Chin. Ann. Math.
29B(5), 2008, 521–530
DOI: 10.1007/s11401-007-0312-y

Chinese Annals of Mathematics, Series B

On Singular Sets of Local Solutions to p-Laplace Equations**

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Abstract The author proves that the right-hand term of a p-Laplace equation is zero on the singular set of a local solution to the equation. Such a result is used to study the existence of an optimal control problem.

Keywords Singular set, p-Laplace equation, Optimal control, Existence
2000 MR Subject Classification 35J70, 49J45

1 Introduction

In this paper, we consider local solutions to the following p-Laplace equation:

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) = f, \quad \text{in } \Omega, \tag{1.1}$$

where $1 < p < +\infty$, $\Omega \subset \mathbb{R}^n$ is a bounded domain, and $f \in L^q(\Omega)$ for some $q \geq 1$. A local solution of (1.1) means that $u \in W^{1,p}_{\rm loc}(\Omega)$ and

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi \,\mathrm{d}x = \int_\Omega f\varphi \,\mathrm{d}x, \quad \forall\, \varphi \in C_c^\infty(\Omega). \tag{1.2}$$

Our main result is the following theorem.

Theorem 1.1 *Suppose that $u \in W^{1,p}_{\rm loc}(\Omega)$ is a local solution to* (1.1), *$f \in L^q(\Omega)$, and*

$$q > \frac{n}{p}, \quad q \geq 2. \tag{1.3}$$

Then

$$f(x) = 0, \quad \text{a.e. } x \in \{\nabla u = 0\}. \tag{1.4}$$

Morrey established the following well-known result (see [7]).

Lemma 1.1 *Suppose $\psi \in W^{1,1}_{\rm loc}(\Omega)$, then*

$$\nabla\psi(x) = 0, \quad \text{a.e. } \{\psi = 0\}.$$

Manuscript received August 10, 2007. Revised March 10, 2008. Published online August 27, 2008.
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**Project supported by the National Natural Science Foundation of China (No. 10671040), the Foundation for the Author of National Excellent Doctoral Dissertation of China (No. 200522) and the Program for New Century Excellent Talents in University of China (No. 06-0359).

The above lemma and its equivalent form

$$\nabla\psi^+(x) = \begin{cases} \nabla\psi(x), & \text{if } \psi(x) > 0, \\ 0, & \text{if } \psi(x) \le 0 \end{cases}$$

are widely used in the theory of partial differential equations (see [3, 4, 10]). When we treat quasilinear partial differential equations, similar results are needed. In Theorem 1.1, if $p = 2$, then $u \in W^{2,q}_{\mathrm{loc}}(\Omega)$. Consequently,

$$\frac{\partial u}{\partial x_i} \in W^{1,1}_{\mathrm{loc}}(\Omega), \quad \forall\, i = 1, 2, \cdots, n,$$

and

$$\frac{\partial^2 u}{\partial x_i^2} = 0, \quad \text{a.e. } \Big\{\frac{\partial u}{\partial x_i} = 0\Big\}$$

by Lemma 1.1. Therefore

$$f = \triangle u = 0, \quad \text{a.e. } \{\nabla u = 0\} = \bigcap_{i=1}^{n} \Big\{\frac{\partial u}{\partial x_i} = 0\Big\}.$$

Thus, in case $p = 2$, Theorem 1.1 is a direct corollary of Lemma 1.1. When $p \ne 2$, the difficulty to get Theorem 1.1 is that one does not know if $|\nabla u|^{p-2}\frac{\partial u}{\partial x_i} \in W^{1,1}_{\mathrm{loc}}(\Omega)$.

As a corollary of Theorem 1.1, we have

Corollary 1.1 *Under the assumptions of Theorem* 1.1*, if for almost all $x \in \Omega$, $f(x) \ne 0$, then the Lebesgue measure of the singular set $\{\nabla u = 0\}$ is zero. In particular, for any $C \in \mathbb{R}$, the level set $\{u = C\}$ has zero measure.*

Remark 1.1 Let $\Omega = \{x \in \mathbb{R}^n \mid 1 < |x| < 2\}$,

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 1, & \text{in } \Omega, \\ u|_{\partial\Omega} = 0. \end{cases}$$

Then

$$\{\nabla u = 0\} = \{x \in \mathbb{R}^n |\, |x| = r\}$$

for some $r \in (1, 2)$. In this case, $\{\nabla u = 0\}$ has positive $(n-1)$-dimensional Hausdorff measure.

Remark 1.2 On the set $\{\nabla u \ne 0\}$, usually u has better regularity than on the set $\{\nabla u = 0\}$. Thus, similar result can be obtained from Lemma 1.1.

We will prove Theorem 1.1 in Section 2 and give an application of Theorem 1.1 in Section 3.

2 Proof of the Main Theorem

To prove Theorem 1.1, we first establish the following lemma.

Lemma 2.1 *Under the assumptions of Theorem* 1.1*, we have*

$$|\nabla u|^{p-1} \in W^{1,2}_{\mathrm{loc}}(\Omega).$$

Proof It suffices to prove that $|\nabla u|^{p-1} \in W^{1,2}_{\rm loc}(B)$ for any ball $B \subset\subset \Omega$.

Since $q > \frac{n}{p}$, $q \geq 2 > 1$, by modifying the proof of Theorem 8.17 in [3], one can easily prove that $u \in L^\infty_{\rm loc}(\Omega)$ by standard Moser iteration (see also [8]). In particular, $u \in L^\infty(\overline{B})$.

For $\varepsilon \in (0,1)$, let $f_\varepsilon \in C^\infty(B)$ satisfy

$$\begin{cases} \|f_\varepsilon\|_{L^q(B)} \leq \|f\|_{L^q(B)} + 1, \\ f_\varepsilon \to f, \quad \text{strongly in } L^q(B). \end{cases} \tag{2.1}$$

Let $u_\varepsilon \in W^{1,p}(B)$ be the unique solution to the following equation:

$$\begin{cases} -\operatorname{div}\,[(\varepsilon^2 + |\nabla u_\varepsilon|^2)^{\frac{p-2}{2}} \nabla u_\varepsilon] = f_\varepsilon, \quad \text{in } B, \\ u_\varepsilon|_{\partial B} = u. \end{cases} \tag{2.2}$$

It is not very hard to prove that $u_\varepsilon \in C^\infty(B)$,

$$u_\varepsilon \to u, \quad \text{strongly in } W^{1,p}(B), \tag{2.3}$$

and

$$\int_B (\varepsilon^2 + |\nabla u_\varepsilon|^2)^{\frac{p}{2}}\, {\rm d}x \leq C, \tag{2.4}$$

here and hereafter, $C > 0$ denotes a constant independent of $\varepsilon \in (0,1)$. By De Giorgi estimate, we can get

$$\|u_\varepsilon\|_{L_\infty(B)} \leq C. \tag{2.5}$$

Now, let

$$G^\varepsilon(x) \equiv \frac{1}{p}(\varepsilon^2 + |x|^2)^{\frac{p}{2}}, \quad \forall\, x \in \mathbb{R}^n.$$

We have

$$\begin{aligned} \nabla G^\varepsilon(x) &= (\varepsilon^2 + |x|^2)^{\frac{p-2}{2}} x, \\ D^2 G^\varepsilon(x) &= (\varepsilon^2 + |x|^2)^{\frac{p-2}{2}} I + (p-2)(\varepsilon^2 + |x|^2)^{\frac{p-4}{2}} x x^\top, \end{aligned}$$

where I denotes the unit matrix. Thus,

$$\lambda|\xi|^2 \leq \frac{\langle D^2 G^\varepsilon(x)\xi, \xi\rangle}{(\varepsilon^2 + |x|^2)^{\frac{p-2}{2}}} \leq \Lambda|\xi|^2, \quad \forall\, \xi \in \mathbb{R}^n, \tag{2.6}$$

where

$$\lambda = \min(p-1, 1) \quad \text{and} \quad \Lambda = \max(p-1, 1).$$

Rewriting (2.2), we get

$$\begin{cases} -\operatorname{div}\,[\nabla G^\varepsilon(\nabla u_\varepsilon)] = f_\varepsilon, \quad \text{in } B, \\ u_\varepsilon|_{\partial B} = u. \end{cases} \tag{2.7}$$

Denote

$$\widetilde{G}^\varepsilon = D^2 G^\varepsilon(\nabla u_\varepsilon).$$

For $k=1,2,\cdots,n$, we have

$$-\operatorname{div}(\widetilde{G}^{\varepsilon}\nabla(D_k u_\varepsilon))=D_k f_\varepsilon,\quad \text{in } B. \tag{2.8}$$

For any domain $\Omega_0\subset\subset B$, let $\eta\in C_c^\infty(\Omega)$ be such that $0\le\eta\le 1$ in B, and $\eta=1$ in Ω_0. Multiply (2.8) by $\eta^2(\varepsilon^2+|\nabla u_\varepsilon|^2)^{\frac{p-2}{2}}D_k u_\varepsilon$, then integrate in B, and summarize from $k=1$ to n. By the divergence theorem, we get

$$\begin{aligned}
0=&\int_B \eta^2(\varepsilon^2+|\nabla u_\varepsilon|^2)^{\frac{p-2}{2}}\ \operatorname{tr}(D^2u_\varepsilon\widetilde{G}^{\varepsilon}D^2u_\varepsilon)\,\mathrm{d}x\\
&+(p-2)\int_B \eta^2(\varepsilon^2+|\nabla u_\varepsilon|^2)^{\frac{p-4}{2}}\ \langle D^2u_\varepsilon\widetilde{G}^{\varepsilon}D^2u_\varepsilon\nabla u_\varepsilon,\nabla u_\varepsilon\rangle\,\mathrm{d}x\\
&+\int_B 2\eta(\varepsilon^2+|\nabla u_\varepsilon|^2)^{\frac{p-2}{2}}\ \langle\widetilde{G}^{\varepsilon}D^2u_\varepsilon\nabla u_\varepsilon,\nabla\eta\rangle\,\mathrm{d}x\\
&+\int_B \eta^2 f_\varepsilon(\varepsilon^2+|\nabla u_\varepsilon|^2)^{\frac{p-2}{2}}\triangle u_\varepsilon\,\mathrm{d}x\\
&+(p-2)\int_\Omega \eta^2 f_\varepsilon(\varepsilon^2+|\nabla u_\varepsilon|^2)^{\frac{p-4}{2}}\ \langle D^2u_\varepsilon\nabla u_\varepsilon,\nabla u_\varepsilon\rangle\,\mathrm{d}x\\
&+\int_B 2\eta f_\varepsilon(\varepsilon^2+|\nabla u_\varepsilon|^2)^{\frac{p-2}{2}}\nabla u_\varepsilon\cdot\nabla\eta\,\mathrm{d}x\\
\equiv&\ \mathrm{I}_1+\mathrm{I}_2+\mathrm{I}_3+\mathrm{I}_4+\mathrm{I}_5+\mathrm{I}_6.
\end{aligned}\tag{2.9}$$

By (2.6), the matrix $D^2u_\varepsilon\widetilde{G}^{\varepsilon}D^2u_\varepsilon$ is positive simi-definite. Thus

$$\begin{aligned}
\mathrm{I}_1+\mathrm{I}_2=&\int_B \eta^2(\varepsilon^2+|\nabla u_\varepsilon|^2)^{\frac{p-2}{2}}\ \operatorname{tr}(D^2u_\varepsilon\widetilde{G}^{\varepsilon}D^2u_\varepsilon)\,\mathrm{d}x\\
&+(p-2)\int_B \eta^2(\varepsilon^2+|\nabla u_\varepsilon|^2)^{\frac{p-4}{2}}\operatorname{tr}(D^2u_\varepsilon\widetilde{G}^{\varepsilon}D^2u_\varepsilon\nabla u_\varepsilon\nabla u_\varepsilon^\top)\,\mathrm{d}x\\
=&\int_B \eta^2\operatorname{tr}(D^2u_\varepsilon\widetilde{G}^{\varepsilon}D^2u_\varepsilon\widetilde{G}^{\varepsilon})\\
\ge&\ \lambda\int_B \eta^2(\varepsilon^2+|\nabla u_\varepsilon|^2)^{\frac{p-2}{2}}\operatorname{tr}(D^2u_\varepsilon\widetilde{G}^{\varepsilon}D^2u_\varepsilon)\\
\ge&\ \lambda^2\int_B \eta^2(\varepsilon^2+|\nabla u_\varepsilon|^2)^{p-2}\|D^2u_\varepsilon\|^2\,\mathrm{d}x,
\end{aligned}\tag{2.10}$$

where we denote $\|A\|\equiv\sqrt{\operatorname{tr}(AA^\top)}$. Using (2.6) again, we get that for any $\delta>0$,

$$\begin{aligned}
|\mathrm{I}_3|\le&\int_B 2\eta(\varepsilon^2+|\nabla u_\varepsilon|^2)^{\frac{p-2}{2}}|\widetilde{G}^{\varepsilon}D^2u_\varepsilon\nabla u_\varepsilon|\,|\nabla\eta|\,\mathrm{d}x\\
\le&\ C\int_B \eta(\varepsilon^2+|\nabla u_\varepsilon|^2)^{p-2}\|D^2u_\varepsilon\|\,|\nabla u_\varepsilon|\,|\nabla\eta|\,\mathrm{d}x\\
\le&\ \delta\int_B \eta^2(\varepsilon^2+|\nabla u_\varepsilon|^2)^{p-2}\|D^2u_\varepsilon\|^2\,\mathrm{d}x+\frac{C}{\delta}\int_B(\varepsilon^2+|\nabla u_\varepsilon|^2)^{p-2}|\nabla u_\varepsilon|^2\,|\nabla\eta|^2\,\mathrm{d}x.
\end{aligned}\tag{2.11}$$

Since $q\ge 2$, we have $\|f_\varepsilon\|_{L^2(B)}\le C$ by (2.1). Consequently,

$$|\mathrm{I}_4+\mathrm{I}_5|\le\delta\int_B \eta^2(\varepsilon^2+|\nabla u_\varepsilon|^2)^{p-2}\|D^2u_\varepsilon\|^2\,\mathrm{d}x+\frac{C}{\delta}, \tag{2.12}$$

$$|\mathrm{I}_6|\le\int_B(\varepsilon^2+|\nabla u_\varepsilon|^2)^{p-2}|\nabla u_\varepsilon|^2\,|\nabla\eta|^2\,\mathrm{d}x+C. \tag{2.13}$$

Combining (2.10)–(2.13) and choosing $\delta = \frac{\lambda^2}{3}$, we obtain

$$\int_B \eta^2(\varepsilon^2+|\nabla u_\varepsilon|^2)^{p-2}\|D^2u_\varepsilon\|^2\,\mathrm{d}x \le C\int_B(\varepsilon^2+|\nabla u_\varepsilon|^2)^{p-2}|\nabla u_\varepsilon|^2|\nabla\eta|^2\,\mathrm{d}x + C. \tag{2.14}$$

We claim that for any $\Omega_0 \subset\subset B$, there exists a $C = C(\Omega_0)$ such that

$$\int_{\Omega_0}(\varepsilon^2+|\nabla u_\varepsilon|^2)^{p-2}\|D^2u_\varepsilon\|^2\,\mathrm{d}x \le C, \tag{2.15}$$

$$\int_{\Omega_0}(\varepsilon^2+|\nabla u_\varepsilon|^2)^{p-1}\,\mathrm{d}x \le C. \tag{2.16}$$

If $1 < p \le 2$, then $0 < p-1 \le \frac{p}{2}$, and (2.16) follows from (2.4). Moreover, by (2.14), we have

$$\begin{aligned}\int_B \eta^2(\varepsilon^2+|\nabla u_\varepsilon|^2)^{p-2}\|D^2u_\varepsilon\|^2\,\mathrm{d}x &\le C\int_B(\varepsilon^2+|\nabla u_\varepsilon|^2)^{p-1}\,\mathrm{d}x + C\\ &\le C\Big[\int_\Omega(\varepsilon^2+|\nabla u_\varepsilon|^2)^{\frac{p}{2}}\,\mathrm{d}x\Big]^{\frac{2(p-1)}{p}} + C \le C.\end{aligned}$$

Therefore, (2.15) holds.

If $2 < p \le 4$, then replacing η by η^2 in (2.14), we have

$$\begin{aligned}\int_B \eta^4(\varepsilon^2+|\nabla u_\varepsilon|^2)^{p-2}\|D^2u_\varepsilon\|^2\,\mathrm{d}x &\le C\int_B \eta^2(\varepsilon^2+|\nabla u_\varepsilon|^2)^{p-2}|\nabla u_\varepsilon|^2|\nabla\eta|^2\,\mathrm{d}x + C\\ &\le C\int_\Omega \eta^2(\varepsilon^2+|\nabla u_\varepsilon|^2)^{p-2}|\nabla u_\varepsilon|^2\,\mathrm{d}x + C.\end{aligned} \tag{2.17}$$

By (2.5) and the divergence theorem, $\forall\,\beta > 0$, we have

$$\begin{aligned}&\int_B \eta^2(\varepsilon^2+|\nabla u_\varepsilon|^2)^{p-2}|\nabla u_\varepsilon|^2\,\mathrm{d}x\\ &= -\int_B \eta^2u_\varepsilon(\varepsilon^2+|\nabla u_\varepsilon|^2)^{p-2}\triangle u_\varepsilon\,\mathrm{d}x - \int_B 2\eta u_\varepsilon(\nabla\eta\cdot\nabla u_\varepsilon)(\varepsilon^2+|\nabla u_\varepsilon|^2)^{p-2}\,\mathrm{d}x\\ &\quad - 2(p-2)\int_B \eta^2u_\varepsilon(\varepsilon^2+|\nabla u_\varepsilon|^2)^{p-3}\langle D^2u_\varepsilon\nabla u_\varepsilon, \nabla u_\varepsilon\rangle\,\mathrm{d}x\\ &\le C\int_B \eta^2(\varepsilon^2+|\nabla u_\varepsilon|^2)^{p-2}\,\|D^2u_\varepsilon\|\,\mathrm{d}x + C\int_B \eta|\nabla u_\varepsilon|(\varepsilon^2+|\nabla u_\varepsilon|^2)^{p-2}\,\mathrm{d}x\\ &\le \beta\int_B \eta^4(\varepsilon^2+|\nabla u_\varepsilon|^2)^{p-2}\,\|D^2u_\varepsilon\|^2\,\mathrm{d}x + \frac{C}{\beta}\int_\Omega(\varepsilon^2+|\nabla u_\varepsilon|^2)^{p-2}\,\mathrm{d}x\\ &\quad + \beta\int_B \eta^2(\varepsilon^2+|\nabla u_\varepsilon|^2)^{p-2}|\nabla u_\varepsilon|^2\,\mathrm{d}x.\end{aligned}$$

Since $0 < p-2 \le \frac{p}{2}$, by (2.4) and the above inequality, we get for $\beta \in (0, \frac{1}{2})$,

$$\int_B \eta^2(\varepsilon^2+|\nabla u_\varepsilon|^2)^{p-2}|\nabla u_\varepsilon|^2\,\mathrm{d}x \le 2\beta\int_B \eta^4(\varepsilon^2+|\nabla u_\varepsilon|^2)^{p-2}\,\|D^2u_\varepsilon\|^2\,\mathrm{d}x + \frac{C}{\beta}. \tag{2.18}$$

Combining it with (2.17), and choosing β sufficiently small, we get

$$\int_B \eta^4(\varepsilon^2+|\nabla u_\varepsilon|^2)^{p-2}\,\|D^2u_\varepsilon\|^2\,\mathrm{d}x \le C. \tag{2.19}$$

That is, (2.15) holds too in this case. On the other hand, by (2.18) and (2.19), we have

$$\int_B \eta^2(\varepsilon^2+|\nabla u_\varepsilon|^2)^{p-2}|\nabla u_\varepsilon|^2\,\mathrm{d}x \le C.$$

Combining it with

$$\int_B (\varepsilon^2+|\nabla u_\varepsilon|^2)^{p-2}\,\mathrm{d}x \le C\Big[\int_B (\varepsilon^2+|\nabla u_\varepsilon|^2)^{\frac{p}{2}}\,\mathrm{d}x\Big]^{\frac{2(p-2)}{p}} \le C,$$

we get (2.16) in the case $2<p\le 4$. By a similar discussion, we can get (2.15)–(2.16) for any $p\in(1,+\infty)$.

Since

$$|\nabla|\nabla u_\varepsilon|^2| = 2|D^2u_\varepsilon\nabla u_\varepsilon| \le 2|\nabla u_\varepsilon|\,\|D^2u_\varepsilon\|,$$

(2.15) implies

$$\int_{\Omega_0} |\nabla[\,(\varepsilon^2+|\nabla u_\varepsilon|^2)^{\frac{p-1}{2}}\,]|^2\,\mathrm{d}x \le C.$$

Combining it with (2.16), we get

$$\|(\varepsilon^2+|\nabla u_\varepsilon|^2)^{\frac{p-1}{2}}\|_{W^{1,2}(\Omega_0)} \le C.$$

Thus, we can suppose that

$$(\varepsilon^2+|\nabla u_\varepsilon|^2)^{\frac{p-1}{2}} \to h, \quad \text{weakly in } W^{1,2}(\Omega_0),\ \text{strongly in } L^2(\Omega_0).$$

Then, by (2.3), we must have $h=|\nabla u|^{p-1}$. Therefore

$$|\nabla u|^{p-1}\in W^{1,2}(\Omega_0).$$

Consequently,

$$|\nabla u|^{p-1}\in W^{1,2}_{\mathrm{loc}}(B),$$

and we get the proof.

Now, we give a proof of Theorem 1.1.

Proof of Theorem 1.1 By Lemma 2.1, we have $|\nabla u|^{p-1}\in W^{1,2}_{\mathrm{loc}}(\Omega)$. Thus, for any $\varepsilon>0$,

$$\frac{|\nabla u|^{p-1}}{\varepsilon+|\nabla u|^{p-1}} \in W^{1,2}_{\mathrm{loc}}(\Omega).$$

Therefore, $\forall\,\varphi\in C_c^\infty(\Omega)$, we have

$$\begin{aligned}\int_\Omega \frac{|\nabla u|^{p-1}}{\varepsilon+|\nabla u|^{p-1}}\varphi f\,\mathrm{d}x &= \int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla\Big(\frac{|\nabla u|^{p-1}}{\varepsilon+|\nabla u|^{p-1}}\varphi\Big)\,\mathrm{d}x\\ &= \int_\Omega |\nabla u|^{p-2}\frac{|\nabla u|^{p-1}}{\varepsilon+|\nabla u|^{p-1}}\nabla u\cdot\nabla\varphi\,\mathrm{d}x\\ &\quad+\int_\Omega |\nabla u|^{p-2}\varphi\frac{\varepsilon}{(\varepsilon+|\nabla u|^{p-1})^2}\nabla(|\nabla u|^{p-1})\cdot\nabla u\,\mathrm{d}x.\end{aligned}$$

Since

$$\Big||\nabla u|^{p-2}\varphi\frac{\varepsilon}{(\varepsilon+|\nabla u|^{p-1})^2}\nabla(|\nabla u|^{p-1})\cdot\nabla u\Big|\le|\varphi\,\nabla|\nabla u|^{p-1}|,$$

and the last term belongs to $L^1(\Omega)$ and is independent of $\varepsilon>0$, letting $\varepsilon\to0^+$, we get

$$\int_{\Omega\setminus\{\nabla u=0\}}\varphi f\,\mathrm{d}x=\int_\Omega|\nabla u|^{p-2}\nabla u\cdot\nabla\varphi\,\mathrm{d}x=\int_\Omega\varphi f\,\mathrm{d}x,\quad\forall\,\varphi\in C_c^\infty(\Omega)$$

by Lebesgue's dominated convergence theorem. Consequently,

$$f(x)=0,\quad\text{a.e. }\{\nabla u=0\}.$$

This completes the proof.

Similarly, we can get

Corollary 2.1 *Let $F\in C[0,+\infty)\cap C^1(0,+\infty)$ be such that*

$$F(0)=0,$$

and

$$\lambda r^{p-2}\le F'(r)\le\Lambda r^{p-2},\quad\forall\,r\in(0,+\infty)$$

for some constants $\Lambda>\lambda>0$. Suppose that $u\in W^{1,p}_{\mathrm{loc}}(\Omega)$ is a local solution to the following equation:

$$-\mathrm{div}\Big(\frac{F(|\nabla u|)}{|\nabla u|}\nabla u\Big)=f,\quad\text{in }\Omega,$$

$f\in L^q(\Omega)$, and q satisfies (1.3). *Then* (1.4) *holds.*

3 An Application in Control Theory

Let $1<p<2$, $U=[0,1]$, Ω be a bounded smooth domain in $\mathbb{R}^n$, and

$$\mathcal{U}_{ad}=\{v:\Omega\to U\,|\,v\text{ is measurable}\}.$$

Consider the following optimal control problem:

Problem 3.1 Find a $\overline{u}(\,\cdot\,)\in\mathcal{U}_{ad}$ such that

$$J(\overline{u}(\,\cdot\,))=\inf_{u(\,\cdot\,)\in\mathcal{U}_{ad}}J(u(\,\cdot\,)),\tag{3.1}$$

where

$$J(u(\,\cdot\,))=\int_\Omega[y^2(x)-u^2(x)]\,\mathrm{d}x,\tag{3.2}$$

and $y(\,\cdot\,)=y(\,\cdot\,;u(\,\cdot\,))\in W_0^{1,p}(\Omega)$ is the solution of the following equation:

$$\begin{cases}-\mathrm{div}(|\nabla y|^{p-2}\nabla y)=u(x),\quad\text{in }\Omega,\\ y|_{\partial\Omega}=0.\end{cases}\tag{3.3}$$

For an optimal control problem, to guarantee the existence of a solution we usually need a Cesari type condition, which is a natural generalization of optimal control problems with linear state equations and convex cost functionals. In our problem, such a condition does not hold. It is an important problem in control theory to study the existence of a solution when Cesari condition does not hold. In the following, we will use Theorem 1.1 and the method we used in [5] to prove that for Problem 3.1, there exists at least one solution. For simplicity, the equation (3.3) and cost functional (3.2) are special, but the method we used here contains some basic ideas to study the existence of minimizers for such kind of optimal control problems.

Let $\mathcal{M}_+^1(U)$ be the set of all probability measures in U, $\mathcal{R}(\Omega,U)$ be the set of all weak measurable probability measure-valued functions on Ω, that is, $\sigma(\,\cdot\,)\in\mathcal{R}(\Omega,U)$ if and only if

$$\sigma(x)\in\mathcal{M}_+^1(U),\quad \text{a.e. } x\in\Omega,$$

and

$$x\mapsto\int_U h(v)\sigma(x)(\mathrm{d}\,v)\text{ is measurable},\quad \forall\,h\in C(U),\tag{3.4}$$

where $C(U)$ is the space of all continuous functions on U. Similarly to [5, Theorem 3.2], we have

$$\overline{\sigma}(\,\cdot\,)\in\mathcal{R}(\Omega,U),$$

such that

$$J(\overline{\sigma}(\,\cdot\,))=\inf_{\sigma(\,\cdot\,)\in\mathcal{R}(\Omega,U)}J(\sigma(\,\cdot\,)),$$

where

$$J(\sigma(\,\cdot\,))\stackrel{\triangle}{=}\int_\Omega\mathrm{d}x\int_U[y_\sigma^2(x)-v^2]\,\sigma(x)(\mathrm{d}\,v),\tag{3.5}$$

and $y_\sigma(\,\cdot\,)$ is a solution of the following equation:

$$\begin{cases}-\mathrm{div}(|\nabla y_\sigma|^{p-2}\nabla y_\sigma)=\displaystyle\int_U v\sigma(x)(\mathrm{d}\,v),\quad \text{in }\Omega,\\ y_\sigma|_{\partial\Omega}=0.\end{cases}\tag{3.6}$$

We mention that $\mathcal{U}_{ad}$ can be imbedded into $\mathcal{R}(\Omega,U)$ by identifying each $u(\,\cdot\,)\in\mathcal{U}_{ad}$ with the Dirac measure-Valued function $\delta_{u(\,\cdot\,)}\in\mathcal{R}(\Omega,U)$. Moreover, $J(\delta_{u(\,\cdot\,)})$ defined by (3.5) coincides with $J(u(\,\cdot\,))$ defined by (3.2). Thus notation $J(\sigma(\,\cdot\,))$ would not cause any confusion. On the other hand, as [1, Theorem 5.2] and [5, Theorem 4.1], we can prove that there exists a $\overline{\varphi}(\,\cdot\,)$, such that for any $\mathcal{Q}\subset\subset\{\nabla\overline{y}\neq0\}$, $\overline{\varphi}(\,\cdot\,)\in W^{1,2}(\mathcal{Q})$,

$$-\mathrm{div}(|\nabla\overline{y}|^{p-2}\nabla\overline{\varphi})-\mathrm{div}[|\nabla\overline{y}|^{p-4}(\nabla\overline{y}\cdot\nabla\overline{\varphi})\nabla\overline{y}]=-\overline{y},\quad\text{in }\mathcal{Q},\tag{3.7}$$

and

$$\mathrm{supp}\;\overline{\sigma}(x)\subseteq\{w\in U\mid w\overline{\varphi}(x)+w^2=\max_{v\in U}[v\overline{\varphi}(x)+v^2]\},\quad\text{a.e. }\Omega,\tag{3.8}$$

where $\overline{y}(\,\cdot\,)$ is the solution of equation (3.6) corresponding to $\overline{\sigma}(\,\cdot\,)$. By the result of [2] (see also [9]), $\overline{y}(\,\cdot\,)\in C^{1,\Gamma}(\Omega)$ for some $\Gamma\in(0,1)$. Then, it is easy to get from (3.6) that

$$\overline{y}(\,\cdot\,)\in W^{2,q}(\mathcal{Q}),\quad\forall\,q\in(1,+\infty),\;\mathcal{Q}\subset\subset\{\nabla\overline{y}\neq0\}.\tag{3.9}$$

Consequently, we can get

$$\overline{\varphi}(\,\cdot\,) \in W^{2,q}(\mathcal{Q}), \quad \forall\, q \in (1, +\infty),\ \mathcal{Q} \subset\subset \{\nabla \overline{y} \neq 0\}. \tag{3.10}$$

Now, we will prove that $\operatorname{supp} \overline{\sigma}(x)$ is a singleton of U for almost all $x \in \Omega$. If

$$\overline{\varphi}(x) \neq -1,$$

then

$$\Big\{ w \in U \,\Big|\, w\overline{\varphi}(x) + w^2 = \max_{v \in U} [v\overline{\varphi}(x) + v^2] \Big\}$$

is a singleton. Thus, by (3.8),

$$\operatorname{supp} \overline{\sigma}(x) \text{ is a singleton for almost all } x \in \{\overline{\varphi} \neq -1\}. \tag{3.11}$$

By (3.10) and Lemma 1.1, we have

$$\nabla \overline{\varphi} = 0, \quad \text{a.e. } \{\overline{\varphi} = -1\} \cap \{\nabla \overline{y} \neq 0\}.$$

Furthermore, by (3.9) and (3.10), we have

$$|\nabla \overline{y}|^{p-2} \frac{\partial \overline{\varphi}}{\partial x_i},\ |\nabla \overline{y}|^{p-4} (\nabla \overline{y} \cdot \nabla \overline{\varphi}) \frac{\partial \overline{y}}{\partial x_i} \in W^{1,1}(\mathcal{Q}), \quad \forall\, i = 1, 2, \cdots, n,\ \mathcal{Q} \subset\subset \{\nabla \overline{y} \neq 0\}.$$

Thus, by Lemma 1.1 and (3.7), we have

$$\overline{y}(x) = 0, \quad \text{a.e. } \{\overline{\varphi} = -1\} \cap \{\nabla \overline{y} \neq 0\}.$$

Consequently, it follows from $\overline{y}(\,\cdot\,) \in W_0^{1,p}(\Omega)$ that

$$\nabla \overline{y} = 0, \quad \text{a.e. } \{\overline{\varphi} = -1\} \cap \{\nabla \overline{y} \neq 0\}.$$

That is, $\{\overline{\varphi}(x) = -1\} \cap \{\nabla \overline{y} \neq 0\}$ has zero measure and therefore

$$\nabla \overline{y} = 0, \quad \text{a.e. } \{\overline{\varphi} = -1\}.$$

By Theorem 1.1 and (3.6) (with $(y_\sigma(\,\cdot\,), \sigma(\,\cdot\,))$ being replaced by $(\overline{y}(\,\cdot\,), \overline{\sigma}(\,\cdot\,))$), we get

$$0 = \int_U v\, \overline{\sigma}(\mathrm{d}v), \quad \text{a.e. } \{\overline{\varphi} = -1\}.$$

Consequently,

$$\operatorname{supp}\, \overline{\sigma}(x) = \{0\}, \quad \text{a.e. } \{\overline{\varphi} = -1\}.$$

Combine it with (3.11), we see that for almost all $x \in \Omega$, $\operatorname{supp}\, \overline{\sigma}(x)$ is a singleton. This means that

$$\overline{\sigma}(x) = \delta_{\overline{u}(x)}, \qquad \text{a.e. } \Omega$$

for some $\overline{u}(\,\cdot\,)$. It follows from (3.4) that $\overline{u}(\,\cdot\,) \in \mathcal{U}_{ad}$. On the other hand, it is easy to verify that

$$J(\overline{u}(\,\cdot\,)) = J(\overline{\sigma}(\,\cdot\,)) = \inf_{\sigma(\,\cdot\,) \in \mathcal{R}(\Omega, U)} J(\sigma(\,\cdot\,)) \le \inf_{u(\,\cdot\,) \in \mathcal{U}_{ad}} J(u(\,\cdot\,)).$$

Therefore, $\overline{u}(\,\cdot\,)$ is a solution of Problem 3.1.

Acknowledgement It is a pleasure to express my thanks to Professor Jiayu Li for many helpful discussions.

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