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Exponential Synchronization of the Linearly Coupled Dynamical Networks with Delays***

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Abstract In this paper, the authors investigate the synchronization of an array of linearly coupled identical dynamical systems with a delayed coupling. Here the coupling matrix can be asymmetric and reducible. Some criteria ensuring delay-independent and delay-dependent global synchronization are derived respectively. It is shown that if the coupling delay is less than a positive threshold, then the coupled network will be synchronized. On the other hand, with the increase of coupling delay, the synchronization stability of the network will be restrained, even eventually de-synchronized.

Keywords Time-delay, Synchronization, Exponential stability, Left eigenvector
2000 MR Subject Classification 17B40, 17B50,

1 Introduction

Recently, an increasing interest has been devoted to the study of complex networks (cf. [1–3]). Among them, arrays of linearly coupled dynamical systems have attracted much attention, since they can exhibit an interesting phenomenon: synchronization. For example, at night in certain parts of southeast Asia, thousands of male fireflies congregate in trees and flash in synchrony, and this phenomenon has been proved to occur in a group of integrate-and-fire cells (cf. [4]). Meantime, synchronization of coupled oscillators can not only explain many natural phenomena, but also have many applications, such as image processing, secure communication (cf. [5]).

An interesting approach to investigate synchronization was proposed by Wu and Chua [6], in which they defined a distance from a point to the synchronization manifold for N oscillators $\sum\limits_{\substack{i,j=1\\ i\neq j}}^{N} m_{ij}^2\|x_i(t)-x_j(t)\|^2$, where m_{ij} is defined by the symmetric and irreducible coupling matrix. But in the real world, the coupling matrix can be asymmetric. Lu and Chen generalized this approach and gave a new criteria for synchronization. Moreover, in [7], Lu and Chen proposed a new approach to investigate the synchronization. They defined a reference vector $x_\xi(t)$, and in order to prove the synchronization, it only needs to check whether $\lim\limits_{t\to\infty}\|x_i(t)-x_\xi(t)\|=0$. In the following, we will also use this method to investigate synchronization with delayed coupling.

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In [8], the authors investigate the synchronization dynamics of the following linearly coupled networks with a delay:

$$\frac{dx_i(t)}{dt} = f(x_i(t), t) + \sum_{\substack{j=1 \\ j \neq i}}^{N} c_{ij}\Gamma[x_j(t-\tau) - x_i(t)], \quad i = 1, \cdots, N.$$

In this paper, we will investigate synchronization dynamics of another general model of complex delayed dynamical networks as well as the effects of time delays. The general linearly coupled networks with a delay can be described as

$$\frac{dx_i(t)}{dt} = f(x_i(t), t) + \sum_{\substack{j=1 \\ j \neq i}}^{N} c_{ij}\Gamma[x_j(t-\tau) - x_i(t-\tau)], \quad i = 1, \cdots, N,$$

or be rewritten as

$$\frac{dx_i(t)}{dt} = f(x_i(t), t) + \sum_{j=1}^{N} c_{ij}\Gamma x_j(t-\tau), \quad i = 1, 2, \cdots, N, \tag{1.1}$$

where $C = (c_{ij}) \in R^{N \times N}$ denotes the coupling configuration of the coupled networks with

$$\begin{cases} c_{ij} \geq 0, & i \neq j, \\ c_{ii} = -\sum\limits_{\substack{j=1 \\ j \neq i}} c_{ij}, & i = 1, 2, \cdots, N, \end{cases} \tag{1.2}$$

and $\Gamma = \mathrm{diag}\{\gamma_1, \gamma_2, \cdots, \gamma_n\}$, where $\gamma_i > 0$, $i = 1, 2, \cdots, n$.

In [9], Zhou and Chen have investigated the synchronization problem when the coupling matrix C is symmetric and irreducible. Here, we will generalize the results given in [9] to the case that the coupling matrix C might be asymmetric and reducible. Moreover, in [9], the authors only investigate the local synchronization. In this paper, we also investigate the global synchronization and give some criteria ensuring the delay-independent and delay-dependent synchronization, respectively.

This paper is organized as follows. In Section 2, some necessary definitions, lemmas and hypotheses are given. In Section 3, the criteria for global delay-independent and delay-dependent synchronization of coupled networks are derived respectively. In Section 4, we give a simulation to verify our theoretical results. We conclude this paper in Section 5.

2 Preliminaries

In this section, we present some definitions and lemmas, which are needed in this paper.

Definition 2.1 *The coupled networks* (1.1) *are said to be synchronized, if*

$$\lim_{t \to \infty} \|x_i(t) - x_j(t)\| = 0, \quad i, j = 1, 2, \cdots, N, \tag{2.1}$$

where $\|\cdot\|$ *denotes some norm and* $x_i(t) = (x_{1,i}(t), x_{2,i}(t), \cdots, x_{n,i}(t))^T \in R^n$. *The set* $\mathbf{S} = \{x = (x_1^T, x_2^T, \cdots, x_N^T)^T \mid x_i = x_j, \ i, j = 1, 2, \cdots, N\}$ *is called the synchronization manifold.*

Definition 2.2 *A matrix* $C = (c_{ij}) \in R^{N\times N}$ *is said:* $C \in A1$, *if*

(1) $c_{ij} \geq 0,\ i \neq j,\ c_{ii} = -\sum\limits_{\substack{j=1\\ j\neq i}}^{N} c_{ij},\ i = 1, 2, \cdots, N,$

(2) *Real part of eigenvalues of* C *are all negative except an eigenvalue* 0 *with multiplicity one.*

Definition 2.3 *Function Class* $\mathrm{QUAD}(P, \Pi)$: *we say* $f \in \mathrm{QUAD}(P, \Pi, \varpi)$, *when* $P = \mathrm{diag}\{p_1, p_2, \cdots, p_n\}$ *is a positive definite diagonal matrix,* $\Pi = \mathrm{diag}\{\pi_1, \pi_2, \cdots, \pi_n\}$ *is a diagonal matrix, and* $\varpi > 0$, *if and only if*

$$(x-y)^T P[f(x,t) - f(y,t) - \Pi x + \Pi y] \leq -\varpi (x-y)^T (x-y) \tag{2.2}$$

holds for any $x, y \in R^n$.

Definition 2.4 *Function* $f \in H(M)$, *where* $M > 0$, *if and only if*

$$[f(x,t) - f(y,t)]^T [f(x,t) - f(y,t)] \leq M^2 (x-y)^T (x-y) \tag{2.3}$$

holds for any $x, y \in R^n$.

Lemma 2.1 (Cf. [10]) *If a matrix* $C_{N\times N} \in A1$, *then*

(1) *The vector* $(1, 1, \cdots, 1)^T$ *is the right eigenvector of* C *corresponding to eigenvalue* 0 *with multiplicity* 1.

(2) *The left eigenvector of* $C: \xi = (\xi_1, \xi_2, \cdots, \xi_N)^T \in R^N$ *corresponding to eigenvalue* 0 *has following properties: it is non-zero and its multiplicity is* 1; *all* $\xi_i \geq 0$, $i = 1, 2, \cdots, N$. *More precisely,*

(a) C *is irreducible if and only if all* $\xi_i > 0$, $i = 1, 2, \cdots, N$.

(b) C *is reducible if and only if for some* i, $\xi_i = 0$. *In such case, by suitable rearrangement, we can assume that* $\xi^T = [\xi_+^T, \xi_0^T]$, *where* $\xi_+ = [\xi_1, \xi_2, \cdots, \xi_r]^T \in R^r$, *with all* $\xi_i > 0$, $i = 1, 2, \cdots, r$; $\xi_0 = [\xi_{r+1}, \xi_{r+2}, \cdots, \xi_N]^T \in R^{N-r}$ *with all* $\xi_j = 0$, $r+1 \leq j \leq N$, *and* C *can be rewritten as* $\left[\begin{smallmatrix} C_{11} & C_{12} \\ C_{21} & C_{22}\end{smallmatrix}\right]$, *where* $C_{11} \in R^{r\times r}$ *is irreducible and* $C_{12} = 0$.

Definition 2.5 $L = \left\{X = (x_1^T, x_2^T, \cdots, x_N^T)^T \in R^{Nn}, \sum\limits_{i=1}^{N} \xi_i x_i = 0\right\}$ *is called the transverse space. For the case of* $n = 1$, $L = \left\{(u_1, u_2, \cdots, u_N)^T \in R^N, \sum\limits_{i=1}^{N} \xi_i u_i = 0\right\}$.

Let

$$x_\xi(t) = \sum_{i=1}^{N} \xi_i x_i(t),$$

where ξ is the left eigenvector corresponding to the eigenvalue 0 defined in Lemma 2.1 and $\sum\limits_{i=1}^{N} \xi_i = 1$.

For $i = 1, \cdots, N$, denote $\overline{x}_i(t) = x_i(t) - x_\xi(t)$, $\overline{X}(t) = (\overline{x}_1^T(t), \overline{x}_2^T(t), \cdots, \overline{x}_N^T(t))^T$, and $X^* = (x_\xi^T(t), x_\xi^T(t), \cdots, x_\xi^T(t))^T$. Then, $\overline{X}(t) \in L$, while $X^* \in \mathbf{S}$. Proving the synchronization of the coupled networks (1.1) is equivalent to proving that $\overline{X}(t) \to 0$.

3 Global Synchronization

It is clear that the dynamical equation for $\overline{x}_i(t) = x_i(t) - x_\xi(t)$ is

$$\frac{d\overline{x}_i(t)}{dt} = f(x_i(t),t) - f(x_\xi(t),t) - \sum_{k=1}^{N} \xi_k(f(x_k(t),t) - f(x_\xi(t),t)) + \sum_{j=1}^{N} c_{ij}\Gamma\,\overline{x}_j(t-\tau).$$

Let

$$\overline{X}(t) = \begin{bmatrix} \overline{x}_1(t) \\ \overline{x}_2(t) \\ \vdots \\ \overline{x}_N(t) \end{bmatrix}, \quad \overline{F}(t) = \begin{bmatrix} f(x_1(t),t) - f(x_\xi(t),t) \\ f(x_2(t),t) - f(x_\xi(t),t) \\ \vdots \\ f(x_N(t),t) - f(x_\xi(t),t) \end{bmatrix}, \quad \Omega = \begin{bmatrix} \xi_1 & \cdots & \xi_N \\ \vdots & \vdots & \vdots \\ \xi_1 & \cdots & \xi_N \end{bmatrix}, \tag{3.1}$$

where $P = \mathrm{diag}\{p_1, p_2, \cdots, p_n\}$ is a positive definite diagonal matrix, $\Pi = \mathrm{diag}\{\pi_1, \pi_2, \cdots, \pi_n\}$ is a diagonal matrix, $\Xi = \mathrm{diag}\{\xi_1, \xi_2, \cdots, \xi_N\}$, I_n (or I_N) is an $n \times n$ (or $N \times N$) identity matrix, and $\otimes$ is the Kronecker product. Moreover, denote

$$\mathbf{\Omega} = \Omega \otimes I_n, \quad \mathbf{C} = C \otimes \Gamma, \quad \mathbf{\Xi} = \Xi \otimes I_n, \quad \mathbf{P} = I_N \otimes P,$$
$$\mathbf{\Gamma} = I_N \otimes \Gamma, \quad \mathbf{\Pi} = I_N \otimes \Pi, \quad \mathbf{I} = I_N \otimes I_n.$$

Then coupled networks (1.1) can be rewritten in the following form:

$$\frac{d\overline{X}(t)}{dt} = (\mathbf{I} - \mathbf{\Omega})\overline{F}(t) + \mathbf{C}\overline{X}(t-\tau). \tag{3.2}$$

3.1 Global delay-independent synchronization

In this section, we discuss the global synchronization, which is independent of the delay.

Theorem 3.1 *If there exist a positive definite diagonal matrix* $P = \mathrm{diag}\{p_1, p_2, \cdots, p_n\}$, *a diagonal matrix* $\Pi = \mathrm{diag}\{\pi_1, \pi_2, \cdots, \pi_n\}$, *and a positive constant* $\varpi > 0$, *such that* $f \in \mathrm{QUAD}(P, \Pi, \varpi)$; *and there exist positive definite matrices* B_j *such that*

$$\begin{pmatrix} 2\pi_j Q^T \Xi Q & \gamma_j Q^T \Xi C Q \\ \gamma_j Q^T C^T \Xi Q & -Q^T B_j Q \end{pmatrix} < 0, \quad j = 1, 2, \cdots, n, \tag{3.3}$$

where

$$Q = \begin{bmatrix} & & I_{N-1} & \\ -\frac{\xi_1}{\xi_N} & -\frac{\xi_2}{\xi_N} & \cdots & -\frac{\xi_{N-1}}{\xi_N} \end{bmatrix} \tag{3.4}$$

and I_{N-1} *denotes the* $(N-1)\times(N-1)$ *identity matrix, then the coupled networks* (1.1) *will be globally exponentially synchronized.*

Proof By the inequality (3.3), we can find a positive constant ε satisfying $-2\varpi I_n + \varepsilon P < 0$ such that

$$T_j = \begin{bmatrix} 2\pi_j \Xi + B_j e^{\varepsilon\tau} & \gamma_j \Xi C \\ \gamma_j C^T \Xi & -B_j \end{bmatrix} \le 0, \quad j = 1, \cdots, n \tag{3.5}$$

holds on the transverse subspace $L \times L$.

Define a Lyapunov function

$$L(t) = \overline{X}^T(t)\mathbf{P\Xi}\overline{X}(t)e^{\varepsilon t} + \sum_{j=1}^{n}\int_{t-\tau}^{t} p_j\widetilde{x}_j^T(s)B_j\widetilde{x}_j(s)e^{\varepsilon(s+\tau)}ds,$$

where $\widetilde{x}_j^T(t) = (x_{j,1}(t), \cdots, x_{j,N}(t))^T,\ j = 1, \cdots, n.$

Differentiating $L(t)$, we have

$$\begin{aligned}
\frac{dL(t)}{dt} &= \varepsilon e^{\varepsilon t}\overline{X}^T(t)\mathbf{P\Xi}\overline{X}(t) + 2e^{\varepsilon t}\overline{X}^T(t)\mathbf{P\Xi}[(\mathbf{I}-\mathbf{\Omega})\overline{F}(t) + \mathbf{C}\overline{X}(t-\tau)] \\
&\quad + \sum_{j=1}^{n} p_j\widetilde{x}_j^T(t)B_j\widetilde{x}_j(t)e^{\varepsilon(t+\tau)} - \sum_{j=1}^{n} p_j\widetilde{x}_j^T(t-\tau)B_j\widetilde{x}_j(t-\tau)e^{\varepsilon t} \\
&= e^{\varepsilon t}\varepsilon\sum_{i=1}^{N}\xi_i\overline{x}_i^T(t)P\overline{x}_i(t) + 2e^{\varepsilon t}\sum_{i=1}^{N}\xi_i\overline{x}_i^T(t)P[f(x_i(t),t) - f(x_\xi(t),t) - \Pi\overline{x}_i(t)] \\
&\quad + 2e^{\varepsilon t}\overline{X}^T(t)\mathbf{P\Xi\Pi}\overline{X}(t) + 2e^{\varepsilon t}\overline{X}^T(t)\mathbf{P\Xi C}\overline{X}(t-\tau) + \sum_{j=1}^{n} p_j\widetilde{x}_j^T(t)B_j\widetilde{x}_j(t)e^{\varepsilon(t+\tau)} \\
&\quad - \sum_{j=1}^{n} p_j\widetilde{x}_j^T(t-\tau)B_j\widetilde{x}_j(t-\tau)e^{\varepsilon t} \\
&\le e^{\varepsilon t}\sum_{i=1}^{N}\xi_i\overline{x}_i^T(t)(-2\varpi I_n + \varepsilon P)\overline{x}_i(t) + e^{\varepsilon t}\sum_{j=1}^{n} p_j[\widetilde{x}_j^T(t)(2\pi_j\Xi + B_je^{\varepsilon\tau})\widetilde{x}_j(t) \\
&\quad + 2\gamma_j\widetilde{x}_j^T(t)\Xi C\widetilde{x}_j(t-\tau) - \widetilde{x}_j^T(t-\tau)B_j\widetilde{x}_j(t-\tau)] \\
&= e^{\varepsilon t}\sum_{i=1}^{N}\xi_i\overline{x}_i^T(t)(-2\varpi I_n + \varepsilon P)\overline{x}_i(t) + e^{\varepsilon t}\sum_{j=1}^{n} p_j[\widetilde{x}_j^T(t), \widetilde{x}_j^T(t-\tau)]T_j\begin{bmatrix}\widetilde{x}_j(t) \\ \widetilde{x}_j(t-\tau)\end{bmatrix} \\
&\le 0.
\end{aligned}$$

Therefore, $L(t) \le L(0)$, which implies that $\overline{X}^T\mathbf{P\Xi}\overline{X} \le L(0)e^{-\varepsilon t}$. Hence, $\overline{X}$ converges to zero exponentially with rate $e^{-\frac{\varepsilon}{2}t}$.

The proof of this theorem is completed.

3.2 Global delay-dependent synchronization

In this section, we discuss the global synchronization, which depends on the delay.

We can rewrite equation (3.2) as follows

$$\frac{d\overline{X}(t)}{dt} = (\mathbf{I}-\mathbf{\Omega})\overline{F}(t) + \mathbf{C}\overline{X}(t) - \mathbf{C}\int_{t-\tau}^{t}\frac{d\overline{X}(s)}{ds}ds. \tag{3.6}$$

Theorem 3.2 *If there exist a positive diagonal matrix* $P = \mathrm{diag}\{p_1, p_2, \cdots, p_n\}$*, a diagonal matrix* $\Pi = \mathrm{diag}\{\pi_1, \pi_2, \cdots, \pi_n\}$*,* $\varpi > 0$*, and* $M > 0$*, such that the following conditions are all satisfied*

(1) $f \in \mathrm{QUAD}(P, \Pi, \varpi)$;

(2) $f \in H(M)$;

(3) $Q^T\{\Xi(\gamma_jC + \pi_jI_N)\}^sQ < 0$ *for* $i = 1, 2, \cdots, n$;

(4) $\tau < \overline{\tau} = \sqrt{\frac{\beta_3}{\beta_1+\beta_2\beta_3}}$, where

$$\beta_1 = 3\|\Xi\|_2^2\|C\|_2^2\|\Gamma\|_2^2\|P\|_2^2(M^2\|I_N - \Xi\|_2^2 + \|C\|_2^2\|\Gamma\|_2^2),$$
$$\beta_2 = 3\|C\|_2^2\|\Gamma\|_2^2,$$
$$\beta_3 = \varpi^2\Big(\min_i \xi_i\Big)^2,$$

where Q is defined by (3.4), *then the coupled networks* (1.1) *will be exponentially synchronized.*

Proof By the Theorem 3.2(4), we can find $\varepsilon > 0$ and $k > 0$, such that

$$\begin{aligned}&\Big(\varepsilon\|P\|_2 \max_i \xi_i - 2\varpi \min_i \xi_i\Big) + \tau\sigma\|\Xi\|_2\|P\|_2\|\Gamma\|_2^2\|C\|_2^2\\&+ 3k\zeta(M^2\|I_N - \Xi\|_2^2 + \|C\|_2^2\|\Gamma\|_2^2) < 0, \qquad (3.7)\\&k > \frac{\|\Xi\|_2\|P\|_2\sigma^{-1}}{1 - 3\tau\zeta\|C\|_2^2\|\Gamma\|_2^2}, \qquad (3.8)\end{aligned}$$

where

$$\zeta = \frac{e^{\varepsilon\tau} - 1}{\varepsilon} \quad \text{and} \quad \sigma = \sqrt{\frac{3\zeta(M^2\|I_N - \Xi\|_2^2 + \|C\|_2^2\|\Gamma\|_2^2)}{\tau\|C\|_2^2\|\Gamma\|_2^2(1 - 3\zeta\tau\|C\|_2^2\|\Gamma\|_2^2)}}.$$

Let

$$L_1(t) = e^{\varepsilon t}\overline{X}^T(t)\mathbf{P\Xi}\overline{X}(t), \quad L_2(t) = k\int_{t-\tau}^t e^{\varepsilon(s+\tau)}ds\int_s^t \frac{d\overline{X}^T(\theta)}{d\theta}\frac{d\overline{X}(\theta)}{d\theta}d\theta.$$

Then, differentiating it, we have

$$\begin{aligned}\frac{dL_1(t)}{dt} &= \varepsilon e^{\varepsilon t}\overline{X}^T(t)\mathbf{P\Xi}\overline{X}(t) + 2e^{\varepsilon t}\overline{X}^T(t)\mathbf{P\Xi}\Big[(\mathbf{I} - \mathbf{\Omega})\overline{F}(t) + \mathbf{C}\overline{X}(t) - \mathbf{C}\int_{t-\tau}^t \frac{d\overline{X}(s)}{ds}ds\Big]\\&\le e^{\varepsilon t}\Big(\varepsilon\|P\|_2 \max_i \xi_i - 2\varpi \min_i \xi_i\Big)\overline{X}^T(t)\overline{X}(t)\\&\quad + 2e^{\varepsilon t}\sum_{j=1}^n p_j\widetilde{x}_j^T(t)\Xi(\gamma_j C + \pi_j I_N)\widetilde{x}_j(t) - 2e^{\varepsilon t}\int_{t-\tau}^t \overline{X}^T(t)\mathbf{P\Xi C}\frac{d\overline{X}(s)}{ds}ds.\end{aligned}$$

By Theorem 3.2(3), we have

$$\widetilde{x}_j^T(t)\{\Xi(\gamma_j C + \pi_j I_N)\}^s\widetilde{x}_j(t) \le 0.$$

Therefore,

$$\begin{aligned}\frac{dL_1(t)}{dt} &\le e^{\varepsilon t}\Big(\varepsilon\|P\|_2 \max_i \xi_i - 2\varpi \min_i \xi_i\Big)\overline{X}^T(t)\overline{X}(t)\\&\quad + e^{\varepsilon t}\|\mathbf{P}\|_2\|\mathbf{\Xi}\|_2\Big(\tau\sigma\|\mathbf{C}\|_2^2\overline{X}^T(t)\overline{X}(t) + \sigma^{-1}\int_{t-\tau}^t \frac{d\overline{X}^T(s)}{ds}\frac{d\overline{X}(s)}{ds}ds\Big)\\&= e^{\varepsilon t}\Big(\varepsilon\|P\|_2 \max_i \xi_i - 2\varpi \min_i \xi_i\Big)\overline{X}^T(t)\overline{X}(t)\\&\quad + e^{\varepsilon t}\|P\|_2\|\Xi\|_2\Big(\tau\sigma\|\Gamma\|_2^2\|C\|_2^2\overline{X}^T(t)\overline{X}(t) + \sigma^{-1}\int_{t-\tau}^t \frac{d\overline{X}^T(s)}{ds}\frac{d\overline{X}(s)}{ds}ds\Big),\end{aligned}$$

$$
\begin{aligned}
\frac{dL_2(t)}{dt} &= -ke^{\varepsilon t}\int_{t-\tau}^{t}\frac{d\overline{X}^T(s)}{ds}\frac{d\overline{X}(s)}{ds}ds + k\zeta e^{\varepsilon t}\frac{d\overline{X}^T(t)}{dt}\frac{d\overline{X}(t)}{dt}\\
&= -ke^{\varepsilon t}\int_{t-\tau}^{t}\frac{d\overline{X}^T(s)}{ds}\frac{d\overline{X}(s)}{ds}ds + k\zeta e^{\varepsilon t}\Big[(\mathbf{I}-\mathbf{\Omega})\overline{F}(t) + \mathbf{C}\overline{X}(t) - \mathbf{C}\int_{t-\tau}^{t}\frac{d\overline{X}(s)}{ds}ds\Big]^T\\
&\quad\times\Big[(\mathbf{I}-\mathbf{\Omega})\overline{F}(t) + \mathbf{C}\overline{X}(t) - \mathbf{C}\int_{t-\tau}^{t}\frac{d\overline{X}(s)}{ds}ds\Big]\\
&\le -ke^{\varepsilon t}\int_{t-\tau}^{t}\frac{d\overline{X}^T(s)}{ds}\frac{d\overline{X}(s)}{ds}ds + 3k\zeta e^{\varepsilon t}\Big[\overline{F}^T(t)(\mathbf{I}-\mathbf{\Omega})^T(\mathbf{I}-\mathbf{\Omega})\overline{F}(t)\\
&\quad+\overline{X}^T(t)\mathbf{C}^T\mathbf{C}\overline{X}(t) + \int_{t-\tau}^{t}\frac{d\overline{X}^T(s)}{ds}ds\mathbf{C}^T\mathbf{C}\int_{t-\tau}^{t}\frac{d\overline{X}(\theta)}{d\theta}d\theta\Big].
\end{aligned}
$$

Since

$$
\begin{aligned}
&\int_{t-\tau}^{t}\frac{d\overline{X}^T(s)}{ds}ds\int_{t-\tau}^{t}\frac{d\overline{X}(\theta)}{d\theta}d\theta\\
&= \int_{t-\tau}^{t}ds\int_{t-\tau}^{t}\frac{d\overline{X}^T(s)}{ds}\frac{d\overline{X}(\theta)}{d\theta}d\theta\\
&\le \frac{1}{2}\Big[\int_{t-\tau}^{t}ds\int_{t-\tau}^{t}\frac{d\overline{X}^T(s)}{ds}\frac{d\overline{X}(s)}{ds}d\theta + \int_{t-\tau}^{t}ds\int_{t-\tau}^{t}\frac{d\overline{X}^T(\theta)}{d\theta}\frac{d\overline{X}(\theta)}{d\theta}d\theta\Big]\\
&= \tau\int_{t-\tau}^{t}\frac{d\overline{X}^T(s)}{ds}\frac{d\overline{X}(s)}{ds}ds,
\end{aligned}
$$

we have

$$
\begin{aligned}
\frac{dL_2(t)}{dt} &\le -ke^{\varepsilon t}\int_{t-\tau}^{t}\frac{d\overline{X}^T(s)}{ds}\frac{d\overline{X}(s)}{ds}ds + 3k\zeta e^{\varepsilon t}\Big[M^2\|I_N-\Xi\|_2^2\overline{X}^T(t)\overline{X}(t)\\
&\quad+\|C\|_2^2\|\Gamma\|_2^2\overline{X}^T(t)\overline{X}(t) + \tau\|C\|_2^2\|\Gamma\|_2^2\int_{t-\tau}^{t}\frac{d\overline{X}^T(s)}{ds}\frac{d\overline{X}(s)}{ds}ds\Big].
\end{aligned}
$$

In summary, we have

$$
\begin{aligned}
\frac{dL(t)}{dt} &= \frac{dL_1(t)}{dt} + \frac{dL_2(t)}{dt}\\
&\le e^{\varepsilon t}\Big[\Big(\varepsilon\|P\|_2\max_i\xi_i - 2\varpi\min_i\xi_i\Big) + \tau\sigma\|\Xi\|_2\|P\|_2\|\Gamma\|_2^2\|C\|_2^2\\
&\quad+3k\zeta M^2\|I_N-\Xi\|_2^2 + 3k\zeta\|C\|_2^2\|\Gamma\|_2^2\Big]\overline{X}^T(t)\overline{X}(t)\\
&\quad+e^{\varepsilon t}[\sigma^{-1}\|\Xi\|_2\|P\|_2 - k + 3k\zeta\tau\|C\|_2^2\|\Gamma\|_2^2]\int_{t-\tau}^{t}\frac{d\overline{X}^T(s)}{ds}\frac{d\overline{X}(s)}{ds}ds\\
&\le 0.
\end{aligned}
$$

Thus the proof is completed.

3.3 Reducible coupling matrix

When C is reducible, without loss of generality, we assume that $C = \left[\begin{smallmatrix} C_{11} & 0 \\ C_{21} & C_{22}\end{smallmatrix}\right]$, where $C_{11} \in R^{r\times r}$ is irreducible. Then the coupled networks (1.1) can be decomposed into two subsystems:

$$\frac{dx_i(t)}{dt} = f(x_i(t),t) + \sum_{j=1}^{r} c_{ij}\Gamma x_j(t-\tau), \quad i = 1,2,\cdots,r, \tag{3.9}$$

$$\frac{dx_i(t)}{dt} = f(x_i(t),t) + \sum_{j=1}^{N} c_{ij}\Gamma x_j(t-\tau), \quad i = r+1,\cdots,N. \tag{3.10}$$

If for the subsystem (3.9), all the conditions in Theorems 3.1 and 3.2 are satisfied, then the coupled subsystem (3.9) is globally exponentially stable. Under the conditions given in Lemma 2.1, $\xi_i > 0$ for $i = 1,2,\cdots,r$; and $\xi_i = 0$ for $i = r+1,\cdots,N$, $x_\xi(t) = \sum_{i=1}^{r} \xi_i x_i(t)$; and $\|x_i(t) - x_\xi(t)\| \to 0$ for $i = 1,2,\cdots,r$.

$$\frac{d\widehat{x}_i(t)}{dt} = f(x_i(t),t) - f(x_\xi(t),t) - c_i\Gamma\widehat{x}_i(t-\tau) + \sum_{j=r+1}^{N} \widehat{c}_{ij}\Gamma\widehat{x}_j(t-\tau), \tag{3.11}$$

where $\widehat{x}_i(t) = x_i(t) - x_\xi(t),\ i = r+1, r+2,\cdots,N$, $c_i = \sum_{j=1}^{r} c_{ij}$ and

$$\widehat{c}_{ij} = \begin{cases} c_{ij}, & j \neq i, \\ c_{ij} + c_i, & j = i. \end{cases}$$

Then for the system (3.11), with the same analysis as above, we can get the corresponding synchronization criteria, and here we omit them.

4 Numerical Examples

As applications of the theoretical criteria given above, chaos synchronization problem of a prototype composing of Chua's circuits is discussed in this section.

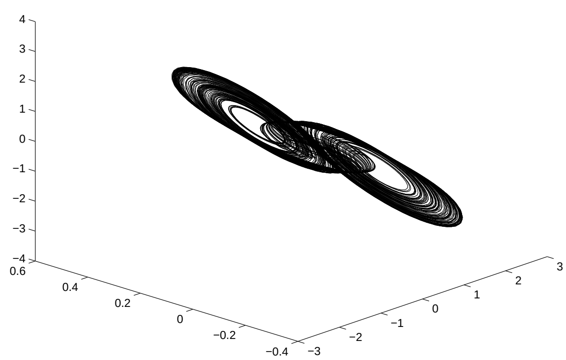

Figure 1 The double-scrolling attractor of Chua's Circuit

The uncoupled system $\dot{s}(t) = f(s(t),t)$ for each node is described by

$$\begin{cases} \dot{s}_1(t) = m[s_2(t) - h(s_1(t))], \\ \dot{s}_2(t) = s_1(t) - s_2(t) + s_3(t), \\ \dot{s}_3(t) = -ns_2(t), \end{cases} \tag{4.1}$$

where $h(s_1(t)) = \frac{2}{7}s_1(t) - \frac{3}{14}[|s_1(t)+1| - |s_1(t)-1|]$, $m = 9$ and $n = 14\frac{2}{7}$. In this case, the system has a double-scroll chaotic attractor, as shown in Figure 1.

Consider four linearly coupled Chua's circuits with a delayed coupling

$$\frac{dx_i(t)}{dt} = f(x_i(t), t) + \sum_{j=1}^{4} c_{ij}\Gamma x_j(t-\tau), \tag{4.2}$$

where $x_i(t) = (x_{1,i}(t), x_{2,i}(t), x_{3,i}(t))^T \in R^3$, $i = 1, 2, 3, 4$, f is defined in (4.1), $\Gamma = 12I_3$, and the coupling matrix

$$C = \begin{pmatrix} -3 & 1 & 1 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & -1 \end{pmatrix}.$$

The left eigenvector of C corresponding to the eigenvalue 0 is $\xi = [\frac{1}{7}, \frac{1}{7}, \frac{2}{7}, \frac{3}{7}]$. We will use Theorem 3.2 to investigate the global synchronization for (4.2).

Direct calculation indicates that the Jacobian matrix Df satisfies

$$\frac{[Df^T(x) + Df(x)]}{2} \le R = \begin{pmatrix} 1.2857 & 5.0000 & 0 \\ 5.0000 & -1.0000 & -6.6429 \\ 0 & -6.6429 & 0 \end{pmatrix},$$

whose eigenvalues are $-8.6325, 0.8107, 8.1075$. Therefore, $f \in \mathrm{QUAD}(I_3, 10I_3, 0.6218)$.

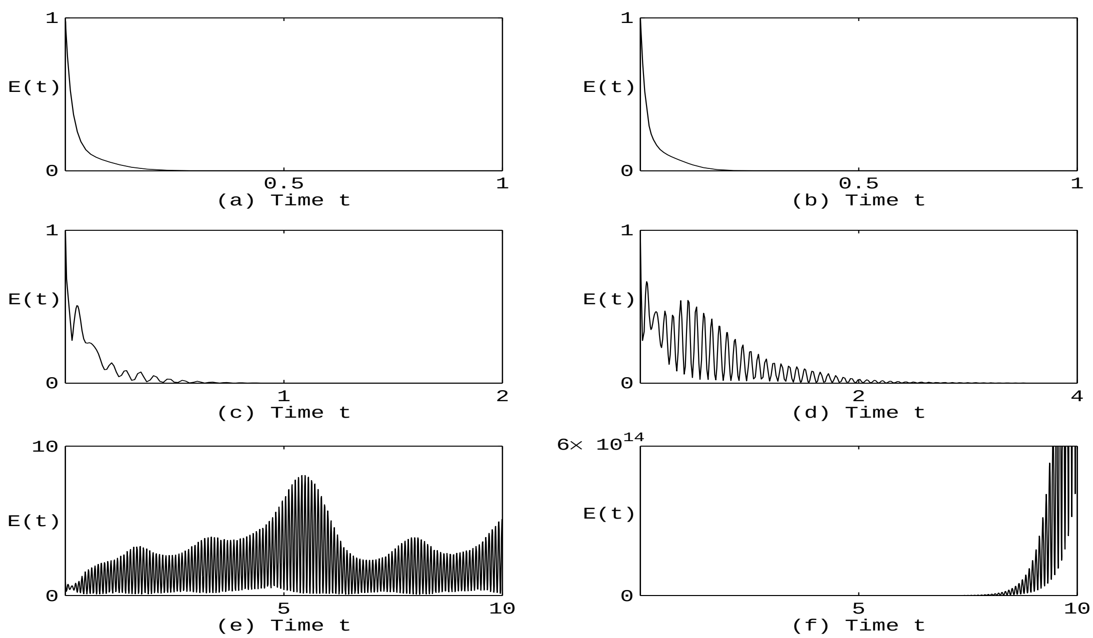

Figure 2 Time responses of $E(t)$ with different values of coupling delays τ

(a) $\tau = 0.00015$ (b) $\tau = 0.01$ (c) $\tau = 0.031$ (d) $\tau = 0.035$ (e) $\tau = 0.037$ (f) $\tau = 0.040$

It can also be checked that $f \in H(M)$ with $M = 16.9754$.

Direct calculation shows that the eigenvalues of $Q^T\{\Xi(12C + 10I_N)\}^s Q$ are -7.2940, -2.0922, and -0.3281. So Theorem 3.2(3) is satisfied.

Finally, using the equations in Theorem 3.2(4), we obtain an upper bound $\overline{\tau} \cong 1.8158 \times 10^{-4}$. Therefore, if the coupling delay is less than $\overline{\tau}$, then the coupled networks (4.2) can be synchronized exponentially and globally.

We introduce a synchronization error as

$$E(t) = \frac{\sum\limits_{i,j} \|x_i(t) - x_j(t)\|_2}{\sum\limits_{i,j} \|x_i(0) - x_j(0)\|_2}.$$

Figure 2 visualizes the dynamics of $E(t)$ in delayed networks (4.2) with different values of coupling delay τ in time interval $[0, 10]$. It seems that $\tau^\star \approx 0.036$ is a critical value of the delay. From the figure, we can see that synchronization stability of the coupled networks (4.2) will decrease as τ increases gradually, even eventually de-synchronized at a certain value.

5 Conclusion

In this paper, we investigate the exponential synchronization of a general model for the linearly coupled dynamical networks with delays and the effects of the delayed coupling. Here we mainly discuss the case that the coupling matrix is asymmetric and reducible. Some simple yet generic criteria for global synchronization manifold to be delay-independent and delay-dependent exponentially stable are derived. It has been shown that if the coupling delay is less than a positive value, the coupling network will be synchronized. On the other hand, along with the coupling delay increases gradually, the synchronization stability of the network becomes restrained, even the network will eventually desynchronize. Simulations verify theoretical results.

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