

The Einstein-Scalar Field Constraints on Asymptotically Euclidean Manifolds****

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*(Dedicated to the memory of Shiing-Shen Chern, with admiration for
his mathematical discoveries and his character)*

Abstract By using the conformal method, solutions of the Einstein-scalar field gravitational constraint equations are obtained. Handling scalar fields is a bit more challenging than handling matter fields such as fluids, Maxwell fields or Yang-Mills fields, because the scalar field introduces three extra terms into the Lichnerowicz equation, rather than just one. The proofs are constructive and allow for arbitrary dimension (> 2) as well as low regularity initial data.

Keywords Einstein-scalar field, Constraints, Asymptotically Euclidian
2000 MR Subject Classification 83C05

1 Introduction

To explain recent observations of far away stars and galaxies, as well as the possible origin of matter elements, it has become more and more relevant in Einsteinian cosmology to admit the existence of a scalar field with a potential which remains to be estimated. On the other hand various considerations, in particular the search for the unification of all the fundamental fields, including gravitation, leads to the belief that the universe has extra dimensions, beyond the usual three space and one time. These extra dimensions would be spacelike, and their extent so small that we do not perceive them at the usual scales of our experiments.

The relevant equations for cosmology would then be the Einstein equations on an $n + 1$ dimensional manifold V , with source a scalar field ψ of potential $V(\psi)$. These equations are, for a metric g on V of Lorentzian signature¹,

$$\text{Einstein}(g) \equiv \text{Ricci}(g) - \frac{1}{2}R(g) = T; \quad (1.1)$$

that is, in a local frame

$$S_{\alpha\beta} \equiv R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R = T_{\alpha\beta}, \quad (1.2)$$

Manuscript received July 5, 2005.

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****Project supported by NSF Grant PHY-0354659 at the University of Oregon, and by NSF Grant DMS-0305048 at the University of Washington.

¹ We choose the signature to be $- + + \cdots +$.

where T is the stress energy tensor of a scalar field ψ with potential $V(\psi)$, i.e.,

$$T_{\alpha\beta} \equiv \partial_\alpha \psi \partial_\beta \psi - \frac{1}{2} g_{\alpha\beta} \partial_\lambda \psi \partial^\lambda \psi - g_{\alpha\beta} V(\psi). \quad (1.3)$$

The Einstein tensor satisfies the contracted Bianchi identities

$$\nabla_\alpha S^{\alpha\beta} \equiv 0. \quad (1.4)$$

The field ψ is supposed to satisfy the semi linear wave equation

$$\nabla^\alpha \partial_\alpha \psi - V'(\psi) = 0, \quad V'(\psi) := \frac{dV(\psi)}{d\psi}. \quad (1.5)$$

The tensor T is then divergence free

$$\nabla_\alpha T^{\alpha\beta} = 0. \quad (1.6)$$

As a consequence of condition (1.6), equations (1.2) are compatible.

The Cauchy problem for the Einstein equations, determination of an Einsteinian spacetime from initial data on a spacelike n dimensional manifold, is a geometric analysis problem. Its solution does not exist for arbitrary initial data, and is not unique from the point of view of analysis due to the invariance of the equations under diffeomorphisms. The geometric initial data are a triple (M, \bar{g}, K) with M an n dimensional manifold, which we suppose to be smooth, \bar{g} a Riemannian metric on M , and K a symmetric 2-tensor on M . The Cauchy data for the scalar field are two functions $\bar{\psi}$ and $\bar{\pi}$. An $n+1$ dimensional spacetime (V, g) together with a scalar function ψ on V is called an Einstein scalar development of these initial data if M can be embedded in V , so that g induces on M the metric \bar{g} and K can be identified with the extrinsic curvature of M as submanifold of (V, g) , while $\bar{\psi}$ is the value of ψ on M , and $\bar{\pi}$ is the value on M of the derivative of ψ in the direction of the unit normal to M in (V, g) .

In Sections 1 to 5 of this article we use the conformal method to obtain an elliptic system for the constraints satisfied by the initial data of an Einstein-scalar field system. In the following sections we prove some existence and uniqueness theorems for their solution in the case where (M, \bar{g}) is asymptotically Euclidean, under low regularity hypothesis.

The cases of compact M and of (M, \bar{g}) asymptotically hyperbolic will be treated elsewhere.

2 Constraints for the Einstein-Scalar Field Equations

The constraint equations are a consequence of the Gauss Codazzi identities satisfied by the Ricci tensor of any pseudo Riemannian manifold.

It is convenient to suppose that $V = M \times R$ and to choose on V a moving frame θ^α , $\alpha = 0, 1, \dots, n$, called a Cauchy adapted frame as long as θ^0 annihilates vectors tangent to submanifolds $M \times \{t\}$. The space time metric is then decomposed as follows

$$g \equiv -N^2(\theta^0)^2 + g_{ij}\theta^i\theta^j \quad \text{with} \quad \theta^0 \equiv dt, \quad \theta^i \equiv dx^i + \beta^i dt, \quad i = 1, \dots, n. \quad (2.1)$$

The function N is called the lapse and the time dependent spatial vector β the shift of the chosen representation of the spacetime metric. In this frame the unit normal n to a submanifold $M \times \{t\}$ has components

$$n^0 = N^{-1}, \quad n_0 = -N, \quad n^i = n_i = 0. \quad (2.2)$$

The derivative of the function ψ in the direction of n is

$$\pi \equiv N^{-1} \partial_0 \psi, \quad (2.3)$$

with ∂_0 the Pfaff derivative with respect to the 1-form θ^0 in the frame θ^α , i.e.

$$\partial_0 \equiv \frac{\partial}{\partial t} - \beta^i \partial_i, \quad \partial_i \equiv \frac{\partial}{\partial x^i}. \quad (2.4)$$

In a Cauchy adapted frame the constraints read as the following equations, where we overbar values induced on M by spacetime quantities, and we set

$$\tau \equiv \text{tr}_{\bar{g}} K := \bar{g}^{ij} K_{ij}, \quad |K|_{\bar{g}}^2 := \bar{g}^{ih} \bar{g}^{jk} K_{ij} K_{hk}. \quad (2.5)$$

- Hamiltonian constraint:

$$R(\bar{g}) - |K|_{\bar{g}}^2 + \tau^2 = 2\rho \equiv 2\bar{N}^{-2} \bar{T}_{00}. \quad (2.6)$$

- Momentum constraint:

$$\bar{\nabla}_i K^{ij} - \bar{g}^{ij} \partial_i \tau = J^j \equiv -\bar{N}^{-1} \bar{T}_0^j. \quad (2.7)$$

In the case under study, where a source is a scalar field ψ , we find that

$$\begin{aligned} 2\bar{N}^{-2} \bar{T}_{00} &= \bar{\pi}^2 + |D\bar{\psi}|_{\bar{g}}^2 + 2V(\bar{\psi}), \\ -\bar{N}^{-1} \bar{T}_0^i &= -\bar{\pi} \bar{g}^{ij} \partial_j \bar{\psi}. \end{aligned}$$

3 Conformal Formulation

3.1 Hamiltonian constraint

In order to turn the Hamiltonian constraint into a semilinear elliptic equation to be solved for a scalar function, one considers the metric \bar{g} as determined only up to a conformal factor. One sets for $n > 2$,

$$\bar{g} = \varphi^{4/(n-2)} \gamma, \quad \text{i.e.} \quad \bar{g}_{ij} = \varphi^{4/(n-2)} \gamma_{ij} \quad (3.1)$$

with γ a given Riemannian metric on M . This particular conformal weight turns into a linear operator the differential operator on φ appearing in (3.2) below.

The scalar curvatures $R(\bar{g})$ and $R(\gamma)$ of the conformal metrics \bar{g} and γ are linked by the formula, where Δ_γ is the Laplace operator in the metric γ ,

$$R(\bar{g}) \equiv \varphi^{-(n+2)/(n-2)} \left(\varphi R(\gamma) - \frac{4(n-1)}{n-2} \Delta_\gamma \varphi \right). \quad (3.2)$$

The Hamiltonian constraint becomes, when γ and K are known, a semi linear elliptic equation for φ with a non linearity of a fairly simple type:

$$\Delta_\gamma \varphi - k_n R(\gamma) \varphi + k_n (|K|_{\bar{g}}^2 - \tau^2 + 2\rho) \varphi^{(n+2)/(n-2)} = 0 \quad (3.3)$$

with

$$k_n = \frac{n-2}{4(n-1)}. \quad (3.4)$$

3.2 Momentum constraint

We can express the momentum constraint in terms of γ , K , ρ , J and φ by using the relations between the connections of two conformally related metrics.

Lemma 3.1 *On an n dimensional manifold, if $\bar{g} = \varphi^{4/(n-2)}\gamma$, and if the covariant derivatives in \bar{g} and γ are written respectively as $\bar{\nabla}$ and D , then the divergences in the metric \bar{g} and γ of an arbitrary contravariant 2-tensor P^{ij} are linked by the identity*

$$\bar{\nabla}_i P^{ij} \equiv \varphi^{-2(n+2)/(n-2)} D_i \{ \varphi^{2(n+2)/(n-2)} P^{ij} \} - \frac{2}{n-2} \varphi^{-1} \gamma^{ij} \partial_i \varphi \operatorname{tr}_\gamma P. \quad (3.5)$$

Proof The proof follows from a simple computation using the identity which links the coefficients of the connections $\bar{\Gamma}$ of \bar{g} and C of γ :

$$\bar{\Gamma}_{jh}^i = C_{jh}^i + \frac{2}{n-2} \varphi^{-1} \{ \delta_j^i \partial_h \varphi + \delta_h^i \partial_j \varphi - \gamma^{ik} \gamma_{jh} \partial_k \varphi \}. \quad (3.6)$$

One sees from the identity (3.5) that it is convenient to split the unknown K into a weighted traceless part and its trace, namely we set

$$K^{ij} = \varphi^{-2(n+2)/(n-2)} \tilde{K}^{ij} + \frac{1}{n} \bar{g}^{ij} \tau. \quad (3.7)$$

Here \tilde{K}^{ij} is a symmetric traceless two tensor, in the sense that

$$\operatorname{tr} \tilde{K} \equiv \bar{g}_{ij} \tilde{K}^{ij} = \gamma_{ij} \tilde{K}^{ij} = 0, \quad (3.8)$$

while τ is the trace.

The momentum constraint (2.7) then becomes

$$D_i \tilde{K}^{ij} = \frac{n-1}{n} \varphi^{2n/(n-2)} \gamma^{ij} \partial_i \tau + \varphi^{2(n+2)/(n-2)} J^j. \quad (3.9)$$

It follows from an elementary computation that

$$|K|_{\bar{g}}^2 \equiv \bar{g}_{ih} \bar{g}_{jk} K^{ij} K^{hk} = \varphi^{-(3n-2)/(n-2)} |\tilde{K}|_\gamma^2 + \frac{1}{n} \tau \quad \text{with} \quad |\tilde{K}|_\gamma^2 \equiv \gamma_{ih} \gamma_{jk} \tilde{K}^{ij} \tilde{K}^{hk}. \quad (3.10)$$

The Hamiltonian constraint therefore reads

$$\begin{aligned} & \Delta_\gamma \varphi - k_n R(\gamma) \varphi + k_n \varphi^{-(3n-2)/(n-2)} |\tilde{K}|_\gamma^2 - \frac{n-2}{4n} \varphi^{(n+2)/(n-2)} \tau^2 \\ &= -\frac{n-2}{2(n-1)} \rho \varphi^{(n+2)/(n-2)}. \end{aligned} \quad (3.11)$$

If γ , \tilde{K} , τ and ρ are specified, this is a semilinear elliptic equation for φ , called a Lichnerowicz equation².

²This equation was derived by Lichnerowicz for $n = 3$ (see [27]). In 1972 York [34] introduced the scaling of the sources, and in 1987 Choquet-Bruhat extended the analysis to general n . In view of this history we refer to (3.11) as the Lichnerowicz equation.

4 Scaling of $\bar{\pi}$

We denote by an overbar the values induced on M by spacetime quantities. The initial data of the scalar field ψ is the value $\bar{\psi}$ induced by ψ on M . It is independent of the choice of the conformal metric γ , but there is an ambiguity for the data of the initial data for π , because π depends on the lapse N : it holds that $\bar{\pi} = \bar{N}^{-1} \bar{\partial}_0 \bar{\psi}$. We associate to the unphysical metric γ an unphysical lapse \tilde{N} , such that \bar{N} and \tilde{N} have the same associated densities respectively for \bar{g} and γ , that is,

$$\bar{N}(\text{Det } \bar{g})^{-\frac{1}{2}} = \tilde{N}(\text{Det } \gamma)^{-\frac{1}{2}}, \quad (4.1)$$

i.e.

$$\bar{N} = \varphi^{2n/(n-2)} \tilde{N}, \quad (4.2)$$

and we suppose that the given initial data is

$$\tilde{\pi} = \tilde{N}^{-1} \bar{\partial}_0 \bar{\psi} = \varphi^{2n/(n-2)} \bar{N}^{-1} \bar{\partial}_0 \bar{\psi} = \varphi^{2n/(n-2)} \bar{\pi}.$$

4.1 Hamiltonian constraint

The energy density on M of a scalar field ψ with potential $V(\psi)$, for an observer at rest in the physical metric \bar{g} reads as follows in terms of the given data:

$$\rho = \frac{1}{2}(\varphi^{-4n/(n-2)} |\tilde{\pi}|^2 + \varphi^{-4/(n-2)} \gamma^{ij} \partial_i \bar{\psi} \partial_j \bar{\psi}) + V(\bar{\psi}). \quad (4.3)$$

We see that the term $|\tilde{\pi}|^2$ adds in the Hamiltonian constraint to $|\tilde{K}|_\gamma^2$, while the term $V(\bar{\psi})$ remains unscaled by a power of φ . The $\partial\psi$ term adds a positive contribution to the φ term, adding to $-R(\gamma)$. The Hamiltonian constraint now reads

$$\mathcal{H} \equiv \Delta_\gamma \varphi - f(\varphi) = 0 \quad (4.4)$$

with

$$f(\varphi) \equiv r\varphi - a\varphi^{-(3n-2)/(n-2)} + b\varphi^{(n+2)/(n-2)},$$

where we have again set $k_n = \frac{n-2}{4(n-1)}$ and where

$$r \equiv k_n[R(\gamma) - |D\bar{\psi}|_\gamma^2], \quad a \equiv k_n(|\tilde{K}|_\gamma^2 + |\tilde{\pi}|^2), \quad b \equiv \frac{n-2}{4n}\tau^2 - \frac{n-2}{(n-1)}V(\bar{\psi}). \quad (4.5)$$

We observe that $a \geq 0$, while $b \leq 0$ if $V(\bar{\psi}) \geq 0$ and $\tau = 0$ (maximal slicing).

We call the equation (4.4) the conformally formulated Hamiltonian constraint, or the Lichnerowicz equation for the Einstein-scalar field theory.

4.2 Momentum constraint

The expression of the scalar field momentum density in terms of the new data is

$$J^i = -\bar{g}^{ij}(\partial_j \bar{\psi})\bar{\pi} = -\varphi^{-2(n+2)/(n-2)}\gamma^{ij}(\partial_j \bar{\psi})\tilde{\pi}. \quad (4.6)$$

The momentum constraint now reads

$$\mathcal{M}^j \equiv D_i \tilde{K}^{ij} - F^j = 0 \quad (4.7)$$

with

$$F^j \equiv \frac{n-1}{n} \varphi^{2n/(n-2)} \gamma^{ij} \partial_i \tau - \gamma^{ij} \partial_j \bar{\psi} \tilde{\pi}.$$

We call this equation the conformally formulated momentum constraint.

We have proved the following theorem.

Theorem 4.1 *The conformally formulated momentum constraint of the Einstein-scalar field system (4.7) is a linear system for \tilde{K} when γ , τ , $\bar{\psi}$ and $\tilde{\pi}$ are given and the function φ is known. It does not contain φ if τ is a constant.*

4.3 Conformal covariance of the constraint equations

It follows from the analysis above that if (φ, \tilde{K}) satisfies the conformally formulated constraints (3.9, 4.6, 3.11), for a specified choice of the free data $(\gamma, \tau, \bar{\psi}, \tilde{\pi})$, then

$$\bar{g}_{ij} = \varphi^{4/(n-2)} \gamma_{ij}, \quad K^{ij} = \varphi^{-2(n+2)/(n-2)} \tilde{K}^{ij} + \frac{1}{n} \bar{g}^{ij} \tau, \quad \bar{\psi}, \quad \bar{\pi} = \varphi^{2n/(n-2)} \tilde{\pi} \quad (4.8)$$

is a solution of the original Einstein-scalar field constraints.

The following conformal covariance result is an immediate corollary.

Theorem 4.2 *Let (φ, \tilde{K}) be a solution of the conformally formulated constraints in the metric γ with data τ , $\bar{\psi}$ and $\tilde{\pi}$. Then $(\varphi' = \theta^{-1} \varphi, \tilde{K}' \equiv \theta^{-2(n+2)/(n-2)} \tilde{K})$ is a solution of the conformally formulated constraints in the metric $\gamma' = \theta^{4/(n-2)} \gamma$ with data $\tau' = \tau$, $\bar{\psi}' = \bar{\psi}$, $\tilde{\pi}' = \theta^{-2n/(n-2)} \tilde{\pi}$.*

5 Solution of the Conformal Momentum Constraint

The general solution of a non homogeneous linear system is obtained by adding a particular solution to the general solution of the associated linear homogeneous system, which, in the case of (3.9), is the following:

$$D_j \tilde{K}^{ij} = 0, \quad \gamma_{ij} \tilde{K}^{ij} = 0. \quad (5.1)$$

Symmetric 2-tensors satisfying (5.1) are called TT tensors (transverse, traceless). As a consequence of Lemma 3.1 the space of TT tensors is the same for two conformal metrics.

We may obtain both the particular solution to (3.9) and the general solution to (5.1) by essentially the same ansatz. One can look for the particular solution of (4.7) as the conformal Lie derivative of a vector field Z , an element of the formal L^2 dual of the space of TT tensors defined by

$$(\mathcal{L}_{\gamma, \text{conf}} Z)_{ij} := D_i Z_j + D_j Z_i - \frac{2}{n} \gamma_{ij} D_h Z^h. \quad (5.2)$$

We look for \tilde{K}_{TT} as the sum of the conformal Lie derivative of a vector Y and an arbitrary traceless symmetric 2 tensor U . Then, setting $X := Z + Y$ it holds that

$$\tilde{K}^{ij} = (\mathcal{L}_{\gamma, \text{conf}} X)^{ij} - U^{ij}, \quad (5.3)$$

with X a vector field solution of the linear system

$$(\Delta_{\gamma, \text{conf}} X)^j := D_i (\mathcal{L}_{\gamma, \text{conf}} Z)^{ij} = D_i U^{ij} + \frac{n-1}{n} \varphi^{2n/(n-2)} \gamma^{ij} \partial_i \tau - \gamma^{ij} \partial_j \bar{\psi} \tilde{\pi}. \quad (5.4)$$

The arbitrary data in the traceless tensor \tilde{K} is the symmetric traceless tensor U .

It has been noted by York [35] that, though the formulation (4.4), (4.7) is invariant in the sense of Theorem 4.2, the splitting of the solution \tilde{K} into a given traceless tensor U and the conformal Lie derivative of an unknown vector X cannot be made conformally invariant. To try to obtain $\tilde{K}' \equiv \theta^{-2(n+2)/(n-2)} \tilde{K}$ given $\gamma' = \theta^{4/(n-2)} \gamma$, we can impose the relation between the given traceless tensors U and U' :

$$U'^{ij} \equiv \theta^{-2(n+2)/(n-2)} U^{ij}, \quad (5.5)$$

however for an arbitrary vector X one has

$$(\mathcal{L}_{\gamma', \text{conf}} X)^{ij} \equiv \theta^{-4/(n-2)} (\mathcal{L}_{\gamma, \text{conf}} X)^{ij}.$$

There is no scaling of X by a power of φ that leads to a vector X' and results in the desired scaling of its conformal Lie derivative. York has proposed to remedy this defect by what he called “the conformal thin sandwich formulation”. Inspired by his work, and by the expression $K^{ij} \equiv N^{-1} \bar{\partial}_0 g^{ij}$, we replace the search for a particular solution as a conformal Lie derivative by the following. For \tilde{N} is a given scalar, we define

$$\tilde{\mathcal{L}}_{\gamma, \text{conf}} X := \tilde{N}^{-1} \mathcal{L}_{\gamma, \text{conf}} X, \quad (5.6)$$

$$(\tilde{\Delta}_{\gamma, \text{conf}} X)^j := D_i (\tilde{\mathcal{L}}_{\gamma, \text{conf}} X)^{ij}. \quad (5.7)$$

The mathematical properties of $\Delta_{\gamma, \text{conf}}$ and $\tilde{\Delta}_{\gamma, \text{conf}}$ are essentially the same.

We choose X to be a solution of the equation

$$(\tilde{\Delta}_{\gamma, \text{conf}} X)^j = D_i U^{ij} + \frac{n-1}{n} \varphi^{2n/(n-2)} \gamma^{ij} \partial_i \tau - \gamma^{ij} \partial_j \psi \tilde{\pi} \quad (5.8)$$

(instead of (5.4)). The tensor \tilde{K} solution of (3.9) is now, instead of (5.3),

$$\tilde{K}^{ij} \equiv (\tilde{\mathcal{L}}_{\gamma, \text{conf}} X)^{ij} - U^{ij}. \quad (5.9)$$

Noting that if we conformally change the metric via $\gamma' = \theta^{4/(n-2)} \gamma$ and the lapse via $\tilde{N} = \theta^{-2n/(n-2)} \tilde{N}'$ then

$$(\tilde{\mathcal{L}}_{\gamma', \text{conf}} X)^{ij} = \theta^{-2(n+2)/(n-2)} (\tilde{\mathcal{L}}_{\gamma, \text{conf}} X)^{ij}, \quad (5.10)$$

we find that K has the required scaling.

We are thus led to the following corollary to Theorem 4.2, under otherwise the same hypothesis.

Corollary 5.1 *If the tensor \tilde{K} , a solution of the momentum constraint conformally formulated in a metric γ , is obtained as the sum of a given traceless tensor U and the product by a given function \tilde{N} of a conformal Lie derivative of a vector X :*

$$\tilde{K}^{ij} \equiv (\tilde{\mathcal{L}}_{\gamma, \text{conf}} X)^{ij} - U^{ij}, \quad (5.11)$$

then the tensor

$$\tilde{K}'^{ij} \equiv (\tilde{\mathcal{L}}_{\gamma', \text{conf}} X)^{ij} - U'^{ij}, \quad U'^{ij} = \theta^{-2(n+2)/(n-2)} U^{ij}, \quad \tilde{N}' = \theta^{2n/(n-2)} \tilde{N} \quad (5.12)$$

is a solution of the momentum constraint conformally formulated in the metric γ' .

6 Asymptotically Euclidean Manifolds

6.1 Definitions

In the following sections we will study the solution of the conformally formulated constraints (4.4) and (4.7) on asymptotically Euclidean manifolds of dimension $n \geq 3$.

The Euclidean space E^n is the manifold R^n endowed with the Euclidean metric, which is $\sum (dx^i)^2$ in canonical coordinates.

A C^∞ , n -dimensional, Riemannian manifold (M, e) is called “Euclidean at infinity” if there exists a compact subset S of M such that $M - S$ is the disjoint union of a finite number of open sets U_i , with each (U_i, e) being isometric to the exterior of a ball in R^n . Each open set $U_i \subset M$ is sometimes called an “end” of M . If M is diffeomorphic to R^n , it has only one end; and we can then take for e the Euclidean metric. Unless otherwise specified our manifolds are without boundary; hence the manifold (M, e) is complete³.

A Riemannian manifold (M, γ) is called **asymptotically Euclidean** if there exists a Riemannian manifold (M, e) , Euclidean at infinity, and if γ tends to e at infinity in each end. Consider one end U and the canonical coordinates x^i in the space R^n which contains the exterior of the ball to which U is diffeomorphic. Set $r \equiv \{\sum (x^i)^2\}^{1/2}$. In the coordinates x^i the metric e has components $e_{ij} = \delta_{ij}$. The metric γ tends to e at infinity if in these coordinates $\gamma_{ij} - \delta_{ij}$ tends to zero. A possible way of making this statement mathematically precise is to use the Nirenberg-Walker weighted Sobolev spaces. One can also use in these elliptic constraint problems weighted Hölder spaces⁴, but they are not well adapted to the related evolution questions.

A **weighted Sobolev space** $W_{s,\delta}^p$, with $1 \leq p \leq \infty$, with s a positive or zero integer, δ a real number, for tensors of some given type on the manifold (M, e) Euclidean at infinity is the space of tensors of that type which admit generalized e -covariant derivatives of order up to s and for which the following norm is finite:

$$\|u\|_{W_{s,\delta}^p} = \left\{ \sum_{0 \leq m \leq s} \int_V |\partial^m u|^p (1 + d^2)^{\frac{1}{2}p(\delta+m)} d\mu \right\}^{1/p}. \quad (6.1)$$

Here ∂ , $|\cdot|$ and $d\mu$ denote the covariant derivative, norm and volume element corresponding to the metric e , and d is the distance in the metric e from a point of M to a fixed point. If (M, e) is an Euclidean space one can choose $d = r$, the Euclidean distance to the origin. The space \mathcal{D} of C^∞ tensors with compact support is dense in $W_{s,\delta}^p$, regardless of what s and δ are, so long as $p < \infty$.

If s and δ are large enough, a function (or tensor field) in $W_{s,\delta}^p$ is continuous and tends to zero at infinity. Specifically if we define C_β^m to be the Banach space of weighted C^m functions (or tensor fields) on (M, e) with norm given by

$$\|u\|_{C_\beta^m} \equiv \sum_{0 \leq \ell \leq m} \sup_M (|\partial^\ell u| (1 + d^2)^{\frac{1}{2}(\beta+\ell)}),$$

then the following inequality holds⁵, with C a number depending only on (M, e) ,

$$\|u\|_{C_\beta^m} \leq C \|u\|_{W_{s,\delta}^p}, \quad \text{if } s > m + \frac{n}{p}, \quad \delta > \beta - \frac{n}{p}. \quad (6.2)$$

³See Chrusciel and Delay [20], and Maxwell [28, 29].

⁴See [13].

⁵For proofs of this embedding and the multiplication rule (6.3), see [16], or [12, II, p. 396].

We see that $u \in W_{s,\delta}^p$ implies that u is continuous and tends to zero at infinity if $s > \frac{n}{p}$ and $\delta > -\frac{n}{p}$.

Let (M, e) be a manifold which is Euclidean at infinity. The Riemannian manifold (M, γ) is said to be (p, σ, ρ) asymptotically Euclidean if $\gamma - e \in W_{\sigma,\rho}^p$. If $\gamma - e \in W_{\sigma,\rho}^p$ with $\sigma > \frac{n}{p}$, and $\rho > -\frac{n}{p}$, then γ is C^0 and $\gamma - e$ tends to zero at infinity. The set of Riemannian metrics (i.e. positive definite symmetric 2-tensors) such that $\gamma - e \in W_{\sigma,\rho}^p$ is denoted by $M_{\sigma,\rho}^p$.

We recall the multiplication lemma

$$W_{s_1,\delta_1}^p \times W_{s_2,\delta_2}^p \subset W_{s,\delta}^p, \quad \text{if } s < s_1 + s_2 - \frac{n}{p}, \quad \delta < \delta_1 + \delta_2 + \frac{1}{p}, \quad (6.3)$$

and the interpolation⁶ inequality: for any $\varepsilon > 0$, there is a $C(\varepsilon)$ such that, for all $u \in W_{m,\delta}^p$, $1 \leq p \leq q \leq \infty$, and $j < m$, one has

$$\|\partial^j u\|_{W_{0,\delta+\frac{n}{p}-\frac{n}{q}+j}^q} \leq C\{\varepsilon\|u\|_{W_{m,\delta}^p} + C_\varepsilon\|u\|_{W_{0,\delta}^p}\}. \quad (6.4)$$

6.2 Linear elliptic systems

We state⁷ the following existence theorem for solutions of linear elliptic PDE's.

Theorem 6.1 *Hypotheses:*

Let (M, e) be a smooth Riemannian manifold Euclidean at infinity. Let

$$Lu \equiv a_2 \partial^2 u + a_1 \partial u + a_0 u \quad (6.5)$$

be a second order linear elliptic operator acting on tensor fields on (M, e) , which in terms of components u^A , $A = 1, \dots, p$, of the tensor u takes the form

$$(Lu)^A \equiv a_{2,B}^{ij,A} \partial_{ij}^2 u^B + a_{1,B}^{i,A} \partial_i u^B + a_{0,B}^A u^B.$$

Let the principal symbol $a_2^{ij} \xi_i \xi_j$ of L be an isomorphism from R^p onto R^p for $\xi \neq 0$. Suppose that the coefficients of L satisfy the following hypotheses

$$a_2 - A \in W_{2,\delta}^p, \quad a_1 \in W_{1,\delta+1}^p, \quad a_0 \in W_{0,\delta+2}^p, \quad (6.6)$$

where $A\partial^2$ is an elliptic operator with C^∞ coefficients, constant in each end of (M, e) and

$$p > \frac{n}{2}, \quad -\frac{n}{p} < \delta < -\frac{n}{p} + n - 2. \quad (6.7)$$

Conclusions:

(1) The operator L is a continuous mapping from $W_{2,\delta}^p$ into $W_{0,\delta+2}^p$.

(2a) There exists a number $C_L > 0$, depending only on A and on the norms of $a_2 - A$, a_1 , a_0 , and a number $\delta' > \delta$ such that the following inequality holds for all $u \in W_{2,\delta}^p$:

$$\|u\|_{W_{2,\delta}^p} \leq C_L \{\|Lu\|_{W_{0,\delta+2}^p} + \|u\|_{W_{1,\delta'}^p}\}. \quad (6.8)$$

⁶See the section entitled "Second order elliptic systems on Riemannian manifolds" in [6].

⁷The theorem relies on previous results of Nirenberg and Walker [32], Cantor [5], Choquet-Bruhat and Christodoulou [16], and the interpolation lemma to lower the regularity required of coefficients (see [6]).

(2b) If in addition L is injective there exists a number C such that the following inequality holds for all $u \in W_{2,\delta}^p$:

$$\|u\|_{W_{2,\delta}^p} \leq C \|Lu\|_{W_{0,\delta+2}^p}. \quad (6.9)$$

(2c) The operator L has finite dimensional kernel and closed range.

(3a) If the adjoint operator L^* is injective on $W_{2,\delta}^p$, then L is surjective from $W_{2,\delta}^p$ onto $W_{0,\delta+2}^p$.

(3b) If L and L^* are both injective then they are isomorphisms from $W_{2,\delta}^p$ onto $W_{0,\delta+2}^p$.

Corollary 6.2 *If, in addition to the previous hypothesis (including injectivity) it holds that*

$$a_2 - A \in W_{s+2,\delta}^p, \quad a_1 \in W_{s+1,\delta+1}^p, \quad a_0 \in W_{s,\delta+2}^p, \quad (6.10)$$

then L is an isomorphism from $W_{s+2,\delta}^p$ onto $W_{s,\delta+2}^p$.

We recall also the following lemma (see [16, Lemma 5.2]).

Lemma 6.3 *Suppose that $u \in W_{2,\delta}^p$ is a solution of the equation*

$$Lu \equiv a_2 \partial^2 u + a_1 \partial u + a_0 u = f, \quad (6.11)$$

where L satisfies the hypotheses of the above theorem and where $f \in W_{0,\delta+2}^p$, $\delta \leq \tilde{\delta}$. Then u is in fact in $W_{2,\tilde{\delta}}^p$, so long as $\tilde{\delta} < n - 2 - \frac{n}{p}$.

6.3 The Poisson operator, $\Delta_\gamma - a$

Theorem 6.4 *Let (M, γ) be a $M_{2,\delta}^p$ manifold⁸, $p > \frac{n}{2}$, $\delta > -\frac{n}{p}$. Let $a \in W_{0,\delta+2}^p$ be given. The Poisson operator $\Delta_\gamma - a$ is an isomorphism from $W_{2,\delta}^p$ onto $W_{0,\delta+2}^p$ if*

$$\int_M \{|\partial u|^2 + au^2\} \mu_\gamma > 0 \quad (6.12)$$

for any $u \in W_{2,\tilde{\delta}}^p$, with $\tilde{\delta}$ some number such that $n - 2 - \frac{n}{p} > \tilde{\delta} > -1 + \frac{n}{2} - \frac{n}{p}$ ($\tilde{\delta} = -1$ if $p = 2$), with $u \not\equiv 0$.

Proof The operator $\Delta_\gamma - a$ is self adjoint. It is an isomorphism $W_{2,\delta}^p \rightarrow W_{0,\delta+2}^p$ if injective. By Lemma 6.3 it is sufficient to prove the injectivity on $W_{2,\tilde{\delta}}^p$ for some $\tilde{\delta}$ such that $\tilde{\delta} < n - 2 - \frac{n}{p}$. The theorem is obtained by integration on M of $u(\Delta_\gamma u - au)$, trivially in the case $p = 2$, $\tilde{\delta} = -1$ (compatible with $p > \frac{n}{2}$ if and only if $n = 3$); and by using either Sobolev embeddings or the Holder inequality in the case $p \neq 2$ and $\tilde{\delta} > -1 + \frac{n}{2} - \frac{n}{p}$.

Theorem 6.5 *Let u satisfy the equation*

$$\Delta_\gamma u - au = -f \quad (6.13)$$

with $\gamma \in M_{2,\delta}^p$, $\delta > -\frac{n}{p}$ and $p > \frac{n}{2}$. Suppose $a \in W_{0,\delta+2}^p$, $u - c \in W_{2,\tilde{\delta}}^p$, where c is a given number, $-1 + \frac{n}{2} - \frac{n}{p} < \tilde{\delta}$ ($\tilde{\delta} \geq -1$ if $p = 2$). Suppose $a \geq 0$. Then $u \geq 0$ on M if $f \geq 0$ and $c \geq 0$. If $f \leq 0$ and $c \leq 0$, then $u \leq 0$ on M . The lower bound of $\tilde{\delta}$ can be weakened to $\delta > -\frac{n}{p}$ if $c = 0$ and $f \in W_{2,\tilde{\delta}}^p$, $\tilde{\delta} > -\frac{n}{p} + \frac{n}{2} - 1$ ($\tilde{\delta} = -1$ if $p = 2$).

⁸The hypothesis $p > n/2$ is stronger than necessary but simplifies the proof, and is needed later in our treatment of non linear equations.

Proof The integration on M of $v(\Delta_\gamma u - au)$, and the choice $v = u^+ = \text{Sup}(u, 0)$, gives $u^+ = \text{constant}$, therefore $u^+ \equiv 0$ since u^+ tends to zero at infinity.

7 Solution of the Momentum Constraint

Given the Riemannian metric γ and the scalar field \tilde{N} the conformally formulated momentum constraint reads

$$D_j(\tilde{\mathcal{L}}X)^{ij} \equiv (\tilde{\Delta}_{\gamma, \text{conf}}X)^i = F^i(\varphi) \quad (7.1)$$

with

$$F^i(\varphi) \equiv D_j U^{ij} + \frac{n-1}{n} \varphi^{2n/(n-2)} \gamma^{ij} \partial_j \tau + \gamma^{ij} (\partial_j \bar{\psi}) \tilde{\pi}, \quad (7.2)$$

where τ is a given function on M and U is a given symmetric traceless 2-tensor field. The sources $\bar{\psi}$ and $\tilde{\pi}$ are given. We suppose momentarily that φ is also a known function. In fact it disappears from the equation if $\partial\tau \equiv 0$.

Lemma 7.1 *Let (M, γ) be a $W_{2,\delta}^p$ asymptotically Euclidean manifold, and let $\tilde{N} = 1 + \nu$, $\nu \in W_{2,\delta}^p$, $\tilde{N} > 0$, be given. Suppose that $p > \frac{n}{2}$, $\delta > -\frac{n}{p}$. Then*

- (1) *The operator $\tilde{\Delta}_{\gamma, \text{conf}}$ is elliptic.*
- (2) *Its kernel in $W_{2,\delta}^p$ is the space of $W_{2,\delta}^p$ conformal Killing vector fields of the metric γ .*

Proof It holds that

$$(\tilde{\Delta}_{\gamma, \text{conf}}Y)^j \equiv \tilde{N}^{-1} D_i \left[D^i Y^j + D^j Y^i - \frac{2}{n} \gamma^{ij} D_k Y^k \right] - \tilde{N}^{-2} (\mathcal{L}_{\gamma, \text{conf}}Y)^{ij} D_i \tilde{N}. \quad (7.3)$$

Using the Ricci identity we find that the principal part is

$$\tilde{N}^{-1} \left[(\Delta_\gamma Y)^j + \left(1 - \frac{2}{n} \right) D^j D_i Y^i \right]. \quad (7.4)$$

The principal symbol is easily checked to be an isomorphism of R^n , for any $n \geq 2$.

(2) We prove the second part of this lemma using integration by parts, using Lemma 6.3.

We can now prove the following theorem.

Theorem 7.2 *Let (M, γ) be a $M_{2,\delta}^p$ asymptotically Euclidean manifold, with $p > \frac{n}{2}$, and $\delta > -\frac{n}{p}$. Let $\bar{\psi} \in W_{2,\delta}^p$ and $U, \tau, \tilde{\pi} \in W_{1,\delta+1}^p$ be given. Suppose also that φ is known, with $\varphi > 0$, and $(1-\varphi) \in W_{2,\delta}^p$. Then the momentum constraint (4.7) has one and only one solution $X \in W_{2,\delta}^p$ if, in addition, $\delta < n - 2 - \frac{n}{p}$.*

If $\partial\tau \equiv 0$, the condition on φ is irrelevant.

Corollary 7.3 *If in addition $\gamma \in M_{2+s,\delta}^p$, $\bar{\psi} \in W_{2+s,\delta}^p$ and $U, \tau, \tilde{\pi} \in W_{1+s,\delta+1}^p$ and $(1-\varphi) \in W_{s+2,\delta}^p$, then the solution X belongs to $W_{s+2,\delta}^p$.*

Proof The given hypothesis and the Sobolev embedding and multiplication properties imply that the coefficients of the operator $\tilde{\Delta}_{\gamma, \text{conf}}$ satisfy the hypotheses of Corollary 6.2 and that $F(\varphi) \in W_{0,\delta+2}^p$. The operator $\tilde{\Delta}_{\gamma, \text{conf}}$ is self adjoint, and its kernel in $W_{2,\delta}^p$ is empty, because there are no such conformal Killing fields on (M, γ) .

8 Solution of the Lichnerowicz Equation

We consider (4.4) (the Lichnerowicz equation)

$$\mathcal{H}(x, X, \varphi) \equiv \Delta_\gamma \varphi - f(x, \varphi) = 0, \quad (8.1)$$

where

$$f(x, \varphi) \equiv r\varphi - a\varphi^{-(3n-2)/(n-2)} + b\varphi^{(n+2)/(n-2)},$$

and

$$r \equiv k_n(R(\gamma) - |D\bar{\psi}|_\gamma^2), \quad a \equiv k_n(|\tilde{K}|_\gamma^2 + |\tilde{\pi}|_\gamma^2) \geq 0, \quad b \equiv \frac{n-2}{4n}\tau^2 - \frac{n-2}{n-1}V(\bar{\psi}). \quad (8.2)$$

We first prove the following lemma.

Lemma 8.1 *If (M, γ) is an $M_{2,\delta}^p$ manifold with $p > \frac{n}{2}$, $\delta > -\frac{n}{p}$ and $\bar{\psi} \in W_{2,\delta}^p$, then $r(\gamma, \bar{\psi}) \in W_{0,\delta+2}^p$.*

Proof $R(\gamma)$ is a sum of terms of the form $\gamma\partial^2\gamma$ and $\gamma\partial\gamma\partial\gamma$ with $\gamma - e \in W_{2,\delta}^p$, $\partial\gamma \in W_{1,\delta+1}^p$ and $\partial^2\gamma \in W_{\sigma-2,\delta+2}^p$. Under the hypotheses made on p and δ , the Sobolev embedding theorem shows that $\gamma - e$ is continuous and bounded on M ; the multiplication theorem completes the proof, also for $|D\bar{\psi}|_\gamma^2$.

8.1 General existence theorem

The following theorem extends to asymptotically Euclidean manifolds a theorem which has been proved for data on compact⁹ manifolds. It can be proved by similar methods.

Theorem 8.2 *Let (M, γ) be in $M_{2,\delta}^p$, $\delta > -\frac{n}{p}$ and $p > \frac{n}{2}$. Suppose that $a, b, r \in W_{0,\delta+2}^p$, and $-1 + \frac{n}{2} - \frac{n}{p} < \delta$ (if $p = 2$ then $\delta = -1$ is admissible). Suppose the Lichnerowicz equation (4.4) admits a subsolution φ_- and a supersolution φ_+ , which are continuous and bounded functions with $\partial\varphi_+, \partial\varphi_- \in W_{1,\delta+1}^p$, such that*

$$\Delta_\gamma \varphi_- \geq f(\cdot, \varphi_-), \quad \Delta_\gamma \varphi_+ \leq f(\cdot, \varphi_+), \quad (8.3)$$

$$\lim_{\infty} \varphi_- \leq 1, \quad \lim_{\infty} \varphi_+ \geq 1 \quad (8.4)$$

and for which there exist numbers ℓ and m , with $\ell > 0$ if $a \neq 0$, such that on M ,

$$\ell \leq \varphi_- \leq \varphi_+ \leq m. \quad (8.5)$$

Then the equation admits a solution φ such that

$$\varphi_- \leq \varphi \leq \varphi_+, \quad 1 - \varphi \in W_{2,\delta}^p \quad (8.6)$$

for

$$\delta < n - 2 - \frac{n}{p}. \quad (8.7)$$

If moreover $\gamma \in M_{2+s,\delta}^p$ and $a, b \in W_{s,\delta+2}^p$, then the solution is such that $1 - \varphi \in W_{s+2,\delta}^p$.

Note that constant sub and super solutions are not natural in the asymptotically Euclidean case. In our application of this theorem to the Lichnerowicz equation, we introduce some intermediate steps to obtain non constant sub and supersolutions.

⁹Results of this sort on compact manifolds were proven by Choquet-Bruhat and Leray [17], using Leray-Schauder degree techniques. In later work (see [25]), sub and super solution techniques have been used.

8.2 Uniqueness theorem

The uniqueness of a solution φ of the Lichnerowicz equation follows from monotonicity if we assume that $r \geq 0$, $a \geq 0$, and $b \geq 0$. A proof of uniqueness can be given under the same hypothesis on a and b , but with no restriction on the sign of r .

Theorem 8.3 *The Lichnerowicz equation (4.4) on (M, γ) , with $\gamma \in M_{2,\delta}^p$, $p > \frac{n}{2}$, $\delta > -\frac{n}{p}$ has at most one positive solution φ , $\varphi - 1 \in W_{2,\delta}^p$, if $a, b, r \in W_{0,\delta+2}^p$, and if $a \geq 0$, and $b \geq 0$.*

Proof Suppose it admits two solutions $\varphi_1 > 0$ and $\varphi_2 > 0$. Using the identity (3.2) we find that, with $\gamma_i := \varphi_i^{4/(n-2)}\gamma$, and $r_i := k_n(R(\gamma_i) - |D\bar{\psi}|_{\gamma_i}^2)$, $i = 1, 2$,

$$\Delta_{\gamma_2}(\varphi_1\varphi_2^{-1}) - (\varphi_1\varphi_2^{-1})r_2 \equiv -(\varphi_1\varphi_2^{-1})^{(n+2)/(n-2)}r_1. \quad (8.8)$$

Since φ_1 is a solution of (8.1), we have

$$\begin{aligned} r_1 &\equiv -\varphi_1^{-(n+2)/(n-2)}\{\Delta_{\gamma}\varphi_1 - \varphi_1 r(\gamma, \bar{\psi})\} \\ &= \varphi_1^{-(n+2)/(n-2)}\{a\varphi_1^{(-3n+2)/(n-2)} - \varphi^{(n+2)/(n-2)}b\} \end{aligned}$$

and an analogous equation for r_2 . Inserting these results in the previous equation gives an equation of the form

$$\Delta_{\gamma_2}(\varphi_1\varphi_2^{-1} - 1) - \lambda\{(\varphi_1\varphi_2^{-1} - 1)\} = 0 \quad (8.9)$$

with

$$\begin{aligned} \lambda &\equiv a\varphi_1^{(-3n+2)/(n-2)}\varphi_2^{-(n+2)/(n-2)}\frac{(\varphi_1\varphi_2^{-1})^{4(n-1)/(n-2)} - 1}{\varphi_1\varphi_2^{-1} - 1} \\ &\quad + b\varphi_1\varphi_2^{-1}\frac{(\varphi_1\varphi_2^{-1})^{4/(n-2)} - 1}{\varphi_1\varphi_2^{-1} - 1}. \end{aligned} \quad (8.10)$$

So long as φ_1 and φ_2 are continuous and positive functions on M , the fractions with denominator $\varphi_1\varphi_2^{-1} - 1$ are continuous and positive functions as well, since the powers of $\varphi_1\varphi_2^{-1}$ appearing in their numerators are greater than 1. Therefore $\lambda \in W_{0,\delta+2}^p$. Noting that by definition $a \geq 0$, it follows that if τ and $V(\bar{\psi})$ are such that $b \geq 0$, then $\lambda \geq 0$. Hence using $\varphi_1\varphi_2^{-1} - 1 \in W_{2,\delta}^p$, and the injectivity of $\Delta_{\gamma} - \lambda$ on $W_{2,\delta}^p$, we have $\varphi_1\varphi_2^{-1} - 1 \equiv 0$.

8.3 Generalized Brill-Cantor theorem

For compact smooth Riemannian manifolds the solutions of the Lichnerowicz equations have been classified by Isenberg [25] through the use of the Yamabe theorem. The Yamabe conformal invariant is defined by

$$\inf_{f \in \mathcal{D}, f \neq 0} \left\{ \int_M \{|Df|^2 + k_n R(\gamma) f^2\} \mu_{\gamma} / \|f\|_{L^{2n/(n-2)}}^2 \right\}.$$

The Yamabe theorem, proved for smooth metrics in an increasing number of cases by Trudinger, Aubin and Schoen, says that any compact Riemannian manifold is conformal to a manifold with constant scalar curvature, $+1$, -1 , or 0 according to the sign of the Yamabe invariant. It is easy to see that this theorem extends to W_2^p metrics, $p > \frac{n}{2}$, in the negative or zero case. In the positive case only a weaker form (proved by Yamabe himself in the smooth case) is proved

to hold, namely that the W_2^p manifold is conformal to a manifold with strictly positive scalar curvature. This property is used in [11] and [30]. Maxwell in particular establishes the classification of solutions of the Lichnerowicz equation using only the sign of the Yamabe invariant and not the full Yamabe theorem. The definition of the Yamabe conformal invariant extends to non compact manifolds (see [28])¹⁰ but there is no theorem for asymptotically Euclidean manifolds analogous to the Yamabe theorem, and the denomination of “positive Yamabe class” for asymptotically Euclidean manifolds with a positive Yamabe invariant is somewhat misleading, as shown by the following theorem, proved¹¹ by Brill and Cantor [4], and generalized in the presence of a scalar field as follows.

Theorem 8.4 *Let (M, γ) be a $(p, 2, \delta)$ asymptotically Euclidean manifold with $p > \frac{n}{2}$, $\delta > -\frac{n}{p}$, and let $\bar{\psi} \in W_{2,\delta}^p$ be a scalar field on M . There exists on M a $(p, 2, \delta)$ asymptotically Euclidean metric γ' conformal to γ such that $r(\gamma', \bar{\psi}) = 0$ if and only if $(M, \gamma, \bar{\psi})$ satisfy the following inequality¹²*

$$\int_M \{|Df|^2 + r(\gamma, \bar{\psi})f^2\} \mu_\gamma > 0 \quad (8.11)$$

for every function f on M with $f \in W_{2,\delta}^p$, $\tilde{\delta} > -\frac{n}{p} + \frac{n}{2} - 1$ ($\tilde{\delta} \geq -1$ if $p = 2$), $f \not\equiv 0$.

Proof (M, γ) is conformal to $(M, \gamma') \in M_{2,\delta}^p$ with $r(\gamma', \bar{\psi}) = 0$ if and only if there exists a function $\varphi > 0$, such that $\gamma' = \varphi^{4/(n-2)}\gamma \in M_{2,\delta}^p$ and

$$\Delta_\gamma \varphi - r(\gamma, \bar{\psi})\varphi = 0. \quad (8.12)$$

Equivalently, setting $\varphi \equiv 1 + u$, (8.12) reads

$$\Delta_\gamma u - r(\gamma, \bar{\psi})u = r(\gamma, \bar{\psi}). \quad (8.13)$$

(1) Suppose that the condition (8.11) is satisfied. The equation (8.13) is linear and elliptic, with $\Delta_\gamma - r(\gamma, \bar{\psi})$ an injective operator on $W_{2,\delta}^p$, and satisfies the hypotheses of Theorem 8.2; therefore it admits a solution $u \in W_{2,\delta}^p \subset C_\alpha^0$. It remains to prove that $\varphi \equiv 1 + u$ is positive, then $\varphi^{4/(n-2)}\gamma \in M_{2,\delta}^p$. One cannot use directly the maximum principle because $r(\gamma, \bar{\psi})$ is not necessarily positive. Inspired by Brill and Cantor (see also [28]) we consider the family of equations

$$\Delta_\gamma \varphi - kr(\gamma, \bar{\psi})\varphi = 0 \quad \text{i.e.} \quad \Delta_\gamma u - kr(\gamma, \bar{\psi})u = kr(\gamma, \bar{\psi}) \quad (8.14)$$

with $k \in [0, 1]$ a number. Each of these equations satisfy the condition (8.11) hence admits a solution $u_k \in W_{2,\delta}^p \subset C_\alpha^0$, and the C_α^0 norm of u_k depends continuously on k . The set $S := \{u_k \in C_\alpha^0, u_k > -1\}$ is open in C_α^0 and non empty because for $k = 0$ it holds that $u_0 = 0$ (i.e. $\varphi_0 = 1$). To show that it is closed, suppose that $u_{k'}$ belongs to its boundary ∂S . Then $u_{k'} \geq -1$, $\varphi_{k'} \geq 0$. Suppose that $\varphi_{k'}$, solution of the elliptic equation (8.12), vanishes at a

¹⁰The definition used by Brill-Cantor in their theorem, carried over in [15],

$$\int_M \{|Df|^2 + k_n R(\gamma)f^2\} \mu_\gamma / \|f\|_{L^{2n/(n-2)}}^2 > 0, \quad f \in \mathcal{D}, f \not\equiv 0$$

was incorrect, because it did not imply this inequality for all $f \in W_{2,\delta}^p$, since the limit of positive functions is not necessarily positive.

¹¹Under more restrictive hypothesis on regularity, and in the case $n = 3$.

¹²This condition, already used to prove injectivity, is implied by the positivity of the Yamabe invariant, because \mathcal{D} is dense in $W_{2,\delta}^p$.

point of M . Then by the weak Harnack inequality (see [33]) there is a ball B_R of center x and a number C such that

$$\|\varphi_{k'}\|_{L^q(B_{2R})} \leq C \inf_{B_R} \varphi_{k'} = 0, \quad (8.15)$$

hence $\varphi_{k'} = 0$ in B_R and also, by continuity, on M . This is impossible because $\varphi_{k'}$ tends to 1 at infinity. Hence $\varphi_{k'} > 0$. The subset S of C_α^0 being both open and closed is all of C_α^0 .

(2) Conversely suppose that $\varphi > 0$ exists and solves the equation satisfying the hypothesis of the theorem. Then we will show that for any $f \not\equiv 0$, $f \in W_{2,\delta}^p$ the inequality (8.11) holds. We set $\theta = f\varphi^{-1}$, then $\theta \in W_{2,\delta}^p \subset C_\alpha^0$. We have by elementary calculus:

$$|Df|^2 = |D\theta|^2\varphi^2 + \varphi D\varphi \cdot D(\theta^2) + \theta^2 |D\varphi|^2. \quad (8.16)$$

The following integration by parts holds for the functions under consideration:

$$\int_M \varphi D\varphi \cdot D(\theta^2) \mu_\gamma = \int_M -\theta^2 D(\varphi D\varphi) \mu_\gamma. \quad (8.17)$$

Therefore

$$\int_M \varphi D\varphi \cdot D(\theta^2) \mu_\gamma = \int_M -\theta^2 (\varphi \Delta_\gamma \varphi + |D\varphi|^2) \mu_\gamma, \quad (8.18)$$

$$\int_M |Df|^2 \mu_\gamma = \int_M \{ |D\theta|^2 \varphi^2 - \theta^2 \varphi \Delta_\gamma \varphi \} \mu_\gamma. \quad (8.19)$$

We have $D\theta \not\equiv 0$ since $\theta \in C_\alpha^0$ tends to zero at infinity and cannot be a constant without being identically zero, which is ruled out by the hypothesis $f \not\equiv 0$. Hence when $\varphi > 0$ satisfies the equation (8.12) the function $f \in W_{2,\delta}^p$, $f \not\equiv 0$ satisfies the inequality

$$\int_M \{ |Df|^2 + r(\gamma, \bar{\psi}) f^2 \} \mu_\gamma > 0. \quad (8.20)$$

Remark The same sort of proof shows that, under the same hypothesis, there exists on M a metric γ' conformal to γ such that $r(\gamma', \bar{\psi}) \leq 0$.

8.4 Existence theorems

Theorem 8.5 *Let (M, γ) be a $M_{2,\delta}^p$ manifold with $p > \frac{n}{2}$. Let $\bar{\psi}$ be a scalar field on M with potential $V(\bar{\psi})$, such that $\bar{\psi} \in W_{2,\delta}^p$ and $V(\bar{\psi}) \in W_{0,\delta+2}^p$. Suppose that (8.11) is satisfied and $b \geq 0$. The Lichnerowicz equation*

$$\Delta_\gamma \varphi - r\varphi + a\varphi^{-(3n-2)/(n-2)} - b\varphi^{(n+2)/(n-2)} = 0, \quad (8.21)$$

$$a, b \in W_{0,\delta+2}^p, \quad \delta > -1 + \frac{n}{2} - \frac{n}{p}, \quad \delta \geq -1 \quad \text{if } p = 2, \quad (8.22)$$

has one and only one solution, $\varphi = 1 + u$, $u \in W_{2,\delta}^p$, if $n - 2 - \frac{n}{p} > \delta > -1 + \frac{n}{2} - \frac{n}{p}$ (extended to $\delta \geq -1$ if $p = 2$). The solution can be obtained by iteration.

Corollary 8.6 *If moreover $\gamma \in M_{2+s,\delta}^p$ and $a, b \in W_{s,\delta+2}^p$, then $u \in W_{s+2,\delta}^p$.*

Proof Uniqueness. This follows from the general theorem 8.3. It can also be proved directly using the monotonicity of the non linear term.

Existence. Since it follows from Theorem 4.2 that the Lichnerowicz equation is conformally invariant, we may, without loss of generality, conformally transform equation (8.21) to one with a metric such that $r(\gamma, \bar{\psi}) = 0$,

$$\Delta_\gamma \varphi + a\varphi^{-(3n-2)/(n-2)} - b\varphi^{(n+2)/(n-2)} = 0. \quad (8.23)$$

(1) We first consider equation (8.23) with $b = 0$:

$$\Delta_\gamma \varphi + a\varphi^{-(3n-2)/(n-2)} = 0. \quad (8.24)$$

This equation admits a constant subsolution $\varphi_- = 1$ but no finite constant supersolution. However, it admits a non constant supersolution, namely the function $\varphi_+ = 1 + u_+$ with $u_+ \in W_{2,\delta}^p$ a solution of the linear equation

$$\Delta_\gamma u_+ = -a; \quad (8.25)$$

indeed the maximum principle shows that $u_+ \geq 0$, hence $\varphi_+ \geq 1$ and

$$\Delta_\gamma \varphi_+ = -a \leq -a\varphi_+^{-(3n-2)/(n-2)}. \quad (8.26)$$

We can use the general existence theorem 8.2 to prove the existence of a solution φ_1 .

(2) We next consider the equation with $a = 0$:

$$\Delta_\gamma \varphi - b\varphi^{(n+2)/(n-2)} = 0. \quad (8.27)$$

This equation admits the subsolution $\varphi_- = 0$ and the supersolution $\varphi_+ = 1$. It admits therefore a solution φ_2 , with $1 - \varphi_2 \in W_{2,\delta}^p$, and $0 \leq \varphi_2 \leq 1$. We prove that $\varphi_2 > 0$ by an argument similar to the one used in the proof of the Brill-Cantor theorem: We consider the family of equations

$$\Delta_\gamma \varphi - kb\varphi^{(n+2)/(n-2)} = 0 \quad (8.28)$$

with $k \in [0, 1]$ a number. Each of these equations admits one solution $\varphi_k = 1 + u_k \geq 0$, with $u_k \in W_{2,\delta}^p \subset C_\alpha^0$, and the C_α^0 norm of u_k depends continuously on k . The proof continues as in the proof of Theorem 8.4.

(3) Consider the general equation (8.23). By the above results this admits φ_1 as a supersolution and φ_2 as a subsolution. Therefore the existence of a solution follows again from the general existence theorem 8.2. The proof of the corollary also follows from this result.

The proof of the corollary follows from that of Theorem 8.2.

We now state two theorems which suppose $b \leq 0$. They can be applied in particular when the scalar field has a non negative potential $V(\bar{\psi})$ and the initial manifold is maximal or has an appropriately small mean extrinsic curvature.

These theorems can also be applied if there exists a density of matter q which is unscaled and non negative. Such a term q adds to $V(\bar{\psi})$.

We first prove a calculus lemma.

Lemma 8.7 *Consider the following algebraic function of y with $a > 0, r > 0$ and $d \geq 0$:*

$$f(y) \equiv dy^{n/(n-2)} - ry^{(n-1)/(n-2)} + a. \quad (8.29)$$

There are two real numbers y_1 and y_2 , such that $0 < y_1 \leq y_2$ and

$$f(y_1) \geq 0, \quad f(y_2) \leq 0 \quad (8.30)$$

if

$$ad^{n-1} < \left[\frac{(n-1)^{n-1}}{n^n} \right] r^n. \quad (8.31)$$

Proof Suppose $d > 0$. The function f starts from $a > 0$ for $y = 0$, decreases when y increases from 0 to $y_m = \left[\frac{(n-1)r}{nd} \right]^{n-2}$, then increases up to infinity with y . The numbers y_1 and y_2 exist with the indicated properties if $f(y_m) < 0$; that is if the inequality (8.31) is satisfied. This inequality always holds if $d = 0$: $f(y)$ starts then from $a > 0$ and decreases to $-\infty$, so we can then verify that the numbers y_1 and y_2 exist.

We use this lemma to prove the following result.

Theorem 8.8 Let (M, γ) be a $M_{2,\delta}^p$ manifold with $p > \frac{n}{2}$. Let $a, b, r \in W_{0,\delta+2}^p$ be given on (M, γ) , with $a \geq 0$, $r \geq 0$, $b \leq 0$ and $\delta > -\frac{n}{p} + \frac{n}{2} - 1$ ($\delta \geq -1$ if $p = 2$). The equation

$$\Delta_\gamma \varphi - r\varphi + a\varphi^{-(3n-2)/(n-2)} - b\varphi^{(n+2)/(n-2)} = 0 \quad (8.32)$$

has a solution $\varphi > 0$, with $1 - \varphi \in W_{2,\delta}^p$, if $\delta < n - 2 - \frac{n}{p}$ and if the inequality (8.31) is satisfied on M , with $d = -b$, and so long as

$$\inf_{x \in M} y_1(x) > 0, \quad \inf_{x \in M} y_2(x) \geq \max \left\{ 1, \sup_{x \in M} z_1(x) \right\}, \quad (8.33)$$

where $y_1(x)$ and $y_2(x)$ are the two positive numbers which annul the algebraic function¹³

$$f_x(z) \equiv -b(x)y^{n/(n-2)} - r(x)y^{(n-1)/(n-2)} + a(x). \quad (8.34)$$

Proof The equation admits a constant subsolution $\varphi_- = \ell > 0$ and a constant supersolution $\varphi_+ = m \geq 1$, $\geq \ell$, and therefore a solution φ with the given properties, so long as almost every $x \in M$ it holds that

$$f_x(\ell^4) \geq 0, \quad f_x(m^4) \leq 0. \quad (8.35)$$

The lemma, and the inequalities (8.33) insure the existence of such numbers ℓ and m , given by

$$\ell = \min \left\{ 1, \inf_{x \in M} z_1(x) \right\}, \quad m = \inf_{x \in M} z_2(x). \quad (8.36)$$

The next theorem does not rely on the sub-super solution method. It supposes that $r \leq 0$, hence applies in particular to data satisfying the generalized positive Yamabe condition, after their conformal transformation to the case $r = 0$. It has a simpler formulation, but it restricts the size of the coefficients a, r and b .

Theorem 8.9 Let (M, γ) be a $M_{2,\delta}^p$ manifold with $p > \frac{n}{2}$. Let $a, b \in W_{0,\delta+2}^p$ be given on (M, γ) , $a \geq 0$, while $b \leq 0$, $r \leq 0$, $\delta > -\frac{n}{p} + \frac{n}{2} - 1$ ($\delta \geq -1$ if $p = 2$). The equation

$$\Delta_\gamma \varphi - r\varphi + a\varphi^{-(3n-2)/(n-2)} - b\varphi^{(n+2)/(n-2)} = 0 \quad (8.37)$$

¹³Polynomial in the case $n = 3$.

has a solution $\varphi > 0$ with $1 - \varphi \in W_{2,\delta}^p$, if $\delta < n - 2 - \frac{n}{p}$ and if a, b and r are small enough in the $W_{0,\delta+2}^p$ norm.

Proof The equation admits the subsolution $\varphi = 1$. We solve it by iteration, starting from $u_0 = 1 - \varphi_0 = 0$. We set

$$\Delta_\gamma u_1 = -a + b + r \leq 0, \quad (8.38)$$

and we see that u_1 exists, $u_1 \geq 0$, $u_1 \in W_{2,\delta}^p$ with

$$\|u_1\|_{W_{2,\delta}^p} \leq C_E M, \quad (8.39)$$

where C_E is a number depending only on γ , through the constant C_E of the elliptic estimate, and where we have set

$$M := A + B + R, \quad A \equiv \|a\|_{W_{0,\delta+2}^p}, \quad B \equiv \|b\|_{W_{0,\delta+2}^p}, \quad R \equiv \|r\|_{W_{0,\delta+2}^p}. \quad (8.40)$$

The Sobolev embedding theorem $W_{2,\delta}^p \subset C_\alpha^0$ implies then the following inequality where C_S is a Sobolev constant

$$\|u_1\|_{C_\alpha^0} \leq C_S \|u_1\|_{W_{2,\delta}^p} \leq CM \quad \text{with} \quad C := C_S C_E. \quad (8.41)$$

This inequality implies that

$$\|\varphi_1\|_{C^0} \leq 1 + M. \quad (8.42)$$

Recursively, we suppose $u_{n-1} \geq 0$ and $\|u_{n-1}\|_{W_{2,\delta}^p} \leq C_E M$; hence $\|u_{n-1}\|_{C_\alpha^0} \leq CM$. The equation defining u_n ,

$$\Delta_\gamma u_n = -r\varphi_{n-1} + a\varphi_{n-1}^{-(3n-2)/(n-2)} - b\varphi_{n-1}^{(n+2)/(n-2)}, \quad (8.43)$$

implies $u_n \geq 0$ and also that

$$\|u_n\|_{W_{2,\delta}^p} \leq C_E \{A + R(1 + CM) + B(1 + CM)^{(n+2)/(n-2)}\}. \quad (8.44)$$

Hence $\|u_n\|_{W_{2,\delta}^p} \leq C_E M \equiv C_E(A + B + R)$ if

$$A + R(1 + M) + B(1 + M)^{(n+2)/(n-2)} \leq A + B + R, \quad (8.45)$$

that is,

$$RM + B[(1 + M)^{(n+2)/(n-2)} - 1] \leq 0. \quad (8.46)$$

This inequality is satisfied if A, B, R are small enough. The sequence u_n is then uniformly bounded in $W_{2,\delta}^p$. The proof can be completed by the usual methods of functional analysis.

9 Uncoupled System of Constraints

The conformally formulated momentum and hamiltonian constraints for the Einstein-scalar field system decouple, in the asymptotically Euclidean case if the initial manifold M is maximal. When the constraints decouple the theorems of the previous sections are sufficient to give existence, non-existence or uniqueness theorems of the systems of constraints. The previously obtained results give, for example, the following theorems under a common hypothesis on the a priori given conformal data.

Theorem 9.1 *Let (M, γ) be an $M_{2,\delta}^p$ manifold; $\bar{\psi} \in W_{2,\delta}^p$ a scalar field with potential $V(\bar{\psi}) \in W_{0,\delta+2}^p$; $\bar{\pi} \in W_{1,\delta+1}^p$ a second scalar field, and $U \in W_{\delta+1}^p$ a symmetric 2-tensor. We assume that $p > \frac{n}{2}$, $\delta > -1 + \frac{n}{2} - \frac{n}{p}$ and $\delta < -2 + n - \frac{n}{p}$ ($\delta = -1$ is admissible if $p = 2$). Then the conformally formulated constraints (7.1) and (8.1) on a maximal submanifold ($\tau = 0$) admit a solution X , $\varphi = 1 + u > 0$, with $X, u \in W_{2,\delta}^p$ if either the hypothesis of Theorem 8.5, or 8.8, or 8.9 are satisfied.*

Proof We have already proven that under the given hypotheses the constraint (7.1) has a unique solution, $X \in W_{2,\delta}^p$, therefore $\tilde{K} \in W_{1,\delta+1}^p$ and $a \in W_{0,\delta+2}^p$ (Sobolev embedding and multiplication (6.2), (6.3)). We know also (see Lemma 8.1) that $r \in W_{0,\delta+2}^p$. Therefore the coefficients of the Lichnerowicz equation (given by equation (4.5) with $\tau = 0$) satisfy the hypothesis required in the quoted theorems. It has a solution $\varphi > 0$, $\varphi - 1 \in W_{2,\delta}^p$, and the pair X, φ satisfies the conformally formulated constraints.

This solution is unique in the cases for which the solution of the Lichnerowicz equation is unique.

Remark 9.2 The theorem still holds if in addition to the scalar field $\bar{\psi}$ there exists unscaled sources with zero momentum and energy density q , and we set $b \equiv -\frac{n-2}{n-1}\{V(\bar{\psi}) + q\}$. This is so because the constraint equations still decouple (assuming $\tau \equiv 0$) when unscaled matter sources are present if these sources have a zero momentum¹⁴.

10 Coupled System of Constraints

In this section we prove a theorem for the case in which the constraints do not decouple. This result is in the spirit of a stability theorem. The use of the implicit function theorem is the simplest way of proving existence of solutions of equations in the neighbourhood of a given one.

We consider as given the $M_{2,\delta}^p$ manifold (M, γ) together with the scalar functions $\bar{\psi}$, $V(\bar{\psi})$, $\bar{\pi}$ and the traceless symmetric 2-tensor U , with $\bar{\psi} \in W_{2,\delta}^p$, $V(\bar{\psi}) \in W_{0,\delta+2}^p$, $\bar{\pi} \in W_{1,\delta+1}^p$, $U \in W_{\delta+1}^p$. We consider the existence of a solution φ and X of the constraints (4.4), (4.7) as we perturb $\tau \in W_{1,\delta+1}^p$ away from zero.

We define as follows a mapping \mathcal{F} from open sets of a pair of Banach spaces into another Banach space:

$$\mathcal{F}: (W_{1,\delta+1}^p; W_{2,\delta}^p \times W_{2,\delta}^p \cap \varphi > 0) \rightarrow W_{0,\delta+2}^p \times W_{0,\delta+2}^p, \quad p > \frac{n}{2}, \delta > -\frac{n}{p} \quad (10.1)$$

by

$$(\tau; X, u \equiv \varphi - 1) \mapsto (\mathcal{H}(\tau; \varphi, X), \mathcal{M}(\tau; \varphi, X)), \quad (10.2)$$

where \mathcal{H} and \mathcal{M} are the left-hand sides of the conformal formulation (4.4), (4.7) of the constraints.

The multiplication properties of weighted Sobolev spaces show that \mathcal{F} is a C^1 mapping. The partial derivative $\mathcal{F}'_{X,u}$ at a point $(0; X, u)$ is the linear mapping from $W_{2,\delta}^p \times W_{2,\delta}^p$ into $W_{0,\delta+2}^p$ given by

$$(\delta X, \delta u) \mapsto (\delta \mathcal{H}, \delta \mathcal{M}), \quad (10.3)$$

¹⁴Dain and Nagy [23] consider unscaled sources with scaled momentum on a maximal submanifold, using H. Friedrich conformal compactification.

where ($\delta b = 0$ because $\tau = 0$ at the considered point and $V(\bar{\psi})$ is fixed)

$$\delta\mathcal{H} \equiv \Delta_\gamma \delta u - \alpha \delta u + \varphi^{-(3n-2)/(n-2)} \delta a \quad (10.4)$$

with

$$\alpha = r + \frac{3n-2}{n-2} a \varphi^{-4(n-1)/(n-2)} + \frac{n+2}{n-2} b \varphi^{4/(n-2)}, \quad (10.5)$$

and, using the expressions for a and \tilde{K} ,

$$\delta a = \frac{n-2}{2(n-1)} \tilde{K}(\tilde{\mathcal{L}}_{\gamma, \text{conf}}) \delta X. \quad (10.6)$$

On the other hand,

$$\delta\mathcal{M} \equiv (\tilde{\Delta}_{\gamma, \text{conf}} \delta X)^i. \quad (10.7)$$

Theorem 10.1 *Specify on the $M_{2,\delta}^p$ manifold (M, γ) the scalar functions $\bar{\psi}$, $V(\bar{\psi})$, $\tilde{\pi}$, \tilde{N} and the traceless symmetric 2-tensor U , with $\bar{\psi} \in W_{2,\delta}^p$, $V(\bar{\psi}) \in W_{0,\delta+2}^p$, $\tilde{\pi} \in W_{1,\delta+1}^p$, $\tilde{N} - 1 \in W_{s+2,\delta}^p$, $\tilde{N} > 0$, $p > \frac{n}{2}$, $-\frac{n}{p} < \delta < n - 2 - \frac{n}{p}$. Let (X_0, φ_0) be a solution of the corresponding constraints with $\tau_0 = 0$. Suppose that for some $\tilde{\delta} > -1 + \frac{n}{2} - \frac{n}{p}$ ($\tilde{\delta} = -1$ if $p = 2$) it holds that*

$$\int_M \{|Df|_\gamma^2 + \alpha_0 f^2\} d\mu_\gamma > 0 \quad \text{for all } f \in W_{2,\tilde{\delta}}^p, f \neq 0 \quad (10.8)$$

with

$$\alpha_0 := r + \frac{3n-2}{n-2} a_0 \varphi_0^{-4(n-1)/(n-2)} - \frac{n+2}{n-2} V(\bar{\psi}) \varphi_0^{4/(n-2)} \geq 0. \quad (10.9)$$

Then there exists a neighbourhood Ω of zero in $W_{1,\delta+1}^p$ such that, if $\tau \in \Omega$, the coupled constraints have one and only one solution (X, φ) , with $\varphi > 0$, and $X, u \equiv \varphi - 1 \in W_{2,\delta}^p$.

Proof Under the hypotheses that we have made, due to properties of elliptic equations discussed above, the partial derivative of \mathcal{F} with respect to the pair (u, X) determines an isomorphism from $W_{2,\delta}^p \times W_{2,\delta}^p$ onto $W_{0,\delta+2}^p$, given by

$$(\delta u, \delta X) \mapsto (\delta\mathcal{M}, \delta\mathcal{H}). \quad (10.10)$$

A straightforward application of the implicit function theorem then completes the proof.

Remark 10.2 The conclusion of the theorem holds in particular if $(M, \gamma, \bar{\psi})$ satisfy the inequality (8.11) and $V(\bar{\psi}) \leq 0$, since then one has always $a_0 \geq 0$.

Remark 10.3 An analogous method can be used to prove the existence of solutions of the coupled system in the additional presence of matter sources with momentum small enough in $W_{0,\delta+2}^p$ norm.

Acknowledgement This work was partially done in the Department of Mathematics of the University of Washington, whose hospitality is gratefully acknowledged by the first two authors.

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