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Large Solutions to Complex Monge-Ampère Equations: Existence, Uniqueness and Asymptotics*

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Abstract The authors consider the complex Monge-Ampère equation $\det(u_{i\overline{j}}) = \psi(z, u, \nabla u)$ in bounded strictly pseudoconvex domains Ω, subject to the singular boundary condition $u = \infty$ on $\partial\Omega$. Under suitable conditions on ψ, the existence, uniqueness and the exact asymptotic behavior of solutions to boundary blow-up problems for the complex Monge-Ampère equations are established.

Keywords Complex Monge-Ampère equation, Boundary blow-up, Plurisubharmonic, Pseudoconvex, Asymptotics
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1 Introduction and Main Results

Let Ω be a domain in C^n and ψ, which is defined in $\Omega \times \mathbb{R} \times \mathbb{R}^{2n}$, be a positive function. We study the boundary blow-up problem for the complex Monge-Ampere equation

$$\begin{cases} \det \dfrac{\partial^2 u}{\partial z_i \partial \overline{z}_j} = \psi(z, u, \nabla u), \quad z \in \Omega, \\ u(z) = \infty, \quad z \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\nabla u(z_0) = p_0$ (seen as a function of $(\mathrm{Re}\, z, \mathrm{Im}\, z)$). The boundary condition means $u(z) \to \infty$ as $d(z) = \mathrm{dist}(z, \partial\Omega) \to 0$.

Problems of this type was considered by Cheng and Yau [3, 4] (with $\psi(u) = \mathrm{e}^{Ku}$ in bounded doamins and with $\psi(u) = \mathrm{e}^{2u}$ in unbounded domains). The real Monge-Ampère equations with the boundary blow-up was treated in [4, 7, 8].

For the complex Monge-Ampère with the boundary blow-up, Ivarsson and Matero [10] proved the existence and regularity result when the right-hand side of the equation in (1.1) is of the form $\psi(z, u(z))$. More general results are obtained in [14].

Cîrstea and Trombetti [6] considered the real Monge-Ampère equation subject to the singular boundary condition and obtained the existence, uniqueness and asymptotics. In this article, we deal with the complex case and obtain the existence in strictly pseudoconvex domain, uniqueness and the exact asymptotic behavior of solutions to boundary blow-up problems.

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Theorem 1.1 (Existence) *Let Ω be a bounded, strictly pseudoconvex domain in C^n. Suppose that $\psi \in C^\infty(\overline{\Omega}, \mathbb{R}, \mathbb{R}^{2n})$ satisfies $\psi > 0$, as well as*

$$M(u^+)^p \leq \psi(z, u, \nabla u), \tag{1.2}$$

where $(z, u, \nabla u) \in \Omega \times \mathbb{R} \times \mathbb{R}^{2n}$. There is a constant C, such that

$$-\psi_u, |\psi_{z_j}|, |\nabla\psi| \leq C\psi^{1-\frac{1}{n}}, \tag{1.3}$$

where $p > n$, $M > 0$, $C > 0$, $u^+ = \max\{u, 0\}$. Then there is a strictly plurisubharmonic solution $u \in C^\infty(\Omega)$ to (1.1).

For the asymptotic behavior near $\partial\Omega$ of the blow-up solutions, we consider the equations

$$\begin{cases} \det \dfrac{\partial^2 u}{\partial z_i \partial \overline{z}_j} = g(z) f(u), & z \in \Omega, \\ u(z) = \infty, & z \in \partial\Omega, \end{cases} \tag{1.4}$$

where $g(z) \in C^\infty(\overline{\Omega})$ is positive in Ω, $f \in C[0, \infty) \cap C^\infty(0, \infty)$ is positive increasing.

Let $\Re_l$ denote the set of all positive non-decreasing C^1 functions m defined on $(0, \mu)$, for some $\mu > 0$, for which there exist

$$\lim_{t \to 0^+} \frac{\int_0^t m(s)\mathrm{d}s}{m(t)} = 0 \quad \text{and} \quad \lim_{t \to 0^+} \frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{\int_0^t m(s)\mathrm{d}s}{m(t)}\Big) = l. \tag{1.5}$$

A complete characterization of $\Re_l$ is provided by [5] (according to $l \neq 0$ or $l = 0$).

One has the following examples for a special l:

(a) $m(t) = (-\frac{1}{\ln t})^p$ with $l = 1$;

(b) $m(t) = t^p$ with $l = \frac{1}{p+1}$;

(c) $m(t) = \mathrm{e}^{-\frac{1}{t^p}}$ with $l = 0$,

where $p > 0$ is arbitrary.

Definition 1.1 *A positive measurable function f defined in $[a, \infty)$, for some $a > 0$, is called regularly varying at infinity with an index q, written $f \in \mathrm{RV}_q$, if for each $\lambda > 0$ and some $q \in \mathbb{R}$,*

$$\lim_{t \to \infty} \frac{f(\lambda t)}{f(t)} = \lambda^q. \tag{1.6}$$

The real number q is called the index of regular variation.

When $q = 0$, we have the next definition.

Definition 1.2 *A positive measurable function L defined in $[a, \infty)$, for some $a > 0$, is called regularly varying at infinity, if for $\lambda > 0$ and some $q \in \mathbb{R}$,*

$$\lim_{t \to \infty} \frac{L(\lambda t)}{L(t)} = 1. \tag{1.7}$$

By Definitions 1.1 and 1.2, if $f \in \mathrm{RV}_q$, it can be represented in the form

$$f(t) = t^q L(t). \tag{1.8}$$

Definition 1.3 *If H is a non-decreasing function on $\mathbb{R}$, then we denote by $H^{\leftarrow}$ the (left-continuous) inverse of H (see* [13]*), that is,*

$$H^{\leftarrow}(y) = \inf\{s : H(s) \geq y\}. \tag{1.9}$$

If $a > 0$ is sufficiently large, we define

$$\wp(u) = \sup\Big\{\frac{f(y)}{y^k} : a \leq y \leq u\Big\} \quad \textit{for } u \geq a. \tag{1.10}$$

Theorem 1.2 (Asymptotic Behaviors) *Let Ω be a bounded, strictly pseudoconvex domain in C^n with a smooth boundary. Assume that $f \in \mathrm{RV}_q$ with $q > n$ and there exists an $m \in \Re_l$, such that*

$$0 < \beta^- = \liminf_{d(z)\to 0} \frac{g(z)}{m^{n+1}(d(z))} \quad \textit{and} \quad \limsup_{d(z)\to 0} \frac{g(z)}{m^{n+1}(d(z))} = \beta^+ < \infty. \tag{1.11}$$

Then every strictly plurisubharmonic blow-up solution u_∞ to (1.4) *satisfies*

$$\xi^- \leq \liminf_{d(z)\to 0} \frac{u_\infty(z)}{\varphi(d(z))} \quad \textit{and} \quad \limsup_{d(z)\to 0} \frac{u_\infty(z)}{\varphi(d(z))} \leq \xi^+, \tag{1.12}$$

where φ is defined by

$$\varphi(t) = \wp^{\leftarrow}\Big(\Big(\int_0^t m(s)\mathrm{d}s\Big)^{-n-1}\Big) \quad \textit{for } t > 0 \text{ small}, \tag{1.13}$$

and $\xi^{\pm}$ are positive constants given by

$$\frac{(\xi^+)^{n-q}}{\lambda_1\beta^-} = \frac{(\xi^-)^{n-q}}{\lambda_n\beta^+} = \frac{[\frac{q-n}{n+1}]^{n+1}}{1 + \frac{l(q-n)}{n+1}}, \tag{1.14}$$

where λ_1, λ_n are positive constants which only depend on the strictly pseudonconvex domain Ω.

Corollary 1.1 *Let Ω be a ball of radius $R > 0$ in C^n and $f \in \mathrm{RV}_q$ with $q > n$. If $g(z) \sim m^{n+1}(d(z))$ as $d(z) \to 0$ for $m \in \Re_l$, then every strictly plurisubharmonic blow-up solution to* (1.4) *satisfies*

$$u(z) \sim \Big\{\frac{[\frac{q-n}{n+1}]^{n+1}R^{n-1}}{1 + \frac{l(q-n)}{n+1}}\Big\}^{\frac{1}{n-q}} \varphi(d(z)). \tag{1.15}$$

Under slightly more restrictive conditions than those in Theorem 1.2, there is at most one strictly plurisubharmonic blow-up solution to (1.4).

Theorem 1.3 (Uniqueness) *Let Ω be a bounded, strictly pseudoconvex domain in C^n with a smooth boundary. Assume $f \in \mathrm{RV}_q$ with $q > n$, and $\frac{f(u)}{u^n}$ is increasing in $(0, \infty)$. Then* (1.4) *has at most one strictly plurisubharmonic blow-up solution, provided that either*

(i) *$g(z)$ is positive on $\overline{\Omega}$, or*

(ii) *$g(z) = 0$ on $\partial\Omega$, Ω is a ball of radius $R > 0$ and $g(z) \sim (m(d(z)))^{n+1}$ as $d(z) \to 0$ for some $m \in \Re_l$.*

2 Preliminaries

As the first step, let us start with some notations that we shall use later. Let $z = (z_1, \cdots, z_n) \in C^n$. For the complex variables, $\partial_{i\bar{j}} u = \frac{\partial^2 u}{\partial z_i \partial z_{\bar{j}}}$, $u_{i\bar{j}} = \partial_{i\bar{j}} u$. A domain $\Omega \subset C^n$ with a smooth boundary $\partial\Omega$ is called strongly pseudoconvex, if there exists a C^∞ function ρ defined on a neighborhood of $\partial\Omega$, such that $d\rho \neq 0$, and $\rho < 0$ in Ω; $\rho = 0$ on $\partial\Omega$; $\rho > 0$ outside of Ω. ρ is strictly plurisubharmonic.

Lemma 2.1 *Let Ω be an open subset of C^n. If $b \in C^2(\Omega)$ and $h \in C^2(R)$, then the following holds:*

$$\begin{aligned}\det \partial_{i\bar{j}} h(b(z)) = {} & [h'(b(z))]^{n-1} h''(b(z)) \langle \mathrm{Co}(\partial_{i\bar{j}} b(z)) \nabla b(z), \overline{\nabla b(z)} \rangle \\ & + [h'(b(z))]^n \det \partial_{i\bar{j}} b(z), \quad \forall\, z \in \Omega,\end{aligned} \tag{2.1}$$

where $\mathrm{Co}(\partial_{i\bar{j}} b(z))$ denotes the cofactor matrix of $\partial_{i\bar{j}} b(z)$.

We need the following lemma in [11].

Lemma 2.2 *Let $\Omega \subset C^n$ be a domain with a C^2 boundary. Let $z_0 \in \partial\Omega$ be a point of strong pseudoconvexity. Then there exists a neighborhood $Z \subset C^n$ of z_0 and a biholomorphic mapping Φ on Z, such that $W = \Phi(Z \cap U)$ is strongly convex.*

Lemma 2.3 *Let $\Omega \subset C^n$ be a strictly pseudoconvex domain with a smooth boundary. $d(z)$ is the distant function for the boundary. Then $-d(z)$ is a smooth defining function for Ω.*

Lemma 2.4 *There exist $\lambda_1 > 0$ and $\lambda_n > 0$, such that*

$$\lambda_1 |z|^2 \leq \sum \frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_k}(z_0) z_j \overline{z}_k \leq \lambda_n |z|^2, \quad z_0 \in \partial\Omega. \tag{2.2}$$

For $\mu > 0$, we set $\Gamma_\mu = \{z \in \overline{\Omega} : d(z) < \mu\}$.

Corollary 2.1 *Let Ω be a bounded, strictly pseudoconvex domain in C^n with a smooth boundary. Assume that $\mu > 0$ is small such that $d \in C^2(\Gamma_\mu)$ and $h \in C^2(0, \mu)$. Let $\widehat{z}_0 \in \Gamma_\mu \setminus \partial\Omega$ and $z_0 \in \partial\Omega$, such that $|\widehat{z}_0 - z_0| = d(z_0)$. Then we have*

$$\begin{aligned}\det \partial_{i\bar{j}} h(d(\widehat{z}_0)) = {} & [-h'(d(\widehat{z}_0))]^{n-1} h''(d(\widehat{z}_0)) \langle \mathrm{Co}(\partial_{i\bar{j}} \rho(\widehat{z}_0)) \nabla \rho(\widehat{z}_0), \overline{\nabla \rho(\widehat{z}_0)} \rangle \\ & + [-h'(d(\widehat{z}_0))]^n \det \partial_{i\bar{j}} \rho(\widehat{z}_0),\end{aligned} \tag{2.3}$$

where $\rho(z) = -d(z)$.

We now give a brief account of the definitions and properties of regularity varying functions; see also [6, 13].

Proposition 2.1 *If L is slowly varying, then $\frac{L(\lambda u)}{L(u)}$ tends to 1 as $u \to \infty$, uniformly on each compact set in $(0, \infty)$.*

Proposition 2.2 *We have*

(i) *if $R \in \mathrm{RV}_q$, then $\lim\limits_{u \to \infty} \frac{R(u)}{\log u} = q$.*

(ii) *if $R_1 \in \mathrm{RV}_{q_1}$ and $R_2 \in \mathrm{RV}_{q_2}$ with $\lim\limits_{u \to \infty} R_2(u) = \infty$, then $R_1 \circ R_2 \in \mathrm{RV}_{q_1 q_2}$.*

(iii) *suppose that R is non-decreasing and $R \in \mathrm{RV}_q$, $0 < q < \infty$. Then $R^{\leftarrow} \in \mathrm{RV}_{q^{-1}}$.*

(iv) *suppose that R_1, R_2 are non-decreasing and q-varying with $q \in (0,\infty)$. Then for $c \in (0,\infty)$,*

$$\lim_{u\to\infty}\frac{R_1(u)}{R_2(u)} = c \Longleftrightarrow \lim_{u\to\infty}\frac{R_1^{\leftarrow}(u)}{R_2^{\leftarrow}(u)} = c^{-\frac{1}{q}}. \tag{2.4}$$

Proposition 2.3 *Let $R \in \mathrm{RV}_q$ and choose $B \geq 0$, such that R is locally bounded on $[B,\infty)$. If $q > 0$, then*

(i) $\sup\{R(y) : B \leq y \leq u\} \sim R(u)$ *as* $u \to \infty$;

(ii) $\inf\{R(y) : y \geq u\} \sim R(u)$ *as* $u \to \infty$.

If $q < 0$, then

(iii) $\sup\{R(y) : y \geq u\} \sim R(u)$ *as* $u \to \infty$;

(iv) $\inf\{R(y) : B \leq y \leq u\} \sim R(u)$ *as* $u \to \infty$.

The following comparison principle is sometimes useful. For the proof, see for example [9].

Proposition 2.4 (Comparison Principle) *Let Ω be a bounded, pseudoconvex domain in C^n. Assume that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a non-negative function which is increasing in the second variable. Let $\underline{u}$, $\overline{u} \in C^\infty(\overline{\Omega} \cap \mathrm{PSH}(\Omega))$ and $u \in C^\infty(\Omega) \cap \mathrm{PSH}(\Omega)$, such that $u(z) = \infty$ for all $z \in \partial\Omega$. Then*

(i) $\det \frac{\partial^2 \overline{u}}{\partial z_i \partial \overline{z}_j} \leq f(z,\overline{u}(z))$, $f(z,\underline{u}(z)) \leq \det \frac{\partial^2 \underline{u}}{\partial z_i \partial \overline{z}_j}$ *and $\underline{u} \leq \overline{u}$ on $\partial\Omega$ imply that $\underline{u} \leq \overline{u}$ in Ω;*

(ii) $\det \frac{\partial^2 u}{\partial z_i \partial \overline{z}_j} \leq f(z,u(z))$ *and* $f(z,\underline{u}(z)) \leq \det \frac{\partial^2 \underline{u}}{\partial z_i \partial \overline{z}_j}$ *imply that $\underline{u} \leq u$ in Ω.*

Lemma 2.5 *Let $m \in \Re_l$ and $f \in \mathrm{RV}_q$ with $q > n$. If φ is defined by (1.13), then there exists a function $\psi \in C^{0,\tau}$ with $\tau > 0$ which satisfies $\lim_{t\to 0}\frac{\psi(t)}{\varphi(t)} = 1$ and the following:*

(i) $\lim_{t\to 0}\frac{\psi(t)\psi''(t)}{[\psi'(t)]^2} = 1 + \frac{(q-n)l}{n+1}$;

(ii) $\lim_{t\to 0}\frac{[-\psi'(t)]^{n-1}\psi''(t)}{m^{n+1}(t)f(\psi(t))} = [\frac{n+1}{q-n}]^{n+1}[1 + \frac{(q-n)l}{n+1}]$.

The proof of Lemma 2.5 is in [6, Lemma 5.1].

3 The Proof of Existence

We obtained the existence in strictly convex in [14]. Now we can extend the existence to the strictly pseudoconvex domain.

Moreover, we will construct some radially symmetric and strictly plurisubharmonic functions that will be used as barriers. The proofs here will be omitted because they bear an analogy to the real case in [8].

Lemma 3.1 *Let $\eta \in C^1(\mathbb{R})$ satisfy $\eta(\phi) > 0$, $\eta'(\phi) \geq 0$ for all $\phi \in \mathbb{R}$. Then for any $a > 0$, there is a strictly plurisubharmonic and radially symmetric function $v \in C^2(B_a(0))$ satisfying*

$$\begin{cases}\det(v_{i\overline{j}}) \geq \mathrm{e}^v\eta(v)(1 + |\nabla v|^n), \quad z \in B_a(0),\\ v = +\infty, \quad z \in \partial B_a(0).\end{cases} \tag{3.1}$$

In the sequel, we will denote the function $v \in C^2(B_a(0))$ in Lemma 3.1 by $v^{a,\eta}$. We will also write $v^{a,\eta}(z) = v^{a,\eta}(|z|)$, since it is radially symmetric.

A straightforward calculation shows that when $p > n$ the function

$$w(z) := (1 - |z|^2)^{\frac{n+1}{n-p}} \tag{3.2}$$

is strictly plurisubharmonic and satisfies the inequality

$$\det(w_{i\bar{j}}) \leq C(n,p) w^p, \quad \text{in } B_1(0), \tag{3.3}$$

where C is a constant only depending on n and p. By rescaling, we have the following lemma.

Lemma 3.2 *Let $a, M > 0$ and $p > n$. Define $w^{a,M} \in C^\infty(B_a(0))$ by*

$$w^{a,M}(z) := \lambda w\Big(\frac{z}{a}\Big), \quad z \in B_a(0), \tag{3.4}$$

where

$$\lambda = \Big(\frac{C(n,p)}{a^{2n} M}\Big)^{\frac{1}{p-n}}. \tag{3.5}$$

Then

$$\det(w^{a,M}_{i\bar{j}}) \leq M (w^{a,M})^p, \quad z \in B_a(0). \tag{3.6}$$

In the sequel, we will denote the function $w^{a,M}$ in Lemma 3.2 by $\overline{h}(a)$. Then we only give the outline of the proof for the existence Theorem 1.1, because the process is similar to those in [14].

Proof of Lemma 3.2 **Step 1** We first assume Ω to be smooth. For each integer $k \geq 1$, consider the Dirichlet problem

$$\begin{cases} \det(u_{i\bar{j}}) = \psi(z, u, \nabla u), & z \in \Omega, \\ u = k, & z \in \partial\Omega. \end{cases} \tag{3.7}$$

We can obtain a strictly plurisubharmonic solution $u_k \in C^\infty(\overline{\Omega})$ to (3.7), which satisfies

$$\|u_k\|_{c^{2,\alpha}(\overline{\Omega})} \leq C(k), \quad k \geq 1, \tag{3.8}$$

where $C(k)$ is a constant depending on k. Moreover, there is an $a > 0$ depending only on Ω and a decreasing sequence $a_k \to a$ $(k \to \infty)$, such that

$$v^{a_k,\eta}(a - d(z)) \leq u_k(z) \leq \overline{h}(d(z)), \quad \forall\, z \in \Omega,\ k \geq 1. \tag{3.9}$$

The definitions of $v^{a_k,\eta}$ and $\overline{h}$ can be found in Lemmas 3.1 and 3.2.

Step 2 We need to prove an a priori interior estimate which is independent of k. Firstly, let h, v_k denote the functions defined in Ω by

$$h(z) := \overline{h}(d(z)), \quad v_k(z) := v^{a_k,\eta}(a - d(z)), \quad z \in \Omega. \tag{3.10}$$

For $l > 0$ and $k \geq 1$, denote

$$H_l := \{z \in \Omega : h(z) < l\}, \quad U_{k,l} := \{z \in \Omega : u_k(z) < l\}, \quad V_{k,l} := \{z \in \Omega : v_k(z) < l\}. \tag{3.11}$$

Now we have $H_l \subset U_{k,l} \subset V_{k,l}$ for each $k \geq 1$.

Let K be a compact subset of Ω. We may choose $l > 0$ and k_0 large enough, such that $K \subset H_{\frac{l}{2}}$ and $\overline{V}_{k_0,4l} \subset \Omega$. From (3.8) and (3.9), we see that

$$|u_k| \leq C_0, \quad \text{in } \overline{U}_{k,2l}, \ \forall\, k \geq k_0, \tag{3.12}$$

where C_0 depends on l, k_0 but is independent of k.

Secondly, by the interior estimates in [1, 2, 7], we have

$$\max_{\overline{U}_{k,2l}} |\nabla u_k| = \max_{\partial U_{k,2l}} |\nabla u_k| \leq \max_{\overline{V}_{k_0,2l}} |\nabla u_k| \leq \frac{\widetilde{C}}{\inf\limits_{z \in V_{k_0,2l}} d_\Omega z} \equiv C_1, \tag{3.13}$$

where $\widetilde{C}$ depends only on the Lipschitz constant of $\psi^{\frac{1}{n}}$ and the diameter of Ω, while C_1 depends on $k_0, l, \widetilde{C}$ but is independent of k.

Thirdly, applying the result of Blocki in [1, 2] or that of Guan and Spruck in [7], we obtain

$$\|u_{i\bar{j}}\|_{(C^\alpha(\overline{H}_l))} \leq \frac{\widehat{C}}{\inf\limits_{z \in V_{k_0,2l}} d_\Omega^2 z} \leq C_3, \quad \forall\, k \geq k_0, \tag{3.14}$$

where C_3 depends on C_0, C_1, k_0, l and the Lipschitz constant of $\psi^{\frac{1}{n}}$, but is independent of k. By the theorem of Evans and Krylov, we obtain

$$\|u_k\|_{C^{2,\alpha}(K)} \leq C,$$

where C is independent of k.

Step 3 By the above steps, there are a subsequence $\{u_{k_j}\}$ and $u \in C^{2,\alpha}(\Omega)$, such that

$$\lim_{j \to \infty} \|u_{k_j} - u\|_{C^{2,\alpha}(K)} = 0 \tag{3.15}$$

for any compact subset $K \subset \Omega$. We see that u is strictly plurisubharmonic and satisfies (1.1). This completes the proof of Theorem 1.1 when Ω is smooth.

Step 4 Suppose that Ω is not smooth. We choose a sequence of smooth, strictly pseudo-convex domains

$$\Omega_1 \subset \cdots \subset \Omega_k \subset \cdots \subset \Omega, \tag{3.16}$$

such that

$$\Omega = \bigcup_{k=1}^{\infty} \Omega_k. \tag{3.17}$$

We can obtain $u \in C^{2,\alpha}(\Omega)$. That $u \in C^\infty(\Omega)$ follows from the elliptic regularity theory.

4 The Proof of Asymptotic Behavior

Fix $\epsilon \in (0, \frac{1}{2})$. We choose $\delta > 0$ small enough, such that

(a) m is non-decreasing on $(0, 2\delta)$;

(b) $\beta^- m^{n+1}(d(z)) \le g(z) \le \beta^+ m^{n+1}(d(z))$ for $z \in \Omega_{2\delta}$, where for $\lambda > 0$, we set $\Omega_\lambda = \{z \in \Omega : d(z) < \lambda\}$;

(c) $d(z)$ is a C^2 function on $\Gamma_{2\delta} = \{z \in \overline{\Omega} : d(z) < 2\delta\}$;

(d) $\psi' < 0$ and $\psi, \psi'' > 0$ on $(0, 2\delta)$, where ψ is as in Lemma 2.5.

Fix $\tau \in (0, \delta)$. With $\xi^\pm$ given by (1.14), we set

$$\eta^\pm = [(1 \mp 2\epsilon)]^{\frac{1}{n-q}} \xi^\pm. \tag{4.1}$$

Let us define

$$\begin{cases} v_\tau^+(z) = \eta^+ \psi(d(z) - \tau), & \forall\, z \in \Omega_{2\delta} \setminus \overline{\Omega_\tau}, \\ v_\tau^-(z) = \eta^- \psi(d(z) + \tau), & \forall\, z \in \Omega_{2\delta-\tau}. \end{cases} \tag{4.2}$$

Step 1 We prove that near the boundary, v_τ^+ (resp. v_τ^-) is an upper (resp. lower) solution to (1.4), that is,

$$\begin{cases} \det \partial_{i\bar{j}} v_\tau^+(z) \le g(z) f(v_\tau^+(z)), & \forall\, z \in \Omega_{2\delta} \setminus \overline{\Omega_\tau}, \\ \det \partial_{i\bar{j}} v_\tau^-(z) \ge g(z) f(v_\tau^-(z)), & \forall\, z \in \Omega_{2\delta-\tau}. \end{cases} \tag{4.3}$$

By (a) and (b), it suffices to show that

$$\begin{cases} \det \partial_{i\bar{j}} v_\tau^+(z) \le \beta^- m^{n+1}(d(z)) f(v_\tau^+(z)), & \forall\, z \in \Omega_{2\delta} \setminus \overline{\Omega_\tau}, \\ \det \partial_{i\bar{j}} v_\tau^-(z) \ge \beta^+ m^{n+1}(d(z)) f(v_\tau^-(z)), & \forall\, z \in \Omega_{2\delta-\tau}. \end{cases} \tag{4.4}$$

Using Corollary 2.1, we obtain

$$\begin{aligned} \det \partial_{i\bar{j}} v_\tau^- &= (\eta^-)^n [-\psi'(d(z)+\tau)]^{n-1} \psi''(d(z)+\tau) \langle \mathrm{Co}(\partial_{i\bar{j}}\rho(z)) \nabla\rho(z), \overline{\nabla\rho(z)} \rangle \\ &\quad + (\eta^-)^n [-\psi'(d(z)+\tau)]^n \det \partial_{i\bar{j}} \rho(z) \\ &\ge (\eta^-)^n [-\psi'(d(z)+\tau)]^{n-1} \psi''(d(z)+\tau) \langle \mathrm{Co}(\partial_{i\bar{j}}\rho(z)) \nabla\rho(z), \overline{\nabla\rho(z)} \rangle \\ &\ge \frac{(\eta^-)^n}{\lambda_n} [-\psi'(d(z)+\tau)]^{n-1} \psi''(d(z)+\tau), \quad \forall\, z \in \Omega_{2\delta-\tau}, \end{aligned} \tag{4.5}$$

$$\begin{aligned} \det \partial_{i\bar{j}} v_\tau^+ &= (\eta^+)^n [-\psi'(d(z)-\tau)]^{n-1} \psi''(d(z)-\tau) \langle \mathrm{Co}(\partial_{i\bar{j}}\rho(z)) \nabla\rho(z), \overline{\nabla\rho(z)} \rangle \\ &\quad + (\eta^+)^n [-\psi'(d(z)-\tau)]^n \det \partial_{i\bar{j}} \rho(z) \\ &= \mathrm{A} + \mathrm{B}, \quad \forall\, z \in \Omega_{2\delta} \setminus \overline{\Omega_\tau}, \end{aligned} \tag{4.6}$$

where

$$\begin{aligned} \mathrm{A} &= (\eta^+)^n [-\psi'(d(z)-\tau)]^{n-1} \psi''(d(z)-\tau) \langle \mathrm{Co}(\partial_{i\bar{j}}\rho(z)) \nabla\rho(z), \overline{\nabla\rho(z)} \rangle, \\ \mathrm{B} &= (\eta^+)^n [-\psi'(d(z)-\tau)]^n \det \partial_{i\bar{j}} \rho(z) \end{aligned}$$

and

$$\begin{aligned} \mathrm{B} &\le (\eta^+)^n (\lambda_n)^n (-\psi')^n \\ &\le (\eta^+)^n (\lambda_n)^n \frac{(-\psi')^{n-1} \psi'' (-\psi')}{\psi''} \\ &= (\eta^+)^n (\lambda_n)^n (-\psi')^{n-1} \psi'' \frac{(-\psi')(-\psi')}{\psi'' \psi} \frac{\psi}{-\psi'}. \end{aligned} \tag{4.7}$$

In the proof of Lemma 2.5, we have (5.4) in [6]

$$\frac{\psi}{-\psi'} \sim \frac{(q-n)}{(n+1)} \frac{\int_0^t m(s)\mathrm{d}s}{m(t)}, \tag{4.8}$$

where

$$\lim_{t\to 0^+} \frac{\int_0^t m(s)\mathrm{d}s}{m(t)} = 0. \tag{4.9}$$

So by Lemma 2.5, $\lim\limits_{d(z)\to 0} \mathrm{B} = 0$. We only need to consider A.

$$\mathrm{A} \le \frac{(\eta^+)^n}{\lambda_1}(-\psi'(d(z)-\tau))^{n-1}\psi''(d(z)-\tau), \tag{4.10}$$

$$\det \partial_{i\bar{j}} v_\tau^+ \le \frac{(\eta^+)^n}{\lambda_1}[-\psi'(d(z)-\tau)]^{n-1}\psi''(d(z)-\tau), \quad \forall\, z \in \Omega_{2\delta} \setminus \overline{\Omega_\tau}. \tag{4.11}$$

Therefore, to deduce (4.4), it is enough to establish

$$\lim_{t\to 0} \frac{(\eta^+)^n}{\lambda_1 \beta^-} \frac{[-\psi'(t)]^{n-1}\psi''(t)}{m^{n+1}(t) f(\eta^+\psi(t))} = 1-2\epsilon, \tag{4.12}$$

$$\lim_{t\to 0} \frac{(\eta^-)^n}{\lambda_n \beta^+} \frac{[-\psi'(t)]^{n-1}\psi''(t)}{m^{n+1}(t) f(\eta^-\psi(t))} = 1+2\epsilon. \tag{4.13}$$

Since $f \in \mathrm{RV}_q$, (4.12) and (4.13) are valid thanks to Lemma 2.5 and our choice of $\eta^\pm$ in (4.1).

Step 2 Every strictly plurisubharmonic blow-up solution u_∞ to (1.4) satisfies (1.12).

Let $C = \max\limits_{d(z)=\delta} u_\infty(z)$. Notice that

$$\begin{cases} v_\tau^+(z) + C = \infty > u_\infty(z), & \forall\, z \in \Omega \text{ with } d(z) = \tau, \\ v_\tau^+(z) + C \ge u_\infty(z), & \forall\, z \in \Omega \text{ with } d(z) = \delta. \end{cases} \tag{4.14}$$

Using (4.3), we deduce that, for every $z \in \Omega_\delta \setminus \overline{\Omega_\tau}$,

$$\det \partial_{i\bar{j}}(v_\tau^+(z) + C) = \det \partial_{i\bar{j}}(v_\tau^+(z)) \le g(z) f(v_\tau^+(z)) \le g(z) f(v_\tau^+(z) + C). \tag{4.15}$$

Since u_∞ is a solution to (1.4), by Proposition 2.4, we find

$$v_\tau^+(z) + C \ge u_\infty(z), \quad \forall\, z \in \Omega_\delta \setminus \overline{\Omega_\tau}. \tag{4.16}$$

We set $C' = \xi^- \psi(\delta)$. Hence, we have $C' \ge v_\tau^-(z)$ for every $z \in \Omega$ with $d(z) = \delta - \tau$. It follows that

$$u_\infty(z) + C' \ge v_\tau^-(z), \quad \forall\, z \in \partial\Omega_{\delta-\tau}. \tag{4.17}$$

We see that, for every $z \in \Omega_{\delta-\tau}$,

$$\det \partial_{i\bar{j}}(u_\infty(z) + C') = \det \partial_{i\bar{j}}(u_\infty(z)) = g(z) f(u_\infty(z)) \le g(z) f(u_\infty(z) + C'), \tag{4.18}$$

while by (4.3) we have

$$\det \partial_{i\bar{j}} v_\tau^-(z) \ge g(z) f(v_\tau^-(z)), \quad \forall\, z \in \Omega_{\delta-\tau}. \tag{4.19}$$

Using again Proposition 2.4, we infer that

$$u_\infty(z) + C' \geq v_\tau^-(z), \quad \forall\, z \in \Omega_{\delta-\tau}. \tag{4.20}$$

By (4.16) and (4.20), letting $\tau \to 0$, we obtain

$$\begin{cases} (1+2\epsilon)^{\frac{1}{n-q}}\xi^-\psi(d(z)) - C' \leq u_\infty(z), & \forall\, z \in \Omega_\delta, \\ u_\infty(z) \leq (1-2\epsilon)^{\frac{1}{n-q}}\xi^+\psi(d(z)) + C, & \forall\, z \in \Omega_\delta. \end{cases} \tag{4.21}$$

Dividing by $\psi(d(z))$ and letting $d(z) \to 0$, we obtain

$$\begin{cases} \liminf\limits_{d(z)\to 0} \dfrac{u_\infty(z)}{\psi(d(z))} \geq (1+2\epsilon)^{\frac{1}{n-q}}\xi^-, \\ \limsup\limits_{d(z)\to 0} \dfrac{u_\infty(z)}{\psi(d(z))} \leq (1-2\epsilon)^{\frac{1}{n-q}}\xi^+. \end{cases} \tag{4.22}$$

Since $\epsilon > 0$ is arbitary, we let $\epsilon \to 0$ and conclude (1.12). This completes the proof of Theorem 1.2.

5 The Proof of Uniqueness

Now we divide the proof of uniqueness into two steps.

Step 1 For the strictly plurisubharmonic blow-up solutions u_1, u_2 to (1.4), it holds that

$$\lim_{d(z)\to 0} \frac{u_1(z)}{u_2(z)} = 1. \tag{5.1}$$

Since u_1, u_2 are arbitrary, it suffices to show that

$$\liminf_{d(z)\to 0} \frac{u_1(z)}{u_2(z)} \geq 1. \tag{5.2}$$

Without loss of generality, we can assume that 0 belongs to Ω.

Case 1 Let $\epsilon \in (0,1)$ be fixed, and let $\lambda > 1$ be close to 1.

We set

$$C_\lambda = \Big[\Big((1+\epsilon)\lambda^{2n} \max_{z\in(\frac{1}{\lambda})\overline{\Omega}} \frac{g(\lambda z)}{g(z)}\Big)\Big]^{\frac{1}{q-n}}, \tag{5.3}$$

where $(\frac{1}{\lambda})\overline{\Omega} = \{(\frac{1}{\lambda})z : z \in \overline{\Omega}\}$. Note that $C_\lambda \to (1+\epsilon)^{\frac{1}{q-n}}$ as $\lambda \to 1$.

Hence by Proposition 2.1 and $\lim\limits_{d(z)\to 0} u_1(z) = \infty$, we deduce that there exists a $\delta = \delta(\epsilon) > 0$, independent of λ, such that

$$C_\lambda^q \frac{f(u_1)}{f(C_\lambda u_1)} \leq 1 + \epsilon, \quad \forall\, z \in \Omega_\delta \text{ and } \lambda \text{ close to } 1. \tag{5.4}$$

We now define U_λ as

$$U_\lambda(z) = C_\lambda u_1(\lambda z), \quad z \in \Big(\frac{1}{\lambda}\Big)\Omega_\delta. \tag{5.5}$$

We infer that

$$\begin{aligned}\det \frac{\partial^2 U_\lambda}{\partial z_i \partial \overline{z}_j} &= \lambda^{2n} C_\lambda^n g(\lambda z) f(u_1(\lambda z)) \\ &\le \lambda^{2n} C_\lambda^{n-q} g(\lambda z) f(C_\lambda u_1(\lambda z)) \\ &\le g(z) f(C_\lambda u_1(\lambda z)) \\ &= g(z) f(U_\lambda(z)), \quad z \in \Big(\frac{1}{\lambda}\Big)\Omega_\delta, \end{aligned} \tag{5.6}$$

that is, $U_\lambda(z)$ is a supersolution of (1.4).

Since f is increasing in $(0, \infty)$, for each constant $M > 0$,

$$\det \frac{\partial^2 (U_\lambda + M)}{\partial z_i \partial \overline{z}_j} = \det \frac{\partial^2 U_\lambda}{\partial z_i \partial \overline{z}_j} \le g(z) f(U_\lambda(z)) \le g(z) f(U_\lambda(z) + M). \tag{5.7}$$

Note that $U_\lambda(z) = \infty > u_2(z)$ for $z \in (\frac{1}{\lambda})\partial\Omega$. Moreover, if we choose $M = \max\limits_{d(z)=\delta} u_2(z)$, by Proposition 2.4, we obtain

$$U_\lambda(z) + M \ge u_2(z), \quad \forall\, z \in \Omega_\delta \cap \Big(\frac{1}{\lambda}\Big)\Omega_\delta. \tag{5.8}$$

Letting $\lambda \to 1$, we find

$$(1+\epsilon)^{\frac{1}{q-n}} u_1(z) + M \ge u_2(z), \quad \forall\, z \in \Omega_\delta, \tag{5.9}$$

which implies that

$$\liminf_{d(z)\to 0} \frac{u_1}{u_2} \ge (1+\epsilon)^{\frac{1}{n-q}}. \tag{5.10}$$

Then, letting $\epsilon \to 0$, we obtain (5.2).

Case 2 $g(z) = 0$ on $\partial\Omega$, by Corollary 1.1, every plurisubharmonic blow-up solution u to (1.4) satisfies

$$\lim_{d(z)\to 0} \frac{u(z)}{\varphi(d(z))} = \Big\{ \frac{[\frac{q-n}{n+1}]^{n+1} R^{n-1}}{1 + \frac{l(q-n)}{n+1}} \Big\}^{\frac{1}{n-q}}. \tag{5.11}$$

Hence, the assertion of Step 1 is proved in both cases.

Step 2 There is at most one strictly plurisubharmonic blow-up solution to (1.4). If u_1, u_2 are arbitrary, strictly plurisubharmonic blow-up solutions to (1.4), it suffices to show that $u_1 \le u_2$ in Ω. Fixing $\delta > 0$, by Step 1, we infer that

$$\lim_{d(z)\to 0} [u_1(z) - (1+\delta) u_2(z)] = -\infty. \tag{5.12}$$

Since $\frac{f(u)}{u^n}$ is increasing, we deduce that

$$\det \frac{\partial^2 (1+\delta) u_2(z)}{\partial z_i \partial \overline{z}_j} \le g(z) f((1+\delta) u_2(z)), \quad \forall\, z \in \Omega. \tag{5.13}$$

By (5.12)–(5.13) and Proposition 2.4, we find $u_1(z) \le (1+\delta) u_2(z)$ in Ω. Letting $\delta \to 0$, we obtain $u_1 \le u_2$ in Ω. This completes the proof of Theorem 1.3.

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