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Compatibility and Schur Complements of Operators on Hilbert C^*-Module*

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Abstract Let E be a Hilbert C^*-module, and $\mathscr{S}$ be an orthogonally complemented closed submodule of E. The authors generalize the definitions of $\mathscr{S}$-complementability and $\mathscr{S}$-compatibility for general (adjointable) operators from Hilbert space to Hilbert C^*-module, and discuss the relationship between each other. Several equivalent statements about $\mathscr{S}$-complementability and $\mathscr{S}$-compatibility, and several representations of Schur complements of $\mathscr{S}$-complementable operators (especially, of $\mathscr{S}$-compatible operators and of positive $\mathscr{S}$-compatible operators) on a Hilbert C^*-module are obtained. In addition, the quotient property for Schur complements of matrices is generalized to the quotient property for Schur complements of $\mathscr{S}$-complementable operators and $\mathscr{S}^*$-complementable operators on a Hilbert C^*-module.

Keywords Hilbert C^*-module, Compatibility, Complementability, Schur complement, Quotient property

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1 Introduction

The Schur complement plays an important role in matrix analysis, statistics, numerical analysis, and many other areas of mathematics and its applications. The Schur complement was first formally introduced by Haynsworth [18], but it had been implicitly used since the beginning of the theory of matrices: given a block matrix $T = \left(\begin{smallmatrix} B & C \\ D & E \end{smallmatrix}\right)$ with B invertible, then $T_{/B} = E - DB^{-1}C$ is the Schur complement of B in T. This notion was then generalized in several directions for bounded linear operators on Hilbert spaces.

Anderson and Trapp [1] first introduced the definition of the Schur complement for positive operators on Hilbert spaces and applied it in electrical network theory. Given a positive operator $T \in L(H)$, where H is a Hilbert space, the shorted operator (just the Schur complement) of T to a closed subspace $\mathscr{S}$ is defined as

$$T_{/\mathscr{S}} = \max\{X \in L(H) : 0 \le X \le T,\ R(X) \subseteq \mathscr{S}^{\perp}\},$$

which actually had been studied as part of the theory of extensions of Hermitian operators by Krein [22].

Later, given a closed subspace $\mathscr{S}$ of Hilbert space H, Ando introduced the definition of the Schur complement for $\mathscr{S}$-complementable bounded operators in [2]. Since then, the Schur

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complement of bounded linear operators has attracted more and more attention (see [5, 9, 10, 16, 28, 29], etc.).

Compatible pairs were recently studied by Hassi and Nordstrom [17] and Corach et al [9] because they defined the classes of projections which have a minimality property that may be relevant to different areas, e.g., approximation theory, abstract splines and least square problems (see [8]). Corach et al [10] showed that a selfadjoint operator $A \in L(H)$ is $\mathscr{S}$-complementable if and only if the pair $(A, \mathscr{S})$ is compatible, i.e., there exists an idempotent $Q \in L(H)$ with range $\mathscr{S}$ which is selfadjoint with respect to the sesquilinear form induced by A:

$$\langle x, y\rangle_A = \langle Ax, y\rangle, \quad x, y \in H.$$

Moreover, Corach et al [9] gave some characterizations of the compatibility of $(A, \mathscr{S})$.

As a natural generalization of the notions of Hilbert space and C^*-algebra, the notion of Hilbert C^*-module is an important tool in the theory of C^*-algebras, especially in the theory of KK-group and in the theory of induced representations (see [12–14, 19, 23, 31]). Therefore, it is meaningful to put forward a generalized version of the previous results in the context of Hilbert C^*-modules.

In this paper, for an orthogonal complemented closed submodule $\mathscr{S}$ of a Hilbert C^*-module, we aim to put forward the definitions of $\mathscr{S}$-compatibility and $\mathscr{S}$-complementability for adjointable operators (not just for selfadjoint operators as in [9]) on Hilbert C^*-modules and discuss the Schur complement of this kind of operators. In Section 2, we obtain some characterizations of the compatibility of $(A, \mathscr{S})$ (in which case A is called a $\mathscr{S}$-compatible operator) as in [9]. In Section 3, we obtain some characterizations of $\mathscr{S}$-complementability and $\mathscr{S}^*$-complementability, the equivalence of $\mathscr{S}$-compatibility and $\mathscr{S}^*$-complementability, and a concrete representation of Schur complements of $\mathscr{S}$-complementable operators. In Section 4, we obtain more concrete representations of Schur complements of positive $\mathscr{S}$-compatible operators. Finally in Section 5, we generalize the quotient property for Schur complements of matrices to the quotient property for Schur complements of $\mathscr{S}$-complementable operators and $\mathscr{S}^*$-complementable operators on a Hilbert C^*-module.

First of all, we recall some knowledge about Hilbert C^*-modules.

Throughout this paper, $\mathcal{A}$ is a C^*-algebra. An inner-product $\mathcal{A}$-module is a linear space E which is a right $\mathcal{A}$-module, together with a map $(x, y) \to \langle x, y\rangle : E \times E \to \mathcal{A}$ such that for any $x, y, z \in E$, $\alpha, \beta \in \mathbb{C}$ and $a \in \mathcal{A}$, the following conditions hold:

(i) $\langle x, \alpha y + \beta z\rangle = \alpha\langle x, y\rangle + \beta\langle x, z\rangle$;

(ii) $\langle x, ya\rangle = \langle x, y\rangle a$;

(iii) $\langle x, y\rangle = \langle y, x\rangle^*$;

(iv) $\langle x, x\rangle \geq 0$, and $\langle x, x\rangle = 0$ if and only if $x = 0$.

An inner-product $\mathcal{A}$-module E is called a (right) Hilbert $\mathcal{A}$-module if it is complete with respect to the induced norm $\|x\| = \|\langle x, x\rangle\|^{\frac{1}{2}}$.

Suppose that E, F are two Hilbert $\mathcal{A}$-modules. We denote by $L_{\mathcal{A}}(E, F)$ the set of all maps $T : E \to F$ which are adjointable in the sense that there is a map $T^* : F \to E$ such that

$$\langle Tx, y\rangle = \langle x, T^*y\rangle \quad \text{for each } x \in E \text{ and } y \in F.$$

It is known that any element T of $L_{\mathcal{A}}(E,F)$ must be a bounded linear operator, and also $\mathcal{A}$-linear in the sense that $T(xa)=T(x)a$ for $x\in E$ and $a\in\mathcal{A}$. We denote by $B_{\mathcal{A}}(E,F)$ the set of all bounded linear $\mathcal{A}$-maps, and therefore $L_{\mathcal{A}}(E,F)\subseteq B_{\mathcal{A}}(E,F)$. For any $T\in B_{\mathcal{A}}(E,F)$, the range space and the null space of T are denoted by $R(T)$ and $N(T)$ respectively. In case $E=F$, $L_{\mathcal{A}}(E)$, to which we abbreviate $L_{\mathcal{A}}(E,E)$, is a C^*-algebra. Then for $A\in L_{\mathcal{A}}(E)$, A is Hermitian (selfadjoint) if and only if $\langle Ax,y\rangle=\langle x,Ay\rangle$ for any $x,y\in E$, and positive if and only if $\langle Ax,x\rangle\geq 0$ for any $x\in E$, in which case, we denote by $A^{\frac{1}{2}}$ the unique positive element B such that $B^2=A$ in the C^*-algebra $L_{\mathcal{A}}(E)$ and then $\overline{R(A)}=\overline{R(A^{\frac{1}{2}})}$. Denote by $L_{\mathcal{A}}(E)_{sa}$ and $L_{\mathcal{A}}(E)_+$ the sets of Hermitian and positive elements of $L_{\mathcal{A}}(E)$ respectively. For any $A,B\in L_{\mathcal{A}}(E)_{sa}$, we say $A\geq B$ if $\langle (A-B)x,x\rangle\geq 0$ for any $x\in E$. For $\mathcal{A}_+$, the set of positive elements of the C^*-algebra $\mathcal{A}$, is a positive cone, we could easily verify that " $\geq$ " is a partial order of $L_{\mathcal{A}}(E)$.

We say that a closed submodule E_1 of E is topologically complemented if there is a closed submodule E_2 of E such that $E_1+E_2=E$ and $E_1\cap E_2=0$, and briefly denote the sum by $E=E_1\widetilde{\oplus}E_2$, called the direct sum of E_1 and E_2. If moreover $E_2=E_1^{\perp}$, where $E_1^{\perp}=\{x\in E:\langle x,y\rangle=0 \text{ for all } y\in E_1\}$, we say that E_1 is orthogonally complemented and briefly denote the sum by $E=E_1\oplus E_2$, called the orthogonal sum of E_1 and E_2. In this case $E_1=E_1^{\perp\perp}$. Let $T\in L_{\mathcal{A}}(E,F)$. Then (1) $N(T)=R(T^*)^{\perp}$ and $N(T)^{\perp}\supseteq\overline{R(T^*)}$; (2) if $R(T)$ is closed, then so is $R(T^*)$, and in this case both $R(T)$ and $R(T^*)$ are orthogonally complemented (see [23, Theorem 3.2]). It is well-known that the range of any idempotent in $B_{\mathcal{A}}(E,E)$ is closed, and therefore orthogonally complemented if this idempotent is adjointable. If $E=E_1\widetilde{\oplus}E_2$, it is known that there exists an idempotent operator $Q\in B_{\mathcal{A}}(E,E)$ with $R(Q)=E_1$ and $N(Q)=E_2$. Moreover, (1) if $E_2=E_1^{\perp}$, then Q is a selfadjoint operator in $L_{\mathcal{A}}(E)$, i.e., a projection in $L_{\mathcal{A}}(E)$; (2) if Q is adjointable, i.e., in $L_{\mathcal{A}}(E)$, then both the closed submodules E_1 and E_2, as the range spaces of Q and $1-Q$ respectively, are orthogonally complemented, in which case we call the direct sum $E=E_1\widetilde{\oplus}E_2$ adjointable. For two closed submodules $\mathscr{S},\mathscr{T}$ of E, we denote $\mathscr{S}\ominus\mathscr{T}=\mathscr{S}\cap(\mathscr{S}\cap\mathscr{T})^{\perp}$. If moreover $\mathscr{S}$ is orthogonally complemented and $\mathscr{T}\subseteq\mathscr{S}$, then $\mathscr{T}$ is orthogonally complemented in E if and only if $\mathscr{T}$ is orthogonally complemented in $\mathscr{S}$. The reader may refer to [19, 23, 31] for details.

In this paper, E,F and G are three Hilbert $\mathcal{A}$-modules.

2 Compatibility of an Operator and a Closed Submodule

Given $A\in L_{\mathcal{A}}(E)$, consider the sesquilinear form in $E\times E$ defined by

$$\langle x,y\rangle_A=\langle Ax,y\rangle \quad \text{for all } x,y\in E.$$

If $\mathscr{S}$ is a closed submodule of E and $A\in L_{\mathcal{A}}(E)$, the A-orthogonal submodule to $\mathscr{S}$ is given by

$$\begin{aligned}\mathscr{S}^{\perp A}&:=\{x\in E:\langle x,s\rangle_A=\langle s,x\rangle_A=0,\ \forall s\in\mathscr{S}\}\\&=(A\mathscr{S})^{\perp}\cap(A^*\mathscr{S})^{\perp}=A^{-1}(\mathscr{S}^{\perp})\cap(A^*)^{-1}(\mathscr{S}^{\perp}).\end{aligned}$$

An operator $T\in L_{\mathcal{A}}(E)$ is called A-selfadjoint if $\langle Tx,y\rangle_A=\langle x,Ty\rangle_A$ for all $x,y\in E$. It is easy to see that $T\in L_{\mathcal{A}}(E)$ is A-selfadjoint if and only if $AT=T^*A$.

Definition 2.1 (see [9]) *Let $A \in L_{\mathcal{A}}(E)$ and $\mathscr{S}$ be an orthogonally complemented closed submodule of E. The pair $(A, \mathscr{S})$ is called compatible if there exists an (adjointable) A-selfadjoint idempotent with range $\mathscr{S}$, i.e., if the set*

$$\mathscr{P}(A, \mathscr{S}) = \{Q \in L_{\mathcal{A}}(E) : Q = Q^2,\ R(Q) = \mathscr{S},\ AQ = Q^*A\}$$

is not empty.

In this case, A is said to be $\mathscr{S}$-compatible.

Remark 2.1 (1) Definition 3.1 in [9] gave the definition of the compatibility of $(A, \mathscr{S})$ only for selfadjoint operators on Hilbert spaces, so even in the case of Hilbert space, the definition here is suitable for more operators, and the concerning proofs have to take more spaces.

(2) Given an orthogonally complemented closed submodule $\mathscr{S}$ of E. Let $P = P_{\mathscr{S}}$ be the projection of E onto $\mathscr{S}$. Under the decomposition $I = P + (I - P)$, each operator $A \in L_{\mathcal{A}}(E)$ is identified with

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}.$$

Suppose that $Q \in L_{\mathcal{A}}(E)$ is an idempotent. Then $R(Q)$ is closed and there exists decomposition $E = R(Q) \oplus R(Q)^{\perp}$, under which Q is of the form $\left(\begin{smallmatrix} U & V \\ 0 & 0 \end{smallmatrix}\right)$. For every $x \in R(Q)$, there exists $y \in E$ such that $x = Qy$. So we get $Qx = Q^2y = Qy = x$ and then $U = I$. Therefore $Q = \left(\begin{smallmatrix} I & V \\ 0 & 0 \end{smallmatrix}\right)$. From this it could be seen that $Q \in L_{\mathcal{A}}(E)$ is an idempotent if and only if $R(Q)$ is closed and $Q = \left(\begin{smallmatrix} I & V \\ 0 & 0 \end{smallmatrix}\right)$ with $V \in L_{\mathcal{A}}(R(Q)^{\perp}, R(Q))$ under the decomposition $E = R(Q) \oplus R(Q)^{\perp}$. Moreover, it is easy to see that $N(Q) = R(I - Q)$ for any idempotent $Q \in L_{\mathcal{A}}(E)$.

Lemma 2.1 (see [15]) *Let $C \in L_{\mathcal{A}}(G, F)$ and $A \in L_{\mathcal{A}}(E, F)$ with $\overline{R(A^*)} \subseteq E$ orthogonally complemented. Then the following statements are equivalent:*

(i) *$CC^* \leq \lambda AA^*$ for some $\lambda > 0$.*

(ii) *There exists $\mu > 0$ such that $\|C^*z\| \leq \mu \|A^*z\|$ for all $z \in F$.*

(iii) *There exists $X \in L_{\mathcal{A}}(G, E)$ such that $C = AX$, i.e., the equation $AX = C$ has a solution.*

(iv) *$R(C) \subseteq R(A)$.*

Moreover, there exists a unique operator $D \in L_{\mathcal{A}}(G, E)$ which satisfies the conditions

$$C = AD, \quad R(D) \subseteq N(A)^{\perp}.$$

In this case,

$$\|D\|^2 = \inf\{\lambda : CC^* \leq \lambda AA^*\} \quad \text{and} \quad R(D) \subseteq \overline{R(A^*)}; \quad N(D) = N(C),$$

and this D is called the reduced solution of the equation $AX = C$.

Lemma 2.2 (see [15]) *Let $A \in L_{\mathcal{A}}(E, F)$ with $\overline{R(A^*)} \subseteq E$ orthogonally complemented and Q be an idempotent in $L_{\mathcal{A}}(F)$ such that $R(QA) \subseteq R(A)$. Then the reduced solution D (i.e., $R(D) \subseteq N(A)^{\perp}$) of $AX = QA$ is an idempotent.*

The following lemma is easy to prove in the case of Hilbert space, but in the case of Hilbert C^*-module for the lack of Riesz representation theory, we have to use the functional calculus of C^*-algebras.

Lemma 2.3 *Let $T \in L_{\mathcal{A}}(E)$ and $\mathscr{S} \subseteq E$ be an orthogonally complemented closed submodule. Suppose that $T = \begin{pmatrix} I & V \\ 0 & 0 \end{pmatrix}$ under the orthogonal decomposition $E = \mathscr{S} \oplus \mathscr{S}^{\perp}$. Then $\|T\|^2 = 1 + \|V\|^2$.*

Proof For $T \in L_{\mathcal{A}}(E)$, which is a C^*-algebra, we know $\|T\|^2 = \|TT^*\|$. It could be seen easily that $TT^* = \begin{pmatrix} I & VV^* \\ 0 & 0 \end{pmatrix}$ and $\|TT^*\| = \|I + VV^*\| = \max\{1 + \lambda \mid \lambda \in \sigma(VV^*)\} = 1 + \|VV^*\| = 1 + \|V\|^2$. So we have $\|T\|^2 = 1 + \|V\|^2$.

Corollary 2.1 *Let $Q \in L_{\mathcal{A}}(E)$ be an idempotent and $\|Q\| \leq 1$. Then $Q^* = Q$.*

The following proposition is similar to [9, Lemma 3.2], but it is for the general (not only selfadjoint) operators on Hilbert C^*-modules.

Proposition 2.1 *Let $A \in L_{\mathcal{A}}(E)$, $\mathscr{S}$ be an orthogonally complemented closed submodule of E and $Q \in L_{\mathcal{A}}(E)$ be an idempotent with range $\mathscr{S}$. Then the following statements are equivalent:*

(i) *$Q \in \mathscr{P}(A, \mathscr{S})$, which is equivalent to $Q^*A = AQ$.*

(ii) *$N(Q) \subseteq \mathscr{S}^{\perp A}$.*

If $A \geq 0$, then (i) *and* (ii) *imply*

(iii) *$Q^*AQ \leq A$.*

If $A \geq 0$ and $\overline{R(A)}$ is orthogonally complemented in E, then (i), (ii) *and* (iii) *are equivalent.*

Proof If $Q \in \mathscr{P}(A, \mathscr{S})$, it is obvious that for any $x, y \in E$,

$$\langle Ax, Qy\rangle = \langle Q^*Ax, y\rangle = \langle AQx, y\rangle = \langle Qx, A^*y\rangle,$$
$$\langle A^*x, Qy\rangle = \langle Q^*A^*x, y\rangle = \langle A^*Qx, y\rangle = \langle Qx, Ay\rangle,$$

so $N(Q) \subseteq A^{-1}(\mathscr{S}^{\perp}) \cap A^{*-1}(\mathscr{S}^{\perp}) = \mathscr{S}^{\perp A}$.

With the preparation of Lemmas 2.1, 2.2 and Corollary 2.1, the rest of the proof could be referred to that of [9, Lemma 3.2].

Corollary 2.2 *Let $A \in L_{\mathcal{A}}(E)$ and $\mathscr{S} \subseteq E$ be an orthogonally complemented closed submodule with the orthogonal projection P. Then $P \in \mathscr{P}(A, \mathscr{S})$ if and only if $\mathscr{S}$ is the reduced submodule (i.e., $A\mathscr{S} \subseteq \mathscr{S}$ and $A(\mathscr{S}^{\perp}) \subseteq \mathscr{S}^{\perp}$) of E, and if and only if $\mathscr{S}^{\perp} \subseteq \mathscr{S}^{\perp A}$.*

Theorem 2.1 *Let $A \in L_{\mathcal{A}}(E)$ and $\mathscr{S} \subseteq E$ be an orthogonally complemented closed submodule with the orthogonal projection P. Suppose that $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ under the decomposition $E = \mathscr{S} \oplus \mathscr{S}^{\perp}$. Then*

(i) *$(A, \mathscr{S})$ is compatible if and only if the equations $A_{11}X = A_{12}$ and $A_{11}^*X = A_{21}^*$ have a common solution $X \in L_{\mathcal{A}}(\mathscr{S}^{\perp}, \mathscr{S})$. Moreover, $Q \in \mathscr{P}(A, \mathscr{S})$ if and only if $Q \in L_{\mathcal{A}}(E)$ is of the form $\begin{pmatrix} I & V \\ 0 & 0 \end{pmatrix}$, where $V \in L_{\mathcal{A}}(\mathscr{S}^{\perp}, \mathscr{S})$ is a common solution of $A_{11}X = A_{12}$ and ${A_{11}}^*X = {A_{21}}^*$.*

(ii) *$(A, \mathscr{S})$ is compatible if and only if there exists an idempotent $Q \in L_{\mathcal{A}}(E)$ with range $\mathscr{S}$ such that $N(Q) \subseteq \mathscr{S}^{\perp A}$.*

(iii) *$(A, \mathscr{S})$ is compatible if and only if there exists an adjointable direct sum $E = \mathscr{S} \widetilde{\oplus} \mathscr{S}'$ such that $\mathscr{S}' \subseteq \mathscr{S}^{\perp A}$.*

(iv) *If $A \geq 0$ and $\overline{R(A)}$ is orthogonally complemented, then $(A, \mathscr{S})$ is compatible if and only if there exists an idempotent $Q \in L_{\mathcal{A}}(E)$ with range $\mathscr{S}$ such that $Q^*AQ \leq A$.*

(v) *If* $\overline{R(A_{11}{}^*)+R(A_{11})} \subseteq E$ *is orthogonally complemented, then* $(A,\mathscr{S})$ *is compatible if and only if* $R\left(\begin{pmatrix} A_{12} \\ A_{21}^* \end{pmatrix}\right) \subseteq R\left(\begin{pmatrix} A_{11} \\ A_{11}^* \end{pmatrix}\right)$, *if and only if* $\begin{pmatrix} A_{11} \\ A_{11}^* \end{pmatrix} X = \begin{pmatrix} A_{12} \\ A_{21}^* \end{pmatrix}$ *has a solution* $X \in L_{\mathcal{A}}(\mathscr{S}^{\perp},\mathscr{S})$, *and in which case we have a unique reduced solution.*

(vi) *If* $A = A^*$ *and* $\overline{R(A_{11})} \subseteq E$ *is orthogonally complemented, then* $(A,\mathscr{S})$ *is compatible if and only if* $R(PA) = R(PAP)$, *and if and only if* $R(A_{12}) \subseteq R(A_{11})$.

Proof Assume that $Q \in \mathscr{P}(A,\mathscr{S})$, and $Q = \left(\begin{smallmatrix} I & V \\ 0 & 0 \end{smallmatrix}\right)$ under the decomposition $E = \mathscr{S} \oplus \mathscr{S}^{\perp}$. Since Q is A-selfadjoint, $Q^*A = AQ$, by direct calculation, it follows that

$$A_{11}V = A_{12}, \quad A_{11}^*V = A_{21}^*,$$

and then V is a common solution to the equations $A_{11}X = A_{12}$ and $A_{11}^*X = A_{21}^*$.

Conversely, if there exists an operator $X \in L_{\mathcal{A}}(\mathscr{S}^{\perp},\mathscr{S})$ such that

$$A_{11}X = A_{12}, \quad A_{11}^*X = A_{21}^*,$$

then set $Q = \left(\begin{smallmatrix} I & X \\ 0 & 0 \end{smallmatrix}\right)$, which is an A-selfadjoint idempotent with range $\mathscr{S}$.

(ii)–(iv) By Proposition 2.1, (ii)–(iv) are obvious.

(v) It is easy to check that $\overline{R\left(\begin{pmatrix} A_{11} \\ A_{11}^* \end{pmatrix}^*\right)} = \overline{R(A_{11}^*) + R(A_{11})}$. Then we can get (v) from Lemma 2.1 and (i).

(vi) With the similar proof to that of Proposition 3.3 in [9], we obtain that $R(A_{12}) \subseteq R(A_{11})$ if and only if $R(PA) = R(PAP)$.

For $A_{12} = A_{21}^*$ and $A_{11} = A_{11}^*$, from (v) and Lemma 2.1, we obtain that $(A,\mathscr{S})$ is compatible if and only if $R(A_{12}) \subseteq R(A_{11})$.

The Proof of Theorem 2.1 is completed.

The following proposition is similar to [9, Theorem 3.6(3)], but it is for the general (not only selfadjoint) operators on Hilbert C^*-modules.

Proposition 2.2 *Let* $A \in L_{\mathcal{A}}(E)$ *and* $\mathscr{S} \subseteq E$ *be an orthogonally complemented closed submodule such that* $(A,\mathscr{S})$ *is compatible. Set* $\mathscr{N} = \mathscr{S} \cap N(A) \cap N(A^*)$. *Then*

$$\mathscr{S} \cap \mathscr{S}^{\perp A} = \mathscr{N}.$$

Proof Obviously, $\mathscr{S} \cap N(A^*) \cap N(A) \subseteq \mathscr{S} \cap \mathscr{S}^{\perp A}$. Since $(A,\mathscr{S})$ is compatible, by Theorem 2.1(i) there exists an operator $V \in L_{\mathcal{A}}(\mathscr{S}^{\perp},\mathscr{S})$ such that $A_{11}V = A_{12}$, $A_{11}^*V = A_{21}^*$, where A_{11}, A_{12}, A_{21} are set as in Theorem 2.1.

For any $x \in \mathscr{S} \cap \mathscr{S}^{\perp A}$ and $y \in \mathscr{S}$, we have $\langle Ax, y\rangle = 0$ and $\langle A^*x, y\rangle = 0$. It follows that $\langle A_{11}x, y\rangle = 0$ and $\langle A_{11}^*x, y\rangle = 0$, so that $A_{11}x = A_{11}^*x = 0$. Thus

$$Ax = A_{11}x + A_{21}x = A_{11}x + V^*A_{11}x = 0,$$
$$A^*x = A_{11}^*x + A_{12}^*x = A_{11}^*x + V^*A_{11}^*x = 0,$$

which is to say $x \in N(A) \cap N(A^*)$, so that $\mathscr{S} \cap \mathscr{S}^{\perp A} \subseteq \mathscr{S} \cap N(A) \cap N(A^*)$.

Remark 2.2 (1) If $A \in L_{\mathcal{A}}(E)$ is positive, then for any $x \in \mathscr{S} \cap \mathscr{S}^{\perp A}$, $\langle A^{\frac{1}{2}}x, A^{\frac{1}{2}}x\rangle = \langle Ax, x\rangle = 0$, so that $\mathscr{S} \cap N(A) = \mathscr{S} \cap \mathscr{S}^{\perp A}$. Therefore, $\mathscr{S} \cap N(A) = \mathscr{S} \cap \mathscr{S}^{\perp A}$ for positive operators without the assumption of $\mathscr{S}$-compatibility.

(2) Suppose that $\mathscr{N}$ is orthogonally complemented in E. Then $\mathscr{S} = \mathscr{N} \oplus (\mathscr{S} \ominus \mathscr{N})$ and $\mathscr{S}^{\perp A} = \mathscr{N} \oplus (\mathscr{S}^{\perp A} \ominus \mathscr{N})$. If additionally, $(A, \mathscr{S})$ is compatible, then $E = \mathscr{S} + \mathscr{S}^{\perp A}$ by Theorem 2.1(iii) and $\mathscr{N} = \mathscr{S} \cap \mathscr{S}^{\perp A}$ by Proposition 2.2. Therefore,

$$E = \mathscr{S} + (\mathscr{N} \oplus (\mathscr{S}^{\perp A} \ominus \mathscr{N})) = \mathscr{S} + (\mathscr{S}^{\perp A} \ominus \mathscr{N}) = \mathscr{S} \widetilde{\oplus} (\mathscr{S}^{\perp A} \ominus \mathscr{N}).$$

Similarly, we have $E = (\mathscr{S} \ominus \mathscr{N}) \widetilde{\oplus} \mathscr{S}^{\perp A}$.

Definition 2.2 (see [9]) *Let $A \in L_{\mathcal{A}}(E)$ and $\mathscr{S} \subseteq E$ be an orthogonally complemented closed submodule with the orthogonal projection P such that $(A, \mathscr{S})$ is compatible. Suppose that $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ under the decomposition $I = P + (I - P)$ and the equation $\begin{pmatrix} A_{11} \\ A_{11}^* \end{pmatrix} X = \begin{pmatrix} A_{12} \\ A_{21}^* \end{pmatrix}$ has the reduced solution $D \in L_{\mathcal{A}}(\mathscr{S}^\perp, \mathscr{S})$ (in particular if $\overline{R(A_{11}{}^*) + R(A_{11})} \subseteq E$ is orthogonally complemented by Theorem 2.1(v)). We define the following idempotent onto $\mathscr{S}$ by $P_{A,\mathscr{S}}$:*

$$P_{A,\mathscr{S}} = \begin{pmatrix} I & D \\ 0 & 0 \end{pmatrix}.$$

Theorem 2.2 (see [9]) *Let $A, \mathscr{S}$ and D be as in Definition 2.2, $\mathscr{N} = \mathscr{S} \cap N(A) \cap N(A^*)$, and let $L_{\mathcal{A}}^{\mathscr{N}}(\mathscr{S}^\perp, \mathscr{S}) = \{T \in L_{\mathcal{A}}(\mathscr{S}^\perp, \mathscr{S}) : R(T) \subseteq \mathscr{N}\}$ be viewed as a subspace of $L_{\mathcal{A}}(E)$. Then*

$$\mathscr{P}(A, \mathscr{S}) = P_{A,\mathscr{S}} + L_{\mathcal{A}}^{\mathscr{N}}(\mathscr{S}^\perp, \mathscr{S}).$$

Moreover, if $\mathscr{N}$ is an orthogonally complemented submodule of E, then

(i) *there is a matrix representation of the above equation: $Q \in \mathscr{P}(A, \mathscr{S})$ if and only if there is a unique $Z \in L_{\mathcal{A}}^{\mathscr{N}}(\mathscr{S}^\perp, \mathscr{S})$ such that*

$$Q = P_{A,\mathscr{S}} + Z = \begin{pmatrix} I & 0 & D \\ 0 & I & Z \\ 0 & 0 & 0 \end{pmatrix}$$

under the decomposition $E = (\mathscr{S} \ominus \mathscr{N}) \oplus \mathscr{N} \oplus \mathscr{S}^\perp$.

(ii) *$P_{A,\mathscr{S}}$ has the minimal norm in $\mathscr{P}(A, \mathscr{S})$, i.e.,*

$$\|P_{A,\mathscr{S}}\| = \min\{\|Q\| : Q \in \mathscr{P}(A, \mathscr{S})\}.$$

Proof It is obvious that $P_{A,\mathscr{S}} \in \mathscr{P}(A, \mathscr{S})$ by Theorem 2.1(i). Moreover, it is easy to check that for any $Z \in L_{\mathcal{A}}^{\mathscr{N}}(\mathscr{S}^\perp, \mathscr{S})$, $Q = P_{A,\mathscr{S}} + Z \in \mathscr{P}(A, \mathscr{S})$.

Now we assume $Q \in \mathscr{P}(A, \mathscr{S})$, just as in the proof of Theorem 3.6(4) in [9]. Replacing A_{11} by $\begin{pmatrix} A_{11} \\ A_{11}^* \end{pmatrix}$, we could obtain some operator $Z \in L_{\mathcal{A}}(\mathscr{S}^\perp, \mathscr{S})$ such that $Q = P_{A,\mathscr{S}} + Z$ and $R(Z) \subseteq N\left(\begin{pmatrix} A_{11} \\ A_{11}^* \end{pmatrix}\right)$, i.e., $R(Z) \subseteq N(A_{11}) \cap N(A_{11}{}^*)$. It is easy to know that $N(A_{11}) \cap N(A_{11}{}^*) = \mathscr{N}$ and so $Z \in L_{\mathcal{A}}^{\mathscr{N}}(\mathscr{S}^\perp, \mathscr{S})$.

Now we assume that $\mathscr{N}$ is an orthogonally complemented submodule of E. Then by Lemma 2.1,

$$R(D) \subseteq N\left(\begin{pmatrix} A_{11} \\ A_{11}{}^* \end{pmatrix}\right)^\perp = (N(A_{11}) \cap N(A_{11}{}^*))^\perp = \mathscr{S} \cap \mathscr{N}^\perp = \mathscr{S} \ominus \mathscr{N}.$$

Thus (i) is proved, and (ii) is obvious by Lemma 2.3.

Corollary 2.3 *Let $A, \mathscr{S}$ and D be as in Definition 2.2, and $\mathscr{N} = \mathscr{S} \cap N(A) \cap N(A^*)$. Then $\mathscr{P}(A, \mathscr{S})$ has a unique element (namely, $P_{A,\mathscr{S}}$) if and only if $\mathscr{S} \widetilde{\oplus} \mathscr{S}^{\perp A} = E$.*

Proof From Theorem 2.1(i) and the definition of $P_{A,\mathscr{S}}$, we know that if $\mathscr{P}(A,\mathscr{S})$ has a unique element, it must be $P_{A,\mathscr{S}}$.

By Theorem 2.2, $\mathscr{P}(A,\mathscr{S})$ has a unique element if and only if for all $Z \in L_{\mathcal{A}}^{\mathscr{N}}(\mathscr{S}^{\perp},\mathscr{S})$, $Z = 0$, which is equivalent to $\mathscr{N} = 0$. Since $(A,\mathscr{S})$ is compatible, $\mathscr{N} = \mathscr{S} \cap \mathscr{S}^{\perp A}$ and $E = \mathscr{S} + \mathscr{S}^{\perp A}$. Therefore, $\mathscr{N} = 0$ if and only if $E = \mathscr{S} \widetilde{\oplus} \mathscr{S}^{\perp A}$.

3 Schur Complements of Operators on Hilbert C^*-Module

The Schur complement was first defined for square matrices in [18]: Given a block matrix $T = \left(\begin{smallmatrix} B & C \\ D & E \end{smallmatrix}\right)$ with B invertible, then $T_{/B} = E - DB^{-1}C$ is the Schur complement of B in T. This notion was then generalized in several directions. In this section, we first put forward this definition in the context of Hilbert C^*-modules following the generalization by Ando [2] in the context of Hilbert spaces, and then obtain some characterizations of Schur complement.

Definition 3.1 (see [2]) *Given an operator $T \in L_{\mathcal{A}}(E)$ and an orthogonally complemented closed submodule $\mathscr{S}$ of E. T is called $\mathscr{S}$-complementable if there exist operators $M_l, M_r \in L_{\mathcal{A}}(E)$ such that*

$$PM_r = M_r, \quad M_lP = M_l, \quad PTM_r = PT, \quad M_lTP = TP,$$

where $P = P_{\mathscr{S}}$ is the projection onto $\mathscr{S}$.

In this case, TM_r is independent of the choice of M_r. The operator $T_{\mathscr{S}} = TM_r$ is defined as the $\mathscr{S}$-compression of T to $\mathscr{S}$, and $T_{/\mathscr{S}} = T - T_{\mathscr{S}}$ is defined as the Schur complement of T to $\mathscr{S}$.

Remark 3.1 (1) From the above analysis we know that $TM_r = M_lPT = M_lT$. Therefore, if T is selfadjoint, then the existence of M_r is sufficient to ensure the $\mathscr{S}$-complementability of T as we can choose $M_l = M_r^*$.

(2) If $E = \mathbb{C}^n$, $\mathscr{S} = \mathbb{C}^k \times \{0\}$ and $\mathscr{S}^{\perp} = \{0\} \times \mathbb{C}^{n-k}$, then every $T \in L_{\mathcal{A}}(E)$ can be written as $T = \left(\begin{smallmatrix} B & C \\ D & E \end{smallmatrix}\right)$. Moreover if B^{-1} is invertible, then $M_r = \left(\begin{smallmatrix} I & B^{-1}C \\ 0 & 0 \end{smallmatrix}\right)$ and $M_l = \left(\begin{smallmatrix} I & 0 \\ DB^{-1} & 0 \end{smallmatrix}\right)$ satisfy the equations in Definition 3.1 and

$$T_{\mathscr{S}} = \begin{pmatrix} B & C \\ D & DB^{-1}C \end{pmatrix}, \quad T_{/\mathscr{S}} = \begin{pmatrix} 0 & 0 \\ 0 & E - DB^{-1}C \end{pmatrix}.$$

Thus $T_{/\mathscr{S}}$ is determined by the classical Schur complement.

Theorem 3.1 *Let $T \in L_{\mathcal{A}}(E)$ and $\mathscr{S} \subseteq E$ be an orthogonally complemented submodule with the orthogonal projection P. Suppose $T = \left(\begin{smallmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{smallmatrix}\right)$ under the decomposition $E = \mathscr{S} \oplus \mathscr{S}^{\perp}$. Then the following statements are equivalent:*

(i) *T is $\mathscr{S}$-complementable.*

(ii) *There exist operators $X, Y \in L_{\mathcal{A}}(\mathscr{S}^{\perp},\mathscr{S})$ such that $T_{11}X = T_{12}$ and ${T_{11}}^*Y = {T_{21}}^*$.*

(iii) *There exist idempotents Q, $R \in L_{\mathcal{A}}(E)$ with ranges $\mathscr{S}$ such that $PTQ = PT$ and $PT^*R = PT^*$, i.e., each of the equations $PTX = PT$ and $PT^*Y = PT^*$ has an idempotent solution in $L_{\mathcal{A}}(E)$ with range $\mathscr{S}$.*

In which case, the Schur complement of T to $\mathscr{S}$ is

$$T_{/\mathscr{S}} = \begin{pmatrix} 0 & 0 \\ 0 & T_{22} - W^*T_{11}V \end{pmatrix} = T(1-Q) = (1-R^*)T,$$

where V and W are the solutions of $T_{11}X = T_{12}$ and $T_{11}{}^*Y = T_{21}{}^*$ in $L_{\mathcal{A}}(\mathscr{S}^\perp, \mathscr{S})$ respectively, and Q, R are given by (iii) *(in particular, $Q = \left(\begin{smallmatrix} I & V \\ 0 & 0 \end{smallmatrix}\right)$ and $R = \left(\begin{smallmatrix} I & W \\ 0 & 0 \end{smallmatrix}\right)$).*

Proof (i)⇒(ii) Let T be $\mathscr{S}$-complementable and

$$M_r = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}, \quad M_l = \begin{pmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{pmatrix}.$$

From $M_r = PM_r$ and $PTM_r = PT$, it could be seen that $T_{11}M_{11} = T_{11},\ T_{11}M_{12} = T_{12}$. Similarly, by $M_lP = M_l,\ M_lTP = TP$, we easily obtain that $N_{11}T_{11} = T_{11},\ N_{21}T_{11} = T_{21}$. Therefore, each of the equations $T_{11}X = T_{12}$ and $T_{11}{}^*Y = T_{21}{}^*$ has a solution in $L_{\mathcal{A}}(\mathscr{S}^\perp, \mathscr{S})$.

(ii)⇒(iii) Let V and W be the solutions of $T_{11}X = T_{12}$ and $T_{11}{}^*Y = T_{21}{}^*$ in $L_{\mathcal{A}}(\mathscr{S}^\perp, \mathscr{S})$ respectively. Set $M_r = \left(\begin{smallmatrix} I & V \\ 0 & 0 \end{smallmatrix}\right),\ M_l = \left(\begin{smallmatrix} I & 0 \\ W^* & 0 \end{smallmatrix}\right)$. It is easy to check that

$$PTM_r = PT, \quad PT^*M_l{}^* = PT^*,$$
$$M_r = M_r{}^2, \quad M_l{}^* = M_l{}^{*2}, \quad R(M_r) = R(M_l{}^*) = \mathscr{S}.$$

Set $Q = M_r$ and $R = M_l{}^*$, we complete the proof.

(iii)⇒(i) Set $M_r = Q,\ M_l = R^*$.

Thus we prove the equivalence of (i)–(iii).

If T is $\mathscr{S}$-complementable, set

$$M_r = \begin{pmatrix} I & V \\ 0 & 0 \end{pmatrix}, \quad M_l = \begin{pmatrix} I & 0 \\ W^* & 0 \end{pmatrix},$$

where V and W are the solutions of $T_{11}X = T_{12}$ and $T_{11}{}^*Y = T_{21}{}^*$ in $L_{\mathcal{A}}(\mathscr{S}^\perp, \mathscr{S})$ respectively. Then by the discussion above and the definition of the Schur complement of T to $\mathscr{S}$, we have

$$T_{/\mathscr{S}} = T(I - M_r) = (I - M_l)T(I - M_r) = \begin{pmatrix} 0 & 0 \\ 0 & T_{22} - W^*T_{11}V \end{pmatrix}.$$

Definition 3.2 *Let $T \in L_{\mathcal{A}}(E)$ and $\mathscr{S} \subseteq E$ be an orthogonally complemented closed submodule. We call T $\mathscr{S}^*$-complementable if there exists an operator $M \in L_{\mathcal{A}}(E)$ such that*

$$PM = M, \quad PTM = PT, \quad PT^*M = PT^*, \quad \text{i.e.,} \quad M^*TP = TP,$$

where $P = P_{\mathscr{S}}$ is the projection of E onto $\mathscr{S}$.

Remark 3.2 Clearly if T is $\mathscr{S}^*$-complementable, it must be $\mathscr{S}$-complementable. Conversely, it may be not true. For example, in Remark 3.1(2), $M_l = M_r^*$ need not be ensured. However, for selfadjoint operators, the two definitions are identical with each other from Remark 3.1(1). Corach et al [10] showed that $\mathscr{S}$-compatibility and $\mathscr{S}$-complementability are equivalent properties for selfadjoint operators on Hilbert spaces, and so $\mathscr{S}$-compatibility and $\mathscr{S}^*$-complementability are equivalent properties for selfadjoint operators on Hilbert spaces too. We could see that this result is also true for general adjointable (maybe not selfadjoint) operators not only on Hilbert spaces but also on Hilbert C^*-modules.

Theorem 3.2 *Let $T \in L_{\mathcal{A}}(E)$ and $\mathscr{S} \subseteq E$ be an orthogonally complemented closed submodule of E. Then the following statements are equivalent:*

(i) *T is $\mathscr{S}^*$-complementable.*

(ii) $(T, \mathscr{S})$ *is compatible.*

(iii) $T_{11}X = T_{12}$ *and* $T_{11}{}^*X = T_{21}{}^*$ *have a common solution* $X \in L_{\mathcal{A}}(\mathscr{S}^\perp, \mathscr{S})$.

(iv) *There exists an idempotent* $Q \in L_{\mathcal{A}}(E)$ *with range* $\mathscr{S}$ *such that* $PTQ = PT$ *and* $PT^*Q = PT^*$, *i.e.,* $PTX = PT$ *and* $PT^*X = PT^*$ *have a common idempotent solution in* $L_{\mathcal{A}}(E)$ *with range* $\mathscr{S}$.

In which case, $Q \in \mathscr{P}(T, \mathscr{S})$ *if and only if* $Q \in L_{\mathcal{A}}(E)$ *is of the form* $\left(\begin{smallmatrix} I & V \\ 0 & 0 \end{smallmatrix}\right)$ *under the decomposition* $E = \mathscr{S} \oplus \mathscr{S}^\perp$, *where* $V \in L_{\mathcal{A}}(\mathscr{S}^\perp, \mathscr{S})$ *is a common solution of* $T_{11}X = T_{12}$, $T_{11}{}^*X = T_{21}{}^*$, *if and only if* $Q \in L_{\mathcal{A}}(E)$ *is an idempotent with range* $\mathscr{S}$ *such that* $PTQ = PT$ *and* $PT^*Q = PT^*$. *Moreover, with* V, Q *as above,*

$$T_{/\mathscr{S}} = \begin{pmatrix} 0 & 0 \\ 0 & T_{22} - V^*T_{11}V \end{pmatrix} = T(1 - Q) = (1 - Q^*)T.$$

Proof By the similar proof as in Theorem 3.1, we could easily obtain the equivalence of (i), (iii) and (iv). The proof is then completed by Theorem 2.1, Remark 2.1 and the proof of Theorem 3.1.

4 Schur Complements of Positive Operators on Hilbert C^*-Module

Corach et al [10] proved that, if H is a Hilbert space, $A \in L(H)_+$ and $(A, \mathscr{S})$ is compatible, then the Schur complement (Ando)

$$A_{/\mathscr{S}} = \max\{X \in L(H)_+ : \ X \le A, \ R(X) \subseteq \mathscr{S}^\perp\},$$

where the maximum in the right-hand side of the equation above does exist for any positive operator A and any closed subspace $\mathscr{S}$ of H, and is defined by Anderson and Trapp [1] as the shorted operator of A by $\mathscr{S}$. Therefore, the Schur complement (Ando) of $\mathscr{S}$-compatible positive operators have the same properties as the shorted operators which were widely studied (see [1, 9, 10, 28]).

Unfortunately, in the case of Hilbert C^*-module, the maximum above maybe does not exist. For instance, let $\mathcal{A} = C_0(-1, 1)$. Then $E = \mathcal{A} \oplus \mathcal{A}$ is a Hilbert $\mathcal{A}$-module under the inner product

$$\langle f_1 \oplus g_1, f_2 \oplus g_2 \rangle = \overline{f}_1 f_2 + \overline{g}_1 g_2 \in \mathcal{A} \quad \text{for } f_1, f_2, g_1, g_2 \in \mathcal{A}.$$

It is well-known that $L_{\mathcal{A}}(E) = M_2(L_{\mathcal{A}}(\mathcal{A})) = M_2(C_b(-1, 1))$, where $C_b(-1, 1)$ is the algebra of continuous bounded functions on $(-1, 1)$. Set $\mathscr{S} = \mathcal{A} \oplus 0$. Then $\mathscr{S} \subseteq E$ is orthogonally complemented and $\mathscr{S}^\perp = 0 \oplus \mathcal{A}$. Define continuous functions as follows:

$$f(x) = \begin{cases} 0, & x \in (-1, 0), \\ x^2, & x \in (0, 1), \end{cases} \qquad h(x) = \begin{cases} 0, & x \in (-1, 0), \\ x, & x \in (0, 1), \end{cases} \qquad g(x) = 2x + 2.$$

Obviously $f, h, g \in C_b(-1, 1)$. Set $A = \left(\begin{smallmatrix} f & h \\ h & g \end{smallmatrix}\right)$ under the orthogonal decomposition $E = \mathscr{S} \oplus \mathscr{S}^\perp$. Then $A \in L_{\mathcal{A}}(E)$. For any $a, b \in \mathcal{A}$, we have

$$\begin{aligned} \left\langle A \begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} a \\ b \end{pmatrix} \right\rangle &= \left\langle \begin{pmatrix} f & h \\ h & g \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} a \\ b \end{pmatrix} \right\rangle \\ &= \langle fa + hb, a \rangle + \langle ha + gb, b \rangle \\ &= fa^2 + 2hab + gb^2. \end{aligned}$$

For any $x \in (-1,0),\ y \in (0,1)$, we have

$$(fa^2+2hab+gb^2)(x) = g(x)b^2 \geq 0,$$
$$(fa^2+2hab+gb^2)(y) = y^2a^2+2yab+g(y)b^2 = (ya+b)^2+(g(y)-1)b^2 \geq 0.$$

Thus we obtain $A \geq 0$. If $X \in L_{\mathcal{A}}(E)_+$ and $R(X) \subseteq \mathscr{S}^\perp$, we note $X = \left(\begin{smallmatrix} 0 & 0 \\ 0 & X_1 \end{smallmatrix}\right)$ and $X_1 \geq 0$. Suppose $X \leq A$. Then we have for any $a, b \in \mathcal{A}$,

$$\left\langle (A-X)\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} a \\ b \end{pmatrix} \right\rangle = fa^2+2hab+(g-X_1)b^2 \geq 0.$$

By previous calculation, we know that $X_1|_{(-1,0)} \leq g = 2x+2$ and $X_1|_{(0,1)} \leq g-1 = 2x+1$. Therefore, we obtain that $X \in L_{\mathcal{A}}(E)_+$ such that $X \leq A$ and $R(X) \subseteq \mathscr{S}^\perp$ if and only if $X = \left(\begin{smallmatrix} 0 & 0 \\ 0 & X_1 \end{smallmatrix}\right)$ with $X_1 \in C_b(-1,1)$ such that $X_1 \geq 0$, $X_1|_{(-1,0)} \leq 2x+2$ and $X_1|_{(0,1)} \leq 2x+1$. From this, we easily know that the maximum of $\{X \in L_{\mathcal{A}}(E)_+ :\ X \leq A,\ R(X) \subseteq \mathscr{S}^\perp\}$ does not exist.

Therefore, we could not define the shorted operator as we did in the case of Hilbert space, and by which study the properties of Schur complements as in [10]. However, we prove that if $(A, \mathscr{S})$ is compatible, then the maximum above exists.

In this section, we want to obtain more representations of the Schur complements for $\mathscr{S}$-compatible positive operators. Throughout this section, A denotes a positive operator in $L_{\mathcal{A}}(E)$ and $\mathscr{S}$ denotes an orthogonally complemented closed submodule in E, and $A = \left(\begin{smallmatrix} A_{11} & A_{12} \\ A_{12}^* & A_{22} \end{smallmatrix}\right)$ under the orthogonal decomposition $E = \mathscr{S} \oplus \mathscr{S}^\perp$.

Lemma 4.1 *There exists a positive number $\lambda > 0$ such that $A_{12}A_{12}^* \leq \lambda A_{11}$. If additionally, $\overline{R(A_{11})}$ is orthogonally complemented in $\mathscr{S}$, then the equation $A_{11}^{\frac{1}{2}}X = A_{12}$ has a reduced solution.*

Proof Set $A = S^*S$ and $S = \left(\begin{smallmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{smallmatrix}\right)$ under the orthogonal decomposition $E = \mathscr{S} \oplus \mathscr{S}^\perp$. Then we have

$$A = \begin{pmatrix} S_{11}^* & S_{21}^* \\ S_{12}^* & S_{22}^* \end{pmatrix}\begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} = \begin{pmatrix} S_{11}^*S_{11}+S_{21}^*S_{21} & S_{11}^*S_{12}+S_{21}^*S_{22} \\ S_{12}^*S_{11}+S_{22}^*S_{21} & S_{12}^*S_{12}+S_{22}^*S_{22} \end{pmatrix}.$$

By direct computation it follows that

$$A_{12}A_{12}^* \leq 2(\|S_{12}\|^2 {S_{11}}^* S_{11} + \|S_{22}\|^2 {S_{21}}^* S_{21}).$$

Set $\lambda = \max\{2\|S_{12}\|^2, 2\|S_{22}\|^2\}$. Then we have

$$A_{12}A_{12}^* \leq \lambda A_{11}.$$

If $\overline{R(A_{11})} = \overline{R(A_{11}^{\frac{1}{2}})}$ is the orthogonally complemented closed submodule of $\mathscr{S}$, by Lemma 2.1, we know that $R(A_{12}) \subseteq R(A_{11}^{\frac{1}{2}})$, and the equation $A_{11}^{\frac{1}{2}}X = A_{12}$ has a reduced solution $X \in L_{\mathcal{A}}(\mathscr{S}^\perp, \mathscr{S})$.

Theorem 4.1 (i) *If $(A, \mathscr{S})$ is compatible, then $A_{11}^{\frac{1}{2}}X = A_{12}$ has the reduced solution $D \in L_{\mathcal{A}}(\mathscr{S}^\perp, \mathscr{S})$ such that $D = A_{11}^{\frac{1}{2}}V$, where $V \in L_{\mathcal{A}}(\mathscr{S}^\perp, \mathscr{S})$ is a solution to $A_{11}X = A_{12}$. In this case,*

$$A_{/\mathscr{S}} = \begin{pmatrix} 0 & 0 \\ 0 & A_{22}-D^*D \end{pmatrix}.$$

(ii) *Suppose that* ${A_{11}}^{\frac{1}{2}}X = A_{12}$ *has the reduced solution* D *in* $L_{\mathcal{A}}(\mathscr{S}^{\perp}, \mathscr{S})$ (*in particular if* $\overline{R(A_{11})}$ *is orthogonally complemented in* $\mathscr{S}$). *Then* $(A, \mathscr{S})$ *is compatible if and only if* ${A_{11}}^{\frac{1}{2}}X = D$ *has a solution.*

(iii) *Suppose that* $R(A_{11}) = R({A_{11}}^{\frac{1}{2}})$, *and* $\overline{R(A_{11})}$ *is orthogonally complemented in* $\mathscr{S}$. *Then* $(A, \mathscr{S})$ *is compatible. In particular, if* A_{11} *is invertible* (*especially if* A *is invertible*), *then* $(A, \mathscr{S})$ *is compatible and* $A_{/\mathscr{S}} = \left(\begin{smallmatrix} 0 & 0 \\ 0 & A_{22} - A_{12}^{*}{A_{11}}^{-1}A_{12} \end{smallmatrix}\right)$.

Proof (i) Let $V \in L_{\mathcal{A}}(\mathscr{S}^{\perp}, \mathscr{S})$ be a solution to $A_{11}X = A_{12}$. Then $A_{11}^{\frac{1}{2}}(A_{11}^{\frac{1}{2}}V) = A_{12}$. Moreover, we know that $R(A_{11}^{\frac{1}{2}}V) \subseteq R(A_{11}^{\frac{1}{2}}) \subseteq N(A_{11}^{\frac{1}{2}})^{\perp}$, so $A_{11}^{\frac{1}{2}}V$ is the reduced solution to $A_{11}^{\frac{1}{2}}X = A_{12}$, which we denote by D. Therefore, the Schur complement of A to $\mathscr{S}$

$$A_{/\mathscr{S}} = \begin{pmatrix} 0 & 0 \\ 0 & A_{22} - V^{*}A_{11}V \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & A_{22} - D^{*}D \end{pmatrix}.$$

(ii) It is clear from (i) and Theorem 3.2.

(iii) From Lemma 4.1 and Theorem 2.1(vi) we obtain that $(A, \mathscr{S})$ is compatible.

If A is invertible, it is easy to know that A_{11} is invertible, then so is $A_{11}^{\frac{1}{2}}$. Let $A_{11}^{-\frac{1}{2}}$ denote the inverse of $A_{11}^{\frac{1}{2}}$ in $L_{\mathcal{A}}(\mathscr{S})$. Then $D = A_{11}^{-\frac{1}{2}}A_{12}$ and

$$A_{/\mathscr{S}} = \begin{pmatrix} 0 & 0 \\ 0 & A_{22} - A_{12}^{*}A_{11}^{-1}A_{12} \end{pmatrix}.$$

Theorem 4.2 *Suppose that* $A_{11}^{\frac{1}{2}}X = A_{12}$ *has the reduced solution* $D \in L_{\mathcal{A}}(\mathscr{S}^{\perp}, \mathscr{S})$ (*in particular if* $\overline{R(A_{11})}$ *is orthogonally complemented in* $\mathscr{S}$, *or if* $(A, \mathscr{S})$ *is compatible*). *Then the following statements are equivalent:*

(i) $\left(\begin{smallmatrix} 0 & 0 \\ 0 & A_{22} - D^{*}D \end{smallmatrix}\right)$ *is the minimum of* $\{Q^{*}AQ : \ Q^2 = Q, \ N(Q) = \mathscr{S}\}$.

(ii) *The equation* ${A_{11}}^{\frac{1}{2}}X = D$ *has a solution* $X \in L_{\mathcal{A}}(\mathscr{S}^{\perp}, \mathscr{S})$.

(iii) $(A, \mathscr{S})$ *is compatible.*

In this case, $A_{/\mathscr{S}} = \left(\begin{smallmatrix} 0 & 0 \\ 0 & A_{22} - D^{*}D \end{smallmatrix}\right)$, *and* $A_{/\mathscr{S}}$ *is also the maximum of* $\{X \in L_{\mathcal{A}}(E)_{+} : X \leq A, \ R(X) \subseteq \mathscr{S}^{\perp}\}$.

Proof We only need to show (i)$\Leftrightarrow$(ii).

By the previous discussion, we know that if Q is an idempotent with $N(Q) = \mathscr{S}$, i.e., $R(I - Q) = \mathscr{S}$, then $Q = \left(\begin{smallmatrix} 0 & X \\ 0 & I \end{smallmatrix}\right)$ under the orthogonal decomposition $E = \mathscr{S} \oplus \mathscr{S}^{\perp}$, where $X \in L_{\mathcal{A}}(\mathscr{S}^{\perp}, \mathscr{S})$. So we have

$$Q^{*}AQ = \begin{pmatrix} 0 & 0 \\ 0 & A_{22} + X^{*}A_{12} + A_{12}^{*}X + X^{*}A_{11}X \end{pmatrix}.$$

In the other hand,

$$\begin{aligned} A_{22} + X^{*}A_{12} + A_{12}^{*}X + X^{*}A_{11}X &= A_{22} + X^{*}A_{11}^{\frac{1}{2}}D + D^{*}A_{11}^{\frac{1}{2}}X + X^{*}A_{11}X \\ &= A_{22} + (X^{*}A_{11}^{\frac{1}{2}})D + D^{*}(A_{11}^{\frac{1}{2}}X) + X^{*}A_{11}X \\ &\geq A_{22} - D^{*}D, \end{aligned}$$

and " = " holds if and only if $A_{11}^{\frac{1}{2}}X = -D$. Set $X_0 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & A_{22} - D^{*}D \end{smallmatrix}\right)$. Then X_0 is the minimum of $\{Q^{*}AQ : \ Q^2 = Q, \ N(Q) = \mathscr{S}\}$ if and only if the equation $A_{11}^{\frac{1}{2}}X = D$ has a solution in $L_{\mathcal{A}}(\mathscr{S}^{\perp}, \mathscr{S})$. Thus (i)$\Leftrightarrow$(ii) is proved.

Now we assume that X_0 is the minimum of $\{Q^*AQ: \ Q^2 = Q, \ N(Q) = \mathscr{S}\}$. It is clear that $R(X_0) \subseteq \mathscr{S}^\perp$, and $A - X_0 = \begin{pmatrix} A_{11}^{\frac{1}{2}} & D \\ 0 & 0 \end{pmatrix}^* \begin{pmatrix} A_{11}^{\frac{1}{2}} & D \\ 0 & 0 \end{pmatrix} \geq 0$.

For any $X \in L_{\mathcal{A}}(E)$ such that $0 \leq X \leq A$ and $R(X) \subseteq \mathscr{S}^\perp$, X is of the form $\begin{pmatrix} 0 & 0 \\ 0 & X_1 \end{pmatrix}$ under the orthogonal decomposition $E = \mathscr{S} \oplus \mathscr{S}^\perp$. Let $Q \in L_{\mathcal{A}}(E)$ with $Q^2 = Q$ and $N(Q) = \mathscr{S}$ and set $Q = \begin{pmatrix} 0 & Y \\ 0 & I \end{pmatrix}$, with the direct computation we know that $Q^*XQ = X$ and

$$0 \leq Q^*(A - X)Q = Q^*AQ - Q^*XQ = Q^*AQ - X.$$

Therefore, $X_0 \geq X$. This completes the proof.

Corollary 4.1 *Suppose that $(A, \mathscr{S})$ is compatible. Then*

$$\begin{aligned} A_{/\mathscr{S}} &= \min\{Q^*AQ: \ Q^2 = Q, \ N(Q) = \mathscr{S}\} \\ &= \max\{X \in L_{\mathcal{A}}(E)_+ : X \leq A, \ R(X) \subseteq \mathscr{S}^\perp\}. \end{aligned}$$

Example 4.1 Let $\mathcal{A} = C_0(0,1)$. Then $E = \mathcal{A} \oplus \mathcal{A}$ is a Hilbert $\mathcal{A}$-module, and $L_{\mathcal{A}}(E) = M_2(C_b(0,1))$. Set $\mathscr{S} = \mathcal{A} \oplus 0$. Then $\mathscr{S} \subseteq E$ is orthogonally complemented and $\mathscr{S}^\perp = 0 \oplus \mathcal{A}$. Suppose $A = \begin{pmatrix} x^2 & x \\ x & 2 \end{pmatrix}$ under the orthogonal decomposition $E = \mathscr{S} \oplus \mathscr{S}^\perp$. As we discussed previously in the case $\mathcal{A} = C_0(-1,1)$, we obtain that (i) $A \in L_{\mathcal{A}}(E)$ and $A \geq 0$; (ii) if $0 \leq X \leq A$ and $R(X) \subseteq \mathscr{S}^\perp$, we may set $X = \begin{pmatrix} 0 & 0 \\ 0 & X_1 \end{pmatrix}$ with $0 \leq X_1 \leq 1$, i.e., $X \leq \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Obviously, $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in L_{\mathcal{A}}(E)_+$ and then $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \max\{X \in L_{\mathcal{A}}(E)_+ : X \leq A, R(X) \subseteq \mathscr{S}\}$.

However, $(A, \mathscr{S})$ is not compatible for $x^{-1} \notin C_b(0,1)$.

Lemma 4.2 *Suppose that $(A, \mathscr{S})$ is compatible and $\overline{R(A)}$ is orthogonally complemented. Set $\mathscr{M} = \overline{A^{\frac{1}{2}}\mathscr{S}}$. Then $\mathscr{M}$ is orthogonally complemented.*

Proof Suppose $Q \in \mathscr{P}(A, \mathscr{S})$. Since $\overline{R(A^{\frac{1}{2}})} = \overline{R(A)}$ is orthogonally complemented, we have $\overline{R(A^{\frac{1}{2}})} \oplus N(A^{\frac{1}{2}}) = E$. For any $x \in \overline{R(A^{\frac{1}{2}})}$, there exist $\{x_n\} \subseteq E$ such that $x = \lim\limits_n A^{\frac{1}{2}} x_n$. By Proposition 2.1, we know that $Q^*AQ \leq A$. So

$$\langle A^{\frac{1}{2}}Q(x_n - x_m), A^{\frac{1}{2}}Q(x_n - x_m)\rangle \leq \langle A^{\frac{1}{2}}(x_n - x_m), A^{\frac{1}{2}}(x_n - x_m)\rangle.$$

Then $A^{\frac{1}{2}}Qx_n$ is convergent. Hence we obtain

$$\overline{R(A^{\frac{1}{2}})} = \overline{R(A^{\frac{1}{2}}Q)} + \overline{R(A^{\frac{1}{2}}(I - Q))},$$

and so $\mathscr{M} + \overline{R(A^{\frac{1}{2}}(I - Q))} + N(A^{\frac{1}{2}}) = E$.

Obviously, $N(A^{\frac{1}{2}}) \subseteq \mathscr{M}^\perp$. For any $x, \ y \in E$, we have

$$\langle A^{\frac{1}{2}}(I - Q)x, A^{\frac{1}{2}}Qy\rangle = \langle A(I - Q)x, Qy\rangle = 0,$$

so that $\overline{R(A^{\frac{1}{2}}(I - Q))} \subseteq \mathscr{M}^\perp$. Therefore $E = \mathscr{M} \oplus \mathscr{M}^\perp$, and $\mathscr{M}$ is orthogonally complemented in E.

Proposition 4.1 *Suppose that $(A, \mathscr{S})$ is compatible and $\overline{R(A)}$ is orthogonally complemented in E. Then*

$$A_{/\mathscr{S}} = A^{\frac{1}{2}}(I - P_{\mathscr{M}})A^{\frac{1}{2}},$$

where $\mathscr{M} = \overline{A^{\frac{1}{2}}\mathscr{S}}$, and $P_{\mathscr{M}} : E \to \mathscr{M}$ is the projection of E onto $\mathscr{M}$.

Proof Suppose $Q \in \mathscr{P}(A, \mathscr{S})$. Given $x \in E$, for $R(A^{\frac{1}{2}}(I-Q)) \subseteq \mathscr{M}^{\perp}$, we have $P_{\mathscr{M}} A^{\frac{1}{2}} x = P_{\mathscr{M}} A^{\frac{1}{2}} Q x = A^{\frac{1}{2}} Q x$. Therefore,

$$A_{/\mathscr{S}} = A(I-Q) = A - A^{\frac{1}{2}} P_{\mathscr{M}} A^{\frac{1}{2}} Q = A^{\frac{1}{2}}(I - P_{\mathscr{M}}) A^{\frac{1}{2}}.$$

Proposition 4.2 *Suppose that $(A, \mathscr{S})$ is compatible and $\overline{R(A)}$ is orthogonally complemented in E. Then*

(i) $R(A_{/\mathscr{S}}) = R(A) \cap \mathscr{S}^{\perp}$,

(ii) $R((A_{/\mathscr{S}})^{\frac{1}{2}}) = R(A^{\frac{1}{2}}) \cap \mathscr{S}^{\perp}$.

Proof (i) Let $x \in E$ with $Ax \in R(A) \cap \mathscr{S}^{\perp}$. For any $y \in \mathscr{S}$, we have $\langle Ax, y \rangle = 0 = \langle A^{\frac{1}{2}} x, A^{\frac{1}{2}} y \rangle$, so that $A^{\frac{1}{2}} x \in \mathscr{M}^{\perp}$, i.e., $(I - P_{\mathscr{M}}) A^{\frac{1}{2}} x = A^{\frac{1}{2}} x$. Then it follows that

$$Ax = A^{\frac{1}{2}} A^{\frac{1}{2}} x = A^{\frac{1}{2}} (I - P_{\mathscr{M}}) A^{\frac{1}{2}} x = A_{/\mathscr{S}} x$$

and so $R(A) \cap \mathscr{S}^{\perp} \subseteq R(A_{/\mathscr{S}})$. The other inclusion is obvious from Theorem 4.1.

(ii) By Proposition 4.1, we know $A_{/\mathscr{S}} = A^{\frac{1}{2}}(I - P_{\mathscr{M}}) A^{\frac{1}{2}} \leq A$. From Lemma 2.1, it could be seen that $R((A_{/\mathscr{S}})^{\frac{1}{2}}) \subseteq R(A^{\frac{1}{2}}) \cap \overline{R(A_{/\mathscr{S}})} \subseteq R(A^{\frac{1}{2}}) \cap \mathscr{S}^{\perp}$.

Conversely, given $x \in R(A^{\frac{1}{2}}) \cap \mathscr{S}^{\perp}$, we assume $x = A^{\frac{1}{2}} y$, where $y \in N(A^{\frac{1}{2}})^{\perp} = \overline{R(A^{\frac{1}{2}})}$. For any $z \in \mathscr{S}$, we get

$$\langle x, z \rangle = 0 = \langle A^{\frac{1}{2}} y, z \rangle = \langle y, A^{\frac{1}{2}} z \rangle.$$

So $y \in \mathscr{M}^{\perp}$, i.e., $P_{\mathscr{M}} y = 0$.

Since $y \in \overline{R(A^{\frac{1}{2}})}$, we set $y = \lim_n A^{\frac{1}{2}} y_n$. Then we have

$$x = A^{\frac{1}{2}} y = A^{\frac{1}{2}} (I - P_{\mathscr{M}}) y = \lim_n A_{/\mathscr{S}} y_n.$$

As $A_{/\mathscr{S}} \leq A$, we obtain that for all $x \in E$, $\langle A_{/\mathscr{S}} x, x \rangle \leq \langle Ax, x \rangle$, i.e.,

$$\langle (A_{/\mathscr{S}})^{\frac{1}{2}} x, (A_{/\mathscr{S}})^{\frac{1}{2}} x \rangle \leq \langle A^{\frac{1}{2}} x, A^{\frac{1}{2}} x \rangle.$$

Therefore, $(A_{/\mathscr{S}})^{\frac{1}{2}} y_n$ is convergent in E in norm topology. So

$$x = \lim_n A_{/\mathscr{S}} y_n = (A_{/\mathscr{S}})^{\frac{1}{2}} \lim_n (A_{/\mathscr{S}})^{\frac{1}{2}} y_n$$

and $R(A^{\frac{1}{2}}) \cap \mathscr{S}^{\perp} \subseteq R((A_{/\mathscr{S}})^{\frac{1}{2}})$.

Magajna and Schweizer showed, respectively, that C^*-algebras of compact operators can be characterized by the property that every closed in norm (coinciding with its biorthogonal complement, respectively) submodule of every Hilbert C^*-module over them is automatically an orthogonal summand (see [25, 30]). Hence, we obtain the following statement.

Theorem 4.3 *Let $A \in L_{\mathcal{A}}(E)_+$ and $\mathscr{S}$ be an orthogonally complemented closed submodule of E. If A is $\mathscr{S}$-compatible, then*

(i) *The equation $A_{11}^{\frac{1}{2}} X = A_{12}$ has the reduced solution $D \in L(\mathscr{S}^{\perp}, \mathscr{S})$, and*

$$A_{/\mathscr{S}} = \begin{pmatrix} 0 & 0 \\ 0 & A_{22} - D^* D \end{pmatrix}.$$

(ii) $A_{/\mathscr{S}} = \min\{Q^* A Q : \ Q^2 = Q, \ N(Q) = \mathscr{S}\}$.

(iii) $A_{/\mathscr{S}} = \max\{X \in L_{\mathcal{A}}(E)_+ : X \leq A,\ R(X) \subseteq \mathscr{S}^{\perp}\}$.

If additionally, $\mathcal{A}$ is an arbitrary C^*-algebra of compact operators, then

(iv) $A_{/\mathscr{S}} = A^{\frac{1}{2}} P_{\mathscr{M}} A^{\frac{1}{2}}$, *where* $\mathscr{M} = (A^{\frac{1}{2}}\mathscr{S})^{\perp}$ *and* $P_{\mathscr{M}}$ *is the projection of* E *onto* $\mathscr{M}$.

(v) $R(A) \cap \mathscr{S}^{\perp} = R(A_{/\mathscr{S}}),\ R(A^{\frac{1}{2}}) \cap \mathscr{S}^{\perp} = R((A_{/\mathscr{S}})^{\frac{1}{2}})$.

5 The Quotient Property for Schur Complements of Operators on Hilbert C^*-Module

An important property of Schur complements of matrices is the quotient property first demonstrated in [11]. Another proof was given in [26] and a simpler one can be found in [4]: given a semi-positive definite matrix $H = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ with A, A_{11} invertible, then $H_{/A} = (H_{/A_{11}})_{/(A_{/A_{11}})}$.

In this section, we attempt to extend the above result for matrices to operators on Hilbert C^*-modules, i.e., to prove the quotient property for Schur complements of $\mathscr{S}$-complementable ($\mathscr{S}^*$-complementable) operators on Hilbert C^*-module E.

Throughout this section, we assume that $T \in L_{\mathcal{A}}(E)$, $\mathscr{S} \subseteq E$ and $\mathscr{S}_1 \subseteq \mathscr{S}$ are orthogonally complemented closed submodules of Hilbert C^*-module E, and $T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$ under the orthogonal decomposition $E = \mathscr{S} \oplus \mathscr{S}^{\perp}$.

Lemma 5.1 *Suppose that T is $\mathscr{S}$-complementable ($\mathscr{S}^*$-complementable), and $U_1, U_2 \in L_{\mathcal{A}}(E)$ ($U_1,\ U_2 \in L_{\mathcal{A}}(E)$ with $U_1 = U_2^*$) have the matrix representations*

$$U_1 = \begin{pmatrix} I & Y \\ 0 & I \end{pmatrix}, \quad U_2 = \begin{pmatrix} I & 0 \\ Z & I \end{pmatrix}$$

under the orthogonal decomposition $E = \mathscr{S} \oplus \mathscr{S}^{\perp}$. Then U_2TU_1 is $\mathscr{S}$-complementable ($\mathscr{S}^*$-complementable) and $U_2TU_{1/\mathscr{S}} = T_{/\mathscr{S}}$.

Proof Suppose that $T \in L_{\mathcal{A}}(E)$ is $\mathscr{S}$-complementable and $V,\ W$ are the solutions of $T_{11}X = T_{12}$ and $T_{11}^*X = T_{21}^*$, respectively. We have by direct computation

$$U_2TU_1 = \begin{pmatrix} T_{11} & T_{11}Y + T_{12} \\ T_{21} + ZT_{11} & ZT_{11}Y + T_{21}Y + ZT_{12} + T_{22} \end{pmatrix},$$

and it could be seen easily that

$$T_{11}(V + Y) = T_{11}Y + T_{12}, \quad T_{11}^*(W + Z^*) = (T_{21} + ZT_{11})^*,$$

i.e., $V + Y,\ W + Z^*$ are the solutions of $T_{11}X = T_{12}$ and $T_{11}^*X = T_{21}^*$, respectively. So by Theorem 3.1, U_2TU_1 is $\mathscr{S}$-complementable and

$$\begin{aligned} U_2TU_{1/\mathscr{S}} &= \begin{pmatrix} 0 & 0 \\ 0 & ZT_{11}Y + T_{21}Y + ZT_{12} + T_{22} - (W + Z^*)^*T_{11}(V + Y) \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & T_{22} - W^*T_{11}V \end{pmatrix} \\ &= T_{/\mathscr{S}}. \end{aligned}$$

About the case of $\mathscr{S}^*$-complementability, it is enough to note that $Y = Z^*$ and $V = W$.

Lemma 5.2 *Suppose that T is $\mathscr{S}$-complementable ($\mathscr{S}^*$-complementable) and T_{11} is $\mathscr{S}_1$-complementable ($\mathscr{S}_1^*$-complementable). Then T is also $\mathscr{S}_1$-complementable ($\mathscr{S}_1^*$-complementable), and there exist invertible operators $U_1, U_2 \in L_{\mathcal{A}}(E)$ ($U_1, U_2 \in L_{\mathcal{A}}(E)$ with $U_1 = U_2^*$) such that*

$$U_2 T_{/\mathscr{S}_1} U_1 = T_{/\mathscr{S}} + T_{11/\mathscr{S}_1},$$

where $T_{11/\mathscr{S}_1}$ is identified with $\begin{pmatrix} T_{11/\mathscr{S}_1} & 0 \\ 0 & 0 \end{pmatrix}$ in $L_{\mathcal{A}}(E)$.

Proof Let P, P_1 denote the orthogonal projections of E onto $\mathscr{S}, \mathscr{S}_1$ respectively. By assumption it could be seen that $P - P_1 \in L_{\mathcal{A}}(E)$ is a projection. We denote $P - P_1$ by P_2, and $R(P_2)$ by $\mathscr{S}_2$, i.e., $\mathscr{S}_2 = \mathscr{S} \ominus \mathscr{S}_1$. Suppose that $J_i : \mathscr{S}_i \to \mathscr{S}$ is the including map, and $P_i{}' : \mathscr{S} \to \mathscr{S}_i$ satisfies that for all $x \in \mathscr{S}$, $P_i{}'x = P_i x$ $(i = 1, 2)$. It is easy to know that $J_i \in L_{\mathcal{A}}(\mathscr{S}_i, \mathscr{S})$ and $J_i{}^* = P_i{}'$.

Suppose

$$T = \begin{pmatrix} T_{11}^1 & T_{11}^2 & T_{12}^1 \\ T_{11}^3 & T_{11}^4 & T_{12}^2 \\ T_{21}^1 & T_{21}^2 & T_{22} \end{pmatrix}$$

under the orthogonal decomposition $P_1 + P_2 + (I - P) = I$, where

$$T_{11} = \begin{pmatrix} T_{11}^1 & T_{11}^2 \\ T_{11}^3 & T_{11}^4 \end{pmatrix}, \quad T_{12} = \begin{pmatrix} T_{12}^1 \\ T_{12}^2 \end{pmatrix} \quad \text{and} \quad T_{21} = \begin{pmatrix} T_{21}^1 & T_{21}^2 \end{pmatrix}.$$

Since $T \in L_{\mathcal{A}}(E)$ is $\mathscr{S}$-complementable, we suppose that $V_1, W_1 \in L_{\mathcal{A}}(\mathscr{S}^{\perp}, \mathscr{S})$ are the solutions of $T_{11}X = T_{12}$ and $T_{11}{}^*Y = T_{21}{}^*$ respectively. So by Theorem 3.1, we have

$$T_{/\mathscr{S}} = \left(I - \begin{pmatrix} 0 & 0 \\ W_1{}^* & I \end{pmatrix}\right) T \left(I - \begin{pmatrix} 0 & V_1 \\ 0 & I \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 0 & T_{22} - W_1{}^* T_{11} V_1 \end{pmatrix}.$$

Replacing T_{11}, T_{12}, T_{21} by $\begin{pmatrix} T_{11}^1 & T_{11}^2 \\ T_{11}^3 & T_{11}^4 \end{pmatrix}, \begin{pmatrix} T_{12}^1 \\ T_{12}^2 \end{pmatrix}, \begin{pmatrix} T_{21}^1 & T_{21}^2 \end{pmatrix}$ in the equations $T_{11}V_1 = T_{12}$ and $T_{11}{}^*W_1 = T_{21}{}^*$, we could obtain the following equations:

$$\begin{cases} T_{11}^1 P_1{}'V_1 + T_{11}^2 P_2{}'V_1 = T_{12}^1, \\ T_{11}^3 P_1{}'V_1 + T_{11}^4 P_2{}'V_1 = T_{12}^2, \\ T_{11}^1{}^* P_1{}'W_1 + T_{11}^3{}^* P_2{}'W_1 = T_{21}^1{}^*, \\ T_{11}^2{}^* P_1{}'W_1 + T_{11}^4{}^* P_2{}'W_1 = T_{21}^2{}^*. \end{cases}$$

Since $T_{11} \in L_{\mathcal{A}}(\mathscr{S})$ is $\mathscr{S}_1$-complementable, we may assume that V_2, $W_2 \in L_{\mathcal{A}}(\mathscr{S}_2, \mathscr{S}_1)$ are the solutions of $T_{11}^1 X = T_{11}^2$, $T_{11}^1{}^* Y = T_{11}^3{}^*$ respectively. Therefore, the Schur complement of T_{11} to $\mathscr{S}_1$ is

$$T_{11/\mathscr{S}_1} = \left(I - \begin{pmatrix} 0 & 0 \\ W_2{}^* & I \end{pmatrix}\right) T_{11} \left(I - \begin{pmatrix} 0 & V_2 \\ 0 & I \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 0 & T_{11}^4 - W_2^* T_{11}^1 V_2 \end{pmatrix} \in L_{\mathcal{A}}(\mathscr{S}).$$

Set

$$V = (V_2,\ P_1{}'V_1 + V_2 P_2{}'V_1), \quad W = (W_2,\ P_1{}'W_1 + W_2 P_2{}'W_1).$$

Then V, $W \in L_{\mathcal{A}}(\mathscr{S}_2 \oplus \mathscr{S}^{\perp}, \mathscr{S}_1)$. Moreover, we have

$$\begin{aligned} T_{11}^1 V &= (T_{11}^1 V_2,\ T_{11}^1 P_1{}'V_1 + T_{11}^1 V_2 P_2{}'V_1) = (T_{11}^2,\ T_{11}^1 P_1{}'V_1 + T_{11}^2 P_2{}'V_1) = (T_{11}^2,\ T_{12}^1), \\ T_{11}^1{}^* W &= (T_{11}^1{}^* W_2,\ T_{11}^1{}^* P_1{}'W_1 + T_{11}^1{}^* W_2 P_2{}'W_1) = (T_{11}^3{}^*,\ T_{11}^1{}^* P_1{}'W_1 + T_{11}^3{}^* P_2{}'W_1) \\ &= (T_{11}^3{}^*,\ T_{21}^1{}^*) = \begin{pmatrix} T_{11}^3 \\ T_{21}^1 \end{pmatrix}^*. \end{aligned}$$

By Theorem 3.1, we know that $T \in L_{\mathcal{A}}(E)$ is $\mathscr{S}_1$-complementable.

Set $M_r = \begin{pmatrix} I & V \\ 0 & 0 \end{pmatrix}$, $M_l = \begin{pmatrix} I & 0 \\ W^* & 0 \end{pmatrix}$. We obtain

$$\begin{aligned} T_{/\mathscr{S}_1} &= (I - M_l)T(I - M_r) \\ &= \begin{pmatrix} 0 & 0 & 0 \\ -{W_2}^* & I & 0 \\ -W_1^* J_1 - W_1^* J_2 W_2^* & 0 & I \end{pmatrix} T \begin{pmatrix} 0 & -V_2 & -{P_1}'V_1 - V_2{P_2}'V_1 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}. \end{aligned}$$

Because

$$\begin{pmatrix} T_{11/\mathscr{S}_1} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} I - M_{l2} & 0 \\ 0 & 0 \end{pmatrix} T \begin{pmatrix} I - M_{r2} & 0 \\ 0 & 0 \end{pmatrix},$$

where $M_{r2} = \begin{pmatrix} I & V_2 \\ 0 & 0 \end{pmatrix}$ and $M_{l2} = \begin{pmatrix} I & 0 \\ {W_2}^* & 0 \end{pmatrix}$, we obtain

$$\begin{aligned} & T_{/\mathscr{S}} + T_{11/\mathscr{S}_1} \\ &= \begin{pmatrix} 0 & 0 \\ -W_1^* & I \end{pmatrix} T \begin{pmatrix} 0 & -V_1 \\ 0 & I \end{pmatrix} + \begin{pmatrix} I - M_{l2} & 0 \\ 0 & 0 \end{pmatrix} T \begin{pmatrix} I - M_{r2} & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} I - M_{l2} & 0 \\ -W_1^* & I \end{pmatrix} T \begin{pmatrix} I - M_{r2} & -V_1 \\ 0 & I \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 \\ -W_2^* & I & 0 \\ -W_1^* J_1 & -W_1^* J_2 & I \end{pmatrix} T \begin{pmatrix} 0 & -V_2 & -{P_1}'V_1 \\ 0 & I & -{P_2}'V_1 \\ 0 & 0 & I \end{pmatrix}. \end{aligned}$$

Set $U_1 = \begin{pmatrix} I & 0 & 0 \\ 0 & I & -{P_2}'V_1 \\ 0 & 0 & I \end{pmatrix}$ and $U_2 = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & -W_1^* J_2 & I \end{pmatrix}$. Therefore, we have

$$\begin{aligned} & T_{/\mathscr{S}} + T_{11/\mathscr{S}_1} \\ &= U_2 \begin{pmatrix} 0 & 0 & 0 \\ -W_2^* & I & 0 \\ -W_1^* J_1 - W_1^* J_2 {W_2}^* & 0 & I \end{pmatrix} T \begin{pmatrix} 0 & -V_2 & -P_1'V_1 - V_2 P_2' V_1 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} U_1 \\ &= U_2 T_{/\mathscr{S}_1} U_1. \end{aligned}$$

In the case of $\mathscr{S}^*$-complementability, it is easy to see that in the proof above, $W_1 = V_1$ and $W_2 = V_2$. Then $W = V$ and $U_2 = {U_1}^*$. Therefore T is $\mathscr{S}_1^*$-complementable.

Lemma 5.3 *$T_{11} \in L_{\mathcal{A}}(\mathscr{S})$ is $\mathscr{S}_1$-complementable if and only if PTP is $\mathscr{S}_1$-complementable. Moreover,*

$$(PTP)_{/\mathscr{S}_1} = \begin{pmatrix} T_{11/\mathscr{S}_1} & 0 \\ 0 & 0 \end{pmatrix}.$$

Proof Let T, P, P_1, P_2 be as in Lemma 5.2. Then under the orthogonal decomposition $P_1 + P_2 + (I - P) = I$,

$$PTP = \begin{pmatrix} T_{11}^1 & T_{11}^2 & 0 \\ T_{11}^3 & T_{11}^4 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

If T_{11} is $\mathscr{S}_1$-complementable, then there exist operators V_1 and $W_1 \in L_{\mathcal{A}}(\mathscr{S}_2, \mathscr{S}_1)$ which are solutions of $T_{11}^1 X = T_{11}^2$ and ${T_{11}^1}^* Y = {T_{11}^3}^*$ respectively. Set $V = (V_1, 0), W = (W_1, 0) \in L_{\mathcal{A}}(\mathscr{S}_2 \oplus \mathscr{S}^{\perp}, \mathscr{S}_1)$. We have $T_{11}V = (T_{11}^2, 0)$, ${T_{11}^1}^* W = ({T_{11}^3}^*, 0) = \begin{pmatrix} T_{11}^3 \\ 0 \end{pmatrix}^*$. Therefore, PTP is $\mathscr{S}_1$-complementable.

Conversely, if PTP is $\mathscr{S}_1$-complementable, there are operators $V, W \in L_{\mathcal{A}}(\mathscr{S}_2 \oplus \mathscr{S}^{\perp}, \mathscr{S}_1)$ such that $T_{11}^1 V = (T_{11}^2, 0)$, ${T_{11}^1}^* W = ({T_{11}^3}^*, 0)$. It is easy to know that $VJ_{\mathscr{S}_2}, WJ_{\mathscr{S}_2} \in L_{\mathcal{A}}(\mathscr{S}_2, \mathscr{S}_1)$, where $J_{\mathscr{S}_2} : \mathscr{S}_2 \to \mathscr{S}_2 \oplus \mathscr{S}^{\perp}$ is the including map. Since $T_{11}^1 VJ_{\mathscr{S}_2} = T_{11}^2$, ${T_{11}^1}^* WJ_{\mathscr{S}_2} = {T_{11}^3}^*$, by Theorem 3.1 we know that T_{11} is $\mathscr{S}_1$-complementable.

Set $V_1 = VJ_{\mathscr{S}_2}, W_1 = WJ_{\mathscr{S}_2}$. It is easy to know that $(V_1, 0)$ and $(W_1, 0) \in L_{\mathcal{A}}(\mathscr{S}_2 \oplus \mathscr{S}^{\perp}, \mathscr{S}_1)$ are solutions to $T_{11}^1 X = (T_{11}^2, 0)$ and ${T_{11}^1}^* Y = ({T_{11}^3}^*, 0)$ respectively. So

$$(PTP)_{/\mathscr{S}_1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & T_{11}^4 - W_1^* T_{11}^1 V_1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} T_{11/\mathscr{S}_1} & 0 \\ 0 & 0 \end{pmatrix}.$$

Combining Lemma 5.2 and Lemma 5.3, we get the following proposition.

Proposition 5.1 *Let $T \in L_{\mathcal{A}}(E)$, $\mathscr{S} \subseteq E$ and $\mathscr{S}_1 \subseteq \mathscr{S}$ be orthogonally complemented closed submodules of Hilbert C^*-module E, and P be the projection of E onto $\mathscr{S}$. If T is $\mathscr{S}$-complementable and PTP is $\mathscr{S}_1$-complementable, then T is $\mathscr{S}_1$-complementable, and there exist invertible operators $U_1, U_2 \in L_{\mathcal{A}}(E)$ such that*

$$U_2 T_{/\mathscr{S}_1} U_1 = T_{/\mathscr{S}} + (PTP)_{/\mathscr{S}_1}.$$

Corollary 5.1 *Let $T \in L_{\mathcal{A}}(E)$, $\mathscr{S} \subseteq E$ and $\mathscr{S}_1 \subseteq \mathscr{S}$ be orthogonally complemented closed submodules of Hilbert C^*-module E, and P be the projection of E onto $\mathscr{S}$. If T is $\mathscr{S}^*$-complementable and PTP is $\mathscr{S}_1^*$-complementable, then T is $\mathscr{S}_1^*$-complementable, and there exists an invertible operator $U \in L_{\mathcal{A}}(E)$ such that*

$$U^* T_{/\mathscr{S}_1} U = T_{/\mathscr{S}} + (PTP)_{/\mathscr{S}_1}.$$

Proof The proof is the same as that of Proposition 5.1. We note that in the proof of Lemma 5.3 we could have $W_1 = V_1$ and $W = V$ in the case of $\mathscr{S}_1^*$-complementability. Then $T_{11} \in L_{\mathcal{A}}(\mathscr{S})$ is $\mathscr{S}_1^*$-complementable if and only if PTP is $\mathscr{S}_1^*$-complementable.

Theorem 5.1 *Let $T \in L_{\mathcal{A}}(E)$, $\mathscr{S} \subseteq E$ and $\mathscr{S}_1 \subseteq \mathscr{S}$ be orthogonally complemented closed submodules of Hilbert C^*-module E, and P be the projection of E onto $\mathscr{S}$. If T is $\mathscr{S}$-complementable and PTP is $\mathscr{S}_1$-complementable, then T is $\mathscr{S}_1$ -complementable, $T_{/\mathscr{S}_1}$ is $\mathscr{S} \ominus \mathscr{S}_1$-complementable, and*

$$(T_{/\mathscr{S}_1})_{/\mathscr{S} \ominus \mathscr{S}_1} = T_{/\mathscr{S}}.$$

As a consequence, $(T_{/\mathscr{S}})_{/\mathscr{S}^{\perp}} = 0$.

Proof From Lemma 5.2, it could be seen that T is $\mathscr{S}_1$-complementable. As in the proof of Lemma 5.2, we set $\mathscr{S}_2 = \mathscr{S} \ominus \mathscr{S}_1$, and

$$U_1 = \begin{pmatrix} I & 0 & 0 \\ 0 & I & -P_2' V_1 \\ 0 & 0 & I \end{pmatrix}, \quad U_2 = \begin{pmatrix} I & 0 & 0 \\ 0 & I & \\ 0 & -W_1^* J_2 & I \end{pmatrix}$$

under the orthogonal decomposition $P_1 + P_2 + (I - P) = I$. So under the orthogonal decomposition $P_2 + P_1 + (I - P) = I$, we obtain the matrix representations

$$U_1 = \begin{pmatrix} I & 0 & -P_2' V_1 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}, \quad U_2 = \begin{pmatrix} I & 0 & 0 \\ 0 & I & \\ -W_1^* J_2 & 0 & I \end{pmatrix}.$$

For the sake of clarity, all matrix representations appeared next are under the orthogonal decomposition $P_2 + P_1 + (I - P) = I$.

It is easy to know ${U_1}^{-1} = \begin{pmatrix} I & 0 & P_2' V_1 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}$, ${U_2}^{-1} = \begin{pmatrix} I & 0 & 0 \\ 0 & I & \\ W_1^* J_2 & 0 & I \end{pmatrix}$. By Proposition 5.1, we have

$$\begin{aligned} T_{/\mathscr{S}_1} &= {U_2}^{-1} U_2 T_{/\mathscr{S}_1} U_1 {U_1}^{-1} = {U_2}^{-1}(T_{/\mathscr{S}} + (PTP)_{/\mathscr{S}_1}){U_1}^{-1} \\ &= {U_2}^{-1} \begin{pmatrix} (T_{11/\mathscr{S}_1})_{22} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & (T_{/\mathscr{S}})_{33} \end{pmatrix} {U_1}^{-1}, \end{aligned}$$

where $(PTP)_{/\mathscr{S}_1} = \begin{pmatrix} (T_{11/\mathscr{S}_1})_{22} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $T_{/\mathscr{S}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & (T_{/\mathscr{S}})_{33} \end{pmatrix}$.

Obviously, $\begin{pmatrix} (T_{11/\mathscr{S}_1})_{22} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & (T_{/\mathscr{S}})_{33} \end{pmatrix}$ is $\mathscr{S}_2$-complementable. By Lemma 5.1, we know that $T_{/\mathscr{S}_1}$ is $\mathscr{S}_2$-complementable, and

$$(T_{/\mathscr{S}_1})_{\mathscr{S}_2} = \begin{pmatrix} (T_{11/\mathscr{S}_1})_{22} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & (T_{/\mathscr{S}})_{33} \end{pmatrix}_{/\mathscr{S}_2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & (T_{/\mathscr{S}})_{33} \end{pmatrix} = T_{/\mathscr{S}}.$$

Proposition 5.2 *Let $T \in L_{\mathcal{A}}(E)$, $\mathscr{S} \subseteq E$ and $\mathscr{S}_1 \subseteq \mathscr{S}$ be orthogonally complemented closed submodules of Hilbert C^*-module E, and P be the projection of E onto $\mathscr{S}$. If T is $\mathscr{S}^*$-complementable and PTP is $\mathscr{S}_1^*$-complementable, then T is $\mathscr{S}_1^*$-complementable, $T_{/\mathscr{S}_1}$ is $(\mathscr{S} \ominus \mathscr{S}_1)^*$-complementable, and*

$$(T_{/\mathscr{S}_1})_{/\mathscr{S} \ominus \mathscr{S}_1} = T_{/\mathscr{S}}.$$

As a consequence, $(T_{/\mathscr{S}})_{/\mathscr{S}^\perp} = 0$.

Proof The proof is the same as that of Theorem 5.1. We note that in the proof of Theorem 5.1 we could have $W_1 = V_1$ and $U_1 = U_2^*$ in the case of $\mathscr{S}^*$-complementability and $\mathscr{S}_1^*$-complementability.

References

[1] Anderson, W. N. and Trapp, G. E., Shorted operators II, *SIAM J. Appl. Math.*, **28**, 1975, 60–71.

[2] Ando, T., Generalized Schur complements, *Linear Algebra Appl.*, **27**, 1979, 173–186.

[3] Andruchow, E., Corach, G. and Stojanoff, D., Geometry of oblique projections, *Studia Math.*, **137**, 1999, 61–79.

[4] Brezinski, C. and Redivo-Zaglia, M., A Schur complement approach to a general extrapolation algorithm, *Linear Algebra Appl.*, **368**, 2003, 279–301.

[5] Carlson, D., What are Schur complements, anyway? *Linear Algebra Appl.*, **74**, 1986, 257–275.

[6] Choi, M. D. and Davis, C., The spectral mapping theorem for joint approximate point spectrum, *Bull. Amer. Math. Soc.*, **80**, 1974, 317–321.

[7] Choi, M. D. and Li, C. K., The ultimate estimate of the upper norm bound for the summation of operators, *J. Funct. Anal.*, **232**, 2006, 455–476.

[8] Corach, G., Maestripieri, A. and Stojanoff, D., Oblique projections and abstract splines, *J. Approx. Theory*, **117**, 2002, 189–206.

[9] Corach, G., Maestripieri, A. and Stojanoff, D., Oblique projections and Schur complements, *Acta Sci. Math.* (*Szeged*), **67**, 2001, 439–459.

[10] Corach, G., Maestripieri, A. and Stojanoff, D., Generalized Schur complements and oblique projections, *Linear Algebra Appl.*, **341**, 2002, 259–272.

[11] Crabtree, D. E. and Haynsworth, E. V., An identity for the Schur complement of a matrix, *Proc. Amer. Math. Soc.*, **22**, 1969, 364–366.

[12] Fang, X. C., The represention of topological groupoid, *Acta Math. Sinica*, **39**, 1996, 6–15.

[13] Fang, X. C., The induced representation of C*-groupoid dynamical system, *Chin. Ann. Math.*, **17B**(1), 1996, 103–114.

[14] Fang, X. C., The realization of multiplier Hilbert bimodule on bidule space and Tietze extension theorem, *Chin. Ann. Math.*, **21B**(3), 2000, 375–380.

[15] Fang, X. C., Yu, J. and Yao, H., Solutions to operator equations on Hilbert C^*-modules, *Linear Algebra Appl.*, **431**, 2009, 2142–2153.

[16] Giribet, J. I., Maestripieri, A. and Peria, F. M., Shorting selfadjoint operators in Hilbert spaces, *Linear Algebra Appl.*, **428**, 2008, 1899–1911.

[17] Hassi, S. and Nordstrom, K., On projections in a space with an indefinite metric, *Linear Algebra Appl.*, **208–209**, 1994, 401–417.

[18] Haynsworth, E., Determination of the inertia of a partitioned Hermitian matrix, *Linear Algebra Appl.*, **1**, 1968, 73–81.

[19] Jensen, K. K. and Thomsen, K., Elements of KK-theory, Birkhauser, Boston, 1991.

[20] Karaev, M. T., Berezin symbol and invertibility of operators on the functional Hilbert spaces, *J. Funct. Anal.*, **238**, 2006, 181–192.

[21] Khatri, C. G. and Mitra, S. K., Hermitian and nonnegative definite solutions of linear matrix equations, *SIAM J. Appl. Math.*, **31**, 1976, 579–585.

[22] Krein, M. G., The theory of self-adjoint extensions of semibounded Hermitian operators and its applications, *Mat. Sb.* (*N. S.*), **20**(62), 1947, 431–495.

[23] Lance, E. C., Hilbert C^*-modules: A toolkit for operator algebraists, Cambridge University Press, Cambridge, 1995.

[24] Lauzon, M. M. and Treil, S., Common complements of two subspaces of a Hilbert space, *J. Funct. Anal.*, **212**, 2004, 500–512.

[25] Magajna, M., Hilbert C^*-modules in which all closed submodules are complemented, *Proc. Amer. Math. Soc.*, **125**, 1997, 849–852.

[26] Ostrowski, A., A new proof of Haynsworths quotient formula for Schur complements, *Linear Algebra Appl.*, **4**, 1971, 389–392.

[27] Pasternak-Winiarski, Z., On the dependence of the orthogonal projector on deformations of the scalar product, *Studia Math.*, **128**, 1998, 1–17.

[28] Pekarev, E. L., Shorts of operators and some extremal problems, *Acta Sci. Math.* (*Szeged*), **56**, 1992, 147–163.

[29] Ptak, V., Extremal operators and oblique projections, *Casopis pro pestovani Matematiky*, **110**, 1985, 343–350.

[30] Schweizer, J., A description of Hilbert C^*-modules in which all closed submodules are orthogonally closed, *Proc. Amer. Math. Soc.*, **127**, 1999, 2123–2125.

[31] Wegge-Olsen, N. E., K-Theory and C^*-Algebras: A Friendly Approach, Oxford University Press, Oxford, 1993.