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On the Well-Posedness for Stochastic Schrödinger Equations with Quadratic Potential*

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Abstract The authors investigate the influence of a harmonic potential and random perturbations on the nonlinear Schrödinger equations. The local and global well-posedness are proved with values in the space $\Sigma(\mathbb{R}^n) = \{f \in H^1(\mathbb{R}^n), |\cdot|f \in L^2(\mathbb{R}^n)\}$. When the nonlinearity is focusing and L^2-supercritical, the authors give sufficient conditions for the solutions to blow up in finite time for both confining and repulsive potential. Especially for the repulsive case, the solution to the deterministic equation with the initial data satisfying the stochastic blow-up condition will also blow up in finite time. Thus, compared with the deterministic equation for the repulsive case, the blow-up condition is stronger on average, and depends on the regularity of the noise. If $\phi = 0$, our results coincide with the ones for the deterministic equation.

Keywords Stochastic Schrödinger equation, Well-posedness, Blow up
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1 Introduction

In this paper, we are concerned with the stochastic Schrödinger equation with harmonic potential

$$\mathrm{i}u_t + \Delta u + \theta|x|^2 u + \lambda|u|^{2\sigma}u = \dot{\xi}, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^n. \tag{1.1}$$

This type of equations rises from both physical and mathematical considerations. When $\theta < 0$, the equation models Bose-Einstein condensations and the sign of λ stands for different chemical elements (see [5, 7]). The nonlinearity describes the interactions between the particles. The additive noise $\dot{\xi}$ expresses the random perturbations. It is a basic generalization to the stochastic case which is more natural in physics because of the effects of the random media. Mathematically, we are familiar with the properties of the solution to the classical NLS without potential (see [8])

$$\mathrm{i}\partial_t u + \Delta u + \lambda|u|^{2\sigma}u = 0. \tag{1.2}$$

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The local well-posedness (LWP) is well-known within the energy space $H^1(\mathbb{R}^n)$. And the global well-posedness (GWP) holds for the defocusing ($\lambda < 0$) or L^2-subcritical ($\sigma < \frac{2}{n}$) nonlinearities. However, when $\lambda > 0$ and $\sigma \geq \frac{2}{n}$, there exist solutions to (1.2) which blow up in finite time. We also know from the work of R. Carles (see [5–6]) that a harmonic potential may strongly enhance or prevent the blow-up phenomenon according to the sign and the strength of the potential. On the other hand, A. de Bouard and A. Debussche studied in [1–3] the equation (1.2) with an additive noise or a multiplicative noise. We are interested in the problem that how it will affect the long-time behavior of the solution when an additive noise acts on the classical NLS with harmonic potential, i.e., (1.1).

We study the Cauchy problem

$$\begin{cases} \mathrm{i}\partial_t u + \Delta u + \theta|x|^2 u + \lambda|u|^{2\sigma}u = \dot{\xi}, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}^n, \\ u(0) = u_0, \end{cases} \tag{1.3}$$

where $\lambda, \theta \in \mathbb{R} \setminus \{0\}$, $\sigma \geq 0$ and $\sigma < \frac{2}{n-2}$ if $n \geq 3$. The noise $\dot{\xi}$ is white in time and colored in space. As in [1–3], we mention that we could not deal with the space-time white noise because of the lack of smoothing effects of the Schrödinger operators. In this paper, we first prove a local existence of the solutions with values in Σ on a stopping time interval, where the maximality of the existence time is relative to the Σ-norm of the initial data. There is no conservation of energy for the stochastic system (1.3). But we can study the evolution of the energy or part of the energy to get the global existence for the defocusing or the L^2-subcritical nonlinearities. To study the blow-up phenomenon, we use "the variance identity" method but in a stochastic version. In the case of a repulsive potential ($\theta > 0$), we prove for a class of initial data in Σ that the solution to (1.3) will blow up in finite time. (However, in a weaker sense, see Theorem 4.1.) We mention that compared with the deterministic equation, this condition is stronger on average, and depends on the regularity of the noise. If $\phi = 0$, our results coincide with the ones for the deterministic equation. In the case of a confining potential ($\theta < 0$), we give a sufficient condition for u_0 under which the solution u will blow up before $T = \frac{\pi}{4\sqrt{-\theta}}$.

This paper is organized as follows. In Section 2, we set some notations and some properties with respect to the operators which play an important role in our proof. In Section 3, we prove the local well-posedness and global existence for equation (1.3) in space $\Sigma(\mathbb{R}^n)$. The results are analogues to the deterministic equation but the proof is more complicate because of the random perturbations. Section 4 is devoted to the blow-up phenomenon. We derive the sufficient conditions for the solution to blow up in finite time for both confining (resp. $\theta < 0$) and repulsive (resp. $\theta > 0$) potential.

2 Notations and Preliminaries

Throughout this paper, we consider the probability space $(\Omega, \mathcal{F}, P)$ endowed with a normal filtration $(\mathcal{F}_t)_{t\geq 0}$. Let $(\beta_k)_{k\in\mathbb{N}}$ be a sequence of independent real valued Brownian motion on $\mathbb{R}^+$ and $(e_k)_{k\in\mathbb{N}}$ be an orthogonormal basis of some Hilbert space U. Then the process

$$W(t,x,\omega) = \sum_{k=0}^{\infty} \beta_k(t,\omega)\phi e_k(x), \quad t \geq 0,\ x \in \mathbb{R}^n,\ \omega \in \Omega$$

is an H-valued Wiener process, where ϕ is a Hilbert-Schmidt operator from U to some Hilbert space H. We denote the space of all the Hilbert-Schmidt operators from U to H by $L_2(U,H)$, with the norm

$$\|\phi\|_{L_2(U,H)^2} = \sum_{l\in\mathbb{N}} \|\phi e_l\|_H^2.$$

Set $\dot{\xi} = \frac{\partial W}{\partial t}$. Then (1.1) can be rewritten as

$$\mathrm{i}\mathrm{d}u + (\Delta u + \theta|x|^2 u + \lambda|u|^{2\sigma}u)\mathrm{d}t = \mathrm{d}W. \tag{2.1}$$

Denote $U_\theta(t) = \exp\{\mathrm{i}t(\Delta + \theta|x|^2)\}$. Then (2.1) can be expressed by Duhamel's principle as

$$u(t) = U_\theta(t)u_0 + \mathrm{i}\lambda\int_0^t U_\theta(t-s)(|u|^{2\sigma}u)(s)\mathrm{d}s - \mathrm{i}\int_0^t U_\theta(t-s)\mathrm{d}W(s). \tag{2.2}$$

We will consider the equation of this mild form. Recall that $U_\theta(t)$ has the following properties (see [5–6]).

Lemma 2.1 (Strichartz Estimates for U_θ)

(1) *If $\theta > 0$, then for any admissible pair (q,r), (q_1,r_1), (q_2,r_2) and any interval I, there exist C_r, C_{r_1,r_2} independent of θ and I, such that*

$$\|U_\theta(\cdot)\varphi\|_{L^q(I,L^r)} \le C_r\|\varphi\|_{L^2}$$

for every $\varphi \in L^2(\mathbb{R}^n)$, and

$$\Big\|\int_{I\cap\{s\le t\}} U_\theta(t-s)F(s)\mathrm{d}s\Big\|_{L^{q_1}(I,L^{r_1})} \le C_{r_1,r_2}\|F\|_{L^{q_2'}(I,L^{r_2'})}$$

for every $F \in L^{q_2'}(I,L^{r_2'})$.

(2) *If $\theta < 0$, then for any interval I contained in $[0,\frac{\pi}{4\nu}]$, where $\nu = \sqrt{-\theta}$, the inequalities stated in* (1) *also hold.*

Here a pair (q,r) is said to be admissible if $2 \le r < \frac{2n}{n-2}$ (resp., $2 \le r \le \infty$ if $n = 1$, $2 \le r < \infty$ if $n = 2$) and

$$\frac{2}{q} = \delta(r) = n\Big(\frac{1}{2} - \frac{1}{r}\Big).$$

The following operators which were introduced by R. Carles in [5] and [6] will be used in our paper. In the case $\theta > 0$, setting $\mu = \sqrt{\theta}$, we use

$$J_+(t) = \mu x\sinh(2\mu t) + \mathrm{i}\cosh(2\mu t)\nabla_x, \quad K_+(t) = x\cosh(2\mu t) + \frac{\mathrm{i}}{\mu}\sinh(2\mu t)\nabla_x. \tag{2.3}$$

In the case $\theta < 0$, setting $\nu = \sqrt{-\theta}$, we use

$$J_-(t) = \nu x\sin(2\nu t) - \mathrm{i}\cos(2\nu t)\nabla_x, \quad K_-(t) = x\cos(2\nu t) + \frac{\mathrm{i}}{\nu}\sin(2\nu t)\nabla_x. \tag{2.4}$$

Recall that $J_\pm(t)$, $K_\pm(t)$ have some nice properties (see [5–6]).

Lemma 2.2 *$J_\pm(t)$, $K_\pm(t)$ satisfy the following properties:*

(1)

$$J_\pm(t) = \pm U_\theta(t)\mathrm{i}\nabla_x U_\theta(-t), \quad K_\pm(t) = U_\theta(t)xU_\theta(-t), \tag{2.5}$$

and they commute with the linear part of (1.1)*, that is,*

$$[\mathrm{i}\partial_t + \Delta + \theta|x|^2, J_\pm(t)] = [\mathrm{i}\partial_t + \Delta + \theta|x|^2, K_\pm(t)] = 0.$$

(2) *If $\theta > 0$, the modified Gagliardo-Nirenberg inequalities hold. For $r \geq 2$ and $r < \frac{2n}{n-2}$ if $n \geq 3$, there exists a C_r such that for any $f \in \Sigma$,*

$$\|f\|_{L^r} \leq \frac{C_r}{(\cosh(2\mu t))^{\delta(r)}} \|f\|_{L^2}^{1-\delta(r)} \|J_+(t)f\|_{L^2}^{\delta(r)}, \quad \forall t \in \mathbb{R}, \tag{2.6}$$

$$\|f\|_{L^r} \leq \frac{C_r}{(\sinh(2\mu t))^{\delta(r)}} \|f\|_{L^2}^{1-\delta(r)} \|K_+(t)f\|_{L^2}^{\delta(r)}, \quad \forall t \neq 0. \tag{2.7}$$

(3) *If $\theta < 0$, with r, f defined as in* (2)*, we have*

$$\|f\|_{L^r} \leq C_r \|f\|_{L^2}^{1-\delta(r)} (\|J_-(t)f\|_{L^2} + \|K_-(t)f\|_{L^2})^{\delta(r)}. \tag{2.8}$$

(4) *If $F \in C^1(\mathbb{C}, \mathbb{C})$ is of the form $F(z) = zG(|z|^2)$, then they act like derivatives on F, that is,*

$$J_\pm(t)F(u) = \partial_z F(u) J_\pm(t)u - \partial_{\overline{z}} F(u) \overline{J_\pm(t)u}, \quad \textit{if } \theta < 0,\ t \notin \frac{\pi}{2\nu}\mathbb{Z}, \tag{2.9}$$

$$K_\pm(t)F(u) = \partial_z F(u) K_\pm(t)u - \partial_{\overline{z}} F(u) \overline{K_\pm(t)u}, \quad \textit{if } \theta < 0,\ t \notin \frac{\pi}{4\nu} + \frac{\pi}{2\nu}\mathbb{Z}. \tag{2.10}$$

Remark 2.1 When we discuss the local well-posedness in Section 3.1, we will not distinguish the proof according to the sign of θ, since it does not change our proof essentially. So for simplicity, we will use J, K to denote J_+ (resp. J_-), K_+ (resp. K_-) when $\theta > 0$ (resp. $\theta < 0$).

We will use some quantities relative to ϕ several times throughout this paper, so we set the following notations for simplicity:

$$c_\phi^\Sigma = \sum_l \int_{\mathbb{R}^n} |x|^2 |\phi e_l|^2 \mathrm{d}x, \quad c_\phi^1 = \sum_l \|\nabla \phi e_l\|_{L^2}^2, \quad c_\phi^2 = \sum_l \mathrm{Im} \int_{\mathbb{R}^n} \overline{\phi e_l} x \cdot \nabla(\phi e_l) \mathrm{d}x.$$

We also use C to denote different constants, and $C(\cdot)$ to emphasize the dependence. ε will denote different quantities which can be chosen arbitrarily small.

3 Some Well-posedness Results

3.1 Local well-posedness

In this section, we prove the local existence and uniqueness of the solution to (2.2) with paths in the following set:

$$\begin{aligned} Y_r(T, M) =& \{f \in C([0,T], \Sigma) : A(\cdot)f \in L^q(0,T; L^r) \cap C([0,T], L^2), \\ & \text{and } \|A(\cdot)u\|_{L_T^q L^r} + \|A(\cdot)u\|_{L_T^\infty L^2} \leq M,\ \forall A(t) \in \{J(t), K(t), \mathrm{Id}\}\} \end{aligned} \tag{3.1}$$

equipped with the distance

$$d_r(T)(u, v) = \|u - v\|_{L_T^\infty L^2} + \|u - v\|_{L_T^q L^r}, \tag{3.2}$$

where $r = 2\sigma + 2$ and (q, r) is an admissible pair. $T, M > 0$ will be fixed later. We claim that $(Y_r(T, M), d_r(T))$ is a complete metric space. Let $Y(T, M)$ be the intersection of the spaces $Y_r(T, M)$ where (q, r) takes values of all admissible pairs. The LWP result is as follows.

Theorem 3.1 *Assume $\sigma \geq 0$ and $\sigma < \frac{2}{n-2}$ if $n \geq 3$, ϕ is a Hilbert-Schmidt operator from $L^2(\mathbb{R}^n)$ into Σ. For any $\mathcal{F}_0$-measurable random variable u_0 with values in Σ, there exists a τ^* and for any $\tau < \tau^*$ a.s., there exists a unique solution u to* (2.1) *in $Y_r(\tau, M)$ such that $u(0) = u_0$ a.s.. Moreover, u belongs to $Y(\tau, M)$ a.s.. Here, $\tau^*(u_0)$ is a stopping time and is maximal in the following sense:*

$$\tau^*(u_0, \omega) = +\infty \quad or \quad \lim_{t \nearrow \tau^*(u_0,\omega)} \|u(t,\omega)\|_\Sigma = +\infty. \tag{3.3}$$

We prove this theorem pathwise, that is, for fixed ω, we prove an existence result in $Y_r(T, M)$, where M and the existence time T depend on ω.

Proof Recall that u satisfing (2.1) is equivalent to that u being the solution to the integral equation (2.2),

$$u(t) = U_\theta(t)u_0 + \mathrm{i}\lambda \int_0^t U_\theta(t-s)(|u|^{2\sigma}u)(s)\mathrm{d}s - \mathrm{i}\int_0^t U_\theta(t-s)\mathrm{d}W(s).$$

Set $v(t) = u(t) + z(t)$, where $z(t) = \mathrm{i}\int_0^t U_\theta(t-s)\mathrm{d}W(s)$. Then $v(t)$ satisfies the integral equation

$$v(t) = U_\theta(t)u_0 + \mathrm{i}\lambda \int_0^t U_\theta(t-s)(|v-z|^{2\sigma}(v-z))(s)\mathrm{d}s. \tag{3.4}$$

It is reduced to proving that for almost all ω, there exists a unique solution to (3.4) with paths in $Y_r(T, M)$ and initial value u_0. We need the following lemma, which will be proved later.

Lemma 3.1 *Under the assumptions of Theorem* 3.1, *$A(\cdot)z \in C([0,\infty), L^2) \cap L^q_{\mathrm{loc}}([0,\infty), L^r)$ almost surely.*

Now we are ready to prove the theorem by using the fixed point argument. Set

$$\mathcal{H}v(t) = U_\theta(t)u_0 + \mathrm{i}\lambda \int_0^t U_\theta(t-s)(|v-z|^{2\sigma}(v-z))(s)\mathrm{d}s. \tag{3.5}$$

Then we need to prove that $\mathcal{H}$ is a strict contraction on $(Y_r(T, M), d_r(T))$, which leads us to estimate the $L^\infty_T L^2$ and the $L^q_T L^r$-norm of $A(\cdot)\mathcal{H}v$. Take the operator $J(t)$ for example. By (1) of Lemma 2.2, we have

$$J(t)\mathcal{H}v(t) = U_\theta(t)\mathrm{i}\nabla_x u_0 + \mathrm{i}\lambda \int_0^t U_\theta(t-s)J(s)(|v-z|^{2\sigma}(v-z))(s)\mathrm{d}s. \tag{3.6}$$

From (4) of Lemma 2.2 we know that $J(t)$ acts on the nonlinearities like derivatives. Then we can estimate with the help of the Strichartz estimate as follows:

$$\begin{aligned} &\|J(\cdot)\mathcal{H}v\|_{L^\infty_T L^2} + \|J(\cdot)\mathcal{H}v\|_{L^q_T L^r} \\ &\leq C\|\nabla u_0\|_{L^2} + CT^{1-\delta(r)}\|v-z\|^{2\sigma}_{L^\infty_T L^r}\|J(\cdot)(v-z)\|_{L^q_T L^r}. \end{aligned} \tag{3.7}$$

Similar estimates also hold for $K(t)$ and Id. For the contraction we have

$$\mathcal{H}w(t) - \mathcal{H}v(t) = \mathrm{i}\lambda \int_0^t U_\theta(t-s)(|w-z|^{2\sigma}(w-z) - |v-z|^{2\sigma}(v-z))(s)\mathrm{d}s \tag{3.8}$$

and

$$\begin{aligned}&\|\mathcal{H}w-\mathcal{H}v\|_{L_T^\infty L^2}+\|\mathcal{H}w-\mathcal{H}v\|_{L_T^q L^r}\\ \le\ & CT^{1-\delta(r)}(\|w-z\|_{L_T^\infty L^r}^{2\sigma}+\|v-z\|_{L_T^\infty L^r}^{2\sigma})\|w-v\|_{L_T^q L^r}.\end{aligned}\tag{3.9}$$

From (3.7) and (3.9), it is not hard to choose the proper T and M with respect to the Σ-norm of u_0, the $L_T^\infty L^2$ and the $L_T^q L^r$-norm of $A(\cdot)z$ for almost all $\omega\in\Omega$ to make sure that $\mathcal{H}$ is a strict contraction on $Y_r(T,M)$. And by the fixed point theorem, we get a unique solution v to (3.4). In fact, by the Strichartz estimate, v is in $Y(T,M)$.

It remains to demonstrate the blow up alternative. From the existence part, we know that M is chosen only depending on the Σ-norm of the initial data and some constants (depending on z). We choose $T_\omega=T_\omega(M)$ small enough to get a unique solution $u\in Y_r(T_\omega,M)$. In particular, u is in $C([0,T_\omega],\Sigma)$. Now we define

$$\tau_\omega^*=\sup\{T_\omega>0,\text{there exists a solution } u \text{ on } [0,T_\omega]\}.\tag{3.10}$$

Then we have $u\in C([0,\tau_\omega^*),\Sigma)$.

Assume that there exists a sequence $t_k\nearrow\tau_\omega^*$ as $k\to\infty$ and $\|u(t_k,\omega)\|_\Sigma\le\widetilde{M}$. We can choose k large enough such that $t_k+T_\omega(\widetilde{M})>\tau_\omega^*$. Then by the local existence argument one can extend the solution u starting from t_k to $t_k+T_\omega(\widetilde{M})>\tau_\omega^*$, which contradicts the maximality of τ_ω^*.

Remark 3.1 One can also solve the Cauchy problem pathwise under the L^2-scheme by the Strichartz estimate without using the operators $J(t)$ and $K(t)$, and get the uniqueness in the space $C([0,T_\omega],L^2)\cap L^q(0,T_\omega;L^r)$ for any $T_\omega<\widetilde{\tau_\omega^*}$, where $\widetilde{\tau_\omega^*}\ge\tau_\omega^*$ a.s.. This implies that the uniqueness showed in Theorem 3.1 holds in the larger space $C([0,T_\omega],L^2)\cap L^q(0,T_\omega;L^r)$ for any $T_\omega<\widetilde{\tau_\omega^*}$. Considering the stochastic nonlinear Schrödinger equation with multiplicative noise, de Bouard and Debussche [4] obtained the global existence and uniqueness of solution in the L^2-scheme.

As the end of this section we prove Lemma 3.1 to complete the proof of Theorem 3.1.

Proof of Lemma 3.1 It is a standard result that $A(\cdot)z$ has paths in $C([0,\infty),L^2)$ with $\phi\in L_2(L^2,\Sigma)$ (see [9, Theorem 6.10]). We only need to show that $A(\cdot)z$ is in $L^q([0,T],L^r)$ almost surely. Set $m\ge q,r$. We estimate the m-th moment for $\|A(t)z\|_{L_T^qL^r}$,

$$\begin{aligned}\|A(t)z\|_{L_\omega^m L_T^q L^r}&\le\|A(t)z\|_{L_T^q L^r L_\omega^m}\\&=\|(E|A(t)z|^m)^{\frac{1}{m}}\|_{L_T^q L^r}\\&\le C\Big\|\Big(\sum_l\int_0^t|U_\theta(t-s)A(s)\phi e_l|^2\mathrm{d}s\Big)^{\frac12}\Big\|_{L_T^q L^r}\\&\le C\Big\|\sum_l\int_0^t\|U_\theta(t-s)A(s)\phi e_l\|_{L^r}^2\mathrm{d}s\Big\|_{L_T^{\frac{q}{2}}}^{\frac12}\\&\le C(T)\Big\|\sum_l\Big(\int_0^t\|U_\theta(t-s)A(s)\phi e_l\|_{L^r}^q\mathrm{d}s\Big)^{\frac{2}{q}}\Big\|_{L_T^{\frac{q}{2}}}^{\frac12}\\&\le C(T)\Big(\sum_{l\in\mathbb{N}}\big(\|U_\theta x\phi e_l\|_{L_T^qL^r}+\|U_\theta\nabla\phi e_l\|_{L_T^qL^r}+\|U_\theta\phi e_l\|_{L_T^qL^r}\big)^2\Big)^{\frac12}\end{aligned}$$

$$\leq C(T)\|\phi\|_{L_2(L^2,\Sigma)}.$$

We have used Burkholder-Davis-Gundy inequality in the third line and the Strichartz estimates in the last one.

3.2 Global well-posedness

In this section, we consider the GWP for (2.1).

For the classical NLS

$$\mathrm{i}u_t + \Delta u + \lambda|u|^{2\sigma}u = 0, \tag{3.11}$$

it is well-known that if we define the energy $H_c(u(t)) = \frac{1}{2}\|\nabla u\|_{L^2}^2 - \frac{\lambda}{2\sigma+2}\|u\|_{L^{2\sigma+2}}^{2\sigma+2}$, there is conservation of energy, i.e., $H_c(u(t)) = H_c(u_0)$, $\forall t < T_{\max}$. When $\lambda < 0$, $\|\nabla u(t)\|_{L^2}$ can be controlled uniformly by $H_c(u_0)$, and we get GWP. When $\lambda > 0$, by Gagliardo-Nirenberg inequality, we know that if $\sigma < \frac{2}{n}$, $\|\nabla u\|_{L^2}^2$ is the main part of energy and can also be controlled uniformly.

For the NLS with harmonic potential, the energy reads

$$H(u(t)) = \frac{1}{2}\|\nabla u\|_{L^2}^2 - \frac{\theta}{2}\int_{\mathbb{R}^n}|xu(x)|^2\mathrm{d}x - \frac{\lambda}{2\sigma+2}\|u\|_{L^{2\sigma+2}}^{2\sigma+2}, \tag{3.12}$$

and there is conservation of energy $H(u(t)) = H(u_0)$. It is not obvious that $\lambda < 0$ indicates a uniform control of $\|u\|_\Sigma$ since θ may be larger than 0. In this case, R. Carles introduced the operators $J_+(t)$ and $K_+(t)$ and split the energy into two parts relative to the operators accordingly,

$$H_1(u(t)) = \frac{1}{2}\|J_+u(t)\|_{L^2}^2 - \frac{\lambda}{2\sigma+2}\cosh^2(2\mu t)\|u(t)\|_{2\sigma+2}^{2\sigma+2}, \tag{3.13}$$

$$H_2(u(t)) = -\frac{\mu^2}{2}\|K_+u(t)\|_{L^2}^2 + \frac{\lambda}{2\sigma+2}\sinh^2(2\mu t)\|u(t)\|_{2\sigma+2}^{2\sigma+2}, \tag{3.14}$$

such that $H(u(t)) = H_1(u(t)) + H_2(u(t))$. And the analysis for H_1 gives the control for the Σ-norm of the solution.

Here we use the method above but in a stochastic version, and get the next result.

Theorem 3.2 *In addition to Theorem* 3.1, *suppose that* $\theta > 0$, *and either* $\sigma < \frac{2}{n}$ *or* $\lambda \leq 0$. *Then for any* $\mathcal{F}_0$*-measurable* u_0, *the solution* u *to* (2.1) *with* $u(0) = u_0$ *given by Theorem* 3.1 *is global, in the sense of* $\tau^*(u_0) = +\infty$, *a.s..*

Proof We assume first that $u_0 \in L^{\frac{4\sigma}{2-n\sigma}+2}(\Omega, L^2(\mathbb{R}^n)) \cap L^2(\Omega, \Sigma(\mathbb{R}^n))$. To prove $\tau^* = +\infty$ a.s., we only need to show that for any $T_0 > 0$ and any $\tau < \inf\{T_0, \tau^*(u_0)\}$ a.s., we have

$$\begin{aligned} E\Big(\sup_{t\leq\tau}\|\nabla u\|_{L^2}^2\Big) &< C(\phi, u_0, T_0), \\ E\Big(\sup_{t\leq\tau}\||\cdot|u\|_{L^2}^2\Big) &< C(\phi, u_0, T_0). \end{aligned} \tag{3.15}$$

From (2.3) we know

$$\begin{aligned} xu &= \cosh(2\mu t)K_+u(t) - \frac{1}{\mu}\sinh(2\mu t)J_+u(t), \\ \nabla u &= -\mathrm{i}\cosh(2\mu t)J_+u(t) + \mathrm{i}\mu\sinh(2\mu t)K_+u(t). \end{aligned}$$

So (3.15) is equivalent to

$$E\Big(\sup_{t\le\tau}\|J_+u(t)\|_{L^2}^2\Big)<C(\phi,u_0,T_0),$$
$$E\Big(\sup_{t\le\tau}\|K_+u(t)\|_{L^2}^2\Big)<C(\phi,u_0,T_0). \tag{3.16}$$

We will need the next two lemmas, which can be checked by Itô's formula and a regularization procedure (see [1]).

Lemma 3.2 *Under the hypothesis of Theorem* 3.1, *for any* $\tau<\tau^*(u_0)$ *a.s., we have*

$$H(u(\tau))=H(u_0)+M(\tau)+R(\tau), \tag{3.17}$$

where

$$M(\tau)=\mathrm{Im}\int_{\mathbb{R}^n}\int_0^\tau(\Delta u+\theta|x|^2u+\lambda|u|^{2\sigma}u)\overline{\mathrm{d}W}\mathrm{d}x, \tag{3.18}$$

$$R(\tau)=\frac{c_\phi^1\tau}{2}-\frac{\theta}{2}\tau c_\phi^\Sigma-\frac{\lambda}{2}\sum_{l\in\mathbb{N}}\int_0^\tau\int_{\mathbb{R}^n}|u|^{2\sigma}|\phi e_l(x)|^2\mathrm{d}x\mathrm{d}t$$
$$-\sigma\lambda\sum_{l\in\mathbb{N}}\int_0^\tau\int_{\mathbb{R}^n}|u|^{2\sigma-2}(\mathrm{Im}(\overline{u}\phi e_l(x)))^2\mathrm{d}x\mathrm{d}t. \tag{3.19}$$

Lemma 3.3 *Let* σ, ϕ, u_0 *be as in Theorem* 3.1. $H_1(u(t))$ *is defined in* (3.13). *For any stopping time* $\tau<\tau^*$ *a.s., we have*

$$H_1(u(\tau))=H_1(u_0)+\frac{(n\sigma-2)\mu\lambda}{2\sigma+2}\int_0^\tau\sinh(4\mu t)\|u(t)\|_{L^{2\sigma+2}}^{2\sigma+2}\mathrm{d}t+M_1(\tau)+R_1(\tau), \tag{3.20}$$

where

$$M_1(\tau)=\mu^2\mathrm{Im}\int_{\mathbb{R}^n}\int_0^\tau\sinh^2(2\mu t)|x|^2\overline{u}\mathrm{d}W\mathrm{d}x-\mathrm{Im}\int_{\mathbb{R}^n}\int_0^\tau\cosh^2(2\mu t)\Delta\overline{u}\mathrm{d}W\mathrm{d}x$$
$$-\mu\mathrm{Re}\int_{\mathbb{R}^n}\int_0^\tau\sinh(4\mu t)x\cdot\nabla u\overline{\mathrm{d}W}\mathrm{d}x$$
$$-\frac{n\mu}{2}\mathrm{Re}\int_{\mathbb{R}^n}\int_0^\tau\sinh(4\mu t)\overline{u}\mathrm{d}W\mathrm{d}x-\lambda\mathrm{Im}\int_{\mathbb{R}^n}\int_0^\tau\cosh^2(2\mu t)|u|^{2\sigma}\overline{u}\mathrm{d}W\mathrm{d}x, \tag{3.21}$$

$$R_1(\tau)=\frac{\mu^2c_\phi^\Sigma}{2}\int_0^\tau\sinh^2(2\mu t)\mathrm{d}t+\frac{c_\phi^1}{2}\int_0^\tau\cosh^2(2\mu t)\mathrm{d}t-\frac{\mu c_\phi^2}{2}\int_0^\tau\sinh(4\mu t)\mathrm{d}t$$
$$-\lambda\sigma\int_0^\tau\cosh^2(2\mu t)\sum_{l\in\mathbb{N}}\int_{\mathbb{R}^n}|u|^{2\sigma-2}(\mathrm{Im}(u\overline{\phi e_l}))^2\mathrm{d}x\mathrm{d}t$$
$$-\frac{\lambda}{2}\int_0^\tau\cosh^2(2\mu t)\sum_{l\in\mathbb{N}}\int_{\mathbb{R}^n}|u|^{2\sigma}|\phi e_l(x)|^2\mathrm{d}x\mathrm{d}t. \tag{3.22}$$

First assume $\sigma<\frac{2}{n}$. By (3.20), we have

$$E\Big(\sup_{t\le\tau}\|J_+u(t)\|_{L^2}^2\Big)=EH_1(u_0)+CE\Big(\sup_{t\le\tau}\int_0^t\sinh(4\mu s)\|u(s)\|_{L^{2\sigma+2}}^{2\sigma+2}\mathrm{d}s\Big)$$
$$+CE\Big(\sup_{t\le\tau}\cosh^2(2\mu t)\|u(t)\|_{L^{2\sigma+2}}^{2\sigma+2}\Big)$$
$$+E\Big(\sup_{t\le\tau}M_1(t)\Big)+E\Big(\sup_{t\le\tau}R_1(t)\Big). \tag{3.23}$$

Since $\sigma < \frac{2}{n}$, by Gagliardo-Nirenberg inequality (2.6) and Young inequality, $\|u(t)\|_{L^{2\sigma+2}}^{2\sigma+2}$ can be controlled by $\|J_+u(t)\|_{L^2}^2$ with an arbitrarily small coefficient ε, and then the RHS except the term $E\Big(\sup\limits_{t\le\tau} M_1(t)\Big)$ can be estimated as

$$C(T_0, \phi, EH_1(u_0), E(\|u_0\|_{L^2}^{\frac{4\sigma}{2-n\sigma}+2})) + \varepsilon C(T_0)E\Big(\sup_{t\le\tau}\|J_+u(t)\|_{L^2}^2\Big), \tag{3.24}$$

where we have used the fact that $E\Big(\sup\limits_{t\le\tau}\|u(t)\|_{L^2}^{\frac{4\sigma}{2-n\sigma}+2}\Big) \le CE(\|u_0\|_{L^2}^{\frac{4\sigma}{2-n\sigma}+2})$ (see [1, Proposition 3.2]).

Now we estimate $E\Big(\sup\limits_{t\le\tau} M_1(t)\Big)$ by martingale inequality and Burkholder-Davis-Gundy inequality as follows:

$$\begin{aligned}
E\Big(\sup_{t\le\tau} M_1(t)\Big) &\le C(T_0, \phi, E\|u_0\|_{L^2}^2) + C(T_0,\phi)E\Big(\sup_{t\le\tau}\||\cdot|u(t)\|_{L^2}\Big)\\
&\quad + C(T_0,\phi)E\Big(\sup_{t\le\tau}\|\nabla u(t)\|_{L^2}\Big) + C(T_0,\phi)E\Big(\sup_{t\le\tau}\|u(t)\|_{L^{2\sigma+2}}^{2\sigma+2}\Big)\\
&\le C(T_0, \phi, E\|u_0\|_{L^2}^2, E(\|u_0\|_{L^2}^{\frac{4\sigma}{2-n\sigma}+2}), \varepsilon)\\
&\quad + \varepsilon E\Big(\sup_{t\le\tau}\|K_+u(t)\|_{L^2}^2\Big) + \varepsilon E\Big(\sup_{t\le\tau}\|J_+u(t)\|_{L^2}^2\Big).
\end{aligned} \tag{3.25}$$

We used Gagliardo-Nirenberg inequality again in the second inequality and ε can be chosen arbitrarily small.

On the other hand, by (3.17) and the similar estimates as above, we get

$$\begin{aligned}
E\Big(\sup_{t\le\tau}\|K_+u(t)\|_{L^2}^2\Big) &\le C(T_0,\phi,u_0,\varepsilon) + E\Big(\sup_{t\le\tau}\|J_+u(t)\|_{L^2}^2\Big)\\
&\quad + \varepsilon E\Big(\sup_{t\le\tau}\|J_+u(t)\|_{L^2}^2\Big) + \varepsilon E\Big(\sup_{t\le\tau}\|K_+u(t)\|_{L^2}^2\Big).
\end{aligned} \tag{3.26}$$

Together with (3.24)–(3.25), we can choose a proper $\varepsilon = \varepsilon(T_0)$, such that

$$E\Big(\sup_{t\le\tau}\|K_+u(t)\|_{L^2}^2\Big) \le C(T_0,\phi,u_0)$$

and

$$E\Big(\sup_{t\le\tau}\|J_+u(t)\|_{L^2}^2\Big) \le C(T_0,\phi,u_0),$$

which implies $\tau^* = +\infty$ a.s..

Now assume that $\sigma \ge \frac{2}{n}$ and $\lambda < 0$. Then $H_1(u(t))$ is the sum of two positive terms. By Lemma 3.3 we have $\forall t < \tau^*$ a.s.,

$$\begin{aligned}
&\frac{1}{2}\|J_+u(t)\|_{L^2}^2 + \frac{|\lambda|}{2\sigma+2}\cosh^2(2\mu t)\|u(t)\|_{L^{2\sigma+2}}^{2\sigma+2}\\
&= H_1(u_0) + \frac{(n\sigma-2)\mu\lambda}{2\sigma+2}\int_0^t \sinh(4\mu s)\|u(s)\|_{L^{2\sigma+2}}^{2\sigma+2}\mathrm{d}s + M_1(t) + R_1(t).
\end{aligned} \tag{3.27}$$

Notice that the second term of RHS is nonpositive so that we can omit it. To control the second term of LHS, we use the martingale inequality and Burkholder-Davis-Gundy inequality again as in (3.25) and get

$$\begin{aligned}&\frac{|\lambda|}{2\sigma+2}E\Big(\sup_{t\leq\tau}\cosh^2(2\mu t)\|u(t)\|^{2\sigma+2}_{L^{2\sigma+2}}\Big)\\ &\leq C(T_0,\phi,u_0,\varepsilon,\widetilde{\varepsilon})+\varepsilon E\Big(\sup_{t\leq\tau}\|J_+u(t)\|^2_{L^2}\Big)+\varepsilon E\Big(\sup_{t\leq\tau}\|K_+u(t)\|^2_{L^2}\Big)\\ &\quad+\widetilde{\varepsilon}E\Big(\sup_{t\leq\tau}\cosh^2(2\mu t)\|u(t)\|^{2\sigma+2}_{L^{2\sigma+2}}\Big).\end{aligned}\tag{3.28}$$

Choosing $\widetilde{\varepsilon}$ properly small and using (3.27) again, we have

$$\begin{aligned}&\frac{1}{2}E\Big(\sup_{t\leq\tau}\|J_+u(t)\|^2_{L^2}\Big)\\ &\leq C(T_0,\phi,u_0,\varepsilon)+\varepsilon E\Big(\sup_{t\leq\tau}\|K_+u(t)\|^2_{L^2}\Big)+\varepsilon E\Big(\sup_{t\leq\tau}\|J_+u(t)\|^2_{L^2}\Big).\end{aligned}\tag{3.29}$$

Again, we estimate (3.17) in the same way and get

$$\begin{aligned}\frac{1}{2}E\Big(\sup_{t\leq\tau}\|K_+u(t)\|^2_{L^2}\Big)&\leq C(T_0,\phi,u_0,\varepsilon)+\Big(\frac{1}{2}+\varepsilon\Big)E\Big(\sup_{t\leq\tau}\|J_+u(t)\|^2_{L^2}\Big)\\ &\quad+\varepsilon E\Big(\sup_{t\leq\tau}\|K_+u(t)\|^2_{L^2}\Big).\end{aligned}\tag{3.30}$$

With $\varepsilon=\varepsilon(T_0)$ properly small, (3.29) and (3.30) imply (3.16), which gives the proof for limited data u_0.

A localization argument will give the conclusion for any $\mathcal{F}_0$-measurable u_0. Indeed, we already have the results on $\Omega_N=\{\omega\in\Omega,\|u_0(\omega)\|_\Sigma\leq N\}$, where we substitute the expectation with $E_N(\cdot)=\frac{E(\cdot\chi_{\Omega_N})}{P(\Omega_N)}$ and the measure with $P_N(\cdot)=\frac{P(\cdot\chi_{\Omega_N})}{P(\Omega_N)}$, respectively. Notice that $P(\Omega_N)\to 1$ as $N\to\infty$, a limit procedure completes the proof.

Remark 3.2 When $\theta<0$ and $\lambda<0$, $H(u)$ gives a uniform control of Σ-norm to the solution $u(t)$, and a priori estimate of (3.17) leads to the global well-posedness. However, if $\lambda>0$ and $\sigma<\frac{2}{n}$, the similar argument as the first case in the proof of Theorem 3.2 implies GWP. We use Gagliardo-Nirenberg inequality (2.8) instead of (2.6) in both cases. So we have the next GWP result.

Corollary 3.1 *Let σ, u_0, ϕ be as in Theorem* 3.1*. Then GWP holds in either of the two cases:* (1) $\lambda\leq 0$*;* (2) $\sigma<\frac{2}{n}$.

4 Finite Time Blow Up

From Section 3, we know that when $\lambda\leq 0$ or $\sigma<\frac{2}{n}$, the solution is global in time. However, it does not tell how it acts when $\lambda>0$ and $\sigma\geq\frac{2}{n}$. In the deterministic case (see [5–6]), R. Carles gave the sufficient condition such that $u(t)$ blows up in finite time. In this section, we will give a sufficient condition for the stochastic case.

Theorem 4.1 *Assume that* $\frac{2}{n} \le \sigma < \frac{2}{n-2}$ *if* $n \ge 3$ *or* $\sigma \ge \frac{2}{n}$ *for* $n = 1, 2$, $\lambda > 0$, $\phi \in L_2(L^2, \Sigma)$ *and the initial data* u_0 *is an* $\mathcal{F}_0$*-measurable random variable with values in* Σ, *such that*

$$E(\|u_0\|_\Sigma^2) < \infty, \quad E(\|u_0\|_{L^{2\sigma+2}}^{2\sigma+2}) < \infty. \tag{4.1}$$

Then blow up phenomenon occurs in the following two cases:

(1) *Case* $\theta > 0$, *if* u_0 *satisfies*

$$\begin{aligned} &E(\||\cdot|u_0\|_{L^2}^2) + \frac{E\|\nabla u_0\|_{L^2}^2}{\mu^2} - \frac{\lambda}{(\sigma+1)\mu^2} E(\|u_0\|_{L^{2\sigma+2}}^{2\sigma+2}) - \frac{2E(\operatorname{Im}\int_{\mathbb{R}^n} u_0 x \cdot \nabla \overline{u}_0 \mathrm{d}x)}{\mu} \\ &< -\frac{c_\phi^\Sigma}{4\mu} - \frac{c_\phi^2}{2\mu^2} - \frac{c_\phi^1}{4\mu^3}, \end{aligned} \tag{4.2}$$

where $\mu = \sqrt{\theta}$, *then there exists a* $T_1 > 0$ *and either of the following happens:*

$$E\int_0^{T_1} (\|\nabla u\|_{L^2}^2 + \|u\|_{L^{2\sigma+2}}^{4\sigma+2}) \mathrm{d}s = \infty$$

or

$$P(\tau^* \le T_1) > 0.$$

(2) *Case* $\theta < 0$, *if* u_0 *satisfies*

$$\frac{1}{2}E\big(\|\nabla u_0\|_{L^2}^2\big) - \frac{\lambda}{2\sigma+2} E\big(\|u_0\|_{L^{2\sigma+2}}^{2\sigma+2}\big) + f_\phi\Big(\frac{\pi}{4\nu}\Big) \le 0, \tag{4.3}$$

where $\nu = \sqrt{-\theta}$, *then we have either*

$$E\int_0^{\frac{\pi}{4\nu}} (\|\nabla u\|_{L^2}^2 + \|u\|_{L^{2\sigma+2}}^{4\sigma+2}) \mathrm{d}s = \infty$$

or

$$P\Big(\tau^* < \frac{\pi}{4\nu}\Big) > 0.$$

Here

$$f_\phi(t) = \frac{1}{4}(\nu^2 c_\phi^\Sigma + c_\phi^1)t + \frac{1}{16}\Big(\frac{c_\phi^1}{\nu} - \nu c_\phi^\Sigma\Big)\sin(4\nu t) - \frac{c_\phi^2}{8\nu}\cos(4\nu t) + \frac{c_\phi^2}{8\nu}. \tag{4.4}$$

Remark 4.1 If $\phi = 0$, conditions (4.2) and (4.3) coincide with the ones for the deterministic equation (see [5–6]). However, for the repulsive harmonic potential, i.e., $\theta > 0$, the existence of the noise term makes the condition even stronger in the average sense. Indeed, notice that $\frac{|c_\phi^2|}{\mu^2} \le \frac{c_\phi^\Sigma}{2\mu} + \frac{c_\phi^1}{2\mu^3}$, RHS of (4.2) is no more than $-\frac{|c_\phi^2|}{2\mu^2} - \frac{c_\phi^2}{2\mu^2} \le 0$. Thus, compared with the deterministic equation, this condition is stronger on average, and depends on the regularity of the noise.

4.1 Stochastic variance identity and square integrable martingales

In this section we generalize "the variance identity" for the deterministic Schrödinger equation to the stochastic case with the help of the Itô's formula. We need the following quantities:

$$\begin{aligned}V(v)&=\int_{\mathbb{R}^n}|x|^2|v|^2\mathrm{d}x,\\ G(v)&=\mathrm{Im}\int_{\mathbb{R}^n}vx\cdot\nabla\overline{v}\mathrm{d}x,\\ M(v)&=\int_{\mathbb{R}^n}|v|^2\mathrm{d}x.\end{aligned}$$

By using the Itô's formula and a regularizing technique (see [1, Propositions 3.2 and 3.3]), we can get the following lemmas.

Lemma 4.1 *Let u_0, σ and ϕ be as in Theorem 4.1. Then for any stopping time $\tau<\tau^*$ a.s., we have $u\in L^\infty((0,\tau),\Sigma)$ and*

$$\begin{aligned}V(u(\tau))&=V(u_0)-4\mathrm{Im}\int_0^\tau\int_{\mathbb{R}^n}ux\cdot\nabla\overline{u}\mathrm{d}x\mathrm{d}t-2\mathrm{Im}\int_{\mathbb{R}^n}\int_0^\tau|x|^2u\overline{\phi}\,\overline{\mathrm{d}W}\mathrm{d}x+\tau c_\phi^\Sigma\\ &=V(u_0)-4\int_0^\tau G(u(t))\mathrm{d}t-2\mathrm{Im}\int_{\mathbb{R}^n}\int_0^\tau|x|^2u\overline{\phi}\,\overline{\mathrm{d}W}\mathrm{d}x+\tau c_\phi^\Sigma.\end{aligned}$$

Lemma 4.2 *Let u_0, σ and ϕ be as in Theorem 4.1. Then for any stopping time $\tau<\tau^*$ a.s., we have*

$$\begin{aligned}G(u(\tau))&=G(u_0)-4\int_0^\tau H(u(t))\mathrm{d}t-4\theta\int_0^\tau\int_{\mathbb{R}^n}|x|^2|u|^2\mathrm{d}x\mathrm{d}t\\ &\quad-\frac{\lambda(2-n\sigma)}{\sigma+1}\int_0^\tau\int_{\mathbb{R}^n}|u|^{2\sigma+2}\mathrm{d}x\mathrm{d}t-\mathrm{Re}\int_{\mathbb{R}^n}\int_0^\tau(2\nabla u\cdot x+nu)\overline{\mathrm{d}W}\mathrm{d}x-\tau c_\phi^2.\end{aligned}$$

As a consequence of Lemma 4.1 and Lemma 4.2, we have the next corollary.

Corollary 4.1 (Stochastic Variance Identity) *Let u_0, σ and ϕ be as in Theorem 4.1. Then for any stopping time $\tau<\tau^*$ a.s., we have*

$$\begin{aligned}V(u(\tau))&=V(u_0)-4G(u_0)\tau+8H(u_0)\tau^2+16\theta\int_0^\tau\int_0^t V(u(s))\mathrm{d}s\mathrm{d}t\\ &\quad+\frac{4\lambda(2-n\sigma)}{\sigma+1}\int_0^\tau\int_0^t\int_{\mathbb{R}^n}|u|^{2\sigma+2}\mathrm{d}x\mathrm{d}s\mathrm{d}t+c_\phi^\Sigma\tau+2c_\phi^2\tau^2+\Big(\frac{4}{3}c_\phi^1-\frac{4}{3}\theta c_\phi^\Sigma\Big)\tau^3\\ &\quad-8\lambda\int_0^\tau\int_0^t\int_0^s\sum_l\int_{\mathbb{R}^n}|u|^{2\sigma}|\phi e_l|^2\mathrm{d}x\mathrm{d}r\mathrm{d}s\mathrm{d}t\\ &\quad-16\sigma\lambda\int_0^\tau\int_0^t\int_0^s\sum_l\int_{\mathbb{R}^n}|u|^{2\sigma-2}(\mathrm{Im}(\overline{u}\phi e_l))^2\mathrm{d}x\mathrm{d}r\mathrm{d}s\mathrm{d}t\\ &\quad-2\mathrm{Im}\int_{\mathbb{R}^n}\int_0^\tau|x|^2u\overline{\mathrm{d}W}\mathrm{d}x+4\mathrm{Re}\int_{\mathbb{R}^n}\int_0^\tau\int_0^t(2\nabla u\cdot x+nu)\overline{\mathrm{d}W}\mathrm{d}t\mathrm{d}x\\ &\quad+16\mathrm{Im}\int_{\mathbb{R}^n}\int_0^\tau\int_0^t\int_0^s(\Delta u+\theta|x|^2u+\lambda|u|^{2\sigma}u)\overline{\mathrm{d}W}\mathrm{d}s\mathrm{d}t\mathrm{d}x.\end{aligned}\tag{4.5}$$

Set

$$N_1(t) = 2\mathrm{Im}\int_{\mathbb{R}^n}\int_0^t |x|^2 u\overline{\mathrm{d}W}\mathrm{d}x,$$
$$N_2(t) = 4\mathrm{Re}\int_{\mathbb{R}^n}\int_0^t (2\nabla u\cdot x + nu)\overline{\phi}\,\overline{\mathrm{d}W}\mathrm{d}x,$$
$$N_3(t) = 16\mathrm{Im}\int_{\mathbb{R}^n}\int_0^t (\Delta u + \theta|x|^2 u + \lambda|u|^{2\sigma}u)\overline{\mathrm{d}W}\mathrm{d}x.$$

We claim that $N_1(t)$, $N_2(t)$ and $N_3(t)$ are square integrable martingales under the assumptions of Theorem 4.1.

Theorem 4.2 *Assume that*

$$E(\|u_0\|_{H^1}^2) < \infty, \quad E(\|u_0\|_{L^{2\sigma+2}}^{2\sigma+2}) < \infty, \quad E(V(u_0)) < \infty, \tag{4.6}$$

and there exists a $T > 0$ with $T < \tau^*(u_0)$ a.s., such that

$$E\Big(\int_0^T (\|\nabla u(s)\|_{L^2}^2 + \|u(s)\|_{L^{2\sigma+2}}^{4\sigma+2})\mathrm{d}s\Big) < \infty. \tag{4.7}$$

Then $\sup\limits_{t\in[0,T]} E(V(u(t))) < \infty$ and N_1, N_2, N_3 are square integrable martingales.

Proof We estimate N_1, N_2 and N_3 first. For $N_1(t) = 2\mathrm{Im}\int_{\mathbb{R}^n}\int_0^t |x|^2 u\overline{\mathrm{d}W}\mathrm{d}x$, we have

$$E(N_1^2(t)) \le CE\int_0^t \|\phi^*(|\cdot|^2 u)\|_{L^2}^2\mathrm{d}s \le C(\phi)E\int_0^t V(u(s))\mathrm{d}s. \tag{4.8}$$

The last inequality makes sense given $\sup\limits_{s\in[0,t]} EV(u(s)) < \infty$.

For $N_2(t) = 4\mathrm{Re}\int_{\mathbb{R}^n}\int_0^t(2\nabla u\cdot x + nu)\overline{\mathrm{d}W}\mathrm{d}x = -4\mathrm{Re}\int_{\mathbb{R}^n}\int_0^t(2ux\cdot\nabla + nu)\overline{\mathrm{d}W}\mathrm{d}x$, we have

$$\begin{aligned}
E(N_2^2(t)) &\le E\Big(\sum_l\int_0^t\int_{\mathbb{R}^n}(2u(s,x)x\cdot\nabla\overline{\phi}\,\overline{e_l(x)} + nu(s,x)\overline{\phi}\,\overline{e_l(x)})\mathrm{d}x\mathrm{d}\beta_l(s)\Big)^2\\
&= E\sum_l\int_0^t\Big(\int_{\mathbb{R}^n}(2u(s,x)x\cdot\nabla\overline{\phi}\,\overline{e_l(x)} + nu(s,x)\overline{\phi}\,\overline{e_l(x)})\mathrm{d}x\Big)^2\mathrm{d}s\\
&\le C(\phi)\Big(\int_0^t E(V(u(s)))\mathrm{d}s + tE\|u_0\|_{L^2}^2\Big),
\end{aligned} \tag{4.9}$$

where in the last inequality we have used the fact that $E\big(\sup\limits_{s\in[0,t]}\|u(s)\|_{L^2}^2\big)$ can be controlled by $E(\|u_0\|_{L^2}^2)$ (see [1, Proposition 3.2]).

For $N_3(t)$, we have

$$\begin{aligned}
E(N_3^2(t)) &\le CE\Big(\sum_l\int_{\mathbb{R}^n}\int_0^t(\Delta u + \theta|x|^2u + \lambda|u|^{2\sigma}u)\overline{\phi}\,\overline{e_l}\mathrm{d}\beta_l(s)\mathrm{d}x\Big)^2\\
&\le CE\sum_l\int_0^t\Big(\int_{\mathbb{R}^n}(\Delta u + \theta|x|^2u + \lambda|u|^{2\sigma}u)\overline{\phi}\,\overline{e_l}\mathrm{d}x\Big)^2\mathrm{d}s\\
&\le CE\sum_l\int_0^t(\|\nabla u\|_{L^2}^2\|\nabla\phi\,e_l\|_{L^2}^2 + V(u(s))\||\cdot|\phi e_l\|_{L^2}^2 + \|u\|_{L^{2\sigma+2}}^{4\sigma+2}\|\phi e_l\|_{L^{2\sigma+2}}^2)\mathrm{d}s.
\end{aligned}$$

If $E\int_0^t(\|\nabla u\|_{L^2}^2+\|u\|_{L^{2\sigma+2}}^{4\sigma+2})\mathrm{d}s<\infty$, then $E(N_3^2(t))$ could be controlled by

$$C(\phi)E\int_0^t(\|\nabla u\|_{L^2}^2+\|u\|_{L^{2\sigma+2}}^{4\sigma+2})\mathrm{d}s+C(\phi)\int_0^t E(V(u(s)))\mathrm{d}s. \tag{4.10}$$

Given $\sup\limits_{t\in[0,T]}E(V(u(t)))<\infty$, from (4.8)–(4.10) we know that $N_1(t)$, $N_2(t)$ and $N_3(t)$ are square integrable martingales on $[0,T]$, where the martingale property is satisfied by the definition of the stochastic integral. Then we only need to show $\sup\limits_{t\in[0,T]}E(V(u(t)))<\infty$.

Set $\tau_k=\inf\{s\in[0,T],V(u(s))\geq k\}$. In stead of τ, in Corollary 4.1 we take the stopping time $t\wedge\tau_k$, where t is in $[0,T]$. Since $\lambda>0$, $\sigma\geq\frac{2}{n}$, the fifth, eighth and tenth terms are nonpositive which we can omit. We also suppose $\theta>0$ and then control the forth term by $16\theta T\int_0^t EV(u(s\wedge\tau_k))\mathrm{d}s$. The case when $\theta<0$ is even simpler since we can omit the forth term directly. We estimate the last two integrals as follows:

$$\begin{aligned}
&E\Big|\mathrm{Re}\int_{\mathbb{R}^n}\int_0^{t\wedge\tau_k}\int_0^s(2\nabla u\cdot x+nu)\overline{\mathrm{d}W}\mathrm{d}s\mathrm{d}x\Big|\\
&=E\Big|\mathrm{Re}\int_{\mathbb{R}^n}\int_0^{t\wedge\tau_k}\int_0^s\sum_l(2ux\cdot\nabla\phi+nu\phi)\overline{e_l}\mathrm{d}\beta_l\mathrm{d}s\mathrm{d}x\Big|\\
&\leq\int_0^t\Big(E\int_0^{s\wedge\tau_k}\Big|\sum_l(2ux\cdot\nabla\phi+nu\phi)\overline{e_l}\Big|_{L^1}^2\mathrm{d}r\Big)^{\frac12}\mathrm{d}s\\
&\leq C\int_0^t\Big(c_\phi^1E\int_0^{s\wedge\tau_k}\int_{\mathbb{R}^n}|x|^2|u(r,x)|^2\mathrm{d}x\mathrm{d}r+\|\phi\|_{L_2(L^2,\Sigma)}^2E\int_0^{s\wedge\tau_k}\int_{\mathbb{R}^n}|u|^2\mathrm{d}x\mathrm{d}r\Big)^{\frac12}\mathrm{d}s\\
&\leq C(\phi)\int_0^t\Big(E\int_0^s V(u(r\wedge\tau_k))\mathrm{d}r+TE\|u_0\|_{L^2}^2\Big)^{\frac12}\mathrm{d}s\\
&\leq C(\phi,T)\Big(\int_0^t EV(u(s\wedge\tau_k))\mathrm{d}s\Big)^{\frac12}+C(\phi,E\|u_0\|_{L^2}^2,T),
\end{aligned}\tag{4.11}$$

and similarly,

$$\begin{aligned}
&E\Big|\mathrm{Im}\int_{\mathbb{R}^n}\int_0^t\int_0^s\int_0^r(\Delta u+\theta|x|^2u+\lambda|u|^{2\sigma}u)\overline{\mathrm{d}W}\mathrm{d}r\mathrm{d}s\mathrm{d}x\Big|\\
&\leq\int_0^t\int_0^s E\Big|\int_{\mathbb{R}^n}\int_0^{r\wedge\tau_k}(\Delta u+\theta|x|^2u+\lambda|u|^{2\sigma}u)\overline{\mathrm{d}W}\mathrm{d}x\Big|\mathrm{d}r\mathrm{d}s\\
&\leq T\int_0^t\Big(E\int_0^{s\wedge\tau_k}|\phi^*(\Delta u+\theta|x|^2u+\lambda|u|^{2\sigma}u)|_{L^2}^2\mathrm{d}r\Big)^{\frac12}\mathrm{d}s\\
&\leq T^2\Big(\|\phi\|_{L_2(L^2,H^1)}^2E\int_0^T(\|\nabla u\|_{L^2}^2+\|u\|_{L^{2\sigma+2}}^{4\sigma+2})\mathrm{d}s+c_\phi^\Sigma E\int_0^{t\wedge\tau_k}V(u(s))\mathrm{d}s\Big)^{\frac12}\\
&\leq C(\phi,T)\Big(E\int_0^T(\|\nabla u\|_{L^2}^2+\|u\|_{L^{2\sigma+2}}^{4\sigma+2})\mathrm{d}s\Big)^{\frac12}+C(\phi,T)\Big(E\int_0^t V(u(s\wedge\tau_k))\mathrm{d}s\Big)^{\frac12}.
\end{aligned}\tag{4.12}$$

So we have

$$\begin{aligned}
&E(V(u(t\wedge\tau_k)))\\
&\leq C(T)\int_0^t EV(u(s\wedge\tau_k))\mathrm{d}s+C\Big(EV(u_0),EG(u_0),EH(u_0),\phi,T,EM(u_0),\\
&\quad E\int_0^T(\|\nabla u\|_{L^2}^2+\|u\|_{L^{2\sigma+2}}^{4\sigma+2})\mathrm{d}s\Big).
\end{aligned}$$

Then by Gronwall's Lemma, we have

$$\begin{aligned} E(V(u(t\wedge\tau_k))) \le C\Big(&EV(u_0), EG(u_0), EH(u_0), \phi, T, EM(u_0), \\ &E\int_0^T (\|\nabla u\|_{L^2}^2 + \|u\|_{L^{2\sigma+2}}^{4\sigma+2})\mathrm{d}s\Big) \end{aligned} \tag{4.13}$$

uniformly for $t \in [0,T]$ and $k \in \mathbb{N}$.

Letting $k \to \infty$ and taking supremum of $t \in [0,T]$, we get the conclusion that

$$\sup_{t\in[0,T]} E(V(u(t))) < \infty,$$

which makes sure that N_1, N_2, N_3 are square integrable martingales.

4.2 Case $\theta > 0$, proof of the first part of Theorem 4.1

We prove by contradiction, i.e., assume for any $T > 0$, both of the following happen:

$$E\int_0^T \big(\|\nabla u\|_{L^2}^2 + \|u\|_{L^{2\sigma+2}}^{4\sigma+2}\big)\mathrm{d}s < \infty$$

and

$$P(\tau^* > T) = 1.$$

Then for any $t \in [0,T]$ we can apply Corollary 4.1 and take expectation of (4.5). To make the expression simpler, we denote

$$\begin{aligned} &Y(t) = E(V(u(t))), \quad A = c_\phi^\Sigma - 4E(G(u_0)), \\ &B = 2c_\phi^2 + 8E(H(u_0)), \quad C = \frac{4}{3}c_\phi^1 - \frac{4}{3}\theta c_\phi^\Sigma, \\ &P = \frac{4\lambda(n\sigma-2)}{\sigma+1}\int_0^t\int_0^s\int_{\mathbb{R}^n} E|u|^{2\sigma+2}\mathrm{d}x\mathrm{d}r\mathrm{d}s \\ &\quad + 8\lambda\int_0^t\int_0^s\int_0^r\sum_l\int_{\mathbb{R}^n} E|u|^{2\sigma}|\phi e_l|^2\mathrm{d}x\mathrm{d}\theta\mathrm{d}r\mathrm{d}s \\ &\quad + 16\sigma\lambda\int_0^t\int_0^s\int_0^r\sum_l\int_{\mathbb{R}^n} E|u|^{2\sigma-2}(\mathrm{Im}(\overline{u}\phi e_l))^2\mathrm{d}x\mathrm{d}\theta\mathrm{d}r\mathrm{d}s. \end{aligned} \tag{4.14}$$

Hence $P > 0$. Notice that under the assumptions, the last three stochastic integrals are square integrable and become zero after taking expectation, that is,

$$\begin{aligned} Y(t) &= Y(0) + At + Bt^2 + Ct^3 + 16\theta\int_0^t\int_0^s Y(r)\mathrm{d}r\mathrm{d}s - P \\ &= Y(0) + At + Bt^2 + Ct^3 + 16\theta\int_0^t (t-s)Y(s)\mathrm{d}s - P, \end{aligned}$$

which deduces

$$
\begin{aligned}
Y(t) &\le Y(0) + At + Bt^2 + Ct^3 + 4\mu \int_0^t \sinh(4\mu(t-s))(Y(0) + As + Bs^2 + Cs^3)\mathrm{d}s \\
&= Y(0)\cosh(4\mu t) + \frac{A}{4\mu}\sinh(4\mu t) - \frac{B}{8\mu^2} + \frac{B}{8\mu^2}\cosh(4\mu t) - \frac{3Ct}{8\mu^2} + \frac{3C}{32\mu^3}\sinh(4\mu t) \\
&= \cosh^2(2\mu t)\Big(2Y(0) + \frac{A}{2\mu}\tanh(2\mu t) + \frac{B}{4\mu^2}\tanh^2(2\mu t) + \frac{3C}{16\mu^3}\tanh(2\mu t)\Big) \\
&\quad - \frac{3Ct}{8\mu^2} - Y(0),
\end{aligned} \tag{4.15}
$$

where $\mu = \sqrt{\theta}$.

Noticing that $\cosh^2(2\mu t)$ increases exponentially, we only need to consider the first term in (4.15). Since $\tanh(2\mu t) \in [0,1]$ as $t \in [0,+\infty)$ and $\tanh(2\mu t) \to 1$ as $t \to +\infty$, if in addition

$$
2Y(0) + \frac{A}{2\mu} + \frac{B}{4\mu^2} + \frac{3C}{16\mu^3} < 0, \tag{4.16}
$$

then there exists a T_1 large enough such that (4.15) is less than zero at $t = T_1$.

Now we claim that (4.16) is the sufficient condition for the solution u to (2.1) to blow up in positive time. Indeed, if (4.2) is satisfied, under the assumptions of Theorem 4.1, if $E\int_0^{T_1}(\|\nabla u\|_{L^2}^2 + \|u\|_{L^{2\sigma+2}}^{4\sigma+2})\mathrm{d}s < \infty$ and $P(T_1 < \tau^*) = 1$, then $Y(T_1) \le (4.15)\big|_{t=T_1} < 0$, which is a contradiction since $Y(t)$ is nonnegative. So we conclude that either

$$
E\int_0^{T_1}(\|\nabla u\|_{L^2}^2 + \|u\|_{L^{2\sigma+2}}^{4\sigma+2})\mathrm{d}s = \infty
$$

or

$$
P(T_1 < \tau^*) < 1,
$$

which completes the first part of the theorem.

4.3 Case $\theta < 0$, proof of the second part of Theorem 4.1

In this part, we use the operators J_- and K_- defined in (2.4). Noticing that

$$
|J_-u|^2 + \nu^2|K_-u|^2 = \nu^2|x|^2|u|^2 + |\nabla u|^2,
$$

we can split the energy into two parts,

$$
\widetilde{H_1}(u(t)) = \frac{1}{2}\|J_-u(t)\|_{L^2}^2 - \frac{\lambda}{2\sigma+2}\cos^2(2\nu t)\|u(t)\|_{L^{2\sigma+2}}^{2\sigma+2}, \tag{4.17}
$$

$$
\widetilde{H_2}(u(t)) = \frac{\nu^2}{2}\|K_-u(t)\|_{L^2}^2 - \frac{\lambda}{2\sigma+2}\sin^2(2\nu t)\|u(t)\|_{L^{2\sigma+2}}^{2\sigma+2}. \tag{4.18}
$$

Instead of $E(V(u))$, we estimate the evolution of $E(\widetilde{H_1}(u))$ here. We need the following lemma, which is not hard to derive with the help of the Itô's formula.

Lemma 4.3 *For any stopping time $\tau < \tau^*$ a.s., we have*

$$
\widetilde{H_1}(u(\tau)) = \widetilde{H_1}(u_0) + \frac{\nu\lambda(2-n\sigma)}{2\sigma+2}\int_0^\tau\int_{\mathbb{R}^n}\sin(4\nu s)|u|^{2\sigma+2}\mathrm{d}x\mathrm{d}s + \widetilde{M_1}(t) + \widetilde{R_1}(t), \tag{4.19}
$$

where

$$\widetilde{M_1}(t) = \frac{1}{2}\mathrm{Re}\int_{\mathbb{R}^n}\int_0^\tau \nu\sin(4\nu s)(2x\cdot\nabla u + nu)\overline{\mathrm{d}W}\mathrm{d}x - \nu^2\mathrm{Im}\int_{\mathbb{R}^n}\int_0^\tau |x|^2\sin^2(2\nu s)u\overline{\mathrm{d}W}\mathrm{d}x$$
$$+ \mathrm{Im}\int_{\mathbb{R}^n}\int_0^\tau \cos^2(2\nu s)\Delta u\overline{\mathrm{d}W}\mathrm{d}x + \lambda\mathrm{Im}\int_{\mathbb{R}^n}\int_0^\tau \cos^2(2\nu s)|u|^{2\sigma}u\overline{\mathrm{d}W}\mathrm{d}x, \tag{4.20}$$

$$\widetilde{R_1}(t) = \frac{1}{4}(\nu^2 c_\phi^\Sigma + c_\phi^1)t + \frac{1}{16}\Big(\frac{c_\phi^1}{\nu} - \nu c_\phi^\Sigma\Big)\sin(4\nu t) - \frac{c_\phi^2}{8\nu}\cos(4\nu t) + \frac{c_\phi^2}{8\nu}$$
$$- \frac{\lambda}{2}\sum_l\int_0^t\int_{\mathbb{R}^n}\cos^2(2\nu s)|u|^{2\sigma}|\phi e_l|^2\mathrm{d}x\mathrm{d}s$$
$$- \sigma\lambda\sum_l\int_0^t\int_{\mathbb{R}^n}\cos^2(2\nu s)|u|^{2\sigma-2}(\mathrm{Im}(\overline{u}\phi e_l))^2\mathrm{d}x\mathrm{d}s. \tag{4.21}$$

Notice that by (4.4)*, we have for any* $t\in\mathbb{R}$*,* $\widetilde{R_1}(t) < f_\phi(t)$ *with probability* 1*.*

Following the proof of Theorem 4.2, we can see, under the same assumptions, $\widetilde{M_1}(t)$ are square integrable martingales.

Corollary 4.2 *Assume that*

$$E(\|u_0\|_{H^1}^2) < \infty, \quad E(\|u_0\|_{L^{2\sigma+2}}^{2\sigma+2}) < \infty, \quad E(V(u_0)) < \infty,$$

and there exists a $T>0$ *with* $T<\tau^*(u_0)$ *a.s., such that*

$$E\Big(\int_0^T(\|\nabla u(s)\|_{L^2}^2 + \|u(s)\|_{L^{2\sigma+2}}^{4\sigma+2})\mathrm{d}s\Big) < \infty. \tag{4.22}$$

Then $\sup\limits_{t\in[0,T]} E(V(u(t))) < \infty$*, and* $\widetilde{M_1}(t)$ *are square integrable martingales.*

Now we can complete the proof of Theorem 4.1. Set $T=\frac{\pi}{4\nu}$, and assume

$$E\int_0^T(\|\nabla u\|_{L^2}^2 + \|u\|_{L^{2\sigma+2}}^{4\sigma+2})\mathrm{d}s < \infty, \tag{4.23}$$
$$P(\tau^* > T) = 1. \tag{4.24}$$

Take expectation of (4.19), where we choose $\tau = T$. Then the stochastic integrals vanish, and for $t\in[0,T]$, we have

$$E(\widetilde{H_1}(u(t))) = E(\widetilde{H_1}(u_0)) + \frac{\nu\lambda(2-n\sigma)}{2\sigma+2}\int_0^t\int_{\mathbb{R}^n}\sin(4\nu s)E(|u(s,x)|^{2\sigma+2})\mathrm{d}x\mathrm{d}s + E(\widetilde{R_1}). \tag{4.25}$$

Since $\sigma\ge\frac{2}{n}$ and $\sin(4\nu t)\ge 0$ when $t\in[0,T]$, we have

$$E(\widetilde{H_1}(u(T))) < E(\widetilde{H_1}(u_0)) + f_\phi(T) \le 0, \tag{4.26}$$

which contradicts the fact that $E(\widetilde{H_1}(u(\frac{\pi}{4\nu}))) = \frac{\nu^2}{2}\|\,|\cdot|u\|_{L^2}^2 \ge 0$. So we come to the conclusion that either

$$E\int_0^{\frac{\pi}{4\nu}}(\|\nabla u\|_{L^2}^2 + \|u\|_{L^{2\sigma+2}}^{4\sigma+2})\mathrm{d}s = \infty \tag{4.27}$$

or

$$P\Big(\tau^*(u_0) < \frac{\pi}{4\nu}\Big) > 0, \tag{4.28}$$

which completes our proof.

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