Chin. Ann. Math.
29B(6), 2008, 687–698
DOI: 10.1007/s11401-007-0234-8

Chinese Annals of Mathematics, Series B

A Remark on Chen's Theorem (II)**

Yingchun CAI*

Abstract Let p denote a prime and P_2 denote an almost prime with at most two prime factors. The author proves that for sufficiently large x, $\sum\limits_{\substack{p\le x\\ p+2=P_2}} 1 > \frac{1.13Cx}{\log^2 x}$, where the constant 1.13 constitutes an improvement of the previous result 1.104 due to J. Wu.

Keywords Chen's theorem, Sieve, Mean value theorem
2000 MR Subject Classification 11N36

1 Introduction

Let p, p' denote primes and P_2 denote an almost prime with at most two prime factors. For sufficiently large x, it is conjectured by Hardy and Littlewood [9] that

$$\sum_{\substack{p\le x\\ p+2=p'}} 1 = (1+o(1))\frac{Cx}{\log^2 x},$$

where

$$C = 2\prod_{p>2}\Big(1-\frac{1}{(p-1)^2}\Big).$$

This conjecture still remains open. The best result in this aspect is due to J. R. Chen [2] who showed in 1973 that

$$\pi_{1,2}(x) > \frac{0.335Cx}{\log^2 x},$$

where

$$\pi_{1,2}(x) = \sum_{\substack{p\le x\\ p+2=P_2}} 1.$$

The constant 0.335 was improved successively to

$$0.3445,\ 0.3772,\ 0.405,\ 0.71,\ 1.015,\ 1.05,\ 1.0974,\ 1.104$$

by Halberstam [7], J. R. Chen [3, 4], Fouvry and Grupp [5], H. Q. Liu [12], J. Wu [14], Y. C. Cai [1] and J. Wu [15] respectively.

In this paper, we obtain the following sharper result.

Manuscript received June 1, 2007. Published online November 5, 2008.
*Department of Mathematics, Tongji University, Shanghai 200092, China.
E-mail: yingchuncai@mail.tongji.edu.cn
**Project supported by the National Natural Science Foundation of China (Nos. 10171060, 10171076, 10471104).

Theorem 1.1

$$\pi_{1,2}(x) > \frac{1.13Cx}{\log^2 x}.$$

2 Some Lemmas

Lemma 2.1 (see [5]) *For $\varepsilon > 0$, let $Q = x^{\frac{4}{7}-\varepsilon}$, and $\lambda(\cdot)$ denote a well-factorable function of level Q. Then, for any given $A > 0$ and $|a| \le \log^A x$, we have*

$$\sum_{(q,a)=1} \lambda(q)\Big(\pi(x;q,a) - \frac{\mathrm{Li}x}{\varphi(q)}\Big) = O_{A,\varepsilon,a}\Big(\frac{x}{\log^A x}\Big).$$

Lemma 2.2 (see [15]) *Let (α_m) and (β_n) be two sequences satisfying the following conditions:*

(A_1) *$M \ge x^\varepsilon$, $\alpha_m = 0$ for $m \notin [M, 2M]$, $|\alpha_m| \le \tau_k(m)$;*

(A_2) *$N \ge x^\varepsilon$, $\beta_n = 0$ for $n \notin [N, 2N]$, $|\beta_n| \le \tau_k(n)$;*

(A_3) *For any given $e \ge 1$, $d \ge 1$, $(d,l) = 1$, $A > 0$,*

$$\sum_{\substack{n\equiv l(d)\\(n,e)=1}} \beta_n = \frac{1}{\varphi(d)} \sum_{(n,de)=1} \beta_n + O\Big(\frac{N\tau(e)^B}{\log^A N}\Big);$$

(A_4) *If $p|n \to p < \exp(\log^{\frac{1}{2}} x)$, then $\beta_n = 0$,*

where k and B are constants. Let $MN \le x$, $v = \frac{\log N}{\log x}$ and $Q = x^{\theta(v)-2\varepsilon}$, where $\theta(v)$ is defined by

$$\theta(v) = \begin{cases} \dfrac{6-5v}{10}, & 0 < v \le \dfrac{1}{15}, \\ \dfrac{1+2v}{2}, & \dfrac{1}{15} \le v \le \dfrac{1}{10}, \\ \dfrac{5-2v}{8}, & \dfrac{1}{10} \le v \le \dfrac{3}{14}, \\ \dfrac{3+2v}{6}, & \dfrac{3}{14} \le v \le \dfrac{1}{4}, \\ \dfrac{2-v}{3}, & \dfrac{1}{4} \le v \le \dfrac{2}{7}, \\ \dfrac{2+v}{4}, & \dfrac{2}{7} \le v \le \dfrac{2}{5}, \\ 1-v, & \dfrac{2}{5} \le v \le \dfrac{1}{2}; \\ \dfrac{1}{2}, & \dfrac{1}{2} \le v < 1. \end{cases}$$

Then for any $A > 0$ and $|a| \le \log^A x$,

$$\sum_{(q,a)=1} \lambda(q)\Big(\sum_{mn\equiv a(q)} \alpha_m\beta_n - \frac{1}{\varphi(q)} \sum_{(mn,q)=1} \alpha_m\beta_n\Big) = O_{A,\varepsilon,k,B}\Big(\frac{x}{\log^A x}\Big).$$

Lemma 2.3 (see [6]) *Let $\xi(\cdot)$ denote an arithmetical function such that*

$$|\xi(q)| \le \log x, \quad \xi(q) = 0 \quad \text{for } q > Q_1.$$

Then

$$\sum_{(qq_1,a)=1} \lambda(q)\xi(q_1)\Big(\pi(x;qq_1,a)-\frac{\mathrm{Li}x}{\varphi(qq_1)}\Big)=O_{A,\varepsilon,a}\Big(\frac{x}{\log^A x}\Big)$$

if either

(1) $Q_1 \le Q,\ Q_1Q \le x^{\frac{4}{7}-\varepsilon}$, *or*
(2) $Q_1 \ge Q,\ Q_1^6Q \le x^{2-\varepsilon}$, *or*
(3) $\xi(q)=\Lambda(q),\ Q_1Q \le x^{\frac{11}{20}-\varepsilon},\ Q_1 \le x^{\frac{1}{3}-\varepsilon}$.

Lemma 2.4 (see [6]) *Let $\eta>0$ and define*

$$g(t)=\begin{cases}\dfrac{4}{7}, & 0\le t\le \dfrac{2}{7}-\eta,\\ \dfrac{11}{20}, & \dfrac{2}{7}-\eta\le t\le \dfrac{1}{3}-\eta,\\ \dfrac{1}{2}, & \dfrac{1}{3}-\eta\le t\le \dfrac{1}{2}-\eta.\end{cases}$$

Then, for any $A>0$, $\varepsilon>0$ and $|a|\le \log^A x$, we have

$$\sum_{x^t\le p<2x^t}\ \sum_{(q,a)=1} \lambda(q)\Big(\pi(x;pq,a)-\frac{\mathrm{Li}x}{\varphi(pq)}\Big)=O_{A,k,a}\Big(\frac{x}{\log^A x}\Big),$$

where $Q=x^{g(t)-t-\varepsilon}$.

Lemma 2.5 (see [11, 13]) *Let*

$$x>1,\quad z=x^{\frac{1}{u}},\quad Q(z)=\prod_{p<z}p.$$

Then, for $u\ge u_0>1$, we have

$$\sum_{\substack{n\le x\\ (n,Q(z))=1}} 1=w(u)\frac{x}{\log z}+O\Big(\frac{x}{\log^2 z}\Big),$$

where $w(u)$ is determined by a differential-difference equation and

$$\begin{cases} w(u)<\dfrac{1}{1.763}, & u\ge 2,\\ w(u)<0.5644, & u\ge 3.\end{cases}$$

3 Weighted Sieve Method

Let x be a sufficiently large real number and put

$$\mathcal{A}=\{a\mid a=p+2,\ p\le x\}, \tag{3.1}$$

$$\mathcal{P}=\{p\mid p>2\}. \tag{3.2}$$

Lemma 3.1 *Let $0 < \alpha < \beta \le \frac{1}{3}$. Then*

$$\pi_{1,2}(x) \ge S(\mathcal{A}, x^{\alpha}) - \frac{1}{2} \sum_{x^{\alpha} \le p < x^{\beta}} S(\mathcal{A}_p, x^{\alpha}) - \frac{1}{2} \sum_{x^{\alpha} \le p_1 < x^{\beta} \le p_2 < (\frac{x}{p_1})^{\frac{1}{2}}} S(\mathcal{A}_{p_1 p_2}; \mathcal{P}(p_1), p_2)$$
$$- \sum_{x^{\beta} \le p_1 < p_2 < (\frac{x}{p_1})^{\frac{1}{2}}} S(\mathcal{A}_{p_1 p_2}; \mathcal{P}(p_1), p_2)$$
$$+ \frac{1}{2} \sum_{x^{\alpha} \le p_1 < p_2 < p_3 < x^{\beta}} S(\mathcal{A}_{p_1 p_2 p_3}; \mathcal{P}(p_1), p_2) + O(x^{1-\alpha}).$$

Proof By the trivial inequality

$$\pi_{1,2}(x) \ge S(\mathcal{A}, x^{\beta}) - \sum_{x^{\beta} \le p_1 < p_2 < (\frac{x}{p_1})^{\frac{1}{2}}} S(\mathcal{A}_{p_1 p_2}; \mathcal{P}(p_1), p_2)$$

and Buchstab's identity, we have

$$\begin{aligned} \pi_{1,2}(x) &\ge S(\mathcal{A}, x^{\beta}) - \sum_{x^{\beta} \le p_1 < p_2 < (\frac{x}{p_1})^{\frac{1}{2}}} S(\mathcal{A}_{p_1 p_2}; \mathcal{P}(p_1), p_2) \\ &= S(\mathcal{A}, x^{\alpha}) - \sum_{x^{\alpha} \le p < x^{\beta}} S(\mathcal{A}_p, x^{\alpha}) + \sum_{x^{\alpha} \le p_1 < p_2 < x^{\beta}} S(\mathcal{A}_{p_1 p_2}, p_1) \\ &\quad - \sum_{x^{\beta} \le p_1 < p_2 < (\frac{x}{p_1})^{\frac{1}{2}}} S(\mathcal{A}_{p_1 p_2}; \mathcal{P}(p_1), p_2). \end{aligned} \tag{3.3}$$

On the other hand, we have the trivial inequality

$$\begin{aligned} \pi_{1,2}(x) \ge S(\mathcal{A}, x^{\alpha}) &- \sum_{x^{\alpha} \le p_1 < p_2 < x^{\beta}} S(\mathcal{A}_{p_1 p_2}; \mathcal{P}(p_1), p_2) - \sum_{x^{\alpha} \le p_1 < x^{\beta} \le p_2 < (\frac{x}{p_1})^{\frac{1}{2}}} S(\mathcal{A}_{p_1 p_2}; \mathcal{P}(p_1), p_2) \\ &- \sum_{x^{\beta} \le p_1 < p_2 < (\frac{x}{p_1})^{\frac{1}{2}}} S(\mathcal{A}_{p_1 p_2}; \mathcal{P}(p_1), p_2). \end{aligned} \tag{3.4}$$

Now by Buchstab's identity we have

$$\begin{aligned} &\sum_{x^{\alpha} \le p_1 < p_2 < x^{\beta}} S(\mathcal{A}_{p_1 p_2}, p_1) - \sum_{x^{\alpha} \le p_1 < p_2 < x^{\beta}} S(\mathcal{A}_{p_1 p_2}; \mathcal{P}(p_1), p_2) \\ &= \sum_{x^{\alpha} \le p_1 < p_2 < p_3 < x^{\beta}} S(\mathcal{A}_{p_1 p_2 p_3}; \mathcal{P}(p_1), p_2) + \sum_{x^{\alpha} \le p_1 < p_2 < x^{\beta}} S(\mathcal{A}_{p_1^2 p_2}, p_1) \\ &= \sum_{x^{\alpha} \le p_1 < p_2 < p_3 < x^{\beta}} S(\mathcal{A}_{p_1 p_2 p_3}; \mathcal{P}(p_1), p_2) + O(x^{1-\alpha}). \end{aligned} \tag{3.5}$$

Now we add (3.3) and (3.4) and by (3.5), Lemma 3.1 follows.

Lemma 3.2

$$2\pi_{1,2}(x) \ge \frac{3}{2} S(\mathcal{A}, x^{\frac{1}{12}}) + \frac{1}{2} S(\mathcal{A}, x^{\frac{1}{7.2}}) + \frac{1}{2} \sum_{x^{\frac{1}{12}} \le p_1 < p_2 < x^{\frac{1}{7.2}}} S(\mathcal{A}_{p_1 p_2}, x^{\frac{1}{12}})$$
$$+ \frac{1}{2} \sum_{x^{\frac{1}{12}} \le p_1 < x^{\frac{1}{7.2}} \le p_2 < \min(x^{\frac{1}{3.5}}, x^{\frac{17}{42}} p_1^{-1})} S(\mathcal{A}_{p_1 p_2}, x^{\frac{1}{12}})$$

$$
\begin{aligned}
&- \frac{1}{2} \sum_{x^{\frac{1}{12}} \le p < x^{\frac{1}{3}}} S(\mathcal{A}_p, x^{\frac{1}{12}}) - \frac{1}{2} \sum_{x^{\frac{1}{12}} \le p < x^{\frac{1}{3.5}}} S(\mathcal{A}_p, x^{\frac{1}{12}}) \\
&- \frac{1}{2} \sum_{x^{\frac{1}{12}} \le p_1 < x^{\frac{1}{3}} \le p_2 < (\frac{x}{p_1})^{\frac{1}{2}}} S(\mathcal{A}_{p_1p_2}; \mathcal{P}(p_1), p_2) \\
&- \frac{1}{2} \sum_{x^{\frac{1}{7.2}} \le p_1 < x^{\frac{1}{3.5}} \le p_2 < (\frac{x}{p_1})^{\frac{1}{2}}} S\Big(\mathcal{A}_{p_1p_2}; \mathcal{P}(p_1), \Big(\frac{x}{p_1p_2}\Big)^{\frac{1}{2}}\Big) \\
&- \sum_{x^{\frac{1}{3.5}} \le p_1 < p_2 < (\frac{x}{p_1})^{\frac{1}{2}}} S(\mathcal{A}_{p_1p_2}; \mathcal{P}(p_1), p_2) \\
&- \frac{1}{2} \sum_{x^{\frac{1}{12}} \le p_1 < p_2 < p_3 < p_4 < x^{\frac{1}{7.2}}} S(\mathcal{A}_{p_1p_2p_3p_4}; \mathcal{P}(p_1), p_2) \\
&- \frac{1}{2} \sum_{x^{\frac{1}{12}} \le p_1 < p_2 < p_3 < x^{\frac{1}{7.2}} \le p_4 < \min(x^{\frac{1}{3.5}}, x^{\frac{17}{42}} p_3^{-1})} S(\mathcal{A}_{p_1p_2p_3p_4}; \mathcal{P}(p_1), p_2) + O(x^{\frac{11}{12}}) \\
= &\frac{1}{2}(3S_{11} + S_{12}) + \frac{1}{2}(S_{21} + S_{22}) - \frac{1}{2}(S_{31} + S_{32}) - \frac{1}{2}(S_{41} + S_{42}) \\
&- S_5 - \frac{1}{2}(S_{61} + S_{62}) + O(x^{\frac{11}{12}}) \\
= &\frac{1}{2}S_1 + \frac{1}{2}S_2 - \frac{1}{2}S_3 - \frac{1}{2}S_4 - S_5 - \frac{1}{2}S_6 + O(x^{\frac{11}{12}}).
\end{aligned}
$$

Proof By Buchstab's identity, we have

$$
\begin{aligned}
\frac{1}{2}S(\mathcal{A}, x^{\frac{1}{7.2}}) = &\frac{1}{2}S(\mathcal{A}, x^{\frac{1}{12}}) - \frac{1}{2} \sum_{x^{\frac{1}{12}} \le p < x^{\frac{1}{7.2}}} S(\mathcal{A}_p, x^{\frac{1}{12}}) + \frac{1}{2} \sum_{x^{\frac{1}{12}} \le p_1 < p_2 < x^{\frac{1}{7.2}}} S(\mathcal{A}_{p_1p_2}, x^{\frac{1}{12}}) \\
&- \frac{1}{2} \sum_{x^{\frac{1}{12}} \le p_1 < p_2 < p_3 < x^{\frac{1}{7.2}}} S(\mathcal{A}_{p_1p_2p_3}, p_1),
\end{aligned} \tag{3.6}
$$

$$
\begin{aligned}
\sum_{x^{\frac{1}{7.2}} \le p < x^{\frac{1}{3.5}}} S(\mathcal{A}_p, x^{\frac{1}{7.2}}) \le &\sum_{x^{\frac{1}{7.2}} \le p < x^{\frac{1}{3.5}}} S(\mathcal{A}_p, x^{\frac{1}{12}}) \\
&- \sum_{x^{\frac{1}{12}} \le p_1 < x^{\frac{1}{7.2}} \le p_2 < \min(x^{\frac{1}{3.5}}, x^{\frac{17}{42}} p_1^{-1})} S(\mathcal{A}_{p_1p_2}, x^{\frac{1}{12}}) \\
&+ \sum_{x^{\frac{1}{12}} \le p_1 < p_2 < x^{\frac{1}{7.2}} \le p_3 < \min(x^{\frac{1}{3.5}}, x^{\frac{17}{42}} p_2^{-1})} S(\mathcal{A}_{p_1p_2p_3}, p_1),
\end{aligned} \tag{3.7}
$$

$$
\begin{aligned}
\sum_{x^{\frac{1}{7.2}} \le p_1 < x^{\frac{1}{3.5}} \le p_2 < (\frac{x}{p_1})^{\frac{1}{2}}} S(\mathcal{A}_{p_1p_2}; \mathcal{P}(p_1), p_2) = &\sum_{x^{\frac{1}{7.2}} \le p_1 < x^{\frac{1}{3.5}} \le p_2 < (\frac{x}{p_1})^{\frac{1}{3}}} S(\mathcal{A}_{p_1p_2}; \mathcal{P}(p_1), p_2) \\
&+ \sum_{\substack{x^{\frac{1}{7.2}} \le p_1 < x^{\frac{1}{3.5}} \\ (\frac{x}{p_1})^{\frac{1}{3}} \le p_2 < (\frac{x}{p_1})^{\frac{1}{2}}}} S(\mathcal{A}_{p_1p_2}; \mathcal{P}(p_1), p_2).
\end{aligned} \tag{3.8}
$$

If $p_2 \le (\frac{x}{p_1})^{\frac{1}{3}}$, then $p_2 \le (\frac{x}{p_1p_2})^{\frac{1}{2}}$ and by Buchstab's identity, we have

$$\begin{aligned}
&\sum_{x^{\frac{1}{7.2}} \le p_1 < x^{\frac{1}{3.5}} \le p_2 < (\frac{x}{p_1})^{\frac{1}{3}}} S(\mathcal{A}_{p_1p_2}; \mathcal{P}(p_1), p_2) \\
&= \sum_{x^{\frac{1}{7.2}} \le p_1 < x^{\frac{1}{3.5}} \le p_2 < (\frac{x}{p_1})^{\frac{1}{3}}} S\Big(\mathcal{A}_{p_1p_2}; \mathcal{P}(p_1p_2), \Big(\frac{x}{p_1p_2}\Big)^{\frac{1}{2}}\Big) \\
&\quad + \sum_{x^{\frac{1}{7.2}} \le p_1 < x^{\frac{1}{3.5}} \le p_2 \le p_3 < (\frac{x}{p_1p_2})^{\frac{1}{2}}} S(\mathcal{A}_{p_1p_2p_3}; \mathcal{P}(p_1p_2), p_3). \qquad (3.9)
\end{aligned}$$

On the other hand, if $p_2 \ge (\frac{x}{p_1})^{\frac{1}{3}}$, then $p_2 \ge (\frac{x}{p_1p_2})^{\frac{1}{2}}$ and we have

$$\begin{aligned}
&\sum_{\substack{x^{\frac{1}{7.2}} \le p_1 < x^{\frac{1}{3.5}} \\ (\frac{x}{p_1})^{\frac{1}{3}} \le p_2 < (\frac{x}{p_1})^{\frac{1}{2}}}} S(\mathcal{A}_{p_1p_2}; \mathcal{P}(p_1), p_2) \\
&\le \sum_{\substack{x^{\frac{1}{7.2}} \le p_1 < x^{\frac{1}{3.5}} \\ (\frac{x}{p_1})^{\frac{1}{3}} \le p_2 < (\frac{x}{p_1})^{\frac{1}{2}}}} S\Big(\mathcal{A}_{p_1p_2}; \mathcal{P}(p_1p_2), \Big(\frac{x}{p_1p_2}\Big)^{\frac{1}{2}}\Big). \qquad (3.10)
\end{aligned}$$

By (3.8)–(3.10), we get

$$\begin{aligned}
&\sum_{x^{\frac{1}{7.2}} \le p_1 < x^{\frac{1}{3.5}} \le p_2 < (\frac{x}{p_1})^{\frac{1}{2}}} S(\mathcal{A}_{p_1p_2}; \mathcal{P}(p_1), p_2) \\
&\le \sum_{x^{\frac{1}{7.2}} \le p_1 < x^{\frac{1}{3.5}} \le p_2 < (\frac{x}{p_1})^{\frac{1}{2}}} S\Big(\mathcal{A}_{p_1p_2}; \mathcal{P}(p_1p_2), \Big(\frac{x}{p_1p_2}\Big)^{\frac{1}{2}}\Big) \\
&\quad + \sum_{x^{\frac{1}{7.2}} \le p_1 < x^{\frac{1}{3.5}} \le p_2 < p_3 < (\frac{x}{p_1p_2})^{\frac{1}{2}}} S(\mathcal{A}_{p_1p_2p_3}; \mathcal{P}(p_1p_2), p_3). \qquad (3.11)
\end{aligned}$$

Now by Buchstab's identity we have

$$\begin{aligned}
&\sum_{x^{\frac{1}{12}} \le p_1 < p_2 < p_3 < x^{\frac{1}{3}}} S(\mathcal{A}_{p_1p_2p_3}; \mathcal{P}(p_1), p_2) - \sum_{x^{\frac{1}{12}} \le p_1 < p_2 < p_3 < x^{\frac{1}{7.2}}} S(\mathcal{A}_{p_1p_2p_3}, p_1) \\
&- \sum_{x^{\frac{1}{12}} \le p_1 < p_2 < x^{\frac{1}{7.2}} \le p_3 < \min(x^{\frac{1}{3.5}}, x^{\frac{17}{42}} p_2^{-1})} S(\mathcal{A}_{p_1p_2p_3}, p_1) \\
&- \sum_{x^{\frac{1}{7.2}} \le p_1 < x^{\frac{1}{3.5}} \le p_2 \le p_3 < (\frac{x}{p_1p_2})^{\frac{1}{2}}} S(\mathcal{A}_{p_1p_2p_3}; \mathcal{P}(p_1p_2), p_3) \\
&\ge - \sum_{x^{\frac{1}{12}} \le p_1 < p_2 < p_3 < p_4 < x^{\frac{1}{7.2}}} S(\mathcal{A}_{p_1p_2p_3}; \mathcal{P}(p_1), p_2) \\
&\quad - \sum_{x^{\frac{1}{12}} \le p_1 < p_2 < p_3 < x^{\frac{1}{7.2}} \le p_4 < \min(x^{\frac{1}{3.5}}, x^{\frac{17}{42}} p_3^{-1})} S(\mathcal{A}_{p_1p_2p_3}; \mathcal{P}(p_1), p_2) + O(x^{\frac{11}{12}}). \qquad (3.12)
\end{aligned}$$

Now by Lemma 3.1 with $(\alpha, \beta) = (\frac{1}{12}, \frac{1}{3})$ and $(\alpha, \beta) = (\frac{1}{7.2}, \frac{1}{3.5})$ and (3.6), (3.7), (3.11) and (3.12), we complete the proof of Lemma 3.2.

4 Proof of Theorem 1.1

In this section, the sets $\mathcal{A}$ and $\mathcal{P}$ are defined by (3.1) and (3.2) respectively.

4.1 Evaluation of S_1 and S_2

Let $Q = x^{\frac{4}{7}-\varepsilon}$. By Lemma 2.1 and the sieve theory with bilinear error term in [10], we get

$$S_{11} \geq 3.5(1+O(\varepsilon))\frac{Cx}{\log^2 x}\Big(\log\frac{41}{7} + \int_2^{\frac{34}{7}} \frac{\log(s-1)}{s}\log\frac{\frac{41}{7}}{s+1}\mathrm{d}s\Big) \geq 6.73740\frac{Cx}{\log^2 x},$$
$$S_{12} \geq 3.5(1+O(\varepsilon))\frac{Cx}{\log^2 x}\Big(\log\frac{21.8}{7} + \int_2^{\frac{14.8}{7}} \frac{\log(s-1)}{s}\log\frac{\frac{21.8}{7}}{s+1}\mathrm{d}s\Big) \geq 3.97613\frac{Cx}{\log^2 x}.$$

Then

$$S_1 = 3S_{11} + S_{12} \geq 24.18833\frac{Cx}{\log^2 x}. \tag{4.1}$$

Let λ_1' and λ_2' denote the characteristic functions of the primes in the intervals $[L_1, L_1')$ and $[L_2, L_2')$ respectively, where $x^{\frac{1}{12}} \leq L_1 < L_1' \leq 2L_1 < x^{\frac{1}{7.2}}$, $x^{\frac{1}{7.2}} \leq L_2 < L_2' \leq 2L_2 < \min(x^{\frac{1}{3.5}-\varepsilon}, x^{\frac{17}{42}-\varepsilon}(2L_1)^{-1})$, and λ denote a well-factorable function of level $Q(L_1L_2)^{-1}$. Then $L_1' < Q(L_1L_2)^{-1}$, $L_2' < QL_2^{-1}$. Thus $\lambda \star \lambda_1'$ is a well-factorable function of level QL_2^{-1}, and $(\lambda\star\lambda_1')\star\lambda_2'$ is a well-factorable function of level Q. By Lemma 2.1 and the bilinear sieve theory in [10], we get

$$\begin{aligned} S_{22} &\geq \sum_{x^{\frac{1}{12}} \leq p_1 < x^{\frac{1}{7.2}} \leq p_2 < \min(x^{\frac{1}{3.5}-\varepsilon}, x^{\frac{17}{42}-\varepsilon}p_1^{-1})} S(\mathcal{A}_{p_1p_2}, x^{\frac{1}{12}}) \\ &\geq (1+O(\varepsilon))\frac{3.5Cx}{\log^2 x}\int_{\frac{1}{12}}^{\frac{1}{7.2}}\int_{\frac{1}{7.2}}^{\min(\frac{1}{3.5},\frac{17}{42}-t_1)} \frac{\log(\frac{41}{7}-12(t_1+t_2))}{t_1t_2(1-1.75(t_1+t_2))}\mathrm{d}t_1\mathrm{d}t_2. \end{aligned}$$

Similarly,

$$S_{21} \geq (1+O(\varepsilon))\frac{3.5Cx}{\log^2 x}\int_{\frac{1}{12}}^{\frac{1}{7.2}}\int_{t_1}^{\frac{1}{7.2}} \frac{\log(\frac{41}{7}-12(t_1+t_2))}{t_1t_2(1-1.75(t_1+t_2))}\mathrm{d}t_1\mathrm{d}t_2.$$

Then

$$\begin{aligned} S_2 &= S_{21} + S_{22} \\ &\geq (1+O(\varepsilon))\frac{3.5Cx}{\log^2 x}\int_{\frac{1}{12}}^{\frac{1}{7.2}}\int_{t_1}^{\min(\frac{1}{3.5},\frac{17}{42}-t_1)} \frac{\log(\frac{41}{7}-12(t_1+t_2))}{t_1t_2(1-1.75(t_1+t_2))}\mathrm{d}t_1\mathrm{d}t_2 \\ &\geq 2.83084\frac{Cx}{\log^2 x}. \end{aligned} \tag{4.2}$$

4.2 Evaluation of S_3

We have

$$\begin{aligned} S_{31} &= \Big(\sum_{x^{\frac{1}{12}} \leq p < x^{\frac{2}{7}-\varepsilon}} + \sum_{x^{\frac{2}{7}-\varepsilon} \leq p < x^{0.29}}\Big)S(\mathcal{A}_p, x^{\frac{1}{12}}) + \Big(\sum_{x^{0.29} \leq p < x^{\frac{1}{3}-\varepsilon}} + \sum_{x^{\frac{1}{3}-\varepsilon} \leq p \leq x^{\frac{1}{3}}}\Big)S(\mathcal{A}_p, x^{\frac{1}{12}}) \\ &= \Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4. \end{aligned} \tag{4.3}$$

By Lemma 2.1 and the arguments used in [14], we get

$$\begin{aligned}\Sigma_1 \le{}& 3.5(1+O(\varepsilon))\frac{Cx}{\log^2 x}\Bigg(\Big(1+\int_2^{\frac{17}{7}}\frac{\log(s-1)}{s}\mathrm{d}s\Big)\log\frac{12-1.75}{3.5-1.75}\\ &+\int_{\frac{17}{7}}^{\frac{34}{7}}\frac{\log(s-1)}{s}\log\frac{\frac{41}{7}(\frac{41}{7}-s)}{s+1}\mathrm{d}s\\ &+\int_2^{\frac{20}{7}}\frac{\log(s-1)}{s}\mathrm{d}s\int_{s+2}^{\frac{34}{7}}\frac{1}{t}\log\frac{t-1}{s+1}\log\frac{\frac{41}{7}(\frac{41}{7}-t)}{t+1}\mathrm{d}t\Bigg)\\ \le{}& 8.37862\frac{Cx}{\log^2 x}.\end{aligned} \tag{4.4}$$

By Lemma 2.3, Lemma 2.4 and the arguments used in [14], we have

$$\begin{aligned}\Sigma_2 &\le (1+O(\varepsilon))\frac{Cx}{\log^2 x}\Bigg(\Big(1+\int_2^{2.12}\frac{\log(s-1)}{s}\mathrm{d}s\Big)\log\frac{29}{26}+\int_{2.12}^{\frac{17}{7}}\frac{\log(s-1)}{s}\log\frac{23-s}{6(s+1)}\mathrm{d}s\Bigg)\\ &\le 0.11104\frac{Cx}{\log^2 x}, &(4.5)\\ \Sigma_3 &\le (1+O(\varepsilon))\frac{40Cx}{11\log^2 x}\Big(\log\frac{52}{37.7}+\int_2^{2.12}\frac{\log(s-1)}{s}\log\frac{26(5.6-s)}{29(s+1)}\mathrm{d}s\Big)\\ &\le 1.16970\frac{Cx}{\log^2 x}. &(4.6)\end{aligned}$$

By a trivial estimation, we have

$$\Sigma_4 = O\Big(\frac{\varepsilon Cx}{\log^2 x}\Big). \tag{4.7}$$

By (4.3)–(4.7), we get

$$\begin{aligned}S_{31} &= \Sigma_1+\Sigma_2+\Sigma_3+\Sigma_4 \le 9.65936\frac{Cx}{\log^2 x},\\ S_{32} &= \Sigma_1 + O\Big(\frac{\varepsilon Cx}{\log^2 x}\Big) \le 8.37862\frac{Cx}{\log^2 x}.\end{aligned} \tag{4.8}$$

Then

$$S_3 = S_{31}+S_{32} \le 18.03798\frac{Cx}{\log^2 x}. \tag{4.9}$$

4.3 Evaluation of S_6

By Lemma 2.2, Lemma 2.5 and the arguments used in [15], we get

$$S_{61} \le (1+O(\varepsilon))C_1\frac{Cx}{\log^2 x} \le 0.05331\frac{Cx}{\log^2 x}, \tag{4.10}$$

where

$$\begin{aligned}C_1 ={}& 4\int_{\frac{1}{12}}^{\frac{1}{10}}\frac{\mathrm{d}t_1}{t_1(1+2t_1)}\int_{t_1}^{\frac{1}{7.2}}\frac{\mathrm{d}t_2}{t_2^2}\int_{t_2}^{\frac{1}{7.2}}\frac{\mathrm{d}t_3}{t_3}\int_{t_3}^{\frac{1}{7.2}}\omega\Big(\frac{1-t_1-t_2-t_3-t_4}{t_2}\Big)\frac{\mathrm{d}t_4}{t_4}\\ &+16\int_{\frac{1}{10}}^{\frac{1}{7.2}}\frac{\mathrm{d}t_1}{t_1(5-2t_1)}\int_{t_1}^{\frac{1}{7.2}}\frac{\mathrm{d}t_2}{t_2^2}\int_{t_2}^{\frac{1}{7.2}}\frac{\mathrm{d}t_3}{t_3}\int_{t_3}^{\frac{1}{7.2}}\omega\Big(\frac{1-t_1-t_2-t_3-t_4}{t_2}\Big)\frac{\mathrm{d}t_4}{t_4}.\end{aligned}$$

By a similar method, we get

$$
\begin{aligned}
S_{62} = & \sum_{x^{\frac{1}{12}} \le p_1 < p_2 < p_3 < x^{\frac{5}{42}} < x^{\frac{1}{7.2}} \le p_4 < x^{\frac{1}{3.5}}} S(\mathcal{A}_{p_1 p_2 p_3 p_4}; \mathcal{P}(p_1), p_2) \\
& + \sum_{x^{\frac{1}{12}} \le p_1 < p_2 < x^{\frac{5}{42}} \le p_3 < x^{\frac{1}{7.2}} \le p_4 < x^{\frac{17}{42}} p_3^{-1}} S(\mathcal{A}_{p_1 p_2 p_3 p_4}; \mathcal{P}(p_1), p_2) \\
& + \sum_{x^{\frac{1}{12}} \le p_1 < x^{\frac{5}{42}} \le p_2 < p_3 < x^{\frac{1}{7.2}} \le p_4 < x^{\frac{17}{42}} p_3^{-1}} S(\mathcal{A}_{p_1 p_2 p_3 p_4}; \mathcal{P}(p_1), p_2) \\
& + \sum_{x^{\frac{5}{42}} \le p_1 < p_2 < p_3 < x^{\frac{1}{7.2}} \le p_4 < x^{\frac{17}{42}} p_3^{-1}} S(\mathcal{A}_{p_1 p_2 p_3 p_4}; \mathcal{P}(p_1), p_2) \\
= & \, S_{62}^1 + S_{62}^2 + S_{62}^3 + S_{62}^4,
\end{aligned} \tag{4.11}
$$

where

$$
\begin{aligned}
S_{62}^1 \le & (1 + O(\varepsilon)) \frac{Cx}{\log^2 x} \\
& \times \Big(4 \int_{\frac{1}{12}}^{\frac{1}{10}} \frac{\mathrm{d}t_1}{t_1(1+2t_1)} \int_{t_1}^{\frac{5}{42}} \frac{\mathrm{d}t_2}{t_2^2} \int_{t_2}^{\frac{5}{42}} \frac{\mathrm{d}t_3}{t_3} \int_{\frac{1}{7.2}}^{\frac{1}{3.5}} \omega\Big(\frac{1 - t_1 - t_2 - t_3 - t_4}{t_2} \Big) \frac{\mathrm{d}t_4}{t_4} \\
& + 16 \int_{\frac{1}{10}}^{\frac{5}{42}} \frac{\mathrm{d}t_1}{t_1(5-2t_1)} \int_{t_1}^{\frac{5}{42}} \frac{\mathrm{d}t_2}{t_2^2} \int_{t_2}^{\frac{5}{42}} \frac{\mathrm{d}t_3}{t_3} \int_{\frac{1}{7.2}}^{\frac{1}{3.5}} \omega\Big(\frac{1 - t_1 - t_2 - t_3 - t_4}{t_2} \Big) \frac{\mathrm{d}t_4}{t_4} \Big) \\
\le & \, 0.10505 \frac{Cx}{\log^2 x},
\end{aligned} \tag{4.12}
$$

$$
\begin{aligned}
S_{62}^2 \le & (1 + O(\varepsilon)) \frac{Cx}{\log^2 x} \\
& \times \Big(4 \int_{\frac{1}{12}}^{\frac{1}{10}} \frac{\mathrm{d}t_1}{t_1(1+2t_1)} \int_{t_1}^{\frac{5}{42}} \frac{\mathrm{d}t_2}{t_2^2} \int_{\frac{5}{42}}^{\frac{1}{7.2}} \frac{\mathrm{d}t_3}{t_3} \int_{\frac{1}{7.2}}^{\frac{17}{42} - t_3} \omega\Big(\frac{1 - t_1 - t_2 - t_3 - t_4}{t_2} \Big) \frac{\mathrm{d}t_4}{t_4} \\
& + 16 \int_{\frac{1}{10}}^{\frac{5}{42}} \frac{\mathrm{d}t_1}{t_1(5-2t_1)} \int_{t_1}^{\frac{5}{42}} \frac{\mathrm{d}t_2}{t_2^2} \int_{\frac{5}{42}}^{\frac{1}{7.2}} \frac{\mathrm{d}t_3}{t_3} \int_{\frac{1}{7.2}}^{\frac{17}{42} - t_3} \omega\Big(\frac{1 - t_1 - t_2 - t_3 - t_4}{t_2} \Big) \frac{\mathrm{d}t_4}{t_4} \Big) \\
\le & \, 0.12188 \frac{Cx}{\log^2 x},
\end{aligned} \tag{4.13}
$$

$$
\begin{aligned}
S_{62}^3 \le & (1 + O(\varepsilon)) \frac{Cx}{\log^2 x} \\
& \times \Big(4 \int_{\frac{1}{12}}^{\frac{1}{10}} \frac{\mathrm{d}t_1}{t_1(1+2t_1)} \int_{\frac{5}{42}}^{\frac{1}{7.2}} \frac{\mathrm{d}t_2}{t_2^2} \int_{t_2}^{\frac{1}{7.2}} \frac{\mathrm{d}t_3}{t_3} \int_{\frac{1}{7.2}}^{\frac{17}{42} - t_3} \omega\Big(\frac{1 - t_1 - t_2 - t_3 - t_4}{t_2} \Big) \frac{\mathrm{d}t_4}{t_4} \\
& + 16 \int_{\frac{1}{10}}^{\frac{5}{42}} \frac{\mathrm{d}t_1}{t_1(5-2t_1)} \int_{\frac{5}{42}}^{\frac{1}{7.2}} \frac{\mathrm{d}t_2}{t_2^2} \int_{t_2}^{\frac{1}{7.2}} \frac{\mathrm{d}t_3}{t_3} \int_{\frac{1}{7.2}}^{\frac{17}{42} - t_3} \omega\Big(\frac{1 - t_1 - t_2 - t_3 - t_4}{t_2} \Big) \frac{\mathrm{d}t_4}{t_4} \Big) \\
\le & \, 0.04359 \frac{Cx}{\log^2 x},
\end{aligned} \tag{4.14}
$$

$$
S_{62}^4 \le (1 + O(\varepsilon)) \frac{Cx}{\log^2 x}
$$

$$\times\Big(16\int_{\frac{5}{42}}^{\frac{1}{7.2}}\frac{\mathrm{d}t_1}{t_1(5-2t_1)}\int_{t_1}^{\frac{1}{7.2}}\frac{\mathrm{d}t_2}{t_2^2}\int_{t_2}^{\frac{1}{7.2}}\frac{\mathrm{d}t_3}{t_3}\int_{\frac{1}{7.2}}^{\frac{17}{42}-t_3}\omega\Big(\frac{1-t_1-t_2-t_3-t_4}{t_2}\Big)\frac{\mathrm{d}t_4}{t_4}\Big)$$
$$\le 0.00608\frac{Cx}{\log^2 x}. \tag{4.15}$$

By (4.11)–(4.15), we get

$$S_{62} \le 0.27656\frac{Cx}{\log^2 x}. \tag{4.16}$$

By (4.10) and (4.16), we obtain

$$S_6 = S_{61} + S_{62} \le 0.32987\frac{Cx}{\log^2 x}. \tag{4.17}$$

4.4 Evaluation of S_4 and S_5

By Lemma 2.2 and the arguments used in [12], we get

$$S_{41} \le (1+O(\varepsilon))C_2\frac{Cx}{\log^2 x} \le \frac{2.02916Cx}{\log^2 x}, \tag{4.18}$$

where

$$\begin{aligned}
C_2 = {} & 4\int_{\substack{\frac{1}{12}\\ \frac{1+2t_1}{2}\ge\frac{2+t_2}{4}}}^{\frac{1}{10}}\mathrm{d}t_1\int_{\frac{1}{3}}^{\frac{2}{5}}\frac{\mathrm{d}t_2}{t_1t_2(1+2t_1)(1-t_1-t_2)} + 8\int_{\substack{\frac{1}{12}\\ \frac{1+2t_1}{2}\le\frac{2+t_2}{4}}}^{\frac{1}{10}}\mathrm{d}t_1\int_{\frac{1}{3}}^{\frac{2}{5}}\frac{\mathrm{d}t_2}{t_1t_2(2+t_2)(1-t_1-t_2)}\\
& + 4\int_{\substack{\frac{1}{12}\\ \frac{1+2t_1}{2}\ge 1-t_2}}^{\frac{1}{10}}\mathrm{d}t_1\int_{\frac{2}{5}}^{\frac{1-t_1}{2}}\frac{\mathrm{d}t_2}{t_1t_2(1+2t_1)(1-t_1-t_2)} + 2\int_{\substack{\frac{1}{12}\\ \frac{1+2t_1}{2}\le 1-t_2}}^{\frac{1}{10}}\mathrm{d}t_1\int_{\frac{2}{5}}^{\frac{1-t_1}{2}}\frac{\mathrm{d}t_2}{t_1t_2(1-t_2)(1-t_1-t_2)}\\
& + 16\int_{\substack{\frac{1}{10}\\ \frac{5-2t_1}{8}\ge\frac{2+t_2}{4}}}^{\frac{1}{5}}\mathrm{d}t_1\int_{\frac{1}{3}}^{\frac{2}{5}}\frac{\mathrm{d}t_2}{t_1t_2(5-2t_1)(1-t_1-t_2)} + 8\int_{\substack{\frac{1}{10}\\ \frac{5-2t_1}{8}\le\frac{2+t_2}{4}}}^{\frac{1}{5}}\mathrm{d}t_1\int_{\frac{1}{3}}^{\frac{2}{5}}\frac{\mathrm{d}t_2}{t_1t_2(2+t_2)(1-t_1-t_2)}\\
& + 16\int_{\substack{\frac{1}{10}\\ \frac{5-2t_1}{8}\ge 1-t_2}}^{\frac{1}{5}}\mathrm{d}t_1\int_{\frac{2}{5}}^{\frac{1-t_1}{2}}\frac{\mathrm{d}t_2}{t_1t_2(5-2t_1)(1-t_1-t_2)} + 2\int_{\substack{\frac{1}{10}\\ \frac{5-2t_1}{8}\le 1-t_2}}^{\frac{1}{5}}\mathrm{d}t_1\int_{\frac{2}{5}}^{\frac{1-t_1}{2}}\frac{\mathrm{d}t_2}{t_1t_2(1-t_2)(1-t_1-t_2)}\\
& + 8\int_{\frac{1}{5}}^{\frac{3}{14}}\mathrm{d}t_1\int_{\frac{1}{3}}^{\frac{1-t_1}{2}}\frac{\mathrm{d}t_2}{t_1t_2(2+t_2)(1-t_1-t_2)} + 8\int_{\frac{3}{14}}^{\frac{1}{4}}\mathrm{d}t_1\int_{\frac{1}{3}}^{\frac{1-t_1}{2}}\frac{\mathrm{d}t_2}{t_1t_2(2+t_2)(1-t_1-t_2)}\\
& + 8\int_{\frac{1}{4}}^{\frac{2}{7}}\mathrm{d}t_1\int_{\frac{1}{3}}^{\frac{1-t_1}{2}}\frac{\mathrm{d}t_2}{t_1t_2(2+t_2)(1-t_1-t_2)} + 8\int_{\frac{1}{3}}^{\frac{2}{7}}\mathrm{d}t_1\int_{\frac{1}{3}}^{\frac{1-t_1}{2}}\frac{\mathrm{d}t_2}{t_1t_2(2+t_2)(1-t_1-t_2)}.
\end{aligned}$$

In a similarly way, we have

$$S_{42} \le (1+O(\varepsilon))C_3\frac{Cx}{\log^2 x} \le \frac{1.77427Cx}{\log^2 x}, \tag{4.19}$$

where

$$\begin{aligned}C_3 = {} & 16\int_{\substack{\frac{1}{7.2}\\ \frac{5-2t_1}{8}\ge\frac{2+t_2}{4}}}^{\frac{1}{5}} \mathrm{d}t_1 \int_{\frac{1}{3.5}}^{\frac{2}{5}} \frac{\mathrm{d}t_2}{t_1t_2(5-2t_1)(1-t_1-t_2)} + 8\int_{\substack{\frac{1}{7.2}\\ \frac{5-2t_1}{8}\le\frac{2+t_2}{4}}}^{\frac{1}{5}} \mathrm{d}t_1 \int_{\frac{1}{3.5}}^{\frac{2}{5}} \frac{\mathrm{d}t_2}{t_1t_2(2+t_2)(1-t_1-t_2)}\\
& + 16\int_{\substack{\frac{1}{7.2}\\ \frac{5-2t_1}{8}\ge 1-t_2}}^{\frac{1}{5}} \mathrm{d}t_1 \int_{\frac{2}{5}}^{\frac{1-t_1}{2}} \frac{\mathrm{d}t_2}{t_1t_2(5-2t_1)(1-t_1-t_2)} + 2\int_{\substack{\frac{1}{7.2}\\ \frac{5-2t_1}{8}\le 1-t_2}}^{\frac{1}{5}} \mathrm{d}t_1 \int_{\frac{2}{5}}^{\frac{1-t_1}{2}} \frac{\mathrm{d}t_2}{t_1t_2(1-t_2)(1-t_1-t_2)}\\
& + 16\int_{\substack{\frac{1}{5}\\ \frac{5-2t_1}{8}\ge\frac{2+t_2}{4}}}^{\frac{3}{14}} \mathrm{d}t_1 \int_{\frac{1}{3.5}}^{\frac{1-t_1}{2}} \frac{\mathrm{d}t_2}{t_1t_2(5-2t_1)(1-t_1-t_2)} + 8\int_{\substack{\frac{1}{5}\\ \frac{5-2t_1}{8}\le\frac{2+t_2}{4}}}^{\frac{3}{14}} \mathrm{d}t_1 \int_{\frac{1}{3.5}}^{\frac{1-t_1}{2}} \frac{\mathrm{d}t_2}{t_1t_2(2+t_2)(1-t_1-t_2)}\\
& + 12\int_{\substack{\frac{3}{14}\\ \frac{3+2t_1}{6}\ge\frac{2+t_2}{4}}}^{\frac{1}{4}} \mathrm{d}t_1 \int_{\frac{1}{3.5}}^{\frac{1-t_1}{2}} \frac{\mathrm{d}t_2}{t_1t_2(3+2t_1)(1-t_1-t_2)} + 8\int_{\substack{\frac{3}{14}\\ \frac{3+2t_1}{8}\le\frac{2+t_2}{4}}}^{\frac{1}{4}} \mathrm{d}t_1 \int_{\frac{1}{3.5}}^{\frac{1-t_1}{2}} \frac{\mathrm{d}t_2}{t_1t_2(2+t_2)(1-t_1-t_2)}\\
& + 6\int_{\substack{\frac{1}{4}\\ \frac{2-t_1}{3}\le\frac{2+t_2}{4}}}^{\frac{1}{3.5}} \mathrm{d}t_1 \int_{\frac{1}{3.5}}^{\frac{1-t_1}{2}} \frac{\mathrm{d}t_2}{t_1t_2(2-t_2)(1-t_1-t_2)} + 8\int_{\substack{\frac{1}{3.5}\\ \frac{2-t_1}{3}\le\frac{2+t_2}{4}}}^{\frac{1}{4}} \mathrm{d}t_1 \int_{\frac{1}{3.5}}^{\frac{1-t_1}{2}} \frac{\mathrm{d}t_2}{t_1t_2(2+t_2)(1-t_1-t_2)}.\end{aligned}$$

By (4.18) and (4.19), we get

$$S_4 = S_{41} + S_{42} \le 3.80343\frac{Cx}{\log^2 x}. \tag{4.20}$$

By Lemma 2.2 and the arguments used in [12], we get

$$S_5 \le 8(1+O(\varepsilon))\frac{Cx}{\log^2 x}\int_{\frac{2}{7}}^{\frac{1}{3}} \frac{\log\left(\frac{1}{t}-2\right)}{t(2+t)(1-t)}\mathrm{d}t \le 0.16203\frac{Cx}{\log^2 x}. \tag{4.21}$$

4.5 Proof of Theorem 1.1

By Lemma 3.2 and (4.1), (4.2), (4.9), (4.17), (4.20) and (4.21), we get

$$\begin{aligned}2\pi_{1,2}(x) &\ge \frac{1}{2}(S_1+S_2) - \frac{1}{2}(S_3+S_4) - S_5 - \frac{1}{2}S_6 + O(x^{\frac{11}{12}})\\
&\ge \Big(\frac{24.18833}{2} + \frac{2.83084}{2} - \frac{18.03798}{2} - \frac{3.80343}{2} - 0.16203 - \frac{0.32987}{2}\Big)\frac{Cx}{\log^2 x}\\
&> \frac{2.26Cx}{\log^2 x},\\
\pi_{1,2}(x) &> \frac{1.13Cx}{\log^2 x}.\end{aligned}$$

The theorem is proved.

References

[1] Cai, Y. C., A remark on Chen's theorem, *Acta Arith.*, **102**(4), 2002, 339–352.

[2] Chen, J. R., On the representation of a large even integer as the sum of a prime and the product of at most two primes, *Sci. Sin.*, **16**, 1973, 157–176.

[3] Chen, J. R., On the representation of a large even integer as the sum of a prime and the product of at most two primes (II), *Sci. Sin.*, **21**, 1978, 421–430.

[4] Chen, J. R., On the representation of a large even integer as the sum of a prime and the product of at most two primes (II) (in Chinese), *Sci. Sin.*, **21**, 1978, 477–494.

[5] Fouvry, E. and Grupp, F., On the switching principle in sieve theory, *J. Reine Angew. Math.*, **370**, 1986, 101–126.

[6] Fouvry, E. and Grupp, F., Weighted sieves and twin prime type equations, *Duke Math. J.*, **58**, 1989, 731–748.

[7] Halberstam, H., A proof of Chen's theorem, *Asterisque*, **24**–**25**, 1975, 281–293.

[8] Halberstam, H. and Richert, H. E., Sieve Methods, Academic Press, London, 1974.

[9] Hardy, G. H. and Littlewood, J. E., Some problems of 'partitio numerorum', III: On the expression of a number as a sum of primes, *Acta Math.*, **44**, 1923, 1–70.

[10] Iwaniec, H., A new form of the error term in the linear sieve, *Acta Arith.*, **114**(3), 1980, 307–320.

[11] Jia, C. H., Almost all short intervals containing prime numbers, *Acta Arith.*, **76**(1), 1996, 21–84.

[12] Liu, H. Q., On prime twins problem, *Sci. Sin.*, **10**, 1989, 1030–1045.

[13] Pan, C. D. and Pan, C. B., Goldbach Conjecture (in Chinese), Science Press, Beijing, 1981.

[14] Wu, J., Sur la suite des nombres premiers jumeaux, *Acta Arith.*, **55**, 1990, 365–394.

[15] Wu, J., Chen's double sieve, Goldbach's conjecture and the twin prime problem, *Acta Arith.*, **114**(3), 2004, 215–273.