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Gorenstein-Projective Modules over $T_{m,n}(A)^*$

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Abstract A new class of Gorenstein algebras $T_{m,n}(A)$ is introduced, their module categories are described, and all the Gorenstein-projective $T_{m,n}(A)$-modules are explicitly determined.

Keywords Gorenstein algebra, Gorenstein-projective module
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1 Introduction

Throughout all algebras A are Artinian, and all modules are finitely generated. Let A-mod be the category of left A-modules. An A-module M is Gorenstein-projective, if there is an exact sequence $\cdots \longrightarrow P_{-1} \longrightarrow P_0 \xrightarrow{d_0} P_1 \longrightarrow \cdots$ of projective modules, which stays exact under $\mathrm{Hom}_A(-, A)$, and such that $M \cong \mathrm{Ker}\, d_0$. This kind of modules is an important ingredient in the Gorenstein homological algebras and in the representation theory of algebras (see, e.g., [1, 5, 7–9]); it plays a central role in the Tate cohomology of algebras (see e.g. [3]), and is widely used such as in derived categories and singularity theory (see e.g. [4, 6, 7]).

Gorenstein-projective modules are especially important to the Gorenstein algebras (see e.g. [5]). An algebra A is Gorenstein if inj.dim ${}_AA < \infty$ and inj.dim $A_A < \infty$. Many important classes of algebras, such as group algebras of finite groups, finite-dimensional Hopf algebras, self-injective algebras, algebras of finite global dimension, the cluster tilted algebras, are Gorenstein. It is then fundamental to construct all the Gorenstein-projective modules of a given algebra (see e.g. [10, 11]). In this note, we introduce a new class of Gorenstein algebras $T_{m,n}(A)$, describe the category $T_{m,n}(A)$-mod, and explicitly determine all the Gorenstein-projective $T_{m,n}(A)$-modules.

Many constructions are inductive. Here we use the upper triangular matrix extensions of algebras via bimodules, i.e., the algebra $\Lambda = \left(\begin{smallmatrix} A & M \\ 0 & B \end{smallmatrix}\right)$ with multiplication given by the one of matrices, where M is an A-B-bimodule. An advantage of such an extension is that a Λ-module is identified with a tripe $\left(\begin{smallmatrix} X \\ Y \end{smallmatrix}\right)_\phi$, or $\left(\begin{smallmatrix} X \\ Y \end{smallmatrix}\right)$ if ϕ is clear, where $X \in A$-mod, $Y \in B$-mod, and

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$\phi : M \otimes_B Y \longrightarrow X$ is an A-map, and that a Λ-map $\left(\begin{smallmatrix} X \\ Y \end{smallmatrix}\right)_\phi \longrightarrow \left(\begin{smallmatrix} X' \\ Y' \end{smallmatrix}\right)_{\phi'}$ is identified with a pair $\left(\begin{smallmatrix} f \\ g \end{smallmatrix}\right)$, where $f \in \mathrm{Hom}_A(X, X')$, $g \in \mathrm{Hom}_B(Y, Y')$, such that $\phi'(\mathrm{Id} \otimes g) = f\phi$ (see [3, p. 73]).

2 Module Category of $T_{m,n}(A)$

This section is to describe the category $T_{m,n}(A)$-mod.

Let A be an algebra. For an integer $m \geq 1$, denote by $T_m(A)$ the upper triangular matrix algebra $\begin{pmatrix} A & A & \cdots & A \\ 0 & A & \cdots & A \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & A \end{pmatrix}_{m \times m}$. For integers m and n with $1 \leq m \leq n$, let $T_{m,n}(A)$ be the algebra given by the block matrix $\left(\begin{smallmatrix} T_m(A) & N \\ 0 & AE \end{smallmatrix}\right)$, where $N = \begin{pmatrix} A & \cdots & A \\ A & \cdots & A \\ \vdots & & \vdots \\ A & \cdots & A \end{pmatrix}_{m \times (n-m)}$ and $AE = \begin{pmatrix} A & 0 & \cdots & 0 \\ 0 & A & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & A \end{pmatrix}_{(n-m) \times (n-m)}$. Then $T_{m,n}(A) = \begin{pmatrix} A & A & \cdots & A & A & \cdots & A \\ 0 & A & \cdots & A & A & \cdots & A \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & A & A & \cdots & A \\ 0 & 0 & \cdots & 0 & A & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & A \end{pmatrix}_{n \times n}$ and $T_{m,m}(A) = T_m(A)$.

In order to describe $T_{m,n}(A)$-mod, we first recall a description of $T_m(A)$-mod from [11]. Define a category $T_m(A)$-rep as follows. An object is $\begin{pmatrix} X_m \\ \vdots \\ X_1 \end{pmatrix}_{(\phi_i)}$, where $X_i \in A$-mod for all i and $\phi_i : X_i \longrightarrow X_{i+1}$ are A-maps for $1 \leq i \leq m-1$. A morphism $\begin{pmatrix} X_m \\ \vdots \\ X_1 \end{pmatrix}_{(\phi_i)} \longrightarrow \begin{pmatrix} Y_m \\ \vdots \\ Y_1 \end{pmatrix}_{(\theta_i)}$ is $\begin{pmatrix} f_m \\ \vdots \\ f_1 \end{pmatrix}$, where $f_i : X_i \longrightarrow Y_i$ are A-maps for all i, such that the following diagram commutes

$$\begin{tikzcd}
X_1 \arrow[r, "\phi_1"] \arrow[d, "f_1"'] & X_2 \arrow[r, "\phi_2"] \arrow[d, "f_2"'] & \cdots \arrow[r, "\phi_{m-1}"] & X_m \arrow[d, "f_m"'] \\
Y_1 \arrow[r, "\theta_1"] & Y_2 \arrow[r, "\theta_2"] & \cdots \arrow[r, "\theta_{n-1}"] & Y_m
\end{tikzcd}$$

Lemma 2.1 (see [11]) (i) *$T_m(A)$ is Gorenstein if and only if A is Gorenstein.*

(ii) *There is an equivalence of categories* $T_m(A)$-mod $\cong T_m(A)$-rep.

(iii) *If A is Gorenstein, then $\begin{pmatrix} X_m \\ \vdots \\ X_1 \end{pmatrix}_{(\phi_i)}$ is a Gorenstein-projective $T_m(A)$-module if and only if X_i are Gorenstin-projective A-modules for all i, $\phi_i : X_i \longrightarrow X_{i+1}$ are monomorphisms and* Coker ϕ_i *are Gorenstin-projective A-modules for $1 \leq i \leq m-1$.*

(iv) *Let A and B be algebras, M an A-B-bimodule with* proj.dim ${}_AM < \infty$. *Then $\Lambda = \left(\begin{smallmatrix} A & M \\ 0 & B \end{smallmatrix}\right)$ is Gorenstein if and only if A and B are Gorenstein and* proj.dim $M_B < \infty$.

Now we define a category $T_{m,n}(A)$-rep as follows. An object is $\begin{pmatrix} X_n \\ \vdots \\ X_{n-m+1} \\ \vdots \\ X_1 \end{pmatrix}_{(\phi_i)}$, where

$X_i \in A$-mod for all i, $\phi_i : X_i \longrightarrow X_{n-m+1}$ $(1 \le i \le n-m)$, and $\phi_j : X_j \longrightarrow X_{j+1}$ $(n-m+1 \le j \le n-1)$ are A-maps. A morphism $\begin{pmatrix} X_n \\ \vdots \\ X_{n-m+1} \\ \vdots \\ X_1 \end{pmatrix}_{(\phi_i)} \longrightarrow \begin{pmatrix} Y_n \\ \vdots \\ Y_{n-m+1} \\ \vdots \\ Y_1 \end{pmatrix}_{(\theta_i)}$ is $\begin{pmatrix} f_n \\ \vdots \\ f_{n-m+1} \\ \vdots \\ f_1 \end{pmatrix}$, where $f_i : X_i \longrightarrow Y_i$ are A-maps for all i, such that the following diagram commutes for $1 \le i \le n-m$

$$\begin{tikzcd} X_i \arrow[r, "\phi_i"] \arrow[d, "f_i"'] & X_{n-m+1} \arrow[r, "\phi_{n-m+1}"] \arrow[d, "f_{n-m+1}"'] & \cdots \arrow[r, "\phi_{n-1}"] & X_n \arrow[d, "f_n"'] \\ Y_i \arrow[r, "\theta_i"] & Y_{n-m+1} \arrow[r, "\theta_{n-m+1}"] & \cdots \arrow[r, "\theta_{n-1}"] & Y_n \end{tikzcd}$$

The main result of this section is as follows.

Theorem 2.1 (i) *$T_{m,n}(A)$ is a Gorenstein algebra if and only if A is a Gorenstein algebra.*

(ii) *There is an equivalence of categories* $T_{m,n}(A)$-mod $\cong T_{m,n}(A)$-rep.

Proof (i) Use induction on n. If $n = m$, then the assertion follows from Lemma 2.1(i). Assume that the assertion is true for $n = m+t$ with $t \ge 0$. We prove that the assertion is true for $n = m+t+1$. Put $P = \begin{pmatrix} A \\ \vdots \\ A \\ 0 \\ \vdots \\ 0 \end{pmatrix}$. Then P has a natural $T_{m,n-1}(A)$-A-bimodule structure such that $T_{m,n}(A) = \begin{pmatrix} T_{m,n-1}(A) & P \\ 0 & A \end{pmatrix}$. Since P is projective as a left $T_{m,n-1}(A)$-module and as a right A-module, the assertion follows directly from Lemma 2.1(iv) and the inductive hypothesis.

(ii) Use induction on n. The assertion for $n = m$ follows from Lemma 2.1(ii). Assume that the assertion holds for $n = m+t$ with $t \ge 0$. We prove that the assertion holds for $n = m+t+1$.

Let $T_{m,n}(A) = \begin{pmatrix} T_{m,n-1}(A) & P \\ 0 & A \end{pmatrix}$. Then a $T_{m,n}(A)$-module X is identified with $\begin{pmatrix} X' \\ X_1 \end{pmatrix}_{\phi}$, where $X' \in T_{m,n-1}$-mod, $X_1 \in A$-mod, and $\phi : P \otimes_A X_1 \longrightarrow X'$ is a $T_{m,n-1}(A)$-map. Using the inductive hypothesis on $T_{m,n-1}$-mod, we have $X' = \begin{pmatrix} X_n \\ \vdots \\ X_{n-m+1} \\ \vdots \\ X_2 \end{pmatrix}_{(\phi_i,\ i\ge 2)}$, where $\phi_i : X_i \longrightarrow X_{n-m+1}$ $(i = 2, \cdots, n-m)$ and $\phi_j : X_j \longrightarrow X_{j+1}$ $(j = n-m+1, \cdots, n-1)$ are A-maps. Since P is a projective $T_{m,n-1}(A)$-module, all A-maps attached to P between A are identities. It follows that all maps attached to $P \otimes_A X_1 = \begin{pmatrix} X_1 \\ \vdots \\ X_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ between X_1 are also identities. Thus

$\phi = \begin{pmatrix} h_n \\ \vdots \\ h_{n-m+1} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, where $h_i : X_1 \longrightarrow X_i$ are A-maps for $n-m+1 \leq i \leq n$, such that the following diagram

$$\begin{tikzcd} X_1 \arrow[r, "\mathrm{Id}_{X_1}"] \arrow[d, "h_{n-m+1}"'] & \cdots \arrow[r, "\mathrm{Id}_{X_1}"] & X_1 \arrow[d, "h_n"] \\ X_{n-m+1} \arrow[r, "\phi_{n-m+1}"] & \cdots \arrow[r, "\phi_{n-1}"] & X_n \end{tikzcd}$$

commutes. Put $\phi_1 = h_{n-m+1}$. Then we get

$$h_{n-m+1} = \phi_1, \quad h_{n-m+2} = \phi_{n-m+1}\phi_1, \quad \cdots, \quad h_n = \phi_{n-1}\cdots\phi_{n-m+1}\phi_1. \tag{2.1}$$

Thus ϕ is uniquely determined by ϕ_i $(1 \leq i \leq n-1)$. And hence all h_i $(n-m+2 \leq i \leq n)$ can be omitted. So X is expressed as $\begin{pmatrix} X_n \\ \vdots \\ X_{n-m+1} \\ \vdots \\ X_1 \end{pmatrix}_{(\phi_i)}$. This completes the proof.

3 Gorenstein-Projective $T_{m,n}(A)$-Modules

In this section, we explicitly describe Gorenstein-projective $T_{m,n}(A)$-modules.

Lemma 3.1 (see [11]) *Let $\Lambda = \left(\begin{smallmatrix} A & M \\ 0 & B \end{smallmatrix}\right)$ be a Gorenstein algebra. If ${}_AM$ and M_B are projective, then $\left(\begin{smallmatrix} X \\ Y \end{smallmatrix}\right)_\phi$ is a Gorenstein-projective Λ-module if and only if $\phi :\ M\otimes_B Y \longrightarrow X$ is a monomorphism, X and* Coker ϕ *are Gorenstein-projective A-modules, and ${}_BY$ is a Gorenstein-projective B-module.*

In what follows, we identify a $T_{m,n}(A)$-module with an object in $T_{m,n}(A)$-rep. So, a $T_{m,n}(A)$-module is written as $\begin{pmatrix} X_n \\ \vdots \\ X_{n-m+1} \\ \vdots \\ X_1 \end{pmatrix}_{(\phi_i)}$, where $\phi_i : X_i \longrightarrow X_{n-m+1}$ $(1 \leq i \leq n-m)$ and $\phi_j : X_j \longrightarrow X_{j+1}$ $(n-m+1 \leq j \leq n-1)$ are A-maps. The main result of this section is as follows.

Theorem 3.1 *Let A be a Gorenstein algebra. Then $X = \begin{pmatrix} X_n \\ \vdots \\ X_{n-m+1} \\ \vdots \\ X_1 \end{pmatrix}_{(\phi_i)}$ is a Gorenstein-projective $T_{m,n}(A)$-module if and only if the following conditions are satisfied:*

(1) *All ϕ_i $(1 \leq i \leq n-1)$ are monomorphisms;*

(2) *All X_i $(1 \leq i \leq n)$ and all* Coker ϕ_i *$(1 \leq i \leq n-1)$ are Gorenstein-projective A-modules;*

(3) *For $1 \leq k \leq n-m$, we have $\mathrm{Im}\phi_k + \cdots + \mathrm{Im}\phi_{n-m} = \mathrm{Im}\phi_k \oplus \cdots \oplus \mathrm{Im}\phi_{n-m}$, and $X_{n-m+1}/(\mathrm{Im}\phi_k + \cdots + \mathrm{Im}\phi_{n-m})$ are Gorenstein-projective A-modules.*

Proof Use induction on n. If $n = m$, then $T_{m,n}(A) = T_m(A)$ and the assertion follows from Lemma 2.1(iii) (in this case condition (3) vanishes). Assume that the assertion holds for $n = m + t$ with $t \geq 0$. We will prove that the assertion holds for $n = m + t + 1$.

Write $T_{m,n}(A)$ as $T_{m,n}(A) = \begin{pmatrix} T_{m,n-1}(A) & P \\ 0 & A \end{pmatrix}$, where P is given as in the proof of Theorem 2.1(i). Then P is a projective right A-module and projective left $T_{m,n-1}(A)$-module. In order to apply Lemma 3.1, we write $X = \begin{pmatrix} X' \\ X_1 \end{pmatrix}_{\phi}$, where $X' = \begin{pmatrix} X_n \\ \vdots \\ X_{n-m+1} \\ \vdots \\ X_2 \end{pmatrix}_{(\phi_i,\ i\geq 2)} \in T_{m,n-1}(A)$-mod, and $\phi = \begin{pmatrix} h_n \\ \vdots \\ h_{n-m+1} \\ 0 \\ \vdots \\ 0 \end{pmatrix} : P \otimes_A X_1 = \begin{pmatrix} X_1 \\ \vdots \\ X_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \longrightarrow \begin{pmatrix} X_n \\ \vdots \\ X_{n-m+1} \\ X_{n-m} \\ \vdots \\ X_2 \end{pmatrix}_{(\phi_i,\ i\geq 2)}$ is a $T_{m,n-1}(A)$-map given by (2.1).

Note that if $n = m+1$ then $X_2, \cdots, X_{n-m}$ vanish. However, the following proof also works in this case. Thus $\operatorname{Coker}\phi = \begin{pmatrix} \operatorname{Coker} h_n \\ \vdots \\ \operatorname{Coker} h_{n-m+1} \\ X_{n-m} \\ \vdots \\ X_2 \end{pmatrix}_{(f_i, i\geq 2)}$, where $f_i : X_i \longrightarrow \operatorname{Coker} h_{n-m+1}$ is the composition of $\phi_i : X_i \longrightarrow X_{n-m+1}$ and the canonical A-map $\pi : X_{n-m+1} \longrightarrow \operatorname{Coker}\phi_1$ for each $2 \leq i \leq n-m$, $f_j : \operatorname{Coker} h_j \longrightarrow \operatorname{Coker} h_{j+1}$ are A-maps induced by the following commutative diagrams for $n - m + 1 \leq j \leq n - 1$

$$\begin{tikzcd} X_1 \arrow[r, "="] \arrow[d, "h_j"] & X_1 \arrow[d, "h_{j+1}"] \\ X_j \arrow[r, "\phi_j"] & X_{j+1} \end{tikzcd} \tag{$*$}$$

By Lemma 3.1, X is a Gorenstein-projective $T_{m,n}(A)$-module if and only if ϕ is a $T_{m,n-1}(A)$-monomorphism, X_1 is a Gorenstein-projective A-module, X' and $\operatorname{Coker}\phi$ are Gorenstein-projective $T_{m,n-1}(A)$-modules. Thus by the inductive hypothesis on X' we know that X is a Gorenstein-projective $T_{m,n}(A)$-module if and only if the following conditions are satisfied:

(i) ϕ is a $T_{m,n-1}(A)$-monomorphism;

(ii) All ϕ_i's are monomorphisms for $i \geq 2$;

(iii) All X_i's are Gorenstein-projective A-modules;

(iv) All $\operatorname{Coker}\phi_i$'s are Gorenstein-projective A-modules for $i \geq 2$;

(v) For $2 \leq k \leq n - m$, we have $\operatorname{Im}\phi_k + \cdots + \operatorname{Im}\phi_{n-m} = \operatorname{Im}\phi_k \oplus \cdots \oplus \operatorname{Im}\phi_{n-m}$, and $X_{n-m+1}/(\operatorname{Im}\phi_k + \cdots + \operatorname{Im}\phi_{n-m})$ are Gorenstein-projective A-modules;

(vi) $\operatorname{Coker}\phi$ is a Gorenstein-projective $T_{m,n-1}(A)$-module.

So, what we need to do now is to prove that the conditions (i)–(vi) are equivalent to the conditions (1)–(3).

First, we prove that conditions (i)–(vi) imply conditions (1)–(3). By (2.1) we see that (i) and (ii) imply (1). Since $\operatorname{Coker}\phi$ is a Gorenstein-projective $T_{m,n-1}(A)$-module, it follows from the inductive hypothesis on $\operatorname{Coker}\phi$ that $\operatorname{Coker}\phi_1$ is a Gorenstein-projective A-module, and hence (2) holds by (iii) and (iv). It remains to prove (3). Since $\operatorname{Coker}\phi$ is a Gorenstein-projective $T_{m,n-1}(A)$-module, by the inductive hypothesis on $\operatorname{Coker}\phi$ we know that f_i are monic for $2 \le i \le n-m$, and $\mathrm{Im}f_2 + \cdots + \mathrm{Im}f_{n-m} = \mathrm{Im}f_2 \oplus \cdots \oplus \mathrm{Im}f_{n-m}$. By construction the fact that f_i is monic means $\mathrm{Im}\phi_i \cap \mathrm{Im}\phi_1 = 0$; while $\mathrm{Im}f_2 + \cdots + \mathrm{Im}f_{n-m} = \mathrm{Im}f_2 \oplus \cdots \oplus \mathrm{Im}f_{n-m}$ means that if $x_2 + \cdots + x_{n-m} \in \mathrm{Im}\phi_1$ with $x_i \in \mathrm{Im}\phi_i$ $(2 \le i \le n-m)$, then $x_i \in \mathrm{Im}\phi_1$ $(2 \le i \le n-m)$. Now assume that $x_1 + x_2 + \cdots + x_{n-m} = 0$ with $x_i \in \mathrm{Im}\phi_i$ $(1 \le i \le n-m)$. This means $x_2 + \cdots + x_{n-m} \in \mathrm{Im}\phi_1$. Then by the above argument we have $x_i \in \mathrm{Im}\phi_1$ $(2 \le i \le n-m)$. Thus $x_i \in \mathrm{Im}\phi_i \cap \mathrm{Im}\phi_1 = 0$ $(2 \le i \le n-m)$. This proves $\mathrm{Im}\phi_1 + \cdots + \mathrm{Im}\phi_{n-m} = \mathrm{Im}\phi_1 \oplus \cdots \oplus \mathrm{Im}\phi_{n-m}$. Since $\operatorname{Coker}\phi$ is a Gorenstein-projective $T_{m,n-1}(A)$-module, it follows from the inductive hypothesis on $\operatorname{Coker}\phi$ that

$$\begin{aligned}
\operatorname{Coker} h_{n-m+1}/(\mathrm{Im}f_2 + \cdots + \mathrm{Im}f_{n-m}) &= \operatorname{Coker}\phi_1/(\mathrm{Im}f_2 + \cdots + \mathrm{Im}f_{n-m}) \\
&= (X_{n-m+1}/\mathrm{Im}\phi_1)/((\mathrm{Im}\phi_1 + \cdots + \mathrm{Im}\phi_{n-m})/\mathrm{Im}\phi_1) \\
&\cong X_{n-m+1}/(\mathrm{Im}\phi_1 + \cdots + \mathrm{Im}\phi_{n-m})
\end{aligned}$$

is a Gorenstein-projective A-module. Now (3) follows from (v).

Conversely, we need to prove that conditions (1)–(3) imply conditions (i)–(vi). From (2.1) we see that (i) and (ii) follow from (1), and (iii)–(v) directly follow from (2) and (3). It remains to prove (vi). By (3) we have $\mathrm{Im}\phi_i \cap \mathrm{Im}\phi_1 = 0$ for $2 \le i \le n-m$. Since ϕ_i is monic for $2 \le i \le n-m$, it follows from the construction that all f_i are monic for $2 \le i \le n-m$. From $(*)$ we get the following commutative diagram for $n-m+1 \le j \le n-1$

$$\begin{tikzcd}
0 \arrow[r] & X_1 \arrow[r, "="] \arrow[d, "h_j"] & X_1 \arrow[r] \arrow[d, "h_{j+1}"] & 0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & X_j \arrow[r, "\phi_j"] & X_{j+1} \arrow[r] & \operatorname{Coker}\phi_j \arrow[r] & 0
\end{tikzcd}$$

Hence by the Snake's Lemma we get an exact sequence

$$0 \longrightarrow \operatorname{Coker} h_j \xrightarrow{f_j} \operatorname{Coker} h_{j+1} \longrightarrow \operatorname{Coker}\phi_j \longrightarrow 0.$$

So f_j is monic for $n-m+1 \le j \le n-1$. Since $h_{n-m+1} = \phi_1$, it follows from the exact sequence above and the Gorensteinness of $\operatorname{Coker}\phi_j$ that all $\operatorname{Coker} h_j$ and $\operatorname{Coker} f_j$ are Gorenstein-projective modules for $n-m+1 \le j \le n-1$. On the other hand, for $2 \le i \le n-m$, we have

$$\begin{aligned}
\operatorname{Coker} f_i &= \operatorname{Coker} h_{n-m+1}/\mathrm{Im}f_i = \operatorname{Coker}\phi_1/\mathrm{Im}f_i \\
&= (X_{n-m+1}/\mathrm{Im}\phi_1)/((\mathrm{Im}\phi_i + \mathrm{Im}\phi_1)/\mathrm{Im}\phi_1) \\
&= X_{n-m+1}/(\mathrm{Im}\phi_i + \mathrm{Im}\phi_1)
\end{aligned}$$

and the exact sequence

$$0 \longrightarrow \Big(\bigoplus_{1\le j\le n-m} \mathrm{Im}\phi_j\Big)\Big/(\mathrm{Im}\phi_i+\mathrm{Im}\phi_1) \longrightarrow \mathrm{Coker}\, f_i \longrightarrow X_{n-m+1}\Big/\Big(\bigoplus_{1\le j\le n-m} \mathrm{Im}\phi_j\Big) \longrightarrow 0.$$

By (3) we know that $X_{n-m+1}\Big/\Big(\bigoplus_{1\le j\le n-m} \mathrm{Im}\phi_j\Big)$ is a Gorenstein-projective A-module. While

$$\Big(\bigoplus_{1\le j\le n-m} \mathrm{Im}\phi_j\Big)\Big/(\mathrm{Im}\phi_i+\mathrm{Im}\phi_1) = \Big(\bigoplus_{1\le j\le n-m} \mathrm{Im}\phi_j\Big)\Big/(\mathrm{Im}\phi_i\oplus\mathrm{Im}\phi_1) = \bigoplus_{2\le t\le n-m, t\ne i} \mathrm{Im}\phi_t$$

is a Gorenstein-projective A-module, so is $\mathrm{Coker}\, f_i$ for $2\le i\le n-m$.

For $2\le k\le n-m$, we still need to prove that $\mathrm{Im}f_k+\cdots+\mathrm{Im}f_{n-m} = \mathrm{Im}f_k\oplus\cdots\oplus \mathrm{Im}f_{n-m}$, and that $\mathrm{Coker}\,\phi_1/(\mathrm{Im}f_k+\cdots+\mathrm{Im}f_{n-m})$ is a Gorenstein-projective A-module. Since $\mathrm{Im}f_k = (\mathrm{Im}\phi_k+\mathrm{Im}\phi_1)/\mathrm{Im}\phi_1$ $(2\le k\le n-m)$, it follows from the direct sum $\mathrm{Im}\phi_1+\cdots+\mathrm{Im}\phi_{n-m} = \mathrm{Im}\phi_1\oplus\cdots\oplus\mathrm{Im}\phi_{n-m}$ in (3) that $\mathrm{Im}f_k+\cdots+\mathrm{Im}f_{n-m} = \mathrm{Im}f_k\oplus\cdots\oplus\mathrm{Im}f_{n-m}$ for $2\le k\le n-m$. Note that $\mathrm{Coker}\,\phi_1/(\mathrm{Im}f_k+\cdots+\mathrm{Im}f_{n-m}) = X_{n-m+1}\Big/\Big(\mathrm{Im}\phi_1\oplus \bigoplus_{k\le i\le n-m}\mathrm{Im}\phi_i\Big)$. Since $\Big(\bigoplus_{1\le i\le n-m}\mathrm{Im}\phi_i\Big)\Big/\Big(\mathrm{Im}\phi_1\oplus\bigoplus_{k\le i\le n-m}\mathrm{Im}\phi_i\Big) = \mathrm{Im}\phi_2\oplus\cdots\oplus\mathrm{Im}\phi_{k-1}$ and $X_{n-m+1}\Big/\Big(\bigoplus_{1\le i\le n-m}\mathrm{Im}\phi_i\Big)$ are Gorenstein-projective A-modules, it follows from the exact sequence

$$\begin{aligned} 0 &\longrightarrow \Big(\bigoplus_{1\le i\le n-m}\mathrm{Im}\phi_i\Big)\Big/\Big(\mathrm{Im}\phi_1\oplus\bigoplus_{k\le i\le n-m}\mathrm{Im}\phi_i\Big) \longrightarrow X_{n-m+1}\Big/\Big(\mathrm{Im}\phi_1\oplus\bigoplus_{k\le i\le n-m}\mathrm{Im}\phi_i\Big)\\ &\longrightarrow X_{n-m+1}\Big/\Big(\bigoplus_{1\le i\le n-m}\mathrm{Im}\phi_i\Big)\longrightarrow 0\end{aligned}$$

that $\mathrm{Coker}\,\phi_1/(\mathrm{Im}f_k+\cdots+\mathrm{Im}f_{n-m})$ is a Gorenstein-projective A-module for $2\le k\le n-m$. Hence $\mathrm{Coker}\,\phi$ is a Gorenstein-projective $T_{m,n-1}(A)$-module by the inductive hypothesis on $\mathrm{Coker}\,\phi$. This proves (vi) and the proof is completed.

Example 3.1 Let A be a Gorenstein algebra.

(i) By Theorem 3.1 a $T_{m,m+1}(A)$-module $X_{(\phi_i)}$ is Gorenstein-projective if and only if ϕ_i $(1\le i\le m)$ are monic, X_i $(1\le i\le m+1)$ and $\mathrm{Coker}\,\phi_i$ $(1\le i\le m)$ are Gorenstein-projective A-modules.

(ii) Note that $\begin{pmatrix} A\\A\\A\\A\end{pmatrix}_{(\mathrm{Id})}$ is not a Gorenstein-projective $T_{2,4}(A)$-module (in fact, $A+A\ne A\oplus A$). But $\begin{pmatrix} A\oplus A\\A\oplus A\\A\\A\end{pmatrix}_{(\binom{1}{0},\binom{0}{1},\mathrm{Id})}$ is a Gorenstein-projective $T_{2,4}(A)$-module.

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