Chin. Ann. Math.
31B(5), 2010, 619–630
DOI: 10.1007/s11401-010-0605-4

Chinese Annals of Mathematics, Series B

Regular Global Attractors of Type III Thermoelastic Extensible Beams****

Michele COTI ZELATI* Vittorino PATA** Ramon QUINTANILLA***

(*Dedicated to Professor Roger Temam on the Occasion of his* 70*th Birthday*)

Abstract For $\beta \in \mathbb{R}$, the authors consider the evolution system in the unknown variables u and α

$$\begin{cases}\partial_{tt}u + \partial_{xxxx}u + \partial_{xxt}\alpha - (\beta + \|\partial_x u\|_{L^2}^2)\partial_{xx}u = f,\\ \partial_{tt}\alpha - \partial_{xx}\alpha - \partial_{xxt}\alpha - \partial_{xxt}u = 0\end{cases}$$

describing the dynamics of type III thermoelastic extensible beams, where the dissipation is entirely contributed by the second equation ruling the evolution of the thermal displacement α. Under natural boundary conditions, the existence of the global attractor of optimal regularity for the related dynamical system acting on the phase space of weak energy solutions is established.

Keywords Type III thermoelastic extensible beam, Lyapunov functional, Global attractor
2000 MR Subject Classification 35B41, 37B25, 74F05, 74K10

1 Introduction

After the pioneering work of Woinowsky-Krieger [28], a great interest has been devoted to the study of nonlinear evolution problems modelling the vibrations of extensible beams. We may mention [1, 5, 6, 13, 25], along with the more recent papers [2, 3, 8, 9] dealing with thermoelastic and viscoelastic beams or plates.

The system of equations describing the transverse deformations of an extensible thermoelastic homogeneous beam of unitary natural length, obtained by combining the ideas of [14] and [28], reads

$$\begin{cases}\partial_{tt}u + \partial_{xxxx}u + \partial_{xx}\theta - \Big(\beta + \displaystyle\int_0^1 |\partial_x u(x,\cdot)|^2 \mathrm{d}x\Big)\partial_{xx}u = f,\\ \partial_t\theta - \partial_{xx}\theta - \partial_{xxt}u = 0,\end{cases} \tag{1.1}$$

Manuscript received June 3, 2010. Published online August 25, 2010.
*Mathematics Department, Indiana University, Rawles Hall, Bloomington, IN 47405, USA.
E-mail: micotize@indiana.edu
**Dipartimento di Matematica "F. Brioschi", Politecnico di Milano, Via Bonardi 9, 20133 Milano, Italy.
E-mail: vittorino.pata@polimi.it
***Matemática Aplicada 2 ETSEIAT-UPC Terrassa, Colom 11, 08222 Terrassa, Barcelona, Spain.
E-mail: ramon.quintanilla@upc.edu
****Project supported by the Spanish Ministry of Science and Technology through the Project "Partial Defferential Equations in Thermomechanics. Theory and Applications" (No. MTM2009-08150).

in the unknowns $u = u(x,t) : [0,1] \times \mathbb{R}^+ \to \mathbb{R}$ and $\theta = \theta(x,t) : [0,1] \times \mathbb{R}^+ \to \mathbb{R}$, having set for simplicity all the physical parameters, except β, equal to one. Here, u represents the vertical deflection of the beam from its configuration reference and θ is the relative temperature from a reference value, while $f = f(x)$ is the external load distribution, assumed constant in time for simplicity. The parameter $\beta \in \mathbb{R}$ accounts for the axial force acting in the reference configuration, positive when the beam is stretched and negative when compressed.

However, classical heat conduction based on the Fourier law has encountered strong criticism, since it predicts the instantaneous propagation of thermal signals. Such an effect, often referred to [7] as the paradox of heat conduction, is physically unrealistic; reason why many alternative theories have been proposed through the years. Among those, we quote the one by Green and Naghdi [10–12]. In fact, they established three different theories, labelled as type I, II and III, respectively, based on an entropy balance law rather than on the usual entropy inequality. Linearizing type I theory, one recovers classical thermoelasticity. Type II thermoelasticity, also known as thermoelasticity without energy dissipation, is fully hyperbolic and the energy is conserved; consequently, thermoelastic disturbances propagate with finite speed. In this theory, the thermal displacement

$$\alpha(x,t) = \alpha(x,0) + \int_0^t \theta(x,s)\mathrm{d}s$$

plays a relevant role. Indeed, its gradient is included in the list of independent variables, whereas the gradient of the temperature is omitted. In type III thermoelasticity, the independent variables are the gradient of the displacement, the temperature and its gradient, and the gradient of the thermal displacement. For homogeneous materials, the constitutive equation for the heat flux vector takes the form

$$q = k_1 \partial_x \alpha + k_2 \partial_x \theta,$$

in place of the classical Fourier law, where $k_\imath > 0$ are constants that we set equal to one in the rest of the paper.

The theories of Green and Naghdi, albeit relatively recent, have become quite popular in the scientific community, especially among mathematicians. We recall results concerning uniqueness of solutions (see [21, 23]) and their spatial behavior (see [20]), as well as several kind of wave propagation phenomena (see [19, 24]). But, probably, the main efforts have been directed towards the asymptotic analysis of the related models (see [15–18, 22, 26, 29, 30]).

Our aim is to proceed along this line, analyzing the longterm behavior of type III thermoelastic extensible beams. Accordingly, we replace the second equation of (1.1), ruling the thermal evolution, with the one arising in the framework of type III thermoelasticity. Therefore, in light of the relation $\partial_t \alpha = \theta$, we are led to the new evolution system

$$\begin{cases} \partial_{tt} u + \partial_{xxxx} u + \partial_{xxt} \alpha - \Big(\beta + \displaystyle\int_0^1 |\partial_x u(x,\cdot)|^2 \mathrm{d}x\Big)\partial_{xx} u = f, \\ \partial_{tt} \alpha - \partial_{xx} \alpha - \partial_{xxt} \alpha - \partial_{xxt} u = 0. \end{cases} \tag{1.2}$$

System (1.2) is supplemented by the initial conditions

$$\begin{cases} u(x,0)=u_0(x), & \partial_t u(x,0)=u_1(x), \\ \alpha(x,0)=\alpha_0(x), & \partial_t \alpha(x,0)=\alpha_1(x), \end{cases} \tag{1.3}$$

where $u_0, u_1, \alpha_0, \alpha_1$ are assigned functions defined on the interval $[0,1]$. Moreover, we assume Dirichlet boundary conditions for the thermal displacement α and hinged boundary conditions for the vertical deflection u; namely, for all $t \geq 0$,

$$\begin{cases} u(0,t)=u(1,t)=\partial_{xx}u(0,t)=\partial_{xx}u(1,t)=0, \\ \alpha(0,t)=\alpha(1,t)=0. \end{cases} \tag{1.4}$$

If $f=f(x)\in L^2(0,1)$, problem (1.2)–(1.4) is shown to generate a strongly continuous semigroup, or dynamical system, $S(t)$ acting on the natural phase space

$$[H^2(0,1)\cap H_0^1(0,1)]\times L^2(0,1)\times H_0^1(0,1)\times L^2(0,1)$$

of weak energy solutions. The main result of this paper read as follows.

Theorem 1.1 *The semigroup $S(t)$ possesses the global attractor.*

In fact, several generalizations of the problem are possible. For instance, one can allow the presence of an external heat supply in the second equation of (1.2), which is easily handled by means of standard techniques. It would be also interesting to carry out the same kind of analysis assuming different boundary conditions for the vertical displacement, such as clamped, or one end clamped and the other one hinged. In that case, the analysis is more complicated, and a major modification of the needed tools is required. A further step is proving the existence of regular exponential attractors having finite fractal dimension. However, this is a relatively simple task; indeed, as shown in the next sections, the dynamical system possesses regular exponentially attracting sets. Finally, one can investigate the nonautonomous case, with a time-dependent external force. We address the reader to the paper [8], where the above issues are discussed in connection with the classical model (1.1).

2 The Abstract Dynamical System

Denoting $H=L^2(0,1)$ with inner product and norm $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively, we introduce the strictly positive selfadjoint operator on H

$$A=\frac{\mathrm{d}^4}{\mathrm{d}x^4}, \quad \mathrm{dom}(A)=\{\phi\in H^4(0,1):\phi(0)=\phi(1)=\phi''(0)=\phi''(1)=0\}.$$

In which case, the powers of A are well-defined; in particular,

$$A^{\frac{1}{2}}=-\frac{\mathrm{d}^2}{\mathrm{d}x^2}, \quad \mathrm{dom}(A^{\frac{1}{2}})=H^2(0,1)\cap H_0^1(0,1).$$

Thus, we can rewrite system (1.2) with boundary conditions (1.4) in the abstract form

$$\begin{cases} \partial_{tt}u+Au-A^{\frac{1}{2}}\partial_t\alpha+(\beta+\|A^{\frac{1}{4}}u\|^2)A^{\frac{1}{2}}u=f, \\ \partial_{tt}\alpha+A^{\frac{1}{2}}\alpha+A^{\frac{1}{2}}\partial_t\alpha+A^{\frac{1}{2}}\partial_t u=0. \end{cases} \tag{2.1}$$

Remark 2.1 We point out that the results of our paper hold for the abstract system (2.1), where A is any strictly positive selfadjoint operator on some separable real Hilbert space H with compact embedding $\mathrm{dom}(A) \Subset H$. This is noteworthy, since (2.1) serves as a model to describe quite general situations, including thermoelastic plates.

For $r \in \mathbb{R}$, we introduce the scale of Hilbert spaces generated by the powers of A

$$H^r = \mathrm{dom}(A^{\frac{r}{4}}), \quad \langle u, v\rangle_r = \langle A^{\frac{r}{4}} u, A^{\frac{r}{4}} v\rangle, \quad \|u\|_r = \|A^{\frac{r}{4}} u\|.$$

We will always omit the index r if $r = 0$. In particular, $H^{r_1} \Subset H^{r_2}$ whenever $r_1 > r_2$. Then, we define the product Hilbert spaces with compact embedding

$$\mathcal{V} := H^4 \times H^2 \times H^3 \times H^2 \Subset \mathcal{H} := H^2 \times H \times H^1 \times H,$$

endowed with the standard inner products and norms. The well-posedness result for (2.1) in the weak energy space $\mathcal{H}$ is stated in the next proposition, whose proof, based on a standard Galerkin approximation scheme, is omitted.

Proposition 2.1 *For every $\beta \in \mathbb{R}$ and $f \in H$, system* (2.1) *generates a semigroup of solutions $S(t) : \mathcal{H} \to \mathcal{H}$ satisfying the joint continuity property*

$$(t, z) \mapsto S(t)z \in \mathcal{C}(\mathbb{R}^+ \times \mathcal{H}, \mathcal{H}).$$

Accordingly, for every $t \geq 0$ and every $z = (u_0, u_1, \alpha_0, \alpha_1) \in \mathcal{H}$,

$$S(t)z = (u(t), \partial_t u(t), \alpha(t), \partial_t \alpha(t))$$

is the unique weak solution at time t to (2.1) with initial datum z.

3 The Lyapunov Functional

The energy corresponding to the initial datum $z = (u_0, u_1, \alpha_0, \alpha_1) \in \mathcal{H}$ at time $t \geq 0$ is given by

$$E(t) = \frac{1}{2}\|S(t)z\|^2_{\mathcal{H}} + \frac{1}{4}(\beta + \|u(t)\|_1^2)^2.$$

Indeed, multiplying the first equation of (2.1) by $\partial_t u$ and the second one by $\partial_t \alpha$, we readily obtain the energy identity

$$\frac{\mathrm{d}}{\mathrm{d}t} E + \|\partial_t \alpha\|_1^2 = \langle f, \partial_t u\rangle. \tag{3.1}$$

The next step is to prove the existence of a Lyapunov functional for the semigroup. This is a function $\mathfrak{L} \in \mathcal{C}(\mathcal{H}, \mathbb{R})$ such that

(i) $\mathfrak{L}(S(t)z) \leq \mathfrak{L}(z)$ for every $z \in \mathcal{H}$ and every $t \geq 0$,

(ii) $\mathfrak{L}(S(t)z) = \mathfrak{L}(z)$ for all $t \geq 0$ implies that $z \in \mathcal{S}$,

where

$$\mathcal{S} = \{\overline{z} \in \mathcal{H} : S(t)\overline{z} = \overline{z},\ \forall\, t \geq 0\}$$

is the set of stationary points of $S(t)$. It is easily seen that $\mathcal{S}$ is made of all vectors of the form $(\overline{u}, 0, 0, 0)$, with $\overline{u}$ solution to the elliptic equation

$$A\overline{u} + (\beta + \|\overline{u}\|_1^2)A^{\frac{1}{2}}\overline{u} = f. \tag{3.2}$$

In particular, $\mathcal{S}$ turns out to be a nonempty bounded subset of $\mathcal{V}$.

Remark 3.1 According to [4], the cardinality of $\mathcal{S}$ does not exceed $4n_\star + 1$ whenever all the eigenvalues λ_n of A satisfying the relation $\beta < -\sqrt{\lambda_n}$ are simple, where $n_\star$ is the (finite) cardinality of the set $\{n \in \mathbb{N} : \beta < -\sqrt{\lambda_n}\}$. This is the case for the concrete realization (1.2)–(1.4) of (2.1), where $\lambda_n = n^4\pi^4$.

For $z = (u_0, u_1, \alpha_0, \alpha_1) \in \mathcal{H}$, we set

$$\mathfrak{L}(z) = \frac{1}{2}\|z\|_{\mathcal{H}}^2 + \frac{1}{4}(\beta + \|u_0\|_1^2)^2 - \langle f, u_0\rangle.$$

Proposition 3.1 *The above-defined $\mathfrak{L}$ is a Lyapunov functional for $S(t)$. Moreover,*

$$\|z\|_{\mathcal{H}} \to \infty \Leftrightarrow \mathfrak{L}(z) \to \infty.$$

Proof Property (i) is a consequence of the energy identity (3.1), which translates into

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathfrak{L}(S(t)z) + \|\partial_t\alpha(t)\|_1^2 = 0. \tag{3.3}$$

Concerning (ii), if $\mathfrak{L}$ is constant along a trajectory, it follows that $\partial_t\alpha(t) = 0$ for all $t \geq 0$. From the second equation of (2.1), we learn that $\partial_{tt}u(t) = 0$. Hence, using the first equation, we conclude that $u(t)$ is constant in time and solves (3.2). Therefore, the second equation reads $A^{\frac{1}{2}}\alpha(t) = 0$, yielding $\alpha(t) = 0$ for all times. Finally, it is immediate to verify that

$$\frac{1}{4}\|z\|_{\mathcal{H}}^2 - c \leq \mathfrak{L}(z) \leq c\|z\|_{\mathcal{H}}^4 + c$$

for some $c > 0$ independent of z, proving the last assertion.

Remark 3.2 A dynamical system $S(t)$ with a Lyapunov functional complying with Proposition 3.1 is commonly called a gradient system.

4 The Global Attractor

In the theory of dynamical systems (see [13, 27]), the global attractor of a semigroup $S(t)$ acting on a Banach space $\mathcal{H}$ is the unique compact set $\mathfrak{A} \subset \mathcal{H}$ at the same time fully invariant and attracting for the semigroup, i.e., $S(t)\mathfrak{A} = \mathfrak{A}$ for all $t \geq 0$, and

$$\lim_{t\to\infty} \mathrm{dist}_{\mathcal{H}}(S(t)\mathcal{B}, \mathfrak{A}) = \prime$$

for every bounded set $\mathcal{B} \subset \mathcal{H}$, where $\mathrm{dist}_{\mathcal{H}}$ denotes the usual Hausdorff semidistance in $\mathcal{H}$, defined as

$$\mathrm{dist}_{\mathcal{H}}(\mathcal{B}_1, \mathcal{B}_2) = \sup_{z_1\in\mathcal{B}_\infty} \inf_{z_2\in\mathcal{B}_2} \|z_1 - z_2\|_{\mathcal{H}}.$$

The global attractor is a crucial object in order to understand the asymptotic dynamics, since it captures all the trajectories in the longterm, uniformly with respect to any bounded set of initial data. When the semigroup, as in our case, possesses a Lyapunov functional, further information on the structure of the attractor can be drawn. Namely, recalling that a function $\zeta : \mathbb{R} \to \mathcal{H}$ is a complete bounded trajectory (c.b.t.) of $S(t)$ when

$$\sup_{\tau \in \mathbb{R}} \|\zeta(\tau)\|_{\mathcal{H}} < \infty$$

and

$$\zeta(t+\tau) = S(t)\zeta(\tau), \quad \forall\, t \geq 0,\ \forall\, \tau \in \mathbb{R},$$

the attractor $\mathfrak{A}$ coincides with the unstable set of $\mathcal{S}$, that is,

$$\mathfrak{A} = \Big\{ \zeta(0) : \zeta \text{ c.b.t. and } \lim_{\tau \to -\infty} \|\zeta(\tau) - \mathcal{S}\|_{\mathcal{H}} = 0 \Big\}.$$

In particular, if $\mathcal{S}$ has finite cardinality,

$$\mathfrak{A} = \Big\{ \zeta(0) : \lim_{\tau \to -\infty} \|\zeta(\tau) - \overline{z}_1\|_{\mathcal{H}} = \lim_{\tau \to \infty} \|\zeta(\tau) - \overline{z}_2\|_{\mathcal{H}} = 0 \Big\},$$

for some $\overline{z}_\imath \in \mathcal{S}$.

We are now in a position to state a generalized version of Theorem 1.1 for the abstract semigroup $S(t)$ on $\mathcal{H}$ generated by system (2.1).

Theorem 4.1 *For every $\beta \in \mathbb{R}$ and $f \in H$, the unstable set $\mathfrak{A}$ of $\mathcal{S}$ is the global attractor of $S(t)$. In addition, $\mathfrak{A}$ is connected and bounded in $\mathcal{V}$.*

Within our hypotheses, the attained regularity of $\mathfrak{A}$ is optimal. On the other hand, the global attractor is as regular as f permits. For instance, if $f \in H^n$ for every $n \in \mathbb{N}$, then each component of $\mathfrak{A}$ is bounded in H^n for every $n \in \mathbb{N}$.

Actually, we establish a stronger result. Indeed, as will be clear in the next sections, we show the existence of a bounded subset of $\mathcal{V}$ attracting all bounded subsets of $\mathcal{H}$ exponentially fast with respect to the Hausdorff semidistance (see Remark 5.1). Incidentally, this allows to demonstrate the existence of regular exponential attractors for $S(t)$ having finite fractal dimension in $\mathcal{H}$. Since the global attractor is the minimal closed attracting set, the fractal dimension of $\mathfrak{A}$ in $\mathcal{H}$ turns out to be finite as well.

Taking advantage of the gradient system structure of the semigroup ensured by Proposition 3.1, a standard way to prove Theorem 4.1 is showing the asymptotic compactness of $S(t)\mathcal{B}$ for any given bounded set $\mathcal{B} \subset \mathcal{H}$. More precisely, we have the following result (see e.g. [13]).

Lemma 4.1 *For every fixed $R \geq 0$, let the semigroup $S(t)$ admit the decomposition*

$$S(t) = S_0(t) + S_1(t),$$

where the one-parameter operators $S_0(t)$ and $S_1(t)$ fulfill, uniformly as $\|z\|_{\mathcal{H}} \leq R$,

$$\lim_{t \to \infty} \|S_0(t)z\|_{\mathcal{H}} = 0 \quad \textit{and} \quad \sup_{t \geq 0} \|S_1(t)z\|_{\mathcal{V}} \leq C,$$

for some $C = C(R) \geq 0$. *Then, the conclusions of Theorem* 4.1 *hold.*

Section 5 is devoted to the proof of Theorem 4.1, carried out by verifying the hypotheses of Lemma 4.1. This method permits to find the attractor without proving first the existence of an absorbing set, i.e., a bounded set $\mathcal{B}_0 \subset \mathcal{H}$ such that, for every bounded set $\mathcal{B} \subset \mathcal{H}$,

$$S(t)\mathcal{B} \subset \mathcal{B}_0, \quad \forall\, t \geq t_0,$$

for some entering time $t_0 = t_0(\mathcal{B}) \geq 0$. In fact, the existence of $\mathcal{B}_0$ is recovered as a byproduct of Theorem 4.1.

5 Proof of Theorem 4.1

Let $R \geq 0$ be fixed. Till the end of the section, $C = C(R)$ will denote a generic positive constant depending only on R, besides the structural quantities of the model. Moreover, let $z = (u_0, u_1, \alpha_0, \alpha_1) \in \mathcal{H}$ be a generic initial datum of the problem subject to the bound $\|z\|_{\mathcal{H}} \leq R$.

Thanks to Proposition 3.1, and integrating (3.3) on $\mathbb{R}^+$, we draw the uniform (with respect to $\|z\|_{\mathcal{H}} \leq R$) controls

$$\|S(t)z\|_{\mathcal{H}} \leq C \tag{5.1}$$

and

$$\int_0^\infty \|\partial_t \alpha(\tau)\|_1^2 \mathrm{d}\tau \leq C. \tag{5.2}$$

The following dissipation integral for the norm of $\partial_t u$ will be crucial in the proofs.

Lemma 5.1 *For every fixed* $\nu \in (0, 1]$, *the integral estimate*

$$\int_s^t \|\partial_t u(\tau)\|^2 \mathrm{d}\tau \leq \nu(t-s) + \frac{C}{\nu} \tag{5.3}$$

holds for all $t > s \geq 0$.

Proof By direct calculations, the functional

$$\Psi(t) = \langle \partial_t u(t), \partial_t \alpha(t)\rangle_{-1} + \langle u(t), \alpha(t)\rangle$$

satisfies the relation

$$\frac{\mathrm{d}}{\mathrm{d}t}\Psi + \|\partial_t u\|^2 = \|\partial_t \alpha\|^2 - \langle u, \partial_t \alpha\rangle_1 - \langle \partial_t u, \partial_t \alpha\rangle + (1 - \beta - \|u\|_1^2)\langle u, \partial_t \alpha\rangle + \langle f, \partial_t \alpha\rangle_{-1}.$$

From (5.1), we learn that $|\Psi| \leq C$, while the right-hand side in the above equality is bounded by

$$C\|\partial_t \alpha\|_1^2 + C\|\partial_t \alpha\|_1 \leq \nu + \frac{C}{\nu}\|\partial_t \alpha\|_1^2.$$

Thus, an integration on the interval (s, t) yields

$$\int_s^t \|\partial_t u(\tau)\|^2 \mathrm{d}\tau \leq \nu(t-s) + C + \frac{C}{\nu}\int_s^t \|\partial_t \alpha(\tau)\|_1^2 \mathrm{d}\tau,$$

and (5.2) completes the argument.

Borrowing a technique from [9], leaning on the interpolation inequality

$$\|u\|_1^2 \le \|u\|\|u\|_2,$$

we now choose $\gamma > 0$ large enough such that

$$\frac{1}{4}\|u\|_2^2 \le \frac{1}{2}\|u\|_2^2 + \beta\|u\|_1^2 + \gamma\|u\|^2. \tag{5.4}$$

Then, we split the solution $S(t)z$ into the sum

$$S(t)z = S_0(t)z + S_1(t)z, \tag{5.5}$$

where

$$S_0(t)z = (v(t), \partial_t v(t), \eta(t), \partial_t \eta(t)) \quad \text{and} \quad S_1(t)z = (w(t), \partial_t w(t), \xi(t), \partial_t \xi(t))$$

solve the Cauchy problems

$$\begin{cases} \partial_{tt} v + Av - A^{\frac{1}{2}}\partial_t \eta + (\beta + \|u\|_1^2)A^{\frac{1}{2}} v + \gamma v = 0, \\ \partial_{tt}\eta + A^{\frac{1}{2}}\eta + A^{\frac{1}{2}}\partial_t\eta + A^{\frac{1}{2}}\partial_t v = 0, \\ (v(0), \partial_t v(0), \eta(0), \partial_t\eta(0)) = z, \end{cases} \tag{5.6}$$

and

$$\begin{cases} \partial_{tt} w + Aw - A^{\frac{1}{2}}\partial_t \xi + (\beta + \|u\|_1^2)A^{\frac{1}{2}} w - \gamma v = f, \\ \partial_{tt}\xi + A^{\frac{1}{2}}\xi + A^{\frac{1}{2}}\partial_t\xi + A^{\frac{1}{2}}\partial_t w = 0, \\ (w(0), \partial_t w(0), \xi(0), \partial_t\xi(0)) = 0. \end{cases} \tag{5.7}$$

We are left to show that $S_0(t)$ and $S_1(t)$ comply with the assumptions of Lemma 4.1.

Lemma 5.2 *There exists* $\omega = \omega(R) > 0$ *such that*

$$E_0(t) := \|S_0(t)z\|_{\mathcal{H}}^2 \le C\mathrm{e}^{-\omega t}.$$

Proof For $\varepsilon > 0$ to be fixed later, introduce the functional

$$\Lambda_0(t) = \Gamma_0(t) + \varepsilon\{\Phi_0(t) + 2\Psi_0(t) + \Theta_0(t)\},$$

having set

$$\begin{aligned} \Gamma_0(t) &= E_0(t) + \beta\|v(t)\|_1^2 + \gamma\|v(t)\|^2 + \|u(t)\|_1^2\|v(t)\|_1^2, \\ \Phi_0(t) &= \langle \partial_t v(t), v(t)\rangle, \\ \Psi_0(t) &= \langle \partial_t v(t), \partial_t\eta(t)\rangle_{-1} + \langle v(t), \eta(t)\rangle, \\ \Theta_0(t) &= \langle \partial_t\eta(t), \eta(t)\rangle + \langle v(t), \eta(t)\rangle_1. \end{aligned}$$

In light of (5.1) and (5.4), for all ε small enough,

$$\frac{1}{2}E_0(t) \le \Lambda_0(t) \le CE_0(t). \tag{5.8}$$

Using the equations of system (5.6) and the uniform bound (5.1), we estimate the time derivative of each single component of Λ_0. The first one reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\Gamma_0 + 2\|\partial_t\eta\|_1^2 = 2\|v\|_1^2\langle A^{\frac{1}{2}}u, \partial_t u\rangle \le C\|\partial_t u\|E_0. \tag{5.9}$$

Concerning Φ_0,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Phi_0 + \|v\|_2^2 + \beta\|v\|_1^2 + \gamma\|v\|^2 + \|u\|_1^2\|v\|_1^2 &= \|\partial_t v\|^2 + \langle v, \partial_t\eta\rangle_1 \\ &\le \frac{1}{4}\|v\|_2^2 + \|\partial_t v\|^2 + C\|\partial_t\eta\|_1^2,\end{aligned}$$

and exploiting (5.4), we arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_0 + \frac{1}{2}\|v\|_2^2 \le \|\partial_t v\|^2 + C\|\partial_t\eta\|_1^2. \tag{5.10}$$

Finally,

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\Psi_0 + \|\partial_t v\|^2 \\ &= -\langle\partial_t v, \partial_t\eta\rangle - \langle v, \partial_t\eta\rangle_1 + \|\partial_t\eta\|^2 + (1-\beta-\|u\|_1^2)\langle v, \partial_t\eta\rangle - \gamma\langle v, \partial_t\eta\rangle_{-1} \\ &\le \frac{1}{16}\|v\|_2^2 + \frac{1}{4}\|\partial_t v\|^2 + C\|\partial_t\eta\|_1^2,\end{aligned} \tag{5.11}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\Theta_0 + \|\eta\|_1^2 = \|\partial_t\eta\|^2 + \langle v, \partial_t\eta\rangle_1 - \langle\eta, \partial_t\eta\rangle_1 \le \frac{1}{8}\|v\|_2^2 + \frac{1}{2}\|\eta\|_1^2 + C\|\partial_t\eta\|_1^2. \tag{5.12}$$

Adding (5.9) and ε-times (5.10)–(5.12), we end up with

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\Lambda_0 + \frac{\varepsilon}{4}(\|v\|_2^2 + 2\|\partial_t v\|^2 + 2\|\eta\|_1^2) + (2-\varepsilon C)\|\partial_t\eta\|_1^2 \\ &\le C\|\partial_t u\|E_0 \le \frac{\varepsilon}{8}E_0 + C\|\partial_t u\|^2 E_0.\end{aligned}$$

In view of (5.8), it is apparent that, up to taking ε small enough (depending on C which in turn depends on R), the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda_0 + 2\omega\Lambda_0 \le \psi\Lambda_0$$

holds for some $\omega = \omega(R) > 0$, where we set $\psi(t) = C\|\partial_t u(t)\|^2$ and fix ν in (5.3) sufficiently small in order to have

$$\int_s^t \psi(\tau)\mathrm{d}\tau \le \omega(t-s) + C.$$

On account of a Gronwall-type lemma (see e.g. [8, 9]), this allows us to conclude that

$$\Lambda_0(t) \le C\Lambda_0(0)\mathrm{e}^{-\omega t}.$$

A further use of (5.8) yields the desired claim.

Lemma 5.3 *We have the uniform estimate*

$$\sup_{t\ge 0}\|S_1(t)z\|_{\mathcal{V}} \le C.$$

Proof For $\varepsilon > 0$ to be fixed later, and setting for simplicity $E_1(t) = \|S_1(t)z\|_{\mathcal{V}}^2$, we define the functional

$$\Lambda_1(t) = \Gamma_1(t) + \varepsilon\{\Phi_1(t) + 2\Psi_1(t) + \Theta_1(t)\},$$

where

$$\begin{aligned}
\Gamma_1(t) &= E_1(t) + \beta\|w(t)\|_3^2 + \|u(t)\|_1^2\|w(t)\|_3^2 - 2\langle f, Aw(t)\rangle,\\
\Phi_1(t) &= \langle \partial_t w(t), w(t)\rangle_2,\\
\Psi_1(t) &= \langle \partial_t w(t), \partial_t \xi(t)\rangle_1 + \langle w(t), \xi(t)\rangle_2,\\
\Theta_1(t) &= \langle \partial_t \xi(t), \xi(t)\rangle_2 + \langle w(t), \xi(t)\rangle_3.
\end{aligned}$$

Due to (5.1) and the bound $\|v\|_2 \le C$, ensured by the previous lemma,

$$\|w\|_3^2 \le \|w\|_2\|w\|_4 \le (\|u\|_2 + \|v\|_2)\|w\|_4 \le C\|w\|_4.$$

It is then an easy matter to see that, for ε small enough,

$$\frac{1}{2}E_1(t) - C \le \Lambda_1(t) \le CE_1(t) + C. \tag{5.13}$$

Again, we evaluate the time derivative of every single functional, making use of the equations of system (5.7). For Γ_1, the standard higher order estimates entail

$$\frac{\mathrm{d}}{\mathrm{d}t}\Gamma_1 + 2\|\partial_t\xi\|_3^2 = 2\gamma\langle v, \partial_t w\rangle_2 + 2\|w\|_3^2\langle A^{\frac12}u, \partial_t u\rangle \le \frac{\varepsilon}{4}\|\partial_t w\|_2^2 + \frac{\varepsilon}{8}\|w\|_4^2 + \frac{C}{\varepsilon}. \tag{5.14}$$

As far as Φ_1 and Ψ_1 are concerned, we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\Phi_1 + \|w\|_4^2 &= \|\partial_t w\|_2^2 + \langle \partial_t \xi, w\rangle_3 + \langle f, Aw\rangle + \gamma\langle v, w\rangle_2 - (\beta + \|u\|_1^2)\|w\|_3^2\\
&\le \|\partial_t w\|_2^2 + \frac{1}{8}\|w\|_4^2 + C\|\partial_t\xi\|_3^2 + C, \qquad (5.15)
\end{aligned}$$

and

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\Psi_1 + \|\partial_t w\|_2^2 &= \|\partial_t \xi\|_2^2 + \langle w, \partial_t\xi\rangle_2 - \langle \partial_t w, \partial_t \xi\rangle_2 - \langle w, \partial_t \xi\rangle_3\\
&\qquad - (\beta + \|u\|_1^2)\langle w, \partial_t\xi\rangle_2 + \gamma\langle v, \partial_t\xi\rangle_1 + \langle f, A^{\frac12}\partial_t\xi\rangle\\
&\le \frac{1}{8}\|\partial_t w\|_2^2 + \frac{1}{16}\|w\|_4^2 + C\|\partial_t\xi\|_3^2 + C. \qquad (5.16)
\end{aligned}$$

Finally,

$$\frac{\mathrm{d}}{\mathrm{d}t}\Theta_1 + \|\xi\|_3^2 = \|\partial_t\xi\|_2^2 + \langle w, \partial_t\xi\rangle_3 - \langle \partial_t\xi, \xi\rangle_3 \le \frac{1}{8}\|w\|_4^2 + \frac{1}{2}\|\xi\|_3^2 + C\|\partial_t\xi\|_3^2. \tag{5.17}$$

Collecting (5.14) and ε-times (5.15)–(5.17), we are led to

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda_1 + \frac{\varepsilon}{2}(\|w\|_4^2 + \|\partial_t w\|_2^2 + \|\xi\|_3^2) + (2 - C\varepsilon)\|\partial_t\xi\|_3^2 \le \frac{C}{\varepsilon}.$$

Thus, fixing $\varepsilon = \varepsilon(R)$ suitably small, we eventually obtain the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda_1 + \frac{\varepsilon}{2}E_1 \le C.$$

Noting that $\Lambda_1(0) = 0$, the conclusion follows from the controls (5.13) together with the usual Gronwall lemma.

In view of the semigroup decomposition (5.5), Lemmas 5.2 and 5.3, we meet the hypotheses of Lemma 4.1. This finishes the proof of Theorem 4.1.

Remark 5.1 Once the existence of the attractor is established by means of Lemma 4.1, every ball $\mathcal{B}_0$ of $\mathcal{H}$ centered in the origin with radius

$$R_0 > \sup \Big\{ \|z\|_{\mathcal{H}} : \mathfrak{L}(z) \leq \max_{\overline{z} \in \mathcal{S}} \mathfrak{L}(\overline{z}) \Big\}$$

is an absorbing set for $S(t)$. Hence, arguing as above with $R = R_0$, we find the exponential attraction property

$$\mathrm{dist}_{\mathcal{H}}(S(t)\mathcal{B}_0, \mathfrak{C}) \leq C_0 \mathrm{e}^{-\omega_0 t},$$

where $\mathfrak{C}$ is a closed ball of $\mathcal{V}$, and the positive constants $C_0 > 0$ and ω_0 can be explicitly estimated in terms of the radius R_0. In turn, this entails the existence of an increasing positive function $\mathfrak{I}$ such that, for every bounded set $\mathcal{B} \subset \mathcal{H}$,

$$\mathrm{dist}_{\mathcal{H}}(S(t)\mathcal{B}, \mathfrak{C}) \leq \mathfrak{I}(\|\mathcal{B}\|_{\mathcal{H}}) \mathrm{e}^{-\omega_0 t}.$$

References

[1] Ball, J. M., Initial-boundary value problems for an extensible beam, *J. Math. Anal. Appl.*, **42**, 1973, 61–90.

[2] Bucci, F. and Chueshov, I., Long-time dynamics of a coupled system of nonlinear wave and thermoelastic plate equations, *Discrete Cont. Dyn. Systems*, **22**, 2008, 557–586.

[3] Chueshov, I. and Lasiecka, I., Attractors and long time behavior of von Karman thermoelastic plates, *Appl. Math. Optim.*, **58**, 2008, 195–241.

[4] Coti Zelati, M., Giorgi, C. and Pata, V., Steady states of the hinged extensible beam with external load, *Math. Models Methods Appl. Sci.*, **20**, 2010, 43–58.

[5] Dickey, R. W., Free vibrations and dynamic buckling of the extensible beam, *J. Math. Anal. Appl.*, **29**, 1970, 443–454.

[6] Eden, A. and Milani, A. J., Exponential attractors for extensible beam equations, *Nonlinearity*, **6**, 1993, 457–479.

[7] Fichera, G., Is the Fourier theory of heat propagation paradoxical? *Rend. Circ. Mat. Palermo*, **41**, 1992, 5–28.

[8] Giorgi, C., Naso, M. G., Pata, V. and Potomkin, M., Global attractors for the extensible thermoelastic beam system, *J. Diff. Eqs.*, **246**, 2009, 3496–3517.

[9] Giorgi, C., Pata, V. and Vuk, E., On the extensible viscoelastic beam, *Nonlinearity*, **21**, 2008, 713–733.

[10] Green, A. E. and Naghdi, P. M., On undamped heat waves in an elastic solid, *J. Thermal Stresses*, **15**, 1992, 253–264.

[11] Green, A. E. and Naghdi, P. M., Thermoelasticity without energy dissipation, *J. Elasticity*, **31**, 1993, 189–208.

[12] Green, A. E. and Naghdi, P. M., A unified procedure for construction of theories of deformable media. I. Classical continuum physics, II. Generalized continua, III. Mixtures of interacting continua, *Proc. Roy. Soc. London*, **448**A, 1995, 335–356, 357–377, 379–388.

[13] Hale, J. K., Asymptotic behavior of dissipative systems, Mathematical Surveys and Monographs, Vol. 25, A. M. S., Providence, RI, 1988.

[14] Lagnese, J. E., Leugering, G. and Schmidt, E. J. P. G., Modelling of dynamic networks of thin thermoelastic beams, *Math. Methods Appl. Sci.*, **16**, 1993, 327–358.

[15] Lazzari, B. and Nibbi, R., On the exponential decay in thermoelasticity without energy dissipation and of type III in presence of an absorbing boundary, *J. Math. Anal. Appl.*, **338**, 2008, 317–329.

[16] Liu, Z. and Quintanilla, R., Analyticity of solutions in type III thermoelastic plates, *IMA J. Appl. Math.*, **75**, 2010, 356–365.

[17] Messaoudi, S. A. and Said-Houari, B., Energy decay in a Timoshenko-type system of thermoelasticity of type III, *J. Math. Anal. Appl.*, **348**, 2008, 298–307.

[18] Messaoudi, S. A. and Soufyane, A., Boundary stabilization of memory type in thermoelasticity of type III, *Appl. Anal.*, **87**, 2008, 13–28.

[19] Puri, P. and Jordan, P. M., On the propagation of plane waves in type-III thermoelastic media, *Proc. Roy. Soc. London*, **460**A, 2004, 3203–3221.

[20] Quintanilla, R., Damping of end effects in a thermoelastic theory, *Appl. Math. Letters*, **14**, 2001, 137–141.

[21] Quintanilla, R., On the impossibility of localization in linear thermoelasticity, *Proc. Roy. Soc. London*, **463**A, 2007, 3311–3322.

[22] Quintanilla, R. and Racke, R., Stability in thermoelasticity of type III, *Discrete Cont. Dyn. Systems Ser.*, **3**B, 2003, 383–400.

[23] Quintanilla, R. and Straughan, B., Growth and uniqueness in thermoelasticity, *Proc. Roy. Soc. London*, **456**A, 2000, 1419–1429.

[24] Quintanilla, R. and Straughan, B., A note on discontinuity waves in type III thermoelasticity, *Proc. Roy. Soc. London*, **460**A, 2004, 1169–1175.

[25] Reiss, E. L. and Matkowsky, B. J., Nonlinear dynamic buckling of a compressed elastic column, *Quart. Appl. Math.*, **29**, 1971, 245–260.

[26] Reissig, M. and Wang, Y. G., Cauchy problems for linear thermoelastic systems of type III in one space variable, *Math. Methods Appl. Sci.*, **28**, 2005, 1359–1381.

[27] Temam, R., Infinite-dimensional Dynamical Systems in Mechanics and Physics, Springer-Verlag, New York, 1997.

[28] Woinowsky-Krieger, S., The effect of an axial force on the vibration of hinged bars, *J. Appl. Mech.*, **17**, 1950, 35–36.

[29] Yang, L. and Wang, Y. G., Well-posedness and decay estimates for Cauchy problems of linear thermoelastic systems of type III in 3-D, *Indiana Univ. Math. J.*, **55**, 2006, 1333–1361.

[30] Zhang, X. and Zuazua, E., Decay of solutions of the system of thermoelasticity of type III, *Commun. Contemp. Math.*, **5**, 2003, 25–83.