

A Mathematical Model with Delays for Schistosomiasis Japonicum Transmission***

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Abstract A dynamic model of schistosoma japonicum transmission is presented that incorporates effects of the prepatent periods of the different stages of schistosoma into Barbour's model. The model consists of four delay differential equations. Stability of the disease free equilibrium and the existence of an endemic equilibrium for this model are stated in terms of a key threshold parameter. The study of dynamics for the model shows that the endemic equilibrium is globally stable in an open region if it exists and there is no delays, and for some nonzero delays the endemic equilibrium undergoes Hopf bifurcation and a periodic orbit emerges. Some numerical results are provided to support the theoretic results in this paper. These results suggest that prepatent periods in infection affect the prevalence of schistosomiasis, and it is an effective strategy on schistosomiasis control to lengthen in prepatent period on infected definitive hosts by drug treatment (or lengthen in prepatent period on infected intermediate snails by lower water temperature).

Keywords A mathematical model, Schistosoma japonicum transmission, Dynamics,
Globally stable, Periodic orbits

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1 Introduction

Schistosoma japonicum causes schistosomiasis which is one of the most prevalent parasitic diseases in the tropical and subtropical regions of the developing nations. In China, an estimated 843 011 people were infected with Schistosoma japonicum in 2003 (see [16]), and schistosomiasis still remains a major public health problem despite the remarkable achievements in schistosomiasis control over the past five decades. Thus, controlling schistosomiasis is a long-term task in the tropical and subtropical regions of the developing nations, and mathematical modeling of Schistosoma japonicum transmission can aid in the development of new strategies for control.

The first mathematical models for schistosomiasis were those developed by Macdonald in [11] and Hairston in [8]. Since then, a number of mathematical models have been developed by a variety of approaches, which made contributions to the understanding of the interplay of biology, transmission dynamics and control of Schistosomiasis (see e.g. [1–3, 5, 6, 10, 12, 14, 15], etc.). In these classic publications, there was a mathematical model given by Barbour in [3]. The model tracks dynamics of infected human population and infected snails in a community.

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For simplicity, he assumed that the total populations of both human and snails are constants without recruitment and death, and let $I_h(t)$ and $I_s(t)$ denote the numbers of infected humans and snails at time t , respectively. Barbour modeled the schistosomiasis transmission in the community by two ordinary differential equations as follows

$$\begin{aligned}\frac{dI_h}{dt} &= \alpha I_s(1 - I_h) - rI_h, \\ \frac{dI_s}{dt} &= \beta I_h(1 - I_s) - d_s I_s,\end{aligned}\tag{1.1}$$

where α (β) is the per capita rate of infection of human (snail, respectively) by one infected snail (man, respectively), r is the per capita rate of recovery in human and d_s is the per capita death rate of infected snails. This model played an important role in epidemiology for evaluating possible control strategies. However, it is known that there are incubations of schistosoma. The aim of this paper is to incorporate effects of the prepatent periods of the different stages of schistosoma into Barbour's model, and estimate the impact of prepatent periods on the schistosomiasis transmission in the community.

Note that an infected snail can not infect susceptible man (or an animal) directly and vice versa. Schistosomiasis are transmitted indirectly between the definitive hosts and intermediate snails in the sense that free-swimming stages (cercariae and miracidia) are interposed. Cercariae emerging from the infected snail are capable of infecting susceptible definitive hosts (human or animals) and miracidia hatching from parasite eggs in feces of infected definitive hosts are capable of infecting susceptible snails. Figure 1 gives a schematic description of the transmission of schistosome japonicum in definitive host (such as human, bovines) and intermediate snail. The parasite eggs hatch into free-swimming larva called miracidia in water, the miracidium then penetrate an appropriate snail within one or two days at suitable temperature. In the infected snail, the miracidium undergoes asexual multiplication through a series of stages called sporocysts, then produces in large number of free-swimming larvae called cercariae. There is data which shows that the shortest incubation of cercarial production in snail was 17–19 days at temperature 30, 31 and 32 Celsius degrees and cercarial development required at least 106–113 days at temperature 18 Celsius degrees (see [13]). Cercariae are shed from the snail and penetrate the skin of a definitive host (such as human) in water within approximately two days. After penetration, the schistosome worm migrates through the hosts circulatory system to the liver where they mate and start laying eggs within 23 to 35 days. The eggs infiltrate through the tissues and are passed in the feces. That finishes schistosomiasis life cycle. From the life cycle, we can see that the developmental times (or prepatent periods) of the different stages of schistosoma is very important for Schistosome japonicum transmission. In this paper, we incorporate effects of the developmental times of the different stages of schistosoma into the model (1.1) and propose a generalized Barbour's model which is a system of four delay differential equations. we study dynamics of the system and obtain the basic reproductive number. When the basic reproductive number is greater than one, the system has an endemic equilibrium. Some conditions are given for global stability or local stability of the endemic equilibrium. And it is shown that the system can undergo Hopf bifurcation and a periodic orbit emerge in the small neighborhood of the endemic equilibrium if the delays take some values, and the basic reproductive number decreases if the prepatent periods on infected hosts or snails are prolonged. This implies that the dynamics of the system depends on the delays,

which is different from the conclusions in [14].

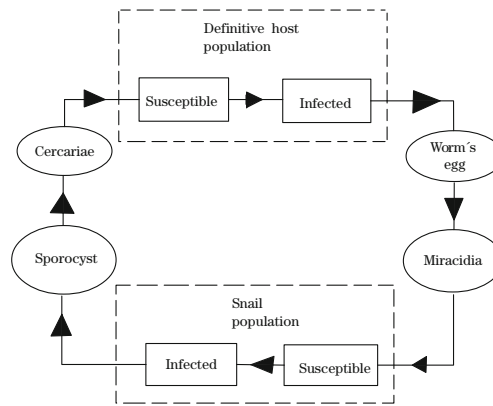


Figure 1 A transmission diagram of *Schistosoma japonicum*

This paper is organized as follows. In Section 2, after stating some assumptions we formulate a mathematical model of the transmission dynamics of *Schistosoma japonicum* with prepatent periods. The qualitative analysis and numerical simulations of the model are presented in Section 3. The paper ends with a brief discussion.

2 Model Formulation

In this section, we incorporate the effect of the developmental times (or prepatent periods) of the different stages of schistosoma into model (1.1), and consider that the numbers of definitive hosts and the intermediate snails are not constants in a community. This approach leads to a system of four delay differential equations.

Consider a relatively isolated community where there are not immigration or emigration, each group of definitive hosts (human or animals) may be infected by *Schistosoma japonicum* in stationary environmental conditions. As we know, a real-world environment is clearly non-stationary, and would include seasonal and weather variations in snail population and contact patterns. Hence, “stationary environmental conditions” implies that we have made the assumption that snail populations and infection rates in the community are independence of environmental factor for simplifying model.

Adapting Barbour’s idea, we divide the definitive hosts population (e.g. human) and the intermediate snails population in the community into two disjoint classes: susceptible (H , S) and infected (I_h , I_s), respectively. Suppose that the infection in the definitive host or intermediate snail does not result death or isolation directly and all newborns are susceptible. For the transmission of the pathogen, it is assumed that a susceptible host can receive the infection only by contacting with water in which there exist cercariae from infected snails, and a susceptible snail can receive the infection only from miracidia hatching from parasite eggs in feces of infected hosts. Assume that the transit time from cercaria in water to schistosomule in host is τ_1 and the transit time from parasite eggs to miracidia to infect snail is τ_3 . It is known that the transit times are very short, i.e., τ_1 and τ_3 are very small. On the other hand, a susceptible host becomes infection for some time and then excretes faeces with parasite eggs,

and a susceptible snail becomes infection for some time and then release cercariae. Assume that the prepatent period in host and snail has duration τ_2 and τ_4 , respectively. It is possible that some hosts (or snail) die due to natural death during this incubation period, respectively. Thus, of those hosts (snails) after τ_2 (τ_4 , respectively) unit times, only $H(t-\tau_2)e^{-d_h\tau_2}$ ($S(t-\tau_4)e^{-d_s\tau_4}$, respectively) is left at the present time t , where d_h (d_s) is the per capita natural death rate of the definitive hosts (intermediate snail, respectively). Then the dynamics of the definitive hosts population and the intermediate snails population in the community is formulated by the following system:

$$\begin{aligned}\frac{dH}{dt} &= \lambda_h - d_h H + r I_h - \alpha I_s(t - \tau_1) H(t - \tau_2) e^{-d_h \tau_2}, \\ \frac{dI_h}{dt} &= \alpha I_s(t - \tau_1) H(t - \tau_2) e^{-d_h \tau_2} - (d_h + r) I_h, \\ \frac{dS}{dt} &= \lambda_s - d_s S - \beta I_h(t - \tau_3) S(t - \tau_4) e^{-d_s \tau_4}, \\ \frac{dI_s}{dt} &= \beta I_h(t - \tau_3) S(t - \tau_4) e^{-d_s \tau_4} - d_s I_s,\end{aligned}\tag{2.1}$$

where $H(t)$ ($S(t)$) is the numbers of susceptible hosts (snails, respectively) and $I_h(t)$ ($I_s(t)$) is the numbers of infected hosts (snails, respectively) at time t in the community. λ_h (λ_s) is the recruitment rate of hosts (snails, respectively), d_h (d_s) is the per capita natural death rate of the definitive hosts (intermediate snail, respectively), r is the per capita rate of recovery in hosts, α is the per capita rate of infection of hosts by cercaria released by a infected snail, β is the per capita rate of infection of snails by miracidia from the parasite eggs from a infected host, and τ_i ($i = 1, 2, 3, 4$) are transit times or prepatent periods described as above.

From biological view, we assume that system (2.1) holds for the time $t > 0$ with given nonnegative initial conditions:

$$\begin{aligned}H(t) &\geq 0, \quad \text{on } [-\tau_2, 0]; & S(t) &\geq 0, \quad \text{on } [-\tau_4, 0]; \\ I_h(t) &\geq 0, \quad \text{on } [-\tau_3, 0]; & I_s(t) &\geq 0, \quad \text{on } [-\tau_1, 0].\end{aligned}\tag{2.2}$$

By qualitative analysis and standard results of functional differential equations in [9], we can see that solutions to system (2.1) with initial conditions (2.2) exist and are unique, and $H(t) \geq 0$, $S(t) \geq 0$, $I_h(t) \geq 0$ and $I_s(t) \geq 0$ for all $t \geq 0$. In the following, we focus on dynamics of system (2.1) in a nonnegative cone

$$D = \{(H(t), I_h(t), S(t), I_s(t)) : H(t) \geq 0, I_h(t) \geq 0, S(t) \geq 0, I_s(t) \geq 0 \text{ for } t \geq 0\}.$$

3 Dynamics of the Model

In this section, we study the dynamics of system (2.1) with conditions (2.2) in the nonnegative cone D for three cases: without all delays, without prepatent period from infected host and without prepatent period from infected snail, discuss the existence and stability of nonnegative equilibria and periodic orbits, and give the basic reproductive number which is an important parameter in the transmission of infectious diseases.

When the infective hosts and the infective snails do not exist, i.e., $I_h = I_s = 0$, then $H = \frac{\lambda_h}{d_h}$ and $S = \frac{\lambda_s}{d_s}$. This is the infection free equilibrium $E_0 = (\frac{\lambda_h}{d_h}, 0, \frac{\lambda_s}{d_s}, 0)$ for schistosomiasis. The

following theorem determines linear stability of E_0 and existence of endemic equilibrium in terms of a threshold parameter

$$R_0 = \frac{\alpha\beta\lambda_h\lambda_s}{(r+d_h)d_hd_s^2e^{d_h\tau_2+d_s\tau_4}}.$$

Theorem 3.1 *If $R_0 \leq 1$, then system (2.1) has a unique equilibrium $E_0 = (\frac{\lambda_h}{d_h}, 0, \frac{\lambda_s}{d_s}, 0)$, and E_0 is linear stable if $R_0 < 1$. If $R_0 > 1$, then system (2.1) has an endemic equilibrium $E_1 = (H^*, I_h^*, S^*, I_s^*)$ except the disease free equilibrium E_0 , where*

$$\begin{aligned} H^* &= \frac{\lambda_h}{d_h} - \frac{\alpha\beta\lambda_h\lambda_s - (r+d_h)d_hd_s^2e^{d_h\tau_2+d_s\tau_4}}{\beta d_h[\alpha\lambda_s + (r+d_h)d_se^{d_h\tau_2}]}, \\ I_h^* &= \frac{\alpha\beta\lambda_h\lambda_s - (r+d_h)d_hd_s^2e^{d_h\tau_2+d_s\tau_4}}{\beta d_h[\alpha\lambda_s + (r+d_h)d_se^{d_h\tau_2}]}, \\ S^* &= \frac{\alpha\lambda_sd_h + (r+d_h)d_hd_se^{d_h\tau_2}}{\alpha(\beta\lambda_he^{-d_s\tau_4} + d_hd_s)}, \\ I_s^* &= \frac{\alpha\beta\lambda_s\lambda_he^{-d_s\tau_4} - (r+d_h)d_hd_s^2e^{d_h\tau_2}}{\alpha d_s(\beta\lambda_he^{-d_s\tau_4} + d_hd_s)}. \end{aligned} \quad (3.1)$$

Proof Computing the nonnegative solutions of the following equations:

$$\begin{aligned} \lambda_h - d_hH + rI_h - \alpha I_s H e^{-d_h\tau_2} &= 0, \\ \alpha I_s H e^{-d_h\tau_2} - (d_h + r)I_h &= 0, \\ \lambda_s - d_sS - \beta I_h S e^{-d_s\tau_4} &= 0, \\ \beta I_h S e^{-d_s\tau_4} - d_sI_s &= 0, \end{aligned} \quad (3.2)$$

we can easily obtain the existence of two equilibria E_0 and E_1 .

The standard approach to studying linear stability of an equilibrium for (2.1) is to compute the linearized operator of (2.1) at the equilibrium and to study the eigenvalues of the operator. The equilibrium is linear stable if all eigenvalues of the operator have negative real parts.

We now calculate the associated characteristic equation of operator of system (2.1) at E_0 and obtain

$$(\lambda + d_h)(\lambda + d_s)[\lambda^2 + \delta_1\lambda + \delta_2 + \delta_3e^{-\lambda\tau}] = 0, \quad (3.3)$$

where $\tau = \tau_1 + \tau_3$, $\delta_1 = d_h + r + d_s$, $\delta_2 = (d_h + r)d_s$ and $\delta_3 = -\frac{\alpha\beta\lambda_h\lambda_s}{d_hd_s}e^{-d_h\tau_2-d_s\tau_4}$. It is obvious that $\lambda_1 = -d_h$ and $\lambda_2 = -d_s$ are two negative characteristic roots of (3.3). Hence, we only need to discuss the roots of the following equation:

$$\lambda^2 + \delta_1\lambda + \delta_2 + \delta_3e^{-\lambda\tau} = 0. \quad (3.4)$$

Let

$$F(\lambda, \tau) = \lambda^2 + \delta_1\lambda + \delta_2 + \delta_3e^{-\lambda\tau}.$$

Then $F(0, \tau) = \delta_2 + \delta_3 = (d_h + r)d_s(1 - R_0) \geq 0$ and

$$\frac{\partial F(\lambda, \tau)}{\partial \lambda} = 2\lambda + d_h + r + d_s + (\tau_1 + \tau_3)\frac{\alpha\beta\lambda_h\lambda_s}{d_hd_s}e^{-d_h\tau_2-d_s\tau_4}e^{-\lambda(\tau_1+\tau_3)} > 0$$

for $\tau \geq 0$ and $\lambda \geq 0$. Thus, (3.4) has no positive root for all positive τ .

Note that all characteristic roots of (3.4) are negative if $\tau = 0$ and $R_0 < 1$. We further claim that any root of (3.4) must have negative real part for all $\tau > 0$ as $R_0 < 1$. Assume that there exists a $\tau_0 > 0$ such that (3.4) has pure imaginary roots $\lambda = \pm i\omega$ ($\omega > 0$). Then we have from (3.4) that

$$\begin{cases} \delta_3 \cos \omega \tau_0 = \omega^2 - \delta_2, \\ \delta_3 \sin \omega \tau_0 = \delta_1 \omega. \end{cases}$$

Adding up the squares of both equations, we obtain

$$\omega^4 + (\delta_1^2 - 2\delta_2)\omega^2 + \delta_2^2 - \delta_3^2 = 0. \quad (3.5)$$

By calculation, we have

$$\delta_1^2 - 2\delta_2 = (d_h + r)^2 + d_s^2 > 0$$

and

$$\delta_2 - \delta_3 = (d_h + r)d_s + \frac{\alpha\beta\lambda_h\lambda_s}{d_h d_s} e^{-d_h\tau_2 - d_s\tau_4} > 0.$$

Thus, $\delta_2^2 - \delta_3^2 \geq 0$, which implies that (3.5) has no positive roots, i.e., τ_0 does not exist. This yields that all roots of (3.4) have negative real parts if $R_0 < 1$. We complete the proof.

According to definition of the basic reproductive number in [7], we can see that R_0 is a basic reproductive number of system (2.1). From Theorem 3.1, we can see that the dynamics of system (2.1) is interesting if $R_0 > 1$. Adding the first two equations of system (2.1), we obtain

$$\frac{d(H + I_h)}{dt} = \lambda_h - d_h(H + I_h).$$

We conclude that $H + I_h = \frac{\lambda_h}{d_h}$ is an invariant attracting manifold of system (2.1) for all $t \geq 0$. Similarly, $S + I_s = \frac{\lambda_s}{d_s}$ is also an invariant attracting manifold of system (2.1) for all $t \geq 0$ by adding the last two equations of system (2.1). Therefore, system (2.1) can be reduced to

$$\begin{aligned} \frac{dI_h}{dt} &= \alpha I_s(t - \tau_1) \left(\frac{\lambda_h}{d_h} - I_h(t - \tau_2) \right) e^{-d_h\tau_2} - (d_h + r)I_h, \\ \frac{dI_s}{dt} &= \beta I_h(t - \tau_3) \left(\frac{\lambda_s}{d_s} - I_s(t - \tau_4) \right) e^{-d_s\tau_4} - d_s I_s. \end{aligned} \quad (3.6)$$

We are interested in what the simplified two dimensional system (3.6) in

$$\overline{D} = \{(I_h(t), I_s(t)) : I_h(t) \geq 0, I_s(t) \geq 0 \text{ for } t \geq 0\}$$

predicts as the long-term dynamics. It is clear that system (3.6) has two equilibria $\overline{E}_0 = (0, 0)$ and $\overline{E}_1 = (I_h^*, I_s^*)$ if $R_0 > 1$. To study the stability of \overline{E}_1 , we calculate the associated characteristic equation of linear operator of system (3.6) at $\overline{E}_1 = (I_h^*, I_s^*)$ and obtain

$$\begin{aligned} &\lambda^2 + (d_h + r + d_s)\lambda + (d_h + r)d_s + \beta I_h^* e^{-d_s\tau_4} (\lambda + d_h + r) e^{-\lambda\tau_4} \\ &+ \alpha I_s^* e^{-d_h\tau_2} (\lambda + d_s) e^{-\lambda\tau_2} + \alpha \beta I_h^* I_s^* e^{-d_h\tau_2 - d_s\tau_4} e^{-\lambda(\tau_2 + \tau_4)} \\ &- \alpha \beta \left(\frac{\lambda_h}{d_h} - I_h^* \right) \left(\frac{\lambda_s}{d_s} - I_s^* \right) e^{-d_h\tau_2 - d_s\tau_4} e^{-\lambda(\tau_1 + \tau_3)} = 0. \end{aligned} \quad (3.7)$$

It is a challenge to compute the roots of (3.7) for all τ_i ($i = 1, 2, 3, 4$). We now study equation (3.7) and the dynamics of system (3.6) in three cases.

3.1 Dynamics of system (3.6) without delays

When $\tau_i = 0$ ($i = 1, 2, 3, 4$) equation (3.7) becomes

$$\begin{aligned} & \lambda^2 + (d_h + r + d_s)\lambda + (d_h + r)d_s + \beta I_h^*(\lambda + d_h + r) + \alpha I_s^*(\lambda + d_s) \\ & + \alpha \beta I_h^* I_s^* - \alpha \beta \left(\frac{\lambda_h}{d_h} - I_h^* \right) \left(\frac{\lambda_s}{d_s} - I_s^* \right) = 0. \end{aligned} \quad (3.8)$$

By a tedious calculation, we can see that two roots of (3.8) have negative real parts if $R_0 > 1$. Thus, equilibrium \bar{E}_1 is locally stable if $R_0 > 1$.

On the other hand, the solutions of system (3.6) are ultimately bounded in the nonnegative quadrate \bar{D} if $\tau_i = 0$ ($i = 1, 2, 3, 4$). By qualitative analysis, we can see that $0 \leq I_h(t) \leq \frac{\lambda_h}{d_h}$ and $0 \leq I_s(t) \leq \frac{\lambda_s}{d_s}$ as $t \geq t_0$ for some nonnegative t_0 .

Theorem 3.2 Assume $\tau_i = 0$ ($i = 1, 2, 3, 4$). Then the equilibrium \bar{E}_0 of system (3.6) is globally stable in \bar{D} if $R_0 < 1$, and the equilibrium \bar{E}_1 of system (3.6) is globally stable in the interior of \bar{D} if $R_0 > 1$ (see Figure 2).

Proof The first assertion follows from the fact that the solutions of system (3.6) are ultimately bounded in \bar{D} and Theorem 3.1.

It is easy to check that equilibrium \bar{E}_0 of system (3.6) is a saddle if $R_0 > 1$. By analysis above, we know that equilibrium \bar{E}_1 is locally stable if $R_0 > 1$ and all solutions of system (3.6) are ultimately bounded in \bar{D} . To prove the second assertion, we only prove that system (3.6) has not periodic orbits in the interior of \bar{D} if $R_0 > 1$. Since the divergence of (3.6) in the interior of \bar{D} is

$$-(d_h + r + d_s + \alpha I_s(t) + \beta I_h(t)) \leq 0,$$

which leads to the nonexistence of periodic orbits in the interior of \bar{D} for system (3.6) by Bendixson theorem, therefore, the proof is completed.

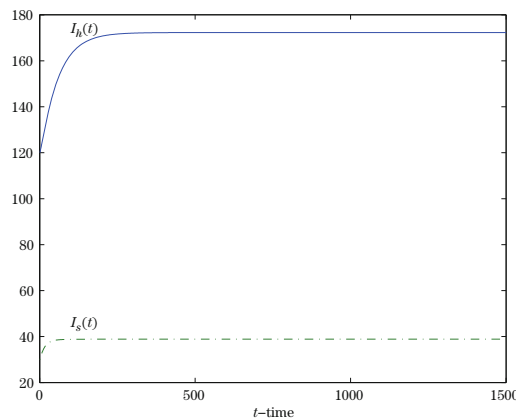


Figure 2 Global stability of \bar{E}_1 for system (3.6) as $\lambda_h = 6$, $d_h = 0.03$, $r = 0.02$, $\alpha = 0.008$, $\lambda_s = 2$, $d_s = 0.05$, $\beta = 0.01$ and $\tau_i = 0$ ($i = 1, 2, 3, 4$).

3.2 Dynamics of system (3.6) without prepatent period on infected snail

Assume that infected snail has no prepatent period. Then $\tau_4 = 0$. We further assume that $\tau_1 + \tau_3 = \tau_2 = \tau$. Hence, equation (3.7) becomes

$$P(\lambda, \tau) + Q(\lambda, \tau)e^{-\lambda\tau} = 0, \quad (3.9)$$

where

$$\begin{aligned} P(\lambda, \tau) &= \lambda^2 + A_1\lambda + A_2, \\ Q(\lambda, \tau) &= A_3e^{-d_h\tau}\lambda + A_4e^{-d_h\tau}. \end{aligned}$$

Here $A_1 = d_h + r + d_s + \beta I_h^*$, $A_2 = (d_h + r)(d_s + \beta I_h^*)$, $A_3 = \alpha I_s^*$ and $A_4 = \alpha d_s I_s^* + \frac{\alpha\beta\lambda_h I_s^*}{d_h} + \frac{\alpha\beta\lambda_s I_h^*}{d_s} - \frac{\alpha\beta\lambda_h\lambda_s}{d_h d_s}$.

When $\tau = 0$, equation (3.9) becomes equation (3.8). All roots of equation (3.9) have negative real parts if $R_0 > 1$ by Theorem 3.2. Note that zero is not a root of (3.8) for all positive τ . In the following, we study whether there exists a pair of purely imaginary roots $\lambda = \pm i\omega$ ($\omega > 0$) of (3.9) for some positive τ . Following a geometrical criterion in [4], to warrant the existence of purely imaginary roots, we need to check some conditions as follows:

- (i) $F(\omega, \tau) = |P(i\omega, \tau)|^2 - |Q(i\omega, \tau)|^2$ has at most a finite number of real zeros on ω ;
- (ii) Each positive root $\omega(\tau)$ of $F(\omega, \tau) = 0$ is continuous and differentiable in τ whenever it exists.

By calculation, we have

$$F(\omega, \tau) = \omega^4 + a_1(\tau)\omega^2 + a_2(\tau),$$

where $a_1(\tau) = A_1^2 - 2A_2 - (A_3e^{-d_h\tau})^2$ and $a_2(\tau) = A_2^2 - (A_4e^{-d_h\tau})^2$. It is obvious that condition (i) holds, and condition (ii) also holds by continuous differentiability of $F(\omega, \tau)$ and Implicit Function Theorem.

Suppose that $\lambda = i\omega$ ($\omega > 0$) is a root of (3.9). Substituting it into (3.9) and separating the real and imaginary parts yield

$$\begin{cases} A_4e^{-d_h\tau} \cos \omega\tau + A_3e^{-d_h\tau} \omega \sin \omega\tau = \omega^2 - A_2, \\ A_3e^{-d_h\tau} \omega \cos \omega\tau - A_4e^{-d_h\tau} \sin \omega\tau = -A_1\omega. \end{cases} \quad (3.10)$$

From (3.10), it follows that

$$\begin{aligned} \sin \omega\tau &= \frac{\omega(A_3e^{-d_h\tau}\omega^2 + A_1A_4e^{-d_h\tau} - A_2A_3e^{-d_h\tau})}{A_3^2e^{-2d_h\tau}\omega^2 + A_4^2e^{-2d_h\tau}}, \\ \cos \omega\tau &= \frac{(A_4e^{-d_h\tau} - A_1A_3e^{-d_h\tau})\omega^2 - A_2A_4e^{-d_h\tau}}{A_3^2e^{-2d_h\tau}\omega^2 + A_4^2e^{-2d_h\tau}}. \end{aligned} \quad (3.11)$$

We rewrite (3.11) into

$$\sin \omega\tau = \operatorname{Im}\left(\frac{P(i\omega, \tau)}{Q(i\omega, \tau)}\right) \quad \text{and} \quad \cos \omega\tau = -\operatorname{Re}\left(\frac{P(i\omega, \tau)}{Q(i\omega, \tau)}\right).$$

Hence, if ω satisfies (3.10), then $\omega(\tau)$ must be a solution to

$$|P(i\omega, \tau)|^2 - |Q(i\omega, \tau)|^2 = \omega^4 + a_1(\tau)\omega^2 + a_2(\tau) = 0, \quad (3.12)$$

which is given by

$$\begin{aligned}\omega_+^2(\tau) &= \frac{-a_1(\tau) + \sqrt{a_1^2(\tau) - 4a_2(\tau)}}{2}, \\ \omega_-^2(\tau) &= \frac{-a_1(\tau) - \sqrt{a_1^2(\tau) - 4a_2(\tau)}}{2}.\end{aligned}\tag{3.13}$$

By calculation, we obtain that $a_1(\tau) > 0$ and $a_2(\tau) > 0$ if $d_h + r > \frac{\alpha\lambda_s}{d_s}$. Hence, equation (3.12) has no positive solution if $d_h + r > \frac{\alpha\lambda_s}{d_s}$, which leads to the fact that (3.9) has not any pure imaginary roots for all $\tau > 0$. Therefore, \overline{E}_1 is asymptotic stability.

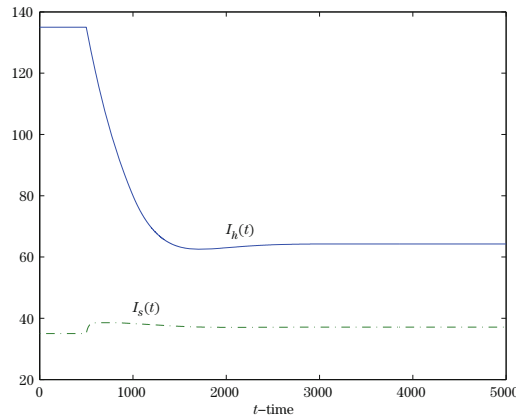


Figure 3 Asymptotic stability of \overline{E}_1 for system (3.6) as $\lambda_h = 6$, $d_h = 0.03$, $r = 0.02$, $\alpha = 0.001$, $\lambda_s = 2$, $d_s = 0.05$, $\beta = 0.01$, $\tau_1 = 6$, $\tau_2 = 15$, $\tau_3 = 9$ and $\tau_4 = 0$.

On the other hand, if $a_2(\tau) < 0$, then (3.12) has a unique positive solution. If $a_1^2(\tau) - 4a_2(\tau) \geq 0$ and $a_1(\tau) < 0$, then (3.12) has at least one positive root.

Suppose that $I \subseteq \mathbb{R}_{0+}$ is the set such that $\omega(\tau)$ is a positive solution of (3.11) for $\tau \in I$. For any $\tau \in I$, we define the angle $\theta(\tau) \in [0, 2\pi]$ such that $\sin \theta(\tau)$ and $\cos \theta(\tau)$ are given by the right-hand sides of (3.11), respectively. And the relation between the argument $\theta(\tau)$ and $\omega(\tau)\tau$ for $\tau \in I$ must be

$$\omega(\tau)\tau = \theta(\tau) + 2n\pi, \quad n \in \mathbb{N}.$$

Hence we can define the maps $\tau_n : I \rightarrow \mathbb{R}_{0+}$ given by

$$\tau_n(\tau) = \frac{\theta(\tau) + 2n\pi}{\omega(\tau)}, \quad n \in \mathbb{N}, \tau \in I,$$

where $\omega(\tau)$ is a positive solution of (3.12). Let us introduce the functions $S_n(\tau) : I \rightarrow \mathbb{R}$,

$$S_n(\tau) = \tau - \tau_n(\tau), \quad n \in \mathbb{N}, \tau \in I,$$

which is continuous and differentiable in τ .

Following the theorem in [4], we have the lemma below.

Lemma 3.1 Assume that $\omega(\tau)$ is a positive solution to (3.12) defined on $\tau \in I$, $I \subseteq \mathbb{R}_{0+}$, and there exists some $\tau^* \in I$ such that $S_n(\tau^*) = 0$ for some $n \in \mathbb{N}$. Then equation (3.9) has a

pair of conjugate pure imaginary roots $\lambda = \pm i\omega(\tau^*)$ at $\tau = \tau^*$, and equation (3.9) has a complex solution $\omega(\tau)$ with positive (negative) real part as $\tau > \tau^*$ if $\delta(\tau^*) > 0$ ($\delta(\tau^*) < 0$, respectively), where

$$\delta(\tau^*) = \text{sign}\{F'_\omega(\omega(\tau^*), \tau^*)\} \text{sign}\left\{\left.\frac{dS_n(\tau)}{d\tau}\right|_{\tau=\tau^*}\right\}.$$

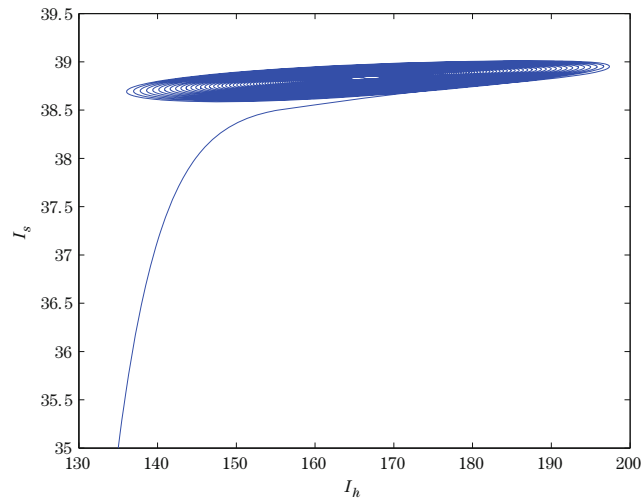


Figure 4 Asymptotic stability of \overline{E}_1 for system (3.6) as $\lambda_h = 6$, $d_h = 0.03$, $r = 0.02$, $\alpha = 0.008$, $\lambda_s = 2$, $d_s = 0.05$, $\beta = 0.01$, $\tau_1 = 4.2$, $\tau_2 = 7.2$, $\tau_3 = 3$ and $\tau_4 = 0$, here $\tau^* = 7.301$.

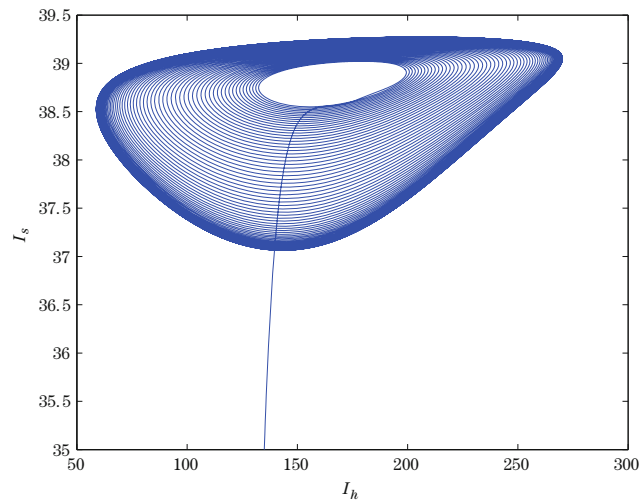


Figure 5 System (3.6) undergoes Hopf bifurcation and a stable periodic orbit emerges as $\lambda_h = 6$, $d_h = 0.03$, $r = 0.02$, $\alpha = 0.008$, $\lambda_s = 2$, $d_s = 0.05$, $\beta = 0.01$, $\tau_1 = 3$, $\tau_2 = 7.5$, $\tau_3 = 4.5$ and $\tau_4 = 0$, here $\tau^* = 7.301$.

Summarizing above discussion, we have the following conclusion by Lemma 3.1 and Hopf bifurcation theorem in [9].

Theorem 3.3 Assume that $R_0 > 1$, $\tau_4 = 0$ and $\tau_1 + \tau_3 = \tau_2$. Then

- (i) the endemic equilibrium \bar{E}_1 of system (3.6) is asymptotically stable for all $\tau \geq 0$ if $d_h + r > \frac{\alpha\lambda_s}{d_s}$; the numerical simulation is provided in Figure 3;
- (ii) there exists a τ^* such that the endemic equilibrium \bar{E}_1 of system (3.6) is asymptotically stable for $0 \leq \tau < \tau^*$, and when $\tau > \tau^*$ system (3.6) undergoes Hopf bifurcation and a stable periodic orbit emerges in the small neighborhood of \bar{E}_1 if either $a_2(\tau) < 0$ or $a_1^2(\tau) - 4a_2(\tau) \geq 0$ and $a_1(\tau) < 0$; the numerical simulations are provided in Figure 4 and Figure 5.

3.3 Dynamics of system (3.6) without prepatent period on infected host

Assume that infected host has no prepatent period. Then $\tau_2 = 0$. We further assume $\tau_1 + \tau_3 = \tau_4 = \tau$. Hence, equation (3.7) becomes

$$P_1(\lambda, \tau) + Q_1(\lambda, \tau)e^{-\lambda\tau} = 0, \quad (3.14)$$

where

$$\begin{aligned} P_1(\lambda, \tau) &= \lambda^2 + B_1\lambda + B_2, \\ Q_1(\lambda, \tau) &= B_3e^{-d_s\tau}\lambda + B_4e^{-d_s\tau}, \end{aligned}$$

where $B_1 = d_h + r + d_s + \alpha I_s^*$, $B_2 = d_s(d_h + r + \alpha I_s^*)$, $B_3 = \beta I_h^*$ and $B_4 = \beta I_h^*(d_h + r) + \frac{\alpha\beta\lambda_h I_s^*}{d_h} + \frac{\alpha\beta\lambda_h I_h^*}{d_s} - \frac{\alpha\beta\lambda_h \lambda_s}{d_h d_s}$.

Similarly to the analysis in Subsection 3.2, we can see that when $\tau = 0$, equation (3.14) becomes equation (3.8). All roots of equation (3.14) have negative real parts if $R_0 > 1$ by Theorem 3.2. Note that zero is not a root of (3.8) for all positive τ .

Let

$$F(\omega, \tau) = |P_1(i\omega, \tau)|^2 - |Q_1(i\omega, \tau)|^2.$$

Then, by calculation, we have

$$F(\omega, \tau) = \omega^4 + \bar{a}_1(\tau)\omega^2 + \bar{a}_2(\tau),$$

where $\bar{a}_1(\tau) = B_1^2 - 2B_2 - (B_3e^{-d_s\tau})^2$ and $\bar{a}_2(\tau) = B_2^2 - (B_4e^{-d_s\tau})^2$.

Suppose that $\lambda = i\omega$ ($\omega > 0$) is a solution to (3.14), then we have

$$\begin{cases} B_4e^{-d_s\tau} \cos \omega\tau + B_3e^{-d_s\tau} \omega \sin \omega\tau = \omega^2 - B_2, \\ B_3e^{-d_s\tau} \omega \cos \omega\tau - B_4e^{-d_s\tau} \sin \omega\tau = -B_1\omega. \end{cases} \quad (3.15)$$

From (3.15), it follows that

$$\begin{aligned} \sin \omega\tau &= \frac{\omega(B_3e^{-d_s\tau}\omega^2 + B_1B_4e^{-d_s\tau} - B_2B_3e^{-d_s\tau})}{B_3^2e^{-2d_s\tau}\omega^2 + B_4^2e^{-2d_s\tau}}, \\ \cos \omega\tau &= \frac{(B_4e^{-d_s\tau} - B_1B_3e^{-d_s\tau})\omega^2 - B_2B_4e^{-d_s\tau}}{B_3^2e^{-2d_s\tau}\omega^2 + B_4^2e^{-2d_s\tau}}. \end{aligned} \quad (3.16)$$

Thus,

$$\sin \omega\tau = \operatorname{Im}\left(\frac{P_1(i\omega, \tau)}{Q_1(i\omega, \tau)}\right) \quad \text{and} \quad \cos \omega\tau = -\operatorname{Re}\left(\frac{P_1(i\omega, \tau)}{Q_1(i\omega, \tau)}\right).$$

Suppose that $\omega(\tau)$ is a solution to (3.15), then $\omega(\tau)$ must satisfy

$$|P_1(i\omega, \tau)|^2 - |Q_1(i\omega, \tau)|^2 = \omega^4 + \bar{a}_1(\tau)\omega^2 + \bar{a}_2(\tau) = 0. \quad (3.17)$$

The solutions of (3.17) are given by

$$\begin{aligned} \omega_+^2(\tau) &= \frac{-\bar{a}_1(\tau) + \sqrt{\bar{a}_1^2(\tau) - 4\bar{a}_2(\tau)}}{2}, \\ \omega_-^2(\tau) &= \frac{-\bar{a}_1(\tau) - \sqrt{\bar{a}_1^2(\tau) - 4\bar{a}_2(\tau)}}{2}. \end{aligned} \quad (3.18)$$

Using the similar arguments in Subsection 3.2, we obtain the following theorem.

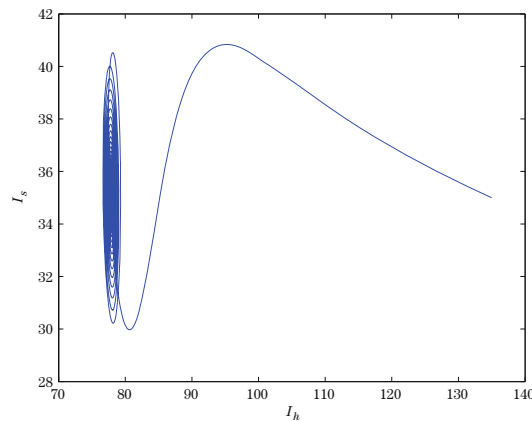


Figure 6 Asymptotic stability of \bar{E}_1 for system (3.6) as $\lambda_h = 3$, $d_h = 0.03$, $r = 0.02$, $\alpha = 0.005$, $\lambda_s = 2$, $d_s = 0.05$, $\beta = 0.006$, $\tau_1 = 1.8$, $\tau_2 = 0$, $\tau_3 = 2.7$ and $\tau_4 = 4.5$, here $\tau^* = 4.7578$.

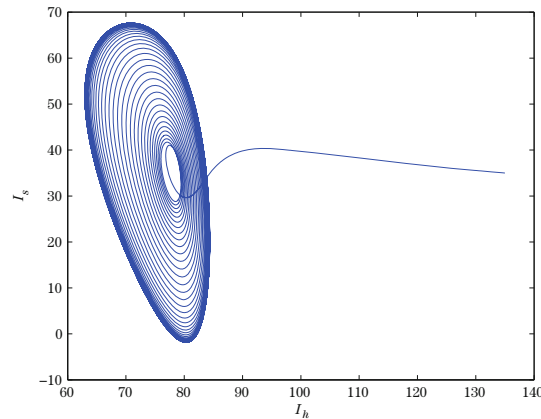


Figure 7 System (3.6) undergoes Hopf bifurcation and a periodic orbit emerges as $\lambda_h = 3$, $d_h = 0.03$, $r = 0.02$, $\alpha = 0.005$, $\lambda_s = 2$, $d_s = 0.05$, $\beta = 0.006$, $\tau_1 = 3$, $\tau_2 = 0$, $\tau_3 = 2.1$ and $\tau_4 = 5.1$, here $\tau^* = 4.7578$.

Theorem 3.4 Assume that $R_0 > 1$, $\tau_2 = 0$ and $\tau_1 + \tau_3 = \tau_4$. Then

(i) the endemic equilibrium \bar{E}_1 of system (3.6) is asymptotically stable for all $\tau \geq 0$ if $\bar{a}_1(\tau) > 0$ and $\bar{a}_2(\tau) > 0$;

(ii) there exists a positive τ^* such that the endemic equilibrium \bar{E}_1 of system (3.6) is asymptotically stable for $0 \leq \tau < \tau^*$, and when $\tau > \tau^*$ system (3.6) undergoes Hopf bifurcation and a stable periodic orbit emerges in the small neighborhood of \bar{E}_1 if either $\bar{a}_2(\tau) < 0$ or $\bar{a}_1^2(\tau) - 4\bar{a}_2(\tau) \geq 0$ and $\bar{a}_1(\tau) < 0$; the numerical simulations are provided in Figure 6 and Figure 7.

4 Discussions

In this paper, we propose a system of delay differential equations as a generalized Barbour's model for schistosomiasis japonicum transmission. The model takes into account the prepatent periods for the transmission of infection between the definitive hosts and the intermediate snails. It is shown that the system has only the infection free equilibrium which is stable if the basic reproductive number R_0 is less than one, and the system has an endemic equilibrium if the basic reproductive number R_0 is greater than one. Some sufficient conditions are given for the asymptotical stable of the endemic equilibrium. Bifurcation analysis indicates that the system can undergo Hopf bifurcation and a periodic orbit emerges in the small neighborhood of the endemic equilibrium if the delays take some values, and the basic reproductive number decreases if the prepatent periods on infected hosts or snails are prolonged. This implies that delays affect the dynamics of the system, which is different to the conclusions in [14]. Our results suggest that it is an effective strategy on schistosomiasis control to lengthen in prepatent period on infected definitive hosts by drug treatment (or lengthen in prepatent period on infected intermediate snails by lower water temperature).

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