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The Non-selfsimilar Riemann Problem for 2-D Zero-Pressure Flow in Gas Dynamics***

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Abstract The non-selfsimilar Riemann problem for two-dimensional zero-pressure flow in gas dynamics with two constant states separated by a convex curve is considered. By means of the generalized Rankine-Hugoniot relation and the generalized characteristic analysis method, the global solution involving delta shock wave and vacuum is constructed. The explicit solution for a special case is also given.

Keywords Zero-pressure flow, Non-selfsimilar Riemann problem, Generalized Rankine-Hugoniot relation, Entropy condition, Delta shock
2000 MR Subject Classification 35L65, 76N10

1 Introduction

The two-dimensional zero-pressure flow in gas dynamics is as follows:

$$\begin{cases}\rho_t + \nabla\cdot(\rho U) = 0,\\ (\rho U)_t + \nabla\cdot(\rho U\otimes U) = 0,\end{cases} \tag{1.1}$$

where $\rho(t,x,y) \geq 0$ and $U = (u,v)$ are density and mean velocity respectively, $\nabla = (\partial_x, \partial_y)$, and $\otimes$ is the tensor product. We consider the non-selfsimilar Riemann problem (1.1) with the initial data

$$(\rho, U)(0,x,y) = \begin{cases}(\rho_1, U_1), & y > \varphi(x),\\ (\rho_2, U_2), & y < \varphi(x),\end{cases} \tag{1.2}$$

where $y = \varphi(x)$ $(-\infty < x < +\infty)$ is a smooth convex curve, and (ρ_i, U_i) $(i = 1,2)$ are two constant states.

System (1.1) can be regarded as the direct result by ignoring the effect of pressure on the isentropic Euler equations in gas dynamics (cf. [1, 4, 6]), which can be used to describe many phenomena, such as aerodynamics, water waves, etc. It can be used to describe the process of free particles sticking under collision in low temperature. There exist delta shock wave and vacuum appearing in the solution. By using the generalized Rankine-Hugoniot relation, the delta shock gets a well description from the sides of location, propagation speed and weight (cf.

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[3, 6]). The global solution is obtained in the sense of a suitable geometrical entropy (cf. [2, 6]).

In 1999, Sheng and Zhang [6] solved the 2-D Riemann problem in which the initial data is separated into four domains by axis. All of the existence, uniqueness and stability for viscous perturbations have been proved analytically. In 2001, Yang [7] studied the multi-dimensional system in gas dynamics with the initial data separated by a superplane into two infinite parts, and through self-similar transformation, the problem got solved actually in the sense of one-dimensional. In 2003, Yang [8] studied a multi-dimensional scalar conservation law with the initial data separated by an $(n-1)$-dimensional manifold, and obtained the solution by constructing implicit function.

So far, the study on the non-selfsimilar Riemann problem for multi-dimensional system of conservation laws is still scare relatively, because it cannot be solved by the existent methods of dealing with one-dimensional cases. In 2005, we studied the two-dimensional non-selfsimilar Riemann problem for zero-pressure flow (1.1) with the initial data separated by a circular curve into two constant states, and constructively obtained the explicit solution (cf. [5]).

It is important to study the non-selfsimilar Riemann problem since the general initial discontinuous curve can be regarded as the combination of convex curves, and it is a further step to study Cauchy problem. It is also useful to test the numerical schemes.

The paper is arranged as follows: In Section 2, some preliminaries are given. The problem (1.1) with initial data separated by a line is presented, the generalized characteristic analysis method and generalized Rankine-Hugoniot relation are also introduced. In Section 3, by the use of the former results, the two-dimensional non-selfsimilar Riemann problem (1.1) and (1.2) is constructively solved. In Section 4, an example with a kind of special initial data is presented.

2 Preliminaries

Consider the Riemann problem (1.1) with the following initial data

$$\rho\big|_{t=t_0} = \overline{\rho} + m_0\delta(\zeta), \quad U|_{t=t_0} = \begin{cases} U_1, & \zeta < 0, \\ U_0, & \zeta = 0, \\ U_2, & \zeta > 0, \end{cases} \tag{2.1}$$

where

$$\overline{\rho}(t_0, x, y) = \begin{cases} \rho_1, & \zeta < 0, \\ \rho_2, & \zeta > 0, \end{cases} \tag{2.2}$$

$m_0 > 0$, $\zeta = (x - x_0, y - y_0) \cdot \vec{n}$, $\vec{n} := (\mu, \nu)$ is the unit normal of the line $\zeta = 0$ which is chosen to point toward side $\zeta > 0$ and $U_0 \cdot \vec{n}$ satisfies entropy condition

$$U_2 \cdot \vec{n} \le U_0 \cdot \vec{n} \le U_1 \cdot \vec{n}. \tag{2.3}$$

Under the condition (2.3), a delta shock wave appears. The delta shock is the concentration of mass. From a physical point of view, the formation of delta shock is the process of the concentration of particles. Particles march straight forward unitl they collide, their trajectories are just the characteristic lines. The collision of particles is equivalent to the fact that the characteristics meet together; therefore we can use this point to construct the solution. At present, the solution should be found in the Borel space in the sense of generalized functional. By the Borel measure theory and Radon-Nykodym's theorem (cf. [3]), we can define the measure solution.

Definition 2.1 *A pair (ρ, U) is called a measure solution of* (1.1) *if it satisfies*

$$\begin{aligned} \int_0^\infty \int_{-\infty}^{+\infty} (\phi_t + U \cdot \nabla\phi) d\rho dt = 0, \\ \int_0^\infty \int_{-\infty}^{+\infty} U(\phi_t + U \cdot \nabla\phi) d\rho dt = 0 \end{aligned} \tag{2.4}$$

for all test function $\phi \in C_0^\infty([0,\infty] \times R^1)$.

We claim that a weak solution of (1.1) and (2.1) in the direction $\vec{n}$ satisfies (2.4) in the measure sense if the following generalized Rankine-Hugoniot relation is satisfied.

$$\begin{cases} \dfrac{d\zeta}{dt} = U_\delta \cdot \vec{n}, \\ \dfrac{dw}{dt} = -[\rho](U_\delta \cdot \vec{n}) + [\rho(U \cdot \vec{n})], \\ \dfrac{d(wU_\delta)}{dt} = -[\rho U](U_\delta \cdot \vec{n}) + [\rho U(U \cdot \vec{n})], \end{cases} \tag{2.5}$$

where ζ, U_δ and w are the location, propagation speed and weight of delta shock respectively, and $[G] = G_1 - G_2$ is the difference across the discontinuity. By virtue of Green formula and partly integration, it is easy to verify that (2.5) is identical with (2.4).

Integrating (2.5) with the initial data (2.1), we get the solution (cf. [5])

$$\zeta(t) = \begin{cases} \dfrac{m_0 U_0 \cdot \vec{n} t + \frac{1}{2}[\rho(U \cdot \vec{n})^2] t^2}{m_0 + [\rho]\zeta_0 + [\rho(U \cdot \vec{n})] t}, & [\rho] = 0, \\ \dfrac{m_0 + [\rho(U \cdot \vec{n})] t - w(t)}{[\rho]}, & [\rho] \neq 0, \end{cases} \tag{2.6}$$

where

$$\begin{aligned} w(t) = \{ & m_0^2 + [\rho]\zeta_0 + \rho_1\rho_2([U] \cdot \vec{n})^2 t^2 + 2m_0[\rho(U \cdot \vec{n})] t \\ & - 2[\rho][\rho(U \cdot \vec{n})]\zeta_0 t - 2m_0[\rho] U_0 \cdot \vec{n} t\}^{1/2}, \end{aligned} \tag{2.7}$$

$$U_\delta(t) = \frac{1}{w(t)} \{ m_0 U_0 + [\rho U]\zeta_0 + [\rho U(U \cdot \vec{n})] t - [\rho U]\zeta \}. \tag{2.8}$$

The solution has the following property

$$\lim_{t \to \infty} U_\delta(t) = \frac{\sqrt{\rho_1} U_1 + \sqrt{\rho_2} U_2}{\sqrt{\rho_1} + \sqrt{\rho_2}} =: U_\delta^0 = (u_\delta^0, v_\delta^0). \tag{2.9}$$

Especially, when $m_0 = 0$ and $U_0 = 0$, we have

$$(\zeta, w, U_\delta)(t) = (U_\delta^0 t, \sqrt{\rho_1\rho_2}([U] \cdot \vec{n}) t, U_\delta^0). \tag{2.10}$$

Then the solution of Riemann problem (1.1) and (2.1) is

$$\rho(t) = \overline{\rho} + \rho\delta(\zeta - \zeta(t)), \quad U = \begin{cases} U_1, & \zeta < \zeta(t), \\ U_\delta^0, & \zeta = \zeta(t), \\ U_2, & \zeta < \zeta(t), \end{cases} \tag{2.11}$$

where

$$\overline{\rho} = \begin{cases} \rho_1, & \zeta < \zeta(t), \\ \rho_2, & \zeta > \zeta(t). \end{cases} \tag{2.12}$$

When $[U] \cdot \vec{n} < 0$, the characteristics will not interact, and then vacuum appears. The solution is

$$(\rho, U)(t, \zeta) = \begin{cases} (\rho_1, U_1), & \zeta < U_1 \cdot \vec{n}t, \\ \text{Vacuum}, & U_1 \cdot \vec{n}t < \zeta < U_2 \cdot \vec{n}t, \\ (\rho_2, U_2), & \zeta > U_2 \cdot \vec{n}t. \end{cases} \tag{2.13}$$

3 Solutions of the Problem (1.1) and (1.2)

For simplicity, we suppose $\varphi''(x) > 0$. Take the normal of $y = \varphi(x)$ as $\vec{n}(x, y) = (\varphi'(x), -1)$. We construct the solution in two cases.

Case 1 $U_1 = U_2,\ \rho_1 \neq \rho_2$

In this case, the solution is constructed by two constant states separated by a contact discontinuity J.

$$(\rho, U)(t, x, y) = \begin{cases} (\rho_1, U_1), & y - v_1 t > \varphi(x - u_1 t), \\ (\rho_2, U_2), & y - v_1 t < \varphi(x - u_1 t). \end{cases} \tag{3.1}$$

Case 2 $U_1 \neq U_2$

In this case, there exists a point $(x_0, y_0) \in \{(x, y) \mid y = \varphi(x)\}$ satisfying $\vec{n}(x_0, y_0) \cdot [U] = 0$. Let $x > x_0$. We have $\vec{n}(x, y) \cdot [U] > 0$. Thus there will be a delta shock appearing at the initial discontinuity $x > x_0$ and vacuum appearing at $x < x_0$. Next we construct the solution pointwisely.

Lemma 3.1 *For any point $(x_1, y_1) \in \{(x, y) \mid y = \varphi(x), x > x_0\}$, there exists a line*

$$\ell: \ \frac{x - x_1}{u_1 - u_2} = \frac{y - y_1}{v_1 - v_2}, \tag{3.2}$$

such that all the characteristics coming from ℓ form a plane in which a delta shock originates from (x_1, y_1) and it keeps constant speed U_δ^0 (see (2.10)) until it meets vacuum.

Proof From the characteristic theory, since the plane formed by the characteristics coming from ℓ in $y > \varphi(x)$ and the plane formed by the characteristics coming from ℓ in $y < \varphi(x)$ have the same normal

$$(v_1 - v_2, u_2 - u_1, u_1 v_2 - u_2 v_1),$$

the coplanarity is completed. The plane is

$$\Gamma : (v_1 - v_2)(x - x_1) - (u_1 - u_2)(y - y_1) + (u_1 v_2 - u_2 v_1)t = 0. \tag{3.3}$$

In Γ, the problem (1.1) and (1.2) is reduced into a non-selfsimilar Riemann problem with three pieces of constants, and a delta shock with speed U_δ^0 originates from (x_1, y_1) until it meets vacuum.

Let ℓ intersect with $y = \varphi(x)$ at the other point (x_1', y_1'). From Lemma 3.1, in Γ we have

$$\begin{cases} \dfrac{x - x_1'}{u_1} = \dfrac{y - y_1'}{v_1} = \dfrac{t}{1}, & \text{characteristics,} \\ \dfrac{x - x_1}{u_\delta^0} = \dfrac{y - y_1}{v_\delta^0} = \dfrac{t}{1}, & \text{delta shock,} \end{cases} \tag{3.4}$$

which has the solution

$$t_0 = (x_1 - x_1') \frac{\sqrt{\rho_1} + \sqrt{\rho_2}}{\sqrt{\rho_2}(u_1 - u_2)}. \tag{3.5}$$

When $t \leq t_0$, the solution of (1.1) and (1.2) is

$$(\rho, U)(t, \zeta) = (\overline{\rho}, \overline{U}) + (w(t)\delta(\zeta - \zeta(t)), U_\delta), \tag{3.6}$$

where

$$(\overline{\rho}, \overline{U})(t, \zeta) = \begin{cases} (\rho_1, U_1), & (t, x, y) \in \Sigma_1, \\ \text{Vacuum}, & (t, x, y) \in \Sigma_2, \\ (\rho_2, U_2), & (t, x, y) \in \ell_1 \setminus \{\Sigma_1 \bigcup \Sigma_2\}, \end{cases} \tag{3.7}$$

where $w(t), U_\delta$ can be expressed in (2.10),

$$\begin{aligned} \Sigma_1 = \{(t, x, y) \mid (t, x, y) \in \ell_1, y - v_\delta^0 t - y_1 > \varphi(x - u_\delta^0 t - x_1) \\ \text{and } y - v_1 t - y_1' > \varphi(x - u_1 t - x_1')\}, \\ \Sigma_2 = \{(t, x, y) \mid (t, x, y) \in \ell_1, y - v_1 t - y_1' < \varphi(x - u_1 t - x_1') \\ \text{and } y - v_2 t - y_1' > \varphi(x - u_2 t - x_1')\}, \end{aligned}$$

and

$$\ell_1 = \left\{(t, x, y) \,\middle|\, \frac{x - x_1' - u_2 t}{u_1 - u_2} = \frac{y - y_1' - v_2 t}{v_1 - v_2} = \frac{t}{0}\right\}. \tag{3.8}$$

When $t \geq t_0$, a new non-selfsimilar Riemann problem appears. By solving the initial value problem (2.5) with the initial data

$$(\zeta, w, U_\delta)\big|_{t=t_0} = (\zeta_0, m_0, U_\delta) = (U_\delta^0 \cdot \vec{n}(x_1, y_1)t_0, \sqrt{\rho_1 \rho_2}([U] \cdot \vec{n}(x_1, y_1))t_0, U_\delta^0), \tag{3.9}$$

the delta shock solution can still be expressed in the form (2.6)–(2.8) just by replacing t by $t - t_0$ and noting that the left side of the delta shock is vacuum, i.e., $(\rho_1, U_1)\big|_{t \geq t_0} = (0, \mathbf{0})$. The solution can be expressed as

$$(\rho, U)(t, \zeta) = (\overline{\rho}, \overline{U}) + (w(t)\delta(\zeta - \zeta(t)), U_\delta), \tag{3.10}$$

where

$$(\overline{\rho},\overline{U})(t,\zeta)=\begin{cases}\text{Vacuum}, & (t,x,y)\in\Sigma,\\ (\rho_2,U_2), & (t,x,y)\in\ell_1\setminus\Sigma,\end{cases} \tag{3.11}$$

where

$$\Sigma=\{(t,x,y)\mid(t,x,y)\in\ell_1 \text{ and } (x_1',y_1')\cdot\vec{n}(x_1,y_1)+U_2\cdot\vec{n}(x_1,y_1)t<\zeta<\zeta(t)\}.$$

When x takes all the values in $\{(x,y)\,|\,y=\varphi(x),x>x_0\}$, the solution construction is completed.

4 An Example with Special Initial Value

In this section, we take $\varphi(x)=x^2$. The normal of the initial discontinuity is $\vec{n}(x,y):=(2x,-1)$. We consider the non-selfsimilar Riemann problem (1.1) with the initial value

$$(\rho,U)=\begin{cases}(\rho_1,u_1,0), & y>x^2,\\ (\rho_2,u_2,0), & y<x^2.\end{cases} \tag{4.1}$$

Furthermore, we suppose $u_1>u_2$. Then point $(0,0)$ satisfies $(2x,-1)\cdot[U]=0$. For $(x_1,y_1)\in\{(x,y)\mid y=x^2,x>0\}$, $[U]\cdot\vec{n}(x_1,y_1)>0$, the delta shock will appear, which can be expressed as

$$x-x_1=\begin{cases}\dfrac{\frac{1}{2}[\rho u^2]t^2+m_0u_0t}{m_0+[\rho u]t}, & [\rho]=0,\\ \dfrac{[\rho u]t+m_0-w}{[\rho]}, & [\rho]\neq 0,\end{cases} \tag{4.2}$$

where

$$w=\{\rho_1\rho_2[u]^2t^2+2m_0t(\rho_1(u_1-u_0)-\rho_2(u_2-u_0))+m_0^2\}^{1/2}, \tag{4.3}$$

$$u_\delta=\frac{1}{w(t)}\{-[\rho u](x-x_1)+[\rho u^2]t+m_0u_0\}. \tag{4.4}$$

Especially, when $m_0=0,u_0=0$, the delta shock is

$$(x,w,u_\delta)(t)=(x_1+u_\delta^0t,\sqrt{\rho_1\rho_2}[u]t,u_\delta^0). \tag{4.5}$$

By virtue of Lemma 3.1, we have

$$\begin{cases}\dfrac{x-x_1}{u_1}=\dfrac{y-y_1}{0}=\dfrac{t}{1},\\ \dfrac{x-x_1}{u_\delta}=\dfrac{y-y_1}{0}=\dfrac{t}{1},\end{cases} \tag{4.6}$$

with the solution

$$t_0=2x_1\frac{\sqrt{\rho_1}+\sqrt{\rho_2}}{\sqrt{\rho_2}(u_1-u_2)}. \tag{4.7}$$

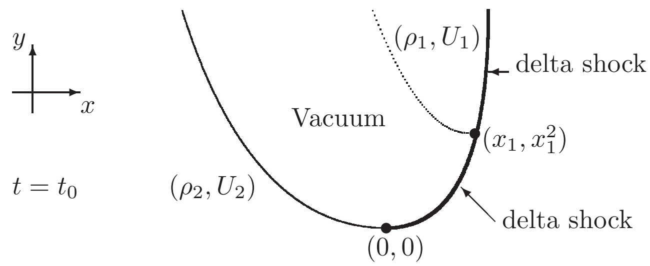

Figure 1

When $t = t_0$, the solution of the problem (1.1) and (4.1) is constructed as follows.

For $(\widetilde{x}, \widetilde{y}) \in \{(x, y) \mid y = x^2, x > x_1\}$, the solution is

$$(\rho, U)(t, x, y)|_{t=t_0} = (\overline{\rho}, \overline{U}\,) + (w\delta(x - x(t), U_\delta)), \tag{4.8}$$

where $w(t), u_\delta$ are expressed as in (4.5),

$$(\overline{\rho}, \overline{U})(t, x, y)|_{t=t_0} = \begin{cases} \text{Vacuum}, & (t, x, y) \in \Sigma_1, \\ (\rho_1, U_1), & (t, x, y) \in \Sigma_2, \\ (\rho_2, U_2), & \ell' \setminus \{\Sigma_1 \cup \Sigma_2\}, \end{cases} \tag{4.9}$$

in which

$$\begin{aligned} &\Sigma_1 = \{(t, x, y) \mid A + \lambda_1(B - A), \lambda_1 \in (0, 1), t = t_0\}, \\ &\Sigma_2 = \{(t, x, y) \mid B + \lambda_2(C - B), \lambda_2 \in (0, 1), t = t_0\}, \\ &A = (-\widetilde{x}, \widetilde{x}^2) + (u_2, 0)t_0, \\ &B = (-\widetilde{x}, \widetilde{x}^2) + (u_1, 0)t_0, \\ &C = (\widetilde{x}, \widetilde{x}^2) + (u_\delta, 0)t_0, \\ &\ell' = \{(t, x, y) \mid y = \widetilde{y}, t = t_0\}. \end{aligned}$$

For $(\widetilde{x}, \widetilde{y}) \in \{(x, y) \mid y = x^2, 0 < x < x_1\}$, where the vacuum approaching the delta shock is before than t_0, the solution is

$$(\rho, U)(t, x, y)|_{t=t_0} = (\overline{\rho}, \overline{U}) + (w(t)\delta(x - x(t)), U_\delta), \tag{4.10}$$

where

$$(\overline{\rho}, \overline{U})(t, x, y)|_{t=t_0} = \begin{cases} \text{Vacuum}, & (t, x, y) \in \Sigma, \\ (\rho_2, U_2), & \ell' \setminus \Sigma, \end{cases} \tag{4.11}$$

in which

$$\begin{aligned} \Sigma &= \{(t, x, y) \mid A + \lambda(C - A), \lambda \in (0, 1), t = t_0\}, \\ C &= (x(t), \widetilde{x}^2), \\ x(t) &= \widetilde{x} + \frac{1}{\rho_2}\Big\{2\rho_2 u_2(x_1 - \widetilde{x})\frac{\sqrt{\rho_1} + \sqrt{\rho_2}}{\sqrt{\rho_2}[u]} - 2\widetilde{x}(\rho_1 + \sqrt{\rho_1\rho_2}) \\ &\quad + (8\widetilde{x}x_1(\rho_1 + \sqrt{\rho_1\rho_2})\sqrt{\rho_1\rho_2} + 4\widetilde{x}\rho_1[\rho])^{\frac{1}{2}}\Big\}. \end{aligned} \tag{4.12}$$

The solution structure is illustrated in Figure 1.

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