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Curvature Estimates for Minimal Submanifolds of Higher Codimension***

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Abstract The authors derive curvature estimates for minimal submanifolds in Euclidean space for arbitrary dimension and codimension via Gauss map. Thus, Schoen-Simon-Yau's results and Ecker-Huisken's results are generalized to higher codimension. In this way, Hildebrandt-Jost-Widman's result for the Bernstein type theorem is improved.

Keywords Curvature estimates, Minimal submanifold, Bernstein type theorem
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1 Introduction

Let f be a smooth function on an open domain $\Omega \subset \mathbb{R}^n$. If f satisfies the minimal surface equation, it defines a minimal hypersurface M in $\mathbb{R}^{n+1}$. If f is an entire solution to the equation, f must be an affine linear function for $n \leq 7$ whose graph is a hyperplane. Those are the classical Bernstein theorem (see [2]) and its higher dimensional generalization which was finally proved by Simons [20]. Counterexamples to the theorems for $n \geq 8$ were given by Bombeiri-De Gorge-Guisti [1].

Heinze [10] considered the minimal graph defined over a disc $D_R \subset \mathbb{R}^2$ and gave curvature estimates. The classical Bernstein theorem can be obtained by letting $R \to +\infty$ in his curvature estimates.

For general minimal surfaces in Euclidean space, so-called parametric case, the Bernstein type results are closely related to the value distribution of the Gauss image. The question of the value of Gauss map for complete minimal surfaces in $\mathbb{R}^3$ is settled in [7, 17, 24].

A minimal graph is area-minimizing. In particular, it is stable in the sense that its second variation of the volume is non-negative on any compact subset of M. Any stable minimal surface in $\mathbb{R}^3$ is a plane (see [4, 6]).

For stable minimal hypersurfaces, Schoen-Simon-Yau gave curvature estimates, which not only gave us a direct proof for Bernstein type theorems for $n \leq 5$ dimensional minimal graphs, but also gave us a new method to obtain curvature estimates.

For any $n \geq 2$, there is a weak version of the Bernstein type theorem. It was Moser [15] who proved that the entire solution f to the minimal surface equation is affine linear, provided

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$|\nabla f|$ is uniformly bounded. There is no dimension limitation. By a geometric approach, Ecker-Huisken [9] obtained the curvature estimates. As a conclusion, Moser's result has been improved for the controlled growth of $|\nabla f|$.

For area-minimizing hypersurfaces with vanishing first Betti number, Solomom [22] was able to give curvature estimates under the hypotheses of Gauss map.

Higher codimensional Bernstein problem becomes more complicated. There are counter-examples given by Lawson-Osserman [13]. On the other hand, Hildebrandt-Jost-Widman [11] generalized Moser's result to higher codimension as follows.

Theorem 1.1 *Let $z^\alpha = f^\alpha(x)$, $\alpha = 1, \cdots, m$, $x = (x^1, \cdots, x^n) \in \mathbb{R}^n$, be the C^2 solution to the system of minimal surface equations. Suppose that there exists β, where*

$$\beta < \cos^{-p}\Big(\frac{\pi}{2\sqrt{pK}}\Big), \quad K = \begin{cases} 1, & \text{if } p = 1, \\ 2, & \text{if } p \ge 2, \end{cases} \quad p = \min(m, n),$$

such that for any $x \in \mathbb{R}^n$,

$$\Delta_f(x) = \{\det(\delta_{ij} + f^s_{x^i}(x) f^s_{x^j}(x))\}^{\frac{1}{2}} \le \beta.$$

Then $f^1, \cdots, f^m$ are affine linear functions on $\mathbb{R}^n$, whose graph is an affine n-plane in $\mathbb{R}^{m+n}$.

The geometric meaning of the condition in the above theorem is that the image under the Gauss map lies in a closed subset of an open geodesic ball of the radius $\frac{\sqrt{2}}{4}\pi$. Later in a joint work of the first author of this paper with Jost [12], we found larger geodesic convex set $B_{JX}(P_0)$, where P_0 denotes a fixed n-plane, and then improved the above theorem. Our bound of slope is 2, larger than $\cos^{-p}(\frac{\pi}{2\sqrt{2p}})$. It should be noted that although $B_{JX}(P_0) \supset B_{\frac{\sqrt{2}}{4}\pi}(P_0)$, they have some common boundary points.

Recently, the first author of this paper and his collaborators studied complete minimal submanifolds with flat normal bundle and positive w-function (see [21, 27]). In this special situation, the Schoen-Simon-Yau type curvature estimates and the Ecker-Huisken type curvature estimates can be carried out. Then the corresponding Bernstein type theorems follow immediately.

In this paper, we study a complete minimal submanifold M in $\mathbb{R}^{m+n}$ with the codimension $m \ge 2$. We have a Bochner type formula for the squared norm of the second fundamental form B. As in the codimension one case, we need a Kato type inequality for $|\nabla B|^2$ in terms of $|\nabla |B||^2$. We derive it for any codimension in Section 2.

For the curved normal bundle, the curvature estimates would be more delicate. We can define Gauss map from M to the Grassmannian manifold $\mathbf{G}_{n,m}$. From the counterexamples of Lawson-Osserman, some additional conditions are needed to study higher codimensional Bernstein problem. The adequate conditions would confine the image of the Gauss map, as in the previous work of Osserman-Xavier-Fujimoto in dimension 2 and in the work of Solomon in higher dimension. Now, in general dimension and codimension, we assume that the image under the Gauss map lies in an open geodesic ball of radius $\frac{\sqrt{2}}{4}\pi$ in $\mathbf{G}_{n,m}$ which is the largest convex geodesic ball in Grassmannian. We find two auxiliary functions h_1 and h_2 on M via Gauss map. Their precise definitions and their properties can be found in Section 3. This technique can also be used in mean curvature flow in higher codimension (see [28]).

(3.4) shows that h_1 can be viewed as a generalized support function. With the aid of the function h_1, we can derive a "strong stability inequality" which enables us to carry out the curvature estimates of Schoen-Simon-Yau type.

Using the function h_2, we can find subharmonic functions on M and carry out the curvature estimates of Ecker-Huisken type in terms of h_2.

By those curvature estimates, we can obtain the following Bernstein type theorems.

Theorem 1.2 *Let M be a complete minimal n-submanifold in $\mathbb{R}^{n+m}$ $(n \le 6)$. If the Gauss image of M is contained in an open geodesic ball of $\mathbf{G}_{n,m}$ centered at P_0 and of radius $\frac{\sqrt{2}}{4}\pi$, then M has to be an affine linear subspace.*

Theorem 1.3 *Let M be a complete minimal n-submanifold in $\mathbb{R}^{n+m}$. If the Gauss image of M is contained in an open geodesic ball of $\mathbf{G}_{n,m}$ centered at P_0 and of radius $\frac{\sqrt{2}}{4}\pi$, and $\big(\frac{\sqrt{2}}{4}\pi - \rho\circ\gamma\big)^{-1}$ has growth*

$$\Big(\frac{\sqrt{2}}{4}\pi - \rho\circ\gamma\Big)^{-1} = o(R), \tag{1.1}$$

where ρ denotes the distance on $\mathbf{G}_{n,m}$ from P_0 and R is the Euclidean distance from any point in M, then M has to be an affine linear subspace.

Theorems 1.2 and 1.3 are closely related to Theorem 1.1. Our results are higher codimensional generalizations of Schoen-Simon-Yau's results and Ecker-Huisken's results, and improve Hildebrandt-Jost-Widman's theorem.

It is worth to note that our method is also suitable for codimension one case. We only need to modify the auxiliary functions h_1 and h_2 in order to recover the known results for minimal hypersurfaces.

2 Preliminaries

Let $M \to \overline{M}$ be an isometric immersion with the second fundamental form B, which can be viewed as a cross-section of the vector bundle $\mathrm{Hom}(\odot^2 TM, NM)$ over M, where TM and NM denote the tangent bundle and the normal bundle along M, respectively. A connection on $\mathrm{Hom}(\odot^2 TM, NM)$ can be induced from those of TM and NM naturally.

For $\nu \in \Gamma(NM)$, the shape operator $A^\nu : TM \to TM$ satisfies

$$\langle B_{XY}, \nu\rangle = \langle A^\nu(X), Y\rangle.$$

We define the mean curvature H to be the trace of the second fundamental form. It is a normal vector field on M in $\overline{M}$.

The second fundamental form, curvature tensor of the submanifold, curvature tensor of the normal bundle and that of the ambient manifold satisfy the Gauss equations, the Codazzi equations and the Ricci equations as follows:

$$\begin{aligned}
&\langle R_{XY}Z, W\rangle = \langle \overline{R}_{XY}Z, W\rangle - \langle B_{XW}, B_{YZ}\rangle + \langle B_{XZ}, B_{YW}\rangle,\\
&(\nabla_X B)_{YZ} - (\nabla_Y B)_{XZ} = -(\overline{R}_{XY}Z)^N,\\
&\langle R_{XY}\mu, \nu\rangle = \langle \overline{R}_{XY}\mu, \nu\rangle + \langle B_{Xe_i}, \mu\rangle\langle B_{Ye_i}, \nu\rangle - \langle B_{Xe_i}, \nu\rangle\langle B_{Ye_i}, \mu\rangle,
\end{aligned}$$

where $\{e_i\}$ is a local orthonormal frame field of M; X, Y and Z are tangent vector fields; μ, ν are normal vector fields in M. Here and in the sequel, we use the summation convention and agree the range of indices,

$$1 \le i, j, s, t \le n, \quad 1 \le \alpha, \beta, \gamma \le m, \quad 1 \le a, b, c \le m + n.$$

There is the trace-Laplace operator ∇^2 acting on any cross-section of a Riemannian vector bundle E over M.

Now, we consider a minimal submanifold M of dimension n in Euclidean $(m + n)$-space $\mathbb{R}^{m+n}$ with $m \ge 2$. We have (see [20])

$$\nabla^2 B = -\widetilde{\mathcal{B}} - \underline{\mathcal{B}}. \tag{2.1}$$

We recall the following notations:

$$\widetilde{\mathcal{B}} \triangleq B \circ B^t \circ B,$$

where B^t is the conjugate map of B;

$$\underline{\mathcal{B}}_{XY} \triangleq \sum_{\alpha=1}^{m} (B_{A^{\nu_\alpha} A^{\nu_\alpha}(X) Y} + B_{X A^{\nu_\alpha} A^{\nu_\alpha}(Y)} - 2B_{A^{\nu_\alpha}(X) A^{\nu_\alpha}(Y)}),$$

where ν_α are basis vectors of normal space. It is obvious that $\underline{\mathcal{B}}_{XY}$ is symmetric in X and Y, which is a cross-section of the bundle $\mathrm{Hom}(\odot^2 TM, NM)$. Simons [20] also gave an estimate

$$\langle \widetilde{\mathcal{B}} + \underline{\mathcal{B}}, B \rangle \le \Big(2 - \frac{1}{m}\Big)|B|^4.$$

It is optimal for the codimension $m = 1$.

In the case when $m \ge 2$, there is a refined estimate (see [5, 14])

$$\langle \widetilde{\mathcal{B}} + \underline{\mathcal{B}}, B \rangle \le \frac{3}{2}|B|^4.$$

Substituting it into (2.1) gives

$$\langle \nabla^2 B, B \rangle \ge -\frac{3}{2}|B|^4.$$

It follows that

$$\Delta |B|^2 \ge -3|B|^4 + 2|\nabla B|^2. \tag{2.2}$$

We need a Kato-type inequality in order to use the formula (2.2). Namely, we would estimate $|\nabla B|^2$ in terms of $|\nabla |B||^2$. Schoen-Simon-Yau [19] did such an estimate for codimension $m = 1$. For any m with flat normal bundle, their technique is also applicable (see [27]). The following lemma is a generalized version of their estimate for any codimension m.

Lemma 2.1

$$|\nabla B|^2 \ge \Big(1 + \frac{2}{mn}\Big)|\nabla |B||^2. \tag{2.3}$$

Proof It is sufficient for us to prove the inequality at the points where $|B|^2 \ne 0$. Choose a local orthonormal tangent frame field $\{e_1, \cdots, e_n\}$ and a local orthonormal normal frame field $\{\nu_1, \cdots, \nu_m\}$ of M near the considered point x, such that

$$\nabla_{e_i} e_j(x) = 0, \quad \nabla_{e_i} \nu_\alpha(x) = 0. \tag{2.4}$$

Denote the shape operator $A^\alpha = A^{\nu_\alpha}$. Then obviously $|B|^2 = \sum_\alpha |A^\alpha|^2$ and

$$\nabla|B|^2 = \sum_\alpha \nabla|A^\alpha|^2,$$
$$|\nabla|B|^2|^2 = \sum_{\alpha,\beta} \nabla|A^\alpha|^2 \cdot \nabla|A^\beta|^2 \le \frac{1}{2}\sum_{\alpha,\beta}(|\nabla|A^\alpha|^2|^2 + |\nabla|A^\beta|^2|^2) \le m\sum_\alpha |\nabla|A^\alpha|^2|^2.$$

Therefore

$$|\nabla|B||^2 = \frac{|\nabla|B|^2|^2}{4|B|^2} \le \frac{m\sum_\alpha |\nabla|A^\alpha|^2|^2}{4\sum_\alpha |A^\alpha|^2}. \tag{2.5}$$

Since $|B|^2 \neq 0$, there exist γ, k, l such that $h_{\gamma kl} \neq 0$, where $h_{\alpha ij} = \langle B_{e_i e_j}, \nu_\alpha\rangle$ for arbitrary α, i, j, then

$$(h_{1kl}, \cdots, h_{mkl}) \in \mathbb{R}^m - \{0\}.$$

Obviously, there exists an $m \times m$ orthogonal matrix U and $z = (z_1, \cdots, z_m) \in \mathbb{R}^m$, such that

$$z_\alpha \neq 0 \quad \text{for every } 1 \le \alpha \le m$$

and

$$z_\alpha = U_\alpha^\beta h_{\beta kl}.$$

Now we define $\widetilde{\nu}_\alpha = U_\alpha^\beta \nu_\beta$. Then $\{\widetilde{\nu}_1, \cdots, \widetilde{\nu}_m\}$ is also a local orthogonal normal frame field satisfying (2.4), and moreover,

$$\widetilde{h}_{\alpha kl} = \langle B_{e_k e_l}, \widetilde{\nu}_\alpha\rangle = U_\alpha^\beta h_{\beta kl} = z_\alpha \neq 0.$$

Define the shape operator $\widetilde{A}^\alpha$ corresponding to $\widetilde{\nu}_\alpha$. Then

$$|\widetilde{A}^\alpha|^2 = \sum_{i,j} \widetilde{h}^2_{\alpha ij} \ge \widetilde{h}^2_{\alpha kl} > 0.$$

Hence we can assume $|A^\alpha|^2 > 0$ for arbitrary α without loss of generality.

Let $1 \le \gamma \le m$ be such that

$$\frac{|\nabla|A^\gamma|^2|^2}{|A^\gamma|^2} = \max_\alpha\left\{\frac{|\nabla|A^\alpha|^2|^2}{|A^\alpha|^2}\right\} < +\infty.$$

Then from (2.5),

$$|\nabla|B||^2 \le \frac{m|\nabla|A^\gamma|^2|^2}{4|A^\gamma|^2}. \tag{2.6}$$

Since $|A^\gamma|^2$ and $\nabla|A^\gamma|^2$ are independent of the choice of $\{e_1, \cdots, e_n\}$, without loss of generality we can assume $h_{\gamma ij} = 0$ whenever $i \neq j$. Then

$$\nabla|A^\gamma|^2 = 2\sum_k\sum_{i,j} h_{\gamma ij}h_{\gamma ijk}e_k = 2\sum_k\sum_i h_{\gamma ii}h_{\gamma iik}e_k,$$
$$|\nabla|A^\gamma|^2|^2 = 4\sum_k\Big(\sum_i h_{\gamma ii}h_{\gamma iik}\Big)^2 \le 4\sum_k\Big(\sum_i h^2_{\gamma ii}\Big)\Big(\sum_i h^2_{\gamma iik}\Big)$$
$$= 4\Big(\sum_i h^2_{\gamma ii}\Big)\Big(\sum_{i,k} h^2_{\gamma iik}\Big) = 4|A^\gamma|^2\sum_{i,k} h^2_{\gamma iik}$$

and

$$
\begin{aligned}
|\nabla|B||^2 &\le \frac{m|\nabla|A^\gamma|^2|^2}{4|A^\gamma|^2} \le m\sum_{i,k} h^2_{\gamma iik} \\
&= m\sum_{i\ne k} h^2_{\gamma iik} + m\sum_i h^2_{\gamma iii} \\
&= m\sum_{i\ne k} h^2_{\gamma iki} + m\sum_i \Big(\sum_{j\ne i} h_{\gamma jji}\Big)^2 \\
&\le m\sum_{i\ne k} h^2_{\gamma iki} + (n-1)m\sum_{i\ne j} h^2_{\gamma jji} \\
&= nm\sum_{i\ne k} h^2_{\gamma iki}, \qquad (2.7)
\end{aligned}
$$

where we used the Codazzi equations and the vanishing mean curvature condition $H = 0$. Please note that $h_{\alpha ijk} = \langle (\nabla_{e_k} B)_{e_i,e_j}, \nu_\alpha\rangle$ for arbitrary α, i, j, k.

On the other hand, a direct calculation shows

$$
|\nabla|B|^2|^2 = \Big|2\sum_k\sum_{\alpha,i,j} h_{\alpha ij}h_{\alpha ijk}e_k\Big|^2 = 4\sum_{\alpha,\beta,i,j,s,t,k} h_{\alpha ij}h_{\alpha ijk}h_{\beta st}h_{\beta stk},
$$

$$
\begin{aligned}
|\nabla B|^2 - |\nabla|B||^2 &= |\nabla B|^2 - \frac{|\nabla|B|^2|^2}{4|B|^2} \\
&= \sum_{\alpha,i,j,k} h^2_{\alpha ijk} - \frac{\sum\limits_{\alpha,\beta,i,j,s,t,k} h_{\alpha ij}h_{\alpha ijk}h_{\beta st}h_{\beta stk}}{\sum\limits_{\beta,s,t} h^2_{\beta st}} \\
&= \frac{\sum\limits_{\alpha,i,j,s,t,k}(h_{\alpha ijk}h_{\beta st} - h_{\beta stk}h_{\alpha ij})^2}{2|B|^2} \\
&\ge \frac{\sum\limits_{\beta,i\ne j,s,t,k} h^2_{\gamma ijk}h^2_{\beta st} + \sum\limits_{\alpha,s\ne t,i,j,k} h^2_{\gamma stk}h^2_{\alpha ij}}{2|B|^2} \\
&= \sum_{i\ne j,k} h^2_{\gamma ijk} \ge \sum_{i\ne k}(h^2_{\gamma iki} + h^2_{\gamma ikk}) \\
&= 2\sum_{i\ne k} h^2_{\gamma iki}. \qquad (2.8)
\end{aligned}
$$

In conjunction with (2.7), we finally arrive at (2.3).

Hence it follows from (2.2) and (2.3) that

$$
\Delta|B|^2 \ge 2\Big(1+\frac{2}{mn}\Big)|\nabla|B||^2 - 3|B|^4. \qquad (2.9)
$$

3 Auxiliary Functions via Gauss Maps

Let $\mathbb{R}^{m+n}$ be an $(m+n)$-dimensional Euclidean space. All oriented n-subspaces constitute the Grassmannian manifold $\mathbf{G}_{n,m}$, which is an irreducible symmetric space of compact type. The canonical Riemannian metric on $\mathbf{G}_{n,m}$ can be expressed in the following way.

Let $\{e_i, e_{n+\alpha}\}$ be a local orthonormal frame field in $\mathbb{R}^{m+n}$. Let $\{\omega_i, \omega_{n+\alpha}\}$ be the dual frame field so that the Euclidean metric is

$$g = \sum_i \omega_i^2 + \sum_\alpha \omega_{n+\alpha}^2.$$

The Levi-Civita connection forms ω_{ab} of $\mathbb{R}^{m+n}$ are uniquely determined by the equations

$$\mathrm{d}\omega_a = \omega_{ab} \wedge \omega_b, \quad \omega_{ab} + \omega_{ba} = 0.$$

Let $P \in \mathbf{G}_{n,m}$ be any point which is spanned by $\{e_1, \cdots, e_n\}$. Then the canonical Riemannian metric on $\mathbf{G}_{n,m}$ can be written as

$$\mathrm{d}s^2 = \sum_{i,\alpha} \omega_{n+\alpha\, i}^2. \tag{3.1}$$

The sectional curvature of the above canonical metric varies in the interval $[0, 2]$ in the case of $\min\{n, m\} \geq 2$. By the standard Hessian comparison theorem, we have

$$\mathrm{Hess}(\rho) \geq \sqrt{2}\, \cot(\sqrt{2}\, \rho)(g - \mathrm{d}\rho \otimes \mathrm{d}\rho), \tag{3.2}$$

where ρ is the distance function from a fixed point in $\mathbf{G}_{n,m}$ and g is the metric tensor on $\mathbf{G}_{n,m}$.

Let 0 be the origin of $\mathbb{R}^{m+n}$, $\mathrm{SO}(m+n)$ be the Lie group consisting of all the orthonormal frames $(0; e_i, e_{n+\alpha})$, $P = \{(x; e_1, \cdots, e_n) : x \in M, e_i \in T_x M\}$ be the principal bundle of orthonormal tangent frames over M, and $Q = \{(x; e_{n+1}, \cdots, e_{n+m}) : x \in M, e_{n+\alpha} \in N_x M\}$ be the principal bundle of orthonormal normal frames over M. Then $\overline{\pi} : P \oplus Q \to M$ is the projection with fiber $\mathrm{SO}(m) \times \mathrm{SO}(n)$ and $i : P \oplus Q \hookrightarrow \mathrm{SO}(m+n)$ is the natural inclusion.

The Gauss map $\gamma : M \to \mathbf{G}_{n,m}$ is defined by

$$\gamma(x) = T_x M \in \mathbf{G}_{n,m}$$

via the parallel translation in $\mathbb{R}^{m+n}$ for any $x \in M$. Thus, the following commutative diagram holds:

$$\begin{array}{ccc} P \oplus Q & \xrightarrow{\ i\ } & \mathrm{SO}(m+n) \\ {\scriptstyle \overline{\pi}}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ M & \xrightarrow{\ \gamma\ } & \mathbf{G}_{n,m} \end{array}$$

From the above diagram we know the energy density of the Gauss map (see [25, Chapter 3, §3.1])

$$e(\gamma) = \frac{1}{2}\langle \gamma_* e_i, \gamma_* e_i \rangle = \frac{1}{2}|B|^2.$$

Ruh-Vilms proved that the mean curvature vector of M is parallel if and only if its Gauss map is a harmonic map (see [18]).

We consider smooth functions on an open geodesic ball $B_{\frac{\sqrt{2}}{4}\pi}(P_0) \subset \mathbf{G}_{n,m}$ of radius $\frac{\sqrt{2}}{4}\pi$ and centered at P_0. Those are useful for our curvature estimates later. Let

$$u = \cos(\sqrt{2}\, \rho),$$

where ρ is the distance function from P_0 in $\mathbf{G}_{n,m}$. We have

$$u' = -\sqrt{2}\, \sin(\sqrt{2}\, \rho), \quad u'' = -2\cos(\sqrt{2}\, \rho).$$

Then

$$\begin{aligned}\mathrm{Hess}(u) &= u'\mathrm{Hess}(\rho) + u''\mathrm{d}\rho\otimes\mathrm{d}\rho\\ &\le -2\cos(\sqrt{2}\,\rho)(g-\mathrm{d}\rho\otimes\mathrm{d}\rho) - 2\cos(\sqrt{2}\,\rho)\mathrm{d}\rho\otimes\mathrm{d}\rho = -2ug.\end{aligned}\tag{3.3}$$

The composition function $h_1 = u\circ\gamma$ of u with the Gauss map γ defines a function on M. Using the composition formula, we have

$$\Delta h_1 = \mathrm{Hess}(u)(\gamma_* e_i, \gamma_* e_i) + \mathrm{d}u(\tau(\gamma)) \le -2|B|^2 h_1,\tag{3.4}$$

where $\tau(\gamma)$ is the tension field of the Gauss map, which is zero, provided M has parallel mean curvature by the Ruh-Vilms theorem mentioned above.

Let

$$h = \sec^2(\sqrt{2}\,\rho),$$

where ρ is the distance function from P_0 in $\mathbf{G}_{n,m}$. We have

$$\begin{aligned}h' &= 2\sqrt{2}\,\sec^2(\sqrt{2}\,\rho)\tan(\sqrt{2}\,\rho),\\ h'' &= 12\sec^2(\sqrt{2}\,\rho)\tan^2(\sqrt{2}\,\rho) + 4\sec^2(\sqrt{2}\,\rho).\end{aligned}$$

Hence

$$\begin{aligned}\mathrm{Hess}(h) &= h'\mathrm{Hess}(\rho) + h''\mathrm{d}\rho\otimes\mathrm{d}\rho\\ &\ge 4\sec^2(\sqrt{2}\,\rho)(g-\mathrm{d}\rho\otimes\mathrm{d}\rho) + (12\sec^2(\sqrt{2}\,\rho)\tan^2(\sqrt{2}\,\rho) + 4\sec^2(\sqrt{2}\,\rho))\mathrm{d}\rho\otimes\mathrm{d}\rho\\ &= 4hg + \frac{3}{2}h^{-1}\mathrm{d}h\otimes\mathrm{d}h.\end{aligned}$$

The composition function $h_2 = h\circ\gamma$ of h with the Gauss map γ defines a function on M. Using the composition formula, we have

$$\Delta h_2 = \mathrm{Hess}(h)(\gamma_* e_i, \gamma_* e_i) + \mathrm{d}h(\tau(\gamma)) \ge 4h_2|B|^2 + \frac{3}{2}h_2^{-1}|\nabla h_2|^2,\tag{3.5}$$

where $\tau(\gamma)$ is the tension field of the Gauss map, which is zero in our consideration.

With the aid of h_1, we immediately have the following lemma.

Lemma 3.1 *Let M be an n-dimensional minimal submanifold of $\mathbb{R}^{n+m}$ (M need not be complete). If the Gauss image of M is contained in an open geodesic ball of radius $\frac{\sqrt{2}}{4}\pi$ in $\mathbf{G}_{n,m}$, then we have*

$$\int_M |\nabla\phi|^2 * 1 \ge 2\int_M |B|^2\phi^2 * 1\tag{3.6}$$

for any function ϕ with compact support $D\subset M$.

Proof Let

$$L\phi = -\Delta\phi - 2|B|^2\phi.$$

Its first eigenvalue with the Dirichlet boundary condition in D is λ_1 and the corresponding eigenfunction is v. Without loss of generality, we assume that v achieves the positive maximum. Consider a C^2 function

$$f = \frac{v}{h_1}.$$

Since $f|_{\partial D}=0$, it achieves the positive maximum at a point $x\in D$. Therefore, at x,

$$\nabla f=0,\quad \Delta f\le 0.$$

It follows that

$$\Delta v=\Delta(fh_1)=\Delta f\cdot h_1+f\Delta h_1+2\nabla f\cdot\nabla h_1\le f\Delta h_1=\frac{v\Delta h_1}{h_1}.$$

Namely, at x,

$$\frac{\Delta v}{v}\le\frac{\Delta h_1}{h_1},$$

$$\frac{\Delta v+2|B|^2v}{v}\le\frac{\Delta h_1+2|B|^2h_1}{h_1}\le 0. \tag{3.7}$$

On the other hand,

$$\frac{\Delta v+2|B|^2v}{v}=-\lambda_1. \tag{3.8}$$

(3.7) and (3.8) implies $\lambda_1\ge 0$. Hence we have

$$0\le\lambda_1=\inf\frac{\int_D\phi L\phi*1}{\int_D\phi^2*1}\le\frac{\int_D\phi L\phi*1}{\int_D\phi^2*1},$$

which shows that (3.6) holds true.

Remark 3.1 For a stable minimal hypersurface there is the stability inequality, which is one of the main ingredients for Schoen-Simon-Yau's curvature estimates for stable minimal hypersurfaces. For minimal submanifolds with the Gauss image restriction, we have stronger inequality as shown in (3.6).

4 Curvature Estimates of Schoen-Simon-Yau Type

Replacing ϕ by $|B|^{1+q}\phi$ in (3.6) gives

$$\begin{aligned}\int_M|B|^{4+2q}\phi^2*1\le&\frac{1}{2}\int_M|\nabla(|B|^{1+q}\phi)|^2*1\\ =&\frac{1}{2}(1+q)^2\int_M|B|^{2q}|\nabla|B||^2\phi^2*1+\frac{1}{2}\int_M|B|^{2+2q}|\nabla\phi|^2*1\\ &+(1+q)\int_M|B|^{1+2q}\nabla|B|\cdot\phi\nabla\phi*1.\end{aligned} \tag{4.1}$$

From (2.9), we can derive

$$\frac{2}{mn}|\nabla|B||^2\le|B|\Delta|B|+\frac{3}{2}|B|^4. \tag{4.2}$$

Multiplying both sides of (4.2) by $|B|^{2q}\phi^2$ and integrating by parts, we have

$$\begin{aligned}\frac{2}{mn}\int_M|B|^{2q}|\nabla|B||^2\phi^2*1\le&\int_M|B|^{1+2q}\Delta|B|\phi^2*1+\frac{3}{2}\int_M|B|^{4+2q}\phi^2*1\\ =&-\int_M\nabla|B|\cdot\nabla(|B|^{1+2q}\phi^2)*1+\frac{3}{2}\int_M|B|^{4+2q}\phi^2*1\\ =&-(1+2q)\int_M|B|^{2q}|\nabla|B||^2\phi^2*1\\ &-2\int_M|B|^{1+2q}\nabla|B|\cdot\phi\nabla\phi*1+\frac{3}{2}\int_M|B|^{4+2q}\phi^2*1.\end{aligned} \tag{4.3}$$

Multiplying both sides of (4.1) by $\frac{3}{2}$ and then adding up both sides of it and (4.3), we have

$$\Big(\frac{2}{mn}+1+2q-\frac{3}{4}(1+q)^2\Big)\int_M |B|^{2q}|\nabla|B||^2\phi^2 * 1$$
$$\le \frac{3}{4}\int_M |B|^{2+2q}|\nabla\phi|^2 * 1+\Big(\frac{3}{2}(1+q)-2\Big)\int_M |B|^{1+2q}\nabla|B|\cdot\phi\nabla\phi * 1. \tag{4.4}$$

By using Young's inequality, we have

$$\Big(\frac{3}{2}(1+q)-2\Big)\int_M |B|^{1+2q}\nabla|B|\cdot\phi\nabla\phi*1 \le \varepsilon\int_M |B|^{2q}|\nabla|B||^2\phi^2*1+C_1(\varepsilon,q)\int_M |B|^{2+2q}|\nabla\phi|^2*1.$$

Then (4.4) becomes

$$\Big(\frac{2}{mn}+1+2q-\frac{3}{4}(1+q)^2-\varepsilon\Big)\int_M |B|^{2q}|\nabla|B||^2\phi^2 * 1 \le C_2(\varepsilon,q)\int_M |B|^{2+2q}|\nabla\phi|^2 * 1. \tag{4.5}$$

When

$$q\in\Big[0,\frac{1}{3}+\frac{2}{3}\sqrt{1+\frac{6}{mn}}\Big), \tag{4.6}$$

we have

$$\frac{2}{mn}+1+2q-\frac{3}{4}(1+q)^2>0.$$

Then we can choose ε sufficiently small, such that

$$\int_M |B|^{2q}|\nabla|B||^2\phi^2 * 1 \le C_3\int_M |B|^{2+2q}|\nabla\phi|^2 * 1, \tag{4.7}$$

where C_3 only depends on n, m and q.

Using Young's inequality again yields

$$|B|^{1+2q}\nabla|B|\cdot\phi\nabla\phi \le \frac{1}{2}(|B|^{2q}|\nabla|B||^2\phi^2+|B|^{2+2q}|\nabla\phi|^2). \tag{4.8}$$

Substituting (4.7) and (4.8) into (4.1) gives

$$\int_M |B|^{4+2q}\phi^2 * 1 \le C_4(n,m,q)\int_M |B|^{2+2q}|\nabla\phi|^2 * 1. \tag{4.9}$$

Replacing ϕ by ϕ^{2+q} in (4.9) gives

$$\int_M |B|^{4+2q}\phi^{4+2q} * 1 \le C_4(2+q)^2\int_M |B|^{2+2q}\phi^{2+2q}|\nabla\phi|^2 * 1.$$

By Hölder's inequality, we have

$$\int_M |B|^{2+2q}\phi^{2+2q}|\nabla\phi|^2 * 1 \le \Big(\int_M |B|^{4+2q}\phi^{4+2q} * 1\Big)^{\frac{1+q}{2+q}}\Big(\int_M |\nabla\phi|^{4+2q} * 1\Big)^{\frac{1}{2+q}}.$$

Therefore

$$\int_M |B|^{4+2q}\phi^{4+2q} * 1 \le C\int_M |\nabla\phi|^{4+2q} * 1, \tag{4.10}$$

where C is a constant only depending on n, m and q.

Replacing ϕ by ϕ^{1+q} in (4.9) gives

$$\int_M |B|^{4+2q}\phi^{2+2q} * 1 \le C_4(1+q)^2 \int_M |B|^{2+2q}\phi^{2q}|\nabla\phi|^2 * 1.$$

By Hölder's inequality, we have

$$\int_M |B|^{2+2q}\phi^{2q}|\nabla\phi|^2 * 1 \le \Big(\int_M |B|^{4+2q}\phi^{2+2q} * 1\Big)^{\frac{q}{1+q}}\Big(\int_M |B|^2|\nabla\phi|^{2+2q} * 1\Big)^{\frac{1}{1+q}}.$$

Therefore

$$\int_M |B|^{4+2q}\phi^{2+2q} * 1 \le C' \int_M |B|^2|\nabla\phi|^{2+2q} * 1, \tag{4.11}$$

where C' is a constant only depending on n, m and q.

Let r be a function on M with $|\nabla r| \le 1$. For any $R \in [0, R_0]$, where $R_0 = \sup_M r$, suppose that

$$M_R = \{x \in M,\ r \le R\}$$

is compact.

(4.10) enables us to prove the following results.

Theorem 4.1 *Let M be an n-dimensional minimal submanifolds of $\mathbb{R}^{n+m}$. If the Gauss image of M_R is contained in an open geodesic ball of radius $\frac{\sqrt{2}}{4}\pi$ in $\mathbf{G}_{n,m}$, then we have the L^p-estimate*

$$\||B|\|_{L^p(M_{\theta R})} \le C(n,m,p)(1-\theta)^{-1}R^{-1}\mathrm{Vol}(M_R)^{\frac{1}{p}} \tag{4.12}$$

for arbitrary $\theta \in (0,1)$ and

$$p \in \Big[4, 4 + \frac{2}{3} + \frac{4}{3}\sqrt{1 + \frac{6}{mn}}\Big).$$

Proof Take $\phi \in C_c^\infty(M_R)$ to be the standard cut-off function such that $\phi \equiv 1$ in $M_{\theta R}$ and $|\nabla\phi| \le C(1-\theta)^{-1}R^{-1}$. Then (4.10) yields

$$\int_{M_{\theta R}} |B|^p * 1 \le C(1-\theta)^{-p}R^{-p}\mathrm{Vol}(M_R),$$

where $p = 4 + 2q$. Thus the conclusion immediately follows from (4.10).

5 Curvature Estimates of Ecker-Huisken Type

From (2.9) and (3.5), we compute

$$\begin{aligned}
\Delta(|B|^{2p}h_2^q) =& \Delta|B|^{2p}\cdot h_2^q + |B|^{2p}\Delta h_2^q + 2\nabla|B|^{2p}\cdot\nabla h_2^q \\
=& (p|B|^{2p-2}\Delta|B|^2 + p(p-1)|B|^{2p-4}|\nabla|B|^2|^2)h_2^q \\
&+ |B|^{2p}(qh_2^{q-1}\Delta h_2 + q(q-1)h_2^{q-2}|\nabla h_2|^2) + 4pq|B|^{2p-1}\nabla|B|\cdot h_2^{q-1}\nabla h_2 \\
\ge& (4q-3p)|B|^{2p+2}h_2^q + 2p\Big(2p - 1 + \frac{2}{mn}\Big)|B|^{2p-2}|\nabla|B||^2h_2^q \\
&+ q\Big(q + \frac{1}{2}\Big)|B|^{2p}h_2^{q-2}|\nabla h_2|^2 + 4pq|B|^{2p-1}\nabla|B|\cdot h_2^{q-1}\nabla h_2.
\end{aligned}$$

By Young's inequality, when $2p\big(2p-1+\frac{2}{mn}\big)\cdot q\big(q+\frac{1}{2}\big)\geq(2pq)^2$, i.e.,

$$p\geq\frac{1}{2}-\frac{1}{mn}+\Big(1-\frac{2}{mn}\Big)q, \tag{5.1}$$

the inequality

$$\Delta(|B|^{2p}h_2^q)\geq(4q-3p)|B|^{2p+2}h_2^q \tag{5.2}$$

holds. Especially,

$$\Delta(|B|^{p-1}h_2^{\frac{p}{2}})\geq\frac{3}{2}|B|^{p+1}h_2^{\frac{p}{2}} \tag{5.3}$$

whenever

$$p\geq mn-1. \tag{5.4}$$

Let η be a smooth function with compact support. Integrating by parts in conjunction with Young's inequality leads to

$$\begin{aligned}\int_M|B|^{2p}h_2^p\eta^{2p}*1\leq&\frac{2}{3}\int_M|B|^{p-1}h_2^{\frac{p}{2}}\eta^{2p}\Delta(|B|^{p-1}h_2^{\frac{p}{2}})*1\\=&-\frac{2}{3}\int_M\nabla(|B|^{p-1}h_2^{\frac{p}{2}}\eta^{2p})\cdot\nabla(|B|^{p-1}h_2^{\frac{p}{2}})*1\\=&-\frac{2}{3}\int_M|\nabla(|B|^{p-1}h_2^{\frac{p}{2}})|^2\eta^{2p}*1\\&-\frac{2}{3}\int_M|B|^{p-1}h_2^{\frac{p}{2}}\cdot2p\eta^{2p-1}\nabla\eta\cdot\nabla(|B|^{p-1}h_2^{\frac{p}{2}})*1\\\leq&-\frac{2}{3}\int_M|\nabla(|B|^{p-1}h_2^{\frac{p}{2}})|^2\eta^{2p}*1+\frac{2}{3}\int_M|\nabla(|B|^{p-1}h_2^{\frac{p}{2}})|^2\eta^{2p}*1\\&+\frac{2}{3}\int_M p^2|B|^{2p-2}h_2^p\eta^{2p-2}|\nabla\eta|^2*1\\=&\frac{2}{3}p^2\int_M|B|^{2p-2}h_2^p\eta^{2p-2}|\nabla\eta|^2*1.\end{aligned} \tag{5.5}$$

By Hölder's inequality, we have

$$\begin{aligned}\int_M|B|^{2p-2}h_2^p\eta^{2p-2}|\nabla\eta|^2*1=&\int_M|B|^{2p-2}h_2^{p-1}\eta^{2p-2}\cdot h_2|\nabla\eta|^2*1\\\leq&\Big(\int_M|B|^{2p}h_2^p\eta^{2p}*1\Big)^{\frac{p-1}{p}}\Big(\int_M h_2^p|\nabla\eta|^{2p}*1\Big)^{\frac{1}{p}}.\end{aligned} \tag{5.6}$$

By (5.5) and (5.6), we finally arrive at

$$\Big(\int_M|B|^{2p}h_2^p\eta^{2p}*1\Big)^{\frac{1}{p}}\leq\frac{2}{3}p^2\Big(\int_M h_2^p|\nabla\eta|^{2p}*1\Big)^{\frac{1}{p}}. \tag{5.7}$$

Take $\eta\in C_c^\infty(M_R)$ to be the standard cut-off function such that $\eta\equiv1$ in $M_{\theta R}$ and $|\nabla\eta|\leq C(1-\theta)^{-1}R^{-1}$. Then, from (5.7), we have the following estimate.

Theorem 5.1 *Let M be an n-dimensional minimal submanifolds of $\mathbb{R}^{n+m}$. If the Gauss image of M_R is contained in an open geodesic ball of radius $\frac{\sqrt{2}}{4}\pi$ in $\mathbf{G}_{n,m}$, then there exists $C_1=C_1(n,m)$, such that*

$$\||B|^2h_2\|_{L^p(M_{\theta R})}\leq C_2(p)(1-\theta)^{-2}R^{-2}\|h_2\|_{L^p(M_R)} \tag{5.8}$$

whenever $p\geq C_1$ and $\theta\in(0,1)$.

Furthermore, the mean value inequality for any subharmonic function on minimal submanifolds in $\mathbb{R}^{m+n}$ (see [3, 16]) can be used to yield an estimate of the upper bound of $|B|^2$.

Let $B_R(x) \subset \mathbb{R}^{m+n}$ be a ball of radius R and centered at $x \in M$. Its restriction on M is denoted by

$$D_R(x) = B_R(x) \cap M.$$

Theorem 5.2 *Let $x \in M$, $R > 0$ such that the image of $D_R(x)$ under the Gauss map lies in an open geodesic ball of radius $\frac{\sqrt{2}}{4}\pi$ in $\mathbf{G}_{n,m}$. Then there exists $C_1 = C_1(n,m)$, such that*

$$|B|^{2p}(x) \le C(n,p) R^{-(n+2p)} \Big(\sup_{D_R(x)} h_2\Big)^p \mathrm{Vol}(D_R(x)) \tag{5.9}$$

for arbitrary $p \ge C_1$.

Proof Choose $q = p \ge mn - 1$ which satisfies (5.1). The inequality (5.2) means that $|B|^{2p} h_2^p$ is a subharmonic function on the minimal submanifold M. By Theorem 5.1 and the mean value inequality, we have

$$\begin{aligned} |B|^{2p} h_2^p(x) &\le \frac{C(n)}{(\frac{R}{2})^n} \int_{D_{\frac{R}{2}}(x)} |B|^{2p} h_2^p * 1 \\ &= \frac{C(n)}{(\frac{R}{2})^n} \||B|^2 h_2\|^p_{L^p(D_{\frac{R}{2}}(x))} \\ &\le \frac{C(n) C_2(p)^p}{(\frac{R}{2})^{n+2p}} \|h_2\|^p_{L^p(D_R(x))} \\ &\le \frac{C(n) C_2(p)^p}{(\frac{R}{2})^{n+2p}} \Big(\sup_{D_R(x)} h_2\Big)^p \mathrm{Vol}(D_R(x)), \end{aligned} \tag{5.10}$$

whenever $p \ge C_1(n,m)$.

6 Geometric Conclusions

Let $P_0 \in \mathbf{G}_{n,m}$ be a fixed point which is described by

$$P_0 = \varepsilon_1 \wedge \cdots \wedge \varepsilon_n,$$

where $\varepsilon_1, \cdots, \varepsilon_n$ are orthonormal vectors in $\mathbb{R}^{m+n}$. Choose complementary orthonormal vectors $\varepsilon_{n+1}, \cdots, \varepsilon_{n+m}$, such that $\{\varepsilon_1, \cdots, \varepsilon_n, \varepsilon_{n+1}, \cdots, \varepsilon_{n+m}\}$ is an orhtonormal base in $\mathbb{R}^{m+n}$.

Let $p : \mathbb{R}^{m+n} \to \mathbb{R}^n$ be the natural projection defined by

$$p(x^1, \cdots, x^n; x^{n+1}, \cdots, x^{m+n}) = (x^1, \cdots, x^n),$$

which induces a map from M to $\mathbb{R}^n$. It is a smooth map from a complete manifold to $\mathbb{R}^n$.

For any point $x \in M$, choose a local orthonormal tangent frame field $\{e_1, \cdots, e_n\}$ near x. Let $v = v_i e_i \in TM$. Its projection is

$$p_* v = \langle v_i e_i, \varepsilon_j \rangle \varepsilon_j = v_i \langle e_i, \varepsilon_j \rangle \varepsilon_j.$$

For any $P \in \gamma(M)$,

$$w \triangleq \langle P, P_0 \rangle = \langle e_1 \wedge \cdots \wedge e_n, \varepsilon_1 \wedge \cdots \wedge \varepsilon_n \rangle = \det W,$$

where $W = (\langle e_i, \varepsilon_j \rangle)$. It is well-known that

$$W^T W = O^T \Lambda O,$$

where O is an orthogonal matrix and

$$\Lambda = \begin{pmatrix} \lambda_1^2 & & 0 \\ & \ddots & \\ 0 & & \lambda_r^2 \end{pmatrix}, \quad r = \min(m, n),$$

where each $0 \le \lambda_i^2 \le 1$.

We now compare the length of any tangent vector v to M with its projection $p_* v$. Since

$$|p_* v|^2 = \sum_{j=1}^{n} (v_i \langle e_i, \varepsilon_j \rangle)^2 = (WV)^T WV,$$

where $V = (v_1, \cdots, v_n)^{\mathrm{T}}$, it follows that

$$|p_* v|^2 \ge (\lambda')^2 |v|^2 \ge w^2 |v|^2 \ge w_0^2 |v|^2, \tag{6.1}$$

where $\lambda' = \min_i \{\lambda_i\}$ and $w_0 = \inf_M w$. The induced metric $\mathrm{d}s^2$ on M from $\mathbb{R}^{m+n}$ is complete, so is the homothetic metric $\widetilde{\mathrm{d}}s^2 = w_0^2 \mathrm{d}s^2$ whenever $w_0 > 0$. (6.1) implies that

$$p : (M, \widetilde{\mathrm{d}}s^2) \to (\mathbb{R}^n, \text{canonical metric})$$

increases the distance. It follows that p is a covering map from a complete manifold into $\mathbb{R}^n$ and a diffeomorphism, since $\mathbb{R}^n$ is simply connected. Hence, the induced Riemannian metric on M can be expressed as $(\mathbb{R}^n, \mathrm{d}s^2)$ with

$$\mathrm{d}s^2 = g_{ij} \mathrm{d}x^i \mathrm{d}x^j.$$

Furthermore, the immersion $F : M \to \mathbb{R}^{m+n}$ is realized by a graph $(x, f(x))$ with $f : \mathbb{R}^n \to \mathbb{R}^m$ and

$$g_{ij} = \delta_{ij} + \frac{\partial f^\alpha}{\partial x^i} \frac{\partial f^\alpha}{\partial x^j}.$$

At each point in M, its image n-plane P under the Gauss map is spanned by

$$f_i = \varepsilon_i + \frac{\partial f^\alpha}{\partial x^i} \varepsilon_\alpha.$$

It follows that

$$|f_1 \wedge \cdots \wedge f_n|^2 = \det \Big(\delta_{ij} + \sum_\alpha \frac{\partial f^\alpha}{\partial x^i} \frac{\partial f^\alpha}{\partial x^j} \Big)$$

and

$$\sqrt{g} = |f_1 \wedge \cdots \wedge f_n|.$$

The n-plane P is also spanned by

$$p_i = g^{-\frac{1}{2n}} f_i.$$

Furthermore, we have

$$|p_1 \wedge \cdots \wedge p_n| = 1.$$

Then we have

$$\langle P, P_0\rangle = \det(\langle \varepsilon_i, p_j\rangle) = \begin{pmatrix} g^{-\frac{1}{2n}} & & 0 \\ & \ddots & \\ 0 & & g^{-\frac{1}{2n}} \end{pmatrix} = \frac{1}{\sqrt{g}} \geq w_0$$

and

$$\sqrt{g} \leq \frac{1}{w_0}. \tag{6.2}$$

Now, set

$$D_R(x) = \{(\widetilde{x}, f(\widetilde{x})) : \widetilde{x} \in \Omega,\ f_1, \cdots, f_m \text{ are smooth functions on } \Omega\},$$

where $\Omega \subset B_R \subset \mathbb{R}^n$. Then (6.2) implies

$$\mathrm{Vol}(D_R(x)) \leq \frac{1}{w_0} \cdot \mathrm{Vol}(\Omega) \leq \frac{1}{w_0} C(n) R^n. \tag{6.3}$$

The previous arguments show the following result.

Proposition 6.1 *Let M be a complete submanifold in $\mathbb{R}^{m+n}$. If the w-function is bounded below by a positive constant w_0, then M is an entire graph with Euclidean volume growth. In particular, if the Gauss image of M is contained in a geodesic ball of radius $\frac{\sqrt{2}}{4}\pi$, then M is an entire graph with Euclidean volume growth.*

Proof Now we consider the case that the image under the Gauss map γ is contained in an open geodesic ball of radius $\frac{\sqrt{2}}{4}\pi$ and centered at P_0. The Jordan angles between P and P_0 are

$$\theta_i = \cos^{-1}(\lambda_i),$$

where λ_i^2 are eigenvalues of the symmetric matrix $W^T W$ (see [23]). We know

$$w = \prod \cos\theta_i.$$

On the other hand, the distance between P_0 and P (see [26, pp. 188–194])

$$d(P_0, P) = \sqrt{\sum \theta_i^2}$$

is less than $\frac{\sqrt{2}}{4}\pi$ by the assumption. It follows that

$$w > w_0 = \Big(\cos\frac{\sqrt{2}}{4}\pi\Big)^r.$$

Theorem 4.1, Schoen-Simon-Yau's type estimates and Proposition 6.1 give us the following Bernstein type theorem.

Theorem 6.1 *Let M be a complete minimal n-dimensional submanifold in $\mathbb{R}^{n+m}$ with $n \leq 6$ and $m \geq 2$. If the Gauss image of M is contained in an open geodesic ball of $\mathbf{G}_{n,m}$ centered at P_0 and of radius $\frac{\sqrt{2}}{4}\pi$, then M has to be an affine linear subspace.*

Proof Now we choose

$$p = 4 + \frac{2}{3} + \frac{4}{3}\sqrt{1 + \frac{6}{mn}} > 6.$$

Fix $x \in M$ and let r be the Euclidean distance function from x and $M_R = D_R(x)$. Hence, letting $R \to +\infty$ in (4.12) yields

$$\||B|\|_{L^p(M)} = 0,$$

i.e., $|B|^2 = 0$. Thus M has to be an affine linear subspace.

Theorem 5.2 and Proposition 6.1 yield a Bernstein type result as follows.

Theorem 6.2 *Let M be a complete minimal n-dimensional submanifold in $\mathbb{R}^{n+m}$. If the Gauss image of M is contained in an open geodesic ball of $\mathbf{G}_{n,m}$ centered at P_0 and of radius $\frac{\sqrt{2}}{4}\pi$, and $\big(\frac{\sqrt{2}}{4}\pi - \rho \circ \gamma\big)^{-1}$ has growth*

$$\Big(\frac{\sqrt{2}}{4}\pi - \rho \circ \gamma\Big)^{-1} = o(R), \tag{6.4}$$

where ρ denotes the distance on $\mathbf{G}_{n,m}$ from P_0 and R is the Euclidean distance from any point in M, then M has to be an affine linear subspace.

Proof Now we claim

$$\sec(\sqrt{2}\,\rho) \le C\Big(\frac{\sqrt{2}}{4}\pi - \rho\Big)^{-1} \tag{6.5}$$

for a positive constant C. It is sufficient to prove that the function

$$t \in \Big[0, \frac{\sqrt{2}}{4}\pi\Big) \mapsto \sec(\sqrt{2}\,t)\Big(\frac{\sqrt{2}}{4}\pi - t\Big)$$

is bounded, which follows from

$$\lim_{t \to (\frac{\sqrt{2}}{4}\pi)^-} \sec(\sqrt{2}\,t)\Big(\frac{\sqrt{2}}{4}\pi - t\Big) = \lim_{t \to (\frac{\sqrt{2}}{4}\pi)^-} \frac{\frac{\sqrt{2}}{4}\pi - t}{\cos(\sqrt{2}\,t)} = \lim_{t \to (\frac{\sqrt{2}}{4}\pi)^-} \frac{-1}{-\sqrt{2}\sin(\sqrt{2}\,t)} = \frac{\sqrt{2}}{2}.$$

Hence we arrive at the inequality

$$h_2 \le C\Big(\frac{\sqrt{2}}{4}\pi - \rho \circ \gamma\Big)^{-2}. \tag{6.6}$$

Thus, for any point $x \in M$, by Theorem 5.2 and Proposition 6.1, we have

$$|B|^{2p}(x) \le C(n,p)R^{-2p}\Big(\frac{\sqrt{2}}{4}\pi - \rho \circ \gamma\Big)^{-2p}.$$

Letting $R \to +\infty$ in the above inequality forces $|B(x)| = 0$.

From (4.11), it is easy to obtain the following result.

Theorem 6.3 *Let M be an n-dimensional complete minimal submanifolds of $\mathbb{R}^{n+m}$. If the Gauss image of M is contained in an open geodesic ball in $\mathbf{G}_{n,m}$ of radius $\frac{\sqrt{2}}{4}\pi$ and M has finite total curvature, then M has to be an affine linear subspace.*

For a minimal n-submanifold in $\mathbb{R}^{m+n}$, if its Gauss image is contained in an open geodesic ball on $\mathbf{G}_{n,m}$ of radius $\frac{\sqrt{2}}{4}\pi$, there is a positive function $h_1 = \cos(\sqrt{2}\,\rho \circ \gamma)$. Then the strong stability inequality (3.6) follows. Besides, its key role in Schoen-Simon-Yau's estimates, there are other applications. We state the following results, whose detailed proof can be found in the previous paper of the first author [27].

Theorem 6.4 *Let M be a complete minimal n-submanifold in $\mathbb{R}^{m+n}$. If the image under the Gauss map is contained in an open geodesic ball in $\mathbf{G}_{n,m}$ of radius $\frac{\sqrt{2}}{4}\pi$, then any L^2-harmonic 1-form vanishes.*

Theorem 6.5 *Let M be one as in Theorem 6.4, and N be a manifold with non-positive sectional curvature. Then any harmonic map $f : M \to N$ with finite energy has to be constant.*

References

[1] Bombieri, E., De Giorgi, E. and Guusti, E., Minimal cones and the Bernstein problem, *Invent. Math.*, **7**(3), 1969, 243–268.

[2] Bernstein, S., Sur un théorème de géométrie et ses application aux équations aux dérivés partielles du type elliptique, *Comm. de la Soc. Math. de Kharkov* (2*éme Sér.*), **15**, 1915–1917, 38–45.

[3] Cheng, S. Y., Li, P. and Yau, S. T., Heat equations on minimal submanifolds and their applications, *Amer. J. Math.*, **106**(5), 1984, 1033–1065.

[4] do Carmo, M. and Peng, C. K., Stable complete minimal surfaces in $\mathbb{R}^3$ are planes, *Bull. Amer. Math. Soc.*, **1**(6), 1979, 903–906.

[5] Chen, Q. and Xu, S., Rigidity of compact minimal submanifolds in a unit sphere, *Geom. Dedicata*, **45**(1), 1993, 83–88.

[6] Fischer-Colbrie, D. and Schoen, R., The structure of complete stable minimal surfaces in 3-manifolds of non-negative scalar curvature, *Comm. Pure Appl. Math.*, **33**(2), 1980, 199–211.

[7] Fujimoto, H., Modified defect relations for Gauss map of minimal surfaces, *J. Diff. Geom.*, **29**(2), 1989, 245–262.

[8] Fischer-Colbrie, D., Some rigidity theorems for minimal submanifolds of the sphere, *Acta Math.*, **145**(1), 1980, 29–46.

[9] Ecker, K. and Huisken, G., A Bernstein result for minimal graphs of controlled growth, *J. Diff. Geom.*, **31**(2), 1990, 397–400.

[10] Heinz, E., Über die lösungen der minimalflächengleichung, *Nachr. Akad. Wiss. Göttingen Math. Phys.*, **K1 II**, 1952, 51–56.

[11] Hildebrandt, S., Jost, J. and Widman, K. O., Harmonic mappings and minimal submanifolds, *Invent. Math.*, **62**(2), 1980, 269–298.

[12] Jost, J. and Xin, Y. L., Bernstein type theorems for higher codimension, *Calc. Var. PDE*, **9**(4), 1999, 277–296.

[13] Lawson, H. B. and Osserman, R., Non-existence, non-uniqueness and irregularity of solutions to the minimal surface system, *Acta Math.*, **139**(1), 1977, 1–17.

[14] Li, A. M. and Li, J. M., An intrinsic rigidity theorem for minimal submanifolds in a sphere, *Arch. Math.*, **58**(6), 1992, 582–594.

[15] Moser, J., On Harnack's theorem for elliptic differential equations, *Comm. Pure Appl. Math.*, **14**, 1961, 577–591.

[16] Ni, L., Gap theorems for minimal submanifolds in $\mathbb{R}^{n+1}$, *Comm. Analy. Geom.*, **9**(3), 2001, 641–656.

[17] Osserman, R., A Survey of Minimal Surfaces, Van Nostrand Reinhold, New York, 1969.

[18] Ruh, E. A. and Vilms, J., The tension field of Gauss map, *Trans. Amer. Math. Soc.*, **149**(2), 1970, 569–573.

[19] Schoen, R., Simon, L. and Yau, S. T., Curvature estimates for minimal hypersurfaces, *Acta Math.*, **134**(1), 1975, 275–288.

[20] Simons, J., Minimal varieties in Riemannian manifolds, *Ann. of Math.*, **88**(1), 1968, 62–105.

[21] Smoczyk, K., Wang, G. F. and Xin, Y. L., Bernstein type theorems with flat normal bundle, *Calc. Var. PDE*, **26**(1), 2006, 57–67.

[22] Solomon, B., On the Gauss map of an area-minimizing hypersurface, *J. Diff. Geom.*, **19**(1), 1984, 221–232.

[23] Wong, Y. C., Differential geometry of Grassmann manifolds, *Proc. NAS*, **57**(3), 1967, 589–594.

[24] Xavier, F., The Gauss map of a complete non-flat minimal surface cannot omit 7 points of the sphere, *Ann. of Math.*, **113**(1), 1981, 211–214.

[25] Xin, Y. L., Geometry of Harmonic Maps, Progress in Nonlinear Differential Equations and Their Applications, **23**, Birkhäuser, Basel, 1996.

[26] Xin, Y. L., Minimal Submanifolds and Related Topics, World Scientific, Singapore, 2003.

[27] Xin, Y. L., Bernstein type theorems without graphic condition, *Asia J. Math.*, **9**(1), 2005, 31–44.

[28] Xin, Y. L., Mean curvature flow with convex Gauss image, *Chin. Ann. Math.*, **29B**(2), 2008, 121–134.