Chin. Ann. Math.
32B(6), 2011, 803–822
DOI: 10.1007/s11401-011-0683-y

Chinese Annals of Mathematics, Series B

Exact Boundary Controllability for a Kind of Second-Order Quasilinear Hyperbolic Systems*

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Abstract Based on the theory of semi-global C^1 solution and the local exact boundary controllability for first-order quasilinear hyperbolic systems, the local exact boundary controllability for a kind of second-order quasilinear hyperbolic systems is obtained by a constructive method.

Keywords First-order quasilinear hyperbolic systems, Second-order quasilinear hyperbolic systems, Exact boundary controllability, Mixed initial-boundary value problem

2000 MR Subject Classification 35L40, 35L51, 35L53, 93B05

1 Introduction

The local exact boundary controllability for the quasilinear wave equation, a special form of second-order quasilinear hyperbolic equations, was obtained in a complete manner (see [1–6, 9]), and the author of [8] obtained its global exact boundary controllability under certain restrictions. For the following 1-D quasilinear hyperbolic equation: $u_{tt} + a(u, u_x, u_t)u_{tx} + b(u, u_x, u_t)u_{xx} = c(u, u_x, u_t)$, where u is the unknown function of (t, x) and $(a^2 - 4b)(0, 0, 0) > 0$, Zhuang and Shang [13] established the corresponding local exact boundary controllability, including the quasilinear wave equation as its special case.

For second-order quasilinear hyperbolic systems, there are few results on the exact boundary controllability. Yu [11] established the local exact boundary controllability for the following second-order quasilinear hyperbolic system: $u_{tt} - A(u, u_x, u_t)u_{xx} = F(u, u_x, u_t)$, where $u = (u_1, \cdots, u_n)^{\mathrm{T}}$ is the unknown vector function of (t, x), and matrix A has only n positive eigenvalues. Later, for second-order quasilinear hyperbolic system $u_{tt} + (A+B)(u, u_x, u_t)u_{tx} + AB(u, u_x, u_t)u_{xx} = F(u, u_x, u_t)$, where $u = (u_1, \cdots, u_n)^{\mathrm{T}}$ is the unknown vector function of (t, x), matrices A and B have only n positive eigenvalues and n negative eigenvalues, respectively, the local exact boundary controllability was obtained by Yu [12]. When matrices A and B satisfy $A = -B$, this conclusion can be obtained from [11].

In this paper, we consider another kind of 1-D second-order quasilinear hyperbolic systems, which can be rewritten in the form of second-order quasilinear hyperbolic systems discussed in [12], but restrictions on the eigenvalues of matrices A and B are much weakened. Based on the

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*Project supported by the Excellent Doctoral Research Foundation for Key Subject of Fudan University (No. EHH1411208).

existence and uniqueness of semi-global C^1 solution and the local exact boundary controllability for first-order quasilinear hyperbolic systems, by a constructive method developed by Li (see [2]), we can obtain the local exact boundary controllability for this kind of second-order quasilinear hyperbolic systems. The conclusions in [11] and [13] are both of its special cases.

2 A Kind of 1-D Second-Order Quasilinear Hyperbolic Systems

We consider the following second-order quasilinear system:

$$u_{tt} + A(u, u_x, u_t)u_{tx} + B(u, u_x, u_t)u_{xx} = C(u, u_x, u_t), \tag{2.1}$$

where $u = (u_1, \cdots, u_n)^{\mathrm{T}}$ is the unknown vector function of (t, x), $A(u, v, w)$ and $B(u, v, w)$ are both $n \times n$ matrices with smooth entries $a_{ij}(u, v, w)$ and $b_{ij}(u, v, w)$ $(i, j = 1, \cdots, n)$, and have n real eigenvalues and a complete set of left eigenvectors on the domain under consideration, respectively. Suppose that

$$AB(u, v, w) = BA(u, v, w), \tag{2.2}$$

and $C = C(u, v, w) = (c_1(u, v, w), \cdots, c_n(u, v, w))^{\mathrm{T}}$ is a smooth vector function with

$$C(0, 0, 0) = 0. \tag{2.3}$$

Condition (2.2) is equivalent to the fact that matrices A and B can be simultaneously diagonalizable (see [10]), namely, there exists an invertible $n \times n$ matrix $L(u, v, w)$, such that

$$LAL^{-1}(u, v, w) = \mathrm{diag}\{\lambda_1, \cdots, \lambda_n\}, \tag{2.4}$$

$$LBL^{-1}(u, v, w) = \mathrm{diag}\{\mu_1, \cdots, \mu_n\}, \tag{2.5}$$

where $\lambda_1, \cdots, \lambda_n$ and $\mu_1, \cdots, \mu_n$ are the real eigenvalues of matrices A and B, respectively, and $L = (l_{ij})$ is the matrix composed by the common left eigenvectors of A and B. Furthermore, we assume that in the domain under consideration

$$\mu_i(u, v, w) \neq 0, \quad i = 1, \cdots, n \tag{2.6}$$

and

$$\lambda_i^2 - 4\mu_i(u, v, w) > 0, \qquad \text{when } \mu_i(u, v, w) > 0, \ i = 1, \cdots, n. \tag{2.7}$$

To illustrate that the second-order quasilinear system under consideration is hyperbolic, setting

$$v_i = \frac{\partial u_i}{\partial x}, \quad w_i = \frac{\partial u_i}{\partial t}, \quad i = 1, \cdots, n, \tag{2.8}$$

$$v = (v_1, \cdots, v_n)^{\mathrm{T}}, \quad w = (w_1, \cdots, w_n)^{\mathrm{T}}, \quad U = \begin{pmatrix} u \\ v \\ w \end{pmatrix}, \tag{2.9}$$

we can reduce system (2.1) to the following first-order quasilinear system:

$$\begin{cases} \dfrac{\partial u}{\partial t} = w, \\ \dfrac{\partial v}{\partial t} - \dfrac{\partial w}{\partial x} = 0, \\ \dfrac{\partial w}{\partial t} + B(u, v, w)\dfrac{\partial v}{\partial x} + A(u, v, w)\dfrac{\partial w}{\partial x} = C(u, v, w). \end{cases} \tag{2.10}$$

It is easy to see that its characteristic equation is

$$\det\begin{vmatrix} \widetilde{\lambda}I_n & 0 & 0 \\ 0 & \widetilde{\lambda}I_n & I_n \\ 0 & -B & \widetilde{\lambda}I_n - A \end{vmatrix} = \widetilde{\lambda}^n \det\begin{vmatrix} \widetilde{\lambda}I_n & I_n \\ -B & \widetilde{\lambda}I_n - A \end{vmatrix} = \widetilde{\lambda}^n \det|\widetilde{\lambda}^2 I_n - \widetilde{\lambda}A + B| = 0. \quad (2.11)$$

By (2.4)–(2.5), $\det|\widetilde{\lambda}^2 I_n - \widetilde{\lambda}A + B| = 0$ can be rewritten as $\widetilde{\lambda}^2 - \lambda_i\widetilde{\lambda} + \mu_i = 0$ $(i = 1, \cdots, n)$, whose solutions are

$$\widetilde{\lambda}_i^{\pm} = \frac{\lambda_i \pm \sqrt{\lambda_i^2 - 4\mu_i}}{2}, \quad i = 1, \cdots, n. \quad (2.12)$$

Obviously,

$$\widetilde{\lambda}_i^{\pm} \neq 0, \quad \widetilde{\lambda}_i^{+} > \widetilde{\lambda}_i^{-}, \quad i = 1, \cdots, n \quad (2.13)$$

and

$$\widetilde{\lambda}_i^{+} + \widetilde{\lambda}_i^{-} = \lambda_i, \quad \widetilde{\lambda}_i^{+}\widetilde{\lambda}_i^{-} = \mu_i, \quad i = 1, \cdots, n. \quad (2.14)$$

Then system (2.10) has $3n$ real eigenvalues

$$\widetilde{\lambda}_i^{-} = \frac{\lambda_i - \sqrt{\lambda_i^2 - 4\mu_i}}{2}, \quad \widetilde{\lambda}_i^{0} \equiv 0, \quad \widetilde{\lambda}_i^{+} = \frac{\lambda_i + \sqrt{\lambda_i^2 - 4\mu_i}}{2}, \quad i = 1, \cdots, n. \quad (2.15)$$

Noting (2.14), it is easy to see that the corresponding left eigenvectors, which constitute a complete set, can be chosen as

$$\widetilde{l}_i^{-} = (\overbrace{0, \cdots, 0}^{n}, \widetilde{\lambda}_i^{+} l_i, l_i), \quad \widetilde{l}_i^{0} = (e_i, \overbrace{0, \cdots, 0}^{2n}), \quad \widetilde{l}_i^{+} = (\overbrace{0, \cdots, 0}^{n}, \widetilde{\lambda}_i^{-} l_i, l_i), \quad i = 1, \cdots, n, \quad (2.16)$$

in which l_i is the ith column vector of matrix L, i.e., the ith common left eigenvector of matrices A and B, and $e_i = (0, \cdots, \overset{(i)}{1}, \cdots, 0)$.

Thus, the first-order quasilinear system (2.10) reduced from the system (2.1) is hyperbolic. Then, the quasilinear system (2.1) under consideration is a kind of second-order quasilinear hyperbolic systems.

Remark 2.1 Let

$$P = L^{-1}\mathrm{diag}\{\widetilde{\lambda}_1^{+}, \cdots, \widetilde{\lambda}_n^{+}\}L, \quad Q = L^{-1}\mathrm{diag}\{\widetilde{\lambda}_1^{-}, \cdots, \widetilde{\lambda}_n^{-}\}L. \quad (2.17)$$

By (2.14), we get $A = P + Q$, $B = PQ$. Then system (2.1) can be rewritten as

$$u_{tt} + (P + Q)(u, u_x, u_t)u_{tx} + PQ(u, u_x, u_t)u_{xx} = C(u, u_x, u_t), \quad (2.18)$$

which is of the same form of the second-order quasilinear hyperbolic system considered in [12], but there are much less restrictions on the eigenvalues of the matrices.

3 Existence and Uniqueness of Semi-Global C^2 Solution

For the second-order quasilinear hyperbolic system (2.1), we give the following initial condition:

$$t=0:\ u=\varphi(x),\quad u_t=\psi(x),\quad 0\le x\le L \tag{3.1}$$

and final condition

$$t=T:\ u=\Phi(x),\quad u_t=\Psi(x),\quad 0\le x\le L, \tag{3.2}$$

where $\varphi=(\varphi_1,\cdots,\varphi_n)^{\mathrm{T}}$ and $\Phi=(\Phi_1,\cdots,\Phi_n)^{\mathrm{T}}$ are given C^2 vector functions, $\psi=(\psi_1,\cdots,\psi_n)^{\mathrm{T}}$ and $\Psi=(\Psi_1,\cdots,\Psi_n)^{\mathrm{T}}$ are given C^1 vector functions.

Let

$$D^{\pm}=\mathrm{diag}\Big\{\frac{\mu_1}{\widetilde{\lambda}_1^{\pm}},\cdots,\frac{\mu_n}{\widetilde{\lambda}_n^{\pm}}\Big\}=\mathrm{diag}\{\widetilde{\lambda}_1^{\mp},\cdots,\widetilde{\lambda}_n^{\mp}\}. \tag{3.3}$$

Based on (2.13), according to different signs of $\widetilde{\lambda}_i^{\pm}$ $(i=1,\cdots,n)$ in a neighborhood of $(u,v,w)=(0,0,0)$, without loss of generality, we only need to discuss the following three typical cases:

(1) $$\widetilde{\lambda}_i^{+}>0,\quad \widetilde{\lambda}_i^{-}<0,\quad i=1,\cdots,n; \tag{3.4}$$

(2) $$\begin{aligned}&\widetilde{\lambda}_j^{\pm}>0,\quad \widetilde{\lambda}_k^{+}>0,\quad \widetilde{\lambda}_k^{-}<0,\quad \widetilde{\lambda}_h^{\pm}<0,\\ &j=1,\cdots,d_1,\quad k=d_1+1,\cdots,d_2,\quad h=d_2+1,\cdots,n,\end{aligned} \tag{3.5}$$

where d_1, d_2 are any given integers satisfying $0\le d_1\le d_2\le n$, but excluding the case $d_1=n$ (which is Case (3)) and the case $d_1=0$ and $d_2=n$ (which is Case (1));

(3) $$\widetilde{\lambda}_i^{\pm}>0,\quad i=1,\cdots,n. \tag{3.6}$$

Case (1) Equation (3.4) is equivalent to

$$\mu_i(u,v,w)<0,\quad i=1,\cdots,n \tag{3.7}$$

in a neighborhood of $(u,v,w)=(0,0,0)$. Then system (2.1) has n positive eigenvalues and n negative eigenvalues. On the ends $x=0$ and $x=L$, we prescribe the following nonlinear boundary conditions, respectively:

$$x=0:\ \begin{cases}G_p(u)=H_p(t), & p=1,\cdots,l,\\ G_q(u,u_x,u_t)=H_q(t), & q=l+1,\cdots,n,\end{cases} \tag{3.8}$$

$$x=L:\ \begin{cases}\overline{G}_r(u)=\overline{H}_r(t), & r=1,\cdots,m,\\ \overline{G}_s(u,u_x,u_t)=\overline{H}_s(t), & s=m+1,\cdots,n,\end{cases} \tag{3.9}$$

where G_p, H_p, $\overline{G}_r$ and $\overline{H}_r$ are all C^2 functions with respect to their arguments, G_q, H_q, $\overline{G}_s$ and $\overline{H}_s$ are all C^1 functions with respect to their arguments. For different needs in further discussions, some or all of the following assumptions will be imposed in different situations:

$$\det\left|\begin{pmatrix}\frac{\partial G_1}{\partial u_1} & \cdots & \frac{\partial G_1}{\partial u_n}\\ \vdots & & \vdots\\ \frac{\partial G_l}{\partial u_1} & \cdots & \frac{\partial G_l}{\partial u_n}\\ \frac{\partial G_{l+1}}{\partial u_{1t}} & \cdots & \frac{\partial G_{l+1}}{\partial u_{nt}}\\ \vdots & & \vdots\\ \frac{\partial G_n}{\partial u_{1t}} & \cdots & \frac{\partial G_n}{\partial u_{nt}}\end{pmatrix}(L^{-1}D^{-})-\begin{pmatrix}0 & \cdots & 0\\ \vdots & & \vdots\\ 0 & \cdots & 0\\ \frac{\partial G_{l+1}}{\partial u_{1x}} & \cdots & \frac{\partial G_{l+1}}{\partial u_{nx}}\\ \vdots & & \vdots\\ \frac{\partial G_n}{\partial u_{1x}} & \cdots & \frac{\partial G_n}{\partial u_{nx}}\end{pmatrix}(L^{-1})\right|_{(0,0,0)}\neq 0, \tag{3.10}$$

$$\det\left|\begin{pmatrix} \frac{\partial G_1}{\partial u_1} & \cdots & \frac{\partial G_1}{\partial u_n} \\ \vdots & & \vdots \\ \frac{\partial G_l}{\partial u_1} & \cdots & \frac{\partial G_l}{\partial u_n} \\ \frac{\partial G_{l+1}}{\partial u_{1t}} & \cdots & \frac{\partial G_{l+1}}{\partial u_{nt}} \\ \vdots & & \vdots \\ \frac{\partial G_n}{\partial u_{1t}} & \cdots & \frac{\partial G_n}{\partial u_{nt}} \end{pmatrix}(L^{-1}D^+) - \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \\ \frac{\partial G_{l+1}}{\partial u_{1x}} & \cdots & \frac{\partial G_{l+1}}{\partial u_{nx}} \\ \vdots & & \vdots \\ \frac{\partial G_n}{\partial u_{1x}} & \cdots & \frac{\partial G_n}{\partial u_{nx}} \end{pmatrix}(L^{-1})\right|_{(0,0,0)} \neq 0, \tag{3.11}$$

$$\det\left|\begin{pmatrix} \frac{\partial \overline{G}_1}{\partial u_1} & \cdots & \frac{\partial \overline{G}_1}{\partial u_n} \\ \vdots & & \vdots \\ \frac{\partial \overline{G}_m}{\partial u_1} & \cdots & \frac{\partial \overline{G}_m}{\partial u_n} \\ \frac{\partial \overline{G}_{m+1}}{\partial u_{1t}} & \cdots & \frac{\partial \overline{G}_{m+1}}{\partial u_{nt}} \\ \vdots & & \vdots \\ \frac{\partial \overline{G}_n}{\partial u_{1t}} & \cdots & \frac{\partial \overline{G}_n}{\partial u_{nt}} \end{pmatrix}(L^{-1}D^-) - \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \\ \frac{\partial \overline{G}_{m+1}}{\partial u_{1x}} & \cdots & \frac{\partial \overline{G}_{m+1}}{\partial u_{nx}} \\ \vdots & & \vdots \\ \frac{\partial \overline{G}_n}{\partial u_{1x}} & \cdots & \frac{\partial \overline{G}_n}{\partial u_{nx}} \end{pmatrix}(L^{-1})\right|_{(0,0,0)} \neq 0, \tag{3.12}$$

$$\det\left|\begin{pmatrix} \frac{\partial \overline{G}_1}{\partial u_1} & \cdots & \frac{\partial \overline{G}_1}{\partial u_n} \\ \vdots & & \vdots \\ \frac{\partial \overline{G}_m}{\partial u_1} & \cdots & \frac{\partial \overline{G}_m}{\partial u_n} \\ \frac{\partial \overline{G}_{m+1}}{\partial u_{1t}} & \cdots & \frac{\partial \overline{G}_{m+1}}{\partial u_{nt}} \\ \vdots & & \vdots \\ \frac{\partial \overline{G}_n}{\partial u_{1t}} & \cdots & \frac{\partial \overline{G}_n}{\partial u_{nt}} \end{pmatrix}(L^{-1}D^+) - \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \\ \frac{\partial \overline{G}_{m+1}}{\partial u_{1x}} & \cdots & \frac{\partial \overline{G}_{m+1}}{\partial u_{nx}} \\ \vdots & & \vdots \\ \frac{\partial \overline{G}_n}{\partial u_{1x}} & \cdots & \frac{\partial \overline{G}_n}{\partial u_{nx}} \end{pmatrix}(L^{-1})\right|_{(0,0,0)} \neq 0. \tag{3.13}$$

Case (2) In this case, system (2.1) has $d_1 + d_2$ positive eigenvalues and $2n - (d_1 + d_2)$ negative eigenvalues. Without loss of generality, we assume

$$d_1 + d_2 \leq n, \tag{3.14}$$

namely, the number of positive eigenvalues is less than or equal to the number of negative eigenvalues. Correspondingly, on the ends $x = 0$ and $x = L$, we prescribe the following nonlinear boundary conditions, respectively:

$$x = 0: \begin{cases} G_p(u) = H_p(t), & p = 1, \cdots, l, \\ G_q(u, u_x, u_t) = H_q(t), & q = l+1, \cdots, d_1 + d_2, \end{cases} \tag{3.15}$$

$$x = L: \begin{cases} \overline{G}_r(u) = \overline{H}_r(t), & r = 1, \cdots, m, \\ \overline{G}_s(u, u_x, u_t) = \overline{H}_s(t), & s = m+1, \cdots, 2n - (d_1 + d_2), \end{cases} \tag{3.16}$$

where G_p, H_p, $\overline{G}_r$ and $\overline{H}_r$ are all C^2 functions with respect to their arguments, G_q, H_q, $\overline{G}_s$ and $\overline{H}_s$ are all C^1 functions with respect to their arguments. For different needs in further discussions, some or all of the following assumptions will be imposed in different situations:

$$\det\left|\begin{pmatrix} \frac{\partial G_1}{\partial u_1} & \cdots & \frac{\partial G_1}{\partial u_n} \\ \vdots & & \vdots \\ \frac{\partial G_l}{\partial u_1} & \cdots & \frac{\partial G_l}{\partial u_n} \\ \frac{\partial G_{l+1}}{\partial u_{1t}} & \cdots & \frac{\partial G_{l+1}}{\partial u_{nt}} \\ \vdots & & \vdots \\ \frac{\partial G_{d_1+d_2}}{\partial u_{1t}} & \cdots & \frac{\partial G_{d_1+d_2}}{\partial u_{nt}} \end{pmatrix}((L^{-1}D^-)_{\{1,d_2\}} \vdots (L^{-1}D^+)_{\{1,d_1\}})\right.$$

$$
\left.-\begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \\ \frac{\partial G_{l+1}}{\partial u_{1x}} & \cdots & \frac{\partial G_{l+1}}{\partial u_{nx}} \\ \vdots & & \vdots \\ \frac{\partial G_{d_1+d_2}}{\partial u_{1x}} & \cdots & \frac{\partial G_{d_1+d_2}}{\partial u_{nx}} \end{pmatrix} ((L^{-1})_{\{1,d_2\}} \vdots (L^{-1})_{\{1,d_1\}}) \right|_{(0,0,0)} \neq 0, \tag{3.17}
$$

$$
\det \left| \begin{pmatrix} \frac{\partial G_1}{\partial u_1} & \cdots & \frac{\partial G_1}{\partial u_n} \\ \vdots & & \vdots \\ \frac{\partial G_l}{\partial u_1} & \cdots & \frac{\partial G_l}{\partial u_n} \\ \frac{\partial G_{l+1}}{\partial u_{1t}} & \cdots & \frac{\partial G_{l+1}}{\partial u_{nt}} \\ \vdots & & \vdots \\ \frac{\partial G_{d_1+d_2}}{\partial u_{1t}} & \cdots & \frac{\partial G_{d_1+d_2}}{\partial u_{nt}} \end{pmatrix} ((L^{-1}D^-)_{\{d_2+1,d_2+d_1\}} \vdots (L^{-1}D^+)_{\{d_1+1,d_1+d_2\}}) \right.
$$

$$
\left.-\begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \\ \frac{\partial G_{l+1}}{\partial u_{1x}} & \cdots & \frac{\partial G_{l+1}}{\partial u_{nx}} \\ \vdots & & \vdots \\ \frac{\partial G_{d_1+d_2}}{\partial u_{1x}} & \cdots & \frac{\partial G_{d_1+d_2}}{\partial u_{nx}} \end{pmatrix} ((L^{-1})_{\{d_2+1,d_2+d_1\}} \vdots (L^{-1})_{\{d_1+1,d_1+d_2\}}) \right|_{(0,0,0)} \neq 0, \tag{3.18}
$$

$$
\det \left| \begin{pmatrix} \frac{\partial \overline{G}_1}{\partial u_1} & \cdots & \frac{\partial \overline{G}_1}{\partial u_n} \\ \vdots & & \vdots \\ \frac{\partial \overline{G}_m}{\partial u_1} & \cdots & \frac{\partial \overline{G}_m}{\partial u_n} \\ \frac{\partial \overline{G}_{m+1}}{\partial u_{1t}} & \cdots & \frac{\partial \overline{G}_{m+1}}{\partial u_{nt}} \\ \vdots & & \vdots \\ \frac{\partial \overline{G}_{2n-(d_1+d_2)}}{\partial u_{1t}} & \cdots & \frac{\partial \overline{G}_{2n-(d_1+d_2)}}{\partial u_{nt}} \end{pmatrix} ((L^{-1}D^-)_{\{d_2+1,n\}} \vdots (L^{-1}D^+)_{\{d_1+1,n\}}) \right.
$$

$$
\left.-\begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \\ \frac{\partial \overline{G}_{m+1}}{\partial u_{1x}} & \cdots & \frac{\partial \overline{G}_{m+1}}{\partial u_{nx}} \\ \vdots & & \vdots \\ \frac{\partial \overline{G}_{2n-(d_1+d_2)}}{\partial u_{1x}} & \cdots & \frac{\partial \overline{G}_{2n-(d_1+d_2)}}{\partial u_{nx}} \end{pmatrix} ((L^{-1})_{\{d_2+1,n\}} \vdots (L^{-1})_{\{d_1+1,n\}}) \right|_{(0,0,0)} \neq 0, \tag{3.19}
$$

where $(L^{-1}D^-)_{\{1,d_1\}}$ indicates the matrix composed of the first column to the d_1th column of matrix $(L^{-1}D^-)$, etc.

Case (3) In this case, system (2.1) has $2n$ positive eigenvalues. Correspondingly, there need only $2n$ boundary conditions on the end $x = 0$:

$$
x = 0: \ u = H(t), \quad u_x = \overline{H}(t), \tag{3.20}
$$

where $H = (H_1, \cdots, H_n)^{\mathrm{T}}$ is a C^2 vector function, $\overline{H} = (\overline{H}_1, \cdots, \overline{H}_n)^{\mathrm{T}}$ is a C^1 vector function.

First of all, for Case (1), we give the following theorem on the existence and uniqueness of semi-global C^2 solution to system (2.1).

Theorem 3.1 *Suppose that a_{ij}, b_{ij}, c_i, λ_i, μ_i, l_{ij}, G_q, $\overline{G}_s$ $(i,j=1,\cdots,n;\ q=l+1,\cdots,n;\ s=m+1,\cdots,n)$ are C^1 functions with respect to their arguments, G_p, $\overline{G}_r$ $(p=1,\cdots,l;\ r=1,\cdots,m)$ are C^2 functions with respect to their arguments, and (H_p,H_q), $(\overline{H}_r,\overline{H}_s)$ $(p=1,\cdots,l;\ q=l+1,\cdots,n;\ r=1,\cdots,m;\ s=m+1,\cdots,n)$ are $C^2\times C^1$ vector functions. Suppose furthermore that (3.7), (3.10) and (3.13) hold, and the conditions of C^2 compatibility are satisfied at the points $(t,x)=(0,0)$ and $(0,L)$, respectively. Then, for any given and possibly quite large $T>0$, the forward mixed initial-boundary value problem (2.1), (3.1) and (3.8)–(3.9) admits a unique semi-global C^2 solution $u=u(t,x)$ on the domain $R(T)=\{(t,x)\mid 0\le t\le T, 0\le x\le L\}$ with small C^2 norm, provided that the norms $\|(\varphi(x),\psi(x))\|_{C^2[0,L]\times C^1[0,L]}$, $\|(H_p(t),H_q(t))\|_{C^2[0,T]\times C^1[0,T]}$, $\|(\overline{H}_r(t),\overline{H}_s(t))\|_{C^2[0,T]\times C^1[0,T]}$ $(p=1,\cdots,l;\ q=l+1,\cdots,n;\ r=1,\cdots,m;\ s=m+1,\cdots,n)$ are small enough.*

Proof Let $U=(u,v,w)^{\mathrm{T}}$ and

$$V_i^-=\widetilde{l}_i^-(U)U,\quad V_i=\widetilde{l}_i^0(U)U,\quad V_i^+=\widetilde{l}_i^+(U)U,\quad i=1,\cdots,n. \tag{3.21}$$

By (2.16), we have

$$\begin{cases}V_i^-=\widetilde{\lambda}_i^+l_iv+l_iw,\\ V_i\ \ =u_i,\\ V_i^+=\widetilde{\lambda}_i^-l_iv+l_iw,\end{cases} \tag{3.22}$$

i.e.,

$$\begin{cases}V^-=D^-Lv+Lw,\\ V\ \ =u,\\ V^+=D^+Lv+Lw.\end{cases} \tag{3.23}$$

It is easy to see that

$$(u,v,w)=0\Leftrightarrow(V^-,V,V^+)=0 \tag{3.24}$$

and

$$\det\Big|\frac{\partial(V^-,V,V^+)}{\partial(u,v,w)}\Big|_{(0,0,0)}=\det|(D^+-D^-)L^2|_{(0,0,0)}\neq 0. \tag{3.25}$$

Hence, by the implicit function theorem, there exist C^1 vector functions $\widetilde{G}=(\widetilde{G}_1,\cdots,\widetilde{G}_n)^{\mathrm{T}}$ and $\widetilde{H}=(\widetilde{H}_1,\cdots,\widetilde{H}_n)^{\mathrm{T}}$ in a neighborhood of $U=0$, such that

$$u=V,\quad v=\widetilde{G}(V^-,V,V^+),\quad w=\widetilde{H}(V^-,V,V^+) \tag{3.26}$$

and

$$\widetilde{G}(0,0,0)=0,\quad \widetilde{H}(0,0,0)=0, \tag{3.27}$$

$$\begin{cases}\det\Big|\dfrac{\partial\widetilde{G}}{\partial V^-}\Big|_{(0,0,0)}=\det|-L^{-1}(D^+-D^-)^{-1}|_{(0,0,0)}\neq 0,\\ \det\Big|\dfrac{\partial\widetilde{G}}{\partial V^+}\Big|_{(0,0,0)}=\det|L^{-1}(D^+-D^-)^{-1}|_{(0,0,0)}\neq 0,\\ \det\Big|\dfrac{\partial\widetilde{H}}{\partial V^-}\Big|_{(0,0,0)}=\det|L^{-1}D^+(D^+-D^-)^{-1}|_{(0,0,0)}\neq 0,\\ \det\Big|\dfrac{\partial\widetilde{H}}{\partial V^+}\Big|_{(0,0,0)}=\det|-L^{-1}D^-(D^+-D^-)^{-1}|_{(0,0,0)}\neq 0.\end{cases} \tag{3.28}$$

Noting the conditions of C^2 compatibility at the points $(t,x)=(0,0)$ and $(0,L)$, respectively, we can replace boundary conditions (3.8) and (3.9) correspondingly by

$$x=0:\ \begin{cases}\displaystyle\sum_{k=1}^{n}\frac{\partial G_p}{\partial u_k}(u)w_k=H_p'(t), & p=1,\cdots,l,\\ G_q(u,v,w)=H_q(t), & q=l+1,\cdots,n\end{cases}\tag{3.29}$$

and

$$x=L:\ \begin{cases}\displaystyle\sum_{k=1}^{n}\frac{\partial \overline{G}_r}{\partial u_k}(u)w_k=\overline{H}_r'(t), & r=1,\cdots,m,\\ \overline{G}_s(u,v,w)=\overline{H}_s(t), & s=m+1,\cdots,n,\end{cases}\tag{3.30}$$

respectively, and reduce the initial condition (3.1) to

$$t=0:\ u=\varphi(x),\quad v=\varphi'(x),\quad w=\psi(x),\quad 0\le x\le L.\tag{3.31}$$

By (3.10) and (3.28), boundary condition (3.29) can be equivalently rewritten as

$$x=0:\ V_i^+=g_i(t,V^-,V)+h_i(t),\quad i=1,\cdots,n,\tag{3.32}$$

in a neighborhood of $(u,v,w)=(0,0,0)$, where g_i, h_i $(i=1,\cdots,n)$ are C^1 functions with respect to their arguments and, without loss of generality,

$$g_i(t,0,0)\equiv 0,\quad i=1,\cdots,n.\tag{3.33}$$

Then

$$\begin{aligned}&\|(H_p,H_q)\|_{C^2[0,T]\times C^1[0,T]}\ (p=1,\cdots,l;\ q=l+1,\cdots,n)\text{ small enough}\\ \Leftrightarrow\ &\|(h_1,\cdots,h_n)\|_{C^1[0,T]}\text{ small enough.}\end{aligned}\tag{3.34}$$

Similarly, by (3.13) and (3.28), boundary condition (3.30) can be equivalently rewritten as

$$x=L:\ V_i^-=\overline{g}_i(t,V,V^+)+\overline{h}_i(t),\quad i=1,\cdots,n\tag{3.35}$$

in a neighborhood of $(u,v,w)=(0,0,0)$, where $\overline{g}_i$, $\overline{h}_i$ $(i=1,\cdots,n)$ are C^1 functions with respect to their arguments and

$$\overline{g}_i(t,0,0)\equiv 0,\quad i=1,\cdots,n.\tag{3.36}$$

Then

$$\begin{aligned}&\|(\overline{H}_r,\overline{H}_s)\|_{C^2[0,T]\times C^1[0,T]}\ (r=1,\cdots,m;\ s=m+1,\cdots,n)\text{ small enough}\\ \Leftrightarrow\ &\|(\overline{h}_1,\cdots,\overline{h}_n)\|_{C^1[0,T]}\text{ small enough.}\end{aligned}\tag{3.37}$$

Moreover, it follows easily from the conditions of C^2 compatibility of the original mixed initial-boundary value problem (2.1), (3.1) and (3.8)–(3.9) that the mixed initial-boundary value problem (2.10), (3.31) and (3.29)–(3.30) satisfies the conditions of C^1 compatibility at the points $(t,x)=(0,0)$ and $(0,L)$, respectively. Applying directly the existence and uniqueness of semi-global C^1 solution to the mixed initial-boundary value problem for first-order quasilinear hyperbolic systems (see [2, 3]), we can obtain the existence and uniqueness of the semi-global C^2 solution to the original mixed initial-boundary value problem (2.1), (3.1) and (3.8)–(3.9) for Case (1), which is the conclusion of Theorem 3.1.

Remark 3.1 For the backward mixed initial-boundary value problem (2.1), (3.2) and (3.8)–(3.9), suppose that (3.7) and (3.11)–(3.12) hold and the conditions of C^2 compatibility are satisfied at the points $(t,x)=(T,0)$ and (T,L), respectively. Similarly, we obtain that there exists a unique semi-global C^2 solution $u=u(t,x)$ on the domain $R(T)=\{(t,x)\mid 0\le t\le T, 0\le x\le L\}$ with small C^2 norm.

Remark 3.2 In particular, if the boundary conditions are given as

$$x=0:\ u_i=H_i(t),\quad i=1,\cdots,n, \tag{3.38}$$

$$x=L:\ u_i=\overline{H}_i(t),\quad i=1,\cdots,n, \tag{3.39}$$

it is easy to see that the assumptions (3.10)–(3.13) are simplified to

$$\det|I_n(L^{-1}D^-)|_{(0,0,0)}\neq 0, \tag*{(3.10)$'$}$$

$$\det|I_n(L^{-1}D^+)|_{(0,0,0)}\neq 0, \tag*{(3.11)$'$}$$

$$\det|I_n(L^{-1}D^-)|_{(0,0,0)}\neq 0, \tag*{(3.12)$'$}$$

$$\det|I_n(L^{-1}D^+)|_{(0,0,0)}\neq 0. \tag*{(3.13)$'$}$$

Obviously, they are automatically satisfied.

For Cases (2) and (3), we can similarly obtain the existence and uniqueness of semi-global C^2 solution.

Theorem 3.2 *Suppose that* a_{ij}, b_{ij}, c_i, λ_i, μ_i, l_{ij}, G_q, $\overline{G}_s$ $(i,j=1,\cdots,n;\ q=l+1,\cdots,d_1+d_2;\ s=m+1,\cdots,2n-(d_1+d_2))$ *are* C^1 *functions with respect to their arguments,* G_p, $\overline{G}_r$ $(p=1,\cdots,l;\ r=1,\cdots,m)$ *are* C^2 *functions with respect to their arguments, and* (H_p,H_q), $(\overline{H}_r,\overline{H}_s)$ $(p=1,\cdots,l;\ q=l+1,\cdots,d_1+d_2;\ r=1,\cdots,m;\ s=m+1,\cdots,2n-(d_1+d_2))$ *are* $C^2\times C^1$ *vector functions. Suppose furthermore that* (3.5), (3.14), (3.17) *and* (3.19) *hold, and the conditions of* C^2 *compatibility are satisfied at the points* $(t,x)=(0,0)$ *and* $(0,L)$*, respectively. Then, for any given and possibly quite large* $T>0$*, the forward mixed initial-boundary value problem* (2.1), (3.1) *and* (3.15)–(3.16) *admits a unique semi-global* C^2 *solution* $u=u(t,x)$ *on the domain* $R(T)=\{(t,x)\mid 0\le t\le T, 0\le x\le L\}$ *with small* C^2 *norm, provided that the norms* $\|(\varphi(x),\psi(x))\|_{C^2[0,L]\times C^1[0,L]}$, $\|(H_p(t),H_q(t))\|_{C^2[0,T]\times C^1[0,T]}$, $\|(\overline{H}_r(t),\overline{H}_s(t))\|_{C^2[0,T]\times C^1[0,T]}$ $(p=1,\cdots,l;\ q=l+1,\cdots,d_1+d_2;\ r=1,\cdots,m;\ s=m+1,\cdots,2n-(d_1+d_2))$ *are small enough.*

Remark 3.3 In particular, if the boundary conditions are given as

$$x=0:\ \begin{cases}(I_{d_2},0)L(0)u=H(t),\\(I_{d_1},0)L(0)u_x=\widetilde{H}(t),\end{cases} \tag{3.40}$$

$$x=L:\ \begin{cases}(0,I_{n-d_2})L(0)u=\overline{H}(t),\\(0,I_{n-d_1})L(0)u_x=\widetilde{\overline{H}}(t),\end{cases} \tag{3.41}$$

where

$$H(t)=(H_1(t),\cdots,H_{d_2}(t)),\qquad \widetilde{H}(t)=(H_{d_2+1}(t),\cdots,H_{d_1+d_2}(t)),$$
$$\overline{H}(t)=(\overline{H}_1(t),\cdots,\overline{H}_{n-d_2}(t)),\quad \widetilde{\overline{H}}(t)=(\overline{H}_{n-d_2+1}(t),\cdots,\overline{H}_{2n-(d_1+d_2)}(t)),$$

then the assumptions (3.17) and (3.19) are automatically satisfied.

Remark 3.4 Under the assumption (3.14), consider the backward mixed initial-boundary value problem for system (2.1) with the final condition (3.2) and the following boundary conditions:

$$x=0:\quad \begin{cases} G_p(u)=H_p(t), & p=1,\cdots,\widetilde{l},\\ G_q(u,u_x,u_t)=H_q(t), & q=\widetilde{l}+1,\cdots,2n-(d_1+d_2), \end{cases} \tag{3.42}$$

$$x=L:\quad \begin{cases} \overline{G}_r(u)=\overline{H}_r(t), & r=1,\cdots,\widetilde{m},\\ \overline{G}_s(u,u_x,u_t)=\overline{H}_s(t), & s=\widetilde{m}+1,\cdots,d_1+d_2, \end{cases} \tag{3.43}$$

where G_p, H_p, $\overline{G}_r$ and $\overline{H}_r$ are C^2 functions with respect to their arguments, G_q, H_q, $\overline{G}_s$ and $\overline{H}_s$ are C^1 functions with respect to their arguments. Suppose furthermore that the following conditions are satisfied:

$$\det\left|\begin{pmatrix} \frac{\partial G_1}{\partial u_1} & \cdots & \frac{\partial G_1}{\partial u_n}\\ \vdots & & \vdots\\ \frac{\partial G_{\widetilde{l}}}{\partial u_1} & \cdots & \frac{\partial G_{\widetilde{l}}}{\partial u_n}\\ \frac{\partial G_{\widetilde{l}+1}}{\partial u_{1t}} & \cdots & \frac{\partial G_{\widetilde{l}+1}}{\partial u_{nt}}\\ \vdots & & \vdots\\ \frac{\partial G_{2n-(d_1+d_2)}}{\partial u_{1t}} & \cdots & \frac{\partial G_{2n-(d_1+d_2)}}{\partial u_{nt}} \end{pmatrix}((L^{-1}D^-)_{\{d_2+1,n\}}\vdots(L^{-1}D^+)_{\{d_1+1,n\}}) - \begin{pmatrix} 0 & \cdots & 0\\ \vdots & & \vdots\\ 0 & \cdots & 0\\ \frac{\partial G_{\widetilde{l}+1}}{\partial u_{1x}} & \cdots & \frac{\partial G_{\widetilde{l}+1}}{\partial u_{nx}}\\ \vdots & & \vdots\\ \frac{\partial G_{2n-(d_1+d_2)}}{\partial u_{1x}} & \cdots & \frac{\partial G_{2n-(d_1+d_2)}}{\partial u_{nx}} \end{pmatrix}((L^{-1})_{\{d_2+1,n\}}\vdots(L^{-1})_{\{d_1+1,n\}})\right|_{(0,0,0)} \neq 0, \tag{3.44}$$

$$\det\left|\begin{pmatrix} \frac{\partial \overline{G}_1}{\partial u_1} & \cdots & \frac{\partial \overline{G}_1}{\partial u_n}\\ \vdots & & \vdots\\ \frac{\partial \overline{G}_{\widetilde{m}}}{\partial u_1} & \cdots & \frac{\partial \overline{G}_{\widetilde{m}}}{\partial u_n}\\ \frac{\partial \overline{G}_{\widetilde{m}+1}}{\partial u_{1t}} & \cdots & \frac{\partial \overline{G}_{\widetilde{m}+1}}{\partial u_{nt}}\\ \vdots & & \vdots\\ \frac{\partial \overline{G}_{d_1+d_2}}{\partial u_{1t}} & \cdots & \frac{\partial \overline{G}_{d_1+d_2}}{\partial u_{nt}} \end{pmatrix}((L^{-1}D^-)_{\{1,d_2\}}\vdots(L^{-1}D^+)_{\{1,d_1\}}) - \begin{pmatrix} 0 & \cdots & 0\\ \vdots & & \vdots\\ 0 & \cdots & 0\\ \frac{\partial \overline{G}_{\widetilde{m}+1}}{\partial u_{1x}} & \cdots & \frac{\partial \overline{G}_{\widetilde{m}+1}}{\partial u_{nx}}\\ \vdots & & \vdots\\ \frac{\partial \overline{G}_{d_1+d_2}}{\partial u_{1x}} & \cdots & \frac{\partial \overline{G}_{d_1+d_2}}{\partial u_{nx}} \end{pmatrix}((L^{-1})_{\{1,d_2\}}\vdots(L^{-1})_{\{1,d_1\}})\right|_{(0,0,0)} \neq 0. \tag{3.45}$$

Suppose that (3.5) holds and the conditions of C^2 compatibility are satisfied at the points $(t,x)=(T,0)$ and (T,L), respectively. Similarly, there exists a unique semi-global C^2 solution $u=u(t,x)$ on the domain $R(T)=\{(t,x)\mid 0\le t\le T, 0\le x\le L\}$ with small C^2 norm.

In particular, if the boundary conditions are given as

$$x=0:\ \begin{cases}(0,I_{n-d_2})L(0)u=H(t),\\ (0,I_{n-d_1})L(0)u_x=\widetilde{H}(t),\end{cases}\tag{3.46}$$

$$x=L:\ \begin{cases}(I_{d_2},0)L(0)u=\overline{H}(t),\\ (I_{d_1},0)L(0)u_x=\widetilde{\overline{H}}(t),\end{cases}\tag{3.47}$$

the assumptions (3.44)–(3.45) are automatically satisfied.

In particular, if the boundary conditions are given as (3.15) and the following boundary conditions:

$$x=0:\ \begin{cases}(0,I_{n-(d_1+d_2)})L(0)u=H(t),\\ (0,I_{n-(d_1+d_2)})L(0)u_x=\widetilde{H}(t),\end{cases}\tag{3.48}$$

$$x=L:\ \begin{cases}(I_{d_2},0)L(0)u=\overline{H}(t),\\ (I_{d_1},0)L(0)u_x=\widetilde{\overline{H}}(t),\end{cases}\tag{3.49}$$

it is easy to see that when (3.18) holds, (3.44)–(3.45) are also satisfied.

Theorem 3.3 *Suppose that a_{ij}, b_{ij}, c_i, λ_i, μ_i, l_{ij} $(i,j=1,\cdots,n)$ are C^1 functions with respect to their arguments, $(H(t),\overline{H}(t))$ is a $C^2\times C^1$ vector function. Suppose furthermore that* (3.6) *holds, and the conditions of C^2 compatibility are satisfied at the points $(t,x)=(0,0)$ and $(0,L)$, respectively. Then, for any given and possibly quite large $T>0$, the forward mixed initial-boundary value problem* (2.1), (3.1) *and* (3.20) *admits a unique semi-global C^2 solution $u=u(t,x)$ on the domain $R(T)=\{(t,x)\mid 0\le t\le T, 0\le x\le L\}$ with small C^2 norm, provided that the norms $\|(\varphi(x),\psi(x))\|_{C^2[0,L]\times C^1[0,L]}$, $\|(H(t),\overline{H}(t))\|_{C^2[0,T]\times C^1[0,T]}$ are small enough.*

Remark 3.5 For the backward mixed initial-boundary value problem (2.1), (3.2) and the following boundary condition:

$$x=L:\ u=\widetilde{H}(t),\quad u_x=\widetilde{\overline{H}}(t),\tag{3.50}$$

where $\widetilde{H}=(\widetilde{H}_1,\cdots,\widetilde{H}_n)^{\rm T}$ is a C^2 vector function, $\widetilde{\overline{H}}=(\widetilde{\overline{H}}_1,\cdots,\widetilde{\overline{H}}_n)^{\rm T}$ is a C^1 vector function, suppose that (3.6) holds and the conditions of C^2 compatibility are satisfied at the points $(t,x)=(T,0)$ and (T,L), respectively. Similarly, there exists a unique semi-global C^2 solution $u=u(t,x)$ on the domain $R(T)=\{(t,x)\mid 0\le t\le T, 0\le x\le L\}$ with small C^2 norm.

4 Local Exact Boundary Controllability

First, for Case (1) we consider the corresponding local exact boundary controllability.

Theorem 4.1 (Local Two-Sided Exact Boundary Controllability) *Suppose that a_{ij}, b_{ij}, c_i, λ_i, μ_i, l_{ij} $(i,j=1,\cdots,n)$ are C^1 functions with respect to their arguments. Suppose furthermore that* (3.7), (3.10) *and* (3.13) *hold. Let*

$$T>L\max_{i=1,\cdots,n}\Big\{\frac{1}{\widetilde{\lambda}_i^+(0,0,0)},\frac{1}{|\widetilde{\lambda}_i^-(0,0,0)|}\Big\}.\tag{4.1}$$

For any given initial data $(\varphi(x),\psi(x))$ and final data $(\Phi(x),\Psi(x))$ with small norms $\|(\varphi(x),\psi(x))\|_{C^2[0,L]\times C^1[0,L]}$ and $\|(\Phi(x),\Psi(x))\|_{C^2[0,L]\times C^1[0,L]}$, there exist boundary controls $(H_p(t),$

$H_q(t))$ and $(\overline{H}_r(t), \overline{H}_s(t))$ with small norms $\|(H_p(t), H_q(t))\|_{C^2[0,T]\times C^1[0,T]}$ and $\|(\overline{H}_r(t), \overline{H}_s(t))\|_{C^2[0,T]\times C^1[0,T]}$ $(p = 1, \cdots, l;\ q = l+1, \cdots, n;\ r = 1, \cdots, m;\ s = m+1, \cdots, n)$, such that the mixed initial-boundary value problem (2.1), (3.1) *and* (3.8)–(3.9) *admits a unique C^2 solution $u = u(t,x)$ with small C^2 norm on the domain $R(T) = \{(t,x) \mid 0 \le t \le T, 0 \le x \le L\}$, which satisfies exactly the final condition* (3.2).

In order to prove Theorem 4.1, it suffices to establish the following lemma (see [2]).

Lemma 4.1 *Under the assumptions of Theorem* 4.1*, for any given initial data (φ, ψ) and final data (Φ, Ψ) with small norms $\|(\varphi(x), \psi(x))\|_{C^2[0,L]\times C^1[0,L]}$ and $\|(\Phi(x), \Psi(x))\|_{C^2[0,L]\times C^1[0,L]}$, system* (2.1) *admits a C^2 solution $u = u(t,x)$ with small C^2 norm on the domain $R(T) = \{(t,x) \mid 0 \le t \le T, 0 \le x \le L\}$, which satisfies simultaneously the initial condition* (3.1) *and the final condition* (3.2).

Proof Noting (4.1), there exists an $\varepsilon > 0$ so small that

$$T > L \max_{i=1,\cdots,n} \Big\{ \max_{|(u,v,w)|\le\varepsilon} \frac{1}{\widetilde{\lambda}_i^+(u,v,w)}, \max_{|(u,v,w)|\le\varepsilon} \frac{1}{|\widetilde{\lambda}_i^-(u,v,w)|} \Big\}. \tag{4.2}$$

Let

$$T_1 = \frac{L}{2} \max_{i=1,\cdots,n} \Big\{ \max_{|(u,v,w)|\le\varepsilon} \frac{1}{\widetilde{\lambda}_i^+(u,v,w)}, \max_{|(u,v,w)|\le\varepsilon} \frac{1}{|\widetilde{\lambda}_i^-(u,v,w)|} \Big\}. \tag{4.3}$$

(1) First, we consider the forward mixed initial-boundary value problem for system (2.1) with the initial condition (3.1) and the following artificial boundary conditions:

$$x = 0: \ u_i = f_i(t), \quad i = 1, \cdots, n, \tag{4.4}$$

$$x = L: \ u_i = g_i(t), \quad i = 1, \cdots, n, \tag{4.5}$$

where f_i, g_i $(i = 1, \cdots, n)$ are any given C^2 functions with small $C^2[0, T_1]$ norms, such that the conditions of C^2 compatibility are satisfied at the points $(t,x) = (0,0)$ and $(0,L)$, respectively. By Theorem 3.1 and Remark 3.2, this forward problem has a unique semi-global C^2 solution $u = u^f(t,x)$ with small C^2 norm on the domain $R_f = \{(t,x) \mid 0 \le t \le T_1, 0 \le x \le L\}$. In particular, we have

$$|(u^f, u_x^f, u_t^f)(t,x)| \le \varepsilon, \quad \forall (t,x) \in R_f. \tag{4.6}$$

Thus, we can determine the corresponding value of (u^f, u_x^f) at $x = \frac{L}{2}$ as

$$x = \frac{L}{2}: \ (u^f, u_x^f) = (a(t), \overline{a}(t)), \quad 0 \le t \le T_1, \tag{4.7}$$

and $\|(a(t), \overline{a}(t))\|_{C^2[0,T_1]\times C^1[0,T_1]}$ is small enough.

(2) Similarly, we consider the backward mixed initial-boundary value problem for system (2.1) with the final condition (3.2) and the following artificial boundary conditions:

$$x = 0: \ u_i = \overline{f}_i(t), \quad i = 1, \cdots, n, \tag{4.8}$$

$$x = L: \ u_i = \overline{g}_i(t), \quad i = 1, \cdots, n, \tag{4.9}$$

where $\overline{f}_i$, $\overline{g}_i$ $(i = 1, \cdots, n)$ are any given C^2 functions with small $C^2[T - T_1, T]$ norms, such that the conditions of C^2 compatibility are satisfied at the points $(t,x) = (T,0)$ and (T,L), respectively. By Remarks 3.1 and 3.2, this backward problem has a unique semi-global C^2 solution $u = u^b(t,x)$ with small C^2 norm on the domain $R_b = \{(t,x) \mid T - T_1 \le t \le T, 0 \le x \le L\}$. In particular, we have

$$|(u^b, u_x^b, u_t^b)(t,x)| \le \varepsilon, \quad \forall (t,x) \in R_b. \tag{4.10}$$

Thus, we can determine the corresponding value of (u^b, u_x^b) at $x = \frac{L}{2}$ as

$$x = \frac{L}{2}: \ (u^b, u_x^b) = (b(t), \overline{b}(t)), \quad T - T_1 \le t \le T, \tag{4.11}$$

and $\|(b(t), \overline{b}(t))\|_{C^2[T-T_1,T]\times C^1[T-T_1,T]}$ is small enough.

(3) Since $2T_1 < T$, the two domains R_f and R_b never intersect each other. Then, we can find a vector function $(c(t), \overline{c}(t)) \in C^2[0,T] \times C^1[0,T]$ with small norm $\|(c(t), \overline{c}(t))\|_{C^2[0,T]\times C^1[0,T]}$, such that

$$(c(t), \overline{c}(t)) = \begin{cases} (a(t), \overline{a}(t)), & 0 \le t \le T_1, \\ (b(t), \overline{b}(t)), & T - T_1 \le t \le T. \end{cases} \tag{4.12}$$

Noting (2.6), we change the role of t and x so that system (2.1) can be equivalently rewritten as

$$u_{xx} + B^{-1} A u_{tx} + B^{-1} u_{tt} = B^{-1} C \tag{4.13}$$

in a neighborhood of $(u, v, w) = (0, 0, 0)$, which still satisfies the commutative condition $B^{-1} B^{-1} A = B^{-1} A B^{-1}$.

Consider the leftward mixed initial-boundary value problem for system (4.13) with the final condition

$$x = \frac{L}{2}: \ (u, u_x) = (c(t), \overline{c}(t)), \quad 0 \le t \le T \tag{4.14}$$

and the boundary conditions coming from the original initial condition (3.1) and final condition (3.2)

$$t = 0: \ u = \varphi(x), \quad 0 \le x \le \frac{L}{2}, \tag{4.15}$$

$$t = T: \ u = \Phi(x), \quad 0 \le x \le \frac{L}{2}. \tag{4.16}$$

By Theorem 3.1 and Remark 3.2, there exists a unique semi-global C^2 solution $u = u^l(t,x)$ with small C^2 norm on the domain $R_l = \{(t,x) \mid 0 \le t \le T, 0 \le x \le \frac{L}{2}\}$. In particular, we have

$$|(u^l, u_x^l, u_t^l)(t,x)| \le \varepsilon, \quad \forall (t,x) \in R_l. \tag{4.17}$$

(4) Similarly, the rightward mixed initial-boundary value problem for system (4.13) with the initial condition (4.14) and the boundary conditions coming from the original initial condition (3.1) and final condition (3.2)

$$t = 0: \ u = \varphi(x), \quad \frac{L}{2} \le x \le L, \tag{4.18}$$

$$t = T: \ u = \Phi(x), \quad \frac{L}{2} \le x \le L \tag{4.19}$$

admits a unique semi-global C^2 solution $u = u^r(t,x)$ with small C^2 norm on the domain $R_r = \{(t,x) \mid 0 \le t \le T, \frac{L}{2} \le x \le L\}$. In particular, we have

$$|(u^r, u^r_x, u^r_t)(t,x)| \le \varepsilon, \quad \forall (t,x) \in R_r. \tag{4.20}$$

(5) Let

$$u(t,x) = \begin{cases} u^l(t,x), & (t,x) \in R_l, \\ u^r(t,x), & (t,x) \in R_r. \end{cases} \tag{4.21}$$

Obviously, $u \in C^2[R(T)]$, and it satisfies system (2.1) on the whole domain $R(T)$.

The only thing left is to prove that $u = u(t,x)$ satisfies the initial condition (3.1) and the final condition (3.2).

In fact, the C^2 solutions $u = u^l(t,x)$ (resp., $u = u^r(t,x)$) and $u = u^f(t,x)$ satisfy simultaneously the one-sided mixed initial-boundary value problem for the same system (2.1) (i.e., (4.13)) with the same final (resp., initial) condition

$$x = \frac{L}{2}: \ (u, u_x) = (a(t), \overline{a}(t)), \quad 0 \le t \le T_1 \tag{4.22}$$

and the same boundary condition (4.15) (resp., (4.18)). Noting the choice of T_1 in (4.3), it is easy to see that the maximum determinate domain of this one-sided mixed problem contains the triangular domain

$$\begin{aligned} &\Big\{(t,x) \Big| \, 0 \le t \le \frac{2T_1 x}{L}, 0 \le x \le \frac{L}{2}\Big\} \\ &\Big(\text{resp., } \Big\{(t,x) \Big| \, 0 \le t \le \frac{2T_1(L-x)}{L}, \frac{L}{2} \le x \le L\Big\}\Big). \end{aligned} \tag{4.23}$$

Then, by the uniqueness of C^2 solution to the one-sided mixed initial-boundary value problem (see [2, 7]), we have that $u(t,x) \equiv u^f(t,x)$ on these domains and, in particular, (3.1) holds. In a similar way, we have (3.2). Thus, $u = u(t,x)$ satisfies all the requirements of Lemma 4.1.

Theorem 4.2 (Local One-Sided Exact Boundary Controllability) *Suppose that a_{ij}, b_{ij}, c_i, λ_i, μ_i and l_{ij} $(i,j = 1, \cdots, n)$ are C^1 functions with respect to their arguments. Suppose furthermore that* (3.7), (3.10)–(3.11) *and* (3.13) *hold. Let*

$$T > L\Big(\max_{i=1,\cdots,n} \frac{1}{\widetilde{\lambda}^+_i(0,0,0)} + \max_{i=1,\cdots,n} \frac{1}{|\widetilde{\lambda}^-_i(0,0,0)|}\Big). \tag{4.24}$$

For any given initial data $(\varphi(x), \psi(x))$ and final data $(\Phi(x), \Psi(x))$ with small norms $\|(\varphi(x), \psi(x))\|_{C^2[0,L]\times C^1[0,L]}$ and $\|(\Phi(x), \Psi(x))\|_{C^2[0,L]\times C^1[0,L]}$, and for any given boundary functions $(H_p(t), H_q(t))$ with small norms $\|(H_p(t), H_q(t))\|_{C^2[0,T]\times C^1[0,T]}$ $(p = 1, \cdots, l;\ q = l+1, \cdots, n)$, such that the conditions of C^2 compatibility are satisfied at the points $(t,x) = (0,0)$ and $(T,0)$, respectively, there exist boundary controls $(\overline{H}_r(t), \overline{H}_s(t))$ with small norms $\|(\overline{H}_r(t), \overline{H}_s(t))\|_{C^2[0,T]\times C^1[0,T]}$ $(r = 1, \cdots, m;\ s = m+1, \cdots, n)$, such that the mixed initial-boundary value problem (2.1), (3.1) *and* (3.8)–(3.9) *admits a unique C^2 solution $u = u(t,x)$ with small C^2 norm in the domain $R(T) = \{(t,x) \mid 0 \le t \le T, 0 \le x \le L\}$, which satisfies exactly the final conditions* (3.2).

In order to prove Theorem 4.2, it suffices to establish the following lemma (see [2]).

Lemma 4.2 *Under the assumptions of Theorem* 4.2*, for any given initial data* (φ, ψ) *and final data* (Φ, Ψ) *with small norms* $\|(\varphi(x), \psi(x))\|_{C^2[0,L]\times C^1[0,L]}$ *and* $\|(\Phi(x), \Psi(x))\|_{C^2[0,L]\times C^1[0,L]}$*, for any given boundary functions* $(H_p(t), H_q(t))$ *with small norms* $\|(H_p(t), H_q(t))\|_{C^2[0,T]\times C^1[0,T]}$ $(p = 1, \cdots, l;\ q = l+1, \cdots, n)$*, such that the conditions of* C^2 *compatibility are satisfied at the points* $(t, x) = (0, 0)$ *and* $(T, 0)$*, respectively, system* (2.1) *admits a* C^2 *solution* $u = u(t, x)$ *with small* C^2 *norm on the domain* $R(T) = \{(t, x) \mid 0 \le t \le T, 0 \le x \le L\}$*, which satisfies simultaneously the initial condition* (3.1)*, the final condition* (3.2) *and the boundary condition* (3.8)*.*

Proof Noting (4.24), there exists an $\varepsilon > 0$ so small that

$$T > L\Big(\max_{\substack{|(u,v,w)|\le\varepsilon\\ i=1,\cdots,n}} \frac{1}{\widetilde{\lambda}_i^+(u,v,w)} + \max_{\substack{|(u,v,w)|\le\varepsilon\\ i=1,\cdots,n}} \frac{1}{|\widetilde{\lambda}_i^-(u,v,w)|}\Big). \tag{4.25}$$

Let

$$T_f = L \max_{\substack{|(u,v,w)|\le\varepsilon\\ i=1,\cdots,n}} \frac{1}{|\widetilde{\lambda}_i^-(u,v,w)|}, \quad T_b = L \max_{\substack{|(u,v,w)|\le\varepsilon\\ i=1,\cdots,n}} \frac{1}{\widetilde{\lambda}_i^+(u,v,w)}. \tag{4.26}$$

(1) First, we consider the forward mixed initial-boundary value problem for system (2.1) with the initial condition (3.1), the boundary condition (3.8) and the artificial boundary condition (4.5) (in which $g_i(t)$ $(i = 1, \cdots, n)$ satisfy the same assumptions). By Theorem 3.1 and Remark 3.2, and noting (3.10), this forward problem has a unique semi-global C^2 solution $u = u^f(t, x)$ with small C^2 norm in the domain $R_f = \{(t, x) \mid 0 \le t \le T_f, 0 \le x \le L\}$, and (4.6) holds. Thus, we can determine the corresponding value of (u^f, u^f_x) at $x = 0$ as

$$x = 0: \ (u^f, u^f_x) = (a(t), \overline{a}(t)), \quad 0 \le t \le T_f, \tag{4.27}$$

$\|(a(t), \overline{a}(t))\|_{C^2[0,T_f]\times C^1[0,T_f]}$ is small enough, and $(a(t), \overline{a}(t), a'(t))$ satisfies the boundary condition (3.8) on $x = 0$ for $0 \le t \le T_f$.

(2) Similarly, considering the backward mixed initial-boundary value problem for system (2.1) with the final condition (3.2), the boundary condition (3.8) and the artificial boundary condition (4.9) (in which $\overline{g}_i(t)$ $(i = 1, \cdots, n)$ satisfy the same assumptions). Noting (3.11), by Remarks 3.1 and 3.2, this backward problem has a unique semi-global C^2 solution $u = u^b(t, x)$ with small C^2 norm on the domain $R_b = \{(t, x) \mid T - T_b \le t \le T, 0 \le x \le L\}$, and (4.10) holds. Thus, we can determine the corresponding value of (u^b, u^b_x) at $x = 0$ as

$$x = 0: \ (u^b, u^b_x) = (b(t), \overline{b}(t)), \quad T - T_b \le t \le T, \tag{4.28}$$

$\|(b(t), \overline{b}(t))\|_{C^2[T-T_b,T]\times C^1[T-T_b,T]}$ is small enough, and $(b(t), \overline{b}(t), b'(t))$ satisfies the boundary condition (3.8) on $x = 0$ for $T - T_b \le t \le T$.

(3) Since $T_f + T_b < T$, the two domains R_f and R_b never intersect each other. Then we can find a vector function $(c(t), \overline{c}(t)) \in C^2[0, T]\times C^1[0, T]$ with small norm $\|(c(t), \overline{c}(t))\|_{C^2[0,T]\times C^1[0,T]}$, such that

$$(c(t), \overline{c}(t)) = \begin{cases} (a(t), \overline{a}(t)), & 0 \le t \le T_f, \\ (b(t), \overline{b}(t)), & T - T_b \le t \le T, \end{cases} \tag{4.29}$$

and $(u, u_x, u_t) = (c(t), \overline{c}(t), c'(t))$ satisfies the boundary condition (3.8) on the whole interval $[0, T]$.

Changing the role of t and x, we now consider the rightward mixed initial-boundary value problem for system (4.13) with the initial condition

$$x=0: \ u=c(t), \quad u_x=\overline{c}(t), \quad 0 \le t \le T \tag{4.30}$$

and the boundary conditions

$$t=0: \quad u=\varphi(x), \quad 0 \le x \le L, \tag{4.31}$$

$$t=T: \ u=\psi(x), \quad 0 \le x \le L. \tag{4.32}$$

By Theorem 3.1 and Remark 3.2, we get a unique semi-global C^2 solution $u=u(t,x)$ with small C^2 norm on the domain $R(T)=\{(t,x) \mid 0 \le t \le T, 0 \le x \le L\}$, and

$$|(u,u_x,u_t)(t,x)| \le \varepsilon, \quad \forall (t,x) \in R(T). \tag{4.33}$$

(4) The C^2 solutions $u=u(t,x)$ and $u=u^f(t,x)$ satisfy simultaneously the one-sided mixed initial-boundary value problem for the same system (2.1) (i.e., (4.13)) with the same initial condition

$$x=0: \ (u,u_x)=(a(t),\overline{a}(t)), \quad 0 \le t \le T_f \tag{4.34}$$

and the same boundary condition (4.31). By the choice of T_f in (4.26), it is easy to see that the maximum determinate domain of this one-sided mixed problem contains the triangular domain

$$\Big\{(t,x) \,\Big|\, 0 \le t \le \frac{T_f}{L}(L-x), \ 0 \le x \le L\Big\}. \tag{4.35}$$

By the uniqueness of C^2 solution to the one-sided mixed initial-boundary value problem (see [2, 7]), we have $u(t,x) \equiv u^f(t,x)$ on this domain and, in particular, (3.1) holds. In a similar way, we have (3.2). Thus, $u=u(t,x)$ satisfies all the requirements of Lemma 4.2.

Remark 4.1 In Case (1), the number of positive eigenvalues is equal to the number of negative eigenvalues for system (2.1), we can still realize the one-sided local exact boundary controllability by suitable boundary controls acting on the end $x=0$, provided that assumptions (3.10)–(3.11) and (3.13) are replaced by assumptions (3.10) and (3.12)–(3.13).

5 Local Two-Sided and One-Sided Exact Boundary Controllabilities for Other Cases

In Case (2), corresponding local two-sided and one-sided exact boundary controllabilities are as follows.

Theorem 5.1 (Local Two-Sided Exact Boundary Controllability) *Suppose that a_{ij}, b_{ij}, c_i, λ_i, μ_i, l_{ij} $(i,j=1,\cdots,n)$ are C^1 functions with respect to their arguments. Suppose furthermore that* (3.5)*,* (3.14)*,* (3.17) *and* (3.19) *hold. Let*

$$T > L \max_{i=1,\cdots,n} \frac{1}{|\widetilde{\lambda}_i^{\pm}(0,0,0)|}. \tag{5.1}$$

For any given initial data $(\varphi(x),\psi(x))$ and final data $(\Phi(x),\Psi(x))$ with small norms $\|(\varphi(x),\psi(x))\|_{C^2[0,L]\times C^1[0,L]}$ and $\|(\Phi(x),\Psi(x))\|_{C^2[0,L]\times C^1[0,L]}$, there exist boundary controls $(H_p(t)$,

$H_q(t)$) and $(\overline{H}_r(t), \overline{H}_s(t))$ with small norms $\|(H_p(t), H_q(t))\|_{C^2[0,T]\times C^1[0,T]}$ and $\|(\overline{H}_r(t), \overline{H}_s(t))\|_{C^2[0,T]\ \times C^1[0,T]}$ $(p = 1, \cdots, l;\ q = l+1, \cdots, d_1+d_2;\ r = 1, \cdots, m;\ s = m+1, \cdots, 2n-(d_1+d_2))$, such that the mixed initial-boundary value problem (2.1), (3.1) *and* (3.15)–(3.16) *admits a unique C^2 solution $u = u(t,x)$ with small C^2 norm on the domain $R(T) = \{(t,x) \mid 0 \le t \le T, 0 \le x \le L\}$, which satisfies exactly the final condition* (3.2).

In order to prove Theorem 5.1, it suffices to establish the following lemma.

Lemma 5.1 *Under the assumptions of Theorem* 5.1*, for any given initial data (φ, ψ) and final data (Φ,Ψ) with small norms $\|(\varphi(x),\psi(x))\|_{C^2[0,L]\times C^1[0,L]}$ and $\|(\Phi(x),\Psi(x))\|_{C^2[0,L]\times C^1[0,L]}$, system* (2.1) *admits a C^2 solution $u = u(t,x)$ with small C^2 norm on the domain $R(T) = \{(t,x) \mid 0 \le t \le T, 0 \le x \le L\}$, which satisfies simultaneously the initial condition* (3.1) *and the final condition* (3.2).

Proof Noting (5.1), there exists an $\varepsilon > 0$ so small that

$$T > L \max_{i=1,\cdots,n} \max_{|(u,v,w)|\le\varepsilon} \frac{1}{|\widetilde{\lambda}_i^{\pm}(u,v,w)|}. \tag{5.2}$$

Let

$$T_1 = \frac{L}{2} \max_{i=1,\cdots,n} \max_{|(u,v,w)|\le\varepsilon} \frac{1}{|\widetilde{\lambda}_i^{\pm}(u,v,w)|}. \tag{5.3}$$

(1) First, we consider the forward mixed initial-boundary value problem for system (2.1) with the initial condition (3.1) and the following artificial boundary conditions:

$$x = 0: \quad \begin{cases} (I_{d_2}, 0)L(0)u = f(t), \\ (I_{d_1}, 0)L(0)u_x = \widetilde{f}(t), \end{cases} \tag{5.4}$$

$$x = L: \quad \begin{cases} (0, I_{n-d_2})L(0)u = \overline{f}(t), \\ (0, I_{n-d_1})L(0)u_x = \widetilde{\overline{f}}(t), \end{cases} \tag{5.5}$$

where

$$\begin{aligned} f(t) &= (f_1(t), \cdots, f_{d_2}(t)), \\ \overline{f}(t) &= (\overline{f}_1(t), \cdots, \overline{f}_{n-d_2}(t)) \end{aligned}$$

are any given C^2 vector functions with small $C^2[0,T_1]$ norms, and

$$\begin{aligned} \widetilde{f}(t) &= (f_{d_2+1}(t), \cdots, f_{d_1+d_2}(t)), \\ \widetilde{\overline{f}}(t) &= (\overline{f}_{n-d_2+1}(t), \cdots, \overline{f}_{2n-(d_1+d_2)}(t)) \end{aligned}$$

are any given C^1 vector functions with small $C^1[0,T_1]$ norms, such that the conditions of C^2 compatibility are satisfied at the points $(t,x) = (0,0)$ and $(0,L)$, respectively. By Theorem 3.2 and Remark 3.3, this forward problem has a unique semi-global C^2 solution $u = u^f(t,x)$ with small C^2 norm on the domain $R_f = \{(t,x) \mid 0 \le t \le T_1, 0 \le x \le L\}$. In particular, we have (4.6). Thus, we can determine the corresponding value of (u^f, u^f_x) at $x = \frac{L}{2}$ as $(a(t), \overline{a}(t))$ $(0 \le t \le T_1)$, which satisfies (4.7) and the corresponding conditions.

(2) Similarly, we consider the backward mixed initial-boundary value problem for system (2.1) with the final condition (3.2) and the following artificial boundary conditions:

$$x = 0: \quad \begin{cases} (0, I_{n-d_2})L(0)u = g(t), \\ (0, I_{n-d_1})L(0)u_x = \widetilde{g}(t), \end{cases} \tag{5.6}$$

$$x = L: \quad \begin{cases} (I_{d_2}, 0)L(0)u = \overline{g}(t), \\ (I_{d_1}, 0)L(0)u_x = \widetilde{\overline{g}}(t), \end{cases} \tag{5.7}$$

where $g(t) = (g_1(t), \cdots, g_{n-d_2}(t))$, $\overline{g}(t) = (\overline{g}_1(t), \cdots, \overline{g}_{d_2}(t))$ are any given C^2 vector functions with small $C^2[T-T_1, T]$ norms, $\widetilde{g}(t) = (g_{n-d_2+1}(t), \cdots, g_{2n-(d_1+d_2)}(t))$, $\widetilde{\overline{g}}(t) = (\overline{g}_{d_2+1}(t), \cdots, \overline{g}_{d_1+d_2}(t))$ are any given C^1 vector functions with small $C^1[T-T_1, T]$ norms, such that the conditions of C^2 compatibility are satisfied at the points $(t, x) = (T, 0)$ and (T, L), respectively. By Remark 3.4, this backward problem has a unique semi-global C^2 solution $u = u^b(t, x)$ with small C^2 norm on the domain $R_b = \{(t, x) \mid T - T_1 \le t \le T, 0 \le x \le L\}$. In particular, we have (4.10). Thus, we can determine the corresponding value of (u^b, u^b_x) at $x = \frac{L}{2}$ as $(b(t), \overline{b}(t))$ $(T - T_1 \le t \le T)$, which satisfies (4.11) and corresponding conditions.

(3) Since $2T_1 < T$, the two domains R_f and R_b never intersect each other. Then, we can find a vector function $(c(t), \overline{c}(t)) \in C^2[0, T] \times C^1[0, T]$ with small norm $\|(c(t), \overline{c}(t))\|_{C^2[0,T]\times C^1[0,T]}$, such that (4.12) holds. Noting (2.6), we change the role of t and x so that system (2.1) can be equivalently rewritten to (4.13) in a neighborhood of $(u, v, w) = (0, 0, 0)$.

Consider the leftward mixed initial-boundary value problem for system (4.13) with the final condition (4.14) and the following boundary conditions coming from the original initial condition (3.1) and final condition (3.2):

$$t = 0: \begin{cases} (0, I_{n-d_2})L(0)u = (0, I_{n-d_2})L(0)\varphi(x), \\ (0, I_{n-d_1})L(0)u_t = (0, I_{n-d_1})L(0)\psi(x), \end{cases} \quad 0 \le x \le \frac{L}{2}, \tag{5.8}$$

$$t = T: \begin{cases} (I_{d_2}, 0)L(0)u = (I_{d_2}, 0)L(0)\Phi(x), \\ (I_{d_1}, 0)L(0)u_t = (I_{d_1}, 0)L(0)\Psi(x), \end{cases} \quad 0 \le x \le \frac{L}{2}. \tag{5.9}$$

By Remark 3.4, there exists a unique semi-global C^2 solution $u = u^l(t, x)$ with small C^2 norm on the domain $R_l = \{(t, x) \mid 0 \le t \le T, 0 \le x \le \frac{L}{2}\}$. In particular, we have (4.17).

(4) Similarly, by Remark 3.3, the rightward mixed initial-boundary value problem for system (4.13) with the initial condition (4.14) and the following boundary conditions coming from the original initial condition (3.1) and final condition (3.2):

$$t = 0: \begin{cases} (I_{d_2}, 0)L(0)u = (I_{d_2}, 0)L(0)\varphi(x), \\ (I_{d_1}, 0)L(0)u_t = (I_{d_1}, 0)L(0)\psi(x), \end{cases} \quad \frac{L}{2} \le x \le L, \tag{5.10}$$

$$t = T: \begin{cases} (0, I_{n-d_2})L(0)u = (0, I_{n-d_2})L(0)\Phi(x), \\ (0, I_{n-d_1})L(0)u_t = (0, I_{n-d_1})L(0)\Psi(x), \end{cases} \quad \frac{L}{2} \le x \le L \tag{5.11}$$

admits a unique semi-global C^2 solution $u = u^r(t, x)$ with small C^2 norm on the domain $R_r = \{(t, x) \mid 0 \le t \le T, \frac{L}{2} \le x \le L\}$. In particular, we have (4.20).

(5) Let $u(t, x)$ be defined by (4.21). Obviously, $u \in C^2[R(T)]$ and it satisfies system (2.1) on the whole domain $R(T)$. The only thing left is to prove that $u = u(t, x)$ satisfies the initial condition (3.1) and the final condition (3.2).

In fact, the C^2 solutions $u = u^l(t,x)$ (resp., $u = u^r(t,x)$) and $u = u^f(t,x)$ satisfy simultaneously the one-sided mixed initial-boundary value problem for the same system (2.1) (i.e., (4.13)) with the same final (resp., initial) condition (4.22) and the same boundary condition (5.8) (resp., (5.10)). Noting the choice of T_1 in (5.3), it is easy to see that the maximum determinate domain of this one-sided mixed problem contains the triangular domain (4.23). Then, by the uniqueness of C^2 solution to the one-sided mixed initial-boundary value problem (see [2, 7]), we have $u(t,x) \equiv u^f(t,x)$ on these domains, and in particular, (3.1) holds. In a similar way, we have (3.2). Thus, $u = u(t,x)$ satisfies all the requirements of Lemma 5.1.

Theorem 5.2 (Local One-Sided Exact Boundary Controllability) *Suppose that a_{ij}, b_{ij}, c_i, λ_i, μ_i and l_{ij} $(i,j = 1,\cdots,n)$ are C^1 functions with respect to their arguments. Suppose furthermore that* (3.5)*,* (3.14) *and* (3.17)–(3.19) *hold. Let*

$$\begin{aligned} T > L\Big(& \max_{\substack{j=1,\cdots,d_1\\ k=d_1+1,\cdots,d_2}} \Big\{\frac{1}{\widetilde{\lambda}_j^{\pm}(0,0,0)}, \frac{1}{\widetilde{\lambda}_k^{+}(0,0,0)}\Big\} \\ & + \max_{\substack{k=d_1+1,\cdots,d_2\\ h=d_2+1,\cdots,n}} \Big\{\frac{1}{|\widetilde{\lambda}_k^{-}(0,0,0)|}, \frac{1}{|\widetilde{\lambda}_h^{\pm}(0,0,0)|}\Big\}\Big). \end{aligned} \tag{5.12}$$

For any given initial data $(\varphi(x), \psi(x))$ and final data $(\Phi(x), \Psi(x))$ with small norms $\|(\varphi(x), \psi(x))\|_{C^2[0,L]\times C^1[0,L]}$ and $\|(\Phi(x), \Psi(x))\|_{C^2[0,L]\times C^1[0,L]}$, and for any given boundary functions $(H_p(t), H_q(t))$ with small norms $\|(H_p(t), H_q(t))\|_{C^2[0,T]\times C^1[0,T]}$ $(p = 1,\cdots,l;\ q = l+1,\cdots,d_1+d_2)$, such that the conditions of C^2 compatibility are satisfied at the points $(t,x) = (0,0)$ and $(T,0)$, respectively, there exist boundary controls $(\overline{H}_r(t), \overline{H}_s(t))$ with small norms $\|(\overline{H}_r(t), \overline{H}_s(t))\|_{C^2[0,T]\times C^1[0,T]}$ $(r = 1,\cdots,m;\ s = m+1,\cdots,2n-(d_1+d_2))$, such that the mixed initial-boundary value problem (2.1), (3.1) *and* (3.15)–(3.16) *admits a unique C^2 solution $u = u(t,x)$ with small C^2 norm on the domain $R(T) = \{(t,x) \mid 0 \le t \le T, 0 \le x \le L\}$, which satisfies exactly the final condition* (3.2).

Remark 5.1 The proof of Theorem 5.2 is similar to that of Theorem 4.2. Noting (3.14) and (3.18), in the second step, we need to solve the backward mixed initial-boundary value problem for system (2.1) with the final condition (3.2), the boundary condition (3.15) and the following artificial boundary conditions:

$$x = 0: \quad \begin{cases} (0, I_{n-(d_1+d_2)})L(0)u = g(t), \\ (0, I_{n-(d_1+d_2)})L(0)u_x = \widetilde{g}(t), \end{cases} \tag{4.9$'$}$$

$$x = L: \quad \begin{cases} (I_{d_2}, 0)L(0)u = \overline{g}(t), \\ (I_{d_1}, 0)L(0)u_x = \widetilde{\overline{g}}(t). \end{cases} \tag{5.13}$$

By Remark 3.4, if the corresponding norms of $g(t), \widetilde{g}(t), \overline{g}(t)$ and $\widetilde{\overline{g}}(t)$ are small enough and the conditions of C^2 compatibility are satisfied at the points $(t,x) = (0,0)$ and $(T,0)$, respectively, then we can get the existence and uniqueness of the semi-global C^2 solution. The other steps are similar to those in the proof of Theorem 4.2.

For Case (3), we need only to consider the local one-sided exact boundary controllability with controls acting on the end $x = 0$. A similar conclusion can be obtained.

Theorem 5.3 (Local One-Sided Exact Boundary Controllability on the End $x = 0$) *Suppose that a_{ij}, b_{ij}, c_i, λ_i, μ_i, l_{ij} $(i,j = 1,\cdots,n)$ are C^1 functions with respect to their arguments.*

Suppose furthermore that (3.6) *holds. Let*

$$T > L \max_{i=1,\cdots,n} \frac{1}{\widetilde{\lambda}_i^-(0,0,0)}. \tag{5.14}$$

For any given initial data $(\varphi(x), \psi(x))$ *and final data* $(\Phi(x), \Psi(x))$ *with small norms* $\|(\varphi(x), \psi(x))\|_{C^2[0,L]\times C^1[0,L]}$ *and* $\|(\Phi(x), \Psi(x))\|_{C^2[0,L]\times C^1[0,L]}$, *there exist boundary controls* $(H(t), \overline{H}(t))$ *with small norm* $\|(H(t), \overline{H}(t))\|_{C^2[0,T]\times C^1[0,T]}$, *such that the mixed initial-boundary value problem* (2.1), (3.1) *and* (3.20) *admits a unique* C^2 *solution* $u = u(t,x)$ *with small* C^2 *norm on the domain* $R(T) = \{(t,x) \mid 0 \le t \le T, 0 \le x \le L\}$, *which satisfies exactly the final condition* (3.2).

Remark 5.2 The second-order quasilinear hyperbolic equation $u_{tt}+a(u,u_x,u_t)u_{tx}+b(u,u_x,u_t)u_{xx} = c(u,u_x,u_t)$ considered in [13], where u is the unknown function of (t,x), is a special form of system (2.1) for $n = 1$.

Remark 5.3 The second-order quasilinear hyperbolic system $u_{tt} - A(u,u_x,u_t)u_{xx} = F(u,u_x,u_t)$ considered in [11], where $u = (u_1,\cdots,u_n)^{\mathrm{T}}$ is the unknown vector function of (t,x), is also a special form of system (2.1). The conclusions obtained in this paper can also be applied to this kind of second-order quasilinear hyperbolic systems.

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