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The Change-Base Issue for Ω-Categories**

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Abstract Let $G : \Omega \to \Omega'$ be a closed unital map between commutative, unital quantales. G induces a functor $\overline{G}$ from the category of Ω-categories to that of Ω'-categories. This paper is concerned with some basic properties of $\overline{G}$. The main results are: (1) when Ω, Ω' are integral, $G : \Omega \to \Omega'$ and $F : \Omega' \to \Omega$ are closed unital maps, $\overline{F}$ is a left adjoint of $\overline{G}$ if and only if F is a left adjoint of G; (2) $\overline{G}$ is an equivalence of categories if and only if G is an isomorphism in the category of commutative unital quantales and closed unital maps; and (3) a sufficient condition is obtained for $\overline{G}$ to preserve completeness in the sense that $\overline{G}A$ is a complete Ω'-category whenever A is a complete Ω-category.

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1 Introduction

Let $(\Omega, *, I)$ (or Ω for short) be a commutative, unital quantale (cf. Definition 2.1). From the point of view of category theory, $(\Omega, *, I)$ is exactly a symmetric, monoidal closed category with the underlying category being a complete lattice. Thus, one can establish a theory of Ω-categories (that is, categories enriched over Ω). If $G : (\Omega, *, I) \to (\Omega', *', I')$ is a closed unital map between commutative, unital quantales (cf. Definition 3.1) and A is an Ω-category, then

$$(\overline{G}A)(a, b) \triangleq G(A(a, b))$$

defines an Ω'-category $\overline{G}A$. This correspondence defines a functor $\overline{G} : \Omega\text{-}\mathbf{Cat} \to \Omega'\text{-}\mathbf{Cat}$ from the category of Ω-categories to that of Ω'-categories (cf. [4, 9]). This functor plays a role of change-base in the theory of Ω-categories. Therefore, the study of the functor $\overline{G}$ is important for Ω-categories. In this paper, we are concerned with some basic questions about $\overline{G}$:

(1) When does $\overline{G}$ have a left adjoint which is also of this form?

(2) When is $\overline{G}$ an equivalence of categories?

(3) When does $\overline{G}$ preserve completeness?

For the first question, a necessary and sufficient condition is obtained in the case that both Ω and Ω' are integral quantales. For the second, it is shown that $\overline{G}$ is an equivalence of categories if and only if G is an isomorphism in the category of commutative, unital quantales and closed

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unital maps. For the third, it is shown that if G has a left adjoint which preserves tensors then $\overline{G}$ preserves completeness.

At the end of this introduction, we recall some basic notions of concrete categories from [1], which shall be needed in the sequel.

By a concrete category over the category **Set** of sets is meant a pair $(\mathbf{A}, U)$, where $\mathbf{A}$ is a category and $U : \mathbf{A} \to \mathbf{Set}$ is a faithful functor. U is called the underlying functor or the forgetful functor. For each object A in $\mathbf{A}$, $U(A)$ (also written as $|A|$) is called the underlying set of A; and for each morphism $f : A \to B$ in $\mathbf{A}$, $U(f)$ is called the underlying function of f. In this paper, by a concrete category we always mean a concrete category over **Set**. A concrete category $(\mathbf{A}, U)$ is often abbreviated to $\mathbf{A}$ if the functor U is obvious. A concrete functor $F : (\mathbf{A}, U) \to (\mathbf{B}, V)$ is a functor $F : \mathbf{A} \to \mathbf{B}$ such that $V \circ F = U$.

2 Basic Ideas of Ω-Categories

Definition 2.1 (cf. [12]) *A (commutative) unital quantale is a triple $(\Omega, *, I)$, where Ω is a complete lattice, I is a fixed element of Ω, and $*$ is a (commutative) associative binary operation on Ω such that $a * (\vee b_i) = \vee(a * b_i)$, $(\vee b_i) * a = \vee(b_i * a)$ and $I * a = a = a * I$ for all $a, b_i \in \Omega$.*

For a unital quantale $(\Omega, *, I)$, the binary operation $*$ is called the tensor on Ω. $(\Omega, *, I)$ is often abbreviated to Ω if there would be no confusion with respect to the unit and the tensor. A unital quantale Ω is integral (cf. [6]) if the unit element coincides with the top element in Ω.

We are mainly concerned with commutative, unital quantales in this article. Given a commutative, unital quantale $(\Omega, *, I)$, let $a \to b = \vee\{c \in \Omega, a * c \le b\}$ for all $a, b \in \Omega$. Then $*$ and $\to$ are interlocked with each other by the adjoint property: $c \le a \to b \Leftrightarrow a * c \le b$. The binary operation $\to$ shall be called the cotensor of Ω (with respect to $*$).

Given a commutative, unital quantale $(\Omega, *, I)$, an Ω-category (cf. [7, 9]) is a set A together with an assignment of an element $A(a, b) \in \Omega$ to every ordered pair $(a, b) \in A \times A$, such that

(1) $I \le A(a, a)$ for every $a \in A$;

(2) $A(a, b) * A(b, c) \le A(a, c)$ for all $a, b, c \in A$.

An Ω-functor (or simply a functor) between Ω-categories A and B is a function $f : A \to B$ such that $A(a, b) \le B(f(a), f(b))$ for all $a, b \in A$. An Ω-functor f is called an Ω-isometry if $A(a, b) = B(f(a), f(b))$ for all $a, b \in A$. A bijective Ω-isometry f is an (Ω-)isomorphism. Ω-functors are composed by composing the underlying functions on sets. Ω-categories and Ω-functors form a concrete category, which is denoted by Ω-**Cat**.

Given an Ω-category A and $x, y \in A$, set $x \le y$ if $I \le A(x, y)$. Then $A_0 \triangleq (A, \le)$ is a preordered set, called the underlying preordered set of A. An Ω-category A is said to be antisymmetric if A_0 is a partially ordered set, that is, if $A(x, y) \ge I$ and $A(y, x) \ge I$ then $x = y$. Some examples of commutative, unital quantales and corresponding Ω-categories are given below.

Example 2.1 (1) Let $\{0, 1\}$ be the two-point lattice ordered by $0 < 1$. Then $\mathbf{2} =$

$(\{0,1\},\wedge,1)$ is a commutative, unital quantale. The category of **2**-categories is exactly the category **PrOrd** of preordered sets and order-preserving functions.

(2) (The Canonical Ω-Category Structure on Ω) Let $\Omega(\alpha,\beta)=\alpha\to\beta$. Then it is easy to check that Ω becomes an antisymmetric Ω-category. We shall write $(\Omega,\to)$ (or Ω for short) for this Ω-category.

(3) Let $\Omega=[0,\infty]^{\mathrm{op}}$ denote the extended interval of all non-negative real numbers with the opposite ordering as real numbers (so 0 is the greatest element). Let $+$ be the usual addition on real numbers extended to cope with infinity such as $x+\infty=\infty$ for every $x\in[0,\infty]$. Then $\Omega=([0,\infty]^{\mathrm{op}},+,0)$ is a commutative, unital quantale. The category of Ω-categories is just the category **GMet** of generalized metric spaces (or pseudo-quasi-metric spaces) and non-expansive functions (cf. [2, 9]).

(4) (Discrete Ω-Categories) Let X be a set. For all $x,y\in X$, let $X(x,y)=I$, the unit element in Ω, if $x=y$; otherwise, let $X(x,y)=0$, the least element in Ω. Then X becomes an Ω-category. Such an Ω-category will be called a discrete Ω-category since every function from such an Ω-category to any other Ω-category is always an Ω-functor.

Definition 2.2 (cf. [7, 16]) *Let K and A be Ω-categories.*

(1) *An element $a\in A$ is called a limit of an Ω-functor $f:K\to A$ weighted by $\psi:K\to\Omega$ if for each $y\in A$,*

$$A(y,a)=\bigwedge_{x\in K}\psi(x)\to A(y,f(x)).$$

(2) *An element $b\in A$ is called a colimit of an Ω-functor $f:K\to A$ weighted by $\phi:K^{\mathrm{op}}\to\Omega$ if for each $y\in A$,*

$$A(b,y)=\bigwedge_{x\in K}\phi(x)\to A(f(x),y).$$

Weighted limits, when they exist, are unique up to isomorphism. Thus, we write $b=\lim_\psi f$ if b is a limit of f weighted by ψ; similarly, we write $b=\mathrm{colim}_\phi f$ if b is a colimit of f weighted by ϕ.

Example 2.2 If A is an Ω-category, let $|A|$ be the discrete Ω-category with the same underlying set of A. If a is a limit of $\mathrm{id}:|A|\to A$ weighted by $\mu:|A|\to\Omega$, then for all $y\in A$,

$$A(y,a)=\bigwedge_{x\in A}\mu(x)\to A(y,x).$$

This equality can be interpreted as that y is smaller than or equal to a if and only if y is smaller than or equal to x whenever x is in μ. Therefore, a is called an infimum (or a greatest lower bound) of μ, denoted by $\inf\mu$.

Similarly, if a is a colimit of $\mathrm{id}:|A|\to A$ weighted by $\mu:|A|\to\Omega$, then for all $y\in A$,

$$A(b,y)=\bigwedge_{x\in A}\mu(x)\to A(x,y).$$

b is called a supremum (or a least upper bound) of μ, denoted by $\sup\mu$.

Definition 2.3 (cf. [3, 7]) *An Ω-category A is said to be complete if for any Ω-functor $f : K \to A$ and any $\psi : K \to \Omega$, the weighted limit* $\lim_\psi f$ *exists. A is said to be cocomplete if for any Ω-functor $f : K \to A$ and any $\phi : K^{\mathrm{op}} \to \Omega$, the weighted colimit* $\mathrm{colim}_\phi f$ *exists.*

Proposition 2.1 (cf. [10, 14]) *Suppose that A is an Ω-category. Then the following conditions are equivalent:*

(1) *A is complete.*

(2) *Every $\mu \in \Omega^A$ has an infimum.*

(3) *Every $\mu \in \Omega^A$ has a supremum.*

(4) *A is cocomplete.*

Definition 2.4 *An Ω-functor $f : A \to B$ is said to preserve weighted limits if for all Ω-functor $g : C \to A$ and $\psi : C \to \Omega$ such that the weighted limit* $\lim_\psi g$ *exists, the weighted limit of $f \circ g : C \to B$ weighted by ψ exists and* $\lim_\psi(fg) = f(\lim_\psi g)$. *Dually, one can define weighted-colimits-preserving Ω-functors. An Ω-functor f is said to be complete if f preserves both weighted limits and weighted colimits.*

A complete Ω-lattice is an antisymmetric, complete Ω-category. All complete Ω-lattices and complete maps form a category, denoted Ω-**CLat**.

Example 2.3 (cf. [3, 10]) $(\Omega, \to)$ is a complete Ω-lattice, since for any Ω-functor $f : K \to \Omega$ and any $\psi : K \to \Omega$ the weighted limit exists and

$$\lim_\psi f = \bigwedge_{x \in K} (\psi(x) \to f(x)).$$

Definition 2.5 (cf. [3, 7, 15]) *Let A be an Ω-category. Then*

(1) *A is said to be tensored if for all $\alpha \in \Omega, x \in A$, there is an element $\alpha \otimes x \in A$ such that*

$$A(\alpha \otimes x, y) = \alpha \to A(x, y)$$

for any $y \in A$. In this case $\alpha \otimes x$ is called the tensor of α with x.

(2) *A is said to be cotensored if for all $\alpha \in \Omega, x \in A$, there is an element $\alpha \rightarrowtail x \in A$ such that*

$$A(y, \alpha \rightarrowtail x) = \alpha \to A(y, x)$$

for any $y \in A$. In this case $\alpha \rightarrowtail x$ is called the cotensor of α with x.

For any $\alpha \in \Omega, x \in A$, define a function $\alpha_x : |A| \to \Omega$ by $\alpha_x(z) = \alpha$ if $z = x$ and $\alpha_x(z) = 0$ if $z \neq x$. Then $\alpha \otimes x$ and $\alpha \rightarrowtail x$ are exactly the supremum and infimum of α_x respectively (cf. [17]). Thus, every complete Ω-category is both tensored and cotensored (cf. [15]).

It is easy to see that if an Ω-category A is tensored then A_0 has a bottom element $\perp$. Similarly, if A is cotensored then A_0 has a top element $\top$.

Proposition 2.2 (cf. [15]) *Suppose that A is an antisymmetric Ω-category.*

(1) *If A is tensored, then the tensor $\otimes : \Omega \times A_0 \to A_0$ satisfies:*

(i) *$0 \otimes x = \perp$, $I \otimes x = x$.*

(ii) $A(\alpha\otimes x,y)=\alpha\to A(x,y)$. *Hence,* $\alpha\otimes x\le y$ *in* $A_0\Leftrightarrow\alpha\le A(x,y)$.

(iii) $(\alpha*\beta)\otimes x=\alpha\otimes(\beta\otimes x)$.

(2) *If A is both tensored and cotensored, then for any* $\alpha\in\Omega$, $\alpha\otimes(-):A_0\to A_0$ *is a left adjoint of* $\alpha\rightarrowtail(-):A_0\to A_0$.

(3) *If A is both tensored and cotensored, then for any* $x\in A$, $(-)\otimes x:\Omega\to A_0$ *is a left adjoint of* $A(x,-):A_0\to\Omega$.

Theorem 2.1 (cf. [15]) *Let A be a both tensored and cotensored Ω-category. Then A is a complete Ω-category if and only if A_0 is a complete preorder.*

Theorem 2.2 (cf. [15]) *Let $f:A\to B$ be an Ω-functor between complete Ω-categories. Then the following conditions are equivalent:*

(1) *f is complete.*

(2) *$f:A_0\to B_0$ preserves meets and joins; and f preserves tensors and cotensors in the sense that* $f(\alpha\otimes x)=\alpha\otimes f(x)$, $f(\alpha\rightarrowtail x)=\alpha\rightarrowtail f(x)$ *for all* $x\in A$, $\alpha\in\Omega$.

A combination of Proposition 2.2 and Theorem 2.1 shows that a complete Ω-lattice is essentially an Ω-module in the category of complete lattices and suprema-preserving functions (cf. [15]). The details are as follows.

Suppose that A is a complete Ω-lattice. Then A_0 is a complete lattice. The set $[A_0,A_0]$ of join-preserving endo-maps on A_0 is a complete lattice under the pointwise ordering. Clearly, $([A_0,A_0],\circ,\mathrm{id})$ becomes a unital quantale, which is not commutative in general. By Proposition 2.2, the function $k:\Omega\to[A_0,A_0]$, $k(\alpha)=\alpha\otimes(-)$, satisfies: (a) k preserves joins; (b) $k(I)=\mathrm{id}$; and (c) $k(\alpha*\beta)=k(\alpha)\circ k(\beta)$. Conversely, given a complete lattice A_0 and a function $k:\Omega\to[A_0,A_0]$ fulfilling the conditions (a)–(c), define $A(x,y)=\vee\{\alpha\in\Omega\mid k(\alpha)(x)\le y\}$ for all $x,y\in A_0$. Then A becomes an Ω-category with A_0 as underlying preorder. Moreover, A is tensored and cotensored, with tensor and cotensor given by $\alpha\otimes x=k(\alpha)(x)$ and $\alpha\rightarrowtail x=\vee\{y\in A_0\mid k(\alpha)(y)\le x\}$. Therefore A is a complete Ω-lattice.

3 The Change-Base Issue

Definition 3.1 (cf. [12]) *A closed unital map* $G:(\Omega,*,I)\to(\Omega',*',I')$ *is a function* $G:\Omega\to\Omega'$ *such that*

(1) *G preserves order*;

(2) $I'\le G(I)$;

(3) $G(a)*'G(b)\le G(a*b)$ *for all* $a,b\in\Omega$.

In terminology of category theory, a closed unital map between commutative unital quantales is a closed functor if we regard commutative unital quantales as symmetric, monoidal closed categories (cf. [4, 7, 9]).

Remark 3.1 In the presence of (1), (3) in the above definition is equivalent to

(3′) $G(a\to b)\le G(a)\to' G(b)$ for all $a,b\in\Omega$.

On one hand, if $(3')$ holds, then $G(b) \leq G(a \to a * b) \leq G(a) \to' G(a * b)$, hence

$$G(a) *' G(b) \leq G(a * b).$$

On the other hand, if (3) holds, then $G(a) *' G(a \to b) \leq G(a * (a \to b)) \leq G(b)$, thus

$$G(a \to b) \leq G(a) \to' G(b).$$

Let $G : (\Omega, *, I) \to (\Omega', *', I')$ be a closed unital map and A be an Ω-category. Then

$$(\overline{G}A)(a, b) \triangleq G(A(a, b))$$

defines an Ω'-category $\overline{G}A$ with the same underlying set of A. Moreover, if $f : X \to Y$ is an Ω-functor, then $f : \overline{G}X \to \overline{G}Y$ is also an Ω'-functor, denoted by $\overline{G}f$. Therefore, we obtain a functor $\overline{G} : \Omega\text{-}\mathbf{Cat} \to \Omega'\text{-}\mathbf{Cat}$, which plays a role of change-base in the theory of Ω-categories.

Example 3.1 (1) For each commutative, unital quantale $(\Omega, *, I)$. Let $[-] : \Omega \to \mathbf{2}$ be given by

$$[x] = \begin{cases} 1, & x \geq I; \\ 0, & x \not\geq I. \end{cases}$$

Then $[-]$ is a closed unital map. For each Ω-category A, $\overline{[A]}$ is exactly the underlying preordered set A_0 of A. Therefore, we write $(-)_0$ to denote the functor $\overline{[-]} : \Omega\text{-}\mathbf{Cat} \to \mathbf{PrOrd}$.

(2) If $e : \mathbf{2} \to \Omega$ is the function given by $e(0) = 0, e(1) = I$. Then e is a closed unital map and the functor $\overline{e} : \mathbf{PrOrd} \to \Omega\text{-}\mathbf{Cat}$ is clearly an embedding. $\overline{e}$ is a left adjoint of $(-)_0$ (cf. Example 3.2).

Theorem 3.1 *Let Ω, Ω' be integral, commutative, unital quantales. If both $G : \Omega \to \Omega'$ and $F : \Omega' \to \Omega$ are closed unital maps, then the following conditions are equivalent:*

(1) *(F, G) is an adjunction.*

(2) *$(\overline{F}, \overline{G})$ is an adjunction.*

Proof (1) $\Rightarrow$ (2) Let A be an Ω-category and B an Ω'-category. Then $f : B \to \overline{G}A$ is an Ω'-functor $\Leftrightarrow B(x, y) \leq G(A(f(x), f(y)))$ for all $x, y \in B \Leftrightarrow F(B(x, y)) \leq A(f(x), f(y))$ for all $x, y \in B \Leftrightarrow f : \overline{F}B \to A$ is an Ω-functor. Therefore $(\overline{F}, \overline{G})$ is an adjunction, and it is indeed a Galois correspondence following the terminology of [1].

(2) $\Rightarrow$ (1) First, let $\overline{\eta} : \mathrm{id} \to \overline{G} \circ \overline{F}$ be the unit of the adjunction $(\overline{F}, \overline{G})$. Then for each Ω'-category B, the underlying function of $\overline{\eta}_B : B \to \overline{G} \circ \overline{F}(B)$ must be the identity function on B. Suppose on the contrary that there is an Ω'-category B such that $\overline{\eta}_B(x) = y$ for some different elements $x, y \in B$. Let $f : B \to B$ be a constant map with value x. Then f is easily checked to be an Ω'-functor since Ω' is integral. Because $\overline{G} \circ \overline{F}(f)(\overline{\eta}_B(x)) = x$ and $(\overline{\eta}_B \circ f)(x) = y$, the diagram

$$\begin{tikzcd} B \arrow[r, "\overline{\eta}_B"] \arrow[d, "f"'] & \overline{G} \circ \overline{F}(B) \arrow[d, "\overline{G} \circ \overline{F}(f)"] \\ B \arrow[r, "\overline{\eta}_B"] & \overline{G} \circ \overline{F}(B) \end{tikzcd}$$

does not commute, a contradiction to that $\overline{\eta}$ is a natural transformation.

Similarly, if $\overline{\varepsilon}$ denotes the co-unit of the adjunction $(\overline{F}, \overline{G})$, then for each Ω-category A, the underlying function of $\overline{\varepsilon}_A : \overline{F} \circ \overline{G}(A) \to A$ is the identity function on A.

Now, we show that (F, G) is an adjunction. Since the underlying functions of $\overline{\eta}_{\Omega'}, \overline{\varepsilon}_{\Omega}$ are both identities, for any $\alpha \in \Omega$ and $\alpha' \in \Omega'$, we have

$$\alpha' = \Omega'(I', \alpha') \le \overline{G} \circ \overline{F}\Omega'(I', \alpha') = GF(\alpha'),$$
$$FG(\alpha) = \overline{F} \circ \overline{G}\Omega(I, \alpha) \le \Omega(I, \alpha) = \alpha.$$

Therefore, (F, G) is an adjunction.

Example 3.2 For each commutative, unital quantale Ω, let $[-]$ and e be defined as in Example 3.1. Then $(e, [-])$ is an adjunction in the category of quantales and closed unital maps. Hence $(\overline{e}, (-)_0)$ is an adjunction.

Theorem 3.2 *If $G : \Omega \to \Omega'$ is a closed unital map, then the following* (1), (2) *and* (3) *are equivalent:*

(1) $\overline{G} : \Omega$-**Cat** $\to \Omega'$-**Cat** *is an equivalence of categories,*

(2) G *is an order isomorphism and preserves tensor,*

(3) $\overline{G} : \Omega$-**Cat** $\to \Omega'$-**Cat** *is an isomorphism of categories.*

Proof (1) ⇒ (2) (i) G is surjective. Since $\overline{G}$ is an equivalence of categories, there is an Ω-category A such that $(\Omega', \to') \cong \overline{G}A$. Let $f : \Omega' \to \overline{G}A$ be an Ω′-isomorphism. Then for any $x' \in \Omega'$, $x' = \Omega'(I', x') = \overline{G}A(f(I'), f(x')) = G(A(f(I'), f(x')))$. Hence, G is surjective.

(ii) G reflects order in the sense that if $G(\alpha) \le G(\beta)$ then $\alpha \le \beta$. In particular, G is injective.

First, we note that $f : A \to B$ is an Ω-functor if and only if $f : \overline{G}A \to \overline{G}B$ is an Ω′-functor since $\overline{G}$ is a full and faithful concrete functor.

Suppose on the contrary that $G(\alpha) \le G(\beta)$ but $\alpha \not\le \beta$. Define two Ω-categories A and B as follows. The underlying set of A is $\{x, y\}$ and that of B is $\{z, w\}$; the hom-functors are given by

$$A(x, y) = \alpha, \quad A(x, x) = A(y, y) = I, \quad A(y, x) = 0$$

and

$$B(z, w) = \beta, \quad B(z, z) = B(w, w) = I, \quad B(w, z) = 0.$$

Let f be the function given by $f(x) = z, f(y) = w$. Then $f : \overline{G}A \to \overline{G}B$ is an Ω′-functor, but $f : A \to B$ is not an Ω-functor, a contradiction.

(iii) It follows from (i) and (ii) that G is an order isomorphism.

(iv) G preserves tensor. Suppose on the contrary that there exist $\alpha, \beta \in \Omega$, such that $G(\alpha * \beta) > G(\alpha) *' G(\beta)$. Define an Ω′-category B as follows: the underlying set is $\{x', y', z'\}$, and the hom-functor is given by

$$B(x', y') = G(\alpha), \quad B(y', z') = G(\beta), \quad B(x', z') = G(\alpha) *' G(\beta)$$

and

$$B(y',x') = B(z',y') = B(z',x') = 0, \quad B(x',x') = B(y',y') = B(z',z') = I'.$$

Since $\overline{G}$ is an equivalence, there is an Ω-category A, such that $\overline{G}A \cong B$. Let $f : \overline{G}A \to B$ be such an Ω'-isomorphism. By definition of $\overline{G}$, the underlying set of A has exactly 3 elements, say, $\{x,y,z\}$. Suppose that $f(x) = x'$, $f(y) = y'$, and $f(z) = z'$. Because G is injective and

$$G(A(x,y)) = \overline{G}A(x,y) = B(f(x),f(y)) = B(x',y') = G(\alpha),$$

we get $A(x,y) = \alpha$. Similarly, it can be checked that $A(y,z) = \beta$. Thus,

$$\begin{aligned} G(\alpha) *' G(\beta) &< G(\alpha * \beta) = G(A(x,y) * A(y,z)) \\ &\le G(A(x,z)) = B(x',z') \\ &= G(\alpha) *' G(\beta), \end{aligned}$$

a contradiction. Hence, G preserves tensor.

(2) $\Rightarrow$ (3) Since G is an order isomorphism, there is a functor $F : \Omega' \to \Omega$, such that $GF = 1_{\Omega'}$, $FG = 1_{\Omega}$.

We check that F preserves tensor and unit, which is thus a closed unital map. Since G preserves tensor, for any $\alpha', \beta' \in \Omega'$, $\alpha' *' \beta' = GF(\alpha') *' GF(\beta') = G(F(\alpha') * F(\beta'))$. Then $F(\alpha' *' \beta') = F(\alpha') * F(\beta')$, i.e., F preserves tensor. F preserves unit because $F(I') = F(I') * I = F(I') * FG(I) = F(I' *' G(I)) = FG(I) = I$.

Thus, both F and G are isomorphisms, inverse to each other, in the category of commutative, unital quantales and unital closed maps. By definition of $\overline{F}$ and $\overline{G}$, it is easy to see that they are inverse to each other. Therefore, $\overline{G}$ is an isomorphism of categories.

(3) $\Rightarrow$ (1) Trivial.

Example 3.3 Let $\Omega = ([0,\infty]^{\mathrm{op}}, +, 0)$, $\Omega' = ([0,1], \cdot, 1)$. Then, $G : \Omega \to \Omega', G(x) = \mathrm{e}^{-x}$ is an order isomorphism and preserves tensor. Hence, $\overline{G} : \Omega\text{-}\mathbf{Cat} \to \Omega'\text{-}\mathbf{Cat}$ is an isomorphism of categories.

4 Preservation of Completeness

In this section a sufficient condition is obtained for $\overline{G}$ to preserve completeness in the sense that $\overline{G}A$ is a complete Ω'-category whenever A is a complete Ω-category.

Theorem 4.1 *Suppose that $G : (\Omega, *, I) \to (\Omega', *', I')$ is a closed unital map with a left adjoint $F : \Omega' \to \Omega$ which preserves tensor in the sense that $F(\alpha') * F(\beta') = F(\alpha' *' \beta')$ for any $\alpha', \beta' \in \Omega'$. Then for any complete Ω-category A, $\overline{G}(A)$ is a complete Ω'-category.*

We prove a lemma first.

Lemma 4.1 *Suppose that $G : (\Omega, *, I) \to (\Omega', *', I')$ is a closed unital map with a left adjoint $F : \Omega' \to \Omega$. Then the following conditions are equivalent:*

(1) *For any* $\alpha', \beta' \in \Omega'$, $F(\alpha') * F(\beta') = F(\alpha' *' \beta')$.
(2) *For any* $\alpha' \in \Omega'$, $\alpha \in \Omega$, $G(F(\alpha') \to \alpha) = \alpha' \to' G(\alpha)$.

Proof (1) ⇒ (2) (F, G) is an adjunction. For any $\alpha \in \Omega$, $\alpha', \beta' \in \Omega'$,

$$\begin{aligned}
\beta' \leq \alpha' \to' G(\alpha) &\Leftrightarrow \beta' *' \alpha' \leq G(\alpha) \\
&\Leftrightarrow F(\beta' *' \alpha') \leq \alpha \\
&\Leftrightarrow F(\beta') * F(\alpha') \leq \alpha \\
&\Leftrightarrow F(\beta') \leq F(\alpha') \to \alpha \\
&\Leftrightarrow \beta' \leq G(F(\alpha') \to \alpha).
\end{aligned}$$

Thus, $\alpha' \to' G(\alpha) = G(F(\alpha') \to \alpha)$.

(2) ⇒ (1) For any $\gamma \in \Omega$,

$$\begin{aligned}
F(\alpha' *' \beta') \leq \gamma &\Leftrightarrow \alpha' *' \beta' \leq G(\gamma) \\
&\Leftrightarrow \alpha' \leq \beta' \to' G(\gamma) = G(F(\beta') \to \gamma) \\
&\Leftrightarrow F(\alpha') \leq F(\beta') \to \gamma \\
&\Leftrightarrow F(\alpha') * F(\beta') \leq \gamma.
\end{aligned}$$

Thus, $F(\alpha') * F(\beta') = F(\alpha' *' \beta')$.

Now we prove Theorem 4.1.

Proof Suppose that A is a complete Ω-category. We show that $\overline{G}A$ is also a complete Ω'-category. To this end, we check that every $\mu' \in \Omega'^{\overline{G}A}$ has an infimum in $\overline{G}A$.

Let $a \in A$ be an infimum of $F \circ \mu' \in \Omega^A$ in A. Then, for any $y \in \overline{G}A$,

$$\begin{aligned}
\overline{G}A(y, a) &= G(A(y, a)) \\
&= G\Big(\bigwedge_{x \in A} F \circ \mu'(x) \to A(y, x) \Big) \\
&= \bigwedge_{x \in \overline{G}A} G(F(\mu'(x)) \to A(y, x)) \\
&= \bigwedge_{x \in \overline{G}A} \mu'(x) \to' G(A(y, x)) \\
&= \bigwedge_{x \in \overline{G}A} \mu'(x) \to' (\overline{G}A)(y, x).
\end{aligned}$$

Therefore, a is an infimum of μ' in $\overline{G}A$.

Example 4.1 (cf. [4, 7]) For each commutative, unital quantale Ω, the closed unital map $[-] : \Omega \to \mathbf{2}$ satisfies the conditions in Theorem 4.1. Thus, A_0 is a complete preorder if A is a complete Ω-category.

Given a closed unital map $G : \Omega \to \Omega'$ with the conditions in Theorem 4.1, the left adjoint F of G is not always a closed unital map. And for an Ω-category A, the underlying preorders of A and $\overline{G}A$ might be different.

Example 4.2 Let $\Omega = (\{0, \frac{1}{2}, 1\}, \wedge, 1)$ and $G : \Omega \to \mathbf{2}$ be given by $G(0) = 0$, $G(\frac{1}{2}) = G(1) = 1$. Then G is a closed unital map. The function $F : \mathbf{2} \to \Omega$ given by $F(0) = 0, F(1) = \frac{1}{2}$ is a left adjoint of G. Obviously, F preserves tensor. Hence, G satisfies the condition in Theorem 4.1. But F is not a closed unital map because $F(1) < 1$. Let $A = (\Omega, \to)$. Then $\overline{G}A(1, \frac{1}{2}) = G(A(1, \frac{1}{2})) = G(\frac{1}{2}) = 1$. Thus, $1 \le \frac{1}{2}$ in $(\overline{G}A)_0$, which is not the case in A_0.

If the left adjoint F in Theorem 4.1 is already a closed unital map, $\overline{G}$ can be described in another equivalent way.

Definition 4.1 (cf. [11]) *Let $F : \Omega \to \Omega'$ be a closed unital map. Then*

(1) *F is strict if $F(I) = I'$ and $F(\alpha * \beta) = F(\alpha) *' F(\beta)$ for all $\alpha, \beta \in \Omega$.*

(2) *F is cocontinuous if F is join-preserving.*

It is easy to check that the right adjoint of every strict, cocontinuous closed map is also a closed unital map. Conversely, if $F : \Omega \to \Omega'$ is at the same time a closed unital map and a left adjoint of a closed unital map $G : \Omega' \to \Omega$, then F is strict and cocontinuous.

Suppose that $F : \Omega' \to \Omega$ is a strict, cocontinuous, closed unital map and that A is a complete Ω-category. Let $k : \Omega \to [A_0, A_0], k(\alpha) = \alpha \otimes (-)$, be the Ω-module representation of A. Then the composition $k \circ F$ defines a complete Ω'-category $\underline{F}A$ in terms of Ω'-modules with $(\underline{F}A)_0 = A_0$. If we denote the tensor and cotensor in A by $\otimes$ and $\rightarrowtail$ respectively; the tensor and cotensor in $\underline{F}A$ by $\otimes'$ and $\rightarrowtail'$ respectively, then we have the following conclusion.

Proposition 4.1 *$\alpha' \otimes' x = F(\alpha') \otimes x$; $\alpha' \rightarrowtail' x = F(\alpha') \rightarrowtail x$.*

It is easy to check that $\underline{F}$ is a functor from the category Ω-**CLat** of complete Ω-lattices and complete maps to the category Ω'-**CLat** of complete Ω'-lattices and complete maps.

Proposition 4.2 *Let $G : \Omega \to \Omega'$ be a closed unital map such that G has a left adjoint $F : \Omega' \to \Omega$ which is a closed unital map. Then, for any complete Ω-lattice A, $\underline{F}A = \overline{G}A$. In particular, $\overline{G}$ preserves completeness.*

Proof For any $\beta' \in \Omega'$ and $x, y \in A$,

$$\begin{aligned} \beta' \le \overline{G}A(x, y) &\Leftrightarrow \beta' \le G(A(x, y)) \\ &\Leftrightarrow F(\beta') \le A(x, y) \\ &\Leftrightarrow F(\beta') \otimes x \le y \\ &\Leftrightarrow \beta' \otimes' x \le y \\ &\Leftrightarrow \beta' \le \underline{F}A(x, y). \end{aligned}$$

Therefore, $\underline{F}A = \overline{G}A$.

Example 4.3 (1) Let $\Omega = ([0, 1], \wedge, 1)$, or $\Omega = ([0, 1], \times, 1)$, and $e : \mathbf{2} \to \Omega$ be the closed unital map given by $e(0) = 0$ and $e(1) = 1$. e has a left adjoint $F : \Omega \to \mathbf{2}$ given by $F(0) = 0$ and $F(x) = 1$ whenever $x \ne 0$. It is easy to see that F is a closed unital map. Thus, for any complete $\mathbf{2}$-category A, $\overline{e}A$ is a complete Ω-category.

(2) A distance distribution function (briefly, a d.d.f.) is a non-decreasing function f defined on $[0,\infty]$ such that $f(0)=0, f(\infty)=1$, and is left continuous on $(0,\infty)$. The set of all d.d.f.'s will be denoted by Δ^+. Clearly, Δ^+ is a complete lattice under the pointwise order with a top element ε_0, where, $\varepsilon_0(0)=0$ and $\varepsilon_0(x)=1$ whenever $x>0$. Suppose that $*$ is a left continuous t-norm on [0, 1]. Let $f \circledast g(t)=\bigvee\{f(r)*g(s) \mid r+s\leq t\}$ for all $f,g\in\Delta^+, t\in[0,\infty]$. Then, $(\Delta^+,\circledast,\varepsilon_0)$ is a commutative, unital quantale. Categories enriched over $(\Delta^+,\circledast,\varepsilon_0)$ are exactly the pseudo-quasi-probabilistic metric spaces (cf. [13]). Define $i:([0,\infty]^{\mathrm{op}},+,0)\to(\Delta^+,\circledast,\varepsilon_0)$ by $i(x)(t)=0$ if $t\leq x$ and $i(x)(t)=1$ if $x<t$. i is clearly a cocontinuous, strict closed unital map. The right adjoint $j:\Delta^+\to[0,\infty]^{\mathrm{op}}$ of i is given by $j(f)=\inf\{x\in[0,\infty]\mid f(x)=1\}$, where the infimum is taken in $[0,\infty]$, not in $[0,\infty]^{\mathrm{op}}$. Then, $j:(\Delta^+,\circledast,\varepsilon_0)\to([0,\infty]^{\mathrm{op}},+,0)$ is a closed unital map and $\overline{j}$ $(=\underline{i})$ preserves completeness by the above proposition.

The following examples show that if the left adjoint F in Theorem 4.1 does not preserve tensor, then $\overline{G}$ does not preserve completeness in general.

Example 4.4 A left continuous t-norm (cf. [8]) on $[0,1]$ is a binary operation $*$ on $[0,1]$ such that $([0,1],*,1)$ becomes a commutative, unital quantale. Let $*$ and $*'$ be two left continuous t-norms on $[0,1]$ such that $x*y\geq x*'y$ for all $x,y\in[0,1]$. Then $G=\mathrm{id}:([0,1],*,1)\to([0,1],*',1)$ is a closed unital map. Clearly, G has a left adjoint which fails to preserve tenor whenever $*'\neq *$. We shall show that $\overline{G}$ does not preserve completeness whenever $*'\neq *$.

For convenience, we write Ω for $([0,1],*,1)$ and Ω' for $([0,1],*',1)$. Let $\to$ and $\to'$ denote the cotensors of Ω and Ω' respectively. Then $A=([0,1],\to)$ is a complete Ω-category. Because $G=\mathrm{id}$, the underlying preorder of $\overline{G}A$ coincides with that of A, which is the usual order on $[0,1]$. If $\overline{G}A$ is a complete Ω'-category, it must be tensored. Denote the tensor on $\overline{G}A$ by $\otimes'$. Then, appealing to Proposition 2.2 (1)(ii), we have

$$\alpha\otimes' x\leq y\Leftrightarrow \alpha\leq \overline{G}A(x,y)=A(x,y)=x\to y\Leftrightarrow \alpha * x\leq y,$$

which implies that $\alpha\otimes' x=\alpha * x$. Therefore, for any $\alpha,\beta\in[0,1]$, by Proposition 2.2 (1)(iii)

$$\alpha *'\beta=(\alpha *'\beta)*1=(\alpha *'\beta)\otimes' 1=\alpha\otimes'(\beta\otimes' 1)=\alpha*(\beta*1)=\alpha*\beta.$$

Example 4.5 Let $\Omega=([0,1],*,1)$, where $*$ is the Łukasiewicz t-norm on $[0,1]$, i.e., $x*y=\max\{x+y-1,0\}$. The left adjoint $F:\Omega\to\mathbf{2}$ of the closed unital map $e:\mathbf{2}\to\Omega$ does not preserve tensor. We say that $\overline{e}:\mathbf{PrOrd}\to\Omega\text{-}\mathbf{Cat}$ does not preserve completeness. Indeed, if A is a complete lattice with at least 2 elements, we show that $\overline{e}A$ is not a complete Ω-category. To see this, let $\mu:\overline{e}A\to[0,1]$ be a constant function with value $\frac{1}{2}$. Then for each $y\in A\backslash\{\bot\}$, where $\bot$ is the least element of A,

$$\bigwedge_{x\in A}\mu(x)\to(\overline{e}A)(y,x)=\bigwedge_{x\in A}\mu(x)\to e(A(y,x))=\frac{1}{2}\to 0=\frac{1}{2}.$$

But, $(\overline{e}A)(y,a)=e(A(y,a))\neq\frac{1}{2}$ for any $a\in A$. Therefore, μ has no infimum.

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