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On Korn's Inequality

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(*Dedicated to Professor Roger Temam on the Occasion of his* 70*th Birthday*)

Abstract The author first reviews the classical Korn inequality and its proof. Following recent works of S. Kesavan, P. Ciarlet, Jr., and the author, it is shown how the Korn inequality can be recovered by an entirely different proof. This new proof hinges on appropriate weak versions of the classical Poincaré and Saint-Venant lemma. *In fine*, both proofs essentially depend on a crucial lemma of J. L. Lions, recalled at the beginning of this paper.

Keywords Korn inequality, J. L. Lions lemma
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1 A Lemma of J. L. Lions

Let Ω be an open subset of $\mathbb{R}^N$. A function $v \in L^2(\Omega)$ being identified with the distribution T_v that it defines, it is clear that $v \in L^2(\Omega)$ implies that

$$v \in H^{-1}(\Omega) \text{ and } \partial_i v \in H^{-1}(\Omega), \quad 1 \le i \le N,$$

since

$$|T_v(\varphi)| = \Big|\int_\Omega v\varphi \,\mathrm{d}x\Big| \le \|v\|_{0,\Omega}\|\varphi\|_{1,\Omega} \quad \text{for all } \varphi \in \mathcal{D}(\Omega),$$
$$|\partial_i T_v(\varphi)| = |-T_v(\partial_i\varphi)| = \Big|-\int_\Omega v\partial_i\varphi \,\mathrm{d}x\Big| \le \|v\|_{0,\Omega}\|\varphi\|_{1,\Omega} \quad \text{for all } \varphi \in \mathcal{D}(\Omega).$$

A domain in $\mathbb{R}^N$ is a bounded and connected open subset Ω of $\mathbb{R}^N$ with a Lipschitz-continuous boundary Γ, the set Ω being locally on the same side of Γ.

It is remarkable, but also remarkably difficult to prove, that, if Ω is a domain, a converse implication holds:

Theorem 1.1 (J. L. Lions Lemma) *Let Ω be a domain in $\mathbb{R}^N$. Let a distribution $v \in \mathcal{D}'(\Omega)$ be such that $\partial_i v \in H^{-1}(\Omega)$, $1 \le i \le N$. Then $v \in L^2(\Omega)$.*

This implication was first established, for domains with smooth boundaries and for functions $v \in L^2(\Omega)$, by Jacques-Louis Lions in 1958, as stated in Footnote (22) of [20]. Its first published proof by J. L. Lions appeared in [5]. Other proofs of the original lemma of J. L. Lions have since then been given, some extending it to genuine domains (i.e., with Lipschitz-continuous

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boundaries, as stated in Theorem 1.1), or to the more general spaces $W^{-1,q}(\Omega)$, $1 < q < \infty$ (see, e.g., [9, 22, 23]). The extension to distributions $v \in \mathcal{D}'(\Omega)$ (instead of distributions in $H^{-1}(\Omega)$ or $W^{-1,q}(\Omega)$) as stated in Theorem 1.1 is due to [2] and [3]. A counterexample to J. L. Lions lemma when Ω is not a domain is given in [8].

J. L. Lions lemma is of fundamental importance: it is in particular the key to proving many fundamental results, such as the Korn's inequality (see Section 2), the weak Poincaré lemma (see Section 3), or the weak Saint-Venant lemma (see Section 4).

Remark 1.1 Although Theorem 1.1 shall be referred to as *"the" lemma of J. L. Lions* in this article, there are other results of his that bear the same name in the literature, such as his "compactness lemmas" (see [17, Proposition 4.1] or [18, Section 5.2]), or his "singular perturbation lemma" (see [19, Lemma 5.1]).

2 Korn's Inequality

The norms in the space $L^2(\Omega)$ and $H^1(\Omega)$ are denoted $\|\cdot\|_{0,\Omega}$ and $\|\cdot\|_{1,\Omega}$, respectively.

Korn's inequality asserts that, given a domain Ω in $\mathbb{R}^N$, there exists a constant C depending solely on Ω such that

$$\Big(\sum_{i=1}^N \|v_i\|_{0,\Omega}^2 + \sum_{i,j=1}^N \|\partial_j v_i\|_{0,\Omega}^2\Big)^{\frac{1}{2}} \le C\Big(\sum_{i=1}^N \|v_i\|_{0,\Omega}^2 + \sum_{i,j=1}^N \|e_{ij}(\boldsymbol{v})\|_{0,\Omega}^2\Big)^{\frac{1}{2}}$$

for all vector fields $\boldsymbol{v} = (v_i)_{i=1}^N \in H^1(\Omega; \mathbb{R}^N)$, where

$$e_{ij}(\boldsymbol{v}) := \frac{1}{2}(\partial_j v_i + \partial_i v_j) \in L^2(\Omega), \quad 1 \le i,\, j \le N.$$

This inequality appeared for the first time, with a proof under the assumption that the vector fields $\boldsymbol{v}$ vanish on the boundary of Ω, in [14, 15, 16]. A second proof, this time under the assumption that the vector fields $\boldsymbol{v}$ satisfy $\int_\Omega \mathbf{curl}\, \boldsymbol{v}\, \mathrm{d}x = 0$, was then given in [7]. The first proof in full generality (based on the Calderón-Zygmund theory of singular integrals), is due to [11].

As is well-known, its special case $N = 3$ is crucial to establishing the existence and uniqueness of the solution to the weak formulation of the boundary value problem of linearized three-dimensional elasticity (as the key to proving the coerciveness of the associated bilinear form).

Korn's inequality thus provides an upper bound for the $L^2(\Omega)$-norms of *all* the N^2 partial derivatives $\partial_j v_i$ of a vector field $\boldsymbol{v} = (v_i) \in H^1(\Omega; \mathbb{R})$ in terms of the $L^2(\Omega)$-norms of *only* $\frac{N(N+1)}{2}$ particular linear combinations of these partial derivatives, namely the functions $e_{ij}(\boldsymbol{v}) = e_{ji}(\boldsymbol{v})$. This truly remarkable feature suggests that none of its various available proofs (see, e.g., the list of references provided in [12]) should be simple. For instance, the proof given below (which is well-known, but is reproduced here for the reader's convenience) is short and illuminating, but it depends on the lemma of J. L. Lions (see Theorem 1.1).

In what follows, spaces of vector-valued (resp. symmetric tensor-valued) fields are denoted by boldface (resp. special roman with s as a subscript) capitals, while the norms are denoted as in the scalar case. Thus, for instance,

$$\|\boldsymbol{v}\|_{1,\Omega} = \Big(\sum_{i=1}^{N} \|v_i\|_{1,\Omega}^2\Big)^{\frac{1}{2}} \qquad \text{for any } \boldsymbol{v} = (v_i) \in \boldsymbol{H}^1(\Omega) := H^1(\Omega; \mathbb{R}^N),$$

$$\|\boldsymbol{e}\|_{0,\Omega} = \Big(\sum_{i,j=1}^{N} \|e_{ij}\|_{0,\Omega}^2\Big)^{\frac{1}{2}} \qquad \text{for any } \boldsymbol{e} = (e_{ij}) \in \mathbb{L}_s^2(\Omega) := L^2(\Omega; \mathbb{S}^N),$$

where $\mathbb{S}^N$ denotes the space of all real $N \times N$ symmetric matrices.

Theorem 2.1 (Korn's Inequality in $\boldsymbol{H}^1(\Omega)$) *Let Ω be a domain in $\mathbb{R}^N$. Then there exists a constant $C = C(\Omega)$ such that*

$$\|\boldsymbol{v}\|_{1,\Omega} \le C(\|\boldsymbol{v}\|_{0,\Omega}^2 + \|\boldsymbol{e}(\boldsymbol{v})\|_{0,\Omega}^2)^{\frac{1}{2}} \quad \textit{for all } \boldsymbol{v} \in \boldsymbol{H}^1(\Omega),$$

where

$$\boldsymbol{e}(\boldsymbol{v}) := (e_{ij}(v)) \quad \textit{with } e_{ij}(\boldsymbol{v}) := \frac{1}{2}(\partial_j v_i + \partial_i v_j), \quad 1 \le i,\, j \le N.$$

Proof The proof given here follows that of Theorem 3.3 in Chapter 3 of [5].

(i) Define the space

$$\boldsymbol{E}(\Omega) := \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega);\ \boldsymbol{e}(\boldsymbol{v}) \in \mathbb{L}_s^2(\Omega)\}.$$

Then, equipped with the norm defined by $\|\boldsymbol{v}\| := (\|\boldsymbol{v}\|_{0,\Omega}^2 + \|\boldsymbol{e}(\boldsymbol{v})\|_{0,\Omega}^2)^{\frac{1}{2}}$, the space $\boldsymbol{E}(\Omega)$ is a Hilbert space.

The relation "$\boldsymbol{e}(\boldsymbol{v}) \in \mathbb{L}_s^2(\Omega)$" appearing in the definition of the space $\boldsymbol{E}(\Omega)$ is understood in the sense of distributions, i.e., it means that there exist functions in the space $L^2(\Omega)$, denoted by $e_{ij}(\boldsymbol{v}) = e_{ji}(\boldsymbol{v})$, such that

$$\int_\Omega e_{ij}(\boldsymbol{v})\varphi \,\mathrm{d}x = -\frac{1}{2}\int_\Omega (v_i \partial_j \varphi + v_j \partial_i \varphi)\,\mathrm{d}x \quad \text{for all } \varphi \in \mathcal{D}(\Omega).$$

Consider a Cauchy sequence $(\boldsymbol{v}^k)_{k=1}^\infty$ of elements $\boldsymbol{v}^k = (v_i^k)_{i=1}^N \in \boldsymbol{E}(\Omega)$. The definition of the norm $\|\cdot\|$ shows that there exist functions $v_i \in L^2(\Omega)$ and $e_{ij} \in L^2(\Omega)$ such that

$$v_i^k \to v_i \quad \text{in } L^2(\Omega), \quad \text{and} \quad e_{ij}(\boldsymbol{v}^k) \to e_{ij} \quad \text{in } L^2(\Omega), \quad \text{as } k \to \infty,$$

since the space $L^2(\Omega)$ is complete. Given a function $\varphi \in \mathcal{D}(\Omega)$, letting $k \to \infty$ in the relations

$$\int_\Omega e_{ij}(\boldsymbol{v}^k)\varphi \,\mathrm{d}x = -\frac{1}{2}\int_\Omega (v_i^k \partial_j \varphi + v_j^k \partial_i \varphi)\,\mathrm{d}x, \quad k \ge 1$$

shows that $e_{ij} = e_{ij}(\boldsymbol{v})$.

(ii) The two spaces $\boldsymbol{E}(\Omega)$ and $\boldsymbol{H}^1(\Omega)$ coincide.

Clearly, $\boldsymbol{H}^1(\Omega) \subset \boldsymbol{E}(\Omega)$. To prove the other inclusion, let $\boldsymbol{v} = (v_i)_{i=1}^N \in \boldsymbol{E}(\Omega)$. Then for $1 \le i, j, k \le N$,

$$\partial_k v_i \in H^{-1}(\Omega), \quad \partial_j(\partial_k v_i) = \{\partial_j e_{ik}(\boldsymbol{v}) + \partial_k e_{ij}(\boldsymbol{v}) - \partial_i e_{jk}(\boldsymbol{v})\} \in H^{-1}(\Omega),$$

since $w \in L^2(\Omega)$ implies $\partial_\ell w \in H^{-1}(\Omega)$, $1 \le \ell \le N$. Hence $\partial_k v_i \in L^2(\Omega)$ by the lemma of J. L. Lions (see Theorem 1.1), and thus $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$.

(iii) Korn's inequality.

The identity mapping ι from $\boldsymbol{H}^1(\Omega)$ equipped with $\|\cdot\|_{1,\Omega}$ into $\boldsymbol{E}(\Omega)$ equipped with $\|\cdot\|$ is injective, continuous (there clearly exists a constant c such that $\|\boldsymbol{v}\| \leq c\|\boldsymbol{v}\|_{1,\Omega}$ for all $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$), and surjective by (ii).

Banach open mapping theorem then shows that the inverse mapping ι^{-1} is also continuous, which is exactly what is expressed by Korn's inequality.

A counterexample showing that the Korn inequality does not necessarily hold if Ω is not a domain is found in [8].

Similar inequalities can be established on a domain Ω in $\mathbb{R}^N$, such as a Korn inequality in $\boldsymbol{W}^{1,p}(\Omega)$, which asserts that for each $1 < p < \infty$, there exists a constant C_p such that (see [9])

$$\|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C_p(\|\boldsymbol{v}\|^p_{\boldsymbol{L}^p(\Omega)} + \|\boldsymbol{e}(\boldsymbol{v})\|^p_{\mathbb{L}^p_s(\Omega)})^{\frac{1}{p}} \quad \text{for all } \boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega),$$

or a Korn inequality in $\boldsymbol{L}^2(\Omega)$, which asserts that there exists a constant C such that (see [1])

$$\|\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)} \leq C\big(\|\boldsymbol{v}\|^2_{\boldsymbol{H}^{-1}(\Omega)} + \|\boldsymbol{e}(\boldsymbol{v})\|^2_{\mathbb{H}^{-1}_s(\Omega)}\big)^{\frac{1}{2}} \quad \text{for all } \boldsymbol{v} \in \boldsymbol{L}^2(\Omega).$$

We now establish an equivalent form of the Korn inequality in $\boldsymbol{H}^1(\Omega)$, this time in a quotient space (see Theorem 2.3). For this purpose, we first need to identify those vector fields $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$ that satisfy $\boldsymbol{e}(\boldsymbol{v}) = \boldsymbol{0}$ in $\mathbb{L}^2_s(\Omega)$ (see Theorem 2.2).

Let $\mathbb{A}^N$ denote the space of all real $N \times N$ antisymmetric matrices.

Theorem 2.2 *Let Ω be a connected open subset $\mathbb{R}^N$. Then*

$$\{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega);\ \boldsymbol{e}(\boldsymbol{v}) = \boldsymbol{0} \text{ in } \Omega\} = \{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega); \text{there exist } \boldsymbol{A} \in \mathbb{A}^N \text{ and } \boldsymbol{c} \in \mathbb{R}^N$$
$$\text{such that } \boldsymbol{v}(x) = \boldsymbol{A}x + \boldsymbol{c} \text{ for almost all } x \in \mathbb{R}^N\}.$$

Proof We first note that, for each $1 \leq i, j, k \leq N$, any vector field $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$ satisfies

$$\int_\Omega (\partial_j v_i)\partial_k \varphi \,\mathrm{d}x = \int_\Omega \{e_{ij}(\boldsymbol{v})\partial_k\varphi + e_{ik}(\boldsymbol{v})\partial_j\varphi - e_{jk}(\boldsymbol{v})\partial_i\varphi\} \,\mathrm{d}x \quad \text{for all } \varphi \in \mathcal{D}(\Omega),$$

since the two sides of this relation are equal to $-\int_\Omega v_i \partial_{kj}\varphi \,\mathrm{d}x$. Consequently,

$$\boldsymbol{e}(\boldsymbol{v}) = 0 \text{ in } \Omega \quad \text{implies} \quad \int_\Omega (\partial_j v_i)\partial_k\varphi \,\mathrm{d}x = 0 \quad \text{for all } \varphi \in \mathcal{D}(\Omega).$$

There thus exist constants b_{ij}, $1 \leq i,\ j \leq N$, such that $\partial_j v_i(x) = b_{ij}$ for almost all $x \in \Omega$. In addition, $e_{ij}(\boldsymbol{v}) = 0$ implies that $b_{ij} = -b_{ji}$.

Let $w_i(x) := \sum_{j=1}^N b_{ij}x_j$ for all $x = (x_j) \in \Omega$, $1 \leq i \leq N$. Then

$$\int_\Omega v_i \partial_j\varphi \,\mathrm{d}x = -\int_\Omega (\partial_j v_i)\varphi \,\mathrm{d}x = -b_{ij}\int_\Omega \varphi \,\mathrm{d}x = -\int_\Omega (\partial_j w_i)\varphi \,\mathrm{d}x = \int_\Omega w_i\partial_j\varphi \,\mathrm{d}x$$

for all $\varphi \in \mathcal{D}(\Omega)$. There thus exist constants c_i such that $(v_i - w_i)(x) = c_i,\ 1 \leq i \leq N$.

We have therefore shown that, if a vector field $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$ satisfies $\boldsymbol{e}(\boldsymbol{v}) = 0$ in Ω, there exist an $N \times N$ antisymmetric matrix $\boldsymbol{A} = (b_{ij})$ and a vector $\boldsymbol{c} \in \mathbb{R}^N$ such that

$$\boldsymbol{v}(x) = \boldsymbol{A}x + \boldsymbol{c} \quad \text{for all } x \in \Omega.$$

Note in passing that part (i) of the above proof implies that, when $N = 3$, a vector field $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$ satisfies $\boldsymbol{e}(\boldsymbol{v}) = \boldsymbol{0}$ in $\mathbb{L}^2_s(\Omega)$ (if and) only if there exist two vectors $\boldsymbol{a} \in \mathbb{R}^3$ and $\boldsymbol{c} \in \mathbb{R}^3$ such that

$$\boldsymbol{v}(x) = \boldsymbol{a} \wedge \boldsymbol{ox} + \boldsymbol{c} \quad \text{for almost all } x \in \Omega.$$

When thought of as a "displacement field" of the set Ω, such a vector field is called an infinitesimal rigid displacement, "infinitesimal" reflecting its relation to a genuine "rigid deformation" of Ω.

Let $\mathbb{M}^N$ denote the space of all $N \times N$ real matrices. Given an open subset Ω of $\mathbb{R}^N$ and a smooth enough vector field $\boldsymbol{v} = (v_i) : \Omega \to \mathbb{R}^N$, the gradient of v is the matrix field $\boldsymbol{\nabla v} : \Omega \to \mathbb{M}^N$ defined by $(\boldsymbol{\nabla v})_{ij} = \partial_j v_i$. Hence the matrix field $\boldsymbol{e}(\boldsymbol{v}) : \Omega^{\sim} \to \mathbb{S}^N$ introduced in this section is also given by

$$\boldsymbol{e}(\boldsymbol{v}) = \frac{1}{2}(\boldsymbol{\nabla v}^{\mathrm{T}} + \boldsymbol{\nabla v}).$$

For this reason, $\boldsymbol{e}(\boldsymbol{v})$ is also called the symmetrized gradient of v and is sometimes (like in the next theorem) denoted by the more "operator-like" notation $\boldsymbol{\nabla}^{\sim}_s \boldsymbol{v}$.

Part (a) of the next theorem is known (see, e.g., [5]). A (different) proof is given here for the reader's convenience. A proof of part (b) was given in [4].

Theorem 2.3 (Korn's Inequality in the Quotient Space $\boldsymbol{H}^1(\Omega)/\mathrm{Ker}\boldsymbol{\nabla}_s$) *Let Ω be a domain in $\mathbb{R}^N$. Define the quotient space*

$$\dot{\boldsymbol{H}}(\Omega) := \boldsymbol{H}^1(\Omega)/\mathrm{Ker}\boldsymbol{\nabla}_s,$$

where

$$\mathrm{Ker}\boldsymbol{\nabla}_s := \Big\{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega);\ \boldsymbol{\nabla}_s \boldsymbol{v} := \frac{1}{2}(\boldsymbol{\nabla v}^{\mathrm{T}} + \boldsymbol{\nabla v}) = \boldsymbol{0} \ \text{in } \Omega\Big\}.$$

Equipped with the quotient norm $\|\cdot\|_{1,\Omega}$ defined by

$$\|\dot{\boldsymbol{v}}\|_{1,\Omega} := \inf_{\boldsymbol{r} \in \mathrm{Ker}\boldsymbol{\nabla}_s} \|\boldsymbol{v} + \boldsymbol{r}\|_{1,\Omega} \quad \text{for all } \dot{\boldsymbol{v}} \in \dot{\boldsymbol{H}}^1(\Omega),$$

the space $\boldsymbol{H}^1(\Omega)$ is thus a Hilbert space. Then:

(a) *There exists a constant $\dot{C} = \dot{C}(\Omega)$ such that the Korn's inequality in $\dot{\boldsymbol{H}}^1(\Omega)$ holds, viz.,*

$$\|\dot{\boldsymbol{v}}\|_{1,\Omega} \le \dot{C}\|\boldsymbol{e}(\dot{\boldsymbol{v}})\|_{0,\Omega} \quad \text{for all } \dot{\boldsymbol{v}} \in \dot{\boldsymbol{H}}^1(\Omega),$$

where $\boldsymbol{e}(\dot{\boldsymbol{v}}) := \boldsymbol{e}(\boldsymbol{w})$ for any $\boldsymbol{w} \in \dot{\boldsymbol{v}}$.

(b) *Conversely, the Korn inequality in $\dot{\boldsymbol{H}}^1(\Omega)$ implies the Korn inequality in $\boldsymbol{H}^1(\Omega)$ (see Theorem* 2.1*).*

Proof By Theorem 2.2, the space $\mathrm{Ker}\boldsymbol{\nabla}_s$ is finite-dimensional and its dimension is $M := \frac{N(N+1)}{2}$.

By the Hahn-Banach theorem in a normed vector space, there exist M continuous linear forms ℓ_α on $\boldsymbol{H}^1(\Omega)$, $1 \le \alpha \le M$, with the following property: An element $\boldsymbol{r} \in \mathrm{Ker}\boldsymbol{\nabla}_s$ is equal to $\boldsymbol{0}$ if and only if $\ell_\alpha(\boldsymbol{r}) = 0$, $1 \le \alpha \le M$. We then claim that there exists a constant D such that

$$\|\boldsymbol{v}\|_{1,\Omega} \le D\Big(\|\boldsymbol{e}(\boldsymbol{v})\|_{0,\Omega} + \sum_{\alpha=1}^{M} |\ell_\alpha(\boldsymbol{v})|\Big) \quad \text{for all } \boldsymbol{v} \in \boldsymbol{H}^1(\Omega).$$

This inequality in turn implies Korn's inequality in $\dot{\boldsymbol{H}}^1(\Omega)$: Given any $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$, let $\boldsymbol{r}(\boldsymbol{v}) \in \mathrm{Ker}\boldsymbol{\nabla}_s$ be such that $\ell_\alpha(\boldsymbol{v} + \boldsymbol{r}(\boldsymbol{v})) = 0$, $1 \le \alpha \le M$; then

$$\|\dot{\boldsymbol{v}}\|_{1,\Omega} = \inf_{\boldsymbol{r} \in \mathrm{Ker}\boldsymbol{\nabla}_s} \|\boldsymbol{v} + \boldsymbol{r}\|_{1,\Omega} \le \|\boldsymbol{v} + \boldsymbol{r}(\boldsymbol{v})\|_{1,\Omega} \le D\|\boldsymbol{e}(\boldsymbol{v})\|_{0,\Omega} = D\|\boldsymbol{e}(\dot{\boldsymbol{v}})\|_{0,\Omega}.$$

To establish the existence of such a constant D, assume the contrary. Then there exist $\boldsymbol{v}^k \in \boldsymbol{H}^1(\Omega)$, $k \ge 1$, such that

$$\|\boldsymbol{v}^k\|_{1,\Omega} = 1 \quad \text{for all } k \ge 1 \quad \text{and} \quad \Big(\|\boldsymbol{e}(\boldsymbol{v}^k)\|_{0,\Omega} + \sum_{\alpha=1}^{M} |\ell_\alpha(\boldsymbol{v}^k)|\Big) \underset{k\to\infty}{\longrightarrow} 0.$$

By the Rellich-Kondrašov theorem, there exists a subsequence $(\boldsymbol{v}^\ell)_{\ell=1}^\infty$ that converges in $\boldsymbol{L}^2(\Omega)$. Since the sequence $(\boldsymbol{e}(\boldsymbol{v}^\ell))_{\ell=1}^\infty$ also converges in $\mathbb{L}^2_s(\Omega)$, the subsequence $(\boldsymbol{v}^\ell)_{\ell=1}^\infty$ is a Cauchy sequence with respect to the norm $\boldsymbol{v} \to \{\|\boldsymbol{v}\|^2_{0,\Omega} + \|\boldsymbol{e}(\boldsymbol{v})\|^2_{0,\Omega}\}^{\frac{1}{2}}$, hence also with respect to the norm $\|\cdot\|_{1,\Omega}$ by Korn's inequality in $\boldsymbol{H}^1(\Omega)$ (see Theorem 2.1). Consequently, there exists $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$ such that

$$\|\boldsymbol{v}^\ell - \boldsymbol{v}\|_{1,\Omega} \underset{\ell\to\infty}{\longrightarrow} 0.$$

But we have $\boldsymbol{v} = \boldsymbol{0}$ since $\boldsymbol{e}(\boldsymbol{v}) = 0$ and $\ell_\alpha(\boldsymbol{v}) = 0$, $1 \le \alpha \le M$, in contradiction with the relations $\|\boldsymbol{v}^\ell\|_{1,\Omega} = 1$ for all $\ell \ge 1$. This proves (a).

We next show that, conversely, Korn's inequality in the quotient space $\dot{\boldsymbol{H}}^1(\Omega)$ implies Korn's inequality in the space $\boldsymbol{H}^1(\Omega)$.

Assume the contrary. Then there exist $\boldsymbol{v}^k \in \boldsymbol{H}^1(\Omega)$, $k \ge 1$, such that

$$\|\boldsymbol{v}^k\|_{1,\Omega} = 1 \quad \text{for all } k \ge 1 \quad \text{and} \quad (\|\boldsymbol{v}^k\|_{0,\Omega} + \|\boldsymbol{e}(\boldsymbol{v}^k)\|_{0,\Omega}) \underset{k\to\infty}{\longrightarrow} 0.$$

Let $\boldsymbol{r}^k \in \mathrm{Ker}\boldsymbol{\nabla}_s$ denote for each $k \ge 1$ the projection of $\boldsymbol{v}^k$ on $\mathrm{Ker}\boldsymbol{\nabla}_s$ with respect to the inner-product of $\boldsymbol{H}^1(\Omega)$, which thus satisfies

$$\|\boldsymbol{v}^k - \boldsymbol{r}^k\|_{1,\Omega} = \inf_{\boldsymbol{r} \in \mathrm{Ker}\boldsymbol{\nabla}_s} \|\boldsymbol{v}^k - \boldsymbol{r}\|_{1,\Omega} \quad \text{and} \quad \|\boldsymbol{v}^k\|^2_{1,\Omega} = \|\boldsymbol{v}^k - \boldsymbol{r}^k\|^2_{1,\Omega} + \|\boldsymbol{r}^k\|^2_{1,\Omega}.$$

The space $\mathrm{Ker}\boldsymbol{\nabla}_s$ being finite-dimensional, the inequalities $\|\boldsymbol{r}^k\|_{1,\Omega} \le 1$ for all $k \ge 1$ imply the existence of a subsequence $(\boldsymbol{r}^\ell)_{\ell=1}^\infty$ that converges in $\boldsymbol{H}^1(\Omega)$ to an element $\boldsymbol{r} \in \mathrm{Ker}\boldsymbol{\nabla}_s$. Besides, Korn's inequality in $\dot{\boldsymbol{H}}^1(\Omega)$ implies that $\|\boldsymbol{v}^\ell - \boldsymbol{r}^\ell\|_{1,\Omega} \underset{\ell\to\infty}{\longrightarrow} 0$, so that $\|\boldsymbol{v}^\ell - \boldsymbol{r}\|_{1,\Omega} \underset{\ell\to\infty}{\longrightarrow} 0$. Hence $\|\boldsymbol{v}^\ell - \boldsymbol{r}\|_{0,\Omega} \underset{\ell\to\infty}{\longrightarrow} 0$, which forces $\boldsymbol{r}$ to be $\boldsymbol{0}$, since $\|\boldsymbol{v}^\ell\|_{0,\Omega} \to 0$ on the other hand. We thus reach the conclusion that $\|\boldsymbol{v}^\ell\|_{1,\Omega} \to 0$, a contradiction.

3 Poincaré Lemma: the Classical and Weak Versions

Given an open subset Ω of $\mathbb{R}^N$, consider the linear operator $\mathbf{grad} : \mathcal{C}^2(\Omega) \to \mathcal{C}^1(\Omega; \mathbb{R}^N)$ defined by

$$p \in \mathcal{C}^2(\Omega) \to \mathbf{grad}\, p := (\partial_i p) \in \mathcal{C}^1(\Omega; \mathbb{R}^N).$$

A natural question then arises, as to whether this linear operator is *invertible*, i.e., whether, given a vector field $\boldsymbol{h} = (h_i) \in \mathcal{C}^1(\Omega; \mathbb{R}^N)$, there exists a function $p \in \mathcal{C}^2(\Omega)$ such that

$$\partial_i p = h_i \quad \text{in } \Omega,\ 1 \le i \le N.$$

Since then $\partial_{ij}p = \partial_{ji}p$ if this is the case, it is clear that the functions h_i must *necessarily* satisfy the compatibility conditions

$$\partial_i h_j - \partial_j h_i = 0 \quad \text{in } \mathcal{C}(\Omega),\ \ 1 \le i, j \le N,$$

or equivalently, in vector form, $\mathbf{curl}\ \boldsymbol{h} = \mathbf{0}$ in $\mathcal{C}(\Omega; \mathbb{R}^N)$.

These necessary conditions become sufficient if the open set Ω is simply-connected: this is the essence of the well-known Poincaré lemma; for a proof, see, e.g., Section 3.6 in [6]. This classical result is recalled in Theorem 3.1 below ("classical", as opposed to the "weak" form of this lemma, established in Theorem 3.2).

Theorem 3.1 (Poincaré Lemma: Classical Version) *Let Ω be a simply-connected open subset of $\mathbb{R}^N$, and let there be given functions $h_i \in \mathcal{C}^1(\Omega)$, $1 \le i \le N$, that satisfy*

$$\partial_j h_i = \partial_i h_j \quad \textit{in } \Omega,\ 1 \le i, j \le N.$$

Then there exists a function $p \in \mathcal{C}^2(\Omega)$ such that

$$\partial_i p = h_i \quad \textit{in } \Omega,\ 1 \le i \le N.$$

Besides, any other solution $\widetilde{p} \in \mathcal{C}^2(\Omega)$ to the equations $\partial_i \widetilde{p} = h_i$ in Ω, $1 \le i \le N$, is of the form $\widetilde{p} = p + C$ for some constant $\mathcal{C}$.

Our second application of J. L. Lions lemma will now consist in showing that Poincaré's lemma still holds under a substantially weaker regularity assumption, viz., that h_i, $1 \le i \le N$, be only distributions in $H^{-1}(\Omega)$. This result is due to [4]. The simpler proof given here is due to [13].

Theorem 3.2 (Poincaré Lemma: Weak Version) *Let Ω be a simply-connected domain in $\mathbb{R}^N$ and let there be given distributions $h_i \in H^{-1}(\Omega)$, $1 \le i \le N$, that satisfy*

$$\partial_j h_i = \partial_i h_j \quad \textit{in } H^{-2}(\Omega),\ 1 \le i, j \le N.$$

Then there exists a function $p \in L^2(\Omega)$ such that

$$\partial_i p = h_i \quad \textit{in } H^{-1}(\Omega),\ 1 \le i \le N.$$

Besides, any other solution $\widetilde{p} \in L^2(\Omega)$ to the equations $\partial_i \widetilde{p} = h_i$ in $H^{-1}(\Omega)$, $1 \le i \le N$, is of the form $\widetilde{p} = p + C$, where C is a constant.

Proof We have to show that, if $\boldsymbol{h} \in \boldsymbol{H}^{-1}(\Omega)$ satisfies $\mathbf{curl}\ \boldsymbol{h} = \mathbf{0}$ in $\boldsymbol{H}^{-2}(\Omega)$, then there exists $p \in L^2(\Omega)$ such that $\boldsymbol{h} = \mathbf{grad}\, p$ in $\boldsymbol{H}^{-1}(\Omega)$. To this end, we proceed in two stages.

(i) By Theorem 5.1 in Chapter 1 of [10], there exist a vector field $\boldsymbol{u} \in \boldsymbol{H}_0^1(\Omega)$ and a function $\pi \in L^2(\Omega)$ such that

$$\begin{aligned} -\Delta \boldsymbol{u} + \mathbf{grad}\, \pi = \boldsymbol{h} \quad & \text{in } \boldsymbol{H}^{-1}(\Omega), \\ \operatorname{div} \boldsymbol{u} = 0 \quad & \text{in } L^2(\Omega) \end{aligned}$$

(the assumptions that Ω is simply-connected and that $\mathbf{curl}\ \boldsymbol{h} = \mathbf{0}$ in $\boldsymbol{H}^{-2}(\Omega)$ are not needed at this stage).

(ii) The assumption $\mathbf{curl}\ \boldsymbol{h} = \mathbf{0}$ in $\boldsymbol{H}^{-2}(\Omega)$, together with the relation

$$\mathbf{curl}\ \mathbf{grad}\, \pi = \mathbf{0}, \quad \text{in } \boldsymbol{\mathcal{D}}'(\Omega) \text{ for any } \pi \in \mathcal{D}'(\Omega),$$

imply that

$$\Delta(\mathbf{curl}\, \boldsymbol{u}) = \mathbf{curl}\, (\Delta \boldsymbol{u}) = \mathbf{curl}\, \boldsymbol{h} - \mathbf{curl}\, \mathbf{grad}\, \pi = \mathbf{0}.$$

Since $\mathbf{curl}\, \boldsymbol{u} \in \boldsymbol{L}^2(\Omega) \subset \boldsymbol{L}^1_{\mathrm{loc}}(\Omega)$, the hypoellipticity of Δ (see, e.g., [24]) shows that $\mathbf{curl}\, \boldsymbol{u} \in \boldsymbol{C}^\infty(\Omega)$, so that $(\partial_j u_i - \partial_i u_j) \in \mathcal{C}^\infty(\Omega)$ for all $1 \le i, j \le N$. Therefore

$$\partial_j(\partial_j u_i - \partial_i u_j) = \Delta u_i - \partial_i(\mathrm{div}\, \boldsymbol{u}) = \Delta u_i \in \mathcal{C}^\infty(\Omega), \quad 1 \le i \le N,$$

since $\mathrm{div}\, \boldsymbol{u} = 0$.

Since $\Delta \boldsymbol{u} \in \mathcal{C}^\infty(\Omega)$ and $\mathbf{curl}\, \Delta \boldsymbol{u} = 0$ in Ω, and Ω is simply-connected, the classical Poincaré lemma (see Theorem 3.1) can be applied, showing that there exists a function $\widetilde{p} \in \mathcal{C}^\infty(\Omega) \subset L^1_{\mathrm{loc}}(\Omega) \subset \mathcal{D}'(\Omega)$ such that

$$\mathbf{grad}\, \widetilde{p} = \Delta \boldsymbol{u} = \mathbf{grad}\, \pi - \boldsymbol{h}, \quad \text{in } \boldsymbol{H}^{-1}(\Omega).$$

Since the distribution $p := \pi - \widetilde{p} \in \mathcal{D}'(\Omega)$ is such that

$$\mathbf{grad}\, p = \mathbf{grad}\, \pi - \mathbf{grad}\, \widetilde{p} = \boldsymbol{h} \in \boldsymbol{H}^{-1}(\Omega),$$

J. L. Lions lemma shows that p is in effect a function in $L^2(\Omega)$.

Let $\pi \in L^2(\Omega)$ be such that $\mathbf{grad}\, \pi = \mathbf{0}$ in $\boldsymbol{H}^{-1}(\Omega)$, which means that

$$\partial_i \pi(\varphi) := -\int_\Omega \pi \partial_i \varphi \,\mathrm{d}x = 0 \quad \text{for all } \varphi \in \mathcal{D}(\Omega),\ 1 \le i \le N.$$

Since the open set Ω is connected, the function π is a constant. Hence the function $p \in L^2(\Omega)$ found above is unique modulo the addition of a constant.

Together with the hypoellipticity of Δ, J. L. Lions lemma thus plays a key role for proving the weak Poincaré lemma. Note that, as shown in [13], the weak Poincaré lemma conversely provides a very simple proof of J. L. Lions lemma (at least for simply-connected domains, but then the extension to non-simply-connected domains is easy). Note also that Poincaré lemma was shown to hold in the even weaker sense of distributions in [21].

4 Saint-Venant Lemma: the Classical and Weak Versions

This section is the "matrix analog" of Section 3, the vector gradient operator

$$\mathbf{grad} : p \in \mathcal{D}'(\Omega) \to \mathbf{grad}\, p \in \mathcal{D}'(\Omega; \mathbb{R}^N)$$

being "replaced" by the matrix symmetrized gradient operator

$$\boldsymbol{\nabla}_s : \boldsymbol{v} \in \mathcal{D}'(\Omega; \mathbb{R}^N) \to \boldsymbol{\nabla}_s \boldsymbol{v} := \boldsymbol{e}(\boldsymbol{v}) = \frac{1}{2}(\boldsymbol{\nabla} \boldsymbol{v}^{\mathrm{T}} + \boldsymbol{\nabla} \boldsymbol{v}) \in \mathcal{D}'(\Omega; \mathbb{S}^N).$$

This explains why the discourse follows the same lines as in Section 3, and why Theorems 4.1 and 4.2 below again crucially depend on Poincaré lemma, in its classical and weak versions.

Given an open subset of $\mathbb{R}^N$, consider the linear operator from the space $\mathcal{C}^3(\Omega; \mathbb{R}^N)$ into the space $\mathcal{C}^2(\Omega; \mathbb{S}^N)$ (these regularity assumptions insure that the compatibility relations satisfied by the functions e_{ij} make sense in the space $\mathcal{C}(\Omega)$) defined by

$$\boldsymbol{v} = (v_i) \in \mathcal{C}^3(\Omega; \mathbb{R}^N) \to \boldsymbol{e}(\boldsymbol{v}) = (e_{ij}(\boldsymbol{v})) \in \mathcal{C}^2(\Omega; \mathbb{S}^N),$$

where

$$e_{ij}(\boldsymbol{v}) := \frac{1}{2}(\partial_j v_i + \partial_i v_j) = e_{ji}(\boldsymbol{v}), \quad 1 \le i, j \le N.$$

A natural question therefore arises, as to whether this linear operator is *invertible*, i.e., whether, given a matrix field $\boldsymbol{e} = (e_{ij}) \in \mathcal{C}^2(\Omega; \mathbb{S}^N)$, there exists a vector field $\boldsymbol{v} \in \mathcal{C}^3(\Omega; \mathbb{R}^N)$ such that

$$\frac{1}{2}(\partial_j v_i + \partial_i v_j) = e_{ij} \quad \text{in } \Omega,\ 1 \le i, j \le N.$$

If this is the case, it is then immediately verified that the functions $e_{ij} = e_{ji} \in \mathcal{C}^2(\Omega)$ must *necessarily* satisfy the Saint-Venant compatibility relations, so named after Adhémar-Jean-Claude Barré de Saint-Venant, who published these relations in 1864:

$$\partial_{\ell j} e_{ik} + \partial_{ki} e_{j\ell} - \partial_{\ell i} e_{jk} - \partial_{kj} e_{i\ell} = 0 \quad \text{in } \mathcal{C}(\Omega),\ 1 \le i, j, k, \ell \le N.$$

It is remarkable that these necessary conditions become *sufficient* if the open set Ω is *simply-connected*. The next proof is well-known.

Theorem 4.1 (Saint-Venant Lemma: Classical Version) *Let Ω be a simply-connected open subset of $\mathbb{R}^N$, and let there be given functions $e_{ij} = e_{ji} \in \mathcal{C}^2(\Omega)$, $1 \le i, j \le N$, that satisfy the Saint-Venant compatibility relations*

$$\partial_{\ell j} e_{ik} + \partial_{ki} e_{j\ell} - \partial_{\ell i} e_{jk} - \partial_{kj} e_{i\ell} = 0 \quad \text{in } \Omega,\ 1 \le i, j, k, \ell \le N.$$

Then there exists a vector field $\boldsymbol{v} = (v_i) \in \mathcal{C}^3(\Omega; \mathbb{R}^N)$ such that

$$\frac{1}{2}(\partial_j v_i + \partial_i v_j) = e_{ij} \quad \text{in } \Omega,\ 1 \le i, j \le N.$$

Besides, any other solution $\widetilde{\boldsymbol{v}} = (\widetilde{v}_i) \in \mathcal{C}^3(\Omega; \mathbb{R}^N)$ to the equations

$$\frac{1}{2}(\partial_j \widetilde{v}_i + \partial_i \widetilde{v}_j) = e_{ij} \quad \text{in } \Omega,\ 1 \le i, j \le N,$$

is of the form $\widetilde{\boldsymbol{v}}(x) = \boldsymbol{v}(x) + \boldsymbol{A}x + \boldsymbol{c}$, $x \in \Omega$, for some $N \times N$ antisymmetric matrix $\boldsymbol{A}$ and vector $\boldsymbol{c} \in \mathbb{R}^N$.

Proof It is implicitly understood that the various relations found in this proof hold for all the values $1, 2, \cdots, N$ of the Latin indices appearing in them. The Saint-Venant compatibility relations may be equivalently rewritten as

$$\partial_\ell h_{ijk} = \partial_k h_{ij\ell} \quad \text{in } \mathcal{C}(\Omega) \text{ with } h_{ijk} := \partial_j e_{ik} - \partial_i e_{jk} \in \mathcal{C}^1(\Omega).$$

Hence the classical Poincaré lemma (see Theorem 3.1) shows that there exist functions $p_{ij} \in \mathcal{C}^2(\Omega)$, unique up to additive constants, such that

$$\partial_k p_{ij} = h_{ijk} = \partial_j e_{ik} - \partial_i e_{jk} \quad \text{in } \mathcal{C}^1(\Omega).$$

Besides, since $\partial_k p_{ij} = -\partial_k p_{ji}$ in $\mathcal{C}^1(\Omega)$, we have the freedom of choosing the functions p_{ij} in such a way that $p_{ij} + p_{ji} = 0$ in $\mathcal{C}^2(\Omega)$.

Noting that the functions $q_{ij} := (e_{ij} + p_{ij}) \in \mathcal{C}^2(\Omega)$ satisfy

$$\partial_k q_{ij} = \partial_k e_{ij} + \partial_k p_{ij} = \partial_k e_{ij} + \partial_j e_{ik} - \partial_i e_{jk} = \partial_j e_{ik} + \partial_j p_{ik} = \partial_j q_{ik} \quad \text{in } \mathcal{C}^1(\Omega),$$

we again resort to the classical Poincaré lemma to assert the existence of functions $v_i \in \mathcal{C}^3(\Omega)$, unique up to additive constants, such that

$$\partial_j v_i = q_{ij} = e_{ij} + p_{ij} \quad \text{in } \mathcal{C}^2(\Omega).$$

Consequently,

$$\frac{1}{2}(\partial_j v_i + \partial_i v_j) = e_{ij} + \frac{1}{2}(p_{ij} + p_{ji}) = e_{ij} \quad \text{in } \mathcal{C}^2(\Omega),$$

as required. That all other solutions are of the indicated form is established like in the proof of Theorem 2.2.

Using the weak version of Poincaré lemma, hence *in fine* using J. L. Lions lemma, we now show that the Saint-Venant lemma still holds under a substantially weaker regularity assumption, viz., that e_{ij}, $1 \le i,j \le N$, be only functions in $L^2(\Omega)$. This result is due to [4].

This "weak version" of the Saint-Venant lemma will in turn provide a new proof of Korn's inequality (see Theorem 4.4).

Theorem 4.2 (Saint-Venant Lemma: Weak Version) *Let Ω be a simply-connected domain in $\mathbb{R}^N$. Let $\boldsymbol{e} = (e_{ij}) \in \mathbb{L}^2_s(\Omega)$ be a symmetric matrix field that satisfies the Saint-Venant compatibility relations:*

$$\partial_{\ell j} e_{ik} + \partial_{ki} e_{j\ell} - \partial_{\ell i} e_{jk} - \partial_{kj} e_{i\ell} = 0 \quad \textit{in } H^{-2}(\Omega),\ 1 \le i,j,k,\ell \le N.$$

Then there exists a vector field $\boldsymbol{v} = (v_i) \in \boldsymbol{H}^1(\Omega)$ such that

$$e_{ij} = \frac{1}{2}(\partial_j v_i + \partial_i v_j) \quad \textit{in } L^2(\Omega),\ 1 \le i,j \le N.$$

Besides, all other solutions $\widetilde{\boldsymbol{v}} = (\widetilde{v}_i) \in \boldsymbol{H}^1(\Omega)$ to the equations $e_{ij} = \frac{1}{2}(\partial_j \widetilde{v}_i + \partial_i \widetilde{v}_j)$ are of the form

$$\widetilde{\boldsymbol{v}}(x) = \boldsymbol{v}(x) + \boldsymbol{A}x + \boldsymbol{c} \quad \textit{for almost } x \in \Omega,$$

for some $N \times N$ antisymmetric matrix $\boldsymbol{A}$ and vector $\boldsymbol{c} \in \mathbb{R}^N$.

Proof The proof is analogous to that of Theorem 4.1, save that it is now the *weak* version of Poincaré lemma (see Theorem 3.2) that is used twice: first, to show that there exist functions $p_{ij} \in L^2(\Omega)$, unique up to additive constants, that satisfy

$$\partial_k p_{ij} = h_{ijk} = \partial_j e_{ik} - \partial_i e_{jk} \quad \text{in } H^{-1}(\Omega),$$

and, second, to show that there exist functions $v_i \in H^1(\Omega)$, again unique up to additive constants, that satisfy $\partial_j v_i = q_{ij} = e_{ij} + p_{ij}$ in $L^2(\Omega)$.

Consequently,

$$\frac{1}{2}(\partial_j v_i + \partial_i v_j) = e_{ij} + \frac{1}{2}(p_{ij} + p_{ji}) = e_{ij} \quad \text{in } L^2(\Omega),$$

as desired. That all other solutions are of the indicated form follows from Theorem 2.2.

Let a symmetric matrix field $\boldsymbol{e} = (e_{ij}) \in \mathbb{L}^2_s(\Omega)$ satisfy

$$\partial_{\ell j} e_{ik} + \partial_{ki} e_{j\ell} - \partial_{\ell i} e_{jk} - \partial_{kj} e_{i\ell} = 0 \quad \text{in } H^{-2}(\Omega),\ 1 \le i,j,k,\ell \le N,$$

i.e., the weak form of Saint Venant's compatibility relations. By Theorem 4.2, there then exists a unique equivalence class $\dot{\boldsymbol{v}} \in \dot{\boldsymbol{H}}^1(\Omega) = \boldsymbol{H}^1(\Omega)/\mathrm{Ker}\boldsymbol{\nabla}_s$ such that $\boldsymbol{e} = \boldsymbol{e}(\dot{\boldsymbol{v}})$ in $\mathbb{L}^2_s(\Omega)$.

We now show that the mapping $\boldsymbol{\mathcal{F}} : \boldsymbol{e} \to \dot{\boldsymbol{v}}$ defined in this fashion is an isomorphism between appropriate Hilbert spaces.

Theorem 4.3 *Let Ω be a simply-connected domain in $\mathbb{R}^N$. Define the space*

$$\mathbb{E}_s(\Omega) := \{\boldsymbol{e} = (e_{ij}) \in \mathbb{L}^2_s(\Omega);\ \partial_{\ell j} e_{ik} + \partial_{ki} e_{j\ell} - \partial_{\ell i} e_{jk} - \partial_{kj} e_{i\ell} = 0 \\ \text{in } H^{-2}(\Omega), 1 \leq i, j, k, \ell \leq N\},$$

and let

$$\boldsymbol{\mathcal{F}} : \mathbb{E}_s(\Omega) \to \dot{\boldsymbol{H}}^1(\Omega)$$

be the linear mapping defined for each $\boldsymbol{e} \in \mathbb{E}_s(\Omega)$ by $\boldsymbol{\mathcal{F}}(\boldsymbol{e}) = \dot{\boldsymbol{v}}$, where $\dot{\boldsymbol{v}}$ is the unique element in the quotient space $\dot{\boldsymbol{H}}^1(\Omega)$ that satisfies $\boldsymbol{e}(\dot{\boldsymbol{v}}) = \boldsymbol{e}$ in $\mathbb{L}^2_s(\Omega)$ (see Theorem 4.2). Then $\boldsymbol{\mathcal{F}}$ is an isomorphism between the Hilbert spaces $\mathbb{E}_s(\Omega)$ and $\dot{\boldsymbol{H}}^1(\Omega)$.

Proof Clearly, $\mathbb{E}_s(\Omega)$ is a Hilbert space as a closed subspace of $\mathbb{L}^2_s(\Omega)$. The mapping $\boldsymbol{\mathcal{F}}$ is injective since $\boldsymbol{\mathcal{F}}(\boldsymbol{e}) = \dot{\boldsymbol{0}}$ means that $\boldsymbol{e} = \boldsymbol{e}(\dot{\boldsymbol{0}}) = \boldsymbol{0}$, and surjective since, given any $\dot{\boldsymbol{v}} \in \dot{\boldsymbol{H}}^1(\Omega)$, the matrix field $\boldsymbol{e}(\dot{\boldsymbol{v}}) \in \mathbb{L}^2_s(\Omega)$ necessarily satisfies $\partial_{\ell j} e_{ik} + \partial_{ki} e_{j\ell} - \partial_{\ell i} e_{jk} - \partial_{kj} e_{i\ell} = 0$ in $H^{-2}(\Omega)$, $1 \leq i, j, k, \ell \leq N$.

Finally, the inverse mapping

$$\boldsymbol{\mathcal{F}}^{-1} : \dot{\boldsymbol{v}} \in \dot{\boldsymbol{H}}^1(\Omega) \to \boldsymbol{e}(\dot{\boldsymbol{v}}) \in \mathbb{E}_s(\Omega)$$

is continuous, since there evidently exists a constant c such that

$$\|\boldsymbol{e}(\dot{\boldsymbol{v}})\|_{0,\Omega} = \|\boldsymbol{e}(\boldsymbol{v} + \boldsymbol{r})\|_{0,\Omega} \leq c\|\boldsymbol{v} + \boldsymbol{r}\|_{1,\Omega}$$

for any $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$ and any $\boldsymbol{r} \in \mathrm{Ker}\boldsymbol{\nabla}_s$, so that

$$\|\boldsymbol{e}(\dot{\boldsymbol{v}})\|_{0,\Omega} \leq c \inf_{\boldsymbol{r} \in \mathrm{Ker}\boldsymbol{\nabla}_s} \|\boldsymbol{v} + \boldsymbol{r}\|_{1,\Omega} = c\|\dot{\boldsymbol{v}}\|_{1,\Omega}.$$

The conclusion thus follows from Banach open mapping theorem.

Remarkably, the Korn's inequalities of Section 2 can now be very simply recovered from Theorem 4.3.

Theorem 4.4 *That the mapping $\boldsymbol{\mathcal{F}} : \mathbb{E}_s(\Omega) \to \dot{\boldsymbol{H}}^1(\Omega)$ is an isomorphism implies Korn's inequalities in both spaces $\boldsymbol{H}^1(\Omega)$ and $\dot{\boldsymbol{H}}^1(\Omega)$ (see Theorems 2.1 and 2.3).*

Proof Since $\boldsymbol{\mathcal{F}}$ is an isomorphism by Theorem 4.3, there exists a constant $\dot{C}$ such that

$$\|\boldsymbol{\mathcal{F}}(\boldsymbol{e})\|_{1,\Omega} \leq \dot{C}\|\boldsymbol{e}\|_{0,\Omega} \quad \text{for all } \boldsymbol{e} \in \mathbb{E}_s(\Omega),$$

or equivalently such that

$$\|\dot{\boldsymbol{v}}\|_{1,\Omega} \leq \dot{C}\|\boldsymbol{e}(\dot{\boldsymbol{v}})\|_{0,\Omega} \quad \text{for all } \dot{\boldsymbol{v}} \in \dot{\boldsymbol{H}}^1(\Omega).$$

But this is exactly the Korn's inequality in the quotient space $\dot{\boldsymbol{H}}^1(\Omega)$, itself equivalent to the Korn's inequality in the space $\boldsymbol{H}^1(\Omega)$ (see Theorem 2.3).

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