Chin. Ann. Math.
31B(4), 2010, 555–578
DOI: 10.1007/s11401-010-0585-4

Chinese Annals of Mathematics, Series B

Carleman Estimates for the Schrödinger Equation and Applications to an Inverse Problem and an Observability Inequality***

Ganghua YUAN* Masahiro YAMAMOTO**

Abstract The authors prove Carleman estimates for the Schrödinger equation in Sobolev spaces of negative orders, and use these estimates to prove the uniqueness in the inverse problem of determining L^p-potentials. An L^2-level observability inequality and unique continuation results for the Schrödinger equation are also obtained.

Keywords Schrödinger equation, Carleman estimate, Observability inequality, Inverse problem, Unique continuation
2000 MR Subject Classification 93B05, 35R30, 35B60

1 Introduction and Main Results

In this paper, we discuss Carleman estimates and apply them to an inverse problem and an observability inequality for the Schrödinger equations. First we derive Carleman estimates for the Schrödinger equations in Sobolev spaces of negative orders.

Throughout this paper, we set

$$\begin{aligned}(Pv)(x,t) &= \mathrm{i}\partial_t v(x,t) + g(x,t)\triangle v(x,t) + \sum_{j=1}^{n} r_{1j}(x,t)\partial_j v(x,t)\\ &\quad + \sum_{j=1}^{n}\int_0^t r_{2j}(x,t,\theta)\partial_j v(x,\theta)\mathrm{d}\theta, \quad (x,t)\in Q=\Omega\times(-T,T),\end{aligned}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with boundary $\partial\Omega\in C^3$. Set $\Sigma=\partial\Omega\times(-T,T)$, $\mathrm{i}=\sqrt{-1}$, $\partial_t=\frac{\partial}{\partial t}$, $\partial_j=\frac{\partial}{\partial x_j}$, $j=1,2,\cdots,n$. Let $\nabla u=(\partial_1 u,\partial_2 u,\cdots,\partial_n u)$, and $\nu(x)$ be the unit outward normal vector to $\partial\Omega$ at x. Moreover, let $\alpha=(\alpha_1,\alpha_2,\cdots,\alpha_n)$ be a multi-index with $\alpha_j\in\mathbb{N}\cup\{0\}$, and let

$$\partial_x^\alpha=\partial_1^{\alpha_1}\partial_2^{\alpha_2}\cdots\partial_n^{\alpha_n}, \quad |\alpha|=\alpha_1+\alpha_2+\cdots+\alpha_n.$$

Manuscript received January 29, 2009. Revised March 22, 2010. Published online June 21, 2010.
*School of Mathematics and Statistics, Northeast Normal University, Changchun 130024, China.
E-mail: yuangh925@nenu.edu.cn
**Graduate School of Mathematical Sciences, The University of Tokyo, 3-8-1 Komaba, Meguro, Tokyo 153, Japan. E-mail: myama@ms.u-tokyo.ac.jp
***Project supported by the Japanese Government Scholarship, the National Natural Science Foundation of China (No. 10801030), the Science Foundation for Young Teachers of Northeast Normal University (No. 20080103), the Japan Society for the Promotion of Science (No. 15340027) and the Grant from the Ministry of Education, Cultures, Sports and Technology of Japan (No. 17654019).

Let $p > 1$, $\gamma > 0$, and let $W^\gamma_p(\Omega)$ and $W^\gamma_{p,0}(\Omega)$ denote usual Sobolev spaces. Set $H^\gamma(\Omega) = W^\gamma_2(\Omega)$ and $H^\gamma_0(\Omega) = W^\gamma_{2,0}(\Omega)$ (see e.g., [39]). Let $\frac{1}{p}+\frac{1}{p'} = 1$. Identifying the dual of $L^p(\Omega)$ with $L^{p'}(\Omega)$, we denote the dual of $W^\gamma_{p,0}(\Omega)$ by $W^{-\gamma}_{p'}(\Omega)$. Set $H^{-\gamma}(\Omega) = W^{-\gamma}_2(\Omega)$.

Assume that r_{1j}, r_{2j}, $1 \le j \le n$ are complex-valued and satisfy

$$r_{1j} \in C^1(\overline{Q}), \quad r_{2j} \in C^1(\overline{\Omega} \times [-T,T] \times [-T,T]), \quad 1 \le j \le n. \tag{1.1}$$

Assume that d and g are real-valued and satisfy

$$\begin{cases} g \in C^2(\overline{Q}), \quad g > 0, \quad \text{on } \overline{Q}, \\ d \in C^4(\overline{\Omega}), \quad |\nabla d| > 0, \quad \text{on } \overline{\Omega}, \\ \nabla \log\left(\frac{1}{g(x,t)}\right) \cdot \nabla d(x) + |\xi \cdot \nabla d(x)|^2 > -2\sum\limits_{j,k=1}^n (\partial_j\partial_k d(x))\xi_j\overline{\xi}_k, \quad (x,t) \in \overline{Q}, \\ \qquad\qquad\qquad\qquad \text{for every } \xi = (\xi_1, \cdots, \xi_n) \in \mathbb{C}^n \text{ satisfying } |\xi| = 1. \end{cases} \tag{1.2}$$

First we show a Carleman estimate whose right-hand side is estimated in $L^2(-T,T;H^{-s}(\Omega))$ with $\frac{3}{4} < s \le 1$.

Theorem 1.1 *Let $\frac{3}{4} < s \le 1$, $0 < T_1 < T$, and $\Omega^0 \subset \Omega$ be a domain satisfying $\overline{\Omega^0} \subset \Omega$. Assume that* (1.1) *and* (1.2) *hold. For positive constants λ and β, we set $\phi(x,t) = \mathrm{e}^{\lambda(d(x)-\beta t^2)}$. Then for large enough $\lambda > 0$, there exists a constant $C_1 = C_1(\lambda) > 0$ such that for all large enough $\tau > 0$,*

$$\tau \int_{-T}^T \|u\mathrm{e}^{\tau\phi}\|^2_{H^{1-s}(\Omega)}\mathrm{d}t \le C_1 \int_{-T}^T \|\mathrm{e}^{\tau\phi}Pu\|^2_{H^{-s}(\Omega)}\mathrm{d}t \tag{1.3}$$

for all $u \in L^2(-T,T;H^{1-s}(\Omega))$ with compact support in $\Omega^0 \times (-T_1,T_1)$ such that $Pu \in L^2(-T,T;H^{-s}(\Omega))$.

Carleman estimates in Sobolev space of negative orders are useful tools for dealing with inverse problems (see e.g., [12, 14]) and unique continuation (see e.g., [38]) for the solutions of partial differential equation with less regular coefficients or non-homogenous terms. For H^{-1} Carleman estimate for hyperbolic equations, we can refer to [11, 12]. For H^{-1} Carleman estimate for parabolic equations, we can refer to [14, 17]. We can refer to [32] for an L^2-energy estimate with right-hand side in a Sobolev space of negative order for a nonconservative Schrödinger equation. The estimate in [32] involves a lower-order term on the right-hand side.

From Theorem 1.1, it is easy to see the following result.

Corollary 1.1 *Let $\frac{3}{4} < s \le 1$, $0 < T_1 < T$, and $\Omega^0 \subset \Omega$ be a domain satisfying $\overline{\Omega^0} \subset \Omega$. Assume that* (1.1) *holds, and real-valued g satisfies*

$$\begin{cases} g \in C^2(\overline{Q}), \quad g > 0, \quad \text{on } \overline{Q}, \\ \nabla \log\left(\dfrac{1}{g(x,t)}\right) \cdot (x - x_0) > -2, \quad (x,t) \in \overline{Q} \end{cases} \tag{1.4}$$

for a fixed $x_0 \in \mathbb{R}^n\backslash\overline{\Omega}$. We set $\phi_1(x,t) = \mathrm{e}^{\lambda(|x-x_0|^2-\beta t^2)}$ for λ, $\beta > 0$. Then for large enough $\lambda > 0$, there exists a constant $C_2 = C_2(\lambda) > 0$ such that for all large enough $\tau > 0$,

$$\tau \int_{-T}^T \|u\mathrm{e}^{\tau\phi_1}\|^2_{H^{1-s}(\Omega)}\mathrm{d}t \le C_2 \int_{-T}^T \|\mathrm{e}^{\tau\phi_1}Pu\|^2_{H^{-s}(\Omega)}\mathrm{d}t \tag{1.5}$$

for all $u \in L^2(-T,T;H^{1-s}(\Omega))$ *with compact support in* $\Omega^0 \times (-T_1,T_1)$ *such that* $Pu \in L^2(-T,T;H^{-s}(\Omega))$.

Moreover, we can prove a Carleman estimate with less regular potential in x.

Corollary 1.2 *Let* $n \geq 2$, $\frac{3}{4} < s < 1$, $0 < T_1 < T$, *and* $\Omega^0 \subset \Omega$ *be a domain satisfying* $\overline{\Omega^0} \subset \Omega$. *Assume that* (1.1) *and* (1.2) *hold. Let* $0 < \gamma < 1-s$ *and* $p > 1$ *satisfy* $1-\gamma > n(1-\frac{1}{p})$, *and let* $\frac{1}{p}+\frac{1}{p'} = 1$. *If* ζ_1 *is complex-valued and* $\zeta_1 \in L^\infty(-T,T;W_{p'}^{-\gamma}(\Omega))$, *then for large enough* $\lambda > 0$, *there exists a constant* $C_3 = C_3(\lambda) > 0$ *such that for all large enough* $\tau > 0$,

$$\tau\|u\mathrm{e}^{\tau\phi}\|^2_{L^2(-T,T;H^{1-s}(\Omega))} \leq C_3\|\mathrm{e}^{\tau\phi}(P+\zeta_1)u\|^2_{L^2(-T,T;H^{-s}(\Omega))} \tag{1.6}$$

for all $u \in L^2(-T,T;H^{1-s}(\Omega))$ *with compact support in* $\Omega^0 \times (-T_1,T_1)$ *such that* $(P+\zeta_1)u \in L^2(-T,T;H^{-s}(\Omega))$.

From Corollary 1.2, we can obtain a unique continuation result for the Schrödinger equation with potential in a Sobolev space in x of negative order.

We consider the Schrödinger equation:

$$(P_1u)(x,t) \equiv \mathrm{i}\partial_t u(x,t) + \mathrm{div}\,(\rho(x,t)\nabla u(x,t)) - \zeta(x,t)u(x,t), \quad (x,t) \in Q. \tag{1.7}$$

For $x_0 \in \mathbb{R}^n\backslash\overline{\Omega}$ and a constant $\beta > 0$, we set $J(x,t) = |x-x_0|^2 - \beta t^2$.

Theorem 1.2 *Let* $n \geq 2$ *and* ρ *satisfy*

$$\begin{cases}\rho \in C^2(\overline{Q}), \quad \rho(x,t) > 0,\\ \nabla \log\left(\dfrac{1}{\rho(x,t)}\right)\cdot(x-x_0) > -2, \quad (x,t) \in \overline{Q}\end{cases} \tag{1.8}$$

for a fixed $x_0 \in \mathbb{R}^n\backslash\overline{\Omega}$. *Let* $\zeta \in L^\infty(-T,T;W_{p'}^{-\gamma}(\Omega))$ *be complex-valued and* $u \in L^2(-T,T;H^{1-s}(\Omega))$, *where* $\frac{3}{4} < s < 1$, $0 < \gamma < 1-s$, $p > 1$ *satisfying* $1-\gamma > n(1-\frac{1}{p})$ *and* $\frac{1}{p}+\frac{1}{p'} = 1$. *If* $P_1u = 0$ *in* Q *and* $u = 0$ *in* $\{(x,t) \in Q; J(x,t) > J(x^1,t^1)\}$ *for a point* $(x^1,t^1) \in Q$, *then there exists a neighbourhood* $\mathcal{U}$ *of* (x^1,t^1) *such that*

$$u(x,t) = 0, \quad (x,t) \in \mathcal{U}.$$

Theorem 1.2 is a consequence of Corollary 1.2 by using a known argument in deriving the unique continuation by a Carleman estimate (see e.g., [9, 10]). We give the details of the proof of Theorem 1.2 in Appendix.

For unique continuation across any non-characteristic surface for the Schrödinger equation with partially analytic coefficients, we can refer to [40]. For the unique continuation of the H^1-solution to the Schrödinger equation with L^∞-potential, we refer to [19, 20, 41], and for the unique continuation of the solution in $L^r(\mathbb{R};W_r^2(\mathbb{R}^n)) \cap W_r^1(\mathbb{R};L^r(\mathbb{R}^n))$ with $r = \frac{2(n+2)}{n+4}$ to the Schrödinger equation with $L^{\frac{n+2}{2}}$-potential, we refer to [23]. We note that Theorem 1.2 is a unique continuation result with potential in a Sobolev space of negative order if $u \in L^2(-T,T;H^{1-s}(\Omega))$.

Next we consider an inverse problem of determining an L^p-potential q locally by some suitable local observation data.

We consider the Schrödinger equation:

$$\begin{cases} \mathrm{i}\partial_t y(x,t) + \operatorname{div}(b(x)\nabla y(x,t)) - q(x)y(x,t) = 0, & \text{in } Q, \\ y = y_\Sigma, & \text{on } \Sigma, \\ y(\cdot,0) = y_0, & \text{in } \Omega. \end{cases} \tag{1.9}$$

First we introduce an admissible set $\mathcal{Q}$ of unknown potentials: let

$$p > \max\{n, 2\}.$$

We set $\mathcal{Q} = \{q \in L^p(\Omega);\, q(x) \geq -c_0,\ x \in \Omega\}$, where $c_0 > 0$ is a constant.

For

$$b \in \mathcal{B} \equiv \Big\{ b \in C^2(\overline{\Omega});\, b(x) > 0 \text{ and } \nabla \log\Big(\frac{1}{b(x)}\Big) \cdot (x - x_0) > -2,\ x \in \overline{\Omega} \Big\},$$

we introduce an admissible set of boundary values:

$$\begin{aligned} \mathcal{V}_b = \{(y_0, y_\Sigma);\ &\text{the solution } y = y(q, b, y_0, y_\Sigma) \text{ of (1.9)} \\ &\text{is in } H^1((-T,T); L^2(\Omega)) \text{ for any } q \in \mathcal{Q}\}. \end{aligned}$$

Remark 1.1 The admissible set $\mathcal{V}_b$ is not empty. For example, if $y_0 \in H^3(\Omega)$ and $y_\Sigma \in H^{\frac{7}{2},\frac{7}{4}}(\Sigma)$ satisfy the compatibility condition $y_\Sigma(x,0) = y_0(x)$, then we can choose a function $\Phi \in H^{4,2}(Q)$ (see e.g., [34, Chapter 4]) such that $\Phi(x,t) = y_\Sigma(x,t)$, $(x,t) \in \Sigma$, and $\Phi(x,0) = y_0(x)$, $x \in \Omega$. By (1.9), we have

$$\begin{cases} \mathrm{i}\partial_t (y - \Phi) + \operatorname{div}(b(x)\nabla(y - \Phi)) - q(x)(y - \Phi) = F, & \text{in } Q, \\ y - \Phi = 0, & \text{on } \Sigma, \\ (y - \Phi)(\,\cdot\,, 0) = 0, & \text{in } \Omega, \end{cases}$$

where $F := -\mathrm{i}\partial_t \Phi - \operatorname{div}(b(x)\nabla\Phi) + q(x)\Phi \in H^1(-T,T; L^2(\Omega))$. Then one can verify that $y \in L^2(-T,T; H^2(\Omega)) \cap H^1(-T,T; L^2(\Omega))$ (see e.g., Theorem 12.1 in Chapter 5 of [34]).

For fixed $x_0 \in \mathbb{R}^n \backslash \overline{\Omega}$, we set $\Omega(\delta) = \{x \in \Omega; |x - x_0| > \delta\}$, where $\delta > 0$ is a constant.

Theorem 1.3 *Let $S \subset \partial\Omega$ be a relatively open subset of $\partial\Omega$ and $\omega_S \subset \Omega$ be a subdomain such that $\partial\omega_S \supset S$, and let a constant $\delta_0 > 0$ satisfy*

$$\overline{\Omega(\delta_0)} \subset (\Omega \cup S). \tag{1.10}$$

Assume that $b \in \mathcal{B}$, $(y_0, y_\Sigma) \in \mathcal{V}_b$, $q_1 \in \mathcal{Q}$, and

$$|y_0(x)| > c > 0, \quad x \in \overline{\Omega(\delta_0)}, \tag{1.11}$$

$$\partial_x^\alpha \partial_t y(q_1, b, y_0, y_\Sigma) \in C(\overline{\Omega(\delta_0)} \times (-T,T)), \quad |\alpha| = 2. \tag{1.12}$$

For $q_2 \in \mathcal{Q}$, if

$$y(q_2, b, y_0, y_\Sigma)(x,t) = y(q_1, b, y_0, y_\Sigma)(x,t), \quad (x,t) \in \omega_S \times (-T,T), \tag{1.13}$$

then

$$q_2(x) = q_1(x), \quad x \in \Omega(\delta_0).$$

We can prove also the stability, but in this paper we omit the details.

The uniqueness in determining the L^∞-potential in the Schrödinger equation was firstly proved by Bukhgeim [5]. Afterwards, the stability was proved by Baudouin and Puel [2]. In those two papers, Carleman estimates in L^2-spaces are used. We further refer to [1] which proved the stability in determining $L^\infty(\Omega)$-potential in the Schrödinger equation with discontinuous principal term, and refer to [36] which established a new Carleman estimate and proved the Lipschitz stability in determining potentials in $L^\infty(\Omega)$. In this paper, we use an H^{-1}-Carleman estimate which leads to the uniqueness in $L^p(\Omega)$ with $p > \max\{n, 2\}$. The method of solving inverse problems by Carleman estimates was firstly introduced by Bukhgeim and Klibanov [6] (see also [25, 26]). As for works on the stability and the uniqueness for inverse problems for hyperbolic or parabolic equations by Carleman estimates, see [3, 4, 13, 15, 16, 18, 20, 24, 28, 43] and the references therein.

Finally, we derive an L^2-level observability inequality for the Schrödinger equations:

$$\begin{cases} \mathrm{i}\partial_t \eta(x,t) + \operatorname{div}(b(x)\nabla\eta(x,t)) - q(x)\eta(x,t) = 0, & \text{in } Q = \Omega\times(-T,T), \\ \eta(x,t) = 0, & \text{on } \Sigma = \partial\Omega\times(-T,T), \\ \eta(\cdot,0) = \eta_0, & \text{in } \Omega. \end{cases} \tag{1.14}$$

Now we present an L^2-level observability inequality for the Schrödinger equation.

Theorem 1.4 *Let $\omega \subset \Omega$ be a neighbourhood of $\partial\Omega$ satisfying $\partial\omega \supset \partial\Omega$. Assume that*

$$b \in C^2(\overline{\Omega}),\ b > 0, \quad \text{on } \overline{\Omega}, \quad \nabla \log\left(\frac{1}{b(x)}\right)\cdot(x - x_0) > -2, \quad x \in \overline{\Omega}$$

and $q \in L^p(\Omega)$ with $p > \max\{n, 2\}$, $q(x) \ge -c_0$, $x \in \Omega$ with some constant $c_0 > 0$. Then for every $T > 0$, there exists a constant $C_4 > 0$ such that

$$\|\eta_0\|^2_{L^2(\Omega)} \le C_4 \int_{-T}^{T}\int_\omega |\eta|^2 \mathrm{d}x\mathrm{d}t \tag{1.15}$$

for every solution $\eta \in L^2(Q)$ to (1.14) with $\eta_0 \in L^2(\Omega)$.

For L^2-level observability inequality for the Schrödinger equation with $b \equiv 1$ and $q = 0$ in (1.14), we refer [35, 37] where methods based on multiplier techniques were used to prove the observability inequalities. In this paper, we derive an L^2-level observability inequality in Theorem 1.4 for the Schrödinger equation with a variable principal term and an L^p-potential. Our method is based on the H^{-1} Carleman estimate (see Theorem 1.1). As for the method of applying Carleman estimate to derive observability inequalities, see [22, 27]. As for observability inequalities by Carleman estimates, see further [7, 8, 17, 21, 28–31, 33].

This paper is composed of five sections. In Section 2, we prove Theorem 1.1 and Corollary 1.2. In Section 3 and Section 4, we prove Theorem 1.3 and Theorem 1.4 respectively. A proof of a Carleman estimate in L^2 space is given in Appendix. We also present the proof of Theorem 1.2 in Appendix.

2 Proofs of Carleman Estimates

For $\delta > 0$, we set $\Omega_\delta = \{x \in \mathbb{R}^n;\ \mathrm{dist}(x,\Omega) < \delta\}$, $Q^\delta = \Omega_\delta \times (-T-\delta, T+\delta)$. Let

$$(P_0 u)(x,t) = \mathrm{i}\partial_t u(x,t) + g(x,t)\triangle u(x,t), \quad (x,t) \in Q$$

where $g \in C^2(\overline{Q})$.

Henceforth, C and C_j denote generic constants which are dependent on Ω, T and λ, but independent of τ. The numbering in C_j can be independent in the different sections and the appendix. By $\overline{c}$, we denote the complex conjugate of $c \in \mathbb{C}$, while we note that $\overline{\Omega}$ means the closure of a domain Ω.

We will prove Carleman estimates with right-hand sides in L^2 and negative-order Sobolev spaces as well.

First we present a Carleman estimate in L^2 space.

Proposition 2.1 *Let $\phi(x,t) = \mathrm{e}^{\lambda(d(x)-\beta t^2)}$, where $\beta > 0$, $\lambda > 0$. Assume that g and d satisfy* (1.2). *Then there exists a constant $\lambda_0 > 0$ such that for arbitrary $\lambda \geq \lambda_0$, we can choose $\tau_0 > 0$ satisfying: there exists a constant $C(\tau_0, \lambda_0) > 0$ such that*

$$\begin{aligned}&\tau \int_Q |\nabla v|^2 \mathrm{e}^{2\tau\phi}\mathrm{d}x\mathrm{d}t + \tau^3 \int_Q |v|^2 \mathrm{e}^{2\tau\phi}\mathrm{d}x\mathrm{d}t \\ &\leq C \int_Q |P_0 v|^2 \mathrm{e}^{2\tau\phi}\mathrm{d}x\mathrm{d}t + C\tau \int_{-T}^{T} \int_{\partial\Omega} \Big|\frac{\partial v}{\partial \nu}\Big|^2 \mathrm{e}^{2\tau\phi}(\nabla d \cdot \nu)\mathrm{d}\Sigma \end{aligned} \tag{2.1}$$

for all $\tau > \tau_0$ and all v satisfying

$$\begin{cases} P_0 v \in L^2(Q), & v \in H^1(Q) \cap L^2(-T,T;H_0^1(\Omega)), \\ \dfrac{\partial v}{\partial \nu} \in L^2(-T,T;L^2(\partial\Omega)), & v(\,\cdot\,,-T) = v(\,\cdot\,,T) = 0. \end{cases} \tag{2.2}$$

We give the proof of Proposition 2.1 in Appendix.

Next we show Carleman estimates in Sobolev spaces of negative orders by an argument similar to [12].

We set

$$\begin{aligned} P_\phi w = {}& \mathrm{i}\partial_t w + g\triangle w - 2\tau g \nabla\phi \cdot \nabla w + \left(\tau^2 |\nabla \phi|^2 - \tau \triangle \phi\right) g w - \mathrm{i}\tau(\partial_t \phi) w \\ & + \sum_{j=1}^n [r_{1j}\partial_j w - \tau r_{1j}(\partial_j\phi)w] + \sum_{j=1}^n \int_0^t [r_{2j}\partial_j w - \tau r_{2j}(\partial_j \phi)w]\mathrm{d}\theta. \end{aligned}$$

Let

$$\begin{aligned} &\xi' = (\xi_0, \xi_1, \xi_2, \cdots, \xi_n), \quad \xi = (\xi_1, \xi_2, \cdots, \xi_n), \\ &\langle \xi \rangle = \sqrt{\xi_1^2 + \xi_2^2 + \cdots + \xi_n^2 + 1}. \end{aligned}$$

For $\vartheta \in \mathbb{R}$ and $f \in L^2(\Omega)$, we define

$$\wedge_\tau^s f = \begin{cases} F^{-1}[(\langle \xi \rangle^s + \tau^s) F f], & s \geq 0, \\ F^{-1}\Big[\Big(\dfrac{1}{\langle \xi \rangle^{-s} + \tau^{-s}}\Big) F f\Big], & s < 0, \end{cases}$$

where F denotes the Fourier transform in $\mathbb{R}^n$ which is defined by

$$(Ff)(\xi) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}x\cdot\xi} f(x)\mathrm{d}x, \quad f \in L^2(\Omega),$$

and F^{-1} denotes its inverse transform. By choosing a sufficiently small $\delta_1 > 0$, we extend d into $\mathbb{R}^n$, g, r_{1j}, $1 \le j \le n$ into $\mathbb{R}^n \times (-T,T)$ and r_{2j}, $1 \le j \le n$ into $\mathbb{R}^n \times (-T,T) \times (-T,T)$, so that they are supported in $\Omega_{3\delta_1}$, $\Omega_{3\delta_1} \times (-T,T)$, $\Omega_{3\delta_1} \times (-T,T)$ and $\Omega_{3\delta_1} \times (-T,T) \times (-T,T)$ respectively, and satisfy

$$\begin{cases} g \in C^1(\mathbb{R}^n \times [-T,T]), \ \sup\limits_{-T\le t\le T} \|g(\,\cdot\,,t)\|_{\mathrm{Lip}(\mathbb{R}^n)} + \sup\limits_{-T\le t\le T} \|\nabla g(\,\cdot\,,t)\|_{\mathrm{Lip}(\mathbb{R}^n)} < +\infty, \\ r_{1j} \in C(\mathbb{R}^n \times [-T,T]), \ \sup\limits_{-T\le t\le T} \|r_{1j}(\,\cdot\,,t)\|_{\mathrm{Lip}(\mathbb{R}^n)} < +\infty, \quad 1 \le j \le n, \\ r_{2j} \in C(\mathbb{R}^n \times (-T,T) \times (-T,T)), \ \sup\limits_{-T\le t,\theta\le T} \|r_{2j}(\,\cdot\,,t,\theta)\|_{\mathrm{Lip}(\mathbb{R}^n)} < +\infty, \quad 1 \le j \le n \end{cases} \tag{2.3}$$

and

$$\begin{cases} g > 0, \quad \text{on } \overline{Q^{2\delta_1}}, \\ d \in C^4(\mathbb{R}^n), \quad |\nabla d| > 0, \quad \text{on } \overline{\Omega_{2\delta_1}}, \\ \nabla \log\Big(\dfrac{1}{g(x,t)}\Big) \cdot \nabla d(x) + |\xi \cdot \nabla d(x)|^2 > -2 \sum\limits_{j,k=1}^{n} (\partial_j\partial_k d(x))\xi_j\overline{\xi}_k, \quad (x,t) \in \overline{Q^{2\delta_1}}, \\ \qquad\qquad\qquad \text{for every } \xi = (\xi_1, \cdots, \xi_n) \in \mathbb{C}^n \text{ satisfying } |\xi| = 1. \end{cases} \tag{2.4}$$

Here and henceforth, we use the same notations to denote the extension of the coefficients, and we set

$$\|f\|_{\mathrm{Lip}(\mathbb{R}^n)} = \sup_{\substack{x^1,x^2\in\mathbb{R}^n \\ x^1\ne x^2}} \frac{|f(x^1) - f(x^2)|}{|x^1 - x^2|}$$

provided that the right-hand side is finite.

In order to prove our Carleman estimates, we need a commutator estimate as follows.

Lemma 2.1 *Let $\frac{3}{4} < s \le 1$, g, r_{1j} and r_{2j} satisfy* (2.3). *Then there exists a constant $C_1 > 0$ such that the following commutator estimate holds:*

$$\int_{-T}^{T} \| \wedge_\tau^{-s} P_\phi v - P_\phi \wedge_\tau^{-s} v\|^2_{L^2(\Omega_{2\delta_1})} \mathrm{d}t \le C_1 \tau^{4-4s} \|v\|^2_{L^2(Q)} + C_1 \tau^{2-2s} \int_{-T}^{T} \|\nabla(\wedge_\tau^{-s} v)\|^2_{L^2(\mathbb{R}^n)} \mathrm{d}t$$

for all $v \in C_0^2(Q)$ and $\tau > 1$.

Proof It is clear that

$$\wedge_\tau^{-s} \mathrm{i}\partial_t v - \mathrm{i}\partial_t \wedge_\tau^{-s} v = 0,$$

and

$$\wedge_\tau^{-s} g(x,t)\partial_j^2 v - g(x,t)\partial_j^2 \wedge_\tau^{-s} v = \wedge_\tau^{-s}\{\partial_j[(g \wedge_0^s - \wedge_0^s g)(\wedge_\tau^{-s}\partial_j v)] + [\wedge_0^s \partial_j g - (\partial_j g)\wedge_0^s](\wedge_\tau^{-s}\partial_j v)\}$$

for $1 \le j \le n$. By the Parseval equality and the definition of $\wedge_\tau^{-s}$, we obtain

$$\int_{-T}^{T} \| \wedge_\tau^{-s} \partial_j[(g \wedge_0^s - \wedge_0^s g)(\wedge_\tau^{-s}\partial_j v)]\|^2_{L^2(\Omega_{2\delta_1})} \mathrm{d}t \le \int_{-T}^{T} \|(g \wedge_0^s - \wedge_0^s g)(\wedge_\tau^{-s}\partial_j v)\|^2_{H^{1-s}(\mathbb{R}^n)} \mathrm{d}t$$

for $1 \le j \le n$. By an argument in Chapter 3 of [42], we can prove an estimate of a commutator of a pseudo-differential operator

$$\int_{-T}^{T} \|(g \wedge_0^s - \wedge_0^s g)(\wedge_\tau^{-s}\partial_j v)\|^2_{H^{1-s}(\mathbb{R}^n)} \mathrm{d}t \le \sup_{-T\le t\le T} \|g(\,\cdot\,,t)\|^2_{\mathrm{Lip}(\mathbb{R}^n)} \int_{-T}^{T} \int_{\mathbb{R}^n} | \wedge_\tau^{-s} \partial_j v|^2 \mathrm{d}x\mathrm{d}t$$

for $1 \le j \le n$. Thus we obtain

$$\int_{-T}^{T} \| \wedge_\tau^{-s} \partial_j[(g \wedge_0^s - \wedge_0^s g)(\wedge_\tau^{-s}\partial_j v)]\|^2_{L^2(\Omega_{2\delta_1})}\mathrm{d}t \le C_2 \int_{-T}^{T}\int_{\mathbb{R}^n} | \wedge_\tau^{-s} \partial_j v|^2 \mathrm{d}x\mathrm{d}t \tag{2.5}$$

for $1 \le j \le n$. On the other hand, by the Parseval equality and an estimate of a commutator of a pseudo-differential operator (see e.g., [42, Subsection 3.6]), we obtain

$$\begin{aligned}
&\int_{-T}^{T} \|[\wedge_0^s \partial_j g - (\partial_j g)\wedge_0^s](\wedge_\tau^{-s}\partial_j v)\|^2_{L^2(\Omega_{2\delta_1})}\mathrm{d}t \\
&\le \sup_{-T\le t\le T} \|\nabla g(\,\cdot\,,t)\|^2_{\mathrm{Lip}(\mathbb{R}^n)} \int_{-T}^{T}\int_{\mathbb{R}^n} | \wedge_\tau^{-s} \partial_j v|^2 \mathrm{d}x\mathrm{d}t
\end{aligned} \tag{2.6}$$

for $1 \le j \le n$. By (2.5) and (2.6), we have

$$\int_{-T}^{T} \| \wedge_\tau^{-s} g(\,\cdot\,,t)\partial_j^2 v - g(\,\cdot\,,t)\partial_j^2 \wedge_\tau^{-s} v\|^2_{L^2(\Omega_{2\delta_1})}\mathrm{d}t \le C_3 \int_{-T}^{T}\int_{\mathbb{R}^n} | \wedge_\tau^{-s} \partial_j v|^2 \mathrm{d}x\mathrm{d}t$$

for $1 \le j \le n$. By the Parseval equality and an estimate of a commutator of a pseudo-differential operator (see e.g., [42, Subsection 3.6]), we obtain

$$\begin{aligned}
&\int_{-T}^{T} \| \wedge_\tau^{-s} \tau^2(\partial_j\phi)^2 g v - \tau^2(\partial_j\phi)^2 g \wedge_\tau^{-s} v\|^2_{L^2(\Omega_{2\delta_1})}\mathrm{d}t \\
&= \int_{-T}^{T} \| \wedge_\tau^{-s} \tau[(\partial_j\phi)^2 g \wedge_0^s - \wedge_0^s (\partial_j\phi)^2 g] \wedge_\tau^{-s} \tau v\|^2_{L^2(\Omega_{2\delta_1})}\mathrm{d}t \\
&\le C_3\tau^{4-4s} \int_Q |v|^2 \mathrm{d}x\mathrm{d}t
\end{aligned}$$

for $1 \le j \le n$. Similarly,

$$\begin{aligned}
&\int_{-T}^{T} \| \wedge_\tau^{-s} 2\tau(\partial_j\phi)g\partial_j v - 2\tau(\partial_j\phi)g\partial_j \wedge_\tau^{-s} v\|^2_{L^2(\Omega_{2\delta_1})}\mathrm{d}t \\
&\le C_4\tau^{2-2s} \int_{-T}^{T}\int_{\mathbb{R}^n} | \wedge_\tau^{-s} \partial_j v|^2 \mathrm{d}x\mathrm{d}t, \\
&\int_{-T}^{T} \| \wedge_\tau^{-s} [\mathrm{i}\tau(\partial_t\phi) - \tau(\nabla\phi)g]v - [\mathrm{i}\tau(\partial_t\phi) - \tau(\nabla\phi)g] \wedge_\tau^{-s} v\|^2_{L^2(\Omega_{2\delta_1})}\mathrm{d}t \\
&\le C_5\tau^{2-2s} \int_Q |v|^2 \mathrm{d}x\mathrm{d}t, \\
&\int_{-T}^{T} \left\| \wedge_\tau^{-s} \int_0^t r_{2j}(\,\cdot\,,t,\theta)\partial_j v(\,\cdot\,,\theta)\mathrm{d}\theta - \int_0^t r_{2j}(\,\cdot\,,t,\theta)\partial_j \wedge_\tau^{-s} v(\,\cdot\,,\theta)\mathrm{d}\theta \right\|^2_{L^2(\Omega_{2\delta_1})}\mathrm{d}t \\
&= \int_{-T}^{T} \left\| \int_0^t (\wedge_\tau^{-s} r_{2j}(\,\cdot\,,t,\theta)\partial_j v(\,\cdot\,,\theta) - r_{2j}(\,\cdot\,,t,\theta)\partial_j \wedge_\tau^{-s} v(\,\cdot\,,\theta))\mathrm{d}\theta \right\|^2_{L^2(\Omega_{2\delta_1})}\mathrm{d}t \\
&\le C_6 \int_{-T}^{T}\int_{-T}^{T} \| \wedge_\tau^{-s} r_{2j}(\,\cdot\,,t,\theta)\partial_j v(\,\cdot\,,\theta) - r_{2j}(\,\cdot\,,t,\theta)\partial_j \wedge_\tau^{-s} v(\,\cdot\,,\theta)\|^2_{L^2(\Omega_{2\delta_1})}\mathrm{d}\theta\mathrm{d}t \\
&\le 2TC_7 \sup_{-T\le t,\theta\le T} \|r_{2j}(\,\cdot\,,t,\theta)\|_{\mathrm{Lip}(\mathbb{R}^n)} \int_{-T}^{T} \| \wedge_\tau^{-s} \partial_j v(\,\cdot\,,\theta)\|^2_{L^2(\Omega_{2\delta_1})}\mathrm{d}\theta \\
&\le C_8 \int_{-T}^{T}\int_{\mathbb{R}^n} | \wedge_\tau^{-s} \partial_j v|^2 \mathrm{d}x\mathrm{d}t,
\end{aligned}$$

$$\int_{-T}^{T}\Big\| \wedge_\tau^{-s}\int_0^t \tau r_{2j}(\,\cdot\,,t,\theta)(\partial_j\phi(\,\cdot\,,\theta))v(\,\cdot\,,\theta)\mathrm{d}\theta$$

$$-\int_0^t \tau r_{2j}(\,\cdot\,,t,\theta)(\partial_j\phi(\,\cdot\,,\theta))\wedge_\tau^{-s} v(\,\cdot\,,\theta)\mathrm{d}\theta\Big\|^2_{L^2(\Omega_{2\delta_1})}\mathrm{d}t \le C_9\tau^{2-2s}\int_Q |v|^2\mathrm{d}x\mathrm{d}t,$$

$$\int_{-T}^{T}\|\wedge_\tau^{-s} r_{1j}\partial_j v - r_{1j}\partial_j\wedge_\tau^{-s} v\|^2_{L^2(\Omega_{2\delta_1})}\mathrm{d}t$$

$$\le \sup_{-T\le t\le T}\|r_{1j}(\,\cdot\,,t)\|_{\mathrm{Lip}(\mathbb{R}^n)}\int_{-T}^{T}\int_{\mathbb{R}^n}|\wedge_\tau^{-s}\partial_j v|^2\mathrm{d}x\mathrm{d}t \le C_{10}\int_{-T}^{T}\int_{\mathbb{R}^n}|\wedge_\tau^{-s}\partial_j v|^2\mathrm{d}x\mathrm{d}t,$$

$$\int_{-T}^{T}\|\wedge_\tau^{-s}\tau r_{1j}(\partial_j\phi)v\mathrm{d}\tau - \tau r_{1j}(\partial_j\phi)\wedge_\tau^{-s} v\mathrm{d}\tau\|^2_{L^2(\Omega_{2\delta_1})}\mathrm{d}t \le C_{10}\tau^{2-2s}\int_Q |v|^2\mathrm{d}x\mathrm{d}t$$

for $j=1,2,\cdots,n$. Consequently we have

$$\int_{-T}^{T}\|\wedge_\tau^{-s}P_\phi v - P_\phi\wedge_\tau^{-s} v\|^2_{L^2(\Omega_{2\delta_1})}\mathrm{d}t \le C_{11}\tau^{4-4s}\int_Q |v|^2\mathrm{d}x\mathrm{d}t + C_{11}\tau^{2-2s}\int_{-T}^{T}\int_{\mathbb{R}^n}|\wedge_\tau^{-s}\partial_j v|^2\mathrm{d}x\mathrm{d}t.$$

Thus the proof of Lemma 2.1 is completed.

Lemma 2.2 *Let $K(x,y;\tau)$ be the Schwartz kernel of the pseudo-differential operator $\wedge_\tau^{-s}$ with $\tau>1$. Then*

$$|\partial_x^\alpha K(x,y;\tau)| \le C_{12}(\mu)\tau^{-2s}|x-y|^{-2n-3}$$

provided that $|\alpha|\le 1$ and $|x-y|\ge\mu>0$.

Proof The Schwartz kernel $K(x,y;\tau)$ is the oscillatory integral

$$\begin{aligned}\int_{\mathbb{R}^n}\mathrm{e}^{\mathrm{i}(x-y)\cdot\xi}(\langle\xi\rangle^s+\tau^s)^{-1}\mathrm{d}\xi &= -\frac{1}{|x-y|^2}\int_{\mathbb{R}^n}(\triangle_\xi \mathrm{e}^{\mathrm{i}(x-y)\cdot\xi})(\langle\xi\rangle^s+\tau^s)^{-1}\mathrm{d}\xi\\ &= -\frac{1}{|x-y|^2}\int_{\mathbb{R}^n}\mathrm{e}^{\mathrm{i}(x-y)\cdot\xi}\triangle_\xi(\langle\xi\rangle^s+\tau^s)^{-1}\mathrm{d}\xi\\ &\;\;\vdots\\ &= \frac{(-1)^\ell}{|x-y|^{2\ell}}\int_{\mathbb{R}^n}\mathrm{e}^{\mathrm{i}(x-y)\cdot\xi}\triangle_\xi^\ell(\langle\xi\rangle^s+\tau^s)^{-1}\mathrm{d}\xi,\quad \ell\in\mathbb{N}.\end{aligned}$$

We have integrated by parts in the above equalities. Observing that

$$|\triangle_\xi^\ell(\langle\xi\rangle^s+\tau^s)^{-1}| \le C_{13}(\ell)(\langle\xi\rangle^s+\tau^s)^{-2}\langle\xi\rangle^{-2\ell+3s},\quad \ell=1,2,\cdots$$

and choosing $\ell=n+2$, we complete the proof of Lemma 2.2.

Lemma 2.3 *Let $\frac{3}{4}<s\le 1$, and g, r_{1j}, r_{2j}, $1\le j\le n$ satisfy* (2.3). *Moreover, assume that g and d satisfy* (2.4). *Then for large enough $\lambda>0$, there exists a constant $C_{14}=C_{14}(\lambda)>0$ such that*

$$\tau\int_{-T}^{T}\|u\mathrm{e}^{\tau\phi}\|^2_{H^{1-s}(\Omega)}\mathrm{d}t \le C_{14}\int_{-T}^{T}\|\mathrm{e}^{\tau\phi}Pu\|^2_{H^{-s}(\mathbb{R}^n)}\mathrm{d}t$$

for all $u\in C_0^2(Q)$, provided that $\tau>0$ is large enough.

Proof By Proposition 2.1, we know that for large enough λ, there exists a constant $C_{15}=C_{15}(\lambda)>0$ such that if τ is large enough, then

$$\sum_{|\alpha|=0}^{1}\tau^{3-2|\alpha|}\int_{Q^{2\delta_1}}|\partial_x^\alpha v_0|^2\mathrm{d}x\mathrm{d}t \le C_{15}\int_{Q^{2\delta_1}}|P_\phi v_0|^2\mathrm{d}x\mathrm{d}t\quad\text{for all } v_0\in C_0^2(Q^{2\delta_1}).$$

Let $\chi \in C_0^\infty(Q^{2\delta_1})$, $\chi = 1$ in Q^{δ_1}. Applying the Carleman estimate to $\chi \wedge_\tau^{-s} v$, we have

$$\begin{aligned}
&\int_{Q^{2\delta_1}} \Big[\tau^3\chi^2 |\wedge_\tau^{-s} v|^2 + \tau \sum_{j=1}^n |\chi\partial_j(\wedge_\tau^{-s} v) + (\partial_j \chi) \wedge_\tau^{-s} v|^2\Big] \mathrm{d}x\mathrm{d}t \\
&\le C_{16} \int_{Q^{2\delta_1}} |P_\phi(\chi \wedge_\tau^{-s} v)|^2 \mathrm{d}x\mathrm{d}t \\
&\le C_{17} \int_{Q^{2\delta_1}} [|P_\phi(\wedge_\tau^{-s} v)|^2 + \tau^2 |\wedge_\tau^{-s} v|^2 + |\nabla(\wedge_\tau^{-s} v)|^2] \mathrm{d}x\mathrm{d}t. \qquad (2.7)
\end{aligned}$$

Furthermore, we can estimate

$$\int_{Q^{2\delta_1}} |\wedge_\tau^{-s} v|^2 \mathrm{d}x\mathrm{d}t \le \frac{1}{\tau^{2s}} \int_{Q^{2\delta_1}} |v|^2 \mathrm{d}x\mathrm{d}t. \qquad (2.8)$$

From (2.7) and (2.8), we have

$$\begin{aligned}
&\int_{Q^{\delta_1}} [\tau^3 |\wedge_\tau^{-s} v|^2 + \tau |\nabla(\wedge_\tau^{-s} v)|^2] \mathrm{d}x\mathrm{d}t \\
&\le C_{18} \int_{Q^{2\delta_1}} [|P_\phi(\wedge_\tau^{-s} v)|^2 + \tau^{2-2s} |v|^2 + |\nabla(\wedge_\tau^{-s} v)|^2] \mathrm{d}x\mathrm{d}t. \qquad (2.9)
\end{aligned}$$

We estimate

$$\begin{aligned}
\tau \int_{-T}^T \int_{\Omega_{\delta_1}} |v|^2 \mathrm{d}x\mathrm{d}t &\le \tau \int_{-T}^T \int_{\mathbb{R}^n} |\wedge_\tau^s \wedge_\tau^{-s} v|^2 \mathrm{d}x\mathrm{d}t \\
&\le C_{19} \int_{-T}^T \int_{\mathbb{R}^n} [\tau^3 |\wedge_\tau^{-s} v|^2 + \tau |\nabla(\wedge_\tau^{-s} v)|^2] \mathrm{d}x\mathrm{d}t. \qquad (2.10)
\end{aligned}$$

We also have

$$\begin{aligned}
&\tau \int_{-T}^T \int_{\mathbb{R}^n} |\wedge_0^{1-s} v|^2 \mathrm{d}x\mathrm{d}t \\
&\le \int_{-T}^T \int_{\mathbb{R}^n} \tau |\wedge_0^{1-s} \wedge_\tau^s \wedge_\tau^{-s} v|^2 \mathrm{d}x\mathrm{d}t \\
&\le \tau \int_{-T}^T \int_{\mathbb{R}^n} |F^{-1}[(\langle\xi\rangle + \langle\xi\rangle^{1-s}\tau^s) F(\wedge_\tau^{-s} v)]|^2 \mathrm{d}x\mathrm{d}t \\
&\le 2\tau \int_{-T}^T \int_{\mathbb{R}^n} |\langle\xi\rangle F(\wedge_\tau^{-s} v)|^2 \mathrm{d}\xi\mathrm{d}t + 2\tau \int_{-T}^T \int_{\mathbb{R}^n} |\langle\xi\rangle^{1-s}\tau^s F(\wedge_\tau^{-s} v)|^2 \mathrm{d}\xi\mathrm{d}t.
\end{aligned}$$

For the last inequality, we have used Parseval's identity. By the Young's inequality, we have

$$\begin{aligned}
\tau \int_{-T}^T \int_{\mathbb{R}^n} |\wedge_0^{1-s} v|^2 \mathrm{d}x\mathrm{d}t &\le 2\tau \int_{-T}^T \int_{\mathbb{R}^n} |\langle\xi\rangle F(\wedge_\tau^{-s} v)|^2 \mathrm{d}\xi\mathrm{d}t \\
&\quad + 2\tau \int_{-T}^T \int_{\mathbb{R}^n} \Big|\Big(\frac{\langle\xi\rangle^{4(1-s)}}{4} + \frac{3}{4}\tau^{\frac{4}{3}s}\Big) F(\wedge_\tau^{-s} v)\Big|^2 \mathrm{d}\xi\mathrm{d}t.
\end{aligned}$$

By noting $0 \le 4(1-s) \le 1$ and $\frac{4}{3}s \le \frac{4}{3}$, we can obtain

$$\tau \int_{-T}^T \int_{\mathbb{R}^n} |\wedge_0^{1-s} v|^2 \mathrm{d}x\mathrm{d}t \le C_{20} \int_{-T}^T \int_{\mathbb{R}^n} [\tau^3 |\wedge_\tau^{-s} v|^2 + \tau |\nabla(\wedge_\tau^{-s} v)|^2] \mathrm{d}x\mathrm{d}t. \qquad (2.11)$$

By (2.9)–(2.11) and Lemma 2.1, we have

$$
\begin{aligned}
&\int_{-T}^{T}\int_{\mathbb{R}^n}[\tau^3|\wedge_\tau^{-s} v|^2+\tau|v|^2+\tau|\wedge_0^{1-s} v|^2+\tau|\nabla(\wedge_\tau^{-s}v)|^2]\mathrm{d}x\mathrm{d}t\\
\le{}& C_{21}\int_{-T}^{T}\int_{\mathbb{R}^n}[\tau^3|\wedge_\tau^{-s} v|^2+\tau|\nabla(\wedge_\tau^{-s}v)|^2]\mathrm{d}x\mathrm{d}t\\
={}& C_{21}\int_{-T}^{T}\int_{\mathbb{R}^n\backslash\Omega_{\delta_1}}[\tau^3|\wedge_\tau^{-s} v|^2+\tau|\nabla(\wedge_\tau^{-s}v)|^2]\mathrm{d}x\mathrm{d}t\\
&+C_{21}\int_{-T}^{T}\int_{\Omega_{\delta_1}}[\tau^3|\wedge_\tau^{-s} v|^2+\tau|\nabla(\wedge_\tau^{-s}v)|^2]\mathrm{d}x\mathrm{d}t\\
\le{}& C_{22}\int_{-T}^{T}\int_{\Omega_{2\delta_1}}|\wedge_\tau^{-s} P_\phi v|^2\mathrm{d}x\mathrm{d}t+C_{22}\tau^{4-4s}\int_Q|v|^2\mathrm{d}x\mathrm{d}t+C_{22}\tau^{2-2s}\int_{-T}^{T}\int_{\mathbb{R}^n}|\nabla(\wedge_\tau^{-s}v)|^2\mathrm{d}x\mathrm{d}t\\
&+C_{22}\int_{-T}^{T}\int_{\mathbb{R}^n\backslash\Omega_{\delta_1}}[\tau^3|\wedge_\tau^{-s} v|^2+\tau|\nabla(\wedge_\tau^{-s}v)|^2]\mathrm{d}x\mathrm{d}t. \qquad (2.12)
\end{aligned}
$$

Now we eliminate the last integral. From Lemma 2.2, we have

$$
\begin{aligned}
|\partial_x^\alpha\wedge_\tau^{-s} v(x,t)|&\le(2\pi)^{-n}\int_\Omega|v(y,t)||\partial_x^\alpha K(x,y;\tau)|\mathrm{d}y\\
&\le C_{23}(\delta_1)\tau^{-2s}\int_\Omega|x-y|^{-2n-3}|v(y,t)|\mathrm{d}y,\quad|\alpha|\le1. \qquad (2.13)
\end{aligned}
$$

Here we take $\mu=\frac{1}{2}\mathrm{dist}(\partial\Omega_{\delta_1},\Omega)=\frac{1}{2}\delta_1>0$ when we apply Lemma 2.1. Since Ω is bounded, we can choose a constant $C_{24}>1$ such that $|y|<\frac{\delta_1 C_{24}}{2}$ for $y\in\Omega$. Since

$$
|x-y|=\frac{1}{2}|x-y|+\frac{1}{2}|x-y|\ge\frac{\delta_1}{2}+\frac{1}{2C_{24}}|x|-\frac{1}{2C_{24}}|y|\ge\frac{\delta_1}{4}+\frac{1}{2C_{24}}|x|\ge\frac{1+|x|}{C_{25}},
$$

for $x\in\mathbb{R}^n\backslash\Omega_{\delta_1}$, $y\in\Omega$, we have

$$
|\partial_x^\alpha\wedge_\tau^{-s} v(x,t)|\le C_{26}\tau^{-2s}(1+|x|)^{-2n-3}\Big(\int_\Omega|v|^2dx\Big)^{\frac{1}{2}},\quad|\alpha|\le1. \qquad (2.14)
$$

Consequently,

$$
\begin{aligned}
\int_{-T}^{T}\int_{\mathbb{R}^n\backslash\Omega_{\delta_1}}|\partial_x^\alpha\wedge_\tau^{-s} v(x,t)|^2\mathrm{d}x\mathrm{d}t&\le C_{27}\tau^{-4s}\int_{\mathbb{R}^n\backslash\Omega_{\delta_1}}(1+|x|)^{-4n-6}\mathrm{d}x\int_{-T}^{T}\int_\Omega|v|^2\mathrm{d}x\mathrm{d}t\\
&\le C_{28}\tau^{-4s}\int_Q|v|^2\mathrm{d}x\mathrm{d}t,\quad|\alpha|\le1. \qquad (2.15)
\end{aligned}
$$

From (2.12), (2.14) and (2.15), we have

$$
\begin{aligned}
&\tau\int_{-T}^{T}\int_{\mathbb{R}^n}|\wedge_0^{1-s} v|^2\mathrm{d}x\mathrm{d}t+\tau\int_Q|v|^2\mathrm{d}x\mathrm{d}t\\
\le{}& C_{29}\int_{-T}^{T}\int_{\mathbb{R}^n}|\wedge_\tau^{-s} P_\phi v|^2\mathrm{d}x\mathrm{d}t+C_{29}\tau^{4-4s}\int_Q|v|^2\mathrm{d}x\mathrm{d}t. \qquad (2.16)
\end{aligned}
$$

Since $4-4s<1$, the last term on the right-hand side of (2.16) can be absorbed by the left-hand side if $\tau>0$ large enough. Then we have

$$
\tau\int_{-T}^{T}\int_{\mathbb{R}^n}|\wedge_0^{1-s} v|^2\mathrm{d}x\mathrm{d}t+\tau\int_Q|v|^2\mathrm{d}x\mathrm{d}t\le C_{29}\int_{-T}^{T}\int_{\mathbb{R}^n}|\wedge_\tau^{-s} P_\phi v|^2\mathrm{d}x\mathrm{d}t \qquad (2.17)
$$

by choosing $\tau > 0$ large enough. By the definitions of $\wedge_0^{1-s}$ and $\wedge_\tau^{-s}$, we see that

$$\begin{cases} \displaystyle\int_{-T}^{T}\int_{\mathbb{R}^n} |\wedge_0^{1-s} v|^2 \mathrm{d}x\mathrm{d}t = \int_{-T}^{T} \|v\|_{H^{1-s}(\mathbb{R}^n)}^2 \mathrm{d}t, \\ \displaystyle\int_{-T}^{T}\int_{\mathbb{R}^n} |\wedge_\tau^{-s} P_\phi v|^2 \mathrm{d}x\mathrm{d}t \le C_{30}\int_{-T}^{T} \|P_\phi v\|_{H^{-s}(\mathbb{R}^n)}^2 \mathrm{d}t. \end{cases} \tag{2.18}$$

Consequently, from (2.17) and (2.18), we have

$$\tau\int_Q |v|^2 \mathrm{d}x\mathrm{d}t + \tau\int_{-T}^{T} \|v\|_{H^{1-s}(\mathbb{R}^n)}^2 \mathrm{d}t \le C_{31}\int_{-T}^{T} \|P_\phi v\|_{H^{-s}(\mathbb{R}^n)}^2 \mathrm{d}t. \tag{2.19}$$

By taking $v = u\mathrm{e}^{\tau\phi}$, we complete the proof of Lemma 2.3.

Proof of Theorem 1.1 We divide the proof of Theorem 1.1 into two steps.

Step 1 We first prove that (1.5) holds for all $u \in C^2(Q)$ with compact support in $\Omega^0 \times (-T_1, T_1)$. Thanks to Lemma 2.3, it suffices to prove that there exists a constant $C_{32} > 0$ such that

$$\|v_1\|_{H^{-s}(\mathbb{R}^n)} \le C_{32}\|v_1\|_{H^{-s}(\Omega)} \tag{2.20}$$

for all $v_1 \in C(\Omega)$ with compact support in Ω^0.

Let $\kappa = \mathrm{dist}(\partial\Omega, \partial\Omega^0)$. We choose $\chi_1 \in C_0^\infty(\Omega)$ such that

$$\chi_1 = \begin{cases} 1, & x \in \left\{x \in \Omega;\ \mathrm{dist}(x, \partial\Omega) \ge \dfrac{2}{3}\kappa\right\}, \\ 0, & x \in \left\{x \in \Omega;\ \mathrm{dist}(x, \partial\Omega) < \dfrac{1}{3}\kappa\right\}, \end{cases}$$

and $\|\nabla\chi_1\|_{L^\infty(\Omega)} \le C_{33}\kappa^{-1}$. For every $\varrho \in H^s(\mathbb{R}^n)$ satisfying $\|\varrho\|_{H^s(\mathbb{R}^n)} = 1$, we set $\varrho_1 = \frac{\chi_1\varrho}{\|\chi_1\varrho\|_{H_0^s(\Omega)}}$. Then $\varrho_1 \in H_0^s(\Omega)$ and $\|\varrho_1\|_{H_0^s(\Omega)} = 1$. Hence we have

$$\begin{aligned} {}_{H^{-s}(\mathbb{R}^n)}\langle v_1, \varrho\rangle_{H^s(\mathbb{R}^n)} &= {}_{H^{-s}(\mathbb{R}^n)}\langle v_1, \chi_1\varrho\rangle_{H^s(\mathbb{R}^n)} \\ &= \|\chi_1\varrho\|_{H_0^s(\Omega)}\ {}_{H^{-s}(\Omega)}\langle v_1, \varrho_1\rangle_{H_0^s(\Omega)} \le C_{34}(1+\kappa^{-1})_{H^{-s}(\Omega)}\langle v_1, \varrho_1\rangle_{H_0^s(\Omega)} \end{aligned}$$

for all $v_1 \in C(\Omega)$ with compact support in Ω^0. Therefore

$$\begin{aligned} \|v_1\|_{H^{-s}(\mathbb{R}^n)} &= \sup_{\|\varrho\|_{H^s(\mathbb{R}^n)}=1} |{}_{H^{-s}(\mathbb{R}^n)}\langle v_1, \varrho\rangle_{H^s(\mathbb{R}^n)}| \\ &\le C_{35}(1+\kappa^{-1}) \sup_{\|\varrho_1\|_{H_0^s(\Omega)}=1} |{}_{H^{-s}(\Omega)}\langle v_1, \varrho_1\rangle_{H^s(\Omega)}| \\ &= C_{35}(1+\kappa^{-1})\|v_1\|_{H^{-s}(\Omega)} \end{aligned}$$

for all $v_1 \in C(\Omega)$ with compact support in Ω^0. Thus (2.20) is proved.

Step 2 We finish the proof of Theorem 1.1 by an argument based on approximation. Let $\varsigma \in C_0^\infty(\mathbb{R}^{n+1})$ be a mollifier such that $\mathrm{supp}\,\varsigma \subset \{(x,t); |x|^2 + t^2 < 1\}$, $\varsigma \ge 0$ and $\int_{\mathbb{R}^{n+1}} \varsigma(x,t)\mathrm{d}x\mathrm{d}t = 1$. We set $\varsigma_\epsilon(x,t) = \epsilon^{-n-1}\varsigma\big(\frac{x}{\epsilon}, \frac{t}{\epsilon}\big)$, $\epsilon > 0$. Let $(\varsigma_\epsilon * u)(x,t)$ be the convolution of ς_ϵ and u (see e.g., [9, Chapter I]). Then we have $\varsigma_\epsilon * u \in C^\infty(\mathbb{R}^{n+1})$ and $\mathrm{supp}(\varsigma_\epsilon * u) \subset \mathrm{supp}\,\varsigma_\epsilon + \mathrm{supp}\,u$. By choosing

$\epsilon > 0$ sufficiently small, we see that $\varsigma_\epsilon * u \in C^\infty(Q)$ with support in $\Omega^0_\epsilon \times (-T_1 - \epsilon, T_1 + \epsilon) \subset Q$. Here $\Omega^0_\epsilon = \{x \in \mathbb{R}^n; \mathrm{dist}(x, \Omega^0) < \epsilon\}$. By the result of Step 1, we have

$$\tau \int_{-T}^{T} \|(\varsigma_\epsilon * u)\mathrm{e}^{\tau\phi}\|^2_{H^{1-s}(\Omega)}\mathrm{d}t \le C_{35} \int_{-T}^{T} \|\mathrm{e}^{\tau\phi} P(\varsigma_\epsilon * u)\|^2_{H^{-s}(\Omega)}\mathrm{d}t. \tag{2.21}$$

By an argument similar to the proof of Friedrichs' lemma (P.9) in [10], we obtain

$$\int_{-T}^{T} \|\mathrm{e}^{\tau\phi} P(\varsigma_\epsilon * u) - \mathrm{e}^{\tau\phi}[\zeta_\epsilon * (Pu)]\|^2_{H^{-s}(\Omega)}\mathrm{d}t \to 0, \quad \epsilon \to 0.$$

Moreover, since it is easily verified that

$$\int_{-T}^{T} \|\mathrm{e}^{\tau\phi}[\varsigma_\epsilon * (Pu)] - \mathrm{e}^{\tau\phi} Pu\|^2_{H^{-s}(\Omega)}\mathrm{d}t \to 0, \quad \epsilon \to 0,$$

we have

$$\int_{-T}^{T} \|\mathrm{e}^{\tau\phi} P(\varsigma_\epsilon * u) - \mathrm{e}^{\tau\phi} Pu\|^2_{H^{-s}(\Omega)}\mathrm{d}t \to 0, \quad \epsilon \to 0.$$

On the other hand, we have

$$\int_{-T}^{T} \|(\varsigma_\epsilon * u)\mathrm{e}^{\tau\phi} - u\mathrm{e}^{\tau\phi}\|^2_{H^{1-s}(\Omega)}\mathrm{d}t \to 0, \quad \epsilon \to 0.$$

Thus, letting $\epsilon \to 0$ in (2.21), we complete the proof of Theorem 1.1.

Proof of Corollary 1.2 By virtue of Theorem 1.1, we see that if $\zeta_1 u\mathrm{e}^{\tau\phi} \in L^2(-T, T; H^{-s}(\Omega))$, then

$$\begin{aligned} &\tau\|u\mathrm{e}^{\tau\phi}\|^2_{L^2(-T,T;H^{1-s}(\Omega))} \\ &\le C_{36}\|\mathrm{e}^{\tau\phi}(P + \zeta_1 - \zeta_1)u\|^2_{L^2(-T,T;H^{-s}(\Omega))} \\ &\le 2C_{36}(\|\mathrm{e}^{\tau\phi}(P + \zeta_1)u\|^2_{L^2(-T,T;H^{-s}(\Omega))} + \|\zeta_1 u\mathrm{e}^{\tau\phi}\|^2_{L^2(-T,T;H^{-s}(\Omega))}) \end{aligned} \tag{2.22}$$

for all $u \in L^2(Q)$ with compact support in $\Omega^0 \times (-T_1, T_1)$ such that $Pu \in L^2(-T, T; H^{-s}(\Omega))$.

Next we estimate the last term on the right-hand side. We see that

$$\begin{aligned} \int_{-T}^{T} {}_{H^{-s}(\Omega)}\langle \zeta_1 u\mathrm{e}^{s\phi}, \mu\rangle_{H^s(\Omega)}\mathrm{d}t &= \int_{-T}^{T} {}_{W^{-\gamma}_{p'}(\Omega)}\langle \zeta_1, u\mathrm{e}^{s\phi}\mu\rangle_{W^{\gamma}_{p}(\Omega)}\mathrm{d}t \\ &\le \int_{-T}^{T} \|\zeta_1\|_{W^{-\gamma}_{p'}(\Omega)} \|u\mathrm{e}^{s\phi}\mu\|_{W^{\gamma}_{p}(\Omega)}\mathrm{d}t, \end{aligned} \tag{2.23}$$

if $u\mathrm{e}^{s\phi}\mu \in W^\gamma_p(\Omega)$. From the corollary in [39, p. 189], we know that if $n \ge 2$, $0 < \gamma < 1 - s$ and $1 - \gamma > n(1 - \frac{1}{p})$, then

$$H^{1-s}(\mathbb{R}^n) \cdot H^s(\mathbb{R}^n) \subset W^\gamma_p(\mathbb{R}^n)$$

for $\frac{3}{4} < s < 1$. By using an argument of taking the 0-extension of μ and $u\mathrm{e}^{s\phi}$ outside $\Omega \times (-T, T)$, it is easy to prove

$$\|u\mathrm{e}^{s\phi}\mu\|_{W^\gamma_p(\Omega)} \le \|u\mathrm{e}^{s\phi}\|_{H^{1-s}(\Omega)} \|\mu\|_{H^s_0(\Omega)}. \tag{2.24}$$

From (2.23) and (2.24), we obtain

$$\begin{aligned}&\int_{-T}^{T} {}_{H^{-s}(\Omega)}\langle \zeta_1 u\mathrm{e}^{s\phi}, \mu\rangle_{H_0^s(\Omega)}\mathrm{d}t\\ \le& \int_{-T}^{T} \|\zeta_1\|_{W_{p'}^{-\gamma}(\Omega)}\|u\mathrm{e}^{s\phi}\|_{H^{1-s}(\Omega)}\|\mu\|_{H_0^s(\Omega)}\mathrm{d}t\\ \le& \|\zeta_1\|_{L^\infty(-T,T;W_{p'}^{-\gamma}(\Omega))}\|u\mathrm{e}^{s\phi}\|_{L^2(-T,T;H^{1-s}(\Omega))}\|\mu\|_{L^2(-T,T;H_0^s(\Omega))}.\end{aligned} \tag{2.25}$$

Inequality (2.25) means $\zeta_1 u\mathrm{e}^{\tau\phi} \in L^2(-T,T;H^{-s}(\Omega))$ and

$$\|\zeta_1 u\mathrm{e}^{\tau\phi}\|_{L^2(-T,T;H^{-s}(\Omega))} \le \|\zeta_1\|_{L^\infty(-T,T;W_{p'}^{-\gamma}(\Omega))}\|u\mathrm{e}^{s\phi}\|_{L^2(-T,T;H^{1-s}(\Omega))}. \tag{2.26}$$

From (2.22) and (2.26), we obtain (1.6) by taking τ large enough. Thus the proof of Corollary 1.2 is completed.

3 Proof of Theorem 1.3

Proof of Theorem 1.3 Let $q(x) = q_1(x) - q_2(x)$, $y_1(x,t) = y(q_1, b, y_0, y_\Sigma)(x,t)$, $y_2(x,t) = y(q_2, b, y_0, y_\Sigma)(x,t)$ and $y(x,t) = y_2(x,t) - y_1(x,t)$. Then we have

$$\begin{cases} \mathrm{i}\partial_t y(x,t) + \mathrm{div}\,(b(x)\nabla y(x,t)) - q_2(x)y(x,t) = qy_1, & \text{in } Q,\\ y = 0, & \text{on } \Sigma,\\ y(\,\cdot\,,0) = 0, & \text{in } \Omega. \end{cases} \tag{3.1}$$

Since $|y_0(x)| > c > 0$, $x \in \overline{\Omega(\delta_0)}$ and $y_1 \in C(\overline{\Omega(\delta_0)} \times (-T,T))$, there exist constants $\epsilon_1 > 0$ and $c_1 > 0$ such that

$$y_1(x,t) > c_1 > 0, \quad (x,t) \in \overline{\Omega(\delta_0)} \times [-\epsilon_1, \epsilon_1].$$

It is easy to see that

$$\frac{\mathrm{i}\partial_t y + \mathrm{div}\,(b(x)\nabla y(x,t)) - q_2(x)y}{y_1} = q(x), \quad \text{in } \Omega(\delta_0) \times (-\epsilon_1, \epsilon_1). \tag{3.2}$$

Because $q(x)$ is independent of t, the differentiation of both sides of (3.2) with respect to t eliminates $q(x)$. Hence

$$\begin{aligned}&\partial_t[\mathrm{i}\partial_t y + \mathrm{div}\,(b(x)\nabla y(x,t)) - q_2(x)y(x,t)]\\ =& \frac{\partial_t y_1}{y_1}[\mathrm{i}\partial_t y + \mathrm{div}\,(b(x)\nabla y(x,t)) - q_2(x)y(x,t)], \quad \text{in } (\mathcal{D}(\Omega(\delta_0) \times (-\epsilon_1, \epsilon_1)))'.\end{aligned} \tag{3.3}$$

We set

$$h_1(x,t) = \frac{\partial_t y_1(x,t)}{y_1(x,t)}, \quad h(x,t) = \partial_t y(x,t) - h_1(x,t)y(x,t).$$

Then we have

$$y(x,t) = \int_0^t \frac{y_1(x,t)}{y_1(x,\theta)} \cdot h(x,\theta)\mathrm{d}\theta. \tag{3.4}$$

Consequently, we have

$$\mathrm{i}\partial_t h + \mathrm{div}\,(b(x)\nabla h) - q_2 h = \int_0^t \Big(\sum_{j=1}^n K_{1j}(x,t,\theta)h(x,\theta) + \sum_{j=1}^n K_{2j}(x,t,\theta)\partial_j h(x,\theta)\Big)\mathrm{d}\theta, \tag{3.5}$$

in $(\mathcal{D}(\Omega(\delta_0)\times(-\epsilon_1,\epsilon_1)))'$, where

$$K_{1j}(x,t,\theta)=2b(x)(\partial_j h_1)\partial_j\Big(\frac{y_1(x,t)}{y_1(x,\theta)}\Big)+(\mathrm{i}\partial_t h_1+\mathrm{div}\,(b(x)\nabla h_1))\frac{y_1(x,t)}{y_1(x,\theta)},$$
$$K_{2j}(x,t,\theta)=2b(x)(\partial_j h_1)\frac{y_1(x,t)}{y_1(x,\theta)}.$$

Let $\delta>0$ be a constant. Define

$$Q(\delta)=\{(x,t)\in Q;\ |x-x_0|^2-\beta t^2>\delta^2\}.$$

Here we choose a constant $\beta>0$ such that $Q(\delta_0)\subset\Omega\times(-\epsilon_1,\epsilon_1)$. We introduce a cut-off function $\chi_2\in C^\infty(Q)$ such that

$$\chi_2(x,t)=\begin{cases}1, & (x,t)\in Q(\delta_0+3\delta_1),\\ 0, & (x,t)\in Q\backslash\overline{Q(\delta_0+2\delta_1)},\end{cases}\tag{3.6}$$

where $\delta_1>0$ is a small constant.

Let $z=\chi_2 h$. By (3.6) and (1.13), we know that $z\in L^2(\Omega(\delta_0+\delta_1)\times(-T,T))$ with $\mathrm{supp}\,z\subset(\Omega(\delta_0+2\delta_1)\backslash\omega)\times(-\epsilon_1,\epsilon_1)$.

From (3.5), we have

$$\begin{aligned}&\mathrm{i}\partial_t z+\mathrm{div}\,(b(x)\nabla z)-\int_0^t\sum_{j=1}^n K_{2j}(x,t,\theta)\partial_j z(x,\theta)\mathrm{d}\theta\\ &=q_2z+\chi_2\int_0^t\sum_{j=1}^n K_{1j}(x,t,\theta)h(x,\theta)\mathrm{d}\theta-\int_0^t\sum_{j=1}^n K_{2j}(x,t,\theta)h(x,\theta)\partial_j\chi_2(x,\theta)\mathrm{d}\theta\\ &\quad+\mathrm{i}(\partial_t\chi_2)h+b(\triangle\chi_2)h+2b\nabla\chi_2\cdot\nabla h+h\nabla b\cdot\nabla\chi_2,\quad\text{in }(\mathcal{D}(\Omega(\delta_0)\times(-\epsilon_1,\epsilon_1)))'.\end{aligned}\tag{3.7}$$

By noting also (1.12), we apply Corollary 1.1 to z. Then we obtain

$$\begin{aligned}&\tau\int_{-T}^T\int_{\Omega(\delta_0+\delta_1)}|z|^2\mathrm{e}^{2\tau\phi_1}\mathrm{d}x\mathrm{d}t\\ &\le\int_{-T}^T\Big\|q_2z\mathrm{e}^{\tau\phi_1}+\int_0^t\sum_{j=1}^n K_{1j}(\,\cdot\,,t,\theta)\chi_2(\,\cdot\,,\theta)h(\,\cdot\,,\theta)\mathrm{e}^{\tau\phi_1}\mathrm{d}\theta\Big\|^2_{H^{-1}(\Omega(\delta_0+\delta_1))}\mathrm{d}t\\ &\quad+\int_{-T}^T\Big\|\int_0^t\sum_{j=1}^n K_{2j}(\,\cdot\,,t,\theta)h(\,\cdot\,,\theta)(\partial_j\chi_2)(\,\cdot\,,\theta)\mathrm{e}^{2\tau\phi_1}\mathrm{d}\theta\Big\|^2_{H^{-1}(\Omega(\delta_0+\delta_1))}\mathrm{d}t\\ &\quad+\int_{-T}^T\|\mathrm{i}(\partial_t\chi_2)h\mathrm{e}^{2\tau\phi_1}+b(\triangle\chi_2)h\mathrm{e}^{2\tau\phi_1}+2b(\nabla\chi_2\cdot\nabla h)\mathrm{e}^{2\tau\phi_1}\\ &\quad+h(\nabla b\cdot\nabla\chi_2)\mathrm{e}^{2\tau\phi_1}\|^2_{H^{-1}(\Omega(\delta_0+\delta_1))}\mathrm{d}t\end{aligned}\tag{3.8}$$

for large λ, $\tau>0$. By (1.12), (3.6) and $q_2\in L^p(\Omega)$ with $p>\max\{n,2\}$, we have

$$\begin{aligned}&\tau\int_{-T}^T\int_{\Omega(\delta_0+\delta_1)}|z|^2\mathrm{e}^{2\tau\phi_1}\mathrm{d}x\mathrm{d}t\\ &\le\int_{-T}^T\Big\|\int_0^t\sum_{j=1}^n K_{1j}(x,t,\theta)\chi_2(x,\theta)h(x,\theta)\mathrm{e}^{\tau\phi_1}\mathrm{d}\theta\Big\|^2_{L^2(\Omega(\delta_0+\delta_1))}\mathrm{d}t\end{aligned}$$

$$
\begin{aligned}
&+\int_{-T}^{T}\|q_2\|_{L^p(\Omega)}^2\|z\mathrm{e}^{\tau\phi_1}\|_{L^2(\Omega(\delta_0+\delta_1))}^2\mathrm{d}t+C_2\tau\mathrm{e}^{2\tau(\delta_0+3\delta_1)}\\
&\le C_3\int_{-T}^{T}\int_{\Omega(\delta_0+\delta_1)}|z|^2\mathrm{e}^{2\tau\phi_1}\mathrm{d}x\mathrm{d}t+C_4\tau\mathrm{e}^{2\tau(\delta_0+3\delta_1)}. \qquad (3.9)
\end{aligned}
$$

By taking τ large enough, the first term on the right-hand side of (3.9) can be absorbed by the term on the left-hand side. Then we have

$$
\tau\int_{-T}^{T}\int_{\Omega(\delta_0+\delta_1)}|h\chi_2|^2\mathrm{e}^{2\tau\phi_1}\mathrm{d}x\mathrm{d}t\le C_5\tau\mathrm{e}^{2\tau(\delta_0+3\delta_1)}
$$

for large $\tau>0$. By (3.6), we obtain

$$
\int_{Q(\delta_0+4\delta_1)}|h|^2\mathrm{d}x\mathrm{d}t\le C_6e^{-2\tau\delta_1}
$$

for large $\tau>0$. As $\tau\to+\infty$, we have $\int_{Q(\delta_0+4\delta_1)}|h|^2\mathrm{d}x\mathrm{d}t=0$. This implies that $h(x,t)=0,\ (x,t)\in Q(\delta_0+4\delta_1)$. By (3.4), we have $y(x,t)=0$ in $Q(\delta_0+4\delta_1)$. Consequently, we have $q(x)=0$ in $Q(\delta_0+4\delta_1)$ by noting (3.2). Since δ_1 is arbitrary, $q_2(x)=q_1(x),\ x\in\Omega(\delta_0)$. Thus the proof of Theorem 1.3 is completed.

4 Proof of Theorem 1.4

Proof of Theorem 1.4 Let $d_0=\inf\limits_{x\in\Omega}\exp\{|x-x_0|^2\}$. We choose $\beta>0$ such that $\sup\limits_{x\in\Omega}|x-x_0|^2<T^2\beta$. For $\lambda>1$, we set $\phi_1(x,t)=\exp\{\lambda(|x-x_0|^2-\beta t^2)\}$. Then we have

$$
\phi_1(x,0)\ge d_0,\quad \phi_1(x,-T)=\phi_1(x,T)<d_0,\quad x\in\overline{\Omega}.
$$

Thus, for given $\epsilon>0$, we choose a sufficiently small $\delta_1=\delta_1(\epsilon)>0$ such that

$$
\phi_1(x,t)\ge d_0-\epsilon,\qquad (x,t)\in\overline{\Omega}\times[-\delta_1,\delta_1], \qquad (4.1)
$$

$$
\phi_1(x,t)\le d_0-2\epsilon,\quad (x,t)\in\overline{\Omega}\times([-T,-T+2\delta_1]\cup[T-2\delta_1,T]). \qquad (4.2)
$$

We introduce a cut-off function χ_3 satisfying $0\le\chi_3\le 1$, $\chi_3\in C^\infty[0,T]$ and

$$
\chi_3(t)=\begin{cases}0, & t\in[-T,-T+\delta_1]\cup[T-\delta_1,T],\\ 1, & t\in[-T+2\delta_1,T-2\delta_1].\end{cases} \qquad (4.3)
$$

Let $\Omega^0\subset\omega$ satisfy $\partial\Omega^0\supset\partial\Omega$. We introduce another cut-off function χ_4 satisfying $0\le\chi_4\le 1$, $\chi_4\in C_0^\infty(\Omega)$ and

$$
\chi_4(x)=\begin{cases}0, & x\in\Omega^0,\\ 1, & x\in\Omega\backslash\omega.\end{cases} \qquad (4.4)
$$

Multiply the first equation in (1.14) by $\overline{\eta}$, take the imaginary parts and integrate by parts. Then we have

$$
\|\eta(\,\cdot\,,t)\|_{L^2(\Omega)}=\|\eta(\,\cdot\,,0)\|_{L^2(\Omega)},\quad t\in[0,T]. \qquad (4.5)
$$

Consequently, we have

$$\int_{-T}^{T}\int_{\omega}|\eta|^2\mathrm{d}x\mathrm{d}t \le C_1\|\eta_0\|^2_{L^2(\Omega)}.$$

By a density argument, we can see that in order to prove Theorem 1.4 it suffices to prove that (1.15) holds for $\eta_0 \in H^2(\Omega)\cap H_0^1(\Omega)$. Then we have

$$\eta \in C([-T,T];H^2(\Omega)\cap H_0^1(\Omega))\cap C^1([-T,T];L^2(\Omega)). \tag{4.6}$$

Let $z=\chi_3\chi_4\eta$. Then by (4.3)–(4.6), we have

$$z\in H^1(Q),\quad \operatorname{supp} z\subset(\Omega\backslash\omega)\times(-T+2\delta_1,T-2\delta_1). \tag{4.7}$$

Since $\mathrm{i}\partial_t z+\mathrm{div}(b(x)\nabla z)\in L^2(Q)$, we apply Corollary 1.1 to z for the case when $s=1$. Then we have

$$\begin{aligned}
\tau\int_Q|z\mathrm{e}^{\tau\phi_1}|^2\mathrm{d}x\mathrm{d}t &\le C_2\int_{-T}^{T}\|\mathrm{e}^{\tau\phi_1}[\mathrm{i}\partial_t z+\mathrm{div}(b(x)\nabla z)]\|^2_{H^{-1}(\Omega)}\mathrm{d}t\\
&= C_2\int_{-T}^{T}\|\mathrm{e}^{\tau\phi_1}\{\chi_3\chi_4[\mathrm{i}\partial_t\eta+\mathrm{div}(b\nabla\eta)]+\mathrm{i}\chi_4\eta\partial_t\chi_3\\
&\quad+2\chi_3\mathrm{div}(b\eta\nabla\chi_4)-\chi_3\eta\nabla b\cdot\nabla\chi_4-\chi_3 b\eta\triangle\chi_4\}\|^2_{H^{-1}(\Omega)}\mathrm{d}t\\
&= C_2\int_{-T}^{T}\|\mathrm{e}^{\tau\phi_1}\{\chi_3\chi_4 q\eta+\mathrm{i}\chi_4\eta\partial_t\chi_3+2\chi_3\mathrm{div}(b\eta\nabla\chi_4)\\
&\quad-\chi_3\eta\nabla b\cdot\nabla\chi_4-\chi_3 b\eta\triangle\chi_4\}\|^2_{H^{-1}(\Omega)}\mathrm{d}t,
\end{aligned} \tag{4.8}$$

provided that $\lambda>1$, $\tau>1$ are large enough. By noting that $\partial_t\chi_3\neq 0$ only in the case where $\phi_1(t,x)\le d_0-2\epsilon$, and $\partial_j\chi_4$, $1\le j\le n$, are supported in ω, we have

$$\begin{aligned}
\tau\int_Q|z\mathrm{e}^{\tau\phi_1}|^2\mathrm{d}x\mathrm{d}t &\le C_3\int_{-T}^{T}\|q\|^2_{L^p(\Omega)}\|z\mathrm{e}^{\tau\phi_1}\|^2_{L^2(\Omega)}\mathrm{d}t+C_3\mathrm{e}^{2\tau(d_0-2\epsilon)}\int_Q|\chi_4\eta|^2\mathrm{d}x\mathrm{d}t\\
&\quad+C_3\mathrm{e}^{2\tau\Phi}\int_{-T}^{T}\|\eta\|^2_{L^2(\omega)}\mathrm{d}t+C_3\int_{-T}^{T}\|\mathrm{e}^{\tau\phi_1}2b\chi_3\mathrm{div}(\eta\nabla\chi_4)\|^2_{H^{-1}(\Omega)}\mathrm{d}t,
\end{aligned} \tag{4.9}$$

where $\Phi=\sup\limits_{(x,t)\in Q}\phi_1(x,t)$. The first term on the right-hand side of (4.9) can be absorbed by the left-hand term when we choose $\tau>1$ large enough. Thus we have

$$\begin{aligned}
\tau\int_Q|z\mathrm{e}^{\tau\phi_1}|^2\mathrm{d}x\mathrm{d}t &\le C_4\mathrm{e}^{2\tau(d_0-2\epsilon)}\int_Q|\chi_4\eta|^2\mathrm{d}x\mathrm{d}t+C_4\mathrm{e}^{2\tau\Phi}\int_{-T}^{T}\|\eta\|^2_{L^2(\omega)}\mathrm{d}t\\
&\quad+C_4\int_{-T}^{T}\|\mathrm{e}^{\tau\phi_1}2\chi_3\mathrm{div}(b\eta\nabla\chi_4)\|^2_{H^{-1}(\Omega)}\mathrm{d}t.
\end{aligned} \tag{4.10}$$

By noting that $\mathrm{e}^{\tau\phi_1}2\chi_3\mathrm{div}(b\eta\nabla\chi_4)=2\sum\limits_{j=1}^{n}\partial_j(\mathrm{e}^{\tau\phi_1}b\chi_3\eta\partial_j\chi_4)-2\mathrm{e}^{\tau\phi_1}\tau b\chi_3\eta(\nabla\phi_1\cdot\nabla\chi_4)$, we can obtain

$$\tau\int_Q|z\mathrm{e}^{\tau\phi_1}|^2\mathrm{d}x\mathrm{d}t\le C_5\mathrm{e}^{2\tau(d_0-2\epsilon)}\int_Q|\chi_4\eta|^2\mathrm{d}x\mathrm{d}t+C_5(1+\tau)\mathrm{e}^{2\tau\Phi}\int_{-T}^{T}\|\eta\|^2_{L^2(\omega)}\mathrm{d}t. \tag{4.11}$$

Noting $z=\chi_3\chi_4\eta$, (4.1) and (4.3), we can obtain

$$\begin{aligned}&\tau\mathrm{e}^{2\tau(d_0-\epsilon)}\int_{-\delta_1}^{\delta_1}\int_\Omega|\chi_4\eta|^2\mathrm{d}x\mathrm{d}t\\ \le{}& C_6\mathrm{e}^{2\tau(d_0-2\epsilon)}\int_Q|\eta|^2\mathrm{d}x\mathrm{d}t+C_6(1+\tau)\mathrm{e}^{2\tau\Phi}\int_{-T}^T\|\eta\|_{L^2(\omega)}^2\mathrm{d}t.\end{aligned}\tag{4.12}$$

By (4.4), we have

$$\begin{aligned}&\tau\mathrm{e}^{2\tau(d_0-\epsilon)}\int_{-\delta_1}^{\delta_1}\int_{\Omega\backslash\omega}|\eta|^2\mathrm{d}x\mathrm{d}t\\ \le{}& C_7\mathrm{e}^{2\tau(d_0-2\epsilon)}\int_Q|\eta|^2\mathrm{d}x\mathrm{d}t+C_7(1+\tau)\mathrm{e}^{2\tau\Phi}\int_{-T}^T\|\eta\|_{L^2(\omega)}^2\mathrm{d}t.\end{aligned}\tag{4.13}$$

Consequently, we have

$$\begin{aligned}&\tau\mathrm{e}^{2\tau(d_0-\epsilon)}\int_{-\delta_1}^{\delta_1}\int_\Omega|\eta|^2\mathrm{d}x\mathrm{d}t\\ \le{}& C_8\mathrm{e}^{2\tau(d_0-2\epsilon)}\int_Q|\eta|^2\mathrm{d}x\mathrm{d}t+C_8[(1+\tau)\mathrm{e}^{2\tau\Phi}+\tau\mathrm{e}^{2\tau(d_0-\epsilon)}]\int_{-T}^T\|\eta\|_{L^2(\omega)}^2\mathrm{d}t\end{aligned}\tag{4.14}$$

for large $\tau>1$. By noting (4.5) we obtain

$$\begin{aligned}&2\tau\delta_1\mathrm{e}^{2\tau(d_0-\epsilon)}\Big(1-\frac{C_9T\delta_1^{-1}}{\tau\mathrm{e}^{2\tau\epsilon}}\Big)\|\eta_0\|_{L^2(\Omega)}^2\mathrm{d}x\mathrm{d}t\\ \le{}& C_9[(1+\tau)\mathrm{e}^{2\tau\Phi}+\tau\mathrm{e}^{2\tau(d_0-\epsilon)}]\int_{-T}^T\|\eta\|_{L^2(\omega)}^2\mathrm{d}t.\end{aligned}\tag{4.15}$$

Taking $\tau>\max\{1,\frac{\log 2C_9T\delta_1^{-1}}{2\epsilon}\}$, we have

$$1-\frac{C_9T\delta_1^{-1}}{\tau\mathrm{e}^{2\tau\epsilon}}>\frac{1}{2}.$$

Thus, by (4.15), we complete the proof of Theorem 1.4.

Appendix

Proof of Proposition 2.1 We first prove (2.1). Let $a(x,t)=\frac{1}{g(x,t)}$, $P^0v=\mathrm{i}a\partial_tv+\triangle v$, $w=v\mathrm{e}^{\tau\phi}$ and $Lw=\mathrm{e}^{\tau\phi}P^0(\mathrm{e}^{-\tau\phi}w)$. Then we have

$$\begin{aligned}&Lw+\mathrm{i}\tau(\partial_t\phi)aw=L_1w+L_2w,\\ &L_1w=\mathrm{i}a\partial_tw+\triangle w+\tau^2|\nabla\phi|^2w,\\ &L_2w=-2\tau\nabla\phi\cdot\nabla w-\tau w\triangle\phi.\end{aligned}\tag{A.1}$$

Then

$$\int_Q|Lw+\mathrm{i}\tau(\partial_t\phi)aw|^2\mathrm{d}x\mathrm{d}t=\int_Q|L_1w|^2\mathrm{d}x\mathrm{d}t+\int_Q|L_2w|^2\mathrm{d}x\mathrm{d}t+2\mathrm{Re}\int_Q L_1w\overline{L_2w}\mathrm{d}x\mathrm{d}t.\tag{A.2}$$

We calculate the last term:

$$\begin{aligned}2\mathrm{Re}\int_Q L_1w\overline{L_2w}\mathrm{d}x\mathrm{d}t &= -4\tau\mathrm{Re}\int_Q \mathrm{i}(\partial_t w)a\nabla\phi\cdot\nabla\overline{w}\mathrm{d}x\mathrm{d}t - 4\tau\mathrm{Re}\int_Q(\triangle w)\nabla\phi\cdot\nabla\overline{w}\mathrm{d}x\mathrm{d}t\\ &\quad - 2\tau\mathrm{Re}\int_Q \mathrm{i}a\overline{w}(\partial_t w)\triangle\phi\mathrm{d}x\mathrm{d}t - 2\tau\mathrm{Re}\int_Q\overline{w}(\triangle w)\triangle\phi\mathrm{d}x\mathrm{d}t\\ &\quad - 4\tau^3\mathrm{Re}\int_Q|\nabla\phi|^2 w\nabla\phi\cdot\nabla\overline{w}\mathrm{d}x\mathrm{d}t - 2\tau^3\int_Q|\nabla\phi|^2(\triangle\phi)|w|^2\mathrm{d}x\mathrm{d}t\\ &= \sum_{j=1}^{6}\mathrm{I}_j. \end{aligned}\tag{A.3}$$

Integrating by parts and using $w|_{\partial\Omega}=0$ and (A.1), we have

$$\begin{aligned}\mathrm{I}_1 &= -2\tau\mathrm{Im}\int_Q\{(-2\lambda^2\beta t)a\phi(\nabla d\cdot\nabla\overline{w})w - a\partial_t\overline{w}(\lambda\phi\triangle d+\lambda^2\phi|\nabla d|^2)w\}\mathrm{d}x\mathrm{d}t\\ &\quad + 2\tau\mathrm{Re}\int_Q\lambda\phi(\nabla\log a\cdot\nabla d)wL_1\overline{w}\mathrm{d}x\mathrm{d}t - 2\tau\mathrm{Im}\int_Q(\partial_t a)w\nabla\overline{w}\cdot\nabla\phi\mathrm{d}x\mathrm{d}t\\ &\quad + 2\tau\mathrm{Re}\int_Q[\lambda\phi w\nabla(\nabla\log a\cdot\nabla d)\cdot\nabla\overline{w}+\lambda^2\phi(\nabla\log a\cdot\nabla d)(\nabla d\cdot\nabla\overline{w})w]\mathrm{d}x\mathrm{d}t\\ &\quad + 2\tau\int_Q\lambda\phi(\nabla\log a\cdot\nabla d)|\nabla w|^2\mathrm{d}x\mathrm{d}t - 2\tau^3\int_Q\lambda^3\phi^3(\nabla\log a\cdot\nabla d)|\nabla d|^2|w|^2\mathrm{d}x\mathrm{d}t,\\ \mathrm{I}_2 &= 2\tau\int_Q -(\lambda\phi\triangle d+\lambda^2\phi|\nabla d|^2)|\nabla w|^2\mathrm{d}x\mathrm{d}t\\ &\quad + 4\tau\mathrm{Re}\int_Q\sum_{j,k=1}^{n}[\lambda\phi\partial_j\partial_k d+\lambda^2\phi(\partial_j d)\partial_k d](\partial_j w)\partial_k\overline{w}\\ &\quad + 2\tau\int_{-T}^{T}\int_{\partial\Omega}\lambda\phi\frac{\partial d}{\partial\nu}|\nabla w|^2\mathrm{d}\Sigma - 4\tau\mathrm{Re}\int_{-T}^{T}\int_{\partial\Omega}\lambda\phi(\nabla d\cdot\nabla\overline{w})\frac{\partial w}{\partial\nu}\mathrm{d}x\mathrm{d}t,\\ \mathrm{I}_3 &= -2\tau\mathrm{Im}\int_Q aw(\partial_t\overline{w})(\lambda\phi\triangle d+\lambda^2\phi|\nabla d|^2)\mathrm{d}x\mathrm{d}t,\\ \mathrm{I}_4 &= 2\tau\int_Q(\lambda\phi\triangle d+\lambda^2\phi|\nabla d|^2)|\nabla w|^2\mathrm{d}x\mathrm{d}t - \tau\lambda\int_Q\phi(\triangle^2 d)|w|^2\mathrm{d}x\mathrm{d}t\\ &\quad - \tau\lambda^2\int_Q\phi(|\triangle d|^2+2\nabla d\cdot\nabla(\triangle d)+\triangle(|\nabla d|^2))|w|^2\mathrm{d}x\mathrm{d}t\\ &\quad - 2\tau\lambda^3\int_Q\phi(|\nabla d|^2\triangle d+\nabla d\cdot\nabla(|\nabla d|^2))|w|^2\mathrm{d}x\mathrm{d}t - \tau\lambda^4\int_Q\phi|\nabla d|^4|w|^2\mathrm{d}x\mathrm{d}t,\\ \mathrm{I}_5 &= 2\tau^3\lambda^3\int_Q\phi^3(|\nabla d|^2\triangle d+\nabla d\cdot\nabla(|\nabla d|^2))|w|^2\mathrm{d}x\mathrm{d}t + 6\tau^3\lambda^4\int_Q\phi^3|\nabla d|^4|w|^2\mathrm{d}x\mathrm{d}t,\\ \mathrm{I}_6 &= -2\tau^3\lambda^3\int_Q\phi^3|\nabla d|^2(\triangle d)|w|^2\mathrm{d}x\mathrm{d}t - 2\tau^3\lambda^4\int_Q\phi^3|\nabla d|^4|w|^2\mathrm{d}x\mathrm{d}t.\end{aligned}$$

Consequently, we have

$$\begin{aligned}&2\mathrm{Re}\int_Q L_1w\overline{L_2w}\mathrm{d}x\mathrm{d}t\\ &= 2\tau\lambda\int_Q\phi\Big[|\nabla w|^2\nabla\log a\cdot\nabla d + 2\sum_{j,k=1}^{n}(\partial_j\partial_k d)(\partial_j w)\partial_k\overline{w}\Big]\mathrm{d}x\mathrm{d}t\end{aligned}$$

$$+ 4\tau\lambda^2 \int_Q \phi|\nabla d\cdot\nabla w|^2 \mathrm{d}x\mathrm{d}t + 4\tau^3\lambda^4 \int_Q \phi^3|\nabla d|^4|w|^2 \mathrm{d}x\mathrm{d}t$$
$$+ 2\tau\int_{-T}^{T}\int_{\partial\Omega} \lambda\phi\frac{\partial d}{\partial\nu}|\nabla w|^2 \mathrm{d}\Sigma - 4\tau\mathrm{Re}\int_{-T}^{T}\int_{\partial\Omega}\lambda\phi(\nabla d\cdot\nabla\overline{w})\frac{\partial w}{\partial\nu}\mathrm{d}\Sigma + X_1 + X_2, \tag{A.4}$$

where

$$\begin{cases} X_1 = -2\tau^3\int_Q \lambda^3\phi^3(\nabla\log a\cdot\nabla d)|\nabla d|^2|w|^2\mathrm{d}x\mathrm{d}t + 2\tau^3\lambda^3\int_Q\phi^3\nabla d\cdot\nabla(|\nabla d|^2)|w|^2\mathrm{d}x\mathrm{d}t, \\ X_2 = -2\tau\mathrm{Im}\int_Q(-2\lambda^2\beta t)a\phi(\nabla d\cdot\nabla\overline{w})w\mathrm{d}x\mathrm{d}t + 2\tau\mathrm{Re}\int_Q\lambda\phi(\nabla\log a\cdot\nabla d)wL_1\overline{w}\mathrm{d}x\mathrm{d}t \\ \qquad +2\tau\mathrm{Re}\int_Q[\lambda\phi w\nabla(\nabla\log a\cdot\nabla d)\cdot\nabla\overline{w} + \lambda^2\phi(\nabla\log a\cdot\nabla d)(\nabla d\cdot\nabla\overline{w})w]\mathrm{d}x\mathrm{d}t \\ \qquad -2\tau\mathrm{Im}\int_Q(\partial_t a)w\nabla\overline{w}\cdot\nabla\phi\mathrm{d}x\mathrm{d}t - \tau\lambda\int_Q\phi(\triangle^2 d)|w|^2\mathrm{d}x\mathrm{d}t \\ \qquad -\tau\lambda^2\int_Q\phi(|\triangle d|^2 + 2\nabla d\cdot\nabla(\triangle d) + \triangle(|\nabla d|^2))|w|^2\mathrm{d}x\mathrm{d}t \\ \qquad -2\tau\lambda^3\int_Q\phi(|\nabla d|^2\triangle d + \nabla d\cdot\nabla(|\nabla d|^2))|w|^2 - \tau\lambda^4\int_Q\phi|\nabla d|^4|w|^2\mathrm{d}x\mathrm{d}t. \end{cases} \tag{A.5}$$

By (1.2), we obtain

$$\int_Q\phi\Big[|\nabla w|^2\nabla\log a\cdot\nabla d + 2\sum_{j,k=1}^n(\partial_j\partial_k d)(\partial_j w)\partial_k\overline{w}\Big]\mathrm{d}x\mathrm{d}t + \int_Q\phi|\nabla d\cdot\nabla w|^2\mathrm{d}x\mathrm{d}t$$
$$\geq C_1\int_Q\phi|\nabla w|^2\mathrm{d}x\mathrm{d}t.$$

By taking $\lambda > \frac{1}{2}$, we have

$$\int_Q\phi\Big[|\nabla w|^2\nabla\log a\cdot\nabla d + 2\sum_{j,k=1}^n(\partial_j\partial_k d)(\partial_j w)\partial_k\overline{w}\Big]\mathrm{d}x\mathrm{d}t + 2\lambda\int_Q\phi|\nabla d\cdot\nabla w|^2\mathrm{d}x\mathrm{d}t$$
$$\geq C_1\int_Q\phi|\nabla w|^2\mathrm{d}x\mathrm{d}t. \tag{A.6}$$

From (A.4) and (A.6), we have

$$\begin{aligned}\int_Q|Lw + \mathrm{i}\tau a(\partial_t\phi)w|^2\mathrm{d}x\mathrm{d}t \geq{}& \int_Q|L_1 w|^2\mathrm{d}x\mathrm{d}t + \int_Q|L_2 w|^2\mathrm{d}x\mathrm{d}t \\ &+ 2\tau\lambda C_1\int_Q\phi|\nabla w|^2\mathrm{d}x\mathrm{d}t + 4\tau^3\lambda^4\int_Q\phi^3|\nabla d|^4|w|^2\mathrm{d}x\mathrm{d}t \\ &+ 2\tau\int_{-T}^{T}\int_{\partial\Omega}\lambda\phi\frac{\partial d}{\partial\nu}|\nabla w|^2\mathrm{d}\Sigma \\ &- 4\tau\lambda\mathrm{Re}\int_{-T}^{T}\int_{\partial\Omega}\phi(\nabla d\cdot\nabla\overline{w})\frac{\partial w}{\partial\nu}\mathrm{d}\Sigma + X_1 + X_2.\end{aligned}$$

Since $|\nabla d| > 0$ on $\overline{\Omega}$, there exists a constant $C_2 > 0$ such that $|\nabla d|^4 > C_2$ on $\overline{\Omega}$. Then we have

$$\begin{aligned}\int_Q |Lw + \mathrm{i}\tau a(\partial_t\phi)w|^2 \mathrm{d}x\mathrm{d}t \geq{}& \int_Q |L_1 w|^2 \mathrm{d}x\mathrm{d}t + \int_Q |L_2 w|^2 \mathrm{d}x\mathrm{d}t \\ &+ 2\tau\lambda C_1 \int_Q \phi|\nabla w|^2 \mathrm{d}x\mathrm{d}t + 4C_2\tau^3\lambda^4 \int_Q \phi^3 |w|^2 \mathrm{d}x\mathrm{d}t \\ &+ 2\tau \int_{-T}^{T}\int_{\partial\Omega} \lambda\phi \frac{\partial d}{\partial\nu}|\nabla w|^2 \mathrm{d}\Sigma \\ &- 4\tau\lambda \mathrm{Re}\int_{-T}^{T}\int_{\partial\Omega} \phi(\nabla d\cdot\nabla\overline{w})\frac{\partial w}{\partial\nu}\mathrm{d}\Sigma + X_1 + X_2. \end{aligned} \tag{A.7}$$

By (A.5) and (A.7), we see that there exists a constant $\lambda_1 > \frac{1}{2}$ such that for arbitrary $\lambda > \lambda_1$, the terms of X_1 can be absorbed by $4C_2\tau^3\lambda^4\int_Q \phi^3|w|^2\mathrm{d}x\mathrm{d}t$, and we have

$$\begin{aligned}&\int_Q |Lw + \mathrm{i}\tau a(\partial_t\phi)w|^2 \mathrm{d}x\mathrm{d}t \\ &\geq \int_Q |L_1 w|^2 \mathrm{d}x\mathrm{d}t + \int_Q |L_2 w|^2 \mathrm{d}x\mathrm{d}t + 2\tau\lambda C_1 \int_Q \phi|\nabla w|^2 \mathrm{d}x\mathrm{d}t + C_3\tau^3\lambda^4 \int_Q \phi^3|w|^2\mathrm{d}x\mathrm{d}t \\ &\quad + 2\tau\int_{-T}^{T}\int_{\partial\Omega}\lambda\phi\frac{\partial d}{\partial\nu}|\nabla w|^2\mathrm{d}\Sigma - 4\tau\lambda\mathrm{Re}\int_{-T}^{T}\int_{\partial\Omega}\phi(\nabla d\cdot\nabla\overline{w})\frac{\partial w}{\partial\nu}\mathrm{d}\Sigma + X_2.\end{aligned}$$

Since $\phi > 0$ on $\overline{Q}$ for $\lambda > \lambda_1$, there exist constants $C_4 = C_4(\lambda)$ and $\tau_1 = \tau_1(\lambda)$ such that for all $\tau > \tau_1$,

$$\begin{aligned}&\int_Q |Lw + \mathrm{i}\tau a(\partial_t\phi)w|^2 \mathrm{d}x\mathrm{d}t \\ &\geq \int_Q |L_1 w|^2 \mathrm{d}x\mathrm{d}t + \int_Q |L_2 w|^2 \mathrm{d}x\mathrm{d}t + C_4(\lambda)\tau \int_Q |\nabla w|^2 \mathrm{d}x\mathrm{d}t + C_4(\lambda)\tau^3 \int_Q |w|^2\mathrm{d}x\mathrm{d}t \\ &\quad + 2\tau\int_{-T}^{T}\int_{\partial\Omega}\lambda\phi\frac{\partial d}{\partial\nu}|\nabla w|^2\mathrm{d}\Sigma - 4\tau\lambda\mathrm{Re}\int_{-T}^{T}\int_{\partial\Omega}\phi(\nabla d\cdot\nabla\overline{w})\frac{\partial w}{\partial\nu}\mathrm{d}\Sigma + X_2.\end{aligned}$$

Then we choose $\tau_2 = \tau_2(\lambda) > 0$ such that all the terms of X_2 can be absorbed into $\|L_1 w\|^2_{L^2(Q)}$, $\|L_2 w\|^2_{L^2(Q)}$, $C_4\|\nabla w\|^2_{L^2(Q)}$ and $C_4\tau^3\|w\|^2_{L^2(Q)}$, if we take $\tau > \tau_2$. Hence since $\nabla w = \frac{\partial w}{\partial\nu}\nu$ by $w = 0$ on $\partial\Omega$ and $\int_Q |Lw + \mathrm{i}\tau a(\partial_t\phi)w|^2\mathrm{d}x\mathrm{d}t \leq 2\int_Q |Lw|^2\mathrm{d}x\mathrm{d}t + C_5\tau^2\int_Q |w|^2\mathrm{d}x\mathrm{d}t$, taking $\tau > 0$ sufficiently large, we obtain

$$\begin{aligned}C_6\int_Q |Lw|^2\mathrm{d}x\mathrm{d}t \geq{}& \int_Q |L_1 w|^2\mathrm{d}x\mathrm{d}t + \int_Q |L_2 w|^2\mathrm{d}x\mathrm{d}t + \tau\int_Q |\nabla w|^2\mathrm{d}x\mathrm{d}t + \tau^3\int_Q |w|^2\mathrm{d}x\mathrm{d}t \\ &- C_7\tau\int_{-T}^{T}\int_{\partial\Omega}\phi(\nabla d\cdot\nu)\Big|\frac{\partial w}{\partial\nu}\Big|^2\mathrm{d}\Sigma.\end{aligned}$$

Now we rewrite our inequality with v instead of w. By $w = v\mathrm{e}^{\tau\phi}$, $|v|^2\mathrm{e}^{2\tau\phi} = |w|^2$, $Lw = \mathrm{e}^{\tau\phi}P^0 v$, and $|\nabla v|^2\mathrm{e}^{2\tau\phi} \leq 2|\nabla w|^2 + 2\tau^2\lambda^2\phi^2|\nabla d|^2|w|^2$, $|\frac{\partial v}{\partial\nu}|^2\mathrm{e}^{2\tau\phi} = |\frac{\partial w}{\partial\nu}|^2$ on $\partial\Omega\times(-T,T)$, we see that there exist positive constants $C_8(\lambda)$ and $\tau_0 > \tau_2(\lambda)$ such that for all $\tau > \tau_0$,

$$\begin{aligned}&\tau\int_Q |\nabla v|^2\mathrm{e}^{2\tau\phi}\mathrm{d}x\mathrm{d}t + \tau^3\int_Q |v|^2\mathrm{e}^{\tau\phi}\mathrm{d}x\mathrm{d}t \\ &\leq C_8\int_Q |P^0 v|^2\mathrm{e}^{2\tau\phi}\mathrm{d}x\mathrm{d}t + C_8\tau\int_{-T}^{T}\int_{\partial\Omega}\phi\Big|\frac{\partial v}{\partial\nu}\Big|^2\mathrm{e}^{2\tau\phi}(\nabla d\cdot\nu)\mathrm{d}\Sigma.\end{aligned}$$

By noting $P^0 u = \frac{1}{g(x,t)} P_0 u$ and $g \in C^1(\overline{Q})$, $g > 0$ on $\overline{Q}$, we complete the proof of (2.1). Thus the proof of Proposition 2.1 is completed.

Proof of Theorem 1.2 Let $0 < T' < T$, and Ω' be a subset of Ω satisfying $(x^1, t^1) \in \Omega' \times (-T', T')$ and $\overline{\Omega' \times (-T', T')} \subset Q$. Let $\mathcal{U}'$ be a neighbourhood of (x^1, t^1) satisfying $\overline{\mathcal{U}'} \subset \Omega' \times (-T', T')$. We introduce a cut-off function $\chi_5 \in C_0^\infty(Q)$ such that $\chi_5(x,t) = 1$, $(x,t) \in \mathcal{U}'$ and $\chi_5(x,t) = 0$, $(x,t) \in Q \backslash \Omega' \times (-T', T')$. We set $z(x,t) = \chi_5(x,t) u(x,t)$. Then $z \in L^2(-T, T; H^{1-s}(\Omega))$ and $\operatorname{supp} z \subset \Omega' \times (-T', T')$. Next we check that $P_1 z \in L^2(-T, T; H^{-s}(\Omega))$. In fact, by (1.7), we obtain

$$\begin{aligned}\|P_1 z\|_{L^2(-T,T;H^{-s}(\Omega))} &= \|\mathrm{i} u \partial_t \chi_5 + \operatorname{div}(\rho u \nabla \chi_5) + \rho \nabla u \cdot \nabla \chi_5\|^2_{L^2(-T,T;H^{-s}(\Omega))} \\ &\le C_1 \|u\|^2_{L^2(-T,T;H^{1-s}(\Omega))} < \infty.\end{aligned}$$

Here we use $\|\nabla v\|_{H^{-s}(\Omega)} \le C_1' \|v\|_{H^{1-s}(\Omega)}$, which can be proved for example by an interpolation inequality (see e.g., [34, Section 5, Chapter 1, Vol. 1]), because ∇ is a bounded operator from $H_0^1(\Omega)$ to $L^2(\Omega)$ and from $L^2(\Omega)$ to $H^{-1}(\Omega)$.

Let $\delta_1 > 0$ be a constant. We take $d(x) = |x - x_0|^2 - \delta_1 |x - x^1|^2$ in Corollary 1.2. From (1.8), by choosing $\delta_1 > 0$ small enough, we apply Corollary 1.2 to P_1 and z. Let

$$\phi_2(x,t) = \mathrm{e}^{\lambda(|x-x_0|^2 - \beta t^2 - \delta_1 |x - x^1|^2)}.$$

Then we have

$$\tau \int_{-T}^{T} \|z \mathrm{e}^{\tau\phi_2}\|^2_{H^{1-s}(\Omega)} \mathrm{d}t \le C_2 \int_{-T}^{T} \|\mathrm{e}^{\tau\phi_2} P_1 z\|^2_{H^{-s}(\Omega)} \mathrm{d}t$$

for large τ, $\lambda > 1$. By noting that $P_1 u = 0$ in Q, we obtain

$$\begin{aligned}\tau \int_{-T}^{T} \|\chi_5 u \mathrm{e}^{\tau\phi_2}\|^2_{H^{1-s}(\Omega)} \mathrm{d}t \le C_3 \int_{-T}^{T} \|[&\mathrm{i}(\partial_t \chi_5) u + (\triangle \chi_5) u \rho \\ &+ 2\rho \nabla \chi_5 \cdot \nabla u + u \nabla \chi_5 \cdot \nabla \rho] \mathrm{e}^{\tau\phi_2}\|^2_{H^{-s}(\Omega)} \mathrm{d}t\end{aligned}$$

for large τ, $\lambda > 1$. Consequently,

$$\begin{aligned}\tau \int_{-T}^{T} \|\chi_5 u \mathrm{e}^{\tau\phi_2}\|^2_{H^{1-s}(\Omega)} \mathrm{d}t &\le C_4 \int_{-T}^{T} \{\|\mathrm{i}(\partial_t \chi_5) u \mathrm{e}^{\tau\phi_2}\|^2_{H^{-s}(\Omega)} + \|(\triangle \chi_5) u \rho \mathrm{e}^{\tau\phi_2}\|^2_{H^{-s}(\Omega)} \\ &\quad + \|\mathrm{e}^{\tau\phi_2} u \nabla \chi_5 \cdot \nabla \rho\|^2_{H^{-s}(\Omega)} + \|2(\nabla \chi_5 \cdot \nabla u) \rho \mathrm{e}^{\tau\phi_2}\|^2_{H^{-s}(\Omega)}\} \mathrm{d}t \\ &\le C_5 \int_{-T}^{T} \{\|\mathrm{i}(\partial_t \chi_5) u \mathrm{e}^{\tau\phi_2}\|^2_{L^2(\Omega)} + \|(\triangle \chi_5) u \mathrm{e}^{\tau\phi_2}\|^2_{L^2(\Omega)} \\ &\quad + \|3 u \mathrm{e}^{\tau\phi_2} \nabla \chi_5 \cdot \nabla \rho\|^2_{L^2(\Omega)} + \sum_{i=1}^{n} \|2(\partial_i \chi_5) u \rho \mathrm{e}^{\tau\phi_2}\|^2_{H^{1-s}(\Omega)} \\ &\quad + \|2(\triangle \chi_5) u \rho \mathrm{e}^{\tau\phi_2}\|^2_{L^2(\Omega)} + \|2\tau(\nabla \chi_5 \cdot \nabla \phi_2) u \rho \mathrm{e}^{\tau\phi_2}\|^2_{L^2(\Omega)}\} \mathrm{d}t. \quad \text{(A.8)}\end{aligned}$$

Since $u = 0$ in $\{(x,t) \in Q;\ \mathrm{e}^{\lambda(|x-x_0|^2 - \beta t^2)} > \mathrm{e}^{\lambda(|x^1 - x_0|^2 - \beta |t^1|^2)}\}$, we see that there exists a constant $\epsilon > 0$ such that either $u = 0$ or all of the derivatives of χ_5 are equal to zero in $\{(x,t) \in Q;\ \phi_2(x,t) > \phi_2(x^1, t^1) - \epsilon\}$. Hence all the terms on the right-hand side of (A.8) are equal to zero in $\{(x,t) \in Q;\ \phi_2(x,t) > \phi_2(x^1, t^1) - \epsilon\}$. Then we obtain

$$\tau \int_{-T}^{T} \int_{\Omega} |\chi_5 u|^2 \mathrm{e}^{2\tau\phi_2} \mathrm{d}x \mathrm{d}t \le C_6 \tau \mathrm{e}^{2\tau \mathrm{e}^{\phi_2(x^1,t^1) - \epsilon}} \int_{-T}^{T} \|u\|^2_{H^{1-s}(\Omega)} \mathrm{d}t.$$

Consequently,

$$\mathrm{e}^{2\tau \mathrm{e}^{\phi_2(x^1,t^1)-\frac{\epsilon}{2}}} \int_{\mathcal{U}'\cap\{(x,t)\in Q;\phi_2(x,t)>\phi_2(x^1,t^1)-\frac{\epsilon}{2}\}} |u|^2 \mathrm{d}x\mathrm{d}t \le C_7 \mathrm{e}^{2\tau \mathrm{e}^{\phi_2(x^1,t^1)-\epsilon}} \int_{-T}^{T} \|u\|^2_{H^{1-s}(\Omega)} \mathrm{d}t.$$

By taking $\tau \to +\infty$, we obtain $u = 0$ in $\mathcal{U}' \cap \{(x,t) \in Q;\ \phi_2(x,t) > \phi_2(x^1,t^1) - \frac{\epsilon}{2}\}$. Hence we can choose a neighbourhood $\mathcal{U}$ of (x^1,t^1) such that $u = 0$ in $\mathcal{U}$. Thus, the proof of Theorem 1.2 is completed.

Acknowledgement The authors thank the anonymous referees for valuable comments.

References

[1] Baudouin, L. and Mercado, A., An inverse problem for Schrödinger equations with discontinuous main coefficient, *Appl. Anal.*, **87**, 2008, 1145–1165.

[2] Baudouin, L. and Puel, J. P., Uniqueness and stability in an inverse problem for the Schrödinger equation, *Inverse Problems*, **18**, 2001, 1537–1554.

[3] Bellassoued, M., Global logarithmic stability in inverse hyperbolic problem by arbitrary boundary observation, *Inverse Problems*, **20**, 2004, 1033–1052.

[4] Bellassoued, M. and Yamamoto, M., Logarithmic stability in determination of a coefficient in an acoustic equation by arbitrary boundary observation, *J. Math. Pures Appl.*, **85**, 2006, 193–224.

[5] Bukhgeim, A. L., Introduction to the Theory of Inverse Problems, VSP, Utrecht, 2000.

[6] Bukhgeim, A. L. and Klibanov, M. V., Global uniqueness of a class of multidimensional inverse problems, *Soviet Math. Dokl.*, **24**, 1981, 244–247.

[7] Cheng, J., Isakov, V., Yamamoto, M. and Zhou, Q., Lipschitz stability in the lateral Cauchy problem for elasticity system, *J. Math. Kyoto Univ.*, **43**, 2003, 475–501.

[8] Dehman, B., Gérard, P. and Lebeau, G., Stabilization and control for the nonlinear Schrödinger equation on a compact surface, *Math. Z.*, **254**, 2006, 729–749.

[9] Hörmander, L., Linear Partial Differential Operators, Springer-Verlag, Berlin, 1963.

[10] Hörmander, L., The Analysis of Partial Differential Operators, III, Springer-Verlag, Berlin, 1985.

[11] Imanuvilov, O. Y., On Carleman estimates for hyperbolic equations, *Asymptotic Analysis*, **32**, 2002, 185–220.

[12] Imanuvilov, O. Y., Isakov, V. and Yamamoto, M., An inverse problem for the dynamical Lamé system with two sets of boundary data, *Comm. Pure Appl. Math.*, **56**, 2003, 1366–1382.

[13] Imanuvilov, O. Y. and Yamamoto, M., Lipschitz stability in inverse parabolic problems by the Carleman estimate, *Inverse Problems*, **14**, 1998, 1229–1245.

[14] Imanuvilov, O. Y. and Yamamoto, M., Carleman estimate for a parabolic equation in a Sobolev spaces of negative order and its applications, Control of Nonlinear Distributed Parameter Systems, Dekker, New York, 2001, 113–137.

[15] Imanuvilov, O. Y. and Yamamoto, M., Global Lipschitz stability in an inverse hyperbolic problem by interior observations, *Inverse Problems*, **17**, 2001, 717–728.

[16] Imanuvilov, O. Y. and Yamamoto, M., Global uniqueness and stability in determining coefficients of wave equations, *Comm. Part. Diff. Eqs.*, **26**, 2001, 1409–1425.

[17] Imanuvilov, O. Y. and Yamamoto, M., Carleman inequalities for parabolic equations in Sobolev spaces of negative order and exact controllability for semilinear parabolic equations, *Publ. Res. Inst. Math. Sci.*, **39**, 2003, 227–274.

[18] Isakov, V., Inverse Source Problems, American Mathematical Society, Providence, RI, 1990.

[19] Isakov, V., Carleman type estimates in an anisotropic case and applications, *J. Diff. Eqs.*, **105**, 1993, 217–238.

[20] Isakov, V., Inverse Problems for Partial Differential Equations, Springer-Verlag, New York, 2006.

[21] Isakov, V. and Yamamoto, M., Carleman estimate with the Neumann boundary condition and its applications to the observability inequality and inverse hyperbolic problems, *Contemp. Math.*, **268**, 2000, 191–225.

[22] Kazemi, M. A. and Klibanov, M. V., Stability estimates for ill-posed Cauchy problems involving hyperbolic equations and inequalities, *Appl. Anal.*, **50**, 1993, 93–102.

[23] Kenig, C. E. and Sogge, C. D., A note on unique continuation for Schrödinger's operator, *Proc. Amer. Math. Soc.*, **103**, 1988, 543–546

[24] Khaĭdarov, A., Carleman estimates and inverse problems for second order hyperbolic equations, *Math. USSR Sbornik*, **58**, 1987, 267–277.

[25] Klibanov, M. V., Inverse problems in the 'large' and Carleman bounds, *Diff. Eqs.*, **20**, 1984, 755–760.

[26] Klibanov, M. V., Inverse problems and Carleman estimates, *Inverse Problems*, **8**, 1992, 575–596.

[27] Klibanov, M. V. and Malinsky, J., Newton-Kantorovich method for 3-dimensional potential inverse scattering problem and stability of the hyperbolic Cauchy problem with time dependent data, *Inverse Problems*, **7**, 1991, 577–595.

[28] Klibanov, M. V. and Timonov, A. A., Carleman Estimates for Coefficient Inverse Problems and Numerical Applications, VSP, Utrecht, 2004.

[29] Lasiecka, I. and Triggiani, R., Optimal regularity, exact controllability and uniform stabilization of Schrödinger equations with Dirichlet control, *Diff. Int. Eqs.*, **5**, 1992, 521–535.

[30] Lasiecka, I. and Triggiani, R., Carleman estimates and exact boundary controllability for a system of coupled, nonconservative second-order hyperbolic equations, Partial Differential Equation Methods in Control and Sharp Analysis, Lecture Notes in Pure and Appl. Math., **188**, Dekker, New York, 1997, 215–243.

[31] Lasiecka, I., Triggiani, R. and Zhang, X., Nonconservative wave equations with unobserved Neumann B. C.: global uniqueness and observability in one shot, *Contemp. Math.*, **268**, 2000, 227–325.

[32] Lasiecka, I., Triggiani, R. and Zhang, X., Global uniqueness, observability and stabilization of nonconservative Schrödinger equations via pointwise Carleman estimates, *J. Inverse Ill-Posed Probl.*, **12**, 2004, 43–123, 183–231.

[33] Li, S. and Yamamoto, M., Carleman estimate for Maxwell's equations in anisotropic media and the observability inequality, *J. Phys. Conf. Ser.*, **12**, 2005, 110–115.

[34] Lions, J. L. and Magenes, E., Non-homogeneous Boundary Value Problems and Applications, Springer-Verlag, Berlin, 1972.

[35] Machtyngier, E., Exact controllability for the Schrödinger equation, *SIAM J. Control Optim.*, **32**, 1994, 24–34.

[36] Mercado, A., Osses, A. and Rosier, L., Inverse problems for the Schrödinger equation via Carleman inequalities with degenerate weights, *Inverse Problems*, **24**, 2008, 015017.

[37] Phung, K. D., Observability and control of Schrodinger equations, *SIAM J. Control Optim.*, **40**, 2001, 211–230

[38] Ruiz, A., Unique continuation for weak solutions of the wave equation plus a potential, *J. Math. Pures Appl.*, **71**, 1992, 445–467.

[39] Runst, T. and Sickel, W., Sobolev Spaces of Fractional Order, Nemytskij Operators, and Nonlinear Partial Differential Equations, Walter de Gruyter, Berlin, 1996.

[40] Tataru, D., Unique continuation for solutions to PDE's; between Hörmander's theorem and Holmgren's theorem, *Comm. Part. Diff. Eqs.*, **20**, 1995, 855–884.

[41] Tataru, D., Carleman estimates and unique continuation for solutions to boundary value problem, *J. Math. Pures Appl.*, **75**, 1996, 367–408.

[42] Taylor, M. E., Pseudodifferential Operators and Nonlinear PDE, Birkhäuser, Boston, 1991.

[43] Yamamoto, M., Uniqueness and stability in multidimensional hyperbolic inverse problems, *J. Math. Pures Appl.*, **78**, 1999, 65–98.