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Generalized Integral Representations for Functions with Values in $C(V_{3,3})$ *

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Abstract By using the solution to the Helmholtz equation $\triangle u - \lambda u = 0$ $(\lambda \geq 0)$, the explicit forms of the so-called kernel functions and the higher order kernel functions are given. Then by the generalized Stokes formula, the integral representation formulas related with the Helmholtz operator for functions with values in $C(V_{3,3})$ are obtained. As application of the integral representations, the maximum modulus theorem for function u which satisfies $Hu = 0$ is given.

Keywords Universal Clifford algebra, Helmholtz equation, Generalized Cauchy-Pompeiu formula
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1 Introduction and Preliminaries

Integral representation formulas in complex analysis and Clifford analysis have been well-developed in [2–9, 11–21, 24–32], etc. These integral representation formulas are powerful tools. In [24], E. Obolashvilli obtained some generalized Cauchy-Pompeiu representation formulas. In [25–26], higher order Cauchy-Pompeiu type representation formulas related to the factors and powers of the Helmholtz operator were given in quaternionic analysis in case of taking $\lambda = |h|^2$ with a complex or real number h. In [8], in case of taking $\lambda = |h|^2$ with a hyper-complex number h, we constructed the kernel functions and then got the higher order integral representation formulas for functions with values in a universal Clifford algebra $C(V_{n,n})$. In this paper, by using the idea in [8, 24, 28], we get the explicit expressions of the kernel functions and then get the explicit integral representation formulas for functions with values in $C(V_{3,3})$. These explicit integral representation formulas will play an important role in studying the further properties of the functions with values in $C(V_{3,3})$. In the last section, as application of the integral representations, we give the maximum modulus theorem for function u which satisfies $Hu = 0$.

Let $V_{n,s}$ $(0 \leq s \leq n)$ be an n-dimensional $(n \geq 1)$ real linear space with basis $\{e_1, e_2, \cdots, e_n\}$, $C(V_{n,s})$ be the 2^n-dimensional real linear space with basis

$$\{e_A, A = \{h_1, \cdots, h_r\} \in \mathcal{P}N, 1 \leq h_1 < \cdots < h_r \leq n\},$$

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where N stands for the set $\{1,\cdots,n\}$ and $\mathcal{P}N$ denotes the family of all order-preserving subsets of N in the above way. we denote $e_\emptyset$ as e_0 and e_A as $e_{h_1\cdots h_r}$ for $A=\{h_1,\cdots,h_r\}\in\mathcal{P}N$. The product on $C(V_{n,s})$ is defined by

$$\begin{cases} e_Ae_B=(-1)^{\#((A\cap B)\setminus S)}(-1)^{P(A,B)}e_{A\triangle B}, & \text{if } A,B\in\mathcal{P}N,\\ \lambda\mu=\sum\limits_{A\in\mathcal{P}N}\sum\limits_{B\in\mathcal{P}N}\lambda_A\mu_Be_Ae_B, & \text{if } \lambda=\sum\limits_{A\in\mathcal{P}N}\lambda_Ae_A,\ \mu=\sum\limits_{B\in\mathcal{P}N}\mu_Be_B, \end{cases} \tag{1.1}$$

where S stands for the set $\{1,\cdots,s\}$, $\#(A)$ is the cardinal number of the set A, the number $P(A,B)=\sum\limits_{j\in B}P(A,j)$, $P(A,j)=\#\{i,\ i\in A,\ i>j\}$, the symmetric difference set $A\triangle B$ is also order-preserving in the above way, and $\lambda_A\in\mathcal{R}$ is the coefficient of the e_A-component of the Clifford number λ. We also denote λ_A as $[\lambda]_A$, $\lambda_{\{i\}}$ as $[\lambda]_i$ and λ_0 as $\operatorname{Re}\lambda$. It follows immediately from the multiplication rule (1.1) that e_0 is the identity element written now as 1 and in particular,

$$\begin{cases} e_i^2=1, & \text{if } i=1,\cdots,s,\\ e_j^2=-1, & \text{if } j=s+1,\cdots,n,\\ e_ie_j=-e_je_i, & \text{if } 1\le i<j\le n,\\ e_{h_1}e_{h_2}\cdots e_{h_r}=e_{h_1h_2\cdots h_r}, & \text{if } 1\le h_1<h_2\cdots<h_r\le n. \end{cases} \tag{1.2}$$

Thus $C(V_{n,s})$ is a real linear, associative, but non-commutative algebra and it is called the universal Clifford algebra over $V_{n,s}$. In particular, suppose $n=s=3$. Then $C(V_{n,s})$ is just the universal Clifford algebra $C(V_{3,3})$.

An involution is defined by

$$\begin{cases} \overline{e}_A=(-1)^{\sigma(A)+\#(A\cap S)}e_A, & \text{if } A\in\mathcal{P}N,\\ \overline{\lambda}=\sum\limits_{A\in\mathcal{P}N}\lambda_A\overline{e}_A, & \text{if } \lambda=\sum\limits_{A\in\mathcal{P}N}\lambda_Ae_A, \end{cases} \tag{1.3}$$

where $\sigma(A)=\frac{\#(A)(\#(A)+1)}{2}$. From (1.1) and (1.3), we have

$$\begin{cases} \overline{e}_i=e_i, & \text{if } i=0,1,\cdots,s,\\ \overline{e}_j=-e_j, & \text{if } j=s+1,\cdots,n,\\ \overline{\lambda\mu}=\overline{\mu}\,\overline{\lambda}, & \text{for any } \lambda,\mu\in C(V_{n,s}). \end{cases} \tag{1.4}$$

The $C\left(V_{n.n}\right)$-valued $(n-1)$-differential form

$$\mathrm{d}\sigma=\sum_{k=1}^{n}(-1)^{k-1}e_k\mathrm{d}\widehat{x}_k^N$$

is exact, where

$$\mathrm{d}\widehat{x}_k^N=\mathrm{d}x^1\wedge\cdots\wedge\mathrm{d}x^{k-1}\wedge\mathrm{d}x^{k+1}\wedge\cdots\wedge\mathrm{d}x^n.$$

In this paper, we shall only consider the special case $s=n=3$. Let Ω be an open non-empty subset of $\mathcal{R}^3$. Functions f defined in Ω and with values in $C(V_{3,3})$ will be considered, i.e., $f:\Omega\to C(V_{3,3})$. $f\in C^{(r)}(\Omega,C(V_{3,3}))$ is clear that all the component functions of $f(x)$ possess the cited property. The operator D which is written as

$$D=\sum_{k=1}^{3}e_k\frac{\partial}{\partial x_k}:C^{(r)}(\Omega,C(V_{3,3}))\to C^{(r-1)}(\Omega,C(V_{3,3})).$$

The operator D acts on the function f from the left and from the right being governed by the rule

$$D[f] = \sum_{k=1}^{3} \sum_{A} e_k e_A \frac{\partial f_A}{\partial x_k}, \quad [f]D = \sum_{k=1}^{3} \sum_{A} e_A e_k \frac{\partial f_A}{\partial x_k}.$$

The classical Riemann boundary value problems and singular integral equations are widely used for the solution to a great field of problems in applied mechanics theory (see [22, 23] etc.). Naturally, it is interesting to generalize the theory of holomorphic functions in the plane to higher dimensions. Especially, it has practical meaning to generalize the classical theory in the plane to the theory of k-regular functions defined in $\mathcal{R}^3$ with values in $C(V_{3,3})$. For example, R_m $(m > 0)$ Riemann boundary value problem for bi-harmonic function was studied in [20], as for the R_m $(m > 0)$ Riemann boundary value problem for bi-harmonic functions, and even more general R_m $(m > 0)$ Riemann boundary value problem for k-regular functions, the higher order integral representation formulas play a very important role. The higher order integral representation formulas provide a special solution to the R_m $(m > 0)$ Riemann boundary value problem. The importance is similar to the classical Cauchy type integral in complex plane. On the other hand, the generalized Liouville theorem for k-regular functions is very important for solving the R_m $(m > 0)$ Riemann boundary value problem. In [1, 10], the generalized Liouville theorem for k-regular functions in Clifford analysis was solved under some growth conditons at infinity. It is a very important result, while it is also valuable to refine the too many growth conditions at infinity especially for Riemann boundary value problems. But how to refine the conditions is difficult. For harmonic functions and bi-harmonic functions defined in $\mathcal{R}^3$ and with values in $C(V_{3,3})$, we know that the growth conditions at infinity of the generalized Liouville theorem can be refined to one of the conditions in [1, 10], and the method depends on the higher order integral representation formulas (see [20]). In view of the above reasons, we are interested in the different types of higher order integral representation formulas in Clifford analysis. As is well-known that the kernel functions play the most important role in obtaining the higher order integral representation formulas. While in many cases, the explicit expressions of the kernel functions are important to study the further properties of functions, for example, the generalized Liouville theorem of k-regular functions in Clifford analysis.

As for the above purposes, we keep our study in the framework of universal Clifford algebra $C(V_{3,3})$ in this paper. On one hand, it has practical meaning in applied mechanics theory; on the other hand, it provides a foundation to study the more general theory of functions with values in $C(V_{n,n})$.

We consider the following operators:

$$L_\lambda u = Du + \lambda u, \quad L_{*\lambda} u = uD - \lambda u, \tag{1.5}$$

$$Lu = Du + uh, \qquad L_* u = uD - hu \tag{1.6}$$

and the Helmholtz operator

$$Hu = (\triangle - |h|^2)u, \tag{1.7}$$

where

$$h = \sum_{i=1}^{3} h_i e_i. \tag{1.8}$$

2 Generalized Integral Representations

Denote the Helmholtz equation as

$$\triangle u-|h|^2u=0. \tag{2.1}$$

Lemma 2.1 *Let* $g_3(|x|,|h|)=\frac{1}{|x|}\mathrm{e}^{-|h||x|}$*. Then* $g_3(|x|,|h|)$ *is the fundamental solution to* (2.1) *in* $\mathcal{R}^3$.

Lemma 2.2 *Let* $g_3(|x|,|h|)$ *be defined as in Lemma* 2.1*. Then*
(i) $D[g_3](|x|,|h|)e_A-g_3(|x|,|h|)e_Ah$ *is the fundamental solution to equation*

$$Lu=Du+uh=0,\quad h=\sum_{i=1}^{3}h_ie_i, \tag{2.2}$$

(ii) $e_AD[g_3](|x|,|h|)+he_Ag_3(|x|,|h|)$ *is the fundamental solution to equation*

$$L_*u=uD-hu=0,\quad h=\sum_{i=1}^{3}h_ie_i. \tag{2.3}$$

In the following, Ω is supposed to be an open bounded non-empty subset of $\mathcal{R}^3$ with a Liapunov boundary $\partial\Omega$.

Lemma 2.3 (see [8]) *Let* $u,v\in C^1(\Omega,C(V_{3,3}))\cap C(\overline{\Omega},C(V_{3,3}))$*. Then*

$$\mathrm{Re}\Big\{\int_\Omega(vL[u]+L_*[v]u)\mathrm{d}y\Big\}=\mathrm{Re}\Big\{\int_{\partial\Omega}v\mathrm{d}\sigma_yu\Big\}. \tag{2.4}$$

Lemma 2.4 *Let* $u,v\in C^1(\Omega,C(V_{3,3}))\cap C(\overline{\Omega},C(V_{3,3}))$*. Then*

$$\int_\Omega(vL_\lambda[u]+L_{*\lambda}[v]u)\mathrm{d}y=\int_{\partial\Omega}v\mathrm{d}\sigma_yu. \tag{2.5}$$

Denote

$$K_1^A(r,h)=\frac{1}{\omega_3}\Big(\frac{(\mathbf{y}-\mathbf{x})e_A}{r^3}+\frac{|h|(\mathbf{y}-\mathbf{x})e_A}{r^2}+\frac{e_Ah}{r}\Big)\mathrm{e}^{-|h|r}, \tag{2.6}$$

$$K_{*1}^{\ A}(r,h)=\frac{1}{\omega_3}\Big(\frac{e_A(\mathbf{y}-\mathbf{x})}{r^3}+\frac{e_A|h|(\mathbf{y}-\mathbf{x})}{r^2}-\frac{he_A}{r}\Big)\mathrm{e}^{-|h|r}, \tag{2.7}$$

$$K_1(r,\lambda)=\frac{1}{\omega_3}\Big(\frac{\mathbf{y}-\mathbf{x}}{r^3}+\frac{\lambda(\mathbf{y}-\mathbf{x})}{r^2}+\frac{\lambda}{r}\Big)\mathrm{e}^{-\lambda r}, \tag{2.8}$$

$$K_{*1}(r,\lambda)=\frac{1}{\omega_3}\Big(\frac{\mathbf{y}-\mathbf{x}}{r^3}+\frac{\lambda(\mathbf{y}-\mathbf{x})}{r^2}-\frac{\lambda}{r}\Big)\mathrm{e}^{-\lambda r}, \tag{2.9}$$

where $\mathbf{y}-\mathbf{x}=\sum\limits_{i=1}^{3}(y_i-x_i)e_i$, $r=|y-x|$ and ω_3 denotes the area of the unit sphere in $\mathcal{R}^3$.

Lemma 2.5 *Let* $K_1^A(r,h)$ *and* $K_{*1}^{\ A}(r,h)$ *be as in* (2.6) *and* (2.7)*, where* $r=|y-x|$*. Then*

$$\begin{cases}L[K_1^A(r,h)]=0, & y\in\mathcal{R}^3_x=\mathcal{R}^3\setminus\{x\},\\ L_*[K_{*1}^{\ A}(r,h)]=0, & y\in\mathcal{R}^3_x=\mathcal{R}^3\setminus\{x\}.\end{cases} \tag{2.10}$$

Lemma 2.6 *Let $K_1(r,\lambda)$ and $K_{*1}(r,\lambda)$ be as in (2.8) and (2.9), where $r=|y-x|$. Then*

$$\begin{cases} L_\lambda[K_1(r,\lambda)] = L_{*-\lambda}[K_1(r,\lambda)] = 0, & y\in\mathcal{R}^3_x=\mathcal{R}^3\setminus\{x\}, \\ L_{-\lambda}[K_{*1}(r,\lambda)] = L_{*\lambda}[K_{*1}(r,\lambda)] = 0, & y\in\mathcal{R}^3_x=\mathcal{R}^3\setminus\{x\}. \end{cases} \tag{2.11}$$

Remark 2.1 By Lemmas 2.5 and 2.6, $K_1^A(r,0)\overline{e}_A$, $\overline{e}_A K_{*1}^{\ A}(r,0)$, $K_1(r,0)$ and $K_{*1}(r,0)$ are the classical kernel functions.

Lemma 2.7 *Let $K_1^A(r,h)$ be as in (2.6). Then*

$$\int_{\partial B(x,\delta)} \mathrm{d}\sigma_y K_1^A(r,h) = (1+\delta|h|)e_A \mathrm{e}^{-\delta|h|}, \tag{2.12}$$

where $B(x,\delta)=\Big\{y, \sum\limits_{i=1}^{3}(y_i-x_i)^2\le\delta^2\Big\}$ and $\partial B(x,\delta)$ is given the induced orientation.

Proof By Stokes formula, we have

$$\begin{aligned}\int_{\partial B(x,\delta)} \mathrm{d}\sigma_y K_1^A(r,h) &= \frac{1}{\omega_3}\int_{\partial B(x,\delta)} \mathrm{d}\sigma_y\Big(\frac{(\mathbf{y}-\mathbf{x})e_A}{\delta^3} + \frac{|h|(\mathbf{y}-\mathbf{x})e_A}{\delta^2} + \frac{e_A h}{\delta}\Big)\mathrm{e}^{-\delta|h|} \\ &= \frac{1}{\omega_3}\int_{B(x,\delta)}\Big(\frac{3}{\delta^3}+\frac{3|h|}{\delta^2}\Big)e_A\mathrm{e}^{-\delta|h|}\mathrm{d}y \\ &= (1+\delta|h|)e_A\mathrm{e}^{-\delta|h|}.\end{aligned}$$

Thus, the result follows.

Lemma 2.8 *Let $K_{*1}^{\ A}(r,h)$ be as in (2.7). Then*

$$\int_{\partial B(x,\delta)} K_{*1}^{\ A}(r,h)\mathrm{d}\sigma_y = e_A(1+\delta|h|)\mathrm{e}^{-\delta|h|}, \tag{2.13}$$

where $B(x,\delta)=\Big\{y, \sum\limits_{i=1}^{3}(y_i-x_i)^2\le\delta^2\Big\}$ and $\partial B(x,\delta)$ is given the induced orientation.

Theorem 2.1 (Generalized Cauchy-Pompeiu Formula) *Let Ω be an open bounded non-empty subset of $\mathcal{R}^3$ with a Liapunov boundary $\partial\Omega$, $u\in C^1(\Omega, C(V_{3,3}))\cap C(\overline{\Omega}, C(V_{3,3}))$. Then for $x\in\Omega$,*

$$\begin{aligned} u(x) &= \frac{1}{\omega_3}\int_{\partial\Omega}\Big(\frac{\mathbf{y}-\mathbf{x}}{r^3}+\frac{|h|(\mathbf{y}-\mathbf{x})}{r^2}\Big)\mathrm{e}^{-|h|r}\mathrm{d}\sigma_y u(y) - \frac{1}{\omega_3}\int_{\partial\Omega}\frac{\mathrm{e}^{-|h|r}}{r}\mathrm{d}\sigma_y u(y)h \\ &\quad - \frac{1}{\omega_3}\int_{\Omega}\Big\{\Big(\frac{\mathbf{y}-\mathbf{x}}{r^3}+\frac{|h|(\mathbf{y}-\mathbf{x})}{r^2}\Big)\mathrm{e}^{-|h|r}L[u](y) - \frac{\mathrm{e}^{-|h|r}}{r}L[u](y)h\Big\}\mathrm{d}y. \end{aligned} \tag{2.14}$$

Proof For $x\in\Omega$, by Lemmas 2.3 and 2.5, we have

$$\mathrm{Re}\int_{\partial\Omega\setminus\partial B(x,\varepsilon)} K_{*1}^{\ A}(r,h)\mathrm{d}\sigma_y u(y) = \mathrm{Re}\int_{\Omega\setminus B(x,\varepsilon)} K_{*1}^{\ A}(r,h)L[u]\mathrm{d}y. \tag{2.15}$$

Obviously,

$$\mathrm{Re}\int_{\partial B(x,\varepsilon)} K_{*1}^{\ A}(r,h)\mathrm{d}\sigma_y u(y)$$

$$= \mathrm{Re}\int_{\partial B(x,\varepsilon)} K_{*1}^{\ A}(r,h)\mathrm{d}\sigma_y(u(y)-u(x)) + \mathrm{Re}\int_{\partial B(x,\varepsilon)} K_{*1}^{\ A}(r,h)\mathrm{d}\sigma_y u(x). \tag{2.16}$$

By Lemma 2.8, it can be proved that

$$\lim_{\varepsilon\to 0}\mathrm{Re}\int_{\partial B(x,\varepsilon)} K_{*1}^{\ A}(r,h)\mathrm{d}\sigma_y u(x) = u_A(x)e_A^2. \tag{2.17}$$

In view of $u \in C^1(\Omega, C(V_{3,3})) \cap C(\overline{\Omega}, C(V_{3,3}))$, by Stokes formula, it can be similarly proved as Lemma 2.8 that

$$\lim_{\varepsilon\to 0}\mathrm{Re}\int_{\partial B(x,\varepsilon)} K_{*1}^{\ A}(r,h)\mathrm{d}\sigma_y(u(y)-u(x)) = 0. \tag{2.18}$$

Taking limit $\varepsilon \to 0$ in (2.15), in view of the weak singularity of $K_{*1}^{\ A}(r,h)$ and (2.16)–(2.18), we have

$$u_A(x)e_A^2 = \mathrm{Re}\int_{\partial\Omega} K_{*1}^{\ A}(r,h)\mathrm{d}\sigma_y u(y) - \mathrm{Re}\int_{\Omega} K_{*1}^{\ A}(r,h)L[u]\mathrm{d}y. \tag{2.19}$$

Thus

$$\begin{aligned} u_A(x) = {}& \frac{e_A^2}{\omega_3}\mathrm{Re}\int_{\partial\Omega} e_A\Big(\frac{\mathbf{y}-\mathbf{x}}{r^3} + \frac{|h|(\mathbf{y}-\mathbf{x})}{r^2}\Big)\mathrm{e}^{-|h|r}\mathrm{d}\sigma_y u(y) - \frac{e_A^2}{\omega_3}\mathrm{Re}\int_{\partial\Omega} he_A\frac{\mathrm{e}^{-|h|r}}{r}\mathrm{d}\sigma_y u(y) \\ & - \frac{e_A^2}{\omega_3}\mathrm{Re}\int_{\Omega}\Big\{e_A\Big(\frac{\mathbf{y}-\mathbf{x}}{r^3} + \frac{|h|(\mathbf{y}-\mathbf{x})}{r^2}\Big)\mathrm{e}^{-|h|r}L[u] - he_A\frac{\mathrm{e}^{-|h|r}}{r}L[u]\Big\}\mathrm{d}y. \end{aligned} \tag{2.20}$$

From (2.20), it is easy to check that

$$\begin{aligned} u_A(x) = {}& \frac{e_A^2}{\omega_3}\mathrm{Re}\int_{\partial\Omega} e_A\Big(\frac{\mathbf{y}-\mathbf{x}}{r^3} + \frac{|h|(\mathbf{y}-\mathbf{x})}{r^2}\Big)\mathrm{e}^{-|h|r}\mathrm{d}\sigma_y u(y) - \frac{e_A^2}{\omega_3}\mathrm{Re}\int_{\partial\Omega} e_A\frac{\mathrm{e}^{-|h|r}}{r}\mathrm{d}\sigma_y u(y)h \\ & - \frac{e_A^2}{\omega_3}\mathrm{Re}\int_{\Omega}\Big\{e_A\Big(\frac{\mathbf{y}-\mathbf{x}}{r^3} + \frac{|h|(\mathbf{y}-\mathbf{x})}{r^2}\Big)\mathrm{e}^{-|h|r}L[u] - e_A\frac{\mathrm{e}^{-|h|r}}{r}L[u]h\Big\}\mathrm{d}y. \end{aligned} \tag{2.21}$$

Then the result follows.

Theorem 2.2 (Generalized Cauchy-Pompeiu Formula) *Let Ω be an open bounded nonempty subset of $\mathcal{R}^3$ with a Liapunov boundary $\partial\Omega$, $u \in C^1(\Omega, C(V_{3,3})) \cap C(\overline{\Omega}, C(V_{3,3}))$. Then for $x \in \Omega$,*

$$\begin{aligned} u(x) = {}& \frac{1}{\omega_3}\int_{\partial\Omega} u(y)\mathrm{d}\sigma_y\Big(\frac{\mathbf{y}-\mathbf{x}}{r^3} + \frac{|h|(\mathbf{y}-\mathbf{x})}{r^2}\Big)\mathrm{e}^{-|h|r} + \frac{1}{\omega_3}\int_{\partial\Omega} hu(y)\mathrm{d}\sigma_y\frac{\mathrm{e}^{-|h|r}}{r} \\ & - \frac{1}{\omega_3}\int_{\Omega}\Big\{L_*[u](y)\Big(\frac{\mathbf{y}-\mathbf{x}}{r^3} + \frac{|h|(\mathbf{y}-\mathbf{x})}{r^2}\Big)\mathrm{e}^{-|h|r} + hL_*[u](y)\frac{\mathrm{e}^{-|h|r}}{r}\Big\}\mathrm{d}y. \end{aligned} \tag{2.22}$$

By Theorems 2.1 and 2.2, we can deduce the following result.

Theorem 2.3 (Generalized Cauchy Integral Formula) *Let Ω be an open bounded nonempty subset of $\mathcal{R}^3$ with a Liapunov boundary $\partial\Omega$, $u \in C^1(\Omega, C(V_{3,3})) \cap C(\overline{\Omega}, C(V_{3,3}))$, and $Lu = 0$ in Ω. Then for $x \in \Omega$,*

$$u(x) = \frac{1}{\omega_3}\int_{\partial\Omega}\Big(\frac{\mathbf{y}-\mathbf{x}}{r^3} + \frac{|h|(\mathbf{y}-\mathbf{x})}{r^2}\Big)\mathrm{e}^{-|h|r}\mathrm{d}\sigma_y u(y) - \frac{1}{\omega_3}\int_{\partial\Omega}\frac{\mathrm{e}^{-|h|r}}{r}\mathrm{d}\sigma_y u(y)h. \tag{2.23}$$

Theorem 2.4 (Generalized Cauchy Integral Formula) *Let Ω be an open bounded non-empty subset of $\mathcal{R}^3$ with a Liapunov boundary $\partial\Omega$, $u \in C^1(\Omega, C(V_{3,3})) \cap C(\overline{\Omega}, C(V_{3,3}))$, and $L_* u = 0$ in Ω. Then for $x \in \Omega$,*

$$u(x) = \frac{1}{\omega_3}\int_{\partial\Omega} u(y)\mathrm{d}\sigma_y\Big(\frac{\mathbf{y}-\mathbf{x}}{r^3} + \frac{|h|(\mathbf{y}-\mathbf{x})}{r^2}\Big)\mathrm{e}^{-|h|r} + \frac{1}{\omega_3}\int_{\partial\Omega} hu(y)\mathrm{d}\sigma_y \frac{\mathrm{e}^{-|h|r}}{r}. \tag{2.24}$$

Remark 2.2 In case of $h = 0$, Theorems 2.1–2.4 are respectively the classical Cauchy-Pompeiu formula and Cauchy integral formula in Clifford analysis.

By Lemmas 2.4 and 2.6, the following theorems can be similarly proved as Theorem 2.1.

Theorem 2.5 *Let Ω be an open bounded non-empty subset of $\mathcal{R}^3$ with a Liapunov boundary $\partial\Omega$, $u \in C^1(\Omega, C(V_{3,3})) \cap C(\overline{\Omega}, C(V_{3,3}))$, $K_{*1}(r, \lambda)$ be as in (2.9). Then*

$$\int_{\partial\Omega} K_{*1}(r,\lambda)\mathrm{d}\sigma_y u(y) - \int_{\Omega} K_{*1}(r,\lambda)L_\lambda[u](y)\mathrm{d}y = \begin{cases} u(x), & x \in \Omega, \\ 0, & x \in \mathcal{R}^3 \setminus \overline{\Omega}. \end{cases} \tag{2.25}$$

Theorem 2.6 *Let Ω be an open bounded non-empty subset of $\mathcal{R}^3$ with a Liapunov boundary $\partial\Omega$, $u \in C^1(\Omega, C(V_{3,3})) \cap C(\overline{\Omega}, C(V_{3,3}))$, $K_1(r, \lambda)$ be as in (2.8). Then*

$$\int_{\partial\Omega} K_1(r,\lambda)\mathrm{d}\sigma_y u(y) - \int_{\Omega} K_1(r,\lambda)L_{-\lambda}[u](y)\mathrm{d}y = \begin{cases} u(x), & x \in \Omega, \\ 0, & x \in \mathcal{R}^3 \setminus \overline{\Omega}. \end{cases} \tag{2.26}$$

Theorem 2.7 *Let Ω be an open bounded non-empty subset of $\mathcal{R}^3$ with a Liapunov boundary $\partial\Omega$, $u \in C^1(\Omega, C(V_{3,3})) \cap C(\overline{\Omega}, C(V_{3,3}))$, $K_{*1}(r, \lambda)$ be as in (2.9). Then*

$$\int_{\partial\Omega} u(y)\mathrm{d}\sigma_y K_{*1}(r,\lambda) - \int_{\Omega} L_{*-\lambda}[u](y)K_{*1}(r,\lambda)\mathrm{d}y = \begin{cases} u(x), & x \in \Omega, \\ 0, & x \in \mathcal{R}^3 \setminus \overline{\Omega}. \end{cases} \tag{2.27}$$

Theorem 2.8 *Let Ω be an open bounded non-empty subset of $\mathcal{R}^3$ with a Liapunov boundary $\partial\Omega$, $u \in C^1(\Omega, C(V_{3,3})) \cap C(\overline{\Omega}, C(V_{3,3}))$, $K_1(r, \lambda)$ be as in (2.8). Then*

$$\int_{\partial\Omega} u(y)\mathrm{d}\sigma_y K_1(r,\lambda) - \int_{\Omega} L_{*\lambda}[u](y)K_1(r,\lambda)\mathrm{d}y = \begin{cases} u(x), & x \in \Omega, \\ 0, & x \in \mathcal{R}^3 \setminus \overline{\Omega}. \end{cases} \tag{2.28}$$

Remark 2.3 In case of $\lambda = 0$, Theorems 2.5–2.8 are respectively the classical Cauchy-Pompeiu formula in Clifford analysis.

From Theorems 2.5–2.8, the following results can be directly proved.

Theorem 2.9 *Let Ω be an open bounded non-empty subset of $\mathcal{R}^3$ with a Liapunov boundary $\partial\Omega$, $u \in C^1(\Omega, C(V_{3,3})) \cap C(\overline{\Omega}, C(V_{3,3}))$, $K_{*1}(r, \lambda)$ be as in (2.9), and $L_\lambda[u] = 0$ in Ω. Then*

$$\int_{\partial\Omega} K_{*1}(r,\lambda)\mathrm{d}\sigma_y u(y) = \begin{cases} u(x), & x \in \Omega, \\ 0, & x \in \mathcal{R}^3 \setminus \overline{\Omega}. \end{cases} \tag{2.29}$$

Theorem 2.10 *Let Ω be an open bounded non-empty subset of $\mathcal{R}^3$ with a Liapunov boundary $\partial\Omega$, $u \in C^1(\Omega, C(V_{3,3})) \cap C(\overline{\Omega}, C(V_{3,3}))$, $K_1(r,\lambda)$ be as in (2.8), and $L_{-\lambda}[u] = 0$ in Ω. Then*

$$\int_{\partial\Omega} K_1(r,\lambda)\mathrm{d}\sigma_y u(y) = \begin{cases} u(x), & x \in \Omega, \\ 0, & x \in \mathcal{R}^3 \setminus \overline{\Omega}. \end{cases} \tag{2.30}$$

Theorem 2.11 *Let Ω be an open bounded non-empty subset of $\mathcal{R}^3$ with a Liapunov boundary $\partial\Omega$, $u \in C^1(\Omega, C(V_{3,3})) \cap C(\overline{\Omega}, C(V_{3,3}))$, $K_{*1}(r,\lambda)$ be as in (2.9), and $L_{*-\lambda}[u] = 0$ in Ω. Then*

$$\int_{\partial\Omega} u(y)\mathrm{d}\sigma_y K_{*1}(r,\lambda) = \begin{cases} u(x), & x \in \Omega, \\ 0, & x \in \mathcal{R}^3 \setminus \overline{\Omega}. \end{cases} \tag{2.31}$$

Theorem 2.12 *Let Ω be an open bounded non empty subset of $\mathcal{R}^3$ with a Liapunov boundary $\partial\Omega$, $u \in C^1(\Omega, C(V_{3,3})) \bigcap C(\overline{\Omega}, C(V_{3,3}))$, $K_1(r,\lambda)$ be as in (2.8), and $L_{*\lambda}[u] = 0$ in Ω, then*

$$\int_{\partial\Omega} u(y)\mathrm{d}\sigma_y K_1(r,\lambda) = \begin{cases} u(x), & x \in \Omega, \\ 0, & x \in \mathcal{R}^3 \setminus \overline{\Omega}. \end{cases} \tag{2.32}$$

Remark 2.4 In case of $\lambda = 0$, Theorems 2.9–2.12 are respectively the classical Cauchy integral formula in Clifford analysis.

Corollary 2.1 *Suppose $\|u\| = O(\mathrm{e}^{\theta\|y\|})$ $(\|y\| \to +\infty,\ y \in \mathcal{R}^3)$, $0 < \theta < |h|$ and denote the conditions as follows:* (I) $Lu = 0$ *in* $\mathcal{R}^3$; (II) $L_* u = 0$ *in* $\mathcal{R}^3$; (III) $L_\lambda[u] = 0$ *in* $\mathcal{R}^3$; (IV) $L_{-\lambda}[u] = 0$ *in* $\mathcal{R}^3$; (V) $L_{*-\lambda}[u] = 0$ *in* $\mathcal{R}^3$; (VI) $L_{*\lambda}[u] = 0$ *in* $\mathcal{R}^3$. *If any of the above conditions is satisfied, then $u \equiv 0$ in $\mathcal{R}^3$.*

Proof By Theorems 2.3, 2.4, 2.9–2.12, for any $x \in \mathcal{R}^3$, replacing $\partial\Omega$ by $\partial B(x,R)$ in the integral representations, taking limit $R \to \infty$, the result follows.

3 Generalized Higher Order Integral Representations

Denote

$$K^A_{m+1}(r,h) = \frac{1}{m!\omega_3}\Big(\frac{(1-m)(\mathbf{y}-\mathbf{x})e_A}{r^{3-m}} + \frac{|h|(\mathbf{y}-\mathbf{x})e_A}{r^{2-m}} - \frac{me_A\hbar}{r^{2-m}} + \frac{e_A h}{r^{1-m}}\Big)\mathrm{e}^{-|h|r}\hbar^m, \tag{3.1}$$

$$K_{*m+1}^{\ A}(r,h) = \frac{(-1)^m\hbar^m}{m!\omega_3}\Big(\frac{(1-m)e_A(\mathbf{y}-\mathbf{x})}{r^{3-m}} + \frac{e_A|h|(\mathbf{y}-\mathbf{x})}{r^{2-m}} + \frac{m\hbar e_A}{r^{2-m}} - \frac{he_A}{r^{1-m}}\Big)\mathrm{e}^{-|h|r}, \tag{3.2}$$

$$K_{m+1}(r,\lambda) = \frac{1}{m!\omega_3}\Big(\frac{(1-m)(\mathbf{y}-\mathbf{x})}{r^{3-m}} + \frac{\lambda(\mathbf{y}-\mathbf{x})}{r^{2-m}} - \frac{m}{r^{2-m}} + \frac{\lambda}{r^{1-m}}\Big)\mathrm{e}^{-\lambda r}, \tag{3.3}$$

$$K_{*m+1}(r,\lambda) = \frac{(-1)^m}{m!\omega_3}\Big(\frac{(1-m)(\mathbf{y}-\mathbf{x})}{r^{3-m}} + \frac{\lambda(\mathbf{y}-\mathbf{x})}{r^{2-m}} + \frac{m}{r^{2-m}} - \frac{\lambda}{r^{1-m}}\Big)\mathrm{e}^{-\lambda r}, \tag{3.4}$$

where $\mathbf{y}-\mathbf{x} = \sum_{i=1}^{3}(y_i - x_i)e_i$, $|h| = \lambda$, $h = |h|\hbar$, $r = |y-x|$, $m \geq 0$ and ω_3 denotes the area of the unit sphere in $\mathcal{R}^3$.

Lemma 3.1 *Let $K^A_{m+1}(r,h)$ and $K_{*}{}^{A}_{m+1}(r,h)$ be as in (3.1) and (3.2), $m\geq 0$. Then*

$$\begin{cases} L[K^A_{m+1}(r,h)] = K^A_m(r,h), & y\in\mathcal{R}^3_x=\mathcal{R}^3\setminus\{x\},\\ L_*[K_{*}{}^{A}_{m+1}(r,h)] = K_{*}{}^{A}_{m}(r,h), & y\in\mathcal{R}^3_x=\mathcal{R}^3\setminus\{x\}.\end{cases} \tag{3.5}$$

Lemma 3.2 *Let $K^A_{m+1}(r,h)$ and $K_{*}{}^{A}_{m+1}(r,h)$ be as in (3.1) and (3.2), $m\geq 0$. Then*

$$\begin{cases} L^{m+1}[K^A_{m+1}(r,h)] = 0, & y\in\mathcal{R}^3_x=\mathcal{R}^3\setminus\{x\},\\ L_*^{m+1}[K_{*}{}^{A}_{m+1}(r,h)] = 0, & y\in\mathcal{R}^3_x=\mathcal{R}^3\setminus\{x\}.\end{cases} \tag{3.6}$$

Lemma 3.3 *Let $K_{m+1}(r,\lambda)$ and $K_{*m+1}(r,\lambda)$ be as in (3.3) and (3.4), $m\geq 0$. Then*

$$\begin{cases} L_\lambda[K_{m+1}(r,\lambda)] = L_{*-\lambda}[K_{m+1}(r,\lambda)] = K_m(r,\lambda), & y\in\mathcal{R}^3_x=\mathcal{R}^3\setminus\{x\},\\ L_{-\lambda}[K_{*m+1}(r,\lambda)] = L_{*\lambda}[K_{*m+1}(r,\lambda)] = K_{*m}(r,\lambda), & y\in\mathcal{R}^3_x=\mathcal{R}^3\setminus\{x\}.\end{cases} \tag{3.7}$$

Lemma 3.4 *Let $K_{m+1}(r,\lambda)$ and $K_{*m+1}(r,\lambda)$ be as in (3.3) and (3.4), $m\geq 0$. Then*

$$\begin{cases} L^{m+1}_\lambda[K_{m+1}(r,\lambda)] = L_{*}{}^{m+1}_{-\lambda}[K_{m+1}(r,\lambda)] = 0, & y\in\mathcal{R}^3_x=\mathcal{R}^3\setminus\{x\},\\ L^{m+1}_{-\lambda}[K_{*m+1}(r,\lambda)] = L_{*}{}^{m+1}_{\lambda}[K_{*m+1}(r,\lambda)] = 0, & y\in\mathcal{R}^3_x=\mathcal{R}^3\setminus\{x\}.\end{cases} \tag{3.8}$$

Suppose $\lambda=0$, then (3.3) and (3.4) can be rewritten as

$$K_{m+1}(r,0) = \frac{1}{m!\omega_3}\Big(\frac{(1-m)(\mathbf{y}-\mathbf{x})}{r^{3-m}} - \frac{m}{r^{2-m}}\Big), \tag{3.9}$$

$$K_{*m+1}(r,0) = \frac{(-1)^m}{m!\omega_3}\Big(\frac{(1-m)(\mathbf{y}-\mathbf{x})}{r^{3-m}} + \frac{m}{r^{2-m}}\Big). \tag{3.10}$$

Lemma 3.5 *Let $K_{m+1}(r,0)$ and $K_{*m+1}(r,0)$ be as in (3.9) and (3.10), $m\geq 0$. Then*

$$\begin{cases} D[K_{m+1}(r,0)] = [K_{m+1}(r,0)]D = K_m(r,0), & y\in\mathcal{R}^3_x=\mathcal{R}^3\setminus\{x\},\\ D[K_{*m+1}(r,0)] = [K_{*m+1}(r,0)]D = K_{*m}(r,0), & y\in\mathcal{R}^3_x=\mathcal{R}^3\setminus\{x\}.\end{cases} \tag{3.11}$$

Lemma 3.6 *Let $K_{m+1}(r,0)$ and $K_{*m+1}(r,0)$ be as in (3.9) and (3.10), $m\geq 0$. Then*

$$\begin{cases} D^{m+1}[K_{m+1}(r,0)] = [K_{m+1}(r,0)]D^{m+1} = 0, & y\in\mathcal{R}^3_x=\mathcal{R}^3\setminus\{x\},\\ D^{m+1}[K_{*m+1}(r,0)] = [K_{*m+1}(r,0)]D^{m+1} = 0, & y\in\mathcal{R}^3_x=\mathcal{R}^3\setminus\{x\}.\end{cases} \tag{3.12}$$

Remark 3.1 By Lemma 3.6, $K_{m+1}(r,0)$ and $K_{*m+1}(r,0)$ are the higher order kernel functions. Comparing the forms with the results in [7, 29–31], these kernel functions are different. In case of $m=0$, $K_{m+1}(r,0)$ and $K_{*m+1}(r,0)$ are the classical kernel functions in Clifford analysis. Thus, (3.3) and (3.4) also provide a way to find the kernel functions for $(m+1)$-regular functions.

Remark 3.2 Denote $H^*_j(x)$, $j\geq 1$ as in [7, 29–31]. Then we have

$$H^*_j(y-x) = \frac{K_j(r,0)+K_{*j}(r,0)}{2}, \quad j\geq 1. \tag{3.13}$$

Theorem 3.1 *Let Ω be an open bounded non-empty subset of $\mathcal{R}^3$ with a Liapunov boundary $\partial\Omega$, $u \in C^k(\Omega, C(V_{3,3})) \cap C^{k-1}(\overline{\Omega}, C(V_{3,3}))$. Then for $x \in \Omega$,*

$$\begin{aligned} u(x) = \frac{1}{\omega_3}\Big[&\sum_{j=1}^{k}\frac{1}{(j-1)!}\int_{\partial\Omega}\Big[\frac{(2-j)(\mathbf{y}-\mathbf{x})}{r^{4-j}}+\frac{|h|(\mathbf{y}-\mathbf{x})}{r^{3-j}}\Big]\mathrm{e}^{-|h|r}\mathrm{d}\sigma_y L^{j-1}[u](y)\hbar^{j-1} \\ &+\sum_{j=1}^{k}\frac{1}{(j-1)!}\int_{\partial\Omega}\Big[\frac{j-1}{r^{3-j}}-\frac{|h|}{r^{2-j}}\Big]\mathrm{e}^{-|h|r}\mathrm{d}\sigma_y L^{j-1}[u](y)\hbar^{j}\Big] \\ &-\frac{1}{(k-1)!\omega_3}\int_{\Omega}\Big[\frac{(2-k)(\mathbf{y}-\mathbf{x})}{r^{4-k}}+\frac{|h|(\mathbf{y}-\mathbf{x})}{r^{3-k}}\Big]\mathrm{e}^{-|h|r}L^{k}[u](y)\hbar^{k-1}\mathrm{d}y \\ &-\frac{1}{(k-1)!\omega_3}\int_{\Omega}\Big[\frac{k-1}{r^{3-k}}-\frac{|h|}{r^{2-k}}\Big]\mathrm{e}^{-|h|r}L^{k}[u](y)\hbar^{k}\mathrm{d}y. \end{aligned} \tag{3.14}$$

Proof For $x \in \Omega$, by Lemmas 2.3, 3.1 and 3.2, we have

$$\begin{aligned} &\mathrm{Re}\Big\{\sum_{j=1}^{k}(-1)^{j-1}\int_{\partial\Omega\setminus\partial B(x,\varepsilon)}K_{*j}^{\ A}(r,h)\mathrm{d}\sigma_y L^{j-1}[u](y)\Big\} \\ &=\mathrm{Re}\Big\{(-1)^{k-1}\int_{\Omega\setminus B(x,\varepsilon)}K_{*k}^{\ A}(r,h)L^{k}[u]\mathrm{d}y\Big\}. \end{aligned} \tag{3.15}$$

In view of $u \in C^k(\Omega, C(V_{3,3})) \cap C^{k-1}(\overline{\Omega}, C(V_{3,3}))$, by Stokes formula, it can be proved that

$$\lim_{\varepsilon\to 0}\mathrm{Re}\Big\{\int_{\partial B(x,\varepsilon)}K_{*j}^{\ A}(r,h)\mathrm{d}\sigma_y L^{j-1}[u](y)\Big\}=0, \quad 2\le j\le k. \tag{3.16}$$

Taking limit $\varepsilon \to 0$ in (3.15), in view of the weak singularity of $K_{*j}^{\ A}(r,h)$ $(j = 1, \cdots, k)$, by (2.16)–(2.18) and (3.16), we have

$$\begin{aligned} &\mathrm{Re}\Big\{\sum_{j=1}^{k}(-1)^{j-1}\int_{\partial\Omega}K_{*j}^{\ A}(r,h)\mathrm{d}\sigma_y L^{j-1}[u](y)\Big\} \\ &=u_A(x)e_A^2+\mathrm{Re}(-1)^{k-1}\int_{\Omega}K_{*k}^{\ A}(r,h)L^{k}[u]\mathrm{d}y. \end{aligned} \tag{3.17}$$

Combining (3.2) with (3.17), by the same technique as in (2.21), (3.14) holds.

By Lemmas 2.3, 3.1 and 3.2, similarly, we have the following theorem.

Theorem 3.2 *Let Ω be an open bounded non-empty subset of $\mathcal{R}^3$ with a Liapunov boundary $\partial\Omega$, $u \in C^k(\Omega, C(V_{3,3})) \cap C^{k-1}(\overline{\Omega}, C(V_{3,3}))$. Then for $x \in \Omega$,*

$$\begin{aligned} u(x) = \frac{1}{\omega_3}\Big[&\sum_{j=1}^{k}\frac{(-1)^{j-1}}{(j-1)!}\int_{\partial\Omega}\hbar^{j-1}L_*^{j-1}[u](y)\mathrm{d}\sigma_y\Big[\frac{(2-j)(\mathbf{y}-\mathbf{x})}{r^{4-j}}+\frac{|h|(\mathbf{y}-\mathbf{x})}{r^{3-j}}\Big]\mathrm{e}^{-|h|r} \\ &-\sum_{j=1}^{k}\frac{(-1)^{j-1}}{(j-1)!}\int_{\partial\Omega}\hbar^{j}L_*^{j-1}[u](y)\mathrm{d}\sigma_y\Big[\frac{j-1}{r^{3-j}}-\frac{|h|}{r^{2-j}}\Big]\mathrm{e}^{-|h|r}\Big] \\ &-\frac{(-1)^{k-1}}{(k-1)!\omega_3}\int_{\Omega}\hbar^{k-1}L_*^{k}[u](y)\Big[\frac{(2-k)(\mathbf{y}-\mathbf{x})}{r^{4-k}}+\frac{|h|(\mathbf{y}-\mathbf{x})}{r^{3-k}}\Big]\mathrm{e}^{-|h|r}\mathrm{d}y \end{aligned}$$

$$-\frac{(-1)^{k-1}}{(k-1)!\omega_3}\int_{\Omega}\hbar^k L_*^k[u](y)\Big[\frac{1-k}{r^{3-k}}+\frac{|h|}{r^{2-k}}\Big]\mathrm{e}^{-|h|r}\mathrm{d}y. \tag{3.18}$$

Remark 3.3 In case of $k=1$, Theorems 3.1 and 3.2 are just Theorems 2.1 and 2.2. Thus we also call Theorems 3.1 and 3.2 the generalized higher order Cauchy-Pompeiu formulas.

By Theorems 3.1 and 3.2, we have the following result.

Theorem 3.3 *Let Ω be an open bounded non-empty subset of $\mathcal{R}^3$ with a Liapunov boundary $\partial\Omega$, $u\in C^k(\Omega,C(V_{3,3}))\cap C^{k-1}(\overline{\Omega},C(V_{3,3}))$, $L^k u=0$ in Ω. Then for $x\in\Omega$,*

$$\begin{aligned}u(x)=&\frac{1}{\omega_3}\Big[\sum_{j=1}^{k}\frac{1}{(j-1)!}\int_{\partial\Omega}\Big[\frac{(2-j)(\mathbf{y}-\mathbf{x})}{r^{4-j}}+\frac{|h|(\mathbf{y}-\mathbf{x})}{r^{3-j}}\Big]\mathrm{e}^{-|h|r}\mathrm{d}\sigma_y L^{j-1}[u](y)\hbar^{j-1}\\&+\sum_{j=1}^{k}\frac{1}{(j-1)!}\int_{\partial\Omega}\Big[\frac{j-1}{r^{3-j}}-\frac{|h|}{r^{2-j}}\Big]\mathrm{e}^{-|h|r}\mathrm{d}\sigma_y L^{j-1}[u](y)\hbar^{j}\Big].\end{aligned} \tag{3.19}$$

Theorem 3.4 *Let Ω be an open bounded non-empty subset of $\mathcal{R}^3$ with a Liapunov boundary $\partial\Omega$, $u\in C^k(\Omega,C(V_{3,3}))\cap C^{k-1}(\overline{\Omega},C(V_{3,3}))$, $L_*^k u=0$ in Ω. Then for $x\in\Omega$,*

$$\begin{aligned}u(x)=&\frac{1}{\omega_3}\Big[\sum_{j=1}^{k}\frac{(-1)^{j-1}}{(j-1)!}\int_{\partial\Omega}\hbar^{j-1}L_*^{j-1}[u](y)\mathrm{d}\sigma_y\Big[\frac{(2-j)(\mathbf{y}-\mathbf{x})}{r^{4-j}}+\frac{|h|(\mathbf{y}-\mathbf{x})}{r^{3-j}}\Big]\mathrm{e}^{-|h|r}\\&-\sum_{j=1}^{k}\frac{(-1)^{j-1}}{(j-1)!}\int_{\partial\Omega}\hbar^{j}L_*^{j-1}[u](y)\mathrm{d}\sigma_y\Big[\frac{j-1}{r^{3-j}}-\frac{|h|}{r^{2-j}}\Big]\mathrm{e}^{-|h|r}\Big].\end{aligned} \tag{3.20}$$

Remark 3.4 In case of $k=1$, Theorems 3.3 and 3.4 are just Theorems 2.3 and 2.4. Thus we also call Theorems 3.3 and 3.4 the generalized higher order Cauchy integral formulas.

By Lemmas 2.4, 3.3 and 3.4, we have the following result.

Theorem 3.5 *Let Ω be an open bounded non-empty subset of $\mathcal{R}^3$ with a Liapunov boundary $\partial\Omega$, $u\in C^k(\Omega,C(V_{3,3}))\cap C^{k-1}(\overline{\Omega},C(V_{3,3}))$, $K_{*j}(r,\lambda)$, $j=1,\cdots,k$ be as in (3.4). Then*

$$\begin{aligned}&\sum_{j=1}^{k}(-1)^{j-1}\int_{\partial\Omega}K_{*j}(r,\lambda)\mathrm{d}\sigma_y L_\lambda^{j-1}[u](y)+(-1)^k\int_{\Omega}K_{*k}(r,\lambda)L_\lambda^k[u](y)\mathrm{d}y\\&=\begin{cases}u(x), & x\in\Omega,\\0, & x\in\mathcal{R}^3\setminus\overline{\Omega}.\end{cases}\end{aligned} \tag{3.21}$$

Theorem 3.6 *Let Ω be an open bounded non-empty subset of $\mathcal{R}^3$ with a Liapunov boundary $\partial\Omega$, $u\in C^k(\Omega,C(V_{3,3}))\cap C^{k-1}(\overline{\Omega},C(V_{3,3}))$, $K_j(r,\lambda)$, $j=1,\cdots,k$ be as in (3.3). Then*

$$\begin{aligned}&\sum_{j=1}^{k}(-1)^{j-1}\int_{\partial\Omega}K_j(r,\lambda)\mathrm{d}\sigma_y L_{-\lambda}^{j-1}[u](y)+(-1)^k\int_{\Omega}K_k(r,\lambda)L_{-\lambda}^k[u](y)\mathrm{d}y\\&=\begin{cases}u(x), & x\in\Omega,\\0, & x\in\mathcal{R}^3\setminus\overline{\Omega}.\end{cases}\end{aligned} \tag{3.22}$$

Theorem 3.7 *Let Ω be an open bounded non-empty subset of $\mathcal{R}^3$ with a Liapunov boundary $\partial\Omega$, $u \in C^k(\Omega, C(V_{3,3})) \cap C^{k-1}(\overline{\Omega}, C(V_{3,3}))$, $K_{*j}(r,\lambda)$, $j = 1, \cdots, k$ be as in* (3.4). *Then*

$$\sum_{j=1}^{k}(-1)^{j-1}\int_{\partial\Omega} L_{*-\lambda}^{\,j-1}[u](y)\mathrm{d}\sigma_y K_{*j}(r,\lambda) + (-1)^k \int_{\Omega} L_{*-\lambda}^{\,k}[u](y)K_{*k}(r,\lambda)\mathrm{d}y = \begin{cases} u(x), & x \in \Omega, \\ 0, & x \in \mathcal{R}^3 \setminus \overline{\Omega}. \end{cases} \tag{3.23}$$

Theorem 3.8 *Let Ω be an open bounded non-empty subset of $\mathcal{R}^3$ with a Liapunov boundary $\partial\Omega$, $u \in C^k(\Omega, C(V_{3,3})) \cap C^{k-1}(\overline{\Omega}, C(V_{3,3}))$, $K_j(r,\lambda)$, $j = 1, \cdots, k$ be as in* (3.3). *Then*

$$\sum_{j=1}^{k}(-1)^{j-1}\int_{\partial\Omega} L_{*\lambda}^{\,j-1}[u](y)\mathrm{d}\sigma_y K_j(r,\lambda) + (-1)^k \int_{\Omega} L_{*\lambda}^{\,k}[u](y)K_k(r,\lambda)\mathrm{d}y = \begin{cases} u(x), & x \in \Omega, \\ 0, & x \in \mathcal{R}^3 \setminus \overline{\Omega}. \end{cases} \tag{3.24}$$

Remark 3.5 In case of $k = 1$, Theorems 3.5–3.8 are just Theorems 2.5–2.8.

Remark 3.6 In case of $\lambda = 0$, Theorems 3.5–3.8 are the other expressions of the higher order Cauchy-Pompeiu formulas which are different from the results in [7, 29–31].

By Theorems 3.5–3.8, we have the following results.

Theorem 3.9 *Let Ω be an open bounded non-empty subset of $\mathcal{R}^3$ with a Liapunov boundary $\partial\Omega$, $u \in C^k(\Omega, C(V_{3,3})) \cap C^{k-1}(\overline{\Omega}, C(V_{3,3}))$, $K_{*j}(r,\lambda)$, $j = 1, \cdots, k$ be as in* (3.4), *$L_{\lambda}^{k}u = 0$ in Ω. Then*

$$\sum_{j=1}^{k}(-1)^{j-1}\int_{\partial\Omega} K_{*j}(r,\lambda)\mathrm{d}\sigma_y L_{\lambda}^{j-1}[u](y) = \begin{cases} u(x), & x \in \Omega, \\ 0, & x \in \mathcal{R}^3 \setminus \overline{\Omega}. \end{cases} \tag{3.25}$$

Theorem 3.10 *Let Ω be an open bounded non-empty subset of $\mathcal{R}^3$ with a Liapunov boundary $\partial\Omega$, $u \in C^k(\Omega, C(V_{3,3})) \cap C^{k-1}(\overline{\Omega}, C(V_{3,3}))$, $K_j(r,\lambda)$, $j = 1, \cdots, k$ be as in* (3.3), *$L_{-\lambda}^{k}u = 0$ in Ω. Then*

$$\sum_{j=1}^{k}(-1)^{j-1}\int_{\partial\Omega} K_j(r,\lambda)\mathrm{d}\sigma_y L_{-\lambda}^{j-1}[u](y) = \begin{cases} u(x), & x \in \Omega, \\ 0, & x \in \mathcal{R}^3 \setminus \overline{\Omega}. \end{cases} \tag{3.26}$$

Theorem 3.11 *Let Ω be an open bounded non-empty subset of $\mathcal{R}^3$ with a Liapunov boundary $\partial\Omega$, $u \in C^k(\Omega, C(V_{3,3})) \cap C^{k-1}(\overline{\Omega}, C(V_{3,3}))$, $K_{*j}(r,\lambda)$, $j = 1, \cdots, k$ be as in* (3.4), *$L_{*-\lambda}^{\,k}u = 0$ in Ω. Then*

$$\sum_{j=1}^{k}(-1)^{j-1}\int_{\partial\Omega} L_{*-\lambda}^{\,j-1}[u](y)\mathrm{d}\sigma_y K_{*j}(r,\lambda) = \begin{cases} u(x), & x \in \Omega, \\ 0, & x \in \mathcal{R}^3 \setminus \overline{\Omega}. \end{cases} \tag{3.27}$$

Theorem 3.12 *Let Ω be an open bounded non-empty subset of $\mathcal{R}^3$ with a Liapunov boundary $\partial\Omega$, $u \in C^k(\Omega, C(V_{3,3})) \cap C^{k-1}(\overline{\Omega}, C(V_{3,3}))$, $K_j(r,\lambda)$, $j = 1, \cdots, k$ be as in* (3.3)*, $L_{*\lambda}^{\,k} u = 0$. Then*

$$\sum_{j=1}^{k} (-1)^{j-1} \int_{\partial\Omega} L_{*\lambda}^{\,j-1}[u](y) \mathrm{d}\sigma_y K_j(r,\lambda) = \begin{cases} u(x), & x \in \Omega, \\ 0, & x \in \mathcal{R}^3 \setminus \overline{\Omega}. \end{cases} \tag{3.28}$$

Remark 3.7 In case of $k = 1$, Theorems 3.9–3.12 are just Theorems 2.9–2.12.

Remark 3.8 In case of $\lambda = 0$, Theorems 3.9–3.12 are the other expressions of the higher order Cauchy integral formulas which are different from the results in [7, 29–31].

4 Maximum Modulus Theorem

In this section, suppose that Ω is a bounded domain in $\mathcal{R}^3$, $Hu = 0$ in Ω. We shall give the Maximum Modulus Theorem for u.

Lemma 4.1 *Let $u, v \in C^2(\Omega, C(V_{3,3})) \cap C^1(\overline{\Omega}, C(V_{3,3}))$. Then*

$$\int_{\Omega} (vH[u] - H[v]u) \mathrm{d}y = \int_{\partial\Omega} v \mathrm{d}\sigma_y L_\lambda[u] - \int_{\partial\Omega} L_{*-\lambda}[v] \mathrm{d}\sigma_y u. \tag{4.1}$$

Remark 4.1 In case of $\lambda = 0$, Lemma 4.1 is just the following well-known Green formula in Clifford analysis:

$$\int_{\Omega} (v\Delta[u] - \Delta[v]u) \mathrm{d}y = \int_{\partial\Omega} v \mathrm{d}\sigma_y D[u] - \int_{\partial\Omega} [v] D \mathrm{d}\sigma_y u. \tag{4.2}$$

Lemma 4.2 *Let Ω be a bounded domain in $\mathcal{R}^3$, $u \in C^2(\Omega, C(V_{3,3}))$, $Hu = 0$ in Ω and $|u(x)| = constant$ in Ω. We have*

(I) *If $\lambda > 0$, then $u(x) \equiv 0$ in Ω,*

(II) *If $\lambda = 0$, then $u(x)$ must be constant.*

Proof Denote $u(x) = \sum_A u_A(x) e_A$. Then $\sum_A u_A^2(x) \equiv C$, from which we have

$$\sum_{i=1}^{3} \sum_A \left(\frac{\partial u_A}{\partial x_i}\right)^2 + \sum_A u_A(x) \Delta[u_A] = 0. \tag{4.3}$$

In view of $Hu = 0$ in Ω, thus $Hu_A = 0$ in Ω. Then more clearly,

$$\Delta[u_A] = \lambda^2 u_A. \tag{4.4}$$

Combining (4.3) with (4.4), we have

$$\sum_{i=1}^{3} \sum_A \left(\frac{\partial u_A}{\partial x_i}\right)^2 + \sum_A \lambda^2 u_A^2(x) = 0. \tag{4.5}$$

Obviously, if $\lambda > 0$, then $u_A(x) \equiv 0$. Thus $u(x) \equiv 0$.

If $\lambda = 0$, then

$$\frac{\partial u_A(x)}{\partial x_i} = 0, \quad i = 1, 2, 3,\ A \in \mathcal{P}N,\ N = \{1, \cdots, 3\},\ x \in \Omega. \tag{4.6}$$

Consequently, $u(x)$ is a constant.

Remark 4.2 Lemma 4.2 is still valid in $\mathcal{R}^3$.

Remark 4.3 By Lemma 4.2, suppose that Ω is a bounded domain in $\mathcal{R}^3$, $u \in C^2(\Omega, C(V_{3,3}))$, $|u(x)| =$ constant in Ω. Denote the conditions as follows: (I) $Lu = 0$ in Ω; (II) $L_*u = 0$ in Ω; (III) $L_\lambda[u] = 0$ in Ω; (IV) $L_{-\lambda}[u] = 0$ in Ω; (V) $L_{*-\lambda}[u] = 0$ in Ω; (VI) $L_{*\lambda}[u] = 0$ in Ω, (VII) $D[u] = 0$ in Ω, (VIII) $[u]D = 0$ in Ω. If any of the above conditions is satisfied, then $u(x) \equiv C$.

Theorem 4.1 *Let Ω be an open bounded non-empty subset of $\mathcal{R}^3$ with a Liapunov boundary $\partial\Omega$, $u \in C^2(\Omega, C(V_{3,3})) \cap C^1(\overline{\Omega}, C(V_{3,3}))$, and $Hu = 0$ in Ω. Then for $x \in \Omega$,*

$$u(x) = \frac{1}{\omega_3}\int_{\partial\Omega}\Big(\frac{\mathbf{y}-\mathbf{x}}{r^3} + \frac{\lambda(\mathbf{y}-\mathbf{x})}{r^2} - \frac{\lambda}{r}\Big)\mathrm{e}^{-\lambda r}\mathrm{d}\sigma_y u(y) + \frac{1}{\omega_3}\int_{\partial\Omega}\frac{\mathrm{e}^{-\lambda r}}{r}\mathrm{d}\sigma_y L_\lambda[u](y). \tag{4.7}$$

Proof By Lemma 4.1, taking $v = -\frac{1}{\omega_3}\frac{\mathrm{e}^{-\lambda r}}{r}$ in (4.1), obviously, we get $L_{*-\lambda}[v] = K_{*1}(r,\lambda)$. Thus for $x \in \Omega$, we have

$$-\frac{1}{\omega_3}\int_{\partial(\Omega\setminus B(x,\delta))}\frac{\mathrm{e}^{-\lambda r}}{r}\mathrm{d}\sigma_y L_\lambda[u](y) - \int_{\partial(\Omega\setminus B(x,\delta))}K_{*1}(r,\lambda)\mathrm{d}\sigma_y u = 0. \tag{4.8}$$

It is easy to check that

$$\lim_{\delta\to 0}\Big[-\frac{1}{\omega_3}\int_{\partial B(x,\delta)}\frac{\mathrm{e}^{-\lambda r}}{r}\mathrm{d}\sigma_y L_\lambda[u](y) - \int_{\partial B(x,\delta)}K_{*1}(r,\lambda)\mathrm{d}\sigma_y u\Big] = -u(x). \tag{4.9}$$

Combining (4.8) with (4.9), (4.7) follows.

Corollary 4.1 *Let Ω be a bounded domain in $\mathcal{R}^3$, $u \in C^2(\Omega, C(V_{3,3}))$ and $Hu = 0$ in Ω. Then for any $x_0 \in \Omega$,*

$$u(x_0) = \frac{1}{\omega_3}\Big(\frac{1}{R^2} + \frac{\lambda}{R}\Big)\mathrm{e}^{-\lambda R}\int_{\partial B(x_0,R)}u(y)\mathrm{d}S + \frac{\lambda^2\mathrm{e}^{-\lambda R}}{\omega_3 R}\int_{B(x_0,R)}u(y)\mathrm{d}y, \tag{4.10}$$

where R is chosen such that $B(x_0,R) \subset \Omega$.

Remark 4.4 In case of $\lambda = 0$, Corollary 4.1 is just the mean value theorem for harmonic funtions.

Remark 4.5 Corollary 4.1 is still valid for the functions satisfying any of the following conditions: (I) $Lu = 0$ in Ω; (II) $L_*u = 0$ in Ω; (III) $L_\lambda[u] = 0$ in Ω; (IV) $L_{-\lambda}[u] = 0$ in Ω; (V) $L_{*-\lambda}[u] = 0$ in Ω; (VI) $L_{*\lambda}[u] = 0$ in Ω; (VII) $D[u] = 0$ in Ω; (VIII) $[u]D = 0$ in Ω.

Theorem 4.2 (Maximum Modulus Theorem) *Suppose that Ω is a bounded domain in $\mathcal{R}^3$, $u \in C^2(\Omega, C(V_{3,3}))$ and $Hu = 0$ in Ω. If there exists an $x_0 \in \Omega$, such that*

$$|u(x_0)| \geq |u(x)|, \quad x \in \Omega, \tag{4.11}$$

then $u(x)$ is a constant. Moreover, if $\lambda > 0$, then $u(x) \equiv 0$.

Proof (I) $\lambda = 0$, by the mean value theorem of harmonic functions and Lemma 4.2, the result follows.

(II) $\lambda > 0$, suppose $|u(x_0)| > 0$. Taking $R > 0$ such that $B(x_0,R) \subset \Omega$, by Corollary 4.1, we have

$$|u(x_0)| \leq \frac{1}{\omega_3}\Big(\frac{1}{R^2} + \frac{\lambda}{R}\Big)\mathrm{e}^{-\lambda R}|u(x_0)|R^2\omega_3 + \frac{\lambda^2\mathrm{e}^{-\lambda R}}{\omega_3 R}|u(x_0)|R^3 V_3, \tag{4.12}$$

more clearly,

$$|u(x_0)| \leq \frac{1 + \lambda R + \frac{(\lambda R)^2}{3}}{\mathrm{e}^{\lambda R}} |u(x_0)| < |u(x_0)|. \tag{4.13}$$

(4.13) implies $|u(x_0)| = 0$. Thus the result follows.

Corollary 4.2 *Suppose that* Ω *is a bounded domain in* $\mathcal{R}^3$, $u \in C^2(\Omega, C(V_{3,3}))$, $Hu = 0$ *in* Ω *and continuous on* $\overline{\Omega}$. *Then*

$$\max\{|u(x)|,\ x \in \overline{\Omega}\} = \max\{|u(x)|,\ x \in \partial\Omega\}. \tag{4.14}$$

Remark 4.6 Theorem 4.2 and Corollary 4.2 are still valid for the functions satisfying any of the following conditions: (I) $Lu = 0$ in Ω; (II) $L_*u = 0$ in Ω; (III) $L_\lambda[u] = 0$ in Ω; (IV) $L_{-\lambda}[u] = 0$ in Ω; (V) $L_{*-\lambda}[u] = 0$ in Ω; (VI) $L_{*\lambda}[u] = 0$ in Ω; (VII) $D[u] = 0$ in Ω, (VIII) $[u]D = 0$ in Ω.

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