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The Nonlinear Schrödinger Equations with Combined Nonlinearities of Power-Type and Hartree-Type*

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Abstract The primary goal of this paper is to present a comprehensive study of the nonlinear Schrödinger equations with combined nonlinearities of the power-type and Hartree-type. Under certain structural conditions, the authors are able to provide a complete picture of how the nonlinear Schrödinger equations with combined nonlinearities interact in the given energy space. The method used in the paper is based upon the Morawetz estimates and perturbation principles.

Keywords Global well-posedness, Scattering, blowup, Morawetz estimates, Perturbation principles
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1 Introduction

Consider the Cauchy problem for the following Schrödinger equations:

$$\begin{cases} \mathrm{i}u_t + \Delta u = \lambda_1 |u|^p u + \lambda_2 (|x|^{-\gamma} * |u|^2) u, \\ u(0,x) = u_0(x), \end{cases} \tag{1.1}$$

where $u(t,x)$ is a complex-valued function in spacetime $\mathbb{R}\times\mathbb{R}^n$ $(n \geq 3)$, λ_1 and λ_2 are nonzero constants, $0 < p \leq \frac{4}{n-2}$, and $\gamma \in (0,4] \cap (0,n)$. When initial data u_0 take value in $H_x^1(\mathbb{R}^n)$ (or $\Sigma = \{u \in H_x^1(\mathbb{R}^n) : |\cdot|u(\cdot) \in L_x^2(\mathbb{R}^n)\}$), this is a Hamiltonian PDE with energy and mass functions

$$E(u(t)) := \frac{1}{2}\int |\nabla u|^2 \,\mathrm{d}x + \frac{\lambda_1}{p+2}\int |u|^{p+2}\,\mathrm{d}x + \frac{\lambda_2}{4}\int (|x|^{-\gamma} * |u|^2)|u|^2\,\mathrm{d}x,$$
$$M(u(t)) := \int |u|^2\,\mathrm{d}x.$$

Since the mass and energy are conservative, we drop the variable t in energy and mass functions, and denote them by $E(u)$ and $M(u)$, respectively.

When one of λ_1 or λ_2 is zero, the problem is well understood. When the initial data taking value in H^1 space, T. Cazenave [2] had a quite thorough study of this case that $\lambda_1 \cdot \lambda_2 = 0$, $0 < p < \frac{4}{n-2}$, $0 < \gamma < 4$: (1) for the defocusing case (i.e. λ_1 or λ_2 is positive), the problem is globally well-posed when $0 < p < \frac{4}{n-2}$ and $0 < \gamma < 4$; (2) for the focusing case (i.e., λ_1 or λ_2

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is negative), it is globally well-posed when $0 < p < \frac{4}{n}$ and $0 < \gamma < 2$. It is well-known that $p = \frac{4}{n-2}$, $\gamma = 4$ and $p = \frac{4}{n}$, $\gamma = 2$ are the critical values for the defocusing case and focusing case, respectively. In fact, at these critical values of λ and p, T. Cazenave obtained the same results under additional assumption of initial data being small. Fortunately, in the past few years, there was some great breakthrough into large initial data at these critical values.

For the energy-critical NLS

$$\begin{cases} \mathrm{i}u_t + \Delta u = \lambda_1 |u|^{\frac{4}{n-2}} u, \\ u(0,x) = u_0(x). \end{cases} \tag{1.2}$$

Firstly, J. Bourgain studied the global existence of the defocusing case ($\lambda_1 > 0$) in $\mathbb{R}^3$ with the radial initial data (see [1]), then J. Colliander, M. Keel, G. Staffilani, H. Takaoka and T. Tao solved the problem with the general initial data in $\mathbb{R}^3$ (see [5]). Thereafter, E. Ryckman and M. Visan extended the results to the higher dimension system ([22, 27]). The focusing case ($\lambda_1 < 0$) was considered by Carlos E. Kenig and F. Merle (see [14]) and then their obtained results were extended to the higher dimension system by R. Killip and M. Visan (see [15]). The name "energy-critical" refers to the fact that the scaling symmetry leaves the equation (1.2) and the energy invariant, where the energy is defined as

$$E(u) = \frac{1}{2}\int |\nabla u|^2 \,\mathrm{d}x + \lambda_1 \frac{2n}{n-2}\int |u|^{\frac{2n}{n-2}} \,\mathrm{d}x.$$

For the focusing mass-critical NLS

$$\begin{cases} \mathrm{i}u_t + \Delta u = \lambda_1 |u|^{\frac{4}{n}} u, \\ u(0,x) = u_0(x), \end{cases} \tag{1.3}$$

T. Tao, M. Visan and X. Y. Zhang [25], R. Killip, M. Visan and X. Y. Zhang [16] considered the problem. Here by "mass-critical", we mean that the scaling symmetry leaves the equation (1.2) and the mass invariant, where the mass is defined as

$$M(u(t)) = \|u(t)\|_{L^2_x}.$$

The relevant results of the system with the Hartree nonlinearity

$$\begin{cases} \mathrm{i}u_t + \Delta u = \lambda_2 (|x|^{-4} * |u|^2) u, \\ u(0,x) = u_0(x), \end{cases} \tag{1.4}$$

$$\begin{cases} \mathrm{i}u_t + \Delta u = \lambda_2 (|x|^{-2} * |u|^2) u, \\ u(0,x) = u_0(x) \end{cases} \tag{1.5}$$

were obtained by C. X. Miao, G. X. Xu and L. F. Zhao [18–21].

From the above cited work, it appears that most recent researches are primarily concentrated on the one nonlinearity. It would be interesting to investigate what would happen if both nonlinearities are combined. When both power-type and Hartree-type nonlinearities are presented in the Cauchy problem, there are three possible cases: (a) none of the nonlinearities is critical, (b) one of them is critical, or (c) both are critical. Cazenave treated case (a) in [2], but he did not consider other cases. In this paper, we will study cases (b) and (c), mainly discussing that one nonlinearity is defocusing and the other is focusing, and we want to give a complete view of how these both nonlinearities interact and impact on the well-posedness. For the case that both nonlinearities are defocusing, we will discuss the Cauchy problem, especially with one nonlinearity being energy-critical. As for other cases, we hope that under certain assumptions on λ and p, the defocusing term may dominate the focusing term so that the whole nonlinearity behavior exhibits the defocusing property. Then one can obtain the global well-posedness result since the defocusing amplifies the dispersive effect of the linear equation, but the focusing usually cancels this effect.

Before we state our first theorem, we introduce the solution of ground state whose properties can be found in Appendix. Let W be the solution of ground state: $\triangle W + (|x|^{-\gamma} * |W|^2)W = \frac{4-\gamma}{\gamma}W$, and define the energy: $\widetilde{E}(W) := \frac{1}{2}\int |\nabla W|^2 \,\mathrm{d}x - \frac{1}{4}\int (|x|^{-\gamma} * |W|^2)|W|^2 \,\mathrm{d}x$. Similarly, let R be the solution of ground state: $\Delta R + |R|^p R = \frac{4-(n-2)p}{np}R$, and define the energy: $\widetilde{E}(R) := \frac{1}{2}\int |\nabla R|^2 \,\mathrm{d}x - \frac{1}{p+2}\int |R|^{p+2} \,\mathrm{d}x$.

Then the main theorem of this paper is stated as follows.

Theorem 1.1 (Global Well-Posedness) *Let $u_0 \in H_x^1$. Then there exists a unique global solution u to* (1.1) *in each of the following cases:*

(1) $\lambda_1, \lambda_2 > 0,\ 0 < p \le \frac{4}{n-2},\ \gamma \in (0,4] \cap (0,n)$ *except* $(p,\gamma) = (\frac{4}{n-2}, 4)$.

(2) $\lambda_1 > 0,\ \lambda_2 < 0,$

(2.1) $0 < p \le \frac{4}{n-2}$, *and* $0 < \gamma < \min\{n, \frac{np}{2}\}$.

(2.2) $\frac{np}{2} \le \gamma < 2$.

(2.3) $\frac{np}{2} \le \gamma = 2$, *and* $\|u_0\|_{L^2}^2 < \frac{1}{|\lambda_2|}\|W\|_{L^2}^2$.

(2.4) $\frac{np}{2} \le \gamma = 4\ (n > 4),\ E < \frac{\widetilde{E}(W)}{|\lambda_2|},\ \|\nabla u_0\|_{L^2}^2 < \frac{1}{|\lambda_2|}\|\nabla W\|_{L^2}^2$ *and u_0 is radial except* $(p,\gamma) = (\frac{4}{n-2}, 4)$.

(2.5) $\frac{np}{2} \le \gamma,\ 2 < \gamma < \min\{4, n\},\ EM^{\frac{4-\gamma}{\gamma-2}} < (\frac{1}{2} - \frac{1}{\gamma})\Big[\frac{2\gamma\widetilde{E}(W)}{|\lambda_2|(\gamma-2)}\Big]^{\frac{2}{\gamma-2}}$ *and* $\|\nabla u_0\|_{L^2}^2 M^{\frac{4-\gamma}{\gamma-2}} < \Big(\frac{\|\nabla W\|_{L^2}^2}{|\lambda_2|}\Big)^{\frac{2}{\gamma-2}}$.

(3) $\lambda_1 < 0,\ \lambda_2 > 0,$

(3.1) $0 < p < \max\{\frac{4}{n}, \frac{4}{2+n-\gamma}\}$, *and* $\gamma \in (0,4] \cap (0,n)$.

(3.2) $p = \frac{4}{n},\ p \ge \frac{4}{2+n-\gamma}$, *and* $\|u_0\|_{L^2} < |\lambda_1|^{-\frac{n}{4}}\|R\|_{L^2}$.

(3.3) $\frac{4}{2+n-\gamma} \le p = \frac{4}{n-2}$ *except for* $(p,\gamma) = (\frac{4}{n-2}, 4)$, *in addition, if* $n \ge 5$, $E < |\lambda_1|^{\frac{2-n}{2}}\widetilde{E}(R)$, $\|\nabla u_0\|_{L^2}^2 < |\lambda_1|^{\frac{2-n}{2}}\|\nabla R\|_{L^2}^2$; *if* $n = 3, 4$, u_0 *is radial.*

(3.4) $\frac{4}{n} < p < \frac{4}{n-2}$, *and* $\frac{4}{2+n-\gamma} \le p$ *with*

$$EM^{\frac{4-(n-2)p}{np-4}} < |\lambda_1|^{\frac{4}{4-np}}\Big(\frac{2np}{np-4}\Big)^{\frac{4-(n-2)p}{np-4}}(\widetilde{E}(R))^{\frac{2p}{np-4}},$$
$$\|\nabla u_0\|_{L^2}^2 M^{\frac{4-(n-2)p}{np-4}} < |\lambda_1|^{\frac{4}{4-np}}\|\nabla R\|_{L^2}^{\frac{4p}{np-4}},$$

(4) $\lambda_1 < 0,\ \lambda_2 < 0,\ 0 < p < \frac{4}{n}$, *and* $0 < \gamma < 2$.

Moreover, for any compact interval I, the global solution is bounded:

$$\|u\|_{S^1(I\times\mathbb{R}^n)} \le C(|I|, E, M). \tag{1.6}$$

Remark 1.1 For case (3.3), R. Killip and M. Visan [15] proved the global well-posedness for (1.2) if the initial data are not radial. However, their approach is not suitable for the low dimension case, and that is why we assume that the initial data are radial when $n = 3, 4$.

The key ingredient in the proof of this theorem is to obtain a bound for $\|u\|_{H_x^1}$ which only depends on the energy and mass. Then use perturbation principles to derive the desired results. As mentioned above, we hope that the defocusing term can control the focusing term, which is not true in general, but we can show that under the assumptions in Cases 2.1 and 3.1 in Theorem 1.1, this is true. For other cases, as shown in the work of T. Cazenave, some assumptions of the smallness about the energy and mass are required. The point is that the smallness is characterized by the ground state. Unfortunately, our method is not applicable to the case that both the power and Hartree nonlinearities are energy-critical.

Next, we consider the asymptotic behavior of global solutions. It is natural to use the unconditional scattering theory for (1.3) and (1.5). However, at least at this moment, we have to assume that the initial data are radial and the size of the mass is smaller than that of the ground state (see [27, 16]). Therefore, we need the following assumptions.

Assumption 1.1 *Let $v_0 \in H_x^1,\ \lambda_1 > 0$. Then there exists a unique global solution v to* (1.3)*, satisfying*

$$\|v\|_{L_{t,x}^{\frac{2(n+2)}{n}}(\mathbb{R}\times\mathbb{R}^n)} \le C(\|v_0\|_{L_x^2}). \tag{1.7}$$

Assumption 1.2 *Let $w_0 \in H_x^1,\ \lambda_2 > 0$. Then there exists a unique global solution w to* (1.5)*, satisfying*

$$\|w\|_{L_t^6 L_x^{\frac{6n}{3n-2}}(\mathbb{R}\times\mathbb{R}^n)} \le C(\|w_0\|_{L_x^2}). \tag{1.8}$$

Now we state the second main theorem of this paper.

Theorem 1.2 (Energy Space Scattering) *Suppose that $u_0 \in H_x^1$, the conditions in Theorem* 1.1 *hold, and u is the unique solution to* (1.1)*. Moreover, if $p = \frac{4}{n}$, Assumption* 1.1 *holds true; if $\gamma = 2$, Assumption* 1.2 *holds true. Then for both of the following cases:*

Case 1 *$\lambda_1,\ \lambda_2 > 0,\ \frac{4}{n} \le p \le \frac{4}{n-2},\ 2 \le \gamma \le 4$ with $\gamma < n$ except $(p,\gamma) = (\frac{4}{n-2}, 4)$, especially, when $(p,\gamma) = (\frac{4}{n}, 2)$, mass is small;*

Case 2 *$\lambda_1 \cdot \lambda_2 < 0,\ \frac{4}{n} \le p \le \frac{4}{n-2},\ 2 \le \gamma \le 4$ with $\gamma < n$, and mass is small except when $(p,\gamma) = (\frac{4}{n-2}, 4)$, there exist $u_+, u_- \in H_x^1$ such that*

$$\|u - \mathrm{e}^{\mathrm{i}t\Delta} u_\pm\|_{H_x^1} \to 0, \quad \textit{as } t \to \pm\infty \tag{1.9}$$

and

$$\|u_+\|_{L^2} = \|u_-\|_{L^2} = \|u_0\|_{L^2} \quad \textit{and} \quad \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u_+|^2 = \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u_-|^2 = E(u_0).$$

We will prove this theorem in Section 5. The primary tools used in the proof are the refined Morawetz estimate and the perturbation principles. To apply the refined Morawetz estimate, we need to assume that $\lambda_1 > 0,\ \lambda_2 > 0,\ p > \frac{4}{n}$ and $\gamma > 2$. When $\lambda_1 \cdot \lambda_2 < 0$, we need to assume that the mass is sufficiently small. The refined Morawetz estimate was first used by T. Tao to prove the dispersive property of the cubic Schrödinger equation in [6] when the space dimension is at least 3. Then, J. Colliander, M. Grillakis and N. Tzirakis obtained refined Morawetz estimates for 1-D and 2-D and the scattering of 2-D power type Schrödinger equation. However, when $\gamma < 2$, the Morawetz estimate is not applicable. Thus, we cannot have scattering for Hartree type or for (1.1). When $p = \frac{4}{n}$ and $\gamma = 2$, i.e., both nonlinearities are mass-critical, the low frequency of the solution may possess an effective control, but not for the higher frequencies. Thus here we view (1.1) as the perturbation of the free Schrödinger equation.

In Section 6, we describe the blowup phenomena with the initial data in Σ space. We believe that the machinery we used there is also suitable for the case that the initial data are radial and in the energy space. We refer readers to [2, Chapter 6] for details.

The major results regarding blowup phenomena are stated as follows.

Theorem 1.3 (Blowup) *Let $u_0 \in \Sigma$. Then blowup occurs in each of the following cases:*
(1) *for $\lambda_1 > 0,\ \lambda_2 < 0$: when $2 \le \gamma \le 4,\ 0 < p \le \frac{4}{n-2}$, $\gamma \ge \frac{np}{2}$, and $E < 0$;*
(2) *for $\lambda_1 < 0,\ \lambda_2 > 0$: when $\frac{4}{n} \le p \le \frac{4}{n-2},\ 0 < \gamma \le \frac{np}{2}$, and $E < 0$;*
(3) *for $\lambda_1 < 0,\ \lambda_2 < 0$:*
(i) *when $\frac{4}{n} < p \le \frac{4}{n-2},\ 0 < \gamma < 2$, and $4npE + C(M) < 0$;*
(ii) *when $0 < p < \frac{4}{n},\ 2 < \gamma \le 4$, and $8\gamma E + C(M) < 0$;*
(iii) *when $\frac{4}{n} \le p \le \frac{4}{n-2},\ 2 \le \gamma \le 4$, and $E < 0$.*

Remark 1.2 The results in Theorems 1.1 and 1.3 are consistent. The energy in Theorem 1.1 is nonnegative. Also notice that we do not study the condition $\frac{np}{2} < \gamma \le 2+n-\frac{4}{p}$ for the case $\lambda_1 < 0,\ \lambda_2 > 0$, because we are not clear about the relationship between $\int (|x|^{-\gamma} * |u|^2)|u|^2\,\mathrm{d}x$ and $\|u\|_{L_x^{p+2}}^{p+2}$. Note that the inequalities

$$\|u\|_{L_x^q}^q \lesssim \int (|x|^{-\gamma} * |u|^2)|u|^2\,\mathrm{d}x, \quad \|u\|_{L_x^{p+2}}^{p+2} \lesssim \|u\|_{L_x^r}^r$$

hold, where $q = \frac{2(4+n-\gamma)}{2+n-\gamma},\ r = \frac{2n+2\gamma}{n}$. If one could prove $\int (|x|^{-\gamma} * |u|^2)|u|^2\,\mathrm{d}x \sim \|u\|_{L_x^s}^s$, for $s > p+2$, one may use the method in Subsection 4.2 for Case 2 to obtain the global well-posedness and scattering. For $s \le p+2$, using the method in Section 6, one can obtain the blowup result in finite time under certain conditions.

2 Notations

In this section, we introduce the notations and several fundamental inequalities needed in this paper.

Definition 2.1 *A pair (q, r) is called Schrödinger-admissible if $\frac{2}{q} + \frac{n}{r} = \frac{n}{2}$ and $2 \le q, r \le \infty$ (if $n = 1$, then $2 \le r \le \infty$, if $n = 2$, then $2 \le r < \infty$).*

Let $I \times \mathbb{R}^n$ be a spacetime slab. We define

$$\|u\|_{\dot S^0(I\times\mathbb{R}^n)} := \sup \|u\|_{L_t^q L_x^r(I\times\mathbb{R}^n)},$$

where the sup *is taken over all admissible pairs (q, r), and*

$$\|u\|_{\dot S^1(I\times\mathbb{R}^n)} := \|\nabla u\|_{\dot S^0(I\times\mathbb{R}^n)}.$$

Denote $\dot N^0(I\times\mathbb{R}^n)$ the dual space of $\dot S^0(I\times\mathbb{R}^n)$, and

$$\dot N^1(I\times\mathbb{R}^n) := \{u : \nabla u \in \dot N^0(I\times\mathbb{R}^n)\}.$$

Furthermore, define the following norms:

$$\|u\|_{U(I)} := \|u\|_{L_t^6 L_x^{\frac{6n}{3n-2}}(I\times\mathbb{R}^n)},$$

$$\|u\|_{V(I)} := \|u\|_{L_{t,x}^{\frac{2(n+2)}{n}}(I\times\mathbb{R}^n)},$$

$$\|u\|_{W(I)} := \|u\|_{L_{t,x}^{\frac{2(n+2)}{n-2}}(I\times\mathbb{R}^n)},$$

$$\|u\|_{Z(I)} := \|u\|_{L_t^{n+1} L_x^{\frac{2(n+1)}{n-1}}(I\times\mathbb{R}^n)},$$

and denote

$$\dot{X}^0(I) = \begin{cases} L_t^q L_x^r(I\times\mathbb{R}^n), & 0<p<\frac{4}{n-2},\\ L_t^{\frac{2(n+2)}{n-2}} L_x^{\frac{2n(n+2)}{n^2+4}}(I\times\mathbb{R}^n)\cap V(I), & p=\frac{4}{n-2},\end{cases}$$

where $q=\frac{4(p+2)}{p(n-2)}$, $r=\frac{n(p+2)}{n+p}$,

$$\dot{Y}^0(I) := \begin{cases} L_t^\infty L_x^2(I\times\mathbb{R}^n), & 0<\gamma\le 2,\\ L_t^\infty L_x^2(I\times\mathbb{R}^n)\cap L_t^\mu L_x^\sigma(I\times\mathbb{R}^n), & 2<\gamma\le 4 \text{ and } \gamma<n,\end{cases}$$

where $\mu=\frac{6}{\gamma-2}$, $\sigma=\frac{6n}{3n+4-2\gamma}$, *and*

$$\dot{B}^0(I) := \dot{X}^0(I)\cap\dot{Y}^0(I),\quad \dot{X}^1(I) := \{u:\nabla u\in\dot{X}^0(I)\},\quad \dot{Y}^1(I) := \{u:\nabla u\in\dot{Y}^0(I)\}.$$

By the Sobolev's embedding theorem, we get the following results.

Lemma 2.1 *For any* $\dot{S}^1$ *function* u *in* $I\times\mathbb{R}^n$, *we have*

$$\begin{aligned}&\|\nabla u\|_{L_t^\infty L_x^2} + \|\nabla u\|_{L_t^{\frac{2(n+2)}{n-2}} L_x^{\frac{2n(n+2)}{n^2+4}}} + \|\nabla u\|_V + \|\nabla u\|_{L_t^2 L_x^{\frac{2n}{n-2}}} + \|\nabla u\|_U\\ &+ \|u\|_{L_t^\infty L_x^{\frac{2n}{n-2}}} + \|u\|_W + \|u\|_{L_t^{\frac{2(n+2)}{n}} L_x^{\frac{2n(n+2)}{n^2-2n-4}}} \lesssim \|u\|_{\dot{S}^1},\end{aligned} \tag{2.1}$$

where all spacetime norms are taken in $I\times\mathbb{R}^n$.

Lemma 2.2 (Strichartz Estimates) *Let* I *be a compact time interval,* $k=0,1$, *and* $u: I\times\mathbb{R}^n\to\mathbb{C}$ *be an* $\dot{S}^k$ *solution to the forced Schrödinger equation*

$$\mathrm{i}u_t+\Delta u = F$$

for a given function F. *Then we have*

$$\|u\|_{\dot{S}^k(I\times\mathbb{R}^n)} \lesssim \|u(t_0)\|_{\dot{H}^k(\mathbb{R}^n)} + \|F\|_{\dot{N}^k(I\times\mathbb{R}^n)} \tag{2.2}$$

for any time $t_0\in I$.

Detailed proof of this lemma can be found in [2, 13].

In addition, we need Littlewood-Paley Theory. Let $\varphi(\xi)$ be a smooth bump function with support $|\xi|\le 2$ and equal to 1 in $|\xi|\le 1$. For each dyadic number $N\in 2^{\mathbb{Z}}$, we can define the Littlewood-Paley operators:

$$\begin{aligned}\widehat{P_{\le N}f}(\xi) &:= \varphi\Big(\frac{\xi}{N}\Big)\widehat{f}(\xi),\\ \widehat{P_{>N}f}(\xi) &:= \Big[1-\varphi\Big(\frac{\xi}{N}\Big)\Big]\widehat{f}(\xi),\\ \widehat{P_N f}(\xi) &:= \Big[\varphi\Big(\frac{\xi}{N}\Big)-\varphi\Big(\frac{2\xi}{N}\Big)\Big]\widehat{f}(\xi).\end{aligned}$$

With these notations in mind, we recall several standard Bernstein type inequalities.

Lemma 2.3 *For any* $1\le p\le q\le\infty$, $s>0$, *we have*

$$\begin{aligned}\|P_{\ge N}f\|_{L_x^p} &\lesssim N^{-s}\||\nabla|^s P_{\ge N}f\|_{L_x^p},\\ \||\nabla|^s P_{\le N}f\|_{L_x^p} &\lesssim N^s\|P_{\le N}f\|_{L_x^p},\\ \||\nabla|^{\pm s}P_N f\|_{L_x^p} &\sim N^{\pm s}\|P_N f\|_{L_x^p},\\ \|P_{\le N}f\|_{L_x^q} &\lesssim N^{\frac{n}{p}-\frac{n}{q}}\|P_{\le N}f\|_{L_x^p},\\ \|P_N f\|_{L_x^q} &\lesssim N^{\frac{n}{p}-\frac{n}{q}}\|P_N f\|_{L_x^p}.\end{aligned}$$

Furthermore, we also need the following maximal estimate, which follows immediately from the sharp Hardy inequality (see [10]).

Lemma 2.4 *Let $0 < \gamma < n$. We have*

$$\| |x|^{-\gamma} * |u|^2 \|_{L_x^\infty} \le C(n,\gamma) \|u\|^2_{\dot H^{\frac{\gamma}{2}}}. \tag{2.3}$$

In this paper, the major task is to control the nonlinearity. Here we use the Morawetz inequality to accomplish this mission, which further means finding the connection between nonlinearity and Morawetz inequality. It turns out that the norm $Z(I)$ is the linkage we are looking for.

Lemma 2.5 *Let $k = 0, 1$, $\frac{4}{n} < p < \frac{4}{n-2}$ and $2 < \gamma < \min\{4, n\}$. Then there exists a large enough $\theta > 0$ such that in each slab $I \times \mathbb{R}^n$, we have*

$$\| |u|^p u \|_{\dot N^k(I\times\mathbb{R}^n)} \lesssim \|u\|_{\dot S^k(I\times\mathbb{R}^n)} \|u\|_{Z(I)}^{\frac{n+1}{2(2\theta+1)}} \|u\|^{\alpha_1(\theta)}_{L_t^\infty L_x^2} \|u\|^{\alpha_2(\theta)}_{L_t^\infty L_x^{\frac{2n}{n-2}}}, \tag{2.4}$$

$$\| (|x|^{-\gamma} * |u|^2) u \|_{\dot N^k(I\times\mathbb{R}^n)} \lesssim \|u\|_{\dot S^k(I\times\mathbb{R}^n)} \|u\|_{Z(I)}^{\frac{n+1}{2(2\theta+1)}} \|u\|^{\beta_1(\theta)}_{L_t^\infty L_x^2} \|u\|^{\beta_2(\theta)}_{L_t^\infty L_x^{\frac{2n}{n-2}}}, \tag{2.5}$$

where

$$\alpha_1(\theta) = p\Big(1 - \frac{n}{2}\Big) + \frac{8\theta+1}{2(2\theta+1)}, \quad \alpha_2(\theta) = \frac{n}{2}\Big(p - \frac{n+8\theta+2}{n(2\theta+1)}\Big),$$

$$\beta_1(\theta) = (3-\gamma) + \frac{4\theta-1}{2(2\theta+1)}, \qquad \beta_2(\theta) = (\gamma-1) - \frac{4\theta+n}{2(2\theta+1)}.$$

Proof The proof of the first inequality is given in [24]. The same method can be used to prove the second. When $\beta_1(\theta)$ and $\beta_2(\theta)$ are positive, from Hölder and Hardy-Littlewood-Sobolev inequality, we have

$$\begin{aligned} \| (|x|^{-\gamma} * |u|^2) u \|_{\dot N^k(I\times\mathbb{R}^n)} &\lesssim \| |\nabla|^k [(|x|^{-\gamma} * |u|^2) u] \|_{L_t^2 L_x^{\frac{2n}{n+2}}(I\times\mathbb{R}^n)} \\ &\lesssim \| |\nabla|^k u \|_{L_t^{2+\frac{1}{\theta}} L_x^{\frac{2n(2\theta+1)}{n(2\theta+1)-4\theta}}} \|u\|_{Z(I)}^{\frac{n+1}{2(2\theta+1)}} \|u\|^{\beta_1(\theta)}_{L_t^\infty L_x^2} \|u\|^{\beta_2(\theta)}_{L_t^\infty L_x^{\frac{2n}{n-2}}}. \end{aligned} \tag{2.6}$$

Notice that $\big(2+\frac{1}{\theta}, \frac{2n(2\theta+1)}{n(2\theta+1)-4\theta}\big)$ is Schrödinger-admissible. When $2 < \gamma < 4$, $\beta_1(\theta)$ and $\beta_2(\theta)$ are positive as long as θ is large enough, because both the functions are increasing in θ and

$$\beta_1(\theta) \to (4-\gamma) > 0, \quad \beta_2(\theta) \to (\gamma-2) > 0,$$

as $\theta \to \infty$.

Lemma 2.6 *Let $I \times \mathbb{R}^n$ be a spacetime slab. Then there exists a small constant $0 < \rho < 1$, such that*

$$\| |u|^{\frac{4}{n-2}} u \|_{\dot N^0(I\times *\mathbb{R}^n)} \lesssim \|u\|^{\rho}_{Z(I)} \|u\|^{\frac{n+2}{n-2}-\rho}_{S^1(I\times *\mathbb{R}^n)}, \tag{2.7}$$

$$\begin{aligned} \| (|x|^{-4} * |u|^2) u \|_{\dot N^0(I\times\mathbb{R}^n)} &\lesssim \|u\|_{L_t^{2+\varepsilon} L_x^{\frac{2n}{n-2-\varepsilon}}} \|u\|^{\rho}_{Z(I)} \|u\|^{\frac{\varepsilon(1+\varepsilon)}{2(2+\varepsilon)}}_{L_t^\infty L_x^2} \|u\|^{2-\frac{\varepsilon(2+\varepsilon+n)}{2(2+\varepsilon)}}_{L_t^\infty L_x^{\frac{2n}{n-2}}} \\ &\lesssim \|u\|^{\rho}_{Z(I)} \|u\|^{3-\rho}_{S^1(I\times\mathbb{R}^n)}, \end{aligned} \tag{2.8}$$

where $\rho = \frac{\varepsilon(n+1)}{2(2+\varepsilon)}$ and ε is a small constant.

Proof The first inequality is proved in [24]. For the other, note that for sufficiently small ε $L_t^{2+\varepsilon}L_x^{\frac{2n}{n-2-\varepsilon}}$ interpolates between the $\dot{S}^0$-norm $L_t^{2+\varepsilon}L_x^{\frac{2n(2+\varepsilon)}{n(2+\varepsilon)-4}}$ and the $\dot{S}^1$-norm $L_t^{2+\varepsilon}L_x^{\frac{2n(2+\varepsilon)}{n(2+\varepsilon)-2(4+\varepsilon)}}$. Then we have

$$\|u\|_{L_t^{2+\varepsilon}L_x^{\frac{2n}{n-2-\varepsilon}}} \lesssim \|u\|_{S^1(I\times\mathbb{R}^n)}.$$

Let $a(\varepsilon)=\frac{\varepsilon(1+\varepsilon)}{2(2+\varepsilon)},\ b(\varepsilon)=2-\frac{\varepsilon(n+2+\varepsilon)}{2(2+\varepsilon)}$. Since the estimate is a simple consequence of Hölder inequality and Hardy-Littlewood-Sobolev inequality, we only need to check that $a(\varepsilon)$ and $b(\varepsilon)$ are positive.

As functions of ε, a is increasing and $a(0)=0$, while b is decreasing and $b(0)=2$. Thus letting $\varepsilon>0$ be sufficiently small, we have $a(\varepsilon)>0,\ b(\varepsilon)>0$. Taking $\rho=\frac{\varepsilon(n+1)}{2(2+\varepsilon)}$, we complete the proof.

Remark 2.1 A byproduct of the proof of Lemma 2.6 is that one can get the estimates for nonlinearities of the form $|u|^{\frac{4}{n-2}}v$ and $(|x|^{-\gamma}*|u|^2)v$. More precisely,

$$\||u|^{\frac{4}{n-2}}v\|_{\dot{N}^0(I\times\mathbb{R}^n)} \lesssim \|u\|_{Z(I)}^{\rho}\|u\|_{S^1(I\times\mathbb{R}^n)}^{\frac{4}{n-2}-\rho}\|v\|_{S^1(I\times\mathbb{R}^n)}, \tag{2.9}$$

$$\|(|x|^{-\gamma}*|u|^2)v\|_{\dot{N}^0(I\times\mathbb{R}^n)} \lesssim \|u\|_{Z(I)}^{\rho}\|u\|_{S^1(I\times\mathbb{R}^n)}^{2-\rho}\|v\|_{S^1(I\times\mathbb{R}^n)}, \tag{2.10}$$

$$\|(|x|^{-\gamma}*(wv))v\|_{\dot{N}^0(I\times\mathbb{R}^n)} \lesssim \|u\|_{S^1(I\times\mathbb{R}^n)}\|w\|_{L_t^\infty L_x^2}^{a(\varepsilon)}\|v\|_{Z(I)}^{\rho}\|v\|_{L_t^\infty L_x^{\frac{2n}{n-2}}}^{b(\varepsilon)}. \tag{2.11}$$

3 Local Theory

In this section we will state the local theory for the initial value problem (1.1). As the results are well-known, we omit the proofs and refer readers to [2–3, 11–13].

Proposition 3.1 (Local Well-Posedness for (1.1) with Energy-Subcritical Nonlinearities) *Let $u_0\in H_x^1$, λ_1 and λ_2 be nonzero real constants and $0<p<\frac{4}{n-2},\ 0<\gamma<\min\{n,4\}$. Then, there exists a $T=T(\|u\|_{H_x^1})$ such that (1.1) admits a unique strong H_x^1-solution u in $[-T,T]$. Let $(-T_{\min},T_{\max})$ be the maximal time interval in which the solution u is well-defined. For every compact time interval $I\subset(-T_{\min},T_{\max})$, we have $u\in S^1(I\times\mathbb{R}^n)$ satisfying the following properties:*

(1) *If $T_{\max}<\infty$ $(T_{\min}<\infty)$, then*

$$\|u(t)\|_{H_x^1}\to\infty,\quad \text{as } t\uparrow T_{\max}\ (\text{as } t\downarrow -T_{\min}).$$

(2) *The solution depends continuously on the initial value. That is, there exists a $T=T(\|u\|_{H_x^1})$ such that if $u_0^{(m)}\to u_0$ in H_x^1 and if $u^{(m)}$ is the solution to (1.1) with initial condition $u_0^{(m)}$, then $u^{(m)}$ is defined in $[-T,T]$ for sufficiently large m and $u^{(m)}\to u$ in $S^1([-T,T]\times\mathbb{R}^n)$.*

Proposition 3.2 (Local Well-Posedness for (1.1) with an Energy-Critical Nonlinearity) *Let $u_0\in H_x^1,\lambda_1$ and λ_2 be nonzero real constants.*

(1) *When $p=\frac{4}{n-2}$ and $0<\gamma<\min\{n,4\}$, for every $T>0$, there exists an $\eta=\eta(T)$ such that, if*

$$\|e^{it\Delta}u_0\|_{\dot{X}^1([-T,T])}\le\eta,$$

then (1.1) admits a unique strong H_x^1-solution u defined in $[-T,T]$.

(2) *When $0<p<\frac{4}{n-2},\ \gamma=4$ and $n\ge5$, for every $T>0$, there exists an $\eta=\eta(T)$ such that, if*

$$\|e^{it\Delta}u_0\|_{\dot{Y}^1([-T,T])}\le\eta,$$

then (1.1) admits a unique strong H_x^1-solution u defined in $[-T,T]$.

(3) *Let $(-T_{\min}, T_{\max})$ be the maximal time interval on which the solution u is well-defined. Then for each compact time interval $I \subset (-T_{\min}, T_{\max})$, $u \in S^1(I \times \mathbb{R}^n)$ and the following blowup alternative holds:*

If $T_{\max} < \infty$ (respectively, if $T_{\min} < \infty$), then either $\|u(t)\|_{H_x^1} \to \infty$ or $\|u(t)\|_{S^1((0,t)\times\mathbb{R}^n)} \to \infty$ as $t \uparrow T_{\max}$ (respectively, as $t \downarrow -T_{\min}$).

Next, we state the stability results for the H_x^1-critical and the L_x^2-critical NLS with Hartree type.

Lemma 3.1 (Short-Time Perturbation) *Let I be a compact interval and $\widetilde{u}$ a function in $I \times \mathbb{R}^n$ which is a near-solution to (1.4) in the sense of that*

$$(\mathrm{i}\partial_t + \Delta)\widetilde{u} = \lambda(|x|^{-4} * |\widetilde{u}|^2)\widetilde{u} + e$$

for some function e, and

$$\|\widetilde{u}\|_{L_t^\infty \dot{H}^1(I\times\mathbb{R}^n)} \le E \tag{3.1}$$

for some $E > 0$.

Furthermore, let $t_0 \in I$ and $u(t_0)$ be close to $\widetilde{u}(t_0)$ in the sense of

$$\|u(t_0) - \widetilde{u}(t_0)\|_{\dot{H}_x^1} \le E' \tag{3.2}$$

for some $E' > 0$. Assume also that

$$\|\nabla\widetilde{u}\|_{U(I)} \le \epsilon_0, \tag{3.3}$$

$$\|\mathrm{e}^{\mathrm{i}(t-t_0)\Delta}\nabla(u(t_0) - \widetilde{u}(t_0))\|_{U(I)} \le \epsilon, \tag{3.4}$$

$$\|e\|_{\dot{N}^1(I\times\mathbb{R}^n)} \le \epsilon, \tag{3.5}$$

for some $0 < \epsilon < \epsilon_0$, where ϵ_0 is a small constant $\epsilon_0 = \epsilon_0(E, E') > 0$.

Then there exists a solution u to (1.4) in $I \times \mathbb{R}^n$ with the special initial data $u(t_0)$ at t_0, and

$$\|u - \widetilde{u}\|_{\dot{S}^1(I\times\mathbb{R}^n)} \lesssim E' + \epsilon, \tag{3.6}$$

$$\|u\|_{\dot{S}^1(I\times\mathbb{R}^n)} \lesssim E' + E, \tag{3.7}$$

$$\|u - \widetilde{u}\|_{L_t^6 L_x^{\frac{6n}{3n-8}}(I\times\mathbb{R}^n)} \lesssim \epsilon, \tag{3.8}$$

$$\|(\mathrm{i}\partial_t + \Delta)(u - \widetilde{u})\|_{\dot{N}^1(I\times\mathbb{R}^n)} \lesssim \epsilon. \tag{3.9}$$

Proof Without loss of generality, we assume $t_0 = \inf I$. Define $z = u - \widetilde{u}$. Then $u = z + \widetilde{u}$,

$$S(t) := \|(\mathrm{i}\partial_t + \Delta)z\|_{\dot{N}^1([t_0,t]\times\mathbb{R}^n)}.$$

On one hand, by Hölder and Hardy-Littlewood-Sobelov inequality, we have

$$\|(|x|^{-4} * (ab))c\|_{\dot{N}^1} \lesssim \|\nabla a\|_{U(I)}\|\nabla b\|_{U(I)}\|\nabla c\|_{U(I)}. \tag{3.10}$$

From (3.3), (3.5) and (3.10), we get

$$\begin{aligned} S(t) &\le \|[|x|^{-4} * (|z|^2 + z\overline{\widetilde{u}} + \overline{z}\widetilde{u})](z + \widetilde{u})\|_{\dot{N}^1} + \|(|x|^{-4} * |\widetilde{u}|^2)z\|_{\dot{N}^1} + \|e\|_{\dot{N}^1} \\ &\lesssim \epsilon + \sum_{j=0}^{2} \|\nabla z\|_{U(I)}^j \|\nabla\widetilde{u}\|_{U(I)}^{3-j} \lesssim \epsilon + \sum_{j=0}^{2} \epsilon_0^{3-j}\|\nabla z\|_{U(I)}^j. \end{aligned}$$

On the other hand, we obtain

$$\|\nabla z\|_{U(I)} \lesssim \|e^{(i(t-t_0)\Delta)}\nabla z(t_0)\|_{U(I)} + S(t) \lesssim S(t) + \epsilon \tag{3.11}$$

and

$$S(t) \lesssim \epsilon + \sum_{j=0}^{2} \epsilon_0^{3-j}(S(t)+\epsilon)^j.$$

By a standard continuity argument, one can show that $S(t) \lesssim \epsilon$. Then from (3.11) and the Sobolev's embedding, we get

$$\begin{aligned}
&\|u-\widetilde{u}\|_{L_t^6 L_x^{\frac{6n}{3n-8}}} \lesssim \epsilon,\\
&\|\widetilde{u}\|_{\dot{S}^1} \lesssim \|\widetilde{u}(t_0)\|_{\dot{H}^1} + \|\nabla\widetilde{u}\|_{U(I)}^3 + \|e\|_{\dot{N}^1} \lesssim E + \epsilon_0^3 + \epsilon \lesssim E,\\
&\|u-\widetilde{u}\|_{\dot{S}^1} \lesssim \|u(t_0)-\widetilde{u}(t_0)\|_{\dot{H}_x^1} + S(t) \lesssim E' + \epsilon.
\end{aligned}$$

Therefore

$$\|u\|_{\dot{S}^1} \lesssim \|u-\widetilde{u}\|_{\dot{S}^1} + \|\widetilde{u}\|_{\dot{S}^1} \lesssim E + E'.$$

Lemma 3.2 (H_x^1-Critical Stability Result for Hartree Type) *Let I be a compact interval, $t_0 \in I$, and $\widetilde{u}$ be a function in $I \times \mathbb{R}^n$ which is a near-solution to* (1.4) *in the sense of that*

$$(i\partial_t + \Delta)\widetilde{u} = \lambda(|x|^{-4} * |\widetilde{u}|^2)\widetilde{u} + e \quad \text{for some function } e,$$

and $u(t_0)$ is close to $\widetilde{u}(t_0)$ in the sense of

$$\|u(t_0)-\widetilde{u}(t_0)\|_{\dot{H}_x^1} \le E' \quad \text{for some } E' > 0. \tag{3.12}$$

Moreover, suppose that

$$\|\widetilde{u}\|_{L_t^\infty \dot{H}^1(I\times\mathbb{R}^n)} \le E \quad \text{for some } E > 0 \tag{3.13}$$

and

$$\|\nabla\widetilde{u}\|_{U(I)} \le M \quad \text{for some } M > 0, \tag{3.14}$$

$$\|e^{i(t-t_0)\Delta}\nabla(u(t_0)-\widetilde{u}(t_0))\|_{U(I)} \le \epsilon, \tag{3.15}$$

$$\|e\|_{\dot{N}^1(I\times\mathbb{R}^n)} \le \epsilon \tag{3.16}$$

for some $0 < \epsilon < \epsilon_0$, where $\epsilon_0 = \epsilon_0(E, E', M) > 0$ is a small constant.

Then there exists a solution u to (1.4) *in $I \times \mathbb{R}^n$ with the special initial data $u(t_0)$ at t_0, satisfying*

$$\|u-\widetilde{u}\|_{\dot{S}^1(I\times\mathbb{R}^n)} \lesssim C(M,E)(E'+\epsilon), \tag{3.17}$$

$$\|u\|_{\dot{S}^1(I\times\mathbb{R}^n)} \lesssim C(M,E',E), \tag{3.18}$$

$$\|u-\widetilde{u}\|_{L_t^6 L_x^{\frac{6n}{3n-8}}(I\times\mathbb{R}^n)} \lesssim C(M,E,E')\epsilon. \tag{3.19}$$

Proof Without loss of generality, we assume $t_0 = \inf I$. Divide I into $J \sim \big(1 + \frac{M}{\epsilon_0}\big)^6$ subintervals I_j such that in each I_j

$$\|\nabla\widetilde{u}\|_{U(I_j)} \le \epsilon_0.$$

Fix $I_0 = [t_0, t_1]$. By the short-time perturbation, one can get

$$\begin{aligned}
&\|u - \widetilde{u}\|_{\dot{S}^1(I_0\times\mathbb{R}^n)} \lesssim E' + \epsilon,\\
&\|u\|_{\dot{S}^1(I_0\times\mathbb{R}^n)} \lesssim E' + E,\\
&\|u - \widetilde{u}\|_{L_t^6 L_x^{\frac{6n}{3n-8}}(I_0\times\mathbb{R}^n)} \lesssim \epsilon,\\
&\|(\mathrm{i}\partial_t + \Delta)(u - \widetilde{u})\|_{\dot{N}^1(I_0\times\mathbb{R}^n)} \lesssim \epsilon.
\end{aligned}$$

Furthermore, we have

$$\|u(t_1) - \widetilde{u}(t_1)\|_{\dot{H}_x^1} \le \|u - \widetilde{u}\|_{\dot{S}_x^1(I_0\times\mathbb{R}^n)} \lesssim E' + \epsilon$$

and

$$\begin{aligned}
&\|\mathrm{e}^{\mathrm{i}(t-t_1)\Delta}\nabla(u(t_1) - \widetilde{u}(t_1))\|_{U(I_1)}\\
\lesssim\ &\|\mathrm{e}^{\mathrm{i}(t-t_0)\Delta}\nabla(u(t_0) - \widetilde{u}(t_0))\|_{U(I_1)} + \|(\mathrm{i}\partial_t + \Delta)(u - \widetilde{u})\|_{N^1(I_0\times\mathbb{R}^n)} \lesssim \epsilon.
\end{aligned}$$

Choosing ϵ to be small enough, from the short-time perturbation, we have that the results also hold in I_1. By induction, we complete the proof of the lemma.

Remark 3.1 Notice that condition (3.15) is weaker than the condition stated in [18] in which it requires

$$\begin{aligned}
&\Big(\sum_N \|P_N \nabla \mathrm{e}^{(\mathrm{i}(t-t_0)\Delta)}(u(t_0) - \widetilde{u}(t_0))\|_{U(I)}^2\Big)^{\frac{1}{2}}\\
&+\Big(\sum_N \|P_N \nabla \mathrm{e}^{(\mathrm{i}(t-t_0)\Delta)}(u(t_0) - \widetilde{u}(t_0))\|^2_{L_t^3 L_x^{\frac{6n}{3n-4}}(I\times\mathbb{R}^n)}\Big)^{\frac{1}{2}} \le \epsilon.
\end{aligned}$$

In fact, for the Hartree type, the nonlinearity and its derivatives are Lipschitz continuous.

The same method can be used to prove the perturbation theory of the L_x^2-critical NLS with Hartree type. Note that by Hölder and Hardy-Littlewood-Sobolev inequality, we have

$$\|(|x|^{-2} * (ab))c\|_{\dot{N}^0} \lesssim \|a\|_{U(I)}\|b\|_{U(I)}\|c\|_{U(I)} \tag{3.20}$$

instead of (3.10). Arguing similarly, we can get the following result.

Lemma 3.3 (L_x^2-Critical Stability Result for Hartree Type) *Let I be a compact interval, $t_0 \in I$, and $\widetilde{u}$ be a function in $I \times \mathbb{R}^n$ which is a near-solution to* (1.5) *in the sense of that*

$$(\mathrm{i}\partial_t + \Delta)\widetilde{u} = \lambda(|x|^{-2} * |\widetilde{u}|^2)\widetilde{u} + e \quad \textit{for some function } e,$$

and $u(t_0)$ be close to $\widetilde{u}(t_0)$ in the sense of

$$\|u(t_0) - \widetilde{u}(t_0)\|_{L_x^2(\mathbb{R}^n)} \le M' \quad \textit{for some } M' > 0. \tag{3.21}$$

Further suppose that

$$\|\widetilde{u}\|_{L_t^\infty L_x^2(I\times\mathbb{R}^n)} \le M \quad \textit{for some } M > 0 \tag{3.22}$$

and

$$\|\widetilde{u}\|_{U(I)} \le L \quad \textit{for some } L > 0, \tag{3.23}$$

$$\|\mathrm{e}^{i(t-t_0)\Delta}(u(t_0) - \widetilde{u}(t_0))\|_{U(I)} \le \epsilon, \tag{3.24}$$

$$\|e\|_{\dot{N}^0(I\times\mathbb{R}^n)} \le \epsilon \tag{3.25}$$

for some $0<\epsilon<\epsilon_1$, where $\epsilon_1=\epsilon_1(M,M',L)>0$ is a small constant.

Then there exists a solution u to (1.5) in $I\times\mathbb{R}^n$ with the special initial data $u(t_0)$ at t_0, and

$$\|u-\widetilde{u}\|_{\dot{S}^0(I\times\mathbb{R}^n)}\lesssim C(L,M,M')(M'+\epsilon), \tag{3.26}$$

$$\|u\|_{\dot{S}^0(I\times\mathbb{R}^n)}\lesssim C(L,M,M'), \tag{3.27}$$

$$\|u-\widetilde{u}\|_{U(I)}\lesssim C(L,M,M')\epsilon. \tag{3.28}$$

The corresponding stability results for the H_x^1-critical and the L_x^2-critical NLS with power type are established in [23, 24]. However, as the derivatives of the nonlinearity are merely Hölder continuous of order $\frac{4}{n-2}$ rather than Lipschitz, the problem becomes more subtle when the dimension n is greater than 6. One can find the details in [23, 24]. Here we simply state their result as follows.

Lemma 3.4 (H_x^1-Critical Stability Result for Power Type) *Let I be a compact interval, $t_0\in I$, and $\widetilde{u}$ be a function in $I\times\mathbb{R}^n$ which is a near-solution to (1.2) meaning*

$$(\mathrm{i}\partial_t+\Delta)\widetilde{u}=\lambda|\widetilde{u}|^{\frac{4}{n-2}}\widetilde{u}+e\quad\text{for some function } e,$$

and $u(t_0)$ be close to $\widetilde{u}(t_0)$, by which we mean

$$\|u(t_0)-\widetilde{u}(t_0)\|_{\dot{H}_x^1}\le E_0'\quad\text{for some } E_0'>0. \tag{3.29}$$

In addition, suppose that

$$\|\widetilde{u}\|_{L_t^\infty\dot{H}^1(I\times\mathbb{R}^n)}\le E_0\quad\text{for some } E_0>0 \tag{3.30}$$

and

$$\|\widetilde{u}\|_{W(I)}\le M_0\quad\text{for some } M_0>0, \tag{3.31}$$

$$\Big(\sum_N\|P_N\nabla\mathrm{e}^{(\mathrm{i}(t-t_0)\Delta)}(u(t_0)-\widetilde{u}(t_0))\|^2_{L_t^{\frac{2(n+2)}{n-2}}L_x^{\frac{2n(n+2)}{n^2+4}}(I\times\mathbb{R}^n)}\Big)^{\frac12}\le\epsilon, \tag{3.32}$$

$$\|e\|_{\dot{N}^1(I\times\mathbb{R}^n)}\le\epsilon \tag{3.33}$$

for some $0<\epsilon<\epsilon_2$, and $\epsilon_2=\epsilon_2(E_0,E_0',M_0)$ is a small constant.

Then there exists a solution u to (1.2) in $I\times\mathbb{R}^n$ with the special initial data $u(t_0)$ at t_0, and

$$\|u-\widetilde{u}\|_{\dot{S}^1(I\times\mathbb{R}^n)}\lesssim C(E_0,E_0',M_0)(E_0'+\epsilon+\epsilon^{\frac{7}{(n-2)^2}}), \tag{3.34}$$

$$\|u\|_{\dot{S}^1(I\times\mathbb{R}^n)}\lesssim C(M_0,E_0',E_0), \tag{3.35}$$

$$\|u-\widetilde{u}\|_{L_t^{\frac{2(n+2)}{n-2}}L_x^{\frac{2n(n+2)}{n^2+4}}(I\times\mathbb{R}^n)}\lesssim C(M_0,E_0,E_0')(\epsilon+\epsilon^{\frac{7}{(n-2)^2}}). \tag{3.36}$$

Lemma 3.5 (L_x^2-Critical Stability Result for Power Type) *Let I be a compact interval, $t_0\in I$, and $\widetilde{u}$ be a function in $I\times\mathbb{R}^n$ which is a near-solution to (1.3) meaning*

$$(\mathrm{i}\partial_t+\Delta)\widetilde{u}=\lambda|\widetilde{u}|^{\frac{4}{n}}\widetilde{u}+e\quad\text{for some function } e,$$

and $u(t_0)$ be close to $\widetilde{u}(t_0)$ in the sense of

$$\|u(t_0)-\widetilde{u}(t_0)\|_{L_x^2(\mathbb{R}^n)}\le M_0'\quad\text{for some } M_0'>0. \tag{3.37}$$

Moreover, assume that

$$\|\widetilde{u}\|_{L_t^\infty L_x^2(I\times\mathbb{R}^n)} \le M_0 \quad \text{for some } M_0 > 0 \tag{3.38}$$

and

$$\|\widetilde{u}\|_{V(I)} \le L_0 \quad \text{for some } L_0 > 0, \tag{3.39}$$

$$\|\mathrm{e}^{i(t-t_0)\Delta}(u(t_0) - \widetilde{u}(t_0))\|_{V(I)} \le \epsilon, \tag{3.40}$$

$$\|e\|_{\dot{N}^0(I\times\mathbb{R}^n)} \le \epsilon \tag{3.41}$$

for some $0 < \epsilon < \epsilon_3$*, where* $\epsilon_3 = \epsilon_3(M_0, M_0', L_0) > 0$ *is a small constant.*

Then there exists a solution u *to* (1.3) *in* $I \times \mathbb{R}^n$ *with the special initial data* $u(t_0)$ *at* t_0 *and*

$$\|u - \widetilde{u}\|_{\dot{S}^0(I\times\mathbb{R}^n)} \lesssim C(L_0, M_0, M_0')M_0', \tag{3.42}$$

$$\|u\|_{\dot{S}^0(I\times\mathbb{R}^n)} \lesssim C(L_0, M_0, M_0'), \tag{3.43}$$

$$\|u - \widetilde{u}\|_{V(I)} \lesssim C(L_0, M_0, M_0')\epsilon. \tag{3.44}$$

To conclude this section, we state the results involving persistence of L^2 or $\dot{H}^1$ regularity for critical NLS with Hartree type or power type as follows.

Lemma 3.6 (Persistence of Regularity) *Let* $k = 0, 1$*,* I *be a compact interval, and* $t_0 \in I$*.*

Case 1 u *is a solution to* (1.2) *in* $I \times \mathbb{R}^n$ *with*

$$\|u\|_{W(I)} \le M.$$

Then, if $u(t_0) \in \dot{H}_x^k$*, we have*

$$\|u\|_{\dot{S}^k(I\times\mathbb{R}^n)} \le C(M)\|u(t_0)\|_{\dot{H}_x^k}.$$

Case 2 *Let* u *be a solution to* (1.3) *in* $I \times \mathbb{R}^n$ *and*

$$\|u\|_{V(I)} \le L.$$

Then, if $u(t_0) \in \dot{H}_x^k$*, we have*

$$\|u\|_{\dot{S}^k(I\times\mathbb{R}^n)} \le C(L)\|u(t_0)\|_{\dot{H}_x^k}.$$

Case 3 u *is a solution to* (1.4) *in* $I \times \mathbb{R}^n$ *satisfying*

$$\|u\|_{L_t^6 L_x^{\frac{6n}{3n-8}}(I\times\mathbb{R}^n)} \le M.$$

Then, if $u(t_0) \in \dot{H}_x^k$*, we have*

$$\|u\|_{\dot{S}^k(I\times\mathbb{R}^n)} \le C(M)\|u(t_0)\|_{\dot{H}_x^k}.$$

Case 4 u *is a solution to* (1.5) *in* $I \times \mathbb{R}^n$ *satisfying*

$$\|u\|_{U(I)} \le L.$$

Then, if $u(t_0) \in \dot{H}_x^k$*, we have*

$$\|u\|_{\dot{S}^k(I\times\mathbb{R}^n)} \le C(L)\|u(t_0)\|_{\dot{H}_x^k}.$$

Proof All four cases can be proved similarly, and hence we only consider Case 1.

Divide the interval I into $N \sim (1 + \frac{M}{\eta})^6$ subintervals $I_j = [t_j, t_{j+1}]$ such that

$$\|u\|_{W(I_j)} \leq \eta,$$

where η is a small positive constant to be chosen later. By Strichartz estimates, in each I_j we obtain

$$\begin{aligned}\|u\|_{\dot{S}^k(I_j\times\mathbb{R}^n)} &\lesssim \|u(t_j)\|_{\dot{H}^k_x} + \|u\|_{\dot{S}^k(I_j\times\mathbb{R}^n)}\|u\|_{W(I_j)}^{\frac{4}{n-2}} \\ &\lesssim \|u(t_j)\|_{\dot{H}^k_x} + \eta^{\frac{4}{n-2}}\|u\|_{\dot{S}^k(I_j\times\mathbb{R}^n)}.\end{aligned}$$

Choosing η to be sufficiently small (say $\eta \leq \frac{1}{2}$), we get

$$\|u\|_{\dot{S}^k(I_j\times\mathbb{R}^n)} \lesssim \|u(t_j)\|_{\dot{H}^k_x}.$$

Next, we consider the relationship between $\|u(t_j)\|_{\dot{H}^k_x}$ and $\|u(t_0)\|_{\dot{H}^k_x}$.

In I_0, we have

$$\|u(t_1)\|_{\dot{H}^k_x} \leq \|u\|_{\dot{S}^k(I_0\times\mathbb{R}^n)} \leq C\|u(t_0)\|_{\dot{H}^k_x}.$$

In I_1, we have

$$\|u(t_2)\|_{\dot{H}^k_x} \leq \|u\|_{\dot{S}^k(I_1\times\mathbb{R}^n)} \leq C\|u(t_1)\|_{\dot{H}^k_x} \leq C^2\|u(t_0)\|_{\dot{H}^k_x}.$$

Likewise, for each I_j we can obtain

$$\|u(t_j)\|_{\dot{H}^k_x} \leq C^j\|u(t_0)\|_{\dot{H}^k_x}.$$

Summing up the estimates over all the subinterval I_j, we obtained the desired results.

4 Global Well-Posedness

We will prove Theorem 1.1 in this section. Due to the conservation of energy and mass, we shall denote the energy $E(u)$ and the mass $M(u)$ by E and M, respectively. In order to prove the global well-posedness of (1.1), we show that the blowup in Propositions 3.1 and 3.2 cannot happen. Suppose that the initial data u_0 of (1.1) are in H^1_x. When $p \neq \frac{4}{n-2}$ and $\gamma \neq 4$, we prove that $\|u(t)\|_{H^1_x}$ is bounded for all t at which the solution is defined. In view of the conservation of mass, we focus on the bounds of $\|u(t)\|_{\dot{H}^1_x}$. When $p = \frac{4}{n-2}$ or $\gamma = 4$, the boundedness of $\|u(t)\|_{H^1_x}$ is not enough to prove Theorem 1.1. So we view the energy-subcritical nonlinearity as a perturbation to the energy-critical NLS, which is known as globally well-posed.

4.1 Kinetic energy control

We will obtain a prior control on the kinetic energy which is bounded for t at which the solution is defined. Moreover, the bound only depends on energy and mass,

$$\|u(t)\|_{\dot{H}^1_x} \leq C(E, M). \tag{4.1}$$

Observing that the energy

$$E(u) = \frac{1}{2}\int |\nabla u|^2 \,\mathrm{d}x + \frac{\lambda_1}{p+2}\int |u|^{p+2} \,\mathrm{d}x + \frac{\lambda_2}{4}\int (|x|^{-\gamma} * |u|^2)|u|^2 \,\mathrm{d}x$$

is conserved. Hence, for Case 1, we have

$$\|u(t)\|_{\dot{H}^1_x} \lesssim E.$$

As for Case 2, from Parseval identity, Hardy-Littlewood-Sobolev inequality and interpolation, we have

$$\begin{aligned}\int(|x|^{-\gamma}*|u|^2)|u|^2\,\mathrm{d}x :&= \int(|\nabla|^{-(n-\gamma)}|u|^2)|u|^2\,\mathrm{d}x = \||\nabla|^{-\frac{n-\gamma}{2}}|u|^2\|_{L^2}^2\\ &\le \|u\|^4_{L^{\frac{4n}{2n-\gamma}}} \le \|u\|_{L^2}^{4(1-\frac{n+\gamma}{2n})}\|u\|^{\frac{2n+2\gamma}{n}}_{L^{\frac{2n+2\gamma}{n}}}. \end{aligned}\tag{4.2}$$

Recall that for any positive constants a, δ, and $p_1 < p_2$, the following inequality

$$a^{p_1+2} \le C(\delta)a^2 + \delta a^{p_2+2} \tag{4.3}$$

holds. Therefore if $\frac{2n+2\gamma}{n} < p+2$, i.e., $\gamma < \frac{np}{2}$,

$$\|u\|^{\frac{2n+2\gamma}{n}}_{L^{\frac{2n+2\gamma}{n}}} \le C(\delta)\|u\|_{L^2}^2 + \delta\|u\|_{L^{p+2}}^{p+2}.$$

Hence

$$E(u) \ge \frac{1}{2}\int|\nabla u|^2\,\mathrm{d}x + \frac{\lambda_1}{p+2}\int|u|^{p+2}\,\mathrm{d}x - C\|u\|_{L^2}^{4(1-\frac{n+\gamma}{2n})}\delta\int|u|^{p+2}\,\mathrm{d}x - C(M).$$

Let δ be sufficiently small. Then we get

$$\|u(t)\|_{\dot{H}^1_x} \le C(E, M).$$

If $\gamma \ge \frac{np}{2}$, using $\lambda_1 > 0$ and Appendix, we obtain

$$E \ge E_1 \ge \frac{1}{2}\int|\nabla u|^2\,\mathrm{d}x - \frac{|\lambda_2|}{4}C_W\|\nabla u\|_{L^2}^{\gamma}\|u\|_{L^2}^{4-\gamma}.$$

For the case $\gamma < 2$, by the Young's inequality, one has

$$E \ge \frac{1}{2}\int|\nabla u|^2\,\mathrm{d}x - \frac{|\lambda_2|}{4}\delta C_W\|\nabla u\|_{L^2}^2 - \frac{|\lambda_2|}{4}C_W C(\delta)\|u\|_{L^2}^{\frac{2(4-\gamma)}{2-\gamma}}.$$

For δ small enough, we obtain

$$\|u(t)\|_{\dot{H}^1_x} \le C(E, M).$$

When $\gamma = 2$, we have

$$E \ge \Big(\frac{1}{2} - \frac{|\lambda_2|}{4}C_W\|u\|_{L^2}^2\Big)\|\nabla u\|_{L^2}^2.$$

If

$$\|u\|_{L^2}^2 < \frac{2}{C_W|\lambda_2|} = \frac{1}{|\lambda_2|}\|W\|_{L^2}^2$$

is true, we can obtain

$$\|u(t)\|_{\dot{H}^1_x} \le C(E, M).$$

For the case $2 < \gamma < 4$, applying Appendix and the conservation of energy and mass, we only need to show that if

$$\|\nabla u_0\|_{L^2}^2(\|u_0\|_{L^2}^2)^{\frac{4-\gamma}{\gamma-2}} < \Big(\frac{\|\nabla W\|_{L^2}^2}{|\lambda_2|}\Big)^{\frac{2}{\gamma-2}},$$

then

$$\|\nabla u\|_{L^2}^2(\|u\|_{L^2}^2)^{\frac{4-\gamma}{\gamma-2}} < \Big(\frac{\|\nabla W\|_{L^2}^2}{|\lambda_2|}\Big)^{\frac{2}{\gamma-2}}.$$

We prove it by the standard continuity argument. Define

$$\Omega = \Big\{t \in I,\ \|\nabla u\|_{L^2}^2(\|u\|_{L^2}^2)^{\frac{4-\gamma}{\gamma-2}} < \Big(\frac{\|\nabla W\|_{L^2}^2}{|\lambda_2|}\Big)^{\frac{2}{\gamma-2}},$$
$$E(u(t))(\|u\|_{L^2}^2)^{\frac{4-\gamma}{\gamma-2}} \le (1-\delta_0)\Big(\frac{1}{2}-\frac{1}{\gamma}\Big)\Big[\frac{2\gamma\widetilde{E}(W)}{|\lambda_2|(\gamma-2)}\Big]^{\frac{2}{\gamma-2}}\Big\}.$$

It suffices to show that Ω is both open and closed. Note that $t_0 \in \Omega$. Ω is obviously open since $u \in C_t^0(I, \dot{H}_x^1)$. Therefore, it remains to prove that Ω is closed. Let $t_n \in \Omega,\ T \in I$ be a sequence such that $t_n \to T$. Then we have

$$\|\nabla u(t_n)\|_{L^2}^2 M^{\frac{4-\gamma}{\gamma-2}} < \Big(\frac{\|\nabla W\|_{L^2}^2}{|\lambda_2|}\Big)^{\frac{2}{\gamma-2}}$$

and

$$E(u(t_n))M^{\frac{4-\gamma}{\gamma-2}} \le (1-\delta_0)\Big(\frac{1}{2}-\frac{1}{\gamma}\Big)\Big[\frac{2\gamma\widetilde{E}(W)}{|\lambda_2|(\gamma-2)}\Big]^{\frac{2}{\gamma-2}}.$$

By Lemma A.2, we have

$$\|\nabla u(t_n)\|_{L^2}^2 M^{\frac{4-\gamma}{\gamma-2}} \le (1-\overline{\delta})\Big(\frac{\|\nabla W\|_{L^2}^2}{|\lambda_2|}\Big)^{\frac{2}{\gamma-2}}.$$

Since $u \in C_t^0(I, \dot{H}_x^1)$, and energy and mass are conserved, we get

$$\|\nabla u(T)\|_{L^2}^2 M^{\frac{4-\gamma}{\gamma-2}} \le (1-\overline{\delta})\Big(\frac{\|\nabla W\|_{L^2}^2}{|\lambda_2|}\Big)^{\frac{2}{\gamma-2}}$$

and

$$E(u(T))M^{\frac{4-\gamma}{\gamma-2}} \le (1-\delta_0)\Big(\frac{1}{2}-\frac{1}{\gamma}\Big)\Big[\frac{2\gamma\widetilde{E}(W)}{|\lambda_2|(\gamma-2)}\Big]^{\frac{2}{\gamma-2}}.$$

This implies that $T \in \Omega$ and $\|u(t)\|_{\dot{H}_x^1} \le C(E, M)$.

Remark 4.1 When $\gamma = \frac{np}{2}$, we have

$$E \ge \frac{1}{2}\int |\nabla u|^2\,\mathrm{d}x + \frac{|\lambda_1|}{p+2}\|u\|_{L^{p+2}}^{p+2} - C\frac{|\lambda_2|}{4}M^{\frac{2-p}{2}}\|u\|_{L^{p+2}}^{p+2}.$$

The condition $n > \gamma = \frac{np}{2}$ implies $p < 2$. If in addition, $\frac{|\lambda_1|}{p+2} > C\frac{|\lambda_2|}{4}M^{\frac{2-p}{2}}$ also holds, i.e., $M < \big(\frac{4|\lambda_1|}{(p+2)C|\lambda_2|}\big)^{\frac{2}{2-p}}$, we can also obtain $\|u(t)\|_{\dot{H}_x^1} \le C(E, M)$.

To prove Case 3, we need the following lemma.

Lemma 4.1

$$\||\nabla|^{-\frac{n-\gamma}{4}} f\|_{L^4} \lesssim \||\nabla|^{-\frac{n-\gamma}{2}}|f|^2\|_{L^2}^{\frac{1}{2}}. \tag{4.4}$$

Remark 4.2 T. Tao proved the inequality for $\gamma = 3$ in [24]. We can show that the same is true for general γ.

Proof It suffices to prove (4.4) for a positive Schwartz function f. In fact, we only need to prove the pointwise inequality

$$S(|\nabla|^{-\frac{n-\gamma}{4}} f)(x) \lesssim [(|\nabla|^{-\frac{n-\gamma}{2}}|f|^2)(x)]^{\frac{1}{2}}, \tag{4.5}$$

where $Sf := \Big(\sum\limits_N |P_N f|^2\Big)^{\frac{1}{2}}$.

It is clear that (4.5) implies (4.4).

$$\||\nabla|^{-\frac{n-\gamma}{4}} f\|_{L^4} \lesssim \|S(|\nabla|^{-\frac{n-\gamma}{4}} f)\|_{L^4} \lesssim \|(|\nabla|^{-\frac{n-\gamma}{2}}|f|^2)^{\frac{1}{2}}\|_{L^4} \lesssim \||\nabla|^{-\frac{n-\gamma}{2}}|f|^2\|_{L^2}^{\frac{1}{2}}.$$

Consequently, we will focus our attention on the estimate for each of the dyadic pieces

$$P_N(|\nabla|^{-\frac{n-\gamma}{4}} f)(x) = \int \mathrm{e}^{2\pi \mathrm{i} x\cdot\xi} \widehat{f}(\xi)|\xi|^{-\frac{n-\gamma}{4}} m\Big(\frac{\xi}{N}\Big)\,\mathrm{d}\xi,$$

where $m(\xi) := \varphi(\xi) - \varphi(2\xi)$ introduced in Section 2.

Since $|\xi|^{-\frac{n-\gamma}{4}} m(\frac{\xi}{N}) \sim N^{-\frac{n-\gamma}{4}} m(\frac{\xi}{N})$, we have

$$P_N(|\nabla|^{-\frac{n-\gamma}{4}} f)(x) \sim N^{\frac{3n+\gamma}{4}} f * \check{m}(Nx) = N^{\frac{3n+\gamma}{4}} \int f(x-y)\check{m}(Ny)\,\mathrm{d}y.$$

Furthermore, m is a Schwartz function, we have

$$|P_N(|\nabla|^{-\frac{n-\gamma}{4}} f)(x)| \lesssim N^{\frac{3n+\gamma}{4}} \int_{|y|\le N^{-1}} f(x-y)\,\mathrm{d}y + N^{\frac{3n+\gamma}{4}} \int_{|y|>N^{-1}} f(x-y)\frac{1}{|Ny|^{\beta}}\,\mathrm{d}y,$$

where β is chosen later.

A simple application of Cauchy-Schwartz yields

$$\begin{aligned}
&S(|\nabla|^{-\frac{n-\gamma}{4}} f)(x)\\
&= \Big(\sum_N |P_N(|\nabla|^{-\frac{n-\gamma}{4}} f)(x)|^2\Big)^{\frac{1}{2}}\\
&\lesssim \Big(\sum_N N^{\frac{3n+\gamma}{2}} \Big|\int_{|y|\le N^{-1}} f(x-y)\,\mathrm{d}y\Big|^2 + \sum_N N^{\frac{3n+\gamma}{2}}\Big|\int_{|y|>N^{-1}} f(x-y)\frac{1}{|Ny|^{\beta}}\,\mathrm{d}y\Big|^2\Big)^{\frac{1}{2}}\\
&\lesssim \Big[\sum_N N^{\frac{3n+\gamma}{2}} N^{-n} \int_{|y|\le N^{-1}} |f(x-y)|^2\,\mathrm{d}y\\
&\quad + \sum_N N^{\frac{3n+\gamma}{2}} \Big(\int_{|y|>N^{-1}} \frac{|f(x-y)|^2}{|y|^{\alpha}}\,\mathrm{d}y\Big)\Big(\int_{|y|>N^{-1}} \frac{|y|^{\alpha}}{|Ny|^{2\beta}}\,\mathrm{d}y\Big)\Big]^{\frac{1}{2}},
\end{aligned}$$

where α is to be decided later.

Note that if α and β are chosen to satisfy $n+\alpha-2\beta<0$, $\frac{\gamma+n}{2}-\alpha<0$, we have

$$\sum_N N^{\frac{n+\gamma}{2}} \chi_{\{|y|\le N^{-1}\}}(y) \lesssim \sum_{|y|\le N^{-1}} N^{\frac{n+\gamma}{2}} \lesssim |y|^{-\frac{n+\gamma}{2}},$$

$$\sum_N N^{\frac{3n+\gamma}{2}} \Big(\int_{|y|>N^{-1}} \frac{|y|^{\alpha}}{|Ny|^{2\beta}}\,\mathrm{d}y\Big)\chi_{\{|y|>N^{-1}\}}(y) \lesssim \sum_{|y|>N^{-1}} N^{\frac{3n+\gamma}{2}} N^{-2\beta} N^{-(n+\alpha-2\beta)} \lesssim |y|^{\alpha-\frac{n+\gamma}{2}}$$

and

$$\begin{aligned}
S(|\nabla|^{-\frac{n-\gamma}{4}} f)(x) &\lesssim \Big(\int_{|y|\le N^{-1}} \frac{|f(x-y)|^2}{|y|^{\frac{n+\gamma}{2}}}\,\mathrm{d}y + \int_{|y|>N^{-1}} \frac{|f(x-y)|^2}{|y|^{\frac{n+\gamma}{2}}}\,\mathrm{d}y\Big)^{\frac{1}{2}}\\
&\sim \Big(\int \frac{|f(x-y)|^2}{|y|^{\frac{n+\gamma}{2}}}\,\mathrm{d}y\Big)^{\frac{1}{2}} \sim [(|\nabla|^{-\frac{n-\gamma}{2}}|f|^2)(x)]^{\frac{1}{2}}.
\end{aligned}$$

This completes the proof.

Using interpolation and the Young's inequality, we get

$$\|u\|_{L^q}^q \lesssim \|\nabla u\|_{L^2}^{\frac{2(n-\gamma)}{2+n-\gamma}} \||\nabla|^{-\frac{n-\gamma}{4}} u\|_{L^4}^{\frac{8}{2+n-\gamma}} \lesssim \varepsilon \|\nabla u\|_{L^2}^2 + C(\varepsilon)\||\nabla|^{-\frac{n-\gamma}{4}} u\|_{L^4}^4,$$

where $q = \frac{2(4+n-\gamma)}{2+n-\gamma}$. Then,

$$\||\nabla|^{-\frac{n-\gamma}{4}} u\|_{L^4}^4 \geq c(\varepsilon)\|u\|_{L^q}^q - c(\varepsilon)\|\nabla u\|_{L^2}^2.$$

On the other hand, in view of

$$\frac{|\lambda_2|}{4}\||\nabla|^{-\frac{n-\gamma}{2}}|u|^2\|_{L^2}^2 \gtrsim \||\nabla|^{-\frac{n-\gamma}{4}} u\|_{L^4}^4 \geq c(\varepsilon)\|u\|_{L^q}^q - c(\varepsilon)\|\nabla u\|_{L^2}^2,$$

from (4.3)–(4.4) and

$$\|u\|_{L^{p+2}}^{p+2} \leq C(\delta)\|u\|_{L^2}^2 + \delta\|u\|_{L^q}^q,$$

we have

$$E \geq \frac{1}{2}\int |\nabla u|^2\, dx + c(\varepsilon)\|u\|_{L^q}^q - c(\varepsilon)\|\nabla u\|_{L^2}^2 - \frac{|\lambda_1|}{p+2}\delta\|u\|_{L^q}^q - \frac{|\lambda_1|}{p+2}C(\delta)\|u\|_{L^2}^2.$$

Choosing ε and $\delta = \delta(\varepsilon)$ to be small enough, we obtain

$$\|u(t)\|_{\dot H_x^1} \leq C(E, M).$$

If $p \geq \frac{4}{2+n-\gamma}$, and notice $\lambda_2 > 0$, applying the same method used in Case 2, under the conditions of Case 3, we have

$$\|u(t)\|_{\dot H_x^1} \leq C(E, M).$$

About Case 4, using (A.1)–(A.2) and the Young's inequality, we have

$$\begin{aligned} E &\geq \frac{1}{2}\int |\nabla u|^2\, \mathrm{d}x - \frac{|\lambda_1|}{p+2}C_R\|u\|_{L^2}^{\frac{4-(n-2)p}{2}}\|\nabla u\|_{L^2}^{\frac{np}{2}} - \frac{|\lambda_2|}{4}C_W\|u\|_{L^2}^{4-\gamma}\|\nabla u\|_{L^2}^{\gamma} \\ &\geq \frac{1}{2}\int |\nabla u|^2\, \mathrm{d}x - \frac{|\lambda_1|}{p+2}C_R\delta\|\nabla u\|_{L^2}^2 - \frac{|\lambda_2|}{4}C_W\delta\|\nabla u\|_{L^2}^2 - C(M), \end{aligned}$$

where δ is chosen to be sufficiently small. We obtain

$$\|u(t)\|_{\dot H_x^1} \leq C(E, M).$$

4.2 Completion of the proof of Theorem 1.1

Now we are in the position to complete the proof of Theorem 1.1. Recall that when both nonlinearities are $\dot H_x^1$-subcritical, by Proposition 3.1, the a prior control on the kinetic and the conservation of mass can conclude that the unique strong solution u to (1.1) is a global solution. In fact, we can find $T = T(\|u_0\|_{H_x^1})$ such that (1.1) admits a unique strong solution $u \in S^1([-T,T]\times\mathbb{R}^n)$ and

$$\|u\|_{S^1([-T,T]\times\mathbb{R}^n)} \leq C(E, M).$$

If we divide the interval I into subintervals of length T, compute S^1-bounds on each subinterval and then sum up the bounds over subintervals, we can get the bound (1.6).

When $p = \frac{4}{n-2}$ or $\gamma = 4$, we treat the other nonlinearity as a perturbation of the energy-critical NLS, which is globally well-posed, [4, 14–15, 18, 22, 27]. Here, we only discuss the case:

$p = \frac{4}{n-2}$ and $0 < \gamma < \min\{n, 4\}$; the case: $0 < p < \frac{4}{n-2},\ \gamma = 4$ with $n \geq 5$ can be discussed similarly.

Let v be the unique strong global solution to the energy-critical equation (1.2) with initial data $v_0 = u_0$ at time $t = 0$. By the main results in [4, 14–15, 22, 27], we know that such a v exists and

$$\|v\|_{\dot{S}^1(\mathbb{R}\times\mathbb{R}^n)} \leq C(\|u_0\|_{\dot{H}^1_x}). \tag{4.6}$$

Furthermore, by Lemma 3.6, we also have

$$\|v\|_{\dot{S}^0(\mathbb{R}\times\mathbb{R}^n)} \leq C(\|u_0\|_{\dot{H}^1_x})\|u_0\|_{L^2} \leq C(E, M).$$

By time reversal symmetry, it suffices to solve the problem forward in time. By (4.6), divide $\mathbb{R}^+$ into $J = J(E, \eta)$ subintervals $I_j = [t_j, t_{j+1}]$ such that

$$\|v\|_{\dot{B}^1(I_j)} \sim \eta \tag{4.7}$$

for some small η to be chosen later (at last we will find η just depending on the energy and mass).

We may assume that there exists $J' < J$ such that for any $0 \leq j \leq J' - 1,\ [0, T] \cap I_j \neq \emptyset$.

Thus, we can write $[0, T] = \bigcup\limits_{j=0}^{J'-1} ([0, T] \cap I_j)$.

According to the Strichartz estimate, Sobolev embedding and (4.7), the free evolution $e^{i(t-t_j)\Delta}v(t_j)$ is small in I_j

$$\begin{aligned}
\|e^{i(t-t_j)\Delta}v(t_j)\|_{\dot{B}^1(I_j)} &\leq \|v\|_{\dot{B}^1(I_j)} + \|\nabla(|v|^{\frac{4}{n-2}}v)\|_{L_{t,x}^{\frac{2(n+2)}{n+4}}(I_j\times\mathbb{R}^n)} \\
&\leq \|v\|_{\dot{B}^1(I_j)} + C\|v\|_{\dot{B}^1(I_j)}^{\frac{n+2}{n-2}} \\
&\leq \eta + C\eta^{\frac{n+2}{n-2}}.
\end{aligned}$$

Thus letting η be sufficiently small, for any $0 \leq j \leq J' - 1$, we obtain

$$\|e^{i(t-t_j)\Delta}v(t_j)\|_{\dot{B}^1(I_j)} \leq 2\eta.$$

On the interval I_0, recalling that $u(0) = v(0) = u_0$, we estimate

$$\begin{aligned}
\|u\|_{\dot{B}^1(I_0)} &\leq \|e^{it\Delta}u_0\|_{\dot{B}^1(I_0)} + CT^\alpha\|u\|^3_{\dot{B}^1(I_0)} + C\|u\|_{\dot{B}^1(I_0)}^{\frac{n+2}{n-2}} \\
&\leq 2\eta + CT^\alpha\|u\|^3_{\dot{B}^1(I_0)} + C\|u\|_{\dot{B}^1(I_0)}^{\frac{n+2}{n-2}},
\end{aligned}$$

where $\alpha = \min\{1,\ 2 - \frac{\gamma}{2}\}$.

Assume that both η and T are sufficiently small. Then a standard continuity argument yields

$$\|u\|_{\dot{B}^1(I_0)} \leq 4\eta.$$

In order to use Lemma 3.4, we notice that (3.29) holds with $E_0' = 0$ and (3.30) holds for $E_0 := C(E, M)$. Furthermore, (3.31) holds on $I := I_0$ for $M_0 := 4C\eta$. We only need to prove that the error, which is the second nonlinearity in this case, is sufficiently small. In fact

$$\|\nabla e\|_{\dot{N}^0(I_0\times\mathbb{R}^n)} \lesssim T^\alpha\|u\|^3_{\dot{Y}^1(I_0)} \lesssim T^\alpha\|u\|^3_{\dot{B}^1(I_0)} \lesssim T^\alpha\eta^3.$$

We see that by choosing T to be sufficiently small depending on ϵ, we get

$$\|\nabla e\|_{\dot{N}^0(I_0\times\mathbb{R}^n)} < \epsilon,$$

where $\epsilon = \epsilon(E, M)$ is a small constant to be chosen later. Thus, with ϵ sufficiently small, the hypotheses of Lemma 3.4 are satisfied, and hence

$$\|u - v\|_{\dot{S}^1(I_0\times\mathbb{R}^n)} \leq C(E, M)\epsilon^c \tag{4.8}$$

for a small positive constant c which depends only on the dimension n.

Strichartz estimates and (4.8) imply

$$\|u(t_1) - v(t_1)\|_{\dot{H}^1_x} \leq C(E, M)\epsilon^c, \tag{4.9}$$

$$\|\mathrm{e}^{\mathrm{i}(t-t_1)\Delta}(u(t_1) - v(t_1))\|_{\dot{B}^1(I_1)} \leq C(E, M)\epsilon^c. \tag{4.10}$$

By (4.9)–(4.10) and Strichartz estimates, we get

$$\begin{aligned}\|u\|_{\dot{B}^1(I_1)} &\leq \|\mathrm{e}^{\mathrm{i}(t-t_1)\Delta}v(t_1)\|_{\dot{B}^1(I_1)} + \|\mathrm{e}^{\mathrm{i}(t-t_1)\Delta}(u(t_1) - v(t_1))\|_{\dot{B}^1(I_1)}\\ &\quad + CT^\alpha\|u\|^3_{\dot{B}^1(I_1)} + C\|u\|^{\frac{n+2}{n-2}}_{\dot{B}^1(I_1)}\\ &\leq 2\eta + C(E, M)\epsilon^c + CT^\alpha\|u\|^3_{\dot{B}^1(I_1)} + C\|u\|^{\frac{n+2}{n-2}}_{\dot{B}^1(I_1)}.\end{aligned}$$

A standard continuity argument then yields

$$\|u\|_{\dot{B}^1(I_0)} \leq 4\eta$$

provided that ϵ is sufficiently small and depends on E and M, which amounts to taking T to be sufficiently small depending on E and M. We apply Lemma 3.4 again to $I := I_1$ to obtain

$$\|u - v\|_{\dot{S}^1(I_1\times\mathbb{R}^n)} \leq C(E, M)\epsilon^c.$$

By induction argument, for every $0 \leq j \leq J' - 1$, we have

$$\|u\|_{\dot{B}^1(I_j)} \leq 4\eta \tag{4.11}$$

provided that ϵ (and hence T) is sufficiently small depending on E and M. Sum (4.11) over all $0 \leq j \leq J' - 1$ and notice that $J' < J = J(E, M)$. We obtain

$$\|u\|_{\dot{B}^1([0,T])} \leq 4J'\eta \leq C(E, M). \tag{4.12}$$

Using Strichartz estimates, (4.12) and $T = T(E, M)$, we get

$$\|u\|_{\dot{S}^1([0,T]\times\mathbb{R}^n)} \lesssim \|u_0\|_{\dot{H}^1_x} + T^\alpha\|u\|^3_{\dot{B}^1([0,T])} + \|u\|^{\frac{n+2}{n-2}}_{\dot{B}^1([0,T])} \leq C(E, M). \tag{4.13}$$

Similarly, we get

$$\|u\|_{\dot{S}^0([0,T]\times\mathbb{R}^n)} \lesssim M^{\frac{1}{2}} + C(E, M)\|u\|^2_{\dot{B}^1([0,T])}\|u\|_{\dot{S}^0([0,T])} + \|u\|^{\frac{4}{n-2}}_{\dot{B}^1([0,T])}\|u\|_{\dot{S}^0([0,T])}.$$

Subdivide $[0, T]$ into $N = N(E, M, \delta)$ subintervals J_k such that

$$\|u\|_{\dot{B}^1(J_k)} \sim \delta$$

for some small constant $\delta > 0$ to be chosen later. Thus, we get

$$\|u\|_{\dot{S}^0(J_k\times\mathbb{R}^n)} \lesssim M^{\frac{1}{2}} + C(E,M)\delta^2\|u\|_{\dot{S}^0(J_k\times\mathbb{R}^n)} + \delta^{\frac{4}{n-2}}\|u\|_{\dot{S}^0(J_k\times\mathbb{R}^n)}.$$

Let $C(E,M)\delta^2 + \delta^{\frac{4}{n-2}} \le \frac{1}{2}$. A standard continuity method then yields

$$\|u\|_{\dot{S}^0(J_k\times\mathbb{R}^n)} \le C(E,M).$$

Sum these bounds over all subintervals J_k. We get

$$\|u\|_{\dot{S}^0([0,T]\times\mathbb{R}^n)} \le C(E,M). \tag{4.14}$$

Combining (4.13) and (4.14), we get

$$\|u\|_{S^1([0,T]\times\mathbb{R}^n)} \le C(E,M),$$

where T only depends on energy and mass. So, if we divide the interval I into subintervals of length T, and sum up the corresponding S^1-bounds in these subintervals, the proof of Theorem 1.1 is completed.

Remark 4.3 If $p = \frac{4}{n-2}$ and $\gamma = 4$, then α is zero and $\|u\|_{\dot{B}^1(I_j)} \le 4\eta$ no longer holds. Therefore the method used in the proof is not applicable in such a case.

5 Results on Scattering

5.1 The interaction Morawetz inequality

First we state a proposition from [24].

Proposition 5.1 (General Interaction Morawetz Inequality)

$$\begin{aligned}
&-(n-1)\int_I\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\Delta\Big(\frac{1}{|x-y|}\Big)|u(y)|^2|u(x)|^2\,\mathrm{d}x\mathrm{d}y\mathrm{d}t\\
&+2\int_I\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|u(t,y)|^2\frac{x-y}{|x-y|}\{N,u\}_p(t,x)\,\mathrm{d}x\mathrm{d}y\mathrm{d}t\\
\le\ & 4\|u\|^3_{L_t^\infty L_x^2(I\times\mathbb{R}^n)}\|\nabla u\|_{L_t^\infty L_x^2(I\times\mathbb{R}^n)}\\
&+4\int_I\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|\{N,u\}_m(t,y)u(t,x)\nabla u(t,x)|\,\mathrm{d}x\mathrm{d}y\mathrm{d}t,
\end{aligned} \tag{5.1}$$

where $N := \lambda_1|u|^pu + \lambda_2(|x|^{-\gamma}*|u|^2)u$, $\{f,g\}_p := \mathrm{Re}(f\nabla\overline{g} - g\nabla\overline{f})$, $\{f,g\}_m = \mathrm{Im}\{f\overline{g}\}$.

Using this proposition, we can show the following result.

Proposition 5.2 (Morawetz Control) *Let I be a compact interval, λ_1 and λ_2 positive real numbers, and u a solution to* (1.1) *on the slab $I\times\mathbb{R}^n$. Then*

$$\|u\|_{Z(I)} \lesssim \|u\|_{L_t^\infty H_x^1(I\times\mathbb{R}^n)}. \tag{5.2}$$

Proof Let $N := \lambda_1|u|^pu + \lambda_2(|x|^{-\gamma}*|u|^2)u$. We have

$$\{N,u\}_m = 0, \quad \{N,u\}_p = -\frac{\lambda_1 p}{p+2}\nabla(|u|^{p+2}) - \lambda_2\mathrm{Re}\{\nabla(|x|^{-\gamma}*|u|^2)|u|^2\}.$$

If one can show that the second term on the left-hand side of (5.1) is positive, then

$$-\int_I\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\Delta\Big(\frac{1}{|x-y|}\Big)|u(y)|^2|u(x)|^2\,\mathrm{d}x\mathrm{d}y\mathrm{d}t \le \|u\|^4_{L_t^\infty H_x^1(I\times\mathbb{R}^n)}. \tag{5.3}$$

When $n=3$, we know $-\Delta\big(\frac{1}{|x|}\big)=4\pi\delta$. Hence (5.3) implies

$$\|u\|^4_{L^4_{t,x}(I\times\mathbb{R}^3)} \lesssim \|u\|^4_{L^\infty_t H^1_x(I\times\mathbb{R}^3)},$$

which is what we want to show.

When $n\geq 4$, we have $-\Delta(\frac{1}{|x|})=\frac{n-3}{|x|^3}$. Similarly (5.3) yields

$$\||\nabla|^{-\frac{n-3}{2}}|u|^2\|_{L^2_{t,x}(I\times\mathbb{R}^n)} \lesssim \|u\|^2_{L^\infty_t H^1_x(I\times\mathbb{R}^n)}. \tag{5.4}$$

By Lemma 4.1 and the above inequality, we can deduce

$$\||\nabla|^{-\frac{n-3}{4}}u\|_{L^4_{t,x}(I\times\mathbb{R}^n)} \lesssim \|u\|_{L^\infty_t H^1_x(I\times\mathbb{R}^n)}. \tag{5.5}$$

The result in this case follows from interpolation between (5.5) and the bound on the kinetic energy

$$\|\nabla u\|_{L^\infty_t L^2_x} \lesssim E^{\frac{1}{2}},$$

which is an immediate consequence of the conservation of energy when both nonlinearities are defocusing.

To show that the second term on the left-hand side of (5.1) is positive, we note that

$$\begin{aligned}
&\int_I\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|u(t,y)|^2\frac{x-y}{|x-y|}\{N,u\}_p(t,x)\,\mathrm{d}x\mathrm{d}y\mathrm{d}t\\
&=-\int_I\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|u(t,y)|^2\frac{x-y}{|x-y|}\frac{\lambda_1 p}{p+2}\nabla(|u|^{p+2})\,\mathrm{d}x\mathrm{d}y\mathrm{d}t\\
&\quad-\lambda_2\mathrm{Re}\int_I\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|u(t,y)|^2\frac{x-y}{|x-y|}\{\nabla(|x|^{-\gamma}*|u|^2)|u|^2\}\,\mathrm{d}x\mathrm{d}y\mathrm{d}t\\
&=(\mathrm{I})+(\mathrm{II}).
\end{aligned}$$

For (I), we have

$$(\mathrm{I})=(n-1)\frac{\lambda_1 p}{p+2}\int_I\int_{\mathbb{R}^n}\frac{|u(t,y)|^2|u(t,x)|^{p+2}}{|x-y|}\,\mathrm{d}x\mathrm{d}y\mathrm{d}t.$$

Note $\lambda_1>0$. Hence (I) is positive.

For (II), define $h(x)=\int_{\mathbb{R}^n}|u(t,y)|^2\frac{x-y}{|x-y|}\,\mathrm{d}y$. Then we have

$$(\mathrm{II})=\frac{1}{2}\lambda_2\gamma\mathrm{Re}\int_I\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{1}{|x-z|^{\gamma+2}}|u(t,z)|^2|u(t,x)|^2[(x-z)(h(x)-h(z))]\,\mathrm{d}x\mathrm{d}y\mathrm{d}t.$$

Notice that

$$(x-z)(h(x)-h(z))=(x-z)\int_{\mathbb{R}^n}|u(t,y)|^2\Big(\frac{x-y}{|x-y|}-\frac{z-y}{|z-y|}\Big)\,\mathrm{d}y \tag{5.6}$$

and denote $a:=x-y,\ b:=z-y$. Then (5.6) equals

$$\int_{\mathbb{R}^n}|u(t,y)|^2(a-b)\Big(\frac{a}{|a|}-\frac{b}{|b|}\Big)\,\mathrm{d}y.$$

Since $(a-b)(\frac{a}{|a|}-\frac{b}{|b|})=(|a||b|-ab)(\frac{1}{|a|}+\frac{1}{|b|})\geq 0$ and $\lambda_2>0$, we have that (II) is positive.

Remark 5.1 When the space dimension $n=2$, we do not know whether $-\Delta(\frac{1}{|x|})$ is positive or not. However, J. Colliander, M. Grillakis and N. Tzirakis used the refined tensor product to prove that (5.4) still holds for $n=2$ (see [4, 9]), and hence the corresponding (5.2) and (2.4) are also true. We can employ the same approach used in Section 5.3 to show the scattering of the power type. However, the corresponding (2.5) no longer holds, for we need $\gamma>2$, but in this case $\gamma<n=2$. Therefore the scattering of the Hartree type cannot be obtained.

5.2 Global bounds in the case: $p = \frac{4}{n}$, $2 < \gamma < \min\{n, 4\}$ and $\lambda_1, \lambda_2 > 0$ or $\frac{4}{n} < p < \frac{4}{n-2}$, $\gamma = 2$ and $\lambda_1, \lambda_2 > 0$

Since both cases can be treated by the same method, here we only discuss the first case. Without loss of generality, let $\lambda_1 = \lambda_2 = 1$.

We view the second nonlinearity as a perturbation to (1.3). By Proposition (5.2) and the conservation of the energy and mass, we have

$$\|u\|_{Z(\mathbb{R})} \lesssim \|u\|_{L_t^\infty H_x^1(\mathbb{R}\times\mathbb{R}^n)} \leq C(E, M).$$

Divide $\mathbb{R}$ into $J = J(E, M, \varepsilon)$ subintervals I_j, $0 \leq j \leq J-1$, such that

$$\|u\|_{Z(I_j)} \sim \varepsilon,$$

where ε is a small positive constant to be chosen later.

In the slab $I \times \mathbb{R}^n$, define

$$\dot{\tilde{X}}^0(I) := L_t^{2+\frac{1}{\theta}} L_x^{\frac{2n(2\theta+1)}{n(2\theta+1)-4\theta}}(I\times\mathbb{R}^n) \cap V(I),$$

where θ is introduced in Lemma 2.5.

In each I_j $(0 \leq j \leq J-1)$, by (2.6) we have

$$\begin{aligned} \|(|x|^{-\gamma} * |u|^2)u\|_{\dot{N}^0(I_j\times\mathbb{R}^n)} &\lesssim \|u\|_{L_t^{2+\frac{1}{\theta}} L_x^{\frac{2n(2\theta+1)}{n(2\theta+1)-4\theta}}(I_j\times\mathbb{R}^n)} \|u\|_{Z(I_j)}^{\frac{n+1}{2(2\theta+1)}} \|u\|_{L_t^\infty H_x^1(I_j\times\mathbb{R}^n)}^{\beta_1(\theta)+\beta_2(\theta)} \\ &\leq C(E, M)\varepsilon^c \|u\|_{\dot{\tilde{X}}^0(I_j)}, \end{aligned} \tag{5.7}$$

where $c = \frac{n+1}{2(2\theta+1)}$.

In the rest of the section, we fix an interval $I_{j_0} = [a, b]$ and prove that u admits good Strichartz estimates in the slab $I_{j_0} \times \mathbb{R}^n$. Let v be a solution to

$$\begin{cases} \mathrm{i}v_t + \Delta v = |v|^{\frac{4}{n}} v, \\ v(a) = u(a). \end{cases}$$

As this initial value problem is globally well-posed in H_x^1, and by Assumption 1.1 and Lemma 3.6, the unique solution v satisfies

$$\|v\|_{\dot{S}^0(\mathbb{R}\times\mathbb{R}^n)} \leq C(M).$$

Subdivide $\mathbb{R}$ into $K = K(M, \eta)$ subinterval J_k such that on each J_k

$$\|v\|_{\dot{\tilde{X}}^0(J_k)} \sim \eta \tag{5.8}$$

for a small constant $\eta > 0$ to be chosen later.

We are only interested in the subintervals $J_k = [t_k, t_{k+1}]$ such that its intersection with I_{j_0} is nonempty. Without loss of generality, we assume that $[a, b] = \bigcup\limits_{k=0}^{k'-1} J_k$, $t_0 = a$, $t_{k'} = b$.

In each J_k, by Strichartz estimates and (5.4), we get

$$\|\mathrm{e}^{\mathrm{i}(t-t_k)\Delta} v(t_k)\|_{\dot{\tilde{X}}^0(J_k)} \leq \|v\|_{\dot{\tilde{X}}^0(J_k)} + C\||v|^{\frac{4}{n}} v\|_{\dot{N}^0(J_k\times\mathbb{R}^n)} \leq \eta + C\|v\|_{V(J_k)}^{1+\frac{4}{n}} \leq \eta + C\eta^{1+\frac{4}{n}}.$$

Choosing η to be sufficiently small, we have

$$\|\mathrm{e}^{\mathrm{i}(t-t_k)\Delta} v(t_k)\|_{\dot{\tilde{X}}^0(J_k)} \leq 2\eta. \tag{5.9}$$

Next we will use Lemma 3.5 to obtain an estimate for the S^1-norm of u in $I_{j_0} \times \mathbb{R}^n$. In the interval J_0, notice that $u(t_0) = v(t_0)$. We apply Strichartz estimates, (5.7) and (5.9) to get

$$\begin{aligned}\|u\|_{\dot{\tilde{X}}^0(J_0)} &\le \|e^{i(t-t_0)\Delta}u(t_0)\|_{\dot{\tilde{X}}^0(J_0)} + C\|u\|^{1+\frac{4}{n}}_{\dot{\tilde{X}}^0(J_0)} + C(E,M)\varepsilon^c\|u\|_{\dot{\tilde{X}}^0(J_0)} \\ &\le 2\eta + C\|u\|^{1+\frac{4}{n}}_{\dot{\tilde{X}}^0(J_0)} + C(E,M)\varepsilon^c\|u\|_{\dot{\tilde{X}}^0(J_0)}.\end{aligned}$$

Using a standard continuity argument, one has

$$\|u\|_{\dot{\tilde{X}}^0(J_0)} \le 4\eta$$

provided η and ε are sufficiently small. In order to use Lemma 3.5, we notice that (3.39) holds in $I := J_0$ for $L_0 := 4\eta$, (3.37) holds with $M_0' = 0$. It suffices to show that the error is sufficiently small. In fact, from

$$\|e\|_{\dot{N}^0(J_0\times\mathbb{R}^n)} \le C(E,M)\varepsilon^c\|u\|_{\dot{\tilde{X}}^0(J_0)} \le C(E,M)\eta\varepsilon^c,$$

and choosing ε to be sufficiently small, we obtain

$$\|u-v\|_{\dot{S}^0(J_0\times\mathbb{R}^n)} \le \varepsilon^{\frac{c}{2}}.$$

By Strichartz estimates, we have

$$\|u(t_1)-v(t_1)\|_{L^2_x} \le \varepsilon^{\frac{c}{2}}, \quad \|e^{i(t-t_1)\Delta}(u(t_1)-v(t_1))\|_{\dot{\tilde{X}}^0(J_1)} \lesssim \varepsilon^{\frac{c}{2}}. \tag{5.10}$$

On the other hand, we have

$$\begin{aligned}\|u\|_{\dot{S}^1(J_0\times\mathbb{R}^n)} &\lesssim \|u(a)\|_{\dot{H}^1_x} + \|u\|^{\frac{4}{n}}_{V(J_0)}\|u\|_{\dot{S}^1(J_0\times\mathbb{R}^n)} + \|(|x|^{-\gamma}*|u|^2)u\|_{\dot{N}^1(I\times\mathbb{R}^n)} \\ &\lesssim C(E) + (4\eta)^{\frac{4}{n}}\|u\|_{\dot{S}^1(J_0\times\mathbb{R}^n)} + C(E,M)\varepsilon^c\|u\|_{\dot{S}^1(J_0\times\mathbb{R}^n)}.\end{aligned}$$

Assuming that η and ε are sufficiently small, we have

$$\|u\|_{\dot{S}^1(J_0\times\mathbb{R}^n)} \le C(E).$$

Again applying Strichartz estimates, we find that (5.7) and (5.10) to the intervals J_1 yield

$$\begin{aligned}\|u\|_{\dot{\tilde{X}}^0(J_1)} \le{}& \|e^{i(t-t_1)\Delta}v(t_1)\|_{\dot{\tilde{X}}^0(J_1)} + \|e^{i(t-t_1)\Delta}(u(t_1)-v(t_1))\|_{\dot{\tilde{X}}^0(J_1)} \\ &+ C\|u\|^{1+\frac{4}{n}}_{\dot{\tilde{X}}^0(J_1)} + C(E,M)\varepsilon^c\|u\|_{\dot{\tilde{X}}^0(J_1)} \\ \le{}& 2\eta + \varepsilon^{\frac{c}{2}} + C\|u\|^{1+\frac{4}{n}}_{\dot{\tilde{X}}^0(J_1)} + C(E,M)\varepsilon^c\|u\|_{\dot{\tilde{X}}^0(J_1)}.\end{aligned}$$

With η and ε sufficiently small, we obtain

$$\|u\|_{\dot{\tilde{X}}^0(J_1)} \le 4\eta.$$

In a way similar to the proof in I_0, we choose a sufficiently small ε and use Lemma 3.5 to derive

$$\|u-v\|_{\dot{S}^0(J_1\times\mathbb{R}^n)} \le \varepsilon^{\frac{c}{4}}.$$

The same arguments yield

$$\|u\|_{\dot{S}^1(J_1\times\mathbb{R}^n)} \le C(E).$$

By induction, for each $0 \le k \le k'-1$, we get

$$\|u-v\|_{\dot{S}^0(J_k\times\mathbb{R}^n)} \le \varepsilon^{\frac{c}{2^{k+1}}},$$
$$\|u\|_{\dot{S}^1(J_k\times\mathbb{R}^n)} \le C(E).$$

Adding these estimates over all the subintervals J_k, we obtain

$$\|u\|_{\dot{S}^0(I_{j_0}\times\mathbb{R}^n)} \le \|v\|_{\dot{S}^0(I_{j_0}\times\mathbb{R}^n)} + \sum_{k=0}^{k'-1}\|u-v\|_{\dot{S}^0(J_k\times\mathbb{R}^n)} \le C(E,M),$$
$$\|u\|_{\dot{S}^1(I_{j_0}\times\mathbb{R}^n)} \le \sum_{k=0}^{k'-1}\|u\|_{\dot{S}^1(J_k\times\mathbb{R}^n)} \le C(E,M).$$

Since intervals I_{j_0} are arbitrary, we have

$$\|u\|_{\dot{S}^0(\mathbb{R}\times\mathbb{R}^n)} \le \sum_{j=0}^{J-1}\|u\|_{\dot{S}^0(I_j\times\mathbb{R}^n)} \le C(E,M),$$
$$\|u\|_{\dot{S}^1(\mathbb{R}\times\mathbb{R}^n)} \le \sum_{j=0}^{J-1}\|u\|_{\dot{S}^1(I_j\times\mathbb{R}^n)} \le C(E,M).$$

Hence

$$\|u\|_{S^1(\mathbb{R}\times\mathbb{R}^n)} \le C(E,M).$$

5.3 Global bounds in the case: $\frac{4}{n} < p < \frac{4}{n-2}$, $2 < \gamma < \min\{n,4\}$ and $\lambda_1, \lambda_2 > 0$

The results were proved in [2] with a more complicated argument. Here we present a simpler proof using the interaction Morawetz estimate.

By Proposition 5.2, we have

$$\|u\|_{Z(\mathbb{R})} \lesssim \|u\|_{L_t^\infty H_x^1(\mathbb{R}\times\mathbb{R}^n)} \le C(E,M).$$

Divide $\mathbb{R}$ into $J = J(E,M,\eta)$ subintervals $I_j = [t_j, t_{j+1}]$ such that

$$\|u\|_{Z(I_j)} \sim \eta,$$

where $\eta > 0$ is a small constant to be chosen later.

Applying Strichartz estimates and Lemma 2.5 to each I_j, we can deduce

$$\begin{aligned}\|u\|_{S^1(I_j\times\mathbb{R}^n)} &\lesssim \|u(t_j)\|_{H_x^1} + \eta^{\frac{n+1}{2(2\theta+1)}}\|u\|^{\alpha_1(\theta)+\alpha_2(\theta)}_{L_t^\infty H_x^1(I_j\times\mathbb{R}^n)}\|u\|_{S^1(I_j\times\mathbb{R}^n)}\\ &\quad + \eta^{\frac{n+1}{2(2\theta+1)}}\|u\|^{\beta_1(\theta)+\beta_2(\theta)}_{L_t^\infty H_x^1(I_j\times\mathbb{R}^n)}\|u\|_{S^1(I_j\times\mathbb{R}^n)}\\ &\lesssim C(E,M) + \eta^{\frac{n+1}{2(2\theta+1)}}C(E,M)\|u\|_{S^1(I_j\times\mathbb{R}^n)}\\ &\quad + \eta^{\frac{n+1}{2(2\theta+1)}}C(E,M)\|u\|_{S^1(I_j\times\mathbb{R}^n)}.\end{aligned}$$

Assuming η to be sufficiently small, we have

$$\|u\|_{S^1(I_j\times\mathbb{R}^n)} \le C(E,M)$$

and

$$\|u\|_{S^1(\mathbb{R}\times\mathbb{R}^n)} \le \sum_{j=0}^{J-1}\|u\|_{S^1(I_j\times\mathbb{R}^n)} \le C(E,M).$$

5.4 Global bounds in the case: $\frac{4}{n} < p < \frac{4}{n-2}$, $\gamma = 4$ with $n \geq 5$ and $\lambda_1, \lambda_2 > 0$ or $p = \frac{4}{n-2}$, $2 < \gamma < \min\{n, 4\}$ and $\lambda_1, \lambda_2 > 0$

Due to the same reason as in Subsection 5.3, we present the first case and the other can be done similarly. In the slab $I \times \mathbb{R}^n$, define

$$\dot{\tilde{Y}}^0(I) := L_t^{2+\frac{1}{\theta}} L_x^{\frac{2n(2\theta+1)}{n(2\theta+1)-4\theta}}(I \times \mathbb{R}^n) \cap L_t^6 L_x^{\frac{6n}{3n-2}}(I \times \mathbb{R}^n),$$

where θ is introduced in Lemma 2.5. Replace $\dot{\tilde{X}}^0(I)$ by $\dot{\tilde{Y}}^0(I)$ in Subsection 5.2, and Lemma 3.5 by Lemma 3.2, then apply the same approach used in Subsection 5.2. One can get

$$\|u\|_{S^1(\mathbb{R}\times\mathbb{R}^n)} \leq C(E, M).$$

5.5 Global bounds in the case: $p = \frac{4}{n-2}$, $\gamma = 2$ and $\lambda_1, \lambda_2 > 0$ or $p = \frac{4}{n}$, $\gamma = 4$ with $n \geq 5$ and $\lambda_1, \lambda_2 > 0$

Similarly, it suffices to discuss the first case. Without loss of generality, let $\lambda_1 = \lambda_2 = 1$. The idea is to decompose u into the low frequency part $u_{\rm lo}$ and the high frequency part $u_{\rm hi}$. One can view the former as a perturbation of mass-critical NLS, and the latter as the H_x^1-critical NLS. Finally, we get the finite global Strichartz bounds in this case.

Let

$$0 < \eta_3 \ll \eta_2 \ll \eta_1 \ll 1,$$

where η_j may depend on the energy, the mass and any η_i greater than η_j. By Proposition 5.2 and conservation of energy and mass, we have

$$\|u\|_{Z(\mathbb{R})} \leq C(E, M).$$

We divide $\mathbb{R}$ into $K = K(E, M, \eta_3)$ subintervals J_k such that in each slab $J_k \times \mathbb{R}^n$ we have

$$\|u\|_{Z(J_k)} \sim \eta_3. \tag{5.11}$$

Fix $J_{k_0} = [a, b]$. For every $t \in J_{k_0}$, write $u(t) = u_{\rm lo}(t) + u_{\rm hi}(t)$, where $u_{\rm lo}(t) := P_{<\eta_2^{-1}} u(t)$, $u_{\rm hi}(t) := P_{\geq \eta_2^{-1}} u(t)$.

In the slab $J_{k_0} \times \mathbb{R}^n$, we view $u_{\rm lo}(t)$ as the solution to the following L_x^2-critical Hartree NLS

$$\begin{cases} (\mathrm{i}\partial_t + \Delta) v = (|x|^{-2} * |v|^2) v, \\ v(a) = u_{\rm lo}(a), \end{cases}$$

which is globally well-posed in H_x^1. Moreover, by Assumption 1.2, one has

$$\|v\|_{U(\mathbb{R})} \leq C(\|u_{\rm lo}(a)\|_{L_x^2}) \leq C(M).$$

By Lemma 3.6, we have

$$\|v\|_{\dot{S}^0(\mathbb{R}\times\mathbb{R}^n)} \leq C(M), \tag{5.12}$$

$$\|v\|_{\dot{S}^1(\mathbb{R}\times\mathbb{R}^n)} \leq C(E, M). \tag{5.13}$$

Furthermore, we divide $J_{k_0} = [a, b]$ into $J = J(M, \eta_1)$ subintervals $I_j = [t_{j-1}, t_j]$ with $t_0 = a$, $t_J = b$ such that

$$\|v\|_{U(I_j)} \sim \eta_1. \tag{5.14}$$

Proceeding inductively for each $j = 1, \cdots, J$, we can establish

$$P(j): \begin{cases} \|u_{\rm lo} - v\|_{\dot S^0([t_0,t_j])} \le \eta_2^{1-2\delta}, \\ \|u_{\rm hi}\|_{\dot S^1(I_l)} \le L(E) \quad \text{for every } 1 \le l \le j, \\ \|u\|_{S^1([t_0,t_j])} \le C(\eta_1, \eta_2), \end{cases} \tag{5.15}$$

where $\delta > 0$ is a small constant, and $L(E)$ is a large quantity to be chosen later which depends only on E (not on any η_j). As the method of checking that (5.15) holds for $j = 1$ is similar to that in inductive step, i.e., showing that $P(j)$ implies $P(j+1)$, we will only prove the latter.

Assume that (5.15) is true for some $1 \le j < J$. We want to show

$$\begin{cases} \|u_{\rm lo} - v\|_{\dot S^0([t_0,t_{j+1}])} \le \eta_2^{1-2\delta}, \\ \|u_{\rm hi}\|_{\dot S^1(I_l)} \le L(E) \quad \text{for every } 1 \le l \le j+1, \\ \|u\|_{S^1([t_0,t_{j+1}])} \le C(\eta_1, \eta_2). \end{cases} \tag{5.16}$$

Let Ω_1 be the set of time $T \in I_{j+1}$ such that

$$\|u_{\rm lo} - v\|_{\dot S^0([t_0,T])} \le \eta_2^{1-2\delta}, \tag{5.17}$$

$$\|u_{\rm hi}\|_{\dot S^1([t_j,T])} \le L(E), \tag{5.18}$$

$$\|u\|_{S^1([t_0,T])} \le C(\eta_1, \eta_2). \tag{5.19}$$

In order to prove $\Omega_1 = I_{j+1}$, we notice that Ω_1 is nonempty (as $t_j \in \Omega_1$) and closed (by Fatou). Let Ω_2 be the set of all times $T \in I_{j+1}$ such that

$$\|u_{\rm lo} - v\|_{\dot S^0([t_0,T])} \le 2\eta_2^{1-2\delta}, \tag{5.20}$$

$$\|u_{\rm hi}\|_{\dot S^1([t_j,T])} \le 2L(E), \tag{5.21}$$

$$\|u\|_{S^1([t_0,T])} \le 2C(\eta_1, \eta_2). \tag{5.22}$$

We will show $\Omega_2 \subset \Omega_1$, which will conclude the argument.

Lemma 5.1 *Let $T \in \Omega_2$. Then, the following properties hold:*

$$\|u_{\rm lo}\|_{U(I)} \lesssim \eta_1, \tag{5.23}$$

$$\|u_{\rm lo}\|_{\dot S^0([t_0,T]\times\mathbb{R}^n)} \le C(M), \tag{5.24}$$

$$\|u_{\rm lo}\|_{W([t_j,T])} \lesssim \eta_2, \tag{5.25}$$

$$\|u_{\rm lo}\|_{\dot S^1(I\times\mathbb{R}^n)} \lesssim E, \tag{5.26}$$

$$\|u_{\rm lo}\|_{\dot S^1([t_0,T]\times\mathbb{R}^n)} \lesssim C(\eta_1)E, \tag{5.27}$$

$$\|u_{\rm hi}\|_{\dot S^0(I\times\mathbb{R}^n)} \lesssim \eta_2 L(E), \tag{5.28}$$

$$\|u_{\rm hi}\|_{\dot S^0([t_0,T]\times\mathbb{R}^n)} \lesssim \eta_2 C(\eta_1) L(E), \tag{5.29}$$

$$\|u_{\rm hi}\|_{\dot S^1([t_0,T]\times\mathbb{R}^n)} \lesssim C(\eta_1) L(E), \tag{5.30}$$

where $I \in \{I_l, 1 \le l \le j\} \cup \{[t_j, T]\}$.

Proof Using (5.12), (5.14), (5.20) and Bernstein inequality, we have

$$\begin{aligned} \|u_{\rm lo}\|_{U(I)} &\le \|u_{\rm lo} - v\|_{U(I)} + \|v\|_{U(I)} \lesssim \eta_2^{(1-2\delta)} + \eta_1 \lesssim \eta_1, \\ \|u_{\rm lo}\|_{\dot S^0([t_0,T]\times\mathbb{R}^n)} &\le \|u_{\rm lo} - v\|_{\dot S^0([t_0,T]\times\mathbb{R}^n)} + \|v\|_{\dot S^0([t_0,T]\times\mathbb{R}^n)} \lesssim \eta_2^{1-2\delta} + C(M) \le C(M), \\ \|u_{\rm hi}\|_{\dot S^0(I\times\mathbb{R}^n)} &\lesssim \eta_2 \|u_{\rm hi}\|_{\dot S^1(I\times\mathbb{R}^n)} \lesssim \eta_2 L(E). \end{aligned}$$

Therefore, (5.23), (5.24) and (5.28) hold. In view of $J = O(\eta_1^{-C})$, we get

$$\begin{aligned}
\|u_{\rm hi}\|_{\dot S^1([t_0,T]\times\mathbb{R}^n)} &\lesssim \sum_{l=1}^{j} \|u_{\rm hi}\|_{\dot S^1(I_l\times\mathbb{R}^n)} + \|u_{\rm hi}\|_{\dot S^1([t_j,T]\times\mathbb{R}^n)} \\
&\le C(\eta_1)L(E) + \eta_2 L(E) \le C(\eta_1)L(E), \\
\|u_{\rm hi}\|_{\dot S^0([t_0,T]\times\mathbb{R}^n)} &\lesssim \eta_2 \|u_{\rm hi}\|_{\dot S^1([t_0,T]\times\mathbb{R}^n)} \le \eta_2 C(\eta_1)L(E).
\end{aligned}$$

Hence, (5.29) and (5.30) hold. In the slab $I\times\mathbb{R}^n$, $u_{\rm lo}$ satisfies the equation

$$u_{\rm lo}(t) = \mathrm{e}^{\mathrm{i}(t-t_l)\Delta} u_{\rm lo}(t_l) - \mathrm{i}\int_{t_l}^{t} \mathrm{e}^{\mathrm{i}(t-s)\Delta} P_{\rm lo}(|u|^{\frac{4}{n-2}}u + (|x|^{-2} * |u|^2)u)(s)\,\mathrm{d}s,$$

where $0\le l\le j$. Then by Strichartz estimate

$$\|u_{\rm lo}\|_{\dot S^1(I\times\mathbb{R}^n)} \lesssim \|u_{\rm lo}(t_l)\|_{\dot H^1_x} + \|P_{\rm lo}(|u|^{\frac{4}{n-2}}u)\|_{\dot N^1(I\times\mathbb{R}^n)} + \|P_{\rm lo}((|x|^{-2} * |u|^2)u)\|_{\dot N^1(I\times\mathbb{R}^n)}.$$

By Bernstein inequality, Lemma 2.6, (5.11) and (5.22), we have

$$\begin{aligned}
\|P_{\rm lo}(|u|^{\frac{4}{n-2}}u)\|_{\dot N^1(I\times\mathbb{R}^n)} &\lesssim \eta_2^{-1} \||u|^{\frac{4}{n-2}}u\|_{\dot N^0(I\times\mathbb{R}^n)} \\
&\lesssim \eta_2^{-1}\|u\|^{\rho}_{Z(I)} \|u\|^{\frac{n+2}{n-2}-\rho}_{S^1(I\times *\mathbb{R}^n)} \lesssim \eta_2^{-1}\eta_3^{\rho} C(\eta_1,\eta_2) \le \eta_2,
\end{aligned}$$

where η_3 is sufficiently small and depends on η_1 and η_2.

Hölder and Hardy-Littlewood-Sobolev inequality, together with (5.21), (5.23) and (5.28), implies

$$\begin{aligned}
\|P_{\rm lo}((|x|^{-2} * |u|^2)u)\|_{\dot N^1(I\times\mathbb{R}^n)} &\lesssim \|u\|^2_{U(I)} \|\nabla u\|_{U(I)} \\
&\lesssim \|u_{\rm lo}\|^2_{U(I)}\|\nabla u_{\rm lo}\|_{U(I)} + \|u_{\rm hi}\|^2_{U(I)}\|\nabla u_{\rm hi}\|_{U(I)} \\
&\quad + \|u_{\rm lo}\|^2_{U(I)}\|\nabla u_{\rm hi}\|_{U(I)} + \|u_{\rm hi}\|^2_{U(I)}\|\nabla u_{\rm lo}\|_{U(I)} \\
&\lesssim \eta_1^2\|u_{\rm lo}\|_{\dot S^1(I\times\mathbb{R}^n)} + (\eta_2 L(E))^2 L(E) \\
&\quad + \eta_1^2 L(E) + (\eta_2 L(E))^2 \|u_{\rm lo}\|_{\dot S^1(I\times\mathbb{R}^n)}.
\end{aligned}$$

Then $\|u_{\rm lo}\|_{\dot S^1(I\times\mathbb{R}^n)} \lesssim E + \eta_2 + (\eta_2 L(E))^2 L(E) + \eta_1^2 L(E) + (\eta_1^2 + (\eta_2 L(E))^2)\|u_{\rm lo}\|_{\dot S^1(I\times\mathbb{R}^n)}$.

Taking η_1 and η_2 to be sufficiently small depending on E, we can derive

$$\|u_{\rm lo}\|_{\dot S^1(I\times\mathbb{R}^n)} \lesssim E.$$

Thus (5.26) holds. Since $J = C(\eta_1)$, (5.27) follows from (5.26).

Finally, we are ready to show that (5.25) is true. We write $u_{\rm lo} = P_{\le\eta_2}u_{\rm lo} + P_{\eta_2<\cdot<\eta_2^{-1}}u_{\rm lo}$.

When $n\ge 5$, by interpolation, Sobolev embedding, Bernstein inequality, (5.11) and (5.26), we have

$$\begin{aligned}
&\|P_{\eta_2<\cdot<\eta_2^{-1}}u_{\rm lo}\|_{W([t_j,T])} \\
\lesssim{}& \|P_{\eta_2<\cdot<\eta_2^{-1}}u_{\rm lo}\|^c_{L_t^{n+1}L_x^{\frac{2n(n+1)}{n^2-n-6}}([t_j,T]\times\mathbb{R}^n)} \|P_{\eta_2<\cdot<\eta_2^{-1}}u_{\rm lo}\|^{1-c}_{L_t^2 L_x^{\frac{2n}{n-4}}([t_j,T]\times\mathbb{R}^n)} \\
\lesssim{}& \||\nabla|^{\frac{3}{n+1}} P_{\eta_2<\cdot<\eta_2^{-1}}u_{\rm lo}\|^c_{Z([t_j,T])} \|u_{\rm lo}\|^{1-c}_{\dot S^1([t_j,T]\times\mathbb{R}^n)} \\
\lesssim{}& \eta_2^{-\frac{3}{n+1}} \|u_{\rm lo}\|^c_{Z([t_j,T])} E^{1-c} \\
\lesssim{}& \eta_2^{-\frac{3}{n+1}} \eta_3^c E^{1-c} \\
\le{}& \eta_2,
\end{aligned}$$

where $c = \frac{4(n+1)}{(n-1)(n+2)}$.

When $n = 4$, using interpolation, Sobolev embedding, Bernstein inequality, the conservation of energy and (5.11), we get

$$\begin{aligned}\|P_{\eta_2<\cdot<\eta_2^{-1}}u_{\mathrm{lo}}\|_{W([t_j,T])} &\lesssim \|P_{\eta_2<\cdot<\eta_2^{-1}}u_{\mathrm{lo}}\|^{\frac{5}{6}}_{L_t^5 L_x^{\frac{20}{3}}([t_j,T]\times\mathbb{R}^n)}\|P_{\eta_2<\cdot<\eta_2^{-1}}u_{\mathrm{lo}}\|^{\frac{1}{6}}_{L_t^\infty L_x^4([t_j,T]\times\mathbb{R}^n)}\\ &\lesssim \||\nabla|^{\frac{3}{5}}P_{\eta_2<\cdot<\eta_2^{-1}}u_{\mathrm{lo}}\|^{\frac{5}{6}}_{Z([t_j,T])}E^{\frac{1}{6}}\\ &\lesssim (\eta_2^{-\frac{3}{5}}\eta_3)^{\frac{5}{6}}E^{\frac{1}{6}}\\ &\le \eta_2.\end{aligned}$$

Similarly, for $n = 3$, we have

$$\begin{aligned}\|P_{\eta_2<\cdot<\eta_2^{-1}}u_{\mathrm{lo}}\|_{W([t_j,T])} &\lesssim \|P_{\eta_2<\cdot<\eta_2^{-1}}u_{\mathrm{lo}}\|^{\frac{2}{5}}_{L_t^4 L_x^\infty([t_j,T]\times\mathbb{R}^n)}\|P_{\eta_2<\cdot<\eta_2^{-1}}u_{\mathrm{lo}}\|^{\frac{3}{5}}_{L_t^\infty L_x^6([t_j,T]\times\mathbb{R}^n)}\\ &\lesssim \|(1+|\nabla|)^{\frac{3}{4}+\epsilon}P_{\eta_2<\cdot<\eta_2^{-1}}u_{\mathrm{lo}}\|^{\frac{2}{5}}_{Z([t_j,T])}E^{\frac{3}{5}}\\ &\lesssim (\eta_2^{-\frac{3}{4}}\eta_3)^{\frac{2}{5}}E^{\frac{3}{5}}\\ &\le \eta_2.\end{aligned}$$

Hence, for any $n \ge 3$, we have

$$\|P_{\eta_2<\cdot<\eta_2^{-1}}u_{\mathrm{lo}}\|_{W([t_j,T])} \le \eta_2.$$

By Sobolev embedding, Bernstein inequality and (5.24), we have

$$\|P_{\le\eta_2}u_{\mathrm{lo}}\|_{W([t_j,T])} \lesssim \|\nabla P_{\le\eta_2}u_{\mathrm{lo}}\|_{L_t^{\frac{2(n+2)}{n-2}} L_x^{\frac{2n(n+2)}{n^2+4}}([t_j,T]\times\mathbb{R}^n)} \lesssim \eta_2\|u_{\mathrm{lo}}\|_{L_t^{\frac{2(n+2)}{n-2}} L_x^{\frac{2n(n+2)}{n^2+4}}([t_j,T]\times\mathbb{R}^n)}.$$

For $n = 3$, by interpolation, (5.24) and the conservation of mass, we get

$$\|P_{\le\eta_2}u_{\mathrm{lo}}\|_{W([t_j,T])} \lesssim \eta_2\|u_{\mathrm{lo}}\|^{\frac{3}{5}}_{U([t_j,T])}\|u_{\mathrm{lo}}\|^{\frac{2}{5}}_{L_t^\infty L_x^2([t_j,T]\times\mathbb{R}^n)} \lesssim \eta_2\eta_1^{\frac{3}{5}}M^{\frac{2}{5}} \le \eta_2$$

provided that η_1 is a sufficiently small constant depending on M.

For $n = 4$, since $L_t^{\frac{2(n+2)}{n-2}} L_x^{\frac{2n(n+2)}{n^2+4}} = U$, we have

$$\|P_{\le\eta_2}u_{\mathrm{lo}}\|_{W([t_j,T])} \lesssim \eta_2\eta_1 \le \eta_2.$$

For $n \ge 5$, by interpolation, (5.23) and (5.24), we have

$$\begin{aligned}\|P_{\le\eta_2}u_{\mathrm{lo}}\|_{W([t_j,T])} &\lesssim \eta_2\|u_{\mathrm{lo}}\|^{\frac{6}{n+2}}_{U([t_j,T])}\|u_{\mathrm{lo}}\|^{\frac{n-4}{n+2}}_{L_t L_x^{\frac{2n}{n-2}}([t_j,T]\times\mathbb{R}^n)}\\ &\lesssim \eta_2\eta_1^{\frac{6}{n+2}}\|u_{\mathrm{lo}}\|^{\frac{n-4}{n+2}}_{\dot S^0([t_j,T]\times\mathbb{R}^n)} \lesssim \eta_2\eta_1^{\frac{6}{n+2}}C(M) \le \eta_2.\end{aligned}$$

Hence, For all $n \ge 3$, we get

$$\|P_{\le\eta_2}u_{\mathrm{lo}}\|_{W([t_j,T])} \le \eta_2.$$

Therefore, by the triangle inequality, (5.25) is true.

Now it remains to show $\Omega_2 \subset \Omega_1$. We will first show (5.15). The method is to treat u_{lo} as v via the perturbation result of Lemma 3.3. Note that u_{lo} satisfies the following initial value problem in the slab $[t_0,T]\times\mathbb{R}^n$,

$$\begin{cases}(\mathrm{i}\partial_t+\Delta)u_{\mathrm{lo}} = (|x|^{-2}*|u_{\mathrm{lo}}|^2)u_{\mathrm{lo}} + P_{\mathrm{lo}}(|u|^{\frac{4}{n-2}}u)\\ \qquad\qquad +P_{\mathrm{lo}}[(|x|^{-2}*|u|^2)u-(|x|^{-2}*|u_{\mathrm{lo}}|^2)u_{\mathrm{lo}}]-P_{\mathrm{hi}}((|x|^{-2}*|u_{\mathrm{lo}}|^2)u_{\mathrm{lo}}),\\ u_{\mathrm{lo}}(t_0) = u_{\mathrm{lo}}(a).\end{cases}$$

Since (5.24) and $v(t_0) = u_{\rm lo}(t_0)$, in order to use Lemma 3.3, we only need to show that the error term

$$e = P_{\rm lo}(|u|^{\frac{4}{n-2}}u) + P_{\rm lo}[(|x|^{-2} * |u|^2)u - (|x|^{-2} * |u_{\rm lo}|^2)u_{\rm lo}] - P_{\rm hi}((|x|^{-2} * |u_{\rm lo}|^2)u_{\rm lo})$$

is small in $\dot{N}^0([t_0, T] \times \mathbb{R}^n)$.

By Lemma 2.6, (5.11) and (5.22), we have

$$\|P_{\rm lo}(|u|^{\frac{4}{n-2}}u)\|_{\dot{N}^0([t_0,T]\times\mathbb{R}^n)} \lesssim \|u\|^{\theta}_{Z([t_0,T])}\|u\|^{\frac{n+2}{n-2}-\theta}_{\dot{S}^1([t_0,T]\times\mathbb{R}^n)} \lesssim \eta_3^{\theta}(C(\eta_1,\eta_2))^{\frac{n+2}{n-2}-\theta} \le \eta_2^{1-\delta},$$

if η_3 is sufficiently small and depends on η_1 and η_2. By Bernstein inequality, Hölder inequality, Hardy-littlewood-Sobolev inequality, (5.24) and (5.27), we have

$$\begin{aligned}\|P_{\rm hi}((|x|^{-2} * |u_{\rm lo}|^2)u_{\rm lo})\|_{\dot{N}^0([t_0,T]\times\mathbb{R}^n)} &\lesssim \eta_2\|u_{\rm lo}\|^2_{U([t_0,T])}\|\nabla u_{\rm lo}\|_{U([t_0,T])}\\ &\lesssim \eta_2\|u_{\rm lo}\|^2_{\dot{S}^0([t_0,T]\times\mathbb{R}^n)}\|\nabla u_{\rm lo}\|_{\dot{S}^1([t_0,T]\times\mathbb{R}^n)}\\ &\lesssim \eta_2 C(M)C(\eta_1)E \le \eta_2^{1-\delta},\end{aligned}$$

whenever η_2 is sufficiently small depending on E, M and η_1. From Hölder inequality, Hardy-littlewood-Sobolev inequality, (5.24) and (5.29), one can get

$$\begin{aligned}&\|P_{\rm lo}[(|x|^{-2} * |u|^2)u - (|x|^{-2} * |u_{\rm lo}|^2)u_{\rm lo}]\|_{\dot{N}^0([t_0,T]\times\mathbb{R}^n)}\\ \lesssim{}& \|(|x|^{-2} * |u_{\rm lo}|^2)u_{\rm hi}\|_{\dot{N}^0([t_0,T]\times\mathbb{R}^n)}\\ &+ \|(|x|^{-2} * |u_{\rm hi}|^2)u_{\rm hi}\|_{\dot{N}^0([t_0,T]\times\mathbb{R}^n)} + \|(|x|^{-2} * |u_{\rm hi}|^2)u_{\rm lo}\|_{\dot{N}^0([t_0,T]\times\mathbb{R}^n)}\\ \lesssim{}& \|u_{\rm lo}\|^2_{\dot{S}^0([t_0,T]\times\mathbb{R}^n)}\|u_{\rm hi}\|_{\dot{S}^0([t_0,T]\times\mathbb{R}^n)}\\ &+ \|u_{\rm hi}\|^2_{\dot{S}^0([t_0,T]\times\mathbb{R}^n)}\|u_{\rm lo}\|_{\dot{S}^0([t_0,T]\times\mathbb{R}^n)} + \|u_{\rm hi}\|^3_{\dot{S}^0([t_0,T]\times\mathbb{R}^n)}\\ \lesssim{}& C(M)\eta_2 C(\eta_1)L(E) + (\eta_2 C(\eta_1)L(E))^2 C(M) + (\eta_2 C(\eta_1)L(E))^3\\ \le{}& \eta_2^{1-\delta}.\end{aligned}$$

Therefore,

$$\|e\|_{\dot{N}^0([t_0,T]\times\mathbb{R}^n)} \le 3\eta_2^{1-\delta},$$

and hence let η_2 be a sufficiently small constant depending on M, we can use Lemma 3.3 to get

$$\|u_{\rm lo} - v\|_{\dot{S}^0([t_0,T]\times\mathbb{R}^n)} \le C(M)\eta_2^{1-\delta} \le \eta_2^{1-2\delta}.$$

Thus (5.15) is true. Next we prove that (5.18) is true. The idea is to view $u_{\rm hi}$ as the solution to the energy-critical NLS

$$\begin{cases} {\rm i}w_t + \Delta w = |w|^{\frac{4}{n-2}}w,\\ w(t_j) = u_{\rm hi}(t_j).\end{cases} \tag{5.31}$$

Based upon the results in [5, 22, 27], we know that (5.31) is globally well-posed and

$$\|w\|_{\dot{S}^1(\mathbb{R}\times\mathbb{R}^n)} \le C(E). \tag{5.32}$$

Using Lemma 3.6 and (5.28), we also get

$$\|w\|_{\dot{S}^0(\mathbb{R}\times\mathbb{R}^n)} \le C(E)\|u_{\rm hi}(t_j)\|_{L_x^2} \lesssim \eta_2 C(E)L(E),$$

where u_{hi} satisfies the following initial value problem in the slab $[t_j,T]\times\mathbb{R}^n$:

$$\begin{cases}(\mathrm{i}\partial_t+\Delta)u_{\mathrm{hi}}=|u_{\mathrm{hi}}|^{\frac{4}{n-2}}u_{\mathrm{hi}}+P_{\mathrm{hi}}((|x|^{-2}*|u|^2)u)\\ \qquad\qquad\qquad +P_{\mathrm{hi}}(|u|^{\frac{4}{n-2}}u-|u_{\mathrm{hi}}|^{\frac{4}{n-2}}u_{\mathrm{hi}})-P_{\mathrm{lo}}(|u_{\mathrm{hi}}|^{\frac{4}{n-2}}u_{\mathrm{hi}}),\\ u_{\mathrm{hi}}(t_j)=u_{\mathrm{hi}}(t_j).\end{cases}$$

In order to use Lemma 3.4, we show that the error term

$$e=P_{\mathrm{hi}}((|x|^{-2}*|u|^2)u)+P_{\mathrm{hi}}(|u|^{\frac{4}{n-2}}u-|u_{\mathrm{hi}}|^{\frac{4}{n-2}}u_{\mathrm{hi}})-P_{\mathrm{lo}}(|u_{\mathrm{hi}}|^{\frac{4}{n-2}}u_{\mathrm{hi}})$$

is small in $\dot{N}^1([t_j,T]\times\mathbb{R}^n)$.

From Hölder, Hardy-Littlewood-Sobolev inequality, (5.21), (5.23), (5.26), (5.29) and (5.30), we have

$$\begin{aligned}&\|P_{\mathrm{hi}}((|x|^{-2}*|u|^2)u\|_{\dot{N}^1([t_j,T]\times\mathbb{R}^n)}\\ \lesssim{}&\|u\|^2_{U([t_j,T])}\|\nabla u\|_{U([t_0,T])}\\ \lesssim{}&\|u_{\mathrm{hi}}\|^2_{\dot{S}^0([t_j,T]\times\mathbb{R}^n)}\|u_{\mathrm{hi}}\|_{\dot{S}^1([t_j,T]\times\mathbb{R}^n)}+\|u_{\mathrm{lo}}\|^2_{\dot{S}^0([t_j,T]\times\mathbb{R}^n)}\|u_{\mathrm{lo}}\|_{\dot{S}^1([t_j,T]\times\mathbb{R}^n)}\\ &+\|u_{\mathrm{lo}}\|^2_{\dot{S}^0([t_j,T]\times\mathbb{R}^n)}\|u_{\mathrm{hi}}\|_{\dot{S}^1([t_j,T]\times\mathbb{R}^n)}+\|u_{\mathrm{hi}}\|^2_{\dot{S}^0([t_j,T]\times\mathbb{R}^n)}\|u_{\mathrm{lo}}\|_{\dot{S}^1([t_j,T]\times\mathbb{R}^n)}\\ \lesssim{}&(\eta_2C(\eta_1)L(E))^2C(\eta_1)L(E)+\eta_1^2E+\eta_1^2L(E)+(\eta_2L(E))^2E\le\eta_2,\end{aligned}$$

if η_2 is sufficiently small depending on E and η_1.

Using Bernstein inequality, Lemma 2.6, (5.11) and (5.22), one has

$$\|P_{\mathrm{lo}}(|u_{\mathrm{hi}}|^{\frac{4}{n-2}}u_{\mathrm{hi}})\|_{\dot{N}^1([t_j,T]\times\mathbb{R}^n)}\lesssim\eta_2^{-1}\|u\|^{\theta}_{Z([t_j,T])}\|u\|^{\frac{n+2}{n-2}-\theta}_{\dot{S}^1([t_j,T]\times\mathbb{R}^n)}\lesssim\eta_2^{-1}\eta_3^{\theta}C(\eta_1,\eta_2)\le\eta_2$$

with the assumption that η_3 is sufficiently small depending on η_1 and η_2.

Now we estimate the last term $\|P_{\mathrm{hi}}(|u|^{\frac{4}{n-2}}u-|u_{\mathrm{hi}}|^{\frac{4}{n-2}}u_{\mathrm{hi}})\|_{\dot{N}^1([t_j,T]\times\mathbb{R}^n)}$. Since the function $z\to|z|^{\frac{4}{n-2}}\frac{z^2}{|z|^2}$ is Hölder continuous of order $\frac{4}{n-2}$, we have

$$\begin{aligned}&\|P_{\mathrm{hi}}(|u|^{\frac{4}{n-2}}u-|u_{\mathrm{hi}}|^{\frac{4}{n-2}}u_{\mathrm{hi}})\|_{\dot{N}^1([t_j,T]\times\mathbb{R}^n)}\\ \lesssim{}&\||u|^{\frac{4}{n-2}}u-|u_{\mathrm{hi}}|^{\frac{4}{n-2}}u_{\mathrm{hi}}\|_{\dot{N}^1([t_j,T]\times\mathbb{R}^n)}\\ \lesssim{}&\||u|^{\frac{4}{n-2}}\nabla u-|u_{\mathrm{hi}}|^{\frac{4}{n-2}}\nabla u_{\mathrm{hi}}\|_{\dot{N}^0([t_j,T]\times\mathbb{R}^n)}+\||u|^{\frac{4}{n-2}}\nabla u_{\mathrm{lo}}\|_{\dot{N}^0([t_j,T]\times\mathbb{R}^n)}\\ &+\Big\|\Big(|u|^{\frac{4}{n-2}}\frac{u^2}{|u|^2}-|u_{\mathrm{hi}}|^{\frac{4}{n-2}}\frac{u_{\mathrm{hi}}^2}{|u_{\mathrm{hi}}|^2}\Big)\nabla u_{\mathrm{hi}}\Big\|_{\dot{N}^0([t_j,T]\times\mathbb{R}^n)}\\ \lesssim{}&\||u|^{\frac{4}{n-2}}\nabla u_{\mathrm{lo}}\|_{\dot{N}^0([t_j,T]\times\mathbb{R}^n)}+\|(|u|^{\frac{4}{n-2}}-|u_{\mathrm{hi}}|^{\frac{4}{n-2}})\nabla u_{\mathrm{hi}}\|_{\dot{N}^0([t_j,T]\times\mathbb{R}^n)}\\ &+\||u_{\mathrm{lo}}|^{\frac{4}{n-2}}\nabla u_{\mathrm{hi}}\|_{\dot{N}^0([t_j,T]\times\mathbb{R}^n)}\\ ={}&(\mathrm{I})+(\mathrm{II})+(\mathrm{III}).\end{aligned}\tag{5.33}$$

For (I) in (5.33), from Remark 2.1, Bernstein inequality, (5.11), (5.19) and (5.26), we have

$$\begin{aligned}\||u|^{\frac{4}{n-2}}\nabla u_{\mathrm{lo}}\|_{\dot{N}^0([t_j,T]\times\mathbb{R}^n)}&\lesssim\|u\|^{\rho}_{Z([t_j,T])}\|u\|^{\frac{4}{n-2}-\rho}_{S^1([t_j,T]\times\mathbb{R}^n)}\|\nabla u_{\mathrm{lo}}\|_{S^1([t_j,T]\times\mathbb{R}^n)}\\ &\lesssim\eta_3^{\rho}C(\eta_1,\eta_2)\eta_2^{-1}\|u_{\mathrm{lo}}\|_{S^1([t_j,T]\times\mathbb{R}^n)}\le\eta_2\end{aligned}$$

as long as η_3 is sufficiently small depending on η_1 and η_2.

For (II) in (5.33), when the dimension $3 \le n < 6$, by Hölder inequality, (5.21), (5.24) and (5.26), we can get

$$\begin{aligned}
&\|(|u|^{\frac{4}{n-2}} - |u_{\rm hi}|^{\frac{4}{n-2}})\nabla u_{\rm hi}\|_{\dot N^0([t_j,T]\times\mathbb{R}^n)}\\
&\lesssim \|(|u|^{\frac{6-n}{n-2}} u_{\rm lo}\nabla u_{\rm hi}\|_{\dot N^0([t_j,T]\times\mathbb{R}^n)}\\
&\lesssim (\|u_{\rm hi}\|^{\frac{6-n}{n-2}}_{\dot S^1([t_j,T]\times\mathbb{R}^n)} + \|u_{\rm lo}\|^{\frac{6-n}{n-2}}_{\dot S^1([t_j,T]\times\mathbb{R}^n)})\|\nabla u_{\rm hi}\|_{\dot S^0([t_j,T]\times\mathbb{R}^n)}\|u_{\rm lo}\|_{W([t_j,T])}\\
&\lesssim (L(E)+E)^{\frac{6-n}{n-2}}\eta_2 L(E) \le \eta_2^{\frac{1}{2}},
\end{aligned}$$

provided that η_2 is sufficiently small depending on E.

When the dimension $n \ge 6$, applying the inequality $(a+b)^p \le a^p + b^p$ as $a, b \ge 0,\ p \le 1$, (5.21) and (5.25), we have

$$\begin{aligned}
\|(|u|^{\frac{4}{n-2}} - |u_{\rm hi}|^{\frac{4}{n-2}})\nabla u_{\rm hi}\|_{\dot N^0([t_j,T]\times\mathbb{R}^n)} &\lesssim \|u_{\rm lo}|^{\frac{4}{n-2}}\nabla u_{\rm hi}\|_{\dot N^0([t_j,T]\times\mathbb{R}^n)}\\
&\lesssim \|u_{\rm hi}\|_{\dot S^1([t_j,T]\times\mathbb{R}^n)}\|u_{\rm lo}\|^{\frac{4}{n-2}}_{W([t_j,T])}\\
&\lesssim L(E)\eta_2^{\frac{4}{n-2}} \le \eta_2^{\frac{3}{n-2}}.
\end{aligned}$$

Then (5.33) is bounded from above by $\eta_2^{\frac{3}{n-2}}$.

Therefore,

$$\|e\|_{\dot N^1([t_j,T]\times\mathbb{R}^n)} \le \eta_2 + \eta_2^{\frac{1}{2}} + 2\eta_2^{\frac{3}{n-2}} \le \eta_2^{\frac{3}{n}}.$$

Taking η_2 to be sufficiently small depending on E, we can use Lemma 3.4 to derive

$$\|u_{\rm hi} - w\|_{\dot S^1([t_j,T]\times\mathbb{R}^n)} \lesssim \eta_1^c$$

for a small constant $c > 0$ depending only on the dimension n. So we obtain

$$\|u_{\rm hi}\|_{\dot S^1([t_j,T]\times\mathbb{R}^n)} \le \|u_{\rm hi} - w\|_{\dot S^1([t_j,T]\times\mathbb{R}^n)} + \|w\|_{\dot S^1([t_j,T]\times\mathbb{R}^n)} \lesssim \eta_1^c + C(E) \le L(E)$$

by choosing $L(E)$ to be sufficiently large.

Finally, (5.19) follows from

$$\begin{aligned}
\|u\|_{S^1([t_0,T]\times\mathbb{R}^n)} &\le \|u_{\rm hi}\|_{S^1([t_0,T]\times\mathbb{R}^n)} + \|u_{\rm lo}\|_{S^1([t_0,T]\times\mathbb{R}^n)}\\
&\le C(M) + C(\eta_1)E + \eta_2 C(\eta_1)L(E) + C(\eta_1)L(E)\\
&\le C(\eta_1,\eta_2).
\end{aligned}$$

This proves that $\Omega_2 \subset \Omega_1$. By induction, we have

$$\|u\|_{S^1(J_{k_0}\times\mathbb{R}^n)} \le C(\eta_1,\eta_2).$$

As J_{k_0} is arbitrary and the total number of intervals J_k is $K = K(E, M, \eta_3)$, putting these bounds together, we obtain

$$\|u\|_{S^1(\mathbb{R}\times\mathbb{R}^n)} \le C(\eta_1,\eta_2,\eta_3) = C(E, M).$$

5.6 Global bounds in the case: $p = \frac{4}{n-2}$, $2 \le \gamma < 4$ with $\gamma < n$ and $\lambda_1 \cdot \lambda_2 < 0$ or $\frac{4}{n} \le p < \frac{4}{n-2}$, $\gamma = 4$ with $\gamma < n$ and $\lambda_1 \cdot \lambda_2 < 0$

Here we only prove the first case, since the other case can be handled similarly. Without loss of generality, let $|\lambda_1| = |\lambda_2| = 1$.

We view u as the perturbation to the energy-critical problem

$$\begin{cases} \mathrm{i}w_t + \Delta w = |w|^{\frac{4}{n-2}} w, \\ w(0) = u_{\mathrm{hi}}(0), \end{cases}$$

which is globally well-posed by [5, 22, 27] and

$$\|w\|_{\dot{S}^1(\mathbb{R}\times\mathbb{R}^n)} \le C(E, M). \tag{5.34}$$

By Lemma 3.6, (5.34) implies

$$\|w\|_{\dot{S}^0(\mathbb{R}\times\mathbb{R}^n)} \le C(E, M)\|u_0\|_{L^2_x} \le C(E, M)M^{\frac{1}{2}}. \tag{5.35}$$

Definition 5.1 $\dot{D}^0(I) := V(I) \cap U(I) \cap L_t^{\frac{2(n+2)}{n-2}} L_x^{\frac{2(n+2)}{n^2+4}}$, *and* $\|u\|_{\dot{D}^k(I)} := \||\nabla|^k u\|_{\dot{D}^0(I)}$.

It is easy to verify that

$$\|(|x|^{-\gamma} * |u|^2)u\|_{\dot{N}^k(I\times\mathbb{R}^n)} \lesssim \|u\|_{\dot{D}^k(I)} \|u\|^{4-\gamma}_{\dot{D}^0(I)} \|u\|^{\gamma-2}_{\dot{D}^1(I)}, \tag{5.36}$$

$$\||u|^{\frac{4}{n-2}} u\|_{\dot{N}^k(I\times\mathbb{R}^n)} \lesssim \|u\|^{\frac{4}{n-2}}_{\dot{D}^1(I)} \|u\|_{\dot{D}^k(I)}, \tag{5.37}$$

where $k = 0, 1$.

As we have done before, we divide $\mathbb{R}$ into $J = J(E, M, \eta)$ subintervals $I_j = [t_j, t_{j+1}]$ such that

$$\|u\|_{\dot{D}^1(I_j)} \sim \eta,$$

where $\eta > 0$ is a small constant to be chosen later.

Moreover, let M be sufficiently small and depend on E and η. In view of (5.35), we may assume

$$\|w\|_{\dot{S}^0(\mathbb{R}\times\mathbb{R}^n)} \le \eta.$$

Then we get

$$\|u\|_{D^1(I_j)} \sim \eta. \tag{5.38}$$

As a matter of fact, in each slab $I_j \times \mathbb{R}^n$, we have

$$\|e^{\mathrm{i}(t-t_j)\Delta} w(t_j)\|_{D^1(I_j)} \le \|w\|_{D^1(I_j)} + C\|w\|^{\frac{n+2}{n-2}}_{D^1(I_j)} \le \eta + C\eta^{\frac{n+2}{n-2}} \le 2\eta, \tag{5.39}$$

if $C\eta^{\frac{n+2}{n-2}} \le \eta$.

Let $I_0 = [t_0, t_1]$. Since $w(t_0) = u(t_0) = u_0$, using Strichartz estimates, (5.36), (5.37) and (5.39), we can deduce

$$\|u\|_{D^1(I_0)} \le 2\eta + C\|w\|^{\frac{n+2}{n-2}}_{D^1(I_0)} + C\|w\|^3_{D^1(I_0)}.$$

By a standard continuity argument, this yields

$$\|u\|_{D^1(I_0)} \le 4\eta \tag{5.40}$$

with η being sufficiently small.

On the other hand, Strichartz estimates, (5.36), (5.37) and (5.40) imply

$$\begin{aligned}\|u\|_{\dot{D}^0(I_0)} &\lesssim M^{\frac{1}{2}} + \|u\|_{\dot{D}^1(I_0)}^{\frac{4}{n-2}}\|u\|_{\dot{D}^0(I_0)} + \|u\|_{\dot{D}^0(I_0)}^{5-\gamma}\|u\|_{\dot{D}^1(I_0)}^{\gamma-2}\\ &\lesssim M^{\frac{1}{2}} + \eta^{\frac{4}{n-2}}\|u\|_{\dot{D}^0(I_0)} + \|u\|_{\dot{D}^0(I_0)}^{5-\gamma}\eta^{\gamma-2}.\end{aligned}$$

Therefore, making η sufficiently small and $\gamma < 4$, we get

$$\|u\|_{\dot{D}^0(I_0)} \lesssim M^{\frac{1}{2}}.$$

In order to apply Lemma 3.4, we need to show that the error $(|x|^{-\gamma} * |u|^2)u$ is small in the norm $\dot{N}^1(I_0 \times \mathbb{R}^n)$. In fact, by

$$\|(|x|^{-\gamma} * |u|^2)u\|_{\dot{N}^1(I_0\times\mathbb{R}^n)} \lesssim \|u\|_{\dot{D}^1(I_0)}^{\gamma-1}\|u\|_{\dot{D}^0(I_0)}^{4-\gamma} \lesssim \eta^{\gamma-1}M^{2-\frac{\gamma}{2}} \le M^{\delta_0}$$

for a small constant $\delta_0 > 0$, together with M being sufficiently small which depends on E and η, and by Lemma 3.4, we get

$$\|u - w\|_{\dot{S}^1(I_0\times\mathbb{R}^n)} \le M^{c\delta_0}$$

for a small constant $c > 0$ that depends only on the dimension n. Strichartz estimate implies

$$\|\mathrm{e}^{\mathrm{i}(t-t_1)\Delta}(u(t_1) - w(t_1))\|_{\dot{S}^1(I_1\times\mathbb{R}^n)} \le M^{c\delta_0}. \tag{5.41}$$

Now we turn to the interval $I_1 = [t_1, t_2]$. Using Strichartz estimate, (5.36), (5.37), (5.39) and (5.41), we can get

$$\begin{aligned}\|u\|_{D^1(I_1)} &\le \|\mathrm{e}^{\mathrm{i}(t-t_1)\Delta}u(t_1)\|_{\dot{D}^0(I_1)} + \|\mathrm{e}^{\mathrm{i}(t-t_1)\Delta}(u(t_1) - w(t_1))\|_{\dot{D}^1(I_1)}\\ &\quad + \|\mathrm{e}^{\mathrm{i}(t-t_1)\Delta}w(t_1)\|_{\dot{D}^1(I_1)} + C\|u\|_{D^1(I_1)}^{\frac{n+2}{n-2}} + C\|u\|_{D^1(I_1)}^3\\ &\lesssim M^{\frac{1}{2}} + M^{c\delta_0} + \eta + \|u\|_{D^1(I_1)}^{\frac{n+2}{n-2}} + \|u\|_{D^1(I_1)}^3.\end{aligned}$$

Assuming η and M to be sufficiently small and by a standard continuity argument, we obtain

$$\|u\|_{D^1(I_1)} \le 4\eta.$$

Moreover, we also get

$$\|u\|_{\dot{D}^0(I_1)} \lesssim M^{\frac{1}{2}}.$$

For an M sufficiently small, we can use Lemma 3.4 to obtain

$$\|u - w\|_{\dot{S}^1(I_1\times\mathbb{R}^n)} \le M^{c\delta_1}$$

for a small constant $0 < \delta_1 < \delta_0$.

By induction argument, choosing M to be the smallest one of above steps, we obtain

$$\|u\|_{D^1(I_j)} \le 4\eta.$$

Summing these estimates over all intervals I_j, and since the total number of these intervals is $J = J(E, M, \eta)$, we get

$$\|u\|_{D^1(\mathbb{R})} \lesssim J\eta \le C(E, M).$$

Using Strichartz estimate, (5.36) and (5.37), we get

$$\|u\|_{S^1(\mathbb{R}\times\mathbb{R}^n)} \lesssim \|u_0\|_{H_x^1} + \|u\|_{D^1(\mathbb{R})}^{\frac{n+2}{n-2}} + \|u\|_{D^1(\mathbb{R})}^3 \lesssim M + E + C(E) \le C(E, M).$$

5.7 Global bounds in the case: $\frac{4}{n} \le p < \frac{4}{n-2}$, $2 \le \gamma < 4$ with $\gamma < n$ and $\lambda_1 \cdot \lambda_2 < 0$ or $p = \frac{4}{n}$, $\gamma = 2$ and $\lambda_1, \lambda_2 > 0$

Both cases can be discussed similarly with the method used in Subsection 5.6. The only difference is that here we treat u as the solution to the free Schrödinger equation

$$\mathrm{i}\widetilde{u}_t + \Delta \widetilde{u} = 0, \quad \widetilde{u}(0) = u_0.$$

By Strichartz estimate, the global solution $\widetilde{u}$ obeys the spacetime estimates

$$\|\widetilde{u}\|_{S^1(\mathbb{R}\times\mathbb{R}^n)} \lesssim \|u_0\|_{\dot{H}^1_x} \le C(E, M)$$

and

$$\|\widetilde{u}\|_{S^0(\mathbb{R}\times\mathbb{R}^n)} \lesssim \|u_0\|_{L^2_x} \lesssim M^{\frac{1}{2}}.$$

Using the similar method used in Subsection 5.6, it is easy to show that

$$\|u\|_{S^1(\mathbb{R}\times\mathbb{R}^n)} \le C(E, M).$$

5.8 Finite global Strichartz norms imply scattering

Finally, we show that finite global Strichartz norms imply scattering. For simplicity, we only construct the scattering state in the positive time direction. Similar arguments can be used to construct the scattering state in the negative direction.

For $0 < t < \infty$, define

$$u_+(t) = u_0 - \mathrm{i} \int_0^t \mathrm{e}^{-\mathrm{i}s\Delta} (\lambda_1 |u|^p u + \lambda_2 (|x|^{-\gamma} * |u|^2) u) \,\mathrm{d}s.$$

Since $u \in S^1(\mathbb{R} \times \mathbb{R}^n)$, Strichartz estimates and Hölder inequality show that $u_+(t) \in H^1_x$ for all $t \in \mathbb{R}^+$, and for $0 < \tau < t$, we have

$$\begin{aligned}
\|u_+(t) - u_+(\tau)\|_{H^1_x} &\lesssim \Big\| \int_\tau^t \mathrm{e}^{\mathrm{i}(t-s)\Delta} (\lambda_1 |u|^p u + \lambda_2 (|x|^{-\gamma} * |u|^2) u) \,\mathrm{d}s \Big\|_{L^\infty_t H^1_x([\tau,t]\times\mathbb{R}^n)} \\
&\lesssim \|u\|^{2-\frac{(n-2)p}{2}}_{V([\tau,t])} \|u\|^{\frac{np}{2}-2}_{W([\tau,t])} \|(1+|\nabla|)u\|_{V([\tau,t])} \\
&\quad + \|u\|^{4-\gamma}_{U([\tau,t])} \|u\|^{\gamma-2}_{L^6_t L^{\frac{6n}{3n-8}}_x([\tau,t]\times\mathbb{R}^n)} \|(1+|\nabla|)u\|_{U([\tau,t])},
\end{aligned}$$

and for $\varepsilon > 0$, there exists a $T_\varepsilon > 0$ such that

$$\|u_+(t) - u_+(\tau)\|_{H^1_x} \le \varepsilon$$

for any $t, \tau > T_\varepsilon$. Thus $u_+(t)$ converges to some function u_+ in H^1_x as $t \to +\infty$. In fact

$$u_+ := u_0 - \mathrm{i} \int_0^\infty \mathrm{e}^{-\mathrm{i}s\Delta} (\lambda_1 |u|^p u + \lambda_2 (|x|^{-\gamma} * |u|^2) u) \,\mathrm{d}s.$$

Then the scattering follows because of

$$\begin{aligned}
\|\mathrm{e}^{-\mathrm{i}t\Delta} u(t) - u_+\|_{H^1_x} &= \Big\| \int_t^\infty \mathrm{e}^{-\mathrm{i}s\Delta} (\lambda_1 |u|^p u + \lambda_2 (|x|^{-\gamma} * |u|^2) u) \,\mathrm{d}s \Big\|_{H^1_x} \\
&= \Big\| \int_t^\infty \mathrm{e}^{\mathrm{i}(t-s)\Delta} (\lambda_1 |u|^p u + \lambda_2 (|x|^{-\gamma} * |u|^2) u) \,\mathrm{d}s \Big\|_{H^1_x} \\
&\lesssim \|u\|^{2-\frac{(n-2)p}{2}}_{V([t,\infty))} \|u\|^{\frac{np}{2}-2}_{W([t,\infty))} \|(1+|\nabla|)u\|_{V([t,\infty))} \\
&\quad + \|u\|^{4-\gamma}_{U([t,\infty))} \|u\|^{\gamma-2}_{L^6_t L^{\frac{6n}{3n-8}}_x([t,\infty)\times\mathbb{R}^n)} \|(1+|\nabla|)u\|_{U([t,\infty))},
\end{aligned}$$

noting that the right-hand side of the above inequality obviously tends to 0 as $t \to +\infty$. The other properties follow from conservation of mass and energy.

6 Blowup Results

From Theorem 1.1, we can see that there are still many regions in which additional conditions, such as small energy and small mass, are required for the problem to be globally well-posed. In this section, we will show that in these regions, under suitable assumptions the solution to (1.1) will blow up at finite time. We follow the approach of Glassey [8], which is essentially a convexity method. Consider the variance

$$f(t) = \int_{\mathbb{R}^n} |x|^2 |u(t,x)|^2 \,\mathrm{d}x.$$

For strong H^1_x-solution u to (1.1) with initial data $u_0 \in \Sigma$, it is well-known that if $f \in C^2(-T_{\min},\, T_{\max})$ we have the following lemma (see, for example, [2, Chapter 6]).

Lemma 6.1 *For all* $t \in (-T_{\min},\, T_{\max})$*, we have*

$$f'(t) = 4\mathrm{Im}\int \overline{u} x \cdot \nabla u \,\mathrm{d}x$$

and

$$f''(t) = 16E + \frac{4np-16}{p+2}\lambda_1 \|u\|^{p+2}_{L^{p+2}_x} + 2\lambda_2(\gamma-2)\int (|x|^{-\gamma} * |u|^2)|u|^2 \,\mathrm{d}x. \tag{6.1}$$

If $f''(t)$ is bounded from above by a constant A in $(-T_{\min},\, T_{\max})$, then we have

$$\|xu\|^2_{L^2} \le \theta(t), \tag{6.2}$$

where

$$\theta(t) = \|x\varphi\|^2_{L^2} + 4t\mathrm{Im}\int \overline{\varphi} x \cdot \nabla\varphi \,\mathrm{d}x + \frac{1}{2}t^2 A.$$

When A is negative, we observe that $\theta(t)$ is a polynomial of degree 2, and thus $\theta(t) < 0$ for large enough $|t|$. Since $\|xu\|^2_{L^2} \ge 0$, we deduce from (6.1) that both $T_{\min}$ and $T_{\max}$ are finite. However, $A < 0$ is not a necessary nor sufficient condition for $\theta(t)$ to be negative. A necessary and sufficient condition for $\theta(t) < 0$ is actually

$$8\Big(\mathrm{Im}\int \overline{\varphi} x \cdot \nabla\varphi \,\mathrm{d}x\Big)^2 > A\|x\varphi\|^2_{L^2}.$$

But in many situations, $T_{\min}$ and $T_{\max}$ are not both finite. Interested readers are referred to Chapter 6 in [2].

In what follows, we will find the negative constant A such that $f''(t) \le A$.

Case 1 $\lambda_1 < 0,\ \lambda_2 > 0,\ \frac{4}{n} \le p \le \frac{4}{n-2},\ 0 < \gamma \le \frac{np}{2}$ and $E < 0$.

By (6.1), the conservation of energy and our assumption, we get

$$\begin{aligned} f''(t) &= 16E + (4np-16)\Big\{E - \frac{1}{2}\|\nabla u\|^2_{L^2} - \frac{\lambda_2}{4}\int (|x|^{-\gamma} * |u|^2)|u|^2 \,\mathrm{d}x\Big\} \\ &\quad + 2\lambda_2(\gamma-2)\int (|x|^{-\gamma} * |u|^2)|u|^2 \,\mathrm{d}x \\ &= 4npE - (2np-8)\|\nabla u\|^2_{L^2} - (np-2\gamma)\lambda_2 \int (|x|^{-\gamma} * |u|^2)|u|^2 \,\mathrm{d}x \\ &\le 4npE \end{aligned} \tag{6.3}$$

and $4npE < 0$, which is the negative constant A.

Case 2 $\lambda_1 > 0,\ \lambda_2 < 0,\ \frac{2\gamma}{n} \le p \le \frac{4}{n-2},\ 2 \le \gamma \le 4$ and $E < 0$.

Similarly, we get

$$\begin{aligned} f''(t) &= 16E + \frac{4np-16}{p+2}\lambda_1\|u\|_{L_x^{p+2}}^{p+2} + 8(\gamma-2)\Big\{E - \frac{1}{2}\|\nabla u\|_{L^2}^2 - \frac{\lambda_1}{p+2}\|u\|_{L_x^{p+2}}^{p+2}\Big\} \\ &= 8\gamma E - 4(\gamma-2)\|\nabla u\|_{L^2}^2 + \frac{4np-8\gamma}{p+2}\lambda_1\|u\|_{L_x^{p+2}}^{p+2} \\ &\le 8\gamma E, \end{aligned} \tag{6.4}$$

where $8\gamma E < 0$ and $A = 8\gamma E$.

Case 3 $\lambda_1 < 0,\ \lambda_2 < 0,\ \frac{4}{n} \le p \le \frac{4}{n-2},\ 2 \le \gamma \le 4$ and $E < 0$.

When $\gamma \ge \frac{np}{2}$, using (6.3) and our assumption, we have

$$f''(t) \le 4npE.$$

When $\gamma < \frac{np}{2}$, from (6.4) and our assumption, we have

$$f''(t) \le 8\gamma E.$$

So we also find the negative constant A.

Case 4 $\lambda_1 < 0,\ \lambda_2 < 0,\ 0 < \gamma < 2,\ \frac{4}{n} < p \le \frac{4}{n-2}$ and $4npE + C(M) < 0$.

Using (A.2) and the Young's inequality, we have that when $\gamma < 2$,

$$\|\nabla u\|_{L^2}^{\gamma} \le \delta\|\nabla u\|_{L^2}^2 + C(\delta).$$

From (6.3) and our assumption, we have

$$f''(t) \le 4npE + [C(np-2\gamma)|\lambda_2|\delta - (2np-8)]\|\nabla u\|_{L^2}^2 + C(np-2\gamma)|\lambda_2|C(\delta)\|u\|_{L^2}^{4-\gamma}.$$

When δ is sufficiently small, we have

$$f''(t) \le 4npE + C(M).$$

Then $A = 4npE + C(M) < 0$.

Case 5 $\lambda_1 < 0,\ \lambda_2 < 0,\ 2 < \gamma \le 4,\ 0 < p < \frac{4}{n}$ and $8\gamma E + C(M) < 0$.

By (A.1) and the Young's inequality, we have, when $p < \frac{4}{n}$,

$$\|\nabla u\|_{L^2}^{\frac{np}{2}} \le \delta\|\nabla u\|_{L^2}^2 + C(\delta).$$

From (6.4) and our assumption, we have

$$f''(t) \le 8\gamma E - 4(\gamma-2)\|\nabla u\|_{L^2}^2 + \Big(\frac{4np-8\gamma}{p+2}\lambda_1 C\delta\|\nabla u\|_{L^2}^2 + \frac{4np-8\gamma}{p+2}\lambda_1 C(\delta)\Big)\|u\|_{L^2}^{\frac{4-(n-2)p}{2}}.$$

With δ being sufficiently small, we have

$$f''(t) \le 8\gamma E + C(M).$$

We can take $A = 8\gamma E + C(M) < 0$, which defines the desired negative constant A.

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Appendix Bound State and Properties

Let $R(x)$ and $W(x)$ be the positive radial Schwartz solutions of the ground state to the elliptic equations, respectively:

$$\Delta R + |R|^p R = \frac{4-(n-2)p}{np} R \quad \text{and} \quad \Delta W + (|x|^{-\gamma} * |W|^2)W = \frac{4-\gamma}{\gamma} W.$$

Based upon the work of [2, 7, 17, 26], we have the following characterization of R and W:

$$\|u\|_{L^{p+2}}^{p+2} \le C_R \|\nabla u\|_{L^2}^{\frac{np}{2}} \|u\|_{L^2}^{\frac{4-(n-2)p}{2}}, \quad \forall\, u, v \in H_x^1, \tag{A.1}$$

$$\|(|x|^{-\gamma} * |v|^2)|v|^2\|_{L^1} \le C_W \|\nabla v\|_{L^2}^{\gamma} \|v\|_{L^2}^{4-\gamma}, \tag{A.2}$$

where

$$C_R = \frac{2(p+2)}{np} \|\nabla R\|_{L^2}^{-p} = \frac{2(p+2)}{np} \|R\|_{L^2}^{-p},$$
$$C_W = \frac{4}{\gamma} \|\nabla W\|_{L^2}^{-2} = \frac{4}{\gamma} \|W\|_{L^2}^{-2},$$

which are the best constants for their inequalities, respectively.

If we define

$$\widetilde{E}(R) : = \frac{1}{2} \int |\nabla R|^2 \,\mathrm{d}x - \frac{1}{p+2} \int |R|^{p+2} \,\mathrm{d}x,$$
$$\widetilde{E}(W) : = \frac{1}{2} \int |\nabla W|^2 \,\mathrm{d}x - \frac{1}{4} \int (|x|^{-\gamma} * |W|^2)|W|^2 \,\mathrm{d}x,$$

then we have

$$\widetilde{E}(R) = \Big(\frac{1}{2} - \frac{2}{np}\Big) \int |\nabla R|^2 \,\mathrm{d}x = \Big(\frac{1}{2} - \frac{2}{np}\Big) \Big(\frac{2(p+2)}{npC_R}\Big)^{\frac{2}{p}},$$
$$\widetilde{E}(W) = \Big(\frac{1}{2} - \frac{1}{\gamma}\Big) \int |\nabla W|^2 \,\mathrm{d}x = \frac{2(\gamma-2)}{\gamma^2 C_W}.$$

Also define $E_1 := \frac{1}{2} \int |\nabla u|^2 \,\mathrm{d}x - \frac{|\lambda_1|}{p+2} \int |u|^{p+2} \,\mathrm{d}x$, where λ_1 is the constant in (1.1).

Lemma A.1 *Assume that*

$$\|\nabla u\|_{L^2}^2 (\|u\|_{L^2}^2)^{\frac{4-(n-2)p}{np-4}} < |\lambda_1|^{\frac{4}{4-np}} \|\nabla R\|_{L^2}^{\frac{4p}{np-4}},$$
$$E_1 \cdot (\|u\|_{L^2}^2)^{\frac{4-(n-2)p}{np-4}} \le (1-\delta_0) |\lambda_1|^{\frac{4}{4-np}} \Big(\frac{2np}{np-4}\Big)^{\frac{4-(n-2)p}{np-4}} (\widetilde{E}(R))^{\frac{2p}{np-4}}, \quad \textit{where } \delta_0 > 0.$$

Then we have that when $\frac{4}{n} < p \le \frac{4}{n-2}$, *there exists a* $\overline{\delta} = \overline{\delta}(\delta_0, n) > 0$, *such that*

$$\|\nabla u\|_{L^2}^2 (\|u\|_{L^2}^2)^{\frac{4-(n-2)p}{np-4}} \le (1-\overline{\delta}) |\lambda_1|^{\frac{4}{4-np}} \|\nabla R\|_{L^2}^{\frac{4p}{np-4}} \quad \textit{and} \quad E_1 \ge 0.$$

Proof By (A.1), we have

$$E_1 \ge \frac{1}{2} \int |\nabla u|^2 \,\mathrm{d}x - \frac{|\lambda_1|}{p+2} C_R \|\nabla u\|_{L^2}^{\frac{np}{2}} \|u\|_{L^2}^{\frac{4-(n-2)p}{2}}.$$

Let

$$f(x) = \frac{1}{2} x - \frac{|\lambda_1|}{p+2} C_R \|u\|_{L^2}^{\frac{4-(n-2)p}{2}} x^{\frac{np}{4}}$$

and $a = \int |\nabla u|^2 \, \mathrm{d}x$. Note that

$$f'(x) = 0 \;\Leftrightarrow\; x = |\lambda_1|^{\frac{4}{4-np}} \|u\|_{L^2}^{-\frac{2[4-(n-2)p]}{np-4}} \|\nabla R\|_{L^2}^{\frac{4p}{np-4}} := x_0$$

and

$$\begin{aligned} &f'(x) > 0 \quad \text{for } x < x_0, \\ &f(0) = 0, \quad f(x_0) = \Big(\frac{1}{2} - \frac{2}{np}\Big)|\lambda_1|^{\frac{4}{4-np}} \Big(\frac{2np}{np-4}\Big)^{\frac{4-(n-2)p}{np-4}} (\|u\|_{L^2}^2)^{-\frac{4-(n-2)p}{np-4}} (\widetilde{E}(R))^{\frac{2p}{np-4}}. \end{aligned}$$

Using the fact that $a \in [0, x_0)$ and the condition $E_1 \le (1-\delta_0) f(x_0)$, we deduce that there exists $\overline{\delta} = \overline{\delta}(\delta_0, n)$, such that

$$a \le (1 - \overline{\delta}) x_0 \quad \text{and} \quad E_1 \ge f(a) \ge 0.$$

Define $E_2 := \frac{1}{2}\int |\nabla v|^2 \, \mathrm{d}x - \frac{|\lambda_2|}{4} \int \big(|x|^{-\gamma} * |v|^2\big) |v|^2 \, \mathrm{d}x$, where λ_2 is the constant in (1.1). We can obtain a similar result for $W(x)$ as follows.

Lemma A.2 *Assume that*

$$\begin{aligned} \|\nabla v\|_{L^2}^2 (\|v\|_{L^2}^2)^{\frac{4-\gamma}{\gamma-2}} &< \Big(\frac{\|\nabla W\|_{L^2}^2}{|\lambda_2|}\Big)^{\frac{2}{\gamma-2}}, \\ E_2 \cdot (\|v\|_{L^2}^2)^{\frac{4-\gamma}{\gamma-2}} &\le (1-\delta_0)\Big(\frac{1}{2} - \frac{1}{\gamma}\Big)\Big[\frac{2\gamma \widetilde{E}(W)}{|\lambda_2|(\gamma-2)}\Big]^{\frac{2}{\gamma-2}}, \end{aligned}$$

where $\delta_0 > 0$. Then when $2 < \gamma \le 4$, there exists a $\overline{\delta} = \overline{\delta}(\delta_0, n) > 0$, such that

$$\|\nabla v\|_{L^2}^2 (\|v\|_{L^2}^2)^{\frac{4-\gamma}{\gamma-2}} \le (1 - \overline{\delta})\Big(\frac{\|\nabla W\|_{L^2}^2}{|\lambda_2|}\Big)^{\frac{2}{\gamma-2}} \quad \textit{and} \quad E_2 \ge 0.$$