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On the Regularity of Shear Thickening Viscous Fluids

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Abstract The aim of this note is to improve the regularity results obtained by H. Beirão da Veiga in 2008 for a class of p-fluid flows in a cubic domain. The key idea is exploiting the better regularity of solutions in the tangential directions with respect to the normal one, by appealing to anisotropic Sobolev embeddings.

Keywords Non-Newtonian fluids, Shear dependent viscosity, Boundary regularity
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1 Introduction

We are concerned with the global regularity problem for weak solutions of the following system describing the motion of a generalized Newtonian fluid

$$\begin{cases} -\nabla \cdot [(\nu_0 + \nu_1 |\mathcal{D}u|^{p-2})\mathcal{D}u] + (u \cdot \nabla)u + \nabla\pi = f, & \text{in } \Omega, \\ \nabla \cdot u = 0, & \text{in } \Omega, \end{cases} \tag{1.1}$$

where u and π denote, respectively, the velocity and the pressure fields, $\frac{1}{2}\mathcal{D}u$ denotes the symmetric gradient of u, i.e.

$$\mathcal{D}u = \nabla u + \nabla u^{\mathrm{T}},$$

ν_0, ν_1 are positive constants, and $p > 1$. We are interested in the case where the fluid shear thickens. Hence, we assume $p > 2$. Note that system (1.1) is just the canonical representative of a wider class of models, to which our proof applies.

An extensive discussion of incompressible fluids with shear dependent viscosity can be found, for instance, in [2, 13, 15, 16, 19].

We are interested in the regularity results up to the boundary for all the second derivatives of the velocity and the first derivatives of the pressure. The general outline of our proof follows that introduced in the pioneering paper [3], where the half-space case $\mathbb{R}^n_+$, $n \geq 3$, is considered (under slip and non-slip boundary conditions), and $p > 2$. More recently, in [4], the presentation of the above method is simplified by considering a three dimensional cubic domain $\Omega = (]0, 1[)^3$ instead of the half space $\mathbb{R}^n_+$. In the sequel, we improve the results stated in this last paper.

On the other hand, in [5], the method is applied to the shear-thinning case, i.e. $p \in (1, 2)$. Further, in [9], by introducing a new device in the proof given in [5], the author improves the results stated in this last paper. More precisely the author succeeds in taking advantage of the better regularity of solutions in the tangential directions, with respect to the normal one. This allows the use of anisotropic Sobolev embedding theorems (see [20]), instead of the

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classical ones. In this note, by applying this last device to shear thickening fluids, we improve the results stated in [4]. Differently from [9], our iteration scheme leads to actually reaching the limiting exponent of the bootstrap technique. This is possible once one observes that the embedding constants, both of the isotropic and the anisotropic embedding theorem, can be uniformly bounded from above.

Finally, we remark that the same improvements also hold in the cases considered in [10, 11], where, in the shear thinning fluids case, we deal with the regularity problem for cylindrical boundary domains. Concerning general nonflat boundaries, the extension of the above kind of results presents new obstacles, in comparison with the classical case $p = 2$. These obstacles have been overcome in [6–8].

2 Notations and Statement of the Main Results

Throughout this paper, Ω denotes a three dimensional cube $\Omega = (]0,1[)^3$. We set

$$\Gamma = \{x : |x_1| < 1, |x_2| < 1, x_3 = 0\} \cup \{x : |x_1| < 1, |x_2| < 1, x_3 = 1\}.$$

We will impose Dirichlet boundary conditions on the two opposite faces defining Γ and periodicity in the other two directions. We set $x' = (x_1, x_2)$ and say that a function is x'-periodic if it is periodic in both the two directions x_1 and x_2. Therefore, we can write the boundary conditions as follows:

$$u|_\Gamma = 0, \quad u \text{ is } x'\text{-periodic}. \tag{2.1}$$

By $L^p(\Omega)$, $p \in [1, \infty]$, we denote the usual Lebesgue space with norm $\|\cdot\|_p$. Further, we set $\|\cdot\| = \|\cdot\|_2$. By $W^{m,p}(\Omega)$, where m is a nonnegative integer and $p \in (1, \infty)$, we denote the usual Sobolev space with norm $\|\cdot\|_{m,p}$. We denote by $W_0^{1,p}(\Omega)$ the closure in $W^{1,p}(\Omega)$ of $C_0^\infty(\Omega)$, and by $W^{-1,p'}(\Omega)$, $p' = \frac{p}{p-1}$, the strong dual of $W_0^{1,p}(\Omega)$ with norm $\|\cdot\|_{-1,p}$. In the notations concerning duality pairings, norms and functional spaces, we will not distinguish between scalar and vector fields.

We set

$$V_q = \{v \in W^{1,q}(\Omega) : \nabla \cdot v = 0, v|_\Gamma = 0, v \text{ is } x'\text{-periodic}\},$$

and by $(V_q)'$ we denote the dual space of V_q. Recall that, by using a Korn's type inequality, one shows that there exists a constant c, such that $\|v\|_q + \|\nabla v\|_q \le c\|\mathcal{D}v\|_q$ for any $v \in V_q$. In the sequel, we shall tacitly appeal to the previous equivalence for the norms in V_q.

We denote by D^2u the set of all the second partial derivatives of u. The symbol D_*^2u may denote any second-order partial derivative $\partial_{ik}^2 u_j$ (with the obvious meaning $\partial_{ik}^2 u_j = \frac{\partial^2 u_j}{\partial x_i \partial x_k}$) except for the derivatives $\partial_{33}^2 u_j$, $j = 1, 2$. Moreover, we set

$$|D_*^2 u|^2 := |\partial_{33}^2 u_3|^2 + \sum_{\substack{i,j,k=1 \\ (i,k)\neq(3,3)}}^{3} |\partial_{ik}^2 u_j|^2.$$

By the symbol $\nabla_*\pi$ we denote the second and the third components of the gradient of π.

We denote by c positive constants, which may have different values even in the same equation.

Definition 2.1 *Assume that $f \in (V_2)'$. We say that (u, π) is a weak solution to problem* (1.1)–(2.1), *if $u \in V_p$ satisfies*

$$\frac{1}{2}\int_\Omega (\nu_0 + \nu_1|\mathcal{D}u|^{p-2})\mathcal{D}u \cdot \mathcal{D}v\,\mathrm{d}x = \int_\Omega f \cdot v\,\mathrm{d}x \tag{2.2}$$

for all $v \in V_p$ and π is a distribution determined, up to a constant, by de Rham's Theorem.

Note that, by replacing v by u in (2.2) and then applying Hölder's and Young's inequalities, it readily follows that

$$\begin{aligned} &\nu_0^2\|\nabla u\|^2 + 2\nu_0\nu_1\|\mathcal{D}u\|_p^p \le c\|f\|^2, \\ &\nu_0\nu_1^{\frac{1}{p-1}}\|\nabla u\|^2 + \nu_1^{p'}\|\mathcal{D}u\|_p^p \le c\|f\|_{p'}^{p'}. \end{aligned} \tag{2.3}$$

The existence and uniqueness of a weak solution is a well-known result. The proof can be obtained by using the theory of monotone operators as described in [14]. As far as the regularity problem is concerned, the best regularity results for the Dirichlet boundary value problem are obtained in [4]. Here we reproduce the main theorems of that paper as Theorems 2.1 and 2.2 below, concerning the steady Stokes and Navier-Stokes type equations, respectively. To this end, let us write system (1.1) without the convective term

$$\begin{cases} -\nabla\cdot[(\nu_0+\nu_1|\mathcal{D}u|^{p-2})\mathcal{D}u] + \nabla\pi = f, & \text{in } \Omega, \\ \nabla\cdot u = 0, & \text{in } \Omega. \end{cases} \tag{2.4}$$

Theorem 2.1 *Assume that $p \in [2,3]$, $f \in L^2(\Omega)$, and let (u,π) be the weak solution to problem* (2.4), (2.1). *Then D_*^2u and $|\mathcal{D}u|^{\frac{p-2}{2}}\nabla_*\mathcal{D}u$ belong to $L^2(\Omega)$ with*

$$\nu_0\|D_*^2u\| + (\nu_0\nu_1)^{\frac{1}{2}}\||\mathcal{D}u|^{\frac{p-2}{2}}\nabla_*\mathcal{D}u\| \le c\|f\|. \tag{2.5}$$

Moreover, D^2u, $|\mathcal{D}u|^{p-2}\nabla_*\mathcal{D}u$ and $\nabla_*\pi$ belong to $L^l(\Omega)$, where

$$l = 3\frac{4-p}{5-p},$$

and

$$\|\nabla_*\pi\|_l + \|D^2u\|_l + \||\mathcal{D}u|^{p-2}\nabla_*\mathcal{D}u\|_l \le c\|f\| + c\|f\|^{\frac{2}{4-p}}. \tag{2.6}$$

Finally, $\partial_3\pi \in L^m(\Omega)$, where

$$m = 6\frac{4-p}{8-p},$$

and

$$\|\nabla\pi\|_m \le c\|f\|^{\frac{2}{p}} + c\|f\|^{\frac{p}{4-p}}. \tag{2.7}$$

Theorem 2.2 *Assume that $p \in [2,3]$, $f \in L^2(\Omega)$, and let (u,π) be the weak solution to problem* (1.1), (2.1) *in Ω. Then all the regularity results of Theorem* 2.1 *still hold. Moreover, all the estimates still hold, provided that one replaces $\|f\|$ by $\|f\| + \|\nabla u\|_p^2$.*

Our aim is to improve the previous regularity results for both the steady Stokes and Navier-Stokes type equations. The main step is to prove Theorem 2.3 below for solutions to system (2.4). The extension of the results obtained for the Stokes type equations to the Navier-Stokes ones is quite natural, by treating the convective term "as a right-hand side".

Set

$$\frac{1}{s(q)} = \frac{p-2}{q} + \frac{1}{2}. \tag{2.8}$$

Theorem 2.3 *Assume that $p \in [2,4]$, $f \in L^2(\Omega)$, and let (u,π) be the weak solution to problem* (2.4), (2.1) *in Ω. Then, D^2u, $\nabla_*\pi$ and $|\mathcal{D}u|^{p-2}\nabla_*\mathcal{D}u$ belong to $L^{\frac{8-p}{3}}(\Omega)$, and $\partial_3\pi \in L^{\overline{s}}(\Omega)$, where*

$$\overline{s} = \min\left\{s(8-p), \frac{8-p}{3}\right\} = s(8-p).$$

Moreover, for $p \in [2, 4)$*, there hold*

$$\|\nabla_* \pi\|_{\frac{8-p}{3}} + \|D^2 u\|_{\frac{8-p}{3}} + \||\mathcal{D}u|^{p-2} \nabla_* \mathcal{D}u\|_{\frac{8-p}{3}} \le c\|f\| + c\|f\|^{\frac{2}{4-p}} \tag{2.9}$$

and

$$\|\nabla \pi\|_{\overline{s}} \le c\|f\| + c\|f\|^{\frac{2p-2}{4-p}}. \tag{2.10}$$

Theorem 2.4 *Assume that* $p \in [2, 4]$*,* $f \in L^2(\Omega)$*, and let* (u, π) *be the weak solution to problem* (1.1), (2.1) *in* Ω*. Then all the regularity results of Theorem* 2.3 *still hold.*

3 Proof of Theorem 2.3

We recall some preliminary results, which play a key role for the proof of the theorem, and introduce some further notations.

In order to make more clear our way of reasoning, it is convenient to recall the main steps and results of [4], without any claim of completeness. The first step is the following basic result (see [4, Theorem 3.1]).

Theorem 3.1 *Assume that* $p \ge 2$*,* $f \in L^2(\Omega)$*, and let* (u, π) *be the weak solution to problem* (2.4), (2.1)*. Then, in addition to* (2.5)*, one has that* $D^2 u$*,* $|\mathcal{D}u|^{p-2} \nabla_* \mathcal{D}u$ *and* $\nabla_* \pi$ *belong to* $L^{p'}(\Omega)$*, and satisfy the following estimate*

$$\|\nabla_* \pi\|_{p'} + \|D^2 u\|_{p'} + \||\mathcal{D}u|^{p-2} \nabla_* \mathcal{D}u\|_{p'} \le \mathcal{K}_p \tag{3.1}$$

with

$$\mathcal{K}_p = c\|f\|(1 + \|\mathcal{D}u\|_p^{\frac{p-2}{2}}).$$

The previous theorem is the first step for a bootstrap argument which leads to estimates (2.6) and (2.7) of Theorem 2.1 (see [4]). We observe that, while the assumption $p \ge 2$ is a crucial point in the previous theorems, Theorem 2.1 still holds if the assumption $p \le 3$ is relaxed to $p < 4$. However, such a theorem would not give improvements to Theorem 3.1 in this case, since there holds $l < p'$ for $p \in (3, 4)$.

The second tool is an "intermediate" regularity result, which gives higher regularity results in the extra-hypothesis of higher integrability of $\mathcal{D}u$ (see [4, Theorem 3.2]).

Theorem 3.2 *Let the assumptions of Theorem* 3.1 *be satisfied. Assume, in addition,* $p \in [2, 4]$ *and*

$$\mathcal{D}u \in L^q(\Omega) \quad \textit{for some } q \in [p, 6]. \tag{3.2}$$

Then, besides estimate (2.5)*, one has that* $D^2 u$*,* $|\mathcal{D}u|^{p-2} \nabla_* \mathcal{D}u$ *and* $\nabla_* \pi$ *belong to* $L^r(\Omega)$*, with*

$$r = r(q) = \frac{2q}{p - 2 + q}. \tag{3.3}$$

More precisely,

$$\|\nabla_* \pi\|_r + \|D^2 u\|_r + \||\mathcal{D}u|^{p-2} \nabla_* \mathcal{D}u\|_r \le \mathcal{K}_q \tag{3.4}$$

with

$$\mathcal{K}_q = c\|f\|(1 + \|\mathcal{D}u\|_q^{\frac{p-2}{2}}). \tag{3.5}$$

The previous theorem is proved in [4] on slightly different assumptions on p and q, i.e., $p \in [2, 3]$ and $q \in [3, 6]$. However, as the author remarks, the result still holds on our assumptions on the ranges of p and q, without any changes in the proof.

Remark 3.1 The assumption $p \geq 2$ implies that r is a nondecreasing function of q. In particular, if $p \in [2, 4]$ and $q \in [p, 6]$, then $r \in [\frac{4}{3}, 2]$. Further, we observe that the positive constants c that appear in the previous Theorems 2.1, 3.1 and 3.2 do not depend on the parameters p, q and r, since in our hypotheses for these parameters, they are bounded from above. For further details in this regard, we refer to the proofs in [4].

Rather than to furnish a regularity result corresponding to stronger integrability hypotheses on $\mathcal{D}u$, the aim of [4, Theorem 3.2] is to allow a bootstrap argument which leads to higher regularity results in the natural hypothesis $u \in W^{1,p}(\Omega)$. In the sequel, we also appeal to a bootstrap argument. However, as already stressed before, the new idea consists in appealing to embedding theorem for anisotropic Sobolev spaces instead of classical Sobolev embeddings. Hence, the main tool of our proof is the following embedding theorem, due to Nikol'skiĭ, Slobodeckiĭ and Troisi. For details, we refer, for instance, to [1, 20]. Let $1 \leq p_1, p_2, p_3 < +\infty$ be real numbers with harmonic mean $\frac{1}{\overline{p}} = \frac{1}{3} \sum_{i=1}^{3} \frac{1}{p_i}$. For $\overline{p} < 3$, let $\overline{p}^*$ be the Sobolev conjugate of $\overline{p}$, i.e.,

$$\frac{1}{\overline{p}^*} = \frac{1}{\overline{p}} - \frac{1}{3},$$

and set

$$W^{1,(p_1,p_2,p_3)}(\Omega) = \{v \in W^{1,1}(\Omega) : \partial_i v \in L^{p_i}(\Omega)\}.$$

Proposition 3.1 *Let $v \in W^{1,(p_1,p_2,p_3)}(\Omega)$ and*

$$q \begin{cases} = \overline{p}^*, & \text{if } \overline{p} < 3, \\ \in [1, +\infty), & \text{if } \overline{p} \geq 3. \end{cases}$$

Moreover, if $\overline{p} < 3$, assume that $\overline{p}^* > p_i$ for any i. Then there exists a constant C, depending on p_1, p_2 and p_3 if $\overline{p} < 3$, and also on q if $\overline{p} \geq 3$, such that

$$\|v\|_{L^q(\Omega)} \leq C \sum_{i=1}^{3} (\|\partial_i v\|_{L^{p_i}(\Omega)} + \|v\|_{L^1(\Omega)}). \tag{3.6}$$

The following lemma is a direct consequence of the application of Proposition 3.1 to our setting and gives the main device of the proof.

Lemma 3.1 *Assume that $D_*^2 u \in L^2(\Omega)$ and $\partial_{33}^2 u_1, \partial_{33}^2 u_2 \in L^\beta(\Omega)$, for some $\beta \in [1, 2]$. Then $\mathcal{D}u \in L^{3\beta}(\Omega)$.*

Proof From the hypotheses, it follows that the derivatives $\partial_1 \nabla u$ and $\partial_2 \nabla u$ belong to $L^2(\Omega)$, while the derivative of ∇u with respect to x_3 belongs to $L^\beta(\Omega)$. Then we can apply Proposition 3.1 with $v = \nabla u$. Since $\frac{1}{\overline{p}} = \frac{1}{3}(1 + \frac{1}{\beta}) > \frac{1}{3}$, we have $\overline{p}^* = 3\beta$ and we obtain $\nabla u \in L^{3\beta}(\Omega)$. This obviously means that $\mathcal{D}u \in L^{3\beta}(\Omega)$ and

$$\|\mathcal{D}u\|_{3\beta} \leq c(\|D_*^2 u\| + \|\partial_{33}^2 u_1\|_\beta + \|\partial_{33}^2 u_2\|_\beta + \|\nabla u\|). \tag{3.7}$$

It is outstanding to observe that the constant c can be uniformly bounded from above, if the exponent $\overline{p}$ is strictly less than the dimension $n = 3$; in this regard, we refer to the proof of the anisotropic embedding theorem given in [20]. This is obviously our case. Hence, in what follows, the constants c will be independent of the parameters.

Proof of Theorem 2.3 For the exponent $p = 4$, the result follows directly from Theorem 3.1. Hence, we assume $p \in [2, 4)$. Set $q_1 = p$. From estimate (2.3) we know that $\mathcal{D}u \in L^{q_1}(\Omega)$. Furthermore, Theorem 3.1 gives $D_*^2 u \in L^2(\Omega)$ and $\partial_{33}u_1, \partial_{33}u_2 \in L^{p'}(\Omega)$. By applying Lemma 3.1 with $\beta = r(q_1) = p'$, we get $\mathcal{D}u \in L^{q_2}(\Omega)$, with $q_2 = 3r(q_1)$. We explicitly observe that $q_2 > q_1$, since $p < 4$. Actually, by combining Theorem 3.1 with Lemma 3.1, we see that $u \in W^{1,q}(\Omega)$ implies $u \in W^{1,3r(q)}(\Omega)$, where $r(q)$ is given by (3.3). Moreover, by (3.4), (3.5) and (3.7),

$$\|u\|_{1,3r(q)} \le c\|f\| + c\|\nabla u\| + c\|f\|\|u\|_{1,q}^{\frac{p-2}{2}}. \tag{3.8}$$

Define the recursive sequence $\{q_m\}$ as

$$\begin{cases} q_1 = p, \\ q_{m+1} = 3r(q_m) = \dfrac{6q_m}{p-2+q_m}, \end{cases}$$

or equivalently,

$$\frac{1}{q_{m+1}} = \frac{1}{3}\Big(\frac{p-2}{2q_m} + \frac{1}{2}\Big).$$

Since $p \in [2, 4)$, such a sequence is increasing and bounded from above. Assuming that $\mathcal{D}u \in L^{q_m}(\Omega)$, from Theorem 3.2 we obtain $\partial_{33}u_1, \partial_{33}u_2 \in L^{r(q_m)}(\Omega)$, with $r(q_m) = \frac{2q_m}{p-2+q_m}$. Hence, by applying Lemma 3.1 with $\beta = r(q_m)$, we get $\mathcal{D}u \in L^{\frac{6q_m}{p-2+q_m}}(\Omega)$. The sequence $\{q_m\}$ monotonically converges to the limit $8 - p$. This shows that $\mathcal{D}u \in L^{\overline{q}}(\Omega)$ for any $\overline{q} < 8 - p$. Moreover, from (3.8),

$$\|u\|_{1,q_{m+1}} \le c\|f\| + c\|f\|\|u\|_{1,q_m}^{\frac{p-2}{2}}, \tag{3.9}$$

where we have used estimate (2.3). Taking into account that, if $p \in [2, 4)$, then $0 \le \frac{p-2}{2} < 1$, and following the arguments used in [4], we can easily verify that $\|u\|_{1,q_m}$ is uniformly bounded, at least for large values of m, as

$$\|u\|_{1,q_m} \le c\|f\| + c\|f\|^{\frac{2}{4-p}}. \tag{3.10}$$

For readers' convenience, we reproduce the proof here. Set $b = \|f\|$ and $\alpha = \frac{p-2}{2}$. Moreover, set $a_m = \|u\|_{1,q_m}$, $b_1 = a_1$ and $b_{m+1} = cb + cbb_m^\alpha$. From (3.9), $a_m \le b_m$ for each m. Denote by λ the unique solution of the equation $\lambda = cb + cb\lambda^\alpha$. If $b_1 < \lambda$, then b_m is an increasing sequence and converges to the fixed point λ. Hence $a_m < \lambda$ for any m. If $b_1 > \lambda$, then the sequence b_m decreases and converges to the value λ. Hence $a_n < 2\lambda$ for large values of m. On the other hand, if $\lambda \le 1$, then $\lambda = cb + cb\lambda^\alpha \le 2cb$; if $\lambda > 1$, then $\lambda = cb + cb\lambda^\alpha \le 2cb\lambda^\alpha$, which gives $\lambda < (2cb)^{\frac{1}{1-\alpha}}$. This shows that $\lambda \le 2cb + (2cb)^{\frac{1}{1-\alpha}}$. Therefore, one obtains (3.10) at least for large values of m.

Hence, the limit exponent $\overline{q} = 8 - p$ can be actually reached. Recalling (2.3), we get

$$\|u\|_{1,\overline{q}} \le c\|f\| + c\|f\|^{\frac{2}{4-p}}. \tag{3.11}$$

Applying once again Theorem 3.2 with $\mathcal{D}u \in L^{\overline{q}}(\Omega)$, we also get $\nabla_*\pi, |\mathcal{D}u|^{p-2}\nabla_*\mathcal{D}u, \partial_{33}u_1$, $\partial_{33}u_2 \in L^{\overline{r}}(\Omega)$ with $\overline{r} = \frac{8-p}{3}$, and, using estimate (3.11), we arrive at estimate (2.9). Finally, $\partial_3\pi \in L^{\overline{s}}$ with $\overline{s} = \min\{s(8-p), \frac{8-p}{3}\} = s(8-p)$ follows from estimate (7.6) in [4] and, using (3.5) and (3.11), we can easily get (2.10). The proof is then accomplished.

4 Proof of Theorem 2.4

In this section, we deal with the full Navier-Stokes type equations (1.1). The proof of Theorem 2.4 follows step by step the proof of the corresponding theorem in [4]. However, we prefer to reproduce it here, in order to be self-contained. The proof is based on the usual strategy of treating the convective term $(u \cdot \nabla)u$ as a right-hand side and deriving a priori estimates. First of all, we observe that the validity of the following identity

$$\int_\Omega (u \cdot \nabla)u \cdot u \mathrm{d}x = 0$$

implies that the estimates (2.3) still hold for weak solutions of the complete system (1.1). Hence, in particular,

$$\|u\|_{1,p} \leq c\|f\|_{p'}^{\frac{1}{p-1}}.$$

Set

$$F = f - (u \cdot \nabla)u$$

and let us estimate the L^2-norm of F.

At first, we assume that $p \in [2,3)$. By Hölder's inequality, there holds

$$\|(u \cdot \nabla)u\| \leq c\|u\|_{p^*}\|\nabla u\|_s,$$

where p^* is the Sobolev embedding exponent of p and $s = \frac{6p}{5p-6}$. Applying the Sobolev embedding $W^{1,p}(\Omega) \subset L^{p^*}(\Omega)$ and the well-known embeddings $W^{\frac{3}{2},p'}(\Omega) \subset W^{1,s}(\Omega)$, $W^{2,p'}(\Omega) \subset W^{\frac{3}{2},p'}(\Omega)$, one has that for any real $\varepsilon > 0$, there exists a constant C_ε, such that

$$\|(u \cdot \nabla)u\| \leq c\|u\|_{1,p}\|u\|_{\frac{3}{2},p'} \leq \|u\|_{1,p}(C_\varepsilon\|u\|_{1,p} + \varepsilon\|u\|_{2,p'}).$$

Hence

$$\|F\| \leq \|f\| + c\|u\|_{1,p}(C_\varepsilon\|u\|_{1,p} + \varepsilon\|u\|_{2,p'}).$$

This enable us to obtain the desired a priori estimate on the $W^{2,p'}$-norm of u in terms of $\|f\|$. Indeed, by (3.1) and the above estimate, we get

$$\begin{aligned}\|u\|_{2,p'} &\leq c(1 + \|\mathcal{D}u\|_p^{\frac{p-2}{2}})\|F\| \\ &\leq c(1 + \|f\|_{p'}^{\frac{p-2}{2(p-1)}})(\|f\| + C_\varepsilon\|u\|_{1,p}^2) + c\varepsilon(1 + \|f\|_{p'}^{\frac{p-2}{2(p-1)}})\|u\|_{1,p}\|u\|_{2,p'},\end{aligned}$$

which, with a suitable choice of ε, gives the estimate for $\|u\|_{2,p'}$ and leads to the boundedness of $\|F\|$.

In the case $p \in [3,4]$, the boundedness of $\|F\|$ comes easily by increasing the convective term as

$$\|(u \cdot \nabla)u\| \leq c\|u\|_{\frac{2p}{p-2}}\|\nabla u\|_p \leq c\|u\|_{1,p}^2,$$

where we have used the Sobolev embeddings for $p = 3$ and $p > 3$.

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