

Laplacian on Complex Finsler Manifolds*

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Abstract In this paper, the Laplacian on the holomorphic tangent bundle $T^{1,0}M$ of a complex manifold M endowed with a strongly pseudoconvex complex Finsler metric is defined and its explicit expression is obtained by using the Chern Finsler connection associated with (M, F) . Utilizing the initiated “Bochner technique”, a vanishing theorem for vector fields on the holomorphic tangent bundle $T^{1,0}M$ is obtained.

Keywords Laplacian, Strongly pseudoconvex complex Finsler metric, Chern Finsler connection

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1 Introduction

It is well-known that the Laplacian plays an important role in the theory of harmonic integral and Bochner technique in both Riemannian and Kähler manifolds (see [1–6]). The Laplacian also makes sense in Finsler cases (see [7–14]). Let M be a complex manifold endowed with a strongly pseudoconvex Finsler metric in the sense of [15]. In [16–17], by using the complex Rund connection, Zhong defined the horizontal and vertical Laplacians in an invariant way on a strongly pseudoconvex Finsler manifold and used the horizontal Laplacian to obtain a vanishing theorem of p -forms on the base manifold M under the assumption that F is a Kähler Finsler metric on M .

As an application of the horizontal Laplacian associated with a complex Finsler manifold (M, F) , the Bochner technique (see [1–3]) or Bochner Kodaira technique (see [18–20]) has also been studied (see [21–22]). In this paper, the authors derive the Laplacian on the holomorphic tangent bundle $T^{1,0}M$ for a strongly pseudoconvex Finsler manifold in terms of the Chern-Finsler connection associated with (M, F) . Furthermore, by using the Chern-Finsler connection, the authors obtain the so-called Weitzenböck formula for the Laplacian. Finally, as an application, the authors obtain a Bochner type vanishing theorem for vector fields on the holomorphic tangent bundle $T^{1,0}M$.

2 Preliminaries

Let M be a complex manifold of complex dimension m . Denote $\pi : T^{1,0}M \rightarrow M$ the holomorphic tangent bundle of M . Note that $T^{1,0}M$ is a non-compact complex manifold, even if M is compact. For a local complex coordinate system $z = (z^1, \dots, z^m)$ on M , a holomorphic

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tangent vector v of M is written as

$$v = v^\mu \partial_\mu, \quad \partial_\mu = \frac{\partial}{\partial z^\mu}, \quad \dot{\partial}_\alpha = \frac{\partial}{\partial v^\alpha},$$

and we take $(z, v) = (z^1, \dots, z^m, v^1, \dots, v^m)$ as local holomorphic coordinate neighborhood of $T^{1,0}M$. Let $\widetilde{M} = T^{1,0}M \setminus \{0\}$ denote $T^{1,0}M$ without the zero section. $\{\partial_\mu, \dot{\partial}_\alpha\}$ gives a local holomorphic frame field of the holomorphic tangent bundle $T^{1,0}\widetilde{M}$ of \widetilde{M} . A complex Finsler metric on a complex manifold M is a continuous function $F : T^{1,0}M \rightarrow R^+$ with the following properties (see [15]):

- (i) $G = F^2$ is smooth on \widetilde{M} ;
- (ii) $F(v) > 0$ for all $v \in \widetilde{M}$ and $F(v) = 0$ for all $v = 0$;
- (iii) $F(\zeta v) = |\zeta|F(v)$ for all $v \in T^{1,0}M$ and $\zeta \in C$.

The pair (M, F) is called a complex Finsler manifold. A complex Finsler metric F is said to be strongly pseudoconvex if the Levi matrix $(G_{\alpha\bar{\beta}})$ is positive definite on \widetilde{M} , where $G_{\alpha\bar{\beta}} = \dot{\partial}_\alpha \dot{\partial}_{\bar{\beta}} G$, and the pair (M, F) is called a strongly pseudoconvex Finsler manifold.

Let $\widetilde{\pi} : T^{1,0}\widetilde{M} \rightarrow \widetilde{M}$ denote the natural projection. The differential $d\pi : T^{\mathbb{C}}\widetilde{M} \rightarrow T^{\mathbb{C}}M$ of $\pi : \widetilde{M} \rightarrow M$ defines the vertical bundle \mathcal{V} over \widetilde{M} by

$$\mathcal{V} = \text{Ker } d\pi \cap T^{1,0}\widetilde{M}, \quad (2.1)$$

which is the holomorphic vector bundle of rank m over \widetilde{M} , and $\{\dot{\partial}_\alpha\}$ gives a local frame for \mathcal{V} . As is defined in [15], there is a horizontal bundle \mathcal{H} over \widetilde{M} such that $T^{1,0}\widetilde{M} = \mathcal{V} \oplus \mathcal{H}$, and the local frame for \mathcal{H} is given by

$$\delta_\mu = \partial_\mu - \Gamma_\mu^\alpha \dot{\partial}_\alpha, \quad \Gamma_\mu^\alpha = G^{\bar{\tau}\alpha} G_{\bar{\tau};\mu}, \quad (2.2)$$

where $(G^{\bar{\tau}\alpha}) = (G_{\alpha\bar{\tau}})^{-1}$, $G_{\bar{\tau};\mu} = \dot{\partial}_{\bar{\tau}} \partial_\mu G$. Thus $\{\delta_\mu, \dot{\partial}_\alpha\}$ gives a local frame for $T^{1,0}\widetilde{M}$. Let $\{dz^\mu, \delta v^\alpha\}$ be the dual frame for $T^{1,0*}\widetilde{M}$, where $\delta v^\alpha = dv^\alpha + \Gamma_\mu^\alpha dz^\mu$. The frames $\{\delta_\mu, \dot{\partial}_\alpha\}$ and $\{dz^\mu, \delta v^\alpha\}$ are called the adapted frames for $T^{1,0}\widetilde{M}$ and $T^{1,0*}\widetilde{M}$ respectively, and the following Lie brackets hold (see [15]):

$$\begin{aligned} [\delta_\mu, \delta_\nu] &= 0, & [\delta_\mu, \dot{\partial}_\alpha] &= \Gamma_{\alpha;\mu}^\sigma \dot{\partial}_\sigma, & [\dot{\partial}_\alpha, \dot{\partial}_\beta] &= 0, \\ [\delta_\mu, \delta_{\bar{\nu}}] &= \delta_{\bar{\nu}}(\Gamma_\mu^\alpha) \dot{\partial}_\alpha - \delta_\mu(\overline{\Gamma_{\bar{\nu}}^\alpha}) \dot{\partial}_{\bar{\alpha}}, & [\delta_\mu, \dot{\partial}_{\bar{\alpha}}] &= \Gamma_{\bar{\alpha};\mu}^\sigma \dot{\partial}_\sigma, & [\dot{\partial}_\alpha, \dot{\partial}_{\bar{\beta}}] &= 0. \end{aligned} \quad (2.3)$$

For a strongly pseudoconvex Finsler metric, there is a unique Chern-Finsler connection D on $T^{1,0}\widetilde{M}$. Its connection form ω_β^α is given by

$$\omega_\beta^\alpha = G^{\bar{\tau}\alpha} \partial G_{\beta\bar{\tau}} = \Gamma_{\beta;\mu}^\alpha dz^\mu + \Gamma_{\beta\gamma}^\alpha \delta v^\gamma,$$

where $\Gamma_{\beta;\mu}^\alpha = G^{\bar{\tau}\alpha} \delta_\mu G_{\beta\bar{\tau}}$, $\Gamma_{\beta\gamma}^\alpha = G^{\bar{\tau}\alpha} \dot{\partial}_\gamma G_{\beta\bar{\tau}}$.

By defining $D(\overline{X}) = \overline{DX}$ and complex linearity, the Chern-Finsler connection D can be extended to the whole complex vector bundle $T^{\mathbb{C}}\widetilde{M}$ and its dual complex vector bundle $T^{\mathbb{C}*}\widetilde{M}$ by requiring $D\varphi(X) + \varphi(DX) = d\varphi(X)$ for every $\varphi \in \chi(T^{\mathbb{C}*}\widetilde{M})$ and $X \in \chi(T^{\mathbb{C}}\widetilde{M})$. Thus the Chern-Finsler connection can also be extended to the complex linear connection $D : \chi(T_{\mathbb{C}}^{r,s}\widetilde{M}) \rightarrow \chi(T_{\mathbb{C}}^{r,s}\widetilde{M} \otimes T_{\mathbb{C}}^{r,s}\widetilde{M})$ in the usual way. All the extended connections are still called the Chern-Finsler connection with the conjugation preserving the type. Let ∇ be the covariant differentiation defined by D . Then, according to the adapted local frame $\{\delta_\alpha, \dot{\partial}_\alpha, \delta_{\bar{\alpha}}, \dot{\partial}_{\bar{\alpha}}\}$

of $T^{\mathbb{C}}(\widetilde{M})$ and the local dual frame $\{dz^\alpha, \delta v^\alpha, d\bar{z}^\alpha, \delta\bar{v}^\alpha\}$ for $T^{\mathbb{C}*}\widetilde{M}$, we have

$$\begin{aligned} \nabla_{\delta_\alpha} dz^\beta &= -\Gamma_{\gamma;\alpha}^\beta dz^\gamma, & \nabla_{\delta_\alpha} d\bar{z}^\beta &= 0, & \nabla_{\delta_\alpha} \delta v^\beta &= -\Gamma_{\gamma;\alpha}^\beta \delta v^\gamma, & \nabla_{\delta_\alpha} \delta\bar{v}^\beta &= 0, \\ \nabla_{\dot{\partial}_\alpha} dz^\beta &= -\Gamma_{\gamma\alpha}^\beta dz^\gamma, & \nabla_{\dot{\partial}_\alpha} d\bar{z}^\beta &= 0, & \nabla_{\dot{\partial}_\alpha} \delta v^\beta &= -\Gamma_{\gamma\alpha}^\beta \delta v^\gamma, & \nabla_{\dot{\partial}_\alpha} \delta\bar{v}^\beta &= 0. \end{aligned} \quad (2.4)$$

Locally, the $(2,0)$ -torsion θ of the connection is given by

$$\theta = \theta^\sigma \otimes \delta_\sigma, \quad (2.5)$$

where $\theta^\sigma = \frac{1}{2}[\Gamma_{\mu;\nu}^\sigma - \Gamma_{\nu;\mu}^\sigma]dz^\mu \wedge dz^\nu + \Gamma_{\gamma\nu}^\sigma \delta v^\gamma \wedge dz^\nu$. The $(1,1)$ -torsion τ for the connection is given by

$$\tau = \tau^\alpha \otimes \dot{\partial}_\alpha, \quad (2.6)$$

where $\tau^\alpha = -\delta_{\bar{\nu}}(\Gamma_\mu^\alpha)dz^\mu \wedge d\bar{z}^\nu - \Gamma_{\beta;\mu}^\alpha dz^\mu \wedge \delta\bar{v}^\beta$. The curvature operator $\Omega = \Omega_\beta^\alpha \otimes [dz^\beta \otimes \delta_\alpha + \delta v^\beta \otimes \dot{\partial}_\alpha]$, where $\Omega_\beta^\alpha = \bar{\partial}\omega_\beta^\alpha$. Then for any $X, Y \in \mathcal{X}(T^{1,0}\widetilde{M})$, we have

$$\nabla_X \nabla_Y - \nabla_Y \nabla_X = \nabla_{[X,Y]}, \quad (2.7)$$

$$\nabla_X \nabla_{\bar{Y}} - \nabla_{\bar{Y}} \nabla_X = \nabla_{[X,\bar{Y}]} + \Omega(X, \bar{Y}). \quad (2.8)$$

3 Decomposition of the Exterior Derivative on \widetilde{M}

Now we consider the space

$$\mathcal{A} = \bigoplus_{p,q,r,s} \mathcal{A}^{p,q;r,s}$$

of C^∞ complex-valued forms with compact support on \widetilde{M} , which is defined by [17, 23, 24], where $\mathcal{A}^{p,q;r,s}$ is the non-zero set of $(p,q;r,s)$ -forms only when they act on p vector fields of h -type, on q -vector fields of \bar{h} -type, on r vector fields of v -type, and on s vector fields of \bar{v} -type. Let $\{dz^\alpha, \delta v^\alpha, d\bar{z}^\alpha, \delta\bar{v}^\alpha\}$ be the adapted local frame of $T^{\mathbb{C}*}\widetilde{M}$. For any $\varphi \in \mathcal{A}^{p,q;r,s}$, it can be represented by

$$\varphi = \frac{1}{p!q!r!s!} \varphi_{A_p \bar{B}_q C_r \bar{D}_s} dz^{A_p} \wedge d\bar{z}^{B_q} \wedge \delta v^{C_r} \wedge \delta\bar{v}^{D_s},$$

where $A_p = (\mu_1, \dots, \mu_p)$, $B_q = (\nu_1, \dots, \nu_q)$, $C_r = (\alpha_1, \dots, \alpha_r)$, $D_s = (\beta_1, \dots, \beta_s)$, and $dz^{A_p} = dz^{\mu_1} \wedge \dots \wedge dz^{\mu_p}$, $d\bar{z}^{B_q} = d\bar{z}^{\nu_1} \wedge \dots \wedge d\bar{z}^{\nu_q}$, $\delta v^{C_r} = \delta v^{\alpha_1} \wedge \dots \wedge \delta v^{\alpha_r}$, $\delta\bar{v}^{D_s} = \delta\bar{v}^{\beta_1} \wedge \dots \wedge \delta\bar{v}^{\beta_s}$.

For the adapted local frame $\{\delta v^\alpha, \delta\bar{v}^\alpha\}$, we have

$$\begin{aligned} \partial(\delta v^\alpha) &= \Gamma_{\beta;\mu}^\alpha \delta v^\beta \wedge dz^\mu, \\ \bar{\partial}(\delta v^\alpha) &= \delta_{\bar{\beta}}(\Gamma_\mu^\alpha) d\bar{z}^\beta \wedge dz^\mu + \dot{\partial}_{\bar{\beta}}(\Gamma_\mu^\alpha) \delta\bar{v}^\beta \wedge dz^\mu, \\ \partial(\delta\bar{v}^\alpha) &= \delta_\beta(\bar{\Gamma}_\mu^\alpha) dz^\beta \wedge d\bar{z}^\mu + \dot{\partial}_\beta(\bar{\Gamma}_\mu^\alpha) \delta v^\beta \wedge d\bar{z}^\mu, \\ \bar{\partial}(\delta\bar{v}^\alpha) &= \bar{\Gamma}_{\beta;\mu}^\alpha \delta\bar{v}^\beta \wedge d\bar{z}^\mu, \end{aligned} \quad (3.1)$$

where in the first equation we have used the identities (see [15])

$$\delta_\beta(\Gamma_\mu^\alpha) - \delta_\mu(\Gamma_\beta^\alpha) = 0, \quad \dot{\partial}_\beta(\Gamma_\mu^\alpha) = \Gamma_{\beta;\mu}^\alpha.$$

Consequently for $\varphi \in \mathcal{A}^{p,q;r,s}$, we get

$$\begin{aligned} \partial\varphi &\in \mathcal{A}^{p+1,q;r,s} \oplus \mathcal{A}^{p,q;r+1,s} \oplus \mathcal{A}^{p+1,q+1;r,s-1} \oplus \mathcal{A}^{p,q+1;r+1,s-1}, \\ \bar{\partial}\varphi &\in \mathcal{A}^{p,q+1;r,s} \oplus \mathcal{A}^{p,q;r,s+1} \oplus \mathcal{A}^{p+1,q+1;r-1,s} \oplus \mathcal{A}^{p+1,q;r-1,s+1}. \end{aligned} \quad (3.2)$$

In the following, we shall rewrite the exterior derivative operators on \widetilde{M} by using the Chern Finsler connection ∇ . For simplicity, we need to introduce some notations: denote $i(X)$ the usual interior product induced by $X \in \mathcal{X}(T^{\mathbb{C}}(T^{1,0}M))$; and denote $e(\varphi)$ the exterior product induced by φ , for $\psi \in \mathcal{A}^{p,q;r,s}$, $e(\varphi)\psi = \varphi \wedge \psi$. Then for any $X, Y \in \mathcal{X}(T^{\mathbb{C}}(T^{1,0}M))$, and $\varphi \in \mathcal{A}^{p,q;r,s}$, we have

$$i(X)i(Y)\varphi = -i(Y)i(X)\varphi, \quad i(\overline{X})\overline{\varphi} = \overline{i(X)\varphi}, \quad (3.3)$$

and for any $\varphi, \psi, \phi \in \mathcal{A}^{p,q;r,s}$, we have

$$\begin{aligned} e(\varphi)e(\psi)\phi &= (-1)^{p+q+r+s}e(\psi)e(\varphi)\phi, \\ i(X)e(\psi)\phi &= [i(X)\psi] \wedge \phi + (-1)^{p+q+r+s}e(\psi)i(X)\phi. \end{aligned} \quad (3.4)$$

Under the adapted local frame,

$$\begin{aligned} i(\delta_\alpha)dz^\beta &= \delta_\alpha^\beta, & i(\delta_\alpha)d\overline{z}^\beta &= 0, & i(\delta_\alpha)\delta v^\beta &= 0, & i(\delta_\alpha)\delta\overline{v}^\beta &= 0, \\ i(\dot{\partial}_\alpha)dz^\beta &= 0, & i(\dot{\partial}_\alpha)d\overline{z}^\beta &= 0, & i(\dot{\partial}_\alpha)\delta v^\beta &= \delta_\alpha^\beta, & i(\dot{\partial}_\alpha)\delta\overline{v}^\beta &= 0 \end{aligned} \quad (3.5)$$

and

$$\begin{aligned} i(\delta_\alpha)e(\delta\overline{v}^\sigma) &= -e(\delta\overline{v}^\sigma)i(\delta_\alpha), & i(\dot{\partial}_\gamma)e(\delta v^\alpha) &= \delta_\gamma^\alpha - e(\delta v^\alpha)i(\dot{\partial}_\gamma), \\ i(\delta_\mu)e(dz^\nu) &= \delta_\mu^\nu - e(dz^\nu)i(\delta_\mu). \end{aligned} \quad (3.6)$$

It is easy to check the following lemma.

Lemma 3.1 *Let $\varphi \in \mathcal{A}^{p,q;r,s}$. Then we have*

$$\begin{aligned} \nabla_{\delta_\alpha}i(\delta_\beta)\varphi &= i(\delta_\beta)\nabla_{\delta_\alpha}\varphi + \Gamma_{\beta;\alpha}^\gamma i(\delta_\gamma)\varphi, \\ \nabla_{\delta_\alpha}i(\delta_{\overline{\beta}})\varphi &= i(\delta_{\overline{\beta}})\nabla_{\delta_\alpha}\varphi, \\ \nabla_{\delta_\alpha}i(\dot{\partial}_\beta)\varphi &= i(\dot{\partial}_\beta)\nabla_{\delta_\alpha}\varphi + \Gamma_{\beta;\alpha}^\gamma i(\dot{\partial}_\gamma)\varphi, \\ \nabla_{\delta_\alpha}i(\dot{\partial}_{\overline{\beta}})\varphi &= i(\dot{\partial}_{\overline{\beta}})\nabla_{\delta_\alpha}\varphi, \\ \nabla_{\dot{\partial}_\alpha}i(\dot{\partial}_\beta)\varphi &= i(\dot{\partial}_\beta)\nabla_{\dot{\partial}_\alpha}\varphi + \Gamma_{\beta\alpha}^\gamma i(\dot{\partial}_\gamma)\varphi, \\ \nabla_{\dot{\partial}_\alpha}i(\dot{\partial}_{\overline{\beta}})\varphi &= i(\dot{\partial}_{\overline{\beta}})\nabla_{\dot{\partial}_\alpha}\varphi, \\ \nabla_{\dot{\partial}_\alpha}i(\delta_\beta)\varphi &= i(\delta_\beta)\nabla_{\dot{\partial}_\alpha}\varphi + \Gamma_{\beta\alpha}^\gamma i(\delta_\gamma)\varphi, \\ \nabla_{\dot{\partial}_\alpha}i(\delta_{\overline{\beta}})\varphi &= i(\delta_{\overline{\beta}})\nabla_{\dot{\partial}_\alpha}\varphi, \\ \nabla_{\delta_\alpha}e(dz^\beta)\varphi &= -e(dz^\beta)\nabla_{\delta_\alpha}\varphi - \Gamma_{\mu;\alpha}^\beta e(dz^\mu)\varphi, \\ \nabla_{\delta_\alpha}e(d\overline{z}^\beta)\varphi &= -e(d\overline{z}^\beta)\nabla_{\delta_\alpha}\varphi, \\ \nabla_{\delta_\alpha}e(\delta v^\beta)\varphi &= -e(\delta v^\beta)\nabla_{\delta_\alpha}\varphi - \Gamma_{\gamma;\alpha}^\beta e(\delta v^\gamma)\varphi, \\ \nabla_{\delta_\alpha}e(\delta\overline{v}^\beta)\varphi &= -e(\delta\overline{v}^\beta)\nabla_{\delta_\alpha}\varphi, \\ \nabla_{\dot{\partial}_\alpha}e(\delta v^\beta)\varphi &= -e(\delta v^\beta)\nabla_{\dot{\partial}_\alpha}\varphi - \Gamma_{\gamma\alpha}^\beta e(\delta v^\gamma)\varphi, \\ \nabla_{\dot{\partial}_\alpha}e(\delta\overline{v}^\beta)\varphi &= -e(\delta\overline{v}^\beta)\nabla_{\dot{\partial}_\alpha}\varphi, \\ \nabla_{\dot{\partial}_\alpha}e(dz^\beta)\varphi &= -e(dz^\beta)\nabla_{\dot{\partial}_\alpha}\varphi - \Gamma_{\gamma\alpha}^\beta e(dz^\gamma)\varphi, \\ \nabla_{\dot{\partial}_\alpha}e(d\overline{z}^\beta)\varphi &= -e(d\overline{z}^\beta)\nabla_{\dot{\partial}_\alpha}\varphi. \end{aligned}$$

Combined with (2.4), the exterior derivative operator $d = \partial + \overline{\partial}$ can be represented by using the Chern Finsler connection

$$\partial = e(dz^\alpha)\nabla_{\delta_\alpha} + \Gamma_{\beta;\alpha}^\gamma e(dz^\beta)i(\delta_\gamma) + e(\delta v^\alpha)\nabla_{\dot{\partial}_\alpha} + \Gamma_{\alpha\beta}^\gamma e(\delta v^\alpha)e(dz^\beta)i(\delta_\gamma)$$

$$+ \delta_\beta(\overline{\Gamma}_\mu^\alpha) e(dz^\beta) e(d\bar{z}^\mu) i(\dot{\partial}_\alpha) + \dot{\partial}_\beta(\overline{\Gamma}_\mu^\alpha) e(\delta v^\beta) e(d\bar{z}^\mu) i(\dot{\partial}_\alpha), \quad (3.7)$$

$$\begin{aligned} \bar{\partial} &= e(d\bar{z}^\alpha) \nabla_{\delta_\alpha} + \overline{\Gamma}_{\beta;\alpha}^\gamma e(d\bar{z}^\alpha) e(d\bar{z}^\beta) i(\delta_\gamma) + e(\delta\bar{v}^\alpha) \nabla_{\dot{\partial}_\alpha} + \overline{\Gamma}_{\alpha\beta}^\gamma e(\delta\bar{v}^\alpha) e(d\bar{z}^\beta) i(\delta_\gamma) \\ &+ \delta_{\bar{\beta}}(\Gamma_\mu^\alpha) e(d\bar{z}^\beta) e(dz^\mu) i(\dot{\partial}_\alpha) + \dot{\partial}_{\bar{\beta}}(\Gamma_\mu^\alpha) e(\delta\bar{v}^\beta) e(dz^\mu) i(\dot{\partial}_\alpha). \end{aligned} \quad (3.8)$$

In accordance with the classification, denote the operators (see [17])

$$D_{\mathcal{H}} = e(dz^\alpha) \nabla_{\delta_\alpha} + \Gamma_{\beta;\alpha}^\gamma e(dz^\alpha) e(dz^\beta) i(\delta_\gamma) : \mathcal{A}^{p,q;r,s} \longrightarrow \mathcal{A}^{p+1,q;r,s}, \quad (3.9)$$

$$\bar{D}_{\mathcal{H}} = e(d\bar{z}^\alpha) \nabla_{\delta_\alpha} + \overline{\Gamma}_{\beta;\alpha}^\gamma e(d\bar{z}^\alpha) e(d\bar{z}^\beta) i(\delta_\gamma) : \mathcal{A}^{p,q;r,s} \longrightarrow \mathcal{A}^{p,q+1;r,s}, \quad (3.10)$$

$$D_{\mathcal{V}} = e(\delta v^\alpha) \nabla_{\dot{\partial}_\alpha} + \Gamma_{\alpha\beta}^\gamma e(\delta v^\alpha) e(dz^\beta) i(\delta_\gamma) : \mathcal{A}^{p,q;r,s} \longrightarrow \mathcal{A}^{p,q;r+1,s}, \quad (3.11)$$

$$\bar{D}_{\mathcal{V}} = e(\delta\bar{v}^\alpha) \nabla_{\dot{\partial}_\alpha} + \overline{\Gamma}_{\alpha\beta}^\gamma e(\delta\bar{v}^\alpha) e(d\bar{z}^\beta) i(\delta_\gamma) : \mathcal{A}^{p,q;r,s} \longrightarrow \mathcal{A}^{p,q;r,s+1}, \quad (3.12)$$

$$D_3 = \delta_\beta(\overline{\Gamma}_\mu^\alpha) e(dz^\beta) e(d\bar{z}^\mu) i(\dot{\partial}_\alpha) : \mathcal{A}^{p,q;r,s} \longrightarrow \mathcal{A}^{p+1,q+1;r,s-1}, \quad (3.13)$$

$$\bar{D}_3 = \delta_{\bar{\beta}}(\Gamma_\mu^\alpha) e(d\bar{z}^\beta) e(dz^\mu) i(\dot{\partial}_\alpha) : \mathcal{A}^{p,q;r,s} \longrightarrow \mathcal{A}^{p+1,q+1;r-1,s}, \quad (3.14)$$

$$D_4 = \dot{\partial}_\beta(\overline{\Gamma}_\mu^\alpha) e(\delta v^\beta) e(d\bar{z}^\mu) i(\dot{\partial}_\alpha) : \mathcal{A}^{p,q;r,s} \longrightarrow \mathcal{A}^{p,q+1;r+1,s-1}, \quad (3.15)$$

$$\bar{D}_4 = \dot{\partial}_{\bar{\beta}}(\Gamma_\mu^\alpha) e(\delta\bar{v}^\beta) e(dz^\mu) i(\dot{\partial}_\alpha) : \mathcal{A}^{p,q;r,s} \longrightarrow \mathcal{A}^{p+1,q;r-1,s+1}. \quad (3.16)$$

Then the exterior derivative operator $d = \partial + \bar{\partial}$ can be rewritten as

$$d = D_{\mathcal{H}} + D_{\mathcal{V}} + D_3 + D_4 + \bar{D}_{\mathcal{H}} + \bar{D}_{\mathcal{V}} + \bar{D}_3 + \bar{D}_4. \quad (3.17)$$

4 The Hodge-Laplace Operator and Weitzenböck Formulas on \widetilde{M}

Let (M, F) be an m -dimensional strongly pseudoconvex complex Finsler manifold, where F is the strongly pseudoconvex complex Finsler metric. Then, F induces a Hermitian metric

$$\widetilde{G} = G_{\alpha\bar{\beta}} dz^\alpha \otimes d\bar{z}^\beta + G_{\alpha\bar{\beta}} \delta v^\alpha \otimes \delta\bar{v}^\beta, \quad (4.1)$$

on $T^{\mathbb{C}}\widetilde{M}$, denoted by $\langle \cdot, \cdot \rangle$. Associated to this metric, we have the Kaehler form $\omega = iG_{\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta + iG_{\alpha\bar{\beta}} \delta v^\alpha \wedge \delta\bar{v}^\beta$ and $\omega^{2m} = (-1)^m (2m)! \mathcal{G}^2 dz^1 \wedge \cdots \wedge dz^m \wedge d\bar{z}^1 \wedge \cdots \wedge d\bar{z}^m \wedge \delta v^1 \wedge \cdots \wedge \delta v^m \wedge \delta\bar{v}^1 \wedge \cdots \wedge \delta\bar{v}^m$, where $\mathcal{G} = \det(G_{\alpha\bar{\beta}})$. It is easy to check that $dV_{\widetilde{M}} = \frac{\omega^{2m}}{(2m)!}$ defines a global invariant volume form of \widetilde{M} . Associated with the decomposition $T^{1,0}\widetilde{M} = \mathcal{H} \oplus \mathcal{V}$, and according to the adapted local frame $\{\delta_\alpha, \dot{\partial}_\alpha, \delta_{\bar{\alpha}}, \dot{\partial}_{\bar{\alpha}}\}$ of $T^{\mathbb{C}}(\widetilde{M})$ and $\{dz^\alpha, \delta v^\alpha, d\bar{z}^\alpha, \delta\bar{v}^\alpha\}$ for $T^{\mathbb{C}*}\widetilde{M}$, we have

$$\begin{aligned} \langle \delta_\alpha, \delta_\beta \rangle &= G_{\alpha\bar{\beta}}, & \langle \delta_\alpha, \dot{\partial}_\beta \rangle &= 0, & \langle \dot{\partial}_\alpha, \dot{\partial}_\beta \rangle &= G_{\alpha\bar{\beta}}, \\ \langle dz^\alpha, dz^\beta \rangle &= G_{\alpha\bar{\beta}}, & \langle dz^\alpha, \delta v^\beta \rangle &= 0, & \langle \delta v^\alpha, \delta v^\beta \rangle &= G_{\alpha\bar{\beta}}. \end{aligned} \quad (4.2)$$

Now, if we denote $G^{\bar{A}_p B_p} = G^{\bar{\alpha}_1 \beta_1} \cdots G^{\bar{\alpha}_2 \beta_2} \cdots G^{\bar{\alpha}_p \beta_p}$ on \widetilde{M} , then for any $\varphi, \psi \in \mathcal{A}^{p,q;r,s}$, at each point $v \in T^{1,0}M$, the inner product can be defined by

$$\langle \varphi, \psi \rangle = \frac{1}{p!q!r!s!} \varphi_{A_p \bar{B}_q C_r \bar{D}_s} \overline{\psi_{\bar{A}_p B_q \bar{C}_r D_s}}, \quad (4.3)$$

where $\psi_{\bar{A}_p B_q \bar{C}_r D_s} = \psi_{E_p \bar{F}_q H_r \bar{L}_s} G^{\bar{A}_p E_p} G^{\bar{F}_q B_q} G^{\bar{C}_r H_r} G^{\bar{L}_s D_s}$.

Definition 4.1 *Let (M, F) be an m -dimensional strongly pseudoconvex compact complex Finsler manifold. The global inner product of two forms $\varphi, \psi \in \mathcal{A}^{p,q;r,s}$ is defined by*

$$(\varphi, \psi) = \int_{\widetilde{M}} \langle \varphi, \psi \rangle \frac{\omega^{2m}}{(2m)!}. \quad (4.4)$$

It is easy to check that (\cdot, \cdot) satisfies the following properties:

- (1) $(\varphi, \varphi) \geq 0$ and $(\varphi, \varphi) = 0$ if and only if $\varphi = 0$;
- (2) $(a\varphi + b\phi, \psi) = a(\varphi, \psi) + b(\phi, \psi)$ for any $\phi, \varphi, \psi \in \mathcal{A}^{p,q;r,s}$ and $a, b \in \mathbb{C}$;
- (3) $\overline{(\varphi, \psi)} = (\psi, \varphi)$.

We define $\|\varphi\| = \sqrt{(\varphi, \varphi)}$ as usual.

As in Hermitian geometry (see [6]), we define the Hodge star operator

$$* : \mathcal{A}^{p,q;r,s} \longrightarrow \mathcal{A}^{m-q,m-p;m-s,m-r},$$

such that for any $\psi \in \mathcal{A}^{p,q;r,s}$,

$$\begin{aligned} *\psi &= (-1)^{m(p+r+1)+(r+s)(p+q)} G_{A_q A_{m-q} \overline{B}_p \overline{B}_{m-p}} G_{C_s C_{m-s} \overline{D}_r \overline{D}_{m-r}} \\ &\quad \cdot \psi \overline{B}_p A_q \overline{D}_r C_s dz^{A_{m-q}} \wedge d\overline{z}^{B_{m-p}} \wedge \delta v^{C_{m-s}} \wedge \delta \overline{v}^{D_{m-r}}. \end{aligned} \quad (4.5)$$

Then we have the result as follows.

Theorem 4.1 *Let (M, F) be an m -dimensional strongly pseudoconvex compact complex Finsler manifold. Then the complex linear map $* : \mathcal{A}^{p,q;r,s} \longrightarrow \mathcal{A}^{m-q,m-p;m-s,m-r}$ satisfies*

- (i) $\langle \varphi, \psi \rangle \frac{\omega^{2m}}{(2m)!} = \varphi \wedge *\overline{\psi}$;
- (ii) $\overline{*\psi} = * \overline{\psi}$ (i.e., $*$ is a real operator);
- (iii) $**\psi = (-1)^{p+q+r+s} \psi$.

Now we define the adjoint operators of the operators (3.9)–(3.16) as follows.

Definition 4.2 $D_{\mathcal{H}}^* = -*\overline{D}_{\mathcal{H}}*$, $D_{\mathcal{V}}^* = -*\overline{D}_{\mathcal{V}}*$, $D_3^* = -*\overline{D}_3*$, $D_4^* = -*\overline{D}_4*$. By conjugation, we have

$$\overline{D}_{\mathcal{H}}^* = -*D_{\mathcal{H}}*, \quad \overline{D}_{\mathcal{V}}^* = -*D_{\mathcal{V}}*, \quad \overline{D}_3^* = -*D_3*, \quad \overline{D}_4^* = -*D_4*.$$

Then we can define the adjoint of d as $d^* = -*d*$. It follows from (3.17) that

$$d^* = D_{\mathcal{H}}^* + D_{\mathcal{V}}^* + D_3^* + D_4^* + \overline{D}_{\mathcal{H}}^* + \overline{D}_{\mathcal{V}}^* + \overline{D}_3^* + \overline{D}_4^*. \quad (4.6)$$

It is easy to see that

$$\begin{aligned} D_{\mathcal{H}}^* &: \mathcal{A}^{p,q;r,s} \longrightarrow \mathcal{A}^{p,q-1;r,s}, \\ \overline{D}_{\mathcal{H}}^* &: \mathcal{A}^{p,q;r,s} \longrightarrow \mathcal{A}^{p-1,q;r,s}, \\ D_{\mathcal{V}}^* &: \mathcal{A}^{p,q;r,s} \longrightarrow \mathcal{A}^{p,q;r,s-1}, \\ \overline{D}_{\mathcal{V}}^* &: \mathcal{A}^{p,q;r,s} \longrightarrow \mathcal{A}^{p,q;r-1,s}, \\ D_3^* &: \mathcal{A}^{p,q;r,s} \longrightarrow \mathcal{A}^{p-1,q-1;r+1,s}, \\ \overline{D}_3^* &: \mathcal{A}^{p,q;r,s} \longrightarrow \mathcal{A}^{p-1,q-1;r,s+1}, \\ D_4^* &: \mathcal{A}^{p,q;r,s} \longrightarrow \mathcal{A}^{p-1,q;r+1,s-1}, \\ \overline{D}_4^* &: \mathcal{A}^{p,q;r,s} \longrightarrow \mathcal{A}^{p,q-1;r-1,s+1}. \end{aligned}$$

Taking $\varphi \in \mathcal{A}^{p-1,q;r,s}$, $\psi \in \mathcal{A}^{p,q;r,s}$, then by type reason we have

$$\begin{aligned} (D_{\mathcal{H}}\varphi, \psi) &= (d\varphi, \psi) = \int_{\widetilde{M}} d\varphi \wedge *\overline{\psi} \\ &= (-1)^{p+q+r+s+1} \int_{\widetilde{M}} \varphi \wedge d*\overline{\psi} + \int_{\widetilde{M}} d(\varphi \wedge *\overline{\psi}) \end{aligned}$$

$$\begin{aligned}
&= (-1)^{p+q+r+s+1} \int_{\widetilde{M}} \varphi \wedge d * \bar{\psi} \\
&= - \int_{\widetilde{M}} \varphi \wedge * (\bar{d} \bar{*}) \bar{\psi} = - \int_{\widetilde{M}} \varphi \wedge * (\bar{d} \bar{*}) \bar{\psi} \\
&= (\varphi, \bar{D}_{\mathcal{H}}^* \psi).
\end{aligned}$$

Similarly,

$$(D_{\mathcal{V}} \varphi, \psi) = (\varphi, \bar{D}_{\mathcal{V}}^* \psi), \quad (D_3 \varphi, \psi) = (\varphi, \bar{D}_3^* \psi), \quad (D_4 \varphi, \psi) = (\varphi, \bar{D}_4^* \psi).$$

By conjugation, we have

$$\begin{aligned}
(\bar{D}_{\mathcal{H}} \varphi, \psi) &= (\varphi, D_{\mathcal{H}}^* \psi), & (\bar{D}_{\mathcal{V}} \varphi, \psi) &= (\varphi, D_{\mathcal{V}}^* \psi), \\
(\bar{D}_3 \varphi, \psi) &= (\varphi, D_3^* \psi), & (\bar{D}_4 \varphi, \psi) &= (\varphi, D_4^* \psi).
\end{aligned}$$

According to (3.9)–(3.16), we get the explicit expressions of the adjoint operators by solving the above formulas.

Theorem 4.2 *Let (M, F) be an m -dimensional strongly pseudoconvex compact complex Finsler manifold. Then we have*

$$\begin{aligned}
D_{\mathcal{H}}^* &= -G^{\bar{\beta}\alpha} \nabla_{\delta_{\alpha}} i(\delta_{\bar{\beta}}) + G^{\bar{\beta}\alpha} [\Gamma_{\alpha;\gamma}^{\gamma} - 2\Gamma_{\gamma;\alpha}^{\gamma}] i(\delta_{\bar{\beta}}) \\
&\quad + \Gamma_{\beta;\alpha}^{\gamma} G^{\bar{\sigma}\alpha} G^{\bar{\sigma}\beta} G_{\gamma\bar{\varepsilon}} e(d\bar{z}^{\varepsilon}) i(\delta_{\bar{\sigma}}) i(\delta_{\bar{\tau}}),
\end{aligned} \tag{4.7}$$

$$\begin{aligned}
\bar{D}_{\mathcal{H}}^* &= -G^{\bar{\beta}\alpha} \nabla_{\delta_{\bar{\beta}}} i(\delta_{\alpha}) + G^{\bar{\beta}\alpha} [\bar{\Gamma}_{\beta;\gamma}^{\gamma} - 2\bar{\Gamma}_{\gamma;\beta}^{\gamma}] i(\delta_{\alpha}) \\
&\quad + \bar{\Gamma}_{\beta;\alpha}^{\gamma} G^{\bar{\sigma}\alpha} G^{\bar{\sigma}\beta} G_{\varepsilon\bar{\gamma}} e(dz^{\varepsilon}) i(\delta_{\sigma}) i(\delta_{\tau}),
\end{aligned} \tag{4.8}$$

$$\begin{aligned}
D_{\mathcal{V}}^* &= -G^{\bar{\beta}\alpha} \nabla_{\dot{\delta}_{\alpha}} i(\dot{\delta}_{\bar{\beta}}) - G^{\bar{\beta}\alpha} \Gamma_{\gamma\alpha}^{\gamma} i(\dot{\delta}_{\bar{\beta}}) \\
&\quad + \Gamma_{\alpha\beta}^{\gamma} G^{\bar{\sigma}\alpha} G^{\bar{\sigma}\beta} G_{\gamma\bar{\varepsilon}} e(d\bar{z}^{\varepsilon}) i(\delta_{\bar{\sigma}}) i(\dot{\delta}_{\bar{\tau}}),
\end{aligned} \tag{4.9}$$

$$\begin{aligned}
\bar{D}_{\mathcal{V}}^* &= -G^{\bar{\beta}\alpha} \nabla_{\dot{\delta}_{\bar{\beta}}} i(\dot{\delta}_{\alpha}) - G^{\bar{\beta}\alpha} \bar{\Gamma}_{\gamma\beta}^{\gamma} i(\dot{\delta}_{\alpha}) \\
&\quad + \bar{\Gamma}_{\alpha\beta}^{\gamma} G^{\bar{\sigma}\alpha} G^{\bar{\sigma}\beta} G_{\varepsilon\bar{\gamma}} e(dz^{\varepsilon}) i(\delta_{\tau}) i(\dot{\delta}_{\sigma}),
\end{aligned} \tag{4.10}$$

$$D_3^* = \delta_{\beta} (\bar{\Gamma}_{\mu}^{\alpha}) G^{\bar{\gamma}\beta} G^{\bar{\mu}\nu} G_{\tau\bar{\alpha}} e(\delta v^{\tau}) i(\delta_{\nu}) i(\delta_{\bar{\tau}}), \tag{4.11}$$

$$\bar{D}_3^* = -\delta_{\bar{\beta}} (\Gamma_{\mu}^{\alpha}) G^{\bar{\beta}\gamma} G^{\bar{\mu}\nu} G_{\alpha\bar{\sigma}} e(\delta \bar{v}^{\sigma}) i(\delta_{\gamma}) i(\delta_{\bar{\nu}}), \tag{4.12}$$

$$D_4^* = \dot{\delta}_{\beta} (\bar{\Gamma}_{\mu}^{\alpha}) G^{\bar{\gamma}\beta} G^{\bar{\mu}\nu} G_{\tau\bar{\alpha}} e(\delta v^{\tau}) i(\delta_{\nu}) i(\dot{\delta}_{\bar{\tau}}), \tag{4.13}$$

$$\bar{D}_4^* = -\dot{\delta}_{\bar{\beta}} (\Gamma_{\mu}^{\alpha}) G^{\bar{\gamma}\beta} G^{\bar{\mu}\nu} G_{\alpha\bar{\sigma}} e(\delta \bar{v}^{\sigma}) i(\dot{\delta}_{\varepsilon}) i(\delta_{\bar{\lambda}}). \tag{4.14}$$

Proof We only need to prove (4.7), and the others are similar to obtain. For any $\varphi, \psi \in \mathcal{A}^{p,q;r,s}$, according to (3.10), we have

$$\begin{aligned}
(\varphi, D_{\mathcal{H}}^* \psi) &= (\bar{D}_{\mathcal{H}} \varphi, \psi) = (e(d\bar{z}^{\alpha}) \nabla_{\delta_{\bar{\alpha}}} \varphi + \bar{\Gamma}_{\beta;\alpha}^{\gamma} e(d\bar{z}^{\alpha}) e(d\bar{z}^{\beta}) i(\delta_{\bar{\gamma}}) \varphi, \psi) \\
&= \int_{\widetilde{M}} \langle e(d\bar{z}^{\alpha}) \nabla_{\delta_{\bar{\alpha}}} \varphi, \psi \rangle dV + \int_{\widetilde{M}} \langle \bar{\Gamma}_{\beta;\alpha}^{\gamma} e(d\bar{z}^{\alpha}) e(d\bar{z}^{\beta}) i(\delta_{\bar{\gamma}}) \varphi, \psi \rangle dV,
\end{aligned}$$

while

$$\begin{aligned}
&\int_{\widetilde{M}} \langle e(d\bar{z}^{\alpha}) \nabla_{\delta_{\bar{\alpha}}} \varphi, \psi \rangle dV \\
&= \int_{\widetilde{M}} G^{\bar{\alpha}\beta} \langle \nabla_{\delta_{\bar{\alpha}}} \varphi, i(\delta_{\bar{\beta}}) \psi \rangle dV
\end{aligned}$$

$$\begin{aligned}
&= \int_{\widetilde{M}} G^{\bar{\alpha}\beta} \delta_{\bar{\alpha}}(\langle \varphi, i(\delta_{\bar{\beta}})\psi \rangle) \mathcal{G}^2(-1)^m dz \wedge d\bar{z} \wedge \delta v \wedge \delta \bar{v} - \int_{\widetilde{M}} G^{\bar{\alpha}\beta} \langle \varphi, \nabla_{\delta_{\alpha}} i(\delta_{\bar{\beta}})\psi \rangle dV \\
&= \int_{\widetilde{M}} \{ \delta_{\bar{\alpha}}[G^{\bar{\alpha}\beta} \langle \varphi, i(\delta_{\bar{\beta}})\psi \rangle \mathcal{G}^2] - \delta_{\bar{\alpha}}(G^{\bar{\alpha}\beta}) \langle \varphi, i(\delta_{\bar{\beta}})\psi \rangle \mathcal{G}^2 - G^{\bar{\alpha}\beta} \langle \varphi, i(\delta_{\bar{\beta}})\psi \rangle \delta_{\bar{\alpha}}(\mathcal{G}^2) \} \\
&\quad \cdot (-1)^m dz \wedge d\bar{z} \wedge \delta v \wedge \delta \bar{v} - \int_{\widetilde{M}} G^{\bar{\alpha}\beta} \langle \varphi, \nabla_{\delta_{\alpha}} i(\delta_{\bar{\beta}})\psi \rangle dV \\
&= \int_{\widetilde{M}} \{ \langle \varphi, G^{\bar{\beta}\gamma} \Gamma_{\gamma;\alpha}^{\alpha} i(\delta_{\bar{\beta}})\psi \rangle - \langle \varphi, 2G^{\bar{\beta}\alpha} \Gamma_{\gamma;\alpha}^{\gamma} i(\delta_{\bar{\beta}})\psi \rangle - \langle \varphi, G^{\bar{\beta}\alpha} \nabla_{\delta_{\alpha}} i(\delta_{\bar{\beta}})\psi \rangle \} dV \\
&= \int_{\widetilde{M}} \langle \varphi, \{ G^{\bar{\beta}\alpha} [\Gamma_{\alpha;\gamma}^{\gamma} - 2\Gamma_{\gamma;\alpha}^{\gamma}] i(\delta_{\bar{\beta}}) - G^{\bar{\beta}\alpha} \nabla_{\delta_{\alpha}} i(\delta_{\bar{\beta}}) \} \psi \rangle dV
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\widetilde{M}} \langle \overline{\Gamma_{\beta;\alpha}^{\gamma}} e(d\bar{z}^{\alpha}) e(d\bar{z}^{\beta}) i(\delta_{\bar{\gamma}})\varphi, \psi \rangle dV \\
&= \int_{\widetilde{M}} \overline{\Gamma_{\beta;\alpha}^{\gamma}} G^{\bar{\alpha}\tau} \langle e(d\bar{z}^{\beta}) i(\delta_{\bar{\gamma}})\varphi, i(\delta_{\bar{\tau}})\psi \rangle dV \\
&= \int_{\widetilde{M}} \overline{\Gamma_{\beta;\alpha}^{\gamma}} G^{\bar{\alpha}\tau} G^{\bar{\beta}\sigma} \langle i(\delta_{\bar{\gamma}})\varphi, i(\delta_{\bar{\sigma}}) i(\delta_{\bar{\tau}})\psi \rangle dV \\
&= \int_{\widetilde{M}} \overline{\Gamma_{\beta;\alpha}^{\gamma}} G^{\bar{\alpha}\tau} G^{\bar{\beta}\sigma} G_{\bar{\gamma}\bar{\tau}} \langle \varphi, e(d\bar{z}^{\varepsilon}) i(\delta_{\bar{\sigma}}) i(\delta_{\bar{\tau}})\psi \rangle dV \\
&= \int_{\widetilde{M}} \langle \varphi, \Gamma_{\beta;\alpha}^{\gamma} G^{\bar{\tau}\alpha} G^{\bar{\sigma}\beta} G_{\bar{\gamma}\bar{\varepsilon}} e(d\bar{z}^{\varepsilon}) i(\delta_{\bar{\sigma}}) i(\delta_{\bar{\tau}})\psi \rangle dV.
\end{aligned}$$

Then

$$\begin{aligned}
(\varphi, D_{\mathcal{H}}^* \psi) &= \int_{\widetilde{M}} \langle \varphi, \{ -G^{\bar{\beta}\alpha} \nabla_{\delta_{\alpha}} i(\delta_{\bar{\beta}}) + G^{\bar{\beta}\alpha} [\Gamma_{\alpha;\gamma}^{\gamma} - 2\Gamma_{\gamma;\alpha}^{\gamma}] i(\delta_{\bar{\beta}}) \\
&\quad + \Gamma_{\beta;\alpha}^{\gamma} G^{\bar{\tau}\alpha} G^{\bar{\sigma}\beta} G_{\bar{\gamma}\bar{\varepsilon}} e(d\bar{z}^{\varepsilon}) i(\delta_{\bar{\sigma}}) i(\delta_{\bar{\tau}}) \} \psi \rangle dV,
\end{aligned}$$

that is

$$D_{\mathcal{H}}^* = -G^{\bar{\beta}\alpha} \nabla_{\delta_{\alpha}} i(\delta_{\bar{\beta}}) + G^{\bar{\beta}\alpha} [\Gamma_{\alpha;\gamma}^{\gamma} - 2\Gamma_{\gamma;\alpha}^{\gamma}] i(\delta_{\bar{\beta}}) + \Gamma_{\beta;\alpha}^{\gamma} G^{\bar{\tau}\alpha} G^{\bar{\sigma}\beta} G_{\bar{\gamma}\bar{\varepsilon}} e(d\bar{z}^{\varepsilon}) i(\delta_{\bar{\sigma}}) i(\delta_{\bar{\tau}}).$$

Furthermore, we have

Theorem 4.3 *Let (M, F) be an m -dimensional strongly pseudoconvex compact complex Finsler manifold. Then for any $\varphi \in \mathcal{A}^{p,q;r,s}$, we have*

$$\begin{aligned}
D_{\mathcal{H}}^* \varphi &= -\frac{1}{p!q!r!s!} \delta_{\alpha}(\varphi_{A_p \bar{B}_q C_r \bar{D}_s}) G^{\bar{\beta}\alpha} i(\delta_{\bar{\beta}}) (dz^{A_p} \wedge d\bar{z}^{B_q} \wedge \delta v^{C_r} \wedge \delta \bar{v}^{D_s}) \\
&\quad - G^{\bar{\beta}\alpha} \Gamma_{\sigma;\alpha}^{\mu} e(dz^{\sigma}) i(\delta_{\bar{\beta}}) i(\delta_{\bar{\mu}}) \varphi - G^{\bar{\beta}\alpha} \Gamma_{\sigma;\alpha}^{\mu} e(\delta v^{\sigma}) i(\delta_{\bar{\beta}}) i(\dot{\delta}_{\bar{\mu}}) \varphi \\
&\quad + G^{\bar{\beta}\alpha} [\Gamma_{\alpha;\gamma}^{\gamma} - 2\Gamma_{\gamma;\alpha}^{\gamma}] i(\delta_{\bar{\beta}}) \varphi + \Gamma_{\beta;\alpha}^{\gamma} G^{\bar{\tau}\alpha} G^{\bar{\sigma}\beta} G_{\bar{\gamma}\bar{\varepsilon}} e(d\bar{z}^{\varepsilon}) i(\delta_{\bar{\sigma}}) i(\delta_{\bar{\tau}}) \varphi, \\
\bar{D}_{\mathcal{H}}^* \varphi &= -\frac{1}{p!q!r!s!} \delta_{\bar{\beta}}(\varphi_{A_p \bar{B}_q C_r \bar{D}_s}) G^{\bar{\beta}\alpha} i(\delta_{\alpha}) (dz^{A_p} \wedge d\bar{z}^{B_q} \wedge \delta v^{C_r} \wedge \delta \bar{v}^{D_s}) \\
&\quad - G^{\bar{\beta}\alpha} \overline{\Gamma_{\sigma;\beta}^{\mu}} e(d\bar{z}^{\sigma}) i(\delta_{\alpha}) i(\delta_{\bar{\mu}}) \varphi - G^{\bar{\beta}\alpha} \overline{\Gamma_{\sigma;\beta}^{\mu}} e(\delta \bar{v}^{\sigma}) i(\delta_{\alpha}) i(\dot{\delta}_{\bar{\mu}}) \varphi \\
&\quad + G^{\bar{\beta}\alpha} [\overline{\Gamma_{\beta;\gamma}^{\gamma}} - 2\overline{\Gamma_{\gamma;\beta}^{\gamma}}] i(\delta_{\alpha}) \varphi + \overline{\Gamma_{\beta;\alpha}^{\gamma}} G^{\bar{\beta}\sigma} G^{\bar{\alpha}\tau} G_{\bar{\varepsilon}\bar{\tau}} e(dz^{\varepsilon}) i(\delta_{\sigma}) i(\delta_{\bar{\tau}}) \varphi, \\
D_{\mathcal{V}}^* \varphi &= -\frac{1}{p!q!r!s!} \dot{\delta}_{\alpha}(\varphi_{A_p \bar{B}_q C_r \bar{D}_s}) G^{\bar{\beta}\alpha} i(\dot{\delta}_{\bar{\beta}}) (dz^{A_p} \wedge d\bar{z}^{B_q} \wedge \delta v^{C_r} \wedge \delta \bar{v}^{D_s})
\end{aligned}$$

$$\begin{aligned}
& -G^{\bar{\beta}\alpha}\Gamma_{\gamma\alpha}^{\mu}e(dz^{\sigma})i(\dot{\partial}_{\bar{\beta}})i(\delta_{\mu})\varphi - G^{\bar{\beta}\alpha}\Gamma_{\gamma\alpha}^{\mu}e(\delta\bar{v}^{\sigma})i(\dot{\partial}_{\bar{\beta}})i(\dot{\partial}_{\mu})\varphi \\
& -G^{\bar{\beta}\alpha}\Gamma_{\gamma\alpha}^{\gamma}i(\dot{\partial}_{\bar{\beta}})\varphi + \Gamma_{\alpha\beta}^{\gamma}G^{\bar{\sigma}\alpha}G^{\bar{\tau}\beta}G_{\gamma\bar{\tau}}e(d\bar{z}^{\varepsilon})i(\delta_{\bar{\tau}})i(\dot{\partial}_{\sigma})\varphi, \\
\bar{D}_{\mathcal{V}}^*\varphi = & -\frac{1}{p!q!r!s!}\dot{\partial}_{\bar{\beta}}(\varphi_{A_p\bar{B}_qC_r\bar{D}_s})G^{\bar{\beta}\alpha}i(\dot{\partial}_{\alpha})(dz^{A_p}\wedge d\bar{z}^{B_q}\wedge \delta v^{C_r}\wedge \delta\bar{v}^{D_s}) \\
& -G^{\bar{\beta}\alpha}\overline{\Gamma_{\gamma\beta}^{\mu}}e(d\bar{z}^{\sigma})i(\dot{\partial}_{\alpha})i(\delta_{\bar{\mu}})\varphi - G^{\bar{\beta}\alpha}\overline{\Gamma_{\gamma\beta}^{\mu}}e(\delta\bar{v}^{\sigma})i(\dot{\partial}_{\alpha})i(\dot{\partial}_{\bar{\mu}})\varphi \\
& -G^{\bar{\beta}\alpha}\overline{\Gamma_{\gamma\beta}^{\gamma}}i(\dot{\partial}_{\alpha})\varphi + \overline{\Gamma_{\alpha\beta}^{\gamma}}G^{\bar{\sigma}\alpha}G^{\bar{\tau}\beta}G_{\varepsilon\bar{\tau}}e(dz^{\varepsilon})i(\delta_{\bar{\tau}})i(\dot{\partial}_{\sigma})\varphi.
\end{aligned}$$

Definition 4.3 We define the following differential operators:

$$\begin{aligned}
\Box_{\mathcal{H}} &= \bar{D}_{\mathcal{H}}D_{\mathcal{H}}^* + D_{\mathcal{H}}^*\bar{D}_{\mathcal{H}}, & \Box_{\mathcal{V}} &= \bar{D}_{\mathcal{V}}D_{\mathcal{V}}^* + D_{\mathcal{V}}^*\bar{D}_{\mathcal{V}}, \\
\Box_3 &= \bar{D}_3D_3^* + D_3^*\bar{D}_3, & \Box_4 &= \bar{D}_4D_4^* + D_4^*\bar{D}_4.
\end{aligned}$$

Associated to the Hermitian metric $\langle \cdot, \cdot \rangle$ of holomorphic tangent bundle \widetilde{M} induced by the strongly pseudoconvex Finsler metric on M , $\Box_{\mathcal{H}}$ and $\Box_{\mathcal{V}}$ are called the horizontal and vertical complex Laplacians respectively, while \Box_3 and \Box_4 are called the mixed complex Laplacians, and they are all type preserving operators from $\mathcal{A}^{p,q;r,s}$ to $\mathcal{A}^{p,q;r,s}$.

Theorem 4.4 Let M be a strongly pseudoconvex compact complex Finsler manifold. Then for any $f \in C^\infty(\widetilde{M})$, by using the Chern Finsler connection, we have

$$\Delta f = d^*df = (\Box_{\mathcal{H}} + \bar{\Box}_{\mathcal{H}} + \Box_{\mathcal{V}} + \bar{\Box}_{\mathcal{V}})f.$$

Proof Since for any $f \in C^\infty(\widetilde{M})$, we have $\Delta f = d^*df$. Thus by (4.7)–(4.14), we get

$$\begin{aligned}
D_{\mathcal{H}}^*df &= \{-G^{\bar{\beta}\alpha}i(\delta_{\bar{\beta}})\nabla_{\delta_{\alpha}} + G^{\bar{\beta}\alpha}[\Gamma_{\alpha;\gamma}^{\gamma} - 2\Gamma_{\gamma;\alpha}^{\gamma}]i(\delta_{\bar{\beta}}) + \Gamma_{\beta;\alpha}^{\gamma}G^{\bar{\tau}\alpha}G^{\bar{\sigma}\beta}G_{\gamma\bar{\tau}}e(d\bar{z}^{\varepsilon})i(\delta_{\bar{\sigma}})i(\delta_{\bar{\tau}})\}(df) \\
&= -G^{\bar{\beta}\alpha}\delta_{\alpha}\delta_{\bar{\beta}}f + G^{\bar{\beta}\alpha}[\Gamma_{\alpha;\gamma}^{\gamma} - 2\Gamma_{\gamma;\alpha}^{\gamma}]\delta_{\bar{\beta}}f = D_{\mathcal{H}}^*\bar{D}_{\mathcal{H}}f = \Box_{\mathcal{H}}f, \\
\bar{D}_{\mathcal{H}}^*df &= -G^{\bar{\beta}\alpha}\delta_{\bar{\beta}}\delta_{\alpha}f + G^{\bar{\beta}\alpha}[\overline{\Gamma_{\beta;\gamma}^{\gamma}} - 2\overline{\Gamma_{\gamma;\beta}^{\gamma}}]\delta_{\alpha}f = \bar{D}_{\mathcal{H}}^*D_{\mathcal{H}}f = \bar{\Box}_{\mathcal{H}}f, \\
D_{\mathcal{V}}^*df &= \{-G^{\bar{\beta}\alpha}i(\dot{\partial}_{\bar{\beta}})\nabla_{\dot{\partial}_{\alpha}} - G^{\bar{\beta}\alpha}\Gamma_{\gamma\alpha}^{\gamma}i(\dot{\partial}_{\bar{\beta}})\}(df) = -G^{\bar{\beta}\alpha}\dot{\partial}_{\alpha}\dot{\partial}_{\bar{\beta}}f - G^{\bar{\beta}\alpha}\Gamma_{\gamma\alpha}^{\gamma}\dot{\partial}_{\bar{\beta}}f \\
&= D_{\mathcal{V}}^*\bar{D}_{\mathcal{V}}f = \Box_{\mathcal{V}}f, \\
\bar{D}_{\mathcal{V}}^*df &= -G^{\bar{\beta}\alpha}\dot{\partial}_{\bar{\beta}}\dot{\partial}_{\alpha}f - G^{\bar{\beta}\alpha}\overline{\Gamma_{\gamma\beta}^{\gamma}}\dot{\partial}_{\alpha}f = \bar{D}_{\mathcal{V}}^*D_{\mathcal{V}}f = \bar{\Box}_{\mathcal{V}}f, \\
D_3^*df &= D_3^*\bar{D}_3f = 0, & \bar{D}_3^*df &= \bar{D}_3^*D_3f = 0, \\
D_4^*df &= D_4^*\bar{D}_4f = 0, & \bar{D}_4^*df &= \bar{D}_4^*D_4f = 0.
\end{aligned}$$

Then, by (4.6) we have

$$\Delta f = d^*df = (\Box_{\mathcal{H}} + \bar{\Box}_{\mathcal{H}} + \Box_{\mathcal{V}} + \bar{\Box}_{\mathcal{V}})f.$$

Remark 4.1 [17, Theorem 5.7] also gives the same formula for a strongly pseudoconvex compact complex Finsler manifold with the associated complex Rund connection D .

Obviously, the type preserving component of Hodge-Laplace operator $\Delta = d^*d + dd^*$ is

$$\Box_{\mathcal{H}} + \bar{\Box}_{\mathcal{H}} + \Box_{\mathcal{V}} + \bar{\Box}_{\mathcal{V}} + \Box_3 + \bar{\Box}_3 + \Box_4 + \bar{\Box}_4. \quad (4.15)$$

Then, for any $\varphi, \psi \in \mathcal{A}^{p,q;r,s}$, we have

$$(\Delta\varphi, \psi) = (\Box_{\mathcal{H}}\varphi, \psi) + (\bar{\Box}_{\mathcal{H}}\varphi, \psi) + (\Box_{\mathcal{V}}\varphi, \psi) + (\bar{\Box}_{\mathcal{V}}\varphi, \psi)$$

$$+ (\square_3 \varphi, \psi) + (\overline{\square}_3 \varphi, \psi) + (\square_4 \varphi, \psi) + (\overline{\square}_4 \varphi, \psi). \quad (4.16)$$

Thus, a form $\varphi \in \mathcal{A}^{p,q;r,s}$ is Δ -harmonic if and only if

$$\begin{aligned} \square_{\mathcal{H}} \varphi &= 0, & \overline{\square}_{\mathcal{H}} \varphi &= 0, & \square_{\mathcal{V}} \varphi &= 0, & \overline{\square}_{\mathcal{V}} \varphi &= 0, \\ \square_3 \varphi &= 0, & \overline{\square}_3 \varphi &= 0, & \square_4 \varphi &= 0, & \overline{\square}_4 \varphi &= 0. \end{aligned} \quad (4.17)$$

Theorem 4.5 *Let M be a strongly pseudoconvex compact complex Finsler manifold. Then by using the Chern Finsler connection, we obtain*

$$\begin{aligned} \square_{\mathcal{H}} &= -G^{\overline{\beta}\alpha} \nabla_{\delta_\alpha} \nabla_{\delta_{\overline{\beta}}} + G^{\overline{\beta}\alpha} e(d\overline{z}^\mu) i(\delta_{\overline{\beta}}) (\nabla_{\delta_\alpha} \nabla_{\delta_{\overline{\mu}}} - \nabla_{\delta_{\overline{\mu}}} \nabla_{\delta_\alpha}) \\ &\quad + G^{\overline{\beta}\alpha} (\Gamma_{\alpha;\gamma}^\gamma - 2\Gamma_{\gamma;\alpha}^\gamma) \nabla_{\delta_{\overline{\beta}}} + G^{\overline{\beta}\alpha} (\overline{\Gamma}_{\beta;\mu}^\gamma - \overline{\Gamma}_{\mu;\beta}^\gamma) e(d\overline{z}^\mu) i(\delta_{\overline{\gamma}}) \nabla_{\delta_\alpha} \\ &\quad + (\Gamma_{\beta;\alpha}^\gamma - \Gamma_{\alpha;\beta}^\gamma) G^{\overline{\tau}\alpha} G^{\overline{\sigma}\beta} G_{\gamma\overline{\tau}} e(d\overline{z}^\varepsilon) i(\delta_{\overline{\sigma}}) \nabla_{\delta_{\overline{\tau}}} \\ &\quad + G^{\overline{\beta}\alpha} (\Gamma_{\alpha;\gamma}^\gamma - 2\Gamma_{\gamma;\alpha}^\gamma) (\overline{\Gamma}_{\mu;\beta}^\omega - \overline{\Gamma}_{\beta;\mu}^\omega) e(d\overline{z}^\mu) i(\delta_{\overline{\omega}}) \\ &\quad + \Gamma_{\beta;\alpha}^\gamma (\overline{\Gamma}_{\sigma;\tau}^\omega - \overline{\Gamma}_{\tau;\sigma}^\omega) G^{\overline{\tau}\alpha} G^{\overline{\sigma}\beta} G_{\gamma\overline{\tau}} e(d\overline{z}^\varepsilon) i(\delta_{\overline{\omega}}) + G^{\overline{\beta}\alpha} \delta_{\overline{\mu}}^\gamma (\Gamma_{\alpha;\gamma}^\gamma - 2\Gamma_{\gamma;\alpha}^\gamma) e(d\overline{z}^\mu) i(\delta_{\overline{\beta}}) \\ &\quad + (\Gamma_{\beta;\alpha}^\gamma - \Gamma_{\alpha;\beta}^\gamma) (\overline{\Gamma}_{\mu;\tau}^\omega - \overline{\Gamma}_{\tau;\mu}^\omega) G^{\overline{\tau}\alpha} G^{\overline{\sigma}\beta} G_{\gamma\overline{\tau}} e(d\overline{z}^\varepsilon) e(d\overline{z}^\mu) i(\delta_{\overline{\sigma}}) i(\delta_{\overline{\omega}}) \\ &\quad + \delta_{\overline{\mu}}^\gamma (\Gamma_{\beta;\alpha}^\gamma) G^{\overline{\tau}\alpha} G^{\overline{\sigma}\beta} G_{\gamma\overline{\tau}} e(d\overline{z}^\mu) e(d\overline{z}^\varepsilon) i(\delta_{\overline{\sigma}}) i(\delta_{\overline{\tau}}) \\ &\quad + \Gamma_{\beta;\alpha}^\gamma \overline{\Gamma}_{\nu;\mu}^\varepsilon G^{\overline{\tau}\alpha} G^{\overline{\sigma}\beta} G_{\gamma\overline{\tau}} e(d\overline{z}^\mu) e(d\overline{z}^\nu) i(\delta_{\overline{\sigma}}) i(\delta_{\overline{\tau}}), \end{aligned} \quad (4.18)$$

$$\begin{aligned} \square_{\mathcal{V}} &= -G^{\overline{\beta}\alpha} \nabla_{\dot{\delta}_\alpha} \nabla_{\dot{\delta}_{\overline{\beta}}} - G^{\overline{\beta}\alpha} \Gamma_{\gamma\alpha}^\gamma \nabla_{\dot{\delta}_{\overline{\beta}}} + G^{\overline{\beta}\alpha} e(\delta\overline{v}^\mu) i(\dot{\delta}_{\overline{\beta}}) (\nabla_{\dot{\delta}_\alpha} \nabla_{\dot{\delta}_{\overline{\mu}}} - \nabla_{\dot{\delta}_{\overline{\mu}}} \nabla_{\dot{\delta}_\alpha}) \\ &\quad - G^{\overline{\beta}\alpha} \dot{\delta}_\alpha (\overline{\Gamma}_{\beta\nu}^\omega) e(d\overline{z}^\nu) i(\delta_{\overline{\omega}}) - G^{\overline{\beta}\alpha} \dot{\delta}_{\overline{\varepsilon}} (\Gamma_{\gamma\alpha}^\gamma) e(\delta\overline{v}^\varepsilon) i(\dot{\delta}_{\overline{\beta}}) - G^{\overline{\beta}\alpha} \overline{\Gamma}_{\beta\nu}^\omega e(d\overline{z}^\nu) i(\delta_{\overline{\omega}}) \nabla_{\dot{\delta}_\alpha} \\ &\quad - G^{\overline{\beta}\alpha} \dot{\delta}_\alpha (\overline{\Gamma}_{\mu\nu}^\omega) e(\delta\overline{v}^\mu) e(d\overline{z}^\nu) i(\dot{\delta}_{\overline{\beta}}) i(\delta_{\overline{\omega}}) + \dot{\delta}_{\overline{\mu}}^\gamma (\Gamma_{\alpha\beta}^\gamma) G^{\overline{\sigma}\alpha} G^{\overline{\tau}\beta} G_{\gamma\overline{\tau}} e(\delta\overline{v}^\mu) e(d\overline{z}^\varepsilon) i(\delta_{\overline{\sigma}}) i(\dot{\delta}_{\overline{\tau}}) \\ &\quad - G^{\overline{\beta}\alpha} \Gamma_{\gamma\alpha}^\gamma \overline{\Gamma}_{\beta\nu}^\omega e(d\overline{z}^\nu) i(\delta_{\overline{\omega}}) + \Gamma_{\alpha\beta}^\gamma G^{\overline{\sigma}\alpha} G^{\overline{\tau}\beta} G_{\gamma\overline{\tau}} e(d\overline{z}^\varepsilon) i(\delta_{\overline{\sigma}}) \nabla_{\dot{\delta}_{\overline{\tau}}} \\ &\quad + \Gamma_{\alpha\beta}^\gamma \overline{\Gamma}_{\sigma\tau}^\omega G^{\overline{\tau}\alpha} G^{\overline{\sigma}\beta} G_{\gamma\overline{\tau}} e(d\overline{z}^\varepsilon) i(\delta_{\overline{\omega}}) - \Gamma_{\alpha\beta}^\gamma \overline{\Gamma}_{\sigma\nu}^\omega G^{\overline{\sigma}\alpha} G^{\overline{\tau}\beta} G_{\gamma\overline{\tau}} e(d\overline{z}^\varepsilon) e(d\overline{z}^\nu) i(\delta_{\overline{\sigma}}) i(\delta_{\overline{\omega}}) \\ &\quad - \Gamma_{\alpha\beta}^\gamma \overline{\Gamma}_{\mu\tau}^\omega G^{\overline{\tau}\alpha} G^{\overline{\sigma}\beta} G_{\gamma\overline{\tau}} e(d\overline{z}^\varepsilon) e(\delta\overline{v}^\mu) i(\dot{\delta}_{\overline{\sigma}}) i(\delta_{\overline{\omega}}) \\ &\quad + \Gamma_{\alpha\beta}^\gamma \overline{\Gamma}_{\mu\nu}^\varepsilon G^{\overline{\tau}\alpha} G^{\overline{\sigma}\beta} G_{\gamma\overline{\tau}} e(\delta\overline{v}^\mu) e(d\overline{z}^\nu) i(\delta_{\overline{\sigma}}) i(\dot{\delta}_{\overline{\tau}}), \end{aligned} \quad (4.19)$$

$$\begin{aligned} \square_3 &= \delta_\sigma (\Gamma_{\overline{\varepsilon}}^\omega) \delta_{\overline{\beta}} (\Gamma_\mu^\alpha) G^{\overline{\beta}\sigma} G^{\overline{\varepsilon}\mu} G_{\tau\overline{\omega}} e(\delta v^\tau) i(\dot{\delta}_\alpha) \\ &\quad - \delta_\sigma (\Gamma_{\overline{\varepsilon}}^\omega) \delta_{\overline{\beta}} (\Gamma_\mu^\alpha) G^{\overline{\tau}\sigma} G^{\overline{\varepsilon}\nu} G_{\alpha\overline{\omega}} e(dz^\mu) e(d\overline{z}^\beta) i(\delta_\nu) i(\delta_{\overline{\gamma}}) \\ &\quad - \delta_\sigma (\Gamma_{\overline{\varepsilon}}^\omega) \delta_{\overline{\beta}} (\Gamma_\mu^\alpha) G^{\overline{\beta}\sigma} G^{\overline{\varepsilon}\nu} G_{\tau\overline{\omega}} e(dz^\mu) e(\delta v^\tau) i(\delta_\nu) i(\dot{\delta}_\alpha) \\ &\quad + \delta_\sigma (\Gamma_{\overline{\varepsilon}}^\omega) \delta_{\overline{\beta}} (\Gamma_\mu^\alpha) G^{\overline{\tau}\sigma} G^{\overline{\varepsilon}\mu} G_{\tau\overline{\omega}} e(d\overline{z}^\beta) e(\delta v^\tau) i(\delta_{\overline{\gamma}}) i(\dot{\delta}_\alpha), \end{aligned} \quad (4.20)$$

$$\begin{aligned} \square_4 &= \dot{\delta}_\beta (\Gamma_{\overline{\mu}}^\alpha) \dot{\delta}_{\overline{\sigma}} (\Gamma_\nu^\varepsilon) G^{\overline{\sigma}\beta} G^{\overline{\mu}\nu} G_{\tau\overline{\alpha}} e(\delta v^\tau) i(\dot{\delta}_\varepsilon) \\ &\quad + \dot{\delta}_\beta (\Gamma_{\overline{\mu}}^\alpha) \dot{\delta}_{\overline{\sigma}} (\Gamma_\omega^\varepsilon) G^{\overline{\sigma}\beta} G^{\overline{\mu}\nu} G_{\tau\overline{\alpha}} e(dz^\omega) e(\delta v^\tau) i(\delta_\nu) i(\dot{\delta}_\varepsilon) \\ &\quad + \dot{\delta}_\beta (\Gamma_{\overline{\mu}}^\alpha) \dot{\delta}_{\overline{\sigma}} (\Gamma_\nu^\varepsilon) G^{\overline{\tau}\beta} G^{\overline{\mu}\nu} G_{\tau\overline{\alpha}} e(\delta v^\tau) e(\delta\overline{v}^\sigma) i(\dot{\delta}_{\overline{\gamma}}) i(\dot{\delta}_\varepsilon) \\ &\quad - \dot{\delta}_\beta (\Gamma_{\overline{\mu}}^\alpha) \dot{\delta}_{\overline{\sigma}} (\Gamma_\omega^\tau) G^{\overline{\tau}\beta} G^{\overline{\mu}\nu} G_{\tau\overline{\alpha}} e(dz^\omega) e(\delta\overline{v}^\sigma) i(\delta_\nu) i(\dot{\delta}_{\overline{\gamma}}). \end{aligned} \quad (4.21)$$

Proof We only calculate $\square_{\mathcal{V}}$, and the others are similar to obtain.

From (3.12) and (4.9), by a direct computation, we have

$$\begin{aligned} D_{\mathcal{V}}^* \overline{D}_{\mathcal{V}} &= -G^{\overline{\beta}\alpha} \nabla_{\dot{\delta}_\alpha} \nabla_{\dot{\delta}_{\overline{\beta}}} + G^{\overline{\beta}\alpha} e(\delta\overline{v}^\mu) i(\dot{\delta}_{\overline{\beta}}) \nabla_{\dot{\delta}_\alpha} \nabla_{\dot{\delta}_{\overline{\mu}}} - G^{\overline{\beta}\alpha} \dot{\delta}_\alpha (\overline{\Gamma}_{\beta\nu}^\omega) e(d\overline{z}^\nu) i(\delta_{\overline{\omega}}) \\ &\quad - G^{\overline{\beta}\alpha} \overline{\Gamma}_{\beta\nu}^\omega e(d\overline{z}^\nu) i(\delta_{\overline{\omega}}) \nabla_{\dot{\delta}_\alpha} - G^{\overline{\beta}\alpha} \dot{\delta}_\alpha (\overline{\Gamma}_{\mu\nu}^\omega) e(\delta\overline{v}^\mu) e(d\overline{z}^\nu) i(\dot{\delta}_{\overline{\beta}}) i(\delta_{\overline{\omega}}) \end{aligned}$$

$$\begin{aligned}
& -G^{\bar{\beta}\alpha}\overline{\Gamma_{\mu\nu}^\omega}e(\delta\bar{v}^\mu)e(d\bar{z}^\nu)i(\dot{\partial}_{\bar{\beta}})i(\delta_{\bar{\omega}})\nabla_{\dot{\partial}_\alpha}-G^{\bar{\beta}\alpha}\Gamma_{\gamma\alpha}^\gamma\nabla_{\dot{\partial}_{\bar{\beta}}}+G^{\bar{\beta}\alpha}\Gamma_{\gamma\alpha}^\gamma e(\delta\bar{v}^\mu)i(\dot{\partial}_{\bar{\beta}})\nabla_{\dot{\partial}_\mu}\\
& -G^{\bar{\beta}\alpha}\Gamma_{\gamma\alpha}^\gamma\overline{\Gamma_{\beta\nu}^\omega}e(d\bar{z}^\nu)i(\delta_{\bar{\omega}})-G^{\bar{\beta}\alpha}\Gamma_{\gamma\alpha}^\gamma\overline{\Gamma_{\mu\nu}^\omega}e(\delta\bar{v}^\mu)e(d\bar{z}^\nu)i(\dot{\partial}_{\bar{\beta}})i(\delta_{\bar{\omega}})\\
& +\Gamma_{\alpha\beta}^\gamma G^{\bar{\sigma}\alpha}G^{\bar{\tau}\beta}G_{\gamma\bar{\varepsilon}}e(d\bar{z}^\varepsilon)i(\delta_{\bar{\tau}})\nabla_{\dot{\partial}_{\bar{\sigma}}}+\Gamma_{\alpha\beta}^\gamma G^{\bar{\sigma}\alpha}G^{\bar{\tau}\beta}G_{\gamma\bar{\varepsilon}}e(d\bar{z}^\varepsilon)e(\delta\bar{v}^\mu)i(\delta_{\bar{\tau}})i(\dot{\partial}_{\bar{\sigma}})\nabla_{\dot{\partial}_\mu}\\
& +\Gamma_{\alpha\beta}^\gamma\overline{\Gamma_{\sigma\tau}^\omega}G^{\bar{\sigma}\alpha}G^{\bar{\tau}\beta}G_{\gamma\bar{\varepsilon}}e(d\bar{z}^\varepsilon)i(\delta_{\bar{\omega}})-\Gamma_{\alpha\beta}^\gamma\overline{\Gamma_{\mu\nu}^\omega}G^{\bar{\sigma}\alpha}G^{\bar{\tau}\beta}G_{\gamma\bar{\varepsilon}}e(d\bar{z}^\varepsilon)e(d\bar{z}^\nu)i(\delta_{\bar{\tau}})i(\delta_{\bar{\omega}})\\
& -\Gamma_{\alpha\beta}^\gamma\overline{\Gamma_{\mu\tau}^\omega}G^{\bar{\sigma}\alpha}G^{\bar{\tau}\beta}G_{\gamma\bar{\varepsilon}}e(d\bar{z}^\varepsilon)e(\delta\bar{v}^\mu)i(\dot{\partial}_{\bar{\sigma}})i(\delta_{\bar{\omega}})\\
& -\Gamma_{\alpha\beta}^\gamma\overline{\Gamma_{\mu\nu}^\omega}G^{\bar{\sigma}\alpha}G^{\bar{\tau}\beta}G_{\gamma\bar{\varepsilon}}e(d\bar{z}^\varepsilon)e(\delta\bar{v}^\mu)e(d\bar{z}^\mu)i(\dot{\partial}_{\bar{\sigma}})i(\delta_{\bar{\omega}})i(\delta_{\bar{\omega}}),\\
\overline{D}_V D_V^* = & -G^{\bar{\beta}\alpha}e(\delta\bar{v}^\mu)i(\dot{\partial}_{\bar{\beta}})\nabla_{\dot{\partial}_\mu}\nabla_{\dot{\partial}_\alpha}-G^{\bar{\beta}\alpha}\dot{\partial}_{\bar{\varepsilon}}(\Gamma_{\gamma\alpha}^\gamma)e(\delta\bar{v}^\varepsilon)i(\dot{\partial}_{\bar{\beta}})\\
& -G^{\bar{\beta}\alpha}\Gamma_{\gamma\alpha}^\gamma e(\delta\bar{v}^\mu)i(\dot{\partial}_{\bar{\beta}})\nabla_{\dot{\partial}_\mu}+\dot{\partial}_{\bar{\mu}}(\Gamma_{\alpha\beta}^\gamma)G^{\bar{\sigma}\alpha}G^{\bar{\tau}\beta}G_{\gamma\bar{\varepsilon}}e(\delta\bar{v}^\mu)e(d\bar{z}^\varepsilon)i(\delta_{\bar{\tau}})i(\dot{\partial}_{\bar{\sigma}})\\
& +\Gamma_{\alpha\beta}^\gamma G^{\bar{\sigma}\alpha}G^{\bar{\tau}\beta}G_{\gamma\bar{\varepsilon}}e(\delta\bar{v}^\mu)e(d\bar{z}^\varepsilon)i(\delta_{\bar{\tau}})i(\dot{\partial}_{\bar{\sigma}})\nabla_{\dot{\partial}_\mu}-G^{\bar{\beta}\alpha}\overline{\Gamma_{\mu\nu}^\omega}e(\delta\bar{v}^\mu)e(d\bar{z}^\nu)i(\delta_{\bar{\omega}})i(\dot{\partial}_{\bar{\beta}})\nabla_{\dot{\partial}_\alpha}\\
& -G^{\bar{\beta}\alpha}\Gamma_{\gamma\alpha}^\gamma\overline{\Gamma_{\mu\nu}^\omega}e(\delta\bar{v}^\mu)e(d\bar{z}^\nu)i(\delta_{\bar{\omega}})i(\dot{\partial}_{\bar{\beta}})+\Gamma_{\alpha\beta}^\gamma\overline{\Gamma_{\sigma\nu}^\omega}G^{\bar{\sigma}\alpha}G^{\bar{\tau}\beta}G_{\gamma\bar{\varepsilon}}e(\delta\bar{v}^\mu)e(d\bar{z}^\nu)i(\delta_{\bar{\tau}})i(\dot{\partial}_{\bar{\sigma}})\\
& -\Gamma_{\alpha\beta}^\gamma\overline{\Gamma_{\mu\nu}^\omega}G^{\bar{\sigma}\alpha}G^{\bar{\tau}\beta}G_{\gamma\bar{\varepsilon}}e(\delta\bar{v}^\mu)e(d\bar{z}^\mu)e(d\bar{z}^\varepsilon)i(\delta_{\bar{\omega}})i(\delta_{\bar{\tau}})i(\dot{\partial}_{\bar{\sigma}}),
\end{aligned}$$

and \square_V can be obtained by plusing the above two formulas.

Remark 4.2 If M is a compact Kähler Finsler manifold, then $\Gamma_{\mu;\nu}^\alpha = \Gamma_{\nu;\mu}^\alpha$. Therefore (see [17])

$$\begin{aligned}
\square_{\mathcal{H}} = & -G^{\bar{\beta}\alpha}\nabla_{\delta_\alpha}\nabla_{\delta_{\bar{\beta}}}+G^{\bar{\beta}\alpha}(\nabla_{[\delta_\alpha,\delta_{\bar{\mu}}]}+\Omega(\delta_\alpha,\delta_{\bar{\mu}}))e(d\bar{z}^\mu)i(\delta_{\bar{\beta}})-G^{\bar{\beta}\alpha}\Gamma_{\alpha;\gamma}^\gamma\nabla_{\delta_{\bar{\beta}}}\\
& -G^{\bar{\beta}\alpha}\delta_{\bar{\mu}}(\Gamma_{\alpha;\gamma}^\gamma)e(d\bar{z}^\mu)i(\delta_{\bar{\beta}})+\delta_{\bar{\mu}}(\Gamma_{\beta;\alpha}^\gamma)G^{\bar{\sigma}\alpha}G^{\bar{\tau}\beta}G_{\gamma\bar{\varepsilon}}e(d\bar{z}^\mu)e(d\bar{z}^\varepsilon)i(\delta_{\bar{\sigma}})i(\delta_{\bar{\tau}})i(\delta_{\bar{\sigma}})i(\delta_{\bar{\tau}})\\
& +\Gamma_{\beta;\alpha}^\gamma\overline{\Gamma_{\nu;\mu}^\varepsilon}G^{\bar{\sigma}\alpha}G^{\bar{\tau}\beta}G_{\gamma\bar{\varepsilon}}e(d\bar{z}^\mu)e(d\bar{z}^\nu)i(\delta_{\bar{\sigma}})i(\delta_{\bar{\tau}}).
\end{aligned}$$

Since the Kähler Finsler condition shows no influence on $\square_V, \square_3, \square_4$ and their conjugations, the expressions of $\square_V, \square_3, \square_4$ are valid for a Kähler Finsler manifold.

Remark 4.3 By conjugations, we can also obtain the complex Laplacians $\overline{\square}_{\mathcal{H}}, \overline{\square}_V, \overline{\square}_3$ and $\overline{\square}_4$.

5 A Vanishing Theorem

Let $dV = (-1)^m \mathcal{G}^2 dz \wedge d\bar{z} \wedge \delta v \wedge \delta \bar{v}$ be the volume form associated to the Hermitian metric \tilde{G} of \widetilde{M} , where $\mathcal{G} = \det(G_{\alpha\bar{\beta}})$. Denote \mathcal{L}_X by the Lie derivative with respect to $X \in \mathcal{X}(T\widetilde{M})$. Then the divergence of X is defined by the equation

$$\mathcal{L}_X dV = (\operatorname{div} X) dV. \quad (5.1)$$

Lemma 5.1 Let (M, F) be a strongly pseudoconvex compact complex Finsler manifold with $\dim_{\mathbb{C}} M = m$. Then for $X = X^\mu \delta_\mu + \overline{X}^\nu \delta_{\bar{\nu}} + \dot{X}^\beta \dot{\partial}_\beta + \overline{\dot{X}}^\beta \dot{\partial}_{\bar{\beta}} \in \mathcal{X}(T\widetilde{M})$, we have

$$\begin{aligned}
\operatorname{div}(X) = & \delta_\alpha(X^\alpha) + \delta_{\bar{\alpha}}(\overline{X}^\alpha) + \dot{\partial}_\alpha(\dot{X}^\alpha) + \dot{\partial}_{\bar{\alpha}}(\overline{\dot{X}}^\alpha) + X^\mu \Gamma_{\alpha;\mu}^\alpha + \overline{X}^\nu \overline{\Gamma}_{\alpha;\nu}^\alpha \\
& + 2\dot{X}^\alpha \Gamma_{\gamma\alpha}^\gamma + 2\overline{\dot{X}}^\alpha \overline{\Gamma}_{\gamma\alpha}^\gamma.
\end{aligned} \quad (5.2)$$

Proof By (2.3), we have

$$[X, \delta_\alpha] = -\delta_\alpha(X^\mu)\delta_\mu - \overline{X}^\nu \delta_{\bar{\nu}}(\Gamma_\alpha^\sigma)\dot{\partial}_\sigma + \overline{X}^\nu \delta_\alpha(\overline{\Gamma}_\nu^\tau)\dot{\partial}_\tau - \delta_\alpha(\overline{X}^\nu)\delta_{\bar{\nu}} - \dot{X}^\beta \Gamma_{\beta;\alpha}^\sigma \dot{\partial}_\sigma$$

$$\begin{aligned}
& -\delta_\alpha(\dot{X}^\beta)\dot{\partial}_\beta - \overline{X}^\beta \Gamma_{\beta;\alpha}^\sigma \dot{\partial}_\sigma - \delta_\alpha(\overline{X}^\beta)\dot{\partial}_{\overline{\beta}}, \\
[X, \delta_{\overline{\alpha}}] &= X^\nu \delta_{\overline{\alpha}}(\Gamma_\nu^\sigma)\dot{\partial}_\sigma - \delta_{\overline{\alpha}}(X^\mu)\delta_\mu - X^\nu \delta_\nu(\overline{\Gamma}_\alpha^\tau)\dot{\partial}_{\overline{\tau}} - \delta_{\overline{\alpha}}(\overline{X}^\nu)\delta_{\overline{\nu}} - \dot{X}^\beta \Gamma_{\beta;\overline{\alpha}}^\sigma \dot{\partial}_\sigma \\
& - \delta_{\overline{\alpha}}(\dot{X}^\beta)\dot{\partial}_\beta - \overline{X}^\beta \overline{\Gamma}_{\beta;\alpha}^\sigma \dot{\partial}_{\overline{\sigma}} - \delta_{\overline{\alpha}}(\overline{X}^\beta)\dot{\partial}_{\overline{\beta}}, \\
[X, \dot{\partial}_\alpha] &= X^\mu \Gamma_{\alpha;\mu}^\sigma \dot{\partial}_\sigma - \dot{\partial}_\alpha(X^\mu)\delta_\mu + \overline{X}^\nu \Gamma_{\alpha;\overline{\nu}}^\sigma \dot{\partial}_{\overline{\sigma}} - \dot{\partial}_\alpha(\overline{X}^\nu)\delta_{\overline{\nu}} - \dot{\partial}_\alpha(\dot{X}^\beta)\dot{\partial}_\beta \\
& - \dot{\partial}_\alpha(\overline{X}^\beta)\dot{\partial}_{\overline{\beta}}, \\
[X, \dot{\partial}_{\overline{\alpha}}] &= X^\mu \Gamma_{\overline{\alpha};\mu}^\sigma \dot{\partial}_\sigma - \dot{\partial}_{\overline{\alpha}}(X^\mu)\delta_\mu + \overline{X}^\nu \overline{\Gamma}_{\alpha;\nu}^\sigma \dot{\partial}_{\overline{\sigma}} - \dot{\partial}_{\overline{\alpha}}(\overline{X}^\nu)\delta_{\overline{\nu}} - \dot{\partial}_{\overline{\alpha}}(\dot{X}^\beta)\dot{\partial}_\beta \\
& - \dot{\partial}_{\overline{\alpha}}(\overline{X}^\beta)\dot{\partial}_{\overline{\beta}},
\end{aligned}$$

and then

$$\begin{aligned}
\operatorname{div}(X)(-1)^m \mathcal{G}^2 &= \operatorname{div}(X) dV(\delta_1, \dots, \delta_m, \delta_{\overline{1}}, \dots, \delta_{\overline{m}}, \dot{\partial}_1, \dots, \dot{\partial}_m, \dot{\partial}_{\overline{1}}, \dots, \dot{\partial}_{\overline{m}}) \\
&= \mathcal{L}_X dV(\delta_1, \dots, \delta_m, \delta_{\overline{1}}, \dots, \delta_{\overline{m}}, \dot{\partial}_1, \dots, \dot{\partial}_m, \dot{\partial}_{\overline{1}}, \dots, \dot{\partial}_{\overline{m}}) \\
&= X(dV(\delta_1, \dots, \delta_m, \delta_{\overline{1}}, \dots, \delta_{\overline{m}}, \dot{\partial}_1, \dots, \dot{\partial}_m, \dot{\partial}_{\overline{1}}, \dots, \dot{\partial}_{\overline{m}})) \\
&\quad - \sum_{\alpha=1}^m dV(\delta_1, \dots, \delta_{\alpha-1}, [X, \delta_\alpha], \delta_{\alpha+1}, \dots, \delta_m, \dots, \dots, \dots) \\
&\quad - \sum_{\alpha=1}^m dV(\dots, \delta_{\overline{1}}, \dots, \delta_{\overline{\alpha-1}}, [X, \delta_{\overline{\alpha}}], \delta_{\overline{\alpha+1}}, \dots, \delta_{\overline{m}}, \dots, \dots) \\
&\quad - \sum_{\alpha=1}^m dV(\dots, \dots, \dot{\partial}_1, \dots, \dot{\partial}_{\alpha-1}, [X, \dot{\partial}_\alpha], \dot{\partial}_{\alpha+1}, \dots, \dot{\partial}_m, \dots) \\
&\quad - \sum_{\alpha=1}^m dV(\dots, \dots, \dots, \dot{\partial}_{\overline{1}}, \dots, \dot{\partial}_{\overline{\alpha-1}}, [X, \dot{\partial}_{\overline{\alpha}}], \dot{\partial}_{\overline{\alpha+1}}, \dots, \dot{\partial}_{\overline{m}}) \\
&= (-1)^m \mathcal{G}^2 \{ \delta_\alpha(X^\alpha) + \delta_{\overline{\alpha}}(\overline{X}^\alpha) + \dot{\partial}_\alpha(\dot{X}^\alpha) + \dot{\partial}_{\overline{\alpha}}(\overline{X}^\alpha) \\
&\quad + X^\mu \Gamma_{\alpha;\mu}^\alpha + \overline{X}^\nu \overline{\Gamma}_{\alpha;\nu}^\alpha + 2\dot{X}^\alpha \Gamma_{\gamma\alpha}^\gamma + 2\overline{X}^\alpha \overline{\Gamma}_{\gamma\alpha}^\gamma \},
\end{aligned}$$

that is,

$$\begin{aligned}
\operatorname{div}(X) &= \delta_\alpha(X^\alpha) + \delta_{\overline{\alpha}}(\overline{X}^\alpha) + \dot{\partial}_\alpha(\dot{X}^\alpha) + \dot{\partial}_{\overline{\alpha}}(\overline{X}^\alpha) + X^\mu \Gamma_{\alpha;\mu}^\alpha + \overline{X}^\nu \overline{\Gamma}_{\alpha;\nu}^\alpha \\
&\quad + 2\dot{X}^\alpha \Gamma_{\gamma\alpha}^\gamma + 2\overline{X}^\alpha \overline{\Gamma}_{\gamma\alpha}^\gamma.
\end{aligned}$$

Theorem 5.1 *Let (M, F) be a strongly pseudoconvex compact complex Finsler manifold with $\dim_{\mathbb{C}} M = m$. Then for any function $f \in C^\infty(\widetilde{M})$, we have*

$$\int_{\widetilde{M}} \{ \Delta f + \overline{\mathcal{B}}^\beta \delta_{\overline{\beta}} f + \mathcal{B}^\alpha \delta_\alpha f \} dV = 0, \quad (5.3)$$

where $\mathcal{B}^\alpha = G^{\overline{\beta}\alpha} \overline{\Gamma}_{\gamma;\beta}^\gamma$.

Proof Let $X = G^{\overline{\beta}\alpha} \delta_{\overline{\beta}} f \delta_\alpha + G^{\overline{\beta}\alpha} \delta_\alpha f \delta_{\overline{\beta}} + G^{\overline{\beta}\alpha} \dot{\partial}_{\overline{\beta}} f \dot{\partial}_\alpha + G^{\overline{\beta}\alpha} \dot{\partial}_\alpha f \dot{\partial}_{\overline{\beta}}$. By (5.2), we have

$$\begin{aligned}
\operatorname{div}(X) &= \delta_\alpha(G^{\overline{\beta}\alpha} \delta_{\overline{\beta}} f) + \delta_{\overline{\beta}}(G^{\overline{\beta}\alpha} \delta_\alpha f) + \dot{\partial}_\alpha(G^{\overline{\beta}\alpha} \dot{\partial}_{\overline{\beta}} f) + \dot{\partial}_{\overline{\beta}}(G^{\overline{\beta}\alpha} \dot{\partial}_\alpha f) \\
&\quad + G^{\overline{\beta}\alpha} \delta_{\overline{\beta}} f \Gamma_{\gamma;\alpha}^\gamma + G^{\overline{\beta}\alpha} \delta_\alpha f \overline{\Gamma}_{\gamma;\beta}^\gamma + 2G^{\overline{\beta}\alpha} \dot{\partial}_{\overline{\beta}} f \Gamma_{\gamma\alpha}^\gamma + 2G^{\overline{\beta}\alpha} \dot{\partial}_\alpha f \overline{\Gamma}_{\gamma\beta}^\gamma
\end{aligned}$$

$$\begin{aligned}
&= G^{\bar{\beta}\alpha}(\delta_\alpha\delta_{\bar{\beta}} + \delta_{\bar{\beta}}\delta_\alpha)f + G^{\bar{\beta}\alpha}(\dot{\partial}_\alpha\dot{\partial}_{\bar{\beta}} + \dot{\partial}_{\bar{\beta}}\dot{\partial}_\alpha)f + G^{\bar{\beta}\alpha}(\Gamma_{\gamma;\alpha}^\gamma - \Gamma_{\alpha;\gamma}^\gamma)\delta_{\bar{\beta}}f \\
&\quad + G^{\bar{\beta}\alpha}(\bar{\Gamma}_{\gamma;\beta}^\gamma - \bar{\Gamma}_{\beta;\gamma}^\gamma)\delta_\alpha f + G^{\bar{\beta}\alpha}\Gamma_{\gamma\alpha}^\gamma\dot{\partial}_{\bar{\beta}}f + G^{\bar{\beta}\alpha}\bar{\Gamma}_{\gamma\beta}^\gamma\dot{\partial}_\alpha f,
\end{aligned}$$

and by Theorem 4.4, we have

$$\begin{aligned}
(\square_{\mathcal{H}} + \bar{\square}_{\mathcal{H}})f &= -G^{\bar{\beta}\alpha}(\delta_\alpha\delta_{\bar{\beta}} + \delta_{\bar{\beta}}\delta_\alpha)f + G^{\bar{\beta}\alpha}(\Gamma_{\alpha;\gamma}^\gamma - 2\Gamma_{\gamma;\alpha}^\gamma)\delta_{\bar{\beta}}f + G^{\bar{\beta}\alpha}(\bar{\Gamma}_{\beta;\gamma}^\gamma - 2\bar{\Gamma}_{\gamma;\beta}^\gamma)\delta_\alpha f, \\
(\square_{\mathcal{V}} + \bar{\square}_{\mathcal{V}})f &= G^{\bar{\beta}\alpha}(\dot{\partial}_\alpha\dot{\partial}_{\bar{\beta}} + \dot{\partial}_{\bar{\beta}}\dot{\partial}_\alpha)f + G^{\bar{\beta}\alpha}\Gamma_{\gamma\alpha}^\gamma\dot{\partial}_{\bar{\beta}}f + G^{\bar{\beta}\alpha}\bar{\Gamma}_{\gamma\beta}^\gamma\dot{\partial}_\alpha f,
\end{aligned}$$

therefore,

$$\Delta f + G^{\bar{\beta}\alpha}\Gamma_{\gamma;\alpha}^\gamma\delta_{\bar{\beta}}f + G^{\bar{\beta}\alpha}\bar{\Gamma}_{\gamma;\beta}^\gamma\delta_\alpha f = -\operatorname{div}(X),$$

that is,

$$\int_{\widetilde{M}} \{\Delta f + G^{\bar{\beta}\alpha}\Gamma_{\gamma;\alpha}^\gamma\delta_{\bar{\beta}}f + G^{\bar{\beta}\alpha}\bar{\Gamma}_{\gamma;\beta}^\gamma\delta_\alpha f\} dV = 0.$$

Theorem 5.2 *Let (M, F) be a strongly pseudoconvex compact complex Finsler manifold with $\dim_{\mathbb{C}} M = m$. For any $\varphi \in \mathcal{A}^{p,0,r,0}$, if*

$$\operatorname{Re}(\langle \varphi, \mathcal{M}\varphi \rangle) > 0, \quad (5.4)$$

where $\mathcal{M} = \mathcal{B}^\beta \nabla_{\dot{\partial}_\beta}$, then there exists no non-zero $\varphi \in \mathcal{A}^{p,0,r,0}$ such that $\bar{\partial}\varphi = 0$.

Proof For all $\varphi \in \mathcal{A}^{p,0,r,0}$ such that $\bar{\partial}\varphi = 0$, we have $\nabla_{\delta_{\bar{\alpha}}}\varphi = 0$, $\nabla_{\dot{\partial}_{\bar{\alpha}}}\varphi = 0$.

If there exists a non-zero $\varphi \in \mathcal{A}^{p,0,r,0}$ such that $\bar{\partial}\varphi = 0$, then

$$\begin{aligned}
\square_{\mathcal{H}}\varphi &= -G^{\bar{\beta}\alpha}\nabla_{\delta_\alpha}\nabla_{\delta_{\bar{\beta}}}\varphi - G^{\bar{\beta}\alpha}\Gamma_{\alpha;\gamma}^\gamma\nabla_{\delta_{\bar{\beta}}}\varphi = 0, \\
\square_{\mathcal{V}}\varphi &= -G^{\bar{\beta}\alpha}\nabla_{\dot{\partial}_\alpha}\nabla_{\dot{\partial}_{\bar{\beta}}}\varphi - G^{\bar{\beta}\alpha}\Gamma_{\gamma\alpha}^\gamma\nabla_{\dot{\partial}_{\bar{\beta}}}\varphi = 0.
\end{aligned}$$

Let $f = |\varphi|^2 = \langle \varphi, \varphi \rangle$. We have

$$\begin{aligned}
G^{\bar{\beta}\alpha}(\delta_\alpha\delta_{\bar{\beta}} + \delta_{\bar{\beta}}\delta_\alpha)|\varphi|^2 &= 2|D_{\mathcal{H}}\varphi|^2 + 2|\bar{D}_{\mathcal{H}}\varphi|^2, \\
G^{\bar{\beta}\alpha}\dot{\partial}_\alpha\dot{\partial}_{\bar{\beta}}|\varphi|^2 &= |D_{\mathcal{V}}\varphi|^2 + |\bar{D}_{\mathcal{V}}\varphi|^2,
\end{aligned}$$

where $|D_{\mathcal{H}}\varphi|^2 = G^{\bar{\beta}\alpha}\langle \nabla_{\delta_\alpha}\varphi, \nabla_{\delta_{\bar{\beta}}}\varphi \rangle$, $|\bar{D}_{\mathcal{H}}\varphi|^2 = G^{\bar{\beta}\alpha}\langle \nabla_{\delta_{\bar{\alpha}}}\varphi, \nabla_{\delta_\beta}\varphi \rangle = 0$, $|D_{\mathcal{V}}\varphi|^2 = G^{\bar{\beta}\alpha}\langle \nabla_{\dot{\partial}_\alpha}\varphi, \nabla_{\dot{\partial}_{\bar{\beta}}}\varphi \rangle$, $|\bar{D}_{\mathcal{V}}\varphi|^2 = G^{\bar{\beta}\alpha}\langle \nabla_{\dot{\partial}_{\bar{\alpha}}}\varphi, \nabla_{\dot{\partial}_\beta}\varphi \rangle = 0$, and therefore, by (5.3)

$$\begin{aligned}
&\int_{\widetilde{M}} \{\Delta f + G^{\bar{\beta}\alpha}\Gamma_{\gamma;\alpha}^\gamma\delta_{\bar{\beta}}f + G^{\bar{\beta}\alpha}\bar{\Gamma}_{\gamma;\beta}^\gamma\delta_\alpha f\} dV \\
&= \int_{\widetilde{M}} \{2|D_{\mathcal{H}}\varphi|^2 + 2|\bar{D}_{\mathcal{H}}\varphi|^2 + 2|D_{\mathcal{V}}\varphi|^2 + 2|\bar{D}_{\mathcal{V}}\varphi|^2 + G^{\bar{\beta}\alpha}\Gamma_{\gamma\alpha}^\gamma\dot{\partial}_{\bar{\beta}}|\varphi|^2 + G^{\bar{\beta}\alpha}\bar{\Gamma}_{\gamma\beta}^\gamma\dot{\partial}_\alpha|\varphi|^2\} dV \\
&= \int_{\widetilde{M}} 2\{|D_{\mathcal{H}}\varphi|^2 + |D_{\mathcal{V}}\varphi|^2 + \operatorname{Re}(G^{\bar{\beta}\alpha}\Gamma_{\gamma\alpha}^\gamma\dot{\partial}_{\bar{\beta}}|\varphi|^2)\} dV \\
&= \int_{\widetilde{M}} 2\{|D_{\mathcal{H}}\varphi|^2 + |D_{\mathcal{V}}\varphi|^2 + \operatorname{Re}(G^{\bar{\beta}\alpha}\Gamma_{\gamma\alpha}^\gamma\langle \varphi, \nabla_{\dot{\partial}_{\bar{\beta}}}\varphi \rangle)\} dV \\
&= 0.
\end{aligned}$$

Thus, if

$$\operatorname{Re}(G^{\bar{\beta}\alpha}\Gamma_{\gamma\alpha}^\gamma\langle \varphi, \nabla_{\dot{\partial}_{\bar{\beta}}}\varphi \rangle) > 0,$$

then there is no non-zero $\varphi \in \mathcal{A}^{p,0,r,0}$ such that $\bar{\partial}\varphi = 0$.

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References

- [1] Bochner, S., Vector fields and Ricci curvature, *Bull. Amer. Math. Soc.*, **52**, 1946, 776–797.
- [2] Bochner, S., Curvature in Hermitian metric, *Bull. Amer. Math. Soc.*, **53**, 1947, 179–195.
- [3] Bochner, S., Curvature and Betti numbers I, II, *Ann. of Math.*, **49**, 1948, 379–390; **50**, 1949, 77–93.
- [4] Yano, K. and Bochner, S., Curvature and Betti Numbers, Princeton University Press, New Jersey, 1953.
- [5] Wu, H.-H., The Bochner Technique in Differential Geometry, Harwood Academic Publishers, London, Paris, 1988.
- [6] Morrow, J. and Kodaira, K., Complex Manifold, Holt, Rinehart and Winston, Inc., New York, 1971.
- [7] Antonelli, P. L. and Lackey, B., The Theory of Finslerian Laplacians and Applications, MAIA, **459**, Kluwer Academic Publishers, Dordrecht, 1998.
- [8] Bao, D. and Lackey, B., Randers surfaces whose Laplacian have completely positive symbol, *Nonlinear Analysis*, **38**, 1999, 27–40.
- [9] Bao, D. and Lackey, B., A Hodge decomposition theorem for Finsler spaces, *C. R. Math. Acad. Sci. Paris*, **323**(1), 1996, 51–56.
- [10] Chern, S. S. and Shen, Z., An Introduction to Riemannian-Finsler Geometry, Springer-Verlag, New York, 2000.
- [11] Munteanu, O., Weitzenböck formulas for horizontal and vertical Laplacians, *Houston J. Math.*, **29**(4), 2003, 889–900.
- [12] Yan, R., Laplace operator on Finsler manifold, *Acta Math. Sci.*, **24A**(4), 2004, 420–425.
- [13] Zhong, C. P. and Zhong, T. D., Horizontal $\bar{\partial}$ -Laplacian on complex Finsler manifolds, *Sci. China Ser. A*, **48**(Supp.), 2005, 377–391.
- [14] Zhong, C. P. and Zhong, T. D., Hodge decomposition theorem on strongly Kähler Finsler manifolds, *Sci. China Ser. A*, **49**(11), 2006, 1696–1714.
- [15] Abate, M. and Patrizio, G., Finsler Metric — A Global Approach, Lect. Notes in Math., **1591**, Springer-Verlag, Berlin, Heidelberg, 1994.
- [16] Zhong, C. P., Laplacians on the holomorphic tangent bundle of a Kaehler manifold, *Sci. China Ser. A*, **52**(12), 2009, 2841–2854.
- [17] Zhong, C. P., A vanishing theorem on Kaehler Finsler manifolds, *Diff. Geom. Appl.*, **27**, 2009, 551–565.
- [18] Kodaira, K., On a differential-geometric method in the theory of analytic stacks, *Proc. Natl. Acad. Sci. USA*, **39**, 1953, 1268–1273.
- [19] Siu, Y.-T., The complex-analyticity of harmonic maps and the strong rigidity of compact Kähler manifolds, *Ann. of Math.*, **112**, 1980, 73–111.
- [20] Siu, Y.-T., Complex analyticity of harmonic maps, vanishing and Lefschetz theorems, *J. Diff. Geom.*, **17**, 1982, 55–138.
- [21] Xiao, J. X., Zhong, T. D. and Qiu, C. H., Bochner technique in strongly Kähler Finsler manifold, *Acta Math. Sci.*, **30B**(1), 2010, 89–106.
- [22] Xiao, J. X., Zhong, T. D. and Qiu, C. H., Bochner-Kodaira techniques on Kähler Finsler manifolds, *J. Geom. Anal.*, submitted.
- [23] Munteanu, O., Complex Spaces in Finsler, Lagrange and Hamilton Geometries, Kluwer Academic Publishers, Dordrecht, 2004.
- [24] Pitis, G. and Munteanu, O., v -Cohomology of complex Finsler manifolds, *Stud. Univ. Babes-Bolyai Math.*, **18**(3), 1998, 889–900.