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Remarks on Vanishing Viscosity Limits for the 3D Navier-Stokes Equations with a Slip Boundary Condition*

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(*Dedicated to Professor Roger Temam on the Occasion of his* 70*th Birthday*)

Abstract The authors study vanishing viscosity limits of solutions to the 3-dimensional incompressible Navier-Stokes system in general smooth domains with curved boundaries for a class of slip boundary conditions. In contrast to the case of flat boundaries, where the uniform convergence in super-norm can be obtained, the asymptotic behavior of viscous solutions for small viscosity depends on the curvature of the boundary in general. It is shown, in particular, that the viscous solution converges to that of the ideal Euler equations in $C([0,T];H^1(\Omega))$ provided that the initial vorticity vanishes on the boundary of the domain.

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1 Introduction

In [9], we investigated the vanishing viscosity limit for the 3D Navier-Stokes equations

$$\partial_t u - \varepsilon\Delta u + (\nabla\times u)\times u + \nabla p = 0, \quad \text{in } \Omega, \tag{1.1}$$

$$\nabla\cdot u = 0, \quad \text{in } \Omega \tag{1.2}$$

with given initial data and the slip boundary condition

$$u\cdot n = 0, \quad (\nabla\times u)\cdot\tau = 0, \quad \text{on } \partial\Omega, \tag{1.3}$$

in a bounded smooth domain Ω, where the positive constant ε is the viscosity coefficient. As was pointed out in [8], Proposition 4.1 in [9] holds only in the case when the boundary is flat. For general domains, it should be replaced by the following proposition.

Proposition 1.1 *Let $u\in D(\Omega)$. Then the boundary condition* (1.3) *is equivalent to $u\cdot n = 0$, $\partial_n(u(x)\cdot\tau) = -u(x)\cdot\partial_\tau n(x)$, where τ is a unit tangential vector on the boundary.*

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As a result, the main convergence results in [9] hold only in the case of flat boundary as we will discuss in Section 2 of this paper.

For general domains, since $\partial_\tau n(x) \neq 0$ in general, Proposition 4.1 and Theorem 4.2 in [9] may not be true. Although the global existence of weak solutions (see [9, Theorem 6.3]) and the local uniqueness of strong solutions in an ε-independent time interval (see [9, Theorem 7.2]) remain valid, yet one cannot establish the uniform convergence results in [9, Theorems 7.3, 7.4 and 8.2] for general domains.

The main purpose of the current paper is to clarify the differences between the vanishing viscosity limits for flat and curved boundaries with the slip boundary condition with the emphasis on domains with curved boundaries. Note that in the case of flat boundaries, the results in [9] (see also Section 2) imply that not only the viscous flows converge uniformly to the corresponding ideal flow in $L^\infty([0,T]\times\Omega)$, but also the Euler flow satisfies the same slip boundary condition (1.3). Thus the boundary layer behavior of the viscous flow is described precisely in this case. On the other hand, for general domains with curved boundaries, this may be different. It should be noted that the vanishing viscosity limit problem with general Navier slip boundary conditions for general 3D smooth domains was studied successfully in the weak norm, $L^\infty([0,T], L^2(\Omega))$ (see [3–4]). Since such a norm is too weak to observe the boundary layer behavior, it is desirable to investigate the vanishing viscosity limit problem in stronger norms.

Recall that the main approach in [9] is based on that the nonlinearity in (1.1) and the Euler system satisfy the same boundary condition (1.3) for the flat boundary (see [9, Theorem 4.2]). Now, this fails to be true in general for domains with curved boundaries, as was shown by a counter-example in [8]. To treat general domains, we observe that if instead of the slip condition (1.3), one considers the following boundary conditions:

$$u\cdot n = 0, \quad \nabla\times u = 0, \quad \text{on } \partial\Omega, \tag{1.4}$$

and in terms of vorticity $w = \nabla\times u$, the nonlinearity in both the Navier-Stokes and the Euler system takes the form

$$(u\cdot\nabla)w - (w\cdot\nabla)u,$$

which vanishes on the boundary if (1.4) holds. Unfortunately, although (1.4) is compatible for the ideal Euler system, the Navier-Stokes system is over-determined under condition (1.4). Thus we first consider a related viscous system with suitable boundary conditions motivated by (1.4), whose solutions are shown to converge uniformly to solutions to the ideal Euler system in $L^p(0,T;H^3(\Omega))\cap C([0,T];H^2(\Omega))$ (for velocity fields) by this new observation and the approach in [9]. Indeed, we are able to prove the convergence of solutions to the following boundary problem:

$$\partial_t\omega - \varepsilon\Delta\omega + (u\cdot\nabla)\omega - (\omega\cdot\nabla)u = 0, \quad \text{in } \Omega, \tag{1.5}$$

$$\nabla\times u = \omega + \nabla p, \quad \text{in } \Omega, \tag{1.6}$$

$$-\Delta p = \nabla\cdot\omega, \quad \text{in } \Omega, \tag{1.7}$$

$$\nabla\cdot u = 0, \quad \text{in } \Omega \tag{1.8}$$

with complementary boundary condition

$$u\cdot n = 0, \quad \omega = 0, \quad p = 0, \quad \text{on } \partial\Omega, \tag{1.9}$$

which can be regarded as a singular perturbation of the following system:

$$\partial_t\omega^0 + (u^0\cdot\nabla)\omega^0 - (\omega^0\cdot\nabla)u^0 = 0, \quad \text{in } \Omega, \tag{1.10}$$

$$\nabla\times u^0 = \omega^0 + \nabla p^0, \quad \text{in } \Omega, \tag{1.11}$$

$$-\Delta p^0 = \nabla\cdot\omega^0, \quad \text{in } \Omega, \tag{1.12}$$

$$\nabla\cdot u^0 = 0, \quad \text{in } \Omega \tag{1.13}$$

with the same boundary condition

$$u^0\cdot n = 0, \quad \omega^0 = 0, \quad p^0 = 0, \quad \text{on } \partial\Omega. \tag{1.14}$$

We remark that if the domain is periodic, and the initial data ω_0 satisfy $\nabla\cdot\omega_0 = 0$, it then follows from (1.5) that $\nabla\cdot\omega$ satisfies

$$\partial_t(\nabla\cdot\omega) - \varepsilon\Delta(\nabla\cdot\omega) + u\cdot\nabla(\nabla\cdot\omega) = 0, \quad \text{in } \Omega, \tag{1.15}$$

which implies $\nabla\cdot\omega = 0$, and then $\omega = \nabla\times u$. This shows that (1.5)–(1.8) is just the vorticity formulation of the Navier-Stokes system for this initial data. For this reason, we call the system (1.5)–(1.8) the relaxed Navier-Stokes system. Although in bounded domains, this system is not equivalent to (1.1), yet the limiting system is indeed the vorticity formulation of the Euler equations provided that $\nabla\cdot w_0 = 0$ initially.

An interesting corollary of this convergence is that solutions to the Euler equations will preserve zero vorticity on the boundary in its evolution if it is initially so (see Corollary 3.1 for details).

To study the strong convergence of the solutions for the Navier-Stokes equations (1.1), we investigate the Navier-Stokes equations

$$\partial_t u - \varepsilon\Delta u + (\nabla\times u)\times u + \nabla p = 0, \quad \text{in } \Omega, \tag{1.16}$$

$$\nabla\cdot u = 0, \quad \text{in } \Omega \tag{1.17}$$

with the slip boundary condition

$$u\cdot n = 0, \quad (\nabla\times(u-u^0))\cdot\tau = 0, \quad \text{on } \partial\Omega \tag{1.18}$$

by estimating its deviation from u^0 directly, where u^0 is the corresponding solution to the Euler equations with the same initial data and the boundary condition $u\cdot n = 0$ on $\partial\Omega$, and we will obtain the following rate of convergence

$$\|u(\varepsilon) - u\|_1^2 + \varepsilon\int_0^T \|u(\varepsilon) - u\|_2^2 \mathrm{d}t \le C\varepsilon^2 \tag{1.19}$$

for the solutions.

In particular, if the initial data are given such that its vorticity vanishes on the boundary, then the vorticity remains to be zero in the time evolution for solutions to the Euler equations (see Corollary 3.1 for details). So, the estimate (1.19) is also valid for the original problem (1.1)–(1.3) for these initial data.

There exist a huge amount of literature on the studies of vanishing viscosity limit problem for various boundary conditions and systems. We refer to [1, 7–9] for more references.

The rest of the paper is arranged as follows. In the next section, we restate the main results in [9] for the case of flat boundaries. In Section 3, we investigate the H^3 convergence for the relaxed system (1.5)–(1.8). In Section 4, we give the estimate of strong convergence rate for the Navier-Stokes equations with the slip boundary condition that its voticity coincides with that of the Euler equations in the tangential directions on the boundary.

2 Flat-Boundaries

Let $\Omega = [0,1]^2_{\text{per}} \times (0,1)$, and denote $\partial\Omega = \{x \in \overline{\Omega};\ x_3 = 0 \text{ or } x_3 = 1\}$. We will use the same notations as in [9]. To avoid confusions in applications, we restate the main results in [9] as follows.

Theorem 2.1 *Let $u_0 \in H^0_\tau(\Omega)$ and $T > 0$. There exists at least one weak solution u to* (1.1)–(1.3) *which satisfies the energy inequality*

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 + 2\varepsilon\|\nabla \times u\|^2 \le 0 \tag{2.1}$$

in the sense of distribution.

Theorem 2.2 *Let $u_0 \in H^1_\tau(\Omega)$. Then there exists a $T^* = T^*(u_0) > 0$, such that the problem* (1.1)–(1.3) *has a unique strong solution u in the interval $[0, T^*)$ satisfying*

$$u \in L^2(0,T;W) \cap C([0,T^*);H^1_\tau(\Omega)), \tag{2.2}$$
$$u' \in L^2(0,T;H^0_\tau(\Omega)), \tag{2.3}$$
$$\|u\|_1 \to \infty, \quad \text{as } t \to T^* \tag{2.4}$$

for any $T \in (0, T^*)$.

It follows that the energy equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\omega\|^2 + 2\varepsilon\|\Delta u\|^2 + 2(\nabla \times B(u), \omega) = 0 \tag{2.5}$$

holds and (2.1) *becomes an equation.*

Theorem 2.3 *The unique strong solution u belongs to $C((0,T^*);W)$. Moreover, if $u_0 \in W$, then*

$$u \in L^2(0,T;H^3(\Omega)) \cap C([0,T^*);W), \tag{2.6}$$
$$u' \in L^2(0,T;H^1_\tau(\Omega)), \tag{2.7}$$

and the energy equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\|v\|^2 + 2\varepsilon\|\nabla \times v\|^2 + 2(-\Delta B(u), v) = 0 \tag{2.8}$$

holds for $v = -\Delta u$ on the time interval.

Theorem 2.4 *The unique strong solution u belongs to $C((0,T^*);H^3(\Omega))$, and if $u_0 \in W \cap H^3(\Omega)$, then*

$$u \in L^2(0,T;H^4(\Omega)) \cap C([0,T^*);H^3(\Omega)), \tag{2.9}$$
$$u' \in L^2(0,T;W) \tag{2.10}$$

and the energy equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla \times v\|^2 + 2\varepsilon\|\Delta v\|^2 + 2(\Delta B(u), \Delta v) = 0 \tag{2.11}$$

holds for $v = -\Delta u$ in the sense of distribution. Moreover, v satisfies

$$(\nabla \times v) \cdot \tau = 0 \tag{2.12}$$

for a.e. $t \in (0, T)$.

Theorem 2.5 *Let* $u_0 \in W \cap H^3(\Omega)$. *Then there exists a* $T_0 > 0$ *such that the strong solution* $u(\varepsilon)$ *to the Navier-Stokes equations with the initial data* u_0 *converges to the unique solution* u *of the Euler equations with the same initial data and the slip boundary condition* $u \cdot n = 0$ *on* $\partial\Omega$ *in the following sense:*

$$u(\varepsilon) \to u, \quad \text{in } L^p(0, T; H^3(\Omega)), \tag{2.13}$$

$$u(\varepsilon) \to u, \quad \text{in } C([0, T]; H^2(\Omega)), \tag{2.14}$$

$1 \le p < \infty$, *as* $\varepsilon \to 0$.

Consequently, we have the following result.

Corollary 2.1 *For initial velocity* $u_0 \in H^3(\Omega) \cap W$, *the unique solution* u *to the Euler equations with the slip boundary condition,* $u \cdot n = 0$ *on* $\partial\Omega$, *satisfies an extra condition* $(\nabla \times u)_\tau = 0$ *on* $\partial\Omega$ *in its maximum existent interval* $[0, \overline{T})$.

Finally, we have the following estimate on the rate of the convergence.

Theorem 2.6 *Let* $u_0 \in H^3(\Omega) \cap W$, *and* $T, \overline{T}$ *be as above. Then it holds that*

$$\|u(\varepsilon) - u\|_2^2 \le C(T)\varepsilon, \tag{2.15}$$

in the interval $[0, \min\{T, \overline{T}\}]$.

3 H^3 Convergence of the Relaxed System

In this section, we consider the following viscous system:

$$\partial_t \omega - \varepsilon\Delta\omega + (u \cdot \nabla)\omega - (\omega \cdot \nabla)u = 0, \quad \text{in } \Omega, \tag{3.1}$$

$$\nabla \times u = \omega + \nabla p, \quad \text{in } \Omega, \tag{3.2}$$

$$-\Delta p = \nabla \cdot \omega, \quad \text{in } \Omega, \tag{3.3}$$

$$\nabla \cdot u = 0, \quad \text{in } \Omega \tag{3.4}$$

with suitable initial data and the boundary condition

$$u \cdot n = 0, \quad \omega = 0, \quad p = 0, \quad \text{on } \partial\Omega. \tag{3.5}$$

By using a similar approach as in [9], we can pass the zero viscosity limit of the solution to this problem to get the solution to the Euler equations that satisfies the standard slip condition and an extra boundary condition of vorticity being zero.

Remark 3.1 The velocity u involved in the system (3.1)–(3.5) satisfies an inhomogeneous Navier-Stokes system with the slip boundary condition (1.3), i.e.,

$$\partial_t u - \varepsilon\Delta u + (\nabla \times u) \times u + \nabla q = F, \quad \text{in } \Omega, \tag{3.6}$$

$$\nabla \cdot u = 0, \quad \text{in } \Omega, \tag{3.7}$$

$$u \cdot n = 0, \quad (\nabla \times u) \cdot \tau = 0, \quad \text{on } \partial\Omega \tag{3.8}$$

with F satisfying

$$\nabla \times F = \partial_t(\nabla p) - \varepsilon\Delta(\nabla p) + (u \cdot \nabla)(\nabla p) - ((\nabla p) \cdot \nabla)u, \quad \text{in } \Omega, \tag{3.9}$$

$$\nabla \cdot F = 0, \quad \text{in } \Omega, \tag{3.10}$$

$$F \cdot n = 0, \quad \text{on } \partial\Omega. \tag{3.11}$$

Denote by $A = -\Delta$ the Laplacian with the domain $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$, and define $P\omega = \omega + \nabla p$ with p determined by

$$-\Delta p = \nabla \cdot \omega, \quad \text{in } \Omega, \tag{3.12}$$

$$p = 0, \quad \text{on } \partial\Omega, \tag{3.13}$$

and $Tg = u$ determined by

$$\nabla \times u = g, \quad \text{in } \Omega, \tag{3.14}$$

$$\nabla \cdot u = 0, \quad \text{in } \Omega, \tag{3.15}$$

$$u \cdot n = 0, \quad \text{on } \partial\Omega. \tag{3.16}$$

Due to the standard elliptic regularity, the linear operator $P : D(A^{\frac{1}{2}}) = H_0^1(\Omega) \to H_n^1(\Omega)$ (see [9] for the definition) is bounded. It follows from Lemma 2.1 in [8] that the linear operator $T : H_n^1(\Omega) \to H_\tau^2(\Omega)$ (see [9] for the definitions) is bounded. Set

$$B(\omega) = (T(P\omega) \cdot \nabla)\omega - (\omega \cdot \nabla)T(P\omega). \tag{3.17}$$

Then Sobolev estimates imply

$$\|(\phi \cdot \nabla)\psi\| \le c\|\phi\|_2\|\psi\|_1 \tag{3.18}$$

and

$$\|(\phi \cdot \nabla)\psi\| \le c\|\phi\|_1\|\psi\|_2. \tag{3.19}$$

Hence, we have the following result.

Lemma 3.1 $B : D(A^{\frac{1}{2}}) \to L^2(\Omega)$ *is a local Lipschitz continuous mapping.*

We then can reduce the problem (3.1)–(3.5) to a semi-linear abstract parabolic system of the form

$$\omega' + \varepsilon A\omega + B(\omega) = 0, \tag{3.20}$$

in $L^2(\Omega)$.

Definition 3.1 $\omega \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega))$ *(or the triple* (ω, u, p)*) is said to be a weak solution to the problem* (3.1)–(3.5) *in the interval* $[0, T)$ *if*

$$(\omega', v) + \varepsilon(\nabla\omega, \nabla v) + (B(\omega), v) = 0 \tag{3.21}$$

holds for all $v \in H_0^1(\Omega)$*, where* p *and* u *are determined by solving* (3.12)–(3.16) *respectively.*

To solve (3.20), one has the Galerkin approximations $\omega_m = \sum\limits_1^m a_j e_j$ with a_j solving the systems

$$a_j' + \varepsilon\lambda_j a_j + g_j = 0, \quad j = 1, \cdots, m, \tag{3.22}$$

where $\{\lambda_i\}$ is the list of all eigenvalues of A and e_i is the corresponding eigenvector, and g_j is given by

$$g_j(t) = (B(\omega_m), e_j). \tag{3.23}$$

Applying the energy equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\omega\|^2 + 2\varepsilon\|\nabla\omega\|^2 + 2(B(\omega),\omega) = 0 \tag{3.24}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\omega\|^2 + 2\varepsilon\|A\omega\|^2 + 2(B(\omega),A\omega) = 0 \tag{3.25}$$

to the Galerkin approximations, one can prove the following theorem by the standard method.

Theorem 3.1 *For given initial data $\omega_0 \in L^2(\Omega)$, there exists a $T_\varepsilon > 0$ such that* (3.1)–(3.5) *has a unique solution $\omega \in C([0,T_\varepsilon);L^2(\Omega)) \cap L^2(0,T_\epsilon;H_0^1(\Omega))$ in $[0,T_\varepsilon)$ with initial data ω_0, and if $T_\varepsilon < \infty$, then*

$$\|\nabla\omega\| \to \infty, \quad \text{as } t \to T_\varepsilon. \tag{3.26}$$

Moreover, if $\omega_0 \in H_0^1(\Omega)$, then the solution $\omega \in C([0,T_\varepsilon),H_0^1(\Omega)) \cap L^2(0,T_\varepsilon;D(A))$.

Now, we derive the $L^\infty(0,T;D(A))$ uniform estimate of the solutions in some fixed time interval independent of ε for the more regular initial data $\omega_0 \in D(A)$, and prove the main result of this section.

Theorem 3.2 *Let $\omega_0 \in D(A)$. Then there exists a $T > 0$ such that the solution $\omega(\varepsilon)$ to the system* (3.1)–(3.5) *with the initial data ω_0 converges to ω^0 as $\varepsilon \to 0^+$ in the following sense:*

$$\omega(\varepsilon) \to \omega^0, \quad \text{in } L^p(0,T;H^2(\Omega)), \tag{3.27}$$

$$\omega(\varepsilon) \to \omega^0, \quad \text{in } C([0,T];H^1(\Omega)) \tag{3.28}$$

with $1 \le p < \infty$, and (ω^0,p^0,u^0) satisfying

$$\partial_t\omega^0 + (u^0\cdot\nabla)\omega^0 - (\omega^0\cdot\nabla)u^0 = 0, \quad \text{in } \Omega, \tag{3.29}$$

$$\nabla\times u^0 = \omega^0 + \nabla p^0, \quad \text{in } \Omega, \tag{3.30}$$

$$-\Delta p^0 = \nabla\cdot\omega^0, \quad \text{in } \Omega, \tag{3.31}$$

$$\nabla\cdot u^0 = 0, \quad \text{in } \Omega \tag{3.32}$$

and the boundary condition

$$u^0\cdot n = 0, \quad \omega^0 = 0, \quad p^0 = 0, \quad \text{on } \partial\Omega. \tag{3.33}$$

We start with the following observation.

Lemma 3.2 *Let ω be smooth and satisfy $\omega = 0$ on the boundary. Then it holds that*

$$B(\omega) = 0, \quad \text{on } \partial\Omega. \tag{3.34}$$

Proof It follows from $\omega = 0$ on the boundary that

$$\omega\cdot\nabla u = 0, \quad \text{on } \partial\Omega. \tag{3.35}$$

For any smooth Φ, it holds that

$$(u\cdot\nabla\omega)\cdot\Phi = u\cdot\nabla(\omega\cdot\Phi) - (u\cdot\nabla\Phi)\cdot\omega. \tag{3.36}$$

Since $u \cdot n = 0$ on $\partial\Omega$, we have

$$u \cdot \nabla(\omega \cdot \Phi) = 0, \tag{3.37}$$

and hence the lemma follows.

Now, we prove Theorem 3.2.

Proof of Theorem 3.2 By a similar argument of [9] and Lemma 3.2, we have that $\Psi = -\Delta\omega$ satisfies

$$\partial_t \Psi - \varepsilon\Delta\Psi - \Delta B(\omega) = 0, \quad \text{in } \Omega, \tag{3.38}$$
$$\Psi = 0, \qquad \text{on } \partial\Omega. \tag{3.39}$$

$-\Delta B(\omega)$ can be calculated by

$$-\Delta B(\omega) = u \cdot \nabla(-\Delta\omega) + \sum_{\alpha=0,1,2} D^\alpha \omega \cdot D^{3-\alpha} u. \tag{3.40}$$

Note that

$$(u \cdot \nabla(-\Delta\omega), -\Delta\omega) = 0 \tag{3.41}$$

and

$$\|u\|_{s+1} \le c\|\omega + \nabla p\|_s \le c\|\omega\|_s. \tag{3.42}$$

It follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta\omega\|^2 + \varepsilon\|\nabla(\Delta\omega)\|^2 \le c\|\Delta\omega\|^3, \tag{3.43}$$

which implies that there is a $T > 0$ such that $T_\varepsilon \ge T$ and

$$\omega(\varepsilon) \text{ uniformly bounded in } C([0,T]; H^2(\Omega)), \tag{3.44}$$
$$\omega'(\varepsilon) \text{ uniformly bounded in } L^2(0,T; H^1(\Omega)). \tag{3.45}$$

Hence, there is a subsequence ε_n of ε and a vector function ω^0 such that

$$\omega_n \to \omega^0 \in L^p(0,T; H^2(\Omega)), \tag{3.46}$$
$$\omega_n \to \omega^0, \quad \text{in } C([0,T]; H^1(\Omega)) \tag{3.47}$$

for all $1 \le p < \infty$, as $\varepsilon_n \to 0$, where $\omega_n = \omega(\varepsilon_n)$ denotes the corresponding solution obtained in Theorem 3.1. Passing to the limit and combining with the continuity of the operators P, T and B show that the triple (ω^0, p^0, u^0) satisfies (3.29)–(3.33). A standard estimate on the difference between two solutions (see also the next section) shows that the solution is unique. Hence, the convergence of the whole sequence can be obtained and the theorem is proved.

Remark 3.2 The convergence in terms of u is given by

$$u(\varepsilon) \to u^0, \quad \text{in } L^p(0,T; H^3(\Omega)), \tag{3.48}$$
$$u(\varepsilon) \to u^0, \quad \text{in } C([0,T]; H^2(\Omega)). \tag{3.49}$$

Next, we claim that the limit (3.29)–(3.33) is just the vorticity form of the Euler equations for the initial data satisfying $\nabla \cdot (\omega_0^0) = 0$. Indeed, note that

$$B(\omega) = \nabla \times (\omega \times u) + (\nabla \cdot \omega)u. \tag{3.50}$$

Taking divergence in (3.29) yields

$$\partial_t(\nabla \cdot \omega^0) + u^0 \cdot \nabla(\nabla \cdot \omega^0) = 0, \quad \text{in } \Omega, \tag{3.51}$$

which implies

$$\nabla \cdot \omega^0 = 0, \quad \text{in } \Omega, \tag{3.52}$$

if it is initially so. Then, $p^0 = 0$, $\nabla \times u^0 = \omega^0$, and then the assertion follows.

The corollary below follows from the uniqueness of the solution to the Euler equations.

Corollary 3.1 *Let u be the smooth solution to the Euler equations in $[0, T]$ with the slip boundary condition $u \cdot n = 0$ on $\partial\Omega$ and the initial data u_0 satisfying $\omega_0 = \nabla \times u_0 = 0$ on the boundary. Then the vorticity satisfies*

$$\omega = \nabla \times u = 0, \quad \text{on } \partial\Omega \tag{3.53}$$

in the interval $[0, T]$.

Remark 3.3 This corollary may also be obtained from the Euler-Lagrangian particle path by considering the ordinary differential equation

$$\frac{D\omega}{\mathrm{d}t} = \omega \cdot \nabla u, \tag{3.54}$$

on the boundary, since the condition $u \cdot n = 0$ on the boundary, so the particle will stay on the boundary (see [2]).

4 Strong Convergence for the Navier-Stokes Equations

In this section, we return to the original vanishing viscosity limit problem. Specifically, we consider the Navier-Stokes equations

$$\partial_t u - \varepsilon\Delta u + (\nabla \times u) \times u + \nabla p = 0, \quad \text{in } \Omega, \tag{4.1}$$

$$\nabla \cdot u = 0, \quad \text{in } \Omega \tag{4.2}$$

with the slip boundary condition that its vorticity coincides with that of the solutions to the Euler equations in the tangential directions, namely

$$u \cdot n = 0, \quad (\nabla \times (u - u^0)) \cdot \tau = 0, \quad \text{on } \partial\Omega, \tag{4.3}$$

where u^0 is the corresponding solution to the Euler equations

$$\partial_t u^0 + (\nabla \times u^0) \times u^0 + \nabla p^0 = 0, \quad \text{in } \Omega, \tag{4.4}$$

$$\nabla \cdot u^0 = 0, \quad \text{in } \Omega, \tag{4.5}$$

$$u^0 \cdot n = 0, \quad \text{on } \partial\Omega. \tag{4.6}$$

Then the main results in this section can be stated as follows.

Theorem 4.1 *Let $u_0 \in H^3$ and $u_0 \cdot n = 0$ on the boundary, and u^0 be the smooth solution to the Euler equations with $u^0(0) = u_0$ in the time interval $[0, T]$. Then for ε small enough, the strong solution to the boundary value problem (4.1)–(4.3) of the Navier-Stokes equations has a unique solution $u = u(\varepsilon) \in C([0, T], H^1(\Omega)) \cap L^2(0, T; H^2(\Omega))$ with the same initial data in the same interval, and*

$$\sup_{0\leq t\leq T} \|u(\varepsilon) - u\|_1^2 + \varepsilon \int_0^T \|u(\varepsilon) - u\|_2^2 \mathrm{d}t \leq C\varepsilon^2. \tag{4.7}$$

Proof We begin by considering the following system:

$$\partial_t w - \varepsilon\Delta w + B_0(w, t) - \varepsilon\Delta u^0 + \nabla q = 0, \quad \text{in } \Omega, \tag{4.8}$$

$$\nabla \cdot w = 0, \quad \text{in } \Omega, \tag{4.9}$$

$$w \cdot n = 0, \quad (\nabla \times w) \cdot \tau = 0, \quad \text{on } \partial\Omega \tag{4.10}$$

with the initial data $w(0) = 0$, where

$$B_0(w, t) = (\nabla \times u^0) \times w + (\nabla \times w) \times u^0 + (\nabla \times w) \times w. \tag{4.11}$$

Note that the solution to the Euler equations u^0 has been determined in $[0, T]$, and the system (4.8)–(4.10) differs from the system (1.1)–(1.3) only in replacing the nonlinearity by a time dependent one. So in the similar way as in the proof of the existence and uniqueness of the strong solution to Navier-stokes equations (1.1)–(1.3) stated in [9], we can show that the problem (4.8)–(4.10) has a unique (strong) solution $w = w(\varepsilon) \in C([0, T_\varepsilon), H^1(\Omega)) \cap L^2((0, T_\varepsilon); H^2(\Omega))$ with $w(0) = 0$ in a time interval $[0, T_\varepsilon)$ for some $T_\varepsilon \leq T$ and if $T_\varepsilon < T$, then $\|w\|_1 \to \infty$ as $t \to T_\varepsilon$.

We then estimate w. Taking the $L^2(\Omega)$ inner product of (4.8) with $-\Delta w$, noting that

$$(\Phi, -\Delta w) = (\nabla \times \Phi, \nabla \times w) \tag{4.12}$$

for all $\Phi \in H^1(\Omega)$ due to the boundary condition $(\nabla \times w) \times n = 0$ and integrating by parts, one has

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla \times w\|^2 + 2\varepsilon\|\Delta w\|^2 + 2(\nabla \times (B_0(w, T), \nabla \times w) = -2\varepsilon((\nabla\times)^3 u^0, \nabla \times w). \tag{4.13}$$

Direct calculation yields

$$\nabla \times (B_0(w, t)) = [w, \nabla \times u^0] + [u^0, \nabla \times w] + [w, \nabla \times w], \tag{4.14}$$

where

$$[\varphi, \psi] = (\varphi \cdot \nabla)\psi - (\psi \cdot \nabla)\varphi. \tag{4.15}$$

Note that

$$((u^0 \cdot \nabla)(\nabla \times w), \nabla \times w) = 0, \tag{4.16}$$

$$((w \cdot \nabla)(\nabla \times w), \nabla \times w) = 0 \tag{4.17}$$

and

$$|((w \cdot \nabla)(\nabla \times u^0), \nabla \times w)| \leq c\|w\|_1^2\|u^0\|_3, \tag{4.18}$$

$$|(((\nabla \times u^0) \cdot \nabla)w, \nabla \times w)| \leq c\|w\|_1^2\|u^0\|_3, \tag{4.19}$$

$$|(((\nabla \times w) \cdot \nabla)u^0, \nabla \times w)| \leq c\|w\|_1^2\|u^0\|_3, \tag{4.20}$$

$$|(((\nabla \times w) \cdot \nabla)w, \nabla \times w)| \leq c\|w\|_1^{\frac{3}{2}}\|w\|_2^{\frac{3}{2}}, \tag{4.21}$$

due to the Sobolev's inequality.

Since u is the smooth solution to the Euler equations so that $\|u^0\|_3 \le C$ in $[0,T]$, it follows that

$$|(\nabla \times (B_0(w,t)), \nabla \times w)| \le C(\|w\|_1^2 + \|w\|_1^{\frac{3}{2}} \|w\|_2^{\frac{3}{2}}). \tag{4.22}$$

So the Young's inequality yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla \times w\|^2 + \varepsilon\|\Delta w\|^2 \le C(\|w\|_1^2 + \varepsilon^{-3}\|w\|_1^6 + \varepsilon^2). \tag{4.23}$$

This, together with $w(0) = 0$, implies

$$\|w\|_1^2 + \varepsilon \int_0^T \|w\|_2^2 \mathrm{d}t \le C\varepsilon^2, \tag{4.24}$$

in the interval $[0,T]$ for ε small enough. Thus, (4.7) follows and the theorem is proved.

Note that Corollary 3.1 implies that the initial zero vorticity on the boundary is preserved by the Euler flows. Consequently, Theorem 4.1 gives the following uniform convergence for the original problem (1.1)–(1.3).

Corollary 4.1 *Let $u_0 \in H^3(\Omega) \cap W$ satisfying $\nabla \times u_0 = 0$ on the boundary, and u^0 be the smooth solution to the Euler equations in the time interval $[0,T]$ with initial data u_0 and the non-slip boundary condition* (4.6). *Then for ε small enough, the strong solution to the problem* (1.1)–(1.3) *of the Navier-Stokes equations with the same initial data exists in the same interval such that*

$$\|u(\varepsilon) - u\|_1^2 + \varepsilon \int_0^T \|u(\varepsilon) - u\|_2^2 \mathrm{d}t \le C\varepsilon^2 \tag{4.25}$$

holds in the interval $[0,T]$.

Remark 4.1 This result is weaker than that for flat boundaries (see Theorem 2.6), but stronger than that was obtained in [3] (see also [4]), where they proved the following convergence estimate:

$$\|u(\varepsilon) - u\|^2 + \varepsilon \int_0^T \|u(\varepsilon) - u\|_1^2 \mathrm{d}t \le C\varepsilon^2 \tag{4.26}$$

for general Navier-slip boundary conditions and general smooth domains.

Remark 4.2 It should be noted that in the proof of Theorem 4.1, only the tangential vorticity of the solution to the Euler equations was used. However, it seems not clear to us whether zero tangential vorticity on the boundary can be preserved by smooth Euler flows with the slip condition (4.6) for general domains. Indeed, let $X(t)$ be any Lagrangian particle path. Since $u \cdot n = 0$ on the boundary, it follows that $X(t)$ remains on the boundary, i.e., $X(t) \in \partial\Omega$ for all $t \in [0,T]$ if $X(0) \in \partial\Omega$. Denote by $\{\tau_1, \tau_2, n\}$ the local coordinate near the boundary. It follows that

$$\frac{D(\omega \cdot \tau_i)}{\mathrm{d}t} = S(u)\omega \cdot \tau_i + \omega \cdot \frac{D\tau_i}{\mathrm{d}t}. \tag{4.27}$$

Note that $\tau_i = \tau_i(X(t))$ does not involve the variable t directly. It follows that

$$\omega \cdot \frac{D\tau_i}{\mathrm{d}t} = \omega \cdot (u \cdot \nabla)\tau_i. \tag{4.28}$$

Write

$$\omega = \omega_T + (\omega \cdot n)n = (\omega \cdot \tau_i)\tau_i + (\omega \cdot n)n. \tag{4.29}$$

Then

$$\frac{D(\omega \cdot \tau_i)}{\mathrm{d}t} = S(u)\omega_T \cdot \tau_i + \omega_T(u \cdot \nabla)\tau_i + N, \tag{4.30}$$

where

$$N = (\omega \cdot n)(S(u)n \cdot \tau_i + n \cdot (u \cdot \nabla)\tau_i). \tag{4.31}$$

Direct calculation gives

$$N = -(\omega \cdot n)(2S(n)u + \omega_T \times n) \cdot \tau_i, \tag{4.32}$$

where $2S = \nabla + \nabla^\perp$. It holds that

$$\frac{D(\omega \cdot \tau_i)}{\mathrm{d}t} = a_{ij}(u)\omega \cdot \tau_j - 2(\omega \cdot n)S(n)u \cdot \tau_i. \tag{4.33}$$

Since $(\omega \cdot n)S(n)u \cdot \tau_i$ may not be zero in general, we have that $\omega \cdot \tau$ may not remain zero even it is zero initially.

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