Chin. Ann. Math.
30B(4), 2009, 421–426
DOI: 10.1007/s11401-008-0283-7

Chinese Annals of Mathematics, Series B

Some Results on the Uniqueness of Meromorphic Mappings***

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Abstract The authors study the problem of uniqueness of meromorphic mappings and obtain two results which partially improve two theorems of Yan and Chen in 2006.

Keywords Meromorphic mapping, Uniqueness theorem, Nevanlinna theory
2000 MR Subject Classification 32H30, 30D35

1 Introduction

Nevanlinna's five-value theorem (see [4]) says that if two meromorphic functions share five values ignoring multiplicity, then these two functions must be identical.

Yang [7] observed that one can weaken the assumption of sharing five values to "partially" sharing five values in Nevanlinna's five-value theorem. We say that a meromorphic function $f(z)$ partially shares a value a with a meromorphic function $g(z)$ if $\overline{E}(a,f) \subseteq \overline{E}(a,g)$, where $\overline{E}(a,h) = \{z \mid h(z) = a\}$ for a meromorphic function $h(z)$. In fact, he proved the following theorem.

Theorem 1.1 *Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions and a_j $(1 \le j \le 5)$ be five distinct values. If*

$$\overline{E}(a_j,f) \subseteq \overline{E}(a_j,g), \quad 1 \le j \le 5 \quad \textit{and} \quad \liminf_{r\to\infty} \sum_{j=1}^{5} N^1_{f-a_j}(r)\Big(\sum_{j=1}^{5} N^1_{g-a_j}(r)\Big)^{-1} > \frac{1}{2},$$

then $f \equiv g$.

In 2006, Yan and Chen [6] generalized the above result to meromorphic mapping, under the inclusive relation of the sets of zero points between two meromorphic mappings. Suppose that $f(z)$ is a meromorphic mapping and H is a hyperplane of $P^N(\mathbb{C})$. Similarly, let $\overline{E}(H,f)$ denote the zero set of (f,H). They proved the following theorem.

Theorem 1.2 *Let $f(z)$ and $g(z)$ be two linearly non-degenerate meromorphic mappings of $\mathbb{C}^n$ into $P^N(\mathbb{C})$ and H_j $(1 \le j \le q)$ be q hyperplanes in general position such that* $\dim f^{-1}(H_i \cap H_j) \le n-2$ *for $i \ne j$. Assume*

$$\overline{E}(H_j,f) \subseteq \overline{E}(H_j,g), \quad 1 \le j \le q$$

Manuscript received July 18, 2008. Published online June 8, 2009.
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***Project supported by the National Natural Science Foundation of China (No. 10371065) and the Nature and Science Fund of China-Russial Federation Basical Research (No. 10911120056).

and $f = g$ *on* $\bigcup_{j=1}^{q} f^{-1}(H_j)$. *If* $q = 3N+2$ *and*

$$\liminf_{r\to\infty} \sum_{j=1}^{3N+2} N^1_{(f,H_j)}(r)\Big(\sum_{j=1}^{3N+2} N^1_{(g,H_j)}(r)\Big)^{-1} > \frac{N}{N+1},$$

then $f \equiv g$.

We will ask whether $\liminf_{r\to\infty} \sum_{j=1}^{3N+2} N^1_{f-a_j}(r)\Big(\sum_{j=1}^{3N+2} N^1_{g-a_j}(r)\Big)^{-1} > \frac{N}{N+1}$ is sharp or not in Theorem 1.2. In this paper, the authors discuss the problem and obtain the following result.

Theorem 1.3 *Let* $f(z)$ *and* $g(z)$ *be two linearly non-degenerate meromorphic mappings of* $\mathbb{C}^n$ *into* $P^N(\mathbb{C})$ *and* H_j $(1 \le j \le q)$ *be q hyperplanes in general position such that* $\dim f^{-1}(H_i \cap H_j) \le n-2$ *for* $i \ne j$. *Assume*

$$\overline{E}(H_j, f) \subseteq \overline{E}(H_j, g), \quad 1 \le j \le q$$

and $f = g$ *on* $\bigcup_{j=1}^{q} f^{-1}(H_j)$. *If* $q \ge 3N+2$, $N \ge 2$ *and*

$$\liminf_{r\to\infty} \sum_{j=1}^{q} N^1_{(f,H_j)}(r)\Big(\sum_{j=1}^{q} N^1_{(g,H_j)}(r)\Big)^{-1} \ge \frac{N}{q-2N-1},$$

then $f \equiv g$.

Remark 1.1 If $\liminf_{r\to\infty} \sum_{j=1}^{q} N^1_{(f,H_j)}(r)\Big(\sum_{j=1}^{q} N^1_{(g,H_j)}(r)\Big)^{-1} > \frac{N}{q-2N-1}$ and $N = 1$ in Theorem 1.3, from the proof of Theorem 1.3, we can see that the conclusion still holds.

Remark 1.2 It is easy to see that Theorem 1.3 partially improves Theorem 1.2.

The second main theorem for moving targets is stated as follows, which was given by Thai and Quang [5].

Theorem 1.4 *Let* $f(z) : \mathbb{C}^n \to P^N(\mathbb{C})$ *be a meromorphic mapping. Let* H_i $(1 \le i \le q)$ *be q moving hyperplanes in general position. Assume that* (f, H_i) *is free for every* H_i $(1 \le i \le q)$. *If* $q \ge 2N+1$, *then*

$$\Big\| \frac{q}{2N+1} T(r,f) \le \sum_{i=1}^{q} N^N_{(f,H_i)}(r) + o(T(r,f)) + O\Big(\max_{1\le i\le q} T(r,a_i)\Big).$$

As usual, by the notation "$\|P$" we mean that the assertion P holds for all $r \in [0,\infty)$ excluding a Borel subset E of the interval $[0,\infty)$ with $\int_E \mathrm{d}r < \infty$.

In [6], Yan and Chen raised a problem of how to generalize Theorem 1.2 to the case of moving targets. Later, they answered the question in [2]. As a matter of fact, we can describe their theorem as follows.

Theorem 1.5 *Let* $f(z)$ *and* $g(z)$ *be two meromorphic mappings of* $\mathbb{C}^n$ *into* $P^N(\mathbb{C})$ *and* H_j $(1 \le j \le q)$ *be q "small" (with respect to f) moving targets in general position such that* $\dim f^{-1}(H_i \cap H_j) \le n-2$ *for* $i \ne j$. *Assume that*

$$\overline{E}(H_j, f) \subseteq \overline{E}(H_j, g), \quad 1 \le j \le q$$

and $f = g$ *on* $\bigcup_{j=1}^{q} f^{-1}(H_j)$*. If* $q = 2N(2N+1)+1$ *and*

$$\liminf_{r\to\infty} \sum_{j=1}^{q} N^1_{(f,H_j)}(r)\Big(\sum_{j=1}^{q} N^1_{(g,H_j)}(r)\Big)^{-1} > \frac{(2N+1)N}{N(2N+1)+1},$$

then $f \equiv g$*.*

Using Theorem 1.4 and the similar method in proving Theorem 1.3, we can obtain the following result, which partially improves Theorem 1.5.

Theorem 1.6 *Let* $f(z)$ *and* $g(z)$ *be two meromorphic mappings of* $\mathbb{C}^n$ *into* $P^N(\mathbb{C})$ *and* H_j $(1 \le j \le q)$ *be* q *"small"* (*with respect to* f) *moving targets in general position such that* $\dim f^{-1}(H_i \cap H_j) \le n-2$ *for* $i \ne j$*. Assume*

$$\overline{E}(H_j, f) \subseteq \overline{E}(H_j, g), \quad 1 \le j \le q$$

and $f = g$ *on* $\bigcup_{j=1}^{q} f^{-1}(H_j)$*. If* $q \ge 2N(2N+1)+1$, $N \ge 2$ *and*

$$\liminf_{r\to\infty} \sum_{j=1}^{q} N^1_{(f,H_j)}(r)\Big(\sum_{j=1}^{q} N^1_{(g,H_j)}(r)\Big)^{-1} \ge \frac{(2N+1)N}{q-(2N+1)N},$$

then $f \equiv g$*.*

2 Preliminaries and Some Lemmas

We set $\|z\| = (|z_1|^2 + \cdots + |z_n|^2)^{\frac{1}{2}}$ for $z = (z_1, \cdots, z_n)$ and define

$$B(r) = \{z \in \mathbb{C}^n : \|z\| < r\}, \quad S(r) = \{z \in \mathbb{C}^n : \|z\| = r\}, \quad 0 < r < \infty,$$
$$\upsilon_{n-1}(z) = (\mathrm{dd}^c\|z\|^2)^{n-1}, \qquad \sigma_n(z) = \mathrm{d}^c \log\|z\|^2 \wedge (\mathrm{dd}^c \log\|z\|^2)^{n-1},$$

on $\mathbb{C}^n \setminus \{0\}$.

Let f be a nonconstant meromorphic mapping of $\mathbb{C}^n$ into $P^N(\mathbb{C})$ and k be a positive integer. We take the holomorphic functions $f_0, \cdots, f_N$ on $\mathbb{C}^n$ such that $\mathcal{I}_f = \{z \in \mathbb{C}^n : f_0(z) = \cdots f_N(z) = 0\}$ is of dimension at most $n-2$ and call $f = \{f_0, \cdots, f_N\}$ on $\mathbb{C}^n$ a reduced representation of f. The characteristic function of f is defined as

$$T(r, f) = \int_{S(r)} \log\|f\|\sigma_n - \int_{S(1)} \log\|f\|\sigma_n.$$

Note that $T(r, f)$ is independent of the choice of the reduced representation of f.

We now define the counting functions. For a divisor ν on $\mathbb{C}^n$ and for positive k, M (or $M = \infty$), we define the counting functions of ν as follows. Set

$$\nu^M(z) = \min\{M, \nu(z)\}, \quad \nu^M_{\le k} = \begin{cases} 0, & \text{if } \nu(z) > k, \\ \nu^M(z), & \text{if } \nu(z) \le k, \end{cases} \quad \nu^M_{>k} = \begin{cases} \nu^M(z), & \text{if } \nu(z) > k, \\ 0, & \text{if } \nu(z) \le k. \end{cases}$$

We define $n(t)$ by

$$n(t) = \begin{cases} \displaystyle\int_{|\nu| \cap B(t)} \nu(z)\upsilon_{n-1}, & \text{if } n \ge 2, \\ \displaystyle\sum_{|z| \le t} \nu(z), & \text{if } n = 1. \end{cases}$$

Similarly, we define $n^M(t)$, $n^M_{\leq k}(t)$, $n^M_{>k}(t)$. Define

$$N(r,\nu)=\int_1^r \frac{n(t)}{t^{2n-1}}\mathrm{d}t, \quad 1<r<\infty.$$

Similarly, we define $N(r,\nu^M)$, $N(r,\nu^M_{\leq k})$, $N(r,\nu^M_{>k})$ and denote them by $N^M(r,\nu)$, $N^M_{\leq k}(r,\nu)$, $N^M_{>k}(r,\nu)$, respectively.

Let $\phi:\mathbb{C}^n\to\mathbb{C}$ be a meromorphic function. Define

$$N_\phi(r)=N(r,\nu_\phi), \quad N^M_\phi(r)=N^M(r,\nu_\phi), \quad N^M_{\phi,\leq k}(r)=N^M_{\leq k}(r,\nu_\phi), \quad N^M_{\phi,>k}(r)=N^M_{>k}(r,\nu_\phi).$$

For brevity we will omit the superscript M if $M=\infty$.

For a hyperplane, we define the proximity function of H by

$$m_{f,H}(r)=\int_{S(r)}\log\frac{\|f\|\|H\|}{|(f,H)|}\sigma_n-\int_{S(1)}\log\frac{\|f\|\|H\|}{|(f,H)|}\sigma_n.$$

Now, take two distinct hyperplanes H_j $(j=1,2)$ and consider a meromorphic function

$$F_f^{H_1,H_2}=\frac{(f,H_1)}{(f,H_2)}.$$

We have

Lemma 2.1 (see [3]) $T(r,F_f^{H_1,H_2})\leq T(r,f)+O(1)$.

Lemma 2.2 (see [3]) *Let $f:\mathbb{C}^n\to P^N(\mathbb{C})$ be a linearly non-degenerate meromorphic mapping and $H_1,\cdots,H_q$ be q hyperplanes in general position in $P^N(\mathbb{C})$. Then*

$$\|(q-N-1)T(r,f)\leq\sum_{i=1}^q N^N_{(f,H_i)}(r)+S(r,f).$$

3 Proof of Theorem 1.3

For brevity we denote $T(r,f)+T(r,g)$ and $S(r,f)+S(r,g)$ by $T(r)$ and $S(r)$, respectively.

Assume $f\not\equiv g$. For some $1\leq j\leq q$, there exists a $c\in\mathbb{C}^{N+1}\backslash\{0\}$ such that $F_f^{H_j,c}-F_g^{H_j,c}\not\equiv 0$. Since $f=g$ on $\bigcup\limits_{i=1}^q f^{-1}(H_i)$, we have $F_f^{H_j,c}-F_g^{H_j,c}(z)=0$ on $\bigcup\limits_{i=1}^q f^{-1}(H_i)$. Then by Lemma 2.1, we have

$$\sum_{i=1}^q N^1_{(f,H_i)}(r)\leq N^1_{F_f^{H_j,c}-F_g^{H_j,c}}(r)\leq T(r,F_f^{H_j,c}-F_g^{H_j,c})+O(1)\leq T(r)+S(r). \tag{3.1}$$

By Lemma 2.2, we have

$$\begin{aligned}(q-N-1)T(r,f)&\leq\sum_{i=1}^q N^N_{(f,H_i)}(r)+S(r)\\&\leq N\sum_{i=1}^q N^1_{(f,H_i)}(r)-\sum_{i=1}^q N^1_{(f,H_i)\leq N-1}(r)+S(r),\end{aligned}\tag{3.2}$$

$$\begin{aligned}(q-N-1)T(r,g)&\leq\sum_{i=1}^q N^N_{(g,H_i)}(r)+S(r)\\&\leq N\sum_{i=1}^q N^1_{(g,H_i)}(r)-\sum_{i=1}^q N^1_{(g,H_i)\leq N-1}(r)+S(r).\end{aligned}\tag{3.3}$$

Noting $\liminf\limits_{r\to\infty} \sum\limits_{i=1}^{q} N^1_{(f,H_i)}(r)\Big(\sum\limits_{i=1}^{q} N^1_{(g,H_j)}(r)\Big)^{-1} \geq \frac{N}{q-2N-1}$, we derive

$$\limsup_{r\to\infty} \sum_{i=1}^{q} N^1_{(g,H_i)}(r)\Big(\sum_{i=1}^{q} N^1_{(f,H_j)}(r)\Big)^{-1} \leq \frac{q-2N-1}{N}.$$

For any $\varepsilon > 0$, we have (for r large enough)

$$\sum_{i=1}^{q} N^1_{(g,H_i)}(r)\Big(\sum_{i=1}^{q} N^1_{(f,H_j)}(r)\Big)^{-1} \leq \frac{q-2N-1}{N} + \varepsilon.$$

From (3.1)–(3.3), we derive

$$\begin{aligned}
&(q-N-1)T(r)\\
&\leq N\sum_{i=1}^{q} N^1_{(f,H_i)}(r) - \sum_{i=1}^{q} N^1_{(f,H_i)\leq N-1}(r) + N\sum_{i=1}^{q} N^1_{(g,H_i)}(r) - \sum_{i=1}^{q} N^1_{(g,H_i)\leq N-1}(r) + S(r)\\
&\leq N\Big(1+\frac{q-2N-1}{N}+\varepsilon\Big)\sum_{i=1}^{q} N^1_{(f,H_i)}(r) - \sum_{i=1}^{q} N^1_{(f,H_i)\leq N-1}(r) - \sum_{i=1}^{q} N^1_{(g,H_i)\leq N-1}(r) + S(r)\\
&\leq N\Big(1+\frac{q-2N-1}{N}+\varepsilon\Big)T(r) - \sum_{i=1}^{q} N^1_{(f,H_i)\leq N-1}(r) - \sum_{i=1}^{q} N^1_{(g,H_i)\leq N-1}(r) + S(r). \qquad (3.4)
\end{aligned}$$

By (3.4), we get

$$\sum_{i=1}^{q} N^1_{(f,H_i)\leq N-1}(r) + \sum_{i=1}^{q} N^1_{(g,H_i)\leq N-1}(r) \leq N\varepsilon T(r) + S(r), \qquad (3.5)$$

$$\sum_{i=1}^{q} N^1_{(f,H_i)}(r) \geq \frac{q-N-1}{q-N-1+N\varepsilon}T(r) + S(r). \qquad (3.6)$$

We know that for each $1 \leq i \leq q$, there exist $1 \leq j \neq i \leq q$ such that

$$P_i = F_f^{H_i,H_j} - F_f^{H_i,H_j} \not\equiv 0. \qquad (3.7)$$

Put

$$\eta_i(z) = \begin{cases} 1, & \text{if } \min\{\nu_{(f,H_i)}, \nu_{(g,H_i)}\} \geq N,\\ 0, & \text{if } \min\{\nu_{(f,H_i)}, \nu_{(g,H_i)}\} \leq N-1. \end{cases}$$

Then $\nu_{P_i}(z) \geq N\eta_i(z) + \sum\limits_{\substack{l=1\\ l\neq i,j}}^{q} \nu^1_{(f,H_l)}(r)$ holds outside a finite union of analytic sets of dimension $\leq n-2$. Hence

$$N_{P_i}(r) \geq NN(r,\eta_i) + \sum_{\substack{l=1\\ l\neq i,j}}^{q} N^1_{(f,H_l)}(r). \qquad (3.8)$$

On the other hand, we have

$$\eta_i(z) \geq \nu^1_{(f,H_i)>N-1}(z) - \nu^1_{(g,H_i)\leq N-1}(z) = \nu^1_{(f,H_i)}(z) - \nu^1_{(f,H_i)\leq N-1}(z) - \nu^1_{(g,H_i)\leq N-1}(z).$$

Hence

$$N(r,\eta_i) \geq N^1_{(f,H_i)}(z) - N^1_{(f,H_i)\leq N-1}(z) - N^1_{(g,H_i)\leq N-1}(z). \qquad (3.9)$$

From (3.7), we have

$$\begin{aligned} N_{P_i}(r) &\le T(r,P_i) + O(1) \le N_{P_i^{-1}}(r) + m(r,P_i) + O(1) \\ &\le N_{P_i^{-1}}(r) + T(r,f) - N_{(f,H_j)}(r) + T(r,g) - N_{(g,H_j)}(r) + S(r). \end{aligned} \tag{3.10}$$

Note

$$N_{P_i^{-1}}(r) \le \max\{N_{(f,H_j)}(r), N_{(g,H_j)}(r)\}.$$

Thus

$$N_{(f,H_j)}(r) + N_{(g,H_j)}(r) - N_{P_i^{-1}}(r) \ge N^1_{(f,H_j)}(r). \tag{3.11}$$

Substituting (3.8), (3.9) and (3.11) into (3.10), we derive

$$\|(N-1)N^1_{(f,H_i)}(r)+\sum_{l=1}^{q} N^1_{(f,H_l)}(r) \le T(r)+N(N^1_{(f,H_i)\le N-1}(z)+N^1_{(g,H_i)\le N-1}(z))+S(r). \tag{3.12}$$

From (3.5), (3.6) and (3.12), we obtain

$$\|(N-1)N^1_{(f,H_i)}(r) \le \frac{N\varepsilon}{q-N-1+N\varepsilon}T(r) + N^2\varepsilon T(r) + S(r).$$

Thus

$$N^1_{(f,H_i)}(r) \le M_i\varepsilon T(r) + S(r), \quad 1 \le i \le q, \tag{3.13}$$

where M_i is a positive number.

Putting (3.13) into (3.6), we can derive that $T(r) = S(r)$, which is a contradiction. Thus, we complete the proof of Theorem 1.3.

We can prove Theorem 1.6 in a similar way. We omit it here.

4 Questions

There are several questions related to the above results.

Question 4.1 Is there any assertion similar to Theorem 1.3 and Theorem 1.6 for $N = 1$?

Question 4.2 Is there any assertion similar to Theorem 1.3 and Theorem 1.6 for $q = 3N+1$ and $q = 2N(2N+1)$, respectively?

Acknowledgement The authors are grateful to the referee for giving very helpful suggestions to the manuscript.

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