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The Existence and Long-Time Behavior of Weak Solution to Bipolar Quantum Drift-Diffusion Model***

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Abstract The authors study the existence and long-time behavior of weak solutions to the bipolar transient quantum drift-diffusion model, a fourth order parabolic system. Using semi-discretization in time and entropy estimate, the authors get the global existence of nonnegative weak solutions to the one-dimensional model with nonnegative initial and homogenous Neumann (or periodic) boundary conditions. Furthermore, by a logarithmic Sobolev inequality, it is proved that the periodic weak solution exponentially approaches its mean value as time increases to infinity.

Keywords Quantum drift-diffusion, Weak solution, Long-time behavior
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1 Introduction

The quantum drift-diffusion model (QDDM) is one of the quantum macroscopic models derived recently to simulate the quantum effects in miniaturized semiconductor devices (see [12, 20]). In this paper, we will consider the bipolar transient case,

$$\begin{cases} n_t = \operatorname{div}\Big[-\varepsilon^2 n\nabla\Big(\dfrac{\triangle\sqrt{n}}{\sqrt{n}}\Big)+\nabla(n^\alpha)-n\nabla V\Big], \\ p_t = \operatorname{div}\Big[-\xi\varepsilon^2 p\nabla\Big(\dfrac{\triangle\sqrt{p}}{\sqrt{p}}\Big)+\nabla(p^\beta)+p\nabla V\Big], \\ \lambda^2\triangle V = n-p-C(x), \end{cases} \tag{1.1}$$

where the electron density n, hole density p and the electrostatic potential V are unknown variables; the doping profile $C(x)$ representing the distribution of charged background ions is supposed to be independent of time t; $1\le\alpha,\beta<\infty$, the scaled Planck constant $\varepsilon>0$, Debye length $\lambda>0$ and the ratio of the effective masses of electrons and holes $\xi>0$ are parameters.

From the mathematical point of view, the main difficulty of (1.1) lies in the fourth order parabolic equations. For the fourth order equation in one-dimensional case with mixed Dirichlet-Neumann boundary condition, [14, 17] obtained the nonnegative global existence and

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exponential decay of weak solution. Similar results for periodic-boundary problem were proved in [2, 3] recently.

For the one space dimensional QDDM, Jüngel and Pinnau [15, 16] first obtained a positivity preserving global weak solution with large enough θ for the mixed Dirichlet-Neumann boundary problem, and in the sequel, a series of works (see [6–8]) investigated this model elaborately with homogeneous Neumann in the aspects of weak existence and semiclassical limit. But to the authors' knowledge, there are no results on the long-time behavior for bipolar transient (isothermal or isentropic) model.

The main task of this paper is to establish the long-time behavior of (1.1) and the existence of weak solution for this isentropic case.

From now on, our discussion will be fixed on one-dimensional space. Since all the results in this paper are obtained for fixed $\lambda, \xi > 0$, for convenience we let $\lambda = \xi = 1$ in the following. To search for solutions which are physically reasonable, namely the solutions for which densities n and p are both nonnegative, we might as well suppose $n = \rho^2$ and $p = \eta^2$. Furthermore, by considering the quantum quasi-Fermi level F and G, we are led to study the system

$$\begin{cases} (\rho^2)_t = (\rho^2 F_x)_x, \\ (\eta^2)_t = (\eta^2 G_x)_x, \\ -\varepsilon^2 \dfrac{\rho_{xx}}{\rho} + \dfrac{\alpha}{\alpha-1}\rho^{2(\alpha-1)} - V = F, \\ -\varepsilon^2 \dfrac{\eta_{xx}}{\eta} + \dfrac{\beta}{\beta-1}\eta^{2(\beta-1)} + V = G, \\ V_{xx} = \rho^2 - \eta^2 - C(x), \end{cases} \tag{1.2}$$

which is equivalent to (1.1).

More precisely, let $\Omega = (0,1)$, $T > 0$ be any fixed constant, $Q_T = (0,T] \times \Omega$ and $\mathbb{T}$ represent one-dimensional flat torus. We will consider (1.2) with the initial homogenous Neumann boundary condition

$$\begin{cases} \rho_x = \eta_x = F_x = G_x = V_x = 0, & \text{in } (0,T] \times \partial\Omega, \\ \rho(0,\cdot) = \rho_0(\cdot),\ \eta(0,\cdot) = \eta_0(\cdot), & \text{in } \Omega, \\ \displaystyle\int_\Omega (\rho_0^2 - \eta_0^2)dx = \int_\Omega C(x)dx, \end{cases} \tag{1.3}$$

and the initial periodic boundary condition

$$\begin{cases} \rho(0,\cdot) = \rho_0(\cdot),\ \eta(0,\cdot) = \eta_0(\cdot), & \text{in } \mathbb{T}, \\ \displaystyle\int_\mathbb{T} (\rho_0^2 - \eta_0^2)dx = \int_\mathbb{T} C(x)dx, \end{cases} \tag{1.4}$$

where

$$\int_\Omega (\rho_0^2 - \eta_0^2)dx = \int_\Omega C(x)dx \quad \text{and} \quad \int_\mathbb{T} (\rho_0^2 - \eta_0^2)dx = \int_\mathbb{T} C(x)dx$$

are necessary conditions such that the Poisson equations in (1.3) and (1.4) are solvable for their corresponding boundary value problems respectively.

The following notations will be used in this paper.

(1) The Sobolev spaces, $W^{m,p}(\Omega)$ $(H^m(\Omega) = W^{m,2}(\Omega))$;

(2) The Hölder spaces, $C^{k,\theta}(\overline{\Omega})$;

(3) Denote $H \triangleq \{u \in H^2(\Omega) \mid u_x \in H_0^1(\Omega)\}$, which is a Hilbert space obviously;

(4) $A \hookrightarrow B$ (or $A \hookrightarrow\hookrightarrow B$) denotes that A is continuously (or compactly) embedded in B;

(5) B' denotes the dual space of B.

Our main results are stated as follows.

Theorem 1.1 (Existence) *Let $C(x) \in L^\infty(\Omega)$ and suppose that $0 \le \rho_0,\ \eta_0 \in H^1(\Omega)$, $\ln\rho_0,\ \ln\eta_0 \in L^1(\Omega)$, and $\int_\Omega(\rho_0^2 - \eta_0^2)dx = \int_\Omega C(x)dx$. Then for any fixed $\varepsilon > 0$, there exists (ρ, η, E) such that*

(i) $0 \le \rho, \eta \in L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H), \quad E \in L^\infty(0,T;L^2(\Omega))$,

$$\rho^2, \eta^2 \in C([0,T];C^{0,\mu}(\overline{\Omega})) \cap L^2(0,T;C^{1,\mu}(\overline{\Omega})) \cap L^{\frac{6}{5}}(0,T;C^{2,\nu}(\overline{\Omega})) \cap H^1(0,T;(H^1(\Omega))'),$$

where $0 < \mu < \frac{1}{2}$, $0 < \nu < \frac{1}{6}$;

$$\text{(ii)} \int_0^T \langle \partial_t\rho^2, \varphi\rangle_{\langle (H^1)', H^1\rangle} dt = \varepsilon^2 \int_0^T\int_\Omega (2\rho\rho_{xxx}\varphi_x + \rho\rho_{xx}\varphi_{xx})dxdt - \int_0^T\int_\Omega [\alpha\rho^{2(\alpha-1)}(\rho^2)_x - \rho^2 E]\varphi_x dxdt, \quad \forall\, \varphi \in L^6(0,T;H), \tag{1.5}$$

$$\int_0^T \langle \partial_t\eta^2, \varphi\rangle_{\langle (H^1)', H^1\rangle} dt = \varepsilon^2 \int_0^T\int_\Omega (2\eta\eta_{xxx}\varphi_x + \eta\eta_{xx}\varphi_{xx})dxdt - \int_0^T\int_\Omega [\beta\eta^{2(\beta-1)}(\eta^2)_x + \eta^2 E]\varphi_x dxdt, \quad \forall\, \varphi \in L^6(0,T;H), \tag{1.6}$$

and

$$\int_\Omega E(t,\cdot)\phi_x dx = \int_\Omega [C(x) - \rho^2(t,\cdot) + \eta^2(t,\cdot)]\phi dx, \quad \forall\, \phi \in H^1(\Omega),\ a.e.,\ \forall\, t \in (0,T). \tag{1.7}$$

Moreover, there exist $P, R \in L^2(Q_T)$ such that

$$\int_0^T \langle \partial_t\rho^2, \psi\rangle_{\langle (H^1)', H^1\rangle} dt = \varepsilon^2 \int_0^T\int_\Omega P\psi_x dxdt - \int_0^T\int_\Omega [\alpha\rho^{2(\alpha-1)}(\rho^2)_x - \rho^2 E]\psi_x dxdt, \quad \forall\, \psi \in L^2(0,T;H^1(\Omega)), \tag{1.8}$$

$$\int_0^T \langle \partial_t\eta^2, \psi\rangle_{\langle (H^1)', H^1\rangle} dt = \varepsilon^2 \int_0^T\int_\Omega R\psi_x dxdt - \int_0^T\int_\Omega [\beta\eta^{2(\beta-1)}(\eta^2)_x + \eta^2 E]\psi_x dxdt, \quad \forall\, \psi \in L^2(0,T;H^1(\Omega)). \tag{1.9}$$

Remark 1.1 In Theorem 1.1, (ρ, η, E) is a weak solution of (1.2)–(1.3) in the sense of (1.5)–(1.9). Note that V_x is replaced by E.

Similar existence results also hold for the periodic boundary condition case.

Theorem 1.2 *Let $C(x) \in L^\infty(\mathbb{T})$ and suppose that $0 \le \rho_0, \eta_0 \in H^1(\mathbb{T})$, $\ln\rho_0, \ln\eta_0 \in L^1(\mathbb{T})$, and $\int_{\mathbb{T}}(\rho_0^2 - \eta_0^2)dx = \int_{\mathbb{T}} C(x)dx$. Then for any fixed $\varepsilon > 0$, there exists (ρ, η, E) such that*

(i) $0 \le \rho, \eta \in L^\infty(0,T; H^1(\mathbb{T})) \cap L^2(0,T; H^2(\mathbb{T}))$, $\quad E \in L^\infty(0,T; L^2(\mathbb{T}))$,

$$\rho^2, \eta^2 \in L^\infty(0,T; C^{0,\mu}(\mathbb{T})) \cap L^{\frac{6}{5}}(0,T; C^{2,\nu}(\mathbb{T})) \cap H^1(0,T; (H^1(\mathbb{T}))')$$

and $\ln\rho^2, \ln\eta^2 \in L^2(0,T; H^2(\mathbb{T}))$, *where* $0 < \mu < \frac{1}{2}$, $0 < \nu < \frac{1}{6}$;

(ii) (ρ, η, E) *is a weak solution of* (1.2), (1.4) *in the form of* (1.5)–(1.9), *where the integral domain Ω needs to be replaced by $\mathbb{T}$.*

Theorem 1.3 (Long-Time Behavior) *Let $C(x) \equiv C$ be a constant, $\overline{\rho_0^2} = \frac{\int_{\mathbb{T}} \rho_0^2 dx}{|\mathbb{T}|}$, $\overline{\eta_0^2} = \frac{\int_{\mathbb{T}} \eta_0^2 dx}{|\mathbb{T}|}$, $0 \le \rho_0$, $\eta_0 \in H^1(\mathbb{T})$, $\ln\rho_0$, $\ln\eta_0 \in L^1(\mathbb{T})$, and $\int_{\mathbb{T}}(\rho_0^2 - \eta_0^2)dx = \int_{\mathbb{T}} C(x)dx$. Then the weak solution (ρ, η) obtained in Theorem* 1.2 *satisfies*

$$\Big\| \rho(t,\cdot) - \sqrt{\overline{\rho_0^2}} \Big\|^2_{L^2(\mathbb{T})} + \Big\| \eta(t,\cdot) - \sqrt{\overline{\eta_0^2}} \Big\|^2_{L^2(\mathbb{T})} \le C_0 e^{-Mt}, \quad \forall\, t \in (0,T], \tag{1.10}$$

where $M = \frac{16\pi^4\varepsilon^2}{|\mathbb{T}|^4}$ *and* $C_0 = \displaystyle\int_{\mathbb{T}} \Big[\rho_0^2 \ln\frac{\rho_0^2}{\overline{\rho_0^2}} + \eta_0^2 \ln\frac{\eta_0^2}{\overline{\eta_0^2}}\Big] dx.$

Remark 1.2 The assumption in Theorem 1.3 implies $C = \overline{\rho_0^2} - \overline{\eta_0^2}$.

Remark 1.3 For the isothermal QDDM, we have a similar long-time behavior result by a similar proof.

There are other quantum macroscopic models, namely quantum hydrodynamic models, quantum energy transport models which were also introduced recently to simulate the miniaturized semiconductor devices where the quantum effects play a leading role. Some mathematical results of these models as well as their derivation could be found in [13, 10] etc.

This article is organized as follows. In Section 2, we will construct the approximation problem by the method of semi-discretization in time, which was used in [14–16, 6] as well as in [4] to deal with strongly coupled parabolic system. Section 3 contains all of the uniform entropy estimates which will be used in the proof of existence. Then in Section 4, we use a compactness argument for fixed $\varepsilon > 0$ to prove Theorem 1.1. This is somewhat motivated by the methods in [11, 18, 19, 22]. Finally in Section 5, by using a logarithmic Sobolev inequality, we will study the long-time behavior and prove Theorem 1.3. Since the proof of Theorem 1.2 is the same as that of Theorem 1.1, we omit the details.

2 Semidiscretization Approximate Problem

Chen and Ju (see [7]) have constructed the approximation problem and given its existence results without detail proof. Here, for completeness, we would like to provide a proof by employing the exponential transformation technique as in [14–16, 6].

Let $\tau > 0$ be such that $T = N\tau$ (without loss of generality. Otherwise, let $N = [\frac{T}{\tau}] + 1$.) Hence $N = N(\tau) \in \mathbb{N}$ depends only on τ. We divide the time interval $(0,T]$ by $(0,T] =$

$\bigcup\limits_{k=1}^{N}((k-1)\tau, k\tau]$. For any $k=1,2,\cdots,N$, given ρ_{k-1} and η_{k-1} such that

$$\int_\Omega(\rho_{k-1}^2-\eta_{k-1}^2)dx=\int_\Omega C(x)dx,$$

we will solve the following problem

$$\begin{cases}\dfrac{\rho_k^2-\rho_{k-1}^2}{\tau}=[\rho_k^2(F_k)_x]_x, & \text{in } \Omega,\\ \dfrac{\eta_k^2-\eta_{k-1}^2}{\tau}=[\eta_k^2(G_k)_x]_x, & \text{in } \Omega,\\ -\varepsilon^2\dfrac{(\rho_k)_{xx}}{\rho_k}+\dfrac{\alpha}{\alpha-1}\rho_k^{2(\alpha-1)}-V_k=F_k, & \text{in } \Omega,\\ -\varepsilon^2\dfrac{(\eta_k)_{xx}}{\eta_k}+\dfrac{\beta}{\beta-1}\eta_k^{2(\beta-1)}+V_k=G_k, & \text{in } \Omega,\\ (V_k)_{xx}=\rho_k^2-\eta_k^2-C(x), & \text{in } \Omega,\\ (\rho_k)_x=(\eta_k)_x=(F_k)_x=(G_k)_x=(V_k)_x=0, & \text{on } \partial\Omega,\\ \displaystyle\int_\Omega(\rho_k^2-\eta_k^2)dx=\int_\Omega C(x)dx. \end{cases} \tag{2.1}$$

Theorem 2.1 *Let $\alpha>1,\ \beta>1,\ 0<\gamma<\frac{1}{2}$, and $C(x)\in L^\infty(\Omega)$. Suppose that $\rho_{k-1},\eta_{k-1}\in C^{0,\gamma}(\overline{\Omega})$, $\min\limits_{\overline{\Omega}}\rho_{k-1}>0$, $\min\limits_{\overline{\Omega}}\eta_{k-1}>0$, and $\int_\Omega(\rho_{k-1}^2-\eta_{k-1}^2)dx=\int_\Omega C(x)dx$. Then (2.1) has a solution $(\rho_k,\eta_k,F_k,G_k,V_k)\in(W^{4,p}(\Omega))^2\times(W^{2,p}(\Omega))^3$ $(\forall\, p>1)$ with $\rho_k\geq c_k>0$ and $\eta_k\geq c_k>0$, where c_k is a constant dependent on k.*

Proof First we solve the following problem

$$\begin{cases}\dfrac{e^{2w_k}-e^{2w_{k-1}}}{\tau}=[e^{2w_k}(F_k)_x]_x, & \text{in } \Omega,\\ \dfrac{e^{2u_k}-e^{2u_{k-1}}}{\tau}=[e^{2u_k}(G_k)_x]_x, & \text{in } \Omega,\\ -\varepsilon^2\dfrac{(e^{w_k})_{xx}}{e^{w_k}}+\dfrac{\alpha}{\alpha-1}e^{2(\alpha-1)w_k}-V_k=F_k, & \text{in } \Omega,\\ -\varepsilon^2\dfrac{(e^{u_k})_{xx}}{e^{u_k}}+\dfrac{\beta}{\beta-1}e^{2(\beta-1)u_k}+V_k=G_k, & \text{in } \Omega,\\ (V_k)_{xx}=e^{2w_k}-e^{2u_k}-C(x), & \text{in } \Omega,\\ (w_k)_x=(u_k)_x=(F_k)_x=(G_k)_x=(V_k)_x=0, & \text{on } \partial\Omega,\\ \displaystyle\int_\Omega(e^{2w_k}-e^{2u_k})dx=\int_\Omega C(x)dx, \end{cases} \tag{2.2}$$

where $w_{k-1}=\ln\rho_{k-1}$ and $u_{k-1}=\ln\eta_{k-1}$ are determined by the given ρ_{k-1} and η_{k-1}. Obviously, (2.2) is equivalent to (2.1) due to the exponential transformation of variables $\rho_k=e^{w_k}$

and $\eta_k = e^{u_k}$. To solve (2.2), we first treat the equivalent problem

$$\begin{cases} \dfrac{e^{2w} - e^{2w_{k-1}}}{\tau} = -\varepsilon^2 (e^{2w} w_{xx})_{xx} + 2\alpha (e^{2\alpha w} w_x)_x - (e^{2w} V_x)_x, & \text{in } \Omega, \\ \dfrac{e^{2u} - e^{2u_{k-1}}}{\tau} = -\varepsilon^2 (e^{2u} u_{xx})_{xx} + 2\beta (e^{2\beta u} u_x)_x + (e^{2u} V_x)_x, & \text{in } \Omega, \\ V_{xx} = e^{2w} - e^{2u} - C(x), & \text{in } \Omega, \\ w_x = w_{xxx} = u_x = u_{xxx} = V_x = 0, & \text{on } \partial\Omega, \\ \int_\Omega (e^{2w} - e^{2u}) dx = \int_\Omega C(x) dx \end{cases} \tag{2.3}$$

by eliminating F_k, G_k. Here and hereafter, we denote $w = w_k, u = u_k, V = V_k$ for convenience.

Obviously, the homogenous Neumann boundary value problem of Poisson equation in (2.3) has a unique (up to an additive constant) classical solution, which satisfies $V_x = \int_0^x (e^{2w} - e^{2u} - C(y)) dy$. Substituting $\int_0^x (e^{2w} - e^{2u} - C(y)) dy$ for V_x, (2.3) is equivalent to

$$\begin{cases} \dfrac{e^{2w} - e^{2w_{k-1}}}{\tau} = -\varepsilon^2 (e^{2w} w_{xx})_{xx} + 2\alpha (e^{2\alpha w} w_x)_x & \\ \qquad\qquad - \Big[e^{2w} \displaystyle\int_0^x (e^{2w} - e^{2u} - C(y)) dy \Big]_x, & \text{in } \Omega, \\ \dfrac{e^{2u} - e^{2u_{k-1}}}{\tau} = -\varepsilon^2 (e^{2u} u_{xx})_{xx} + 2\beta (e^{2\beta u} u_x)_x & \\ \qquad\qquad + \Big[e^{2u} \displaystyle\int_0^x (e^{2w} - e^{2u} - C(y)) dy \Big]_x, & \text{in } \Omega, \\ w_x = w_{xxx} = u_x = u_{xxx} = 0, & \text{on } \partial\Omega, \\ \displaystyle\int_\Omega (e^{2w} - e^{2u}) dx = \int_\Omega C(x) dx. \end{cases} \tag{2.4}$$

We will use Leray-Schauder fixed point theorem to establish the existence of weak solution to (2.4). Let $(\overline{w}, \overline{u}) \in H^1(\Omega) \times H^1(\Omega)$, and $\sigma \in [0, 1]$. We solve the linear problem

$$\begin{aligned} \frac{1}{\tau} \Big(\frac{e^{2\overline{w}} - 1}{\overline{w}} w + \sigma - \sigma e^{2w_{k-1}} \Big) &= -\varepsilon^2 (e^{2\overline{w}} w_{xx})_{xx} + 2\alpha (e^{2\alpha\overline{w}} w_x)_x \\ &\quad - \sigma \Big[e^{2\overline{w}} \int_0^x (e^{2\overline{w}} - e^{2\overline{u}} - C(y)) dy \Big]_x, \end{aligned} \tag{2.5}$$

$$\begin{aligned} \frac{1}{\tau} \Big(\frac{e^{2\overline{u}} - 1}{\overline{u}} u + \sigma - \sigma e^{2u_{k-1}} \Big) &= -\varepsilon^2 (e^{2\overline{u}} u_{xx})_{xx} + 2\beta (e^{2\beta\overline{u}} u_x)_x \\ &\quad + \sigma \Big[e^{2\overline{u}} \int_0^x (e^{2\overline{w}} - e^{2\overline{u}} - C(y)) dy \Big]_x, \end{aligned} \tag{2.6}$$

in $H \times H$. For this purpose, we introduce the bilinear form

$$a(w, \phi) = \int_\Omega \Big(\varepsilon^2 e^{2\overline{w}} w_{xx} \phi_{xx} + 2\alpha e^{2\alpha\overline{w}} w_x \phi_x + \frac{1}{\tau} \frac{e^{2\overline{w}} - 1}{\overline{w}} w \phi \Big) dx \quad \text{for any } w, \phi \in H$$

and the linear functional

$$F(\phi) = \sigma \int_\Omega \Big[e^{2\overline{w}} \int_0^x (e^{2\overline{w}} - e^{2\overline{u}} - C(y)) dy \phi_x + \frac{1}{\tau} (e^{2w_{k-1}} - 1) \phi \Big] dx \quad \text{for any } \phi \in H.$$

It is easy to check that for any $w, \phi \in H$, we have

$$a(w,w) \geq C\|w\|^2_{H^2(\Omega)}, \quad |a(w,\phi)| \leq C\|w\|_{H^2(\Omega)}\|\phi\|_{H^2(\Omega)} \quad \text{and} \quad |F(\phi)| \leq C\|\phi\|_{H^2(\Omega)},$$

where $C > 0$ is a constant depending on $\overline{w}$, $\overline{u}$ and w_{k-1}. By Lax-Milgram theorem, there exists a unique $w \in H$ such that $a(w,\phi) = F(\phi)$ for any $\phi \in H$. Hence, $w \in H$ solves (2.5). Similarly, we get a unique $u \in H$ which solves (2.6). Thus we can define the mapping A: $H^1(\Omega) \times H^1(\Omega) \times [0,1] \to H^1(\Omega) \times H^1(\Omega)$ by $A((\overline{w},\overline{u}),\sigma) = (w,u)$.

It is obvious that $A((\overline{w},\overline{u}),0) = 0$ for any $(\overline{w},\overline{u}) \in H^1(\Omega)\times H^1(\Omega)$. Moreover, it can be easily seen that A is continuous and also compact because of the compact embedding $H^2(\Omega) \hookrightarrow\hookrightarrow H^1(\Omega)$. To show the uniform bound of the fixed-points, let $((w,u),\sigma) \in H^1(\Omega) \times H^1(\Omega) \times [0,1]$ such that $A((w,u),\sigma) = (w,u)$. Using $\phi = 1 - e^{-2w} \in H$ as a test function, we deduce, in view of the inequalities $e^x - x - 1 \geq 0$ and $e^x - x \geq |x|$ for all $x \in R$, that

$$\begin{aligned} &2\varepsilon^2 \int_\Omega w_{xx}^2 dx + 4\alpha \int_\Omega e^{2(\alpha-1)w} w_x^2 dx + \frac{2}{\tau}\int_\Omega |w| dx \\ &\leq \frac{1}{\tau}\int_\Omega (e^{2w_{k-1}} - 2w_{k-1}) dx + 2\sigma \int_\Omega \Big[\int_0^x (e^{2w} - e^{2u} - C(y)) dy\Big] w_x dx. \end{aligned} \tag{2.7}$$

Similarly, we have

$$\begin{aligned} &2\varepsilon^2 \int_\Omega u_{xx}^2 dx + 4\beta \int_\Omega e^{2(\beta-1)u} u_x^2 dx + \frac{2}{\tau}\int_\Omega |u| dx \\ &\leq \frac{1}{\tau}\int_\Omega \big(e^{2u_{k-1}} - 2u_{k-1}\big) dx - 2\sigma \int_\Omega \Big[\int_0^x (e^{2w} - e^{2u} - C(y)) dy\Big] u_x dx. \end{aligned} \tag{2.8}$$

Since $1 \in H$, choosing $\phi = 1$ as a test function, we see that $\int_\Omega (e^{2w} - 1 + \sigma - \sigma e^{2w_{k-1}}) dx = 0$ and $\int_\Omega (e^{2u} - 1 + \sigma - \sigma e^{2u_{k-1}}) dx = 0$, which, together with the assumption $\int_\Omega (e^{2w_{k-1}} - e^{2u_{k-1}}) dx = \int_\Omega C(x) dx$, implies

$$\int_\Omega (e^{2w} - e^{2u}) dx = \sigma \int_\Omega (e^{2w_{k-1}} - e^{2u_{k-1}}) dx = \sigma \int_\Omega C(x) dx \leq C,$$

where C is a constant independent of w, u and τ. Hence (2.7) and (2.8) lead us to

$$\frac{2}{\tau}\|w\|_{L^1(\Omega)} + 2\varepsilon^2 \|w_{xx}\|^2_{L^2(\Omega)} \leq C(\tau) + C\|w_x\|_{L^1(\Omega)} \tag{2.9}$$

and

$$\frac{2}{\tau}\|u\|_{L^1(\Omega)} + 2\varepsilon^2 \|u_{xx}\|^2_{L^2(\Omega)} \leq C(\tau) + C\|u_x\|_{L^1(\Omega)}, \tag{2.10}$$

where $C(\tau)$ is a constant depending only on τ, w_{k-1} and u_{k-1}, while C is a constant independent of w, u and τ. For any small $\tau > 0$, we have from (2.9) and (2.10) that $\|w\|_{H^2(\Omega)} \leq C$ and $\|u\|_{H^2(\Omega)} \leq C$, where C is a constant independent of w, u.

Then by Leray-Schauder fixed point theorem, there exists $(w_k, u_k) \in H \times H$ such that $A((w_k,u_k),1) = (w_k,u_k)$. So $(w_k,u_k) \in H \times H$ is a weak solution of (2.4) and hence $(\rho_k,\eta_k) = (e^{w_k}, e^{u_k}) \in H \times H$ is a solution of (2.1). Since $H^2(\Omega) \hookrightarrow C^{1,\gamma}(\overline{\Omega})$, we have $0 < c_k \leq \rho_k, \eta_k \leq c'_k < \infty$, where c_k and c'_k are constants dependent only on k. Hence, the former four equations in (2.1) are all uniformly elliptic. Therefore, the standard elliptic estimates tell us that $C(x) \in L^\infty(\Omega)$ implies $V_k, F_k, G_k \in W^{2,p}(\Omega)$ (for any $p > 1$). Hence $\rho_k, \eta_k \in W^{4,p}(\Omega)$. Then the proof of Theorem 2.1 is completed.

3 Uniform Estimates of Approximate Solution

Suppose that $\rho_0, \eta_0 \in H^1(\Omega)$ and $C \in L^\infty(\Omega)$ satisfy the assumption of Theorem 2.1 for $k = 1$, i.e.,

$$\min_{\overline{\Omega}} \rho_0 > 0, \quad \min_{\overline{\Omega}} \eta_0 > 0 \quad \text{and} \quad \int_\Omega (\rho_0^2 - \eta_0^2) dx = \int_\Omega C(x) dx.$$

Let $V_0 \in W^{2,p}(\Omega)$ (for any $p > 1$) be the unique (up to an additive constant) solution of $(V_0)_{xx} = \rho_0^2 - \eta_0^2 - C(x)$ with the initial and boundary conditions $(V_0)_x = 0$ on $\partial\Omega$ and

$$\int_\Omega (\rho_0^2 - \eta_0^2) dx = \int_\Omega C(x) dx.$$

We use Theorem 2.1 iteratively to obtain a sequence of approximate solution $(\rho_k, \eta_k, F_k, G_k, V_k) \in (W^{4,p}(\Omega))^2 \times (W^{2,p}(\Omega))^3$ $(k = 1, 2, \cdots, N)$. In this section, we focus on the uniform estimates for the approximate solution.

From now on, C is supposed to be a constant dependent only on $\varepsilon, T, \gamma, \|C(\cdot)\|_{L^\infty(\Omega)}$, $\|\rho_0\|_{H^1(\Omega)}$, $\|\ln\rho_0\|_{L^1(\Omega)}$, $\|\eta_0\|_{H^1(\Omega)}$, and $\|\ln\eta_0\|_{L^1(\Omega)}$. By using the test functions $F_k, G_k, \ln\rho_k$, $\ln\eta_k$ and $1 - \frac{1}{\rho_k^2}, 1 - \frac{1}{\eta_k^2}$, respectively, Chen and Ju [7] have obtained the estimates in the following Lemmas 3.1–3.3. We will get another estimate (see Lemma 3.4) by employing the other two test functions $-\frac{(\rho_k)_{xx}}{\rho_k}$ and $-\frac{(\eta_k)_{xx}}{\eta_k}$ which was first used in [6, 5].

Lemma 3.1

$$\begin{aligned}
&\varepsilon^2 \int_\Omega (|(\rho_k)_x|^2 + |(\eta_k)_x|^2) dx + \int_\Omega \Big(\frac{1}{\alpha - 1}\rho_k^{2\alpha} + \frac{1}{\beta - 1}\eta_k^{2\beta}\Big) dx \\
&+ \frac{1}{2}\int_\Omega |(V_k)_x|^2 dx + \tau \int_\Omega (\rho_k^2 |(F_k)_x|^2 + \eta_k^2 |(G_k)_x|^2) dx \\
\le\, &\varepsilon^2 \int_\Omega (|(\rho_{k-1})_x|^2 + |(\eta_{k-1})_x|^2) dx + \int_\Omega \Big(\frac{1}{\alpha - 1}\rho_{k-1}^{2\alpha} + \frac{1}{\beta - 1}\eta_{k-1}^{2\beta}\Big) dx + \frac{1}{2}\int_\Omega |(V_{k-1})_x|^2 dx.
\end{aligned}$$

Lemma 3.2

$$\begin{aligned}
&\int_\Omega [\rho_k^2(\ln\rho_k^2 - 1) + 1] dx + \int_\Omega [\eta_k^2(\ln\eta_k^2 - 1) + 1] dx \\
&+ 2\varepsilon^2\tau \int_\Omega (|(\rho_k)_{xx}|^2 + |(\eta_k)_{xx}|^2) dx + \frac{32}{3}\varepsilon^2\tau \int_\Omega (|(\sqrt{\rho_k})_x|^4 + |(\sqrt{\eta_k})_x|^4) dx \\
&+ 4\alpha\tau \int_\Omega \rho_k^{2(\alpha-1)} |(\rho_k)_x|^2 dx + 4\beta\tau \int_\Omega \eta_k^{2(\beta-1)} |(\eta_k)_x|^2 dx \\
\le &\int_\Omega [\rho_{k-1}^2(\ln\rho_{k-1}^2 - 1) + 1] dx + \int_\Omega [\eta_{k-1}^2(\ln\eta_{k-1}^2 - 1) + 1] dx \\
&+ \tau \int_\Omega [C(x) - (\rho_k^2 - \eta_k^2)](\rho_k^2 - \eta_k^2) dx.
\end{aligned} \tag{3.1}$$

Lemma 3.3

$$\int_\Omega [(\rho_k^2 - \ln\rho_k^2) + (\eta_k^2 - \ln\eta_k^2)] dx + 2\varepsilon^2\tau \int_\Omega (|(\ln\rho_k)_{xx}|^2 + |(\ln\eta_k)_{xx}|^2) dx$$

$$+4\tau\int_\Omega[\alpha\rho_k^{2(\alpha-1)}|(\ln\rho_k)_x|^2+\beta\eta_k^{2(\beta-1)}|(\ln\eta_k)_x|^2]dx+\tau\int_\Omega(\rho_k^2-\eta_k^2)(\ln\rho_k^2-\ln\eta_k^2)dx$$
$$\le\int_\Omega[(\rho_{k-1}^2-\ln\rho_{k-1}^2)+(\eta_{k-1}^2-\ln\eta_{k-1}^2)]dx+C\tau\int_\Omega(|\ln\rho_k|+|\ln\eta_k|)dx.$$

Lemma 3.4

$$\int_\Omega(|(\rho_k)_x|^2+|(\eta_k)_x|^2)dx+\frac{1}{2}\varepsilon^2\tau\int_\Omega\Big[\rho_k^2\Big|\Big(\frac{(\rho_k)_{xx}}{\rho_k}\Big)_x\Big|^2+\eta_k^2\Big|\Big(\frac{(\eta_k)_{xx}}{\eta_k}\Big)_x\Big|^2\Big]dx$$
$$\le\int_\Omega(|(\rho_{k-1})_x|^2+|(\eta_{k-1})_x|^2)dx+C\tau\int_\Omega(\rho_k^{4(\alpha-1)}|(\rho_k)_x|^2+\eta_k^{4(\beta-1)}|(\eta_k)_x|^2)dx$$
$$+C\tau\int_\Omega[(\rho_k)_x^2\eta_k^2+(\eta_k)_x^2\rho_k^2+\rho_k^2+\eta_k^2+|(\rho_k)_x|^2+|(\eta_k)_x|^2+|(\rho_k)_{xx}|^2+|(\eta_k)_{xx}|^2]dx. \quad (3.2)$$

Proof Multiplying the first equation of (2.1) by $-\frac{(\rho_k)_{xx}}{\rho_k}$, we have

$$\int_\Omega\frac{\rho_k^2-\rho_{k-1}^2}{\tau}\Big(-\frac{(\rho_k)_{xx}}{\rho_k}\Big)dx=\int_\Omega[\rho_k^2(F_k)_x]_x\Big(-\frac{(\rho_k)_{xx}}{\rho_k}\Big)dx.$$

By integral by parts and Cauchy inequality, we get

$$\int_\Omega\frac{\rho_k^2-\rho_{k-1}^2}{\tau}\Big(-\frac{(\rho_k)_{xx}}{\rho_k}\Big)dx\ge\frac{1}{\tau}\Big[\int_\Omega|(\rho_k)_x|^2dx-\int_\Omega|(\rho_{k-1})_x|^2dx\Big].$$

Then integrating by parts again, we deduce by Young inequality that

$$\frac{1}{\tau}\Big[\int_\Omega|(\rho_k)_x|^2dx-\int_\Omega|(\rho_{k-1})_x|^2dx\Big]\le\int_\Omega[\rho_k^2(F_k)_x]_x\Big(-\frac{(\rho_k)_{xx}}{\rho_k}\Big)dx$$
$$=-\varepsilon^2\int_\Omega\rho_k^2\Big|\Big(\frac{(\rho_k)_{xx}}{\rho_k}\Big)_x\Big|^2dx+2\alpha\int_\Omega\rho_k^{2\alpha-1}(\rho_k)_x\Big(\frac{(\rho_k)_{xx}}{\rho_k}\Big)_xdx-\int_\Omega\rho_k^2(V_k)_x\Big(\frac{(\rho_k)_{xx}}{\rho_k}\Big)_xdx$$
$$\le-\frac{1}{2}\varepsilon^2\int_\Omega\rho_k^2\Big|\Big(\frac{(\rho_k)_{xx}}{\rho_k}\Big)_x\Big|^2dx+C\int_\Omega\rho_k^{4(\alpha-1)}|(\rho_k)_x|^2dx-\int_\Omega\big[(\rho_k)_x^2-\rho_k(\rho_k)_{xx}\big](V_k)_{xx}dx.$$

Multiplying the second equation of (2.1) by $-\frac{(\eta_k)_{xx}}{\eta_k}$, we get the similar inequality

$$\frac{1}{\tau}\Big[\int_\Omega|(\eta_k)_x|^2dx-\int_\Omega|(\eta_{k-1})_x|^2dx\Big]$$
$$\le-\frac{1}{2}\varepsilon^2\int_\Omega\eta_k^2\Big|\Big(\frac{(\eta_k)_{xx}}{\eta_k}\Big)_x\Big|^2dx+C\int_\Omega\eta_k^{4(\beta-1)}|(\eta_k)_x|^2dx+\int_\Omega\big[(\eta_k)_x^2-\eta_k(\eta_k)_{xx}\big](V_k)_{xx}dx.$$

Hence

$$\frac{1}{\tau}\int_\Omega(|(\rho_k)_x|^2+|(\eta_k)_x|^2)dx+\frac{1}{2}\varepsilon^2\int_\Omega\Big[\rho_k^2\Big|\Big(\frac{(\rho_k)_{xx}}{\rho_k}\Big)_x\Big|^2+\eta_k^2\Big|\Big(\frac{(\eta_k)_{xx}}{\eta_k}\Big)_x\Big|^2\Big]dx$$
$$\le\frac{1}{\tau}\int_\Omega(|(\rho_{k-1})_x|^2+|(\eta_{k-1})_x|^2)dx+C\int_\Omega(\rho_k^{4(\alpha-1)}|(\rho_k)_x|^2+\eta_k^{4(\beta-1)}|(\eta_k)_x|^2)dx+I,$$

where

$$I=-\int_\Omega[(\rho_k)_x^2-\rho_k(\rho_k)_{xx}](V_k)_{xx}dx+\int_\Omega[(\eta_k)_x^2-\eta_k(\eta_k)_{xx}](V_k)_{xx}dx.$$

By Cauchy inequality and integral by parts, we have

$$
\begin{aligned}
I &= \int_\Omega [\rho_k(\rho_k)_{xx} - (\rho_k)_x^2 + (\eta_k)_x^2 - \eta_k(\eta_k)_{xx}](\rho_k^2 - \eta_k^2 - C(x))dx \\
&\le \int_\Omega [-4\rho_k^2(\rho_k)_x^2 - 4\eta_k^2(\eta_k)_x^2 + 2(\rho_k)_x^2\eta_k^2 + 2\rho_k^2(\eta_k)_x^2 + 4\rho_k(\rho_k)_x\eta_k(\eta_k)_x]dx \\
&\quad + C\int_\Omega [\rho_k^2 + \eta_k^2 + |(\rho_k)_x|^2 + |(\eta_k)_x|^2 + |(\rho_k)_{xx}|^2 + |(\eta_k)_{xx}|^2]dx \\
&\le C\int_\Omega [(\rho_k)_x^2\eta_k^2 + (\eta_k)_x^2\rho_k^2 + \rho_k^2 + \eta_k^2 + |(\rho_k)_x|^2 + |(\eta_k)_x|^2 + |(\rho_k)_{xx}|^2 + |(\eta_k)_{xx}|^2]dx.
\end{aligned}
$$

Consequently, (3.2) is proved.

Definition 3.1 *Let $(\rho_k, \eta_k, F_k, G_k, V_k) \in (W^{4,p}(\Omega))^2 \times (W^{2,p}(\Omega))^3$ be the solution in Theorem 2.1, $k = 1, 2, \cdots, N$. We define $\rho_\tau(t,x) \triangleq \rho_k(x)$ for $x \in \Omega$, $t \in ((k-1)\tau, k\tau]$, and similarly $\eta_\tau, F_\tau, G_\tau, V_\tau$.*

Remark 3.1 In Definition 3.1, $0 < \rho_\tau(t,\cdot), \eta_\tau(t,\cdot) \in H$ for any $t \in (0,T]$ (in fact, by Theorem 2.1, we have $0 < \rho_k, \eta_k \in H$ for any $1 \le k \le N$).

Using Lemmas 3.1–3.4 and copying the arguments of [6], we obtain the following uninform estimates.

Theorem 3.1

$$
\begin{aligned}
&\|\rho_\tau\|_{L^\infty(0,T;H^1(\Omega))\bigcap L^2(0,T;H^2(\Omega))} + \|\eta_\tau\|_{L^\infty(0,T;H^1(\Omega))\bigcap L^2(0,T;H^2(\Omega))} \\
&+ \|(V_\tau)_x\|_{L^\infty(0,T;L^2(\Omega))} + \|\rho_\tau(F_\tau)_x\|_{L^2(Q_T)} + \|\eta_\tau(G_\tau)_x\|_{L^2(Q_T)} \\
&+ \Big\|\rho_\tau\Big(\frac{(\rho_\tau)_{xx}}{\rho_\tau}\Big)_x\Big\|_{L^2(Q_T)} + \Big\|\eta_\tau\Big(\frac{(\eta_\tau)_{xx}}{\eta_\tau}\Big)_x\Big\|_{L^2(Q_T)} \\
&+ \|\ln\rho_\tau\|_{L^\infty(0,T;L^1(\Omega))} + \|\ln\eta_\tau\|_{L^\infty(0,T;L^1(\Omega))} + \|(\ln\rho_\tau)_{xx}\|_{L^2(Q_T)} \\
&+ \|(\ln\eta_\tau)_{xx}\|_{L^2(Q_T)} + \|(\rho_\tau)_{xxx}\|_{L^{\frac{6}{5}}(Q_T)} + \|(\eta_\tau)_{xxx}\|_{L^{\frac{6}{5}}(Q_T)} \le C(\varepsilon). \qquad (3.3)
\end{aligned}
$$

4 Existence of Weak Solution

Throughout this section, let ρ_τ, η_τ, F_τ, G_τ, V_τ be the functions in Definition 3.1, which satisfy Theorem 3.1. Using a compactness argument and Aubin-Lions lemma (see [21]) and repeating the arguments of [6], we can prove the following convergence of ρ_τ, η_τ, V_τ (see Theorem 4.1), which will complete the proof of Theorem 1.1.

Definition 4.1 *We define the linear interpolation and difference quotient of ρ_τ^2 by*

$$
\widetilde{\rho_\tau^2}(t,x) \triangleq \begin{cases} \dfrac{t-(k-1)\tau}{\tau}(\rho_k^2(x) - \rho_{k-1}^2(x)) + \rho_{k-1}^2(x) & \text{for } x \in \Omega,\ t \in ((k-1)\tau, k\tau], \\ \rho_0^2(x) & \text{for } x \in \Omega,\ t = 0 \end{cases}
$$

and

$$
\partial_t^\tau \rho_\tau^2(t,x) \triangleq \begin{cases} \dfrac{\rho_k^2(x) - \rho_{k-1}^2(x)}{\tau} & \text{for } x \in \Omega,\ t \in ((k-1)\tau, k\tau], \\ \rho_0^2(x) & \text{for } x \in \Omega,\ t = 0, \end{cases}
$$

respectively. The linear interpolation and difference quotient of η_τ^2 are similarly defined.

Theorem 4.1 *Let $0 < \mu < \frac{1}{2}$ and $0 < \nu < \frac{1}{6}$. Then for any fixed $\varepsilon > 0$, as $\tau \to 0$, there exists a subsequence of $\{(\rho_\tau, V_\tau, \widetilde{\rho_\tau^2}, \partial_t^\tau \rho_\tau^2)\}_{\tau>0}$ which is not relabeled, such that*

$$\rho_\tau \rightharpoonup \rho, \qquad \text{in } L^2(0,T;H), \tag{4.1}$$

$$\rho_\tau \overset{*}{\rightharpoonup} \rho, \qquad \text{in } L^\infty(0,T;H^1(\Omega)), \tag{4.2}$$

$$\rho_\tau^2 \to \rho^2, \qquad \text{in } L^p(0,T;C^{0,\alpha}(\overline{\Omega}))\ (\forall\, 1 \le p < \infty), \tag{4.3}$$

$$\rho_\tau^2 \to \rho^2, \qquad \text{in } L^2(0,T;C^{1,\alpha}(\overline{\Omega})) \cap L^{\frac{6}{5}}(0,T;C^{2,\beta}(\overline{\Omega})), \tag{4.4}$$

$$\widetilde{\rho_\tau^2} \to \rho^2, \qquad \text{in } C([0,T];C^{0,\alpha}(\overline{\Omega})), \tag{4.5}$$

$$\partial_t^\tau \rho_\tau^2 \rightharpoonup \partial_t \rho^2, \qquad \text{in } L^2(0,T;(H^1(\Omega))'), \tag{4.6}$$

$$\rho_\tau(\rho_\tau)_{xx} \rightharpoonup \rho\rho_{xx}, \qquad \text{in } L^2(Q_T), \tag{4.7}$$

$$\rho_\tau(\rho_\tau)_{xxx} \rightharpoonup \rho\rho_{xxx}, \qquad \text{in } L^{\frac{6}{5}}(Q_T), \tag{4.8}$$

$$(\rho_\tau^2)_x \to (\rho^2)_x, \qquad \text{in } L^2(Q_T), \tag{4.9}$$

$$\rho_\tau^{2(\alpha-1)}(\rho_\tau^2)_x \overset{*}{\rightharpoonup} \rho^{2(\alpha-1)}(\rho^2)_x, \qquad \text{in } L^\infty(0,T;L^2(\Omega)), \tag{4.10}$$

$$(V_\tau)_x(t,\cdot) \rightharpoonup E(t,\cdot), \qquad \text{in } L^2(\Omega),\ \text{for a.e. } t \in (0,T), \tag{4.11}$$

$$\rho_\tau^2(V_\tau)_x \overset{*}{\rightharpoonup} \rho^2 E, \qquad \text{in } L^\infty(0,T;L^2(\Omega)), \tag{4.12}$$

$$\rho_\tau^2\Big(\frac{(\rho_\tau)_{xx}}{\rho_\tau}\Big)_x \rightharpoonup P, \qquad \text{in } L^2(Q_T), \tag{4.13}$$

where $\rho \ge 0$. There are similar convergent results for η_τ.

Proof of Theorem 1.1 The proof is obvious due to Theorems 2.1 and 4.1.

5 Long-Time Behavior

In this section, we try to study the long-time behavior of (1.2) and (1.4) by using a logarithmic Sobolev inequality. Let $\mathbb{T}$ be parameterized by $x \in [0, L]$.

Proof of Theorem 1.2 Since the proof is the same as that of Theorem 1.1, we omit the details.

Remark 5.1 In particular, (3.1), (3.3) and (4.3) also hold in the periodic case, which would be used in the following.

Lemma 5.1 (See [9]) *Let $\mathcal{H} = \{u \in H^2(\mathbb{T}) \,|\, u_x \ne 0\}$ and $\overline{u^2} = \frac{\int_{\mathbb{T}} u^2 dx}{L}$. Then*

$$\inf_{u\in\mathcal{H}} \frac{\int_{\mathbb{T}} u_{xx}^2 dx}{\int_{\mathbb{T}} u^2 \ln\big(\frac{u^2}{\overline{u^2}}\big) dx} = \frac{8\pi^4}{L^4}.$$

Lemma 5.2 *The assumption of Theorem* 1.3 *implies that the weak solution (ρ, η) obtained* *in Theorem* 1.2 *satisfies for* $M = \frac{16\pi^4\varepsilon^2}{L^4}$,

$$\int_{\mathbb{T}} \Big[\rho^2(t,\cdot)\ln\frac{\rho^2(t,\cdot)}{\overline{\rho_0^2}} + \eta^2(t,\cdot)\ln\frac{\eta^2(t,\cdot)}{\overline{\eta_0^2}}\Big]dx \le e^{-Mt}\int_{\mathbb{T}}\Big[\rho_0^2\ln\frac{\rho_0^2}{\overline{\rho_0^2}} + \eta_0^2\ln\frac{\eta_0^2}{\overline{\eta_0^2}}\Big]dx, \quad \forall\, t\in(0,T], \tag{5.1}$$

Proof By Lemma 5.1, the approximate solutions of (1.4) satisfies

$$\frac{8\pi^4}{L^4}\int_{\mathbb{T}}\rho_k^2\ln\frac{\rho_k^2}{\overline{\rho_k^2}}dx \le \int_{\mathbb{T}}|(\rho_k)_{xx}|^2dx \quad \text{and} \quad \frac{8\pi^4}{L^4}\int_{\mathbb{T}}\eta_k^2\ln\frac{\eta_k^2}{\overline{\eta_k^2}}dx \le \int_{\mathbb{T}}|(\eta_k)_{xx}|^2dx,$$

where $\overline{\rho_k^2} = \frac{\int_{\mathbb{T}}\rho_k^2dx}{L}$, $\overline{\eta_k^2} = \frac{\int_{\mathbb{T}}\eta_k^2dx}{L}$. Then it follows from (3.1) in periodic case that

$$\begin{aligned}&\frac{1}{\tau}\int_{\mathbb{T}}[\rho_k^2(\ln\rho_k^2-1)+\eta_k^2(\ln\eta_k^2-1)]dx + 2\varepsilon^2\int_{\mathbb{T}}[|(\rho_k)_{xx}|^2+|(\eta_k)_{xx}|^2]dx \\ &\le \frac{1}{\tau}\int_{\mathbb{T}}[\rho_{k-1}^2(\ln\rho_{k-1}^2-1)+\eta_{k-1}^2(\ln\eta_{k-1}^2-1)]dx + \int_{\Omega}[C(x)-(\rho_k^2-\eta_k^2)](\rho_k^2-\eta_k^2)dx.\end{aligned}$$

Since

$$C(x)\equiv C,\quad \int_{\mathbb{T}}(\rho_k^2-\eta_k^2)dx = \int_{\mathbb{T}}C(x)dx,\quad \int_{\mathbb{T}}\rho_k^2dx = \int_{\mathbb{T}}\rho_{k-1}^2dx \quad\text{and}\quad \int_{\mathbb{T}}\eta_k^2dx = \int_{\mathbb{T}}\eta_{k-1}^2dx,$$

we have

$$\overline{\rho_k^2} = \overline{\rho_0^2},\quad \overline{\eta_k^2} = \overline{\eta_0^2} \quad\text{and}\quad \int_{\mathbb{T}}[C(x)-(\rho_k^2-\eta_k^2)](\rho_k^2-\eta_k^2)dx \le 0,$$

so that

$$\begin{aligned}&\frac{1}{\tau}\int_{\mathbb{T}}\Big[\rho_k^2\ln\frac{\rho_k^2}{\overline{\rho_0^2}} - \rho_{k-1}^2\ln\frac{\rho_{k-1}^2}{\overline{\rho_0^2}}\Big]dx + \frac{1}{\tau}\int_{\mathbb{T}}\Big[\eta_k^2\ln\frac{\eta_k^2}{\overline{\eta_0^2}} - \eta_{k-1}^2\ln\frac{\eta_{k-1}^2}{\overline{\eta_0^2}}\Big]dx \\ &+ \frac{16\pi^4\varepsilon^2}{L^4}\int_{\mathbb{T}}\Big[\rho_k^2\ln\frac{\rho_k^2}{\overline{\rho_0^2}} + \eta_k^2\ln\frac{\eta_k^2}{\overline{\eta_0^2}}\Big]dx \le 0.\end{aligned}$$

Let

$$E_k = \int_{\mathbb{T}}\Big[\rho_k^2\ln\frac{\rho_k^2}{\overline{\rho_0^2}} + \eta_k^2\ln\frac{\eta_k^2}{\overline{\eta_0^2}}\Big]dx \quad\text{and}\quad M = \frac{16\pi^4\varepsilon^2}{L^4}.$$

Then we have

$$(1+\tau M)E_k \le E_{k-1}$$

and hence

$$E_k \le E_0(1+\tau M)^{-k}.$$

Therefore

$$E_k \le E_0(1+\tau M)^{-t/\tau},\quad \forall\, t\in((k-1)\tau, k\tau],$$

namely

$$\int_{\mathbb{T}}\rho_\tau^2\ln\frac{\rho_\tau^2}{\overline{\rho_0^2}}dx + \int_{\mathbb{T}}\eta_\tau^2\ln\frac{\eta_\tau^2}{\overline{\eta_0^2}}dx \le E_0(1+\tau M)^{-t/\tau},\quad \forall\, t\in((k-1)\tau,k\tau]. \tag{5.2}$$

In view of (4.3) in periodic case, as $\tau \to 0$, we obtain $\rho_\tau^2 \to \rho^2$ a.e. in $(0,T)\times\mathbb{T}$, so that $\rho_\tau^2 \ln\frac{\rho_\tau^2}{\overline{\rho_0^2}} \to \rho^2 \ln\frac{\rho^2}{\overline{\rho_0^2}}$ a.e. in $(0,T)\times\mathbb{T}$. Using the convexity of $f(z)=z\ln z$ in $(0,\infty)$ and (3.3) in periodic case, we have $\big|\rho_\tau^2 \ln\frac{\rho_\tau^2}{\overline{\rho_0^2}}\big| \le C$ in $(0,T)\times\mathbb{T}$, where C is a constant independent of τ, t, x. Similarly, one has $\big|\eta_\tau^2 \ln\frac{\eta_\tau^2}{\overline{\eta_0^2}}\big| \le C$ in $(0,T)\times\mathbb{T}$. Consequently, letting $\tau\to 0$ in (5.2), we obtain (5.1) by Lebesgue dominated convergence theorem.

Proof of Theorem 1.3 By the inequality $a-b+(\sqrt{a}-\sqrt{b}\,)^2 \le a\ln\frac{a}{b}$, $\forall\, a,b>0$, we have

$$\rho^2 - \overline{\rho_0^2} + \Big(\rho - \sqrt{\overline{\rho_0^2}}\,\Big)^2 \le \rho^2 \ln\frac{\rho^2}{\overline{\rho_0^2}}. \tag{5.3}$$

In view of $\overline{\rho_k^2} = \overline{\rho_0^2}$ obtained in the proof of Lemma 5.2, we obtain $\int_{\mathbb{T}} \rho^2 dx = \int_{\mathbb{T}} \overline{\rho_0^2} dx$ with the aid of (3.3) and (4.3) in periodic case. Hence, by integrating (5.3), we deduce that

$$\int_{\mathbb{T}} \Big(\rho(t,\,\cdot\,) - \sqrt{\overline{\rho_0^2}}\,\Big)^2 dx \le \int_{\mathbb{T}} \rho^2 \ln\frac{\rho^2}{\overline{\rho_0^2}} dx, \quad \forall\, t\in(0,T].$$

Similarly, we get

$$\int_{\mathbb{T}} \Big(\eta(t,\,\cdot\,) - \sqrt{\overline{\eta_0^2}}\,\Big)^2 dx \le \int_{\mathbb{T}} \eta^2 \ln\frac{\eta^2}{\overline{\eta_0^2}} dx, \quad \forall\, t\in(0,T].$$

This proves (1.10) by Lemma 5.2.

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