Chin. Ann. Math.
29B(5), 2008, 501–520
DOI: 10.1007/s11401-007-0380-z

Chinese Annals of Mathematics, Series B

Propagation of Density-Oscillations in Solutions to the Compressible Navier-Stokes-Poisson System***

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Abstract Concerning a bounded sequence of finite energy weak solutions to the compressible Navier-Stokes-Poisson system (denoted by CNSP), which converges up to extraction of a subsequence, the limit system may not be the same system. By introducing Young measures as in [6, 15], the authors deduce the system (HCNSP) which the limit functions must satisfy. Then they solve this system in a subclass where Young measures are convex combinations of Dirac measures, to give the information on the propagation of density-oscillations. The results for strong solutions to (CNSP) (see Corollary 6.1) are also obtained.

Keywords Compressible fluids, Navier-Stokes-Poisson equations, Young measures, Propagation of oscillations, Strong solutions
2000 MR Subject Classification 35A05, 35M10, 76M50, 76N10

1 Introduction

The motion of a compressible viscous isentropic fluid, confined in a bounded smooth domain $\Omega \subset \mathbb{R}^3$, flowing under the self-gravitational force can be described as the system of the Navier-Stokes-Poisson equations, i.e. for $(t,x) \in (0,T) \times \Omega$,

$$\rho_t + \operatorname{div}(\rho\mathbf{u}) = 0, \tag{1.1}$$

$$(\rho\mathbf{u})_t + \operatorname{div}(\rho\mathbf{u} \otimes \mathbf{u}) + \nabla p - \mu\Delta\mathbf{u} - (\lambda+\mu)\nabla(\operatorname{div}\mathbf{u}) = \rho\nabla\Phi, \tag{1.2}$$

$$-\Delta\Phi = 4\pi g\Big(\rho - \frac{1}{|\Omega|}\int_\Omega \rho\Big), \tag{1.3}$$

where the unknown functions $\rho(t,x)$, $\mathbf{u}(t,x)$, $p(t,x) = P(\rho) = a\rho^\gamma$, $\Phi(t,x)$ denote the density, velocity, pressure and Newtonian gravitational potential of the fluid, respectively. The viscosity coefficients μ, λ satisfy

$$\mu > 0, \quad \lambda + \frac{2}{3}\mu \geq 0.$$

$g > 0$ is the gravitational constant, $a > 0$ is a constant, and the adiabatic constant $\gamma > 1$.

Manuscript received September 23, 2007. Published online August 27, 2008.
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***Supported by the National Natural Science Foundation of China (No. 10531020) and the Program of 985 Innovation Engineering on Information in Xiamen University (2004–2007) and the New Century Excellent Talents in Xiamen University.

We complement this system with the initial-boundary conditions

$$\begin{aligned} &\rho|_{t=0} = \rho^0, \quad (\rho\mathbf{u})|_{t=0} = \mathbf{q}^0 \quad \text{in } \Omega, \\ &\mathbf{u} = 0, \quad \frac{\partial \Phi}{\partial \nu} = 0 \quad \text{on } (0,T)\times\partial\Omega \end{aligned} \tag{1.4}$$

and refer to the system (1.1)–(1.4) as (CNSP).

The compressible fluids have been studied by many authors (see [3–7, 9–16, 19–20]). In particular, for the system without Poisson term (i.e. Φ), namely Navier-Stokes equations, Lions [12] proved the existence of weak solutions for $\gamma > \frac{9}{5}$ for the general large initial data, and Feireisl [4] presented some ideas to extend Lions's result to the optimal constraint $\gamma > \frac{3}{2}$. While for the full system (CNSP), using the techniques introduced in Lions [12], Feireisl [4] and additional regularity results on the elliptic equation (1.3), Kobayashi [10] proved that there exists a finite energy weak solution globally in time under the constraint $\gamma > \frac{3}{2}$. Here, a finite energy weak solution in $(0,T)$ means a triplet of functions $(\rho, \mathbf{u}, \Phi)$ satisfying

(1) $\rho \geq 0$, $\rho \in L^\infty(0,T;L^\gamma(\Omega))$, $\mathbf{u} \in L^2(0,T;H_0^1(\Omega))$.

(2) $E = E(t) \in L^1_{\mathrm{loc}}(0,T)$ and $\frac{\mathrm{d}}{\mathrm{d}t}E(t) + \int_\Omega \mu|\nabla\mathbf{u}|^2 + (\lambda+\mu)|\mathrm{div}\mathbf{u}|^2 \leq 0$ holds in $\mathcal{D}'(0,T)$, where E is the total energy defined as

$$E(t) = \int_\Omega \frac{1}{2}\rho|\mathbf{u}|^2 + \frac{a}{\gamma-1}\rho^\gamma - \frac{1}{8\pi g}|\nabla\Phi|^2 \mathrm{d}x.$$

(3) Equations (1.1) and (1.2) are satisfied in $\mathcal{D}'((0,T)\times\Omega)$. Moreover, provided that ρ and $\mathbf{u}$ are prolonged to be zero on $\mathbb{R}^3\backslash\Omega$, (1.1) holds in $\mathcal{D}'((0,T)\times\mathbb{R}^3)$.

(4) $\Phi(t,\cdot) = g\int_\Omega G(\cdot,y)\rho(t,y)\mathrm{d}y$ for a.e. $t \in (0,T)$, where $G = G(x,y)$ denotes the Green's function of the Poisson part.

(5) Moreover, equation (1.1) is satisfied in the sense of renormalized solutions, i.e.,

$$(b(\rho))_t + \mathrm{div}(b(\rho)\mathbf{u}) + (b'(\rho)\rho - b(\rho))\mathrm{div}\mathbf{u} = 0, \quad \text{in } \mathcal{D}'((0,T)\times\Omega) \tag{1.5}$$

for any $b \in C^1(\mathbb{R})$ such that

$$b'(z) = 0 \quad \text{for } |z| \text{ large enough.} \tag{1.6}$$

On the other hand, since the time evolution of the density ρ is governed by the hyperbolic equation (1.1), it is plausible to expect the oscillations in initial data will be transported by the flow (see [4]). So, if we select a bounded sequence of finite energy weak solutions $(\rho_n, \mathbf{u}_n, \Phi_n)$ which converges to a triplet $(\rho, \mathbf{u}, \Phi)$ up to the extraction if necessary, the limit functions may not satisfy (CNSP), for instance, when densities oscillate faster and faster (see [12]). For this, in Section 3 we will introduce Young measures, as done in [6], to deduce that the limit functions must satisfy a "homogenized system" (HCNSP) (see Theorem 3.1), whose solutions are the triplets of $(\nu, \mathbf{u}, \Phi)$, where

$$\nu = \{\nu_{(t,x)}\}_{(t,x)\in(0,T)\times\Omega}$$

is the Young measure (a family of probability measures, see Definition 3.1) associated with the sequence $\{\rho_n\}_{n\in N}$. Next, we intend to solve the system (HCNSP) in a subclass of Young measures that ν can be expressed as a convex combination of finite Dirac measures, i.e.,

$$\nu_{(t,x)} = \sum_{i=1}^k \alpha_i(t,x)\delta_{\rho_i(t,x)}, \quad \forall\,(t,x)\in(0,T)\times\Omega. \tag{1.7}$$

In Section 4, by a crucial technique, we instead begin to treat a reduced system (PHCNSP) whose solutions a priori are solutions to (HCNSP) and we will prove that system (HCNSP) admits at most one solution and actually is the only one solution to (PHCNSP), if the latter exists (see Theorem 4.1). Then we will prove the local existence of solutions to (PHCNSP) with initial data sufficiently regular in Section 5 and the global existence with small data in Section 6. The theorems obtained in our paper give the information on the possible persistence of oscillations in solutions to (CNSP).

2 Preliminaries

In this note, we denote the norms of the spaces $L^p(\Omega)$, $W^{m,p}(\Omega)$ and $H^m(\Omega)$ by $|\cdot|_p$, $\|\cdot\|_{m,p}$ and $\|\cdot\|_m$ respectively, while the norms of $L^q(0,T;L^p(\Omega))$, $L^q(0,T;W^{m,p}(\Omega))$ and $L^q(0,T;H^m(\Omega))$ are denoted by $|||\cdot|||_{q,0,p}$, $|||\cdot|||_{q,m,p}$ and $|||\cdot|||_{q,m}$ respectively. The symbol C is a positive generic constant depending at most on λ,μ,a,g,T and Ω which may take different values in different formulas and C_0 is a positive constant depending only on C and initial data. We will point out special dependencies if necessary. We introduce the duality bracket $\langle\cdot,\cdot\rangle$ between measures and bounded continuous functions, and the functions I and P stand for the identity on $\mathbb{R}$ and $x\to ax^\gamma$ respectively. Let Y denote the set of continuous functions b defined on $\mathbb{R}$ satisfying (1.6). For simplification of notations, we denote the operator $\mathcal{L}\mathbf{u}=-\mu\Delta\mathbf{u}-(\lambda+\mu)\nabla(\mathrm{div}\mathbf{u})$. Finally, for a vector-function $\mathbf{r}=(r_1,\cdots,r_m)$ defined over a domain $\mathcal{O}$, we shall use

$$\overline{\mathbf{r}}:=\sup_{i=1,\cdots,m}\sup_{y\in\mathcal{O}}r_i(y),\quad \underline{\mathbf{r}}:=\inf_{i=1,\cdots,m}\inf_{y\in\mathcal{O}}r_i(y).$$

We consider a bounded sequence of global finite energy weak solutions $(\rho_n,\mathbf{u}_n,\Phi_n)$ which converges to a triplet $(\rho,\mathbf{u},\Phi)$ in some sense (see Theorem 3.1). In this paper, convergences of sequences are implicitly considered up to the extraction of a subsequence. To get the system (HCNSP) which the limit functions satisfy, we need the following lemmas.

Lemma 2.1 *Given $b\in Y$ with compact support, let $b(\rho_n)\to\overline{b}$ weakly-$*$ in $L^\infty((0,T)\times\Omega)$. Then for all $\phi\in\mathcal{D}((0,T)\times\Omega)$, we have*

$$\begin{aligned}&\lim_{n\to\infty}\int_0^T\int_\Omega[(P(\rho_n)-(\lambda+2\mu)\mathrm{div}(\mathbf{u}_n))b(\rho_n)]\phi(t,x)\mathrm{d}x\mathrm{d}t\\ =&\int_0^T\int_\Omega[(q-(\lambda+2\mu)\mathrm{div}\mathbf{u})\overline{b}]\phi(t,x)\mathrm{d}x\mathrm{d}t,\end{aligned}\tag{2.1}$$

where q and $\mathbf{u}$ are the limit functions which will be found in Theorem 3.3*.*

Proof See [10, Lemma 4.2] for details, just replacing T_k there by b without any other modifications.

Lemma 2.2 *Let (X,μ) be a measurable set with finite measure. Assume that $f_n:X\to\mathbb{R}$ is a sequence in $L^\alpha(X,\mu)$ $(1\le\alpha<\infty)$ converging weakly to f in this space. Then*

$$\lim_{k\to\infty}\limsup_{n\to\infty}\int_{\{|f_n|>k\}}|f_n|\mathrm{d}\mu=0.\tag{2.2}$$

Proof See [6, Lemma 2].

3 Homogenized System (HCNSP)

In this section, we will introduce Young measures to derive the system (HCNSP) (to be explained later), and this will be done in the following theorems.

Theorem 3.1 *Let $\gamma > \frac{3}{2}$. Assume $(\rho_n, \mathbf{u}_n, \Phi_n)$ to be a bounded sequence of global finite energy weak solutions, and the limit functions*

$$\rho \in L^\infty(0,T;L^\gamma(\Omega)), \quad \mathbf{u} \in L^2(0,T;H_0^1(\Omega)), \quad \Phi \in L^\infty(0,T;W^{2,\gamma}(\Omega)).$$

Then there exists a Young measure, $\nu = \{\nu_{(t,x)}\}_{(t,x)\in(0,T)\times\Omega}$, a family of probability measures, such that

(i) *we have*

$$\langle \nu, I\rangle = \rho \text{ and } \langle \nu, P\rangle = q \text{ for some } q, \text{ in a sense to be precise;} \tag{3.1}$$

(ii) *for all $b \in C(\mathbb{R}^+)$, smooth, with compact support,*

$$\begin{aligned} &(\langle \nu, b\rangle)_t + \mathrm{div}(\langle \nu, b\rangle \mathbf{u}) + \langle \nu, (Ib' - b)\rangle \mathrm{div}\mathbf{u} \\ &= \frac{\langle \nu, (Ib'-b)\rangle q - \langle \nu, (Ib'-b)P\rangle}{\lambda + 2\mu}, \quad \text{in } \mathcal{D}'((0,T)\times\Omega); \end{aligned} \tag{3.2}$$

(iii) *$\rho, q, \mathbf{u}$ and Φ satisfy*

$$\begin{cases} \rho_t + \mathrm{div}(\rho\mathbf{u}) = 0, \\ (\rho\mathbf{u})_t + \mathrm{div}(\rho\mathbf{u}\otimes\mathbf{u}) + \nabla q + \mathcal{L}\mathbf{u} = \rho\nabla\Phi, \quad \text{in } \mathcal{D}'((0,T)\times\Omega), \\ -\Delta\Phi = 4\pi g\Big(\rho - \frac{1}{|\Omega|}\int_\Omega \rho\Big) \end{cases} \tag{3.3}$$

together with the boundary conditions

$$\mathbf{u} = 0, \quad \frac{\partial\Phi}{\partial\nu} = 0, \quad \text{on } (0,T)\times\partial\Omega. \tag{3.4}$$

Proof The uniform bounds on the solutions imply that there exists a triplet

$$\rho \in L^\infty(0,T;L^\gamma(\Omega)), \quad \mathbf{u} \in L^2(0,T;H_0^1(\Omega)), \quad \Phi \in L^\infty(0,T;W^{2,\gamma}(\Omega))$$

such that

$$\begin{aligned} &\rho_n \to \rho, \quad \text{in } C([0,T];L^\gamma_{\text{weak}}(\Omega)), \quad \mathbf{u}_n \to \mathbf{u}, \quad \text{weakly in } L^2(0,T;H_0^1(\Omega)), \\ &\rho_n\mathbf{u}_n \to \rho\mathbf{u}, \quad \text{in } C([0,T];L^{\frac{2\gamma}{\gamma+1}}_{\text{weak}}(\Omega)). \end{aligned}$$

Then, for $\gamma > \frac{3}{2}$, $\frac{2\gamma}{\gamma+1} > \frac{6}{5}$, $L^{\frac{2\gamma}{\gamma+1}}(\Omega) \subset\subset H^{-1}(\Omega)$, we have

$$\rho_n\mathbf{u}_n\otimes\mathbf{u}_n \to \rho\mathbf{u}\otimes\mathbf{u}, \quad \text{in } \mathcal{D}'((0,T)\times\Omega).$$

While the elliptic regularity guarantees $\nabla\Phi_n \to \nabla\Phi$ in $C([0,T];W^{1,\gamma}_{\text{weak}}(\Omega))$, then if $\gamma \geq 3$, it is easy to obtain $\nabla\Phi_n \to \nabla\Phi$ in $C([0,T];L^\gamma(\Omega))$, hence

$$\rho_n\nabla\Phi_n \to \rho\nabla\Phi, \quad \text{in } C([0,T];L^{\frac{\gamma}{2}}_{\text{weak}}(\Omega));$$

if $\frac{3}{2} < \gamma < 3$, since $W^{1,\gamma}(\Omega) \subset\subset L^{\frac{3\gamma}{3-\gamma}}(\Omega)$, we have $\nabla\Phi_n \to \nabla\Phi$ in $C([0,T]; L^{\frac{3\gamma}{3-\gamma}}(\Omega))$, hence

$$\rho_n \nabla\Phi_n \to \rho\nabla\Phi, \quad \text{in } C([0,T]; L^{\frac{3\gamma}{6-\gamma}}_{\text{weak}}(\Omega)).$$

Both cases imply that

$$\rho_n \nabla\Phi_n \to \rho\nabla\Phi, \quad \text{in } \mathcal{D}'((0,T)\times\Omega).$$

On the other hand, using the regularity properties of the Bogovskii operator, one may improve integrability of ρ_n that $P(\rho_n)$ is bounded in $L^{\frac{\gamma+\theta}{\gamma}}((0,T)\times\Omega)$, where $\theta = \frac{2}{3}\gamma - 1$ (see [10, Lemma 4.1] for more details). Then, $P(\rho_n) \to q$ weakly in $L^{\frac{\gamma+\theta}{\gamma}}((0,T)\times\Omega)$.

Consequently, we can pass to the limit in (1.1)–(1.3) in the sense of distributions (at least) to obtain (3.3), and the boundary conditions (3.4) follows directly from the convergences of $\mathbf{u}_n$ and Φ_n. This proves (iii).

Now, we introduce Young measures. By the definition of Y, for any $b \in Y$, the sequence $\{b(\rho_n)\}_{n\in N}$ is bounded in $L^\infty((0,T)\times\Omega)$ uniformly with respect to n. Then

$$b(\rho_n) \to \overline{b}, \quad \text{weakly-}* \text{ in } L^\infty((0,T)\times\Omega).$$

For almost every $(t,x) \in (0,T)\times\Omega$, the functional: $b \to \overline{b}(t,x)$ is a positive linear mapping which maps the constant function $\mathbb{R} \to \{1\}$ to 1 . Noticing that $(Y, |\cdot|_\infty)$ is separable, we may in this way define a family of probability measures $\{\nu_{(t,x)}\}_{(t,x)\in(0,T)\times\Omega}$ such that

$$\overline{b}(t,x) = \langle \nu_{(t,x)}, b\rangle \equiv \int_{\mathbb{R}} b(y)\mathrm{d}\nu_{(t,x)}(y), \quad \text{a.e. } (t,x) \in (0,T)\times\Omega. \tag{3.5}$$

Definition 3.1 *We call* $\nu := \{\nu_{(t,x)}\}_{(t,x)\in(0,T)\times\Omega}$ *the Young measure associated with the sequence* $\{\rho_n\}_{n\in N}$.

For more general theory about Young measures, we let the readers refer to [2, 17, 18] for instance.

Next, for any smooth $b \in C(\mathbb{R}^+)$, with compact support, obviously $b \in Y$. As $(\rho_n, \mathbf{u}_n)$ is a renormalized solution to (1.1), we have

$$b(\rho_n)_t + \mathrm{div}(b(\rho_n)\mathbf{u}_n) = (b(\rho_n) - b'(\rho_n)\rho_n)\mathrm{div}(\mathbf{u}_n), \quad \text{in } \mathcal{D}'((0,T)\times\Omega). \tag{3.6}$$

Hence, $b(\rho_n)$ and $b(\rho_n)_t$ are bounded in $L^\infty(\Omega)$ and $L^2(0,T;H^{-1}(\Omega))$ respectively. Applying the Aubin-Lions Lemma, we see that $b(\rho_n)$ converges to $\overline{b}$ in $L^2(0,T;H^{-1}(\Omega))$. Then, $b(\rho_n)\mathbf{u}_n$ converges to $\overline{b}\mathbf{u}$ in the sense of distributions, at least. We can rewrite the right-hand side of (3.6) as

$$\mathrm{RHS}_n = \frac{(b(\rho_n) - b'(\rho_n)\rho_n)P(\rho_n) - (b(\rho_n) - b'(\rho_n)\rho_n)(P(\rho_n) - (\lambda+2\mu)\mathrm{div}\mathbf{u}_\mathrm{n})}{\lambda+2\mu},$$

where $(Ib' - b)P \in Y$ has compact support. Consequently, with Lemma 2.1, when $n \to \infty$, RHS_n converges, in the sense of distributions, to

$$\mathrm{RHS}_\infty = \frac{\langle \nu, (b - Ib')P\rangle - \langle \nu, (b - Ib')\rangle(q - (\lambda+2\mu)\mathrm{div}\mathbf{u})}{\lambda+2\mu}$$

and thus we have (3.2). This proves (ii).

Finally, we give a sense to (i) in the following. As I and P do not belong to Y, we introduce a family of truncation functions $T_k:\mathbb{R}^+\to\mathbb{R}^+$ defined by $T_k(z):=\min(z,k)$. Then T_k and $T_k\circ P$ are two families of elements of Y. As $T_k(z)=z$ for $|z|\le k$, by Lemma 2.2, we have

$$\lim_{k\to\infty}\int_\Omega|\langle\nu,T_k\circ P\rangle-q|\le\lim_{k\to\infty}\varliminf_{n\to\infty}\int_{\{a\rho_n^\gamma>k\}}|k-a\rho_n^\gamma|\mathrm{d}x\le\lim_{k\to\infty}2\varliminf_{n\to\infty}\int_{\{a\rho_n^\gamma>k\}}|a\rho_n^\gamma|\mathrm{d}x=0.$$

Then we obtain $\lim\limits_{k\to\infty}\langle\nu,T_k\circ P\rangle=q$. Similarly, we can prove $\lim\limits_{k\to\infty}\langle\nu,T_k\circ I\rangle=\rho$. We may in this way set $\langle\nu,I\rangle:=\lim\limits_{k\to\infty}\langle\nu,T_k\circ I\rangle$, $\langle\nu,P\rangle:=\lim\limits_{k\to\infty}\langle\nu,T_k\circ P\rangle$ to obtain (i) and the proof of Theorem 3.1 completed.

On the other hand, noticing that

$$\rho_n\to\rho\in C([0,T];L^\gamma_{\text{weak}}(\Omega))\quad\text{and}\quad\rho_n\mathbf{u}_n\to\rho\mathbf{u}\in C([0,T];L^{\frac{2\gamma}{\gamma+1}}_{\text{weak}}(\Omega)),$$

we may give initial conditions for the finite energy weak solutions, and initial data ρ_n^0 (resp. $\mathbf{q}_n^0$) converges also weakly in $L^\gamma(\Omega)$ (resp. $L^{\frac{2\gamma}{\gamma+1}}(\Omega)$) to some ρ^0 (resp. $\mathbf{q}^0$) such that $\rho(0,\cdot)=\rho^0$ (resp. $\rho\mathbf{u}(0,\cdot)=\mathbf{q}^0$). Introducing $\overline{b}^0:=\lim b(\rho_n^0)$, we obtain

Theorem 3.2 *Under the assumptions of Theorem* 3.1*, there exists a family of probability measures* $\{\nu_x^0\}_{x\in\Omega}$*, such that, for all smooth* $b\in Y$ *with compact support,*

$$\langle\nu,b\rangle(0,\cdot)=\langle\nu^0,b\rangle,\quad a.e.\ x\in\Omega,\tag{3.7}$$

where ν *is the Young measure derived in Theorem* 3.1*.*

From now on, the system (3.2), (3.3) together with initial data (3.7), $(\rho\mathbf{u})_{t=0}=\mathbf{q}^0$, boundary conditions (3.4) and compatibility condition (3.1) is referred to as (HCNSP), standing for Homogenized Compressible Navier-Stokes-Poisson equations.

In the sequel, we search solutions for the homogenized system (HCNSP), denoted by $(\nu,\mathbf{u},\Phi)$. Firstly, notice that Φ is determined by ρ in accordance with $(3.3)_3$, so we search solutions to (HCNSP) for $(\nu,\mathbf{u},\Phi)$, implicitly $(\nu,\mathbf{u})$. As the density satisfies a hyperbolic equation, initial oscillations may persist in time. So, if initial density oscillates between k values, it seems reasonable to assume that it will oscillate as well at least locally in time. More precisely, as in [6], let us introduce S_0^k, the set of pairs $(\nu^0,\mathbf{u}^0)$, such that $\mathbf{u}^0\in H_0^1(\Omega)\cap H^2(\Omega)$ and ν^0 is the convex combination, with weights $(\alpha_i^0)_{i=1,\cdots,k}\in H^2(\Omega)$ of Dirac measures located in $(\rho_i^0)_{i=1,\cdots,k}\in H^2(\Omega)$, i.e.

$$\nu^0(x)=\sum_{i=1}^k\alpha_i^0(x)\delta_{\rho_i{}^0(x)}.$$

Moreover, we assume $\underline{\mathbf{r}}^0>0$, $\rho_i{}^0(x)\neq\rho_j{}^0(x)$, $\forall\,i\neq j$, $\forall\,x\in\Omega$. The no-vacuum hypothesis is added here for technical purpose. Consequently, the initial condition on $\rho\mathbf{u}$ is translated into a condition on $\mathbf{u}$. Then by the motivation mentioned before, we plan to search solutions to (HCNSP) (with initial data in S_0^k) in the subset. We denote by S_T^k the set of pairs $(\nu,\mathbf{u})$ where

$$\mathbf{u}\in C([0,T];H_0^1(\Omega)\cap H^2(\Omega))\cap L^2(0,T;H^3(\Omega))$$

and ν is the convex combination, with weights $(\alpha_i)_{i=1,\cdots,k}\in C([0,T];H^2(\Omega))$ of Dirac measures located in $(\rho_i)_{i=1,\cdots,k}\in C([0,T];H^2(\Omega))$, i.e.

$$\nu(t,x)=\sum_{i=1}^k\alpha_i(t,x)\delta_{\rho_i(t,x)}.$$

Moreover, we assume $\underline{\mathbf{r}} > 0,\ \rho_i(t,x) \neq \rho_j(t,x),\ \forall\, i \neq j,\ \forall\,(t,x) \in (0,T) \times \Omega$.

The above formulas define a mapping $(\mathbf{u}, \mathbf{r} := (\rho_1, \cdots, \rho_k), \mathbf{a} := (\alpha_1, \cdots, \alpha_k)) \to (\nu, \mathbf{u})$ that enables us to identify S_T^k (resp. S_o^k) with

$$X_T^k := (C([0,T]; H_0^1(\Omega) \cap H^2(\Omega)) \cap L^2(0,T; H^3(\Omega))) \times C([0,T]; H^2(\Omega))^{2k}$$

(resp. $X_0^k := (H_0^1(\Omega) \cap H^2(\Omega)) \times H^2(\Omega)^{2k}$) with the corresponding restrictions to densities $(\rho_i \neq \rho_j)$ and weights $(\sum \alpha_i = 1,\ \alpha_i \geq 0)$.

Now, we can search solutions for (HCNSP) inside X_T^k. And on such a subclass of solutions, we can reduce the system (HCNSP) as follows.

Theorem 3.3 *Given* $(\mathbf{u}, \mathbf{r}, \mathbf{a}) \in X_T^k$*, we denote*

$$f_{\alpha_i} := \frac{\alpha_i(a\rho_i^\gamma - q)}{\lambda + 2\mu}, \quad f_{\rho_i} := \frac{\rho_i(q - a\rho_i^\gamma)}{\lambda + 2\mu}, \quad i = 1, \cdots, k. \tag{3.8}$$

Then, $(\mathbf{u}, \mathbf{r}, \mathbf{a})$ *is a solution to (HCNSP) with initial data* $(\mathbf{u^0}, \mathbf{r^0}, \mathbf{a^0}) \in X_0^k$*, if and only if*

$$\rho = \sum_{i=1}^k \alpha_i \rho_i, \quad q = a \sum_{i=1}^k \alpha_i \rho_i^\gamma, \tag{3.9}$$

and

$$\begin{cases} (\alpha_i)_t + \mathbf{u} \cdot \nabla \alpha_i = f_{\alpha_i}, \\ \alpha_i((\rho_i)_t + \operatorname{div}(\rho_i \mathbf{u})) = \alpha_i f_{\rho_i}, \\ (\rho \mathbf{u})_t + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla q + \mathcal{L}\mathbf{u} = \rho \nabla \Phi, \\ -\Delta \Phi = 4\pi g\Big(\rho - \dfrac{1}{|\Omega|} \displaystyle\int_\Omega \rho\Big) \end{cases} \tag{3.10}$$

with initial-boundary conditions

$$\begin{aligned} &\mathbf{u}|_{\partial\Omega} = 0, \quad \frac{\partial \Phi}{\partial \nu}\Big|_{\partial\Omega} = 0, \\ &(\mathbf{u})|_{t=0} = \mathbf{u}^0, \quad (\alpha_i)|_{t=0} = \alpha_i^0, \quad (\rho_i)|_{t=0} = \rho_i^0. \end{aligned} \tag{3.11}$$

Proof We prove that (HCNSP) implies this new system. For this, let $(\mathbf{u}, \mathbf{r}, \mathbf{a})$ be a solution to (HCNSP). First, as ν satisfies (3.1), (3.9) holds. Next, for any smooth $b \in Y$ with compact support, let us denote $b_i := b(\rho_i)$ and $b_i' := b'(\rho_i)$. Applying (3.2) with b and replacing, one recovers that

$$\sum_{i=1}^k \{((\alpha_i)_t + \mathbf{u} \cdot \nabla \alpha_i - f_{\alpha_i}) b_i + (\alpha_i((\rho_i)_t + \operatorname{div}(\rho_i \mathbf{u})) - \alpha_i f_{\rho_i}) b_i'\} = 0. \tag{3.12}$$

Because $\rho_i(t,x) \neq \rho_j(t,x),\ \forall\, i \neq j,\ \forall\,(t,x) \in (0,T) \times \Omega$, and $(b_i, b_i')_{i=1,\cdots,k}$ are independent functions, (3.12) implies $(3.11)_1$ and $(3.10)_2$. Other equations or conditions can be obtained by immediate calculus and converse implication also.

Consequently, in the following, instead of searching $(\nu, \mathbf{u})$ solutions to (HCNSP), we look for $(\mathbf{u}, \mathbf{r}, \mathbf{a})$ solutions to (HCNSP) in X_T^k identifying implicitly the system (3.8)–(3.12).

4 Uniqueness for (HCNSP)

We prove in this section an interesting uniqueness result for the system (HCNSP). For this, first of all, notice that in (HCNSP) the equation (i.e. $(3.10)_2$) concerning ρ_i is multiplied by α_i. It seems difficult to treat ρ_i if α_i disappears. However, contrary to the equation (1.1) on ρ in (CNSP), $\mathbf{u}$ is only involved through its characteristics, and we may reach a uniqueness result. Motivated by this, we plan to study (HCNSP) replacing $(3.10)_2$ by

$$(\rho_i)_t + \operatorname{div}(\rho_i \mathbf{u}) = f_{\rho_i}, \quad i = 1, \cdots, k,$$

and we complement the system where $\alpha_i = 0$ by the same equation. A solution to this new system is a fortiori a solution to (HCNSP). Then, we prove that it is actually the only solution to (HCNSP) in X_T^k if it exists.

Let us recall precisely the system under consideration

$$\begin{cases} (\alpha_i)_t + \mathbf{u} \cdot \nabla \alpha_i = f_{\alpha_i}, \\ (\rho_i)_t + \operatorname{div}(\rho_i \mathbf{u}) = \mathbf{f}_{\rho_\mathbf{i}} \end{cases} \tag{4.1}$$

together with

$$\begin{cases} (\rho \mathbf{u})_t + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla q + \mathcal{L}\mathbf{u} = \rho \nabla \Phi, \\ -\Delta \Phi = 4\pi g\Big(\rho - \dfrac{1}{|\Omega|} \displaystyle\int_\Omega \rho\Big), \end{cases} \tag{4.2}$$

where ρ and q are defined by (3.9) and f_{α_i}, f_{ρ_i} by (3.8) accordingly. Complementing the system with initial-boundary conditions (3.11), we denote this system by (PHCNSP), standing for Positive Homogenized Compressible Navier-Stokes-Poisson equations.

We will prove that if the system (PHCNSP) has a solution in X_T^k then the system (HCNSP) has only one solution, actually the unique solution to (PHCNSP). This can be deduced from the fact that in our case Young measures are convex combinations of Dirac measures together with the following uniqueness result.

Theorem 4.1 *Assume that* $(\mathbf{u}, \mathbf{r}, \mathbf{a})$ *and* $(\widehat{\mathbf{u}}, \widehat{\mathbf{r}}, \widehat{\mathbf{a}}) \in X_T^k$ *are solutions to* (*HCNSP*) *and* (*PHCNSP*) *respectively. Then*

$$\mathbf{u} = \widehat{\mathbf{u}}, \quad \mathbf{a} = \widehat{\mathbf{a}}, \quad (\rho_i)|_{\alpha_i > 0} = (\widehat{\rho}_i)|_{\widehat{\alpha}_i > 0}, \quad \forall\, i = 1, \cdots, k.$$

Proof First of all, denoting ρ, $\widehat{\rho}$, q and $\widehat{q}$ as in (3.9) respectively and setting $\mathbf{w} := \mathbf{u} - \widehat{\mathbf{u}}$, we can notice that

$$\rho - \widehat{\rho} = \sum_{i=1}^k (\alpha_i \rho_i - \widehat{\alpha}_i \widehat{\rho}_i), \quad q - \widehat{q} = \sum_{i=1}^k (\alpha_i \rho_i - \widehat{\alpha}_i \widehat{\rho}_i) \widehat{\rho}_i^{\gamma-1} + \alpha_i \rho_i (\rho_i^{\gamma-1} - \widehat{\rho}_i^{\gamma-1}).$$

It is easy to check that for $i = 1, \cdots, k$,

$$(\alpha_i \rho_i)_t + \operatorname{div}(\alpha_i \rho_i \mathbf{u}) = 0, \quad (\widehat{\alpha}_i \widehat{\rho}_i)_t + \operatorname{div}(\widehat{\alpha}_i \widehat{\rho}_i \widehat{\mathbf{u}}) = 0.$$

Take the difference of these two equations, multiply it by $(\alpha_i \rho_i - \widehat{\alpha}_i \widehat{\rho}_i)$, and then integrate over Ω (by parts). We obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{|\alpha_i \rho_i - \widehat{\alpha}_i \widehat{\rho}_i|_2^2}{2}\Big) + \int_\Omega \operatorname{div}((\alpha_i \rho_i - \widehat{\alpha}_i \widehat{\rho}_i)\mathbf{u})(\alpha_i \rho_i - \widehat{\alpha}_i \widehat{\rho}_i) + \operatorname{div}(\widehat{\alpha}_i \widehat{\rho}_i \mathbf{w})(\alpha_i \rho_i - \widehat{\alpha}_i \widehat{\rho}_i) \mathrm{d}x = 0.$$

Noticing that $\mathbf{w}|_{\partial\Omega} = \mathbf{u}|_{\partial\Omega} = \widehat{\mathbf{u}}|_{\partial\Omega} = 0$ and $0 \le \alpha_i, \widehat{\alpha}_i \le 1$, we obtain for arbitrary $\varepsilon > 0$,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{|\alpha_i\rho_i - \widehat{\alpha}_i\widehat{\rho}_i|_2^2}{2}\Big)\\
&\le \Big|\int_\Omega \frac{1}{2}\mathrm{div}\mathbf{u}(\alpha_i\rho_i - \widehat{\alpha}_i\widehat{\rho}_i)^2\mathrm{d}x\Big| + \Big|\int_\Omega \mathrm{div}\mathbf{w}\widehat{\alpha}_i\widehat{\rho}_i(\alpha_i\rho_i - \widehat{\alpha}_i\widehat{\rho}_i)\mathrm{d}x\Big|\\
&\quad + \Big|\int_\Omega \mathbf{w}\cdot\nabla(\widehat{\alpha}_i\widehat{\rho}_i)(\alpha_i\rho_i - \widehat{\alpha}_i\widehat{\rho}_i)\mathrm{d}x\Big|\\
&\le C(|\mathrm{div}\mathbf{u}|_\infty|\alpha_i\rho_i - \widehat{\alpha}_i\widehat{\rho}_i|_2^2 + |\widehat{\alpha}_i\widehat{\rho}_i|_\infty|\mathrm{div}\mathbf{w}|_2|\alpha_i\rho_i - \widehat{\alpha}_i\widehat{\rho}_i|_2 + |\mathbf{w}|_6|\nabla(\widehat{\alpha}_i\widehat{\rho}_i)|_3|\alpha_i\rho_i - \widehat{\alpha}_i\widehat{\rho}_i|_2)\\
&\le C(\|\mathbf{u}\|_3|\alpha_i\rho_i - \widehat{\alpha}_i\widehat{\rho}_i|_2^2 + \|\widehat{\alpha}_i\widehat{\rho}_i\|_2|\nabla\mathbf{w}|_2|\alpha_i\rho_i - \widehat{\alpha}_i\widehat{\rho}_i|_2)\\
&\le C(\varepsilon)(\|\mathbf{u}\|_3 + \|\widehat{\alpha}_i\widehat{\rho}_i\|_2^2)|\alpha_i\rho_i - \widehat{\alpha}_i\widehat{\rho}_i|_2^2 + \varepsilon|\nabla\mathbf{w}|_2^2.
\end{aligned} \tag{4.3}$$

As $\min(\underline{\mathbf{r}}, \widehat{\underline{\mathbf{r}}}) > 0$, there exists a constant C_0 depending on this minimum, such that

$$|q - \widehat{q}|^2 \le C_0 \sum_{i=1}^k (|\alpha_i\rho_i - \widehat{\alpha}_i\widehat{\rho}_i|^2 + \alpha_i|\rho_i - \widehat{\rho}_i|^2). \tag{4.4}$$

We also notice from (4.1) and (3.10) that

$$\begin{aligned}
&\alpha_i(\rho_i - \widehat{\rho}_i)_t + \alpha_i\mathrm{div}((\rho_i - \widehat{\rho}_i)\mathbf{u}) + \alpha_i\mathrm{div}(\widehat{\rho}_i\mathbf{w})\\
&= \frac{\alpha_i[(\rho_i - \widehat{\rho}_i)(q - a\rho_i^\gamma) + \widehat{\rho}_i((q - \widehat{q}) - a(\rho_i^\gamma - \widehat{\rho}_i^\gamma))]}{\lambda + 2\mu}.
\end{aligned}$$

Multiplying it by $(\rho_i - \widehat{\rho}_i)$ and then integrating over Ω, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\int_\Omega \frac{\alpha_i(\rho_i - \widehat{\rho}_i)^2}{2}\mathrm{d}x\Big) = \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3 + \mathrm{I}_4,$$

where

$$\begin{aligned}
|\mathrm{I}_1| :&= \Big|\int_\Omega (\alpha_i)_t\frac{(\rho_i - \widehat{\rho}_i)^2}{2}\mathrm{d}x - \int_\Omega \alpha_i\mathrm{div}((\rho_i - \widehat{\rho}_i)\mathbf{u})(\rho_i - \widehat{\rho}_i)\mathrm{d}x\Big|\\
&= \Big|\int_\Omega ((\alpha_i)_t + \mathbf{u}.\nabla\alpha_i)\frac{(\rho_i - \widehat{\rho}_i)^2}{2}\mathrm{d}x - \int_\Omega \mathrm{div}\mathbf{u}\frac{\alpha_i(\rho_i - \widehat{\rho}_i)^2}{2}\mathrm{d}x\Big|\\
&= \Big|\int_\Omega \Big(\frac{\alpha_i(a\rho_i^\gamma - q)}{\lambda + 2\mu} - \mathrm{div}\mathbf{u}\Big)\frac{\alpha_i(\rho_i - \widehat{\rho}_i)^2}{2}\mathrm{d}x\Big|\\
&\le (C_0 + |\mathrm{div}\mathbf{u}|_\infty)\int_\Omega \frac{\alpha_i(\rho_i - \widehat{\rho}_i)^2}{2}\mathrm{d}x,\\
|\mathrm{I}_2| :&= \Big|\int_\Omega \alpha_i\mathrm{div}(\widehat{\rho}_i\mathbf{w})(\rho_i - \widehat{\rho}_i)\mathrm{d}x\Big|\\
&\le \Big|\int_\Omega \sqrt{\alpha_i}\,\mathrm{div}\mathbf{w}\widehat{\rho}_i\sqrt{\alpha_i}\,(\rho_i - \widehat{\rho}_i)\mathrm{d}x\Big| + \Big|\int_\Omega \sqrt{\alpha_i}\,\nabla\widehat{\rho}_i\cdot\mathbf{w}\sqrt{\alpha_i}\,(\rho_i - \widehat{\rho}_i)\mathrm{d}x\Big|\\
&\le |\widehat{\rho}_i|_\infty|\nabla\mathbf{w}|_2|\sqrt{\alpha_i}\,(\rho_i - \widehat{\rho}_i)|_2 + |\nabla\widehat{\rho}_i|_3|\mathbf{w}|_6|\sqrt{\alpha_i}\,(\rho_i - \widehat{\rho}_i)|_2\\
&\le C\|\widehat{\rho}_i\|_2|\sqrt{\alpha_i}\,(\rho_i - \widehat{\rho}_i)|_2|\nabla\mathbf{w}|_2\\
&\le C(\varepsilon)\|\widehat{\rho}_i\|_2^2|\sqrt{\alpha_i}\,(\rho_i - \widehat{\rho}_i)|_2^2 + \varepsilon|\nabla\mathbf{w}|_2^2,
\end{aligned}$$

$$|\mathrm{I}_3|:=\Big|\int_\Omega \frac{\alpha_i(\rho_i-\widehat{\rho}_i)^2(q-a\rho_i^\gamma)}{\lambda+2\mu}\mathrm{d}x\Big|\le C_0\int_\Omega\frac{\alpha_i(\rho_i-\widehat{\rho}_i)^2}{2}\mathrm{d}x,$$
$$|\mathrm{I}_4|:=\Big|\int_\Omega \frac{\alpha_i(\rho_i-\widehat{\rho}_i)\widehat{\rho}_i((q-\widehat{q})+a(\rho_i^\gamma-\widehat{\rho}_i^\gamma))}{\lambda+2\mu}\mathrm{d}x\Big|\le C(\varepsilon,C_0)\int_\Omega\frac{\alpha_i(\rho_i-\widehat{\rho}_i)^2}{2}\mathrm{d}x+\varepsilon|q-\widehat{q}|_2^2.$$

Here C_0 depends on $\underline{\mathbf{r}}$, $\overline{\mathbf{r}}$, $\underline{\widehat{\mathbf{r}}}$ and $\overline{\widehat{\mathbf{r}}}$.

Gathering up all the estimates above, we obtain for any $\varepsilon>0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}(|\sqrt{\alpha_i}\,(\rho_i-\widehat{\rho}_i)|_2^2)\le C(\varepsilon,C_0)(1+\|\mathbf{u}\|_3+\|\widehat{\rho}_i\|_2^2)|\sqrt{\alpha_i}\,(\rho_i-\widehat{\rho}_i)|_2^2+\varepsilon|\nabla\mathbf{w}|_2^2+\varepsilon|q-\widehat{q}|_2^2. \quad (4.5)$$

Finally, according to (5.28) and (5.30), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{|\sqrt{\rho}\,\mathbf{w}|_2^2}{2}\Big)+\frac{\mu}{2}|\nabla\mathbf{w}|_2^2\le C(m+M^2+\|(\widehat{\mathbf{u}})_t\|_1^2)|\rho-\widehat{\rho}|_2^2+C|q-\widehat{q}|_2^2+CmM|\mathbf{w}|_2^2, \quad (4.6)$$

where m,M are the same as that of Lemma 5.4. If we denote

$$N=|\sqrt{\rho}\,\mathbf{w}|_2^2+\sum_{i=1}^k(|\alpha_i\rho_i-\widehat{\alpha}_i\widehat{\rho}_i|_2^2+|\sqrt{\alpha_i}\,(\rho_i-\widehat{\rho}_i)|_2^2),$$

by (4.3), (4.5), (4.6) and taking ε small enough, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}N(t)\le C_0\Big(1+\|\widehat{\mathbf{u}}_t\|_1^2+\|\mathbf{u}\|_3+\sum_{i=1}^k(\|\widehat{\alpha}_i\widehat{\rho}_i\|_2^2+\|\widehat{\rho}_i\|_2^2)\Big)N(t).$$

Noticing that $N(0)=0$, and the term by which N is multiplied in the right-hand side of this inequality is in $L^1(0,T)$, we deduce that $N(t)=0$ in $[0,T]$ by the Gronwall's inequality. This completes the proof.

With the help of this uniqueness criterion, in the sequel, in order to prove the existence (local or global in time) of solutions to (HCNSP), it is enough to study the existence of solutions to (PHCNSP) instead.

5 Local Existence for (PHCNSP)

In this Section, as mentioned in Section 4, we manage to prove the local existence of strong solutions to the system (PHCNSP), and the global results will be settled in the next section. Indeed, we shall prove

Theorem 5.1 *Given* $(\mathbf{u^0},\mathbf{r^0},\mathbf{a^0})\in X_0^k$, *there exists a time* $T_*>0$ *depending on* $(\|\mathbf{u}^0\|_2$, $\|\mathbf{a}^0\|_2$, $\|\mathbf{r}^0\|_2)$ *and* $\underline{\mathbf{r}}^0$, *such that there is a unique solution to (PHCNSP) in* $X_{T_*}^k$.

The uniqueness part was proved as argued in the previous section. (Actually, it follows from Theorem 4.1 that the system (PHCNSP) admits at most one solution in X_T^k, for any $T>0$.) It remains to prove the existence part and this will be done by linearizing the system (PHCNSP) and then employing the Schauder fixed-point theorem. More precisely, the proof of Theorem 5.1 will be done in three steps. First, given

$$\mathbf{u}\in C([0,T];H_0^1(\Omega)\cap H^2(\Omega))\cap L^2(0,T;H^3(\Omega)),$$

we study the solution $(\mathbf{r}, \mathbf{a})$ to (4.1). Next, given $(\mathbf{v}, \mathbf{r}, \mathbf{a}) \in X_T^k$, which satisfies compatibility condition

$$\rho_t + \operatorname{div}(\rho \mathbf{v}) = 0, \tag{5.1}$$

with ρ defined as in (3.9), we consider the following linearization of (4.2):

$$\begin{cases} (\rho \mathbf{u})_t + \operatorname{div}(\rho \mathbf{v} \otimes \mathbf{u}) + \nabla q + \mathcal{L}\mathbf{u} = \rho \nabla \Phi, \\ -\Delta \Phi = 4\pi g \Big(\rho - \dfrac{1}{|\Omega|} \displaystyle\int_\Omega \rho \Big), \\ \mathbf{u}|_{\partial\Omega} = 0, \quad \dfrac{\partial \Phi}{\partial \nu}\Big|_{\partial\Omega} = 0, \end{cases} \tag{5.2}$$

with q defined as in (3.9). Finally, we employ the Schauder fixed-point theorem to conclude the local existence of solutions, and complete the proof of Theorem 5.1.

5.1 Linearized hyperbolic problem

We quote here the results derived in [6], which concern the existence, uniqueness and regularities of solutions to the system (4.1).

Lemma 5.1 *Assume* $\mathbf{u} \in C([0,T]; H_0^1(\Omega) \cap H^2(\Omega)) \cap L^2(0,T; H^3(\Omega))$*. Then there exists a small time* T_**, depending only on the norms of* $(\mathbf{u}^0, \mathbf{r}^0, \mathbf{a}^0)$ *and* $\underline{\mathbf{r}}^0$*, such that there is a (unique) solution* $(\mathbf{r}, \mathbf{a}) \in C([0,T_*]; H^2(\Omega))$ *to the system* (4.1) *with initial data* $(\mathbf{r}^0, \mathbf{a}^0)$*. Moreover,*

$$(\mathbf{r}_t, \mathbf{a}_t) \in C([0,T_*]; H^1(\Omega)) \quad \text{and} \quad \rho_i(t,x) \neq \rho_j(t,x), \quad \forall\, i \neq j,\ \forall\, (t,x) \in (0,T) \times \Omega.$$

Restricting a little smaller T_* *if necessary, but keeping the same dependencies, we have the following estimates:*

$$\frac{1}{2}\underline{\mathbf{r}}^0 \le \underline{\mathbf{r}} \le \overline{\mathbf{r}} \le 2\overline{\mathbf{r}}^0, \tag{5.3}$$

$$|||\mathbf{a}|||_{\infty,2}^2 + |||\mathbf{r}|||_{\infty,2}^2 \le 4(\|\mathbf{a}^0\|_2^2 + \|\mathbf{r}^0\|_2^2), \tag{5.4}$$

$$|||\mathbf{a}_t|||_{\infty,1}^2 + |||\mathbf{r}_t|||_{\infty,1}^2 \le C_0(\|\mathbf{a}^0\|_2^2 + \|\mathbf{r}^0\|_2^2)(\|\mathbf{a}^0\|_2^2 + \|\mathbf{r}^0\|_2^2 + |||\mathbf{u}|||_{\infty,2}^2 + 1), \tag{5.5}$$

where C_0 *is a constant depending on* $(\overline{\mathbf{r}}^0, \underline{\mathbf{r}}^0)$*.*

Proof See [6, Lemma 4].

Lemma 5.2 *Given* $(\mathbf{u}_1, \mathbf{u}_2) \in C([0,T]; H_0^1(\Omega) \cap H^2(\Omega)) \cap L^2(0,T; H^3(\Omega))$*, let* $(T_*^1, (\mathbf{r}_1, \mathbf{a}_1))$ *and* $(T_*^2, (\mathbf{r}_2, \mathbf{a}_2))$ *be the times and solutions to* (4.1) *obtained in Lemma* 5.1 *with data* $\mathbf{u}_1$, $\mathbf{u}_2$ *respectively. Denote* $T_* = \min(T_*^1, T_*^2)$*. Then there exists* $K > 0$*, depending only on* $(|||\mathbf{u}_1|||_{2,3}, T_*)$*, such that inside* $(0, T_*)$*,*

$$|||\mathbf{r}_1 - \mathbf{r}_2|||_{\infty,0}^2 + |||\mathbf{a}_1 - \mathbf{a}_2|||_{\infty,0}^2 \le K |||\mathbf{u}_1 - \mathbf{u}_2|||_{2,1}^2. \tag{5.6}$$

Proof See [6, Lemma 5].

5.2 Linearized parabolic-elliptic problem

In this subsection, that we say $\mathbf{u}$ is a solution to the linearized system (5.2) always implicitly means that a pair of functions $(\mathbf{u}, \Phi)$ satisfy $(5.2)_1$, where Φ is in accordance with $(5.2)_2$. Concerning the linearized system (5.2), we can prove

Lemma 5.3 *Let* $(\rho, q, \mathbf{v}) \in C([0,T]; H^2(\Omega))^2 \times C([0,T]; H_0^1(\Omega) \cap H^2(\Omega)) \cap L^2(0,T; H^3(\Omega))$, *satisfy compatibility condition* (5.1), *and assume further that*

$$\frac{1}{2}\underline{\mathbf{r}}^0 \leq \underline{\rho} \leq \overline{\rho} \leq 2\overline{\mathbf{r}}^0, \quad (\rho_t, q_t) \in C([0,T]; H^1(\Omega)), \quad \mathbf{v}_t \in L^\infty(0,T; L^2(\Omega)) \cap L^2(0,T; H^1(\Omega)).$$

Then there exists a unique solution $\mathbf{u}$ *to* (5.2) *satisfying*

$$\mathbf{u} \in C([0,T]; H_0^1(\Omega) \cap H^2(\Omega)) \cap L^2(0,T; H^3(\Omega)), \quad \mathbf{u}_t \in L^\infty(0,T; L^2(\Omega)) \cap L^2(0,T; H^1(\Omega)).$$

Proof For this ρ, by the classical elliptic theory, there is a unique solution Φ to $(5.2)_2$, with the regularity, $\Phi \in C([0,T]; H^4(\Omega))$ and $\Phi_t \in C([0,T]; H^3(\Omega))$. Notice that $\rho \geq \underline{\rho} > 0$ and with (5.1), the linearized momentum equation $(5.2)_1$ can be written as a linear parabolic system

$$\mathbf{u}_t + \mathbf{v}.\nabla \mathbf{u} + \rho^{-1}\mathcal{L}\mathbf{u} = F, \tag{5.7}$$

where

$$F = -\rho^{-1}\nabla q - \nabla\Phi \quad \text{and} \quad F_t = \rho^{-1}\nabla q_t - \rho^{-2}\rho_t \nabla q - \nabla\Phi_t.$$

It is a simple matter to verify $F \in C([0,T]; H^1(\Omega))$ and $F_t \in L^\infty(0,T; L^2(\Omega))$. Then the existence and expected regularity of the unique solution $\mathbf{u}$ to (5.7) can be proved by applying classical methods, for instance, the method of continuity (see [19]).

Lemma 5.4 *Under the same assumptions of Lemma* 5.3, *moreover, if assume that there exists* (m, M), *with* M *sufficiently big with respect to initial data and* m, *such that*

$$\begin{cases} \mathbf{v}|_{t=0} = \mathbf{u}^0, \\ |||\mathbf{v}|||_{\infty,2}^2 + |||\mathbf{v}|||_{2,3}^2 + |||\mathbf{v}_t|||_{\infty,0}^2 + |||\mathbf{v}_t|||_{2,1}^2 \leq M, \\ |||\rho|||_{\infty,2}^2 + |||q|||_{\infty,2}^2 \leq m, \\ |||\rho_t|||_{\infty,1}^2 + |||q_t|||_{\infty,1}^2 \leq (M+m)^2, \end{cases}$$

then there exists a small time T_* *depending only on the initial data and* (m, M), *such that in* $(0, T_*)$,

$$|||\mathbf{u}|||_{\infty,2}^2 + |||\mathbf{u}|||_{2,3}^2 + |||\mathbf{u}_t|||_{\infty,0}^2 + |||\mathbf{u}_t|||_{2,1}^2 \leq M,$$

where $\mathbf{u}$ *is the solution to* (5.2) *obtained in previous lemma.*

Proof of Lemma 5.4 Step 1 First of all, the regularity results on the elliptic equation $(5.2)_2$ together with Sobolev's inequality yield

$$|\nabla\Phi|_2 \leq \|\Phi\|_2 \leq C|\rho|_2 \quad \text{and also} \quad |\nabla\Phi_t|_2 \leq C|\rho_t|_2. \tag{5.8}$$

In view of (5.1), the linearized momentum equation $(5.2)_1$ can be rewritten as the form

$$\rho\mathbf{u}_t + \rho\mathbf{v}.\nabla\mathbf{u} + \rho\nabla\Phi + \nabla q + \mathcal{L}\mathbf{u} = 0. \tag{5.9}$$

Multiplying (5.9) by $\mathbf{u}$ and integrating over Ω, also using (5.1), we have

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\int_\Omega \rho|\mathbf{u}|^2\mathrm{d}x\Big) + \int_\Omega \mu|\nabla\mathbf{u}|^2 + (\lambda+\mu)|\mathrm{div}\mathbf{u}|^2\mathrm{d}x \\ =& -\int_\Omega (\nabla q\cdot\mathbf{u} + \rho\nabla\Phi\cdot\mathbf{u})\mathrm{d}x \\ \leq& C|q|_2^2 + (\lambda+\mu)|\mathrm{div}\mathbf{u}|_2^2 + \overline{\rho}|\nabla\Phi|_2|\mathbf{u}|_2 \\ \leq& C|q|_2^2 + (\lambda+\mu)|\mathrm{div}\mathbf{u}|_2^2 + C\overline{\rho}^2|\rho|_2^2 + \frac{\mu}{2}|\nabla\mathbf{u}|_2^2. \end{aligned}$$

Integrating this inequality directly in time, one recovers that

$$|||\sqrt{\rho}\,\mathbf{u}|||_{\infty,0}^2 + \mu|||\nabla\mathbf{u}|||_{2,0}^2 \le |\sqrt{\rho^0}\,\mathbf{u}^0|_2^2 + C(1+\overline{\rho}^2)mT.$$

Consequently, choosing T small enough with respect to m and initial data, we obtain

$$|||\mathbf{u}|||_{\infty,0}^2 + |||\nabla\mathbf{u}|||_{2,0}^2 \le C_0. \tag{5.10}$$

Next, multiplying (5.9) by $\mathbf{u}_t$ and integrating over Ω, we have

$$\begin{aligned}
&\int_\Omega \rho|\mathbf{u}_t|^2\mathrm{d}x + \frac{\mathrm{d}}{\mathrm{d}t}\Big(\int_\Omega \frac{\mu}{2}|\nabla\mathbf{u}|^2 + \frac{\lambda+\mu}{2}|\mathrm{div}\mathbf{u}|^2 - q\mathrm{div}\mathbf{u}\mathrm{d}x\Big)\\
&= \int_\Omega -(\rho\mathbf{v}.\nabla\mathbf{u} + \rho\nabla\Phi)\mathbf{u}_t - q_t\mathrm{div}\mathbf{u}\mathrm{d}x\\
&\le \int_\Omega |q_t||\nabla\mathbf{u}| + |\sqrt{\rho}\,\mathbf{v}||\nabla\mathbf{u}||\sqrt{\rho}\,\mathbf{u}_t| + \sqrt{\rho}\,|\nabla\Phi||\sqrt{\rho}\,\mathbf{u}_t|\mathrm{d}x\\
&\le |q_t|_2^2 + \frac14|\nabla\mathbf{u}|_2^2 + \overline{\rho}|\mathbf{v}|_\infty^2|\nabla\mathbf{u}|_2^2 + \frac14|\sqrt{\rho}\,\mathbf{u}_t|_2^2 + C\overline{\rho}|\nabla\Phi|_2^2 + \frac14|\sqrt{\rho}\,\mathbf{u}_t|_2^2\\
&\le \frac12|\sqrt{\rho}\,\mathbf{u}_t|_2^2 + |q_t|_2^2 + C\overline{\rho}|\rho|_2^2 + \Big(C\overline{\rho}\|\mathbf{v}\|_2^2 + \frac14\Big)|\nabla\mathbf{u}|_2^2.
\end{aligned}$$

Then, integrating in time, with the domination

$$\Big|\int_\Omega q\mathrm{div}\mathbf{u}\mathrm{d}x\Big| \le C|q|_2^2 + \frac{\lambda+\mu}{2}|\mathrm{div}\mathbf{u}|_2^2$$

and (5.10), we obtain

$$\begin{aligned}
&|||\sqrt{\rho}\,\mathbf{u}_t|||_{2,0}^2 + \mu|||\nabla\mathbf{u}|||_{\infty,0}^2\\
&\le C|||q|||_{\infty,0}^2 + C(|||q_t|||_{\infty,0}^2 + \overline{\rho}|||\rho|||_{\infty,0}^2)T + \Big(C\overline{\rho}|||\mathbf{v}|||_{\infty,2}^2 + \frac12\Big)|||\nabla\mathbf{u}|||_{2,0}^2\\
&\le Cm + C[(M+m)^2 + \overline{\rho}m]T + (\overline{\rho}M + 1)C_0.
\end{aligned}$$

Consequently, choosing T small enough with respect to (m, M) and initial data, we obtain

$$|||\mathbf{u}_t|||_{2,0}^2 + |||\nabla\mathbf{u}|||_{\infty,0}^2 \le C_0(m+M+1). \tag{5.11}$$

To derive higher regularity estimates, we differentiate (5.9) with respect to t and obtain

$$\rho\mathbf{u}_{tt} + \rho\mathbf{v}.\nabla\mathbf{u}_t + L\mathbf{u}_t + \nabla q_t = -\rho_t\mathbf{u}_t - \rho_t\mathbf{v}.\nabla\mathbf{u} - \rho\mathbf{v}_t.\nabla\mathbf{u} - \rho_t\nabla\Phi - \rho\nabla\Phi_t. \tag{5.12}$$

Multiplying this equation by $\mathbf{u}_t$ and integrating over Ω, as $(\mathbf{u}_t)|_{\partial\Omega} = 0$, we obtain

$$\begin{aligned}
&\frac12\frac{\mathrm{d}}{\mathrm{d}t}\Big(\int_\Omega \rho|\mathbf{u}_t|^2\mathrm{d}x\Big) + \int_\Omega \mu|\nabla\mathbf{u}_t|^2 + (\lambda+\mu)|\mathrm{div}\mathbf{u}_t|^2\mathrm{d}x\\
&= \int_\Omega -\rho_t(\mathbf{u}_t + \mathbf{v}.\nabla\mathbf{u} + \nabla\Phi)\mathbf{u}_t - \rho(\mathbf{v}_t.\nabla\mathbf{u} + \nabla\Phi_t)\mathbf{u}_t + q_t\mathrm{div}\mathbf{u}_t\mathrm{d}x\\
&\le \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3 + \mathrm{I}_4 + \mathrm{I}_5 + \mathrm{I}_6,
\end{aligned}$$

where

$$\begin{aligned}
\mathrm{I}_1 &:= \Big|\int_\Omega \rho_t|\mathbf{u}_t|^2\mathrm{d}x\Big| \le |\rho_t|_2|\mathbf{u}_t|_2^{\frac12}|\mathbf{u}_t|_6^{\frac32} \le C|\rho_t|_2|\mathbf{u}_t|_2^{\frac12}|\nabla\mathbf{u}_t|_2^{\frac32} \le C|\rho_t|_2^4|\mathbf{u}_t|_2^2 + \frac{\mu}{12}|\nabla\mathbf{u}_t|_2^2,\\
\mathrm{I}_2 &:= \Big|\int_\Omega \rho_t\mathbf{v}.\nabla\mathbf{u}\cdot\mathbf{u}_t\Big|\mathrm{d}x \le |\rho_t|_3|\mathbf{v}|_\infty|\nabla\mathbf{u}|_2|\mathbf{u}_t|_6 \le C\|\rho_t\|_1\|\mathbf{v}\|_2|\nabla\mathbf{u}|_2|\nabla\mathbf{u}_t|_2\\
&\le C\|\rho_t\|_1^2\|\mathbf{v}\|_2^2|\nabla\mathbf{u}|_2^2 + \frac{\mu}{12}|\nabla\mathbf{u}_t|_2^2,\\
\mathrm{I}_3 &:= \Big|\int_\Omega \rho_t\nabla\Phi\cdot\mathbf{u}_t\Big|\mathrm{d}x \le |\rho_t|_3|\nabla\Phi|_2|\mathbf{u}_t|_6 \le C\|\rho_t\|_1|\rho|_2|\nabla\mathbf{u}_t|_2 \le C\|\rho_t\|_1^2|\rho|_2^2 + \frac{\mu}{12}|\nabla\mathbf{u}_t|_2^2,\\
\mathrm{I}_4 &:= \Big|\int_\Omega \rho\mathbf{v}_t.\nabla\mathbf{u}\cdot\mathbf{u}_t\Big|\mathrm{d}x \le |\rho|_\infty|\mathbf{v}_t|_3|\nabla\mathbf{u}|_2|\mathbf{u}_t|_6 \le C\|\rho\|_2|\mathbf{v}_t|_2^{\frac12}|\mathbf{v}_t|_6^{\frac12}|\nabla\mathbf{u}|_2|\nabla\mathbf{u}_t|_2\\
&\le C\|\rho\|_2^2|\mathbf{v}_t|_2\|\mathbf{v}_t\|_1|\nabla\mathbf{u}|_2^2 + \frac{\mu}{12}|\nabla\mathbf{u}_t|_2^2,\\
\mathrm{I}_5 &:= \Big|\int_\Omega \rho\nabla\Phi_t\cdot\mathbf{u}_t\Big|\mathrm{d}x \le |\rho|_3|\nabla\Phi_t|_2|\mathbf{u}_t|_6 \le C\|\rho\|_2|\rho_t|_2|\nabla\mathbf{u}_t|_2 \le C\|\rho\|_2^2|\rho_t|_2^2 + \frac{\mu}{12}|\nabla\mathbf{u}_t|_2^2,\\
\mathrm{I}_6 &:= \Big|\int_\Omega q_t\mathrm{div}\mathbf{u}_t\mathrm{d}x\Big| \le C|q_t|_2^2 + \frac{\mu}{12}|\nabla\mathbf{u}_t|_2^2.
\end{aligned}$$

Summing up all the estimates I_1–I_6 together with (5.10) and (5.11) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\int_\Omega \rho|\mathbf{u}_t|^2\mathrm{d}x\Big) + \int_\Omega \mu|\nabla\mathbf{u}_t|^2 \le C_1\int_\Omega \rho|\mathbf{u}_t|^2\mathrm{d}x + C_2\|\mathbf{v}_t\|_1 + C_3, \tag{5.13}$$

where

$$C_1 = C_0(M+m)^4, \quad C_2 = C_0mM^{\frac12}(m+M+1), \quad C_3 = C_0(m+M+1)^4.$$

Applying the Gronwall Lemma, we have for any $\tau \le t \le T$,

$$\Big(\int_\Omega \rho|\mathbf{u}_t|^2\mathrm{d}x\Big)(t) \le \mathrm{e}^{C_1T}\Big[\Big(\int_\Omega \rho|\mathbf{u}_t|^2\mathrm{d}x\Big)(\tau) + C_2M^{\frac12}\sqrt{T} + C_3T\Big]. \tag{5.14}$$

On the other hand, as $\rho > 0$, we observe from (5.9) that

$$\begin{aligned}
\int_\Omega \rho|\mathbf{u}_t|^2\mathrm{d}x &\le C\int_\Omega \rho|\mathbf{v}|^2|\nabla\mathbf{u}|^2 + \rho^{-1}|\mathcal{L}\mathbf{u} + \nabla q|^2 + \rho|\nabla\Phi|^2\mathrm{d}x\\
&\le C\int_\Omega \rho|\mathbf{v}|^2|\nabla\mathbf{u}|^2 + \rho^{-1}|\mathcal{L}\mathbf{u} + \nabla q|^2 + \overline{\rho}\rho^2\mathrm{d}x
\end{aligned}$$

and thus there exists a constant C_0 depending only initial data such that

$$\limsup_{\tau\to0}\int_\Omega \rho|\mathbf{u}_t|^2\mathrm{d}x(\tau) \le C_0. \tag{5.15}$$

Letting $\tau \to 0$ in (5.14) with (5.15) in mind, we can choose T small enough with respect to (m, M) and initial data, such that

$$\int_\Omega \rho|\mathbf{u}_t|^2\mathrm{d}x(t) \le C_0, \quad \forall\, t \in [0, T]. \tag{5.16}$$

Substituting (5.16) into (5.13) and integrating over $(0, t)$, $0 \le t \le T$ with T chosen before, we obtain

$$|||\mathbf{u}_t|||_{\infty,0}^2 + |||\mathbf{u}_t|||_{2,1}^2 \le C_0, \tag{5.17}$$

where C_0 depends only on initial data. So if choosing $M \geq 3C_0$, we have

$$|||\mathbf{u}_t|||_{\infty,0}^2 + |||\mathbf{u}_t|||_{2,1}^2 \leq \frac{M}{3}.$$

Step 2 Recall that $\mathcal{L}$, with Dirichlet boundary condition, is an elliptic operator (see [1]). Then

$$\begin{aligned}|||\mathbf{u}|||_{\infty,2}^2 &\leq C(|||\mathbf{u}|||_{\infty,0}^2 + |||\rho\mathbf{u}_t|||_{\infty,0}^2 + |||\rho\mathbf{v}.\nabla\mathbf{u}|||_{\infty,0}^2 + |||\nabla q|||_{\infty,0}^2 + |||\rho\nabla\Phi|||_{\infty,0}^2)\\ &\leq C(|||\mathbf{u}|||_{\infty,0}^2 + \overline{\rho}^2|||\mathbf{u}_t|||_{\infty,0}^2 + \overline{\rho}^2|||\mathbf{v}.\nabla\mathbf{u}|||_{\infty,0}^2 + |||q|||_{\infty,1}^2 + \overline{\rho}^2|||\rho|||_{\infty,0}^2), \qquad (5.18)\end{aligned}$$

where we may dominate (recall that $\mathbf{v}|_{t=0} = \mathbf{u}^0$)

$$\begin{aligned}|||\mathbf{v}.\nabla\mathbf{u}|||_{\infty,0} &\leq |\mathbf{u}^0.\nabla\mathbf{u}^0|_2 + \int_0^T |(\mathbf{v}.\nabla\mathbf{u})_t|_2 \mathrm{d}t\\ &\leq C(\|\mathbf{u}^0\|_2^2 + |||\mathbf{v}_t|||_{2,1}|||\mathbf{u}|||_{\infty,2}\sqrt{T} + |||\mathbf{v}|||_{\infty,2}|||\mathbf{u}_t|||_{2,1}\sqrt{T}\,). \qquad (5.19)\end{aligned}$$

Combining (5.17)–(5.19) and choosing T small enough with respect to M and initial data, we have

$$|||\mathbf{u}|||_{\infty,2}^2 \leq C_0(1+m). \qquad (5.20)$$

If we have chosen M such that M is bigger than three times the right-hand side of this inequality, then

$$|||\mathbf{u}|||_{\infty,2}^2 \leq \frac{M}{3}.$$

Step 3 Due to the same elliptic argument about $\mathcal{L}$ together with (5.17) and (5.20), we obtain

$$\begin{aligned}&|||\mathbf{u}|||_{2,3}^2\\ &\leq C(|||\mathbf{u}|||_{2,1}^2 + |||\rho\mathbf{u}_t|||_{2,1}^2 + |||\rho\mathbf{v}.\nabla\mathbf{u}|||_{2,1}^2 + |||\nabla q|||_{2,1}^2 + |||\rho\nabla\Phi|||_{2,1}^2)\\ &\leq C(|||\mathbf{u}|||_{2,1}^2 + |||\rho|||_{\infty,2}^2|||\mathbf{u}_t|||_{2,1}^2 + |||\rho|||_{\infty,2}^2|||\mathbf{v}.\nabla\mathbf{u}|||_{2,1}^2 + |||q|||_{2,2}^2 + |||\rho|||_{\infty,2}^2|||\rho|||_{2,1}^2)\\ &\leq C(|||\mathbf{u}|||_{\infty,2}^2T + |||\rho|||_{\infty,2}^2|||\mathbf{u}_t|||_{2,1}^2 + |||\rho|||_{\infty,2}^2|||\mathbf{v}|||_{\infty,2}^2|||\mathbf{u}|||_{\infty,2}^2T + |||q|||_{\infty,2}^2T + |||\rho|||_{\infty,2}^4T)\\ &\leq C((m+M+1)^2T + mC_0).\end{aligned}$$

By a similar argument as did for $|||\mathbf{u}|||_{\infty,2}$, we can choose T sufficiently small with respect to m, M and initial data, M large enough with respect to m and initial data such that

$$|||\mathbf{u}|||_{2,3}^2 \leq \frac{M}{3}.$$

The proof of Lemma 5.4 is completed.

On the other hand, for this linearized parabolic-elliptic system, we can also prove the following continuity property.

Lemma 5.5 *Suppose that* $(\rho_1, \mathbf{v}_1, q_1)$ *and* $(\rho_2, \mathbf{v}_2, q_2)$ *satisfy the hypothesis of Lemma* 5.4*. Let* T_* *be the time obtained before. Let* $\mathbf{u}_1$ *and* $\mathbf{u}_2$ *be respectively the solutions to* (5.2)*. Then there exists* $K > 0$ *such that in* $(0, T_*)$*,*

$$|||\mathbf{u}_1 - \mathbf{u}_2|||_{\infty,0}^2 + |||\mathbf{u}_1 - \mathbf{u}_2|||_{2,1}^2 \leq K(|||\rho_1 - \rho_2|||_{\infty,0}^2 + |||q_1 - q_2|||_{\infty,0}^2 + |||\mathbf{v}_1 - \mathbf{v}_2|||_{\infty,0}^2). \qquad (5.21)$$

Proof First recall that $\mathbf{u}_1$ and $\mathbf{u}_2$ satisfy

$$\rho_1(\mathbf{u}_1)_t + \rho_1\mathbf{v}_1.\nabla\mathbf{u}_1 + \nabla q_1 + \mathcal{L}\mathbf{u}_1 = \rho_1\nabla\Phi_1, \tag{5.22}$$

$$\rho_2(\mathbf{u}_2)_t + \rho_2\mathbf{v}_2.\nabla\mathbf{u}_2 + \nabla q_2 + \mathcal{L}\mathbf{u}_2 = \rho_2\nabla\Phi_2, \tag{5.23}$$

$$-\Delta(\Phi_1 - \Phi_2) = 4\pi g\Big[(\rho_1 - \rho_2) - \frac{1}{|\Omega|}\int_\Omega(\rho_1 - \rho_2)\Big]. \tag{5.24}$$

If we set $\mathbf{w} = \mathbf{u}_1 - \mathbf{u}_2$, taking the difference of (5.22) and (5.23), we have

$$\rho_1\mathbf{w}_t + \rho_1\mathbf{v}_1\nabla\mathbf{w} + \nabla(q_1 - q_2) + \mathcal{L}\mathbf{w} = \mathbf{e} + (\rho_1 - \rho_2)\nabla\Phi_1 + \rho_2\nabla(\Phi_1 - \Phi_2), \tag{5.25}$$

where $\mathbf{e} = -\rho_1(\mathbf{v}_1 - \mathbf{v}_2)\nabla\mathbf{u}_2 - (\rho_1 - \rho_2)(\mathbf{u}_2)_t - (\rho_1 - \rho_2)\mathbf{v}_2.\nabla\mathbf{u}_2$.

Multiplying (5.25) by $\mathbf{w}$ and integrating over Ω, with the domination

$$\Big|\int_\Omega \nabla(q_1 - q_2)\mathbf{w}\mathrm{d}x\Big| \le C|q_1 - q_2|_2^2 + (\lambda + \mu)|\mathrm{div}\mathbf{w}|_2^2,$$

we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{|\sqrt{\rho_1}\,\mathbf{w}|_2^2}{2}\Big) + \mu|\nabla\mathbf{w}|_2^2\\
&\le C|q_1 - q_2|_2^2 + \Big|\int_\Omega \mathbf{e}\mathbf{w}\mathrm{d}x\Big| + \Big|\int_\Omega(\rho_1 - \rho_2)\nabla\Phi_1\mathbf{w}\mathrm{d}x\Big| + \Big|\int_\Omega \rho_2\nabla(\Phi_1 - \Phi_2)\mathbf{w}\mathrm{d}x\Big|\\
&:= C|q_1 - q_2|_2^2 + \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3.
\end{aligned} \tag{5.26}$$

With the classical elliptic estimates, as often done in our paper, we have

$$|\nabla\Phi_1|_3 \le C|\rho_1|_2, \quad |\nabla(\Phi_1 - \Phi_2)|_2 \le C|\rho_1 - \rho_2|_2. \tag{5.27}$$

By Lemma 5.4 and (5.27), we can dominate

$$\begin{aligned}
\mathrm{I}_1 &\le \int_\Omega |\rho_1||\mathbf{v}_1 - \mathbf{v}_2||\nabla\mathbf{u}_2||\mathbf{w}| + |\rho_1 - \rho_2||(\mathbf{u}_2)_t||\mathbf{w}| + |\rho_1 - \rho_2||\mathbf{v}_2||\nabla\mathbf{u}_2||\mathbf{w}|\mathrm{d}x\\
&\le |\rho_1|_\infty|\mathbf{v}_1 - \mathbf{v}_2|_2|\nabla\mathbf{u}_2|_3|\mathbf{w}|_6 + |\rho_1 - \rho_2|_2|(\mathbf{u}_2)_t|_3|\mathbf{w}|_6 + |\rho_1 - \rho_2|_2|\mathbf{v}_2|_\infty|\nabla\mathbf{u}_2|_3|\mathbf{w}|_6\\
&\le C\|\rho_1\|_2|\mathbf{v}_1 - \mathbf{v}_2|_2\|\mathbf{u}_2\|_2|\nabla\mathbf{w}|_2 + |\rho_1 - \rho_2|_2(\|(\mathbf{u}_2)_t\|_1 + \|\mathbf{v}_2\|_2\|\nabla\mathbf{u}_2\|_2)|\nabla\mathbf{w}|_2\\
&\le C(\|(\mathbf{u}_2)_t\|_1^2 + M^2)|\rho_1 - \rho_2|_2^2 + CmM|\mathbf{v}_1 - \mathbf{v}_2|_2^2 + \frac{3\mu}{10}|\nabla\mathbf{w}|_2^2,\\
\mathrm{I}_2 &\le |\nabla\Phi_1|_3|\rho_1 - \rho_2|_2|\mathbf{w}|_6 \le C|\rho_1|_2|\rho_1 - \rho_2|_2|\nabla\mathbf{w}|_2 \le Cm|\rho_1 - \rho_2|_2^2 + \frac{\mu}{10}|\nabla\mathbf{w}|_2^2,\\
\mathrm{I}_3 &\le |\rho_2|_3|\nabla(\Phi_1 - \Phi_2)|_2|\mathbf{w}|_6 \le C\|\rho_2\|_1|\rho_1 - \rho_2|_2|\nabla\mathbf{w}|_2 \le Cm|\rho_1 - \rho_2|_2^2 + \frac{\mu}{10}|\nabla\mathbf{w}|_2^2.
\end{aligned}$$

Gathering up these estimates and substituting them into (5.26), we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{|\sqrt{\rho_1}\,\mathbf{w}|_2^2}{2}\Big) + \frac{\mu}{2}|\nabla\mathbf{w}|_2^2\\
&\le C(m + M^2 + \|(\mathbf{u}_2)_t\|_1^2)|\rho_1 - \rho_2|_2^2 + C|q_1 - q_2|_2^2 + CmM|\mathbf{v}_1 - \mathbf{v}_2|_2^2.
\end{aligned} \tag{5.28}$$

We notice that $\mathbf{w}|_{t=0} = 0$. Integrating (5.28) in time yields in $(0, T_*)$,

$$\begin{aligned}
&\Big|\Big|\Big|\frac{\sqrt{\rho_1}\,\mathbf{w}}{2}\Big|\Big|\Big|_{\infty,0}^2 + \frac{\mu}{2}|||\nabla\mathbf{w}|||_{2,0}^2\\
&\le K(m, M)\Big(|||\rho_1 - \rho_2|||_{\infty,0}^2\int_0^{T_*}(\|(\mathbf{u}_2)_t\|_1^2 + 1)\mathrm{d}t + |||q_1 - q_2|||_{\infty,0}^2T_* + |||\mathbf{v}_1 - \mathbf{v}_2|||_{\infty,0}^2T_*\Big)\\
&\le K(m, M, T_*)(|||\rho_1 - \rho_2|||_{\infty,0}^2 + |||q_1 - q_2|||_{\infty,0}^2 + |||\mathbf{v}_1 - \mathbf{v}_2|||_{\infty,0}^2).
\end{aligned}$$

We recall that $\mathbf{w} = \mathbf{u}_1 - \mathbf{u}_2$, $\rho_1 \ge \underline{\mathbf{r}}^0$. Then the desired result follows.

5.3 Fixed-point argument

All the results obtained before guarantee the local existence of solutions to (PHCNSP), and this will be done by a fixed-point argument via the Schauder fixed-point theorem. For this, let $\mathbf{V}$ denote the set of velocity fields defined in [0,T] satisfying

$$\begin{cases} \mathbf{v} \in C([0,T]; H_0^1(\Omega) \cap H^2(\Omega)) \cap L^2(0,T; H^3(\Omega)), \\ \mathbf{v}_t \in L^\infty(0,T; L^2(\Omega)) \cap L^2(0,T; H^1(\Omega)), \\ |||\mathbf{v}|||_{\infty,2}^2 + |||\mathbf{v}|||_{2,3}^2 + |||\mathbf{v}_t|||_{\infty,0}^2 + |||\mathbf{v}_t|||_{2,1}^2 \le M, \\ \mathbf{v}|_{t=0} = \mathbf{u}^0. \end{cases}$$

By the Aubin-Lions Lemma, we know that $\mathbf{V}$ is a compact space with respect to the topology associated with the norm $|||\cdot|||_{\infty,0} + |||\cdot|||_{2,1}$. By Lemma 5.1, there exists a time T_*^1 such that for any $\mathbf{v} \in \mathbf{V}$, there is a unique solution $(\mathbf{r}, \mathbf{a})$ to (4.1). Moreover this solution satisfies the estimates (5.3)–(5.5). Therefore there exist $K_0^1,\ K_0^2$, depending on the initial data, such that

$$|||\rho|||_{\infty,2}^2 + |||q|||_{\infty,2}^2 \le K_0^1, \quad |||\rho_t|||_{\infty,1}^2 + |||q_t|||_{\infty,1}^2 \le K_0^1(K_0^2 + M), \tag{5.29}$$

where ρ and q are defined by (3.9). Next, applying Lemmas 5.3 and 5.4, to this $(\mathbf{v}, \rho, q)$, we see that there exists a time T_*^2 such that there is a unique solution to (5.2) denoted by $\Gamma(\mathbf{v})$ where Γ is a well-defined mapping. Notice that m in Lemma 5.4 can be fixed by initial date (see (5.29)), so if we have chosen M sufficiently big with respect to initial data and let $T = T_* = \min(T_*^1, T_*^2)$, we have $\Gamma(\mathbf{v}) \in \mathbf{V}$. Finally, for $\mathbf{v}_1, \mathbf{v}_2 \in \mathbf{V}$, applying Lemmas 5.2 and 5.5 to them respectively, we obtain $(\mathbf{r}^1, \mathbf{a}^1, \rho_1, q_1, \Gamma(\mathbf{v}^1))$ and $(\mathbf{r}^2, \mathbf{a}^2, \rho_2, q_2, \Gamma(\mathbf{v}^2))$. Moreover, from (3.9) we have

$$|||\rho_1 - \rho_2|||_{\infty,0}^2 + |||q_1 - q_2|||_{\infty,0}^2 \le K_0^3(|||\mathbf{r}_1 - \mathbf{r}_2|||_{\infty,0}^2 + |||\mathbf{a}_1 - \mathbf{a}_2|||_{\infty,0}^2),$$

where K_0^3 depends only on $\mathbf{r}^0$. Then by this together with Lemmas 5.2 and 5.5, we find

$$|||\Gamma(\mathbf{v}_1) - \Gamma(\mathbf{v}_2)|||_{\infty,0}^2 + |||\Gamma(\mathbf{v}_1) - \Gamma(\mathbf{v}_2)|||_{2,1}^2 \le K_0^4(|||\mathbf{v}_1 - \mathbf{v}_2|||_{\infty,0}^2 + |||\mathbf{v}_1 - \mathbf{v}_2|||_{2,1}^2),$$

where K_0^4 is a constant depending only on initial data. That is to say, the mapping Γ is Lipschitz in $\mathbf{V}$ with respect to the norm $|||\cdot|||_{\infty,0} + |||\cdot|||_{2,1}$. Consequently, as a result of application of the Schauder fixed-point theorem there is a fixed point to Γ in $\mathbf{V}$, denoted by $\mathbf{u}$. Then there is a unique solution $(\mathbf{r}, \mathbf{a})$ to (4.1) with data $\mathbf{u}$. By the definition of $\mathbf{u}$ the triplet $(\mathbf{u}, \mathbf{r}, \mathbf{a})$ is a solution to (PHCNSP) in $X_{T_*}^k$. Moreover,

$$\begin{cases} \mathbf{u} \in C([0,T_*]; H_0^1(\Omega) \cap H^2(\Omega)) \cap L^2(0,T_*; H^3(\Omega)), \\ \mathbf{u}_t \in L^\infty(0,T; L^2(\Omega)) \cap L^2(0,T; H^1(\Omega)), \\ (\mathbf{r}, \mathbf{a}) \in C([0,T_*]; H^2(\Omega)),\ (\mathbf{r}_t, \mathbf{a}_t) \in C([0,T_*]; H^1(\Omega)) \end{cases} \tag{5.30}$$

and $\underline{\mathbf{r}} > 0,\ \rho_i(t,x) \ne \rho_j(t,x),\ \forall\, i \ne j,\ \forall\, (t,x) \in (0,T) \times \Omega.$

Up to now, the proof of Theorem 5.1 is completed.

6 Global Existence for (PHCNSP)

In this section, the aim is to prove the global existence of solutions to (PHCNSP) for small initial data. In the following, let $\widetilde{\rho}$ be a strictly positive constant. We consider initial conditions $(\mathbf{u^0}, \mathbf{r^0}, \mathbf{a^0}) \in X_0^k$, and set $\mathbf{r}^0 = \widetilde{\rho} + \mathbf{s}^0$, i.e. $\rho_i^0 = \widetilde{\rho} + \sigma_i^0,\ i = 1, \cdots, k$. We have

Theorem 6.1 *Given $\widetilde{\rho} > 0$ and $\mathbf{a^0} \in H^2(\Omega)$, there exists a strictly positive constant $C(\widetilde{\rho}, \mathbf{a^0})$ such that, for any initial data $(\mathbf{u^0}, \mathbf{r^0} := \widetilde{\rho} + \mathbf{s^0}, \mathbf{a^0}) \in X_0^k$ satisfying*

$$\|\mathbf{u}^0\|_2^2 + \|\mathbf{s}^0\|_2^2 \leq C(\widetilde{\rho}, \mathbf{a^0}),$$

there is a unique global solution to (PHCNSP).

The same as local case, it is enough to prove the existence part. The key point in our proof is to rewrite (PHCNSP) as a perturbation form in accordance with [6] that enable us employ almost all the estimates presented in [6, Section 5] to complete our proof. For this, let $(\mathbf{u}, \mathbf{r}, \mathbf{a}) \in X_T^k$ be a solution to (PHCNSP) satisfying (5.30). We rewrite $\mathbf{r} := \widetilde{\rho} + \mathbf{s}$ with $\mathbf{s} := (\sigma_1, \cdots, \sigma_k)$, and by $(4.1)_2$, we have

$$(\sigma_i)_t + \mathbf{u} \cdot \nabla \sigma_i + \widetilde{\rho}\,\mathrm{div}(\mathbf{u}) = f_{\sigma_i} - \sigma_i \mathrm{div}\mathbf{u}, \quad \forall\, i = 1, \cdots, k, \tag{6.1}$$

where

$$f_{\sigma_i} := \frac{\rho_i(q - a\rho_i^\gamma)}{\lambda + 2\mu}, \quad q := a\sum_{i=1}^k \alpha_i \rho_i^\gamma,$$

and the perturbation $\sigma = \sum\limits_{i=1}^k \alpha_i \sigma_i$ is a solution to

$$\sigma_t + \mathbf{u} \cdot \nabla \sigma + \widetilde{\rho}\,\mathrm{div}\mathbf{u} = f_\sigma, \tag{6.2}$$

where $f_\sigma = -\sigma \mathrm{div}\mathbf{u}$. We rewrite $(4.2)_1$ as

$$\mathbf{u}_t - \widetilde{A}\mathbf{u} + q_1 \nabla \sigma = \mathbf{f} + \mathbf{b} \tag{6.3}$$

with $\widetilde{A}\mathbf{u} := \widetilde{\mu}\Delta\mathbf{u} + \widetilde{\beta}\nabla\mathrm{div}\mathbf{u}$, where $\widetilde{\mu} = \frac{\mu}{\widetilde{\rho}}$, $\widetilde{\beta} = \frac{\lambda+\mu}{\widetilde{\rho}}$ and

$$\mathbf{f} := -(\mathbf{u}.\nabla\mathbf{u}) - \frac{\widetilde{\mu}\sigma}{\rho}\Delta\mathbf{u} - \frac{\widetilde{\beta}\sigma}{\rho}\nabla\mathrm{div}(\mathbf{u}) + q_1\nabla\sigma + \nabla\Phi - \frac{\nabla P(\rho)}{\rho},$$

where $q_1 := a\gamma\widetilde{\rho}^{\gamma-2}$ and $\mathbf{b} := \frac{\nabla P(\rho) - \nabla q}{\rho}$.

Consequently, we can rewrite the (PHCNSP) as

$$\begin{cases} \sigma_t + \mathbf{u} \cdot \nabla \sigma + \widetilde{\rho}\,\mathrm{div}\mathbf{u} = f_\sigma, \\ \mathbf{u}_t - \widetilde{A}\mathbf{u} + q_1 \nabla\sigma = \mathbf{f} + \mathbf{b}, \\ \Delta\Phi = 4\pi g\Big(\sigma - \dfrac{1}{|\Omega|}\displaystyle\int_\Omega \sigma\Big), \\ \mathbf{u}|_{\partial\Omega} = 0, \quad \dfrac{\partial\Phi}{\partial\nu}\Big|_{\partial\Omega} = 0. \end{cases} \tag{6.4}$$

Notice carefully that the system (6.4) is similar to the system (44)–(46) of [6] except the term $\mathbf{f}$ in which a Poisson term Φ satisfying the elliptic equation $(6.4)_3$ is added. Thus all the estimates in [6] which do not involve $\mathbf{f}$ are still available for us, namely [6, Lemma 8] concerning the estimates on σ_i and [6, Lemma 9] concerning the estimates on α_i in [6]. Since [6, Lemma 10] involves the term $\mathbf{f}$, we have to prove that it holds also for our system. Indeed, we prove

Lemma 6.1 *There exist two norms we denote by $]\cdot[_m$ (resp. $[\cdot]_m$) equivalent to (resp. dominated by) $\|\cdot\|_m$, satisfying the following property:*

Assume that $(\sigma,\mathbf{u})\in C([0,T];H^2(\Omega))\times C([0,T];H_0^1(\Omega)\cap H^2(\Omega))\cap L^2(0,T;H^3(\Omega))$ satisfies (6.4) and assume that $\widetilde{\rho}$ satisfies

$$\frac{\widetilde{\rho}}{4}\le\widetilde{\rho}+\sigma\le 3\widetilde{\rho},\quad in\ (0,T)\times\Omega.$$

Then there exist constants $c_i, i=1,\cdots,5$, such that, if we denote

$$\phi(t):=]\mathbf{u}[_1^2+]\sigma[_2^2+c_1]\mathbf{u}_t[_0^2+c_2]\sigma_t[_0^2+c_3[\mathbf{u}]_2^2,\quad \psi(t):=\|\mathbf{u}\|_3^2+\|\sigma\|_2^2+\|\mathbf{u}_t\|_1^2+\|\sigma_t\|_1^2,$$

we have

$$\frac{\mathrm{d}\phi}{\mathrm{d}t}+\psi\le c_4\psi(\phi+\phi^2)+c_5(\|\mathbf{b}\|_1^2+\|\mathbf{b}_t\|_{-1}^2). \tag{6.5}$$

Proof The notations $]\cdot[_m$ and $[\cdot]_m$ come from [19] (see (4.40) and (4.42) there, respectively). First notice that our aim inequality (6.5) is exactly (4.49) in [19] and all the computations in [19, Section 4] from (4.6) until (4.42) are also true for our case since they do not involve the expression of the term $\mathbf{f}$. So in order to prove (6.5) we only have to prove that the estimates (4.43) and (4.44) in [19] are also true here. But since the Poisson term Φ in $\mathbf{f}$ which yields the difference satisfies the elliptic equation $(6.4)_3$, it is easy to check the estimates (4.43) and (4.44) in [19] by using the classical regularity results to dominate Φ in term of σ. Consequently, we can complete our proof by the same method as in [19], just with a slightly modification.

Since Lemma 6.1 is proved, [6, Lemmas 11–13] are also available for us. Thus we are able to return to the proof of Theorem 6.1.

Proof of Theorem 6.1 The situation considered here is the same as [6, Section 5], so we can combine our Lemma 6.1 with [6, Lemmas 8, 9 and 11–13], and then repeat the argument presented in the last paragraph of [6, Section 5] without any modification to complete the proof of Theorem 6.1 (see [6, Section 5] for more details).

Up to now, we have proved the local and global existence of solutions to (PHCNSP) with different initial data respectively. By the arguments in Section 4, the solution is actually the only solution to (HCNSP).

Finally, notice that if there is no oscillation, i.e. $k=1$ in (1.7), the Young measure is a deterministic family of Dirac measures. Specially, we obtain the following results for strong solutions to (CNSP).

Corollary 6.1 *Let $\gamma>1$, $\rho_0\in H^2(\Omega)$, $\mathbf{u}^0\in H_0^1(\Omega)\cap H^2(\Omega)$ and $\rho_0\ge\delta>0$ in Ω. There exists $T_*>0$ such that there is a unique strong solution to (CNSP) in $[0,T_*]$ satisfying*

$$\begin{cases}\mathbf{u}\in C([0,T_*];H_0^1(\Omega)\cap H^2(\Omega))\cap L^2(0,T_*;H^3(\Omega)),\\ \mathbf{u}_t\in L^\infty(0,T_*;L^2(\Omega))\cap L^2(0,T_*;H^1(\Omega)),\\ \rho\in C([0,T_*];H^2(\Omega)),\quad \rho_t\in C([0,T_*];H^1(\Omega))\end{cases}$$

and $\rho > 0$ *in* $(0, T_*) \times \Omega$. *Moreover, given a constant* $\widetilde{\rho} > 0$, *there exists* $C(\widetilde{\rho}) > 0$ *such that for any initial data* $(\rho_0 = \sigma_0 + \widetilde{\rho}, \mathbf{u}^0)$ *satisfying*

$$\|\mathbf{u}^0\|_2^2 + \|\sigma_0\|_2^2 \leq C(\widetilde{\rho}),$$

the unique strong solution is global in time.

Acknowledgement The authors thank the referees for their valuable comments and suggestions.

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