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String Equations of the *q*-KP Hierarchy*

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Abstract Based on the Lax operator L and Orlov-Shulman's M operator, the string equations of the q-KP hierarchy are established from the special additional symmetry flows, and the negative Virasoro constraint generators $\{L_{-n}, n \geq 1\}$ of the 2-reduced q-KP hierarchy are also obtained.

Keywords q-KP hierarchy, Additional symmetry, String equations, Virasoro constraints
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1 Introduction

The q-deformed integrable system (also called the q-analogue or q-deformation of classical integrable system) is defined by means of q-derivative ∂_q (see [1–2]) instead of usual derivative ∂ with respect to x in a classical system. It reduces to a classical integrable system as $q \to 1$. Recently, the q-deformed Kadomtsev-Petviashvili (q-KP) hierarchy is a subject of intensive study in the literature from [3] to [14]. Its infinite conservation laws, bi-Hamiltonian structure, τ function, additional symmetries and its constrained sub-hierarchy have already been reported in [4–5, 11–12, 14].

The additional symmetries, string equations and Virasoro constraints of the KP hierarchy are important as they are involved in the matrix models of the string theory (see [15]). For example, there are several new works [16–20] on this topic. The additional symmetries were discovered independently at least twice by Sato School [21] and Orlov-Shulman [22], in quite different environments and philosophy although they are essentially equivalent. It is well-known that L. A. Dickey [23] presented a very elegant and compact proof of Adler-Shiota-van Moerbeke (ASvM) formula (see [24–25]) based on the Lax operator L and Orlov-Shulman's M operator (see [22]), and gave the string equation and the action of the additional symmetries on the τ function of the classical KP hierarchy. S. Panda and S. Roy gave the Virasoro and W-constraints on the τ function of the p-reduced KP hierarchy by expanding the additional symmetry operator in terms of the Lax operator (see [26–27]). It is quite interesting to study the analogous properties of q-deformed KP hierarchy by this expanding method. The main purpose of this article is to give the string equations of the q-KP hierarchy, and then study the negative Virasoro constraint generators $\{L_{-n}, n \geq 1\}$ of 2-reduced q-KP hierarchy.

The organization of this paper is as follows. We recall some basic results and additional symmetries of the q-KP hierarchy in Section 2. The string equations are given in Section 3.

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The Virasoro constraints on the τ function of the 2-reduced (q-KdV) hierarchy are studied in Section 4. Section 5 is devoted to the conclusions and discussions.

At the end of the this section, we shall collect some useful facts of q-calculus (see [2]) to make this paper self-contained. The q-derivative ∂_q is defined by

$$\partial_q(f(x)) = \frac{f(qx) - f(x)}{x(q-1)}, \tag{1.1}$$

and the q-shift operator is

$$\theta(f(x)) = f(qx). \tag{1.2}$$

$\partial_q(f(x))$ recovers the ordinary differentiation $\partial_x(f(x))$ as q goes to 1. Let ∂_q^{-1} denote the formal inverse of ∂_q. In general, the following q-deformed Leibniz rule holds:

$$\partial_q^n \circ f = \sum_{k\geq 0} \binom{n}{k}_q \theta^{n-k}(\partial_q^k f)\partial_q^{n-k}, \quad n \in \mathbb{Z}, \tag{1.3}$$

where the q-number and the q-binomial are defined by

$$(n)_q = \frac{q^n - 1}{q - 1},$$
$$\binom{n}{k}_q = \frac{(n)_q(n-1)_q \cdots (n-k+1)_q}{(1)_q(2)_q \cdots (k)_q}, \qquad \binom{n}{0}_q = 1.$$

For a q-pseudo-differential operator (q-PDO) of the form $P = \sum\limits_{i=-\infty}^{n} p_i \partial_q^i$, we separate P into the differential part $P_+ = \sum\limits_{i\geq 0} p_i \partial_q^i$ and the integral part $P_- = \sum\limits_{i\leq -1} p_i \partial_q^i$. The conjugate operation "$*$" for P is defined by $P^* = \sum\limits_i (\partial_q^*)^i p_i$ with $\partial_q^* = -\partial_q \theta^{-1} = -\frac{1}{q}\partial_{\frac{1}{q}}$, $(\partial_q^{-1})^* = (\partial_q^*)^{-1} = -\theta\partial_q^{-1}$.

The q-exponent e_q^x is defined as follows:

$$\mathrm{e}_q^x = \sum_{n=0}^{\infty} \frac{x^n}{(n)_q!}, \quad (n)_q! = (n)_q(n-1)_q(n-2)_q \cdots (1)_q.$$

Its equivalent expression is of the form

$$\mathrm{e}_q^x = \exp\Big(\sum_{k=1}^{\infty} \frac{(1-q)^k}{k(1-q^k)} x^k\Big), \tag{1.4}$$

which is crucial to developing the τ function of the q-KP hierarchy (see [11]).

2 *q*-KP Hierarchy and Its Additional Symmetries

Similar to the general way of describing the classical KP hierarchy (see [21, 28]), we first give a brief introduction to the q-KP hierarchy and its additional symmetries based on [11–12].

Let L be one q-PDO given by

$$L = \partial_q + u_0 + u_{-1}\partial_q^{-1} + u_{-2}\partial_q^{-2} + \cdots, \tag{2.1}$$

which is called the Lax operator of q-KP hierarchy. There exist infinite numbers of q-partial differential equations related to dynamical variables $\{u_i(x, t_1, t_2, t_3, \cdots), i = 0, -1, -2, -3, \cdots\}$ and can be deduced from the generalized Lax equation

$$\frac{\partial L}{\partial t_n} = [B_n, L], \quad n = 1, 2, 3, \cdots, \tag{2.2}$$

which are called the q-KP hierarchy. Here $B_n = (L^n)_+ = \sum_{i=0}^{n} b_i \partial_q^i$ and $L^n_- = L^n - L^n_+$. L in (2.1) can be generated by dressing operator $S = 1 + \sum_{k=1}^{\infty} s_k \partial_q^{-k}$ in the following way:

$$L = S \circ \partial_q \circ S^{-1}. \tag{2.3}$$

Dressing operator S satisfies Sato equation

$$\frac{\partial S}{\partial t_n} = -(L^n)_- S, \quad n = 1, 2, 3, \cdots. \tag{2.4}$$

The q-wave function $w_q(x, t; z)$ and the q-adjoint function $w_q^*(x, t; z)$ are given by

$$w_q = S \mathrm{e}_q^{xz} \exp\Big(\sum_{i=1}^{\infty} t_i z^i\Big),$$

$$w_q^*(x, t; z) = (S^*)^{-1}|_{\frac{x}{q}} \mathrm{e}_{\frac{1}{q}}^{-xz} \exp\Big(-\sum_{i=1}^{\infty} t_i z^i\Big),$$

which satisfy the following linear q-differential equations:

$$L w_q = z w_q, \quad L^*|_{\frac{x}{q}} w_q^* = z w_q^*.$$

Here the notation $P|_{\frac{x}{t}} = \sum_i P_i(\frac{x}{t}) t^i \partial_q^i$ is used for $P = \sum_i p_i(x) \partial_q^i$.

Furthermore, $w_q(x, t; z)$ and $w_q^*(x, t; z)$ can be expressed by the sole function $\tau_q(x; t)$ (see [11]) as

$$\begin{aligned} w_q &= \frac{\tau_q(x; t - [z^{-1}])}{\tau_q(x; \overline{t})} \mathrm{e}_q^{xz} \exp\Big(\sum_{i=1}^{\infty} t_i z^i\Big) = \frac{\mathrm{e}_q^{xz} \mathrm{e}^{\xi(t,z)} \mathrm{e}^{-\sum_{i=1}^{\infty} \frac{z^{-i}}{i} \partial_i} \tau_q}{\tau_q}, \\ w_q^* &= \frac{\tau_q(x; t + [z^{-1}])}{\tau_q(x; t)} \mathrm{e}_{\frac{1}{q}}^{-xz} \exp\Big(-\sum_{i=1}^{\infty} t_i z^i\Big) = \frac{\mathrm{e}_{\frac{1}{q}}^{-xz} \mathrm{e}^{-\xi(t,z)} \mathrm{e}^{+\sum_{i=1}^{\infty} \frac{z^{-i}}{i} \partial_i} \tau_q}{\tau_q}, \end{aligned} \tag{2.5}$$

where

$$[z] = \Big(z, \frac{z^2}{2}, \frac{z^3}{3}, \cdots\Big).$$

The following lemma shows that there exists an essential correspondence between the q-KP hierarchy and the KP hierarchy.

Lemma 2.1 (see [11]) *Let $L_1 = \partial + u_{-1}\partial^{-1} + u_{-2}\partial^{-2} + \cdots$, where $\partial = \frac{\partial}{\partial x}$, be a solution to the classical KP hierarchy and τ be its τ function. Then*

$$\tau_q(x, t) = \tau(t + [x]_q)$$

is a τ function of the q-KP hierarchy associated with Lax operator L in (2.1)*, where*

$$[x]_q = \Big(x, \frac{(1-q)^2}{2(1-q^2)} x^2, \frac{(1-q)^3}{3(1-q^3)} x^3, \cdots, \frac{(1-q)^i}{i(1-q^i)} x^i, \cdots\Big).$$

Define Γ_q and Orlov-Shulman's M operator

$$\Gamma_q = \sum_{i=1}^{\infty}\Big(it_i + \frac{(1-q)^i}{(1-q^i)}x^i\Big)\partial_q^{i-1}, \tag{2.6}$$

$$M = S\Gamma_q S^{-1}. \tag{2.7}$$

Dressing $[\partial_k - \partial_q^k, \Gamma_q] = 0$ gives

$$\partial_k M = [B_k, M]. \tag{2.8}$$

(2.2) together with (2.8) implies that

$$\partial_k(M^m L^n) = [B_k, M^m L^n]. \tag{2.9}$$

Define the additional flows for each pair m, n as follows:

$$\frac{\partial S}{\partial t^*_{m,n}} = -(M^m L^n)_- S, \tag{2.10}$$

or equivalently

$$\frac{\partial L}{\partial t^*_{m,n}} = -[(M^m L^n)_-, L], \tag{2.11}$$

$$\frac{\partial M}{\partial t^*_{m,n}} = -[(M^m L^n)_-, M]. \tag{2.12}$$

The additional flows $\partial^*_{mn} = \frac{\partial}{\partial t^*_{m,n}}$ commute with the hierarchy, i.e., $[\partial^*_{mn}, \partial_k] = 0$ but do not commute with each other. So they are additional symmetries (see [12]). $(M^m L^n)_-$ serves as the generator of the additional symmetries along the trajectory parametrized by $t^*_{m,n}$.

3 String Equations of the q-KP Hierarchy

In this section, we shall get string equations for the q-KP hierarchy from special additional symmetry flows. For this, we need a lemma.

Lemma 3.1 *The following equation*

$$[M, L] = -1 \tag{3.1}$$

holds.

Proof Direct calculations show that

$$\begin{aligned}
[\Gamma_q, \partial_q] &= \Big[\sum_{i=1}^{\infty}\Big(it_i + \frac{(1-q)^i}{1-q^i}x^i\Big)\partial_q^{i-1}, \partial_q\Big] \\
&= \sum_{i=1}^{\infty}\Big[\frac{(1-q)^i}{1-q^i}x^i\partial_q^{i-1}, \partial_q\Big] \\
&= \sum_{i=1}^{\infty}\frac{(1-q)^i}{1-q^i}(x^i\partial_q^i - (\partial_q \circ x^i)\partial_q^{i-1}) \\
&= \sum_{i=1}^{\infty}\frac{(1-q)^i}{1-q^i}(x^i\partial_q^i - ((\partial_q x^i) + q^i x^i \partial_q)\partial_q^{i-1})
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{\infty} \frac{(1-q)^i}{1-q^i} \Big((1-q^i) x^i \partial_q^i - \frac{1-q^i}{1-q} x^{i-1} \partial_q^{i-1} \Big) \\
&= \sum_{i=1}^{\infty} ((1-q)^i x^i \partial_q^i - (1-q)^{i-1} x^{i-1} \partial_q^{i-1}) \\
&= -1,
\end{aligned}
$$

where we have used $[t_i, \partial_q] = 0$ in the second step and $\partial_q \circ x^i = (\partial_q x^i) + q^i x^i \partial_q$ in the fourth step. Then

$$[M, L] = [S\Gamma_q S^{-1}, S\partial_q S^{-1}] = S[\Gamma_q, \partial_q]S^{-1} = -1.$$

By virtue of Lemma 3.1, we have the following corollary.

Corollary 3.1 $[M, L] = -1$ *implies* $[M, L^n] = -nL^{n-1}$. *Therefore,*

$$[ML^{-n+1}, L^n] = -n. \tag{3.2}$$

The action of additional flows $\partial^*_{1,-n+1}$ on L^n is $\partial^*_{1,-n+1} L^n = -[(ML^{-n+1})_-, L^n]$, which can be written as

$$\partial^*_{1,-n+1} L^n = [(ML^{-n+1})_+, L^n] + n. \tag{3.3}$$

The following theorem holds by virtue of (3.3).

Theorem 3.1 *If an operator L does not depend on the parameters t_n and the additional variables $t^*_{1,-n+1}$, then L^n is a purely differential operator, and the string equations of the q-KP hierarchy are given by*

$$\Big[L^n, \frac{1}{n}(ML^{-n+1})_+\Big] = 1, \quad n = 2, 3, 4, \cdots. \tag{3.4}$$

In view of the additional symmetries and string equations, we can get the following corollary, which plays a crucial role in the study of the constraints on the τ function of the p-reduced q-KP hierarchy.

Corollary 3.2 *If L^n is a differential operator and $\partial^*_{1,-n+1} S = 0$, then*

$$(ML^{-n+1})_- = \frac{n-1}{2} L^{-n}, \quad n = 2, 3, 4, \cdots. \tag{3.5}$$

Proof Since $[M, L] = -1$, it is not difficult to obtain

$$[M, L^{-n+1}] = (n-1)L^{-n}.$$

Hence

$$(ML^{-n+1})_- - (L^{-n+1}M)_- = (n-1)L^{-n}. \tag{3.6}$$

Noticing $[(n-1)L^{-n}, L^n] = 0$, we have

$$[(ML^{-n+1})_- - (L^{-n+1}M)_-, L^n] = 0, \quad \text{i.e.,} \quad [(ML^{-n+1})_-, L^n] = [(L^{-n+1}M)_-, L^n].$$

Thus

$$\partial^*_{1,-n+1} L^n = -[(L^{-n+1}M)_-, L^n] = -\frac{1}{2}[(ML^{-n+1})_- + (L^{-n+1}M)_-, L^n],$$

or equivalently

$$\partial^*_{1,-n+1}S = -\frac{1}{2}(ML^{-n+1} + L^{-n+1}M)_- S.$$

Therefore, it follows from the equation $\partial^*_{1,-n+1}S = 0$ that

$$(ML^{-n+1} + L^{-n+1}M)_- = 0.$$

Combining this with (3.6) finishes the proof.

4 Constraints on the τ Function of the q-KdV Hierarchy

In this section, we mainly study the associated constraints on τ function of the 2-reduced q-KP (q-KdV) hierarchy from string equations (3.4). To this end, we first define residue $\operatorname{res} L = u_{-1}$ of L given by (2.1) and state two very useful lemmas.

Lemma 4.1 *For $n = 1, 2, 3, \cdots$,*

$$\operatorname{res} L^n = \frac{\partial^2 \log \tau_q}{\partial t_1 \partial t_n}, \tag{4.1}$$

where τ_q is the τ function of the q-KP hierarchy.

Proof Taking the residue of $\frac{\partial S}{\partial t_n} = -(L^n)_- S$, we get

$$\frac{\partial s_1}{\partial t_n} = -\operatorname{res}((L^n)_-(1 + s_1\partial_q^{-1} + s_2\partial_q^{-2} + \cdots)) = -\operatorname{res}(L^n)_- = -\operatorname{res} L^n.$$

Noting that $u_0 = s_1 - \theta(s_1) = -x(q-1)\partial_q s_1 = x(q-1)\partial_q\partial_{t_1}\log\tau_q,\ s_1 = -\frac{\partial \log \tau_q}{\partial t_1}$ (see [14]), we have

$$\operatorname{res} L^n = -\frac{\partial s_1}{\partial t_n} = \frac{\partial^2 \log \tau_q}{\partial t_1 \partial t_n}.$$

Lemma 4.2 *Orlov-Shulman's M operator has the expansion of the form*

$$M = \sum_{i=1}^{\infty}\Big(it_i + \frac{(1-q)^i}{1-q^i}x^i\Big)L^{i-1} + \sum_{i=1}^{\infty} V_{i+1}L^{-i-1}, \tag{4.2}$$

where

$$V_{i+1} = -i \sum_{a_1+2a_2+3a_3+\cdots=i} (-1)^{a_1+a_2+\cdots}\frac{(\partial t_1)^{a_1}}{a_1!}\frac{(\frac{1}{2}\partial t_2)^{a_2}}{a_2!}\frac{(\frac{1}{3}\partial t_3)^{a_3}}{a_3!}\cdots \log\tau_q.$$

Proof First, we assert $Mw_q = \frac{\partial w_q}{\partial z}$. Indeed, from the identity $\partial_q^{i-1}\mathrm{e}_q^{xz} = z^{i-1}\mathrm{e}_q^{xz}$, we have

$$Mw_q = S\Gamma_q S^{-1}S\mathrm{e}_q^{xz}\mathrm{e}^{\xi(t,z)} = S\Big(\sum_{i=1}^{\infty}\Big(it_i + \frac{(1-q)^i}{1-q^i}x^i\Big)z^{i-1}\Big)\mathrm{e}_q^{xz}\mathrm{e}^{\xi(t,z)},$$

where $\xi(t,z) = \sum\limits_{i=1}^{\infty} t_i z^i$. On the other hand,

$$\begin{aligned}\frac{\partial w_q}{\partial z} &= \frac{\partial(S\mathrm{e}_q^{xz}\mathrm{e}^{\xi(t,z)})}{\partial z} = S\Big(\frac{\partial \mathrm{e}_q^{xz}}{\partial z}\mathrm{e}^{\xi(t,z)} + \mathrm{e}_q^{xz}\frac{\partial \mathrm{e}^{\xi(t,z)}}{\partial z}\Big)\\ &= S\Big(\sum_{i=1}^{\infty}\Big(it_i + \frac{(1-q)^i}{1-q^i}x^i\Big)z^{i-1}\Big)\mathrm{e}_q^{xz}\mathrm{e}^{\xi(t,z)}.\end{aligned}$$

Thus the assertion is verified. Next, by a direct calculation from (1.4) and (2.5), we have

$$\log w_q = \sum_{k=1}^{\infty} \frac{(1-q)^k}{k(1-q^k)}(xz)^k + \sum_{n=1}^{\infty} t_n z^n + \sum_{N=0}^{\infty} \frac{1}{N!}\Big(-\sum_{i=1}^{\infty} \frac{z^{-i}}{i}\partial_i\Big)^N \log \tau_q - \log \tau_q. \tag{4.3}$$

Let $M = \sum\limits_{n=1}^{\infty} a_n L^{n-1} + \sum\limits_{n=1}^{\infty} b_n L^{-n}$. Then in light of $Lw_q = zw_q$ and the assertion mentioned above, we obtain

$$\frac{\partial w_q}{\partial z} = Mw_q = \Big(\sum_{n=1}^{\infty} a_n L^{n-1} + \sum_{n=1}^{\infty} b_n L^{-n}\Big) w_q,$$

and hence

$$\frac{\partial \log w_q}{\partial z} = \frac{1}{w_q}\frac{\partial w_q}{\partial z} = \sum_{n=1}^{\infty} a_n z^{n-1} + \sum_{n=1}^{\infty} b_n z^{-n}. \tag{4.4}$$

Thus by comparing the coefficients of z in $\frac{\partial \log w_q}{\partial z}$ given by (4.3) and (4.4), a_i and b_i are determined such that M is obtained as (4.2).

To be an intuitive glance, the first few V_{i+1} are given as follows:

$$\begin{aligned}
V_2 &= \frac{\partial \log \tau_q}{\partial t_1}, \\
V_3 &= \frac{\partial \log \tau_q}{\partial t_2} - \frac{\partial^2 \log \tau_q}{\partial t_1^2}, \\
V_4 &= \Big(\frac{1}{2}\frac{\partial^3}{\partial t_1^3} - \frac{3}{2}\frac{\partial^2}{\partial t_1 \partial t_2} + \frac{\partial}{\partial t_3}\Big) \log \tau_q, \\
V_5 &= \Big(-\frac{1}{3!}\frac{\partial^4}{\partial t_1^4} - \frac{1}{2}\frac{\partial^2}{\partial t_2^2} - \frac{4}{3}\frac{\partial^2}{\partial t_1 \partial t_3} + \frac{\partial}{\partial t_4}\Big) \log \tau_q, \\
V_6 &= \Big(\frac{1}{4!}\frac{\partial^5}{\partial t_1^5} - \frac{5}{12}\frac{\partial^4}{\partial t_1^3 \partial t_3} + \frac{5}{6}\frac{\partial^3}{\partial t_1^2 \partial t_3} - \frac{5}{4}\frac{\partial^2}{\partial t_1 \partial t_4} - \frac{5}{6}\frac{\partial^2}{\partial t_2 \partial t_3} + \frac{\partial}{\partial t_5}\Big) \log \tau_q.
\end{aligned}$$

Now we consider the 2-reduced q-KP hierarchy (q-KdV hierarchy), by setting $L_-^2 = 0$ or setting

$$L^2 = \partial_q^2 + (q-1)xu\partial_q + u. \tag{4.5}$$

To make the following theorem be a compact form, we introduce

$$L_{-n} = \frac{1}{2} \sum_{\substack{i=2n+1 \\ i \neq 0 \pmod 2}}^{\infty} i\widetilde{t}_i \frac{\partial}{\partial \widetilde{t}_{i-2n}} + \frac{1}{4}\sum_{k+l=n+1} (2k-1)(2l-1)\widetilde{t}_{2k-1}\widetilde{t}_{2k-1} \tag{4.6}$$

and

$$\widetilde{t}_i = t_i + \frac{(1-q)^i}{i(1-q^i)}x^i, \quad i = 1, 2, 3, \cdots. \tag{4.7}$$

Theorem 4.1 *If L^2 satisfies* (3.4), *the Virasoro constraints imposed on the τ function of the q-KdV hierarchy are*

$$L_{-n}\tau_q = 0, \quad n = 1, 2, 3, \cdots, \tag{4.8}$$

and the Virasoro commutation relations

$$[L_{-n}, L_{-m}] = (-n+m)L_{-(n+m)}, \quad m,n = 1,2,3,\cdots \tag{4.9}$$

hold.

Proof For $n = 1,2,3,\cdots$, we have

$$\operatorname{res}(ML^{-2n+1}) = \operatorname{res}(ML^{-2n+1})_- = \operatorname{res}\Big(-\frac{2n+1}{2}L^{-2n}\Big)_- = 0 \tag{4.10}$$

with the help of (3.5). Substituting the expansion of M in (4.2) into (4.10), we have

$$\sum_{i=1}^{\infty}\Big(it_i + \frac{(1-q)^i}{1-q^i}x^i\Big)\operatorname{res} L^{i-2n} + \sum_{i=1}^{\infty}\operatorname{res}(V_{i+1}L^{-i-2n}) = 0,$$

which implies

$$\sum_{\substack{i=2n+1\\ i\neq 0 \pmod 2}}^{\infty}\Big(it_i + \frac{(1-q)^i}{1-q^i}x^i\Big)\operatorname{res} L^{i-2n} + (2n-1)t_{2n-1} + \frac{(1-q)^{2n-1}}{1-q^{2n-1}}x^{2n-1} = 0. \tag{4.11}$$

Substituting $\operatorname{res} L^{i-2n} = \frac{\partial^2 \log\tau_q}{\partial t_1 \partial t_{i-2n}}$ into (4.11), then performing an integration with respect to t_1 and multiplying by $\frac{\tau_q}{2}$, it becomes

$$\widetilde{L}_{-n}\tau_q = 0, \quad n = 1,2,3,\cdots,$$

where

$$\begin{aligned}\widetilde{L}_{-n} = {} & \frac{1}{2}\sum_{\substack{i=2n+1\\ i\neq 0 \pmod 2}}^{\infty}\Big(it_i + \frac{(1-q)^i}{1-q^i}x^i\Big)\frac{\partial}{\partial t_{i-2n}} + \frac{(1-q)^{2n-1}}{1-q^{2n-1}}\cdot\frac{1}{2}t_1x^{2n-1} \\ & + \frac{1}{2}(2n-1)t_1t_{2n-1} + C(t_2,t_3,\cdots;x).\end{aligned} \tag{4.12}$$

The integration constant $C(t_2,t_3,\cdots;x)$ with respect to t_1 could be the arbitrary function with the parameters $(t_2,t_3,\cdots;x)$. What we shall do is to determine $C(t_2,t_3,\cdots;x)$ such that $\widetilde{L}_{-n}$ satisfy Virasoro commutation relations.

Let

$$\widetilde{t}_i = t_i + \frac{(1-q)^i}{i(1-q^i)}x^i, \quad i = 1,2,3,\cdots,$$

and choose $C(t_2,t_3,\cdots;x)$ as

$$\begin{aligned}C(t_2,t_3,\cdots;x) = {} & -\frac{1}{4}\sum_{k=3}^{2n-3}(2k-1)(2n-2k+1)\Big(t_{2k-1} + \frac{(1-q)^{2k-1}}{(2k-1)(1-q^{2k-1})}x^{2k-1}\Big) \\ & \cdot\Big(t_{2n-2k+1} + \frac{(1-q)^{2n-2k+1}}{(2n-2k+1)(1-q^{2n-2k+1})}x^{2n-2k+1}\Big) \\ & - \frac{1}{2}(2n-1)x\Big(t_{2n-1} + \frac{(1-q)^{2n-1}}{(2n-1)(1-q^{2n-1})}x^{2n-1}\Big).\end{aligned}$$

Then

$$\widetilde{L}_{-n} = \frac{1}{2}\sum_{\substack{i=2n+1\\ i\neq 0 \pmod 2}}^{\infty} i\widetilde{t}_i\frac{\partial}{\partial\widetilde{t}_{i-2n}} + \frac{1}{4}\sum_{k+l=n+1}(2k-1)(2l-1)\widetilde{t}_{2k-1}\widetilde{t}_{2k-1} \equiv L_{-n}$$

and

$$L_{-n}\tau_q = 0, \quad n = 1, 2, 3, \cdots$$

as we expected. By a straightforward and tedious calculation, the Virasoro commutation relations

$$[L_{-n}, L_{-m}] = (-n+m)L_{-(n+m)}, \quad m, n = 1, 2, 3, \cdots$$

can be verified.

Remark 4.1 As we know, the q-deformed KP hierarchy reduces to the classical KP hierarchy when $q \to 1$ and $u_0 = 0$. The parameters $(\widetilde{t}_1, \widetilde{t}_2, \cdots, \widetilde{t}_i, \cdots)$ in (4.6) tend to $(t_1 + x, t_2, \cdots, t_i, \cdots)$ as $q \to 1$. One can further identify $t_1 + x$ with x in the classical KP hierarchy, i.e., $t_1 + x \to x$, and therefore the Virasoro generators L_{-n} in (4.6) of the 2-reduced q-KP hierarchy tend to

$$\widehat{L}_{-n} = \frac{1}{2} \sum_{\substack{i=2n+1 \\ i \neq 0 \ (\mathrm{mod}\, 2)}}^{\infty} it_i \frac{\partial}{\partial t_{i-2n}} + \frac{1}{4} \sum_{k+l=n+1} (2k-1)(2l-1)t_{2k-1}t_{2k-1}, \quad n = 2, 3, \cdots \tag{4.13}$$

and

$$\widehat{L}_{-1} = \frac{1}{2} \sum_{\substack{i=3 \\ i \neq 0 \ (\mathrm{mod}\, 2)}}^{\infty} it_i \frac{\partial}{\partial t_{i-2}} + \frac{1}{4}x^2, \tag{4.14}$$

which are identical with the results of the classical KP hierarchy given by L. A. Dickey [29] and S. Panda, S. Roy [26].

5 Conclusions and Discussions

To summarize, we have derived the string equations in (3.4) and the negative Virasoro constraint generators on the τ function of 2-reduced q-KP hierarchy in (4.8) in Theorem 4.1. The results of this paper show obviously that the Virasoro generators $\{L_{-n}, n \geq 1\}$ of the q-KP hierarchy are different from the $\{\widehat{L}_{-n}, n \geq 1\}$ of the KP hierarchy, although they satisfy the common Virasoro commutation relations. Furthermore, one can find the following interesting relation between the q-KP hierarchy and the KP hierarchy

$$L_{-n} = \widehat{L}_{-n}\big|_{t_i \to \widetilde{t}_i = t_i + \frac{(1-q)^i}{i(1-q^i)}x^i},$$

and it seems to demonstrate that q-deformation is a non-uniform transformation for coordinates $t_i \to \widetilde{t}_i$, which is consistent with the results on τ function (see [11]) and the q-soliton (see [14]) of the q-KP hierarchy.

For the p-reduced ($p \geq 3$) q-KP hierarchy, which is the q-KP hierarchy satisfying the reduction condition $(L^p)_- = 0$, we can obtain $(ML^{pn+1})_- = 0$. Using the similar technique in q-KdV hierarchy, we can deduce the Virasoro constraints on the τ function of the p-reduced q-KP hierarchy for $p \geq 3$. Moreover, for $\{L_n, n \geq 0\}$ we find a subtle point at the calculation of $\mathrm{res}\,(V_{i+1}L^{-i+2n})$, and shall try to study it in the future.

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