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Relative $\boldsymbol{T}$-Injective Modules and Relative $\boldsymbol{T}$-Flat Modules

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Abstract Let T be a Wakamatsu tilting module. A module M is called (n,T)-copure injective (resp. (n,T)-copure flat) if $\mathcal{E}_T^1(N,M)=0$ (resp. $\Gamma_1^T(N,M)=0$) for any module N with T-injective dimension at most n (see Definition 2.2). In this paper, it is shown that M is (n,T)-copure injective if and only if M is the kernel of an $\mathcal{I}_n(T)$-precover $f:A\to B$ with $A\in\operatorname{Prod}T$. Also, some results on $\operatorname{Prod}T$-syzygies are presented. For instance, it is shown that every nth $\operatorname{Prod}T$-syzygy of every module, generated by T, is (n,T)-copure injective.

Keywords Wakamatsu tilting module, (n,T)-Copure injective module, (n,T)-Copure flat module, T-Projective dimension, T-Injective dimension
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1 Introduction

The study of tilting theory has become an exciting subject in homological algebra. Many subjects in homological algebra are based on the properties of tilting and cotilting modules (see [1, 2, 4, 8, 9] for instance). Throughout this paper, R is an associative ring with non-zero identity, all modules are unitary R-modules and T is a fixed R-module. We denote by $\operatorname{Add}T$ ($\operatorname{Prod}T$) the class of modules isomorphic to direct summands of direct sum (direct product) of copies of T, by Pres^nT and $\operatorname{Pres}^\infty T$ the set of all modules M such that there exist the exact sequences

$$T_n\longrightarrow T_{n-1}\longrightarrow\cdots\longrightarrow T_1\longrightarrow M\longrightarrow 0\quad\text{and}$$
$$\cdots\longrightarrow T_n\longrightarrow T_{n-1}\longrightarrow\cdots\longrightarrow T_1\longrightarrow T_0\longrightarrow M\longrightarrow 0,$$

respectively, where $T_i\in\operatorname{Add}T$ for every $i\ge 1$. A module M is said to be generated by T, denoted by $M\in\operatorname{Gen}T$ (resp. cogenerated by T, denoted by $M\in\operatorname{Cogen}T$) if there exists an exact sequence $T^n\longrightarrow M\longrightarrow 0$ (resp. $M\longrightarrow T^n\longrightarrow 0$), for some positive integer n. Let $\mathcal{C}$ be a class of modules and M be a module. A left (resp. right) $\mathcal{C}$-resolution of M is a long exact sequence $\cdots\longrightarrow C_1\longrightarrow C_0\longrightarrow M\longrightarrow 0$ (resp. $0\longrightarrow M\longrightarrow C_0\longrightarrow C_1\longrightarrow\cdots$), where $C_i\in\mathcal{C}$ for every $i\ge 0$. A module T is called Wakamatsu tilting if $\operatorname{Ext}^i(T,T)=0$ for every $i\ge 1$, and there exists a long exact sequence

$$0\longrightarrow R\xrightarrow{f_0}T_0\xrightarrow{f_1}T_1\xrightarrow{f_2}\cdots,$$

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where $T_i \in \mathrm{Add}\, T$ and $\mathrm{Ext}^1(\mathrm{Coker} f_i, T) = 0$ for every $i \geq 0$. We refer the reader to [4, 8] for more details. In fact, the concept of a Wakamatsu tilting module generalizes both tilting and cotilting modules (see [8, Proposition 2.1]). Let T be a Wakamatsu tilting module.

In Section 2, some relative homological dimensions and derived functors are introduced. The existence of $\mathrm{Add}\, T$-resolutions and $\mathrm{Prod}\, T$-resolutions and some properties of their syzygies will be studied, too. For every $M \in \mathrm{Gen}\, T$ (resp. $M \in \mathrm{Cogen}\, T$), we define T-projective (resp. T-injective) dimension of M to be the length of a left $\mathrm{Add}\, T$-resolution (resp. right $\mathrm{Prod}\, T$-resolution) of M. We denote by $\mathcal{P}_n(T)$ and $\mathcal{I}_n(T)$ the class of modules with T-projective dimension at most n and the class of modules with T-injective dimension at most n, respectively. If T is a 1-quasi-projective module (see [9, Definition 2.1]), then T-projective dimension of a module equals its T-dimension which has been studied by the authors in [6]. For any homomorphism f of R-modules, we denote by $\mathrm{Ker}\, f$ and $\mathrm{Im}\, f$, the kernel and the image of f, respectively. Let B and M be modules. If $M \in \mathrm{Gen}\, T$, then we define $\Gamma_n^T(M, B) = \frac{\mathrm{Ker}(\delta_n \otimes 1_B)}{\mathrm{Im}(\delta_{n+1} \otimes 1_B)}$, where

$$\cdots \longrightarrow T_2 \xrightarrow{\delta_2} T_1 \xrightarrow{\delta_1} T_0 \xrightarrow{\delta_0} M \longrightarrow 0$$

is a left $\mathrm{Add}\, T$-resolution of M. Also, if $M \in \mathrm{Cogen}\, T$, then we define $\mathcal{E}_T^n(C, M) = \frac{\mathrm{Ker}\, \delta_*^n}{\mathrm{Im}\, \delta_*^{n-1}}$, where

$$0 \longrightarrow M \xrightarrow{\delta^0} T^0 \xrightarrow{\delta^1} T^1 \xrightarrow{\delta^2} T^2 \longrightarrow \cdots$$

is a right $\mathrm{Prod}\, T$-resolution of M and $\delta_*^n = \mathrm{Hom}(\delta_n, T)$.

A module M is said to be (n, T)-copure injective (resp. (n, T)-copure flat) if $\mathcal{E}_T^1(N, M) = 0$ (resp. $\Gamma_1^T(N, M) = 0$) for every $N \in \mathcal{I}_n(T)$. Let $\mathcal{C}$ be a class of R-modules. Recall that an epimorphism $\phi : C \longrightarrow M$ with $C \in \mathcal{C}$ is a $\mathcal{C}$-precover of M if for every homomorphism $f : C' \longrightarrow M$ with $C' \in \mathcal{C}$, there exists a homomorphism $g : C' \longrightarrow C$ such that $f = \phi g$. Moreover, if $C' = C$ implies that g is an automorphism, then $\phi : C \longrightarrow M$ is called a $\mathcal{C}$-cover of M. Preenvelopes and envelopes are defined dually (see [3] for more details).

Section 3 is devoted to some characterization of (n, T)-copure injective modules and (n, T)-copure flat modules. For instance, it is shown that a module is an (n, T)-copure injective if and only if it is the Kernel of an $\mathcal{I}_n(T)$-precover $f : A \longrightarrow B$ with $A \in \mathrm{Prod}\, T$. Also it is proved that a module M is (n, T)-copure injective (resp. (n, T)-copure flat) if and only if $\mathrm{Hom}(T^0, M)$ (resp. $T^0 \otimes M$) is (n, T)-copure injective (resp. (n, T)-copure flat), for any $T^0 \in \mathrm{Prod}\, T$. Among other results, we study Wakamatsu tilting modules with finite T-injective dimension.

2 Relative Homological Dimensions and Derived Functors

In this section, we give basic notions and results and we recall some relevant background in tilting theory from [2, 4, 8, 9]. First let us recall the following definition of (not necessarily finitely generated) tilting modules (see [2]).

A module M is called tilting (1-tilting) if it satisfies the following conditions:

(1) $\mathrm{pd}(T) \leq 1$, where $\mathrm{pd}(T)$ denotes the projective dimension of T;

(2) $\mathrm{Ext}^i(T, T^{(\lambda)}) = 0$ for every $i > 0$ and for every cardinal λ;

(3) There exists an exact sequence $0 \longrightarrow R \longrightarrow T_0 \longrightarrow T_1 \longrightarrow 0$, where $T_0, T_1 \in \mathrm{Add}\, T$.

The 1-cotilting module is defined dually (see [2] for more details). Wakamatsu generalized the concept of the tilting module in [8]. An R-module T is said to be a Wakamatsu tilting module if $\mathrm{Ext}^i(T,T)=0$ for every $i\geq 1$, and there exists a long exact sequence

$$0 \longrightarrow R \xrightarrow{f_0} T_0 \xrightarrow{f_1} T_1 \xrightarrow{f_2} \cdots,$$

where $T_i \in \mathrm{Add}\,T$ and $\mathrm{Ext}^1(\mathrm{Coker}f_i, T)=0$ for every $i\geq 0$. A Wakamatsu cotilting module is defined dually.

Let n be a positive integer. A module T is said to be n-quasi-projective if for any exact sequence $0\longrightarrow L\longrightarrow T_0\longrightarrow N\longrightarrow 0$ with $T_0\in\mathrm{Add}\,T$ and $L\in\mathrm{Pres}^nT$, the induced sequence $0\longrightarrow\mathrm{Hom}(T,L)\longrightarrow\mathrm{Hom}(T,T_0)\longrightarrow\mathrm{Hom}(T,N)\longrightarrow 0$ is also exact (see [9, Definition 2.1]). Also, T is called an n-star module if T is $(n+1)$-quasi-projective and $\mathrm{Pres}^nT=\mathrm{Pres}^{n+1}T$ (see [9, Definition 3.1]).

Proposition 2.1 *If M is a generated* (*resp. cogenerated*) *module by a Wakamatsu tilting module T, then M has a left* $\mathrm{Add}\,T$*-resolution* (*resp. right* $\mathrm{Prod}\,T$*-resolution*).

Proof Since T is tilting, [2, Theorem 3.11] implies that it is 1-star and $\mathrm{Gen}\,T=\mathrm{Pres}^{\infty}T$. So $M\in\mathrm{Pres}^{\infty}T$. This shows that M has a left $\mathrm{Add}\,T$-resolution. Similarly, one can show that any module $M\in\mathrm{Cogen}\,T$ has a right $\mathrm{Prod}\,T$-resolution.

Remark 2.1 (1) If T is a tilting module, then it is a 1-star module by [9, Theorem 4.3], and hence it is 1-quasi-projective by [9, Definition 3.1]. So, if $M\in\mathrm{Gen}\,T$ and $0\longrightarrow K_1\longrightarrow T_1\longrightarrow M\longrightarrow 0$ and $0\longrightarrow K_2\longrightarrow T_2\longrightarrow M\longrightarrow 0$ are two short exact sequences such that $T_1,T_2\in\mathrm{Add}\,T$, then by [9, Lemma 2.3], we deduce that $K_1\oplus T_2\cong K_2\oplus T_1$.

(2) Consider the following exact sequences:

$$0\longrightarrow K\longrightarrow T_{n-1}\longrightarrow\cdots\longrightarrow T_1\longrightarrow T_0\longrightarrow M\longrightarrow 0,$$
$$0\longrightarrow K'\longrightarrow T'_{n-1}\longrightarrow\cdots\longrightarrow T'_1\longrightarrow T'_0\longrightarrow M\longrightarrow 0,$$

in which $T_i,T'_i\in\mathrm{Add}\,T$ for every i $(0\leq i\leq n-1)$. Then we have

$$K\oplus T'_{n-1}\oplus\cdots\cong K'\oplus T_{n-1}\oplus\cdots.$$

The dual of Remark 2.1 is also true. The next definition is a generalization of the derived functors Ext and Tor.

Definition 2.1 *Let T be a* (*Wakamatsu*) *tilting module.*

(1) *For any $M\in\mathrm{Gen}\,T$, we define $\Gamma_n^T(M,B):=\frac{\mathrm{Ker}(\delta_n\otimes 1_B)}{\mathrm{Im}(\delta_{n+1}\otimes 1_B)}$, where*

$$\cdots \xrightarrow{\delta_2} T_1 \xrightarrow{\delta_1} T_0 \xrightarrow{\delta_0} M \longrightarrow 0$$

is a left $\mathrm{Add}\,T$*-resolution of M.*

(2) *For any $M\in\mathrm{Cogen}\,T$, we define $\mathcal{E}_T^n(C,M):=\frac{\mathrm{Ker}\,\delta_*^n}{\mathrm{Im}\,\delta_*^{n-1}}$, where*

$$0 \longrightarrow M \xrightarrow{\delta^0} T^0 \xrightarrow{\delta^1} T^1 \xrightarrow{\delta^2} \cdots$$

is a right $\mathrm{Prod}\,T$*-resolution of M and $\delta_*^n=\mathrm{Hom}(\delta_n,T)$.*

A similar proof to that of [6, Proposition 2.3] shows that the definition of $\Gamma_n^T(M,B)$ (resp. $\mathcal{E}_T^n(C,M)$) is independent from the choice of left $\mathrm{Add}\,T$-resolutions (resp. right $\mathrm{Prod}\,T$-resolutions).

Definition 2.2 *Let T be a Wakamatsu tilting module.*

(1) *If $M \in \operatorname{Gen} T$, then we say that M is of T-projective dimension n (briefly,* $\mathrm{T.p.dim}(M) = n$*) if n is the least non-negative integer such that there exists a long exact sequence*

$$0 \longrightarrow T_n \longrightarrow T_{n-1} \longrightarrow \cdots \longrightarrow T_1 \longrightarrow T_0 \longrightarrow M \longrightarrow 0$$

with $T_i \in \operatorname{Add} T$ for each $i \geq 0$.

(2) *If $M \in \operatorname{Cogen} T$, then we say that M is of T-injective dimension n if n is the least non-negative integer such that there exists a long exact sequence*

$$0 \longrightarrow M \longrightarrow T^0 \longrightarrow T^1 \longrightarrow \cdots \longrightarrow T^n \longrightarrow 0$$

with $T^i \in \operatorname{Prod} T$ for each $i \geq 0$.

(3) *A module M is called (n, T)-projective (resp. (n, T)-injective) if* $\mathrm{T.p.dim}(M) \leq n$ *(resp.* $\mathrm{T.i.dim}(M) \leq n$*). We denote the class of all (n, T)-projective (resp. (n, T)-injective) modules by $\mathcal{P}_n(T)$ (resp. $\mathcal{I}_n(T)$).*

In particular, if $T = R$, then M is called n-projective (resp. n-injective). The class of n-projective modules was studied in [5].

Remark 2.2 Let T be a tilting module. Then for every $M \in \operatorname{Gen} T$, the following statements are equivalent:

(1) $\mathrm{T.p.dim}(M) \leq n$;

(2) For every $\operatorname{Add} T$-resolution

$$T_{n-1} \longrightarrow T_{n-2} \longrightarrow \cdots \longrightarrow T_1 \longrightarrow T_0 \longrightarrow M \longrightarrow 0,$$

$\operatorname{Ker}(T_{n-1} \longrightarrow T_{n-2})$ belongs to $\operatorname{Add} T$;

(3) $\mathcal{E}_T^i(M, B) = 0$ for every $i > n$ and every module B.

Replacing T by R as an R-module, we see that T-projective dimension and T-dimension are the same as projective dimension and injective dimension, respectively.

Let M and N be two modules. From [6, Lemma 2.11], we know that $\mathcal{E}_T^0(M, N) \cong \operatorname{Hom}(M, N)$. Similarly, it is seen that $\Gamma_0^T(M, N) \cong M \otimes N$. If $\mathcal{E}_T^1(M, -) = 0$, then $M \in \operatorname{Add} T$. If $\mathcal{E}_T^1(-, N) = 0$, then $N \in \operatorname{Prod} T$. Let $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ be a short exact sequence. Then for every module M and every non-negative integer n, the following long exact sequences exist:

$$\begin{aligned}
&\cdots \longrightarrow \mathcal{E}_T^n(M, A) \longrightarrow \mathcal{E}_T^n(M, B) \longrightarrow \mathcal{E}_T^n(M, C) \longrightarrow \mathcal{E}_T^{n+1}(M, A) \longrightarrow \cdots, \\
&\cdots \longrightarrow \mathcal{E}_T^n(C, M) \longrightarrow \mathcal{E}_T^n(B, M) \longrightarrow \mathcal{E}_T^n(A, M) \longrightarrow \mathcal{E}_T^{n+1}(C, M) \longrightarrow \cdots, \\
&\cdots \longrightarrow \Gamma_{n+1}^T(M, A) \longrightarrow \Gamma_{n+1}^T(M, B) \longrightarrow \Gamma_{n+1}^T(M, C) \longrightarrow \Gamma_n^T(M, A) \longrightarrow \cdots.
\end{aligned}$$

It is natural to define $\mathrm{T.f.dim}\,(M)$ (T-flat dimension of M) to be the least nonnegative integer n such that for every module B, $\Gamma_n^T(M, B) = 0$.

We denote by $\mathcal{F}_n(T)$ the class of all modules with T-flat dimension at most n.

Let $\mathcal{C}$ be a class of modules and M be an arbitrary module. If

$$\cdots \longrightarrow C_2 \longrightarrow C_1 \longrightarrow C_0 \longrightarrow M \longrightarrow 0$$

and

$$0 \longrightarrow M \longrightarrow C^0 \longrightarrow C^1 \longrightarrow C^2 \longrightarrow \cdots$$

are left and right $\mathcal{C}$-resolutions of M, respectively, then the module $K_n = \mathrm{Ker}(C_n \longrightarrow C_{n-1})$ is called nth $\mathcal{C}$-syzygy of M and $L^n = \mathrm{Coker}(C^n \longrightarrow C^{n+1})$ is called nth $\mathcal{C}$-cosyzygy of M. We refer the reader to [3] for more information.

Proposition 2.2 *Consider the following* $\mathrm{Add}\, T$*-resolution:*

$$\cdots \longrightarrow T_2 \xrightarrow{\delta_2} T_1 \xrightarrow{\delta_1} T_0 \xrightarrow{\delta_0} M \longrightarrow 0.$$

If K_i is an ith $\mathrm{Add}\, T$-syzygy of M, for $i \geq 0$, then the following statements hold:

(1) $\Gamma^T_{n+1}(M, B) \cong \Gamma^T_n(K_0, B) \cong \cdots \cong \Gamma^T_1(K_{n-1}, B)$;

(2) $\mathcal{E}^{n+1}_T(M, B) \cong \mathcal{E}^n_T(K_0, B) \cong \cdots \cong \mathcal{E}^1_T(K_{n-1}, B)$.

Proof It is clear that $\cdots \longrightarrow T_2 \longrightarrow T_1 \longrightarrow K_0 \longrightarrow 0$ is an $\mathrm{Add}\, T$-resolution of K_0. Define $S_{n-1} = T_n$ and $\Delta_{n-1} = \delta_n$ for each $n \geq 1$. The $\mathrm{Add}\, T$-resolution now reads

$$\cdots \longrightarrow S_2 \longrightarrow S_1 \longrightarrow S_0 \longrightarrow K_0 \longrightarrow 0.$$

By definition, we get

$$\Gamma^T_n(K_0, B) \cong \frac{\mathrm{Ker}(\Delta_n \otimes 1_B)}{\mathrm{Im}(\Delta_{n-1} \otimes 1_B)} = \frac{\mathrm{Ker}(\delta_{n+1} \otimes 1_B)}{\mathrm{Im}(\delta_n \otimes 1_B)} = \Gamma^T_{n+1}(M, B).$$

This proves (1), and the proof of (2) is similar to that of (1).

3 (n, T)-Copure Injective Modules and (n, T)-Copure Flat Modules

Unless otherwise stated, throughout this section, T will be a Wakamatsu tilting module. We give a generalization of copure injective modules and copure flat modules, and then we study some of their properties.

Definition 3.1 *Let n be a fixed nonnegative integer. Then $M \in \mathrm{Gen}\, T$ is called (n, T)-copure injective (resp. (n, T)-copure flat) if $\mathcal{E}^1_T(N, M) = 0$ (resp. $\Gamma^T_1(M, N) = 0$), for any $N \in \mathcal{I}_n(T)$.*

In the first theorem of this section, we give some characterizations of (n, T)-copure injective modules. Before embarking this characterization, we need the following proposition.

Proposition 3.1 *The following statements are true:*

(1) *If $\mathcal{E}^i_T(N, M) = 0$ for any i $(1 \leq i \leq n + 1)$ and any $N \in \mathrm{Prod}\, T$, then every kth $\mathrm{Prod}\, T$-cosyzygy of M is $(n-k, T)$-copure injective. In particular, M is (n, T)-copure injective*;

(2) *If $\Gamma^T_1(M, N) = 0$ for any i $(1 \leq i \leq n + 1)$ and any $N \in \mathrm{Prod}\, T$, then every kth $\mathrm{Add}\, T$-syzygy of M is $(n - k, T)$-copure flat with $0 \leq k \leq n$. In particular, M is (n, T)-copure flat.*

Proof Let k be an integer with $0 \leq k \leq n$, L^k be the kth $\mathrm{Prod}\, T$-cosyzygy of M and $N \in \mathcal{I}_{n-k}(T)$. Then $\mathcal{E}^1_T(N, L^k) \cong \mathcal{E}^{k+1}_T(N, M)$. On the other hand, there is an exact sequence

$$0 \longrightarrow N \longrightarrow T^0 \longrightarrow T^1 \longrightarrow \cdots \longrightarrow T^{n-k} \longrightarrow 0,$$

where $T^i \in \mathrm{Prod}\, T$ for every i $(0 \leq i \leq n - k)$, and so $\mathcal{E}^{k+1}_T(N, M) \cong \mathcal{E}^{n+1}_T(T^{n-k}, M) = 0$ by assumption. Thus $\mathcal{E}^1_T(N, L^k) = 0$ and hence L^k is $(n - k, T)$-copure injective. This proves (1). The proof of (2) is similar to that of (1).

Theorem 3.1 *If $M \in \operatorname{Gen} T$, then the following statements are equivalent:*

(1) *M is an (n,T)-copure injective module*;

(2) *For every exact sequence $0 \longrightarrow M \longrightarrow I \longrightarrow L \longrightarrow 0$ with $I \in \mathcal{I}_n(T)$, $I \longrightarrow L$ is an $\mathcal{I}_n(T)$-precover of L*;

(3) *M is the Kernel of an $\mathcal{I}_n(T)$-precover $f : A \longrightarrow B$ with $A \in \operatorname{Prod} T$.*

Proof (1) $\Rightarrow$ (2) Let $I' \in \mathcal{I}_n(T)$. Since $\mathcal{E}^1_T(I', M) = 0$, we obtain the exact sequence $\operatorname{Hom}(I', I) \longrightarrow \operatorname{Hom}(I', L) \longrightarrow 0$. Thus $I \longrightarrow L$ is an $\mathcal{I}_n(T)$-precovere of L.

(2) $\Rightarrow$ (3) Consider the short exact sequence $0 \longrightarrow M \longrightarrow I \longrightarrow \frac{I}{M} \longrightarrow 0$, where I is an $\mathcal{I}_n(T)$-preenvelope of M. Then (3) follows from (2).

(3) $\Rightarrow$ (1) Let M be the kernel of an $I_n(T)$-precover $f : A \longrightarrow B$ with $A \in \operatorname{Prod} T$. Then we naturally have an exact sequence $0 \longrightarrow M \longrightarrow A \longrightarrow \frac{A}{M} \longrightarrow 0$. Therefore, by (3), the sequence $\operatorname{Hom}(N, A) \longrightarrow \operatorname{Hom}(N, \frac{A}{M}) \longrightarrow 0$ is exact for every $N \in \mathcal{I}_n(T)$. Thus $\mathcal{E}^1_T(N, M) = 0$ and so (1) follows.

Now, let us give some sufficient conditions under which Prod$T$-syzygies are (n,T)-copure injective.

Proposition 3.2 *Every nth $\operatorname{Prod} T$-syzygy of every generated module by T is (n,T)-copure injective.*

Proof Let $M \in \operatorname{Gen} T$. Then by Proposition 2.1, M has a $\operatorname{Prod} T$-resolution, say

$$\cdots \longrightarrow U_1 \longrightarrow U_0 \longrightarrow M = U_{-1} \longrightarrow 0.$$

For every nonnegative integer n, set $K_n = \operatorname{Ker}(U_{n-1} \longrightarrow U_{n-2})$. We use induction to prove that $\text{T.i.dim}(M) \leq n$ if and only if $\operatorname{Hom}(M, U_n) \longrightarrow \operatorname{Hom}(M, K_n) \longrightarrow 0$ is exact. By Proposition 2.1, there is a short exact sequence $0 \longrightarrow M \longrightarrow U \longrightarrow M' \longrightarrow 0$ with $U \in \operatorname{Prod} T$. The following two commutative diagrams with exact rows are obtained:

$$\begin{tikzcd}
\operatorname{Hom}(U,U_n) \arrow[r] \arrow[d] & \operatorname{Hom}(U,K_n) \arrow[r] \arrow[d] & 0 \\
\operatorname{Hom}(M,U_n) \arrow[r] \arrow[d] & \operatorname{Hom}(M,K_n) & \\
0 & &
\end{tikzcd}$$

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \operatorname{Hom}(M',K_n) \arrow[r] \arrow[d] & \operatorname{Hom}(M',U_{n-1}) \arrow[r] \arrow[d] & \operatorname{Hom}(M',K_{n-1}) \arrow[d] & \\
0 \arrow[r] & \operatorname{Hom}(U,K_n) \arrow[r] \arrow[d] & \operatorname{Hom}(U,U_{n-1}) \arrow[r] \arrow[d] & \operatorname{Hom}(U,K_{n-1}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \operatorname{Hom}(M,K_n) \arrow[r] & \operatorname{Hom}(M,U_{n-1}) \arrow[r] \arrow[d] & \operatorname{Hom}(M,K_{n-1}) \arrow[r] & 0 \\
 & & 0 & &
\end{tikzcd}$$

If $n = 0$, then $K_0 = M$ and so from the first diagram, we deduce that $\mathrm{Hom}(M, U_0) \longrightarrow \mathrm{Hom}(M, M)$ is surjective. This means that $\mathrm{Hom}(U, M) \longrightarrow \mathrm{Hom}(M, M)$ is surjective. Thus $M \in \mathrm{Prod}\, T$ and so $\mathrm{T.i.dim}(M) = 0$. The converse is trivial. Thus we can suppose that $n \geq 1$. It is seen that $\mathrm{T.i.dim}(M) \leq n$ if and only if $\mathrm{T.i.dim}(M') \leq n - 1$, by dimension shifting, if and only if $\mathrm{Hom}(M', U_{n-1}) \longrightarrow \mathrm{Hom}(M', K_{n-1})$ is surjective, by induction, if and only if $\mathrm{Hom}(U, K_n) \longrightarrow \mathrm{Hom}(M, K_n)$ is surjective, by the second diagram, if and only if $\mathrm{Hom}(M, U_n) \longrightarrow \mathrm{Hom}(M, K_n)$ is surjective, by the first diagram. Now, we return to the main proof. The above inductive proof shows that $U_n \longrightarrow K_n$ is an $\mathcal{I}_n(T)$-precover, where K_n is the nth $\mathrm{Prod}\, T$-syzygy of M. Thus by Proposition 3.1, nth $\mathrm{Prod}\, T$-syzygy of M is (n, T)-copure injective and so we are done.

Recall that the character module of a non-zero R-module M is defined to be $\mathrm{Hom}_{\mathbb{Z}}(M, \frac{Q}{\mathbb{Z}})$ and it is denoted by M^+ (see also [3, Definition 3.2.7]).

Proposition 3.3 *If T is a Wakamatsu tilting module and $M \in \mathrm{Gen}\, T$, then the following statements are equivalent:*

(1) *M is (n, T)-copure flat;*

(2) *M^+ is (n, T)-copure injective;*

(3) *$\mathcal{E}^1_T(M, B^+) = 0$ for every $B \in \mathcal{I}_n(T)$;*

(4) *The tensor functor, $M \otimes -$, preserves the exactness of every exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ with $C \in \mathcal{I}_n(T)$.*

Proof A similar proof to that of [7, p. 360] shows that for every $N \in \mathrm{Gen}\, T$, $\mathcal{E}^1_T(N, M^+) \cong \Gamma^T_1(M, N)^+ \cong \mathcal{E}^1_T(M, N^+)$. Thus the implications (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) follows. (1) $\Leftrightarrow$ (4) is easy to prove.

Proposition 3.4 *Let n be a positive integer.*

(1) *If $M \in \mathrm{Gen}\, T$, then $\mathrm{T.i.dim}(M) \leq n$ if and only if M is (n, T)-copure injective and $\mathrm{T.i.dim}(M) \leq n + 1$.*

(2) *If $N \in \mathrm{Cogen}\, T$, then $\mathrm{T.f.dim}(N) \leq n$ if and only if N is (n, T)-copure flat and $\mathrm{T.f.dim}(N) \leq n + 1$.*

Proof (1) Consider the exact sequence

$$0 \longrightarrow M \longrightarrow E_T(M) \longrightarrow \frac{E_T(M)}{M} \longrightarrow 0,$$

where $E_T(M)$ is a $\mathrm{Prod}\, T$-envelope of M. Then for every module N, we obtain the induced exact sequence

$$0 \longrightarrow \mathcal{E}^{n+1}_T(N, M) \longrightarrow \mathcal{E}^{n+1}_T(N, E_T(M)) \longrightarrow \mathcal{E}^{n+1}_T\Big(N, \frac{E_T(M)}{M}\Big) \longrightarrow \cdots .$$

Since $\mathrm{T.i.dim}(M) \leq n + 1$, dimension shifting implies that $\mathrm{T.i.dim}(\frac{E_T(M)}{M}) \leq n$ and so we have $\mathcal{E}^{n+1}_T(N, \frac{E_T(M)}{M}) = 0$. Also, from $E_T(M) \in \mathrm{Prod}\, T$ we deduce that $\mathcal{E}^{n+1}_T(N, E_T(M)) = 0$. Hence $\mathcal{E}^{n+1}_T(N, M) = 0$ and so $\mathrm{T.i.dim}(M) \leq n$. The converse is trivial.

(2) Let N be an (n, T)-copure flat module with $\mathrm{T.f.dim}(N) \leq n + 1$. Then N^+ is (n, T)-copure injective by Proposition 3.3. Since $\mathrm{T.i.dim}(N^+) \leq n+1$, (1) implies that $\mathrm{T.i.dim}(N^+) \leq n$. Hence $\mathrm{T.f.dim}(N) \leq n$. The converse is trivial.

Theorem 3.2 *Let T be a Wakamatsu tilting R-module such that $R \in \operatorname{Prod} T$ and $\mathcal{I}_n(T) \subseteq \operatorname{Gen} T$. Then the following statements hold:*

(1) *M is (n, T)-copure injective if and only if $\operatorname{Hom}(T^0, M)$ is (n, T)-copure injective, for every $T^0 \in \operatorname{Prod} T$;*

(2) *M is (n, T)-copure flat if and only if $T^0 \otimes M$ is (n, T)-copure flat, for every $T^0 \in \operatorname{Prod} T$.*

Proof (1) Let $T^0 \in \operatorname{Prod} T$ and $U \in \mathcal{I}_n(T)$. Then U has T-injective dimension at most n and so $U \in \operatorname{Gen} T$. Since T is a tilting module, by using [9, Definition 3.1 and Theorem 4.3], we have $U \in \operatorname{Pres}^{\infty} T$. Therefore, we can consider the exact sequence $0 \longrightarrow K \longrightarrow T_0 \longrightarrow U \longrightarrow 0$ with $T_0 \in \operatorname{Add} T$, which gives rise to the exactness of

$$0 \longrightarrow K \otimes T^0 \longrightarrow T_0 \otimes T^0 \longrightarrow U \otimes T^0 \longrightarrow 0.$$

Since $T^0 \in \operatorname{Prod} T$, we deduce that $U \otimes T^0 \in \mathcal{I}_n(T)$. Thus we have the exact sequence

$$\operatorname{Hom}(T_0 \otimes T^0, M) \longrightarrow \operatorname{Hom}(K \otimes T^0, M) \longrightarrow \mathcal{E}_T^1(U \otimes T^0, M) = 0.$$

Therefore, by [7, Theorem 2.75], we obtain the exact sequence

$$\operatorname{Hom}(T_0, \operatorname{Hom}(T^0, M)) \longrightarrow \operatorname{Hom}(K, \operatorname{Hom}(T^0, M)) \longrightarrow 0.$$

On the other hand, the sequence

$$\operatorname{Hom}(K, \operatorname{Hom}(T^0, M)) \longrightarrow \mathcal{E}_T^1(U, \operatorname{Hom}(T^0, M)) \longrightarrow \mathcal{E}_T^1(T_0, \operatorname{Hom}(T^0, M)) = 0$$

is exact. Thus $\mathcal{E}_T^1(U, \operatorname{Hom}(T^0, M)) = 0$, that is, $\operatorname{Hom}(T^0, M)$ is (n, T)-copure injective. The converse holds by letting $T^0 = R$.

(2) Since $T^0 \in \operatorname{Prod} T$, we only need to show that $(T^0 \otimes M)^+$ is (n, T)-copure injective by Proposition 3.3. But we have $(T^0 \otimes M)^+ \cong \operatorname{Hom}(T^0, M^+)$ and it is (n, T)-copure injective by (1). The converse holds by letting $T^0 = R$.

Proposition 3.5 *Let T be a Wakamatsu tilting module such that $\operatorname{Pres}^1 T = \operatorname{Pres}^2 T$. Then every infinite module in $\operatorname{Gen} T$ has an $\mathcal{F}_n(T)$-preenvelope.*

Proof Let $M \in \operatorname{Gen} T$ with $\operatorname{Card}(M) = \aleph_\beta$. It is not hard to prove that there exists an infinite cardinal number $\aleph_\alpha$ such that if $F \in \mathcal{F}_n(T)$ and S is a submodule of F with $\operatorname{Card}(S) \leq \aleph_\beta$, then there exists a submodule G of F with $S \subseteq G$ and $\operatorname{Card}(G) \leq \aleph_\alpha$. Therefore, M has an $\mathcal{F}_n(T)$-preenvelope, by [3, Corollary 6.2.2]. This fact that $\operatorname{Pres}^1 T = \operatorname{Pres}^2 T$ guarantees that $\mathcal{F}_n(T)$ is closed under direct products.

The following proposition gives a method to construct many examples of (n, T)-copure flat modules.

Proposition 3.6 *Let M be the cokernel of an $\mathcal{F}_n(T)$-preenvelope $K \longrightarrow F$ of K. Then M is (n, T)-copure flat.*

Proof Let $K \longrightarrow F$ be an $\mathcal{F}_n(T)$-preenvelope of K and $M = \operatorname{Coker}(K \longrightarrow F)$. Then we obtain the exact sequence $0 \longrightarrow K \longrightarrow F \longrightarrow M \longrightarrow 0$. Choose $E \in \mathcal{F}_n(T)$. Then it is not hard to show that $E^+ \in \mathcal{I}_n(T)$. So we have the exact sequence

$$0 \longrightarrow \operatorname{Hom}(M, E^+) \longrightarrow \operatorname{Hom}(F, E^+) \longrightarrow \operatorname{Hom}(K, E^+) \longrightarrow 0.$$

Thus by [7, Theorem 2.75],

$$0 \longrightarrow (M \otimes E)^{+} \longrightarrow (F \otimes E)^{+} \longrightarrow (K \otimes E)^{+} \longrightarrow 0$$

is an exact sequence which induces the exact sequence

$$0 \longrightarrow K \otimes E \longrightarrow F \otimes E \longrightarrow M \otimes E \longrightarrow 0. \tag{3.1}$$

On the other hand, we have the exact sequence

$$\Gamma_1^T(M, E) \longrightarrow K \otimes E \longrightarrow F \otimes E \longrightarrow M \otimes E \longrightarrow 0. \tag{3.2}$$

Therefore, by comparing the exact sequences (3.1) and (3.2), we deduce that $\Gamma_1^T(M, E) = 0$ and hence M is (n, T)-copure flat.

Finally, we close this paper with the following result about Wakamatsu tilting modules with finite T-injective dimension.

Theorem 3.3 *If* $\mathrm{T.i.dim}(T) \leq n$*, then the following statements hold:*

(1) *If* $M \in \mathrm{Gen}\, T$ *is an* $(n-1, T)$*-copure injective module, then there is an exact sequence* $0 \longrightarrow K \longrightarrow T^0 \longrightarrow M \longrightarrow 0$ *such that* $T^0 \in \mathrm{Prod}\, T$ *and* K *is* (n, T)*-copure injective*;

(2) *If* $N \in \mathrm{Cogen}\, T$ *is an* $(n-1, T)$*-copure flat module, then there is an exact sequence* $0 \longrightarrow N \longrightarrow F \longrightarrow L \longrightarrow 0$ *such that* $F \in \mathcal{F}_0(T)$ *and* L *is* (n, T)*-copure flat.*

Proof (1) Since $M \in \mathrm{Gen}\, T$, one can obtain the exact sequence

$$0 \longrightarrow N \longrightarrow T_0 \longrightarrow M \longrightarrow 0,$$

where $T_0 \in \mathrm{Add}\, T$. Now, consider the following commutative diagram with exact rows:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & N \arrow[r] \arrow[d, equal] & T_0 \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & N \arrow[r] & E_T(T_0) \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d] & 0 \\
 & & C \arrow[r, equal] \arrow[d] & C \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

where $T_0 \longrightarrow E_T(T_0)$ is an $\mathcal{I}_0(T)$-envelope and the square $T_0 M Q E_T(T_0)$ is a push out diagram. Since $\mathrm{T.i.dim}(T_0) \leq n$, we deduce that $\mathrm{T.i.dim}(T) \leq n$ and so shifting dimension implies that $\mathrm{T.i.dim}(C) \leq n-1$. Thus $\mathcal{E}_T^1(C, M) = 0$. Now, consider the exact sequence

$$0 \longrightarrow K \longrightarrow T^0 \xrightarrow{\alpha} M \longrightarrow 0$$

in which α is a $\mathrm{Prod}\, T$-cover of M. To complete the proof of (1), we show that K is (n, T)-copure injective. To see this, let $X \in \mathcal{I}_n(T)$ and consider the exact sequence

$$0 \longrightarrow X \xrightarrow{\beta} E_T(X) \xrightarrow{\gamma} D \longrightarrow 0,$$

where β is a $\mathrm{Prod}\,T$-envelope of X. Then $D \in \mathcal{I}_{n-1}(T)$, by shifting dimension. Thus we get the induced exact sequence

$$0 = \mathcal{E}_T^1(D, M) \longrightarrow \mathcal{E}_T^2(D, K) \longrightarrow \mathcal{E}_T^2(D, T^0) = 0.$$

Therefore, $\mathcal{E}_T^2(D, K) = 0$. On the other hand, the sequence

$$0 \longrightarrow X \longrightarrow E_T(X) \longrightarrow D \longrightarrow 0$$

induces the exact sequence

$$0 = \mathcal{E}_T^1(E_T(X), K) \longrightarrow \mathcal{E}_T^1(X, K) \longrightarrow \mathcal{E}_T^2(D, K) = 0,$$

and hence $\mathcal{E}_T^1(X, K) = 0$, as desired.

(2) Let N be an $(n-1, T)$-copure flat module. Then N^+ is $(n-1, T)$-copure injective, by Proposition 3.3. Thus by (1), there is an exact sequence $T^0 \longrightarrow N^+ \longrightarrow 0$ with $T^0 \in \mathrm{Prod}\,T$ and so $0 \longrightarrow N^{++} \longrightarrow T^{0+}$ is an exact sequence. So N is embedded in a module which belongs to $\mathcal{F}_0(T)$. Now, consider the exact sequence

$$0 \longrightarrow N \xrightarrow{\delta} F \longrightarrow L \longrightarrow 0,$$

where δ is an $\mathcal{F}_0(T)$-preenvelope of N. By Proposition 3.5, L is $(1, T)$-copure flat. Applying an argument similar to that in the proof of (1), we conclude that L is (n, T)-copure flat.

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References

[1] Angleri-Hugel, L. and Coelho, F. U., Infinitely generated tilting modules of finite projective dimension, *Forum Math.*, **13**, 2001, 239–250.

[2] Bazzoni, S., A characterization of n-cotilting and n-tilting modules, *J. Algebra*, **273**, 2005, 359–372.

[3] Enochs, E. E. and Jenda, O. M. G., Relative Homological Algebra, Walter de Gruyter, Berlin, New York, 2000.

[4] Green, E. L., Reiten, I. and Solberg, O., Dualities on Generalized Koszul Algebras, Mem. Amer. Math. Soc., **159**, A. M. S., Providence, RI, 2002, 754.

[5] Nikmehr, M. J., Shaveisi, F. and Nikandish, R., n-Projective modules, *Algebras Groups Geom.*, **24**, 2007, 447–454.

[6] Nikmehr, M. J. and Shaveisi, F., T-Dimension and $(n + \frac{1}{2}, T)$-projective modules, *Southeast Asian Bull. Math.*, **35**, 2011, 1–11.

[7] Rotman, J. J., An Introduction to Homological Algebra, Academic Press, New York, 1979.

[8] Wakamatsu, T., On modules with trivial self-extension, *J. Algebra*, **114**, 1998, 106–114.

[9] Wei, J., n-Star modules and n-tilting modules, *J. Algebra*, **283**, 2005, 711–722.