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On the Volume Formulas of Cones and Orthogonal Multi-cones in $S^n(1)$ and $H^n(-1)$

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(*Dedicated to the memory of Shiing-Shen Chern*)

Abstract In the study of n-dimensional spherical or hyperbolic geometry, $n \geq 3$, the volume of various objects such as simplexes, convex polytopes, etc. often becomes rather difficult to deal with. In this paper, we use the method of infinitesimal symmetrization to provide a systematic way of obtaining volume formulas of cones and orthogonal multiple cones in $S^n(1)$ and $H^n(-1)$.

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1 Introduction

In the Euclidean plane geometry, one has the well-known area formula of triangles which is of basic importance because all the other basic theorems in *quantitative plane geometry* such as Pythagoras' theorem, similar triangle theorem, etc. can be deduced directly from such a simple formula. In the case of solid geometry, the corresponding formula that the volume of a cone should be equal to *one third of its base area times its height* had been realized for a long time before a proof of such a basic formula was finally achieved by Eudoxus, using the *method of exhaustion* invented by himself. His remarkable proof is, historically, *the first* application of what, nowadays, called integration. In the case of n-dimensional Euclidean space, it is straightforward to extend Eudoxus' proof to show that the volume of a cone is equal to one n-th of the $(n-1)$-dimensional volume of its base times its height.

For a given point p in a given Riemannian manifold M^n, the *local isometry group of M^n at p*, denoted by $\mathrm{ISO}(M^n, p)$, consists of those isometries fixing p. It is easy to see that the mapping of such an isometry $g \in \mathrm{ISO}(M^n, p)$ to its induced action on T_pM^n (i.e. $dg|_p$) is an *injective* isomorphism. Thus, $\mathrm{ISO}(M^n, p)$ can always be identified to a subgroup of $O(n)$. The three kinds of classical spaces (i.e. Euclidean, spherical and hyperbolic n-spaces) are exactly those simply connected Riemannian manifolds whose local isometry groups are everywhere isomorphic to the maximal possibility of $O(n)$. In short, they are the three kinds of *most symmetric geometries.* It is natural to expect that many basic formulas in Euclidean geometry such as the volume formula of cones should have their useful generalizations in both the spherical and the hyperbolic geometries, and such generalizations will always be a natural way of providing further

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understandings of these two additional kinds of most symmetric geometries of fundamental importance.

In this paper, we shall first study the generalizations of the volume formula of cones in the spherical (resp. hyperbolic) space and then further extending such volume formulas to that of *orthogonal multiple cones* (cf. §5). Up to a scaling factor, it suffices to study the above problems in the *normalized* cases of $S^n(1)$ and $H^n(-1)$, namely, of constant sectional curvatures of ± 1 respectively.

We shall first derive the volume formulas of cones in $S^n(1)$ (resp. $H^n(-1)$) for the beginning cases of $n = 2$ and 3 in §2, and then proceed to solve the problem in its full generality of $n \geq 4$ in §3 (cf. formulas (38), (38′), (39) and (39′) of Theorem 1).

In general, the volume of a cone in $S^n(1)$ (resp. $H^n(-1)$) is given by the above mentioned *integral formulas*, which is the integration of the weight function $w_n(k,\rho)$ (resp. $\widetilde{w}_n(k,\rho)$) of (38′) (resp. (39′)) over the base Ω. In the most symmetric situation that the base, Ω, is a ball of radius r centered at O, the integral formulas (38) and (39) can be explicitly integrated, thus providing a family of simple *volume functions* for such a special family of cones that we shall call them *orthospindles*. Such explicit integrations have been carried out in §4, and the volume functions of orthospindles, denoted by $\psi_{n,1}(k,r)$ (resp. $\tilde{\psi}_{n,1}(k,r)$) will play the central role for generalizing the volume formulas of cones to that of *orthogonal double cones* in §5 (cf. formulas (67) and (67′) of Theorem 2).

In the study of Euclidean, spherical and hyperbolic geometries, cones naturally constitutes a useful family of simple, basic objects, while *orthogonal cone decomposition* often providing canonical ways of reducing the computation or estimation of volumes of more general geometric bodies to that of cones. However, in the case of higher dimensional geometric problems, it is often necessary to apply this kind of cone decomposition several times in order to obtain pieces with certain kinds of technical simplicity. Therefore, it is useful and technically necessary to further extend the volume formulas of cones and double cones to *orthogonal multiple cones* (cf. §5 for the definition of orthogonal multiple cones). This is achieved in §6 and the main results are stated as Theorem 3 and Theorem 4, in which the volume functions of multiple orthospindles (denoted by $\psi_{n,\ell}$ and $\tilde{\psi}_{n,\ell}$) play the central role.

We refer to [2], [3], etc. for some of the applications of the volume formulas of this paper. In fact, it was those applications which motivated the author to develop such a family of volume formulas, because they are exactly what are needed in order to provide the kind of volume estimates necessary for solving those problems of [2], [3], etc.

2 Volume Formulas of Cones in $S^n(1)$ (Resp. $H^n(-1)$) for the Cases of $n = 2$ and 3

2.1 The case of $n = 2$

Even in the beginning case of $n = 2$, triangles in $S^2(1)$ (resp. $H^2(-1)$) with the same base length and the same height are *no longer* of the same area, but rather, its area will depend on the *relative position* between its *base interval* and its *height interval.* Thus, one needs to have an area formula taking account of the effect of their relative position. Due to the simplicity of the area formula of spherical (resp. hyperbolic) triangles, it is quite simple to derive such a formula as follows:

Let $\overline{AB}$ and $\overline{OC}$ be the base interval and the height interval of a given spherical (resp. hyperbolic) triangle $\triangle ABC$, as indicated in Figure 1. Let x be the arclength parameter on $\overrightarrow{AB}$ with its values at O, A, B equal to 0, $a < b$ respectively. Set $A(x)$ to be the oriented area of $\triangle OXC$. Then the area of $\triangle ABC$ is equal to $A(b) - A(a)$. Therefore, it suffices to compute $A(x)$ in terms of x and k, namely, the area of a spherical (resp. hyperbolic) right-angle triangle with x and k as the pair of side-lengths (adjacent to the right-angle). We state it as the following lemma.

Lemma 1 *Let $A(x)$ (resp. $\widetilde{A}(x)$) be the oriented area of the right-angle spherical (resp. hyperbolic) triangle $\triangle OXC$ as indicated in Figure 1. Then*

$$\cos A(x) = \frac{\cos k + \cos x}{1 + \cos k \cos x}, \qquad \sin A(x) = \frac{\sin k \sin x}{1 + \cos k \cos x} \tag{1}$$

$$\Big(\text{resp.} \quad \cos \widetilde{A}(x) = \frac{\cosh k + \cosh x}{1 + \cosh x \cosh x}, \quad \sin \widetilde{A}(x) = \frac{\sinh k \sinh x}{1 + \cosh k \cosh x}\Big).$$

Proof Set ℓ (resp. ξ, β) to be the length (resp. the angles) of $\overline{CX}$ (resp. at C, X). Then, by the laws of spherical and hyperbolic trigonometries,

$$\begin{aligned}
A(x) &= \xi + \beta - \frac{\pi}{2} && \Big(\text{resp. } \widetilde{A}(x) = \frac{\pi}{2} - (\xi + \beta)\Big),\\
\cos \ell &= \cos k \cos x && (\text{resp. } \cosh \ell = \cosh k \cosh x),\\
\sin \beta &= \frac{\sin k}{\sin \ell} && \Big(\text{resp. } \sin \beta = \frac{\sinh k}{\sinh \ell}\Big),\\
\cos \beta &= \frac{\tan x}{\tan \ell} && \Big(\text{resp. } \cos \beta = \frac{\tanh x}{\tanh \ell}\Big), \text{ etc.}
\end{aligned} \tag{2}$$

Therefore

$$\begin{aligned}
\cos A(x) &= \sin(\xi + \beta) = \sin \xi \cos \beta + \cos \xi \sin \beta\\
&= \frac{1}{\sin \ell \tan \ell}\{\sin x \tan x + \sin k \tan k\}\\
&= \frac{\cos k \cos x}{\sin^2 \ell}\{\sin x \tan x + \sin k \tan k\}\\
&= \frac{(1 - \cos k \cos x)(\cos k + \cos x)}{(1 + \cos \ell)(1 - \cos \ell)} = \frac{\cos k + \cos x}{1 + \cos k \cos x}
\end{aligned} \tag{3}$$

$$\Big(\text{resp.} \quad \cos \widetilde{A}(x) = \frac{\cosh k + \cosh x}{1 + \cosh k \cosh x}\Big) \tag{$\tilde{3}$}$$

and similar computations will also show that

$$\sin A(x) = \frac{\sin k \sin x}{1 + \cos k \cos x} \quad \Big(\text{resp. } \sin \widetilde{A}(x) = \frac{\sinh k \sinh x}{1 + \cosh k \cosh x}\Big).$$

Corollary *The area of the spherical (resp. hyperbolic) triangle with height k and its base interval $[a, b]$ is given by the following formula, namely*

$$A(b) - A(a) = \int_a^b A'(x)dx = \int_a^b \frac{\sin k\, dx}{1 + \cos k \cos x}, \tag{4}$$

(resp.)

$$\widetilde{A}(b) - \widetilde{A}(a) = \int_a^b \widetilde{A}'(x)dx = \int_a^b \frac{\sinh k\, dx}{1 + \cosh k \cosh x}. \tag{$\tilde{4}$}$$

Proof The differentiation of (3) gives

$$\begin{aligned} -\sin A(x)A'(x) &= \frac{-\sin x(1+\cos k\cos x) + (\cos k + \cos x)\cos k \sin x}{(1+\cos k \cos x)^2} \\ &= \frac{-\sin x \sin^2 k}{(1+\cos k\cos x)^2}. \end{aligned} \tag{5}$$

Therefore, by (1)

$$A'(x) = \frac{\sin k}{1+\cos k \cos x}. \tag{5$'$}$$

Similar computations also show that

$$\widetilde{A}'(x) = \frac{\sinh k}{1+\cosh k\cosh x}. \tag{6}$$

2.2 The case of $S^3(1)$

In view of the fact that the *volume functions* of *spherical* (resp. *hyperbolic*) *tetrahedra* are rather complicated and transcendental (cf. [1]), one expects that the volume formulas of cones in $S^3(1)$ (resp. $H^3(-1)$) will be considerably more difficult to obtain. Although the cases of $S^3(1)$ and $H^3(-1)$ will, again, be achieved by a kind of parallel computations, it is better to present them consecutively, instead of presenting them jointly as we did in §2.1. Therefore, we shall first study the case of $S^3(1)$ in this subsection.

Let Ω be a domain inside the equatorial $S^2(1)$ and V be a point in the upper hemisphere. We shall denote the cone in $S^3(1)$ with Ω as its base and V as its vertex by $C^V(\Omega)$. In the special case that V is situated at the north pole, the volume of $C^V(\Omega)$ is simply equal to $\frac{\pi}{4}$ times the area of Ω. Thus, we shall only discuss the case that V is *not* situated at the north pole in the following:

Let O be the (unique) intersection point of the *longitude* passing through V and the equator $S^2(1)$, namely, $\overline{OV}$ is exactly the *height interval* of $C^V(\Omega)$ and its length is the height of the

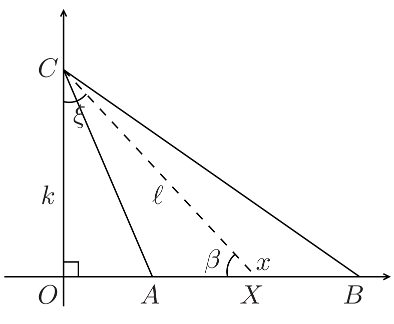


Figure 1

cone. Let (r, θ) be a chosen spherical polar coordinate system on $S^2(1)$ with O as the origin and $d\sigma$ be its "*area element*", namely, $d\sigma = \sin r\, dr \wedge d\theta$ as indicated in Figure 2. It follows from the rotational symmetry of $S^3(1)$ with $\overline{OV}$ as the axis that the volume of $C^V(d\sigma)$ is equal to a multiple of $d\sigma$ (up to second order of infinitesimal) which only depends on the height k and the distance between $d\sigma$ and O (i.e. r). In other words, there exists a function $w(k, r)$ such that

$$\operatorname{vol} C^V(d\sigma) \equiv w(k, r) d\sigma \pmod{o(d\sigma)}, \tag{7}$$

while the volume formula of $C^V(\Omega)$ that we are seeking is given by

$$\operatorname{vol} C^V(\Omega) = \int_\Omega w(k, r) \sin r\, dr \wedge d\theta. \tag{8}$$

Anyhow, the task of finding out such a volume formula lies in the determination of such a function $w(k, r)$ satisfying (7).

Lemma 2 *The function $w(k, r)$ satisfying* (7) *is given by*

$$w(k, r) = \frac{\sin k \sec^3 r}{2(\tan^2 r + \sin^2 k)^{\frac{3}{2}}} \{\cos^{-1}(\cos k \cos r) - \cos k \cos r \sqrt{1 - \cos^2 k \cos^2 r}\,\}. \tag{9}$$

Proof Set $\Gamma(r_0, dr)$ to be the set of points on $S^2(1)$ whose polar coordinates satisfying $r_0 \le r \le r_0 + dr$, as indicated by the shaded ring in Figure 2. Then

$$\operatorname{vol} C^V(\Gamma(r_0, dr)) \equiv 2\pi w(k, r_0) \sin r_0\, dr \pmod{dr^2}. \tag{10}$$

Set $\ell(r)$ to be the hypotenuse-length of the right-angle spherical triangle with k and r as its other two side-lengths and $\lambda(r)$ to be the angle opposite to the side of length r. Then

$$\cos \ell(r) = \cos k \cos r, \quad \tan \lambda(r) = \frac{\tan r}{\sin k}. \tag{11}$$

Differentiate the second equation of (11), one gets

$$\left.\frac{d\lambda}{dr}\right|_{r_0} = \frac{\cos^2 \lambda(r_0) \sec^2 r_0}{\sin k} = \frac{\sin k \sec^2 r_0}{\tan^2 r_0 + \sin^2 k}, \tag{12}$$

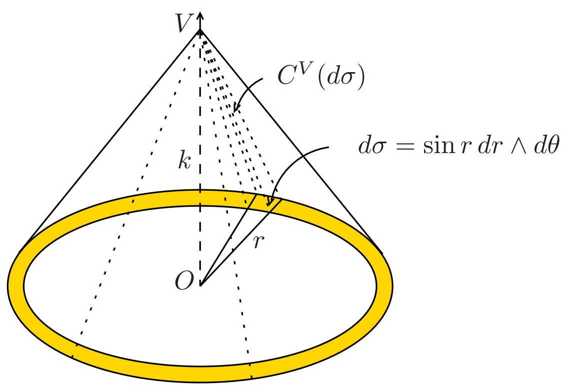


Figure 2

while the amount of solid angle at the vertex of $C^V(\Gamma(r_0, dr))$ is equal to

$$2\pi \sin\lambda(r_0) \frac{d\lambda}{dr}\bigg|_{r_0} dr = \frac{2\pi \sin k \tan r_0 \sec^2 r_0}{(\tan^2 r_0 + \sin^2 k)^{\frac{3}{2}}} dr \pmod{dr^2}. \tag{13}$$

Note that, modulo dr^2, the volume of $C^V(\Gamma(r_0, dr))$ is also equal to that of the *portion* of the spherical ball of radius $\ell(r_0)$ in $S^3(1)$ center at V with exactly the *same solid angle*, while the total volume of such a spherical ball of radius $\ell(\lambda_0)$ is equal to

$$4\pi \int_0^{\ell(r_0)} \sin^2 \rho \, d\rho = 2\pi(\ell(r_0) - \sin \ell(r_0) \cos \ell(r_0)). \tag{14}$$

Therefore, it follows from the symmetry property of spherical balls that the volume of such a portion is equal to

$$2\pi(\ell(r_0) - \sin \ell(r_0) \cos \ell(r_0)) \frac{2\pi}{4\pi} \frac{\sin k \tan r_0 \sec^2 r_0}{(\tan^2 r_0 + \sin^2 k)^{\frac{3}{2}}} dr \pmod{dr^2}. \tag{15}$$

Hence, the equality between (15) and $2\pi\omega(k, r_0) \sin r_0 \, dr$ modulo dr^2 will readily give the formula (9) for $r = r_0$ (i.e. an arbitrary given value of r). This proves (9) holds in general.

Corollary *The volume of* $C^V(\Omega)$ *is given by the following integral formula, namely*

$$\operatorname{vol} C^V(\Omega) = \int_\Omega w(k, r) \sin r \, dr \wedge d\theta, \tag{16}$$

where $w(k, r)$ *is given by* (9).

2.3 The case of $H^3(-1)$

The hyperbolic 3-space $H^3(-1)$ has the same kind of local symmetry as that of $S^3(1)$, while the *hyperbolic trigonometry* has parallel laws and formulas as that of the spherical case. Thus, it is quite straightforward to follow the same geometric ideas and to carry out the same kind of computations as that of the previous subsection, in order to derive the corresponding volume formula for cones in $H^3(-1)$.

Let Ω be a domain in a given hyperbolic plane $H^2(-1)$ and V be a point in the upper half-space. The reflection symmetry of $H^3(-1)$ with respect to the given $H^2(-1)$ maps V to V' in the lower half-space, while the unique geodesic interval $\overline{VV'}$ intersects perpendicularly with $H^2(-1)$ at its middle point, say denoted by O. Again, let (r, θ) be a chosen hyperbolic polar coordinate on $H^2(-1)$ centered at O and $d\tilde{\sigma}$ be its *area element*, $d\tilde{\sigma} = \sinh r \, dr \wedge d\theta$. It follows from the rotational symmetry of $H^3(-1)$ with the geodesic line of VV' as the axis that there exists a function $\widetilde{w}(k, r)$ such that

$$\operatorname{vol} C^V(d\tilde{\sigma}) \equiv \widetilde{w}(k, r) d\tilde{\sigma} \pmod{o(d\tilde{\sigma})}, \tag{17}$$

while the volume formula of $C^V(\Omega)$ in $H^3(-1)$ that we are seeking is given by

$$\operatorname{vol} C^V(\Omega) = \int_\Omega \widetilde{w}(k, r) \sinh r \, dr \wedge d\theta. \tag{18}$$

Lemma $\tilde{2}$ *The function $\widetilde{w}(k,r)$ for the above volume formula in $H^3(-1)$ is given by*

$$\widetilde{w}(k,r) = \frac{\sinh k\,\mathrm{sech}^3 r}{2(\tanh^2 r + \sinh^2 k)^{\frac{3}{2}}}\{\cosh^{-1}(\cosh k\cosh r) - \cosh k\cosh r\sqrt{1-\cosh^2 k\cosh^2 r}\,\}. \tag{19}$$

Proof Essentially, the proof of Lemma $\tilde{2}$ is almost the same as that of Lemma 2. One simply replaces those spherical trigonometric formulas used in the proof of Lemma 2 by their corresponding ones in hyperbolic trigonometry.

3 Volume Formulas of Cones in $S^n(1)$ (Resp. $H^n(-1)$) for $n \geq 4$

3.1 The basic geometric setting

In this section, we shall proceed to derive the volume formula of cones in both the spherical and the hyperbolic n-space for the general case of $n \geq 4$. For the sake of simplicity both in notations and presentations, we shall use the following set of notations in this section:

First of all, we shall denote $S^n(1)$ (resp. $H^n(-1)$) simply by Σ^n (resp. $\widetilde{\Sigma}^n$), and a chosen, fixed hyperplane in it by Σ^{n-1} (resp. $\widetilde{\Sigma}^{n-1}$). Let Ω be a given region in Σ^{n-1} (resp. $\widetilde{\Sigma}^{n-1}$) and V a given point in the upper half-space. Then the cone with V as its vertex and Ω as its base will again be denoted by $C^V(\Omega)$. The reflection symmetry of Σ^n (resp. $\widetilde{\Sigma}^n$) with respect to the chosen Σ^{n-1} (resp. $\widetilde{\Sigma}^{n-1}$) maps V to V' in the lower half-space. In the *unique special case* of Σ^n with V situated at the *north pole* (corresponding to Σ^{n-1} as the *equator*), one has the following very simple volume formula, namely

$$\mathrm{vol}_n\, C^V(\Omega) = \frac{\omega_{n+1}}{2\omega_n}\mathrm{vol}_{n-1}(\Omega) \quad (\text{if } V \text{ is the north pole}), \tag{20}$$

where ω_{n+1} (resp. ω_n) are the total volume of Σ^n (resp. Σ^{n-1}). Thus, we shall always assume that V is *not* situated at the north pole in the case of Σ^n. Hence, there is a *unique* shortest geodesic interval linking V to its symmetric point V', say denoted by $\overline{VV'}$, which intersects perpendicularly with Σ^{n-1} (resp. $\widetilde{\Sigma}^{n-1}$) at its middle point O. We shall denote the local isometry group of Σ^n (resp. $\widetilde{\Sigma}^n$) at V, i.e. $\mathrm{ISO}(\Sigma^n, V)$ (resp. $\mathrm{ISO}(\widetilde{\Sigma}^n, V)$), simply by $O(n)$ and its subgroup fixing all points on the geodesic line VV' simply by $O(n-1)$, which can also be identified with $\mathrm{ISO}(\Sigma^{n-1}, O)$ (resp. $\mathrm{ISO}(\widetilde{\Sigma}^{n-1}, O)$).

Set $\mathbb{R}^{n-1}$ to be the tangent space of Σ^{n-1} (resp. $\widetilde{\Sigma}^{n-1}$) at O and $S^{n-2}(1)$ to be the unit sphere in $\mathbb{R}^{n-1}$, representing those *unit* tangent vectors. The *exponential map* based at O:

$$\mathrm{Exp}: \mathbb{R}^{n-1} \to \Sigma^{n-1} \quad (\text{resp. } \widetilde{\Sigma}^{n-1}) \tag{21}$$

is bijective in the case of $\widetilde{\Sigma}^{n-1}$, while in the case of Σ^{n-1}, it maps the open ball of radius π bijectively onto Σ^{n-1} with the *antipodal* point of O deleted, but maps the whole sphere of radius π into the antipodal point of O. Anyhow, we shall use the bijective part of the above exponential map to transplant the Euclidean spherical coordinate system on $\mathbb{R}^{n-1}$ onto $\Sigma^{n-1}\setminus\{\mathrm{pt}\}$ (resp. $\widetilde{\Sigma}^{n-1}$) and simply call them the *spherical coordinate system of* Σ^{n-1} (resp. $\widetilde{\Sigma}^{n-1}$) with O as the origin. We shall denote the spherical coordinate of a point P by (ρ,ξ), in which ρ is the *distance* between O and P and ξ is the *initial direction* of the shortest geodesic interval linking O toward P, i.e. $\xi \in S^{n-2}(1)$.

Let $d\sigma$ (resp. $d\tilde{\sigma}$, dv) be the *volume element* of Σ^{n-1} (resp. $\widetilde{\Sigma}^{n-1}$, $S^{n-2}(1)$). Then, it is well known that

$$\begin{aligned} d\sigma|_{(\rho,\xi)} &= \sin^{n-2}\rho\, d\rho \wedge dv|_\xi, \\ d\tilde{\sigma}|_{(\rho,\xi)} &= \sinh^{n-2}\rho\, d\rho \wedge dv|_\xi. \end{aligned} \tag{22}$$

Moreover, it follows directly from the $O(n-1)$-*symmetry* that there exist suitable functions $w_n(k,\rho)$ (resp. $\widetilde{w}_n(k,\rho)$), $k =$ the length of $\overline{OV}$, such that

$$\begin{aligned} \operatorname{vol} C^V(d\sigma|_{(\rho,\xi)}) &\equiv w_n(k,\rho) d\sigma|_{(\rho,\xi)} \pmod{o(d\sigma|_{(\rho,\xi)})}, \\ \operatorname{vol} C^V(d\tilde{\sigma}|_{(\rho,\xi)}) &\equiv \widetilde{w}_n(k,\rho) d\tilde{\sigma}|_{(\rho,\xi)} \pmod{o(d\tilde{\sigma}|_{(\rho,\xi)})}, \end{aligned} \tag{23}$$

while the volume formulas that we are seeking will be given by

$$\begin{aligned} \operatorname{vol} C^V(\Omega) &= \int_\Omega w_n(k,\rho) \sin^{n-2}\rho\, d\rho \wedge dv, \\ \operatorname{vol} C^V(\Omega) &= \int_\Omega \widetilde{w}_n(k,\rho) \sinh^{n-2}\rho\, d\rho \wedge dv. \end{aligned} \tag{24}$$

Of course, the task here is the actual determinations of $w_n(k,\rho)$ and $\widetilde{w}_n(k,\rho)$ analytically.

3.2 The determinations of w_n and $\widetilde{w}_n$ via symmetry and the method of infinitesimal symmetrization

Let $\mathcal{S}(\rho_0, d\rho)$ (resp. $\widetilde{\mathcal{S}}(\rho_0, d\rho)$) be the *spherical shell* in Σ^{n-1} (resp. $\widetilde{\Sigma}^{n-1}$) centered at O with *thickness* $d\rho$ and *inside radius* equal to ρ_0, namely

$$\begin{aligned} &\mathcal{S}(\rho_0, d\rho) \quad (\text{resp. } \widetilde{\mathcal{S}}(\rho_0, d\rho)) \\ &= \{P(\rho,\xi);\ \rho_0 \le \rho \le \rho_0 + d\rho,\ \xi \in S^{n-2}(1)\}. \end{aligned} \tag{25}$$

Then, it is quite simple to see that

$$\begin{aligned} \operatorname{vol} C^V(\mathcal{S}(\rho_0, d\rho)) &\equiv w_n(k,\rho) \cdot \omega_{n-1} \sin^{n-2}\rho_0\, d\rho \pmod{d\rho^2}, \\ \operatorname{vol} C^V(\widetilde{\mathcal{S}}(\rho_0, d\rho)) &\equiv \widetilde{w}_n(k,\rho) \cdot \omega_{n-1} \sinh^{n-2}\rho_0\, d\rho \pmod{d\rho^2}. \end{aligned} \tag{26}$$

Set $\ell(\rho)$ (resp. $\tilde{\ell}(\rho)$) to be the hypotenuse-length of the right-angle spherical (resp. hyperbolic) triangle with k and ρ as its other two side-lengths and $\lambda(\rho)$ (resp. $\tilde{\lambda}(\rho)$) to be the angle opposite to the side of length ρ. Then, by the laws of spherical (resp. hyperbolic) trigonometry,

$$\cos\ell(\rho) = \cos k \cos\rho, \quad \cosh\tilde{\ell}(\rho) = \cosh k \cosh\rho, \tag{27}$$

$$\tan\lambda(\rho) = \frac{\tan\rho}{\sin k}, \qquad \tan\tilde{\lambda}(\rho) = \frac{\tanh\rho}{\sinh k}. \tag{28}$$

Hence, by differentiating (28) with respect to ρ,

$$\left.\frac{d\lambda}{d\rho}\right|_{\rho_0} = \frac{\sin k \sec^2\rho_0}{\tan^2\rho_0 + \sin^2 k}, \quad \left.\frac{d\tilde{\lambda}}{d\rho}\right|_{\rho_0} = \frac{\sinh k\, \mathrm{sech}^2\rho_0}{\tanh^2\rho_0 + \sinh^2 k}. \tag{29}$$

Set $B_n(\ell(\rho_0))$ (resp. $\widetilde{B}_n(\tilde{\ell}(\rho_0))$) to be the ball of radius $\ell(\rho_0)$ (resp. $\tilde{\ell}(\rho_0)$) in Σ^n (resp. $\widetilde{\Sigma}^n$) centered at V. Then, it is not difficult to see that the volume of the portion of $B(\ell(\rho_0))$ (resp.

$\widetilde{B}(\tilde{\ell}(\rho_0)))$ inside of the *solid angle cone* spanned by $\mathcal{S}(\rho_0, d\rho)$ (resp. $\widetilde{\mathcal{S}}(\rho_0, d\rho)$) and the volume of $C^V(\mathcal{S}(\rho_0, d\rho))$ (resp. $C^V(\widetilde{\mathcal{S}}(\rho_0, d\rho)))$ are equal to each other modulo $d\rho^2$. Moreover, the total volumes of $B_n(\ell(\rho_0))$ (resp. $\widetilde{B}_n(\tilde{\ell}(\rho_0)))$ can be computed by the following integrations, namely

$$\begin{aligned}\operatorname{vol} B_n(\ell(\rho_0)) &= \omega_n \int_0^{\ell(\rho_0)} \sin^{n-1}\rho\, d\rho,\\ \operatorname{vol} \widetilde{B}_n(\tilde{\ell}(\rho_0)) &= \omega_n \int_0^{\tilde{\ell}(\rho_0)} \sinh^{n-1}\rho\, d\rho,\end{aligned} \tag{30}$$

(cf. Lemma 3 for their explicit (i.e. integrated) formula), while the volume of such a portion is proportionate to the *amount of its solid angle* at the center V, which is given by

$$\begin{aligned}&\omega_{n-1}\sin^{n-2}\lambda(\rho_0)\frac{d\lambda}{d\rho}\bigg|_{\rho_0} d\rho \equiv \frac{\omega_{n-1}\sin k \tan^{n-2}\rho_0 \sec^2\rho_0\, d\rho}{(\tan^2\rho_0+\sin^2 k)^{\frac{n}{2}}} \qquad (\mathrm{mod}\ d\rho^2),\\ \text{(resp.)}\quad &\omega_{n-1}\sin^{n-2}\tilde{\lambda}(\rho_0)\frac{d\tilde{\lambda}}{d\rho}\bigg|_{\rho_0} d\rho \equiv \frac{\omega_{n-1}\sinh k \tanh^{n-2}\rho_0\, \mathrm{sech}^2\rho_0\, d\rho}{(\tanh^2\rho_0+\sinh^2 k)^{\frac{n}{2}}} \qquad (\mathrm{mod}\ d\rho^2).\end{aligned} \tag{31}$$

Lemma 3 *Set $\psi_{n,0}(r)$ (resp. $\tilde{\psi}_{n,0}(r)$) to be the volume of the ball of radius r in Σ^n (resp. $\widetilde{\Sigma}^n$). Then $\psi_{n,0}(r) = \omega_n J_n$, $\tilde{\psi}_{n,0}(r) = \omega_n \widetilde{J}_n$, where J_n and $\widetilde{J}_n$ are given by the following formulas:*

$$\begin{aligned}J_{2k} &= \sum_{i=0}^{k-1}(-1)^i\frac{1}{2i+1}\binom{k-1}{i} - \sum_{i=0}^{k-1}(-1)^i\frac{1}{2i+1}\binom{k-1}{i}\cos^{2i+1} r,\\ \widetilde{J}_{2k} &= \sum_{i=0}^{k-1}(-1)^{k-i-1}\binom{k-1}{i}\frac{1}{2i+1}\cosh^{2i+1} r - \sum_{i=0}^{k-1}(-1)^{k-i-1}\frac{1}{2i+1}\binom{k-1}{i},\\ J_{2k+1} &= \binom{k-\frac12}{k} r - \frac{\cos r}{2k}\bigg\{\sin^{2k-1} r + \frac{2k-1}{2k-2}\sin^{2k-3} r\\ &\quad + \frac{(2k-1)(2k-3)}{(2k-2)(2k-4)}\sin^{2k-5} r + \cdots + \binom{k-\frac12}{k-1}\sin r\bigg\},\\ \widetilde{J}_{2k+1} &= (-1)\binom{k-\frac12}{k} r + \frac{\cosh r}{2k}\bigg\{\sinh^{2k-1} r - \frac{2k-1}{2k-2}\sinh^{2k-3} r\\ &\quad + \frac{(2k-1)(2k-3)}{(2k-2)(2k-4)}\sinh^{2k-5} r + \cdots + (-1)^{k-1}\binom{k-\frac12}{k-1}\sinh r\bigg\}.\end{aligned} \tag{32}$$

Proof Set

$$J_n = \int_0^r \sin^{n-1}\rho\, d\rho, \quad \widetilde{J}_n = \int_0^r \sinh^{n-1}\rho\, d\rho. \tag{33}$$

Then the above formulas of (32) are simply the results of explicit integrations. For example

$$\begin{aligned}J_{2k} &= \int_0^r \sin^{2k-1}\rho\, d\rho = \int_{\cos r}^1 (1-u^2)^{k-1} du \quad (u=\cos\rho)\\ &= \bigg\{\sum_{i=0}^{k-1}(-1)^i\frac{1}{2i+1}\binom{k-1}{i}u^{2i+1}\bigg\}\bigg|_{\cos r}^1,\end{aligned} \tag{34}$$

which verifies $(32)_1$. Similarly, one verifies $(32)_2$ as follows:

$$\begin{aligned}\widetilde{J}_{2k} &= \int_0^r \sinh^{2k-1}\rho\,d\rho = \int_0^r (\cosh^2\rho - 1)^{k-1}\sinh\rho\,d\rho \\ &= \int_1^{\cosh r}(u^2-1)^{k-1}du \quad (u = \cosh\rho) \\ &= \left\{\sum_{i=0}^{k-1}(-1)^{k-i-1}\binom{k-1}{i}\frac{u^{2i+1}}{2i+1}\right\}\Bigg|_1^{\cosh r}. \end{aligned} \tag{35}$$

Applying integration by parts to J_{2k+1} and $\widetilde{J}_{2k+1}$, one gets

$$\begin{aligned} J_{2k+1} &= \int_0^r \sin^{2k}\rho\,d\rho = -\sin^{2k-1} r\cos r + (2k-1)\int_0^r \sin^{2k-2}\rho\cos^2\rho\,d\rho \\ &= -\sin^{2k-1} r\cos r + (2k-1)J_{2k-1} - (2k-1)J_{2k+1} \end{aligned}$$

$$\Rightarrow\ J_{2k+1} = \frac{1}{2k}(-\sin^{2k-1} r\cos r) + \frac{2k-1}{2k}J_{2k-1}, \tag{36}$$

$$\widetilde{J}_{2k+1} = \int_0^r \sinh^{2k}\rho\,d\rho = \sinh^{2k-1} r\cosh r - (2k-1)\int_0^r \sinh^{2k-2}\rho\cosh^2\rho\,d\rho$$

$$\Rightarrow\ \widetilde{J}_{2k+1} = \frac{1}{2k}\sinh^{2k-1} r\cosh r - \frac{2k-1}{2k}\widetilde{J}_{2k-1}. \tag{37}$$

Formulas $(32)_3$ and $(32)_4$ follow directly from the recursive formulas of (36) and (37) respectively.

Theorem 1 *The volume of a cone $C^V(\Omega)$ in Σ^n, $n \geq 3$, is given by*

$$\mathrm{vol}\,(C^V(\Omega)) = \int_\Omega w_n(k,\rho)\sin^{n-2}\rho\,d\rho\wedge dv, \tag{38}$$

where

$$w_n(k,\rho) = \frac{1}{\omega_n}\psi_{n,0}(\ell(\rho))\frac{\sin k\sec^n\rho}{(\tan^2\rho+\sin^2 k)^{\frac{n}{2}}} \tag{38$'$}$$

and $\psi_{n,0}(\ell(\rho))$ is given by Lemma 3 and (27).

The volume of a cone $C^V(\Omega)$ in $\widetilde{\Sigma}_n$, $n \geq 3$, is given by

$$\mathrm{vol}\,(C^V(\Omega)) = \int_\Omega \widetilde{w}_n(k,\rho)\sinh^{n-2}\rho\,d\rho\wedge dv, \tag{39}$$

where

$$\widetilde{w}_n(k,\rho) = \frac{1}{\omega_n}\tilde{\psi}_{n,0}(\tilde{\ell}(\rho))\frac{\sinh k\,\mathrm{sech}^n\rho}{(\tanh^2\rho+\sinh^2 k)^{\frac{n}{2}}} \tag{39$'$}$$

and $\tilde{\psi}_{n,0}(\tilde{\ell}(\rho))$ is given by Lemma 3 and (27).

Proof The volume of the portion of $B_n(\ell(\rho_0))$ inside of the solid angle cone of $C^V(\mathcal{S}(\rho_0, d\rho))$ is proportionate to the amount of its solid angle given by (31). Therefore

$$\begin{aligned} & w_n(k,\rho_0)\omega_{n-1}\sin^{n-2}\rho_0\,d\rho \\ &\equiv \psi_{n,0}(\ell(\rho_0))\cdot\frac{\omega_{n-1}}{\omega_n}\cdot\frac{\sin k\tan^{n-2}\rho_0\sec^2\rho_0}{(\tan^2\rho_0+\sin^2 k)^{\frac{n}{2}}}d\rho \quad (\mathrm{mod}\ d\rho^2) \end{aligned} \tag{40}$$

and hence (38′) and (38) follows readily.

The proof of (39′) and (39) is completely parallel to that of the spherical case, namely, it follows from

$$\begin{aligned}&\widetilde{w}_n(k,\rho_0)\omega_{n-1}\sinh^{n-2}\rho_0\,d\rho\\ \equiv\ &\tilde{\psi}_{n,0}(\tilde{\ell}(\rho_0))\cdot\frac{\omega_{n-1}}{\omega_n}\cdot\frac{\sinh k\tanh^{n-2}\rho_0\,\mathrm{sech}^2\rho_0}{(\tanh^2\rho_0+\sinh^2 k)^{\frac{n}{2}}}d\rho\quad(\mathrm{mod}\ d\rho^2).\end{aligned}\tag{$\widetilde{40}$}$$

4 The Volume Functions of Orthospindles

In the most symmetric situation that the base-region Ω is a ball of radius r center at O, $C^V(\Omega)$ is $O(n-1)$-symmetric and one naturally expects that the volumes of such cones should have a nice formula. We shall call this family of $O(n-1)$-symmetric cones *orthospindles* and apply the results of §2 and §3 to compute their volume functions.

An orthospindle in Σ^n (resp. $\widetilde{\Sigma}^n$) is uniquely determined, up to congruence, by its *height* k and the *radius* r of its base. We shall denote the volume of such an orthospindle in Σ^n (resp. $\widetilde{\Sigma}^n$) by $\psi_{n,1}(k,r)$ (resp. $\tilde{\psi}_{n,1}(k,r)$). Let $\ell(\rho)$, $\tilde{\ell}(\rho)$, $\lambda(\rho)$ and $\tilde{\lambda}(\rho)$ be the same as in §3.2. Then

$$\mathrm{vol}\,C^V(\mathcal{S}(\rho_0,d\rho))\equiv\psi_{n,0}(\ell(\rho_0))\frac{\omega_{n-1}}{\omega_n}\sin^{n-2}\lambda(\rho_0)\left.\frac{d\lambda}{d\rho}\right|_{\rho_0}d\rho\quad(\mathrm{mod}\ d\rho^2),\tag{41}$$

$$\mathrm{vol}\,C^V(\widetilde{\mathcal{S}}(\rho_0,d\rho))\equiv\tilde{\psi}_{n,0}(\tilde{\ell}(\rho_0))\frac{\omega_{n-1}}{\omega_n}\sin^{n-2}\tilde{\lambda}(\rho_0)\left.\frac{d\tilde{\lambda}}{d\rho}\right|_{\rho_0}d\rho\quad(\mathrm{mod}\ d\rho^2).\tag{$\widetilde{41}$}$$

Therefore, it suffices to compute the integrals over the interval of $0\le\rho\le r$ with the RHS of (41) (resp. ($\widetilde{41}$)) as the integrands. Moreover, it is natural to use λ instead of ρ for the computation of such integrals, namely

Lemma 4 *Let $\psi_{n,1}(k,r)$ (resp. $\tilde{\psi}_{n,1}(k,r)$) be the volumes of the orthospindles in Σ^n (resp. $\widetilde{\Sigma}^n$) with k as the height and r as the base-radius. Set*

$$\begin{aligned}&\lambda_0=\tan^{-1}\frac{\tan r}{\sin k}\qquad\Big(\textit{resp.}\ \tilde{\lambda}=\tan^{-1}\frac{\tanh r}{\sinh k}\Big),\\ &\ell(\lambda)=\tan^{-1}(\tan k\sec\lambda)\quad(\textit{resp.}\ \tilde{\ell}(\lambda)=\tanh^{-1}(\tanh k\sec\lambda)).\end{aligned}\tag{42}$$

Then

$$\begin{aligned}\psi_{n,1}(k,r)&=\frac{\omega_{n-1}}{\omega_n}\int_0^{\lambda_0}\psi_{n,0}(\ell(\lambda))\sin^{n-2}\lambda\,d\lambda,\\ \tilde{\psi}_{n,1}(k,r)&=\frac{\omega_{n-1}}{\omega_n}\int_0^{\tilde{\lambda}_0}\tilde{\psi}_{n,0}(\tilde{\ell}(\lambda))\sin^{n-2}\lambda\,d\lambda.\end{aligned}\tag{43}$$

Proof Direct consequence of (41) and ($\widetilde{41}$), and the change of variables from ρ to λ.

Next let us proceed to actually compute the above integrals, beginning with the case of $n=3$.

Case 1 $n = 3$.

$$\begin{aligned}\psi_{3,1}(k,r) &= \frac{1}{2}\int_0^{\lambda_0} \psi_{3,0}(\ell(\lambda))\sin\lambda\, d\lambda \\ &= \pi\int_0^{\lambda_0}\Big\{\tan^{-1}\frac{\tan k}{\cos\lambda} - \frac{\tan k\cos\lambda}{\cos^2\lambda+\tan^2 k}\Big\}\sin\lambda\, d\lambda \\ &= \pi\Big(-\cos\lambda\tan^{-1}\frac{\tan k}{\cos\lambda}\Big)\Big|_0^{\lambda_0} = \pi(k-\cos\lambda_0\,\ell(\lambda_0)),\end{aligned} \tag{44}$$

and similarly for the hyperbolic case, namely

$$\begin{aligned}\tilde\psi_{3,1}(k,r) &= \frac{1}{2}\int_0^{\tilde\lambda_0} \tilde\psi_{3,0}(\tilde\ell(\lambda))\sin\lambda\, d\lambda \\ &= \pi\int_0^{\tilde\lambda_0}\Big\{\frac{\tanh k\cos\lambda}{\cos^2\lambda-\tanh^2 k} - \tanh^{-1}\frac{\tanh k}{\cos\lambda}\Big\}\sin\lambda\, d\lambda \\ &= \pi\Big(\cos\lambda\tanh^{-1}\frac{\tanh k}{\cos\lambda}\Big)\Big|_0^{\tilde\lambda_0} = \pi(\cos\tilde\lambda_0\,\tilde\ell(\tilde\lambda_0)-k).\end{aligned} \tag{45}$$

Case 2 $n = 2m$, $m \geq 2$.

Let us first discuss the spherical subcase:

$$\begin{aligned}\psi_{2m,1}(k,r) &= \frac{\omega_{2m-1}}{\omega_{2m}}\int_0^{\lambda_0} \psi_{2m,0}(\ell(\lambda))\sin^{2m-2}\lambda\, d\lambda \\ &= \omega_{2m-1}\int_0^{\lambda_0} J_{2m}(\ell(\lambda))\sin^{2m-2}\lambda\, d\lambda,\end{aligned} \tag{46}$$

where

$$J_{2m}(\ell(\lambda)) = \sum_{i=0}^{m-1}\frac{(-1)^i}{2i+1}\binom{m-1}{i} - \sum_{i=0}^{m-1}\frac{(-1)^i}{2i+1}\binom{m-1}{i}\Big(\frac{\cos k\cos\lambda}{\sqrt{1-\cos^2 k\sin^2\lambda}}\Big)^{2i+1}. \tag{46$'$}$$

Therefore, it suffices to compute the integration of each term, namely

$$\int_0^{\lambda_0}\Big(\frac{\cos k\cos\lambda}{\sqrt{1-\cos^2 k\sin^2\lambda}}\Big)^{2i+1}\sin^{2m-2}\lambda\, d\lambda, \quad 0\leq i\leq m-1. \tag{47$_i$}$$

Set $\sin u = \cos k\sin\lambda$. Then

$$du = \frac{\cos k\cos\lambda\, d\lambda}{\sqrt{1-\cos^k\sin^2\lambda}}, \quad \sin u_0 = \cos k\sin\lambda_0 = \frac{\cos k\sin r}{\sqrt{1-\cos^2 k\cos^2 r}}. \tag{48}$$

Thus, direct substitution and algebraic computation will show that the integral of $(47)_i$ is equal to the following, namely

$$\int_0^{u_0}\tan^{2i}u(\sec^2 k\sin^2 u)^{m-i-1}(1-\sec^2 k\sin^2 u)^i du, \quad 0\leq i\leq m-1. \tag{47$'_i$}$$

Therefore, the computations of the above collection of integrals can easily be reduced to that of the following, namely

$$F_{i,j}(u_0) = \int_0^{u_0}\tan^{2i}u\sin^{2j}u\, du. \tag{49}$$

In case both i and j are at least 1, one has

$$F_{i,j}(u_0) = \int_0^{u_0} \tan^{2(i-1)} u(\sec^2 u - 1)\sin^{2j} u\,du = F_{i,j-1} - F_{i-1,j}. \tag{50}$$

Hence, it is easy to write down $F_{i,j}$ with positive i, j as an integer linear combination of thsoe F's with either i or j being zero. Moreover,

$$\begin{aligned} F_{0,j}(u_0) &= \int_0^{u_0} \sin^{2j} u\,du = J_{2j+1}(u_0) \quad \text{(cf. Lemma 3)}, \\ F_{i,0}(u_0) &= \int_0^{u_0} \tan^{2i} u\,du = \int_0^{u_0} \tan^{2(i-1)} u \sec^2 u\,du - F_{i-1,0}(u_0) \\ &= \frac{1}{2i-1}\tan^{2i-1} u_0 - F_{i-1,0}(u_0). \end{aligned} \tag{51}$$

Therefore, $\psi_{2m,1}(k,r)$ can be explicitly expressed as a linear combination of $\{J_{2j+1}(u_0)$ and $\tan^{2i-1} u_0,\ i,j \le n\}$, with trigonometric functions of k as the coefficients.

Next, let us consider the other subcase of $\tilde{\psi}_{2m,1}(k,r)$:

$$\tilde{\psi}_{2m,1}(k,r) = \frac{\omega_{2m-1}}{\omega_{2m}} \int_0^{\lambda_0} \tilde{\psi}_{2m,0}(\tilde{\ell}(\lambda))\sin^{2m-2}\lambda\,d\lambda = \omega_{2m-1}\int_0^{\lambda_0} \widetilde{J}_{2m}(\tilde{\ell}(\lambda))\sin^{2m-2}\lambda\,d\lambda, \tag{52}$$

where

$$\widetilde{J}_{2m}(\tilde{\ell}(\lambda)) = \sum_{i=0}^{m-1} \frac{(-1)^{m-i-1}}{2i+1}\binom{m-1}{i}\cosh^{2i+1}\tilde{\ell}(\lambda) - \sum_{i=0}^{m-1}\frac{(-1)^{m-i-1}}{2i+1}\binom{m-1}{i}. \tag{52$'$}$$

Note that

$$\cosh\tilde{\ell}(\lambda) = \frac{\cos\lambda}{\sqrt{\cos^2\lambda - \tanh^2 k}} = \frac{\cosh k\cos\lambda}{\sqrt{1-\cosh^2 k\sin^2\lambda}}. \tag{53}$$

Therefore, the only difference between

$$\int_0^{\lambda_0}\left(\frac{\cosh k\cos\lambda}{\sqrt{1-\cosh^2 k\sin^2\lambda}}\right)^{2i+1}\sin^{2(m-1)}\lambda\,d\lambda, \quad 0 \le i \le m-1 \tag{53$_i$}$$

and that of $(47)_i$ is that $\cos k$ have been replaced by $\cosh k$. Hence, the substitution $\sin u = \cosh k \sin\lambda$ will again transform $(53)_i$ to

$$\int_0^{\tilde{u}_0} \tan^{2i} u(\operatorname{sech}^2 k\sin^2 u)^{m-i-1}(1 - \operatorname{sech}^2 k\sin^2 u)^i du, \quad 0 \le i \le m-1. \tag{53$'_i$}$$

Thus, the hyperbolic subcase is, indeed, parallel to the spherical subcase.

Case 3 $n = 2m+1$, $m \ge 2$.

Again, let us first discuss the *spherical* subcase as follows:

$$\psi_{2m+1,1}(k,r) = \omega_{2m}\int_0^{\lambda_0} J_{2m+1}(\ell(\lambda))\sin^{2m-1}\lambda\,d\lambda, \tag{54}$$

where

$$J_{2m+1}(\ell(\lambda)) = \binom{m-\frac{1}{2}}{m}\ell(\lambda) - \frac{\cos\ell(\lambda)}{2m}\Big\{\sin^{2m-1}\ell(\lambda) + \frac{2m-1}{2m-2}\sin^{2m-3}\ell(\lambda)$$
$$+\frac{(2m-1)(2m-3)}{(2m-2)(2m-4)}\sin^{2m-5}\ell(\lambda) + \cdots + \binom{m-\frac{1}{2}}{m-1}\sin\ell(\lambda)\Big\}, \tag{54$'$}$$

$$\ell(\lambda) = \tan^{-1}\frac{\tan k}{\cos\lambda}, \quad \cos\ell(\lambda) = \frac{\cos k\cos\lambda}{\sqrt{1-\cos^2 k\sin^2\lambda}}, \quad \sin\ell(\lambda) = \frac{\sin k}{\sqrt{1-\cos^2 k\sin^2\lambda}}.$$

Thus, it suffices to compute the following integrals separately, namely

$$\int_0^{\lambda_0}\tan^{-1}\frac{\tan k}{\cos\lambda}\sin^{2m-1}\lambda\,d\lambda, \tag{55}$$

$$\int_0^{\lambda_0}\cos\ell(\lambda)\sin^{2i+1}\ell(\lambda)\sin^{2m-1}\lambda\,d\lambda, \quad 0\le i\le m-1. \tag*{$(56)_i$}$$

Apply integration by parts to the integral of (55), one has

$$\int_0^{\lambda_0}\tan^{-1}\frac{\tan k}{\cos\lambda}\sin^{2m-1}\lambda\,d\lambda$$
$$= \left\{\tan^{-1}\frac{\tan k}{\cos\lambda}\sum_{i=0}^{m-1}\frac{(-1)^{i+1}}{2i+1}\binom{m-1}{i}\cos^{2i+1}\lambda\right\}\Bigg|_0^{\lambda_0}$$
$$+\sum_{i=0}^{m-1}\frac{(-1)^i}{2i+1}\binom{m-1}{i}\frac{\tan k}{2}\int_0^{\lambda_0}\frac{\cos^{2i}\lambda\cdot 2\cos\lambda\sin\lambda\,d\lambda}{\cos^2\lambda+\tan^2 k}. \tag{57}$$

Set $\cos^2\lambda+\tan^2 k = u$, $du = -2\cos\lambda\sin\lambda\,d\lambda$. Then

$$\int_0^{\lambda_0}\frac{\cos^{2i}\lambda\cdot 2\cos\lambda\sin\lambda\,d\lambda}{\cos^2\lambda+\tan^2 k} = \int_{\tan^2 k+\cos^2\lambda_0}^{\sec^2 k}\frac{1}{u}(u-\tan^2 k)^i du$$
$$= \left\{\sum_{j=0}^{i-1}\binom{i}{j}(-1)^j\tan^{2j}k\frac{u^{(i-j)}}{(i-j)} + (-1)^i\tan^{2i}k\ln u\right\}\Bigg|_{\tan^2 k+\cos^2\lambda_0}^{\sec^2 k}. \tag{58}$$

Next let us compute the integrals of $(56)_i$, $0\le i\le m-1$.

$$\int_0^{\lambda_0}\cos\ell(\lambda)\sin^{2i+1}\ell(\lambda)\sin^{2m-1}\lambda\,d\lambda$$
$$= \int_0^{\lambda_0}\frac{\cos\lambda\tan^{2i+1}k\sin^{2m-1}\lambda\,d\lambda}{(\cos^2\lambda+\tan^2 k)^{i+1}}$$
$$= \frac{\tan^{2i+1}k}{2}\int_{\tan^2 k+\cos^2\lambda_0}^{\sec^2 k}\frac{1}{u^{i+1}}(\sec^2 k-u)^{m-1}du$$
$$= \frac{\tan^{2i+1}k}{2}\Big\{\sum_{j\ne i}(-1)^j\binom{m-1}{j}\sec^{2(m-j-1)}k\frac{u^{(j-i)}}{(j-i)}$$
$$+(-1)^i\binom{m-1}{i}\sec^{2(m-i-1)}k\ln u\Big\}\Bigg|_{\tan^2 k+\cos^2\lambda_0}^{\sec^2 k}. \tag{59}$$

Specially, we list the explicit formula of $\psi_{3,1}(k,r)$ as follows

$$\begin{aligned}\psi_{3,1}(k,r) &= \pi(k-\cos\lambda_0\,\ell(\lambda_0))\\ &= \pi\left(k-\frac{\sin k}{\sqrt{\tan^2 r+\sin^2 k}}\tan^{-1}\frac{\tan r\sqrt{\tan^2 r+\sin^2 k}}{\sin k}\right).\end{aligned} \tag{60}$$

Remark As it turns out, it is always simpler to express $\psi_{n,1}(k,r)$ (resp. $\tilde{\psi}_{n,1}(k,r)$) as functions of k and $\lambda_0=\tan^{-1}\frac{\tan r}{\sin k}$ instead of k and r. Therefore, from now on, we shall denote the volume of spherical (resp. hyperbolic) orthospindles either by $\psi_{n,1}(k,r)$ (resp. $\tilde{\psi}_{n,1}(k,r)$) or by $\varphi_{n,1}(k,\lambda_0)$ (resp. $\tilde{\varphi}_{n,1}(k,\lambda_0)$).

5 Volume Formulas of Orthogonal Double Cones in $S^n(1)$ (Resp. $H^n(-1)$) for $n\geq 4$

In the study of various geometric problems in both the spherical geometry and hyperbolic geometry, cones constitutes a useful family of simple, basic objects, and suitably chosen decompositions of more general geometric objects into cones often provide some advantageous ways of computing, understanding or estimating their volumes. However, in the case of higher dimensions (i.e. $n>3$), it is often necessary to apply such suitably chosen cone-decomposition several times in order to obtain pieces with certain kind of technical simplicity. Therefore, it is useful to extend the volume formula of cones to that of *orthogonal multiple cones*:

Orthogonal double cones: Let Σ^n be either $S^n(1)$ or $H^n(-1)$, Σ^{n-1} be a given hyperplane in Σ^n, and Σ^{n-2} be a given hyperplane of Σ^{n-1}. Let Ω be a given region in Σ^{n-2} and V_1 (resp. V_2) be given points of $\Sigma^{n-1}\setminus\Sigma^{n-2}$ (resp. $\Sigma^n\setminus\Sigma^{n-1}$) such that $\overline{V_1V}_2$ is a *shortest parth* between V_2 and Σ^{n-1}. Then $C^{V_1}(\Omega)$ is a cone in Σ^{n-1} and $C^{V_2}(C^{V_1}(\Omega))$ is called an *orthogonal double cone* with Ω as its base and V_1, V_2 as its successive vertices. (It is called "orthogonal" because $\overline{V_1V}_2$ is orthogonal to Σ^{n-1}.)

Orthogonal ℓ-multiple cones: Let Σ^ℓ and $\Sigma^{n-\ell}$ be a pair of ℓ-dimensional and $(n-\ell)$-dimensional subspaces in Σ^n, *normal* to each other at O. Let $O\in\Sigma^1\subset\Sigma^2\subset\cdots\subset\Sigma^\ell$ be a "*flag*" in Σ^ℓ at O and $\{V_i,\ 1\leq i\leq\ell\}$ be a chosen sequence of points such that $V_i\in\Sigma^i\setminus\Sigma^{i-1}$ and $\overline{V_iV}_{i-1}$ is the shortest path from V_i to Σ^{i-1}, $1\leq i\leq\ell$. Then, to a given region $\Omega\subset\Sigma^{n-\ell}$, the following successive cone constructions

$$C^{V_1}(\Omega),\ C^{V_2}(C^{V_1}(\Omega)),\cdots,C^{V_\ell}(C^{V_{\ell-1}}(\cdots(C^{V_2}(C^{V_1}(\Omega)))\cdots))$$

provide an orthogonal ℓ-multiple cone with Ω as its *base* and $\{\overline{OV}_1,\overline{V_1V}_2,\cdots,\overline{V_{\ell-1}V}_\ell\}$ as its *successive height-intervals* while their lengths are called its *successive heights*, say denoted by $\{k_i,\ 1\leq i\leq\ell\}$. We shall denote the above multiple cone by $C^{V_\ell,\cdots,V_1}(\Omega)$.

In the well-known simple case of $\mathbb{E}^n$ (i.e. Euclidean n-space) the volume of such an orthogonal ℓ-multiple cone is equal to the product of its ℓ height and the $(n-\ell)$-dimensional volume of its base *divided* by $n(n-1)\cdots(n-\ell+1)$. In the study of problems of spherical and hyperbolic geometries of higher dimensions, extensions of the volume formulas of cones to that of orthogonal multiple cones will certainly provide a set of powerful tools of computing or estimating volumes, which often constitutes a major difficulty in solving many higher dimensional spherical or hyperbolic geometric problems.

In this section, we shall generalize the volume formulas of cones to the next case of orthogonal double cones. Here, we shall use (ρ, ξ) to denote the spherical coordinates on Σ^{n-2} (resp. $\widetilde{\Sigma}^{n-2}$) with O as the origin (cf. §3 for the spherical coordinate system on Σ^{n-1} (resp. $\widetilde{\Sigma}^{n-1}$)). Let $d\sigma$ (resp. $d\tilde{\sigma}$, dv) be the *volume element* of Σ^{n-2} (resp. $\widetilde{\Sigma}^{n-2}$, $S^{n-3}(1)$). Then, we have

$$\begin{aligned} d\sigma|_{(\rho,\xi)} &= \sin^{n-3}\rho\, d\rho \wedge dv|_{\xi}, \\ d\tilde{\sigma}|_{(\rho,\xi)} &= \sinh^{n-3}\rho\, d\rho \wedge dv|_{\xi}. \end{aligned} \tag{61}$$

Let $O(n-2)$ be the subgroup of isometries fixing all points of the Σ^2 (resp. $\widetilde{\Sigma}^2$) normal to Σ^{n-2} (resp. $\widetilde{\Sigma}^{n-2}$) at O. Then, it follows directly from the $O(n-2)$-invariance that there again exist suitable functions $w_{n,2}(k_1,k_2,\rho)$ (resp. $\widetilde{w}_{n,2}(k_1,k_2,\rho)$) such that

$$\begin{aligned} \operatorname{vol} C^{V_2,V_1}(d\sigma) &\equiv w_{n,2}(k_1,k_2,\rho) d\sigma|_{(\rho,\xi)} \quad (\mathrm{mod}\ o(d\sigma|_{(\rho,\xi)})), \\ \operatorname{vol} C^{V_2,V_1}(d\tilde{\sigma}) &\equiv \widetilde{w}_{n,2}(k_1,k_2,\rho) d\sigma|_{(\rho,\xi)} \quad (\mathrm{mod}\ o(d\tilde{\sigma}|_{(\rho,\xi)})), \end{aligned} \tag{62}$$

while the volume formula of orthogonal double cones will be given by

$$\begin{aligned} \operatorname{vol} C^{V_2,V_1}(\Omega) &= \int_{\Omega} w_{n,2}(k_1,k_2,\rho) \sin^{n-3}\rho\, d\rho \wedge dv, \\ \operatorname{vol} C^{V_2,V_1}(\Omega) &= \int_{\Omega} \widetilde{w}_{n,2}(k_1,k_2,\rho) \sinh^{n-3}\rho\, d\rho \wedge dv. \end{aligned} \tag{63}$$

Set $\mathcal{S}(\rho_0, d\rho)$ (resp. $\widetilde{\mathcal{S}}(\rho_0, d\rho)$) to be the *spherical shell* in Σ^{n-2} (resp. $\widetilde{\Sigma}^{n-2}$)) centered at O with $d\rho$ as its thickness and ρ_0 as its inside radius (cf. §3.2). Then, it again follows from the $O(n-2)$-symmetry that

$$\begin{aligned} \operatorname{vol} C^{V_2,V_1}(\mathcal{S}(\rho_0, d\rho)) &\equiv w_{n,2}(k_1,k_2,\rho_0)\omega_{n-2} \sin^{n-3}\rho_0\, d\rho \quad (\mathrm{mod}\ d\rho^2), \\ \operatorname{vol} C^{V_2,V_1}(\widetilde{\mathcal{S}}(\rho_0, d\rho)) &\equiv \widetilde{w}_{n,2}(k_1,k_2,\rho_0)\omega_{n-2} \sinh^{n-3}\rho_0\, d\rho \quad (\mathrm{mod}\ d\rho^2) \end{aligned} \tag{64}$$

on the one hand, and on the other hand, it follows from the method of infinitesimal symmetrization, the volume of the above double cones of $\mathcal{S}(\rho_0, d\rho)$ (resp. $\widetilde{\mathcal{S}}(\rho_0, d\rho)$) is equal to that of the *portion* of the spindle $C^{V_2}(B_{n-1}(\ell(\rho_0)))$ (resp. $C^{V_2}(B_{n-1}(\tilde{\ell}(\rho_0))))$ with the *same span of solid angle along* $\overline{V_1V}_2$ as that of the above double cones, *modulo* $d\rho^2$, namely

Lemma 5 *Set*

$$\begin{aligned} \ell(\rho_0) &= \cos^{-1}(\cos k_1 \cos \rho_0), \qquad & \lambda(\rho_0) &= \tan^{-1}\frac{\tan \rho_0}{\sin k_1}, \\ \tilde{\ell}(\rho_0) &= \cosh^{-1}(\cosh k_1 \cosh \rho_0), \qquad & \tilde{\lambda}(\rho_0) &= \tan^{-1}\frac{\tanh \rho_0}{\sinh k_1}. \end{aligned} \tag{65}$$

Then, modulo $d\rho^2$, the volumes of $C^{V_2,V_1}(\mathcal{S}(\rho_0, d\rho))$ (resp. $C^{V_2,V_1}(\widetilde{\mathcal{S}}(\rho_0, d\rho))$) are also equal to

$$\begin{aligned} &\psi_{n,1}(k_2, \ell(\rho_0)) \cdot \frac{\omega_{n-2}}{\omega_{n-1}} \sin^{n-3}\lambda(\rho_0) \frac{d\lambda}{d\rho}\bigg|_{\rho_0} d\rho, \\ (\mathit{resp.}) \quad &\tilde{\psi}_{n,1}(k_2, \tilde{\ell}(\rho_0)) \cdot \frac{\omega_{n-2}}{\omega_{n-1}} \sin^{n-3}\tilde{\lambda}(\rho_0) \frac{d\tilde{\lambda}}{d\rho}\bigg|_{\rho_0} d\rho, \end{aligned} \tag{66}$$

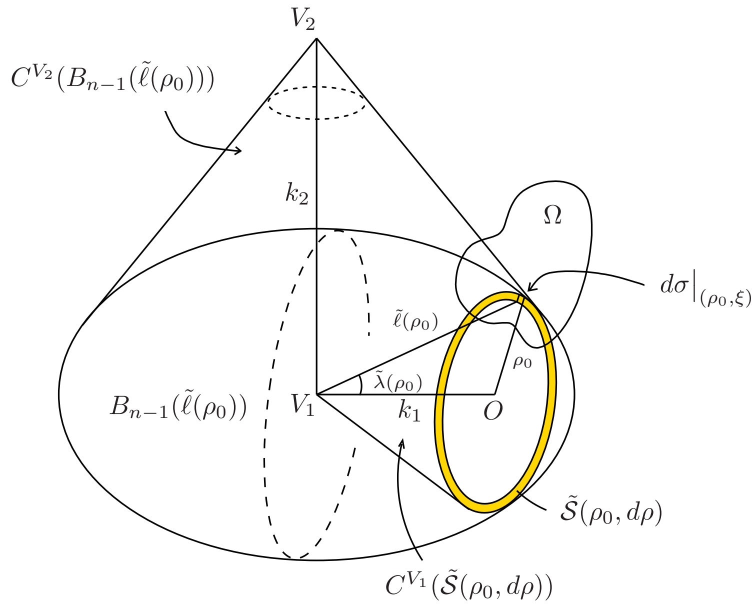


Figure 3

where $\psi_{n,1}(k_2, \ell(\rho_0))$ *(resp.* $\tilde{\psi}_{n,1}(k_2, \tilde{\ell}(\rho_0))$*) are the volumes of the orthospindles in* Σ^n *(resp.* $\widetilde{\Sigma}^n$*) with* k_2 *as the height and* $\ell(\rho_0)$ *(resp.* $\tilde{\ell}(\rho_0)$*) as the base-radii.*

Theorem 2 *The volume of* $C^{V_2,V_1}(\Omega)$ *in* Σ^n *(resp.* $\widetilde{\Sigma}^n$*) is given by*

$$\begin{aligned} &\int_\Omega w_{n,2}(k_1,k_2,\rho)\sin^{n-3}\rho\, d\rho \wedge dv, \\ (\textit{resp.}) \quad &\int_\Omega \widetilde{w}_{n,2}(k_1,k_2,\rho)\sinh^{n-3}\rho\, d\rho \wedge dv, \end{aligned} \tag{67}$$

where

$$\begin{aligned} w_{n,2}(k_1,k_2,\rho) &= \frac{1}{\omega_{n-1}}\psi_{n,1}(k_2,\ell(\rho))\frac{\sin k_1 \sec^{n-1}\rho}{(\tan^2\rho + \sin^2 k_1)^{\frac{n-1}{2}}}, \\ (\textit{resp.}) \quad \widetilde{w}_{n,2}(k_1,k_2,\rho) &= \frac{1}{\omega_{n-1}}\tilde{\psi}_{n,1}(k_2,\tilde{\ell}(\rho))\frac{\sinh k_1 \mathrm{sech}^{n-1}\rho}{(\tanh^2\rho + \sinh^2 k_1)^{\frac{n-1}{2}}}, \\ \ell(\rho) &= \cos^{-1}(\cos k_1 \cos\rho), \quad \tilde{\ell}(\rho) = \cosh^{-1}(\cosh k_1 \cosh\rho). \end{aligned} \tag{67$'$}$$

Proof The proofs for the spherical case and the hyperbolic case are completely parallel. Thus, we shall only exhibit the proof of the *hyperbolic* case in the following. Moreover, the underlying geometric idea of such a proof is essentially the same as that of Theorem 1, namely, via effective usage of symmetry and the method of infinitesimal symmetrization of the integrand. However, due to the high dimensional geometric situation, such a geometric idea can only be illustrated by the following much simplified diagram rather than a realistic picture. As indicated in Figure 3, $d\sigma\big|_{(\rho_0,\xi)}$ is an infinitesimal piece of volume element in $\widetilde{\Sigma}^{n-2}$, inside of the given region Ω, $\widetilde{\mathcal{S}}(\rho_0, d\rho)$ is the spherical shell in $\widetilde{\Sigma}^{n-2}$ centered at O and with thickness $d\rho$, inner radius ρ_0, consisting of all those $O(n-2)$-orbits passing through $d\sigma\big|_{(\rho_0,\xi)}$. Moreover, $\tilde{\ell}(\rho_0)$ is the distance between V_1 and (ρ_0,ξ), while $\widetilde{B}_{n-1}(\tilde{\ell}(\rho_0))$ is the $(n-1)$-ball of radius $\tilde{\ell}(\rho_0)$ in $\widetilde{\Sigma}^{n-1}$ centered at V_1, which is, of course, $O(n-1)$-symmetric. Therefore, the ratio between the $(n-1)$-dimensional volume of $C^{V_1}(\widetilde{\mathcal{S}}(\rho_0,d\rho))$ and that of $\widetilde{B}_{n-1}(\tilde{\ell}(\rho_0))$ is equal to the ratio between the $(n-1)$-dimensional solid angle spanned by $\widetilde{\mathcal{S}}(\rho_0,d\rho)$ at V_1 and the totality of $(n-1)$-dimensional solid angle (i.e. ω_{n-1}). Hence, modulo $d\rho^2$, the n-dimensional volume of the orthogonal double cone with $\widetilde{\mathcal{S}}(\rho_0,d\rho)$ as its base is equal to $\tilde{\psi}_{n,1}(k_2,\tilde{\ell}(\rho_0))$ times that ratio, namely

$$\operatorname{vol} C^{V_2,V_1}(\widetilde{\mathcal{S}}(\rho_0,d\rho)) \equiv \tilde{\psi}_{n,1}(k_2,\tilde{\ell}(\rho_0))\cdot\frac{\omega_{n-2}}{\omega_{n-1}}\sin^{n-3}\tilde{\lambda}(\rho_0)\frac{d\tilde{\lambda}}{d\rho}\bigg|_{\rho_0} d\rho. \tag{68}$$

Hence

$$w_{n,2}(k_1,k_2,\rho)=\frac{1}{\omega_{n-1}}\tilde{\psi}_{n,1}(k_2,\tilde{\ell}(\rho))\frac{\sinh k_1\operatorname{sech}^{n-1}\rho}{(\tanh^2\rho+\sinh^2 k_1)^{\frac{n-1}{2}}} \tag{69}$$

and

$$\operatorname{vol} C^{V_2,V_1}(\Omega)=\int_\Omega \widetilde{w}_{n,2}(k_1,k_2,\rho)\sinh^{n-3}\rho\, d\rho\wedge dv. \tag{70}$$

This proves the hyperbolic case of Theorem 2, while that of the spherical case is completely parallel to the above proof.

6 Volume Formulas of Orthogonal Multiple Spindles and Multiple Cones in $S^n(1)$ (Resp. $H^n(-1)$)

Among all orthogonal ℓ-multiple cones in Σ^n (resp. $\widetilde{\Sigma}^n$), the special case that the region $\Omega\subset\Sigma^{n-\ell}$ (resp. $\widetilde{\Sigma}^{n-\ell}$) happens to be an $(n-\ell)$-ball centered at O is undoubtedly the simplest and also the most symmetric one. We shall call such particularly nice multiple cones *orthogonal* (n,ℓ)*-spindles* or (n,ℓ)*-orthospindles*. The volume of an orthogonal (n,ℓ)-spindle is, of course, a function of its ℓ *successive heights* and the *radius of its base*, which already constitute a complete set congruence invariants of such spindles. We shall denote this function by $\psi_{n,\ell}(k_1,\cdots,k_\ell;r)$ (resp. $\tilde{\psi}_{n,\ell}(k_1,\cdots,k_\ell;r)$). For example, $\psi_{n,0}(r)$ (resp. $\tilde{\psi}_{n,0}(r)$) are just the volumes of the n-balls of radius r in Σ^n (resp. $\widetilde{\Sigma}^n$); $\psi_{n,1}(k_1,r)$ (resp. $\tilde{\psi}_{n,1}(k_1,r)$) are the volumes of orthogonal cones with height k and base-radius r in Σ^n (resp. $\widetilde{\Sigma}^n$) (cf. §4). Moreover, the results of §3 and §5 show that $\psi_{n,0}(r)$ (resp. $\tilde{\psi}_{n,0}(r)$) plays the central role in the derivation of the volume formulas of cones in Σ^n (resp. $\widetilde{\Sigma}^n$), while $\psi_{n,1}(k,r)$ (resp. $\tilde{\psi}_{n,1}(k,r)$) also plays the central role in the derivation of the volume formulas of orthogonal double cones in Σ^n (resp. $\widetilde{\Sigma}^n$).

In this section, we shall prove the same kind of volume formulas for orthogonal ℓ-multiple cones as that of Theorem 2 for orthogonal double cones, in which the volume functions of orthogonal $(n,\ell-1)$-spindles, i.e. $\psi_{n,\ell-1}$ (resp. $\tilde{\psi}_{n,\ell-1}$) will again play the central role. Moreover, such volume formulas will also provide an inductive way of computing $\psi_{n,\ell}$ (resp. $\tilde{\psi}_{n,\ell}$) via a specific integral formula involving $\psi_{n,\ell-1}$ (resp. $\tilde{\psi}_{n,\ell-1}$), namely

Theorem 3 *Let $\psi_{n,\ell-1}(k_1,\cdots,k_{\ell-1},r)$ (resp. $\tilde{\psi}_{n,\ell-1}(k_1,\cdots,k_{\ell-1},r)$) be the volume functions of orthogonal $(n,\ell-1)$-spindles in Σ^n (resp. $\widetilde{\Sigma}^n$). Then the volume of an orthogonal*

ℓ-multiple cone in Σ^n (resp. $\widetilde{\Sigma}^n$) is given by the following integral formulas,

$$\begin{aligned}
&\int_\Omega w_{n,\ell}(k_1,\cdots,k_\ell,\rho)\sin^{(n-\ell-1)}\rho\,d\rho\wedge dv,\\
(\textit{resp.})\quad &\int_\Omega \widetilde{w}_{n,\ell}(k_1,\cdots,k_\ell,\rho)\sinh^{(n-\ell-1)}\rho\,d\rho\wedge dv,
\end{aligned}\tag{71}$$

where $\{k_i, 1\le i\le \ell\}$ are its successive heights, ρ is the distance between O and $P(\rho,\xi)\in\Omega$, dv is the volume element of $S^{(n-\ell-1)}(1)$ at ξ and the "weight functions" $w_{n,\ell}$ (resp. $\widetilde{w}_{n,\ell}$) are given as follows:

$$\begin{aligned}
&w_{n,\ell}(k_1,\cdots,k_\ell,\rho)=\frac{1}{\omega_{n-\ell+1}}\psi_{n,\ell-1}(k_2,\cdots,k_\ell,\ell(\rho))\frac{\sin k_1\sec^{n-\ell+1}\rho}{(\tan^2\rho+\sin^2 k_1)^{\frac{n-\ell+1}{2}}},\\
&\widetilde{w}_{n,\ell}(k_1,\cdots,k_\ell,\rho)=\frac{1}{\omega_{n-\ell+1}}\tilde{\psi}_{n,\ell-1}(k_2,\cdots,k_\ell,\tilde{\ell}(\rho))\frac{\sinh k_1\,\mathrm{sech}^{n-\ell+1}\rho}{(\tanh^2\rho+\sinh^2 k_1)^{\frac{n-\ell+1}{2}}},\\
&\ell(\rho)=\cos^{-1}(\cos k_1\cos\rho),\quad \tilde{\ell}(\rho)=\cosh^{-1}(\cosh k_1\cosh\rho).
\end{aligned}\tag{71$'$}$$

Proof In the beginning cases of $\ell=1$ and 2, Theorem 3 is exactly Theorem 1 and Theorem 2 respectively. Thus, Theorem 3 is their generalization for the general case of $\ell\le n$. In fact, the following proof is also a straightforward generalization of that of Theorem 1 and Theorem 2, namely, the above volume formulas (71) and (71$'$) can again be derived via the same kind of effective usage of the $O(n-\ell)$-symmetry and the method of infinitesimal symmetrization of the integrand. Since the proofs for the spherical case and the hyperbolic case are completely parallel, we shall only discuss here the proof of the hyperbolic case in the following:

Let (ρ,ξ), $\xi\in S^{n-\ell-1}(1)$, be the spherical coordinate system on $\widetilde{\Sigma}^{n-\ell}$ with O as the origin, and $d\sigma|_{(\rho_0,\xi)}$ be an infinitesimal piece of volume element inside of the given region Ω. It follows from the $O(n-\ell)$-symmetry (i.e. of successive cone construction) that the *ratio* between the n-dimensional volume of the orthogonal ℓ-multiple cone with $d\sigma|_{(\rho_0,\xi)}$ as its *base* and the $(n-\ell)$-dimensional volume of $d\sigma|_{(\rho_0,\xi)}$, *modulo* $o(d\sigma)$, only depends on the successive heights $\{k_i, 1\le i\le\ell\}$ and the coordinate ρ_0, namely, there *exists* a weight function $\widetilde{w}_{n,\ell}(k_1,\cdots,k_\ell,\rho)$ such that the volume of the orthogonal ℓ-multiple cone with Ω as its base is given by (71). The remarkable part of Theorem 3 is that such a weight function is, in fact, given by the formula (71$'$). Thus, the most crucial part of the proof is the derivation of (71$'$) by the method of infinitesimal symmetrization.

Let $d\sigma^*$ be the $(n-\ell+1)$-dimensional *solid angle* spanned by $d\sigma|_{(\rho_0,\xi)}$ at V_1. Then, the $(n-\ell+1)$-dimensional volume of the portion $\widetilde{B}_{n-\ell+1}(\tilde{\ell}(\rho_0))$ *inside* of *the solid angle cone* of $d\sigma^*$ is equal to $\frac{d\sigma^*}{\omega_{n-\ell+1}}$ times the volume of $\widetilde{B}_{n-\ell+1}(\tilde{\ell}(\rho_0))$ on the one hand, and on the other hand, it is also equal to the $(n-\ell+1)$-dimensional volume of $C^{V_1}(d\sigma|_{(\rho_0,\xi)})$, *modulo* $o(d\sigma)$. Therefore, the n-dimensional volume of the orthogonal ℓ-multiple cone with $d\sigma\big|_{(\rho_0,\xi)}$ as its base is equal to that of the orthogonal $(\ell-1)$-multiple cone with the above portion of $\widetilde{B}_{n-\ell+1}(\tilde{\ell}(\rho_0))$ inside of $d\sigma^*$ as its base *modulo* $o(d\sigma|_{(\rho_0,\xi)})$, namely

$$\widetilde{w}_{n,\ell}(k_1,\cdots,k_\ell,\rho_0)\,d\sigma|_{(\rho_0,\xi)}\equiv\tilde{\psi}_{n,\ell-1}(k_2,\cdots,k_\ell,\tilde{\ell}(\rho_0))\frac{d\sigma^*}{\omega_{n-\ell+1}}\quad(\mathrm{mod}\ o(d\sigma)).\tag{72}$$

Therefore, the derivation of $(71')$ is reduced to the computation of the ratio between $d\sigma^*$ and $d\sigma|_{(\rho_0,\xi)}$. The same kind of computation as that of §3.2 will again show that

$$d\sigma^* \equiv \frac{\sinh k_1}{\sinh^{n-\ell+1} \tilde{\ell}(\rho)} \, d\sigma\bigg|_{(\rho_0,\xi)}. \tag{73}$$

This proves Theorem 3 for the hyperbolic case, and that of the spherical case is completely parallel to the above proof.

Applying Theorem 3 to the special case that Ω is the $(n-\ell)$-ball of radius r centered at O, one gets the following Theorem 4 which provides an inductive way of computing the basic volume functions $\psi_{n,\ell}$ and $\tilde{\psi}_{n,\ell}$, namely

Theorem 4 *Set*

$$\begin{aligned} \lambda_0 &= \tan^{-1}\frac{\tan r}{\sin k_1}, \qquad \ell(\lambda) = \tan^{-1}(\tan k_1 \sec\lambda), \\ (\textit{resp.}) \quad \tilde{\lambda}_0 &= \tan^{-1}\frac{\tanh r}{\sinh k_1}, \qquad \tilde{\ell}(\lambda) = \tanh^{-1}(\tanh k_1 \sec\lambda). \end{aligned} \tag{74}$$

Then

$$\begin{aligned} \psi_{n,\ell}(k_1,\cdots,k_\ell,r) &= \frac{\omega_{n-\ell}}{\omega_{n-\ell+1}}\int_0^{\lambda_0} \psi_{n,\ell-1}(k_2,\cdots,k_\ell,\ell(\lambda))\sin^{n-\ell-1}\lambda\, d\lambda, \\ \tilde{\psi}_{n,\ell}(k_1,\cdots,k_\ell,r) &= \frac{\omega_{n-\ell}}{\omega_{n-\ell+1}}\int_0^{\tilde{\lambda}_0} \tilde{\psi}_{n,\ell-1}(k_2,\cdots,k_\ell,\tilde{\ell}(\lambda))\sin^{n-\ell-1}\lambda\, d\lambda. \end{aligned} \tag{75}$$

Proof Theorem 4 can be deduced from Theorem 3 by integration, using the simplification provided by the $O(n-\ell)$-symmetry of Ω in the case $\Omega = B_{n-\ell}(r)$ centered at O. However, it might as well to give a direct proof by the same method of infinitesimal symmetrization.

Let $O(n-\ell)$ be the isometry subgroup of Σ^n, fixing all points of the ℓ-dimensional normal plane, Σ^ℓ, of $\Sigma^{n-\ell}$ at O. Let

$$\mathcal{S}(k_1,\cdots,k_\ell;B_{n-\ell}(r))$$

be an orthogonal (n,ℓ)-spindle with $\{k_i, 1\le i\le \ell\}$ as its successive heights and $B_{n-\ell}(r)$ as its base, which is clearly $O(n-\ell)$-invariant. Therefore, such a set can be concisely represented by its image set at the level of the orbit space $\Sigma^n/O(n-\ell)$ (cf. Appendix). As indicated in Figure 4, the orbit space, $\Sigma^n/O(n-\ell)$, equipped with the *orbital distance metric* is isometric to a half space in $\Sigma^{\ell+1}$, while the image set of $B_{n-\ell}(r)$ (resp. $\mathcal{S}(k_1,\cdots,k_\ell;B_{n-\ell}(r))$) is the interval $\overline{OP}$ of length r on the half line $\Sigma^{n-\ell}/O(n-\ell)$ (resp. the $(\ell+1)$-dimensional "orthoscheme" which is exactly the convex hull of $\{P,O,V_1,\cdots,V_\ell\}$ in $\Sigma_+^{\ell+1}$, cf. Appendix).

Let $\mathcal{S}(\rho_0,d\rho)$ be the *spherical shell* in $\Sigma^{n-\ell}$ of inner radius ρ_0 and thickness $d\rho$, centered at O, whose image in the orbit space is the sub-interval $[\rho_0,\rho_0+d\rho]$ on the interval $\overline{OP}$. Thus, the subdivision of $\overline{OP}$ into such *infinitesimal segments* corresponding to the subdivision of $B_{n-\ell}(r)$ into such spherical shells of *infinitesimal thickness*. Note that the given orthogonal $(n-\ell)$-spindle is exactly the orthogonal ℓ-multiple cone with $B_{n-\ell}(r)$ as its base and $\{V_i, 1\le i\le\ell\}$ as its successive vertices. Therefore, the orthogonal ℓ-multiple cones with such *spherical shells* of *infinitesimal thickness* as their bases and the same $\{V_i,1\le i\le\ell\}$ as their successive vertices also constitutes a *subdivision* of the given (n,ℓ)-spindle. Hence, the volume of such an (n,ℓ)-

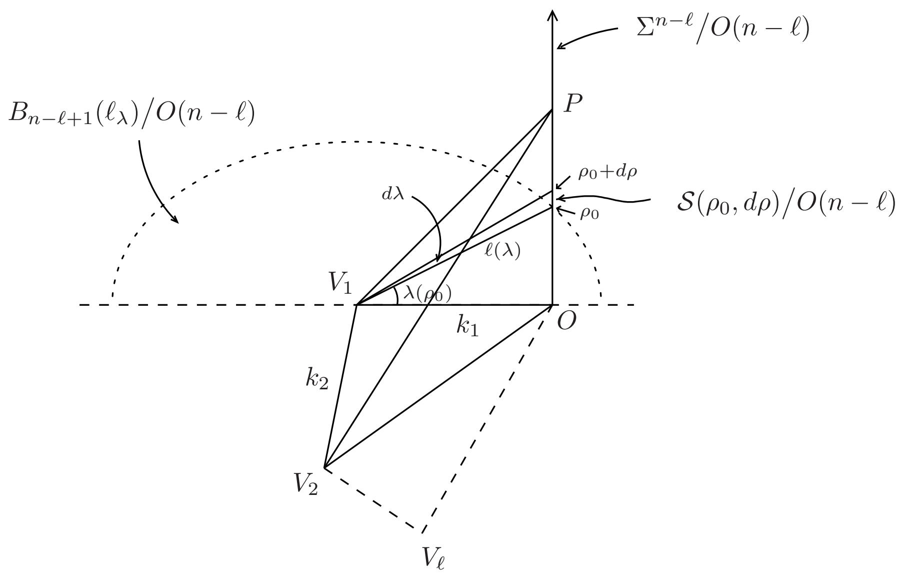


Figure 4

spindle is equal to the *integration* with the volumes of those ℓ-multiple cones of spherical shells as the "*integrand*", namely

$$\mathrm{vol}(\mathcal{S}(k_1, \cdots, k_\ell, B_{n-\ell}(r))) = \int_0^r \mathrm{vol}\ (\ell\text{-multiple cones of } \mathcal{S}(\rho, d\rho)). \tag{76}$$

In integration, two integrands which are *equal to each other modulo higher order infinitesimal* will always produce the *same value* of integration. The orthogonal ℓ-multiple cone of $\mathcal{S}(\rho_0, d\rho)$ is, by definition, the orthogonal $(\ell-1)$-multiple cone of $C^{V_1}(\mathcal{S}(\rho_0, d\rho))$. If we replace $C^{V_1}(\mathcal{S}(\rho_0, d\rho))$ by its subset lying inside of the $(n-\ell+1)$-ball of radius $\ell(\lambda)$ in $\Sigma^{n-\ell+1}$ centered at V_1, say denoted by $C_*^{V_1}(\mathcal{S}(\rho_0, d\rho))$, then it is easy to see that the volumes of their orthogonal $(\ell-1)$-cones are equal to each other modulo $d\rho^2$. On the other hand, it follows from the $O(n-\ell+1)$-symmetry of the orthogonal $(\ell-1)$-fold cone construction, the ratio between the volume of the orthogonal $(\ell-1)$-cone with $C_*^{V_1}(\mathcal{S}(\rho_0, d\rho))$ as its base and that with $B_{n+\ell+1}(\ell(\lambda))$ as its base (i.e. $\psi_{n,\ell-1}(k_2, \cdots, k_\ell, \ell(\lambda))$) is equal to the ratio between the $(n-\ell+1)$-dimensional solid angle spanned by $\mathcal{S}(\rho_0, d\rho)$ at V_1 and the total amount $\omega_{n-\ell+1}$, namely

$$\omega_{n-\ell} \sin^{n-\ell-1} \lambda(\rho_0) \frac{d\lambda}{d\rho}\bigg|_{\rho_0} d\rho \bigg/ \omega_{n-\ell+1}. \tag{77}$$

Hence

$$\begin{aligned}
\psi_{n,\ell}(k_1, \cdots, k_\ell, r) &= \mathrm{vol}\, \mathcal{S}(k_1, \cdots, k_\ell, B_{n-\ell}(r)) \\
&= \int_0^r \mathrm{vol}\ (\ell\text{-multiple cones of } \mathcal{S}(\rho, d\rho)) \\
&= \int_0^r \mathrm{vol}\ ((\ell-1)\text{-multiple cones of } C_*^{V_1}(\mathcal{S}(\rho, d\rho)))
\end{aligned}$$

$$= \frac{\omega_{n-\ell}}{\omega_{n-\ell+1}} \int_0^r \psi_{n,\ell-1}(k_2, \cdots, k_\ell, \ell(\lambda)) \sin^{n-\ell-1} \lambda \frac{d\lambda}{d\rho}\, d\rho$$
$$= \frac{\omega_{n-\ell}}{\omega_{n-\ell+1}} \int_0^{\lambda_0} \psi_{n,\ell-1}(k_2, \cdots, k_\ell, \ell(\lambda)) \sin^{n-\ell-1} \lambda \, d\lambda. \tag{78}$$

Examples: Explicit formulas of $\psi_{n,2}$ (resp. $\tilde{\psi}_{n,2}$) for $n = 3$ and 4

The integral formulas of (75) in Theorem 4 provides a straightforward way of computing $\psi_{n,\ell}$ and $\tilde{\psi}_{n,\ell}$ inductively by integration. The starting cases of $\psi_{n,0}(r)$ (resp. $\tilde{\psi}_{n,0}(r)$) are given by Lemma 3, and the beginning step of such an inductive computation, namely, from $\ell = 0$ to $\ell = 1$, has already been discussion in §4. In concluding this section, we shall discuss the next step of from $\ell = 1$ to $\ell = 2$ as an example. Since the computations of the hyperbolic case is completely parallel to that of the spherical case, while the computations for higher dimensions, say $n \geq 5$, will only encounter the same kinds of integrations as that of the lower dimensional cases. We shall only discuss a few typical cases as examples of such computations in the following:

Example 1 $\psi_{3,2}$.

In the lowest dimensional case of $n = 3$, $\psi_{3,2}(k_1, k_2, r)$ is equal to twice of the volume of the spherical orthoscheme with $\{k_1, k_2, r\}$ as its sequence of lengths, namely, the tetrahedron spanned by $\{P, O, V_1, V_2\}$ as indicated in Figure 5.

Set $\lambda_0 = \angle V_1V_2O$, $\lambda_1 = \angle OV_1P$, $\alpha(\theta) = \angle V_1V_2X_\theta$, $\gamma(\theta) = \angle OV_2X_\theta$ and $\ell(\theta) = \overline{V_2X_\theta}$. Then it follows from the orthogonality that $\theta = \angle OV_1X_\theta$ is exactly the dihedral angle between $\triangle V_2V_1O$ and $\triangle V_2V_1X_\theta$, while the two spherical triangles indicated in Figure 5 are both right-angle ones. Hence

$$\begin{aligned} &\tan\gamma(\theta) = \sin\lambda_0 \tan\theta, \quad \tan\ell(\theta) = \frac{\tan k_2}{\cos\alpha(\theta)}, \\ &\cos\alpha(\theta) = \cos\lambda_0\cos\gamma(\theta) = \frac{\sqrt{1-\sin^2\lambda_0}}{\sqrt{1+\sin^2\lambda_0\tan^2\theta}}. \end{aligned} \tag{79}$$

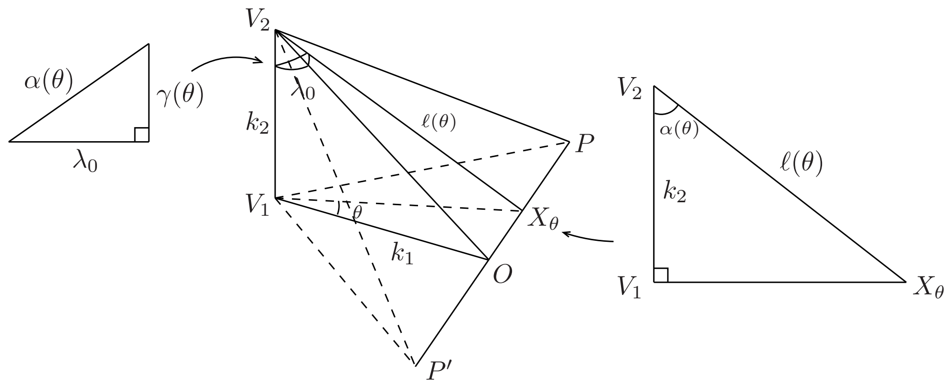


Figure 5

Therefore, by (60) and (75),

$$\psi_{3,2}(k_1,k_2,r)=\int_0^{\lambda_1}(k_2-\cos\alpha(\theta)\ell(\theta))d\theta$$

$$=\lambda_1k_2-\int_0^{\lambda_1}\frac{\sqrt{1-\sin^2\lambda_0}}{\sqrt{1+\sin^2\lambda_0\tan^2\theta}}\tan^{-1}\left\{\frac{\tan k_2\sqrt{1+\sin^2\lambda_0\tan^2\theta}}{\sqrt{1-\sin^2\lambda_0}}\right\}d\theta \tag{80}$$

$$(\,=\varphi_{3,2}(k_2,\lambda_0,\lambda_1)).$$

We include here the following useful lemma on the above function $\varphi_{3,2}(k_2,\lambda_0,\lambda_1)$ given by the RHS of (80), namely

Lemma 6 *Let $\varphi_{3,2}(k_2,\lambda_0,\lambda_1)$ be the function given by the RHS of* (80). *Then*

$$\frac{\partial\varphi_{3,2}}{\partial k_2}=\frac{\sin\lambda_0\tan k_2}{\sqrt{1-\sin^2\lambda_0+\tan^2k_2}}\tan^{-1}\frac{\sin\lambda_0\tan\lambda_1\tan k_2}{\sqrt{1-\sin^2\lambda_0+\tan^2k_2}}. \tag{81}$$

Proof Set $\tan k_2=\tau$, $\sin\lambda_0=c$ and consider the RHS of (80) as a function of the "parameter" τ. Then

$$\frac{\partial\varphi_{3,2}}{\partial\tau}=\frac{\lambda_1}{1+\tau^2}-\int_0^{\lambda_1}\frac{(1-c^2)d\theta}{1-c^2+\tau^2+c^2\tau^2\tan^2\theta}. \tag{82}$$

Set $A=1-c^2+\tau^2$, $B=c^2\tau^2$ and $u=\tan\theta$. One has

$$\frac{\partial\varphi_{3,2}}{\partial\tau}=\frac{\lambda_1}{1+\tau^2}-(1-c^2)\int_0^{\tan\lambda_1}\frac{du}{(A+Bu^2)(1+u^2)}$$

$$=\frac{\lambda_1}{1+\tau^2}-\frac{1-c^2}{A-B}\int_0^{\tan\lambda_1}\left\{\frac{1}{1+u^2}-\frac{B}{A+Bu^2}\right\}du$$

$$=\frac{c\tau}{(1+\tau^2)\sqrt{1-c^2+\tau^2}}\tan^{-1}\frac{c\tau\tan\lambda_1}{\sqrt{1-c^2+\tau^2}}. \tag{83}$$

Hence

$$\frac{\partial\varphi_{3,2}}{\partial k_2}=(1+\tau^2)\frac{\partial\varphi_{3,2}}{\partial\tau}=\frac{c\tau}{\sqrt{1-c^2+\tau^2}}\tan^{-1}\frac{c\tan\lambda_1\tau}{\sqrt{1-c^2+\tau^2}}. \tag{84}$$

Example 2 $\psi_{4,2}$.

A $(4,2)$-orthospindle with k_1, k_2 as its successive heights and r as its base-radius is an $O(2)$-invariant body in $S^4(1)$ whose image in the orbit space $S^4(1)/O(2)\cong S^3_+(1)$ is an orthoscheme with $\{k_2,k_1,r\}$ as its sequence of lengths, as indicated in Figure 5. Therefore, it is advantageous to use the same notations as that of Example 1, in which formula (75) can be rewritten as follows:

$$\psi_{4,2}(k_1,k_2,r)=\frac{1}{2}\int_0^{\lambda_1}\varphi_{4,1}(k_2,\alpha(\theta))\sin\theta\,d\theta$$

$$=2\pi\int_0^{\lambda_1}\left\{\frac{1}{3}\alpha(\theta)-3\sin\alpha(\theta)\cos\alpha(\theta)-\frac{1}{3}\sin^{-1}(\cos k_2\sin\alpha(\theta))\right.$$

$$\left.+3\cos k_2\sin\alpha(\theta)\sqrt{1-\cos^2k_2\sin^2\alpha(\theta)}-\frac{\tan k_2\sin k_2\sin\alpha(\theta)}{\sqrt{1-\cos^2k_2\sin^2\alpha(\theta)}}\right\}\sin\theta\,d\theta, \tag{85}$$

where

$$\alpha(\theta) = \tan^{-1}\Big(\frac{\tan\lambda_0}{\cos\theta}\Big), \quad \sin\alpha(\theta) = \frac{\tan\lambda_0}{\sqrt{\cos^2\theta + \tan^2\lambda_0}}, \quad \cos\alpha(\theta) = \frac{\cos\theta}{\sqrt{\cos^2\theta + \tan^2\lambda_0}}. \quad (85')$$

It is not difficult to show that each term of the integrals of the RHS of (85) can in fact be explicitly integrated in terms of elementary functions. For example,

$$\begin{aligned} &\int_0^{\lambda_1} \alpha(\theta)\sin\theta\, d\theta = \int_0^{\lambda_1} \tan^{-1}\Big(\frac{\tan\lambda_0}{\cos\theta}\Big)\sin\theta\, d\theta \\ &= \Big\{ -\cos\theta\tan^{-1}\Big(\frac{\tan\lambda_0}{\cos\theta}\Big)\Big\}\Big|_0^{\lambda_1} + \int_0^{\lambda_1} \frac{\tan\lambda_0\cos\theta\sin\theta\, d\theta}{\cos^2\theta + \tan^2\lambda_0} \\ &= \lambda_0 - \cos\lambda_1\tan^{-1}\Big(\frac{\tan\lambda_0}{\cos\lambda_1}\Big) + \frac{1}{2}\ln\frac{\sec^2\lambda_0}{\sec^2\lambda_0 - \sin^2\lambda_1}. \end{aligned} \quad (86)$$

Remarks (i) The explicit computations for $\psi_{n,2}$ (resp. $\tilde{\psi}_{n,2}$) for odd n are similar to that of $\psi_{3,2}$, while that of even n are similar to the case of $\psi_{4,2}$.

(ii) It is always advantageous to express the volume formulas of orthospindles (and orthoschemes) in terms of the last height $k_{\ell-1}$ and the angular parameters $\{\lambda_0, \cdots, \lambda_{\ell-2}\}$, namely, $\varphi_{n,\ell}(k_\ell, \lambda_0, \cdots, \lambda_{\ell-1})$ instead of $\psi_{n,\ell}(k_1, \cdots, k_\ell, r)$. Therefore, the integral formulas of (75) should be rewritten in terms of $\varphi_{n,\ell-1}$ (resp. $\tilde{\varphi}_{n,\ell-1}$) as follows, namely

$$\begin{aligned} \varphi_{n,\ell}(k_\ell, \lambda_0, \cdots, \lambda_{\ell-1}) &= \frac{\omega_{n-\ell}}{\omega_{n-\ell+1}} \int_0^{\lambda_{\ell-1}} \varphi_{n,\ell-1}(k_\ell, \lambda_0, \cdots, \lambda_{\ell-2}, \alpha(\theta))\sin^{n-\ell-1}\theta\, d\theta, \\ \tilde{\varphi}_{n,\ell}(k_\ell, \tilde{\lambda}_0, \cdots, \tilde{\lambda}_{\ell-2}) &= \frac{\omega_{n-\ell}}{\omega_{n-\ell+1}} \int_0^{\tilde{\lambda}_{\ell-1}} \tilde{\varphi}_{n,\ell-1}(k_\ell, \tilde{\lambda}_0, \cdots, \tilde{\lambda}_{\ell-2}, \tilde{\alpha}(\theta))\sin^{n-\ell-1}\theta\, d\theta, \end{aligned} \quad (75^*)$$

where $\alpha(\theta) = \tan^{-1}(\frac{\tan\lambda_{\ell-2}}{\cos\theta})$, $\tilde{\alpha}(\theta) = \tan^{-1}(\frac{\tan\tilde{\lambda}_{\ell-2}}{\cos\theta})$ and the relationship between the length parameters $\{k_\ell, k_{\ell-1}, \cdots, k_1, r\}$ and the angular parameters $\{\lambda_0, \lambda_1, \cdots, \lambda_{\ell-1}\}$ are given by (93) of Appendix.

7 Concluding Remarks

1. In the study of n-dimensional Euclidean (resp. spherical, hyperbolic) geometries, the volume of an n-simplex is the top dimensional geometric invariant which is clearly of fundamental importance, just as the importance of the area formula of triangles in the study of 2-dimensional geometry. It is well-known that the dimension of the moduli space of congruence classes of n-simplexes in $\mathbb{E}^n$ (resp. $S^n(1)$, $H^n(-1)$) is equal to $\frac{n(n+1)}{2}$, thus needing a set of at least $\frac{n(n+1)}{2}$ invariants, such as the set of $\frac{n(n+1)}{2}$ edge-lengths, in order to specify a congruence class. Hence, the volume function of n-simplexes is a function of $\frac{n(n+1)}{2}$ independent variables depending on the choice of such a complete set of invariants, such as the edge-lengths or the dihedral angles in the cases of $S^n(1)$ and $H^n(-1)$. Anyhow, the volume functions of n-simplexes for the cases of $S^n(1)$ and $H^n(-1)$ with $n \geq 3$ are known to be analytically difficult to deal with, even the number of variables already increases rapidly as n increases. Of course, if one restricts to certain special kind of n-simplexes such as the *orthoschemes* (cf. Appendix), then

the number of variables of their volume function will stay to be n, while a general n-simplexes can be geometrically decomposed into the non-overlapping union or differences of such special kind of n-simplexes, often by means of orthogonal multiple cone-constructions. However, the volume functions of orthoschemes in $S^n(1)$ and $H^n(-1)$ are still highly transcendental and difficult to deal with analytically. In fact, this is often the *main source of difficulty* in the study of various high dimensional problems in spherical and hyperbolic geometries.

2. The volume function of n-simplexes in $\mathbb{E}^n$ can be expressed by determinant on the one hand, but on the other hand, the volume functions of n-simplexes in $S^n(1)$ (resp. $H^n(-1)$) becomes highly transcendental. Such a drastic contrast in their analytical behaviors is, of course, coming from the *curvature* of the latter. However, the three kinds of classical geometries are commonly characterized by the *maximality* of their *local isometry groups* (i.e. $\mathrm{ISO}(M^n, \mathrm{pt}) = O(n)$ everywhere). In fact, even the definition of simplexes and the congruence properties of simplexes are all depending on the above common symmetry property of the three kinds of geometries. Therefore, it is natural to seek understanding of the volume in $S^n(1)$ (resp. $H^n(-1)$) by studying the *interaction* between "volume" and symmetry. From this viewpoint, one expects that the volume functions of more symmetric bodies should be considerably simpler than the volume functions of less symmetric bodies. For example, the volume functions of an n-ball of radius r in $S^n(1)$ (resp. $H^n(-1)$) are indeed rather simple because it is $O(n)$-symmetric. Moreover, the families of orthogonal (n, ℓ)-spindles, $0 \le \ell \le (n-1)$, constitutes a sequence of $O(n-\ell)$-symmetric families. Their volume function $\psi_{n,\ell}$ (resp. $\tilde{\psi}_{n,\ell}$) play the central role of providing the powerful formulas for orthogonal $(\ell+1)$-multiple cones in $S^n(1)$ (resp. $H^n(-1)$). Examplified by the remarkable achievement of Eudoxus, integration has always been *indispensible* analytic technique in the study of volume. Thus, it is natural to study how symmetry interacts with integration? The method of *infinitesimal symmetrization* was first introduced by the author in [1] for the study of volume formula of tetrahedrons in $S^3(1)$ and $H^3(-1)$. In this paper, we make a more systematic application of this method, thus obtaining a much more systematic understanding on volumes in $S^n(1)$ and $H^n(-1)$ such as Theorem 3 and Theorem 4.

3. In the study of various kinds of spherical (resp. hyperbolic) geometric problems, in which the volume plays an important role, what one often needs is *not* the numerical computation of the volumes of certain specific bodies, but rather the *estimation* of the volumes of certain specific *family of bodies*. It is in this aspect that the volume formulas of this paper provide a powerful set of tools for proving such estimates. We refer to [2], [3], etc. for examples of this kind of applications.

4. Orthogonality and the $O(n)$-symmetry (i.e. that of the local isometry groups) are intimately related in the geometry of $\mathbb{E}^n$ (resp. S^n, H^n). The results of this paper demonstrates how they can be jointly used to provide understanding on volumes in such spaces.

Appendix Volumes of Orthoschemes and Orthospindles as Multiple Integrals

Let O be a chosen *base point* in Σ^n (resp. $\widetilde{\Sigma}^n$) and $\{\mathbf{e}_i, 1 \le i \le n\}$ be a chosen orthonormal basis of the tangent space at O. Then

$$\{0\} \subset <\mathbf{e}_1> \subset <\mathbf{e}_1, \mathbf{e}_2> \subset \cdots \subset <\mathbf{e}_i, 1 \le i \le k> \subset \cdots \subset T_O \tag{87}$$

constitutes a *flag* in T_O, while their images under the exponential map constitutes a spherical (resp. hyperbolic) flag of subspaces in Σ^n (resp. $\widetilde{\Sigma}^n$), namely

$$\begin{aligned} & O \in \Sigma^1 \subset \Sigma^2 \subset \cdots \subset \Sigma^k \subset \cdots \subset \Sigma^n \\ (\text{resp.} \quad & O \in \widetilde{\Sigma}^1 \subset \widetilde{\Sigma}^2 \subset \cdots \subset \widetilde{\Sigma}^k \subset \cdots \subset \widetilde{\Sigma}^n). \end{aligned} \tag{88}$$

Set $\Sigma_*^{(n-2)}$ to be the set of points in Σ^n whose distances to the above Σ^1 of (88) are equal to $\frac{\pi}{2}$. In the following discussion, we shall always restrict to points *not* belonging $\Sigma_*^{(n-2)}$ in the case of Σ^n.

A.1 Spherical (resp. hyperbolic) cartesian coordinate system on $\boldsymbol{\Sigma^n \setminus \Sigma_*^{(n-2)}}$ (resp. $\boldsymbol{\widetilde{\Sigma}^n}$)

To a given point $P \in \Sigma^n \setminus \Sigma_*^{(n-2)}$ (resp. $\widetilde{\Sigma}^n$), there exists a unique sequence of points $P_k \in \Sigma^k$ (resp. $\widetilde{\Sigma}^k$), $1 \leq k \leq n-1$, such that $\overline{PP}_{n-1}$ and $\overline{P_kP}_{k-1}$, $(n-1) \geq k \geq 2$, are respectively the unique shortest paths between P and Σ^{n-1}; P_k and Σ^{k-1}. Set $x_1, x_2, \cdots, x_k$ and x_n to be the *oriented lengths* of $\overrightarrow{OP}_1$, $\overrightarrow{P_1P_2}$, $\cdots$, $\overrightarrow{P_{k-1}P_k}$ and $\overrightarrow{P_{n-1}P}$ respectively. It is easy to show that such an n-tuple $(x_1, \cdots, x_n)$ and $P \in \Sigma^n \setminus \Sigma_*^{(n-2)}$ (resp. $\widetilde{\Sigma}^n$) determine each other uniquely. (Note that $-\frac{\pi}{2} < x_i < \frac{\pi}{2}$ in the spherical case.) We shall call it the spherical (resp. hyperbolic) cartesian coordinate system associated to the chosen spherical (resp. hyperbolic) flag of (88). As one can see readily, the *orthogonality* plays the major role in the construction of such a coordinate system on the one hand, and on the other hand, it will naturally provide an advantageous framework for the study of those geometric objects and problems having the orthogonality as an important feature of their structures such as the *orthoschemes*, *orthospindles* and *multiple orthogonal cones*.

A.2 Coordinate curves and volume element of spherical (resp. hyperbolic) cartesian coordinate system

In the well-known case of Euclidean geometry, the vector field $\frac{\partial}{\partial x_1}$ is the Killing vector field generated by the *translation symmetry* in the x_1-direction. In the case of Σ^n (resp. $\widetilde{\Sigma}^n$) the corresponding 1-parameter subgroup of isometry is the *transvections* along the geodesic line Σ^1 (resp. $\widetilde{\Sigma}^1$).

Sublemma 1 *The length of* $\dfrac{\partial}{\partial x_1}(P)$ *is given by*

$$\begin{aligned} & \left|\frac{\partial}{\partial x_1}(P)\right| = \cos d(P, \Sigma^1) = \prod_{i=2}^{n} \cos x_i \\ \Big(\textit{resp.} \quad & \left|\frac{\partial}{\partial x_1}(P)\right| = \cosh d(P, \widetilde{\Sigma}^1) = \prod_{i=2}^{n} \cosh x_i\Big). \end{aligned} \tag{89}$$

Proof Let $\Sigma^2(P)$ (resp. $\widetilde{\Sigma}^2(P)$) be the "plane" spanned by P and Σ^1 (resp. $\widetilde{\Sigma}^1$) in Σ^n (resp. $\widetilde{\Sigma}^n$). Then the proof of (89) can be reduced to that of 2-dimensional spherical (resp. hyperbolic space), which can be readily verified by straightforward computations of spherical (resp. hyperbolic) trigonometry.

Corollary *For $k \leq n-1$, one has*

$$\Big|\frac{\partial}{\partial x_k}(P)\Big| = \cos d(P, \Sigma^k) = \prod_{i=k+1}^{n} \cos x_i \tag{90}$$

$$\Big(\textit{resp.} \quad = \cosh d(P, \widetilde{\Sigma}^k) = \prod_{i=k+1}^{n} \cosh x_i\Big).$$

$\Big|\frac{\partial}{\partial x_n}(P)\Big| = 1.$

Sublemma 2 *In terms of the cartesian coordinates, the volume elements of Σ^n (resp. $\widetilde{\Sigma}^n$) is given by*

$$\begin{aligned} d\sigma &= \det\Big(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\Big) dx_1 \wedge \cdots \wedge dx_n \\ &= \Big(\prod_{i=2}^{n} \cos^{(i-1)} x_i\Big) dx_1 \wedge \cdots \wedge dx_n \\ \Big(\textit{resp.} \quad d\tilde{\sigma} &= \Big(\prod_{i=2}^{n} \cosh^{(i-1)} x_i\Big) dx_1 \wedge \cdots \wedge dx_n\Big). \end{aligned} \tag{91}$$

Proof It follows directly from Sublemma 1, the Corollary and the orthogonality of $\{\frac{\partial}{\partial x_i}, 1 \leq i \leq n\}$.

A.3 Spherical (resp. hyperbolic) orthoschemes

Let $\{A_i, 0 \leq i \leq k\}$ be a sequence of $(k+1)$ points in Σ^n (resp. $\widetilde{\Sigma}^n$) satisfying:

(i) the lengths ℓ_i of $\overline{A_{i-1}A_i}$, $1 \leq i \leq \ell$, are all positive and less than $\frac{\pi}{2}$ in the spherical case,

(ii) $\overline{A_iA_{i+1}}$ is perpendicular to the i-dimensional subspace spanned by $\{A_0, \cdots, A_i\}$.

Then the k-simplex spanned by such a set is called a *spherical* (resp. *hyperbolic*) *k-orthoscheme*. For examples, a 2-orthoscheme is just a right-angle triangle, a 3-orthoscheme is a doubly orthogonal tetrahedron.

Note that the $(k-1)$-simplices spanned by $\{A_i, 0 \leq i \leq k-1\}$ and $\{A_i, 0 \leq i \leq k-2,\ A_k\}$ are both $(k-1)$-orthoschemes with the $(k-2)$-orthoscheme spanned by $\{A_i, 0 \leq i \leq (k-2)\}$ as their common face, while the *dihedral angle* between them is equal to $\angle A_{k-1}A_{k-2}A_k$. Set

$$\lambda_j = \angle A_{j+1}A_jA_{j+2}, \quad 0 \leq j \leq (k-2). \tag{92}$$

Then

$$\lambda_j = \tan^{-1}(\tan \ell_{j+2} \csc \ell_{j+1}) \quad (\text{resp.} \quad \tan^{-1}(\tanh \ell_{j+2} \operatorname{csch} \ell_{j+1})). \tag{93}$$

A k-orthoscheme in Σ^n (resp. $\widetilde{\Sigma}^n$) is uniquely determined up to congruence by the sequence of lengths $\{\ell_i = \ell(\overline{A_{i-1}A_i}), 1 \leq i \leq k\}$ or ℓ_1 and the sequence of $(k-1)$ dihedral angles $\{\lambda_j, 0 \leq j \leq (k-2)\}$, while the above two sets of congruence invariants are related by (93).

In the top dimensional case of $k = n$, let $\{A_i, 0 \leq i \leq n\}$ be the sequence of $(n+1)$ points whose coordinates are given by

$$A_0 = (0, \cdots, 0),\ A_i = (\ell_1, \cdots, \ell_i, 0, \cdots, 0), \quad 1 \leq i \leq n. \tag{94}$$

Then the n-simplex spanned by them is clearly an n-orthoscheme with $\{\ell_i, 1 \le i \le n\}$ as its sequence of lengths. Conversely, to any given n-orthoscheme with $\{\ell_i, 1 \le i \le n\}$ as its sequence of lengths, it is straightforward to choose the corresponding cartesian coordinate system such that the coordinates of its sequence of vertices $\{A_i, 0 \le i \le n\}$ are as given by (94).

Sublemma 3 *Let $\{A_i, 0 \le i \le n\}$ be the $(n+1)$ points given by* (94) *and $\{\lambda_j, 0 \le j \le n-2\}$ be sequence of $(n-1)$ angles given by* (93). *Then the n-orthoscheme spanned by $\{A_i, 0 \le i \le n\}$ in Σ^n (resp. $\widetilde{\Sigma}^n$) consists of those points whose coordinates coordinates $(x_1, \cdots, x_n)$ are specified by the following set of inequalities, namely*

$$\begin{aligned} &0 \le x_1 \le \ell_1, \quad \textit{and for} \ \ 0 \le j \le (n-2), \\ &0 \le x_{j+2} \le \tan^{-1}(\tan\lambda_j \sin x_{j+1}) \quad (\textit{resp.} \quad \tanh^{-1}(\tan\lambda_j \sinh x_{j+1})). \end{aligned} \tag{95}$$

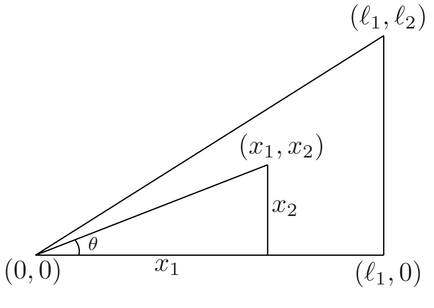
Proof We shall prove by induction on n. The beginning case of $n=1$ is trivial, while the next case of $n=2$ is the well-known description of a right-angle spherical (resp. hyperbolic) triangle, as indicated in Figure 6.

Note that the subset with $x_n = 0$ is an $(n-1)$-orthoscheme in Σ^{n-1} (resp. $\widetilde{\Sigma}^{n-1}$), which is geometrically the image of orthogonal projection of the given n-orthoscheme. Let $P = (a_1, \cdots, a_n)$ be an arbitrary point of the n-orthoscheme. Then $P' = (a_1, \cdots, a_{n-1}, 0)$ is its image of orthogonal projection. Hence, by induction assumption

$$\begin{aligned} &0 \le a_1 \le \ell_1, \\ &0 \le a_{j+2} \le \tan^{-1}(\tan\lambda_j \sin a_{j+1}) \quad (\text{resp.} \quad \tanh^{-1}(\tan\lambda_j \sinh a_{j+1})) \end{aligned} \tag{96}$$

for $0 \le j \le (n-3)$. Thus, it suffices to prove that a_n satisfies the last inequality of (95). Set $P'' = (a_1, \cdots, a_{n-2}, 0, 0)$ and $\Sigma^2_+(P'')$ (resp. $\widetilde{\Sigma}^2_+(P'')$) to be the half-plane perpendicular to Σ^{n-2} (resp. $\widetilde{\Sigma}^{n-2}$) at P''. Then the intersection of $\Sigma^2_+(P'')$ (resp. $\widetilde{\Sigma}^2_+(P'')$) and the given n-orthoscheme is a right-angle spherical (resp. hyperbolic) triangle with λ_{n-2} as its angle at P''. Hence, it follows from the same kind of geometry as indicated in Figure 6 that

$$\begin{gathered} \tan a_n = \tan\theta \sin a_{n-1} \le \tan\lambda_{n-2} \sin a_{n-1} \\ (\text{resp.} \quad \tanh a_n = \tan\theta \sinh a_{n-1} \le \tan^{-1}(\tan\lambda_{n-2} \sin a_{n-1})) \end{gathered} \tag{97}$$



$$\begin{aligned} &0 \le x_1 \le \ell_1, \\ &0 \le \theta \le \lambda_0 \\ \Leftrightarrow \ &0 \le \tan x_2 \le \tan\lambda_0 \sin x_1 \\ (\text{resp.} \ \ &0 \le \tanh x_2 \le \tan\lambda_0 \sinh x_1 \ ) \end{aligned}$$

Figure 6

A volume formula of n-orthoscheme: Let $\Omega(\ell_1, \cdots, \ell_n)$ be the n-orthoscheme with $\{\ell_i, 1 \leq i \leq n\}$ as the sequence of lengths and $\{\lambda_j, 0 \leq j \leq (n-2)\}$ be its sequence of angular invariants given by (93). Then, its volume is equal to the following n-multiple integral, namely

$$\int_0^{\ell_1} dx_1 \Big\{ \int_0^{\varphi_1(x_1)} \cos x_2 dx_2 \Big\{ \int_0^{\varphi_2(x_2)} \cos^2 x_3 dx_3 \Big\{ \cdots \Big\{ \int_0^{\varphi_{n-1}(x_{n-1})} \cos^{n-1} x_n dx_n \Big\} \cdots \Big\}\Big\}\Big\}, \quad (98)$$

where

$$\varphi_k(x_k) = \tan^{-1}(\tan \lambda_{k-1} \sin x_k), \quad 1 \leq k \leq (n-1), \quad (98'),$$

(resp.)

$$\int_0^{\ell_1} dx_1 \Big\{ \int_0^{\tilde{\varphi}_1(x_1)} \cosh x_2 dx_2 \Big\{ \int_0^{\tilde{\varphi}_2(x_2)} \cosh^2 x_3 dx_3 \Big\{ \cdots \Big\{ \int_0^{\tilde{\varphi}_{n-1}(x_{n-1})} \cosh^{n-1} x_n dx_n \Big\} \cdots \Big\}\Big\}\Big\}, \quad (99)$$

where

$$\tilde{\varphi}_k(x_k) = \tanh^{-1}(\tan \lambda_{k-1} \sinh x_k), \quad 1 \leq k \leq (n-1). \quad (99')$$

Proof Direct consequence of Sublemma 2 and Sublemma 3.

A.4 Spherical (resp. hyperbolic) orthospindles

Let $O(n)$ be the local isometry group of Σ^n (resp. $\widetilde{\Sigma}^n$) at a chosen base point O and $O(m)$ be a subgroup of $O(n)$ whose induced action on the tangent space at O fixed all directions of an $(n-m)$-dimensional linear subspace. The fixed point set of such an $O(m)$ is an $(n-m)$-dimensional subspace Σ^{n-m} (resp. $\widetilde{\Sigma}^{n-m}$), while its normal m-dimensional subspace at O, say denoted by Σ^m (resp. $\widetilde{\Sigma}^m$) is $O(m)$-invariant. Let $O(m-1)$ be the subgroup of $O(m)$ which fixes an additional direction. Then, the fixed point set of $O(m-1)$ is an $(n-m+1)$-dimensional subspace Σ^{n-m+1} (resp. $\widetilde{\Sigma}^{n-m+1}$) which intersect every $O(m)$-orbit perpendicularly and moreover, there is a natural bijection between the $O(1)$-orbits of Σ^{n-m+1} (resp. $\widetilde{\Sigma}^{n-m+1}$) and the $O(m)$-orbits of Σ^n (resp. $\widetilde{\Sigma}^n$), namely

$$\begin{array}{ccc} \Sigma^{n-m+1} & \xrightarrow{\subset} & \Sigma^n \\ \downarrow & & \downarrow \\ \Sigma^{n-m+1}/O(1) & \xrightarrow{\cong} & \Sigma^n/O(m), \end{array}$$

(100)

(resp.)

$$\begin{array}{ccc} \widetilde{\Sigma}^{n-m+1} & \xrightarrow{\subset} & \widetilde{\Sigma}^n \\ \downarrow & & \downarrow \\ \widetilde{\Sigma}^{n-m+1}/O(1) & \xrightarrow{\cong} & \widetilde{\Sigma}^n/O(m). \end{array}$$

Hence, the orbit spaces $\Sigma^n/O(m)$ (resp. $\widetilde{\Sigma}^n/O(m)$) equipped with the orbital distance metrics are isometric to that of $\Sigma^{n-m+1}/O(1)$ (resp. $\widetilde{\Sigma}^{n-m+1}/O(1)$), while the $O(1)$-action on Σ^{n-m+1} (resp. $\widetilde{\Sigma}^{n-m+1}$) is exactly the reflection symmetry with respect to Σ^{n-m} (resp. $\widetilde{\Sigma}^{n-m}$). Thus $\Sigma^n/O(m)$ (resp. $\widetilde{\Sigma}^n/O(m)$) is isometric to the *closed half space* Σ_+^{n-m+1} (resp. $\widetilde{\Sigma}_+^{n-m+1}$) with Σ^{n-m} (resp. $\widetilde{\Sigma}^{n-m}$) as its boundary. Set $k = n - m$ and choose a cartesian coordinate system $(x_1, \cdots, x_{k+1})$ on Σ^{k+1} (resp. $\widetilde{\Sigma}^{k+1}$) such that the above half spaces Σ_+^{k+1} (resp. $\widetilde{\Sigma}_+^{k+1}$) are given by $x_{k+1} \geq 0$.

Definition *Let* $\{A_i, 0 \leq i \leq k+1\}$ *be a sequence of* $k+2$ *points of* Σ_+^{k+1} (*resp.* $\widetilde{\Sigma}_+^{k+1}$) *whose coordinates are given by* (94) *and* $p : \Sigma^n$ (*resp.* $\widetilde{\Sigma}^n$) $\to \Sigma_+^{k+1}$ (*resp.* $\widetilde{\Sigma}_+^{k+1}$) *be the* orbital projection map. *The* inverse image *of the* $(k+1)$*-orthoscheme with* $\{A_i\}$ *as its vertices is called a spherical* (*resp. hyperbolic*) (n,k)*-orthospindle.*

Geometrically, every point $(x_1, \cdots, x_k, r) \in \Sigma_+^{k+1}$ (resp. $\widetilde{\Sigma}_+^{k+1}$) represents an $O(m)$-orbit which is isometric to an Euclidean $(m-1)$-sphere of radius $\sin r$ (resp. $\sinh r$), thus having its $(m-1)$-dimensional volume equal to

$$v(r) = \omega_m \sin^{m-1} r \quad (\text{resp.} \quad \omega_m \sinh^{m-1} r). \tag{101}$$

A volume formula of (n,k)-orthospindle: Let $\mathcal{S}(\ell_k, \ell_{k-1}, \cdots, \ell_1, B_m(r))$ (resp. $\widetilde{\mathcal{S}}(\ell_k, \ell_{k-1}, \cdots, \ell_1, B_m(r))$) be the (n,k)-orthospindle with $\{\ell_k, \ell_{k-1}, \cdots, \ell_1\}$ as its successive heights and r as its base-radius (cf. §6). Then, it is an $O(m)$-invariant body in Σ^n (resp. $\widetilde{\Sigma}^n$) whose image under orbital projection is an $(k+1)$-orthoscheme with $\{\ell_1, \cdots, \ell_{k-1}, \ell_k, r\}$ as the sequence of lengths, in the orbit space Σ_+^{k+1} (resp. $\widetilde{\Sigma}_+^{k+1}$). Therefore, its volume is given by the integration of $v(r)$ over the $(k+1)$-orthoscheme $\Omega(\ell_1, \cdots, \ell_k, r)$, namely

$$\begin{aligned} &\operatorname{vol}(\mathcal{S}(\ell_k, \cdots, \ell_1, B_m(r))) = \int_{\Omega(\ell_1, \cdots, \ell_k, r)} \omega_m \sin^{m-1} x_{k+1} d\sigma \\ &= \int_0^{\ell} dx_1 \Big\{ \int_0^{\varphi_1(x_1)} \cos x_2 dx_2 \Big\{ \cdots \Big\{ \int_0^{\varphi_k(x_k)} \omega_m \sin^{m-1} x_{k+1} \cos^k x_{k+1} dx_{k+1} \Big\} \cdots \Big\}\Big\}, \end{aligned} \tag{102}$$

$$\begin{aligned} (\text{resp.}) \qquad &\operatorname{vol}(\widetilde{\mathcal{S}}(\ell_k, \cdots, \ell_1, B_m(r))) = \int_{\Omega(\ell_1, \cdots, \ell_k, r)} \omega_m \sinh^{m-1} x_{k+1} d\sigma \\ &= \int_0^{\ell_1} dx_1 \Big\{ \int_0^{\tilde{\varphi}_1(x_1)} \cos x_2 dx_2 \Big\{ \cdots \Big\{ \int_0^{\tilde{\varphi}_k(x_k)} \omega_m \sinh^{m-1} x_{k+1} \cosh^k x_{k+1} dx_{k+1} \Big\} \cdots \Big\}\Big\}. \end{aligned} \tag{103}$$

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