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# Exponential and Strong Ergodicity for Markov Processes with an Application to Queues**

Yuanyuan LIU* Zhenting HOU*

**Abstract** For an ergodic continuous-time Markov process with a particular state in its space, the authors provide the necessary and sufficient conditions for exponential and strong ergodicity in terms of the moments of the first hitting time on the state. An application to the queue length process of M/G/1 queue with multiple vacations is given.

**Keywords** Markov processes, Queueing theory, Exponential ergodicity, Strong ergodicity
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## 1 Introduction

Necessary and sufficient criteria of exponential and strong ergodicity for continuous-time Markov chains (i.e. continuous-time Markov processes on a countable space), based on the moments of the first hitting time, have been developed in [1, 2], while for continuous-time Markov processes on a general space, the given criteria (see [3, 4]) are sufficient but not necessary. In this paper, we aim to find the necessary and sufficient conditions for both forms of ergodicity for Markov processes with a particular state in their spaces, using different methods, and apply the results to the study of the queue length of the M/G/1 queue with vacations.

Throughout the paper, we denote by $R_+$ the non-negative real number set, $Z_+$ the non-negative integer set and $N_+$ the positive integer set. Let $(\Phi_t)_{t\in R_+}$ be a time-homogeneous continuous-time Markov process on a locally compact separable metric space $X$, endowed with the Borel $\sigma$-field $\mathcal{B}(X)$. We denote by $P(t,x,A),\ t\in R_+, A\in\mathcal{B}(X)$ the transition probability function of the Markov process:

$$P(t,x,A)=P_x[\Phi_t\in A]=E_x[I_{\{\Phi_t\in A\}}].$$

Here, $P_x$ and $E_x$ denote respectively the probability and expectation of the Markov process $\Phi_t$ under the initial condition $\Phi_0=x$. We write $P(t,x,x)=P(t,x,\{x\})$.

The Markov process $\Phi_t$ is said to be ergodic if there exists (the unique) invariant probability measure $\pi$ such that

$$\lim_{t\to\infty}\|P(t,x,\,\cdot\,)-\pi\|=0 \tag{1.1}$$

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*School of Mathematics, Railway Campus, Central South University, Changsha 410075, China.
E-mail: liuyy@csu.edu.cn zthou@csu.edu.cn
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for all $x \in X$, where $\|\cdot\|$ denotes the usual total variation norm; exponentially ergodic if it is ergodic, and there exists some $r > 0$ such that

$$\lim_{t\to\infty} \mathrm{e}^{rt}\|P(t,x,\,\cdot\,)-\pi\| = 0 \tag{1.2}$$

for $\pi$-a.e. $x \in X$; strongly ergodic if it is ergodic, and

$$\lim_{t\to\infty} \sup_{x\in X} \|P(t,x,\,\cdot\,)-\pi\| = 0. \tag{1.3}$$

To obtain our results, we shall make the following assumption on the Markov process.

**Assumption 1.1** *There exists a state $x_0$ such that whenever the Markov process $\Phi_t$ hits $x_0$, it will sojourn there for a random time that is positive and finite with probability* 1.

The assumption seems a little strong, but there are still plenty of Markov processes satisfying it, for example many queueing processes and all the totally stable continuous-time Markov chains. For a Markov process satisfying the assumption, due to the Markov property and the homogeneity, it can be easily proved that the sojourn time $T_{x_0}$ in $x_0$ is exponentially distributed with some parameter $\lambda$, $0 < \lambda < \infty$. Z. T. Hou, et al [5] first investigated subgeometric convergence for such a process. Roughly speaking, subgeometric convergence is a kind of convergence quicker than ordinary ergodicity and slower than exponential ergodicity. As its continuation, we study exponential and strong ergodicity in the paper.

## 2 Exponential and Strong Ergodicity

In this section, we study exponential and strong ergodicity for Markov processes in terms of its discrete-time skeleton chains. In the following, we first review some definitions and results of discrete-time Markov chains.

Let $\Phi_n$ be a discrete-time Markov chain on $X$ and define $\tau_x = \inf\{n \in N_+ : \Phi_n = x\}$ to be the first return time to the state $x$. The chain $\Phi_n$ is called ergodic, geometrically ergodic, and strongly ergodic if (1.1)–(1.3) hold for $t = n$, respectively. The following proposition states the known criteria of geometric and strong ergodicity, of which part (i) is from [6, Proposition 1] and part (ii) is from [7, Theorem 16.0.2].

**Proposition 2.1** *Let $\Phi_n$ be an ergodic Markov chain on $X$ with invariant probability measure $\pi$. Suppose that there exists a state $x_0 \in X$ such that $\pi_{x_0} > 0$. Then*

(i) *$\Phi_n$ is geometrically ergodic if and only if $E_{x_0}[\mathrm{e}^{\alpha\tau_{x_0}}] < \infty$ for some $\alpha > 0$,*

(ii) *$\Phi_n$ is strongly ergodic if and only if $\sup_{x\in X} E_x[\tau_{x_0}] < \infty$.*

For the Markov process $\Phi_t$, we define $\tau_x = \inf\{t > 0 : \Phi_t = x\}$ to be the first hitting time on $x$, and $\delta_x = \inf\{t > J_1 : \Phi_t = x\}$ to be the first return time to $x$, where $J_1$ is the first jump time of $\Phi_t$. Define $\tau_{x_0}(h) = h\inf\{n \in N_+ : \Phi_{nh} = x_0\}$ to be the first hitting time on $x_0$ of its skeleton chain $\Phi_{nh}$. Due to the convention, we adopt the notation $\tau_x$ for both discrete-time chains and continuous-time processes. The notation's meaning in the paper is clear and should not cause any confusion for understanding.

The following lemma reveals the relationship between the moments of the first hitting time of the Markov process $\Phi_t$ and those of its skeleton chain $\Phi_{nh}$, which plays a crucial role in proving Theorem 2.1.

**Lemma 2.1** *Suppose that the Markov process* $\Phi_t$ *satisfies Assumption* 1.1. *Then*

(i) $E_{x_0}[e^{r\delta_{x_0}}] < \infty$ *for some* $r > 0$ *if and only if* $E_{x_0}[e^{\alpha\delta_{x_0}(h)}] < \infty$ *for some* $\alpha > 0$ *and* $h > 0$;

(ii) $\sup\limits_{x\in X} E_x[\tau_{x_0}] < \infty$ *if and only if* $\sup\limits_{x\in X} E_x[\tau_{x_0}(h)] < \infty$ *for any* $h > 0$.

**Proof** Due to the assumption on the process, we know that $P(t, x_0, x_0) \geq e^{-\lambda t} > 0$ for all $t > 0$. We now prove that $P(t, x_0, x_0) < 1$ for all $t > 0$. We conversely assume that $P(\widehat{t}, x_0, x_0) = 1$ for some $\widehat{t} > 0$. Then for $s < \widehat{t}$, by Chapman-Kolmogorov equation we have

$$\begin{aligned} 0 &= P(\widehat{t}, x_0, X - \{x_0\}) \\ &= \int_X P(\widehat{t} - s, x_0, \mathrm{d}y) P(s, y, X - \{x_0\}) \\ &\geq P(\widehat{t} - s, x_0, x_0) P(s, x_0, X - \{x_0\}). \end{aligned}$$

Since $P(\widehat{t} - s, x_0, x_0) > 0$, it implies that $P(s, x_0, X - \{x_0\}) = 0$, so $P(s, x_0, x_0) = 1$. And for $s > \widehat{t}$, choose some $n$ such that $\frac{s}{n} < \widehat{t}$, we have $P(s, x_0, x_0) \geq [P(\frac{s}{n}, x_0, x_0)]^n = 1$. Hence, we get that $P(t, x_0, x_0) = 1$ for all $t > 0$, thus it conflicts Assumption 1.1.

(1) With the proved fact that $0 < P(t, x_0, x_0) < 1$ for all $t > 0$, we can prove part (i) by copying the proof of Lemma 6.2 in [1, Chapter 6], so we omit the proof.

(2) To prove (ii), we first prove the sufficiency. Suppose that $\sup\limits_{x\in X} E_x[\tau_{x_0}(h)] < \infty$ for any $h > 0$. It is possible that the skeleton chain $\Phi_{nh}$ can miss visits of the continuous-time process to $x_0$, and so result in $\tau_{x_0} \leq \tau_{x_0}(h)$. Hence we have $\sup\limits_{x\in X} E_x[\tau_{x_0}] < \infty$.

Next prove the necessity. Suppose that $\sup\limits_{x\in X} E_x[\tau_{x_0}] < \infty$. Then we have

$$E_{x_0}[\delta_{x_0}] = \int_{X\setminus x_0} P(J_1, x_0, \mathrm{d}y) E_y[\tau_{x_0}] + E_{x_0}[J_1] \leq \sup_{x\in X} E_x[\tau_{x_0}] + \frac{1}{\lambda} < \infty. \tag{2.1}$$

Assume that $\Phi_0 = x$. Once the process $\Phi_t$ arrives at $x_0$, it must stay at $x_0$ for a positive length, and then repeat leaving and returning infinitely. Let $D_k$ be the $k$th sojourn time in $x_0$ and $W_k$ be the length of the interval between the $k$th exit from $x_0$ and the next visit to $x_0$.

Note that $W_k$ are independent and that $D_k$ are independent of each other and the $W_k$. Moreover, $D_k$ are identically exponentially distributed with parameter $\lambda$. Define $N = \min\{n \geq 1 \mid \text{the } h\text{-skeleton is in state } x_0 \text{ during the interval } D_n\}$. Then we have

$$\begin{aligned} E_x[\tau_{x_0}(h)] &\leq E_x[\tau_{x_0}] + E_{x_0}\Big[\sum_{i=1}^{N-1}(D_i + W_i) + h\Big] \\ &\leq (h + E_x[\tau_{x_0}]) + \sum_{n=1}^{\infty} E_{x_0}\Big[\sum_{i=1}^{n-1}(D_i + W_i) I_{\{N=n\}}\Big] \\ &\leq (h + E_x[\tau_{x_0}]) + \sum_{n=1}^{\infty} E_{x_0}\Big[\sum_{i=1}^{n-1}(D_i + W_i) I_{\bigcap\limits_{k=1}^{n-1}\{D_k \leq h\}}\Big] \end{aligned}$$

$$\le (h+E_x[\tau_{x_0}]) + \sum_{n=1}^{\infty} E_{x_0}\Big[(n-1)h + \sum_{i=1}^{n-1} W_i\Big](1-e^{-\lambda h})^{n-1}$$
$$\le (h+E_x[\tau_{x_0}]) + \sum_{n=1}^{\infty}(n-1)(h+E_{x_0}[\delta_{x_0}])(1-e^{-\lambda h})^{n-1}. \tag{2.2}$$

From (2.1) and (2.2), we have that $\sup\limits_{x\in X} E_x[\tau_{x_0}(h)] < \infty$.

It was shown by [5] that the Markov process $\Phi_t$ is ergodic, subgeometrically convergent if and only if so is any skeleton chain $\Phi_{nh}$ of $\Phi_t$. Combing this fact with the following lemma, we can say that the Markov process $\Phi_t$ has almost the same convergence as any skeleton chain of $\Phi_t$.

**Lemma 2.2** *The Markov process $\Phi_t$ is exponentially* (*resp. strongly*) *ergodic if and only if any skeleton chain $\Phi_{nh}$ of $\Phi_t$ is geometrically* (*resp. strongly*) *ergodic.*

**Proof** The necessity is obvious. In fact, if $\Phi_t$ is exponentially (resp. strongly) ergodic, by putting $t = nh$ in (1.2) (resp. (1.3)), then we get that the skeleton chain $\Phi_{nh}$ is geometrically (resp. strongly) ergodic.

For the sufficiency, [6, Theorem 1] has shown that if any skeleton chain $\Phi_{nh}$ of $\Phi_t$ is geometrically ergodic, then $\Phi_t$ is exponentially ergodic. Similarly, we can prove that if the skeleton chain $\Phi_{nh}$ of $\Phi_t$ is strongly ergodic, then so is $\Phi_t$.

We are now in a position to establish our main result.

**Theorem 2.1** *Suppose that the Markov process $\Phi_t$ is ergodic and satisfies Assumption* 1.1. *Then*

(i) $\Phi_t$ *is exponentially ergodic if and only if* $E_{x_0}[e^{r\delta_{x_0}}] < \infty$ *for some* $r > 0$;

(ii) $\Phi_t$ *is strongly ergodic if and only if* $\sup\limits_{x\in X} E_x[\tau_{x_0}] < \infty$.

**Proof** (i) In the proof Theorem 2.1 of [5], we have shown that $\pi_{x_0} > 0$. Suppose that $\Phi_t$ is exponentially ergodic. By Lemma 2.2, we know that any skeleton chain $\Phi_{nh}$ of $\Phi_t$ is geometrically ergodic, and by Proposition 2.1 we know that there exists some $\alpha > 0$ such that $E_{x_0}[e^{\alpha\delta_{x_0}(h)}] < \infty$. Hence, it follows from Lemma 2.1 that there exists some $r > 0$ such that $E_{x_0}[e^{r\delta_{x_0}}] < \infty$.

Conversely, suppose that $E_{x_0}[e^{r\delta_{x_0}}] < \infty$ for some $r > 0$. By Lemma 2.1 we know that there exists some $\alpha > 0$, $h > 0$ such that $E_{x_0}[e^{\alpha\tau_{x_0}(h)}] < \infty$, and by Proposition 2.1 and Lemma 2.2 we know that $\Phi_t$ is exponentially ergodic.

(ii) Following the same lines as the proof of part (i), we easily have that part (ii) holds from Proposition 2.1, Lemmas 2.1 and 2.2.

**Remark 2.1** Part (i) of Theorem 2.1 is not an entirely new result. The sufficiency of part (i) can also be proved with different methods. (For more details, see [4, Theorem 5.2 and Theorem 6.2].)

## 3 Length of the M/G/1 Queue with Multiple Vacations

There is much literature (e.g. [8, 9]) on M/G/1 queues with vacations. These queues include, for example the M/G/1 queue with step-up time, with $N$-policy, with single vacation, or with multiple vacations. In this section, we only study the most complicated case: the M/G/1 queue with multiple vacations, which is denoted simply by M/G/1(E, MV) (see e.g. [10–12]), and the corresponding results for the other queues can be easily obtained by the same method.

M/G/1(E, MV) is gotten by introducing the strategy of exhaustive service and multiple vacations to the classical M/G/1 queue: once the system has no customers, the server begins a vacation of random length $V$ immediately. If, when the vacation ends, the system still has no customers, then the server continues with further independent, identically distributed vacations that do not end until the system has customers queueing when a vacation ends. Here $V$ is always assumed to be a non-negative random variable, with distribution function $V(x)$, that has finite first moment, i.e., $E[V] < \infty$. For M/G/1(E, MV), the customers arrive according to a Poisson process with the parameter $\lambda$, $0 < \lambda < \infty$ and the service time $B$ has a general distribution $B(x)$.

Let $Q_b$ be the number of customers in the system when one busy period begins. Then

$$P[Q_b = j] = \frac{v_j}{1 - v_0}, \quad j \in N_+,$$

where

$$v_j = \int_0^\infty \frac{(\lambda t)^j}{j!} e^{-\lambda t} \mathrm{d}V(t), \quad j \in Z_+$$

is the probability that $j$ customers join the queue during a vacation. Denote by $D_v$ the busy period of M/G/1(E, MV) and by $D$ the busy period of the classical M/G/1 queue. It is easy to see that

$$\{D_v \mid Q_b = k\} = \{D_1 + D_2 + \cdots + D_k\}, \tag{3.1}$$

where $D_k$ is the busy period of the classical M/G/1 queue caused by the $k$th customer and $D_i$'s are independent and identically distributed. Note that $D_k$ has the same distribution as $D$. Let $J$ be the number of vacations during a series of consecutive vacations. Then

$$P[J = j] = v_0^{j-1}(1 - v_0), \quad j \in N_+.$$

Let $V_v$ be the vacation period of M/G/1(E, MV). Then

$$\{V_v \mid J = j\} = \{V_1 + V_2 + \cdots + V_j\}, \tag{3.2}$$

where $V_i$ denotes the $i$th vacation and the $V_i$'s are independent and identically distributed. Define

$$\frac{1}{\mu} = \int_0^\infty x \,\mathrm{d}B(x) \quad \text{and} \quad \rho = \frac{\lambda}{\mu}.$$

Let $L_t$ be the queue length process of M/G/1(E, MV). It is known that $L_t$ is not a Markov process unless $B(x)$ is exponentially distributed. We introduce a supplementary variable as follows:

$$\theta_t = \text{the elapsed service time of the customer being served at time } t$$
$$(= 0 \text{ if the server is idle at time } t).$$

Then $(L_t, \theta_t)$ becomes a continuous-time Markov process on the two-dimensional state space $X = Z_+ \times R_+$. It is easy to see that when $(L_t, \theta_t)$ hits the state $(0, 0)$, it will stay there for a random length which is exponentially distributed with the parameter $\lambda$, so $(L_t, \theta_t)$ satisfies Assumption 1.1 with $x_0 = (0, 0)$. Several types of ergodicity for the discrete-time embedded chain of $L(t)$ were studied in [13], and polynomial convergence for $(L_t, \theta_t)$ was investigated in [5]. By [5, Theorem 3.1], we know that $(L_t, \theta_t)$ is ergodic if and only if $\rho < 1$.

For a given constant $r > 0$, denote by $\mathcal{G}^+(r)$ the set of all distributions such that

$$\int_0^\infty \mathrm{e}^{rx} \mathrm{d}F(x) < \infty,$$

and by $\mathcal{G}^+$ the set of all nonnegative distributions with finite exponential moments, i.e.

$$\mathcal{G}^+ = \bigcup_{r>0} \mathcal{G}^+(r).$$

**Lemma 3.1** *Suppose that $\rho < 1$ for the classical* M/G/1 *queue. Then its busy period distribution $D(x) \in \mathcal{G}^+$ if and only if its service time distribution $B(x) \in \mathcal{G}^+$.*

**Proof** Let $W(t)$ be the virtual waiting time of the classical M/G/1 queue. Then $W(t)$ is a Markov process satisfying Assumption 1.1 and the state 0 is the particular state $x_0$. From [6, Theorem 2], we know that $W(t)$ is ergodic if and only if $\rho < 1$. Moreover, $W(t)$ is exponentially ergodic if and only if $B(x)$ is in $\mathcal{G}^+$. Now suppose that $\rho < 1$. Then it follows from Theorem 2.1 that $W(t)$ is exponentially ergodic if and only if $E_0[\mathrm{e}^{r\delta_0}] = E[\mathrm{e}^{rD}] < \infty$, or equivalently, $D(x)$ is in $\mathcal{G}^+$. Hence, $D(x)$ is in $\mathcal{G}^+$ if and only if $B(x)$ is in $\mathcal{G}^+$.

**Remark 3.1** Lemma 3.1 has been obtained with a different method (see, e.g. [14]), which is important in the queue literature. Here, we display a new and short proof of it.

**Theorem 3.1** *Suppose that $(L_t, \theta_t)$ is ergodic. Then*
(i) *$(L_t, \theta_t)$ is exponentially ergodic if and only if both $V(x)$ and $B(x)$ are in $\mathcal{G}^+$,*
(ii) *$(L_t, \theta_t)$ is not strongly ergodic.*

**Proof** (i) If both $V(x)$ and $B(x)$ belong to $\mathcal{G}^+$, then there exists some $r > 0$, such that

$$E[\mathrm{e}^{rB}] < +\infty \quad \text{and} \quad E[\mathrm{e}^{rV}] < +\infty.$$

Since $E[\mathrm{e}^{rB}] < +\infty$, it implies from Lemma 3.1 that there exists some $r_1 > 0$, such that $E[\mathrm{e}^{r_1 D}] < \infty$. Thus the functions $E[\mathrm{e}^{sD}]$ and $E[\mathrm{e}^{sV}]$ are continuous in $s$ when $0 \le s \le \min\{r, r_1\}$, so we can choose an appropriate $r_2$ that is greater than, but sufficiently close to, 0 such that

$$\lambda(E[\mathrm{e}^{r_2 D}] - 1) < r, \quad v_0 E[\mathrm{e}^{r_2 V}] < 1.$$

Thus from (3.1), we have

$$
\begin{aligned}
E[\mathrm{e}^{r_2 D_v}] &= \sum_{k=1}^{\infty} P\{Q_b = k\} E[\mathrm{e}^{r_2(D_1+D_2+\cdots+D_{Q_b})} \mid Q_b = k] \\
&= \sum_{k=1}^{\infty} \frac{v_k}{1-v_0} (E[\mathrm{e}^{r_2 D}])^k \\
&= \sum_{k=1}^{\infty} \frac{(E[\mathrm{e}^{r_2 D}])^k}{1-v_0} \int_0^{\infty} \frac{(\lambda t)^k}{k!} \mathrm{e}^{-\lambda t} \mathrm{d}V(t) \\
&\le \frac{1}{1-v_0} \int_0^{\infty} \mathrm{e}^{\lambda(E[\mathrm{e}^{r_2 D}]-1)t} \mathrm{d}V(t) \\
&< \infty,
\end{aligned}
\tag{3.3}
$$

and from (3.2), we get

$$
\begin{aligned}
E[\mathrm{e}^{r_2 V_v}] &= \sum_{j=1}^{\infty} P\{J = j\} E[\mathrm{e}^{r_2(V_1+V_2+\cdots+V_J)} \mid J = j] \\
&= \sum_{j=1}^{\infty} (1-v_0) v_0^{j-1} (E[\mathrm{e}^{r_2 V}])^j \\
&< \infty.
\end{aligned}
\tag{3.4}
$$

Hence,

$$
E_{(0,0)}[\mathrm{e}^{r_2 \delta_{(0,0)}}] = E_{(0,0)}[\mathrm{e}^{r_2(D_v+V_v)}] = E[\mathrm{e}^{r_2 V_v}] E[\mathrm{e}^{r_2 D_v}] < \infty, \tag{3.5}
$$

and by Theorem 2.1 we see that $(L_t, \theta_t)$ is exponentially ergodic.

On the other hand, if $(L_t, \theta_t)$ is exponentially ergodic, then by Theorem 2.1 we know that for some $r > 0$,

$$
E_{(0,0)}[\mathrm{e}^{r \delta_{(0,0)}}] < \infty.
$$

We get from (3.3)–(3.5) that both $V(x)$ and $B(x)$ are in $\mathcal{G}^+$.

(ii) Since

$$
\sup_{x \in Z_+ \times R_+} E_x[\delta_{(0,0)}] \ge \sup_{i \in Z_+} E_{(i,0)}[\delta_{(0,0)}] \ge \sup_{i \in Z_+} iE[D] = \infty,
$$

it follows from Theorem 2.1 that $(L_t, \theta_t)$ is not strongly ergodic.

**Remark 3.2** Combing Theorem 3.1 with [5, Theorem 3.3], we know that exponential (resp. polynomial) moments of $V$ and $B$ determine the corresponding convergence of $(L_t, \theta_t)$. It should be noted that usually queue length processes are not strongly ergodic.

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