Chin. Ann. Math.
32B(6), 2011, 905–916
DOI: 10.1007/s11401-011-0675-y

Chinese Annals of Mathematics, Series B

Homoclinic Flip Bifurcations Accompanied by Transcritical Bifurcation*

Xingbo LIU[1]

Abstract The bifurcations of orbit flip homoclinic loop with nonhyperbolic equilibria are investigated. By constructing local coordinate systems near the unperturbed homoclinic orbit, Poincaré maps for the new system are established. Then the existence of homoclinic orbit and the periodic orbit is studied for the system accompanied with transcritical bifurcation.

Keywords Transcritical bifurcation, Homoclinic orbit, Periodic orbit, Local coordinate system, Poincaré maps
2000 MR Subject Classification 34C23, 34C37, 37C29

1 Introduction

Homoclinic or heteroclinic orbits have tremendous potential for applications in many important areas. Therefore, bifurcations of homoclinic or heteroclinic orbits were studied extensively in the literature (see [1–10] and the references therein). In recent years, many papers focus on the following three different codimension two bifurcations: resonant leading eigenvalues, an inclination-flip condition or an orbit-flip condition. The first possibility was worked out in [11], the second in [12], and the third in [13, 14]. But the corresponding problems with nonhyperbolic equilibrium are rarely investigated (see [15]) where the bifurcation of the inclination-flip homoclinic orbit associated to a saddle-node singularity was studied. It is well-known that nonhyperbolic equilibrium is unstable and always undergoes saddle-node, transcritical or pitchfork bifurcation or other bifurcation phenomena. Obviously, the bifurcation problems of orbits joining nonhyperbolic equilibria are much more difficult and challenging. Knobloch [16] considered bifurcations of the homoclinic orbits to saddle-center. Wagenknecht [17] studied the homoclinic pitchfork bifurcation in a reversible system. For the other bifurcations involving nonhyperbolic equilibria, we refer to [18–25] and the references therein.

In this paper, we are interested in orbit flip homoclinic orbit converging to a nonhyperbolic equilibrium, which undergoes a transcritical bifurcation. We introduce the method originally established in [26] and then improved in [27–28], that is, choosing fundamental solutions of variational equations as a new local coordinate system along the homoclinic orbit and then constructing a Poincaré map to induce bifurcation equations. It is divided into two key steps.

Manuscript received May 15, 2011.
[1]Department of Mathematics, East China Normal University, Shanghai 200241, China.
E-mail: xbliu@math.ecnu.edu.cn
*Project supported by the National Natural Science Foundation of China (No. 10801051) and the Shanghai Leading Academic Discipline Project (No. B407).

The first one is to find fundamental solutions to linear variational equations to divide the tangent space, use the corresponding fundamental solutions as an active coordinate system along the homoclinic orbit, and define two Poincaré sections spanned by the new coordinate system. Then using the new coordinates, we establish the map induced by the flow between two sections outside the small neighborhood of the equilibrium. The second step is to construct the map between these two sections, which is induced by the flow in the small neighborhood of the equilibrium. Then the whole Poincaré map is obtained by composing these maps.

The rest of the paper is arranged as follows. In Section 2, we present a qualitative analysis of the system. Using the invariant manifold theory, we give the local normal form of the system. In Section 3, we construct the local coordinate system along the homoclinic orbit, and then establish the corresponding Poincaré map. In Section 4, we give the sufficient condition for the persistence of the homoclinic orbit and the existence of the periodic orbit bifurcated from the homoclinic orbit.

2 Hypotheses and Normal Form

Consider the following $\mathbb{C}^r$ system:

$$\dot{w} = G(w, \lambda, \varepsilon), \tag{2.1}$$

and its unperturbed system:

$$\dot{w} = F(w), \tag{2.2}$$

where $w \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$, $\varepsilon \in \mathbb{R}$, $0 \le |\varepsilon| \ll 1$, $F, G \in \mathbb{C}^r$, $r \ge 3$, $G(w, 0, 0) = F(w)$.

Assume that system (2.2) has a homoclinic orbit Γ: $\gamma(t)$ connecting the origin with $\gamma(\pm\infty) = 0$. The linearization $D_w F(0)$ has simple real eigenvalues at the equilibrium O: $-\rho_1$, 0, λ_1, λ_2 satisfying $-\rho_1 < 0 < \lambda_1 < \lambda_2$. For convenience, we assume $\rho_1 > \lambda_1$ (the case $\rho_1 < \lambda_1$ can be discussed similarly). Obviously, nonhyperbolic equilibrium O has a 1-dimensional stable manifold W^s, a 2-dimensional unstable manifold W^u and a 1-dimensional center manifold W^c, which are all $\mathbb{C}^r$.

Further, we need the following hypotheses:

(H$_1$) $\dim(T_{\gamma(t)} W^c \cap T_{\gamma(t)} W^u) = \dim(T_{\gamma(t)} W^{cs} \cap T_{\gamma(t)} W^u) = 1$,
where W^{cs} is the center-stable manifold of O.

(H$_2$) Let $e_u^+ = \lim\limits_{t \to -\infty} \frac{\dot{\gamma}(t)}{|\dot{\gamma}(t)|}$, $e_c^- = \lim\limits_{t \to +\infty} \frac{\dot{\gamma}(t)}{|\dot{\gamma}(t)|}$, where $e_u^+ \in T_0 W^{uu}$, $e_c^- \in T_0 W^c$ are unit eigenvectors corresponding to λ_2, 0, respectively, and W^{uu} is the strong unstable manifold of O.

From the above hypotheses, we know that the homoclinic orbit Γ enters the equilibrium O along the center manifold W^c as $t \to +\infty$ and enters the equilibrium O along the strong unstable direction of $T_0 W^u$ as $t \to -\infty$, which means that Γ is a homoclinic orbit with an orbit flip (see [12, 14]), so (H$_2$) is a nongeneric hypothesis.

Let e^+, e_s^- be unit eigenvectors corresponding to λ_1, $-\rho_1$, respectively.

(H$_3$) span $\{T_{\gamma(t)} W^u, T_{\gamma(t)} W^{cs}, e^+\} = \mathbb{R}^4$, $t \gg 1$.
span $\{T_{\gamma(t)} W^u, T_{\gamma(t)} W^{cs}, e_c^-\} = \mathbb{R}^4$, $t \ll -1$.

Hypothesis (H_3) is called the strong inclination property, which is equivalent to

$$\lim_{t\to+\infty} T_{\gamma(t)}W^u = \mathrm{span}\{e_c^-, e_u^+\}, \quad \lim_{t\to-\infty} T_{\gamma(t)}W^{cs} = \mathrm{span}\{e_u^+, e_s^-\}.$$

Remark 2.1 From [29], we know (H_3) is generic.

(H_4) Let y be the coordinate of system in center orientation, $\theta(y, \alpha, \mu)$ be the vector field on the center manifold. We may assume

$$\theta(0,\lambda,\mu) = 0, \quad \frac{\partial\theta}{\partial y}(0,0,0) = 0, \quad \frac{\partial^2\theta}{\partial y^2}(0,0,0) < 0,$$
$$\frac{\partial^2\theta}{\partial y\partial\lambda}(0,0,0) > 0, \quad \frac{\partial^2\theta}{\partial y\partial\varepsilon}(0,0,0) = 0.$$

From [30], under the above assumptions, the origin O is a transcritical bifurcation point, and λ is the control parameter of transcritical bifurcation. Under small perturbation, when $\lambda > 0$, the origin O is perturbed into two hyperbolic equilibria p_0, p_1. Let $w = (x, y, z)$, where $x = (x_1, x_2)$ denotes the unstable space, y denotes the center space, and z denotes the stable space. Now p_0, p_1 can be written as

$$p_0 = 0, \quad p_1 = p_0 + (0, \theta_0\lambda, 0) + O(\lambda^2) + O(\lambda\varepsilon),$$

where

$$\theta_0 = \frac{\frac{-\partial^2\theta}{\partial y\partial\lambda}(0,0,0)}{\frac{\partial^2\theta}{\partial y^2}}(0,0,0),$$

p_0 has a 3-dimensional unstable manifold and a 1-dimensional stable manifold, and p_1 has a 2-dimensional unstable manifold and a 2-dimensional stable manifold. To simplify the calculation, we make a scaling transformation to remove the coefficients $\frac{\partial^2\theta}{\partial y\partial\lambda}$, $-\frac{\partial^2\theta}{\partial y^2}$ of λy and y^2, respectively.

Based on the invariant manifold theory and the hypotheses (H_1)–(H_4), there exists a $\mathbb{C}^r$ coordinate change to flatten the stable manifold W^s, the unstable manifold W^u, the strong unstable manifold W^{uu}, and the center manifold W^c (see [28]), so that the flattened local invariant manifolds can be expressed as follows:

$$W^u_{\mathrm{loc}} = \{(x,y,z) : y = 0, z = 0\}, \quad W^{uu}_{\mathrm{loc}} = \{(x,y,z) : x_1 = 0, y = 0, z = 0\},$$
$$W^c_{\mathrm{loc}} = \{(x,y,z) : x = 0, z = 0\}, \quad W^s_{\mathrm{loc}} = \{(x,y,z) : x = 0, y = 0\}.$$

Now system (2.1) is changed into the following $\mathbb{C}^{r-1}$ system:

$$\begin{aligned} \dot{x}_1 &= f_{11}(x,y,z,\lambda,\varepsilon), \quad \dot{y} = f_{21}(x,y,z,\lambda,\varepsilon), \\ \dot{x}_2 &= f_{12}(x,y,z,\lambda,\varepsilon), \quad \dot{z} = f_{22}(x,y,z,\lambda,\varepsilon), \end{aligned} \tag{2.3}$$

which has the following form in U_0:

$$\begin{aligned} \dot{x}_1 &= [\lambda_1(\alpha) + \cdots]x_1 + (O(y) + O(z)) \cdot O(x_2), \\ \dot{y} &= \lambda y - y^2 + \varepsilon h(x,y,z) + \text{h.o.t.}, \\ \dot{x}_2 &= [\lambda_2(\alpha) + \cdots]x_2 + x_1[O(x_1) + O(y) + O(z)], \\ \dot{z} &= [-\rho_1(\alpha) + \cdots]z, \end{aligned} \tag{2.4}$$

where $\alpha = (\lambda, \varepsilon)$, $h|_{x=z=0} = O(y^2)$.

3 Local Coordinate System and the Poincaré Map

In this section, we will establish the Poincaré map, given as compositions of the local transition map (using the flow near the origin) and the global transition map. Then we use the Poincaré map to produce the successor functions.

After the straightening transformations, obviously, we may select the time T large enough such that $\gamma(\pm T) \subset U_0$ and

$$\gamma(-T) = (0, 0, \delta, 0), \quad \gamma(T) = (0, \delta, 0, 0),$$

where $\delta > 0$ is small enough such that $\{(x, y, z) : |x|, |y|, |z| < 2\delta\} \subset U_0$.

Consider the linear variational system

$$\dot{\Phi} = A(t)\Phi \tag{3.1}$$

and its adjoint system

$$\dot{\Psi} = -A^*(t)\Psi, \tag{3.2}$$

where $A(t) = Df(\gamma(t))$, $f = (f_{11}, f_{21}, f_{12}, f_{22})$.

Lemma 3.1 *If hypotheses* (H_1)–(H_4) *hold, we can choose a fundamental solution matrix to* (3.1)

$$\Phi(t) = (\phi_1(t), \phi_2(t), \phi_3(t), \phi_4(t))$$

satisfying

$$\begin{aligned}
&\phi_1(t) \in (T_{\gamma(t)}W^{cs})^c \cap (T_{\gamma(t)}W^u)^c,\\
&\phi_2(t) = -\frac{\dot{\gamma}(t)}{|\dot{\gamma}(T)|} \in [T_{\gamma(t)}W^{cs} \cap T_{\gamma(t)}W^u],\\
&\phi_3(t) \in \frac{T_{\gamma(t)}W^u}{T_{\gamma(t)}W^{cs} \cap T_{\gamma(t)}W^u},\\
&\phi_4(t) \in \frac{T_{\gamma(t)}W^{cs}}{T_{\gamma(t)}W^c},
\end{aligned}$$

such that

$$\Phi(T) = \begin{pmatrix} 1 & 0 & \phi_{31} & 0 \\ 0 & 1 & \phi_{32} & 0 \\ 0 & 0 & \phi_{33} & 0 \\ \phi_{14} & 0 & \phi_{34} & 1 \end{pmatrix}, \quad \Phi(-T) = \begin{pmatrix} \phi_{11} & 0 & 1 & \phi_{41} \\ \phi_{12} & 0 & 0 & \phi_{42} \\ \phi_{13} & \phi_{23} & 0 & \phi_{43} \\ 0 & 0 & 0 & \phi_{44} \end{pmatrix},$$

where $\det \phi_{ii} \neq 0$, $i = 3, 4$, $\phi_{12} \neq 0$ *and* $|\phi_{ij}\phi_{ii}^{-1}| \ll 1$, $i = 3, 4$, $j = 1, 2, 3, 4$, $i \neq j$.

Proof From the discussion above, the existence of $\phi_2(t)$, $\phi_3(t)$ and $\phi_4(t)$ with the given expression at $t = \pm T$ is clear. Based on (H_3), we take $\widetilde{\phi}_1(t) \in (T_{\gamma(t)}W^{cs})^c \cap (T_{\gamma(t)}W^u)^c$ satisfying

$$\widetilde{\phi}_1(T) = (1, 0, 0, 0)^*, \quad \widetilde{\phi}_1(-T) = (\widetilde{\phi}_{11}, \widetilde{\phi}_{12}, \widetilde{\phi}_{13}, \widetilde{\phi}_{14})^*.$$

Now let $\phi_1(t) = \widetilde{\phi}_1(t) + \phi_4(t)\phi_{14}$, where $\phi_{14} = -\phi_{44}^{-1}\widetilde{\phi}_{14}$. We have $\phi_1(t) \in (T_{\gamma(t)}W^{cs})^c \cap (T_{\gamma(t)}W^u)^c$, and

$$\phi_1(T) = (1, 0, 0, \phi_{14})^*, \quad \phi_1(-T) = (\phi_{11}, \phi_{12}, \phi_{13}, 0)^*,$$

where $\phi_{1i} = \widetilde{\phi}_{1i} - \widetilde{\phi}_{14}\phi_{44}^{-1}\phi_{4i}$ $(i = 1, 2, 3)$. According to Liouville's formula, $\det \Phi(T) \neq 0$ implies $\det \Phi(-T) \neq 0$, so we must have $\phi_{12} \neq 0$. Similar to that of [13, 28], the remaining can be proved.

Denote $\Psi(t) = (\psi_1^*, \cdots, \psi_4^*) = \Phi^{-1*}(t)$, which means that $\Psi(t)$ is a fundamental solution matrix to system (3.2), since $\Psi^*(t)\Phi(t)$=Id. By the exponential trichotomy theory and hypotheses (H_1)–(H_2), it is easy to see that there exists a $\beta > 0$ such that $\mathrm{e}^{-\beta t}\phi_1(t) \to 0$, $\mathrm{e}^{\beta t}\psi_1(t) \to 0$ as $t \to -\infty$. While as $t \to +\infty$, we have $\phi_1(t) \to \infty$ and $\psi_1(t) \to 0$ exponentially.

Now we choose $(\phi_1(t), \phi_2(t), \phi_3(t), \phi_4(t))$ as the local coordinate system of (2.3) along Γ. The relation between the original and the new coordinates can be defined as follows:

$$s(t) = \gamma(t) + \phi_1(t)n_1 + \phi_3(t)n_3 + \phi_4(t)n_4, \tag{3.3}$$

where n_1, n_3, n_4 are the coordinates in the new coordinate system.

Let

$$\begin{aligned} S_0 &= \{(x, y, z) = s(T) : \ |n_i| < \delta\}, \\ S_1 &= \{(x, y, z) = s(-T) : \ |n_i| < \delta\}. \end{aligned}$$

It is obvious that S_0 and S_1 are two Poincaré sections of (2.3) at $\gamma(T)$ and $\gamma(-T)$, respectively, where δ is small enough such that $S_0, S_1 \subset U_0$.

3.1 Establishment of the regular map P_1

First we use the flow of (2.3) to establish the regular map $P_1 : S_1 \to S_0$ in the tubular neighborhood of Γ. Make a coordinate change

$$(x, y, z)^* = s(t) = \gamma(t) + \phi_1(t)n_1 + \phi_3(t)n_3 + \phi_4(t)n_4, \tag{3.4}$$

where $t \in [-T, T]$. Substituting it into (2.3), we get

$$\begin{aligned} &\dot{\gamma}(t) + \sum_{i\neq 2} \dot{\phi}_i(t)n_i + \sum_{i\neq 2} \phi_i(t)\dot{n}_i \\ = &F(\gamma(t)) + DF(\gamma(t)) \sum_{i\neq 2} \phi_i(t)n_i + \lambda G_\lambda(\gamma(t), 0, 0) + \varepsilon G_\varepsilon(\gamma(t), 0, 0) + \text{h.o.t.} \end{aligned}$$

Multiplying both sides of the above equation by ψ_1^*, ψ_3^* and ψ_4^*, respectively, and using $\Psi^*(t)\Phi(t) = I$, we obtain

$$\dot{n}_i = \psi_i^*[\lambda G_\lambda(\gamma(t), 0, 0) + \varepsilon G_\varepsilon(\gamma(t), 0, 0)] + O(2), \tag{3.5}$$

where $i = 1, 3, 4$. Now integrating (3.5) from $-T$ to T, we get the regular map

$$P_1 : \ S_1 \to S_0, \quad (n_1(-T), n_3(-T), n_4(-T)) \mapsto (n_1(T), n_3(T), n_4(T)),$$

which can be expressed as

$$n_i(T) = n_i(-T) + \lambda M_{i\lambda} + \varepsilon M_{i\varepsilon} + \text{h.o.t.}, \tag{3.6}$$

where $M_{i\lambda} = \int_{-T}^{T} \psi_i^* G_\lambda(\gamma(t), 0, 0)\mathrm{d}t$, $M_{i\varepsilon} = \int_{-T}^{T} \psi_i^* G_\varepsilon(\gamma(t), 0, 0)\mathrm{d}t$, $i = 1, 3, 4$.

Due to

$$\begin{aligned} &\gamma(t) = (0, 0, x_2(t), 0)^*, \quad t \le -T, \\ &\gamma(t) = (0, y(t), 0, 0)^*, \quad\ \ t \ge T \end{aligned}$$

and the special form of (2.3) at the neighborhood of origin, it is easy to see that

$$\begin{aligned} M_{1\lambda} &= \int_{-T}^{T} \psi_1^* G_\lambda(\gamma(t), 0, 0)\mathrm{d}t = \int_{-\infty}^{+\infty} \psi_1^* G_\lambda(\gamma(t), 0, 0)\mathrm{d}t + \text{h.o.t.}, \\ M_{1\varepsilon} &= \int_{-T}^{T} \psi_1^* G_\varepsilon(\gamma(t), 0, 0)\mathrm{d}t = \int_{-\infty}^{+\infty} \psi_1^* G_\varepsilon(\gamma(t), 0, 0)\mathrm{d}t + \text{h.o.t.} \end{aligned}$$

The details of the deduction may refer to [13, 28].

3.2 Establishment of the singular map P_2

In this section, we set up the singular map P_2 from S_0 to S_1 induced by the flow of (2.4) in U_0. Consider the map

$$P_2 : S_0 \to S_1, \quad q_0(x_{10}, y_{10}, x_{20}, z_0) \mapsto q_1(x_{11}, y_{11}, x_{21}, z_1).$$

Let $\tau = \tau(q_0)$ be the flying time from $q_0 \in S_0$ to $q_1 \in S_1$. In order to guarantee the differentiability of the map at the origin, we set $s = \exp\{-\lambda_1(\alpha)\tau\}$, which is called Silnikov time (see [4, 26]). Using the method of variation of constants, we can get the following expression in U_0:

$$\begin{aligned} &x_{10} = s x_{11} + \text{h.o.t.}, &&y_{11} = h^{-1}(s) y_{10} + \text{h.o.t.}, \\ &x_{20} = s^{\frac{\lambda_2(\alpha)}{\lambda_1(\alpha)}} x_{21} + \text{h.o.t.}, &&z_1 = s^{\frac{\rho_1(\alpha)}{\lambda_1(\alpha)}} z_0 + \text{h.o.t.}, \end{aligned} \tag{3.7}$$

where

$$h(s) = \begin{cases} s^{\frac{\lambda}{\lambda_1}} + \lambda^{-1} y_{10}(1 - s^{\frac{\lambda}{\lambda_1}}), & \lambda \ne 0, \\ 1 - \lambda_1^{-1} y_{10} \ln s, & \lambda = 0. \end{cases} \tag{3.8}$$

Notice that, in (3.7), $y_{11} = h^{-1}(s) y_{10} + \text{h.o.t.}$ holds only for $y_{11} \ge \lambda$. When $y_{11} \in (-\rho, \lambda)$ ($\rho \ll 1$), P_2 is meaningful only for $s = 0$. So we extend the definition of map P_2 defined by (3.7) as follows:

$$y_{10} = \delta, \quad s = 0, \quad \text{when } y_{11} \in (-\rho, \lambda). \tag{3.9}$$

In order to compose Poincaré maps P_1 and P_2, we need to change the coordinates of q_0 and q_1 as follows:

$$\begin{aligned} q_0 &= (x_{10}, y_{10}, x_{20}, z_0)^* = \gamma(T) + \Phi(T)(n_{10}, 0, n_{30}, n_{40})^*, \\ q_1 &= (x_{11}, y_{11}, x_{21}, z_1)^* = \gamma(-T) + \Phi(-T)(n_1(-T), 0, n_3(-T), n_4(-T))^*. \end{aligned}$$

Using $\gamma(-T)=(0,0,\delta,0)^*$, $\gamma(T)=(0,\delta,0,0)^*$ and the expressions of $\Phi(T)$, $\Phi(-T)$, we get

$$\begin{aligned}
&n_{10}=x_{10}-\phi_{31}\phi_{33}^{-1}x_{20},\\
&n_{30}=\phi_{33}^{-1}x_{20},\\
&n_{40}=z_0-\phi_{14}x_{10}+(\phi_{14}\phi_{31}-\phi_{34})\phi_{33}^{-1}x_{20},\\
&n_1(-T)=\phi_{12}^{-1}y_{11}-\phi_{12}^{-1}\phi_{42}\phi_{44}^{-1}z_1,\\
&n_3(-T)=x_{11}-\phi_{11}\phi_{12}^{-1}y_{11}+(\phi_{11}\phi_{12}^{-1}\phi_{42}\phi_{44}^{-1}-\phi_{41}\phi_{44}^{-1})z_1,\\
&n_4(-T)=\phi_{44}^{-1}z_1,\\
&y_{10}\approx\delta,\quad x_{21}\approx\delta.
\end{aligned}\tag{3.10}$$

Now we can set up the Poincaré map $P=P_1\cdot P_2:S_0\to S_0$ by (3.5)–(3.10), which is given by

$$\begin{aligned}
n_1(T)&=\phi_{12}^{-1}y_{11}-\phi_{12}^{-1}\phi_{42}\phi_{44}^{-1}s^{\frac{\rho_1}{\lambda_1}}z_0+\lambda M_{1\lambda}+\varepsilon M_{1\varepsilon}+\text{h.o.t.},\\
n_3(T)&=x_{11}-\phi_{11}\phi_{12}^{-1}y_{11}+(\phi_{11}\phi_{12}^{-1}\phi_{42}\phi_{44}^{-1}-\phi_{41}\phi_{44}^{-1})s^{\frac{\rho_1}{\lambda_1}}z_0\\
&\qquad+\lambda M_{3\lambda}+\varepsilon M_{3\varepsilon}+\text{h.o.t.},\\
n_4(T)&=\phi_{44}^{-1}s^{\frac{\rho_1}{\lambda_1}}z_0+\lambda M_{4\lambda}+\varepsilon M_{4\varepsilon}+\text{h.o.t.},
\end{aligned}\tag{3.11}$$

and its associated successor function

$$H(s,x_{11},z_0)=(H_1,H_3,H_4)=(P-I)(n_{10},n_{30},n_{40})$$

is given by

$$\begin{aligned}
H_1&=-sx_{11}+\phi_{31}\phi_{33}^{-1}s^{\frac{\lambda_2(\alpha)}{\lambda_1(\alpha)}}x_{21}+\phi_{12}^{-1}y_{11}-\phi_{12}^{-1}\phi_{42}\phi_{44}^{-1}s^{\frac{\rho_1(\alpha)}{\lambda_1(\alpha)}}z_0\\
&\qquad+\lambda M_{1\lambda}+\varepsilon M_{1\varepsilon}+\text{h.o.t.},\\
H_3&=-\phi_{33}^{-1}s^{\frac{\lambda_2(\alpha)}{\lambda_1(\alpha)}}x_{21}+x_{11}-\phi_{11}\phi_{12}^{-1}y_{11}+as^{\frac{\rho_1(\alpha)}{\lambda_1(\alpha)}}z_0+\lambda M_{3\lambda}+\varepsilon M_{3\varepsilon}+\text{h.o.t.},\\
H_4&=(\phi_{44}^{-1}s^{\frac{\rho_1(\alpha)}{\lambda_1(\alpha)}}-1)z_0+\phi_{14}sx_{11}-(\phi_{14}\phi_{31}-\phi_{34})\phi_{33}^{-1}s^{\frac{\lambda_2(\alpha)}{\lambda_1(\alpha)}}x_{21}\\
&\qquad+\lambda M_{4\lambda}+\varepsilon M_{4\varepsilon}+\text{h.o.t.},
\end{aligned}\tag{3.12}$$

where y_{11} is given by (3.7)–(3.9), $a=\phi_{11}\phi_{12}^{-1}\phi_{42}\phi_{44}^{-1}-\phi_{41}\phi_{44}^{-1}\ll 1$.

4 Bifurcation Analysis

In this section, we use the above successor function to discuss the homoclinic bifurcation problem accompanied with transcritical bifurcation. Now we use (3.12) to study the existence and the uniqueness of 1-homoclinic (1-heteroclinic) and 1-periodic orbits.

It is easy to see that system (2.1) has a homoclinic or heteroclinic orbit (resp. periodic orbit) near Γ if and only if $H=0$ has solutions satisfying $s=0$ (resp. $s>0$).

Consider the solutions of equation $H(s,x_{11},z_0)=0$. It is easy to see that $\mathrm{H}_3=0$, $\mathrm{H}_4=0$ have unique solutions $x_{11}=x(s,y_{11})$, $z_0=z(s,y_{11})$ as λ, ε sufficiently small and $0<s\ll 1$. Substituting it into $H_1=0$, we have

$$\begin{aligned}
&-\phi_{12}^{-1}\phi_{11}sy_{11}+\delta\phi_{31}\phi_{33}^{-1}s^{\frac{\lambda_2(\alpha)}{\lambda_1(\alpha)}}+\phi_{12}^{-1}y_{11}+\lambda M_{1\lambda}+\varepsilon M_{1\varepsilon}\\
&+s\lambda M_{3\lambda}+s\varepsilon M_{3\varepsilon}+\text{h.o.t.}=0.
\end{aligned}\tag{4.1}$$

We call (4.1) the bifurcation equation of the nongeneric homoclinic bifurcation accompanied with the transcritical bifurcation. It is obvious that $H = 0$ has solutions satisfying $s = 0$ or $s > 0$ if and only if

$$H_1(s, x(s, y_{11}), z(s, y_{11})) = 0$$

has solutions satisfying $s = 0$ or $s > 0$. So we only need to study the solutions to (4.1).

First we consider the homoclinic bifurcation as $\lambda = 0$ and ε is small enough. Now the transcritical bifurcation does not happen. From (3.7)–(3.8), we know that (4.1) can be written as

$$\begin{aligned} &-\phi_{12}^{-1}\phi_{11}s\frac{\delta}{1-\lambda_1^{-1}\delta\ln s} + \delta\phi_{31}\phi_{33}^{-1}s^{\frac{\lambda_2(\alpha)}{\lambda_1(\alpha)}} + \phi_{12}^{-1}\frac{\delta}{1-\lambda_1^{-1}\delta\ln s} \\ &+ s\varepsilon M_{3\varepsilon} + \varepsilon M_{1\varepsilon} + \text{h.o.t.} = 0. \end{aligned} \tag{4.2}$$

Since $s^{\frac{\lambda_2(\alpha)}{\lambda_1(\alpha)}} = o(\frac{1}{1-\lambda_1^{-1}\delta\ln s})$, $\lim\limits_{s\to 0} s(1-\lambda_1^{-1}\delta\ln s) = 0$, now (4.2) can be changed into

$$\frac{\delta}{1-\lambda_1^{-1}\delta\ln s} + \phi_{12}s\varepsilon M_{3\varepsilon} + \phi_{12}\varepsilon M_{1\varepsilon} + \text{h.o.t.} = 0. \tag{4.3}$$

Let

$$\begin{aligned} L(s,\varepsilon) &= -\phi_{12}s\varepsilon M_{3\varepsilon} - \phi_{12}\varepsilon M_{1\varepsilon} + \text{h.o.t.}, \\ N(s,\varepsilon) &= \tfrac{\delta}{1-\lambda_1^{-1}\delta\ln s} + \text{h.o.t.} \end{aligned}$$

Then we have

$$N(0,\varepsilon) = 0, \quad L(0,\varepsilon) = -\phi_{12}\varepsilon M_{1\varepsilon} + o(\varepsilon^2) \neq 0, \quad 0 < |\varepsilon| \ll 1 \tag{4.4}$$

and

$$N'(s,\varepsilon) = \frac{\delta^2}{\lambda_1 s(1-\lambda_1^{-1}\delta\ln s)^2} + \text{h.o.t.}, \quad L'(s,\varepsilon) = -\phi_{12}\varepsilon M_{3\varepsilon} + \text{h.o.t.}$$

Now we have the following results.

Theorem 4.1 *Suppose that* (H_1)–(H_4) *are valid,* $\lambda = 0$, $0 < |\varepsilon| \ll 1$, $M_{1\varepsilon} \neq 0$, $M_{3\varepsilon} \neq 0$. *We have*

(i) *if* $\phi_{12}\varepsilon M_{1\varepsilon} < 0$, $\phi_{12}\varepsilon M_{3\varepsilon} > 0$, *then system* (2.1) *has a unique* 1*-period orbit near* Γ.

(ii) *if* $\phi_{12}\varepsilon M_{1\varepsilon} < 0$, $\phi_{12}\varepsilon M_{3\varepsilon} < 0$, *there exist three parameter curves* $\Sigma_0(\varepsilon)$, $\Sigma_1(\varepsilon)$ *and* $\Sigma_2(\varepsilon)$, *such that*

system (2.1) *does not have any* 1*-period orbit or* 1*-homoclinic orbit as* $\varepsilon \in \Sigma_0(\varepsilon)$;

system (2.1) *has a unique two-fold* 1*-period orbit near* Γ *as* $\varepsilon \in \Sigma_1(\varepsilon)$;

system (2.1) *has exactly two* 1*-period orbits near* Γ *as* $\varepsilon \in \Sigma_2(\varepsilon)$.

(iii) *if* $\phi_{12}\varepsilon M_{1\varepsilon} > 0$, $\phi_{12}\varepsilon M_{3\varepsilon} > 0$, *then there is no* 1*-period orbit near* Γ.

(iv) *if* $\phi_{12}\varepsilon M_{1\varepsilon} > 0$, $\phi_{12}\varepsilon M_{3\varepsilon} < 0$, *then there is an* $\overline{s} = -\frac{M_{1\varepsilon}}{M_{3\varepsilon}} + \text{h.o.t.}$ *for* $0 < s < \overline{s}$, $0 < |\varepsilon| \ll 1$, *and system* (2.1) *has no* 1*-period orbit near* Γ.

Proof Case (i) In this case, it is easy to see that

$$N'(s,\varepsilon) > 0, \quad N''(s,\varepsilon) = -\frac{\delta^2[(1-\lambda_1^{-1}\delta\ln s) - 2\lambda_1^{-1}\delta]}{\lambda_1 s^2(1-\lambda_1^{-1}\delta\ln s)^3} + \text{h.o.t.} < 0 \quad \text{for } 0 < s \ll 1,\ |\varepsilon| \ll 1,$$

while $L(0,\varepsilon) > 0$, $L'(s,\varepsilon) < 0$, which means that $L(s,\varepsilon) = N(s,\varepsilon)$ has a unique solution. Case (i) follows immediately.

Case (ii) Notice that $L(s,\varepsilon)$ is tangent to $N(s,\varepsilon)$ at some point s, if and only if the following hold

$$L(s,\varepsilon) = N(s,\varepsilon), \quad L'(s,\varepsilon) = N'(s,\varepsilon),$$

that is

$$-\phi_{12}s\varepsilon M_{3\varepsilon} - \phi_{12}\varepsilon M_{1\varepsilon} = \tfrac{\delta}{1-\lambda_1^{-1}\delta\ln s} + \text{h.o.t.},$$
$$-\phi_{12}\varepsilon M_{3\varepsilon} = \tfrac{\delta^2}{\lambda_1 s(1-\lambda_1^{-1}\delta\ln s)^2} + \text{h.o.t.},$$

namely

$$(\phi_{12}s\varepsilon M_{3\varepsilon} + \phi_{12}\varepsilon M_{1\varepsilon})^2 + \text{h.o.t.} = -\lambda_1 s\phi_{12}\varepsilon M_{3\varepsilon} + \text{h.o.t.}$$

Based on the implict function theorem, there exists a function curve $s = s_0(\varepsilon)$ such that

$$L(s_0,\varepsilon) = N(s_0,\varepsilon), \quad L'(s_0,\varepsilon) = N'(s_0,\varepsilon).$$

Let

$$h(s,\varepsilon) = L(s,\varepsilon) - N(s,\varepsilon).$$

Take $h(s_0(\varepsilon),\varepsilon) \triangleq \Delta(\varepsilon)$. Then, if $\varepsilon \in \Sigma_0(\varepsilon) = \{\varepsilon \mid \Delta(\varepsilon) = 0\}$, the straight line $L(s,\varepsilon)$ intersects the curve $N(s,\varepsilon)$ at a unique point $s = s_0(\varepsilon)$, which implies that system (1.1) has a unique two-fold 1-periodic orbit near Γ, where $\Delta(\varepsilon) = 0$ is the saddle-node bifurcation curve. If $\varepsilon \in \Sigma_1(\varepsilon) = \{\varepsilon \mid \Delta(\varepsilon) > 0\}$, then the straight line $L(s,\varepsilon)$ does not intersect the curve $N(s,\varepsilon)$. Now system (1.1) has not any 1-periodic orbit near Γ. If $\varepsilon \in \Sigma_2(\varepsilon) = \{\varepsilon \mid \Delta(\varepsilon) < 0\}$, then the straight line $L(s,\varepsilon)$ intersects the curve $N(s,\varepsilon)$ at two exact points $s = s_1, s_2$, and $0 < s_1 < s_0 < s_2$, which means that system (1.1) has exactly two 1-periodic orbits near Γ.

Case (iii) Based on the hypotheses and (4.4), we have

$$L(0,\varepsilon) < 0, \quad L'(s,\varepsilon) < 0, \quad N(0,\varepsilon) = 0, \quad N'(s,\varepsilon) > 0.$$

It is easy to see that the straight line $L(s,\varepsilon)$ does not intersect the curve $N(s,\varepsilon)$. Hence there is no 1-periodic orbit near Γ.

Case (iv) Let $L(s,\varepsilon) = 0$. Then we can solve that $\overline{s} = -\frac{M_{1\varepsilon}}{M_{3\varepsilon}} + \text{h.o.t.}$ It is clear that there is no 1-periodic orbit near Γ for $0 < s < \overline{s}$ and $0 < |\varepsilon| \ll 1$. The proof is completed.

Now, we turn to discuss the bifurcations of the homoclinic orbit when the origin undergoes the transcritical bifurcation, namely, $\lambda > 0$. In this case, (4.1) becomes

$$\begin{aligned}&-\phi_{12}^{-1}\phi_{11}s\frac{y_{10}}{s^{\frac{\lambda}{\lambda_1}} + \lambda^{-1}y_{10}(1 - s^{\frac{\lambda}{\lambda_1}})} + \phi_{12}^{-1}\frac{y_{10}}{s^{\frac{\lambda}{\lambda_1}} + \lambda^{-1}y_{10}(1 - s^{\frac{\lambda}{\lambda_1}})}\\&+ \lambda(M_{1\lambda} + sM_{3\lambda}) + \varepsilon(M_{1\varepsilon} + sM_{3\varepsilon}) + \text{h.o.t.} = 0.\end{aligned} \tag{4.5}$$

Let $r = s^{\frac{\lambda}{\lambda_1}}$. It follows from (4.5) that

$$\begin{aligned}&(-\phi_{12}^{-1}\phi_{11}r^{\frac{\lambda_1}{\lambda}} + \phi_{12}^{-1})(\lambda + \lambda\delta^{-1}(\delta - \lambda)r) + \lambda(M_{1\lambda} + r^{\frac{\lambda_1}{\lambda}}M_{3\lambda})\\&+ \varepsilon(M_{1\varepsilon} + r^{\frac{\lambda_1}{\lambda}}M_{3\varepsilon}) + \text{h.o.t.} = 0.\end{aligned} \tag{4.6}$$

Assume $r=0$ in (4.6). Then

$$\phi_{12}^{-1}\lambda+\lambda M_{1\lambda}+\varepsilon M_{1\varepsilon}+\text{h.o.t.}=0.$$

If $M_{1\varepsilon}\neq 0$, the above equation determines a curve

$$L_1^\lambda=\{\varepsilon(\lambda):\varepsilon M_{1\varepsilon}+\phi_{12}^{-1}\lambda+\lambda M_{1\lambda}+\text{h.o.t.}=0\},$$

such that (4.6) has a solution $r=s=0$ as $\varepsilon\in L_1^\lambda$ and $0<|\varepsilon|\ll 1$, i.e., system (2.1) has a homoclinic orbit near Γ.

When $r\neq 0$, system (4.6) has a solution

$$\varepsilon(r,\lambda)=\frac{-\lambda[\phi_{12}^{-1}+\delta^{-1}(\delta-\lambda)\phi_{12}^{-1}r+M_{1\lambda}+(M_{3\lambda}-\phi_{12}^{-1}\phi_{11})r^{\frac{\lambda_1}{\lambda}}]+\text{h.o.t.}}{r^{\frac{\lambda_1}{\lambda}}M_{3\varepsilon}+M_{1\varepsilon}}.$$

Then differentiating $\varepsilon(r,\lambda)$ with respect to r, and after some simple calculation, we have

$$\frac{\partial\varepsilon}{\partial r}=\frac{-1}{[r^{\frac{\lambda_1}{\lambda}}M_{3\varepsilon}+M_{1\varepsilon}]^2}[\lambda\delta^{-1}(\delta-\lambda)\phi_{12}^{-1}M_{1\varepsilon}+\text{h.o.t.}].$$

It is obvious that $\varepsilon(r,\lambda)$ is monotonous with respect to r when $M_{1\varepsilon}\neq 0$. Moreover, we see

$$\varepsilon(r,\lambda)\to\varepsilon(0,\lambda)=\frac{-\phi_{12}^{-1}\lambda-\lambda M_{1\lambda}}{M_{1\varepsilon}}+\text{h.o.t.},\quad r\to 0.$$

Notice that $s=r^{\frac{\lambda_1}{\lambda}}$ increases monotonously with respect to r as $0<\lambda\ll 1$, and then we can get the following conclusions.

Theorem 4.2 *Suppose that hypotheses* (H_1)–(H_4) *hold, and* $0<\lambda\ll 1$, $M_{1\varepsilon}\neq 0$. *Then for* $\varepsilon_1>0$ *small enough, we have*

(i) *when* $\phi_{12}M_{1\varepsilon}<0$, $\varepsilon(0,\lambda)<\varepsilon<\varepsilon_1$, *or* $\phi_{12}M_{1\varepsilon}>0$, $-\varepsilon_1<\varepsilon<\varepsilon(0,\lambda)$, *system* (2.1) *has a unique* 1*-periodic orbit near* Γ;

(ii) *when* $\varepsilon=\varepsilon(0,\lambda)$, *namely* $\varepsilon\in L_1^\lambda$, *system* (2.1) *has a unique* 1*-homoclinic orbit* Γ_ε *connecting* p_1 *near* Γ. *Also in addition to the homoclinic orbit* Γ_ε, *the system has no periodic orbit as* $\varepsilon\in L_1^\lambda$.

Next we consider the case of heteroclinic bifurcation for $\lambda>0$. From (3.9), we know that when $y_{11}<\lambda$, based on (4.1), the bifurcation equation is changed into

$$\phi_{12}^{-1}y_{11}+\lambda M_{1\lambda}+\varepsilon M_{1\varepsilon}+\text{h.o.t.}=0.$$

If we notice that when $\lambda>0$, after the creation of p_0 and p_1, there always exists a straight segment orbit heteroclinic to p_0 and p_1, with length λ, then we have the following conclusions.

Theorem 4.3 *Suppose that hypotheses* (H_1)–(H_4) *are valid,* $0<\lambda\ll 1$, $M_{1\varepsilon}\neq 0$. *Then we have for* $\phi_{12}M_{1\varepsilon}<0$, $-\frac{\lambda M_{1\lambda}}{M_{1\varepsilon}}<\varepsilon<\frac{-\lambda}{M_{1\varepsilon}}(\phi_{12}^{-1}+M_{1\lambda})$, *or* $\phi_{12}M_{1\varepsilon}>0$, $\frac{-\lambda}{M_{1\varepsilon}}(\phi_{12}^{-1}+M_{1\lambda})<\varepsilon<-\frac{\lambda M_{1\lambda}}{M_{1\varepsilon}}$, *there exist two heteroclinic orbits connecting* p_0 *with* p_1.

Remark 4.1 Let $\phi_{12} = \triangle|\phi_{12}|$. When $\triangle = 1$, we call Γ nontwisted, and when $\triangle = -1$, we call Γ twisted. In general, when we study the problem for homoclinic or heteroclinic bifurcation connecting hyperbolic equilibrium, the bifurcation is more complicated as Γ is twisted. But when we study the bifurcation of homoclinic orbits accompanied with transcritical bifurcation, from Theorems 4.1–4.3, we know that the influence of the jump component on the center manifold is stronger than that of ϕ_{12} which reflects twisting or not of Γ. So twisting or not will not affect the bifurcation results.

Acknowledgement The author would like to thank the referees for their helpful comments and suggestions.

References

[1] Aronson, D. G., Krupa, M. and van Gils, S. A., Homoclinic twist bifurcations with $\mathbb{Z}_2$ symmetry, *J. Non. Sci.*, **4**, 1994, 195–219.

[2] Bonatti, C., Diaz, L. J. and Marcelo, V., Dynamics Beyond Uniform Hyperbolicity: A Global Geometric and Probabilistic Perspective, Springer-Verlag, Berlin, 2005.

[3] Battelli, F. and Palmer, K., Shilnikov Saddle-Focus Homoclinic Orbits in Singularly Perturbed Systems in Dimension Higher than Three, ENOC-2008, Saint Peterburg, Russia, 2008.

[4] Homburg, A. J., Singular heteroclinic cycles, *J. Diff. Eqs.*, **161**, 2000, 358–402.

[5] Li, J. B., Zhao, X. H. and Chen, G. R., Breaking wave solutions to the second class of singular nonlinear traveling wave equations, *Internat. J. Bifur. Chaos Appl. Sci. Engrg.*, **19**(4), 2009, 1289–1306.

[6] Homoburg, A. J. and Knobloch, J., Multiple homoclinic orbits in conservative and reversible systems, *Trans. Amer. Math. Soc.*, **358**, 2006, 1715–1740.

[7] Morales, C. A. and Pacifico, M. J., Inclination-flip homoclinic orbits arising from orbit-flip, *Nonlinearity*, **14**, 2001, 379–393.

[8] Rademacher, J. D. M., Homoclinic orbits near heteroclinic cycles with one equilibrium and one periodic orbit, *J. Diff. Eqs.*, **218**, 2005, 390–443.

[9] Yagasaki, K., The method of Melnikov for perturbations of multi-degree-of-freedom Hamiltonian systems, *Nonlinearity*, **12**, 1999, 799–822.

[10] Knobloch, J. and Wagenknecht, T., Homoclinic snaking near a heteroclinic cycle in reversible systems, *Physica D*, **1**, 2005, 82–93.

[11] Robinson, R. C., Nonsymmetric Lorenz attaractors from a homoclinic bifurcation, *SIAM J. Math. Anal.*, **32**, 2000, 119–141.

[12] Homburg, A. J. and Krauskopf, B., Resonant homoclinic flip bifurcations, *J. Dynam. Diff. Eqs.*, **12**(4), 2000, 807–850.

[13] Shui, S. L., Codimension 3 non-resonant bifurcations of rough heteroclinic loops with one orbit flip, *Chin. Ann. Math.*, **27B**(6), 2006, 657–674.

[14] Naudot, V., A strange attractor in the unfolding of an orbit-flip homoclinic orbit, *Dyn. Syst.*, **17**(1), 2002, 45–63.

[15] Morales, C. A., On inclination-flip homoclinic orbit associated to a saddle-node singularity, *Bol. Soc. Bras. Math.*, **27**(2), 1996, 145–160.

[16] Klaus, J. and Knobloch, J., Bifurcation of homoclinic orbits to a saddle-center in reversible systems, *Inter. J. Bifur. Chaos*, **13**(9), 2003, 2603–2622.

[17] Wagenknecht, T., Two-heteroclinic orbits emerging in the reversible homoclinic pitchfork bifurcation, *Nonlinearity*, **18**(2), 2005, 527–542.

[18] Deng, B., Homoclinic bifurcations with nonhyperbolic equilibria, *SIAM J. Math. Anal.*, **21**, 1990, 693–720.

[19] Chow, S. N. and Lin, X. B., Bifurcation of a homoclinic orbit with a saddle node equilibrium, *Diff. Integral Eqs.*, **3**, 1990, 435–466.

[20] Krauskopf, B. and Oldeman, B. E., Bifurcations of global reinjection orbits near a saddle-node Hopf bifurcation, *Nonlinearity*, **19**, 2006, 2149–2167.

[21] Dumortier, F., Roussarie, R. and Sotomayor, J., Generic 3-parameter families of vector fields on the plane, unfolding a singularity with nilpotent linear part: The cusp case of codimension 3, *Ergodic Theory Dynam. Syst.*, **7**, 1987, 375–413.

[22] Lamb, J. S. W., Teixeira, M. A. and Kevin, N. W., Heteroclinic bifurcations near Hopf-zero bifurcation in reversible vector fields in R^3, *J. Diff. Eqs.*, **219**, 2005, 78–115.

[23] Fečkan, M. and Gruendler, J., Homoclinic-Hopf interaction: an autoparametric bifurcation, *Proc. Roy. Soc. Edinburgh*, **130**, 2000, 999–1015.

[24] Champneys, A. R., Härterich, J. and Sandstede, B. A., Non-trans verse homoclinic orbit to a saddle-node equilibrium, *Ergodic Theory Dyn. Syst.*, **16**, 1996, 431–450.

[25] Ye, Z. Y. and Han, M. A., Bifurcations of subharmonic solutions and invariant tori for a singularly perturbed system, *Chin. Ann. Math.*, **28B**(2), 2007, 135–147.

[26] Zhu, D. M. and Xia, Z. H., Bifurcations of heteroclinic loops, *Sci. in China, Ser. A*, **41**(8), 1998, 837–848.

[27] Geng, F. J. and Zhu, D. M., Bifurcations of generic heteroclinic loop accompanied by transcritical bifurcation, *Inter. J. Bifur. Chaos*, **18**, 2008, 1069–1083.

[28] Zhang, T. S. and Zhu, D. M., Codimension 3 homoclinic bifurcation of orbit flip with resonant eigenvalues corresponding to the tangent directions, *Inter. J. Bifur. Chaos*, **14**, 2004, 4161–4175.

[29] Deng, B., The Silnikov problem, exponential expansion, strong λ-Lemma, C^1-linearization, and homoclinic bifurcation, *J. Diff. Eqs.*, **79**, 1989, 189–231.

[30] Wiggins, S., Introduction to Applied Nonlinear Dynamical Systems and Chaos, Springer-Verlag, New York, 1990.