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# Inverse Coefficient Problems for Elliptic Hemivariational Inequalities***

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**Abstract** This paper is devoted to a class of inverse coefficient problems for nonlinear elliptic hemivariational inequalities. The unknown coefficient of elliptic hemivariational inequalities depends on the gradient of the solution and belongs to a set of admissible coefficients. It is shown that the nonlinear elliptic hemivariational inequalities are uniquely solvable for the given class of coefficients. The result of existence of quasisolutions of the inverse problems is obtained.

**Keywords** Elliptic hemivariational inequality, Inverse coefficient problem, Existence of quasisolution
**2000 MR Subject Classification** 35J85, 49J20, 49J45

## 1 Introduction

In this paper, we study the problem of identification of an unknown coefficient $k$ in an elliptic hemivariational inequality. The elliptic hemivariational inequality under consideration is as follows:

$$\begin{cases} -\nabla(k(|\nabla u|^2)\nabla u) + w(x) = f(x), & x \in \Omega, \\ w(x) \in \partial G(u), & \\ u(x) \le \varphi(x),\ k(|\nabla u|^2)\dfrac{\partial u}{\partial n} \le 0,\ [u-\varphi]k(|\nabla u|^2)\dfrac{\partial u}{\partial n} = 0, & x \in \Gamma_0, \\ u(x) = 0, & x \in \Gamma_u, \\ k(|\nabla u|^2)\dfrac{\partial u}{\partial n} = \psi(x), & x \in \Gamma_\sigma, \end{cases} \tag{1.1}$$

where the open domain $\Omega \subset R^N$ $(N \ge 2)$ is assumed to be bounded simply connected with a piecewise smooth boundary $\partial\Omega$ and $\Gamma_0 \cap \Gamma_u = \emptyset,\ \Gamma_0 \cap \Gamma_\sigma = \emptyset,\ \Gamma_u \cap \Gamma_\sigma = \emptyset,\ \overline{\Gamma}_0 \cup \overline{\Gamma}_u \cup \overline{\Gamma}_\sigma = \partial\Omega,$ $\operatorname{meas}\Gamma_0 > 0,\ \operatorname{meas}\Gamma_u > 0,\ \operatorname{meas}\Gamma_\sigma > 0.$ $\partial G$ denotes the generalized Clarke subdifferential of a locally Lipschitzian functional $G$.

Let $V = \{v \in H^1(\Omega) : \gamma v = 0 \text{ on } \Gamma_u \subset \partial\Omega\}$, where $\gamma : H^1(\Omega) \to L^2(\partial\Omega)$ denotes the trace operator and $H^1(\Omega)$ is the usual Sobolev space (see [6]). Applying the Poincaré inequality we may define the Hilbert space $V$ with the norm of $\|u\|_V = (\int_\Omega |\nabla u|^2 \mathrm{d}x)^{\frac{1}{2}}$. Identifying $H = L^2(\Omega)$

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with its dual, we have an evolution triple $V \subset H \subset V^*$ with dense, continuous and compact embeddings (see [19]). For conveniences, we denote by $\langle \cdot\,, \cdot \rangle_B$ the duality of $B$ and its dual $B^*$ as well as by $\| \cdot \|_B$ the norm for any Banach space $B$.

For a locally Lipschitzian functional $h : V \to R$, we denote by $h^0(u, v)$ the Clarke generalized directional derivative of $h$ at $u$ in the direction $v$, that is,

$$h^0(u,v) := \limsup_{\substack{\lambda \to 0+ \\ w \to u}} \frac{h(w + \lambda v) - h(w)}{\lambda}.$$

Recall also at this point that

$$\partial h(u) := \{u^* \in V^* \mid h^0(u, v) \geq \langle u^*, v \rangle_V, \ \forall\, v \in V\} \tag{1.2}$$

denotes the generalized Clarke subdifferential (see [3] for details).

The following situation for the locally Lipschitzian functional $G : L^2(\Omega) \to R$ is of particular interest in applications (see [9, 10, 13]):

$$G(u) = \int_\Omega g(u(x))\mathrm{d}x, \quad u \in L^2(\Omega), \tag{1.3}$$

where $g : R \to R$ is the function $g(t) = \int_0^t \theta(\tau)\mathrm{d}\tau$, $t \in R$, which corresponds to a function $\theta : R \to R$ satisfying the assumption: there exist two positive constants $a, c_0$ independent of $\tau$ such that $|\theta(\tau)| \leq a + c_0|\tau|$, $\forall\, \tau \in R$. It is known that under the assumption above the functional $G$ in (1.3) is locally Lipschitz. We set for $\delta > 0$,

$$\theta_\delta^-(s) = \operatorname*{ess\,inf}_{|t-s|\leq\delta} \theta(t), \quad \theta_\delta^+(s) = \operatorname*{ess\,sup}_{|t-s|\leq\delta} \theta(t),$$
$$\theta^-(s) = \lim_{\delta\to 0^+} \theta_\delta^-(s), \quad \theta^+(s) = \lim_{\delta\to 0^+} \theta_\delta^+(s).$$

From [3], it follows $\partial g(t) = [\theta^-(t), \theta^+(t)]$. Therefore, for any $w \in \partial g(t)$, one obtains

$$|w| \leq \max\{|\theta^-(t)|, |\theta^+(t)|\} \leq a + c_0|t|, \quad \forall\, t \in R. \tag{1.4}$$

By the same argument as in [2], we obtain the following characteristics of the generalized gradient $\partial G(u)$:

$$\partial(G|_V)(u) \subset \partial(G|_{L^2(\Omega)})(u), \quad \forall\, u \in V. \tag{1.5}$$

If $w \in \partial G|_{L^2(\Omega)}(u)$, then $w \in L^2(\Omega)$ satisfies

$$w(x) \in [\theta^-(u(x)), \theta^+(u(x))] \quad \text{for a.e. } x \in \Omega. \tag{1.6}$$

In the sequel, we need the following assumption of relaxed monotonicity ($\mathrm{H}_1$):

$$\langle u^* - v^*, u - v \rangle_{L^2(\Omega)} \geq -m\|u - v\|^2_{L^2(\Omega)}, \quad \forall\, u, v \in L^2(\Omega)$$

for any $u^* \in \partial G|_{L^2(\Omega)}(u)$ and $v^* \in \partial G|_{L^2(\Omega)}(v)$, where $m$ is a positive constant (see [11]).

With respect to coefficients $k = k(s)$ we assume the following assumptions:

($\mathrm{A}_1$) $k \in C[0, \infty)$ and $c_1 \leq k(s) \leq c_2$, $\forall s \in [0, \infty)$;

($\mathrm{A}_2$) $\sum\limits_{i=1}^N [k(|\xi|^2)\xi_i - k(|\xi'|^2)\xi'_i](\xi_i - \xi'_i) \geq c_3|\xi - \xi'|^2$, $\forall\, \xi = (\xi_1, \cdots, \xi_N)$, $\xi' = (\xi'_1, \cdots, \xi'_N) \in R^N$, where $c_1, c_2, c_3$ are positive constants such that

$$c_3 > m[C(\Omega)]^2, \tag{1.7}$$

where $m$ is the constant of Assumption ($\mathrm{H}_1$) and $C(\Omega)$ is the Poincaré constant, i.e.,

$$\Big(\int_\Omega |u|^2 \mathrm{d}x\Big)^{\frac{1}{2}} \le C(\Omega)\Big(\int_\Omega |\nabla u|^2 \mathrm{d}x\Big)^{\frac{1}{2}}, \quad \forall\, u \in V. \tag{1.8}$$

The set $K$ of coefficients satisfying assumptions ($\mathrm{A}_1$) and ($\mathrm{A}_2$) is called the set of admissible coefficients for the inverse coefficient problems under consideration.

With respect to the given data we also assume that

$$f \in L^2(\Omega), \quad \varphi \in H^{\frac{1}{2}}(\Gamma_0), \quad \psi \in L^2(\Gamma_\sigma). \tag{1.9}$$

Let $V_0 = \{v \in V : \gamma v(x) \le \varphi(x),\, x \in \Gamma_0\}$, which is a closed convex subset in $V$.

Define operator $A : V \to V^*$ and a bounded linear functional $\ell$ on $V$ as follows:

$$\langle Au, v\rangle_V = \int_\Omega k(|\nabla u|^2)\nabla u \nabla v \mathrm{d}x, \quad \ell(v) = \int_\Omega f(x)v(x)\mathrm{d}x + \int_{\Gamma_\sigma} \psi(x)\gamma v(x)\mathrm{d}x.$$

Then it is easy to see that $u \in V_0$ is a weak solution of problem (1.1) means that there is a $w(x) \in \partial G|_V(u)$ such that the following hemivariational inequality holds

$$\langle Au + w, v - u\rangle_V \ge \ell(v - u), \quad \forall\, v \in V_0. \tag{1.10}$$

If $G \equiv 0$ in (1.1), the classical form of (1.1) is known as the Signorini problem of elasticity. This describes the equilibrium position of an elastic body which is supposed at its boundary by a rigid frictionless constraint surface. For more details of the deformation theory of plasticity, we refer to [1, 16]. Let us introduce the set $\Gamma_c = \{x \in \Gamma_0 : v(x) = \varphi(x)\}$ and assume that $\Gamma_c$ is a nonempty, simply connected subset of $\Gamma_0$. Note that the main distinction of the elliptic variational inequality (1.1) is that the part $\Gamma_c$ of the boundary is unknown in advance and for this reason problem (1.1) is nonlinear even for a linear elliptic operator. If $\Gamma_c$ were known, the solution could be more easily obtained by solving the corresponding mixed Dirichlet-Neumann boundary value problem. But finding this unknown $\Gamma_c$ is the essential problem here, analogous to various problems with a "free boundary".

In this paper, we are interested in the inverse problem consisting of the recovery of the coefficient $k$ from some class of admissible coefficients $K$ by using a measured data on the part of the boundary $\partial\Omega$. Let $\Gamma_1 \subset \partial\Omega$ be an accessible part of the boundary and $\Gamma_1 \cap \Gamma_c = \emptyset$, $\Gamma_1 \cap \Gamma_u = \emptyset$, $\operatorname{meas}\Gamma_1 \ne 0$. Assume that

$$\gamma u(x) = g(x), \quad x \in \Gamma_1 \tag{1.11}$$

is a given Dirichlet-type measured data. Then the problem of finding the coefficient $k$ from (1.1) and (1.11) we denote as an inverse coefficient problem (the ICP) with Dirichlet data (1.11) for the nonlinear hemivariational inequality (1.10).

The determination of unknown coefficients in variational inequalities from overspecified data measured on the boundary is a problem of some importance in applied mathematics. Such so-called inverse coefficient problems (ICPs) arise naturally, for example, in modeling nonlinear diffusion and flow in porous media. Direct measurement of the quantities represented by the unknown coefficients often requires very difficult physical experiments. The point of the inverse problems is to replace a difficult physical experiment by a mathematical problem for which the input is easy to measure. The ease of measurement requirement suggests that the data be measured on the boundary.

ICPs for partial differential equations have been studied by many authors (see [4, 5, 7, 8, 12, 17]). ICPs for elliptic hemivariational inequalities were considered by Migorski and Ochal [14, 15].

For a given coefficient $k = k(s)$, we sometimes call problem (1.1) (or (1.10)) as the direct problem (DP). Denote the solution of DP by $u[x;k]$. Then from the additional condition (1.11), it is seen that the ICP1 consists of solving the following nonlinear functional equation $\gamma u[x;k] = g(x),\ x \in \Gamma_1$ for a given data $g = g(x)$, over the solution $u = u[x;k]$ of the elliptic hemivariational inequalities (1.10).

In applications, instead of the measured data (1.11) on the boundary, one may get the nonlocal measured data

$$\int_{\Omega_1} u(x)\mathrm{d}x = \Phi, \tag{1.12}$$

where $\Omega_1 \subseteq \Omega$ is a given set. In this case we define the problem of determining a solution $k = k(s)$ of nonlinear functional equation $\int_{\Omega_1} u[x;k]\mathrm{d}x = \Phi$ over the solution of the DP as a nonlocal inverse coefficient problem.

In the practical solution of such ICPs, instead of solving, for example, the functional equations above, one may usually try to find the solutions of the minimization problems

$$I_1(k) = \min_{\widetilde{k}\in K} I_1(\widetilde{k}), \tag{1.13}$$

where $I_1(\widetilde{k}) = \int_{\Gamma_1} |\gamma u[x;\widetilde{k}] - g(x)|^2\mathrm{d}x$ is an auxiliary functional, $K$ is a set of admissible coefficients. According to [18], a solution of the minimization problems (1.13) is called a quasisolution of the ICP1.

For the ICP2, a quasisolution can be defined as a solution of the following minimization problems

$$I_2(k) = \min_{\widetilde{k}\in K} I_2(\widetilde{k}), \tag{1.14}$$

where $I_2(\widetilde{k}) = |\int_{\Omega_1} u[x;\widetilde{k}]\mathrm{d}x - \Phi|$.

## 2 The Inverse Coefficient Problems

To formulate our main results, let us recall that a multivalued operator $T$ is said to be bounded if it maps bounded sets into bounded sets. We say that $T : V \to 2^{V^*}$ is pseudomonotone if and only if the following three conditions are fulfilled:

(a) For each $u \in V$, the set $Tu$ be nonempty, bounded, closed and convex,

(b) The restriction of $T$ to any finite dimensional subspace $S$ of $V$ is weakly u.s.c. as an operator from $S$ to $V^*$,

(c) If $\{u_n\} \subset V$ such that $u_n \rightharpoonup u$ in $V$, $u_n^* \in T(u_n)$ $(n = 1, 2, \cdots)$ and $\limsup\limits_{n\to\infty}\langle u_n^*, u_n - u\rangle_V \le 0$, then for each $v \in V$ there exists $u^*(v) \in T(u)$ with the property that

$$\liminf_{n\to\infty}\langle u_n^*, u_n - v\rangle_V \ge \langle u^*(v), u - v\rangle_V.$$

The first theorem we intend to prove is the following

**Theorem 2.1** *If $k \in K$, then the elliptic hemivariational inequality* (1.10) *has a unique solution $u \in V$. For any solution of the hemivariational inequalities of* (1.10)*, there exists a constant $c > 0$ (which is independent of $k \in K$) such that*

$$\|u\|_V \le c(1 + \|f\|_{L^2(\Omega)} + \|\psi\|_{L^2(\Gamma_\sigma)}). \tag{2.1}$$

**Proof** In virtue of Assumptions $(\mathrm{H}_1), (\mathrm{A}_1), (\mathrm{A}_2)$, we can easily show that the sum operator $A+\partial G|_V : V \to 2^{V^*}$ is coercive and pseudomonotone. Applying a well-known existence theorem for pseudomonotone operators, we readily obtain that the hemivariational inequality (1.10) has at least a solution in $V$ for any $k \in K$ (see, for example, [9–11, 13]).

For a given $k \in K$, let $u_1, u_2$ be two solutions to (1.10). Then there exist $w_i \in \partial G|_V(u_i),\ i = 1, 2$ such that

$$\begin{aligned} &\langle Au_1, u_2 - u_1\rangle_V + \langle w_1, u_2 - u_1\rangle_V \geq \ell(u_2 - u_1),\\ &\langle Au_2, u_1 - u_2\rangle_V + \langle w_2, u_1 - u_2\rangle_V \geq \ell(u_1 - u_2), \end{aligned}$$

which imply

$$\langle Au_2 - Au_1, u_2 - u_1\rangle_V + \langle w_2 - w_1, u_2 - u_1\rangle_V \leq 0. \tag{2.2}$$

From (1.5) we have $w_i \in \partial(G|_{L^2(\Omega)})(u_i),\ i = 1, 2$, which implies $w_i \in L^2(\Omega),\ i = 1, 2$. Hence

$$\langle w_i, v\rangle_V = \langle w_i, v\rangle_{L^2(\Omega)}, \quad \forall\, v \in V,\ i = 1, 2. \tag{2.3}$$

By use of (2.2) and (2.3), we obtain from $(\mathrm{H}_1)$ and $(\mathrm{A}_2)$ that

$$\begin{aligned} 0 &\geq \langle Au_2 - Au_1, u_2 - u_1\rangle_V + \langle w_2 - w_1, u_2 - u_1\rangle_{L^2(\Omega)}\\ &\geq \int_\Omega [k(|\nabla u_2|^2)\nabla u_2 - k(|\nabla u_1|^2)\nabla u_1]\nabla(u_2 - u_1)\mathrm{d}x - m\|u_2 - u_1\|^2_{L^2(\Omega)}\\ &\geq c_3\|u_2 - u_1\|^2_V - m\|u_2 - u_1\|^2_{L^2(\Omega)}\\ &\geq (c_3 - mC(\Omega)^2)\|u_2 - u_1\|^2_V. \end{aligned} \tag{2.4}$$

Therefore, we have $u_1 = u_2$, which completes the proof of uniqueness.

Suppose that $u$ is a solution of (1.10). Then there exists $w \in \partial G|_V(u)$ such that for any fixed $v_0 \in V_0$,

$$\int_\Omega k(|\nabla u|^2)\nabla u\nabla(v_0 - u)\mathrm{d}x + \langle w, v_0 - u\rangle_V \geq \int_\Omega f(v_0 - u)\mathrm{d}x + \int_{\Gamma_\sigma} \psi(\gamma(v_0 - u))\mathrm{d}x. \tag{2.5}$$

Similarly, we have from (2.5) for any $w_0 \in \partial G(v_0)$ that

$$\begin{aligned} &(c_3 - mC(\Omega)^2)\|u - v_0\|^2_V\\ \leq &\int_\Omega [k(|\nabla u|^2)\nabla u - k(|\nabla v_0|^2)\nabla v_0]\nabla(u - v_0)\mathrm{d}x + \langle w - w_0, u - v_0\rangle_V\\ \leq &\int_\Omega k(|\nabla v_0|^2)|\nabla v_0||\nabla(u - v_0)|\mathrm{d}x + |\langle w_0, u - v_0\rangle_V|\\ &+ \int_\Omega f(u - v_0)\mathrm{d}x + \int_{\Gamma_\sigma} \psi(\gamma(u - v_0))\mathrm{d}x. \end{aligned} \tag{2.6}$$

By the Hölder inequality, we get from $(\mathrm{A}_1)$ that

$$\begin{aligned} &\int_\Omega k(|\nabla v_0|^2)|\nabla v_0||\nabla(u - v_0)|\mathrm{d}x + |\langle w_0, u - v_0\rangle_V|\\ \leq\ &c_2\int_\Omega |\nabla v_0||\nabla(u - v_0)|\mathrm{d}x + \|w_0\|_{V^*}\|u - v_0\|_V\\ \leq\ &C_1\|u - v_0\|_V, \end{aligned} \tag{2.7}$$

where the positive constant $C_1$ may depend on the norm of $v_0$ and its subdifferential $\partial G|_V(v_0)$. Similarly, we have

$$\Big|\int_\Omega f(u-v_0)\mathrm{d}x\Big| \le \|f\|_{L^2(\Omega)}\|u-v_0\|_{L^2(\Omega)} \le C(\Omega)\|f\|_{L^2(\Omega)}\|u-v_0\|_V. \tag{2.8}$$

By virtue of the boundedness of the trace operator $\gamma$ and the Hölder inequality again, we obtain

$$\begin{aligned}\Big|\int_{\Gamma_\sigma} \psi(\gamma(u-v_0))\mathrm{d}x\Big| &\le \|\psi\|_{L^2(\Gamma_\sigma)}\|\gamma(u-v_0)\|_{L^2(\Gamma_\sigma)}\\ &\le \|\psi\|_{L^2(\Gamma_\sigma)}\|\gamma\|\|u-v_0\|_V. \end{aligned}\tag{2.9}$$

From (1.7) and (2.6)–(2.9), we may choose a fixed element $v_0\in V_0$ and readily deduce (2.1). This completes the proof.

In the following text, we analyze the class of admissible coefficients and prove the coefficient stability and then obtain the main result — the existence theorem for the inverse problem. As seen above, the assumptions $(\mathrm{H}_1)$, $(\mathrm{A}_1)$ and $(\mathrm{A}_2)$ guarantee the solvability of the nonlinear DP in $V$. Therefore, to define a set of admissible coefficients for the ICPs under consideration, some conditions are already given. On the other hand, it is natural to endeavour to obtain a solution of any ICP with minimal requirements on the desired coefficient. Unfortunately, in many cases the given conditions (physical or mathematical, such as the DP solvability conditions $(\mathrm{A}_1)$ and $(\mathrm{A}_2)$) do not guarantee the compactness of the set of admissible coefficients in the suitable space. Therefore, the main problem is to construct a compact set of admissible coefficients with minimal additional conditions with respect to $k=k(s)$. Now we turn to the solvability of inverse coefficient problems. In order to obtain the existence theorems of quasisolutions for the inverse coefficient problems, we need the following result.

**Theorem 2.2** *Suppose that a sequence of coefficients $\{k_m\}\in K$ converges pointwise in $[0,\infty)$ to a function $k\in K$. Then the sequence of solutions $u_m=u(x;k_m)$ converges to the solution $u=u(x;k)$ in $V$.*

**Proof** Since $k, k_m\in K$ $(m=1,2.\cdots)$, by Theorem 2.1 the solutions $u, u_m$ $(m=1,2,3,\cdots)$ are well-defined. By the definition of solutions for (1.10), there exist $w\in\partial G|_V(u)$, $w_m\in\partial G|_V(u_m)$ such that

$$\int_\Omega k(|\nabla u|^2)\nabla u\nabla(u_m-u)\mathrm{d}x+\langle w,u_m-u\rangle_V \ge \int_\Omega f(u_m-u)\mathrm{d}x+\int_{\Gamma_\sigma}\psi(\gamma(u_m-u))\mathrm{d}x,$$

$$\int_\Omega k_m(|\nabla u_m|^2)\nabla u_m\nabla(u-u_m)\mathrm{d}x+\langle w_m,u-u_m\rangle_V \ge \int_\Omega f(u-u_m)\mathrm{d}x+\int_{\Gamma_\sigma}\psi(\gamma(u-u_m))\mathrm{d}x,$$

which imply

$$\begin{aligned}0 &\ge \int_\Omega [k_m(|\nabla u_m|^2)\nabla u_m - k(|\nabla u|^2)\nabla u]\nabla(u_m-u)\mathrm{d}x+\langle w_m-w,u_m-u\rangle_V\\ &= \int_\Omega [k_m(|\nabla u_m|^2)\nabla u_m - k_m(|\nabla u|^2)\nabla u]\nabla(u_m-u)\mathrm{d}x+\langle w_m-w,u_m-u\rangle_V\\ &\quad+\int_\Omega [k_m(|\nabla u|^2)-k(|\nabla u|^2)]\nabla u\nabla(u_m-u)\mathrm{d}x. \end{aligned}\tag{2.10}$$

Similarly to (2.4), we get from the Hölder inequality that

$$
\begin{aligned}
& (c_3 - mC(\Omega)^2)\|u_m - u\|_V^2 \\
\leq & \int_\Omega [k_m(|\nabla u_m|^2)\nabla u_m - k_m(|\nabla u|^2)\nabla u]\nabla(u_m - u)\mathrm{d}x + \langle w_m - w, u_m - u\rangle_V \\
\leq & \int_\Omega [k_m(|\nabla u|^2) - k(|\nabla u|^2)]\nabla u \nabla(u - u_m)\mathrm{d}x \\
\leq & \Big\{ \int_\Omega |k_m(|\nabla u|^2) - k(|\nabla u|^2)|^2 |\nabla u|^2 \mathrm{d}x \Big\}^{\frac{1}{2}} \|u_m - u\|_V,
\end{aligned}
$$

which implies

$$
(c_3 - mC(\Omega)^2)\|u_m - u\|_V \leq \int_\Omega |k_m(|\nabla u|^2) - k(|\nabla u|^2)|^2 |\nabla u|^2 \mathrm{d}x. \tag{2.11}
$$

By Assumption ($\mathrm{A}_1$), we have

$$
|k_m(|\nabla u|^2) - k(|\nabla u|^2)|^2 |\nabla u|^2 \leq (c_2 - c_1)^2 |\nabla u|^2. \tag{2.12}
$$

In virtue of (2.12), the assumption of the theorem and the Lebegue's dominated convergence theorem, we obtain $\lim\limits_{m\to\infty} \int_\Omega |k_m(|\nabla u|^2) - k(|\nabla u|^2)|^2 |\nabla u|^2 \mathrm{d}x = 0$. Therefore, by using inequality (2.11), it is easy to see that

$$
\lim_{m\to\infty} \|u_m - u\|_V = 0. \tag{2.13}
$$

The proof of the theorem is complete.

Next we study the existence of a quasisolution of the inverse problems ICP1 and ICP2. For this reason we need a compact set of coefficients and continuity of the functionals $I_1(k), I_2(k)$ defined in previous section, respectively. First we note the two assumptions ($\mathrm{A}_1$), ($\mathrm{A}_2$) that compose the set of admissible coefficients $K$ arise as solvability conditions for the problem DP. In virtue of Theorem 2.2, it is natural to construct a compactness set of admissible coefficients in $C[0,\infty)$. For this reason, in addition to Assumptions ($\mathrm{A}_1$), ($\mathrm{A}_2$), we assume that the subset $K_c$ of $K$ has the equicontinuity, i.e., $K_c \subset K$ and for every $\epsilon > 0$, there exists $\delta > 0$ such that if $\forall k \in K_c, s_1, s_2 \in [0,\infty)$ and $|s_1 - s_2| < \delta$, then $|k(s_1) - k(s_2)| < \epsilon$.

In the sequel, we also need the following generalized Ascoli-Arzela theorem.

**Theorem 2.3** *Let $K_c$ be an equicontinuous subset of $K$. Then for any sequence $\{k_m\}$ of coefficients in $K_c$, there exists a subsequence, still denoted by $\{k_m\}$, such that $\lim\limits_{m\to\infty} k_m(s) = k(s),\ \forall s \in [0,\infty)$ and $k \in K_c$.*

**Proof** The idea of the proof is similar to that of Ascoli-Arzela theorem. So we omit the detailed proof. For instance, one may refer to [17].

**Remark 2.1** Let $K_h$ be a uniformly Hölder continuous subset of $K$. Then $K_h$ is equicontinuous. Especially any subsets of $K$ which are bounded in $H^1([0,\infty))$ are equicontinuous.

Using the compactness of the class of admissible coefficients $K_c \subset K$, we can prove the following existence theorems for the problem ICPs.

**Theorem 2.4** *Both ICP1 and ICP2 have at least one quasisolution in the set of admissible coefficients $K_c$.*

**Proof** Let $\{k_m\} \subset K_c$ be a minimizing sequence of the functional $I_1$ on $K_c$ defined by (1.13). Due to Theorem 2.3, we may assume that $k_m(s) \to k(s)$, as $m \to \infty$, $\forall\, s \in [0, \infty)$. By using Theorem 2.2, the sequence $u_m = u(x; k_m)$ converges to $u = u(x; k)$ in $V$. Applying the trace theorem (see [6, Theorem 6.5]), we conclude that the sequence $\{u_m\}$ converges to $u$ in $L^2(\Gamma_1)$. Therefore, we have $\min\limits_{\widetilde{k} \in K_c} I_1(\widetilde{k}) = \lim\limits_{m\to\infty} I_1(k_m) = I_1(k)$. Similarly, we can get the existence of quaisolutions of $I_2$ on $K_c$. The proof is complete.

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