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REGULAR ARTICLE

Dispersion of growth paths of macroeconomic models in thermodynamic limits: two-parameter Poisson–Dirichlet models

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Abstract This paper discusses dispersion of growth patterns of macroeconomic models in thermodynamic limits. More specifically, the paper shows that the coefficients of variations of the total numbers of clusters and the numbers of clusters of specific sizes of one- and two-parameter Poisson–Dirichlet models behave qualitatively differently in the thermodynamic limits. The coefficients of variations of the numbers of clusters in the former class of distributions are all self-averaging, while the those in the latter class are all non-self averaging. In other words, dispersions or variations of growth rates about the means do not vanish in the two-parameter version of the model, while they do in the one-parameter version in the thermodynamic limits. The paper ends by pointing out other models, such as triangular urn models, may converge to Mittag–Leffler distributions which exhibit non-self-averaging behavior for certain parameter combinations.

Keywords Non-self averaging · Coefficients of variations · Poisson–Dirichlet distributions · Mittag–Leffler distributions · Power laws

1 Introduction

In a book entitlted *Growth Theory* and in a more recent paper, Solow (2000, 2004) has expressed his misgivings about the common practice by the mainstream growth economists of focussing almost exclusively on exponential growth rates, and wondered about their adverse influences on growth policy.

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In short, he apparently feels that growth economists have centered their attention on steady-state exponential growth, and that they made special assumptions for convenience to gauarantee the existence of exponential steady states. He then worries that the set of their assumptions has become standard as if they have some independent validation for policy makers to speak of their intention of raising the growth rate. According to him, the very vocabulary of growth policy becomes identified with moving the mean (exponential) growth rates. He condems this pattern as being unnecessary and dysfunctional both for theory and for policy.[1]

Why should economists limit their attention to only exponential growth models? Recent developments in growth theory covering human capital, endogenous technology, new consumer goods and Schumperterian ideas, all elucidated in Chaps 8 through 11 in Solow (2000), suggest to this author at least that stochasic analysis of formations and dissolutions of clusters of economic agents or other resources, and their dynamics or evolution of interactions and growth processes are needed, even though their dynamics may not be exponential.

There are disciplines which deal with non-exponential growths. For example, Ewens (1972) and Kingman (1978, 1980) have examined non-exponential growth patterns in population genetics by focussing on formations or dissolutions of clusters or partitions of agents of different types. In statistical physics literature Higgs (1995), Mekjian (1991), and Mekjian and Chase (1997) have noted similarities of cluster distributions in physics with those in population genetics.

This paper uses two classes of stochastic models, known as Poisson–Dirichlet one- and two-parameter processes in the population genetics literature, to model formations of new clusters (sectors) due to inventions or innovations, and growth of existing sectors by incorporating new arrivals as growth.

In other words, we model growth of existing sectors (firms) and creation of new sectors (firms) as the Poisson–Dirichlet processes. We then show that these two types of models have qualitatively different patterns of dispersions around the means when measured by means of coefficients of variation.

This notion is commonly used in econometrics but less so in growth literature. We find that this concept is useful in discussing how policies may affect not just the means but also dispersions of growth paths, as suggested by some preliminary esxamination of examples in Aoki (2002, Sects. 7.4, 8.6) and Aoki and Yoshikawa (2007a, Chaps. 6, 7) suggest. See also Aoki (2008).

One parameter Poisson–Dirichlet models are well behaved in the sense that the models' coefficients of variation tend to zero as model sizes grows unboundedly.[2] The coefficients of variation of the two-parameter models, on the other hand, do not go to zero in the thermodynamic limits. This distinction implies that growth patterns of the two-parameter Poisson–Dirichlet models are more unpredictable, and history-dependent, than those of the one-parameter models. In the two-parameter version effects in changes in growth paths, intentional or accidental, will affect future growth patterns.

[1] Aoki and Yoshikawa (2007a) has cosidered one growth model that escapes his criticism, since it does not have a constant exponential growth rate as time goes to infinity.

[2] This manner of taking limits is called thermodynamic limits in physics literature.

The paper is organized as follows: Poisson–Dirichlet models are introduced first. We describe how an arrival of innovations initiates a cluster (of size one initially). An agents of known type, that is, not new, joins one of the already exisiting clusters, thus increasing the cluster size by one. The number of total clusters suitably normalized converges to known distribution. In this context the family of Mittag–Leffler probability densities and their generalization are introduced. They generalize the well-known exponential density, and appear in models of sluggish reponse patterns. A short summary of Mittag–Leffler probability density is in Appendix.

2 Poisson–Dirichlet models: a non-exponential growth model example

Agents or factors of production of different characteristics or strategies belong to different types and form separate clusters (firsm, sectors). These clusters jointly affect aggregate behavior.

Kingman (1978) invented the one-parameter Poisson–Dirichlet distribution to describe random partitions of populations of heterogeneous agents into distinct clusters. Models of this class are also known as Ewens models. See Aoki (2000a,b) for further explanation.

This one-parameter model was extended to a two-parameter version by Pitman. See Kingman (1993), Carlton (1999), Feng and Hoppe (1998), Pitman (1999), Pitman (2002), and Pitman and Yor (1996), among others.[3]

If the coefficient of variation of an extensive random variable X does not approach zero but goes to some positive number or tends towards infinity as the size of "clusters" or model becomes very large, then the behavior of X remains sample-dependent even when the sample size approaches infinity.[4]

This paper shows that variables in the two-parameter Poisson–Dirichlet model, denoted by $PD(\alpha, \theta)$, with positive α less than 1, and $\theta + \alpha > 0$ have non-vanishing coefficients of variations as the number of samples approaches infinity, while the corresponding variables in the one-parameter Poisson–Dirichlet distribution with $\alpha = 0$, denoted by $PD(\theta)$, also known as Ewens model, does not. The former distribution is called non-self averaging, while the latter is self-averaging.

2.1 Cluster generating processes

Given n basic units (called agents for short), let K_n be the number of clusters formed by n agents. When a new agent arrives on the scene, it either forms a new cluster of size 1, or join one of the existing K_n clusters. Even though these models are not deterministic exponential growth models familiar to economists, the number of clusters grow in general. We examine dispersion around the mean path of the number of clusters, suitably normalized.

[3] In physics literature, Mekjian and Chase (1997) have used two-parameter models. They refer to Pitman (1996).

[4] The square of the coefficient of variation is called the measure of non-self averaging in the physics literature, Sornette (2000).

In the two-parameter version, an entering agent either joins a cluster of size n_i with rate

$$p_i = \frac{n_i - \alpha}{n + \theta},$$

$i = 1, \ldots, k$, where $K_n = k$, $n_i > 0$, and θ is a positive number, and $0 < \alpha < 1$, or a new agent creates a new cluster (with initial size 1) with rate

$$1 - \sum_{i=1}^{k} p_i = \frac{\theta + k\alpha}{n + \theta},$$

where $\sum_{i=1}^{k} n_i = n$ and $K_n = k$ is the number of existing clusters, and where α is positive, less than $1, \theta > 0$.

We can state the above succinctly as

$$\Pr(K_{n+1} = k + 1 | K_1, K_2, \ldots, K_n = k) = \frac{\theta + k\alpha}{n + \theta},$$

and

$$\Pr(K_{n+1} = k | K_1, K_2, \ldots, K_n = k) = \frac{n - k\alpha}{n + \theta}.$$

From these we obtain the basic recursion formula for the expected value of the number of clusters after n agents have arrived:

$$E(K_{n+1}) = \frac{\theta}{n + \theta} + \left\{ 1 + \frac{\alpha}{n + \theta} \right\} E(K_n). \tag{1}$$

Straightforward calculations show that the correlation between K_n and K_{n+1} are positive even when α is zero. With positive α the correlation is increased. After we introduce coefficient of variations next, we show that the coefficient of variation, however, is zero with zero α and positive only with non-zero α. We show this after we formally define the coefficient of variation next.

3 Coefficient of variation: a measure of dispersion or uncertainty

The coefficient of variation of a random variable X, denoted by $\mathrm{cv}(X)$, is defined by

$$\mathrm{cv}(X) = \frac{\sqrt{\mathrm{variance}(X)}}{\mathrm{mean}(X)}.$$

To be concrete we use the number of clusters as X, which is an extensive random variable[5] in this paper.

[5] A variable is extensive if it scales with the "size" of the model.

Some extensive random variable X of the model, such as the number of sectors, is called non-self averaging if it has the coefficient of variation that does not converge to zero as model size goes to infinity, i.e., in the thermodynamic limit.

4 Asymptotic behavior of cluster sizes

Let K_n be the number of clusters after n agents entered the model.

4.1 $PD(\theta)$ models

First, we discuss the case with $\alpha = 0$. This class of models is called Ewens models. It is known that

$$\frac{K_n - \theta \log(n)}{\sqrt{\theta \log(n)}} \to N(0, 1).$$

that is,

$$E(K_n) = \theta \log(n),$$

and

$$\text{var}(K_n) = \theta \log(n).$$

See Carlton (1999), or Pitman (2002, p. 69) for example.

Hence, this class of model has vanishing coefficient of variations as model sizes grow unboundedly:

$$\text{cv}(K_n) = (\theta \log(n))^{-1/2} \to 0,$$

as n tends to infinity in the Ewens model. This model is therefore self-averaging.

4.2 $PD(\alpha, \theta)$ models

Solving the recursion equation (1), we obtain

$$E(K_n) = \frac{\theta}{\alpha} \left\{ \frac{(\theta + \alpha)^{[n]}}{\theta^{[n]}} - 1 \right\}, \tag{2}$$

where we use the notation $x^{[j+1]} := x^{[j]}(x + j)$ for positive integer j and a positive number x.

Note that

$$\frac{(\theta + \alpha)^{[n]}}{\theta^{[n]}} = \frac{\Gamma(\theta)}{\Gamma(\theta + \alpha)} \frac{\Gamma(\theta + \alpha + n)}{\Gamma(\theta + n)}. \tag{3}$$

We denote the asymptotic equality of two sequences by $a_n \asymp b_n$. It means that the ratio a_n/b_n goes to 1 as n tends to infinity.

Substituting (3) into (2), and from the asymptotic expression of Gamma function in Abramovitz and Stegun (1968),

$$\frac{\Gamma(n+\alpha)}{\Gamma(n)} \asymp n^{\alpha},$$

we have

$$E\left(\frac{K_n}{n^{\alpha}}\right) \asymp \frac{\Gamma(\theta+1)}{\alpha\Gamma(\theta+\alpha)}.$$

See Yamato and Sibuya (2000).

Yamato and Sibuya also obtained the asymptotic expression of the variance of K_n/n^{α} to be

$$\operatorname{var}(K_n/n^{\alpha}) \asymp \frac{\Gamma(\theta+1)}{\alpha^2}\gamma_{\alpha,\theta},$$

where

$$\gamma_{\alpha,\theta} := \frac{\theta+\alpha}{\Gamma(\theta+2\alpha)} - \frac{\Gamma(\theta+1)}{\Gamma(\theta+\alpha)^2}.$$

Note that $\gamma_{\alpha,\theta}$ vanishes for $\alpha = 0$.

We thus deduce the thermodynamic limit is

$$\operatorname{cv}\left(\frac{K_n}{n^{\alpha}}\right) \to \Gamma(\theta+\alpha)\sqrt{\frac{\gamma(\alpha,\theta)}{\Gamma(\theta+1)}} > 0. \tag{4}$$

This expression is simplified to

$$\operatorname{cv}(K_n/n^{\alpha}) \asymp \sqrt{\frac{\alpha}{\theta}} + o(\alpha). \tag{5}$$

We record this as

Proposition *The two-parameter Poisson–Dirichlet model is non-self averaging.*

4.3 The partition vector **a**

For simpler presentation we have just discussed the random variable K_n, even though the components of the partition vector, i.e., the number of clusters of size j, denoted by a_j, and the total size of clusters of size j, ja_j can be analogously treated.

Components of partition vector **a** has expected value

$$E(a_j) = \frac{n!}{j!(n-j)!} \frac{(\theta+\alpha)^{[n-j]}}{(1-\alpha)^{[j-1]}(\theta+1)^{[n-1]}}.$$

We can show that

$$\frac{a_j(n)}{K_n} \to \frac{\alpha}{j!} P_{\alpha,j},$$

a.s., where

$$P_{\alpha,j} = \frac{\Gamma(j-\alpha)}{\Gamma(1-\alpha)}.$$

4.4 Mittag–Leffler distributions

Yamato and Sibuya denote by μ'_r the limit of $E\left(\frac{K_n}{n^\alpha}\right)^r$, as n tends to infinity, for $r = 1, 2, \ldots$, and noted that μ'_r is the r moment of the generalized Mittag–Leffler distribution with density

$$g_{\alpha,\theta} := \frac{\Gamma(\theta+1)}{\Gamma(\theta/\alpha+1)} x^{\frac{\theta}{\alpha}} g_\alpha(x),$$

where $\theta/\alpha > -1$, and where $g_\alpha(x)$ is the Mittag–Leffler (α) density function. See Appendix for the expression of $g_\alpha(\cdot)$. See also Erdélyi (1955), or Pitman (2002).

Its moments are given by

$$\int_0^\infty x^p g_\alpha(x) dx = \frac{\Gamma(p+1)}{\Gamma(p\alpha+1)},$$

for all $p > -1$.

Generally speaking, the fact that all moments of two distributions defined on infinite domain $[0, \infty)$ match does not imply that the distributions are the same. There is, however, a sufficient condition on the moments that the distribution functions are uniquely determined by the equalities of all the moments. This condition is satisfied for the problem at hand.[6]

There is another example in which Mittag–Leffler distributions appear as limiting distributions, and for which non-self averaging phenomena are observed for some model parameter combinations. One class of such problems are that of triangular urn models, in which two types (colors) of balls are involved and the rule for extracting balls and replacing them after examining the color of the balls is expressed by 2 by 2 triangular matrix.

[6] See Bingham et al. (1987) for example.

See Fabritiis et al. (2003), Janson (2006), Puyhaubert (2005). Under certain parameter combinations, the triangular urn models they examine all exhibit limiting distributions that are Mittag–Leffler distributions. Therefore, we can verify conditions for the existence of non-self averaging behavior just as we have done in this paper, even though some authors may not be aware that their urn models have Mittag–Leffler distributions as limiting distributions.

See Appendix, and also Blumenfeld and Mandelbrot (1997) who credit Feller (1949) as the original source.

5 Potential applications: waiting time distributions

It is known that Mittag–Leffler functions generically appear in situations where Darling–Kac theorem applies. See Bingham et al. (1987). For example waiting time distribution problems in the econo-physics literature are such examples. Waiting time situations arise also in macroeconomics. For example, the entry and exit problem discussed by Dixit (1989) in exchange rate pass-through can be phrased more correctly as waiting time problem.

In view of these results, we conjecture that the model of this paper can be used with minor changes to analyze effects of various growth policies to determine how they affect growth patterns, and characterize their effects in terms of the coefficients of variation, for example.

6 Concluding discussions

In physics phenomena with non-vanishing coefficients of variation abound. Derrida (1994) is one example. In traditional microeconomic foundations of economics, one deals almost exclusively with well-posed optimization problems for the representative agents with well defined peaks and valleys of the cost functions. It is also taken for granted that as the number of agents goes to infinity, any unpleasant fluctuations vanish and well defined deterministic macroeconomic relations prevail. In other words, non-self-averaging phenomena are not in the mental pictures of average macro- or microeconomists.

We know, however, that in problems where agents must solve some combinatorial optimization problems, this nice picture may disappear. In the limit of the number of agents going to infinity some results remain sample-dependent and deterministic results will not follow. Some of this type of phenomena have been reported in Aoki (1996, Sect. 7.1.7) and also in Aoki (1996, p. 225) where Derrida's random energy model, Derrida (1981), was introduced to the economic audience. Unfortunately, it did not catch the attention of the economic audiences. See also Mertens (2000).

This paper is another attempt at exposing non-self-averaging phenomena in economics. There are other types of models with non-self averaging behavior. Aoki and Yoshikawa (2007b) have shown such a growth model, and certain balanced triangular urns in which balls of one color are non-self averaging.

What are the implications if some economic models have non-self averaging property? For one thing, it means that we cannot blindly try for larger size samples in the hope that we obtain better estimates.

The examples above are just a hint of the potential of this approach of using exchangeable random partition methods. It is the opinion of this author that subjects such as in the papers by Fabritiis et al. (2003), or by Amaral et al. (1998) could be re-examined from the random combinatorial partition approach with profit. Another example is Sutton (2002). He modeled independent business in which the business sizes vary by partitions of integers to discuss the dependence of variances of firm growth rates. He assumed each partition is equally likely, however. Use of random partitions discussed in this paper may provide more realistic or flexible framework for the question he examined.

Finally, one key question in applications to macroeconomic or financial modelings of the random partition approach is "What are the most likely combinations of the values of $K_n = k$, a_j, and ja_j all suitably normalized ?" See also Aoki and Yoshikawa (2007b).

Appendix

Pitman showed that

$$K_n/n^\alpha \to \mathcal{L},$$

in distribution and Pitman (2002, Sect. 3) has stronger result of convergence a.s. See Yamato and Sibuya also. The random variable $\mathcal{L}$ has the density

$$\frac{d}{ds} P_{\alpha,\theta}(\mathcal{L} \in ds) = g_{\alpha,\theta}$$

where letting $\eta = \frac{\theta}{\alpha}$ we define

$$g_{\alpha,\theta}(s) := \frac{\Gamma(\theta+1)}{\Gamma(\eta+1)} s^\eta g_\alpha(s),$$

where $s > 0$, and where $g_\alpha = g_{\alpha,0}$ is the Mittag–Leffler density

$$g_\alpha(s) = \frac{1}{\pi} \sum_{k=1}^{\infty} \left[\frac{\Gamma(k\alpha)}{\Gamma(k)} \sin(k\pi\alpha)(-s)^{k-1} \right].$$

See Blumenfeld and Mandelbrot (1997), or Pitman (2002) for example.

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