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Global Existence of Strong Solutions of Navier-Stokes-Poisson Equations for One-Dimensional Isentropic Compressible Fluids**

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Abstract The authors prove two global existence results of strong solutions of the isentropic compressible Navier-Stokes-Poisson equations in one-dimensional bounded intervals. The first result shows only the existence. And the second one shows the existence and uniqueness result based on the first result, but the uniqueness requires some compatibility condition. In this paper the initial vacuum is allowed, and T is bounded.

Keywords Global strong solutions, Navier-Stokes-Poisson equations, Existence and uniqueness

2000 MR Subject Classification 35A05, 35Q30

1 Introduction

In this paper, we consider the system:

$$\begin{cases} \rho_t + (\rho u)_x = 0, & \text{in } (0,T)\times\Omega, \quad (1.1)\\ (\rho u)_t + (\rho u^2)_x + \rho\Phi_x - \lambda u_{xx} + p_x = \rho f, & \text{in } (0,T)\times\Omega, \quad (1.2)\\ \Phi_{xx} = 4\pi g\Big(\rho - \dfrac{1}{|\Omega|}\displaystyle\int_\Omega \rho \mathrm{d}x\Big), & \text{in } (0,T)\times\Omega, \quad (1.3) \end{cases}$$

where $p = a\rho^\gamma$ $(a > 0,\ \gamma > 1)$, $\lambda > 0$. In this paper, we only consider that Ω is a one-dimensional bounded interval. For simplicity we only consider $\Omega = (0,1)$, $T < +\infty$. The initial and boundary conditions are

$$\rho|_{t=0} = \rho_0(x) \geq 0, \quad u|_{t=0} = u_0, \quad \forall\, x \in (0,1), \tag{1.4}$$

$$u(0,t) = u(1,t) = 0, \quad \Phi(0,t) = \Phi(1,t) = 0, \quad \forall\, t > 0. \tag{1.5}$$

For the vacuum case, in [1], Takayuki Kobayshi and Takashi Suzuki proved the existence of weak solution to Navier-Stokes-Poisson equations. Their methods are similar to Feireisl's methods (see [2]). But as for Navier-Stokes-Poisson systems, the results of strong solutions are few. However, as for the existence, uniqueness, or other virtues of the strong solutions of Navier-Stokes equations, we may refer to [3–10].

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Remark 1.1 The problem about the radially solutions of the Navier-Stokes-Poisson equations is worthy to consider.

1.1 Main results

Theorem 1.1 *Assume that the initial conditions and f satisfy*

$$\rho_0 \in H^1(0,1), \quad u_0 \in H_0^1(0,1), \quad f \in L^2_{\rm loc}(0,\infty; L^{\frac{2\gamma}{\gamma-1}}(0,1)). \tag{1.6}$$

Then there is a global strong solution (ρ, u, Φ) of (1.1)–(1.5), such that for all $T \in (0,\infty)$, we have

$$\begin{aligned}
&\rho \in L^\infty(0,T;H^1(0,1)), \quad \rho_t \in L^\infty(0,T;L^2(0,1)),\\
&\Phi \in L^\infty(0,T;H^3(0,1)), \quad \Phi_t \in L^\infty(0,T;H^2(0,1)),\\
&u \in L^\infty(0,T;H_0^1(0,1)), \quad (\rho u)_t, u_{xx} \in L^2(0,T;L^2(0,1)).
\end{aligned}$$

Theorem 1.2 *Assume that the initial conditions and f satisfy*

$$\begin{aligned}
&\rho_0 \in H^1(0,1), \quad u_0 \in H_0^1(0,1) \cap H^2(0,1),\\
&f \in L^2_{\rm loc}(0,\infty; L^{\frac{2\gamma}{\gamma-1}}(0,1)), \quad f_x, f_t \in L^2_{\rm loc}(0,\infty; L^2(0,1))
\end{aligned} \tag{1.7}$$

and compatibility condition

$$\lambda(u_0)_{xx} - (a\rho_0^\gamma)_x = \rho^{\frac{1}{2}} g \quad \text{for some } g \in L^2(0,1). \tag{1.8}$$

Then there is a unique strong solution (ρ, u, Φ), such that for all $T \in (0,\infty)$, we have

$$\begin{aligned}
&\rho \in L^\infty(0,T;H^1(0,1)), \quad u \in L^\infty(0,T;H^2(0,1)), \quad \Phi \in L^\infty(0,T;H^3(0,1)),\\
&\rho_t, \sqrt{\rho}\, u_t \in L^\infty(0,T;L^2(0,1)), \quad u_t, G_x \in L^2(0,T;H^1(0,1)), \quad \Phi_t \in L^\infty(0,T;H^2(0,1)).
\end{aligned}$$

Remark 1.2 When $G = \lambda u_x - p$ is an effective flux, we can easily get $\rho \in C([0,T] \times (0,1))$, $u \in C([0,T]; H_0^1(0,1))$, $\Phi \in C([0,T]; H^2(0,1))$, and combining the equations (1.1)–(1.3) and the effective viscous flux, we may obtain other regularity.

2 A priori Estimates for Smooth Solutions

To get the existence of strong solutions, obviously, we require some more regular estimates. So we provide that (ρ, u, Φ) is a smooth solution of (1.1)–(1.5), $\rho > 0$, and $T \in (0,\infty)$ is some fixed time. Moreover we may let $m_0 := \int_0^1 \rho_0(x)\mathrm{d}x$ be initial mass and $m_0 > 0$. To simplify, we let $\lambda \equiv 1$. In fact, as we can deal with approximate system, we only consider initial nonvacuum. Combining the classical results of (1.3) with our correlated uniform estimates, we may get the existence of strong solutions of our system. To prove uniqueness, we use the classical method.

Lemma 2.1

$$\sup_{0\le t\le T} \int_0^1 (\rho u^2 + p)\mathrm{d}x + \int_0^T\!\!\int_0^1 u_x^2 \mathrm{d}x\mathrm{d}t \le C, \tag{2.1}$$

where C is dependent on $|\rho_0|_{H^1(0,1)}$, $|u_0|_{H_0^1(0,1)}$ and $|f|_{L^2(0,T;L^{\frac{2\gamma}{\gamma-1}}(0,1))}$, but is independent of the lower bound of ρ_0.

Proof Firstly, we introduce the energy formula

$$E(t)=\int_0^1\Big(\frac{1}{2}\rho|u|^2+\frac{a}{\gamma-1}\rho^\gamma\Big)\mathrm{d}x-\frac{1}{8\pi g}\int_0^1|\Phi_x|^2\mathrm{d}x,$$
$$E(0)=\int_0^1\Big(\frac{1}{2}\rho_0|u_0|^2+\frac{a}{\gamma-1}\rho_0^\gamma\Big)\mathrm{d}x-\frac{1}{8\pi g}\int_0^1|\Phi_{0x}|^2\mathrm{d}x,$$

where $E(0)$ is the initial energy. It follows from (1.1) and (1.2) that

$$\rho u_t+(\rho u)u_x+\rho\Phi_x-\lambda u_{xx}+P_x=\rho f.$$

Multiplying this equation by u, integrating (by parts) over $(0,1)$, and combining the equations (1.1) and (1.3), we can deal with each term as follows:

$$\int_0^1\rho u_t u\mathrm{d}x+\int_0^1(\rho u)u_x u\mathrm{d}x=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1\rho u^2\mathrm{d}x,$$
$$\int_0^1\rho\Phi_x u\mathrm{d}x=-\int_0^1(\rho u)_x\Phi\mathrm{d}x=\int_0^1\rho_t\Phi\mathrm{d}x=\frac{1}{4\pi g}\int_0^1\Phi_{xxt}\Phi\mathrm{d}x=-\frac{1}{8\pi g}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1\Phi_x^2\mathrm{d}x,$$
$$\int_0^1 a\gamma\rho^{\gamma-1}\rho_x u\mathrm{d}x=\int_0^1 a\gamma\rho^{\gamma-1}(-\rho_t-\rho u_x)\mathrm{d}x=-\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 a\rho^\gamma\mathrm{d}x+\gamma\int_0^1 P_x u\mathrm{d}x.$$

Then

$$\int_0^1 P_x u\mathrm{d}x=\frac{a}{\gamma-1}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1\rho^\gamma\mathrm{d}x.$$

Combining these estimates, we can conclude that

$$\frac{\mathrm{d}E(t)}{\mathrm{d}t}+\lambda|u_x|^2_{L^2(0,1)}\le\int_0^1\rho uf\mathrm{d}x\mathrm{d}t. \tag{2.2}$$

We deal with $\int_0^1|\Phi_x|^2\mathrm{d}x$ of $E(t)$. Multiplying (1.3) by Φ and integrating over $(0,1)$, we get

$$\int_0^1\Phi_{xx}\Phi\mathrm{d}x=4\pi g\Big(\int_0^1\rho\Phi\mathrm{d}x-m_0\int_0^1\Phi\mathrm{d}x\Big) \tag{2.3}$$

and

$$\begin{aligned}4\pi g\Big(\int_0^1\rho|\Phi|\mathrm{d}x-m_0\int_0^1\Phi\mathrm{d}x\Big)&\le 8\pi g m_0|\Phi|_{L^\infty(0,1)}\le 8\pi g m_0|\Phi_x|_{L^2(0,1)}\\&\le\frac{1}{2}|\Phi_x|^2_{L^2(0,1)}+32\pi^2g^2m_0^2.\end{aligned}$$

Consequently,

$$\int_0^1\Phi_x^2\mathrm{d}x\le C\Big(\int_0^1\rho\mathrm{d}x\Big)^2\le C(m_0). \tag{2.4}$$

Integrating (2.2) over $(0,t)$, we have

$$E(t)+\lambda\int_0^t|u_x|^2\mathrm{d}s\le E(0)+\int_0^t\int_0^1\rho|u||f|\mathrm{d}x\mathrm{d}s. \tag{2.5}$$

Combining (2.4) and the form of $E(t)$, we obtain

$$\begin{aligned}
&\frac{1}{2}|\sqrt{\rho}u(t)|^2_{L^2(0,1)}+\frac{a}{4(\gamma-1)}|\rho|^{\gamma}_{L^{\gamma}(0,1)}+\lambda\int_0^t u_x^2\mathrm{d}x\\
&\le C+\int_0^t\int_0^1 \rho|u||f|\mathrm{d}x\mathrm{d}s\\
&\le C+C\int_0^t |f|_{L^{\frac{2\gamma}{\gamma-1}}(0,1)}|\rho|^{\frac{1}{2}}_{L^{\gamma}(0,1)}|\sqrt{\rho}\,u|_{L^2(0,1)}\mathrm{d}s\\
&\le C+C\int_0^t |f|_{L^{\frac{2\gamma}{\gamma-1}}(0,1)}(1+|\rho|^{\gamma}_{L^{\gamma}(0,1)}+|\sqrt{\rho}\,u|^2_{L^2(0,1)})\mathrm{d}s\\
&\le C+C\int_0^t |f|_{L^{\frac{2\gamma}{\gamma-1}}(0,1)}\mathrm{d}s+C\int_0^t |f|_{L^{\frac{2\gamma}{\gamma-1}}(0,1)}(|\rho|^{\gamma}_{L^{\gamma}(0,1)}+|\sqrt{\rho}\,u|^2_{L^2(0,1)})\mathrm{d}s.
\end{aligned}$$

Using Gronwall's inequality, we get

$$\sup_{0\le t\le T}(|\sqrt{\rho}\,u|^2_{L^2(0,1)}+|\rho|^{\gamma}_{L^{\gamma}(0,1)})+\lambda\int_0^T\int_0^1 u_x^2\mathrm{d}x\mathrm{d}t\le C(m_0,f), \tag{2.6}$$

where C is independent of the lower bound of ρ_0.

Lemma 2.2

$$\sup_{0\le t\le T}|\rho(t)|_{L^{\infty}(0,1)}\le C, \tag{2.7}$$

where C is dependent on the initial mass m_0, $|\rho_0|_{H^1(0,1)}$, $|u_0|_{H_0^1(0,1)}$ and $|f|_{L^2(0,T;L^{\frac{2\gamma}{\gamma-1}}(0,1))}$, but is independent of the lower bound of ρ_0.

Proof Consider Lagrangian flow $X=X(t,x)$ of u and define

$$\begin{cases}\dfrac{\partial X}{\partial t}=u(t,X(t,x)),\\ X(0,x)=x\in[0,1].\end{cases}$$

Then we only require to prove $\rho(t,X(t,x))\le C$, for any $(t,x)\in(0,T]\times(0,1)$. Let $t_0\in(0,T]$ be any fixed time. Combining $C^{-1}\le\int_0^1\rho_0(x)\mathrm{d}x=m_0\le C$, the conservative mass, and the means of the Lagrangian flow $X=X(t,x)$, proving by contradiction, we can easily find some $x_1\in(0,1)$, such that

$$C^{-1}\le\rho_0(x_1)\quad\text{and}\quad\rho(t_0,X(t_0,x_1))\le C. \tag{2.8}$$

Furthermore, if we take some subset $(a,b)\subset(0,1)$ such that $C^{-1}\le\rho_0(x)$, then combining the means of X and the conservative mass, we get

$$\int_{X(t_0,a)}^{X(t_0,b)}\rho(t_0,x)\mathrm{d}x=\int_a^b\rho_0(x)\mathrm{d}x\le C.$$

Next, we prove that for any $x_2\in(0,1)$, $\rho(t_0,X(t_0,x_2))\le C$ holds. Let $X_j(t):=X(t,x_j)$, $j=1,2$, and $L(t)=\log\rho(t,X_2(t))-\log\rho(t,X_1(t))$. Then using (1.1)–(1.3), we get

$$\begin{aligned}
\frac{\mathrm{d}L}{\mathrm{d}t}&=\frac{1}{\rho(t,X_2(t))}\Big(\rho_t(t,X_2(t))+\rho_x\frac{\mathrm{d}X_2}{\mathrm{d}t}\Big)-\frac{1}{\rho(t,X_1(t))}\Big(\rho_t(t,X_1(t))+\rho_x\frac{\mathrm{d}X_1}{\mathrm{d}t}\Big)\\
&=\frac{1}{\rho(t,X_2(t))}(-\rho_x(t,X_2)u(t,X_2)-\rho(t,X_2)u_x(t,X_2)+\rho_x u(t,X_2))
\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{\rho(t,X_1(t))}(-\rho_x(t,X_1)u(t,X_1)-\rho(t,X_1)u_x(t,X_1)+\rho_x u(t,X_1))\\
&=-(u_x(t,X_2)-u_x(t,X_1))=-\int_{X_1(t)}^{X_2(t)}u_{xx}\mathrm{d}x\\
&=-\int_{X_1(t)}^{X_2(t)}[(\rho u)_t+(\rho u^2)_x+p_x+\rho\Phi_x-\rho f]\mathrm{d}x\\
&=-\int_{X_1(t)}^{X_2(t)}[-(\rho u)_x u+\rho u_t+(\rho u)_x u+\rho u u_x+p_x+\rho\Phi_x-\rho f]\mathrm{d}x\\
&=-\int_{X_1(t)}^{X_2(t)}(\rho u_t+\rho u u_x+p_x+\rho\Phi_x-\rho f)\mathrm{d}x. \qquad (2.9)
\end{aligned}
$$

Let $U(t)=\int_{X_1(t)}^{X_2(t)}\rho u(t,x)\mathrm{d}x$. Then

$$
\begin{aligned}
\frac{\mathrm{d}U(t)}{\mathrm{d}t}&=\rho u(t,X_2(t))\frac{\mathrm{d}X_2}{\mathrm{d}t}-\rho u(t,X_1(t))\frac{\mathrm{d}X_1}{\mathrm{d}t}+\int_{X_1(t)}^{X_2(t)}(\rho u(t,x))_t\mathrm{d}x\\
&=\rho u^2(t,X_2(t))-\rho u^2(t,X_1(t))+\int_{X_1(t)}^{X_2(t)}[-(\rho u)_x u+\rho u_t]\mathrm{d}x\\
&=\int_{X_1(t)}^{X_2(t)}\rho u(t,x)u_x(t,x)\mathrm{d}x+\int_{X_1(t)}^{X_2(t)}\rho u_t(t,x)\mathrm{d}x. \qquad (2.10)
\end{aligned}
$$

Substituting (2.10) into (2.9), we have

$$
\frac{\mathrm{d}L(t)}{\mathrm{d}t}+\frac{\mathrm{d}U(t)}{\mathrm{d}t}=-\Big(\int_{X_1(t)}^{X_2(t)}p_x\mathrm{d}x+\int_{X_1(t)}^{X_2(t)}\rho\Phi_x\mathrm{d}x-\int_{X_1(t)}^{X_2(t)}\rho f\mathrm{d}x\Big). \qquad (2.11)
$$

Let $\alpha(t)=\frac{p(\rho(t,X_2))-p(\rho(t,X_1))}{L(t)}$. We easily find $\alpha(t)>0$. Thus (2.11) becomes

$$
\frac{\mathrm{d}L(t)}{\mathrm{d}t}+\frac{\mathrm{d}U(t)}{\mathrm{d}t}=-\alpha(L+U)+\alpha U-\int_{X_1(t)}^{X_2(t)}(\rho\Phi_x-\rho f)\mathrm{d}x. \qquad (2.12)
$$

Using the theorem of ODE, we get

$$
L(t)+U(t)=\mathrm{e}^{-\int_0^t\alpha(s)\mathrm{d}s}(L(0)+U(0))+\int_0^t\mathrm{e}^{-\int_s^t\alpha(\tau)\mathrm{d}\tau}(\alpha(s)U(s)-\Psi(s))\mathrm{d}s,
$$

where $\Psi(s)=\int_{X_2(s)}^{X_1(s)}(\rho\Phi_x-\rho f)\mathrm{d}x$. Combining $L(0)\le C$ $(|\rho_0(x)|_{L^\infty(0,1)}\le C)$, we have

$$
L(t)\le C+|U(0)|+|U(t)|+\int_0^t\mathrm{e}^{-\int_s^t\alpha(\tau)\mathrm{d}\tau}(\alpha(s)|U(s)|+|\Psi(s)|)\mathrm{d}s.
$$

Particularly, we let $t=t_0$. Thus

$$
L(t_0)=C+|U(0)|+\sup_{0\le t\le t_0}|U(t)|+\int_0^{t_0}|\Psi(s)|\mathrm{d}s. \qquad (2.13)
$$

We deal with the latter terms as follows:

$$
\begin{aligned}
&|U(0)|\le\int_{x_1}^{x_2}\rho_0|u(0,x)|\mathrm{d}x\le C, \qquad (2.14)\\
&|U(t)|\le\int_{X_1(t)}^{X_2(t)}\rho|u(t,x)|\mathrm{d}x\le\Big(\int_{X_1(t)}^{X_2(t)}\rho\mathrm{d}x\Big)^{\frac12}\Big(\int_{X_1(t)}^{X_2(t)}\rho u^2\mathrm{d}x\Big)^{\frac12}.
\end{aligned}
$$

Consequently,

$$\sup_{0\le t\le t_0}|U(t)|\le \sup_{0\le t\le t_0}\Big(\int_0^1 \rho \mathrm{d}x\Big)^{\frac12}\Big(\int_0^1 \rho u^2\mathrm{d}x\Big)^{\frac12}.$$

Combining (2.6), we get

$$\sup_{0\le t\le t_0}|U(t)|\le C, \tag{2.15}$$

$$\int_0^t\int_{X_1(s)}^{X_2(s)}\rho|f|\mathrm{d}x\mathrm{d}s\le C\int_0^t\Big(\int_0^1\rho^\gamma\mathrm{d}x\Big)^{\frac1\gamma}\Big(\int_0^1|f|^{\frac{2\gamma}{\gamma-1}}\mathrm{d}x\Big)^{\frac{\gamma-1}{2\gamma}}\mathrm{d}s\le C(T),$$

$$\int_0^t\int_{X_1(s)}^{X_2(s)}\rho|\Phi_x|\mathrm{d}x\mathrm{d}s\le C\int_0^t\Big(\int_0^1\rho^\gamma\mathrm{d}x\Big)^{\frac1\gamma}\Big(\int_0^1|\Phi_x|^{\frac{2\gamma}{\gamma-1}}\mathrm{d}x\Big)^{\frac{\gamma-1}{2\gamma}}\mathrm{d}s.$$

Combining imbedded theorem and the estimates of Poisson equation, we get

$$\int_0^t\int_{X_1(s)}^{X_2(s)}\rho|\Phi_x|\mathrm{d}x\mathrm{d}s\le C\int_0^t\int_0^1|\Phi_{xx}|\mathrm{d}x\mathrm{d}s\le C(m_0,T). \tag{2.16}$$

From (2.13)–(2.16), we have $L(t_0)\le C$. Then

$$\log\rho(t_0,X(t_0,x_2))=\log\rho(t_0,X(t_0,x_1))+L(t_0)\le C. \tag{2.17}$$

Because t_0, x_2 are arbitrary, Lemma 2.2 is proved.

To get the higher estimates, the effective viscous flux is very important. In the proof below, usually we will use the regularity of $G_x=u_{xx}-p_x$.

Lemma 2.3

$$\sup_{0\le t\le T}(|u|_{L^\infty(0,1)}+|u_x|_{L^2(0,1)})+\int_0^T|\sqrt{\rho}\,u_t|^2_{L^2(0,1)}\mathrm{d}t\le C, \tag{2.18}$$

where C is dependent on $|\rho_0|_{H^1(0,1)}$, $|u_0|_{H_0^1(0,1)}$ and $|f|_{L^2(0,T;L^{\frac{2\gamma}{\gamma-1}}(0,1))}$, but is independent of the lower bound of ρ_0.

Proof Multiplying (1.2) by u_t and integrating over $(0,1)$, we get

$$\int_0^1\rho u_t^2\mathrm{d}x+\int_0^1\rho u u_x u_t\mathrm{d}x+\int_0^1\rho\Phi_x u_t\mathrm{d}x-\lambda\int_0^1 u_{xx}u_t\mathrm{d}x=\int_0^1\rho f u_t\mathrm{d}x-\int_0^1 p_x u_t\mathrm{d}x.$$

Thus

$$\int_0^1\rho u_t^2\mathrm{d}x+\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1\frac12 u_x^2\mathrm{d}x\le C\Big(\int_0^1\rho u^2u_x^2\mathrm{d}x+\int_0^1\Phi_x^2\mathrm{d}x+\int_0^1 p u_{xt}\mathrm{d}x+\int_0^1\rho|f|^2\mathrm{d}x\Big). \tag{2.19}$$

Next, we deal with each of the above terms as follows:

$$\begin{aligned}\int_0^1 p u_{xt}\mathrm{d}x&=\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 p u_x\mathrm{d}x-\int_0^1 p_t u_{xt}\mathrm{d}x=\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 p u_x\mathrm{d}x-\int_0^1 a\gamma\rho^{\gamma-1}\rho_t u_{xt}\mathrm{d}x\\&=\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 p u_x\mathrm{d}x-\int_0^1 p u u_{xx}\mathrm{d}x+(\gamma-1)\int_0^1 p u_x^2\mathrm{d}x\\&=\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 p u_x\mathrm{d}x-\int_0^1 p u(G_x+p_x)\mathrm{d}x+(\gamma-1)\int_0^1 p(G+p)^2\mathrm{d}x,\end{aligned} \tag{2.20}$$

where

$$\begin{aligned} -\int_0^1 pup_x \mathrm{d}x &= \frac{-1}{2(2\gamma-1)}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 p^2\mathrm{d}x; \\ (\gamma-1)\int_0^1 pu_x^2\mathrm{d}x &= (\gamma-1)\int_0^1 p(G+p)^2\mathrm{d}x \\ &= (\gamma-1)\int_0^1 pG^2\mathrm{d}x + 4(\gamma-1)\int_0^1 pp_xu\mathrm{d}x - (\gamma-1)\int_0^1 p^3\mathrm{d}x. \end{aligned}$$

Then, (2.20) becomes

$$\begin{aligned} \int_0^1 pu_{xt}\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 pu_x\mathrm{d}x - \frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \frac{4\gamma-3}{2(2\gamma-1)}p^2\mathrm{d}x \\ + (\gamma-1)\int_0^1 p(G^2-p^2)\mathrm{d}x - \int_0^1 puG_x\mathrm{d}x. \end{aligned} \tag{2.21}$$

Substituting (2.21) into (2.19) and integrating over $(0,t)$, we get

$$\begin{aligned} &\int_0^t\int_0^1 \rho u_t^2\mathrm{d}x\mathrm{d}s + \int_0^1 \frac{1}{2}u_x^2(t)\mathrm{d}x - \int_0^1 \frac{1}{2}u_x^2(0)\mathrm{d}x \\ &\le C + \int_0^t\int_0^1 (\rho u^2u_x^2 + pG^2 + p|u||G_x|)\mathrm{d}x\mathrm{d}s + \int_0^t\int_0^1 (\Phi_x^2 + \rho f^2)\mathrm{d}x\mathrm{d}s \\ &\quad + \int_0^1 (pu_x)(t)\mathrm{d}x - \int_0^1 (pu_x)(0)\mathrm{d}x - \int_0^1 \frac{4\gamma-3}{2(2\gamma-1)}p^2(t)\mathrm{d}x \\ &\quad + \int_0^1 \frac{4\gamma-3}{2(2\gamma-1)}p^2(0)\mathrm{d}x - (\gamma-1)\int_0^t\int_0^1 p^3\mathrm{d}x\mathrm{d}s \\ &\le C + \int_0^t\int_0^1 (\rho u^2u_x^2 + pG^2 + p|u||G_x|)\mathrm{d}x\mathrm{d}s. \end{aligned} \tag{2.22}$$

Now, we deal with each of the right terms of (2.22) as follows:

$$\begin{aligned} \int_0^t\int_0^1 \rho u^2u_x^2\mathrm{d}x\mathrm{d}s &\le \int_0^t |\rho|_{L^\infty(0,1)}|u|^2_{L^\infty(0,1)}|u_x|^2_{L^2(0,1)}\mathrm{d}s \le \int_0^t |u_x|^4_{L^2(0,1)}\mathrm{d}s, \\ \int_0^t\int_0^1 pG^2\mathrm{d}x\mathrm{d}s &\le 2\int_0^t\int_0^1 p(u_x^2+p^2)\mathrm{d}x\mathrm{d}s \le C, \\ \int_0^t\int_0^1 p|u||G_x|\mathrm{d}x\mathrm{d}s &\le C\int_0^t |\rho|^{\gamma-\frac{1}{2}}_{L^\infty(0,1)}|\sqrt{\rho}\,u|_{L^2(0,1)}|G_x|_{L^2(0,1)}\mathrm{d}s \le C\int_0^t |G_x|_{L^2(0,1)}\mathrm{d}s, \end{aligned} \tag{2.23}$$

and

$$G_x = u_{xx} - p_x = \rho u_t + \rho uu_x + \rho\Phi_x - \rho f.$$

Consequently,

$$|G_x|_{L^2(0,1)} \le C(|\sqrt{\rho}\,u_t|_{L^2(0,1)} + |uu_x|_{L^2(0,1)} + |\Phi_x|_{L^2(0,1)} + |\rho f|_{L^2(0,1)}).$$

Then (2.23) becomes

$$\begin{aligned} \int_0^t\int_0^1 p|u||G_x|\mathrm{d}x\mathrm{d}s &\le C\int_0^t (|\sqrt{\rho}\,u_t|_{L^2(0,1)} + |uu_x|_{L^2(0,1)} + |\Phi_x|_{L^2(0,1)} + |\rho f|_{L^2(0,1)})\mathrm{d}s \\ &\le C + C\int_0^t |u_x|^4_{L^2(0,1)}\mathrm{d}s + \frac{1}{2}\int_0^t\int_0^1 \rho u_t^2\mathrm{d}x\mathrm{d}s. \end{aligned}$$

Combining the above estimates, we get

$$\int_0^t (|\sqrt{\rho}\, u_t|^2_{L^2(0,1)} + |u_x|^2_{L^2(0,1)})\mathrm{d}s \le C + C\int_0^t |u_x|^4_{L^2(0,1)}\mathrm{d}s.$$

Using Gronwall's inequality, we obtain

$$\int_0^t (|\sqrt{\rho}\, u_t|^2_{L^2(0,1)} + |u_x|^2_{L^2(0,1)})\mathrm{d}s \le C.$$

Combining $u|_{\partial\Omega} = 0$ and embedding theorem, we have

$$\sup_{0\le t\le T}(|u|_{L^\infty(0,1)} + |u_x|^2_{L^2(0,1)}) + \int_0^t |\sqrt{\rho}\, u_t|^2_{L^2(0,1)}\mathrm{d}t \le C.$$

Lemma 2.4

$$\int_0^T (|u_x|^2_{L^\infty(0,1)} + |G_x|^2_{L^2(0,1)})\mathrm{d}t \le C, \tag{2.24}$$

where C is dependent on $|\rho_0|_{H^1(0,1)}$, $|u_0|_{H^1_0(0,1)}$ and $|f|_{L^2(0,T;L^{\frac{2\gamma}{\gamma-1}}(0,1))}$, but is independent of the lower bound of ρ_0.

Proof

$$|u_x|^2_{L^\infty(0,1)} \le 2(|G|^2_{L^\infty(0,1)} + |p|^2_{L^\infty(0,1)}) \le 2(|G|^2_{L^2(0,1)} + |G_x|^2_{L^2(0,1)} + |p|^2_{L^\infty(0,1)}),$$

and

$$\begin{aligned}\int_0^T |G|^2_{L^2(0,1)}\mathrm{d}t &\le 2\int_0^T\int_0^1 |u_x|^2\mathrm{d}x\mathrm{d}t + \int_0^T\int_0^1 |p|^2\mathrm{d}x\mathrm{d}t \le C,\\ \int_0^T |G_x|^2_{L^2(0,1)}\mathrm{d}t &\le C\int_0^T (C + |\sqrt{\rho}\, u_t|^2_{L^2(0,1)} + |uu_x|^2_{L^2(0,1)}|\sqrt{\rho} f|^2_{L^2(0,1)})\mathrm{d}s\\ &\le C + C\int_0^T |u_x|^2_{L^2(0,1)}\mathrm{d}s \le C.\end{aligned}$$

From the above estimates, we have

$$\sup_{0\le t\le T}(|\rho|_{L^\infty(0,1)} + |u_x|_{L^2(0,1)}) + \int_0^T (|\sqrt{\rho}\, u_t|^2_{L^2(0,1)} + |u_x|^2_{L^\infty(0,1)} + |G_x|^2_{L^2(0,1)})\mathrm{d}t \le C, \tag{2.25}$$

where C is dependent on $|\rho|_{H^1(0,1)}$, $|u_0|_{H^1_0(0,1)}$ and $|f|_{L^2(0,T;L^{\frac{2\gamma}{\gamma-1}}(0,1))}$.

Lemma 2.5

$$\sup_{0\le t\le T}|\rho_x|_{L^2(0,1)} \le C(T), \tag{2.26}$$

where C is dependent on $|\rho_0|_{H^1(0,1)}$, $|u_0|_{H^1_0(0,1)}$, $|f|_{L^2(0,T;L^{\frac{2\gamma}{\gamma-1}}(0,1))}$ and T, but is independent of the lower bound of ρ_0.

Proof Differentiating (1.1) with respect to x gives

$$(\rho_x)_t + (\rho u)_{xx} = 0.$$

Multiplying it by ρ_x and integrating over $(0,1)$, we get

$$\begin{aligned}
&\int_0^1 (\rho_x)_t\rho_x \mathrm{d}x + \int_0^1 (\rho u)_{xx}\rho_x \mathrm{d}x = 0,\\
&\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_0^1 (\rho_x)^2 \mathrm{d}x + \int_0^1 [(\rho_x u) + \rho u_x]_x \rho_x \mathrm{d}x = 0, \qquad (2.27)\\
&\int_0^1 (\rho_x u)_x\rho_x \mathrm{d}x = \int_0^1 (\rho_{xx}u + \rho_x u_x)\rho_x \mathrm{d}x = \int_0^1 \rho_{xx}\rho_x u \mathrm{d}x + \int_0^1 \rho_x^2 u_x \mathrm{d}x.
\end{aligned}$$

However

$$\int_0^1 \rho_{xx}\rho_x u\mathrm{d}x = -\int_0^1 \rho_x(\rho_x u)_x \mathrm{d}x = -\int_0^1 \rho_x\rho_{xx}u\mathrm{d}x - \int_0^1 \rho_x^2 u_x \mathrm{d}x.$$

Thus

$$\int_0^1 \rho_{xx}\rho_x u\mathrm{d}x = -\frac{1}{2}\int_0^1 \rho_x^2 u_x \mathrm{d}x,$$

but

$$\begin{aligned}
\int_0^1 \rho_x^2 u_x \mathrm{d}x &\le |u_x|_{L^\infty(0,1)}\int_0^1 \rho_x^2 \mathrm{d}x,\\
\int_0^1 (\rho u_x)_x\rho_x \mathrm{d}x &= \int_0^1 \rho_x^2 u_x \mathrm{d}x + \int_0^1 \rho u_{xx}\rho_x \mathrm{d}x\\
&\le |u_x|_{L^\infty(0,1)}\int_0^1 \rho_x^2 \mathrm{d}x + \int_0^1 \rho(G_x + p_x)\rho_x \mathrm{d}x\\
&\le C\Big(|u_x|_{L^\infty(0,1)}\int_0^1 \rho_x^2 \mathrm{d}x + \int_0^1 G_x^2 \mathrm{d}x\int_0^1 \rho_x^2 \mathrm{d}x + \int_0^1 a\gamma\rho^\gamma\rho_x^2 \mathrm{d}x\Big)\\
&\le C\Big(|u_x|_{L^\infty(0,1)}\int_0^1 \rho_x^2 \mathrm{d}x + \int_0^1 G_x^2 \mathrm{d}x\int_0^1 \rho_x^2 \mathrm{d}x + \int_0^1 \rho_x^2 \mathrm{d}x\Big). \qquad (2.28)
\end{aligned}$$

Combining (2.27) and the above estimates, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_0^1 (\rho_x)^2 \mathrm{d}x \le C\Big(|u_x|_{L^\infty(0,1)}\int_0^1 \rho_x^2 \mathrm{d}x + \int_0^1 G_x^2 \mathrm{d}x\int_0^1 \rho_x^2 \mathrm{d}x + \int_0^1 \rho_x^2 \mathrm{d}x\Big).$$

From the above lemmas and using Gronwall's inequality, we have

$$\sup_{0\le t\le T}|\rho_x|_{L^2(0,1)} \le C(T).$$

Lemma 2.6

$$\sup_{0\le t\le T}(|\rho|_{H^1(0,1)} + |\rho_t|_{L^2(0,1)} + |u_x|_{L^2(0,1)}) + \int_0^T (|(\rho u)_t|^2_{L^2(0,1)} + |u_{xx}|^2_{L^2(0,1)})\mathrm{d}s \le C(T),$$

where C is dependent on $|\rho_0|_{H^1(0,1)}$, $|u_0|_{H_0^1(0,1)}$, $|f|_{L^2(0,T;L^{\frac{2\gamma}{\gamma-1}}(0,1))}$ and T, but is independent of the lower bound of ρ_0.

Proof From equation (1.1), we have $\rho_t^2 = [-(\rho_x u + \rho u_x)]^2$. Integrating it over $(0,1)$, we get

$$\begin{aligned}
\int_0^1 \rho_t^2 \mathrm{d}x &= \int_0^1 \rho_x^2 u^2 \mathrm{d}x + 2\int_0^1 \rho_x\rho u u_x \mathrm{d}x + \int_0^1 \rho^2 u_x^2 \mathrm{d}x\\
&\le |u|^2_{L^\infty(0,1)}\int_0^1 \rho_x^2 \mathrm{d}x + 2|\rho|_{L^\infty(0,1)}|u|_{L^\infty(0,1)}|\rho_x|_{L^2(0,1)}|u_x|^2_{L^2(0,1)} + |\rho|^2_{L^\infty(0,1)}\int_0^1 u_x^2 \mathrm{d}x.
\end{aligned}$$

Thus

$$\begin{aligned}\sup_{0\le t\le T}\int_0^1 \rho_t^2 \mathrm{d}x \le& \sup_{0\le t\le T}|u|^2_{L^\infty(0,1)}\sup_{0\le t\le T}\int_0^1\rho_x^2\mathrm{d}x\\ &+2\sup_{0\le t\le T}|\rho|_{L^\infty(0,1)}\sup_{0\le t\le T}|u|_{L^\infty(0,1)}\sup_{0\le t\le T}|\rho_x|_{L^2(0,1)}\sup_{0\le t\le T}|u_x|_{L^2(0,1)}\\ &+\sup_{0\le t\le T}|\rho|^2_{L^\infty(0,1)}\sup_{0\le t\le T}|u_x|^2_{L^2(0,1)}\le C.\end{aligned}\tag{2.29}$$

Consequently

$$\begin{aligned}\int_0^T\int_0^1(\rho u)_t^2\mathrm{d}x\mathrm{d}t &\le 2\int_0^T\int_0^1(\rho_t u)^2\mathrm{d}x\mathrm{d}t+2\int_0^T\int_0^1(\rho u_t)^2\mathrm{d}x\mathrm{d}t\\ &\le 2\sup_{0\le t\le T}|u|^2_{L^\infty(0,1)}\int_0^T\int_0^1(\rho_t)^2\mathrm{d}x\mathrm{d}t\\ &\quad+2|\rho^{\frac12}|_{L^\infty(0,T)\times(0,1)}\int_0^T\int_0^1(\sqrt{\rho}\,u_t)^2\mathrm{d}x\mathrm{d}t\le C.\end{aligned}\tag{2.30}$$

Combining the momentum equation and Poisson equation, we have

$$u_{xx}=(\rho u)_t+(\rho u^2)_x+p_x+\rho\Phi_x-\rho f.\tag{2.31}$$

Using (2.30) and $(\rho u_t)\in L^2((0,T)\times(0,1))$, we can easily get $p_x\in L^2((0,T)\times(0,1))$, and

$$[(\rho u^2)_x]^2=(\rho_x u+2\rho u u_x)^2\le 2(\rho_x^2u^4+4\rho^2u^2u_x^2).$$

Combining $|u|_{L^\infty((0,T)\times(0,1))}\le C$, $|\rho|_{L^\infty((0,T)\times(0,1))}\le C$, $|\rho_x|_{L^\infty(0,T;L^2(0,1))}\le C$ and $|u_x|_{L^\infty(0,T;L^2(0,1))}\le C$, we see that $(\rho u^2)_x\in L^2(0,T)\times(0,1)$. In fact, we have

$$\begin{aligned}&(\rho u^2)_x\in L^\infty(0,T;L^2(0,1)),\\ &\int_0^T\int_0^1|\rho\Phi_x|^2\mathrm{d}x\mathrm{d}t\le|\rho|^2_{L^\infty((0,T)\times(0,1))}\int_0^T\int_0^1|\Phi_x|^2\mathrm{d}x\mathrm{d}t\le C,\\ &\rho f\in L^2((0,T)\times(0,1)).\end{aligned}$$

Combining the above estimates and (2.31), we have

$$\sup_{0\le t\le T}(|\rho|_{H^1(0,1)}+|\rho_t|_{L^2(0,1)}+|u_x|_{L^2(0,1)})+\int_0^T(|(\rho u)_t|^2_{L^2(0,1)}+|u_{xx}|^2_{L^2(0,1)})\mathrm{d}t\le C(T).\tag{2.32}$$

To prove Theorem 1.2, we must deal with the following estimate.

Lemma 2.7

$$\begin{aligned}&|\sqrt{\rho}\,u_t(t)|^2_{L^2(0,1)}+\int_\tau^t|u_{tx}|^2_{L^2(0,1)}\mathrm{d}s\\ &\le C+|\sqrt{\rho}\,u_t(\tau)|^2_{L^2(0,1)}+C\int_0^t|u_x|_{L^\infty(0,1)}|\sqrt{\rho}\,u_t|^2_{L^2(0,1)}\mathrm{d}s,\end{aligned}\tag{2.33}$$

where C is dependent on $|\rho_0|_{H^1(0,1)}$, $|u_0|_{H^1_0(0,1)}$, $|f|_{L^2(0,T;L^{\frac{2\gamma}{\gamma-1}}(0,1))}$, $|f_x|_{L^2(0,T;L^2(0,1))}$ and $|f_t|_{L^2(0,T;L^2(0,1))}$, but is independent of the lower bound of ρ_0.

Proof Combining the momentum equation (1.2) and the mass equation (1.1), we can easily get

$$\rho u_t + \rho u u_x + \rho\Phi_x + p_x - u_{xx} = \rho f.$$

Differentiating it with respect to time, we have

$$\rho u_{tt} + \rho_t u_t + (\rho u)u_{xt} + \rho_t u u_x + \rho u_t u_x + \rho_t\Phi_x + \rho\Phi_{xt} + p_{xt} - u_{xxt} = \rho_t f + \rho f_t,$$

that is,

$$\rho u_{tt} + \rho u u_{xt} + p_{xt} - u_{xxt} = -\rho_t(u_t + uu_x + \Phi_x - f) - \rho u_t u_x - \rho\Phi_{xt} + \rho f_t.$$

Multiplying it by u_t and integrating over $(0,1)$, we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \frac{1}{2}\rho u_t^2 \mathrm{d}x - \int_0^1 \frac{1}{2}\rho_t u_t^2 \mathrm{d}x + \int_0^1 \frac{1}{2}(\rho u u_t^2)_x \mathrm{d}x \\
&- \int_0^1 \frac{1}{2}(\rho u)_x u_t^2 \mathrm{d}x - \int_0^1 p_t u_{xt} \mathrm{d}x + \int_0^1 u_{xt}^2 \mathrm{d}x \\
= &\int_0^1 (\rho u)_x(u_t^2 + uu_x u_t + \Phi_x u_t - f u_t)\mathrm{d}x - \int_0^1 \rho u_t^2 u_x \mathrm{d}x - \int_0^1 \rho\Phi_{tx} u_t \mathrm{d}x + \int_0^1 \rho f_t u_t \mathrm{d}x.
\end{aligned}$$

From equation (1.1), we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \frac{1}{2}\rho u_t^2 \mathrm{d}x + \int_0^1 u_{xt}^2 \mathrm{d}x - \int_0^1 p_t u_{xt} \mathrm{d}x \\
= &-\int_0^1 \rho u(u_t^2 + uu_x u_t + \Phi_x u_t - f u_t)_x \mathrm{d}x \\
&- \int_0^1 \rho u_t^2 u_x \mathrm{d}x - \int_0^1 \rho\Phi_{tx} u_t \mathrm{d}x + \int_0^1 \rho f_t u_t \mathrm{d}x.
\end{aligned} \tag{2.34}$$

We deal with each term of (2.34) as follows:

$$\begin{aligned}
-\int_0^1 p_t u_{xt} \mathrm{d}x &= \int_0^1 a\gamma\rho^{\gamma-1}(\rho u)_x u_{xt} \mathrm{d}x = \int_0^1 (p_x u + \gamma p u_x) u_{tx} \mathrm{d}x \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \frac{\gamma}{2} p u_x^2 \mathrm{d}x - \int_0^1 \frac{\gamma}{2} p_t u_x^2 \mathrm{d}x + \int_0^1 p_x u u_{tx} \mathrm{d}x \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \frac{\gamma}{2} p u_x^2 \mathrm{d}x + \int_0^1 p_x u u_{tx} \mathrm{d}x + \frac{\gamma}{2}\int_0^1 (p_x u + \gamma p u_x) u_x^2 \mathrm{d}x \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \frac{\gamma}{2} p u_x^2 \mathrm{d}x + \int_0^1 p_x u u_{tx} \mathrm{d}x + \frac{\gamma}{2}\int_0^1 \gamma p u_x^3 \mathrm{d}x \\
&\quad - \frac{\gamma}{2}\int_0^1 p u_x^3 \mathrm{d}x - \frac{\gamma}{2}\int_0^1 p u (u_x^2)_x \mathrm{d}x \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \frac{\gamma}{2} p u_x^2 \mathrm{d}x + \int_0^1 p_x u u_{tx} \mathrm{d}x + \frac{\gamma}{2}\int_0^1 (-pu(u_x^2)_x + (\gamma-1) p u_x^3) \mathrm{d}x.
\end{aligned}$$

Substituting it into (2.34), we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1\Big(\frac{1}{2}\rho u_t^2+\frac{\gamma}{2}pu_x^2\Big)\mathrm{d}x+\int_0^1 u_{xt}^2\mathrm{d}x\\
&\le\int_0^1|p_x||u||u_{xt}|\mathrm{d}x+\gamma\int_0^1 p|u_x||u||u_{xx}|\mathrm{d}x+\frac{\gamma}{2}\int_0^1(\gamma-1)p|u_x|^3\mathrm{d}x\\
&\quad+2\int_0^1\rho|u_t||u||u_{xt}|\mathrm{d}x+\int_0^1\rho|u_t||u||u_x|^2\mathrm{d}x+\int_0^1\rho|u_t||u|^2|u_{xx}|\mathrm{d}x+\int_0^1\rho|u_t||u|^2|u_x|\mathrm{d}x\\
&\quad-\int_0^1\rho u(\Phi_x u_t)_x\mathrm{d}x+\int_0^1\rho|f_x||u||u_t|\mathrm{d}x+\int_0^1\rho|f||u||u_{xt}|\mathrm{d}x\\
&\quad+\int_0^1\rho|u_t|^2|u_x|\mathrm{d}x-\int_0^1\rho\Phi_{xt}u_t\mathrm{d}x+\int_0^1\rho|f_t||u_t|\mathrm{d}x\equiv\sum_{j=1}^{13}\mathrm{I}_j. \qquad (2.35)
\end{aligned}$$

Next, we deal with I_1–I_{13}:

$$\begin{aligned}
\mathrm{I}_1&=\int_0^1|p_x||u||u_{xt}|\mathrm{d}x\le\varepsilon|u_{xt}|^2_{L^2(0,1)}+C(\varepsilon)|p_x|^2_{L^2(0,1)};\\
\mathrm{I}_2&=\gamma\int_0^1 p|u_x||u||u_{xx}|\mathrm{d}x\le C\int_0^1|u_x||u_{xx}|\mathrm{d}x\le C(|u_x|^2_{L^\infty(0,1)}+|u_{xx}|^2_{L^2(0,1)});\\
\mathrm{I}_3&=\frac{\gamma}{2}\int_0^1(\gamma-1)p|u_x|^3\mathrm{d}x\le C|u_x|_{L^2(0,1)}|u_x|^2_{L^\infty(0,1)}\le C|u_x|^2_{L^\infty(0,1)};\\
\mathrm{I}_4&=2\int_0^1\rho|u_t||u||u_{xt}|\mathrm{d}x\le C|\sqrt{\rho}\,u_t|_{L^2(0,1)}|u_{xt}|_{L^2(0,1)}\le\varepsilon|u_{xt}|^2_{L^2(0,1)}+C(\varepsilon)|\sqrt{\rho}\,u_t|^2_{L^2(0,1)};\\
\mathrm{I}_5&=\int_0^1\rho|u_t||u||u_x|^2\mathrm{d}x\le C|\sqrt{\rho}\,u_t|_{L^2(0,1)}|u_x|_{L^\infty(0,1)}|u_x|_{L^2(0,1)}\\
&\le C|\sqrt{\rho}\,u_t|^2_{L^2(0,1)}+C|u_x|^2_{L^\infty(0,1)};\\
\mathrm{I}_6&=\int_0^1\rho|u_t||u|^2|u_{xx}|\mathrm{d}x\le C|\sqrt{\rho}\,u_t|_{L^2(0,1)}|u_{xx}|_{L^2(0,1)}\le C(|\sqrt{\rho}\,u_t|^2_{L^2(0,1)}+|u_{xx}|^2_{L^2(0,1)});\\
\mathrm{I}_7&=\int_0^1\rho|u_t||u|^2|u_x|\mathrm{d}x\le C(|\sqrt{\rho}\,u_t|^2_{L^2(0,1)}+|u_x|^2_{L^2(0,1)});\\
\mathrm{I}_8&=\Big|\int_0^1\rho u(\Phi_x u_t)_x\mathrm{d}x\Big|\le\int_0^1\rho|u||\Phi_x||u_{xt}|dx+\int_0^1\rho|u||\Phi_{xx}||u_t|\mathrm{d}x\\
&\le C|\Phi_x|^2_{L^2(0,1)}+\varepsilon|u_{xt}|^2_{L^2(0,1)}+C|\sqrt{\rho}\,u_t|^2_{L^2(0,1)};\\
\mathrm{I}_9&=\int_0^1\rho|f_x||u||u_t|\mathrm{d}x\le C|\sqrt{\rho}\,u_t|^2_{L^2(0,1)}+C|f_x|^2_{L^2(0,1)};\\
\mathrm{I}_{10}&=\int_0^1\rho|f||u||u_{xt}|\mathrm{d}x\le C|f|^2_{L^2(0,1)}+\varepsilon|u_{xt}|^2_{L^2(0,1)};\\
\mathrm{I}_{11}&=\int_0^1\rho|u_t|^2|u_x|\mathrm{d}x\le C|u_x|_{L^\infty(0,1)}|\sqrt{\rho}\,u_t|^2_{L^2(0,1)};\\
\mathrm{I}_{12}&=\Big|\int_0^1\rho\Phi_{xt}u_t\mathrm{d}x\Big|\le C|\sqrt{\rho}\,u_t|^2_{L^2(0,1)}+C\Big(\int_0^1|\Phi_{xt}|^2\mathrm{d}x\Big).
\end{aligned}$$

We deal with the estimate of Φ_{xt}.

Differentiating (1.3) with respect to time, multiplying it by Φ_t and integrating over $(0,1)$, we get

$$\int_0^1 \Phi_{xxt}\Phi_t \mathrm{d}x = 4\pi g \int_0^1 \rho_t \Phi_t \mathrm{d}x.$$

Then

$$\int_0^1 |\Phi_{xt}|^2 \mathrm{d}x \le C|\rho_t|_{L^2(0,1)}|\Phi_t|_{L^2(0,1)} \le C|\rho_t|^2_{L^2(0,1)} + \varepsilon|\Phi_t|^2_{L^2(0,1)}.$$

Thus

$$\int_0^1 |\Phi_{xt}|^2 \mathrm{d}x \le C|\rho_t|^2_{L^2(0,1)}.$$

Consequently, I_{12} becomes

$$\begin{aligned}
&\mathrm{I}_{12} \le C(|\sqrt{\rho}\, u_t|^2_{L^2(0,1)} + |\rho_t|^2_{L^2(0,1)}),\\
&\mathrm{I}_{13} = \int_0^1 \rho|f_t||u_t|\mathrm{d}x \le C|f_t|^2_{L^2(0,1)} + C|\sqrt{\rho}\, u_t|^2_{L^2(0,1)}.
\end{aligned}$$

From the estimates of I_1–I_{13}, and (2.35), we obtain

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \Big(\frac{1}{2}\rho u_t^2 + \frac{\gamma}{2} p u_x^2\Big)\mathrm{d}x + \int_0^1 u_{xt}^2 \mathrm{d}x &\le C(1 + |\sqrt{\rho}\, u_t|^2_{L^2(0,1)} + |\rho_t|^2_{L^2(0,1)} + |u_x|^2_{L^2(0,1)}\\
&\quad + |u_x|^2_{L^\infty(0,1)} + |f_t|^2_{L^2(0,1)} + |f_x|^2_{L^2(0,1)} + |p_x|^2_{L^2(0,1)})\\
&\quad + C|u_x|_{L^\infty(0,1)}|\sqrt{\rho}\, u_t|^2_{L^2(0,1)}. \qquad (2.36)
\end{aligned}$$

Integrating it over $(\tau, t) \subset (0, T)$, we conclude that

$$\begin{aligned}
&\int_0^1 \rho u_t^2(t)\mathrm{d}x + \int_\tau^t\int_0^1 u_{xt}^2 \mathrm{d}x\mathrm{d}s\\
&\le C + C\Big(\int_0^1 \rho u_t^2(\tau)\mathrm{d}x + \int_0^1 p u_x^2(\tau)\mathrm{d}x\Big) + C\int_\tau^t |u_x|_{L^\infty(0,1)}|\sqrt{\rho}\, u_s|^2_{L^2(0,1)}\mathrm{d}s.
\end{aligned}$$

Using Lemma 2.3, we get

$$\begin{aligned}
&\int_0^1 \rho u_t^2(t)\mathrm{d}x + \int_\tau^t\int_0^1 u_{xt}^2 \mathrm{d}x\mathrm{d}s\\
&\le C + C|\sqrt{\rho}\, u_t(\tau)|^2_{L^2(0,1)} + C\int_\tau^t |u_x|_{L^\infty(0,1)}|\sqrt{\rho}\, u_s|^2_{L^2(0,1)}\mathrm{d}s. \qquad (2.37)
\end{aligned}$$

Combining the momentum equation (1.2) and compatibility condition, we have

$$\rho u_t + \rho u u_x + \rho \Phi_x + p_x - u_{xx} = \rho f.$$

Thus

$$\rho^{\frac{1}{2}} u_t = -(\rho^{\frac{1}{2}} u u_x + \rho^{\frac{1}{2}}\Phi_x + \rho^{-\frac{1}{2}} p_x - \rho^{-\frac{1}{2}} u_{xx}) + \rho^{\frac{1}{2}} f.$$

Consequently

$$|\sqrt{\rho}\, u_t(\tau)|^2_{L^2(0,1)} \le C(|\sqrt{\rho}\, f|^2_{L^2(0,1)} + |\sqrt{\rho}\,\Phi_x|^2_{L^2(0,1)} + |\sqrt{\rho}\, u u_x|^2_{L^2(0,1)} + |\rho^{-\frac{1}{2}} p_x - \rho^{-\frac{1}{2}} u_{xx}|^2_{L^2(0,1)}).$$

Combining the above estimates $f_t \in L^2(0,1;L^2(0,1))$, $\sup_{0\le t\le T} |f|^2_{L^2(0,1)} \le C$, $|\Phi_x|^2_{L^2(0,1)} \le C|\rho|^\gamma_{L^\gamma(0,1)} + C$, $|\rho^{\frac{1}{2}} u u_x|^2_{L^2(0,1)} \le C|u_x|^2_{L^2(0,1)}$ and $\sup_{0\le t\le T} |u_x|^2_{L^2(0,1)} \le C$, we obtain

$$|\sqrt{\rho}\, u_t(\tau)|^2_{L^2(0,1)} \le C + C|\rho^{-\frac{1}{2}} p_x(\tau) - \rho^{-\frac{1}{2}} u_{xx}(\tau)|^2_{L^2(0,1)}.$$

Substituting it into (2.37) and letting $\tau \to 0$, we get

$$\int_0^1 \frac{1}{2}\rho u_t^2(t)\mathrm{d}x + \int_0^t\int_0^1 u_{xt}^2 \mathrm{d}x\mathrm{d}s \le C + C\int_0^t |u_x|_{L^\infty(0,1)}|\sqrt{\rho}\,u_s|^2_{L^2(0,1)}\mathrm{d}s.$$

Using Gronwall's inequality and $\int_0^T |u_x|_{L^\infty(0,1)}\mathrm{d}t \le C$, we get

$$\int_0^1 \rho u_t^2(t)\mathrm{d}x + \int_0^t\int_0^1 u_{xt}^2\mathrm{d}x\mathrm{d}s \le C. \tag{2.38}$$

Next, we construct the approximate systems to deal with the existence.

3 Proof of the Existence

Our method that constructed approximate systems is similar to that in [9]. We take a semi-discrete Galerkin scheme. We take our basic function space as $X = H_0^1(0,1)\cap H^2(0,1)$ and the finite-dimensional subspaces as $X^m = \mathrm{span}\{\varphi^1, \varphi^2, \cdots, \varphi^m\} \subset X \cap C^2([0,1])$. Here φ^m is the mth eigenfunction of the strongly elliptic operator $A = -\frac{\partial^2}{\partial x^2}$ defined on X.

Let ρ_0, u_0 and f satisfy the hypotheses of Theorem 1.1 or Theorem 1.2. Assume for the moment that $\rho_0^\delta \in C^1([0,1])$ and $\rho_0^\delta \ge \delta$ in $(0,1)$ (for some constant $\delta > 0$). We may construct an approximate solution for any $\upsilon \in X^m, \varphi \in C^2([0,1])$

$$\begin{cases} \displaystyle\int_\Omega (\rho^m u_t^m + \rho^m u^m \cdot u_x^m + Au^m + p_x^m + \rho^m \Phi_x^m)\cdot \upsilon \mathrm{d}x = \int_\Omega \rho^m f^\delta \cdot \upsilon \mathrm{d}x, \\ \displaystyle\int_\Omega \rho_t^m \varphi \mathrm{d}x + \int_\Omega (\rho^m u^m)_x \varphi \mathrm{d}x = 0, \\ \displaystyle\int_\Omega \Phi_{xx}^m \varphi \mathrm{d}x = 4\pi g \int_\Omega \Big(\rho^m - \frac{m_0}{|\Omega|}\Big)\varphi \mathrm{d}x, \end{cases}$$

where $f^\delta \in C^1((0,T)\times(0,1))$ and $f^\delta \to f$ in $L^2(0,T; L^{\frac{2\gamma}{\gamma-1}}(0,1))$. The initial and boundary conditions are

$$u_0^m \equiv \sum_{k=1}^m (u_0, \varphi^k)_{L^2(\Omega)}\varphi^k \quad \text{and} \quad \rho^m(0) = \rho_0^\delta > \delta, \quad \rho^\delta(0) < |\rho_0|_{L^\infty} + 1,$$
$$|\rho_0^\delta - \rho_0|_{H^1(0,1)} \to 0, \quad u^m(0,x) = u^m(1,x) = 0, \quad \Phi^m(0,x) = \Phi^m(1,x) = 0.$$

Under the hypotheses of Theorem 1.1, similarly, for any fixed $\delta > 0$, we may get the similar estimate of Lemmas 2.1–2.6.

$$\sup_{0\le t\le T}(|\rho_\delta^m|_{L^\infty(0,1)} + |u_{x\delta}^m|_{L^2(0,1)}) + \int_0^T (|\sqrt{\rho_\delta^m}\,u_{\delta t}^m|^2_{L^2(0,1)} + |u_{x\delta}^m|^2_{L^2(0,1)} + |G_{x\delta}^m|^2_{L^2(0,1)})\mathrm{d}t \le C(T),$$
$$\sup_{0\le t\le T}(|\rho_\delta^m|_{H^1(0,1)} + |\rho_{\delta t}^m|_{L^2(0,1)} + |u_{\delta x}^m|_{L^2(0,1)}) + \int_0^T (|(\rho_\delta^m u_\delta^m)_t|^2_{L^2(0,1)} + |u_{xx\delta}^m|^2_{L^2(0,1)})\mathrm{d}t \le C(T).$$

Combining the course of estimates and the initial condition of approximate system, we can easily deduce that C is dependent on T, ρ_0, u_0 and f. Moreover, because the constants C of Lemmas 2.1–2.6 are independent of the lower bound of ρ_0. Here, $C(T)$ does not depend on δ and m (for any $m \ge M$, M is dependent on the approximate velocity of initial condition). Thus, we can deduce from the two above estimates that (ρ^m, u^m, Φ^m) converges, up to an extraction of subsequences, to some limit $(\rho_\delta, u_\delta, \Phi_\delta)$ in the obvious weak sense, and there are estimates:

$$\sup_{0\le t\le T}(|\rho_\delta|_{L^\infty(0,1)} + |u_{x\delta}|_{L^2(0,1)}) + \int_0^T (|\sqrt{\rho_\delta}\, u_{\delta t}|^2_{L^2(0,1)} + |u_{x\delta}|^2_{L^2(0,1)} + |G_{x\delta}|^2_{L^2(0,1)})\mathrm{d}t \le C(T),$$

$$\sup_{0\le t\le T}(|\rho_\delta|_{H^1(0,1)} + |\rho_{\delta t}|_{L^2(0,1)} + |u_{\delta x}|_{L^2(0,1)}) + \int_0^T (|(\rho_\delta u_\delta)_t|^2_{L^2(0,1)} + |u_{xx\delta}|^2_{L^2(0,1)})\mathrm{d}t \le C(T).$$

Because $C(T)$ is independent of δ, when $\delta \to 0$, we can deduce that $(\rho_\delta, u_\delta, \Phi_\delta)$ converges, up to an extraction of subsequences, to some limit (ρ, u, Φ) in weak sense, and

$$\sup_{0\le t\le T}(|\rho|_{L^\infty(0,1)} + |u_x|_{L^2(0,1)}) + \int_0^T (|\sqrt{\rho}\, u_t|^2_{L^2(0,1)} + |u_x|^2_{L^2(0,1)} + |G_x|^2_{L^2(0,1)})\mathrm{d}t \le C(T),$$

$$\sup_{0\le t\le T}(|\rho|_{H^1(0,1)} + |\rho_t|_{L^2(0,1)} + |u_x|_{L^2(0,1)}) + \int_0^T (|(\rho u)_t|^2_{L^2(0,1)} + |u_{xx}|^2_{L^2(0,1)})\mathrm{d}t \le C(T).$$

From the L^p-strong estimates of Poisson equation, we can easily get the regularity in Theorem 1.1.

As for Theorem 1.2, we can deal with it similarly, but the initial and outer power conditions are

$$\delta \le \rho^\delta(0) \le |\rho_0|_{L^\infty} + 1, \quad \rho_0^\delta \in C^2([0,1]), \quad g^\delta \in C_c^2(0,1), \quad |g_0^\delta - g|_{L^2(0,1)} \to 0,$$
$$f^\delta \in C_c^2((0,T)\times(0,1)), \quad |\rho_0^\delta - \rho_0|_{H^1(0,1)} \to 0, \quad |(f^\delta, f_x^\delta, f_t^\delta) - (f, f_x, f_t)|_{L^2_{\mathrm{loc}}(0,T;L^2(0,1))} \to 0.$$

Because we have compatibility condition, we let $u_0^\delta \in C^3[0,1]$ be the solution of the following elliptic equation

$$u_{0xx}^\delta = (p^\delta)_x + \rho_0^{\frac{1}{2}} g^\delta, \quad 0 < x < 1, \quad u_0^\delta(0) = u_0^\delta(1) = 0.$$

Combining the classical stableness results of the elliptic equation and the compatibility condition of Theorem 1.2, we deduce that $u_0^\delta \to u_0$ in $H^2(0,1)$, and u_0 satisfies the compatibility of Theorem 1.2.

For any fixed $\delta > 0$, similarly, we may get the similar results with Lemmas 2.1–2.7, and we have estimates

$$\sup_{0\le t\le T}(|\rho^\delta|_{H^1(0,1)} + |\rho_t^\delta|_{L^2(0,1)} + |u_x^\delta|_{L^2(0,1)}) + \int_0^T (|u_x^\delta|^2_{L^\infty(0,1)} + |u_{xx}^\delta|^2_{L^2(0,1)})\mathrm{d}t \le C(T),$$

$$|\sqrt{\rho^\delta}\, u_t^\delta(t)|^2_{L^2(0,1)} + \int_0^T |u_{xt}^\delta|^2_{L^2(0,1)}\mathrm{d}t \le C(T),$$

where C is independent of δ. Thus when $\delta \to 0$, we can deduce from the above estimates that $(\rho_\delta, u_\delta, \Phi_\delta)$ converges to some limit (ρ, u, Φ), and we have estimates

$$\sup_{0\le t\le T}(|\rho|_{H^1(0,1)} + |\rho_t|_{L^2(0,1)} + |u_x|_{L^2(0,1)}) + \int_0^T (|u_x|^2_{L^\infty(0,1)} + |u_{xx}|^2_{L^2(0,1)})\mathrm{d}t \le C(T),$$

$$|\sqrt{\rho}\, u_t(t)|^2_{L^2(0,1)} + \int_0^T |u_{xt}|^2_{L^2(0,1)}\mathrm{d}t \le C(T).$$

From L^p-strong estimates of Poisson equation, we can easily get the regularity in Theorem 1.2.

4 Proof of the Uniqueness

Let (ρ, u, Φ) and $(\overline{\rho}, \overline{u}, \overline{\Phi})$ be two solutions that satisfy the same initial condition. Then combining (1.1) and (1.2), we have

$$\rho u_t + \rho u u_x + \rho \Phi_x + p_x - u_{xx} = \rho f, \quad \overline{\rho u}_t + \overline{\rho u u}_x + \overline{\rho}\overline{\Phi}_x + \overline{p}_x - \overline{u}_{xx} = \overline{\rho} f.$$

Thus

$$\begin{aligned}&\rho(u-\overline{u})_t + \rho u(u-\overline{u})_x - (u-\overline{u})_{xx}\\ =&(\rho-\overline{\rho})(f-\overline{u}_t - \overline{u}\,\overline{u}_x - \overline{\Phi}_x) - (p-\overline{p})_x - \rho(\Phi-\overline{\Phi})_x - \rho(u-\overline{u})\overline{u}_x.\end{aligned} \tag{4.1}$$

Multiplying it by $(u-\overline{u})$ and integrating over $(0,1)$, we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \frac{1}{2}\rho(u-\overline{u})^2\mathrm{d}x + \frac{1}{2}\int_0^1 (\rho u)_x (u-\overline{u})^2\mathrm{d}x + \int_0^1 \rho u (u-\overline{u})_x (u-\overline{u})\mathrm{d}x + \int_0^1 (u-\overline{u})_x^2\mathrm{d}x\\ \le&\int_0^1 |\rho-\overline{\rho}||f-\overline{u}_t - \overline{u}\overline{u}_x - \overline{\Phi}_x||u-\overline{u}|\mathrm{d}x + \int_0^1 |p-\overline{p}||(u-\overline{u})_x|\mathrm{d}x\\ &+\int_0^1 \rho|(\Phi-\overline{\Phi})_x||u-\overline{u}|\mathrm{d}x + \int_0^1 \rho|u-\overline{u}|^2|\overline{u}_x|\mathrm{d}x,\end{aligned}$$

that is,

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \frac{1}{2}\rho(u-\overline{u})^2\mathrm{d}x + \int_0^1 (u-\overline{u})_x^2\mathrm{d}x\\ \le&|\rho-\overline{\rho}|_{L^2(0,1)}|f-\overline{u}_t - \overline{u}\overline{u}_x - \overline{\Phi}_x|_{L^2(0,1)}|u-\overline{u}|_{L^\infty(0,1)} + |p-\overline{p}|_{L^2(0,1)}|(u-\overline{u})_x|_{L^2(0,1)}\\ &+C|(\Phi-\overline{\Phi})_x|_{L^2(0,1)}|u-\overline{u}|_{L^2(0,1)} + |\sqrt{\rho}|u-\overline{u}||^2_{L^2(0,1)}|\overline{u}_x|_{L^\infty(0,1)}\\ \le&\varepsilon|(u-\overline{u})_x|^2_{L^2(0,1)} + |\rho-\overline{\rho}|^2_{L^2(0,1)}(C+C|\overline{u}_t|^2_{L^2(0,1)})\\ &+|p-\overline{p}|^2_{L^2(0,1)} + |\sqrt{\rho}|u-\overline{u}||^2_{L^2(0,1)}|\overline{u}_x|_{L^\infty(0,1)}.\end{aligned}$$

Consequently,

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \frac{1}{2}\rho(u-\overline{u})^2\mathrm{d}x + \int_0^1 (u-\overline{u})_x^2\mathrm{d}x\\ \le&|\rho-\overline{\rho}|_{L^2(0,1)}(C+C|\overline{u}_t|^2_{L^2(0,1)}) + |p-\overline{p}|^2_{L^2(0,1)} + |\sqrt{\rho}|u-\overline{u}||^2_{L^2(0,1)}|\overline{u}_x|_{L^\infty(0,1)}.\end{aligned} \tag{4.2}$$

Moreover, from the conservative mass equation, we have

$$\rho_t + \rho_x u + \rho u_x = 0, \quad \overline{\rho}_t + \overline{\rho}_x\overline{u} + \overline{\rho}\,\overline{u}_x = 0.$$

Then

$$(\rho-\overline{\rho})_t + (\rho-\overline{\rho})_x u + \overline{\rho}_x u - \overline{\rho}_x\overline{u} + (\rho-\overline{\rho})u_x + \overline{\rho}u_x - \overline{\rho}\,\overline{u}_x = 0,$$

that is,

$$(\rho-\overline{\rho})_t + (\rho-\overline{\rho})_x u + \overline{\rho}_x(u-\overline{u}) + (\rho-\overline{\rho})u_x + \overline{\rho}(u_x-\overline{u}_x) = 0.$$

Multiplying it by $(\rho-\overline{\rho})$ and integrating over $(0,1)$, we get

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \frac{1}{2}(\rho-\overline{\rho})^2\mathrm{d}x - \int_0^1 \frac{1}{2}(\rho-\overline{\rho})^2 u_x\mathrm{d}x + \int_0^1 \overline{\rho}_x(u-\overline{u})(\rho-\overline{\rho})\mathrm{d}x\\ &+\int_0^1 (\rho-\overline{\rho})^2 u_x\mathrm{d}x + \int_0^1 \overline{\rho}(u-\overline{u})_x(\rho-\overline{\rho})\mathrm{d}x = 0.\end{aligned}$$

Thus

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1\frac{1}{2}(\rho-\overline{\rho})^2\mathrm{d}x\\ \le& C\Big(\int_0^1(\rho-\overline{\rho})^2|u_x|\mathrm{d}x+\int_0^1|\overline{\rho}_x||(u-\overline{u})||(\rho-\overline{\rho})|\mathrm{d}x+\int_0^1\overline{\rho}|(u-\overline{u})_x||(\rho-\overline{\rho})|\mathrm{d}x\Big)\end{aligned}$$

that is,

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1\frac{1}{2}(\rho-\overline{\rho})^2\mathrm{d}x\\ \le& C\Big(|u_x|_{L^\infty(0,1)}\int_0^1(\rho-\overline{\rho})^2\mathrm{d}x+\Big(\int_0^1\rho_x^2\mathrm{d}x\Big)^{\frac{1}{2}}|u-\overline{u}|_{L^\infty(0,1)}\Big)\Big(\int_0^1(\rho-\overline{\rho})^2\mathrm{d}x\Big)^{\frac{1}{2}}\\ &+\Big(\int_0^1|(u-\overline{u})_x|^2\mathrm{d}x\Big)^{\frac{1}{2}}\Big(\int_0^1(\rho-\overline{\rho})^2\mathrm{d}x\Big)^{\frac{1}{2}}\\ \le& C\Big(|u_x|_{L^\infty(0,1)}+C(\varepsilon)+C(\varepsilon)\int_0^1\rho_x^2\mathrm{d}x\Big)\int_0^1(\rho-\overline{\rho})^2\mathrm{d}x+\varepsilon|(u-\overline{u})_x|^2_{L^2(0,1)}.\end{aligned}\tag{4.3}$$

Moreover, multiplying (1.1) by $a\gamma\rho^{\gamma-1}$, we get

$$p_t+p_xu+\gamma pu_x=0,\quad \overline{p}_t+\overline{p}_x\overline{u}+\gamma\overline{p}\overline{u}_x=0.$$

Similarly, we get

$$(p-\overline{p})_t+(p-\overline{p})_xu+\overline{p}_xu-\overline{p}_x\overline{u}+\gamma(p-\overline{p})u_x+\gamma\overline{p}u_x-\gamma\overline{p}\overline{u}_x=0,$$

that is,

$$(p-\overline{p})_t+(p-\overline{p})_xu+\overline{p}_x(u-\overline{u})+\gamma(p-\overline{p})u_x+\gamma\overline{p}(u-\overline{u})_x=0.$$

Multiplying it by $(p-\overline{p})$ and integrating over $(0,1)$, we get

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1\frac{1}{2}(p-\overline{p})^2\mathrm{d}x-\int_0^1\frac{1}{2}(p-\overline{p})^2\mathrm{d}x+\int_0^1\overline{p}(u-\overline{u})(p-\overline{p})\mathrm{d}x\\ &+\gamma\int_0^1(p-\overline{p})^2u_x\mathrm{d}x+\gamma\int_0^1\overline{p}(u-\overline{u})_x(p-\overline{p})\mathrm{d}x=0.\end{aligned}$$

Then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1(p-\overline{p})^2\mathrm{d}x\le& C|u_x|_{L^\infty(0,1)}|p-\overline{p}|^2_{L^2(0,1)}+|\overline{p}_x|_{L^2(0,1)}|u-\overline{u}|_{L^\infty(0,1)}|p-\overline{p}|_{L^2(0,1)}\\ &+\varepsilon|(u-\overline{u})_x|^2_{L^2(0,1)}+C(\varepsilon)|p-\overline{p}|^2_{L^2(0,1)}\\ \le& C(|u_x|_{L^\infty(0,1)}+|\overline{p}_x|_{L^2(0,1)}+1)|p-\overline{p}|^2_{L^2(0,1)}+2\varepsilon|(u-\overline{u})_x|^2_{L^2(0,1)}.\end{aligned}\tag{4.4}$$

From (4.2)–(4.4), we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1(\rho(u-\overline{u})^2+(\rho-\overline{\rho})^2+(p-\overline{p})^2)\mathrm{d}x+\int_0^1(u-\overline{u})_x^2\mathrm{d}x\\ \le& C(1+|\overline{u}_t|^2_{L^2(0,1)}+|\overline{u}_x|_{L^\infty(0,1)}+|\rho_x|^2_{L^2(0,1)}+|u_x|_{L^\infty(0,1)}+|\overline{p}_x|^2_{L^2(0,1)})\\ &\cdot(|\sqrt{\rho}(u-\overline{u})|^2_{L^2(0,1)}+|\rho-\overline{\rho}|^2_{L^2(0,1)}+|p-\overline{p}|^2_{L^2(0,1)}).\end{aligned}$$

Using Gronwall's inequality and the above regularity of strong solution, we have

$$|\sqrt{\rho}(u-\overline{u})|^2_{L^2(0,1)} + |\rho-\overline{\rho}|^2_{L^2(0,1)} + |p-\overline{p}|^2_{L^2(0,1)} = 0,$$

that is, $u=\overline{u}$, $\rho=\overline{\rho}$ in $L^2(0,1)$. From the classical theorems of Poisson equation, we get $|\Phi-\overline{\Phi}|_{W^{2,2}(0,1)}=0$. Finally, we get the uniqueness.

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