}(\overline{1})$) and call it the set of even (resp. odd) arrows. To exhibit the parity of arrows intuitively on quivers, one can assign to the arrows with different colors, say black for the even and red for the odd. Hence superquivers also appear as colored quivers in literature. In this paper, we adapt the convention of Han and Zhao [10] in which even arrows are set to be solid and odd arrows dotted.

Given a superquiver $(Q, \mathfrak{p})$, we can associate to it a natural supercoalgebra structure as follows. Let kQ be the k-space spanned by the set of paths (defined as the usual quivers). The counit and the comultiplication maps are defined by $\varepsilon(g) = 1,\ \Delta(g) = g \otimes g$ for each $g \in Q_0$, and for each nontrivial path $p = a_n \cdots a_1,\ \varepsilon(p) = 0$,

$$\Delta(a_n \cdots a_1) = p \otimes s(a_1) + \sum_{i=1}^{n-1} a_n \cdots a_{i+1} \otimes a_i \cdots a_1 + t(a_n) \otimes p.$$

For the parity map, we extend the map $\mathfrak{p} : Q_1 \to \mathbb{Z}_2$ by setting $\mathfrak{p}(p) = \sum_{i=1}^{n} \mathfrak{p}(a_i)$ for each path $p = a_n \cdots a_1$. It is routine to verify that, with comultiplication, counit and parity maps defined as above, kQ becomes a supercoalgebra. This is called the associated path supercoalgebra of the superquiver $(Q, \mathfrak{p})$ and denoted by $(kQ, \mathfrak{p})$.

Note that, if we ignore the parity, then kQ with the same comultiplication and counit is the usual path coalgebra of Q (see [4, 6]). Moreover, the length of paths gives naturally a positive gradation to the path supercoalgebra. Let Q_n denote the set of paths of length n in Q. Then $kQ = \bigoplus_{n \geq 0} kQ_n$ and $\Delta(kQ_n) \subseteq \bigoplus_{n=i+j} kQ_i \otimes kQ_j$. Clearly kQ has the following coradical filtration

$$kQ_0 \subseteq kQ_0 \oplus kQ_1 \subseteq kQ_0 \oplus kQ_1 \oplus kQ_2 \subseteq \cdots .$$

Hence kQ is coradically graded.

2.3 Pointed supercoalgebras

A supercoalgebra is said to be pointed, if it is a pointed coalgebra (see [16]) when we ignore its super structure. Clearly, path supercoalgebras are pointed. The Gabriel type theorem for pointed coalgebras due to Chin and Montgomery [4] can be generalized to the super situation without difficulty.

Proposition 2.1 *Assume that C is a pointed supercoalgebra. Then there exist a unique superquiver $(Q, \mathfrak{p})$ and a supercoalgebra embedding $\mathrm{I} : C \hookrightarrow kQ$ such that $\mathrm{I}(C) \supseteq kQ_0 \oplus kQ_1$.*

A Hopf superalgebra is called pointed, if its underlying supercoalgebra is so. Let H be a pointed Hopf superalgebra. Denote its coradical filtration by $\{H_n\}_{n=0}^{\infty}$. Define

$$\mathrm{gr}(H) = H_0 \oplus H_1/H_0 \oplus H_2/H_1 \oplus \cdots$$

as the corresponding coradically graded supercoalgebra. Then one can show that $\mathrm{gr}(H)$ inherits from H a coradically graded Hopf superalgebra structure. See [16, Lemma 5.2.8] for a proof of the case of the usual Hopf algebras, in which the arguments can be easily extended to the super case.

2.4 Hopf quivers

According to Cibils and Rosso [6], a quiver Q is said to be a Hopf quiver if the corresponding path coalgebra kQ admits a graded Hopf algebra structure. Hopf quivers can be determined by ramification data of groups. Let G be a group, $\mathcal{C}$ the set of conjugacy classes. A ramification datum R of the group G is a formal sum $\sum\limits_{C\in\mathcal{C}} R_C C$ of conjugacy classes with coefficients in $\mathbb{N}=\{0,1,2,\cdots\}$. The corresponding Hopf quiver $Q=Q(G,R)$ is defined as follows: the set of vertices Q_0 is G, and for each $x\in G$ and $c\in C$, there are R_C arrows going from x to cx. For a given Hopf quiver Q, the set of graded Hopf structures on kQ is in one-to-one correspondence with the set of kQ_0-Hopf bimodule structures on kQ_1.

The graded Hopf structures are obtained from Hopf bimodules via the quantum shuffle product (see [18]). Suppose that Q is a Hopf quiver with a necessary kQ_0-Hopf bimodule structure on kQ_1. Let $p\in Q_l$ be a path. An n-thin split of it is a sequence $(p_1,\ \cdots,\ p_n)$ of vertices and arrows such that the concatenation $p_n\cdots p_1$ is exactly p. These n-thin splits are in one-to-one correspondence with the n-sequences of $(n-l)$ 0's and l 1's. Denote the set of such sequences by D_l^n. Clearly $|D_l^n|=\binom{n}{l}$. For $d=(d_1,\ \cdots,\ d_n)\in D_l^n$, the corresponding n-thin split is written as $dp=((dp)_1,\ \cdots,\ (dp)_n)$, in which $(dp)_i$ is a vertex if $d_i=0$ and an arrow if $d_i=1$. Let $\alpha=a_m\cdots a_1$ and $\beta=b_n\cdots b_1$ be paths of length m and n respectively. Let $d\in D_m^{m+n}$ and $\overline{d}\in D_n^{m+n}$ the complement sequence which is obtained from d by replacing each 0 by 1 and each 1 by 0. Define an element

$$(\alpha\cdot\beta)_d=[(d\alpha)_{m+n}.(\overline{d}\beta)_{m+n}]\cdots[(d\alpha)_1.(\overline{d}\beta)_1] \tag{2.3}$$

in Q_{m+n}, where $[(d\alpha)_i.(\overline{d}\beta)_i]$ is understood as the action of kQ_0-Hopf bimodule on kQ_1 and these terms in different brackets are put together linearly by concatenation. In terms of these notations, the formula of the product of α and β is given as follows:

$$\alpha\cdot\beta=\sum_{d\in D_m^{m+n}}(\alpha\cdot\beta)_d. \tag{2.4}$$

3 Hopf Superquivers

In this section, we determine those superquivers whose path coalgebras admit graded Hopf superalgebra structures. In addition, we give a complete classification of such structures via representations of groups.

3.1 Firstly, we generalized the notion of Hopf quivers to fit for the super situation. Let G be a group and $\mathcal{C}$ the set of its conjugacy classes. A super ramification datum R of G is a pair of formal sums

$$\Big(R_{\overline{0}}=\sum_{C\in\mathcal{C}}R_{C,\overline{0}}C,\ R_{\overline{1}}=\sum_{C\in\mathcal{C}}R_{C,\overline{1}}C\Big)$$

of conjugacy classes with non-negative integer coefficients. We call $R_{\overline{0}}$ (resp. $R_{\overline{1}}$) the even (resp. odd) part of R.

Definition 3.1 *Let G be a group and $R = (R_{\overline{0}}, R_{\overline{1}})$ be a super ramification datum with $R_{\overline{0}} = \sum_{C\in\mathcal{C}} R_{C,\overline{0}}C$ and $R_{\overline{1}} = \sum_{C\in\mathcal{C}} R_{C,\overline{1}}C$. The associated Hopf superquiver $(Q(G, R_{\overline{0}}, R_{\overline{1}}), \mathfrak{p})$ is defined as follows:*

(1) *the set of vertices is G;*

(2) *for each $x \in G$ and $c \in C$, there are $R_{C,\overline{0}}$ even arrows and $R_{C,\overline{1}}$ odd arrows with x as source and cx as target.*

Clearly, a Hopf superquiver is a usual Hopf quiver if we neglect its parity map. By the definition, the even (resp. odd) part of the super ramification datum corresponds to the set of even (resp. odd) arrows of the associated Hopf superquiver.

3.2 Let $(Q, \mathfrak{p})$ be a superquiver and $(kQ, \mathfrak{p})$ the associated path supercoalgebra. If there exists a superalgebra structure which respects the length-gradation of kQ and is compatible with the path supercoalgebra, then we say that the path supercoalgebra $(kQ, \mathfrak{p})$ admits a graded superbialgebra structure. If, in addition, the superbialgebra admits a gradation-preserving antipode, then we say $(kQ, \mathfrak{p})$ admits a graded Hopf superalgebra structure.

The following is a super analogue of a theorem of Cibils and Rosso (see [6, Theorem 3.3]).

Proposition 3.1 *Let $(Q, \mathfrak{p})$ be a superquiver. Then the path supercoalgebra $(kQ, \mathfrak{p})$ admits a graded Hopf superalgebra structure if and only if $(Q, \mathfrak{p})$ is a Hopf superquiver of some group with respect to a super ramification datum.*

Proof Firstly assume that the path supercoalgebra $(kQ, \mathfrak{p})$ admits a graded Hopf superalgebra structure. Then the degree 0 part kQ_0 has a sub Hopf superalgebra structure. It is in fact a usual Hopf algebra since by assumption the elements of degree 0 are even. Note that the elements of Q_0 are group-like, hence the set Q_0 has a group structure, denoted by G, and kQ_0 is the corresponding group algebra kG. Clearly the graded Hopf superalgebra structure on kQ induces kG-Hopf bimodule structures on $kQ_{1,\overline{0}}$ and $kQ_{1,\overline{1}}$. Now use the proof of Theorem 3.3 in [6], one can conclude that there is a super ramification datum R of the group G such that $(Q, \mathfrak{p}) = Q(G, R, \mathfrak{p})$.

Conversely, assume that $(Q, \mathfrak{p})$ is a Hopf superquiver, say $Q = Q(G, R_{\overline{0}}, R_{\overline{1}})$. Let M denote kQ_1 and write $M_{\overline{0}} = kQ_{1,\overline{0}}$, $M_{\overline{1}} = kQ_{1,\overline{1}}$. Now choose kG-Hopf bimodule structures on $M_{\overline{0}}$ and $M_{\overline{1}}$. According to [5, Proposition 3.3] and [6, Theorem 3.3], this is always possible. Now by extending Theorem 3.8 in [6] to our super setting, it is not hard to give a graded Hopf superalgebra structure on kQ. For completeness, we give the construction in more detail. Define the unit map $u : k \to kQ$ by $\lambda \mapsto \lambda 1_G$. For a pair of paths α and β of length m and n respectively, the multiplication is defined by

$$\alpha \cdot \beta = \sum_{d\in D_m^{m+n}} (-1)^{\sum_{i=1}^{m+n-1} \mathfrak{p}((d\alpha)_i) \sum_{j=i+1}^{m+n} \mathfrak{p}((\overline{d}\beta)_j)} (\alpha \cdot \beta)_d. \tag{3.1}$$

Here we use the notations of Subsection 2.4. Finally, we use induction on the length of paths to define the antipode map $\mathcal{S} : kQ \to kQ$ as follows: for each $g \in Q_0$, set $\mathcal{S}(g) = g^{-1}$; for each $a \in Q_1$, set $\mathcal{S}(a) = -t(a)^{-1} \cdot a \cdot s(a)^{-1}$; for a general path $p = a_n \cdots a_1$ of length $n \geq 2$, set inductively

$$\mathcal{S}(p) = -t(a_n)^{-1} \cdot p \cdot s(a_1)^{-1} - \sum_{i=1}^{n-1} t(a_n)^{-1} \cdot a_n \cdots a_{i+1} \cdot \mathcal{S}(a_i \cdots a_1). \tag{3.2}$$

One can show that $M,\ u,\ S$ are compatible multiplication, unit and antipode maps which make the path supercoalgebra $(kQ, \mathfrak{p})$ a graded Hopf superalgebra.

The proof is completed.

It is of interest to note that our multiplication formula (3.1) is exactly a super version of the usual one (2.4).

3.3 Let $(Q, \mathfrak{p}) = Q(G, R_{\overline{0}}, R_{\overline{1}})$ be a Hopf superquiver. The proof of Proposition 3.1 also implies that the set of graded Hopf structures on the path supercoalgebra $(kQ, \mathfrak{p})$ is in one-to-one correspondence with the set of pairs of kG-Hopf bimodules on $kQ_{1,\overline{0}}$ and $kQ_{1,\overline{1}}$.

Now we are ready to give a complete classification of graded Hopf superalgebra structures on the path supercoalgebras of Hopf superquivers.

Corollary 3.1 *Let G be a group, $\mathcal{C}$ its set of conjugacy classes, and for each $C \in \mathcal{C}$, let Z_C denote the centralizer of one of its elements. Let $R = (R_{\overline{0}}, R_{\overline{1}})$ be a super ramification datum of G with $R_{\overline{0}} = \sum_{C\in\mathcal{C}} R_{C,\overline{0}}C$ and $R_{\overline{1}} = \sum_{C\in\mathcal{C}} R_{C,\overline{1}}C$. Denote the associated Hopf superquiver $(Q(G, R_{\overline{0}}, R_{\overline{1}}), \mathfrak{p})$ as $(Q, \mathfrak{p})$. Then the set of graded Hopf superalgebra structures on $(kQ, \mathfrak{p})$ with $Q_0 \cong G$ as groups is in one-to-one correspondence with the set of pairs of collections $\{(V_{C,\overline{0}})_{C\in\mathcal{C}}, (V_{C,\overline{1}})_{C\in\mathcal{C}}\}$ in which $V_{C,\overline{0}}$ (resp. $V_{C,\overline{1}}$), is a kZ_C-module of dimension $R_{C,\overline{0}}$ (resp. $R_{C,\overline{1}}$) for all $C \in \mathcal{C}$.*

Proof By Proposition 3.1, the classification of graded Hopf superalgebras on $(kQ, \mathfrak{p})$ with $Q_0 \cong G$ is equivalent to that of kG-Hopf bimodules on $kQ_{1,\overline{0}}$ and $kQ_{1,\overline{1}}$. For this, we make use of the classification result of Hopf bimodules over group algebras in [5] due to Cibils and Rosso. Recall that, for a group G, the category of kG-Hopf bimodules is equivalent to the product of the categories of usual module categories $\prod_{C\in\mathcal{C}} kZ_C$-mod, where $\mathcal{C}$ is the set of conjugacy classes and Z_C is the centralizer of one of the elements in the class $C \in \mathcal{C}$. By their correspondence, the set of kG-Hopf bimodules on $kQ_{1,\overline{0}}$ (resp. $kQ_{1,\overline{1}}$) should associate with the collection $(V_{C,\overline{0}})_{C\in\mathcal{C}}$ (resp. $(V_{C,\overline{1}})_{C\in\mathcal{C}}$) in which $V_{C,\overline{0}}$ (resp. $V_{C,\overline{1}}$) is a kZ_C-module of dimension $R_{C,\overline{0}}$ (resp. $R_{C,\overline{1}}$) for all $C \in \mathcal{C}$. Now we are done.

4 Superquiver Setting for Pointed Hopf Superalgebras

The aim of this section is to build up a superquiver setting for general pointed Hopf superalgebras. In particular, we give a super version of the Gabriel type theorem and the Cartier-Gabriel decomposition theorem.

4.1 It was shown in [19, Theorem 4.5] that there is a Gabriel type theorem for pointed Hopf algebras. That is, any coradically graded pointed Hopf algebra can be viewed as a large sub Hopf algebra of some graded Hopf structure on some unique Hopf quiver. Here by "large" we mean that the sub Hopf algebra contains the space spanned by the set of vertices and arrows of the quiver.

In the following, we show that an analogous Gabriel type theorem holds for pointed Hopf superalgebras. This provides a handy quiver setting for Hopf superalgebras.

Proposition 4.1 *Let H be a pointed Hopf superalgebra and $\operatorname{gr} H$ its coradically graded version as mentioned in Subsection 2.3. Then there exists a unique Hopf superquiver $(Q, \mathfrak{p})$ and a graded Hopf superalgebra structure on $(kQ, \mathfrak{p})$ such that $\operatorname{gr} H$ is isomorphic to a sub Hopf superalgebra of $(kQ, \mathfrak{p})$ which contains $kQ_0 \oplus kQ_1$.*

Proof Let $\{H_n\}_{n=0}^{\infty}$ be the coradical filtration of H and let M denote H_1/H_0. Assume that G is the set of group-like elements of H. Since H is pointed by assumption, we have $H_0 = kG$. Write $M = M_{\overline{0}} \oplus M_{\overline{1}}$ as the decomposition into even and odd subspaces. The coradically graded Hopf superalgebra structure induces on M a kG-Hopf bimodule. Since the elements of G are even, the even part $M_{\overline{0}}$ and the odd part $M_{\overline{1}}$ are kG-Hopf bimodules themselves. Now use the decomposition of $M_{\overline{0}}$ (resp. $M_{\overline{1}}$) into isotypic components $\bigoplus\limits_{x,y\in G} {}^{x}M_{\overline{0}}^{y}$ $\Big($resp. $\bigoplus\limits_{x,y\in G} {}^{x}M_{\overline{1}}^{y}\Big)$. They have the same properties as given in the proof of Proposition 3.1. Let $\mathcal{C}$ be the set of the conjugacy classes of G. For each $C \in \mathcal{C}$, fix an element $c \in C$ and set $R_{C,\overline{0}} = \dim_k {}^{c}M_{\overline{0}}^{1}$ and $R_{C,\overline{1}} = \dim_k {}^{c}M_{\overline{1}}^{1}$. Take a super ramification datum of G as $R = \Big(\sum\limits_{C\in\mathcal{C}} R_{C,\overline{0}}C, \sum\limits_{C\in\mathcal{C}} R_{C,\overline{1}}C\Big)$. Let Q denote the Hopf superquiver $(Q(G, R_{\overline{0}}, R_{\overline{1}}), \mathfrak{p})$. By Proposition 2.1, the Gabriel type theorem for pointed Hopf supercoalgebras, one has that $\operatorname{gr} H$ can be viewed as a large sub supercoalgebra of $(kQ, \mathfrak{p})$. Note that such quiver Q is unique. On the other hand, we take the kG-Hopf bimodule on $kQ_{1,\overline{0}}$ (resp. $kQ_{1,\overline{1}}$) as that on $M_{\overline{0}}$ (resp. $M_{\overline{1}}$). This provides a graded Hopf superalgebra structure on $(kQ, \mathfrak{p})$. It is routine to show that the supercoalgebra imbedding $\operatorname{gr} H \hookrightarrow (kQ, \mathfrak{p})$ respects their Hopf superalgebra structures. This is the desired Hopf superalgebra embedding. The proof is completed.

The preceding result implies that one can construct pointed Hopf superalgebras exhaustively on Hopf superquivers. Firstly one needs to classify all the possible coradically graded large sub Hopf superalgebras of those on Hopf superquivers. Then one needs to carry out a deformation procedure, or the lifting in the sense of [1], on them to get general non-graded ones.

4.2 Let G be a group and $R = (R_{\overline{0}}, R_{\overline{1}})$ be a super ramification datum with $R_{\overline{0}} = \sum\limits_{C\in\mathcal{C}} R_{C,\overline{0}}C$ and $R_{\overline{1}} = \sum\limits_{C\in\mathcal{C}} R_{C,\overline{1}}C$. Then the associated Hopf superquiver $(Q, \mathfrak{p}) = (Q(G, R_{\overline{0}}, R_{\overline{1}}), \mathfrak{p})$ is connected if and only if the union of the conjugacy classes with $R_{C,\overline{0}} + R_{C,\overline{1}} \neq 0$ in the super ramification datum R generates G. In general, let $Q(e)$ denote the connected component containing the unit $e \in G$. Then $Q(e)_0$ is a normal subgroup of G and as a superquiver Q is equivalent to $\bigcup\limits_{x\in G/Q(e)_0} Q(x)$ where $Q(x)$ is identical to $Q(e)$ graphically for all x. By the construction of graded Hopf superalgebra structures on kQ as given in the proof of Proposition 3.1, it is ready to see that $kQ(e)$ is closed under the multiplication and becomes a sub Hopf superalgebra of kQ. And the Hopf superalgebra on kQ can also be decomposed as a crossed product (see e.g. [16]) of $kQ(e)$ with the quotient group algebra $k[G/Q(e)_0]$. This enables us to work on only connected Hopf superquivers.

In the following, we show that such decomposition holds for general pointed Hopf superalgebras. Let H be a pointed Hopf superalgebra, G its set of group-likes and Q its superquiver. Denote the unit of G by e and the connected component of Q containing e by $Q(e)$. The supercoalgebra embedding $H \hookrightarrow kQ$ decomposes H into blocks in traditional terms of algebra (see [9]), or link-indecomposable components in the sense of Montgomery (see [17]). For each $g \in G$, let $Q(g)$ be the connected component of Q containing g, and $H_{(g)}$ the image of H in $kQ(g)$, i.e., the block (or indecomposable component) of H containing g. For any $g, h \in G$, restrict the multiplication to $H_{(g)} \otimes H_{(h)} \to H_{(g)}H_{(h)}$. Apparently, this is a supercoalgebra map and the image falls into the block $H_{(gh)}$. It is also easy to see that the antipode maps $H_{(g)}$ to $H_{(g^{-1})}$. It follows in particular that $H_{(e)}$ is a sub Hopf superalgebra. Follow the traditional terminology, we call it the principle block of H.

Let N denote $Q(e)_0$. Take a set T of distinct coset representatives of N in G. In particular, for the coset N itself, we take e as its representative. For any $g \in G$, write $\overline{g} \in T$ as the the representative of the coset in that g lies. Then there is a 2-cocycle $\sigma : \ G/N \times G/N \to N$ such that $\overline{u}\,\overline{v} = \sigma(\overline{u}, \overline{v})\overline{uv}$ for any $\overline{u}, \overline{v} \in T$. Notice that $H = \bigoplus_{\overline{u} \in T} H_{(\overline{u})}$. Define for each $\overline{u} \in T$ the translation map $\mathrm{Tr}_{\overline{u}} : \ H_{(e)} \to H_{(\overline{u})}$ by $p \mapsto p\overline{u}$. This is a natural isomorphism map of supercoalgebras. It follows that the blocks are identical. Hence we can write H as $\bigoplus_{\overline{u} \in T} H_{(e)}\overline{u}$. Now the multiplication of H can be transported to $\bigoplus_{\overline{u} \in T} H_{(e)}\overline{u}$ as

$$(p\overline{u})(q\overline{v}) = p(\overline{u}.q)\sigma(\overline{u}, \overline{v})\overline{uv},$$

where we have written $\overline{u}.q = \overline{u}q\overline{u}^{-1}$. It follows that the Hopf superalgebra H is the crossed product of the sub Hopf superalgebra $H_{(e)}$ and the quotient group algebra $k[G/N]$.

We summarize the previous arguments as follows.

Proposition 4.2 *Keep the notations as above.*

(1) *The map* $\mathrm{Tr}_g : \ H_{(e)} \to H_{(g)}$ *defined by* $p \mapsto pg$ *is a supercoalgebra isomorphism.*

(2) $H_{(g)} \cdot H_{(h)} \subseteq H_{(gh)}$ *and* $\mathcal{S}(H_{(g)}) \subseteq H_{(g^{-1})}$. *In particular,* $H_{(e)}$ *is itself a Hopf superalgebra.*

(3) *There is a Hopf superalgebra isomorphism* $H \cong H_{(e)}\#_\sigma k[G/N]$, *where* $\sigma : G/N \times G/N \to N$ *is a* 2*-cocycle and* $H_{(e)}\#_\sigma k[G/N]$ *is a crossed product.*

This gives a super analogue of a result of Montgomery [17, Theorem 3.2], which is a generalization of the well-known decomposition theorem of Cartier-Gabriel (see [7]). With this observation, the study of pointed Hopf superalgebras can be reduced to their principle blocks, whose superquivers are connected.

5 Examples

In this section, we give some examples to elucidate our quiver techniques.

5.1 Let $G = \{1\}$ be the unit group and $R = (m1, n1)$ a super ramification datum. Then the associated Hopf superquiver, denoted by Q for short, has a single vertex and m even loops and n odd loops. By Corollary 3.1, there is a unique graded Hopf superalgebra algebra on the path supercoalgebra $(kQ, \mathfrak{p})$. For an even loop x, by (3.1) we have

$$x^n = \overbrace{x \cdot x \cdot \cdots \cdot x}^{n} = n! x^{(n)},$$

where $x^{(n)}$ is the path $\overbrace{xx \cdots x}^{n}$. While for an odd loop y, we have by (3.1)

$$y^2 = yy + (-1)^{\mathfrak{p}(y)\mathfrak{p}(y)} yy = 0.$$

For any two loops x and y, we have

$$x \cdot y = xy + (-1)^{\mathfrak{p}(x)\mathfrak{p}(y)} yx.$$

Let $\mathfrak{g}$ denote the space kQ_1 and define

$$[x, y] = x \cdot y - (-1)^{\mathfrak{p}(x)\mathfrak{p}(y)} y \cdot x$$

for any two loops x, y. Clearly, $(\mathfrak{g}, [\ ,\ ])$ is an abelian Lie superalgebra (see e.g. [11]), that is, $[x, y] = 0$ for all x, y. The sub Hopf superalgebras generated by the loops of Q is the universal enveloping algebra $U(\mathfrak{g})$, which is a super polynomial algebra with m even and n odd variables. In particular, if the loops are odd, i.e., $m = 0$, then this is exactly the exterior algebra in n variables.

5.2 Now we assume the ground field k is algebraically closed of characteristic 0. Let $\mathbb{Z}_n = \langle g \mid g^n = 1 \rangle$ be the cyclic group of order $n > 1$ and consider the Hopf superquiver $Q = Q(\mathbb{Z}_n, 0, g)$. It is a basic cycle of length n. For each integer i modulo n, let a_i denote the arrow $g^i \to g^{i+1}$. The quiver Q has the following form:

$$\begin{tikzcd} & 1 \arrow[dr, dotted, "a_0"] & \\ g^{n-1} \arrow[ur, dotted, "a_{n-1}"] & \cdots \arrow[l, dotted, "a_{n-2}"] & g \arrow[l, dotted, "a_1"] \end{tikzcd}$$

Recall that, odd arrows are drawn as dotted. Let p_i^l denote the path with source g^i and length l.

According to Corollary 3.1, the set of graded Hopf superalgebra structures on the path supercoalgebra kQ is in one-to-one correspondence with the set of 1-dimensional $k\mathbb{Z}_n$-modules, and hence with the set of n-th roots of unity in k. Let q be an n-th root of unity. Then there is a corresponding $k\mathbb{Z}_n$-Hopf bimodule structure on kQ_1 given by

$$g.a_i = q a_{i+1}, \quad a_i.g = a_{i+1}$$

for all i. The multiplication formula of the associated graded Hopf superalgebra can be given via (3.1).

As preparation, we recall some notations of Gauss binomial coefficients. For any $\hbar \in k$, integers $l, m \geq 0$, set

$$l_\hbar = 1 + \hbar + \cdots + \hbar^{l-1}, \quad l!_\hbar = 1_\hbar \cdots l_\hbar, \quad \binom{l+m}{l}_\hbar = \frac{(l+m)!_\hbar}{l!_\hbar m!_\hbar}.$$

Lemma 5.1 *Keep the previous assumptions and notations. Then we have*

$$p_i^l \cdot p_j^m = q^{im} \binom{l+m}{l}_{(-q)} p_{i+j}^{l+m}. \tag{5.1}$$

Proof Firstly, we compute a_0^l. We claim

$$a_0^l = l!_{(-q)} p_0^l. \tag{5.2}$$

For $l \leq 2$, this is clear. By induction, we have

$$a_0^l = a_0 \cdot a_0^{l-1} = (l-1)!_{(-q)} a_0 \cdot p_0^{l-1}.$$

By (3.1), one has

$$a_0 \cdot p_0^{l-1} = (1 - q + q^2 - \cdots + (-q)^{l-1}) p_0^l.$$

Therefore, $a_0^l = l!_{(-q)} p_0^l$.

Note that $p_i^l = p_0^l \cdot g^i$ and $g^i \cdot p_0^m = q^{im} p_0^m \cdot g^i$. Therefore

$$p_i^l \cdot p_j^m = (p_0^l \cdot g^i) \cdot (p_0^m \cdot g^j) = q^{im} (p_0^l \cdot p_0^m) \cdot g^{i+j}.$$

Applying (5.2), we have

$$(l+m)!_{(-q)} p_0^{l+m} = a_0^{l+m} = a_0^l \cdot a_0^m = l!_{(-q)} m!_{(-q)} p_0^l \cdot p_0^m,$$

and hence

$$p_0^l \cdot p_0^m = \binom{l+m}{l}_{(-q)} p_0^{l+m}.$$

Now (5.1) follows.

It is interesting to compare the formula (5.1) with the one of usual Hopf algebras on kQ as given in [6, Proposition 3.17]. Consider the sub Hopf superalgebras generated by vertices and arrows. Clearly, it is generated by g and a_0. Note

$$a_0^l = l!_{(-q)} p_0^l.$$

If $\operatorname{order}(q) = d$, then $a_0^d = 0$. Let $C_d(n)$ denote the d-truncated sub supercoalgebra of kQ, namely it is spanned by the set $\{p_i^l \mid 0 \leq i \leq n-1,\ 0 \leq l \leq d-1\}$ of paths of length less than d. Now we see that the sub Hopf superalgebras generated by vertices and arrows is exactly $C_d(n)$ with multiplication given by (5.1).

It was shown in [3] that $C_d(n)$ admits a usual Hopf structure if and only if $d \mid n$. The arguments there can be adjusted to the super case. We give the result without repeating the detail. Note that some new phenomenon arises.

Proposition 5.1 *If n is odd, then $C_d(n)$ admits a graded Hopf superalgebra structure if and only if d is even and $\frac{d}{2} \,\Big|\, n$. If n is even, then $C_d(n)$ admits a graded Hopf superalgebra structure if and only if d is even and $d \mid n$.*

3.3 Let $\mathbb{Z} = \langle K \rangle$ be the infinite cyclic group and consider the Hopf superquiver $Q = Q(\mathbb{Z}, 0, 2K)$. Then Q looks as follows

$$\cdots \;\; \overset{}{\underset{}{\rightrightarrows}} K^{-2} \underset{F_{-2}}{\overset{E_{-2}}{\rightrightarrows}} K^{-1} \underset{F_{-1}}{\overset{E_{-1}}{\rightrightarrows}} 1 \underset{F_{0}}{\overset{E_{0}}{\rightrightarrows}} K \underset{F_{1}}{\overset{E_{1}}{\rightrightarrows}} K^{2} \underset{F_{2}}{\overset{E_{2}}{\rightrightarrows}} \;\; \cdots$$

By Corollary 3.1, graded Hopf superalgebra structures on kQ correspond to 2-dimensional representations (on the space $kE_0 \oplus kF_0$) of the group $\mathbb{Z}$.

Consider the representation given by

$$K \triangleright E_0 = qE_0, \quad K \triangleright F_0 = q^{-1}F_0$$

with $0 \neq q \in k$. We can extend this to a $k\mathbb{Z}$-Hopf bimodule on kQ_1 given as follows

$$K.E_i = qE_{i+1}, \quad E_i.K = E_{i+1}, \quad K.F_i = q^{-1}F_{i+1}, \quad F_i.K = F_{i+1}$$

for all i. The associated graded Hopf superalgebra has the following multiplication formulae.

Lemma 5.2

$$E_0^n = n!_{(-q)}E_{n-1}\cdots E_1E_0, \qquad F_0^n = n!_{(-q^{-1})}F_{n-1}\cdots F_1F_0, \tag{5.3}$$

$$E_0 \cdot F_{-1} = E_0F_{-1} - q^{-1}F_0E_{-1}, \quad F_{-1} \cdot E_0 = -E_0F_{-1} + q^{-1}F_0E_{-1}. \tag{5.4}$$

The formulae are obtained by direct computation via (3.1). Note that (5.4) implies

$$E_0 \cdot F_{-1} + F_{-1} \cdot E_0 = 0.$$

Now it is easy to see that, if q is generic (i.e., not roots of unity), then the sub Hopf superalgebra generated by vertices and arrows is isomorphic to the coradically graded version of the well-known quantum supergroup $U_q(\mathrm{osp}(1|2))$ (see for instance [13, 20]). Of course, if q is a root of unity, then one gets the corresponding small quantum supergroup.

We remark that general quantum supergroups can also be obtained via Hopf superquivers in this way. This is left for future work.

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