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On the Cauchy Problem of Evolution p-Laplacian Equation with Nonlinear Gradient Term***

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Abstract The authors study the existence of solution to p-Laplacian equation with nonlinear forcing term under optimal assumptions on the initial data, which are assumed to be measures. The existence of local solution is obtained.

Keywords p-Laplacian equation, Nonlinear gradient term, Measures initial data, Local solution
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1 Introduction and Main Results

In this note we consider the existence (and nonexistence) of solutions to the Cauchy problem

$$u_t - \operatorname{div}(|Du|^{p-2}Du) = |Du^q|^\nu, \quad \text{in } S_T = \mathbb{R}^N \times (0,T),\ T>0, \tag{1.1}$$

$$u(x,0) = \mu, \quad x \in \mathbb{R}^N, \tag{1.2}$$

where $p > 2$, $0 < \nu < p$, $q\nu \geq p-1$, $N \geq 1$, and μ is a nonnegative Radon measure in $\mathbb{R}^N$. Equation (1.1) is a class of degenerate parabolic equation with nonlinear forcing term. They appear in the theory of non-Newtonian fluids. The main feature of this class of equation is the interplay between the degeneracy in the principal part of the equation and the nonlinear forcing term, where the latter depends on the space gradient of a power of the solution.

In this paper, our interest is mainly focussed on the optimal condition of initial data μ, for the existence of solutions of (1.1)–(1.2).

It is well-known (see, e.g., [1]) that for the Cauchy problem of p-Laplacian equation

$$u_t - \operatorname{div}(|Du|^{p-2}Du) = 0, \quad \text{in } S_T \tag{1.3}$$

to have a solution, the optimal condition on initial data is

$$\sup_{\rho\geq\gamma} \rho^{-\frac{\kappa}{p-2}} \int_{B_\rho} |\mathrm{d}\mu| < \infty$$

for some $\gamma > 0$, $\kappa = N(p-2)+p$. For the Cauchy problem of p-Laplacian equation with strongly nonlinear sources

$$u_t - \operatorname{div}(|Du|^{p-2}Du) = u^q, \quad \text{in } S_T = \mathbb{R}^N \times (0,T),\ p>2 \tag{1.4}$$

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to have a nonnegative solution, the optimal condition on measure initial data is

$$\sup_{x\in\mathbb{R}^N}\fint_{B_\rho(x)}\mathrm{d}\mu<\infty,$$

where $1<q<p-1+\frac{p}{N}$ is assumed (see [2]). The problem of existence of solution to (1.4) with μ a measure and $q>p-1+\frac{p}{N}$ was still open. In this paper, by introducing Morrey norms as in [4], we first investigate the existence of nonnegative solution to Cauchy problem (1.1)–(1.2). Here we do not place any restriction on the growth condition of μ. Next using the same method, we give a sufficient condition to ensure the existence of solution to (1.4) corresponding to data measure in the case $q>p-1+\frac{p}{N}$. Moreover, we prove also that the sufficient condition is actually optimal for the existence of solution to (1.4) and (1.2).

For the Porous Medium equation with nonlinear forcing term, similar problems were considered by D. Andreucci and E. DiBenedetto (see [3, 4]). Here we use some ideas in [4].

Since equation (1.1) is a degenerate parabolic equation, problem (1.1)–(1.2) does not in general have classical solutions. We now define the weak solution to (1.1)–(1.2).

Definition 1.1 *A nonnegative measurable function $u(x,t)$ defined in S_T is said to be a weak solution to problem* (1.1)–(1.2), *if*

$$u\in L^\infty_{\mathrm{loc}}(S_T)\cap C((0,T);L^p_{\mathrm{loc}}(\mathbb{R}^N)),$$
$$|Du|^p\in L^1_{\mathrm{loc}}(S_T),\quad |Du^q|^\nu\in L^1_{\mathrm{loc}}(S_T),\tag{1.5}$$
$$\iint_{S_T}\{-u\phi_t+|Du|^{p-2}DuD\phi\}\mathrm{d}x\mathrm{d}t=\iint_{S_T}|Du^q|^\nu\phi\mathrm{d}x\mathrm{d}t\tag{1.6}$$

for all $\phi\in C^1_0(S_T)$. Moreover,

$$\lim_{t\to0}\int_{\mathbb{R}^N}u(x,t)\eta(x)\mathrm{d}x=\int_{\mathbb{R}^N}\eta(x)\mathrm{d}\mu\tag{1.7}$$

for all $\eta\in C^1_0(\mathbb{R}^N)$.

Weak subsolution (resp. supersolution) is defined in the same way except that the = in (1.6) is replaced by $\le$ (resp. $\ge$) and ϕ is taken to be nonnegative.

We introduce the following notation as in [4].

Let μ be a nonnegative Radon measure in $\mathbb{R}^N$, and $u\in L^\infty_{\mathrm{loc}}(S_T)$, $u\ge0$ with $S_T=\mathbb{R}^N\times(0,T)$, $T>0$. Let also $0\le\theta\le N$ be given. We use the following notations throughout the paper:

$$[\mu]=\sup_{x\in\mathbb{R}^N}\sup_{0<\rho<1}\rho^\theta\fint_{B_\rho(x)}\mathrm{d}\mu,$$
$$[u]_t=\sup_{0<\tau<t}[u(\,\cdot\,,\tau)],\quad 0<t<T,$$

where we let

$$\fint_E\mathrm{d}\mu=\frac{1}{|E|}\int_E\mathrm{d}\mu,\quad |E|=\text{lebesgue-measure of }E.$$

First we state our existence theorem.

Theorem 1.1 *Let $[\mu]<\infty$ with $\theta[q\nu-(p-1)]<(p-\nu)$. Then there exists a solution u to* (1.1)–(1.2) *defined in $S_{T_0}=\mathbb{R}^N\times(0,T_0)$,where $T_0=T_0(N,p,q,\nu,\theta,[\mu])$, such that for all*

$0 < t < T_0$,

$$[u]_t \le \gamma[\mu], \tag{1.8}$$

$$\|u(\cdot,t)\|_{\infty,\mathbb{R}^N} \le \gamma t^{-\frac{\theta}{p+\theta(p-2)}}[\mu] \tag{1.9}$$

for some $\gamma = \gamma(N,p,q,\nu,\theta)$.

Next we prove that the critical threshold for θ in Theorem 1.1 is actually optimal for the existence of solutions in the class considered here.

Theorem 1.2 *Let* $\nu \in [1,\infty)$ *and* u *be a solution to* (1.1) *in* $\mathbb{R}^N \times (0,T)$ *such that* $[u]_t < \infty$, $0 < t < T$ *and*

$$\mu(B_\rho(x_0)) \ge \varepsilon\rho^{N-\theta}, \quad 0 < \rho < \varepsilon,$$

where $\varepsilon > 0$, $x_0 \in \mathbb{R}^N$ *are given. Then* $\theta[q\nu - (p-1)] < (p-\nu)_+$.

The problem of existence of solutions in the limiting case $\theta[q\nu - (p-1)] = (p-\nu)$ is still open.

It is known (see [2]) that any initial datum μ to a nonnegative solution of (1.4) with $q < p - 1 + \frac{p}{N}$ must fulfill the following necessary condition:

$$\mu(B_\rho) \le \gamma\rho^N.$$

Using the techniques introduced in the proof of Theorem 1.1, we can prove the following theorem.

Theorem 1.3 *Let* $[\mu] < \infty$, $q < p - 1 + \frac{p}{\theta}$. *Then there exists a solution* u *to* (1.4) *and* (1.2) *defined in* $S_{T_0} = \mathbb{R}^N \times (0,T_0)$, *where* $T_0 = T_0(N,p,q,\theta,[\mu])$, *such that for all* $0 < t < T_0$,

$$[u]_t \le \gamma[\mu], \tag{1.10}$$

$$\|u(\cdot,t)\|_{\infty,\mathbb{R}^N} \le \gamma t^{-\frac{\theta}{p+\theta(p-2)}}[\mu] \tag{1.11}$$

for some $\gamma = \gamma(N,p,q,\theta)$.

The following theorem shows that the bound $q < p - 1 + \frac{p}{\theta}$ in Theorem 1.3 is optimal.

Theorem 1.4 *Let* u *be a nonnegative solution to* (1.4) *in* S_T *such that* $[u]_t < \infty$, $t \in (0,T)$ *and*

$$\mu(B_\rho(x_0)) \ge \epsilon\rho^{N-\theta}, \quad 0 < \rho < \varepsilon,$$

where $\epsilon > 0$, $x_0 \in \mathbb{R}^N$ *are given. Then* $\theta(q-p+1) \le p$.

Throughout the paper, we use $\gamma_i(a_1,a_2,\cdots,a_n)$ to denote positive constants that can be determined a priori and only depend on specified quantities $a_1,a_2,\cdots,a_n$.

This paper is organized as follows. In Section 2, we collect the a priori estimates needed to prove Theorem 1.1, whose proof is given in Section 3. Theorem 1.2 is proven in Section 4. Theorems 1.3 and 1.4 are proven in Sections 5 and 6.

2 Some Estimates

For technical convenience, we need to define, for $u \in L^\infty_{\text{loc}}(S_{T^*})$, $u \ge 0$, with $S_{T^*} = \mathbb{R}^N \times (0,T^\star)$,

$$\langle u\rangle_t = \sup_{0<\tau<t}\sup_{x\in\mathbb{R}^N}\sup_{R(\tau)<\rho<1}\rho^\theta \fint_{B_\rho(x)} u(y,\tau)\mathrm{d}y, \quad R(t) = \Gamma t^{\frac{1}{p+\theta(p-2)}}$$

for all $0 < t < T^*$, where Γ is a positive constant which can be chosen a priori (as a function of N, p, q, θ and $[\mu]$). We also need to assume that

$$T^* \text{ is chosen so that } R(T^*) = 1.$$

Finally we define the constants

$$\kappa = N(p-2) + p, \quad \Theta = \frac{\nu}{p-\nu}[pq - (p-1)] - 1.$$

We can check at once that the assumption $\theta[q\nu - (p-1)] < (p-\nu)$ may be written as $\theta\Theta < p + \theta(p-2)$, and that $\Theta > 0$ follows from our assumptions.

First we prove sup estimate.

Lemma 2.1 *Let u be a continuous nonnegative subsolution of* (1.1) *in S_{T^*}. Also assume that a time $0 < T < T^*$ is given, so that*

$$\Gamma^{-p} t^{\frac{\theta(p-2)}{p+\theta(p-2)}} \|u(\,\cdot\,,t)\|_{\infty,\mathbb{R}^N}^{p-2} + t\|u(\,\cdot\,,t)\|_{\infty,\mathbb{R}^N}^{\Theta} \le 1, \quad 0 < t < T. \tag{2.1}$$

Then

$$\|u(\,\cdot\,,t)\|_{\infty,\mathbb{R}^N} \le \gamma t^{-\frac{\theta}{p+\theta(p-2)}} \Gamma^{\frac{p}{\kappa}(N-\theta)} \langle u\rangle_t^{\frac{p}{\kappa}}, \quad 0 < t < T, \tag{2.2}$$

where $\gamma = \gamma(N,p,q,\nu)$.

Proof Fix a ball $B_\rho \subset \mathbb{R}^N$, and $0 < t < T$ with $R(t) \le \rho$. Let $k > 0$ be chosen and for $n = 0, 1, 2, \cdots$, set

$$\begin{aligned} &\rho_n = \frac{\rho}{2} + \frac{\rho}{2^{n+1}}, \quad t_n = \frac{t}{2} - \Big(\frac{1}{2^{n+1}}\Big)^p t, \quad k_n = k - \frac{k}{2^{n+1}}, \\ &B_n = B_{\rho_n}(x), \quad Q_n = B_n \times (t_n, t), \quad 0 < t_n < t \le T. \end{aligned}$$

Let $\xi_n(x,t)$ be a smooth cutoff function in Q_n such that

$$\xi_n = 1 \text{ on } Q_{n+1}, \quad 0 \le \frac{\partial \xi_n}{\partial t} \le \gamma\frac{2^{np}}{t}, \quad |D\xi_n| \le \gamma\frac{2^n}{\rho}.$$

After a Steklov averaging process, from Definition 1.1, we obtain

$$\begin{aligned} &\frac{1}{2}\int_{B_n(t)} (u-k_{n+1})_+^2 \xi_n^p \mathrm{d}x + \iint_{Q_n} |D(u-k_{n+1})_+|^p \xi_n^p \mathrm{d}x\mathrm{d}\tau \\ =& \frac{p}{2}\iint_{Q_n} (u-k_{n+1})_+^2 \xi_n^{p-1}\xi_{nt}\mathrm{d}x\mathrm{d}\tau + \iint_{Q_n} |Du^q|^\nu (u-k_{n+1})_+ \xi_n^p \mathrm{d}\tau \\ &- p\iint_{Q_n} (u-k_{n+1})_+ \xi_n^{p-1}|Du|^{p-2}Du\cdot D\xi_n \mathrm{d}x\mathrm{d}\tau. \end{aligned} \tag{2.3}$$

By Young inequality, we have

$$\begin{aligned} &\Big|p\iint_{Q_n} (u-k_{n+1})_+ \xi_n^{p-1}|Du|^{p-2}Du\cdot D\xi_n \mathrm{d}x\mathrm{d}\tau\Big| \\ \le& \frac{1}{3}\iint_{Q_n} |D(u-k_{n+1})_+|^p \xi_n^p \mathrm{d}x\mathrm{d}\tau + \gamma\frac{2^{np}}{\rho^p}\|u\|_{\infty,Q_n}^{p-2}\iint_{Q_n} (u-k_{n+1})_+^2 \mathrm{d}x\mathrm{d}\tau. \end{aligned} \tag{2.4}$$

Note that if $\frac{u}{2} > k_n$, then

$$(u-k_n)_+^2 \ge (u-k_n)_+^\alpha \Big(\frac{u}{2}\Big)^{2-\alpha} \ge C(u-k_{n+1})_+^\alpha u^{2-\alpha}, \quad \alpha \in (0,1);$$

if $k_{n+1} \le u \le 2k_n$, then

$$(u-k_n)_+^2 \ge (u-k_n)^\alpha (k_{n+1}-k_n)^{2-\alpha} \ge 2^{-(n+3)(2-\alpha)} u^{2-\alpha}(u-k_{n+1})^\alpha.$$

Thus we have

$$\iint_{Q_n} |Du^q|^\nu (u-k_{n+1})_+ \xi_n^p \mathrm{d}x\mathrm{d}\tau \le \frac{1}{3}\iint_{Q_n} |D(u-k_{n+1})_+|^p \xi_n^p \mathrm{d}x\mathrm{d}\tau + \gamma \|u\|_{\infty,Q_n}^\Theta \iint_{Q_n} \max\{1, 2^{n(2-\frac{p}{p-\nu})}\}(u-k_n)_+^2 \xi_n^p \mathrm{d}x\mathrm{d}\tau. \quad (2.5)$$

Collecting (2.3)–(2.5), we have

$$\sup_{0<\tau<t}\int_{B_n} [(u-k_{n+1})_+^2 \xi_n^p](x,\tau)\mathrm{d}x + \iint_{Q_n} |D(u-k_{n+1})_+|^p \xi_n^p \mathrm{d}x\mathrm{d}\tau \le \gamma \frac{2^{np}}{t}(1+M)\iint_{Q_n}(u-k_n)_+^2 \mathrm{d}x\mathrm{d}\tau, \quad (2.6)$$

where $M = \frac{t}{\rho^p}\|u\|_{\infty,Q_n}^{p-2} + t\|u\|_{\infty,Q_n}^\Theta$. Hence, as in [2, Proposition 3.1], we can obtain

$$\|u(\,\cdot\,,t)\|_{\infty,B_{\frac{\rho}{2}}} \le \gamma t^{-\frac{N+p}{\kappa}}\Big(\int_0^t\int_{B_\rho} u\mathrm{d}x\mathrm{d}\tau\Big)^{\frac{p}{\kappa}} \quad (2.7)$$

for all $0<t<T$, $B_\rho \subset \mathbb{R}^N$, $\rho = R(t)$. By virtue of (2.7) and the definition of $\langle u\rangle_t$, we get

$$\|u(\,\cdot\,,t)\|_{\infty,B_{\frac{\rho}{2}}} \le \gamma t^{-\frac{N+p}{\kappa}}\Big(\int_0^t\int_{B_{R(t)}} u\mathrm{d}x\mathrm{d}\tau\Big)^{\frac{p}{\kappa}} \le \gamma t^{-\frac{N}{\kappa}} R(t)^{\frac{p}{\kappa}(N-\theta)}\langle u\rangle_t^{\frac{p}{\kappa}}. \quad (2.8)$$

Lemma 2.1 is proved.

We now estimate $|Du^q|^\nu$ in (1.6).

For $\lambda \in (0,p)$, $\sigma > 0$, define

$$H(\lambda,\sigma) = \frac{\lambda}{p-\lambda}[p\sigma - (p-1)] - 1.$$

Lemma 2.2 *Assume that the assumptions of Lemma* 2.1 *are fulfilled and that* $H>0$ *and* $\theta H < p + \theta(p-2)$. *Then we have for all* $B_\rho \subset \mathbb{R}^N$, $0<t<T$, $R(t)\le \rho \le 1$,

$$\int_0^t\int_{B_{\frac{\rho}{2}}} |Du^\sigma|^\lambda \mathrm{d}x\mathrm{d}\tau \le \gamma G(t)\{\Gamma^{\frac{p}{\kappa}(N-\theta)H}\langle u\rangle_t^{\frac{pH}{\kappa}} t^{1-\frac{\theta H}{p+\theta(p-2)}}\}^{1-\frac{\lambda}{p}}, \quad (2.9)$$

where $G(t) = \sup_{0<\tau<t}\|u(\,\cdot\,,\tau)\|_{1,B_\rho}$, $\gamma = \gamma(N,p,q,\nu,\lambda,\sigma,\theta)$.

Proof Let $R(t)\le\rho\le 1$, and $\zeta = \zeta(x)$ be a smooth cutoff function in B_ρ such that

$$0\le \zeta(x)\le 1 \text{ in } B_\rho, \quad \zeta(x) = 1 \text{ in } B_{\frac{\rho}{2}}, \quad |D\zeta| \le \frac{\gamma}{\rho}.$$

Choose as a testing in (1.6) $\phi = t^\beta u^r \zeta^p$, where $\beta>0$ and $r>0$ are to be chosen. After standard calculations, we can obtain

$$\frac{1}{r+1}\int_{B_{\rho(t)}} \tau^\beta u^{r+1}\zeta^p \mathrm{d}x + r\int_0^t\int_{B_\rho} |Du|^p \tau^\beta u^{r-1}\zeta^p \mathrm{d}x\mathrm{d}\tau - \frac{\beta}{r+1}\int_0^t\int_{B_\rho}\tau^{\beta-1}u^{r+1}\zeta^p\mathrm{d}x\mathrm{d}\tau + p\int_0^t\int_{B_\rho}|Du|^{p-2}Du\cdot D\zeta \tau^\beta u^r \zeta^{p-1}\mathrm{d}x\mathrm{d}\tau = \int_0^t\int_{B_\rho}|Du^q|^\nu \tau^\beta u^r \zeta^p \mathrm{d}x\mathrm{d}\tau. \quad (2.10)$$

By Young inequality, we have

$$
\begin{aligned}
&\Big|p\int_0^t\int_{B_\rho}|Du|^{p-2}Du\cdot D\zeta\tau^\beta u^r\zeta^{p-1}\mathrm{d}x\mathrm{d}\tau\Big|\\
\le{}&\frac{r}{3}\int_0^t\int_{B_\rho}|Du|^p\tau^\beta u^{r-1}\zeta^p\mathrm{d}x\mathrm{d}\tau+\gamma\int_0^t\int_{B_\rho}\tau^\beta u^{r+p-1}|D\zeta|^p\mathrm{d}x\mathrm{d}\tau, \qquad (2.11)\\
&\int_0^t\int_{B_\rho}|Du^q|^\nu\tau^\beta u^r\zeta^p\mathrm{d}x\mathrm{d}\tau\\
\le{}&\frac{r}{3}\int_0^t\int_{B_\rho}|Du|^p\tau^\beta u^{r-1}\zeta^p\mathrm{d}x\mathrm{d}\tau+\gamma\int_0^t\int_{B_\rho}\tau^\beta u^{\Theta+r+1}\zeta^p\mathrm{d}x\mathrm{d}\tau. \qquad (2.12)
\end{aligned}
$$

Noting that

$$\int_{B_\rho}\tau^\beta u^{r+1}\zeta^p\mathrm{d}x\ge 0,$$

from (2.10)–(2.12) and (2.1), we get

$$
\begin{aligned}
\int_0^t\int_{B_\rho}|Du|^p\tau^\beta u^{r-1}\zeta^p\mathrm{d}x\mathrm{d}\tau&\le\gamma\int_0^t\int_{B_\rho}\tau^{\beta-1}u^{r+1}\zeta^p\mathrm{d}x\mathrm{d}\tau+\gamma\int_0^t\int_{B_\rho}\tau^\beta u^{r+p-1}|D\zeta|^p\mathrm{d}x\mathrm{d}\tau\\
&\quad+\gamma\int_0^t\int_{B_\rho}\tau^\beta u^{\Theta+r+1}\zeta^p\mathrm{d}x\mathrm{d}\tau\\
&\le\gamma\int_0^t\int_{B_\rho}\tau^{\beta-1}u^{r+1}(1+\tau\rho^{-p}u^{p-2}+\tau u^\Theta)\mathrm{d}x\mathrm{d}\tau\\
&\le\gamma\int_0^t\int_{B_\rho}\tau^{\beta-1}u^{r+1}\mathrm{d}x\mathrm{d}\tau\triangleq\mathrm{I}_1. \qquad (2.13)
\end{aligned}
$$

By Lemma 2.1, we may further bound I_1 above by

$$\mathrm{I}_1\le\gamma G(t)\int_0^t\tau^{\beta-1}\|u(\,\cdot\,,\tau)\|^r_{\infty,B_\rho}\mathrm{d}\tau\le\gamma G(t)\Gamma^{\frac{p}{\kappa}(N-\theta)r}\langle u\rangle_t^{\frac{p}{\kappa}r}t^{\beta-\frac{\theta r}{p+\theta(p-2)}}\triangleq\mathrm{I}_2, \qquad (2.14)$$

provided

$$\beta>\frac{\theta r}{p+\theta(p-2)}. \qquad (2.15)$$

Next by Hölder inequality and (2.13)–(2.14), we have

$$
\begin{aligned}
\int_0^t\int_{B_\rho}|Du^\sigma|^\lambda\zeta^p\mathrm{d}x\mathrm{d}\tau&\le\gamma\Big(\int_0^t\int_{B_\rho}|Du|^p\tau^\beta u^{r-1}\zeta^p\mathrm{d}x\mathrm{d}\tau\Big)^{\frac{\lambda}{p}}\Big(\int_0^t\int_{B_\rho}u^{1+H-\frac{r\lambda}{p-\lambda}}\tau^{-\frac{\lambda\beta}{p-\lambda}}\mathrm{d}x\mathrm{d}\tau\Big)^{1-\frac{\lambda}{p}}\\
&\le\gamma\mathrm{I}_2^{\frac{\lambda}{p}}\Big(\int_0^t\int_{B_\rho}u^{1+H-\frac{r\lambda}{p-\lambda}}\tau^{-\frac{\lambda\beta}{p-\lambda}}\mathrm{d}x\mathrm{d}\tau\Big)^{1-\frac{\lambda}{p}}. \qquad (2.16)
\end{aligned}
$$

Assume from now on that

$$H\ge\frac{r\lambda}{p-\lambda}. \qquad (2.17)$$

Then, we get

$$
\begin{aligned}
\int_0^t\int_{B_\rho}u^{1+H-\frac{r\lambda}{p-\lambda}}\tau^{-\frac{\lambda\beta}{p-\lambda}}\mathrm{d}x\mathrm{d}\tau&\le G(t)\int_0^t\|u(\,\cdot\,,\tau)\|^{H-\frac{r\lambda}{p-\lambda}}_{\infty,B_\rho}\tau^{-\frac{\lambda\beta}{p-\lambda}}\mathrm{d}x\mathrm{d}\tau\\
&\le\gamma G(t)\Gamma^{\frac{p}{\kappa}(N-\theta)(H-\frac{r\lambda}{p-\lambda})}\langle u\rangle_t^{\frac{p}{\kappa}(H-\frac{r\lambda}{p-\lambda})}t^\alpha, \qquad (2.18)
\end{aligned}
$$

provided

$$\alpha = 1 - \frac{\theta\Theta}{p+\theta(p-2)} - \frac{\lambda}{p-\lambda}\Big(\beta - \frac{r\theta}{p+\theta(p-2)}\Big) > 0. \tag{2.19}$$

Substituting I_2 and (2.18) into (2.16), we get (2.9). We now show that β and r can be chosen, such that (2.15), (2.17) and (2.19) hold.

If $\theta = 0$, then (2.15) and (2.19) follow from $0 < \beta < \frac{p-\lambda}{\lambda}$.

Assume that $\theta > 0$, and fix arbitrarily

$$0 < \beta < \frac{p-\lambda}{\lambda}. \tag{2.20}$$

We write (2.19) as

$$r > \Big[\frac{p-\lambda}{\lambda}\Big(\frac{\theta H}{p+\theta(p-2)} - 1\Big) + \beta\Big]\frac{p+\theta(p-2)}{\theta} \triangleq z_1, \tag{2.21}$$

and combine (2.15) and (2.17) in the form

$$r < \min\Big\{\frac{p+\theta(p-2)}{\theta}\beta, \frac{p-\lambda}{\lambda}H\Big\} \triangleq z_2. \tag{2.22}$$

It is easy to check that $z_1 < z_2$, as a consequence of the assumption $\theta H < p + \theta(p-2)$ and (2.20). Since we can fix $r > 0$ such that $r \in (z_1, z_2)$, Lemma 2.2 is proved.

From now on, we choose specially the constant Γ appearing in the definition of $R(t)$ as

$$\Gamma = C[\mu]^{\frac{p-2}{p+\theta(p-2)}}, \tag{2.23}$$

where $C > 0$ will be chosen below as a function of N, p, q, ν and θ. Obviously we may assume $[\mu] > 0$ throughout.

Remark 2.1 By definition, it holds that $\langle u\rangle_t \le [u]_t$. Moreover, using Lemma 2.1 to estimate the integrals over B_ρ with $0 < \rho < R(\tau)$, appearing in the definition of $[u]_t$, we find also

$$[u]_t \le \langle u\rangle_t + \gamma(C)[\mu]^{\frac{N}{\kappa}(p-2)}\langle u\rangle_t^{\frac{p}{\kappa}}, \quad 0 < t < T,$$

where T is chosen as in Lemma 2.1, and we have used (2.23) too.

Lemma 2.3 *Let $u \ge 0$ be a uniformly continuous and bounded solution to* (1.1) *in S_{T^*}. Then if $\theta[q\nu - (p-1)] < (p-\nu)$, there exists a $T_0 > 0$, $T_0 = T_0(N, p, q, \nu, \theta, [\mu])$, such that*

$$[u]_t \le \gamma[\mu], \quad 0 < t < T_0,$$

and (2.1)–(2.2) *hold for all $0 < t < T_0$, where $\gamma = \gamma(N, p, q, \nu, \theta)$.*

Proof Define

$$t_0 = \sup\{0 < T < T^* \mid (2.1) \text{ holds, where } \Gamma \text{ is given by } (2.23)\}.$$

Choose $0 < t < t_0$, and let $B_\rho(x_0) \subset \mathbb{R}^N$ be any ball with radius $R(t) \le \rho \le 1$, centered at an arbitrarily fixed $x_0 \in \mathbb{R}^N$. Let $\zeta = \zeta(x)$ be a smooth cutoff function in B_ρ such that

$$\zeta(x) \equiv 1 \text{ in } B_{\frac{\rho}{2}}, \quad |D\zeta| \le \frac{\gamma}{\rho}.$$

We find after straightforward calculations that

$$\int_{B_{\frac{\rho}{2}}} u(x,t)\mathrm{d}x \le \int_{B_\rho} \mathrm{d}\mu + \frac{\gamma}{\rho}\int_0^t\int_{B_\rho} |Du|^{p-1}\mathrm{d}x\mathrm{d}\tau + \int_0^t\int_{B_\rho} |Du^q|^\nu \mathrm{d}x\mathrm{d}\tau.$$

Multiplying both sides of above inequality by $\rho^\theta |B_\rho|^{-1}$, we find by using Lemma 2.2 that

$$\rho^\theta \fint_{B_{\frac{\rho}{2}}} u(x,t)\mathrm{d}x \le 2^N[\mu] + \gamma\langle u\rangle_t\{A(C,\langle u\rangle_t) + B(C,t,\langle u\rangle_t)\} \tag{2.24}$$

for all $0 < t < t_0$, $R(t) \le \rho \le 1$. Here

$$A(C,\langle u\rangle_t) = C^{-\frac{p+\theta(p-2)}{\kappa}}\left\{\frac{\langle u\rangle_t}{[\mu]}\right\}^{\frac{p-2}{\kappa}},$$
$$B(C,t,\langle u\rangle_t) = \{\Gamma^{\frac{p}{\kappa}(N-\theta)\Theta}\langle u\rangle_t^{\frac{p\Theta}{\kappa}} t^{1-\frac{\theta\Theta}{p+\theta(p-2)}}\}^{1-\frac{\nu}{p}},$$

where C is the constant in (2.23).

(2.24) implies

$$\langle u\rangle_t \le \gamma_1[\mu] + \gamma_2\langle u\rangle_t\{A(C,\langle u\rangle_t) + B(C,t,\langle u\rangle_t)\}, \tag{2.25}$$

where γ_2 depends on the same quantities determining the constants γ in Lemma 2.1 and Lemma 2.2, but it does not depend on C. Next we define

$$t_1 = \sup\{0 < t < T^* \mid \langle u\rangle_t \le 4\gamma_1[\mu]\},$$
$$t_2 = \sup\{0 < t < T^* \mid B(C,t,\langle u\rangle_t) < \delta\},$$

where the (small) constant $\delta > 0$ is to be chosen. Note that t_1 and t_2 are well-defined because the stipulated assumptions make sure that $\langle u\rangle_t$ is continuous in $[0,T^*]$, and that the exponent of t in B is positive. Let

$$t_3 = \min\{t_0, t_1, t_2\}.$$

Then for all $0 < t < t_3$, since γ_1 and γ_2 do not depend on C, we have

$$\gamma_2 A(C,\langle u\rangle_t) \le \gamma_2 C^{-\frac{p+\theta(p-2)}{\kappa}}(4\gamma_1)^{\frac{p-2}{\kappa}} \le \frac{1}{4}, \tag{2.26}$$

provided C is suitably chosen. Then, if we also choose $\delta < (4\gamma_2)^{-1}$, we have

$$\gamma_2 B(C,t,\langle u\rangle_t) \le \frac{1}{4}. \tag{2.27}$$

From (2.25)–(2.27), we obtain

$$\langle u\rangle_t \le 2\gamma_1[\mu], \quad 0 < t < t_3. \tag{2.28}$$

Now we choose $0 < T_0 \le t_3$ to get

$$[u]_t \le 2\gamma_1[\mu] + \gamma(C)[\mu]^{\frac{N}{\kappa}(p-2)}[\mu]^{\frac{p}{\kappa}} \le \gamma[\mu], \quad 0 < t < T_0 \le t_3.$$

Here we have used Remark 2.1 and (2.28).

Finally, we show that a quantitative estimate below $T_0 \le t_3$ with T_0 as above can be found. As a matter of fact, for $0 < t < T_0 \le t_3 \le t_0$ and $x \in \mathbb{R}^N$, Lemma 2.1 together with (2.28) implies

$$\begin{aligned}\Gamma^{-p} t^{\frac{\theta(p-2)}{p+\theta(p-2)}} u^{p-2}(x,t) + t u^{\Theta}(x,t) &\le \gamma C^{-\frac{p}{\kappa}[p+\theta(p-2)]} + \gamma (B(C,t,\langle u\rangle_t))^{\frac{p}{p-\nu}} \\ &\le \gamma (C^{-\frac{p}{\kappa}[p+\theta(p-2)]} + \delta^{\frac{p}{p-\nu}}) \le \frac{1}{2}, \end{aligned} \tag{2.29}$$

if C is chosen large enough, and then δ in the definition of t_2 is chosen small enough. Let us remark that this can be done safely, as the constant γ in (2.29) is known a priori, and, especially, it does not depend on C or δ. Therefore (2.1) and (2.2) hold for $0 < t < T_0$. Lemma 2.3 is proved.

3 Proof of Theorem 1.1

Consider the family of approximating problem

$$u_{nt} - \operatorname{div}(|Du_n|^{p-2} Du_n) = \min(|Du_n^q|^{\nu}, n), \quad \text{in } \mathbb{R}^N \times (0,\infty), \tag{3.1}$$

$$u_n(x,0) = u_{0n}(x), \quad x \in \mathbb{R}^N, \tag{3.2}$$

where

$$\begin{aligned} &u_{0n} \ge 0, \quad u_{0n} \in C^\infty(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N), \\ &\lim_{n\to\infty} \int_{\mathbb{R}^N} u_{0n} \eta(x) \mathrm{d}x = \int_{\mathbb{R}^N} \eta(x) \mathrm{d}\mu, \quad \forall\, \eta(x) \in C_0^1(\mathbb{R}^N). \end{aligned}$$

We can assume without loss of generality that $[u_{0n}] \le \gamma(N)[\mu]$. By the result of [5] and [6], there exists a solution $u_n \in C(\overline{S_T}) \cap L^\infty(S_T) \cap L^p(0,T;W^{1,p}(\mathbb{R}^N))$ to (3.1)–(3.2). Let us remark that the estimates in Section 2 can applied to u_n, because it is a continuous and bounded solution to (3.1). We consider the strip $S_{T_0} = \mathbb{R}^N \times (0,T_0)$, where T_0 is defined as in Lemma 2.3. By Lemmas 2.1 and 2.3, we have

$$\|u_n\|_{\infty,K} \le \gamma(K,[\mu]) \tag{3.3}$$

for any compact set $K \subset S_{T_0}$. By the mentioned a priori estimates and standard calculations, we find

$$\int_{B_R} u_n^{h+1}(x,t)\mathrm{d}x + \int_0^t \int_{B_R} |Du_n^{\frac{h+p-1}{p}}|^p \mathrm{d}x\mathrm{d}\tau \le \gamma(R,[\mu],h), \quad t \in (0,T_0).$$

Hence

$$\|Du_n^{\frac{h+p-1}{p}}\|_{p,K} \le \gamma(K,[\mu],h), \quad \forall\, h \ge 0. \tag{3.4}$$

Note that T_0 and the bounds in (3.3) and (3.4) do not depend on n. As a consequence of the quoted result of [5], there is a subsequence of $\{u_n\}$, which is denoted by $\{u_n\}$ again, and a function $u \in C(S_{T_0}) \cap L^p_{\mathrm{loc}}((0,T_0);W^{1,p}_{\mathrm{loc}}(\mathbb{R}^N))$, such that in each compact set K contained in S_{T_0},

$$u_n \to u, \quad \text{uniformly on } K, \tag{3.5}$$

$$Du_n \rightharpoonup Du, \quad \text{weakly in } L^p(K) \tag{3.6}$$

as $n \to \infty$. We now prove

$$Du_n \to Du, \quad \text{strongly in } L^p(K). \tag{3.7}$$

Let $\varphi \in C_0^1(S_{T_0})$, $\varphi \geq 0$. We can assume without loss of generality that $u_{nt} \in L^2(S_{T_0})$, otherwise we can use Steklov averages approximation. Multiplying (3.1) by $(u_n - u_m)\varphi$ and integrating it over S_{T_0}, we have

$$\begin{aligned}
&\iint_{S_{T_0}} \varphi u_{nt}(u_n - u_m)\mathrm{d}x\mathrm{d}t + \iint_{S_{T_0}} \varphi |Du_n|^{p-2} Du_n (Du_n - Du_m)\mathrm{d}x\mathrm{d}t \\
&+ \iint_{S_{T_0}} |Du_n|^{p-2} Du_n \cdot D\varphi (u_n - u_m)\mathrm{d}x\mathrm{d}t \\
= &\iint_{S_{T_0}} \min\{|Du_n^q|^\nu, n\}(u_n - u_m)\varphi \mathrm{d}x\mathrm{d}t. \qquad (3.8)
\end{aligned}$$

Adding (3.8) to the similar equality obtained by interchanging u_n and u_m, we find

$$\begin{aligned}
&\Big| \iint_{S_{T_0}} \phi(|Du_n|^{p-2}Du_n - |Du_m|^{p-2}Du_m)(Du_n - Du_m)\mathrm{d}x\mathrm{d}t \Big| \\
\leq &\gamma \iint_{S_{T_0}} \{|\phi_t||u_m - u_n| + |D\phi|(|Du_m|^{p-1} + |Du_n|^{p-1}) \\
&+ |\phi|(|Du_m^q|^\nu + |Du_n^q|^\nu)\}|u_n - u_m|\mathrm{d}x\mathrm{d}t. \qquad (3.9)
\end{aligned}$$

Note that

$$(|Du_n|^{p-2}Du_n - |Du_m|^{p-2}Du_m)(Du_n - Du_m) \geq \gamma |Du_n - Du_m|^p \qquad (3.10)$$

for some $\gamma > 0$. We may derive (3.7) from (3.5), (3.6), (3.9) and (3.10). Moreover, from (3.5) and (3.7), we can obtain

$$|Du_n^q|^\nu \to |Du^q|^\nu, \quad \text{as } n \to \infty.$$

Now, it follows from a standard limiting process, that u satisfies (1.6).

Finally we prove (1.7). For any $\eta \in C_0^1(\mathbb{R}^N)$, by Lemma 2.2,

$$\begin{aligned}
\int_{\mathbb{R}^N} |u_n(x,t) - u_n(x,0)|\eta \mathrm{d}x &\leq \iint_{S_{T_0}} |Du_n|^{p-1}|D\eta|\mathrm{d}x\mathrm{d}t + \iint_{S_{T_0}} |Du_n^q|^\nu \eta \mathrm{d}x\mathrm{d}t \\
&\leq \gamma_1 t^{\frac{1}{p+\theta(p-2)}} + \gamma_2 t^{[1-\frac{\theta\Theta}{p+\theta(p-2)}]\frac{p-\nu}{p}} \to 0, \quad \text{as } t \to 0, \qquad (3.11)
\end{aligned}$$

where γ_1 and γ_2 do not depend on t. By (3.2), (3.5) and (3.11), we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} u(x,t)\eta(x)\mathrm{d}x - \int_{\mathbb{R}^N} \eta(x)\mathrm{d}\mu \\
\leq &\int_{\mathbb{R}^N} |u(x,t) - u_n(x,t)|\eta(x)\mathrm{d}x + \int_{\mathbb{R}^N} |u_n(x,t) - u_n(x,0)|\eta(x)\mathrm{d}x \\
&+ \Big| \int_{\mathbb{R}^N} u_n(x,0)\eta(x)\mathrm{d}x - \int_{\mathbb{R}^N} \eta(x)\mathrm{d}\mu \Big| \to 0, \quad \text{as } n \to \infty,\ t \to 0.
\end{aligned}$$

This implies (1.7). Then the existence of solution to (1.1)–(1.2) is proved.

4 Proof of Theorem 1.2

Lemma 4.1 *Let u be a continuous nonnegative supersolution of* (1.3) *with $p \geq 2$, and $u(\cdot,t) \to \mu$ as $t \to 0$, in sense of measures, with*

$$\mu(B_\rho(0)) \geq \varepsilon \rho^{N-\theta}, \quad 0 < \rho < \sigma, \qquad (4.1)$$

where $\sigma > 0$, $\varepsilon > 0$ and $0 \le \theta \le N$ are given. Then

$$u(x,t) \ge \gamma_0 \varepsilon^{\frac{p}{p+\theta(p-2)}} t^{-\frac{\theta}{p+\theta(p-2)}}, \tag{4.2}$$

where $|x| \le \gamma_0 \varepsilon^{p-\frac{2}{p+\theta(p-2)}} t^{\frac{1}{p+\theta(p-2)}} < \sigma$, $\gamma_0 t = \varepsilon^{-(p-2)}\rho^{p+\theta(p-2)}$, $\gamma_0 = \gamma_0(N,p,\theta)$, $t < T$.

Proof If $p > 2$, it is well-known that the solution of (1.3) satisfies Harnack inequality (see [5, Theorem 7.1])

$$\oint_{B_\rho(0)} u(x,\tau)\mathrm{d}x \le \gamma\Big\{\Big(\frac{\rho^p}{t}\Big)^{\frac{1}{p-2}} + \Big(\frac{t}{\rho^p}\Big)^{\frac{N}{p}}\Big[\inf_{|x|<\rho} u(x,t+\tau)\Big]^{\frac{\kappa}{p}}\Big\}, \tag{4.3}$$

where $\kappa = N(p-2)+p$, $0 < 2\tau < t < T$, $\rho > 0$. Hence, letting $\tau \to 0$ and choosing $\gamma_0 t = \varepsilon^{-(p-2)}\rho^{p+\theta(p-2)}$, γ_0 suitably small, we get (4.2). If $p = 2$, (4.3) follows from the Harnack inequality for nonnegative solutions to the heat equation in S_T (see [7]).

Remark 4.1 By comparison principle, Lemma 4.1 holds for the solution of (1.1)–(1.2).

Proof of Theorem 1.2 We may assume $\theta > 0$, $q\nu > p-1$ and $x_0 = 0$. From now on, all the balls are understood to be centered at 0.

Let $\xi \in C_0^1(B_\rho(0))$ be a cutoff function in B_ρ, $\xi = 1$ in $B_{\frac{\rho}{2}}$, $|D\xi| \le \frac{\gamma}{\rho}$. The next calculations are formal, in which the solution u of (1.1) is required to be strictly positive and $u_t \in L^2_{\mathrm{loc}}(S_T)$. They can be rigorous by replacing u with $u+\epsilon$ and letting $\epsilon \to 0$, and $u_t \in L^2_{\mathrm{loc}}(S_T)$ can be made by Steklov average technic. We choose $\phi = u^{s-1}\xi^b$ in (1.6), where $s \in (0,1)$ and b is large enough, to obtain, $\forall\, 0 < t_0 < t < T$,

$$\begin{aligned}&\frac{1}{s}\int_{B_\rho} u^s(x,t)\xi^b\mathrm{d}x + \int_{t_0}^t\int_{B_\rho} b\xi^{b-1}u^{s-1}|Du|^{p-2}Du\cdot D\xi\mathrm{d}x\mathrm{d}\tau\\ =&\frac{1}{s}\int_{B_\rho} u^s(x,t_0)\xi^b\mathrm{d}x + (1-s)\int_{t_0}^t\int_{B_\rho} u^{s-2}|Du|^p\xi^b\mathrm{d}x\mathrm{d}\tau + \int_{t_0}^t\int_{B_\rho}\xi^b u^{s-1}|Du^q|^\nu\mathrm{d}x\mathrm{d}\tau.\end{aligned}$$

By Young inequality, we get

$$\begin{aligned}\frac{1}{s}\int_{B_\rho} u^s(x,t)\xi^b\mathrm{d}x \ge &\frac{1}{s}\int_{B_\rho} u^s(x,t_0)\xi^b\mathrm{d}x - \frac{\gamma}{\rho^p}\int_{t_0}^t\int_{B_\rho}\xi^{b-p}u^{p+s-2}\mathrm{d}x\mathrm{d}\tau\\ &+\int_{t_0}^t\int_{B_\rho}\xi^b u^{s-1}|Du^q|^\nu\mathrm{d}x\mathrm{d}\tau.\end{aligned} \tag{4.4}$$

Noting that $[u]_t < \infty$, we have

$$\int_{B_\rho} u(x,t)\mathrm{d}x \le \gamma\rho^{N-\theta}, \quad 0 < t < t_*,\ 0 < \rho < 1 \tag{4.5}$$

for a suitable $t_* > 0$. Define

$$A(t) = \{x \in B_\rho \mid u(x,t) > \omega\rho^{-\theta}\}, \quad E(t) = B_\rho \setminus A(t),$$

where $\omega > 0$ is to be chosen. Hence, for $0 < t < t_*$, (4.5) implies

$$|A(t)| \le \gamma\omega^{-1}|B_\rho|, \quad 0 < t < t_*,\ 0 < \rho < 1.$$

As a consequence,

$$|E(t)\cap B_{\frac{\rho}{2}}| \ge \frac{1}{2}|B_{\frac{\rho}{2}}|, \quad 0 < t < t_*,\ 0 < \rho < 1,$$

if ω is large enough.

We need the following result, which is proved in [8, Lemma 5.1].

Let $v \in W^1_m(B_\rho)$, $v \geq 0$, $m \geq 1$. Then for all measurable sets $\Sigma \subset B_{\frac{\rho}{2}}$, $v \equiv 0$ in Σ, we have

$$\int_{B_\rho} v^m(x)G(|x|)\mathrm{d}x \leq \gamma(N,m)\Big(\rho^{N+1}|\Sigma|^{-1}G(0)G^{-1}\Big(\frac{\rho}{2}\Big)\Big)^m \int_{B_\rho} |Dv(x)|^m G(|x|)\mathrm{d}x,$$

where $G : [0, \rho) \to (0, \infty)$ is nonincreasing.

Let $v = (u - \omega\rho^{-\theta})^r_+$, $r = q + \frac{s-1}{\nu}$, $m = \nu$, $\Sigma = E(t) \cap B_{\frac{\rho}{2}}$, $G(|x|) = \xi^p(x)$. We find

$$\gamma \int_{t_0}^t \int_{B_\rho} |Du^r|^\nu \xi^p \mathrm{d}x\mathrm{d}\tau \geq \rho^{-\nu} \int_{t_0}^t \int_{B_\rho} (u - \omega\rho^{-\theta})^{r\nu}_+ \xi^p \mathrm{d}x\mathrm{d}\tau \tag{4.6}$$

for $0 < t_0 < t < t_*$, $0 < \rho < 1$. Moreover, we note that (4.2) implies

$$u(x,\tau) \geq 2\omega\rho^{-\theta}, \quad \frac{t}{2} < \tau < t, \quad |x| \leq \gamma_0\rho \quad \text{for } \rho = \rho(t) = P_0 t^{\frac{1}{p+\theta(p-2)}}, \tag{4.7}$$

provided $P_0 > 0$ is fixed and large enough (in (4.7), γ_0 depends on P_0). From now on, we choose ρ as in (4.7). We also redefine t^* to ensure $\rho < 1$ for $t < t^*$. Thus straightforward calculations lead us to

$$\gamma \int_{B_\rho} (u(x,\tau) - \omega\rho^{-\theta})^{r\nu}_+ \xi^p \mathrm{d}x \geq \int_{B_\rho} u^{r\nu}(x,\tau)\xi^p \mathrm{d}x, \quad \frac{t}{2} < \tau < t < t^*. \tag{4.8}$$

Combining (4.4), (4.6) and (4.8), we find

$$\begin{aligned}\int_{B_\rho} u^s(x,z)\xi^b \mathrm{d}x - \int_{B_\rho} u^s\Big(x,\frac{t}{2}\Big)\xi^b \mathrm{d}x \geq &-\frac{\gamma}{\rho^p}\int_{\frac{t}{2}}^z \int_{B_\rho} \xi^{b-p}u^{p+s-2}\mathrm{d}x\mathrm{d}\tau \\ &+ \gamma\rho^{-\nu}\int_{\frac{t}{2}}^z \int_{B_\rho} u^{\nu(q+\frac{s-1}{\nu})}\xi^b \mathrm{d}x\mathrm{d}\tau \quad \text{for } \frac{t}{2} < z < t.\end{aligned}$$

Using Young inequality, we get

$$\begin{aligned}\int_{B_\rho} u^s(x,z)\xi^b \mathrm{d}x - \int_{B_\rho} u^s\Big(x,\frac{t}{2}\Big)\xi^b \mathrm{d}x \geq &-\gamma_1\rho^{-\frac{p(\nu q+s-1)-\nu(p+s-2)}{\nu q-p+1}+N}\Big(z-\frac{t}{2}\Big) \\ &+ \gamma_2\rho^{-\nu}\int_{\frac{t}{2}}^z \int_{B_\rho} u^{\nu(q+\frac{s-1}{\nu})}\xi^b \mathrm{d}x\mathrm{d}\tau \quad \text{for } \frac{t}{2} < z < t.\end{aligned} \tag{4.9}$$

Note that $\int_{B_\rho} u^s(x,z)\xi^b \mathrm{d}x$ is a supersolution of the following problem:

$$\begin{aligned}&y' = \gamma_2\rho^{-(\nu+\frac{N(q\nu-1)}{s})}y^{\frac{\nu q+s-1}{s}} - \gamma_1\rho^{-\frac{p(\nu q+s-1)-\nu(p+s-2)}{\nu q-p+1}+N}, \quad \text{in } \Big(\frac{t}{2},t\Big),\\ &y\Big(\frac{t}{2}\Big) = \int_{B_\rho} u^s\Big(x,\frac{t}{2}\Big)\xi^b \mathrm{d}x.\end{aligned}$$

Hence, it follows (see [9, Lemma 4.1]) that

$$\int_{B_\rho} u^s\Big(x,\frac{t}{2}\Big)\xi^b \mathrm{d}x \leq \gamma_2 \max\{\rho^{\frac{s\nu}{q\nu-1}+N}t^{-\frac{s}{q\nu-1}}, \rho^{(\nu-\frac{p(\nu q+s-1)-\nu(p+s-2)}{q\nu-(p-1)})\frac{s}{\nu q+s-1}+N}\}. \tag{4.10}$$

On the other hand, by (4.7) we have

$$\int_{B_\rho} u^s\Big(x,\frac{t}{2}\Big)\xi^p \mathrm{d}x \geq \gamma_0\rho^{-s\theta+N} \tag{4.11}$$

for small t and $\rho = \rho(t)$ defined in (4.7). Let $t \to 0$. (4.10) and (4.11) imply $\nu < p$, and

$$\theta \le \min\Big\{\frac{(p-\nu)_+}{\nu q - p + 1}, -\frac{\nu}{q\nu + s - 1} + \frac{p(\nu q + s - 1) - \nu(p + s - 2)}{(q\nu - (p-1))(\nu q + s - 1)}\Big\}. \tag{4.12}$$

Since $-\frac{\nu}{q\nu+s-1} + \frac{p(\nu q+s-1)-\nu(p+s-2)}{(q\nu-(p-1))(\nu q+s-1)}$ is increasing in $s \in (0,1)$ and equals to $\frac{(p-\nu)_+}{\nu q-p+1}$ as $s = 1$, (4.12) implies

$$\theta(q\nu - p + 1) < (p-\nu)_+.$$

5 Proof of Theorem 1.3

First we prove the sup estimate.

Lemma 5.1 *Let u be a continuous nonnegative subsolution of* (1.4) *in S_{T^*}. Also assume that a time $0 < T < T^*$ is given, so that*

$$\Gamma^{-p} t^{\frac{\theta(p-2)}{p+\theta(p-2)}} \|u(\,\cdot\,,t)\|_{\infty,\mathbb{R}^N}^{p-2} + t\|u(\,\cdot\,,t)\|_{\infty,\mathbb{R}^N}^{q-1} \le 1, \quad 0 < t < T. \tag{5.1}$$

Then

$$\|u(\,\cdot\,,t)\|_{\infty,\mathbb{R}^N} \le \gamma t^{-\frac{\theta}{p+\theta(p-2)}} \Gamma^{\frac{p}{\kappa}(N-\theta)} \langle u\rangle_t^{\frac{p}{\kappa}}, \quad 0 < t < T, \tag{5.2}$$

where $\gamma = \gamma(N,p,q)$.

Proof As in [2, Proposition 3.1], we can obtain

$$\|u(\,\cdot\,,t)\|_{\infty,B_{\frac{\rho}{2}}} \le \gamma t^{-\frac{N+p}{\kappa}} \Big(\int_0^t \int_{B_\rho} u \mathrm{d}x\mathrm{d}\tau\Big)^{\frac{p}{\kappa}} \tag{5.3}$$

for all $0 < t < T$, $B_\rho \subset \mathbb{R}^N$, $\rho = R(t)$. By virtue of (5.3) and the definition of $\langle u\rangle_t$, we get

$$\|u(\,\cdot\,,t)\|_{\infty,B_{\frac{\rho}{2}}} \le \gamma t^{-\frac{N+p}{\kappa}} \Big(\int_0^t \int_{B_{R(t)}} u \mathrm{d}x\mathrm{d}\tau\Big)^{\frac{p}{\kappa}} \le \gamma t^{-\frac{N}{\kappa}} R(t)^{\frac{p}{\kappa}(N-\theta)} \langle u\rangle_t^{\frac{p}{\kappa}}. \tag{5.4}$$

This implies (5.2).

Lemma 5.2 *Assume that the assumptions of Lemma* 5.1 *are fulfilled. Then for all $B_\rho \subset \mathbb{R}^N$, $0 < t < T$, $R(t) \le \rho \le 1$, we have*

$$\int_0^t \int_{B_{\frac{\rho}{2}}} |Du|^{p-1} \mathrm{d}x\mathrm{d}\tau \le \gamma G(t) \{\Gamma^{\frac{p}{\kappa}(N-\theta)(p-2)} \langle u\rangle_t^{\frac{p(p-2)}{\kappa}} t^{1-\frac{\theta(p-2)}{p+\theta(p-2)}}\}^{\frac{1}{p}}, \tag{5.5}$$

where $G(t) = \sup\limits_{0<\tau<t} \|u(\,\cdot\,,\tau)\|_{1,B_\rho}$, $\gamma = \gamma(N,p,q,\theta)$.

Proof Let $R(t) \le \rho \le 1$. Similar to the argument in Lemma 2.2, we get

$$\begin{aligned}
\int_0^t \int_{B_\rho} |Du|^p \tau^\beta u^{r-1} \zeta^p \mathrm{d}x\mathrm{d}\tau &\le \gamma \int_0^t \int_{B_\rho} \tau^{\beta-1} u^{r+1} \zeta^p \mathrm{d}x\mathrm{d}\tau + \gamma \int_0^t \int_{B_\rho} \tau^\beta u^{r+p-1} |D\zeta|^p \mathrm{d}x\mathrm{d}\tau \\
&\quad + \gamma \int_0^t \int_{B_\rho} \tau^\beta u^{q+r} \zeta^p \mathrm{d}x\mathrm{d}\tau \\
&\le \gamma \int_0^t \int_{B_\rho} \tau^{\beta-1} u^{r+1} (1 + \tau\rho^{-p} u^{p-2} + \tau u^{q-1}) \mathrm{d}x\mathrm{d}\tau \\
&\le \gamma \int_0^t \int_{B_\rho} \tau^{\beta-1} u^{r+1} \mathrm{d}x\mathrm{d}\tau \triangleq \mathrm{I}_1.
\end{aligned} \tag{5.6}$$

By Lemma 5.1, we may further bound I_1 above by

$$\mathrm{I}_1 \le \gamma G(t)\int_0^t \tau^{\beta-1}\|u(\,\cdot\,,\tau)\|_{\infty,B_\rho}^r \mathrm{d}\tau \le \gamma G(t)\Gamma^{\frac{p}{\kappa}(N-\theta)r}\langle u\rangle_t^{\frac{p}{\kappa}r} t^{\beta-\frac{\theta r}{p+\theta(p-2)}} \triangleq \mathrm{I}_2, \tag{5.7}$$

provided

$$\beta > \frac{\theta r}{p+\theta(p-2)}. \tag{5.8}$$

Next by Hölder inequality and (5.6)–(5.7), we have

$$\begin{aligned}
&\int_0^t\int_{B_\rho} |Du|^{p-1}\zeta^p \mathrm{d}x\mathrm{d}\tau \\
&\le \gamma\Big(\int_0^t\int_{B_\rho} |Du|^p\tau^\beta u^{r-1}\zeta^p \mathrm{d}x\mathrm{d}\tau\Big)^{\frac{p-1}{p}}\Big(\int_0^t\int_{B_\rho} u^{(p-1)(1-r)}\tau^{-(p-1)\beta}\mathrm{d}x\mathrm{d}\tau\Big)^{\frac{1}{p}} \\
&\le \gamma \mathrm{I}_2^{\frac{p-1}{p}}\Big(\int_0^t\int_{B_\rho} u^{(p-1)(1-r)}\tau^{-(p-1)\beta}\mathrm{d}x\mathrm{d}\tau\Big)^{\frac{1}{p}}.
\end{aligned} \tag{5.9}$$

From now on, we assume

$$p-2 > r(p-1). \tag{5.10}$$

Then, we get

$$\begin{aligned}
\int_0^t\int_{B_\rho} u^{(p-1)(1-r)}\tau^{-\beta(p-1)}\mathrm{d}x\mathrm{d}\tau &\le G(t)\int_0^t \|u(\,\cdot\,,\tau)\|_{\infty,B_\rho}^{(p-2)-r(p-1)}\tau^{-\beta(p-1)}\mathrm{d}x\mathrm{d}\tau \\
&\le \gamma G(t)\Gamma^{\frac{p}{\kappa}(N-\theta)(p-2-r(p-1))}\langle u\rangle_t^{\frac{p}{\kappa}(p-2-r(p-1))} t^\alpha,
\end{aligned} \tag{5.11}$$

provided

$$\alpha = 1-\frac{\theta(p-2)}{p+\theta(p-2)}-(p-1)\big(\beta-\frac{r\theta}{p+\theta(p-2)}\big) > 0. \tag{5.12}$$

Substituting I_2 and (5.11) into (5.9), we get (5.5). We now show that β, r can be chosen, such that (5.8), (5.10) and (5.12) hold.

If $\theta = 0$, (5.8) and (5.12) follow from $0 < \beta < \frac{1}{p-1}$.

Assume that $\theta > 0$ and arbitrarily fix

$$0 < \beta < \frac{1}{p-1}. \tag{5.13}$$

We write (5.12) as

$$r > \Big[\frac{1}{p-1}\Big(\frac{\theta(p-2)}{p+\theta(p-2)}-1\Big)+\beta\Big]\frac{p+\theta(p-2)}{\theta} \triangleq z_1, \tag{5.14}$$

and we combine (5.8) and (5.10) in the form

$$r < \min\Big\{\frac{p+\theta(p-2)}{\theta}\beta, \frac{p-2}{p-1}\Big\} \triangleq z_2. \tag{5.15}$$

It is easy to check that $z_1 < z_2$, as a consequence of assumption (5.13). Since we can fix $r > 0$ such that $r \in (z_1, z_2)$, Lemma 5.2 is proved.

Lemma 5.3 *Let $u \geq 0$ be a uniformly continuous and bounded solution to* (1.1) *in S_{T^*}. Then if $q < p - 1 + \frac{p}{\theta}$, there exists a $T_0 > 0$, $T_0 = T_0(N, p, q, \theta, [\mu])$, such that*

$$[u]_t \leq \gamma[\mu], \quad 0 < t < T_0,$$

and (5.1)–(5.2) *hold for all $0 < t < T_0$, where $\gamma = \gamma(N, p, q, \theta)$.*

Proof Define

$$t_0 = \sup\{0 < T < T^* \mid (5.1) \text{ holds, where } \Gamma \text{ is given by } (2.23)\}.$$

Choose $0 < t < t_0$, and let $B_\rho(x_0) \subset \mathbb{R}^N$ be any ball with radius $R(t) \leq \rho \leq 1$, centered at an arbitrarily fixed $x_0 \in \mathbb{R}^N$. Let $\zeta = \zeta(x)$ be a smooth cutoff function in B_ρ such that

$$\zeta(x) \equiv 1 \quad \text{in } B_{\frac{\rho}{2}}, \quad |D\zeta| \leq \frac{\gamma}{\rho}.$$

After straightforward calculations, we find

$$\int_{B_{\frac{\rho}{2}}} u(x,t)\mathrm{d}x \leq \int_{B_\rho} \mathrm{d}\mu + \frac{\gamma}{\rho}\int_0^t\int_{B_\rho} |Du|^{p-1}\mathrm{d}x\mathrm{d}\tau + \int_0^t\int_{B_\rho} u^q\mathrm{d}x\mathrm{d}\tau.$$

Multiplying both sides of above inequality by $\rho^\theta |B_\rho|^{-1}$ and using Lemma 5.1 and Lemma 5.2, we find

$$\rho^\theta \fint_{B_{\frac{\rho}{2}}} u(x,t)\mathrm{d}x \leq 2^N[\mu] + \gamma\langle u\rangle_t\{A(C, \langle u\rangle_t) + B(C, t, \langle u\rangle_t)\} \tag{5.16}$$

for all $0 < t < t_0$, $R(t) \leq \rho \leq 1$. Here

$$A(C, \langle u\rangle_t) = C^{-\frac{p+\theta(p-2)}{\kappa}}\Big\{\frac{\langle u\rangle_t}{[\mu]}\Big\}^{\frac{p-2}{\kappa}},$$

$$B(C, t, \langle u\rangle_t) = \Gamma^{\frac{p}{\kappa}(N-\theta)(q-1)}\langle u\rangle_t^{\frac{p(q-1)}{\kappa}} t^{1-\frac{\theta(q-1)}{p+\theta(p-2)}},$$

where C is the constant in (2.23).

(5.16) implies

$$\langle u\rangle_t \leq \gamma_1[\mu] + \gamma_2\langle u\rangle_t\{A(C, \langle u\rangle_t) + B(C, t, \langle u\rangle_t)\}, \tag{5.17}$$

where γ_2 depends on the same quantities determining the constants γ in Lemmas 5.1 and 5.2, but it does not depend on C. Hence similarly to the argument in Lemma 2.3, we can prove Lemma 5.3.

Similar to the proof of Theorem 1.1, we can prove Theorem 1.3.

6 Proof of Theorem 1.4

We may assume $\theta > 0$, $q > p - 1$ and $x_0 = 0$. From now on, all the balls are understood to be centered at 0.

Let $\xi \in C_0^1(B_\rho(0))$ be a cutoff function in B_ρ, $\xi = 1$ in $B_{\frac{\rho}{2}}$, $|D\xi| \leq \frac{\gamma}{\rho}$. We choose $\phi = u^{s-1}\xi^b$ in the definition of the solution of (1.4), where $s \in (0, 1)$ and b is large enough, to obtain, $\forall\, 0 < t_0 < t < T$,

$$\begin{aligned}\frac{1}{s}\int_{B_\rho} u^s(x,t)\xi^b\mathrm{d}x \geq{}& \frac{1}{s}\int_{B_\rho} u^s(x,t_0)\xi^b\mathrm{d}x - \frac{\gamma}{\rho^p}\int_{t_0}^t\int_{B_\rho} \xi^{b-p}u^{p+s-2}\mathrm{d}x\mathrm{d}\tau \\ &+ \int_{t_0}^t\int_{B_\rho} \xi^b u^{q+s-1}\mathrm{d}x\mathrm{d}\tau.\end{aligned} \tag{6.1}$$

Using Young inequality, we get

$$\begin{aligned}&\int_{B_\rho} u^s(x,z)\xi^b \mathrm{d}x - \int_{B_\rho} u^s\Big(x,\frac{t}{2}\Big)\xi^b \mathrm{d}x\\ &\geq -\gamma_1\rho^{N-\frac{p(q+s-1)}{q-p+1}}\Big(z-\frac{t}{2}\Big) + \gamma_2\int_{\frac{t}{2}}^{z}\int_{B_\rho} u^{q+s-1}\xi^b \mathrm{d}x\mathrm{d}\tau \quad \text{for } \frac{t}{2}<z<t. \end{aligned}\tag{6.2}$$

Note that $\int_{B_\rho} u^s(x,z)\xi^b \mathrm{d}x$ is a supersolution of the following problem

$$\begin{aligned}&y' = \gamma_2\rho^{-\frac{N(q-1)}{s}} y^{\frac{q+s-1}{s}} - \gamma_1\rho^{N-\frac{p(q+s-1)}{q-p+1}}, \quad \text{in } \Big(\frac{t}{2},t\Big),\\ &y\Big(\frac{t}{2}\Big) = \int_{B_\rho} u^s\Big(x,\frac{t}{2}\Big)\xi^b \mathrm{d}x.\end{aligned}$$

Hence it follows (see [9, Lemma 4.1]) that

$$\int_{B_\rho} u^s\Big(x,\frac{t}{2}\Big)\xi^b \mathrm{d}x \leq \gamma_2 \max\{\rho^N t^{-\frac{s}{q-1}}, \rho^{N-\frac{ps}{q-p+1}}\}. \tag{6.3}$$

On the other hand, by (4.7), we have

$$\int_{B_\rho} u^s\Big(x,\frac{t}{2}\Big)\xi^p \mathrm{d}x \geq \gamma_0\rho^{-s\theta+N} \tag{6.4}$$

for small t and $\rho=\rho(t)$ defined in (4.7). Let $t\to 0$. (6.3) and (6.4) imply

$$\theta \leq \min\Big\{\frac{p+\theta(p-2)}{q-1}, \frac{p}{q-(p-1)}\Big\}.$$

This implies

$$\theta(q-p+1)\leq p.$$

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