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Weighted Inequalities for the Generalized Maximal Operator in Martingale Spaces*

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Abstract The generalized maximal operator $\mathcal{M}$ in martingale spaces is considered. For $1 < p \le q < \infty$, the authors give a necessary and sufficient condition on the pair $(\widehat{\mu}, v)$ for $\mathcal{M}$ to be a bounded operator from martingale space $L^p(v)$ into $L^q(\widehat{\mu})$ or weak-$L^q(\widehat{\mu})$, where $\widehat{\mu}$ is a measure on $\Omega \times \mathbb{N}$ and v a weight on Ω. Moreover, the similar inequalities for usual maximal operator are discussed.

Keywords Martingale, Maximal operator, Weighted inequality, Carleson measure
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1 Introduction

Let $\mathbb{R}^n$ be the n-dimensional real Euclidean space and f a real valued measurable function. The classical Hardy-Littlewood maximal operator M is defined by

$$Mf(x) = \sup_{x\in Q} \frac{1}{|Q|} \int_Q |f(y)| \mathrm{d}y,$$

where Q runs over the class of non-degenerate cubes with sides parallel to the coordinate axes and $|Q|$ is the Lebesgue measure of Q. The generalized maximal operator $\mathcal{M}$ is defined by

$$\mathcal{M}f(x,t) = \sup_{Q} \frac{1}{|Q|} \int_Q |f(y)| \mathrm{d}y,$$

where the supremum is taken over the cubes Q in $\mathbb{R}^n$, containing x and having side length at least t.

Let u, v be two weights, i.e., positive measurable functions. As well known, for $p > 1$, B. Muckenhoupt [12] showed that the inequality

$$\int_{\mathbb{R}^n} (Mf(x))^p v(x) \mathrm{d}x \le C \int_{\mathbb{R}^n} |f(x)|^p v(x) \mathrm{d}x, \quad \lambda > 0,\ f \in L^p(v)$$

holds if and only if v satisfies

$$\sup_Q \Big(\frac{1}{|Q|}\int_Q v(x)\mathrm{d}x\Big)\Big(\frac{1}{|Q|}\int_Q v(x)^{-\frac{1}{p-1}}\mathrm{d}x\Big)^{p-1} < \infty.$$

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On the other hand, E. T. Sawyer [15] obtained a characterization for the weak-type inequality

$$\lambda|\{Mf>\lambda\}|_u^{\frac{1}{q}} \le C\Big(\int_{\mathbb{R}^n}|f(x)|^p v(x)\mathrm{d}x\Big)^{\frac{1}{p}}, \quad f\in L^p(v).$$

Also, when $1<p\le q<\infty$, a necessary and sufficient condition was established in order that the weighted inequality

$$\Big(\int_{\mathbb{R}^n}(Mf(x))^q u(x)\mathrm{d}x\Big)^{\frac{1}{q}} \le C\Big(\int_{\mathbb{R}^n}|f(x)|^p v(x)\mathrm{d}x\Big)^{\frac{1}{p}}, \quad f\in L^p(v)$$

hold (see [3, 16]).

Moreover, let $\widehat{\mu}$ be a measure on $\mathbb{R}_+^{n+1}$ and v a weight on $\mathbb{R}^n$. For $p>1$, a characterization of those pairs $(\widehat{\mu},v)$ was found in [13] for which the generalized maximal operator $\mathcal{M}$ is bounded from $L^p(\mathbb{R}^n,v)$ into weak-$L^p(\mathbb{R}_+^{n+1},\widehat{\mu})$, and a characterization of those pairs $(\widehat{\mu},v)$ was given in [14] for which $\mathcal{M}$ is bounded from $L^p(\mathbb{R}^n,v)$ into $L^p(\mathbb{R}_+^{n+1},\widehat{\mu})$. In [1] and [4], the analogues of the above results have been developed in spaces of (non-)homogeneous type.

Now let $(\Omega,\mathcal{F},\mu)$ be a complete probability space and $(\mathcal{F}_n)_{n\ge0}$ an increasing sequence of sub-σ-fields of $\mathcal{F}$ with $\mathcal{F}=\bigvee_{n\ge0}\mathcal{F}_n$. The maximal operator M for martingale $f=(f_n)$ is defined by $Mf=\sup_{n\ge0}|f_n|$. A weight u is a random variable with $u>0$ and $E(u)<\infty$. In this paper, for $p\ge1$, a martingale $f=(f_n)_{n\ge0}\in L^p(\omega)$ is meant as $f_n=E(f\mid\mathcal{F}_n),\ f\in L^p(\omega)$.

As well known, in regular martingale spaces, M. Izumisawa and N. Kazamaki [5] characterized the inequality

$$\Big(\int_\Omega (Mf)^p v\mathrm{d}\mu\Big)^{\frac{1}{p}} \le C\Big(\int_\Omega |f|^p v\mathrm{d}\mu\Big)^{\frac{1}{p}},$$

where $p>1$ and v is a weight. Comparing with [5], it was found that the regularity is superfluous (see [10] or [6]). In addition, some characterizations were obtained (see [11]) for the inequalities

$$\lambda|\{Mf>\lambda\}|_u^{\frac{1}{q}} \le C\Big(\int_\Omega |f|^p v\mathrm{d}\mu\Big)^{\frac{1}{p}}, \quad f=(f_n)\in L^p(v)$$

and

$$\Big(\int_\Omega (Mf)^q u\mathrm{d}\mu\Big)^{\frac{1}{q}} \le C\Big(\int_\Omega |f|^p v\mathrm{d}\mu\Big)^{\frac{1}{p}}, \quad f=(f_n)\in L^p(v),$$

where $1<p\le q<\infty$ and (u,v) is a pair of weights. In [9] and [8], interpolation theorems for the weighted Hardy martingale spaces were also discussed.

In this paper, we discuss some weighted inequalities in martingale setting. In Section 2, we deal with the generalized maximal operator $\mathcal{M}$ associated with Carleson measures. To get our result, we adopt a direct method instead of Marcinkiewicz's interpolation method. In Section 3, we consider the boundedness of maximal operator by the tailed maximal operator. In fact, Theorem 3.1 shows that some assumptions in [2] are superfluous. The rest of Section 1 consists of the preliminaries for the whole paper.

In [10] and [7], Carleson measure was discussed by using martingale theory. Recall that a nonnegative measure $\widehat{\mu}$ on $\Omega\times\mathbb{N}$ is said to be a Carleson measure, if

$$\sup_{\tau\in\mathcal{T}}|\{\tau<\infty\}|_\mu^{-1}|\widehat{\{\tau<\infty\}}|_{\widehat{\mu}}<\infty, \tag{1.1}$$

where

$$\widehat{\{\tau<\infty\}}=\{(\omega,k):\ (\omega,k)\in\Omega\times\mathbb{N},\ k\ge\tau(\omega)\}$$

and $\mathcal{T}$ is the set of all stopping times with respect to $(\mathcal{F}_n)_{n\geq 0}$. Let

$$\mathcal{M}^{(n)}f(\omega) = \sup_{n\geq m\geq 0} |f_m(\omega)|, \quad \forall(\omega,n)\in\Omega\times\mathbb{N}.$$

It is clear that $\mathcal{M}^{(\cdot)}f(\cdot)$ is a measurable function on $\Omega\times\mathbb{N}$ for every martingale f, and it is said to be f's generalized maximal function and denoted by $\mathcal{M}f$.

Fix $\lambda>0$, and let $\tau=\inf\{n:|f_n|>\lambda\}$. Then $\widehat{\{\tau<\infty\}}=\{\mathcal{M}f>\lambda\}$. Throughout this paper, we always suppose that $|B\times\mathbb{N}|_{\widehat{\mu}}=0$, if $|B|_\mu=0$.

In addition, Chang [2] introduced the tailed maximal operator. For $n\in\mathbb{N}$, let

$$^*M_nf = \sup_{m\geq n}|f_m| \quad (\text{or } {}^*f_n = \sup_{m\geq n}|f_m|).$$

$^*M_nf$ (or $^*f_n$) is said to be the n-th tailed maximal operator.

2 The Generalized Maximal Operator

Theorem 2.1 *Given p and q with $1<p\leq q<\infty$ and a weight v on Ω, suppose that $\sigma=v^{-\frac{1}{p-1}}\in L^1(\mu)$ and $\widehat{\mu}$ is a nonnegative measure on $\Omega\times\mathbb{N}$. Then the following statements are equivalent:*

(1) *There exists a positive constant C_1 such that*

$$\Big(\int_{\widehat{\{\tau<\infty\}}}(\mathcal{M}(\sigma\chi_{\{\tau<\infty\}}))^q\mathrm{d}\widehat{\mu}\Big)^{\frac{1}{q}}\leq C_1\Big(\int_{\{\tau<\infty\}}\sigma\mathrm{d}\mu\Big)^{\frac{1}{p}}, \quad \forall\tau\in\mathcal{T}; \tag{2.1}$$

(2) *There exists a positive constant C_2 such that*

$$\Big(\int_{\Omega\times\mathbb{N}}(\mathcal{M}f)^q\mathrm{d}\widehat{\mu}\Big)^{\frac{1}{q}}\leq C_2\Big(\int_{\Omega}|f|^pv\mathrm{d}\mu\Big)^{\frac{1}{p}}, \quad \forall f=(f_n)\in L^p(v). \tag{2.2}$$

Proof Note that $\widehat{\{\tau<\infty\}}\subset\Omega\times\mathbb{N}$, and then the necessity of inequality (2.1) follows immediately if we substitute $f=\sigma\chi_{\{\tau<\infty\}}$ into (2.2).

To show (1) $\Rightarrow$ (2). Fix $f\in L^p(v)$. For each $n\in\mathbb{N}$, $k\in\mathbb{Z}$ and $j\in\mathbb{Z}$, define $\tau_k^{(n)}=\inf\{m\leq n: |f_m|>2^k\}$, and set

$$A_{k,j}^{(n)}=\{\tau_k^{(n)}<\infty\}\cap\{2^j<E(\sigma\mid\mathcal{F}_{\tau_k^{(n)}})\leq 2^{j+1}\},$$
$$B_{k,j}^{(n)}=\{\tau_k^{(n)}<\infty,\tau_{k+1}^{(n)}=\infty\}\cap\{2^j<E(\sigma\mid\mathcal{F}_{\tau_k^{(n)}})\leq 2^{j+1}\}.$$

Trivially, $A_{k,j}^{(n)}\in\mathcal{F}_{\tau_k^{(n)}}$, $B_{k,j}^{(n)}\subseteq A_{k,j}^{(n)}$ and $\{\tau_k^{(n)}<\infty\}=\{\mathcal{M}^{(n)}f>2^k\}$. Obviously, for each $n\in\mathbb{N}$, $\{B_{k,j}^{(n)}\}_{k,j}$ is a family of disjoint sets and

$$\{2^k<\mathcal{M}^{(n)}f\leq 2^{k+1}\}=\{\tau_k^{(n)}<\infty,\tau_{k+1}^{(n)}=\infty\}=\bigcup_{j\in\mathbb{Z}}B_{k,j}^{(n)}.$$

Let $\widehat{E}(\cdot\mid\mathcal{F}_{\tau_k^{(n)}})$ be the expectation with respect to $\big(\Omega,\mathcal{F}_{\tau_k^{(n)}},\frac{\sigma}{E(\sigma)}\mathrm{d}\mu\big)$, we have

$$f_{\tau_k^{(n)}}=E(f\mid\mathcal{F}_{\tau_k^{(n)}})=\widehat{E}(f\sigma^{-1}\mid\mathcal{F}_{\tau_k^{(n)}})E(\sigma\mid\mathcal{F}_{\tau_k^{(n)}}).$$

It follows that

$$\begin{aligned}
2^{kq} &\le \operatorname*{essinf}_{A_{k,j}^{(n)}} |f_{\tau_k^{(n)}}|^q \\
&\le \operatorname*{essinf}_{A_{k,j}^{(n)}} \widehat{E}(|f|\sigma^{-1} \mid \mathcal{F}_{\tau_k^{(n)}})^q \operatorname*{esssup}_{A_{k,j}^{(n)}} E(\sigma \mid \mathcal{F}_{\tau_k^{(n)}})^q \\
&\le 2^q \operatorname*{essinf}_{A_{k,j}^{(n)}} \widehat{E}(|f|\sigma^{-1} \mid \mathcal{F}_{\tau_k^{(n)}})^q |B_{k,j}^{(n)} \times \{n\}|_{\widehat{\mu}}^{-1} \int_{B_{k,j}^{(n)}\times\{n\}} E(\sigma \mid \mathcal{F}_{\tau_k^{(n)}})^q \mathrm{d}\widehat{\mu},
\end{aligned}$$

provided $|B_{k,j}^{(n)} \times \{n\}|_{\widehat{\mu}} \neq 0$.

We now estimate $\int_{\Omega\times\mathbb{N}} (\mathcal{M}f)^q \mathrm{d}\widehat{\mu}$ as follows:

$$\begin{aligned}
\int_{\Omega\times\mathbb{N}} (\mathcal{M}f)^q \mathrm{d}\widehat{\mu} &= \sum_{n\in\mathbb{N}} \int_{\Omega\times\{n\}} (\mathcal{M}^{(n)}f)^q \mathrm{d}\widehat{\mu} \\
&= \sum_{\substack{n\in\mathbb{N}\\ k\in\mathbb{Z}}} \int_{\{2^k<\mathcal{M}^{(n)}f\le 2^{k+1}\}\times\{n\}} (\mathcal{M}^{(n)}f)^q \mathrm{d}\widehat{\mu} \\
&\le 2^q \sum_{\substack{n\in\mathbb{N}\\ k\in\mathbb{Z}}} \int_{\{2^k<\mathcal{M}^{(n)}f\le 2^{k+1}\}\times\{n\}} 2^{kq} \mathrm{d}\widehat{\mu} \\
&= 2^q \sum_{\substack{n\in\mathbb{N}\\ k\in\mathbb{Z}\\ j\in\mathbb{Z}}} \int_{B_{k,j}^{(n)}\times\{n\}} 2^{kq} \mathrm{d}\widehat{\mu} \\
&\le 4^q \sum_{\substack{n\in\mathbb{N}\\ k\in\mathbb{Z}\\ j\in\mathbb{Z}}} \operatorname*{essinf}_{A_{k,j}^{(n)}} \widehat{E}(|f|\sigma^{-1} \mid \mathcal{F}_{\tau_k^{(n)}})^q \int_{B_{k,j}^{(n)}\times\{n\}} E(\sigma \mid \mathcal{F}_{\tau_k^{(n)}})^q \mathrm{d}\widehat{\mu}.
\end{aligned}$$

It is clear that

$$\vartheta(n,k,j) = \int_{B_{k,j}^{(n)}\times\{n\}} E(\sigma \mid \mathcal{F}_{\tau_k^{(n)}})^q \mathrm{d}\widehat{\mu}$$

is a measure on $X = \mathbb{N}\times\mathbb{Z}\times\mathbb{Z}$. For the above $f \in L^p(v)$, define

$$Tf(n,k,j) = \operatorname*{essinf}_{A_{k,j}^{(n)}} \widehat{E}(|f|\sigma^{-1} \mid \mathcal{F}_{\tau_k^{(n)}})^q,$$

and denote

$$\begin{aligned}
E_\lambda^{(n)} &= \{(k,j) : \operatorname*{essinf}_{A_{k,j}^{(n)}} \widehat{E}(|f|\sigma^{-1} \mid \mathcal{F}_{\tau_k^{(n)}})^q > \lambda\}, \\
G_\lambda^{(n)} &= \bigcup_{(k,j)\in E_\lambda} A_{k,j}^{(n)}, \\
\tau &= \inf\{m : \widehat{E}(|f|\sigma^{-1} \mid \mathcal{F}_m) > \lambda^{\frac{1}{q}}\} \quad \text{for each } \lambda > 0.
\end{aligned}$$

Note that $\tau_k^{(n)} \le n$ on $A_{k,j}^{(n)}$. Then

$$E(\sigma\chi_{G_\lambda^{(n)}} \mid \mathcal{F}_{\tau_k^{(n)}})\chi_{A_{k,j}^{(n)}} \le \mathcal{M}^{(n)}(\sigma\chi_{G_\lambda^{(n)}})\chi_{A_{k,j}^{(n)}}.$$

Thus

$$\begin{aligned}
|\{Tf>\lambda\}|_{\vartheta} &= \sum_{n\in\mathbb{N}}\sum_{(k,j)\in E_\lambda^{(n)}}\int_{B_{k,j}^{(n)}\times\{n\}} E(\sigma \mid \mathcal{F}_{\tau_k^{(n)}})^q \mathrm{d}\widehat{\mu} \\
&= \sum_{n\in\mathbb{N}}\sum_{(k,j)\in E_\lambda^{(n)}}\int_{B_{k,j}^{(n)}\times\{n\}} E(\sigma\chi_{G_\lambda^{(n)}} \mid \mathcal{F}_{\tau_k^{(n)}})^q \mathrm{d}\widehat{\mu} \\
&\le \sum_{n\in\mathbb{N}}\sum_{(k,j)\in E_\lambda^{(n)}}\int_{B_{k,j}^{(n)}\times\{n\}} (\mathcal{M}^{(n)}(\sigma\chi_{G_\lambda^{(n)}}))^q \mathrm{d}\widehat{\mu} \\
&\le \sum_{n\in\mathbb{N}}\int_{G_\lambda^{(n)}\times\{n\}} (\mathcal{M}^{(n)}(\sigma\chi_{G_\lambda^{(n)}}))^q \mathrm{d}\widehat{\mu}.
\end{aligned}$$

When $(k,j)\in E_\lambda^{(n)}$, we observe that

$$\begin{aligned}
\lambda^{\frac{1}{q}}\chi_{A_{k,j}^{(n)}} &\le \widehat{E}(|f|\sigma^{-1} \mid \mathcal{F}_{\tau_k^{(n)}})\chi_{A_{k,j}^{(n)}} \\
&= \widehat{E}(|f|\sigma^{-1} \mid \mathcal{F}_{\tau_k^{(n)}})\chi_{A_{k,j}^{(n)}}\chi_{\{\tau_k^{(n)}\le n\}} \\
&= \Big(\sum_{i=0}^{n}\widehat{E}(|f|\sigma^{-1} \mid \mathcal{F}_i)\chi_{\{\tau_k^{(n)}=i\}}\Big)\chi_{A_{k,j}^{(n)}}.
\end{aligned}$$

Then

$$\lambda^{\frac{1}{q}}\chi_{\{\tau_k^{(n)}=i\}}\chi_{A_{k,j}^{(n)}} \le \widehat{E}(|f|\sigma^{-1} \mid \mathcal{F}_i)\chi_{\{\tau_k^{(n)}=i\}}\chi_{A_{k,j}^{(n)}}, \quad 0\le i\le n,$$

which implies

$$\tau(\omega)\le n, \quad \omega\in A_{k,j}^{(n)}, \quad \text{a.e..}$$

Consequently

$$\begin{aligned}
|\{Tf>\lambda\}|_{\vartheta} &\le \sum_{n\in\mathbb{N}}\int_{G_\lambda^{(n)}\times\{n\}} (\mathcal{M}^{(n)}(\sigma\chi_{G_\lambda^{(n)}}))^q \mathrm{d}\widehat{\mu} \\
&\le \sum_{n\in\mathbb{N}}\int_{G_\lambda^{(n)}\times\{n\}} (\mathcal{M}^{(n)}(\sigma\chi_{\{\tau<\infty\}}))^q \mathrm{d}\widehat{\mu} \\
&= \sum_{n\in\mathbb{N}}\int_{G_\lambda^{(n)}\times\{n\}} (\mathcal{M}(\sigma\chi_{\{\tau<\infty\}}))^q \mathrm{d}\widehat{\mu} \\
&\le \int_{\widehat{\{\tau<\infty\}}} (\mathcal{M}(\sigma\chi_{\{\tau<\infty\}}))^q \mathrm{d}\widehat{\mu}.
\end{aligned}$$

It follows from (2.1) that

$$|\{Tf>\lambda\}|_{\vartheta} \le C_1^q\Big(\int_{\{\tau<\infty\}}\sigma\mathrm{d}\mu\Big)^{\frac{q}{p}} = C_1^q|\{\tau<\infty\}|_\sigma^{\frac{q}{p}} = C_1^q|\{(\widehat{M}(|f|\sigma^{-1}))^q>\lambda\}|_\sigma^{\frac{q}{p}},$$

where $\widehat{M}(\cdot)=\sup\limits_{n\ge 0}\widehat{E}(\cdot \mid \mathcal{F}_n)$. Therefore

$$\begin{aligned}
\int_{\Omega\times\mathbb{N}}(\mathcal{M}f)^q\mathrm{d}\widehat{\mu} \le 4^q\int_X Tf\mathrm{d}\vartheta &= 4^q\int_0^\infty |\{Tf>\lambda\}|_\vartheta \mathrm{d}\lambda \\
&= 4^q\sum_{l\in\mathbb{Z}}\int_{2^l}^{2^{l+1}} |\{Tf>\lambda\}|_\vartheta \mathrm{d}\lambda
\end{aligned}$$

$$
\begin{aligned}
&\le 4^q \sum_{l\in\mathbb{Z}} 2^l |\{Tf > 2^l\}|_\vartheta \\
&\le 4^q C_1^q \sum_{l\in\mathbb{Z}} 2^l |\{(\widehat{M}(|f|\sigma^{-1}))^q > 2^l\}|_\sigma^{\frac{q}{p}} \\
&\le 4^q C_1^q \sum_{l\in\mathbb{Z}} (2^{\frac{p}{q}\cdot l} |\{(\widehat{M}(|f|\sigma^{-1}))^p > 2^{\frac{p}{q}\cdot l}\}|_\sigma)^{\frac{q}{p}} \\
&\le 4^q C_1^q \Big(\sum_{l\in\mathbb{Z}} 2^{\frac{p}{q}\cdot l} |\{(\widehat{M}(|f|\sigma^{-1}))^p > 2^{\frac{p}{q}\cdot l}\}|_\sigma\Big)^{\frac{q}{p}} \\
&= 4^q \Big(\frac{2^{\frac{p}{q}}}{2^{\frac{p}{q}}-1}\Big)^{\frac{q}{p}} C_1^q \Big(\sum_{l\in\mathbb{Z}} (2^{\frac{p}{q}\cdot l} - 2^{\frac{p}{q}(l-1)}) |\{(\widehat{M}(|f|\sigma^{-1}))^p > 2^{\frac{p}{q}\cdot l}\}|_\sigma\Big)^{\frac{q}{p}} \\
&\le 4^q \Big(\frac{2^{\frac{p}{q}}}{2^{\frac{p}{q}}-1}\Big)^{\frac{q}{p}} C_1^q \Big(\sum_{l\in\mathbb{Z}} \int_{2^{\frac{p}{q}(l-1)}}^{2^{\frac{p}{q}\cdot l}} |\{(\widehat{M}(|f|\sigma^{-1}))^p > \lambda\}|_\sigma \mathrm{d}\lambda\Big)^{\frac{q}{p}} \\
&\le 4^q \Big(\frac{2^{\frac{p}{q}}}{2^{\frac{p}{q}}-1}\Big)^{\frac{q}{p}} C_1^q \Big(\int_0^\infty |\{(\widehat{M}(|f|\sigma^{-1}))^p > \lambda\}|_\sigma \mathrm{d}\lambda\Big)^{\frac{q}{p}} \\
&= 4^q \Big(\frac{2^{\frac{p}{q}}}{2^{\frac{p}{q}}-1}\Big)^{\frac{q}{p}} C_1^q \Big(\int_\Omega (\widehat{M}(|f|\sigma^{-1}))^p \sigma \mathrm{d}\mu\Big)^{\frac{q}{p}}, \qquad (2.3)
\end{aligned}
$$

where we have used $p \le q$.

For $p > 1$, in virtue of Doob's inequality, we have

$$
\begin{aligned}
\int_{\Omega\times\mathbb{N}} (\mathcal{M}f)^q \mathrm{d}\widehat{\mu} &\le (4p')^q \Big(\frac{2^{\frac{p}{q}}}{2^{\frac{p}{q}}-1}\Big)^{\frac{q}{p}} C_1^q \Big(\int_\Omega (|f|^p \sigma^{-p}) \sigma \mathrm{d}\mu\Big)^{\frac{q}{p}} \\
&= (4p')^q \Big(\frac{2^{\frac{p}{q}}}{2^{\frac{p}{q}}-1}\Big)^{\frac{q}{p}} C_1^q \Big(\int_\Omega |f|^p v \mathrm{d}\mu\Big)^{\frac{q}{p}}. \qquad (2.4)
\end{aligned}
$$

Whence (2.2) is valid with $C_2 = 4p'\big(\frac{2^{\frac{p}{q}}}{2^{\frac{p}{q}}-1}\big)^{\frac{1}{p}} C_1$.

Corollary 2.1 *Given p, $1 < p < \infty$, suppose that $v \equiv 1$ and $\widehat{\mu}$ is a nonnegative measure on $\Omega \times \mathbb{N}$. Then the following statements are equivalent:*

(1) *There exists a positive constant C_1 such that*

$$
\int_{\widehat{\{\tau<\infty\}}} (\mathcal{M}(\chi_{\{\tau<\infty\}}))^p \mathrm{d}\widehat{\mu} \le C_1^p |\{\tau < \infty\}|_\mu, \quad \forall \tau \in \mathcal{T}; \qquad (2.5)
$$

(2) *There exists a positive constant C_2 such that*

$$
\Big(\int_{\Omega\times\mathbb{N}} (\mathcal{M}f)^p \mathrm{d}\widehat{\mu}\Big)^{\frac{1}{p}} \le C_2 \Big(\int_\Omega |f|^p \mathrm{d}\mu\Big)^{\frac{1}{p}}, \quad \forall f = (f_n) \in L^p. \qquad (2.6)
$$

Obviously, Corollary 2.1 is a special case of Theorem 2.1. Since

$$
\mathcal{M}(\chi_{\{\tau<\infty\}}) \chi_{\widehat{\{\tau<\infty\}}} = \chi_{\widehat{\{\tau<\infty\}}},
$$

we have that (2.5) is (1.1), that is, $\widehat{\mu}$ is a Carleson measure on $\Omega \times \mathbb{N}$.

Theorem 2.2 *Given p and q with $1 < p \le q < \infty$ and a weight v on Ω, suppose that $\sigma = v^{-\frac{1}{p-1}} \in L^1(\mu)$ and $\widehat{\mu}$ is a nonnegative measure on $\Omega \times \mathbb{N}$. Then the following statements are equivalent:*

(1) *There exists a positive constant C_3 such that*

$$\Big(\int_{\widehat{\{\tau<\infty\}}} |f_\tau|^q \mathrm{d}\widehat{\mu}\Big)^{\frac{1}{q}} \le C_3\Big(\int_\Omega |f|^p v \mathrm{d}\mu\Big)^{\frac{1}{p}}, \quad \forall f=(f_n)\in L^p(v),\ \tau\in\mathcal{T}; \tag{2.7}$$

(2) *There exists a positive constant C_4 such that*

$$\Big(\int_{\widehat{\{\tau<\infty\}}} \sigma_\tau^q \mathrm{d}\widehat{\mu}\Big)^{\frac{1}{q}} \le C_4\Big(\int_{\{\tau<\infty\}} \sigma \mathrm{d}\mu\Big)^{\frac{1}{p}}, \quad \forall \tau\in\mathcal{T}; \tag{2.8}$$

(3) *There exists a positive constant C_5 such that*

$$\lambda|\{\mathcal{M}f>\lambda\}|_{\widehat{\mu}}^{\frac{1}{q}} \le C_5\Big(\int_\Omega |f|^p v \mathrm{d}\mu\Big)^{\frac{1}{p}}, \quad \forall f=(f_n)\in L^p(v),\ \lambda>0. \tag{2.9}$$

Proof We shall follow the scheme: (2) $\Leftrightarrow$ (1) $\Leftrightarrow$ (3).

(1) $\Rightarrow$ (2) Substituting $f=\sigma\chi_{\{\tau<\infty\}}$ into (2.7), we have (2.8).

(2) $\Rightarrow$ (1) If $\tau\equiv n$ for some $n\in\mathbb{N}$, we shall show that (2.7) is valid. Fix $f\in L^p(v)$. For each $k\in\mathbb{Z}$ and $j\in\mathbb{Z}$, let

$$B_{k,j}=\{2^k<|f_n|\le 2^{k+1}\}\cap\{2^j<E(\sigma\mid\mathcal{F}_n)\le 2^{j+1}\}.$$

Then $B_{k,j}\in\mathcal{F}_n$. Moreover, $\{B_{k,j}\}_{k,j}$ is a family of disjoint sets and

$$\{2^k<|f_n|\le 2^{k+1}\}=\bigcup_{j\in\mathbb{Z}} B_{k,j}.$$

It is easy to check that

$$\begin{aligned} 2^{kq} &\le \operatorname*{essinf}_{B_{k,j}}|f_n|^q \le \operatorname*{essinf}_{B_{k,j}}\widehat{E}(|f|\sigma^{-1}\mid\mathcal{F}_n)^q \operatorname*{esssup}_{B_{k,j}}E(\sigma\mid\mathcal{F}_n)^q \\ &\le 2^q \operatorname*{essinf}_{B_{k,j}}\widehat{E}(|f|\sigma^{-1}\mid\mathcal{F}_n)^q |B_{k,j}\times\{m:m\ge n\}|_{\widehat{\mu}}^{-1}\int_{B_{k,j}\times\{m:m\ge n\}} E(\sigma\mid\mathcal{F}_n)^q \mathrm{d}\widehat{\mu}, \end{aligned}$$

provided $|B_{k,j}^{(n)}\times\{m:m\ge n\}|_{\widehat{\mu}}\ne 0$.

We now estimate $\int_{\widehat{\{\tau<\infty\}}}|f_\tau|^q\mathrm{d}\widehat{\mu}$. Note that

$$\begin{aligned} \int_{\widehat{\{\tau<\infty\}}}|f_\tau|^q\mathrm{d}\widehat{\mu} &= \int_{\Omega\times\{m:m\ge n\}}|f_n|^q\mathrm{d}\widehat{\mu} \le 2^q\sum_{k\in\mathbb{Z},\,j\in\mathbb{Z}}\int_{B_{k,j}\times\{m:m\ge n\}}2^{kq}\mathrm{d}\widehat{\mu} \\ &\le 4^q\sum_{k\in\mathbb{Z},\,j\in\mathbb{Z}}\operatorname*{essinf}_{B_{k,j}}\widehat{E}(|f|\sigma^{-1}\mid\mathcal{F}_n)^q\int_{B_{k,j}\times\{m:m\ge n\}}E(\sigma\mid\mathcal{F}_n)^q\mathrm{d}\widehat{\mu}. \end{aligned}$$

It is obvious that

$$\vartheta(k,j)=\int_{B_{k,j}\times\{m:m\ge n\}}E(\sigma\mid\mathcal{F}_n)^q\mathrm{d}\widehat{\mu}$$

is a measure on $X=\mathbb{Z}^2$.

For the above $f\in L^p(v)$, define

$$Tf(k,j)=\operatorname*{essinf}_{B_{k,j}}\widehat{E}(|f|\sigma^{-1}\mid\mathcal{F}_n)^q$$

and denote

$$E_\lambda = \{(k,j) : \operatorname*{essinf}_{B_{k,j}} \widehat{E}(|f|\sigma^{-1} \mid \mathcal{F}_n)^q > \lambda\} \quad \text{and} \quad G_\lambda = \bigcup_{(k,j)\in E_\lambda} B_{k,j} \quad \text{for each } \lambda > 0.$$

Trivially, $B_{k,j} \times \{m : m \geq n\} = \{\widehat{\tau_{k,j} < \infty}\}$, where $\tau_{k,j} = n\chi_{B_{k,j}} + \infty\chi_{\{\Omega\setminus B_{k,j}\}}$. Thus

$$|\{Tf > \lambda\}|_\vartheta = \sum_{(k,j)\in E_\lambda} \int_{B_{k,j}\times\{m:m\geq n\}} \sigma_n^q \mathrm{d}\widehat{\mu} = \sum_{(k,j)\in E_\lambda} \int_{\{\widehat{\tau_{k,j}<\infty}\}} \sigma_n^q \mathrm{d}\widehat{\mu}.$$

It follows from (2.8) that

$$\begin{aligned} |\{Tf > \lambda\}|_\vartheta &\leq C_4^q \sum_{(k,j)\in E_\lambda} |B_{k,j}|_\sigma^{\frac{q}{p}} \leq C_4^q \Big(\sum_{(k,j)\in E_\lambda} |B_{k,j}|_\sigma \Big)^{\frac{q}{p}} \\ &= C_4^q |G_\lambda|_\sigma^{\frac{q}{p}} \leq C_4^q |\{(\widehat{M}(|f|\sigma^{-1}))^p > \lambda\}|_\sigma^{\frac{q}{p}}. \end{aligned}$$

As we have done in Theorem 2.1, (2.7) is valid with $C_3 = 4p'\big(\frac{2^{\frac{p}{q}}}{2^{\frac{p}{q}}-1}\big)^{\frac{1}{p}} C_4$.

If $\tau \in \mathcal{T}$ is arbitrary, we shall show that (2.7) is still valid. Fix $\tau \in \mathcal{T}$, and let

$$B_k = \{\tau = k\} \quad \text{and} \quad \tau_k \equiv k, \quad k \in \mathbb{N}.$$

Using $p \leq q$, we obtain that

$$\begin{aligned} \int_{\{\widehat{\tau<\infty}\}} |f_\tau|^q \mathrm{d}\widehat{\mu} &= \sum_{k\in\mathbb{N}} \int_{B_k\times\{m:m\geq k\}} |f_k|^q \mathrm{d}\widehat{\mu} = \sum_{k\in\mathbb{N}} \int_{\Omega\times\{m:m\geq k\}} |f_k\chi_{B_k}|^q \mathrm{d}\widehat{\mu} \\ &= \sum_{k\in\mathbb{N}} \int_{\{\widehat{\tau_k<\infty}\}} E(|f\chi_{B_k}| \mid \mathcal{F}_k)^q \mathrm{d}\widehat{\mu} \leq C_3^q \sum_{k\in\mathbb{N}} \Big(\int_\Omega |f\chi_{B_k}|^p v \mathrm{d}\mu \Big)^{\frac{q}{p}} \\ &\leq C_3^q \Big(\int_\Omega |f|^p v \mathrm{d}\mu \Big)^{\frac{q}{p}}. \end{aligned}$$

(1) $\Rightarrow$ (3) Fix $f \in L^p(v)$ and $\lambda > 0$. Let $\tau = \inf\{n : |f_n| > \lambda\}$. It follows from (2.7) that

$$\lambda^q |\{\mathcal{M}f > \lambda\}|_{\widehat{\mu}} \leq \int_{\{\widehat{\tau<\infty}\}} |f_\tau|^q \mathrm{d}\widehat{\mu} \leq C_3^q \Big(\int_\Omega |f|^p v \mathrm{d}\mu \Big)^{\frac{q}{p}}.$$

Thus (2.9) holds with $C_3 = C_5$.

(3) $\Rightarrow$ (1) It suffices to prove that (2.7) holds for $\tau \equiv n$, for every $n \in \mathbb{N}$. Fix $n \in N$. For $B \in \mathcal{F}_n$, let $g = f\chi_B$. Trivially, $|g_n| = |f_n|\chi_B$. Thus

$$(\{|f_n| > \lambda\} \cap B) \times \{m : m \geq n\} \subset \{\mathcal{M}g > \lambda\}.$$

In virtue of (2.9), we have

$$\lambda^q \int_{(\{|f_n|>\lambda\}\cap B)\times\{m:m\geq n\}} \mathrm{d}\widehat{\mu} \leq \lambda^q \int_{\{\mathcal{M}g>\lambda\}} \mathrm{d}\widehat{\mu} \leq C_5^q \Big(\int_\Omega |g|^p v \mathrm{d}\mu \Big)^{\frac{q}{p}} \leq C_5^q \Big(\int_B |f|^p v \mathrm{d}\mu \Big)^{\frac{q}{p}}.$$

Thus

$$\begin{aligned}
\int_{\widehat{\{\tau<\infty\}}} |f_\tau|^q \mathrm{d}\widehat{\mu} &= \int_{\Omega\times\{m:m\geq n\}} |f_n|^q \mathrm{d}\widehat{\mu} \\
&= \sum_{k\in\mathbb{Z}} \int_{\{2^k<|f_n|\leq 2^{k+1}\}\times\{m:m\geq n\}} |f_n|^q \mathrm{d}\widehat{\mu} \\
&\leq 2^q \sum_{k\in\mathbb{Z}} 2^{kq} \int_{\{2^k<|f_n|\leq 2^{k+1}\}\times\{m:m\geq n\}} \mathrm{d}\widehat{\mu} \\
&= 2^q \sum_{k\in\mathbb{Z}} 2^{kq} \int_{(\{|f_n|>2^k\}\cap\{2^k<|f_n|\leq 2^{k+1}\})\times\{m:m\geq n\}} \mathrm{d}\widehat{\mu} \\
&\leq 2^q C_5^q \sum_{k\in\mathbb{Z}} \Big(\int_{\{2^k<|f_n|\leq 2^{k+1}\}} |f|^p v \mathrm{d}\mu\Big)^{\frac{q}{p}} \\
&\leq 2^q C_5^q \Big(\sum_{k\in\mathbb{Z}} \int_{\{2^k<|f_n|\leq 2^{k+1}\}} |f|^p v \mathrm{d}\mu\Big)^{\frac{q}{p}} \\
&\leq 2^q C_5^q \Big(\int_\Omega |f|^p v \mathrm{d}\mu\Big)^{\frac{q}{p}},
\end{aligned}$$

which implies (2.7) with $C_3 = 2C_5$.

3 The Maximal Operator

Theorem 3.1 *Given p, $1<p<\infty$ and a pair of weights (u,v), suppose that $\sigma = v^{-\frac{1}{p-1}} \in L^1(\mu)$. Then the following statements are equivalent:*

(1) *There exists a positive constant C_1 such that*

$$\int_{\{\tau<\infty\}} (M(\sigma\chi_{\{\tau<\infty\}}))^p u \mathrm{d}\mu \leq C_1 \int_{\{\tau<\infty\}} \sigma \mathrm{d}\mu, \quad \forall\tau\in\mathcal{T}; \tag{3.1}$$

(2) *There exists a positive constant C_2 such that*

$$E({}^*\sigma_n^p u \mid \mathcal{F}_n) \leq C_2\sigma_n, \quad \forall n\in\mathbb{N}; \tag{3.2}$$

(3) *There exists a positive constant C_3 such that*

$$\int_\Omega (Mf)^p u \mathrm{d}\mu \leq C_3 \int_\Omega |f|^p v \mathrm{d}\mu, \quad \forall f=(f_n)\in L^p(v). \tag{3.3}$$

Proof We shall follow the scheme: (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1).

(1) $\Rightarrow$ (2) For $B\in\mathcal{F}_n$, let $\tau = n\chi_B + \infty\chi_{\{\Omega\setminus B\}}$. Then

$${}^*\sigma_n\chi_B = {}^*M_n(\sigma\chi_B)\chi_B = {}^*M_n(\sigma\chi_{\{\tau<\infty\}})\chi_B \leq M(\sigma\chi_{\{\tau<\infty\}}).$$

Following (3.1), we have

$$\begin{aligned}
\int_B E({}^*\sigma_n^p u \mid \mathcal{F}_n)\mathrm{d}\mu &= \int_B {}^*\sigma_n^p u \mathrm{d}\mu = \int_B {}^*\sigma_n^p \chi_B u \mathrm{d}\mu \\
&\leq \int_{\{\tau<\infty\}} (M(\sigma\chi_{\{\tau<\infty\}}))^p u \mathrm{d}\mu \\
&\leq C_1 \int_{\{\tau<\infty\}} \sigma \mathrm{d}\mu = C_1 \int_B \sigma \mathrm{d}\mu.
\end{aligned}$$

Noting that B is arbitrary, we obtain (3.2) with $C_2 = C_1$.

(2) $\Rightarrow$ (3) Trivially, (3.2) implies

$$E((^*M_\tau\sigma)^p u \mid \mathcal{F}_\tau)\chi_{\{\tau<\infty\}} \leq C_2\sigma_\tau\chi_{\{\tau<\infty\}} \quad \text{for all } \tau \in \mathcal{T}. \tag{3.4}$$

Fix $f \in L^p(v)$. For $k \in \mathbb{Z}$, define stopping times $\tau_k = \inf\{n : |f_n| > 2^k\}$. Set

$$A_{k,j} = \{\tau_k < \infty\} \cap \{2^j < E(\sigma \mid \mathcal{F}_{\tau_k}) \leq 2^{j+1}\},$$
$$B_{k,j} = \{\tau_k < \infty, \tau_{k+1} = \infty\} \cap \{2^j < E(\sigma \mid \mathcal{F}_{\tau_k}) \leq 2^{j+1}\}.$$

Then $A_{k,j} \in \mathcal{F}_{\tau_k}$, $B_{k,j} \subseteq A_{k,j}$. Moreover, $\{B_{k,j}\}_{k,j}$ is a family of disjoint sets and

$$\{2^k < Mf \leq 2^{k+1}\} = \{\tau_k < \infty, \tau_{k+1} = \infty\} = \bigcup_{j\in\mathbb{Z}} B_{k,j}.$$

Obviously, we have

$$f_{\tau_k} = E(f \mid \mathcal{F}_{\tau_k}) = \widehat{E}(f\sigma^{-1} \mid \mathcal{F}_{\tau_k})E(\sigma \mid \mathcal{F}_{\tau_k})$$

and

$$\begin{aligned}2^{kp} &\leq \operatorname*{essinf}_{A_{k,j}} |f_{\tau_k}|^p \leq \operatorname*{essinf}_{A_{k,j}} \widehat{E}(|f|\sigma^{-1} \mid \mathcal{F}_{\tau_k})^p \operatorname*{esssup}_{A_{k,j}} E(\sigma \mid \mathcal{F}_{\tau_k})^p \\ &\leq 2^p \operatorname*{essinf}_{A_{k,j}} \widehat{E}(|f|\sigma^{-1} \mid \mathcal{F}_{\tau_k})^p |B_{k,j}|_u^{-1} \int_{B_{k,j}} E(\sigma \mid \mathcal{F}_{\tau_k})^p u \mathrm{d}\mu,\end{aligned}$$

provided $|B_{k,j}|_u \neq 0$.

To estimate $\int_\Omega (Mf)^p u\mathrm{d}\mu$, firstly we have

$$\begin{aligned}\int_\Omega (Mf)^p u\mathrm{d}\mu &= \sum_{k\in\mathbb{Z}} \int_{\{2^k<Mf\leq 2^{k+1}\}} (Mf)^p u\mathrm{d}\mu \\ &\leq 2^p \sum_{k\in\mathbb{Z}} \int_{\{2^k<Mf\leq 2^{k+1}\}} 2^{kp} u\mathrm{d}\mu = 2^p \sum_{k\in\mathbb{Z}, j\in\mathbb{Z}} \int_{B_{k,j}} 2^{kp} u\mathrm{d}\mu \\ &\leq 4^p \sum_{\substack{k\in\mathbb{Z}\\ j\in\mathbb{Z}}} \operatorname*{essinf}_{A_{k,j}} \widehat{E}(f\sigma^{-1} \mid \mathcal{F}_{\tau_k})^p \int_{B_{k,j}} E(\sigma \mid \mathcal{F}_{\tau_k})^p u\mathrm{d}\mu.\end{aligned}$$

It is obvious that

$$\vartheta(k,j) = \int_{B_{k,j}} E(\sigma \mid \mathcal{F}_{\tau_k})^p u\mathrm{d}\mu$$

is a measure on $X = \mathbb{Z}^2$. For the above $f \in L^p(v)$, we define

$$Tf(k,j) = \operatorname*{essinf}_{A_{k,j}} \widehat{E}(|f|\sigma^{-1} \mid \mathcal{F}_{\tau_k})^p$$

and denote

$$\begin{aligned}E_\lambda &= \{(k,j) : \operatorname*{essinf}_{A_{k,j}} \widehat{E}(|f|\sigma^{-1} \mid \mathcal{F}_{\tau_k})^p > \lambda\}, \\ G_\lambda &= \bigcup_{(k,j)\in E_\lambda} A_{k,j}, \\ \tau &= \inf\{m : \widehat{E}(|f|\sigma^{-1} \mid \mathcal{F}_m) > \lambda^{\frac{1}{q}}\} \quad \text{for each } \lambda > 0.\end{aligned}$$

When $(k,j) \in E_\lambda$, we observe that $\tau\chi_{\{\tau_k<\infty\}} \leq \tau_k\chi_{\{\tau_k<\infty\}}$. Thus

$$\begin{aligned}|\{Tf > \lambda\}|_\vartheta &= \sum_{(k,j)\in E_\lambda} \int_{B_{k,j}} E(\sigma \mid \mathcal{F}_{\tau_k})^p u\mathrm{d}\mu \\ &= \sum_{(k,j)\in E_\lambda} \int_{B_{k,j}} E(\sigma\chi_{G_\lambda} \mid \mathcal{F}_{\tau_k})^p u\mathrm{d}\mu \\ &\leq \int_{G_\lambda} ({}^*M_\tau(\sigma\chi_{G_\lambda}))^p u\mathrm{d}\mu. \end{aligned} \tag{3.5}$$

It follows from $G_\lambda \subseteq \{\tau < \infty\}$ and (3.4) that

$$|\{Tf > \lambda\}|_\vartheta \leq C_2 \int_{\{\tau<\infty\}} \sigma\mathrm{d}\mu \ = C_2|\{(\widehat{M}(|f|\sigma^{-1}))^p > \lambda\}|_\sigma.$$

Therefore

$$\begin{aligned}\int_\Omega (Mf)^p u\mathrm{d}\mu &\leq 4^p \sum_{k\in\mathbb{Z},j\in\mathbb{Z}} \operatorname*{essinf}_{A_{k,j}} \widehat{E}(|f|\sigma^{-1} \mid \mathcal{F}_{\tau_k})^p \int_{B_{k,j}} E(\sigma \mid \mathcal{F}_{\tau_k})^p u\mathrm{d}\mu \\ &= 4^p \int_X Tf\mathrm{d}\vartheta = 4^p C_2 \int_0^\infty |Tf > \lambda|_\vartheta \mathrm{d}\lambda \\ &\leq 4^p C_2 \int_0^\infty |(\widehat{M}(|f|\sigma^{-1}))^p > \lambda|_\sigma \mathrm{d}\lambda = 4^p C_2 \int_\Omega (\widehat{M}(|f|\sigma^{-1}))^p \sigma\mathrm{d}\mu \\ &\leq (4p')^p C_2 \int_\Omega (|f|\sigma^{-1})^p \sigma\mathrm{d}\mu = (4p')^p C_2 \int_\Omega |f|^p \sigma^{1-p}\mathrm{d}\mu \\ &= (4p')^p C_2 \int_\Omega |f|^p v\mathrm{d}\mu,\end{aligned}$$

which is (3.3) with $C_3 = (4p')^p C_2$.

(3) $\Rightarrow$ (1) It is trivial.

The proof is completed.

Theorem 3.2 *Given p and q with $1 < p \leq q < \infty$ and a pair of weights (u, v), suppose $\sigma = v^{-\frac{1}{p-1}} \in L^1(\mu)$. Then the following statements are equivalent:*

(1) *There exists a positive constant C_1 such that*

$$\Big(\int_{\{\tau<\infty\}} (M(\sigma\chi_{\{\tau<\infty\}}))^q u\mathrm{d}\mu\Big)^{\frac{1}{q}} \leq C_1\Big(\int_{\{\tau<\infty\}} \sigma\mathrm{d}\mu\Big)^{\frac{1}{p}}, \quad \forall\tau \in \mathcal{T}; \tag{3.6}$$

(2) *There exists a positive constant C_2 such that*

$$\Big(\int_{\{\tau<\infty\}} ({}^*M_\tau(\sigma\chi_{\{\tau<\infty\}}))^q u\mathrm{d}\mu\Big)^{\frac{1}{q}} \leq C_2\Big(\int_{\{\tau<\infty\}} \sigma\mathrm{d}\mu\Big)^{\frac{1}{p}}, \quad \forall\tau \in \mathcal{T}; \tag{3.7}$$

(3) *There exists a positive constant C_3 such that*

$$\Big(\int_\Omega (Mf)^q u\mathrm{d}\mu\Big)^{\frac{1}{q}} \leq C_3\Big(\int_\Omega |f|^p v\mathrm{d}\mu\Big)^{\frac{1}{p}}, \quad \forall f = (f_n) \in L^p(v). \tag{3.8}$$

Proof It is easy to check that (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (1) and we omit them.

For (2) $\Rightarrow$ (3), we proceed like in the proof of Theorem 3.1 just up to (3.5) with slight modifications. After this we continue as follows:

$$\begin{aligned}|\{Tf > \lambda\}|_\vartheta &\le \int_{G_\lambda} ({}^*M_\tau(\sigma\chi_{G_\lambda}))^p u \mathrm{d}\mu \\ &\le \int_{\{\tau<\infty\}} ({}^*M_\tau(\sigma\chi_{\{\tau<\infty\}}))^p u \mathrm{d}\mu \\ &\le C_2^q\Big(\int_{\{\tau<\infty\}} \sigma \mathrm{d}\mu\Big)^{\frac{q}{p}} = C_2^q|\{\tau<\infty\}|_\sigma^{\frac{q}{p}}.\end{aligned}$$

In the same way as (2.3) and (2.4), we have

$$\int_\Omega (Mf)^q u \mathrm{d}\mu \le 4^q \int_X Tf \mathrm{d}\vartheta \le (4p')^q\Big(\frac{2^{\frac{p}{q}}}{2^{\frac{p}{q}}-1}\Big)^{\frac{q}{p}} C_2^q \Big(\int_\Omega |f|^p v \mathrm{d}\mu\Big)^{\frac{q}{p}},$$

which implies (3.8) with $C_3 = 4p'\big(\frac{2^{\frac{q}{q}}}{2^{\frac{p}{q}}-1}\big)^{\frac{1}{p}} C_2$.

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