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A Sixth-Order Parabolic System in Semiconductors*

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Abstract The authors investigate the global existence and semiclassical limit of weak solutions to a sixth-order parabolic system, which is a quantum-corrected macroscopic model derived recently to simulate the quantum effects in miniaturized semiconductor devices.

Keywords Weak solution, Semiclassical limit, Sixth-order parabolic system
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1 Introduction

In this paper, we consider the system

$$\begin{cases} n_t = \Big[\varepsilon^4 n\Big(\dfrac{(n(\ln n)_{xx})_{xx}}{n} + \dfrac{((\ln n)_{xx})^2}{2}\Big)_x - \dfrac{\varepsilon^2}{2}(n(\ln n)_{xx})_x + n_x + nV_x\Big]_x, \\ -\lambda^2 V_{xx} = n - C(x), \end{cases} \tag{1.1}$$

where the electron density n and the electrostatic potential V are unknown variables; the doping profile $C(x)$ representing the distribution of charged background ions is supposed to be independent of time t; the scaled Planck constant $\varepsilon > 0$ and Debye length $\lambda > 0$ are parameters. For smooth positive solutions, (1.1) is equivalent to

$$\begin{cases} n_t = \varepsilon^4[(n(\ln n)_{xxx})_{xxx} + 2(n((\ln n)_{xx})^2)_{xx}] - \dfrac{\varepsilon^2}{2}(n(\ln n)_{xx})_{xx} + n_{xx} + (nV_x)_x, \\ -\lambda^2 V_{xx} = n - C(x). \end{cases} \tag{1.2}$$

Since all the results in this paper are obtained for fixed $\lambda > 0$, for convenience, we let $\lambda = 1$ in the following. To search for solutions which are physically reasonable, namely the solutions in which density n are nonnegative, we might as well suppose $n = \rho^4$. Moreover, let $T > 0$ be any fixed constant and $Q_T = (0, T] \times \mathbb{T}$ where $\mathbb{T}$ represents one dimensional flat torus. We consider

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the initial-periodic boundary value problem

$$\begin{cases} (\rho^4)_t = \varepsilon^4[(\rho^4(\ln\rho^4)_{xxx})_{xxx} + 2(\rho^4((\ln\rho^4)_{xx})^2)_{xx}] \\ \qquad -\dfrac{\varepsilon^2}{2}(\rho^4(\ln\rho^4)_{xx})_{xx} + (\rho^4)_{xx} + (\rho^4V_x)_x, & \text{in } (0,T]\times\mathbb{T}, \\ -V_{xx} = \rho^4 - C(x), & \text{in } (0,T]\times\mathbb{T}, \\ \rho(0,\cdot) = \rho_0(\,\cdot\,), & \text{in } \mathbb{T}, \\ \displaystyle\int_{\mathbb{T}}\rho_0^4\mathrm{d}x = \int_{\mathbb{T}}C(x)\mathrm{d}x, \end{cases} \tag{1.3}$$

where $\int_{\mathbb{T}}\rho_0^4\mathrm{d}x = \int_{\mathbb{T}}C(x)\mathrm{d}x$ is a necessary condition such that the Poisson equation in (1.3) is solvable.

The sixth-order parabolic equation in (1.1) was introduced recently by A. Jüngel and D. Matthes [24], and the weak solutions of its special case

$$n_t = n\Big(\frac{(n(\ln n)_{xx})_{xx}}{n} + \frac{((\ln n)_{xx})^2}{2}\Big)_x \tag{1.4}$$

were obtained by A. Jüngel and J. Milišić [25]. This system could be regarded naturally as a sixth-order correction of quantum drift-diffusion model. Quantum drift-diffusion model, quantum hydrodynamic model and quantum energy transport model are three quantum macroscopic models, which were introduced to simulate the quantum effects in miniaturized semiconductor devices. Some derivations of these models could be found in [14, 15, 26] etc., and some mathematical results of them were given in a series of works [1–13, 16–24, 29] etc. While for the sixth-order parabolic system (1.1), up to the authors' knowledge, no theoretic results were shown. The main task of this paper is to establish the global existence and semiclassical limit of weak solutions to (1.1).

Firstly, invoking semi-discretization method and compactness argument, we obtain the global weak solutions to (1.3) with nonnegative large initial value, where moreover the regularity of these solutions is better than that of [25] on the simple sixth-order equation (1.4). In the proof, we make use of two techniques. On one hand, we cope with the higher-order terms as three separate parts for the time-discretization weak solutions $n_\tau = \rho_\tau^4$. On the other hand, for any smooth function u,

$$u^4(\ln u^4)_{xxx} = 2u^2(u^2)_{xxx} + 2(u^2)_x(u^2)_{xx} - 16u(u^2)_xu_{xx}, \tag{1.5}$$

$$u^4((\ln u^4)_{xx})^2 = 4((u^2)_{xx})^2 - 64u(u_x)^2u_{xx}, \tag{1.6}$$

$$(u^4)_{xxx} = 2u^2(u^2)_{xxx} + 6(u^2)_x(u^2)_{xx},$$

$$u^4(\ln u^4)_{xx} = 2u^2(u^2)_{xx} - 2((u^2)_x)^2. \tag{1.7}$$

These four equivalent transformations permit us to obtain the key convergent results for the three separate higher-order parts, that is,

$$\begin{aligned} n_\tau(\ln n_\tau)_{xxx} &\overset{\text{weakly}}{\rightharpoonup} n(\ln n)_{xxx}, \quad && \text{in } L^{\frac{12}{7}}(0,T;L^2(\mathbb{T})), \\ n_\tau(\ln n_\tau)_{xx}^2 &\overset{\text{weakly}}{\rightharpoonup} n(\ln n)_{xx}^2, && \text{in } L^{\frac{3}{2}}(0,T;H^{-1}(\mathbb{T})), \\ n_\tau(\ln n_\tau)_{xx} &\overset{\text{weakly}}{\rightharpoonup} n(\ln n)_{xx}, && \text{in } L^{\frac{12}{7}}(0,T;L^2(\mathbb{T})). \end{aligned}$$

Secondly, in the proof of semiclassical limit, one needs that the higher order quantum terms, especially the sixth-order terms vanish in some sense as $\varepsilon \to 0$. To deal with this difficulty, we have to make full use of delicate interpolation skills and the equivalent transformations (1.5) and (1.6). More precisely, by the Gagliardo-Nirenberg inequality, one has $\varepsilon^{\frac{2}{3}}(\sqrt{n})_x \in L^6(0,T;L^2(\mathbb{T}))$, $\varepsilon^{\frac{4}{3}}(\sqrt{n})_{xx} \in L^3(0,T;L^2(\mathbb{T}))$, $\varepsilon^{\frac{5}{3}}(\sqrt{n})_{xx} \in L^{\frac{12}{5}}(0,T;L^\infty(\mathbb{T}))$ uniformly in ε. These and (1.5)–(1.6) imply $\varepsilon^{\frac{8}{3}}n(\ln n)^2_{xx} \in L^{\frac{3}{2}}(0,T;L^1(\mathbb{T}))$, $\varepsilon^{\frac{7}{3}}n(\ln n)_{xxx} \in L^{\frac{12}{7}}(0,T;L^2(\mathbb{T}))$ uniformly in ε, which permit us to show the vanishing of the higher order terms in the discussion.

The following notations will be used in this paper. $W^{m,p}(\mathbb{T})$ ($H^m(\mathbb{T}) = W^{m,2}(\mathbb{T})$) and $C^{k,\theta}(\mathbb{T})$ denote the Sobolev spaces and Hölder spaces, respectively; $H^{-m}(\mathbb{T})$ denotes the dual space of $H^m(\mathbb{T})$; $\mathscr{D}'(Q_T)$ represents the set of all distributions on Q_T; $A \hookrightarrow B$ (or $A \hookrightarrow\hookrightarrow B$) denotes A being continuously (or compactly) embedded in B; $f_x^\alpha = (f_x)^\alpha$, $f_{xx}^\alpha = (f_{xx})^\alpha$, $f_{xxx}^\alpha = (f_{xxx})^\alpha$, $\alpha \in \mathbb{R}$; $f_\tau \to$ ($\rightharpoonup$ or $\overset{*}{\rightharpoonup}$) f in A denotes a sequence $\{f_\tau\}_{\tau>0} \subset A$ converging strongly (weakly or weakly star) to $f \in A$ as $\tau \to 0$.

Our main results are stated as follows.

Theorem 1.1 *Let $C(x) \in L^\infty(\mathbb{T})$ and ρ_0 be nonnegative measurable function on $\mathbb{T}$ with $\int_{\mathbb{T}} \rho_0^4 \mathrm{d}x = \int_{\mathbb{T}} C(x)\mathrm{d}x$ and $\int_{\mathbb{T}}[\rho_0^4(\ln \rho_0^4 - 1) + 1]\mathrm{d}x < \infty$. Then for any fixed $\varepsilon > 0$, there exists a weak solution (ρ, V) to* (1.3) *such that $0 \le \rho \in L^\infty(0,T;L^4(\mathbb{T}))$, $\rho^2 \in L^2(0,T;H^3(\mathbb{T}))$, $\rho^4 \in W^{1,\frac{3}{2}}(0,T;H^{-3}(\mathbb{T}))$, $\int_{\mathbb{T}} \rho^4 \mathrm{d}x = \int_{\mathbb{T}} C(x)\mathrm{d}x$, $V \in L^6(0,T;H^2(\mathbb{T}))$ and*

$$
\begin{aligned}
&\int_0^T \langle \partial_t \rho^4, \varphi\rangle_{\langle H^{-3},H^3\rangle}\mathrm{d}t \\
&= -\varepsilon^4 \int_0^T\int_{\mathbb{T}} \rho^4(\ln \rho^4)_{xxx}\varphi_{xxx}\mathrm{d}x\mathrm{d}t + 2\varepsilon^4 \int_0^T\int_{\mathbb{T}} \rho^4(\ln \rho^4)^2_{xx}\varphi_{xx}\mathrm{d}x\mathrm{d}t \\
&\quad - \frac{\varepsilon^2}{2}\int_0^T\int_{\mathbb{T}} \rho^4(\ln \rho^4)_{xx}\varphi_{xx}\mathrm{d}x\mathrm{d}t - \int_0^T\int_{\mathbb{T}}[(\rho^4)_x + \rho^4 V_x]\varphi_x\mathrm{d}x\mathrm{d}t
\end{aligned} \tag{1.8}
$$

for any $\varphi \in L^3(0,T;H^3(\mathbb{T}))$ and $-V_{xx} = \rho^4 - C(x)$ a.e. in Q_T, where the initial value is satisfied in the sense of $H^{-3}(\mathbb{T})$.

Remark 1.1 In (1.8), the expressions $\rho^4(\ln \rho^4)_{xxx}$, $\rho^4(\ln \rho^4)^2_{xx}$, $\rho^4(\ln \rho^4)_{xx}$ are in reality in the sense of (1.5)–(1.7). For the sake of brevity, in the following these expressions would also be used.

Since the solution depends on ε, for the sake of explicitness, we relabel ρ, V as $\rho_\varepsilon, V_\varepsilon$ in the following theorem.

Theorem 1.2 *For those solutions obtained in Theorem* 1.1, *$\{(\rho_\varepsilon, V_\varepsilon)\}_{\varepsilon>0}$, as $\varepsilon \to 0$, there exists a subsequence which is not relabeled, such that*

$$\rho_\varepsilon^4 \rightharpoonup n, \quad \text{in } L^{\frac{24}{13}}(0,T;W^{1,\frac{12}{11}}(\mathbb{T})) \cap L^{12}(0,T;L^{\frac{6}{5}}(\mathbb{T})), \tag{1.9}$$

$$\rho_\varepsilon^4 \to n, \quad \text{in } L^{\frac{24}{13}}(0,T;C^{0,\lambda}(\mathbb{T})) \quad \Big(\forall\, 0 < \lambda < \frac{1}{12}\Big), \tag{1.10}$$

$$V_\varepsilon \to V, \quad \text{in } L^{\frac{24}{13}}(0,T;H^2(\mathbb{T})), \tag{1.11}$$

$$\partial_t\rho_\varepsilon^4 \rightharpoonup \partial_t n, \quad \text{in } L^{\frac{3}{2}}(0,T;H^{-3}(\mathbb{T})), \tag{1.12}$$

$$\varepsilon^4\rho_\varepsilon^4(\ln \rho_\varepsilon^4)_{xxx} \to 0, \quad \text{in } L^{\frac{12}{7}}(0,T;L^2(\mathbb{T})), \tag{1.13}$$

$$\varepsilon^4\rho_\varepsilon^4(\ln \rho_\varepsilon^4)^2_{xx} \to 0, \quad \text{in } L^{\frac{3}{2}}(0,T;H^{-1}(\mathbb{T})), \tag{1.14}$$

$$\varepsilon^2\rho_\varepsilon^4(\ln\rho_\varepsilon^4)_{xx}\to 0, \qquad in\ L^{\frac{12}{7}}(0,T;L^2(\mathbb{T})), \tag{1.15}$$

and (n, V) *is a weak solution of the classical drift-diffusion model in the following sense:*

$$\begin{cases}\displaystyle\int_0^T\langle\partial_t n,\varphi\rangle_{\langle H^{-3},H^3\rangle}\mathrm{d}t=-\int_0^T\int_{\mathbb{T}}(n_x+nV_x)\varphi_x\mathrm{d}x\mathrm{d}t, \quad \forall\,\varphi\in L^3(0,T;H^3(\mathbb{T})),\\ -V_{xx}=n-C(x), \quad a.e.\ in\ Q_T.\end{cases} \tag{1.16}$$

This article is organized as follows. In Section 2, we show the semi-discretization approximation problem and its solutions. Section 3 contains the uniform entropy estimate which would be used in the proof of existence. Then in Section 4, we use a compactness argument for fixed $\varepsilon>0$ to prove Theorem 1.1. Furthermore in Section 5, we obtain the semiclassical limit by the uniform estimates in ε. The final section is a summary.

2 Approximate Problem

In this section, we introduce the semi-discretization approximate problem of (1.3) and prove the following Proposition 2.1 by employing the idea of [25] where the approximate problem of (1.4) was treated. More precisely, let $\tau>0$ such that $T=N\tau$ (without loss of generality, otherwise, let $N=[\frac{T}{\tau}]+1$). Hence $N=N(\tau)\in\mathbb{N}$ depends only on τ. We divide the time interval $(0,T]$ by $(0,T]=\bigcup\limits_{k=1}^{N}((k-1)\tau,k\tau]$. For any $k=1,2,\cdots,N$, given ρ_{k-1} such that $\int_{\mathbb{T}}\rho_{k-1}^4\mathrm{d}x=\int_{\mathbb{T}}C(x)\mathrm{d}x$, we will solve the following problem:

$$\begin{cases}\dfrac{\rho_k^4-\rho_{k-1}^4}{\tau}=\varepsilon^4[(\rho_k^4(\ln\rho_k^4)_{xxx})_{xxx}+2(\rho_k^4(\ln\rho_k^4)_{xx}^2)_{xx}]\\ \qquad\qquad\qquad -\dfrac{\varepsilon^2}{2}(\rho_k^4(\ln\rho_k^4)_{xx})_{xx}+(\rho_k^4)_{xx}+(\rho_k^4(V_k)_x)_x, \quad \text{in } \mathbb{T},\\ -(V_k)_{xx}=\rho_k^4-C(x), \quad \text{in } \mathbb{T},\\ \displaystyle\int_{\mathbb{T}}\rho_k^4\mathrm{d}x=\int_{\mathbb{T}}C(x)\mathrm{d}x.\end{cases} \tag{2.1}$$

Proposition 2.1 *Let* $C(x)\in L^\infty(\mathbb{T})$ *and* ρ_{k-1} *be nonnegative measurable function on* $\mathbb{T}$ *with* $\int_{\mathbb{T}}\rho_{k-1}^4\mathrm{d}x=\int_{\mathbb{T}}C(x)\mathrm{d}x$ *and* $\int_{\mathbb{T}}[\rho_{k-1}^4(\ln\rho_{k-1}^4-1)+1]\mathrm{d}x<\infty$. *Then* (2.1) *has a weak solution* $(\rho_k,V_k)\in(H^3(\mathbb{T}))^2$ *satisfying that for any* $\rho_k\geq 0$, $\int_{\mathbb{T}}\rho_k^4\mathrm{d}x=\int_{\mathbb{T}}C(x)\mathrm{d}x$, *and in the sense that, for any* $\varphi\in H^3(\mathbb{T})$,

$$\begin{aligned}\int_{\mathbb{T}}\frac{\rho_k^4-\rho_{k-1}^4}{\tau}\varphi\mathrm{d}x=&-\varepsilon^4\int_{\mathbb{T}}\rho_k^4(\ln\rho_k^4)_{xxx}\varphi_{xxx}\mathrm{d}x+2\varepsilon^4\int_{\mathbb{T}}\rho_k^4(\ln\rho_k^4)_{xx}^2\varphi_{xx}\mathrm{d}x\\ &-\frac{\varepsilon^2}{2}\int_{\mathbb{T}}\rho_k^4(\ln\rho_k^4)_{xx}\varphi_{xx}\mathrm{d}x-\int_{\mathbb{T}}[(\rho_k^4)_x+\rho_k^4(V_k)_x]\varphi_x\mathrm{d}x,\end{aligned} \tag{2.2}$$

and V_k *is a strong solution to* $-(V_k)_{xx}=\rho_k^4-C(x)$ *in* $\mathbb{T}$.

Proof Using a general version of Leray-Schauder fixed-point theorem (see [28, Theorem

B.5, p. 262]), we could solve the following regularized problem:

$$\begin{cases} \dfrac{\rho_k^4-\rho_{k-1}^4}{\tau}=\varepsilon^4[(\rho_k^4(\ln\rho_k^4)_{xxx})_{xxx}+2(\rho_k^4(\ln\rho_k^4)_{xx}^2)_{xx}]-\dfrac{\varepsilon^2}{2}(\rho_k^4(\ln\rho_k^4)_{xx})_{xx} \\ \qquad\qquad +\delta[(\ln\rho_k^4)_{xxxxxx}-\ln\rho_k^4]+(\rho_k^4)_{xx}+(\rho_k^4(V_k)_x)_x, \quad \text{in } \mathbb{T}, \\ -(V_k)_{xx}=\rho_k^4-C(x), \quad \text{in } \mathbb{T}, \\ \displaystyle\int_{\mathbb{T}}\rho_k^4\mathrm{d}x=\int_{\mathbb{T}}C(x)\mathrm{d}x. \end{cases} \tag{2.3}$$

Let $\overline{\rho}\in W^{2,4}(\mathbb{T})\hookrightarrow L^\infty(\mathbb{T})$ and $\sigma\in[0,1]$. Then there is a unique (up to an additive constant) solution $\overline{V}\in H^2(\mathbb{T})$ to the Poisson problem in (2.3). We introduce

$$\begin{aligned} a(w,\phi)=&\int_{\mathbb{T}}\Big[\varepsilon^4\overline{\rho}^4w_{xxx}\phi_{xxx}+\frac{\varepsilon^2}{2}\overline{\rho}^4w_{xx}\phi_{xx} \\ &+\delta(w_{xxx}\phi_{xxx}+w\phi)+\overline{\rho}^4w_x\phi_x\Big]\mathrm{d}x, \quad \forall\, w,\phi\in H^3(\mathbb{T}), \\ F(\phi)=&-\sigma\int_{\mathbb{T}}\Big[-2\varepsilon^4\overline{\rho}^4(\ln\overline{\rho}^4)_{xx}^2\phi_{xx}+\overline{\rho}^4\overline{V}_x\phi_x+\frac{\overline{\rho}^4-\rho_{k-1}^4}{\tau}\phi\Big]\mathrm{d}x, \quad \forall\,\phi\in H^3(\mathbb{T}). \end{aligned}$$

Then by Lax-Milgram theorem, there exists a unique $w\in H^3(\mathbb{T})$ such that $a(w,\phi)=F(\phi)$ for any $\phi\in H^3(\mathbb{T})$. Thus we can define a mapping G: $W^{2,4}(\mathbb{T})\times[0,1]\to W^{2,4}(\mathbb{T})$ by $G(\overline{\rho},\sigma)=\rho=\mathrm{e}^{\frac{w}{4}}\in H^3(\mathbb{T})$. It is easy to check that $G(\overline{\rho},0)\equiv 1$ for any $\overline{\rho}\in W^{2,4}(\mathbb{T})$ and G is continuous and compact. We are left to show the uniform bound of fixed-points. Let $(\rho,\sigma)\in W^{2,4}(\mathbb{T})\times[0,1]$ such that $G(\rho,\sigma)=\rho\in H^3(\mathbb{T})$, i.e., $(\rho,V)\in H^3(\mathbb{T})\times H^2(\mathbb{T})$, where for the sake of explicitness, we relabel it as (ρ_δ,V_δ), satisfies

$$-(V_\delta)_{xx}={\rho_\delta}^4-C(x), \quad \int_{\mathbb{T}}{\rho_\delta}^4\mathrm{d}x=\int_{\mathbb{T}}C(x)\mathrm{d}x \tag{2.4}$$

and for any $\phi\in H^3(\mathbb{T})$,

$$\begin{aligned} &\sigma\int_{\mathbb{T}}\frac{{\rho_\delta}^4-\rho_{k-1}^4}{\tau}\phi\mathrm{d}x \\ =&-\int_{\mathbb{T}}\Big[\varepsilon^4{\rho_\delta}^4(\ln{\rho_\delta}^4)_{xxx}\phi_{xxx}+\frac{\varepsilon^2}{2}{\rho_\delta}^4(\ln{\rho_\delta}^4)_{xx}\phi_{xx}+\sigma{\rho_\delta}^4(V_\delta)_x\phi_x \\ &+({\rho_\delta}^4)_x\phi_x-2\sigma\varepsilon^4{\rho_\delta}^4(\ln{\rho_\delta}^4)_{xx}^2\phi_{xx}+\delta((\ln{\rho_\delta}^4)_{xxx}\phi_{xxx}+\phi\ln{\rho_\delta}^4)\Big]\mathrm{d}x. \end{aligned} \tag{2.5}$$

Using $\ln{\rho_\delta}^2\in H^3(\mathbb{T})$ as a test function and integration by parts, we deduce that

$$\begin{aligned} &\frac{\sigma}{\tau}\int_{\mathbb{T}}[\rho_\delta^4(\ln\rho_\delta^4-1)+1]\mathrm{d}x+\varepsilon^4\int_{\mathbb{T}}[\rho_\delta^4(\ln\rho_\delta^4)_{xxx}^2-2\sigma\rho_\delta^4(\ln\rho_\delta^4)_{xx}^3]\mathrm{d}x+\frac{\varepsilon^2}{2}\int_{\mathbb{T}}\rho_\delta^4(\ln\rho_\delta^4)_{xx}^2\mathrm{d}x \\ &+\delta\int_{\mathbb{T}}[|(\ln\rho_\delta^4)_{xxx}|^2\mathrm{d}x+|\ln\rho_\delta^4|^2]\mathrm{d}x+4\int_{\mathbb{T}}|({\rho_\delta}^2)_x|^2\mathrm{d}x+\sigma\int_{\mathbb{T}}{\rho_\delta}^8\mathrm{d}x \\ \le&\frac{\sigma}{\tau}\int_{\mathbb{T}}[\rho_{k-1}^4(\ln\rho_{k-1}^4-1)+1]\mathrm{d}x+C\int_{\mathbb{T}}{\rho_\delta}^4\mathrm{d}x\le C. \end{aligned} \tag{2.6}$$

Here and in the following, C is a constant independent of δ. Observe from [25, Lemma 2.1] that

$$\begin{aligned}\int_{\mathbb{T}}[\rho_\delta^4(\ln\rho_\delta^4)_{xxx}^2-2\sigma\rho_\delta^4(\ln\rho_\delta^4)_{xx}^3]\mathrm{d}x &\geq \int_{\mathbb{T}}[\rho_\delta^4(\ln\rho_\delta^4)_{xxx}^2-2\rho_\delta^4(\ln\rho_\delta^4)_{xx}^3]\mathrm{d}x\\ &\geq 2\int_{\mathbb{T}}[|(\rho_\delta^2)_{xxx}|^2+|(\rho_\delta)_x(\rho_\delta)_{xx}|^2+|(\rho_\delta^{\frac{2}{3}})_x|^6]\mathrm{d}x. \qquad (2.7)\end{aligned}$$

Therefore in view of the standard elliptic theory,

$$\begin{aligned}&\|\rho_\delta^2(\ln\rho_\delta^4)_{xx}\|_{L^2(\mathbb{T})}+\|\rho_\delta^2\|_{H^3(\mathbb{T})}+\|(\rho_\delta)_x(\rho_\delta)_{xx}\|_{L^2(\mathbb{T})}\\ &+\|(\rho_\delta^{\frac{2}{3}})_x\|_{L^6(\mathbb{T})}+\delta^{\frac{1}{2}}\|\ln\rho_\delta^4\|_{H^3(\mathbb{T})}+\|V_\delta\|_{H^2(\mathbb{T})}\leq M. \qquad (2.8)\end{aligned}$$

This implies $\|\ln\rho_\delta\|_{H^3(\mathbb{T})}\leq M_\delta$, where M_δ is a constant independent of σ, and hence $\|\rho_\delta\|_{H^3(\mathbb{T})}\leq M_\delta$ which establishes the uniform bound of fixed-points. So by Leray-Schauder fixed-point theorem for any $\delta>0$, we obtain a solution ρ_δ to $G(\rho_\delta,1)=\rho_\delta$, that is, ρ_δ satisfies (2.4) and (2.5) for $\sigma=1$. Hence, we obtain the weak solution to (2.3).

Moreover, we also have the same uniform estimates as (2.8) by repeating the proofs above provided $\sigma=1$ in (2.6). By noting the compact embedding $H^{1+i}(\mathbb{T})\hookrightarrow\hookrightarrow C^{i,\lambda}(\mathbb{T})\,(0<\lambda<\frac{1}{2})$, $i=1,2$, we establish the convergent results as follows, for a subsequence which is not relabeled,

$$\begin{aligned}\rho_\delta^2&\rightharpoonup\rho^2, &&\text{in } H^3(\mathbb{T}),\\ V_\delta&\rightharpoonup V, &&\text{in } H^2(\mathbb{T}),\\ \rho_\delta^2&\to\rho^2, &&\text{in } C^{2,\lambda}(\mathbb{T}),\\ V_\delta&\to V, &&\text{in } C^{1,\lambda}(\mathbb{T}),\\ \delta\ln\rho_\delta^4&\to 0, &&\text{in } H^3(\mathbb{T}).\end{aligned} \qquad (2.9)$$

These formulas imply

$$\begin{aligned}\rho_\delta^4(\ln\rho_\delta^4)_{xxx}&\rightharpoonup\rho^4(\ln\rho^4)_{xxx}, &&\text{in } L^2(\mathbb{T}),\\ \rho_\delta^4(\ln\rho_\delta^4)_{xx}^2&\rightharpoonup\rho^4(\ln\rho^4)_{xx}^2, &&\text{in } L^2(\mathbb{T}),\\ \rho_\delta^4(\ln\rho_\delta^4)_{xx}&\rightharpoonup\rho^4(\ln\rho^4)_{xx}, &&\text{in } L^2(\mathbb{T}).\end{aligned} \qquad (2.10)$$

On one hand,

$$\begin{aligned}\rho_\delta^4(\ln\rho_\delta^4)_{xxx}&\rightharpoonup\rho^4(\ln\rho^4)_{xxx}, &&\text{in } \mathscr{D}'(\mathbb{T}),\\ \rho_\delta^4(\ln\rho_\delta^4)_{xx}^2&\rightharpoonup\rho^4(\ln\rho^4)_{xx}^2, &&\text{in } \mathscr{D}'(\mathbb{T}),\\ \rho_\delta^4(\ln\rho_\delta^4)_{xx}&\rightharpoonup\rho^4(\ln\rho^4)_{xx}, &&\text{in } \mathscr{D}'(\mathbb{T}).\end{aligned} \qquad (2.11)$$

Since the proof of (2.11) is similar to but easier than that of (4.11)–(4.13) in the following Proposition 4.1, we omit the details here for the sake of brevity. On the other hand, we check the L^2-boundedness by employing (1.5)–(1.6), (2.8) and the embedding $H^3(\mathbb{T})\hookrightarrow W^{2,\infty}(\mathbb{T})$.

$$\begin{aligned}\|\rho_\delta^4(\ln\rho_\delta^4)_{xxx}\|_{L^2(\mathbb{T})}&=\|2\rho_\delta^2(\rho_\delta^2)_{xxx}+2(\rho_\delta^2)_x(\rho_\delta^2)_{xx}-32\rho_\delta^2(\rho_\delta)_x(\rho_\delta)_{xx}\|_{L^2(\mathbb{T})}\\ &\leq C(\|\rho_\delta^2\|_{L^\infty(\mathbb{T})}\|\rho_\delta^2\|_{H^3(\mathbb{T})}+\|(\rho_\delta^2)_x\|_{L^\infty(\mathbb{T})}\|\rho_\delta^2\|_{H^2(\mathbb{T})}\\ &\quad+\|\rho_\delta^2\|_{L^\infty(\mathbb{T})}\|(\rho_\delta)_x(\rho_\delta)_{xx}\|_{L^2(\mathbb{T})})\leq C,\\ \|\rho_\delta^4(\ln\rho_\delta^4)_{xx}^2\|_{L^2(\mathbb{T})}&=\|4(\rho_\delta^2)_{xx}^2-32(\rho_\delta^2)_x(\rho_\delta)_x(\rho_\delta)_{xx}\|_{L^2(\mathbb{T})}\\ &\leq C(\|(\rho_\delta^2)_{xx}\|_{L^4(\mathbb{T})}^2+\|(\rho_\delta^2)_x\|_{L^\infty(\mathbb{T})}\|(\rho_\delta)_x(\rho_\delta)_{xx}\|_{L^2(\mathbb{T})})\leq C,\\ \|\rho_\delta^4(\ln\rho_\delta^4)_{xx}\|_{L^2(\mathbb{T})}&\leq C(\|\rho_\delta^2\|_{L^\infty(\mathbb{T})}\|\rho_\delta^2(\ln\rho_\delta^4)_{xx}\|_{L^2(\mathbb{T})}\leq C,\end{aligned}$$

where C is a constant independent of δ. These complete the proof of (2.10).

Passing to the limit $\delta \to 0$ in the weak form of (2.3), we complete the proof.

3 Uniform Estimates

Suppose that ρ_0 and $C(x)$ satisfy the assumption of Proposition 2.1 for $k = 1$. We use Proposition 2.1 iteratively to obtain a sequence of approximate solution $(\rho_k, V_k) \in (H^3(\mathbb{T}))^2$ $(k = 1, 2, \cdots, N)$. From now on, C (or C_ε) is supposed to be a constant dependent only on T, $\|C(\cdot)\|_{L^\infty(\mathbb{T})}$, $\|\rho_0^4(\ln\rho_0^4 - 1) + 1\|_{L^1(\mathbb{T})}$ (and ε).

Lemma 3.1

$$\begin{aligned}&\int_{\mathbb{T}}[\rho_k^4(\ln\rho_k^4 - 1) + 1]\mathrm{d}x + 2\varepsilon^4\tau\int_{\mathbb{T}}[|(\rho_k^2)_{xxx}|^2 + |(\rho_k)_x(\rho_k)_{xx}|^2 + |(\rho_k^{\frac{2}{3}})_x|^6]\mathrm{d}x\\&+\frac{\varepsilon^2}{2}\tau\int_{\mathbb{T}}|\rho_k^2(\ln\rho_k^4)_{xx}|^2\mathrm{d}x + 4\tau\int_{\mathbb{T}}|(\rho_k^2)_x|^2\mathrm{d}x + \tau\int_{\mathbb{T}}\rho_k^8\mathrm{d}x\\&\le \int_{\mathbb{T}}[\rho_{k-1}^4(\ln\rho_{k-1}^4 - 1) + 1]\mathrm{d}x + C\tau\int_{\mathbb{T}}\rho_k^4\mathrm{d}x.\end{aligned}\tag{3.1}$$

Proof Using (2.6) and (2.7) with $\sigma = 1$, we have

$$\begin{aligned}&\frac{1}{\tau}\int_{\mathbb{T}}[\rho_\delta^4(\ln\rho_\delta^4 - 1) + 1]\mathrm{d}x + 2\varepsilon^4\int_{\mathbb{T}}[|(\rho_\delta^2)_{xxx}|^2 + |(\rho_\delta)_x(\rho_\delta)_{xx}|^2 + |(\rho_\delta^{\frac{2}{3}})_x|^6]\mathrm{d}x\\&+\frac{\varepsilon^2}{2}\int_{\mathbb{T}}\rho_\delta^4(\ln\rho_\delta^4)_{xx}^2\mathrm{d}x + \delta\int_{\mathbb{T}}[|(\ln\rho_\delta^4)_{xxx}|^2\mathrm{d}x + |\ln\rho_\delta^4|^2]\mathrm{d}x + 4\int_{\mathbb{T}}|({\rho_\delta}^2)_x|^2\mathrm{d}x + \int_{\mathbb{T}}{\rho_\delta}^8\mathrm{d}x\\&\le \frac{1}{\tau}\int_{\mathbb{T}}[\rho_{k-1}^4(\ln\rho_{k-1}^4 - 1) + 1]\mathrm{d}x + C\int_{\mathbb{T}}{\rho_\delta}^4\mathrm{d}x.\end{aligned}\tag{3.2}$$

From (2.9), one has $(\rho_\delta^{\frac{2}{3}})_x \rightharpoonup (\rho_k^{\frac{2}{3}})_x$ in $L^6(\mathbb{T})$ and $(\rho_\delta)_x(\rho_\delta)_{xx} \rightharpoonup (\rho_k)_x(\rho_k)_{xx}$ in $L^2(\mathbb{T})$ by [25, Proposition 6.1]. Therefore, letting $\delta \to 0$ in (3.2), we complete the proof.

Define the piecewise function in t in the following sense, $\rho_\tau(t,x) \triangleq \rho_k(x)$, $V_\tau(t, x) \triangleq V_k(x)$ for $x \in \mathbb{T}$, $t \in ((k-1)\tau, k\tau]$ $(k = 1, 2, \cdots, N)$. We now obtain the uniform estimates in τ for ρ_τ, V_τ.

Proposition 3.1

$$\begin{aligned}&\|\rho_\tau\|_{L^\infty(0,T;L^4(\mathbb{T}))\cap L^8(Q_T)} + \|(\rho_\tau^2)_x\|_{L^2(Q_T)} + \|\varepsilon^2(\rho_\tau^2)_{xxx}\|_{L^2(Q_T)}\\&+\|\varepsilon^2(\rho_\tau^{\frac{2}{3}})_x\|_{L^6(Q_T)} + \|\varepsilon^2(\rho_\tau)_x(\rho_\tau)_{xx}\|_{L^2(Q_T)} + \|\varepsilon\rho_\tau^2(\ln\rho_\tau^4)_{xx}\|_{L^2(Q_T)} \le C,\end{aligned}\tag{3.3}$$

$$\begin{aligned}&\|\rho_\tau^2\|_{L^2(0,T;H^1(\mathbb{T}))} + \|\varepsilon^2\rho_\tau^2\|_{L^2(0,T;H^3(\mathbb{T}))} + \|\varepsilon^2\rho_\tau^{\frac{2}{3}}\|_{L^6(0,T;W^{1,6}(\mathbb{T}))}\\&+\|\varepsilon^{\frac{1}{3}}\rho_\tau^2\|_{L^{12}(0,T;L^\infty(\mathbb{T}))} + \|\varepsilon^{\frac{4}{3}}\rho_\tau^4(\ln\rho_\tau^4)_{xx}\|_{L^{\frac{12}{7}}(0,T;L^2(\mathbb{T}))} \le C,\end{aligned}\tag{3.4}$$

$$\|\varepsilon^{\frac{8}{3}}\rho_\tau^4(\ln\rho_\tau^4)_{xx}^2\|_{L^{\frac{3}{2}}(0,T;L^1(\mathbb{T}))} + \|\varepsilon^{\frac{7}{3}}\rho_\tau^4(\ln\rho_\tau^4)_{xxx}\|_{L^{\frac{12}{7}}(0,T;L^2(\mathbb{T}))} \le C,\tag{3.5}$$

$$\|\rho_\tau^2\|_{L^{24}(0,T;L^4(\mathbb{T}))} + \|V_\tau\|_{L^6(0,T;H^2(\mathbb{T}))} \le C_\varepsilon.\tag{3.6}$$

Proof Using the Gronwall's inequality and Lemma 3.1, we could establish (3.3). Obviously, we have from (3.3) that

$$\|\rho_\tau^2\|_{L^2(0,T;H^1(\mathbb{T}))} + \|\varepsilon^2\rho_\tau^2\|_{L^2(0,T;H^3(\mathbb{T}))} + \|\varepsilon^2\rho_\tau^{\frac{2}{3}}\|_{L^6(0,T;W^{1,6}(\mathbb{T}))} \le C.\tag{3.7}$$

Using the Gagliardo-Nirenberg inequality, we have

$$\|\rho_\tau^2\|_{L^\infty(\mathbb{T})} \le C\|\rho_\tau^2\|_{H^3(\mathbb{T})}^{\frac{1}{6}}\|\rho_\tau^2\|_{L^2(\mathbb{T})}^{\frac{5}{6}},$$
$$\|\rho_\tau^2\|_{L^4(\mathbb{T})} \le C\|\rho_\tau^2\|_{H^3(\mathbb{T})}^{\frac{1}{12}}\|\rho_\tau^2\|_{L^2(\mathbb{T})}^{\frac{11}{12}}.$$

This formula and the Hölder inequality, together with (3.3) and (3.7), imply

$$\|\varepsilon^{\frac{1}{3}}\rho_\tau^2\|_{L^{12}(0,T;L^\infty(\mathbb{T}))} \le C\|\varepsilon^2\rho_\tau^2\|_{L^2(0,T;H^3(\mathbb{T}))}^{\frac{1}{6}}\|\rho_\tau^2\|_{L^\infty(0,T;L^2(\mathbb{T}))}^{\frac{5}{6}} \le C,$$
$$\|\varepsilon^{\frac{1}{6}}\rho_\tau^2\|_{L^{24}(0,T;L^4(\mathbb{T}))} \le C\|\varepsilon^2\rho_\tau^2\|_{L^2(0,T;H^3(\mathbb{T}))}^{\frac{1}{12}}\|\rho_\tau^2\|_{L^\infty(0,T;L^2(\mathbb{T}))}^{\frac{11}{12}} \le C.$$

Then

$$\|\varepsilon^{\frac{4}{3}}\rho_\tau^4(\ln\rho_\tau^4)_{xx}\|_{L^{\frac{12}{7}}(0,T;L^2(\mathbb{T}))} \le C\|\varepsilon\rho_\tau^2(\ln\rho_\tau^4)_{xx}\|_{L^2(Q_T)}\|\varepsilon^{\frac{1}{3}}\rho_\tau^2\|_{L^{12}(0,T;L^\infty(\mathbb{T}))} \le C.$$

Obviously, we have $\|\rho_\tau^4\|_{L^6(0,T;L^\infty(\mathbb{T}))} \le C_\varepsilon$. Consequently, by the standard elliptic estimates, one has $\|V_\tau\|_{L^6(0,T;H^2(\mathbb{T}))} \le C_\varepsilon$. This completes the proof of (3.4) and (3.6). Now we are left to investigate (3.5). First, we establish some estimates about ρ_τ^2. Employing the Gagliardo-Nirenberg inequality, one has

$$\|(\rho_\tau^2)_x\|_{L^2(\mathbb{T})} \le C\|\rho_\tau^2\|_{H^3(\mathbb{T})}^{\frac{1}{3}}\|\rho_\tau^2\|_{L^2(\mathbb{T})}^{\frac{2}{3}},$$
$$\|(\rho_\tau^2)_{xx}\|_{L^2(\mathbb{T})} \le C\|\rho_\tau^2\|_{H^3(\mathbb{T})}^{\frac{2}{3}}\|\rho_\tau^2\|_{L^2(\mathbb{T})}^{\frac{1}{3}},$$
$$\|(\rho_\tau^2)_{xx}\|_{L^\infty(\mathbb{T})} \le C\|\rho_\tau^2\|_{H^3(\mathbb{T})}^{\frac{5}{6}}\|\rho_\tau^2\|_{L^2(\mathbb{T})}^{\frac{1}{6}}.$$

These inequalities and (3.3)–(3.4) imply

$$\|\varepsilon^{\frac{2}{3}}(\rho_\tau^2)_x\|_{L^6(0,T;L^2(\mathbb{T}))} \le C\|\varepsilon^2\rho_\tau^2\|_{L^2(0,T;H^3(\mathbb{T}))}^{\frac{1}{3}}\|\rho_\tau^2\|_{L^\infty(0,T;L^2(\mathbb{T}))}^{\frac{2}{3}} \le C, \tag{3.8}$$
$$\|\varepsilon^{\frac{4}{3}}(\rho_\tau^2)_{xx}\|_{L^3(0,T;L^2(\mathbb{T}))} \le C\|\varepsilon^2\rho_\tau^2\|_{L^2(0,T;H^3(\mathbb{T}))}^{\frac{2}{3}}\|\rho_\tau^2\|_{L^\infty(0,T;L^2(\mathbb{T}))}^{\frac{1}{3}} \le C, \tag{3.9}$$
$$\|\varepsilon^{\frac{5}{3}}(\rho_\tau^2)_{xx}\|_{L^{\frac{12}{5}}(0,T;L^\infty(\mathbb{T}))} \le C\|\varepsilon^2\rho_\tau^2\|_{L^2(0,T;H^3(\mathbb{T}))}^{\frac{5}{6}}\|\rho_\tau^2\|_{L^\infty(0,T;L^2(\mathbb{T}))}^{\frac{1}{6}} \le C. \tag{3.10}$$

Observe

$$\begin{aligned}
\rho_\tau^4(\ln\rho_\tau^4)_{xx}^2 &= 16[\rho_\tau^2(\rho_\tau)_{xx}^2 + (\rho_\tau)_x^4 - 2\rho_\tau(\rho_\tau)_x^2(\rho_\tau)_{xx}] \\
&= 16[\rho_\tau^2(\rho_\tau)_{xx}^2 + (\rho_\tau)_x^4 + 2\rho_\tau(\rho_\tau)_x^2(\rho_\tau)_{xx}] - 64\rho_\tau(\rho_\tau)_x^2(\rho_\tau)_{xx} \\
&= 4(\rho_\tau^2)_{xx}^2 - 64\rho_\tau(\rho_\tau)_x^2(\rho_\tau)_{xx}, \qquad (3.11)\\
\rho_\tau^4(\ln\rho_\tau^4)_{xxx} &= 2\rho_\tau^2(\rho_\tau^2)_{xxx} - 6(\rho_\tau^2)_x(\rho_\tau^2)_{xx} + 16(\rho_\tau^2)_x(\rho_\tau)_x^2 \\
&= 2\rho_\tau^2(\rho_\tau^2)_{xxx} + 2(\rho_\tau^2)_x(\rho_\tau^2)_{xx} - 8(\rho_\tau^2)_x[(\rho_\tau^2)_{xx} - 2(\rho_\tau)_x^2] \\
&= 2\rho_\tau^2(\rho_\tau^2)_{xxx} + 2(\rho_\tau^2)_x(\rho_\tau^2)_{xx} - 16\rho_\tau(\rho_\tau^2)_x(\rho_\tau)_{xx}. \qquad (3.12)
\end{aligned}$$

Thus one has

$$\begin{aligned}
&\|\varepsilon^{\frac{8}{3}}\rho_\tau^4(\ln\rho_\tau^4)_{xx}^2\|_{L^{\frac{3}{2}}(0,T;L^1(\mathbb{T}))} \\
\le\ & 4\|\varepsilon^{\frac{4}{3}}(\rho_\tau^2)_{xx}\|_{L^3(0,T;L^2(\mathbb{T}))}^2 + 32\|\varepsilon^{\frac{2}{3}}(\rho_\tau^2)_x\|_{L^6(0,T;L^2(\mathbb{T}))}\|\varepsilon^2(\rho_\tau)_x(\rho_\tau)_{xx}\|_{L^2(Q_T)} \le C.
\end{aligned}$$

In view of (3.3)–(3.4), it follows that

$$\begin{aligned}
&\|\varepsilon^{\frac{7}{3}}\rho_\tau^4(\ln\rho_\tau^4)_{xxx}\|_{L^{\frac{12}{7}}(0,T;L^2(\mathbb{T}))}\\
&\le 32\|\varepsilon^{\frac{1}{3}}\rho_\tau^2\|_{L^{12}(0,T;L^\infty(\mathbb{T}))}(\|\varepsilon^2(\rho_\tau^2)_{xxx}\|_{L^2(Q_T)}+\|\varepsilon^2(\rho_\tau)_x(\rho_\tau)_{xx}\|_{L^2(Q_T)})\\
&\quad+2\|\varepsilon^{\frac{2}{3}}(\rho_\tau^2)_x\|_{L^6(0,T;L^2(\mathbb{T}))}\|\varepsilon^{\frac{5}{3}}(\rho_\tau^2)_{xx}\|_{L^{\frac{12}{5}}(0,T;L^\infty(\mathbb{T}))}\le C.
\end{aligned}$$

So, (3.5) follows.

4 Weak Existence

Using a compactness argument and Aubin-Lions lemma (see [27]), we can prove the following convergent results for ρ_τ, V_τ (see Proposition 4.1) which will complete the proof of Theorem 1.1. Let $\partial_t^\tau\rho_\tau^4(t,x)$ be the difference quotient of ρ_τ^4.

Proposition 4.1 *For any fixed $\varepsilon>0$, there exists a subsequence of $\{(\rho_\tau,\,V_\tau,\,\partial_t^\tau\rho_\tau^4)\}_{\tau>0}$ as $\tau\to 0$, which is not relabeled, such that*

$$\rho_\tau^2\rightharpoonup\rho^2,\qquad \textit{in } L^2(0,T;H^3(\mathbb{T})),\tag{4.1}$$
$$\rho_\tau\overset{*}{\rightharpoonup}\rho\ge 0,\qquad \textit{in } L^\infty(0,T;L^4(\mathbb{T})),\tag{4.2}$$
$$V_\tau\rightharpoonup V,\qquad \textit{in } L^6(0,T;H^2(\mathbb{T})),\tag{4.3}$$
$$\rho_\tau^4\to\rho^4,\qquad \textit{in } L^{\frac{3}{2}}(0,T;C^{2,\lambda}(\mathbb{T}))\ \Big(\forall\ 0<\lambda<\frac{1}{2}\Big),\tag{4.4}$$
$$V_\tau\to V,\qquad \textit{in } L^{\frac{3}{2}}(0,T;H^2(\mathbb{T})),\tag{4.5}$$
$$\partial_t^\tau\rho_\tau^4\rightharpoonup\partial_t\rho^4,\qquad \textit{in } L^{\frac{3}{2}}(0,T;H^{-3}(\mathbb{T})),\tag{4.6}$$
$$\rho_\tau^4(\ln\rho_\tau^4)_{xxx}\rightharpoonup\rho^4(\ln\rho^4)_{xxx},\qquad \textit{in } L^{\frac{12}{7}}(0,T;L^2(\mathbb{T})),\tag{4.7}$$
$$\rho_\tau^4(\ln\rho_\tau^4)_{xx}^2\rightharpoonup\rho^4(\ln\rho^4)_{xx}^2,\qquad \textit{in } L^{\frac{3}{2}}(0,T;H^{-1}(\mathbb{T})),\tag{4.8}$$
$$\rho_\tau^4(\ln\rho_\tau^4)_{xx}\rightharpoonup\rho^4(\ln\rho^4)_{xx},\qquad \textit{in } L^{\frac{12}{7}}(0,T;L^2(\mathbb{T})).\tag{4.9}$$

Proof By Proposition 3.1, we deduce that there exists $\rho\ge 0$ satisfying (4.1)–(4.3). Let $0<\tau<1$ be fixed. Then for any $0<h<\tau$, we have

$$\|\pi_h\rho_\tau^4-\rho_\tau^4\|^{\frac{3}{2}}_{L^{\frac{3}{2}}(0,T-h;H^{-3}(\mathbb{T}))}=h\sum_{k=1}^{N-1}\|\rho_{k+1}^4-\rho_k^4\|^{\frac{3}{2}}_{H^{-3}(\mathbb{T})},$$

where $(\pi_hf)(t)=f(t+h)$, and for any $1\le k\le N-1$,

$$\begin{aligned}
\Big\|\frac{\rho_{k+1}^4-\rho_k^4}{\tau}\Big\|_{H^{-3}(\mathbb{T})}&\le C(\|\rho_{k+1}^4(\ln\rho_{k+1}^4)_{xxx}\|_{L^2(\mathbb{T})}+\|\rho_{k+1}^4(\ln\rho_{k+1}^4)_{xx}^2\|_{L^1(\mathbb{T})}\\
&\quad+\|\rho_{k+1}^4(\ln\rho_{k+1}^4)_{xx}\|_{L^1(\mathbb{T})}+\|(\rho_{k+1}^4)_x+\rho_{k+1}^4(V_{k+1})_x\|_{L^1(\mathbb{T})}).
\end{aligned}\tag{4.10}$$

Consequently, it follows from (3.1) that $\|\pi_h\rho_\tau^4-\rho_\tau^4\|_{L^{\frac{3}{2}}(0,T-h;H^{-3}(\mathbb{T}))}\le C_\varepsilon h^{\frac{2}{3}}$. We deduce from $(\rho_\tau^4)_{xxx}=2\rho_\tau^2(\rho_\tau^2)_{xxx}+6(\rho_\tau^2)_x(\rho_\tau^2)_{xx}$, (3.8) and Proposition 3.1 that $\|(\rho_\tau^4)_{xxx}\|_{L^{\frac{3}{2}}(0,T;L^2(\mathbb{T}))}\le C_\varepsilon$ and hence $\|\rho_\tau^4\|_{L^{\frac{3}{2}}(0,T;H^3(\mathbb{T}))}\le C_\varepsilon$. Thus we establish (4.4) by Aubin-Lions lemma and the compact embedding $H^3(\mathbb{T})\hookrightarrow\hookrightarrow C^{2,\lambda}(\mathbb{T})$ $(0<\lambda<\frac{1}{2})$. Furthermore, one has (4.5) from (4.4) by standard elliptic estimates. The estimate (4.10) implies $\|\partial_t^\tau\rho_\tau^4\|_{L^{\frac{3}{2}}(0,T;H^{-3}(\mathbb{T}))}\le C_\varepsilon$. The

Mean Value Theorem of differentials and (4.4) show that $\partial_t^\tau \rho_\tau^4 \rightharpoonup \partial_t \rho^4$ in $\mathscr{D}'(Q_T)$. Therefore (4.6) is proved.

We next prove (4.7)–(4.9). Obviously, (3.3) and (3.5) yields

$$\begin{aligned} \rho_\tau^4(\ln\rho_\tau^4)_{xxx} &\rightharpoonup J, \quad && \text{in } L^{\frac{12}{7}}(0,T;L^2(\mathbb{T})),\\ \rho_\tau^4(\ln\rho_\tau^4)_{xx}^2 &\rightharpoonup K, \quad && \text{in } L^{\frac{3}{2}}(0,T;H^{-1}(\mathbb{T})),\\ \rho_\tau^4(\ln\rho_\tau^4)_{xx} &\rightharpoonup L, \quad && \text{in } L^{\frac{12}{7}}(0,T;L^2(\mathbb{T})). \end{aligned}$$

Hence the proof is completed, provided that

$$\rho_\tau^4(\ln\rho_\tau^4)_{xxx} \rightharpoonup \rho^4(\ln\rho^4)_{xxx}, \quad \text{in } \mathscr{D}'(Q_T), \tag{4.11}$$

$$\rho_\tau^4(\ln\rho_\tau^4)_{xx}^2 \rightharpoonup \rho^4(\ln\rho^4)_{xx}^2, \quad \text{in } \mathscr{D}'(Q_T), \tag{4.12}$$

$$\rho_\tau^4(\ln\rho_\tau^4)_{xx} \rightharpoonup \rho^4(\ln\rho^4)_{xx}, \quad \text{in } \mathscr{D}'(Q_T). \tag{4.13}$$

Now we check these. Firstly, it follows from (4.4) and $(a-b)^2 \le |a^2-b^2|$ $(\forall\, a,b>0)$ that

$$\|\rho_\tau^2-\rho^2\|_{L^3(0,T;L^\infty(\mathbb{T}))}^2 = \|(\rho_\tau^2-\rho^2)^2\|_{L^{\frac{3}{2}}(0,T;L^\infty(\mathbb{T}))} \le \|\rho_\tau^4-\rho^4\|_{L^{\frac{3}{2}}(0,T;L^\infty(\mathbb{T}))} \to 0, \tag{4.14}$$

$$\|\rho_\tau-\rho\|_{L^6(0,T;L^\infty(\mathbb{T}))}^2 = \|(\rho_\tau-\rho)^2\|_{L^3(0,T;L^\infty(\mathbb{T}))} \le \|\rho_\tau^2-\rho^2\|_{L^3(0,T;L^\infty(\mathbb{T}))} \to 0. \tag{4.15}$$

Applying the Gagliardo-Nirenberg inequality, one has

$$\begin{aligned} \|(\rho_\tau^2-\rho^2)_x\|_{L^6(\mathbb{T})} &\le C\|\rho_\tau^2-\rho^2\|_{H^3(\mathbb{T})}^{\frac{1}{3}}\|\rho_\tau^2-\rho^2\|_{L^\infty(\mathbb{T})}^{\frac{2}{3}},\\ \|(\rho_\tau^2-\rho^2)_{xx}\|_{L^3(\mathbb{T})} &\le C\|\rho_\tau^2-\rho^2\|_{H^3(\mathbb{T})}^{\frac{2}{3}}\|\rho_\tau^2-\rho^2\|_{L^\infty(\mathbb{T})}^{\frac{1}{3}}. \end{aligned}$$

These estimates and Proposition 3.1 imply the convergent results

$$\begin{aligned} \|(\rho_\tau^2-\rho^2)_x\|_{L^{\frac{18}{7}}(0,T;L^6(\mathbb{T}))} &\le C\|\rho_\tau^2-\rho^2\|_{L^2(0,T;H^3(\mathbb{T}))}^{\frac{1}{3}}\|\rho_\tau^2-\rho^2\|_{L^3(0,T;L^\infty(\mathbb{T}))}^{\frac{2}{3}}\\ &\le C_\varepsilon\|\rho_\tau^2-\rho^2\|_{L^3(0,T;L^\infty(\mathbb{T}))}^{\frac{2}{3}} \to 0, \end{aligned} \tag{4.16}$$

$$\begin{aligned} \|(\rho_\tau^2-\rho^2)_{xx}\|_{L^{\frac{9}{4}}(0,T;L^3(\mathbb{T}))} &\le C\|\rho_\tau^2-\rho^2\|_{L^2(0,T;H^3(\mathbb{T}))}^{\frac{2}{3}}\|\rho_\tau^2-\rho^2\|_{L^3(0,T;L^2(\mathbb{T}))}^{\frac{1}{3}}\\ &\le C_\varepsilon\|\rho_\tau^2-\rho^2\|_{L^3(0,T;L^\infty(\mathbb{T}))}^{\frac{1}{3}} \to 0. \end{aligned} \tag{4.17}$$

Secondly, we infer from (3.3) that

$$\|(\rho_\tau^2)_x(\rho_\tau)_{xx}\|_{L^{\frac{24}{13}}(0,T;L^2(\mathbb{T}))} \le 2\|\rho_\tau\|_{L^{24}(0,T;L^\infty(\mathbb{T}))}\|(\rho_\tau)_x(\rho_\tau)_{xx}\|_{L^2(Q_T)} \le C_\varepsilon, \tag{4.18}$$

$$\begin{aligned} \|(\rho_\tau)_x\|_{L^6(0,T;L^4(\mathbb{T}))} &= \frac{3}{2}\|(\rho_\tau^{\frac{2}{3}})_x\rho_\tau^{\frac{1}{3}}\|_{L^6(0,T;L^4(\mathbb{T}))}\\ &\le C\|(\rho_\tau^{\frac{2}{3}})_x\|_{L^6(Q_T)}\|\rho_\tau\|_{L^\infty(0,T;L^4(\mathbb{T}))}^{\frac{1}{3}} \le C_\varepsilon. \end{aligned} \tag{4.19}$$

Thus

$$\|(\rho_\tau)_x^2(\rho_\tau)_{xx}\|_{L^{\frac{3}{2}}(0,T;L^{\frac{4}{3}}(\mathbb{T}))} \le \|(\rho_\tau)_x\|_{L^6(0,T;L^4(\mathbb{T}))}\|(\rho_\tau)_x(\rho_\tau)_{xx}\|_{L^2(Q_T)} \le C_\varepsilon. \tag{4.20}$$

In light of (4.1), (4.14), (4.16) and (4.17) it suffice to verify that

$$(\rho_\tau^2)_x^2 \rightharpoonup (\rho^2)_x^2, \quad (\rho_\tau^2)_{xx}^2 \rightharpoonup (\rho^2)_{xx}^2, \quad \rho_\tau^2(\rho_\tau^2)_{xx} \rightharpoonup \rho^2(\rho^2)_{xx}, \quad \text{in } \mathscr{D}'(Q_T), \tag{4.21}$$

$$\rho_\tau^2(\rho_\tau^2)_{xxx} \rightharpoonup \rho^2(\rho^2)_{xxx}, \quad (\rho_\tau^2)_x(\rho_\tau^2)_{xx} \rightharpoonup (\rho^2)_x(\rho^2)_{xx}, \quad \text{in } \mathscr{D}'(Q_T) \tag{4.22}$$

and also

$$(\rho_\tau^2)_x(\rho_\tau)_{xx} \rightharpoonup (\rho^2)_x\rho_{xx}, \quad (\rho_\tau)_x(\rho_\tau^2)_{xx} \rightharpoonup \rho_x(\rho^2)_{xx}, \quad \text{in } \mathscr{D}'(Q_T) \tag{4.23}$$

with the help of (4.15) and (4.19). The first convergence result of (4.23) is worthy of being proved. In fact, for any $\varphi \in \mathscr{D}(Q_T)$,

$$\begin{aligned}
&\Big|\int_{Q_T}[(\rho_\tau^2)_x(\rho_\tau)_{xx} - (\rho^2)_x\rho_{xx}]\varphi \mathrm{d}x\mathrm{d}t\Big| \\
\le &\Big|\int_{Q_T}[(\rho_\tau^2)_x - (\rho^2)_x](\rho_\tau)_{xx}\varphi \mathrm{d}x\mathrm{d}t\Big| + \Big|\int_{Q_T}[(\rho_\tau)_{xx} - \rho_{xx}](\rho^2)_x\varphi \mathrm{d}x\mathrm{d}t\Big| = \mathrm{J}_1 + \mathrm{J}_2.
\end{aligned}$$

Integrating by parts, we have from (4.15)–(4.17) and (4.1) that

$$\begin{aligned}
\mathrm{J}_1 &= \Big|\int_{Q_T}[((\rho_\tau^2)_{xx} - (\rho^2)_{xx})\varphi + ((\rho_\tau^2)_x - (\rho^2)_x)\varphi_x](\rho_\tau)_x \mathrm{d}x\mathrm{d}t\Big| \\
&\le \Big|\int_{Q_T}((\rho_\tau^2)_{xx} - (\rho^2)_{xx})\varphi(\rho_\tau)_x \mathrm{d}x\mathrm{d}t\Big| + \Big|\int_{Q_T}((\rho_\tau^2)_x - (\rho^2)_x)\varphi_x(\rho_\tau)_x \mathrm{d}x\mathrm{d}t\Big| \\
&\le C\|(\rho_\tau)_x\|_{L^6(0,T;L^4(\mathbb{T}))}(\|(\rho_\tau^2)_{xx} - (\rho^2)_{xx}\|_{L^{\frac{9}{4}}(0,T;L^3(\mathbb{T}))} \\
&\quad + \|(\rho_\tau^2)_x - (\rho^2)_x\|_{L^{\frac{18}{7}}(0,T;L^6(\mathbb{T}))}) \to 0, \\
\mathrm{J}_2 &= \Big|\int_{Q_T}(\rho_\tau - \rho)[(\rho^2)_x\varphi]_{xx} \mathrm{d}x\mathrm{d}t\Big| \le C\|\rho^2\|_{L^2(0,T;H^3(\mathbb{T}))}\|\rho_\tau - \rho\|_{L^6(0,T;L^\infty(\mathbb{T}))} \\
&\le C_\varepsilon\|\rho_\tau - \rho\|_{L^6(0,T;L^\infty(\mathbb{T}))} \to 0.
\end{aligned}$$

Note that $6(\rho_\tau)_x^2(\rho_\tau)_{xx} = [(\rho_\tau)_x(\rho_\tau^2)_{xx} - (\rho_\tau^2)_x(\rho_\tau)_{xx}]_x$. Then one has from (4.23) that

$$(\rho_\tau)_x^2(\rho_\tau)_{xx} \rightharpoonup \rho_x^2\rho_{xx}, \quad \text{in } \mathscr{D}'(Q_T). \tag{4.24}$$

Hence, in view of (4.20), we get

$$(\rho_\tau)_x^2(\rho_\tau)_{xx} \rightharpoonup \rho_x^2\rho_{xx}, \quad \text{in } L^{\frac{3}{2}}(0,T;L^{\frac{4}{3}}(\mathbb{T})). \tag{4.25}$$

Additionally, (4.18) and (4.23) imply

$$(\rho_\tau^2)_x(\rho_\tau)_{xx} \rightharpoonup (\rho^2)_x\rho_{xx}, \quad \text{in } L^{\frac{24}{13}}(0,T;L^2(\mathbb{T})). \tag{4.26}$$

Employing (4.15), (4.25) and (4.26), we see that

$$\rho_\tau(\rho_\tau^2)_x(\rho_\tau)_{xx} \rightharpoonup \rho(\rho^2)_x\rho_{xx}, \quad \text{in } \mathscr{D}'(Q_T), \tag{4.27}$$

$$\rho_\tau(\rho_\tau)_x^2(\rho_\tau)_{xx} \rightharpoonup \rho\rho_x^2\rho_{xx}, \quad \text{in } \mathscr{D}'(Q_T). \tag{4.28}$$

Finally, recalling (3.11)–(3.12) and observing $\rho_\tau^4(\ln\rho_\tau^4)_{xx} = 2\rho_\tau^2(\rho_\tau^2)_{xx} - 2(\rho_\tau^2)_x^2$, we deduce (4.11)–(4.13) directly from (4.21), (4.22), (4.27) and (4.28).

Proof of Theorem 1.1 Using Proposition 2.1 and Proposition 4.1, it is easy to complete the proof of Theorem 1.1. The initial value is satisfied in the sense of $H^{-3}(\mathbb{T})$ since $W^{1,\frac{3}{2}}(0,T;H^{-3}(\mathbb{T})) \hookrightarrow C([0,T];H^{-3}(\mathbb{T}))$.

5 Semiclassical Limit

Let us turn to discuss the semiclassical limit of the weak solution (ρ, V) obtained in Theorem 1.1. Since the solution depends on ε, to be precise, we relabel it as $(\rho_\varepsilon, V_\varepsilon)$ throughout this section. Then we could establish the uniform estimates in ε.

Lemma 5.1

$$\begin{aligned}&\|\rho_\varepsilon^2\|_{L^\infty(0,T;L^2(\mathbb{T}))\cap L^2(0,T;H^1(\mathbb{T}))} + \|\varepsilon^{\frac{7}{3}}\rho_\varepsilon^4(\ln\rho_\varepsilon^4)_{xxx}\|_{L^{\frac{12}{7}}(0,T;L^2(\mathbb{T}))}\\&+\|\varepsilon^{\frac{8}{3}}\rho_\varepsilon^4(\ln\rho_\varepsilon^4)_{xx}^2\|_{L^{\frac{3}{2}}(0,T;H^{-1}(\mathbb{T}))} + \|\varepsilon^{\frac{4}{3}}\rho_\varepsilon^4(\ln\rho_\varepsilon^4)_{xx}\|_{L^{\frac{12}{7}}(0,T;L^2(\mathbb{T}))} \le C.\end{aligned} \tag{5.1}$$

Furthermore,

$$\|\rho_\varepsilon^4\|_{L^{\frac{24}{13}}(0,T;W^{1,\frac{12}{11}}(\mathbb{T}))\cap L^{12}(0,T;L^{\frac{6}{5}}(\mathbb{T}))} + \|V_\varepsilon\|_{L^{\frac{24}{13}}(0,T;H^2(\mathbb{T}))} + \|\partial_t\rho_\varepsilon^4\|_{L^{\frac{3}{2}}(0,T;H^{-3}(\mathbb{T}))} \le C. \tag{5.2}$$

Proof In view of the weakly lower semi-continuity of norm, we obtain (5.1) from Propositions 3.1 and 4.1. It follows from the Gagliardo-Nirenberg inequality and (5.1) that $\|\rho_\varepsilon^2\|_{L^{\frac{12}{5}}(\mathbb{T})} \le C\|\rho_\varepsilon^2\|_{H^1(\mathbb{T})}^{\frac{1}{12}}\|\rho_\varepsilon^2\|_{L^2(\mathbb{T})}^{\frac{11}{12}}$. Hence

$$\|\rho_\varepsilon^2\|_{L^{24}(0,T;L^{\frac{12}{5}}(\mathbb{T}))} \le C\|\rho_\varepsilon^2\|_{L^2(0,T;H^1(\mathbb{T}))}^{\frac{1}{12}}\|\rho_\varepsilon^2\|_{L^\infty(0,T;L^2(\mathbb{T}))}^{\frac{11}{12}} \le C.$$

Consequently, we deduce from the Hölder inequality and (5.1) that

$$\begin{aligned}\|(\rho_\varepsilon^4)_x\|_{L^{\frac{24}{13}}(0,T;L^{\frac{12}{11}}(\mathbb{T}))} &= 2\|\rho_\varepsilon^2(\rho_\varepsilon^2)_x\|_{L^{\frac{24}{13}}(0,T;L^{\frac{12}{11}}(\mathbb{T}))}\\&\le C\|\rho_\varepsilon^2\|_{L^{24}(0,T;L^{\frac{12}{5}}(\mathbb{T}))}\|(\rho_\varepsilon^2)_x\|_{L^2(Q_T)} \le C.\end{aligned}$$

Thus $\|\rho_\varepsilon^4\|_{L^{\frac{24}{13}}(0,T;W^{1,\frac{12}{11}}(\mathbb{T}))} \le C$. As a result, using standard elliptic estimates and the embedding $W^{1,\frac{12}{11}}(\mathbb{T}) \hookrightarrow L^\infty(\mathbb{T})$, we find that V_ε is uniformly bounded in $L^{\frac{24}{13}}(0,T;H^2(\mathbb{T}))$. Therefore in light of (1.8) and (5.1), it suffices to verify that $\|\partial_t\rho_\varepsilon^4\|_{L^{\frac{3}{2}}(0,T;H^{-3}(\mathbb{T}))} \le C$.

Proof of Theorem 1.2 From (5.2), we deduce that there exists $n \ge 0$ satisfying (1.9). Also invoking (5.2) we infer from the compact embedding $W^{1,\frac{12}{11}}(\mathbb{T}) \hookrightarrow\hookrightarrow C^{0,\lambda}(\mathbb{T})$ $(\forall\, 0 < \lambda < \frac{1}{12})$ and Aubin-Lions lemma; we obtain (1.10) and hence (1.12). Applying standard elliptic estimates, (1.10) implies (1.11). In addition, we have the convergence results (1.13)–(1.15) directly from (5.1). Letting $\varepsilon \to 0$ in (1.8), we complete the proof with the help of Lemma 5.1.

6 Summary

We summarize in this section some new points (both results and methods) in this paper and some interesting problems (physically or mathematically motivated) in this area.

6.1 New Points

(1) New a priori estimates. The only information from the system is from entropy inequality which was first derived in [25] with only 6th order term. In order to investigate the existence and semiclassical limit, the first hand estimates from entropy like [25] did are not enough. Without any additional information, we are able to derive more estimates only by the Sobolev type inequalities, especially the uniform estimates in ε for classical limit.

(2) Global existence of weak solution for the whole system. As stated in the introduction, the leading order equation was studied in [25], where they introduced the main entropy inequality. But the results obtained there are relatively rough, which means that further estimates are possible by some delicate Sobolev inequalities and consequently relatively better solutions can be obtained. We get a priori estimates for the whole system, including lower order terms and the coupling with Poisson. With more analysis, making use of various versions of the 6th order term, we are able to prove the global existence of weak solution for the whole system, including 6th order, 4th order and classical parts.

(3) Semiclassical limits. No any semiclassical limit result is obtained for 6th order system so far. It is a physically interesting topic describing the relation between classical and quantum models. Mathematically, it reveals the connection between second order PDEs and their higher order corrections. By new a priori estimates stated above, we are able to show the semiclassical limit in the weak sense.

6.2 Some interested future problems

(1) Convergence rate estimates. Semiclassical limit shows that the quantum corrected drift diffusion model converges to the classical drift diffusion model, as discussed in [4] for $O(\varepsilon^2)$ order correction and the results in this paper for $O(\varepsilon^4)$ order correction, both are in the case of "good" boundary conditions. While a further interesting problem is the convergence rate analysis, whether the difference between $O(\varepsilon^2)$ corrected solution and the limiting solution can be controlled by $O(\varepsilon^2)$? And going even further, whether the asymptotic relation holds for the solutions of quantum corrected drift diffusion. Mathematically, this kind of results will show the relations between higher order PDEs and their corresponding lower order ones.

(2) Boundary layer analysis. In most cases, we do not have "good" enough boundary conditions. More interesting problems are the boundary layer analysis. Physically, quantum effect usually only occurs in a very thin layer in real devices, such as inversion layers in MOS transistors. Thus, to derive the layer equations for the limiting profile, the reduced problem is the first step in this direction. Then matched asymptotic analysis and uniform estimates hopefully will give the mathematical analysis.

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