Chin. Ann. Math.
27B(2), 2006, 113–120
DOI: 10.1007/s11401-005-0253-2

Chinese Annals of Mathematics, Series B
© The Editorial Office of CAM and Springer-Verlag Berlin Heidelberg 2006

Ricci Curvature and Fundamental Group**

Yuanlong XIN*

(*Dedicated to the memory of Shiing-Shen Chern*)

Abstract By refined volume estimates in terms of Ricci curvature, the two results due to J. Milnor (1968) are generalized.

Keywords Ricci curvature, Fundamental group, Volume growth
2000 MR Subject Classification 53C20, 20F34

1 Introduction

Let M be a compact Riemannian manifold with negative sectional curvature. By the classical theorem of A. Preissmann [6], every nonidentity abelian subgroup of its fundamental group $\Pi_1(M)$ is infinitely cyclic.

Furthermore any fundamental group of a compact manifold is a finitely generated group which can be associated with the growth function $\gamma(s)$. It is proved by J. Milnor [5] that the growth function of the fundamental group is at least exponential for any compact Riemannian manifold with negative sectional curvature. In the same paper, he conjectured: "Perhaps the hypothesis of negative definite mean curvature would already suffice?"

On the other hand, from the work of Gao-Yau [3] and J. Lohkamp [4] we know that there is no topological obstruction for metrics of negative Ricci curvature. Hence, besides the negative Ricci curvature some additional conditions would be necessary to ensure exponential growth of the fundamental group. This is the case for nonnegative sectional curvature and negative Ricci curvature (see [8]).

This note is to pursue weaker additional assumptions. The key point is a volume estimate from below in terms of negative upper bound of Ricci curvature. Once such an estimate is available our result follows immediately:

Theorem 1.1 *Let M be a compact manifold with negative Ricci curvature. Its universal covering $\widetilde{M}$ possesses a pole and any geodesic sphere centered at the pole is convex or concave. Then the growth function of the fundamental group $\Pi_1(M)$ is at least exponential.*

Remark 1.1 If M is a complete simply connected manifold without focal points, then the distance function from any point is convex (see for example, [7, Proposition 5.8]). So the geodesic sphere is convex in this case. We know that Cartan-Hadamard manifolds have no focal

Manuscript received June 17, 2005.
*School of Mathematical Sciences and the Key Laboratory of Mathematics for Nonlinear Sciences, Fudan University, Shanghai 200433, China. E-mail: ylxin@fudan.edu.cn
**Project supported by the National Natural Science Foundation of China (No.10531090) and the Science Foundation of the Ministry of Education of China.

points automatically. If M is a rotationally symmetric manifold in the sense of Greene-Wu [2], then any geodesic sphere centered at the pole is either convex or concave.

In the same paper, Milnor proved that any finitely generated subgroup of the fundamental group of a complete nonnegative Ricci curvature is at most polynomial growth. This result can also be generalized.

Let M be a complete Riemannian manifold of dimension n. Fixing a point $p \in M$, at any point $x \in M$ we have a minimal geodesic $\gamma(t)$ between p and x. Denote the direction along $\gamma(t)$ by $\frac{\partial}{\partial t}$. If

$$\mathrm{Ric}\left(\frac{\partial}{\partial r}\right) \geq -\frac{A(n-1)}{1+r^2},$$

where r denotes the distance $d(p,x)$ between p and x, $A > 0$ is a constant, then M is called a manifold with asymptotically nonnegative Ricci curvature.

By using a model space in [2] we can generalize the volume comparison result of [1]. We obtain a volume estimate from above for manifolds with asymptotically nonnegative Ricci curvature, and the following result.

Theorem 1.2 *Let M be a complete Riemannian manifold with asymptotically nonnegative Ricci curvature. Then any finitely generated subgroup of the fundamental group $\Pi_1(M)$ has at most polynomial growth.*

2 A Model Space

Let M be an n-dimensional rotationally symmeteric manifold, namely, a complete manifold with a pole $p \in M$ and endowed with the metric

$$ds^2 = dr^2 + f^2(r)\, d\Theta^2$$

in the geodesic polar coordinates around p. We know that

$$f(0) = 0, \quad f'(0) = 1 \quad \text{and} \quad f(r) > 0 \quad \text{for all} \quad r > 0,$$

and that f satisfies the Jacobi equation

$$f'' = -K\, f,$$

where $K : [0, \infty) \to \mathbb{R}$ is the sectional curvature $K(t)$ of any 2-plane including the radial direction $\frac{\partial}{\partial r}$.

If $K = -b^2$, then the volume element

$$v(r) = \left(\frac{1}{b}\sinh(br)\right)^{n-1}.$$

If $K(r) = -\frac{A}{1+r^2}$ for all r with constant $A > 0$, then it is proved in [2] that

$$c_1\, r^{(n-1)\beta} \geq v(r) \geq c_2\, r^{(n-1)\gamma}, \tag{2.1}$$

where

$$\beta = \frac{1+\sqrt{1+2A}}{2},$$

γ is any constant with $0 < \gamma < \beta$, c_1 and c_2 are positive constants.

3 Volume Estimates

For any $p \in M$, there is a geodesic polar coordinate neighborhood $(r, \theta^i),\ i = 1, \cdots, n-1,\ r$ is the distance from p, θ^i are local coordinates on the unit sphere in $\mathbb{R}^n$. The metric in this coordinates can be written as

$$ds^2 = dr^2 + g_{ij}(r,\theta)d\theta^i d\theta^j.$$

Then

$$\Delta r = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^a}\Big(\sqrt{g}g^{ab}\frac{\partial r}{\partial x^b}\Big) = \frac{1}{\sqrt{g}}\frac{\partial\sqrt{g}}{\partial r},$$

where $\sqrt{g} = \sqrt{\det(g_{ab})}$. It follows that

$$\frac{\partial\sqrt{g}}{\partial r} = \sqrt{g}\ \Delta r. \tag{3.1}$$

Let $B(r)$ be a geodesic ball of radius r and centered at p. Its boundary is a geodesic sphere $S(r)$. Choose a local orthonormal frame field on $S(r)$. Then by parallel translating along the geodesic rays to get $\{e_i\}$. Thus, $\{\frac{\partial}{\partial r}, e_i\}$ forms a local frame field on $B(r)$. Since

$$\frac{\partial}{\partial r}\Delta r = \Big\langle \nabla_{\frac{\partial}{\partial r}}\nabla_{e_i}\frac{\partial}{\partial r}, e_i\Big\rangle = -\Big\langle R\Big(\frac{\partial}{\partial r}, e_i\Big)\frac{\partial}{\partial r}, e_i\Big\rangle + \Big\langle \nabla_{[\frac{\partial}{\partial r}, e_i]}\frac{\partial}{\partial r}, e_i\Big\rangle,$$

we then have

$$\frac{\partial}{\partial r}\Delta r = -\mathrm{Ric}\Big(\frac{\partial}{\partial r}\Big) - |\mathrm{Hess}\,(r)|^2. \tag{3.2}$$

Denote $\sqrt{g}$ by v. Differentiating both sides of (3.1) and considering (3.2) yield

$$\frac{\partial^2 v}{\partial r^2} = -v\,\mathrm{Ric}\Big(\frac{\partial}{\partial r}\Big) + v\,((\Delta r)^2 - |\mathrm{Hess}\,(r)|^2). \tag{3.3}$$

Suppose that M is complete and for any r the non-zero principal curvatures of the geodesic sphere $S(r)$ have the same sign. Thus $(\Delta r)^2 \geq |\mathrm{Hess}\,(r)|^2$, and (3.3) becomes

$$\frac{\partial^2 v}{\partial r^2} \geq -v\,\mathrm{Ric}\Big(\frac{\partial}{\partial r}\Big). \tag{3.4}$$

On the other hand, for a manifold $\overline{M}$ with constant sectional curvature $-b^2$, denote the volume element by $\bar{v}$ in a geodesic polar coordinate system. Its geodesic sphere is totaly umbilical and

$$(\Delta r)^2 = (n-1)\,|\mathrm{Hess}\,(r)|^2. \tag{3.5}$$

From (3.1), (3.3) and (3.5) we have

$$\frac{\partial^2 \bar{v}}{\partial r^2} = -\bar{v}\,\overline{\mathrm{Ric}}\Big(\frac{\partial}{\partial r}\Big) + \frac{n-2}{n-1}\frac{(\bar{v}')^2}{\bar{v}}. \tag{3.6}$$

Here $\frac{\partial\bar{v}}{\partial r}$ is denoted by $\bar{v}'$ and the meaning of v' is similar in the sequel. Set $D = v\,\bar{v}^{-\frac{1}{n-1}}$. We have

$$D' = D\Big(v^{-1}v' - \frac{1}{n-1}\bar{v}^{-1}\bar{v}'\Big) = D\Big(\Delta r - \frac{1}{n-1}\widetilde{\Delta} r\Big). \tag{3.7}$$

From (3.1) and (3.7) we have

$$D'' = D\Big(\frac{v'}{v} - \frac{1}{n-1}\frac{\bar{v}'}{\bar{v}}\Big)^2 + D\Big[\frac{v''}{v} - \Big(\frac{v'}{v}\Big)^2 - \frac{1}{n-1}\frac{\bar{v}''}{\bar{v}} + \frac{1}{n-1}\Big(\frac{\bar{v}'}{\bar{v}}\Big)^2\Big].$$

By using (3.4) and (3.6) we have

$$D'' \geq D\Big(\frac{v'}{v} - \frac{1}{n-1}\frac{\bar{v}'}{\bar{v}}\Big)^2 + D\Big[-\operatorname{Ric}\Big(\frac{\partial}{\partial r}\Big) - \Big(\frac{v'}{v}\Big)^2 + \frac{1}{n-1}\overline{\operatorname{Ric}}\Big(\frac{\partial}{\partial r}\Big) + \frac{1}{(n-1)^2}\Big(\frac{\bar{v}'}{\bar{v}}\Big)^2\Big].$$

If we assume that the radial Ricci curvature is bounded from above by a negative constant:

$$\operatorname{Ric}\Big(\frac{\partial}{\partial r}\Big) \leq \frac{1}{n-1}\overline{\operatorname{Ric}}\Big(\frac{\partial}{\partial r}\Big) = -b^2,$$

the above expression becomes

$$D'' \geq \frac{-2}{n-1}\frac{\bar{v}'}{\bar{v}}\,D\Big(\frac{v'}{v} - \frac{1}{n-1}\frac{\bar{v}'}{\bar{v}}\Big) = -\frac{2}{n-1}\frac{\bar{v}'}{\bar{v}}\,D',$$

where the last equality comes from (3.7). It follows that

$$(\bar{v}^{\frac{2}{n-1}}\,D')' = \frac{2}{n-1}\,\bar{v}^{\frac{2}{n-1}-1}\,\bar{v}'\,D' + \bar{v}^{\frac{2}{n-1}}\,D'' = \bar{v}^{\frac{2}{n-1}}\,\Big(D'' + \frac{2}{n-1}\frac{\bar{v}'}{\bar{v}}\,D'\Big) \geq 0.$$

We thus prove that $\bar{v}^{\frac{2}{n-1}}\,D'$ is monotone increasing in r. Hence, for any $r \geq \epsilon$,

$$\begin{aligned} D'(r) &\geq \bar{v}^{-\frac{2}{n-1}}(r)\,\bar{v}^{\frac{2}{n-1}}(\epsilon)\,D'(\epsilon) \\ &\geq \lim_{\epsilon\to 0}\bar{v}^{-\frac{2}{n-1}}(r)\,\bar{v}^{\frac{2}{n-1}}(\epsilon)\,v(\epsilon)\bar{v}^{-\frac{1}{n-1}}(\epsilon)\Big(v^{-1}(\epsilon)v'(\epsilon) - \frac{1}{n-1}\bar{v}^{-1}(\epsilon)\bar{v}'(\epsilon)\Big) = 0, \end{aligned}$$

since $v(\epsilon,\theta) \sim \epsilon^{n-1}$ and $\bar{v}(\epsilon) \sim \epsilon^{n-1}$ when $\epsilon \to 0$. This means that

$$D(r) = v\,\bar{v}^{-\frac{1}{n-1}}$$

is also monotone increasing in r. We know that

$$\bar{v}(r) = \Big(\frac{1}{b}\sinh(br)\Big)^{n-1}.$$

For any $r > \epsilon$,

$$\frac{v(r,\theta)}{\frac{1}{b}\sinh(br)} \geq \frac{b\,v(\epsilon,\theta)}{\sinh(b\epsilon)} \sim \epsilon^{n-2} \quad \text{as} \quad \epsilon \to 0.$$

It follows that

$$v(r,\theta) \geq c\,\epsilon^{n-2}\sinh(br),$$

where c is constant. In summary, we obtain a volume estimate

$$\operatorname{vol}(t) = \int_0^t dr\int_{S(r)} v(r,\theta)d\theta \geq c\,\omega^{n-1}\epsilon^{n-2}\int_0^t \sinh(br)dr = c'\,\omega^{n-1}\,\epsilon^{n-2}\exp(bt), \tag{3.8}$$

when t is sufficiently large, where ω^{n-1} stands for the volume of the unit sphere S^{n-1} and c' is a constant. In summary we have the following result.

Theorem 3.1 *Let M be a manifold with a pole and with radial Ricci curvature bounded from above by a negative constant. Suppose that for any r, the geodesic sphere of radius r and centered at the pole is either convex or concave. Then the volume of the geodesic ball centered at the pole grows at least exponentially.*

Now, let us estimate the upper bound of the volume.

By the Schwarz inequality

$$(\Delta r)^2 \leq (n-1)\,|\mathrm{Hess}\,(r)|^2,$$

substituting it into (3.3) yields

$$\frac{\partial^2 v}{\partial r^2} \leq -v\ \mathrm{Ric}\Big(\frac{\partial}{\partial r}\Big) + \frac{n-2}{(n-1)\,v}\Big(\frac{\partial v}{\partial r}\Big)^2. \tag{3.9}$$

We now assume that M is an asymptotically nonnegative Ricci curved manifold. Let $\overline{M}$ be a rotationally symmetric manifold with volume element $\bar{v}$ in the geodesic polar coordinates around the pole.

Let $D = v^{\frac{1}{n-1}}\bar{v}^{-\frac{1}{n-1}}$. We have

$$D' = \frac{1}{n-1}\,D(\Delta r - \overline{\Delta} r). \tag{3.10}$$

From (3.1), (3.6), (3.9) and (3.10), also by the curvature assumption we have

$$D'' \leq -\Big(\frac{2}{n-1}\Big)\,D'\,\frac{\bar{v}'}{\bar{v}}.$$

As the above reason we have $(\bar{v}^{\frac{2}{n-1}}\,D')' \leq 0$ and $D'(r) \leq 0$. This means that

$$r \to \frac{v(r,\theta)}{\bar{v}(r)}$$

is monotone decreasing, where $(r,\theta) \in \Sigma_p$, inside of the cut locus of p. Let χ be the characteristic function of Σ_p, which is monotone decreasing in r. Define $\tilde{v} = \chi \cdot v$. Then

$$r \to \frac{\tilde{v}(r,\theta)}{\bar{v}(r)} = \frac{v(r,\theta)}{\bar{v}(r)} \cdot \chi(r,\theta) \tag{3.11}$$

is monotone decreasing. Let $S(r)$ be the geodesic sphere centered at p with radius r. Define

$$\int_{S(r)} f(r,\theta)d\sigma = \int_{S^{n-1}} f(r,\theta)\chi(r,\theta)d\theta,$$

which implies the integration over the subset of the unit sphere $C(r) \subset S^{n-1}$, such that for any $\theta \in C(r)$, the geodesic $\gamma(s) = \exp_p(s\,\theta)$ is minimizing up to $s = r$. Thus, for any $r_1 \leq r_2$,

$$\int_{S(r_2)} \tilde{v}(r_1,\theta)\,d\theta = \int_{S^{n-1}} \tilde{v}(r_1,\theta)\chi(r_2,\theta)d\theta \leq \int_{S^{n-1}} \tilde{v}(r_1,\theta)\chi(r_1,\theta)d\theta = \int_{S(r_1)} \tilde{v}(r_1,\theta)d\theta. \tag{3.12}$$

From (3.11) we have

$$\tilde{v}(r_1,\theta)\bar{v}(r_2) \geq \tilde{v}(r_2,\theta)\bar{v}(r_1). \tag{3.13}$$

(3.12) and (3.13) give

$$\int_{S(r_1)} \tilde{v}(r_1,\theta)d\theta\,\bar{v}(r_2) \geq \int_{S(r_2)} \tilde{v}(r_1,\theta)d\theta\,\bar{v}(r_2) \geq \int_{S(r_2)} \tilde{v}(r_2,\theta)d\theta\,\bar{v}(r_1)$$

and

$$\frac{\int_{S(r_1)} \tilde{v}(r_1,\theta)d\theta}{\bar{v}(r_1)} \ge \frac{\int_{S(r_2)} \tilde{v}(r_2,\theta)d\theta}{\bar{v}(r_2)}.$$

Since $\overline{M}$ is rotationally symmetric,

$$\int_{S(r)} \bar{v}(r)d\theta = \bar{v}(r)\,\omega^{n-1},$$

where ω^{n-1} denotes the volume of $(n-1)$-dimensional sphere. Thus

$$r \to \frac{\int_{S(r)} \tilde{v}(r,\theta)d\theta}{\int_{S(r)} \bar{v}(r)d\theta}$$

is monotone decreasing. Denote

$$\mathrm{vol}\,(r) = \mathrm{vol}\,(B_p(r))$$

and use similar notation for $\overline{\mathrm{vol}}\,(r)$. We can easily obtain that

$$r \to \frac{\mathrm{vol}\,(r)}{\overline{\mathrm{vol}}\,(r)}$$

is also monotone decreasing.

In the present situation $\overline{M}$ is a model manifold as described in the previous section. From (2.1) we know that

$$\overline{\mathrm{vol}}\,(r) \le c_1\, r^{(n-1)\beta+1},$$

where $\beta = \frac{1+\sqrt{1+2A}}{2}$. It follows that

$$\mathrm{vol}\,(r) \le \overline{\mathrm{vol}}\,(r) \lim_{t\to 0} \frac{\mathrm{vol}\,(t)}{\overline{\mathrm{vol}}\,(t)} \le c_1\, r^{(n-1)\beta+1}. \tag{3.14}$$

Therefore, we obtain the following result.

Theorem 3.2 *Let M be a complete manifold with asymptotic nonnegative Ricci curvature. Then, M has at most polynomial volume growth.*

4 Bounds of the Growth Function

Once we have estimates (3.8) and (3.14), the same argument as that in [5] gives the proofs of the theorems. For completeness we write down it in detail.

Let M be a compact Riemannian manifold, $\Pi : \widetilde{M} \to M$ the universal covering, Γ the group of the deck transformations which act on $\widetilde{M}$ properly continuously. Γ is transitive on fibers of $\widetilde{M} \to M$.

For any $p \in M$, $\Sigma(p) \subset T_pM$ denotes the inside of the tangential cut locus, $\Sigma_p = \exp_p \Sigma(p)$ for the inside of cut locus of p. Introduce the pull back metric on $\widetilde{M}$, such that $\Pi : \widetilde{M} \to M$ is a local isometry. Take any $\tilde{p} \in \Pi^{-1}(p)$. Note that a local isometry maps any geodesic to a geodesic, and that any geodesic is determined by the initial conditions. We have the following commutative diagram

$$\begin{array}{ccc} T_{\tilde{p}}\widetilde{M} & \xrightarrow{\Pi_*} & T_pM \\ \downarrow{\scriptstyle \exp_{\tilde{p}}} & & \downarrow{\scriptstyle \exp_p} \\ \widetilde{M} & \xrightarrow[\Pi]{} & M. \end{array}$$

In particular, we have the restriction of the above diagram

$$\begin{array}{ccc} \Pi_*^{-1}(\Sigma(p)) & \xrightarrow{\Pi_*} & \Sigma(p) \\ {\scriptstyle \exp_{\tilde p}}\downarrow & & \downarrow{\scriptstyle \exp_p} \\ \Omega & \xrightarrow[\Pi]{} & \Sigma_p. \end{array}$$

Thus, Σ_p can be chosen as a distinguished neighborhood of the covering map Π, such that

$$\Pi^{-1}(\Sigma_p) = \bigcup_{\alpha\in\Gamma} \alpha\Omega \quad \text{and} \quad \alpha\Omega \cap \beta\Omega = \emptyset, \quad \text{when} \quad \alpha \neq \beta.$$

Hence

$$\Pi(\overline{\Omega}) = \overline{\Sigma}_p = M, \quad \bigcup_{\alpha\in\Gamma} \alpha\overline{\Omega} = \widetilde{M}.$$

$\overline{\Omega}$ is the fundamental domain of the deck transformation Γ.

Let $r = d(p, \Sigma_p) > 0$. Then $B_p(r) \subset \Sigma_p$. Let $d = \sup_{x,y\in\Sigma_p} d(x, y)$ which is finite when M is compact.

Define $\Gamma_1 = \{\alpha \in \Gamma,\ d(\tilde p, \alpha\tilde p) \le 2d + r\}$, which is finite. Otherwise, $B_{\tilde p}(2d + r)$ contains infinitely many $\alpha\tilde p$ with $\alpha \in \Gamma_1$. We know that $\alpha B_{\tilde p}(r) \subset \alpha\overline{\Omega}$ are disjoint, and therefore $B_{\tilde p}(2d + 2r)$ has infinite volume since it contains infinitely many disjoint balls $\alpha B_{\tilde p}(r)$. This is impossible and Γ_1 is finite.

We claim that Γ can be generated by Γ_1. In fact, for any $\alpha \in \Gamma$ let $\tilde q \in \alpha\overline{\Omega}$, connect $\tilde p$ and $\tilde q$ by a minimal geodesic $\tilde\rho$. Divide it to s arcs, such that each arc has length $< r$, namely, let

$$\tilde p = \tilde q_1, \cdots, \tilde q_i, \tilde q_{i+1}, \cdots, \tilde q_{s+1} = \tilde q.$$

Assume that $\tilde q_i \in \beta_i\overline{\Omega}$. Then $\beta_1 = \mathrm{id}$, $\beta_{s+1} = \alpha$. Set $\alpha_i = \beta_i^{-1}\beta_{i+1}$. Then $\alpha = \alpha_1 \cdots \alpha_s$. Furthermore, each α_i satisfies

$$d(\tilde p, \alpha_i\tilde p) = d(\beta_i\tilde p, \beta_{i+1}\tilde p) \le d(\beta_i\tilde p, \tilde q_i) + d(\tilde q_i, \tilde q_{i+1}) + d(\tilde q_{i+1}, \beta_{i+1}\tilde p) \le 2d + r,$$

which means $\alpha_i \in \Gamma_1$ and we verify our claim.

For a finitely generated group G and a specific choice of generators, the growth function γ can be defined as follows. For an element $g \in G$, the word-length $l(g)$ is defined as the minimum number of words in the generators and their inverses. For each positive integer s, define the growth function

$$\gamma(s) = \#\{g \in G,\ l(g) \le s\}. \tag{4.1}$$

Milnor [5] proved that the asymptotic behavior of $\gamma(s)$ as $s \to \infty$ is independent of the particular choice of generators.

For any $t \in Z^+$, take a geodesic ball $B_{\tilde p}(tr)$. For any $\tilde q \in B_{\tilde p}(tr)$, assume $\tilde q \in \alpha\overline{\Omega}$. Connect $\tilde p$ and $\tilde q$ by a minimal geodesic, then divide it into t arcs, namely $\tilde p = \tilde q_1, \cdots, \tilde q_i, \cdots, \tilde q_{t+1} = \tilde q$, and $d(\tilde q_i, \tilde q_{i+1}) < r$, $\tilde q_i \in \beta_i\overline{\Omega}$. Set $\alpha_i = \beta_i^{-1}\beta_{i+1}$. Then $\alpha_i \in \Gamma_1$ and $\alpha = \alpha_1 \cdots \alpha_t$. This means that

$$B_{\tilde p}(tr) \subset \bigcup_{\alpha\in\Gamma, l(\alpha)\le t} \alpha\overline{\Omega}.$$

Therefore,

$$\mathrm{vol}\,(tr) \le \gamma(t)\mathrm{vol}\,(\overline{\Omega}) = \gamma(t)\mathrm{vol}\,(\Omega) = \gamma(t)\mathrm{vol}\,(M),$$

and

$$\gamma(t) \ge \frac{1}{\mathrm{vol}\,(M)}\mathrm{vol}\,(tr). \tag{4.2}$$

From (3.8) and (4.2) we obtain the proof of Theorem 1.1.

Now, let us prove Theorem 1.2.

Let $G \subset \Gamma$ be any finitely generated subgroup of the fundamental group with generates $g_1, \cdots, g_q$. Let $m = \max_i d(\tilde{p}, g_i\tilde{p})$. For any $\alpha \in G$ with $l(\alpha) \le t$,

$$\alpha = g_1' \cdots g_t', \quad g_k' \in \{q_1, \cdots, g_q, g_1^{-1}, \cdots, g_q^{-1}\}$$

and

$$\begin{aligned} d(\tilde{p}, \alpha\tilde{p}) = d(\tilde{p}, g_1' \cdots g_t'\tilde{p}) &\le d(\tilde{p}, g_1'\tilde{p}) + d(g_1'\tilde{p}, g_1' \cdots g_t'\tilde{p}) \le \cdots \\ &\le d(\tilde{p}, g_1'\tilde{p}) + \cdots + d(\tilde{p}, g_t'\tilde{p}) \le t\,m. \end{aligned}$$

This means that $\alpha\,\tilde{p} \in B_{\tilde{p}}(tm)$ and $B_{\tilde{p}}(tm)$ contains at least $\gamma(t)$ distinct points in G. Choose $r' < r$. Then

$$\alpha\, B_{\tilde{p}}(r') \cap \beta B_{\tilde{p}}(r') = \emptyset \quad \text{for} \quad \alpha \ne \beta.$$

It follows that $B_{\tilde{p}}(t\,m + r')$ contains at least $\gamma(t)$ distinct balls of radius r'. Hence

$$\mathrm{vol}\,(t\,m + r') \ge \gamma(t)\mathrm{vol}\,(r')$$

and

$$\gamma(t) \le \frac{1}{\mathrm{vol}\,(r')}\,\mathrm{vol}\,(t\,m + r'). \tag{4.3}$$

From (3.14) and (4.3) we obtain

$$\gamma(t) \le \frac{1}{\mathrm{vol}\,(r')}\,c\,(t\,m + r')^{(n-1)\beta+1}$$

for any $r' < r$. We complete the proof of Theorem 1.2.

References

[1] Bishop, R. L., A relation between volume, mean curvature, and diameter, *Notices Amer. Math. Soc.*, **10**, 1963, 364.

[2] Greene, R. E. and Wu, H., Function theory on manifold which possess a pole, Lecture Notes in Math., **699**, Springer-Verlag, 1979.

[3] Gao, L. Z. and Yau, S.-T., The estimates of negative Ricci curved metrics on three manifolds, *Invent. Math.*, **85**, 1986, 637–652.

[4] Lohkamp, J., Metric of negative Ricci curvature, *Ann. Math.*, **140**, 1994, 655–683.

[5] Milnor, J., A note on curvature and fundamental group, *J. Diff. Geom.*, **2**, 1968, 1–7.

[6] Preissman, A., Quelques propriétés globales des espaces de Riemann, *Comment. Math. Helv.*, **15**, 1942-43, 175–216.

[7] Xin, Y. L., Geometry of Harmonic Maps, Birkhäuser, 1996.

[8] Yang, Y.-H., On the growth of fundamental groups on nonpositive curvature manifolds, *Bull. Austral. Math.*, **54**, 1996, 483–487.