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On Asymptotic Behavior for Singularities of the Powers of Mean Curvature Flow**

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Abstract Let M^n be a smooth, compact manifold without boundary, and $F_0 : M^n \to R^{n+1}$ a smooth immersion which is convex. The one-parameter families $F(\cdot, t) : M^n \times [0, T) \to R^{n+1}$ of hypersurfaces $M_t^n = F(\cdot, t)(M^n)$ satisfy an initial value problem $\frac{\mathrm{d}F}{\mathrm{d}t}(\cdot, t) = -H^k(\cdot, t)\nu(\cdot, t)$, $F(\cdot, 0) = F_0(\cdot)$, where H is the mean curvature and $\nu(\cdot, t)$ is the outer unit normal at $F(\cdot, t)$, such that $-H\nu = \overrightarrow{H}$ is the mean curvature vector, and $k > 0$ is a constant. This problem is called H^k-flow. Such flow will develop singularities after finite time. According to the blow-up rate of the square norm of the second fundamental forms, the authors analyze the structure of the rescaled limit by classifying the singularities as two types, i.e., Type I and Type II. It is proved that for Type I singularity, the limiting hypersurface satisfies an elliptic equation; for Type II singularity, the limiting hypersurface must be a translating soliton.

Keywords H^k-Curvature flow, Type I singularities, Type II singularities
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1 Introduction

Evolving a hypersurface in Euclidean space (or some Riemannian manifolds) in the direction of its inner normal with speed given by some curvature function deduces a large class of parabolic equations which have been studied extensively before. The simplest case in this class is the mean curvature flow, in which the curvature function is the mean curvature. G. Huisken [15] showed that the convex hypersurfaces moving under such equations contract to points in finite time, and that the hypersurfaces become spherical in shape in the process. This argument has been extended to many processes where convex hypersurfaces move with speeds given by homogeneous degree one, concave or convex monotone symmetric functions of the principal curvatures: B. Chow considered flows by the nth root of the Gauss curvature (see [7]) and the square root of the scalar curvature (see [8]), and B. Andrews considered a general class of such evolution equations (see [1]). Corresponding results for flows where the speed has other positive degrees of homogeneity in the curvature seem much harder to prove. K. Tso [25] and B. Chow [7] have shown that the hypersurfaces moving with speed equal to any positive power k of the Gauss curvature contract to points in finite time. In [3], B. Andrews proved that the limit of the solutions evolve purely by homothetic contraction to the final point for $k \in (\frac{1}{n+2}, \frac{1}{n}]$. For the noncompact solutions, Urbas considered the same evolution equations, and proved the existence of the solutions which evolve by homothetically expanding or translating (see [26]).

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For $k = \frac{1}{n+2}$, the evolution equation becomes affine invariant. B. Andrews proved in [5] that the convex initial data contract smoothly to a point in finite time, with ellipsoids as the natural unique limiting shape. In the case of curves in the plane, M. E. Gage and R. Hamilton [11] showed that convex curves contract to points in finite time and become round under the curve shortening flow (where the speed of motion equals the curvature), and M. Grayson extended this by showing that any embedded curve eventually becomes convex (see [13]). This was extended to include anisotropic analogues of the curve-shortening flow by M. E. Gage in the convex case (see [10]), and by J. Oaks [23], K. S. Chou and X. P. Zhu [6] for nonconvex curves. B. Andrews [4] showed the classification of limiting shapes for the initial convex data.

On the other hand, in the case of mean curvature flow, it is well-known that for closed initial hypersurfaces the solution to the mean curvature flow exists on a maximal time interval $[0, T)$, $0 < T \leq \infty$. If $T < \infty$, the curvature of the hypersurfaces becomes unbounded for $t \to T$. One would like to understand the singular behavior for $t \to T$ in detail. The structure of the rescaled limit depends on the blow-up rate of the singularity. If the quantity $\sup(T-t)|A|^2$ is uniformly bounded (Type I singularity), the rescaling yields a selfsimilar, homothetically shrinking solution of the flow which is completely classified in the case of positive mean curvature (see [16]). If the quantity $\sup(T-t)|A|^2$ is unbounded (Type II singularity), the rescaling of the singularity can be done in such a way that an "eternal solution" of the mean curvature flow results where the maximum of the curvature is attained on the hypersurface. In the convex case, such solutions were shown by R. Hamilton to move isometrically by translations (see [14]). G. Huisken and C. Sinestrari [18, 19] showed a description of all possible singularities (Types I and II) in the mean convex case. For further description of the singularities of the mean curvature flow, see [28, 29], etc.

In this paper, we consider the following problem. Let M^n be a smooth, compact manifold without boundary, and $F_0 : M^n \to R^{n+1}$ a smooth immersion which is convex. The one-parameter families $F(\,\cdot\,, t) : M^n \times [0, T) \to R^{n+1}$ of hypersurfaces $M_t^n = F(\,\cdot\,, t)(M^n)$ satisfy an initial value problem

$$\begin{aligned} \frac{\mathrm{d}F}{\mathrm{d}t}(\,\cdot\,, t) &= -H^k(\,\cdot\,, t)\nu(\,\cdot\,, t), \\ F(\,\cdot\,, 0) &= F_0(\,\cdot\,), \end{aligned} \tag{1.1}$$

where H is the mean curvature and $\nu(\,\cdot\,, t)$ is the outer unit normal at $F(\,\cdot\,, t)$, such that $-H\nu = \overrightarrow{H}$ is the mean curvature vector, and $k > 0$ is a constant. This problem has been considered in [2, 17, 24], etc. In [24], F. Schulze called this flow H^k-flow. F. Schulze proved the following theorem.

Theorem 1.1 (see [24]) *Let $F_0 : M^n \to R^{n+1}$ be a smooth immersion, where $H(F_0(M^n)) > 0$. Then there exists a unique, smooth solution to the initial value problem* (1.1) *on a maximal finite time interval* $[0, T)$. *For $k \geq 1$, we have the bound $T \geq C(k, n)^{-1}\Big(\max\limits_{p \in M} |A|(p, 0)\Big)^{-(k+1)}$. In the case that*

(i) *$F_0(M^n)$ is strictly convex for $0 < k < 1$,*

(ii) *$F_0(M^n)$ is weakly convex for $k \geq 1$,*

then the surfaces $F(M^n, t)$ are strictly convex for all $t > 0$ and contract for $t \to T$ to a point in R^{n+1}.

In this paper, we will analyze the the structure of the rescaled limit, according to the blow-up rate of the singularity under the conditions in the above theorem. We will prove that if the solution $F(\,\cdot\,,t)$ of the flow (1.1) is convex and converges to a point when $t \to T$ and $T < +\infty$, then there exists a constant $C(k,n)$ such that

$$\max_{F(\,\cdot\,,t)} |A|^2 \geq \frac{C(k,n)}{(T-t)^{\frac{2}{k+1}}}$$

(see Proposition 3.1). Modifying the classification of singularities in the mean curvature flow (see [16]), we may say that the H^k-flow is of Type I, if there is a constant C_0 such that

$$\max_{F(\,\cdot\,,t)} |A|^2 \leq \frac{C_0}{(T-t)^{\frac{2}{k+1}}}$$

for all $t \in [0,T)$. Otherwise it is said to be of Type II (see Section 3 for details). We get the following

Theorem 1.2 *Let $F_0 : M^n \to R^{n+1}$ be a smooth immersion, where $F_0(M^n)$ is strictly convex for $0 < k < 1$, and $F_0(M^n)$ is weakly convex for $k \geq 1$. For the Type* I *flow, after rescaling the H^k-flow* (1.1) *by setting*

$$\widetilde{F}(x,\tau) = (F(x,t) - F(x,T))[(k+1)(T-t)]^{-\frac{1}{k+1}},$$

where

$$\tau = -\frac{1}{(k+1)} \log\left(\frac{T-t}{T}\right) \in [0,+\infty)$$

and $[0,T)$ is the maximal finite time interval that the solution to the flow (1.1) *exists, the limiting hypersurface $\widetilde{F}_\infty$, as $\tau \to \infty$, satisfies*

$$\widetilde{H}^{\frac{k+1}{2}} + |\overrightarrow{\widetilde{F}}|^{\frac{k-1}{2}} \langle \overrightarrow{\widetilde{F}}, \overrightarrow{n} \rangle = 0, \tag{1.2}$$

where $\overrightarrow{n}$ is the inner normal vector of the rescaled surface, and $\widetilde{H}$ is the mean curvature of $\widetilde{F}_\infty$. For the Type II *flow, after rescaling, the limiting hypersurface must be a translating soliton.*

Remark 1.1 We may write (1.2) as

$$\langle \overrightarrow{\widetilde{F}}, \overrightarrow{n} \rangle + \sigma \widetilde{H} = 0,$$

where $\sigma = \frac{\widetilde{H}^{\frac{k-1}{2}}}{|\widetilde{F}|^{\frac{k-1}{2}}}$. For $k = 1$, $\sigma \equiv 1$, it was proved by G. Huisken [16] that $\widetilde{F}$ is a round sphere. For $k \neq 1$ and $k > 0$, we do not know if the limiting hypersurface is a round sphere.

2 The H^k-Flow for Convex Hypersurfaces

Let M^n be an n-dimensional smooth manifold and

$$F : M^n \to R^{n+1}$$

be a smooth hypersurface immersion in R^{n+1}. In a local coordinate system $\{x^i\}$, $1 \leq i \leq n$, the metric and the second fundamental form on $F(M^n)$ can be expressed as

$$g_{ij} = \left\langle \frac{\partial F}{\partial x^i}, \frac{\partial F}{\partial x^j} \right\rangle$$

and

$$h_{ij} = \Big\langle \nu, \frac{\partial^2 F}{\partial x^i \partial x^j} \Big\rangle,$$

where ν is the unit outer normal vector to the hypersurface. The mean curvature of the hypersurface $F(M^n)$ is denoted by $H = g^{ij} h_{ij}$, and $|A|^2$ denotes the square norm of the second fundamental forms. It is obvious that $|\nabla H|^2 \le n|\nabla A|^2$. By [15], we also have

$$\begin{aligned}
&|\nabla A|^2 \ge \frac{3}{n+2} |\nabla H|^2,\\
&|\nabla A|^2 - \frac{|\nabla H|^2}{n} \ge \frac{2(n-1)}{3n} |\nabla A|^2.
\end{aligned}$$

Now we consider the H^k-flow (1.1). We first derive the evolution equations for g_{ij}, ν, h_{ij}, h^i_j, H, $|A|^2$ and $\langle F, \nu \rangle$. By [24], we have

Lemma 2.1

(i) $\dfrac{\partial}{\partial t} g_{ij} = -2H^k h_{ij}$,

(ii) $\dfrac{\partial}{\partial t} \nu = kH^{k-1} \nabla H$,

(iii) $\dfrac{\partial}{\partial t} h_{ij} = kH^{k-1} \Delta h_{ij} + k(k-1) H^{k-2} \nabla_i H \nabla_j H - (k+1) H^k h_{jl} g^{lm} h_{mi} + kH^{k-1} |A|^2 h_{ij}$,

(iv) $\dfrac{\partial}{\partial t} h^i_j = kH^{k-1} \Delta h^i_j + k(k-1) H^{k-2} \nabla^i H \nabla_j H - (k-1) H^k h^i_l h^l_j + kH^{k-1} |A|^2 h^i_j$,

(v) $\dfrac{\partial}{\partial t} H = kH^{k-1} \Delta H + k(k-1) H^{k-2} |\nabla H|^2 + |A|^2 H^k$,

(vi) $\dfrac{\partial}{\partial t} \langle F, \nu \rangle = kH^{k-1} \Delta \langle F, \nu \rangle - (k+1) H^k + kH^{k-1} |A|^2 \langle F, \nu \rangle$,

(vii) $$\begin{aligned}\frac{\partial}{\partial t} |A|^2 = {}& kH^{k-1} \Delta |A|^2 + 2k(k-1) H^{k-2} h_{lm} \nabla_i H \nabla_j H g^{il} g^{jm}\\ &- 2kH^{k-1} |\nabla A|^2 - 2(k-1) H^k C + 2kH^{k-1} |A|^4,\end{aligned}$$

where $C = \operatorname{tr} A^3$.

The eigenvalues of the second fundamental form h_{ij} with respect to the metric g_{ij} (i.e., the eigenvalues of the matrix $h_{ij} g^{jk}$) are called the principal curvatures of the hypersurface F, and are denoted by $\kappa_1, \kappa_2, \cdots, \kappa_n$. In particular, the mean curvature is given by $H = \kappa_1 + \cdots + \kappa_n$. We call $\frac{1}{\kappa_1}, \cdots, \frac{1}{\kappa_n}$ the principal radii of the hypersurface F. We denote $\lambda_i = \frac{1}{\kappa_i}$ $(i = 1, \cdots, n)$, which are the eigenvalues of the map $W_p^{-1} = \{b^i_j\} : T_p M \to T_p M$.

By the strong maximum principle to the evolution equation of H, we know that $H > 0$ on $M^n \times [0, T)$. By [24], we have

Lemma 2.2 *Let $F_0(M^n)$ be strictly convex and $F : M^n \times [0, T) \to R^{n+1}$ be an H^k-flow, $k > 0$. Then all M_t, $t \in [0, T)$ are strictly convex and $\kappa_{\min}(t)$ is monotonically increasing.*

For convex surfaces, i.e., $h_{ij} \ge 0$, the full second fundamental form is controlled by its trace: $|A| \le H$. We have (see [24])

Lemma 2.3 *Let $F : M^n \times [0, T) \to R^{n+1}$ be an H^k-flow and $F_0(M^n)$ be weakly convex. Then $F(M^n, t)$ is weakly convex for all $t \in [0, T)$ and $T_{\max} \ge \frac{1}{k+1} (H_{\max}(0))^{-(k+1)}$.*

For $k \ge 1$ weakly hypersurfaces, we have

Lemma 2.4 *Let $F_0(M^n)$ be a weakly convex hypersurface with $H(F_0) \geq \delta > 0$, and $F : M^n \times [0,T) \to R^{n+1}$ be the corresponding H^k-flow with $k > 1$. Then M_t is strictly convex for all $t \in (0,T)$.*

Example 2.1 For the H^k-flow of a sphere $S^n(R_0)$ with radius R_0, we obtain

$$R(t) = (R_0^{k+1} - (1+k)n^k t)^{\frac{1}{k+1}},$$

which implies a maximal existence time $T = \frac{R_0^{k+1}}{n^k(1+k)}$.

If the hypersurface M_t is strictly convex, we can study the inverse $W_p^{-1} = \{b_j^i\} : T_pM \to T_pM$ of the Weingarten map W, i.e., $b_l^i h_j^l = \delta_j^i$. By [24], we have

Lemma 2.5 *Let $k > 0$ and M_t be a flow of strictly convex hypersurfaces. Then*

$$\begin{aligned}\frac{\partial}{\partial t} b_j^i &= kH^{k-1}\Delta b_j^i - 2kH^{k-1}h_m^l \nabla_p b_l^i \nabla^p b_j^m \\ &\quad -k(k-1)H^{k-2}(b_l^i \nabla^l H)(b_j^m \nabla_m H) + (k-1)H^k \delta_j^i - kH^{k-1} b_j^i |A|^2 \\ &\leq kH^{k-1}\Delta b_j^i + (k-1)H^k \delta_j^i - kH^{k-1} b_j^i |A|^2. \end{aligned} \tag{2.1}$$

Furthermore, we have

Lemma 2.6 *Let M_t be a strictly convex solution to the H^k-flow* (1.1). *If $0 < k < 1$, we have*

$$\frac{\partial}{\partial t} b_j^i \geq kH^{k-1}\Delta b_j^i - \frac{4k^2}{k+1} H^{k-1} h_m^l \nabla_p b_l^i \nabla^p b_j^m + (k-1)H^k \delta_j^i - kH^{k-1} b_j^i |A|^2.$$

Proof For $0 < k < 1$, we employ the following method in [24]. At first we write $H^k(\kappa) = (Q_n^k(\lambda))^{-1}$, where $Q_n(\lambda) = \frac{S_n(\lambda)}{S_{n-1}(\lambda)}$, and $S_l(\lambda) = \sum\limits_{1 \leq i_1 < \cdots < i_l \leq n} \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_l}$ for $\lambda = (\lambda_1, \cdots, \lambda_n)$, where $\lambda_1, \cdots, \lambda_n$ are the principle curvature radii. It is well-known that $Q_n(\lambda)$ is concave on the positive cone $\Gamma_n = \{\lambda \in R^n \mid \lambda_1 > 0, \cdots, \lambda_n > 0\}$. Now we have

$$\begin{aligned} k(k-1)H^{k-2}\delta_l^m \delta_p^q &= \frac{\partial^2 (H^k(\kappa))}{\partial h_l^m \partial h_p^q} \\ &= 2k^2 H^{k-2}\delta_l^m \delta_p^q - H^{2k}\frac{\partial^2 (Q_n^k(\lambda))}{\partial b_o^n \partial b_s^r} b^{nq} b_{op} b^{rm} b_{sl} \\ &\quad - kH^{k-1}\delta_{lp} b^{mq} - kH^{k-1}\delta^{mq} b_{lp}. \end{aligned}$$

By multiplication with $\nabla^v h_l^m \nabla_w h_p^q$ and summation,

$$\begin{aligned} k(k-1)H^{k-2}\nabla^v H \nabla_w H &= 2k^2 H^{k-2}\nabla^v H \nabla_w H - H^{2k}\frac{\partial^2 (Q_n^k(\lambda))}{\partial b_o^n \partial b_s^r}(b^{nq} b_{op}\nabla_w h_q^p)(b^{rm} b_{sl}\nabla^v h_m^l) \\ &\quad - 2kH^{k-1} b^{mq}\nabla_w h_q^p \nabla^v h_{pm}. \end{aligned}$$

Using the concavity of $Q_n^k(\lambda)$ for $0 < k \leq 1$, we have

$$-k(k+1)H^{k-2}\nabla^v H \nabla_w H \geq -2kH^{k-1} b^{mq} \nabla_p h_m^v \nabla^p h_{qw}.$$

From this, we have

$$k(k+1)H^{k-2}(b_l^i \nabla^l H)(b_j^m \nabla_m H) \leq 2kH^{k-1} h_m^l \nabla_p b_l^i \nabla^p b_j^m.$$

Substituting this inequality into b_j^i's evolution equation, we may get the desired inequality.

3 Rescaling the Singularity

In the first step, we prove a lower bound for the blow-up rate of the curvature.

Proposition 3.1 *If the solution $F(\,\cdot\,,t)$ of the flow* (1.1) *is convex and converges to a point when $t \to T$ and $T < +\infty$, then there exists a constant $C(k,n)$ such that*

$$\max_{F(\,\cdot\,,t)} |A|^2 \geq \frac{C(k,n)}{(T-t)^{\frac{2}{k+1}}}. \tag{3.1}$$

Proof We denote $H_{\max}(t) = \max\limits_{p\in F(\,\cdot\,,t)} H(p,t)$. By Lemma 2.1(v), we have

$$\frac{\mathrm{d}}{\mathrm{d}t} H_{\max}(t) \leq |A|^2(p_0,t) H_{\max}^k(t) \leq H_{\max}^{k+2}(t),$$

where $H(p_0,t) = \max\limits_{p\in F(\,\cdot\,,t)} H(p,t)$. Then

$$-\frac{1}{k+1}\mathrm{d} H_{\max}^{-(k+1)}(t) \leq \mathrm{d}t,$$

and

$$\frac{1}{k+1} H_{\max}^{-(k+1)}(t) \leq (T-t).$$

Thus we have

$$H_{\max}(t) \geq [(k+1)(T-t)]^{-\frac{1}{k+1}}$$

and

$$\max_{F(\,\cdot\,,t)} |A|^2 \geq \frac{1}{n} H_{\max}^2(t) \geq \frac{1}{n}[(k+1)(T-t)]^{-\frac{2}{k+1}}.$$

A point $P \in R^{n+1}$ is said to be a singularity of the flow (1.1), if there is $x \in M^n$ such that

(i) $F(x,t) \to P$ as $t \to T$, and

(ii) $|A(x,t)|$ becomes unbounded as t tends to T.

We say that the flow is of Type I, if there is a constant C_0 such that

$$\max_{F(\,\cdot\,,t)} |A|^2 \leq \frac{C_0}{(T-t)^{\frac{2}{k+1}}} \tag{3.2}$$

for all $t \in [0,T)$. Otherwise it is said to be of Type II.

Here we concentrate on the case of Type I. In this case we rescale the flow by setting

$$\widetilde{F}(x,\tau) = (F(x,t) - F(x,T))[(k+1)(T-t)]^{-\frac{1}{k+1}}, \tag{3.3}$$

where

$$\tau = -\frac{1}{(k+1)} \log\Big(\frac{T-t}{T}\Big) \in [0,+\infty).$$

Then

$$\frac{\partial t}{\partial \tau} = [(k+1)(T-t)]$$

and

$$\widetilde{H}(\,\cdot\,,t) = [(k+1)(T-t)]^{\frac{1}{k+1}} H(\,\cdot\,,t).$$

The evolution equation for $\widetilde{F}(x,\tau)$ is

$$\frac{\partial \widetilde{F}(x,\tau)}{\partial \tau} = \widetilde{F}(x,\tau) - \widetilde{H}^k(\,\cdot\,,\tau)\widetilde{\nu}(\,\cdot\,,\tau) \tag{3.4}$$

for $(x,\tau) \in M^n \times [0,\infty)$, where $\widetilde{H}$ and $\widetilde{\nu}$ are the mean curvature and the unit outer normal of $\widetilde{F}(\,\cdot\,,\tau)$ respectively. It is easy to see that

$$\begin{aligned} \widetilde{g}_{ij} &= [(k+1)(T-t)]^{-\frac{2}{k+1}} g_{ij}, \\ \widetilde{h}_{ij} &= [(k+1)(T-t)]^{-\frac{1}{k+1}} h_{ij}, \\ |\widetilde{A}|^2_{\widetilde{g}} &= [(k+1)(T-t)]^{\frac{2}{k+1}} |A|^2_g. \end{aligned}$$

In the case of Type I, we have

$$|\widetilde{A}|^2_{\widetilde{g}}(x,\tau) \le C_0 \tag{3.5}$$

for all $(x,\tau) \in M^n \times [0,\infty)$. From (3.1) and (3.2), we also have $|\widetilde{A}|^2_{\widetilde{g}}(x,\tau) \ge C > 0$ for all $(x,\tau) \in M^n \times [0,\infty)$.

4 Gradient Estimates

In this section, we will show that all higher derivatives of the second fundamental form $\widetilde{A}$ are bounded. We discuss H^k-flow (1.1) at first. We have

Proposition 4.1 *Let $F_0(M^n)$ be strictly convex and $F : M^n \times [0,T) \to R^{n+1}$ be a H^k-flow (1.1), $k > 0$. If the norm of the second fundamental form of the solution $F(\,\cdot\,,t)$ is uniformly bounded on $M^n \times [0,T]$, that is,*

$$|A|^2(x,t) \le C_0 \quad \text{for } (x,t) \in M^n \times [0,T],$$

then $|\nabla A|^2$ is also bounded.

Proof By Lemma 2.2, we may assume $0 < a \le \kappa_{\min} \le \kappa_{\max} \le b < +\infty$. Then we have $na^2 \le |A|^2 \le nb^2$ and $\frac{n}{b^2} \le |B|^2 \le \frac{n}{a^2}$, where $|B|^2 = b^i_j b^j_i$. Instead of $|A|^2$, we consider the quantity $|B|^2$. By Lemma 2.5, we have

$$\begin{aligned} \frac{\partial}{\partial t}|B|^2 &= 2\Big(\frac{\partial}{\partial t} b^i_j\Big) b^j_i \\ &= 2(kH^{k-1}\Delta b^i_j - k(k-1)H^{k-2}(b^i_l \nabla^l H)(b^m_j \nabla_m H) \\ &\quad - 2kH^{k-1} h^l_m \nabla_p b^i_l \nabla^p b^m_j + (k-1)H^k \delta^i_j - kH^{k-1} b^i_j |A|^2) b^j_i \\ &= kH^{k-1}\Delta|B|^2 - 2kH^{k-1}|\nabla B|^2 - 2k(k-1)H^{k-2} b^j_i (b^i_l \nabla^l H)(b^m_j \nabla_m H) \\ &\quad - 4kH^{k-1} h^l_m b^j_i \nabla_p b^i_l \nabla^p b^m_j + 2(k-1)H^k(\operatorname{tr} B) - 2kH^{k-1}|A|^2|B|^2. \end{aligned}$$

If $k \ge 1$, we have

$$\frac{\partial}{\partial t}|B|^2 \le kH^{k-1}\Delta|B|^2 - 2kH^{k-1}|\nabla B|^2 + 2(k-1)H^k(\operatorname{tr} B) - 2kH^{k-1}|A|^2|B|^2.$$

If $0 < k < 1$, by the following inequality

$$-k(k+1)H^{k-2}\nabla^v H \nabla_w H \ge -2kH^{k-1} b^{mq} \nabla_p h^v_m \nabla^p h_{qw},$$

we have

$$k(k+1)H^{k-2}(b^i_v\nabla^v H)(b^w_j\nabla_w H)\le 2kH^{k-1}h^l_m\nabla_p b^i_l\nabla^p b^m_j.$$

(See [24] for details.) Then

$$\begin{aligned}-k(k-1)H^{k-2}b^j_i(b^i_v\nabla^v H)(b^w_j\nabla_w H)&=\frac{1-k}{1+k}k(k+1)H^{k-2}b^j_i(b^i_v\nabla^v H)(b^w_j\nabla_w H)\\&\le 2k\frac{1-k}{1+k}H^{k-1}h^l_m b^j_i\nabla_p b^i_l\nabla^p b^m_j\\&=2k\Big(1-\frac{2k}{1+k}\Big)H^{k-1}h^l_m b^j_i\nabla_p b^i_l\nabla^p b^m_j.\end{aligned}$$

We then have

$$\frac{\partial}{\partial t}|B|^2\le kH^{k-1}\Delta|B|^2-2kH^{k-1}|\nabla B|^2+2(k-1)H^k(\operatorname{tr}B)-2kH^{k-1}|A|^2|B|^2.$$

The covariant derivative ∇ involves the Christoffel symbols Γ^i_{jk}, and their time derivative is

$$\begin{aligned}\frac{\partial}{\partial t}\Gamma^i_{jm}&=\frac{1}{2}g^{il}\{\nabla_j g'_{ml}+\nabla_m g'_{jl}-\nabla_l g'_{jm}\}\\&=-g^{il}\{\nabla_j(H^k h_{ml})+\nabla_m(H^k h_{jl})-\nabla_l(H^k h_{jm})\}\\&=-kH^{k-1}\{\nabla_j H h^i_m+\nabla_m H h^i_j-\nabla^i H h_{jm}\}-H^k\nabla_j h^i_m.\end{aligned}$$

Here we use the normal coordinate at a fixed point. Now we consider

$$\begin{aligned}\frac{\partial}{\partial t}(\nabla_l b^i_j)&=\frac{\partial}{\partial t}(\partial_l b^i_j+b^p_j\Gamma^i_{lp}-b^i_p\Gamma^p_{lj})\\&=\partial_l\frac{\partial}{\partial t}(b^i_j)+\frac{\partial}{\partial t}(b^p_j)\Gamma^i_{lp}+b^p_j\frac{\partial}{\partial t}(\Gamma^i_{lp})-\frac{\partial}{\partial t}(b^i_p)\Gamma^p_{lj}-b^i_p\frac{\partial}{\partial t}(\Gamma^p_{lj})\\&=\nabla_l\Big[\frac{\partial}{\partial t}(b^i_j)\Big]+b^p_j\frac{\partial}{\partial t}(\Gamma^i_{lp})-b^i_p\frac{\partial}{\partial t}(\Gamma^p_{lj})\\&=\nabla_l[kH^{k-1}\Delta b^i_j-2kH^{k-1}h^q_m\nabla_p b^i_q\nabla^p b^m_j+(k-1)H^k\delta^i_j\\&\quad-k(k-1)H^{k-2}(b^i_q\nabla^q H)(b^m_j\nabla_m H)-kH^{k-1}b^i_j|A|^2]\\&\quad-kH^{k-1}\{\nabla_l H\delta^i_j+\nabla_p H h^i_l b^p_j-\nabla^i H\delta_{lj}\}-H^k b^p_j\nabla_l h^i_p\\&\quad+kH^{k-1}\{\nabla_l H\delta^i_j+\nabla_j H\delta^i_l-\nabla^p H h_{lj}b^i_p\}+H^k b^i_p\nabla_l h^p_j\\&=kH^{k-1}\{\nabla_l\Delta b^i_j-2\nabla_l h^q_m\nabla_p b^i_q\nabla^p b^m_j+(k-1)H^{-1}\nabla_l H\Delta b^i_j\\&\quad-2(k-1)H^{-1}\nabla_l H h^q_m\nabla_p b^i_q\nabla^p b^m_j-4h^q_m\nabla_l\nabla_p b^i_q\nabla^p b^m_j\\&\quad-2(k-1)H^{-1}(\nabla_l b^i_q\nabla^q H+b^i_q\nabla_l\nabla^q H)(b^m_j\nabla_m H)\\&\quad-(k-1)(k-2)H^{-2}\nabla_l H(b^i_q\nabla^q H)(b^m_j\nabla_m H)\}+\cdots,\end{aligned}$$

where $\cdots$ denotes the terms composed by at most two of $\nabla_l b^i_j$. Then

$$\begin{aligned}\frac{\partial}{\partial t}|\nabla B|^2&=\frac{\partial}{\partial t}(g_{ip}g^{jq}g^{lm}\nabla_l b^i_j\nabla_m b^p_q)\\&=-2H^k h_{ip}g^{jq}g^{lm}\nabla_l b^i_j\nabla_m b^p_q+2H^k g_{ip}h^{jq}g^{lm}\nabla_l b^i_j\nabla_m b^p_q\\&\quad+2H^k g_{ip}g^{jq}h^{lm}\nabla_l b^i_j\nabla_m b^p_q+2g_{ip}g^{jq}g^{lm}\Big(\frac{\partial}{\partial t}(\nabla_l b^i_j)\Big)\nabla_m b^p_q\\&=2kH^{k-1}g_{ip}g^{jq}g^{lm}\nabla_m b^p_q\{\nabla_l\Delta b^i_j-2\nabla_l h^s_u\nabla_r b^i_s\nabla^r b^u_j+(k-1)H^{-1}\nabla_l H\Delta b^i_j\\&\quad-2(k-1)H^{-1}\nabla_l H h^u_r\nabla_v b^i_u\nabla^v b^r_j-2(k-1)H^{-1}(\nabla_l b^i_u\nabla^u H+b^i_u\nabla_l\nabla^u H)(b^r_j\nabla_r H)\end{aligned}$$

$$
\begin{aligned}
&- (k-1)(k-2)H^{-2}\nabla_l H(b^i_u\nabla^u H)(b^v_j\nabla_v H) - 4h^u_r\nabla_l\nabla_v b^i_u\nabla^v b^r_j + \cdots\}\\
&+ 2H^k g_{ip}g^{jq}h^{lm}\nabla_l b^i_j\nabla_m b^p_q.
\end{aligned}
$$

Since

$$
\Delta|\nabla B|^2 = \nabla_r(2g_{ip}g^{jq}g^{lm}\nabla_m b^p_q\nabla_r\nabla_l b^i_j) = 2g_{ip}g^{jq}g^{lm}\nabla_m b^p_q\Delta\nabla_l b^i_j + 2|\nabla^2 B|^2,
$$

we have

$$
\begin{aligned}
\frac{\partial}{\partial t}|\nabla B|^2 = {}& kH^{k-1}\{\Delta|\nabla B|^2 - 2|\nabla^2 B|^2\} + 2kH^{k-1}g_{ip}g^{jq}g^{lm}\nabla_m b^p_q\{2h^s_v h^w_u\nabla_l b^v_w\nabla_r b^i_s\nabla^r b^u_j\\
&+ (k-1)H^{-1}\nabla_l H\Delta b^i_j - 2(k-1)H^{-1}\nabla_l H h^u_r\nabla_v b^i_u\nabla^v b^r_j\\
&- (k-1)(k-2)H^{-2}\nabla_l H(b^i_u\nabla^u H)(b^v_j\nabla_v H)\\
&- 2(k-1)H^{-1}(\nabla_l b^i_u\nabla^u H + b^i_u\nabla_l\nabla^u H)(b^r_j\nabla_r H) - 4h^u_r\nabla_l\nabla_v b^i_u\nabla^v b^r_j + \cdots\}\\
&+ 2H^k g_{ip}g^{jq}h^{lm}\nabla_l b^i_j\nabla_m b^p_q\\
\le{}& kH^{k-1}\{\Delta|\nabla B|^2 - 2|\nabla^2 B|^2\} + 4kH^{k-1}|A|^2|\nabla B|^4 + 2k|k-1|H^{k-1}|A||\nabla^2 B||\nabla B|^2\\
&+ 4k|k-1|H^{k-1}|A|^2|\nabla B|^4 + 8kH^{k-1}|A||\nabla^2 B||\nabla B|^2 + 4k|k-1|H^{k-1}|A|^2|\nabla B|^4\\
&+ k|k-1||k-2|H^{k-1}|A|^2|\nabla B|^4 + 4k|k-1|H^{k-1}|A||\nabla^2 B||\nabla B|^2 + O(|\nabla B|^2)\\
\le{}& kH^{k-1}\{\Delta|\nabla B|^2 - 2|\nabla^2 B|^2 + C_1|A||\nabla^2 B||\nabla B|^2 + C_2|A|^2|\nabla B|^4 + C_3|A|^2|\nabla B|^2\}\\
\le{}& kH^{k-1}\{\Delta|\nabla B|^2 + \overline{C}_2|A|^2|\nabla B|^4 + C_3|A|^2|\nabla B|^2\},
\end{aligned}
$$

where C_1, C_2, $\overline{C}_2$ and C_3 are all positive constants depending only on n and k. Consider $G(x,t) = (1 + \frac{1}{2}|\nabla B|^2)\mathrm{e}^{\phi(|B|^2)}$, where $\phi = \phi(s)$ is a smooth function defined on $[\frac{n}{b^2}, \frac{n}{a^2}]$. Then there exists a point $(x_0, t_0) \in M \times [0,T]$ such that $\max\limits_{M\times[0,T]} G = G(x_0, t_0)$. Now at (x_0, t_0),

$$
0 = \nabla_i G = \langle\nabla_i\nabla B, \nabla B\rangle\mathrm{e}^{\phi(|B|^2)} + 2\Big(1 + \frac{1}{2}|\nabla B|^2\Big)\mathrm{e}^{\phi(|B|^2)}\phi' b^m_n b^n_{m},_i\,.
$$

From this, we have

$$
\langle\nabla_i\nabla B, \nabla B\rangle = b^m_n,_{pi}\, b^n_{m,p} = -2\Big(1 + \frac{1}{2}|\nabla B|^2\Big)\phi' b^m_n b^n_{m},_i\,.
$$

The second derivative of G at (x_0, t_0) gives

$$
\begin{aligned}
0 \ge{}& \nabla_j\nabla_i G\\
={}& (b^m_n,_{pi}\, b^n_{m,pj} + b^m_n,_{pij}\, b^n_{m,p})\mathrm{e}^{\phi(|B|^2)}\\
&+ 2(b^m_n,_{pi}\, b^n_{m,p})\mathrm{e}^{\phi(|B|^2)}\phi' b^q_r b^r_q,_j + 2(b^m_n,_{pj}\, b^n_{m,p})\mathrm{e}^{\phi(|B|^2)}\phi' b^q_r b^r_q,_i\\
&+ 2\Big(1 + \frac{1}{2}|\nabla B|^2\Big)\mathrm{e}^{\phi}(2b^m_n b^n_{m,j}(\phi')^2 b^q_r b^r_q,_i + 2\phi'' b^m_n b^n_{m,i} b^q_r b^r_q,_j + \phi' b^m_{n,j} b^n_{m,i} + \phi' b^m_n b^n_{m,ij})\\
={}& (b^m_n,_{pi}\, b^n_{m,pj} + b^m_n,_{pij}\, b^n_{m,p})\mathrm{e}^{\phi(|B|^2)} + 4\Big(1 + \frac{1}{2}|\nabla B|^2\Big)\mathrm{e}^{\phi} b^m_n b^n_m,_i\, b^q_r b^r_q,_j\,(\phi'' - (\phi')^2)\\
&+ 2\Big(1 + \frac{1}{2}|\nabla B|^2\Big)\mathrm{e}^{\phi}\phi'(b^m_{n,j} b^n_{m,i} + b^m_n b^n_{m,ij})\\
={}& \frac{1}{2}(\nabla_j\nabla_i|\nabla B|^2)\mathrm{e}^{\phi(|B|^2)} + \Big(1 + \frac{1}{2}|\nabla B|^2\Big)\mathrm{e}^{\phi}\phi'\nabla_j\nabla_i|B|^2\\
&+ 4\Big(1 + \frac{1}{2}|\nabla B|^2\Big)\mathrm{e}^{\phi} b^m_n b^n_m,_i\, b^q_r b^r_q,_j\,(\phi'' - (\phi')^2).
\end{aligned}
\tag{4.1}
$$

At (x_0,t_0), we also have

$$
\begin{aligned}
0\le\frac{\partial}{\partial t}G&=\frac12\Big(\frac{\partial}{\partial t}|\nabla B|^2\Big)\mathrm{e}^{\phi(|B|^2)}+\Big(1+\frac12|\nabla B|^2\Big)\mathrm{e}^{\phi}\phi'\frac{\partial}{\partial t}|B|^2\\
&\le\frac12kH^{k-1}\{\Delta|\nabla B|^2+\overline{C}_2|A|^2|\nabla B|^4+C_3|A|^2|\nabla B|^2\}\mathrm{e}^{\phi}\\
&\quad+\Big(1+\frac12|\nabla B|^2\Big)\mathrm{e}^{\phi}\phi' kH^{k-1}\Big(\Delta|B|^2-2|\nabla B|^2+2\frac{k-1}{k}H(\mathrm{tr}B)-2|A|^2|B|^2\Big).
\end{aligned}
$$

Then by (4.1), we have

$$
\begin{aligned}
0&\ge kH^{k-1}g^{ij}\nabla_j\nabla_iG\\
&\ge-\frac12kH^{k-1}\{\overline{C}_2|A|^2|\nabla B|^4+C_3|A|^2|\nabla B|^2\}\mathrm{e}^{\phi}+2\Big(1+\frac12|\nabla B|^2\Big)\mathrm{e}^{\phi}\phi' kH^{k-1}|\nabla B|^2\\
&\quad-\Big(1+\frac12|\nabla B|^2\Big)\mathrm{e}^{\phi}\phi' kH^{k-1}\Big(2\frac{k-1}{k}H(\mathrm{tr}B)-2|A|^2|B|^2\Big)\\
&\quad+4\Big(1+\frac12|\nabla B|^2\Big)\mathrm{e}^{\phi}kH^{k-1}|\nabla|B|^2|^2(\phi''-(\phi')^2).
\end{aligned}
$$

That is

$$
\begin{aligned}
0\ge&-\frac{\overline{C}_2}{2}|A|^2|\nabla B|^4+2\Big(1+\frac12|\nabla B|^2\Big)\phi'|\nabla B|^2+4\Big(1+\frac12|\nabla B|^2\Big)|\nabla|B|^2|^2(\phi''-(\phi')^2)\\
&-\frac12C_3|A|^2|\nabla B|^2-\Big(1+\frac12|\nabla B|^2\Big)\phi'\Big(2\frac{k-1}{k}H(\mathrm{tr}B)-2|A|^2|B|^2\Big).
\end{aligned}\tag{4.2}
$$

Now let $L=\overline{C}_2nb^2$ and $\theta=\arctan L$. It is clear that $0<\theta<\frac{\pi}{2}$. If $\frac{n}{a^2}-\frac{n}{b^2}<\frac{\pi}{2}-\theta$, we may choose $\theta_0=$ const. such that $\frac{n}{b^2}+\theta_0=\theta$ and let $\phi(s)=\log\sec(s+\theta_0)$ for $s\in[\frac{n}{b^2},\frac{n}{a^2}]$. Then

$$\phi''(s)-(\phi')^2(s)=\sec^2(s+\theta_0)-\tan^2(s+\theta_0)=1>0$$

and

$$\phi'(s)=\tan(s+\theta_0)\ge\tan\theta=L=\overline{C}_2nb^2.$$

Then by (4.2), we have

$$C|\nabla B|^4\le C|\nabla B|^2+C$$

for some positive constant C. Therefore we get $|\nabla B|^2\le C$ for some constant C depending only on n, k and the bounds of $|B|^2$. If $\frac{n}{a^2}-\frac{n}{b^2}\ge\frac{\pi}{2}-\theta$, we may make dilation for the hypersurfaces $F(\,\cdot\,,t)$ so that

$$\widehat{F}(\,\cdot\,,t)=\tau F(\,\cdot\,,t)\tag{4.3}$$

for some positive constant τ depending on n, k and the bounds of $|B|^2$ such that

$$|\widehat{B}|^2_{\widehat{g}}\in\Big[\tau^2\frac{n}{b^2},\tau^2\frac{n}{a^2}\Big]$$

and

$$\tau^2\frac{n}{a^2}-\tau^2\frac{n}{b^2}<\frac{\pi}{2}-\theta.$$

After making this dilation, the inequality (4.2) becomes

$$
\begin{aligned}
0\ge&-\frac{\overline{C}_2}{2}\tau^2|\widehat{A}|^2_{\widehat{g}}|\widehat{\nabla}\widehat{B}|^4_{\widehat{g}}+2\Big(1+\frac12|\widehat{\nabla}\widehat{B}|^2_{\widehat{g}}\Big)\phi'|\widehat{\nabla}\widehat{B}|^2_{\widehat{g}}\\
&+4\tau^{-2}\Big(1+\frac12|\widehat{\nabla}\widehat{B}|^2_{\widehat{g}}\Big)|\widehat{\nabla}|\widehat{B}|^2_{\widehat{g}}|^2_{\widehat{g}}(\phi''-(\phi')^2)-\frac12C_3\tau^2|\widehat{A}|^2_{\widehat{g}}|\widehat{\nabla}\widehat{B}|^2_{\widehat{g}}\\
&-\Big(1+\frac12|\widehat{\nabla}\widehat{B}|^2_{\widehat{g}}\Big)\phi'\Big(2\frac{k-1}{k}\widehat{H}(\mathrm{tr}\widehat{B})-2|\widehat{A}|^2_{\widehat{g}}|\widehat{B}|^2_{\widehat{g}}\Big).
\end{aligned}\tag{4.4}
$$

Since $\overline{C}_2\tau^2|\widehat{A}|^2_{\widehat{g}} = \overline{C}_2|A|^2 \le L$ and $|\widehat{\nabla}\widehat{B}|^2_{\widehat{g}} = |\nabla B|^2$, according to the previous discussion, we still have $|\nabla B|^2 \le C$ by choosing $\phi(s) = \log\sec(s+\theta_0)$ for $s \in [\tau^2\frac{n}{b^2}, \tau^2\frac{n}{a^2}]$. Since $\nabla_l h^i_j = -h^i_p(\nabla_l b^p_q)h^q_j$, we have $|\nabla A|^2 \le |A|^4|\nabla B|^2 \le C$ for some positive constant C depending on n, k and a, b.

Next we consider the rescaling flow (3.4). In the case of Type I, we have

Proposition 4.2 *For each $m \ge 0$, there exists a constant $C(m)$ depending only on m, n, C_0, k and the initial hypersurface such that*

$$|\widetilde{\nabla}^m \widetilde{A}|^2 \le C(m)$$

on $M^n \times [1, +\infty)$.

Proof At first, by the same discussion as Proposition 4.1, we have $|\widetilde{\nabla}\widetilde{A}|^2 \le C$, where C depends on n, C_0, k and the initial hypersurface F_0. Next we consider the high order derivatives of $\widetilde{A}$. If S and T are two tensors, we write $S * T$ for any linear combination of tensors formed by contraction on $S_{i\cdots j}T_{k\cdots l}$ using g^{ik}. Then

$$\begin{aligned}
\frac{\partial}{\partial t}\nabla A &= \nabla\Big(\frac{\partial}{\partial t}A\Big) + H^{k-1}\nabla A * A * A + H^k\nabla A * A \\
&= \nabla(kH^{k-1}\Delta A + H^{k-2}\nabla A * \nabla A + H^{k-1}A * A * A) + H^{k-1}\nabla A * A * A + H^k\nabla A * A \\
&= kH^{k-1}\Delta\nabla A + kH^{k-1}\nabla A * A^2 + k(k-1)H^{k-2}\nabla A * \nabla^2 A \\
&\quad + k(k-1)(k-2)H^{k-3}\nabla A * \nabla A * \nabla A + k(k+1)H^{k-1}\nabla A * A^2 \\
&\quad + (k+1)H^k\nabla A * A + k(k-1)H^{k-2}\nabla A * A^3 + H^{k-1}\nabla A * A * A + H^k\nabla A * A \\
&= kH^{k-1}\Big(\Delta\nabla A + H^{-2}\sum_{i+j=2}\nabla^i A * \nabla^j A * \nabla A + \nabla A * A^2\Big)
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial}{\partial t}\nabla^2 A &= \frac{\partial}{\partial t}\nabla(\nabla A) = \nabla\Big(\frac{\partial}{\partial t}\nabla A\Big) + \Big(\frac{\partial}{\partial t}\Gamma\Big) * \nabla A \\
&= \nabla\Big(kH^{k-1}\Big(\Delta\nabla A + H^{-2}\sum_{i+j=2}\nabla^i A * \nabla^j A * \nabla A + \nabla A * A^2\Big)\Big) + H^{k-1}\nabla A * A * \nabla A \\
&= kH^{k-1}\Big[\Delta\nabla^2 A + \nabla^2 A * A^2 + H^{-1}\nabla A * \nabla^3 A + H^{-3}(\nabla A)^2 * \sum_{i+j=2}\nabla^i A * \nabla^j A \\
&\quad + \sum_{i+j=2}H^{-2}\nabla^i A * \nabla^j A * \nabla^2 A + \sum_{i+j=3}H^{-2}\nabla^i A * \nabla^j A * \nabla A + H^{-1}(\nabla A)^2 * A^2 \\
&\quad + \nabla^2 A * A^2 + \nabla A * \nabla A * A\Big] \\
&= kH^{k-1}\Big(\Delta\nabla^2 A + H^{-3}\sum_{\substack{i+j+l+m=4\\ i,j,l,m<4}}\nabla^i A * \nabla^j A * \nabla^l A * \nabla^m A \\
&\quad + \nabla A * \nabla A * A + \nabla^2 A * A^2\Big).
\end{aligned}$$

In general, we may get

$$\frac{\partial}{\partial t}\nabla^m A = kH^{k-1}\Delta\nabla^m A + H^{k-m-2}\sum_{\substack{i_1+\cdots+i_{m+2}=m+2\\ i_1,\cdots,i_{m+2}<m+2}}\nabla^{i_1}A * \cdots * \nabla^{i_{m+1}}A * \nabla^{i_{m+2}}A$$

$$+ H^{k-1}\sum_{j_1+j_2+j_3=m}\nabla^{j_1}A*\nabla^{j_2}A*\nabla^{j_3}A + H^{k-1}\nabla^m A*A^2.$$

Then

$$\begin{aligned}\frac{\partial}{\partial t}|\nabla^m A|^2 = kH^{k-1}\Big(&\Delta|\nabla^m A|^2 - 2|\nabla^{m+1}A|^2\\ &+ H^{-m-1}\sum_{\substack{i_1+\cdots+i_{m+2}=m+2\\ i_1,\cdots,i_{m+2}<m+2}}\nabla^m A*\nabla^{i_1}A*\cdots*\nabla^{i_{m+1}}A*\nabla^{i_{m+2}}A\\ &+\sum_{j_1+j_2+j_3=m}\nabla^m A*\nabla^{j_1}A*\nabla^{j_2}A*\nabla^{j_3}A\\ &+\nabla^m A*\nabla^m A*A^2 + H\nabla^m A*\nabla^m A*A\Big).\end{aligned}$$

Thus for the rescale hypersurfaces $\widetilde{F}(\,\cdot\,,\tau)$,

$$\begin{aligned}\frac{\partial}{\partial\tau}|\widetilde{\nabla}^m\widetilde{A}|^2 &= \frac{\partial}{\partial t}|\widetilde{\nabla}^m\widetilde{A}|^2\frac{\partial t}{\partial\tau}\\ &\le k\widetilde{H}^{k-1}\{\widetilde{\Delta}|\widetilde{\nabla}^m\widetilde{A}|^2 - 2|\widetilde{\nabla}^{m+1}\widetilde{A}|^2 + \widetilde{H}^{-1}|\widetilde{\nabla}^{m+1}\widetilde{A}*\widetilde{\nabla}^m\widetilde{A}*\widetilde{\nabla}\widetilde{A}|\\ &\quad +C(m,n,k)(1+|\widetilde{\nabla}^m\widetilde{A}|^2)\}\\ &\le k\widetilde{H}^{k-1}[\widetilde{\Delta}|\widetilde{\nabla}^m\widetilde{A}|^2 + E(1+|\widetilde{\nabla}^m\widetilde{A}|^2)] \end{aligned}\tag{4.5}$$

for some positive constants $C(m,n,k)$ and E. In the last two inequalities, we have employed the Cauchy inequality, the convexity fact of $\widetilde{F}(\,\cdot\,,\tau)$ and the inductive assumption that $|\widetilde{\nabla}^i\widetilde{A}|^2 \le C(i)$ for $i \le m-1$. From this, we have

$$\begin{aligned}&\frac{\partial}{\partial\tau}\Big(\frac{\tau}{\tau+1}|\widetilde{\nabla}^m\widetilde{A}|^2 + E|\widetilde{\nabla}^{m-1}\widetilde{A}|^2\Big)\\ &\le k\widetilde{H}^{k-1}\Big\{\frac{\tau}{\tau+1}[\widetilde{\Delta}|\widetilde{\nabla}^m\widetilde{A}|^2 + E(1+|\widetilde{\nabla}^m\widetilde{A}|^2)] + \frac{1}{(\tau+1)^2}|\widetilde{\nabla}^m\widetilde{A}|^2\\ &\quad + E\Big[\Big(\widetilde{\Delta}|\widetilde{\nabla}^{m-1}\widetilde{A}|^2 - \frac{3}{2}|\widetilde{\nabla}^m\widetilde{A}|^2\Big) + E_1\Big]\Big\}\\ &\le k\widetilde{H}^{k-1}\widetilde{\Delta}\Big[\frac{\tau}{\tau+1}|\widetilde{\nabla}^m\widetilde{A}|^2 + E|\widetilde{\nabla}^{m-1}\widetilde{A}|^2\Big] - \frac{E_3}{2}(E-1)|\widetilde{\nabla}^m\widetilde{A}|^2 + E_2\end{aligned}$$

for some positive constants E_1, E_2 and E_3. Thus we have the desired estimates from the maximum principle and an induction argument. (In fact, E_3 is determined by the lower bound of $k\widetilde{H}^{k-1}$, E_1 is from the first inequality of (4.5), and $E_2 = kEE_1(nC_0)^{\frac{k-1}{2}}$, where C_0 is from (3.5). We denote $f(\tau) = \max\limits_{\widetilde{F}(\,\cdot\,,\tau)}[\frac{\tau}{\tau+1}|\widetilde{\nabla}^m\widetilde{A}|^2 + E|\widetilde{\nabla}^{m-1}\widetilde{A}|^2]$. Then $\frac{\mathrm{d}f}{\mathrm{d}\tau} \le k\widetilde{H}^{k-1}\widetilde{\Delta}f - \frac{E_3}{2}(E-1)|\widetilde{\nabla}^m\widetilde{A}|^2 + E_2 \le -\frac{E_3}{2}(E-1)|\widetilde{\nabla}^m\widetilde{A}|^2 + E_2$. If $-\frac{E_3}{2}(E-1)|\widetilde{\nabla}^m\widetilde{A}|^2 + E_2 \le 0$, then the function f is monotonically decreasing. By an induction argument, we get the result. If $-\frac{E_3}{2}(E-1)|\widetilde{\nabla}^m\widetilde{A}|^2 + E_2 \ge 0$, we also get the desired inequality.)

Corollary 4.1 *For each sequence $\tau_j \to +\infty$, there is a subsequence τ_{j_k} such that $\widetilde{F}(\,\cdot\,,\tau_{j_k})$ converges smoothly to an immersed nonempty limiting hypersurface $\widetilde{F}_\infty$.*

Proof By the above proposition, we only need to show that the limit is nonempty. Since

$$\begin{aligned}|F(x,t)-F(x,T-0)| &\le \int_t^T |H^k(x,\tau)|\mathrm{d}\tau \le \int_t^T |[(k+1)(T-\tau)]^{-\frac{k}{k+1}}\widetilde{H}^k(\,\cdot\,,\tau)|\mathrm{d}\tau\\ &\le C\int_t^T [(k+1)(T-\tau)]^{-\frac{k}{k+1}}\mathrm{d}\tau \le C[(k+1)(T-t)]^{\frac{1}{k+1}},\end{aligned}$$

by (3.2), we have

$$|\widetilde{F}(x,\tau)| \le C.$$

Thus $\widetilde{F}(x,\tau)$ remains bounded as $\tau \to +\infty$.

5 The Monotonicity Formula of the H^k-Flow

To understand the structure of $\widetilde{F}_\infty$, we need the following monotonicity formula. It was obtained by G. Huisken [16] in the case of mean curvature flow.

Theorem 5.1 *If $\widetilde{F}(x,\tau)$ satisfies the rescaled evolution equation* (3.3), *then we have*

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\int_{\widetilde{F}(x,\tau)} \widetilde{\rho}\,\mathrm{d}\widetilde{\mu}_\tau \le -\int_{\widetilde{F}(x,\tau)} \widetilde{\rho}|\widetilde{F}|^{k-1}|\langle \widetilde{F}, \overrightarrow{n}\rangle + \sigma \widetilde{H}|^2 \mathrm{d}\widetilde{\mu}_\tau,$$

where $\widetilde{\rho}(\widetilde{F}) = \exp(-\frac{1}{k+1}|\widetilde{F}|^{k+1})$, $\mathrm{d}\widetilde{\mu}_\tau$ is the volume element of $\widetilde{F}(x,\tau)$, and $\sigma = \frac{\widetilde{H}^{\frac{k-1}{2}}}{|\widetilde{F}|^{\frac{k-1}{2}}}$. Here $\overrightarrow{n}$ is the inner normal vector of the rescaled surface, and $\widetilde{H} > 0$.

Proof First of all, we get the evolution equation of the metric

$$\begin{aligned}
\frac{\partial}{\partial\tau}\widetilde{g}_{ij} &= \frac{\partial}{\partial t}\Big(\frac{g_{ij}}{\varphi^2}\Big)\frac{\partial t}{\partial \tau}\\
&= \Big(\frac{-2H^k h_{ij}}{\varphi^2} + \frac{2\varphi^{1-k} g_{ij}}{\varphi^4}\Big)(k+1)(T-\tau)\\
&= -2H^k h_{ij}\varphi^{k-1} + \frac{2g_{ij}}{\varphi^2}\\
&= -2\widetilde{H}^k \widetilde{h}_{ij} + 2\widetilde{g}_{ij}.
\end{aligned}$$

Here we denote $\varphi = ((k+1)(T-t))^{\frac{1}{k+1}}$. From this, we have

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\mathrm{d}\widetilde{\mu}_\tau = (-\widetilde{H}^{k+1} + n)\mathrm{d}\widetilde{\mu}_\tau$$

and

$$\frac{\partial}{\partial\tau}\widetilde{\rho} = -\widetilde{\rho}\frac{1}{k+1}\frac{k+1}{2}\langle \widetilde{F}, \widetilde{F}\rangle^{\frac{k-1}{2}} \cdot 2\Big\langle \frac{\partial \widetilde{F}}{\partial\tau}, \widetilde{F}\Big\rangle = -\widetilde{\rho}|\widetilde{F}|^{k+1} - \widetilde{\rho}\widetilde{H}^k|\widetilde{F}|^{k-1}\langle \widetilde{F}, \overrightarrow{n}\rangle.$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\int_{\widetilde{F}(x,\tau)} \widetilde{\rho}\,\mathrm{d}\widetilde{\mu}_\tau = \int_{\widetilde{F}(x,\tau)} -\widetilde{\rho}[|\widetilde{F}|^{k+1} + \widetilde{H}^{k-1}|\widetilde{F}|^{k-1}\langle \widetilde{F}, \overrightarrow{\widetilde{H}}\rangle + \widetilde{H}^{k+1} - n]\mathrm{d}\widetilde{\mu}_\tau.$$

For a position vector $\widetilde{F} = (x_1, \cdots, x_{n+1})$ and any tangent vector $\overrightarrow{v} = \sum\limits_{\alpha=1}^{n+1} v^\alpha e_\alpha$ of $\widetilde{F}(\cdot,\tau)$ at this position,

$$D_{\overrightarrow{v}}\widetilde{F} = \sum_{\alpha,\beta=1}^{n+1} v^\alpha D_{e_\alpha} x^\beta e_\beta = \sum_{\alpha,\beta=1}^{n+1} v^\alpha \delta_\alpha^\beta e_\beta = \overrightarrow{v}.$$

Now by the first variational formula,

$$-\int_{\widetilde{F}(x,\tau)} \operatorname{div}\overrightarrow{Y}\,\mathrm{d}\widetilde{\mu}_\tau = \int_{\widetilde{F}(x,\tau)} \langle \overrightarrow{\widetilde{H}}, \overrightarrow{Y}\rangle \mathrm{d}\widetilde{\mu}_\tau.$$

Let $\overrightarrow{Y} = \widetilde{\rho}\widetilde{F}$. Then for any orthonormal basis $\overrightarrow{v}_1, \cdots, \overrightarrow{v}_n$ for the tangent space of $\widetilde{F}(\cdot, \tau)$,

$$\begin{aligned}\int_{\widetilde{F}(x,\tau)} \langle \widetilde{\rho}\widetilde{F}, \overrightarrow{\widetilde{H}} \rangle \mathrm{d}\widetilde{\mu}_\tau &= -\int_{\widetilde{F}} \mathrm{div}\,(\widetilde{\rho}\widetilde{F})\mathrm{d}\widetilde{\mu}_\tau \\ &= -\int_{\widetilde{F}} \sum v_i D_{v_i}(\widetilde{\rho}\widetilde{F})\mathrm{d}\widetilde{\mu}_\tau \\ &= \int_{\widetilde{F}} \widetilde{\rho}\Big[|\widetilde{F}|^{k-1}\sum_i \langle v_i, \widetilde{F}\rangle^2 - n\Big]\mathrm{d}\widetilde{\mu}_\tau.\end{aligned}$$

Therefore

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\tau}\int_{\widetilde{F}(x,\tau)} \widetilde{\rho}\mathrm{d}\widetilde{\mu}_\tau &= \int_{\widetilde{F}} -\widetilde{\rho}\Big[|\widetilde{F}|^{k+1} + \widetilde{H}^{k-1}|\widetilde{F}|^{k-1}\langle \widetilde{F}, \overrightarrow{\widetilde{H}}\rangle + \langle \widetilde{F}, \overrightarrow{\widetilde{H}}\rangle \\ &\quad + \widetilde{H}^{k+1} - |\widetilde{F}|^{k-1}\sum_i \langle v_i, \widetilde{F}\rangle^2\Big]\mathrm{d}\widetilde{\mu}_\tau \\ &\le \int_{\widetilde{F}} -\widetilde{\rho}\Big[|\widetilde{F}|^{k+1} + 2\widetilde{H}^{\frac{k-1}{2}}|\widetilde{F}|^{\frac{k-1}{2}}\langle \widetilde{F}, \overrightarrow{\widetilde{H}}\rangle + \widetilde{H}^{k+1} - |\widetilde{F}|^{k-1}\sum_i \langle v_i, \widetilde{F}\rangle^2\Big]\mathrm{d}\widetilde{\mu}_\tau \\ &= \int_{\widetilde{F}} -\widetilde{\rho}|\widetilde{F}|^{k-1}\Big[|\widetilde{F}|^2 + 2\sigma\langle \widetilde{F}, \overrightarrow{\widetilde{H}}\rangle + \sigma^2\widetilde{H}^2 - \sum_i \langle v_i, \widetilde{F}\rangle^2\Big]\mathrm{d}\widetilde{\mu}_\tau \\ &= \int_{\widetilde{F}} -\widetilde{\rho}|\widetilde{F}|^{k-1}\Big[|\widetilde{F} + \sigma\overrightarrow{\widetilde{H}}|^2 - \sum_i \langle v_i, \widetilde{F}\rangle^2\Big]\mathrm{d}\widetilde{\mu}_\tau \\ &= \int_{\widetilde{F}} -\widetilde{\rho}|\widetilde{F}|^{k-1}|\langle \widetilde{F}, \overrightarrow{n}\rangle + \sigma\widetilde{H}|^2\mathrm{d}\widetilde{\mu}_\tau.\end{aligned}$$

Thus from the previous Corollary 4.1, we know that every limit hypersurface $\widetilde{F}_\infty$ satisfies the equation

$$\langle \overrightarrow{\widetilde{F}}, \overrightarrow{n}\rangle + \sigma\widetilde{H} = 0,$$

i.e.,

$$\widetilde{H}^{\frac{k+1}{2}} + |\overrightarrow{\widetilde{F}}|^{\frac{k-1}{2}}\langle \overrightarrow{\widetilde{F}}, \overrightarrow{n}\rangle = 0. \tag{5.1}$$

Therefore, we have

Theorem 5.2 *Each limiting hypersurface* $\widetilde{F}_\infty$ *as obtained in Corollary* 4.1 *satisfies equation* (5.1).

6 Type II Singularities

In this section, we discuss the Type II singularities. We will prove the following

Theorem 6.1 *Let* $F(\cdot, t)$, $t \in [0, T)$, *be a maximal solution of the* H^k*-flow, and* $k > 0$. *Assume that the initial hypersurface* $F_0 : M^n \to R^{n+1}$ $(n \ge 2)$ *is compact and convex* (*as in Theorem* 1.1), *and the flow will develop Type* II *singularities. Then after rescaling, the limit of the solution must be translating soliton.*

By Theorem 1.1, $M_t = F_t(M)$ is always convex for $t \in (0, T)$. It shows that H^2 and $|A|^2$ have the same blow-up rate. We choose a sequence $\{(x_i, t_i)\}$ as follows. For each integer $i \ge 1$, let $t_i \in [0, T - \frac{1}{i}]$, $x_i \in M^n$, be such that

$$H^2(x_i, t_i)\Big(T - \frac{1}{i} - t_i\Big)^{\frac{2}{k+1}} = \max_{\substack{t \le T - \frac{1}{i} \\ x \in M^n}} H^2(x, t)\Big(T - \frac{1}{i} - t\Big)^{\frac{2}{k+1}}. \tag{6.1}$$

Dilate the solution $F(\cdot,t)$, $t\in[0,T)$ into

$$F_i(\cdot,\tau)=\frac{F(\cdot,t_i+\varepsilon_i^{k+1}\tau)-F(x_i,t_i)}{\varepsilon_i}\quad \text{for } \tau\in\Big[-\frac{t_i}{\varepsilon_i^{k+1}},\frac{T-t_i-\frac{1}{i}}{\varepsilon_i^{k+1}}\Big],$$

where $\varepsilon_i=(H(x_i,t_i))^{-1}$.

Since we assume that the singularity is Type II, the right-hand side of (6.1) tends to $+\infty$ as $i\to+\infty$. This shows that

$$\frac{T-t_i-\frac{1}{i}}{\varepsilon_i^{k+1}}\to+\infty,\quad \text{as } i\to+\infty.$$

Then for any fixed $\tau\in\Big[-\frac{t_i}{\varepsilon_i^{k+1}},\frac{T-t_i-\frac{1}{i}}{\varepsilon_i^{k+1}}\Big)$, the mean curvature H_i of the rescaled hypersurface F_i satisfies

$$H_i^2(\cdot,\tau)\le\Big(\frac{T-\frac{1}{i}-t_i}{T-\frac{1}{i}-t}\Big)^{\frac{2}{k+1}}=\Big(\frac{T-\frac{1}{i}-t_i}{T-\frac{1}{i}-t_i-\varepsilon_i^{k+1}\tau}\Big)^{\frac{2}{k+1}}\to 1,\quad \text{as } i\to+\infty.$$

It follows that for any $\omega>0$ and $\varepsilon>0$, there exists k_0 such that

$$\max_{M^n}H_i(\cdot,\tau)\le 1+\varepsilon \tag{6.2}$$

for any $i\ge k_0$, $\tau\in[-\omega,\omega]$.

We have already shown that the curvature bound in (6.2) implies analogous bounds on the second fundamental form and all its covariant derivatives (see Section 4). Then by standard method, based on the Arzela-Ascoli theorem, we can extract a subsequence of $F_i(\cdot,\tau)$ converging uniformly on compact subsets of $R^{n+1}\times R^1$ to a limiting solution $F_\infty(\cdot,\tau)$ of the mean curvature flow. Theorem 1.1 shows that the limit must be convex. So we get the following result.

Proposition 6.1 *Let $F(\cdot,t)$, $t\in[0,T)$ be a maximal solution of the H^k-flow. Assume that the initial hypersurface $F_0:M^n\to R^{n+1}$ $(n\ge 2)$ is compact and convex* (*as in Theorem* 1.1), *and that the flow develops a singularity of Type* II *as $t\to T$. Then a sequence of the rescaled flow $F_i(\cdot,\tau)$ converges smoothly on every compact set to an H^k-flow $F_\infty(\cdot,\tau)$, defined for all $\tau\in(-\infty,+\infty)$. Moreover, the mean curvature H_∞ of the limit flow satisfies $0<H_\infty\le 1$ everywhere and is equal to* 1 *at least at one point.*

Next we need to classify all such solutions. In [2], B. Andrews obtained Li-Yau-Hamilton type inequalities for a class of curvature flows. His result holds for compact hypersurfaces. Recently, J. Wang [27] proved a similar inequality which holds not only for compact hypersurfaces but also for complete case. As its application, he obtained the following

Proposition 6.2 (See [27]) *Any strictly convex solution $F(\cdot,t)$ to the H^k-flow, for $k>0$ if $F(\cdot,t)$ is compact, for $k>1$ if $F(\cdot,t)$ is complete, where $t\in(-\infty,+\infty)$, and the mean curvature assumes its maximum value at a point in space-time, must be a strictly convex translating soliton.*

Now Theorem 6.1 follows from Propositions 6.1 and 6.2, and Theorem 1.2 follows from Theorems 5.2 and 6.1.

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