

Sharp Growth Theorems and Coefficient Bounds for Starlike Mappings in Several Complex Variables***

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Abstract Let B be the unit ball in a complex Banach space. Let $S_{k+1}^*(B)$ be the family of normalized starlike mappings f on B such that $z = 0$ is a zero of order $k + 1$ of $f(z) - z$. The authors obtain sharp growth and covering theorems, as well as sharp coefficient bounds for various subsets of $S_{k+1}^*(B)$.

Keywords Sharp coefficient bound, Sharp covering theorem, Sharp growth theorem, Starlike mapping, Zero of order k

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1 Introduction

In the case of one complex variable, the following growth theorem and de Branges theorem are well-known (cf. [28]).

Theorem 1.1 Let $f(z) = z + \sum_{m=2}^{\infty} a_m z^m$ be a normalized univalent holomorphic function on the unit disc U in \mathbb{C} . Then

$$(i) \quad \frac{|z|}{(1 + |z|)^2} \leq |f(z)| \leq \frac{|z|}{(1 - |z|)^2}. \quad (1.1)$$

$$(ii) \quad |a_m| \leq m.$$

However, in the case of several complex variables, Cartan [3] pointed out that the above theorem does not hold.

Barnard, FitzGerald and Gong [2], Chuaqui [4] extended the growth theorem (1.1) to normalized starlike mappings on the Euclidean unit ball in \mathbb{C}^n . Kohr [20] obtained a sharp growth theorem for normalized starlike mappings of order α on the Euclidean unit ball in \mathbb{C}^n . Liu and Liu [24] generalized these results and obtained a sharp growth theorem for a normalized starlike mapping f or a normalized starlike mapping f of order α on the unit ball B in a complex Banach space such that $f(z) - z$ has a zero of order $k + 1$ at $z = 0$. On the other hand, Graham, Hamada and Kohr [8] obtained a growth theorem for the set $S_g^0(B)$ of mappings which have

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g -parametric representation on the unit ball in \mathbb{C}^n with respect to an arbitrary norm. Hamada, Honda and Kohr [12] generalized this result to the set $S_{g,k+1}^0(B)$ of mappings f for which there exists a g -Loewner chain $f(z, t)$ such that $\{e^{-t}f(z, t)\}_{t \geq 0}$ is a normal family on the unit ball B in \mathbb{C}^n with respect to an arbitrary norm, $f = f(\cdot, 0)$ and $z = 0$ is a zero of order $k + 1$ of $e^{-t}f(z, t) - z$ for each $t \geq 0$. For the growth theorem of convex mappings, see [6, 15, 18, 34].

Concerning the bounds for coefficients of subclasses of normalized biholomorphic mappings, Kohr [21] obtained a sharp bound for the second coefficient of starlike mappings or starlike mappings of order α on the Euclidean unit ball in \mathbb{C}^n . Gong [7] obtained bounds for the second and third coefficients of starlike mappings on the unit polydisc in \mathbb{C}^n . Liu and Liu [25] obtained bounds for the m -th coefficient ($m = k + 1, \dots, 2k$) of starlike mappings or starlike mappings of order α on the unit ball in a complex Banach space. Especially, the bound is sharp when $m = k + 1$. On the other hand, Graham, Hamada and Kohr [8] obtained a bound for the second coefficient of mappings which have g -parametric representation on the unit ball in \mathbb{C}^n with respect to an arbitrary norm. Hamada, Honda and Kohr [12] obtained a bound for the $(k + 1)$ -th coefficient of mappings in the set $S_{g,k+1}^0(B)$, where B is the unit ball in \mathbb{C}^n with respect to an arbitrary norm.

We do not know whether the growth theorems and coefficient bounds in [8] and [12] are sharp or not. In this paper, we shall obtain sharp growth and covering theorems, as well as sharp coefficient bounds for various subsets of $S_{k+1}^*(B)$, where B is the unit ball in a complex Banach space and $S_{k+1}^*(B)$ is the family of normalized starlike mappings f on B such that $z = 0$ is a zero of order $k + 1$ of $f(z) - z$. These results are generalizations of the above sharp results.

2 Preliminaries

Let X be a complex Banach space with respect to a norm $\|\cdot\|$. Let $B_r = \{z \in X : \|z\| < r\}$ and $B = B_1$. When $X = \mathbb{C}$, B_r is denoted by U_r and B_1 by U . For a domain $G \subset X$, let $H(G)$ be the set of holomorphic mappings from G into X . When $f \in H(B)$, we say that f is biholomorphic on B if $f(B)$ is a domain and the inverse exists and is holomorphic on $f(B)$. Let $L(X, Y)$ denote the set of continuous linear operators from X into a complex Banach space Y . Let I be the identity in $L(X, X)$. For each $z \in X \setminus \{0\}$, let

$$T(z) = \{l_z \in L(X, \mathbb{C}) : l_z(z) = \|z\|, \|l_z\| = 1\}.$$

This set is nonempty by the Hahn-Banach theorem.

When $f \in H(B)$, we say that f is normalized if $f(0) = 0$ and $Df(0) = I$. When $f \in H(B)$, we say that f is starlike if f is biholomorphic on B and $f(B)$ is starlike with respect to the origin. Let $S^*(B)$ be the set of normalized starlike mappings on B . When $B = U$, the set $S^*(U)$ is denoted by S^* .

Assumption 2.1 *Let $g : U \rightarrow \mathbb{C}$ be a univalent holomorphic function such that $g(0) = 1$, $g(\bar{\zeta}) = \overline{g(\zeta)}$ for $\zeta \in U$ (so, g has real coefficients in its power series expansion), $g'(0) < 0$ and*

$\Re g(\zeta) > 0$ on U . We assume that g satisfies the conditions

$$\begin{cases} \min_{|\zeta|=r} \Re g(\zeta) = g(r), \\ \max_{|\zeta|=r} \Re g(\zeta) = g(-r) \end{cases}$$

for $r \in (0, 1)$.

We mention that there are many functions which satisfy the above assumption (cf. [8]).

The following set \mathcal{M} of normalized mappings of “positive real part” on B plays a fundamental role in the study of the Loewner differential equations. Let

$$\mathcal{M} = \{p \in H(B) : p(0) = 0, Dp(0) = I, \Re l_z(p(z)) > 0, z \in B \setminus \{0\}, l_z \in T(z)\}.$$

As in [8, 12, 22], we shall introduce various subsets of \mathcal{M} . Let

$$\mathcal{M}_g = \left\{ p \in H(B) : p(0) = 0, Dp(0) = I, \frac{1}{\|z\|} l_z(p(z)) \in g(U), z \in B \setminus \{0\}, l_z \in T(z) \right\}.$$

If $g(\zeta) = \frac{1-\zeta}{1+\zeta}$, then we obtain $\mathcal{M}_g = \mathcal{M}$. However, there are other choices of g which provide interesting properties of the set \mathcal{M}_g .

Let $f \in H(B)$ be a normalized locally biholomorphic mapping. Then f is starlike if and only if $[Df(z)]^{-1}f(z) \in \mathcal{M}$ (cf. [10, 16, 33]). Let $S_g^*(B)$ denote the subset of $S^*(B)$ consisting of those normalized locally biholomorphic mappings f such that $[Df(z)]^{-1}f(z) \in \mathcal{M}_g$.

Now, we will give particular subsets of $S^*(B)$.

Definition 2.1 Let $0 \leq p < 1$. A normalized locally biholomorphic mapping $f \in H(B)$ is said to be starlike of order p if

$$[Df(z)]^{-1}f(z) \in \mathcal{M}_g,$$

where $g(\zeta) = \frac{1-\zeta}{1-(2p-1)\zeta}$.

We denote by $S_p^*(B)$ the set of all starlike mappings of order p on B .

Definition 2.2 Let $0 < \alpha \leq 1$. A normalized locally biholomorphic mapping $f \in H(B)$ is said to be strongly starlike of order α if

$$[Df(z)]^{-1}f(z) \in \mathcal{M}_g,$$

where $g(\zeta) = \frac{(1-\zeta)^\alpha}{(1+\zeta)^\alpha}$ and the branches of the power functions are chosen such that $(1-\zeta)^\alpha|_{\zeta=0} = (1+\zeta)^\alpha|_{\zeta=0} = 1$.

We denote by $SS_\alpha^*(B)$ the set of all strongly starlike mappings of order α on B .

Definition 2.3 Let $0 \leq \alpha < 1$. A normalized locally biholomorphic mapping $f \in H(B)$ is said to be almost starlike of order α if

$$[Df(z)]^{-1}f(z) \in \mathcal{M}_g,$$

where $g(\zeta) = (1-\alpha)\frac{1-\zeta}{1+\zeta} + \alpha$.

We denote by $AS_\alpha^*(B)$ the set of all almost starlike mappings of order α on B .

Definition 2.4 *Let*

$$q_\rho(\zeta) = 1 + \frac{4(1-\rho)}{\pi^2} \left(\log \frac{1+\sqrt{\zeta}}{1-\sqrt{\zeta}} \right)^2,$$

where we choose the branch of the square root such that $\sqrt{\zeta}|_{\zeta=1} = 1$ and the branch of logarithm function such that $\log 1 = 0$, $0 \leq \rho < 1$. A normalized locally biholomorphic mapping $f \in H(B)$ is said to be parabolic starlike of order ρ if

$$[Df(z)]^{-1}f(z) \in \mathcal{M}_g,$$

where $g = \frac{1}{q_\rho}$.

We denote by $PS_\rho^*(B)$ the set of all parabolic starlike mappings of order ρ on B .

When $B = U$, $S^*(U)$ (respectively $S_g^*(U)$, $S_p^*(U)$, $SS_\alpha^*(U)$, $AS_\alpha^*(U)$, $PS_\rho^*(U)$) is denoted by S^* (respectively S_g^* , S_p^* , SS_α^* , AS_α^* , PS_ρ^*).

Let $f \in H(B)$ and let k be a positive integer. We say that $z = 0$ is a zero of order k of $f(z)$ if $f(0) = 0, \dots, D^{k-1}f(0) = 0$ and $D^k f(0) \neq 0$ (compare with [26]).

Also, we denote by $S_{k+1}^*(B)$ (respectively $S_{g,k+1}^*(B)$, $S_{p,k+1}^*(B)$, $SS_{\alpha,k+1}^*(B)$, $AS_{\alpha,k+1}^*(B)$, $PS_{\rho,k+1}^*(B)$) the subset of $S^*(B)$ (respectively $S_g^*(B)$, $S_p^*(B)$, $SS_\alpha^*(B)$, $AS_\alpha^*(B)$, $PS_\rho^*(B)$) of mappings f such that $z = 0$ is a zero of order $k+1$ of $f(z) - z$. When $B = U$, $S_{k+1}^*(U)$ (respectively $S_{g,k+1}^*(U)$, $S_{p,k+1}^*(U)$, $SS_{\alpha,k+1}^*(U)$, $AS_{\alpha,k+1}^*(U)$, $PS_{\rho,k+1}^*(U)$) is denoted by S_{k+1}^* (respectively $S_{g,k+1}^*$, $S_{p,k+1}^*$, $SS_{\alpha,k+1}^*$, $AS_{\alpha,k+1}^*$, $PS_{\rho,k+1}^*$).

3 Sharpness of Growth Theorems

For the set $S_{g,k+1}^*(B)$, we obtain the following growth result by an argument similar to that in the proof of [12, Theorem 10] (cf. [8, Theorem 2.2] and [22, Theorem 2.3]).

Theorem 3.1 *Let $g : U \rightarrow \mathbb{C}$ satisfy the conditions of Assumption 2.1 and $f \in S_{g,k+1}^*(B)$. Then*

$$\|z\| \exp \int_0^{\|z\|} \left[\frac{1}{g(-x^k)} - 1 \right] \frac{dx}{x} \leq \|f(z)\| \leq \|z\| \exp \int_0^{\|z\|} \left[\frac{1}{g(x^k)} - 1 \right] \frac{dx}{x}, \quad z \in B. \quad (3.1)$$

Later, we will show that the above estimations (3.1) are sharp. First, we give a lemma.

Let $b \in S_g^*$ be defined by $b(0) = b'(0) - 1 = 0$ and

$$\frac{\zeta b'(\zeta)}{b(\zeta)} = \frac{1}{g(\zeta)}, \quad \zeta \in U.$$

For a positive integer k , let

$$b_k(\zeta) = \zeta[\varphi(\zeta^k)]^{\frac{1}{k}},$$

where

$$\varphi(\zeta) = \frac{b(\zeta)}{\zeta}.$$

The branches of the power functions are chosen so that

$$(\varphi(\zeta^k))^{\frac{1}{k}}|_{\zeta=0} = 1.$$

Also, for $u \in \partial B$, let

$$f_u(z) = \frac{b_k(l_u(z))}{l_u(z)}z, \quad z \in B. \quad (3.2)$$

Then, we obtain the following lemma by direct computations.

Lemma 3.1 *Let b_k and f_u be as above. Then*

(i) $b_k(\zeta) = \zeta - \frac{1}{k}g'(0)\zeta^{k+1} + \dots$, $\zeta \in U$ and

$$\frac{\zeta b'_k(\zeta)}{b_k(\zeta)} = \frac{1}{g(\zeta^k)}, \quad \zeta \in U.$$

Thus, $b_k \in S_{g,k+1}^*$ and $b_k(0) = b'_k(0) - 1 = 0$.

(ii) $f_u \in S_{g,k+1}^*(B)$ and

$$f_u(\zeta u) = b_k(\zeta)u = \left(\zeta - \frac{1}{k}g'(0)\zeta^{k+1} + \dots\right)u, \quad \zeta \in U.$$

Now, we obtain the following equivalent formulation of Theorem 3.1 (cf. [1, Theorem 2.5], [13] and [27]).

Theorem 3.2 *Let $g : U \rightarrow \mathbb{C}$ satisfy the conditions of Assumption 2.1. If $f \in S_{g,k+1}^*(B)$, then*

$$e^{-\frac{\pi i}{k}}b_k(e^{\frac{\pi i}{k}}\|z\|) \leq \|f(z)\| \leq b_k(\|z\|), \quad z \in B. \quad (3.3)$$

These estimations are sharp.

Proof From (3.1) and Lemma 3.1(i), we obtain

$$\exp \int_0^{\|z\|} \left[\frac{x \tilde{b}'_k(x)}{\tilde{b}_k(x)} - 1 \right] \frac{dx}{x} \leq \frac{\|f(z)\|}{\|z\|} \leq \exp \int_0^{\|z\|} \left[\frac{x b'_k(x)}{b_k(x)} - 1 \right] \frac{dx}{x}$$

for $z \in B$, where $\tilde{b}_k(\zeta) = e^{-\frac{\pi i}{k}}b_k(e^{\frac{\pi i}{k}}\zeta)$. Then, we obtain

$$\exp \left[\log \frac{\tilde{b}_k(\|z\|)}{\|z\|} - \log \tilde{b}'_k(0) \right] \leq \frac{\|f(z)\|}{\|z\|} \leq \exp \left[\log \frac{b_k(\|z\|)}{\|z\|} - \log b'_k(0) \right]$$

for $z \in B$, since $\tilde{b}_k(x), b_k(x) > 0$ for $x > 0$. This implies (3.3).

Next, we will show that the estimations (3.3) are sharp. Let $f_u \in S_{g,k+1}^*(B)$ be as in (3.2). Since $\|f_u(ru)\| = b_k(r)$ and $\|f_u(e^{\frac{\pi i}{k}}ru)\| = |b_k(e^{\frac{\pi i}{k}}r)|$, the equalities of the estimations (3.3) hold. This completes the proof.

Remark 3.1 The equivalence of (3.1) and (3.3) implies that the estimations (3.1) are sharp.

Now, we obtain the following corollaries from Theorems 3.1 or 3.2.

For $g(\zeta) = \frac{1-\zeta}{1+\zeta}$, we have the following sharp growth and covering results for the set $S_{k+1}^*(B)$ due to Liu and Liu [24, Theorem 1] (cf. [2, 4, 11, 35]).

Corollary 3.1 *If $f \in S_{k+1}^*(B)$, then*

$$\frac{\|z\|}{(1 + \|z\|^k)^{\frac{2}{k}}} \leq \|f(z)\| \leq \frac{\|z\|}{(1 - \|z\|^k)^{\frac{2}{k}}}, \quad z \in B.$$

Consequently, $f(B) \supset B_{2^{-\frac{2}{k}}}$. These estimations are sharp.

For $g(\zeta) = \frac{1-\zeta}{1-(2p-1)\zeta}$, where $0 \leq p < 1$, we have the following sharp growth and covering results for the set $S_{p,k+1}^*(B)$ due to Liu and Liu [24, Theorem 2] (cf. [17, 20]).

Corollary 3.2 *If $f \in S_{p,k+1}^*(B)$, then*

$$\frac{\|z\|}{(1 + \|z\|^k)^{\frac{2(1-p)}{k}}} \leq \|f(z)\| \leq \frac{\|z\|}{(1 - \|z\|^k)^{\frac{2(1-p)}{k}}}, \quad z \in B.$$

Consequently, $f(B) \supset B_{2^{-\frac{2(1-p)}{k}}}$. These estimations are sharp.

For $g(\zeta) = \frac{(1-\zeta)^\alpha}{(1+\zeta)^\alpha}$, where $0 < \alpha \leq 1$, we have the following sharp growth and covering results for the set $SS_{\alpha,k+1}^*(B)$ (cf. [23]).

Corollary 3.3 *Let $f \in SS_{\alpha,k+1}^*(B)$. Then*

$$\|z\| \exp \int_0^{\|z\|} \left[\left(\frac{1-x^k}{1+x^k} \right)^\alpha - 1 \right] \frac{dx}{x} \leq \|f(z)\| \leq \|z\| \exp \int_0^{\|z\|} \left[\left(\frac{1+x^k}{1-x^k} \right)^\alpha - 1 \right] \frac{dx}{x}, \quad z \in B.$$

Consequently, $f(B) \supset B_r$, where

$$r = \exp \int_0^1 \left[\left(\frac{1-x^k}{1+x^k} \right)^\alpha - 1 \right] \frac{dx}{x}.$$

These estimations are sharp.

For $g(\zeta) = (1-\alpha)\frac{1-\zeta}{1+\zeta} + \alpha$, where $0 \leq \alpha < 1$, we have the following sharp growth and covering results for the set $AS_{\alpha,k+1}^*(B)$ (cf. [5, 19]).

Corollary 3.4 *Let $f \in AS_{\alpha,k+1}^*(B)$.*

(i) *If $\alpha \in [0, 1)$ and $\alpha \neq \frac{1}{2}$, then*

$$\frac{\|z\|}{(1 + (1-2\alpha)\|z\|^k)^{\frac{2(1-\alpha)}{k(1-2\alpha)}}} \leq \|f(z)\| \leq \frac{\|z\|}{(1 - (1-2\alpha)\|z\|^k)^{\frac{2(1-\alpha)}{k(1-2\alpha)}}}$$

for $z \in B$. Consequently, $f(B) \supset B_r$, where

$$r = \frac{1}{(2-2\alpha)^{\frac{2(1-\alpha)}{k(1-2\alpha)}}}.$$

These estimations are sharp.

(ii) *If $\alpha = \frac{1}{2}$, then*

$$\|z\| \exp \left(-\frac{1}{k} \|z\|^k \right) \leq \|f(z)\| \leq \|z\| \exp \left(\frac{1}{k} \|z\|^k \right), \quad z \in B.$$

Consequently, $f(B) \supset B_r$, where

$$r = \exp \left(-\frac{1}{k} \right).$$

These estimations are sharp.

For $g = \frac{1}{q_\rho}$, where

$$q_\rho(\zeta) = 1 + \frac{4(1-\rho)}{\pi^2} \left(\log \frac{1+\sqrt{\zeta}}{1-\sqrt{\zeta}} \right)^2$$

and $0 \leq \rho < 1$, we have the following sharp growth and covering results for the set $PS_{\rho,k+1}^*(B)$ (cf. [1, Theorem 2.5], [13], [27] and [31, Theorem 2.3]).

Corollary 3.5 *If $f \in PS_{\alpha, k+1}^*(B)$, then*

$$-\frac{16(1-\rho)}{\pi^2} \int_0^{\|z\|} \frac{1}{x} (\tan^{-1} \sqrt{x^k})^2 dx \leq \log \frac{\|f(z)\|}{\|z\|} \leq \frac{4(1-\rho)}{\pi^2} \int_0^{\|z\|} \frac{1}{x} \left(\log \frac{1+\sqrt{x^k}}{1-\sqrt{x^k}} \right)^2 dx.$$

Consequently, $f(B) \supset B_r$, where

$$r = \exp \left(-\frac{16(1-\rho)}{\pi^2} \int_0^1 \frac{1}{x} (\tan^{-1} \sqrt{x^k})^2 dx \right).$$

These estimations are sharp.

4 Coefficient Bounds

We now obtain an estimate for the $(k+1)$ -th order coefficients of mappings in the set $S_{g, k+1}^*(B)$ as in [12, Theorem 24] (cf. [8, Theorem 2.14], [22, Theorem 2.4] and [29, Theorem 3]). Moreover, we will show that this estimation is sharp.

Theorem 4.1 *Let $g : U \rightarrow \mathbb{C}$ satisfy the conditions of Assumption 2.1 and $f \in S_{g, k+1}^*(B)$. Then*

$$\left| \frac{1}{(k+1)!} l_w(D^{k+1}f(0)(w^{k+1})) \right| \leq \frac{1}{k} |g'(0)|, \quad \|w\| = 1, \quad l_w \in T(w). \quad (4.1)$$

This estimation is sharp.

Proof Since $f \in S_{g, k+1}^*(B)$, f has the Taylor expansion

$$f(z) = z + \frac{1}{(k+1)!} D^{k+1}f(0)(z^{k+1}) + \cdots, \quad z \in B.$$

Let $h(z) = [Df(z)]^{-1}f(z)$. Then h has the Taylor expansion

$$h(z) = z - \frac{k}{(k+1)!} D^{k+1}f(0)(z^{k+1}) + \cdots, \quad z \in B.$$

Fix $w \in \partial B$, $l_w \in T(w)$. Let

$$p(\zeta) = \begin{cases} \frac{1}{\zeta} l_w(h(\zeta w)), & \zeta \in U \setminus \{0\}, \\ 1, & \zeta = 0. \end{cases}$$

Then p is a holomorphic function on U and has the Taylor expansion

$$p(\zeta) = 1 - \frac{k}{(k+1)!} l_w(D^{k+1}f(0)(w^{k+1})) \zeta^k + \cdots, \quad \zeta \in U.$$

Therefore, we obtain

$$p^{(k)}(0) = -\frac{k}{k+1} l_w(D^{k+1}f(0)(w^{k+1})).$$

Also, there exists a holomorphic function \tilde{p} on U such that $p(\zeta) = 1 + \zeta^k \tilde{p}(\zeta)$ for $\zeta \in U$. Since $h \in \mathcal{M}_g$, we deduce that $p(\zeta) \in g(U)$ for $\zeta \in U$. Therefore, $g^{-1} \circ p(\cdot) : U \rightarrow U$ and $g^{-1} \circ p(0) = 0$. Since $g^{-1}(1) = 0$, there exists a holomorphic function $G(w)$ on a neighborhood

of 1 such that $g^{-1}(w) = (w - 1)G(w)$. Therefore, we obtain $g^{-1} \circ p(\zeta) = \zeta^k \tilde{p}(\zeta)G(p(\zeta))$ on a neighborhood of 0. Then, by the Schwarz lemma, we obtain $|g^{-1} \circ p(\zeta)| \leq |\zeta|^k$ for $\zeta \in U$. Thus, there exists a holomorphic function $H(\zeta)$ on U such that $g^{-1} \circ p(\zeta) = \zeta^k H(\zeta)$ on U and $|H(\zeta)| \leq 1$ on U . Then, $p(\zeta) = g(\zeta^k H(\zeta))$. Hence we obtain $|p^{(k)}(0)| \leq k!|g'(0)|$. Thus, we obtain

$$\left| \frac{1}{(k+1)!} l_w(D^{k+1}f(0)(w^{k+1})) \right| = \frac{1}{k} \frac{1}{k!} |p^{(k)}(0)| \leq \frac{1}{k} |g'(0)|.$$

We will show that the estimation (4.1) is sharp. Let f_u be as in (3.2). Then

$$f_w(\zeta w) = \zeta w - \frac{1}{k} g'(0) \zeta^{k+1} w + \dots$$

by Lemma 3.1(ii). Therefore,

$$\left| \frac{1}{(k+1)!} l_w(D^{k+1}f_w(0)(w^{k+1})) \right| = \frac{1}{k} |g'(0)|.$$

This completes the proof.

Moreover, if g is convex, then we obtain the following theorem.

Theorem 4.2 *Let $g : U \rightarrow \mathbb{C}$ be a convex function which satisfies the conditions of Assumption 2.1 and $f \in S_{g,k+1}^*(B)$. Then*

$$\left| \frac{1}{m!} l_w(D^m f(0)(w^m)) \right| \leq \frac{1}{m-1} |g'(0)|, \quad \|w\| = 1, \quad l_w \in T(w)$$

for $m = k+1, \dots, 2k$.

Proof Let $h(z) = [Df(z)]^{-1}f(z)$ and

$$p(\zeta) = \begin{cases} \frac{1}{\zeta} l_w(h(\zeta w)), & \zeta \in U \setminus \{0\}, \\ 1, & \zeta = 0. \end{cases}$$

Since g is convex and p is subordinate to g , we obtain

$$\left| \frac{p^{(m)}(0)}{m!} \right| \leq |g'(0)|, \quad m \geq 1 \quad (4.2)$$

by Rogosinski's Theorem (cf. [30]). Since

$$p(\zeta) = \sum_{m=1}^{\infty} l_w\left(\frac{1}{m!} D^m h(0)(w^m)\right) \zeta^{m-1},$$

we obtain

$$\left| l_w\left(\frac{1}{m!} D^m h(0)(w^m)\right) \right| \leq |g'(0)| \quad (4.3)$$

by (4.2). Since $f(z) - z$ has a zero of order $k+1$ at $z = 0$,

$$\frac{1}{m!} D^m f(0) = \frac{-1}{m-1} \frac{1}{m!} D^m h(0), \quad m = k+1, \dots, 2k \quad (4.4)$$

by [25, (2.3)]. From (4.3) and (4.4), we obtain

$$\left| \frac{1}{m!} l_w(D^m f(0)(w^m)) \right| \leq \frac{1}{m-1} |g'(0)|, \quad \|w\| = 1, \quad l_w \in T(w)$$

for $m = k + 1, \dots, 2k$. This completes the proof.

Now, we obtain the following corollaries from Theorem 4.2. For the norm of the m -th order Fréchet derivative of a mapping in $S_{g,k+1}^*(B)$, where $m = k + 1, \dots, 2k$, we have the following estimate by an argument similar to that in the proof of [12, Corollary 25] (cf. [8, Corollary 2.15]).

Corollary 4.1 *Let $g : U \rightarrow \mathbb{C}$ be a convex function which satisfies the conditions of Assumption 2.1 and $f \in S_{g,k+1}^*(B)$. Then*

$$\left\| \frac{1}{m!} D^m f(0) \right\| \leq c_m |g'(0)|$$

for $m = k + 1, \dots, 2k$, where $c_m = \frac{m^{\frac{m}{m-1}}}{m-1}$.

For the mappings in $S_{k+1}^*(B)$ (respectively $S_{p,k+1}^*(B)$, $SS_{\alpha,k+1}^*(B)$, $AS_{\alpha,k+1}^*(B)$, $PS_{\rho,k+1}^*(B)$), we obtain the following estimates (cf. [12, Corollaries 26 and 27], [13], [21] and [25, Theorems 4 and 5]).

Corollary 4.2 *For $m = k + 1, \dots, 2k$, $\|w\| = 1$ and $l_w \in T(w)$, we have the following estimates:*

(i) *If $f \in S_{k+1}^*(B)$, then*

$$\left| \frac{1}{m!} l_w(D^m f(0)(w^m)) \right| \leq \frac{2}{m-1}.$$

(ii) *If $f \in S_{p,k+1}^*(B)$, then*

$$\left| \frac{1}{m!} l_w(D^m f(0)(w^m)) \right| \leq \frac{2(1-p)}{m-1}.$$

(iii) *If $f \in SS_{\alpha,k+1}^*(B)$, then*

$$\left| \frac{1}{m!} l_w(D^m f(0)(w^m)) \right| \leq \frac{2\alpha}{m-1}.$$

(iv) *If $f \in AS_{\alpha,k+1}^*(B)$, then*

$$\left| \frac{1}{m!} l_w(D^m f(0)(w^m)) \right| \leq \frac{2(1-\alpha)}{m-1}.$$

(v) *If $f \in PS_{\rho,k+1}^*(B)$, then*

$$\left| \frac{1}{m!} l_w(D^m f(0)(w^m)) \right| \leq \frac{16(1-\rho)}{\pi^2(m-1)}.$$

These estimations are sharp for $m = k + 1$.

Corollary 4.3 *Let c_m be as in Corollary 4.1. For $m = k + 1, \dots, 2k$, we have the following estimates:*

(i) *If $f \in S_{k+1}^*(B)$, then*

$$\left\| \frac{1}{m!} D^m f(0) \right\| \leq 2c_m.$$

(ii) If $f \in S_{p,k+1}^*(B)$, then

$$\left\| \frac{1}{m!} D^m f(0) \right\| \leq 2(1-p)c_m.$$

(iii) If $f \in SS_{\alpha,k+1}^*(B)$, then

$$\left\| \frac{1}{m!} D^m f(0) \right\| \leq 2\alpha c_m.$$

(iv) If $f \in AS_{\alpha,k+1}^*(B)$, then

$$\left\| \frac{1}{m!} D^m f(0) \right\| \leq 2(1-\alpha)c_m.$$

(v) If $f \in PS_{\rho,k+1}^*(B)$, then

$$\left\| \frac{1}{m!} D^m f(0) \right\| \leq \frac{16(1-\rho)c_m}{\pi^2}.$$

Let B be the unit polydisc U^n in \mathbb{C}^n . Since

$$\begin{aligned} \left\| \frac{1}{m!} D^m f(0) \right\| &= \sup_{\|w\|=1} \left\| \frac{1}{m!} D^m f(0)(w^m) \right\| \\ &= \sup_{|w_1|=|w_2|=\dots=|w_m|=1} \left\| \frac{1}{m!} D^m f(0)(w^m) \right\| \\ &= \sup_{|w_1|=|w_2|=\dots=|w_m|=1} \max_{1 \leq i \leq n} \left| l_i \left(\frac{1}{m!} D^m f(0)(w^m) \right) \right|, \end{aligned}$$

where $l_i \in T(w)$ with $l_i(y) = \frac{y_i}{w_i}$, we obtain the following corollaries (cf. [25, Theorems 1 and 2]).

Corollary 4.4 *Let $g : U \rightarrow \mathbb{C}$ be a convex function which satisfies the conditions of Assumption 2.1 and $f \in S_{g,k+1}^*(U^n)$. Then*

$$\left\| \frac{1}{m!} D^m f(0) \right\| \leq \frac{1}{m-1} |g'(0)|, \quad \|w\| = 1$$

for $m = k+1, \dots, 2k$. This estimation is sharp for $m = k+1$.

Corollary 4.5 *For $m = k+1, \dots, 2k$, we have the following estimates:*

(i) If $f \in S_{k+1}^*(U^n)$, then

$$\left\| \frac{1}{m!} D^m f(0) \right\| \leq \frac{2}{m-1}.$$

(ii) If $f \in S_{p,k+1}^*(U^n)$, then

$$\left\| \frac{1}{m!} D^m f(0) \right\| \leq \frac{2(1-p)}{m-1}.$$

(iii) If $f \in SS_{\alpha,k+1}^*(U^n)$, then

$$\left\| \frac{1}{m!} D^m f(0) \right\| \leq \frac{2\alpha}{m-1}.$$

(iv) If $f \in AS_{\alpha, k+1}^*(U^n)$, then

$$\left\| \frac{1}{m!} D^m f(0) \right\| \leq \frac{2(1-\alpha)}{m-1}.$$

(v) If $f \in PS_{\rho, k+1}^*(U^n)$, then

$$\left\| \frac{1}{m!} D^m f(0) \right\| \leq \frac{16(1-\rho)}{\pi^2(m-1)}.$$

These estimations are sharp for $m = k + 1$.

Let g be a function which satisfies the conditions of Assumption 2.1. Moreover, we assume that $\frac{1}{g}$ is convex. In this case, we will obtain the estimation of $\|\frac{1}{2}D^2 f(0)\|$ and $\|\frac{1}{3!}D^3 f(0)\|$ for $S_g^*(U^n)$ (cf. [1, Theorem 3.4], [7, Theorem 5.3.1], [13] and [32, Theorem 5]). First, we give a lemma.

Lemma 4.1 *Let g be a function which satisfies the conditions of Assumption 2.1. Moreover, we assume that $\frac{1}{g}$ is convex. Let $f \in S_g^*(U^n)$, $h(z) = [Df(z)]^{-1}f(z)$ and let the Taylor expansion of $\frac{z_i}{h_i(z)}$ at $z = 0$ be*

$$\frac{z_i}{h_i(z)} = 1 + \sum_{m=1}^{\infty} Q_i^{(m)}(z),$$

where $Q_i^{(m)}$ is a homogeneous polynomial of degree m in z . Then

$$|Q_i^{(m)}(z)| \leq |g'(0)| \|z\|^m \quad (4.5)$$

holds for all $m = 1, 2, 3, \dots$ and $z \in U^n$.

Proof Let $z \in U^n \setminus \{0\}$ be fixed. We may assume that $|z_i| = \|z\|$ for all i by the maximum principle. Let

$$\psi(\zeta) = \frac{\zeta \frac{z_i}{\|z\|}}{h_i(\zeta \frac{z}{\|z\|})}, \quad \zeta \in U.$$

Since

$$l_a(y) = \|a\| \frac{y_i}{a_i} \in T(a)$$

for all $a \in U^n \setminus \{0\}$ with $|a_i| = \|a\|$, we have

$$\frac{1}{\|a\|} l_a(h(a)) = \frac{h_i(a)}{a_i} \in g(U)$$

for all $a \in U^n \setminus \{0\}$ with $|a_i| = \|a\|$. This implies that $\psi(\zeta) \in \frac{1}{g}(U)$ for all $\zeta \in U$. Since $\psi(0) = \frac{1}{g}(0) = 1$, we have $\psi \prec \frac{1}{g}$. Since $\frac{1}{g}$ is convex and

$$\psi(\zeta) = 1 + \sum_{m=1}^{\infty} Q_i^{(m)}\left(\frac{z}{\|z\|}\right) \zeta^m,$$

we obtain

$$\left| Q_i^{(m)}\left(\frac{z}{\|z\|}\right) \right| \leq |g'(0)|$$

by Rogosinski's theorem (cf. [30]). This implies the inequality (4.5). This completes the proof.

Theorem 4.3 *Let g be a function which satisfies the conditions of Assumption 2.1. Moreover, we assume that $\frac{1}{g}$ is convex. If $f \in S_g^*(U^n)$, then*

$$\left\| \frac{1}{2!} D^2 f(0) \right\| \leq |g'(0)|, \quad \left\| \frac{1}{3!} D^3 f(0) \right\| \leq \frac{|g'(0)|}{2} (1 + |g'(0)|).$$

Proof Let

$$\varphi^{(2)}(z) = \frac{1}{2!} D^2 f(0)(z^2), \quad \varphi^{(3)}(z) = \frac{1}{3!} D^3 f(0)(z^3).$$

By [7, (5.3.8)],

$$\varphi^{(2)}(z) = (Q_1^{(1)}(z)z_1, \dots, Q_n^{(1)}(z)z_n).$$

Then, by Lemma 4.1, we obtain

$$\|\varphi^{(2)}(z)\| \leq |g'(0)| \|z\|^2.$$

By the proof of [7, Theorem 5.3.1],

$$2\varphi^{(3)}(z) = \begin{pmatrix} Q_1^{(2)}(z)z_1 \\ \vdots \\ Q_n^{(2)}(z)z_n \end{pmatrix} + \begin{pmatrix} a_{11}z_1\varphi_1^{(2)}(z) + \dots + a_{1n}z_1\varphi_n^{(2)}(z) \\ \vdots \\ a_{n1}z_n\varphi_1^{(2)}(z) + \dots + a_{nn}z_n\varphi_n^{(2)}(z) \end{pmatrix},$$

where

$$Q_i^{(1)}(z) = a_{i1}z_1 + \dots + a_{in}z_n.$$

By Lemma 4.1, we obtain

$$2\|\varphi^{(3)}(z)\| \leq |g'(0)| \|z\|^3 + |g'(0)|^2 \|z\|^3.$$

This completes the proof.

Corollary 4.6 (i) *If $f \in S^*(U^n)$, then*

$$\left\| \frac{1}{3!} D^3 f(0) \right\| \leq 3. \quad (4.6)$$

This estimation is sharp.

(ii) *If $f \in S_p^*(U^n)$, then*

$$\left\| \frac{1}{3!} D^3 f(0) \right\| \leq (1-p)(3-2p). \quad (4.7)$$

This estimation is sharp.

(iii) *If $f \in SS_\alpha^*(U^n)$, then*

$$\left\| \frac{1}{3!} D^3 f(0) \right\| \leq \alpha(1+2\alpha).$$

(iv) *If $f \in AS_\alpha^*(U^n)$, then*

$$\left\| \frac{1}{3!} D^3 f(0) \right\| \leq (1-\alpha)(3-2\alpha).$$

(v) *If $f \in PS_\rho^*(U^n)$, then*

$$\left\| \frac{1}{3!} D^3 f(0) \right\| \leq \frac{8(1-\rho)}{\pi^2} \left(1 + \frac{16(1-\rho)}{\pi^2} \right).$$

Proof It suffices to show that the estimations (4.6) and (4.7) are sharp. We can verify that the mapping

$$f(z) = \left(\frac{z_1}{(1 - z_1)^{2(1-p)}}, z_2, \dots, z_n \right), \quad z = (z_1, \dots, z_n) \in U^n$$

attains the equalities in (4.6) and (4.7).

Remark 4.1 The upper bounds in (iii), (iv) and (v) of Corollary 4.6 may not be sharp. The reason is that the sharp upper bounds for the third coefficients of functions in SS_α^* , AS_α^* and PS_ρ^* are less than the bounds in Corollary 4.6 (cf. [27, Theorem 3]).

For $S_{g,k+1}^*(U^n)$, we obtain the following estimation.

Theorem 4.4 *Let g be a function which satisfies the conditions of Assumption 2.1. Moreover, we assume that $\frac{1}{g}$ is convex. If $f \in S_{g,k+1}^*(U^n)$, then*

$$\left\| \frac{1}{m!} D^m f(0) \right\| \leq \frac{1}{m-1} |g'(0)|$$

for $m = k+1, \dots, 2k$. This estimation is sharp for $m = k+1$.

Proof Let

$$\varphi^{(m)}(z) = \frac{1}{m!} D^m f(0)(z^m).$$

Since $f(z) - z$ has a zero of order $k+1$ at $z = 0$, we have $\varphi^{(m)} = 0$ for $m = 2, \dots, k$. Then we obtain

$$\begin{aligned} \sum_{m=k+1}^{\infty} (m-1) \varphi^{(m)}(z) &= [(I + J_{\varphi^{(k+1)}}(z) + \dots) \\ &\quad \times \begin{pmatrix} \sum_{m=1}^{\infty} Q_1^{(m)}(z) & & 0 \\ & \ddots & \\ 0 & & \sum_{m=1}^{\infty} Q_n^{(m)}(z) \end{pmatrix} \\ &\quad \times (I - J_{\varphi^{(k+1)}}(z) + \dots)] \cdot (z + \varphi^{(k+1)}(z) + \dots) \end{aligned} \quad (4.8)$$

from [7, p. 173]. Comparing m -th degree terms on both sides of (4.8), where $m = 2, \dots, k$, we obtain

$$Q_i^{(m)}(z) = 0 \quad \text{for } i = 1, \dots, n \text{ and } m = 1, \dots, k-1.$$

Therefore, comparing m -th degree terms on both sides of (4.8), where $m = k+1, \dots, 2k$, we obtain

$$(m-1) \varphi^{(m)}(z) = (Q_1^{(m-1)}(z) z_1, \dots, Q_n^{(m-1)}(z) z_n)$$

for $m = k+1, \dots, 2k$. Then by Lemma 4.1, we obtain

$$\|\varphi^{(m)}(z)\| \leq \frac{1}{m-1} |g'(0)| \|z\|^m$$

for $m = k+1, \dots, 2k$. This completes the proof.

5 Examples

Let $B^n(p)$ denote the unit ball in \mathbb{C}^n with respect to a p -norm $\|\cdot\|$, $1 \leq p \leq \infty$, where

$$\|z\| = \begin{cases} \left[\sum_{j=1}^n |z_j|^p \right]^{\frac{1}{p}}, & 1 \leq p < \infty, \\ \max_{1 \leq j \leq n} |z_j|, & p = \infty. \end{cases}$$

In Lemma 3.1, we give an example of a mapping for $S_{g,k+1}^*(B)$. In this section, we will give other examples in the case $B = B^n(p)$ and $g(U)$ is a starlike domain with respect to 1.

Let $\alpha \in [0, 1]$. Hamada, Honda and Kohr [13] showed that if f is a parabolic starlike mapping of order ρ on U , then $\Psi_{n,\alpha}(f)$ is a parabolic starlike mapping of order ρ on $B^n(p)$, where

$$\Psi_{n,\alpha}(f)(z) = \left(f(z_1), \tilde{z} \left(\frac{f(z_1)}{z_1} \right)^\alpha \right)$$

for $z = (z_1, \tilde{z}) \in B^n(p)$. The branch of the power function is chosen so that

$$\left(\frac{f(z_1)}{z_1} \right)^\alpha \Big|_{z_1=0} = 1.$$

We will generalize the above result to $f \in S_{g,k+1}^*$ in the case where $g(U)$ is a starlike domain with respect to 1. This result gives many examples of mappings in $S_{g,k+1}^*(B^n(p))$.

Theorem 5.1 *Assume that g satisfies Assumption 2.1 and $g(U)$ is a starlike domain with respect to 1. Let $\alpha \in [0, 1]$. Let $\Psi_{n,\alpha}(f)$ be as above. If $f \in S_{g,k+1}^*$, then $F_\alpha = \Psi_{n,\alpha}(f) \in S_{g,k+1}^*(B^n(p))$, where $1 \leq p \leq \infty$.*

Proof When $1 < p < \infty$, $T(z)$ ($z \neq 0$) consists of those functionals given by

$$l_z(w) = \sum_{j=1}^n \frac{1}{\|z\|^{p-1}} |z_j|^p \frac{w_j}{z_j}$$

(cf. [33]). Since $f \in S_{g,k+1}^*$, we obtain

$$h(z_1) = \frac{f(z_1)}{z_1 f'(z_1)} \in g(U). \quad (5.1)$$

By direct computations, we have

$$[DF_\alpha(z)]^{-1} F_\alpha(z) = (z_1 h(z_1), (1 - \alpha + \alpha h(z_1)) \tilde{z})$$

for $z = (z_1, \tilde{z}) \in B^n(p)$. Then

$$\frac{1}{\|z\|} l_z([DF_\alpha(z)]^{-1} F_\alpha(z)) = \frac{|z_1|^p + \alpha \|\tilde{z}\|^p}{\|z\|^p} h(z_1) + \frac{(1 - \alpha) \|\tilde{z}\|^p}{\|z\|^p} \in g(U)$$

by (5.1). Therefore, $F_\alpha \in S_{g,k+1}^*(B^n(p))$. The proof for the case $p = 1$ or $p = \infty$ is similar. This completes the proof.

Theorem 5.2 *Assume that g satisfies Assumption 2.1 and $g(U)$ is starlike with respect to 1. Then $f_1 \in S_{g,k+1}^*$ if and only if $f(z) = (f_1(z_1), z_2, \dots, z_n) \in S_{g,k+1}^*(B^n(p))$, where $1 \leq p \leq \infty$.*

Proof When $p = \infty$, $T(z)$ ($z \neq 0$) consists of those functionals l_z given by

$$l_z(w) = \sum_{|z_k|=\|z\|} t_k \|z\| \frac{w_k}{z_k}, \quad t_k \geq 0, \quad \sum_{|z_k|=\|z\|} t_k = 1$$

(cf. [33]). Then

$$\frac{1}{\|z\|} l_z([Df(z)]^{-1} f(z)) = \begin{cases} 1, & |z_1| \neq \|z\|, \\ t_1 \frac{f_1}{z_1 f'_1} + (1 - t_1) 1, & |z_1| = \|z\|. \end{cases}$$

Since $g(U)$ is a starlike domain with respect to 1, $f_1 \in S_{g,k+1}^*$ if and only if $f \in S_{g,k+1}^*(B^n(p))$. The proof in the case $p < \infty$ is similar.

Furthermore, if $g(U)$ is convex, then we obtain the following theorem.

Theorem 5.3 Assume that g satisfies Assumption 2.1 and $g(U)$ is convex. Then $f_1, f_2, \dots, f_n \in S_{g,k+1}^*$ if and only if $f(z) = (f_1(z_1), f_2(z_2), \dots, f_n(z_n)) \in S_{g,k+1}^*(B^n(p))$, where $1 \leq p \leq \infty$.

Proof When $p = 1$, $T(z)$ ($z \neq 0$) consists of those functionals given by

$$l_z(w) = \sum_{z_j \neq 0} \frac{|z_j|}{z_j} w_j + \sum_{z_j=0} \alpha_j w_j, \quad |\alpha_j| \leq 1.$$

(cf. [33]). Then

$$\frac{1}{\|z\|} l_z([Df(z)]^{-1} f(z)) = \sum_{z_j \neq 0} \frac{|z_j|}{\|z\|} \frac{f_j(z_j)}{z_j f'_j(z_j)}.$$

Since $g(U)$ is convex, $f_1, f_2, \dots, f_n \in S_{g,k+1}^*$ if and only if $f \in S_{g,k+1}^*(B^n(p))$. The proof in the case $p > 1$ is similar.

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