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A Construction of the Rational Function Sheaves on Elliptic Curves***

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Abstract The authors introduce an effective method to construct the rational function sheaf $\mathcal{K}$ on an elliptic curve $\mathbb{E}$, and further study the relationship between $\mathcal{K}$ and any coherent sheaf on $\mathbb{E}$. Finally, it is shown that the category of all coherent sheaves of finite length on $\mathbb{E}$ is completely characterized by $\mathcal{K}$.

Keywords Elliptic curve, Coherent sheaf, Rational function sheaf
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1 Introduction

Let $\mathbb{E}$ be an elliptic curve over an algebraically closed field k. A rational function sheaf $\mathcal{K}$ on $\mathbb{E}$ is the constant sheaf having section the function field of $\mathbb{E}$. It is known that $\mathcal{K}$ is a quasi-coherent sheaf, but not a coherent sheaf. By [15] the rational function sheaf $\mathcal{K}$ on $\mathbb{E}$ is the unique big injective sheaf, i.e., $\mathcal{K}$ is the unique indecomposable injective sheaf such that $\mathrm{End}\mathcal{K}$ is a division ring and every quasi-coherent sheaf on $\mathbb{E}$ is a subquotient of a direct sum of copies of $\mathcal{K}$. In particular, each coherent sheaf is a subquotient of a finite direct sum of copies of $\mathcal{K}$, and every simple sheaf is a subquotient of $\mathcal{K}$. In [5], we proved that the rational function sheaf $\mathcal{K}$ is a generic sheaf, i.e., for all coherent sheaves $\mathcal{F}$, both $\mathrm{Hom}(\mathcal{F},\mathcal{K})$ and $\mathrm{Ext}^1(\mathcal{F},\mathcal{K})$ have finite $\mathrm{End}\mathcal{K}$-length. Therefore, it is significant to study the rational function sheaf on $\mathbb{E}$.

C. M. Ringel [14, Proposition 5.2] provided a method to construct the unique indecomposable torsionfree divisible module over the ring of tame representation type. Geigle-Lenzing [7] and Lenzing-Meltzer [11] pointed out that there is a classification of finite dimensional modules over a tubular algebra which is closely related to the Atiyah's classification of vector bundle on an elliptic curve (see [1]). All these encourage us to consider the problems: How many indecomposable torsionfree divisible objects are there in the category of quasi-coherent sheaves on an elliptic curve? And how do we construct them?

It is pleased that many main statements which have been proved in [14] for module categories also hold in the category $\mathrm{Qcoh}\mathbb{E}$ of quasi-coherent sheaves on $\mathbb{E}$ by some corresponding relationship. We show in this paper that $\mathcal{K}$ is the only indecomposable torsionfree divisible object in $\mathrm{Qcoh}\mathbb{E}$, and we can construct the rational function sheaf $\mathcal{K}$ in a way similar to Ringel's method (see [14]). Indeed, the rank functor plays an important role in this construction. Using this construction, we study the relationship between $\mathcal{K}$ and any coherent sheaf on $\mathbb{E}$, and then

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prove that the category of all coherent sheaves of finite length on $\mathbb{E}$ is completely characterized by $\mathcal{K}$.

2 The Category of Coherent Sheaves on an Elliptic Curve

By definition, an elliptic curve $\mathbb{E}$ over an algebraically closed field k is a smooth plane projective curve of genus one admitting a k-rational point p_0. Every quasi-coherent sheaf on $\mathbb{E}$ is a direct limit of coherent sheaves, and the category $\mathrm{Qcoh}\mathbb{E}$ of quasi-coherent sheaves on $\mathbb{E}$ is a locally noetherian Grothendieck category. Hence, the structure of a quasi-coherent sheaf on $\mathbb{E}$ much depends on that of coherent sheaves on $\mathbb{E}$. In this section, we recall some well-known results on the category $\mathrm{coh}\mathbb{E}$ of coherent sheaves on $\mathbb{E}$.

Lemma 2.1 (see [10]) *Let $\mathcal{H} = \mathrm{coh}\mathbb{E}$ be the category of coherent sheaves on $\mathbb{E}$.*

(1) *$\mathcal{H}$ is an Abelian, Ext-finite, noetherian, hereditary and Krull-Schmidt k-category.*

(2) *$\mathcal{H}$ is a 1-Calabi-Yau category, that is, for any two coherent sheaves $\mathcal{F}$ and $\mathcal{G}$, there is an isomorphism $\mathrm{Hom}(\mathcal{F},\mathcal{G}) \cong \mathrm{DExt}^1(\mathcal{G},\mathcal{F})$, where $\mathrm{D} = \mathrm{Hom}_k(-,k)$.*

(3) *$\mathcal{H} = \mathcal{H}_+ \bigvee \mathcal{H}_0$, that is, each indecomposable object of $\mathcal{H}$ lies either in $\mathcal{H}_+$ or in $\mathcal{H}_0$, and there are no nonzero morphisms from $\mathcal{H}_0$ to $\mathcal{H}_+$, where $\mathcal{H}_+$ denotes the full subcategory of $\mathcal{H}$ consisting of all objects which do not have a simple subobject, and $\mathcal{H}_0$ denotes the full subcategory of $\mathcal{H}$ consisting of all objects of finite length.*

Remark 2.1 From [13], we know that $\mathrm{Qcoh}\mathbb{E}$ is also hereditary.

There is an additive function $\mathrm{rk} : \mathcal{H} \to \mathbb{Z}$, called rank function, separating the objects of $\mathcal{H}_+$ and $\mathcal{H}_0$, that is, an object in $\mathcal{H}_+$ has rank > 0 and in $\mathcal{H}_0$ has rank 0. Objects of $\mathcal{H}_+$ are called bundles and those of rank one are called line bundles. In particular, $\mathcal{O}_{\mathbb{E}}$ is a line bundle. It is known that if $\mathcal{L}$ is a line bundle and $\mathcal{S}$ is a simple sheaf, then $\mathrm{Hom}(\mathcal{L},\mathcal{S}) \cong k$ (see [10]).

Lemma 2.2 (see [10]) *Line bundles have the following properties.*

(1) *Each nonzero morphism from a line bundle to any bundle is a monomorphism. In particular, the endomorphism ring of a line bundle is isomorphic to k.*

(2) *Each bundle $\mathcal{F}$ with rank n has a line bundle filtration, that is, a chain*

$$0 = \mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}_{n-1} \subseteq \mathcal{F}_n = \mathcal{F}$$

of subobjects of $\mathcal{F}$, satisfying each quotient $\mathcal{F}_{i+1}/\mathcal{F}_i$ is isomorphic to a line bundle.

Lemma 2.3 (see [10]) *$\mathcal{H}_0$ has the following characteristics.*

(1) *$\mathcal{H}_0$ is a hereditary Abelian length category with Serre duality.*

(2) *$\mathcal{H}_0$ is uniserial, and decomposes into a coproduct $\coprod\limits_{x\in\mathbb{E}} \mathcal{U}_x$ of connected uniserial subcategories, whose associated quivers are homogeneous tubes, and the mouth of each homogeneous tube is a simple sheaf.*

For objects $\mathcal{F}$, $\mathcal{G} \in \mathcal{H}$, we define

$$\langle \mathcal{F},\mathcal{G}\rangle = \dim_k \mathrm{Hom}(\mathcal{F},\mathcal{G}) - \dim_k \mathrm{Ext}^1(\mathcal{F},\mathcal{G}).$$

Then the slope of a coherent sheaf $\mathcal{F}$ is an element in $\mathbb{Q} \cup \{\infty\}$ defined as $\mu(\mathcal{F}) = \frac{\chi(\mathcal{F})}{\mathrm{rk}(\mathcal{F})}$, where $\chi(\mathcal{F}) = \langle \mathcal{O}_{\mathbb{E}}, \mathcal{F}\rangle$.

Lemma 2.4 (Riemann-Roch Formula) *For any two coherent sheaves $\mathcal{F}$ and $\mathcal{G}$ on an elliptic curve $\mathbb{E}$, we have*

$$\langle \mathcal{F}, \mathcal{G} \rangle = \chi(\mathcal{G})\mathrm{rk}(\mathcal{F}) - \chi(\mathcal{F})\mathrm{rk}(\mathcal{G}).$$

In particular, $\langle \mathcal{F}, \mathcal{G} \rangle = -\langle \mathcal{G}, \mathcal{F} \rangle$.

A coherent sheaf $\mathcal{F}$ is called stable (resp. semistable) if for any nontrivial exact sequence $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$, $\mu(\mathcal{F}') < \mu(\mathcal{F})$ (resp. $\mu(\mathcal{F}') \le \mu(\mathcal{F})$) holds.

Lemma 2.5 (see [1, 2]) *Any indecomposable coherent sheaf $\mathcal{F}$ on $\mathbb{E}$ is semistable. If two semistable coherent sheaves $\mathcal{F}$, $\mathcal{H} \in \mathrm{coh}\mathbb{E}$ satisfy $\mu(\mathcal{F}) > \mu(\mathcal{H})$, then* $\mathrm{Hom}(\mathcal{F}, \mathcal{H}) = 0$.

Lemma 2.6 (see [2, 4]) *$\mathcal{H}$ has the following detailed description.*

(1) *Let $\mathrm{coh}^{\infty}\mathbb{E}$ be the category of semistable sheaves of slope ∞. Then $\mathrm{coh}^{\infty}\mathbb{E}$ is just $\mathcal{H}_0$. The category of simple sheaves is precisely $\{k(x)\}_{x\in\mathbb{E}}$, where $k(x)$ is a skyscraper sheaf supported at x and it is the mouth of a homogeneous tube $\mathcal{T}_x$ which is the associated quiver of $\mathcal{U}_x$.*

(2) *The indecomposable objects of $\mathcal{H}$ are semistable, and*

$$\mathcal{H} = \mathrm{add}\Big(\bigcup_{q\in\mathbb{Q}\cup\{\infty\}} \mathrm{coh}^q\mathbb{E} \Big),$$

where $\mathrm{coh}^q\mathbb{E} = \{$semistable sheaves of slope $q\}$.

(3) *For any $p \in \mathbb{Q}\cup\{\infty\}$, there is an equivalence of Abelian categories $\mathrm{coh}^p\mathbb{E} \cong \mathrm{coh}^{\infty}\mathbb{E}$ induced by an autoequivalence of $D^b(\mathrm{coh}\mathbb{E})$.*

3 A Construction of the Rational Function Sheaf on Elliptic Curves

First, we extend the notions of torsion sheaves and torsionfree sheaves in [9] to $\mathrm{coh}\mathbb{E}$.

Definition 3.1 *For a quasi-coherent sheaf $\mathcal{F}$, its torsion part $t\mathcal{F}$ is defined to be the sum of all subsheaves of $\mathcal{F}$ having finite length. If $t\mathcal{F} = \mathcal{F}$, then $\mathcal{F}$ is called a torsion sheaf. If $t\mathcal{F} = 0$, i.e., $\mathrm{Hom}(\mathcal{S}, \mathcal{F}) = 0$ for each simple sheaf $\mathcal{S}$, then $\mathcal{F}$ is called torsionfree.*

It is easy to see that each object in $\mathcal{H}_0$ is torsion and each object in $\mathcal{H}_+$ is torsionfree. And the class of torsion sheaves is closed under quotients and extensions; the class of torsionfree sheaves is closed under subsheaves and extensions.

For a quasi-coherent sheaf $\mathcal{F}$ on $\mathbb{E}$, the torsion part $t\mathcal{F}$ is always a pure subsheaf of $\mathcal{F}$ (see [6]). In particular, if $\mathcal{F}$ is a coherent sheaf, $t\mathcal{F}$ is a direct summand of $\mathcal{F}$ (see [10]).

By definition, the following lemma is easy.

Lemma 3.1 *In $\mathrm{Qcoh}\mathbb{E}$, there is no nonzero morphism from a torsion sheaf to a torsionfree sheaf.*

Proof Suppose that there exist a torsion sheaf $\mathcal{F}$ and a torsionfree sheaf $\mathcal{E}$ satisfying $\mathrm{Hom}(\mathcal{F}, \mathcal{E}) \neq 0$. Each $0 \neq f \in \mathrm{Hom}(\mathcal{F}, \mathcal{E})$ induces two short exact sequences:

$$0 \longrightarrow \mathrm{Ker}f \longrightarrow \mathcal{F} \longrightarrow \mathrm{Im}f \longrightarrow 0$$

and

$$0 \longrightarrow \mathrm{Im}f \longrightarrow \mathcal{E} \longrightarrow \mathrm{Coker}f \longrightarrow 0.$$

The first short exact sequence implies that $\mathrm{Im}f$ is torsion, but the second one implies that $\mathrm{Im}f$ is torsionfree. Then $\mathrm{Im}f = t\mathrm{Im}f = 0$, a contradiction.

Let $\mathcal{F}$ be a coherent sheaf, and $\mathcal{S}$ be a simple sheaf. Then the dimension of $\mathrm{Ext}^1(\mathcal{S},\mathcal{F})$ as $\mathrm{End}\mathcal{S}$-vector space is finite. We set

$$e_{_{\mathcal{S}\mathcal{F}}} = \dim \mathrm{Ext}^1(\mathcal{S},\mathcal{F})_{\mathrm{End}\mathcal{S}}.$$

Since $\mathrm{End}\mathcal{S} \cong k$, we write $e_{_{\mathcal{S}\mathcal{F}}} = \dim_k \mathrm{Ext}^1(\mathcal{S},\mathcal{F})$.

The following lemma shows that [14, Lemma 5.2] also holds in $\mathrm{coh}\mathbb{E}$ by some corresponding relationships.

Lemma 3.2 *Let $\mathcal{F}$ be a bundle, and $\mathcal{S}$ be a simple sheaf. If there exists an exact sequence*

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}' \longrightarrow \oplus_m \mathcal{S} \longrightarrow 0,$$

where $\mathcal{F}'$ is a bundle and $\oplus_m\mathcal{S}$ denotes the direct sum of m copies of $\mathcal{S}$, then $m \le e_{_{\mathcal{S}\mathcal{F}}}$. Conversely, for $m \le e_{_{\mathcal{S}\mathcal{F}}}$, there exists such an exact sequence with $\mathcal{F}'$ being a bundle.

Proof See the proof of [14, Lemma 5.2], and we only need to replace P, X and S by $\mathcal{F}, \mathcal{F}'$ and $\mathcal{S}$ respectively.

Next, we extend the notion of divisible in [14, Definition 4.6] to $\mathrm{Qcoh}\mathbb{E}$.

Definition 3.2 *A quasi-coherent sheaf $\mathcal{F}$ is called divisible if $\mathrm{Ext}^1(\mathcal{S},\mathcal{F}) = 0$ for all simple sheaves $\mathcal{S}$.*

It is easy to check that the class of divisible sheaves is closed under quotients and extensions. And then the class of torsionfree divisible sheaves is closed under direct summands.

Lemma 3.3 *A quasi-coherent sheaf $\mathcal{I}$ is divisible if and only if it is an injective sheaf.*

Proof By definition, an injective sheaf is obviously a divisible sheaf. And by Baer's test, it is not hard to see that the "only if" part holds.

Now we can show that [14, Lemma 5.1] also holds in $\mathrm{Qcoh}\mathbb{E}$.

Lemma 3.4 *Let $\mathcal{E}, \mathcal{F}$ be torsionfree divisible sheaves, $\mathcal{E}' \subseteq \mathcal{E}$, $\mathcal{F}' \subseteq \mathcal{F}$ be subsheaves such that $\mathcal{E}/\mathcal{E}'$ and $\mathcal{F}/\mathcal{F}'$ are torsion sheaves. Then any homomorphism $\varphi' : \mathcal{E}' \to \mathcal{F}'$ has a unique extension $\varphi : \mathcal{E} \to \mathcal{F}$. In particular, if φ' is an isomorphism, then its extension φ is an isomorphism.*

Proof Consider the following two short exact sequences

$$0 \longrightarrow \mathcal{E}' \xrightarrow{\alpha} \mathcal{E} \xrightarrow{\pi} \mathcal{E}/\mathcal{E}' \longrightarrow 0$$

and

$$0 \longrightarrow \mathcal{F}' \xrightarrow{\beta} \mathcal{F} \xrightarrow{\sigma} \mathcal{F}/\mathcal{F}' \longrightarrow 0.$$

Forming the pushout of α and $\beta\varphi'$ induces the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{E}' \arrow[r, "\alpha"] \arrow[d, "\beta\varphi'"'] & \mathcal{E} \arrow[r, "\pi"] \arrow[d, "\gamma"'] & \mathcal{E}/\mathcal{E}' \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{F} \arrow[r, "\alpha'"] & \mathcal{F}'' \arrow[r, "\pi'"] & \mathcal{E}/\mathcal{E}' \arrow[r] & 0.
\end{tikzcd}$$

According to Lemma 3.3, $\mathcal{F}$ is an injective sheaf. Then $\mathrm{Ext}^1(\mathcal{E}/\mathcal{E}', \mathcal{F}) = 0$. This shows that there exists a homomorphism $\delta : \mathcal{F}'' \to \mathcal{F}$ such that $\delta\alpha' = \mathrm{id}_{\mathcal{F}}$. Let $\varphi = \delta\gamma$. Then φ is an extension of φ' and $\varphi\alpha = \delta\gamma\alpha = \delta\alpha'\beta\varphi' = \beta\varphi'$.

In order to prove the uniqueness, it is sufficient to show that the extension of zero homomorphism must be zero. If $\varphi' = 0$, then $\varphi\alpha = \beta\varphi' = 0$. Thus, there exists a unique homomorphism $\theta : \mathcal{E}/\mathcal{E}' \to \mathcal{F}$ such that $\varphi = \theta\pi$. However, $\mathcal{E}/\mathcal{E}'$ is torsion and $\mathcal{F}$ is torsionfree, so $\theta = 0$, and then $\varphi = 0$.

Now assume that φ' is an isomorphism. Since $\varphi\alpha = \beta\varphi'$, we have the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{E}' \arrow[r, "\alpha"] \arrow[d, "\varphi'"'] & \mathcal{E} \arrow[r, "\pi"] \arrow[d, "\varphi"'] & \mathcal{E}/\mathcal{E}' \arrow[r] \arrow[d, "\varphi''"'] & 0 \\
0 \arrow[r] & \mathcal{F}' \arrow[r, "\beta"] & \mathcal{F} \arrow[r] & \mathcal{F}/\mathcal{F}' \arrow[r] & 0.
\end{tikzcd}$$

Since φ' is an isomorphism, there exists a homomorphism $\psi' : \mathcal{F}' \to \mathcal{E}'$ such that $\psi'\varphi' = \mathrm{id}_{\mathcal{E}'}$ and $\varphi'\psi' = \mathrm{id}_{\mathcal{F}'}$. Using a similar argument as above, we see that ψ' has a unique extension $\psi : \mathcal{F} \to \mathcal{E}$ such that $\psi\beta = \alpha\psi'$. Therefore, $\psi'\varphi'$ has an extension $\psi\varphi$. But $\psi'\varphi' = \mathrm{id}_{\mathcal{E}'}$ has an extension $\mathrm{id}_{\mathcal{E}}$, so $\psi\varphi = \mathrm{id}_{\mathcal{E}}$ holds by the uniqueness of the extension. Similarly, we have $\varphi\psi = \mathrm{id}_{\mathcal{F}}$, and then φ is an isomorphism.

Under the previous groundwork, and by a little change of the proof of [14, Proposition 5.2], it is not hard to obtain the following theorem.

Theorem 3.1 *Let $\mathcal{F}$ be a bundle. Then there exists a torsionfree divisible sheaf $\mathcal{G}_{\mathcal{F}}$ with an exact sequence*

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G}_{\mathcal{F}} \longrightarrow \oplus_{\mathcal{S}} \oplus_{e_{\mathcal{SF}}} \mathcal{S}_\infty \longrightarrow 0,$$

where $\mathcal{S}$ runs through all simple sheaves, $\mathcal{S}_\infty$ is the direct limit of the homogeneous tube whose mouth is $\mathcal{S}$.

Proof Let $\mathcal{S}$ be a simple sheaf. According to the structure of $\mathrm{coh}\mathbb{E}$, we have $\mathrm{Ext}^1(\mathcal{S}, \mathcal{F}) \cong \mathrm{Hom}(\mathcal{F}, \mathcal{S}) \neq 0$. By Lemma 3.2, there exists a short exact sequence

$$\xi'_{\mathcal{S}} : 0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'_{\mathcal{S}} \longrightarrow \oplus_{e_{\mathcal{SF}}} \mathcal{S} \longrightarrow 0,$$

such that $\mathcal{F}'_{\mathcal{S}}$ is a bundle. Since the inclusion $\gamma_{\mathcal{S}} : \oplus_{e_{\mathcal{SF}}} \mathcal{S} \to \oplus_{e_{\mathcal{SF}}} \mathcal{S}_\infty$ induces an epimorphism $\mathrm{Ext}^1(\oplus_{e_{\mathcal{SF}}} \mathcal{S}_\infty, \mathcal{F}) \to \mathrm{Ext}^1(\oplus_{e_{\mathcal{SF}}} \mathcal{S}, \mathcal{F})$, we choose $\xi_{\mathcal{S}} \in \mathrm{Ext}^1(\oplus_{e_{\mathcal{SF}}} \mathcal{S}_\infty, \mathcal{F})$ corresponding to $\xi'_{\mathcal{S}}$. Thus, we have the following commutative diagram

$$\begin{tikzcd}
\xi'_{\mathcal{S}} : \; 0 \arrow[r] & \mathcal{F} \arrow[r] \arrow[d, equal] & \mathcal{F}'_{\mathcal{S}} \arrow[r, "\pi'_{\mathcal{S}}"] \arrow[d, "\gamma_{\mathcal{S}'}"'] & \oplus_{e_{\mathcal{SF}}} \mathcal{S} \arrow[r] \arrow[d, "\gamma_{\mathcal{S}}"'] & 0 \\
\xi_{\mathcal{S}} : \; 0 \arrow[r] & \mathcal{F} \arrow[r] & \mathcal{F}''_{\mathcal{S}} \arrow[r, "\pi_{\mathcal{S}}"] & \oplus_{e_{\mathcal{SF}}} \mathcal{S}_\infty \arrow[r] & 0.
\end{tikzcd}$$

Let $\xi = (\xi_{\mathcal{S}})_{\mathcal{S}} \in \Pi_{\mathcal{S}} \mathrm{Ext}^1(\oplus_{e_{\mathcal{SF}}} \mathcal{S}_\infty, \mathcal{F}) = \mathrm{Ext}^1(\oplus_{\mathcal{S}} \oplus_{e_{\mathcal{SF}}} \mathcal{S}_\infty, \mathcal{F})$ be as follows:

$$0 \longrightarrow \mathcal{F} \xrightarrow{\alpha} \mathcal{G}_{\mathcal{F}} \xrightarrow{\pi} \oplus_{\mathcal{S}} \oplus_{e_{\mathcal{SF}}} \mathcal{S}_\infty \longrightarrow 0,$$

where $\mathcal{S}$ runs through all simples. Then for each $\xi_{\mathcal{S}}$, we have the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F} \arrow[r, "\alpha"] \arrow[d, equal] & \mathcal{G}_{\mathcal{F}} \arrow[r, "\pi"] \arrow[d, "\sigma'_{\mathcal{S}}"'] & \oplus_{\mathcal{S}} \oplus_{e_{\mathcal{S}\mathcal{F}}} \mathcal{S}_\infty \arrow[r] \arrow[d, "\sigma_{\mathcal{S}}"'] & 0 \\
0 \arrow[r] & \mathcal{F} \arrow[r] & \mathcal{F}''_{\mathcal{S}} \arrow[r, "\pi_{\mathcal{S}}"] & \oplus_{e_{\mathcal{S}\mathcal{F}}} \mathcal{S}_\infty \arrow[r] & 0.
\end{tikzcd}$$

In order to show that $\mathcal{G}_{\mathcal{F}}$ is torsionfree, suppose that there is a nonzero monomorphism $\iota : \mathcal{T} \to \mathcal{G}_{\mathcal{F}}$, where $\mathcal{T}$ is a simple sheaf. It is easy to see from $\mathrm{Hom}(\mathcal{T}, \mathcal{F}) = 0$ that $\pi\iota \neq 0$. According to the structure of coh$\mathbb{E}$, we know $\pi\iota \in \mathrm{Hom}(\mathcal{T}, \oplus_{e_{\mathcal{T}\mathcal{F}}} \mathcal{T}_\infty)$. Thus, there is a monomorphism $\iota'' : \mathcal{T} \to \oplus_{e_{\mathcal{T}\mathcal{F}}} \mathcal{T}$ such that $\gamma_{\mathcal{T}}\iota'' = \sigma_{\mathcal{T}}\pi\iota$. By the universal property of pullback, there exists a nonzero homomorphism $f : \mathcal{T} \to \mathcal{F}'_{\mathcal{T}}$ such that $\gamma'_{\mathcal{T}} f = \sigma'_{\mathcal{T}}\iota$ and $\pi'_{\mathcal{T}} f = \iota''$, a contradiction. Thus we conclude that $\mathcal{G}_{\mathcal{F}}$ is torsionfree.

Now we are going to show that $\mathcal{G}_{\mathcal{F}}$ is divisible. Suppose that there exists a simple sheaf $\mathcal{T}$ such that $\mathrm{Ext}^1(\mathcal{T}, \mathcal{G}_{\mathcal{F}}) \neq 0$. Then a nonzero element ξ in $\mathrm{Ext}^1(\mathcal{T}, \mathcal{G}_{\mathcal{F}})$,

$$\xi : 0 \longrightarrow \mathcal{G}_{\mathcal{F}} \xrightarrow{\beta} \mathcal{E} \xrightarrow{\gamma} \mathcal{T} \longrightarrow 0,$$

induces the following commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & \mathcal{F} \arrow[r, equal] \arrow[d] & \mathcal{F} \arrow[d] & & \\
0 \arrow[r] & \mathcal{G}_{\mathcal{F}} \arrow[r, "\beta"] \arrow[d, "\pi"'] & \mathcal{E} \arrow[r, "\gamma"] \arrow[d, "\pi'"'] & \mathcal{T} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \oplus_{\mathcal{S}} \oplus_{e_{\mathcal{S}\mathcal{F}}} \mathcal{S}_\infty \arrow[r, "\beta'"] \arrow[d] & \mathcal{E}' \arrow[r, "\gamma'"] \arrow[d] & \mathcal{T} \arrow[r] & 0. \\
 & 0 & 0 & &
\end{tikzcd}$$

Since $\mathrm{Ext}^1(\mathcal{T}, \oplus_{\mathcal{S}} \oplus_{e_{\mathcal{S}\mathcal{F}}} \mathcal{S}_\infty) = 0$, the lower short exact sequence splits. Consequently, $\mathcal{E}'$ has $(\oplus_{e_{\mathcal{T}\mathcal{F}}} \mathcal{T}) \oplus \mathcal{T}$ as a subsheaf. Considering the pullback of ι' and π', where $\iota' : (\oplus_{e_{\mathcal{T}\mathcal{F}}} \mathcal{T}) \oplus \mathcal{T} \to \mathcal{E}'$ is the embedding, we have the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F} \arrow[r] \arrow[d, equal] & \mathcal{E}'' \arrow[r] \arrow[d, "\iota''"'] & (\oplus_{e_{\mathcal{T}\mathcal{F}}} \mathcal{T}) \oplus \mathcal{T} \arrow[r] \arrow[d, "\iota'"'] & 0 \\
0 \arrow[r] & \mathcal{F} \arrow[r] & \mathcal{E} \arrow[r] & \mathcal{E}' \arrow[r] & 0.
\end{tikzcd}$$

The fact that ι' is a monomorphism implies that so is ι''. On the other hand, $\mathcal{E}''$ has a subsheaf isomorphic to $\mathcal{T}$ by Lemma 3.2. Thus we have the inclusion $\sigma : \mathcal{T} \to \mathcal{E}$. It is obvious that $\gamma\sigma \neq 0$ and γ is a split monomorphism, a contradiction.

Now we begin to prove that there is only one indecomposable torsionfree divisible sheaf.

Lemma 3.5 *Let $\mathcal{L}$ be a line bundle, $\mathcal{G}$ be a torsionfree divisible sheaf. Then* $\mathrm{Hom}(\mathcal{L}, \mathcal{G}) \neq 0$.

Proof It is sufficient to prove that $\mathrm{Hom}(\mathcal{L}, \mathcal{G}) = 0$ implies $\mathcal{G} = 0$.

Suppose $\mathcal{G} \neq 0$ and $\mathrm{Hom}(\mathcal{L}, \mathcal{G}) = 0$. Since $\mathcal{G}$ is torsionfree, all the coherent subsheaves of $\mathcal{G}$ are bundles. Note that each bundle has a line bundle filtration. Then $\mathcal{G}$ has a subsheaf $\mathcal{L}'$ which is a line bundle with the inclusion $\iota : \mathcal{L}' \to \mathcal{G}$. Now we consider three possibilities.

(1) $\mu(\mathcal{L}') < \mu(\mathcal{L})$. Then $\mathrm{Hom}(\mathcal{L}', \mathcal{L}) \neq 0$, and there is a short exact sequence

$$0 \longrightarrow \mathcal{L}' \xrightarrow{\alpha} \mathcal{L} \longrightarrow \mathcal{E} \longrightarrow 0.$$

Note that $\mathcal{E} \in \mathcal{H}_0$ since $\mathrm{rk}(\mathcal{E}) = \mathrm{rk}(\mathcal{L}) - \mathrm{rk}(\mathcal{L}') = 0$. The pushout of ι and α induces the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{L}' \arrow[r, "\alpha"] \arrow[d, "\iota"'] & \mathcal{L} \arrow[r] \arrow[d, "\iota'"'] & \mathcal{E} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{G} \arrow[r, "\alpha'"] & \mathcal{F} \arrow[r] & \mathcal{E} \arrow[r] & 0.
\end{tikzcd}$$

Since $\mathcal{G}$ is a divisible sheaf, we get $\mathrm{Ext}^1(\mathcal{E}, \mathcal{G}) = 0$. Then there exists a homomorphism $\pi : \mathcal{F} \to \mathcal{G}$ such that $\pi\alpha' = \mathrm{id}_{\mathcal{G}}$. Let $f = \pi\iota' : \mathcal{L} \to \mathcal{G}$. We have $\iota = \pi\alpha'\iota = \pi\iota'\alpha = f\alpha$, which implies $f \neq 0$. This contradicts the fact that $\mathrm{Hom}(\mathcal{L}, \mathcal{G}) = 0$.

(2) $\mu(\mathcal{L}) < \mu(\mathcal{L}')$. Then $\mathrm{Hom}(\mathcal{L}, \mathcal{L}') \neq 0$, and there is a monomorphism $\iota'' : \mathcal{L} \to \mathcal{L}'$. Thus, $0 \neq \iota\iota'' : \mathcal{L} \to \mathcal{G}$ implies that $\mathrm{Hom}(\mathcal{L}, \mathcal{G}) \neq 0$. It is a contradiction.

(3) $\mu(\mathcal{L}') = \mu(\mathcal{L})$. Let $\mathcal{L}''$ be a line bundle with $\mu(\mathcal{L}'') < \mu(\mathcal{L})$. Similarly to (2), we can regard $\mathcal{L}''$ as a subsheaf of $\mathcal{G}$. Then the result is true by an analogue to case (1).

Lemma 3.6 *Let $\mathcal{G}$ be an indecomposable torsionfree divisible sheaf, $\mathcal{Q}$ be a torsionfree divisible sheaf which has no subsheaf isomorphic to $\mathcal{G}$. Then* $\mathrm{Hom}(\mathcal{G}, \mathcal{Q}) = 0$.

Proof Suppose $\mathrm{Hom}(\mathcal{G}, \mathcal{Q}) \neq 0$. Then, for each $0 \neq f \in \mathrm{Hom}(\mathcal{G}, \mathcal{Q})$, there are two short exact sequences

$$0 \longrightarrow \mathrm{Ker} f \longrightarrow \mathcal{G} \longrightarrow \mathrm{Im} f \longrightarrow 0 \tag{3.1}$$

and

$$0 \longrightarrow \mathrm{Im} f \longrightarrow \mathcal{Q} \longrightarrow \mathrm{Coker} f \longrightarrow 0 \tag{3.2}$$

We consider two possibilities.

(1) $\mathrm{Ker} f = 0$. Then $\mathcal{G} \cong \mathrm{Im} f$. Thus, $\mathcal{Q}$ has a subsheaf $\mathrm{Im} f$ isomorphic to $\mathcal{G}$. It is a contradiction.

(2) $\mathrm{Ker} f \neq 0$. Then $\mathrm{Im} f$ is divisible according to (3.1), and is torsionfree according to (3.2). Let $\mathcal{S}$ be any simple sheaf. Applying $\mathrm{Hom}(\mathcal{S}, -)$ to (3.1), we obtain the long exact sequence

$$\begin{aligned} 0 &\longrightarrow \mathrm{Hom}(\mathcal{S}, \mathrm{Ker} f) \longrightarrow \mathrm{Hom}(\mathcal{S}, \mathcal{G}) \longrightarrow \mathrm{Hom}(\mathcal{S}, \mathrm{Im} f) \\ &\longrightarrow \mathrm{Ext}^1(\mathcal{S}, \mathrm{Ker} f) \longrightarrow \mathrm{Ext}^1(\mathcal{S}, \mathcal{G}) \longrightarrow \mathrm{Ext}^1(\mathcal{S}, \mathrm{Im} f) \longrightarrow 0. \end{aligned}$$

Then $\mathrm{Ker} f$ is torsionfree divisible since $\mathcal{G}$ and $\mathrm{Im} f$ are torsionfree divisible. Thus, (3.1) splits, and then $\mathrm{Ker} f$ is a direct summand of $\mathcal{G}$. Hence $\mathrm{Ker} f \cong \mathcal{G}$ since $\mathcal{G}$ is indecomposable. This means $\mathrm{Im} f = 0$, which contradicts $f \neq 0$.

Lemma 3.7 *Let $\mathcal{L}$ be a line bundle, and $\mathcal{G}$ be an indecomposable torsionfree divisible sheaf. Then $\mathcal{G}/\mathcal{L}$ is a torsion sheaf.*

Proof By Lemma 3.5, there exists a short exact sequence

$$0 \longrightarrow \mathcal{L} \xrightarrow{\iota} \mathcal{G} \xrightarrow{\pi} \mathcal{G}/\mathcal{L} \longrightarrow 0.$$

Set $\mathcal{G}_1 = \mathcal{G}/\mathcal{L}$. Suppose that $\mathcal{G}_1$ is not a torsion sheaf, i.e., $t\mathcal{G}_1 \neq \mathcal{G}_1$. Then there is a short exact sequence

$$0 \longrightarrow t\mathcal{G}_1 \xrightarrow{\alpha} \mathcal{G}_1 \xrightarrow{\pi'} \mathcal{G}_1/t\mathcal{G}_1 \longrightarrow 0.$$

Thus, the composition $\pi'\pi : \mathcal{G} \to \mathcal{G}_1/t\mathcal{G}_1$ is an epimorphism. Note that the class of divisible sheaves is closed under quotients. We have that $\mathcal{G}_1/t\mathcal{G}_1$ is a torsionfree divisible sheaf. By Lemma 3.6, $\mathcal{G}_1/t\mathcal{G}_1$ has a subsheaf isomorphic to $\mathcal{G}$. Thus, we can regard $\mathcal{G}$ as a direct summand of $\mathcal{G}_1/t\mathcal{G}_1$. This induces an epimorphism $\beta : \mathcal{G}_1/t\mathcal{G}_1 \to \mathcal{G}$ which gives a short exact sequence

$$0 \longrightarrow \mathrm{Ker}\beta\pi'\pi \longrightarrow \mathcal{G} \xrightarrow{\beta\pi'\pi} \mathcal{G} \longrightarrow 0.$$

If $\mathrm{Ker}\beta\pi'\pi \neq 0$, then $\mathrm{Ker}\beta\pi'\pi$ is torsionfree divisible since $\mathcal{G}$ is torsionfree divisible. Thus, $\mathrm{Ker}\beta\pi'\pi$ is a direct summand of $\mathcal{G}$, and then $\mathrm{Ker}\beta\pi'\pi \cong \mathcal{G}$. This is impossible. Hence $\mathrm{Ker}\beta\pi'\pi = 0$ and then $\beta\pi'\pi$ is an isomorphism. This implies that π is a monomorphism and then π is an isomorphism. This is a contradiction. Therefore, we conclude that $\mathcal{G}_1$ is a torsion sheaf.

Theorem 3.2 *There exists a unique indecomposable torsionfree divisible sheaf. Its endomorphism ring is a division ring.*

Proof Let $\mathcal{L}$ be a line bundle. By Theorem 3.1, there exists a short exact sequence

$$0 \longrightarrow \mathcal{L} \xrightarrow{\alpha} \mathcal{G}_{\mathcal{L}} \xrightarrow{\pi} \oplus_{\mathcal{S}} \oplus_{e_{\mathcal{S}\mathcal{L}}} \mathcal{S}_\infty \longrightarrow 0,$$

where $\mathcal{G}_{\mathcal{L}}$ is a torsionfree divisible sheaf. According to the structure of coh$\mathbb{E}$, we have that $\dim \mathrm{Ext}^1(\mathcal{S},\mathcal{L})_{\mathrm{End}\mathcal{S}} = \dim_k \mathrm{Ext}^1(\mathcal{S},\mathcal{L}) = \dim_k \mathrm{Hom}(\mathcal{L},\mathcal{S}) = 1$ for each simple sheaf $\mathcal{S}$, i.e., $e_{\mathcal{S}\mathcal{L}} = 1$. Thus, the above exact sequence is just

$$0 \longrightarrow \mathcal{L} \xrightarrow{\alpha} \mathcal{G}_{\mathcal{L}} \xrightarrow{\pi} \oplus_{\mathcal{S}}\mathcal{S}_\infty \longrightarrow 0.$$

We claim that $\mathcal{G}_{\mathcal{L}}$ is the unique indecomposable torsionfree divisible sheaf. Denote $\mathcal{G}_{\mathcal{L}}$ by $\mathcal{G}$.

First, we prove that $\mathcal{G}$ is indecomposable. Let $\mathcal{G}_1$ be a direct summand of $\mathcal{G}$. Then there exist homomorphisms $\iota : \mathcal{G}_1 \to \mathcal{G}$ and $\sigma : \mathcal{G} \to \mathcal{G}_1$ such that $\sigma\iota = \mathrm{id}_{\mathcal{G}_1}$. If $\pi\iota = 0$, then there exists a homomorphism $f : \mathcal{G}_1 \to \mathcal{L}$ such that $\iota = \alpha f$ and $\sigma\alpha f = \sigma\iota = \mathrm{id}_{\mathcal{G}_1}$. This means $\mathcal{L} \cong \mathcal{G}_1$ and then $\mathcal{L}$ is a direct summand of $\mathcal{G}$. This contradicts the fact that $\mathcal{L}$ is not divisible. Therefore $\pi\iota \neq 0$. If $\mathrm{Ker}\pi\iota = 0$, then we can regard $\mathcal{G}_1$ as a subsheaf of $\oplus_{\mathcal{S}}\mathcal{S}_\infty$. This contradicts the fact that $\mathcal{G}_1$ is a torsionfree divisible. If $\mathrm{Ker}\pi\iota \neq 0$, then we have $\pi\iota\alpha_1 = 0$, where $\alpha_1 : \mathrm{Ker}\pi\iota \to \mathcal{G}_1$ is the inclusion. Thus, there exists a monomorphism $\iota' : \mathrm{Ker}\pi\iota \to \mathcal{L}$ such that $\iota\alpha_1 = \alpha\iota'$. This induces the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ker}\pi\iota \arrow[r, "\iota'"] \arrow[d, "\alpha_1"'] & \mathcal{L} \arrow[r] \arrow[d, "\alpha"'] & \mathcal{L}/\mathrm{Ker}\pi\iota \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{G}_1 \arrow[r, "\iota"] & \mathcal{G} \arrow[r, "\pi'"] & \mathcal{G}/\mathcal{G}_1 \arrow[r] & 0.
\end{tikzcd}$$

It is obvious that $\mathrm{Ker}\pi\iota$ is a line bundle and that $\mathrm{rk}(\mathcal{L}/\mathrm{Ker}\pi\iota) = \mathrm{rk}\mathcal{L} - \mathrm{rk}(\mathrm{Ker}\pi\iota) = 0$. This means that $\mathcal{L}/\mathrm{Ker}\pi\iota$ is torsion. Note that $\mathcal{G} \cong \mathcal{G}_1 \oplus \mathcal{G}/\mathcal{G}_1$. We know that $\mathcal{G}/\mathcal{G}_1$ is torsionfree and that $\mathrm{Hom}(\mathcal{L}/\mathrm{Ker}\pi\iota, \mathcal{G}/\mathcal{G}_1) = 0$. This implies that $\pi'\alpha = 0$. So there exists a homomorphism $\beta : \oplus_{\mathcal{S}}\mathcal{S}_\infty \to \mathcal{G}/\mathcal{G}_1$ such that $\pi' = \beta\pi$. Thus $\pi' = 0$ since $\mathrm{Hom}(\oplus_{\mathcal{S}}\mathcal{S}_\infty, \mathcal{G}/\mathcal{G}_1) = 0$. This means $\mathcal{G} \cong \mathcal{G}_1$. Therefore, $\mathcal{G}$ is indecomposable.

Next, we show that the endomorphism ring End$\mathcal{G}$ of $\mathcal{G}$ is a division ring. For a homomorphism $0 \neq \varphi : \mathcal{G} \to \mathcal{G}$, we claim that $\varphi(\mathcal{L}) \neq 0$. In fact, otherwise, $\varphi(\mathcal{L}) = 0$ implies $\varphi\alpha = 0$. This induces a homomorphism $\gamma : \oplus_{\mathcal{S}}\mathcal{S}_\infty \to \mathcal{G}$ such that $\varphi = \gamma\pi$. This contradicts the fact that $\mathrm{Hom}(\oplus_{\mathcal{S}}\mathcal{S}_\infty, \mathcal{G}) = 0$. Thus $\varphi(\mathcal{L}) \neq 0$. Thus, $\varphi(\mathcal{L})$, as a subsheaf of $\mathcal{G}$, is torsionfree. Note that a nonzero homomorphism from a line bundle to a torsionfree sheaf is a monomorphism. We have that $\varphi|_{\mathcal{L}} : \mathcal{L} \to \varphi(\mathcal{L})$ is an isomorphism. Following Lemma 3.4, the fact that φ is an extension of $\varphi|_{\mathcal{L}}$ implies that φ is an isomorphism. Consequently, End$\mathcal{G}$ is a division ring.

Finally, we show the uniqueness. Let $\mathcal{G}, \mathcal{G}'$ be two indecomposable torsionfree divisible sheaves. For any line bundle $\mathcal{L}$, both $\mathcal{G}/\mathcal{L}$ and $\mathcal{G}'/\mathcal{L}$ are torsion sheaves by Lemma 3.7. Thus, by Lemma 3.4, $\mathrm{id}_{\mathcal{L}}$ induces the isomorphism $\varphi : \mathcal{G} \to \mathcal{G}'$.

As we know, the rational function sheaf $\mathcal{K}$ on $\mathbb{E}$ is the constant sheaf having section the function field of $\mathbb{E}$, and $\mathcal{K}$ is an indecomposable injective sheaf. In addition, by [15], $\mathcal{K}$ is torsionfree. Thus, the uniqueness in Theorem 3.2 implies that $\mathcal{K}$ coincides with $\mathcal{G}_{\mathcal{L}}$ constructed in the proof of Theorem 3.2. In other words, the proof of Theorem 3.2 in fact gives a construction of $\mathcal{K}$.

4 The Relationship Between $\mathcal{K}$ and Coherent Sheaves

In this section, we use the construction to study the relationship between $\mathcal{K}$ and coherent sheaves on $\mathbb{E}$.

Lemma 4.1 *$\mathcal{K}$ is a direct summand of any torsionfree divisible sheaf.*

Proof Let $\mathcal{G}$ be a torsionfree divisible sheaf, and $\mathcal{L}$ be a line bundle. By Lemma 3.5, there are monomorphisms $\alpha : \mathcal{L} \to \mathcal{G}$ and $\iota : \mathcal{L} \to \mathcal{K}$. Considering the pushout of α and ι, we have the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{L} \arrow[r, "\alpha"] \arrow[d, "\iota"'] & \mathcal{G} \arrow[r] \arrow[d, "\iota'"'] & \mathcal{G}/\mathcal{L} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{K} \arrow[r, "\alpha'"] & \mathcal{G}' \arrow[r] & \mathcal{G}/\mathcal{L} \arrow[r] & 0.
\end{tikzcd}$$

Then ι' is a monomorphism. Since $\mathcal{G}$ is divisible, i.e., it is injective, there exists a homomorphism $\pi : \mathcal{G}' \to \mathcal{G}$ such that $\pi\iota' = \mathrm{id}_{\mathcal{G}}$. Set $\beta = \pi\alpha'$. We have $\beta\iota = \pi\alpha'\iota = \pi\iota'\alpha = \alpha$, and then $\beta \neq 0$. By Lemma 3.6, there is a monomorphism $\beta' : \mathcal{K} \to \mathcal{G}$. Since $\mathcal{K}$ is injective, we obtain that $\mathcal{K}$ is a direct summand of $\mathcal{G}$.

Theorem 4.1 *Any torsionfree divisible sheaf is a direct sum of copies of $\mathcal{K}$.*

Proof Let $\mathcal{G}$ be a torsionfree divisible sheaf. By transfinite induction, we shall construct a torsionfree divisible subsheaf $\mathcal{G}_\lambda$ with $\mathcal{G}_\lambda \cong \mathcal{G}_{\lambda-1} \oplus \mathcal{K}$ for any ordinal λ, and $\mathcal{G}_\lambda = \bigcup\limits_{\mu<\lambda} \mathcal{G}_\mu$ for any limit ordinal λ, such that for any λ, $\mathcal{G} \cong \mathcal{G}_\lambda \oplus \mathcal{H}_\lambda$, where $\mathcal{H}_\lambda$ is a torsionfree divisible sheaf. The construction will stop when $\mathcal{H}_\lambda = 0$.

Let $\mathcal{G}_0 = 0$. Assume that $\mathcal{G}_\lambda$ has been defined for an ordinal λ, with $\mathcal{G} \cong \mathcal{G}_\lambda \oplus \mathcal{H}_\lambda$, where $\mathcal{H}_\lambda$ is a nonzero torsionfree divisible sheaf. By Lemma 4.1, $\mathcal{K}$ is a direct summand of $\mathcal{H}_\lambda$. This means that there exists a torsionfree divisible sheaf $\mathcal{H}_{\lambda+1}$ such that $\mathcal{H}_\lambda \cong \mathcal{K} \oplus \mathcal{H}_{\lambda+1}$. Let $\mathcal{G}_{\lambda+1} = \mathcal{G}_\lambda \oplus \mathcal{K}$. Then $\mathcal{G}_{\lambda+1}$ is also a torsionfree divisible sheaf and $\mathcal{G} \cong \mathcal{G}_{\lambda+1} \oplus \mathcal{H}_{\lambda+1}$. Assume that $\mathcal{G}_\mu$ has been defined for all $\mu < \lambda$. Let $\mathcal{G}_\lambda = \bigcup\limits_{\mu<\lambda} \mathcal{G}_\mu$. We claim that $\mathcal{G}_\lambda$ is also a torsionfree divisible subsheaf of $\mathcal{G}$. In fact, suppose that there is a simple sheaf $\mathcal{S}$ satisfying $\mathrm{Hom}(\mathcal{S},\mathcal{G}_\lambda)\neq 0$; then there exists $\mu<\lambda$ such that $\mathrm{Hom}(\mathcal{S},\mathcal{G}_\mu)\neq 0$. This is a contradiction. Suppose that there is a simple sheaf $\mathcal{T}$ satisfying $\mathrm{Ext}^1(\mathcal{T},\mathcal{G}_\lambda)\neq 0$. Since $\mathrm{Qcoh}\mathbb{E}$ is a hereditary Abelian category, we have $\mathrm{Ext}^1(\mathcal{T},\mathcal{G}_\lambda)\cong \mathrm{Hom}(\mathcal{T},\mathcal{G}_\lambda[1])$ in the derived category of $\mathrm{Qcoh}\mathbb{E}$, where [1] is the translation functor (see [10, Theorem 2.1]). Then there exists $\nu<\lambda$ with $\mathrm{Hom}(\mathcal{T},\mathcal{G}_\nu[1])\neq 0$ and then $\mathrm{Ext}^1(\mathcal{T},\mathcal{G}_\nu)\neq 0$. This is also a contradiction. Thus, $\mathcal{G}_\lambda$ is also a torsionfree divisible subsheaf of $\mathcal{G}$. Then there exists a torsionfree divisible sheaf $\mathcal{H}_\lambda$ such that $\mathcal{G}\cong\mathcal{G}_\lambda\oplus\mathcal{H}_\lambda$. By the construction, we know that any $\mathcal{G}_\lambda$ is the direct sum of copies of $\mathcal{K}$. The proof is completed.

By Theorem 3.1, for each bundle $\mathcal{F}$, there exists a short exact sequence

$$0\longrightarrow\mathcal{F}\longrightarrow\mathcal{G}_{\mathcal{F}}\longrightarrow\oplus_{\mathcal{S}}\oplus_{e_{\mathcal{SF}}}\mathcal{S}_\infty\longrightarrow 0.$$

Theorem 4.2 *Let $\mathcal{F}$ be a bundle with $\mathrm{rk}\mathcal{F}=n$. Then $\mathcal{G}_{\mathcal{F}}\cong\oplus_n\mathcal{K}$.*

Proof In view of the structure of $\mathrm{coh}\mathbb{E}$, $\mathcal{F}$ has a line bundle filtration. That is, there exists a chain

$$0=\mathcal{F}_0\subseteq\mathcal{F}_1\subseteq\cdots\subseteq\mathcal{F}_{n-1}\subseteq\mathcal{F}_n=\mathcal{F}$$

of subsheaves of $\mathcal{F}$, such that each filtration quotient $\mathcal{F}_{i+1}/\mathcal{F}_i$ is isomorphic to a line bundle, denoted by $\mathcal{L}_{i+1}$. Then there are short exact sequences

$$0\longrightarrow\mathcal{F}_i\xrightarrow{\alpha_i}\mathcal{F}_{i+1}\xrightarrow{\beta_i}\mathcal{L}_{i+1}\longrightarrow 0,\quad 0\leq i\leq n-1.$$

According to Lemma 3.4, it is sufficient to construct the following exact sequence

$$0\longrightarrow\mathcal{F}_j\xrightarrow{\iota_j}\oplus_j\mathcal{K}\xrightarrow{\pi_j}\mathcal{D}_j\longrightarrow 0,\quad 0\leq j\leq n,$$

with $\mathcal{D}_j$ torsion by using induction on j. By the construction of $\mathcal{K}$, the assertion holds obviously in case $i=1$. Assume that there is a short exact sequence

$$0\longrightarrow\mathcal{F}_i\xrightarrow{\iota_i}\oplus_i\mathcal{K}\xrightarrow{\pi_i}\mathcal{D}_i\longrightarrow 0$$

with $\mathcal{D}_i$ torsion. Considering the pushout of ι_i and α_i, we have the following commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \mathcal{F}_i \arrow[r, "\alpha_i"] \arrow[d, "\iota_i"'] & \mathcal{F}_{i+1} \arrow[r] \arrow[d, "\sigma_{i+1}"'] & \mathcal{L}_{i+1} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \oplus_i\mathcal{K} \arrow[r] \arrow[d] & \mathcal{E}_{i+1} \arrow[r] \arrow[d] & \mathcal{L}_{i+1} \arrow[r] & 0. \\
 & \mathcal{D}_i \arrow[r, equal] \arrow[d] & \mathcal{D}_i \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

Then $\mathcal{E}_{i+1}/\mathcal{F}_{i+1} \cong \mathcal{D}_i$, and $\mathcal{E}_{i+1}/\mathcal{F}_{i+1}$ is torsion. In addition, the short exact sequence

$$0 \longrightarrow \oplus_i \mathcal{K} \longrightarrow \mathcal{E}_{i+1} \longrightarrow \mathcal{L}_{i+1} \longrightarrow 0$$

splits since $\oplus_i \mathcal{K}$ is an injective sheaf. This induces a short exact sequence

$$0 \longrightarrow \mathcal{L}_{i+1} \xrightarrow{\gamma_{i+1}} \mathcal{E}_{i+1} \longrightarrow \oplus_i \mathcal{K} \longrightarrow 0.$$

Again by the construction of $\mathcal{K}$, there is a short exact sequence

$$0 \longrightarrow \mathcal{L}_{i+1} \xrightarrow{\tau_{i+1}} \mathcal{K} \longrightarrow \mathcal{K}/\mathcal{L}_{i+1} \longrightarrow 0,$$

where $\mathcal{K}/\mathcal{L}_{i+1}$ is torsion. Considering the pushout of τ_{i+1} and γ_{i+1}, we have the following commutative diagram

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \mathcal{L}_{i+1} \arrow[r, "\gamma_{i+1}"] \arrow[d, "\tau_{i+1}"'] & \mathcal{E}_{i+1} \arrow[r] \arrow[d, "\sigma'_{i+1}"'] & \oplus_i \mathcal{K} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{K} \arrow[r] \arrow[d] & \mathcal{E}'_{i+1} \arrow[r] \arrow[d] & \oplus_i \mathcal{K} \arrow[r] & 0. \\
& \mathcal{K}/\mathcal{L}_{i+1} \arrow[r, equal] \arrow[d] & \mathcal{K}/\mathcal{L}_{i+1} \arrow[d] & & \\
& 0 & 0 & &
\end{tikzcd}$$

Then $\mathcal{E}'_{i+1}/\mathcal{E}_{i+1} \cong \mathcal{K}/\mathcal{L}_{i+1}$, and $\mathcal{E}'_{i+1}/\mathcal{E}_{i+1}$ is torsion. In addition, the short exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{E}'_{i+1} \longrightarrow \oplus_i \mathcal{K} \longrightarrow 0$$

splits, and $\mathcal{E}'_{i+1} \cong \oplus_{i+1} \mathcal{K}$. Set $\iota_{i+1} = \sigma'_{i+1}\sigma_{i+1}$. We have the exact sequence

$$0 \longrightarrow \mathcal{F}_{i+1} \xrightarrow{\iota_{i+1}} \oplus_{i+1} \mathcal{K} \xrightarrow{\pi_{i+1}} \oplus_{i+1} \mathcal{K}/\mathcal{F}_{i+1} \longrightarrow 0.$$

Since there is a short exact sequence

$$0 \longrightarrow \mathcal{E}_{i+1}/\mathcal{F}_{i+1} \longrightarrow \mathcal{E}'_{i+1}/\mathcal{F}_{i+1} \longrightarrow \mathcal{E}'_{i+1}/\mathcal{E}_{i+1} \longrightarrow 0$$

and the class of torsion sheaves is closed under extensions, we see that $\mathcal{E}'_{i+1}/\mathcal{F}_{i+1}$ is torsion, i.e., $\oplus_{i+1} \mathcal{K}/\mathcal{F}_{i+1}$ is torsion. This finishes the proof.

Remark 4.1 The proof of Theorem 4.2 can be simplified by taking into account the quotient category $\mathrm{Qcoh}\mathbb{E}/\mathrm{Qcoh}_0\mathbb{E}$, where $\mathrm{Qcoh}_0\mathbb{E}$ is the full subcategory of $\mathrm{Qcoh}\mathbb{E}$ consisting of all torsion sheaves. In fact, by [12], $\mathrm{Qcoh}\mathbb{E}/\mathrm{Qcoh}_0\mathbb{E} \cong \mathrm{Mod}(K)$, where K is the function field of $\mathbb{E}$, and then the bundles of rank n become an n-dimensional vector space in the quotient category. The proof we present above is more constructible.

By Theorem 4.2, $\mathrm{Hom}(\mathcal{F}, \mathcal{K}) \neq 0$ for each bundle $\mathcal{F}$. The following consequence is obvious, which is formulated in [9, Theorem 5.2] where the context is that of tubular weighted projective line.

Theorem 4.3 *We have* $\mathcal{H}_0 = {}^{\perp}\mathcal{K} \cap \mathrm{coh}(\mathbb{E})$, *where* ${}^{\perp}\mathcal{K} = \{\mathcal{F} \in \mathrm{Qcoh}\mathbb{E} \mid \mathrm{Hom}(\mathcal{F}, \mathcal{K}) = 0\}$ *is the left perpendicular category of* $\mathcal{K}$.

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