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Fractional Moments of Automorphic L-Functions on GL(m)*

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Abstract Let π be an irreducible unitary cuspidal representation of $\mathrm{GL}_m(\mathbb{A}_{\mathbb{Q}})$, $m \geq 2$. Assume that π is self-contragredient. The author gets upper and lower bounds of the same order for fractional moments of automorphic L-function $L(s, \pi)$ on the critical line under Generalized Ramanujan Conjecture; the upper bound being conditionally subject to the truth of Generalized Riemann Hypothesis.

Keywords Moments, Automorphic L-functions, GRC
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1 Introduction

Understanding moments of families of L-functions on the critical line has long been an important subject in number theory. According to conjectures of Langlands, the general L-functions should be expressed as products of primitive L-functions $L(s, \pi)$ attached to cuspidal automorphic representations of $\mathrm{GL}_m(\mathbb{A}_{\mathbb{Q}})$. For $m = 1$, these are the Riemann zeta function $\zeta(s)$ and Dirichlet L-functions $L(s, \chi)$ with χ primitive. It is conjectured that

$$\int_0^T \Big|\zeta\Big(\frac{1}{2} + \mathrm{i}t\Big)\Big|^{2k} \mathrm{d}t \sim C_k T (\log T)^{k^2}, \tag{1.1}$$

where $k \geq 0$ and $C_k > 0$ is a constant. For $k = 1$ and $k = 2$, (1.1) was proved by Hardy and Littlewood [1] in 1918 and Ingham [2] in 1926, respectively. However, no unconditional asymptotic formula has yet been proved for any other k.

It is of interest therefore to ask for the weaker result

$$T(\log T)^{k^2} \ll \int_0^T \Big|\zeta\Big(\frac{1}{2} + \mathrm{i}t\Big)\Big|^{2k} \mathrm{d}t \ll T\,(\log T)^{k^2}. \tag{1.2}$$

In this direction, Ramacharadra [3–4] and Heath-Brown [5–6] proved that the lower bound in (1.2) holds for all rational $k \geq 0$, and the upper bound holds for $k = \frac{1}{v}$, where v is a positive integer. Moreover, under Riemann Hypothesis (RH in brief), they showed that the lower bound holds for all real $k \geq 0$, and the upper bound holds for $0 \leq k \leq 2$. Recently, Soundararajan [7] showed that, under RH,

$$\int_0^T \Big|\zeta\Big(\frac{1}{2} + \mathrm{i}t\Big)\Big|^{2k} \mathrm{d}t \ll T\,(\log T)^{k^2+\varepsilon}$$

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for any $\varepsilon > 0$ and any $k > 0$. For Dirichlet L-functions $L(s,\chi)$, we refer to [8]. For $m = 2$, the asymptotic formula for the second moment of automorphic L-functions $L(s, f)$ attached to a holomorphic cusp form f for $\mathrm{SL}_2(\mathbb{Z})$ was obtained by Good [9]. Recently, using Heath-Brown's method [5], Laurinčikas and Steuding [10] showed that

$$\int_0^T \left|L\Big(\frac{1}{2}+\mathrm{i}t, f\Big)\right|^{k^2} \mathrm{d}t \gg T\,(\log T)^{k^2} \quad \text{for } k = \frac{1}{v} \text{ with } v \in \mathbb{N},$$
$$\int_0^T \left|L\Big(\frac{1}{2}+\mathrm{i}t, f\Big)\right|^{k^2} \mathrm{d}t \ll T\,(\log T)^{k^2} \quad \text{for } k = \frac{1}{v} \text{ with } 2\,|\,v \text{ and } v \in \mathbb{N},$$

where the upper bound was proved under Generalized Riemann Hypothesis (GRH in brief). Lü and Sun [11] further improved the lower bound by extending the range of k to $k = \frac{u}{v} \le \frac{1}{2}$ with $u, v \in \mathbb{N}$.

In this paper, we are concerned with the fractional moments of automorphic L-functions $L(s,\pi)$, where $\pi = \otimes\pi_p$ is an irreducible unitary cuspidal automorphic representation of $\mathrm{GL}_m(\mathbb{A}_{\mathbb{Q}})$, where, throughout the paper, $m \ge 2$. To this end, we recall some background on automorphic L-functions.

Let $s = \sigma + \mathrm{i}t$. For $\sigma > 1$, $L(s,\pi)$ is defined by the products of local factors,

$$L(s,\pi) = \prod_{p<\infty} L_p(s,\pi_p), \quad L_p(s,\pi_p) = \prod_{j=1}^{m} \Big(1 - \frac{\alpha_\pi(p,j)}{p^s}\Big)^{-1}.$$

The complete L-function attached to π is defined by

$$\Phi(s,\pi) = L_\infty(s,\pi_\infty)L(s,\pi), \tag{1.3}$$

where $L_\infty(s,\pi_\infty) = \prod\limits_{j=1}^{m} \Gamma_{\mathbb{R}}(s+\mu_\pi(j))$. Here $\Gamma_{\mathbb{R}}(s) = \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)$, and $\alpha_\pi(p,j)$ and $\mu_\pi(j)$ ($j = 1,\cdots,m$) are complex numbers associated with π_p and π_∞, respectively. It is well-known that all the non-trivial zeros of $L(s,\pi)$ are in the critical strip $0 < \sigma < 1$, while GRH predicts that they lie on the vertical line $\sigma = \frac{1}{2}$. For $m \ge 2$, $\Phi(s,\pi)$ is entire and satisfies a functional equation

$$\Phi(s,\pi) = \varepsilon(s,\pi)\Phi(1-s,\widetilde{\pi}), \tag{1.4}$$

with $\widetilde{\pi}$ the representation contragredient to π and $\varepsilon(s,\pi) = \varepsilon_\pi N_\pi^{\frac{1}{2}-s}$, where ε_π is the root number and $N_\pi > 1$ is the conductor. For any $p \le \infty$, $\widetilde{\pi}_p$ is equivalent to the complex conjugate $\overline{\pi}_p$, and thus

$$\{\alpha_{\widetilde{\pi}}(j,p)\} = \{\overline{\alpha_\pi(j',p)}\}, \quad \{\mu_{\widetilde{\pi}}(j)\} = \{\overline{\mu_\pi(j')}\}. \tag{1.5}$$

Denote

$$a_\pi(p^\ell) = \sum_{1\le j\le m} \alpha_\pi(p,j)^\ell. \tag{1.6}$$

Then for $\sigma > 1$, we have

$$\frac{\mathrm{d}}{\mathrm{d}s}\log L(s,\pi) = -\sum_{n=1}^{\infty} \frac{\Lambda(n)a_\pi(n)}{n^s},$$

where $\Lambda(n) = \log p$ is $n = p^\ell$ and if 0 otherwise.

In this paper, we assume the Generalized Ramanujan Conjecture (GRC in brief) which states that for any unramified p,

$$|\alpha_\pi(p, j)| = 1 \quad \text{and} \quad \operatorname{Re}\mu_\pi(j) = 0, \quad j = 1, \cdots, m. \tag{1.7}$$

An important consequence of GRC is Selberg's orthogonality conjecture proposed by Selberg [12] in 1989, which states as follows. Let π and π' be automorphic irreducible cuspidal representations of the groups $\mathrm{GL}_m(\mathbb{A}_\mathbb{Q})$ and $\mathrm{GL}_{m'}(\mathbb{A}_\mathbb{Q})$, respectively. Then

$$\sum_{p\le x} \frac{a_\pi(p)\overline{a}_{\pi'}(p)}{p} = \begin{cases} \log\log x + O(1), & \text{if } \pi \cong \pi', \\ O(1), & \text{if } \pi \not\cong \pi'. \end{cases} \tag{1.8}$$

(1.8) was proved by Rudnick and Sarnak [13] under the hypothesis

$$\sum_p \frac{|a_\pi(p^\ell)|^2(\log p)^2}{p^\ell} < \infty,$$

which is an easy consequence of GRC. Under GRC, Liu and Ye [14] proved (1.8) in a more precise form. Precisely, they proved the following result.

Proposition 1.1 *Let π and π' be automorphic irreducible cuspidal representations of the groups $\mathrm{GL}_m(\mathbb{A}_\mathbb{Q})$ and $\mathrm{GL}_{m'}(\mathbb{A}_\mathbb{Q})$, respectively, such that at least one of π and π' is self contragredient: $\pi \cong \widetilde{\pi}$ or $\pi' \cong \widetilde{\pi}'$. Assume GRC for both π and π'. Then*

$$\sum_{p\le x} \frac{a_\pi(p)\overline{a}_{\pi'}(p)}{p} = \begin{cases} \log\log x + C_1 + O(\exp\{-c\sqrt{\log x}\,\}), & \pi' \cong \pi, \\ C_2 + \mathrm{Ei}(\mathrm{i}\tau_0 \log x) + O(\exp\{-c\sqrt{\log x}\,\}), & \pi' \cong \pi \otimes \alpha^{\mathrm{i}\tau_0} \textit{ for some } \tau_0 \in \mathbb{R}^\times, \\ C_3 + O(\exp\{-c\sqrt{\log x}\,\}), & \pi' \not\cong \pi \otimes \alpha^{\mathrm{i}\tau} \textit{ for any } \tau \in \mathbb{R}, \end{cases}$$

where Ei *is the exponential integral, and C_1, C_2, C_3 are positive constants.*

For $k \ge 0$ and $\sigma \ge \frac{1}{2}$, we define $I_k(\sigma, T, \pi) = \int_0^T |L(\sigma + \mathrm{i}t, \pi)|^{2k}\,\mathrm{d}t$. For brevity, we denote

$$I_k\Big(\frac{1}{2}, T\Big) := I_k\Big(\frac{1}{2}, T, \pi\Big) = \int_0^T \Big|L\Big(\frac{1}{2} + \mathrm{i}t, \pi\Big)\Big|^{2k} \mathrm{d}t.$$

Theorem 1.1 *Assume GRC. Let π be an automorphic irreducible cuspidal representation of $\mathrm{GL}_m(\mathbb{A}_\mathbb{Q})$ such that π is self-contragredient. Let $k \in \mathbb{Q}$, $k \ge 0$. Then as $T \to \infty$, we have $I_k(\frac{1}{2}, T) \gg T(\log T)^{k^2}$. Under GRH, the range of k can be extended to $k \in \mathbb{R}$, $k \ge 0$.*

Theorem 1.2 *Assume GRC and GRH. Let π be as in Theorem* 1.1. *Let $0 \le k \le \frac{2}{m} - \varepsilon$ for any $\varepsilon > 0$. Then as $T \to \infty$, we have $I_k\big(\frac{1}{2}, T\big) \ll T(\log T)^{k^2}$.*

Theorems 1.1 and 1.2 are proved by Heath-Brown [5]. Since GRC was proved by Deligne [15] for π being representations corresponding to holomorphic cusp forms, Theorems 1.1 and 1.2 hold without assuming GRC for $m = 2$. Thus Theorems 1.1 and 1.2 improve Laurinčikas and Steuding's result and Lü and Sun's result. We also note that Theorems 1.1 and 1.2 generalize the recent results of Fomenko [16].

In the sequel, we will use c_1, c_2, $\cdots$ to denote positive constants and the implied constants in "$\ll$" and "$\gg$" depending on m, k and π.

2 Proof of Theorems 1.1 and 1.2

Let $w(t,T)=\int_T^{2T}\mathrm{e}^{-(t-\tau)^2}\,\mathrm{d}\tau$. Then $w(t,T)$ has the following properties:

$$\begin{cases} w(t,T)\ll 1 & \text{for all } t,\\ w(t,T)\gg 1 & \text{for } \dfrac{4T}{3}\le t\le \dfrac{5T}{3}. \end{cases} \tag{2.1}$$

Define

$$J(\sigma,T):=J_k(\sigma,T,\pi)=\int_{-\infty}^{\infty}|L(\sigma+\mathrm{i}t,\pi)|^{2k}\,w(t,T)\mathrm{d}t.$$

By (2.1), we have $J(\frac12,T)\ll I_k(\frac12,3T)$. Thus Theorem 1.1 follows if under GRC,

$$J\Big(\frac12,T\Big)\gg T\,(\log T)^{k^2}\quad \text{for } k\in\mathbb{Q},\ k\ge 0, \tag{2.2}$$

and under GRH, it holds for $k\ge 0$.

On the other hand, if we can establish the upper bound

$$J\Big(\frac12,T\Big)\ll T(\log T)^{k^2}\quad \text{for } 0\le k\le \tfrac{2}{m}-\varepsilon \text{ for any } \varepsilon>0, \tag{2.3}$$

under GRC and GRH, then by (2.1), we also have

$$I_k\Big(\frac12,\frac{5T}{3}\Big)-I_k\Big(\frac12,\frac{4T}{3}\Big)\ll J\Big(\frac12,T\Big)\ll T(\log T)^{k^2}.$$

Replacing T by $\big(\frac45\big)^jT$ and summing up over $j=1,2,\cdots$, we obtain Theorem 1.2.

The following three sections will be devoted to the proof of (2.2) and (2.3).

3 The Coefficients of $L(s,\pi)^k$

Let $s=\sigma+\mathrm{i}t$. For $\sigma>1$, we define a branch of the multi-valued function $L(s,\pi)^k$ by

$$L(s,\pi)^k=\exp\{k\log L(s,\pi)\}=\exp\Big\{k\sum_p\sum_{j=1}^m\sum_{\ell=1}^{\infty}\frac{\alpha_\pi(p,j)^\ell}{\ell p^{\ell s}}\Big\}=\prod_p\prod_{j=1}^m\Big(1-\frac{\alpha_\pi(p,j)}{p^s}\Big)^{-k}.$$

For $|z|<1$, we have $(1-z)^{-k}=\sum\limits_{\ell=0}^{\infty}\frac{\Gamma(k+\ell)}{\Gamma(k)\ell!}z^\ell$. For positive integers ℓ, define

$$d_k(p^\ell)=\frac{\Gamma(k+\ell)}{\Gamma(k)\ell!}=\frac{k(k+1)\cdots(k+\ell-1)}{\ell!}.$$

Then for $\sigma>1$, we have

$$L(s,\pi)^k=\prod_p\sum_{\ell=0}^{\infty}\frac{h_k(p^\ell)}{p^{\ell s}}=\sum_{n=1}^{\infty}\frac{h_k(n)}{n^s},$$

where $h_k(n)$ is the multiplicative function given by

$$h_k(p^\ell)=\sum_{\substack{\ell_1+\cdots+\ell_m=\ell\\ \ell_j\ge 0}}d_k(p^{\ell_1})\alpha_\pi(p,1)^{\ell_1}\cdots d_k(p^{\ell_m})\alpha_\pi(p,m)^{\ell_m}\quad \text{for } \ell\in\mathbb{N}. \tag{3.1}$$

Lemma 3.1 (see [5]) *The multiplicative function $d_k(n)$ satisfies the following properties:*

(1) *For $k \geq 0$ and $n \geq 1$, we have $d_k(n) \geq 0$;*

(2) *For fixed $k \geq 0$ and $\varepsilon > 0$, we have $d_k(n) \ll n^{\varepsilon}$;*

(3) *If ℓ is an integer, then $d_{k\ell}(n) = \sum\limits_{n_1 \cdots n_\ell = n} d_k(n_1) \cdots d_k(n_\ell)$.*

Lemma 3.2 (see [17, 18]) *Let $f(n) \geq 0$ be a multiplicative function satisfying*

(i) $\sum\limits_{p \leq x} \frac{\log p}{p} f(p) \sim \tau \log x$ *for some $\tau > 0$;*

(ii) $f(p) \ll 1$;

(iii) $\sum\limits_{p, \ell \geq 2} \frac{f(p^\ell)}{p^\ell} < \infty$;

(iv) $\sum\limits_{\substack{p, \ell \geq 2 \\ p^\ell \leq x}} f(p^\ell) \ll \frac{x}{\log x}$.

Then

$$\sum_{n \leq x} f(n) \sim \frac{\mathrm{e}^{-\gamma_0 \tau}}{\Gamma(\tau)} \frac{x}{\log x} \prod_{p \leq x} \Big(1 + \sum_{\ell=1}^{\infty} \frac{f(p^\ell)}{p^\ell}\Big),$$

where γ_0 is Euler's constant.

Lemma 3.3 *Assume GRC. Let π be an automorphic irreducible cuspidal representation of $\mathrm{GL}_m(\mathbb{A}_{\mathbb{Q}})$, such that π is self-contragredient. We have*

$$\sum_{n \leq x} |h_k(n)|^2 \sim \frac{\mathrm{e}^{-\gamma_0 k^2}}{\Gamma(k^2)} \frac{x}{\log x} \prod_{p \leq x} \Big(1 + \sum_{\ell=1}^{\infty} \frac{\big|h_k(p^\ell)\big|^2}{p^\ell}\Big),$$

where γ_0 is Euler's constant.

Proof Note that $d_k(p) = k$. By (3.1) and (1.6), we have $h_k(p) = d_k(p)(\alpha_\pi(p,1) + \cdots + \alpha_\pi(p,m)) = k a_\pi(p)$. By Proposition 1.1,

$$\sum_{p \leq x} \frac{|a_\pi(p)|^2}{p} = \log\log x + C_1 + R(x),$$

where $R(x) \ll \exp\{-c\sqrt{\log x}\}$. Integrating by parts, we get

$$\begin{aligned}
\sum_{p \leq x} \frac{\log p}{p} |h_k(p)|^2 &= k^2 \int_2^x \log u \,\mathrm{d} \sum_{p \leq u} \frac{|a_\pi(p)|^2}{p} \\
&= k^2 \int_2^x \log u \,\mathrm{d} \log\log u + k^2 \int_2^x \log u \,\mathrm{d}R(u) \\
&= k^2 \log x + O(1) \\
&\sim k^2 \log x, \quad \text{as } x \to \infty.
\end{aligned} \tag{3.2}$$

Next, by (1.7), we have

$$|h_k(p)|^2 = k^2 |a_\pi(p)|^2 \leq k^2 m^2 \ll_{k,m} 1. \tag{3.3}$$

Moreover, by (1.7), (3.1) and Lemma 3.1(3), we have for any $\varepsilon > 0$,

$$\big|h_k(p^\ell)\big| \leq \sum_{\substack{\ell_1 + \cdots + \ell_m = \ell \\ \ell_j \geq 0}} d_k(p^{\ell_1}) \cdots d_k(p^{\ell_m}) = d_{km}(p^\ell) \ll p^{\varepsilon \ell}. \tag{3.4}$$

Thus, for $\varepsilon > 0$ sufficiently small,

$$\sum_{p,\ell\geq 2} \frac{\left|h_k(p^\ell)\right|^2}{p^\ell} \ll \sum_{p,\ell\geq 2} \frac{1}{p^{\ell(1-\varepsilon)}} = \sum_{p=2}^{\infty} \frac{1}{p^{1-\varepsilon}(p^{1-\varepsilon}-1)} < \infty. \tag{3.5}$$

Finally,

$$\sum_{\substack{p,\ell\geq 2\\ p^\ell\leq x}} \left|h_k(p^\ell)\right|^2 \ll \sum_{\substack{p,\ell\geq 2\\ p^\ell\leq x}} p^{2\varepsilon\ell} \ll \sum_{p\leq\sqrt{x}} \sum_{\ell\leq\frac{\log x}{\log p}} p^{2\varepsilon\ell} \ll x^{2\varepsilon}\log x \sum_{p\leq\sqrt{x}} 1 \ll x^{\frac{1}{2}+3\varepsilon} \ll \frac{x}{\log x}.$$

Combined with this estimate, Lemma 3.3 follows from (3.2), (3.4), (3.5) and Lemma 3.2.

Lemma 3.4 *Assume GRC. For any fixed $k > 0$, there exists a constant $C > 0$, such that*

$$\Big(\sigma-\frac{1}{2}\Big)^{-k^2} \ll \sum_{n\leq N} \frac{\left|h_k(n)\right|^2}{n^{2\sigma}} \ll \Big(\sigma-\frac{1}{2}\Big)^{-k^2}, \quad \textit{uniformly for } \frac{1}{2}+\frac{C}{\log N} \leq \sigma \leq 1,$$

$$(\log N)^{k^2} \ll \sum_{n\leq N} \frac{\left|h_k(n)\right|^2}{n} \ll (\log N)^{k^2}.$$

Proof By Lemma 3.3, we have the asymptotic formula

$$\sum_{n\leq x} \left|h_k(n)\right|^2 \sim \frac{\mathrm{e}^{-\gamma_0 k^2}}{\Gamma(k^2)} \frac{x}{\log x} \prod_{p\leq x} \Big(1+\sum_{\ell=1}^{\infty} \frac{\left|h_k(p^\ell)\right|^2}{p^\ell}\Big).$$

By (3.3), (3.4) and Proposition 1.1, we have

$$\begin{aligned}
\prod_{p\leq x} \Big(1+\sum_{\ell=1}^{\infty} \frac{\left|h_k(p^\ell)\right|^2}{p^\ell}\Big) &= \exp\Big\{\sum_{p\leq x} \log\Big(1+\frac{\left|h_k(p)\right|^2}{p}+O\Big(\frac{1}{p^{2(1-\varepsilon)}}\Big)\Big)\Big\}\\
&= \exp\Big\{\sum_{p\leq x} \frac{\left|h_k(p)\right|^2}{p}+O(1)\Big\}\\
&= \exp\Big\{k^2 \sum_{p\leq x} \frac{\left|a_\pi(p)\right|^2}{p}+O(1)\Big\}\\
&= \mathrm{e}^{O(1)}\,(\log x)^{k^2}.
\end{aligned}$$

Therefore, there exist positive constants $c_1 < c_2$, such that

$$c_1 x\,(\log x)^{k^2-1} \leq \sum_{n\leq x} \left|h_k(n)\right|^2 \leq c_2 x\,(\log x)^{k^2-1}.$$

By partial summation, the first assertion of the lemma follows. The second assertion follows from the first one since, for $\sigma = \frac{1}{2}+\frac{C}{\log N}$ and $1 \leq n \leq N$, we have $n^{-1} \ll n^{2\sigma} \ll n^{-1}$.

Define $S_N(s) = \sum\limits_{n\leq N} \frac{h_k(n)}{n^s}$ and $H(\sigma,T) = \int_{-\infty}^{\infty} \left|S_N(\sigma+\mathrm{i}t)\right|^2 w(t,T)\,\mathrm{d}t$.

Lemma 3.5 *Assume GRC. Let $N \ll T$ and $\log N \gg \log T$. Then for $\frac{1}{2}+\frac{C}{\log N} \leq \sigma \leq \frac{3}{4}$, we have*

$$T\Big(\sigma-\frac{1}{2}\Big)^{-k^2} \ll H(\sigma,T) \ll T\Big(\sigma-\frac{1}{2}\Big)^{-k^2},$$

$$T\,(\log T)^{k^2} \ll H\Big(\frac{1}{2},T\Big) \ll T\,(\log T)^{k^2}.$$

Proof Note that $w(t,T) \ll \mathrm{e}^{-c_3(t^2+T^2)}$ for $t \le 0$ and $t \ge 3T$. By (3.4), we have $S_N(s) \ll N^{\frac{1}{2}+\varepsilon}$ for $\sigma \ge \frac{1}{2}$ and any $\varepsilon > 0$. Thus, by (2.1) and the Montgomery-Vaughan Theorem (see [19]), we find that

$$H(\sigma,T) \ll \int_0^{3T} |S_N(\sigma+\mathrm{i}t)|^2 \,\mathrm{d}t \ll (T+N)\sum_{n\le N}\frac{|h_k(n)|^2}{n^{2\sigma}}.$$

On the other hand, by (2.1), we have

$$H(\sigma,T) \gg \int_{\frac{4T}{3}}^{\frac{5T}{3}} |S_N(\sigma+\mathrm{i}t)|^2 \,\mathrm{d}t \gg T\sum_{n\le N}\frac{|h_k(n)|^2}{n^{2\sigma}}.$$

Therefore, Lemma 3.5 follows from Lemma 3.4.

4 Applications of Gabriel's Inequality

We need the following Gabriel's inequality.

Lemma 4.1 (see [20] or [5]) *Let $G(s)$ be regular in the strip $\{s\in\mathbb{C}:\alpha<\sigma<\beta\}$ and continuous in the closed strip $\{s\in\mathbb{C}:\alpha\le\sigma\le\beta\}$. Moreover, assume that $G(s)\to 0$ as $t\to\infty$ uniformly in $\{s\in\mathbb{C}:\alpha\le\sigma\le\beta\}$. Then, for $\alpha\le\gamma\le\beta$ and any $\theta>0$,*

$$\int_{-\infty}^{\infty} |G(\gamma+\mathrm{i}t)|^{\theta}\,\mathrm{d}t \le \Big(\int_{-\infty}^{\infty} |G(\alpha+\mathrm{i}t)|^{\theta}\,\mathrm{d}t\Big)^{\frac{\beta-\gamma}{\beta-\alpha}}\Big(\int_{-\infty}^{\infty} |G(\beta+\mathrm{i}t)|^{\theta}\,\mathrm{d}t\Big)^{\frac{\gamma-\alpha}{\beta-\alpha}}.$$

Lemma 4.2 *Let $\frac{1}{2}\le\sigma\le\frac{3}{4}$ and $T\ge 2$. Then for all $k>0$,*

$$J\Big(\frac{1}{2},T\Big) \ll J(\sigma,T)T^{km(\sigma-\frac{1}{2})} + \mathrm{e}^{-c_4T^2}.$$

Proof Applying Lemma 4.1 with $\gamma=\frac{1}{2}$, $\alpha=1-\sigma$, $\beta=\sigma$, $\frac{1}{2}\le\sigma\le\frac{3}{4}$, $\theta=2k$, and

$$G(s) = L(s,\pi)\mathrm{e}^{\frac{(s-\mathrm{i}\tau)^2}{2k}}, \tag{4.1}$$

we obtain

$$\begin{aligned}&\int_{-\infty}^{\infty}\Big|L\Big(\frac{1}{2}+\mathrm{i}t,\pi\Big)\Big|^{2k}\mathrm{e}^{-(t-\tau)^2}\,\mathrm{d}t\\ \ll{}& \Big(\int_{-\infty}^{\infty}|L(1-\sigma+\mathrm{i}t,\pi)|^{2k}\,\mathrm{e}^{-(t-\tau)^2}\,\mathrm{d}t\Big)^{\frac{1}{2}}\Big(\int_{-\infty}^{\infty}|L(\sigma+\mathrm{i}t,\pi)|^{2k}\,\mathrm{e}^{-(t-\tau)^2}\,\mathrm{d}t\Big)^{\frac{1}{2}}.\end{aligned}$$

By the functional equation (1.3)–(1.5) and the Stirling's formula, we have

$$|L(1-\sigma+\mathrm{i}t,\pi)| \ll |L(\sigma+\mathrm{i}t,\pi)|\,(1+|t|)^{m(\sigma-\frac{1}{2})},$$

where the implied constant depends on $\mathrm{Im}\mu_\pi(j)$, $j=1,\cdots,m$. It follows that

$$\begin{aligned}&\int_{-\infty}^{\infty}|L(1-\sigma+\mathrm{i}t,\pi)|^{2k}\,\mathrm{e}^{-(t-\tau)^2}\,\mathrm{d}t\\ \ll{}& \Big\{\int_{-\infty}^{\frac{\tau}{2}}+\int_{\frac{\tau}{2}}^{\frac{3\tau}{2}}+\int_{\frac{3\tau}{2}}^{\infty}\Big\}|L(\sigma+\mathrm{i}t,\pi)|^{2k}\,(1+|t|)^{km(2\sigma-1)}\,\mathrm{e}^{-(t-\tau)^2}\,\mathrm{d}t\\ \ll{}& \tau^{km(2\sigma-1)}\int_{-\infty}^{\infty}|L(\sigma+\mathrm{i}t,\pi)|^{2k}\,\mathrm{e}^{-(t-\tau)^2}\,\mathrm{d}t + \mathrm{e}^{-c_5\tau^2},\end{aligned}$$

where we have used the bound $L(\sigma+\mathrm{i}t,\pi)\ll|t|^{\frac{m}{4}}$ for $\sigma\ge\frac12$. Therefore,

$$\int_{-\infty}^{\infty}\Big|L\Big(\frac12+\mathrm{i}t,\pi\Big)\Big|^{2k}\mathrm{e}^{-(t-\tau)^2}\,\mathrm{d}t\ll\tau^{km(\sigma-\frac12)}\int_{-\infty}^{\infty}\left|L\left(\sigma+\mathrm{i}t,\pi\right)\right|^{2k}\mathrm{e}^{-(t-\tau)^2}\,\mathrm{d}t+\mathrm{e}^{-c_6\tau^2}.$$

Now integration respect to τ on $[T,2T]$ completes the proof of Lemma 4.2.

Lemma 4.3 *Let $\frac12\le\sigma\le\frac34$ and $T\ge2$. Then for all $k>0$, $J(\sigma,T)\ll J\big(\frac12,T\big)^{\frac32-\sigma}T^{\sigma-\frac12}$.*

Proof Applying Lemma 4.1 with $\gamma=\sigma$, $\frac12\le\sigma\le\frac34$, $\alpha=\frac12$, $\beta=\frac32$, $\theta=2k$ and $G(s)$ as in (4.1), we obtain

$$\int_{-\infty}^{\infty}\left|L\left(\sigma+\mathrm{i}t,\pi\right)\right|^{2k}\mathrm{e}^{-(t-\tau)^2}\,\mathrm{d}t\ll\Big(\int_{-\infty}^{\infty}\Big|L\Big(\frac12+\mathrm{i}t,\pi\Big)\Big|^{2k}\mathrm{e}^{-(t-\tau)^2}\,\mathrm{d}t\Big)^{\frac32-\sigma}.$$

Here we have used the fact that $\int_{-\infty}^{\infty}\left|L\left(\frac32+\mathrm{i}t,\pi\right)\right|^{2k}\mathrm{e}^{-(t-\tau)^2}\,\mathrm{d}t\ll1$. Now integration respect to τ on $[T,2T]$ and the Jensen's inequality give the assertion of Lemma 4.3.

For the proof of Theorem 1.1, in what follows, we will take $k=\frac{u}{v}$ with $u,v\in\mathbb{N}$ and $(u,v)=1$; under GRH, we can take $k=\frac{u}{v}$ with $v=1$ and $u=k\ge0$. For the proof of Theorem 1.2, we will take $k=\frac{u}{v}$ with $0\le u=k\le\frac{2}{m}-\varepsilon$ for any $\varepsilon>0$, and $v=1$.

Define $g(s,\pi)=L\left(s,\pi\right)^u-S_N(s)^v$ and $K(\sigma,T)=\int_{-\infty}^{\infty}\left|g\left(\sigma+\mathrm{i}t,\pi\right)\right|^{\frac2v}w(t,T)\,\mathrm{d}t$.

Lemma 4.4 *Assume GRC. Let $\frac12\le\sigma\le\frac34$, $N\ll T$ and $T\ge2$. Then for any $\varepsilon>0$,*

$$K\left(\sigma,T\right)\ll K\Big(\frac12,T\Big)^{\frac{5-4\sigma}{3}}\big(TN^{-\frac1v(\frac32-\varepsilon)}\big)^{\frac{4\sigma-2}{3}}.$$

Proof Applying Lemma 4.1 with $\gamma=\sigma$, $\frac12\le\sigma\le\frac34$, $\alpha=\frac12$, $\beta=\frac54$, $\theta=\frac2v$ and $G(s)=g(s,\pi)\mathrm{e}^{\frac{v(s-\mathrm{i}\tau)^2}{2}}$, we obtain

$$\begin{aligned}&\int_{-\infty}^{\infty}|g(\sigma+\mathrm{i}t,\pi)|^{\frac2v}\,\mathrm{e}^{-(t-\tau)^2}\,\mathrm{d}t\\&\ll\Big(\int_{-\infty}^{\infty}\Big|g\Big(\frac12+\mathrm{i}t,\pi\Big)\Big|^{\frac2v}\mathrm{e}^{-(t-\tau)^2}\,\mathrm{d}t\Big)^{\frac{5-4\sigma}{3}}\Big(\int_{-\infty}^{\infty}\Big|g\Big(\frac54+\mathrm{i}t,\pi\Big)\Big|^{\frac2v}\mathrm{e}^{-(t-\tau)^2}\,\mathrm{d}t\Big)^{\frac{4\sigma-2}{3}}.\end{aligned}$$

Recall that $S_N(s)\ll N^{\frac12+\varepsilon}$ for $\sigma\ge\frac12$ and any $\varepsilon>0$. Thus $g(s,\pi)\ll N^{(\frac12+\varepsilon)v}+|t|^{\frac{mu}{4}}$ for $\sigma\ge\frac12$. This gives

$$\int_{-\infty}^{\infty}\Big|g\Big(\frac54+\mathrm{i}t,\pi\Big)\Big|^{\frac2v}\mathrm{e}^{-(t-\tau)^2}\,\mathrm{d}t\ll\int_{\frac{\tau}{2}}^{\frac{3\tau}{2}}\Big|g\Big(\frac54+\mathrm{i}t,\pi\Big)\Big|^{\frac2v}\mathrm{e}^{-(t-\tau)^2}\,\mathrm{d}t+N^{1+2\varepsilon}\mathrm{e}^{-c_7\tau^2}.$$

Therefore,

$$\begin{aligned}&\int_{-\infty}^{\infty}|g(\sigma+\mathrm{i}t,\pi)|^{\frac2v}\,\mathrm{e}^{-(t-\tau)^2}\,\mathrm{d}t\\&\ll\Big(\int_{-\infty}^{\infty}\Big|g\Big(\frac12+\mathrm{i}t,\pi\Big)\Big|^{\frac2v}\mathrm{e}^{-(t-\tau)^2}\,\mathrm{d}t\Big)^{\frac{5-4\sigma}{3}}\Big(\int_{\frac{\tau}{2}}^{\frac{3\tau}{2}}\Big|g\Big(\frac54+\mathrm{i}t,\pi\Big)\Big|^{\frac2v}\mathrm{e}^{-(t-\tau)^2}\,\mathrm{d}t\Big)^{\frac{4\sigma-2}{3}}\\&\quad+\Big(\int_{-\infty}^{\infty}\Big|g\Big(\frac12+\mathrm{i}t,\pi\Big)\Big|^{\frac2v}\mathrm{e}^{-(t-\tau)^2}\,\mathrm{d}t\Big)^{\frac{5-4\sigma}{3}}\big(N^{1+2\varepsilon}\mathrm{e}^{-c_7\tau^2}\big)^{\frac{4\sigma-2}{3}}.\end{aligned}$$

Now integration respect to τ over $[T, 2T]$ and the Hölder's inequality give us

$$\begin{aligned} K(\sigma, T) &\ll K\Big(\frac{1}{2}, T\Big)^{\frac{5-4\sigma}{3}} \Big(\int_T^{2T} \int_{\frac{\tau}{2}}^{\frac{3\tau}{2}} \Big|g\Big(\frac{5}{4}+\mathrm{i}t, \pi\Big)\Big|^{\frac{2}{v}} \mathrm{e}^{-(t-\tau)^2}\,\mathrm{d}t\,\mathrm{d}\tau\Big)^{\frac{4\sigma-2}{3}} \\ &\quad + \mathrm{e}^{-c_8 T^2} N^{\frac{4\sigma-2+\varepsilon}{3}} K\Big(\frac{1}{2}, T\Big)^{\frac{5-4\sigma}{3}} \\ &\ll K\Big(\frac{1}{2}, T\Big)^{\frac{5-4\sigma}{3}} \Big(\int_T^{2T} \Big|g\Big(\frac{5}{4}+\mathrm{i}t, \pi\Big)\Big|^{\frac{2}{v}}\,\mathrm{d}t\Big)^{\frac{4\sigma-2}{3}} + \mathrm{e}^{-c_9 T^2} K\Big(\frac{1}{2}, T\Big)^{\frac{5-4\sigma}{3}}. \end{aligned}$$

It remains to establish the bound

$$\int_T^{2T} \Big|g\Big(\frac{5}{4}+\mathrm{i}t, \pi\Big)\Big|^{\frac{2}{v}}\,\mathrm{d}t \ll TN^{-k(\frac{3}{2}-\varepsilon)}. \tag{4.2}$$

The function $g\big(\frac{5}{4}+\mathrm{i}t, \pi\big)$ has a representation as an absolutely convergent Dirichlet series. In view of the identity

$$h_{k\ell}(n) = \sum_{n=n_1\cdots n_\ell} h_k(n_1)\cdots h_k(n_\ell), \quad \ell \in \mathbb{N},$$

we find that

$$g\Big(\frac{5}{4}+\mathrm{i}t, \pi\Big) = L\Big(\frac{5}{4}+\mathrm{i}t, \pi\Big)^u - S_N\Big(\frac{5}{4}+\mathrm{i}t\Big)^v = \sum_{n=N}^{\infty} \frac{a(n)}{n^{\frac{5}{4}+\mathrm{i}t}},$$

where

$$\begin{aligned} a(n) &= h_u(n) - \sum_{\substack{n=n_1\cdots n_v \\ n_j\le N,\, j=1,\cdots,v}} h_k(n_1)\cdots h_k(n_v) \\ &= \sum_{\substack{n=n_1\cdots n_v \\ \exists n_j > N}} h_k(n_1)\cdots h_k(n_v) \\ &\ll \sum_{n=n_1\cdots n_v} |h_k(n_1)|\cdots|h_k(n_v)| \\ &\ll \sum_{n=n_1\cdots n_v} \widehat{h}_k(n_1)\cdots\widehat{h}_k(n_v) = \widehat{h}_u(n), \end{aligned}$$

where $\widehat{h}_u(n)$ is the nth coefficient of the Dirichlet series expansion of the function

$$\widehat{L}(s,\pi)^u = \Big(\prod_p \prod_{1\le j\le m} \Big(1 - \frac{|\alpha_\pi(p,j)|}{p^s}\Big)^{-1}\Big)^u, \quad \sigma > \frac{1}{2}.$$

By (1.7), we have $\widehat{h}_u(n) = d_u(n)$. Thus $a(n) \ll d_u(n) \ll n^\varepsilon$ for any $\varepsilon > 0$. By Montgomery-Vaughan Theorem (see [19]), we have for $N \ll T$,

$$\int_{\frac{T}{2}}^{\frac{3T}{2}} \Big|g\Big(\frac{5}{4}+\mathrm{i}t, \pi\Big)\Big|^2\,\mathrm{d}t \ll T \sum_{n\ge N} \frac{|a(n)|^2}{n^{\frac{5}{2}}} \ll TN^{-\frac{3}{2}+\varepsilon}.$$

By the Jensen's inequality,

$$\int_{\frac{T}{2}}^{\frac{3T}{2}} \Big|g\Big(\frac{5}{4}+\mathrm{i}t, \pi\Big)\Big|^{\frac{2}{v}}\,\mathrm{d}t \ll T\Big(\frac{1}{T}\int_0^{3T} \Big|g\Big(\frac{5}{4}+\mathrm{i}t, \pi\Big)\Big|^2\,\mathrm{d}t\Big)^{\frac{1}{v}} \ll TN^{-\frac{1}{v}(\frac{3}{2}-\varepsilon)}.$$

This proves (4.2) and thus completes the proof of Lemma 4.4.

5 Proof of (2.2) and (2.3)

First, we prove (2.2). By the definition of $g(s,\pi)$,

$$|S_N(s)|^2 = |S_N(s)^v|^{\frac{2}{v}} = |L(s,\pi)^u - g(s,\pi)|^{\frac{2}{v}} \ll |L(s,\pi)|^{2k} + |g(s,\pi)|^{\frac{2}{v}}.$$

Hence

$$H(\sigma,T) \ll J(\sigma,T) + K(\sigma,T). \tag{5.1}$$

Similarly,

$$K(\sigma,T) \ll H(\sigma,T) + J(\sigma,T). \tag{5.2}$$

If $K\big(\frac{1}{2},T\big) \le T$, then by Lemma 3.5 and (5.1) with $\sigma = \frac{1}{2}$, we have

$$T\,(\log T)^{k^2} \ll H\Big(\frac{1}{2},T\Big) \ll J\Big(\frac{1}{2},T\Big) + K\Big(\frac{1}{2},T\Big) \ll J\Big(\frac{1}{2},T\Big) + T,$$

i.e., $J\big(\frac{1}{2},T\big) \gg T\,(\log T)^{k^2}$. If $K\big(\frac{1}{2},T\big) > T$, then by Lemma 4.4, we see that

$$K\,(\sigma,T) \ll K\Big(\frac{1}{2},T\Big)N^{-\frac{1}{v}\frac{4\sigma-2}{3}(\frac{3}{2}-\varepsilon)} \ll K\Big(\frac{1}{2},T\Big)N^{\frac{1}{v}(1-\varepsilon)(1-2\sigma)}. \tag{5.3}$$

Now we choose $N = T^{\frac{1}{2}}$ and $\varepsilon = \frac{1}{2}$. By (5.1)–(5.3), we have

$$H(\sigma,T) \ll J(\sigma,T) + K\Big(\frac{1}{2},T\Big)T^{\frac{1}{2v}(\frac{1}{2}-\sigma)} \ll J(\sigma,T) + \Big[H\Big(\frac{1}{2},T\Big) + J\Big(\frac{1}{2},T\Big)\Big]T^{\frac{1}{2v}(\frac{1}{2}-\sigma)}.$$

Hence, either

$$H(\sigma,T) \ll H\Big(\frac{1}{2},T\Big)T^{\frac{1}{2v}(\frac{1}{2}-\sigma)} \tag{5.4}$$

or

$$H(\sigma,T) \ll J(\sigma,T) + J\Big(\frac{1}{2},T\Big)T^{\frac{1}{2v}(\frac{1}{2}-\sigma)}. \tag{5.5}$$

Take $\sigma = \frac{1}{2} + \frac{C}{\log T}$. By Lemma 3.5 and (5.4), we have

$$C^{-k^2}T\,(\log T)^{k^2} = T\Big(\sigma - \frac{1}{2}\Big)^{-k^2} \ll H(\sigma,T) \ll H\Big(\frac{1}{2},T\Big)T^{\frac{1}{2v}(\frac{1}{2}-\sigma)} \ll T\,(\log T)^{k^2}\,\mathrm{e}^{-\frac{C}{2v}}.$$

Thus, $\mathrm{e}^{\frac{C}{2v}} \le c_{10}^{k^2}$ for some $c_{10} > 0$, which is impossible when C is sufficiently large. Therefore, (5.5) is valid. Now (5.5) and Lemma 4.3 imply

$$H(\sigma,T) \ll J\Big(\frac{1}{2},T\Big)^{\frac{3}{2}-\sigma}T^{\sigma-\frac{1}{2}} + J\Big(\frac{1}{2},T\Big)T^{\frac{1}{2v}(\frac{1}{2}-\sigma)}. \tag{5.6}$$

Taking $\sigma = \frac{1}{2} + \frac{C}{\log T}$ in (5.6) and applying Lemma 3.5, we get

$$T\,(\log T)^{k^2} \ll H\Big(\frac{1}{2},T\Big) \ll J\Big(\frac{1}{2},T\Big)^{1-\frac{C}{\log T}} + J\Big(\frac{1}{2},T\Big)\mathrm{e}^{-\frac{C}{2v}} \ll J\Big(\frac{1}{2},T\Big).$$

This completes the proof of (2.2).

Next we prove (2.3). Note that in this case $k=\frac{u}{v}$ with $0\le u=k\le\frac{2}{m}-\varepsilon$ for any $\varepsilon>0$, and $v=1$. By the definition of $g(s,\pi)$, we have

$$J(\sigma,T)\ll H(\sigma,T)+K(\sigma,T). \tag{5.7}$$

If $K(\frac{1}{2},T)\le T$, then by Lemma 3.5 and (5.7) with $\sigma=\frac{1}{2}$, we have

$$J\Big(\frac{1}{2},T\Big)\ll H\Big(\frac{1}{2},T\Big)+K\Big(\frac{1}{2},T\Big)\ll T\,(\log T)^{k^2}.$$

If $K(\frac{1}{2},T)>T$, then by Lemma 4.4, we see that

$$K\,(\sigma,T)\ll K\Big(\frac{1}{2},T\Big)N^{-\frac{4\sigma-2}{3}(\frac{3}{2}-\varepsilon)}\ll K\Big(\frac{1}{2},T\Big)N^{2(1-\frac{2}{3}\varepsilon)(\frac{1}{2}-\sigma)}.$$

This estimate combined with (5.7) and (5.2) gives

$$\begin{aligned}J(\sigma,T)&\ll H(\sigma,T)+K\Big(\frac{1}{2},T\Big)N^{2(1-\frac{2}{3}\varepsilon)(\frac{1}{2}-\sigma)}\\&\ll H(\sigma,T)+\Big[H\Big(\frac{1}{2},T\Big)+J\Big(\frac{1}{2},T\Big)\Big]N^{2(1-\frac{2}{3}\varepsilon)(\frac{1}{2}-\sigma)}.\end{aligned}$$

Hence, either

$$J(\sigma,T)\ll J\Big(\frac{1}{2},T\Big)N^{2(1-\frac{2}{3}\varepsilon)(\frac{1}{2}-\sigma)} \tag{5.8}$$

or

$$J(\sigma,T)\ll H(\sigma,T)+H\Big(\frac{1}{2},T\Big)N^{2(1-\frac{2}{3}\varepsilon)(\frac{1}{2}-\sigma)}. \tag{5.9}$$

Now we take $N=T$ and $\sigma=\frac{1}{2}+\frac{C}{\log T}$. Recall that $k\le\frac{2}{m}-\varepsilon$. Then by (5.8) and Lemma 4.2, we have

$$J\Big(\frac{1}{2},T\Big)\ll J(\sigma,T)T^{mk(\sigma-\frac{1}{2})}\ll J\Big(\frac{1}{2},T\Big)T^{\varepsilon(m-\frac{4}{3})(\frac{1}{2}-\sigma)},$$

i.e., $\mathrm{e}^{\varepsilon(m-\frac{4}{3})C}\le C(m,k)$, which is false for C sufficiently large. Therefore, (5.9) holds. By Lemma 3.5 and (5.9) with $\sigma=\frac{1}{2}+\frac{C}{\log T}$, we have $J\big(\frac{1}{2},T\big)\ll H\big(\frac{1}{2},T\big)\ll T(\log T)^{k^2}$. This completes the proof of (2.3).

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