Chin. Ann. Math.
29B(3), 2008, 223–238
DOI: 10.1007/s11401-007-0242-8

Chinese Annals of Mathematics, Series B

Stability of Multidimensional Phase Transitions in a Steady van der Waals Flow

Shuyi ZHANG*

Abstract In this paper, the author studies the multidimensional stability of subsonic phase transitions in a steady supersonic flow of van der Waals type. The viscosity capillarity criterion (in "Arch. Rat. Mech. Anal., **81**(4), 1983, 301–315") is used to seek physical admissible planar waves. By showing the Lopatinski determinant being non-zero, it is proved that subsonic phase transitions are uniformly stable in the sense of Majda (in "Mem. Amer. Math. Soc., **41**(275), 1983, 1–95") under both one dimensional and multidimensional perturbations.

Keywords Supersonic flows, Subsonic phase transitions, Euler equations, Multi-dimensional stability
2000 MR Subject Classification 35L45, 35L50

1 Introduction

The motion of an isothermal (or isentropic) 3-dimensional steady flow is governed by the following well-known Euler equations

$$\begin{cases}\partial_x(\rho u)+\partial_y(\rho v)+\partial_y(\rho w)=0,\\ \partial_x(\rho u^2+p(\rho))+\partial_y(\rho uv)+\partial_y(\rho uw)=0,\\ \partial_x(\rho uv)+\partial_y(\rho v^2+p(\rho))+\partial_y(\rho vw)=0,\\ \partial_x(\rho uw)+\partial_y(\rho vw)+\partial_y(\rho w^2+p(\rho))=0,\end{cases}\tag{1.1}$$

where ρ is the density of the flow, $(u,v,w)^T$ is the velocity of the flow and p is the pressure which is a function of ρ. Denote by $U=(\rho,u,v,w)^T$,

$$F_0(U)=\begin{pmatrix}\rho u\\ \rho u^2+p\\ \rho uv\\ \rho uw\end{pmatrix},\quad F_1(U)=\begin{pmatrix}\rho v\\ \rho uv\\ \rho v^2+p\\ \rho vw\end{pmatrix},\quad F_2(U)=\begin{pmatrix}\rho w\\ \rho uw\\ \rho vw\\ \rho w^2+p\end{pmatrix}$$

Manuscript received June 7, 2007. Published online April 23, 2008.
*Department of Mathematics and Physics, Shanghai Institute of Technology, Shanghai 200235, China. E-mail: zh_shuyi@yahoo.com.cn

and

$$
\begin{aligned}
A_1(U) &= (\nabla_U F_0(U))^{-1}\nabla_U F_1(U)\\
&= \frac{1}{\rho u(u^2-c^2)}\begin{pmatrix} \rho u^2 v & -\rho^2 uv & \rho^2 u^2 & 0\\ -uvc^2 & \rho u^2 v & -\rho uc^2 & 0\\ (u^2-c^2)c^2 & 0 & \rho v(u^2-c^2) & 0\\ 0 & 0 & 0 & \rho v(u^2-c^2)\end{pmatrix},\\
A_2(U) &= (\nabla_U F_0(U))^{-1}\nabla_U F_2(U)\\
&= \frac{1}{\rho u(u^2-c^2)}\begin{pmatrix} \rho u^2 w & -\rho^2 uw & 0 & \rho^2 u^2\\ -uwc^2 & \rho u^2 w & 0 & -\rho uc^2\\ 0 & 0 & \rho w(u^2-c^2) & 0\\ (u^2-c^2)c^2 & 0 & 0 & \rho w(u^2-c^2)\end{pmatrix},
\end{aligned}
$$

where $c^2 = d_\rho p(\rho)$ is the sound speed. The Euler equations (1.1) can be rewritten in the following abstract form

$$\partial_x F_0(U) + \partial_y F_1(U) + \partial_z F_2(U) = 0 \tag{1.2}$$

or

$$\partial_x U + A_1(U)\partial_y U + A_2(U)\partial_z U = 0. \tag{1.3}$$

When the flow is supersonic, namely

$$u^2 + v^2 + w^2 > c^2, \tag{1.4}$$

the system (1.1) is a hyperbolic conservation law, which is the case we are dealing with in this paper. In such case, nonlinear waves such as shock waves, rarefaction waves and contact discontinuities usually appear in a γ-pressure law flow. A vast literature has been devoted to such topics and there still remain interesting open problems. See [3, 4, 9, 10, 12, 19] and references therein.

However, in a supersonic flow of van der Waals type, besides the three kinds of nonlinear waves mentioned in the above, subsonic phase transitions also exist due to the non-monotonicity of the state equation, which is given by

$$p(\tau) = \frac{R\theta}{\tau - b} - \frac{a}{\tau^2}, \tag{1.5}$$

where $\tau \equiv \rho^{-1}$ is the specific volume of the fluid, θ is the temperature of the fluid which is assumed to be a constant in an isothermal fluid, R is the perfect gas constant and a, b are positive constants. For $\frac{a}{4bR} < \theta < \frac{8a}{27bR}$, the state equation (1.5) is not monotonic with respect to τ. Precisely speaking, we can find $\tau_*, \tau^* \in (b, +\infty)$ such that

$$\begin{cases} d_\tau p(\tau) < 0, & \tau \in (b, \tau_*) \cup (\tau^*, +\infty),\\ d_\tau p(\tau) > 0, & \tau \in (\tau_*, \tau^*). \end{cases} \tag{1.6}$$

From the physical point of view, the fluid is in liquid phase for $\tau \in (b, \tau_*)$, while it is in vapor phase for $\tau \in (\tau^*, +\infty)$. The region (τ_*, τ^*) is a highly unstable region, where non state can be found in experiment (see [5]).

A subsonic phase transition is a discontinuous solution to the Euler equation (1.1) with a single discontinuity, which changes phases across the discontinuity and satisfies certain subsonic condition on both sides of the discontinuity. To explain the concept with more detail, let us consider the following planar subsonic phase transition

$$U(x,y,z) = \begin{cases} U_- = (\rho_-, u_-, v_-, w_0), & y < \sigma x, \\ U_+ = (\rho_+, u_+, v_+, w_0), & y > \sigma x, \end{cases} \tag{1.7}$$

where $\rho_\pm, u_\pm, v_\pm, w_0$ are constant states of the flow, σ is the constant speed of the discontinuity $\{y = \sigma x\}$ and $\rho_\pm$ belong to different phases. The solution (1.7) satisfies the Rankine-Hugoniot condition

$$\sigma[F_0(U)] - [F_1(U)] = 0, \tag{1.8}$$

and the subsonic condition

$$M_\pm = \Big| \frac{\sigma u_\pm - v_\pm}{c_\pm \sqrt{1+\sigma^2}} \Big| < 1, \tag{1.9}$$

where $[\cdot]$ denotes the difference of a function across the discontinuity $\{y = \sigma x\}$, $M_\pm$ and $c_\pm^2 = d_\rho p(\rho_\pm)$ are the Mach numbers and the sound speeds on each side of the discontinuity $\{y = \sigma x\}$ respectively.

Due to the subsonic property (1.9), the well-known Lax entropy inequality for classical shock waves is violated, which will be stated in detail in Section 2. Hence, other admissible criterion is needed to single out physical admissible subsonic phase transitions. There are several candidates, among which the viscosity capillarity criterion is an important one. The viscosity capillarity criterion was first introduced by Slemrod [14] to study phase transitions in an unsteady van der Waals fluid. Ever since, the study of unsteady van der Waals fluid, especially on problems in one dimensional spaces, has been carried out in many works. See [14, 13, 8, 5] and references therein. There are also works concerning multidimensional problems in an unsteady van der Waals fluid. See [1, 2, 15–17] and references therein.

However, there is not much knowledge on steady van der Waals fluid. The purpose of this paper is to reveal some insights of subsonic phase transitions in a steady supersonic flow of van der Waals type. The viscosity capillarity criterion will be applied to select physical admissible solution. Then, we will prove the uniform stability of multidimensional subsonic phase transitions by showing the validity of Lopatinski condition (see [6, 11]). Without giving much detail, here we briefly state the main result of this paper.

Theorem 1.1 *There exists $\nu_1 > 0$ depending on the bounds of $U_\pm$ and σ given in* (1.7)*, such that for $0 < \nu < \nu_1$, the ν-admissible phase transition* (1.7) *is uniformly stable.*

The definition of the parameter ν and ν-admissible will be given in Section 2, and the uniform stability will be described in detail in Section 4.

The rest of this paper is arranged as follows. In Section 2, the viscosity capillarity is introduced. In Section 3, we derive the linearized problem and prove the one dimensional stability. The multidimensional stability is proved in Section 4.

2 Admissible Criterion

In this section, we explain how subsonic phase transitions violate the Lax entropy inequality (see [7]) in a supersonic flow. Then we introduce the viscosity capillarity criterion and the additional jump condition derived by such criterion.

Compared with subsonic phase transitions in an unsteady fluid, those in a steady flow do not satisfy the Lax entropy inequality either. Considering the planar wave (1.7), we assume that the following supersonic condition is always valid in the coming arguments

$$u_{\pm}^2 - c_{\pm}^2 > 0. \tag{2.1}$$

Denote by

$$\begin{aligned}
\lambda_1^{\pm} &= \frac{1}{u_{\pm}^2 - c_{\pm}^2}(u_{\pm}v_{\pm} - c_{\pm}\sqrt{\triangle_{\pm}}\,),\\
\lambda_2^{\pm} &= \frac{v_{\pm}}{u_{\pm}},\\
\lambda_3^{\pm} &= \frac{1}{u_{\pm}^2 - c_{\pm}^2}(u_{\pm}v_{\pm} + c_{\pm}\sqrt{\triangle_{\pm}}\,),
\end{aligned}$$

the eigenvalues of $A_1(U_{\pm})$ respectively with $\triangle_{\pm} = u_{\pm}^2 + v_{\pm}^2 - c_{\pm}^2$, which satisfy

$$\lambda_1^{\pm} < \lambda_2^{\pm} < \lambda_3^{\pm}. \tag{2.2}$$

Then, the following theorem shows how the Lax inequality is violated.

Theorem 2.1 *The subsonic condition* (1.9) *is equivalent to*

$$\lambda_1^{\pm} < \sigma < \lambda_3^{\pm} \tag{2.3}$$

for the planar subsonic phase transition (1.7).

Obviously, the Lax inequality for 1st-shocks (3rd-shocks), $\lambda_1^{+} < \sigma < \lambda_1^{-}$ ($\lambda_3^{+} < \sigma < \lambda_3^{-}$) is no longer valid.

In order to single out physical admissible solution, Slemrod [14] proposed the viscosity capillarity criterion for one dimensional unsteady fluids under Lagrange coordinates. Motivated by the study of multidimensional problems, Benzoni-Gavage [1, 2] applied this criterion to unsteady fluids under Euclid coordinates. Here, we also follow the viscosity capillarity criterion to seek physical admissible phase transitions in a steady flow. For the simplicity of notations, we will need the following quantities in the coming arguments. Considering the planar subsonic phase transition (1.7), we denote by $u_{n\pm} = \frac{\sigma u_{\pm} - v_{\pm}}{\sqrt{1+\sigma^2}}$ and $u_{\tau} = \frac{u_{\pm} + \sigma v_{\pm}}{\sqrt{1+\sigma^2}}$ the normal velocity and the tangential velocity on each side of the discontinuity $\{y = \sigma x\}$ respectively, $j = \rho_{\pm}u_{n\pm}$ the mass transfer flux, and $\pi = p_{\pm} + j^2\tau_{\pm}$. Then, the Rankine-Hugoniot condition (1.8) and the subsonic condition (1.9) can be rewritten as

$$[j] = 0, \quad [\pi] = 0, \quad [u_{\tau}] = 0, \tag{2.4}$$

and

$$\Big|\frac{u_{n\pm}}{c_{\pm}}\Big| < 1 \quad \text{or} \quad \Big|\frac{j^2}{d_{\tau}p_(\tau_{\pm})}\Big| < 1, \tag{2.5}$$

respectively.

Analogue to the traveling wave method for viscous shocks, the viscosity capillarity criterion is used to find the planar wave (1.7) which admits the existence of the following traveling wave

$$U(\xi) = U\Big(\frac{y - \sigma x}{\epsilon}\Big) \tag{2.6}$$

satisfying $U(\pm\infty) = U_\pm$ and the Navier-Stokes equations

$$\begin{cases} \partial_x(\rho u) + \partial_y(\rho v) + \partial_z(\rho w) = 0, \\ \partial_x(\rho u^2 + p(\rho)) + \partial_y(\rho uv) + \partial_y(\rho uw) = \epsilon\nu\triangle u - \epsilon^2\partial_x\triangle(\rho^{-1}), \\ \partial_x(\rho uv) + \partial_y(\rho v^2 + p(\rho)) + \partial_y(\rho vw) = \epsilon\nu\triangle v - \epsilon^2\partial_y\triangle(\rho^{-1}), \\ \partial_x(\rho uw) + \partial_y(\rho vw) + \partial_z(w^2 + p(\rho)) = \epsilon\nu\triangle w - \epsilon^2\partial_z\triangle(\rho^{-1}), \end{cases} \tag{2.7}$$

where $\triangle = \partial_x^2 + \partial_y^2 + \partial_z^2$ is the Laplace operator, $\epsilon\nu$ is the viscosity coefficient and ϵ^2 is the capillarity coefficient with $\epsilon \geq 0$, $\nu > 0$. Substituting (2.6) into (2.7) and employing the Rankine-Hugoniot condition (2.4), we get the following hecteroclinic problem for the unknown function $\tau(\xi) \equiv \frac{1}{\rho(\xi)}$

$$\begin{cases} \tau'' = \nu j\tau' + \pi - p(\tau) - j^2\tau, \\ \tau(\pm\infty) = \tau_\pm, \end{cases} \tag{2.8}$$

where the prime $'$ denotes the derivative of a function with respect to ξ. As in [2], the admissibility of subsonic phase transitions can be defined by

Definition 2.1 *The planar subsonic phase transition* (1.7) *is admissible if and only if the problem* (2.8) *has a solution. The solution* $\tau(\xi)$ *is called the viscosity capillarity profile or* ν*-profile for simplicity. The pair* (τ_-, τ_+) *is called* ν*-admissible.*

One can find that the hecteroclinic problem (2.8) is exactly the same one for unsteady fluids (see [2]). Thus, we can take advantage of the known results in [2]. Denote by $\{\tau_m, \tau_M\}$ the Maxwell equilibrium defined by the equal area rule

$$\int_{\tau_m}^{\tau_M} (p(\tau_m) - p(\tau))\mathrm{d}\tau = 0.$$

Then, there exists $\tau_1 \in (\tau_M, +\infty)$ such that the chord connecting $(\tau_1, p(\tau_1))$ and $(\tau_m, p(\tau_m))$ is tangent to the graph of $p = p(\tau)$ at $(\tau_1, p(\tau_1))$. With τ_1 and τ_m, we define

$$j_1^2 = \frac{p(\tau_1) - p(\tau_m)}{\tau_m - \tau_1}.$$

When $\nu = 0$, the 0-profile satisfies

$$\begin{cases} \tau'' = \pi - p(\tau) - j^2\tau, \\ \tau(\pm\infty) = \tau_\pm. \end{cases} \tag{2.9}$$

As in [2], a 0-profile $\overline{\tau}(\xi; j)$ satisfying the first equation of (2.9) can be found by the generalized equal area rule, which means

$$\int_{\tau_-}^{\tau_+} (\pi - p(\tau) - j^2\tau)\mathrm{d}\tau = 0.$$

Moreover, for every $\overline{j}$ $(0 < \overline{j}^2 \leq j_1^2)$, a unique pair $(\overline{\tau}_-(\overline{j}), \overline{\tau}_+(\overline{j}))$ can be found such that $\overline{\tau}_-$ and $\overline{\tau}_+$ can be connected by the 0-profile with the parameters j. With the above results, Benzoni-Gavage [2] proved the structural stability and the existence of traveling waves for small ν by the center manifold method.

Theorem 2.2 *For $0 < \overline{j}^2 \le j_1^2$, there exist $\nu_0 > 0$ and neighborhoods $\mathscr{J}_0$, $\mathscr{V}_0$ of $\overline{j}$, $(\overline{\tau}_-(\overline{j}\,), \overline{\tau}_+(\overline{j}\,))$ respectively, such that, for $(j,\nu) \in \mathscr{J}_0 \times [0,\nu_0]$, there are unique pair $(\tau_-,\tau_+) \in \mathscr{V}_0$, for which τ_- and τ_+ are ν admissible with the parameters j.*

Moreover, an additional jump condition can be derived from the above result for the subsonic phase transition (1.7). As we can see from Proposition 2.1, a subsonic phase transition has one more characteristic going out of the free boundary than a shock wave. Hence, the Rankine-Hugoniot condition is not sufficient to guarantee the well-posedness of the corresponding initial boundary value problem. Nevertheless, the viscosity capillarity criterion can provide the following additional jump condition. By multiplying the equation in (2.8) with $\tau'(\xi)$ and integrating from $-\infty$ to $+\infty$ with respect to ξ, we get

$$\Big[f + \pi\tau - \frac{j^2}{2}\tau^2\Big] = -\nu a(j,\nu), \tag{2.10}$$

where $f = -\frac{a}{\tau} - R\theta\ln(\tau - b)$ is the free energy of the fluid and $a(j,\nu) = j\int_{-\infty}^{+\infty}(\tau'(\xi;j,\nu))^2\mathrm{d}\xi$ with $\tau(\xi;j,\nu)$ being the ν-profile. Noticing $a(j,\nu)$ being a nonlocal term, we have the following lemma in [1].

Lemma 2.1 *For all $\nu \in [0,\nu_0]$, the functions $a(j,\nu)$ is continuously differentiable. Moreover, its derivatives are continuous with respect to ν at $\nu = 0$ and are bounded depending on the bounds of $U_\pm$ given in* (1.7). *There exists $\alpha > 0$ such that for all $j \in \mathscr{J}$*

$$\lim_{\nu\to 0}\frac{\partial}{\partial j}a(j,\nu) \ge \alpha > 0. \tag{2.11}$$

3 Linear Problem and 1-Dimensional Stability

In this section, we propose the nonlinear problem for a multidimensional subsonic phase transition and derive the corresponding linearized problem. Then we prove the 1-dimensional stability for the linear problem.

3.1 Linear problem

Compared with the unsteady fluid, in a steady supersonic flow, the variable x can be regarded as the time variable (see [4]). Thus, we can endow the Euler equations with the initial data

$$U(0,y,z) = \begin{cases} U_-^0(y,z), & y < \varphi_0(z), \\ U_+^0(y,z), & y > \varphi_0(z), \end{cases} \tag{3.1}$$

which changes phases across the discontinuity $\{y = \varphi_0(z)\}$. With certain compatibility conditions on the initial data (3.1), we can expect to construct the multidimensional subsonic phase transition

$$U(x,y,z) = \begin{cases} U_-(x,y,z), & y < \varphi(x,z), \\ U_+(x,y,z), & y > \varphi(x,z), \end{cases} \tag{3.2}$$

which satisfies the following nonlinear initial boundary value problem

$$\begin{cases} \partial_x U_\pm + A_1(U_\pm)\partial_y U_\pm + A_2(U_\pm)\partial_z U_\pm = 0, & x>0,\ \pm(y-\varphi(x,z))>0, \\ \varphi_x[F_0(U)] - [F_1(U)] + \varphi_z[F_2(U)] = 0, & y=\varphi(x,z), \\ \Big[I(\rho) + \dfrac{(\varphi_x u - v + \varphi_z w)^2}{2(1+\varphi_x^2+\varphi_z^2)}\Big] = -\nu a(j,\nu), & y=\varphi(x,z), \\ U_\pm(0,y,z) = U_\pm^0(y,z), \quad \varphi(0,z)=\varphi_0(z), \end{cases} \tag{3.3}$$

where the second equation is the Rankine-Hugoniot condition, the third equation is a reformulation of the jump condition (2.10) with $I(\rho) = f + p\tau$ and $a(j,\nu) = j\int_{-\infty}^{+\infty}(\tau'(\xi;j,\nu))^2 \mathrm{d}\xi$ with $j = \frac{\rho_\pm(\varphi_x u_\pm - v_\pm + \varphi_z w_\pm)}{\sqrt{1+\varphi_x^2+\varphi_z^2}}\big|_{y=\varphi(x,z)}$ and $\tau(\xi;j,\nu)$ satisfying

$$\begin{cases} \tau'' = \nu j\tau' + \pi - p(\tau) - j^2\tau, \\ \tau(\pm\infty) = \tau_\pm|_{y=\varphi(x,z)}. \end{cases}$$

Following Majda's approach (see [11]), we use the following transformation

$$\begin{cases} \widetilde{x} = x, \\ \widetilde{y} = \pm(y-\varphi(x,z)), \quad \pm(y-\varphi(x,z))>0, \\ \widetilde{z} = z, \\ \widetilde{U}(\widetilde{x},\widetilde{y},\widetilde{z}) = U(x,y,z) \end{cases} \tag{3.4}$$

to map the free boundary $\{y=\varphi(x,z)\}$ to the fixed boundary $\{\widetilde{y}=0\}$. Then the problem (3.3) becomes

$$\begin{cases} \partial_x U_\pm \pm (A_1(U_\pm) - \varphi_x I - \varphi_z A_2(U_\pm))\partial_y U_\pm + A_2(U_\pm)\partial_z U_\pm = 0, & x,y>0, \\ \varphi_x[F_0(U)] - [F_1(U)] + \varphi_z[F_2(U)] = 0, & y=0, \\ \Big[I(\rho) + \dfrac{(\varphi_x u - v + \varphi_z w)^2}{2(1+\varphi_x^2+\varphi_z^2)}\Big] = -\nu a(j,\nu), & y=0, \\ U_\pm(0,y,z) = U_\pm^0(y,z), \quad \varphi(0,z)=\varphi_0(z), \end{cases} \tag{3.5}$$

where we have dropped the tildes for simplicity.

Consider the perturbation, $(U_+^\epsilon, U_-^\epsilon, \varphi^\epsilon)$, of the planar phase transition (1.7), which satisfies the problem (3.5) and $(U_+^\epsilon, U_-^\epsilon, \varphi^\epsilon)|_{\epsilon=0} = (U_+, U_-, \sigma x)$. Denote

$$(V_+, V_-, \psi) = \frac{\mathrm{d}}{\mathrm{d}\epsilon}(U_+^\epsilon, U_-^\epsilon, \varphi^\epsilon)\Big|_{\epsilon=0}.$$

Then, the following linearized problem for the unknowns (V_+, V_-, ψ) can be derived from (3.5),

$$\begin{cases} \partial_x V_\pm \pm (A_1(U_\pm) - \sigma I)\partial_y V_\pm + A_2(U_\pm)\partial_z V_\pm = f_\pm, & x,y>0, \\ b_0\psi_x + b_1\psi_z + \mathscr{M}_+V_+ + \mathscr{M}_-V_- = g, & y=0, \\ (V_+, V_-, \psi)|_{x<0} \quad \text{vanishes}, \end{cases} \tag{3.6}$$

where

$$b_0 = \begin{pmatrix} [F_0(U)] \\ \frac{u_\tau}{1+\sigma^2}([u_n] + \widetilde{\nu}\rho_+) \end{pmatrix}, \qquad b_1 = \begin{pmatrix} [F_2(U)] \\ \frac{w_0}{\sqrt{1+\sigma^2}}([u_n] + \widetilde{\nu}\rho_+) \end{pmatrix},$$

$$\mathscr{M}_+ = \begin{pmatrix} \sigma F_0'(U_+) - F_1'(U_+) \\ l_+ \end{pmatrix}, \quad \mathscr{M}_- = \begin{pmatrix} -\sigma F_0'(U_-) + F_1'(U_-) \\ l_- \end{pmatrix},$$

where

$$l_+ = \Big(\frac{c_+^2 + \widetilde{\nu} j}{\rho_+}, \frac{\sigma(u_{n+} + \widetilde{\nu}\rho_+)}{\sqrt{1+\sigma^2}}, -\frac{u_{n+} + \widetilde{\nu}\rho_+}{\sqrt{1+\sigma^2}}, 0\Big),$$
$$l_- = \Big(-\frac{c_-^2}{\rho_-}, -\frac{\sigma u_{n-}}{\sqrt{1+\sigma^2}}, \frac{u_{n-}}{\sqrt{1+\sigma^2}}, 0\Big),$$

with $u_{n\pm} = \frac{(\sigma u_\pm - v_\pm)}{\sqrt{1+\sigma^2}}$, $u_\tau = \frac{(u_\pm + \sigma v_\pm)}{\sqrt{1+\sigma^2}}$, $j = \rho_\pm u_{n\pm}$ and $\widetilde{\nu} = \nu \partial_j a(j, \nu)$.

3.2 1-dimensional stability

The 1-dimensional stability is concerned with the case that (V_+, V_-, ψ) does not depend on the variable z, which indicates the main problem is the problem (3.6) without derivatives with respect to z, namely

$$\begin{cases} \partial_x V_\pm \pm (A_1(U_\pm) - \sigma I)\partial_y V_\pm = f_\pm, & x, y > 0, \\ b_0 \psi_x + \mathscr{M}_+ V_+ + \mathscr{M}_- V_- = g, & y = 0, \\ (V_+, V_-, \psi)|_{x<0} \quad \text{vanishes}. \end{cases} \tag{3.7}$$

The above problem is essentially a one dimensional initial boundary value problem, for which we have the following result on the stability (see also [18]).

Theorem 3.1 *There exists $\nu_1 > 0$ depending on the bounds of $U_\pm$ and σ, such that for $0 < \nu < \nu_1$, the subsonic phase transition* (1.7) *is stable with respect to perturbations in the y-direction, which means the problem* (3.7) *being well-posed.*

Proof Without loss of generality, we assume

$$j = \rho_\pm u_{n\pm} > 0 \quad \text{and} \quad u_\pm > 0, \tag{3.8}$$

and other cases can be studied similarly. The main idea of the proof is to show that the boundary values of outgoing characteristics and the free boundary can be determined by the boundary conditions, for which we need to calculate the eigenvalues and eigenvectors of the matrix $A_1(U_\pm) - \sigma I$. The eigenvalues of $A_1(U_\pm) - \sigma I$ are

$$\lambda_1^\pm = \frac{1}{u_\pm^2 - c_\pm^2}(u_\pm v_\pm - c_\pm \sqrt{\triangle_\pm}\,) - \sigma,$$
$$\lambda_3^\pm = \frac{1}{u_\pm^2 - c_\pm^2}(u_\pm v_\pm + c_\pm \sqrt{\triangle_\pm}\,) - \sigma$$

of multiplicity 1 and

$$\lambda_2^\pm = \frac{v_\pm}{u_\pm} - \sigma$$

of multiplicity 2, where $\triangle_\pm = u_\pm^2 + v_\pm^2 - c_\pm^2$. The corresponding right eigenvectors are

$$r_1^\pm = \Big(-\frac{\rho_\pm(u_\pm\sqrt{\triangle_\pm} - v_\pm c_\pm)}{c_\pm(u_\pm^2 - c_\pm^2)}, -\frac{u_\pm v_\pm - c_\pm\sqrt{\triangle_\pm}}{u_\pm^2 - c_\pm^2}, 1, 0\Big)^T,$$
$$r_3^\pm = \Big(\frac{\rho_\pm(u_\pm\sqrt{\triangle_\pm} + v_\pm c_\pm)}{c_\pm(u_\pm^2 - c_\pm^2)}, -\frac{u_\pm v_\pm + c_\pm\sqrt{\triangle_\pm}}{u_\pm^2 - c_\pm^2}, 1, 0\Big)^T$$

and

$$r_{21}^{\pm} = (0, u_{\pm}, v_{\pm}, 0)^T, \quad r_{22}^{\pm} = (0,0,0,1)^T$$

respectively. Denote by

$$V_{\pm} = v_1^{\pm} r_1^{\pm} + v_{21}^{\pm} r_{21}^{\pm} + v_{22}^{\pm} r_{22}^{\pm} + v_3^{\pm} r_3^{\pm}$$

the decompositions of $V_{\pm}$ on the bases $(r_1^{\pm}, r_{21}^{\pm}, r_{22}^{\pm}, r_3^{\pm})$ respectively. Noting the subsonic condition (2.3) and the assumption (3.8), we have

$$\lambda_2^{\pm} < 0 < \lambda_3^{\pm}.$$

Accordingly, the boundary conditions of (3.7) can be rewritten as

$$(b_0, \mathscr{M}_- r_1^-, \mathscr{M}_- r_{21}^-, \mathscr{M}_- r_{22}^-, \mathscr{M}_+ r_3^+) \begin{pmatrix} \psi_x \\ v_1^- \\ v_{21}^- \\ v_{22}^- \\ v_3^+ \end{pmatrix} = g - (M_1 r_3^-, \mathscr{M}_+ r_1^+, \mathscr{M}_+ r_{21}^+, \mathscr{M}_+ r_{22}^+) \begin{pmatrix} v_3^- \\ v_1^+ \\ v_{21}^+ \\ v_{21}^+ \end{pmatrix} \tag{3.9}$$

to separate the outgoing characteristics together with the free boundary from the incoming characteristics. Obviously, the necessary and sufficient condition for the well-posedness of the problem (3.7) is that the determinant

$$\mathscr{D} \equiv \det\left(b_0, \mathscr{M}_- r_1^-, \mathscr{M}_- r_{21}^-, \mathscr{M}_- r_{22}^-, \mathscr{M}_+ r_3^+\right) \tag{3.10}$$

does not vanish. Direct computation yields

$$\begin{aligned}
\mathscr{D} =& \frac{\lambda_1^- {\lambda_2^-}^2 \lambda_3^+ \rho_- u_-}{\sqrt{1+\sigma^2}} \begin{vmatrix} [\rho u] & \frac{\rho_-}{c_-}\sqrt{\triangle_-} & \rho_- u_- & 0 & \frac{\rho_+}{c_+}\sqrt{\triangle_+} \\ [\rho u^2 + p] & \frac{\rho_-}{c_-}(u_-\sqrt{\triangle_-} + v_- c_-) & 2\rho_- u_-^2 & 0 & \frac{\rho_+}{c_+}(u_+\sqrt{\triangle_+} - v_+ c_+) \\ [\rho u] w_0 & \frac{\rho_- w_0}{c_-}\sqrt{\triangle_-} & \rho_- u_- w_0 & 1 & \frac{\rho_+ w_0}{c_+}\sqrt{\triangle_+} \\ [\rho u v] & \frac{\rho_-}{c_-}(v_-\sqrt{\triangle_-} - u_- c_-) & 2\rho_- u_- v_- & 0 & \frac{\rho_+}{c_+}(v_+\sqrt{\triangle_+} + u_+ c_+) \\ \frac{u_\tau([u_n] + \widetilde{\nu}\rho_+)}{\sqrt{1+\sigma^2}} & u_\tau & u_- u_{n-} & 0 & -u_\tau + \widetilde{\nu}\frac{\rho_+}{c_+}\sqrt{\triangle_+} \end{vmatrix} \\
=& \frac{\lambda_1^- {\lambda_2^-}^2 \lambda_3^+ \rho_- u_-}{\sqrt{1+\sigma^2}} \begin{vmatrix} [\rho u] & \frac{\rho_-}{c_-}\sqrt{\triangle_-} & \rho_- u_- & \frac{\rho_+}{c_+}\sqrt{\triangle_+} \\ [\rho u^2 + p] & \frac{\rho_-}{c_-}(u_-\sqrt{\triangle_-} + v_- c_-) & 2\rho_- u_-^2 & \frac{\rho_+}{c_+}(u_+\sqrt{\triangle_+} - v_+ c_+) \\ [\rho u v] & \frac{\rho_-}{c_-}(v_-\sqrt{\triangle_-} - u_- c_-) & 2\rho_- u_- v_- & \frac{\rho_+}{c_+}(v_+\sqrt{\triangle_+} + u_+ c_+) \\ \frac{u_\tau([u_n] + \widetilde{\nu}\rho_+)}{\sqrt{1+\sigma^2}} & u_\tau & u_- u_{n-} & -u_\tau + \widetilde{\nu}\frac{\rho_+}{c_+}\sqrt{\triangle_+} \end{vmatrix} \\
\equiv& \frac{\lambda_1^- {\lambda_2^-}^2 \lambda_3^+ \rho_- u_-}{\sqrt{1+\sigma^2}} (\mathrm{I} + \widetilde{\nu}\mathrm{II}),
\end{aligned}$$

where

$$\mathrm{I} = \begin{vmatrix} [\rho u] & \frac{\rho_-}{c_-}\sqrt{\triangle_-} & \rho_- u_- & \frac{\rho_+}{c_+}\sqrt{\triangle_+} \\ [\rho u^2 + p] & \frac{\rho_-}{c_-}(u_-\sqrt{\triangle_-} + v_- c_-) & 2\rho_- u_-^2 & \frac{\rho_+}{c_+}(u_+\sqrt{\triangle_+} - v_+ c_+) \\ [\rho u v] & \frac{\rho_-}{c_-}(v_-\sqrt{\triangle_-} - u_- c_-) & 2\rho_- u_- v_- & \frac{\rho_+}{c_+}(v_+\sqrt{\triangle_+} + u_+ c_+) \\ \frac{u_\tau [u_n]}{\sqrt{1+\sigma^2}} & u_\tau & u_- u_{n-} & -u_\tau \end{vmatrix}$$

$$= \frac{\rho_- u_- [\rho]^2}{\sqrt{1+\sigma^2}} \Big(\frac{\sqrt{\triangle_+}\sqrt{\triangle_-}}{c_+ c_-} u_{n+} u_{n-} + u_\tau^2 \Big) (u_-^2 + v_-^2) > 0,$$

$$\mathrm{II} = \begin{vmatrix} [\rho u] & \frac{\rho_-}{c_-}\sqrt{\triangle_-} & \rho_- u_- & \frac{\rho_+}{c_+}\sqrt{\triangle_+} \\ [\rho u^2 + p] & \frac{\rho_-}{c_-}(u_-\sqrt{\triangle_-} + v_- c_-) & 2\rho_- u_-^2 & \frac{\rho_+}{c_+}(u_+\sqrt{\triangle_+} - v_+ c_+) \\ [\rho u v] & \frac{\rho_-}{c_-}(v_-\sqrt{\triangle_-} - u_- c_-) & 2\rho_- u_- v_- & \frac{\rho_+}{c_+}(v_+\sqrt{\triangle_+} + u_+ c_+) \\ \frac{u_\tau \rho_+}{\sqrt{1+\sigma^2}} & 0 & 0 & \frac{\rho_+}{c_+}\sqrt{\triangle_+} \end{vmatrix}$$

$$= \frac{\rho_+^2 \rho_-^2 u_-}{c_+^2 c_-^2 j \sqrt{1+\sigma^2}} \Big(\Big(\frac{\sqrt{\triangle_+}}{c_+} u_{n+} + u_\tau \Big) \Big(u_\tau - \frac{\sqrt{\triangle_-}}{c_-} u_{n-} \Big) (u_\tau^2 + u_{n+} u_{n-})$$

$$+ \Big(\frac{\sqrt{\triangle_+}\sqrt{\triangle_-}}{c_+ c_-} u_{n+} u_{n-} - 2 \frac{\sqrt{\triangle_+}}{c_+} u_{n+} u_\tau - u_\tau^2 \Big) (u_-^2 + v_-^2) \Big), \tag{3.11}$$

which implies II is a bounded term depending on the bounds of $U_\pm$ and σ. Therefore, we claim that for sufficiently small ν, the determinant $\triangle$ is nonzero.

4 Multidimensional Stability

First, let us introduce the uniform stability in [11]. Denote $V = (V_+, V_-)^T$ and denote by

$$\widehat{V}(s,\omega,y) = \frac{1}{(2\pi)^2} \int_0^\infty \int_{-\infty}^\infty \mathrm{e}^{-(sx + \mathrm{i}\omega z)} V(x,y,z) \mathrm{d}z \mathrm{d}x$$

the Laplace-Fourier transform of V in (x,z) with $\mathrm{Re}s > 0$. Then from (3.6), we know that $\widehat{V}$ satisfies

$$\frac{\partial \widehat{V}}{\partial y} = B(s,\omega)\widehat{V} + \widehat{f}, \tag{4.1}$$

where

$$B(s,\omega) = \begin{pmatrix} -(A_1(U_+) - \sigma I)^{-1}(sI + \mathrm{i}\omega A_2(U_+)) & 0 \\ 0 & (A_1(U_-) - \sigma I)^{-1}(sI + \mathrm{i}\omega A_2(U_-)) \end{pmatrix}$$

and $\widehat{f} = ((A_1(U_+) - \sigma I)^{-1}\widehat{f}_+, -(A_1(U_-) - \sigma I)^{-1}\widehat{f}_-)^T$.

Denote by $\{\lambda_j\}_{j=1}^l$ all distinct eigenvalues of $B(s,\omega)$ with multiplicity being m_j. Obviously, we have

$$\mathbb{C}^8 = \bigoplus_{j=1}^l \mathrm{Ker}[(\lambda_j I - B(s,\omega))^{m_j}].$$

Introduce

$$E^+(s,\omega) = \{w_j \in \mathrm{Ker}[(\lambda_j I - B(s,\omega))^{m_j}] \mid \mathrm{Re}\lambda_j < 0,\ 1 \le j \le l\},$$

the space of boundary values of all bounded solutions of the special form

$$\widehat{V}(s,\omega,y) = \sum_{\mathrm{Re}\lambda_j < 0} \mathrm{e}^{\lambda_j y} \sum_{p=0}^{m_j - 1} \frac{y^p}{p!} (\lambda_j I - B(s,\omega))^p w_j$$

to (4.1) with $\widehat{f} \equiv 0$. Then the uniform stability result can be stated in detail by the following theorem.

Theorem 4.1 *There exists $\nu_1 > 0$ depending on the bounds of $U_\pm$ and σ given in* (1.7) *such that for $0 < \nu < \nu_1$, the ν-admissible subsonic phase transition* (1.7) *is uniformly stable, i.e. there exists $\eta > 0$ such that the estimate*

$$\inf_{\substack{\mathrm{Re}s\geq 0\\ |s|^2+\omega^2=1}} |(b_0 s + \mathrm{i}b_1\omega)\mu + \mathscr{M}_+V_+ + \mathscr{M}_-V_-|^2 \geq \eta^2(|V_+|^2 + |V_-|^2 + \mu^2) \tag{4.2}$$

holds for all $V = (V_+, V_-) \in E^+(s,\omega)$ and $\mu \in \mathbb{R}$.

4.1 The space $E^+(s,\omega)$

In order to establish the estimate (4.2), we need to investigate the structure of the space $E^+(s,\omega)$. For simplicity, we only consider the case that

$$j = \rho_\pm u_{n\pm} > 0 \quad \text{and} \quad u_\pm > 0, \tag{4.3}$$

and other cases can be studied similarly.

Taking the Laplace-Fourier transform on the equation of (3.6) with $f_\pm = 0$ and making the transformation $\widehat{V}_\pm = T_\pm \widehat{Z}_\pm$ with

$$T_\pm = \begin{pmatrix} 0 & 0 & \frac{\rho_\pm}{c_\pm} & 0\\ \frac{1}{\sqrt{1+\sigma^2}} & \frac{\sigma}{\sqrt{1+\sigma^2}} & -\frac{c_\pm\sigma}{u_{n\pm}\sqrt{1+\sigma^2}} & 0\\ \frac{1}{\sqrt{1+\sigma^2}} & -\frac{1}{\sqrt{1+\sigma^2}} & \frac{c_\pm}{u_{n\pm}\sqrt{1+\sigma^2}} & 0\\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{4.4}$$

yield

$$\frac{\partial \widehat{Z}_\pm}{\partial y} = \mp N_\pm(s,\omega)\widehat{Z}_\pm, \tag{4.5}$$

where

$$N_\pm = \begin{pmatrix} -s\sigma - \frac{\widetilde{s}}{u_{n\pm}} & 0 & \frac{c_\pm s}{u_{n\pm}} & 0\\ 0 & -s\sigma - \frac{\widetilde{s}}{u_{n\pm}} & \frac{c_\pm \widetilde{s}}{u_{n\pm}^2} & 0\\ -\frac{c_\pm u_{n\pm} s}{u_{n\pm}^2 - c_\pm^2} & \frac{c_\pm \widetilde{s}}{u_{n\pm}^2-c_\pm^2} & -s\sigma - \frac{(u_{n\pm}^2+c_\pm^2)\widetilde{s}}{(u_{n\pm}^2-c_\pm^2)u_{n\pm}} & -\frac{\mathrm{i}\omega c_\pm u_{n\pm}\sqrt{1+\sigma^2}}{u_{n\pm}^2-c_\pm^2}\\ 0 & 0 & -\frac{\mathrm{i}\omega c_\pm\sqrt{1+\sigma^2}}{u_{n\pm}} & -s\sigma - \frac{\widetilde{s}}{u_{n\pm}} \end{pmatrix}$$

with $\widetilde{s} = su_\tau + \mathrm{i}\omega w_0\sqrt{1+\sigma^2}$.

The eigenvalues of $N_-(s,\omega)$ with negative real part for $\mathrm{Re}s > 0$ are

$$\lambda_1^- = -s\sigma - \frac{\widetilde{s}}{u_{n\pm}}$$

of multiplicity 2 and

$$\lambda_2^- = \frac{-u_{n-}\widetilde{s} - c_- \sqrt[+]{D_-}}{u_{n-}^2 - c_-^2}$$

of multiplicity 1, where $\sqrt[+]{\cdot}$ denotes the positive real part square root of a complex value and

$$D_- = \widetilde{s}^2 + (s^2 - \omega^2(1+\sigma^2))(u_{n-}^2 - c_-^2),$$

the corresponding right eigenvectors are

$$e_{11}^- = (0, \mathrm{i}\omega u_{n-}\sqrt{1+\sigma^2}, 0, \widetilde{s})^T, \tag{4.6}$$

$$e_{12}^- = (\mathrm{i}\omega\sqrt{1+\sigma^2}, 0, 0, -s)^T, \tag{4.7}$$

and

$$e_2^- = \Big(\frac{c_- s}{u_{n-}}, -\frac{c_-\widetilde{s}}{u_{n-}^2}, \frac{c_-}{u_{n-}^2 - c_-^2}\Big(\frac{c_-\widetilde{s}}{u_{n-}} - \sqrt[+]{D_-}\Big), \frac{\mathrm{i}\omega c_-\sqrt{1+\sigma^2}}{u_{n-}}\Big)^T. \tag{4.8}$$

The eigenvalue of $-N_+(s,\omega)$ with a negative real part for Re$s > 0$ is

$$\lambda_3^+ = \frac{-u_{n+}\widetilde{s} + c_+\sqrt[+]{D_+}}{u_{n+}^2 - c_+^2},$$

where

$$D_+ = \widetilde{s}^2 + (s^2 - \omega^2(1+\sigma^2))(u_{n+}^2 - c_+^2)$$

and corresponding right eigenvector is

$$e_3^+ = \Big(\frac{c_+ s}{u_{n+}}, -\frac{c_-\widetilde{s}}{u_{n+}^2}, \frac{c_+}{u_{n+}^2 - c_+^2}\Big(\frac{c_+\widetilde{s}}{u_{n+}} + \sqrt[+]{D_+}\Big), \frac{\mathrm{i}\omega c_-\sqrt{1+\sigma^2}}{u_{n+}}\Big)^T. \tag{4.9}$$

Remark 4.1 The above eigenvalues and eigenvectors can be continuously extended to the case Re$s \geq 0$, where the notation $\sqrt[+]{\cdot}$ is still used without causing any confusion.

Considering the above eigenvectors, we have

Lemma 4.1 $(e_{11}^-, e_{12}^-, e_2^-, e_3^+)$ *are linearly dependent for* $|s|^2 + \omega^2 = 1$ *and* Re$s \geq 0$ *except at* $\{(s,\omega) \mid \widetilde{s}^2 + u_{n-}^2(s^2 - \omega^2(1+\sigma^2)) = 0$ *or* $\omega = 0\}$.

Proof When $\omega = 0$, the vectors e_{11}^- and e_{12}^- are obviously linear dependent.

When $\widetilde{s}^2 + u_{n-}^2(s^2 - \omega^2(1+\sigma^2)) = 0$, the third component of the vector e_2^- is zero. Therefore, the determinant

$$\begin{vmatrix} 0 & \mathrm{i}\omega\sqrt{1+\sigma^2} & \frac{c_- s}{u_{n-}} \\ \mathrm{i}\omega u_{n-}\sqrt{1+\sigma^2} & 0 & -\frac{c_-\widetilde{s}}{u_{n-}^2} \\ \widetilde{s} & s & \frac{\mathrm{i}\omega c_-\sqrt{1+\sigma^2}}{u_{n-}} \end{vmatrix} = -\frac{\mathrm{i}\omega c_-\sqrt{1+\sigma^2}}{u_{n-}}(\widetilde{s}^2 + u_{n-}^2(s^2 - \omega^2(1+\sigma^2))) = 0$$

tells that the vectors $e_{11}^-, e_{12}^-, e_2^-$ are linearly dependent in this case.

In the above critical cases, the following lemmas help us to find the bases of $E^+(s,\omega)$.

Lemma 4.2 *When* $\omega = 0$, *the vectors*

$$e_{11}^- = (u_\tau, u_{n-}, 1, 0)^T, \tag{4.10}$$

$$e_{12}^- = (0, 0, 0, 1)^T, \tag{4.11}$$

together with (4.8) *and* (4.9) *are linearly independent.*

Lemma 4.3 *When* $\widetilde{s}^2 + u_{n-}^2(s^2 - \omega^2(1+\sigma^2)) = 0$ *and* $\omega \neq 0$, *we have*

$$s = \frac{\omega\sqrt{1+\sigma^2}}{u_-^2 + v_-^2}\Big(-u_\tau w_0 \mathrm{i} \pm u_{n-}\sqrt{u_-^2 + v_-^2 + w_0^2}\Big)$$

and $\lambda_1^- = \lambda_2^-$. *Then the vector*

$$e_2^- = (0, 2c_-, u_{n-}, 0)^T \tag{4.12}$$

together with (4.6), (4.7) *and* (4.9) *is linearly independent.*

Combining the above propositions, if we naturally expand the eigenvectors as

$$\begin{pmatrix} e_{11}^- \\ 0 \end{pmatrix}, \quad \begin{pmatrix} e_{12}^- \\ 0 \end{pmatrix}, \quad \begin{pmatrix} e_2^- \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ e_3^+ \end{pmatrix},$$

then the bases of $E^+(s,\omega)$ are given for $|s|^2+\omega^2=1$ and Re$s \geq 0$.

4.2 Lopatinski determinant

Now, we are ready to prove the uniform stability of subsonic phase transitions.

Proof of Theorem 4.1 Taking the Laplace-Fourier transformation on the boundary condition in (3.5) with $g=0$ yields

$$(sb_0 + \mathrm{i}\omega b_1)\widehat{\psi} + \mathscr{M}_+\widehat{V}_+ + \mathscr{M}_-\widehat{V}_- = 0. \tag{4.13}$$

Employing the transformation (4.4) and multiplying the boundary condition (4.13) from the left by the invertible matrix

$$\begin{pmatrix} \frac{1}{\sqrt{1+\sigma^2}} & 0 & 0 & 0 & 0 \\ -\frac{u_\tau}{\sqrt{1+\sigma^2}} & \frac{1}{1+\sigma^2} & \frac{\sigma}{1+\sigma^2} & 0 & 0 \\ 0 & -\frac{\sigma}{1+\sigma^2} & \frac{1}{1+\sigma^2} & 0 & 0 \\ -w_0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

we get the following boundary condition

$$c\widehat{\psi} + \widetilde{\mathscr{M}}_+\widehat{Z}_+ + \widetilde{\mathscr{M}}_-\widehat{Z}_- = 0, \tag{4.14}$$

where

$$c = \begin{pmatrix} \frac{\widetilde{s}[\rho]}{1+\sigma^2} \\ \frac{s[p]}{1+\sigma^2} \\ 0 \\ \mathrm{i}\omega[p] \\ \frac{\widetilde{s}}{1+\sigma^2}\left([u_n]+\widetilde{\nu}\rho_+\right) \end{pmatrix}, \quad \widetilde{\mathscr{M}}_\pm = \begin{pmatrix} \pm\mathscr{M}_\pm^0 \\ \widetilde{l}_\pm \end{pmatrix}$$

with

$$\mathscr{M}_\pm^0 = \begin{pmatrix} 0 & \rho_\pm & \frac{\rho_\pm(u_{n\pm}^2-c_\pm^2)}{c_\pm u_{n\pm}} & 0 \\ j & 0 & 0 & 0 \\ 0 & -2j & -\frac{\rho_\pm(u_{n\pm}^2-c_\pm^2)}{c_\pm} & 0 \\ 0 & 0 & 0 & j\sqrt{1+\sigma^2} \end{pmatrix},$$

$$\widetilde{l}_+ = \left(0, u_{n+}+\widetilde{\nu}\rho_+, \widetilde{\nu}\frac{\rho_+(u_{n+}^2-c_+^2)}{u_{n+}c_+}, 0\right),$$

$$\widetilde{l}_- = (0, -u_{n-}, 0, 0).$$

To establish the estimate (4.2), we need to prove the determinant

$$\mathscr{L} \equiv \det(c, \widetilde{\mathscr{M}}_+ e_3^+, \widetilde{\mathscr{M}}_- e_{11}^-, \widetilde{\mathscr{M}}_- e_{12}^-, \widetilde{\mathscr{M}}_- e_2^-)$$

nonzero, for which we consider the following three cases.

(i) $\omega = 0$.

This case is indeed one dimensional, which is already proved in Theorem 3.1. Here we omit the detail.

(ii) $\widetilde{s}^2 + u_{n-}^2(s^2 - \omega^2(1+\sigma^2)) = 0$ and $\omega \neq 0$.

In this case, we have

$$\begin{aligned}\mathscr{L} &= (1+\sigma^2)\\ &\cdot\begin{vmatrix} \frac{\widetilde{s}[\rho]}{1+\sigma^2} & \frac{\rho\sqrt[+]{D_+}}{u_{n+}} & \mathrm{i}\omega\rho_- u_{n-} & 0 & \frac{\rho_-}{c_-}(u_{n-}^2 + c_-^2) \\ \frac{s[p]}{1+\sigma^2} & \rho_+ c_+ s & 0 & \mathrm{i}\omega\rho_- u_{n-} & 0 \\ 0 & \rho_+\big(\frac{c_+\widetilde{s}}{u_{n+}} - \sqrt[+]{D_+}\big) & -2\mathrm{i}\omega\rho_- u_{n-}^2 & 0 & -\frac{\rho_- u_{n-}}{c_-}(u_{n-}^2 + 3c_-^2) \\ \mathrm{i}\omega[p] & \mathrm{i}\omega\rho_+ c_+(1+\sigma^2) & \widetilde{s}\rho_- u_{n-} & -s\rho_- u_{n-} & 0 \\ \frac{\widetilde{s}}{1+\sigma^2}([u_n] + \widetilde{\nu}\rho_+) & -\frac{c_+\widetilde{s}}{u_{n+}} + \widetilde{\nu}\frac{\rho_+\sqrt[+]{D_+}}{u_{n+}} & \mathrm{i}\omega u_{n-}^2 & 0 & 2\rho_- u_{n-}^2 c_- \end{vmatrix}\\ &\equiv \mathrm{I} + \widetilde{\nu}O(1),\end{aligned}$$

where

$$\mathrm{I} = \frac{2\mathrm{i}\omega\rho_+ j^3 c_+ c_- [\tau]\widetilde{s}^2}{1+\sigma^2}\Big(\widetilde{s}^2 + \frac{u_{n+}}{c_+}\sqrt[+]{D_+}\Big)$$

is nonzero and $O(1)$ is a bounded term depending on the bounds of $U_\pm$ and σ given in (1.7). Therefore, in this case, we claim that there exits $\nu_1 > 0$, such that for $0 < \nu < \nu_1$, $\mathscr{L}$ is nonzero.

(iii) $\omega \neq 0$ and $\widetilde{s}^2 + u_{n-}^2(s^2 - \omega^2(1+\sigma^2)) \neq 0$.

In this case, direct calculation yields

$$\begin{aligned}\mathscr{L} &= (1+\sigma^2)\\ &\cdot\begin{vmatrix} \frac{\widetilde{s}[\rho]}{1+\sigma^2} & \frac{\rho\sqrt[+]{D_+}}{u_{n+}} & \mathrm{i}\omega\rho_- u_{n-} & 0 & -\frac{\rho_-\sqrt[+]{D_-}}{u_{n-}} \\ \frac{s[p]}{1+\sigma^2} & \rho_+ c_+ s & 0 & \mathrm{i}\omega\rho_- u_{n-} & 0 \\ 0 & \rho_+\big(\frac{c_+\widetilde{s}}{u_{n+}} - \sqrt[+]{D_+}\big) & -2\mathrm{i}\omega\rho_- u_{n-}^2 & 0 & \rho_-\big(\frac{c_-\widetilde{s}}{u_{n-}} + \sqrt[+]{D_-}\big) \\ \mathrm{i}\omega[p] & \mathrm{i}\omega\rho_+ c_+(1+\sigma^2) & \widetilde{s}\rho_- u_{n-} & -s\rho_- u_{n-} & \mathrm{i}\omega\rho_- c_-(1+\sigma^2) \\ \frac{\widetilde{s}}{1+\sigma^2}([u_n] + \widetilde{\nu}\rho_+) & -\frac{c_+\widetilde{s}}{u_{n+}} + \widetilde{\nu}\frac{\rho_+\sqrt[+]{D_+}}{u_{n+}} & \mathrm{i}\omega u_{n-}^2 & 0 & -\frac{c_-\widetilde{s}}{u_{n-}} \end{vmatrix}\\ &\equiv \frac{\mathrm{i}\omega\rho_+\rho_-(\widetilde{s}^2 + u_{n-}^2(s^2 - \omega(1+\sigma^2)))}{c_+ c_- u_{n+} u_{n-}}(\mathrm{I} + \widetilde{\nu}\,\mathrm{II}),\end{aligned}$$

where

$$\mathrm{I} = -[\tau]^2\Big(\widetilde{s}^2 + \frac{u_{n+}u_{n-}}{c_+ c_-}\sqrt[+]{D_+}\sqrt[+]{D_-}\Big),$$

$$\mathrm{II} = [u_n]\tau_-(s^2 - \omega^2(1+\sigma^2))(u_{n-}^2 - c_-^2)\frac{\widetilde{s} + \frac{u_{n+}}{c_+}\sqrt[+]{D_+}}{\widetilde{s} + \frac{u_{n-}}{c_-}\sqrt[+]{D_-}} + \frac{\widetilde{s}}{j}\Big(\frac{u_{n-}}{c_-}\sqrt[+]{D_-} + \frac{u_{n+}}{c_+}\sqrt[+]{D_+}\Big).$$

Let us solve the equation $\mathrm{I} = 0$ on the compact set $\{(s,\omega) \mid |s|^2 + \omega^2 = 1, \mathrm{Re}s \geq 0\}$, which means solving the equation

$$c_+ c_- \widetilde{s} = -u_{n+} u_{n-} \sqrt[+]{D_+} \sqrt[+]{D_-} \tag{4.15}$$

for the unknown (s,ω). Denote

$$\alpha = \frac{s^2 - \omega^2(1+\sigma^2)}{\widetilde{s}^2}. \tag{4.16}$$

Taking the square on both sides of (4.15) yields the following equation for the unknown α

$$\alpha^2 c_+^2 c_-^2 (M_+^2 - 1)(M_-^2 - 1) + \alpha(c_+^2(M_+^2-1) + c_-^2(M_-^2-1)) + \Big(1 - \frac{1}{M_+^2 M_-^2}\Big) = 0,$$

which has two real roots

$$\alpha = \frac{1}{2c_+^2 c_-^2 (M_+^2-1)(M_-^2-1)}(-(c_+^2(M_+^2-1) + c_-^2(M_-^2-1)) \pm \sqrt{\triangle}\,) \tag{4.17}$$

with

$$\triangle = (c_+^2(M_+^2-1) + c_-^2(M_-^2-1))^2 - 4c_+^2 c_-^2 (M_+^2-1)(M_-^2-1)\Big(1 - \frac{1}{M_+^2 M_-^2}\Big),$$

where the one with the minus sign should be neglected. From (4.16) and $|s|^2 + \omega^2 = 1$, we get the zero points of I

$$\begin{aligned}(s_{1,2}, \omega_{1,2}) &= \Big(\pm \frac{\kappa_+}{\sqrt{1+|\kappa_+|^2}}, \pm \frac{1}{\sqrt{1+|\kappa_+|^2}}\Big),\\ (s_{3,4}, \omega_{3,4}) &= \Big(\pm \frac{\kappa_-}{\sqrt{1+|\kappa_-|^2}}, \pm \frac{1}{\sqrt{1+|\kappa_-|^2}}\Big),\end{aligned} \tag{4.18}$$

where

$$\kappa_\pm = \frac{\mathrm{i}\sqrt{1+\sigma^2}}{\alpha u_\tau^2 - 1}\big(- \alpha u_\tau w_0 \pm \sqrt{\alpha(u_\tau^2 + w_0^2) - 1}\,\big). \tag{4.19}$$

For every $(s,\omega) \in \{(s,\omega) \mid |s|^2 + \omega^2 = 1, \mathrm{Re}s \geq 0\}$ away from (s_i, ω_i) $(i = 1,2,3,4)$, the item I is nonzero, which indicates that we can find $\nu_{(s,\omega)} > 0$, $M_{(s,\omega)} > 0$ and open neighborhood $\mathscr{O}_{(s,\omega)}$ of (s,ω) such that for $0 < \nu < \nu_{(s,\omega)}$, the estimate

$$|\mathrm{I} + \widetilde{\nu}\,\mathrm{II}| > M_{(s,\omega)} \tag{4.20}$$

holds for $(s,\omega) \in \mathscr{O}_{(s,\omega)}$.

When $(s,\omega) \in \bigcup\limits_{i=1}^{4} \{(s_i,\omega_i)\}$, the item I vanishes. However, in such situation, the imaginary part of $\mathrm{I} + \widetilde{\nu}\,\mathrm{II}$ satisfies

$$|\mathrm{Im}(\mathrm{I} + \widetilde{\nu}\,\mathrm{II})| = \widetilde{\nu}\,\Big|\frac{\widetilde{s}}{j}\Big(\frac{u_{n-}}{c_-}\sqrt[+]{D_-} + \frac{u_{n+}}{c_+}\sqrt[+]{D_+}\Big)\Big| > 0$$

for $\nu > 0$. Thus we can find $\nu_{(s_i,\omega_i)} = \nu_0 > 0$ with ν_0 given in Theorem 2.2, $M_{(s_i,\omega_i)} > 0$ and open neighborhoods $\mathscr{O}_{(s_i,\omega_i)}$ of (s_i,ω_i) $(i = 1,2,3,4)$, such that for $0 < \nu < \nu_{(s_i,\omega_i)}$ the estimate

$$|\mathrm{I} + \widetilde{\nu}\,\mathrm{II}| > M_{(s_i,\omega_i)} \tag{4.21}$$

holds for $(s,\omega) \in \mathscr{O}_{(s_i,\omega_i)}$.

Therefore, we can find a finite subset $\{\mathscr{O}_{(s_k,\omega_k)}\}_{k=1}^N$ of $\{\mathscr{O}_{(s,\omega)} \mid |s|^2+\omega^2=1, \mathrm{Re}s \geq 0\}$ covering the compact set $\{(s,\omega) \mid |s|^2+\omega^2=1, \mathrm{Re}s \geq 0\}$. By setting $\nu_a = \min_{k=1,\cdots,N}\{\nu_{(s_k,\omega_k)}\}$ and $M_a = \min_{k=1,\cdots,N}\{M_{(s_k,\omega_k)}\}$, we claim that for $0<\nu<\nu_a$ the estimate

$$|\mathrm{I}+\widetilde{\nu}\,\mathrm{II}| > M_a \tag{4.22}$$

holds for $(s,\omega) \in \{(s,\omega) \mid |s|^2+\omega^2=1, \mathrm{Re}s \geq 0\}$.

Combining the three cases, we draw the conclusion of Theorem 4.1.

Acknowledgement The author thanks Professor Shuxing Chen and Professor Yaguang Wang for their encouragement and stimulating discussion.

References

[1] Benzoni-Gavage, S., Stability of multi-dimensional phase transitions in a van der Waals fluid, *Nonlinear Anal.*, **31**(1-2), 1998, 243–263.

[2] Benzoni-Gavage, S., Stability of subsonic planar phase boundaries in a van der Waals fluid, *Arch. Rat. Mech. Anal.*, **150**(1), 1999, 23–55.

[3] Chen, S., Global existence of supersonic flow past a curved convex wedge, *J. Partial Diff. Eqs*, **11**, 1998, 43–62.

[4] Courant, R. and Friedrichs, K. O., Supersonic Flow and Shock Waves, Springer-Verlag, New York, 1977.

[5] Fan, H. and Slemrod, M., Dynamic flows with liquid/vapor phase transitions, Handbook of Mathematical Fluid Dynamics, Vol. I, North-Holland, Amsterdam, 2002, 373–420.

[6] Kreiss, H. O., Initial boundary value problems for hyperbolic systems, *Comm. Pure Appl. Math.*, **23**, 1970, 227–298.

[7] Lax, P. D., Hyperbolic systems of conservation laws. II, *Comm. Pure Appl. Math.*, **10**, 1957, 537–467.

[8] LeFloch, P. G., Propagating phase boundaries: formulation of the problem and existence via the Glimm method, *Arch. Rat. Mech. Anal.*, **123**(2), 1993, 153–197.

[9] Li, T. and Wang, L., Inverse piston problem for the system of one-dimensional isentropic flow, *Chin. Ann. Math.*, **28B**(3), 2007, 265–282.

[10] Lien, W. and Liu, T. P., Nonlinear stability of a self-similar 3-dimensional gas flow, *Commun. Math. Phys.*, **204**, 1999, 525–549.

[11] Majda, A., The stability of multi-dimensional shock fronts, *Mem. Amer. Math. Soc.*, **41**(275), 1983, 1–95.

[12] Morawetz, C. S., On a weak solution for transonic flow problem, *Comm. Pure Appl. Math.*, **38**, 1985, 423–443.

[13] Shearer, M., Nonuniqueness of admissible solutions of Riemann initial value problem for a system of conservation laws of mixed type, *Arch. Rat. Mech. Anal.*, **93**(1), 1986, 45–59.

[14] Slemrod, M., Admissibility criteria for propagating phase boundaries in a van der Waals fluid, *Arch. Rat. Mech. Anal.*, **81**(4), 1983, 301–315.

[15] Wang, Y. G. and Xin, Z., Stability and existence of multidimensional subsonic phase transitions, *Acta Math. Appl. Sin. Engl. Ser.*, **19**(4), 2003, 529–558.

[16] Zhang, S. Y., Discontinuous solutions to the Euler equations in a van der Waals fluid, *Appl. Math. Letters*, **20**(2), 2007, 170–176.

[17] Zhang, S. Y., Existence of travelling waves in non-isothermal phase dynamics, *J. Hyperbolic Diff. Eqs.*, **4**(3), 2007, 391–400.

[18] Zhang, S. Y., Stability of Phase Transitions in a Steady van der Waals Fluid, preprint.

[19] Zhang, Y., Global existence of steady supersonic potential flow past a curved wedge with a piecewise smooth boundary, *SIAM J. Math. Anal.*, **31**, 1999, 166–183.