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Injectivity Radius and Cartan Polyhedron for Simply Connected Symmetric Spaces**

Ling YANG*

Abstract The author explores the relationship between the cut locus of an arbitrary simply connected and compact Riemannian symmetric space and the Cartan polyhedron of corresponding restricted root system, and computes the injectivity radius and diameter for every type of irreducible ones.

Keywords Cartan polyhedron, Restricted root system, Orthogonal involutive Lie algebra, Dynkin diagram, Stake diagram
2000 MR Subject Classification 53C35

0 Introduction

Let (M, g) be an n-dimensional Riemannian manifold. It is well known that for each $p \in M$ the exponential map is injective on a sufficiently small ball in M_p. Then there is a natural question to be taken up: how can we determine the maximum radius of such a ball (i.e., the injectivity radius of M, denoted by $i(M)$)? Meanwhile, if we assume M to be compact, then the length of an arbitrary minimal geodesic in M has a least upper bound (i.e., the diameter of M, denoted by $d(M)$). The injectivity radius and diameter have a close relationship with the curvature of M, which could be easily seen from Bonnet-Myers Theorem and Klingenberg Theorem: the former gives an upper bound of $d(M)$ when the Ricci curvature of M has a positive lower bound; the latter tells us that $i(M)$ is no less than π when M is simply connected, $n \geq 3$ and $\frac{1}{4} < K \leq 1$. Cheeger, Toponogov, Berger, Grove and Shiohama have made a contribution to this topic (see [3, Chapters 5–6] and [6]).

The purpose of this paper is to determine injectivity radius and diameter for an arbitrary simply connected and compact Riemannian symmetric space explicitly. The author hopes that the results would be beneficial to further research for geometric properties on symmetric spaces of compact type.

Our computation is based on the work of Richard Crittenden, who discussed conjugate points and cut points in symmetric spaces in [5]. In the paper, he claimed that the conjugate locus is determined by the diagram of a single Cartan subalgebra and the isotropy group, and proved that the cut locus of p coincides with the first conjugate locus of p for every $p \in M$ using algebraic method. (Cheeger proved the same conclusion using geometrical method (see [4]).) But he did not describe the first conjugate locus precisely. In Section 1, we explore

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*School of Mathematical Sciences, Fudan University, Shanghai 200433, China.
E-mail: 051018016@fudan.edu.cn
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the relationship between the first conjugate locus and the Cartan polyhedron of corresponding restricted root system after summarizing the results due to Richard Crittenden. Then we obtain the main theorem (i.e., Theorem 1.3) about the cut locus; our notations are mainly from [7] and [2]. (Theorem 1.1 is the same as Proposition 3.1 in [7, p. 294], but our method is different.)

Theorem 1.3 and the definitions of $i(M)$ and $d(M)$ tell us that the two geometrical quantities can be determined from researching into the properties of Cartan polyhedron, and they depend on the type of the restricted root system Σ, the type of the corresponding orthogonal involutive Lie algebra, and the metric on M. So we divide the process of computing $i(M)$ and $d(M)$ into three steps. Section 2 is an independent section, in which the subject we deal with is an arbitrary abstract irreducible root system Φ; we define two new variables, i.e., $i(\Phi)$ and $d(\Phi)$, which only depend on Φ, show that $i(\Phi)$ is equal to the reciprocal of the length of the highest root, and explore the relationship between $d(\Phi)$ and the length of the highest root for every type of irreducible root system. Section 3 is the complement of Section 2, in which the subject we discuss is the restricted root system Σ, and the inner product on it is induced from the Killing form on the orthogonal involutive Lie algebra associated with M; we compute the squared length of the highest root of Σ, which depends on the type of the orthogonal involutive Lie algebra but has no direct relationship with the type of Σ; the computation is on the basis of the Satake diagram given by Araki [1]. (The work is firstly done by X. S. Liu [11], but our method is different.) In Section 4, the subject we research into is an arbitrary simply connected, compact and irreducible Riemannian symmetric space; at the beginning of the section, we define a parameter $\epsilon > 0$ which only depends on the metric of M and show the relationship between ϵ and the Ricci curvature of M; then we claim $i(M) = \pi\kappa^{1/2}$, where κ denotes the maximum of the sectional curvatures of M, and list $i(M)$ and $d(M)$ for every type of them when $\epsilon = 1$, $\mathrm{Ric} = \frac{1}{2}$ in Table 4.1 and Table 4.2 on the basis of what we have done in Sections 2 and 3; at last we spend a little effort to discuss the reducible cases.

1 Conjugate Locus and Cut Locus of an Arbitrary Compact and Simply Connected Riemannian Symmetric Space

Let (M, g) be an arbitrary Riemannian locally symmetric space with non-negative sectional curvature, i.e., $\nabla R = 0$ and $K \geq 0$, where ∇ is Levi-Civita connection corresponding to g, and R is corresponding curvature tensor field on $M(R(X,Y) = -[\nabla_X, \nabla_Y] + \nabla_{[X,Y]})$. For arbitrary $X \in T_oM$, let $\gamma : (-\infty, \infty) \to M$ be a geodesic satisfying $\dot{\gamma}(0) = X$ (i.e., $\gamma(t) = \exp_o(tX)$). A vector field U along γ is called a Jacobi field if it satisfies the Jacobi equation:

$$\ddot{U} + R_{\dot{\gamma}U}\dot{\gamma} = 0. \tag{1.1}$$

Define a self-adjoint map $T_X : (T_oM, \langle\, ,\, \rangle) \to (T_oM, \langle\, ,\, \rangle)$ $Y \mapsto R_{X,Y}X$, where $\langle Y, Z\rangle = g(Y, Z)$ for every $Y, Z \in T_oM$; denote by $\lambda_1, \cdots, \lambda_m \geq 0$ the eigenvalues of T_X, by $(T_X)_i$ the eigenspace with respect to λ_i. Then

$$T_oM = \bigoplus_{i=1}^{m} (T_X)_i. \tag{1.2}$$

For arbitrary $Y_i \in (T_X)_i$, let $Y_i(t)$ be the vector field obtained by parallel translation of Y_i along

γ. Then the Jacobi field satisfying $U_i(0)=0$ and $\dot{U}_i(0)=Y_i$ is

$$U_i(t)=\begin{cases} tY_i(t), & \text{when } \lambda=0,\\ \frac{1}{\sqrt{\lambda}}\sin(\sqrt{\lambda}t)Y_i(t), & \text{when } \lambda>0 \end{cases} \tag{1.3}$$

(cf. [12, p. 195]). And moreover, the Jacobi field satisfying $U(0)=0$ and $\dot{U}(0)=Y=\sum\limits_{i=1}^{m} Y_i$, where $Y_i\in(T_X)_i$, is $U=\sum\limits_{i=1}^{m}U_i$.

X is called a conjugate point in T_oM, if and only if there exists a nonzero Jacobi field U along $\gamma(t)=\exp_o(tX)$, such that $U(0)=U(1)=0$. By (1.2) and (1.3), we immediately obtain the following proposition.

Proposition 1.1 *Let (M,g) be a Riemannian locally symmetric space with non-negative sectional curvature. Fix $o\in M$ and $X\in T_oM$. Denote $T_X(Y)=R_{X,Y}X$. Let $\lambda_1,\cdots,\lambda_m$ be the eigenvalues of T_X. Then X is a conjugate point in T_oM if and only if there exists at least one positive eigenvalue λ_i of T_X such that $\sqrt{\lambda_i}\in\pi\mathbb{Z}$.*

Denote the first conjugate locus of o in T_oM by $K(o)$. By the definition of $K(o)$, $X\in K(o)$ if and only if X is a conjugate point and tX is not conjugate point for every $t\in(0,1)$. Notice $T_{tX}=t^2T_X$. Then applying Proposition 1.1, we have

Proposition 1.2 *The assumption and the notations are similar to those in Proposition* 1.1*. Then $X\in K(o)$ if and only if $\max\limits_{1\le i\le m}\sqrt{\lambda_i}=\pi$.*

Before applying Propositions 1.1 and 1.2 to compact symmetric spaces, we recall several basic concepts about restricted root systems.

Let $\mathfrak{u}$ be a compact semisimple Lie algebra and θ an involutive automorphism of $\mathfrak{u}$. Then θ extends uniquely to a complex involutive automorphism of $\mathfrak{g}$, the complexification of $\mathfrak{u}$. We have then the direct decompositions

$$\mathfrak{u}=\mathfrak{k}_0\oplus\mathfrak{p}_*,\quad \text{where } \mathfrak{k}_0=\{X\in\mathfrak{u}:\theta(X)=X\},\ \mathfrak{p}_*=\{X\in\mathfrak{u}:\theta(X)=-X\}. \tag{1.4}$$

Let $\langle\,,\,\rangle$ be an inner product on $\mathfrak{p}_*$ invariant under $\mathrm{Ad}\,\mathfrak{k}_0$. Then $(\mathfrak{u},\theta,\langle\,,\,\rangle)$ is an orthogonal involutive Lie algebra; without loss of generality we can assume that it is reduced (cf. [2, pp. 20–21]). Let $M=U/K$ with U-invariant metric g is a compact Riemannian symmetric space which associates with $(\mathfrak{u},\theta,\langle\,,\,\rangle)$. Then there is a natural correspondence between (T_oM,g) and $(\mathfrak{p}_*,\langle\,,\,\rangle)$, where $o=eK$. In the following text we identify T_oM and $\mathfrak{p}_*$.

Let $\mathfrak{h}_{\mathfrak{p}_*}$ denote an arbitrary maximal abelian subspace of $\mathfrak{p}_*$, $\mathfrak{h}_{\mathfrak{k}_0}$ be an abelian subalgebra of $\mathfrak{k}_0$ such that $\mathfrak{h}_{\mathfrak{k}_0}\oplus\mathfrak{h}_{\mathfrak{p}_*}$ is a maximal abelian subalgebra of $\mathfrak{u}$, and $\mathfrak{h}$ denote the subalgebra of $\mathfrak{g}$ generated by $\mathfrak{h}_{\mathfrak{k}_0}\oplus\mathfrak{h}_{\mathfrak{p}_*}$. Denote $\mathfrak{p}_0=\sqrt{-1}\,\mathfrak{p}_*$, $\mathfrak{p}=\mathfrak{p}_*\otimes\mathbb{C}$, $\mathfrak{k}=\mathfrak{k}_0\otimes\mathbb{C}$, $\mathfrak{h}_{\mathfrak{p}_0}=\sqrt{-1}\,\mathfrak{h}_{\mathfrak{p}_*}$, $\mathfrak{h}_{\mathfrak{p}}=\mathfrak{h}_{\mathfrak{p}_*}\otimes\mathbb{C}$. Then the Killing form $(\,,\,)=B(\,,\,)$ is positive on $\sqrt{-1}\,\mathfrak{h}_{\mathfrak{k}_0}\oplus\mathfrak{h}_{\mathfrak{p}_0}$. Let Δ be the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Then $\sqrt{-1}\,\mathfrak{h}_{\mathfrak{k}_0}\oplus\mathfrak{h}_{\mathfrak{p}_0}$ is the real linear space generated by Δ, which is denoted by $\mathfrak{h}_{\mathbb{R}}$. Denote by Δ^+ the subset of Δ formed by the positive roots with respect to a lexicographic ordering of Δ. For every $\alpha\in\Delta$, denote $\alpha^\theta=\theta(\alpha)$, and denote by $\overline{\alpha}=\frac{1}{2}(\alpha-\alpha^\theta)$ the orthogonal projection of α into $\mathfrak{p}_0$. Denote $\Delta_0=\{\alpha\in\Delta:\overline{\alpha}=0\}$, $\Delta_{\mathfrak{p}}=\{\alpha\in\Delta:\overline{\alpha}\neq0\}$, $P_+=\Delta^+\cap\Delta_{\mathfrak{p}}$; denote by $\Sigma=\{\overline{\alpha}:\alpha\in\Delta_{\mathfrak{p}}\}$ the restricted root system.

Σ has a compatible ordering with Δ, and $\Sigma^+ = \{\overline{\alpha} : \alpha \in P_+\}$. Denote

$$\mathfrak{g}_\gamma = \{x \in \mathfrak{g} : [H, x] = (H, \gamma)x,\ H \in \mathfrak{p}\}, \qquad \gamma \in \Sigma, \tag{1.5}$$

$$\mathfrak{k}_\gamma = (\mathfrak{g}_\gamma \oplus \mathfrak{g}_{-\gamma}) \cap \mathfrak{k}, \quad \mathfrak{p}_\gamma = (\mathfrak{g}_\gamma \oplus \mathfrak{g}_{-\gamma}) \cap \mathfrak{p}, \quad \gamma \in \Sigma^+, \tag{1.6}$$

and denote by $m_\gamma = \dim_{\mathbb{C}} \mathfrak{g}_\gamma$ the multiplicity of γ. Then

$$\mathfrak{p} = \mathfrak{h}_{\mathfrak{p}} \oplus \Big(\bigoplus_{\gamma \in \Sigma^+} \mathfrak{p}_\gamma \Big), \tag{1.7}$$

$$m_\gamma = |\{\alpha \in \Delta_{\mathfrak{p}} : \overline{\alpha} = \gamma\}|, \quad \dim \mathfrak{k}_\gamma = \dim \mathfrak{p}_\gamma = m_\gamma \tag{1.8}$$

(cf. [7, pp. 283–293]).

It is well known that

$$R_{X,Y} Z = \mathrm{ad}\,[X, Y] Z, \quad X, Y, Z \in \mathfrak{p}_* \tag{1.9}$$

(cf. [10, p. 231], in which the curvature tensor is defined by $R(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$); i.e., $T_X(Y) = R_{X,Y} X = -(\mathrm{ad}\, X)^2 Y$, $T_X = -(\mathrm{ad}\, X)^2$.

For every $X \in \mathfrak{p}_*$, there exist $k \in K$ and $H \in \mathfrak{h}_{\mathfrak{p}_*}$, such that $X = \mathrm{Ad}\,(k)H$ (cf. [2, p. 31]). For arbitrary $u \in \mathfrak{p}_\gamma$, $-(\mathrm{ad}\, H)^2 u = (\mathrm{ad}\,(-\sqrt{-1}\, H))^2 u = (-\sqrt{-1}\, H, \gamma)^2 u$; $-\sqrt{-1}\, H, \gamma \in \mathfrak{h}_{\mathfrak{p}_0}$ yields $(-\sqrt{-1}\, H, \gamma)^2 \geq 0$. Then by (1.5) and (1.7), the eigenvalues of $T_H = -(\mathrm{ad}\, H)^2$ include

$$0, \quad (-\sqrt{-1}\, H, \gamma)^2, \quad \gamma \in \Sigma^+. \tag{1.10}$$

Since $X = \mathrm{Ad}\,(k)H$, we have $\mathrm{ad}\, X = \mathrm{Ad}\,(k) \circ \mathrm{ad}\, H \circ \mathrm{Ad}\,(k)^{-1}$ and moreover $T_X = \mathrm{Ad}\,(k) \circ T_H \circ \mathrm{Ad}\,(k)^{-1}$; therefore the eigenvalues of T_X coincide with the eigenvalues of T_H. Applying Proposition 1.1, we have

Theorem 1.1 *Let $M = U/K$ be a compact Riemannian symmetric space such that U is a semi-simple and compact Lie group, and the denotations of $\mathfrak{p}_*, \mathfrak{k}_0, \mathfrak{h}_{p_*}, \Sigma$ are similar to the above. Then for every $X = \mathrm{Ad}\,(k)H \in \mathfrak{p}_*$, where $k \in K, H \in \mathfrak{h}_{p_*}$, X is a conjugate point in $T_o M$ if and only if there exists at least one $\gamma \in \Sigma$, such that $(H, \gamma) \in \pi\sqrt{-1}\,(\mathbb{Z} - 0)$.*

Now we denote by C the Weyl chamber with respect the ordering of Σ, i.e., $C = \{x \in \mathfrak{h}_{\mathfrak{p}_0} : (x, \gamma) > 0 \text{ for every } \gamma \in \Sigma^+\}$, by Π the set of simple roots. Recall that the planes $(x, \gamma) \in \mathbb{Z}$ $(\gamma \in \Sigma)$ in $\mathfrak{h}_{\mathfrak{p}_0}$ constitute the diagram $D(\Sigma)$ of Σ, and the closure of a connected component of $\mathfrak{h}_{\mathfrak{p}_0} - D(\Sigma)$ will be called a Cartan polyhedron. Especially, let B be the set of maximal roots. Then the inequalities $(x, \gamma) \geq 0$ $(\gamma \in \Pi)$, $(x, \beta) \leq 1$ $(\beta \in B)$ define a Cartan polyhedron, which is denoted by $\triangle$ (see [2, p. 10]). Obviously $\triangle \subset \overline{C}$, where $\overline{C}$ denotes the closure of C in $\mathfrak{h}_{\mathfrak{p}_0}$. Since Weyl group W permutes Weyl chamber in a simply transitive manner and every element of Weyl group can be extended to $\mathrm{Ad}_{\mathfrak{u}}(\mathfrak{k}_0)$ (see [7, pp. 288–290]), for every $X \in \mathfrak{p}_*$, there exist $k \in K$ and $H \in \sqrt{-1}\ \overline{C}$ such that $X = \mathrm{Ad}\,(k)H$. By Proposition 1.2 and (1.10), $X \in K(o)$ if and only if

$$\pi = \max_{\gamma \in \Sigma^+} |(-\sqrt{-1}\, H, \gamma)| = \max_{\beta \in B} |(-\sqrt{-1}\, H, \beta)|, \quad \text{since } -\sqrt{-1}\, H \in \overline{C}, \tag{1.11}$$

i.e., $H \in \pi\sqrt{-1}\, \triangle$ and $(-\sqrt{-1}\, H, \beta) = \pi$ for some $\beta \in B$.

For convenience, we bring in new notation:

Notation 1.1 *Denote $\triangle' = \{x \in \triangle : (x, \beta) = 1$ for some $\beta \in B\}$, i.e., $\triangle'$ is the union of the facets of $\triangle$ which do not contain* 0.

Then we have

Theorem 1.2 *The assumption is the same as that in Theorem* 1.1. *Then*

$$K(o) = \mathrm{Ad}(K)(\pi\sqrt{-1}\,\triangle').$$

On cut locus, in 1962, Richard Crittenden proved the following proposition (see [5]).

Lemma 1.1 *Let M be a simply connected complete symmetric space. For every $p \in M$, the cut locus of p coincides with the first conjugate locus of p.*

For every $p \in M$, denote by $C(p)$ the cut locus of p in T_pM. Let $F : M \to M$ be an isometry. Then for any $p \in M$ and $X \in T_pM$, $d(p, \exp_p(X)) = |X|$ if and only if $d(F(p), \exp_{F(p)}((dF)_pX)) = |(dF)_pX|$, which yields $C(q) = (dF)_pC(p)$. Then by Theorem 1.2 and Lemma 1.1, we have

Theorem 1.3 *Let $M = U/K$ be a simply connected and compact Riemannian symmetric space such that U is a semi-simple and compact Lie group, and the denotations of $\mathfrak{p}_*, \mathfrak{k}_0, \mathfrak{h}_{p_*}, \Sigma, \triangle'$ are similar to the above. Then the cut locus of o in $T_oM = \mathfrak{p}_*$ is $C(o) = \mathrm{Ad}(K)(\pi\sqrt{-1}\triangle')$; and for any $p = aK \in M$, $C(p) = (dL_a)_oC(o)$, where $a \in U$ and L_a is an isometry of M satisfying $L_a(bK) = ab\,K$.*

2 Some Computation on Cartan Polyhedron

In the section, we assume $\Phi \subset V$ to be an irreducible abstract root system with an ordering, where V is an l-dimensional real vector space with inner product (,). Denote by Φ^+ and $\Pi = \{\alpha_1, \cdots, \alpha_l\}$ respectively the positive root system and the simple root system, and by ψ the highest root. Let $d_1, \cdots, d_l \in \mathbb{Z}^+$ be such that $\psi = \sum\limits_{i=1}^{l} d_i\alpha_i$ (cf. [2, pp. 9–10]). The definitions of $\triangle$ and $\triangle'$ are similar to those in Section 1 (notice that in this case $B = \{\psi\}$). Since Φ is irreducible, $\triangle$ is a simplex, whose vertices are $0, e_1, \cdots, e_l$, which satisfy

$$(e_j, \alpha_i) = \frac{1}{d_j}\delta_{ij}. \tag{2.1}$$

Define

$$i(\Phi) = \min_{x\in\triangle'}(x, x)^{1/2}, \quad d(\Phi) = \max_{x\in\triangle'}(x, x)^{1/2}. \tag{2.2}$$

And in what follows, we will compute $i(\Phi)$ and $d(\Phi)$.

For every $x \in \triangle'$, $1 = (x, \psi) \le (x, x)^{1/2}(\psi, \psi)^{1/2}$, and the equal sign holds if and only if $x = \psi/(\psi, \psi) \in \triangle'$ (since $(\psi, \alpha_i) \ge 0$ for every $\alpha_i \in \Pi$). Thus

$$i(\Phi) = (\psi, \psi)^{-1/2}. \tag{2.3}$$

$x \mapsto (x, x)^{1/2}$ is a function on $\triangle'$. Since for any $t \in (0, 1)$,

$$\begin{aligned}(tx_1 + (1-t)x_2, tx_1 + (1-t)x_2)^{1/2} &= (t^2(x_1, x_1) + (1-t)^2(x_2, x_2) + 2t(1-t)(x_1, x_2))^{1/2}\\ &\le t(x_1, x_1)^{1/2} + (1-t)(x_2, x_2)^{1/2},\end{aligned} \tag{2.4}$$

the function reaches its maximum at the vertices of $\triangle'$, including $e_1, \cdots, e_l$. Denote $\Omega_{ij} = (\alpha_i, \alpha_j)$ and $e_j = \alpha_k A_j^k$. (2.1) yields

$$\frac{1}{d_j}\delta_{ij} = (e_j, \alpha_i) = (\alpha_k A_j^k, \alpha_i) = \Omega_{ik} A_j^k,$$

so $A_j^k = \frac{1}{d_j}(\Omega^{-1})_{ki}\delta_{ij} = \frac{1}{d_j}(\Omega^{-1})_{kj}$ and

$$(e_i, e_j) = (\alpha_k A_i^k, e_j) = \frac{1}{d_j}\delta_{jk} A_i^k = \frac{1}{d_j}\delta_{jk}\frac{1}{d_i}(\Omega^{-1})_{ki} = \frac{1}{d_j d_i}(\Omega^{-1})_{ji}.$$

Especially

$$(e_j, e_j) = \frac{1}{d_j^2}(\Omega^{-1})_{jj}. \tag{2.5}$$

Thus

$$d(\Phi) = \max_{1 \le j \le l}(e_j, e_j)^{1/2} = \max_{1 \le j \le l} \frac{1}{d_j}(\Omega^{-1})_{jj}{}^{1/2}. \tag{2.6}$$

A root system Φ is said to be reduced, if and only if for every $\alpha, \beta \in \Phi$ which are proportional, we have $\alpha = \pm\beta$. It is well known that the root systems $\mathfrak{a}_l$ $(l \ge 1)$, $\mathfrak{b}_l$ $(l \ge 2)$, $\mathfrak{c}_l$ $(l \ge 3)$, $\mathfrak{d}_l$ $(l \ge 4)$, $\mathfrak{e}_6$, $\mathfrak{e}_7$, $\mathfrak{e}_8$, $\mathfrak{f}_4$, $\mathfrak{g}_2$ exhaust all irreducible reduced root systems, and every irreducible reduced root system associates with a unique Dynkin diagram. Otherwise, every irreducible nonreduced root system is isomorphic to $(\mathfrak{bc})_l$ $(l \ge 1)$. The set of indivisible roots ($\alpha \in \Phi$ is called indivisible if and only if $\frac{1}{2}\alpha \notin \Phi$) in $(\mathfrak{bc})_l$ is isomorphic to $\mathfrak{b}_l$ (cf. [7, pp. 474–475]). In what follows, we give the detail of computing $d(\Phi)$ for every type of root systems.

$\Phi = \mathfrak{a}_l$: The corresponding Dynkin diagram is


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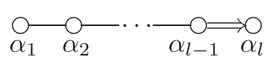
Then $\psi = \sum_{i=1}^{l} \alpha_i$, $d_i = 1$ for every i. Denote $\alpha_1 = x_1 - x_2, \cdots, \alpha_l = x_l - x_{l+1}$. Then $\psi = x_1 - x_{l+1}$ and therefore $(x_i, x_j) = \frac{1}{2}(\psi, \psi)\delta_{ij}$. By (2.1) and (2.6), we obtain

$$e_j = \frac{2}{(\psi,\psi)(l+1)}\Big((l+1-j)\sum_{k=1}^{j} x_k - j\sum_{k=j+1}^{l+1} x_k\Big), \quad 1 \le j \le l, \tag{2.7}$$

$$d(\mathfrak{a}_l) = \max_{1 \le j \le l}(e_j, e_j)^{1/2} = \begin{cases} \frac{\sqrt{2}}{2}(\psi,\psi)^{-1/2}(l+1)^{1/2}, & l \text{ is odd}, \\ \frac{\sqrt{2}}{2}(\psi,\psi)^{-1/2}(l(l+2))^{1/2}(l+1)^{-1/2}, & l \text{ is even}. \end{cases} \tag{2.8}$$

$\Phi = \mathfrak{b}_l$: The corresponding Dynkin diagram is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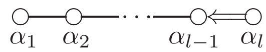
Then $\psi = \alpha_1 + 2\sum_{i=2}^{l} \alpha_i$, $d_1 = 1$ and $d_i = 2$ for every $2 \le i \le l$. Denote $\alpha_1 = x_1 - x_2$, $\cdots$, $\alpha_{l-1} = x_{l-1} - x_l$, $\alpha_l = x_l$. Then $\psi = x_1 + x_2$ and therefore $(x_i, x_j) = \frac{1}{2}(\psi, \psi)\delta_{ij}$. By (2.1) and (2.6), we obtain

$$e_1 = \frac{2}{(\psi,\psi)}x_1, \quad e_j = \frac{1}{(\psi,\psi)}\sum_{k=1}^{j} x_k, \quad 2 \le j \le l, \tag{2.9}$$

$$d(\mathfrak{b}_l) = \max_{1 \le j \le l}(e_j, e_j)^{1/2} = \begin{cases} \sqrt{2}(\psi,\psi)^{-1/2}, & l \le 3, \\ \frac{\sqrt{2}}{2}(\psi,\psi)^{-1/2} l^{1/2}, & l \ge 4. \end{cases} \tag{2.10}$$

$\Phi = \mathfrak{c}_l$: The corresponding Dynkin diagram is



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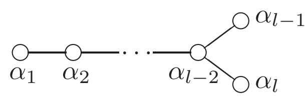
Then $\psi = 2\sum_{i=1}^{l-1}\alpha_i + \alpha_l$, $d_l = 1$ and $d_i = 2$ for every $1 \le i \le l-1$. Denote $\alpha_1 = x_1 - x_2, \cdots, \alpha_{l-1} = x_{l-1} - x_l, \alpha_l = 2x_l$. Then $\psi = 2x_1$ and therefore $(x_i, x_j) = \frac{1}{4}(\psi,\psi)\delta_{ij}$. By (2.1) and (2.6), we obtain

$$e_j = \frac{2}{(\psi,\psi)}\sum_{k=1}^{j} x_k, \quad 1 \le j \le l, \tag{2.11}$$

$$d(\mathfrak{c}_l) = \max_{1\le j\le l}(e_j, e_j)^{1/2} = (\psi,\psi)^{-1/2} l^{1/2}. \tag{2.12}$$

$\Phi = \mathfrak{d}_l$: The corresponding Dynkin diagram is



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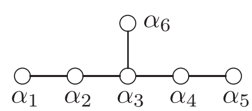
Then $\psi = \alpha_1 + 2\sum_{i=2}^{l-2}\alpha_i + \alpha_{l-1} + \alpha_l$, $d_1 = d_{l-1} = d_l = 1$ and $d_i = 2$ for every $2 \le i \le l-2$. Denote $\alpha_1 = x_1 - x_2,\ \cdots,\ \alpha_{l-1} = x_{l-1} - x_l,\ \alpha_l = x_{l-1} + x_l$. Then $\psi = x_1 + x_2$ and therefore $(x_i, x_j) = \frac{1}{2}(\psi,\psi)\delta_{ij}$. By (2.1) and (2.6), we obtain

$$\begin{aligned} &e_1 = \frac{2}{(\psi,\psi)}x_1, \quad e_j = \frac{1}{(\psi,\psi)}\sum_{k=1}^{j} x_k, \quad 2 \le j \le l-2, \\ &e_{l-1} = \frac{1}{(\psi,\psi)}\Big(\sum_{k=1}^{l-1} x_k - x_l\Big), \quad e_l = \frac{1}{(\psi,\psi)}\sum_{k=1}^{l} x_k, \end{aligned} \tag{2.13}$$

$$d(\mathfrak{d}_l) = \max_{1\le j\le l}(e_j, e_j)^{1/2} = \frac{\sqrt{2}}{2}(\psi,\psi)^{-1/2} l^{1/2}. \tag{2.14}$$

$\Phi = \mathfrak{e}_6$: The corresponding Dynkin diagram is



.

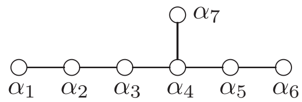
Then $\psi = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5 + 2\alpha_6$, $d_1 = d_5 = 1$, $d_2 = d_4 = d_6 = 2$ and $d_3 = 3$. Since all the roots have the same length,

$$\Omega = \frac{1}{2}(\psi,\psi)\begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & -1 \\ & & -1 & 2 & -1 & \\ & & & -1 & 2 & \\ & & -1 & & & 2 \end{pmatrix}.$$

Then by (2.6),

$$d(\mathfrak{e}_6) = \max_{1\le j\le 6}\frac{1}{d_j}{(\Omega^{-1})_{jj}}^{1/2} = \frac{2\sqrt{6}}{3}(\psi,\psi)^{-1/2}. \tag{2.15}$$

$\Phi = \mathfrak{e}_7$: The corresponding Dynkin diagram is



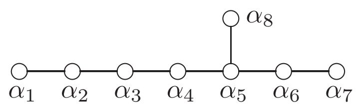
Then $\psi = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + 2\alpha_7$, $d_1 = 1$, $d_2 = d_6 = d_7 = 2$, $d_3 = d_5 = 3$ and $d_4 = 4$. Since all the roots have the same length,

$$\Omega = \frac{1}{2}(\psi,\psi)\begin{pmatrix} 2 & -1 & & & & & \\ -1 & 2 & -1 & & & & \\ & -1 & 2 & -1 & & & \\ & & -1 & 2 & -1 & & -1 \\ & & & -1 & 2 & -1 & \\ & & & & -1 & 2 & \\ & & & -1 & & & 2 \end{pmatrix}.$$

Then by (2.6),

$$d(\mathfrak{e}_7) = \max_{1\le j\le 7}\frac{1}{d_j}{(\Omega^{-1})_{jj}}^{1/2} = \sqrt{3}\,(\psi,\psi)^{-1/2}. \tag{2.16}$$

$\Phi = \mathfrak{e}_8$: The corresponding Dynkin diagram is



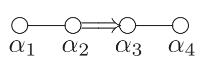
Then $\psi = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 5\alpha_4 + 6\alpha_5 + 4\alpha_6 + 2\alpha_7 + 3\alpha_8$, $d_1 = d_7 = 2$, $d_2 = d_8 = 3$, $d_3 = d_6 = 4$, $d_4 = 5$ and $d_5 = 6$. Since all the roots have the same length,

$$\Omega = \frac{1}{2}(\psi,\psi)\begin{pmatrix} 2 & -1 & & & & & & \\ -1 & 2 & -1 & & & & & \\ & -1 & 2 & -1 & & & & \\ & & -1 & 2 & -1 & & & \\ & & & -1 & 2 & -1 & & -1 \\ & & & & -1 & 2 & -1 & \\ & & & & & -1 & 2 & \\ & & & & -1 & & & 2 \end{pmatrix}.$$

Then by (2.6),

$$d(\mathfrak{e}_8) = \max_{1\le j\le 8}\frac{1}{d_j}{(\Omega^{-1})_{jj}}^{1/2} = \sqrt{2}\,(\psi,\psi)^{-1/2}. \tag{2.17}$$

$\Phi = \mathfrak{f}_4$: The corresponding Dynkin diagram is



Then $\psi = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$, $d_1 = d_4 = 2$, $d_2 = 3$ and $d_3 = 4$. Since $(\psi,\psi) = (\alpha_1,\alpha_1) = (\alpha_2,\alpha_2) = 2(\alpha_3,\alpha_3) = 2(\alpha_4,\alpha_4)$,

$$\Omega = \frac{1}{4}(\psi,\psi)\begin{pmatrix} 4 & -2 & & \\ -2 & 4 & -2 & \\ & -2 & 2 & -1 \\ & & -1 & 2 \end{pmatrix}.$$

Then by (2.6),

$$d(\mathfrak{f}_4) = \max_{1 \le j \le 4} \frac{1}{d_j} {(\Omega^{-1})_{jj}}^{1/2} = \sqrt{2}\,(\psi,\psi)^{-1/2}. \tag{2.18}$$

$\Phi = \mathfrak{g}_2$: The corresponding Dynkin diagram is

$$\underset{\alpha_1}{\circ} \Rrightarrow \underset{\alpha_2}{\circ} \; .$$

Then $\psi = 2\alpha_1 + 3\alpha_2$, $d_1 = 2$ and $d_2 = 3$. Since $(\psi,\psi) = (\alpha_1,\alpha_1) = 3(\alpha_2,\alpha_2)$,

$$\Omega = \frac{1}{6}(\psi,\psi)\begin{pmatrix} 6 & -3 \\ -3 & 2 \end{pmatrix},$$

and by (2.6),

$$d(\mathfrak{g}_2) = \max_{1 \le j \le 2} \frac{1}{d_j} {(\Omega^{-1})_{jj}}^{1/2} = \frac{2\sqrt{3}}{3}(\psi,\psi)^{-1/2}. \tag{2.19}$$

$\Phi = (\mathfrak{bc})_l$: The corresponding Dynkin diagram of the set of indivisible roots in Φ is

$$\begin{array}{ll} \underset{\alpha_1}{\circ} - \underset{\alpha_2}{\circ} - \cdots - \underset{\alpha_{l-1}}{\circ} \Rightarrow \underset{\alpha_l}{\circ} \;, & \text{when } l \ge 2, \\ \underset{\alpha_1}{\circ} \;, & \text{when } l = 1. \end{array}$$

Then $\psi = 2\sum_{i=1}^{l} \alpha_i$, $d_i = 2$ for every $1 \le i \le l$. When $l \ge 2$, denote $\alpha_1 = x_1 - x_2, \ \cdots, \ \alpha_{l-1} = x_{l-1} - x_l, \ \alpha_l = x_l$. Then $\psi = 2x_1$ and therefore $(x_i, x_j) = \frac{1}{4}(\psi,\psi)\delta_{ij}$. By (2.1) and (2.6), we obtain

$$e_j = \frac{2}{(\psi,\psi)} \sum_{k=1}^{j} x_k, \quad 2 \le j \le l, \tag{2.20}$$

$$d((\mathfrak{bc})_l) = \max_{1 \le j \le l} (e_j, e_j)^{\frac{1}{2}} = (\psi,\psi)^{-1/2} l^{1/2}. \tag{2.21}$$

When $l = 1$, $\psi = 2\alpha_1$, $e_1 = (\psi,\psi)^{-1}\psi$, so $d((\mathfrak{bc}_1)) = (e_1,e_1)^{1/2} = (\psi,\psi)^{-1/2}$. The result coincides with (2.21).

3 The Squared Length of the Highest Restricted Root

In this section, we assume $(\mathfrak{u}, \theta, \langle\, , \rangle)$ to be a reduced, irreducible, semi-simple and compact orthogonal involutive Lie algebra, $(\, ,) = B(\, ,)$ be the Killing form on $\mathfrak{u}$. The denotation of $\Delta, \Sigma, \mathfrak{g}, \mathfrak{h}, \mathfrak{h}_{\mathbb{R}}, \mathfrak{h}_{\mathfrak{p}_0}, \Delta_0, m_\gamma$ $(\gamma \in \Sigma)$ is the same as in Section 1, and denote by l and l_1 respectively the rank of Δ and Σ, by $\psi \in \Sigma$ the highest restricted root. Then $(\mathfrak{u}, \theta, \langle\, , \rangle)$ belongs to one of the two following types: (I) $\mathfrak{u}$ is compact and simple, θ is an involution; (II) $\mathfrak{u}$ is a product of two compact simple algebras exchanged by θ (see [2, p. 28]).

Type I In the case, Δ and Σ are both irreducible. Denote by δ the highest root of Δ. Since the orderings of Δ and Σ are compatible (i.e., $\alpha \ge \beta$ yields $\overline{\alpha} \ge \overline{\beta}$ for arbitrary $\alpha, \beta \in \Delta$), $\overline{\delta}$ is the highest root of Σ, i.e., $\psi = \overline{\delta}$.

Denote $\delta^\perp = \{x \in \mathfrak{h}_{\mathbb{R}} : (x, \delta) = 0\}$. Then $\Delta \cap \delta^\perp$ is obviously a subsystem of Δ with an induced ordering. Denote by $\Pi = \{\alpha_1, \cdots, \alpha_l\}$ the set of simple roots in Δ. Since $(\alpha_i, \delta) \ge 0$, we have $\alpha = \sum_{i=1}^{l} a_i\alpha_i \in \Delta \cap \delta^\perp$ if and only if $a_j = 0$ for every $\alpha_j \notin \Pi \cap \delta^\perp$; therefore $\Pi \cap \delta^\perp$ is the simple root system of $\Delta \cap \delta^\perp$.

$\alpha_i \in \Pi \cap \delta^\perp$ if and only if $\delta - \alpha_i \notin \Delta \cup \{0\}$. Then from the Dynkin diagram of Δ, which was described in Section 2, we can clarify $\Pi \cap \delta^\perp$ and $\Delta \cap \delta^\perp$:

$$
\begin{aligned}
&\Delta = \mathfrak{a}_l : \Pi \cap \delta^\perp = \{\alpha_i : i \neq 1, l\}, \quad \Delta \cap \delta^\perp = \mathfrak{a}_{l-2} \ (\emptyset \text{ when } l = 1, 2);\\
&\Delta = \mathfrak{b}_l : \Pi \cap \delta^\perp = \{\alpha_i : i \neq 2\},\\
&\qquad\quad \Delta \cap \delta^\perp = \mathfrak{a}_1 \oplus \mathfrak{b}_{l-2} \ (\mathfrak{a}_1 \text{ when } l = 2, \mathfrak{a}_1 \oplus \mathfrak{a}_1 \text{ when } l = 3);\\
&\Delta = \mathfrak{c}_l : \Pi \cap \delta^\perp = \{\alpha_i : i \neq 1\}, \quad \Delta \cap \delta^\perp = \mathfrak{c}_{l-1} \ (\mathfrak{b}_2 \text{ when } l = 3);\\
&\Delta = \mathfrak{d}_l : \Pi \cap \delta^\perp = \{\alpha_i : i \neq 2\},\\
&\qquad\quad \Delta \cap \delta^\perp = \mathfrak{a}_1 \oplus \mathfrak{d}_{l-2} \ (\mathfrak{a}_1 \oplus \mathfrak{a}_1 \oplus \mathfrak{a}_1 \text{ when } l = 4, \ \mathfrak{a}_1 \oplus \mathfrak{a}_3 \text{ when } l = 5);\\
&\Delta = \mathfrak{c}_l : \Pi \cap \delta^\perp = \{\alpha_i : i \neq 1\}, \quad \Delta \cap \delta^\perp = \mathfrak{c}_{l-1} \ (\mathfrak{b}_2 \text{ when } l = 3);\\
&\Delta = \mathfrak{d}_l : \Pi \cap \delta^\perp = \{\alpha_i : i \neq 2\},\\
&\qquad\quad \Delta \cap \delta^\perp = \mathfrak{a}_1 \oplus \mathfrak{d}_{l-2} \ (\mathfrak{a}_1 \oplus \mathfrak{a}_1 \oplus \mathfrak{a}_1 \text{ when } l = 4, \ \mathfrak{a}_1 \oplus \mathfrak{a}_3 \text{ when } l = 5);\\
&\Delta = \mathfrak{e}_6 : \Pi \cap \delta^\perp = \{\alpha_i : i \neq 6\}, \quad \Delta \cap \delta^\perp = \mathfrak{a}_5;\\
&\Delta = \mathfrak{e}_7 : \Pi \cap \delta^\perp = \{\alpha_i : i \neq 6\}, \quad \Delta \cap \delta^\perp = \mathfrak{d}_6;\\
&\Delta = \mathfrak{e}_8 : \Pi \cap \delta^\perp = \{\alpha_i : i \neq 1\}, \quad \Delta \cap \delta^\perp = \mathfrak{e}_7;\\
&\Delta = \mathfrak{f}_4 : \Pi \cap \delta^\perp = \{\alpha_i : i \neq 1\}, \quad \Delta \cap \delta^\perp = \mathfrak{c}_3;\\
&\Delta = \mathfrak{g}_2 : \Pi \cap \delta^\perp = \{\alpha_2\}, \quad \Delta \cap \delta^\perp = \mathfrak{a}_1.
\end{aligned} \tag{3.1}
$$

On $\Delta \cap \delta^\perp$, we have the following lemmas.

Lemma 3.1 $(\delta, \delta) = 4(|\Delta| - |\Delta \cap \delta^\perp| + 6)^{-1}$.

Proof For every $\alpha \in \Delta^+ - \{\delta\}$, we have $\alpha + \delta > \delta$ and $\alpha - 2\delta < -\delta$, which yields $p_{\alpha,\delta} \leq 1$ and $q_{\alpha,\delta} = 0$. (For arbitrary $\alpha, \beta \in \Delta$, the β-string through α is denoted by $\alpha - p_{\alpha,\beta}, \cdots, \alpha, \cdots, \alpha + q_{\alpha,\beta}\beta$, where $p_{\alpha,\beta}, q_{\alpha,\beta} \in \mathbb{Z}^+$ satisfy $p_{\alpha,\beta} - q_{\alpha,\beta} = 2(\alpha,\beta)/(\beta,\beta)$ (cf. [2, pp. 9–10])). Thus

$$\frac{2(\alpha,\delta)}{(\delta,\delta)} = p_{\alpha,\delta} - q_{\alpha,\delta} = 1 \text{ or } 0, \quad \frac{2(\alpha,\delta)}{(\delta,\delta)} = 1 \quad \text{if and only if} \quad (\alpha,\delta) \neq 0, \text{ i.e., } \alpha \notin \Delta \cap \delta^\perp.$$

By the definition of Killing form,

$$
\begin{aligned}
(\delta,\delta) = \operatorname{tr}(\operatorname{ad}\delta)^2|_{\mathfrak{g}} &= \sum_{\alpha\in\Delta} \dim \mathfrak{g}_\alpha (\alpha,\delta)^2 = \sum_{\alpha\in\Delta} (\alpha,\delta)^2\\
&= \sum_{\alpha\in\Delta-(\Delta\cap\delta^\perp)} (\alpha,\delta)^2 = 2(\delta,\delta)^2 + \frac{1}{4}(|\Delta| - |\Delta\cap\delta^\perp| - 2)(\delta,\delta)^2,
\end{aligned}
$$

that is $(\delta,\delta) = 4(|\Delta| - |\Delta \cap \delta^\perp| + 6)^{-1}$.

Lemma 3.2 $(\overline{\delta}, \overline{\delta}) = (\delta, \delta)$ *or* $\frac{1}{2}(\delta,\delta)$*, and the following conditions are equivalent:*

(a) $(\overline{\delta}, \overline{\delta}) = (\delta, \delta)$;

(b) $\delta^\theta = -\delta$;

(c) $\Pi_0 \subset \Pi \cap \delta^\perp$*, where* $\Pi_0 = \Pi \cap \Delta_0$;

(d) $m_{\overline{\delta}} = 1$.

Proof If $\delta^\theta = -\delta$, then $\overline{\delta} = \delta$ and $(\overline{\delta}, \overline{\delta}) = (\delta, \delta)$. Otherwise $\delta^\theta \neq -\delta$, i.e., $\delta + \delta^\theta \neq 0$. Araki proved $\alpha + \alpha^\theta \notin \Delta$ for every $\alpha \in \Delta$ in [1], especially $\delta + \delta^\theta \notin \Delta$; if $\delta - \delta^\theta = 2\overline{\delta} \in \Delta \cup \{0\}$, then $2\overline{\delta} \in \Sigma^+$, which contradicts the assumption that $\overline{\delta}$ is the highest root of Σ. So $(\delta, \delta^\theta) = 0$ and

$$(\overline{\delta}, \overline{\delta}) = \Big(\frac{\delta - \delta^\theta}{2}, \frac{\delta - \delta^\theta}{2}\Big) = \frac{1}{4}((\delta, \delta) + (\delta^\theta, \delta^\theta)) = \frac{1}{2}(\delta, \delta).$$

So $(\overline{\delta}, \overline{\delta}) = (\delta, \delta)$ or $\frac{1}{2}(\delta, \delta)$ and we have proved (a) $\Leftrightarrow$ (b).

(b)$\Rightarrow$(c) Suppose that there exists $\alpha_i \in \Pi_0$ satisfying $\alpha_i \notin \Pi \cap \delta^\perp$. Then $\overline{\alpha}_i = 0$ and $(\delta, \alpha_i) \neq 0$, which yields $\delta - \alpha_i \in \Delta^+ \cup \{0\}$ and $\theta(\alpha_i) = \alpha_i$. Therefore $\theta(\delta - \alpha_i) \in \Delta \cup \{0\}$ and on the other hand $\theta(\delta - \alpha_i) = -\delta - \alpha_i < -\delta$, which causes a contradiction.

(c)$\Rightarrow$(d) Suppose $m_{\overline{\delta}} > 1$. Then there exists $\alpha \in \Delta^+ - \{\delta\}$ such that $\overline{\alpha} = \overline{\delta}$. By the properties of root system, there exists $\beta_1, \cdots, \beta_k \in \Pi$ such that $\alpha = \delta - \sum\limits_{i=1}^{k} \beta_i$ and

$$\delta - \sum_{i=1}^{j} \beta_i \in \Delta^+ \quad \text{for every } 1 \leq j \leq k.$$

Since $\overline{\alpha} = \overline{\delta}$ and $\overline{\beta}_i \in \Sigma^+ \cup \{0\}$, we have $\beta_i \in \Pi_0$ for every $1 \leq i \leq k$, especially $\beta_1 \in \Pi_0$ and $\beta_1 \notin \Pi \cap \delta^\perp$, which contradict (c).

(d)$\Rightarrow$(b) Since

$$\overline{(-\delta^\theta)} = \frac{-\delta^\theta + \theta^2(\delta)}{2} = \frac{\delta - \delta^\theta}{2} = \overline{\delta}$$

and $m_{\overline{\delta}} = 1$, we have $\delta^\theta = -\delta$ by (1.8).

By Lemma 3.1, from (3.1) and the well-known facts that $|\mathfrak{a}_l| = l(l+1)$, $|\mathfrak{b}_l| = 2l^2$, $|\mathfrak{c}_l| = 2l^2$, $|\mathfrak{d}_l| = 2l(l-1)$, $|\mathfrak{e}_6| = 72$, $|\mathfrak{e}_7| = 126$, $|\mathfrak{e}_8| = 240$, $|\mathfrak{f}_4| = 48$, $|\mathfrak{g}_2| = 12$ (see [7, pp. 461–474]), we can obtain (δ, δ) for every type of irreducible and reduced root systems:

$$\begin{array}{lll} \Delta = \mathfrak{a}_l : (\delta, \delta) = \dfrac{1}{l+1}; & \Delta = \mathfrak{b}_l : (\delta, \delta) = \dfrac{1}{2l-1}; & \Delta = \mathfrak{c}_l : (\delta, \delta) = \dfrac{1}{l+1}; \\ \Delta = \mathfrak{d}_l : (\delta, \delta) = \dfrac{1}{2l-2}; & \Delta = \mathfrak{e}_6 : (\delta, \delta) = \dfrac{1}{12}; & \Delta = \mathfrak{e}_7 : (\delta, \delta) = \dfrac{1}{18}; \\ \Delta = \mathfrak{e}_8 : (\delta, \delta) = \dfrac{1}{30}; & \Delta = \mathfrak{f}_4 : (\delta, \delta) = \dfrac{1}{9}; & \Delta = \mathfrak{g}_2 : (\delta, \delta) = \dfrac{1}{4}. \end{array} \tag{3.2}$$

Given $\mathfrak{u}, \Pi, \theta, \Pi_0, \Sigma$, we can define the Satake diagram of (Π, θ) as follows. Every root of Π_0 is denoted by a black circle $\bullet$ and every root of $\Pi - \Pi_0$ by a white circle $\circ$. If $\overline{\alpha}_i = \overline{\alpha}_j$ for $\alpha_i, \alpha_j \in \Pi - \Pi_0$, then α_i and α_j are joined by a curved arrow. In [1], Araki gave the Satake diagram of (Π, θ) and the Dynkin diagram of Σ for all types of irreducible, simple and compact orthogonal involutive Lie algebras, i.e., A I $-$ A III, BD I, C I $-$ C II, D III, E I $-$ E IX, F I $-$ F II, G. Then by Lemma 3.2, from the Satake diagram and (3.1), we can justify whether $(\overline{\delta}, \overline{\delta}) = (\delta, \delta)$ or $(\overline{\delta}, \overline{\delta}) = \frac{1}{2}(\delta, \delta)$. For example, the Satake diagram of A II is

$$\underset{\alpha_1}{\bullet} - \underset{\alpha_2}{\circ} - \underset{\alpha_3}{\bullet} - \cdots - \underset{\alpha_{l-1}}{\circ} - \underset{\alpha_l}{\bullet}\,, \quad l \geq 3 \text{ is odd},$$

which yields $\Pi_0 = \{\alpha_i : i \text{ is odd}\}$. Thus $\alpha_1 \in \Pi_0$ but $\alpha_1 \notin \Pi \cap \delta^\perp$ and Lemma 3.2 tells us $(\overline{\delta}, \overline{\delta}) = \frac{1}{2}(\delta, \delta)$. The ultimate results are: $(\overline{\delta}, \overline{\delta}) = \frac{1}{2}(\delta, \delta)$ when $(\mathfrak{u}, \theta)$ belongs to A II, C II, E IV, F II or $(\mathfrak{u},\theta)$ belongs to BD I and $l_1=1$, otherwise $(\overline{\delta},\overline{\delta})=(\delta,\delta)$. Combining the results with (3.2), we can compute $(\overline{\delta},\overline{\delta})$, i.e., (ψ,ψ).

Type II In this case, we denote $\mathfrak{u}=\mathfrak{v}\oplus\mathfrak{v}$, where $\mathfrak{v}$ is a compact and simple Lie algebra. Then $\theta(X,Y)=(Y,X)$ for arbitrary $X,Y\in\mathfrak{v}$, $\mathfrak{k}_0=\{(X,X):X\in\mathfrak{v}\}$, $\mathfrak{p}_*=\{(X,-X):X\in\mathfrak{v}\}$. Let $\mathfrak{t}$ be a maximal abelian subalgebra of $\mathfrak{v}$, $\mathfrak{t}_0=\sqrt{-1}\,\mathfrak{t}$, $\Delta^*\subset\mathfrak{t}_0$ be the root system of $\mathfrak{v}\otimes\mathbb{C}$ with respect to $\mathfrak{t}\otimes\mathbb{C}$ with an ordering. Then $\mathfrak{h}_{\mathfrak{p}_*}=\{(X,-X):X\in\mathfrak{t}\}$ is a maximal abelian space of $\mathfrak{p}_*$ and we can assume $\mathfrak{h}_{\mathfrak{k}_0}=\{(X,X):X\in\mathfrak{t}\}$; thus $\mathfrak{h}_{\mathfrak{p}_0}=\{(x,-x):x\in\mathfrak{t}_0\}$, $\mathfrak{h}_{\mathbb{R}}=\{(x,y):x,y\in\mathfrak{t}_0\}$ and

$$\Delta=(\Delta^*,0)\cup(0,\Delta^*),\quad \Sigma=\Big\{\Big(\frac{1}{2}\alpha,-\frac{1}{2}\alpha\Big):\alpha\in\Delta^*\Big\}. \tag{3.3}$$

Δ has a lexicographic ordering induced by the ordering of Δ^*, and we can define an ordering on Σ: $(\frac{1}{2}\alpha,-\frac{1}{2}\alpha)>0$ if and only if $\alpha>0$. Obviously Δ and Σ have compatible orderings. Denote by δ the highest root of Δ^*. Then $\psi=(\frac{1}{2}\delta,-\frac{1}{2}\delta)$ and

$$(\psi,\psi)=\Big(\Big(\frac{1}{2}\delta,-\frac{1}{2}\delta\Big),\Big(\frac{1}{2}\delta,-\frac{1}{2}\delta\Big)\Big)=\frac{1}{2}(\delta,\delta), \tag{3.4}$$

i.e., the squared length of the highest restricted root is a half of the squared length of the highest root of Δ^*.

4 Computation of Injectivity Radius and Diameter

In this section, we assume $U=G/K$ to be irreducible. It is well known that there is a one-to-one correspondence between all compact, simply connected and irreducible Riemannian symmetric spaces and all semisimple, compact, reduced and irreducible orthogonal involutive Lie algebras (cf. [7, pp. 438–443]). Denote by $(\mathfrak{u},\theta,\langle\,,\,\rangle)$ the orthogonal involutive Lie algebras corresponding to $U=G/K$. Then

$$\langle\,,\,\rangle=-\epsilon(\,,\,) \tag{4.1}$$

for some positive instant ϵ (cf. [2, pp. 23–26]).

Remark 4.1 ϵ has geometrical significance: there is a close relationship between ϵ and the Ricci curvature of M.

The denotation of $\mathfrak{k}_0,\mathfrak{p}_*$ is similar to that in Section 1. Let $\{Y_1,\cdots,Y_m\}$ be a basis of $\mathfrak{k}_0$, $\{X_1,\cdots,X_n\}$ be a basis of $\mathfrak{p}_*$. (1.4) tells us $[\mathfrak{k}_0,\mathfrak{p}_*]\subset\mathfrak{p}_*$ and $[\mathfrak{p}_*,\mathfrak{p}_*]\subset\mathfrak{k}_0$. Then for arbitrary $X\in\mathfrak{p}_*$, the matrix of $\mathrm{ad}X|_{\mathfrak{u}}$ with respect to the basis $\{Y_1,\cdots,Y_m,X_1,\cdots,X_n\}$ is

$$\begin{pmatrix} & C\\ D & \end{pmatrix},\quad C\text{ is an } m\times n \text{ matrix},\ D \text{ is an } n\times m\text{ matrix}.$$

Then

$$(\mathrm{ad}X)^2=\begin{pmatrix} CD & \\ & DC\end{pmatrix}$$

and therefore

$$(X,X)=\mathrm{tr}(\mathrm{ad}X)^2|_{\mathfrak{u}}=\mathrm{tr}(CD)+\mathrm{tr}(DC)=2\ \mathrm{tr}(DC)=2\ \mathrm{tr}(\mathrm{ad}X)^2|_{\mathfrak{p}_*}. \tag{4.2}$$

By (1.9), we have

$$\begin{aligned}\mathrm{Ric}(X,X) &= R_{X,X_i}X, X_i = \langle \mathrm{ad}[X,X_i]X, X_i\rangle = \langle -(\mathrm{ad}\ X)^2X_i, X_i\rangle \\ &= -\mathrm{tr}(\mathrm{ad}\ X)^2|_{\mathfrak{p}_*} = -\frac{1}{2}(X,X) = \frac{1}{2\epsilon}\langle X,X\rangle, \qquad (4.3)\end{aligned}$$

i.e., M is an Einstein manifold with Ricci curvature $\frac{1}{2\epsilon}$.

By the definition of injectivity radius and diameter,

$$i(M) = \min_{\substack{p\in M\\ X\in C(p)}} |X| \quad \text{and} \quad d(M) = \max_{\substack{p\in M\\ X\in C(p)}} |X|.$$

By Theorem 3.1, for every $X \in C(p)$ where $p = aK$, we have

$$X = (dL_a)_o(\mathrm{Ad}(k)(\pi\sqrt{-1}\,x))$$

for some $k \in K$ and $x \in \triangle'$, which yields $|X| = \pi\epsilon^{1/2}(x,x)^{1/2}$ and

$$i(M) = \pi\epsilon^{1/2} i(\Sigma), \quad d(M) = \pi\epsilon^{1/2} d(\Sigma). \qquad (4.4)$$

($i(\Sigma)$ and $d(\Sigma)$ are defined in (2.2), and notice that the inner product (,) on Σ is induced by the Killing form of $\mathfrak{u}$.)

According to (4.4), we have the following theorem on $i(M)$.

Theorem 4.1 *Let M be a simply-connected, compact and irreducible Riemannian symmetric space, κ be the maximum of the sectional curvatures of M. Then $i(M) = \pi\kappa^{-1/2}$.*

Proof When $\langle\, ,\rangle = -(\, ,)$, i.e., $\epsilon = 1$, it is known that $\kappa = (\psi,\psi)$, where ψ denotes the highest restricted root (see [7, p. 334]). Then for general cases such that $\epsilon \neq 1$, we have $\kappa = \epsilon^{-1}(\psi,\psi)$. On the other hand, by (4.4) and (2.3), $i(M) = \pi\epsilon^{1/2}(\psi,\psi)^{-1/2}$. Thus $i(M) = \pi\kappa^{-1/2}$.

Moreover, from the results obtained in Section 2 and Section 3, we can compute $i(M)$ and $d(M)$ for every type of compact, simply connected and irreducible Riemannian symmetric spaces and list the results in Tables 4.1 and 4.2.

Table 4.1 The injectivity radius and diameter of compact, simply connected and irreducible Riemannian symmetric spaces of Type I when $\epsilon = 1$, i.e., Ric $= \frac{1}{2}$

Type	M	Σ	$(\overline{\delta},\overline{\delta})$	$i(M)$	$d(M)$
A I	$\mathrm{SU}(n)/\mathrm{SO}(n)$	$\mathfrak{a}_{n-1}$	$\frac{1}{n}$	$\pi n^{1/2}$	$\frac{\sqrt{2}}{2}\pi n$ (n is even) $\frac{\sqrt{2}}{2}\pi(n^2-1)^{1/2}$ (n is odd)
A II	$\mathrm{SU}(2n)/\mathrm{Sp}(n)$	$\mathfrak{a}_{n-1}$	$\frac{1}{4n}$	$2\pi n^{1/2}$	$\sqrt{2}\pi n$ (n is even) $\sqrt{2}\pi(n^2-1)^{1/2}$ (n is odd)
A III	$G_{p,q}(\mathbb{C})$ $(p \le q)$	$(\mathfrak{bc})_p$ $(2 \le p < q)$ $\mathfrak{c}_p$ $(p = q \ge 2)$ $(\mathfrak{bc})_1$ $(p = 1)$	$\frac{1}{p+q}$	$\pi(p+q)^{1/2}$	$\pi(p+q)^{1/2}p^{1/2}$
C I	$\mathrm{SP}(n)/\mathrm{U}(n)$	$\mathfrak{c}_n$	$\frac{1}{n+1}$	$\pi(n+1)^{1/2}$	$\pi(n+1)^{1/2}n^{1/2}$

Table 4.1 (continued)

Type	M	Σ	$(\overline{\delta}, \overline{\delta})$	$i(M)$	$d(M)$
C II	$G_{p,q}(\mathbb{H})$ $(p \leq q)$	$(\mathfrak{bc})_p$ $(2 \leq p < q)$ $\mathfrak{c}_p$ $(p = q \geq 2)$ $(\mathfrak{bc})_1$ $(p = 1)$	$\frac{1}{2(p+q+1)}$	$\sqrt{2}\pi(p+q+1)^{1/2}$	$\sqrt{2}\pi(p+q+1)^{1/2}p^{1/2}$
BD I	$G_{p,q}(\mathbb{R})$ $(p \leq q)$	$\mathfrak{b}_p$ $(2 \leq p < q)$	$\frac{1}{p+q-2}$	$\pi(p+q-2)^{1/2}$	$\sqrt{2}\pi(p+q-2)^{1/2}$ $(p \leq 3)$ $\frac{\sqrt{2}}{2}\pi(p+q-2)^{1/2}p^{1/2}$ $(p \geq 4)$
		$\mathfrak{d}_p$ $(4 \leq p = q)$	$\frac{1}{2p-2}$	$\sqrt{2}\pi(p-1)^{1/2}$	$\pi(p-1)^{1/2}p^{1/2}$
		$\mathfrak{a}_1$ $(1 = p < q)$	$\frac{1}{2q-2}$	$\sqrt{2}\pi(q-1)^{1/2}$	$\sqrt{2}\pi(q-1)^{1/2}$
D III	$\mathrm{SO}(2n)/\mathrm{U}(n)$	$\mathfrak{c}_{\frac{n}{2}}$ (n is even)	$\frac{1}{2n-2}$	$\sqrt{2}\pi(n-1)^{1/2}$	$\pi(n-1)^{1/2}n^{1/2}$
		$(\mathfrak{bc})_{\frac{n-1}{2}}$ (n is odd)	$\frac{1}{2n-2}$	$\sqrt{2}\pi(n-1)^{1/2}$	$\pi(n-1)$
E I	$(\mathfrak{e}_6, \mathrm{sp}(4))$	$\mathfrak{e}_6$	$\frac{1}{12}$	$2\sqrt{3}\pi$	$4\sqrt{2}\pi$
E II	$(\mathfrak{e}_6, \mathrm{su}(6) \oplus \mathrm{su}(2))$	$\mathfrak{f}_4$	$\frac{1}{12}$	$2\sqrt{3}\pi$	$2\sqrt{6}\pi$
E III	$(\mathfrak{e}_6, \mathrm{so}(10) \oplus \mathbb{R})$	$(\mathfrak{bc})_2$	$\frac{1}{12}$	$2\sqrt{3}\pi$	$2\sqrt{6}\pi$
E IV	$(\mathfrak{e}_6, \mathfrak{f}_4)$	$\mathfrak{a}_2$	$\frac{1}{24}$	$2\sqrt{6}\pi$	$4\sqrt{2}\pi$
E V	$(\mathfrak{e}_7, \mathrm{su}(8))$	$\mathfrak{e}_7$	$\frac{1}{18}$	$3\sqrt{2}\pi$	$3\sqrt{6}\pi$
E VI	$(\mathfrak{e}_7, \mathrm{so}(12) \oplus \mathrm{su}(2))$	$\mathfrak{f}_4$	$\frac{1}{18}$	$3\sqrt{2}\pi$	6π
E VII	$(\mathfrak{e}_7, \mathfrak{e}_6 \oplus \mathbb{R})$	$\mathfrak{c}_3$	$\frac{1}{18}$	$3\sqrt{2}\pi$	$3\sqrt{6}\pi$
E VIII	$(\mathfrak{e}_8, \mathrm{so}(16))$	$\mathfrak{e}_8$	$\frac{1}{30}$	$\sqrt{30}\pi$	$2\sqrt{15}\pi$
E IX	$(\mathfrak{e}_8, \mathfrak{e}_7 \oplus \mathrm{su}(2))$	$\mathfrak{f}_4$	$\frac{1}{30}$	$\sqrt{30}\pi$	$2\sqrt{15}\pi$
F I	$(\mathfrak{f}_4, \mathrm{sp}(3) \oplus \mathrm{su}(2))$	$\mathfrak{f}_4$	$\frac{1}{9}$	3π	$3\sqrt{2}\pi$
F II	$(\mathfrak{f}_4, \mathrm{so}(9))$	$(\mathfrak{bc})_1$	$\frac{1}{18}$	$3\sqrt{2}\pi$	$3\sqrt{2}\pi$
G	$(\mathfrak{g}_2, \mathrm{su}(2) \oplus \mathrm{su}(2))$	$\mathfrak{g}_2$	$\frac{1}{4}$	2π	$\frac{4\sqrt{3}}{3}\pi$

Table 4.2 The injectivity radius and diameter of compact, simply connected and irreducible Riemannian symmetric spaces of Type II when $\epsilon = 1$, i.e., Ric $= \frac{1}{2}$

M	Δ^*	(δ, δ)	$i(M)$	$d(M)$
SU(n)	$\mathfrak{a}_{n-1}$	$\frac{1}{n}$	$\sqrt{2}\pi n^{1/2}$	πn (n is even) $\pi(n^2-1)^{1/2}$ (n is odd)
Spin$(2n+1)$	$\mathfrak{b}_n$	$\frac{1}{2n-1}$	$\sqrt{2}\pi(2n-1)^{1/2}$	$2\pi(2n-1)^{1/2}$ $(n \leq 3)$ $\pi(2n-1)^{1/2}n^{1/2}$ $(n \geq 4)$
Sp(n)	$\mathfrak{c}_n$	$\frac{1}{n+1}$	$\sqrt{2}\pi(n+1)^{1/2}$	$\sqrt{2}\pi(n+1)^{1/2}n^{1/2}$
Spin$(2n)$	$\mathfrak{d}_n$	$\frac{1}{2n-2}$	$2\pi(n-1)^{1/2}$	$\sqrt{2}\pi(n-1)^{1/2}n^{1/2}$
E_6	$\mathfrak{e}_6$	$\frac{1}{12}$	$2\sqrt{6}\pi$	8π
E_7	$\mathfrak{e}_7$	$\frac{1}{18}$	6π	$6\sqrt{3}\pi$
E_8	$\mathfrak{e}_8$	$\frac{1}{30}$	$2\sqrt{15}\pi$	$2\sqrt{30}\pi$
F_4	$\mathfrak{f}_4$	$\frac{1}{9}$	$3\sqrt{2}\pi$	6π
G_2	$\mathfrak{g}_2$	$\frac{1}{4}$	$2\sqrt{2}\pi$	$\frac{4\sqrt{6}}{3}\pi$

Remark 4.2 In Table 4.1, Σ denotes the restricted root system, $\overline{\delta}$ denotes the highest restricted root, $(\overline{\delta}, \overline{\delta})$ denotes the squared length of $\overline{\delta}$, $i(M)$ and $d(M)$ denote the injectivity radius and the diameter of M, respectively. In Table 4.2, M is a compact, simply connected and simple Lie group with bi-invariant metric, $\mathfrak{v}$ is the Lie algebra associated to M, $\mathfrak{t}$ is a maximal abelian subalgebra of $\mathfrak{v}$, Δ^* denotes the root system of $\mathfrak{v} \otimes \mathbb{C}$ with respect to $\mathfrak{t} \otimes \mathbb{C}$, and δ denotes the highest root of Δ^* (cf. Section 3).

Remark 4.3 In Table 4.1 and Table 4.2, we assume $\epsilon = 1$, i.e., the K-invariant metric on $M = U/K$ is induced by $-(\, , \,)$ on $\mathfrak{u}$, and Ric $= \frac{1}{2}$. For general cases such that $\epsilon \neq 1$, we should multiply the corresponding results in Table 4.1 or Table 4.2 by $\epsilon^{1/2}$.

For example, let $U = \mathrm{SO}(p+q)$, $K = \mathrm{SO}(p) \times \mathrm{SO}(q)$, $M = U/K = G_{p,q}(\mathbb{R})$ $(p \leq q)$. Then $\mathfrak{u} = \mathrm{so}(p+q)$, $\mathfrak{k}_0 = \mathrm{so}(p) \oplus \mathrm{so}(q)$ and

$$\mathfrak{p}_* = \left\{ \begin{pmatrix} 0 & -X^T \\ X & 0 \end{pmatrix} \in \mathrm{so}(p+q) : X \text{ is a } q \times p \text{ matrix} \right\}. \tag{4.5}$$

Denote $\langle A, B \rangle = -\frac{1}{2}\mathrm{tr}(AB)$ for every $A, B \in \mathfrak{p}_*$. Then it is easily seen that $\langle \, , \, \rangle$ is invariant under K; the canonical metric on $G_{p,q}(\mathbb{R})$ is induced by $\langle \, , \, \rangle$ and it is U-invariant (cf. [10, pp. 271–273]). It is well known that $(A, B) = (p+q-2)\mathrm{tr}(AB)$ for every $A, B \in \mathrm{so}(p+q)$. So $\epsilon = \frac{1}{2(p+q-2)}$ and furthermore, from Table 4.1, we have

$$i(G_{p,q}(\mathbb{R})) = \begin{cases} \frac{\sqrt{2}}{2}\pi, & p \geq 2, \\ \pi, & p = 1, \end{cases} \quad d(G_{p,q}(\mathbb{R})) = \begin{cases} \pi, & p = 1 \text{ or } 2 \leq p \leq 3 \text{ and } q > p, \\ \frac{1}{2}\pi p^{1/2}, & \text{otherwise.} \end{cases} \tag{4.6}$$

Remark 4.4 If (M,g) is a compact, simply connected and reducible Riemannian symmetric space, then by de Rham decomposition theorem (see [10, pp. 210–216]), $(M,g)=(M_1,g_1)\times\cdots\times(M_r,g_r)$, where $(M_1,g_1),\cdots,(M_r,g_r)$ are all compact, simply connected and irreducible. For every $p=(p_1,\cdots,p_r)\in M$ and $X=(X_1,\cdots,X_r)\in T_pM$, where $p_i\in M_i$, $X_i\in T_{p_i}M_i$, $d(p,\exp_p(X))=|X|$ if and only if $d(p_i,\exp_{p_i}(X_i))=|X_i|$ for every $1\le i\le r$; therefore

$$i(M)=\min_{1\le i\le r} i(M_i),\quad d(M)=\Big(\sum_{i=1}^{r} d(M_i)^2\Big)^{1/2}. \tag{4.7}$$

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