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Almost Global Strong Solutions to Quasilinear Dissipative Evolution Equations**

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Abstract The author proves a global existence result for strong solutions to the quasilinear dissipative hyperbolic equation (1.1) below, corresponding to initial values and source terms of arbitrary size, provided that the hyperbolicity parameter ε is sufficiently small. This implies a corresponding global existence result for the reduced quasilinear parabolic equation (1.4) below.

Keywords Quasilinear evolution equation, A priori estimates, Global existence, Small parameter
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1 Introduction

1.1 In this paper, we consider the Cauchy problem for the quasilinear dissipative hyperbolic evolution equation

$$\varepsilon u_{tt} + u_t - a_{ij}(\nabla u)\partial_i\partial_j u = f, \tag{1.1}$$

with $\varepsilon \in]0,1]$, $f = f(t,x)$, $t \geq 0$, $x \in \mathbb{R}^N$, and u subject to the initial conditions

$$u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x). \tag{1.2}$$

In (1.1), as well as in the sequel, summation for i, j from 1 to N is understood. Our goal is to prove an almost global existence result for strong solutions of (1.1), corresponding to initial data u_0, u_1, and source term f, of arbitrary size, at least if the hyperbolicity parameter ε is sufficiently small. More precisely, given $T > 0$ and a set of data $\{u_0, u_1, f\}$, we show that there is $\varepsilon_0 \in]0,1]$ such that if $\varepsilon \in]0,\varepsilon_0]$, then problem (1.1)–(1.2) admits a solution u, defined in all of $[0,T]$. The spaces in which the data are taken, and the solution is found, are specified, respectively, in (2.14), (2.15), and (2.16) below.

The choice of ε_0 depends, in general, not only on the data $\{f, u_0, u_1\}$, but also on T. As far as we can show, the latter dependence is explicit, in the sense that we can define a function $T \mapsto \varepsilon_0(T)$, but for this function,

$$\liminf_{T\to+\infty} \varepsilon_0(T) = 0. \tag{1.3}$$

Thus, at the moment, the question of the asymptotic behavior of u is open; this is why we call our result an "almost global" existence one. Nevertheless, we would like to emphasize the

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fact that we do not put any restriction on the size of the data (except that, if the data depend on ε, they should be bounded as $\varepsilon \to 0$). Indeed, if the data are sufficiently small, a direct application of Matsumura's technique of [7] shows that problem (1.1) is globally solvable; on the other hand, if the data are large, blow up of solutions of nonlinear hyperbolic problems in finite time may in general be expected.

1.2 The major interest of this result resides in our previous result of [9], which related the global solvability of (1.1), for data $\{u_0, u_1, f\}$ of arbitrary size, to that of the corresponding parabolic quasilinear equation

$$u_t - a_{ij}(\nabla u)\partial_i\partial_j u = g, \tag{1.4}$$

with initial condition

$$u(0, \cdot\,) = v_0, \tag{1.5}$$

for data $\{v_0, g\}$ also of arbitrary size. More precisely, in [9] we were able to show that these two problems are equivalent, in the sense that equation (1.4) is globally solvable for data $\{v_0, g\}$ of arbitrary size, if and only if equation (1.1) is globally solvable for data $\{u_0, u_1, f\}$ of arbitrary size, and ε is sufficiently small. However, we were not able to provide a global solvability result for either problem. In contrast, for the corresponding initial-boundary value problems in a bounded domain of $\mathbb{R}^N$, with homogeneous Dirichlet boundary conditions, we were able not only to prove the analogous equivalence result (see [10]), but also, independently, the global solvability of (1.4) (see [10]). The result presented here, together with the equivalence result of [9], allows us therefore to show that the parabolic initial value problem is also globally solvable, for data $\{v_0, g\}$ of arbitrary size; in fact, the solution of (1.4) can be obtained by a singular convergence process as $\varepsilon \to 0$, taking the same first initial value $u_0 = v_0$, an arbitrary u_1 independent of ε, and for f a suitable regularization of g. On the other hand, while in this paper we focus on the hyperbolic equation (1.1), it will be clear that the proof we give carries over in a straightforward manner to the parabolic equation (1.4). Thus, it is possible to present a unified approach to the question of (almost) global existence for either problem.

1.3 This paper is organized as follows. In Section 2 we recall a local existence result for solutions to (1.1), in a nested family $(\mathcal{X}_m)_{m\geq 0}$ of function spaces, the case $m = 0$ corresponding to that of so-called "minimal" regularity, and state our global existence result. In Section 3 we report a basic energy estimate on the solutions of (1.1), together with a Schauder estimate for classical solutions of the parabolic equation (1.4), which we prove in Section 4. In Section 5 we consider more regular solutions (i.e., in $\mathcal{X}_m$, $m \geq 4$) and, writing (1.1) in the form

$$u_t - a_{ij}(\nabla u)\partial_i\partial_j u = f - \varepsilon u_{tt}, \tag{1.6}$$

use the Schauder estimate of Section 3 to deduce time-independent bounds for $\partial_i\partial_j u$, if ε is sufficiently small (so as to take advantage of the term εu_{tt} at the right side of (1.6)). This second set of estimates allows us to deduce that these more regular solutions of (1.1) can be extended to all of $[0, T]$. In Section 6 we resort to an approximation argument to deduce a corresponding almost global existence result for solutions of minimal regularity.

1.4 We conclude this introduction by mentioning that the possibility of resorting to classical estimates for the parabolic equation (1.4) highlights once more the essentially parabolic nature of the dissipative equation (1.1), when ε is small. Another aspect of this feature is described by the so-called diffusion phenomenon of hyperbolic waves, which can sometimes be proven for equations similar to (1.1) and (1.4). For an example concerning the study of this phenomenon for quasilinear equations in divergence form, we refer e.g. to [15]. Finally, we refer to [9] for a more detailed discussion of our main motivations and possible applications of our results.

2 Preliminaries

2.1 Notations and function spaces

2.1.1 We adopt the following notations throughout this paper. Bounded intervals of $\mathbb{R}$ are denoted by $[a,b]$ if closed, $]a,b[$ if open, $[a,b[$ or $]a,b]$ otherwise. For unbounded intervals, we occasionally adopt the notation $\mathbb{R}_{\geq a} := [a,+\infty[$, $\mathbb{R}_{>a} :=]a,+\infty[$, and similarly ($a \in \mathbb{R}$). In analogy, we abbreviate $\mathbb{N}_{\geq m} := \{n \in \mathbb{N} \mid n \geq m\}$, etc. If $(t,x) \mapsto u(t,x)$ is a smooth function, we denote its partial derivatives with respect to t by u_t, u_{tt}, etc., and with respect to the space variables by $\partial_j u$, $\partial_i\partial_j u$, etc. We also set $\nabla u := (\partial_1 u, \cdots, \partial_N u)$ and $Du := \{u_t, \nabla u\}$. More generally, given a multi-index $\alpha = (\alpha_1, \cdots, \alpha_N) \in \mathbb{N}^N$, we denote by $|\alpha| := \alpha_1 + \cdots + \alpha_N$ its length, and set $\partial^\alpha u := \partial_1^{\alpha_1} \cdots \partial_N^{\alpha_N} u$. Given a positive integer k, we denote by $\partial_x^k u$ and $\partial_t^k u$ the set of all derivatives of u of order k, with respect to the space or the time variables.

2.1.2 For $m \in \mathbb{N}$, we denote by $C_{\mathrm{b}}^m(\mathbb{R}^N)$ the Banach space of all m-times continuously differentiable functions on $\mathbb{R}^N$ which are bounded, together with all their derivatives of order up to m, with norm

$$|f|_{C_{\mathrm{b}}^m(\mathbb{R}^N)} := \max_{|\alpha|\leq m} \sup_{x\in\mathbb{R}^N} |\partial^\alpha u(x)|. \tag{2.1}$$

Given also $\alpha \in]0,1[$, we consider the Hölder spaces on $\mathbb{R}^N$

$$C^{(m,\alpha)}(\mathbb{R}^N) := \{f \in C_{\mathrm{b}}^m(\mathbb{R}^N) \mid H_\alpha(\partial_x^m f) < \infty\}, \tag{2.2}$$

where

$$H_\alpha(f) := \sup_{\substack{x,y\in\mathbb{R}^N \\ x\neq y}} \frac{|f(x)-f(y)|}{|x-y|^\alpha}. \tag{2.3}$$

These are also Banach spaces, with norm

$$|f|_{(m,\alpha)} := |f|_{C_{\mathrm{b}}^m(\mathbb{R}^N)} + H_\alpha(\partial_x^m f). \tag{2.4}$$

An analogous definition holds for the Banach spaces $C_{\mathrm{b}}^m(Q)$, where $Q :=]0,T[\,\times\mathbb{R}^N$, $T > 0$; in addition, for $\alpha \in]0,1[$ we consider the Hölder spaces on Q

$$C^{[0,\alpha]}(Q) := \{f \in C_{\mathrm{b}}^0(Q) \mid \widetilde{H}_\alpha(f) < \infty\}, \tag{2.5}$$

$$C^{[2,\alpha]}(Q) := \{f \in C_{\mathrm{b}}^1(Q) \mid \partial_x^2 f \in C_{\mathrm{b}}(Q),\, \widetilde{H}_\alpha(f_t),\, \widetilde{H}_\alpha(\partial_x^2 f) < \infty\}, \tag{2.6}$$

where

$$\widetilde{H}_\alpha(f) := \sup_{\substack{(t,x),(s,y)\in Q \\ (t,x)\neq(s,y)}} \frac{|f(t,x)-f(s,y)|}{(|t-s|+|x-y|^2)^{\frac{\alpha}{2}}}. \tag{2.7}$$

These are again Banach spaces, with norms

$$|f|_{[0,\alpha]} := |f|_{C^0_{\mathrm{b}}(Q)} + \widetilde{H}_\alpha(f), \tag{2.8}$$

$$|f|_{[2,\alpha]} := |f|_{C^1_{\mathrm{b}}(Q)} + |\partial_x^2 f|_{C^0_{\mathrm{b}}(Q)} + \widetilde{H}_\alpha(f_t) + \widetilde{H}_\alpha(\partial_x^2 f). \tag{2.9}$$

For the main properties of these spaces, we refer e.g. to Krylov, [4, §8.5] (where these spaces are denoted by $C^{\frac{\alpha}{2},\alpha}(Q)$ and $C^{1+\frac{\alpha}{2},2+\alpha}(Q)$).

2.1.3 For $1 \le p \le +\infty$, we denote by $|\cdot|_p$ the norm in the Lebesgue space $L^p := L^p(\mathbb{R}^N)$, and abbreviate $\|\cdot\| = \|\cdot\|_0 = |\cdot|_2$ for the norm in L^2. For $m \in \mathbb{N}$, we denote by H^m the usual Sobolev space $W^{m,2}(\mathbb{R}^N)$ of those functions in L^2, whose distributional derivatives of order up to m are again in L^2. We denote its norm by $\|\cdot\|_m$, identify $H^0 = L^2$, and denote the scalar products in H^m and L^2, respectively, by $\langle\,\cdot\,,\cdot\,\rangle_m$ and $\langle\,\cdot\,,\cdot\,\rangle$. We recall that if $\ell \in \mathbb{R}_{>\frac{N}{2}}$, the continuous embedding

$$H^\ell(\mathbb{R}^N) \hookrightarrow C^{(r,\alpha)}(\mathbb{R}^N) \tag{2.10}$$

holds, with $r = \lfloor \ell - \frac{N+1}{2} \rfloor$ and $0 < \alpha \le \alpha_0 := \lfloor \frac{N}{2} \rfloor + 1 - \frac{N}{2}$ (see e.g. [1]; $\lfloor x \rfloor$ denotes the integer part of x); we call C_S the norm of the imbedding (2.10). Finally, we need the following so-called "calculus inequality", for a proof of which we refer e.g. to Racke, [13, Lemma 4.7].

Proposition 2.1 *Let $m \in \mathbb{N}_{\ge 1}$, $\varphi \in C^m(\mathbb{R})$, and $u \in H^m \cap L^\infty$. Then, $\varphi(u) \in L^\infty$, $\nabla\varphi(u) \in H^{m-1}$, and the estimate*

$$|\partial^\alpha \varphi(u)|_2 \le \max_{1\le k\le m} |\varphi^{(k)}(|u|_\infty)|(1 + |u|_\infty^{m-1})|\partial^{|\alpha|}u|_2 \tag{2.11}$$

holds, for all $\alpha \in \mathbb{N}^N$, with $1 \le |\alpha| \le m$.

2.2 Assumptions

2.2.1 We set $s := \lfloor \frac{N}{2} \rfloor + 2$, and assume that the coefficients a_{ij} of equation (1.1) satisfy the following conditions.

(A1) Each $a_{ij} \in C^{s+m}(\mathbb{R}^N, \mathbb{R})$ for some $m \in \mathbb{N}$, with derivatives satisfying a general growth assumption

$$|\partial_p^k a_{ij}(p)| \le \alpha_k(|p|), \quad 0 \le k \le s + m, \tag{2.12}$$

for suitable continuous, nondecreasing functions α_k.

(A2) The matrix $A(p) := [a_{ij}(p)]$ is symmetric for all $p \in \mathbb{R}^N$, and satisfies the uniformly strong ellipticity condition

$$\exists\, \nu_0 > 0,\ \forall\, p,\, q \in \mathbb{R}^N, \quad a_{ij}(p)q^i q^j \ge \nu_0 |q|^2. \tag{2.13}$$

Without loss of generality, we can assume $\nu_0 = 1$. We also assume that

$$f \in Z_{s+m}(T) := \{u \in L^2(0,T;H^{s+m}) \mid \partial_t^{s+m} u \in L^2(0,T;L^2)\}, \tag{2.14}$$

$$u_0 \in H^{s+1+m}, \quad u_1 \in H^{s+m}; \tag{2.15}$$

correspondingly, we look for solutions of (1.1) in the anisotropic Sobolev space

$$\mathcal{X}_{s+m}(T) := \bigcap_{j=0}^{s+1+m} C^j([0,T]; H^{s+1-j+m}). \tag{2.16}$$

Solutions of "minimal regularity" correspond to the case $m = 0$. In the sequel, we set $d := \{u_0,\, u_1,\, f\}$, and consider the corresponding data space

$$\mathcal{D}_{s+m} := H^{s+1+m} \times H^{s+m} \times Z_{s+m}(T), \tag{2.17}$$

endowed with the graph norm

$$\|d\|^2_{\mathcal{D}_{s+m}} := \|u_0\|^2_{s+1+m} + \|u_1\|^2_{s+m} + \sum_{j=0}^{s+m} \int_0^T \|\partial_t^j f\|^2_{s-j+m} \mathrm{d}t. \tag{2.18}$$

2.2.2 Global solutions of (1.1) are obtained by the usual continuation method; that is, by extending local solutions to maximal ones, defined on some interval $[0, T_\varepsilon[\subset [0, T]$, and then establishing time-independent a priori estimates on these maximal solutions. To this end, we first note that, given $\tau \in]0, T]$, it is sufficient to estimate the norm of u in the subspace

$$\mathcal{Y}_{s+m}(\tau) := C([0, \tau]; H^{s+1+m}) \cap C^1([0, \tau]; H^{s+m}), \tag{2.19}$$

with bounds independent of τ. This is because if we do have such time-independent estimates on u and u_t, then we can derive time-independent estimates of the higher order derivatives, i.e. of $\partial_t^j u$, $2 \le j \le s+1+m$, in $C([0, \tau]; H^{s+1-j+m})$, directly from equation (1.1), using the algebra properties of the Sobolev spaces.

Thus, we take the product $H^{s+1+m} \times H^{s+m}$ as the underlying phase space. For $v \in H^{s+1-m}$, $\tau \in]0, T]$, $u \in \mathcal{Y}_{s+m}(\tau)$, $t \in [0, \tau]$ and $\varepsilon \in]0, 1]$, we set

$$Q_{s+m}(\nabla v) := \sum_{|\alpha| \le s+m} \langle a_{ij}(\nabla v) \partial_i \partial^\alpha v, \partial_j \partial^\alpha v \rangle, \tag{2.20}$$

$$\begin{aligned} N_{s+m}(u(t)) := {} & \|\varepsilon u_t(t)\|^2_{s+m} + \langle u(t), \varepsilon u_t(t) \rangle_{s+m} + \frac{1}{2} \|u(t)\|^2_{s+m} \\ & + \varepsilon Q_{s+m}(\nabla u(t)) + \frac{1}{2} \int_0^t (\varepsilon \|u_t\|^2_{s+m} + Q_{s+m}(\nabla u)) \mathrm{d}\theta, \end{aligned} \tag{2.21}$$

$$|\!|\!| u |\!|\!|_{\mathcal{Y}_{s+m}(\tau)} := \max_{0 \le t \le \tau} \sqrt{N_{s+m}(u(t))}, \tag{2.22}$$

and note that (2.13) implies that for all $v \in H^{s+1+m}$,

$$Q_{s+m}(\nabla v) \ge \|\nabla v\|^2_{s+m}. \tag{2.23}$$

2.2.3 Given the data u_0, u_1 and f as in (2.14) and (2.15), we define $u_2 \in H^{s-1+m}$ and $u_3 \in H^{s-2+m}$ by

$$\varepsilon u_2 := f(0) + a_{ij}(\nabla u_0) \partial_i \partial_j u_0 - u_1, \tag{2.24}$$

$$\varepsilon u_3 := f_t(0) + a_{ij}(\nabla u_0) \partial_i \partial_j u_1 + a'_{ij}(\nabla u_0) \cdot \nabla u_1 \partial_i \partial_j u_0 - u_2; \tag{2.25}$$

note that if (1.1) has a solution $u \in \mathcal{X}_{s+m}(\tau)$, for some $\tau \in]0, T]$, then $u_2 = u_{tt}(0)$ and $u_3 = u_{ttt}(0)$. As we have stated above, our goal is to establish time-uniform bounds on $N_{s+m}(u(t))$;

as it turns out, we are able to obtain such estimates, in terms of the following quantities:

$$D_{m1}^2 := \|\varepsilon u_1\|_{s+m}^2 + \langle u_0, \varepsilon u_1\rangle_{s+m} + \frac{1}{2}\|u_0\|_{s+m}^2 + \varepsilon Q_{s+m}(\nabla u_0) + 5\int_0^T \|f\|_{s+m}^2 \mathrm{d}t, \tag{2.26}$$

$$D_{m2} := \|\varepsilon u_2\|_{s-1+m}, \tag{2.27}$$

$$D_{m3} := \varepsilon\|\varepsilon u_3\|_{s-2+m}, \tag{2.28}$$

$$F_m^2 := \int_0^T (\|f\|_{s+m}^2 + \|f_t\|_{s-1+m}^2 + \|f_{tt}\|_{s-2+m}^2)\mathrm{d}t. \tag{2.29}$$

All these quantities depend only on the data u_0, u_1, f, and remain bounded as $\varepsilon \to 0$. We will denote by ψ_j, $j = 0, 1, \cdots$, various functions of F_m and D_{m1}, D_{m2}, D_{m3}, which can always be determined explicitly, with formulas that either involve these quantities directly, or through previously defined functions ψ's. In general, these functions depend on such universal constants as those in the Sobolev imbeddings (2.10), or in the trace inequalities

$$\|f\|_{C([0,T];H^r)} \le C_{tr}(\|f\|_{L^2(0,T;H^{r+1})}^{\frac{1}{2}}\|f_t\|_{L^2(0,T;H^r)}^{\frac{1}{2}} + \|f\|_{L^2(0,T;H^r)}) \tag{2.30}$$

(see e.g. [6, Chapter 1, §3]). We denote such generic constants by C, with the understanding that such C's may vary from formula to formula, or even within the same formula.

2.3 Local existence and extension results

2.3.1 Local solutions to the hyperbolic problem (1.1)–(1.2) are provided by the following result, which can be proven e.g. as in Kato [3].

Theorem 2.1 *Let $m \ge 0$, and assume that the coefficients a_{ij} satisfy conditions* (A1) *and* (A2)*, and the data u_0, u_1, f, are as in* (2.14), (2.15)*. There exists $\tau_0 \in]0, \frac{1}{2}T]$, independent of m, such that for all $\varepsilon \in]0,1]$, problem* (1.1)–(1.2) *has a unique solution $u \in \mathcal{X}_{s+m}(2\tau_0)$. This solution satisfies the estimate*

$$\max_{0\le t\le 2\tau_0} N_{s+m}(u(t)) \le 4D_{m1}^2. \tag{2.31}$$

Remark 2.1 (1) As mentioned in Subsection 2.2.2 above, in the proof of Theorem 2.1 one first establishes the existence of $u \in \mathcal{Y}_{s+m}(2\tau_0)$, satisfying (2.31), and then deduces the time regularity by differentiation of (1.1).

(2) The value of τ_0 depends only (in a usually decreasing fashion) on the value of D_{01}; thus, we explicitly point out that τ_0 can be determined independently of ε. In addition, this means that more regular solutions are defined on the same time interval $[0, 2\tau_0]$. This non-trivial result is explicitly discussed in [3], and follows from estimates similar to (3.2) below. More precisely, we can prove

Theorem 2.2 *Assume that* (2.14) *and* (2.15) *hold for some $m \ge 1$, and that problem* (1.1)–(1.2) *has a corresponding solution $u \in \mathcal{X}_{s+m}(\tau)$, for some $\varepsilon \in]0,1]$ and $\tau \in]0,T]$. Then, u satisfies an estimate of the form*

$$|||u|||_{\mathcal{Y}_{s+m}(\tau)} \le \|d\|_{\mathcal{D}_{s+m}}\varphi(|||u|||_{\mathcal{Y}_s(\tau)}), \tag{2.32}$$

where $\varphi : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ is continuous and nondecreasing.

(3) An analogous local existence result holds for the parabolic problem (1.4), albeit in somewhat different function spaces.

2.3.2 By a standard continuation argument, we can extend u to a maximal interval $[0, T_\varepsilon[\subset [0,T]$, with $T_\varepsilon := \sup \mathcal{T}$,

$$\mathcal{T} := \{\tau \in [0,T] \mid (1.1) \text{ has a solution } u \in \mathcal{X}_{s+m}(\tau)\}; \tag{2.33}$$

this set is not empty, since it contains $2\tau_0$. Global existence consists in showing that $T_\varepsilon = \max \mathcal{T}$, since in this case $u \in \mathcal{X}_{s+m}(T_\varepsilon)$, and $T_\varepsilon = T$, because if it were $T_\varepsilon < T$, the local existence theorem applied to (1.1) with initial values at $t = T_\varepsilon$ would yield an extension of u beyond $[0, T_\varepsilon]$, contradicting T_ε being a supremum. In turn, to show that $T_\varepsilon = \max \mathcal{T}$ it is sufficient to show that the function $t \mapsto N_{s+m}(u(t))$ and its time derivative admit an upper bound in $[0, T_\varepsilon[$. In fact, it is sufficient to bound $N_{s+m}(u(t))$, since then $\frac{\mathrm{d}}{\mathrm{d}t} N_{s+m}(u(t))$ can be bounded by means of estimates like (3.3) below. Our goal is then to prove

Theorem 2.3 *Let assumptions* (A1), (A2), (2.14) *and* (2.15) *hold, for some* $m \geq 0$. *There exists* $\varepsilon_m \in]0,1]$ *such that for all* $\varepsilon \in]0, \varepsilon_m]$, *problem* (1.1)–(1.2) *admits a unique solution* $u \in \mathcal{X}_{s+m}(T)$.

2.3.3 The uniqueness part in Theorem 2.3 is known, and can be proven as in Theorem 2.1, since the argument is independent of the size of the interval where solutions are defined. As for the existence, we proceed in two steps, first considering more regular solutions, corresponding to $m \geq 4$, and then (in Section 6) resorting to an approximation argument to deal with the remaining cases $0 \leq m \leq 3$.

As we mentioned in Remark 2.1, the availability of a time-independent bound on $N_{s+m_0}(u(t))$ for a particular value $m = m_0$ implies that of time-independent bounds on $N_{s+m}(u(t))$ for all $m \geq m_0$. In other words, global existence in $\mathcal{X}_{s+m_0}(T)$ implies global existence in $\mathcal{X}_{s+m}(T)$. We take then $m_0 = 4$, and claim:

Theorem 2.4 *Let assumptions* (A1), (A2), (2.14) *and* (2.15) *hold, for* $m = 4$. *There exists a number* $R_0 \geq 2$, *depending only on* T *and the data* u_0, u_1 *and* f, *through the quantities* F_4, D_{41}, D_{42} *and* D_{43} *of Subsection* 2.2.3, *and a corresponding* $\varepsilon_4 \in]0,1]$, *such that for all* $\varepsilon \in]0, \varepsilon_4]$, *and all* $t \in [0, T_\varepsilon[$,

$$N_{s+4}(u(t)) \leq (R_0 D_{41})^2. \tag{2.34}$$

Consequently, if $\varepsilon \leq \varepsilon_4$, u *can be extended to a solution of* (1.1) *in* $\mathcal{X}_{s+4}(T)$.

We prove Theorem 2.4 in Section 5.

3 A Priori Estimates

The proof of Theorem 2.4 is based on an energy estimate on strong solutions to the hyperbolic equation (1.1), and on a Schauder estimate on classical solutions to the parabolic equation (1.4).

3.1 The hyperbolic estimate

Theorem 3.1 *Let $m \geq 0$, and set*

$$\alpha(r) := \max_{0\leq k\leq s+m} \alpha_k(r), \quad \beta(r) := \alpha(r)(1+r^{s-1}). \tag{3.1}$$

Assume that problem (1.1)–(1.2) *has a solution $u \in \mathcal{Y}_{s+m}(\tau)$, for some $\tau \in]0,T]$. Then, for all $t \in [0,\tau]$,*

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t} N_{s+m}(u) + \frac{1}{2}(\varepsilon\|u_t\|^2_{s+m} + Q_{s+m}(\nabla u))\\ &\leq \langle f, 2\varepsilon u_t + u\rangle_{s+m} + \beta(|\nabla u|_\infty)|\partial_x^2 u|_\infty \|\nabla u\|_{s+m}(\varepsilon\|u_t\|_{s+m} + \|u\|_{s+m})\\ &\quad + \varepsilon\beta(|\nabla u|_\infty)|\nabla u_t|_\infty \|\nabla u\|^2_{s+m}.\end{aligned} \tag{3.2}$$

This result is proven in [9]. Formally, (3.2) is obtained by multiplying (1.1) by $2\varepsilon u_t + u$ in H^{s+m}, and resorting to the so-called "calculus inequalities" (see [12]) to estimate the resulting terms $\partial^\beta(a_{ij}(\nabla u))\partial_i\partial_j\partial^{\alpha-\beta}u$, $1 \leq |\beta| \leq |\alpha| \leq s+m$.

Recalling (2.23), by the (weighted) Cauchy-Schwartz inequality we obtain from (3.2)

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t} N_{s+m}(u) + \frac{1}{4}\|\nabla u\|^2_{s+m}\\ &\leq 5\|f\|^2_{s+m} + \frac{1}{2}\|u\|^2_{s+m} + (\beta(|\nabla u|_\infty)|\partial_x^2 u|_\infty)^2(\varepsilon\|\nabla u\|^2_{s+m} + \|u\|^2_{s+m})\\ &\quad + \varepsilon\beta(|\nabla u|_\infty)|\nabla u_t|_\infty\|\nabla u\|^2_{s+m};\end{aligned} \tag{3.3}$$

we will use this estimate, with $m = 4$, in Subsection 5.1.2 below.

3.2 The parabolic estimate

Theorem 3.2 *Let the coefficients a_{ij} satisfy* (A1) *and* (A2)*, for some $m \geq 0$. Assume that $g \in C^1_\mathrm{b}(Q)$, that $u_0 \in C^{(2,\alpha)}(\mathbb{R}^N)$ for some $\alpha \in]0,1[$, and that* (1.4) *admits a corresponding solution $u \in C^{[2,\alpha]}(Q)$. Let $\tau \in]0,T]$, and set $Q_\tau :=]0,\tau[\times\mathbb{R}^N$. There exist $C_D > 0$ and $\gamma \in]0,\alpha]$, depending on the norms of g in $C^1_\mathrm{b}(Q_\tau)$ and u_0 in $C^2_\mathrm{b}(\mathbb{R}^N)$, but independent of u, such that*

$$|u|_{C^{[2,\gamma]}(Q_\tau)} \leq C_D(|g|_{C^{[0,\alpha]}(Q_\tau)} + |u_0|_{C^{(2,\alpha)}(\mathbb{R}^N)}). \tag{3.4}$$

Estimates like (3.4) are generally considered as well-known; however, we have not been able to find in the literature an explicit proof of (3.4) for quasilinear equations in the whole space $\mathbb{R}^N$. Thus, for the reader's convenience, we give a proof of Theorem 3.2 in Section 4, putting together various results of O. A. Ladyzenskaya et al [5] and N. V. Krylov [4]. Note that (3.4) reads almost exactly as the estimate reported in Krylov's Theorem 8.9.2, which is established for a linear Cauchy problem with coefficients in $C^{[0,\alpha]}(Q)$, whose norm determines the constant C_D (called N there). In the present situation, of course, since we are assuming that (1.4) has a solution $u \in C^{[2,\alpha]}(Q)$, we can consider (1.4) as a linear equation, with known coefficients $\widetilde{a}_{ij}(t,x) = a_{ij}(\nabla u(t,x))$; however, the constant C_D would then depend on the Hölder norm of ∇u, which can generally be estimated only in terms of the Hölder norm of ∇g and the coefficients $\widetilde{a}_{ij}$ again. Since we intend to apply (3.4) to equation (1.6), where $g = f - \varepsilon u_{tt}$, our only concern in Theorem 3.2 is to confirm that the modulus of continuity of ∇u can be estimated "only" in terms of the norms of g in $C^1_{\mathrm{b}}(Q_\tau)$ and of u_0 in $C^2_{\mathrm{b}}(\mathbb{R}^N)$. In particular, C_D depends on τ only through $|g|_{C^1_{\mathrm{b}}(Q_\tau)}$.

3.3 Time-independent estimates

We will obtain time-independent estimates on $N_{s+4}(u(t))$ in two steps. At first, we use the smallness of ε to absorb the two terms with $\varepsilon\|\nabla u\|_{s+4}$ at the right side of (3.3) into the term $\frac{1}{4}\|\nabla u\|^2_{s+4}$ at its left; then, we resort to the parabolic estimate of Theorem 3.2, to obtain an estimate of the coefficient of the term with $\|u\|_{s+4}$ at the right side of (3.3). This estimate will be in terms of $g = f - \varepsilon u_{tt}$, and we use again the smallness of ε to estimate the term εu_{tt} and its first order derivatives. This allows us to deduce from (3.3) a time-independent estimate on $N_{s+4}(u(t))$, via Gronwall's inequality. In conclusion, to prove Theorem 2.4 (and, in fact, Theorem 2.3 as well), it would be sufficient to establish a time-independent estimate on $|\nabla u(t,\cdot)|_\infty$ and $|\partial_x^2 u(t,\cdot)|_\infty$. This is of course well-known (see e.g. [14, Chapter 5]).

4 Proof of Theorem 3.2

We follow O. A. Ladyzenskaya et al [5, Chapter VI] and N. V. Krylov [4, Chapter 8].

4.1 Our first step is to estimate $|u|$ and $|\nabla u|$ in Q_τ by the maximum principle. As we have stated at the end of Section 3.2, we can consider the equation of (1.4) as linear, with known coefficients $\widetilde{a}_{ij}(t,x) = a_{ij}(\nabla u(t,x))$ which are bounded, because we are assuming that (1.4) has a solution $u \in C^{[2,\alpha]}(Q)$. Hence, we can apply [4, Corollary 8.1.5], which yields the explicit estimate

$$\sup_{Q_\tau}|u| \le \tau \sup_{Q_\tau}|g| + \sup_{\mathbb{R}^N}|u_0| \le T|g|_{C^0_{\mathrm{b}}(Q_\tau)} + |u_0|_{C^0_{\mathrm{b}}(\mathbb{R}^N)} =: C_0. \tag{4.1}$$

Next, we differentiate the equation of (1.4): setting $\partial_0 := \frac{\partial}{\partial t}$, we see that, for $0 \le k \le N$, each function $v^k := \partial_k u$ satisfies the linear equation

$$v^k_t - a_{ij}(\nabla u)\partial_i\partial_j v^k - a'_{ij}(\nabla u)\cdot\nabla v^k\partial_i\partial_j u = \partial_k g. \tag{4.2}$$

The coefficients of this equation are also bounded (including those of the lower order terms $v \mapsto a'_{ij}(\nabla u)\cdot(\nabla v)\partial_i\partial_j u$); hence, by the same corollary, as in (4.1),

$$\sup_{Q_\tau}|\nabla u| \le \tau \sup_{Q_\tau}|\nabla g| + \sup_{\mathbb{R}^N}|\nabla u_0| \le T|g|_{C^1_{\mathrm{b}}(Q_\tau)} + |u_0|_{C^1_{\mathrm{b}}(\mathbb{R}^N)} =: C_1, \tag{4.3}$$

and, since u_t satisfies the initial condition

$$u_t(0) = u_1 := g(0,\cdot) + a_{ij}(\nabla u_0)\partial_i\partial_j u_0, \tag{4.4}$$

recalling (2.12), we have

$$\sup_{Q_\tau}|u_t| \le \tau \sup_{Q_\tau}|g_t| + \sup_{\mathbb{R}^N}|u_1| \le (T+1)|g|_{C^1_{\mathrm{b}}(Q_\tau)} + \alpha_0(|u_0|_{C^1_{\mathrm{b}}(\mathbb{R}^N)})|u_0|_{C^2_{\mathrm{b}}(\mathbb{R}^N)} =: C_2. \tag{4.5}$$

4.2 Our second step is to estimate $\widetilde{H}_\alpha(\nabla u)$. To this end, we resort to Lemma 3.1 of Ladyzenskaya-Solonnikov-Ural'tseva [5, Chapter II, §3], which states that ∇u will satisfy a

Hölder condition in t, uniformly in x, if it satisfies a Hölder condition in x, uniformly in t, and u satisfies a Hölder condition in t, uniformly in x. The latter is clearly implied by (4.5); to show the former, we consider $t \in [0,\tau]$ as fixed, and $u(t,\,\cdot\,)$ as a solution of the quasilinear elliptic equation

$$-a_{ij}(\nabla u)\partial_i\partial_j u = g - u_t =: \widetilde{g}, \tag{4.6}$$

and invoke a classical result on the Hölder continuity of the gradient of the solutions to such equations. More precisely, for fixed $t \in [0,\tau]$ and arbitrary $x,\, y \in \mathbb{R}^N$, with $x \neq y$, we wish to estimate the ratio

$$h_\beta(\nabla u(t); x, y) := \frac{|\nabla u(t,x) - \nabla u(t,y)|}{|x-y|^\beta}, \tag{4.7}$$

for suitable $\beta \in]0,\alpha]$. If $|x-y| \geq 1$, (4.3) yields

$$h_\beta(\nabla u(t); x, y) \leq |\nabla u(t,x) - \nabla u(t,y)| \leq 2C_1 \tag{4.8}$$

for all $\beta \in]0,1[$. If instead $0 < |x-y| < 1$, we consider concentric balls B_k with center x and radii respectively equal to $k=1$, $k=2$ and $k=3$, and choose a cut-off function $\zeta \in C_0^\infty(\mathbb{R}^N)$, such that $0 \leq \zeta(y) \leq 1$ for all $y \in \mathbb{R}^N$, $\zeta(y) \equiv 1$ on B_1, $\zeta(y) \geq \frac{1}{2}$ in B_2, and $\zeta(y) \equiv 0$ off B_3. In B_2, the function $v := \zeta u$ satisfies the quasilinear elliptic equation

$$-\overline{a}_{ij}(y,v,\nabla v)\partial_i\partial_j v = \zeta\,\widetilde{g} + (a_{ij}(\nabla u)\partial_i\partial_j\zeta)u + a_{ij}(\nabla u)\partial_i\zeta\partial_j u =: b, \tag{4.9}$$

in which, for $y \in B_2$, $p \in \mathbb{R}$ and $q \in \mathbb{R}^N$, the coefficients

$$\overline{a}_{ij}(y,p,q) := a_{ij}((\zeta(y))^{-1}q - p(\zeta(y))^{-2}\nabla\zeta(y)) \tag{4.10}$$

are of class C^1 in $B_2 \times \mathbb{R} \times \mathbb{R}^N$. Furthermore, the function $y \mapsto b(t,y)$ is in $C^0(\overline{B}_2)$, and its norm in this space can be estimated in terms of C_0 and C_1, because of (4.1) and (4.3). Thus, we can apply Theorem 13.6 of Gilbarg-Trudinger [2, Chapter 13, §4], with $\Omega = B_2$ and $\Omega' = B_1$, to deduce the estimate

$$h_\beta(\nabla v(t); y, y') \leq C_3 d^{-\beta}, \quad y,\, y' \in B_1,\ y \neq y', \tag{4.11}$$

where $d = \operatorname{dist}(B_1, \partial B_2) = 1$, and both $\beta \in]0,1[$ and C_3 depend on the fixed constants N, ν, $\operatorname{diam}(B_2)$ and, more essentially, on $K(t) := |v(t,\,\cdot\,)|_{C^1(\overline{B}_2)}$. We can of course choose $\beta \leq \alpha$. Again by (4.1) and (4.3), $K(t)$ can be estimated in terms of C_0 and C_1, uniformly in $t \in [0,\tau]$. Since $v = u$ in B_1, (4.11) implies that, for all $y \in B_1 \setminus \{x\}$,

$$h_\beta(\nabla u(t); x, y) \leq C_3, \tag{4.12}$$

which yields the desired estimate of (4.7) when $0 < |x-y| < 1$. In conclusion, from (4.8) and (4.12) it follows that for all $x,\, y \in \mathbb{R}^N$, with $x \neq y$,

$$h_\beta(\nabla u(t); x, y) \leq \max\{2C_1,\, C_3\} =: C_4, \tag{4.13}$$

where β is determined in (4.11); as we have observed, C_4 can be estimated in terms of C_0 and C_1. Estimate (4.13) provides the Hölder condition in x, uniformly in t, for ∇u, which is required in [5, Lemma 3.1], to obtain the estimate

$$\frac{|\nabla u(t,x) - \nabla u(s,x)|}{|t-s|^{\frac{\gamma}{2}}} \leq C_5 \tag{4.14}$$

for suitable $\gamma \le \beta \le \alpha$. In (4.14), C_5 is independent of $x \in \mathbb{R}^N$, and can be estimated in terms of C_0, C_1, C_2 and C_4. From (4.13) and (4.14), we deduce that, for (t,x), $(s,y) \in Q_\tau$, with $(t,x) \neq (s,y)$,

$$\frac{|\nabla u(t,x) - \nabla u(s,y)|}{(|t-s| + |x-y|^2)^{\frac{\gamma}{2}}} \le \frac{|\nabla u(t,x) - \nabla u(t,y)|}{|x-y|^\gamma} + \frac{|\nabla u(t,y) - \nabla u(s,y)|}{|t-s|^{\frac{\gamma}{2}}} \le C_4 + C_5 =: C_6, \tag{4.15}$$

from which we conclude that

$$\widetilde{H}_\gamma(\nabla u) \le C_6. \tag{4.16}$$

4.3 Clearly, (4.1), (4.3), (4.16) and (2.12) imply that the coefficients $\widetilde{a}_{ij} = a_{ij}(\nabla u)$ satisfy the estimate

$$|\widetilde{a}_{ij}|_{C^{[0,\gamma]}(Q_\tau)} \le \alpha_0(C_1) + C_6 =: C_7. \tag{4.17}$$

We are then in a position to apply [4, Theorem 8.9.2] (with $K = C_7$ of (4.17)), and deduce the estimate

$$|u|_{C^{[2,\gamma]}(Q_\tau)} \le C_K |g - u|_{C^{[0,\gamma]}(Q_\tau)}; \tag{4.18}$$

note that C_K can indeed be estimated in terms of the norm of g in $C^1_{\mathrm{b}}(Q_\tau)$ and of u_0 in $C^2_{\mathrm{b}}(\mathbb{R}^N)$, as claimed. By the interpolation inequality

$$\widetilde{H}_\gamma(u) \le \eta(\widetilde{H}_\gamma(u_t) + \widetilde{H}_\gamma(\partial_x^2 u)) + C\eta^{-\frac{\gamma}{2}} \sup_{Q_\tau} |u|, \quad \eta > 0, \tag{4.19}$$

(see [4, Chapter 8, §8]), we obtain

$$\begin{aligned} |u|_{C^{[0,\gamma]}(Q_\tau)} &\le \sup_{Q_\tau} |u| + \widetilde{H}_\gamma(u) \\ &\le \eta(\widetilde{H}_\gamma(u_t) + \widetilde{H}_\gamma(\partial_x^2 u)) + C_\eta \sup_{Q_\tau} |u| \\ &\le \eta |u|_{C^{[2,\gamma]}(Q_\tau)} + C_\eta C_0, \end{aligned} \tag{4.20}$$

having recalled (4.1). Taking η sufficiently small, and recalling that $\gamma \le \alpha$, we deduce from (4.18), (4.20) and (4.1) that

$$|u|_{C^{[2,\gamma]}(Q_\tau)} \le C_K(|g|_{C^{[0,\gamma]}(Q_\tau)} + C_0) \le C_K(|g|_{C^{[0,\alpha]}(Q_\tau)} + |u_0|_{(0,\alpha)}), \tag{4.21}$$

from which (3.4) follows. This concludes the proof of Theorem 3.2.

5 Proof of Theorem 2.4

We proceed by contradiction; thus, we assume that for all $\varepsilon_4 \in]0,1]$, there exists $\varepsilon \in]0, \varepsilon_4]$ such that $T_\varepsilon < T$ and

$$\limsup_{t \to T_\varepsilon^-} N_{s+4}(u(t)) = +\infty. \tag{5.1}$$

It follows that for all $R_0 \ge 2$, there is $t_\varepsilon \in]0, T_\varepsilon[$ such that

$$N_{s+4}(u(t_\varepsilon)) > (R_0 D_{41})^2. \tag{5.2}$$

Certainly, $t_\varepsilon \in]2\tau_0, T_\varepsilon[$, since on $[0, 2\tau_0]$ the local estimate (2.31) implies that for all $t \in [0, 2\tau_0]$,

$$N_{s+4}(u(t)) \le 4D_{41}^2 \le (R_0 D_{41})^2. \tag{5.3}$$

In fact, since $N_{s+4}(u(0)) \le D_{41}^2$, by (5.1) there is a largest interval $[0,\tau_\varepsilon]$, $t_\varepsilon \le \tau_\varepsilon < T_\varepsilon$, with the property that for all $t \in [0,\tau_\varepsilon]$,

$$N_{s+4}(u(t)) \le ((R_0+1)D_{41})^2 = N_{s+4}(u(\tau_\varepsilon)). \tag{5.4}$$

That is: any specific choice of R_0 and ε_4 determines an $\varepsilon \in]0,\varepsilon_4]$, and a corresponding $\tau_\varepsilon \in]0,T_\varepsilon[$, such that (5.4) holds. Thus, τ_ε depends on R_0 and ε: we write $\tau_\varepsilon = \rho(R_0,\varepsilon)$.

Our argument will run as follows. We first establish estimates on $N_{s+4}(u(t))$, which hold for arbitrary R_0, small ε, and $t \in [0,\tau_\varepsilon] = [0,\rho(R_0,\varepsilon)]$. These estimates involve eight quantities ψ_j of the type described in Subsection 2.2.3; that is, the ψ_j's can all be determined, explicitly and a priori, in terms of the data f, u_0 and u_1 (and, of course, universal constants). It is crucial to note that, while the ψ_j's depend also on R_0, they are independent of τ_ε. We make then a specific choice of R_0 (in (5.49)), depending only on T and the data f, u_0, u_1, via the quantities $D_{41},\cdots,F_4$ of Subsection 2.2.3. This choice of R_0 completely determines the quantities $\psi_1,\cdots,\psi_8$; in turn, these determine the choice of ε_4, by means of four restrictions: the first is in (5.19) below; the second is $2\varepsilon_4 < \tau_0$ (recall that τ_0 can be determined independently of $\varepsilon \in]0,1]$); the third is in (5.33), and the last in (5.44). With such R_0 and ε_4, we consider the corresponding $\varepsilon \in]0,\varepsilon_4]$, and $\tau_\varepsilon = \rho(R_0,\varepsilon)$, such that (5.4) holds, and show that, for these choices of R_0, ε and τ_ε, the corresponding solution of (1.1) satisfies estimate (5.3) in $[0,\tau_\varepsilon]$. For $t = \tau_\varepsilon$, this yields a contradiction to (5.4).

5.1 Higher order energy estimates

5.1.1 We will use the following estimates on $[0,\tau_\varepsilon]$, derived from (5.4) by means of Schwartz' inequality:

$$\|u(t)\|_{s+4}^2 \le 4N_{s+4}(u(t)) \le 4((R_0+1)D_{41})^2, \tag{5.5}$$

$$\|\nabla u(t)\|_{s+4}^2 \le \frac{1}{\varepsilon}N_{s+4}(u(t)) \le \frac{1}{\varepsilon}((R_0+1)D_{41})^2, \tag{5.6}$$

$$\|u_t(t)\|_{s+4}^2 \le \frac{2}{\varepsilon^2}N_{s+4}(u(t)) \le \frac{2}{\varepsilon^2}((R_0+1)D_{41})^2. \tag{5.7}$$

Also, by Proposition 2.1 and (5.5), for $1 \le m \le s+3$, omitting the variable t, we have

$$\begin{aligned} |a_{ij}(\nabla u)|_\infty + \|\nabla(a_{ij}(\nabla u))\|_{m-1} &\le \alpha(|\nabla u|_\infty)(1+|\nabla u|_\infty^{m-1})\|\nabla u\|_m \\ &\le \alpha(\|u\|_s)(1+\|u\|_s^{m-1})\|u\|_{m+1} \\ &\le \beta(2(R_0+1)D_{41})2(R_0+1)D_{41} =: \psi_1. \end{aligned} \tag{5.8}$$

A crucial remark is that (5.5) also allows us to give estimates of lower order norms of u_t which are, in terms of boundedness as $\varepsilon \to 0$, better than (5.7). Indeed, multiplying (1.1) by $2u_t$ in H^{s+2} yields

$$\varepsilon\frac{\mathrm{d}}{\mathrm{d}t}\|u_t\|_{s+2}^2 + \|u_t\|_{s+2}^2 \le \|f + a_{ij}(\nabla u)\partial_i\partial_j u\|_{s+2}^2. \tag{5.9}$$

Since H^{s+1} is an algebra under pointwise multiplication, by (5.8) and (5.5) we deduce from (5.9) the exponential inequality

$$\begin{aligned} \varepsilon\frac{\mathrm{d}}{\mathrm{d}t}\|u_t\|_{s+2}^2 + \|u_t\|_{s+2}^2 &\le 2\|f\|_{s+2}^2 + 2(|a_{ij}(\nabla u)|_\infty + \|\nabla(a_{ij}(\nabla u))\|_{s+1})^2\|u\|_{s+4}^2 \\ &\le 2\|f\|_{s+3}^2 + 2\psi_1^2 4((R_0+1)D_{41})^2 =: \psi_2^2. \end{aligned} \tag{5.10}$$

We can assume without loss of generality that $\psi_2 \geq \|u_1\|_{s+2}$; then, (5.10) implies that, for all $t \in [0, \tau_\varepsilon]$,

$$\|u_t(t)\|_{s+2} \leq \psi_2. \tag{5.11}$$

In the same way, but using (5.6) instead of (5.5), we also deduce that

$$\|u_t(t)\|_{s+3} \leq \frac{1}{\sqrt{\varepsilon}}\psi_2. \tag{5.12}$$

5.1.2 We now go back to estimate (3.3), with $m = 4$ and $t \in [0, \tau_\varepsilon]$. Since $H^{s-1} \hookrightarrow L^\infty$, we obtain from (5.5) and (5.11)

$$|\nabla u(t)|_\infty + |\partial_x^2 u(t)|_\infty \leq 2C_S\|u(t)\|_{s+1} \leq C(R_0 + 1)D_{41}, \tag{5.13}$$

$$|\nabla u_t|_\infty \leq C_S\|u_t\|_s \leq C_S\psi_2. \tag{5.14}$$

Consequently, setting

$$\psi_3 := \beta(CD_{41}(R_0 + 1))CD_{41}(R_0 + 1), \tag{5.15}$$

$$\psi_4 := \psi_3^2 + \beta(CD_{41}(R_0 + 1))C_S\psi_2, \tag{5.16}$$

$$\gamma_\infty(\partial u) := 4(\beta(|\nabla u|_\infty)|\partial_x^2 u|_\infty)^2, \tag{5.17}$$

we deduce from (3.3) that, for $t \in [0, \tau_\varepsilon]$,

$$\frac{\mathrm{d}}{\mathrm{d}t}N_{s+4}(u) + \frac{1}{4}\|\nabla u\|_{s+4}^2 \leq 5\|f\|_{s+4}^2 + \varepsilon\psi_4\|\nabla u\|_{s+4}^2 + \frac{1}{4}\gamma_\infty(\partial u)\|u\|_{s+4}^2. \tag{5.18}$$

Thus, if we choose $\varepsilon_4 \in]0, 1]$ so small that

$$4\varepsilon_4\psi_4 \leq 1, \tag{5.19}$$

recalling the first inequality of (5.5) we obtain from (5.18) that, for the corresponding $\varepsilon \leq \varepsilon_4$ and $\tau_\varepsilon = \rho(R_0, \varepsilon)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}N_{s+4}(u) \leq 5\|f\|_{s+4}^2 + \gamma_\infty(\partial u)N_{s+4}(u). \tag{5.20}$$

By Gronwall's inequality and the local estimate (2.31), we conclude then that for $t \in [\tau_0, \tau_\varepsilon]$,

$$\begin{aligned} N_{s+4}(u(t)) &\leq \Big(N_{s+4}(u(\tau_0)) + 5\int_0^T \|f\|_{s+4}^2 \mathrm{d}t\Big) \exp\Big(\int_{\tau_0}^t \gamma_\infty(\partial u)\mathrm{d}\theta\Big) \\ &\leq 5D_{41}^2 \exp\Big(\int_{\tau_0}^t \gamma_\infty(\partial u)\mathrm{d}\theta\Big). \end{aligned} \tag{5.21}$$

5.2 L^∞ estimates

5.2.1 To estimate $\gamma_\infty(\partial u)$ in $[\tau_0, \tau_\varepsilon]$, we resort to the parabolic estimates of Theorem 3.2, considering u as solution of equation (1.6), with initial value at $t = \tau_0$. Let $Q_0 :=]\tau_0, \tau_\varepsilon[\times \mathbb{R}^N$. By (3.4),

$$|\nabla u|_{L^\infty(Q_0)} + |\partial^2 u|_{L^\infty(Q_0)} \leq |u|_{C^{[2,\gamma]}(Q_0)} \leq C_D(|f - \varepsilon u_{tt}|_{C^{[0,\alpha]}(Q_0)} + |u(\tau_0)|_{C^{(2,\alpha)}(\mathbb{R}^N)}), \tag{5.22}$$

where the constant C_D depends on the norms of $f-\varepsilon u_{tt}$ in $C^1_{\mathrm{b}}(Q_0)$ and of $u(\tau_0)$ in $C^2_{\mathrm{b}}(\mathbb{R}^N)$, but not explicitly on τ_ε. Our goal is to show that, if ε is sufficiently small, the right side of (5.22), including C_D, can be estimated independently of τ_ε.

At first, by the imbedding (2.10) and the local estimate (2.31),

$$|u(\tau_0)|_{(2,\alpha)} \le C_S\|u(\tau_0)\|_{s+1} \le C\sqrt{N_{s+1}(u(\tau_0))} \le 4CD_{41}, \tag{5.23}$$

a constant independent of τ_ε. Next,

$$|f-\varepsilon u_{tt}|_{C^{[0,\alpha]}(Q_0)} \le |f-\varepsilon u_{tt}|_{C^1_{\mathrm{b}}(Q_0)} \le \|f-\varepsilon u_{tt}\|_{C([\tau_0,\tau_\varepsilon];H^s)} + \|f_t-\varepsilon u_{ttt}\|_{C([\tau_0,\tau_\varepsilon];H^{s-1})}. \tag{5.24}$$

We estimate the terms with f by means of the trace inequality (2.30): recalling (2.29),

$$\|f\|^2_{C([\tau_0,\tau_\varepsilon];H^s)} + \|f_t\|^2_{C([\tau_0,\tau_\varepsilon];H^{s-1})} \le C_S\int_\varepsilon^\tau (\|f\|^2_{s+1}+\|f_t\|^2_s+\|f_{tt}\|^2_{s-1})\mathrm{d}t \le CF_1^2. \tag{5.25}$$

5.2.2 We estimate the term with εu_{tt} at the right side of (5.24) in the higher norm of H^{s+1}, which we need for later purposes. We differentiate (1.1) with respect to t, and multiply the resulting identity

$$\varepsilon u_{ttt} + u_{tt} = f_t + \underbrace{a_{ij}(\nabla u)\partial_i\partial_j u_t}_{=:h_1} + \underbrace{a'_{ij}(\nabla u)\cdot\nabla u_t\partial_i\partial_j u}_{=:h_2} \tag{5.26}$$

in H^{s+1} by $2u_{tt}$, to obtain

$$\varepsilon\frac{\mathrm{d}}{\mathrm{d}t}\|u_{tt}\|^2_{s+1} + \|u_{tt}\|^2_{s+1} \le 3(\|f_t\|^2_{s+1}+\|h_1\|^2_{s+1}+\|h_2\|^2_{s+1}). \tag{5.27}$$

By Proposition 2.1, as in (5.8)

$$\|h_1\|_{s+1} \le (|a_{ij}(\nabla u)|_\infty + \|\nabla(a_{ij}(\nabla u))\|_s)\|\partial_i\partial_j u_t\|_{s+1} \le \beta(|\nabla u|_\infty)\|\nabla u\|_{s+1}\|u_t\|_{s+3}; \tag{5.28}$$

thus, by (5.5), (5.12) and (5.15), for $t\in[0,\tau_\varepsilon]$,

$$\|h_1(t)\|_{s+1} \le \psi_3\frac{1}{\sqrt{\varepsilon}}\psi_2 =: \frac{1}{\sqrt{\varepsilon}}\psi_5. \tag{5.29}$$

Likewise, using (5.11), we have

$$\|h_2\|_{s+1} \le \|a'_{ij}(\nabla u)\|_{s+1}\|\nabla u_t\|_{s+1}\|\partial_i\partial_j u\|_{s+1} \le \psi_5; \tag{5.30}$$

thus, from (5.27) and (5.25) we deduce

$$\varepsilon\frac{\mathrm{d}}{\mathrm{d}t}\|u_{tt}\|^2_{s+1} + \|u_{tt}\|^2_{s+1} \le \frac{3}{\varepsilon}(CF_1^2+2\psi_5^2). \tag{5.31}$$

Integrating this exponential inequality over $[0,\tau_\varepsilon]$, multiplying by $\varepsilon\mathrm{e}^{-\frac{t}{\varepsilon}}$, and recalling that $u_{tt}(0)=u_2$, by (2.27) we obtain

$$\|\varepsilon u_{tt}\|^2_{s+1} \le D_{22}^2\mathrm{e}^{-\frac{t}{\varepsilon}} + 3\varepsilon(CF_1^2+2\psi_5^2). \tag{5.32}$$

Thus, if we choose ε_4 so small that, in addition to (5.19),

$$3\varepsilon_4(CF_1^2+2\psi_5^2) \le D_{22}^2, \tag{5.33}$$

for the corresponding $\varepsilon < \varepsilon_4$ and $\tau_\varepsilon = \rho(R_0, \varepsilon)$ we deduce from (5.32) that for all $t \in [0, \tau_\varepsilon]$,

$$\|\varepsilon u_{tt}(t)\|_{s+1} \le 2D_{22}. \tag{5.34}$$

5.2.3 We proceed to estimate the term with εu_{ttt} at the right side of (5.24). Differentiating (5.26) yields

$$\begin{aligned}\varepsilon u_{tttt} + u_{ttt} \;&=\; f_{tt} + a_{ij}(\nabla u)\partial_i\partial_j u_{tt} + 2a'_{ij}(\nabla u)\cdot\nabla u_t\partial_i\partial_j u_t + a'_{ij}(\nabla u)\cdot\nabla u_{tt}\partial_i\partial_j u\\ &\quad + a''_{ij}(\nabla u)(\nabla u_t, \nabla u_t)\partial_i\partial_j u\\ &=: f_{tt} + \sum_{k=3}^{6} h_k,\end{aligned} \tag{5.35}$$

from which, multiplying in H^{s-1} by $2u_{ttt}$, we get

$$\varepsilon\frac{\mathrm{d}}{\mathrm{d}t}\|u_{ttt}\|^2_{s-1} + \|u_{ttt}\|^2_{s-1} \le 5\Big(\|f_{tt}\|^2_{s-1} + \sum_{k=3}^{6}\|h_k\|^2_{s-1}\Big). \tag{5.36}$$

Acting as in (5.28), we have

$$\|h_3\|_{s-1} \le 2(|a_{ij}(\nabla u)|_\infty + \|\nabla(a_{ij}(\nabla u))\|_{s-2})\|\partial_i\partial_j u_t\|_{s-1} \le \beta(|\nabla u|_\infty)\|u\|_s\|u_{tt}\|_{s+1}; \tag{5.37}$$

from (5.32), we also deduce that

$$\|u_{tt}(t)\|^2_{s+1} \le \frac{1}{\varepsilon^2}D^2_{22}\mathrm{e}^{-\frac{t}{\varepsilon}} + \frac{3}{\varepsilon}(CF_1^2 + 2\psi_5^2); \tag{5.38}$$

thus, from (5.37) we conclude that

$$\|h_3\|_{s-1} \le \psi_6\Big(\frac{1}{\varepsilon}\mathrm{e}^{-\frac{t}{2\varepsilon}} + \frac{1}{\sqrt{\varepsilon}}\Big) \tag{5.39}$$

for a suitable function ψ_6. It is not difficult to see that h_5 can be estimated in the same way, and that h_4 and h_6 satisfy a simpler estimate, of the form

$$\|h_4\|_{s-1} + \|h_6\|_{s-1} \le 2\psi_7; \tag{5.40}$$

thus, from (5.35)–(5.40), it follows that

$$\varepsilon\frac{\mathrm{d}}{\mathrm{d}t}\|u_{ttt}\|^2_{s-1} + \|u_{ttt}\|^2_{s-1} \le \psi_8^2\Big(\frac{1}{\varepsilon} + \frac{1}{\varepsilon^2}\mathrm{e}^{-\frac{t}{\varepsilon}}\Big) \tag{5.41}$$

for a suitable function ψ_8. Integrating this exponential inequality over $[0, t]$, $0 < t \le \tau_\varepsilon$, multiplying by ε, and recalling that $u_{ttt}(0) = u_3$, we obtain that for all $t \in [0, \tau_\varepsilon]$,

$$\|\varepsilon u_{ttt}(t)\|^2_{s-1} \le \|\varepsilon u_3\|^2_{s-1}\mathrm{e}^{-\frac{t}{\varepsilon}} + \varepsilon\psi_8^2 + \frac{1}{\varepsilon}\psi_8^2 t\mathrm{e}^{-\frac{t}{\varepsilon}}; \tag{5.42}$$

consequently, by (2.28), if $\tau_0 \le t \le \tau_\varepsilon$,

$$\|\varepsilon u_{ttt}(t)\|^2_{s-1} \le \frac{1}{\varepsilon^2}D^2_{13}\mathrm{e}^{-\frac{\tau_0}{\varepsilon}} + \varepsilon\psi_8^2 + \frac{1}{\varepsilon}\psi_8^2 T\mathrm{e}^{-\frac{\tau_0}{\varepsilon}}. \tag{5.43}$$

The function $\varepsilon \mapsto \frac{1}{\varepsilon^2}\mathrm{e}^{-\frac{\tau_0}{\varepsilon}}$ is increasing on $]0, \frac{1}{2}\tau_0[$; hence, if we choose $\varepsilon_4 < \frac{1}{2}\tau_0$ (recall that τ_0 can be determined independently of ε) such that, in addition to (5.19) and (5.33),

$$\frac{1}{\varepsilon_4^2} D_{13}^2 \mathrm{e}^{-\frac{\tau_0}{\varepsilon_4}} + \varepsilon_4 \psi_8^2 + \frac{1}{\varepsilon_4}\psi_8^2 T \mathrm{e}^{-\frac{\tau_0}{\varepsilon_4}} \le D_{22}^2, \tag{5.44}$$

then, for the corresponding $\varepsilon \le \varepsilon_4$ and $\tau_\varepsilon = \rho(R_0, \varepsilon)$, we deduce from (5.43) that

$$\|\varepsilon u_{ttt}(t)\|_{s-1} \le D_{22}. \tag{5.45}$$

5.2.4 From (5.34) and (5.45), we conclude that if $\varepsilon_4 < \frac{1}{2}\tau_0$, and satisfies the three restrictions (5.19), (5.33) and (5.44), for the corresponding $\varepsilon \le \varepsilon_4$ and $\tau_\varepsilon = \rho(R_0, \varepsilon)$,

$$\|\varepsilon u_{tt}\|_{C([\tau_0, \tau_\varepsilon]; H^{s+1})} + \|\varepsilon u_{ttt}\|_{C([\tau_0, \tau_\varepsilon]; H^{s-1})} \le 3D_{22}, \tag{5.46}$$

a constant independent of τ_ε and R_0. Putting (5.46), together with (5.25), into (5.24), and recalling (5.23), we see that all terms at the right side of (5.22), including C_D, can be estimated independently of τ_ε; that is, there is $M > 0$, independent of τ_ε and R_0, such that

$$|\nabla u|_{L^\infty(Q_0)} + |\partial^2 u|_{L^\infty(Q_0)} \le M. \tag{5.47}$$

In fact, M depends only on T, F_1, D_{41} and D_{22}. Recalling then (5.17), we deduce from (5.21) that, with a slight abuse of notation,

$$N_{s+4}(u(t)) \le 5D_{41}^2 \exp(\gamma_\infty(M)T) \tag{5.48}$$

for all $t \in [\tau_0, \tau_\varepsilon]$. At this point, keeping in mind that R_0 is arbitrary and M is independent of R_0, we choose

$$R_0 := \sqrt{5}\mathrm{e}^{\gamma_\infty(M)\frac{T}{2}} \quad (\ge 2), \tag{5.49}$$

and deduce from (5.48), for $t = \tau_\varepsilon$, that

$$N_{s+4}(u(\tau_\varepsilon)) \le R_0^2 D_{41}^2. \tag{5.50}$$

Since (5.50) contradicts (5.4), this completes the proof of Theorem 2.4. Consequently, Theorem 2.3 is proven for $m \ge 4$, with $\varepsilon_m = \varepsilon_4$.

Remark 5.1 As we immediately realize, the proof of Theorem 2.4 carries over to the case $\varepsilon = 0$ (except, of course, for Subsections 5.2.2 and 5.2.3, instead of which we use Theorem 3.2 directly). By the results of Section 6 below, the same holds for the minimal regularity case $m = 0$. This yields a global existence result for the parabolic initial value problem (1.4)–(1.5), with

$$u \in L^2(0, T; H^{s+1}), \quad u_t \in L^2(0, T; H^{s-1}), \quad s \ge \left\lfloor \frac{N}{2} \right\rfloor + 2, \tag{5.51}$$

if $u_0 \in H^{s+1}$ and $f \in Z_s(T)$, as per (2.15) and (2.14). In fact, we can easily prove that, in addition to (5.51),

$$u \in L^2(0, T; H^{s+2}), \quad u_t \in L^2(0, T; H^s), \tag{5.52}$$

or, in alternative, that (5.51) holds under the weaker conditions $u_0 \in H^s$ and $f \in Z_{s-1}(T)$. Actually, with the methods of [11], one can prove that the solution of the parabolic equation enjoys the same regularity of that of the hyperbolic equation for t away from 0; namely, that for all $\tau \in]0,T[$,

$$u \in \bigcap_{j=0}^{s+1} C([\tau,T]; H^{s+1-j}). \tag{5.53}$$

6 Minimal Regularity

In this section we prove Theorem 2.3 for the minimal regularity case $m = 0$, by means of an approximation argument based on Theorem 2.4. As we have already remarked, the regularity Theorem 2.2 implies then the validity of Theorem 2.3 for the intermediate cases $m = 1, 2, 3$ as well.

6.1 We construct u as the limit, in $\mathcal{Y}_s(T)$, of a recursive sequence $(u^j)_{j\geq 0} \subset \mathcal{Y}_{s+4}(T)$ of more regular solutions to problem (1.1)–(1.2), corresponding to suitably chosen approximating data $(d^j)_{j\geq 0} \subset \mathcal{D}_{s+4}$. The local existence Theorem 2.1 yields a local solution $u^j \in \mathcal{X}_{s+4}(\tau_j)$, for some $\tau_j \in]0,T]$ independent of $\varepsilon \in]0,1]$; by the global existence Theorem 3.1, with $m = 4$, each u^j can be extended to all of $[0,T]$, with $u^j \in \mathcal{X}_{s+4}(T)$ if ε does not exceed some value $\varepsilon_j \in]0,1]$. In general, the sequence $(\varepsilon_j)_{j\geq 0}$ is infinitesimal; thus, we propose to show that we can choose the data $(d^j)_{j\geq 0}$ so that the corresponding local solution u^j can be extended to $[0,T]$, with

(P1) $u^j \in \mathcal{X}_{s+4}(T)$, at least if $\varepsilon_j < \varepsilon \leq \varepsilon_0$;

(P2) the sequence $(u^j)_{j\geq 0}$ is bounded in $\mathcal{Y}_s(T)$.

To this end, we need, in addition to the regularity result of Theorem 2.2, the following stability result, which can be proven as in [8]:

Theorem 6.1 *Let $\varepsilon \in]0,1]$, $0 < \tau \leq \widetilde{\tau} \leq T$, and $u \in \mathcal{X}_s(\tau)$, $\widetilde{u} \in \mathcal{X}_{s+1}(\widetilde{\tau})$ be solutions of* (1.1)–(1.2)*, corresponding respectively to data $d \in \mathcal{D}_s$ and $\widetilde{d} \in \mathcal{D}_{s+1}$. Then, the difference $u - \widetilde{u}$ satisfies an estimate of the form*

$$|||u - \widetilde{u}|||_{\mathcal{Y}_s(\tau)} \leq \|d - \widetilde{d}\|_{\mathcal{D}_s} \Phi(|||u|||_{\mathcal{Y}_s(\tau)}, |||\widetilde{u}|||_{\mathcal{Y}_{s+1}(\tau)}), \tag{6.1}$$

where $\Phi : \mathbb{R}_{>0} \times \mathbb{R}_{>0} \to \mathbb{R}_{\geq 1}$ is continuous and nondecreasing with respect to its arguments.

6.2 Given the "original" data $d = \{u_0,\, u_1,\, f\} \in \mathcal{D}_s$, we fix arbitrary $\eta \in]0,1[$ and choose data $d^0 = \{u_0^0,\, u_1^0,\, f^0\} \in \mathcal{D}_{s+4}$ such that

$$\|d - d^0\|_{\mathcal{D}_s} \leq \eta. \tag{6.2}$$

By Theorem 2.4, there is $\varepsilon_0 \in]0,1]$ such that for all $\varepsilon \in]0,\varepsilon_0]$, problem (1.1)–(1.2), with data d^0, has a global solution $u^0 \in \mathcal{X}_{s+4}(T)$. Correspondingly, we set

$$M_0 := \max\{\|d^0\|_{\mathcal{D}_s},\, |||u^0|||_{\mathcal{Y}_{s+4}(T)},\, 1\}, \tag{6.3}$$

and proceed to construct a sequence $(u^j)_{j\geq 0}$ of global solutions u^j to (1.1)–(1.2), such that for all $j \geq 0$ and $\varepsilon \in]0,\varepsilon_0]$,

$$u^j \in \mathcal{X}_{s+4}(T) \quad \text{and} \quad |||\, u^j|||_{\mathcal{Y}_s(T)} \leq 2M_0. \tag{6.4}$$

Note that once each such solution u^j is constructed satisfying (6.4), then, by Theorem 2.2,

$$|||u^j|||_{\mathcal{Y}_{s+1}(T)} \leq \|d^j\|_{\mathcal{D}_{s+1}} \varphi(|||u^j|||_{\mathcal{Y}_s(T)}) \leq \|d^j\|_{\mathcal{D}_{s+1}} \varphi(2M_0) =: \omega_j; \tag{6.5}$$

in general, $\omega_j \to +\infty$ as $j \to +\infty$. Fix now $\gamma \in]0,1[$ such that

$$\frac{\gamma}{1-\gamma} \leq \eta < 1 \leq M_0. \tag{6.6}$$

Proceeding recursively, at each step $j \geq 1$ we choose data d^j, depending on the previous choices $d_0, \cdots, d^{j-1}$, such that

$$\|d^j - d^{j-1}\|_{\mathcal{D}_s} \leq \gamma^j (\Phi(4M_0, \omega_{j-1}))^{-1}, \tag{6.7}$$

where Φ and ω_{j-1} are as in (6.1) and (6.5). We claim that for all $\varepsilon \in]0, \varepsilon_0]$, the corresponding local solution u^j can be extended to all of $[0,T]$ so as to satisfy (6.4).

6.3 We prove this claim by induction. For $j = 0$, the solution u^0 satisfies (6.4) by construction, since

$$|||u^0|||_{\mathcal{Y}_s(T)} \leq |||u^0|||_{\mathcal{Y}_{s+4}(T)} \leq M_0. \tag{6.8}$$

Thus, assume that, for $0 \leq j \leq r$, we have solutions u^j, corresponding to data d^j, satisfying respectively (6.4) and (6.7). Choose data d^{r+1} satisfying (6.7) ($j = r+1$); then, by (6.7), (6.6) and (6.3), since $\Phi \geq 1$,

$$\begin{aligned} \|d^{r+1}\|_{\mathcal{D}_s} &\leq \sum_{j=0}^{r} \|d^{j+1} - d^j\|_{\mathcal{D}_s} + \|d^0\|_{\mathcal{D}_s} \\ &\leq \sum_{j=0}^{r} \gamma^{j+1} + \|d^0\|_{\mathcal{D}_s} \leq \frac{\gamma}{1-\gamma} + M_0 \leq 2M_0. \end{aligned} \tag{6.9}$$

Let T_ε^{r+1} denote the life-span of the solution u^{r+1}, as defined in Subsection 2.3.2; by Theorem 2.4, $T_\varepsilon^{r+1} = T$ if ε does not exceed some value $\varepsilon_{r+1} \in]0,1]$. We want to show that $T_\varepsilon^{r+1} = T$ also for $\varepsilon_{r+1} < \varepsilon \leq \varepsilon_0$; by the regularity Theorem 2.2, this would follow from the estimate

$$|||u^{r+1}|||_{\mathcal{Y}_s(t)} \leq 2M_0, \quad 0 \leq t < T_\varepsilon^{r+1}, \tag{6.10}$$

which clearly implies (6.4) for $j = r+1$ as well.

6.4 We prove (6.10) by contradiction; thus, we assume that either $T_\varepsilon^{r+1} < T$, so that, as in (5.1),

$$\limsup_{t \to T_\varepsilon^{r+1}} |||u^{r+1}|||_{\mathcal{Y}_s(t)} = +\infty, \tag{6.11}$$

or $T_\varepsilon^{r+1} = T$, but

$$|||u^{r+1}|||_{\mathcal{Y}_s(T)} > 2M_0. \tag{6.12}$$

In the first case, there would be $\theta \in]0, T_\varepsilon^{r+1}[$ such that

$$|||u^{r+1}|||_{\mathcal{Y}_s(\theta)} = 4M_0. \tag{6.13}$$

By Theorem 6.1, with $\widetilde{u} = u^j$, $\widetilde{\tau} = T$, $u = u^{j+1}$, $\tau = \theta < T$, $0 \le j \le r$,

$$\begin{aligned}
|||u^{r+1}|||_{\mathcal{Y}_s(\theta)} &\le \sum_{j=0}^{r} |||u^{j+1} - u^j\,|||_{\mathcal{Y}_s(\theta)} + |||\,u^0\,|||_{\mathcal{Y}_s(t)} \\
&\le \sum_{j=0}^{r} \|d^{j+1} - d^j\|_{\mathcal{D}_s} \Phi(|||u^{j+1}|||_{\mathcal{Y}_s(\theta)}, |||u^j|||_{\mathcal{Y}_{s+1}(T)}) + |||u^0\,|||_{\mathcal{Y}_s(T)}\,. \qquad (6.14)
\end{aligned}$$

By (6.13) and the induction assumption (6.4) ($0 \le j \le r$), and then by (6.5)–(6.7) and (6.3), we proceed from (6.14) with

$$\begin{aligned}
|||u^{r+1}|||_{\mathcal{Y}_s(\theta)} &\le \sum_{j=0}^{r} \|d^{j+1} - d^j\|_{\mathcal{D}_s} \Phi(4M_0, \omega_j) + |||u^0\,|||_{\mathcal{Y}_s(T)} \\
&= \sum_{j=0}^{r} \gamma^{j+1} + |||u^0\,|||_{\mathcal{Y}_s(T)} \\
&\le \frac{\gamma}{1-\gamma} + M_0 \le 2M_0, \qquad (6.15)
\end{aligned}$$

which contradicts (6.13). The other possibility is (6.12); but then, since by (6.9),

$$|||u^{r+1}|||_{\mathcal{Y}_s(0)} \le \|u_0^{r+1}\|_{s+1} + \|u_1^{r+1}\|_s \le \|d^{r+1}\|_{\mathcal{D}_s} \le 2M_0, \qquad (6.16)$$

there still would be $\theta \in\,]0, T[$ such that

$$2M_0 < |||u^{r+1}|||_{\mathcal{Y}_s(\theta)} \le 4M_0. \qquad (6.17)$$

But then, we can repeat exactly the same estimates, which show that, in fact,

$$|||u^{r+1}|||_{\mathcal{Y}_s(\theta)} \le 2M_0, \qquad (6.18)$$

contradicting (6.17). In conclusion, (6.10) holds. It follows that (6.4) does hold, for all $\varepsilon \in\,]0, \varepsilon_0]$ and $j \ge 0$.

6.5 Because of (6.4), the sequence $(u^j)_{j\ge 0}$ is in a bounded set of $\mathcal{Y}_s(T)$. Thus, there are $u \in \mathcal{Y}_s(T)$ and a subsequence, still denoted by $(u^j)_{j\ge 0}$, such that

$$u^j \to u \quad \text{weakly in } \mathcal{Y}_s(T). \qquad (6.19)$$

It is then straightforward to show (see e.g. [9]) that $(u^j)_{j\ge 0}$ is a Cauchy sequence in $\mathcal{Y}_0(T)$; thus, by interpolation, $(u^j)_{j\ge 0}$ is also a Cauchy sequence in $\mathcal{Y}_k(T)$, $0 \le k \le s-1$. Since $s - 1 > \frac{N}{2}$, it follows that the convergence $\partial_t^k u^j \to \partial_t^k u$, $\partial_x^k u^j \to \partial_x^k u$, $k = 0, 1, 2$, is uniform in $[0, T] \times \mathbb{R}^N$. This allows us to deduce that u is the desired solution of equation (1.1), provided that $f^j \to f$ in $Z_s(T)$. To show this, as well as that u takes on the correct data, it is sufficient to note that, by (6.9), (6.6) and (6.2),

$$\begin{aligned}
\|d^{r+1} - d\|_{\mathcal{D}_s} &\le \sum_{j=0}^{r} \|d^{j+1} - d^j\|_{\mathcal{D}_s} + \|d^0 - d\|_{\mathcal{D}_s} \\
&\le \frac{\gamma}{1-\gamma} + \|d^0 - d\|_{\mathcal{D}_s} \le 2\eta; \qquad (6.20)
\end{aligned}$$

since η is arbitrary, it follows that $d^j \to d$ in $\mathcal{D}_s$, as desired. This yields a proof of Theorem 2.3 in the cases $0 \leq m \leq 3$, with $\varepsilon_m = \varepsilon_0$, as determined above.

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