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REGULAR ARTICLE

Monopolistic competition and new products: a conjectural equilibrium approach

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Abstract In this paper we generalize the heterogeneous risk adverse agents model of diffusion of new products in a multi-firm, heterogeneous and interacting agents environment. We use a model of choice under uncertainty based on Bayesian theory. We discuss the possibility of product failures, the set of equilibria, their stability and some welfare properties depending on the degree of diversification.

Keywords Product diffusion · Risk aversion · Lock-in · Monopolistic competition · Multiple equilibria

JEL Classification L15 · D81 · O33

1 Introduction

The analysis of diffusion processes is interesting under at least two different perspectives. First of all, scholars usually concentrate on new *products*, but it is possible to generalize many conclusions to the adoption of new technologies, behaviours, fashions and strategies (in the game-theoretic sense), so enlarging the focus significantly. Second, diffusion is in essence a multi-disciplinary matter: the literature that has

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studied the problem spans from management to sociology, from psychology to physics including, obviously, alternative economic approaches.[1]

The literature has discussed both the conditions that favour or hamper diffusion —bringing eventually to failure or success—and the speed of diffusion, looking at the factors giving rise to different possible patterns, and in particular to an epidemiologic-like S-shaped curve.

A satisfactory picture should be grounded on some essential building blocks. The first one is *uncertainty*: the very novelty of goods (ideas, technologies, behaviours, etc.) implies that agents must act using conjectures over some unknown feature, as in standard Bayesian approaches (Jensen 1982; Feder and O'Mara 1982; Tsur et al. 1990; Chatterjee and Eliashberg 1990; Young 2005). The second block is *heterogeneity*: individual models are necessarily different at the outset, since they summarize personal conjectures, previous learning and a priori ideas (Abrahamson and Rosenkopf 1993; Cowan and Jonard 2003, 2004; Lopez Pintado and Watts 2006). The third block is *interaction*: the learning activity on the part of agents exploits past observations, stemming mainly from other agents' choices. Interaction thus shapes the overall process, making it path dependent. Coupling all this with some degree of *non-linearity* might finally allow for multiple equilibria, and hence non-uniqueness of outcomes (lock-in: see Arthur 1994; Amable 1992; Agliardi 1998; Aoki and Yoshikawa 2002; Young 2007).

In Bogliacino and Rampa (2008) we developed a setup which includes risk aversion and the interaction between demand for and supply of a single new product. The presence of *both* aspects distances that paper from other existing models. Risk aversion is relevant, because during the learning process the emergence of information shapes the confidence of agents (as captured by individual precisions), so altering their willingness to pay. This aspect was already recognised by some,[2] but never worked out in a fully fledged dynamical learning model, as done instead in Bogliacino and Rampa (2008). Demand-supply interaction allows one to free the analysis from the single-sided approach prevailing in the literature;[3] in addition, this allows to model explicitly firms' uncertainty over demand. The main results proved in that paper were: a positive probability of failure of an otherwise 'good' product, depending on consumers' priors; the possibility of different shapes of the diffusion curve of the new product (besides the usual S-shaped one), depending again on consumers' priors; the existence of a continuum of market equilibria, depending on the firm's prior on demand; the

[1] The milestone for the literature on diffusion is the Bass model of epidemiologic diffusion pattern (Bass 1969). There is a sociological strand of literature focussed on heterogeneity and social effects, e.g. Granovetter (1978), Macy (1991), Abrahamson and Rosenkopf (1993), Valente (1996), Lopez Pintado and Watts (2006). The orthodox Economics literature is more interested in grounding the choice process on robust roots, using Bayesian theory (Jensen 1982; Feder and O'Mara 1982; Bikhchandani et al. 1992; Bergemann and Välimäki 1997; Vettas 1998), but some discussion on more general behaviour rules can be found in Nelson et al. (2004) and Geroski (2000). An excellent review is Hall (2005); an overall discussion of the properties of diffusion curves under alternative setups is Young (2007). On the physics side, one should consider percolation theory as a model of diffusion of ideas and innovations in networks: see e.g. Grimmett (1999) and, as an economic application, Duffie and Manso (2006); an econophysics example is offered by Yanagita and Onozaki (2008).

[2] Roberts and Urban (1988).

[3] Some noteworthy exceptions are Bergemann and Välimäki (1997) and Vettas (1998).

possibility that a part of those equilibria are dynamically unstable under learning, especially if the firm is highly uncertain about demand; and finally a monotonic relation between the degree of stability and welfare effects.

In the present paper we generalize the previous results analyzing a *multiple good* case, i.e. abandoning monopoly and moving to monopolistic competition, but still retaining consumers' and firms' heterogeneity and uncertainty, and consumers' risk aversion. As in the first paper, we provide purely analytical results, characterizing the full set of equilibria of the diffusion process together with their stability properties under the learning dynamics, without relying on simple simulations exercises which in the end give only a partial understanding of the overall process. To our knowledge this is the first article that analyzes analytically the property of a model with multiple goods, two-side uncertainty and heterogeneous agents. Moreover, in the end of the paper we endogenize the number of firms, i.e. we discuss *entry*, while in Proposition 1 we discuss failure, i.e. *exit*. This means that this paper is launching the basis for an overall treatment of the establishment of a new market under heterogeneity and interaction, without relying simply on simulations.

The paper proceeds as follows: Sect. 2 describes the demand side, Sect. 3 the supply one, Sect. 4 presents and discusses the main results, Sect. 5 concludes. Proofs are collected in "Appendix".

2 Consumers

The individual consumer j $(j = 1, \ldots, M)$ maximizes her expected utility choosing the level of consumption of each new good i $(i = 1, \ldots, n)$, over whose qualities she is uncertain. Qualities are independent normal variables, with known precision and unknown mean. Following a standard Bayesian setting, we assume consumers to be endowed with a prior over the unknown mean quality of each good, defined by two hyper-parameters $\mu_{j,i,t}$ and $\tau_{j,i,t}$, respectively the mean and the precision (the inverse of the variance, see De Groot 1970) that evolve through time being updated using Bayes' rule. We assume additively separable preferences. From now on t denotes time, ranging discretely from zero onwards.

We represent the consumer problem in the following way:

$$\begin{aligned}\max_{\{x_{j,i,t}\}_{i=1,\ldots,n}} \quad & E[U(x_{j,i,t}, \lambda_i)|\mu_{j,i,t-1}, \tau_{j,i,t-1}] \\ &= E\left[\sum_{i=1}^{n} u(x_{j,i,t}) f(\lambda_i)|\mu_{j,i,t-1}, \tau_{j,i,t-1}\right] \qquad (1)\end{aligned}$$

such that $\sum_{i=1}^{n} p_{i,t} x_{j,i,t} \leq w$, where $p_{i,t}$ is the price of goof i at time t, and w is the income endowment, for simplicity equal in time and through all consumers.[4] U is an

[4] In (1) agents take expectations with respect to all the available information at time t, which obviously includes the information revealed by the market in the previous period, thus we use the time subscript $t-1$ for hyper-parameters. The reason will become clear in a while.

additively separable utility, where u is the sub-utility[5] function (i.e. the utility of the single good), and $x_{j,i,t}$ the quantity of good i consumed by agent j at time t. The function $f(\cdot)$ is the way the quality of each good (λ_i) is incorporated into agents' preferences.

In particular, as in the single good framework of Bogliacino and Rampa (2008), we assume that U satisfies (i) $\frac{\partial^2 U}{\partial x_{j,i,t}\partial\lambda_i} > 0$, meaning that the consumer wishes to purchase more if quality is higher, for given price; and (ii) $\frac{\partial^3 U}{\partial x_{j,i,t}(\partial\lambda_i)^2} < 0$, i.e. consumers are risk averse in quality:[6] this suggests that a higher variance of quality tends to depress (expected) marginal utility and hence consumption, for given price.

As in Bogliacino and Rampa (2008), we posit $u(\cdot) = (\cdot)^\delta$, and $f(\lambda_i) = A - \exp(-\lambda_i)$; we assume in addition $\lambda_i \sim N(\mu_i, r)$, due to random production and/or delivery factors, where the true mean μ_i is unknown, and r is known, to consumers; the different qualities are statistically independent. The individual prior, defined over the mean of each quality, is also assumed normal, which allows us to use the properties of the conjugate family.[7] The advantage of these assumptions is threefold: first, they satisfy the two conditions (i)–(ii) above; second, they allow us to "pass through" the expected value operator using the fact that, owing to normality and to the exponential, $f(\lambda_i)$ is log-normal; finally, they imply, as we shall see, that consumers are not bound to buy a positive quantity of each good. This last property is useful to study the effects of noisy quality signals on consumers' choices, addressing the possibility of lock-in, i.e. the failure of a diffusion of a "good" product.

As regards the timing of events, the consumer makes her choice at time t using all information available at that time, which is captured through her posterior, and before knowing the others' choices at t. All the new information refers then to choices made at $t-1$, hence the hyper-parameters relevant for the choice at t are $\mu_{j,i,t-1}$ and $\tau_{j,i,t-1}$.

This given, standard maximization[8] implies the following individual demand curve for each good j:

$$x_{j,i,t} = \frac{p_{i,t}^{1/(\delta-1)}\left[\delta\left(A - \exp\left(-\mu_{j,i,t-1} + \frac{\tau_{j,i,t-1}^{-1}+r^{-1}}{2}\right)\right)\right]^{1/(1-\delta)}}{\sum_{i=1}^{n} p_{i,t}^{\delta/(\delta-1)}\left[\delta\left(A - \exp\left(-\mu_{j,i,t-1} + \frac{\tau_{j,i,t-1}^{-1}+r^{-1}}{2}\right)\right)\right]^{1/(1-\delta)}} w \tag{2}$$

5 The term sub-utility is standard in the literature that uses additively separable structure, either summing through time of through goods. In our setup it simply expresses the utility of the consumption of one specific good.

6 In the standard choice theory, risk aversion is deemed as negativity of the *second* derivative. In our setup, this property obviously holds for quality, since $u(\cdot)$ is strictly increasing and the utility function U is multiplicatively separable in quality and quantity. However, we preferred to present this characteristic in terms of *third* cross-derivative, because we want to stress the implication for the *quantity* purchased.

7 A utility function similar to that used in the present setup was proposed also by Roberts and Urban (1988), who however did not explore analytically the dynamic implications of learning and of demand-supply interaction, limiting themselves to simulations exercises.

8 We refer to the solution of the problem (1), i.e. maximization of expected utility under a budget constraint, where the choice variables are the quantities of each good, and the functional forms of u and f are as said above. The problem is strictly concave on a compact feasible set, which implies that the solution is unique.

where one must intend $x_{j,i,t} = 0$ whenever $A - \exp(\cdot) \leq 0$.[9] If $A - \exp(\cdot) > 0$, we say that consumer j is *active* on market i at time t.

The interpretation is straightforward: each consumer spends a share of its total income on good i, depending on the ratio of its price-quality term to that of the whole bundle of goods. Total actual market demand for good i, $Q^D_{i,t}$, is simply the summation over the j index.

After buying the chosen quantity, each active consumer receives a quality signal that she publicly announces to *all* consumers: these signals are used by each of them to update her conjecture. Using the properties of conjugate families (De Groot 1970), the posterior parameters for the normal-normal couple (respectively, the likelihood and the prior) are calculated simply as:

$$\mu_{j,i,t} = \frac{\tau_{j,i,t-1}\mu_{j,i,t-1} + rM_{i,t}\bar{\lambda}_{i,t}}{\tau_{j,i,t-1} + rM_{i,t}}, \quad \tau_{j,i,t} = \tau_{j,i,t-1} + rM_{i,t} \tag{3}$$

where $\bar{\lambda}_{i,t}$ is the quality sample mean, computed from the announced perceived qualities, and $M_{i,t} \leq M$ is the number of active buyers at date t. Notice that consumers treat qualities as independent, and update their conjectures accordingly (that is, separately for each good).

The above equation simply tells us that consumers average their own prior opinions and the sample mean of quality from the new observations, the weight being the relative precisions of the two measures. Moreover, through time individual precisions grow linearly: as one can imagine, given the assumptions of quality-risk-aversion, this fact tends to raise demand in time, due to a simple informational effect.

3 Firms

Firms interact in monopolistic competition, each producing a new good at a constant marginal cost c_i: since each firm corresponds to a single product, as in standard monopolistic competition, we use the i index to define a firm. Every firm is uncertain over its own demand. To make things as simple as possible, we assume that it conjectures a linear demand defined by two parameters: more precisely, given the price $p_{i,t}$, firm i believes that its demand is a random normal variable with mean $Q_{i,t} = a_i - b_i p_{i,t}$ and precision equal to 1. In addition, firm i does not know a_i and b_i, and maintains the hypothesis that the distribution of the two parameters is a normal bivariate: the mean and the precision hyper-parameters of this distribution at date t are as follows:[10]

$$\mathbf{m}_{i,t} = \begin{bmatrix} \alpha_{i,t} \\ \beta_{i,t} \end{bmatrix}, \quad \mathbf{\Gamma}_{i,t} = \begin{bmatrix} \gamma_{i,1,t} & \gamma_{i,12,t} \\ \gamma_{i,21,t} & \gamma_{i,2,t} \end{bmatrix} \tag{4}$$

[9] In fact, although the sub-utility $u(\cdot) = (\cdot)^{\delta}$ satisfies Inada conditions, when this condition holds, the per-period utility becomes negative, except if the quantity is zero: thus not buying becomes the rational choice.

[10] This derives from our assumption that the conditional distribution of $Q_{i,t}$ has known precision equal to 1; if this precision were different from 1, the precision matrix $\mathbf{\Gamma}_{i,t}$ would be multiplied by its value. Things could be generalized, but this would be immaterial for our results, since firm's expected profit does not depend on precisions, given risk neutrality.

where $\gamma_{i,1,t}$ and $\gamma_{i,2,t}$ are positive. Since the firm has surely no reason to conjecture any particular initial value for the correlation among the two mean hyper-parameters, we assume $\gamma_{i,12,0} = \gamma_{i,21,0} = 0$. Define also $\gamma_{i,k} \equiv \gamma_{i,k,0}$, $k = 1, 2$ as the firm's initial precisions of the mean parameters.

As in the consumer case, the timing is as follows: the firm announces the price before observing demand, hence it uses its $(t-1)$-conjecture, formed observing demand at time $t-1$. The firm chooses the price so as to maximize expected profit. Therefore, from standard First-Order Condition in monopoly, the price announced at date t is:

$$p_{i,t} = \frac{\alpha_{i,t-1}}{2\beta_{i,t-1}} + \frac{c_i}{2} \tag{5}$$

and expects the following demand:

$$Q^e_{i,t}(p_{i,t}) = \frac{\alpha_{i,t-1} - c_i\beta_{i,t-1}}{2} \tag{6}$$

We neglect any capacity constraint, and assume that the firm can meet all demand.[11]

The updating process on the part of firm i follows, again, standard Bayesian rules: using primes to denote transposed vectors, define the row vector $\mathbf{x}'_{i,t} \equiv [1 \; -p_{i,t}]$. Given our assumptions, one has (De Groot 1970, Chap 11):

$$\mathbf{m}_{i,t} = \left[\mathbf{\Gamma}_{i,t-1} + \mathbf{x}_{i,t-1}\mathbf{x}'_{i,t-1}\right]^{-1}\left[\mathbf{\Gamma}_{i,t-1}\mathbf{m}_{i,t-1} + \mathbf{x}_{i,t-1}Q^D_{i,t}\right] \tag{7}$$

and

$$\mathbf{\Gamma}_{i,t} = \left[\mathbf{\Gamma}_{i,t-1} + \mathbf{x}_{i,t-1}\mathbf{x}'_{i,t-1}\right] \tag{8}$$

By simple algebra, (7) can be rewritten as:

$$\begin{aligned}
\mathbf{m}_{i,t} &= \left[\mathbf{\Gamma}_{i,t-1} + \mathbf{x}_{i,t-1}\mathbf{x}'_{i,t-1}\right]^{-1}\Big[(\mathbf{\Gamma}_{i,t-1} + \mathbf{x}_{i,t-1}\mathbf{x}'_{i,t-1} - \mathbf{x}_{i,t-1}\mathbf{x}'_{i,t-1}) \\
&\quad \times \mathbf{m}_{i,t-1} + \mathbf{x}_{i,t-1}Q^D_{i,t}\Big] \\
&= \left[\mathbf{\Gamma}_{i,t-1} + \mathbf{x}_{i,t-1}\mathbf{x}'_{i,t-1}\right]^{-1}\Big[(\mathbf{\Gamma}_{i,t-1} + \mathbf{x}_{i,t-1}\mathbf{x}'_{i,t-1})\mathbf{m}_{i,t-1} + \mathbf{x}_{i,t-1} \\
&\quad \times \left(Q^D_{i,t} - \mathbf{x}'_{i,t-1}\mathbf{m}_{i,t-1}\right)\Big] \\
&= \mathbf{m}_{i,t-1} + \left[\mathbf{\Gamma}_{i,t-1} + \mathbf{x}_{i,t-1}\mathbf{x}'_{i,t-1}\right]^{-1}\left[\mathbf{x}_{i,t-1}\left(Q^D_{i,t} - \mathbf{x}'_{i,t-1}\mathbf{m}_{i,t-1}\right)\right]
\end{aligned} \tag{9}$$

[11] An interesting aspect of our setup is the possibility of analysing disequilibrium processes leaving its main features unaltered. In fact, assuming for instance production lags, i.e. the need for the firms to decide quantity *and* price, then equilibrium, as defined short below, is also a *market* equilibrium in the standard sense. From (9) it is clear that all that is needed to discuss the disequilibrium path and the convergence to a market equilibrium is the possibility for firms to observe the true demand for given price in each period and to adjust supply accordingly, e.g. through the use of inventories. However this full characterization is beyond the scope of the present paper, being a question more related to discussion of the features of a general equilibrium.

In a nutshell, the above expression tells us that the new mean parameters are equal to the previous period's ones, plus a correction term depending the prediction error[12] and adjusted for the new precision matrix.

4 Equilibria: main results

The system can be fully characterized in terms of firms' and consumers' hyper-parameters.

Define $\boldsymbol{\mu}_{j,t} = [\mu_{j,1,t} \ldots \mu_{j,n,t}]'$ and $\boldsymbol{\tau}_{j,t} = [\tau_{j,1,t} \ldots \tau_{j,n,t}]'$ as the vectors of consumer j's hyper-parameters at time t. Then define $\boldsymbol{\mu}_t = [\boldsymbol{\mu}'_{1,t} \ldots \boldsymbol{\mu}'_{M,t}]'$ and $\boldsymbol{\tau}_t = [\boldsymbol{\tau}'_{1,t} \ldots \boldsymbol{\tau}'_{M,t}]'$ for all consumers. As regards firms, call $\boldsymbol{\gamma}_{i,t} = [\gamma_{i,1,t}, \gamma_{i,12,t}\ \gamma_{i,21,t}, \gamma_{i,2,t}]'$ the vectorization of the precision matrix of firm i's conjecture at time t; posit finally $\boldsymbol{\gamma}_t = [\boldsymbol{\gamma}'_{1,t} \ldots \boldsymbol{\gamma}'_{n,t}]'$, and $\mathbf{m}_t = [\mathbf{m}'_{1,t} \ldots \mathbf{m}'_{n,t}]'$.

Defining $\mathbf{y}_t = [\boldsymbol{\mu}'_t\ \boldsymbol{\tau}'_t\ \mathbf{m}'_t\ \boldsymbol{\gamma}'_t]'$, we compact all the updating equations[13] in the following system of $2nM + 6n$ first-order difference equations:

$$\mathbf{y}_t = F(\mathbf{y}_{t-1}) \tag{10}$$

which completely describes the learning and diffusion dynamics.

Risk aversion on the part of consumers makes them sensitive to all piece of information available: as time goes by, new information can increase precisions and raise their demand, ceteris paribus. For this reason the system shows path dependence and irreversibility. The relevant equilibrium concept is thus a steady state one, meaning the agents' conjectures remain fixed in time. We use in fact a *conjectural equilibrium* notion: a conjectural equilibrium is a fixed point of (10).

One might think that a conjectural equilibrium requires that all consumers have necessarily learnt the true qualities of the goods. In fact, *if* new information keeps arriving, the Law of Large Numbers implies that consumers are bound to learn the true qualities. It is also possible, however, that consumers are endowed initially with pessimistic conjectures about one of the goods, so demanding a null quantity of it: a null demand, in turn, implies that no signal will arrive at next date, and conjectures remain unchanged (*lock-in*). More importantly, it might happen that, even starting from a positive demand at date t, a highly biased signal switches demand off at date $t + 1$: we term "*failure*" this phenomenon.

As regards this last point, we recall one of the results of Bogliacino and Rampa (2008).

Proposition 1 *Suppose that demand for good i is positive at time t. Then, there exists positive probability of failure of the ith product at time $t + 1$.*

Proof See Bogliacino and Rampa (2008), Proposition 1. □

[12] Notice in fact that $\mathbf{x}'_{i,t-1}\mathbf{m}_{i,t-1}$ is expected demand, given the prior.

[13] Taking account of (2) and (5).

The argument runs as follows: at every time t we can build a complete *ordering* over the set of consumers in terms of a function of their mean and precision hyper-parameters (the indicator being $\tau_{ijt-1}(\mu_{ijt-1} - \tau_{ijt-1}^{-1}/2 - B)$, where $B = r^{-1} - \log(A)$): the higher its value, the higher a consumer's 'optimism'. If a signal is so biased as to drive the most optimistic consumer below a certain threshold (recall that $A < \exp(-\lambda_i)$ implies no purchase), then all demands are driven to zero. But then no information is made available to update conjectures, and consumers are locked-in at zero demand.[14] Finally, using our assumptions on distributions, we can prove that the probability of such an event is always positive. We do not report the complete proof here, since it is quite long and in addition the mentioned ordering has a number of interesting implications that are worth investigating on their own.

We come now to a different set of results, assuming that failure does not occur. In this case a conjectural equilibrium is a situation in which consumers' conjectured means have converged to the true mean qualities, *and* in addition firms' conjectures are confirmed by the true demands, so that prediction errors are zero and firms' conjectures remain unchanged at subsequent dates.[15] We can fix the ideas taking $\mu_{i,j,t} = \mu_i, \forall i, j$, and studying the dynamics in *expected value* terms,[16] i.e. with the signals always equal to the true qualities, so that demands stay constant for given prices (and consumers' precisions are free to diverge as in the standard Bayesian setting).

This given, define

$$g_i(\mathbf{m}_{i,t-1}) = Q_{i,t}^D\left[p_{i,t}\left(\mathbf{m}_{i,t-1}\right), \mu_i\right] - \mathbf{x}'_{i,t}\mathbf{m}_{i,t-1}$$

as the excess of actual demand over expected one for good i; thus the equilibrium condition can be written as follows:

$$g(\mathbf{m}_t) = [g_1(\mathbf{m}_{1,t}), \ldots, \ g_n\ (\mathbf{m}_{n,t})]' = \mathbf{0}, \tag{11}$$

a set of n equations. Then the following Proposition holds.

Proposition 2 *There exists a n-dimensional equilibrium manifold in the space of firms' parameters.*

Proof (11) is a system of n equations in $2n$ variables, hence the set of solutions is generically a n-dimensional manifold. □

Conjectural equilibria, then, form a *continuum*: there is not a unique steady state that can be attained by the system. Although mathematically simple, this proposition states that the reliance on "fundamentals equilibrium reasoning" by standard theory, leading

14 One can also, as in Bogliacino and Rampa (2008), study the diffusion dynamics and microfound logistic or concave diffusion patterns depending on initial consumer conjectures, an aim which is beyond the scope of this paper. This question is a key one in the literature on diffusion, looking for an explanation of aggregate specific patterns of diffusion through time based on individual choice, and not single-equation aggregate behaviour as in the case of Bass (1969). In Bogliacino and Rampa (2008) we show how the hyper-parameters can be used to characterize individual conjectures giving rise to the different results.

15 See expression (9) above.

16 With respect to the true distribution of μ_i.

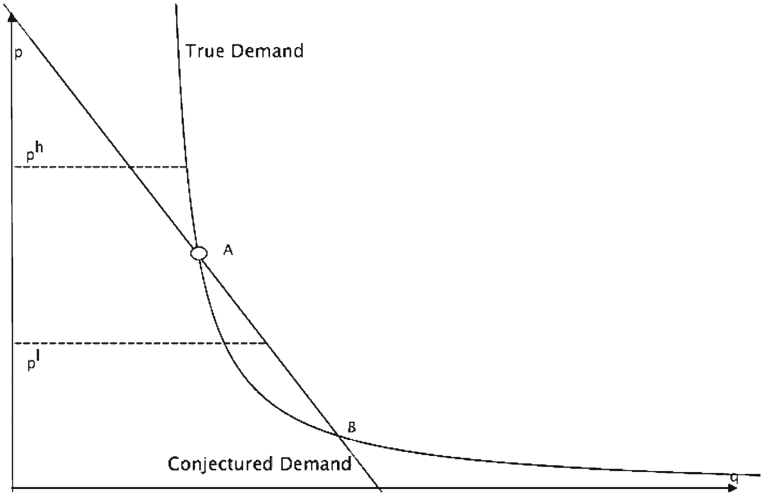

Fig. 1 An equilibrium where conjectured demand is more elastic than the true one

to uniqueness results, should be taken with caution, since it is not robust to the weakening of the common prior assumption. Notice that different conjectural equilibria differ among themselves as regards prices and quantities, and hence welfare as well, not as regards the qualities perceived by consumers, since we are assuming that these have been learned perfectly.

A natural question is now the *stability* of equilibria along the manifold. Given the continuum, we must speak of *Lyapunov* stability: that is, stable equilibria are not asymptotically (locally) stable, since a small displacement from one stable equilibrium to another does not cause convergence back to the former. In addition, in the case of stability, different initial conditions lead to different final states.

We study stability of equilibria at any finite time, recalling that we are assuming $\mu_{i,j,t} = \mu_i, \forall i, j$ and are working in expected values. Hence the stability of equilibria depends entirely on the firms' parameters: indeed we can prove the following Proposition.

Proposition 3 *The equilibria where conjectured demand is more elastic than the true one are locally unstable.*

Proof See Appendix A.1. □

The intuition for this result can be seen using Fig. 1. A possible equilibrium position is A, where the firm maximizes profits, given its conjecture, and there is no prediction error. From the definition of equilibrium, price and quantity are common to both the true and the conjectured demand, so the condition of Proposition 3 implies that the derivative of the conjectured demand is higher (in absolute value) than that of true demand. On the contrary, a B-like equilibrium is one where the true demand is less rigid than the conjectured one.

Look at expression (9), and at how it can be rewritten according to Appendix A.2.: in the presence of excess demand a firm updates its parameters in such a way that the

α parameter grows and the β parameter decreases.[17] Hence, using (5), it follows that firm will raise its price at the subsequent date. The opposite holds in the presence of excess supply.

Consider now what is happening in a neighbourhood of A; a higher (resp. smaller) price, such as p^h (resp. p^l) generates excess supply (resp. demand), thus inducing the firm to raising (resp. lowering) further its price. It is then apparent that the system moves away from the A equilibrium. A similar reasoning for a B-like equilibrium shows that in this case there can exist a basin of attraction (unless there is overshooting, a possibility shown by Proposition 4 below). This type of instability is obviously local, since we can only study linear approximations.

In a B-like equilibrium we could still observe local instability at some finite time, instability being of the oscillatory type. This property, however, is smoothed by the passing of time and the instability is rapidly reabsorbed. In fact we have the following Proposition.

Proposition 4 *In an equilibrium where the true elasticity is high and the demand conjectured by a single firm is more rigid than the true one, there can exist oscillatory instability as long as t is small, and provided that the firms' initial precisions are low.*

Proof See Appendix A.2. □

Under the condition of this Proposition, if the system starts in a neighbourhood of some equilibrium the variables will be pushed away from it, and, given the continuum of equilibria, the location of the steady state depends on initial conditions. Observe however that the same unstable equilibria are turned into stable ones by the passing of time, that has the effect of increasing firms' precisions, as apparent from the proof of Proposition 4.

We can finally add some further results in terms of welfare. In Bogliacino and Rampa (2008), studying a single firm, we analyzed the relation between welfare and stability along the equilibrium manifold. In the present context the higher dimensionality makes things more complex: it is not so easy to identify how individual parameters change together along the manifold; and we cannot block $n-1$ firms, trying to concentrate on a single one, since changing one price implies obviously changes in all expenditure shares. We leave this point for further research.

Our multiple-good setup, however, allows us to analyze the degree of diversification of the decentralized economy and its welfare properties, although under some stricter assumptions. This is a fairly standard procedure in Monopolistic Competition literature: we need to endogenize the number of firms (i.e. the number of varieties) by means of a fixed cost of entry (see Dixit and Stiglitz 1977; Tirole 1988; Bertoletti et al. 2008), then free entry implies a zero profit condition, which closes the model. Indeed, the following Proposition holds.

[17] In A.1. it is shown that (9) is equivalent to $\mathbf{m}_{i,t} = \mathbf{m}_{i,t-1} + \mathbf{C}_{i,t}\ \mathbf{g}_i(\cdot)$, where $\mathbf{C}_{i,t} = d_{i,t}\begin{bmatrix} \frac{\gamma_{i,2}}{p_{i,t}} + tp_{i,t} & tp_{i,t} \\ t & \gamma_{i,1}+t \end{bmatrix}$, $d_{i,t} > 0$, $\mathbf{g}_i(\cdot) \equiv \left[g_i(\cdot) \quad -g_i(\cdot)\right]'$, and $g_i(\cdot)$ was defined before expression (11) above: see (16) and (23) in that Appendix. As a consequence, one can easily check that if $g_i(\cdot) > 0$, that is, if true demand exceeds conjectured demand, then the first element of $\mathbf{C}_{i,t}\ \mathbf{g}_i(\cdot)$ is positive, while the second is negative.

Proposition 5 *In equilibrium with endogenous number of firms (assuming a positive fixed cost of entry) and identical marginal cost and qualities of goods, there is over (resp. under) diversification, if for the marginal firms—defined as that who fix the price at the lowest level in equilibrium—the true elasticity is greater (resp. lower) than the conjectured one.*

Proof See Appendix A.3. □

The interpretation is fairly obvious. Define ε_T and ε_C to be the true elasticity and that conjectured by firms: when the full information case is characterised by efficiency, in equilibrium the condition $\frac{\varepsilon_T}{\varepsilon_C} > 1$ makes firms less able to appropriate surplus, since it pushes entry. The opposite holds for $\frac{\varepsilon_T}{\varepsilon_C} < 1$. Since the full information case is efficient, when conjectured and true elasticity are equal the actual degree of diversification is equal to the optimal one. Thus, interestingly, not only the case $\frac{\varepsilon_T}{\varepsilon_C} = 1$ is stable, as implied by Propositions 3 and 4 taken together: it is also efficient in terms of diversification.

A caveat about this result: it is partly dependent on the particular form of the utility function. In general, the relation between the optimal degree of diversification and that prevailing under perfect information depends on how consumers' preferences affect the mark-up, since the latter is related to the ability of firms to appropriate the surplus (see Dixit and Stiglitz 1977). In our case, the iso-elastic assumption guarantees efficiency. However in the general case the ratio among the true elasticity and the conjectured one still allows us to characterize over and under diversification with respect to the perfect information case; of course one cannot say any longer that the degree of diversification prevailing under perfect information is also optimal.

5 Conclusions

This work studied a monopolistic competitive market, where firms innovate introducing new products and are uncertain about demand; at the same time, consumers are heterogeneous as regards their expectations on product qualities, which they are uncertain about. A key feature of the setup is quality risk aversion on the part of consumers, affecting their willingness to pay for products, due to their degree of uncertainty that in turn depends on past choices of all agents. Indeed, there is interaction in time among and between the market sides: such interaction shapes the learning process and the final pattern observed. In spite of the seeming complexity, we are able to characterize analytically some relevant properties of the stationary states of the dynamics without resorting to simulations, as is instead common in many studies of product diffusion.

The main results can be summarised as follows. First, there is positive probability of lock-in, that is high-quality products can fail to diffuse while lower-quality ones can succeed: this does not depend on some 'objective' increasing returns (as in, e.g., Arthur 1994), but on the constellation of consumers' priors, coupled with learning and risk aversion. Second, differently from the "fundamentals equilibrium reasoning" of standard theory, we find a continuum of conjectural equilibria, i.e. stationary states of the learning process. These two results are common to the single-firm case studied by Bogliacino and Rampa (2008): however, in the present monopolistic-competition setup the topological *dimension* of the equilibrium manifold is higher, since it equals the number of firms or goods; in other terms, we have a higher degree of indeterminacy.

The multi-firm case analysed in this work shows in addition that the (local) stability properties of conjectural equilibria, under the learning dynamics, can be multifarious: if a firm conjectures a demand curves that is more elastic than the true one, then we have *monotonically unstable* equilibria (Proposition 3). However, it is *not* always the case that, if conjectured demands are all less elastic than true ones, then equilibria are stable: indeed, if the true elasticity is high, and if a single firm conjectures a low elasticity but at the same time is very uncertain, then we could observe *unstable oscillations* around equilibria (Proposition 4). These unstable equilibria, that in view of the subsequent Proposition 5 are somehow inefficient, turn into stable ones when firms' uncertainty decreases, i.e. as time elapses. This means that it not true that learning or 'evolution' weeds out inefficiency.

Finally, the setup has been used to study endogenous entry and optimal diversification, at least under some further assumptions. We proved that the case when firms conjecture an elasticity equal to the true one in equilibrium is not only dynamically stable, but is also efficient in terms of diversification. In our decentralised environment, however, the learning process can approach any one of a very large number of stationary states, not only that efficient result, depending on firms' priors about demand. This observation makes room for possible corrective policies. But also the above-mentioned lock-in problem might highlight the necessity of policies aimed at disseminating greater information on quality among consumers.

To our knowledge this is the first article that analyzes analytically the property of a model with multiple goods, two-side uncertainty and heterogeneous agents. Moreover, we endogenize the number of firms, i.e. we discuss *entry*, and in addition we discuss failure, i.e. *exit*. This means that this paper is launching the basis for an overall treatment of the establishment of a new market under heterogeneity and interaction.

Further research includes the use of more sophisticated firms (oligopoly or conjectural variations models), and the characterization of the welfare properties along the continuum of equilibria. Of course the model could be simulated to study the shapes of alternative diffusion curves, and to enquire how final outcomes depend on initial conditions. As regards this last point, we claim that our model can be a workhorse for scholars aiming at simulating the diffusion patterns of multiple goods considering explicitly the role of supply and not only of demand. This setup is sufficiently flexible to account for a *network topology* of the updating process (indeed, an agent might receive information from a *subset* of agents), and for many kinds of noisy disturbances. An interesting aspect of the story is that rationality of agents populating these artificial worlds is not so bounded as to abandon Bayesian decision theory altogether.

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Appendix A

A.1. Proof of Proposition 3

The system is highly non-linear, so we should limit ourselves to discuss local stability, using a linear approximation in a neighbourhood of one equilibrium. The Jacobian matrix of $\mathbf{y}_t = F(\mathbf{y}_{t-1})$ is easily checked to be the following one:

$$\mathbf{J} = \begin{bmatrix} \left[\frac{\partial \mu_t}{\partial \mu_{t-1}}\right] & \left[\frac{\partial \mu_t}{\partial \tau_{t-1}}\right] & \mathbf{0}_{nM,2n} & \mathbf{0}_{nM,4n} \\ \mathbf{0}_{nM,nM} & \mathbf{I}_{nM} & \mathbf{0}_{nM,2n} & \mathbf{0}_{nM,4n} \\ \left[\frac{\partial \mathbf{m}_t}{\partial \mu_{t-1}}\right] & \left[\frac{\partial \mathbf{m}_t}{\partial \tau_{t-1}}\right] & \left[\frac{\partial \mathbf{m}_t}{\partial \mathbf{m}_{t-1}}\right] & \left[\frac{\partial \mathbf{m}_t}{\partial \gamma_{t-1}}\right] \\ \mathbf{0}_{4n,nM} & \mathbf{0}_{4n,nM} & \left[\frac{\partial \gamma_t}{\partial \mathbf{m}_{t-1}}\right] & \mathbf{I}_{4n} \end{bmatrix} \tag{12}$$

where $\mathbf{I}_k$ is the k-identity matrix and $\mathbf{0}_{k,p}$ is a k-by-p null matrix.

The stability condition is that all the eigenvalues of $\mathbf{J}$, evaluated at an equilibrium, do not lie outside the unit circle. We need some preliminary results.

Claim 1 *At an equilibrium, the eigenvalues of* $\mathbf{J}$ *are those of the four blocks along its main diagonal.*

Proof We need simply to prove that $\frac{\partial \mathbf{m}_t}{\partial \gamma_{t-1}} = \mathbf{0}$. Define first

$$\left[\boldsymbol{\Gamma}_{i,t-1} + \mathbf{x}_{i,t}\mathbf{x}'_{i,t}\right] = \left[\boldsymbol{\Gamma}_{i,t-1} + \begin{bmatrix} 1 & -p_{i,t} \\ -p_{i,t} & p_{i,t}^2 \end{bmatrix}\right] \equiv \mathbf{A}_{i,t}(\boldsymbol{\Gamma}_{i,t-1}, \mathbf{m}_{i,t-1}) \tag{13}$$

and

$$\begin{aligned} \left[\mathbf{x}_{i,t}(Q_{i,t}^D - \mathbf{x}'_{i,t}\mathbf{m}_{i,t-1})\right] &= \begin{bmatrix} 1 & 0 \\ 0 & p_t \end{bmatrix} \begin{bmatrix} (Q_{i,t}^D - \mathbf{x}'_{i,t}\mathbf{m}_{i,t-1}) \\ -(Q_{i,t}^D - \mathbf{x}'_{i,t}\mathbf{m}_{i,t-1}) \end{bmatrix} \\ &\equiv \mathbf{B}_{i,t}(\mathbf{m}_{i,t-1})\ \mathbf{g}_i(\mathbf{m}_{i,t-1}, \boldsymbol{\mu}_{t-1}, \boldsymbol{\tau}_{t-1}) \end{aligned} \tag{14}$$

where $\mathbf{g}_i(\cdot) \equiv [g_i(\cdot) - g_i(\cdot)]'$ and $g_i(\cdot)$ was defined before expression (11). Define finally

$$\mathbf{C}_{i,t}(\boldsymbol{\Gamma}_{i,t-1}, \mathbf{m}_{i,t-1}) \equiv [\mathbf{A}_{i,t}(\boldsymbol{\Gamma}_{i,t-1}, \mathbf{m}_{i,t-1})]^{-1}\mathbf{B}_{i,t}(\mathbf{m}_{i,t-1}) \tag{15}$$

Summing up, firm i's updating formula can be written as

$$\mathbf{m}_{i,t} = \mathbf{m}_{i,t-1} + \mathbf{C}_{i,t}(\boldsymbol{\Gamma}_{i,t-1}, \mathbf{m}_{i,t-1})\ \mathbf{g}_i(\mathbf{m}_{i,t-1}) \tag{16}$$

and the block which interests us now is:

$$\left[\frac{\partial \mathbf{m}_t}{\partial \gamma_{t-1}}\right] = \begin{bmatrix} \left[\frac{\partial \mathbf{m}_{1,t}}{\partial \gamma_{t-1}}\right] \\ \vdots \\ \left[\frac{\partial \mathbf{m}_{n,t}}{\partial \gamma_{t-1}}\right] \end{bmatrix} = \begin{bmatrix} \left[\frac{\partial \mathbf{C}_{1,t}}{\partial \gamma_{t-1}}\right] \mathbf{g}_1 + \mathbf{C}_{1,t} \left[\frac{\partial \mathbf{g}_1}{\partial \gamma_{t-1}}\right] \\ \vdots \\ \left[\frac{\partial \mathbf{C}_{n,t}}{\partial \gamma_{t-1}}\right] \mathbf{g}_n + \mathbf{C}_{n,t} \left[\frac{\partial \mathbf{g}_n}{\partial \gamma_{t-1}}\right] \end{bmatrix} \tag{17}$$

which is clearly equal to zero, since from (11) $\mathbf{g}_i(\cdot) = 0$ in equilibrium, and the $\mathbf{g}_i$'s themselves do not depend on firms' precisions. □

We can thus concentrate on the four principal blocks of $\mathbf{J}$. The NW block has eigenvalues lower than one, and tending to one as time goes to infinity: they are the weights attached to consumers' prior means in the updating formulae: see (3) above. The second and fourth blocks give rise to respectively nM and $4n$ eigenvalues equal to one: they relate to the updating of consumers' and firms' precisions, and are immaterial for stability. In fact changes in the precisions do not affect the equilibrium itself, being more important in the initial, rather than the final, phases of the learning process (Rampa 1989).

We are thus left with the $2n$ eigenvalues of the block $\left[\frac{\partial \mathbf{m}_t}{\partial \mathbf{m}_{t-1}}\right]$.

Claim 2 *The eigenvalues of* $\left[\frac{\partial \mathbf{m}_t}{\partial \mathbf{m}_{t-1}}\right]$ *are as follows*:

(i) *n eigenvalues are equal to one, implied by the continuum of equilibria*;
(ii) *the other n eigenvalues are equal to $\rho(\mathbf{D}_t \mathbf{G}_t) + \mathbf{1}$, where $\rho(\cdot)$ is the column vector of the eigenvalues of the argument, $\mathbf{1}$ is a column vector of ones, $\mathbf{D}_t$ is a diagonal matrix with positive diagonal elements, and $\mathbf{G}_t$ is a matrix with positive extra-diagonal elements.*

Proof (i) Define the following matrix:

$$\mathbf{C}_t = \begin{bmatrix} \mathbf{C}_{1,t} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{C}_{n,t} \end{bmatrix} \tag{18}$$

and

$$\mathbf{g} = [\mathbf{g}_1(\mathbf{m}_{t-1})' \quad \dots \quad \mathbf{g}_n(\mathbf{m}_{t-1})']'. \tag{19}$$

Given (16), and given that $\mathbf{g}_i(\cdot) = 0$ in equilibrium, one deduces:

$$\left[\frac{\partial \mathbf{m}_t}{\partial \mathbf{m}_{t-1}}\right] = \mathbf{I}_{2n} + \mathbf{C}_t \left[\frac{\partial \mathbf{g}}{\partial \mathbf{m}_{t-1}}\right] \tag{20}$$

This matrix has $2n$ eigenvalues equal to 1 plus those of the second term. Since by construction $\mathbf{g}(\cdot)$ is formed by $2n$ terms, n of which are the opposite of the remaining n, $\left[\frac{\partial \mathbf{g}}{\partial \mathbf{m}_{t-1}}\right]$ has rank n, and the same is generically true for

$\mathbf{C}_t\left[\frac{\partial \mathbf{g}}{\partial \mathbf{m}_{t-1}}\right]$: hence the latter has n eigenvalues equal to zero. Thus, we can conclude that n eigenvalues of $\left[\frac{\partial\ \mathbf{m}_t}{\partial\ \mathbf{m}_{t-1}}\right]$ are unitary. These n unitary eigenvalues correspond precisely to the very existence of the n-dimensional continuum of equilibria: a move along this continuum is followed neither by divergence nor by convergence to the previous point. This completes the proof of part (i) of Claim 2.

(ii) In order to study the remaining n eigenvalues of $\mathbf{C}_t\left[\frac{\partial \mathbf{g}}{\partial \mathbf{m}_{t-1}}\right]$, we can write:

$$\left[\frac{\partial \mathbf{g}}{\partial \mathbf{m}_{t-1}}\right]=\left[\frac{\partial \mathbf{g}}{\partial \mathbf{p}_t}\right]\left[\frac{\partial \mathbf{p}_t}{\partial \mathbf{m}_{t-1}}\right] \tag{21}$$

where $\left[\frac{\partial \mathbf{g}}{\partial \mathbf{p}_t}\right]$ is $2n$ by n, and $\left[\frac{\partial \mathbf{p}_t}{\partial \mathbf{m}_{t-1}}\right]$ is n by $2n$, and $\mathbf{p}_t=[p_{1,t},\ldots,p_{n,t}]'$. It is well known that the non-zero eigenvalues of $\mathbf{C}_t\left[\frac{\partial \mathbf{g}}{\partial \mathbf{p}_t}\right]\left[\frac{\partial \mathbf{p}_t}{\partial \mathbf{m}_{t-1}}\right]$ are the same as those of $\left[\frac{\partial \mathbf{p}_t}{\partial \mathbf{m}_{t-1}}\right]\mathbf{C}_t\left[\frac{\partial \mathbf{g}}{\partial \mathbf{p}_t}\right]$. Exploiting (5), $\frac{\partial \mathbf{p}_t}{\partial \mathbf{m}_{t-1}}$ has the following expression:

$$\begin{aligned}\left[\frac{\partial \mathbf{p}_t}{\partial \mathbf{m}_{t-1}}\right]&=\begin{bmatrix}\frac{\partial p_1}{\partial \alpha_1} & \frac{\partial p_1}{\partial \beta_1} & 0 & 0 & \cdots & \cdots & 0 & 0\\ 0 & 0 & \frac{\partial p_2}{\partial \alpha_2} & \frac{\partial p_2}{\partial \beta_2} & \cdots & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & \cdots & \cdots & \frac{\partial p_n}{\partial \alpha_n} & \frac{\partial p_n}{\partial \beta_n}\end{bmatrix}\\ &=\frac{1}{2}\begin{bmatrix}\frac{1}{\beta_1} & \frac{-\alpha_1}{\beta_1^2} & 0 & 0 & \cdots & \cdots & 0 & 0\\ 0 & 0 & \frac{1}{\beta_2} & \frac{-\alpha_2}{\beta_2^2} & \cdots & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & \cdots & \cdots & \frac{1}{\beta_n} & \frac{-\alpha_n}{\beta_n^2}\end{bmatrix}\end{aligned} \tag{22}$$

Each of the diagonal blocks of matrix $\mathbf{C}_t$, in turn, can be written[18] as:

$$\mathbf{C}_{i,t}=d_{i,t}\begin{bmatrix}\frac{\gamma_{i,2}}{p_{i,t}}+tp_{i,t} & tp_{i,t}\\ t & \gamma_{i,1}+t\end{bmatrix} \tag{23}$$

where $d_{i,t}=\frac{p_{i,t}}{\gamma_{i,1}\gamma_{i,2}+t\left(\gamma_{i,2}+\gamma_{i,1}p_{i,t}^2\right)}>0$ for $t<\infty$, and recalling that $\gamma_{i,k}\equiv\gamma_{i,k,0}$, $k=1,2$.

[18] See Bogliacino and Rampa (2008), expression C.2 of the Appendix.

Hence, the product $\mathbf{C}_t \left[\frac{\partial \mathbf{g}}{\partial \mathbf{m}_{t-1}}\right]$ is easily seen to be a $2n$-by-n matrix composed by n-by-n column vectors of the following form:

$$d_{i,t}\begin{bmatrix}\frac{\gamma_{i,2}}{p_{i,t}}\\ -\gamma_{i,1}\end{bmatrix}\frac{\partial g_i}{\partial p_{k,t}} \tag{24}$$

As a result of (22)–(24), the product $\left[\frac{\partial \mathbf{p}_t}{\partial \mathbf{m}_{t-1}}\right] \mathbf{C}_t \left[\frac{\partial \mathbf{g}}{\partial \mathbf{p}_t}\right]$ is a n-by-n matrix with elements:

$$\frac{1}{2}d_{i,t}\left(\frac{\gamma_{i2}}{\beta_{i1,t}p_{i,t}}+\frac{\alpha_{i,t}}{\beta_{i,t}^2}\gamma_{i1}\right)\frac{\partial g_i}{\partial p_{k,t}}=\frac{d_{i,t}}{2\beta_{i1,t}}\left(\frac{\gamma_{i2}}{p_{i,t}}+\frac{\alpha_{i,t}}{\beta_{i,t}}\gamma_{i1}\right)\frac{\partial g_i}{\partial p_{k,t}} \tag{25}$$

We can thus write the expression:

$$\left[\frac{\partial \mathbf{p}_t}{\partial \mathbf{m}_{t-1}}\right]\mathbf{C}_t\left[\frac{\partial \mathbf{g}}{\partial \mathbf{p}_t}\right]=\begin{bmatrix}\frac{d_{1,t}}{2\beta_{1,1,t}}\left(\frac{\gamma_{1,2}}{p_{1,t}}+\frac{\alpha_{1,t}}{\beta_{1,t}}\gamma_{1,1}\right) & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \frac{d_{n,t}}{2\beta_{n1,t}}\left(\frac{\gamma_{n,2}}{p_{n,t}}+\frac{\alpha_{n,t}}{\beta_{n,t}}\gamma_{n,1}\right)\end{bmatrix}$$
$$\times\begin{bmatrix}\frac{\partial g_1}{\partial p_{1,t}} & \cdots & \frac{\partial g_1}{\partial p_{n,t}}\\ \vdots & \ddots & \vdots\\ \frac{\partial g_n}{\partial p_{1,t}} & \cdots & \frac{\partial g_n}{\partial p_{n,t}}\end{bmatrix}\equiv \mathbf{D}_t\mathbf{G}_t \tag{26}$$

where $\mathbf{D}_t$ is a diagonal matrix with positive diagonal elements.

We need now to prove that the extra-diagonal elements of the matrix $\mathbf{G}_t \equiv \left[\frac{\partial g_i}{\partial p_{k,t}}\right]$ are positive. Using the definition of $\mathbf{g}_i(\cdot)$ after expression (14), one can see that the elements outside the main diagonal of $\mathbf{G}_t$ are the derivatives of the demand w.r.t. prices of the other goods $\left(\frac{\partial x_i}{\partial p_k},\ i\neq k\right)$. Remind that the consumer's problem is:

$$\max\sum_{i=1}^{n}u(x_i)f(\lambda_i)\quad \text{s.t.}\ \sum_{i=1}^{n}p_ix_i\leq w \tag{27}$$

whose first-order condition is (calling z the Lagrange multiplier):

$$u'(x_i)f(\lambda_i)-zp_i=0. \tag{28}$$

If we calculate $\frac{\partial x_i}{\partial p_k}$ using the implicit function theorem, we get $\frac{\partial x_i}{\partial p_k}=\frac{p_k}{u''(x_i)f(\lambda_i)}\frac{\partial z}{\partial p_k}$, where $\frac{\partial z}{\partial p_k}$ is defined by the boundary condition:

$$\sum_{i=1}^{n}p_i\left[u'^{-1}\left(\frac{p_iz}{f(\lambda_i)}\right)\right]-w=0 \tag{29}$$

Thus we have:

$$\frac{\partial z}{\partial p_k} = -\frac{u'^{-1}\left(\frac{p_i z}{f(\lambda_i)}\right) + \frac{p_i z}{f(\lambda_k)}\left[\left[u'^{-1}\right]'\left(\frac{p_k z}{f(\lambda_k)}\right)\right]}{\sum_{i=1}^{n} \frac{p_i z}{f(\lambda_k)}\left[\left[u'^{-1}\right]'\left(\frac{p_i z}{f(\lambda_i)}\right)\right]} \tag{30}$$

Simple manipulation of the numerator above, in particular using the Inverse Function Theorem, shows that the numerator itself is negative as long as:

$$\left|\frac{u'\left(x_i^*\right)}{u''\left(x_i^*\right)x_i^*}\right| > 1 \tag{31}$$

where x_i^* is a solution to (28). With additively separable preferences the LHS is nothing else than the elasticity of demand (Bertoletti et al. 2008), so (31) is certainly satisfied with our formulation[19] $u(x) = x^{\delta}$. So (30) is negative, and we can conclude that $\frac{\partial x_i}{\partial p_k} > 0, i \neq k$. This completes the proof of Claim 2. □

Claim 3 *If at an equilibrium the elasticity of the conjectured demand is greater than the elasticity of the true demand,* $\mathbf{D}_t\mathbf{G}_t$ *has at least one positive eigenvalue.*

Proof As we said, the elements of $\mathbf{D}_t$ are positive for $t < \infty$. From the definition of $g_i(\cdot)$ and from (26) it follows that the i-th element along the main diagonal of $\mathbf{G}_t$ can be written as

$$\frac{\partial g_i}{\partial p_i} = \left.\frac{\partial x_i}{\partial p_i}\right|_{\text{TrueDemand}} - \left.\frac{\partial x_i}{\partial p_i}\right|_{\text{ConjecturedDemand}} \tag{32}$$

By definition of elasticity, using the fact that at an equilibrium the price-quantity couple is the same for the true and the conjectured demand, the assumption of Claim 3 is equivalent to

$$\left|\frac{\partial x_i}{\partial p_i}\right|_{\text{TrueDemand}} - \left|\frac{\partial x_i}{\partial p_i}\right|_{\text{ConjecturedDemand}} < 0. \tag{33}$$

Since both true and conjectured demand are negatively sloped, (33) implies that (32) is positive.

Using the results of Claim 2, part (ii), the fact that (29) is positive, and finally the fact that $\mathbf{D}_t$ is a positive diagonal matrix, we conclude that all elements of $\mathbf{D}_t\mathbf{G}_t$ are positive. Claim 3 then follows from the Perron-Frobenius Theorem.[20] □

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[19] Indeed one can argue that the property is completely general, since a firm will never find optimal to fix a price where the elasticity of demand is lower than one. However, this condition is true only for conjectured demand, and not for the true one.

[20] See Lancaster and Tismenetsky (1985), Theorem 1 on page 536.

We can finally complete the proof of Proposition 2. From Claim 3 and Claim 2, part (ii), it follows that $\left[\frac{\partial \mathbf{m}_t}{\partial \mathbf{m}_{t-1}}\right]$ has an eigenvalue greater than one. Claim 1 says that the eigenvalues of $\left[\frac{\partial \mathbf{m}_t}{\partial \mathbf{m}_{t-1}}\right]$ are also eigenvalues of $\mathbf{J}$; hence $\mathbf{J}$ has an eigenvalue greater than one. □

A.2. Proof of Proposition 4

Using Claim 2, part (*ii*), we need to prove that $\mathbf{D}_t\mathbf{G}_t$ can have a real eigenvalue lower than –1. In what follows we will drop the time subscripts for easiness of notation, writing $\mathbf{DG}$ instead of $\mathbf{D}_t\mathbf{G}_t$: in fact we are evaluating the jacobian matrix $\mathbf{J}$ at an equilibrium (all consumers have converged to the true quality value and prices are fixed at the equilibrium values). We start once more from some preliminary results.

Claim 4 *The following statements hold*:

(i) $\mathbf{G}$ *is symmetric, and one has* $\mathbf{G} = \mathbf{G}_1 + \mathbf{G}_2$, *where* $\mathbf{G}_1$ *is diagonal and* $\mathbf{G}_2$ *has rank* 1;
(ii) $\mathbf{DG}$ *has the same eigenvalues as* $\mathbf{D}^{1/2}\mathbf{G}\mathbf{D}^{1/2}$;
(iii) $\mathbf{D}^{1/2}\mathbf{G}\mathbf{D}^{1/2} = \mathbf{DG}_1 + \mathbf{D}^{1/2}\mathbf{G2}\mathbf{D}^{1/2}$.

Proof (i) To find the elements of matrix $\mathbf{G}$, we differentiate the equilibrium conditions (11) w.r.t. prices, using the definition of demand (3) and imposing equilibrium condition. We get:

$$\begin{cases} \frac{\partial g_i}{\partial p_i} = \frac{1}{p_i(\delta-1)} Q_i^D + \beta_i + \frac{1}{(1-\delta)} \frac{\delta Q_i^D}{Mw} Q_i^D & i = k \\ \frac{\partial g_i}{\partial p_k} = \frac{1}{(1-\delta)} \frac{\delta Q_i^D}{Mw} Q_k^D > 0 & i \neq k \end{cases} \tag{34}$$

which implies that $\mathbf{G}$ is symmetric.[21]
Hence, $\mathbf{G} = \mathbf{G}_1 + \mathbf{G}_2$: the first matrix is a diagonal matrix with elements $\frac{1}{p_i(\delta-1)} Q_i^D + \beta_i$, and $\mathbf{G}_2 = \frac{\delta}{1-\delta} \frac{1}{Mw} \mathbf{QQ}'$, where $\mathbf{Q}$ is the vector of equilibrium quantities. But the non-zero eigenvalues of $\mathbf{G}_2$ are the same as those of $\frac{\delta}{1-\delta} \frac{1}{Mw} \mathbf{Q}'\mathbf{Q}$, so $\mathbf{G}_2$ has rank 1.
(ii) We know from Claim 2, part ii, that $\mathbf{D}$ is diagonal and non singular, thus it admits $\mathbf{D}^{1/2}$. But $\mathbf{DG}$ is similar to $\mathbf{D}^{-1/2}(\mathbf{DG})\mathbf{D}^{1/2} = \mathbf{D}^{1/2}\mathbf{G}\mathbf{D}^{1/2}$, and the two matrices have the same eigenvalues.
(iii) Given part (i) above, and since diagonal matrices commute, one can write

$$\mathbf{D}^{1/2}\mathbf{G}\mathbf{D}^{1/2} = \mathbf{D}^{1/2}\mathbf{G}_1\mathbf{D}^{1/2} + \mathbf{D}^{1/2}\mathbf{G}_2\mathbf{D}^{1/2} = \mathbf{DG}_1 + \mathbf{D}^{1/2}\mathbf{G}_2\mathbf{D}^{1/2} \tag{35}$$

[21] The $\frac{\partial g_i}{\partial p_k}$, $i \neq k$, are equal to the extra-diagonal terms of the Jacobian of the demand functions, thus in the general case symmetry holds only for compensated demands (see Theorem 1: McKenzie (2002), p. 10), and not for the Marshallian ones, because of income effects (McKenzie 2002, p. 12). However the condition holds under our present assumptions.

where the first term is a diagonal matrix, while the second term is $\frac{\delta}{1-\delta}\frac{1}{Mw}$ times the external product of $\mathbf{D}^{1/2}\mathbf{Q}$ and itself, hence is symmetric. $\square$

Define now the following terms: ϕ_i, as the i-th element of the main diagonal of $\mathbf{D}$; $\nu_i \equiv \gamma_{i1}/\gamma_{i2}$, as the ratio between firm i's initial precisions; and $s_i \equiv \varepsilon_T/\varepsilon_i$, as the ratio between the true elasticity and the conjectured elasticity in the market for product i. We proceed with the following

Claim 5 *The following statements hold*:

(i) *the eigenvalues of*

$$\mathbf{DG}_1 \; are \; (1-s_i)\,\frac{1}{2}\frac{1+\nu_i p_i^2\left(1+\frac{1}{\varepsilon_i}\right)}{\gamma_{i1}+t\left(1+\nu_i p_i^2\right)}, \quad i=1,\ldots,n \tag{36}$$

(i) *the non-zero eigenvalue of*

$$\mathbf{D}^{1/2}\mathbf{G}_2\mathbf{D}^{1/2} \; is \; \frac{1}{2}\delta\sum_k\left(\frac{p_k Q_k}{\sum_k p_k Q_k}s_k\frac{1+\nu_k p_k^2\left(1+\frac{1}{\varepsilon_k}\right)}{\gamma_{k1}+t\left(1+\nu_k p_k^2\right)}\right) \tag{37}$$

Proof (i) We concentrate first on the i-diagonal element of the diagonal matrix $\mathbf{D}$, ϕ_i. From the definitions after (23) we have $\phi_i = \frac{d_i}{2\beta_i}\left(\frac{\gamma_{i2}}{p_i}+\frac{\alpha_i}{\beta_i}\gamma_{i1}\right) = \frac{1}{2\beta_i}\frac{\gamma_{i2}+\gamma_{i1}p_i\frac{\alpha_i}{\beta_i}}{\gamma_{i1}\gamma_{i2}+t\left(\gamma_{i2}+\gamma_{i1}p_i^2\right)}$. From (5) one deduces in addition that $\frac{\alpha_i}{\beta_i} = 2p_i - c_i$, hence one can write $p_i\frac{\alpha_i}{\beta_i} = p_i^2 + (p_i - c_i)\,p_i$. But the monopoly pricing rule (Tirole 1988) is that the *Lerner Index* is equal to the inverse of the conjectured elasticity, whence $(p_i - c_i) = \frac{p_i}{\varepsilon_i}$; then we have $p_i\frac{\alpha_i}{\beta_i} = p_i^2 + \frac{p_i^2}{\varepsilon_i}$. Given the definition $\nu_i \equiv \frac{\gamma_{i1}}{\gamma_{i2}}$, we get thus $\phi_i = \frac{1}{2\beta_i}\frac{1+\nu_i p_i^2\left(1+\frac{1}{\varepsilon_i}\right)}{\gamma_{i1}+t\left(1+\nu_i p_i^2\right)}$.
Now, recall from the proof of Claim 4, part (i), that the i-diagonal element of the diagonal matrix $\mathbf{G}_1$ is $\frac{1}{p_i(\delta-1)}Q_i+\beta_i$. In equilibrium one has $Q_i = (p_i - c_i)\,\beta_i$, which is equal to $\frac{p_i}{\varepsilon_i}\beta_i$ by the monopoly pricing rule. Using the true elasticity $\varepsilon_T = \frac{1}{1-\delta}$ and the definition $s_i \equiv \frac{\varepsilon_T}{\varepsilon_i}$, we obtain $\frac{1}{p_i(\delta-1)}Q_i + \beta_i = \beta_i\,(1-s_i)$. The i-th diagonal element of the diagonal matrix $\mathbf{DG}_1$, that is its i-th eigenvalue, is then equal to $\beta_i\,(1-s_i)\,\phi_i$. Substituting the value of ϕ_i found above, we get (36).

(ii) Using Claim 4, we write $\mathbf{D}^{1/2}\mathbf{G}_2\mathbf{D}^{1/2} = \frac{\delta}{1-\delta}\frac{1}{Mw}\mathbf{D}^{1/2}\mathbf{Q}\mathbf{Q}'\mathbf{D}^{1/2}$, a matrix that has rank 1 and thus a single non-zero eigenvalue. It is easily checked that this matrix has the same non-zero eigenvalue as, $\frac{\delta}{1-\delta}\frac{1}{Mw}\mathbf{Q}'\mathbf{D}\mathbf{Q} = \frac{\delta}{1-\delta}\frac{1}{Mw}\sum_k\phi_k Q_k^2$. Exploit again the equilibrium fact $Q_k = (p_k - c_k)\,\beta_k$, the monopoly pricing rule $(p_k - c_k) = \frac{p_k}{\varepsilon_k}$, and the definitions $s_k \equiv \frac{\varepsilon_T}{\varepsilon_k}$ and $\varepsilon_T = \frac{1}{1-\delta}$, which together imply $\frac{1}{1-\delta}Q_k^2 = Q_k p_k s_k \beta_k$. Substituting finally the value of ϕ_i found in part (i) above, and using the budget constraint $Mw = \sum_k p_k Q_k$, one gets (37). $\square$

We will use the following result, which we call Claim 6.

Claim 6 *If* $\mathbf{A}$ *and* $\mathbf{B}$ *are two symmetric matrices and if rank*$(\mathbf{B}) = 1$, *then the i-th eigenvalue of* $\mathbf{A}+\mathbf{B}$, *say* $\rho_i(\mathbf{A}+\mathbf{B})$, *is equal to* $\rho_i(\mathbf{A}) + m_i \cdot \rho(\mathbf{B})$, *where* $m_i \in [0, 1]$, *and* $\rho(\mathbf{B})$ *is the only non-zero eigenvalue of* $\mathbf{B}$.

Proof See Wilkinson (1965), pp. 97–98. □

We are finally ready to complete the proof of **Proposition 4**, stating that an eigenvalue of $\mathbf{J}$, say $\rho_i(\mathbf{J})$, can be lower than –1. By Claim 2, part (ii), this means $\rho_i(\mathbf{DG}) < -2$. We will posit sufficient conditions for this result.

We know from Claim 4 that $\mathbf{DG}$ has the same eigenvalues as $\mathbf{D}^{1/2}\mathbf{G}\mathbf{D}^{1/2}$, and that $\mathbf{D}^{1/2}\mathbf{G}\mathbf{D}^{1/2} = \mathbf{DG}_1 + \mathbf{D}^{1/2}\mathbf{G}_2\mathbf{D}^{1/2}$, the sum of two symmetric matrices. Claim 5, in turn, gives expressions for the eigenvalues of $\mathbf{DG}_1$and$\mathbf{D}^{1/2}\mathbf{G}_2\mathbf{D}^{1/2}$. Claim 6 asserts finally that $\rho_i(\mathbf{DG}) = \rho_i(\mathbf{DG}_1) + m_i \cdot \rho\left(\mathbf{D}^{1/2}\mathbf{G}_2\mathbf{D}^{1/2}\right)$, with $m_i \in [0, 1]$, implying $\rho_i(\mathbf{DG}) \leq \rho_i(\mathbf{DG}_1) + \rho\left(\mathbf{D}^{1/2}\mathbf{G}_2\mathbf{D}^{1/2}\right)$.

Suppose now that the true elasticity ε_T is very high, implying $\delta \approx 1$ (recall that $\delta < 1$ anyway), and that all firms but the i-th one conjecture an elasticity ε_k very near to the true one, while the i-th firm conjectures a low elasticity ε_i. This implies $s_k \approx 1$ and $1/\varepsilon_k \approx 0$ for all $k \neq i$; at the same time, the i-th firm will price very high, so that (2) implies a low share of consumers' expenditure on good i; in addition, s_i is very high. Suppose further that all firms have low initial precisions of their α parameters, so that γ_{k1} is near to zero, $\forall k$. Finally, consider the system at the very start of the learning process, meaning $t = 1$. Looking carefully at (37), all this implies that $\rho\left(\mathbf{D}^{1/2}\mathbf{G}_2\mathbf{D}^{1/2}\right) \approx 1/2$.

This given, Claim 6 ca be written as $\rho_i(\mathbf{DG}) \leq (1 - s_i)\,\frac{1}{2}\,\frac{1+\nu_i p_i^2\left(1+\frac{1}{\varepsilon_i}\right)}{1+\nu_i p_i^2} + \frac{1}{2}$ for the i-th eigenvalue of $\mathbf{DG}$.

We need to have $(1 - s_i)\,\frac{1}{2}\,\frac{1+\nu_i p_i^2\left(1+\frac{1}{\varepsilon_i}\right)}{1+\nu_i p_i^2} + \frac{1}{2} < -2$, meaning $(1 - s_i)\,\frac{1+\nu_i p_i^2\left(1+\frac{1}{\varepsilon_i}\right)}{1+\nu_i p_i^2} < -5$. This might well be the case, given our current assumptions of a low ε_i and a high s_i, and if in addition one assumes that ν_i is high, i.e. firm i is initially more uncertain on the β parameter than on the α parameter. Notice that, as time passes $(t > 1)$ and hence γ_{k1} grows above zero, the result does not hold any longer.

This completes the proof of Proposition 4. □

A.3. Proof of Proposition 5

Let introduce a fixed cost of entry equal to F. In order to calculate the optimal degree of diversification, we need to fix price equal to marginal cost, introduce lump sum taxation for an amount $\frac{nF}{M}$ for each consumer (reducing her income to $w - \frac{nF}{M}$) and maximize the indirect utility function in n. The demand for goods of identical quality is:

$$Q^D = M(w - nF/M)\frac{p^{1/(1-\delta)}\,[\delta f(\lambda)]^{1/(1-\delta)}}{np^{\delta/(1-\delta)}\,[\delta f(\lambda)]^{1/(1-\delta)}} = \frac{M(w - nF/M)}{np} \tag{38}$$

Replacing the price equal to marginal cost, the indirect utility function is given by $n^{1-\delta}c^{-\delta}M^{\delta}(w - nF/M)^{\delta}$, which must be maximized in n, considered as a real variable for simplicity. The first-order condition is also sufficient, due to the strict concavity of the indirect utility function, and is the following:

$$n^e = (1-\delta)Mw/F = \frac{Mw}{F\varepsilon_T} \tag{39}$$

The equilibrium condition with endogenous number of firms is a zero profit condition for the marginal firm, defined by the price $\overline{p} = \min\{p_i | p_i = p_i^*\}$ (where p_i^* are equilibrium prices), given the equality of marginal cost and quality through firms (and convergence of consumers' conjectures in equilibrium). Hence:

$$(\overline{p} - c)\, Q = F\ \frac{\alpha - c\beta}{2\beta}\frac{Mw}{n\overline{p}} = F \tag{40}$$

By simple algebra we get

$$n^* = \frac{\alpha - c\beta}{\alpha + c\beta}\frac{Mw}{F} = \frac{Mw}{F\varepsilon_C} \tag{41}$$

Over (respectively under) diversification is the case $n^* > n^e$ (respectively $n^* < n^e$). Replacing with (39) and (41) completes the proof. □

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