Chin. Ann. Math.
32B(4), 2011, 521–526
DOI: 10.1007/s11401-011-0652-5

Chinese Annals of Mathematics, Series B

Sharpness on the Lower Bound of the Lifespan of Solutions to Nonlinear Wave Equations*

Yi ZHOU[1] Wei HAN[2]

Abstract This paper is devoted to proving the sharpness on the lower bound of the lifespan of classical solutions to general nonlinear wave equations with small initial data in the case $n = 2$ and cubic nonlinearity (see the results of T. T. Li and Y. M. Chen in 1992). For this purpose, the authors consider the following Cauchy problem:

$$\begin{cases}\Box u = (u_t)^3, \quad n = 2,\\ t = 0: \ u = 0, \quad u_t = \varepsilon g(x), \quad x \in \mathbb{R}^2,\end{cases}$$

where $\Box = \partial_t^2 - \sum\limits_{i=1}^{n} \partial_{x_i}^2$ is the wave operator, $g(x) \not\equiv 0$ is a smooth non-negative function on $\mathbb{R}^2$ with compact support, and $\varepsilon > 0$ is a small parameter. It is shown that the solution blows up in a finite time, and the lifespan $T(\varepsilon)$ of solutions has an upper bound $T(\varepsilon) \le \exp(A\varepsilon^{-2})$ with a positive constant A independent of ε, which belongs to the same kind of the lower bound of the lifespan.

Keywords Nonlinear wave equation, Cauchy problem, Lifespan
2000 MR Subject Classification 35L45, 35L60

1 Introduction and Main Results

Consider the Cauchy problem for the following n-dimensional nonlinear wave equation

$$\begin{cases}u_{tt} - \Delta u = F(Du, D_x Du),\\ t = 0: \ u = \varepsilon f(x), \quad u_t = \varepsilon g(x),\end{cases} \tag{1.1}$$

where $x = (x_1, \cdots, x_n)$, $\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$,

$$Du = (u_t, u_{x_1}, \cdots, u_{x_n}) = (u_{x_0}, u_{x_1}, \cdots, u_{x_n}),$$
$$D_x Du = (u_{x_i x_j}, \ i, j = 0, 1, \cdots, n, \ i + j \ge 1),$$

$f(x), g(x) \in C_0^\infty(\mathbb{R}^n)$, and $\varepsilon > 0$ is a small parameter. Here, for simplicity of notations, we write $x_0 = t$.

Manuscript received June 23, 2010. Revised September 28, 2010.
[1]Nonlinear Mathematical Modeling and Methods Laboratory; Shanghai Key Laboratory for Contemporary Applied Mathematics; School of Mathematical Sciences, Fudan University, Shanghai 200433, China. E-mail: yizhou@fudan.ac.cn
[2]School of Mathematical Sciences, Fudan University, Shanghai 200433, China; Department of Mathematics, North University of China, Taiyuan 030051, China. E-mail: sh_hanweiwei1@126.com
*Project supported by the National Natural Science Foundation of China (No. 10728101), the Basic Research Program of China (No. 2007CB814800), the Doctoral Program Foundation of the Ministry of Education of China, the "111" Project (No. B08018) and SGST (No. 09DZ2272900).

Let

$$\widehat{\lambda} = ((\lambda_i),\ i = 0, 1, \cdots, n;\ (\lambda_{ij}),\ i, j = 0, 1, \cdots, n,\ i + j \geq 1).$$

Suppose that in a neighbourhood of $\widehat{\lambda} = 0$, for $|\widehat{\lambda}| \leq 1$, the nonlinear term $F = F(\widehat{\lambda})$ in equation (1.1) is a sufficiently smooth function with

$$F(\widehat{\lambda}) = O(|\widehat{\lambda}|^{1+\alpha}),$$

where α is an integer and $\alpha \geq 1$.

The lifespan $T(\varepsilon)$ of classical solutions to problem (1.1) is defined to be the supremum of all $\tau > 0$, such that there exists a classical solution to (1.1) for $x \in \mathbb{R}^2$ on $0 \leq t \leq \tau$. Li and Chen [5] used a unified and simple method suggested by Li and Yu [6, 7] to get a complete result concerning the lower bound of the lifespan of classical solutions to (1.1) for all integers α, n with $\alpha \geq 1$ and $n \geq 1$ as follows:

$$T(\varepsilon) \geq \begin{cases} +\infty, & \text{if } K_0 > 1, \\ \exp\{a\varepsilon^{-\alpha}\}, & \text{if } K_0 = 1, \\ b\varepsilon^{-\frac{\alpha}{1-K_0}}, & \text{if } 0 \leq K_0 < 1, \end{cases} \tag{1.2}$$

where $K_0 \triangleq \frac{n-1}{2}\alpha$, and a, b are positive constants depending only on α and n.

As stated in [5], all lower bounds in (1.2), except the case $n = 2$ and $\alpha = 2$, are known to be sharp due to Lax [4] (for $n = 1$ and $\alpha = 1$), John [1] and Zhou [13] (for $n = 2, 3$ and $\alpha = 1$), Kong [3] (for $n = 1$ and $\alpha \geq 1$) and Zhou [13] (for $n \geq 1$ and odd $\alpha \geq 1$). However, up to now there is no sharpness result on the lower bound of the lifespan

$$T(\varepsilon) \geq \exp\{a\varepsilon^{-2}\} \tag{1.3}$$

for solutions to problem (1.1) in the case $n = 2$ and $\alpha = 2$. The aim of this paper is to show the sharpness of (1.3) for small $\varepsilon > 0$ in the case $n = 2$ and $\alpha = 2$.

For this purpose, we consider the following Cauchy problem for the nonlinear wave equation in two space dimensions:

$$\begin{cases} \Box u = (u_t)^3, \quad n = 2, \\ t = 0:\ u = 0, \quad u_t = \varepsilon g(x), \quad x \in \mathbb{R}^2, \end{cases} \tag{1.4}$$

where $\Box = \partial_t^2 - \sum\limits_{i=1}^{n} \partial_{x_i}^2$ is the wave operator, $g(x)$ is a smooth non-negative and radially symmetric function on $\mathbb{R}^2$ with compact support and $g(x) \not\equiv 0$. We will prove that the lifespan $T(\varepsilon)$ of classical solutions to (1.4) possesses an upper bound estimate belonging to the same kind of the lower bound of the lifespan.

For the Cauchy problem

$$\begin{cases} \Box u = |u_t|^p, \\ t = 0:\ u = \varepsilon f(x), \quad u_t = \varepsilon g(x), \quad x \in \mathbb{R}^n, \end{cases} \tag{1.5}$$

Zhou [13] and Takamura [12] obtained the blow-up result, and gave the estimate on the lifespan. In particular, when $p = 3$ and $n = 2$, (1.5) becomes

$$\begin{cases} \Box u = |u_t|^3, \\ t = 0:\ u = \varepsilon f(x), \quad u_t = \varepsilon g(x), \quad x \in \mathbb{R}^2. \end{cases} \tag{1.6}$$

However, from that result, we cannot get the desired sharpness of the lifespan.

The main result of this paper is as follows.

Theorem 1.1 *Suppose that $g(x)$ is a smooth non-negative and radially symmetric function on $\mathbb{R}^2$ with compact support*

$$\operatorname{supp} g \subseteq \{x : |x| \le 1\} \tag{1.7}$$

and $g(x) \not\equiv 0$. If $u = u(t,x)$ is a non-trivial C^2-solution to the Cauchy problem (1.4), then $u = u(t,x)$ blows up in a finite time, and there exists a positive constant A independent of ε, such that the lifespan $T(\varepsilon)$ satisfies

$$T(\varepsilon) \le \exp(A\varepsilon^{-2}). \tag{1.8}$$

The related studies on the blow-up of solutions to nonlinear wave equations can be found in [1–13].

We will give the proof of Theorem 1.1 in Section 2.

2 Proof of Theorem 1.1

Consider the following Cauchy problem:

$$\begin{cases} \Box u = (u_t)^3, \quad n = 2, \\ t = 0: \ u = 0, \quad u_t = \varepsilon g(x), \quad x \in \mathbb{R}^2. \end{cases} \tag{2.1}$$

We first prove that in the domain $r > t$, for the solution $u = u(t,x)$ to Cauchy problem (2.1), we have $u \ge 0$ and $u_t \ge 0$.

By the local existence of classical solutions, the solution to Cauchy problem (2.1) can be approximated by Picard iteration. Let

$$u^{(0)} = 0$$

and

$$\begin{cases} \Box u^{(m)} = (u_t^{(m-1)})^3, \quad n = 2, \\ t = 0: \ u^{(m)} = 0, \quad u_t^{(m)} = \varepsilon g(x), \quad x \in \mathbb{R}^2. \end{cases} \tag{2.2}$$

Then $\{u^{(m)}(t,x)\}$ is a series of approximate solutions to (2.1).

Since $u^{(0)} = 0$, we have $u_t^{(0)} = 0$. As an induction hypothesis, we may suppose that $u^{(m-1)} \ge 0$, $u_t^{(m-1)} \ge 0$ in the domain $r > t$.

By the Duhamel principle, the solution to the Cauchy problem (2.2) of the above two-space-dimensional inhomogeneous wave equation can be expressed as

$$\begin{aligned} u^{(m)}(t,x) = \frac{1}{2\pi}\Big[&\int_{\{y:|y-x|\le t\}} \frac{\varepsilon g(y)}{\sqrt{t^2 - |y-x|^2}} \mathrm{d}y \\ &+ \int_0^t \int_{\{y:|y-x|\le t-\tau\}} \frac{(u_t^{(m-1)}(\tau,y))^3}{\sqrt{(t-\tau)^2 - |y-x|^2}} \mathrm{d}y \mathrm{d}\tau \Big]. \end{aligned} \tag{2.3}$$

Since

$$u_t^{(m-1)} \ge 0, \quad r > t,$$

and $g(x)$ is non-negative, it is easy to see that

$$u^{(m)} \geq 0, \quad r > t. \tag{2.4}$$

Let

$$r = |x|, \quad x \in \mathbb{R}^2.$$

The radially symmetric form of problem (1.4) can be written as

$$\begin{cases} u_{tt} - u_{rr} - \dfrac{u_r}{r} = (u_t)^3, \\ t = 0: \ u = 0, \quad u_t = \varepsilon g(r). \end{cases}$$

In order to estimate $u_t^{(m)}$, we transform (2.2) into the following form:

$$\begin{cases} u_{tt}^{(m)} - u_{rr}^{(m)} - \dfrac{u_r^{(m)}}{r} = (u_t^{(m-1)})^3, \\ t = 0: \ u = 0, \quad u_t = \varepsilon g(r), \end{cases} \tag{2.5}$$

where $r = |x|$ and $x \in \mathbb{R}^2$. It follows from (2.5) that

$$\begin{cases} (\partial_t^2 - \partial_r^2)(r^{\frac{1}{2}} u^{(m)}) = \frac{1}{4} r^{-\frac{3}{2}} u^{(m)} + r^{\frac{1}{2}} (u_t^{(m-1)})^3, \\ t = 0: \ r^{\frac{1}{2}} u^{(m)} = 0, \quad r^{\frac{1}{2}} u_t^{(m)} = \varepsilon r^{\frac{1}{2}} g(r). \end{cases} \tag{2.6}$$

By d'Alembert's formula, in the domain $r > t$, we have

$$\begin{aligned} r^{\frac{1}{2}} u^{(m)}(t,r) = {} & \frac{1}{2} \int_{r-t}^{r+t} \varepsilon \lambda^{\frac{1}{2}} g(\lambda) \mathrm{d}\lambda + \frac{1}{8} \int_0^t \int_{r-(t-\tau)}^{r+t-\tau} \frac{u^{(m)}(\tau,\lambda)}{\lambda^{\frac{3}{2}}} \mathrm{d}\lambda \mathrm{d}\tau \\ & + \frac{1}{2} \int_0^t \int_{r-(t-\tau)}^{r+t-\tau} \lambda^{\frac{1}{2}} (u_t^{(m-1)})^3(\tau,\lambda) \mathrm{d}\lambda \mathrm{d}\tau. \end{aligned} \tag{2.7}$$

Let $G(r) = \frac{1}{2} r^{\frac{1}{2}} g(r)$. Then, in the domain $r > t$, we get

$$\begin{aligned} r^{\frac{1}{2}} u_t^{(m)}(t,r) = {} & \varepsilon G(t+r) + \varepsilon G(r-t) + \frac{1}{8} \int_0^t \Big[\frac{u^{(m)}(\tau,\lambda)}{\lambda^{\frac{3}{2}}} \Big|_{\lambda = r+t-\tau} + \frac{u^{(m)}(\tau,\lambda)}{\lambda^{\frac{3}{2}}} \Big|_{\lambda = r-(t-\tau)} \Big] \mathrm{d}\tau \\ & + \frac{1}{2} \int_0^t [(\lambda^{\frac{1}{2}} (u_t^{(m-1)}(\tau,\lambda))^3)|_{\lambda = r+t-\tau} + (\lambda^{\frac{1}{2}} (u_t^{(m-1)}(\tau,\lambda))^3)|_{\lambda = r-(t-\tau)}] \mathrm{d}\tau. \end{aligned} \tag{2.8}$$

Noting

$$u^{(m)} \geq 0, \quad u_t^{(m-1)} \geq 0, \quad g(r) = g(|x|) \geq 0, \quad \text{in the domain } r > t,$$

we see that

$$r^{\frac{1}{2}} u_t^{(m)} \geq 0, \quad \text{in the domain } r > t.$$

Then

$$u_t^{(m)} \geq 0, \quad \text{in the domain } r > t. \tag{2.9}$$

By means of the estimates on higher-order derivatives (see [11, Chapter 1, p. 23]), it follows from the Sobolev imbedding theorem that $u^{(m)}$ and $u_t^{(m)}$ pointwisely converge to u and u_t, respectively. Taking the limit of (2.4) and (2.9) as $m \to \infty$, we get

$$u = \lim_{m \to \infty} u^{(m)} \geq 0, \quad \text{when } r > t$$

and

$$u_t = \lim_{m\to\infty} u_t^{(m)} \geq 0, \quad \text{when } r > t.$$

Similarly, taking the limit of (2.8) as $m \to \infty$, in the domain $r > t$, we get

$$\begin{aligned} r^{\frac{1}{2}} u_t(t,r) &= \varepsilon G(t+r) + \varepsilon G(r-t) + \frac{1}{8}\int_0^t \Big[\Big(\frac{u(\tau,\lambda)}{\lambda^{\frac{3}{2}}}\Big)\Big|_{\lambda=r+t-\tau} + \Big(\frac{u(\tau,\lambda)}{\lambda^{\frac{3}{2}}}\Big)\Big|_{\lambda=r-(t-\tau)}\Big]\mathrm{d}\tau \\ &\quad + \frac{1}{2}\int_0^t [(\lambda^{\frac{1}{2}}(u_t(\tau,\lambda))^3)|_{\lambda=r+t-\tau} + (\lambda^{\frac{1}{2}}(u_t(\tau,\lambda))^3)|_{\lambda=r-(t-\tau)}]\mathrm{d}\tau. \end{aligned} \tag{2.10}$$

Thus, in the domain $r > t$, we have

$$r^{\frac{1}{2}} u_t(t,r) \geq \varepsilon G(r-t) + \frac{1}{2}\int_0^t (\lambda^{\frac{1}{2}}(u_t(\tau,\lambda))^3)|_{\lambda=r-(t-\tau)}\mathrm{d}\tau. \tag{2.11}$$

Noting (1.7) and that $g(r)$ is a nontrivial smooth function, we have that there exists a $\sigma_0 \in (0,1)$, such that $G(\sigma_0) > 0$. Along the line $r = t + \sigma_0$, we let

$$v(t) = (t+\sigma_0)^{\frac{1}{2}} u_t(t, t+\sigma_0). \tag{2.12}$$

Obviously, $v(t) \geq 0$ for $t \geq 0$.

By (2.11), we have

$$v(t) \geq \varepsilon G(\sigma_0) + \frac{1}{2}\int_0^t (\tau+\sigma_0)^{-1}\cdot v^3(\tau)\mathrm{d}\tau, \quad t \geq 0. \tag{2.13}$$

Now, let $w(t)$ satisfy the following integral equation:

$$w(t) = \varepsilon G(\sigma_0) + \frac{1}{2}\int_0^t (\tau+\sigma_0)^{-1}\cdot w^3(\tau)\mathrm{d}\tau, \quad t \geq 0. \tag{2.14}$$

It follows that

$$v(t) \geq w(t), \quad t \geq 0.$$

$w = w(t)$ is a solution to the following Cauchy problem:

$$\begin{cases} w'(t) = \dfrac{w^3(t)}{2(t+\sigma_0)}, \quad t > 0, \\ w(0) = G(\sigma_0)\varepsilon. \end{cases} \tag{2.15}$$

We have

$$w(t) = \Big[(G(\sigma_0)\varepsilon)^{-2} - \ln\Big(\frac{t+\sigma_0}{\sigma_0}\Big)\Big]^{-\frac{1}{2}}. \tag{2.16}$$

Hence, the lifespan $T(\varepsilon)$ of w and then of v satisfies

$$T(\varepsilon) \leq \exp(A\varepsilon^{-2}),$$

where A is a positive constant independent of ε. This completes the proof of Theorem 1.1.

Acknowledgement The authors would like to express their sincere gratitude to Professor Ta-Tsien Li for his patience in revising the early version of this paper and for his invaluable suggestions on various things as well as for his encouragement.

References

[1] John, F., Non-existence of Global Solutions of $\Box u = \frac{\partial}{\partial t} F(u_t)$ in Two and Three Space Dimensions, MRC Technical Summary Report, 1984.

[2] Klainerman, S., Uniform decay estimates and the Lorentz invariance of the classical wave equation, *Comm. Pure Appl. Math.*, **38**, 1985, 321–332.

[3] Kong, D. X., Life span of classical solutions to reducible quasilinear hyperbolic systems, *Chin. Ann. Math.*, **13B**(2), 1992, 155–164.

[4] Lax, P. D., Development of singularities of nonlinear hyperbolic partial differential equations, *J. Math. Phys.*, 1964, 611–613.

[5] Li, T. T. and Chen, Y. M., Global Classical Solutions for Nonlinear Evolution Equations, Pitman Monographs and Surveys in Pure and Applied Mathematics, Vol. 45, Longman Scientific Technical, New York, 1992.

[6] Li, T. T. and Yu, X., Durée die vie des solutions régulières pour les équations des ondes non linéaires, *C. R. Acad. Sci. Paris*, Série I, **309**, 1989, 469–472.

[7] Li, T. T. and Yu, X., Life-span of classical solutions to fully nonlinear wave equation, *Comm. Part. Diff. Eqs.*, **6**(6), 1991, 909–940.

[8] Lindblad, H. and Sogge, C. D., Long-time existence for small amplitude semilinear wave equations, *Amer. J. Math.*, **118**, 1996, 1047–1135.

[9] Masuda, K., Blow-up of solutions for quasi-linear wave equations in two space dimensions, *Lect. Notes Num. Appl. Anal.*, **6**, 1983, 87–91.

[10] Sideris, T. C., Nonexistence of global solutions to semilinear wave equations in high dimensions, *J. Diff. Eqs.*, **52**, 1984, 378–406.

[11] Sogge, C. D., Lectures on nonlinear wave equations, Monographs in Analysis, International Press Incorporated, Boston, 1995.

[12] Takamura, H., Blow-up for nonlinear wave equations with slowly decaying data, *Math. Z.*, **217**, 1994, 567–576.

[13] Zhou, Y., Blow up of solutions to the Cauchy problem for nonlinear wave equations, *Chin. Ann. Math.*, **22B**(3), 2001, 275–280.