Chin. Ann. Math.
29B(5), 2008, 531–542
DOI: 10.1007/s11401-007-0295-8

Chinese Annals of Mathematics, Series B

Some Uniqueness Theorems with Truncated Multiplicities of Meromorphic Mappings**

Qiming YAN*

Abstract In this article, some uniqueness theorems of meromorphic mappings in several complex variables sharing hyperplanes in general position are proved with truncated multiplicities.

Keywords Value distribution theory, Uniqueness theorem, Meromorphic mapping
2000 MR Subject Classification 32H30

1 Introduction

It is well-known that two non-constant polynomials f, g over an algebraic closed field of characteristic zero are identical if there exist two distinct values a, b such that $f^{-1}(a) = g^{-1}(a)$ and $f^{-1}(b) = g^{-1}(b)$.

In 1926, R. Nevanlinna [1] extended the above result to meromorphic functions. He showed that, for two distinct non-constant meromorphic functions f and g on the complex plane $\mathbb{C}$, they can not have the same inverse images for five distinct values, and g is a special type of linear fractional transformation of f if they have the same inverse counted with multiplicities for four distinct values.

Over the last few decades, there have been several generalizations of Nevanlinna's result to the case of meromorphic mappings from $\mathbb{C}^n$ into the complex projective space $\mathbb{P}^N(\mathbb{C})$.

The study of uniqueness theorems of meromorphic mappings from $\mathbb{C}^n$ into $\mathbb{P}^N(\mathbb{C})$ on a finite set of hyperplanes in $\mathbb{P}^N(\mathbb{C})$ began about 30 years ago and now has ample results.

Some of the first results concerning this research are due to H. Fujimoto [2, 3]. Consider two distinct meromorphic mappings f and g from $\mathbb{C}^n$ into $\mathbb{P}^N(\mathbb{C})$ satisfying the condition that $\nu_{(f,H_j)} = \nu_{(g,H_j)}$ for q hyperplanes $H_1, \cdots, H_q$ in $\mathbb{P}^N(\mathbb{C})$ located in general position, where we denote by $\nu_{(f,H_j)}$ the map of $\mathbb{C}^n$ into $\mathbb{Z}$ whose value $\nu_{(f,H_j)}(z)$ $(z \in \mathbb{C}^n)$ is the intersection multiplicity of the images f and H_j at $f(z)$. He proved the following brilliant theorems.

Theorem A *Let f and g be two non-constant meromorphic mappings from $\mathbb{C}^n$ into $\mathbb{P}^N(\mathbb{C})$. Suppose that there exist $3N+1$ hyperplanes $H_j,\ 1 \le j \le 3N+1$, in $\mathbb{P}^N(\mathbb{C})$ located in general*

Manuscript received July 23, 2007. Published online August 27, 2008.
*School of Mathematical Sciences, Fudan University, Shanghai 200433, China.
E-mail: yan_qiming@hotmail.com
**Project supported by the National Natural Science Foundation of China (No. 10571135), the Doctoral Program Foundation of the Ministry of Education of China (No. 20050240711) and the Foundation of the Committee of Science and Technology of Shanghai (No. 03JC14027).

position such that $f(\mathbb{C}^n) \not\subset H_j$, $g(\mathbb{C}^n) \not\subset H_j$ and $\nu_{(f,H_j)} = \nu_{(g,H_j)}$, $1 \le j \le 3N+1$. Then there is a projective linear transformation L of $\mathbb{P}^N(\mathbb{C})$ such that $L(f) = g$.

Theorem B *Let f and g be two non-constant meromorphic mappings from $\mathbb{C}^n$ into $\mathbb{P}^N(\mathbb{C})$, at least one of which is linearly non-degenerate. Suppose that there exist $3N+2$ hyperplanes H_j, $1 \le j \le 3N+2$, in $\mathbb{P}^N(\mathbb{C})$ located in general position such that $f(\mathbb{C}^n) \not\subset H_j$, $g(\mathbb{C}^n) \not\subset H_j$ and $\nu_{(f,H_j)} = \nu_{(g,H_j)}$, $1 \le j \le 3N+2$. Then $f \equiv g$.*

Theorem C *Let f and g be two non-constant meromorphic mappings from $\mathbb{C}^n$ into $\mathbb{P}^N(\mathbb{C})$, at least one of which is algebraically non-degenerate. Suppose that there exist $2N+3$ hyperplanes H_j, $1 \le j \le 2N+3$, in $\mathbb{P}^N(\mathbb{C})$ located in general position such that $f(\mathbb{C}^n) \not\subset H_j$, $g(\mathbb{C}^n) \not\subset H_j$ and $\nu_{(f,H_j)} = \nu_{(g,H_j)}$, $1 \le j \le 2N+3$. Then $f \equiv g$.*

To state some results of the uniqueness problem with truncated multiplicities, we take a linearly non-degenerate meromorphic mapping $f : \mathbb{C}^n \to \mathbb{P}^N(\mathbb{C})$, a positive integer d and q hyperplanes $H_1, \cdots, H_q$ in general position.

For $1 \le j \le q$, we set

$$\nu_{(f,H_j),\le k}(z) = \begin{cases} 0, & \text{if } \nu_{(f,H_j)}(z) > k, \\ \nu_{(f,H_j)}(z), & \text{if } \nu_{(f,H_j)}(z) \le k, \end{cases}$$

$$\nu_{(f,H_j),> k}(z) = \begin{cases} 0, & \text{if } \nu_{(f,H_j)}(z) \le k, \\ \nu_{(f,H_j)}(z), & \text{if } \nu_{(f,H_j)}(z) > k, \end{cases}$$

where k is a positive integer or $k = +\infty$.

Assume that $\dim\{z \in \mathbb{C}^n \mid \nu_{(f,H_i),\le k}(z) > 0 \text{ and } \nu_{(f,H_j),\le k}(z) > 0\} \le n-2$, $1 \le i < j \le q$.

Consider the set $\mathcal{F}(f, \{H_j\}_{j=1}^q, k, d)$ of all linearly non-degenerate meromorphic mappings $g : \mathbb{C}^n \to \mathbb{P}^N(\mathbb{C})$ satisfying the conditions:

(a) $\min\{\nu_{(f,H_j),\le k}, d\} = \min\{\nu_{(g,H_j),\le k}, d\}$, $1 \le j \le q$,

(b) $f(z) = g(z)$ on $\bigcup_{j=1}^{q} \{z \in \mathbb{C}^n \mid \nu_{(f,H_j),\le k}(z) > 0\}$.

L. Smiley [4] gave the following uniqueness theorem:

Theorem D *If $q \ge 3N+2$, then $\sharp\mathcal{F}(f, \{H_j\}_{j=1}^q, +\infty, 1) = 1$.*

H. Fujimoto [5] proved the following result:

Theorem E *If $q = 3N+1$, then $\sharp\mathcal{F}(f, \{H_j\}_{j=1}^q, +\infty, 2) \le 2$.*

There are several open problems related to the above results (cf. [5]). One of them is the following:

Is it still true if the number q in Theorems D and E is replaced by a smaller one?

In [6], Thai and Quang improved the above results as follows:

Theorem F *If $N \ge 2$, then $\sharp\mathcal{F}(f, \{H_j\}_{j=1}^{3N+1}, +\infty, 1) = 1$.*
If $N \ge 4$, then $\sharp\mathcal{F}(f, \{H_j\}_{j=1}^{3N-1}, +\infty, 2) \le 2$.

Theorem G *If $N \ge 2$, then $\sharp\mathcal{F}(f, \{H_j\}_{j=1}^{3N+1}, k, 2) = 1$ for $k > \frac{(3N+1)(N+2)}{2}$.*
If $N \ge 3$, then $\sharp\mathcal{F}(f, \{H_j\}_{j=1}^{3N}, k, 2) = 1$ for $k > 3N^2 + 12N + 23 + \frac{48}{N-2}$.

In [7], Dethloff and Tan obtained a uniqueness theorem for the case $q = 3N + 1 - x$.

Theorem H *Let f, g be two linearly non-degenerate meromorphic mappings of $\mathbb{C}^n$ into $\mathbb{P}^N(\mathbb{C})$ and $\{H_j\}_{j=1}^q$ be $q = 3N + 1 - x$ hyperplanes in $\mathbb{P}^N(\mathbb{C})$ in general position. Set*

$$E_f^j = \{z \in \mathbb{C}^n \mid 0 \le \nu_{(f,H_j)}(z) \le k\}, \quad {}^*E_f^j = \{z \in \mathbb{C}^n \mid 0 < \nu_{(f,H_j)}(z) \le k\},$$

and similarly for E_g^j, ${}^*E_g^j$, $j = 1, \cdots, q$. Assume that

(a) $\min\{\nu_{(f,H_j)}, 1\} = \min\{\nu_{(g,H_j)}, 1\}$ *on* $E_f^j \cap E_g^j$, $N + 2 + y \le j \le q$,
$\min\{\nu_{(f,H_j)}, p\} = \min\{\nu_{(g,H_j)}, p\}$ *on* $E_f^j \cap E_g^j$, $1 \le j \le N + 1 + y$,

where $1 \le y \le 2N$ *and* $2 \le p \le N$,

(b) $\dim({}^*E_f^i \cap {}^*E_f^j) \le n - 2$, $\dim({}^*E_g^i \cap {}^*E_g^j) \le n - 2$, $1 \le i < j \le q$,

(c) $f = g$ *on* $\bigcup_{j=1}^{q} ({}^*E_f^j \cap {}^*E_g^j)$.

If $0 \le x < \min\{2N - y + 1, \frac{(p-1)y}{N+1+y}\}$, *then* $f \equiv g$ *for* $k \ge \frac{2N(N+1+y)(3N+p-x)}{(p-1)y-x(N+1+y)}$.

Particularly, take $N \ge 2$, $y = 1$, $p = 2$ and $x = 0$.

Corollary A *Let f, g be two linearly non-degenerate meromorphic mappings of $\mathbb{C}^n$ into $\mathbb{P}^N(\mathbb{C})$ $(N \ge 2)$ and $\{H_j\}_{j=1}^{3N+1}$ be hyperplanes in $\mathbb{P}^N(\mathbb{C})$ in general position. Assume that*

(a) $\min\{\nu_{(f,H_j)}, 1\} = \min\{\nu_{(g,H_j)}, 1\}$ *on* $E_f^j \cap E_g^j$, $N + 3 \le j \le 3N + 1$,
$\min\{\nu_{(f,H_j)}, 2\} = \min\{\nu_{(g,H_j)}, 2\}$ *on* $E_f^j \cap E_g^j$, $1 \le j \le N + 2$,

(b) $\dim({}^*E_f^i \cap {}^*E_f^j) \le n - 2$, $\dim({}^*E_g^i \cap {}^*E_g^j) \le n - 2$, $1 \le i < j \le 3N + 1$,

(c) $f = g$ *on* $\bigcup_{j=1}^{3N+1} ({}^*E_f^j \cap {}^*E_g^j)$.

Then $f \equiv g$ *for* $k \ge N(N + 2)(6N + 4)$.

Take $N \ge 3$, $y = N + 2$, $p = 3$ and $x = 1$.

Corollary B *Let f, g be two linearly non-degenerate meromorphic mappings of $\mathbb{C}^n$ into $\mathbb{P}^N(\mathbb{C})$ $(N \ge 3)$ and $\{H_j\}_{j=1}^{3N}$ be hyperplanes in $\mathbb{P}^N(\mathbb{C})$ in general position. Assume that*

(a) $\min\{\nu_{(f,H_j)}, 1\} = \min\{\nu_{(g,H_j)}, 1\}$, $2N + 4 \le j \le 3N$,
$\min\{\nu_{(f,H_j)}, 3\} = \min\{\nu_{(g,H_j)}, 3\}$, $1 \le j \le 2N + 3$,

(b) $\dim(f^{-1}(H_i) \cap f^{-1}(H_j)) \le n - 2$, $1 \le i < j \le 3N$,

(c) $f = g$ *on* $\bigcup_{j=1}^{3N} f^{-1}(H_j)$.

Then $f \equiv g$.

Take $N \ge 2$, $y = I(\sqrt{2N(N+1)}\,)$, $p = N$ and $x = 2N - I(\sqrt{2N(N+1)}\,)$, where $I(m) := \min\{k \in \mathbb{N} \mid k > m\}$ for a positive constant m.

Corollary C *Let f, g be two linearly non-degenerate meromorphic mappings of $\mathbb{C}^n$ into $\mathbb{P}^N(\mathbb{C})$ $(N \ge 2)$ and $\{H_j\}_{j=1}^{N+I(\sqrt{2N(N+1)}\,)+1}$ be hyperplanes in $\mathbb{P}^N(\mathbb{C})$ in general position. Assume that*

(a) $\min\{\nu_{(f,H_j)}, N\} = \min\{\nu_{(g,H_j)}, N\}$, $1 \le j \le N + I(\sqrt{2N(N+1)}\,) + 1$,

(b) $\dim(f^{-1}(H_i) \cap f^{-1}(H_j)) \le n - 2$, $i \ne j$,

(c) $f = g$ *on* $\bigcup_{j=1}^{N+I(\sqrt{2N(N+1)}\,)+1} f^{-1}(H_j)$.

Then $f \equiv g$.

In this paper, we will improve the result given by Dethloff and Tan [7].

Our main results are stated as follows.

Theorem 1.1 *With the same assumptions as in Theorem H, if* $0 \leq x < \min\{2N - y + 1, \frac{(p-1)y}{N+1+y}\}$, *then* $f \equiv g$ *for* $k \geq \frac{2N(N+1+y)(3N+\frac{p}{2}-x)}{(p-1)y-x(N+1+y)}$.

For $q = 3N$, we obtain

Theorem 1.2 *Let* f, g *be two linearly non-degenerate meromorphic mappings of* $\mathbb{C}^n$ *into* $\mathbb{P}^N(\mathbb{C})$ $(N \geq 3)$ *and* $\{H_j\}_{j=1}^{3N}$ *be hyperplanes in* $\mathbb{P}^N(\mathbb{C})$ *in general position such that*

$$\dim\{z \in \mathbb{C}^n \mid \nu_{(f,H_i),\leq k}(z) > 0 \quad \text{and} \quad \nu_{(f,H_j),\leq k}(z) > 0\} \leq n-2, \quad 1 \leq i < j \leq 3N.$$

Assume that

(a) $\min\{\nu_{(f,H_j),\leq k}, p_j\} = \min\{\nu_{(g,H_j),\leq k}, p_j\}$, $1 \leq j \leq 3N$,

(b) $f(z) = g(z)$ *on* $\bigcup\limits_{j=1}^{3N}\{z \in \mathbb{C}^n \mid \nu_{(f,H_j),\leq k}(z) > 0\}$.

Then there exist N *indices* $j_{2N+1}, \cdots, j_{3N}$ *with*

$$p_{j_{2N+1}} = \cdots = p_{j_{3N}} = 1 \quad \text{and} \quad p_{j_1} = \cdots = p_{j_{2N}} = 3,$$

such that $f \equiv g$ *is still valid for* $k \geq N^2 + 6N + 12 + \frac{24}{N-2}$.

Finally, we give a uniqueness theorem for $2N+3$ hyperplanes.

Theorem 1.3 *Let* f, g *be two linearly non-degenerate meromorphic mappings of* $\mathbb{C}^n$ *into* $\mathbb{P}^N(\mathbb{C})$ *and* $\{H_j\}_{j=1}^{2N+3}$ *be hyperplanes in* $\mathbb{P}^N(\mathbb{C})$ *in general position such that*

$$\dim(f^{-1}(H_i) \cap f^{-1}(H_j)) \leq n-2, \quad 1 \leq i < j \leq 2N+3.$$

Assume that

(a) $\min\{\nu_{(f,H_j)}, p_j\} = \min\{\nu_{(g,H_j)}, p_j\}$, $1 \leq j \leq 2N+3$,

(b) $f = g$ *on* $\bigcup\limits_{j=1}^{2N+3} f^{-1}(H_j)$.

Then there exist 3 *indices* $j_{2N+1}, j_{2N+2}, j_{2N+3}$ *with*

$$p_{j_{2N+1}} = p_{j_{2N+2}} = p_{j_{2N+3}} = 1 \quad \text{and} \quad p_{j_1} = \cdots = p_{j_{2N}} = N,$$

such that $f \equiv g$ *is still valid.*

We note that Theorem 1.1 is an improvement of Theorem H, and Theorem 1.2 (Theorem 1.3) is a kind of improvement of Corollary B (Corollary C). Particularly, for $N = 1$, Theorem 1.3 is just the Nevanlinna Five Values Theorem.

2 Preliminaries and Some Lemmas

We first introduce some preliminaries from Nevanlinna theory.

Let $F(z)$ be a nonzero entire function on $\mathbb{C}^n$. For $a \in \mathbb{C}^n$, set $F(z) = \sum\limits_{m=0}^{+\infty} P_m(z-a)$, where the term $P_m(z)$ is either identically zero or a homogeneous polynomial of degree m. The number $\nu_F(a) := \min\{m \mid P_m \not\equiv 0\}$ is said to be the zero-multiplicity of F at a. Set $\operatorname{supp}\nu_F := \overline{\{z \in \mathbb{C}^n \mid \nu_F(z) \neq 0\}}$.

Let φ be a nonzero meromorphic function on $\mathbb{C}^n$. For each $a \in \mathbb{C}^n$, we choose nonzero holomorphic functions F and G on a neighborhood U of a such that $\varphi = \frac{F}{G}$ on U and $\dim(F^{-1}(0) \cap G^{-1}(0)) \leq n-2$, and we define $\nu_\varphi := \nu_F$ and $\nu_\varphi^\infty := \nu_G$, which are independent of the choices of F and G.

For $z = (z_1, \cdots, z_n) \in \mathbb{C}^n$, we set $\|z\| = (|z_1|^2 + \cdots + |z_n|^2)^{\frac{1}{2}}$. For $r > 0$, define $B(r) = \{z \in \mathbb{C}^n \mid \|z\| < r\}$, $S(r) = \{z \in \mathbb{C}^n \mid \|z\| = r\}$, $d^c = (4\pi\sqrt{-1})^{-1}(\partial - \overline{\partial})$, $v = (dd^c\|z\|^2)^{n-1}$ and $\sigma = d^c \log\|z\|^2 \wedge (dd^c \log\|z\|^2)^{n-1}$.

Let $f : \mathbb{C}^n \to \mathbb{P}^N(\mathbb{C})$ be a meromorphic mapping. We can choose holomorphic functions $f_0, \cdots, f_N$ on $\mathbb{C}^n$ such that

$$I_f := \{z \in \mathbb{C}^n \mid f_0(z) = \cdots = f_N(z) = 0\}$$

is of dimension at most $n-2$, and such that $f = [f_0, \cdots, f_N]$. Usually, $[f_0, \cdots, f_N]$ is called a reduced representation of f. The characteristic function of f is defined by

$$T(r,f) = \int_{S(r)} \log\|f\|\sigma - \int_{S(1)} \log\|f\|\sigma, \quad r > 1.$$

Note that $T(r,f)$ is independent of the choice of the reduced representation of f.

We now define counting function. For a hyperplane $H = \{(x_0 : \cdots : x_N) \in \mathbb{P}^N(\mathbb{C}) \mid a_0x_0 + \cdots + a_Nx_N = 0\}$, (f,H) is said to be free if $(f,H) = \sum\limits_{i=0}^{N} a_i f_i \not\equiv 0$. Under the assumption that (f,H) is free, we define

$$\nu^M_{(f,H)}(z) = \min\{M, \nu_{(f,H)}(z)\},$$
$$\nu^M_{(f,H),\leq k}(z) = \begin{cases} 0, & \text{if } \nu_{(f,H)}(z) > k, \\ \nu^M_{(f,H)}(z), & \text{if } \nu_{(f,H)}(z) \leq k, \end{cases}$$
$$\nu^M_{(f,H),>k}(z) = \begin{cases} 0, & \text{if } \nu_{(f,H)}(z) \leq k, \\ \nu^M_{(f,H)}(z), & \text{if } \nu_{(f,H)}(z) > k \end{cases}$$

for positive integers k, M (or $k, M = +\infty$). Set

$$n(t) = \begin{cases} \displaystyle\int_{\operatorname{supp}\nu_{(f,H)} \cap B(t)} \nu_{(f,H)}(z) v, & \text{if } n \geq 2, \\ \displaystyle\sum_{|z| \leq t} \nu_{(f,H)}(z), & \text{if } n = 1. \end{cases}$$

Similarly, we define $n^M(t)$, $n^M_{\leq k}(t)$ and $n^M_{>k}(t)$.

Define

$$N_{(f,H)}(r) = \int_1^r \frac{n(t)}{t^{2n-1}} \mathrm{d}t, \quad 1 < r < +\infty.$$

Similarly, we define $N^M_{(f,H)}(r), N^M_{(f,H),\le k}(r)$ and $N^M_{(f,H),>k}(r)$. For a nonzero meromorphic function φ, we can define the counting function $N(r,\nu^\infty_\varphi)$ similarly.

We define the proximity function of H by

$$m_{f,H}(r)=\int_{S(r)}\log\frac{\|f\|\|H\|}{|(f,H)|}\sigma-\int_{S(1)}\log\frac{\|f\|\|H\|}{|(f,H)|}\sigma,\quad r>1,$$

and the proximity function of a meromorphic function φ on $\mathbb{C}^n$ is defined by

$$m(r,\varphi)=\int_{S(r)}\log^+|\varphi|\sigma.$$

Theorem 2.1 (First Main Theorem) $T(r,f)=m_{f,H}(r)+N_{(f,H)}(r)+O(1)$.

Theorem 2.2 (Second Main Theorem) *Let f be a linearly non-degenerate meromorphic mapping of $\mathbb{C}^n$ into $\mathbb{P}^N(\mathbb{C})$ and $H_1,\cdots,H_q$ be hyperplanes in general position. Then*

$$\|(q-N-1)T(r,f)\le\sum_{j=1}^q N^N_{(f,H_j)}(r)+o(T(r,f)),$$

where "$\|$" means the estimate holds for all large r outside a set of finite Lebesgue measure.

In [7], the second main theorem for multiple value is given as follows:

$$\Big\|\frac{(q-N-1)(k+1)-qN}{k}T(r,f)\le\sum_{j=1}^q N^N_{(f,H_j),\le k}(r)+o(T(r,f)),\quad k\ge N-1.\tag{2.1}$$

In [6], Thai and Quang proved that

$$\Big\|\frac{(q-N-1)(k+1)-qN}{k+1-N}T(r,f)\le\sum_{j=1}^q N^N_{(f,H_j),\le k}(r)+o(T(r,f)),\quad k\ge N-1.\tag{2.2}$$

We include the proof of (2.2) here for the completeness.

Proof By the second main theorem, we have

$$\begin{aligned}\|(q-N-1)T(r,f)&\le\sum_{j=1}^q N^N_{(f,H_j)}(r)+o(T(r,f))\\&\le\sum_{j=1}^q N^N_{(f,H_j),\le k}(r)+\sum_{j=1}^q\frac{N}{k+1}N_{(f,H_j),>k}(r)+o(T(r,f))\\&=\sum_{j=1}^q N^N_{(f,H_j),\le k}(r)+\sum_{j=1}^q\frac{N}{k+1}(N_{(f,H_j)}(r)-N_{(f,H_j),\le k}(r))+o(T(r,f))\\&\le\sum_{j=1}^q\Big(1-\frac{N}{k+1}\Big)N^N_{(f,H_j),\le k}(r)+\frac{Nq}{k+1}T(r,f)+o(T(r,f)).\end{aligned}$$

It implies that

$$\Big\|\Big(q-N-1-\frac{Nq}{k+1}\Big)T(r,f)\le\sum_{j=1}^q\Big(1-\frac{N}{k+1}\Big)N^N_{(f,H_j),\le k}(r)+o(T(r,f)).$$

Hence, we have (2.2).

Now we give some useful lemmas.

Take two distinct hyperplanes $H_j,\ j=1,2$ and consider a meromorphic function

$$F_f^{H_1H_2}=\frac{(f,H_1)}{(f,H_2)}.$$

We have

Lemma 2.1 (cf. [8]) $T(r,F_f^{H_1H_2})\le T(r,f)+O(1)$.

Denote by $\mathcal{S}$ the set of all $c\in\mathbb{C}^{N+1}\setminus\{0\}$ such that

$$\dim\{z\in\mathbb{C}^n\mid (f,H_j)(z)=(f,c)(z)=0\}\le n-2,$$
$$\dim\{z\in\mathbb{C}^n\mid (g,H_j)(z)=(g,c)(z)=0\}\le n-2\quad \text{for } 1\le j\le q.$$

We have

Lemma 2.2 (cf. [9]) *$\mathcal{S}$ is dense in $\mathbb{C}^{N+1}$.*

3 Proof of Main Results

For f,g, we set $T(r)=T(r,f)+T(r,g)$.

Proof of Theorem 1.1 Assume that $f\not\equiv g$. For any fixed $j_0,\ 1\le j_0\le N+1+y$, there exists $c\in\mathcal{S}$ such that $F_f^{H_{j_0}c}-F_g^{H_{j_0}c}\not\equiv 0$ by Lemma 2.2.

Since $\min\{\nu_{(f,H_{j_0})},p\}=\min\{\nu_{(g,H_{j_0})},p\}$ on $E_f^{j_0}\cap E_g^{j_0}$, we see that a zero point z_0 of (f,H_{j_0}) with multiplicity $\le k$ is either a zero point of (g,H_{j_0}) with multiplicity $\le k$ or a zero point of (g,H_{j_0}) with multiplicity $>k$. Then z_0 is a zero point of $F_f^{H_{j_0}c}-F_g^{H_{j_0}c}$ with multiplicity $\ge\min\{\nu_{(f,H_{j_0})},p\}$ outside an analytic set of codimension ≥ 2.

For any $j\in\{1,\cdots,q\}\setminus\{j_0\}$, by $f=g$ on $\bigcup\limits_{j=1}^{q}({}^*E_f^j\cap{}^*E_g^j)$, we have that a zero point of (f,H_j) with multiplicity $\le k$ is either a zero point of $F_f^{H_{j_0}c}-F_g^{H_{j_0}c}$ or a zero point of (g,H_j) with multiplicity $>k$ outside an analytic set of codimension ≥ 2.

Hence

$$N^p_{(f,H_{j_0}),\le k}(r)+\sum_{\substack{1\le j\le q\\ j\ne j_0}}N^1_{(f,H_j),\le k}(r)\le N_{F_f^{H_{j_0}c}-F_g^{H_{j_0}c}}(r)+\sum_{\substack{1\le j\le q\\ j\ne j_0}}N^1_{(g,H_j),>k}(r).\tag{3.1}$$

Remark 3.1 (3.1) improves (3.5) in [7].

By the first main theorem and Lemma 2.1, we get

$$\begin{aligned}
&N_{F_f^{H_{j_0}c}-F_g^{H_{j_0}c}}(r)+\sum_{\substack{1\le j\le q\\ j\ne j_0}}N^1_{(g,H_j),>k}(r)\\
&\le T(r,F_f^{H_{j_0}c}-F_g^{H_{j_0}c})+\frac{1}{k+1}\sum_{\substack{1\le j\le q\\ j\ne j_0}}N_{(g,H_j)}(r)+O(1)\\
&\le T(r)+\frac{q-1}{k+1}T(r,g)+O(1).
\end{aligned}$$

By (3.1), we have

$$N^p_{(f,H_{j_0}),\le k}(r)+\sum_{\substack{1\le j\le q\\ j\ne j_0}}N^1_{(f,H_j),\le k}(r)\le T(r)+\frac{q-1}{k+1}T(r,g)+O(1).$$

Similarly

$$N^p_{(g,H_{j_0}),\le k}(r)+\sum_{\substack{1\le j\le q\\ j\ne j_0}}N^1_{(g,H_j),\le k}(r)\le T(r)+\frac{q-1}{k+1}T(r,f)+O(1).$$

Note that $p\le N$. Hence

$$\begin{aligned}&\frac{p}{N}(N^N_{(f,H_{j_0}),\le k}(r)+N^N_{(g,H_{j_0}),\le k}(r))+\frac{1}{N}\sum_{\substack{1\le j\le q\\ j\ne j_0}}(N^N_{(f,H_j),\le k}(r)+N^N_{(g,H_j),\le k}(r))\\ \le\ &\frac{2(k+1)+q-1}{k+1}T(r)+O(1)\le\frac{2(k+1)+q-1}{k}T(r)+O(1).\\ &\frac{p-1}{N}(N^N_{(f,H_{j_0}),\le k}(r)+N^N_{(g,H_{j_0}),\le k}(r))\\ \le\ &\frac{2(k+1)+q-1}{k}T(r)-\frac{1}{N}\sum_{j=1}^{q}(N^N_{(f,H_j),\le k}(r)+N^N_{(g,H_j),\le k}(r))+O(1).\end{aligned}\tag{3.2}$$

Using (2.1), we have

$$\Big\|\frac{(q-N-1)(k+1)-qN}{k}T(r)\le\sum_{j=1}^{q}(N^N_{(f,H_j),\le k}(r)+N^N_{(g,H_j),\le k}(r))+o(T(r)).\tag{3.3}$$

By (3.2) and (3.3), we have

$$\begin{aligned}&\Big\|\frac{p-1}{N}(N^N_{(f,H_{j_0}),\le k}(r)+N^N_{(g,H_{j_0}),\le k}(r))+o(T(r))\\ \le\ &\Big(\frac{2(k+1)+q-1}{k}-\frac{(q-N-1)(k+1)-qN}{Nk}\Big)T(r).\end{aligned}$$

It means that

$$\|(N^N_{(f,H_{j_0}),\le k}(r)+N^N_{(g,H_{j_0}),\le k}(r))+o(T(r))\le\frac{(3N+1-q)(k+1)+(2q-1)N}{k(p-1)}T(r)$$

for all j_0, $1\le j_0\le N+1+y$.

So

$$\begin{aligned}&\Big\|\sum_{j=1}^{N+1+y}(N^N_{(f,H_j),\le k}(r)+N^N_{(g,H_j),\le k}(r))+o(T(r))\\ \le\ &\frac{(N+1+y)((3N+1-q)(k+1)+(2q-1)N)}{k(p-1)}T(r).\end{aligned}\tag{3.4}$$

Using (2.1) again, we have

$$\Big\|\frac{y(k+1)-N(N+1+y)}{k}T(r)\le\sum_{j=1}^{N+1+y}(N^N_{(f,H_j),\le k}(r)+N^N_{(g,H_j),\le k}(r))+o(T(r)).\tag{3.5}$$

From (3.4) and (3.5), we obtain

$$\begin{aligned}&\Big\| \frac{y(k+1)-N(N+1+y)}{k}T(r)+o(T(r))\\ &\le \frac{(N+1+y)((3N+1-q)(k+1)+(2q-1)N)}{k(p-1)}T(r).\end{aligned}$$

It implies that

$$(p-1)(y(k+1)-N(N+1+y)) \le (N+1+y)(x(k+1)+(6N+1-2x)N).$$

(Note that $q=3N+1-x$.)

We have $k+1 \le \frac{2N(N+1+y)(3N+\frac{p}{2}-x)}{(p-1)y-x(N+1+y)}$, which is a contradiction. Thus, we have $f\equiv g$.

Proof of Theorem 1.2 Assume that $f \not\equiv g$. By changing indices, if necessary, we may assume that

$$\begin{aligned}&\underbrace{\frac{(f,H_1)}{(g,H_1)} \equiv \frac{(f,H_2)}{(g,H_2)} \equiv \cdots \equiv \frac{(f,H_{k_1})}{(g,H_{k_1})}}_{\text{group } 1} \not\equiv \underbrace{\frac{(f,H_{k_1+1})}{(g,H_{k_1+1})} \equiv \cdots \equiv \frac{(f,H_{k_2})}{(g,H_{k_2})}}_{\text{group } 2}\\ &\qquad \not\equiv \cdots \not\equiv \underbrace{\frac{(f,H_{k_{s-1}+1})}{(g,H_{k_{s-1}+1})} \equiv \cdots \equiv \frac{(f,H_{k_s})}{(g,H_{k_s})}}_{\text{group } s},\end{aligned}$$

where $k_s=3N$.

Since $f\not\equiv g$, the number of elements of every group is at most N. For each i, $1\le i\le N$, we set $j=i+N$. Hence $\frac{(f,H_i)}{(g,H_i)}$ and $\frac{(f,H_j)}{(g,H_j)}$ belong to distinct groups, so that $\frac{(f,H_i)}{(g,H_i)} \not\equiv \frac{(f,H_j)}{(g,H_j)}$.

Let

$$p_{2N+1}=\cdots=p_{3N}=1 \quad \text{and} \quad p_1=\cdots=p_{2N}=3.$$

Fixing an index i with $1\le i\le N$, we consider $F_f^{H_iH_j}-F_g^{H_iH_j}\not\equiv 0$. Then, the following four inequalities hold:

$$\begin{aligned}&N^3_{(f,H_i),\le k}(r)+\sum_{\substack{1\le v\le 3N\\ v\ne i,j}} N^1_{(f,H_v),\le k}(r) \le N_{F_f^{H_iH_j}-F_g^{H_iH_j}}(r),\\ &N_{F_f^{H_iH_j}-F_g^{H_iH_j}}(r) \le T(r,F_f^{H_iH_j}-F_g^{H_iH_j})\\ &\qquad\qquad = N(r,\nu^\infty_{F_f^{H_iH_j}-F_g^{H_iH_j}})+m(r,F_f^{H_iH_j}-F_g^{H_iH_j})+O(1),\\ &N(r,\nu^\infty_{F_f^{H_iH_j}-F_g^{H_iH_j}}) \le N(r,\nu_i), \quad \text{where } \nu_i(z)=\max\{\nu_{(f,H_j)}(z),\nu_{(g,H_j)}(z)\},\\ &m(r,F_f^{H_iH_j}-F_g^{H_iH_j}) \le m\Big(r,\frac{(f,H_i)}{(f,H_j)}\Big)+m\Big(r,\frac{(g,H_i)}{(g,H_j)}\Big)+O(1)\\ &\qquad\qquad \le T(r,f)+T(r,g)-N_{(f,H_j)}(r)-N_{(g,H_j)}(r)+O(1).\end{aligned}$$

By the above four inequalities, it follows that

$$\begin{aligned}&N^3_{(f,H_i),\le k}(r)+(N_{(f,H_j)}(r)+N_{(g,H_j)}(r)-N(r,\nu_i))+\sum_{\substack{1\le v\le 3N\\ v\ne i,j}} N^1_{(f,H_v),\le k}(r)\\ &\le T(r)+O(1). \qquad (3.6)\end{aligned}$$

On the other hand, it is easy to see that

$$N_{(f,H_j)}(r) + N_{(g,H_j)}(r) - N(r,\nu_i) \geq N^3_{(f,H_j),\leq k}(r). \tag{3.7}$$

From (3.6) and (3.7), we get

$$N^3_{(f,H_i),\leq k}(r) + N^3_{(f,H_j),\leq k}(r) + \sum_{\substack{1\leq v\leq 3N \\ v\neq i,j}} N^1_{(f,H_v),\leq k}(r) \leq T(r) + O(1).$$

Similarly

$$N^3_{(g,H_i),\leq k}(r) + N^3_{(g,H_j),\leq k}(r) + \sum_{\substack{1\leq v\leq 3N \\ v\neq i,j}} N^1_{(g,H_v),\leq k}(r) \leq T(r) + O(1).$$

Hence

$$\begin{aligned}
&\frac{3}{N}(N^N_{(f,H_i),\leq k}(r) + N^N_{(g,H_i),\leq k}(r) + N^N_{(f,H_j),\leq k}(r) + N^N_{(g,H_j),\leq k}(r)) \\
&+ \frac{1}{N}\sum_{\substack{1\leq v\leq 3N \\ v\neq i,j}} (N^N_{(f,H_v),\leq k}(r) + N^N_{(g,H_v),\leq k}(r)) \\
\leq\ & 2T(r) + O(1).
\end{aligned}$$

$$\begin{aligned}
&\frac{2}{N}(N^N_{(f,H_i),\leq k}(r) + N^N_{(g,H_i),\leq k}(r) + N^N_{(f,H_j),\leq k}(r) + N^N_{(g,H_j),\leq k}(r)) \\
\leq\ & 2T(r) - \frac{1}{N}\sum_{v=1}^{3N}(N^N_{(f,H_v),\leq k}(r) + N^N_{(g,H_v),\leq k}(r)) + O(1).
\end{aligned} \tag{3.8}$$

Using (2.2), we have

$$\Big\| \frac{(2N-1)(k+1) - 3N^2}{k+1-N} T(r) \leq \sum_{v=1}^{3N}(N^N_{(f,H_v),\leq k}(r) + N^N_{(g,H_v),\leq k}(r)) + o(T(r)). \tag{3.9}$$

By (3.8) and (3.9), it implies that

$$\begin{aligned}
&\Big\| \frac{2}{N}(N^N_{(f,H_i),\leq k}(r) + N^N_{(g,H_i),\leq k}(r) + N^N_{(f,H_j),\leq k}(r) + N^N_{(g,H_j),\leq k}(r)) \\
\leq\ & \frac{(k+1)+N^2}{N(k+1-N)} T(r) + o(T(r)).
\end{aligned}$$

Taking summing-up of the above inequality over $1 \leq i \leq N$, we have

$$\begin{aligned}
&\Big\| 2\sum_{i=1}^{N}(N^N_{(f,H_i),\leq k}(r) + N^N_{(g,H_i),\leq k}(r) + N^N_{(f,H_j),\leq k}(r) + N^N_{(g,H_j),\leq k}(r)) \\
=\ & 2\sum_{v=1}^{2N}(N^N_{(f,H_v),\leq k}(r) + N^N_{(g,H_v),\leq k}(r)) \\
\leq\ & \frac{N(k+1)+N^3}{k+1-N} T(r) + o(T(r)).
\end{aligned}$$

(Note that $j = i + N$.)

Using (2.2) again, we get

$$\Big\|\frac{(N-1)(k+1)-2N^2}{k+1-N}T(r) \leq \sum_{v=1}^{2N}(N^N_{(f,H_v),\leq k}(r) + N^N_{(g,H_v),\leq k}(r)) + o(T(r)).$$

Hence

$$\Big\|\frac{(N-1)(k+1)-2N^2}{k+1-N}T(r) \leq \frac{N(k+1)+N^3}{2(k+1-N)}T(r) + o(T(r)).$$

This means that

$$2(N-1)(k+1) - 4N^2 \leq N(k+1) + N^3.$$

We have $k+1 \leq N^2 + 6N + 12 + \frac{24}{N-2}$, which is a contradiction. Thus, we have $f \equiv g$.

Proof of Theorem 1.3 Assume that $f \not\equiv g$. Repeating the argument in the proof of Theorem 1.2, without loss of generality, we may assume that

$$p_{2N+1} = p_{2N+2} = p_{2N+3} = 1 \quad \text{and} \quad p_1 = \cdots = p_{2N} = N.$$

Fixing an index i with $1 \leq i \leq N$, we consider $F_f^{H_iH_j} - F_g^{H_iH_j} \not\equiv 0$, where $j = i + N$. Then, we have

$$\begin{aligned}&\frac{N-1}{N}(N^N_{(f,H_i)}(r) + N^N_{(g,H_i)}(r) + N^N_{(f,H_j)}(r) + N^N_{(g,H_j)}(r))\\ &\leq 2T(r) - \frac{1}{N}\sum_{v=1}^{2N+3}(N^N_{(f,H_v)}(r) + N^N_{(g,H_v)}(r)) + O(1).\end{aligned}$$

(The proof is similar to (3.8).)

Using the second main theorem, we get

$$\|(N+2)T(r) \leq \sum_{v=1}^{2N+3}(N^N_{(f,H_v)}(r) + N^N_{(g,H_v)}(r)) + o(T(r)).$$

It implies that

$$\Big\|\frac{N-1}{N}(N^N_{(f,H_i)}(r) + N^N_{(g,H_i)}(r) + N^N_{(f,H_j)}(r) + N^N_{(g,H_j)}(r)) \leq \frac{N-2}{N}T(r) + o(T(r)).$$

Taking summing-up of the above inequality over $1 \leq i \leq N$, we have

$$\Big\|(N-1)\sum_{j=1}^{2N}(N^N_{(f,H_j)}(r) + N^N_{(g,H_j)}(r)) \leq (N^2 - 2N)T(r) + o(T(r)).$$

Using the second main theorem again, we get

$$\|(N-1)T(r) \leq \sum_{j=1}^{2N}(N^N_{(f,H_j)}(r) + N^N_{(g,H_j)}(r)) + o(T(r)).$$

Hence

$$\|(N-1)^2T(r) \leq (N^2 - 2N)T(r) + o(T(r)).$$

Letting $r \to +\infty$, we get a contradiction. Thus, we have $f \equiv g$.

References

[1] Nevanlinna, R., Einige Eidentigkeitssätze in der Theorie der meromorphen Funktionen, *Acta. Math.*, **48**, 1926, 367–391.

[2] Fujimoto, H., The uniqueness problem of meromorphic maps into the complex projective spaces, *Nagoya Math. J.*, **58**, 1975, 1–23.

[3] Fujimoto, H., A uniqueness theorem for algebraically non-degenerate meromorphic maps into $\mathbb{P}^N(\mathbb{C})$, *Nagoya Math. J.*, **64**, 1976, 117–147.

[4] Smiley, L., Geometric conditions for unicity of holomorphic curves, *Contemp. Math.*, **25**, 1983, 149–154.

[5] Fujimoto, H., Uniqueness problem with truncated multiplicities in value distribution theory, *Nagoya Math. J.*, **152**, 1998, 131–152.

[6] Thai, D. D. and Quang, S. D., Uniqueness problem with truncated multiplicities of meromorphic mappings in several complex variables, *Internat. J. Math.*, **17**(10), 2006, 1223–1257.

[7] Dethloff, G. and Tan, V. T., Uniqueness problem for meromorphic mappings with truncated multiplicities and few targets, *Ann. Fac. Sci. Toulouse Math.* (6), **15**(2), 2006, 217–242.

[8] Ru, M., Nevanlinna Theory and Its Relation to Diophantine Approximation, World Scientific, Singapore, 2001.

[9] Ji, S., Uniqueness problem without multiplicities in value distribution theory, *Pacific J. Math.*, **135**, 1988, 323–348.