Chin. Ann. Math.
32B(1), 2011, 59–68
DOI: 10.1007/s11401-010-0624-1

Chinese Annals of Mathematics, Series B

Transference on Some Non-convolution Operators from Euclidean Spaces to Torus*

Yandan ZHANG[1] Dashan FAN[2] Jiecheng CHEN[3]

Abstract The authors prove the certain de Leeuw type theorems on some non-convolution operators, and give some applications on certain known results.

Keywords n-Torus, de Leeuw's theorem, Commutator
2000 MR Subject Classification 42B05, 42B15

1 Introduction

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space. Suppose that $\mathcal{S}(\mathbb{R}^n)$ is the space of all Schwartz and λ is an L^∞ function on $\mathbb{R}^n$. For $\epsilon > 0$, the multiplier operator T_ϵ, with symbols $\lambda(\epsilon\xi)$, is initially defined on $f \in \mathcal{S}(\mathbb{R}^n)$ by

$$T_\epsilon f(x) = K_\epsilon * f(x), \tag{1.1}$$

where $\{K_\epsilon\}$ is the family of kernels whose Fourier transforms are $\lambda(\epsilon\xi)$. More explicitly, we have

$$T_\epsilon f(x) = \int_{\mathbb{R}^n} \lambda(\epsilon\xi)\widehat{f}(\xi)\mathrm{e}^{2\pi\mathrm{i}x\cdot\xi}\mathrm{d}\xi. \tag{1.2}$$

Analogously, we can define the corresponding multiplier family $\widetilde{T}_\epsilon$, $\epsilon > 0$, on the n-dimensional torus. The n-torus $\mathbb{T}^n$ can be identified with $\mathbb{R}^n \setminus \Lambda$, where Λ is the unit lattice which is an additive group of points in $\mathbb{R}^n$ having integer coordinates. The multiplier operators $\widetilde{T}_\epsilon$, $\epsilon > 0$, on $\mathbb{T}^n$ associated with the symbol function is defined by

$$\widetilde{T}_\epsilon(g)(x) = \sum_{k\in\Lambda} \lambda(\epsilon k)a_k\mathrm{e}^{2\pi\mathrm{i}x\cdot k}, \tag{1.3}$$

on any $g \in C^\infty(\mathbb{T}^n)$, where $\sum\limits_{k\in\Lambda} a_k\mathrm{e}^{2\pi\mathrm{i}x\cdot k}$ is the Fourier series of g. Again, we denote $T_\epsilon = T$ if $\epsilon = 1$. The relation of the L^p boundedness between T and $\widetilde{T}_\epsilon$ can be briefly concluded by a

Manuscript received October 21, 2009. Revised June 13, 2010. Published online December 28, 2010.
[1]Department of Mathematics, Zhejiang University of Science and Technology, Hangzhou 310023, China.
E-mail: yandzhang@163.com
[2]Department of Mathematics, University of Wisconsin-Milwaukee, Milwaukee 53711, USA.
[3]Corresponding author. Department of Mathematics, Zhejiang University, Hangzhou 310017, China.
E-mail: jcchen@zju.edu.cn
*Project supported by the National Natural Science Foundation of China (Nos. 10931001 and 10871173), the Educational Science Foundation of Zhejiang (No. Z201017584) and the Science Foundation of Zhejiang University of Science and Technology (No. F501108A02) .

theorem of de Leeuw [1], which says that if λ is a continuous function on $\mathbb{R}^n$ and if $1 \leq p \leq \infty$, then T is bounded on $L^p(\mathbb{R}^n)$ if and only if $\widetilde{T}_\epsilon$ is uniformly bounded on $L^p(\mathbb{T}^n)$ for $\epsilon > 0$ (also see [2, p. 260]). This theorem, as well as a theorem on the corresponding maximal operator T^* (see [3]), was later extended to many different function spaces. Readers can see [3–8] for further details of these generalizations. More recently, Fan and Sato [9] extended the de Leeuw theorem to the multilinear multiplier operators and obtained on the torus a analog of a famous theorem of Lacey and Thiele [10] about the bilinear Hilbert transform.

On the other hand, we notice that multipliers are convolution operators while many important operators in harmonic analysis are not convolution. Among these operators, singular integrals with variable kernels and commutators are two important classes. In recent years, fruitful results of these two operators on $\mathbb{R}^n$ were obtained, see [11–13]. Thus, a natural question is if we can obtain certain de Leeuw type theorems on these operators by transferring some known boundedness results from $\mathbb{R}^n$ to obtain boundedness of their corresponding operators in the n-torus. With this motivation, in this paper, we first study the de Leeuw theorem on the first commutators, as well as an application on the Bochner-Riesz means in the second section. The extensions to higher orders are studied in Section 3. In Section 4, we study the de Leeuw theorem on singular integrals with rough variable kernels. The basic methods used in this paper is based on ideas in [5], with some technical modifications.

In this paper, we use the notation $A \simeq B$ to mean that there are two positive constants c_1 and c_2 such that $c_1 A \leq B \leq c_2 A$.

2 Commutators

In this section, we study the commutator $T_b f = [b, T]f$ with b being a BMO function. We begin with reviewing the definition of BMO.

Suppose that b is a locally integrable function on $\mathbb{R}^n$. Define

$$\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} = \sup_Q \frac{1}{|Q|} \int_Q |b(x) - b_Q| \mathrm{d}x, \tag{2.1}$$

where the supreme is taken over all cubes $Q \subset \mathbb{R}^n$ and

$$b_Q = \frac{1}{|Q|} \int_Q b(y) \mathrm{d}y. \tag{2.2}$$

For a multiplier operator T with kernel K, its commutator with b is defined by

$$T_{b,\epsilon}(f)(x) = [T_\epsilon, b]f(x) = T_\epsilon(bf)(x) - b(x)T_\epsilon f(x) = \int_{\mathbb{R}^n} K_\epsilon(x-y)(b(x) - b(y))f(y)\mathrm{d}y. \tag{2.3}$$

Thus, we define its kth commutator by

$$T^m_{\vec{b},\epsilon}(f)(x) = \int_{\mathbb{R}^n} K_\epsilon(x-y) \prod_{k=1}^m (b_k(x) - b_k(y)) f(y) \mathrm{d}y, \tag{2.4}$$

where $b_k \in \mathrm{BMO}$, $1 \leq k \leq m$. Since the BMO norm is dilation invariant, by scaling, it is easy to check that

$$\|T^m_{\vec{b}}(f)\|_{L^p(\mathbb{R}^n)} \leq C \prod_{k=1}^m \|b_k\|_{\mathrm{BMO}(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^n)}, \tag{2.5}$$

if and only if

$$\|T^m_{\vec{b},\epsilon}(f)\|_{L^p(\mathbb{R}^n)} \le C \prod_{k=1}^m \|b_k\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f\|_{L^p(\mathbb{R}^n)} \quad \text{for some } \epsilon > 0, \tag{2.6}$$

and they have the same norm. For simplicity, in this section, we only study the case $k = 1$. But the general case can be achieved by the same methods. We now define the commutators on the torus. Suppose that $b \in \mathrm{BMO}$ is a periodic function. The commutator associated to multiplier operators $\widetilde{T}_\epsilon$ ($\epsilon > 0$) on $\mathbb{T}^n$ with the symbol λ is defined by for all where

$$\widetilde{T}_{b,\epsilon}(g)(x) = b(x)\widetilde{T}_{b,\epsilon}(g)(x) - \widetilde{T}_\epsilon(gb)(x), \tag{2.7}$$

on any $g \in C^\infty(\mathbb{T}^n)$. We have the following theorem.

Theorem 2.1 *Let $1 \le p \le \infty$, $\lambda \in L^\infty \cap C(\mathbb{R}^n)$ and $\epsilon > 0$. Suppose that T_ϵ is bounded on $L^p(\mathbb{R}^n)$. Moreover, if*

$$\|T_{b,\epsilon}(f)\|_{L^p(\mathbb{R}^n)} \le C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f\|_{L^p(\mathbb{R}^n)} \tag{2.8}$$

for all $f \in L^p(\mathbb{R}^n)$ and $b \in \mathrm{BMO}$, then

$$\|\widetilde{T}_{b,\epsilon}(g)\|_{L^p(\mathbb{T}^n)} \le C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}\|g\|_{L^p(\mathbb{T}^n)} \tag{2.9}$$

for all $g \in L^p(\mathbb{T}^n)$ and all periodic BMO *functions b.*

Proof Without loss of generality, we may assume $g, b \in C^\infty(\mathbb{T}^n)$. Also, we show the case $\epsilon = 1$. Let Ψ be a function in $\mathcal{S}(\mathbb{R}^n)$ satisfying $0 \le \Psi(x) \le 1$, $\mathrm{supp}(\Psi) \subset (-1, 1]^n$, and $\Psi(x) \equiv 1$ on $(-\frac{1}{2}, \frac{1}{2}]^n$. We denote $\Psi^{\frac{1}{N}}(x) = \Psi\big(\frac{x}{N}\big)$ for $N \in \mathbb{Z}^+$. Noting that $\widetilde{T}_b(g)(x)$ is a periodic function, we have

$$\|\widetilde{T}_b(g)\|^p_{L^p(\mathbb{T}^n)} = \frac{1}{N^n}\int_{[-\frac{N}{2},\frac{N}{2}]^n} \Big|\Psi\Big(\frac{x}{N}\Big)\widetilde{T}_b(g)(x)\Big|^p \mathrm{d}x. \tag{2.10}$$

Set

$$E_N(x) = \Psi\Big(\frac{x}{N}\Big)\widetilde{T}_b(g)(x) - T_b(\Psi^{\frac{1}{N}}g). \tag{2.11}$$

We claim that $E_N(x)$ goes to 0 uniformly on x as $N \to \infty$. If the claim holds, we have

$$\begin{aligned}&\frac{1}{N^n}\int_{[-\frac{N}{2},\frac{N}{2}]^n} \Big|\Psi\Big(\frac{x}{N}\Big)\widetilde{T}_b(g)(x)\Big|^p \mathrm{d}x \\ &\le C\frac{1}{N^n}\int_{\mathbb{R}^n} |T_b(\Psi^{\frac{1}{N}}g)(x)|^p \mathrm{d}x + o(1), \quad \text{as } N \to \infty.\end{aligned} \tag{2.12}$$

By the assumption, we obtain

$$\begin{aligned}\|\widetilde{T}_b(g)\|^p_{L^p(\mathbb{T}^n)} &\le C\|b\|^p_{\mathrm{BMO}}\frac{1}{N^n}\int_{\mathbb{R}^n} |\Psi^{\frac{1}{N}}(x)g(x)|^p \mathrm{d}x + o(1) \\ &\le C\|b\|^p_{\mathrm{BMO}}\frac{1}{N^n}\int_{[-N,N]} |g(x)|^p \mathrm{d}x + o(1) \\ &= C\|b\|^p_{\mathrm{BMO}}\|g\|^p_{L^p(\mathbb{T}^n)} + o(1).\end{aligned}$$

Letting $N \to \infty$, we prove the theorem.

Thus, to complete the proof of the theorem, it remains to show the claim. Note that

$$\Psi\Big(\frac{x}{N}\Big)\widetilde{T}_b(g)(x) = \Psi\Big(\frac{x}{N}\Big)b(x)\sum_{k\in\Lambda}\lambda(k)a_k \mathrm{e}^{2\pi \mathrm{i}x\cdot k} - \Psi\Big(\frac{x}{N}\Big)\sum_{k\in\Lambda}\lambda(k)b_k \mathrm{e}^{2\pi \mathrm{i}x\cdot k}, \tag{2.13}$$

where $\{a_k\}$ is the set of Fourier coefficients of $g(x)$ and $\{b_k\}$ is the set of Fourier coefficients of $(bg)(x)$. Both sequences decay rapidly by our assumption. By [2], we know that

$$\Psi\Big(\frac{x}{N}\Big)\sum_{k\in\Lambda}\lambda(k)b_k \mathrm{e}^{2\pi \mathrm{i}x\cdot k} = T(b\Psi^{\frac{1}{N}}g)(x) + A_N(x) \tag{2.14}$$

and

$$\Psi\Big(\frac{x}{N}\Big)\sum_{k\in\Lambda}\lambda(k)a_k \mathrm{e}^{2\pi \mathrm{i}x\cdot k} = T(\Psi^{\frac{1}{N}}g)(x) + B_N(x), \tag{2.15}$$

where $A_N(x)$ and $B_N(x)$ go to zero uniformly on x as $N \to \infty$.

To eliminate the assumption on b, we need the following steps.

Step 1 For $b \in L^\infty$, we construct $b_n \in C^\infty(\mathbb{T}^n)$ satisfying $\|b_n\|_{\mathrm{BMO}} \le C\|b\|_{\mathrm{BMO}}$, $\|b_n\|_{L^\infty} \le C\|b\|_{L^\infty}$ and $b_n \overset{\text{a.e.}}{\to} b$ $(n \to \infty)$. Thus, by Lebesgue's dominated convergent theorem, it is easy to check $\widetilde{T}_{b_n}g \overset{\text{a.e.}}{\to} \widetilde{T}_b g$ $(n \to \infty)$. Then, using Fatou's Lemma, we have

$$\begin{aligned}\|\widetilde{T}_b(g)\|_{L^p(\mathbb{T}^n)} &= \|\lim_{n\to\infty}\widetilde{T}_{b_n}(g)\|_{L^p(\mathbb{T}^n)}\\ &\le \liminf_{n\to\infty}\|\widetilde{T}_{b_n}(g)\|_{L^p(\mathbb{T}^n)}\\ &\le C\liminf_{n\to\infty}\|b_n\|_{\mathrm{BMO}}\|g\|_{L^p(\mathbb{T}^n)}\\ &\le C\|b\|_{\mathrm{BMO}}\|g\|_{L^p(\mathbb{T}^n)}.\end{aligned}$$

In fact, let $b_n(x) = P_{\frac{1}{n}} * b(x)$, where P is the Poisson kernel. Thus, we prove the theorem for $b \in L^\infty$.

Step 2 For $b \in \mathrm{BMO}$, set $b_n = \max\{\min\{b, n\}, -n\}$. Obviously, $b \in L^\infty$ and satisfies that $\|b_n\|_{\mathrm{BMO}} \le C\|b\|_{\mathrm{BMO}}$, $|b_n(x)| \le |b(x)|$ and $b_n \overset{\text{a.e.}}{\to} b$ $(n \to \infty)$. By applying the result in Step 1 and the same argument as Step 1, we prove the theorem.

We now study an application of the above theorem. Recall that the commutator of the Bochner-Riesz means is defined by

$$\widetilde{B}^\alpha_{b,\epsilon}(g)(x) = b(x)\widetilde{B}^\alpha_\epsilon(g)(x) - \widetilde{B}^\alpha_\epsilon(gb)(x), \tag{2.16}$$

where B^α_ϵ, the Bochner-Riesz operator of order α, is a multiplier with symbol $m_\alpha(\xi) = (1 - |\epsilon\xi|^2)^\alpha_+$. By our transference result and results in [12], we have the following theorems.

Theorem 2.2 *Let* $0 < \alpha < \frac{1}{2}$ *and* $\max\{1, \frac{4}{3+2\alpha}\} < p < \frac{4}{1-2\alpha}$. *Suppose that* b *is any periodic* BMO *function. Then*

$$\|\widetilde{B}^\alpha_{b,\epsilon}(g)\|_{L^p(\mathbb{T}^2)} \le C\|b\|_{\mathrm{BMO}}\|g\|_{L^p(\mathbb{T}^2)}, \tag{2.17}$$

uniformly on $\epsilon > 0$.

Theorem 2.3 *Suppose that b is any periodic* BMO *function. Then there exists a positive constant C such that*

$$\|\widetilde{B}^{\alpha}_{b,\epsilon}(g)\|_{L^p(\mathbb{T}^n)} \le C\|b\|_{\mathrm{BMO}}\|g\|_{L^p(\mathbb{T}^n)}, \quad g \in C^\infty(\mathbb{T}^n), \tag{2.18}$$

uniformly on $\epsilon > 0$, provided $\frac{n-1}{2n+2} < \alpha < \frac{n-1}{2}$ and $\frac{2n}{n+1+2\alpha} < p < \frac{4}{n-1-2\alpha}$.

3 Extensions

First, we extend Theorem 2.1 to $T^m_{\vec{b},\epsilon}$ for $m \ge 2$. Before the proof is given, we give the following notations.

Fixing $m \in \mathbb{N}$ for $1 \le j \le m$, we denote C^m_j by the family of all finite subsets $\sigma = \{\sigma(1), \cdots, \sigma(j)\}$ of $\{1, \cdots, m\}$, where σ is made up of j different elements. For any $\sigma \in C^m_j$, the complementary sequence σ' is given by $\sigma' = \{1, \cdots, m\} \setminus \sigma$. Let $\vec{b} = (b_1, \cdots, b_m)$. For any $\sigma = \{\sigma(1), \cdots, \sigma(j)\}$, we denote $\vec{b}_\sigma$ by $\vec{b}_\sigma = (b_{\sigma(1)}, \cdots, b_{\sigma(j)})$.

Now we begin the proof. As in the proof of Theorem 2.1, we assume g and $b_j \in C^\infty(\mathbb{T}^n)$, $1 \le j \le m$. We also show the case $\epsilon = 1$ only. It is noted that

$$\prod_{j=1}^m (b_j(x)) - b_j(y) = \sum_{j=0}^m \sum_{\sigma \in C^m_j} (-1)^{m-j} (\vec{b}(x))_\sigma (\vec{b}(y))_{\sigma'}.$$

Thus, we write

$$\begin{aligned} T^m_{\vec{b}}(f)(x) &= \sum_{j=0}^m \sum_{\sigma \in C^m_j} (-1)^{m-j} (\vec{b}(x))_\sigma \int_{\mathbb{R}^n} (\vec{b}(y))_{\sigma'} K(x-y) f(y) \mathrm{d}y \\ &= \sum_{j=0}^m \sum_{\sigma \in C^m_j} (-1)^{m-j} (\vec{b}(x))_\sigma T((\vec{b})_{\sigma'} f)(x) \end{aligned} \tag{3.1}$$

for $f \in \mathcal{S}(\mathbb{R}^n)$. For $g \in C^\infty(\mathbb{T}^n)$, we have

$$\widetilde{T}^m_{\vec{b}}(g)(x) = \sum_{j=0}^m \sum_{\sigma \in C^m_j} (-1)^{m-j} (\vec{b}(x))_\sigma \widetilde{T}((\vec{b})_{\sigma'} g)(x) \tag{3.2}$$

by the way.

Let Ψ be as in Theorem 2.1. Set

$$E_N(x) = \Psi\Big(\frac{x}{N}\Big) \widetilde{T}^m_{\vec{b}}(g)(x) - T^m_{\vec{b}}(\Psi^{\frac{1}{N}} g)(x). \tag{3.3}$$

By the same argument as Theorem 2.1, the proof is completed if $E_N(x)$ goes to zero uniformly on x as $N \to \infty$. In fact, due to (3.1) and (3.2), we get

$$E_N(x) = \sum_{j=0}^m \sum_{\sigma \in C^m_j} (-1)^{m-j} (\vec{b}(x))_\sigma \Big[\Psi\Big(\frac{x}{N}\Big) \widetilde{T}((\vec{b})_{\sigma'} g)(x) - T(\Psi^{\frac{1}{N}} (\vec{b})_{\sigma'} g)(x)\Big]. \tag{3.4}$$

Fixed $\sigma \in C^m_j$ $(0 \le j \le m)$, by checking Fourier transform, it is easy to see that

$$\Big|\Psi\Big(\frac{x}{N}\Big) \widetilde{T}((\vec{b})_{\sigma'} g)(x) - T(\Psi^{\frac{1}{N}} (\vec{b})_{\sigma'} g)(x)\Big| \le \sum_{k \in \Lambda} |c_k| \int_{\mathbb{R}^n} \Big|\widehat{\Psi}(\xi)\Big(m(k) - m\Big(k + \frac{\xi}{N}\Big)\Big)\Big| \mathrm{d}\xi, \tag{3.5}$$

where $\{c_k\}$ is the set of Fourier coefficients of $((\vec{b})_{\sigma'}g)(x)$, and this sequence decays rapidly by our assumption.

Since $\widehat{\Psi}$ is integrable, and since m is bounded and continuous, the last quantity converges to zero as $N \to \infty$. Noting that b_j is bounded, $1 \le j \le m$, we finish the proof.

Second, we extend Theorem 2.1 to multilinear commutators. Recall m-linear commutators defined on $\mathbb{R}^n$ by

$$T_{\vec{b}}^m(\vec{f})(x) = \sum_{j=1}^m T_{\vec{b}}^j(\vec{f})(x), \tag{3.6}$$

where $T_{\vec{b}}^j(\vec{f})(x) = b_jT(f_1, \cdots, f_j, \cdots, f_m)(x) - T(f_1, \cdots, b_jf_j, \cdots, f_m)(x)$. This definition coincides with the linear commutator $[b, T]$ when $m = 1$. The m-linear commutators were considered by Pérez and Torres in [14].

Now we begin our proof. We only need to consider the operator with one symbol by linearity and the case $m = 2$ for simplicity. Without loss of generality, we study $T_b(\vec{f}) = bT(f_1, f_2) - T(bf_1, f_2)$. Define the corresponding operator $\widetilde{T}_b(\vec{g})$ on $\mathbb{T}^n$ by $\widetilde{T}_b(\vec{g}) = b\widetilde{T}(g_1, g_2) - \widetilde{T}(bg_1, g_2)$. Choose the function Ψ in Section 2, the difference $E_N(x)$ of $\Psi^{\frac{1}{N}}\widetilde{T}_b(\vec{g})$ and $T_b(\Psi^{\frac{1}{N}}\vec{g})$ is

$$\begin{aligned} |E_N(x)| &= |b[\Psi^{\frac{1}{N}}\widetilde{T}(g_1, g_2) - T(\Psi^{\frac{1}{N}}(g_1, g_2))]| + |\Psi^{\frac{1}{N}}\widetilde{T}(bg_1, g_2) - T(\Psi^{\frac{1}{N}}(bg_1, g_2))| \\ &= C_N(x) + D_N(x). \end{aligned} \tag{3.7}$$

By the proof of Theorem 3 in [9], $C_N(x)$ and $D_N(x)$ go to zero uniformly on x as $N \to \infty$. Thus we obtain the desired result.

4 Variable Rough Kernels

In this section, we study the operator P of a kernel K that is defined by

$$P(f)(x) = \int_{\mathbb{R}^n} K(x, x - y)f(y)\mathrm{d}y. \tag{4.1}$$

If we apply the Plancherel theorem on the y-variable, then formally

$$P(f)(x) = \int_{\mathbb{R}^n} m(x, \xi)\widehat{f}(\xi)\mathrm{e}^{2\pi\mathrm{i}x\cdot\xi}\mathrm{d}\xi, \tag{4.2}$$

where $m(x, \xi) = \int_{\mathbb{R}^n} K(x, y)\mathrm{e}^{2\pi\mathrm{i}y\cdot\xi}\mathrm{d}y$. If $m(x, \cdot)$ is periodic on the x-variable, we define the corresponding operator $\widetilde{T}$ on the n-torus by

$$\widetilde{P}(g)(x) = \sum_{k\in\Lambda} m(x, k)a_k\mathrm{e}^{2\pi\mathrm{i}x\cdot k} \tag{4.3}$$

for all $C^\infty(\mathbb{T}^n)$ function g, where $g(x) = \sum\limits_{k\in\Lambda} a_k\mathrm{e}^{2\pi\mathrm{i}x\cdot k}$. We have the following theorem.

Theorem 4.1 *Let T be an operator with symbol $m(x, \xi) = \lambda_x(\xi)$. Suppose that $\lambda_x(\xi) \in L^\infty \cap C(\mathbb{R}^n)$ uniformly on x. If*

$$\|P(f)\|_{L^p(\mathbb{R}^n)} \le C\|f\|_{L^p(\mathbb{R}^n)} \tag{4.4}$$

for all $f \in L^p(\mathbb{R}^n)$*, then*

$$\|\widetilde{P}(g)\|_{L^p(\mathbb{T}^n)} \leq C\|g\|_{L^p(\mathbb{T}^n)} \tag{4.5}$$

for all $g \in L^p(\mathbb{T}^n)$*.*

Proof Without loss of generality, we assume $g, b \in C^\infty(\mathbb{T}^n)$. Choose the function Ψ as in Section 2. We have

$$\|\widetilde{P}(g)\|^p_{L^p(\mathbb{T}^n)} = \frac{1}{N^n}\int_{[-\frac{N}{2},\frac{N}{2}]^n} \Big|\Psi\Big(\frac{x}{N}\Big)\widetilde{P}(g)(x)\Big|^p \mathrm{d}x. \tag{4.6}$$

Similarly to the proof of Theorem 2.1, to prove the theorem we need to show

$$\Psi\Big(\frac{x}{N}\Big)\widetilde{P}(g)(x) = P(\Psi^{\frac{1}{N}}g)(x) + E_N(x), \tag{4.7}$$

where $E_N(x)$ is the error term. Recall

$$\Psi\Big(\frac{x}{N}\Big)\widetilde{P}(g)(x) = \Psi\Big(\frac{x}{N}\Big)\sum_{k\in\Lambda} m(x,k)a_k \mathrm{e}^{2\pi \mathrm{i}x\cdot k}, \tag{4.8}$$

where $\{a_k\}$ is the set of Fourier coefficients of g and it decays rapidly to 0. First, we notice

$$\begin{aligned}\Psi\Big(\frac{x}{N}\Big)\widetilde{P}(g)(x) &= \Psi\Big(\frac{x}{N}\Big)\sum_{k\in\Lambda} m(x,k)a_k \mathrm{e}^{2\pi \mathrm{i}x\cdot k}\\ &= \sum_{k\in\Lambda} a_k N^n \int_{\mathbb{R}^n} m(x,k)\widehat{\Psi}(Ny)\mathrm{e}^{2\pi \mathrm{i}x\cdot(y+k)}\mathrm{d}y. \end{aligned}\tag{4.9}$$

On the other hand, we recall that

$$P(\Psi^{\frac{1}{N}}g)(x) = \int_{\mathbb{R}^n} m(x,\xi)\widehat{\Psi^{\frac{1}{N}}g}(\xi)\mathrm{e}^{2\pi\mathrm{i}x\xi}\mathrm{d}\xi, \tag{4.10}$$

where we easily compute $\widehat{\Psi^{\frac{1}{N}}g}(\xi) = \sum\limits_{k\in\Lambda} a_k N^n \widehat{\Psi}(N(\xi-k))$. We have

$$\begin{aligned}P(\Psi^{\frac{1}{N}}g)(x) &= \sum_{k\in\Lambda} a_k N^n \int_{\mathbb{R}^n} m(x,\xi)\widehat{\Psi}(N(\xi-k))\mathrm{e}^{2\pi\mathrm{i}x\xi}\mathrm{d}\xi\\ &= \sum_{k\in\Lambda} a_k N^n \int_{\mathbb{R}^n} m(x,y+k)\widehat{\Psi}(Ny)\mathrm{e}^{2\pi\mathrm{i}x\cdot(y+k)}\mathrm{d}y.\end{aligned}$$

Thus the difference of $P(\Psi^{\frac{1}{N}}g)(x)$ and $\widehat{\Psi}^{\frac{1}{N}}g(x)$ is

$$|E_N(x)| \leq \sum_{k\in\Lambda}|a_k|\int_{\mathbb{R}^n}\Big|m\Big(x,\frac{y}{N}+k\Big) - m(x,k)\Big||\widehat{\Psi}(y)|\mathrm{d}y. \tag{4.11}$$

For arbitrary $\epsilon > 0$, there is an $M > 0$ such that

$$\begin{aligned}&\sum_{k\in\Lambda}|a_k|\int_{|y|>M}\Big|m\Big(x,\frac{y}{N}+k\Big) - m(x,k)\Big||\widehat{\Psi}(y)|\mathrm{d}y\\ &\leq \|m\|_{L^\infty}\sum_{k\in\Lambda}|a_k|\int_{|y|>M}|\widehat{\Psi}(y)|\mathrm{d}y < \delta.\end{aligned}\tag{4.12}$$

Then recalling (4.6), we have

$$\begin{aligned}\|\widetilde{P}(g)\|_{L^p(\mathbb{T}^n)}^p &= \frac{1}{N^n}\int_{[-\frac{N}{2},\frac{N}{2}]^n}\Big|\Psi\Big(\frac{x}{N}\Big)\widetilde{P}(g)(x)\Big|^p\mathrm{d}x \\ &\le \frac{1}{N^n}\int_{[-\frac{N}{2},\frac{N}{2}]^n}|P(\Psi^{\frac{1}{N}}g)(x)|^p\mathrm{d}x + \frac{1}{N^n}\int_{[-\frac{N}{2},\frac{N}{2}]^n}|E_N(x)|^p\mathrm{d}x \\ &\le \frac{1}{N^n}\int_{\mathbb{R}^n}|(\Psi^{\frac{1}{N}}g)(x)|^p\mathrm{d}x + \frac{1}{N^n}\int_{[-\frac{N}{2},\frac{N}{2}]^n}|E_N(x)|^p\mathrm{d}x \\ &= \mathrm{J}_1 + \mathrm{J}_2.\end{aligned}$$

By the choice of Ψ, we have

$$\mathrm{J}_1 \le C\frac{1}{N^n}\int_{(-N,N]^n}|(\Psi^{\frac{1}{N}}g)(x)|^p\mathrm{d}x \le C\|g\|_{L^p(\mathbb{T}^n)}^p. \tag{4.13}$$

Also, we have

$$\mathrm{J}_2 \le \frac{\sum\limits_{k\in\Lambda}|a_k|}{N^n}\int_{[-\frac{N}{2},\frac{N}{2}]^n}\sup_{|y|\le M}\Big|m\Big(x,\frac{y}{N}+k\Big)-m(x,k)\Big|^p\mathrm{d}x + \epsilon.$$

Letting $N\to\infty$, we obtain

$$\|P(g)\|_{L^p(\mathbb{T}^n)} \le C\|g\|_{L^p(\mathbb{T}^n)}, \tag{4.14}$$

$$\frac{1}{N^n}\int_{[-\frac{N}{2},\frac{N}{2}]^n}\Big|\sum_{k\in\Lambda}|a_k|\int_{|y|\le M}\Big|m\Big(x,\frac{y}{N}+k\Big)-m(x,k)\Big||\widehat{\Psi}(y)|\mathrm{d}y\Big|^p\mathrm{d}x \le \delta. \tag{4.15}$$

As an application of the above theorem, we study the rough singular integral with a variable kernel

$$T_\Omega(f)(x) = \text{p.v.}\int_{\mathbb{R}^n}K(x,y)f(y)\mathrm{d}y, \quad f\in\mathcal{S}(\mathbb{R}^n), \tag{4.16}$$

where

$$K(x,y) = \text{p.v.}\frac{\Omega(x,y')}{|y|^n} \quad \text{with } \int_{\mathbb{S}^n}\Omega(x,y')\mathrm{d}\sigma(y') = 0. \tag{4.17}$$

It is a well-known result of Calderón-Zygmund that for $1<p\le 2$,

$$\|T_\Omega(f)\|_{L^p(\mathbb{R}^n)} \le C\|f\|_{L^p(\mathbb{R}^n)}, \tag{4.18}$$

if

$$\Omega\in L^\infty(\mathbb{R}^n)\cap L^r(\mathbb{S}^n) \quad \text{with } r > \frac{p'(n-1)}{n}. \tag{4.19}$$

To transfer this result to $\mathbb{T}^n$, by checking the proof of Theorem 4.1, we need to check that

$$\frac{\sum\limits_{k\in\Lambda}|a_k|}{N^n}\int_{[-\frac{N}{2},\frac{N}{2}]^n}\int_{\mathbb{R}^n}\Big|m\Big(x,\frac{y}{N}+k\Big)-m(x,k)\Big|^p\mathrm{d}x = o(1), \tag{4.20}$$

where

$$m(x,k) = \text{p.v.} \int_{\mathbb{R}^n} \frac{\Omega(x,y')}{|y|^n} \mathrm{e}^{2\pi \mathrm{i} y\cdot k} \mathrm{d}y. \tag{4.21}$$

By a similar calculation as in ([15, p. 39]), we get

$$m(x,k) = \int_{\mathbb{S}^{n-1}} \Omega(x,y') \Big[\log\Big(\frac{1}{|y'\cdot k|}\Big) - \frac{\pi \mathrm{i}}{2} \mathrm{sign}(y'\cdot k) \Big] \mathrm{d}\sigma(y') \tag{4.22}$$

for $k \neq 0$. Because $\Omega \in L^\infty(\mathbb{R}^n) \times L^r(\mathbb{S}^{n-1})$, we know that $m(x,\cdot\,)$ is bounded and $m(x,\cdot\,)$ is a continuous function on $\mathbb{R}^n \setminus \{0\}$ uniformly on x. Write

$$\begin{aligned} |E_N(x)| &= \sum_{k\in\Lambda} |a_k| \int_{\mathbb{R}^n} \Big| \widehat{\Psi}(\xi) \Big(m(k) - m\Big(k + \frac{\xi}{N}\Big)\Big) \Big| \mathrm{d}\xi \\ &\le C \Big[a_0 + \sum_{k\in\Lambda\setminus\{0\}} |a_k| \int_{\mathbb{R}^n} \Big| \widehat{\Psi}(\xi) \Big(m(k) - m\Big(k + \frac{\xi}{N}\Big)\Big) \Big| \mathrm{d}\xi \Big], \end{aligned} \tag{4.23}$$

where $\{a_k\}$ is the set of Fourier coefficients of $g(x)$ and this sequence decays rapidly to zero by our assumption. By the proof of Theorem 4.1, the second term of (4.23) goes to zero uniformly on x as $N \to \infty$. We also note that

$$\|a_0\|_{L^p(\mathbb{T}^n)} \le \|g\|_{L^p(\mathbb{T}^n)}, \tag{4.24}$$

and this ends the proof of the desired result.

For $b \in \mathrm{BMO}$, the commutator associated to pseudo-differential operators $P_{b,\epsilon}$ is defined by

$$P_{b,\epsilon}(f)(x) = b(x)P(f)(x) - P(bf)(x), \quad f \in \mathcal{S}(\mathbb{R}^n). \tag{4.25}$$

Suppose that $b \in \mathrm{BMO}$ is a periodic function. The commutator $\widetilde{P}_{b,\epsilon}$ ($\epsilon > 0$) on $\mathbb{T}^n$ is defined by

$$\widetilde{P}_{b,\epsilon}(f)(x) = b(x)\widetilde{P}(f)(x) - \widetilde{P}(bf)(x), \quad g \in C^\infty(\mathbb{T}^n). \tag{4.26}$$

Combining ideas of Theorems 2.1 and 4.1, we can extend Theorem 4.1 to $P_{b,\epsilon}$. Applying this transference result and the result in [13], we obtain the following theorem.

Theorem 4.2 *If* $1 < p < \infty$, $\Omega \in L^\infty(\mathbb{R}^n) \times L^r(\mathbb{S}^{n-1})$, $r > \frac{2(n-1)}{n}$, *then for all periodic* BMO *functions* b, $\widetilde{T}_{\Omega,b}$ *extends to an operator bounded from* $L^2(\mathbb{T}^n)$ *to itself, where* $\widetilde{T}_{\Omega,b}(f)(x) = b(x)\widetilde{T}_\Omega(f)(x) - \widetilde{T}_\Omega(bf)(x)$ *on any* $g \in C^\infty(\mathbb{T}^n)$.

We omit the proof.

Acknowledgement The authors express their thanks to the referee for his/her very valuable comments.

References

[1] de Leeuw, K., On L^p multiplier, *Ann. of Math.,* **91**, 1965, 364–379.

[2] Stein, E. M. and Weiss, G., Introduction to Fourier Analysis on Euclidean Spaces, Princeton University Press, Princeton, 1971.

[3] Kenig, C. and Thomas, P., Maximal operators defined by Fourier multiplier, *Studia Math.,* **68**, 1980, 79–83.

[4] Auscher, P. and Carro, M. J., On relations between operators on $\mathbb{R}^n$, $\mathbb{T}^n$ and $\mathbb{Z}^n$, *Studia Math.,* **101**, 1992, 165–182.

[5] Fan, D., Multipliers on certain function spaces, *Circolo Matematico Di Palermo,* **Tomo XLIII**, 1994, 449–463.

[6] Kaneko, M. and Sato, E., Notes on transference of continuity from maximal Fourier multiplier operators on R^n to those on T^n, *Interdiscip. Inform. Sci.,* **4**, 1998, 97–107.

[7] Liu, Z. and Lu, S., Transference and restriction of maximal multiplier operators on Hardy spaces, *Studia Math.,* **105**, 1993, 121–134.

[8] Chen, D. and Fan, D., Multiplier transformations on H^p spaces, *Studia Math.,* **131**, 1998, 189–204.

[9] Fan, D. and Sato, S., Transference on certain multilinear multiplier operators, *J. Austral. Math. Soc.,* **70**, 2001, 37–55.

[10] Lacey, M. and Thiele, C., L^p estimates on the bilinear Hilbert transform, *Ann. of Math.,* **146**(2), 1997, 693–724.

[11] Calderón, A. P. and Zygmund, A., On singular integrals with variable kernels, *Applicable Anal.,* **7**, 1977/1978, 221–238.

[12] Hu, G. and Lu, S., The commutator of the Bochner-Riesz operator, *Tohoku Math J.,* **48**, 1996, 259–266.

[13] Chen, Y. and Ding, Y., L^2 boundedness for commutators of rough singular integral with variable kernel, *Rev. Math. Ibe.,* **24**, 2008, 531–547.

[14] Pérez, C. and Torres, R., Sharp maximal function estimates for multilinear singular integral operators, *Contemp. Math.,* **320**, 2003, 323–331.

[15] Stein, E. M., Singular Integrals and Differentiability Properties of Functions, Princeton University Press, Princeton, 1970.