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# Recovering of Damping Coefficients for a System of Coupled Wave Equations with Neumann Boundary Conditions: Uniqueness and Stability*

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**Abstract** The authors study the inverse problem of recovering damping coefficients for two coupled hyperbolic PDEs with Neumann boundary conditions by means of an additional measurement of Dirichlet boundary traces of the two solutions on a suitable, explicit subportion $\Gamma_1$ of the boundary $\Gamma$, and over a computable time interval $T > 0$. Under sharp conditions on $\Gamma_0 = \Gamma\backslash\Gamma_1$, $T > 0$, the uniqueness and stability of the damping coefficients are established. The proof uses critically the Carleman estimate due to Lasiecka et al. in 2000, together with a convenient tactical route "post-Carleman estimates" suggested by Isakov in 2006.

**Keywords** Inverse problem, Coupled wave equations, Carleman estimate
**2000 MR Subject Classification** 35R30, 35L10, 49K20

## 1 Introduction — Problem Formulation, Assumptions

### 1.1 Setting and Problem formulation

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be an open bounded domain with boundary $\Gamma = \partial\Omega$ of class $C^2$, consisting of the closure of two disjoint parts: $\Gamma_0$ (the uncontrolled or unobserved part) and $\Gamma_1$ (the controlled or observed part), both relatively open in $\Gamma$. Namely, $\Gamma = \partial\Omega = \overline{\Gamma_0 \cup \Gamma_1}$, $\Gamma_0 \cap \Gamma_1 = \emptyset$. Let $\nu = [\nu_1, \cdots, \nu_n]$ be the unit outward normal vector on $\Gamma$, and let $\frac{\partial}{\partial\nu} = \nabla \cdot \nu$ denote the corresponding normal derivative.

### 1.2 Main geometrical assumptions

Following [25, Section 5], [9] and [16], throughout this paper, we make the following assumptions:

(A.1) Given the triple $\{\Omega, \Gamma_0, \Gamma_1\}$, $\partial\Omega = \overline{\Gamma_0 \cup \Gamma_1}$, $\Gamma_0 = \Gamma \setminus \Gamma_1$, there exists a strictly convex (real-valued) non-negative function $d : \overline{\Omega} \to \mathbb{R}^+$, of class $C^3(\overline{\Omega})$, such that, if we introduce the (conservative) vector field $h(x) = [h_1(x), \cdots, h_n(x)] \equiv \nabla d(x)$, $x \in \Omega$, then the following two properties hold true:

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(i)

$$\frac{\partial d}{\partial \nu}\Big|_{\Gamma_0} = \nabla d \cdot \nu = h \cdot \nu = 0, \quad \text{on } \Gamma_0 = \Gamma \setminus \Gamma_1; \quad h \equiv \nabla d; \tag{1.1}$$

(ii) the (symmetric) Hessian matrix $\mathcal{H}_d$ of $d(x)$ (i.e., the Jacobian matrix $J_h$ of $h(x)$) is strictly positive definite on $\overline{\Omega}$: there exists a constant $\rho > 0$, such that for all $x \in \overline{\Omega}$,

$$\mathcal{H}_d(x) = J_h(x) = \begin{bmatrix} d_{x_1x_1} & \cdots & d_{x_1x_n} \\ \vdots & & \vdots \\ d_{x_nx_1} & \cdots & d_{x_nx_n} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial h_1}{\partial x_1} & \cdots & \dfrac{\partial h_1}{\partial x_n} \\ \vdots & & \vdots \\ \dfrac{\partial h_n}{\partial x_1} & \cdots & \dfrac{\partial h_n}{\partial x_n} \end{bmatrix} \geq \rho I. \tag{1.2}$$

(A.2) $d(x)$ has no critical point on $\overline{\Omega}$:

$$\inf_{x\in\Omega} |h(x)| = \inf_{x\in\Omega} |\nabla d(x)| = s > 0. \tag{1.3}$$

**Remark 1.1** Assumption (A.1) is due to the Neumann boundary conditions of the hyperbolic problem to follow. It was introduced in [25, Section 5]. Assumption (A.2) is needed for the validity of the pointwise Carleman estimate in Section 3 below (it will imply that the constant $\beta$ is positive, $\beta > 0$, in estimate (3.11) below). Actually, as noted in [16, Remark 1.1.3, p. 229], assumption (A.2) = (1.3) is needed to hold true only with the infimum computed for $x \in \Gamma_0$ (the uncontrolled or unobserved part of the boundary $\Gamma$). Moreover, (A.2) can, in effect, be dispensed with [16, Section 10] by use of two vector fields. For the sake of keeping the exposition simpler, we shall not exploit here this substantial generalization. Assumptions (A.1) and (A.2) hold true for large classes of triples $\{\Omega, \Gamma_0, \Gamma_1\}$ (see Appendix in [16] or [20]). One canonical case is that $\Gamma_0$ is flat: here then we can take $d(x) = |x - x_0|^2$, with $x_0$ collocated on the hyperplane containing $\Gamma_0$ and outside $\Omega$. Then $h(x) = \nabla d(x) = 2(x - x_0)$ is radial. Another case is where $\Gamma_0$ is either convex or concave and subtended by a common point; more precisely, see [16, Theorem A.4.1, p. 301], in which case, the corresponding required $d(\,\cdot\,)$ can also be explicitly constructed. See illustrative configurations in the Appendix.

### 1.3 The coupled hyperbolic system with two unknown damping coefficients

Following [14], we consider the following coupled system of two second-order hyperbolic equations in the unknowns $w = w(x,t)$ and $z = z(x,t)$ on $Q = \Omega \times [0,T]$:

$$\begin{cases} w_{tt} = \Delta w + q(x) z_t, \quad z_{tt} = \Delta z + p(x) w_t, & \text{in } Q, & (1.4\text{a}) \\ w\Big(\,\cdot\,, \dfrac{T}{2}\Big) = w_0(x), \ w_t\Big(\,\cdot\,, \dfrac{T}{2}\Big) = w_1(x), & \text{in } \Omega, & (1.4\text{b}) \\ z\Big(\,\cdot\,, \dfrac{T}{2}\Big) = z_0(x), \ z_t\Big(\,\cdot\,, \dfrac{T}{2}\Big) = z_1(x), & \text{in } \Omega, & (1.4\text{c}) \\ \dfrac{\partial w}{\partial \nu}\Big|_{\Sigma} = \mu_1(x,t), \quad \dfrac{\partial z}{\partial \nu}\Big|_{\Sigma} = \mu_2(x,t), & \text{in } \Sigma. & (1.4\text{d}) \end{cases}$$

Here $q(x)$, $p(x)$ are time-independent unknown damping coefficients. Instead, $[w_0, w_1, z_0, z_1]$ are the given initial conditions and $\mu_1$, $\mu_2$ are the given Neumann boundary conditions. We shall denote by $\{w(q,p), z(q,p)\}$ the solution to problem (1.4a)–(1.4d) due to the damping coefficients

$\{q, p\}$ (and fixed data $\{w_0, w_1, z_0, z_1, \mu_1, \mu_2\}$). A sharp interior and boundary regularity theory of the corresponding coupled mixed problem (1.4a)–(1.4d) under a variety of classes of data may be given, following the single-equation case in [10, 11] (see also [12], [15, Chapter 8, Section 8A, p. 755]) and [24]. To this theory, we shall critically appeal in Section 6.

**Remark 1.2** We could, in effect, add the term $r_1(x) \cdot \nabla w(x,t)$ on the RHS of the $w$-equation in (1.4a) and similarly $r_2(x) \cdot \nabla z(x,t)$ on the RHS of the $z$-equation in (1.4a), with known coefficients $r_i(x)$ satisfying $|r_i(x)| \in L^\infty(\Omega)$, $i = 1, 2$. The proofs remain essentially unchanged, as [16] covers this case as well.

We note that the map $\{q, p\} \to \{w, z\}$ is nonlinear and hence consider the following nonlinear inverse problem.

**(I) Nonlinear inverse problem for system (1.4)**

Let $\{w = w(q,p), z = z(q,p)\}$ be a solution to system (1.4). Under geometrical conditions on $\Gamma_0$, is it possible to retrieve $q(x)$ and $p(x)$, $x \in \Omega$, from measurement of the Dirichlet traces of $w(q,p)$ and $z(q,p)$ on the observed part of the boundary $\Gamma_1 \times [0,T]$ over a sufficiently large time interval $T$? This problem comprises two basic issues: uniqueness and stability. More precisely, we consider

**(I1) Uniqueness in the nonlinear inverse problem for system $\{w, z\}$ in (1.4)**

Let $\{w = w(q,p), z = z(q,p)\}$ be the solution to system (1.4). Under geometrical conditions on $\Gamma_0$, do the Dirichlet boundary traces $w|_{\Gamma_1 \times [0,T]}$ and $z|_{\Gamma_1 \times [0,T]}$ determine $q(x)$ and $p(x)$ uniquely? In other words,

$$\text{does } \begin{cases} w(q_1,p_1)|_{\Gamma_1\times[0,T]} = w(q_2,p_2)|_{\Gamma_1\times[0,T]}, \\ z(q_1,p_1)|_{\Gamma_1\times[0,T]} = z(q_2,p_2)|_{\Gamma_1\times[0,T]} \end{cases} \text{ imply } \begin{cases} q_1(x) = q_2(x), \\ p_1(x) = p_2(x) \end{cases} \text{ a.e. in } \Omega? \tag{1.5}$$

**Remark 1.3** As in exact controllability/uniform stabilization theories in [5], one expects that geometrical conditions are needed only in the complementary part $\Gamma_0$ of that part $\Gamma_1$ of the boundary where measurement takes place.

Assuming that the answer to the aforementioned uniqueness question (1.5) is in the affirmative, we then ask the following more demanding, quantitative estimate.

**(I2) Stability in the nonlinear inverse problem for system $\{w, z\}$ in (1.4)**

In the above setting, let $\{w(q_1,p_1), z(q_1,p_1)\}$, $\{w(q_2,p_2), z(q_2,p_2)\}$ be solutions to (1.4) due to corresponding damping coefficients $\{q_1, p_1\}$, $\{q_2, p_2\}$ and fixed common data $\{w_0, w_1, z_0, z_1, \mu_0, \mu_1\}$. Under geometric conditions on the complimentary unobserved part of the boundary $\Gamma_0 = \Gamma \setminus \Gamma_1$, is it possible to estimate the norms $\|q_1 - q_2\|_{L^2(\Omega)}$, $\|p_1 - p_2\|_{L^2(\Omega)}$ in terms of suitable norms of the Dirichlet traces $(w(q_1,p_1) - w(q_2,p_2))|_{\Gamma_1\times[0,T]}$ and $(z(q_1,p_1) - z(q_2,p_2))|_{\Gamma_1\times[0,T]}$?

**(II) The corresponding homogeneous linear inverse problem**

As usual, the nonlinear inverse problem is converted into a linear inverse problem for an auxiliary, corresponding problem. Let

$$\begin{aligned} &f(x) = q_1(x) - q_2(x), \quad g(x) = p_1(x) - p_2(x), \\ &R_1(x,t) = z_t(q_2,p_2)(x,t), \quad R_2(x,t) = w_t(q_2,p_2)(x,t), \end{aligned} \tag{1.6a}$$

$$u(x,t) = w(q_1,p_1)(x,t) - w(q_2,p_2)(x,t), \quad v(x,t) = z(q_1,p_1)(x,t) - z(q_2,p_2)(x,t). \tag{1.6b}$$

Then $\{u(x,t), v(x,t)\}$ satisfies the following homogeneous system:

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) - q(x)v_t(x,t) = f(x)R_1(x,t), & \text{in } Q, \quad (1.7a)\\ v_{tt}(x,t) - \Delta v(x,t) - p(x)u_t(x,t) = g(x)R_2(x,t), & \text{in } Q, \quad (1.7b)\\ u\Big(\cdot, \dfrac{T}{2}\Big) = 0,\ u_t\Big(\cdot, \dfrac{T}{2}\Big) = 0; \quad v\Big(\cdot, \dfrac{T}{2}\Big) = 0,\ v_t\Big(\cdot, \dfrac{T}{2}\Big) = 0, & \text{in } \Omega, \quad (1.7c)\\ \dfrac{\partial u}{\partial \nu}\Big|_{\Sigma} = 0; \quad \dfrac{\partial v}{\partial \nu}\Big|_{\Sigma} = 0, & \text{in } \Sigma. \quad (1.7d) \end{cases}$$

The above serves only as a motivation. Henceforth, we shall consider the $\{u,v\}$-problem, with damping coefficients $q, p \in L^\infty(\Omega)$ as given, and terms $R_1(x,t)$, $R_2(x,t)$ fixed and suitable while the terms $f(x)$, $g(x)$ are unknown time-independent coefficients. The $\{u,v\}$-problem has the advantage over the original $\{w,z\}$-problem in (1.4) that the map $\{f,g\} \to \{u,v\}$ is linear.

**(II1) Uniqueness in the linear inverse problem for system $\{u,v\}$ in (1.7)**

Let $\{u = u(f,g), v = v(f,g)\}$ be the solution to system (1.7). Under geometrical conditions on $\Gamma_0$, do the Dirichlet traces $u|_{\Gamma_1\times[0,T]}$ and $v|_{\Gamma_1\times[0,T]}$ determine $f(x)$ and $g(x)$ uniquely? In other words, by linearity,

$$\text{does } \begin{cases} u(f,g)|_{\Gamma_1\times[0,T]} = 0, \\ v(f,g)|_{\Gamma_1\times[0,T]} = 0 \end{cases} \text{ imply } \begin{cases} f(x) = 0, \\ g(x) = 0 \end{cases} \text{ a.e. in } \Omega? \tag{1.8}$$

Assuming that the answer to the aforementioned uniqueness question (1.8) is in the affirmative, we then ask the following more demanding, quantitative estimate.

**(II2) Stability in the linear inverse problem for system $\{u,v\}$ in (1.7)**

In the above setting, let $\{u(f,g), v(f,g)\}$ be solution to (1.7). Under geometric conditions on the complimentary unobserved part of the boundary $\Gamma_0 = \Gamma \setminus \Gamma_1$, is it possible to estimate the norms $\|f\|_{L^2(\Omega)}$, $\|g\|_{L^2(\Omega)}$ in terms of suitable norms of the Dirichlet traces $u(f,g)|_{\Gamma_1\times[0,T]}$ and $v(f,g)|_{\Gamma_1\times[0,T]}$?

The goal of the present paper is to give affirmative and quantitative answer to the above uniqueness and stability questions for the linear and nonlinear inverse problems.

**Remark 1.4** In models (1.4) and (1.7), we regard $t = \frac{T}{2}$ as the initial time. This is not essential, as the change of variable $t \to t - \frac{T}{2}$ transforms $t = \frac{T}{2}$ to $t = 0$. However, this present choice is convenient in order to apply the Carleman estimate established in [16], which uses the pseudo-convex function $\varphi(x,t)$ in (3.1a) centered around $\frac{T}{2}$.

## 2 Main Results on Uniqueness and Stability

We begin with a uniqueness result for the linear inverse problem (1.8) involving the $\{u,v\}$-system (1.7).

**Theorem 2.1** (Uniqueness of Linear Inverse Problem) *Assume the preliminary geometric assumptions* (A.1) *and* (A.2). *Let*

$$T > T_0 \equiv 2\sqrt{\max_{x\in\overline{\Omega}} d(x)}. \tag{2.1}$$

*With reference to the* $\{u, v\}$*-system* (1.7)*, let the fixed data* $\{q, p\}$*,* $\{R_1, R_2\}$ *and unknown terms* $f$ *and* $g$ *satisfy the following regularity properties:*

$$q, p \in L^\infty(\Omega); \quad R_i, R_{it}, R_{itt} \in L^\infty(Q), \quad R_{ix_j}\Big(\,\cdot\,, \frac{T}{2}\Big) \in L^\infty(\Omega); \quad f, g \in L^2(\Omega), \tag{2.2}$$

$i = 1, 2,\ j = 1, \cdots, n$*, as well as the following positivity conditions:*

$$\Big|R_1\Big(x, \frac{T}{2}\Big)\Big| \geq r_1 > 0, \quad \Big|R_2\Big(x, \frac{T}{2}\Big)\Big| \geq r_2 > 0, \quad x \in \overline{\Omega} \tag{2.3}$$

*for some positive constants* $r_1, r_2$*. If the solution* $\{u = u(f, g), v = v(f, g)\}$ *to system* (1.7) *satisfies the additional homogeneous Dirichlet boundary trace condition*

$$u(f, g)(x, t) = v(f, g)(x, t) = 0, \quad x \in \Gamma_1,\ t \in [0, T], \tag{2.4}$$

*over the observed part* $\Gamma_1$ *of the boundary* $\Gamma$ *and over the time interval* $T$ *as in* (2.1)*, then, in fact*

$$f(x) = g(x) \equiv 0, \quad a.e.\ x \in \Omega. \tag{2.5}$$

Next, we provide the stability result for the linear inverse problem involving the $\{u, v\}$-system (1.7a)–(1.7c), and the determination of the terms $f(\,\cdot\,)$, $g(\,\cdot\,)$ in (1.7a)–(1.7b). We shall seek $f$ and $g$ in $L^2(\Omega)$. We first state the stability of the linear inverse problem.

**Theorem 2.2** (Stability of the Linear Inverse Problem) *Assume the preliminary geometric assumptions* (A.1) *and* (A.2)*. Consider problem* (1.7a)–(1.7d) *on* $[0, T]$ *with* $T > T_0$*, as in* (2.1) *and data satisfying properties* (2.2) *where, moreover,* $R_1$*,* $R_2$ *satisfy the positivity condition* (2.3) *at the initial time* $t = \frac{T}{2}$*. Then there exists a constant* $C = C(\Omega, T, \Gamma_1, \varphi, q, p, R_1, R_2) > 0$*, i.e., depending on the data of problem* (1.7a)–(1.7d)*, but not on the unknown coefficients* $f$ *and* $g$*, such that*

$$\begin{aligned} &\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\Omega)} \\ &\leq C(\|u_t(f)\|_{L^2(\Sigma_1)} + \|u_{tt}(f)\|_{L^2(\Sigma_1)} + \|v_t(f)\|_{L^2(\Sigma_1)} + \|v_{tt}(f)\|_{L^2(\Sigma_1)}) \end{aligned} \tag{2.6}$$

*for all* $f, g \in L^2(\Omega)$*.*

Next we give the corresponding uniqueness and stability results to the nonlinear inverse problem invoking the $\{w, z\}$-system (1.4).

**Theorem 2.3** (Uniqueness of Nonlinear Inverse Problem) *Assume the preliminary geometric assumptions* (A.1) *and* (A.2)*. Let* $T$ *be as in* (2.1)*. Assume further the following a-priori regularity of the unknown coefficients for* (1.4)*:*

$$q_1, q_2, p_1, p_2 \in L^\infty(\Omega). \tag{2.7}$$

*Assume further that the initial conditions of* (1.4) *satisfy the following regularity and positivity conditions:*

$$\begin{aligned} &\{w_0, w_1\}, \{z_0, z_1\} \in \mathcal{D}(A_N^{k+\frac{1}{2}}) \times \mathcal{D}(A_N^k),\ k > \frac{\dim\Omega}{4} + 1, \\ &w_{1x_j}, z_{1x_j} \in L^\infty(\Omega),\ j = 1, \cdots, n, \end{aligned} \tag{2.8}$$

$$|w_1(x)| \geq w_1 > 0, \quad |z_1(x)| \geq z_1 > 0, \quad x \in \overline{\Omega} \tag{2.9}$$

*for some positive constants* $w_1, z_1$, *and the nonhomogemeous boundary conditions satisfy the following*

$$\begin{cases} \mu_i \in H^m(0,T;L^2(\Gamma)) \cap C([0,T];H^{\alpha-\frac{1}{2}+(m-1)}(\Gamma)), \\ \alpha = \dfrac{2}{3} \text{ for a general domain; } \alpha = \dfrac{3}{4} \text{ for a parallelepiped,} \\ \text{with Compatibility Relations,} \\ \mu_i\Big(\dfrac{T}{2}\Big) = \dot{\mu}_i\Big(\dfrac{T}{2}\Big) = \cdots = \mu_i^{(m-1)}\Big(\dfrac{T}{2}\Big) = 0, \quad i = 1,2 \end{cases} \tag{2.10}$$

*for*

$$m > \frac{\dim \Omega}{2} + 3 - \alpha. \tag{2.11}$$

*Finally, if the solutions* $\{w(q_1,p_1), z(q_1,p_1)\}$ *and* $\{w(q_2,p_2),\, z(q_2,p_2)\}$ *to system* (1.4) *have the same Dirichlet boundary traces on* $\Sigma_1 = \Gamma_1 \times [0,T]$:

$$w(q_1,p_1)(x,t) = w(q_2,p_2)(x,t), \quad z(q_1,p_1)(x,t) = z(q_2,p_2)(x,t), \quad x \in \Gamma_1,\ t \in [0,T], \tag{2.12}$$

*then, in fact, the respective damping coefficients coincide*

$$q_1(x) = q_2(x), \quad p_1(x) = p_2(x), \quad a.e.\ x \in \Omega. \tag{2.13}$$

Finally, we state the stability result for the nonlinear inverse problem involving the $\{w,z\}$-problem (1.4a)–(1.4c) with damping coefficients $q(\,\cdot\,)$ and $p(\,\cdot\,)$.

**Theorem 2.4** (Stability of Nonlinear Inverse Problem) *Assume preliminary geometric assumptions* (A.1) *and* (A.2). *Consider problem* (1.4a)–(1.4d) *on* $[0,T]$, *with* $T > T_0$ *as in* (2.1), *one time with damping coefficients* $q_1, p_1 \in L^\infty(\Omega)$, *and one time with damping coefficients* $q_2, p_2 \in L^\infty(\Omega)$, *and let* $w(q_1,p_1)$, $w(q_2,p_2)$ *denote the corresponding solutions. Assume the regularity and positivity conditions* (2.8)–(2.9) *on the initial data and regularity property* (2.10) *on the boundary data. Then, the following stability result holds true for the* $w$*-problem* (1.4a)–(1.4d): *there exists a constant* $C = C(\Omega, T, \Gamma_1, \varphi, M, w_0, w_1, z_0, z_1, \mu_1, \mu_2) > 0$, *i.e., depending on the data of problem* (1.4a)–(1.4d) *and on the* $L^\infty(\Omega)$*-norm of the damping coefficients, such that*

$$\begin{aligned} &\|q_1 - q_2\|_{L^2(\Omega)} + \|p_1 - p_2\|_{L^2(\Omega)} \\ &\leq C(\|w_t(q_1,p_1) - w_t(q_2,p_2)\|_{L^2(\Sigma_1)} + \|w_{tt}(q_1,p_1) - w_{tt}(q_2,p_2)\|_{L^2(\Sigma_1)} \\ &\quad + \|z_t(q_1,p_1) - z_t(q_2,p_2)\|_{L^2(\Sigma_1)} + \|z_{tt}(q_1,p_1) - z_{tt}(q_2,p_2)\|_{L^2(\Sigma_1)}) \end{aligned} \tag{2.14}$$

*for all coefficients* $q_1, p_1, q_2, p_2 \in \{q \in L^\infty(\Omega) \mid \|q\|_{L^\infty} \leq M\}$.

The uniqueness of the multidimensional inverse problem with a single boundary observation, was pioneered by Bukhgeim and Klibanov in [1], where the authors provide a methodology based on a type of exponential weighted estimates, which is usually referred to as the Carleman estimates, initiated by Carleman [3] for a problem with two variables. After [1], several papers concerning inverse problems by using Carleman estimates were published, most of which focused on the Dirichlet problem, however. In [20], we gave an extensive review of the literature in multidimensional inverse problems for hyperbolic PDEs. For single equations, see [7, 8, 26] and the references therein for recovering the potential coefficient and [2] for recovering the damping coefficient. Reference [19] recovered two coefficients — damping and source — for a single equation. Instead, reference [21] treated an inverse problem for two coupled Schrödinger equations. Finally, our geometrical conditions — even for the uniqueness problem of the damping coefficient (energy level term) of a single hyperbolic equation with Neumann boundary conditions — are sharper/optimal with respect to those in the literature, where, instead, uniqueness of the potential coefficient (lower order term) is generally claimed. For example, for $\Omega$ = 2D-disk, we need measurement of the Dirichlet trace only on $\epsilon$-more than $\frac{1}{2}$ circumference, not $\epsilon$-more than $\frac{3}{4}$ circumference (see [8]). We have already noted that our present treatment can readily treat also very strongly coupled systems with gradient terms even time- and space-dependent.

## 3 First Basic Step of Proofs: A Carleman Estimate and a Continuous Observability Inequality at the $H^1 \times L^2$-Level

In this section, we recall from [16] a Carleman estimate at the $H^1 \times L^2$-level for a single hyperbolic equation, that plays a key role in the proof of Theorem 2.1. The Carleman estimate in [16], inspired by [17], is "pointwise" and removes lower order terms. The prior Carleman estimates in [14] or in [23] are in an "integral" form and contain lower order terms which can then be eliminated by appealing to [16] via a compactness-uniqueness argument.

### 3.1 Pseudo-convex function (see [16, p. 230])

Choosing the strictly convex potential function $d(x)$ satisfying (A.1)–(A.2) and $d(x) \geq m > 0$, we next introduce the pseudo-convex function $\varphi(x,t)$ defined by

$$\varphi(x,t) = d(x) - c\Big(t - \frac{T}{2}\Big)^2, \quad x \in \Omega,\ t \in [0,T], \tag{3.1}$$

where $T > T_0$, as in (2.1) and $0 < c < 1$ is selected as follows: By (2.1), there exists a $\delta > 0$, such that

$$T^2 > 4 \max_{x \in \overline{\Omega}} d(x) + 4\delta. \tag{3.2}$$

For this $\delta > 0$, there exists a constant $c$, $0 < c < 1$, such that

$$cT^2 > 4 \max_{x \in \overline{\Omega}} d(x) + 4\delta. \tag{3.3}$$

This function $\varphi(x,t)$ has the following properties:

(a) For the constant $\delta > 0$ fixed in (3.2) and for any $t > 0$,

$$\varphi(x,0) \equiv \varphi(x,T) = d(x) - c\,\frac{T^2}{4} \leq \max_{x \in \overline{\Omega}} d(x) - c\,\frac{T^2}{4} \leq -\delta, \quad \text{uniformly in } x \in \Omega, \tag{3.4a}$$

$$\varphi(x,t) \leq \varphi\Big(x, \frac{T}{2}\Big) \quad \text{for any } t > 0 \text{ and any } x \in \Omega. \tag{3.4b}$$

(b) There are $t_0$ and $t_1$, with $0 < t_0 < \frac{T}{2} < t_1 < T$, such that

$$\min_{\substack{x\in\overline{\Omega}\\ t\in[t_0,t_1]}} \varphi(x,t) \geq \sigma, \quad \text{where } 0<\sigma<m=\min_{x\in\overline{\Omega}} d(x). \tag{3.5}$$

Moreover, we recall the subset $Q(\sigma)$ of $\Omega\times[0,T]$ defined by [16, (1.1.19), p. 232]:

$$Q(\sigma) = \{(x,t) \mid \ x\in\Omega,\ 0\leq t\leq T,\ \varphi(x,t)\geq\sigma>0\}. \tag{3.6}$$

An important property of $Q(\sigma)$ [16, (1.1.20), p. 232] is

$$[t_0,t_1]\times\Omega \subset Q(\sigma)\subset[0,T]\times\Omega. \tag{3.7}$$

### 3.2 Carleman estimate at the $H^1\times L^2$-level

We consider the wave equation of the form

$$y_{tt}(x,t) - \Delta y(x,t) = F(x,t), \quad x\in\Omega,\ t\in[0,T] \tag{3.8}$$

at first without imposing boundary conditions. We shall consider initially solutions $y(x,t)$ to (3.8) in the class

$$y\in H^{2,2}(Q) \equiv L^2(0,T;H^2(\Omega))\cap H^2(0,T;L^2(\Omega)). \tag{3.9}$$

For such solutions to (even to a far more general equation than) (3.8), the following Carleman estimate holds true.

**Theorem 3.1** (see [16, p. 255]) *Assume* (A.1) *and* (A.2). *Let $\varphi(x,t)$ be defined as in* (3.1). *Let $y\in H^2(Q)$ be a solution to equation* (3.8) *where $F\in L^2(Q)$. Then the following one parameter family of estimates holds true, with $\rho>0$, $\beta>0$ (by* (A.2) $=$ (1.3)*), for all $\tau>0$ sufficiently large and $\epsilon>0$ small:*

$$\begin{aligned} &BT|_{\Sigma}(y) + 2\int_Q \mathrm{e}^{2\tau\varphi}|F|^2\mathrm{d}Q + C_{1,T}\mathrm{e}^{2\tau\sigma}\int_Q y^2\mathrm{d}Q \\ \geq{}& C_{1,\tau}\int_Q \mathrm{e}^{2\tau\varphi}[y_t^2+|\nabla y|^2]\mathrm{d}Q + C_{2,\tau}\int_{Q(\sigma)} \mathrm{e}^{2\tau\varphi}y^2\mathrm{d}x\mathrm{d}t - c_T\tau^3\mathrm{e}^{-2\tau\delta}[E_y(0)+E_y(T)], \end{aligned} \tag{3.10}$$

$$C_{1,\tau} = \tau\epsilon\rho, \quad C_{2,\tau} = 2\tau^3\beta + \mathcal{O}(\tau^2). \tag{3.11}$$

*Here $\delta>0$, $\sigma>0$ are the constants in* (3.2) *and* (3.5)*, while $c_T$ and $C_{1,T}$ are positive constants depending on $T$, but not on $\tau$. In addition, the boundary terms $BT|_{\Sigma}(y)$, $\Sigma=\Gamma\times[0,T]$, are given explicitly by, recalling also* (A.1):

$$\begin{aligned} BT|_{\Sigma}(y) ={}& 2\tau\int_0^T\int_{\Gamma_1} \mathrm{e}^{2\tau\varphi}(y_t^2-|\nabla y|^2)h\cdot\nu\mathrm{d}\Gamma\mathrm{d}t + 8c\tau\int_0^T\int_{\Gamma}\mathrm{e}^{2\tau\varphi}\Big(t-\frac{T}{2}\Big)y_t\ \frac{\partial y}{\partial\nu}\mathrm{d}\Gamma\mathrm{d}t \\ &+ 4\tau\int_0^T\int_{\Gamma}\mathrm{e}^{2\tau\varphi}(h\cdot\nabla y)\frac{\partial y}{\partial\nu}\mathrm{d}\Gamma\mathrm{d}t \\ &+\ 4\tau^2\int_0^T\int_{\Gamma}\mathrm{e}^{2\tau\varphi}\Big[|h|^2-4c^2\Big(t-\frac{T}{2}\Big)^2+\frac{\alpha}{2\tau}\Big]y\ \frac{\partial y}{\partial\nu}\mathrm{d}\Gamma\mathrm{d}t \\ &+\ 2\tau\int_0^T\int_{\Gamma_1}\mathrm{e}^{2\tau\varphi}\Big[2\tau^2\Big(|h|^2-4c^2\Big(t-\frac{T}{2}\Big)^2\Big)+\tau(\alpha-\Delta d-2c)\Big]y^2h\cdot\nu\mathrm{d}\Gamma\mathrm{d}t, \end{aligned} \tag{3.12}$$

*where* $h(x) = \nabla d(x)$, $\alpha(x) = \Delta d(x) - 2c - 1 + k$ *for* $0 < k < 1$ *a constant. Moreover, the energy function* $E_y(t)$ *is defined as*

$$E_y(t) = \int_\Omega [y^2(x,t) + y_t^2(x,t) + |\nabla y(x,t)|^2] \mathrm{d}\Omega. \tag{3.13}$$

For what follows, it is relevant to recall also the following extension of the Carleman estimate (3.10) to finite energy solutions. To this end, we introduce the following class of solutions to (3.8) with $F \in L^2(Q)$:

$$\begin{cases} y \in H^{1,1}(Q) = L^2(0,T;H^1(\Omega)) \cap H^1(0,T;L^2(\Omega)), & \text{(3.14a)} \\ y_t \in L^2(0,T;L^2(\Gamma)), \quad \dfrac{\partial y}{\partial \nu} \in L^2(\Sigma) \equiv L^2(0,T;L^2(\Gamma)). & \text{(3.14b)} \end{cases}$$

**Theorem 3.2** (see [16, Theorem 8.2, p. 266]) *Assume* $F \in L^2(Q)$. *Let* $y \in H^{2,2}(Q)$ *be a solution to* (3.8) *for which inequality* (3.10) *holds true, at least as guaranteed by Theorem* 3.1. *Let* $y$ *be a solution to* (3.8) *in the class defined by* (3.14a)–(3.14b). *Then, estimate* (3.10) *is satisfied by such solution* $y$ *as well.*

### 3.3 A continuous observability estimate at the $H^1 \times L^2$-level

An additional result that is needed in the proof of Lipschitz stability in Section 6 is the following continuous observability estimate. It is a consequence of the Carleman estimate above. Consider the following initial/boundary value problem:

$$\begin{cases} y_{tt} = \Delta y + q(x)\psi_t, \quad \psi_{tt} = \Delta\psi + p(x)y_t, & \text{in } Q, & \text{(3.15a)} \\ y\Big(\cdot, \dfrac{T}{2}\Big) = y_0(x),\ y_t\Big(\cdot, \dfrac{T}{2}\Big) = y_1(x), & \text{in } \Omega, & \text{(3.15b)} \\ \psi\Big(\cdot, \dfrac{T}{2}\Big) = \psi_0(x),\ \psi_t\Big(\cdot, \dfrac{T}{2}\Big) = \psi_1(x), & \text{in } \Omega, & \text{(3.15c)} \\ \dfrac{\partial y}{\partial \nu}\Big|_\Sigma = 0, \quad \dfrac{\partial \psi}{\partial \nu}\Big|_\Sigma = 0, & \text{in } \Sigma & \text{(3.15d)} \end{cases}$$

with initial conditions and damping coefficients

$$\{y_0, y_1\}, \{\psi_0, \psi_1\} \in H^1(\Omega) \times L^2(\Omega) \quad \text{and} \quad q, p \in L^\infty(\Omega). \tag{3.16}$$

Then, its solution satisfies

$$\{y, y_t, \psi, \psi_t\} \in C([0,T]; H^1(\Omega) \times L^2(\Omega) \times H^1(\Omega) \times L^2(\Omega)), \tag{3.17}$$

a-fortiori $\{y, \psi\} \in H^{1,1}(Q) \times H^{1,1}(Q)$, continuously.

**Theorem 3.3** *Assume hypothesis* (A.1)–(A.2). *For problem* (3.15) *with data as assumed in* (3.16), *the following continuous observability inequality holds true:*

$$\begin{aligned} &\|y_0\|^2_{H^1(\Omega)} + \|y_1\|^2_{L^2(\Omega)} + \|\psi_0\|^2_{H^1(\Omega)} + \|\psi_1\|^2_{L^2(\Omega)} \\ &\le C_T \int_0^T \int_{\Gamma_1} [y^2 + y_t^2 + \psi^2 + \psi_t^2] \mathrm{d}\Gamma_1 \, \mathrm{d}t, \end{aligned} \tag{3.18}$$

*whenever the right-hand side is finite. Here, $T > T_0$, with $T_0$ defined by* (2.1)*; $\Gamma_1$, is the controlled or observed portion of the boundary, with $\Gamma_0 = \Gamma\backslash\Gamma_1$ satisfying* (1.1)*, and $C_T > 0$ is a positive constant depending on $T$ and on the $L^\infty(\Omega)$-norm of the data $q$ and $p$.*

**Proof** The continuous observability inequality for a single wave equation was proved in [16] by using the Carleman estimate above. For the case of coupled system (3.15), a similar estimate of (3.18) was established in [14, Theorem 1.3, Case 2] by using a different Carleman estimate (with lower order terms). However, we can still invoke the methodology provided in [14, Section 3] for proving the estimate (3.18) by using the Carleman estimate (without lower order terms) Theorem 3.1 above, and hence we omit the proof here for the sake of simplicity. In addition, as noted in both [14, p. 217] and [16, Theorem 9.2, p. 269], Theorem 3.3 is also critically based on [13, Section 7.2] for a sharp trace theory result that expresses the tangential derivative in terms of the normal derivative and the boundary velocity, modulo interior lower order terms. Its proof is by microlocal analysis. A counterpart with an energy level term (rather than a lower order term), is given in [16, Lemma 8.1, p. 265].

## 4 Uniqueness of Linear Inverse Problem for the $\{u, v\}$-System (1.4): Proof of Theorem 2.1

**Step 1** We obtain the following proposition.

**Proposition 4.1** *Assume* (A.1)–(A.2), (2.1)*, $q, p \in L^\infty(\Omega)$, $R_i \in L^\infty(Q)$, $i = 1, 2$, $f, g \in L^2(\Omega)$. Then, the following one-parameter family of energy estimates holds true for the $\{u, v\}$-system* (1.4) *satisfying also the Dirichlet boundary conditions* (2.4)*, for all $\tau > 0$ sufficiently large:*

$$
\begin{aligned}
&C_{1,\tau}\int_Q \mathrm{e}^{2\tau\varphi}[|\nabla u|^2 + u_t^2 + |\nabla v|^2 + v_t^2]\mathrm{d}Q + C_{2,\tau}\int_{Q(\sigma)} \mathrm{e}^{2\tau\varphi}[u^2+v^2]\mathrm{d}x\mathrm{d}t \\
&\le C_{p,q}\int_Q \mathrm{e}^{2\tau\varphi}[u_t^2+v_t^2]\mathrm{d}Q + C_{1,T}\mathrm{e}^{2\tau\sigma}\int_Q [u^2+v^2]\mathrm{d}Q + 4\int_Q \mathrm{e}^{2\tau\varphi}[|fR_1|^2 + |gR_2|^2]\mathrm{d}Q \\
&\quad + c_T\tau^3\mathrm{e}^{-2\tau\delta}\{[E_u(0)+E_u(T)+E_v(0)+E_v(T)]\}. \qquad (4.1)
\end{aligned}
$$

**Proof** Under present assumptions on $p, q, f, g, R_i$, system (2.2), rewritten here as

$$
\begin{bmatrix} u_{tt} \\ v_{tt} \end{bmatrix} = \begin{bmatrix} \Delta & 0 \\ 0 & \Delta \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} 0 & q(x) \\ p(x) & 0 \end{bmatrix}\begin{bmatrix} u_t \\ v_t \end{bmatrix} + \begin{bmatrix} f(x)R_1(x,t) \\ g(x)R_2(x,t) \end{bmatrix} \tag{4.2}
$$

with zero initial conditions as in (1.7c) and homogeneous boundary conditions (1.7d) possesses a-fortiori the regularity $\{u, v\} \in H^1(Q) \times H^1(Q)$. Moreover, also because of (2.4) and $h \cdot \nu = 0$ on $\Gamma_0$ in (1.1), we have that, in view of Theorem 3.2, we can apply the Carleman estimate (3.10) of Theorem 3.1 to the $u$-equation (1.4a) and the $v$-equation (1.4b) separately, where — to fit model (3.8) — we have

$$
F^u(x,t) = q(x)v_t(x,t) + f(x)R_1(x,t)
$$

and

$$
F^v(x,t) = p(x)u_t(x,t) + g(x)R_2(x,t)
$$

respectively. We then obtain

$$\begin{aligned}&BT|_\Sigma(u)+2\int_Q \mathrm{e}^{2\tau\varphi}|qv_t+fR_1|^2\mathrm{d}Q+C_{1,T}\mathrm{e}^{2\tau\sigma}\int_Q u^2\mathrm{d}Q\\ &\ge C_{1,\tau}\int_Q \mathrm{e}^{2\tau\varphi}[u_t^2+|\nabla u|^2]\mathrm{d}Q+C_{2,\tau}\int_{Q(\sigma)}\mathrm{e}^{2\tau\varphi}u^2dx\mathrm{d}t-c_T\tau^3\mathrm{e}^{-2\tau\delta}[E_u(0)+E_u(T)],\end{aligned} \tag{4.3}$$

$$\begin{aligned}&BT|_\Sigma(v)+2\int_Q \mathrm{e}^{2\tau\varphi}|pu_t+gR_2|^2\mathrm{d}Q+C_{1,T}\mathrm{e}^{2\tau\sigma}\int_Q v^2\mathrm{d}Q\\ &\ge C_{1,\tau}\int_Q \mathrm{e}^{2\tau\varphi}[v_t^2+|\nabla v|^2]\mathrm{d}Q+C_{2,\tau}\int_{Q(\sigma)}\mathrm{e}^{2\tau\varphi}v^2\mathrm{d}x\mathrm{d}t-c_T\tau^3\mathrm{e}^{-2\tau\delta}[E_v(0)+E_v(T)]\end{aligned} \tag{4.4}$$

respectively, with boundary terms defined by (3.12), which, in fact, now vanish by (1.7d), (2.4) and $h\cdot\nu=0$ on $\Gamma_0$. We obtain, recalling $q,p\in L^\infty(\Omega)$,

$$BT|_\Sigma(u)\equiv 0,\quad \int_Q \mathrm{e}^{2\tau\varphi}|qv_t+fR_1|^2\mathrm{d}Q\le C_q\int_Q \mathrm{e}^{2\tau\varphi}|v_t|^2\mathrm{d}Q+2\int_Q \mathrm{e}^{2\tau\varphi}|fR_1|^2\mathrm{d}Q, \tag{4.5}$$

$$BT|_\Sigma(v)\equiv 0,\quad \int_Q \mathrm{e}^{2\tau\varphi}|pu_t+gR_2|^2\mathrm{d}Q\le C_p\int_Q \mathrm{e}^{2\tau\varphi}|u_t|^2\mathrm{d}Q+2\int_Q \mathrm{e}^{2\tau\varphi}|gR_2|^2\mathrm{d}Q. \tag{4.6}$$

Adding (4.3) and (4.4), and taking into account (4.5)–(4.6) yields (4.1).

**Step 2** We differentiate system (1.7) in $t$, supplemented by the over-determined boundary conditions (2.4) and obtain, invoking also the initial conditions (1.7c):

$$\begin{cases}(u_t)_{tt}(x,t)-\Delta(u_t)(x,t)-q(x)(v_t)_t(x,t)=f(x)R_{1t}(x,t), & \text{in } Q, \quad (4.7\text{a})\\ (v_t)_{tt}(x,t)-\Delta(v_t)(x,t)-p(x)(u_t)_t(x,t)=g(x)R_{2t}(x,t), & \text{in } Q, \quad (4.7\text{b})\\ (u_t)\Big(\cdot,\dfrac{T}{2}\Big)=0,\quad (u_t)_t\Big(\cdot,\dfrac{T}{2}\Big)=f(x)R_1\Big(x,\dfrac{T}{2}\Big)\in L^2(\Omega), & \text{in } \Omega, \quad (4.7\text{c})\\ (v_t)\Big(\cdot,\dfrac{T}{2}\Big)=0,\quad (v_t)_t\Big(\cdot,\dfrac{T}{2}\Big)=g(x)R_2\Big(x,\dfrac{T}{2}\Big)\in L^2(\Omega), & \text{in } \Omega, \quad (4.7\text{d})\\ \dfrac{\partial}{\partial\nu}(u_t)(x,t)=0,\quad \dfrac{\partial}{\partial\nu}(v_t)(x,t)=0,\quad u_t=0,\quad v_t=0, & \text{in } \Sigma,\Sigma_1. \quad (4.7\text{e})\end{cases}$$

**Proposition 4.2** *Assume the hypotheses of Proposition* 4.1 *with $R_i\in L^\infty(Q)$ replaced now by $R_{it}\in L^\infty(Q)$, $i=1,2$. Then, the following one-parameter family of energy estimates holds true for the $\{u_t,v_t\}$-system* (4.7)*, for all $\tau>0$ sufficiently large:*

$$\begin{aligned}&C_{1,\tau}\int_Q \mathrm{e}^{2\tau\varphi}[|\nabla u_t|^2+u_{tt}^2+|\nabla v_t|^2+v_{tt}^2]\mathrm{d}Q+C_{2,\tau}\int_{Q(\sigma)}\mathrm{e}^{2\tau\varphi}[u_t^2+v_t^2]\mathrm{d}x\mathrm{d}t\\ &\le C_{p,q}\int_Q \mathrm{e}^{2\tau\varphi}[u_{tt}^2+v_{tt}^2]\mathrm{d}Q+C_{1,T}\mathrm{e}^{2\tau\sigma}\int_Q[u_t^2+v_t^2]\mathrm{d}Q+4\int_Q \mathrm{e}^{2\tau\varphi}[|fR_{1t}|^2+|gR_{2t}|^2]\mathrm{d}Q\\ &\quad +c_T\tau^3\mathrm{e}^{-2\tau\delta}\{[E_{u_t}(0)+E_{u_t}(T)+E_{v_t}(0)+E_{v_t}(T)]\}.\end{aligned} \tag{4.8}$$

**Proof** We now have $\{u_t,v_t\}\in H^1(Q)\times H^1(Q)$, since $f(x)R_1(x,t)$, $g(x)R_2(x,t)\in L^2(\Omega)$ under present assumptions. Thus the same proof of Proposition 4.1 applies, based on Theorem 3.2, as $u_t$ and $v_t$ both vanish on $\Gamma_1\times[0,T]$ as in (4.7e).

**Step 3** We differentiate system (4.7) in $t$ one more time and obtain, invoking also the initial conditions (4.7c)–(4.7d):

$$\begin{cases} (u_{tt})_{tt}(x,t) - \Delta(u_{tt})(x,t) - q(x)(v_{tt})_t(x,t) = f(x)R_{1tt}(x,t), & \text{in } Q, \quad (4.9a) \\ (v_{tt})_{tt}(x,t) - \Delta(v_{tt})(x,t) - p(x)(u_{tt})_t(x,t) = g(x)R_{2tt}(x,t), & \text{in } Q, \quad (4.9b) \\ (u_{tt})\Big(\cdot, \dfrac{T}{2}\Big) = f(x)R_1\Big(x, \dfrac{T}{2}\Big), & \text{in } \Omega, \\ (u_{tt})_t\Big(\cdot, \dfrac{T}{2}\Big) = f(x)R_{1t}\Big(x, \dfrac{T}{2}\Big) + q(x)g(x)R_2\Big(x, \dfrac{T}{2}\Big), & \text{in } \Omega, \quad (4.9c) \\ (v_{tt})\Big(\cdot, \dfrac{T}{2}\Big) = g(x)R_2\Big(x, \dfrac{T}{2}\Big), & \text{in } \Omega, \\ (v_{tt})_t\Big(\cdot, \dfrac{T}{2}\Big) = g(x)R_{2t}\Big(x, \dfrac{T}{2}\Big) + p(x)f(x)R_1\Big(x, \dfrac{T}{2}\Big), & \text{in } \Omega, \quad (4.9d) \\ \dfrac{\partial}{\partial\nu}(u_{tt})(x,t) = 0,\ \dfrac{\partial}{\partial\nu}(v_{tt})(x,t) = 0, \quad u_{tt} = 0,\ v_{tt} = 0, & \text{in } \Sigma, \Sigma_1. \quad (4.9e) \end{cases}$$

We note that, under present assumptions $f, g \in L^2(\Omega)$ and $p, q, R_1(\cdot, \frac{T}{2}), R_2(\cdot, \frac{T}{2}) \in L^\infty(\Omega)$, we have

$$(u_{tt})_t\Big(\cdot, \frac{T}{2}\Big) \in L^2(\Omega), \quad (v_{tt})_t\Big(\cdot, \frac{T}{2}\Big) \in L^2(\Omega) \tag{4.10}$$

as desired; however, $(u_{tt})(\cdot, \frac{T}{2})$ and $(v_{tt})(\cdot, \frac{T}{2})$ are only in $L^2(\Omega)$ (see (4.7c)–(4.7d)), and not in $H^1(\Omega)$, as needed to invoke Theorem 3.1.

**Orientation** Henceforth, we shall proceed with the proof under the following provisional restrictions on the data:

$$f(x)R_1\Big(x, \frac{T}{2}\Big) \in H^1(\Omega), \text{ so that } (u_{tt})\Big(\cdot, \frac{T}{2}\Big) \in H^1(\Omega), \tag{4.11}$$

$$g(x)R_2\Big(x, \frac{T}{2}\Big) \in H^1(\Omega), \text{ so that } (v_{tt})\Big(\cdot, \frac{T}{2}\Big) \in H^1(\Omega). \tag{4.12}$$

The regularity properties in (4.11) and (4.12) hold true provided that, respectively

$$f(x) \in H^1(\Omega), \text{ and } R_1\Big(x, \frac{T}{2}\Big) \text{ is a multiplier } H^1(\Omega) \to H^1(\Omega), \tag{4.13}$$

$$g(x) \in H^1(\Omega), \text{ and } R_2\Big(x, \frac{T}{2}\Big) \text{ is a multiplier } H^1(\Omega) \to H^1(\Omega) \tag{4.14}$$

for which a characterization is given in [22, Theorem 1 with $m = l = 1$, $p = 2$, p. 243]. More direct sufficient conditions for (4.13), respectively (4.14), to hold are

$$f(x) \in H^1(\Omega), \quad g(x) \in H^1(\Omega), \quad R_{ix_j}\Big(x, \frac{T}{2}\Big) \in L^\infty(\Omega), \quad i = 1, 2,\ j = 1, \cdots, n. \tag{4.15}$$

We shall first prove the uniqueness property (2.5): $f(x) = g(x) \equiv 0$ of the present Theorem 2.1 under the provisional restrictions (4.11)–(4.12) and in particular (4.15). Then, we shall extend the result to all $f(x), g(x) \in L^2(\Omega)$, with $R_{ix_j}$ as in (4.15) as assumed in (2.2), by using (i) the continuity of the map $\{f, g\} \to \{u(f, g)|_{\Sigma_1}, v(f, g)|_{\Sigma_1}\}$ and (ii) the denseness of $H^1(\Omega)$ in $L^2(\Omega)$.

Thus, under the provisional restrictions (4.11)–(4.12) and by virtue also of (4.10), we have the following regularity for problem (4.9):

$$\{u_{tt}, v_{tt}\} \in H^1(Q) \times H^1(Q) \tag{4.16}$$

so that we can apply Theorem 3.1 to problem (4.9). We obtain the result below.

**Proposition 4.3** *Assume* (A.1)–(A.2), $q, p \in L^\infty(\Omega)$, *as well as* (4.15) *and* $R_{1tt}, R_{2tt} \in L^\infty(Q)$. *Then, the following one-parameter family of energy estimates holds true for the* $\{u_{tt}, v_{tt}\}$*-system* (4.9), *for all* $\tau > 0$ *sufficiently large:*

$$\begin{aligned}
&C_{1,\tau}\int_Q e^{2\tau\varphi}[|\nabla u_{tt}|^2 + u_{ttt}^2 + |\nabla v_{tt}|^2 + v_{ttt}^2]dQ + C_{2,\tau}\int_{Q(\sigma)} e^{2\tau\varphi}[u_{tt}^2 + v_{tt}^2]dxdt\\
&\le C_{p,q}\int_Q e^{2\tau\varphi}[u_{ttt}^2 + v_{ttt}^2]dQ + C_{1,T}e^{2\tau\sigma}\int_Q [u_{tt}^2 + v_{tt}^2]dQ + 4\int_Q e^{2\tau\varphi}[|fR_{1_{tt}}|^2 + |gR_{2_{tt}}|^2]dQ\\
&\quad + c_T\tau^3 e^{-2\tau\delta}\{[E_{u_{tt}}(0) + E_{u_{tt}}(T) + E_{v_{tt}}(0) + E_{v_{tt}}(T)]\}.
\end{aligned}\tag{4.17}$$

**Step 4** Under the assumptions of Proposition 4.1 through 4.3 cumulatively, that is, (2.2) and $f, g \in H^1(\Omega)$, we sum up (4.1), (4.8) and (4.17) to obtain the next proposition.

**Proposition 4.4** *Assume* (A.1)–(A.2), (2.1)–(2.2) *and* $f, g \in H^1(\Omega)$, *as in* (4.13)–(4.14). *Then the following one-parameter family of energy estimates holds true for the* $\{u, v\}$*-system* (1.4), *for all* $\tau > 0$ *sufficiently large:*

$$\begin{aligned}
&C_{1,\tau}\int_Q e^{2\tau\varphi}[|\nabla u_{tt}|^2 + |\nabla u_t|^2 + |\nabla u|^2 + u_{ttt}^2 + u_{tt}^2 + u_t^2 + |\nabla v_{tt}|^2 + |\nabla v_t|^2\\
&\quad + |\nabla v|^2 + v_{ttt}^2 + v_{tt}^2 + v_t^2]dQ + C_{2,\tau}\int_{Q(\sigma)} e^{2\tau\varphi}\{[u_{tt}^2 + u_t^2 + u^2] + [v_{tt}^2 + v_t^2 + v^2]\}dxdt\\
&\le C_{p,q}\int_Q e^{2\tau\varphi}\{[u_{ttt}^2 + u_{tt}^2 + u_t^2] + [v_{ttt}^2 + v_{tt}^2 + v_t^2]\}dQ + C_{1,T}e^{2\tau\sigma}\int_Q\{[u_{tt}^2 + u_t^2 + u^2]\\
&\quad + [v_{tt}^2 + v_t^2 + v^2]\}dQ + 4\int_Q e^{2\tau\varphi}\{[|fR_{1tt}|^2 + |fR_{1t}|^2 + |fR_1|^2]\\
&\quad + [|gR_{2tt}|^2 + |gR_{2t}|^2 + |gR_2|^2]\}dQ + c_T\tau^3 e^{-2\tau\delta}[\mathcal{E}_{u,v}]_0^T,
\end{aligned}\tag{4.18}$$

$$\begin{aligned}
[\mathcal{E}_{u,v}]_0^T = &\{[E_{u_{tt}}(0) + E_{u_{tt}}(T)] + [E_{v_{tt}}(0) + E_{v_{tt}}(T)] + [E_{u_t}(0) + E_{u_t}(T)]\\
&+ [E_{v_t}(0) + E_{v_t}(T)] + [E_u(0) + E_u(T)] + [E_v(0) + E_v(T)]\}.
\end{aligned}\tag{4.19}$$

**Step 5** In this step, we follow an idea of [8, Theorem 8.2.2, p. 231] (see points (2)–(3) below).

**Proposition 4.5** *With reference to the third integral term on the RHS of estimate* (4.18), *assume* (2.2) *as well as* (2.3). *Then we have*

$$\begin{aligned}
(1)\quad &\int_Q e^{2\tau\varphi}[|fR_1|^2 + |gR_2|^2 + |fR_{1t}|^2 + |gR_{2t}|^2 + |fR_{1tt}|^2 + |gR_{2tt}|^2]dQ\\
&\le C_{\boldsymbol{R}}\int_Q e^{2\tau\varphi}[|f|^2 + |g|^2]dQ;
\end{aligned}\tag{4.20}$$

$$(2)\quad \int_Q e^{2\tau\varphi}|f|^2 dQ \le \Big\{\Big(\frac{T}{r_1^2}\Big)(2cT\tau + 1)\Big\}\int_\Omega\int_0^{\frac{T}{2}} e^{2\tau\varphi(x,s)}|u_{tt}(x,s)|^2 ds d\Omega$$

$$+\frac{T}{r_1^2}\int_\Omega\int_0^{\frac{T}{2}} \mathrm{e}^{2\tau\varphi(x,s)}|u_{ttt}(x,s)|^2\mathrm{d}s\mathrm{d}\Omega+\frac{T}{r_1^2}\int_\Omega|u_{tt}(x,0)|^2\mathrm{d}\Omega; \quad (4.21)$$

$$(3)\quad \int_Q \mathrm{e}^{2\tau\varphi}|g|^2\mathrm{d}Q\le\Big\{\Big(\frac{T}{r_2^2}\Big)(2cT\tau+1)\Big\}\int_\Omega\int_0^{\frac{T}{2}}\mathrm{e}^{2\tau\varphi(x,s)}|v_{tt}(x,s)|^2\mathrm{d}s\mathrm{d}\Omega$$

$$+\frac{T}{r_2^2}\int_\Omega\int_0^{\frac{T}{2}} \mathrm{e}^{2\tau\varphi(x,s)}|v_{ttt}(x,s)|^2\mathrm{d}s\mathrm{d}\Omega+\frac{T}{r_2^2}\int_\Omega|v_{tt}(x,0)|^2\mathrm{d}\Omega. \quad (4.22)$$

**Proof** (1) is obvious, recalling assumption (2.2) on $R_i, R_{it}, R_{itt}\in L^\infty(Q)$, $i=1,2$. For (2), we return to (1.7a)–(1.7b), evaluate at the initial time $\frac{T}{2}$, use (1.7c) and obtain

$$u_{tt}\Big(x,\frac{T}{2}\Big)=f(x)R_1\Big(x,\frac{T}{2}\Big),\quad v_{tt}\Big(x,\frac{T}{2}\Big)=g(x)R_2\Big(x,\frac{T}{2}\Big). \quad (4.23)$$

Recalling assumption (2.3), we have

$$|f(x)|\le\frac{1}{r_1}\Big|u_{tt}\Big(x,\frac{T}{2}\Big)\Big|,\quad |g(x)|\le\frac{1}{r_2}\Big|v_{tt}\Big(x,\frac{T}{2}\Big)\Big|,\quad x\in\Omega. \quad (4.24)$$

By virtue of (4.24), we compute, recalling also property (3.4b) and $\frac{\mathrm{d}}{\mathrm{d}s}\varphi(x,s)=2c\big(\frac{T}{2}-s\big)$:

$$\begin{aligned}
\int_Q \mathrm{e}^{2\tau\varphi}|f|^2\mathrm{d}Q&=\int_0^T\int_\Omega \mathrm{e}^{2\tau\varphi(x,t)}|f(x)|^2\mathrm{d}\Omega\mathrm{d}t\le\frac{1}{r_1^2}\int_0^T\int_\Omega\mathrm{e}^{2\tau\varphi(x,t)}\Big|u_{tt}\Big(x,\frac{T}{2}\Big)\Big|^2\mathrm{d}\Omega\mathrm{d}t\\
&\le\frac{T}{r_1^2}\int_\Omega \mathrm{e}^{2\tau\varphi\left(x,\frac{T}{2}\right)}\Big|u_{tt}\Big(x,\frac{T}{2}\Big)\Big|^2\mathrm{d}\Omega\quad\text{(by (3.4b))}\\
&=\frac{T}{r_1^2}\Big(\int_\Omega\int_0^{\frac{T}{2}}\frac{d}{ds}(\mathrm{e}^{2\tau\varphi(x,s)}|u_{tt}(x,s)|^2)\mathrm{d}s\mathrm{d}\Omega+\int_\Omega\mathrm{e}^{2\tau\varphi(x,0)}|u_{tt}(x,0)|^2\mathrm{d}\Omega\Big)\\
&\le\frac{T}{r_1^2}\Big(4c\tau\int_\Omega\int_0^{\frac{T}{2}}\Big(\frac{T}{2}-s\Big)\mathrm{e}^{2\tau\varphi(x,s)}|u_{tt}(x,s)|^2\mathrm{d}s\mathrm{d}\Omega\\
&\quad+2\int_\Omega\int_0^{\frac{T}{2}}\mathrm{e}^{2\tau\varphi}|u_{tt}(x,s)||u_{ttt}(x,s)|\mathrm{d}s\mathrm{d}\Omega+\int_\Omega\mathrm{e}^{2\tau\varphi(x,0)}|u_{tt}(x,0)|^2\mathrm{d}\Omega\Big)\\
&\le\frac{T}{r_1^2}\Big((2cT\tau)\int_\Omega\int_0^{\frac{T}{2}}\mathrm{e}^{2\tau\varphi}|u_{tt}|^2\mathrm{d}t\mathrm{d}\Omega+\int_\Omega\int_0^{\frac{T}{2}}\mathrm{e}^{2\tau\varphi}(|u_{tt}|^2+|u_{ttt}|)^2\mathrm{d}t\mathrm{d}\Omega\\
&\quad+\int_\Omega|u_{tt}(x,0)|^2\mathrm{d}\Omega\Big),
\end{aligned} \quad (4.25)$$

by using in the last step $\varphi(x,0)\le-\delta$ by (3.4a), so that $\mathrm{e}^{2\tau\varphi(x,0)}\le1$. Then (4.21) follows from (4.25). The proof of (3) is similar by using (4.24) on $g$.

**Step 6** By substituting estimates (4.20)–(4.22) on the RHS of (4.18), we obtain the following result.

**Proposition 4.6** *Assume the hypotheses of Theorem* 2.1 *and moreover* $f,g\in H^1(\Omega)$, *as in* (4.13)–(4.14). *Then, the following one-parameter family of energy estimates holds true for the* $\{u,v\}$*-system* (1.4), *for all* $\tau>0$ *sufficiently large:*

$$C_{1,\tau}\int_Q\mathrm{e}^{2\tau\varphi}\{[|\nabla u_{tt}|^2+|\nabla u_t|^2+|\nabla u|^2]+[|\nabla v_{tt}|^2+|\nabla v_t|^2+|\nabla v|^2]\}\mathrm{d}Q$$

$$
\begin{aligned}
&+\Big[C_{1,\tau}-\frac{C_{\mathbf{R}}T}{r_0^2}-C_{p,q}\Big]\int_Q \mathrm{e}^{2\tau\varphi}[u_{ttt}^2+v_{ttt}^2]\mathrm{d}Q\\
&+[C_{1,\tau}-C_{p,q}]\int_Q \mathrm{e}^{2\tau\varphi}[u_{tt}^2+u_t^2+v_{tt}^2+v_t^2]\mathrm{d}Q\\
&+C_{2,\tau}\int_{Q(\sigma)}\mathrm{e}^{2\tau\varphi}\{[u_{tt}^2+u_t^2+u^2]+[v_{tt}^2+v_t^2+v^2]\}\mathrm{d}Q\\
\le{}& \widetilde{C}_{\mathbf{R}}T(2cT\tau+1)\int_Q \mathrm{e}^{2\tau\varphi}[u_{tt}^2+v_{tt}^2]\mathrm{d}Q+C_{1,T}\mathrm{e}^{2\tau\sigma}k_{u,v}+a_{u_{tt},v_{tt}}+c_T\tau^3\mathrm{e}^{-2\tau\delta}[\mathcal{E}_{u,v}]_0^T, \quad (4.26)
\end{aligned}
$$

$$
k_{u,v}=k_{u,v;u_t,v_t;u_{tt},v_{tt}}\equiv\int_Q[(u_{tt}^2+u_t^2+u^2)+(v_{tt}^2+v_t^2+v^2)]\mathrm{d}Q, \tag{4.27}
$$

$$
\widetilde{C}_{\mathbf{R}}=C_{\mathbf{R}}\Big(\frac{1}{r_1^2}+\frac{1}{r_2^2}\Big),\quad a_{u_{tt},v_{tt}}=\frac{T}{r_1^2}\int_\Omega|u_{tt}(x,0)|^2\mathrm{d}\Omega+\frac{T}{r_2^2}\int_\Omega|v_{tt}(x,0)|^2\mathrm{d}\Omega, \tag{4.28}
$$

with constants depending on the solution $\{u,v\}$, $r_0=\min\{r_1,r_2\}$.

**Step 7** Recalling now $\mathrm{e}^{2\tau\varphi}<\mathrm{e}^{2\tau\sigma}$ on $Q\setminus Q(\sigma)$ by (3.6), we obtain the following estimate for the integral terms on the RHS of inequality (4.26):

$$
\begin{aligned}
\int_Q \mathrm{e}^{2\tau\varphi}[u_{tt}^2+v_{tt}^2]\mathrm{d}Q&=\int_{Q(\sigma)}\mathrm{e}^{2\tau\varphi}[u_{tt}^2+v_{tt}^2]\mathrm{d}t\mathrm{d}x+\int_{Q\setminus Q(\sigma)}\mathrm{e}^{2\tau\varphi}[u_{tt}^2+v_{tt}^2]\mathrm{d}x\mathrm{d}t\\
&\le\int_{Q(\sigma)}\mathrm{e}^{2\tau\varphi}[u_{tt}^2+v_{tt}^2]\mathrm{d}t\mathrm{d}x+\mathrm{e}^{2\tau\sigma}\int_{Q\setminus Q(\sigma)}[u_{tt}^2+v_{tt}^2]\mathrm{d}x\mathrm{d}t. \qquad (4.29)
\end{aligned}
$$

Substituting inequality (4.29) in the integral term on the RHS of estimate (4.26) and recalling (4.27), we thus obtain the final sought-after estimate.

**Theorem 4.7** *Assume the hypotheses of Theorem* 2.1 *and moreover* $f,g\in H^1(\Omega)$ *as in* (4.13)–(4.14)*. Then, the following one-parameter family of energy estimates holds true for the* $\{u,v\}$*-system* (1.4)*, for all* $\tau>0$ *sufficiently large:*

$$
\begin{aligned}
&C_{1,\tau}\int_Q \mathrm{e}^{2\tau\varphi}\{[|\nabla u_{tt}|^2+|\nabla u_t|^2+|\nabla u|^2]+[|\nabla v_{tt}|^2+|\nabla v_t|^2+|\nabla v|^2]\}\mathrm{d}Q\\
&+\Big[C_{1,\tau}-\frac{C_{\mathbf{R}}T}{r_0^2}-C_{p,q}\Big]\int_Q \mathrm{e}^{2\tau\varphi}[u_{ttt}^2+v_{ttt}^2]\mathrm{d}Q\\
&+[C_{1,\tau}-C_{p,q}]\int_Q \mathrm{e}^{2\tau\varphi}[u_{tt}^2+u_t^2+v_{tt}^2+v_t^2]\mathrm{d}Q\\
&+[C_{2,\tau}-\widetilde{C}_{\mathbf{R}}T(2cT\tau+1)-C_{p,q}]\int_{Q(\sigma)}\mathrm{e}^{2\tau\varphi}\{[u_{tt}^2+u_t^2+u^2]+[v_{tt}^2+v_t^2+v^2]\}\mathrm{d}Q\\
\le{}&[C_{1,T}+\widetilde{C}_{\mathbf{R}}T]\mathrm{e}^{2\tau\sigma}(2cT\tau+1)\cdot\int_{Q\setminus Q(\sigma)}[u_{tt}^2+u_t^2+u^2+v_{tt}^2+v_t^2+v^2]\mathrm{d}Q\\
&+c_T\tau^3\mathrm{e}^{-2\tau\delta}[\mathcal{E}_{u,v}]_0^T. \qquad (4.30)
\end{aligned}
$$

**Step 8** The "final" estimate (4.30) is more than we need to conclude the argument. First, as all coefficients of the integral terms on the LHS of estimate (4.30) are positive for $\tau>0$ sufficiently large, we can drop all these terms save the term $\int_{Q(\sigma)}$ and obtain

$$
[C_{2,\tau}-\widetilde{C}_{\mathbf{R}}T(2cT\tau+1)-C_{p,q}]\int_{Q(\sigma)}\mathrm{e}^{2\tau\varphi}\{[u_{tt}^2+u_t^2+u^2]+[v_{tt}^2+v_t^2+v^2]\}\mathrm{d}Q
$$

$$\leq (2cT\tau+1)\mathrm{e}^{2\tau\sigma}\widetilde{k}_{u,v} + c_T\tau^3\mathrm{e}^{-2\tau\delta}[\mathcal{E}_{u,v}]_0^T, \tag{4.31}$$

$$\widetilde{k}_{u,v} = \text{constant depending on solution } \{u,v\} \text{ and data.} \tag{4.32}$$

But on $Q(\sigma)$, we have $\mathrm{e}^{2\tau\varphi} \geq \mathrm{e}^{2\tau\sigma}$ by (3.6). Using this, in the LHS integral of (4.31) and dividing (4.31) across by $(2cT\tau+1)\mathrm{e}^{2\tau\sigma}$, we obtain for all $\tau > 0$ sufficiently large:

$$\frac{1}{2cT\tau+1}[C_{2,\tau} - \widetilde{C}_{\mathbf{R}}T(2cT\tau+1) - C_{p,q}]\int_{Q(\sigma)}\{[u_{tt}^2+u_t^2+u^2]+[v_{tt}^2+v_t^2+v^2]\}\mathrm{d}Q$$
$$\leq \widetilde{k}_{u,v} + \frac{c_T\tau^3\mathrm{e}^{-2\tau\delta}}{(2cT\tau+1)\mathrm{e}^{2\tau\sigma}}[\mathcal{E}_{u,v}]_0^T \leq \mathrm{Const}_{u,v,\mathrm{data}}. \tag{4.33}$$

Letting $\tau \to +\infty$ in (4.33), and recalling that $C_{2,\tau}$ grows as $\tau^3$, we obtain

$$u(x,t) \equiv 0, \quad v(x,t) \equiv 0, \quad (x,t) \in Q(\sigma). \tag{4.34}$$

Then (4.34) implies $u_{tt} \equiv 0$, $\Delta u \equiv 0$, $v_{tt} \equiv 0$, $\Delta v \equiv 0$ all in $Q(\sigma)$. Thus, returning to (1.7a)–(1.7b), we then obtain

$$f(x)R_1(x,t) \equiv 0, \quad g(x)R_2(x,t) \equiv 0, \quad \text{in } \; Q(\sigma). \tag{4.35}$$

Recalling now from (3.7) that $[t_0,t_1]\times\Omega \subset Q(\sigma)$ and from (3.4b) that $t_0 < \frac{T}{2} < t_1$, we see that (4.35) in particular implies

$$f(x)R_1\Big(x,\frac{T}{2}\Big) \equiv 0, \quad g(x)R_2\Big(x,\frac{T}{2}\Big) \equiv 0, \quad x \in \Omega. \tag{4.36}$$

Thus by use of assumption (2.3), (4.36) implies

$$f(x) \equiv 0, \quad g(x) \equiv 0, \quad \text{a.e. } x \in \Omega. \tag{4.37}$$

**Step 9** Problem (1.7) = (4.2), with $R_1(x,t), R_2(x,t) \in L^\infty(Q)$ as assumed, yields the standard results:

$$\text{map } \{f,g\} \to \{u,v\}:$$
$$\text{continuous } L^2(\Omega)\times L^2(\Omega) \to C([0,T]; H^1(\Omega)\times H^1(\Omega)), \tag{4.38}$$
$$\text{map } \mathcal{T}: \{f,g\} \to \mathcal{T}\{f,g\} = \{u|_\Sigma, v|_\Sigma\}:$$
$$\text{continuous } L^2(\Omega)\times L^2(\Omega) \to C([0,T]; H^{\frac12}(\Gamma)\times H^{\frac12}(\Gamma)). \tag{4.39}$$

The interior regularity (4.38) follows at once by using the variation of the parameter formula in (4.2) with the solution "sine" operator; trace theory yields then (4.39). Finally, since $H^1(\Omega)$ is dense in $L^2(\Omega)$, the conclusion of (4.37) that $\mathcal{T}\{f,g\} = 0$ for $f,g \in H^1(\Omega)$ can be extended to $\mathcal{T}\{f,g\} = 0$ for all $f,g \in L^2(\Omega)$ (and $R_{ix_j}(x,\frac{T}{2}) \in L^\infty(\Omega)$, $i=1,2$, $j = 1\cdots,n$). The proof of Theorem 2.1 is completed.

## 5 Stability of Linear Inverse Problem for the $\{u,v\}$-System (1.7): Proof of Theorem 2.2

**Step 1** Let $\{u = u(f,g), v = v(f,g)\}$ be the solution to problem (1.7a)–(1.7d), with data

$$\begin{cases} q,p \in L^\infty(\Omega),\ f,g \in L^2(\Omega),\ R_i, R_{it}, R_{itt} \in L^\infty(Q), & (5.1\text{a}) \\ \Big|R_i\Big(x,\frac{T}{2}\Big)\Big| \geq r_i > 0,\ x \in \overline{\Omega},\ R_{ix_j}\Big(x,\frac{T}{2}\Big) \in L^\infty(\Omega), \quad i=1,2,\ j=1,\cdots,n & (5.1\text{b}) \end{cases}$$

from assumptions (2.2) and (2.3). Consider again the $\{u_t, v_t\}$-system (4.7), which we rewrite here for convenience

$$\begin{cases} (u_t)_{tt}(x,t) - \Delta(u_t)(x,t) - q(x)(v_t)_t(x,t) = f(x)R_{1t}(x,t), & \text{in } Q, & (5.2a)\\ (v_t)_{tt}(x,t) - \Delta(v_t)(x,t) - p(x)(u_t)_t(x,t) = g(x)R_{2t}(x,t), & \text{in } Q, & (5.2b)\\ (u_t)\Big(\,\cdot\,,\dfrac{T}{2}\Big) = 0, \quad (u_t)_t\Big(\,\cdot\,,\dfrac{T}{2}\Big) = f(x)R_1\Big(x,\dfrac{T}{2}\Big) \in L^2(\Omega), & \text{in } \Omega, & (5.2c)\\ (v_t)\Big(\,\cdot\,,\dfrac{T}{2}\Big) = 0, \quad (v_t)_t\Big(\,\cdot\,,\dfrac{T}{2}\Big) = g(x)R_2\Big(x,\dfrac{T}{2}\Big) \in L^2(\Omega), & \text{in } \Omega, & (5.2d)\\ \dfrac{\partial}{\partial\nu}(u_t)(x,t) = 0, \quad \dfrac{\partial}{\partial\nu}(v_t)(x,t) = 0, & \text{in } \Sigma, & (5.2e)\end{cases}$$

so that $f(x)R_1(x,\frac{T}{2}),\ g(x)R_2\left(x,\frac{T}{2}\right) \in L^2(\Omega)$. Accordingly, by linearity, we split the problem $\{u_t, v_t\}$ into two components

$$u_t = \overline{u}_t + \widetilde{u}_t, \quad v_t = \overline{v}_t + \widetilde{v}_t, \tag{5.3}$$

where $\{\overline{u}_t, \overline{v}_t\}$ satisfies problem (4.7), however, with homogeneous forcing terms

$$\begin{cases} (\overline{u}_t)_{tt}(x,t) - \Delta(\overline{u}_t)(x,t) - q(x)(\overline{v}_t)_t(x,t) = 0, & \text{in } Q, & (5.4a)\\ (\overline{v}_t)_{tt}(x,t) - \Delta(\overline{v}_t)(x,t) - p(x)(\overline{u}_t)_t(x,t) = 0, & \text{in } Q, & (5.4b)\\ (\overline{u}_t)\Big(\,\cdot\,,\dfrac{T}{2}\Big) = 0, \quad (\overline{u}_t)_t\Big(\,\cdot\,,\dfrac{T}{2}\Big) = f(x)R_1\Big(x,\dfrac{T}{2}\Big), & \text{in } \Omega, & (5.4c)\\ (\overline{v}_t)\Big(\,\cdot\,,\dfrac{T}{2}\Big) = 0, \quad (\overline{v}_t)_t\Big(\,\cdot\,,\dfrac{T}{2}\Big) = g(x)R_2\Big(x,\dfrac{T}{2}\Big), & \text{in } \Omega, & (5.4d)\\ \dfrac{\partial}{\partial\nu}(\overline{u}_t)(x,t) = 0, \quad \dfrac{\partial}{\partial\nu}(\overline{v}_t)(x,t) = 0, & \text{in } \Sigma, & (5.4e)\end{cases}$$

while $\{\widetilde{u}_t, \widetilde{v}_t\}$ satisfies the same problem (4.7), however, with homogeneous initial conditions

$$\begin{cases} (\widetilde{u}_t)_{tt}(x,t) - \Delta(\widetilde{u}_t)(x,t) - q(x)(\widetilde{v}_t)_t(x,t) = f(x)R_{1t}(x,t), & \text{in } Q, & (5.5a)\\ (\widetilde{v}_t)_{tt}(x,t) - \Delta(\widetilde{v}_t)(x,t) - p(x)(\widetilde{u}_t)_t(x,t) = g(x)R_{2t}(x,t), & \text{in } Q, & (5.5b)\\ (\widetilde{u}_t)\Big(\,\cdot\,,\dfrac{T}{2}\Big) = 0, \quad (\widetilde{u}_t)_t\Big(\,\cdot\,,\dfrac{T}{2}\Big) = 0, & \text{in } \Omega, & (5.5c)\\ (\widetilde{v}_t)\Big(\,\cdot\,,\dfrac{T}{2}\Big) = 0, \quad (\widetilde{v}_t)_t\Big(\,\cdot\,,\dfrac{T}{2}\Big) = 0, & \text{in } \Omega, & (5.5d)\\ \dfrac{\partial}{\partial\nu}(\widetilde{u}_t)(x,t) = 0, \quad \dfrac{\partial}{\partial\nu}(\widetilde{v}_t)(x,t) = 0, & \text{in } \Sigma. & (5.5e)\end{cases}$$

**Step 2** Here we apply the continuous observability inequality, Theorem 3.3, (3.18), to the $\{\overline{u}_t, \overline{v}_t\}$-problem (5.4a)–(5.4c), as assumptions (3.16) are satisfied. Accordingly, there is a constant $C_{T,q,p} > 0$ depending on $T$ and on the $L^\infty(\Omega)$-norm of the data $q$ and $p$ but not on $f$ and $g$, such that

$$\Big\|f(\,\cdot\,)R_1\Big(\,\cdot\,,\frac{T}{2}\Big)\Big\|^2_{L^2(\Omega)} + \Big\|g(\,\cdot\,)R_2\Big(\,\cdot\,,\frac{T}{2}\Big)\Big\|^2_{L^2(\Omega)}$$

$$\leq C_{T,q,p}^2 \int_0^T \int_{\Gamma_1} [\overline{u}_t^2 + \overline{u}_{tt}^2 + \overline{v}_t^2 + \overline{v}_{tt}^2] \mathrm{d}\Gamma_1 \, \mathrm{d}t, \tag{5.6}$$

whenever the RHS is finite, where $T > T_0$ (see (2.1)), as assumed. Since $\left|R_i\left(x, \frac{T}{2}\right)\right| \geq r_i > 0$, $x \in \overline{\Omega}$, $i = 1, 2$, by assumption (2.3), we then obtain from (5.6) by use of (5.3), the triangle inequality, with constant $C = C_{T,q,p,r_1,r_2}$:

$$\begin{aligned}
&\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\Omega)} \\
&\leq C(\|\overline{u}_t\|_{L^2(\Gamma_1\times[0,T])} + \|\overline{u}_{tt}\|_{L^2(\Gamma_1\times[0,T])} + \|\overline{v}_t\|_{L^2(\Gamma_1\times[0,T])} + \|\overline{v}_{tt}\|_{L^2(\Gamma_1\times[0,T])}) \\
&\leq C(\|u_t - \widetilde{u}_t\|_{L^2(\Gamma_1\times[0,T])} + \|u_{tt} - \widetilde{u_{tt}}\|_{L^2(\Gamma_1\times[0,T])} \\
&\quad + \|v_t - \widetilde{v}_t\|_{L^2(\Gamma_1\times[0,T])} + \|v_{tt} - \widetilde{v_{tt}}\|_{L^2(\Gamma_1\times[0,T])}) \\
&\leq C(\|u_t\|_{L^2(\Gamma_1\times[0,T])} + \|u_{tt}\|_{L^2(\Gamma_1\times[0,T])} + \|v_t\|_{L^2(\Gamma_1\times[0,T])} + \|v_{tt}\|_{L^2(\Gamma_1\times[0,T])}) \\
&\quad + C(\|\widetilde{u}_t\|_{L^2(\Gamma_1\times[0,T])} + \|\widetilde{u}_{tt}\|_{L^2(\Gamma_1\times[0,T])} + \|\widetilde{v}_t\|_{L^2(\Gamma_1\times[0,T])} + \|\widetilde{v}_{tt}\|_{L^2(\Gamma_1\times[0,T])}).
\end{aligned} \tag{5.7}$$

Inequality (5.7) is the desired, sought-after estimate (2.6) of Theorem 2.2, modulo (polluted by) the $\widetilde{u}_t$, $\widetilde{u}_{tt}$- and $\widetilde{v}_t$, $\widetilde{v}_{tt}$-terms. Such terms will be next absorbed by a compactness-uniqueness argument.

**Step 3** To carry this through, we need the following lemma.

**Lemma 5.1** *Consider the* $\{\widetilde{u}_t, \widetilde{v}_t\}$*-system* (5.5a)–(5.5c) *with data*

$$q, p \in L^\infty(\Omega), \quad f, g \in L^2(\Omega), \quad R_{it}, R_{itt} \in L^\infty(Q), \quad i = 1, 2. \tag{5.8}$$

*Define the following operators* $K$, $K_1$, $L$ *and* $L_1$:

$$\begin{cases}
(K\{f,g\})(x,t) = \widetilde{u}_t(x,t)|_{\Sigma_1} : \ L^2(\Omega) \to L^2(\Gamma_1 \times [0,T]), & (5.9a) \\
(K_1\{f,g\})(x,t) = \widetilde{u}_{tt}(x,t)|_{\Sigma_1} : \ L^2(\Omega) \to L^2(\Gamma_1 \times [0,T]), & (5.9b) \\
(L\{f,g\})(x,t) = \widetilde{v}_t(x,t)|_{\Sigma_1} : \ L^2(\Omega) \to L^2(\Gamma_1 \times [0,T]), & (5.9c) \\
(L_1\{f,g\})(x,t) = \widetilde{v}_{tt}(x,t)|_{\Sigma_1} : \ L^2(\Omega) \to L^2(\Gamma_1 \times [0,T]), & (5.9d)
\end{cases}$$

*where* $\{\widetilde{u}_t, \widetilde{v}_t\}$ *is the unique solution to problem* (5.5a)–(5.5d)*. Then,*

$$K, \ K_1, \ L \text{ and } L_1 \text{ are compact operators.} \tag{5.10}$$

**Proof** First, under present assumptions (5.8) with zero initial conditions (5.5c)–(5.5d) and homogeneous boundary conditions (5.5e), system (5.5) (= (4.2)) possesses a-fortiori the regularity,

$$\{\widetilde{u}_t, \widetilde{v}_t\} \in H^1(Q) \times H^1(Q). \tag{5.11}$$

Moreover, differentiate the system (5.5) in time: we obtain the $\{\widetilde{u}_{tt}, \widetilde{v}_{tt}\}$-system which contains forcing terms $f(x)R_{1tt}(x,t)$, $g(x)R_{2tt}(x,t) \in L^2(Q)$ and non-zero initial velocity $f(x)R_{1t}(x, \frac{T}{2})$, $g(x)R_{2t}(x, \frac{T}{2}) \in L^2(\Omega)$ under the present assumptions (5.8). Therefore a-fortiori we obtain also

$$\{\widetilde{u}_{tt}, \widetilde{v}_{tt}\} \in H^1(Q) \times H^1(Q). \tag{5.12}$$

**Preliminaries** We shall invoke sharp (Dirichlet) trace theory results for the Neumann hyperbolic problem (5.5a)–(5.5d) from [11]. More precisely, regarding the $\{\widetilde{u}_t, \widetilde{v}_t\}$-problem (5.5) (= (4.2)), the following Dirichlet trace results hold true:

(a) Assumptions $f(x), g(x) \in L^2(\Omega)$, $R_{it} \in L^\infty(Q)$, $i = 1, 2$ as in (5.8) and properties (5.11)–(5.12) imply

$$f(x)R_{1t}(x,t) + q(x)\widetilde{v}_{tt}(x,t) \in L^2(Q), \quad g(x)R_{2t}(x,t) + p(x)\widetilde{u}_{tt}(x,t) \in L^2(Q), \tag{5.13}$$

and then from [11], see below

$$f(x)R_{1t}(x,t) + q(x)\widetilde{v}_{tt}(x,t) \in L^2(Q) \Rightarrow \widetilde{u}_t|_\Sigma \in H^\beta(\Sigma) \text{ continuously}, \tag{5.14}$$

$$g(x)R_{2t}(x,t) + p(x)\widetilde{u}_{tt}(x,t) \in L^2(Q) \Rightarrow \widetilde{v}_t|_\Sigma \in H^\beta(\Sigma) \text{ continuously}. \tag{5.15}$$

(b) Assumption (5.8) as well as the regularity properties (5.11) and (5.12) imply

$$\begin{aligned} &f(x)R_{1t}(x,t) + q(x)\widetilde{v}_{tt}(x,t) \in H^1(0,T;L^2(\Omega)), \\ &g(x)R_{2t}(x,t) + p(x)\widetilde{u}_{tt}(x,t) \in H^1(0,T;L^2(\Omega)), \end{aligned} \tag{5.16}$$

and then

$$f(x)R_{1t}(x,t) + q(x)\widetilde{v}_{tt}(x,t) \in H^1(0,T;L^2(\Omega)) \Rightarrow D_t^1 \widetilde{u}_t|_\Sigma = \widetilde{u}_{tt}|_\Sigma \in H^\beta(\Sigma), \tag{5.17}$$

$$g(x)R_{2t}(x,t) + p(x)\widetilde{u}_{tt}(x,t) \in H^1(0,T;L^2(\Omega)) \Rightarrow D_t^1 \widetilde{v}_t|_\Sigma = \widetilde{v}_{tt}|_\Sigma \in H^\beta(\Sigma), \tag{5.18}$$

continuously with $\beta$ the following constant:

$$\begin{aligned} &\beta = \frac{3}{5} \quad \text{for a general } \Omega; \quad \beta = \frac{2}{3}, \quad \text{if } \Omega \text{ is a sphere}, \\ &\beta = \frac{3}{4} - \epsilon, \quad \text{if } \Omega \text{ is a parallelepiped}. \end{aligned} \tag{5.19}$$

The regularity properties (5.14)–(5.15) are from [11, Theorem B, (1), p. 118, for problem (2.5a)–(2.5c), p. 123], which is the same as [11, Theorem 2.0, Part II, (2.10), p. 124].

Then implications (5.17)–(5.18) are immediate consequences of implications (5.14) and (5.15) for problem (5.5a)–(5.5d), as then one applies the regularity properties (5.14)–(5.15) to $\{\widetilde{u}_{tt}, \widetilde{v}_{tt}\}$, solution to the problem obtained from (5.5a)–(5.5d), after differentiating in time once.

(c) By interpolation between (5.14) and (5.17), and between (5.15) and (5.18), one obtains, for $0 \le \theta \le 1$, still under the hypotheses (5.8) and regularity properties (5.11)–(5.12):

$$fR_{1t} + q\widetilde{v}_{tt} \in H^\theta(0,T;L^2(\Omega)) \Rightarrow D_t^\theta \widetilde{u}_t|_\Sigma \in H^\beta(\Sigma) \Rightarrow D_t^\theta \widetilde{u}_{tt}|_\Sigma \in H^{\beta-1}(\Sigma), \tag{5.20}$$

$$gR_{2t} + p\widetilde{u}_{tt} \in H^\theta(0,T;L^2(\Omega)) \Rightarrow D_t^\theta \widetilde{v}_t|_\Sigma \in H^\beta(\Sigma) \Rightarrow D_t^\theta \widetilde{v}_{tt}|_\Sigma \in H^{\beta-1}(\Sigma), \tag{5.21}$$

equivalently,

$$f(x)R_{1t}(x,t) + q(x)\widetilde{v}_{tt}(x,t) \in H^\theta(0,T;L^2(\Omega)) \Rightarrow \widetilde{u}_{tt}|_\Sigma \in H^{\beta+\theta-1}(\Sigma), \tag{5.22}$$

$$g(x)R_{2t}(x,t) + p(x)\widetilde{u}_{tt}(x,t) \in H^\theta(0,T;L^2(\Omega)) \Rightarrow \widetilde{v}_{tt}|_\Sigma \in H^{\beta+\theta-1}(\Sigma), \tag{5.23}$$

continuously. In particular, for $\theta = 1-\beta+\epsilon$,

$$f(x)R_{1t}(x,t) + q(x)\widetilde{v}_{tt}(x,t) \in H^{1-\beta+\epsilon}(0,T;L^2(\Omega)) \Rightarrow \widetilde{u}_{tt}|_\Sigma \in H^\epsilon(\Sigma), \tag{5.24}$$

$$g(x)R_{2t}(x,t) + p(x)\widetilde{u}_{tt}(x,t) \in H^{1-\beta+\epsilon}(0,T;L^2(\Omega)) \Rightarrow \widetilde{v}_{tt}|_\Sigma \in H^\epsilon(\Sigma), \tag{5.25}$$

continuously, for any $\epsilon > 0$.

After these preliminaries, we can now draw the desired conclusions on the compactness of the operators $K$, $K_1$, $L$ and $L_1$ in (5.9a)–(5.9d).

**Compactness of $K$, $L$** According to (5.14) and (5.15), it suffices to have $R_{it}, R_{itt} \in L^\infty(Q)$, $i = 1,2$, in order to have that the map

$$f, g \in L^2(\Omega) \to K\{f,g\} = \widetilde{u}_t|_\Sigma \in H^{\beta-\epsilon}(\Sigma), \quad L\{f,g\} = \widetilde{v}_t|_\Sigma \in H^{\beta-\epsilon}(\Sigma) \text{ are compact}, \tag{5.26}$$

$\forall\ \epsilon > 0$ sufficiently small, for then $f(x)R_{1t}(x,t) + q(x)\widetilde{v}_{tt}(x,t), g(x)R_{2t}(x,t) + p(x)\widetilde{u}_{tt}(x,t) \in L^2(Q)$ as required, by (5.14) and (5.15).

**Compactness of $K_1$, $L_1$** We have seen in (c) above that, under the hypotheses $R_{it}, R_{itt} \in L^\infty(Q)$, $i = 1,2$ in (5.8), (5.24) and (5.25) imply that the maps in (5.9),

$$\begin{aligned} &f, g \in L^2(\Omega) \to K_1\{f,g\} = \widetilde{u}_{tt} \in H^\epsilon(\Sigma) \text{ are continuous} \\ &\Rightarrow K_1\{f,g\} = \widetilde{u}_{tt} \in L^2(\Sigma) \text{ are compact}, \end{aligned} \tag{5.27}$$

$$\begin{aligned} &f, g \in L^2(\Omega) \to L_1\{f,g\} = \widetilde{v}_{tt} \in H^\epsilon(\Sigma) \text{ are continuous} \\ &\Rightarrow L_1\{f,g\} = \widetilde{v}_{tt} \in L^2(\Sigma) \text{ are compact} \end{aligned} \tag{5.28}$$

for then $f(x)R_{1t}(x,t) + q(x)\widetilde{v}_{tt}(x,t), g(x)R_{2t}(x,t) + p(x)\widetilde{u}_{tt}(x,t) \in H^{1-\beta+\epsilon}(0,T;L^2(\Omega))$, as required by (5.24) and (5.25). This completes the proof of Lemma 5.1.

**Step 4** Lemma 5.1 will allow us to absorb the terms

$$\begin{aligned} &\|Kf = \widetilde{u}_t\|_{L^2(\Gamma_1\times[0,T])}, \quad \|K_1 f = \widetilde{u}_{tt}\|_{L^2(\Gamma_1\times[0,T])}, \\ &\|Lf = \widetilde{v}_t\|_{L^2(\Gamma_1\times[0,T])}, \quad \|L_1 f = \widetilde{v}_{tt}\|_{L^2(\Gamma_1\times[0,T])} \end{aligned} \tag{5.29}$$

on the RHS of estimate (5.7), by a compactness-uniqueness argument, as usual.

**Proposition 5.2** *Consider the* $\{u,v\}$*-problem* (1.7a)–(1.7d) *with* $T > T_0$ *in* (2.1) *under assumption* (2.2) = (5.8) *for its data* $q(\cdot)$, $p(\cdot)$, $f(\cdot)$, $g(\cdot)$ *and* $R_i(\cdot,\cdot)$, *with* $R_i$ *satisfying also* (2.3), *so that both estimate* (5.7) *and Lemma* 5.1 *hold true. Then, the terms* $Kf = \widetilde{u}_t|_{\Sigma_1}$, $K_1 f = \widetilde{u}_{tt}|_{\Sigma_1}$, $Lf = \widetilde{v}_t|_{\Sigma_1}$ *and* $L_1 f = \widetilde{v}_{tt}|_{\Sigma_1}$ *measured in the* $L^2(\Gamma_1\times[0,T])$*-norm can be omitted from the RHS of inequality* (6.7) (*for a suitable constant* $C_{T,r_i,\cdots}$ *independent of the solution* $\{u,v\}$), *so that the desired conclusion,* (2.6), *of Theorem* 2.2 *holds true:*

$$\|f\|^2_{L^2(\Omega)} + \|g\|^2_{L^2(\Omega)} \le C_{T,\mathrm{data}}\left\{\int_0^T\int_{\Gamma_1}[u_t^2 + u_{tt}^2 + v_t^2 + v_{tt}^2]\mathrm{d}\Gamma_1\,\mathrm{d}t\right\} \tag{5.30}$$

*for all* $f, g \in L^2(\Omega)$, *with* $C_{T,\mathrm{data}}$ *independent of* $f$ *and* $g$.

**Proof Step (i)** Suppose, by contradiction, that inequality (5.30) is false. Then, there exist sequences $\{f_n\}_{n=1}^{\infty}$, $\{g_n\}_{n=1}^{\infty}$, $f_n, g_n \in L^2(\Omega)$, such that

$$\begin{cases} \|f_n\|_{L^2(\Omega)} = \|g_n\|_{L^2(\Omega)} \equiv 1, \quad n = 1, 2, \cdots, & (5.31a) \\ \lim\limits_{n\to\infty} (\|u_t(f_n)\|_{L^2(\Sigma_1)} + \|u_{tt}(f_n)\|_{L^2(\Sigma_1)} + \|v_t(f_n)\|_{L^2(\Sigma_1)} + \|v_{tt}(f_n)\|_{L^2(\Sigma_1)}) = 0, & (5.31b) \end{cases}$$

where $\{u(f_n, g_n), v(f_n, g_n)\}$ solves problem (1.7a)–(1.7d) with $f = f_n$, $g = g_n$:

$$\begin{cases} u(f_n, g_n)_{tt}(x,t) - \Delta u(f_n, g_n)(x,t) - q(x)v(f_n, g_n)_t(x,t) = f_n(x)R_1(x,t), & \text{in } Q, \quad (5.32a) \\ v(f_n, g_n)_{tt}(x,t) - \Delta v(f_n, g_n)(x,t) - p(x)u(f_n, g_n)_t(x,t) = g_n(x)R_2(x,t), & \text{in } Q, \quad (5.32b) \\ u(f_n, g_n)\Big(\cdot, \dfrac{T}{2}\Big) = u(f_n, g_n)_t\Big(\cdot, \dfrac{T}{2}\Big) = 0, & \text{in } \Omega; \\ v(f_n, g_n)\Big(\cdot, \dfrac{T}{2}\Big) = v_t(f_n, g_n)\Big(\cdot, \dfrac{T}{2}\Big) = 0, & \text{in } \Omega, \quad (5.32c) \\ \dfrac{\partial}{\partial \nu} u(f_n, g_n)\Big|_{\Sigma} = 0, \quad \dfrac{\partial}{\partial \nu} v(f_n, g_n)\Big|_{\Sigma} = 0, & \text{in } \Sigma. \quad (5.32d) \end{cases}$$

In view of (5.31a), there exist subsequences, still denoted by $f_n$ and $g_n$, such that

$$\{f_n, g_n\} \text{ converges weakly in } L^2(\Omega) \text{ to some } \{f_0, g_0\} \in L^2(\Omega). \tag{5.33}$$

Moreover, since the operators $K$, $K_1$, $L$ and $L_1$ are all compact (see Lemma 5.1), it then follows by (5.33) that we have strong convergence

$$\lim_{m,n\to+\infty} \|K\{f_n, g_n\} - K\{f_m, g_m\}\|_{L^2(\Sigma_1)} = \|K_1\{f_n, g_n\} - K_1\{f_m, g_m\}\|_{L^2(\Sigma_1)} = 0, \tag{5.34a}$$

$$\lim_{m,n\to+\infty} \|L\{f_n, g_n\} - L\{f_m, g_m\}\|_{L^2(\Sigma_1)} = \|L_1\{f_n, g_n\} - L_1\{f_m, g_m\}\|_{L^2(\Sigma_1)} = 0. \tag{5.34b}$$

**Step (ii)** On the other hand, since the map $\{f, g\} \to \{u(f, g), v(f, g)\}$ is linear, and recalling the definition of the operators $K$, $K_1$, $L$ and $L_1$ in (5.9), it follows from estimate (5.7) that

$$\begin{aligned} &\|f_n - f_m\|_{L^2(\Omega)} + \|g_n - g_m\|_{L^2(\Omega)} \\ \le\, &C(\|u_t(f_n, g_n) - u_t(f_m, g_m)\|_{L^2(\Gamma_1)} + \|u_{tt}(f_n, g_n) - u_{tt}(f_m, g_m)\|_{L^2(\Sigma_1)} \\ &+ \|v_t(f_n, g_n) - v_t(f_m, g_m)\|_{L^2(\Sigma_1)} + \|v_{tt}(f_n, g_n) - v_{tt}(f_m, g_m)\|_{L^2(\Sigma_1)}) \\ &+ C(\|K\{f_n, g_n\} - K\{f_m, g_m\}\|_{L^2(\Sigma_1)} + \|K_1\{f_n, g_n\} - K_1\{f_m, g_m\}\|_{L^2(\Sigma_1)} \\ &+ \|L\{f_n, g_n\} - L\{f_m, g_m\}\|_{L^2(\Sigma_1)} + \|L_1\{f_n, g_n\} - L_1\{f_m, g_m\}\|_{L^2(\Sigma_1)}) \\ \le\, &C(\|u_t(f_n, g_n)\|_{L^2(\Sigma_1)} + \|u_{tt}(f_n, g_n)\|_{L^2(\Sigma_1)} + \|u_t(f_m, g_m)\|_{L^2(\Sigma_1)} + \|u_{tt}(f_m, g_m)\|_{L^2(\Sigma_1)}) \\ &+ C(\|v_t(f_n, g_n)\|_{L^2(\Sigma_1)} + \|v_{tt}(f_n, g_n)\|_{L^2(\Sigma_1)} + \|v_t(f_m, g_m)\|_{L^2(\Sigma_1)} + \|v_{tt}(f_m, g_m)\|_{L^2(\Sigma_1)}) \\ &+ C(\|K\{f_n, g_n\} - K\{f_m, g_m\}\|_{L^2(\Sigma_1)} + \|K_1\{f_n, g_n\} - K_1\{f_m, g_m\}\|_{L^2(\Sigma_1)} \\ &+ \|L\{f_n, g_n\} - L\{f_m, g_m\}\|_{L^2(\Sigma_1)} + \|L_1\{f_n, g_n\} - L_1\{f_m, g_m\}\|_{L^2(\Sigma_1)}), \end{aligned} \tag{5.35}$$

where again the constant $C = C_{T,q,p,r_1,r_2}$, but is independent of $f$ and $g$. It then follows from (5.31b) and (5.34) as applied to the RHS of (5.35) that

$$\lim_{m,n\to+\infty} \|f_n - f_m\|_{L^2(\Omega)} = 0, \quad \lim_{m,n\to+\infty} \|g_n - g_m\|_{L^2(\Omega)} = 0. \tag{5.36}$$

Thus, $\{f_n\}$, $\{g_n\}$ are Cauchy sequences in $L^2(\Omega)$. By uniqueness of the limit, recalling (5.33), it then follows that

$$\lim_{n\to\infty} \|f_n - f_0\|_{L^2(\Omega)} = 0, \quad \lim_{n\to\infty} \|g_n - g_0\|_{L^2(\Omega)} = 0. \tag{5.37}$$

Thus, in view of (5.31a), then (5.37) implies

$$\|f_0\|_{L^2(\Omega)} = \|g_0\|_{L^2(\Omega)} = 1. \tag{5.38}$$

**Step (iii)** We now apply to $\{u, v\}$-problem (1.7a)–(1.7d) the same trace theorem as [11, Theorem B(1), p. 118] and [11, Theorem 2.0, Part II, (2.10), p. 124] that we have invoked in (5.14) (and (5.15)) for the $\{\widetilde{u}_t, \widetilde{v}_t\}$-problem (5.5a)–(5.5d)), that is, as $f, g \in L^2(\Omega)$, $R_i, R_{it} \in L^\infty(Q)$ by assumption and $\{u, v\}, \{u_t, v_t\} \in H^1(Q) \times H^1(Q)$ a-fortiori due to the $L^2$ forcing terms and initial velocities as well as the homogeneous boundary conditions:

$$fR_1 + qv_t,\ gR_2 + pu_t \in L^2(Q) \Rightarrow \{u|_\Sigma, v|_\Sigma\} \in H^\beta(\Sigma) \times H^\beta(\Sigma), \tag{5.39}$$

$$fR_1 + qv_t,\ gR_2 + pu_t \in H^1(0,T; L^2(\Omega)) \Rightarrow \{u_t|_\Sigma, v_t|_\Sigma\} \in H^\beta(\Sigma) \times H^\beta(\Sigma), \tag{5.40}$$

continuously, hence by interpolation

$$f(x)R_1(x,t) + q(x)v_t(x,t) \in H^{1-\beta}(0,T; L^2(\Omega)) \Rightarrow u_t|_\Sigma \in L^2(\Sigma), \tag{5.41}$$

$$g(x)R_2(x,t) + p(x)u_t(x,t) \in H^{1-\beta}(0,T; L^2(\Omega)) \Rightarrow v_t|_\Sigma \in L^2(\Sigma). \tag{5.42}$$

Here $\beta$ is defined in (5.19).

**Step (iv)** Thus, since $R_i \in L^\infty(Q)$, $i = 1, 2$, we deduce from (5.39) that

$$f, g \in L^2(\Omega) \to u(f,g)|_\Sigma \in H^\beta(\Sigma),\ v(f,g)|_\Sigma \in H^\beta(\Sigma) \text{ continuously}, \tag{5.43}$$

i.e.,

$$\|u(f,g)|_\Sigma\|_{H^\beta(\Sigma)},\ \|v(f,g)|_\Sigma\|_{H^\beta(\Sigma)} \le C_{R_1,R_2}(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\Omega)}) \tag{5.44}$$

with $C_{R_1,R_2} = \max\{\|R_1\|_{L^\infty(Q)}, \|R_2\|_{L^\infty(Q)}\}$.

As the map $\{f, g\} \to \{u(f,g), v(f,g)\}|_\Sigma$ is linear, it then follows in particular from (5.44), since $f_n, g_n, f_0, g_0 \in L^2(\Omega)$,

$$\||u(f_n,g_n)|_{\Sigma_1} - u(f_0,g_0)|_{\Sigma_1}\|_{H^\beta(\Sigma_1)} \le \widetilde{C}_{R_1,R_2}(\|f_n - f_0\|_{L^2(\Omega)} + \|g_n - g_0\|_{L^2(\Omega)}), \tag{5.45}$$

$$\||v(f_n,g_n)|_{\Sigma_1} - v(f_0,g_0)|_{\Sigma_1}\|_{H^\beta(\Sigma_1)} \le \widetilde{C}_{R_1,R_2}(\|f_n - f_0\|_{L^2(\Omega)} + \|g_n - g_0\|_{L^2(\Omega)}). \tag{5.46}$$

Recalling (5.37) on the RHS of (5.45) and (5.46), we conclude first that

$$\lim_{n\to\infty} \|u(f_n,g_n)|_{\Sigma_1} - u(f_0,g_0)|_{\Sigma_1}\|_{H^\beta(\Sigma_1)}$$

$$= \lim_{n\to\infty} \|v(f_n,g_n)|_{\Sigma_1} - v(f_0,g_0)|_{\Sigma_1}\|_{H^\beta(\Sigma_1)} = 0, \tag{5.47}$$

and next that

$$\begin{aligned} &\lim_{n\to\infty} \|u(f_n,g_n)|_{\Sigma_1} - u(f_0,g_0)|_{\Sigma_1}\|_{C([0,T];L^2(\Gamma_1))} \\ &= \lim_{n\to\infty} \|v(f_n,g_n)|_{\Sigma_1} - v(f_0,g_0)|_{\Sigma_1}\|_{C([0,T];L^2(\Gamma_1))} = 0, \end{aligned} \tag{5.48}$$

since $\beta > \frac{1}{2}$, so that $H^\beta(0,T)$ embeds in $C[0,T]$.

**Step (v)** Similarly, from (5.40) and recalling (5.37), where in addition, $R_{it} \in L^\infty(Q)$, $i = 1,2$, we deduce likewise in addition that

$$\begin{aligned} &\lim_{n\to\infty} \|u_t(f_n,g_n)|_{\Sigma_1} - u_t(f_0,g_0)|_{\Sigma_1}\|_{C([0,T];L^2(\Gamma_1))} \\ &= \lim_{n\to\infty} \|v_t(f_n,g_n)|_{\Sigma_1} - v_t(f_0,g_0)|_{\Sigma_1}\|_{C([0,T];L^2(\Gamma_1))} = 0. \end{aligned} \tag{5.49}$$

Then, by virtue of (5.31b), combined with (5.49), we obtain in $t \in [0,T]$ that

$$\begin{aligned} &u_t(f_0,g_0)|_{\Sigma_1} = v_t(f_0,g_0)|_{\Sigma_1} \equiv 0, \text{ or} \\ &u(f_0,g_0)|_{\Sigma_1} \text{ and } v(f_0,g_0)|_{\Sigma_1} \text{are functions of } x \in \Gamma_1. \end{aligned} \tag{5.50}$$

**Step (vi)** We return to problem (5.32). With $f_n, g_n \in L^2(\Omega)$ and data $q,p \in L^\infty(\Omega)$, $R_1, R_2 \in L^\infty(Q)$, we have the following regularity results, continuously:

$$\{u(f_n,g_n), u_t(f_n,g_n), v(f_n,g_n), v_t(f_n,g_n)\} \in C([0,T]; H^1(\Omega)\times L^2(\Omega)\times H^1(\Omega)\times L^2(\Omega)), \tag{5.51}$$

$$\{u(f_n,g_n), v(f_n,g_n)\}|_\Sigma \in H^\beta(\Sigma)\times H^\beta(\Sigma). \tag{5.52}$$

Again, the sharp trace regularity (5.52) is the same result noted in (5.14)–(5.15), and quoted from [11, Theorem B(1), p. 118] and [11, Theorem 2.0, (2.10), p. 124], with $\beta$ the constant in (5.19). As a consequence of (5.37), we also have via (5.51)–(5.52)

$$\begin{aligned} &\{u(f_n,g_n), u_t(f_n,g_n)\} \to \{u(f_0,g_0), u_t(f_0,g_0)\}, \\ &\{v(f_n,g_n), v_t(f_n,g_n)\} \to \{v(f_0,g_0), v_t(f_0,g_0)\}, \quad \text{in } C([0,T;H^1(\Omega)\times L^2(\Omega)), \end{aligned} \tag{5.53}$$

$$\{u(f_n,g_n), v(f_n,g_n)\} \to \{u(f_0,g_0), v(f_0,g_0)\}, \quad \text{in } H^\beta(\Sigma)\times H^\beta(\Sigma). \tag{5.54}$$

On the other hand, recalling the initial conditions (5.32c)–(5.32d), we have $u(f_n,g_n)(x,\frac{T}{2}) = v(f_n,g_n)(x,\frac{T}{2}) \equiv 0$, $x \in \overline{\Omega}$ and hence

$$u(f_n,g_n)\Big(x,\frac{T}{2}\Big) = v(f_n,g_n)\Big(x,\frac{T}{2}\Big) \equiv 0, \quad x \in \Gamma_1 \tag{5.55}$$

in the sense of trace in $H^{\frac{1}{2}}(\Gamma_1)$. Then (5.55) combined with (5.47)–(5.48) yields a-fortiori

$$u(f_0,g_0)\Big(x,\frac{T}{2}\Big) = v(f_0,g_0)\Big(x,\frac{T}{2}\Big) \equiv 0, \quad x \in \Gamma_1, \tag{5.56}$$

and next, by virtue of (5.50), the desired conclusion,

$$u(f_0,g_0)|_{\Sigma_1} = v(f_0,g_0)|_{\Sigma_1} \equiv 0. \tag{5.57}$$

Here, $\{u(f_0,g_0),v(f_0,g_0)\}$ satisfies weakly the limit problem, via (5.37), (5.51)–(5.52) applied to (5.32a)–(5.32d):

$$\begin{cases} u_{tt}(f_0,g_0)(x,t)-\Delta u(f_0,g_0)(x,t)-q(x)v_t(f_0,g_0)(x,t)=f_0(x)R_1(x,t), & \text{in } Q, \quad (5.58a)\\ v_{tt}(f_0,g_0)(x,t)-\Delta v(f_0,g_0)(x,t)-p(x)u_t(f_0,g_0)(x,t)=g_0(x)R_2(x,t), & \text{in } Q, \quad (5.58b)\\ u(f_0,g_0)\Big(\cdot,\dfrac{T}{2}\Big)=u_t(f_0,g_0)\Big(\cdot,\dfrac{T}{2}\Big)=0, & \text{in } \Omega;\\ v(f_0,g_0)\Big(\cdot,\dfrac{T}{2}\Big)=v_t(f_0,g_0)\Big(\cdot,\dfrac{T}{2}\Big)=0, & \text{in } \Omega, \quad (5.58c)\\ \dfrac{\partial}{\partial\nu}u(f_0,g_0)\Big|_\Sigma=0, \quad \dfrac{\partial}{\partial\nu}v(f_0,g_0)\Big|_\Sigma=0, & \text{in } \Sigma, \quad (5.58d)\\ u(f_0,g_0)|_{\Sigma_1}=v(f_0,g_0)|_{\Sigma_1}=0, & \text{in } \Sigma_1, \quad (5.58e) \end{cases}$$

via also (5.57), where $f_0,g_0\in L^2(\Omega)$ and $q,p,R_1,R_2$ satisfy the assumptions (2.2)–(2.3). By virtue of (5.52) and assumption (6.1) = (2.2) + (2.3), thus, the uniqueness Theorem 2.1 applies and yields

$$f_0(x)=g_0(x)\equiv 0, \qquad \text{a.e. } x\in\Omega. \tag{5.59}$$

Then (5.59) contradicts (5.38). Thus, assumption (5.31) is false and inequality (5.30) holds true, and Proposition 5.2 and Theorem 2.2 are then established.

## 6 Uniqueness and Stability of the Nonlinear Inverse Problem for the $\{w,z\}$-system (1.4) — Proof of Theorems 2.3 and 2.4

The proof of Theorem 2.3 (uniqueness of the nonlinear inverse problem for the $\{w,z\}$-dynamics (1.4)) is reduced to Theorem 2.1 (uniqueness of the linear inverse problem for the $\{u,v\}$-dynamics (1.7)), and the proof of Theorem 2.4 (stability of the nonlinear inverse problem for the $\{w,z\}$-dynamics (1.4)) is reduced to Theorem 2.2 (stability of the linear inverse problem for the $\{u,v\}$-dynamics (1.7)). In fact, as in (1.6a)–(1.6b), set

$$\begin{aligned} &f(x)=q_1(x)-q_2(x), \quad g(x)=p_1(x)-p_2(x),\\ &R_1(x,t)=z_t(q_2,p_2)(x,t), \quad R_2(x,t)=w_t(q_2,p_2)(x,t), \end{aligned} \tag{6.1a}$$

$$\begin{aligned} &u(x,t)=w(q_1,p_1)(x,t)-w(q_2,p_2)(x,t),\\ &v(x,t)=z(q_1,p_1)(x,t)-z(q_2,p_2)(x,t). \end{aligned} \tag{6.1b}$$

Then, as noted in Section 1, the variables $u(x,t)$, $v(x,t)$ solve problem (1.7). By virtue of assumption (2.6), we then have via (6.1a) that $f(x),g(x)\in L^\infty(\Omega)$.

**Step 1** We rewrite the coupled problem (1.4) as in (4.2):

$$\begin{bmatrix} w_{tt}\\ z_{tt}\end{bmatrix}=\begin{bmatrix} -\widetilde{A}_N & 0\\ 0 & -\widetilde{A}_N\end{bmatrix}\begin{bmatrix} w\\ z\end{bmatrix}+\begin{bmatrix} 0 & q(\cdot)\\ p(\cdot) & 0\end{bmatrix}\begin{bmatrix} w_t\\ z_t\end{bmatrix}=-A_N\begin{bmatrix} w\\ z\end{bmatrix}+\Pi\begin{bmatrix} w_t\\ z_t\end{bmatrix}, \tag{6.2}$$

where $\widetilde{A}_N=-\Delta$ with Neumann boundary conditions, non-negative self-adjoint on $L^2(\Omega)$, $\Pi$ is a bounded perturbation of $-A_N$ which will not affect the regularity of the solutions (see [4])

$$w(t)=C(t)w_0+S(t)w_1, \quad w_t(t)=A_NS(t)w_0+C(t)w_1, \tag{6.3}$$

$$w_{tt}(t) = A_N C(t) w_0 + A_N S(t) w_1, \quad w_{ttt}(t) = A_N^{\frac{3}{2}} A_N^{\frac{1}{2}} S(t) w_0 + A_N C(t) w_1 \tag{6.4}$$

and similarly for $z(t)$, $z_t(t)$, $z_{tt}(t)$, $z_{ttt}(t)$. Here $C(t)$ is the cosine operator on $L^2(\Omega) \times L^2(\Omega)$ generated by $-A_N$, $\mathcal{D}(A_N) = \mathcal{D}(\widetilde{A}_N) \times \mathcal{D}(\widetilde{A}_N)$, and $S(t)$ its corresponding sine operator (see [4]). We have from (5.3)–(5.4) that

$$\{w_0, w_1\}, \{z_0, z_1\} \in \mathcal{D}(A_N^{k+\frac{1}{2}}) \times \mathcal{D}(A_N^k) \subset H^{2k+1}(\Omega) \times H^{2k}(\Omega) \tag{6.5}$$

implies

$$w_{ttt}, z_{ttt} \in C([0,T]; \mathcal{D}(A_N^{k-1})) \subset C([0,T]; H^{2(k-1)}(\Omega)), \tag{6.6}$$

$$w_{ttt}, z_{ttt} \in C([0,T]; C(\Omega)), \quad \text{provided } k > \frac{\dim\Omega}{4} + 1. \tag{6.7}$$

**Step 2** In terms of the boundary data, we have the next proposition.

**Proposition 6.1** *We return to the* $\{w, z\}$*-problem* (1.4a)–(1.4d).

(a) *Under the following assumptions on the data:*

$$q(\,\cdot\,), p(\,\cdot\,) \in L^\infty(\Omega); \tag{6.8}$$

$$\begin{cases} \mu_i \in H^m(0,T; L^2(\Gamma)) \cap C([0,T]; H^{\alpha-\frac{1}{2}+(m-1)}(\Gamma)), \\ \alpha = \dfrac{2}{3} \text{ for a general domain;} \quad \alpha = \dfrac{3}{4} \text{ for a parallelepiped,} \\ \text{with Compatibility Relations} \\ \mu_i\Big(\dfrac{T}{2}\Big) = \dot{\mu}_i\Big(\dfrac{T}{2}\Big) = \cdots = \mu_i^{(m-1)}\Big(\dfrac{T}{2}\Big) = 0, \quad i = 1, 2. \end{cases} \tag{6.9}$$

*The regularity in* (6.7) *is a-fortiori implied by*

$$\mu_i \in H^{m(2\alpha-1),m}(\Sigma) = L^2(0,T; H^{m(2\alpha-1)}(\Gamma)) \cap H^m(0,T; L^2(\Gamma)), \tag{6.10}$$

*via* [18, *Theorem* 3.1, *p.* 19] (*see* [11, *Remark* 3.1, *p.* 130 *for* $m = 1$; *Remark* 3.4, *p.* 133 *for* $m = 2$]). *Then the solution* $\{w = w(q,p), z = z(q,p)\}$ *satisfies the following regularity property:*

$$\begin{aligned} &\{w, w_t, w_{tt}, w_{ttt}\}, \{z, z_t, z_{tt}, z_{ttt}\} \\ &\in C([0,T]; H^{\alpha+m}(\Omega) \times H^{\alpha+(m-1)}(\Omega) \times H^{\alpha+(m-2)}(\Omega) \times H^{\alpha+(m-3)}(\Omega), \end{aligned} \tag{6.11}$$

*continuously.*

(b) *If, moreover,*

$$m > \frac{\dim\Omega}{2} + 3 - \alpha, \tag{6.12}$$

*then a-fortiori, properties* (6.7) *are fulfilled,*

$$w_t, w_{tt}, w_{ttt}, z_t, z_{tt}, z_{ttt} \in L^\infty(Q). \tag{6.13}$$

**Proof** (a) The result in (a) relies critically on sharp regularity results, contained in [10] in terms of a parameter $\alpha$, which was specified as follows: $\alpha = \frac{2}{5} - \epsilon$ for a general domain, $\alpha = \frac{2}{3}$

for a sphere and certain other domains; $\alpha = \frac{3}{4}$ for a parallelepiped (see [10, Counterexample, p. 294] showed that $\alpha = \frac{3}{4} + \epsilon$ is impossible). Later, Tataru [24] refined this result by obtaining $\alpha = \frac{2}{3}$ for a general domain, except for $\alpha = \frac{3}{4}$ for a parallelepiped. More precisely,

**Case $m = 1$**

Let

$$\mu_i \in H^1(0,T;L^2(\Gamma)) \cap C([0,T];H^{\alpha-\frac{1}{2}}(\Gamma)) \quad \text{with compatibility relations } \mu_i\left(\frac{T}{2}\right) = 0. \tag{6.14}$$

Then [10, Theorem A(2), p. 117], repeated as [10, Theorem 3.1, (3.6), (3.9)–(3.10), p. 129], (7.28) implies that

$$\{w, w_t, w_{tt}\}, \{z, z_t, z_{tt}\} \in C([0,T];H^{\alpha+1}(\Omega) \times H^{\alpha}(\Omega) \times H^{\alpha-1}(\Omega)), \tag{6.15}$$

continuously. (6.15) is result (a), (6.11), for $m = 1$, except for $w_{ttt}, z_{ttt}$.

**Case $m = 2$**

Let now

$$\begin{aligned} &\mu_i \in H^2(0,T;L^2(\Gamma)) \cap C([0,T];H^{\alpha+\frac{1}{2}}(\Gamma)) \\ &\text{with compatibility relations } \mu_i\left(\frac{T}{2}\right) = \dot{\mu}_i\left(\frac{T}{2}\right) = 0, \end{aligned} \tag{6.16}$$

and then [10, Theorem A(4), p. 118], repeated as [10, Theorem 3.2, (3.28), (3.30), (3.32), p. 132] implies that

$$\{w, w_t, w_{tt}\}, \{z, z_t, z_{tt}\} \in C([0,T];H^{\alpha+2}(\Omega) \times H^{\alpha+1}(\Omega) \times H^{\alpha}(\Omega)), \tag{6.17}$$

continuously. (6.17) is result (a), (6.11), for $m = 2$, except for $w_{ttt}, z_{ttt}$.

**General case $m$**

As noted in [11], the general case is similar and yields

$$\begin{aligned} &\mu_i \text{ as in (6.9)} \\ &\Rightarrow \{w, w_t, w_{tt}\}, \{z, z_t, z_{tt}\} \in C([0,T];H^{\alpha+m}(\Omega) \times H^{\alpha+(m-1)}(\Omega) \times H^{\alpha+(m-2)}(\Omega)), \end{aligned} \tag{6.18}$$

continuously, to which we add

$$w_{ttt}, z_{ttt} \in C([0,T];H^{\alpha+(m-3)}(\Omega)), \tag{6.19}$$

as the above theorems for the map $\mu_i \to \{w, w_t, w_{tt}, z, z_t, z_{tt}\}$ (with zero initial conditions) can be applied now to the map $\mu_{it} \to \{w_t, w_{tt}, w_{ttt}, z_t, z_{tt}, z_{ttt}\}$ (still with zero initial conditions), as $q(\cdot)$, $p(\cdot)$ are time-independent. Thus (6.11) is proved.

(b) If $\alpha + (m-3) > \frac{\dim\Omega}{2}$, then from [18, Corollary 9.1, p. 96] the following embedding holds:

$$H^{\alpha+(m-3)}(\Omega) \hookrightarrow C(\overline{\Omega}) \subset L^{\infty}(\Omega), \tag{6.20}$$

which, along with properties (6.11), yields (6.13) under (6.12).

**Step 3** Thus, under assumption (6.5), with $k$ in (6.7) on the initial conditions and assumption (6.9) with $m$ in (6.12) on the boundary conditions, we have that $R_i(x,t)$, $i = 1, 2$ satisfy assumption (2.2); moreover, so do

$$R_1\Big(x, \frac{T}{2}\Big) = z_t(q_2, p_2)\Big(x, \frac{T}{2}\Big) = z_1(x), \quad R_2\Big(x, \frac{T}{2}\Big) = w_t(q_2, p_2)\Big(x, \frac{T}{2}\Big) = w_1(x). \tag{6.21}$$

Thus, assumptions (2.8), (2.10)–(2.11) of Theorem 2.3 imply assumption (2.2) of Theorem 2.1. Moreover, assumption (2.9) of Theorem 2.3 implies assumption (2.3) of Theorem 2.1. In addition, the present assumption (2.12) that

$$w(q_1, p_1)(x,t) = w(q_2, p_2)(x,t), \quad z(q_1, p_1)(x,t) = z(q_2, p_2)(x,t), \quad x \in \Gamma_1, \ t \in [0,T] \tag{6.22}$$

implies via (6.1b) that $u(f,g)(x,t) = 0$, $v(f,g)(x,t) = 0$, $x \in \Gamma_1$, $t \in [0,T]$. Therefore, Theorem 2.1 applies, and we conclude that $f(x) = q_1(x) - p_1(x) = 0$ and $g(x) = q_2(x) - p_2(x) = 0$, that is, $q_1(x) = p_1(x)$, $q_2(x) = p_2(x)$ a.e. $x \in \Omega$. Similarly, Theorem 2.2 also applies and we then obtain for $f(x) = q_1(x) - q_2(x)$, $g(x) = p_1(x) - p_2(x)$, the desired stability estimate (2.14).

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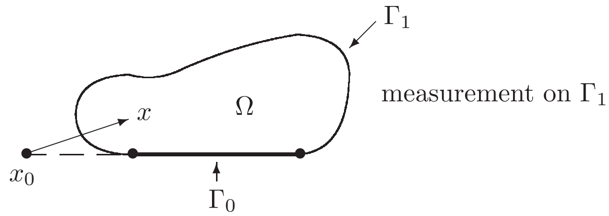
## Appendix Admissible Geometrical Configurations

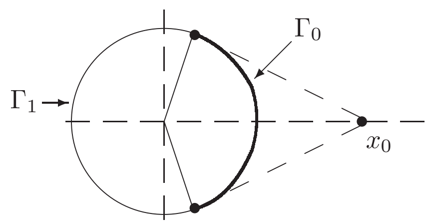
**Example A.1** For any dimension $\geq 2$, $\Gamma_0$ is flat.




Let $x_0 \in$ hyperplane containing $\Gamma_0$. Then

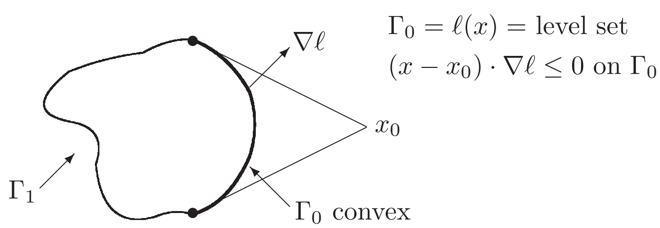
$$d(x) = \|x - x_0\|^2, \quad h(x) = \nabla d(x) = 2(x - x_0).$$

**Example A.2** For a ball of any dimension $\geq 2$, let us see $d(x)$ in [16, Theorem A.4.1, p. 301].

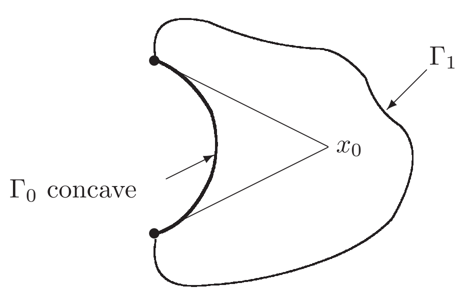



Measurement on $\Gamma_1 > \frac{1}{2}$ circumference (as in the Dirichlet case), or as for controllability $\cdots$

**Example A.3** For the generalizing Example A.2: a domain $\Omega$ of any dimension $\geq 2$ with the unobserved portion $\Gamma_0$ convex, subtended by a common point $x_0$, let us see $d(x)$ in [16, Theorem. A.4.1, p. 301].



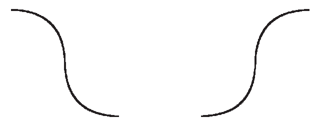

**Example A.4** For a domain $\Omega$ of any dimension $\geq 2$ with the unobserved portion $\Gamma_0$ concave, subtended by a common point $x_0$, let us see $d(x)$ in [16, Theorem A.4.1, p. 301].



**Example A.5** For dim $= 2$, $\Gamma_0$ is neither convex or concave. $\Gamma_0$ is described by graph

$$y = \begin{cases} f_1(x), & x_0 \le x \le x_1,\ y \ge 0, \\ f_2(x), & x_2 \le x \le x_1,\ y < 0, \end{cases}$$

$f_1, f_2$ logarithmic concave on $x_0 < x < x_1$, e.g., $\sin x$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$; $\cos x$, $0 < x < \pi$.



Function $d(x)$ is given in [16, Equation (A.2.7), p. 289].