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Gröbner-Shirshov Basis of Quantum Group of Type $\mathbb{D}_4$*

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Abstract The authors take all isomorphism classes of indecomposable representations as new generators, and obtain all skew-commutators between these generators by using the Ringel-Hall algebra method. Then they prove that the set of these skew-commutators is a Gröbner-Shirshov basis for quantum group of type $\mathbb{D}_4$.

Keywords Ringel-Hall algebra, Indecomposable modules, Gröbner-Shirshov basis, Compositions
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1 Introduction

The Gröbner basis theory for commutative algebras was introduced by Buchberger [4], and provided a solution to the reduction problem for commutative algebras. It gives an algorithm of computing a set of generators for a given ideal of a commutative ring which can be used to determine the reduced elements with respect to the relations given by the ideal. In [1], Bergman generalized the Gröbner basis theory to associative algebras by providing the Diamond Lemma.

The Gröbner basis theory for Lie algebras was developed by Shirshov [12]. The key ingredient of the theory is the so-called Composition Lemma which characterizes the leading terms of elements in the given ideal. In [2], Bokut noticed that Shirshov's method works for associative algebras as well. For this reason, Shirshov's theory for Lie algebras and their universal enveloping algebras is called the Gröbner-Shirshov basis theory.

In [3], Bokut and Malcolmson developed the theory of Gröbner-Shirshov basis for the quantum enveloping algebras, or the so-called quantum groups, and by using the Jimbo relations given by Yamane [13], they explicitly constructed the basis for the quantum group of type $\mathbb{A}_n$ for $q^8 \neq 1$. The Gröbner-Shirshov basis for quantum groups of other types is not known. The main reason for this, from our point of view, may be that the construction of the so-called Jimbo relations for other types by the method of Yamane is very difficult.

In [10], for constructing a PBW type basis for quantum groups, Ringel constructed a generating sequence for Ringel-Hall algebras and some skew commutator relations for these generators by using the Auslander-Reiten theory. In this paper, by using the Ringel's method, we compute all skew-commutator relations for the quantum group of type $\mathbb{D}_4$. Then using the canonical

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isomorphism between the positive parts of quantum groups and the Ringel-Hall algebras, we give a Gröbner-Shirshov basis for quantum group of type $\mathbb{D}_4$. This may give a new idea to get a Gröbner-Shirshov basis for the quantum group of type $\mathbb{D}_n$.

2 Some Preliminaries

In this section, we recall some notions about Gröbner-Shirshov basis theory of quantum groups and the Ringel-Hall algebras, respectively.

First, we recall some basic notions about Gröbner-Shirshov basis theory from [3]. Let k be a field and X a non-empty set of alphabets. Let $\langle X\rangle$ and $k\langle X\rangle$ be the free semigroup with 1 and the free algebra generated by X, respectively. We choose a monomial ordering $<$ on $\langle X\rangle$ in order to determine the leading term $\overline{f}$ for each element $f \in k\langle X\rangle$. An element $f \in k\langle X\rangle$ is called monic if the coefficient of the leading term $\overline{f}$ is $1 \in k$. If f and g are monic elements in $k\langle X\rangle$ with leading terms $\overline{f}$ and $\overline{g}$, there is a so-called composition of intersection if there are a and b in $\langle X\rangle$ such that $\overline{f}a = b\overline{g} = \omega$ with the total length of $\overline{f}$ being larger than that of b. We write $(f,g)_\omega = fa - bg$ in that case and note that the leading term $\overline{(f,g)_\omega} < \omega$. There is a composition of inclusion if there are a and b in $\langle X\rangle$ such that $\overline{f} = a\overline{g}b = \omega$. We write $(f,g)_\omega = f - agb$ in that case and again note that the leading term is less than ω.

Let us take some sets of relations $S \subseteq k\langle X\rangle$ (which, we assume, consists of monic elements). Let us denote by (S) the ideal generated by S in $k\langle X\rangle$. Let $p, q \in k\langle X\rangle$ and $\omega \in \langle X\rangle$. We define an equivalence relation on $k\langle X\rangle$ as follows: $p \equiv q \bmod(S;\omega)$ if and only if $p - q = \sum \alpha_i a_i s_i b_i$, where $\alpha_i \in k$, $a_i, b_i \in \langle X\rangle$, $s_i \in S$, $\overline{a_i s_i b_i} < \omega$. We say that S is closed under composition if for any $f, g \in S$ we have $(f,g)_\omega \equiv 0 \bmod (S;\omega)$, whenever the composition $(f,g)_\omega$ is defined. In this case, we say that the composition $(f,g)_\omega$ is trivial with respect to S. If S is not closed under composition, then we need to expand S by including all nontrivial compositions (inductively) to obtain a completion S^c. If S is complete (i.e., closed under composition) in this sense ($S^c = S$), then Shirshov's Lemma (see [12]) tells us that any monic element $f \in (S)$ has a reducible leading term $\overline{f} = a\overline{s}b$, where $s \in S$ and $a, b \in \langle X\rangle$. That lemma also tells us that a linear basis for the factor algebra $k\langle X\rangle/(S)$ (i.e., as a vector space over k) may be obtained by taking the set of irreducible monomials in $\langle X\rangle$.

The set S is then referred to as a Gröbner-Shirshov basis for the ideal (S). By abusing the definition, we may also refer to S as a Gröbner-Shirshov basis for the factor algebra $k\langle X\rangle/(S)$. The set S is called a minimal Gröbner-Shirshov basis if there is no inclusion composition in S.

Next, we recall the definition of quantum groups from [6] and [8].

Let $A = (a_{ij})$ be an integral symmetrizable $N \times N$ Cartan matrix, so that $a_{ii} = 2$, $a_{ij} \leq 0$ $(i \neq j)$, and there exists a diagonal matrix D with nonzero integer diagonal entries d_i such that the product DA is symmetric. Let q be a nonzero element of k so that $q^{4d_i} \neq 1$ for each i. Then the quantum group $U_q(A)$ is the k-algebra generated by $4N$ elements $E_i, K_i^{\pm 1}, F_i$, subject to the following set of relations (for $1 \leq i, j \leq N$):

$$
\begin{aligned}
K = \{&K_iK_j - K_jK_i,\ K_iK_i^{-1} - 1,\ K_i^{-1}K_i - 1,\ E_jK_i^{\pm 1} - q^{\pm d_i a_{ij}}K_i^{\pm 1}E_j,\\
&K_i^{\pm 1}F_j - q^{\pm d_i a_{ij}}F_jK_i^{\pm 1}\},\\
T = \Big\{&E_iF_j - F_jE_i - \delta_{ij}\frac{K_i^2 - K_i^{-2}}{q^{2d_i} - q^{-2d_i}}\Big\},
\end{aligned}
$$

$$S^+ = \Big\{ \sum_{\mu=0}^{1-a_{ij}} (-1)^\mu \begin{bmatrix} 1-a_{ij} \\ \mu \end{bmatrix}_t E_i^{1-a_{ij}-\mu} E_j E_i^\mu \,\Big|\, i \neq j,\ t = q^{2d_i} \Big\},$$

$$S^- = \Big\{ \sum_{\mu=0}^{1-a_{ij}} (-1)^\mu \begin{bmatrix} 1-a_{ij} \\ \mu \end{bmatrix}_t F_i^{1-a_{ij}-\mu} F_j F_i^\mu \,\Big|\, i \neq j,\ t = q^{2d_i} \Big\},$$

where

$$\begin{bmatrix} m \\ n \end{bmatrix}_t = \begin{cases} \prod_{i=1}^{n} \dfrac{t^{m-i+1} - t^{i-m-1}}{t^i - t^{-i}} & \text{for } m > n > 0, \\ 1 & \text{for } n = 0 \text{ or } n = m. \end{cases}$$

Let $U_q^0(A)$ be the subalgebra of $U_q(A)$ generated by $K_i^{\pm 1}$. Let $U_q^+(A)$ (resp. $U_q^-(A)$) be the subalgebra of $U_q(A)$ generated by E_i (resp. F_i). Then we have the following triangular decomposition of $U_q(A)$ (see [11]):

$$U_q(A) \cong U_q^+(A) \otimes U_q^0(A) \otimes U_q^-(A).$$

The main result in [3] is as follows.

Theorem 2.1 *If the set S^{+c} (resp. S^{-c}) is a Gröbner-Shirshov basis of $U_q^+(A)$ (resp. $U_q^-(A)$), then the set $S^{+c} \cup K \cup T \cup S^{-c}$ is a Gröbner-Shirshov basis of $U_q(A)$.*

Finally, we recall some basic notions about the twisted generic Ringel-Hall algebras. Because we only consider the quantum group of type $\mathbb{D}_4$ in this paper, we recall the relevant notions directly for the finite dimensional hereditary algebra of Dynkin type from [5].

Let $\mathbb{F}$ be a finite field, $\overrightarrow{Q}$ a (connected) quiver with the underlying graph Q of Dynkin type, that is, $Q \in \{\mathbb{A}_n, \mathbb{D}_n, \mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8\}$. Then it is well-known that the path algebra $\Lambda(\mathbb{F}, \overrightarrow{Q}) = \mathbb{F}\overrightarrow{Q}$ is a finite dimensional hereditaty $\mathbb{F}$-algebra of finite representation type. By $\Lambda(\mathbb{F}, \overrightarrow{Q})$-mod, we denote the category of finite dimensional right $\Lambda(\mathbb{F}, \overrightarrow{Q})$-modules. For $M, N_1, \cdots, N_t \in \Lambda(\mathbb{F}, \overrightarrow{Q})$-mod, let $F^M_{N_1,\dots,N_t}$ be the number of filtrations

$$M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_{t-1} \supseteq M_t = 0,$$

such that $M_{i-1}/M_i \cong N_i$ for all $1 \le i \le t$.

For each $M \in \Lambda(\mathbb{F}, \overrightarrow{Q})$-mod, we denote by $[M]$ the isomorphism class of M and by $\mathbf{dim}\, M$ the dimension vector of the $\Lambda(\mathbb{F}, \overrightarrow{Q})$-module M. We have the well-known Euler form $\langle -, - \rangle$ defined by

$$\langle \mathbf{dim}\, M, \mathbf{dim}\, N \rangle = \mathbf{dim}\, \mathrm{Hom}_\Lambda(M, N) - \mathbf{dim}\, \mathrm{Ext}^1_\Lambda(M, N).$$

Note that $(-, -)$ is the symmetrization of $\langle -, - \rangle$.

Let υ be an indeterminate and $\mathbb{Q}(\upsilon)$ be the rational function field of υ over the field $\mathbb{Q}$ of rational numbers and set $\upsilon^2 = q$. In order to define the twisted generic Ringel-Hall algebra, we recall the notion of Hall polynomials.

For a Dynkin diagram Q, there is the corresponding semisimple Lie algebra $\mathfrak{g}$. Let Φ^+ be the set of positive roots of $\mathfrak{g}$. According to [7], $\mathbf{dim}$ is a bijection between the set of the isomorphism classes of the indecomposable modules and the set of positive roots Φ^+ of $\mathfrak{g}$. For each $\alpha \in \Phi^+$, let $M_{\mathbb{F}}(\alpha)$ denote the corresponding indecomposable $\Lambda(\mathbb{F}, \overrightarrow{Q})$-module; thus $\mathbf{dim}\, M_{\mathbb{F}}(\alpha) = \alpha$. By the

Krull-Schmidt theorem, every $\Lambda(\mathbb{F},\overrightarrow{Q})$-module $M_{\mathbb{F}}$ is isomorphic to $M_{\mathbb{F}}(\lambda)=\bigoplus\limits_{\alpha\in\Phi^+}\lambda(\alpha)M_{\mathbb{F}}(\alpha)$ for some function $\lambda:\ \Phi^+\longrightarrow\mathbb{N}$. Thus, isoclasses of $\Lambda(\mathbb{F},\overrightarrow{Q})$-modules are indexed by the set

$$\mathfrak{B}=\mathfrak{B}(\overrightarrow{Q})=:\{\lambda\mid\lambda:\ \Phi^+\longrightarrow\mathbb{N}\},$$

which is independent of the finite field $\mathbb{F}$. To be consistent, we view each $\alpha\in\Phi^+$ as the function $\Phi^+\longrightarrow\mathbb{N},\ \beta\longmapsto\delta_{\alpha,\beta}$. For later use, we denote by α_i the ith simple root in Φ^+ and λ_i the function $\Phi^+\longrightarrow\mathbb{N},\ \beta\longmapsto\delta_{\alpha_i,\beta}$. For any finite field $\mathbb{F}$ and $\lambda,\mu\in\mathfrak{B}(\overrightarrow{Q})$, we define

$$\langle\lambda,\mu\rangle=\langle\mathbf{dim}\,M_{\mathbb{F}}(\lambda),\mathbf{dim}\,M_{\mathbb{F}}(\mu)\rangle.$$

Then we have the result below.

Theorem 2.2 (see [9]) *Assume that $\overrightarrow{Q}$ is a Dynkin quiver. For any $\lambda,\mu,\rho\in\mathfrak{B}=\mathfrak{B}(\overrightarrow{Q})$, there exists a polynomial $\varphi^{\lambda}_{\mu,\rho}(T)\in\mathbb{Z}[T]$, such that*

$$\varphi^{\lambda}_{\mu,\rho}(|\mathbb{F}|)=F^{M_{\mathbb{F}}(\lambda)}_{M_{\mathbb{F}}(\mu),M_{\mathbb{F}}(\rho)}$$

holds for each finite field $\mathbb{F}$.

Now, we are ready to define the twisted generic Ringel-Hall algebra.

Definition 2.1 *The twisted generic Ringel-Hall algebra $\mathcal{H}(\overrightarrow{Q})$ of Dynkin quiver $\overrightarrow{Q}$ is the free $\mathbb{Q}(\upsilon)$-module having basis $\{u_\lambda\mid\lambda\in\mathfrak{B}(\overrightarrow{Q})\}$ with multiplication defined by*

$$u_\mu u_\rho=\upsilon^{\langle\mu,\rho\rangle}\sum_{\lambda\in\mathfrak{B}(\overrightarrow{Q})}\varphi^{\lambda}_{\mu,\rho}(\upsilon^2)u_\lambda.$$

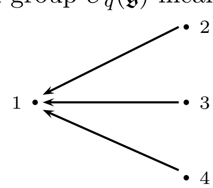
Then $\mathcal{H}(\Lambda)$ is an associative algebra with identity $1=u_0$, where 0 denotes the zero function in $\mathfrak{B}(\overrightarrow{Q})$.

From now on, we fix $k=\mathbb{Q}(\upsilon)$. Let $\overrightarrow{Q}$ be a Dynkin quiver with an underlying graph Q and $\mathfrak{g}$ the corresponding semisimple Lie algebra. Then the main result in [9] is as follows.

Theorem 2.3 *The map $\eta:\ U_q^+(\mathfrak{g})\longrightarrow\mathcal{H}(\overrightarrow{Q})$ given by $\eta(E_i)=u_{[\lambda_i]}$ is a $\mathbb{Q}(\upsilon)$-algebra isomorphism.*

3 Gröbner-Shirshov Basis of Quantum Group of Type $\mathbb{D}_4$

Throughout this section, quantum group $U_q(\mathfrak{g})$ means the quantum group U_q of type $\mathbb{D}_4$:



The corresponding Cartan matrix A is

$$A=\begin{bmatrix}2&-1&-1&-1\\-1&2&0&0\\-1&0&2&0\\-1&0&0&2\end{bmatrix}$$

and the Auslander-Reiten quiver of the algebra given by $\mathbb{D}_4$ is the following:

$$
\begin{tikzcd}
 & \begin{smallmatrix}1\\1\,0\\0\end{smallmatrix} \arrow[dr] & & \begin{smallmatrix}0\\1\,1\\1\end{smallmatrix} \arrow[dr] & & \begin{smallmatrix}1\\0\;\,0\\0\end{smallmatrix} = e_2 \\
e_1 = \begin{smallmatrix}0\\1\,0\\0\end{smallmatrix} \arrow[ur] \arrow[r] \arrow[dr] & \begin{smallmatrix}0\\1\,1\\0\end{smallmatrix} \arrow[r] & \begin{smallmatrix}1\\2\,1\\1\end{smallmatrix} \arrow[ur] \arrow[r] \arrow[dr] & \begin{smallmatrix}1\\1\,0\\1\end{smallmatrix} \arrow[r] & \begin{smallmatrix}1\\1\,1\\1\end{smallmatrix} \arrow[ur] \arrow[r] \arrow[dr] & \begin{smallmatrix}0\\0\;\,1\\0\end{smallmatrix} = e_3 \\
 & \begin{smallmatrix}0\\1\,0\\1\end{smallmatrix} \arrow[ur] & & \begin{smallmatrix}1\\1\,1\\0\end{smallmatrix} \arrow[ur] & & \begin{smallmatrix}0\\0\;\,0\\1\end{smallmatrix} = e_4
\end{tikzcd}
$$

where e_1, e_2, e_3, e_4 are the dimension vectors of simple representations. By abuse of notations, we also denote by e_1, e_2, e_3, e_4 the simple representations.

In the Ringel-Hall algebra $\mathcal{H}(\overrightarrow{\mathbb{D}}_4)$, we consider the following elements:

$$M_{11} = \upsilon^{\frac{1}{2}(\mathbf{dim}\, e_1, \mathbf{dim}\, e_1) - \dim_k e_1} u_{[e_1]} = u_{[e_1]},$$

$$M_{12} = \upsilon^{\frac{1}{2}(\mathbf{dim}(e_1+e_2), \mathbf{dim}(e_1+e_2)) - \dim_k(e_1+e_2)} u_{[e_1+e_2]} = \upsilon^{-1} u_{[e_1+e_2]},$$

$$M_{13} = \upsilon^{\frac{1}{2}(\mathbf{dim}(e_1+e_3), \mathbf{dim}(e_1+e_3)) - \dim_k(e_1+e_3)} u_{[e_1+e_3]} = \upsilon^{-1} u_{[e_1+e_3]},$$

$$M_{14} = \upsilon^{\frac{1}{2}(\mathbf{dim}(e_1+e_4), \mathbf{dim}(e_1+e_4)) - \dim_k(e_1+e_4)} u_{[e_1+e_4]} = \upsilon^{-1} u_{[e_1+e_4]},$$

$$M_{21} = \upsilon^{\frac{1}{2}(\mathbf{dim}(2e_1+e_2+e_3+e_4), \mathbf{dim}(2e_1+e_2+e_3+e_4)) - \dim_k(2e_1+e_2+e_3+e_4)} u_{[2e_1+e_2+e_3+e_4]}$$

$$= \upsilon^{-4} u_{[2e_1+e_2+e_3+e_4]},$$

$$M_{22} = \upsilon^{\frac{1}{2}(\mathbf{dim}(e_1+e_3+e_4), \mathbf{dim}(e_1+e_3+e_4)) - \dim_k(e_1+e_3+e_4)} u_{[e_1+e_3+e_4]} = \upsilon^{-2} u_{[e_1+e_3+e_4]},$$

$$M_{23} = \upsilon^{\frac{1}{2}(\mathbf{dim}(e_1+e_2+e_4), \mathbf{dim}(e_1+e_2+e_4)) - \dim_k(e_1+e_2+e_4)} u_{[e_1+e_2+e_4]} = \upsilon^{-2} u_{[e_1+e_2+e_4]},$$

$$M_{24} = \upsilon^{\frac{1}{2}(\mathbf{dim}(e_1+e_2+e_3), \mathbf{dim}(e_1+e_2+e_3)) - \dim_k(e_1+e_2+e_3)} u_{[e_1+e_2+e_3]} = \upsilon^{-2} u_{[e_1+e_2+e_3]},$$

$$M_{31} = \upsilon^{\frac{1}{2}(\mathbf{dim}(e_1+e_2+e_3+e_4), \mathbf{dim}(e_1+e_2+e_3+e_4)) - \dim_k(e_1+e_2+e_3+e_4)} u_{[e_1+e_2+e_3+e_4]}$$

$$= \upsilon^{-3} u_{[e_1+e_2+e_3+e_4]},$$

$$M_{32} = \upsilon^{\frac{1}{2}(\mathbf{dim}\, e_2, \mathbf{dim}\, e_2) - \dim_k e_2} u_{[e_2]} = u_{[e_2]},$$

$$M_{33} = \upsilon^{\frac{1}{2}(\mathbf{dim}\, e_3, \mathbf{dim}\, e_3) - \dim_k e_3} u_{[e_3]} = u_{[e_3]},$$

$$M_{34} = \upsilon^{\frac{1}{2}(\mathbf{dim}\, e_4, \mathbf{dim}\, e_4) - \dim_k e_4} u_{[e_4]} = u_{[e_4]}.$$

For convenience, we use the following notations:

$$C_1 = \{((m,n)(i,j)) \mid m = i \in \{1,2,3\},\ n \in \{3,4\},\ j \in \{2,3\} \text{ and } n > j\},$$

$$C_2 = \{((m,n)(i,j)) \mid m = i \in \{1,2,3\},\ n \in \{2,3,4\},\ j = 1\},$$

$$C_3 = \{((m,n)(i,j)) \mid m = 3,\ i = 1,\ n = j \in \{2,3,4\}\},$$

$$C_4 = \{((m,n)(i,j)) \mid m \in \{2,3\},\ i = m-1,\ n \in \{1,2,3,4\},\ j \in \{2,3,4\} \text{ and } n \neq j\},$$

$$C_5 = \{((m,n)(i,j)) \mid m = 3,\ n \in \{2,3,4\},\ i = j = 1\},$$

$$C_6 = \{((m,n)(i,j)) \mid m=3,\ i=1,\ n,j \in \{2,3,4\} \text{ and } n \neq j\},$$

$$C_7 = \{((m,n)(i,j)) \mid m \in \{2,3\},\ i=m-1,\ n=j \in \{2,3,4\}\},$$

$$C_8 = \{((m,n)(i,j)) \mid m=3,\ n=i=1,\ j \in \{2,3,4\}\},$$

$$C_9 = \{((m,n)(i,j)) \mid m \in \{2,3\},\ i=m-1,\ n \in \{2,3,4\},\ j=1\},$$

$$C_{10} = \{((m,n)(i,j)) \mid m \in \{2,3\},\ i=m-1,\ n=j=1\},$$

$$C_{11} = \{((m,n)(i,j)) \mid m=3,\ i=n=j=1\}.$$

Then by using the Auslander-Reiten quiver, we get the following relations:

$$\begin{aligned}
M_{mn}M_{ij} &= M_{ij}M_{mn}, & &((m,n)(i,j)) \in C_1,\\
M_{mn}M_{ij} &= \upsilon M_{ij}M_{mn}, & &((m,n)(i,j)) \in C_2 \cup C_3 \cup C_4,\\
M_{mn}M_{ij} &= \upsilon^{-1}M_{ij}M_{mn} + M_{1n}, & &((m,n)(i,j)) \in C_5,\\
M_{mn}M_{ij} &= \upsilon^{-1}M_{ij}M_{mn} + M_{2r}, & &((m,n)(i,j)) \in C_6,
\end{aligned}$$

where $r \in \{2,3,4\}$ and $r \neq n,\ r \neq j,$

$$\begin{aligned}
M_{mn}M_{ij} &= \upsilon^{-1}M_{ij}M_{mn} + M_{m1}, & &((m,n)(i,j)) \in C_7,\\
M_{mn}M_{ij} &= M_{ij}M_{mn} + (\upsilon - \upsilon^{-1})M_{2r}M_{2s}, & &((m,n)(i,j)) \in C_8,
\end{aligned}$$

where $r,s \in \{2,3,4\},$ and $j \neq r,\ j \neq s,\ r < s,$

$$M_{mn}M_{ij} = M_{ij}M_{mn} + (\upsilon - \upsilon^{-1})M_{ir}M_{is}, \quad ((m,n)(i,j)) \in C_9,$$

where $r,s \in \{2,3,4\}$ and $n \neq r,\ n \neq s,\ r < s,$

$$\begin{aligned}
M_{mn}M_{ij} &= \upsilon M_{ij}M_{mn} + (\upsilon^2 - 2 + \upsilon^{-2})M_{i2}M_{i3}M_{i4}, \quad ((m,n)(i,j)) \in C_{10},\\
M_{mn}M_{ij} &= \upsilon^{-1}M_{ij}M_{mn} + (\upsilon - 2\upsilon^{-1})M_{21} + (1-\upsilon^{-2})M_{12}M_{22}\\
&\qquad + (1-\upsilon^{-2})M_{13}M_{23} + (1-\upsilon^{-2})M_{14}M_{24}, \quad ((m,n)(i,j)) \in C_{11}.
\end{aligned}$$

Since e_1, e_2, e_3 and e_4 are the simple modules corresponding to vertices 1, 2, 3 and 4, respectively, it follows that $M_{11} = u_{[e_1]},\ M_{32} = u_{[e_2]},\ M_{33} = u_{[e_3]},\ M_{34} = u_{[e_4]}$. Let

$$\begin{aligned}
&E_1 = E_{11} = \eta^{-1}(M_{11}), && E_4 = E_{34} = \eta^{-1}(M_{34}), && E_{14} = \eta^{-1}(M_{14}), && E_{23} = \eta^{-1}(M_{23}),\\
&E_2 = E_{32} = \eta^{-1}(M_{32}), && E_{12} = \eta^{-1}(M_{12}), && E_{21} = \eta^{-1}(M_{21}), && E_{24} = \eta^{-1}(M_{24}),\\
&E_3 = E_{33} = \eta^{-1}(M_{33}), && E_{13} = \eta^{-1}(M_{13}), && E_{22} = \eta^{-1}(M_{22}), && E_{31} = \eta^{-1}(M_{31}),
\end{aligned}$$

where η is the isomorphism in Theorem 2.3, and let

$$X = \{E_{11}, E_{12}, E_{13}, E_{14}, E_{21}, E_{22}, E_{23}, E_{24}, E_{31}, E_{32}, E_{33}, E_{34}\}.$$

Then we have the following relations (under the isomorphism η^{-1} in Theorem 2.3):

$$\begin{aligned}
E_{mn}E_{ij} &= E_{ij}E_{mn}, && ((m,n)(i,j)) \in C_1,\\
E_{mn}E_{ij} &= \upsilon E_{ij}E_{mn}, && ((m,n)(i,j)) \in C_2,\\
E_{mn}E_{ij} &= \upsilon E_{ij}E_{mn}, && ((m,n)(i,j)) \in C_3,\\
E_{mn}E_{ij} &= \upsilon E_{ij}E_{mn}, && ((m,n)(i,j)) \in C_4,\\
E_{mn}E_{ij} &= \upsilon^{-1} E_{ij}E_{mn} + E_{1n}, && ((m,n)(i,j)) \in C_5,\\
E_{mn}E_{ij} &= \upsilon^{-1} E_{ij}E_{mn} + E_{2r}, && ((m,n)(i,j)) \in C_6,\\
E_{mn}E_{ij} &= \upsilon^{-1} E_{ij}E_{mn} + E_{m1}, && ((m,n)(i,j)) \in C_7,\\
E_{mn}E_{ij} &= E_{ij}E_{mn} + (\upsilon - \upsilon^{-1})E_{2r}E_{2s}, && ((m,n)(i,j)) \in C_8,\\
E_{mn}E_{ij} &= E_{ij}E_{mn} + (\upsilon - \upsilon^{-1})E_{ir}E_{is}, && ((m,n)(i,j)) \in C_9,\\
E_{mn}E_{ij} &= \upsilon E_{ij}E_{mn} + (\upsilon^2 - 2 + \upsilon^{-2})E_{i2}E_{i3}E_{i4}, && ((m,n)(i,j)) \in C_{10},\\
E_{mn}E_{ij} &= \upsilon^{-1} E_{ij}E_{mn} + (\upsilon - 2\upsilon^{-1})E_{21} + (1-\upsilon^{-2})E_{12}E_{22}\\
&\quad +(1-\upsilon^{-2})E_{13}E_{23} + (1-\upsilon^{-2})E_{14}E_{24}, && ((m,n)(i,j)) \in C_{11}.
\end{aligned} \tag{3.1}$$

Note that the relations (3.1) include the Serre relations S^+. So $U_q^+(A)$ can be viewed as a factor algebra $\mathbb{Q}(\upsilon)\langle X\rangle/I$, where I is the ideal generated by the relations (3.1).

We define an ordering

$$E_{11} < E_{12} < E_{13} < E_{14} < E_{21} < E_{22} < E_{23} < E_{24} < E_{31} < E_{32} < E_{33} < E_{34}$$

for the elements $E_{11}, E_{12}, E_{13}, E_{14}, E_{21}, E_{22}, E_{23}, E_{24}, E_{31}, E_{32}, E_{33}, E_{34}$, and then this ordering induces a degree-lexicographical ordering on the monomials of these elements. For convenience, we denote by $r_1, \cdots, r_{11}$, respectively, the polynomials obtained from the relations in (3.1) by subtracting the right-hand side from the left-hand side, and let $S^{+c} = \{r_1, r_2, \cdots, r_{11}\}$. Then, of course, $S^+ \subset S^{+c}$, and we have the following theorem.

Theorem 3.1 *The set S^{+c} is a Gröbner-Shirshov basis of the algebra $U_q^+(A)$.*

Proof The possible compositions between the elements of S^{+c} can be divided into 32 cases. We only prove the triviality of three cases, and the proofs of other cases are similar.

Case 1 Let $f = r_4 = E_{mn}E_{ij} - \upsilon E_{ij}E_{mn},\ g = r_4 = E_{ij}E_{kl} - \upsilon E_{kl}E_{ij},\ \omega = E_{mn}E_{ij}E_{kl}$, where $((m,n)(i,j)), ((i,j)(k,l)) \in C_4$ and $((m,n)(k,l)) \in C_3, C_6$ or C_8. We consider the following different cases:

(1.1) If $((m,n)(i,j)), ((i,j)(k,l)) \in C_4$ and $((m,n)(k,l)) \in C_3$, then

$$\begin{aligned}
(f,g)_\omega &\equiv -\upsilon E_{ij}E_{mn}E_{kl} + \upsilon E_{mn}E_{kl}E_{ij} \mod(S^{+c}, E_{mn}E_{ij}E_{kl})\\
&\equiv -\upsilon^2 E_{ij}E_{kl}E_{mn} + \upsilon^2 E_{kl}E_{mn}E_{ij} \mod(S^{+c}, E_{mn}E_{ij}E_{kl})\\
&\equiv -\upsilon^3 E_{kl}E_{ij}E_{mn} + \upsilon^3 E_{kl}E_{ij}E_{mn} \mod(S^{+c}, E_{mn}E_{ij}E_{kl})\\
&\equiv 0 \mod(S^{+c}, E_{mn}E_{ij}E_{kl}).
\end{aligned}$$

(1.2) If $((m,n)(i,j)), ((i,j)(k,l)) \in C_4$ and $((m,n)(k,l)) \in C_6$, then

$$\begin{aligned}(f,g)_\omega &\equiv -\upsilon E_{ij}E_{mn}E_{kl} + \upsilon E_{mn}E_{kl}E_{ij} \quad \mathrm{mod}(S^{+c}, E_{mn}E_{ij}E_{kl})\\ &\equiv -\upsilon E_{ij}[\upsilon^{-1}E_{kl}E_{mn} + E_{2r}] + \upsilon[\upsilon^{-1}E_{kl}E_{mn} + E_{2r}]E_{ij} \quad \mathrm{mod}(S^{+c}, E_{mn}E_{ij}E_{kl})\\ &\equiv -\upsilon E_{kl}E_{ij}E_{mn} - \upsilon E_{ij}E_{2r} + \upsilon E_{kl}E_{ij}E_{mn} + \upsilon E_{2r}E_{ij} \quad \mathrm{mod}(S^{+c}, E_{mn}E_{ij}E_{kl})\\ &\equiv 0 \quad \mathrm{mod}(S^{+c}, E_{mn}E_{ij}E_{kl}),\end{aligned}$$

where $i = 2$, $j = r$.

(1.3) If $((m,n)(i,j)), ((i,j)(k,l)) \in C_4$, $((i,j)(2,r)), ((i,j)(2,s)) \in C_1$ and $((m,n)(k,l)) \in C_8$, then

$$\begin{aligned}(f,g)_\omega &\equiv -\upsilon E_{ij}E_{mn}E_{kl} + \upsilon E_{mn}E_{kl}E_{ij} \quad \mathrm{mod}(S^{+c}, E_{mn}E_{ij}E_{kl})\\ &\equiv -\upsilon E_{ij}[E_{kl}E_{mn} + (\upsilon-\upsilon^{-1})E_{2r}E_{2s}]\\ &\quad + \upsilon[E_{kl}E_{mn} + (\upsilon-\upsilon^{-1})E_{2r}E_{2s}]E_{ij} \quad \mathrm{mod}(S^{+c}, E_{mn}E_{ij}E_{kl})\\ &\equiv -\upsilon^2 E_{kl}E_{ij}E_{mn} - (\upsilon^2-1)E_{ij}E_{2r}E_{2s}\\ &\quad + \upsilon^2 E_{kl}E_{ij}E_{mn} + (\upsilon^2-1)E_{2r}E_{2s}E_{ij} \quad \mathrm{mod}(S^{+c}, E_{mn}E_{ij}E_{kl})\\ &\equiv -(\upsilon^2-1)E_{2r}E_{2s}E_{ij} + (\upsilon^2-1)E_{2r}E_{2s}E_{ij} \quad \mathrm{mod}(S^{+c}, E_{mn}E_{ij}E_{kl})\\ &\equiv 0 \quad \mathrm{mod}(S^{+c}, E_{mn}E_{ij}E_{kl}).\end{aligned}$$

Case 2 Let $f = r_9 = E_{mn}E_{ij} - E_{ij}E_{mn} - (\upsilon-\upsilon^{-1})E_{ir}E_{is}$, $g = r_4 = E_{ij}E_{kl} - \upsilon E_{kl}E_{ij}$, $\omega = E_{mn}E_{ij}E_{kl}$, where $((m,n)(i,j)) \in C_9$, $((i,j)(k,l)) \in C_4$ and $((m,n)(k,l)) \in C_3$ or C_6. We consider the following different cases:

(2.1) If $((m,n)(k,l)) \in C_3$, $((2,r)(k,l)), ((2,s)(k,l)) \in C_4$, then

$$\begin{aligned}(f,g)_\omega &\equiv -E_{ij}E_{mn}E_{kl} - (\upsilon-\upsilon^{-1})E_{ir}E_{is}E_{kl} + \upsilon E_{mn}E_{kl}E_{ij} \quad \mathrm{mod}(S^{+c}, E_{mn}E_{ij}E_{kl})\\ &\equiv -\upsilon E_{ij}E_{kl}E_{mn} - (\upsilon-\upsilon^{-1})E_{ir}E_{is}E_{kl} + \upsilon^2 E_{kl}E_{mn}E_{ij} \quad \mathrm{mod}(S^{+c}, E_{mn}E_{ij}E_{kl})\\ &\equiv -\upsilon^2 E_{kl}E_{ij}E_{mn} - (\upsilon-\upsilon^{-1})E_{ir}E_{is}E_{kl} + \upsilon^2 E_{kl}E_{ij}E_{mn}\\ &\quad + (\upsilon^3-\upsilon)E_{kl}E_{ir}E_{is} \quad \mathrm{mod}(S^{+c}, E_{mn}E_{ij}E_{kl})\\ &\equiv -(\upsilon-\upsilon^{-1})\upsilon E_{ir}E_{kl}E_{is} + (\upsilon^3-\upsilon)E_{kl}E_{ir}E_{is} \quad \mathrm{mod}(S^{+c}, E_{mn}E_{ij}E_{kl})\\ &\equiv -(\upsilon-\upsilon^{-1})\upsilon^2 E_{kl}E_{ir}E_{is} + (\upsilon^3-\upsilon)E_{kl}E_{ir}E_{is} \quad \mathrm{mod}(S^{+c}, E_{mn}E_{ij}E_{kl})\\ &\equiv 0 \quad \mathrm{mod}(S^{+c}, E_{mn}E_{ij}E_{kl}).\end{aligned}$$

(2.2) If $((m,n)(k,l)) \in C_6$, then

$$\begin{aligned}(f,g)_\omega &\equiv -E_{ij}E_{mn}E_{kl} - (\upsilon-\upsilon^{-1})E_{ir}E_{is}E_{kl} + \upsilon E_{mn}E_{kl}E_{ij} \quad \mathrm{mod}(S^{+c}, E_{mn}E_{ij}E_{kl})\\ &\equiv -E_{ij}[\upsilon^{-1}E_{kl}E_{mn} + E_{2t}] - (\upsilon-\upsilon^{-1})E_{ir}E_{is}E_{kl}\end{aligned}$$

$$+\,\upsilon[\upsilon^{-1}E_{kl}E_{mn}+E_{2t}]E_{ij} \quad \mathrm{mod}(S^{+c},E_{mn}E_{ij}E_{kl})$$

$$\equiv -E_{kl}E_{ij}E_{mn}-E_{ij}E_{2t}-(\upsilon-\upsilon^{-1})E_{ir}E_{is}E_{kl}+E_{kl}E_{ij}E_{mn}$$

$$+\,(\upsilon-\upsilon^{-1})E_{kl}E_{ir}E_{is}+\upsilon E_{2t}E_{ij} \quad \mathrm{mod}(S^{+c},E_{mn}E_{ij}E_{kl})$$

$$\equiv -E_{ij}E_{2t}-(\upsilon-\upsilon^{-1})E_{ir}E_{is}E_{kl}+(\upsilon-\upsilon^{-1})E_{kl}E_{ir}E_{is}$$
$$+\,\upsilon E_{2t}E_{ij} \quad \mathrm{mod}(S^{+c},E_{mn}E_{ij}E_{kl}).$$

If $((i,r)(k,l))\in C_7$, $((i,s)(k,l))\in C_4$, $((2,t)(i,j))\in C_2$, then

$$(f,g)_\omega \equiv -E_{ij}E_{2t}-(\upsilon^2-1)E_{ir}E_{kl}E_{is}+(\upsilon-\upsilon^{-1})E_{kl}E_{ir}E_{is}$$

$$+\,\upsilon^2E_{ij}E_{2t} \quad \mathrm{mod}(S^{+c},E_{mn}E_{ij}E_{kl})$$

$$\equiv -(\upsilon-\upsilon^{-1})E_{kl}E_{ir}E_{is}-(\upsilon^2-1)E_{i1}E_{is}+(\upsilon-\upsilon^{-1})E_{kl}E_{ir}E_{is}$$

$$+\,(\upsilon^2-1)E_{ij}E_{2t} \quad \mathrm{mod}(S^{+c},E_{mn}E_{ij}E_{kl})$$

$$\equiv 0 \quad \mathrm{mod}(S^{+c},E_{mn}E_{ij}E_{kl}),$$

where $i=2$, $j=1$, $t=s$.

If $((i,r)(k,l))\in C_4$, $((i,s)(k,l))\in C_7$, $((2,t)(i,j))\in C_2$, $((i,r)(i,1))\in C_2$, then

$$(f,g)_\omega \equiv -E_{ij}E_{2t}-(\upsilon-\upsilon^{-1})E_{ir}[\upsilon^{-1}E_{kl}E_{is}+E_{i1}]$$

$$+\,(\upsilon-\upsilon^{-1})E_{kl}E_{ir}E_{is}+\upsilon^2E_{ij}E_{2t} \quad \mathrm{mod}(S^{+c},E_{mn}E_{ij}E_{kl})$$

$$\equiv -E_{ij}E_{2t}-(\upsilon-\upsilon^{-1})E_{kl}E_{ir}E_{is}-(\upsilon-\upsilon^{-1})E_{ir}E_{i1}$$

$$+\,(\upsilon-\upsilon^{-1})E_{kl}E_{ir}E_{is}+\upsilon^2E_{ij}E_{2t} \quad \mathrm{mod}(S^{+c},E_{mn}E_{ij}E_{kl})$$

$$\equiv -E_{ij}E_{2t}-(\upsilon^2-1)E_{i1}E_{ir}+\upsilon^2E_{ij}E_{2t} \quad \mathrm{mod}(S^{+c},E_{mn}E_{ij}E_{kl})$$

$$\equiv 0 \quad \mathrm{mod}(S^{+c},E_{mn}E_{ij}E_{kl}),$$

where $i=2$, $j=1$, $t=r$.

Case 3 Let

$$f=r_{10}=E_{31}E_{21}-\upsilon E_{21}E_{31}-(\upsilon^2-2+\upsilon^{-2})E_{22}E_{23}E_{24}$$

and

$$g=r_{10}=E_{21}E_{11}-\upsilon E_{11}E_{21}-(\upsilon^2-2+\upsilon^{-2})E_{12}E_{13}E_{14}, \quad w=E_{31}E_{21}E_{11}.$$

Then

$$(f,g)_\omega \equiv -\upsilon E_{21}E_{31}E_{11}-(\upsilon^2-2+\upsilon^{-2})E_{22}E_{23}E_{24}E_{11}+\upsilon E_{31}E_{11}E_{21}$$

$$+\,(\upsilon^2-2+\upsilon^{-2})E_{31}E_{12}E_{13}E_{14} \quad \mathrm{mod}(S^{+c},E_{mn}E_{ij}E_{kl})$$

$$\equiv -\upsilon E_{21}[\upsilon^{-1}E_{11}E_{31}+(\upsilon-2\upsilon^{-1})E_{21}+(1-\upsilon^{-2})E_{12}E_{22}+(1-\upsilon^{-2})E_{13}E_{23}$$

$$+\,(1-\upsilon^{-2})E_{14}E_{24}]-(\upsilon^2-2+\upsilon^{-2})E_{22}E_{23}[E_{11}E_{24}+(\upsilon-\upsilon^{-1})E_{12}E_{13}]$$

$$
\begin{aligned}
&+ \upsilon[\upsilon^{-1}E_{11}E_{31} + (\upsilon - 2\upsilon^{-1})E_{21} + (1-\upsilon^{-2})E_{12}E_{22} \\
&+ (1-\upsilon^{-2})E_{13}E_{23} + (1-\upsilon^{-2})E_{14}E_{24}]E_{21} \\
&+ (\upsilon^2 - 2 + \upsilon^{-2})[E_{12}E_{31} + (\upsilon - \upsilon^{-1})E_{23}E_{24}]E_{13}E_{14} \quad \mathrm{mod}(S^{+c}, E_{mn}E_{ij}E_{kl}) \\
\equiv\ & -E_{21}E_{11}E_{31} - (\upsilon^2-2)E_{21}E_{21} - (\upsilon-\upsilon^{-1})E_{21}E_{12}E_{22} - (\upsilon-\upsilon^{-1})E_{21}E_{13}E_{23} \\
&- (\upsilon-\upsilon^{-1})E_{21}E_{14}E_{24} - (\upsilon^2-2+\upsilon^{-2})E_{22}[E_{11}E_{23} + (\upsilon-\upsilon^{-1})E_{12}E_{14}]E_{24} \\
&- (\upsilon^3 - 3\upsilon + 3\upsilon^{-1} - \upsilon^{-3})E_{22}E_{23}E_{12}E_{13} + E_{11}E_{31}E_{21} + (\upsilon^2-2)E_{21}E_{21} \\
&+ (\upsilon-\upsilon^{-1})E_{12}E_{22}E_{21} + (\upsilon-\upsilon^{-1})E_{13}E_{23}E_{21} + (\upsilon-\upsilon^{-1})E_{14}E_{24}E_{21} \\
&+ (\upsilon^2-2+\upsilon^{-2})E_{12}[E_{13}E_{31} + (\upsilon-\upsilon^{-1})E_{22}E_{24}]E_{14} \\
&+ (\upsilon^3 - 3\upsilon + 3\upsilon^{-1} - \upsilon^{-3})E_{23}E_{24}E_{13}E_{14} \quad \mathrm{mod}(S^{+c}, E_{mn}E_{ij}E_{kl}) \\
\equiv\ & -\upsilon E_{11}E_{21}E_{31} - (\upsilon^2-2+\upsilon^{-2})E_{12}E_{13}E_{14}E_{31} - (\upsilon^2-1)E_{12}E_{21}E_{22} \\
&- (\upsilon^2-1)E_{13}E_{21}E_{23} - (\upsilon^2-1)E_{14}E_{21}E_{24} - (\upsilon^2-2+\upsilon^{-2})[E_{11}E_{22} \\
&+ (\upsilon-\upsilon^{-1})E_{13}E_{14}]E_{23}E_{24} - (\upsilon^3 - 3\upsilon + 3\upsilon^{-1} - \upsilon^{-3})E_{22}E_{12}E_{14}E_{24} \\
&- (\upsilon^4 - 3\upsilon^2 + 3 - \upsilon^{-2})E_{22}E_{12}E_{23}E_{13} + \upsilon E_{11}E_{21}E_{31} \\
&+ (\upsilon^2-2+\upsilon^{-2})E_{11}E_{22}E_{23}E_{24} + (\upsilon^2-1)E_{12}E_{21}E_{22} + (\upsilon^2-1)E_{13}E_{21}E_{23} \\
&+ (\upsilon^2-1)E_{14}E_{21}E_{24} + (\upsilon^2-2+\upsilon^{-2})E_{12}E_{13}[E_{14}E_{31} + (\upsilon-\upsilon^{-1})E_{22}E_{23}] \\
&- (\upsilon^3 - 3\upsilon + 3\upsilon^{-1} - \upsilon^{-3})E_{12}E_{22}E_{24}E_{14} \\
&+ (\upsilon^4 - 3\upsilon^2 + 3 - \upsilon^{-2})E_{23}E_{13}E_{24}E_{14} \quad \mathrm{mod}(S^{+c}, E_{mn}E_{ij}E_{kl}) \\
\equiv\ & -(\upsilon^2-2+\upsilon^{-2})E_{12}E_{13}E_{14}E_{31} - (\upsilon^2-2+\upsilon^{-2})E_{11}E_{22}E_{23}E_{24} \\
&- (\upsilon^3 - 3\upsilon + 3\upsilon^{-1} - \upsilon^{-3})E_{13}E_{14}E_{23}E_{24} - (\upsilon^3 - 3\upsilon + 3\upsilon^{-1} - \upsilon^{-3})[\upsilon^{-1}E_{12}E_{22} \\
&+ E_{21}]E_{14}E_{24} - (\upsilon^4 - 3\upsilon^2 + 3 - \upsilon^{-2})[\upsilon^{-1}E_{12}E_{22} + E_{21}][\upsilon^{-1}E_{13}E_{23} + E_{21}] \\
&+ (\upsilon^2-2+\upsilon^{-2})E_{11}E_{22}E_{23}E_{24} + (\upsilon^2-2+\upsilon^{-2})E_{12}E_{13}E_{14}E_{31} \\
&+ (\upsilon^3 - 3\upsilon + 3\upsilon^{-1} - \upsilon^{-3})E_{12}E_{13}E_{22}E_{23} \\
&+ (\upsilon^3 - 3\upsilon + 3\upsilon^{-1} - \upsilon^{-3})E_{12}E_{22}[\upsilon^{-1}E_{14}E_{24} + E_{21}] \\
&- (\upsilon^4 - 3\upsilon^2 + 3 - \upsilon^{-2})[\upsilon^{-1}E_{13}E_{23} + E_{21}][\upsilon^{-1}E_{14}E_{24} + E_{21}] \quad \mathrm{mod}(S^{+c}, E_{mn}E_{ij}E_{kl}) \\
\equiv\ & -(\upsilon^3 - 3\upsilon + 3\upsilon^{-1} - \upsilon^{-3})E_{13}E_{14}E_{23}E_{24} - (\upsilon^3 - 3\upsilon + 3\upsilon^{-1} - \upsilon^{-3})E_{12}E_{14}E_{22}E_{24} \\
&- (\upsilon^4 - 3\upsilon^2 + 3 - \upsilon^{-2})E_{14}E_{21}E_{24} - (\upsilon^3 - 3\upsilon + 3\upsilon^{-1} - \upsilon^{-3})E_{12}E_{13}E_{22}E_{23} \\
&- (\upsilon^4 - 3\upsilon^2 + 3 - \upsilon^{-2})E_{12}E_{21}E_{22} - (\upsilon^4 - 3\upsilon^2 + 3 - \upsilon^{-2})E_{13}E_{21}E_{23}
\end{aligned}
$$

$$
\begin{aligned}
&-(v^4-3v^2+3-v^{-2})E_{21}E_{21}+(v^3-3v+3v^{-1}-v^{-3})E_{12}E_{13}E_{22}E_{23}\\
&+(v^3-3v+3v^{-1}-v^{-3})E_{12}E_{14}E_{22}E_{24}+(v^4-3v^2+3-v^{-2})E_{12}E_{21}E_{22}\\
&+(v^3-3v+3v^{-1}-v^{-3})E_{13}E_{14}E_{23}E_{24}+(v^4-3v^2+3-v^{-2})E_{13}E_{21}E_{23}\\
&+(v^4-3v^2+3-v^{-2})E_{14}E_{21}E_{24}+(v^4-3v^2+3-v^{-2})E_{21}E_{21}\\
\equiv\ &0 \quad \mathrm{mod}(S^{+c}, E_{mn}E_{ij}E_{kl}).
\end{aligned}
$$

Dually, by replacing all E's in (3.1) by F's, we get similar relations, say $(3.1)'$, for the generators

$$Y=\{F_{11},F_{12},F_{13},F_{14},F_{21},F_{22},F_{23},F_{24},F_{31},F_{32},F_{33},F_{34}\}$$

of subalgebra $U_q^-(A)$. It is easy to see that relations $(3.1)'$ include the Serre relations S^-. So, similarly, if J is the ideal generated by the relations $(3.1)'$, then the negative part $U_q^-(A)$ of quantum group $U_q(A)$ can be viewed as a factor algebra $\mathbb{Q}(v)\langle Y\rangle/J$ of the free algebra $\mathbb{Q}(v)\langle Y\rangle$ generated by the set Y.

We define an ordering

$$F_{11}<F_{12}<F_{13}<F_{14}<F_{21}<F_{22}<F_{23}<F_{24}<F_{31}<F_{32}<F_{33}<F_{34}$$

for the elements $F_{11},F_{12},F_{13},F_{14},F_{21},F_{22},F_{23},F_{24},F_{31},F_{32},F_{33},F_{34}$. Then this ordering induces a degree-lexicographical ordering on the monomials of these elements. In a way similar to the discussions in the positive part, we denote the polynomials obtained from the relations in $(3.1)'$ by $f_1,\cdots,f_{11}$, and let $S^{-c}=\{f_1,f_2,\cdots,f_{11}\}$. Then, of course, $S^-\subset S^{-c}$, and we have the following theorem.

Theorem 3.2 *The set S^{-c} is a Gröbner-Shirshov basis of the algebra $U_q^-(A)$.*

If we define an ordering

$$
\begin{aligned}
&E_{11}<E_{12}<E_{13}<E_{14}<E_{21}<E_{22}<E_{23}<E_{24}<E_{31}<E_{32}<E_{33}<E_{34}<K_1<K_2\\
&<K_3<K_4<F_{11}<F_{12}<F_{13}<F_{14}<F_{21}<F_{22}<F_{23}<F_{24}<F_{31}<F_{32}<F_{33}<F_{34}
\end{aligned}
$$

for the elements $E_{11},E_{12},E_{13},E_{14},E_{21},E_{22},E_{23},E_{24},E_{31},E_{32},E_{33},E_{34},K_1,K_2,K_3,K_4,\ F_{11}, F_{12},F_{13},F_{14},F_{21},F_{22},F_{23},F_{24},F_{31},F_{32},F_{33},F_{34}$, then this ordering induces a degree-lexicographical ordering on the monomials of these elements. Now, by [3, Theorem 2.7], we are able to state our main result.

Theorem 3.3 *The set $S^{+c}\cup K\cup T\cup S^{-c}$ is a Gröbner-Shirshov basis of the quantum group $U_q(A)$.*

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