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A New 4×4 AKNS Spectral Problem and Its Associated Integrable Decomposition of the AKNS Equation****

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Abstract A new approach to construct a new 4×4 matrix spectral problem from a normal 2×2 matrix spectral problem is presented. AKNS spectral problem is discussed as an example. The isospectral evolution equation of the new 4×4 matrix spectral problem is nothing but the famous AKNS equation hierarchy. With the aid of the binary nonlinearization method, the authors get new integrable decompositions of the AKNS equation. In this process, the r-matrix is used to get the result.

Keywords Spectral problem, Integrable decomposition, r-matrix
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1 Introduction

Constructing high-order matrix spectral problem and searching for new integrable hierarchies of its corresponding soliton equations is an important and interesting topic in soliton theory. Many studies are devoted to obtaining coupling integrable systems and coupling soliton equations by formulating high-order matrix spectral problem (see [1–9]).

Especially, W. X. Ma, B. Fuchssteiner and W. Oevel [10] presented a 3×3 matrix spectral problem for AKNS hierarchy; Z. N. Zhu, H. C. Huang, W. M. Xue and X. N. Wu [11] presented a 3×3 matrix spectral problem for Kaup-Newell hierarchy; M. Blaszak and W. X. Ma [12] showed us a way to construct a new 3×3 matrix spectral problem from a normal 2×2 matrix spectral problem.

AKNS equations can describe the wave phenomena observed in fluid dynamics, plasma and elastic media. We can get many famous equations from the AKNS equations with different reductions, Korteweg-de Vries equation, etc. The nonlinearization methods including mono-nonlinearization method (see [13–15]) and binary nonlinearization method (see [1, 3, 4, 7, 8, 12, 17]) presented a way to obtain the integrable Hamiltonian systems and solutions of a partial differential equation by decomposing it into two ordinary differential equations. Hence, we can

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obtain new solutions of AKNS equation from new high-order matrix spectral problem associated with new constraints.

In Section 2, we recall some results of the hierarchy of AKNS equation obtained from the AKNS spectral problem in 2×2 matrix representation as a preparatory knowledge for further consideration. In Section 3, we formulize a new 4×4 spectral problem from a normal 2×2 AKNS spectral problem. It is found that the isospectral evolution equation hierarchy of the new 4×4 spectral problem turns out to be the famous AKNS equation hierarchy. The binary restricted AKNS flows and their Lax representation are presented. The r-matrix formulation is developed and an involutive system of $4N$ functionally independent integrals of motion generated from the Lax representation is deduced, the decomposition of the AKNS equation is integrable in the sense of Liouville.

2 The Hierarchy of the AKNS Equation

Consider spectral problem (see [3])

$$\Phi_x = U(\lambda, u) = \begin{pmatrix} -\lambda & q \\ r & \lambda \end{pmatrix} \Phi, \quad u = \begin{pmatrix} q \\ r \end{pmatrix}, \quad \Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}. \tag{2.1}$$

Take

$$V = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = \sum_{i=0}^{\infty} \begin{pmatrix} a_i & b_i \\ c_i & -a_i \end{pmatrix} \lambda^{-i}. \tag{2.2}$$

Then, we find that the adjoint representation equation $V_x = [U, V] \equiv UV - VU$ yields

$$\begin{aligned} &a_{0_x} = qc_0 - rb_0, \quad b_0 = 0, \quad c_0 = 0, \quad a_{ix} = qc_i - rb_i, \\ &b_{ix} = -2(b_{i+1} - qa_i), \quad c_{ix} = 2(ra_i - c_{i+1}), \quad i \geq 1. \end{aligned} \tag{2.3}$$

Choose $a_0 = -1$. Then $b_0 = 0$, $c_0 = 0$, $a_1 = 0$, $b_1 = q$, $c_1 = r$, $b_2 = -\frac{1}{2}q_x$, $c_2 = \frac{1}{2}r_x$, $a_2 = \frac{1}{2}qr$,

$$\begin{pmatrix} c_{n+1} \\ b_{n+1} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \partial - 2r\partial^{-1}q & 2r\partial^{-1}r \\ -2q\partial^{-1}q & -\partial + 2q\partial^{-1}r \end{pmatrix} \begin{pmatrix} c_n \\ b_n \end{pmatrix} = \mathcal{L} \begin{pmatrix} c_n \\ b_n \end{pmatrix}, \tag{2.4}$$

$$a_n = \partial^{-1}(qc_n - rb_n), \quad n = 0, 1, 2, 3 \cdots .$$

Consider the auxiliary spectral problem $\Phi_{t_n} = V^{(n)}\Phi$, where

$$V^{(n)} = (\lambda^n V)_+ = \sum_{i=0}^{n} \begin{pmatrix} a_i & b_i \\ c_i & -a_i \end{pmatrix} \lambda^{n-i}. \tag{2.5}$$

And the zero curvature equation $U_{t_n} - V_x^{(n)} + [U, V^{(n)}] = 0$ gives

$$u_{t_n} = \begin{pmatrix} q_{t_n} \\ r_{t_n} \end{pmatrix} = \pi \mathcal{L}^n \begin{pmatrix} r \\ q \end{pmatrix} = \pi \frac{\delta H_n}{\delta u}, \tag{2.6}$$

where (see [15])

$$\pi = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}, \quad H_n = \int \frac{2a_{n+2}}{n+1} \mathrm{d}x, \quad \begin{pmatrix} c_n \\ b_n \end{pmatrix} = \frac{\delta H_n}{\delta u}, \quad n \geq 0. \tag{2.7}$$

For $n = 2$, we obtain the famous AKNS equation

$$\begin{cases} q_{t_1} = -\dfrac{1}{2}q_{xx} + q^2 r, \\ r_{t_1} = \dfrac{1}{2}r_{xx} - qr^2, \end{cases} \tag{2.8}$$

and the corresponding Lax operators are

$$V^{(1)} = \begin{pmatrix} -\lambda & q \\ r & \lambda \end{pmatrix} \equiv U, \quad V^{(2)} = \begin{pmatrix} -\lambda^2 + \frac{1}{2}qr & q\lambda - \frac{1}{2}q_x \\ r\lambda + \frac{1}{2}r_x & \lambda^2 - \frac{1}{2}qr \end{pmatrix}. \tag{2.9}$$

3 A New 4 × 4 Spectral Problem and Its Associated Integrable Decomposition of AKNS Equation

3.1 A new 4 × 4 spectral problem

Assuming $(\overline{\phi}_1, \overline{\phi}_2, \overline{\phi}_3, \overline{\phi}_4) = (\phi_1^2\phi_2, \phi_1\phi_2^2, \phi_2^3, \phi_1^3)$, we can get

$$\begin{aligned} \overline{\phi}_{1x} &= -\lambda\overline{\phi}_1 + 2q\overline{\phi}_2 + r\overline{\phi}_4, \\ \overline{\phi}_{2x} &= 2r\overline{\phi}_1 + \lambda\overline{\phi}_2 + q\overline{\phi}_3, \\ \overline{\phi}_{3x} &= 3r\overline{\phi}_2 + 3\lambda\overline{\phi}_3, \\ \overline{\phi}_{4x} &= 3q\overline{\phi}_1 - 3\lambda\overline{\phi}_4. \end{aligned}$$

A new 4 × 4 spectral problem is obtained as follows:

$$\overline{\Phi}_x = \overline{U}(u,\lambda)\overline{\Phi}, \quad \overline{U}(u,\lambda) = \begin{pmatrix} -\lambda & 2q & 0 & r \\ 2r & \lambda & q & 0 \\ 0 & 3r & 3\lambda & 0 \\ 3q & 0 & 0 & -3\lambda \end{pmatrix}, \quad \overline{\Phi} = \begin{pmatrix} \overline{\phi}_1 \\ \overline{\phi}_2 \\ \overline{\phi}_3 \\ \overline{\phi}_4 \end{pmatrix}, \tag{3.1}$$

where q, r are still potentials and λ is a spectral parameter. The parameters and variants in this article are all constants. We will show that the isospectral evolution equations of the new 4 × 4 spectral problem (3.1) is nothing but the famous AKNS equation hierarchy.

Take

$$\overline{V} = \begin{pmatrix} a & 2b & 0 & c \\ 2c & -a & b & 0 \\ 0 & 3c & -3a & 0 \\ 3b & 0 & 0 & 3a \end{pmatrix} = \sum_{i\geq 0} \begin{pmatrix} a_i & 2b_i & 0 & c_i \\ 2c_i & -a_i & b_i & 0 \\ 0 & 3c_i & -3a_i & 0 \\ 3b_i & 0 & 0 & 3a_i \end{pmatrix} \lambda^{-i}.$$

By simple calculation, the adjoint representation equation $\overline{V}_x = [\overline{U}, \overline{V}]$ leads to (2.3) and (2.4).

Take the auxiliary spectral problem as

$$\overline{\Phi}_{t_n} = \overline{V}^{(n)} \begin{pmatrix} \overline{\phi}_1 \\ \overline{\phi}_2 \\ \overline{\phi}_3 \\ \overline{\phi}_4 \end{pmatrix}, \quad \overline{V}^{(n)} = \sum_{i=0}^{n} \begin{pmatrix} a_i & 2b_i & 0 & c_i \\ 2c_i & -a_i & b_i & 0 \\ 0 & 3c_i & -3a_i & 0 \\ 3b_i & 0 & 0 & 3a_i \end{pmatrix} \lambda^{n-i}. \tag{3.2}$$

The zero curvature equation $\overline{U}_{t_n} - \overline{V}_x^{(n)} + [\overline{U}, \overline{V}^{(n)}] = 0$ just leads to the AKNS equation hierarchies (2.6).

3.2 The binary restricted AKNS flows

Consider N distinct constants $\lambda_1, \ldots, \lambda_N$ and the corresponding spectral problem and the adjoint spectral problem

$$\begin{cases} (\overline{\phi}_{1_j}, \overline{\phi}_{2_j}, \overline{\phi}_{3_j}, \overline{\phi}_{4_j})_x = \overline{U}(\widehat{u}, \lambda_j)(\overline{\phi}_{1_j}, \overline{\phi}_{2_j}, \overline{\phi}_{3_j}, \overline{\phi}_{4_j}), & 1 \le j \le N, \\ (\overline{\psi}_{1_j}, \overline{\psi}_{2_j}, \overline{\psi}_{3_j}, \overline{\psi}_{4_j})_x = -\overline{U}^T(\widehat{u}, \lambda_j)(\overline{\psi}_{1_j}, \overline{\psi}_{2_j}, \overline{\psi}_{3_j}, \overline{\psi}_{4_j}), & 1 \le j \le N. \end{cases} \tag{3.3}$$

From [10], we have

$$\frac{\delta \lambda_j}{\delta u} = \begin{pmatrix} \frac{\delta \lambda_j}{\delta q} \\ \frac{\delta \lambda_j}{\delta r} \end{pmatrix} = \begin{pmatrix} 3\overline{\phi}_{1j}\overline{\psi}_{4j} + 2\overline{\phi}_{2j}\overline{\psi}_{1j} + \overline{\phi}_{3j}\overline{\psi}_{2j} \\ 3\overline{\phi}_{2j}\overline{\psi}_{3j} + 2\overline{\phi}_{1j}\overline{\psi}_{2j} + \overline{\phi}_{4j}\overline{\psi}_{1j} \end{pmatrix}. \tag{3.4}$$

As the usual approach of binary nonlinearization of Lax pairs, we consider the constraints

$$\frac{\delta H_k}{\delta u} = \begin{pmatrix} c_k \\ b_k \end{pmatrix} = \sum_{i=1}^{N} \frac{\delta \lambda_i}{\delta u} = \begin{pmatrix} 3\langle \overline{Q}_1, \overline{P}_4 \rangle + 2\langle \overline{Q}_2, \overline{P}_1 \rangle + \langle \overline{Q}_3, \overline{P}_2 \rangle \\ 3\langle \overline{Q}_2, \overline{P}_3 \rangle + 2\langle \overline{Q}_1, \overline{P}_2 \rangle + \langle \overline{Q}_4, \overline{P}_1 \rangle \end{pmatrix}, \tag{3.5}$$

which lead to finite-dimensional integrable systems, where

$$\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_N), \quad \overline{Q}_i = (\overline{\phi}_{i_i}, \cdots, \overline{\phi}_{i_N})^T, \quad \overline{P}_i = (\overline{\psi}_{i_1}, \cdots, \overline{\psi}_{i_N})^T, \quad i = 1, 2, 3, 4.$$

For $k = 0$, the constraint is

$$\frac{\delta H_0}{\delta u} = \begin{pmatrix} c_0 \\ b_0 \end{pmatrix} = \begin{pmatrix} 3\langle \overline{Q}_1, \overline{P}_4 \rangle + 2\langle \overline{Q}_2, \overline{P}_1 \rangle + \langle \overline{Q}_3, \overline{P}_2 \rangle \\ 3\langle \overline{Q}_2, \overline{P}_3 \rangle + 2\langle \overline{Q}_1, \overline{P}_2 \rangle + \langle \overline{Q}_4, \overline{P}_1 \rangle \end{pmatrix} = \begin{pmatrix} \overline{r} \\ \overline{q} \end{pmatrix}. \tag{3.6}$$

Under this constraint and

$$\begin{cases} (\overline{\phi}_{1_j}, \overline{\phi}_{2_j}, \overline{\phi}_{3_j}, \overline{\phi}_{4_j})_{t_n} = \overline{V}^{(n)}(\widehat{u}, \lambda_j)(\overline{\phi}_{1_j}, \overline{\phi}_{2_j}, \overline{\phi}_{3_j}, \overline{\phi}_{4_j}), \\ (\overline{\psi}_{1_j}, \overline{\psi}_{2_j}, \overline{\psi}_{3_j}, \overline{\psi}_{4_j})_{t_n} = -(\overline{V}^{(n)})^T(\widehat{u}, \lambda_j)(\overline{\psi}_{1_j}, \overline{\psi}_{2_j}, \overline{\psi}_{3_j}, \overline{\psi}_{4_j}), \end{cases} \tag{3.7}$$

we can obtain a hierarchy of finite-dimensional Hamiltonian systems.

For $n = 1$, the spectral problems (3.7) are nonlinearized to the following Hamiltonian systems, x-flows

$$\overline{Q}_{i_x} = \frac{\partial \overline{H}_1}{\partial \overline{P}_i}, \quad \overline{P}_{i_x} = -\frac{\partial \overline{H}_1}{\partial \overline{Q}_i}, \quad i = 1, 2, 3, 4, \tag{3.8}$$

where

$$\begin{aligned} \overline{H}_1 = & -\langle \Lambda \overline{Q}_1, \overline{P}_1 \rangle + \langle \Lambda \overline{Q}_2, \overline{P}_2 \rangle + 3\langle \Lambda \overline{Q}_3, \overline{P}_3 \rangle - 3\langle \Lambda \overline{Q}_4, \overline{P}_4 \rangle \\ & + (3\langle \overline{Q}_1, \overline{P}_4 \rangle + 2\langle \overline{Q}_2, \overline{P}_1 \rangle + \langle \overline{Q}_3, \overline{P}_2 \rangle) \\ & \cdot (3\langle \overline{Q}_2, \overline{P}_3 \rangle + 2\langle \overline{Q}_1, \overline{P}_2 \rangle + \langle \overline{Q}_4, \overline{P}_1 \rangle). \end{aligned}$$

For $n = 2$, the spectral problems (3.7) are nonlinearized to the following Hamiltonian systems, t-flows

$$\overline{Q}_{it_2} = \frac{\partial \overline{H}_2}{\partial \overline{P}_i}, \quad \overline{P}_{it_2} = -\frac{\partial \overline{H}_2}{\partial \overline{Q}_i}, \quad i = 1, 2, 3, 4, \tag{3.9}$$

where

$$\begin{aligned}\overline{H}_2 =& -\langle \Lambda^2\overline{Q}_1,\overline{P}_1\rangle + \langle \Lambda^2\overline{Q}_2,\overline{P}_2\rangle + 3\langle \Lambda^2\overline{Q}_3,\overline{P}_3\rangle - 3\langle \Lambda^2\overline{Q}_4,\overline{P}_4\rangle \\ &+ (3\langle \Lambda\overline{Q}_1,\overline{P}_4\rangle + 2\langle \Lambda\overline{Q}_2,\overline{P}_1\rangle + \langle \Lambda\overline{Q}_3,\overline{P}_2\rangle)(3\langle \overline{Q}_2,\overline{P}_3\rangle + 2\langle \overline{Q}_1,\overline{P}_2\rangle + \langle \overline{Q}_4,\overline{P}_1\rangle) \\ &+ (3\langle \overline{Q}_1,\overline{P}_4\rangle + 2\langle \overline{Q}_2,\overline{P}_1\rangle + \langle \overline{Q}_3,\overline{P}_2\rangle)\cdot(3\langle \Lambda\overline{Q}_2,\overline{P}_3\rangle + 2\langle \Lambda\overline{Q}_1,\overline{P}_2\rangle + \langle \Lambda\overline{Q}_4,\overline{P}_1\rangle) \\ &+ \frac{1}{2}(3\langle \overline{Q}_1,\overline{P}_4\rangle + 2\langle \overline{Q}_2,\overline{P}_1\rangle + \langle \overline{Q}_3,\overline{P}_2\rangle)\cdot(3\langle \overline{Q}_2,\overline{P}_3\rangle + 2\langle \overline{Q}_1,\overline{P}_2\rangle + \langle \overline{Q}_4,\overline{P}_1\rangle) \\ &\cdot(\langle \overline{Q}_1,\overline{P}_1\rangle - \langle \overline{Q}_2,\overline{P}_2\rangle - 3\langle \overline{Q}_3,\overline{P}_3\rangle + 3\langle \overline{Q}_4,\overline{P}_4\rangle).\end{aligned}$$

We can prove that if $\overline{P}$, $\overline{Q}$ satisfy (3.8) and (3.9), then $\overline{q},\overline{r}$ defined by constraint (3.6) solves the AKNS equation (2.8). A new decomposition of the AKNS equation (2.8) is presented.

3.3 Lax representation and r-matrix formulation

Through a long but direct calculation (see [17]), we can get the following proposition.

Proposition 3.1 *Hamiltonian systems* (3.8) *and* (3.9) *have the following Lax representations respectively*

$$L(\lambda)_x = [\widetilde{\overline{U}}(\lambda), L(\lambda)], \quad L(\lambda)_t = [\widetilde{\overline{V}^{(2)}}(\lambda), L(\lambda)], \tag{3.10}$$

where

$$L(\lambda) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix} + \sum_{j=1}^{N} \frac{1}{\lambda-\lambda_j} \begin{pmatrix} A_j & 2B_j & 0 & C_j \\ 2C_j & -A_j & B_j & 0 \\ 0 & 3C_j & -3A_j & 0 \\ 3B_j & 0 & 0 & 3A_j \end{pmatrix},$$

$$\widetilde{\overline{U}} = \begin{pmatrix} -\lambda & 2u_{12} & 0 & u_{14} \\ 2u_{14} & \lambda & u_{12} & 0 \\ 0 & 3u_{14} & 3\lambda & 0 \\ 3u_{12} & 0 & 0 & -3\lambda \end{pmatrix},$$

$$\widetilde{\overline{V}^{(2)}} = \begin{pmatrix} -\lambda^2+\frac{1}{2}\overline{qr} & 2v_{12} & 0 & v_{14} \\ 2v_{14} & \lambda^2-\frac{1}{2}\overline{qr} & v_{12} & 0 \\ 0 & 3v_{14} & 3\lambda^2-\frac{3}{2}\overline{qr} & 0 \\ 3v_{12} & 0 & 0 & -3\lambda^2+\frac{3}{2}\overline{qr} \end{pmatrix},$$

where the $\overline{q}$ *and* $\overline{r}$ *satisfy the constraint* (3.9),

$$\begin{aligned} u_{12} &= 3\langle \overline{Q}_2,\overline{P}_3\rangle + 2\langle \overline{Q}_1,\overline{P}_2\rangle + \langle \overline{Q}_4,\overline{P}_1\rangle, \\ u_{14} &= 3\langle \overline{Q}_1,\overline{P}_4\rangle + 2\langle \overline{Q}_2,\overline{P}_1\rangle + \langle \overline{Q}_3,\overline{P}_2\rangle, \\ v_{12} &= 2\lambda\overline{q} + \frac{1}{2}\overline{q}(\langle \overline{Q}_1,\overline{P}_1\rangle - \langle \overline{Q}_2,\overline{P}_2\rangle - 3\langle \overline{Q}_3,\overline{P}_3\rangle - 3\langle \overline{Q}_4,\overline{P}_4\rangle), \\ v_{14} &= 2\lambda\overline{r} + \frac{1}{2}\overline{r}(\langle \overline{Q}_1,\overline{P}_1\rangle - \langle \overline{Q}_2,\overline{P}_2\rangle - 3\langle \overline{Q}_3,\overline{P}_3\rangle - 3\langle \overline{Q}_4,\overline{P}_4\rangle), \\ 2A_j &= \overline{\phi}_{1j}\overline{\psi}_{1j} - \overline{\phi}_{2j}\overline{\psi}_{2j} - 3\overline{\phi}_{3j}\overline{\psi}_{3j} + 3\overline{\phi}_{4j}\overline{\psi}_{4j}, \\ B_j &= 3\overline{\phi}_{2j}\overline{\psi}_{3j} + 2\overline{\phi}_{1j}\overline{\psi}_{2j} + \overline{\phi}_{4j}\overline{\psi}_{1j}, \quad C_j = 3\overline{\phi}_{1j}\overline{\psi}_{4j} + 2\overline{\phi}_{2j}\overline{\psi}_{1j} + \overline{\phi}_{3j}\overline{\psi}_{2j}. \end{aligned}$$

It is convenient to assume

$$L(\lambda)=\begin{pmatrix} A(\lambda) & 2B(\lambda) & 0 & C(\lambda)\\ 2C(\lambda) & -A(\lambda) & B(\lambda) & 0\\ 0 & 3C(\lambda) & -3A(\lambda) & 0\\ 3B(\lambda) & 0 & 0 & 3A(\lambda)\end{pmatrix}, \tag{3.11}$$

where

$$A(\lambda)=-1+\sum_{j=1}^{N}\frac{1}{\lambda-\lambda_j}A_j,\quad B(\lambda)=\sum_{j=1}^{N}\frac{1}{\lambda-\lambda_j}B_j,\quad C(\lambda)=\sum_{j=1}^{N}\frac{1}{\lambda-\lambda_j}C_j.$$

After a long but direct calculation, we arrive at

$$\{A(\lambda),A(\mu)\}=\{B(\lambda),B(\mu)\}=\{C(\lambda),C(\mu)\}=0,\quad \{A(\lambda),B(\mu)\}=\frac{1}{\mu-\lambda}(B(\mu)-B(\lambda)),$$
$$\{A(\lambda),C(\mu)\}=\frac{1}{\mu-\lambda}(C(\lambda)-C(\mu)),\quad \{B(\lambda),C(\mu)\}=\frac{2}{\mu-\lambda}(A(\mu)-A(\lambda)).$$

From the above relations, we get the following result.

Proposition 3.2 *With the standard Poisson bracket, the Lax matrix $L(\lambda)$ admits the r-matrix representation:*

$$\{L_1(\lambda)\otimes L_2(\mu)\}=[r_{12}(\lambda,\mu),L_1(\lambda)]+[r_{21}(\lambda,\mu),L_2(\mu)], \tag{3.12}$$

where

$$r_{12}(\lambda,\mu)=\frac{1}{2(\mu-\lambda)}(\sigma_1\otimes\sigma_1+\sigma_2\otimes\sigma_3+2\sigma_3\otimes\sigma_2),$$
$$r_{21}(\lambda,\mu)=\frac{1}{2(\mu-\lambda)}(\sigma_1\otimes\sigma_1+2\sigma_2\otimes\sigma_3+\sigma_3\otimes\sigma_2),$$
$$L_1(\lambda)=L(\lambda)\otimes E,\quad L_2(\lambda)=E\otimes L(\lambda),$$

where

$$\sigma_1=\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&-3&0\\0&0&0&3\end{pmatrix},\quad \sigma_2=\begin{pmatrix}0&2&0&0\\0&0&0&1\\0&0&0&0\\3&0&0&0\end{pmatrix},\quad \sigma_3=\begin{pmatrix}0&0&0&1\\2&0&0&0\\0&0&0&0\\0&3&0&0\end{pmatrix},$$

E is the 4×4 unit matrix.

We can find an interesting fact that $\overline{U}(u,\lambda)=\lambda\sigma_1+q\sigma_2+r\sigma_3$.

By the r-matrix theory (see [18]), we have

$$\mathrm{tr}\{L^m(\lambda)\otimes L^n(\mu)\}=mn\{\mathrm{tr}L(\lambda)\otimes\mathrm{tr}L(\mu)\}=0. \tag{3.13}$$

Therefore, $\mathrm{tr}L^k(\lambda)$ can generate the involutive integrals of motion. We remark that

$$\mathrm{tr}L(\lambda)=\mathrm{tr}L^3(\lambda)=0.$$

The involutive integrals of motion can be generated from $\mathrm{tr}L^2(\lambda)$ only.

Assume

$$\frac{1}{20}\mathrm{tr}L^2(\lambda) = A^2 + BC = 1 + \sum_{k=1}^{N} \frac{I_k}{\lambda - \lambda_k} = \sum_{k=1}^{\infty} \overline{F}_k \lambda^{-k}, \tag{3.14}$$

where

$$I_k = -2A_k + \sum_{\substack{j=1 \\ j \neq k}}^{N} \frac{A_j A_k + B_k C_j}{\lambda_k - \lambda_j}. \tag{3.15}$$

We can prove that $\overline{F}_i$ $(i = 1, 2, \cdots, 4N)$ are functionally independent on a certain region of $\mathbb{R}^{8N}$ (see [18, 4]). We get the following theorem.

Theorem 3.1 *The spatial constraint flows* (3.8) *and the temporal constrained flows* (3.9) *of the AKNS equation* (2.8) *are completely integrable in the sense of Liouville. The decomposition* (3.8) *and* (3.9) *of the AKNS equation* (2.8) *is an integrable decomposition.*

4 Remark

With the use of this approach, we can construct new higher spectral problems and get new integrable decompositions of the corresponding soliton equations, such as Kaup-Newell equation, mKdV equation and WKI equation.

In fact, when we assume

$$(\overline{\phi}_1, \overline{\phi}_2, \cdots, \overline{\phi}_{m-1}, \overline{\phi}_m) = (\phi_1^{m-2}\phi_2, \phi_1^{m-3}\phi_2^2, \cdots, \phi_2^{m-1}, \phi_1^{m-1}), \quad m \geq 4,$$

we can obtain an $m \times m$ matrix spectral problem as follows:

$$\overline{\Phi}_x = \overline{U}(u, \lambda)\overline{\Phi}, \quad \overline{\Phi} = (\overline{\phi}_1, \overline{\phi}_2, \overline{\phi}_3, \cdots, \overline{\phi}_{m-2}, \overline{\phi}_{m-1}\, \overline{\phi}_m)^T, \tag{4.1}$$

where

$$\overline{U}(u, \lambda) = \begin{pmatrix} -(m-3)\lambda & (m-2)q & 0 & 0 & \cdots & 0 & 0 & 0 & r \\ 2r & -(m-5)\lambda & (m-3)q & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 3r & -(m-7)\lambda & (m-4)q & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & (m-2)r & (m-3)\lambda & q & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & (m-2)r & (m-1)\lambda & 0 \\ (m-1)q & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & -(m-1)\lambda \end{pmatrix}.$$

The studies of spectral problem (4.1) is similar to the process of the spectral problem (3.1).

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