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Indecomposable Calabi-Yau Objects in Stable Module Categories of Finite Type*

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Abstract The authors give a discription of the stable categories of selfinjective algebras of finite representation type over an algebraically closed field, which admits indecomposable Calabi-Yau obdjects. For selfinjective algebras with such properties, the ones whose stable categories are not Calabi-Yau are determined. For the remaining ones, i.e., those selfinjective algebras whose stable categories are actually Calabi-Yau, the difference between the Calabi-Yau dimensions of the indecomposable Calabi-Yau objects and the Calabi-Yau dimensions of the stable categories is described.

Keywords Selfinjective algebra, Stable category, Calabi-Yau category, Indecomposable Calabi-Yau object
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1 Introduction

Calabi-Yau categories were introduced by Kontsevich [9]. Such kind of categories have some good properties as something like "global naturality". However, in some non-Calabi-Yau categories, there are objects having similar properties. In other words, they enjoy some kind of "local naturality". Inspired by this, Cibils and Zhang introduced the concept of Calabi-Yau objects in [4].

Let k be a field. For the term algebra, throughout this paper, we mean a finite dimensional associative k-algebra with identity. For an algebra A, we denote by $\operatorname{mod} A$ the full subcategory of the module category over A consisting of finitly generated left modules, and by $\underline{\operatorname{mod}}\, A$ the stable category of A. If A is selfinjective, then $\underline{\operatorname{mod}}\, A$ is a triangulated category, where the shift functor is given by the inverse of syzygy functor Ω_A^{-1}, and the distinguished triangles are given by the exact sequences in $\operatorname{mod} A$ (see [8]). An important class of selfinjective algebras are those of finite representation type. It is known that such selfinjective algebras can be divided into two disjoint classes (see [10]): the standard ones which admit simply connected Galois coverings, and the remaining non-standard algebras. For the standard ones, there are some invariants (up to stable equivalence) associated to them. To be more presicely, if A is a non-simple standard

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selfinjective algebra of finite representation type, we may associate a Dynkin graph $\Delta(A)$ and integers $s(A) \geq 1$, $t(A) \in \{1, 2, 3\}$ to A (see Section 2). For a Dynkin diagram Δ, we denote by h_Δ the Coxeter number of Δ. The Coxeter numbers of Dynkin diagrams are as follows: $h_{A_n} = n+1$, $h_{D_n} = 2n-2$, $h_{E_6} = 12$, $h_{E_7} = 18$, $h_{E_8} = 30$. The stable categories of the non-standard selfinjective algebras are all Calabi-Yau (see [3]). Therefore, we only need to investigate the standard cases.

In [3], Bialkowski and Skowroński gave the classification of selfinjective algebras of finite representation type whose stable categories are Calabi-Yau. In [4], the authors gave a description of Calabi-Yau objects in a hom-finite triangulated category. In particular, they classified all the d-th Calabi-Yau objects in the stable categories of selfinjective Nakayama algebras for any integer d, and determined all the selfinjective Nakayama algebras whose stable categories have indecomposable Calabi-Yau objects. Recently, in [6], Dugas computed the Calabi-Yau dimension of the stable categories of selfinjective algebras of finite representation type and made corrections to some results in [3].

Let A be a non-simple standard selfinjective algebra of finite representation type. It is easy to see that $\underline{\mathrm{mod}}\, A$ always has Calabi-Yau objects. The aim of this paper is to give the following theorem answering the question when $\underline{\mathrm{mod}}\, A$ has indecomposable Calabi-Yau objects. This includes the related results in [4]. In the following, a stable category of finite type means a stable category of a selfinjective algebra of finite representation type.

Theorem 1.1 *Let A be a non-simple connected standard selfinjective algebra of finite representation type. Then $\underline{\mathrm{mod}}\, A$ has indecomposable Calabi-Yau objects if and only if $t(A) \leq 2$ and $\gcd(\widetilde{h}_\Delta, s) = 1$, where $\widetilde{h}_\Delta = h_\Delta$ for $\Delta = A_n$ (n even), and $\widetilde{h}_\Delta = \frac{h_\Delta}{2}$ for the other cases.*

This theorem will be proved in Section 3. After that we shall show that when the stable categories of finite type is not Calabi-Yau, but they have indecomposable Calabi-Yau objects. Of course, if a stable category is Calabi-Yau, then every object is a Calabi-Yau object. In this case, we are interested in the difference between the Calabi-Yau dimension of the category and the ones of Calabi-Yau objects.

2 Preliminaries

The functor $\mathrm{Hom}_k(-, k)$ is denoted by D in this paper. For an algebra A, we denote by $\upsilon_A : \underline{\mathrm{mod}}\, A \to \underline{\mathrm{mod}}\, A$ the Nakayama functor $D\,\mathrm{Hom}_A(-, A)$, and by τ_A the Auslander-Reiten translation DTr. It is known that $\tau_A \cong \Omega_A^2\, \upsilon_A \cong \upsilon_A\, \Omega_A^2$, if A is selfinjective.

Let $\mathscr{C}$ be a Hom-finite k-category. Recall that a k-linear functor $S : \mathscr{C} \to \mathscr{C}$ is called a right Serre functor if there are k-isomorphisms $\eta_{A,B} : \mathrm{Hom}_{\mathscr{C}}(A,\, B) \to D\,\mathrm{Hom}_{\mathscr{C}}(B,\, SA)$ for all A, B in $\mathscr{C}$, which are natural both at A and B. S is called a Serre functor if it is an equivalence. Such a functor is unique up to a natural isomorphism. If $\mathscr{C}$ is a triangulated category with Serre functor S, then S is a triangle functor. Let $\mathscr{C}$ be a Hom-finite triangulated k-category with Serre functor S. Denote by [1] the shift functor of $\mathscr{C}$. The category $\mathscr{C}$ is called a Calabi-Yau category if there is a natural isomorphism $S \cong [d]$ of functors for some $d \in \mathbb{Z}$ (see [4]). Denote by $o([1])$ the order of $[1]$. If $o([1]) = \infty$, then the integer d is unique and is called the Calabi-Yau dimension of $\mathscr{C}$, otherwise, the Calabi-Yau dimension of $\mathscr{C}$ is defined to be the minimal non-negative integer d such that $S \cong [d]$. We denote the Calabi-Yau dimension by CY-dim.

It is known that $S = \Omega_A^{-1}\,\tau_A = \Omega\,\upsilon_A$ is a Serre functor on $\underline{\mathrm{mod}}\,A$, where A is a selfinjective algebra. For a Hom-finite triangulated k-category $\mathscr{C}$, a non-zero object X is called a Calabi-Yau object (or, more precisely a d-th Calabi-Yau object) (see [4]), if there is a natural isomorphism $\mathrm{Hom}_{\mathscr{C}}(X,-) \cong D\,\mathrm{Hom}_{\mathscr{C}}(-,X[d])$ for some $d \in \mathbb{Z}$. Such a d is unique up to a multiple of the relative order $o([1]_X)$ of $[1]$ with respect to X, where $o([1]_X)$ is the minimal positive integer such that $X[o([1]_X)] \cong X$, otherwise, $o([1]_X) = \infty$. Similar to the definition of Calabi-Yau dimension of a Calabi-Yau category, the Calabi-Yau dimension of a Calabi-Yau object is defined to be the unique integer d in the equation above, if $o([1]_X) = \infty$ and the minimal non-negative integer d satisfies the equation otherwise. The Calabi-Yau dimension of an object X is denoted by CY-dim(X).

From now on, by an algebra, we also assume that it is non-simple, basic and connected. Moreover, the field k is assumed to be algebraically closed.

We denote by $\widehat{B}$ the repetitive algebra of an algebra B (see [8]). It is a locally bounded algebra with a complete set of primitive orthogonal idempotents $\{e_\lambda\}_{\lambda\in\Lambda}$. We define an automorphism of $\widehat{B}$ to be a k-algebra automorphism of $\widehat{B}$, which fixes the set $\{e_\lambda\}_{\lambda\in\Lambda}$. A group G of automorphisms of $\widehat{B}$ is said to be admissible, if the action of G on $\{e_\lambda\}_{\lambda\in\Lambda}$ is free and has finite many orbits. For example, if we define Nakayama automorphism $\upsilon_{\widehat{B}}$ of $\widehat{B}$ to be the automorphism of $\widehat{B}$ whose restriction on $B_k \oplus D(B_k)$ is the identity $B_k \oplus D(B_k) \to B_{k+1} \oplus D(B_{k+1})$, then the cyclic group generated by $\upsilon_{\widehat{B}}$ is admissible. In fact, $\upsilon_{\widehat{B}}$ induces the Nakayama action on $\mathrm{mod}\,\widehat{B}$. We can get an orbit algebra $\widehat{B}/G$ for any admissible group of automorphisms of $\widehat{B}$ (for more details about orbit algebras, we refer to [10]). $\widehat{B}/\langle \upsilon_{\widehat{B}} \rangle$ is just the trivial extension $TB = B \ltimes D(B)$, which is symmetric. An automorphism φ of $\widehat{B}$ is said to be positive (respectively, rigid), if $\varphi(B_k) \subseteq \sum\limits_{i \geq k} B_i$ (respectively, $\varphi(B_k) = B_k$) for all $k \in \mathbb{Z}$. An automorphism φ is said to be strictly positive, if φ is positive but not rigid.

The following theorem gives the classification of standard selfinjective algebras of finite representation type.

Theorem 2.1 (see [10]) *Let A be a non-simple standard selfinjective algebra. The following conditions are equivalent:*

(i) *A is of finite representation type;*

(ii) *$A \cong \widehat{B}/\sigma\varphi^s$, where B is a tilted algebra of Dynkin type Δ, φ is a strictly positive primitive root of the Nakayama automorphism $\upsilon_{\widehat{B}}$ and σ is a rigid automorphism of $\widehat{B}$ of finite order.*

Therefore, we may associate to any standard selfinjective algebra $A \cong \widehat{B}/\sigma\varphi^s$ of finite representation type the following data: the type of graph $\Delta(A) = \Delta(B)$ ($\Delta(B)$ is the Dynkin graph corresponding to B), the degree $e(A)$ of primitive root φ of $\upsilon_{\widehat{B}}$, the order $t(A)$ of the automorphism σ, and the power $s(A) = s$ of φ. We define $f(A) = s(A)/e(A)$ to be the frequency of A, and type$(A) = (\Delta(A), f(A), t(A))$ to be the type of A.

Following from [1], the standard selfinjective algebras of finite representation type are determined up to stable equivalence by their types. Moreover, there are only finite possibles for the types of standard selfinjective algebras of finite representation type.

Proposition 2.1 (see [1]) *The set of types of non-simple standard selfinjective algebras of finite representation type is the disjoint union of the following sets:*

$\{(A_n, s/n, 1) \mid n, s \in \mathbb{N}\}$; $\{(A_{2p+1}, s, 2) \mid p, s \in \mathbb{N}\}$; $\{(D_n, s, 1) \mid n, s \in \mathbb{N}, n \geq 4\}$;
$\{(D_{3m}, s/3, 1) \mid m, s \in \mathbb{N}, m \geq 2, 3 \nmid s\}$; $\{(D_n, s, 2) \mid n, s \in \mathbb{N}, n \geq 4\}$;
$\{(D_4, s, 3) \mid s \in \mathbb{N}\}$; $\{(E_n, s, 1) \mid n = 6, 7, 8, s \in \mathbb{N}\}$; $\{(E_6, s, 2) \mid s \in \mathbb{N}\}$.

3 Stable Categories with Indecomposable Calabi-Yau Objects

For a Hom-finite triangulated category $\mathscr{C}$ with Serre functor S, in [4], the authors gave a description of Calabi-Yau objects via the minimal ones (those Calabi-Yau objects whose proper direct summands are not Calabi-Yau objects). Actually, there is a bijection between the set of isomorphism classes of minimal d-th Calabi-Yau objects and the set of all finite $([-d] \circ S)$-orbits of indecomposable objects in $\mathscr{C}$, and the direct sum of all the objects in the finite orbit of an indecomposable object in $\mathscr{C}$ is a minimal d-th Calabi-Yau object.

For a standard selfinjective algebra A of finite representaton type, the stable category $\underline{\mathrm{mod}}\, A$ is a triangulated category with shift Ω_A^{-1} and Serre functor $S = \Omega_A^{-1}\tau_A = \Omega_A\,\upsilon_A$. To investigate the d-th Calabi-Yau objects in these categories is to investigate the orbits of $\Omega_A^{d-1}\tau_A$ or $\Omega_A^{d+1}\upsilon_A$. So we need a description of the actions of Ω_A, τ_A and υ_A. By Theorem 2.1 and Proposition 3.1, the question is reduced to giving a description of the actions of such functors for the associated algebra $\widehat{B}$, which will be given in Proposition 3.2.

Proposition 3.1 (see [3]) *Let B be a tilted algebra of Dynkin type, G an admissible infinite cyclic group of automorphisms of $\widehat{B}$ and $A = \widehat{B}/G$. Then the following statements hold:*

(i) $F_\lambda : \mathrm{mod}\,\widehat{B} \to \mathrm{mod}\,A$ *is exact, dense and induces a bijection between the G-orbits of isomorphism classes of indecomposable finite dimensional $\widehat{B}$-modules and the isomorphism classes of indecomposable finite dimensional A-modules;*

(ii) $F_\lambda\Omega_{\widehat{B}} \cong \Omega_A F_\lambda$;

(iii) $F_\lambda\tau_{\widehat{B}} \cong \tau_A F_\lambda$.

For a Dynkin diagram Δ, we put $m_\Delta = h_\Delta - 1$.

Proposition 3.2 (see [3, 7]) *Let B be a tilted algebra of Dynkin type Δ. Then we have equivalences of functors on the category $\underline{\mathrm{mod}}\,\widehat{B}$:*

(i) $\Omega_{\widehat{B}} \cong \tau_{\widehat{B}}^{h_\Delta/2}$ *for* $\Delta = A_1, D_n$ *(n even)*, E_7, E_8;

(ii) $\Omega_{\widehat{B}} \cong \sigma\tau_{\widehat{B}}^{h_\Delta/2}$ *for* $\Delta = A_n$ *($n \geq 3$ odd)*, D_n *(n odd)*, E_6 *and an automorphism σ of order* 2;

(iii) $\Omega_{\widehat{B}} \cong \rho\tau_{\widehat{B}}^{m_\Delta/2}$ *for* $\Delta = A_n$ *(n even) and an automorphism ρ with $\rho^2 = \tau_{\widehat{B}}$.*

The sketch of the proof was given in [7]. We shall give the proof in details for the convenience of the readers.

Proof Let B be a tilted algebra of Dynkin type, that is, $B = \mathrm{End}_H T$ where H is a path algebra whose underlying graph is a Dynkin diagram and T is a tilting H-module. By [8], we get $D^b(B) \simeq D^b(H)$. On the other hand, $D^b(H) \simeq D^b(\widetilde{H})$, where $\widetilde{H}$ is the path algebra whose underlying graph is the same as the one of H, while the orientation is given by the bipartition (see [2]). Since B is of finite global dimension, $D^b(B) \simeq \underline{\mathrm{mod}}\,\widehat{B}$ (see [8]). Therefore, $\underline{\mathrm{mod}}\,\widehat{B} \simeq D^b(\widetilde{H})$. $\widetilde{H}$ is a hereditary algebra, and the AR-quiver of $D^b(\widetilde{H})$ can be deduced from the one of $\widetilde{H}$ (see [8]). Given the AR-quiver of $\underline{\mathrm{mod}}\,\widehat{B}$, it will be easy to describe the action of $\Omega_{\widehat{B}}$ through $\tau_{\widehat{B}}$.

Since $\upsilon_{\widehat{B}} = \tau_{\widehat{B}}\Omega_{\widehat{B}}^{-2}$, the following proposition can be viewed as a corollary of the above proposition.

Proposition 3.3 (see [3]) *Let B be a tilted algebra of Dynkin type Δ. Then $\upsilon_{\widehat{B}} \cong \tau_{\widehat{B}}^{-m_\Delta}$ as endofunctors on $\underline{\mathrm{mod}}\,\widehat{B}$.*

From the description of the types of the non-simple standard selfinjective algebras of finite representation type, we can see that the number $m_{\Delta(A)}/e(A)$ are integers for all cases. Therefore, we may associate the number $o(A) = s(A)t(A)m_{\Delta(A)}/e(A)$ to any non-simple standard selfinjective algebra of finite representation type. It is easy to check that $o(A)$ is the smallest non-negative integer, such that $\tau_A^{o(A)}X \cong X$ for any object X in $\underline{\mathrm{mod}}\,A$ (see also [3]).

Now, we can prove the main theorem of this paper.

Proof of Theorem 1.1 Let $A = \widehat{B}/\sigma\varphi^s$, with $\Delta = A_n$ (n odd), D_n (n odd), E_6. We know that $\Omega_A \cong \sigma_A \tau_A^{h_\Delta/2}$ for some automorphism σ_A of order 2. If $t(A) = 2$, then it can be seen from the AR-quiver that the action on $\underline{\mathrm{mod}}\,A$ induced from the automorphism σ of $\widehat{B}$ is just the automorphism σ_A of modules mentioned above. In the following, we shall denote by σ both the automorphism of algebras and the automorphism of modules.

Case 1 $\Delta = A_n$ (n even), i.e., A is of the type $(A_n, s/n, 1)$, $n, s \in \mathbb{N}$.

In this case, $o(A) = s$ and $\widetilde{h}_\Delta = h_\Delta$. If there is an indecomposable Calabi-Yau object X in $\underline{\mathrm{mod}}\,A$ with dimension d, then $\Omega_A^{d-1}\tau_A X \cong X$. On the other hand, $\Omega_A^{d-1}\tau_A X = (\rho\tau_A^{n/2})^{d-1}\tau_A X = \rho^{h_{A_n}(d-1)+2}X$. So $\rho^{h_{A_n}(d-1)+2}X \cong X$, $o(A)_X \mid h_{A_n}(d-1)/2+1$, where $o(A)_X$ denotes the relative order of the Auslander-Reiten translation with respect to X. From the AR-quiver, we can see that $o(A)_X = o(A)$, and hence $s \mid h_{A_n}(d-1)/2+1$. So $\gcd(h_{A_n}, s) = 1$. Conversely, if $\gcd(h_{A_n}, s) = 1$, then there is some d_1, such that $s \mid hd_1 + 1$. Let $d = 2d_1 + 1$. Then for any object X in $\underline{\mathrm{mod}}\,A$,

$$\begin{aligned}\Omega_A^{d-1}\tau_A X &= (\rho\tau_A^{n/2})^{d-1}\tau_A X = \rho^{d-1}\tau_A^{n(d-1)/2+1}X = \rho^{2d_1}\tau_A^{nd_1+1}X \\ &= \tau_A^{d_1}\tau_A^{nd_1+1}X = \tau_A^{hd_1+1}X \cong X.\end{aligned}$$

This shows that every object in $\underline{\mathrm{mod}}\,A$ is a Calabi-Yau object.

Case 2 $\Delta = A_1, D_n$ (n even), E_7, E_8.

Here, $\widetilde{h}_\Delta = h_\Delta/2$, and $o(A)_X = o(A)$ for all indecomposable objects in $\underline{\mathrm{mod}}\,A$.

(i) $t(A) = 1$. If there is an indecomposable Calabi-Yau object X in $\underline{\mathrm{mod}}\,A$ with dimension d, then by Propositions 3.2 and 3.3 and a similar discussion in the previous case, we can conclude that $o(A) \mid h_\Delta(d-1)/2+1$, hence, $s \mid h_\Delta(d-1)/2+1$, so $\gcd(\widetilde{h}_\Delta, s) = 1$. Conversely, suppose $\gcd(\widetilde{h}_\Delta, s) = 1$, i.e., $\gcd(h_\Delta/2, s) = 1$. Since $\gcd(h_\Delta/2, m_\Delta) = 1$, we have $\gcd(h_\Delta/2, o(A)) = 1$. There is some d_1 such that $o(A) \mid d_1 h_\Delta/2+1$. Put $d = d_1 + 1$. For any object X in $\underline{\mathrm{mod}}\, A$,

$$\Omega_A^{d-1}\tau_A X = (\tau_A^{h_\Delta/2})^{d-1}\tau_A X = \tau_A^{(d-1)h_\Delta/2+1} X \cong X.$$

We conclude that every object in $\underline{\mathrm{mod}}\, A$ is a Calabi-Yau object.

(ii) $t(A) = 2$. A is of type $(A_1, s, 2)$, $s \in \mathbb{N}$ or $(D_n, s, 2)$, $n, s \in \mathbb{N}, n \geq 4$.

$\widetilde{h}_\Delta = h_\Delta/2$ is odd, hence $\gcd(s, \widetilde{h}) = 1$ if and only if $\gcd(st(A), \widetilde{h}) = 1$ if and only if $\gcd(o(A), \widetilde{h}) = 1$. By a similar discussion, we can conclude that $\underline{\mathrm{mod}}\, A$ has indecomposable Calabi-Yau objects if and only if $\gcd(s, \widetilde{h}) = 1$.

(iii) $t(A) = 3$, i.e., A is of type $(D_4, s, 3)$, $s \in \mathbb{N}$.

We need to prove that there is no Calabi-Yau object in this case. $\widetilde{h} = h_{D_4}/2 = 3$, $m_{D_4} = 5$. If there is some indecomposable object X, which is a Calabi-Yau object with dimension d, then we have $\Omega_A^{d-1}\tau_A X = (\tau_A^3)^{d-1}\tau_A X = \tau_A^{3(d-1)+1} X \cong X$. But $o(A)_X = o(A) = st(A)m_{D_4} = 15s$ which cannot divide $3(d-1)+1$, which is a contradiction. Therefore, $\underline{\mathrm{mod}}\, A$ has no Calabi-Yau objects.

Case 3 $\Delta = A_n$ (n odd), D_n (n odd), E_6.

In this case, $\widetilde{h}_\Delta = h_\Delta/2$.

(i) $t(A) = 1$. Here, $o(A)_X = o(A)$ for all indecomposable objects in $\underline{\mathrm{mod}}\, A$.

Let X be an indecomposable Calabi-Yau object with dimension d. One case is that $\sigma X \cong X$. We have $\Omega_A^{d-1}\tau_A X = \tau_A^{(d-1)h_\Delta/2+1} X$, and hence $o(A)_X \mid (d-1)h_\Delta/2+1$ and $\gcd(o(A)_X, h_\Delta/2) = 1$, so $\gcd(s, h_\Delta/2) = 1$. The other case is that $\sigma X \not\cong X$. We have $\Omega_A^{d-1}\tau_A X = (\sigma\tau_A^{h_\Delta/2})^{d-1}\tau_A X = \sigma^{d-1}\tau_A^{(d-1)h_\Delta/2+1} X \cong X$. So $d-1$ is even, and $o(A)_X \mid (d-1)h_\Delta/2+1$. We can also conclude that $\gcd(h_\Delta/2, s) = 1$. Conversely, if $\gcd(h_\Delta/2, s) = 1$, then $\gcd(h_\Delta/2, o(A)) = 1$. There is some integer d_1, such that $o(A) \mid d_1 h_\Delta/2+1$. Put $d = d_1 + 1$. Choose an indecomposable object X in $\underline{\mathrm{mod}}\, A$ such that $\sigma X \cong X$. Then $\Omega_A^{d-1}\tau_A X = \tau_A^{(d-1)h_\Delta/2+1} X = \tau_A^{d_1 h_\Delta/2+1} X \cong X$. Hence X is an indecomposable object in $\underline{\mathrm{mod}}\, A$.

(ii) $t(A) = 2$. Here $o(A)_X \lneq o(A)$ for some indecomposable object in $\underline{\mathrm{mod}}\, A$. Let X be an indecomposable Calabi-Yau object. If $\sigma X \cong X$, then $\Omega_A^{d-1}\tau_A X = \tau_A^{(d-1)h_\Delta/2+1} X$, and hence $o(A)_X \mid (d-1)h_\Delta/2+1$. Actually, $o(A)_X = sm_\Delta/2$. So $\gcd(o(A)_X, h_\Delta/2) = 1$ induces $\gcd(s, h_\Delta/2) = 1$. Otherwise, $\sigma X \not\cong X$. Here $o(A)_X = o(A)$. We have $\Omega_A^{d-1}\tau_A X = (\sigma\tau_A^{h_\Delta/2})^{d-1}\tau_A X = \sigma^{d-1}\tau_A^{(d-1)h_\Delta/2+1} X \cong X$, hence $\tau_A^{2((d-1)h_\Delta/2+1)} X \cong X$. We have $o(A) = 2sm_\Delta/e$. Then $sm_\Delta/e \mid (d-1)h_\Delta/2+1$, which induces $\gcd(s, h_\Delta/2) = 1$. Conversely, if $\gcd(h_\Delta/2, s) = 1$, then $\gcd(h_\Delta/2, o(A)_X) = 1$ for some indecomposable object X with $\sigma X \cong X$. By a similar discussion as above, we can see that there is some integer d, such that $\Omega_A^{d-1}\tau_A X \cong X$. So X is an indecomposable Calabi-Yau object.

Now we have exhibited all cases of non-simple standard selfinjective algebras. Hence the proof is completed.

Corollary 3.1 *Let A be a non-simple standard selfinjective algebra. If the type of A is one of the two following cases:*

(1) $\{(A_n, s/n, 1)\}$, $s \in \mathbb{N}$, $n = 4l-3$, *for integer* $l \geq 2$, $\gcd(s, 4l-2) \neq 1$, $\gcd(s, 2l-1) = 1$*;*

(2) $\{(A_{2p+1}, s, 2)\}$, $s \in \mathbb{N}$ *and is an odd,* $p = 2l$, *for integer* $l \geq 1$, $\gcd(s, 2l+1) = 1$, *the stable category* $\underline{\mathrm{mod}}\, A$ *is not a Calabi-Yau category, but it has indecomposable Calabi-Yau objects.*

Proof We keep the notations as in the proof of Theorem 1.1. It is sufficient to prove that if the type of A is one of the types mentioned in the corollary, then there are indecomposable objects in $\underline{\mathrm{mod}}\, A$ which are not Calabi-Yau objects.

(1) Let A be of type $(A_n, s/n, 1)$ such that $n = 4l-3$, $l \geq 2$, $\gcd(s, 4l-2) \neq 1$ and $\gcd(s, 2l-1) = 1$. We claim that if X is an indecomposable object in $\underline{\mathrm{mod}}\, A$ such that $\sigma X \ncong X$, then X is not a Calabi-Yau object. Let d be an integer. We have

$$\Omega_A^{d-1}\tau_A X = (\sigma\tau_A^{h_\Delta/2})^{d-1}\tau_A X = \sigma^{d-1}\tau_A^{(d-1)h_\Delta/2+1}X.$$

Therefore, if $\Omega_A^{d-1}\tau_A X \cong X$, then $d-1$ is even and $o(A)_X \mid (d-1)h_\Delta/2+1$. We have

$$o(A)_X = o(A) = sn.$$

Since $\gcd(s, 4l-2) \neq 1$ and $\gcd(s, 2l-1) = 1$, the integer s is even. Thus s cannot divide $(d-1)h_\Delta/2+1$. So X is not a Calabi-Yau object.

(2) Let A be of type $(A_{2p+1}, s, 2)$ such that s is odd, $p = 2l$ for some integer $l \geq 1$ and $\gcd(s, 2l+1) = 1$. Let X be an indecomposable object in $\underline{\mathrm{mod}}\, A$ such that $\sigma X \ncong X$. Similarly, if X is a Calabi-Yau object of dimension d, then

$$X \cong \Omega_A^{d-1}\tau_A X = \sigma^{d-1}\tau_A^{(d-1)h_\Delta/2+1}X = \sigma^{d-1}\tau_A^{(d-1)(p+1)+1}X.$$

So there is some integer a, such that $(d-1)(p+1)+1 = as(2p+1)$ and a and $d-1$ have the same parity. But this cannot happen when s is odd and p is even. Therefore, X is not a Calabi-Yau object.

Remark 3.1 $\{(A_n, s/n, 1)\}$ is just the Nakayama selfinjective algebra $\Lambda(s, n+1)$ investigated in [4]. The conclusions we get here coincide with the ones there.

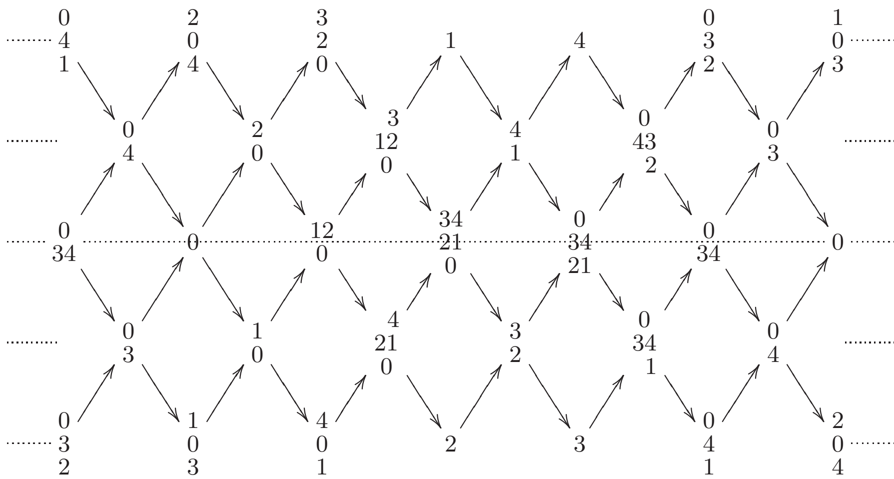
Let us give an example as below.

Example 3.1 $A = k\Gamma/I$, where Γ is the quiver as follows:

$$\begin{tikzcd}
1 \arrow[dd, "\alpha_2"'] & & -1 \arrow[dd, "\alpha_{-2}"] \\
& 0 \arrow[ul, "\alpha_1"'] \arrow[ur, "\alpha_{-1}"] & \\
2 \arrow[ur, "\beta_2"'] & & -2 \arrow[ul, "\beta_{-2}"]
\end{tikzcd}$$

and I is the ideal generated by relations: $\beta_2\alpha_2\alpha_1 - \beta_{-2}\alpha_{-2}\alpha_{-1}$, $\alpha_{-2}\alpha_{-1}\beta_2\alpha_2$, $\alpha_2\alpha_1\beta_{-2}\alpha_{-2}$, $\alpha_1\beta_2$ and $\alpha_{-1}\beta_{-2}$. Then A is of type $(A_5, 1, 2)$.

$\underline{\mathrm{mod}}\, A$ is not a Calabi-Yau category, but it has indecomposable Calabi-Yau objects. The AR-quiver of $\underline{\mathrm{mod}}\, A$ is as follows (the indecomposable modules are denoted by their composition factors):



The indecomposable Calabi-Yau objects are just the modules in the dotted line, and their Calabi-Yau dimensions are 4.

Remark 3.2 Let A be a selfinjective algebra with $\Delta = D_n\,(n \text{ odd}), E_6, A_n\,(n = 4k-1,\ k \geq 1)$. Comparing the main theorem with [6, Propositions 7.3, 7.4 and 9.6], we obtain that $\underline{\mathrm{mod}}\,A$ is a Calabi-Yau category if and only if it has indecomposable Calabi-Yau objects. However, there are examples that the Calabi-Yau dimension of the category and the Calabi-Yau dimension of some indecomposable Calabi-Yau objects are different. Actually, we have the following description of the Calabi-Yau dimension of the categories and the indecomposable Calabi-Yau objects respectively. The Calabi-Yau dimension of the categories was given in [6].

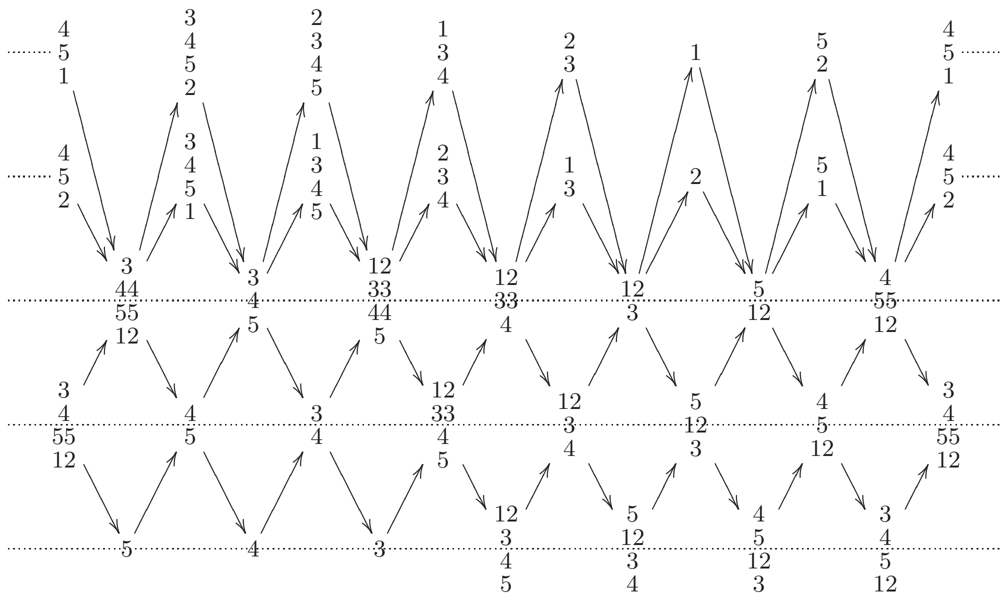
Let $A = \widehat{B}/\sigma\varphi^s$ be a selfinjective algebra in the above cases. If $t(A) = 1$, then CY-dim($\underline{\mathrm{mod}}\,A$) $= 2d_1 + 1$, where d_1 is the least non-negative integer, such that $m_\Delta s/e$ divides $h_\Delta d_1 + 1$. However, for an indecomposable module X such that $\sigma X = X$ (σ is the automorphism in Proposition 3.2), the Calabi-Yau dimension of X is $d_2 + 1$, where d_2 is the least non-negative integer such that $m_\Delta s/e$ divides $h_\Delta d_2/2 + 1$. For the case $t(A) = 2$, there are also indecomposable Calabi-Yau objects X of CY-dim$(X) = d_2 + 1$ with d_2 the least non-negative integer, such that $m_\Delta s/e$ divides $h_\Delta d_2/2 + 1$. If $\Delta(A) \neq A_n\ (n = 4l-3,\ l \geq 1)$, then CY-dim($\underline{\mathrm{mod}}\,A$) $= 2d_1$, where d_1 is the least non-negative integer such that $m_\Delta s/e$ divides $h_\Delta(2d_1-1)/2 + 1$. If $\Delta(A) = A_n\ (n = 4l-3,\ l \geq 1)$, then CY-dim($\underline{\mathrm{mod}}\,A$) $= d_1 + 1$, where d_1 is the least nonnegtive odd integer which can be written as the form $lm_\Delta s/e$ for some odd integer l.

Example 3.2 If $A = k\Gamma/I$, where Γ is the quiver as follows:

$$\begin{tikzcd}
1 \arrow[drr, "\alpha_1"] & & \\
 & 2 \arrow[r, "\alpha_2"'] & 3 \arrow[d, "\alpha_3"] \\
 & 5 \arrow[u, "\beta_2"'] \arrow[uul, "\beta_1"] & 4 \arrow[l, "\alpha_4"]
\end{tikzcd}$$

and I is the ideal generated by relations: all path of length 5 and $\alpha_1\beta_1 - \alpha_2\beta_2$, $\beta_2\alpha_4\alpha_3\alpha_1$ and $\beta_1\alpha_4\alpha_3\alpha_2$, then A is of type $(D_5, 1, 1)$.

$\underline{\mathrm{mod}}\, A$ is Calabi-Yau with CY-dim$(\underline{\mathrm{mod}}\, A) = 13$, but the indecomposable objects on the dotted lines have Calabi-Yau dimension 6.



Let A be a selfinjective algebra. Note that even if every object in $\underline{\mathrm{mod}}\, A$ is a Calabi-Yau object of dimension d, the category $\underline{\mathrm{mod}}\, A$ is not necessarily a Calabi-Yau category of dimension d. We have the following example from [5].

Example 3.3 Let $A = kQ/I$, where Q is the following quiver:

$$\begin{tikzcd} 1 \arrow[loop left, "\alpha"] \arrow[r, bend left, "\beta"] & 2 \arrow[l, bend left, "\gamma"] \end{tikzcd}$$

and I is the ideal generated by the relations: $\alpha^2 - \gamma\beta$ and $\beta\gamma$. The type of A is $(D_6, 1/3, 1)$.

The algebra A is symmetric. Every object X in $\underline{\mathrm{mod}}\, A$ satisfies that $\Omega^3 X \cong X$. If the characteristic of the field is not 2, then we have $\Omega^3 \ncong \mathrm{id}$, $\Omega^6 \cong \mathrm{id}$. Therefore, we have that $\underline{\mathrm{mod}}\, A$ is a Calabi-Yau category of dimension 5, but every object in $\underline{\mathrm{mod}}\, A$ is a Calabi-Yau object of dimension 2.

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