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L^p Estimates for Singular Radon Transforms with Rough Kernels**

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Abstract The purpose of this paper is to study the mapping properties of the singular Radon transforms with rough kernels. Such singular integral operators are proved to be bounded on Lebesgue spaces.

Keywords Singular Radon transform, Rough kernel, Dyadic decomposition
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1 Introduction

Let $r_t(x) = r(x,t)$ be a C^∞ function, defined in a neighborhood of the point $(x_0, 0)$ in $\mathbb{R}^n \times \mathbb{R}^k$, taking values in $\mathbb{R}^n$ with $r(x,0) = x$. The map $x \mapsto r_t(x)$ is a family of local diffeomorphisms of $\mathbb{R}^n$, depending smoothly on the parameter $t \in \mathbb{R}^k$. The differential $\frac{\partial r}{\partial t}$ has rank k when $t = 0$. We also assume that ψ is a suitable C^∞ cut-off function supported near x_0 and a is a small positive constant. The singular Radon transform under consideration in this paper is of the form

$$T(f)(x) = \psi(x) \int_{|t|\le a} f(r(x,t))K(t)\mathrm{d}t, \tag{1.1}$$

where $K(t)$ is a homogeneous kernel on $\mathbb{R}^k$, so that $K(st) = s^{-k}K(t)$ for $s > 0$ and $t \in \mathbb{R}^n$ and satisfies

$$\int_{|t|=1} K(t)\mathrm{d}\sigma(t) = 0.$$

$T(f)(x)$ may be viewed as averaging f in the t variable over surface $r(x,t)$ with respect to the singular integral kernel $K(t)$. They arise naturally in many different areas of analysis and geometry (see [2, 4]).

In the study of singular Radon transform, a curvature condition has been developed for proving boundedness of these operators. It was proved in [2] that there exists a unique collection of vector fields X_α defined in some neighborhood of x_0 with $(\alpha_1 \cdots \alpha_k) \neq 0$, such that

$$r(x,t) \sim \exp\Big(\sum_\alpha \frac{t^\alpha}{\alpha!} X_\alpha\Big)(x), \quad \text{as } t \to 0. \tag{1.2}$$

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The curvature condition of [2] at a point $x_0 \in \mathbb{R}^n$ can be stated as follows.

Curvature Condition *r satisfies curvature condition* (C) *at x_0, if there exists $m > 0$ such that*

$$r(x,t) = \exp\Big(\sum_{0<|\alpha|\le m} \frac{t^\alpha}{\alpha!} X_\alpha\Big) + R(x,t),$$

where $R(x,t) = O(|t|^{m+1})$, and the vector fields $\{X_\alpha : |\alpha| \le m\}$ together with all their iterated commutators of degree $\le m$ span the tangent space to $\mathbb{R}^n$ at x_0.

Let $K_0(t)$ be the restriction of $K(t)$ to the annulas $A_0 = \{t \in R^k : \frac{a}{2} \le |t| \le a\}$. If $K_0(t) \in C^1(A_0)$, the L^p boundedness of T in (1.1) was shown by Christ, Nagel, Stein and Wainger [2] when the curvature condition (C) holds. In this note, we study singular Radon transforms with rough kernels. We improve the main results in [2] by proving L^p-boundedness of T in (1.1) under some weaker assumptions on the regularity of the kernel. Our results are as follows.

Theorem 1.1 *Suppose that $r(x,t)$ satisfies the curvature condition* (C) *at x_0 and K_0 is in the Orlics space $L\log L(A_0)$. Then the operator T in* (1.1) *is bounded on $L^2(\mathbb{R}^n)$.*

Theorem 1.2 *Suppose that $r(x,t)$ satisfies the curvature condition (C) at x_0 and $K_0 \in L^q(A_0)$ for some $1 < q < \infty$. Then the operator T in* (1.1) *is bounded from $L^p(\mathbb{R}^n)$ to itself for every $1 < p < \infty$.*

2 Proof of Main Results

Inspired by the arguments presented in [2, 5], we reduce the study of the singular Radon transform T defined by (1.1) to one on nilpotent Lie group by a lifting technique, where proving L^p-boundedness is less complicated. One of the consequences of this lifting is that we have local dilations, which allow us to re-scale crucial estimates on the annulas A_0 (see [2]). Let $\widetilde{X}_\alpha$ denote corresponding lifted vector fields defined in an open subset of the extended space $\mathbb{R}^d = \mathbb{R}^n \times \mathbb{R}^{d-n}$. When t is small, we define $\widetilde{r}$ by

$$\widetilde{r}(x,z,t) = \exp\Big(\sum_{0<|\alpha|\le |m|} \frac{\widetilde{X}_\alpha}{\alpha!} t^\alpha\Big)(x,z) + (R(x,t),0),$$

where $R(x,t) = O(|t|^{m+1})$.

Denote by $\pi : \mathbb{R}^n \times \mathbb{R}^{d-n} \mapsto \mathbb{R}^n$ the projection $\pi(x,z) = x$. Then we have $\pi(\widetilde{r}(x,z,t)) = r(x,t)$ and let $\psi(x,z)$ be a cut-off function in $\mathbb{R}^d$. We define

$$\widetilde{T}(F)(x,z) = \psi(x,z) \int_{|t|\le a} F(\widetilde{r}_t(x,z))K(t)\mathrm{d}t.$$

By the argument used in [2], the L^p-boundedness of $\widetilde{T}$ implies the corresponding results for T. Therefore it suffices to prove Theorems 1.1 and 1.2 in the extended spaces. For simplification we still write X_α instead of $\widetilde{X}_\alpha$, $\widetilde{r} = r$ and $d = n$.

Let g be the relatively free nilpotent Lie algebra that consists of the vector fields $\{X_\alpha\}$ and their commutators of degree $\leq m$. There is a unique simply connected Lie group G, whose Lie algebra is g. We may identify the Lie group G with $\mathbb{R}^n$. We can also choose a basis of g consisting of $\{X_I\}$, which is called basic (see [2]). For each $x \in \mathbb{R}^n$, we define the mapping $y \mapsto Q_x(y)$ from a neighborhood of x to a neighborhood of the origin, given by

$$Q_x(y) = u = (u_I),$$

where $y = \exp\Big(\sum\limits_{I \text{ basic}} u_I X_I\Big)(x)$. There are dilations δ_r^x, centered at x, given by

$$\delta_r^x(y) = \exp\Big(\sum_{I \text{ basic}} u_I r^{|I|} X_I\Big)(x).$$

A related quasi-distance is defined by $d(x,y) = \rho(Q_x(y))$ with $\rho(u) = \sum\limits_{I \text{ basic}} |u_I|^{\frac{1}{|I|}}$. The homogeneous dimension of $\mathbb{R}^n$ equals $\sum\limits_{I \text{ basic}} |I| = Q$.

We now may proceed to the proof of our main results. We start by decomposing the kernel K dyadically. Let φ be a bump function with $\mathrm{supp}\varphi \subseteq (1,2)$, such that $\sum\limits_j 2^{-j} t \cdot \varphi(2^{-j}t) = \frac{1}{\ln 2}$. Set

$$F_j K(t) = 2^{-j} \int \varphi(2^{-j}s) s^{-k} K\Big(\frac{t}{s}\Big) \mathrm{d}s.$$

From the identity

$$K(t) = \frac{1}{\ln 2} \int s^{-k} K_0\Big(\frac{t}{s}\Big) \frac{\mathrm{d}s}{s},$$

we have the decomposition $K = \sum\limits_{j=0}^{\infty} F_j K_0$.

For $K_0 \in L \log L(A_0)$, that is, $\|K_0\|_{L \log L} = \int_{A_0} |K_0(t)| \log(2 + |K_0(t)|)\mathrm{d}t < \infty$, we have to decompose it further. Let $\overline{K}_0^m$ be the portion of K_0 on the set

$$E_m = \{t \in A_0 : 2^{2^m} \leq 2 + |K_0(t)| < 2^{2^{m+1}}\}.$$

Write $K_0^m = \overline{K}_0^m - \dfrac{\chi_{A_0}}{|A_0|} \int_{A_0} \overline{K}_0^m(t)\mathrm{d}t$. Then each K_0^m has mean zero and $K_0 = \sum\limits_{m \geq 0} K_0^m$. It is easy to see that

$$\sum_{m \geq 0} 2^m \|K_0^m\|_1 \leq C \sum_{m \geq 0} 2^m \|\overline{K}_0^m\|_1 \leq C \|K_0\|_{L \log L}. \tag{2.1}$$

For compactly supported $f \in C^1(\mathbb{R}^n)$, let

$$T_j(f)(x) = \psi(x) \int f(r_t(x)) F_j K_0(t) \mathrm{d}t,$$
$$T_{j,m}(f)(x) = \psi(x) \int f(r_t(x)) F_j K_0^m(t) \mathrm{d}t.$$

Then

$$T(f) = \sum_{j=0}^{\infty} T_j(f) = \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} T_{j,m}(f).$$

Let $r^{-1}(x,t)$ be the inverse mapping of $r(x,t)$, defined by (1.2). For $\tau=(\tau_1,\tau_2,\cdots\tau_{2N})$, $\tau_i\in\mathbb{R}^k, 1\le i\le 2N$, we set

$$\Gamma_N(x,\tau)=r^{-1}_{\tau_{2N}}\cdot r_{\tau_{2N-1}}\cdots r^{-1}_{\tau_2}\cdot r_{\tau_1}(x).$$

Then the operator $(T_jT_j^*)^N$ may be written as

$$(T_{j,m}T^*_{j,m})^N(f)(x)=\int f(\delta^x_{2^{-j}}(\Gamma^{(j)}_N(x,\tau))\psi(x,\tau))\prod_{\nu=1}^{2N}(F_0K_0^m)(\tau)\mathrm{d}\tau, \tag{2.2}$$

where $\Gamma^{(j)}(x,\tau)=\delta^x_{2^j}(\Gamma_N(x,2^{-j}\tau))$, $\psi(x,\tau)\in C^\infty$ has compact support with respect to x and τ. If we substitute K_0^m by K_0, the representation of the operator $(T_jT_j^*)^N$ may be obtained similarly.

To prove our main results, we need to estimate the kernel of $(T_{j,m}T^*_{j,m})^N$ and $(T_jT_j^*)^N$. By the argument used in [2], we shall consider the smoothness of measures transported by the map Φ_x, where $\Phi_x:\tau\mapsto\Phi(x,\tau)$ will be a given C^∞ mapping from a closed finite ball $\overline{B}$ in $\mathbb{R}^{2Nk}$ to $\mathbb{R}^n$ with $N\ge n$. Let $J(x,\tau)$ be the determinant of some $n\times n$ sub-matrix of the Jacobian matrix $\dfrac{\partial\Phi}{\partial\tau}$. Assume that for some α,

$$\partial^\alpha_\tau J(x,\tau)\neq 0 \tag{2.3}$$

for every $\tau\in\overline{B}$. We define the operator T_Φ by

$$T_\Phi(f)(x)=\int_{\overline{B}}f(\Phi(x,\tau))\psi(x,\tau)\prod_{\nu=1}^{2N}(F_0K_0)(\tau_i)\mathrm{d}\tau_1\cdots\mathrm{d}\tau_{2N}$$

and its kernel is given by

$$K_\Phi(x,y)=\int_{\overline{B}}\delta(y-\Phi(x,\tau))\psi(x,\tau)\prod_{\nu=1}^{2N}(F_0K_0)(\tau_i)\mathrm{d}\tau_1\cdots\mathrm{d}\tau_{2N}, \tag{2.4}$$

where δ is the Dirac measure at the origin.

We shall interpret the smoothness of K_Φ by L^1-Lipschitz norm, which is defined by

$$\|f\|_{\Lambda^\delta}=\|f\|_{L^1}+\sup_{z\neq0}|z|^{-\delta}\int|f(y-z)-f(y)|\mathrm{d}y.$$

Proposition 2.1 *Suppose that the mapping Φ_x is as described above and satisfies* (2.3). *Then there exists a constant $\delta>0$ such that the following results hold:*

(i) *If $K_0\in L^q(A_0), 1<q<\infty$, we have*

$$\|K_\Phi(x,\cdot)\|_{\Lambda^\delta}\le C\|K_0\|^{2N}_{L^q(A_0)} \tag{2.5}$$

uniformly in x;

(ii) *If K_0 is replaced by K_0^m, we have*

$$\|K_\Phi(x,\cdot)\|_{\Lambda^\delta}\le C\|K_0^m\|^{2N}_{L^\infty(A_0)} \tag{2.6}$$

uniformly in x.

Proof Since $J(x,\tau)$ satisfies (2.3), by [2], we have

$$\int_{\overline{B}} |J(x,\tau)|^{-\sigma}\mathrm{d}\tau \le A < \infty \tag{2.7}$$

for any $\sigma < \frac{1}{k}$, where $k = |\alpha|$. It follows that $Z = \{\tau \in \overline{B} : J(\tau) = 0\}$ has Lebesgue measure zero.

We are able to cover $\overline{B}/Z$ by a collection of ball $\{B_j(\tau_j, r_j)\}$ with $r_j = c|J(x,\tau_j)|$. If the constant c is small enough, there will exist two positive constants A_1, A_2 such that

(i) $A_1 \le \left|\frac{J(x,\tau)}{J(x,\tau_j)}\right| \le A_2$, whenever $\tau \in B_j(\tau_j, 2r_j)$;

(ii) $\{B_j(\tau_j, 2r_j)\}_j$ have the bounded intersection property.

Let $\{\eta_j\}$ be a smooth partition of unity on $\overline{B}/Z$ subordinated to the covering $\{B_j(\tau_j, r_j)\}_j$ and satisfying $|\nabla\eta_j| \le cr_j^{-1}$. We write

$$K_\Phi(x,y) = \sum_j \int \delta(y - \Phi(x,\tau))\eta_j(\tau)\psi(x,\tau)\prod_{\nu=1}^{2N}(F_0K_0)(\tau_i)\mathrm{d}\tau_1\cdots\mathrm{d}\tau_{2N} = \sum_j K_\Phi^{(j)}(x,y).$$

Then

$$\begin{aligned}\int |K_\Phi^{(j)}(x,y)|\mathrm{d}y &\le C\int_{B_j(\tau_j,2r_j)}\left|\prod_{\nu=1}^{2N}(F_0K_0)(\tau_\nu)\right|\mathrm{d}\tau_1\cdots\mathrm{d}\tau_{2N} \\ &\le C\int_{B_j(\tau_j,2r_j)}\prod_{\nu=1}^{2N}|K_0(\tau_\nu)|\mathrm{d}\tau_1\cdots\mathrm{d}\tau_{2N},\end{aligned} \tag{2.8}$$

$$\int |K_\Phi(x,y)|\mathrm{d}y \le C\int_{\overline{B}}\left|\prod_{\nu=1}^{2N}(F_0K_0)(\tau_i)\right|\mathrm{d}\tau_1\cdots\mathrm{d}\tau_{2N} \le C\|K_0\|_{L^q(A_0)}^{2N}. \tag{2.9}$$

Without loss of generality, we can assume that $J(x,\tau)$ is determined by the first n rows in the matrix $\frac{\partial\Phi}{\partial\tau}$. For $\tau \in \mathbb{R}^{2Nk}$, we write $\tau = (\tau', \tau'') \in \mathbb{R}^n \times \mathbb{R}^{2Nk-n}$. If $B_j(\tau_j', r_j) \subset \mathbb{R}^n, B_j(\tau_j'', r_j) \subset \mathbb{R}^{2Nk-n}$, we then have

$$B_j(\tau_j, r_j) \subset B_j(\tau_j', r_j) \times B_j(\tau_j'', r_j) \subseteq B_j(\tau_j, 2r_j).$$

For fixed $\tau'' \in B_j(\tau_j'', r_j)$, $\Phi_{\tau''}(0,\tau') = \Phi(x,\tau)|_{B_j(\tau',r_j)\times\{\tau''\}}$ is one to one.

By the chain rule, we have

$$\nabla_{\tau'}\delta(y - \Phi_{\tau''}(\tau')) = -\nabla_y\delta(y - \Phi_{\tau''}(\tau'))D\Phi_{\tau''}(\tau').$$

Therefore

$$\nabla_y\delta(y - \Phi_{\tau''}(\tau')) = -\nabla_{\tau'}\delta(y - \Phi_{\tau''}(\tau'))D^{-1}\Phi_{\tau''}(\tau').$$

For $\tau' = (\rho_1, \cdots, \rho_n) \in \mathbb{R}^n$, we get

$$\partial_{y_i}\delta(y - \Phi_{\tau''}(\tau')) = -\sum_{q=1}^n \frac{g(x,\tau)}{J(x,\tau)}\partial\rho_q\delta(y - \Phi_{\tau''}(\tau')),$$

where $g(x,\tau)$ is determined by the minors of order $(n-1)$ of the matrix $D\Phi_{\tau''}(\tau')$. For $\tau \in B_j(\tau_j', r_j)$, we find that

$$\Big|\partial_{\tau_q'}\Big(\frac{g(x,\tau)}{J(x,\tau)}\Big)\Big| \le \frac{C}{r_j^2}.$$

Moreover, if ρ_q is one of the components of τ_{i_0}, we use the integration by parts to get

$$\Big|\partial\rho_q\Big(\prod_{\nu=1}^{2N}(F_0K_0)\Big)(\tau_\nu)\Big| \le C\Big(\int_1^2(|\varphi'(s)|+|\varphi(s)|)s^{-k}\Big|k\Big(\frac{\tau_{i_0}}{s}\Big)\Big|\mathrm{d}s\Big)\prod_{\nu\neq i_0}(F_0K_0)(\tau_\nu).$$

We integrate by parts in the τ_ν variable. Then

$$\int|\nabla_yK_\Phi^{(j)}(x,y)|\mathrm{d}y \le Cr_j^{-2}\int_{B_j(\tau_j,2r_j)}\prod_{\nu=1}^{2N}|K_0(\tau_\nu)|\mathrm{d}\tau_1\cdots\mathrm{d}\tau_{2N}. \tag{2.10}$$

From (2.8) and (2.10), it follows that if $0<\delta<1$,

$$\begin{aligned}
&\sup_{|z|\neq 0}|z|^{-\delta}\int|K_\Phi^{(j)}(y-z)-K_\Phi^{(j)}(y)|\mathrm{d}y\\
&\le C\sup_{|z|\neq 0}(r_j^{-2}|z|^{1-\delta}+|z|^{-\delta})\int_{B_j(\tau_j,2r_j)}\prod_{\nu=1}^{2N}|K_0(\tau_\nu)|\mathrm{d}\tau_1\cdots\mathrm{d}\tau_{2N}\\
&\le Cr_j^{-2\delta}\int_{B_j(\tau_j,2r_j)}\prod_{\nu=1}^{2N}|K_0(\tau_\nu)|\mathrm{d}\tau_1\cdots\mathrm{d}\tau_{2N}.
\end{aligned}$$

By the property of $\{B_j(\tau_j,2r_j)\}$ and (2.5), as long as δ is chosen such that $\delta<\frac{q}{2k(q-1)}$, we have

$$\begin{aligned}
&\sup_{|z|\neq 0}|z|^{-\delta}\int|K_\Phi(x,y-z)-K_\Phi(x,y)|\mathrm{d}y\\
&\le C\sum_j r_j^{-2\delta}\int_{B_j(\tau_j,2r_j)}\prod_{\nu=1}^{2N}|K_0(\tau_\nu)|\mathrm{d}\tau_1\cdots\mathrm{d}\tau_{2N}\\
&\le C\int_{\overline{B}}|J(x,\tau)|^{-2\delta}\prod_{\nu=1}^{2N}|K_0(\tau_\nu)|\mathrm{d}\tau_1\cdots\mathrm{d}\tau_{2N}\\
&\le C\Big(\int_{\overline{B}}|J(x,\tau)|^{-2\delta p}\mathrm{d}\tau\Big)^{\frac{1}{p}}\cdot\|K_0\|_{L^q(A_0)}^{2N}\\
&\le C\|K_0\|_{L^q(A_0)}^{2N}.
\end{aligned}$$

Therefore we get (2.5).

If we replace K_0 by K_0^m, the analogues of inequalities (2.8) and (2.10) are given, respectively, by

$$\int|K_\Phi^{(j)}(x,y)|\mathrm{d}y \le Cr_j^{2Nk}\|K_0^m\|_{L^\infty(A_0)}$$

and

$$\int|\nabla_yK_\Phi^{(j)}(x,y)|\mathrm{d}y \le Cr_j^{2Nk-2}\|K_0^m\|_{L^\infty(A_0)}.$$

These imply (2.6).

We continue to prove Theorem 1.1. Since $x \mapsto r_t(x)$ is a diffeomorphism from a neighborhood of the support of ψ into $\mathbb{R}^n$ for all such t and $1 \le p \le \infty$, by applying the Minkowski integral inequality, we have

$$\|T_{j,m}f\|_p \le C\|F_jK_0^m\|_1 \cdot \|f\|_p \le C\|K_0^m\|_1 \cdot \|f\|_p. \tag{2.11}$$

The same result holds with the adjoint operator $T_{j,m}^*$ substituted for $T_{j,m}$. By the almost orthogonality argument in [6], we will show that

$$\|T_{j,m}^*T_{i,m}\|_2 + \|T_{j,m}T_{i,m}^*\|_2 \le C2^{2^{m+1}} \cdot 2^{-\varepsilon|i-j|} \tag{2.12}$$

for some $\varepsilon > 0$. We shall prove

$$\|T_{j,m}^*T_{i,m}\|_2 \le C2^{2^{m+1}} \cdot 2^{-\varepsilon|i-j|}. \tag{2.13}$$

The boundedness for the second term can be obtained by the same argument. Without loss of generality, we may assume that $i \ge j$ in (2.13).

We need the elementary fact (see [1])

$$\|T_{j,m}^*T_{i,m}\|_2 \le \|T_{i,m}\|_2^{1-2^{-l}} \cdot \|(T_{j,m}T_{j,m}^*)^{2^{l-1}} \cdot T_{i,m}\|_2^{2^{-l}}.$$

We can take $N = 2^{l-1} \ge n$. It suffices to show that

$$\|(T_{j,m}T_{j,m}^*)^NT_{i,m}\|_2 \le C2^{(2N+1)2^{m+1}}2^{-\varepsilon|i-j|}. \tag{2.14}$$

By the formula (2.2), the kernel $K(x,y)$ associated to the operator $(T_{j,m}T_{j,m}^*)^NT_i$ is

$$K(x,y) = \int \delta(y - r_t(\Gamma_N(x,\tau)))\psi(x,\tau)\prod_{\nu=1}^{2N}(F_jK_0^m)(\tau_i)\cdot(F_iK_0^m)(t)\mathrm{d}\tau\mathrm{d}t.$$

Since, as functions of x, $r_t(\Gamma_N(x,\tau))$ have Jacobian determinants near 1, we have

$$\sup_y\int|K(x,y)|\mathrm{d}x \le C\int\prod_{\nu=1}^{2N}(F_jK_0^m)(\tau_i)\cdot(F_iK_0^m)(t)\mathrm{d}\tau\mathrm{d}t \le C2^{(2N+1)2^{m+1}}. \tag{2.15}$$

On the other hand, since each mapping $\tau \mapsto \Gamma^{(j)}(x,\tau) = \delta_{2^j}^x(\Gamma_N(x,2^{-j}\tau))$ satisfies the hypotheses of Proposition 2.1, we may rewrite

$$K(x,y) = \int \delta(y - r_t(\delta_{2^{-j}}^x(z')))K_j(x,z')(F_iK_0^m)(t)\mathrm{d}z'\mathrm{d}t,$$

where $z' \mapsto K_j(x,z')$ belongs to the L^1-Lipschtz space. By the change of variables $z = \delta_{2^{-j}}^x(z')$, the formula of $F(x,y)$ becomes

$$K(x,y) = 2^{jQ}\int \delta(y - r_t(z))K_j(x,\delta_{2^j}^x(z))J(x,z)(F_iK_0^m)(t)\mathrm{d}z\mathrm{d}t,$$

where $2^{jQ}J(x,z)$ is the associated Jacobian determinant.

Substituting $u = r_t(z)$, and noting that $\det\big(\frac{\partial z}{\partial u}\big) = 1 + O(t)$, we have

$$K(x,y) = 2^{jQ} \int \delta(y-u) K_j(x, \delta^x_{2^j}(r_t^{-1}(u))) J(x, r_t^{-1}(u))(1+O(t))(F_i K_0^m)(t) \mathrm{d}u \mathrm{d}t.$$

By Propositions 2.1 and 3.1 of [2], we get

$$\begin{aligned}
&\sup_x \int |K(x,y)| \mathrm{d}y \\
&\le C \sup_x 2^{jQ} \int |J(x, r_t^{-1}(u))[K_j(x, \delta^x_{2^j}(r_t^{-1}(u))) - K_j(x, \delta^x_{2^j}(u))](F_i K_0^m)(t)| \mathrm{d}u \mathrm{d}t \\
&\quad + \sup_x 2^{jQ} \int |J(x, r_t^{-1}(u)) K_j(x, \delta^x_{2^j}(r_t^{-1}(u))) \cdot t (F_i K_0^m)(t)| \mathrm{d}u \mathrm{d}t \\
&\le C 2^{(2N+1)2^{m+1}} 2^{-\delta|i-j|} + C 2^{-i} 2^{(2N+1)2^{m+1}} \\
&\le C 2^{(2N+1)2^{m+1}} 2^{-\delta|i-j|}.
\end{aligned}$$

Therefore,

$$\sup_x \int |K(x,y)| \mathrm{d}y \le C 2^{(2N+1)2^{m+1}} 2^{-\delta|i-j|}. \tag{2.16}$$

By making use of Schur's Lemma (see [6]), we obtain (2.14), and then (2.13) holds.

Setting $U_{i,m} = \sum\limits_{j=i2^m}^{(i+1)2^m-1} T_{j,m}$, then for the fixed m, by (2.12), we have the estimates

$$\|U_{i,m}\|_2 \le 2^m \|T_{j,m}\|_2 \le C 2^m \|K_0^m\|_1, \tag{2.17}$$

$$\|U^*_{i,m} U_{j,m}\|_2 + \|U_{i,m} U^*_{j,m}\|_2 \le C 2^{2m} 2^{2^{m+1}} 2^{-\varepsilon 2^m |i-j|}. \tag{2.18}$$

If $|i-j| \ge k_0$ where $k_0 > \frac{2}{\varepsilon}$, we have

$$\|U^*_{i,m} U_{j,m}\|_2 + \|U_{i,m} U^*_{j,m}\|_2 \le C 2^{2m} 2^{-\delta 2^m |i-j|} \tag{2.19}$$

for some constant $\delta > 0$. Therefore, by the Cotlar-Stein almost orthogonality and (2.1), we have for some $\delta_1 > 0$,

$$\|T\|_2 = \Big\| \sum_{m\ge 0} \sum_i U_{i,m} \Big\|_2 \le C \sum_{m \ge 0} (2^m \|K_0^s\|_1 + 2^m 2^{-\delta_1 2^m}) < \infty,$$

which concludes the proof of Theorem 1.1.

Proof of Theorem 1.2 Let $\Phi \in C_0^\infty(\mathbb{R}^n)$ be an even nonnegative function supported in $|x| < a$, and equal to 1 near the origin. It satisfies $\int_{\mathbb{R}^n} \Phi(x) \mathrm{d}x = 1$. Set $\Phi_j(u) = 2^{iQ} \Phi(\delta_{2^j}(u))$, and fix a cutoff function $x \in C_0^\infty(\mathbb{R}^n)$ which is equal to 1 near x_0. We define

$$S_j(f)(x) = \chi_0(x) \int \Phi_j(Q_x(y)) \chi_0(y) f(y) \mathrm{d}y,$$

where $\chi_0(x) = \chi(x) J(x,x)^{\frac{1}{2}}$ with $J(x,y) = \big|\det\big(\frac{\partial Q_x(y)}{\partial y}\big)\big|$.

Finally define $R_j = S_{j+1} - S_j$, and let $R_j(x,y)$ be its kernel. From the result of [2], we have

$$\int |R_j(x,y)|\mathrm{d}x < C, \tag{2.20}$$

$$\int |R_{j+l}(x,y_1) - R_{j+l}(x,y_2)|\mathrm{d}x \le C2^{(j+l)/m} d(y_1,y_2)^{\frac{1}{m}} \tag{2.21}$$

for all y, y_1, y_2.

Since $I = S_j + \sum\limits_{l=0}^{\infty} R_{j+l}$ (see [2]), we may write

$$T = \sum_{j=0}^{\infty} T_j = \sum_{j=0}^{\infty} T_j S_j + \sum_{l=0}^{\infty} U_l,$$

where $U_l = \sum\limits_{j=0}^{\infty} T_j R_{j+l}$. If $K_0 \in L^q(A_0),\ 1 < q < \infty$, we shall show that T is bounded on each L^p for $1 < p < \infty$ as well.

By using Proposition 2.1 and adaptations of the arguments in [2], we obtain

$$\|U_l\|_2 \le C2^{-l\varepsilon}\|K_0\|_q \tag{2.22}$$

for some $\varepsilon > 0$, and

$$\Big\|\sum_{j=0}^{\infty} T_j S_j\Big\|_2 \le C\|K_0\|_q. \tag{2.23}$$

Let $L_i(x,y)$ and $P(x,y)$ be the kernels of U_i and $\sum\limits_{j=0}^{\infty} T_j S_j$ respectively. Next, define the ball $B(x,r) = \{y \in R^n : d(x,y) < r\}$. To apply the Calderón-Zygmund theory, we need the following estimates.

Lemma 2.1 *There exists a constant $\delta > 1$ such that if $y_2 \in B(y_1, r)$, then*

$$\int_{x\notin B(y_1,\delta r)} |U_l(x,y_1) - U_l(x,y_2)|\mathrm{d}x \le Cl\|K_0\|_q \tag{2.24}$$

and

$$\int_{x\notin B(y_1,\delta r)} |P(x,y_1) - P(x,y_2)|\mathrm{d}x \le C\|K_0\|_q \tag{2.25}$$

hold for all $r > 0$.

Proof The proofs of (2.24) and (2.25) are similar, so we only prove the inequality (2.24) here. Let $U_{j,l}$ be the kernel of $T_j R_{j+l}$. This can be written as (see [2])

$$U_{j,l}(x,y) = \psi(x)\int R_{j+l}(r_t(x),y)(F_j K_0)(t)\mathrm{d}t.$$

The left-hand side of (2.24) is at most

$$\sum_{j=0}^{\infty}\int_{x\notin B(y_1,\delta r)} |U_{j,l}(x,y_1) - U_{j,l}(x,y_2)|\mathrm{d}x,$$

and it equals

$$\sum_{j=0}^{\infty} \int_{x \notin B(y_1,\delta r)} \Big|\psi(x) \int (R_{j+l}(r_t(x), y_1) - R_{j+l}(r_t(x), y_2))(F_j K_0)(t) \mathrm{d}t\Big| \mathrm{d}x. \tag{2.26}$$

Let j_0 be an integer such that $2^{-j_0-1} \leq r \leq 2^{-j_0}$. If $j > j_0$ we conclude that the expression $R_{j+l}(x,y)$ appearing in (2.26) are zero provided that $x \notin B(y_1, \delta r)$ and δ is chosen appropriately large. In fact, under these conditions, from the observation $d(r_t(x), x) = O(t)$ (see [2]), we have

$$d(r_t(x), y_1) \geq C(d(x, y_1) - d(r_t(x), x)) \geq C \cdot 2^{-j_0}.$$

Similarly, we get $d(r_t(x), y_2) \geq C \cdot 2^{-j_0}$. Therefore, inequalities (2.19) and (2.20) give that (2.26) is at most a constant times

$$\sum_{j=0}^{j_0-l-1} 2^{\frac{j+l}{m}} 2^{-\frac{j_0}{m}} \int |(F_j K_0)(t)| \mathrm{d}t + \sum_{j=j_0-l}^{j_0} \int |R_{j+l}(x,y)| \mathrm{d}x \int |(F_j K_0)(t)| \mathrm{d}t \leq Cl \|K_0\|_q.$$

Hence (2.24) is established and Lemma 2.1 is proved.

By the generalized Calderón-Zygmund theorem (see [6, p. 19]), Lemma 2.2 and (2.22) imply that $\|U_l\|_p \leq Cl$ for $1 < p < 2$. Then real interpolation gives $\|U_l\|_p \leq C 2^{-\varepsilon' l}$ for some $\varepsilon' > 0$. We conclude $\big\| \sum\limits_{l=0}^{\infty} U_l \big\|_p \leq C$. Take into account the inequalities (2.23) and (2.25), the boundedness of operator $\sum\limits_{j=0}^{\infty} T_j S_j$ on L^p, for $1 < p < 2$, is deduced from the Calderón-Zygmund theorem directly. Therefore T is bounded on L^p for $1 < p < 2$. The result for $2 < p < \infty$ follows by duality. We finish the proof of Theorem 1.2.

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