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REGULAR ARTICLE

# Asset allocation and multivariate position based trading

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**Abstract** I add a second risky asset and a risk free bond to the univariate artificial market investigated by Lux and Marchesi (Int J Theor Appl Finance 3(4):675–702, 2000), keeping track of traders aggregate positions and wealth. Asset allocation and security selection are modeled as separate decision processes, as is common practice in financial institutions. Introducing position based trading avoids inconsistencies in traders inventories resulting from the order based setup of the original model, while preserving its ability to reproduce the stylized facts of financial return series.

**Keywords** Asset allocation · Heterogeneous agents · Multivariate price dynamics · Position based trading

**JEL Classification** C61 · D40 · D84 · G11 · G12

## 1 Introduction

Lux (1998) provides a model of a single asset market, in which traders switch between a fundamentalist strategy and a combination of trend chasing and imitative behaviour. Stochastic fluctuations in the population size of different trader types lead to occasional destabilisations of an otherwise stable market, when the number of chartists becomes large. The simulation study by Lux and Marchesi (2000) shows that such a model is capable of producing the main stylized facts of financial markets, e.g. a leptokurtic return distribution with a realistic tail index and long memory in clustered return volatility, even when the fundamental price is assumed constant and parameters are chosen such that the dynamics remains in the vicinity of the stable fundamental equilibrium.

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I extend this univariate model into a multivariate setup containing a second risky asset and a riskless bond. The investment process will be split up into asset allocation and security selection, as is common practice in financial institutions.[1]

This study is not the first to tackle the impact of heterogeneous expectations upon more that one risky asset. In particular Westerhoff (2004) considers the interaction of chartists and fundamentalists on multiple assets and generates return series similar to those observed in real markets. My main contribution relative to his study and those by Lux and Marchesi consists in removing inconsistencies concerning traders inventories resulting from the order-based setup of their models. Both Lux and Westerhoff consider trading at disequilibrium prices in order-driven markets following the tradition initiated by Beja and Goldman (1980) and Day and Huang (1990). That is, traders place orders proportional to the expected profits of their investments, while a market maker adjusts prices proportional to net excess demand, filling any imbalances between demand and supply from his inventory. The consequences of such a setup upon traders inventories remained unexplored until Farmer and Joshi (2002) pointed out that the uncoupling of orders and acquired positions may lead to unbounded inventories. The underlying reason (unmentioned in Farmer and Joshi) is the following: orders, if filled, are derivatives of traders holdings with respect to time; or stated the other way round, traders holdings are the integral of filled orders over time. Now if orders, rather than holdings, are assumed to follow a level stationary process, as is generally the case in the order-based literature, then integrating over these orders in order to obtain traders inventories will generate integrated and therefore unbounded holdings, unless the stationary series which the trading decisions were based upon were already over-differenced. Unbounded inventories are of course, as Farmer and Joshi point out, unacceptable from a risk management point of view, and should therefore be avoided in any model of financial markets.

Order-based trading appears also unrealistic because it is well established standard in the academic literature at least since Markowitz (1959), that investors consider portfolio holdings rather than orders as the relevant object of profit and risk considerations. The inconsistencies of an order-based setup become particularly obvious when extending a univariate model into a multi asset framework. Suppose for example that a trader has favored asset $A$ over asset $B$ for a while, but receives now a signal which favors asset $B$ over asset $A$. A consistent model would require the trader to close or at least diminish his position in asset $A$ before entering a new position in asset $B$. That is, a new signal favoring $B$ over $A$ would not only generate buying orders for $B$, but also selling orders for $A$, until the desired new positions in assets $A$ and $B$ are established. This is not achieved by naïvely extending the order-based setup by Beja and Goldman to multiple assets, as it would falsely neglect any acquired position in $A$ when producing new orders for asset $B$.

The traders in my model use therefore position-based rather than order-based trading strategies. That is, they choose portfolio holdings (rather than producing orders)

[1] see e.g. Davis and Steil (2001) for a detailed account of the institutional investment process.

proportional to expected investment profits.[2] Trading orders are generated only when target portfolios change, as is expressed by the derivatives of target holdings with respect to time. I will show that such a setup implies level stationary inventories for all market participants, while still being able to produce the main stylized facts for the returns of both individual stocks and the stock index, provided that the stock and strategy selection process is augmented with an independent asset allocation process between stocks and bonds.

Simply extending Lux and Marchesi (2000) to multiple assets without separate asset allocation does not reproduce the sylized facts with position based trading for the following reason: position based trading is less aggressive than order based trading in the sense that a larger number of irrational investors by itself does not necessarily lead to large returns, since for price changes it is the rate of change in trading strategy populations and not their level that matters. With position based trading, large returns can only be brought about by sudden changes in traders populations or large mispricings, which are unlikely to occur in the vicinity of the fundamental equilibrium. But a separate asset allocation process between stocks and bonds with faster dynamics than the stock and strategy selection within equities may occasionally lead to such sudden changes in the traders populations and corresponding mispricings with the associated clusters of asset return volatility.

The actual process how volatility clustering comes about in this model is rather complex. It will be shown in Sect. 4 that there exists a distribution of trading strategy populations which implies a fundamental equilibrium of the dynamics in the sense that the market prices of both stocks equal on average their fundamental values. The actual trader populations fluctuate around this equilibrium, but without separate asset allocation, a random surplus of investors in any of the stocks is corrected before anything interesting happens at the return level. Stock price changes are dominated by stochastic fluctuations in the number of chartists, with only a negligible price impact of fundamentalist orders, since a substantial mispricing never comes about.

The superposition of an independent asset allocation process with faster dynamics than the strategy and security selection within stocks appears to slow down the mean reverting process within the equity investor subpopulations, bringing it closer to a random walk like dynamics. This is presumably so, because sudden changes in the total number of equity investors introduce extra noise to the population size of each investor type, despite leaving their relative composition intact. Now with slower mean reverting stock investor populations, a random surplus of investors in stock $A$, say, may prevail long enough to attract an increasing number of chartists to that stock, as a result of their herding behaviour. At the same time the price in stock $A$ drifts further and further away from its fundamental value due to their accumulated orders. As long as the mispricing is still moderate, fundamentalists switch to the chartist strategy, which implies larger and larger price changes, since the fundamentalist base to absorb orders resulting from fluctuations in the strategy populations has been reduced. With increased mispricing, fundamentalists start having an impact. This is however

[2] This implies that traders will in general change their target holdings when they switch investment strategy even if they decide to remain invested in the same stock, because fundamentalist holdings are sensitive to the degree of mispricing whereas chartist holdings are not.

stabilizing only in the long run as the increased benefit of fundamental investment reduces the number of chartists. On a shorter timescale, fundamentalist orders increase stock price volatility even further, since stochastic fluctuations in their population brought about by the faster asset allocation process generate large orders in the presence of large mispricings. Only the gradual rebound in chartist and fundamentalist populations due to the increased profitability of fundamentalist investment reduces return volatility back to normal levels.

A separate asset allocation process with the associated capital flows in and out of equities is also necessary in order to avoid negative correlations between the individual stock returns. A pure infection dynamics between different equity investments implies that each increase in holders of stock $A$ is accompanied by a corresponding decrease in holders of stock $B$. This leads necessarily to negative stock return correlations, since each stock return correlates positively with the change in the number of its holders. The bond holders in my model serve as an additional pool of potential equity investors, who may decide to invest in or withdraw from the equity market as a whole. This synchronized investment decision with simultaneous impact on both stocks may compensate for, or even outweigh, the negative return correlations generated by the internal infection dynamics of the stock selection process.

## 2 The problem with order based strategies: an example

As an example for an order-driven model consider the simulations in Lux and Marchesi (2000). The model consists of $n_c$ chartists and $n_f$ fundamentalists, where the chartists may be further subdivided into $n_+$ bullish and $n_-$ bearish speculators. The number of the three kind of traders evolves through time according to the perceived investment returns of the respective groups. Lux models this infection process using the master equation approach from synergetic originally developed for physical systems by Haken (1983) and extended to the social sciences by Weidlich and Haag (1983).

The link between population sizes and prices is order-driven. That is, traders continue buying as long as they consider an asset under priced and keep on selling as long as they consider it over priced, regardless of their accumulated inventories. In Lux' model each bullish (bearish) chartist buys (sells) $t_c$ shares, such that the net excess demand of all chartists $ED_c$ aggregates to

$$ED_c = (n_+ - n_-)t_c, \tag{1}$$

with $n_+$ ($n_-$) denoting the number of bullish (bearish) chartists.

Each fundamentalist is assumed to trade $t_f$ shares per unit mispricing as measured by the difference between fundamental value $p_f$ and market price $p$, such that the excess demand $ED_f$ aggregated over all $n_f$ fundamentalists becomes

$$ED_f = n_f t_f (p_f - p). \tag{2}$$

Note that $t_c$ and $t_f$ denote funds to *reallocate* rather than target positions to *hold*. A market maker adjusts prices in the direction of net excess demand absorbing any imbalances from his inventory

$$\dot{p} := dp/dt = \beta(ED_c + ED_f), \tag{3}$$

where $\beta$ denotes the finite speed of price adjustment by the market maker in the spirit of Beja and Goldman (1980) and Day and Huang (1990).

Farmer and Joshi (2002) consider order based trading unrealistic from a risk management point of view, as the uncoupling of orders and acquired positions may lead to unbounded inventories. In order to illustrate their claim, I have replicated the simulations by Lux and Marchesi (2000) over the same number of observations and using the same parameter sets as they did,[3] keeping track of the market participants inventories and wealth. Consider first the aggregate holdings of chartists (dark solid line) and fundamentalists (light dotted line) over 20,000 observations in Fig. 1. During that time, traders accumulate inventories of the same order of magnitude as the number of observations, with close to symmetric portfolio holdings for chartists and fundamentalists. With the possible exception of parameter set IV, inventories appear to follow rather random walk like than mean reverting processes.

This view is confirmed by inspecting the results of augmented Dickey–Fuller tests upon traders inventories in Table 1. None of the tests led to a rejection of a unit root in inventories even in these very large samples. The only sample which comes somewhat close to a rejection of a unit root is parameter set IV with $p$ values around 0.2 for both the holdings of chartists and fundamentalists. Also, the holdings do not visually appear unbounded in this sample. One might therefore argue, that even longer data sets would finally lead to a rejection of a unit root at least for parameter set IV. However, the closeness to level-stationary holdings in this parameter set could well be due to the periodicity of the random number generator. The generation of 20,000 data points required $8 \times 10^7$ calls of the random number generator. This is only by a factor of 25 below the largest positive value representable by signed 32-bit integers of $2.1 \times 10^9$, which is an upper limit for the period of any random number generator of the form $X_{n+1} = f(X_n)$ on 32-bit computers.[4]

Consider next the marketmaker inventories in Fig. 2. The results of the augmented Dickey–Fuller tests together with the deterministic trend parameters are depicted in Table 2. All series have either a unit root or a highly significant deterministic trend, consistent with the before mentioned possibility of building up unbounded inventories.

For completeness, we shall in the following consider the wealth dynamics for the different kind of traders. Strictly speaking, there is not much of a point in discussing the wealth dynamics of chartists or fundamentalists as individuals, since individual

[3] Lux and Marchesi (2000) used the following parameters. Parameter Set I: $v_1 = 3$, $v_2 = 2$, $\beta = 6$, $T_c = 10$, $T_f = 5$, $\alpha_1 = 0.6$, $\alpha_2 = 0.2$, $\alpha_3 = 0.5$, $s = 0.75$, $\sigma = 0.05$. Parameter Set II: $v_1 = 4$, $v_2 = 1$, $\beta = 4$, $T_c = 7.5$, $T_f = 5$, $\alpha_1 = 0.9$, $\alpha_2 = 0.25$, $\alpha_3 = 1$, $s = 0.75$, $\sigma = 0.1$. Parameter Set III: $v_1 = v_2 = 0.5$, $\beta = 2$, $T_c = T_f = 10$, $\alpha_1 = 0.75$, $\alpha_2 = 0.25$, $\alpha_3 = 0.75$, $s = 0.8$, $\sigma = 0.1$. Parameter Set IV: $v_1 = 2$, $v_2 = 0.6$, $\beta = 4$, $T_c = T_f = 5$, $\alpha_1 = 0.8$, $\alpha_2 = 0.2$, $\alpha_3 = 1$, $s = 0.75$, $\sigma = 0.05$. The following parameters are identical in all four sets: $N = 500$, $p_f = 10$, $r = 0.004$, $R = 0.0004$.

[4] See for example Robert and Casella (2004).

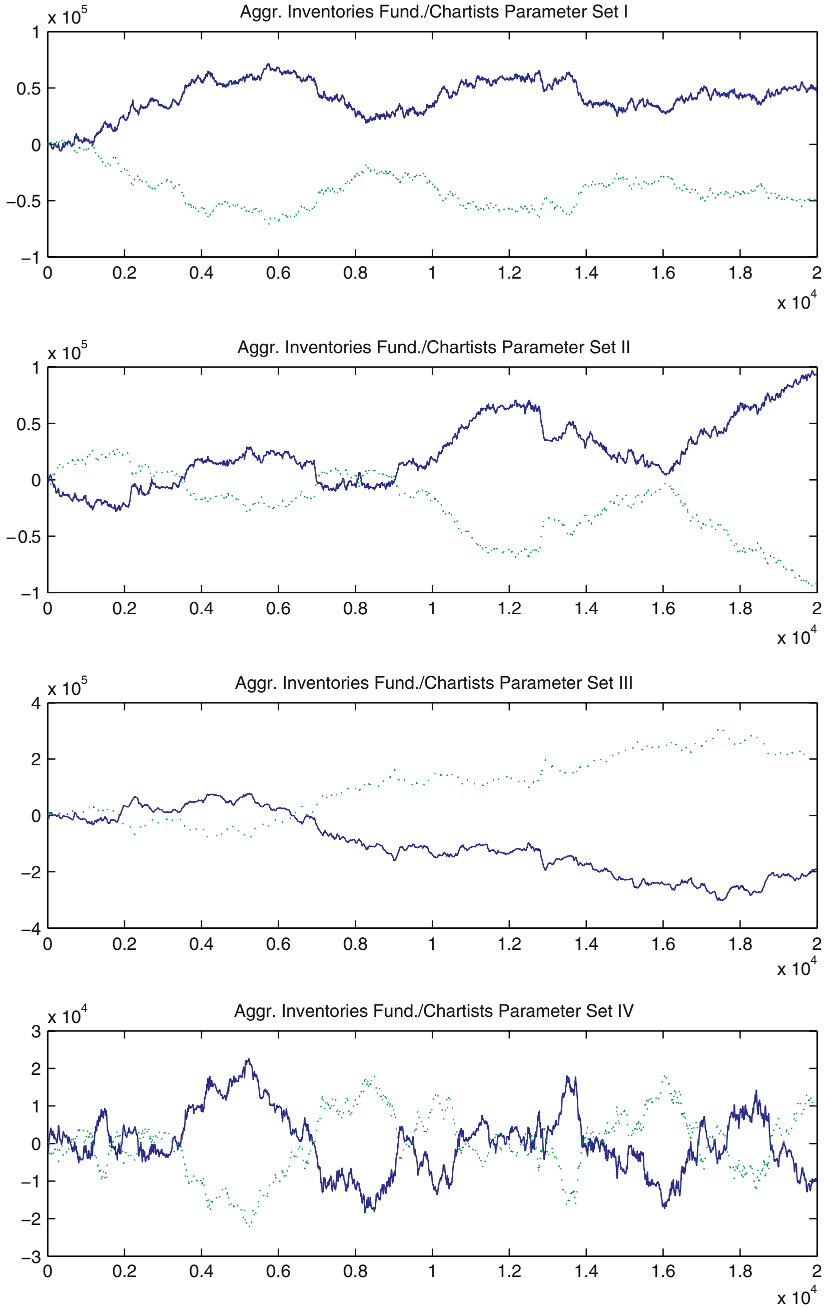


**Fig. 1** Aggregate holdings of chartists (*dark solid line*) and fundamentalists (*light dotted line*) over 20,000 observations using the parameters denoted in Footnote 3. Both types of traders accumulate close to symmetric unbounded inventories

**Table 1** Results of unit-root tests on aggregate traders holdings for the parameter sets given in Lux and Marchesi (2000). All tests accept the null of a unit-root in traders inventories

| Parameters | $\hat{\rho}$ Chartists | $\hat{\rho}$ Fundam. | $p$-val Chartists | $p$-val Fundam. |
|---|---|---|---|---|
| Parameter Set I | 0.999841 | 0.999841 | 0.3585 | 0.3552 |
| Parameter Set II | 0.999946 | 0.999847 | 0.8365 | 0.8413 |
| Parameter Set III | 0.999978 | 0.999971 | 0.5293 | 0.5362 |
| Parameter Set IV | 0.999838 | 0.999837 | 0.2034 | 0.1961 |

traders change between a chartist and a fundamentalist strategy all the time. However, it is still possible to investigate how much wealth a certain trader group as a whole has accumulated, even though the size of that subpopulation may vary over time. This is done for chartists and fundamentalists in Fig. 3.[5]

It is immediately evident from this figure that chartists loose their money to fundamentalists for all parameter sets considered. While one may be tempted to conclude that this will cause chartists to die out in course of time, this need not necessarily be so for at least three reasons:

1. Since traders are constantly changing between a chartist and a fundamentalist strategy, traders may well recover losses experienced while using a chartist strategy from the profits made when trading as fundamentalists.
2. It is reasonable to assume that fundamentalism is costlier then chartism in the sense that figuring out the true value of an asset requires more resources than just following a trend. These costs might just offset the profits fundamentalists make relative to the losses of chartists.
3. If the costs of market entry are lower for chartists than for fundamentalists, it is reasonable to assume that bankrupt chartists are replaced by new chartists entering the market.

Consider finally the wealth dynamics for the marketmaker depicted in Fig. 4. In all four cases market makers incur losses at close to constant rates, which appear harder to justify than those of the chartists because market makers don't change strategy. However, the market maker could well charge a fee from his trading partners in order to repair his losses, for example in form of a bid-ask spread as is common practice in financial markets.

Overall, we can at least not reject the concerns brought forward by Farmer and Joshi, that order based strategies violate fundamental risk management constraints by

[5] Inventories and wealth of the market participants have been calculated as follows. Once the excess demand has been determined, chartists inventories are increased by $ED_c = (n_+ - n_-)t_c$ and fundamentalist inventories by $ED_f = n_f t_f (p_f - p)$. The corresponding amounts of cash, $p \cdot ED_{c/f}$, are subtracted from their wealth after the new price $p$ has been determined using Eq. (3). The aggregate wealth of the two trader subpopulations equal their aggregate inventories evaluated at market price plus their cash. Since the market maker has to supply the shares demanded by the fundamentalist and chartist trader populations starting with zero inventories and cash, her holdings and cash equal the traders aggregate inventories and cash, however with opposite signs.

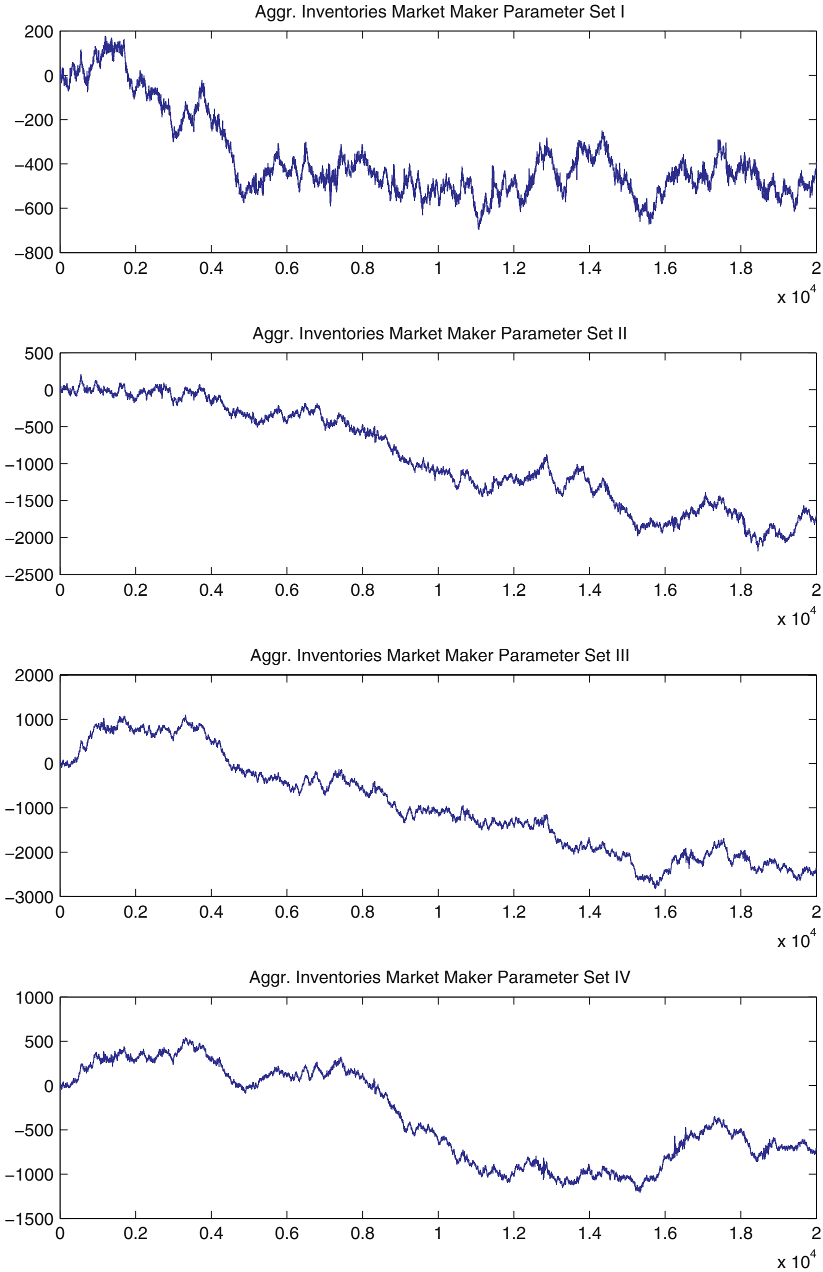


**Fig. 2** Aggregate inventories of the market maker over 20,000 observations using the parameters denoted in Footnote 3. All series contain either a unit root or have a significant deterministic trend (see Table 2)

their implicit tendency to build up infinite holdings over time. In our example, the problems of order based trading became particularly evident for the inventories and wealth of the market maker. This may be not so surprising, given that the market makers

**Table 2** Results of unit-root tests and deterministic trend in market maker inventories for the parameter sets given in Lux and Marchesi (2000). None of the series are bounded, as they contain either a unit root (Parameter sets I, IV) or have a significant deterministic trend (Parameter sets II, III)

| Parameters | $\hat{\rho}$ | ($p$-value) | trend | ($p$-value) |
|---|---|---|---|---|
| Parameter Set I | 0.99859 | (0.228) | −0.000022 | (0.1957) |
| Parameter Set II | 0.99833 | (0.0411) | −0.000187 | (0.0009) |
| Parameter Set III | 0.998992 | (0.0661) | −0.000191 | (0.0011) |
| Parameter Set IV | 0.999719 | (0.856) | −0.00002 | (0.2616) |

wealth and inventories provide the loophole for the modeller to replace equilibrium with disequilibrium trading.

Obviously, it would require considerable additional effort to include market makers wealth and positions into a consistent model of the price discovery process. One may also ask whether the disequilibrium trading provided by the market maker is really such an important feature of financial markets to model for return periods of a full trading day and above, as was the intention in Lux' model. In markets without 24 h trading such as stock markets closing prices must be quite close to equilibrium prices, since otherwise market participants would not be prepared to sleep with them until next morning.

I shall therefore drop the marketmaker in a simplified version of Lux' model with position-based trading in the next section. Because the model is position based, it is easy to generalize to multiple assets, avoiding the inconsistencies of order based trading discussed in the introduction and demonstrated in this section. I will also include a riskless bond (cash) to the model and separate the security selection decision between different stocks from the asset allocation decision between equity and bonds. This will bring the model closer to real life investment decisions and at the same time generate the necessary capital in- and outflows for the equity market as a whole, which are required for generating the stylized facts with position based trading as explained in the introduction.

## 3 Multivariate price dynamics: conservation of shares in position based trading

In this section we shall investigate the price discovery process for multiple assets in position based trading without a market maker. Consider for that purpose an investment community of $N$ portfolio managers or traders. They hold individually only one of three assets, either one of two risky stock issues or a bond issue in infinite supply (cash). The logarithmic trading prices of the stocks are denoted by $p_1$ and $p_2$, and the logarithm of their fundamental values by $p_{f1}$ and $p_{f2}$. Portfolio managers holding stocks may choose one of two investment strategies, fundamentalist or chartist. Fundamentalists hold long (short) positions in a stock because its trading price is below (above) its fundamental price, to which they expect the trading price to converge in the long-run. Chartists wish to hold a stock because most market participants already own

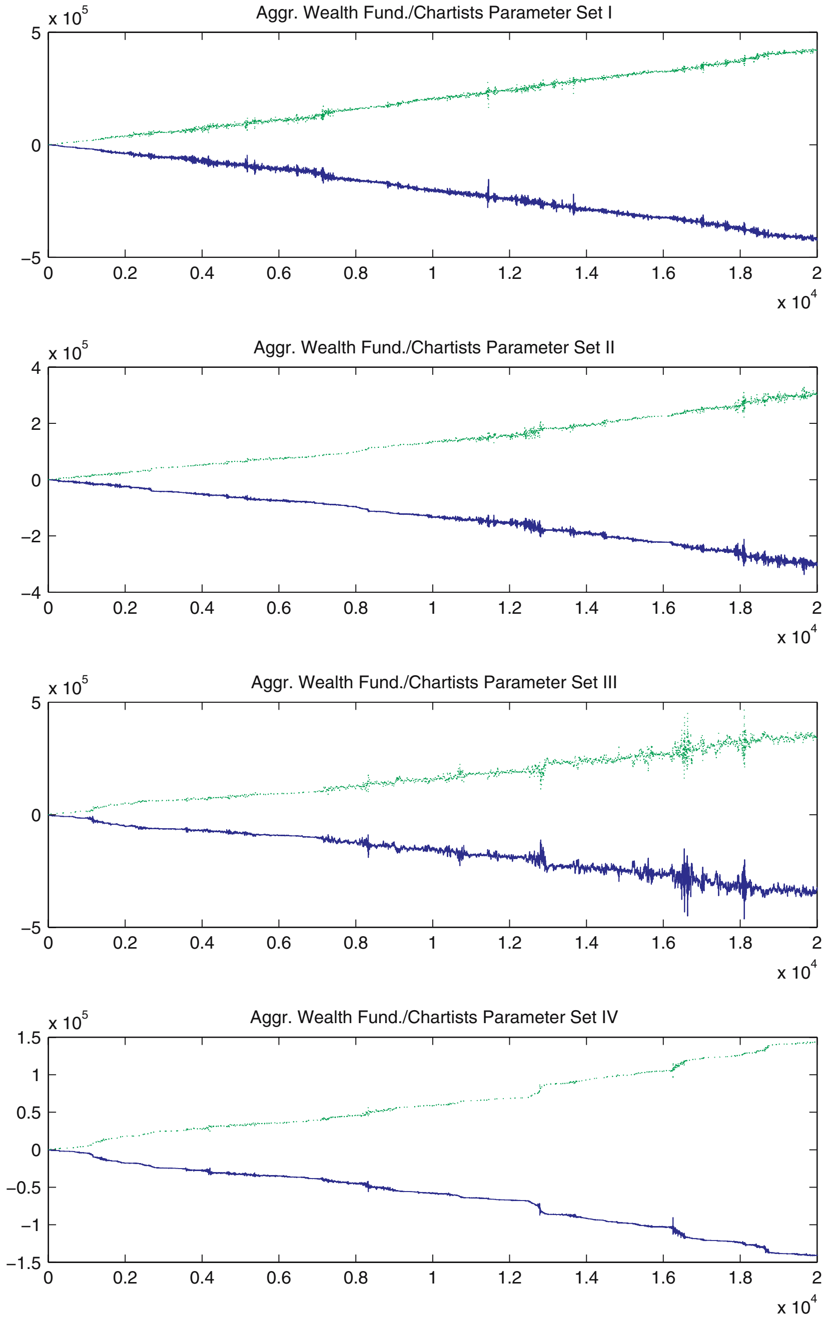


**Fig. 3** Aggregate wealth of the fundamentalist (*light dotted line*) and the chartist (*dark solid line*) trader subgroups over 20,000 observations using the parameters denoted in Footnote 3

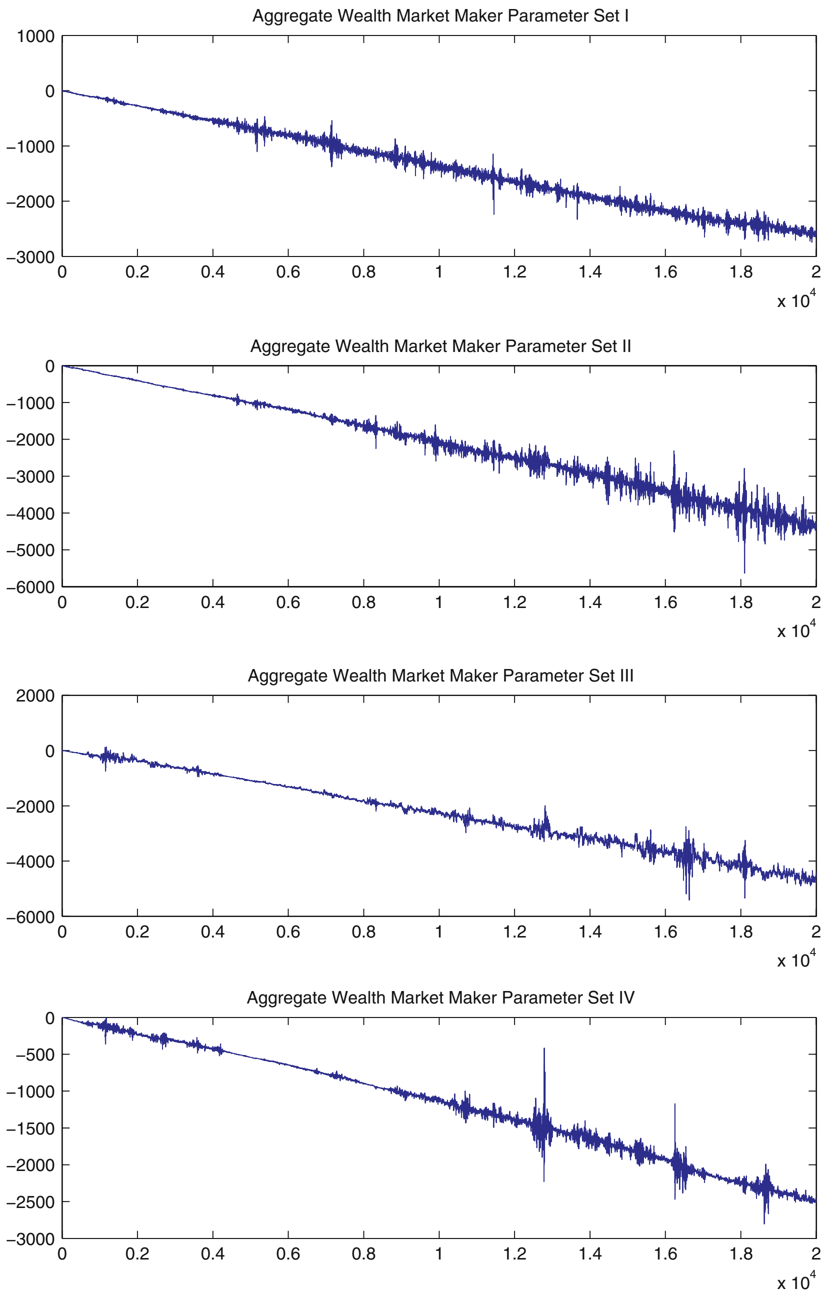


**Fig. 4** Aggregate wealth of the market maker over 20,000 observations using the parameters denoted in Footnote 3

it (herding). This simplifies Lux' original setup by not explicitly including the trend of the stock price itself as a motive for holding stocks, and is done here in order to keep the mathematical formulation of the multivariate setup concise. Herding, rather than riding a price trend, was also the numerically dominant trading motive for chartists in Lux' parameter sets.

Denote the number of chartists invested in stock 1 or 2 with $n_{c1}$ respectively $n_{c2}$ and the number of fundamentalists invested in stock 1 or 2 with $n_{f1}$ respectively $n_{f2}$. Each chartist wishes to hold $t_c$ issues of her favorite stock, whereas the desired holdings of fundamentalists are proportional to the mispricing of the stock they wish to hold. Denoting fundamentalists target holdings per unit mispricing with $t_f$, the aggregate target holding in either stock is

$$E_i = n_{ci}t_c + n_{fi}t_f(p_{fi} - p_i), \quad i = 1, 2, \tag{4}$$

where the first and second term denote aggregate target exposure in stock $i$ by chartists and fundamentalists, respectively. This equation may be seen as a multivariate generalization of the net excess demand $ED_c + ED_f$ in Lux' model, as repeated here in Eqs. (1)–(3). The key difference is that Lux follows the order-based literature in using this expression to describe stocks to *trade* rather than target positions in stocks to *hold*, as is the case here.

I assume the target holding parameters $t_c$ and $t_f$ and the fundamental prices $p_{f1}$ and $p_{f2}$ to be constant over the time period considered. Trading demand for the stocks is generated by changes in desired aggregate holdings due to changes in mispricing or the composition of traders

$$ED_i := \frac{d}{dt}E_i = \dot{n_{ci}}t_c + \dot{n_{fi}}t_f(p_{fi} - p_i) - n_{fi}t_f\dot{p}_i, \quad i = 1, 2. \tag{5}$$

Market clearing ($ED_i = 0$) yields for the logarithmic trading prices of the stocks

$$\dot{p}_i = \frac{1}{n_{fi}}\left(\dot{n_{ci}}\frac{t_c}{t_f} + \dot{n_{fi}}(p_{fi} - p_i)\right), \quad i = 1, 2. \tag{6}$$

We see from Eq. (6) that fast changes in the composition of traders and large mispricings speed up price changes, whereas large fundamentalist populations slow them down. On the chartist side, the speed of price adjustment depends on the target exposures of chartists relative to fundamentalists. Large chartist exposures speed up price changes whereas large fundamentalist exposures have the opposite effect. Overall, we recover the recurrent theme from the interacting agent literature, that fundamentalists have a stabilizing effect and that noise traders have a destabilizing effect upon prices, without having made any specific assumptions yet about how to model changes in the traders populations.

Another important conclusion from Eqs. (4)–(6) is that our setup implies conservation of shares, a feature not present in order based models including a market maker. This may be seen as follows: because we assume market clearing, the aggregate target holdings $E_1$ and $E_2$ in Eq. (4) must equal the number of shares issued by companies 1 and 2. The condition $dE_i/dt = 0$ for market clearing implies then, that the respective number of shares remains constant through time.

## 4 Population dynamics and asset allocation

The synergetic literature models interactions between members of the population in terms of Markov chains. That is, for each member of the population it postulates a transition probability to change its state of behaviour, or equivalently to move to another subpopulation, which depends only upon the systems current state. Suppose there are $M$ subpopulations (trader types) $n_1, \ldots, n_i, \ldots, n_j, \ldots, n_M$ and denote the transition probability to move from subpopulation $i$ to subpopulation $j$ as $p_{ij}$. It is then possible to describe the evolution of expected population sizes through time in terms of a closed system of first order differential equations, the so called *quasi-mean value equations*[6]

$$\dot{n}_i = \sum_{j \neq i}^{M} (n_j p_{ji} - n_i p_{ij}), \quad i = 1, \ldots, M. \tag{7}$$

The quasi-mean value equations have a very intuitive interpretation. They simply state that the expected change in population size $n_i$ consists of expected population inflows from all other states $\sum n_j p_{ji}$ minus all expected population outflows into other states $\sum n_i p_{ij}$. In our case we have $M = 5$ subpopulations: two chartist populations of size $n_{c1}$ and $n_{c2}$, two fundamentalist populations of size $n_{f1}$ and $n_{f2}$, and one bondholder population of size

$$n_B := N - n_E, \quad \text{where } n_E := n_{c1} + n_{c2} + n_{f1} + n_{f2} \tag{8}$$

denotes the number of equity investors. Our task is now to specify the transition probabilities $p_{ij}$ according to which traders change from one subgroup to another. As in Lux, it is assumed that traders change their strategy according to the perceived profits of the other strategies compared to their own.[7] The perceived profits or benefits $F_i$ of fundamentalists holding a position in stock $i$ are modeled as

$$F_i = s|p_{fi} - p_i|, \quad i = 1, 2, \tag{9}$$

[6] The quasi-mean value equations describe the dynamics of expected population sizes only for unimodal probability distributions of the system. If the system bifurcates into a multimodal probability distribution, the evolution of individual systems is no longer meaningfully described by unconditional expected values, as they lie somewhere between the states of maximal probability. But the quasi-mean value equations (7) describe still the most probable dynamics of individual systems given their current states, see Weidlich (2002, Chap. 12).

[7] Using standard portfolio optimization theory in order to obtain portfolio weights would lead to the uninteresting result of equal holdings in both stocks regardless of their correlation, because their long term expected return and variance are the same. While it will be shown in Proposition 1 that at fundamental equilibrium there are equally many holders in both stocks, only the alternative behavioristic assumption of agents changing their holdings based upon comparing the perceived benefits of alternative trading strategies implies the necessary fluctuations in order to replicate the stylized facts of financial returns.

where $s$ is a discount factor, since reversals to the fundamental price are expected to occur only in the future. The fundamentalist benefit $F_i$ is thus proportional to the logarithmic mispricing in stock $i$.

The benefit of chartists is assumed as

$$C_i = \frac{n_{ci} + n_{fi} - n_{cj} - n_{fj}}{N}, \quad i, j = 1, 2, \; i \neq j. \tag{10}$$

This generalizes the opinion index $x$ designed to proxy for herding in Lux' model to multiple assets, as it describes the scaled difference between equity investors in stock $i$ and $j$. The more traders there are invested in stock $i$ relative to stock $j$, the more attractive stock $i$ becomes relative to stock $j$ (herding), and the higher the chartist benefit $C_i$ in stock $i$ will be relative to the chartist benefit $C_j$ in stock $j$.

I follow Lux in assuming that the relative change in probability to switch from one strategy to another is proportional to the difference between the benefits of the respective strategies, i.e.

$$\frac{1}{p_{ij}} \frac{d\, p_{ij}}{d(U_j - U_i)} = \alpha \quad \text{with } U_i, U_j \in \{C_1, C_2, F_1, F_2\}, \tag{11}$$

where $\alpha$ measures the strength of attraction which apparently more profitable strategies exert upon the trader. Inserting the benefits (9) and (10) into (11) yields for the transition probabilities between the trader types

$$\begin{aligned} p_{cicj} &= v e^{\alpha(C_j - C_i)}, \quad p_{fifj} = v e^{\alpha(F_j - F_i)}, \quad i, j = 1, 2, \; i \neq j \\ p_{cifj} &= v e^{\alpha(F_j - C_i)}, \quad p_{ficj} = v e^{\alpha(C_j - F_i)}, \quad i, j = 1, 2, \end{aligned} \tag{12}$$

where $p_{cicj}$ and $p_{fifj}$ denote transitions from stock $i$ to stock $j$ within the chartist and the fundamentalist subgroup, respectively, and $p_{cifj}$ and $p_{ficj}$ denote transitions from chartists to fundamentalists and vice versa. The speed of adjustment parameter $v$ measures the frequency at which equity investors reconsider their investment strategy and depends therefore upon the time unit chosen in the description of the dynamic process.

Consider next the transitions between bond and equity investors as illustrated in Fig. 5. We assume that asset allocation and security selection are performed by separate entities, as is common practice in financial institutions.[8] That is, the individual trader or portfolio manager has no freedom to decide whether to invest in stocks or bonds, but chooses only specific securities within his asset class. This corresponds to portfolio managers in the majority of financial institutions, managing either an equity or a fixed income portfolio.

The decision how to split up traders between the equity and the fixed income side is done by a separate entity, which we shall call the asset allocator or sponsor. He or she is often an external client with little market information who wishes to delegate

[8] An in-depth treatment of the institutional investment process is provided by Davis and Steil (2001).

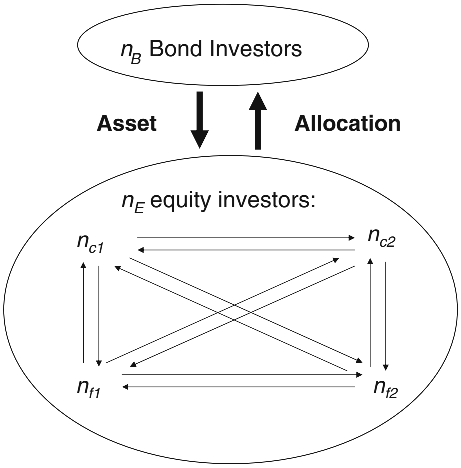


**Fig. 5** Security selection and asset allocation are modeled as separate decision processes. The sponsor decides how many traders to put onto the fixed income as opposed to the equity side (asset allocation). Portfolio managers decide about the stock to invest in and the trading strategy to use (security selection)

the investment management to professionals, whereas the before mentioned security selection is usually done by professional portfolio managers in house. Even when both asset allocation and security selection decisions are made in the same financial institution, the former are generally done by upper hierarchy levels. These have usually more duties than just making asset allocation decisions, which may prevent them from processing valuation relevant information as efficiently as their portfolio managers at security selection level do. The benefit of equity investments for the sponsor is therefore modeled in the same spirit as that of the chartists as

$$E = \frac{n_E - n_B}{N}. \tag{13}$$

That is, the more equity (bond) investors there are already in the market, the more attractive equity (fixed income) investment becomes for the sponsor.

For the sake of simplicity, the perfectly elastically supplied bond (cash) is assumed to pay no interest, such that its benefit is zero. The resulting transition rates between equities and bonds read then in analogy to (12)

$$p_{BE} = v_B e^{\alpha_B (n_E - n_B)/N} \quad \text{and} \quad p_{EB} = v_B e^{-\alpha_B (n_E - n_B)/N}, \tag{14}$$

where $\alpha_B$ is the strength of infection parameter between equity and bonds and $v_B$ denotes the frequency at which asset allocators reconsider their strategy.

In the next step we need to specify, how the transitions between equity and bonds on asset allocation level translate into transition probabilities between the individual stock investors and the bondholders. Keeping in mind that asset allocation and security selection are to be modeled as separate processes, I shall assume here that the asset allocation decision leaves the internal composition of stock investors unchanged. That is, the transition rates from each individual stock investor to bondholders equal just the transition rates between equity and bonds

$$p_{ciB} = p_{fiB} = p_{EB}, \quad i = 1, 2, \tag{15}$$

whereas transitions from the bondholders to the equity investors must be weighted by the relative frequency of the relevant stock investor type

$$p_{Bci} = \frac{n_{ci}}{n_E} p_{BE}, \quad p_{Bfi} = \frac{n_{fi}}{n_E} p_{BE}, \; i = 1, 2. \tag{16}$$

These may then be inserted into the quasi-mean value equations (7) in order to obtain for the population dynamics:

$$\begin{aligned} \dot{n_{c1}} = v_B\, n_{c1} &\left( \frac{n_B}{n_E} e^{\alpha_B (n_E - n_B)/N} - e^{-\alpha_B (n_E - n_B)/N} \right) \\ &+ v \Big[ n_{c2} e^{\alpha(C_1 - C_2)} + n_{f1} e^{\alpha(C_1 - F_1)} + n_{f2} e^{\alpha(C_1 - F_2)} \\ &\quad - n_{c1} e^{\alpha(C_2 - C_1)} - n_{c1} e^{\alpha(F_1 - C_1)} - n_{c1} e^{\alpha(F_2 - C_1)} \Big] \end{aligned} \tag{17a}$$

$$\begin{aligned} \dot{n_{c2}} = v_B\, n_{c2} &\left( \frac{n_B}{n_E} e^{\alpha_B (n_E - n_B)/N} - e^{-\alpha_B (n_E - n_B)/N} \right) \\ &+ v \Big[ n_{c1} e^{\alpha(C_2 - C_1)} + n_{f1} e^{\alpha(C_2 - F_1)} + n_{f2} e^{\alpha(C_2 - F_2)} \\ &\quad - n_{c2} e^{\alpha(C_1 - C_2)} + n_{c2} e^{\alpha(F_1 - C_2)} - n_{c2} e^{\alpha(F_2 - C_2)} \Big] \end{aligned} \tag{17b}$$

$$\begin{aligned} \dot{n_{f1}} = v_B\, n_{f1} &\left( \frac{n_B}{n_E} e^{\alpha_B (n_E - n_B)/N} - e^{-\alpha_B (n_E - n_B)/N} \right) \\ &+ v \Big[ n_{c1} e^{\alpha(F_1 - C_1)} - n_{c2} e^{\alpha(F_1 - C_2)} + n_{f2} e^{\alpha(F_1 - F_2)} \\ &\quad - n_{f1} e^{\alpha(C_1 - F_1)} - n_{f1} e^{\alpha(C_2 - F_1)} - n_{f1} e^{\alpha(F_2 - F_1)} \Big] \end{aligned} \tag{17c}$$

$$\begin{aligned} \dot{n_{f2}} = v_B\, n_{f2} &\left( \frac{n_B}{n_E} e^{\alpha_B (n_E - n_B)/N} - e^{-\alpha_B (n_E - n_B)/N} \right) \\ &+ v \Big[ n_{c1} e^{\alpha(F_2 - C_1)} - n_{c2} e^{\alpha(F_2 - C_2)} + n_{f1} e^{\alpha(F_2 - F_1)} \\ &\quad - n_{f2} e^{\alpha(C_1 - F_2)} - n_{f2} e^{\alpha(C_2 - F_2)} - n_{f2} e^{\alpha(F_1 - F_2)} \Big] \end{aligned} \tag{17d}$$

Combining the time development of the asset prices (6) with the population dynamics (17) one obtains a closed systems of highly non-linear differential equations with state variables $p_1$, $p_2$, $n_{c1}$, $n_{c2}$, $n_{f1}$ and $n_{f2}$. It turns out that this system has a "fundamental"equilibrium, in which the trading prices of both assets equal their respective fundamental values with balanced disposition among traders as detailed below.

**Proposition 1 (Existence of a fundamental equilibrium)**
*The market with separate asset allocation has a fundamental equilibrium at*

$$n_B = n_E = N/2, \quad n_{c1} = n_{c2} = n_{f1} = n_{f2} = n_E/4 = N/8.$$

*Proof* see Appendix.

Intuitively, the equilibrium conditions follow quite naturally from the structure of the quasi-mean value equations (7) as follows. Consider first the subdynamics of the equity investor populations. At fundamental equilibrium both fundamentalist benefits equal zero, because all trading prices equal their fundamental values. Also the chartist benefits equal zero when there are equally many equity investors in stock 1 and 2. All transition probabilities between equity investments in the quasi-mean value equations equal then $v$, such that (7) simplifies for the subdynamics between equity investors to

$$\dot{n_i} = v \cdot \sum_{j \neq i}^{4} (n_j - n_i), \quad n_i, n_j = n_{c1}, n_{c2}, n_{f1}, n_{f2}. \tag{18}$$

It is then immediately clear from (18) that zero expected changes for all trader populations imply that there are equally many investors in each of the equity strategy subpopulations. The same argument applies for the asset allocation subdynamics, thereby implying equally many equity and bond investors.

Employment of absolute values in the fundamentalist utilities (9) implies that the system of differential equations (6) and (17) contains four subregimes ($p_1 > p_{f1},\ p_2 > p_{f2}$), ($p_1 < p_{f1},\ p_2 < p_{f2}$), ($p_1 > p_{f1},\ p_2 < p_{f2}$), and ($p_1 < p_{f1},\ p_2 > p_{f2}$). Necessary conditions for simultaneous stability of the fundamental equilibrium with respect to the regime-specific dynamics are detailed in Proposition 2 below. Note however, that stability with respect to the regime-specific dynamics is in general neither a sufficient nor necessary condition for stability of the overall dynamics. E.g. Honkapohja and Ito (1983) provide several examples demonstrating that stable regimes may very well be patched into an unstable system when a trajectory crosses boundaries at a series of points which become further and further displaced from the equilibrium, or a solution path slides along a boundary in a direction divergent from the equilibrium point. Proposition 2 serves therefore only as a general guideline, which factors may have an impact upon local stability of the fundamental equilibrium within the overall dynamics.[9]

**Proposition 2 (Local stability with respect to regime-specific dynamics)**
*The following are necessary conditions for simultaneous local stability of the fundamental equilibrium with respect to all four subregimes:*

[9] Simulations at the suspected rim of stability suggest that condition 1 ($\alpha_B \leq 1$) of Proposition 2 does indeed mark the boundary between stability and instability of the fundamental equilibrium. Large strength of infection parameters $\alpha$ within the equity investor subpopulations (condition 2), on the other hand, were found to converge to a continuum of additional equilibria. (I am grateful to one of the referees, whose request for simulations at the suspected rim of stability led to their discovery.) These are characterized by balanced populations of bond and equity investors as well as holders of stock 1 and stock 2, but unbalanced disposition of traders following a chartist or a fundamentalist strategy and trading prices deviating from their fundamental values. The exact value of $\alpha$ where this happens is difficult to determine, because both the mispricing and the difference between the strategy subpopulations are continuous functions of $\alpha$ and coincide with the fundamental equilibrium for $\alpha = 0$. Note also that the simulated Markov chain with transition probabilities (12) and (14)–(16) adds many random variables to the deterministic quasi-mean value dynamics (17), for which the stability conditions in Proposition 2 have been derived. I have therefore no experimental evidence for or against validity of condition 2.

1. $$\alpha_B \le 1,$$

2. $$\begin{array}{ll} \alpha(1+\sqrt{1+16ls}) \le 4 & \textit{for } ls \le 3/2, \\ \alpha ls \le 1 & \textit{for } ls \ge 3/2, \end{array}$$

$$\textit{with} \quad l := t_c/t_f$$

*Proof* see Appendix.

The above conditions for local stability of the fundamental equilibrium with respect to the regime-specific dynamics conform with intuition. Large strength of attraction parameters imply that small deviations from equilibrium trigger fast changes in the trader populations, leading to fast price changes as well. Large holdings of chartists relative to fundamentalists speed up price changes as was already mentioned in the discussion of (6). Large discount factors have a similar effect in speeding up population changes by their inclusion into the transition rates between equity investors (12) through the fundamentalist benefits (9).

## 5 Simulation analysis

We shall in the following simulate the artificial market defined by Eqs. (6) and (17) along the same lines as in Lux and Marchesi (2000). That is, we consider an ensemble of $N = 500$ traders with asynchronous updating of strategies approximated by finite time increments of size $\Delta t = 0.002$ in the domain of attraction of the fundamental equilibrium. In order to initialize the simulations both trading prices are set to their fundamental value, while the numbers of chartists and fundamentalists in each stock are set to 62 and 63, respectively, close to their fundamental equilibrium value of $500/8 = 62.5$ identified in Proposition 1. Similar as in Lux, the parameter set depicted in Table 3 has been chosen using the criterion that the bandwidth for returns over unit time steps should roughly conform to what one usually observes for daily data in financial markets. Note that asset allocators operate on a faster time scale than equity portfolio managers.

Consider first the plot of logarithmic trading prices in Fig. 6, where the logarithmic fundamental prices of both stocks were set to zero. Obviously the model is capable of generating both severe crashes and long lasting bubbles. At its most extreme observation, stock 2 trades at almost ten times its intrinsic value. Substantial deviations between fundamental and trading price may occur for several hundred observations in a row, corresponding to time spans of a year and above in real markets.

Figure 7 contains the simulated logreturns of the two stocks in the upper panels together with the logreturns of the equal and the capitalization weighted index in the

**Table 3** Parameter set used in the simulations

| $p_{f1}$ | $p_{f2}$ | $v$ | $v_B$ | $\alpha$ | $\alpha_B$ | $l$ | s |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 0.001 | 0.04 | 0.1 | 0.4 | 0.5 | 0.8 |

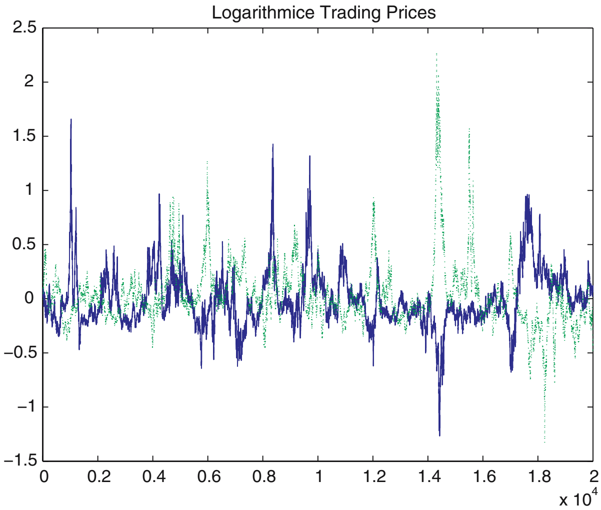


**Fig. 6** Logarithmic trading prices $p_1$ (*dark solid line*) and $p_2$ (*light dotted line*) for the two risky stocks over 20,000 observations using the parameters denoted in Table 3. The logarithmic intrinsic values of both stocks are $p_{f1} = p_{f2} = 0$. Substantial deviations between fundamental and trading prices may occur for several hundred observations in a row

lower panels. All time series are clearly heteroscedastic exhibiting the same intermittent behaviour as those presented in Lux and Marchesi (2000) and frequently observed in financial markets.

Table 4 contains summary statistics for the above mentioned return series. All time series are close to symmetric, with daily absolute returns in the range of 1–2 percent and an annual volatility in the range between 30 and 40 percent. Because the individual stock returns are cross-sectionally close to uncorrelated ($\rho = -0.0195$),[10] the variance of the equal weighted index is about half the variance of the individual stock returns. All time series are heavily leptokurtic with double digit coefficients of kurtosis.

Cross-sectionally uncorrelated returns are of course not a realistic feature of financial markets. Note however, that this simplistic model assumes constant fundamental values, which are not realistic either. It can be easily extended along the same lines as in Lux and Marchesi (1999) by assuming that the intrinsic values of both stocks follow correlated unit root processes. Uncorrelatedness in the current setup would then presumably translate into identical correlations between fundamental and trading returns of the unit root price processes, which may very well be consistent with the positively correlated returns observed in real markets, as far as fundamental values are positively correlated. I am not aware of any order-based study, which would have

[10] Correlation coefficients marginally below zero are representative for asset allocation processes with faster dynamics than the security and strategy selection process within equities. Aligning the speed of both processes or even eliminating asset allocation leads to clearly negative correlations ($\approx -0.3$). On the other hand, eliminating the security selection process but leaving the asset allocation intact, may generate positive correlations between the individual stock returns.

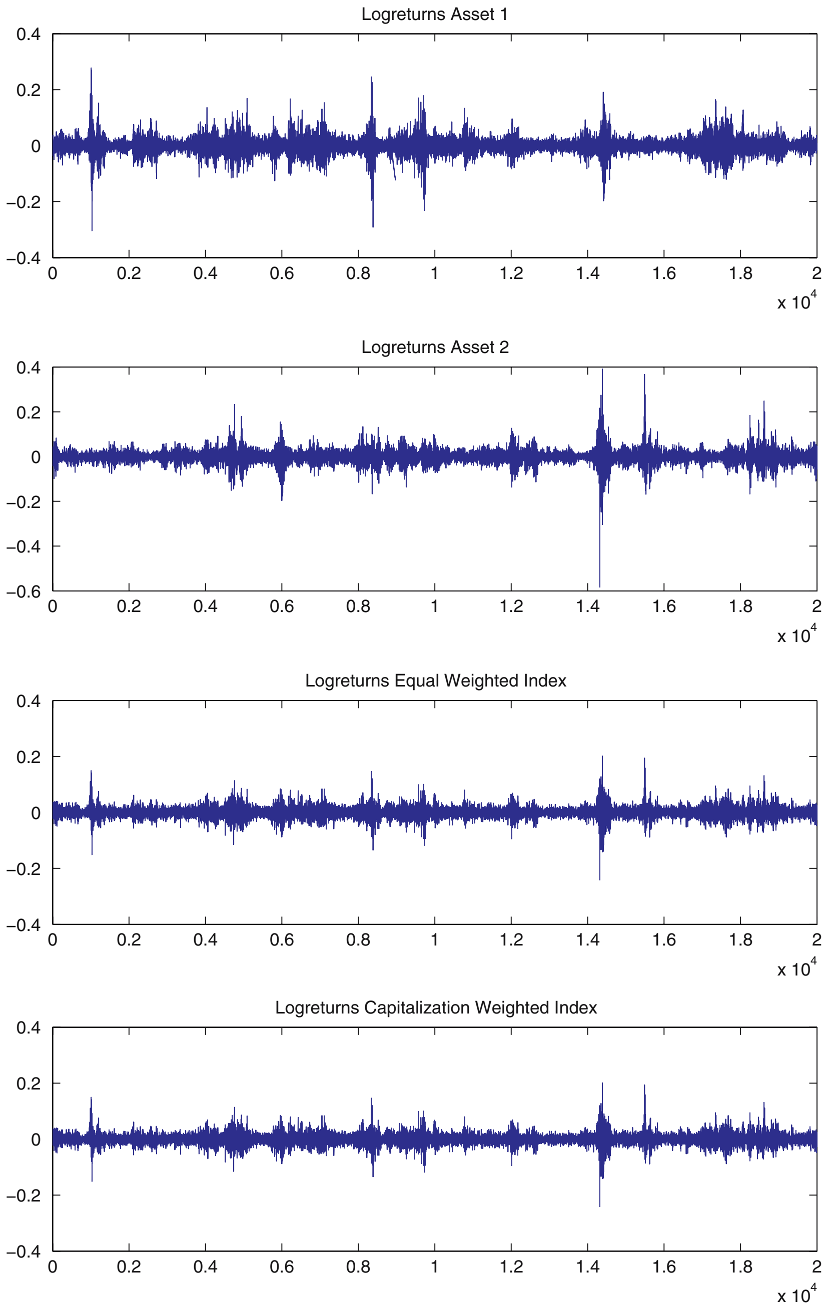


**Fig. 7** Logreturns of the two stocks, the equal weighted index, and the capitalization weighted index over 20,000 observations using the parameters denoted in Table 3. All time series exhibit volatility clustering similar to empirically observed financial market returns

**Table 4** Summary statistics for the simulated logreturns of the two stocks, the equal weighted index and the capitalization weighted index

| Asset | Avg. raw Ret.$\times 10^{-3}$ | Avg. abs. Return | Return Var.$\times 10^{-3}$ | Return Skewness | Return Kurtosis |
|---|---|---|---|---|---|
| Stock 1 | 0.0040 | 0.0180 | 0.7071 | 0.0773 | 12.07 |
| Stock 2 | −0.0231 | 0.0179 | 0.7251 | −0.1424 | 25.88 |
| Index (EW) | 0.1728 | 0.0133 | 0.3514 | 0.1331 | 10.47 |
| Index (CW) | −0.0077 | 0.0144 | 0.4886 | −0.3443 | 37.58 |

reported cross-sectionally uncorrelated or even positively correlated return series, as empirically observed equity market data would require.

Lux and Marchesi demonstrate for their market that outbreaks of volatility are related to the fraction of traders acting as chartists. A similar relation holds here between volatility and the number of stocks owned by chartists, as is illustrated in Fig. 8. The upper two panels show chartists and fundamentalists holdings normalized at $t_c = 1$ (choosing any other value of $t_c$ just changes the scale of the plot) and the lower panel shows chartists and fundamentalist holdings as a fraction of total equity investment. Comparing Figs. 7 and 8 reveals that periods of high volatility tend to coincide with above average chartist positions. Volatility clusters occur typically for large chartist holdings because these usually coincide with only a few investors pursuing a fundamentalist strategy in the relevant stock, which was seen in Eq. (6) to speed up price changes.

Note that contrary to the simulations of the order-based setup discussed in Sect. 2 the trader holdings are stationary in levels in this model, because traders holdings rather than orders have been linked to the level stationary mispricings and trader populations in Eq. (4). The presence of a unit root is strongly rejected in all tests of any traders positions (results not shown here). This confirms our conjecture that the risk of building up infinite inventories is not present in our model, consistent with the conservation of the number of shares discussed in Sect. 3.

Consider finally the wealth dynamics for the two types of traders illustrated in Fig. 9.[11] The upper panel contains the amount of cash aggregated by the chartist and fundamentalist subpopulations, whereas the lower panel includes also the market value of the stocks. It is immediately evident from both plots that chartists loose their money to fundamentalists, as was the case in the original model by Lux and Marchesi. While one might again be tempted to conclude that chartist will go bankrupt and disappear, this need not be so for the same reasons as discussed already in Sect. 2.

## 6 The stylized facts

In order to demonstrate concordance of our simulated time series with the stylized facts of financial returns, we shall in the following apply the same battery of tests to

[11] The accumulated wealth of time-varying trader subpopulation does not coincide with the wealth of traders pursuing the same strategy forever, because the former takes the average timing decision of agents entering and leaving the strategy into account, which the latter does not.

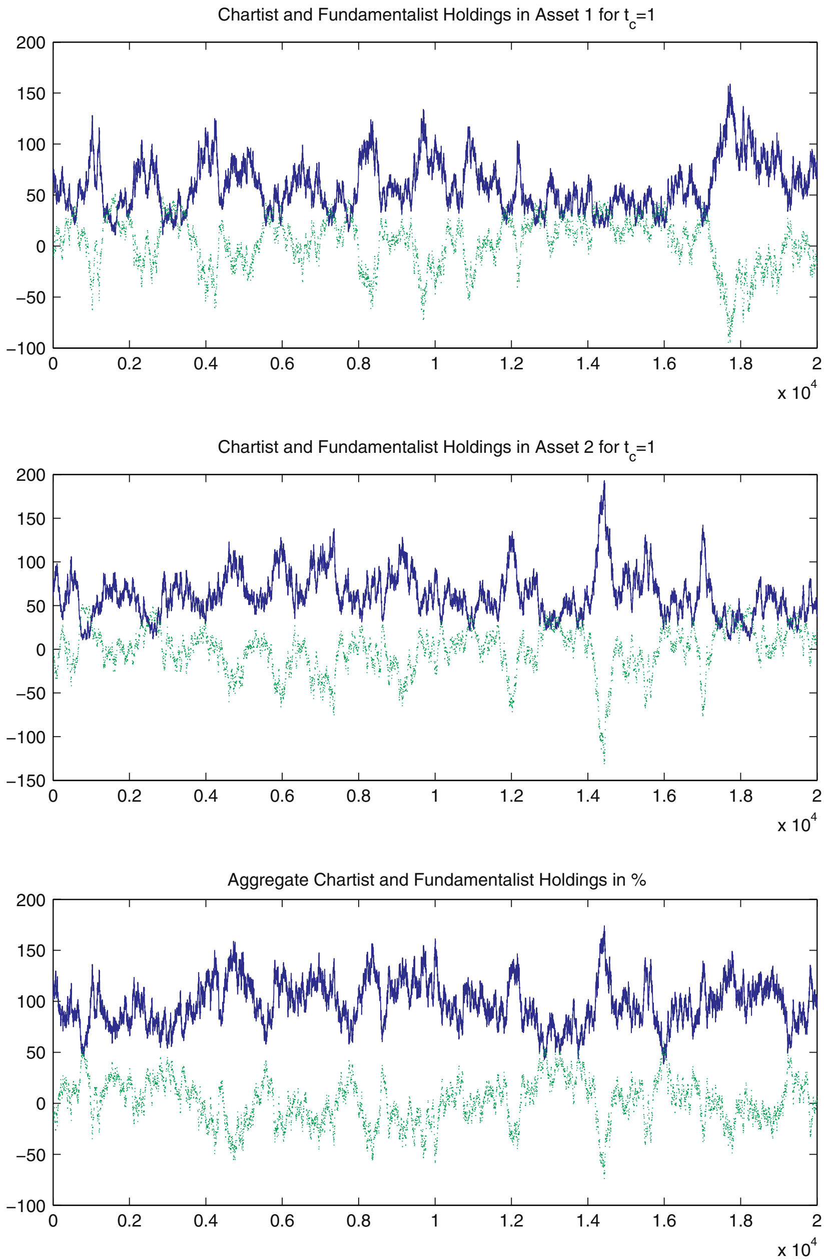


**Fig. 8** Chartist and fundamentalist holdings in the individual stocks normalized at $t_c = 1$ (*upper two panels*), and as a fraction of total equity investment (*lower panel*) using the parameters denoted in Table 3. The *dark solid lines* denote chartist holdings, and the *light dotted lines* fundamentalist holdings

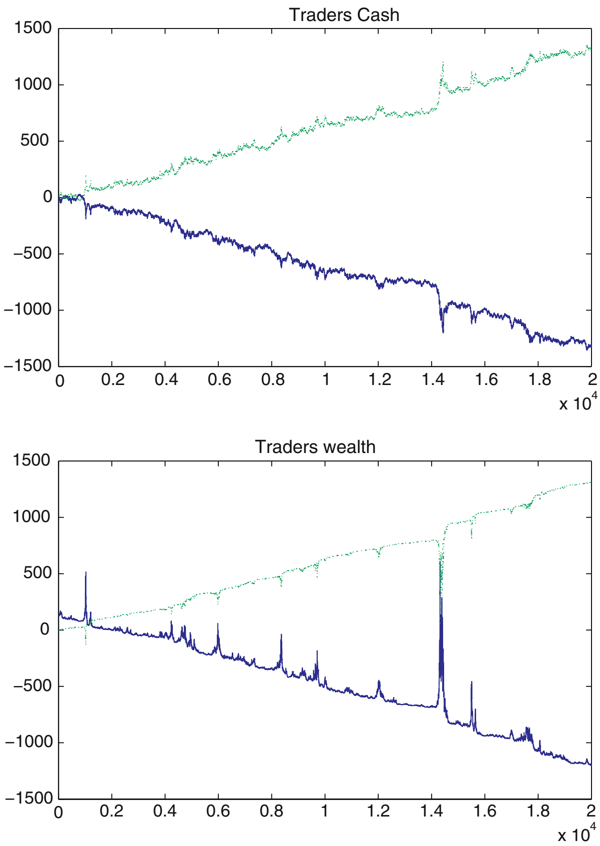


**Fig. 9** Traders cash and total wealth for fundamentalists (*light dotted line*) and chartists (*dark solid line*) using the parameters denoted in Table 3. Chartists loose their money to fundamentalists

them as in Lux and Marchesi (2000).[12] Consider first the fat tail property. The kurtosis for the return series of the returns series of the individual stocks and the stock indices were already given in Table 4. All of them were double digit numbers consistent with empirical findings.

As regards the tail index of the series, we follow Lux and Marchesi in splitting each of our datasets into 10 subsamples of 2,000 observations, and applying the Hill estimator with varying cut-off values to each of them. The results are presented in Table 5. We find tail indices somewhere between 2 and 5 with increasing estimates for decreasing tail size, just like in their study and in harmony with empirical findings.

[12] We shall not repeat the application of the Dickey–Fuller test to simulated prices. Lux and Marchesi claimed erroneously the existence of a unit root in their simulated price process, which was due to a flawed application of the Dickey–Fuller test (personal communication with Thomas Lux). Both their and my model generate level stationary trading prices under the assumption of constant intrinsic values, contrary to empirical findings. Non stationarity of trading prices can however easily be introduced by assuming unit root processes for the fundamental values as in Lux and Marchesi (1999).

**Table 5** Median estimates of the tail index over 10 samples of 2,000 observations each and the range of estimates for common choices of the tail region

| Asset | 2.5% tail | | | 5% tail | | | 10% tail | | |
|---|---|---|---|---|---|---|---|---|---|
| | Min | Median | Max | Min | Median | Max | Min | Median | Max |
| Stock 1 | 2.55 | 4.49 | 5.71 | 2.25 | 3.68 | 4.53 | 1.95 | 3.15 | 3.64 |
| Stock 2 | 2.76 | 4.28 | 5.16 | 2.74 | 3.70 | 4.37 | 2.13 | 3.15 | 3.60 |
| Index (EW) | 2.68 | 4.41 | 5.58 | 2.89 | 3.82 | 4.81 | 2.27 | 3.18 | 3.66 |
| Index (CW) | 1.92 | 4.30 | 5.62 | 2.13 | 3.72 | 4.34 | 1.85 | 2.88 | 3.55 |

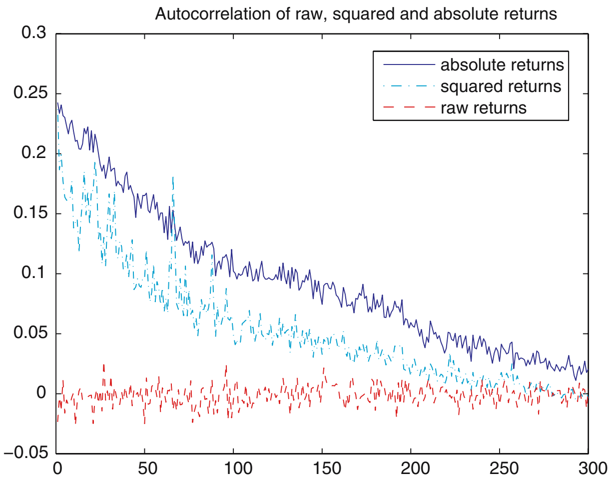


**Fig. 10** Autocorrelation diagram of absolute (*dark solid line*), squared (*light dashed line*) and raw returns (*dark dashed line*) over 300 lags using the parameters denoted in Table 3. The slow decay in the autocorrelation coefficients of absolute and squared returns is indicative of long memory in volatility

Consider next the autocorrelation diagram of raw, squared and absolute returns of the equal weighted index in Fig. 10. Similar to Lux and Marchesi and consistent with empirical findings, squared and absolute returns show much higher autocorrelation than raw returns with only minor fluctuations around zero. Autocorrelation coefficients of absolute returns do not even decay to zero when considering 300 lags, which is consistent with both empirically observed data and long memory in return volatility.

In order to test formally for long memory in return volatility, I follow again Lux and Marchesi in subdividing all datasets into 10 subsamples of 2,000 observations and applying the Geweke/Porter–Hudak estimator of the fractional differencing parameter $d$ of a fractionally integrated ARMA model to each of them. Similar to their study, the results presented in Table 6 indicate evidence for long-term dependence with $d$ estimated significantly larger than zero for most of the tests. Many of the estimates fall even into the region $d > 0.5$ indicating explosive volatility processes as in their study.

**Table 6** Estimates of the long memory parameter $d$ for squared returns (upper panel) and absolute returns (lower panel) for both stocks, the equal weighted index, and the capitalization weighted index over the full sample of 20,000 observations and over 10 subsamples of 2,000 observations each

| | $\hat{d}$ full sample | $\max(\hat{d})$ | 10 samples median$(\hat{d})$ | $\min(\hat{d})$ | # $\hat{d}$ sign. $> 0$ in 10 samples |
|---|---|---|---|---|---|
| Squared returns | | | | | |
| Stock 1 | 0.27 | 0.31 | 0.56 | 0.90 | 8 |
| Stock 2 | 0.40 | 0.27 | 0.56 | 0.77 | 9 |
| Index (EW) | 0.35 | 0.21 | 0.51 | 0.74 | 8 |
| Index (CW) | 0.37 | 0.29 | 0.58 | 0.69 | 8 |
| Absolute returns | | | | | |
| Stock 1 | 0.46 | 0.40 | 0.65 | 0.73 | 9 |
| Stock 2 | 0.49 | 0.41 | 0.60 | 0.86 | 9 |
| Index (EW) | 0.48 | 0.22 | 0.55 | 0.74 | 7 |
| Index (CW) | 0.41 | 0.33 | 0.61 | 0.83 | 7 |

The last column contains the number of significantly positive estimated long memory parameters at a significance level of 5%. All estimates of $d$ over the full sample are significantly positive

However, when using the full datasets, all estimates remain below 0.5 but significantly positive, as they should for stationary long memory processes.

## 7 Conclusion

I have extended the univariate artificial market by Lux and Marchesi into a multivariate setup by including a second risky asset and a bond. The order-based trading strategies of their model were replaced by corresponding position-based strategies in order avoid inconsistencies of the original model, which became first evident after inspecting traders inventories in the original setup. In order to add further realism to the model, asset allocation and security selection have been modeled as two separate decision processes, in line with common practice in financial institutions.

The simulated return series of this artificial market share the stylized facts of financial returns. Serially uncorrelated returns with volatility clustering, leptokurtic return distributions with realistic tail indexes, and long memory in squared and absolute returns were all observed, both for the individual stocks and for the stock indexes. Assuming constant intrinsic values for both stocks, the individual stock returns were found to be close to cross-sectionally uncorrelated. Future work will extend the results of this study to multiple stocks, whose intrinsic values follow unit root processes of varying correlation structure.

**Acknowledgments** I wish to thank Thomas Lux for his patience in explaining to me the working of his model, Wolfgang Weidlich for clearing away my doubts regarding the applicability of the quasi-mean value equations, Matti Laaksonen for always being there, and Seppo Pynnönen for his continuous effort to keep me on track. Financial support from Osuuspankkiryhmän Tutkimussäätiö, Ella ja Georg Ehrnroothin Säätiö and a travel grant from Magnus Ehrnroothin Säätiö are gratefully acknowledged.

## Appendix

### A Proof of Proposition 1

A fundamental equilibrium requires

$$\dot{n_{c1}} = \dot{n_{c2}} = \dot{n_{f1}} = \dot{n_{f2}} = 0 \quad \text{at } p_1 \equiv p_{f1} \quad \text{and} \quad p_2 \equiv p_{f2}. \tag{A.1}$$

Using the identity

$$\begin{aligned} & n_j e^{\alpha(U_i - U_j)} - n_i e^{\alpha(U_j - U_i)} \\ & \quad = (n_i + n_j)\left[\tanh(\alpha(U_i - U_j)) - \frac{n_i - n_j}{n_i + n_j}\right]\cosh(\alpha(U_i - U_j)) \end{aligned} \tag{A.2}$$

the equations of motion for the trader populations (17) may be rewritten in terms of hyperbolic funcions as

$$\begin{aligned} \dot{n_{c1}} = {} & v_B\, n_{c1}\left[\frac{n_B}{n_E} e^{\alpha_B (n_E - n_B)/N} - e^{-\alpha_B (n_E - n_B)/N}\right] \\ & + v\left\{ (n_{c1} + n_{c2})\left[\tanh(\alpha(C_1 - C_2)) - \frac{n_{c1} - n_{c2}}{n_{c1} + n_{c2}}\right]\cosh(\alpha(C_1 - C_2)) \right. \\ & \quad + (n_{c1} + n_{f1})\left[\tanh(\alpha(C_1 - F_1)) - \frac{n_{c1} - n_{f1}}{n_{c1} + n_{f1}}\right]\cosh(\alpha(C_1 - F_1)) \\ & \quad \left. + (n_{c1} + n_{f2})\left[\tanh(\alpha(C_1 - F_2)) - \frac{n_{c1} - n_{f2}}{n_{c1} + n_{f2}}\right]\cosh(\alpha(C_1 - F_2)) \right\} \end{aligned} \tag{A.3a}$$

$$\begin{aligned} \dot{n_{c2}} = {} & v_B\, n_{c2}\left[\frac{n_B}{n_E} e^{\alpha_B (n_E - n_B)/N} - e^{-\alpha_B (n_E - n_B)/N}\right] \\ & + v\left\{ (n_{c2} + n_{c1})\left[\tanh(\alpha(C_2 - C_1)) - \frac{n_{c2} - n_{c1}}{n_{c2} + n_{c1}}\right]\cosh(\alpha(C_2 - C_1)) \right. \\ & \quad + (n_{c2} + n_{f1})\left[\tanh(\alpha(C_2 - F_1)) - \frac{n_{c2} - n_{f1}}{n_{c2} + n_{f1}}\right]\cosh(\alpha(C_2 - F_1)) \\ & \quad \left. + (n_{c2} + n_{f2})\left[\tanh(\alpha(C_2 - F_2)) - \frac{n_{c2} - n_{f2}}{n_{c2} + n_{f2}}\right]\cosh(\alpha(C_2 - F_2)) \right\} \end{aligned} \tag{A.3b}$$

$$
\begin{aligned}
\dot{n_{f1}} = v_B\, n_{f1} & \left[\frac{n_B}{n_E} e^{\alpha_B (n_E - n_B)/N} - e^{-\alpha_B (n_E - n_B)/N}\right] \\
& + v \left\{ (n_{f1} + n_{c1}) \left[\tanh(\alpha(F_1 - C_1)) - \frac{n_{f1} - n_{c1}}{n_{f1} + n_{c1}}\right] \cosh(\alpha(F_1 - C_1)) \right. \\
& \quad + (n_{f1} + n_{c2}) \left[\tanh(\alpha(F_1 - C_2)) - \frac{n_{f1} - n_{c2}}{n_{f1} + n_{c2}}\right] \cosh(\alpha(F_1 - C_2)) \\
& \quad \left. + (n_{f1} + n_{f2}) \left[\tanh(\alpha(F_1 - F_2)) - \frac{n_{f1} - n_{f2}}{n_{f1} + n_{f2}}\right] \cosh(\alpha(F_1 - F_2)) \right\}
\end{aligned} \tag{A.3c}
$$

$$
\begin{aligned}
\dot{n_{f2}} = v_B\, n_{f2} & \left[\frac{n_B}{n_E} e^{\alpha_B (n_E - n_B)/N} - e^{-\alpha_B (n_E - n_B)/N}\right] \\
& + v \left\{ (n_{f2} + n_{c1}) \left[\tanh(\alpha(F_2 - C_1)) - \frac{n_{f2} - n_{c1}}{n_{f2} + n_{c1}}\right] \cosh(\alpha(F_2 - C_1)) \right. \\
& \quad + (n_{f2} + n_{c2}) \left[\tanh(\alpha(F_2 - C_2)) - \frac{n_{f2} - n_{c2}}{n_{f2} + n_{c2}}\right] \cosh(\alpha(F_2 - C_2)) \\
& \quad \left. + (n_{f2} + n_{f1}) \left[\tanh(\alpha(F_2 - F_1)) - \frac{n_{f2} - n_{f1}}{n_{f2} + n_{f1}}\right] \cosh(\alpha(F_2 - F_1)) \right\}
\end{aligned} \tag{A.3d}
$$

In order to fulfil the condition (A.1) it suffices that all squared brackets above equal zero. That is the case when both

$$
n_{c1} = n_{c2} = n_{f1} = n_{f2} = n_E/4 \quad \text{and} \quad n_B = n_E = N/2, \quad \text{as claimed.}
$$

## B Proof of Proposition 2

Local stability with respect to regime-specific dynamics will be considered by inspecting the Jacobian of the system of differential equations for the trader populations and prices.[13] We will for that purpose reformulate the population dynamics (A.3) in terms of the new variables

$$
c_1 := \frac{n_{c1}}{N}, \quad c_2 := \frac{n_{c2}}{N}, \quad f_1 := \frac{n_{f1}}{N}, \quad f_2 := \frac{n_{f2}}{N} \tag{B.1}
$$

[13] see e.g. Gandolfo (1996).

as

$$
\begin{aligned}
\dot{c_1} = v_B\ c_1 &\left[\frac{1-c_1-c_2-f_1-f_2}{c_1+c_2+f_1+f_2} e^{\alpha_B(2(c_1+c_2+f_1+f_2)-1)} - e^{-\alpha_B(2(c_1+c_2+f_1+f_2)-1)}\right] \\
&+ v\left\{ (c_1+c_2)\left[\tanh(\alpha(C_1-C_2)) - \frac{c_1-c_2}{c_1+c_2}\right]\cosh(\alpha(C_1-C_2)) \right. \\
&+ (c_1+f_1)\left[\tanh(\alpha(C_1-F_1)) - \frac{c_1-f_1}{c_1+f_1}\right]\cosh(\alpha(C_1-F_1)) \\
&\left. +(c_1+f_2)\left[\tanh(\alpha(C_1-F_2)) - \frac{c_1-f_2}{c_1+f_2}\right]\cosh(\alpha(C_1-F_2))\right\} \qquad \text{(B.2a)} \\
\dot{c_2} = v_B\ c_2 &\left[\frac{1-c_1-c_2-f_1-f_2}{c_1+c_2+f_1+f_2} e^{\alpha_B(2(c_1+c_2+f_1+f_2)-1)} - e^{-\alpha_B(2(c_1+c_2+f_1+f_2)-1)}\right] \\
&+ v\left\{ (c_2+c_1)\left[\tanh(\alpha(C_2-C_1)) - \frac{c_2-c_1}{c_2+c_1}\right]\cosh(\alpha(C_2-C_1)) \right. \\
&+ (c_2+f_1)\left[\tanh(\alpha(C_2-F_1)) - \frac{c_2-f_1}{c_2+f_1}\right]\cosh(\alpha(C_2-F_1)) \\
&\left. +(c_2+f_2)\left[\tanh(\alpha(C_2-F_2)) - \frac{c_2-f_2}{c_2+f_2}\right]\cosh(\alpha(C_2-F_2))\right\} \qquad \text{(B.2b)} \\
\dot{f_1} = v_B\ f_1 &\left[\frac{1-c_1-c_2-f_1-f_2}{c_1+c_2+f_1+f_2} e^{\alpha_B(2(c_1+c_2+f_1+f_2)-1)} - e^{-\alpha_B(2(c_1+c_2+f_1+f_2)-1)}\right] \\
&+ v\left\{ (f_1+c_1)\left[\tanh(\alpha(F_1-C_1)) - \frac{f_1-c_1}{f_1+c_1}\right]\cosh(\alpha(F_1-C_1)) \right. \\
&+ (f_1+c_2)\left[\tanh(\alpha(F_1-C_2)) - \frac{f_1-c_2}{f_1+c_2}\right]\cosh(\alpha(F_1-C_2)) \\
&\left. +(f_1+f_2)\left[\tanh(\alpha(F_1-F_2)) - \frac{f_1-f_2}{f_1+f_2}\right]\cosh(\alpha(F_1-F_2))\right\} \qquad \text{(B.2c)} \\
\dot{f_2} = v_B\ f_2 &\left[\frac{1-c_1-c_2-f_1-f_2}{c_1+c_2+f_1+f_2} e^{\alpha_B(2(c_1+c_2+f_1+f_2)-1)} - e^{-\alpha_B(2(c_1+c_2+f_1+f_2)-1)}\right] \\
&+ v\left\{ (f_2+c_1)\left[\tanh(\alpha(F_2-C_1)) - \frac{f_2-c_1}{f_2+c_1}\right]\cosh(\alpha(F_2-C_1)) \right. \\
&+ (f_2+c_2)\left[\tanh(\alpha(F_2-C_2)) - \frac{f_2-c_2}{f_2+c_2}\right]\cosh(\alpha(F_2-C_2)) \\
&\left. +(f_2+f_1)\left[\tanh(\alpha(F_2-F_1)) - \frac{f_2-f_1}{f_2+f_1}\right]\cosh(\alpha(F_2-F_1))\right\}. \qquad \text{(B.2d)}
\end{aligned}
$$

The price dynamics (6) may be rewritten in terms of (B.1) as

$$\dot{p}_i = \frac{1}{f_i}\left(l\dot{c}_i + (p_{fi} - p_i)\dot{f}_i\right), \quad i = 1, 2, \qquad \text{(B.3)}$$

where we have used the leverage parameter $l := t_c/t_f$ introduced in Proposition 2 in order to express the relation of chartist relative to fundamentalist target holdings. The entries of the Jakobian matrix

$$J = \begin{pmatrix} \frac{\partial \dot{c}_1}{\partial c_1} & \frac{\partial \dot{c}_1}{\partial c_2} & \frac{\partial \dot{c}_1}{\partial f_1} & \frac{\partial \dot{c}_1}{\partial f_2} & \frac{\partial \dot{c}_1}{\partial p_1} & \frac{\partial \dot{c}_1}{\partial p_2} \\ \frac{\partial \dot{c}_2}{\partial c_1} & \frac{\partial \dot{c}_2}{\partial c_2} & \frac{\partial \dot{c}_2}{\partial f_1} & \frac{\partial \dot{c}_2}{\partial f_2} & \frac{\partial \dot{c}_2}{\partial p_1} & \frac{\partial \dot{c}_2}{\partial p_2} \\ \frac{\partial \dot{f}_1}{\partial c_1} & \frac{\partial \dot{f}_1}{\partial c_2} & \frac{\partial \dot{f}_1}{\partial f_1} & \frac{\partial \dot{f}_1}{\partial f_2} & \frac{\partial \dot{f}_1}{\partial p_1} & \frac{\partial \dot{f}_1}{\partial p_2} \\ \frac{\partial \dot{f}_2}{\partial c_1} & \frac{\partial \dot{f}_2}{\partial c_2} & \frac{\partial \dot{f}_2}{\partial f_1} & \frac{\partial \dot{f}_2}{\partial f_2} & \frac{\partial \dot{f}_2}{\partial p_1} & \frac{\partial \dot{f}_2}{\partial p_2} \\ \frac{\partial \dot{p}_1}{\partial c_1} & \frac{\partial \dot{p}_1}{\partial c_2} & \frac{\partial \dot{p}_1}{\partial f_1} & \frac{\partial \dot{p}_1}{\partial f_2} & \frac{\partial \dot{p}_1}{\partial p_1} & \frac{\partial \dot{p}_1}{\partial p_2} \\ \frac{\partial \dot{p}_2}{\partial c_1} & \frac{\partial \dot{p}_2}{\partial c_2} & \frac{\partial \dot{p}_2}{\partial f_1} & \frac{\partial \dot{p}_2}{\partial f_2} & \frac{\partial \dot{p}_2}{\partial p_1} & \frac{\partial \dot{p}_2}{\partial p_2} \end{pmatrix} \tag{B.4}$$

of the coupled system (B.2) and (B.3) evaluated at the fundamental equilibrium

$$c_1 = c_2 = f_1 = f_2 = 1/8 \tag{B.5}$$

read for the population subdynamics

$$\frac{\partial \dot{c}_1}{\partial c_1} = \frac{\partial \dot{c}_2}{\partial c_2} = (\alpha - 3)v + (\alpha_B - 1)\frac{v_B}{2} \tag{B.6a}$$
$$\frac{\partial \dot{c}_1}{\partial c_2} = \frac{\partial \dot{c}_1}{\partial f_2} = \frac{\partial \dot{c}_2}{\partial c_1} = \frac{\partial \dot{c}_2}{\partial f_1} = (\alpha - 1)v + (\alpha_B - 1)\frac{v_B}{2} \tag{B.6b}$$
$$\frac{\partial \dot{c}_1}{\partial f_1} = \frac{\partial \dot{c}_2}{\partial f_2} = (\alpha + 1)v + (\alpha_B - 1)\frac{v_B}{2} \tag{B.6c}$$
$$\frac{\partial \dot{c}_1}{\partial p_1} = \frac{\partial \dot{c}_2}{\partial p_1} = \frac{\partial \dot{f}_2}{\partial p_1} = -\frac{\alpha v}{4} {F_1}'(p_{f1}) \tag{B.6d}$$
$$\frac{\partial \dot{c}_1}{\partial p_2} = \frac{\partial \dot{c}_2}{\partial p_2} = \frac{\partial \dot{f}_1}{\partial p_2} = -\frac{\alpha v}{4} {F_2}'(p_{f2}) \tag{B.6e}$$
$$\frac{\partial \dot{f}_1}{\partial c_1} = \frac{\partial \dot{f}_1}{\partial c_2} = \frac{\partial \dot{f}_1}{\partial f_2} = \frac{\partial \dot{f}_2}{\partial c_1} = \frac{\partial \dot{f}_2}{\partial c_2} = \frac{\partial \dot{f}_2}{\partial f_1} = v + (\alpha_B - 1)\frac{v_B}{2} \tag{B.6f}$$
$$\frac{\partial \dot{f}_1}{\partial f_1} = \frac{\partial \dot{f}_2}{\partial f_2} = (\alpha_B - 1)\frac{v_B}{2} - 3v \tag{B.6g}$$
$$\frac{\partial \dot{f}_1}{\partial p_1} = \frac{3}{4}\alpha v {F_1}'(p_{f1}) \tag{B.6h}$$
$$\frac{\partial \dot{f}_2}{\partial p_2} = \frac{3}{4}\alpha v {F_2}'(p_{f2}) \tag{B.6i}$$

Application of the chain rule to (B.3) at $p_{1/2} = p_{f1/2}$ yields for the price subdynamics

$$\frac{\partial \dot{p}_1}{\partial c_1} = 8l\frac{\partial \dot{c}_1}{\partial c_1}, \quad \frac{\partial \dot{p}_1}{\partial c_2} = 8l\frac{\partial \dot{c}_1}{\partial c_2}, \quad \ldots \quad \frac{\partial \dot{p}_2}{\partial p_2} = 8l\frac{\partial \dot{c}_2}{\partial p_2}. \tag{B.7}$$

A complication arises from the fact that $F_1{}'(p_{f1})$ and $F_2{}'(p_{f2})$ are not defined because

$$F_{1/2}(p_{1/2}) = s|p_{f1/2} - p_{1/2}| \tag{B.8}$$

implies a jump of the derivative $F'_{1/2}$ at the respective fundamental price

$$F_{1/2}{}'(p_{1/2}) = \pm s \quad \text{for } p_{f1/2} \lessgtr p_{1/2}. \tag{B.9}$$

It is therefore necessary to examine each of the regimes $(p_1 > p_{f1},\ p_2 > p_{f2})$, $(p_1 < p_{f1},\ p_2 < p_{f2})$, and $(p_1 \gtrless p_{f1},\ p_2 \lessgtr p_{f2})$ separately. Furthermore, stability with respect to regime-specific dynamics is in general neither a sufficient nor necessary condition for stability of the overall dynamics Honkapohja and Ito (1983). The following analysis serves therefore only as a general guideline, which factors may have an impact upon local stability of the fundamental equilibrium within the overall dynamics. All four regimes share the common eigenvalues:

$$\lambda_1 = \lambda_2 = 0, \quad \text{and} \tag{B.10a}$$
$$\lambda_3 = 2(\alpha_B - 1)v_b. \tag{B.10b}$$

The remaining eigenvalues differ from one regime to another. Consider first the case where the differences between fundamental and trading prices have the same sign:

$$\lambda_4 = -4v(1 \pm \alpha l s), \tag{B.11a}$$
$$\lambda_5 = v\left[\left(1 + \sqrt{1 \pm 16ls}\right)\alpha - 4\right], \tag{B.11b}$$
$$\lambda_6 = v\left[\left(1 - \sqrt{1 \pm 16ls}\right)\alpha - 4\right], \tag{B.11c}$$

where the plus sigs apply to $(p_1 > p_{f1},\ p_2 > p_{f2})$ and the minus signs to the regime $(p_1 < p_{f1},\ p_2 < p_{f2})$. The last three eigenvalues for the regimes $(p_1 \gtrless p_{f1},\ p_2 \lessgtr p_{f2})$ read

$$\lambda_{4,\pm\mp} = \frac{2}{3}v\left\{\alpha\left[1 + f(ls)^{1/3} + \frac{1}{f(ls)^{1/3}}\right] - 6\right\}, \tag{B.12a}$$
$$\lambda_{5/6,\pm\mp} = \frac{1}{3}v\left\{\alpha\left[\left(2 - f(ls)^{1/3} - \frac{1}{f(ls)^{1/3}}\right) \pm i\sqrt{3}\left(f(ls)^{1/3} - \frac{1}{f(ls)^{1/3}}\right)\right] - 12\right\}, \tag{B.12b}$$

with

$$f(ls) := 1 - 108(ls)^2 + 6ls\sqrt{324(ls)^2 - 6}, \tag{B.12c}$$

the real parts of which are given by

$$\mathrm{Re}(\lambda_{4,\pm\mp}) = \frac{2}{3}v\left\{\alpha\left[1+\left(|f(ls)|^{1/3}+\frac{1}{|f(ls)|^{1/3}}\right)\cos\left(\frac{1}{3}\arg(f(ls))\right)\right]-6\right\}, \tag{B.13a}$$

$$\begin{aligned}\mathrm{Re}(\lambda_{5/6,\pm\mp}) = \frac{1}{3}v\Bigg\{\alpha\Bigg[2-\left(|f(ls)|^{1/3}+\frac{1}{|f(ls)|^{1/3}}\right)\\ \times\left(\cos\left(\frac{1}{3}\arg(f(ls))\right)\pm\sqrt{3}\sin\left(\frac{1}{3}\arg(f(ls))\right)\right)\Bigg]-12\Bigg\},\end{aligned} \tag{B.13b}$$

none of which exceed

$$\mathrm{Re}(\lambda_{\pm\mp})_{\max} = \frac{2}{3}v\left\{\alpha\left[1+|f(ls)|^{1/3}+\frac{1}{|f(ls)|^{1/3}}\right]-6\right\}, \tag{B.14}$$

which is again smaller than the largest eigenvalues in those regimes where the signs of the mispricings in both stocks coincide,

$$\mathrm{Re}(\lambda_{\pm\pm})_{\max} = \begin{cases} v\left[\left(1+\sqrt{1+16ls}\right)\alpha-4\right], & \text{for } ls \le 3/2, \\ 4v(\alpha ls-1), & \text{for } ls \ge 3/2. \end{cases} \tag{B.15}$$

The necessary conditions for stability with respect to the regime-specific dynamics follow then from requiring the real part of all eigenvalues not to exceed zero in any of the four regimes, that is $\lambda_3 \le 0$ and $\mathrm{Re}(\lambda_{\pm\pm})_{\max} \le 0$.

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