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Solutions to Some Open Problems in Fluid Dynamics

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Abstract Let $u = u(x, t, u_0)$ represent the global solution of the initial value problem for the one-dimensional fluid dynamics equation

$$u_t - \varepsilon u_{xxt} + \delta u_x + \gamma H u_{xx} + \beta u_{xxx} + f(u)_x = \alpha u_{xx}, \quad u(x, 0) = u_0(x),$$

where $\alpha > 0$, $\beta \geq 0$, $\gamma \geq 0$, $\delta \geq 0$ and $\varepsilon \geq 0$ are constants. This equation may be viewed as a one-dimensional reduction of n-dimensional incompressible Navier-Stokes equations. The nonlinear function satisfies the conditions $f(0) = 0$, $|f(u)| \to \infty$ as $|u| \to \infty$, and $f \in C^1(\mathbb{R})$, and there exist the following limits

$$L_0 = \limsup_{u \to 0} \frac{f(u)}{u^3} \quad \text{and} \quad L_\infty = \limsup_{u \to \infty} \frac{f(u)}{u^5}.$$

Suppose that the initial function $u_0 \in L^1(\mathbb{R}) \cap H^2(\mathbb{R})$. By using energy estimates, Fourier transform, Plancherel's identity, upper limit estimate, lower limit estimate and the results of the linear problem

$$v_t - \varepsilon v_{xxt} + \delta v_x + \gamma H v_{xx} + \beta v_{xxx} = \alpha v_{xx}, \quad v(x, 0) = v_0(x),$$

the author justifies the following limits (with sharp rates of decay)

$$\lim_{t \to \infty} \Big[(1 + t)^{m+1/2} \int_{\mathbb{R}} |u_{x^m}(x, t)|^2 \mathrm{d}x\Big] = \frac{1}{2\pi} \Big(\frac{\pi}{2\alpha}\Big)^{1/2} \frac{m!!}{(4\alpha)^m} \Big[\int_{\mathbb{R}} u_0(x) \mathrm{d}x\Big]^2,$$

if

$$\int_{\mathbb{R}} u_0(x) \mathrm{d}x \neq 0,$$

where $0!! = 1$, $1!! = 1$ and $m!! = 1 \cdot 3 \cdot \cdots \cdot (2m - 3) \cdot (2m - 1)$. Moreover

$$\lim_{t \to \infty} \Big[(1 + t)^{m+3/2} \int_{\mathbb{R}} |u_{x^m}(x, t)|^2 \mathrm{d}x\Big] = \frac{1}{2\pi} \Big(\frac{\pi}{2\alpha}\Big)^{1/2} \frac{(m + 1)!!}{(4\alpha)^{m+1}} \Big[\int_{\mathbb{R}} \rho_0(x) \mathrm{d}x\Big]^2,$$

if the initial function $u_0(x) = {\rho_0}'(x)$, for some function $\rho_0 \in C^1(\mathbb{R}) \cap L^1(\mathbb{R})$ and

$$\int_{\mathbb{R}} \rho_0(x) \mathrm{d}x \neq 0.$$

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1 Introduction

1.1 Model equations

Let $u = u(x, t, u_0)$ denote the global solution of the Cauchy problem for the one-dimensional fluid dynamics equation

$$\frac{\partial u}{\partial t} - \varepsilon \frac{\partial^3 u}{\partial x^2 \partial t} + \delta \frac{\partial u}{\partial x} + \gamma H \frac{\partial^2 u}{\partial x^2} + \beta \frac{\partial^3 u}{\partial x^3} + \frac{\partial f(u)}{\partial x} = \alpha \frac{\partial^2 u}{\partial x^2}, \tag{1.1}$$

$$u(x, 0) = u_0(x). \tag{1.2}$$

This equation may be viewed as the one-dimensional reduction of the n-dimensional incompressible Navier-Stokes equations. In this equation, $u = u(x, t, u_0)$ is a real-valued function of $x \in \mathbb{R}$, $t > 0$ and u_0, $f(u)$ is a smooth function of u, typically, $f(u) = au^3 + bu^4 + cu^5$, for three real constants a, b and c. The parameters $\alpha > 0$, $\beta \geq 0$, $\gamma \geq 0$, $\delta \geq 0$ and $\varepsilon \geq 0$ are real, and H stands for the Hilbert transform, which is defined by

$$Hu(x, t) = \frac{1}{\pi} \,\text{P.V.} \int_{\mathbb{R}} \frac{u(y, t)}{y - x} \,\mathrm{d}y,$$

where P.V. denotes the Cauchy principal value of the singular integral. The global solution of the Cauchy problem satisfies the boundary conditions

$$\lim_{x \to \pm\infty} u(x, t) = 0, \quad \lim_{x \to \pm\infty} u_x(x, t) = 0, \quad \lim_{x \to \pm\infty} u_{xx}(x, t) = 0 \quad \text{for all } t > 0.$$

In this paper, we are going to investigate the exact limit

$$\lim_{t \to \infty} \Big\{ (1 + t)^{m+1/2} \int_{\mathbb{R}} |u_{x^m}(x, t)|^2 \mathrm{d}x \Big\} \quad \text{or} \quad \lim_{t \to \infty} \Big\{ (1 + t)^{m+3/2} \int_{\mathbb{R}} |u_{x^m}(x, t)|^2 \mathrm{d}x \Big\}.$$

Equation (1.1) contains the following equations as special examples.

(I) When $\beta = \gamma = \varepsilon = 0$ and $\delta = 1$, (1.1) becomes the Burgers equation

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} + \frac{\partial f(u)}{\partial x} = \alpha \frac{\partial^2 u}{\partial x^2}.$$

(II) When $\beta = \gamma = 0$ and $\delta = \varepsilon = 1$, (1.1) becomes the Benjamin-Bona-Mahony-Burgers equation

$$\frac{\partial u}{\partial t} - \frac{\partial^3 u}{\partial x^2 \partial t} + \frac{\partial u}{\partial x} + \frac{\partial f(u)}{\partial x} = \alpha \frac{\partial^2 u}{\partial x^2}.$$

(III) When $\beta = \varepsilon = 0$ and $\gamma = \delta = 1$, (1.1) becomes the Benjamin-Ono-Burgers equation

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} + H \frac{\partial^2 u}{\partial x^2} + \frac{\partial f(u)}{\partial x} = \alpha \frac{\partial^2 u}{\partial x^2}.$$

(IV) When $\beta = \delta = 1$, and $\gamma = \varepsilon = 0$, (1.1) becomes the Korteweg-de Vries-Burgers equation

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} + \frac{\partial f(u)}{\partial x} = \alpha \frac{\partial^2 u}{\partial x^2}.$$

We will use the famous Fourier splitting technique, which is attributable primarily to the original ideas of Maria E. Schonbek (see [12, 13]). Specifically, the method is postulated to accomplish sharp rate of decay of global solutions of dissipative equations.

1.2 Some known results

Here are some well-known results related to problem (1.1)–(1.2).

In 1989, Amick, Bona and Schonbek [4] established the exact limit of the global solution of the following Cauchy problem

$$u_t - u_{xxt} + u_x + uu_x = \alpha u_{xx}, \quad u(x,0) = u_0(x).$$

They proved that

$$\lim_{t\to\infty}\Big[(1+t)^{1/2}\int_{\mathbb{R}}|u(x,t)|^2\mathrm{d}x\Big] = \frac{4\alpha^2\mu^2}{2\pi\sqrt{\alpha}}\int_{\mathbb{R}}\frac{\exp(-2x^2)}{[1+\frac{\mu}{\sqrt{\pi}}\int_x^\infty \exp(-\xi^2)\mathrm{d}\xi]^2}\,\mathrm{d}x,$$

where

$$\mu = \exp\Big[-\frac{1}{2\alpha}\int_{\mathbb{R}}u_0(x)\mathrm{d}x\Big] - 1.$$

Motivated by this result, we study the exact limit of solutions of the more general nonlinear, dispersive, dissipative wave equation (1.1), which may be derived from fluid dynamics.

Theorem 1.1 (Existence and Uniqueness) *Let the initial function $u_0 \in L^1(\mathbb{R}) \cap H^2(\mathbb{R})$. Suppose that the nonlinear function f is sufficiently smooth, satisfying the following estimates*

$$|f(u)| \le C_1|u|^3 \quad \textit{for all } |u| \le 1,$$
$$|f(u)| \le C_2|u|^5 \quad \textit{for all } |u| \ge 1$$

for some positive constants $C_1 > 0$ and $C_2 > 0$. A concrete example of the nonlinear function is $f(u) = u^3 + u^4 + u^5$. Then there exists a unique global strong solution $u \in L^\infty(\mathbb{R}^+, H^2(\mathbb{R})) \cap L^2_{\mathrm{loc}}(\mathbb{R}^+, H^3(\mathbb{R}))$ to the Cauchy problem (1.1)–(1.2). *There also hold the uniform energy estimates*

$$\sup_{t>0}\int_{\mathbb{R}}[|u(x,t)|^2 + \varepsilon|u_x(x,t)|^2]\mathrm{d}x \le \int_{\mathbb{R}}[|u_0(x)|^2 + \varepsilon|u_{0x}(x)|^2]\mathrm{d}x,$$
$$2\alpha\int_0^\infty\int_{\mathbb{R}}|u_x(x,t)|^2\mathrm{d}x\mathrm{d}t \le \int_{\mathbb{R}}[|u_0(x)|^2 + \varepsilon|u_{0x}(x)|^2]\mathrm{d}x.$$

The existence and uniqueness of the global strong solution of problem (1.1)–(1.2) can be demonstrated by applying Leray-Schauder's fixed point theorem. We omit the details of the proof. The strong solution is also smooth, due to the presence of the dissipation.

Formally, the global solution of equation (1.1) may be represented by

$$\begin{aligned} u(x,t) = {} & \frac{1}{2\pi}\int_{\mathbb{R}}\exp(\mathrm{i}\,x\xi)\exp\Big[-\frac{\alpha|\xi|^2 - \mathrm{i}\,\xi(\beta|\xi|^2+\gamma|\xi|-\delta)}{1+\varepsilon|\xi|^2}t\Big]\widehat{u}_0(\xi)\mathrm{d}\xi \\ & - \frac{1}{2\pi}\int_0^t\Big\{\int_{\mathbb{R}}\exp(\mathrm{i}\,x\xi)\exp\Big[-\frac{\alpha|\xi|^2 - \mathrm{i}\,\xi(\beta|\xi|^2+\gamma|\xi|-\delta)}{1+\varepsilon|\xi|^2}(t-\tau)\Big] \\ & \cdot\frac{\mathrm{i}\,\xi}{1+\varepsilon|\xi|^2}\widehat{f(u)}(\xi,\tau)\mathrm{d}\xi\Big\}\mathrm{d}\tau. \end{aligned}$$

Define the semigroup operator

$$[S_t u_0](x) = \frac{1}{2\pi}\int_{\mathbb{R}} \exp(\mathrm{i}\,x\xi)\exp[-\alpha|\xi|^2 t + \mathrm{i}\,\xi(\beta|\xi|^2 + \gamma|\xi| - \delta)t]\widehat{u}_0(\xi)\mathrm{d}\xi.$$

If $\varepsilon = 0$, then we have the solution representation

$$u(x,t) = [S_t u_0](x) - \int_0^t [S_{t-\tau} f(u(\,\cdot\,,\tau))](x)\mathrm{d}\tau.$$

Theorem 1.2 (Decay Estimates with Sharp Rates of Decay)

(I) *Let $u_0 \in H^2(\mathbb{R})$. Then the unique global solution of the Cauchy problem* (1.1)–(1.2) *enjoys the limit*

$$\lim_{t\to\infty}\int_{\mathbb{R}} |u(x,t)|^2 \mathrm{d}x = 0.$$

However, here the rate of convergence may be arbitrarily slow.

(II) *Let $u_0 \in L^1(\mathbb{R}) \cap H^2(\mathbb{R})$ and*

$$\int_{\mathbb{R}} u_0(x)\mathrm{d}x \neq 0.$$

Then the unique global solution enjoys the decay estimates

$$C_1 \leq (1+t)^{1/2}\int_{\mathbb{R}} |u(x,t)|^2 \mathrm{d}x + (1+t)^{3/2}\int_{\mathbb{R}} |u_x(x,t)|^2 \mathrm{d}x \leq C_2.$$

In these estimates, C_1 and C_2 are positive constants, independent of time.

(III) *Let $u_0 \in L^1(\mathbb{R}) \cap H^2(\mathbb{R})$ and*

$$\int_{\mathbb{R}} u_0(x)\mathrm{d}x = 0, \quad \int_{\mathbb{R}} \rho_0(x)\mathrm{d}x \neq 0,$$

where $u_0(x) = \frac{\mathrm{d}}{\mathrm{d}x}\rho_0(x)$. Then there hold the decay estimates

$$C_3 \leq (1+t)^{3/2}\int_{\mathbb{R}} |u(x,t)|^2 \mathrm{d}x + (1+t)^{5/2}\int_{\mathbb{R}} |u_x(x,t)|^2 \mathrm{d}x \leq C_4,$$

where C_3 and C_4 are also positive constants, independent of time.

Amick, Bona and Schonbek [4], Bona and Luo [5–6], Dix [8–10] and Zhang [15–20] established these results for various similar/simpler model equations. Theorem 1.2 is a summary of these known decay results.

1.3 Main goal

In this paper, we will evaluate the following limits explicitly, in terms of the integral of the initial data and the model parameters.

(I) $\lim_{t\to\infty}\Big[(1+t)^{1/2}\int_{\mathbb{R}} |u(x,t)|^2 \mathrm{d}x\Big], \quad \lim_{t\to\infty}\Big[(1+t)^{m+1/2}\int_{\mathbb{R}} |u_{x^m}(x,t)|^2 \mathrm{d}x\Big],$
where $m \geq 1$ is any integer, the initial data $u_0 \in L^1(\mathbb{R}) \cap H^2(\mathbb{R})$ and

$$\int_{\mathbb{R}} u_0(x)\mathrm{d}x \neq 0.$$

$$\text{(II)}\quad \lim_{t\to\infty}\Big[(1+t)^{3/2}\int_{\mathbb{R}}|u(x,t)|^2\mathrm{d}x\Big],\quad \lim_{t\to\infty}\Big[(1+t)^{m+3/2}\int_{\mathbb{R}}|u_{x^m}(x,t)|^2\mathrm{d}x\Big],$$

where the initial data $u_0\in L^1(\mathbb{R})\cap H^2(\mathbb{R})$, $u_0(x)=\frac{\mathrm{d}}{\mathrm{d}x}\rho_0(x)$, and $\rho_0\in C^1(\mathbb{R})\cap L^1(\mathbb{R})$. Moreover

$$\int_{\mathbb{R}}u_0(x)\mathrm{d}x=0,\quad \int_{\mathbb{R}}\rho_0(x)\mathrm{d}x\neq 0.$$

1.4 The main results

Here are the main results of this paper. These results may be applied to dynamical systems, in particular, to Hausdorff dimension of global attractors of the model equations.

Theorem 1.3 *Suppose that the initial data* $u_0\in L^1(\mathbb{R})\cap H^2(\mathbb{R})$*, and that*

$$\int_{\mathbb{R}}u_0(x)\mathrm{d}x\neq 0.$$

Then the unique global solution of the Cauchy problem (1.1)–(1.2) *enjoys the limits*

$$\lim_{t\to\infty}\Big\{(1+t)^{1/2}\int_{\mathbb{R}}|u(x,t)|^2\mathrm{d}x\Big\}=\frac{1}{2\pi}\Big(\frac{\pi}{2\alpha}\Big)^{1/2}\Big[\int_{\mathbb{R}}u_0(x)\mathrm{d}x\Big]^2,$$
$$\lim_{t\to\infty}\Big\{(1+t)^{m+1/2}\int_{\mathbb{R}}|u_{x^m}(x,t)|^2\mathrm{d}x\Big\}=\frac{1}{2\pi}\Big(\frac{\pi}{2\alpha}\Big)^{1/2}\frac{m!!}{(4\alpha)^m}\Big[\int_{\mathbb{R}}u_0(x)\mathrm{d}x\Big]^2.$$

Theorem 1.4 *Suppose that* $u_0\in L^1(\mathbb{R})\cap H^2(\mathbb{R})$*, and that* $u_0(x)=\rho_0'(x)$*,* $\rho_0\in C^1(\mathbb{R})\cap L^1(\mathbb{R})$*, with*

$$\int_{\mathbb{R}}\rho_0(x)\mathrm{d}x\neq 0.$$

Then the unique global solution of the Cauchy problem (1.1)–(1.2) *enjoys the limits*

$$\lim_{t\to\infty}\Big\{(1+t)^{3/2}\int_{\mathbb{R}}|u(x,t)|^2\mathrm{d}x\Big\}=\frac{1}{2\pi}\Big(\frac{\pi}{2\alpha}\Big)^{1/2}\frac{1}{4\alpha}\Big[\int_{\mathbb{R}}\rho_0(x)\mathrm{d}x\Big]^2,$$
$$\lim_{t\to\infty}\Big\{(1+t)^{m+3/2}\int_{\mathbb{R}}|u_{x^m}(x,t)|^2\mathrm{d}x\Big\}=\frac{1}{2\pi}\Big(\frac{\pi}{2\alpha}\Big)^{1/2}\frac{(m+1)!!}{(4\alpha)^{m+1}}\Big[\int_{\mathbb{R}}\rho_0(x)\mathrm{d}x\Big]^2.$$

These results will be established in Section 2.

The rates of decay in both Theorem 1.3 and Theorem 1.4 are optimal. Even if the initial data u_0 satisfies

$$\int_{\mathbb{R}}u_0(x)\mathrm{d}x=0,\quad \int_{\mathbb{R}}xu_0(x)\mathrm{d}x=0,$$

the decay rates in Theorem 1.4 cannot be improved simply because the equation is nonlinear. The precise limits are new.

Remark 1.1 Suppose that $u_0\in L^1(\mathbb{R})\cap H^2(\mathbb{R})$. By using similar ideas as those displayed in this work, we may be able to establish the limits

$$\lim_{t\to\infty}(1+t)\Big\{(1+t)^{m+1/2}\int_{\mathbb{R}}|u_{x^m}(x,t)|^2\mathrm{d}x-\frac{1}{2\pi}\Big(\frac{\pi}{2\alpha}\Big)^{1/2}\frac{m!!}{(4\alpha)^m}\Big[\int_{\mathbb{R}}u_0(x)\mathrm{d}x\Big]^2\Big\},$$

if $\int_{\mathbb{R}}u_0(x)\mathrm{d}x\neq 0$;

$$\lim_{t\to\infty}(1+t)^2\Big\{(1+t)^{m+3/2}\int_{\mathbb{R}}|u_{x^m}(x,t)|^2\mathrm{d}x-\frac{1}{2\pi}\Big(\frac{\pi}{2\alpha}\Big)^{1/2}\frac{(m+1)!!}{(4\alpha)^{m+1}}\Big[\int_{\mathbb{R}}\rho_0(x)\mathrm{d}x\Big]^2\Big\},$$

if $\int_{\mathbb{R}} u_0(x)\mathrm{d}x = 0$. We will use the same letters C, C_1, C_2, $\cdots$ to denote many positive, time-independent constants and they may be different from one place to another.

2 The Mathematical Analysis

2.1 Linear analysis

Let $v = v(x,t,v_0)$ be the global solution of the initial value problem for the linear equation

$$v_t - \varepsilon v_{xxt} + \delta v_x + \gamma H v_{xx} + \beta v_{xxx} = \alpha v_{xx}, \tag{2.1}$$

$$v(x,0) = v_0(x). \tag{2.2}$$

Lemma 2.1 (I) *Suppose that the initial data* $v_0 \in L^1(\mathbb{R}) \cap H^2(\mathbb{R})$. *Then the unique global solution of the Cauchy problem* (2.1)–(2.2) *enjoys the limit*

$$\lim_{t\to\infty} \Big\{(1+t)^{m+1/2} \int_{\mathbb{R}} |v_{x^m}(x,t)|^2 \mathrm{d}x\Big\} = \frac{1}{2\pi}\Big(\frac{\pi}{2\alpha}\Big)^{1/2} \frac{m!!}{(4\alpha)^m}\Big[\int_{\mathbb{R}} v_0(x)\mathrm{d}x\Big]^2,$$

if $\int_{\mathbb{R}} v_0(x)\mathrm{d}x \neq 0$. *Furthermore,*

$$\lim_{t\to\infty} \Big\{(1+t)^{m+3/2} \int_{\mathbb{R}} |v_{x^m}(x,t)|^2 \mathrm{d}x\Big\} = \frac{1}{2\pi}\Big(\frac{\pi}{2\alpha}\Big)^{1/2} \frac{(m+1)!!}{(4\alpha)^{m+1}}\Big[\int_{\mathbb{R}} \sigma_0(x)\mathrm{d}x\Big]^2,$$

if $v_0(x) = \sigma_0'(x)$, $\sigma_0 \in C^1(\mathbb{R}) \cap L^1(\mathbb{R})$, *and*

$$\int_{\mathbb{R}} \sigma_0(x)\mathrm{d}x \neq 0.$$

(II) *Suppose now that* $v_0 \in L^1(\mathbb{R}) \cap H^2(\mathbb{R})$, *and*

$$\int_{\mathbb{R}} |x v_0(x)|\mathrm{d}x + \int_{\mathbb{R}} |x^3 v_0(x)|\mathrm{d}x < \infty.$$

Then

$$\begin{aligned}&\lim_{t\to\infty} \Big\{(1+t)\Big[(1+t)^{m+1/2} \int_{\mathbb{R}} |v_{x^m}(x,t)|^2 \mathrm{d}x - \frac{1}{2\pi}\Big(\frac{\pi}{2\alpha}\Big)^{1/2} \frac{m!!}{(4\alpha)^m}\Big(\int_{\mathbb{R}} v_0(x)\mathrm{d}x\Big)^2\Big]\Big\}\\ &= \frac{1}{2\pi}\Big(\frac{\pi}{2\alpha}\Big)^{1/2} \frac{(m+1)!}{(4\alpha)^{m+1}}\Big\{\Big[\int_{\mathbb{R}} x v_0(x)\mathrm{d}x\Big]^2 - \Big[\int_{\mathbb{R}} v_0(x)\mathrm{d}x\Big]\Big[\int_{\mathbb{R}} x^2 v_0(x)\mathrm{d}x\Big]\Big\}.\end{aligned}$$

Furthermore, if

$$\int_{\mathbb{R}} v_0(x)\mathrm{d}x = 0, \quad \int_{\mathbb{R}} x v_0(x)\mathrm{d}x = 0,$$

then

$$\begin{aligned}&\lim_{t\to\infty} \Big\{(1+t)^2\Big[(1+t)^{m+3/2} \int_{\mathbb{R}} |v_{x^m}(x,t)|^2 \mathrm{d}x - \frac{1}{2\pi}\Big(\frac{\pi}{2\alpha}\Big)^{1/2} \frac{(m+1)!!}{(4\alpha)^{m+1}}\Big(\int_{\mathbb{R}} \sigma_0(x)\mathrm{d}x\Big)^2\Big]\Big\}\\ &= \frac{1}{8\pi}\Big(\frac{\pi}{2\alpha}\Big)^{1/2} \frac{(m+2)!}{(4\alpha)^{m+2}}\Big\{\Big[\int_{\mathbb{R}} x \sigma_0(x)\mathrm{d}x\Big]^2 - \Big[\int_{\mathbb{R}} \sigma_0(x)\mathrm{d}x\Big]\Big[\int_{\mathbb{R}} x^2 \sigma_0(x)\mathrm{d}x\Big]\Big\}.\end{aligned}$$

Proof Let $v_0 \in L^1(\mathbb{R}) \cap H^2(\mathbb{R})$. Then the Fourier transform $\widehat{v}_0$ is continuous everywhere. It is straightforward to prove that the Fourier transform of the global solution of the linear problem (2.1)–(2.2) is given by

$$\widehat{v}(\xi,t) = \exp\Big[-\frac{\alpha|\xi|^2 - \mathrm{i}\,\xi(\beta|\xi|^2 + \gamma|\xi| - \delta)}{1+\varepsilon|\xi|^2}t\Big]\widehat{v}_0(\xi).$$

Then

$$\begin{aligned}
& t^{m+1/2}\int_{\mathbb{R}} |v_{x^m}(x,t)|^2 \mathrm{d}x \\
= & \frac{t^{m+1/2}}{2\pi}\int_{\mathbb{R}} |\xi|^{2m} |\widehat{v}(\xi,t)|^2 \mathrm{d}\xi \\
= & \frac{t^{m+1/2}}{2\pi}\int_{\mathbb{R}} |\xi|^{2m} \exp\Big(-\frac{2\alpha|\xi|^2}{1+\varepsilon|\xi|^2}t\Big)|\widehat{v}_0(\xi)|^2 \mathrm{d}\xi \\
= & \frac{1}{2\pi}\int_{\mathbb{R}} |\eta|^{2m} \exp\Big(-\frac{2\alpha|\eta|^2}{1+\varepsilon|\eta|^2/t}\Big)|\widehat{v}_0(t^{-1/2}\eta)|^2 \mathrm{d}\eta \\
\to & \frac{1}{2\pi}\int_{\mathbb{R}} |\eta|^{2m} \exp(-2\alpha|\eta|^2)|\widehat{v}_0(0)|^2 \mathrm{d}\eta \\
= & \frac{1}{2\pi}\Big(\frac{\pi}{2\alpha}\Big)^{1/2}\frac{m!!}{(4\alpha)^m}\Big[\int_{\mathbb{R}} v_0(x)\mathrm{d}x\Big]^2,
\end{aligned}$$

as $t \to \infty$, where $\eta = t^{1/2}\xi$, $0!! = 1$, $m!! = 1 \cdot 3 \cdot \cdots \cdot (2m-3)\cdot(2m-1)$. Now, let $v_0 \in L^1(\mathbb{R}) \cap H^2(\mathbb{R})$, $v_0(x) = \sigma_0'(x)$, and

$$\int_{\mathbb{R}} \sigma_0(x)\mathrm{d}x \neq 0.$$

Then $\widehat{v}_0(\xi) = \mathrm{i}\,\xi\widehat{\sigma}_0(\xi)$. Very similar to the above, we obtain

$$\begin{aligned}
& t^{m+3/2}\int_{\mathbb{R}} |v_{x^m}(x,t)|^2 \mathrm{d}x \\
= & \frac{t^{m+3/2}}{2\pi}\int_{\mathbb{R}} |\xi|^{2m} |\widehat{v}(\xi,t)|^2 \mathrm{d}\xi \\
= & \frac{t^{m+3/2}}{2\pi}\int_{\mathbb{R}} |\xi|^{2m+2} \exp\Big(-\frac{2\alpha|\xi|^2}{1+\varepsilon|\xi|^2}t\Big)|\widehat{\sigma}_0(\xi)|^2 \mathrm{d}\xi \\
= & \frac{1}{2\pi}\int_{\mathbb{R}} |\eta|^{2m+2} \exp\Big(-\frac{2\alpha|\eta|^2}{1+\varepsilon|\eta|^2/t}\Big)|\widehat{\sigma}_0(t^{-1/2}\eta)|^2 \mathrm{d}\eta \\
\to & \frac{1}{2\pi}\int_{\mathbb{R}} |\eta|^{2m+2} \exp(-2\alpha|\eta|^2)|\widehat{\sigma}_0(0)|^2 \mathrm{d}\eta \\
= & \frac{1}{2\pi}\Big(\frac{\pi}{2\alpha}\Big)^{1/2}\frac{(m+1)!!}{(4\alpha)^{m+1}}\Big[\int_{\mathbb{R}} \sigma_0(x)\mathrm{d}x\Big]^2,
\end{aligned}$$

as $t \to \infty$, where $\eta = t^{1/2}\xi$. Note that

$$\lim_{t\to\infty}\Big[(1+t)^{m+1/2}\int_{\mathbb{R}} |v_{x^{m+1}}(x,t)|^2 \mathrm{d}x\Big] = 0,$$

if $\int_{\mathbb{R}} v_0(x)\mathrm{d}x \neq 0$; and

$$\lim_{t\to\infty}\Big[(1+t)^{m+3/2}\int_{\mathbb{R}} |v_{x^{m+1}}(x,t)|^2 \mathrm{d}x\Big] = 0,$$

if $\int_{\mathbb{R}} v_0(x)\mathrm{d}x = 0$. Noting also that for any initial data $v_0 \in L^1(\mathbb{R}) \cap H^2(\mathbb{R})$, we have the Fourier transform

$$\widehat{v}_0(\xi) = \int_{\mathbb{R}} \exp(-\mathrm{i}\, x\xi) v_0(x)\mathrm{d}x.$$

Suppose now that

$$\int_{\mathbb{R}} |xv_0(x)|\mathrm{d}x + \int_{\mathbb{R}} |x^3 v_0(x)|\mathrm{d}x < \infty.$$

Thus, for any integer $m \geq 1$, formally we get

$$\begin{aligned}
\frac{\mathrm{d}^m \widehat{v}_0}{\mathrm{d}\xi^m}(0) &= (-\mathrm{i})^m \int_{\mathbb{R}} x^m v_0(x)\mathrm{d}x, \\
\frac{\mathrm{d}^{2m} \widehat{v}_0}{\mathrm{d}\xi^{2m}}(0) &= (-1)^m \int_{\mathbb{R}} x^{2m} v_0(x)\mathrm{d}x, \\
\frac{\mathrm{d}^{2m+1} \widehat{v}_0}{\mathrm{d}\xi^{2m+1}}(0) &= (-1)^{m+1}\mathrm{i} \int_{\mathbb{R}} x^{2m+1} v_0(x)\mathrm{d}x.
\end{aligned}$$

Note that even-order derivatives are real while odd-order derivatives are purely imaginary. We want to isolate the real part from the imaginary part in the Taylor expansion. Thus we have

$$\begin{aligned}
\widehat{v}_0(\xi) &= \sum_{m=0}^{\infty} \frac{1}{(2m)!} \frac{\mathrm{d}^{2m} \widehat{v}_0}{\mathrm{d}\xi^{2m}}(0)\xi^{2m} + \sum_{m=0}^{\infty} \frac{1}{(2m+1)!} \frac{\mathrm{d}^{2m+1} \widehat{v}_0}{\mathrm{d}\xi^{2m+1}}(0)\xi^{2m+1} \\
&= \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!} \xi^{2m} \Big[\int_{\mathbb{R}} x^{2m} v_0(x)\mathrm{d}x \Big] - \mathrm{i} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} \xi^{2m+1} \Big[\int_{\mathbb{R}} x^{2m+1} v_0(x)\mathrm{d}x \Big].
\end{aligned}$$

Replacing ξ by $t^{-1/2}\eta$ in this above equation, where $t > 0$, we have

$$\begin{aligned}
&t[|\widehat{v}_0(t^{-1/2}\eta)|^2 - |\widehat{v}_0(0)|^2] \\
&= t\Big\{ \Big| \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!} \frac{\eta^{2m}}{t^m} \Big[\int_{\mathbb{R}} x^{2m} v_0(x)\mathrm{d}x \Big] \Big|^2 - |\widehat{v}_0(0)|^2 \Big\} \\
&\quad + \Big| \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} \frac{\eta^{2m+1}}{t^m} \Big[\int_{\mathbb{R}} x^{2m+1} v_0(x)\mathrm{d}x \Big] \Big|^2 \\
&= \Big\{ \sum_{m=1}^{\infty} \frac{(-1)^m}{(2m)!} \frac{\eta^{2m}}{t^{m-1}} \Big[\int_{\mathbb{R}} x^{2m} v_0(x)\mathrm{d}x \Big] \Big\} \Big\{ 2\widehat{v}_0(0) + \sum_{m=1}^{\infty} \frac{(-1)^m}{(2m)!} \frac{\eta^{2m}}{t^m} \Big[\int_{\mathbb{R}} x^{2m} v_0(x)\mathrm{d}x \Big] \Big\} \\
&\quad + \Big| \eta \int_{\mathbb{R}} x v_0(x)\mathrm{d}x + \sum_{m=1}^{\infty} \frac{(-1)^m}{(2m+1)!} \frac{\eta^{2m+1}}{t^m} \Big[\int_{\mathbb{R}} x^{2m+1} v_0(x)\mathrm{d}x \Big] \Big|^2.
\end{aligned}$$

Therefore

$$\lim_{t\to\infty} \{ t[|\widehat{v}_0(t^{-1/2}\eta)|^2 - |\widehat{v}_0(0)|^2] \} = \eta^2 \Big\{ \Big[\int_{\mathbb{R}} x v_0(x)\mathrm{d}x \Big]^2 - \Big[\int_{\mathbb{R}} v_0(x)\mathrm{d}x \Big] \Big[\int_{\mathbb{R}} x^2 v_0(x)\mathrm{d}x \Big] \Big\}.$$

If

$$\int_{\mathbb{R}} v_0(x)\mathrm{d}x = 0, \quad v_0(x) = \sigma_0'(x),$$

then

$$\lim_{t\to\infty}\{t[|\widehat{\sigma}_0(t^{-1/2}\eta)|^2-|\widehat{\sigma}_0(0)|^2]\}=\Big[\int_{\mathbb{R}}x\sigma_0(x)\mathrm{d}x\Big]^2-\Big[\int_{\mathbb{R}}\sigma_0(x)\mathrm{d}x\Big]\Big[\int_{\mathbb{R}}x^2\sigma_0(x)\mathrm{d}x\Big].$$

These calculations are formally correct, because we are not sure if the improper integrals $\int_{\mathbb{R}}|x|^m v_0(x)\mathrm{d}x$ are convergent or not when m is large. However, we may use the following Taylor formula

$$\widehat{v}_0(\xi)=\int_{\mathbb{R}}v_0(x)\mathrm{d}x-\mathrm{i}\xi\int_{\mathbb{R}}xv_0(x)\mathrm{d}x-\frac{\xi^2}{2}\int_{\mathbb{R}}x^2v_0(x)\mathrm{d}x+\frac{\mathrm{i}\xi^3}{6}\int_{\mathbb{R}}x^3\exp(-\mathrm{i}x\zeta)v_0(x)\mathrm{d}x,$$

where $0<|\zeta|<|\xi|$.

Now we get (without loss of generality, let $\varepsilon=0$ in this part)

$$\begin{aligned}
&t\Big\{t^{m+1/2}\int_{\mathbb{R}}|v_{x^m}(x,t)|^2\mathrm{d}x-\frac{1}{2\pi}\Big(\frac{\pi}{2\alpha}\Big)^{1/2}\frac{m!!}{(4\alpha)^m}\Big[\int_{\mathbb{R}}v_0(x)\mathrm{d}x\Big]^2\Big\}\\
&=\frac{1}{2\pi}\int_{\mathbb{R}}|\eta|^{2m}\exp\Big(-\frac{2\alpha|\eta|^2}{1+\varepsilon|\eta|^2/t}\Big)\{t[|\widehat{v}_0(t^{-1/2}\eta)|^2-|\widehat{v}_0(0)|^2]\}\mathrm{d}\eta\\
&\to\frac{1}{2\pi}\int_{\mathbb{R}}|\eta|^{2m+2}\exp(-2\alpha|\eta|^2)\Big\{\Big[\int_{\mathbb{R}}xv_0(x)\mathrm{d}x\Big]^2-\Big[\int_{\mathbb{R}}v_0(x)\mathrm{d}x\Big]\Big[\int_{\mathbb{R}}x^2v_0(x)\mathrm{d}x\Big]\Big\}\mathrm{d}\eta\\
&=\frac{1}{2\pi}\Big(\frac{\pi}{2\alpha}\Big)^{1/2}\frac{(m+1)!!}{(4\alpha)^{m+1}}\Big\{\Big[\int_{\mathbb{R}}xv_0(x)\mathrm{d}x\Big]^2-\Big[\int_{\mathbb{R}}v_0(x)\mathrm{d}x\Big]\Big[\int_{\mathbb{R}}x^2v_0(x)\mathrm{d}x\Big]\Big\},
\end{aligned}$$

as $t\to\infty$. Now suppose that

$$\int_{\mathbb{R}}v_0(x)\mathrm{d}x=0.$$

Let $v_0(x)=\sigma_0'(x)$, where $\sigma_0\in C^1(\mathbb{R})\cap L^1(\mathbb{R})$. Then

$$\begin{aligned}
&t^2\Big\{t^{m+3/2}\int_{\mathbb{R}}|v_{x^m}(x,t)|^2\mathrm{d}x-\frac{1}{2\pi}\Big(\frac{\pi}{2\alpha}\Big)^{1/2}\frac{(m+1)!!}{(4\alpha)^{m+1}}\Big[\int_{\mathbb{R}}\sigma_0(x)\mathrm{d}x\Big]^2\Big\}\\
&=\frac{1}{2\pi}\int_{\mathbb{R}}|\eta|^{2m+2}\exp\Big(-\frac{2\alpha|\eta|^2}{1+\varepsilon|\eta|^2/t}\Big)\{t^2[|\widehat{\sigma}_0(t^{-1/2}\eta)|^2-|\widehat{\sigma}_0(0)|^2]\}\mathrm{d}\eta\\
&\to\frac{1}{2\pi}\int_{\mathbb{R}}|\eta|^{2m+2}\exp(-2\alpha|\eta|^2)\Big\{\Big[\int_{\mathbb{R}}x\sigma_0(x)\mathrm{d}x\Big]^2-\Big[\int_{\mathbb{R}}\sigma_0(x)\mathrm{d}x\Big]\Big[\int_{\mathbb{R}}x^2\sigma_0(x)\mathrm{d}x\Big]\Big\}\\
&=\frac{1}{8\pi}\Big(\frac{\pi}{2\alpha}\Big)^{1/2}\frac{(m+2)!}{(4\alpha)^{m+2}}\Big\{\Big[\int_{\mathbb{R}}x\sigma_0(x)\mathrm{d}x\Big]^2-\Big[\int_{\mathbb{R}}\sigma_0(x)\mathrm{d}x\Big]\Big[\int_{\mathbb{R}}x^2\sigma_0(x)\mathrm{d}x\Big]\Big\},
\end{aligned}$$

as $t\to\infty$. The proof of Lemma 2.1 is completed.

2.2 Nonlinear analysis

Define $w(x,t)=u(x,t)-v(x,t)$ and $w_0(x)=u_0(x)-v_0(x)$. Then we find that

$$w_t-\varepsilon w_{xxt}+\delta w_x+\gamma Hw_{xx}+\beta w_{xxx}+f(u)_x=\alpha w_{xx}, \tag{2.3}$$

$$w(x,0)=w_0(x). \tag{2.4}$$

Lemma 2.2 *The Fourier transform of w is*

$$\widehat{w}(\xi,t)=\exp\Big[-\frac{\alpha|\xi|^2-\mathrm{i}\,\xi(\beta|\xi|^2+\gamma|\xi|-\delta)}{1+\varepsilon|\xi|^2}t\Big]\widehat{w}_0(\xi)$$
$$-\int_0^t\Big\{\exp\Big[-\frac{\alpha|\xi|^2-\mathrm{i}\,\xi(\beta|\xi|^2+\gamma|\xi|-\delta)}{1+\varepsilon|\xi|^2}(t-\tau)\Big]\frac{\mathrm{i}\,\xi}{1+\varepsilon|\xi|^2}\widehat{f(u)}(\xi,\tau)\Big\}\mathrm{d}\tau.$$

Additionally, if $|f(u)|\le C|u|^{3+\kappa}$ on $\mathbb{R}$, for two constants $C>0$ and $0\le\kappa\le 2$, then there hold the estimates

$$|\widehat{w}(\xi,t)|\le|\widehat{w}_0(\xi)|+C|\xi|\Big\{\int_0^t\Big[\int_{\mathbb{R}}|u(x,\tau)|^2\mathrm{d}x\Big]^{(5+\kappa)/(3-\kappa)}\mathrm{d}\tau\Big\}^{(3-\kappa)/4}$$
$$\cdot\Big\{\int_0^t\Big[\int_{\mathbb{R}}|u_x(x,\tau)|^2\mathrm{d}x\Big]\mathrm{d}\tau\Big\}^{(1+\kappa)/4}$$
$$\le\begin{cases}|\widehat{w}_0(\xi)|+C|\xi|(1+t)^{1/8}, & \text{if } \displaystyle\int_{\mathbb{R}}u_0(x)\mathrm{d}x\ne 0,\\ |\widehat{w}_0(\xi)|+C|\xi|, & \text{if } \displaystyle\int_{\mathbb{R}}u_0(x)\mathrm{d}x=0.\end{cases}$$

Suppose now that $\widehat{u}_0(0)=0$ and $w_0=0$. Then there holds the estimate

$$|\widehat{w}(\xi,t)|\le\min\Big\{C|\xi|,\ \frac{C}{|\xi|}\Big\},\quad\forall\,t>0.$$

Proof The representation of the Fourier transform of the global solution of problem (2.3)–(2.4) is very easy to verify. We skip the details. As before, let $|f(u)|\le C|u|^{3+\kappa}$, for all $u\in\mathbb{R}$. Furthermore, by using Gagliardo-Nirenberg's interpolation inequality and Hölder's inequality, we have the following estimates (below $0\le\kappa\le 2$)

$$\int_0^t\Big[\int_{\mathbb{R}}|u(x,\tau)|^{3+\kappa}\mathrm{d}x\Big]\mathrm{d}\tau$$
$$\le\int_0^t\Big[\|u(\,\cdot\,,\tau)\|^{1+\kappa}_{L^\infty(\mathbb{R})}\int_{\mathbb{R}}|u(x,\tau)|^2\mathrm{d}x\Big]\mathrm{d}\tau$$
$$\le\int_0^t\Big[\int_{\mathbb{R}}|u(x,\tau)|^2\mathrm{d}x\Big]^{(5+\kappa)/4}\Big[\int_{\mathbb{R}}|u_x(x,\tau)|^2\mathrm{d}x\Big]^{(1+\kappa)/4}\mathrm{d}\tau$$
$$\le\Big\{\int_0^t\Big[\int_{\mathbb{R}}|u(x,\tau)|^2\mathrm{d}x\Big]^{(5+\kappa)/(3-\kappa)}\mathrm{d}\tau\Big\}^{(3-\kappa)/4}\Big[\int_0^t\int_{\mathbb{R}}|u_x(x,\tau)|^2\mathrm{d}x\mathrm{d}\tau\Big]^{(1+\kappa)/4}.$$

Therefore, by the results of Theorem 1.2(II), we obtain the following estimates

$$|\widehat{w}(\xi,t)|\le|\widehat{w}_0(\xi)|+|\xi|\int_0^t|\widehat{f(u)}(\xi,\tau)|\mathrm{d}\tau$$
$$\le|\widehat{w}_0(\xi)|+C|\xi|\Big\{\int_0^t\Big[\int_{\mathbb{R}}|u(x,\tau)|^2\mathrm{d}x\Big]^{(5+\kappa)/(3-\kappa)}\mathrm{d}\tau\Big\}^{(3-\kappa)/4}$$
$$\cdot\Big[\int_0^t\int_{\mathbb{R}}|u_x(x,\tau)|^2\mathrm{d}x\mathrm{d}\tau\Big]^{(1+\kappa)/4}$$
$$\le|\widehat{w}_0(\xi)|+C|\xi|(1+t)^{1/8}.$$

If $\widehat{u}_0(0)=0$ and $w_0=0$, then by Theorem 1.2(III), we have $|\widehat{w}(\xi,t)|\le C|\xi|$. Performing the Fourier transform to equation (2.3), we have

$$(1+\varepsilon|\xi|^2)\widehat{w}_t(\xi,t)+[\alpha|\xi|^2-\mathrm{i}\,\xi(\beta|\xi|^2+\gamma|\xi|-\delta)]\widehat{w}(\xi,t)+\widehat{f(u)}_x(\xi,t)=0.$$

By using this equation, we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}[|\widehat{w}(\xi,t)|^2]&=\frac{\mathrm{d}}{\mathrm{d}t}[\widehat{w}(\xi,t)\overline{\widehat{w}}(\xi,t)]=\widehat{w}_t(\xi,t)\overline{\widehat{w}}(\xi,t)+\widehat{w}(\xi,t)\overline{\widehat{w}}_t(\xi,t)\\&=-\frac{2\alpha|\xi|^2}{1+\varepsilon|\xi|^2}|\widehat{w}(\xi,t)|^2-\frac{2}{1+\varepsilon|\xi|^2}\operatorname{Re}[\widehat{f(u)}_x(\xi,t)\overline{\widehat{w}}(\xi,t)].\end{aligned}$$

Equivalently, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big\{\exp\Big(\frac{2\alpha|\xi|^2}{1+\varepsilon|\xi|^2}t\Big)|\widehat{w}(\xi,t)|^2\Big\}+\frac{2}{1+\varepsilon|\xi|^2}\exp\Big(\frac{2\alpha|\xi|^2}{1+\varepsilon|\xi|^2}t\Big)\operatorname{Re}[\widehat{f(u)}_x(\xi,t)\overline{\widehat{w}}(\xi,t)]=0.$$

Integrating in time and using $w_0=0$, we find

$$\exp\Big(\frac{2\alpha|\xi|^2}{1+\varepsilon|\xi|^2}t\Big)|\widehat{w}(\xi,t)|^2=-\frac{2}{1+\varepsilon|\xi|^2}\int_0^t\exp\Big(\frac{2\alpha|\xi|^2}{1+\varepsilon|\xi|^2}\tau\Big)\operatorname{Re}[\widehat{f(u)}_x(\xi,\tau)\overline{\widehat{w}}(\xi,\tau)]\mathrm{d}\tau.$$

Applying Gronwall's inequality, (The Gronwall's inequality: Let the nonnegative functions g and h satisfy the given inequality $[g(t)]^2\le C^2+2\int_0^t g(\tau)h(\tau)\mathrm{d}\tau$ for all $t>0$, where $C\ge 0$ is a constant. Then $g(t)\le C+\int_0^t h(\tau)\mathrm{d}\tau$.) we obtain the estimate

$$\begin{aligned}&\exp\Big(\frac{\alpha|\xi|^2}{1+\varepsilon|\xi|^2}t\Big)|\widehat{w}(\xi,t)|\\&\le\frac{C|\xi|}{1+\varepsilon|\xi|^2}\int_0^t\exp\Big(\frac{\alpha|\xi|^2}{1+\varepsilon|\xi|^2}\tau\Big)\Big[\int_{\mathbb{R}}|u(x,\tau)|^2\mathrm{d}x\Big]^{(5+\kappa)/4}\Big[\int_{\mathbb{R}}|u_x(x,\tau)|^2\mathrm{d}x\Big]^{(1+\kappa)/4}\mathrm{d}\tau\\&\le\frac{C}{|\xi|}\exp\Big(\frac{\alpha|\xi|^2}{1+\varepsilon|\xi|^2}t\Big)\quad\text{for all }t>0.\end{aligned}$$

The proof of Lemma 2.2 is completed.

Lemma 2.3 *Let the initial data satisfy* $u_0=v_0\in L^1(\mathbb{R})\cap H^2(\mathbb{R})$ *and*

$$\int_{\mathbb{R}}u_0(x)\mathrm{d}x\ne 0.$$

Then the Cauchy problem (2.3)–(2.4) *enjoys the decay estimates*

$$(1+t)^{1/2}\int_{\mathbb{R}}[|u(x,t)-v(x,t)|^2+\varepsilon|u_x(x,t)-v_x(x,t)|^2]\mathrm{d}x\le\frac{C}{(1+t)^{1/2}},\tag{2.5}$$

$$(1+t)^{m+1/2}\int_{\mathbb{R}}[|u_{x^m}(x,t)-v_{x^m}(x,t)|^2+\varepsilon|u_{x^{m+1}}(x,t)-v_{x^{m+1}}(x,t)|^2]\mathrm{d}x\le\frac{C_m}{(1+t)^{1/2}},\tag{2.6}$$

where the positive constants C *and* C_m *are independent of time.*

Proof Multiplying equation (2.3) by $2w$ and integrating the result with respect to x over $\mathbb{R}$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}[|w(x,t)|^2+\varepsilon|w_x(x,t)|^2]\mathrm{d}x+2\alpha\int_{\mathbb{R}}|w_x(x,t)|^2\mathrm{d}x+2\int_{\mathbb{R}}w(x,t)f(u)_x\mathrm{d}x=0,$$

where, for all $t > 0$,

$$\int_{\mathbb{R}} w(x,t)w_x(x,t)\mathrm{d}x = 0, \quad \int_{\mathbb{R}} w(x,t)Hw_{xx}(x,t)\mathrm{d}x = 0,$$
$$\int_{\mathbb{R}} w(x,t)w_{xxx}(x,t)\mathrm{d}x = 0, \quad \int_{\mathbb{R}} u(x,t)f(u(x,t))_x\mathrm{d}x = 0.$$

Applying the famous Plancherel's identity to this energy equation yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}(1+\varepsilon|\xi|^2)|\widehat{w}(\xi,t)|^2\mathrm{d}\xi + 2\alpha\int_{\mathbb{R}}|\xi|^2|\widehat{w}(\xi,t)|^2\mathrm{d}\xi + 2\int_{\mathbb{R}}\mathrm{i}\,\xi\widehat{v}(\xi,t)\overline{\widehat{f(u)}}\mathrm{d}\xi = 0,$$

or equivalently, we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\Big\{(1+t)^2\int_{\mathbb{R}}(1+\varepsilon|\xi|^2)|\widehat{w}(\xi,t)|^2\mathrm{d}\xi\Big\} + 2\alpha(1+t)^2\int_{\mathbb{R}}|\xi|^2|\widehat{w}(\xi,t)|^2\mathrm{d}\xi\\ &= 2(1+t)\int_{\mathbb{R}}(1+\varepsilon|\xi|^2)|\widehat{w}(\xi,t)|^2\mathrm{d}\xi - 2(1+t)^2\int_{\mathbb{R}}\mathrm{i}\,\xi\widehat{v}(\xi,t)\overline{\widehat{f(u)}}(\xi,t)\mathrm{d}\xi.\end{aligned}$$

Let $t > \frac{\varepsilon}{\alpha}$. Define a small, time-dependent interval

$$B(t) = \{\xi \in \mathbb{R} : \alpha(1+t)|\xi|^2 \le 1+\varepsilon|\xi|^2\}.$$

Then

$$\begin{aligned}&2\alpha(1+t)^2\int_{\mathbb{R}}|\xi|^2|\widehat{w}(\xi,t)|^2\mathrm{d}\xi\\ &= 2\alpha(1+t)^2\int_{B(t)}|\xi|^2|\widehat{w}(\xi,t)|^2\mathrm{d}\xi + 2\alpha(1+t)^2\int_{B(t)^c}|\xi|^2|\widehat{w}(\xi,t)|^2\mathrm{d}\xi\\ &\ge 2(1+t)\int_{B(t)^c}(1+\varepsilon|\xi|^2)|\widehat{w}(\xi,t)|^2\mathrm{d}\xi\\ &= 2(1+t)\int_{\mathbb{R}}(1+\varepsilon|\xi|^2)|\widehat{w}(\xi,t)|^2\mathrm{d}\xi - 2(1+t)\int_{B(t)}(1+\varepsilon|\xi|^2)|\widehat{w}(\xi,t)|^2\mathrm{d}\xi.\end{aligned}$$

Thus

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\Big\{(1+t)^2\int_{\mathbb{R}}(1+\varepsilon|\xi|^2)|\widehat{w}(\xi,t)|^2\mathrm{d}\xi\Big\}\\ &\le 2(1+t)\int_{B(t)}(1+\varepsilon|\xi|^2)|\widehat{w}(\xi,t)|^2\mathrm{d}\xi + 2(1+t)^2\Big|\int_{\mathbb{R}}\xi\widehat{v}(\xi,t)\overline{\widehat{f(u)}}(\xi,t)\mathrm{d}\xi\Big|.\end{aligned}$$

By using some well-known decay estimates (see Lemma 2.1 and Theorem 1.2), we get

$$\begin{aligned}&\Big|\int_{\mathbb{R}}\xi\widehat{v}(\xi,t)\overline{\widehat{f(u)}}(\xi,t)\mathrm{d}\xi\Big| = 2\pi\Big|\int_{\mathbb{R}}v_xf(u)\mathrm{d}x\Big|\\ &\le C\|v_x(\,\cdot\,,t)\|_{L^\infty(\mathbb{R})}\int_{\mathbb{R}}|u(x,t)|^{3+\kappa}\mathrm{d}x\\ &\le C\|v_x(\,\cdot\,,t)\|_{L^\infty(\mathbb{R})}\Big[\int_{\mathbb{R}}|u(x,t)|^2\mathrm{d}x\Big]^{(5+\kappa)/4}\Big[\int_{\mathbb{R}}|u_x(x,t)|^2\mathrm{d}x\Big]^{(1+\kappa)/4}\\ &\le C(1+t)^{-2-\kappa/2}.\end{aligned}$$

Furthermore, by using Lemma 2.2, we find

$$2(1+t)\int_{B(t)}(1+\varepsilon|\xi|^2)|\widehat{w}(\xi,t)|^2\mathrm{d}\xi \le C(1+t)^{1/4}\int_{B(t)}\mathrm{d}\xi \le C(1+t)^{-1/4}.$$

Therefore

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Big\{(1+t)^2\int_{\mathbb{R}}(1+\varepsilon|\xi|^2)|\widehat{w}(\xi,t)|^2\mathrm{d}\xi\Big\} &\le 2(1+t)\int_{B(t)}(1+\varepsilon|\xi|^2)|\widehat{w}(\xi,t)|^2\mathrm{d}\xi + C(1+t)^{-\kappa/2}\\ &\le C(1+t)^{-1/4}+C(1+t)^{-\kappa/2}\le C.\end{aligned}$$

Integrating this inequality with respect to time leads to

$$(1+t)^2\int_{\mathbb{R}}(1+\varepsilon|\xi|^2)|\widehat{w}(\xi,t)|^2\mathrm{d}\xi \le \int_{\mathbb{R}}(1+\varepsilon|\xi|^2)|\widehat{w}_0(\xi)|^2\mathrm{d}\xi + C(1+t).$$

Now (2.5) is proved. The estimate (2.6) can be established similarly. The proof of Lemma 2.3 is completed now.

Lemma 2.4 *Let* $u_0 = v_0 \in L^1(\mathbb{R})\cap H^2(\mathbb{R})$, $u_0(x)=\rho_0'(x)$, $\rho_0\in C^1(\mathbb{R})\cap L^1(\mathbb{R})$, *and*

$$\int_{\mathbb{R}}\rho_0(x)\mathrm{d}x \ne 0.$$

Then, there exists a positive number λ, *such that the Cauchy problem* (2.3)–(2.4) *enjoys the decay estimates*

$$(1+t)^{3/2}\int_{\mathbb{R}}[|u(x,t)-v(x,t)|^2+\varepsilon|u_x(x,t)-v_x(x,t)|^2]\mathrm{d}x \le \frac{C}{[1+\lambda\ln(1+t)]^{\lambda}}, \quad (2.7)$$

$$(1+t)^{m+3/2}\int_{\mathbb{R}}[|u_{x^m}(x,t)-v_{x^m}(x,t)|^2+\varepsilon|u_{x^{m+1}}(x,t)-v_{x^{m+1}}(x,t)|^2]\mathrm{d}x \le \frac{C_m}{[1+\lambda\ln(1+t)]^{\lambda}}, \quad (2.8)$$

where the positive constants C *and* C_m *are independent of time.*

Proof The proof is rather similar to that of Lemma 2.3. The main difference is the way to split the frequency space into two time-dependent regions. As what we did in Lemma 2.3, we may obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\Big\{(1+t)^4\int_{\mathbb{R}}(1+\varepsilon|\xi|^2)|\widehat{w}(\xi,t)|^2\mathrm{d}\xi\Big\}+2\alpha(1+t)^4\int_{\mathbb{R}}|\xi|^2|\widehat{w}(\xi,t)|^2\mathrm{d}\xi\\ &=4(1+t)^3\int_{\mathbb{R}}(1+\varepsilon|\xi|^2)|\widehat{w}(\xi,t)|^2\mathrm{d}\xi - 2(1+t)^4\int_{\mathbb{R}}\mathrm{i}\,\xi\widehat{v}(\xi,t)\overline{\widehat{f(u)}}(\xi,t)\mathrm{d}\xi.\end{aligned}$$

Let $t>\frac{2\varepsilon}{\alpha}$. Define

$$\begin{aligned}\mathcal{D}(t)&=\{\xi\in\mathbb{R}:\alpha(1+t)|\xi|^2\le 2(1+\varepsilon|\xi|^2)\},\\ \Omega(t)&=\{\xi\in\mathbb{R}:\alpha(1+t)|\xi|^2\omega(t)\le 2(1+\varepsilon|\xi|^2)\},\end{aligned}$$

where

$$\omega(t)=\frac{1}{[1+\lambda\ln(1+t)]^{2\lambda}},$$

and λ is a positive number to be determined later. It is easy to see that

$$2\alpha(1+t)^4\int_{\mathcal{D}(t)}|\xi|^2|\widehat{w}(\xi,t)|^2\mathrm{d}\xi \le 4(1+t)^3\int_{\mathcal{D}(t)}(1+\varepsilon|\xi|^2)|\widehat{w}(\xi,t)|^2\mathrm{d}\xi,$$
$$2\alpha(1+t)^4\int_{\mathcal{D}(t)^c}|\xi|^2|\widehat{w}(\xi,t)|^2\mathrm{d}\xi \ge 4(1+t)^3\int_{\mathcal{D}(t)^c}(1+\varepsilon|\xi|^2)|\widehat{w}(\xi,t)|^2\mathrm{d}\xi.$$

Choose $\lambda = 2$. Now we have

$$\begin{aligned}
&2\alpha(1+t)^4\int_{\Omega(t)^c}|\xi|^2|\widehat{w}(\xi,t)|^2\mathrm{d}\xi - 4(1+t)^3\int_{\Omega(t)^c}(1+\varepsilon|\xi|^2)|\widehat{w}(\xi,t)|^2\mathrm{d}\xi\\
&= 2\alpha(1+t)^4\int_{\mathcal{D}(t)^c}|\xi|^2|\widehat{w}(\xi,t)|^2\mathrm{d}\xi - 4(1+t)^3\int_{\mathcal{D}(t)^c}(1+\varepsilon|\xi|^2)|\widehat{w}(\xi,t)|^2\mathrm{d}\xi\\
&\quad + 2\alpha(1+t)^4\int_{\mathcal{D}(t)\cap\Omega(t)^c}|\xi|^2|\widehat{w}(\xi,t)|^2\mathrm{d}\xi - 4(1+t)^3\int_{\mathcal{D}(t)\cap\Omega(t)^c}(1+\varepsilon|\xi|^2)|\widehat{w}(\xi,t)|^2\mathrm{d}\xi \ge 0.
\end{aligned}$$

Let $t > 0$ be sufficiently large. Overall, we find that

$$2\alpha(1+t)^4\int_{\Omega(t)^c}|\xi|^2|\widehat{w}(\xi,t)|^2\mathrm{d}\xi \ge 4(1+t)^3\int_{\Omega(t)^c}(1+\varepsilon|\xi|^2)|\widehat{w}(\xi,t)|^2\mathrm{d}\xi.$$

Thus

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\Big\{(1+t)^4\int_{\mathbb{R}}(1+\varepsilon|\xi|^2)|\widehat{w}(\xi,t)|^2\mathrm{d}\xi\Big\}\\
&\le 4(1+t)^3\int_{\Omega(t)}(1+\varepsilon|\xi|^2)|\widehat{w}(\xi,t)|^2\mathrm{d}\xi + 2(1+t)^4\Big|\int_{\mathbb{R}}\xi\widehat{v}(\xi,t)\overline{\widehat{f(u)}}(\xi,t)\mathrm{d}\xi\Big|.
\end{aligned}$$

By using some well-known decay estimates (see Lemma 2.1 and Theorem 1.2), we get

$$\begin{aligned}
&\Big|\int_{\mathbb{R}}\xi\widehat{v}(\xi,t)\overline{\widehat{f(u)}}(\xi,t)\mathrm{d}\xi\Big| = 2\pi\Big|\int_{\mathbb{R}}v_x f(u)\mathrm{d}x\Big|\\
&\le C\|v_x(\,\cdot\,,t)\|_{L^\infty(\mathbb{R})}\int_{\mathbb{R}}|u(x,t)|^{3+\kappa}\mathrm{d}x\\
&\le C\|v_x(\,\cdot\,,t)\|_{L^\infty(\mathbb{R})}\Big[\int_{\mathbb{R}}|u(x,t)|^2\mathrm{d}x\Big]^{(5+\kappa)/4}\Big[\int_{\mathbb{R}}|u_x(x,t)|^2\mathrm{d}x\Big]^{(1+\kappa)/4}\\
&\le C(1+t)^{-4-\kappa}.
\end{aligned}$$

Furthermore, by using Lemma 2.2, we find

$$4(1+t)^3\int_{\Omega(t)}(1+\varepsilon|\xi|^2)|\widehat{w}(\xi,t)|^2\mathrm{d}\xi \le C(1+t)^2\int_{\Omega(t)}\mathrm{d}\xi \le \frac{C(1+t)^{3/2}}{[1+\lambda\ln(1+t)]^\lambda}.$$

Therefore

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\Big\{(1+t)^4\int_{\mathbb{R}}(1+\varepsilon|\xi|^2)|\widehat{w}(\xi,t)|^2\mathrm{d}\xi\Big\} &\le 4(1+t)^3\int_{\Omega(t)}(1+\varepsilon|\xi|^2)|\widehat{w}(\xi,t)|^2\mathrm{d}\xi + C(1+t)^{-\kappa}\\
&\le \frac{C(1+t)^{3/2}}{[1+\lambda\ln(1+t)]^\lambda}.
\end{aligned}$$

Integrating this inequality with respect to time leads to

$$(1+t)^4\int_{\mathbb{R}}(1+\varepsilon|\xi|^2)|\widehat{w}(\xi,t)|^2\mathrm{d}\xi \le \int_{\mathbb{R}}(1+\varepsilon|\xi|^2)|\widehat{w_0}(\xi)|^2\mathrm{d}\xi + \frac{C(1+t)^{5/2}}{[1+\lambda\ln(1+t)]^\lambda}.$$

Now (2.7) is proved. The estimate (2.8) can be established similarly. The proof of Lemma 2.4 is completed.

Lemma 2.5 (I) *Suppose that* $u_0 = v_0 \in L^1(\mathbb{R}) \cap H^2(\mathbb{R})$ *and*

$$\int_{\mathbb{R}} u_0(x)\mathrm{d}x \neq 0.$$

Then the unique global solution of the Cauchy problem (1.1)–(1.2) *enjoys the limits*

$$\begin{aligned}
&\lim_{t\to\infty}\Big\{(1+t)^{1/2}\int_{\mathbb{R}}[|u(x,t)|^2+\varepsilon|u_x(x,t)|^2]\mathrm{d}x\Big\}\\
&=\lim_{t\to\infty}\Big\{(1+t)^{1/2}\int_{\mathbb{R}}[|v(x,t)|^2+\varepsilon|v_x(x,t)|^2]\mathrm{d}x\Big\},\\
&\lim_{t\to\infty}\Big\{(1+t)^{m+1/2}\int_{\mathbb{R}}[|u_{x^m}(x,t)|^2+\varepsilon|u_{x^{m+1}}(x,t)|^2]\mathrm{d}x\Big\}\\
&=\lim_{t\to\infty}\Big\{(1+t)^{m+1/2}\int_{\mathbb{R}}[|v_{x^m}(x,t)|^2+\varepsilon|v_{x^{m+1}}(x,t)|^2]\mathrm{d}x\Big\}.
\end{aligned}$$

(II) *Suppose that* $u_0 = v_0 \in L^1(\mathbb{R}) \cap H^2(\mathbb{R})$, $u_0(x) = \rho_0'(x)$, $\rho_0 \in C^1(\mathbb{R}) \cap L^1(\mathbb{R})$, *and*

$$\int_{\mathbb{R}} \rho_0(x)\mathrm{d}x \neq 0.$$

Then

$$\begin{aligned}
&\lim_{t\to\infty}\Big\{(1+t)^{3/2}\int_{\mathbb{R}}[|u(x,t)|^2+\varepsilon|u_x(x,t)|^2]\mathrm{d}x\Big\}\\
&=\lim_{t\to\infty}\Big\{(1+t)^{3/2}\int_{\mathbb{R}}[|v(x,t)|^2+\varepsilon|v_x(x,t)|^2]\mathrm{d}x\Big\},\\
&\lim_{t\to\infty}\Big\{(1+t)^{m+3/2}\int_{\mathbb{R}}[|u_{x^m}(x,t)|^2+\varepsilon|u_{x^{m+1}}(x,t)|^2]\mathrm{d}x\Big\}\\
&=\lim_{t\to\infty}\Big\{(1+t)^{m+3/2}\int_{\mathbb{R}}[|v_{x^m}(x,t)|^2+\varepsilon|v_{x^{m+1}}(x,t)|^2]\mathrm{d}x\Big\}.
\end{aligned}$$

Proof Below we are going to use the notation $\|u_{x^m}(\,\cdot\,,t)\|_\varepsilon$ defined by

$$\|u_{x^m}(\,\cdot\,,t)\|_\varepsilon^2 = \|u_{x^m}(\,\cdot\,,t)\|_{L^2(\mathbb{R})}^2 + \varepsilon\|u_{x^{m+1}}(\,\cdot\,,t)\|_{L^2(\mathbb{R})}^2.$$

If $\varepsilon = 0$ and $m = 0$, then it is just the regular L^2-norm. If $\varepsilon > 0$ and $m = 0$, then it is equivalent to the H^1-norm. If $\varepsilon \geq 0$ and $m \geq 1$, then $\|u_{x^m}(\,\cdot\,,t)\|_\varepsilon$ is a seminorm and the triangle inequality is still valid. By using triangle inequality, first of all, we have the upper bound estimate

$$\begin{aligned}
&(1+t)^{m/2+1/4}\|u_{x^m}(\,\cdot\,,t)\|_\varepsilon\\
&=(1+t)^{m/2+1/4}\|v_{x^m}(\,\cdot\,,t)+u_{x^m}(\,\cdot\,,t)-v_{x^m}(\,\cdot\,,t)\|_\varepsilon\\
&\leq(1+t)^{m/2+1/4}\|v_{x^m}(\,\cdot\,,t)\|_\varepsilon+(1+t)^{m/2+1/4}\|u_{x^m}(\,\cdot\,,t)-v_{x^m}(\,\cdot\,,t)\|_\varepsilon\\
&\leq(1+t)^{m/2+1/4}\|v_{x^m}(\,\cdot\,,t)\|_\varepsilon+\mathcal{C}(t),
\end{aligned}$$

where we have taken the results of Lemma 2.3 into account. Recall that $\mathcal{C}(t) \equiv \frac{C}{(1+t)^{1/4}} \to 0$ as $t \to \infty$. Now we get

$$\limsup_{t\to\infty}[(1+t)^{m/2+1/4}\|u_{x^m}(\,\cdot\,,t)\|_\varepsilon] \le \lim_{t\to\infty}[(1+t)^{m/2+1/4}\|v_{x^m}(\,\cdot\,,t)\|_\varepsilon].$$

On the other hand, we have the lower bound estimate

$$\begin{aligned}
&(1+t)^{m/2+1/4}\|u_{x^m}(\,\cdot\,,t)\|_\varepsilon \\
&= (1+t)^{m/2+1/4}\|v_{x^m}(\,\cdot\,,t)+u_{x^m}(\,\cdot\,,t)-v_{x^m}(\,\cdot\,,t)\|_\varepsilon \\
&\ge (1+t)^{m/2+1/4}\|v_{x^m}(\,\cdot\,,t)\|_\varepsilon-(1+t)^{m/2+1/4}\|u_{x^m}(\,\cdot\,,t)-v_{x^m}(\,\cdot\,,t)\|_\varepsilon \\
&\ge (1+t)^{m/2+1/4}\|v_{x^m}(\,\cdot\,,t)\|_\varepsilon-\mathcal{C}(t).
\end{aligned}$$

Therefore, we also get

$$\liminf_{t\to\infty}[(1+t)^{m/2+1/4}\|u_{x^m}(\,\cdot\,,t)\|_\varepsilon] \ge \lim_{t\to\infty}[(1+t)^{m/2+1/4}\|v_{x^m}(\,\cdot\,,t)\|_\varepsilon].$$

By coupling these two estimates together, we obtain

$$\limsup_{t\to\infty}[(1+t)^{m/2+1/4}\|u_{x^m}(\,\cdot\,,t)\|_\varepsilon] \le \liminf_{t\to\infty}[(1+t)^{m/2+1/4}\|u_{x^m}(\,\cdot\,,t)\|_\varepsilon].$$

Therefore, there exists the limit

$$\lim_{t\to\infty}[(1+t)^{m+1/2}\|u_{x^m}(\,\cdot\,,t)\|_\varepsilon^2] = \lim_{t\to\infty}[(1+t)^{m+1/2}\|v_{x^m}(\,\cdot\,,t)\|_\varepsilon^2].$$

The other case may be proved very similarly, where we may take

$$\mathcal{C}(t) \equiv \frac{C}{[1+2\ln(1+t)]^2} \to 0,$$

as $t\to\infty$. Now the proof of Lemma 2.5 is completed.

3 The Main Results and Proofs

Theorem 3.1 *Suppose that* $u_0 = v_0 \in L^1(\mathbb{R}) \cap H^2(\mathbb{R})$, *and*

$$\int_{\mathbb{R}} u_0(x)\mathrm{d}x \ne 0.$$

Then the unique global solution of the Cauchy problem (1.1)–(1.2) *enjoys the limits*

$$\lim_{t\to\infty}\Big\{(1+t)^{1/2}\int_{\mathbb{R}}[|u(x,t)|^2+\varepsilon|u_x(x,t)|^2]\mathrm{d}x\Big\} = \frac{1}{2\pi}\Big(\frac{\pi}{2\alpha}\Big)^{1/2}\Big[\int_{\mathbb{R}} u_0(x)\mathrm{d}x\Big]^2,$$

$$\lim_{t\to\infty}\Big\{(1+t)^{m+1/2}\int_{\mathbb{R}}[|u_{x^m}(x,t)|^2+\varepsilon|u_{x^{m+1}}(x,t)|^2]\mathrm{d}x\Big\} = \frac{1}{2\pi}\Big(\frac{\pi}{2\alpha}\Big)^{1/2}\frac{m!!}{(4\alpha)^m}\Big[\int_{\mathbb{R}} u_0(x)\mathrm{d}x\Big]^2.$$

Proof Note that for any integer $m \ge 0$, we have

$$\lim_{t\to\infty}\Big[(1+t)^{m+1/2}\int_{\mathbb{R}}|u_{x^{m+1}}(x,t)|^2\mathrm{d}x\Big] = \lim_{t\to\infty}\Big[(1+t)^{m+1/2}\int_{\mathbb{R}}|v_{x^{m+1}}(x,t)|^2\mathrm{d}x\Big] = 0.$$

Therefore, there exists the limit

$$\begin{aligned}\lim_{t\to\infty}[(1+t)^{m+1/2}\|u_{x^m}(\,\cdot\,,t)\|_{\varepsilon}^2] &= \lim_{t\to\infty}[(1+t)^{m+1/2}\|v_{x^m}(\,\cdot\,,t)\|_{\varepsilon}^2]\\ &= \lim_{t\to\infty}[(1+t)^{m+1/2}\|v_{x^m}(\,\cdot\,,t)\|_{L^2(\mathbb{R})}^2]\\ &= \frac{1}{2\pi}\Big(\frac{\pi}{2\alpha}\Big)\frac{m!!}{(4\alpha)^m}\Big|\int_{\mathbb{R}}u_0(x)\mathrm{d}x\Big|^2.\end{aligned}$$

The proof of Theorem 3.1 is completed.

Theorem 3.2 *Suppose that* $u_0\in L^1(\mathbb{R})\cap H^2(\mathbb{R})$, *and that* $u_0(x)=\rho_0'(x)$, $\rho_0\in C^1(\mathbb{R})\cap L^1(\mathbb{R})$, *with*

$$\int_{\mathbb{R}}\rho_0(x)\mathrm{d}x\neq 0.$$

Then the unique global solution enjoys the limits

$$\lim_{t\to\infty}\Big\{(1+t)^{3/2}\int_{\mathbb{R}}[|u(x,t)|^2+\varepsilon|u_x(x,t)|^2]\mathrm{d}x\Big\}=\frac{1}{2\pi}\Big(\frac{\pi}{2\alpha}\Big)^{1/2}\frac{1}{4\alpha}\Big[\int_{\mathbb{R}}\rho_0(x)\mathrm{d}x\Big]^2,$$

$$\lim_{t\to\infty}\Big\{(1+t)^{m+3/2}\int_{\mathbb{R}}[|u_{x^m}(x,t)|^2+\varepsilon|u_{x^{m+1}}(x,t)|^2]\mathrm{d}x\Big\}=\frac{1}{2\pi}\Big(\frac{\pi}{2\alpha}\Big)^{1/2}\frac{(m+1)!!}{(4\alpha)^{m+1}}\Big[\int_{\mathbb{R}}\rho_0(x)\mathrm{d}x\Big]^2.$$

Proof It is very similar to the proof of Theorem 3.1.

The proofs of Theorems 1.3 and 1.4 may be accomplished by coupling the results of Theorems 3.1 and 3.2, respectively, and the following limits

$$\lim_{t\to\infty}\Big\{(1+t)^{m+1/2}\int_{\mathbb{R}}|u_{x^{m+1}}(x,t)|^2\mathrm{d}x\Big\}=0,\quad\text{if }\int_{\mathbb{R}}u_0(x)\mathrm{d}x\neq 0.$$

and

$$\lim_{t\to\infty}\Big\{(1+t)^{m+3/2}\int_{\mathbb{R}}|u_{x^{m+1}}(x,t)|^2\mathrm{d}x\Big\}=0,\quad\text{if }\int_{\mathbb{R}}u_0(x)\mathrm{d}x=0,$$

respectively.

Given two initial functions u_0 and $\widetilde{u}_0$ in $L^1(\mathbb{R})\cap H^2(\mathbb{R})$, there exist two solutions $u=u(x,t)$ and $\widetilde{u}=\widetilde{u}(x,t)$ of problem (1.1)–(1.2) corresponding to u_0 and $\widetilde{u}_0$, respectively.

Theorem 3.3 (I) *There holds the limit*

$$\begin{aligned}&\lim_{t\to\infty}\Big\{(1+t)^{m+1/2}\int_{\mathbb{R}}[|u_{x^m}(x,t)-\widetilde{u}_{x^m}(x,t)|^2+\varepsilon|u_{x^{m+1}}(x,t)-\widetilde{u}_{x^{m+1}}(x,t)|^2]\mathrm{d}x\Big\}\\ &=\frac{1}{2\pi}\Big(\frac{\pi}{2\alpha}\Big)^{1/2}\frac{m!!}{(4\alpha)^m}\Big\{\int_{\mathbb{R}}[u_0(x)-\widetilde{u}_0(x)]\mathrm{d}x\Big\}^2,\end{aligned}$$

if $\int_{\mathbb{R}}[u_0(x)-\widetilde{u}_0(x)]\mathrm{d}x\neq 0$.

(II) *There holds the limit*

$$\begin{aligned}&\lim_{t\to\infty}\Big\{(1+t)^{m+3/2}\int_{\mathbb{R}}[|u_{x^m}(x,t)-\widetilde{u}_{x^m}(x,t)|^2+\varepsilon|u_{x^{m+1}}(x,t)-\widetilde{u}_{x^{m+1}}(x,t)|^2]\mathrm{d}x\Big\}\\ &=\frac{1}{2\pi}\Big(\frac{\pi}{2\alpha}\Big)^{1/2}\frac{(m+1)!!}{(4\alpha)^{m+1}}\Big\{\int_{\mathbb{R}}\rho_0(x)\mathrm{d}x\Big\}^2,\end{aligned}$$

if $\int_{\mathbb{R}}[u_0(x)-\widetilde{u}_0(x)]\mathrm{d}x=0$, $u_0(x)-\widetilde{u}_0(x)=\rho_0'(x)$ *and* $\rho_0\in C^1(\mathbb{R})\cap L^1(\mathbb{R})$.

Proof The idea of the proof is very similar to those presented in the proofs of Lemmas 2.3–2.5 and Theorems 3.1–3.2. Define $\psi(x,t)=u(x,t)-\widetilde{u}(x,t)$, $\phi(x,t)=v(x,t)-\widetilde{v}(x,t)$, and $\psi_0(x)=\phi_0(x)=u_0(x)-\widetilde{u}_0(x)$. Then ψ and ϕ satisfy

$$\begin{aligned}&\psi_t-\varepsilon\psi_{xxt}+\delta\psi_x+\gamma H\psi_{xx}+\beta\psi_{xxx}+[f(u)-f(\widetilde{u})]_x=\alpha\psi_{xx},\\&\psi(x,0)=\psi_0(x),\end{aligned}$$

and

$$\begin{aligned}&\phi_t-\varepsilon\phi_{xxt}+\delta\phi_x+\gamma H\phi_{xx}+\beta\phi_{xxx}=\alpha\phi_{xx},\\&\phi(x,0)=\phi_0(x).\end{aligned}$$

Then, very similar to the above, we obtain the estimates

$$(1+t)^{m+1/2}\int_{\mathbb{R}}[|\psi_{x^m}(x,t)-\phi_{x^m}(x,t)|^2+\varepsilon|\psi_{x^{m+1}}(x,t)-\phi_{x^{m+1}}(x,t)|^2]\mathrm{d}x\le\frac{C}{(1+t)^{1/2}},$$

if

$$\int_{\mathbb{R}}[u_0(x)-\widetilde{u}_0(x)]\mathrm{d}x\ne 0$$

and

$$(1+t)^{m+3/2}\int_{\mathbb{R}}[|\psi_{x^m}(x,t)-\phi_{x^m}(x,t)|^2+\varepsilon|\psi_{x^{m+1}}(x,t)-\phi_{x^{m+1}}(x,t)|^2]\mathrm{d}x\le\frac{C}{[1+2\ln(1+t)]^2},$$

if

$$\int_{\mathbb{R}}[u_0(x)-\widetilde{u}_0(x)]\mathrm{d}x=0.$$

The other similar details are omitted.

The results of Theorem 3.3 imply that for each fixed integer $m\ge 0$, the nonlinear operator

$$\mathcal{L}_m:u_0\mapsto\lim_{t\to\infty}\Big\{(1+t)^{m+1/2}\int_{\mathbb{R}}[|u_{x^m}(x,t)|^2+\varepsilon|u_{x^{m+1}}(x,t)|^2]\mathrm{d}x\Big\}^{1/2}$$

is Lipschitz continuous. Note that the unique solution u depends on the initial data u_0. Indeed, for any two initial functions u_0 and $\widetilde{u}_0$ in $L^1(\mathbb{R})\cap H^2(\mathbb{R})$, there holds the estimate

$$|\mathcal{L}_m u_0-\mathcal{L}_m\widetilde{u}_0|\le\Big[\frac{1}{2\pi}\Big(\frac{\pi}{2\alpha}\Big)^{1/2}\frac{m!!}{(4\alpha)^m}\Big]^{1/2}\Big[\int_{\mathbb{R}}|u_0(x)-\widetilde{u}_0(x)|\mathrm{d}x\Big].$$

Remark 3.1 For $|f(u)|=\mathcal{O}(|u|^{3+\kappa})$, as $|u|\to\infty$, with large κ, if the initial data is sufficiently small, then the existence and uniqueness of the global solution of problem (1.1)–(1.2) are also true. We focused on the exact limits of the global solutions for the case $0\le\kappa\le 2$, with arbitrarily large initial data. The existence and uniqueness of the global strong solution are open for (1.1)–(1.2) with large κ and large u_0.

Remark 3.2 The model equation (1.1) can be written as

$$\mathcal{P}u_t + \mathcal{Q}u + \mathcal{R}u_x + \mathcal{F}(u, u_x, u_{xx}) = 0,$$

where $\mathcal{F}(u, u_x, u_{xx})$ is a nonlinear function of u, u_x and u_{xx}, $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{R}$ are linear differential operators, specifically, $\mathcal{P}$ and $\mathcal{Q}$ are dissipative operators, $\partial_x \mathcal{R}$ is a dispersive operator. Additionally,

$$\begin{aligned}
&\widehat{\mathcal{P}u}(\xi,t) = p(\xi)\widehat{u}(\xi,t), \quad p(0)=1, \quad p(\xi) \geq 1, \quad \forall\, \xi \in \mathbb{R},\\
&\widehat{\mathcal{Q}u}(\xi,t) = q(\xi)\widehat{u}(\xi,t), \quad q(0)=0, \quad q(\xi) > 0, \quad \forall\, \xi \neq 0,\\
&\widehat{\mathcal{R}u}(\xi,t) = r(\xi)\widehat{u}(\xi,t), \quad r(0) \in \mathbb{R}, \quad r(\xi) \in \mathbb{R}, \quad \forall\, \xi \in \mathbb{R},
\end{aligned}$$

where p, q and r are even functions of ξ. In this paper, we only considered the particular case $\mathcal{P}u = u - \varepsilon u_{xx}$, $\mathcal{Q}u = -\alpha u_{xx}$, $\mathcal{R}u = \delta + \gamma H u_x + \beta u_{xx}$, and $\mathcal{F}(u, u_x) = f(u)_x$.

Remark 3.3 Many mathematicians have investigated the existence of global attractors and the existence of inertial manifolds of infinite-dimensional dynamical systems governed by dissipative nonlinear partial differential equations, including our model equation (1.1), the Navier-Stokes equations, the Benard flow problem and the Bingham fluid. Inertial manifold is a finite-dimensional invariant Lipschitz manifold which attracts exponentially all orbits and contains the global attractor. Estimates of Hausdorff dimension and fractal dimension of the global attractor and inertial manifold have been obtained by using various advanced techniques. Some authors have constructed finite-dimensional manifolds (approximate finite-dimensional inertial manifolds) and applied the results to reaction diffusion equations. An approximate inertial manifold can be defined as a finite-dimensional Lipschitz manifold and a thin surrounding neighborhood into which any orbit enters in a finite time. It is clear that the global attractor lies in this neighborhood. The lowest dimension of inertial manifolds for some particular system (e.g. the one-dimensional Kuramoto-Sivashinski equation) have also been established. The finite-dimensional global attractor and the inertial manifold open an important way for the reduction of the dynamics of infinite-dimensional dissipative differential equations to a finite-dimensional system. More precisely, people consider a finite-dimensional system that will capture all the asymptotic behavior of the original system. It has been suggested and expected that the limit will determine the Hausdorff dimension and the fractal dimension of the global attractor and the inertial manifold. Therefore, we study the evolution of the functions $(1+t)^{m+1/2}\int_{\mathbb{R}} |u_{x^m}(x,t)|^2 \mathrm{d}x$, where the integer $m \geq 0$. In particular, we investigate their limits as $t \to \infty$. It turns out that the sharp Hausdorff dimension and fractal dimension of the global attractor depend on the exact limit of a physically important quantity. The exact limits of the physical quantities may play significant roles in the evaluations of the Hausdorff dimension and fractal dimension of the global attractor and the inertial manifold of the infinite-dimensional dynamical systems.

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