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Maximal B-Regular Integro-Differential Equation

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Abstract By using Fourier multiplier theorems, the maximal B-regularity of ordinary integro-differential operator equations is investigated. It is shown that the corresponding differential operator is positive and satisfies coercive estimate. Moreover, these results are used to establish maximal regularity for infinite systems of integro-differential equations.

Keywords Banach-valued Besov spaces, Operator-valued multipliers, Boundary value problems, Integro-differential equations
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1 Introduction, Notations and Background

In recent years, the maximal regularity of differential operator equations has been studied extensively, e.g. in [1–4, 7–9]. Moreover, integro-differential equations (IDEs) have been studied, e.g. in [6, 10–12] and the reference therein. However, the integro-differential operator equation (IDOE) is a relatively less investigated subject. The main aim of present paper is to establish the maximal regularity of convolution differential operator equation

$$Lu = \sum_{k=0}^{l} a_k * \frac{\mathrm{d}^k u}{\mathrm{d}x^k} + A * u = f(x)$$

in E-valued Besov spaces, where E is an arbitrary Banach space, $A = A(x)$ is a possible unbounded operator in E, $a_k = a_k(x)$ are complex-valued functions. Particularly, we prove that the differential operator generated by this equation is a generator of analytic semigroup.

Let $x = (x_1, x_2, \cdots, x_n) \in \Omega \subset R^n$. $L_p(\Omega; E)$ denotes the space of all strongly measurable E-valued functions that are defined on the measurable subset $\Omega \subset R^n$ with the norm

$$\|f\|_{L_p(\Omega;E)} = \Big(\int \|f(x)\|_E^p \mathrm{d}x \Big)^{\frac{1}{p}}, \quad 1 \le p \le \infty,$$
$$\|f\|_{L_\infty(\Omega;E)} = \operatorname*{ess\,sup}_{x\in\Omega} [\|f(x)\|_E].$$

Let $S = S(R^n; E)$ denote a Schwartz class, i.e., a space of E-valued rapidly decreasing smooth functions on R^n and $S'(R^n; E)$ denotes the space of E-valued tempered distributions. Let $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$, where α_i are integers. An E-valued generalized function $D^\alpha f$ is called

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a generalized derivative in the sense of Schwartz distributions of the function $f \in S'(R^n, E)$, if the equality

$$\langle D^\alpha f, \varphi \rangle = (-1)^{|\alpha|} \langle f, D^\alpha \varphi \rangle$$

holds for all $\varphi \in S$.

It is known that

$$F(D_x^\alpha f) = (\mathrm{i}\xi_1)^{\alpha_1} \cdots (\mathrm{i}\xi_n)^{\alpha_n} \widehat{f},$$
$$D_\xi^\alpha (F(f)) = F[(-\mathrm{i}x_n)^{\alpha_1} \cdots (-\mathrm{i}x_n)^{\alpha_n} f]$$

for all $f \in S'(R^n; E)$.

Let E_1 and E_2 be two Banach spaces. A function $\Psi \in L_\infty(R^n; B(E_1, E_2))$ is called a multiplier from $B^s_{p,\theta}(R^n; E_1)$ to $B^s_{q,\theta}(R^n; E_2)$ for $p \in (1, \infty)$ and $q \in [1, \infty]$ if the map $u \to Ku = F^{-1}\Psi(\xi)Fu$, $u \in S(R^n; E_1)$ is well defined and extends to a bounded linear operator

$$K : B^s_{p,\theta}(R^n; E_1) \to B^s_{q,\theta}(R^n; E_2).$$

Definition 1.1 *Let X be a Banach space and $1 \le p \le 2$. Let X be such that*

$$\|Ff\|_{L_{p'}(R^n, X)} \le C\|f\|_{L_p(R^n, X)} \quad \text{for each } f \in S(R^n, X),$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. Then the space X is said to be the Fourier type p.

Let $\mathbf{C}$ be the set of complex numbers and

$$S_\varphi = \{\lambda : \lambda \in \mathbf{C}, |\arg \lambda| \le \varphi\} \cup \{0\}, \quad 0 \le \varphi \le \pi.$$

A linear operator $A = A(x)$ is said to be uniformly positive in a Banach space E, if $D(A(x))$ is dense in E and does not depend on x,

$$\|(A(x) + \lambda I)^{-1}\|_{B(E)} \le M(1 + |\lambda|)^{-1} \quad \text{for every } x$$

with $M > 0$, $\lambda \in S_\varphi$, $\varphi \in [0, \pi)$, where I is the identity operator in E and $B(E)$ is the space of all bounded linear operators in E. Sometimes instead of $A + \lambda I$ we will write $A + \lambda$ and denote it by A_λ.

Let $E(A)$ denote the space $D(A)$ with graphical norm

$$\|u\|_{E(A)} = (\|u\|^p + \|Au\|^p)^{\frac{1}{p}}, \quad 1 \le p < \infty.$$

Definition 1.2 *Let $A = A(t) \in S'(R; B(D(A), E))$. Then the Fourier transformation of $A(t)$ in the sense of Schwartz distributions is defined as follows:*

$$\langle \widehat{A}u, \varphi \rangle = \langle Au, \widehat{\varphi} \rangle, \quad u \in D(A) \text{ and } \varphi \in S(R).$$

For details see [2, p. 7].

Definition 1.3 *Let $A = A(t)$ be a uniformly positive operator in E. Then, it is differentiable if for all $u \in E(A)$,*

$$\Big(\frac{\mathrm{d}}{\mathrm{d}t} A\Big) u = A'(t)u = \lim_{h \to 0} \frac{\|A(t+h)u - A(t)u\|_E}{h} < \infty.$$

Definition 1.4 *Let $A = A(t)$ be a uniformly positive operator in E and $u \in B^s_{p,q}(R; E(A))$ and*

$$(A * u)(t) = \int_R A(t-y)u(y)\mathrm{d}y.$$

Let $y \in R$, $m \in N$ and e_i, $i = 1, 2, \cdots, n$, be standard unit vectors of R^n. Let

$$\begin{aligned}\Delta_i(y)f(x) &= f(x + ye_i) - f(x), \cdots, \Delta_i^m(y)f(x) \\ &= \Delta_i(y)[\Delta_i^{m-1}(y)f(x)] \\ &= \sum_{k=0}^{m} (-1)^{m+k} C_m^k f(x + kye_i).\end{aligned}$$

Let

$$\Delta_i(\Omega, y) = \begin{cases} \Delta_i(y)f(x) & \text{for } [x, x + mye_i] \subset \Omega, \\ 0 & \text{for } [x, x + mye_i] \notin \Omega. \end{cases}$$

Let m_i be integers, s_i be positive numbers and

$$m_i > s_i, \quad s = (s_1, s_2, \cdots, s_n), \quad 1 \le p \le \infty,\ 1 \le q \le \infty,\ y_0 > 0.$$

The space $B^s_{p,q}(\Omega; E)$ is E-valued Besov space, i.e.,

$$\begin{aligned} B^s_{p,q}(\Omega; E) = \Big\{ f : f \in L_p(\Omega; E), \\ \|f\|_{B^s_{p,q}(\Omega;E)} = \sum_{i=1}^{n} \Big(\int_0^{y_0} y^{-(s_i q + 1)} \|\Delta_i^{m_i}(y, \Omega)f(x)\|^q_{L_p(\Omega;E)} \mathrm{d}y \Big)^{\frac{1}{q}} < \infty \Big\}, \end{aligned}$$

$$\|f\|_{B^s_{p,\infty}(\Omega;E)} = \sum_{i=1}^{n} \sup_{0<y\le y_0} \frac{\|\Delta_i^{m_i}(y, \Omega)f(x)\|_{L_p(\Omega;E)}}{y^{s_i}}, \quad 1 \le p \le \infty,\ 1 \le q < \infty.$$

Let

$$\begin{aligned} B^{l,s}_{p,q}(R; E_0, E) = \Big\{ u : u \in B^s_{p,q}(R; E_0),\ D^l u \in B^s_{p,q}(R; E), \\ \|u\|_{B^{l,s}_{p,q}(R;E_0,E)} = \|u\|_{B^s_{p,q}(R;E_0)} + \|D^l u\|_{B^s_{p,q}(R;E)} < \infty \Big\}. \end{aligned}$$

The spaces $C(\Omega; E)$ and $C^{(m)}(\Omega; E)$ will denote the spaces of E-valued bounded, continuous and m-times continuously differentiable functions on Ω, respectively, and $D(\Omega; E)$ will denote the collection of infinitely differentiable E-valued functions with compact support on Ω.

2 Integro-Differential Operator Equations

Let us first recall an important fact (see [5, Corollary 4.11]) that will be used in this section.

Theorem 2.1 *Let $p, r \in [1, \infty]$. If $m \in C^l(R^n, B(X, Y))$ satisfies, for some constant A,*

$$\sup_{t \in R^n} \|(1 + |t|)^{|\alpha|} D^\alpha m(t)\|_{B(X,Y)} \le A$$

for each multi-index α with $|\alpha| \le \delta$, then m is Fourier multiplier from $B^s_{p,r}(R^n, X)$ to $B^s_{p,r}(R^n, Y)$, provided one of the following conditions holds:

(a) *X and Y are arbitrary Banach spaces and $\delta = n+1$.*

(b) *X and Y are uniformly convex Banach spaces and $\delta = n$.*

(c) *X and Y have Fourier type p and $\delta = \lceil \frac{n}{p} \rceil + 1$.*

Through this section the Fourier transform of a function f will be denoted by $\widehat{f}$ and $\frac{\mathrm{d}}{\mathrm{d}\xi}A(\xi)$ by $A'(\xi)$.

Condition 2.1 *Suppose*

$$a_k \in L_1(R), \quad L(\xi) = \sum_{k=0}^{l} \widehat{a}_k(\xi)(\mathrm{i}\xi)^k \in S(\varphi_1), \quad \varphi_1 + \varphi < \pi, \quad |L(\xi)| \geq C|\xi|^l \sum_{k=0}^{l} |\widehat{a}_k(\xi)|.$$

Lemma 2.1 *Let Condition 2.1 be satisfied and $A(\xi)$ be a uniformly φ-positive ($\varphi \in [0, \pi)$) operator in a Banach space E, $\lambda \in S(\varphi)$. Then, operator functions*

$$\begin{aligned}
\sigma_0(\xi,\lambda) &= \lambda[A(\xi) + (\lambda + L(\xi))]^{-1}, \\
\sigma_1(\xi,\lambda) &= A(\xi)[A(\xi) + (\lambda + L(\xi))]^{-1}, \\
\sigma_2(\xi,\lambda) &= \sum_{k=0}^{l} |\lambda|^{1-\frac{k}{l}} \widehat{a}_k(\xi)(\mathrm{i}\xi)^k [A(\xi) + (\lambda + L(\xi))]^{-1}
\end{aligned}$$

are uniformly bounded.

Proof Let us note that for the sake of simplicity we shall not change constants in every step. By using the resolvent properties of positive operators we obtain

$$\begin{aligned}
\|\sigma_0(\xi,\lambda)\|_{B(E)} &\leq M|\lambda|(1 + |\lambda + L(\xi)|)^{-1} \leq M, \\
\|\sigma_1(\xi,\lambda)\|_{B(E)} &= \|A(\xi)[A(\xi) + (\lambda + L(\xi))]^{-1}\|_{B(E)} \\
&= \|I - (\lambda + L(\xi))[A(\xi) + (\lambda + L(\xi))]^{-1}\|_{B(E)} \\
&\leq 1 + |\lambda + L(\xi)| \|[A(\xi) + (\lambda + L(\xi))]^{-1}\|_{B(E)} \\
&\leq 1 + M|\lambda + L(\xi)|(1 + |\lambda + L(\xi)|)^{-1} \\
&\leq 1 + M.
\end{aligned}$$

Next, let us consider σ_2. It is clear to see that

$$\begin{aligned}
\|\sigma_2(\xi,\lambda)\|_{B(E)} &= \Big\| \sum_{k=0}^{l} |\lambda|^{1-\frac{k}{l}} \widehat{a}_k(\xi)(\mathrm{i}\xi)^k [A(\xi) + (\lambda + L(\xi))]^{-1} \Big\|_{B(E)} \\
&\leq C \sum_{k=0}^{l} |\lambda||\widehat{a}_k|[|\xi||\lambda|^{-\frac{1}{l}}]^k \|[A(\xi) + (\lambda + L(\xi))]^{-1}\|_{B(E)}.
\end{aligned}$$

Therefore, $\sigma_2(\xi,\lambda)$ is bounded if

$$\|I\|_{B(E)} = \sum_{k=0}^{l} |\lambda||\widehat{a}_k|[|\xi||\lambda|^{-\frac{1}{l}}]^k \|[A(\xi) + (\lambda + L(\xi))]^{-1}\|_{B(E)} \leq C.$$

Since A is a uniformly φ-positive and $L(\xi) \in S(\varphi_1)$ for all $\xi \in R$, we have

$$\|I\|_{B(E)} \leq C \sum_{k=0}^{l} |\lambda||\widehat{a}_k|[1 + |\xi|^l|\lambda|^{-1}][1 + |\lambda + L(\xi)|]^{-1}.$$

Since $a_k \in L_1(R)$, Housdorff-Youngs inequality implies that $\sum_{k=0}^{l} |\widehat{a}_k| \le C$. Taking into account Condition 2.1 and by using [4, Lemma 2.3], we get

$$\begin{aligned}
\|\sigma_2(\xi,\lambda)\|_{B(E)} &\le C\|I\|_{B(E)} \\
&\le C\sum_{k=0}^{l}[|\lambda||\widehat{a}_k| + |\widehat{a}_k||\xi|^l][1+|\lambda|+|L(\xi)|]^{-1} \\
&\le C\Big[|\lambda| + |\xi|^l \sum_{k=0}^{l} |\widehat{a}_k|\Big]\Big[1+|\lambda|+\Big|\sum_{k=0}^{l}\widehat{a}_k(\xi)(\mathrm{i}\xi)^k\Big|\Big]^{-1} \\
&\le C.
\end{aligned}$$

Lemma 2.2 *Let Condition* 2.1 *be satisfied,* $A(\xi)$ *be a uniformly* φ*-positive operator in a Banach space* E*,* $\lambda \in S(\varphi)$ *for* $|\lambda| \ge |\lambda_0|$ *and*

$$\begin{aligned}
&\widehat{a}_k \in C^{(m)}(R), \quad k = 0,1,\cdots,l,\ m = 1,2, \\
&A(\xi)A^{-1}(\xi_0) \in C^{(m)}(R; B(E)), \quad \xi_0 \in R.
\end{aligned}$$

Suppose that there are positive constants C_i*,* $i = 1,\cdots,4$ *such that*

$$\|A^{(m)}(\xi)A^{-1}(\xi)\|_{B(E)} \le C_1, \quad \|\xi^m A^{(m)}(\xi)A^{-1}(\xi)\|_{B(E)} \le C_2, \tag{2.1}$$

$$|\xi^m \widehat{a}_k(\xi)| \le M, \quad \Big|\frac{\mathrm{d}^m}{\mathrm{d}\xi^m}\widehat{a}_k(\xi)\Big| \le C_3, \quad \Big|\xi^m \frac{\mathrm{d}^m}{\mathrm{d}\xi^m}\widehat{a}_k(\xi)\Big| \le C_4. \tag{2.2}$$

Then, operator functions $\frac{\mathrm{d}^m}{\mathrm{d}\xi^m}\sigma_i(\xi,\lambda)$*,* $i = 0,1,2$*, are uniformly bounded.*

Proof Let us first prove the case of $\frac{\mathrm{d}}{\mathrm{d}\xi}\sigma_1(\xi,\lambda)$. Really,

$$\Big\|\frac{\mathrm{d}}{\mathrm{d}\xi}\sigma_1(\xi,\lambda)\Big\|_{B(E)} \le \|\mathrm{I}_1\|_{B(E)} + \|\mathrm{I}_2\|_{B(E)} + \|\mathrm{I}_3\|_{B(E)},$$

where

$$\begin{aligned}
\mathrm{I}_1 &= A'(\xi)[A(\xi) + (\lambda + L(\xi))]^{-1}, \\
\mathrm{I}_2 &= A(\xi)A'(\xi)[A(\xi) + (\lambda + L(\xi))]^{-2}, \\
\mathrm{I}_3 &= A(\xi)L'(\xi)[A(\xi) + (\lambda + L(\xi))]^{-2}.
\end{aligned}$$

By using (2.1), we get

$$\begin{aligned}
\|\mathrm{I}_1\|_{B(E)} &= \|A'(\xi)A^{-1}(\xi)A(\xi)[A(\xi) + (\lambda + L(\xi))]^{-1}\|_{B(E)} \\
&\le \|A'(\xi)A^{-1}(\xi)\|_{B(E)}\|A(\xi)[A(\xi) + (\lambda + L(\xi))]^{-1}\|_{B(E)} \\
&\le C.
\end{aligned}$$

Taking into account the fact that A is closed and linear operator and by using (2.1), we obtain

$$\|\mathrm{I}_2\|_{B(E)} \le \|A'(\xi)[A(\xi) + (\lambda + L(\xi))]^{-1}\|_{B(E)}\|A(\xi)[A(\xi) + (\lambda + L(\xi))]^{-1}\|_{B(E)} \le C.$$

Since $A(\xi)$ is a uniformly φ-positive, $L(\xi) \in S(\varphi_1)$ and $\lambda \in S(\varphi)$ with $\varphi_1 + \varphi < \pi$, we get

$$\begin{aligned}\|\mathrm{I}_3\|_{B(E)} &\leq |L'(\xi)| \|[A(\xi) + (\lambda + L(\xi))]^{-1}\|_{B(E)} \|A(\xi)[A(\xi) + (\lambda + L(\xi))]^{-1}\|_{B(E)} \\ &\leq C|L'(\xi)|[1 + |\lambda + L(\xi)|]^{-1}.\end{aligned}$$

Then, by using [4, Lemma 2.3], we have

$$\|\mathrm{I}_3\|_{B(E)} \leq C|L'(\xi)| \Big[1 + |\lambda| + \Big|\sum_{k=0}^{l} \widehat{a}_k(\xi)(\mathrm{i}\xi)^k\Big|\Big]^{-1}.$$

It is clear to see that

$$|L'(\xi)| \leq \Big|\sum_{k=0}^{l} \frac{\mathrm{d}}{\mathrm{d}\xi}\widehat{a}_k(\xi)(\mathrm{i}\xi)^k\Big| + C\Big|\sum_{k=1}^{l} \widehat{a}_k(\xi)(\mathrm{i}\xi)^{k-1}\Big|. \tag{2.3}$$

By using (2.2) and (2.3), we obtain

$$\begin{aligned}\Big|\sum_{k=0}^{l} \frac{\mathrm{d}}{\mathrm{d}\xi}\widehat{a}_k(\xi)(\mathrm{i}\xi)^k\Big| &\leq C\Big[1 + |\lambda| + \Big|\sum_{k=0}^{l} \widehat{a}_k(\xi)(\mathrm{i}\xi)^k\Big|\Big], \\ \Big|\sum_{k=1}^{l} \widehat{a}_k(\xi)(\mathrm{i}\xi)^{k-1}\Big| &\leq C\Big[1 + |\lambda| + \Big|\sum_{k=0}^{l} \widehat{a}_k(\xi)(\mathrm{i}\xi)^k\Big|\Big].\end{aligned}$$

That implies

$$\|\mathrm{I}_3\|_{B(E)} \leq C\sum_{k=0}^{l} \Big|\frac{\mathrm{d}}{\mathrm{d}\xi}[\widehat{a}_k(\xi)\xi^k]\Big| \Big[1 + |\lambda| + \Big|\sum_{k=0}^{l} \widehat{a}_k(\xi)(\mathrm{i}\xi)^k\Big|\Big]^{-1} \leq C. \tag{2.4}$$

Next we shall prove that $\frac{\mathrm{d}}{\mathrm{d}\xi}\sigma_2(\xi, \lambda)$ is uniformly bounded. Similarly,

$$\Big\|\frac{\mathrm{d}}{\mathrm{d}\xi}\sigma_2(\xi, \lambda)\Big\|_{B(E)} \leq \|\mathrm{J}_1\|_{B(E)} + \|\mathrm{J}_2\|_{B(E)} + \|\mathrm{J}_3\|_{B(E)} + \|\mathrm{J}_4\|_{B(E)},$$

where

$$\begin{aligned}\mathrm{J}_1 &= \sum_{k=0}^{l} |\lambda|^{1-\frac{k}{l}} \frac{\mathrm{d}}{\mathrm{d}\xi}\widehat{a}_k(\xi)(\mathrm{i}\xi)^k[A(\xi) + (\lambda + L(\xi))]^{-1}, \\ \mathrm{J}_2 &= \sum_{k=0}^{l} |\lambda|^{1-\frac{k}{l}} \widehat{a}_k(\xi)\mathrm{i}k(\mathrm{i}\xi)^{k-1}[A(\xi) + (\lambda + L(\xi))]^{-1}, \\ \mathrm{J}_3 &= \sum_{k=0}^{l} |\lambda|^{1-\frac{k}{l}} \widehat{a}_k(\xi)(\mathrm{i}\xi)^k L'(\xi)[A(\xi) + (\lambda + L(\xi))]^{-2}, \\ \mathrm{J}_4 &= \sum_{k=0}^{l} |\lambda|^{1-\frac{k}{l}} \widehat{a}_k(\xi)(\mathrm{i}\xi)^k A'(\xi)[A(\xi) + (\lambda + L(\xi))]^{-2}.\end{aligned}$$

Let us first show that J_1 is uniformly bounded. Since

$$\|\mathrm{J}_1\|_{B(E)} \leq \sum_{k=0}^{l} \Big|\frac{\mathrm{d}}{\mathrm{d}\xi}\widehat{a}_k(\xi)\Big| \||\lambda|^{1-\frac{k}{l}}(\mathrm{i}\xi)^k[A(\xi) + (\lambda + L(\xi))]^{-1}\|_{B(E)}$$

by virtue of (2.2) and (2.4), we obtain $\|\mathrm{J}_1\|_{B(E)} \leq C$. Second, with the help of (2.2) and the fact that $|\lambda| \geq |\lambda_0|$, we get

$$\begin{aligned}
\|\mathrm{J}_2\|_{B(E)} &\leq \sum_{k=1}^{l} |\lambda_0|^{-\frac{1}{l}} |\widehat{a}_k(\xi)||\lambda|^{1-\frac{k-1}{l}} (\mathrm{i}\xi)^{k-1} \|[A(\xi) + (\lambda + L(\xi))]^{-1}\|_{B(E)} \\
&\leq C \sum_{k=1}^{l} |\lambda||\widehat{a}_k(\xi)|[|\lambda|^{-\frac{1}{l}} |(\mathrm{i}\xi)|]^{k-1} \|[A(\xi) + (\lambda + L(\xi))]^{-1}\|_{B(E)} \\
&\leq C.
\end{aligned}$$

Similarly, with the help of (2.3), we have

$$\begin{aligned}
\|\mathrm{J}_3\|_{B(E)} &\leq C|L'(\xi)| \|[A(\xi) + (\lambda + L(\xi))]^{-1}\|_{B(E)} \\
&\quad \cdot \sum_{k=0}^{l} |\lambda||\widehat{a}_k(\xi)|[|\lambda|^{-\frac{1}{l}} |(\mathrm{i}\xi)|]^{k} \|[A(\xi) + (\lambda + L(\xi))]^{-1}\|_{B(E)} \\
&\leq C. \qquad (2.5)
\end{aligned}$$

Last, we need to show that J_4 is uniformly bounded. Really, by using (2.3) and (2.4), we obtain

$$\begin{aligned}
\|\mathrm{J}_4\|_{B(E)} &\leq C \Big\| \frac{\mathrm{d}}{\mathrm{d}\xi} A(\xi) A^{-1}(\xi) A(\xi)[A(\xi) + (\lambda + L(\xi))]^{-1} \Big\|_{B(E)} \\
&\quad \times \sum_{k=0}^{l} |\lambda||\xi|^k |\lambda|^{\frac{-k}{l}} \|[A(\xi) + (\lambda + L(\xi))]^{-1}\|_{B(E)} \\
&\leq C \|A'(\xi) A^{-1}(\xi)\|_{B(E)} \|A(\xi)[A(\xi) + (\lambda + L(\xi))]^{-1}\|_{B(E)} \\
&\leq C.
\end{aligned}$$

Hence, operator functions $\frac{\mathrm{d}}{\mathrm{d}\xi}\sigma_i(\xi, \lambda)$ are uniformly bounded. In a similar way the boundedness of $\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\sigma_i(\xi, \lambda)$ are obtained.

Lemma 2.3 *Let all conditions of Lemma* 2.2 *be satisfied. Then the following estimates hold:*

$$\Big\| |\xi|^m \frac{\mathrm{d}^m}{\mathrm{d}\xi^m} \sigma_i(\xi, \lambda) \Big\|_{L_\infty(B(E))} \leq A_i, \quad m, i = 0, 1, 2.$$

Proof As a matter of fact, it is enough to prove

$$|\xi| \|\mathrm{I}_i\|_{B(E)} \leq C_i \quad \text{and} \quad |\xi| \|\mathrm{J}_j\|_{B(E)} \leq D_j$$

for some constants C_i and D_j, $i = 1, 2, 3$, $j = 1, 2, 3, 4$. It is easy to see from the proof of Lemma 2.2 that

$$\begin{aligned}
|\xi| \|\mathrm{I}_1\|_{B(E)} &\leq C_1 \|\xi A'(\xi) A^{-1}(\xi)\|_{B(E)} \|A(\xi)[A(\xi) + (\lambda + L(\xi))]^{-1}\|^2_{B(E)}, \\
|\xi| \|\mathrm{I}_2\|_{B(E)} &\leq C_2 \|\xi A'(\xi) A^{-1}(\xi)\|_{B(E)} \|A(\xi)[A(\xi) + (\lambda + L(\xi))]^{-1}\|_{B(E)}, \\
|\xi| \|\mathrm{I}_3\|_{B(E)} &\leq C_3 |\xi||L'(\xi)| \Big[1 + |\lambda| + \sum_{k=0}^{l} |\widehat{a}_k(\xi)||\xi|^k\Big]^{-1}.
\end{aligned}$$

From resolvent properties of positive operators, it follows that $\xi \mathrm{I}_1$ and $\xi \mathrm{I}_2$ are uniformly bounded. By using (2.4) and (2.5), we obtain

$$|\xi| \|\mathrm{I}_3\|_{B(E)} \le C_3 \sum_{k=0}^{l} |\widehat{a}_k(\xi)||\xi|^k \Big[1+|\lambda|+\sum_{k=0}^{l}|\widehat{a}_k(\xi)||\xi|^k\Big]^{-1} \le C_3.$$

Similarly, from the proof of Lemma 2.2, it follows that

$$\begin{aligned}
|\xi| \|\mathrm{J}_1\|_{B(E)} &\le \sum_{k=0}^{l} \Big|\xi \frac{\mathrm{d}}{\mathrm{d}\xi}\widehat{a}_k(\xi)\Big| \| |\lambda|^{1-\frac{k}{l}} (\mathrm{i}\xi)^k [A(\xi)+(\lambda+L(\xi))]^{-1}\|_{B(E)},\\
|\xi| \|\mathrm{J}_2\|_{B(E)} &\le \sum_{k=0}^{l} |\widehat{a}_k(\xi)| \| |\lambda|^{1-\frac{k}{l}} (\mathrm{i}\xi)^k [A(\xi)+(\lambda+L(\xi))]^{-1}\|_{B(E)},\\
|\xi| \|\mathrm{J}_3\|_{B(E)} &\le C|\xi L'(\xi)| \| [A(\xi)+(\lambda+L(\xi))]^{-1}\|_{B(E)}\\
&\quad \cdot \sum_{k=0}^{l} |\lambda||\xi|^k |\lambda|^{\frac{-k}{l}} \|[A(\xi)+(\lambda+L(\xi))]^{-1}\|_{B(E)},\\
|\xi| \|\mathrm{J}_4\|_{B(E)} &\le C\|\xi A'(\xi) A^{-1}(\xi)\|_{B(E)} \|A(\xi)[A(\xi)+(\lambda+L(\xi))]^{-1}\|_{B(E)}.
\end{aligned}$$

Using (2.1), (2.2), (2.4) and the fact that $\widehat{a}_k(\xi) \in L_\infty(R)$, we obtain uniformly boundedness of $|\xi| \|\mathrm{J}_1\|_{B(E)}$, $|\xi| \|\mathrm{J}_2\|_{B(E)}$, $|\xi| \|\mathrm{J}_3\|_{B(E)}$ and $|\xi| \|\mathrm{J}_4\|_{B(E)}$ respectively. In a similar way, the boundedness of $|\xi|^2 \frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\sigma_i(\xi,\lambda)$ is obtained.

Corollary 2.1 *Let all conditions of Lemma* 2.3 *be satisfied. Then operator-functions* $\sigma_i(\xi,\lambda)$ *are uniformly bounded multipliers in* $B^s_{p,r}(R;E)$.

Proof To prove that $\sigma_i(\xi,\lambda)$ are uniformly bounded multipliers in $B^s_{p,r}(R;E)$, we need to show that $\sigma_i \in C^{(m)}(R;B(E))$ and there exists a constant $K>0$ such that

$$\Big\|(1+|\xi|)^\beta \frac{\mathrm{d}^\beta}{\mathrm{d}\xi^\beta}\sigma_i(\xi,\lambda)\Big\|_{L_\infty(R;B(E))} \le K$$

for each $\beta \le 2$. From Lemmas 2.1–2.3, it follows that $\sigma_i \in C^1(R;B(E))$ and

$$\Big\|\frac{\mathrm{d}^m}{\mathrm{d}\xi^m}\sigma_i(\xi,\lambda)\Big\|_{L_\infty(B(E))} \le A_1, \quad \Big\||\xi|^m \frac{\mathrm{d}^m}{\mathrm{d}\xi^m}\sigma_i(\xi,\lambda)\Big\|_{L_\infty(B(E))} \le A_2$$

for every $i,m=0,1,2$. Hence, $\sigma_i(\xi,\lambda)$ are Fourier multipliers in $B^s_{p,r}(R;E)$.

Now, let us consider an ordinary convolution differential operator equation

$$(L+\lambda)u = \sum_{k=0}^{l} a_k * \frac{\mathrm{d}^k u}{\mathrm{d}x^k} + A_\lambda * u = f(t) \tag{2.6}$$

in $B^s_{p,r}(R;E)$, where $A_\lambda = A+\lambda$, $A=A(t)$ is a possible unbounded operator in E,

$$a_k = a_k(t)$$

are complex valued functions.

Theorem 2.2 *Let $f \in B^s_{p,r}(R;E)$ for $p,\ r \in [1,\infty]$. Then equation (2.6) has a unique solution $u \in B^{l,s}_{p,r}(R;E(A),E)$ and the following coercive uniform estimate holds for $\lambda \in S(\varphi)$ ($\varphi \in [0,\pi)$ and $|\lambda| \geq |\lambda_0|$) :*

$$\sum_{k=0}^{l} |\lambda|^{1-\frac{k}{l}} \Big\| a_k * \frac{\mathrm{d}^k u}{\mathrm{d}x^k} \Big\|_{B^s_{p,r}(R;E)} + \|A * u\|_{B^s_{p,r}(R;E)} + |\lambda| \|u\|_{B^s_{p,r}(R;E)} \leq C \|f\|_{B^s_{p,r}(R;E)}, \quad (2.7)$$

provided the below conditions are satisfied:

(1) *E is a Banach space.*

(2) *Condition* 2.1 *holds and*

$$\begin{aligned} &\widehat{a}_k \in C^{(m)}(R), \quad k = 0,1,\cdots,l, \\ &\widehat{A}(\xi)\widehat{A}^{-1}(\xi_0) \in C^{(m)}(R;B(E)), \quad \xi_0 \in R. \end{aligned}$$

(3) *$\widehat{A}(\xi)$ is a uniformly φ-positive ($\varphi \in [0,\pi)$) operator in E. Moreover, there are positive constants C_i, $i = 1,\cdots,4$ such that for $m = 0,1,2$,*

$$\begin{aligned} &|\xi^m \widehat{a}_k(\xi)| \leq M, \quad \Big|\frac{\mathrm{d}^m}{\mathrm{d}\xi^m}\widehat{a}_k(\xi)\Big| \leq C_1, \quad \Big|\xi^m \frac{\mathrm{d}^m}{\mathrm{d}\xi^m}\widehat{a}_k(\xi)\Big| \leq C_2, \\ &\|\widehat{A}^{(m)}(\xi)\widehat{A}^{-1}(\xi)\|_{B(E)} \leq C_3, \quad \|\xi^m \widehat{A}^{(m)}(\xi)\widehat{A}^{-1}(\xi)\|_{B(E)} \leq C_4. \end{aligned}$$

Proof By applying the Fourier transform to equation (2.6), we obtain

$$[\widehat{A}(\xi) + (L(\xi) + \lambda)]u\hat{}(\xi) = f\hat{}(\xi).$$

Since $L(\xi) \in S(\varphi_1)$ for all $\xi \in R$ and $\widehat{A}$ is positive, the operator $\widehat{A}(\xi) + (L(\xi) + \lambda)$ is invertible in E. So we obtain that the solution of equation (2.6) can be represented in the form

$$u(x) = F^{-1}[\widehat{A}(\xi) + (\lambda + L(\xi))]^{-1} f\hat{}. \quad (2.8)$$

By using (2.8), we obtain

$$\begin{aligned} &\|A * u\|_{B^s_{p,r}(R;E)} = \|F^{-1}[\sigma_1(\xi,\lambda) f\hat{}\,]\|_{B^s_{p,r}(R;E)}, \\ &\sum_{k=0}^{l} |\lambda|^{1-\frac{k}{l}} \Big\| a_k * \frac{\mathrm{d}^k u}{\mathrm{d}x^k} \Big\|_{B^s_{p,r}(R;E)} = \|F^{-1}[\sigma_2(\xi,\lambda) f\hat{}\,]\|_{B^s_{p,r}(R;E)}, \end{aligned}$$

where

$$\begin{aligned} &\sigma_0(\xi,\lambda) = \lambda[\widehat{A}(\xi) + (\lambda + L(\xi))]^{-1}, \\ &\sigma_1(\xi,\lambda) = \widehat{A}(\xi)[\widehat{A}(\xi) + (\lambda + L(\xi))]^{-1}, \\ &\sigma_2(\xi,\lambda) = \sum_{k=0}^{l} |\lambda|^{1-\frac{k}{l}} \widehat{a}_k(\xi)(\mathrm{i}\xi)^k [\widehat{A}(\xi) + (\lambda + L(\xi))]^{-1}. \end{aligned}$$

By using Corollary 2.1, we obtain that operator-functions $\sigma_i(\xi,\lambda)$ are uniformly bounded multipliers in $B^s_{p,r}(R;E)$.

Since

$$\|A*u\|_{B^s_{p,r}(R;E)} \leq C_1\|f\|_{B^s_{p,r}(R;E)},$$
$$\sum_{k=0}^{l}|\lambda|^{1-\frac{k}{l}}\left\|a_k*\frac{\mathrm{d}^k u}{\mathrm{d}x^k}\right\|_{B^s_{p,r}(R;E)} \leq C_2\|f\|_{B^s_{p,r}(R;E)},$$

we obtain that for all $f \in B^s_{p,r}(R;E)$, there is a unique solution to equation (2.6) in the form

$$u(x) = F^{-1}[A+(\lambda+L(\xi))]^{-1}f^{\wedge}$$

and the estimate (2.7) holds.

Let Q be the operator generated by problem (2.6), i.e.,

$$D(Q) = B^{l,s}_{p,r}(R;E(A),E), \quad Qu = \sum_{k=0}^{l} a_k*\frac{\mathrm{d}^k u}{\mathrm{d}x^k} + A_\lambda * u.$$

Result 2.1 Assume that all conditions of Theorem 2.2 hold. Then, for all $\lambda \in S(\varphi)$, the resolvent of operator Q exists and the following estimate holds:

$$\sum_{k=0}^{l}|\lambda|^{1-\frac{k}{l}}\left\|a_k*\left[\frac{\mathrm{d}^k}{\mathrm{d}x^k}(Q+\lambda)^{-1}\right]\right\|_{B(B^s_{p,r}(R;E))}$$
$$\cdot\,\|\lambda(Q+\lambda)^{-1}\|_{B(B^s_{p,r}(R;E))} + \|A*(Q+\lambda)^{-1}\|_{B(B^s_{p,r}(R;E))} \leq C.$$

Remark 2.1 Result 2.1 particularly implies that the operator $Q+a$, $a>0$ is positive in $B^s_{p,r}(R;E)$, i.e., if $\widehat{A}$ is uniformly R-positive for $\varphi \in (\frac{\pi}{2},\pi)$, then (see e.g. [4]) the operator $Q+a$ is a generator of analytic semigroup in $B^s_{p,r}(R;E)$.

3 Infinite Systems of IDEs

Consider the following infinity system of convolution equation:

$$\sum_{k=0}^{l} a_k*\frac{\mathrm{d}^k u_m}{\mathrm{d}x^k} + \sum_{j=1}^{\infty}(d_j+\lambda)*u_j(x) = f_m(x), \quad x\in R,\ m=1,2,\cdots,\infty. \tag{3.1}$$

Condition 3.1 *There are positive constants C_1 and C_2 such that for $\{d_j(x)\}_1^\infty \in l_q$, for all $x \in R$ and some $x_0 \in R$,*

$$C_1|d_j(x_0)| \leq |d_j(x)| \leq C_2|d_j(x_0)|.$$

Suppose that $\widehat{a}_k, \widehat{d}_m \in C^{(1)}(R)$ and there are positive constants M_i, $i=1,\cdots,4$ such that

$$\left|\xi^j\frac{\mathrm{d}^j}{\mathrm{d}\xi^j}\widehat{a}_k(\xi)\right| \leq M_1, \quad |\xi^j\widehat{a}_k(\xi)| \leq M_2,$$
$$d^j_m(\xi)d^{-1}_m(\xi) \leq M_3, \quad |\xi|^j d^j_m(\xi)d^{-1}_m(\xi) \leq M_4, \quad j=0,1,2.$$

Let

$$D(x) = \{d_m(x)\}, \quad d_m > 0, \quad u = \{u_m\}, \quad D * u = \{d_m * u_m\}, \quad m = 1, 2, \cdots \infty,$$

$$l_q(D) = \Big\{u : u \in l_q, \|u\|_{l_q(D)} = \|D * u\|_{l_q} = \Big(\sum_{m=1}^{\infty} |d_m * u_m|^q\Big)^{\frac{1}{q}} < \infty\Big\}, \quad 1 < q < \infty.$$

Let Q be a differential operator in $B^s_{p,r}(R; l_q)$ generated by the boundary value problem (3.1). Let

$$B = B(B^s_{p,r}(R; l_q)).$$

Theorem 3.1 *Suppose that Conditions* 2.1 *and* 3.1 *are satisfied. Then*

(a) *For all* $f(x) = \{f_m(x)\}_1^\infty \in B^s_{p,r}(R; l_q(D))$, *for* $\lambda \in S(\varphi)$, $\varphi \in [0, \pi)$ *and for sufficiently large* $|\lambda|$, *problem* (3.1) *has a unique solution* $u(x) = \{u_m(x)\}_1^\infty$ *that belongs to space* $B^{l,s}_{p,r}(R; l_q(D), l_q)$ *and the coercive uniform estimate*

$$\sum_{k=0}^{l} |\lambda|^{1-\frac{k}{l}} \Big\|a_k * \frac{\mathrm{d}^k u}{\mathrm{d}x^k}\Big\|_{B^s_{p,r}(R;l_q)} + \|D * u\|_{B^s_{p,r}(R^n;l_q)} + |\lambda| \|u\|_{B^s_{p,r}(R;l_q)} \le C\|f\|_{B^s_{p,r}(R;l_q)} \tag{3.2}$$

holds for the solution to problem (3.1).

(b) *For sufficiently large* $|\lambda| > 0$, *there exists a resolvent* $(Q + \lambda)^{-1}$ *of operator* Q *and*

$$\sum_{k=0}^{l} |\lambda|^{1-\frac{k}{l}} \Big\|a_k * \Big[\frac{\mathrm{d}^k}{\mathrm{d}x^k}(Q + \lambda)^{-1}\Big]\Big\|_B + \|D * (Q + \lambda)^{-1}\|_B + \|(1 + |\lambda|)(Q + \lambda)^{-1}\|_B \le C. \tag{3.3}$$

Proof Really, let $E = l_q$, A be infinite matrices, such that

$$A = [d_m(t)\delta_{jm}], \quad m, j = 1, 2, \cdots \infty.$$

It is clear to see that the operator A is uniformly positive in l_q. Therefore, by virtue of Theorem 2.2 and Result 2.1, we obtain that, for all $f \in B^s_{p,r}(R; l_q)$, for $\lambda \in S(\varphi)$, $\varphi \in (0, \pi)$ and sufficiently large $|\lambda|$, problem (3.1) has a unique solution u that belongs to space $B^{l,s}_{p,r}(G; l_q(D), l_q)$ and estimates (3.2) and (3.3) are satisfied.

Remark 3.1 There are a lot of positive operators in concrete Banach spaces. Therefore, putting concrete Banach spaces instead of E and concrete positive differential, pseudo differential operators, or finite, infinite matrices, etc. instead of operator A on (2.6), we can obtain the maximal regularity of different class of convolution equations or system of equations by virtue of Theorem 2.2.

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