Chin. Ann. Math.
29B(6), 2008, 663–678
DOI: 10.1007/s11401-007-0293-x

Chinese Annals of Mathematics, Series B

Jump Type Cahn-Hilliard Equations with Fractional Noises****

Lijun BO* Kehua SHI** Yongjin WANG***

Abstract The authors explore a class of jump type Cahn-Hilliard equations with fractional noises. The jump component is described by a (pure jump) Lévy space-time white noise. A fixed point scheme is used to investigate the existence of a unique local mild solution under some appropriate assumptions on coefficients.

Keywords Cahn-Hilliard equations, Fractional noises, Lévy space-time white noise, Local mild solution
2000 MR Subject Classification 60H15, 60G18, 35R60

1 Introduction

It is well-known that a classical model for the process of the spinodal decomposition can be described by a Cahn-Hilliard equation on the domain $[0,T]\times[0,\pi]^d$ (see [7, 16] and their references therein) as follows:

$$\frac{\partial u}{\partial t}+\Delta^2 u-\Delta f(u)=0, \tag{1.1}$$

which describes the complicated phase separation and coarsening phenomena in a melted alloy. Here the mapping f is the derivative of the homogeneous free energy $\widetilde{F}$, which contains a logarithmic term. In some cases, $\widetilde{F}$ can be approximated by an even-degree polynomial with positive dominant coefficient. A standard choice for f is a cubic polynomial such as $f(u)=u-u^3$.

This paper deals with the following jump type Cahn-Hilliard equations with fractional noise potentials:

$$\begin{cases}\Box u(t,x)=\Delta b(u(t,x))+\dot{B}^H(x,t)+a(u(t,x))\dot{F}(x,t), & \text{in } [0,T]\times D,\\ u(0)=\psi, & \\ \dfrac{\partial u}{\partial n}=\dfrac{\partial\Delta u}{\partial n}=0, & \text{on } [0,T]\times\partial D,\end{cases} \tag{1.2}$$

where the operator $\Box:=\frac{\partial}{\partial t}+\Delta^2$ with the Laplace operator Δ, and the domain $D=[0,\pi]^d$. In addition, $\dot{B}^H$ denotes a fractional noise on $D\times[0,\infty)$ with Hurst parameter $H>\frac12$, and $\dot{F}$ is

Manuscript received July 23, 2007. Revised June 27, 2008. Published online November 5, 2008.
*Department of Mathematics, Xidian University, Xi'an 710071, China. E-mail: bolijunnk@yahoo.com.cn
**Corresponding author. School of Mathematical Sciences, Nankai University, Tianjin 300071, China. E-mail: kehuashink@gmail.com
***School of Mathematical Sciences, Nankai University, Tianjin 300071, China. E-mail: yjwang@nankai.edu.cn
****Project supported by the National Natural Science Foundation of China (No. 10871103) and the LPMC at Nankai University.

a (pure jump) Lévy space-time white noise on $D \times [0, \infty)$. The nonlinear drift $b : \mathbf{R} \to \mathbf{R}$ is a polynomial of degree 3 with positive dominant coefficients and $a : \mathbf{R} \to \mathbf{R}$ is a measurable map satisfying additional assumptions (see Section 4 below).

Second order heat equations with fractional noises have been investigated in the literature (see [9, 11, 14, 17, 21]). Among them, Duncan et al. [9] and Tindel et al. [21] investigated a class of parabolic equations with linear fractional noise terms, where the Hurst parameter H in [9] was restricted to $H > \frac{1}{2}$, and the later treats both cases $H > \frac{1}{2}$ and $H < \frac{1}{2}$. The heat equations with a multiplicative fractional noise of Hurst parameter $H = (h_0, \cdots, h_d)$ on $[0, \infty) \times \mathbf{R}^d$ were proposed by Hu [11] and the author established the existence and uniqueness of mild solutions to the equation under some assumptions on H, through chaos expansion. For a nonlinear evolution equation in some Hilbert space, Maslowski and Nualart [14] proved the existence and uniqueness of mild solutions for the equation with a cylindrical fractional Brownian motion (FBM) under $H > \frac{1}{2}$. This leads one to define stochastic integrals with respect to FBM in a pathwise way (see also [18]). In [17], Nualart and Ouknine discussed a quasi-linear parabolic equation driven by an additive fractional noise on $[0, \infty) \times [0, 1]$.

Recently, a class of stochastic Cahn-Hilliard equations with Gaussian noise perturbations were introduced in [8, 5, 6], respectively. Furthermore, Bo and Wang [4] established a unique local solution to a stochastic Cahn-Hilliard equation driven by a Lévy space-time white noise, in which a new version of Burkholder-Davis-Gundy inequality (B-D-G inequality) played a key role (see also Proposition 3.2 below). In [2], Bo et al formulated a fourth-order Anderson model with double-parameter fractional noises on one-dimensional space by employing the Skorohod integral. In the present paper, we are going to develop several different versions of B-D-G inequalities for treating the jump component of (1.2). For the fractional noise term, we will limit our consideration on the linear additive fractional noises with Hurst parameter $H > \frac{1}{2}$ as proposed by Nualart and Ouknine [17]. Our aim is to establish the existence of a unique local mild solution to (1.2).

The outline of this paper is as follows. In the coming section, we will give the definitions of the fractional B^H and the pure jump Lévy noise F, respectively. In Section 3, several different B-D-G inequalities are presented. The statement of main result and its proof will be given in Section 4.

2 Fractional and Lévy Noises

In this section, we will present the definitions of the fractional noises, Lévy space-time white noises and stochastic integrals with respect to them in the respective subsections.

2.1 Fractional noises

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$ be a complete probability space with the filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions, on which $(B^H(A \times [0, t]))_{(t,A)\in[0,T]\times\mathcal{B}(D)}$ is a centered Gaussian family of random variables with the covariance, for $H \in (0, 1)$,

$$\mathbf{E}[B^H(A \times [0, t])B^H(B \times [0, s])] = |A \cap B| R_H(t, s), \quad s, t \in [0, T], \ A, B \in \mathcal{B}(D),$$

with the covariance kernel

$$R_H(t,s) = \frac{1}{2}[t^{2H} + s^{2H} - |t-s|^{2H}].$$

Here $|A|$ denotes Lebesgue measure of the set $A \in \mathcal{B}(D)$.

We denote by ϵ the set of step functions on $D \times [0,T]$. Let $\mathcal{H}$ be the Hilbert space defined as the closure of ϵ with respect to the scalar product

$$\langle \mathbf{1}_{[0,t]\times A}, \mathbf{1}_{[0,s]\times B}\rangle_{\mathcal{H}} = |A \cap B| R_H(t,s).$$

Thus the mapping $\mathbf{1}_{[0,t]\times A} \to B^H(A \times [0,t])$ is an isometry between $\mathcal{E}$ and the linear space $\mathrm{span}\{B^H(A\times[0,t]), A \in \mathcal{B}(D), t \in [0,T]\}$, a subspace of $L^2(\Omega)$. Moreover, the mapping can be extended to an isometry from $\mathcal{H}$ to Gaussian space associated with B^H. This isometry will be denoted by $\varphi \to B^H(\varphi)$ for $\varphi \in \mathcal{H}$. Therefore, we can regard $B^H(\varphi)$ as the stochastic integral with respect to B^H. In general, we use the notation

$$\int_{[0,T]\times D} \varphi(y,s) B^H(\mathrm{d}y, \mathrm{d}s)$$

to represent $B^H(\varphi)$. On the other hand, it is known that the covariance kernel $R_H(t,s)$ satisfies

$$R_H(t,s) = \int_0^{t\wedge s} K_H(t,u) K_H(s,u)\mathrm{d}u,$$

where the kernel

$$K_H(t,s) = c_H(t-s)^{H-\frac{1}{2}} + c_H\Big(\frac{1}{2} - H\Big)\int_s^t (u-s)^{H-\frac{3}{2}}\Big(1 - \Big(\frac{s}{u}\Big)^{\frac{1}{2}-H}\Big)\mathrm{d}u$$

for some constant c_H. In particular, if $H > \frac{1}{2}$, then

$$R_H(t,s) = H(2H-1)\int_0^t\int_0^s |u-v|^{2H-2}\mathrm{d}u\mathrm{d}v. \tag{2.1}$$

Define a linear operator $K_H^* : \mathcal{E} \to L^2([0,T])$ by

$$(K_H^*\psi)(s,x) = K_H(T,s)\psi(s,x) + \int_s^T (\psi(u,x) - \psi(s,x))\frac{\partial K_H}{\partial u}(u,s)\mathrm{d}u. \tag{2.2}$$

Then the operator K_H^* gives an isometry from $\mathcal{H}$ to $L^2([0,T]\times D)$ (see [17, 21]). Consequently,

$$W(t,A) := B^H((K_H^*)^{-1}(\mathbf{1}_{[0,t]\times A})), \quad (t,A) \in [0,T]\times\mathcal{B}(D)$$

defines a space-time white noise. Moreover we can regard B^H as

$$B^H(A\times[0,t]) = \int_0^t\int_D K_H(t,s)W(\mathrm{d}y,\mathrm{d}s).$$

2.2 Lévy space-time white noises

Let $(E_i, \mathcal{E}_i, \mu_i)$, $i = 1, 2$ be two σ-finite measurable spaces. We call $N : (E_1, \mathcal{E}_1, \mu_1) \times (E_2, \mathcal{E}_2, \mu_2) \times (\Omega, \mathcal{F}, \mathbf{P}) \to \mathbf{N} \cup \{0\} \cup \{\infty\}$ a Poisson noise on $(E_1, \mathcal{E}_1, \mu_1)$, if for all $A \in \mathcal{E}_1$, $B \in \mathcal{E}_2$ and $n \in \mathbf{N} \cup \{0\} \cup \{\infty\}$,

$$\mathbf{P}(\omega \in \Omega : N(A, B, \omega) = n) = \frac{\mathrm{e}^{-\mu_1(A)\mu_2(B)}[\mu_1(A)\mu_2(B)]^n}{n!}. \tag{2.3}$$

In particular, when $(E_1, \mathcal{E}_1, \mu_1) = ([0, \infty) \times D, \mathcal{B}([0, \infty) \times D), \mathrm{d}t \times \mathrm{d}x)$, we can define the compensated random martingale measure

$$M(B, A, t) = N([0, t] \times A, B) - \mu_1([0, t] \times A)\mu_2(B) \tag{2.4}$$

by assuming that $\mu_1([0, t] \times A)\mu_2(B) < \infty$ for all $(t, A, B) \in [0, \infty) \times \mathcal{B}(D) \times \mathcal{E}_2$. Moreover, let $f : E_1 \times E_2 \times \Omega \to \mathbf{R}$ be an $(\mathcal{F}_t)_{t \geq 0}$-predictable random process satisfying

$$\mathbf{E}\Big[\int_0^t\int_A\int_B |f(s, x, y)|^2 \mu_2(\mathrm{d}y)\mathrm{d}x\mathrm{d}s\Big] < \infty \tag{2.5}$$

for all $t > 0$ and $(A, B) \in \mathcal{E}_1 \times \mathcal{E}_2$. Then the stochastic integral process

$$\Big(R_t := \int_0^{t+}\int_A\int_B f(s, x, y)M(\mathrm{d}y, \mathrm{d}x, \mathrm{d}s)\Big)_{t \geq 0} \tag{2.6}$$

is a square integrable $(\mathcal{F}_t)_{t \geq 0}$-martingale. It is well-known that a (pure jump) Lévy space-time white noise admits the following structure:

$$\dot{F}(x, t) = \int_{U_0} h_1(t, x, y)\dot{M}(\mathrm{d}y, x, t) + \int_{E_2 \setminus U_0} h_2(t, x, y)\dot{N}(\mathrm{d}y, x, t) \tag{2.7}$$

for some $U_0 \in \mathcal{E}_2$ such that $\mu_2(E_2 \setminus U_0) < \infty$. Here $h_1, h_2 : [0, \infty) \times D \times E_2 \to \mathbf{R}$ are measurable maps; $\dot{M}$ and $\dot{N}$ denote the Radon-Nikodym derivatives

$$\dot{M}(\mathrm{d}y, x, t) := \frac{M(\mathrm{d}y, \mathrm{d}x, \mathrm{d}t)}{\mathrm{d}t \times \mathrm{d}x}, \quad \dot{N}(\mathrm{d}y, x, t) := \frac{N(\mathrm{d}t \times \mathrm{d}x, \mathrm{d}y)}{\mathrm{d}t \times \mathrm{d}x} \tag{2.8}$$

with $(t, x, y) \in [0, \infty) \times D \times E_2$ (see [22]).

3 Burkholder-Davis-Gundy Inequalities

In order to estimate the higher order moments of mild solutions to (1.2), we need several different versions of Burkholder-Davis-Gundy inequalities. Those are quoted from [13, Theorem 4.1] and [10, Corollary 3.1], respectively. Let us first recall the usual Burkholder's inequality (see [19]).

Proposition 3.1 *Let $f : [0, \infty) \times D \times E_2 \times \Omega \to \mathbf{R}$ be an $(\mathcal{F}_t)_{t \geq 0}$-predictable process satisfying* (2.5). *Denote by X the integral process*

$$\Big(X_t := \int_0^{t+}\int_D\int_{E_2} f(s, y, z)M(\mathrm{d}z, \mathrm{d}y, \mathrm{d}s)\Big)_{t \geq 0}.$$

Then for $T > 0$ and $q \geq 1$, there exists a constant $C_1(q) > 0$ such that

$$\mathbf{E}\Big[\sup_{t\in[0,T]}|X_t|^q\Big] \leq C_1(q)\mathbf{E}[X,X]_T^{\frac{q}{2}}, \tag{3.1}$$

where $[X,X]_t = \int_0^t\int_D\int_{E_2}|f(s,y,z)|^2 N(\mathrm{d}s\times\mathrm{d}y,\mathrm{d}z)$ is the quadratic variation process of X.

Remark 3.1 Note that, in Proposition 3.1,

$$\mathbf{E}[X,X]_t = \int_0^t\int_D\int_{E_2}|f(s,y,z)|^2\mu_2(\mathrm{d}z)\mathrm{d}y\mathrm{d}s.$$

Then Jensen's inequality yields, for $q \in [1,2]$,

$$\mathbf{E}\Big[\sup_{t\in[0,T]}|X_t|^q\Big] \leq C_1(q)\Big[\int_0^T\int_D\int_{E_2}\mathbf{E}|f(s,y,z)|^2\mu_2(\mathrm{d}z)\mathrm{d}y\mathrm{d}s\Big]^{\frac{q}{2}}. \tag{3.2}$$

Proposition 3.2 *Let $(X_t)_{t\geq 0}$ be defined as in Proposition 3.1. Then for $T > 0$ and $q \geq 2$, there exists a constant $C_2(q) > 0$ such that*

$$\sup_{t\in[0,T]}\mathbf{E}[|X_t|^q] \leq C_2(q)\Big[\int_0^T\int_D\int_{E_2}(\mathbf{E}|f(s,y,z)|^q)^{\frac{2}{q}}\mu_2(\mathrm{d}z)\mathrm{d}y\mathrm{d}s\Big]^{\frac{q}{2}}. \tag{3.3}$$

On the other hand, let $\mathcal{L}^{\mathrm{sym}}(E_2,\mathcal{E}_2)$ denote the total of all symmetric Lévy measure on $(E_2,\mathcal{E}_2)$ (see [10, Definition 2.2]). If the measure $\mu_2 \in \mathcal{L}^{\mathrm{sym}}(E_2,\mathcal{E}_2)$ for the separable Banach space E_2 and if $q \in [2,4]$, then there exists $C_3(q) > 0$ such that

$$\sup_{t\in[0,T]}\mathbf{E}[|X_t|^q] \leq C_3(q)\int_0^T\int_D\int_{E_2}\mathbf{E}|f(s,y,z)|^q\mu_2(\mathrm{d}z)\mathrm{d}y\mathrm{d}s. \tag{3.4}$$

In particular, if $q = p^n$ for some $n \in \mathbf{N}$ and $1 \leq p \leq 2$, then there exists $C_4(q) > 0$ such that

$$\mathbf{E}\Big[\sup_{t\in[0,T]}|X_t|^q\Big] \leq C_4(q)\sum_{k=1}^n\Big[\int_0^T\int_D\int_{E_2}\mathbf{E}|f(s,y,z)|^{p^k}\mu_2(\mathrm{d}z)\mathrm{d}y\mathrm{d}s\Big]^{p^{n-k}}. \tag{3.5}$$

In what follows, we turn to the definition of the solutions to (1.2). An $(\mathcal{F}_t)_{t\geq 0}$-adapted random field $u = (u(t,x))_{(t,x)\in[0,T]\times D}$ is called a weak solution of (1.2), if for all $\varphi \in C_0^\infty([0,T]\times\mathbf{R}^d)$ with $\frac{\partial\varphi}{\partial n}|_{[0,T]\times\partial D} = \frac{\partial\Delta\varphi}{\partial n}|_{[0,T]\times\partial D} = 0$, it holds that

$$\begin{aligned}\langle u(t),\varphi(t)\rangle = \langle\psi,\varphi(0)\rangle &+ \int_0^t\Big\langle\Big(\frac{\partial}{\partial t}-\Delta^2\Big)\varphi(s),u(s)\Big\rangle\mathrm{d}s\\ &+\int_0^t\langle\Delta\varphi(s),b(u)(s)\rangle\mathrm{d}s + \int_0^t\int_D\varphi(s,x)B^H(\mathrm{d}x,\mathrm{d}s)\\ &+\int_0^t\int_D\varphi(s,x)a(u(s,x))F(\mathrm{d}x,\mathrm{d}s),\end{aligned} \tag{3.6}$$

where $\langle\cdot,\cdot\rangle$ denotes the usual inner product of $L^2(D)$.

Let $G(t,x,y) : [0,t]\times D^2 \to \mathbf{R}$ be the Green kernel of the operator $\frac{\partial}{\partial t}+\Delta^2$ with the homogeneous Neumann boundary condition as in (1.2) (see Appendix). Then by virtue of the proof of [20, Theorem 2.1] (or see [5, (1.9)–(1.10)]), (3.6) is equivalent to the following mild form in the sense of Walsh [23]. For $(t,x)\in[0,T]\times D$,

$$\begin{aligned} u(t,x) &= \int_D G(t,x,y)\psi(y)\mathrm{d}y + \int_0^t\int_D b(u(s,y))\Delta_y G(t-s,x,y)\mathrm{d}y\mathrm{d}s \\ &\quad + \int_0^t\int_D G(t-s,x,y)a(u(s,y))g(s,y)\mathrm{d}y\mathrm{d}s + \int_0^t\int_D G(t-s,x,y)B^H(\mathrm{d}y,\mathrm{d}s) \\ &\quad + \int_0^{t+}\int_D\int_{E_2} G(t-s,x,y)a(u(s-,y))h(s,y,z)M(\mathrm{d}z,\mathrm{d}y,\mathrm{d}s), \end{aligned} \tag{3.7}$$

where the maps g,h are given respectively by

$$\begin{aligned} g(t,y) &= \int_{E_2\setminus U_0} h_2(t,y,z)\mu_2(\mathrm{d}z), \\ h(t,y,z) &= h_1(t,y,z)\mathbf{1}_{U_0}(z) + h_2(t,y,z)\mathbf{1}_{E_2\setminus U_0}(z), \end{aligned}$$

with indicator $\mathbf{1}_A(\,\cdot\,)$ of the set $A\in\mathcal{E}_2$. On the other hand, as in Section 2, the fractional integral term in (3.7) can be represented as

$$\int_0^t\int_D G(t-s,x,y)B^H(\mathrm{d}y,\mathrm{d}s) = \int_0^t\int_D [K_H^* G(t-\,\cdot\,,x,\,\cdot\,)](s,y)W(\mathrm{d}y,\mathrm{d}s), \tag{3.8}$$

with the space-time white noise $(W(t,x))_{(t,x)\in[0,T]\times D}$ mentioned in Section 2. We mainly study the existence of a local mild solution of (3.7). To achieve it, let $\|\cdot\|_q$ denote the usual norm of $L^q(D)$ with $q\in[1,\infty)$. Given $n\in\mathbf{N}$, define a C^1-function $\Psi_n:[0,\infty)\to[0,\infty)$ by

$$\Psi_n(x) = \begin{cases} 1, & \text{if } x<n, \\ 0, & \text{if } x\ge n+1, \end{cases} \tag{3.9}$$

and $\|\Psi_n'\|_\infty := \sup\limits_{x\ge 0}|\Psi_n'(x)|\le 2$. Let the random field $(u_n(t,x))_{(t,x)\in[0,T]\times D}$ be a unique solution of the following:

$$\begin{aligned} u_n(t,x) &= \int_D G(t,x,y)\psi(y)\mathrm{d}y + \int_0^t\int_D G(t-s,x,y)B^H(\mathrm{d}y,\mathrm{d}s) \\ &\quad + \int_0^t\int_D b(u_n(s,y))\Delta_y G(t-s,x,y)\Psi_n(\|u_n(s,\,\cdot\,)\|_q)\mathrm{d}y\mathrm{d}s \\ &\quad + \int_0^t\int_D G(t-s,x,y)a(u_n(s,y))g(s,y)\Psi_n(\|u_n(s,\,\cdot\,)\|_q)\mathrm{d}y\mathrm{d}s \\ &\quad + \int_0^{t+}\int_D\int_{E_2} G(t-s,x,y)a(u_n(s-,y))h(s,y,z)M(\mathrm{d}z,\mathrm{d}y,\mathrm{d}s). \end{aligned} \tag{3.10}$$

Define $\tau_n=\inf\{t>0;\|u_n(t,\,\cdot\,)\|_q\ge n\}$ with $n\in\mathbf{N}$. Then, on the event $\{t<\tau_n\}$, $u_1(t,x)=u_2(t,x)=\cdots=u_n(t,x)$ is a solution of (3.10). Let $\tau=\lim\limits_{n\to\infty}\tau_n$, and define $u(t,\,\cdot\,)=u_n(t,\,\cdot\,)$, on the event $\{t<\tau_n<\tau\}$. Therefore $u(t,\,\cdot\,)$ is a solution of (3.10) on $\{t<\tau\}$. We call it a local mild solution of (1.2). In the following section, we will prove that such a local solution as in (3.10) exists and it is unique.

4 Main Results and Proofs

At the beginning, we state the main result of the paper as follows. Let the interval $(\alpha,\beta]=(\alpha,\beta]$ if $\alpha<\beta$, and $\emptyset$ otherwise.

Theorem 4.1 *Let $H\in(\frac{1}{2},1)$ and $d<\frac{4H}{2-H}$ with $d\in\mathbf{N}$. Suppose that the following conditions are satisfied:*

(i) *b is a polynomial of degree 3 with positive dominant coefficients.*

(ii) *a is Lipschitzian and has linear growth on $\mathbf{R}$, i.e. there exists a constant $C>0$ such that $|a(x)|\le C(1+|x|)$, for all $x\in\mathbf{R}$.*

(iii) *For g,h and μ_2,*

$$\sup_{t\in[0,T]}\|g(t,\,\cdot\,)\|_q<\infty\quad\text{with } q>d+2. \tag{4.1}$$

(1) *For $q\in(d+2,4]$,*

(a) $V_q:=\sup_{(s,y)\in[0,T]\times D}\|h(s,y,\,\cdot\,)s^{-\frac{d}{4}}\|^q_{L^q(E_2,\mu_2)}<\infty$;

(b) *$\mu_2\in\mathcal{L}^{\rm sym}(E_2,\mathcal{E}_2)$ with separable Banach space E_2.*

(2) *For $q>(d+2)\vee 4$,*

(a′) *$G(t-s,x,y)h(s,y,z)$ is $L^q([0,t]\times D^2\times E_2,\mathrm{d}s\times\mathrm{d}x\times\mathrm{d}y\times\mu_2(\mathrm{d}z))$ integrable, for $0\le t\le T$;*

(b′) *$\mu_2(E_2)<\infty$.*

Then for every $\mathcal{F}_0$-adapted initial process $\psi:D\times\Omega\to\mathbf{R}$ satisfying $\mathbf{E}\|\psi(\,\cdot\,)\|_q^q<\infty$, there exists a unique local solution $(u(t,x))_{(t,x)\in[0,T]\times D}$ for (3.7) and there exists a stopping time τ such that

$$\sup_{t\in[0,T]}\mathbf{E}\|u(t\wedge\tau,\,\cdot\,)\|_q^q<\infty\quad\text{for all } q>d+2.$$

Let Λ_q be the space of all $L^q(D)$-valued $\mathcal{F}_t$-adapted RCLL processes $u(t,\,\cdot\,)$. For fixed $\lambda>0$ and $q\in[2,\infty)$, define a norm $\|\cdot\|_{\Lambda_q}$ (depending on (λ,q)) on Λ_q by

$$\|u\|_{\Lambda_q}=\Big[\sup_{t\in[0,T]}\mathrm{e}^{-\lambda t}\mathbf{E}\|u(t,\,\cdot\,)\|_q^q\Big]^{\frac{1}{q}}<\infty, \tag{4.2}$$

with $\|\cdot\|_q$ the usual norm of $L^q(D)$. Then $(\Lambda_q,\|\cdot\|_{\Lambda_q})$ forms a Banach space. Let $\psi\in\Lambda_q$. Recall (1.2) or (3.10). For $(t,x)\in[0,T]\times D$,

$$\begin{aligned}
u_n(t,x)\ &=\int_D G(t,x,y)\psi(y)\mathrm{d}y+\int_0^t\!\!\int_D G(t-s,x,y)B^H(\mathrm{d}y,\mathrm{d}s)\\
&\quad+\int_0^t\!\!\int_D b(u_n(s,y))\Delta_y G(t-s,x,y)\Psi_n(\|u_n(s,\,\cdot\,)\|_q)\mathrm{d}y\mathrm{d}s\\
&\quad+\int_0^t\!\!\int_D G(t-s,x,y)a(u_n(s,y))g(s,y)\Psi_n(\|u_n(s,\,\cdot\,)\|_q)\mathrm{d}y\mathrm{d}s\\
&\quad+\int_0^{t+}\!\!\int_D\int_{E_2}G(t-s,x,y)a(u_n(s-,y))h(s,y,z)M(\mathrm{d}z,\mathrm{d}y,\mathrm{d}s)\\
&:=\mathcal{A}_0(\phi)(t,x)+\sum_{i=1}^4\mathcal{A}_i(u_n)(t,x).
\end{aligned} \tag{4.3}$$

According to (4.3), we have

Proposition 4.1 *Under the assumptions of Theroem* 4.1*, for each* $q > d+2$ *and* $u \in \Lambda_q$*, it holds that* $\mathcal{A}_i(u) \in \Lambda_q$*,* $i = 0, \cdots, 4$.

Proof From (A.2), Minkovski's inequality and Young's inequality for $\frac{1}{q} = 1 + \frac{1}{q} - 1$, it follows that

$$\begin{aligned}\|\mathcal{A}_0(\phi)(t,\,\cdot\,)\|_q &\le Kt^{-\frac{d}{4}}\Big\|\int_D \exp\Big(-C\frac{|\,\cdot - y|^{\frac{4}{3}}}{t^{\frac{1}{3}}}\Big)\psi(y)\mathrm{d}y\Big\|_q \\ &\le Kt^{-\frac{d}{4}}\Big\|\Big(\exp\Big(-C\frac{|\,\cdot\,|^{\frac{4}{3}}}{t^{\frac{1}{3}}}\Big) * \psi(\,\cdot\,)\Big)(\,\cdot\,)\Big\|_q \\ &\le Kt^{-\frac{d}{4}}\Big\|\exp\Big(-C\frac{|\,\cdot\,|^{\frac{4}{3}}}{t^{\frac{1}{3}}}\Big)\Big\|_1 \|\psi(\,\cdot\,)\|_q \\ &= C\|\psi(\,\cdot\,)\|_q. \end{aligned} \tag{4.4}$$

Therefore $\mathcal{A}_0(\phi) \in \Lambda_q$ if $\mathbf{E}\|\psi(\,\cdot\,)\|_q^q < \infty$. Next we turn to $\mathcal{A}_1(u)$. Let $\frac{1}{r_1} = \frac{1}{q} - \frac{1}{\rho} + 1 \in [0,1]$. Applying (A.6), we conclude that

$$\|\mathcal{A}_1(u)(t,\,\cdot\,)\|_q \le C\int_0^t (t-s)^{-\frac{d+2}{4}+\frac{d}{4r_1}}\|b(u(s,\,\cdot\,))\Psi_n(\|u(s,\,\cdot\,)\|_q)\|_\rho \mathrm{d}s. \tag{4.5}$$

In particular, let $\rho = \frac{q}{3}$. Then by the assumption (i) in Theorem 4.1,

$$\|b(u(s,\,\cdot\,))\|_\rho \le C[\|u(s,\,\cdot\,)\|_q + \|u(s,\,\cdot\,)\|_q^2 + \|u(s,\,\cdot\,)\|_q^3].$$

Consequently,

$$\|\mathcal{A}_1(u)(t,\,\cdot\,)\|_q \le C_n\int_0^t (t-s)^{-\frac{d+2}{4}+\frac{d}{4r_1}}\mathrm{d}s, \tag{4.6}$$

which is finite if $\frac{d}{4r_1} - \frac{d+2}{4} > -1$. Since $\frac{1}{r_1} = \frac{q-2}{q}$, we have $\mathcal{A}_1(u) \in \Lambda_q$ for $q > d$. As for $\mathcal{A}_2(u)$, by virtue of (A.5), we have for $\frac{1}{r_2} = \frac{1}{q} - \frac{2}{q} + 1 = -\frac{1}{q} + 1 \in [0,1]$,

$$\begin{aligned}\|\mathcal{A}_2(u)(t,\,\cdot\,)\|_q &\le C\int_0^t (t-s)^{\frac{d}{4r_2}-\frac{d}{4}}\|g(s,\,\cdot\,)a(u(s,\,\cdot\,))\Psi_n(\|u(s,\,\cdot\,)\|_q)\|_{\frac{q}{2}}\mathrm{d}s \\ &\le C\int_0^t (t-s)^{\frac{d}{4r_2}-\frac{d}{4}}\|g(s,\,\cdot\,)(1+|u(s,\,\cdot\,)|)\Psi_n(\|u(s,\,\cdot\,)\|_q)\|_{\frac{q}{2}}\mathrm{d}s \\ &\le C\int_0^t (t-s)^{\frac{d}{4r_2}-\frac{d}{4}}[\|g(s,\,\cdot\,)\|_{\frac{q}{2}} + \|g(s,\,\cdot\,)u(s,\,\cdot\,)\|_{\frac{q}{2}}]\Psi_n(\|u(s,\,\cdot\,)\|_q)\mathrm{d}s \\ &\le C_q\int_0^t (t-s)^{\frac{d}{4r_2}-\frac{d}{4}}[\|g(s,\,\cdot\,)\|_q + \|g(s,\,\cdot\,)\|_q\|u(s,\,\cdot\,)\|_q]\Psi_n(\|u(s,\,\cdot\,)\|_q)\mathrm{d}s \\ &\le C_q\int_0^t (t-s)^{\frac{d}{4r_2}-\frac{d}{4}}[\|g(s,\,\cdot\,)\|_q + (n+1)\|g(s,\,\cdot\,)\|_q]\mathrm{d}s \\ &\le C_{n,q}\sup_{t\in[0,T]}\|g(t,\,\cdot\,)\|_q\int_0^t (t-s)^{\frac{d}{4r_2}-\frac{d}{4}}\mathrm{d}s \\ &< \infty, \end{aligned} \tag{4.7}$$

provided $q > \frac{d}{4}$. Therefore $\mathcal{A}_2(u) \in \Lambda_q$ as $q > \frac{d}{4}$. In what follows, let us consider $\mathcal{A}_3(u)$. Applying the B-D-G inequality (3.1), we conclude that,

$$\begin{aligned}
\mathbf{E}\|\mathcal{A}_3(u)(t,\cdot)\|_q^q &= \int_D \mathbf{E}\Big|\int_0^t\int_D G(t-s,x,y)B^H(\mathrm{d}y,\mathrm{d}s)\Big|^q \mathrm{d}x \\
&= \int_D \mathbf{E}\Big|\int_0^t\int_D (K_H^* G(t-\cdot,x,\cdot))(s,y)W(\mathrm{d}y,\mathrm{d}s)\Big|^q \mathrm{d}x \\
&\le C_q \int_D \mathbf{E}\Big(\int_0^t\int_D (K_H^* G(t-\cdot,x,\cdot))^2(s,y)\mathrm{d}y\mathrm{d}s\Big)^{\frac{q}{2}} \mathrm{d}x \\
&= C_q \int_D \langle (K_H^* G(t-\cdot,x,\cdot))(\cdot,\cdot), (K_H^* G(t-\cdot,x,\cdot))(\cdot,\cdot)\rangle_{L^2([0,T]\times D)}^{\frac{q}{2}} \mathrm{d}x \\
&= C_q \int_D \langle G(t-\cdot,x,\cdot), G(t-\cdot,x,\cdot)\rangle_{\mathcal{H}}^{\frac{q}{2}} \mathrm{d}x \\
&\le C_q \int_D \|G(t-\cdot,x,\cdot)\|^q_{L^{\frac{2}{H}}([0,T]\times D)} \mathrm{d}x, \qquad (4.8)
\end{aligned}$$

where we have used the fact that $L^{\frac{2}{H}}([0,T]\times D) \subset \mathcal{H}$ when $H > \frac{1}{2}$ (see [15, Theorem 1], but we need to modify their proof which is given in Appendix below). Note that

$$\begin{aligned}
\|G(t-\cdot,x,\cdot)\|^q_{L^{\frac{2}{H}}([0,T]\times D)} &= \Big[\int_0^T\int_D |G(t-s,x,y)|^{\frac{2}{H}}\mathrm{d}y\mathrm{d}s\Big]^{\frac{qH}{2}} \\
&= \Big[\int_0^t\int_D |G(t-s,x,y)|^{\frac{2}{H}}\mathrm{d}y\mathrm{d}s\Big]^{\frac{qH}{2}} \\
&\le \Big[\int_0^t (t-s)^{-\frac{d}{2H}}\int_D \exp\Big(-C_H\frac{|x-y|^{\frac{4}{3}}}{(t-s)^{\frac{1}{3}}}\Big)\mathrm{d}y\mathrm{d}s\Big]^{\frac{qH}{2}} \\
&\le C_H\Big[\int_0^t (t-s)^{\frac{d}{4}-\frac{d}{2H}}\mathrm{d}s\Big]^{\frac{qH}{2}} \\
&\le C_H T^{\frac{qH}{2}(1+\frac{d}{4}-\frac{d}{2H})} \\
&< \infty, \qquad (4.9)
\end{aligned}$$

under the assumption $d < \frac{4H}{2-H}$ of Theorem 4.1. So we have $A_3(u) \in \Lambda_q$ for $q \ge 2$. Now we estimate $\mathcal{A}_4(u)$. In the case of $q \in (d+2, 4]$, the inequality (3.4) of Proposition 3.2 yields

$$\begin{aligned}
\|\mathcal{A}_4(u)\|^q_{\Lambda_q} &= \sup_{t\in[0,T]} \mathrm{e}^{-\lambda t}\mathbf{E}\|\mathcal{A}_4(u)(t,\cdot)\|_q^q \\
&= \sup_{t\in[0,T]} \mathrm{e}^{-\lambda t}\int_D \mathbf{E}\Big(\Big|\int_0^{t+}\int_D\int_{E_2} G(t-s,x,y)h(s,y,z) \\
&\quad \times a(u(s-,y))M(\mathrm{d}z,\mathrm{d}y,\mathrm{d}s)\Big|^q\Big)\mathrm{d}x \\
&\le C_q \sup_{t\in[0,T]} \mathrm{e}^{-\lambda t}\int_D \Big(\int_0^t\int_D\int_{E_2} |G(t-s,x,y)h(s,y,z)|^q \\
&\quad \times \mathbf{E}[|a(u(s,y))|^q]\mu_2(\mathrm{d}z)\mathrm{d}y\mathrm{d}s\Big)\mathrm{d}x \\
&= C_q \sup_{t\in[0,T]} \mathrm{e}^{-\lambda t}\int_0^t\int_D \Big(\int_D\int_{E_2} |G(t-s,x,y)h(s,y,z)|^q \mu_2(\mathrm{d}z)\mathrm{d}x\Big)
\end{aligned}$$

$$
\begin{aligned}
&\times \mathbf{E}[|a(u(s,y))|^q]\mathrm{d}y\mathrm{d}s\\
&\le C_q V_q \sup_{t\in[0,T]} \mathrm{e}^{-\lambda t}\int_0^t |t-s|^{\frac{d}{4}}\mathbf{E}\Big[\int_D |a(u(s,y))|^q\mathrm{d}y\Big]\mathrm{d}s\\
&\le C_q V_q \sup_{t\in[0,T]} \mathrm{e}^{-\lambda t}\int_0^t |t-s|^{\frac{d}{4}}\mathbf{E}[\|a(u(s,\cdot))\|_q^q]\mathrm{d}s\\
&\le C_q V_q \sup_{t\in[0,T]} \int_0^t |t-s|^{\frac{d}{4}}\mathrm{e}^{-\lambda(t-s)}(1+\mathrm{e}^{-\lambda s}\mathbf{E}[\|u(s,\cdot)\|_q^q])\mathrm{d}s\\
&\le C_q V_q (1+\|u\|_{\Lambda_q}^q)\int_0^T s^{\frac{d}{4}}\mathrm{e}^{-\lambda s}\mathrm{d}s\\
&\le C_q V_q (1+\|u\|_{\Lambda_q}^q)\frac{\Gamma(\frac{d}{4}+1)}{\lambda^{\frac{d}{4}+1}}\\
&< \infty,
\end{aligned}
\tag{4.10}
$$

where V_q is defined by Theorem 4.1(iii)(1)(a) and $\Gamma(\cdot)$ denotes the Gamma function.

As for $q > (d+2)\vee 4$, by the hypotheses Theorem 4.1(ii), (iii) and the inequality (3.3), it follows that

$$
\begin{aligned}
\|\mathcal{A}_4(u)\|_{\Lambda_q}^q &= \sup_{t\in[0,T]} \mathrm{e}^{-\lambda t}\mathbf{E}\|\mathcal{A}_4(u)(t,\cdot)\|_q^q\\
&= \sup_{t\in[0,T]} \mathrm{e}^{-\lambda t}\int_D \mathbf{E}\Big(\Big|\int_0^{t+}\int_D\int_{E_2} G(t-s,x,y)h(s,y,z)\\
&\quad \times a(u(s-,y))M(\mathrm{d}z,\mathrm{d}y,\mathrm{d}s)\Big|^q\Big)\mathrm{d}x\\
&\le C_q \sup_{t\in[0,T]} \mathrm{e}^{-\lambda t}\int_D\Big(\int_0^t\int_D\int_{E_2}(\mathbf{E}|G(t-s,x,y)h(s,y,z)a(u(s,y))|^q)^{\frac{2}{q}}\\
&\quad \times \mu_2(\mathrm{d}z)\mathrm{d}y\mathrm{d}s\Big)^{\frac{q}{2}}\mathrm{d}x\\
&\le C_q \sup_{t\in[0,T]} \mathrm{e}^{-\lambda t}\int_D\Big(\int_0^t\int_D\int_{E_2}|G(t-s,x,y)h(s,y,z)|^2\mathrm{e}^{\frac{2\lambda s}{q}}\\
&\quad \times (\mathrm{e}^{-\lambda s}\mathbf{E}(1+|u(s,y)|)^q)^{\frac{2}{q}}\mu_2(\mathrm{d}z)\mathrm{d}y\mathrm{d}s\Big)^{\frac{q}{2}}\mathrm{d}x\\
&\le C_q \sup_{t\in[0,T]} \int_D\Big(\int_0^t\int_D\int_{E_2}\mathrm{e}^{-\lambda s}\mathbf{E}(1+|u(s,y)|)^q\mu_2(\mathrm{d}z)\mathrm{d}y\mathrm{d}s\Big)\\
&\quad \times \Big(\int_0^t\int_D\int_{E_2}|G(t-s,x,y)h(s,y,z)|^{\frac{2q}{q-2}}\mathrm{e}^{-\frac{2\lambda(t-s)}{q-2}}\mu_2(\mathrm{d}z)\mathrm{d}y\mathrm{d}s\Big)^{\frac{q-2}{2}}\mathrm{d}x\\
&\le C_q\mu_2(E_2)\Big(T|D|+\int_0^T \mathrm{e}^{-\lambda s}\mathbf{E}\|u(s,\cdot)\|_q^q\mathrm{d}s\Big)\\
&\quad \times \sup_{t\in[0,T]}\int_D\Big(\int_0^t\int_D\int_{E_2}|G(t-s,x,y)h(s,y,z)|^{\frac{2q}{q-2}}\mathrm{e}^{-\frac{2\lambda(t-s)}{q-2}}\mu_2(\mathrm{d}z)\mathrm{d}y\mathrm{d}s\Big)^{\frac{q-2}{2}}\mathrm{d}x\\
&\le C_q\mu_2(E_2)(T|D|+T\cdot\|u\|_{\Lambda_q}^q)\\
&\quad \times \int_D\int_0^T\int_D\int_{E_2}|G(t-s,x,y)h(s,y,z)|^q\mathrm{e}^{-\lambda(t-s)}\mu_2(\mathrm{d}z)\mathrm{d}y\mathrm{d}s\mathrm{d}x\\
&< \infty.
\end{aligned}
\tag{4.11}
$$

This shows that $\mathcal{A}_4(u) \in \Lambda_q$. Thus we complete the proof of the proposition.

We can now define an operator $\mathcal{K}$ on Λ_q by

$$\mathcal{K}(u)(t,x) = \mathcal{A}_0(\phi)(t,x) + \sum_{i=1}^{4} \mathcal{A}_i(u)(t,x), \quad (t,x) \in [0,T] \times D. \tag{4.12}$$

In what follows, we will prove that the operator $\mathcal{K} : \Lambda_q \to \Lambda_q$ is a contract mapping.

Theorem 4.2 *For $q > d+2$, the operator $\mathcal{K}$ defined by* (4.12) *is a contraction on Λ_q under the conditions of Theorem* 4.1. *In other words, there exists a constant $\varrho \in (0,1)$ such that $\|\mathcal{K}(u) - \mathcal{K}(v)\|_{\Lambda_q} \le \varrho \|u - v\|_{\Lambda_q}$ for $u, v \in \Lambda_q$.*

Proof Suppose that ψ_1, ψ_2 are initials of $(\mathcal{F}_t)_{t\ge 0}$-adapted random fields $u, v \in \Lambda_q$ such that $\psi_1 = \psi_2$. Let us begin by considering $\mathcal{A}_1$. Note that for $\rho = \frac{q}{3}$,

$$\|\Psi_n(\|u(s,\cdot)\|_q) b(u(s,\cdot)) - \Psi_n(\|v(s,\cdot)\|_q) b(v(s,\cdot))\|_\rho \le C_n \|u(s,\cdot) - v(s,\cdot)\|_\rho. \tag{4.13}$$

By virtue of (A.6), we have for $\frac{1}{r} = \frac{1}{q} - \frac{1}{\rho} + 1 = -\frac{2}{q} + 1 \in [0,1]$,

$$\begin{aligned}
\|\mathcal{A}_1(u) - \mathcal{A}_1(v)\|_{\Lambda_q}^q &= \sup_{t\in[0,T]} \mathrm{e}^{-\lambda t} \mathbf{E}(\|\mathcal{A}_1(u)(t,\cdot) - \mathcal{A}_1(v)(t,\cdot)\|_q^q) \\
&\le \sup_{t\in[0,T]} \mathrm{e}^{-\lambda t} \mathbf{E}\Big(\int_0^t (t-s)^{\frac{d}{4r} - \frac{d+2}{4}} \\
&\quad \times \|\Psi_n(\|u(s,\cdot)\|_q) b(u(s,\cdot)) - \Psi_n(\|v(s,\cdot)\|_q) b(v(s,\cdot))\|_\rho \mathrm{d}s \Big)^q \\
&\le C_n \sup_{t\in[0,T]} \mathrm{e}^{-\lambda t} \mathbf{E}\Big(\int_0^t (t-s)^{\frac{d}{4r} - \frac{d+2}{4}} \|u(s,\cdot) - v(s,\cdot)\|_q \mathrm{d}s \Big)^q \\
&\le C_{n,q} \sup_{t\in[0,T]} \mathbf{E}\Big(\int_0^t \mathrm{e}^{\frac{-\lambda(t-s)}{q}} (t-s)^{\frac{d}{4r} - \frac{d+2}{4}} \mathrm{e}^{\frac{-\lambda s}{q}} \|u(s,\cdot) - v(s,\cdot)\|_q \mathrm{d}s \Big)^q \\
&\le C_{n,q} \sup_{t\in[0,T]} \mathbf{E}\Big(\int_0^t \mathrm{e}^{-\lambda s} \|u(s,\cdot) - v(s,\cdot)\|_q^q \mathrm{d}s \\
&\quad \times \Big[\int_0^t (\mathrm{e}^{\frac{-\lambda(t-s)}{q}} (t-s)^{\frac{d}{4r} - \frac{d+2}{4}})^{\frac{q}{q-1}} \mathrm{d}s \Big]^{q-1} \Big) \\
&\le C_{n,q} \sup_{t\in[0,T]} \Big(\int_0^t \mathrm{e}^{-\lambda s} \mathbf{E} \|u(s,\cdot) - v(s,\cdot)\|_q^q \mathrm{d}s \\
&\quad \times \Big[\int_0^t (\mathrm{e}^{\frac{-\lambda(t-s)}{q-1}} (t-s)^{\frac{q}{q-1}(\frac{d}{4r} - \frac{d+2}{4})}) \mathrm{d}s \Big]^{q-1} \Big) \\
&\le C_{n,q} T \|u - v\|_{\Lambda_q}^q \Psi(d,q,T),
\end{aligned} \tag{4.14}$$

where

$$\Psi(d,q,T) = \Big[\int_0^T \mathrm{e}^{\frac{-\lambda(t-s)}{q-1}} (t-s)^{\frac{q}{q-1}(\frac{d}{4r} - \frac{d+2}{4})} \mathrm{d}s \Big]^{q-1}.$$

Let

$$k = \frac{q}{q-1} \Big(\frac{d}{4r} - \frac{d+2}{4} \Big).$$

Then

$$\Psi(d,q,T) \le \Big[\frac{(q-1)^{k+1}}{\lambda^{k+1}}\int_0^\infty \mathrm{e}^{-s}s^k\mathrm{d}s\Big]^{q-1} = \Big[\frac{(q-1)^{k+1}\Gamma(k+1)}{\lambda^{k+1}}\Big]^{q-1} < \infty, \tag{4.15}$$

when $q > d+2$. Therefore,

$$\begin{aligned}\|\mathcal{A}_1(u)-\mathcal{A}_1(v)\|_{\Lambda_q} &\le C(n,q)T^{\frac{1}{q}}\Big[\frac{(q-1)^{\frac{k+1}{q}}\Gamma(k+1)^{\frac{1}{q}}}{\lambda^{\frac{k+1}{q}}}\Big]^{q-1}\|u-v\|_{\Lambda_q}\\ &\le \varrho_1\|u-v\|_{\Lambda_q},\end{aligned} \tag{4.16}$$

where $\varrho_1 \in (0,1)$ by choosing λ large enough.

Next we consider $\mathcal{A}_4(u)$. In the case of $q \in (d+2,4]$, from a similar argument as in (4.10), it follows that

$$\begin{aligned}\|\mathcal{A}_4(u)-\mathcal{A}_4(v)\|_{\Lambda_q}^q &= \sup_{t\in[0,T]}\mathrm{e}^{-\lambda t}\mathbf{E}\|\mathcal{A}_4(u)(t,\cdot)-\mathcal{A}_4(v)(t,\cdot)\|_q^q\\ &= \sup_{t\in[0,T]}\mathrm{e}^{-\lambda t}\int_D\mathbf{E}\Big[\Big|\int_0^{t+}\int_D\int_{E_2}G(t-s,x,y)h(s,y,z)\\ &\quad\times a((u(s-,y))-a(v(s-,y)))M(\mathrm{d}z,\mathrm{d}y,\mathrm{d}s)\Big|^q\Big]\mathrm{d}x\\ &= C_q\sup_{t\in[0,T]}\mathrm{e}^{-\lambda t}\int_0^t\int_D\Big(\int_D\int_{E_2}|G(t-s,x,y)h(s,y,z)|^q\mu_2(\mathrm{d}z)\mathrm{d}x\Big)\\ &\quad\times\mathbf{E}|u(s,y)-v(s,y)|^q\mathrm{d}y\mathrm{d}s\\ &\le C_qV_q\|u-v\|_{\Lambda_q}^q\frac{\Gamma(\frac{d}{4}+1)}{\lambda^{\frac{d}{4}+1}}\\ &< \infty.\end{aligned} \tag{4.17}$$

For $q > (d+2)\vee 4$, thanks to Proposition 3.1, we derive from the assumptions (ii) and (iii) of Theorem 4.1 that

$$\begin{aligned}&\|\mathcal{A}_4(u)-\mathcal{A}_4(v)\|_{\Lambda_q}^q\\ &= \sup_{t\in[0,T]}\mathrm{e}^{-\lambda t}\mathbf{E}\|\mathcal{A}_4(u)(t,\cdot)-\mathcal{A}_4(v)(t,\cdot)\|_q^q\\ &= \sup_{t\in[0,T]}\mathrm{e}^{-\lambda t}\int_D\mathbf{E}\Big|\int_0^{t+}\int_D\int_{E_2}G(t-s,x,y)h(s,y,z)\\ &\quad\times(a(u(s-,y))-a(v(s-,y)))M(\mathrm{d}z,\mathrm{d}y,\mathrm{d}s)\Big|^q\mathrm{d}x\\ &\le C_q\sup_{t\in[0,T]}\mathrm{e}^{-\lambda t}\int_D\Big(\int_0^t\int_D\int_{E_2}(\mathbf{E}|G(t-s,x,y)h(s,y,z)\\ &\quad\times(a(u(s,y))-a(v(s,y)))|^q)^{\frac{2}{q}}\mu_2(\mathrm{d}z)\mathrm{d}y\mathrm{d}s\Big)^{\frac{q}{2}}\mathrm{d}x\\ &\le C_q\sup_{t\in[0,T]}\mathrm{e}^{-\lambda t}\int_D\Big(\int_0^t\int_D\int_{E_2}|G(t-s,x,y)h(s,y,z)|^2\\ &\quad\times(\mathbf{E}|u(s,y)-v(s,y)|^q)^{\frac{2}{q}}\mu_2(\mathrm{d}z)\mathrm{d}y\mathrm{d}s\Big)^{\frac{q}{2}}\mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
&= C_q \sup_{t\in[0,T]} \mathrm{e}^{-\lambda t} \int_D \Big(\int_0^t\int_D\int_{E_2} |G(t-s,x,y)h(s,y,z)|^2 \mathrm{e}^{\frac{2\lambda s}{q}} \\
&\quad \times (\mathrm{e}^{-\lambda s}\mathbf{E}|u(s,y)-v(s,y)|^q)^{\frac{2}{q}} \mu_2(\mathrm{d}z)\mathrm{d}y\mathrm{d}s\Big)^{\frac{q}{2}} \mathrm{d}x \\
&\le C_q \sup_{t\in[0,T]} \int_D \Big(\int_0^t\int_D\int_{E_2} \mathrm{e}^{-\lambda s}\mathbf{E}|u(s,y)-v(s,y)|^q \mu_2(\mathrm{d}z)\mathrm{d}y\mathrm{d}s\Big) \\
&\quad \times \Big(\int_0^t\int_D\int_{E_2} |G(t-s,x,y)h(s,y,z)|^{\frac{2q}{q-2}} \mathrm{e}^{-\frac{2\lambda(t-s)}{q-2}} \mu_2(\mathrm{d}z)\mathrm{d}y\mathrm{d}s\Big)^{\frac{q-2}{2}} \mathrm{d}x \\
&\le C_q T\mu_2(E_2)\|u-v\|_{\Lambda_q}^q \\
&\quad \times \sup_{t\in[0,T]} \int_D\int_0^t\int_D\int_{E_2} |G(t-s,x,y)h(s,y,z)|^q \mathrm{e}^{-\lambda(t-s)} \mu_2(\mathrm{d}z)\mathrm{d}y\mathrm{d}s\mathrm{d}x. \qquad (4.18)
\end{aligned}
$$

Take into account

$$
V_q \frac{\Gamma(\frac{d}{4}+1)}{\lambda^{\frac{d}{4}+1}} \to 0, \quad \text{as } \lambda\to\infty
$$

and

$$
\sup_{t\in[0,T]} \int_D\int_0^t\int_D\int_{E_2} |G(t-s,x,y)h(s,y,z)|^q \mathrm{e}^{-\lambda(t-s)} \mu_2(\mathrm{d}z)\mathrm{d}y\mathrm{d}s\mathrm{d}x \to 0, \quad \text{as } \lambda\to\infty.
$$

Then by (4.17) and (4.18), together with the assumption (iii) of Theorem 4.1, $\mathcal{A}_4$ is a contraction on Λ_q, for $\lambda>0$ large enough.

For $q>d+2$, a similar procedure as (4.14)–(4.16) yields that $\mathcal{A}_2$ is a contraction on Λ_q, by letting $\lambda>0$ large enough. Therefore, it follows from (4.12) that $\mathcal{K}(\cdot)$ is a contraction on Λ_q if $\lambda>0$ large enough. Thus the proof of Theorem 4.2 is completed.

We note that $h\to\frac{4h}{2-h}$ is an increasing function with the range $(\frac{4}{3},4)$ on $h\in(\frac{1}{2},1)$. Hence for $q>d+2$, applying the fixed point principal on the set $\{u\in\Lambda_q : u(0)=\psi\}$, we conclude that (3.10) admits a unique solution $u\in\Lambda_q$. Thus the conclusion of Theorem 4.1 follows.

Appendix

Firstly, we will give a short proof for the assertion that $L^{\frac{2}{H}}([0,T]\times D)\subset\mathcal{H}$ if $H>\frac{1}{2}$.

Let

$$
f\in L^{\frac{2}{H}}([0,T]\times D) \quad\text{and}\quad d_H = H(2H-1).
$$

Then Theorem 1 in [15] implies that there exists a positive constant $C(H,T,d)$ depending on H,T,d such that

$$
\begin{aligned}
\|f\|_{\mathcal{H}}^2 &:= d_H \int_0^T\int_0^T\int_D f(u,x)f(v,x)|u-v|^{2H-2}\mathrm{d}x\mathrm{d}u\mathrm{d}v \\
&= d_H \int_D \Big[\int_0^T\int_0^T f(u,x)f(v,x)|u-v|^{2H-2}\mathrm{d}u\mathrm{d}v\Big]\mathrm{d}x
\end{aligned}
$$

$$\begin{aligned}
&\leq C_H \int_D \Big[\int_0^T |f(u,x)|^{\frac{1}{H}}\mathrm{d}u\Big]^{2H}\mathrm{d}x \\
&\leq C_{H,T} \int_D \Big[\int_0^T |f(u,x)|^{\frac{2}{H}}\mathrm{d}u\Big]^{H}\mathrm{d}x \\
&\leq C_{H,T,d}\Big[\int_D\int_0^T |f(u,x)|^{\frac{2}{H}}\mathrm{d}u\mathrm{d}x\Big]^{H} \\
&= C_{H,T,d}\Big[\int_D\int_0^T |f(u,x)|^{\frac{2}{H}}\mathrm{d}u\mathrm{d}x\Big]^{\frac{H}{2}\times 2} \\
&= C_{H,T,d}\|f\|^2_{L^{\frac{2}{H}}([0,T]\times D)}, \qquad\qquad (A.1)
\end{aligned}$$

where we have used Hölder inequality twice. This shows the continuity of the embedding.

In the following, we will give some estimates on the Green kernel $G(t,x,y)$ corresponding to the operator $\frac{\partial}{\partial t}+\Delta^2$ on the domain $[0,\infty)\times D$. As in [8], the Green function $G(t,x,y)$ admits the following expansion. Let $A=-\Delta$ be defined on $D(A)=\{u\in H^2(D):\frac{\partial u}{\partial n}|_{\partial D}=0\}$ and let $(\Theta_k)_{k\in\mathbf{N}^d}$ be the basis of eigenfunctions of A in $L^2(D)$, which can be written as

$$\Theta_k(x)=\prod_{i=1}^d\theta_{k_i}(x_i),$$

where $k=(k_1,\cdots,k_d)\in\mathbf{N}^d$, $x=(x_1,\cdots,x_d)\in D$. Moreover,

$$\begin{cases}\theta_{k_i}(x_i)=\sqrt{\dfrac{2}{\pi}}\cos(k_ix_i), & k_i\neq 0,\\ \theta_0(x_i)=\dfrac{1}{\sqrt{\pi}}, & k_i=0\end{cases}$$

with $i=1,\cdots,d$, and $\Big(\lambda_k=\sum\limits_{i=1}^d k_i^2\Big)_{k\in\mathbf{N}^d}$ are the eigenvalues corresponding to the eigenfunctions. Therefore the Green function $G(t,x,y)$ on $[0,\infty)\times D^2$ can be expressed as

$$G(t,x,y)=\sum_{k\in\mathbf{N}^d}\mathrm{e}^{-\lambda_k^2t}\Theta_k(x)\Theta_k(y)$$

with $(t,x,y)\in[0,\infty)\times D^2$.

Lemma A.1 *There exist $K>0$ and $C>0$ such that for all $t\in(0,T]$, $x,y\in D$,*

$$|G(t,x,y)|\leq\frac{K}{t^{\frac{d}{4}}}\exp\Big(-C\frac{|x-y|^{\frac{4}{3}}}{t^{\frac{1}{3}}}\Big), \tag{A.2}$$

$$|\Delta_yG(t,x,y)|\leq\frac{K}{t^{\frac{d+2}{4}}}\exp\Big(-C\frac{|x-y|^{\frac{4}{3}}}{t^{\frac{1}{3}}}\Big), \tag{A.3}$$

$$\Big|\frac{\partial G(t,x,y)}{\partial t}\Big|\leq\frac{K}{t^{\frac{d+4}{4}}}\exp\Big(-C\frac{|x-y|^{\frac{4}{3}}}{t^{\frac{1}{3}}}\Big). \tag{A.4}$$

Lemma A.2 *For $v\in L^1([0,T],L^\rho(D))$, $0\leq t_0\leq t\leq T$ and $x\in D$, define*

$$J(v)(t_0,t,x)=\int_{t_0}^t\int_D H(t-s,x,y)v(s,y)\mathrm{d}y\mathrm{d}s.$$

Then for any $\rho \in [1, \infty)$, $\frac{1}{r} = \frac{1}{q} - \frac{1}{\rho} + 1 \in [0, 1]$. J *is a bounded operator from* $L^1([0, T], L^\rho(D))$ *to* $L^\infty([0, T], L^q(D))$. *Furthermore,*

(1) *If* $H(t-s, x, y) = G(t-s, x, y)$, *there exists a constant* $C > 0$ *such that*

$$\|J(v)(t_0, t, \cdot)\|_q \leq C \int_{t_0}^{t} (t-s)^{\frac{d}{4r} - \frac{d}{4}} \|v(s, \cdot)\|_\rho \mathrm{d}s. \tag{A.5}$$

(2) *If* $H(t-s, x, y) = \Delta_y G(t-s, x, y)$, *there exists a constant* $C > 0$ *such that*

$$\|J(v)(t_0, t, \cdot)\|_q \leq C \int_{t_0}^{t} (t-s)^{\frac{d}{4r} - \frac{d+2}{4}} \|v(s, \cdot)\|_\rho \mathrm{d}s, \tag{A.6}$$

where we assume that $r \neq \infty$ *if* $d = 2$, *and* $r < 3$, *if* $d = 3$.

(3) *If* $H(t-s, x, y) = G^2(t-s, x, y)$, *there exists a constant* $C > 0$ *such that*

$$\|J(v)(t_0, t, \cdot)\|_q \leq C \int_{t_0}^{t} (t-s)^{\frac{d}{4r} - \frac{d}{2}} \|v(s, \cdot)\|_\rho \mathrm{d}s, \tag{A.7}$$

where we assume that $r \neq \infty$ *if* $d = 2$, *and* $r < \frac{3}{2}$, *if* $d = 3$.

Acknowledgement The authors would like to thank the referees and editors for their carefully reading the manuscript and for their helpful comments.

References

[1] Albeverio, J., Wu, J. and Zhang, T., Parabolic SPDEs driven by Poisson white noise, *Stoch. Proc. Appl.*, **74**, 1998, 21–36.

[2] Bo, L., Jiang, Y. and Wang, Y., On a class of stochastic Anderson models with fractional noises, *Stoch. Anal. Appl.*, **26**, 2008, 256–273.

[3] Bo, L., Shi, K. and Wang, Y., On a nonlocal stochastic Kuramoto-Sivashinsky equation with jumps, *Stoch. Dyn.*, **7**, 2007, 439–457.

[4] Bo, L. and Wang, Y., Stochastic Cahn-Hilliard partial differential equations with Lévy spacetime white noises, *Stoch. Dyn.*, **6**(2), 2006, 229–244.

[5] Cardon-Weber, C., Cahn-Hilliard stochastic equation: existence of the solution and of its density, *Bernoulli*, **7**(5), 2001, 777–816.

[6] Cardon-Weber, C. and Millet, A., On strongly Petrovskiĭ's parabolic SPDEs in arbitrary dimension and application to the stochastic Cahn-Hilliard equation, *J. Th. Prob.*, **17**(1), 2001, 1–49.

[7] Cahn, J. and Hilliard, J., Free energy for a nonuniform system I, Interfacial free energy, *J. Chem. Phys.*, **2**, 1958, 258–267.

[8] Da Prato, G. and Debussche, A., Stochastic Cahn-Hilliard equation, *Nonlinear Anal. Th. Meth. Appl.*, **26**(2), 1996, 241–263.

[9] Duncan, E., Maslowski, B. and Pasik-Duncan, B., Fractional Brownian motion and stochastic equations in Hilbert space, *Stoch. Dyn.*, **2**, 2002, 225–250.

[10] Hausenblas, E., Existence, uniqueness and regularity of parabolic SPDEs driven by Poisson random measure, *Electron. J. Prob.*, **10**(46), 2005, 1496–1546.

[11] Hu, Y., Heat equation with fractional white noise potentials, *Appl. Math. Optim.*, **43**, 2001, 221–243.

[12] Knoche, C., SPDEs in infinite dimension with Poisson noise, *C. R. Math. Acad. Sci. Paris*, **339**(9), 2004, 647–652.

[13] Knoche, C., Mild solutions of SPDEs driven by Poisson noise in infinite dimensions and their dependence on initial conditions, Dissertation zur Erlangung des Doktorgrades, Fakultät für Mathematik, Universität Bielefeld, Bielefeld, 2005.

[14] Maslowski, B. and Nualart, D., Evolution equations driven by a fractional Brownian motion, *J. Funct. Anal.*, **202**, 2003, 277–305.

[15] Mémin, J., Mishura, Y. and Valkeila, E., Inequality for the moments of Wiener integrals with respect to a fractional Brownain, *Stat. Prob. Lett.*, **51**, 2001, 197–206.

[16] Novick-Cohen, A. and Segel, L., Nonlinear aspects of the Cahn-Hilliard eqaution, *Phy. D.*, **10**, 1984, 277–298.

[17] Nualart, D. and Ouknine, Y., Regularization of quasilinear heat equations by a fractional noise, *Stoch. Dyn.*, **4**(2), 2004, 201–221.

[18] Nualart, D. and Rascanu, A., Differential equations driven by fractional Brownian motion, *Collectanea Math.*, **53**, 2002, 55–81.

[19] Protter, P., Stochastic Integration and Differential Equations, Springer, Berlin, 1990.

[20] Shiga, T., Two contrasting properties of solutions for one-dimension stochastic partial differential equations, *Can. J. Math.*, **46**, 1994, 415–437.

[21] Tindel, S., Tudor, A. and Viens, E., Stochastic evolution equations with fractional Brownian motion, *Prob. Th. Relat. Fields*, **127**, 2003, 186–204.

[22] Truman, A. and Wu, J., Stochastic Burgers equation with Lévy space-time white noise, Probabilistic Methods in Fluids, I. M. Davies et al (eds.), World Sci. Publ., River Edge, NJ, 2003, 298–323.

[23] Walsh, J., An Introduction to Stochastic Partial Differential Equations, Lect. Notes in Math., **1180**, Springer, Berlin, 1986, 265–439.