Chin. Ann. Math.
29B(6), 2008, 611–622
DOI: 10.1007/s11401-008-0009-x

Chinese Annals of Mathematics, Series B

Delta Shocks and Vacuum States in Vanishing Pressure Limits of Solutions to the Relativistic Euler Equations***

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Abstract The Riemann problems for the Euler system of conservation laws of energy and momentum in special relativity as pressure vanishes are considered. The Riemann solutions for the pressureless relativistic Euler equations are obtained constructively. There are two kinds of solutions, the one involves delta shock wave and the other involves vacuum. The authors prove that these two kinds of solutions are the limits of the solutions as pressure vanishes in the Euler system of conservation laws of energy and momentum in special relativity.

Keywords Relativistic Euler equations in special relativity, Pressureless relativistic Euler equations, Delta shock waves, Vacuum, Vanishing pressure limits
2000 MR Subject Classification 35L65, 35L67, 76L05, 76N10

1 Introduction

Consider the Euler system of conservation laws of energy and momentum in special relativity

$$\begin{cases} \partial_t\Big(\dfrac{p+\rho c^2}{c^2}\dfrac{v^2}{c^2-v^2}+\rho\Big)+\partial_x\Big((p+\rho c^2)\dfrac{v}{c^2-v^2}\Big)=0,\\ \partial_t\Big((p+\rho c^2)\dfrac{v}{c^2-v^2}\Big)+\partial_x\Big((p+\rho c^2)\dfrac{v^2}{c^2-v^2}+p\Big)=0, \end{cases} \tag{1.1}$$

where $\rho(t,x)$, $p(t,x)$ and $v(t,x)$ represent the proper mass-energy density, the pressure, and the particle speed respectively, and the constant c is the speed of light. The equation of state is

$$p=p(\rho),$$

where $p(\rho)$ is a smooth function of ρ and satisfies $p(0)=0$, and for $\rho>0$,

$$p(\rho)>0,\quad p'(\rho)>0\quad \text{(hyperbolicity)}$$

and

$$p''(\rho)\geq 0.$$

Manuscript received January 4, 2008. Revised May 12, 2008. Published online November 5, 2008.
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***Project supported by the National Natural Science Foundation of China (No. 10671120) and the Shanghai Leading Academic Discipline Project (No. J50101).

For a perfect fluid,

$$p(\rho) = \kappa^2 \rho^\gamma, \quad \gamma \geq 1,$$

where $\gamma = 1$ models an isothermal gas and $\gamma > 1$ a polytropic gas, and κ is the speed of sound satisfying $\kappa < c$.

System (1.1) models the dynamics of plane waves in special relativistic fluids (see [1, 3, 5, 11, 14, 16–26, 29–32]) in a two-dimensional Minkowski space-time (x^0, x^1):

$$\mathrm{div} T = 0,$$

with the stress-energy tensor for the fluid:

$$T^{ij} = (p + \rho c^2) u^i u^j + p \eta^{ij},$$

where all indices run from 0 to 1 with $x^0 = ct$, and

$$\eta^{ij} = \eta_{ij} = \mathrm{diag}(-1, 1)$$

denotes the flat Minkowski metric, u the 2-velocity of the fluid particle, and ρ the mass-energy density of the fluid measured in units of mass in a reference frame moving with the fluid particle; and $p = p(\rho, S)$ is the pressure with the specific entropy S.

In a Lorents transformation, the barred frame $(\overline{t}, \overline{x})$ moves with velocity τ measured in the unbarred frame (t, x), and if v denotes the velocity of a particle measured in the unbarred frame, and $\overline{v}$ the velocity of the same particle measured in the barred frame, then $v = \frac{\tau + \overline{v}}{1 + \frac{\tau \overline{v}}{c^2}}$. It is not difficult to check that system (1.1) is invariant under the Lorents transformation.

Formally, system (1.1) in the Newtonian limit reduces to the classical isentropic Euler equations for compressible fluids (see [31]):

$$\begin{cases} \partial_t \rho + \partial_x (\rho v) = 0, \\ \partial_t (\rho v) + \partial_x (\rho v^2 + p(\rho)) = 0, \end{cases} \tag{1.2}$$

which was studied systematically in the last decades (see [2, 9, 12, 13, 27, 28]). Thus system (1.1) can be viewed as a relativistic version of (1.2).

In 1993, T. Smoller and B. Temple [29] obtained the existence of solutions with shocks with the equation of state $p = \kappa^2 \rho$ and the geometric properties of nonlinear wave curves. In 2001, C. H. Hsu, S. S. Lin and T. Makino [10] proved the existence of a weak solution for the system (1.1) by generalizing a method of DiPerna and G. Q. Chen under a mild hypothesis on the initial data. G. Q. Chen and Y. C. Li [4] established the uniqueness of Riemann solutions to the system (1.1) in the class of entropy solution in $L^\infty \cap BV_{\mathrm{loc}}$ with arbitrarily large oscillation, and then considered the uniqueness and stability of Riemann solutions with vacuum in the class of entropy solutions in L^∞ with large oscillation. In 2005, Y. C. Li et al [15] obtained the global existence of solutions to the isentropic relativistic Euler system using the Glimm scheme.

In 1999, using viscous vanishing method, W. C. Sheng and T. Zhang [27] obtained the Riemann solutions for the transportation equations of zero-pressure flow in gas dynamics constructively, in which delta shock wave and vacuum appeared. J. Q. Li [13] and G. Q. Chen and H. L. Liu [6, 7] presented the asymptotic properties of solution of the compressible Euler equations as pressure vanishes.

In this paper, we consider the Riemann problem for the relativistic Euler equations with zero-pressure and derive the behavior of the solution as κ drops to zero (i.e., temperature goes to zero) to the relativistic Euler equations in special relativity with the equation of state of the form

$$p(\rho) = \kappa^2 \rho. \tag{1.3}$$

For a polytropic gas $\gamma > 1$, one can obtain the same results in a similar way, by a bit more complicated calculations.

This paper is organized as follows. In Section 2, we display the Riemann solutions to the relativistic Euler equations (1.1). Then, in Section 3, we solve the Riemann problem for the relativistic Euler equations with zero-pressure, and construct the solutions with delta shock waves or vacuums. Finally, in Section 4, we develop the behavior of solutions to (1.1) as pressure vanishes.

2 Riemann Problems for the Relativistic Euler Equations

Consider the Riemann problem for system (1.1) with initial data:

$$(\rho, v)(0, x) = (\rho_\pm, v_\pm), \quad \pm x > 0 \tag{2.1}$$

with the equation of state $p = p(\rho)$, where $\rho_\pm, v_\pm$ are constants with $\rho_\pm > 0$.

We can easily obtain the eigenvalues of system (1.1) in the form

$$\lambda_1 = \frac{c^2(v - \sqrt{p'})}{c^2 - v\sqrt{p'}}, \quad \lambda_2 = \frac{c^2(v + \sqrt{p'})}{c^2 + v\sqrt{p'}} \tag{2.2}$$

and the corresponding right-eigenvectors

$$r_j(\rho, v) = \alpha_j(\rho, v)\Big(\frac{(-1)^j}{c^2 - v^2}, \frac{\sqrt{p'}}{p + \rho c^2}\Big)^\top, \quad j = 1, 2, \tag{2.3}$$

where

$$\alpha_j(\rho, v) = \frac{2(c^2 + (-1)^{j+1} v\sqrt{p'})^2 (p + \rho c^2)\sqrt{p'}}{c^2(\rho p'' + 2p') + pp'' - 2p'^2} > 0, \quad j = 1, 2. \tag{2.4}$$

Direct calculations give

$$\nabla\lambda_j(\rho, v) \cdot r_j(\rho, v) = 1, \quad j = 1, 2.$$

So both families of (1.1) are genuinely nonlinear.

Denote 1- (2-) rarefaction wave curve by R_1 (R_2), which means a curve consisting of all the possible states (ρ, v) that can be connected on the right to the left state (ρ_l, v_l) by 1- (2-) rarefaction wave.

Lemma 2.1 (see [15, 29]) *The* 1- (*and* 2-) *rarefaction wave curves are given by*

$$\begin{aligned} R_1: &\quad \frac{c}{2}\ln\frac{c+v}{c-v} + c^2\int_0^\rho \frac{\sqrt{p'(s)}}{p(s) + c^2 s}\,\mathrm{d}s = \text{const.}, \\ R_2: &\quad \frac{c}{2}\ln\frac{c+v}{c-v} - c^2\int_0^\rho \frac{\sqrt{p'(s)}}{p(s) + c^2 s}\,\mathrm{d}s = \text{const.} \end{aligned} \tag{2.5}$$

In addition, $\frac{\mathrm{d}v}{\mathrm{d}\rho} < 0$ (> 0) *on* R_1 (R_2).

Next we discuss the shock curves. Given a left state (ρ_l, v_l), we consider all the states (ρ, v) that can be connected to (ρ_l, v_l) on the right by a shock wave curve. The Rankine-Hugoniot condition gives

$$\begin{aligned}\sigma\Big[\frac{(p+\rho c^2)v}{c^2-v^2}\Big]&=\Big[\frac{(p+\rho c^2)v^2}{c^2-v^2}+p\Big],\\ \sigma\Big[\frac{(p+\rho c^2)v^2}{c^2(c^2-v^2)}+\rho\Big]&=\Big[\frac{(p+\rho c^2)v}{c^2-v^2}\Big],\end{aligned} \tag{2.6}$$

where $[s]=s(\sigma-0)-s(\sigma+0)$. We give the following two lemmas without proof (see [15, 29] for more details).

Lemma 2.2 *The shock curves are given by*

$$\frac{v-v_l}{c^2-v_l^2}=-\frac{\sqrt{\Theta(\rho,\rho_l)}}{1-v_l\sqrt{\Theta(\rho,\rho_l)}},\quad v<v_l, \tag{2.7}$$

with $\rho>\rho_l$ for a 1-shock curve S_1, and $\rho<\rho_l$ for a 2-shock curve S_2, where

$$\Theta(\rho,\rho_l)=\frac{(p-p_l)(\rho-\rho_l)}{(p+\rho_l c^2)(p_l+\rho c^2)}$$

with $p_l=p(\rho_l)$. Furthermore, $\frac{\mathrm{d}v}{\mathrm{d}\rho}<0$ (>0) on S_1 (S_2).

Lemma 2.3 *Assume that (ρ_l,v_l) and (ρ,v) satisfy (2.7) for system (1.1) with equation of state (1.3). Then the following relations hold:*

$$\frac{\rho}{\rho_l}=1+\beta\pm\sqrt{\beta^2+2\beta}, \tag{2.8}$$

where

$$\beta=\frac{(\kappa^2+c^2)^2}{2\kappa^2}\frac{(v-v_l)^2}{(c^2-v^2)(c^2-v_l^2)}, \tag{2.9}$$

and the plus sign and minus sign correspond respectively to S_1 and S_2.

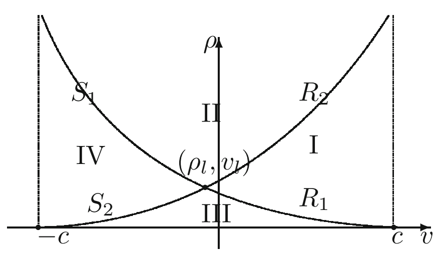

Figure 2.1

The solution of the Riemann problem for system (1.1), consisting of constant states, the rarefaction wave curves R_1,R_2 and the shock curves S_1,S_2, can be sketched in the (ρ,v) plane. The region $\Omega=\{(\rho,v)\mid \rho\ge 0,\ -c<v<c\}$ can be divided into four parts $\mathrm{I}(\rho_l,v_l)$, $\mathrm{II}(\rho_l,v_l)$, $\mathrm{III}(\rho_l,v_l)$ and $\mathrm{IV}(\rho_l,v_l)$ by the rarefaction wave curves, R_1 and R_2, and the shock wave curves, S_1 and S_2 (see Figure 2.1).

3 Riemann Problem for the Zero-Pressure Relativistic Euler Equations

Consider the relativistic zero-pressure Euler equations

$$\begin{cases}\partial_t\Big(\dfrac{\rho}{c^2-v^2}\Big)+\partial_x\Big(\dfrac{\rho v}{c^2-v^2}\Big)=0,\\ \partial_t\Big(\dfrac{\rho v}{c^2-v^2}\Big)+\partial_x\Big(\dfrac{\rho v^2}{c^2-v^2}\Big)=0,\end{cases} \tag{3.1}$$

with the initial condition

$$(\rho, v)(0, x) = (\rho_\pm, v_\pm), \quad \pm x > 0. \tag{3.2}$$

We seek the self-similar solution $(\rho(\xi), v(\xi))$ $(\xi = \frac{x}{t})$, due to the invariant of (3.1) and (3.2). Then (3.1) and (3.2) become

$$\begin{cases} -\xi\partial_\xi\Big(\dfrac{\rho}{c^2-v^2}\Big) + \partial_\xi\Big(\dfrac{\rho v}{c^2-v^2}\Big) = 0, \\ -\xi\partial_\xi\Big(\dfrac{\rho v}{c^2-v^2}\Big) + \partial_\xi\Big(\dfrac{\rho v^2}{c^2-v^2}\Big) = 0 \end{cases} \tag{3.3}$$

and

$$(\rho, v)(\pm\infty) = (\rho_\pm, v_\pm). \tag{3.4}$$

(3.1) can be rewritten as $AU_\xi = 0$ for smooth solutions, where

$$A = \begin{pmatrix} \dfrac{v-\xi}{c^2-v^2} & \dfrac{\rho(c^2+v^2-2v\xi)}{(c^2-v^2)^2} \\ \dfrac{v(v-\xi)}{c^2-v^2} & \dfrac{\rho(2c^2v-(c^2+v^2)\xi)}{(c^2-v^2)^2} \end{pmatrix}, \quad U = \begin{pmatrix} \rho \\ v \end{pmatrix}.$$

It provides that the smooth solutions of (3.1) involve general solutions

$$(\rho(\xi), v(\xi)) = \text{const.}, \quad \rho > 0$$

and singular solutions

$$\begin{cases} \rho = 0, \\ v = v(\xi), \end{cases}$$

where $v(\xi)$ is an arbitrary smooth function.

For a bounded discontinuity at $\xi = \sigma$, the Rankine-Hugoniot conditions read:

$$\begin{cases} -\sigma\Big[\dfrac{\rho}{c^2-v^2}\Big] + \Big[\dfrac{\rho v}{c^2-v^2}\Big] = 0, \\ -\sigma\Big[\dfrac{\rho v}{c^2-v^2}\Big] + \Big[\dfrac{\rho v^2}{c^2-v^2}\Big] = 0. \end{cases} \tag{3.5}$$

Solving (3.5), we obtain

$$\xi = \sigma = v_l (= \lambda_l) = v_r (= \lambda_r). \tag{3.6}$$

It is a slip line, denoted by J. Then two states (ρ_l, v_l) and (ρ_r, v_r) can be connected by J, if and only if $v_l = v_r$.

We now construct the solutions of the Riemann problem (3.3) and (3.4) with constant states, vacuum and contact discontinuity. In the case $v_- < v_+$, we can draw contact discontinuity curve $v = v_-$ from (ρ_-, v_-) in (ρ, v) plane, and it can be extended at $(\rho, v) = (0, v_-)$ with vacuum curve $\rho = 0$ $(v_- < v < c)$. Similarly, we can draw contact discontinuity curve $v = v_+$ from (ρ_+, v_+) and extend it at $(\rho, v) = (0, v_+)$ with vacuum curve $\rho = 0$ $(-c < v < v_+)$ (see Figure 3.1).

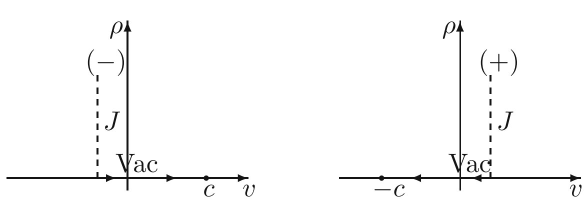

Figure 3.1

Then we can obtain the solution consisting of two contact discontinuities and a vacuum state besides two constant states $(\rho_\pm, v_\pm)$, which can be written in the form

$$(\rho(\xi), v(\xi)) = \begin{cases} (\rho_-, v_-), & -\infty < \xi \le v_-, \\ (0, v(\xi)), & v_- \le \xi \le v_+, \\ (\rho_+, v_+), & v_+ \le \xi < +\infty, \end{cases} \tag{3.7}$$

where $v(\xi)$ is an arbitrary smooth function satisfying $v(v_-) = v_-$ and $v(v_+) = v_+$.

In the case $v_- > v_+$, the characteristic lines from initial data will overlap in the domain Ω shown in Figure 3.2, which shows that singularity must happen in Ω. Consider a piecewise smooth solution of (3.3) and (3.4) of the form

$$(\rho, v)(t,x) = \begin{cases} (\rho_-, v_-)(t,x), & x < x(t), \\ (\omega(t)\delta(x - x(t)), v_\delta(t)), & x = x(t), \\ (\rho_+, v_+)(t,x), & x > x(t), \end{cases} \tag{3.8}$$

where $(\rho_\pm, v_\pm)(t,x) \in C^1$, $x(t), v_\delta(t) \in C^1$. $\omega(t)\delta(x - x(t))$ denotes the weighted delta function supported on a smooth curve $x = x(t)$.

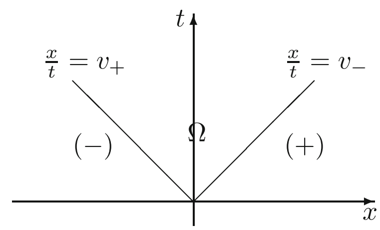

Figure 3.2

It can be verified that (3.8) is a delta shock wave of (3.3) if the generalized Rankine-Hugoniot condition

$$\begin{cases} \dfrac{\mathrm{d}x(t)}{\mathrm{d}t} = v_\delta(t), \\ \dfrac{\mathrm{d}}{\mathrm{d}t}\Big(\dfrac{\omega(t)}{c^2 - v_\delta^2(t)}\Big) = \Big[\dfrac{\rho}{c^2 - v^2}\Big] v_\delta(t) - \Big[\dfrac{\rho v}{c^2 - v^2}\Big], \\ \dfrac{\mathrm{d}}{\mathrm{d}t}\Big(\dfrac{\omega(t) v_\delta(t)}{c^2 - v_\delta^2(t)}\Big) = \Big[\dfrac{\rho v}{c^2 - v^2}\Big] v_\delta(t) - \Big[\dfrac{\rho v^2}{c^2 - v^2}\Big] \end{cases} \tag{3.9}$$

and the generalized entropy condition

$$\lambda_+ < \frac{\mathrm{d}x(t)}{\mathrm{d}t} < \lambda_-,$$

which means that in (x,t)-plane the characteristic lines on both sides of delta wave are incoming, are satisfied. By solving (3.9), we can obtain the solution of Riemann problem (3.3) and (3.4), which is of the form (3.8) with $x(t), v_\delta(t)$ and $\omega(t)$ satisfying

$$\begin{cases} x(t) = \dfrac{v_+\sqrt{\frac{\rho_+}{c^2 - v_+^2}} + v_-\sqrt{\frac{\rho_-}{c^2 - v_-^2}}}{\sqrt{\frac{\rho_+}{c^2 - v_+^2}} + \sqrt{\frac{\rho_-}{c^2 - v_-^2}}} t, \\ v_\delta(t) =: v_\delta = \dfrac{v_+\sqrt{\frac{\rho_+}{c^2 - v_+^2}} + v_-\sqrt{\frac{\rho_-}{c^2 - v_-^2}}}{\sqrt{\frac{\rho_+}{c^2 - v_+^2}} + \sqrt{\frac{\rho_-}{c^2 - v_-^2}}}, \\ \omega(t) = \sqrt{\dfrac{\rho_+}{c^2 - v_+^2} \dfrac{\rho_-}{c^2 - v_-^2}} (v_- - v_+)(c^2 - v_\delta^2) t. \end{cases} \tag{3.10}$$

The reason that we choose $v_\delta(t), x(t)$ and $\omega(t)$ as in (3.10) is that $\frac{\mathrm{d}x(t)}{\mathrm{d}t} = v_\delta(t)$ should be located in the region Ω (see Figure 3.2).

Above discussions immediately give the following results.

Theorem 3.1 *The Riemann problem for the zero-pressure relativistic Euler equations* (3.1) *and* (3.2) *can be solved constructively:*

(i) *When $v_- < v_+$, the solution is of the form in* (3.7)*, consisting of two contact discontinuities and a vacuum state besides two constant states $(\rho_\pm, v_\pm)$.*

(ii) *When $v_- > v_+$, the solution, which is a delta shock wave, is of the form* (3.8) *with $x(t)$, $v_\delta(t)$ and $\omega(t)$ satisfying* (3.10).

4 The Limits of Solutions of (1.1) and (2.1) as $\kappa \to 0$

In this section, we consider the behavior of the solution to the Euler system of conservation laws of energy and momentum in special relativity (1.1) as $\kappa \to 0$.

We divide our issues into two different cases.

Case 1 $v_- > v_+$

Theorem 4.1 *If $v_- > v_+$, then there exists $\kappa_0 > 0$ such that $(\rho_+, v_+) \in \mathrm{IV}(\rho_-, v_-)$ when $0 < \kappa < \kappa_0$, where $\mathrm{IV}(\rho_-, v_-) = \{(v, \rho) \mid \rho_2 < \rho < \rho_1, -c < v < v_-\}$, and $(v_i, \rho_i) \in S_i$ (see Figure* 2.1).

Proof By (2.8) and (2.9), all possible states (ρ, v) that can be connected on the right side to the left state (ρ_-, v_-) by a backward shock wave S_1 or a forward shock wave S_2 satisfy

$$S_1: \quad \frac{\rho}{\rho_-} = 1 + \beta + \sqrt{\beta^2 + 2\beta}, \quad \rho \geq \rho_-$$

or

$$S_2: \quad \frac{\rho}{\rho_-} = 1 + \beta - \sqrt{\beta^2 + 2\beta}, \quad \rho \leq \rho_-,$$

where

$$\beta = \frac{(\kappa^2 + c^2)^2}{2\kappa^2} \frac{(v - v_-)^2}{(c^2 - v^2)(c^2 - v_-^2)}.$$

If $\rho_- = \rho_+$, the conclusion is obviously true. Otherwise, we can obtain the result by taking κ_0 as the form

$$\kappa_0 = \left(M_0 - c^2 - \sqrt{M_0(M_0 - 2c^2)}\,\right)^{\frac{1}{2}},$$

where

$$M_0 = \frac{(\rho_+ - \rho_-)^2(c^2 - v_-^2)(c^2 - v_+^2)}{2\rho_+\rho_-(v_+ - v_-)^2} \geq c^2.$$

From this theorem, we observe that the curves S_1 and S_2 become steeper when κ is much smaller. As $0 < \kappa \leq \kappa_0$, the solution consists of two constant states $(\rho_\pm, v_\pm)$, an intermediate state (ρ_*, v_*) and two shock curves $S_{1,2}$. According to Lemma 2.3, we have

$$S_1: \quad \begin{cases} \dfrac{\rho_*}{\rho_-} = 1 + \beta_1 + \sqrt{\beta_1^2 + 2\beta_1}, & \rho_* \geq \rho_-, \\ \beta_1 = \dfrac{(\kappa^2 + c^2)^2}{2\kappa^2} \dfrac{(v_* - v_-)^2}{(c^2 - v_*^2)(c^2 - v_-^2)}, \end{cases} \tag{4.1}$$

$$S_2: \quad \begin{cases} \dfrac{\rho_+}{\rho_*} = 1 + \beta_2 - \sqrt{\beta_2^2 + 2\beta_2}, & \rho_* \geq \rho_+, \\ \beta_2 = \dfrac{(\kappa^2 + c^2)^2}{2\kappa^2} \dfrac{(v_* - v_+)^2}{(c^2 - v_*^2)(c^2 - v_+^2)}. \end{cases} \tag{4.2}$$

We will now discuss the behavior of the two shock wave curves S_1 and S_2 and the intermediate state (ρ_*, v_*) when $\kappa \to 0$. We have the following theorem.

Theorem 4.2

$$\lim_{\kappa\to 0} \sigma_1 = \lim_{\kappa\to 0} \sigma_2 = \lim_{\kappa\to 0} v_* = \frac{v_+\sqrt{\frac{\rho_+}{c^2-v_+^2}} + v_-\sqrt{\frac{\rho_-}{c^2-v_-^2}}}{\sqrt{\frac{\rho_+}{c^2-v_+^2}} + \sqrt{\frac{\rho_-}{c^2-v_-^2}}}. \tag{4.3}$$

Proof According to (4.1) and (4.2), we have

$$\frac{\rho_+}{\rho_-} = (1 + \beta_1 + \sqrt{\beta_1^2 + 2\beta_1}\,)(1 + \beta_2 - \sqrt{\beta_2^2 + 2\beta_2}\,), \tag{4.4}$$

where the expressions of β_1 and β_2 are the same as those in (4.1) and (4.2). Observe that

$$(1 + \beta + \sqrt{\beta^2 + 2\beta})(1 + \beta - \sqrt{\beta^2 + 2\beta}) = 1.$$

Then (4.4) becomes

$$\frac{\rho_+}{\rho_-} = \frac{1 + \beta_2 - \sqrt{\beta_2^2 + 2\beta_2}}{1 + \beta_1 - \sqrt{\beta_1^2 + 2\beta_1}}.$$

That is

$$\begin{aligned} & l(c^2 - v_*^2) + \frac{kl}{c^2 - v_-^2}(v_* - v_-)^2 - l\sqrt{\frac{k^2}{(c^2 - v_-^2)^2}(v_* - v_-)^4 + \frac{2k}{c^2 - v_-^2}(c^2 - v_*^2)(v_* - v_-)^2} \\ &= c^2 - v_*^2 + \frac{k}{c^2 - v_+^2}(v_* - v_+)^2 - \sqrt{\frac{k^2}{(c^2 - v_+^2)^2}(v_* - v_+)^4 + \frac{2k}{c^2 - v_+^2}(c^2 - v_*^2)(v_* - v_+)^2}, \end{aligned}$$

where $k = \frac{(\kappa^2+c^2)^2}{2\kappa^2}$, $l = \frac{\rho_+}{\rho_-}$.

By some direct calculations, we have

$$\begin{aligned} 0 = {} & (l-1)^4(c^2 - v_*^2)^2 + \frac{4k^2l^2}{(c^2 - v_-^2)^2}(v_* - v_-)^4 + \frac{4k^2l^2}{(c^2 - v_+^2)^2}(v_+ - v_*)^4 \\ & - \frac{4kl(l-1)^2}{c^2 - v_-^2}(v_* - v_-)^2(c^2 - v_*^2) - \frac{4kl(l-1)^2}{c^2 - v_+^2}(c^2 - v_*^2)(v_+ - v_*)^2 \\ & - \frac{4k^2l(l^2+1)}{(c^2 - v_-^2)(c^2 - v_+^2)}(v_* - v_-)^2(v_* - v_+)^2. \end{aligned}$$

It is equivalent to

$$\begin{aligned} 0 = {} & \Big((l-1)^2(c^2 - v_*^2) - \frac{2kl}{c^2 - v_-^2}(v_* - v_-)^2 - \frac{2kl}{c^2 - v_+^2}(v_* - v_+)^2 \\ & + \frac{2kl(l+1)}{\sqrt{l}\sqrt{(c^2 - v_-^2)(c^2 - v_+^2)}}(v_* - v_-)(v_+ - v_*)\Big)\Big((l-1)^2(c^2 - v_*^2) - \frac{2kl}{c^2 - v_-^2}(v_* - v_-)^2 \\ & - \frac{2kl}{c^2 - v_+^2}(v_* - v_+)^2 - \frac{2kl(l+1)}{\sqrt{l}\sqrt{(c^2 - v_-^2)(c^2 - v_+^2)}}(v_* - v_-)(v_+ - v_*)\Big). \end{aligned}$$

Because $v_+ < v_* < v_-$, we take v_* as the form

$$v_* = \frac{b + \sqrt{b^2 - 4ac}}{2a},$$

where

$$\begin{aligned}
a &= (l-1)^2 + \frac{2kl}{c^2 - v_-^2} + \frac{2kl}{c^2 - v_+^2} + \frac{2kl(l+1)}{\sqrt{l}\sqrt{(c^2 - v_-^2)(c^2 - v_+^2)}},\\
b &= \frac{4klv_-}{c^2 - v_-^2} + \frac{4klv_+}{c^2 - v_+^2} + \frac{2kl(l+1)(v_- + v_+)}{\sqrt{l(c^2 - v_-^2)(c^2 - v_+^2)}},\\
c &= \frac{2klv_-^2}{c^2 - v_+^2} + \frac{2klv_+^2}{c^2 - v_+^2} + \frac{2kl(l+1)v_-v_+}{\sqrt{l(c^2 - v_-^2)(c^2 - v_+^2)}} - (l-1)^2c^2.
\end{aligned}$$

Let $\kappa \to 0$. Then

$$\lim_{\kappa\to 0} v_* = \frac{v_+\sqrt{\frac{\rho_+}{c^2 - v_+^2}} + v_-\sqrt{\frac{\rho_-}{c^2 - v_-^2}}}{\sqrt{\frac{\rho_+}{c^2 - v_+^2}} + \sqrt{\frac{\rho_-}{c^2 - v_-^2}}}. \tag{4.5}$$

Based on the Rankine-Hugoniot condition (3.5), we have

$$\sigma_1 = \frac{\frac{(\kappa^2 + c^2)\rho_* v_*^2}{c^2 - v_*^2} - \frac{(\kappa^2 + c^2)\rho_- v_-^2}{c^2 - v_-^2} + \kappa^2(\rho_* - \rho_-)}{\frac{(\kappa^2 + c^2)\rho_* v_*}{c^2 - v_*^2} - \frac{(\kappa^2 + c^2)\rho_- v_-}{c^2 - v_-^2}}, \tag{4.6}$$

$$\sigma_2 = \frac{\frac{(\kappa^2 + c^2)\rho_* v_*^2}{c^2 - v_*^2} - \frac{(\kappa^2 + c^2)\rho_+ v_+^2}{c^2 - v_+^2} + \kappa^2(\rho_* - \rho_+)}{\frac{(\kappa^2 + c^2)\rho_* v_*}{c^2 - v_*^2} - \frac{(\kappa^2 + c^2)\rho_+ v_+}{c^2 - v_+^2}}, \tag{4.7}$$

which leads to

$$\lim_{\kappa\to 0}\sigma_1 = \lim_{\kappa\to 0}\frac{\frac{\kappa^2(\kappa^2 + c^2)\rho_* v_*^2}{c^2 - v_*^2} + \kappa^4\rho_*}{\frac{\kappa^2(\kappa^2 + c^2)\rho_* v_*}{c^2 - v_*^2}} = \lim_{\kappa\to 0} v_* = \lim_{\kappa\to 0}\sigma_2. \tag{4.8}$$

From (4.5) and (4.8), we arrive at (4.4).

This proposition shows that the two shock curves S_1 and S_2 coincide as κ drops to zero.

Next we want to see the distribution of density on this coincidental shock.

Theorem 4.3

$$\lim_{\kappa\to 0}\int_{\sigma_1 t}^{\sigma_2 t}\rho_* \mathrm{d}x = \sqrt{\frac{\rho_+}{c^2 - v_+^2}\frac{\rho_-}{c^2 - v_-^2}}(v_- - v_+)(c^2 - v_\delta^2)t.$$

Proof From (4.6) and (4.7), we deduce

$$\begin{aligned}
\sigma_2 - \sigma_1 = {} & \frac{1}{(\kappa^2 + c^2)^2(b_*\rho_* - b_-)(b_*\rho_* - b_+)}((\kappa^2 + c^2)^2(a_*b_+ - a_+b_* + a_-b_* - b_-a_*)\rho_*\\
& + \kappa^2(\kappa^2 + c^2)(\rho_- - \rho_+)b_*\rho_* + \kappa^2(\kappa^2 + c^2)(b_+ - b_-)\rho_* + (\kappa^2 + c^2)^2(a_-b_+ - a_+b_-)),
\end{aligned}$$

where

$$a_* = \frac{v_*^2}{c^2 - v_*^2}, \quad b_* = \frac{v_*}{c^2 - v_*^2}, \quad a_\pm = \frac{\rho_\pm v_\pm^2}{c^2 - v_\pm^2}, \quad b_\pm = \frac{\rho_\pm v_\pm}{c^2 - v_\pm^2}.$$

Then, we have

$$\begin{aligned}\lim_{\kappa\to 0}\int_{\sigma_1 t}^{\sigma_2 t}\rho_*\mathrm{d}x &= \lim_{\kappa\to 0}\rho_*(\sigma_2-\sigma_1)t\\ &= \lim_{\kappa\to 0}\frac{b_+a_*-b_-a_*+a_-b_*-a_+b_*}{b_*^2}t\\ &= \lim_{\kappa\to 0}\frac{[\frac{\rho v}{c^2-v^2}]v_\delta(t)-[\frac{\rho v^2}{c^2-v^2}]}{\frac{v_*}{c^2-v_*^2}}t\\ &= \sqrt{\frac{\rho_+}{c^2-v_+^2}\frac{\rho_-}{c^2-v_-^2}}(v_--v_+)(c^2-v_\delta^2)t.\end{aligned}$$

We find from this proposition that the measure of ρ on the coincidental shock does not vanish as κ drops to zero. That is to say, the density, which is a linear function of t, becomes a singular measure as $\kappa=0$. Therefore, the solution is no longer self-similar, although the equations (1.1) and the initial data (2.1) are invariant under the self-similar transformation. Furthermore, we observe that the velocity, which is the weighted average of two initial states, still keeps bounded. This brings us naturally to recall the results of the relativistic zero-pressure Euler equations (3.1), which exactly verify the conclusions above.

Case 2 $v_-<v_+$

In this case, the solution involves the vacuum. The results are in the following theorems.

Theorem 4.4 *If $v_-<v_+$, then there exists $\kappa_0>0$ such that $(\rho_+,v_+)\in \mathrm{I}(\rho_-,v_-)$ when $0<\kappa<\kappa_0$, where $\mathrm{I}(\rho_-,v_-)=\{(v,\rho)\mid \rho_1<\rho<\rho_2, v_-<v<-c\}$, and $(v_i,\rho_i)\in R_i$ (see Figure* 2.1).

Proof All possible states (ρ,v) that can be connected on the right side to the left state (ρ_-,v_-) by a backward rarefaction wave R_1 or a forward rarefaction wave R_2 satisfy

$$R_1:\quad \frac{c}{2}\ln\frac{(c-v_-)(c+v)}{(c+v_-)(c-v)}+\frac{\kappa}{1+\frac{\kappa^2}{c^2}}\ln\frac{\rho}{\rho_-}=0,\quad \rho\le\rho_-$$

and

$$R_2:\quad \frac{c}{2}\ln\frac{(c-v_-)(c+v)}{(c+v_-)(c-v)}-\frac{\kappa}{1+\frac{\kappa^2}{c^2}}\ln\frac{\rho}{\rho_-}=0,\quad \rho\ge\rho_-$$

according to (2.5). The conclusion is obviously true when $\rho_-=\rho_+$. If $\rho_-\ne\rho_+$, we can reach the conclusion by taking

$$\kappa_0=\Big(\frac{1}{2}M-\sqrt{\frac{1}{4}M^2-1}\Big)c,$$

where

$$M=\frac{2\ln\frac{\rho_+}{\rho_-}}{\ln\frac{(c+v_+)(c-v_-)}{(c-v_+)(c+v_-)}}\ge 2.$$

This theorem shows that when $0<\kappa<\kappa_0$, the solution consists of two rarefaction waves $R_{1,2}$ and an intermediate state (ρ_*,v_*) besides two constant states $(\rho_\pm,v_\pm)$. By Lemma 2.1, they satisfy the following expressions:

$$R_1:\quad\begin{cases}\lambda_1=\dfrac{v-\kappa}{1-\frac{\kappa v}{c^2}},\\ \dfrac{c}{2}\ln\dfrac{(c-v_-)(c+v)}{(c+v_-)(c-v)}+\dfrac{\kappa}{1+\frac{\kappa^2}{c^2}}\ln\dfrac{\rho}{\rho_-}=0,\quad \rho_*\le\rho\le\rho_-,\end{cases}\tag{4.9}$$

$$R_2: \quad \begin{cases} \lambda_2 = \dfrac{v+\kappa}{1+\frac{\kappa v}{c^2}}, \\ \dfrac{c}{2}\ln\dfrac{(c+v_+)(c-v)}{(c-v_+)(c+v)} - \dfrac{\kappa}{1+\frac{\kappa^2}{c^2}}\ln\dfrac{\rho_+}{\rho} = 0, \quad \rho_* \le \rho \le \rho_+. \end{cases} \tag{4.10}$$

In virtue of the above formulas (4.9) and (4.10), we consider the properties of the two rarefaction wave curves R_1 and R_2, and the behavior of the solution when κ drops to zero. We have the theorem below.

Theorem 4.5 *As κ drops to zero, ρ_* vanishes and two rarefaction waves, R_1 and R_2, become two contact discontinuities connecting the two constant states $(\rho_\pm, v_\pm)$ and the vacuum $(\rho_* = 0)$.*

Proof (4.9) and (4.10) imply

$$\Big(\frac{c-v_-}{c+v_-}\frac{c+v_+}{c-v_+}\Big)^{\frac{c}{2}} = \Big(\frac{\rho_-\rho_+}{\rho_*^2}\Big)^{\frac{\kappa}{1+\frac{\kappa^2}{c^2}}}.$$

So the intermediate state (ρ_*, v_*) can be expressed as

$$\rho_* = \sqrt{\rho_-\rho_+}\Big(\frac{c-v_-}{c+v_-}\frac{c+v_+}{c-v_+}\Big)^{\frac{\left(1+\frac{\kappa^2}{c^2}\right)c}{4\kappa}}.$$

It is easy to know that

$$0 < \frac{c-v_-}{c+v_-}\frac{c+v_+}{c-v_+} < 1, \quad v_- < v_+,$$

and $\frac{\left(1+\frac{\kappa^2}{c^2}\right)c}{4\kappa} \to +\infty$ when $\kappa \to 0$, which shows that $\rho_* \to 0$ and $\lambda_1, \lambda_2 \to v$ as $\kappa \to 0$. That is to say, as κ drops to zero, ρ_* vanishes and two rarefaction waves R_1 and R_2 become two contact discontinuities connecting the constant states $(\rho_\pm, v_\pm)$ and the vacuum $(\rho_* = 0)$.

As a result, the solution of (1.1) and (2.1) accords with that of (3.1) and (3.2), as κ goes to zero, and it presents us two extreme states: one turns into delta shock wave, and the other involves the vacuum. Therefore, system (3.1) can be viewed as the limit of the relativistic Euler equations (1.1) as κ goes to zero.

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