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Property A and Uniform Embeddability of Metric Spaces Under Decompositions of Finite Depth***

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Abstract Property A and uniform embeddability are notions of metric geometry which imply the coarse Baum-Connes conjecture and the Novikov conjecture. In this paper, the authors prove the permanence properties of property A and uniform embeddability of metric spaces under large scale decompositions of finite depth.

Keywords Metric space, Uniform embedding, Property A, Large scale decomposition, Permanence property
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1 Introduction

M. Gromov introduced the following notion of uniform embeddability of metric spaces into Hilbert space.

Definition 1.1 (see [6]) *A map $f : X \to H$ from a metric space X to a Hilbert space H is said to be a uniform embedding if there exist two non-decreasing functions ρ_1 and ρ_2 on $[0, +\infty)$, such that*
(1) $\lim\limits_{r\to+\infty} \rho_1(r) = +\infty$,
(2) $\rho_1(d(x, y)) \le \|f(x) - f(y)\| \le \rho_2(d(x, y))$ *for all* $x, y \in X$.

In the context of coarse geometry, a uniform embedding $f : X \to H$ is a large scale equivalence of X and $f(X)$ (see [6,17]). M. Gromov suggested that coarse embeddability of a discrete group into Hilbert space might be relevant to solve the Novikov conjecture (see [6]). G. Yu subsequently proved the coarse Baum-Connes conjecture (resp. the Novikov conjecture) for bounded geometry discrete metric spaces (resp. groups) which are uniformly embeddable into a Hilbert space (see [21]). This remarkable result leads to the verification of the coarse Baum-Connes conjecture (resp. the Novikov conjecture) for large classes of discrete metric spaces (resp. groups). In the same paper (see [21]), G. Yu introduced a property, called property A,

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on discrete metric spaces and groups, which is a weak form of amenability and which implies uniform embeddability of a metric space.

Definition 1.2 (see [21]) *A discrete metric space (X, d) is said to have property A if for any $R > 0$ and $\varepsilon > 0$, there exists a family $\{A_x\}_{x\in X}$ of finite non-empty subsets of $X \times \mathbb{N}$ such that*

(1) *for all $x, y \in X$ with $d(x, y) < R$, we have*

$$\frac{\#(A_x \triangle A_y)}{\#(A_x \cup A_y)} < \varepsilon;$$

(2) *there exists $S > 0$ such that for each $x \in X$, if $(y, n) \in A_x$, then $d(x, y) \le S$.*

Since the appearance of Yu's work, uniform embeddability and property A have been intensely studied, and many permanence properties on them for metric spaces and group operations have been established (see e.g. [1, 7, 8, 10, 15–17, 19–21]). It turns out that the class of uniformly embeddable groups shares many permanence properties with the class of property A groups. For instance, both classes are closed under taking subgroups, products, direct limits, free products with amalgam, and extensions by property A groups (see [4]).

On the other hand, another notion introduced by M. Gromov (see [6]), called finite asymptotic dimension of a metric space, has also important applications in geometry and topology. Recall that a metric space X is said to have finite asymptotic dimension if there is an integer $n \ge 0$ such that for any (large) number $r > 0$ the space X may be written as a union of $n + 1$ subspaces X_i, each of which may be further decomposed as an r-disjoint union:

$$X = \bigcup_{i=0}^{n} X_i, \quad X_i = \bigsqcup_{j=1}^{\infty} X_{ij}, \quad \operatorname{dist}(X_{ij}, X_{ij'}) > r,$$

in which the metric family $\{X_{ij} : i = 0, 1, 2, \cdots, n, j = 1, 2, 3, \cdots\}$ is bounded, i.e., $S := \sup\limits_{i,j} \operatorname{diam}(X_{ij}) < \infty$.

Inspired by the feature of finite asymptotic dimension, E. Guentner, R. Tessera and G. Yu introduced very recently a measure of computational complexity of metric spaces under large scale decompositions of finite depth to study the stable Borel conjecture (see [9]). This is the so-called property Q.

Definition 1.3 *A metric space X is said to have property Q if there is an integer $m \ge 0$, such that we have m levels of decomposition as follows:*

(1) *there exists an integer $n_0 \ge 0$ such that for any $r_1 > 0$, we have*

$$X = \bigcup_{i_1=0}^{n_0} X_{i_1}, \quad X_{i_1} = \bigsqcup_{r_1\text{-}disjoint} X_{i_1 j_1},$$

where the subscript j_1 runs through a countable set;

(2) *there exists an integer $n_1 = n_1(n_0, r_1) \ge 0$ such that for any $r_2 > 0$ and any $X_{i_1 j_1}$, we have*

$$X_{i_1 j_1} = \bigcup_{i_2=0}^{n_1} X_{i_1 j_1 i_2}, \quad X_{i_1 j_1 i_2} = \bigsqcup_{r_2\text{-}disjoint} X_{i_1 j_1 i_2 j_2},$$

where the subscript j_2 runs through a countable set;

$\cdots\cdots$

(m) *there exists an integer* $n_{m-1} = n_{m-1}(n_0, \cdots, n_{m-2}, r_1, \cdots, r_{m-1}) \geq 0$ *such that for any* $r_m > 0$ *and any* $X_{i_1 j_1 \cdots i_{m-1} j_{m-1}}$*, we have*

$$X_{i_1 j_1 \cdots i_{m-1} j_{m-1}} = \bigcup_{i_m=0}^{n_{m-1}} X_{i_1 j_1 \cdots i_{m-1} j_{m-1} i_m},$$
$$X_{i_1 j_1 \cdots i_{m-1} j_{m-1} i_m} = \bigsqcup_{r_m\text{-disjoint}} X_{i_1 j_1 \cdots i_m j_m},$$

and the family of metric spaces $\{X_{i_1 j_1 \cdots i_m j_m}\}_{i_1, j_1, \cdots, i_m, j_m}$ *is uniformly bounded, i.e.,* $S := \sup_{i_1, j_1, \cdots, i_m, j_m} \operatorname{diam}(X_{i_1 j_1 \cdots i_m j_m}) < \infty$.

Guentner-Tessera-Yu [9] proved that the stable Borel conjecture holds for aspherical manifolds whose fundamental groups have property Q, and that all countable solvable groups and countable subgroups of $\mathrm{SL}_2(K)$, where K is a field, have property Q.

In this paper, we shall regard the formation of the above property Q as an operation of metric spaces, and study permanence properties of uniform embeddability and property A under this large scale decomposition operation of finite depth. To do this, we shall introduce two notions, called property $\mathrm{Q_A}$ and property $\mathrm{Q_{UE}}$ respectively, by replacing the requirement "the family of metric spaces $\{X_{i_1 j_1 \cdots i_m j_m}\}_{i_1, j_1, \cdots, i_m, j_m}$ is uniformly bounded" in the above definition by the requirements that this family has "equi-property A" or "equi-uniform embeddability", respectively. For a discrete metric space X of bounded geometry, we show in Section 2 that if X has property $\mathrm{Q_A}$ then X has property A. For an arbitrary metric space X, we show in Section 3 that if X has property $\mathrm{Q_{UE}}$ then X is uniformly embeddable into Hilbert space. It turns that the proofs of both permanence properties share again close similarities as with the case for groups mentioned above. We remark that P. Nowak (see [14]) gave the first counterexample of discrete metric space which is uniformly embeddable into Hilbert space but does not have property A. However, this counterexample does not have bounded geometry. So far, no such counterexample in the world of bounded geometry metric spaces has been known.

2 Property $\mathrm{Q_A}$

In this section, we first briefly review an equivalent characterization of property A and the notion of "equi-property A", and then introduce the notion of property $\mathrm{Q_A}$ for metric spaces. Finally, we show that if a bounded geometry discrete metric space X has property $\mathrm{Q_A}$, then it has property A.

Let X be a discrete metric space with bounded geometry, i.e., $\forall\, r > 0$, $\exists\, N(r) > 0$, such that $\forall\, x \in X$, the number of elements $\#B_X(x, r)$ in the ball $B_X(x, r)$ is less than $N(r)$. It follows that X is countable. Denote

$$\ell_1(X)_+ := \Big\{ f : X \to \mathbb{R} \;\Big|\; f(x) \geq 0,\ \sum_{x \in X} f(x) < \infty \Big\}.$$

Proposition 2.1 (see [10, 19]) *Let X be a discrete metric space with bounded geometry. Then X has property A if and only if for all $R > 0$ and $\varepsilon > 0$, there exist a map $\xi : X \to \ell_1(X)_+$,*

$\{\xi_x\}_{x\in X}$, *and a constant* $S>0$ *such that for all* $x,y\in X$, *we have* $\|\xi_x\|_1=1$, *and*

(1) $d(x,y)\le R \Longrightarrow \|\xi_x-\xi_y\|_1\le\varepsilon$;

(2) $\operatorname{Supp}\xi_x\subset B_X(x,S)$.

The "degree" of property A was studied by G. Bell [1], and M. Dadarlat and E. Guentner [5].

Definition 2.1 (see [1,5]) *A family of metric spaces* $\{X_i\}_{i\in I}$ *is said to have equi-property A if for all* $R>0$ *and* $\varepsilon>0$, *there exist a family of maps* $\xi^i: X_i\to \ell_1(X_i)_+$ $(i\in I)$ *and a common constant* $S>0$ *such that for all* $i\in I$ *and all* $x,y\in X_i$, *we have* $\|\xi^i_x\|_1=1$, *and*

(1) $d(x,y)\le R \Longrightarrow \|\xi^i_x-\xi^i_y\|_1\le\varepsilon$;

(2) $\operatorname{Supp}\xi^i_x\subset B_{X_i}(x,S)$.

Now we introduce our property $\mathrm{Q_A}$ as follows.

Definition 2.2 *A discrete metric space* (X,d) *is said to have property* Q_A *if there exists an integer* $m\ge 0$ *such that we have* m *levels of decomposition as follows:*

(1) *there exists an integer* $n_0\ge 0$ *such that for any* $r_1>0$, *we have*

$$X=\bigcup_{i_1=0}^{n_0}X_{i_1},\quad X_{i_1}=\bigsqcup_{r_1\text{-}disjoint}X_{i_1j_1};$$

(2) *there exists an integer* $n_1=n_1(n_0,r_1)\ge 0$ *such that for any* $r_2>0$ *and any* $X_{i_1j_1}$, *we have*

$$X_{i_1j_1}=\bigcup_{i_2=0}^{n_1}X_{i_1j_1i_2},\quad X_{i_1j_1i_2}=\bigsqcup_{r_2\text{-}disjoint}X_{i_1j_1i_2j_2};$$

$\cdots\cdots$

(m) *there exists an integer* $n_{m-1}=n_{m-1}(n_0,\cdots,n_{m-2},r_1,\cdots,r_{m-1})\ge 0$ *such that for any* $r_m>0$ *and any* $X_{i_1j_1\cdots i_{m-1}j_{m-1}}$, *we have*

$$X_{i_1j_1\cdots i_{m-1}j_{m-1}}=\bigcup_{i_m=0}^{n_{m-1}}X_{i_1j_1\cdots i_{m-1}j_{m-1}i_m},$$
$$X_{i_1j_1\cdots i_{m-1}j_{m-1}i_m}=\bigsqcup_{r_m\text{-}disjoint}X_{i_1j_1\cdots i_mj_m},$$

and the family of metric spaces $\{X_{i_1j_1\cdots i_mj_m}\}_{i_1,j_1,\cdots,i_m,j_m}$ *has equi-property A.*

The main result of this section is the following permanence property for property A under large scale decompositions of finite depth.

Theorem 2.1 *Let* X *be a discrete metric space with bounded geometry. Then* X *has property A if and only if* X *has property* Q_A.

The necessity is immediate since any family of subspaces of a property A space has equi-property A. To show the sufficiency, we need the following two lemmas.

Lemma 2.1 (see [1]) *Let* $\mathcal{U}=\{U\}$ *be a cover of a metric space* X *with multiplicity at most* $k+1$ $(k\ge 0)$ *and Lebesgue number* $L>0$. *For* $U\in\mathcal{U}$, *define*

$$\phi_U(x)=\frac{d(x,X\setminus U)}{\sum\limits_{V\in\mathcal{U}}d(x,X\setminus V)}.$$

Then $(\phi_U)_{U\in\mathcal{U}}$ is a partition of unity on X subordinate to the cover $\mathcal{U}$. Moreover, each ϕ_U satisfies

$$|\phi_U(x)-\phi_U(y)|\leq \frac{2k+3}{L}d(x,y), \quad \forall\, x,y\in X,$$

and the family $(\phi_U)_{U\in\mathcal{U}}$ satisfies

$$\sum_{U\in\mathcal{U}}|\phi_U(x)-\phi_U(y)|\leq \frac{(2k+2)(2k+3)}{L}d(x,y), \quad \forall\, x,y\in X.$$

The finite union theorem and certain infinite union theorem for property A, established by G. Bell [1], and M. Dadarlat and E. Guentner [5], played an important role in studying permanence properties. Next, we prove a finer "quantitative version of finite union theorem".

Lemma 2.2 *Let X be a discrete metric space of bounded geometry, expressed as a union of finitely many subspaces $X=\bigcup_{i=0}^{n}X_i$. If $R>0$, $\varepsilon>0$ and $S>0$ are any constants such that there exist $n+1$ maps $\xi^i: X_i\to \ell_1(X_i)_+$ $(i=0,1,\cdots,n)$ satisfying that for all $i=0,1,2,\cdots,n$ and all $x,y\in X_i$, we have $\|\xi^i_x\|_1=1$ and*

(1) $d(x,y)\leq R+2(L+R)\Longrightarrow \|\xi^i_x-\xi^i_y\|_1\leq \frac{\varepsilon}{2}$, *where*

$$L=\frac{2(2n+2)(2n+3)R}{\varepsilon};$$

(2) $\mathrm{Supp}(\xi^i_x)\subset B_{X_i}(x,S)$.

Then there exists a map $\eta: X\to \ell_1(X)_+$ such that $\|\eta_x\|_1=1$ for all $x\in X$, and

(1) $d(x,y)\leq R\Longrightarrow \|\eta_x-\eta_y\|_1\leq\varepsilon$ *for all $x,y\in X$;*

(2) $\mathrm{Supp}(\eta_x)\subset B_X(x,S+L+R)$.

Proof Let $R>0$, $\varepsilon>0$ and $S>0$ be given as above. Set

$$N_L(X_i)=\{x\in X_i: d(x,X_i)\leq L\}.$$

Then we have

$$X=\bigcup_{i=0}^{n}N_L(X_i),$$

the multiplicity of the cover $\{N_L(X_i)\}_{i=0}^n$ is at most $n+1$, and the Lebesgue number of $\{N_L(X_i)\}_{i=0}^n$ is at least L.

By Lemma 2.1, there is a partition of unity $\{\phi_i\}_{i=0}^n$ subordinated to the cover $\{N_L(X_i)\}_{i=0}^n$ such that

$$\sum_{i=0}^{n}|\phi_i(x)-\phi_i(y)|\leq \frac{(2n+2)(2n+3)}{L}d(x,y), \quad \forall\, x,y\in X.$$

For each $i=0,1,\cdots,n$ and any $x\in N_{L+R}(X_i)$, choose a point $p(x)\in X_i$ such that $d(x,p(x))\leq 2d(x,X)\leq 2(L+R)$. Define a map

$$\eta^i: N_{L+R}(X_i)\to \ell_1(N_{L+R}(X_i))_+$$

by $\eta^i_x=\xi^i_{p(x)}$. We have $\|\eta^i_x\|_1=\|\xi^i_{p(x)}\|_1=1$ for all $x\in N_{L+R}(X_i)$, and

$$\mathrm{Supp}(\eta^i_x)\subset B_{N_{L+R}(X_i)}(x,S+L+R).$$

For any $x, y \in N_{L+R}(X_i)$ with $d(x,y) \leq R$, we have $d(p(x), p(y)) \leq R + 2(L+R)$. Thus,

$$\|\eta_x^i - \eta_y^i\|_1 = \|\xi_{p(x)}^i - \xi_{p(y)}^i\|_1 \leq \frac{\varepsilon}{2}.$$

Note that $\ell_1(X_i)_+$ can be naturally regarded as a subspace of $\ell_1(N_{L+R}(X_i))_+$. Define

$$\eta : X \to \ell_1(X)_+$$

by

$$\eta_x = \sum_{i=0}^{n} \phi_i(x)\eta_x^i, \quad \forall\, x \in X.$$

Then we claim that η is the desired map. Indeed, firstly we observe

$$\|\eta_x\|_1 = \Big\| \sum_{i=0}^{n} \phi_i(x)\eta_x^i \Big\|_1 = \sum_{i=0}^{n} \phi_i(x) \sum_{y \in N_{L+R}(X)} \eta_x^i(y) = \sum_{i=0}^{n} \phi_i(x)\|\eta_x^i\|_1 = \sum_{i=0}^{n} \phi_i(x) = 1,$$

and

$$\mathrm{Supp}(\eta_x) \subset \bigcup_{i=0}^{n} B_{N_{L+R}(X_i)}(x, S+L+R) = B_X(x, S+L+R).$$

Moreover, for all $x, y \in X$ with $d(x,y) \leq R$, we have

$$\begin{aligned}
\|\eta_x - \eta_y\|_1 &= \Big\| \sum_{i=0}^{n} \phi_i(x)\eta_x^i - \sum_{i=0}^{n} \phi_i(y)\eta_y^i \Big\|_1 \\
&\leq \Big\| \sum_{i=0}^{n} (\phi_i(x) - \phi_i(y))\eta_x^i \Big\|_1 + \Big\| \sum_{i=0}^{n} \phi_i(y)(\eta_x^i - \eta_y^i) \Big\|_1 \\
&\leq \sum_{i=0}^{n} |\phi_i(x) - \phi_i(y)| + \sum_{i=0}^{n} \phi_i(y)\|\eta_x^i - \eta_y^i\|_1 \\
&\leq \frac{\varepsilon}{2} + \sum_{i=0}^{n} \phi_i(y)\frac{\varepsilon}{2} \\
&\leq \varepsilon.
\end{aligned}$$

This completes the proof.

Proof of Theorem 2.1 Let X be a bounded geometry discrete metric space with property $\mathrm{Q_A}$. We show that X has property A.

Let $R > 0$ and $\varepsilon > 0$ be given. By the definition of property $\mathrm{Q_A}$, there is an integer $m \geq 0$ such that

(1) there exists an integer $n_0 \geq 0$ such that for the number $r_1 := R_1 + 1 := R_0 + 2(L_1 + R_0) + 1$, we have

$$X = \bigcup_{i_1=0}^{n_0} X_{i_1}, \quad X_{i_1} = \bigsqcup_{r_1\text{-disjoint}} X_{i_1 j_1},$$

where $R_0 = R$, $L_1 = \frac{2(2n_0+2)(2n_0+3)R_0}{\varepsilon}$ and $R_1 = R_0 + 2(L_1 + R_0)$;

(2) there exists an integer $n_1 = n_1(n_0, r_1) \geq 0$ such that for $r_2 := R_2 + 1 := R_1 + 2(L_2 + R_1) + 1$ and for any $X_{i_1 j_1}$, we have

$$X_{i_1 j_1} = \bigcup_{i_2=0}^{n_1} X_{i_1 j_1 i_2}, \quad X_{i_1 j_1 i_2} = \bigsqcup_{r_2\text{-disjoint}} X_{i_1 j_1 i_2 j_2},$$

where $L_2 = \frac{2^2(2n_1+2)(2n_1+3)R_1}{\varepsilon}$ and $R_2 = R_1 + 2(L_2 + R_1)$;

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(m) there exists an integer $n_{m-1} = n_{m-1}(n_0, \cdots, n_{m-2}, r_1, \cdots, r_{m-1}) \geq 0$ such that for $r_m := R_m + 1 := R_{m-1} + 2(L_m + R_{m-1}) + 1$ and for any $X_{i_1 j_1 \cdots i_{m-1} j_{m-1}}$, we have

$$X_{i_1 j_1 \cdots i_{m-1} j_{m-1}} = \bigcup_{i_m=0}^{n_{m-1}} X_{i_1 j_1 \cdots i_{m-1} j_{m-1} i_m},$$
$$X_{i_1 j_1 \cdots i_{m-1} j_{m-1} i_m} = \bigsqcup_{r_m\text{-disjoint}} X_{i_1 j_1 \cdots i_m j_m},$$

and the family of metric spaces $\{X_{i_1 j_1 \cdots i_m j_m}\}_{i_1, j_1, \cdots, i_m, j_m}$ has equi-property A, where

$$L_m = \frac{2^m(2n_{m-1}+2)(2n_{m-1}+3)R_{m-1}}{\varepsilon}, \quad R_m = R_{m-1} + 2(L_m + R_{m-1}).$$

Hence, by the definition of equi-property A, for the constants $R_m \geq 0$ and the above $\varepsilon > 0$, there exist a constant $S > 0$ and a family of maps

$$\xi^{i_1 j_1 \cdots i_m j_m} : X_{i_1 j_1 \cdots i_m j_m} \to \ell_1(X_{i_1 j_1 \cdots i_m j_m})_+$$

such that for all $x, y \in X_{i_1 j_1 \cdots i_m j_m}$, we have $\|\xi_x^{i_1 j_1 \cdots i_m j_m}\|_1 = 1$ and

(1) $d(x,y) \leq R_m \Longrightarrow \|\xi_x^{i_1 j_1 \cdots i_m j_m} - \xi_y^{i_1 j_1 \cdots i_m j_m}\|_1 \leq \frac{\varepsilon}{2^m}$;

(2) $\mathrm{Supp}(\xi_x^{i_1 j_1 \cdots i_m j_m}) \subset B_{X_{i_1 j_1 \cdots i_m j_m}}(x, S)$.

Since

$$X_{i_1 j_1 \cdots i_m} = \bigsqcup_{r_m\text{-disjoint}} X_{i_1 j_1 \cdots i_m j_m},$$

we naturally define

$$\xi^{i_1 j_1 \cdots i_m} : X_{i_1 j_1 \cdots i_m} \to \ell_1(X_{i_1 j_1 \cdots i_m})_+ = \bigoplus_{j_m} \ell_1(X_{i_1 j_1 \cdots i_m j_m})_+$$

by

$$\xi_x^{i_1 j_1 \cdots i_m} = \begin{cases} \xi_x^{i_1 j_1 \cdots i_m j_m}, & \text{if } x \in X_{i_1 j_1 \cdots i_m j_m}, \\ 0, & \text{otherwise} \end{cases}$$

for all $x \in X_{i_1 j_1 \cdots i_m}$. Note that for any $x \in X_{i_1 j_1 \cdots i_m}$, there exists a unique $X_{i_1 j_1 \cdots i_m \tilde{j}_m}$, such that $x \in X_{i_1 j_1 \cdots i_m \tilde{j}_m}$. Thus $\|\xi_x^{i_1 j_1 \cdots i_m}\|_1 = \|\xi_x^{i_1 j_1 \cdots i_m \tilde{j}_m}\|_1 = 1$, and for all $x, y \in X_{i_1 j_1 \cdots i_m}$, we have

(1) $d(x,y) \leq R_m = R_{m-1} + 2(L_m + R_{m-1}) \Longrightarrow$

$$\|\xi_x^{i_1 j_1 \cdots i_m} - \xi_y^{i_1 j_1 \cdots i_m}\|_1 = \|\xi_x^{i_1 j_1 \cdots i_m j_m} - \xi_y^{i_1 j_1 \cdots i_m j_m}\|_1 \leq \frac{\varepsilon}{2^m};$$

(2) $\mathrm{Supp}(\xi_x^{i_1 j_1 \cdots i_m}) \subset B_{X_{i_1 j_1 \cdots i_m}}(x, S)$.

By Lemma 2.2, we obtain a family of maps

$$\xi^{i_1 j_1 \cdots i_{m-1} j_{m-1}} : X_{i_1 j_1 \cdots i_{m-1} j_{m-1}} \to \ell_1(X_{i_1 j_1 \cdots i_{m-1} j_{m-1}})_+$$

such that for all $x, y \in X_{i_1 j_1 \cdots i_{m-1} j_{m-1}}$, we have $\|\xi_x^{i_1 j_1 \cdots i_{m-1} j_{m-1}}\|_1 = 1$ and

(1) $d(x,y) \leq R_{m-1} \Longrightarrow \|\xi_x^{i_1 j_1 \cdots i_{m-1} j_{m-1}} - \xi_y^{i_1 j_1 \cdots i_{m-1} j_{m-1}}\|_1 \leq \frac{\varepsilon}{2^{m-1}}$;

(2) $\mathrm{Supp}(\xi_x^{i_1j_1\cdots i_{m-1}j_{m-1}}) \subset B_{X_{i_1j_1\cdots i_{m-1}j_{m-1}}}(x, S + L_m + R_{m-1})$.

Now we have moved from the m-th level of decomposition back to the $(m-1)$-th level, and are in the situation as required by the assumption of Lemma 2.2. Repeating the above process by using Lemma 2.2 for m-times, we conclude that, for any $R > 0$ and $\varepsilon > 0$, there exist a map

$$\xi : X \to \ell_1(X)_+$$

and a constant $S' = S + \sum_{i=1}^{m} L_j + \sum_{j=0}^{m-1} R_j$ such that for all $x, y \in X$, we have $\|\xi_x\|_1 = 1$, and

(1) $d(x, y) \le R \Longrightarrow \|\xi_x - \xi_y\|_1 \le \varepsilon$;

(2) $\mathrm{Supp}(\xi_x) \subset B_X(x, S')$.

That is, X has property A, as expected. The proof is completed.

3 Property $\mathbf{Q_{UE}}$

In this section, we first briefly review an equivalent characterization of uniform embeddability (see [4]) and the notion of "equi-uniform embeddability" (see [5]) due to M. Dadarlat and E. Guentner, and then introduce the notion of "property $\mathrm{Q_{UE}}$" for arbitrary metric spaces (without the assumption of bounded geometry). Finally, we show that if a metric space X has property $\mathrm{Q_{UE}}$, then X is uniformly embeddable into Hilbert space.

Proposition 3.1 (see [4]) *Let X be a metric space. Then X is uniformly embeddable into a Hilbert space if and only if for every $R > 0$ and $\varepsilon > 0$ there exists a Hilbert space valued map $\xi : X \to H$, $(\xi_x)_{x\in X}$, such that $\|\xi_x\| = 1$ and, for all $x, y \in X$, we have*

(1) $d(x, y) \le R \Longrightarrow \|\xi_x - \xi_y\| \le \varepsilon$;

(2) $\lim_{S\to\infty} \sup\{|\langle \xi_x, \xi_y\rangle| : d(x, y) \ge S,\ x, y \in X\} = 0$.

Definition 3.1 (see [5]) *A family $\{X_i\}_{i\in I}$ of metric spaces is equi-uniformly embeddable into Hilbert space if for every $R > 0$ and $\varepsilon > 0$ there exists a family $\{\xi^i\}_{i\in I}$ of Hilbert space valued maps $\xi^i : X_i \to H_i$ for all $i \in I$, such that $\|\xi^i_x\| = 1$ for all $x \in X_i$, and*

(1) $\forall\, i \in I,\ \forall\, x, y \in X_i,\ d(x, y) \le R \Longrightarrow \|\xi^i_x - \xi^i_y\| \le \varepsilon$;

(2) $\lim_{S\to\infty} \sup_{i\in I} \sup\{|\langle \xi^i_x, \xi^i_y\rangle| : d(x, y) \ge S,\ x, y \in X_i\} = 0$.

Now we introduce our property $\mathrm{Q_{UE}}$ as follows.

Definition 3.2 *A metric space (X, d) is said to have property Q_{UE} if there exists an integer $m \ge 0$ such that we have m levels of decomposition as follows:*

(1) *there exists an integer $n_0 \ge 0$ such that for any $r_1 > 0$, we have*

$$X = \bigcup_{i_1=0}^{n_0} X_{i_1}, \quad X_{i_1} = \bigsqcup_{r_1\text{-}disjoint} X_{i_1j_1};$$

(2) *there exists an integer $n_1 = n_1(n_0, r_1) \ge 0$ such that for any $r_2 > 0$ and any $X_{i_1j_1}$, we have*

$$X_{i_1j_1} = \bigcup_{i_2=0}^{n_1} X_{i_1j_1i_2}, \quad X_{i_1j_1i_2} = \bigsqcup_{r_2\text{-}disjoint} X_{i_1j_1i_2j_2};$$

......

(m) there exists an integer $n_{m-1} = n_{m-1}(n_0, \cdots, n_{m-2}, r_1, \cdots, r_{m-1}) \geq 0$ *such that for any* $r_m > 0$ *and any* $X_{i_1 j_1 \cdots i_{m-1} j_{m-1}}$*, we have*

$$X_{i_1 j_1 \cdots i_{m-1} j_{m-1}} = \bigcup_{i_m=0}^{n_{m-1}} X_{i_1 j_1 \cdots i_{m-1} j_{m-1} i_m},$$
$$X_{i_1 j_1 \cdots i_{m-1} j_{m-1} i_m} = \bigsqcup_{r_m\text{-}disjoint} X_{i_1 j_1 \cdots i_m j_m},$$

and the family of metric spaces $\{X_{i_1 j_1 \cdots i_m j_m}\}_{i_1, j_1, \cdots, i_m, j_m}$ *is equi-uniformly embeddable into Hilbert space.*

The main result of this section is the following permanence property of uniform embeddability of metric spaces into Hilbert space under large scale decompositions of finite depth.

Theorem 3.1 *A metric space* X *has property* Q_{UE} *if and only if* X *is uniformly embeddable into Hilbert space.*

The necessity is immediate since any family of subspaces of a uniformly embeddable metric space is equi-uniformly embeddable. To show the sufficiency, we need the following "quantitative version of finite union theorem" for uniform embeddings.

Lemma 3.1 *Let* X *be a metric space expressed as a union of finitely many subspaces, say,* $X = \bigcup_{i=0}^{n} X_i$*. Let* $R > 0$ *and* $\varepsilon > 0$ *be any constants such that there exist Hilbert space valued maps* $\xi^i : X_i \to H_i$ $(i = 0, 1, \cdots, n)$ *satisfying* $\|\xi^i_x\| = 1$ *for all* $x \in X_i$*, and*

(1) *for each* i *and all* $x, y \in X_i$*,*

$$d(x, y) \leq R + 2(L + R) \Longrightarrow \|\xi^i_x - \xi^i_y\| \leq \varepsilon,$$

where $L = \frac{(2n+2)(2n+3)R}{\varepsilon^2}$*;*

(2) *for each* i*, we have*

$$\lim_{S \to \infty} \sup\{|\langle \xi^i_x, \xi^i_y \rangle| : d(x, y) \geq S,\ x, y \in X_i\} = 0.$$

Then there is a map $\zeta : X \to H = \bigoplus_{i=0}^{n} H_i$ *such that* $\|\zeta_x\| = 1$ *for all* $x \in X$*, and*

(1) *for all* $x, y \in X$*, we have* $d(x, y) \leq R \Longrightarrow \|\zeta_x - \zeta_y\| \leq 2\varepsilon$*;*

(2) $\lim_{T \to \infty} \sup\{|\langle \zeta_x, \zeta_y \rangle| : d(x, y) \geq T,\ x, y \in X\} = 0.$

Proof Let $R > 0$ and $\varepsilon > 0$ be given as in the assumption. Set $N_L(X_i) = \{x \in X : d(x, X_i) \leq L\}$. Then

(1) $X = \bigcup_{i=0}^{n} N_L(X_i)$;

(2) multiplicity $\{N_L(X_i)\}_{i=0}^n \leq n + 1$;

(3) Lebesgue $\{N_L(X_i)\}_{i=0}^n \geq L$.

By Lemma 2.1, there exists a partition of unity $\{\phi_i\}_{i=0}^n$ subordinate to the cover $\{N_L(X_i)\}_{i=0}^n$ such that

$$\sum_{i=0}^{n} |\phi_i(x) - \phi_i(y)| \leq \frac{(2n+2)(2n+3)}{L} d(x, y), \quad \forall\, x, y \in X.$$

For any $x \in N_{L+R}(X_i)$, choose a point $p(x) \in X_i$ satisfying $d(x, p(x)) \le 2d(x, X) \le 2(L+R)$. Define

$$\eta^i : N_{L+R}(X_i) \to H_i$$

by $\eta^i_x = \xi^i_{p(x)}$. Then we have $\|\eta^i_x\| = \|\xi^i_{p(x)}\| = 1$ for any $x \in N_{L+R}(X_i)$.

Moreover, for each $i = 0, 1, \cdots, n$ and any $x, y \in N_{L+R}(X_i)$ such that $d(x, y) \le R$, we have $d(p(x), p(y)) \le R + 2(L+R)$ so that

$$\|\eta^i_x - \eta^i_y\| = \|\xi^i_{p(x)} - \xi^i_{p(y)}\| \le \varepsilon.$$

Let

$$T = S + 2(L+R).$$

For any $x, y \in N_{L+R}(X_i)$ with $d(x, y) \ge T$, we have $d(p(x), p(y)) \ge S$. Hence, for each $i = 0, 1, \cdots, n$, we have

$$\lim_{T\to\infty} \sup\{|\langle \eta^i_x, \eta^i_y\rangle| : d(x, y) \ge T,\ x, y \in N_{L+R}(X_i)\} = 0.$$

Now, define $\zeta : X \to H = \bigoplus_{i=0}^{n} H_i$ by

$$\zeta_x = \bigoplus_{i=0}^{n} (\phi_i(x)^{\frac{1}{2}} \eta^i_x).$$

Then $\|\zeta_x\| = 1$ for each $x \in X$. For any $x, y \in X$, consider $\alpha(x, y) = \bigoplus_{i=0}^{n} \alpha_i(x, y) \in H$ and $\beta(x, y) = \bigoplus_{i=0}^{n} \beta_i(x, y) \in H$ with components

$$\alpha_i(x, y) = \phi_i(x)^{\frac{1}{2}} (\eta^i_x - \eta^i_y) \in H_i, \quad \beta_i(x, y) = (\phi_i(x)^{\frac{1}{2}} - \phi_i(y)^{\frac{1}{2}}) \eta^i_y \in H_i.$$

On one hand, $\|\alpha(x, y)\|^2 = \sum_{i=0}^{n} \phi_i(x) \|\eta^i_x - \eta^i_y\|^2$. If $d(x, y) \le R$ and $x \in N_L(X_i)$, then $y \in N_{L+R}(X_i)$, so that we obtain $\|\alpha(x, y)\| \le \varepsilon$. On the other hand, since $|a^{\frac{1}{2}} - b^{\frac{1}{2}}|^2 \le |a - b|$, we have

$$\begin{aligned}
\|\beta(x, y)\|^2 &= \sum_{i=0}^{n} \|(\phi_i(x)^{\frac{1}{2}} - \phi_i(y)^{\frac{1}{2}}) \eta^i_y\|^2 \\
&\le \sum_{i=0}^{n} |\phi_i(x)^{\frac{1}{2}} - \phi_i(y)^{\frac{1}{2}}|^2 \\
&\le \sum_{i=0}^{n} |\phi_i(x) - \phi_i(y)| \\
&\le \varepsilon^2.
\end{aligned}$$

That is, $\|\beta(x, y)\| \le \varepsilon$. Therefore,

$$\|\zeta_x - \zeta_y\| = \|\alpha(x, y) + \beta(x, y)\| \le \|\alpha(x, y)\| + \|\beta(x, y)\| \le 2\varepsilon,$$

whenever $d(x, y) \leq R$. Furthermore, since ϕ_i vanishes outside $N_L(X_i)$, for any $x, y \in X$ with $d(x, y) \geq T$, we have

$$|\langle \zeta_x, \zeta_y \rangle| \leq \sum_{i=0}^{n} \phi_i(x)^{\frac{1}{2}} \phi_i(y)^{\frac{1}{2}} |\langle \eta_x^i, \eta_y^i \rangle| \\ \leq \max_{i=0,1,\cdots,n} \sup\{|\langle \eta_{x'}^i, \eta_{y'}^i \rangle| : d(x', y') \geq T,\ x', y' \in N_L(X_i)\}.$$

Since for each $i = 0, 1, \cdots, n$, we have

$$\lim_{T\to\infty} \sup\{|\langle \eta_x^i, \eta_y^i \rangle| : d(x, y) \geq T,\ x, y \in N_{L+R}(X_i)\} = 0,$$

it follows that

$$\lim_{T\to\infty} \sup\{|\langle \zeta_x, \zeta_y \rangle| : d(x, y) \geq T,\ x, y \in X\} = 0,$$

as desired. This completes the proof.

We will actually need the following "equi-version" of Lemma 3.1 for a sequence of metric spaces.

Lemma 3.2 *Let $n \geq 0$ be an integer. Let $\{X_j\}_{j=0}^{\infty}$ be a sequence of metric spaces, each of which can be expressed as a union of $n+1$ subspaces $X_j = \bigcup_{i=0}^{n} X_{ji}$. Let $R > 0$ and $\varepsilon > 0$ be any constants such that there exist Hilbert space valued maps $\xi^{ji} : X_{ji} \to H_{ji}$ satisfying $\|\xi_x^{ji}\| = 1$ for all $x \in X_{ji}$, and*

(1) *for all j, i and all $x, y \in X_{ji}$,*

$$d(x, y) \leq R + 2(L + R) \Longrightarrow \|\xi_x^{ji} - \xi_y^{ji}\| \leq \varepsilon,$$

where $L = \frac{(2n+2)(2n+3)R}{\varepsilon^2}$;

(2) $\lim_{S\to\infty} \sup_{j,i} \sup\{|\langle \xi_x^{ji}, \xi_y^{ji} \rangle| : d(x, y) \geq S,\ x, y \in X_{ji}\} = 0.$

Then there is a sequence of maps $\zeta^j : X_j \to H_j := \bigoplus_{i=0}^{n} H_{ji}$ such that $\|\zeta_x^j\| = 1$ for all $x \in X_j$, and

(a) *for all j and all $x, y \in X_j$, we have*

$$d(x, y) \leq R \Longrightarrow \|\zeta_x^j - \zeta_y^j\| \leq 2\varepsilon;$$

(b) $\lim_{T\to\infty} \sup_{j} \sup\{|\langle \zeta_x^j, \zeta_y^j \rangle| : d(x, y) \geq T,\ x, y \in X_j\} = 0.$

Proof Let $R > 0$ and $\varepsilon > 0$ be given as in the assumption. For any $\delta > 0$, there exists a constant $S_0 > 0$ by condition (2) such that, for all j, i and all $x, y \in X_{ji}$, we have

$$d(x, y) \geq S_0 \Longrightarrow |\langle \xi_x^{ji}, \xi_y^{ji} \rangle| < \delta.$$

Set $T_0 = S_0 + 2(L + R)$. It follows from the above proof of Lemma 3.1 applied to all X_j that, for all j and all $x, y \in X_j$, we have

$$d(x, y) \geq T_0 \Longrightarrow |\langle \zeta_x^j, \zeta_y^j \rangle| < \delta.$$

The proof is completed.

Proof of Theorem 3.1 Let X be a metric space with property $\mathrm{Q_{UE}}$. We show that X is uniformly embeddable into Hilbert space by using Proposition 3.1. Let $R>0$ and $\varepsilon>0$ be given. By the definition of property $\mathrm{Q_{UE}}$, there is an integer $m\ge 0$ such that

(1) there exists an integer $n_0\ge 0$ such that for the number $r_1:=R_1+1:=R_0+2(L_1+R_0)+1$, we have

$$X=\bigcup_{i_1=0}^{n_0}X_{i_1},\quad X_{i_1}=\bigsqcup_{r_1\text{-disjoint}}X_{i_1j_1},$$

where $R_0=R$, $L_1=\frac{(2n_0+2)(2n_0+3)R_0}{(\frac{\varepsilon}{2})^2}$ and $R_1=R_0+2(L_1+R_0)$;

(2) there exists an integer $n_1=n_1(n_0,r_1)\ge 0$ such that for $r_2:=R_2+1:=R_1+2(L_2+R_1)+1$ and for any $X_{i_1j_1}$, we have

$$X_{i_1j_1}=\bigcup_{i_2=0}^{n_1}X_{i_1j_1i_2},\quad X_{i_1j_1i_2}=\bigsqcup_{r_2\text{-disjoint}}X_{i_1j_1i_2j_2},$$

where $L_2=\frac{(2n_1+2)(2n_1+3)R_1}{(\frac{\varepsilon}{4})^2}=\frac{(2n_1+2)(2n_1+3)R_1}{(\frac{\varepsilon}{2^2})^2}$ and $R_2=R_1+2(L_2+R_1)$;

$\cdots\cdots$

(m) there exists an integer $n_{m-1}=n_{m-1}(n_0,\cdots,n_{m-2},r_1,\cdots,r_{m-1})\ge 0$ such that for $r_m:=R_m+1:=R_{m-1}+2(L_m+R_{m-1})+1$ and for any $X_{i_1j_1\cdots i_{m-1}j_{m-1}}$, we have

$$X_{i_1j_1\cdots i_{m-1}j_{m-1}}=\bigcup_{i_m=0}^{n_{m-1}}X_{i_1j_1\cdots i_{m-1}j_{m-1}i_m},$$
$$X_{i_1j_1\cdots i_{m-1}j_{m-1}i_m}=\bigsqcup_{r_m\text{-disjoint}}X_{i_1j_1\cdots i_mj_m},$$

and the family of metric spaces $\{X_{i_1j_1\cdots i_mj_m}\}_{i_1,j_1,\cdots,i_m,j_m}$ is equi-uniformly embeddable into Hilbert space, where

$$L_m=\frac{(2n_{m-1}+2)(2n_{m-1}+3)R_{m-1}}{(\frac{\varepsilon}{2^m})^2},\quad R_m=R_{m-1}+2(L_m+R_{m-1}).$$

Hence, by the definition of equi-uniform embeddability, for the constant $R_m\ge 0$ and the above $\varepsilon>0$, there exists a family of Hilbert space valued maps

$$\xi^{i_1j_1\cdots i_mj_m}:X_{i_1j_1\cdots i_mj_m}\to H_{i_1j_1\cdots i_mj_m}$$

such that for all $x,y\in X_{i_1j_1\cdots i_mj_m}$, we have $\|\xi_x^{i_1j_1\cdots i_mj_m}\|=1$, and

(1) $d(x,y)\le R_m\Longrightarrow \|\xi_x^{i_1j_1\cdots i_mj_m}-\xi_y^{i_1j_1\cdots i_mj_m}\|\le\frac{\varepsilon}{2^m}$;

(2) $\lim\limits_{S\to\infty}\sup\limits_{i_1,j_1,\cdots,i_m,j_m}\sup\{|\langle\xi_x^{i_1j_1\cdots i_mj_m},\xi_y^{i_1j_1\cdots i_mj_m}\rangle|:d(x,y)\ge S,\ x,y\in X_{i_1j_1\cdots i_mj_m}\}=0.$

Since

$$X_{i_1j_1\cdots i_m}=\bigsqcup_{r_m\text{-disjoint}}X_{i_1j_1\cdots i_mj_m},$$

we naturally define

$$\xi^{i_1j_1\cdots i_m}:X_{i_1j_1\cdots i_m}\to H_{i_1j_1\cdots i_m}=\bigoplus_{j_m}H_{i_1j_1\cdots i_mj_m}$$

by

$$\xi_x^{i_1j_1\cdots i_m}=\begin{cases}\xi_x^{i_1j_1\cdots i_mj_m}, & \text{if } x\in X_{i_1j_1\cdots i_mj_m},\\ 0, & \text{otherwise}\end{cases}$$

for all $x\in X_{i_1j_1\cdots i_m}$. Note that for any $x\in X_{i_1j_1\cdots i_m}$, there exists a unique $X_{i_1j_1\cdots i_m\widetilde{j}_m}$ such that $x\in X_{i_1j_1\cdots i_m\widetilde{j}_m}$. Thus $\|\xi_x^{i_1j_1\cdots i_m}\|=\|\xi_x^{i_1j_1\cdots i_m\widetilde{j}_m}\|=1$, and for all $x,y\in X_{i_1j_1\cdots i_m}$ we have

(1) $d(x,y)\le R_m=R_{m-1}+2(L_m+R_{m-1})\Longrightarrow$

$$\|\xi_x^{i_1j_1\cdots i_m}-\xi_y^{i_1j_1\cdots i_m}\|=\|\xi_x^{i_1j_1\cdots i_mj_m}-\xi_y^{i_1j_1\cdots i_mj_m}\|\le\frac{\varepsilon}{2^m};$$

(2) $\lim\limits_{S\to\infty}\sup\limits_{i_1,j_1,\cdots,j_{m-1},i_m}\sup\{|\langle\xi_x^{i_1j_1\cdots i_m},\xi_y^{i_1j_1\cdots i_m}\rangle|:d(x,y)\ge S,\ x,y\in X_{i_1j_1\cdots i_m}\}=0.$

By Lemma 3.2, we obtain a family of maps

$$\xi^{i_1j_1\cdots i_{m-1}j_{m-1}}:X_{i_1j_1\cdots i_{m-1}j_{m-1}}\to H_{i_1j_1\cdots i_{m-1}j_{m-1}}=\bigoplus_{i_m=0}^{n_{m-1}}H_{i_1j_1\cdots i_{m-1}j_{m-1}i_m}$$

such that for all $x,y\in X_{i_1j_1\cdots i_{m-1}j_{m-1}}$, we have $\|\xi_x^{i_1j_1\cdots i_{m-1}j_{m-1}}\|=1$, and

(1) $d(x,y)\le R_{m-1}\Longrightarrow\|\xi_x^{i_1j_1\cdots i_{m-1}j_{m-1}}-\xi_y^{i_1j_1\cdots i_{m-1}j_{m-1}}\|\le\frac{\varepsilon}{2^{m-1}}$;

(2) $\lim\limits_{S_1\to\infty}\sup\limits_{i_1,j_1,\cdots i_{m-1},j_{m-1}}\sup\{|\langle\xi_x^{i_1j_1\cdots i_{m-1}j_{m-1}},\xi_y^{i_1j_1\cdots,i_{m-1}j_{m-1}}\rangle|:d(x,y)\ge S_1,\ x,y\in X_{i_1j_1\cdots i_{m-1}j_{m-1}}\}=0$, where the running variables S_1 and S can be compared with each other as in $S_1=S+2(L_m+R_{m-1})$.

Now we have moved from the m-th level of decomposition back to the $(m-1)$-th level, and are in the situation as required by the assumption of Lemma 3.2. Repeating the above process by using Lemma 3.2 for m-times, we conclude that, for any $R>0$ and $\varepsilon>0$, there exists a map

$$\xi:X\to H=\bigoplus_{i_1=0}^{n_0}H_{i_1}$$

such that for all $x,y\in X$, we have $\|\xi_x\|=1$, and

(1) $d(x,y)\le R\Longrightarrow\|\xi_x-\xi_y\|\le\varepsilon$;

(2) $\lim\limits_{S_m\to\infty}\sup\{|\langle\xi_x,\xi_y\rangle|:d(x,y)\ge S_m,\ x,y\in X\}=0$, where the running variable S_m is related with the previous running variables $S_0=S,S_1,\cdots,S_{m-1}$ by

$$S_m=S_{m-1}+2(L_1+R_0)=S_0+2\sum_{i=1}^m(L_i+R_{i-1}).$$

Hence, by Proposition 3.1, X is uniformly embeddable into Hilbert space. The proof is completed.

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