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Qualitative Analysis on a Reaction-Diffusion Prey-Predator Model and the Corresponding Steady-States**

Qunyi BIE* Rui PENG*

Abstract The authors study a diffusive prey-predator model subject to the homogeneous Neumann boundary condition and give some qualitative descriptions of solutions to this reaction-diffusion system and its corresponding steady-state problem. The local and global stability of the positive constant steady-state are discussed, and then some results for non-existence of positive non-constant steady-states are derived.

Keywords Prey-predator model, Steady-state, Global stability, Non-existence
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1 Introduction

In this paper, we study the following diffusive prey-predator system:

$$\begin{cases} \dfrac{\partial u}{\partial t} - d_1\Delta u = u(a-u-bv), & \text{in } \Omega\times(0,\infty),\\ \dfrac{\partial v}{\partial t} - d_2\Delta v = v\Big(c-\dfrac{v}{m+u}\Big), & \text{in } \Omega\times(0,\infty),\\ \partial_\nu u = \partial_\nu v = 0, & \text{on } \partial\Omega\times(0,\infty),\\ u(x,0)=u_0(x)\ge 0, \not\equiv 0, \quad v(x,0)=v_0(x)\ge 0,\not\equiv 0, & \text{on } \overline{\Omega}, \end{cases} \tag{1.1}$$

where $u(x,t)$ and $v(x,t)$ respectively represent the species densities of the prey and predator. d_i $(i=1,2)$ is the diffusion coefficient corresponding to u and v. Here, $\Omega\subset\mathbb{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$, and ν is the outward unit normal vector on $\partial\Omega$ and $\partial_\nu=\frac{\partial}{\partial\nu}$. The admissible initial data $u_0(x)$ and $v_0(x)$ are continuous functions on $\overline{\Omega}$ and all the parameters appearing in model (1.1) are assumed to be positive constants. The homogeneous Neumann boundary condition means that (1.1) is self-contained and has no population flux across the boundary $\partial\Omega$. For the more detailed biological implication for the model, one may further refer to [1–3, 5, 8, 11, 14, 15, 17], etc.

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*Department of Mathematics, College of Science, China Three Gorges University, Yichang 443002, Hubei, China. E-mail: qybie@126.com pengrui_seu@163.com
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System (1.1) is based on the following prey-predator model

$$
\begin{cases}
\dfrac{\partial u}{\partial t} - d_1 \Delta u = u\Big(a_1 - b_1 u - \dfrac{c_1 v}{1 + m_1 u}\Big), & \text{in } \Omega \times (0,\infty),\\
\dfrac{\partial v}{\partial t} - d_2 \Delta v = v\Big(a_2 - \dfrac{c_2 v}{m_2 + u}\Big), & \text{in } \Omega \times (0,\infty),\\
\partial_\nu u = \partial_\nu v = 0, & \text{on } \partial\Omega \times (0,\infty),\\
u(x,0) = u_0(x) \ge 0, \not\equiv 0, \quad v(x,0) = v_0(x) \ge 0, \not\equiv 0, & \text{on } \overline{\Omega},
\end{cases}
\tag{1.2}
$$

where a_1, a_2, b_1, c_1, c_2 are positive constants, and m_1, m_2 are non-negative constants.

When $m_1 = m_2 = 0$, in [2, 3], the authors studied model (1.2). They paid more attention to the steady-state problem of (1.2) in heterogeneous environment, and observed some quite interesting phenomena of pattern formation.

If $m_1 > 0, m_2 = 0$, the functional response is of Holling-Tanner type. In [11, 12], the authors analyzed the global stability of the unique positive constant steady-state and established some results for the existence and non-existence of positive non-constant steady-states.

In this paper, we investigate the case $m_1 = 0$, $m_2 > 0$. Under the scaling

$$
u \mapsto b_1 u, \quad v \mapsto \frac{1}{b_1 c_2} v,
$$

we obtain the form of system (1.1), where $a = a_1$, $b = c_1$, $c = a_2$, $m = b_1 m_2$.

First of all, we note that (1.1) has three trivial non-negative constant steady states, namely, $E_0 = (0,0)$, $E_1 = (a,0)$ and $E_2 = (0, cm)$. A simple analysis shows that model (1.1) has the only positive constant steady-state solution if and only if $bc < \frac{a}{m}$. We denote this steady state by (u^*, v^*), where

$$
u^* = \frac{a - bcm}{1 + bc} \quad \text{and} \quad v^* = \frac{c(a+m)}{1+bc}.
$$

Another aspect of our goal is to investigate the corresponding steady-state problem of the reaction-diffusion system (1.1), which may display the dynamical behavior of solutions to (1.1) as time goes to infinity. This steady-state problem satisfies

$$
\begin{cases}
-d_1 \Delta u = u(a - u - bv), & \text{in } \Omega,\\
\partial_\nu u = 0, & \text{on } \partial\Omega,\\
-d_2 \Delta v = v\Big(c - \dfrac{v}{m+u}\Big), & \text{in } \Omega,\\
\partial_\nu v = 0, & \text{on } \partial\Omega.
\end{cases}
\tag{1.3}
$$

It is clear that only non-negative solutions of (1.3) are of realistic interest. For this system, we will establish some a priori estimates for positive solutions. Based on these, using two different mathematical techniques, we will discuss the non-existence of positive non-constant solutions as the diffusion coefficient d_1 or d_2 is sufficiently large. Some of our mathematical techniques are different from those in [11, 12]. For example, to obtain the improved global stability of (u^*, v^*), we shall use the iteration argument. Moreover, in the course of the proofs of the main results, the details of our analysis are more involved.

The remaining content in our paper is organized as follows. In Section 2, we mainly analyze the local and global stability of (u^*, v^*) for (1.1). Then, in Section 3, we give a priori estimates of upper and lower bounds for positive solutions of (1.3), and finally in Section 4 we derive some non-existence results of positive non-constant solutions of (1.3).

2 Some Properties of Solutions to (1.1) and Stability of (u^*, v^*)

In this section, we are mainly concerned with some simple properties of solutions to (1.1) and the global stability of (u^*, v^*) for system (1.1). Throughout this section, let $(u(x,t), v(x,t))$ be the unique solution of (1.1). It is easily seen that $(u(x,t), v(x,t))$ exists globally and is positive, namely, $u(x,t), v(x,t) > 0$ for all $x \in \overline{\Omega}$ and $t > 0$.

2.1 Some simple properties of the solutions to (1.1)

Lemma 2.1 *For $0 < \varepsilon \ll 1$, there exists a $t_0 \gg 1$, such that the non-negative solution $(u(x,t), v(x,t))$ of* (1.1) *satisfies*

$$u(x,t) < a + \varepsilon, \quad cm - \varepsilon < v(x,t) < c(a+m) + \varepsilon \tag{2.1}$$

for all $x \in \overline{\Omega}$ and $t \geq t_0$.

Proof For $0 < \varepsilon \ll 1$, from system (1.1), it follows that there exists a $t_0 \gg 1$, such that $u(x,t) < a + \varepsilon$ and $v(x,t) > cm - \varepsilon$ for all $x \in \overline{\Omega}$ and $t \geq t_0$, by the comparison principle for the parabolic equation. Hence, $v(x,t)$ is a lower solution of the following problem:

$$\begin{cases} \dfrac{\partial z}{\partial t} - d_2 \Delta z = \dfrac{cm + c(a+\varepsilon) - z}{m + (a+\varepsilon)} z, & \text{in } \Omega \times (t_0, \infty), \\ \partial_\nu z = 0, & \text{on } \partial\Omega \times (t_0, \infty), \\ z(x, t_0) = v(x, t_0) > 0, & \text{on } \overline{\Omega}. \end{cases} \tag{2.2}$$

Let $v(t)$ be the unique positive solution of the problem

$$\begin{cases} w_t = \dfrac{cm + c(a+\varepsilon) - w}{m + (a+\varepsilon)} w, & \text{in } (t_0, \infty), \\ w(t_0) = \max\limits_{\overline{\Omega}} v(x, t_0) > 0. \end{cases}$$

Then $v(t)$ is an upper solution of (2.2). As $\lim\limits_{t\to\infty} v(t) = c(a+m) + c\varepsilon$, taking larger t_0 if necessary, from the comparison principle, we can get

$$v(x,t) < v(t) + \varepsilon < c(a+m) + (c+1)\varepsilon \quad \text{for all } x \in \overline{\Omega},\ t \geq t_0.$$

The proof is complete.

Theorem 2.1 *Let $(u(x,t), v(x,t))$ be the solution to* (1.1).

(i) *Assume $bc \geq \frac{a}{m}$. Then*

$$(u(x,t), v(x,t)) \to (0, cm), \quad \textit{uniformly on } \overline{\Omega} \textit{ as } t \to \infty. \tag{2.3}$$

(ii) *Assume $bc < \frac{a}{m}$. Then, for $0 < \varepsilon \ll 1$, there exists a $t_0 \gg 1$, such that the solution $(u(x,t), v(x,t))$ of* (1.1) *satisfies*

$$u(x,t) < K + \varepsilon, \quad v(x,t) < c(m+K) + \varepsilon \tag{2.4}$$

for all $x \in \overline{\Omega}$ *and* $t \geq t_0$*, where* $K = a - bcm$*.*

(iii) *Assume* $bc < \min\{\frac{a}{m}, 1\}$*. Then, for* $0 < \varepsilon \ll 1$*, there exists a* $t_0 \gg 1$*, such that*

$$u(x,t) > L - \varepsilon, \quad v(x,t) > c(m+L) - \varepsilon \tag{2.5}$$

for all $x \in \overline{\Omega}$ *and* $t \geq t_0$*, where* $L = a - bc(m+K)$*.*

Proof The idea for our proof comes from [9]. We only prove (2.3) and the first inequality of (2.4). The rest of our conclusions can be established in a similar manner as that of Lemma 2.1.

For $0 < \varepsilon \ll 1$, by Lemma 2.1 there exists a $t_0 \gg 1$, such that $v(x,t) > cm - \varepsilon$ for all $x \in \overline{\Omega}$ and $t \geq t_0$. Hence, $u(x,t)$ is a lower solution of the following problem

$$\begin{cases} \dfrac{\partial z}{\partial t} - d_1 \Delta z = (a - bcm + b\varepsilon - z)z, & \text{in } \Omega \times (t_0, \infty), \\ \partial_\nu z = 0, & \text{on } \partial\Omega \times (t_0, \infty), \\ z(x, t_0) = u(x, t_0) > 0, & \text{on } \overline{\Omega}. \end{cases} \tag{2.6}$$

If $bc \geq \frac{a}{m}$, from (2.6), the simple comparison argument shows that

$$0 < u(x,t) < \varepsilon \quad \text{uniformly on } \overline{\Omega} \text{ as } t \to \infty.$$

As a result, using the second equation in (1.1), one easily knows that

$$v(x,t) \to cm \quad \text{uniformly on } \overline{\Omega} \text{ as } t \to \infty.$$

The proof of (2.3) is complete.

If $bc < \frac{a}{m}$, let $u(t)$ be the unique positive solution of the problem

$$\begin{cases} w_t = (a - bcm + b\varepsilon - w)w, & \text{in } (t_0, \infty), \\ w(t_0) = \max\limits_{\overline{\Omega}} u(x, t_0) > 0. \end{cases}$$

Then $u(t)$ is an upper solution of (2.6). As $\lim\limits_{t\to\infty} u(t) = (a - bcm) + b\varepsilon$, taking larger t_0 if necessary, we can get from the comparison principle that

$$u(x,t) < u(t) + \varepsilon < (a - bcm) + (b+1)\varepsilon,$$

for all $x \in \overline{\Omega}$ and $t \geq t_0$. Thus, the proof is complete.

Remark 2.1 Theorem 2.1 shows that for any small $\varepsilon > 0$, the rectangle $[0, K+\varepsilon) \times (cm - \varepsilon, c(a+m)+\varepsilon)$ is a global attractor of system (1.1) in $\mathbb{R}^2_+$. If $bc < \min\{\frac{a}{m}, 1\}$ holds, the solution of system (1.1) has the persistence property. Furthermore, from Theorem 2.3, under this condition, the solution (u^*, v^*) of system (1.1) is globally asymptotically stable in $\mathbb{R}^2_+$.

2.2 Local stability of (u^*, v^*) to system (1.1)

From Theorem 2.1(i), we see that if $bc \geq \frac{a}{m}$, then $(0, cm)$ is the unique non-negative solution of (1.3). Then, from now on, without special statement, we always assume that $bc < \frac{a}{m}$, which guarantees the existence of (u^*, v^*). In this subsection, we will analyze the local stability of (u^*, v^*) to (1.1). To this end, we first introduce some notations.

In the following, we always let $0 = \mu_0 < \mu_1 < \mu_2 < \cdots$ be the eigenvalues of the operator $-\Delta$ on Ω with the homogeneous Neumann boundary condition. Set

$$\mathbf{X} = \Big\{(u, v) \in [C^1(\overline{\Omega})]^2 \,\Big|\, \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \text{ on } \partial\Omega\Big\},$$

and consider the decomposition $\mathbf{X} = \bigoplus\limits_{j=0}^{\infty} \mathbf{X}_j$, where $\mathbf{X}_j$ is the eigenspace corresponding to μ_j.

Theorem 2.2 *The positive constant solution* (u^*, v^*) *to system* (1.1) *is uniformly asymptotically stable, provided that* $bc < \frac{a}{m}$ (*in the sense of* [4]).

Proof The proof is similar to that of [16, Theorem 2.1]. The linearization of (1.1) at (u^*, v^*) is

$$\frac{\partial}{\partial t}\begin{pmatrix} u \\ v \end{pmatrix} = \mathcal{L}\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} f_1(u - u^*, v - v^*) \\ f_2(u - u^*, v - v^*) \end{pmatrix},$$

where $f_i(z_1, z_2) = O(z_1^2 + z_2^2)$, $i = 1, 2$, and

$$\mathcal{L} = \begin{pmatrix} d_1\Delta - \dfrac{a - bcm}{1 + bc} & \dfrac{-b(a - bcm)}{1 + bc} \\ c^2 & d_2\Delta - c \end{pmatrix}.$$

For each j, $j = 0, 1, 2, \cdots$, X_j is invariant under the operator $\mathcal{L}$, and ξ is an eigenvalue of $\mathcal{L}$ on X_j if and only if ξ is an eigenvalue of the matrix

$$A_j = \begin{pmatrix} -d_1\mu_j - \dfrac{a - bcm}{1 + bc} & \dfrac{-b(a - bcm)}{1 + bc} \\ c^2 & -d_2\mu_j - c \end{pmatrix},$$

$$\det A_j = d_1 d_2 \mu_j^2 + \Big(d_1 c + \frac{d_2(a - bcm)}{1 + bc}\Big)\mu_j + c(a - bcm),$$

$$\operatorname{tr} A_j = -(d_1 + d_2)\mu_j - c - \frac{a - bcm}{1 + bc} \le -c - \frac{a - bcm}{1 + bc},$$

where $\det A_j$ and $\operatorname{tr} A_j$ are respectively the determinant and trace of A_j. It is easy to check that $\det A_j > 0$ and $\operatorname{tr} A_j < 0$. Therefore, the two eigenvalues ξ_j^+ and ξ_j^- have negative real parts. Note that $\xi_0^\pm < 0$. For any $j \ge 1$, the following hold:

(i) If $(\operatorname{tr} A_j)^2 - 4\det A_j \le 0$, then

$$\operatorname{Re}\xi_j^\pm = \frac{1}{2}\operatorname{tr} A_j \le \frac{1}{2}\Big(-c - \frac{a - bcm}{1 + bc}\Big) < 0;$$

(ii) If $(\operatorname{tr} A_j)^2 - 4\det A_j > 0$, then

$$\operatorname{Re}\xi_j^- = \frac{1}{2}\Big\{\operatorname{tr} A_j - \sqrt{(\operatorname{tr} A_j)^2 - 4\det A_j}\Big\} \le \frac{1}{2}\operatorname{tr} A_j \le \frac{1}{2}\Big(-c - \frac{a - bcm}{1 + bc}\Big) < 0,$$

$$\operatorname{Re}\xi_j^+ = \frac{1}{2}\Big\{\operatorname{tr} A_j + \sqrt{(\operatorname{tr} A_j)^2 - 4\det A_j}\Big\} = \frac{2\det A_j}{\operatorname{tr} A_j - \sqrt{(\operatorname{tr} A_j)^2 - 4\det A_j}} \le \frac{\det A_j}{\operatorname{tr} A_j} < -\delta$$

for some positive δ which is independent of j.

This shows that there exists a positive constant δ, which is independent of j, such that $\operatorname{Re}\xi_j^\pm < -\delta$, $\forall\, j$. Consequently, the spectrum of $\mathcal{L}$ lies in $\{\operatorname{Re}\xi < -\delta\}$ (since the spectrum of $\mathcal{L}$ consists of eigenvalues), and we conclude the proof.

2.3 Global stability of (u^*, v^*) to system (1.1)

This subsection is devoted to the global stability of (u^*, v^*) for system (1.1).

Proposition 2.1 *Assume that*

$$bc < \min\Big\{\frac{a}{m}, \frac{4}{a^2}[2m^2 + 2ma + (2m+a)\sqrt{m(m+a)}\,]\Big\}. \tag{2.7}$$

Then (u^*, v^*) is globally asymptotically stable.

Proof In order to give the proof, we need to construct a Lyapunov function. First, we define

$$E(u)(t) = \int_\Omega \Big\{u(x,t) - u^* - u^* \ln \frac{u(x,t)}{u^*}\Big\}\mathrm{d}x,$$
$$E(v)(t) = \int_\Omega \Big\{v(x,t) - v^* - v^* \ln \frac{v(x,t)}{v^*}\Big\}\mathrm{d}x.$$

We note that $E(u)(t)$ and $E(v)(t)$ are non-negative, $E(u)(t) = 0$ and $E(v)(t) = 0$ if and only if $(u(x,t), v(x,t)) = (u^*, v^*)$. Furthermore, easy computations yield

$$\begin{aligned}\frac{\mathrm{d}E(u)}{\mathrm{d}t} &= \int_\Omega \Big(1 - \frac{u^*}{u}\Big)u_t \mathrm{d}x = \int_\Omega \Big\{-d_1 \frac{u^*|\nabla u|^2}{u^2} + (u-u^*)(a-u-bv)\Big\}\mathrm{d}x \\ &= \int_\Omega \Big\{-d_1 \frac{u^*|\nabla u|^2}{u^2} + (u-u^*)(u^*+bv^*-u-bv)\Big\}\mathrm{d}x \\ &= \int_\Omega \Big\{-d_1 \frac{u^*|\nabla u|^2}{u^2} - (u-u^*)^2 - b(u-u^*)(v-v^*)\Big\}\mathrm{d}x.\end{aligned}$$

Similarly,

$$\begin{aligned}\frac{\mathrm{d}E(v)}{\mathrm{d}t} &= \int_\Omega \Big(1 - \frac{v^*}{v}\Big)v_t \mathrm{d}x = \int_\Omega \Big\{-d_2 \frac{v^*|\nabla v|^2}{v^2} + (v-v^*)\Big(c - \frac{v}{m+u}\Big)\Big\}\mathrm{d}x \\ &= \int_\Omega \Big\{-d_2 \frac{v^*|\nabla v|^2}{v^2} + (v-v^*)\Big(\frac{v^*}{m+u^*} - \frac{v}{m+u}\Big)\Big\}\mathrm{d}x \\ &= \int_\Omega \Big\{-d_2 \frac{v^*|\nabla v|^2}{v^2} - \frac{1}{m+u}(v-v^*)^2 + \frac{v^*}{(m+u)(m+u^*)}(u-u^*)(v-v^*)\Big\}\mathrm{d}x.\end{aligned}$$

Now define

$$E(t) = E(u)(t) + \lambda E(v)(t),$$

where the constant λ satisfies $\lambda > 0$ and will be determined later. Set $\xi = u - u^*$, $\eta = v - v^*$. We have

$$\begin{aligned}\frac{\mathrm{d}E(t)}{\mathrm{d}t} &= \frac{\mathrm{d}E(u)(t)}{\mathrm{d}t} + \lambda \frac{\mathrm{d}E(v)(t)}{\mathrm{d}t} \\ &= \int_\Omega \Big\{-d_1 \frac{u^*|\nabla u|^2}{u^2} - \xi^2 - b\xi\eta - d_2\lambda \frac{v^*|\nabla v|^2}{v^2} - \frac{\lambda}{m+u}\eta^2 + \frac{\lambda c}{m+u}\xi\eta\Big\}\mathrm{d}x \\ &\le \int_\Omega \Big\{-\xi^2 + \Big(\frac{\lambda c}{m+u} - b\Big)\xi\eta - \frac{\lambda}{m+u}\eta^2\Big\}\mathrm{d}x.\end{aligned} \tag{2.8}$$

If the inequality

$$\Big(\frac{\lambda c}{m+u} - b\Big)^2 - \frac{4\lambda}{m+u} < 0 \tag{2.9}$$

holds, from (2.8), it is easy to see that

$$-\xi^2 + \Big(\frac{\lambda c}{m+u} - b\Big)\xi\eta - \frac{\lambda}{m+u}\eta^2$$

takes negative values unless $u = u^*$ and $v = v^*$.

Next, we will show that under some conditions, it is possible to choose a suitable $\lambda > 0$ such that (2.9) holds. To this end, we rewrite (2.9) as

$$\frac{c^2}{(m+u)^2}\lambda^2 - \frac{2(bc+2)}{m+u}\lambda + b^2 < 0. \tag{2.10}$$

We find that (2.10) holds if and only if $\lambda \in (\lambda^-, \lambda^+)$, where

$$\lambda^- = \lambda^-(u) = \frac{m+u}{c^2}(bc + 2 - 2\sqrt{1+bc}\,),$$
$$\lambda^+ = \lambda^+(u) = \frac{m+u}{c^2}(bc + 2 + 2\sqrt{1+bc}\,).$$

In order to find a fixed constant $\lambda > 0$ such that $\lambda \in (\lambda^-, \lambda^+)$ holds, it suffices to require $\lambda^-(a) < \lambda^+(0)$, that is

$$\frac{m+a}{c^2}(bc + 2 - 2\sqrt{1+bc}\,) < \frac{m}{c^2}(bc + 2 + 2\sqrt{1+bc}\,),$$

which is equivalent to

$$a^2(bc)^2 - 16m(m+a)bc - 16m(m+a) < 0. \tag{2.11}$$

This holds if (2.7) is satisfied.

Furthermore, we can choose a small $\varepsilon > 0$, such that $\lambda^-(a+\varepsilon) < \lambda^+(0)$, and thus there exists a fixed constant $\lambda > 0$ satisfying $\lambda^-(a+\varepsilon) < \lambda < \lambda^+(0)$. Hence

$$\lambda^-(u) \le \lambda^-(a+\varepsilon) < \lambda < \lambda^+(0) \le \lambda^+(u), \quad \forall\, u \in [0, a+\varepsilon].$$

As a consequence, for any $u \in [0, a+\varepsilon]$, it follows that $\frac{\mathrm{d}E(t)}{\mathrm{d}t} \le 0$. Using Lemma 2.1, we can find a large $T > 0$, such that $u(x,t) \le a + \varepsilon$ for all $t > T$ and $x \in \overline{\Omega}$. Therefore, $\frac{\mathrm{d}E(t)}{\mathrm{d}t} \le 0$ for all $t > T$, and the equality holds if and only if $(u,v) = (u^*, v^*)$. Hence, the standard arguments together with Theorem 2.1(ii) and Theorem 2.2 deduce that (u^*, v^*) attracts all solutions of (1.1). The proof is complete.

In the following, we employ comparison argument and iteration technique to improve the above result.

Proposition 2.2 *Assume that* $bc < \min\{\frac{a}{m}, 1\}$. *Then* (u^*, v^*) *for system* (1.1) *is globally asymptotically stable in* $\mathbb{R}^2_+$.

Proof The proof is similar to that of Theorem 2.1. Let (u,v) be any solution of (1.1). By (2.5), for any $0 < \varepsilon \ll 1$, there exists a $t_0 \gg 1$, such that $u(x,t) > L - \varepsilon$, $v(x,t) > c(m+L) - \varepsilon$ for all $x \in \overline{\Omega}$ and $t \ge t_0$. Hence, $u(x,t)$ is a lower solution of the following problem:

$$\begin{cases} \dfrac{\partial z}{\partial t} - d_1 \Delta z = (a - z - bc(m+L) + b\varepsilon)z, & \text{in } \Omega \times (t_0, \infty), \\ \partial_\nu z = 0, & \text{on } \partial\Omega \times (t_0, \infty), \\ z(x, t_0) = u(x, t_0) > 0, & \text{on } \overline{\Omega}. \end{cases} \tag{2.12}$$

Let $u(t)$ be the solution of the problem

$$\begin{cases} w_t = (a - w - bc(m + L) + b\varepsilon)w, & \text{in } (t_0, \infty), \\ w(t_0) = \max_{\overline{\Omega}} u(x, t_0) > 0. \end{cases}$$

Then $u(t)$ is an upper solution of (2.12). As $\lim_{t\to\infty} u(t) = a - bc(m + L) + b\varepsilon$, we deduce that for $0 < \varepsilon \ll 1$, $x \in \overline{\Omega}$ and $t \geq t_0$,

$$u(x, t) < a - bc(m + L) + \varepsilon := K_1 + \varepsilon.$$

Hence, applying the equation for $v(x, t)$ as above, we have

$$v(x, t) < c(m + K_1) + \varepsilon.$$

As a result, for any $0 < \varepsilon \ll 1$, there exists a $t_0 > 0$, such that

$$u(x, t) > L - \varepsilon \quad \text{and} \quad v(x, t) < c(m + K_1) + \varepsilon$$

for all $x \in \overline{\Omega}$ and $t \geq t_0$. Hence, $u(x, t)$ is an upper solution of the following problem:

$$\begin{cases} \dfrac{\partial z}{\partial t} - d_1 \Delta z = (a - z - bc(m + K_1) - b\varepsilon)z, & \text{in } \Omega \times (t_0, \infty), \\ \partial_\nu z = 0, & \text{on } \partial\Omega \times (t_0, \infty), \\ z(x, t_0) = u(x, t_0) > 0, & \text{on } \overline{\Omega}. \end{cases} \tag{2.13}$$

Let $u(t)$ be the solution of the problem

$$\begin{cases} w_t = (a - w - bc(m + K_1) - b\varepsilon)w, & \text{in } (t_0, \infty), \\ w(t_0) = \min_{\overline{\Omega}} u(x, t_0) > 0. \end{cases}$$

Then $u(t)$ is a lower solution of (2.13). As $\lim_{t\to\infty} u(t) = a - bc(m+K_1) - b\varepsilon$, we get, for $0 < \varepsilon \ll 1$, $x \in \overline{\Omega}$ and $t \geq t_0$,

$$u(x, t) > a - bc(m + K_1) - \varepsilon := L_1 - \varepsilon,$$

and in turn

$$v(x, t) > c(m + L_1) - \varepsilon.$$

It is clear to see that $L < L_1 < K_1 < K$. Repeating the above arguments, inductively, for $i \geq 1$, we see that there exists an increasing sequence $\{L_i\}$ and a decreasing sequence $\{K_i\}$ satisfying

$$\begin{aligned} &L_i = a - bc(m + K_i), \quad K_{i+1} = a - bc(m + L_i), \\ &L < L_1 < \cdots < L_i < L_{i+1} < \cdots < K_{i+1} < K_i < \cdots < K_1 < K. \end{aligned}$$

Hence, we have

$$\lim_{t\to\infty} (L_i, K_i) = (\widetilde{L}, \widetilde{K}).$$

Moreover, $(\widetilde{L}, \widetilde{K})$ satisfies

$$\widetilde{L} = a - bc(m + \widetilde{K}), \quad \widetilde{K} = a - bc(m + \widetilde{L}),$$

since $bc < 1$. Furthermore, $\widetilde{L} - \widetilde{K} = bc(\widetilde{L} - \widetilde{K})$, so

$$\widetilde{L} = \widetilde{K} = \frac{a - bcm}{1 + bc} = u^*.$$

This shows that $u \to u^*$ uniformly on $\overline{\Omega}$ as $t \to \infty$. Owing to the comparison principle, we get $v \to v^*$ uniformly on $\overline{\Omega}$ as $t \to \infty$, which ends the proof.

From Propositions 2.1 and 2.2, we have the following result.

Theorem 2.3 *Assume that*

$$bc < \frac{a}{m} \quad \textit{and} \quad bc < \max\Big\{1, \frac{4}{a^2}[2m^2 + 2ma + (2m + a)\sqrt{m(m + a)}\,]\Big\}. \tag{2.14}$$

Then (u^*, v^*) *for system* (1.1) *is globally asymptotically stable in* $\mathbb{R}^2_+$.

Remark 2.2 From Theorem 2.3, a meticulous computation gives that (u^*, v^*) for system (1.1) is globally asymptotically stable, if one of the cases holds:

(1) $\frac{a}{m} \le 1$ and $bc < \frac{a}{m}$,

(2) $1 < \frac{a}{m} \le 16 + 12\sqrt{2}$ and $bc < \min\{\frac{a}{m}, \frac{4}{a^2}[2m^2 + 2ma + (2m + a)\sqrt{m(m + a)}\,]\}$,

(3) $\frac{a}{m} > 16 + 12\sqrt{2}$ and $bc < 1$.

3 A priori Estimates for Positive Solutions to (1.3)

From now on, our aim is to investigate the steady-state problem (1.3). In this section, we will deduce a priori estimates of positive upper and lower bounds for positive solutions of (1.3). In order to obtain the desired bounds, we need to use the following Harnack inequality due to [6].

Lemma 3.1 (Harnack Inequality) *Let* $w \in C^2(\Omega) \cap C^1(\overline{\Omega})$ *be a positive solution to* $\Delta w(x) + c(x)w(x) = 0$ *in* Ω *subject to the homogeneous Neumann boundary condition, where* $c(x) \in C(\overline{\Omega})$. *Then there exists a positive constant* $C^* = C^*(\|c\|_\infty, \Omega)$, *such that*

$$\max_{\overline{\Omega}} w \le C^* \min_{\overline{\Omega}} w.$$

Theorem 3.1 *Assume that* $bc \neq \frac{a}{m}$, *and let* d *be an arbitrary fixed positive number. Then there exists a positive constant* $\underline{C}$ *only depending on* a, b, c, m, d *and* Ω, *such that if* $d_1 \ge d$, *any positive solution* (u, v) *of* (1.3) *satisfies*

$$\underline{C} < u(x) < a, \quad cm < v(x) < c(a + m).$$

Proof Simple comparison argument shows $u(x) < a$ and $cm < v(x) < c(m + a)$. Now, it suffices to verify the lower bounds of $u(x)$. We shall prove by contradiction.

Suppose that Theorem 3.1 is not true. Then there exists a sequence $\{d_{1,i}\}_{i=1}^{\infty}$ with $d_{1,i} \geq d$ and the positive solution (u_i, v_i) of (1.3) corresponding to $d_1 = d_{1,i}$, such that

$$\min_{\overline{\Omega}} u_i(x) \to 0, \quad \text{as } i \to \infty. \tag{3.1}$$

By the Harnack inequality, we know that there is a positive constant C independent of i, such that $\max_{\overline{\Omega}} u_i(x) \leq C \min_{\overline{\Omega}} u_i(x)$. Consequently,

$$u_i(x) \to 0 \quad \text{uniformly on } \overline{\Omega} \text{ as } i \to \infty. \tag{3.2}$$

Let $w_i = \frac{u_i}{\|u_i\|_\infty}$ and (w_i, v_i) satisfy the following elliptic model:

$$\begin{cases} -d_{1,i}\Delta w_i = w_i(a - u_i - bv_i), & \text{in } \Omega, \\ \partial_\nu w_i = 0, & \text{on } \partial\Omega, \\ -d_2\Delta v_i = v_i\Big(c - \dfrac{v_i}{m + u_i}\Big), & \text{in } \Omega, \\ \partial_\nu v_i = 0, & \text{on } \partial\Omega. \end{cases} \tag{3.3}$$

Moreover, integrating over Ω by parts, we have

$$\int_\Omega w_i(a - u_i - bv_i)\mathrm{d}x = 0, \quad \int_\Omega v_i\Big(c - \frac{v_i}{m + u_i}\Big)\mathrm{d}x = 0. \tag{3.4}$$

The embedding theory and the standard regularity theory of elliptic equations guarantee that there is a subsequence of (w_i, v_i) also denoted by itself, and two non-negative functions $w, v \in C^2(\overline{\Omega})$, such that $(w_i, v_i) \to (w, v)$ in $[C^2(\overline{\Omega})]^2$ as $i \to \infty$. Since $\|w_i\|_\infty = 1$, we have $\|w\|_\infty = 1$. Since (w_i, v_i) satisfies (3.4), so does (w, v). It follows from the second integral identity of (3.4) that $v = cm$. In view of $bc \neq \frac{a}{m}$, the first integral identity of (3.4) yields $\int_\Omega w\mathrm{d}x = 0$, which implies a contradiction. The proof is complete.

4 Non-existence of Positive Non-constant Solutions to (1.3)

In this section, based on the a priori estimates in Section 3 for positive solutions to (1.3), we present some results for non-existence of positive non-constant solutions of (1.1) as the diffusion coefficient d_1 or d_2 is sufficiently large.

Note that μ_1 is the smallest positive eigenvalue of the operator $-\Delta$ in Ω subject to the homogeneous Neumann boundary condition. Now, using the energy estimates, we can claim

Theorem 4.1 (i) *There exists a positive constant $\widetilde{d}_1 = \widetilde{d}_1(a, b, c, m, \Omega)$, such that* (1.3) *has no non-constant positive solutions, provided that $\mu_1 d_1 > \widetilde{d}_1$;*

(ii) *There exists a positive constant $\widetilde{d}_2 = \widetilde{d}_2(a, b, c, m, \Omega)$, such that* (1.3) *has no non-constant positive solutions, provided that $\mu_1 d_2 > \widetilde{d}_2$ and $\mu_1 d_1 > a$.*

Proof Let (u, v) be any positive solution of (1.3) and denote

$$\overline{g} = \frac{1}{|\Omega|}\int_\Omega g\mathrm{d}x.$$

Then, multiplying the corresponding equation in (1.3) by $u-\overline{u}$ and $\frac{v-\overline{v}}{v}$ respectively, integrating over Ω, we obtain

$$\begin{aligned}
d_1\int_\Omega |\nabla u|^2 \mathrm{d}x &= \int_\Omega (au-u^2-buv)(u-\overline{u})\mathrm{d}x \\
&= \int_\Omega [a(u-\overline{u})-(u^2-\overline{u}^2)-(buv-b\overline{uv})](u-\overline{u})\mathrm{d}x \\
&= \int_\Omega [a-(u+\overline{u})-b\overline{v}](u-\overline{u})^2\mathrm{d}x - b\int_\Omega u(u-\overline{u})(v-\overline{v})\mathrm{d}x \\
&\le [a+C(\varepsilon,a,b,c,m,\Omega)]\int_\Omega (u-\overline{u})^2\mathrm{d}x+\varepsilon\int_\Omega (v-\overline{v})^2\mathrm{d}x, \\
d_2\int_\Omega \frac{m|\nabla v|^2}{c(a+m)^2}\mathrm{d}x \le d_2\int_\Omega \frac{\overline{v}|\nabla v|^2}{v^2}\mathrm{d}x &= \int_\Omega \Big(c-\frac{v}{m+u}+\frac{\overline{v}}{m+\overline{u}}\Big)(v-\overline{v})\mathrm{d}x \\
&= \int_\Omega -\frac{1}{m+u}(v-\overline{v})^2\mathrm{d}x + \int_\Omega \frac{\overline{v}}{(m+u)(m+\overline{u})}(u-\overline{u})(v-\overline{v})\mathrm{d}x \\
&\le \int_\Omega \Big(-\frac{1}{m+u}+\varepsilon\Big)(v-\overline{v})^2\mathrm{d}x + C(\varepsilon,a,b,c,m,\Omega)\int_\Omega (u-\overline{u})^2\mathrm{d}x.
\end{aligned}$$

Consequently, there exists a $0<\varepsilon\ll 1$, which depends only on a, b, c, m and Ω, such that

$$\int_\Omega \{d_1|\nabla(u-\overline{u})|^2+d_2|\nabla(v-\overline{v})|^2\}\mathrm{d}x \le C(a,b,c,m,\Omega)\int_\Omega (u-\overline{u})^2\mathrm{d}x. \tag{4.1}$$

Thanks to the well-known Poincaré inequality

$$\mu_1\int_\Omega (g-\overline{g})^2\mathrm{d}x \le \int_\Omega |\nabla(g-\overline{g})|^2\mathrm{d}x,$$

from (4.1), we have

$$\mu_1\int_\Omega \{d_1(u-\overline{u})^2+d_2(v-\overline{v})^2\}\mathrm{d}x \le C(a,b,c,m,\Omega)\int_\Omega (u-\overline{u})^2\mathrm{d}x.$$

It is clear that there exists a $\widetilde{d_1}$ depending only on a, b, c, m and Ω, such that when $d_1>\widetilde{d_1}$, $u\equiv\overline{u}$ =const., in turn, $v\equiv\overline{v}$ =const., which asserts our result (i).

As above, we have

$$\mu_1\int_\Omega \{d_1(u-\overline{u})^2+d_2(v-\overline{v})^2\}\mathrm{d}x \le (a+\varepsilon)\int_\Omega (u-\overline{u})^2\mathrm{d}x + C(\varepsilon,a,b,c,m,\Omega)\int_\Omega (v-\overline{v})^2\mathrm{d}x.$$

The remaining arguments are rather similar as above. The proof is complete.

Next, we will improve the result (ii) in Theorem 4.1 in some cases by applying the implicit function theorem. Our idea comes from [11]. For our purpose, we first have to state a lemma.

Lemma 4.1 *Fix d_1, a, b, c, m, and assume that $bc<\frac{a}{m}$ holds. Let (u_i,v_i) be the positive solution of* (1.3) *with $d_2=d_{2,i}$ and $d_{2,i}\to\infty$ as $i\to\infty$. Then $(u_i,v_i)\to(u^*,v^*)$ in $[C^2(\overline{\Omega})]^2$ as $i\to\infty$.*

Proof By Theorem 3.1, the embedding theory and the standard regularity theory of elliptic equations, there is a subsequence of (u_i,v_i) also labeled by itself, such that $(u_i,v_i)\to(u,v)$ in

$[C^2(\overline{\Omega})]^2$ as $i \to \infty$. Moreover, $v \equiv \delta$, where δ is a positive constant and $\delta \le c(a+m)$, $u > 0$ on $\overline{\Omega}$, and (u, δ) solves

$$\begin{cases} -d_1 \Delta u = u(a - u - b\delta), & \text{in } \Omega, \\ \partial_\nu u = 0, & \text{on } \partial\Omega, \\ \displaystyle\int_\Omega \Big(c - \frac{\delta}{m+u}\Big) \mathrm{d}x = 0. \end{cases} \tag{4.2}$$

Hence, together with [11, Lemma 3.2], from the first equation in (4.2), a simple analysis shows that u must be a positive constant, and so we see $(u, v) = (u^*, v^*)$ through the second equation in (4.2). This ends our proof.

Now, on the base of the above lemma, we can obtain the following result.

Theorem 4.2 *Assume that* $bc < \frac{a}{m}$ *and let* ε *be an arbitrary positive number. Then, there exists a large positive constant* $D_2 = D_2(\varepsilon, a, b, c, m, \Omega)$*, such that* (1.3) *has no positive non-constant solution when* $d_1 > \varepsilon$ *and* $d_2 > D_2$*.*

Proof By Theorem 4.1(ii), for a fixed large constant D_1 depending only on a, b, c, m and Ω, there exists a $D_2 = D_2(a, b, c, m, \Omega)$, such that (1.3) has no positive non-constant solution if $d_1 > D_1$, $d_2 > D_2$. Therefore, in the following, it suffices to consider the case of $d_1 \in [\varepsilon, D_1]$.

We make the decomposition

$$v = w + \xi, \quad \text{where } \int_\Omega w \mathrm{d}x = 0, \ \xi \in \mathbb{R}^+.$$

We observe that finding the positive solution of (1.3) is equivalent to solving the following problem:

$$\begin{cases} d_1 \Delta u + u(a-u) - bu(w+\xi) = 0, & \text{in } \Omega, \\ \partial_\nu u = 0, & \text{on } \partial\Omega, \\ \Delta w + \rho \mathbf{P}\Big\{(w+\xi)\Big(c - \dfrac{w+\xi}{m+u}\Big)\Big\} = 0, & \text{in } \Omega, \\ \partial_\nu w = 0, & \text{on } \partial\Omega, \\ \displaystyle\int_\Omega (w+\xi)\Big(c - \frac{w+\xi}{m+u}\Big) \mathrm{d}x = 0, \quad \xi > 0, \ u > 0, & \text{on } \overline{\Omega}, \end{cases} \tag{4.3}$$

where $\rho = d_2^{-1}$ and $\mathbf{P}z = z - \frac{1}{|\Omega|}\int_\Omega z \mathrm{d}x$, i.e., $\mathbf{P}$ is the projective operator from $L^2(\Omega)$ to $L^2_0(\Omega) = \{g \in L^2(\Omega) \mid \int_\Omega g \mathrm{d}x = 0\}$. Clearly, $(u, w, \xi) = (u^*, 0, v^*)$ is a solution of (4.3) for $\rho > 0$.

From the above analysis, to verify our assertion, by the finite covering argument, it is enough to prove that for any fixed $\widetilde{d}_1 \in [\varepsilon, D_1]$, there exists a small positive constant δ_0, such that if $\rho \in (0, \delta_0)$, $d_1 \in (\widetilde{d}_1 - \delta_0, \widetilde{d}_1 + \delta_0)$, then $(u^*, 0, v^*)$ is the unique solution of (4.3). Define

$$\begin{aligned} F(d_1, \rho, u, w, \xi) &= (f_1, f_2, f_3)(\rho, u, w, \xi), \\ f_1(d_1, \rho, u, w, \xi) &= d_1 \Delta u + u(a-u) - bu(w+\xi), \\ f_2(d_1, \rho, u, w, \xi) &= \Delta w + \rho \mathbf{P}\Big\{(w+\xi)\Big(c - \frac{w+\xi}{m+u}\Big)\Big\}, \\ f_3(d_1, \rho, u, w, \xi) &= \int_\Omega (w+\xi)\Big(c - \frac{w+\xi}{m+u}\Big) \mathrm{d}x. \end{aligned}$$

Then

$$F : \mathbb{R}^+ \times \mathbb{R}^+ \times W_\nu^{2,2} \times (L_0^2(\Omega) \cap W_\nu^{2,2}(\Omega)) \times \mathbb{R}^+ \to L^2(\Omega) \times L_0^2(\Omega) \times \mathbb{R},$$

where

$$W_\nu^{2,2}(\Omega) = \{g \in W^{2,2}(\Omega) \mid \partial_\nu g = 0 \text{ on } \partial\Omega\}.$$

Clearly, (4.3) is equivalent to solving $F(d_1, \rho, u, w, \xi) = 0$. Moreover, (4.3) has a unique solution $(u, w, \xi) = (u^*, 0, v^*)$ when $\rho = 0$. By simple computations, we have

$$\Phi \equiv D_{(u,w,\xi)} F(\widetilde{d}_1, 0, u^*, 0, v^*) : W_\nu^{2,2} \times (L_0^2(\Omega) \cap W_\nu^{2,2}(\Omega)) \times \mathbb{R}^+ \to L^2(\Omega) \times L_0^2(\Omega) \times \mathbb{R},$$

where

$$\Phi(y, z, \tau) = \begin{pmatrix} \widetilde{d}_1 \Delta y - \dfrac{a - bcm}{1 + bc} y - \dfrac{b(a - bcm)}{1 + bc}(z + \tau) \\ \Delta z \\ \displaystyle\int_\Omega \{c^2 y - c(z + \tau)\} \mathrm{d}x \end{pmatrix}.$$

In order to use the implicit function theorem, we have to verify that Φ is both invertible and surjective. In fact, assume that $\Phi(y, z, \tau) = (0, 0, 0)$, then $z \equiv 0$. Thus, $\tau \in \mathbb{R}$ implies that y must be a constant through the first equation in y. Thus, the integral equation in $\Phi(y, z, \tau) = (0, 0, 0)$ yields $\tau = cy$. On the other hand, note that $a - bcm > 0$ due to $bc < \frac{a}{m}$. Then, by the first equation in y again, it is easily verified that $y = \tau = 0$ and so Φ is invertible. Similarly, we also easily see that Φ is a surjection.

By the implicit function theorem, there exist positive constants ρ_0 and δ_0, such that for each $\rho \in [0, \rho_0]$ and $d_1 \in (\widetilde{d}_1 - \delta_0, \widetilde{d}_1 + \delta_0)$, $(u^*, 0, v^*)$ is the unique solution of $F(d_1, \rho, u, w, \xi) = 0$ in $B_{\delta_0}(u^*, 0, v^*)$, where $B_{\delta_0}(u^*, 0, v^*)$ is the ball in $W_\nu^{2,2}(\Omega) \times (L_0^2(\Omega) \cap W_\nu^{2,2}(\Omega)) \times \mathbb{R}$ centered at $(u^*, 0, v^*)$ with radius δ_0. Taking smaller ρ_0 and δ_0 if necessary, we can deduce the conclusion in Theorem 4.2 by use of Lemma 4.1.

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