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Note on a Conjecture of Gopakumar-Vafa

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(*Dedicated to the memory of Shiing-Shen Chern*)

Abstract We rephrase the Gopakumar-Vafa conjecture on genus zero Gromov-Witten invariants of Calabi-Yau threefolds in terms of the virtual degree of the moduli of pure dimension one stable sheaves and investigate the conjecture for K3 fibred local Calabi-Yau threefolds.

Keywords Calabi-Yau threefold, Gromov-Witten invariants, Moduli of stable sheaves
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1 Introduction

Gromov-Witten invariants are virtual enumeration of holomorphic curves in projective (or symplectic) manifolds. Their mysteries, rich structures, and wide applications to many disciplines of mathematics as well as Super-String theories led to a surge of research interest to this subject. Of the Gromov-Witten invariants of various varieties, those of Calabi-Yau threefolds pose exceptional challenge, both because of their conjectured rich structures and their resistance to yield to mathematical proofs.

Two recent conjectures offer a unifying structure of these invariants. One is the recent revitalized Donaldson-Thomas invariants of Calabi-Yau threefolds by the work of Maulik-Nekrasov-Okunkov-Pandharipande [16]. The other is an earlier conjecture of Gopakumar-Vafa that, based on the yet to be fully understood notion of D-branes, put forward a striking structure of the generating functions of the Gromov-Witten invariants of all Calabi-Yau threefolds. In this note, we will reformulate the genus zero Gopakumar-Vafa conjecture in terms of the moduli of stable sheaves of pure dimension one and argue for the validity of the conjecture by looking into a class of K3 fibred Calabi-Yau threefolds.

Given a smooth projective Calabi-Yau threefold and a curve class $\beta \in H_2(X,\mathbb{Z})$, the Gromov-Witten invariants assign to each genus g the virtual number $N_g(\beta)$ of stable maps from genus g curves to X of topological class β. Summing over β gives rise to the genus g generating function

$$\mathcal{F}_{X,g}(q) = \sum_{\beta\in H_2(X,\mathbb{Z})} N_g(\beta) q^\beta;$$

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summing over g defines the master generating function

$$\mathcal{F}_X(q,\lambda)=\sum_{g\geq 0}\mathcal{F}_{X,g}(q)\lambda^{2g-2}.$$

Based on investigating string duality between type IIA and M-Theory, Gopakumar-Vafa conjectured that

Conjecture 1.1 *There are integers $n_h(\beta)$ such that*

$$\mathcal{F}_X(q,\lambda)=\sum_{h\geq 0,k>0,\beta\in H_2(X,\mathbb{Z})} n_h(\beta)\frac{1}{k}\Big(2\sin\frac{k\lambda}{2}\Big)^{2h-2}q^{k\beta}.$$

Indeed, they argued that the collection $n_h(\beta)$ can be defined geometrically using the space of "D-branes". As suggested by many, the mathematical counter-part of D-branes should be stable sheaves of pure dimension one. Following this suggestion, to each class β we shall substitute the space of "D-branes" by the moduli $\mathcal{M}_X(\beta,1)$ of pure dimension one stable sheaves $\mathcal{E}$ on X with second Chern class $c_2(\mathcal{E})=-\beta$ and Euler number $\chi(\mathcal{E})=1$. Then according to Gopakumar-Vafa, the number $n_0(\beta)$, which is the expected number of rational curves in X of class β, should be the signed Euler number of $\mathcal{M}_X(\beta,1)$ when the later is smooth; and the $n_h(\beta)$ of high genus $h>0$ should be detectible based on certain subtle intrinsic geometry of $\mathcal{M}_X(\beta,1)$.

At this point, formulating a mathematical conjecture on $n_h(\beta)$ for $h>0$ remains out of our reach. (See [8] for relationship with intersection cohomology.) As to genus 0, one can make rigorous the conjecture by replacing the Euler number by the degree of the virtual cycle of $\mathcal{M}_X(\beta,1)$ (see [20]). This makes sense because the moduli space $\mathcal{M}_X(\beta,1)$ has self-dual obstruction theory and its virtual cycle $[\mathcal{M}_X(\beta,1)]^{\mathrm{vir}}$ is a dimension zero cycle.

As observed in the recent preprint of Katz [10], we can rephrase the genus zero Gopakumar-Vafa conjecture as

Conjecture 1.2 *Let $n_0(\beta)=\deg[\mathcal{M}_X(\beta,1)]^{\mathrm{vir}}$. Then the genus 0 Gromov-Witten invariant generating function has the form*

$$\mathcal{F}_{X,0}(q)=\sum_{\beta\in H_2(X,\mathbb{Z})}\Big(\sum_{k|\beta}\frac{1}{k^3}\,n_0(\beta/k)\Big)q^{\beta}.$$

To prove this conjecture we need to find a way to relate the moduli of sheaves to that of rational curves. This relation is most transparent when all rational curves in a Calabi-Yau threefold turns out to be smooth $(-1,-1)$-curves. In this case, the virtual number coincides with the actual number of the rational curves, and the genus zero Gromov-Witten invariant follows from these numbers through the multi-cover formula of Aspinwall-Morrison. Likewise, the moduli of pure dimension one sheaves splits into two parts: one consists of sheaves over those rational curves; the other consists of sheaves supported over high genus curves. Because the set of sheaves over high genus curves admits free S^1-actions, they do not contribute to the virtual degree of the moduli space. Consequently, this virtual degree is entirely contributed by the first part of the moduli space, one for each smooth rational curves. Combined, this would establish the desired relationship conjectured by that of Gopakumar-Vafa.

Proving this conjecture for general Calabi-Yau threefolds requires an understanding of the virtual fundamental cycles of the moduli of sheaves and of stable maps deeper than we currently have. In this note, we will look into a simpler case: the local K3 fibred Calabi-Yau threefolds X that are triples $\pi\colon X \to \Delta$ of proper submersive holomorphic maps to the complex disk Δ from smooth three dimensional complex manifolds X with trivial canonical line bundle K_X. Such triples $\pi\colon X \to \Delta$ provide us interesting testing cases in case the central fibers $X_0 = \pi^{-1}(0)$ contain fiberwise-rigid curve classes $\beta \in H_2(X_0, \mathbb{Z})$ in the sense that their Poincaré duals $\beta^\vee \in H^{1,1}(X_0, \mathbb{R})$ but not so in all nearby fibers. Henceforth, all stable maps with fundamental class β or pure dimension one sheaves of second Chern class $-\beta$ must be confined to X_0, thus making the two relevant invariants well defined, the two invariants being the Gromov-Witten invariants and the virtual degrees of the moduli of sheaves of pure dimension one.

We now confine ourselves to the case of a local K3 fibred Calabi-Yau threefold $\pi\colon X \to \Delta$ together with a fibrewise rigid curve class β in X_0.

Theorem 1.3 *Suppose* $\operatorname{Pic}(X_0) \cong \mathbb{Z}$ *and is generated by* β_0*. Suppose further that for an integer* $d \le 5$ *all* (*reduced*) *rational curves in the linear series* $|k\beta_0|$ *for* $k \le d$ *are nodal. Then the conjecture* 1.2 *holds up to* $q^{d\beta_0}$:

$$\mathcal{F}_{X,0}(q) \equiv \sum_{\beta \in H_2(X,\mathbb{Z})} \Big(\sum_{k|\beta} \frac{1}{k^3}\, n_0(\beta/k) \Big) q^\beta \mod q^{(d+1)\beta_0}.$$

In the sequel of this paper, we shall employ a deformation argument to remove the technical condition on nodal rational curves in the linear series.

Because the moduli space $\mathcal{M}_X(\beta,1)$ is a deformation of the Hilbert scheme of point of the K3 surface X_0, by the work of Goettsche the Euler number $n_0(\beta) = e(\mathcal{M}_X(\beta,1))$ is the degree $1 + \frac{\beta^2}{2}$ term of the generating function

$$q\eta(q)^{-24} = \prod_{m=1}^{\infty} (1-q^m)^{-24} = 1 + 24q + 324q^2 + 3200q^3 + 25650q^4 + \cdots .$$

Thus the Gopakumar-Vafa conjecture provides a closed formula of the genus zero Gromov-Witten invariants of a local K3 fibered Calabi-Yau threefolds.

2 K3 Fibred Local Calabi-Yau Threefolds

We begin recalling the notion of fibrewise rigid curve class in the central fiber X_0 of a local K3 fibred Calabi-Yau threefold $\pi\colon X \to \Delta$.

Definition 2.1 *We call a class* $\beta \in H_2(X, \mathbb{Z})$ *a curve class if it can be represented by a proper one-dimensional subscheme* $D \subset X$ (*we assume* D *has no embedded points*); *we call the curve class* β *a curve class in* X_0 *if the subscheme* D *can be chosen as a subscheme of* X_0; *we call a curve class* β *in* X_0 *fibrewise rigid if all infinitesimal deformations of* D *in* X *that represent the class* β *must be contained in* X_0.

Here by an infinitesimal deformation of a curve $D \subset X$ we mean a flat family of curves $\mathcal{D} \subset X \times_D S$ over the spectrum $S = \operatorname{Spec} A$ of a local Artin ring A so that its only closed fiber is the curve $D \subset X$ we begin with.

For the compact K3 surface X_0, curve classes are exactly those in $H_2(X_0, \mathbb{Z})$ whose Poincaré duals are of $(1,1)$-types. By Hodge theory, they are exactly those whose pairings with the $(2,0)$-form Ω of X_0 vanish. Adopting this principle to family cases, we obtain the following topological criterion.

Lemma 2.2 *Let $K_{X/\Delta} = \wedge^2 T^\vee_{X/\Delta}$ be the relative canonical line bundle and let $\Omega \in \Gamma(X, K_{X/\Delta})$ be a nowhere vanishing global section, viewed as a family of $(2,0)$-forms on X_t that depends holomorphically on t. Then a curve class $\beta \in H_2(X;\mathbb{Z})$ in X_0 is fibrewise rigid if and only if the first derivative at $t = 0$ of the pairing (β, Ω_t) is non-zero.*

Proof This follows immediately from the Hodge theory, and shall be omitted. See also [13].

By the Torelli theorem of K3 surfaces, for any curve class β of a K3 surface X_0, we can embed the later as the central fiber of a local K3 fibred Calabi-Yau threefold so that β is fiberwise rigid.

We next look at the notion and the moduli of pure dimension one sheaves on X. Recall that a sheaf of $\mathcal{O}_X$-modules has pure dimension one if the support of every element of that sheaf has dimension one. For instance, for any pair of an immersion $\iota\colon C \to X$ of a curve and a sheaf $\mathcal{F}$ of $\mathcal{O}_C$-modules, the direct image $\iota_*\mathcal{F}$ is of pure dimension one if and only if $\mathcal{F}$ is torsion free.

In [19], Simpson generalized the notion of Gieseker stability to sheaves of pure dimensions by introducing the notion of normalized Hilbert polynomials. Specifically, for a pure dimension d sheaf $\mathcal{E}$ of $\mathcal{O}_W$-modules over a polarized projective scheme (W, H), he defines its normalized Hilbert polynomial be the quotient of the Hilbert polynomial $\chi(\mathcal{E} \otimes H^n)$ by its own leading coefficient:

$$p_{\mathcal{E}}(n) = \frac{\chi(\mathcal{E} \otimes H^n)}{\text{l.c.}\,\chi(\mathcal{E} \otimes H^n)}.$$

He calls $\mathcal{E}$ stable if its normalized Hilbert polynomial is always less than the normalized Hilbert polynomial of any of its proper pure dimension d quotient sheaf.

According to Simpson, the moduli functor of equivalence classes of stable sheaves of pure dimensions d and of fixed Chern classes is coarsely represented by a quasi-projective scheme; and is finely represented by a projective scheme in case the Chern classes are relatively prime.

This construction immediately applies to moduli of sheaves over X_0 once we know that X_0 is projective with a polarization H_0, a condition we will impose throughout this paper. Imposing similar condition on X will be too restrictive to our discussion. As an alternative, we argue that fixing the polarization H_0 on X_0 will be sufficient to define stability of sheaves of pure dimension one on X with $c_2 = -\beta$.

For this, to any pure dimension one sheaf of $\mathcal{O}_X$-modules $\mathcal{E}$ we let its support closed subscheme $\Lambda_{\mathcal{E}}$ be the one defined by the ideal sheaf generated by all regular functions annihilating the sheaf $\mathcal{E}$. Because $\mathcal{E}$ has pure dimension one, $\Lambda_{\mathcal{E}}$ is a curve without embedded points. We suppose $\Lambda_{\mathcal{E}}$ is proper, and denote by $\iota : \Lambda_{\mathcal{E}} \to X$ the immersion. Then there is a sheaf of $\mathcal{O}_{\Lambda_{\mathcal{E}}}$-modules $\mathcal{E}'$ so that $\mathcal{E} = \iota_*\mathcal{E}'$. To proceed, we further assume that $\Lambda_{\mathcal{E}}$ is connected and $c_2(\mathcal{E}) = -\beta$. Because the support of any stable sheaves is always connected, this connectedness requirement does not impose any additional constraint to the sheaves we are interested. On the other hand, once we know $\Lambda_{\mathcal{E}}$ is connected, as a set it must be contained in a single fiber of $X \to \Delta$. Adding that $c_2(\mathcal{E}) = -\beta$ and that β is a curve class only in the central fiber X_0, that fiber must be X_0.

We next pick a line bundle $\mathcal{L}$ on $\Lambda_{\mathcal{E}}$ whose restriction to $\Lambda_{\mathcal{E},\mathrm{red}}$, which is $\Lambda_{\mathcal{E}}$ with the reduced scheme structure, is numerically equivalent to $H_0|_{\Lambda_{\mathcal{E},\mathrm{red}}}$. Because H_0 is ample, $\mathcal{L}$ is ample. We then define the Hilbert polynomial of $\mathcal{E}$ be $\chi(\mathcal{E}' \otimes \mathcal{L}^{\otimes n})$, a linear function in n, and define the normalized Hilbert polynomial of $\mathcal{E}$ be

$$p_{\mathcal{E}}(n) = \frac{\chi(\mathcal{E}' \otimes \mathcal{L}^{\otimes n})}{\mathrm{l.c.}\, \chi(\mathcal{E}' \otimes \mathcal{L}^{\otimes n})}.$$

Definition 2.3 *Let $\mathcal{E}$ be a sheaf of $\mathcal{O}_X$-modules of pure dimension one with connected and proper support subscheme and having $c_2(\mathcal{E}) = -\beta$. We say $\mathcal{E}$ is stable (w.r.t. H_0 on X_0) if for any pure dimension one proper quotient $\mathcal{E} \to \mathcal{F}$ we have*

$$p_{\mathcal{E}}(n) < p_{\mathcal{F}}(n), \quad \forall\, n \gg 1.$$

The so defined stable sheaves enjoy the usual properties satisfied by ordinary stable sheaves, including that their automorphism groups are isomorphic to $\mathbb{C}^*$.

Constructing the moduli of stable sheaves of $\mathcal{O}_X$-modules poses another challenge because of the lack of the projectivity of X. We will overcome this difficulty by showing that the moduli functors of sheaves of $\mathcal{O}_X$-modules and of sheaves of $\mathcal{O}_{X_0}$-modules are identical.

For starters, we form the moduli functor $\mathfrak{F}_{X_0}(\beta, 1)$ of stable pure dimension one sheaves of $\mathcal{O}_{X_0}$-modules of first Chern class $c_1 = \beta$ and Euler characteristic number $\chi = 1$. It assigns to each scheme S the set of equivalence classes of flat S-families of such sheaves over X_0 in which two S-families $\mathcal{E}_1$ and $\mathcal{E}_2$ are equivalent if there is an invertible sheaf of $\mathcal{O}_S$-modules $\mathcal{P}$ so that $\mathcal{E}_1 \cong \mathcal{E}_2 \otimes \pi_S^* \mathcal{P}$. Because X_0 is projective and because $\chi = 1$, the works of Simpson and Maruyama [19, 15] ensure that this functor is finely represented by a projective scheme and its accompanying universal family.

As to sheaves of $\mathcal{O}_X$-modules, we similarly form the moduli functor $\mathfrak{F}_X(\beta, 1)$ of stable pure dimension one sheaves of $\mathcal{O}_X$-modules of connected support, of $c_2 = -\beta$ and Euler characteristic number $\chi = 1$. This new functor will be represented by the same moduli scheme once we prove that

Lemma 2.4 *The functors $\mathfrak{F}_{X_0}(\beta, 1)$ and $\mathfrak{F}_X(\beta, 1)$ are naturally equivalent.*

Proof First of all, pushing forward via the inclusion morphism $\iota \colon X_0 \to X$ transforms any S-family in $\mathfrak{F}_{X_0}(\beta, 1)(S)$ to a family in $\mathfrak{F}_X(\beta, 1)(S)$. Because this transformation preserves the equivalence relation, the stability condition and the pull-back operation, it defines a natural faithful transformation of functors

$$\mathfrak{F}_{X_0}(\beta, 1) \Longrightarrow \mathfrak{F}_X(\beta, 1).$$

Thus to prove the lemma we only need to show that for any scheme S the inclusion of sets

$$\eta_S : \mathfrak{F}_{X_0}(\beta, 1)(S) \longrightarrow \mathfrak{F}_X(\beta, 1)(S) \tag{2.1}$$

is surjective.

Let $\mathcal{E}$ be any family in $\mathfrak{F}_X(\beta,1)(S)$. Because it is a sheaf of $\mathcal{O}_{X\times S}$-modules, multiplying by $t\in\Gamma(\mathcal{O}_D)$ defines a sheaf homomorphism

$$t:\mathcal{E}\longrightarrow\mathcal{E}.$$

Clearly, should it be zero, then $\mathcal{E}$ lies in the image of η_S. Therefore we only need to check the vanishing of this homomorphism.

Suppose not, then after restricting the above homomorphism to the formal completion $\widehat{S}$ of S along some closed point $s\in S$ remains non-trivial. Because $\widehat{S}$ is the pro-limit of the spectrums of local Artin rings, for some local Artin ring A and embedding $\operatorname{Spec}A\subset\widehat{S}$, the parallel homomorphism

$$t:\mathcal{E}\otimes\mathcal{O}_{X\times\operatorname{Spec}A}\longrightarrow\mathcal{E}\otimes\mathcal{O}_{X\times\operatorname{Spec}A}\tag{2.2}$$

stays non-zero. Therefore we only need to prove the surjectivity of (2.2) for $S=\operatorname{Spec}A$ with A a local Artin ring.

Let $S=\operatorname{Spec}A$ with A a local Artin ring A; let $\mathcal{E}$ be a family in $\mathfrak{F}_X(\beta,1)(S)$. We first show that $\mathcal{E}_s$, the restriction of $\mathcal{E}$ to the fiber over the only closed $s\in S$, is a sheaf of $\mathcal{O}_{X_0}$-modules. As argued before, multiplying by t defines an endomorphism of the sheaf $\mathcal{E}_s$. Because $\mathcal{E}_s$ is stable, $\operatorname{End}(\mathcal{E}_s)=\mathbb{C}$. Hence there is a scalar $c\in\mathbb{C}$ so that multiplying by $c-t$ defines a zero endomorphism of $\mathcal{E}_s$, a condition equivalent to that $\mathcal{E}_s$ is a sheaf of $\mathcal{O}_{X_c}$-modules. On the other hand, since $c_2(\mathcal{E}_s)=-\beta\in H_2(X_c,\mathbb{Z})$ is a curve class, by our choice of $X\to\Delta$ this is possible only if $c=0$. Hence $\mathcal{E}_s\in\mathfrak{F}_{X_0}(\beta,1)(s)$.

As to the family $\mathcal{E}$ over S, we form the annihilator ideal sheaf

$$\operatorname{Ann}_{\Delta\times S}(\mathcal{E})=\{f\in\mathcal{O}_\Delta\times A\mid f\cdot\mathcal{E}=0\subset\mathcal{E}\}\subset\mathcal{O}_{\Delta\times S}$$

and its associated subscheme $\Gamma\subset\Delta\times S$. Because $t\cdot\mathcal{E}_s=0\subset\mathcal{E}_s$,

$$t\in\operatorname{Ann}_{\Delta\times S}(\mathcal{E})+\mathfrak{m}$$

for the maximal ideal $\mathfrak{m}\subset A$. This shows that $\Gamma\times_S s$ is the simple point $(0,s)\in\Delta\times S$, which in turn implies that the morphism $\Gamma\to S$ induced by the second projection $D\times S\to S$ is étale at $(0,s)\in\Gamma$. On the other hand, $\Gamma\to S$ is surjective because $\mathcal{E}$ is a sheaf of $\mathcal{O}_{X\times_\Delta\Gamma}$-modules and is flat over S. Therefore, $\Gamma\to S$ is an isomorphism.

This way the subscheme $\Gamma\subset\Delta\times S$ becomes a graph that defines a morphism $j:S\to\Delta$. We let $Y_S=X\times_\Delta S$ and let $\jmath: Y_S\to X\times S$ be the closed immersion. By our construction of Γ, the sheaf $\mathcal{E}$ is isomorphic to a direct image sheaf $\jmath_*\mathcal{E}'$ for a sheaf $\mathcal{E}'$ of $\mathcal{O}_{Y_S}$-modules. Because $\mathcal{E}$ is flat over S, so is $\mathcal{E}'$.

Being a flat family of pure dimension one sheaves over a family of smooth surfaces, we can define its divisor classes $\operatorname{Det}(\mathcal{E}')$. Since Y_S is a family of smooth surfaces and $\mathcal{E}'$ is flat over S, we can cover Y_S by opens U_i so that over each U_i the sheaf $\mathcal{E}'$ admits a locally free resolution

$$0\longrightarrow\mathcal{O}_{U_i}^{\oplus n_2}\longrightarrow\mathcal{O}_{U_i}^{\oplus n_1}\longrightarrow\mathcal{O}_{U_i}^{\oplus n_0}\longrightarrow\mathcal{E}'|_{U_i}\longrightarrow 0.$$

Following [18], we define

$$\det \mathcal{E}'|_{U_i} \triangleq \bigotimes_{i=0}^{2} (\wedge^{n_i} \mathcal{O}_{U_i}^{\oplus n_i})^{(-1)^i}.$$

Obviously, away from the support of $\mathcal{E}'$, the above exact sequence defines canonically a nowhere vanishing section of $\det \mathcal{E}'|_{U_i - \mathrm{Supp}(\mathcal{E}')}$, which extends to a regular section $s_i \in \Gamma(\det \mathcal{E}'|_{U_i})$ that vanishes along $\mathrm{Supp}(\mathcal{E}') \cap U_i$. Then following the argument in [18], the determinant line bundles $\det \mathcal{E}'|_{U_i}$ glue canonically to form a determinant line bundle $\det \mathcal{E}'$ on X, and the individual sections s_i patch together to form a global section

$$s \in \Gamma(Y_S, \det \mathcal{E}')$$

whose vanishing defines a divisor in Y_S flat over S, the divisor we will denote by $\mathrm{Det}(\mathcal{E}')$ and call by the determinant divisor of $\mathcal{E}'$.

But then because β is a fiberwise rigid curve class, the divisor $\mathrm{Det}(\mathcal{E}')$ must be contained entirely in $X_0 \times S$. Hence $Y_S = X_0 \times S$ and $\mathcal{E}'$ is a flat S-family of sheaves over X_0. This proves that $\mathcal{E}$ is in the image of $\mathfrak{F}_{X_0}(\beta, 1)(S)$, the statement that completes the proof of the lemma.

Using the determinant divisor $\mathrm{Det}(\mathcal{E})$ we can define the multiplicity of $\mathcal{E}$ along any irreducible component $D \subset \Lambda_{\mathcal{E}}$ to be the multiplicity of $\mathrm{Det}(\mathcal{E})$ along D. We denote this multiplicity by $\mathrm{mult}_D \mathcal{E}$.

Combining the work of Simpson and Maruyama [19, 15], the moduli functor $\mathfrak{F}_{X_0}(\beta, 1)$ is finely represented by a projective scheme $\mathcal{M}_{X_0}(\beta, 1)$ and a universal family $\mathcal{U}$ over $X_0 \times \mathcal{M}_{X_0}(\beta, 1)$. By the previous lemma, the functor $\mathfrak{F}_X(\beta, 1)$ is represented by the same scheme $\mathfrak{M}_X(\beta, 1) = \mathfrak{M}_{X_0}(\beta, 1)$ and the universal family that is the push forward $\jmath_* \mathcal{U}$ by the closed immersion

$$\jmath : X_0 \times \mathfrak{M}_X(\beta, 1) \to X \times \mathfrak{M}_X(\beta, 1).$$

We will now rephrase the Gopakumar-Vafa conjecture for the $\pi : X \to \Delta$ and the rigid β. Following the work of Mukai [17], the moduli space $\mathfrak{M}_{X_0}(\beta, 1)$ is smooth, of dimension $\beta^2 + 2$, and of tangent sheaf isomorphic to $\mathcal{E}xt^1_{\pi_2}(\mathcal{U}, \mathcal{U})$, the relative extension sheaf of the second projection

$$\pi_2 : X \times \mathfrak{M}_{X_0}(\beta, 1) \to \mathfrak{M}_{X_0}(\beta, 1).$$

As to the moduli space $\mathcal{M}_X(\beta, 1)$, its tangent sheaf and obstruction sheaf a priori are defined as the relative extension sheaves

$$\mathcal{E}xt^1_{p_2}(\jmath_* \mathcal{U}, \jmath_* \mathcal{U}) \quad \text{and} \quad \mathcal{E}xt^2_{p_2}(\jmath_* \mathcal{U}, \jmath_* \mathcal{U})$$

for the second projection p_2 of $X \times \mathcal{M}_X(\beta, 1)$, since $\mathcal{M}_X(\beta, 1) = \mathcal{M}_{X_0}(\beta, 1)$, the two relative extension sheaves isomorphic to their tangent sheaves must canonically isomorphic:

$$\mathcal{E}xt^1_{p_2}(\jmath_* \mathcal{U}, \jmath_* \mathcal{U}) \cong \mathcal{E}xt^1_{\pi_2}(\mathcal{U}, \mathcal{U}).$$

On the other hand, should X be projective, then the Serre duality would provide us a canonical isomorphism

$$\mathcal{E}xt^2_{p_2}(\jmath_* \mathcal{U}, \jmath_* \mathcal{U}) \cong \mathcal{E}xt^1_{p_2}(\jmath_* \mathcal{U}, \jmath_* \mathcal{U})^\vee.$$

(This makes sense since both sheaves are locally free in this case.) This suggests us that even in case X is a local Calabi-Yau threefold, we should still take the dual of the tangent sheaf of the *smooth* moduli space as its obstruction sheaf.

Definition 2.5 *We define the virtual cycle of $\mathcal{M}_X(\beta,1)$ be the top Chern class of the cotangent bundle of $\mathcal{M}_X(\beta,1)$.*

Because $\dim \mathcal{M}_X(\beta,1) = \beta^2+2$ is even, the degree of the virtual cycle of $\mathcal{M}_X(\beta,1)$ is identical to its Euler number.

To rephrase the Gopakumar-Vafa conjecture, we recall the notion of Gromov-Witten invariants of X. For $g=0$, we denote the moduli of genus 0 stable maps to X_0 with fundamental class β by $\mathfrak{M}_0(X_0,\beta)$; denote similar moduli of stable maps to X by $\mathfrak{M}_0(X,\beta)$. Because X_0 is a subvariety of X, the former is a closed substack of the later.

Proposition 2.6 *The canonical inclusion of the stacks $\mathcal{M}_0(X_0,\beta) \subset \mathcal{M}_0(X,\beta)$ is an isomorphism.*

Proof This is true because β is a fiberwise rigid curve class. We shall omit the proof since it is parallel to the case of moduli of sheaves proved in this section.

Because X_0 is projective, $\mathcal{M}_0(X_0,\beta)$ is proper; hence $\mathcal{M}_0(X,\beta)$ is proper too. Let $[\mathcal{M}_0(X,\beta)]^{\mathrm{vir}}$ be the virtual cycle of $\mathcal{M}_0(X,\beta)$. Since X is a Calabi-Yau threefold, it is a zero dimensional cycle whose degree is the genus zero Gromov-Witten invariants of the local Calabi-Yau X.

With the Gromov-Witten invariants and the virtual degree of the moduli of pure dimension one sheaves at hand, we can phrase the genus zero Gopakumar-Vafa conjecture for X as follows:

Conjecture 2.7 *Let $n_0(\beta) = \deg[\mathcal{M}_X(\beta,1)]^{\mathrm{vir}}$ and let $N_0(\beta) = \deg[\mathcal{M}_0(X,\beta)]^{\mathrm{vir}}$. Then*

$$\sum_{k\geq 1} N_0(k\beta)q^k = \sum_{l,k\geq 1} \frac{1}{k^3}\, n_0(l\beta)q^{kl}.$$

3 Localized Along Rational Curves

Following the work of Yau-Zaslow, we can localize the Euler number of $\mathcal{M}_{X_0}(\beta,1)$ to those sheaves that are over rational curves.

To begin with, the associated determinant divisor morphism of the universal family $\mathcal{U}$ over $X_0 \times \mathcal{M}_{X_0}(\beta,1)$ defines a morphism

$$\rho : \mathcal{M}_{X_0}(\beta,1) \longrightarrow |\beta|$$

to the space of divisors in X_0 representing β. Because X_0 is a K3 surface, $|\beta|$ is a $\frac{1}{2}\beta^2+1$ dimensional projective space. It is divided into two parts: one consists of those divisors in $|\beta|$ whose irreducible components have geometric genus zero, the divisors we call rational; the other consists of all remainder elements.

For those divisors that belong to the second set, we have the following vanishing result.

Lemma 3.1 *The fiber $\rho^{-1}(D)$ of ρ over non-rational $D \in |\beta|$ all have vanishing Euler numbers.*

Proof We will prove this vanishing by constructing an S^1-action on $\rho^{-1}(D)$ with finite stabilizers everywhere. We first introduce a group action on $\rho^{-1}(D)$. Let $\mathrm{Pic}^0(D)$ be the group of invertible sheaves of $\mathcal{O}_D$-modules that have vanishing degrees along each irreducible component of D. Because elements of $\rho^{-1}(D)$ can be viewed as sheaves of $\mathcal{O}_D$-modules, we can tensor these sheaves by $\mathcal{L} \in \mathrm{Pic}^0(D)$ to obtain new sheaves of $\mathcal{O}_D$-modules that, when considered as sheaves of $\mathcal{O}_{X_0}$-modules, do belong to $\rho^{-1}(D)$. This tensor product defines a group action of $\mathrm{Pic}^0(D)$ on $\rho^{-1}(D)$,

$$(\mathcal{L}, \mathcal{U}) \in \mathrm{Pic}^0(D) \times \rho^{-1}(D) \longmapsto \mathcal{U} \otimes \mathcal{L} \in \rho^{-1}(D).$$

To deduce the vanishing of $e(\rho^{-1}(D))$ for non-rational D, we need an S^1-subgroup of $\mathrm{Pic}^0(D)$ whose action on $\rho^{-1}(D)$ has finite stabilizers everywhere. To this end, we let D_0 be the normalization of a non-rational irreducible component of D. Because $g(D_0) \geq 1$, $\mathrm{Pic}^0(D_0)$ contains nontrivial subgroups $S^1 \subset \mathrm{Pic}^0(D_0)$. Because the tautological $\mathrm{Pic}^0(D) \to \mathrm{Pic}^0(D_0)$ is surjective, there are S^1-subgroups $S^1 \subset \mathrm{Pic}^0(D)$ whose images in $\mathrm{Pic}^0(D_0)$ are non-trivial subgroups in $\mathrm{Pic}^0(D_0)$. We pick one such S^1-subgroup in $\mathrm{Pic}^0(D)$, denote by S^1 as well. We claim that this S^1 action on $\rho^{-1}(D)$ has finite stabilizers everywhere. Indeed, let $\mathcal{L} \in \mathrm{Pic}^0(D)$ and $\mathcal{U} \in \rho^{-1}(D)$ be any pair that fits into the isomorphism $\mathcal{U} \otimes \mathcal{L} \cong \mathcal{U}$. Let $i : D_0 \to D$ be the tautological morphism. Then under the pull back, the mentioned isomorphism infers

$$i^*\mathcal{U} \cong i^*(\mathcal{U} \otimes \mathcal{L}) \cong i^*\mathcal{U} \otimes i^*\mathcal{L}.$$

Since $\Lambda_{\mathcal{U}} = D$, $i^*\mathcal{U}$ has positive rank $r > 0$. Thus taking the determinant line bundles of both sides, we obtain

$$\det(i^*\mathcal{U}) \cong \det(i^*\mathcal{U}) \otimes i^*\mathcal{L}^{\otimes r}, \tag{3.1}$$

and hence $i^*\mathcal{L}^{\otimes r} \cong \mathcal{O}_{D_0}$. Because $\{i^*\mathcal{L} \mid \mathcal{L} \in S^1\}$ forms a non-trivial subgroup of $\mathrm{Pic}^0(D_0)$, this isomorphism guarantees that there are only finitely many $\mathcal{L} \in S^1$ that satisfies (3.1). This proves that $\#\mathrm{Stab}_{S^1}([\mathcal{U}]) < \infty$. Consequently, following a standard argument in topology, the Euler number of $\rho^{-1}(D)$ must vanish.

Because X_0 is a K3 surface, the first set, the set of rational curves in $|\beta|$, contains finitely many elements. We list them as $C_1, \cdots, C_l$.

Corollary 3.2 *The Euler number of $\mathcal{M}_{X_0}(\beta, 1)$ decomposes*

$$e(\mathcal{M}_{X_0}(\beta, 1)) = \sum_{i=1}^{l} e(\rho^{-1}(C_i)).$$

Proof The proof is a standard application of the Mayer-Vietories sequence in cohomologies, and shall be omitted here.

We can similarly localize the contribution $\deg[\mathcal{M}_0(X, \beta)]^{\mathrm{vir}}$ of the moduli of stable morphisms to individual curves $C_1, \cdots, C_l$. Let $f : \Sigma \to X$ be a stable map in $\mathcal{M}_0(X, \beta)$. Because the image $f(\Sigma)$ lies in the surface X_0, we can define the image divisor of f as a weighted sum

$$\mathcal{D}(f) = \sum_{\Sigma' \text{ irre. components of } \Sigma} \deg(f|_{\Sigma'}) \cdot f(\Sigma')$$

of the reduced image divisors $f(\Sigma')$ by the multiplicities $\deg(f|_{\Sigma'})$ that are the degrees of the restriction morphisms $f|_{\Sigma'} \colon \Sigma' \to f(\Sigma')$. Since $\mathcal{D}(f) \in |\beta|$ and is rational, each

$$\mathcal{M}_0(X,\beta)_{C_i} = \{f \in \mathcal{M}_0(X,\beta) \mid \mathcal{D}(f) = C_i\}$$

is an open and closed subset of $\mathcal{M}_0(X,\beta)$. Summing over all C_i provides an open-closed decomposition

$$\mathcal{M}_0(X,\beta) = \coprod_{i=1}^{l} \mathcal{M}_0(X,\beta)_{C_i}.$$

Accordingly, we can define the localized virtual class $[\mathcal{M}_0(X,\beta)_{C_i}]^{\mathrm{vir}}$. Their degrees sum up to the total degree

$$\deg[\mathcal{M}_0(X,\beta)]^{\mathrm{vir}} = \sum_{i=1}^{l} \deg[\mathcal{M}_0(X,\beta)_{C_i}]^{\mathrm{vir}}.$$

The genus zero Gopakumar-Vafa conjecture for X will follow from its local version:

Conjecture 3.3 *Let $X \to \Delta$ be a* K3 *fibred local Calabi-Yau threefold, and let β be a fiberwise rigid curve class in X_0. Then for any rational $C \in |\beta|$,*

$$\deg[\mathcal{M}_0(X,\beta)_C]^{\mathrm{vir}} = \sum_{k|C} \frac{1}{k^3} e(\rho^{-1}(C/k)).$$

Here we say $k|C$ if the divisor C is a k-multiple of an integral divisor.

4 Invertible Sheaves on Planar Nodal Rational Curves

We shall resort to the old technique of enumerating the fixed points of certain compact group action on a space to evaluate its Euler number. As a first step, we need to construct the group of invertible sheaves in $\mathrm{Pic}^0(C)$. In case C is a reduced nodal rational curve, we know that $\mathrm{Pic}^0(C) \cong (\mathbb{C}^*)^{\times r}$ for some integer r. In general, we can lift the maximal compact subgroup of $\mathrm{Pic}^0(C_{\mathrm{red}})$ to $\mathrm{Pic}^0(C)$. Because we need the explicit form of these invertible sheaves, we shall construct them here in details.

We shall work with planar nodal rational curves, those that embed in some analytic, not necessary compact, complex surfaces.

Definition 4.1 *We call a proper one dimensional scheme planar curve if its formal completion along any of its closed point is isomorphic to* $\mathrm{Spec}\, k[\![z_1,z_2]\!]/(p)$ *for some non-zero polynomial p in (z_1,z_2); we call such curve nodal if the polynomial p can be chosen to be either $z_1^{m_1}$ or $z_1^{m_1}z_2^{m_2}$ for some positive integers m_1 and m_2; we call such curve rational if each irreducible component is rational.*

The planar curves we adopt here admit no embedded points. They are also embedable as proper divisors of analytic surfaces. Such embeddings allow us to work with analytic neighborhoods of these curves. For instance, for a planar curve C embedded in a surface S defined by the vanishing of a section f of a line bundle on S, its analytic neighborhoods are its intersections with the analytic neighborhoods of S. In case $\mathcal{U}$ is an open subset in S that is isomorphic to a disk in $\mathbb{C}^2$, then we can define the ring of analytic functions over $C\cap\mathcal{U}$ be

$$\Gamma(C\cap\mathcal{U},\mathcal{O}_{C\cap\mathcal{U}})=\Gamma(\mathcal{U},\mathcal{O}_{\mathcal{U}})/(f|_{\mathcal{U}}).$$

This analytic embedding also allows us to resolve nodal singularity of C one at a time, while retaining its nilpotent structure. For instance, to resolve a node α in D embedded in S, we can pick an analytic chart (z_1,z_2) of $\alpha\in S$ over an open $\alpha\in\mathcal{U}\subset S$ so that $C\cap\mathcal{U}$ is defined by $z_1^{m_1}z_2^{m_2}=0$. By shrinking S and rescaling z_i, we can arrange $\mathcal{U}$ so that

$$\mathcal{U}=\{(z_1,z_2)\in\mathbb{C}^2\mid |z_1|<1 \text{ and } |z_2|<4, \text{ or, } |z_1|<4 \text{ and } |z_2|<1\}, \tag{4.1}$$

a union of two stripes, one horizontal and one vertical, in $\mathbb{C}^2$, and that the closure

$$\operatorname{closure}(\mathcal{U})\cap(S-\mathcal{U})=\{|z_1|<1 \text{ and } |z_2|=4\}\cup\{|z_1|=4 \text{ and } |z_2|<1\}.$$

To resolve the node α, we first replace S by a new surface S_α resulting from first gluing the (horizontal) stripe

$$V_1=\{(u_1,u_2)\in\mathbb{C}^2\mid |u_1|<4 \text{ and } |u_2|<1\}$$

to $S_0=S-\{|z_1|<1 \text{ and } |z_2|<1\}$ by the rule $z_1=u_1$ and $z_2=u_2$ and followed by gluing the (vertical) stripe

$$V_2=\{(v_1,v_2)\in\mathbb{C}^2\mid |v_1|<1 \text{ and } |v_2|<4\}$$

to the resulting surface by the rule $z_1=v_1$ and $z_2=v_2$. Because of our choice of $\mathcal{U}$, the resulting surface S_α is smooth. Further, the closed subscheme $C\cap S_0$ can be extended to a proper closed subscheme in S_α by adding $u_1^{m_1}=0$ in V_1 and $v_2^{m_2}=0$ in V_2. We denote by $C_\alpha\subset S_\alpha$ the resulting subscheme. By its construction, it is planar; it admits a projection

$$\psi_\alpha\colon C_\alpha\to C;$$

and it reproduces C after gluing

$$A_{\alpha,1}\triangleq\operatorname{Spec}\mathbb{C}[u_1,u_2]/(u_1^{m_1},u_2^{m_2})\quad\text{with}\quad A_{\alpha,2}\triangleq\operatorname{Spec}\mathbb{C}[v_1,v_2]/(v_1^{m_1},v_2^{m_2})$$

following $u_1=v_1$ and $u_2=v_2$. We let B_α be the closed subscheme $z_1^{m_1}=z_2^{m_2}=0$ in S. It is clear that B_α is independent of the choice of (z_1,z_2) and is isomorphic to $A_{\alpha,i}$ canonically; we let $\phi_{\alpha,i}\colon B_\alpha\to A_{\alpha,i}$ be such isomorphisms.

Our group of invertible sheaves on C will be constructed by modifying the canonical exact sequence associated to the push forward $\psi_{\alpha*}\mathcal{O}_{C_\alpha}$:

$$0\longrightarrow\mathcal{O}_C\longrightarrow\psi_{\alpha*}\mathcal{O}_{C_\alpha}\xrightarrow{u_\alpha(1)}\mathcal{O}_{B_\alpha}\longrightarrow 0. \tag{4.2}$$

Tensoring it by $\mathcal{O}_{B_\alpha}$, we see that the arrow $\psi_{\alpha*}\mathcal{O}_{C_\alpha} \to \mathcal{O}_{B_\alpha}$ is induced by the composite

$$u_\alpha(1) : \psi_{\alpha*}\mathcal{O}_{C_\alpha} \longrightarrow \psi_{\alpha*}\mathcal{O}_{C_\alpha} \otimes \mathcal{O}_{B_\alpha} \equiv \mathcal{O}_{A_{\alpha,1}} \oplus \mathcal{O}_{A_{\alpha,2}} \longrightarrow \mathcal{O}_{B_\alpha} \tag{4.3}$$

in which the last arrow is the difference $\phi^*_{\alpha,1} - \phi^*_{\alpha,2}$ of the tautological $\phi^*_{\alpha,i} \colon \mathcal{O}_{A_{\alpha,i}} \to \mathcal{O}_{B_\alpha}$. To construct the desired invertible sheaves $\mathcal{L}_\alpha(\sigma_\alpha)$, for $\sigma_\alpha \in \mathbb{C}^*$, we shall define new homomorphism $u_\alpha(\sigma_\alpha)$ that is the composition of the first arrow in (4.3) with

$$\phi^*_{\alpha,1} - \sigma_\alpha \phi^*_{\alpha,2} : \mathcal{O}_{A_{\alpha,1}} \oplus \mathcal{O}_{A_{\alpha,2}} \longrightarrow \mathcal{O}_{B_\alpha},$$

and define $\mathcal{L}_\alpha(\sigma_\alpha)$ by the same exact sequence (4.2) with $u_\alpha(1)$ replaced by $u_\alpha(\sigma_\alpha)$.

It is direct to check that all $\mathcal{L}_\alpha(\sigma_\alpha) \in \mathrm{Pic}^0(C)$; and for $\sigma_\alpha, \sigma'_\alpha \in \mathbb{C}^*$,

$$\mathcal{L}_\alpha(\sigma_\alpha) \otimes \mathcal{L}_\alpha(\sigma'_\alpha) \cong \mathcal{L}_\alpha(\sigma_\alpha \sigma'_\alpha).$$

Therefore, the correspondence $\sigma_\alpha \mapsto \mathcal{L}_\alpha(\sigma_\alpha)$ defines a homomorphism of groups $\mathbb{C}^* \to \mathrm{Pic}^0(C)$. Consequently, we will denote by G_α the group $\mathbb{C}^*$ together with this homomorphism; More conveniently, we shall form the Cartesian product

$$G(C) = \prod_{\alpha \in \Lambda} G_\alpha \cong (\mathbb{C}^*)^\Lambda$$

(that is the space of all maps from Λ to $\mathbb{C}^*$) together with the tautological homomorphism

$$(\sigma_\alpha)_{\alpha \in \Lambda} \in (\mathbb{C}^*)^\Lambda \longmapsto \bigotimes_{\alpha \in \Lambda} \mathcal{L}_\alpha(\sigma_\alpha) \in \mathrm{Pic}^0(C).$$

For convenience, we shall abbreviate $(\sigma_\alpha)_{\sigma \in \Lambda}$ to a single σ and abbreviate the tensor product $\otimes_\alpha \mathcal{L}_\alpha(\sigma_\alpha)$ to $\mathcal{L}(\sigma)$. This way, the above map can be rephrased as

$$\sigma \in (\mathbb{C}^*)^\Lambda \longmapsto \mathcal{L}(\sigma) \in \mathrm{Pic}^0(C). \tag{4.4}$$

We also comment that here we build the curve C in the notation $G(C)$ because later we need to work with different curves, thus with different groups. In case there is no confusion with other groups, we shall abbreviate $G(C)$ to G.

One property of this group that follows from the construction is its universal family and multiplication morphism. Namely, there is an invertible sheaf $\mathcal{L}$ of $\mathcal{O}_{C \times G}$-modules so that to each $\sigma \in G$ its restriction to the slice $C \times \sigma$ is $\mathcal{L}|_{C \times \sigma} \cong \mathcal{L}(\sigma)$. This family also admits a lifting of the multiplication morphism $\mathfrak{m} \colon G \times G \to G$:

$$\mathfrak{m}^* : (1_C \times \mathfrak{m})^* \mathcal{L} \xrightarrow{\cong} \pi^*_{12}\mathcal{L} \otimes \pi^*_{13}\mathcal{L}, \text{}^{1} \tag{4.5}$$

an isomorphism of sheaves over $C \times G \times G$ whose restriction to $C \times \sigma \times \sigma'$ is the group law of $G(C)$: $\mathcal{L}(\sigma) \otimes \mathcal{L}(\sigma') \cong \mathcal{L}(\sigma\sigma')$.

The other useful property of this family $\mathcal{L}$ that follows from the construction is its local trivializations over $U = C - \Lambda$ and near each node of C. First, restricting to U the defining

[1]We shall adopt the convention that π_{ij} shall always mean the projection from the product to the product of the i-th and the j-th factors; same to π_i and π_{ijk}.

exact sequences (4.2) are reduced to canonical isomorphisms $\mathcal{O}_U \cong \mathcal{L}_\alpha(\sigma_\alpha)|_U$, canonical in that they form an isomorphism on $U \times G$:

$$\eta_U : \mathcal{O}_{U\times G} \longrightarrow \mathcal{L}|_{U\times G} \tag{4.6}$$

that, if we substitute the sheaves $\mathcal{L}|_{U\times G}$ by $\mathcal{O}_{U\times G}$, then the resulting isomorphism in (4.5) is the one that sends the constant section 1 to the tensor product of constant sections $1 \otimes 1$.

Trivializing $\mathcal{L}$ near a node $\alpha \in C$ requires some care. We begin with the (analytic) open neighborhood of $\alpha \in C$ using (4.1):

$$V_\alpha = \mathcal{U} \cap C = \{(z_1, z_2) \mid z_1^{m_1} z_2^{m_2} = 0,\ |z_1| + |z_2| < 1\}. \tag{4.7}$$

The complement $V_\alpha - \alpha$, or the intersection $V_\alpha \cap U$, is the union of

$$V_{\alpha,1} = \{(z_1, z_2) \in V_\alpha \mid 0 < |z_1| < 1\} \ \text{ and } \ V_{\alpha,2} = \{(z_1, z_2) \in V_\alpha \mid 0 < |z_2| < 1\}.$$

We then pick a trivialization $\eta_V : \mathcal{O}_{(V\cap U)\times G} \to \mathcal{L}|_{(V\cap U)\times G}$ whose restriction to $V_{\alpha,1} \times G$ coincides with that of η_U, and whose restriction to $V_{\alpha,2} \times G$ is a scalar product of η_U by the function σ_α^{-1}:

$$\eta_V|_{V_{\alpha,2}\times G} = \sigma_\alpha^{-1} \cdot \eta_U|_{V_{\alpha,2}\times G}. \tag{4.8}$$

Comparing with the defining exact sequence of $\mathcal{L}$, we see immediately that the trivialization η_V extends to a trivialization of $\mathcal{L}$ over $V \times G$, which we still denote by η_V.

The proof of these statements are straightforward and shall be omitted.

In the following, we will use spectral decomposition to characterize those sheaves in the moduli space that are fixed by the compact group G constructed in the previous section. To this end, we need to introduce more notations that will be used in the remainder of this paper. As before, we let $\jmath\colon C \subset S$ be a planar nodal rational curve with Λ the set of nodes of C and U the complement $C - \Lambda$. Eventually we will assume S is K3, in the sense that the canonical line bundle K_S is trivial, and call C a K3-planar nodal rational curve. Note that then all the curves C_α, those derived after resolving the node α of C, remains K3-planar nodal.

After fixing an ample line bundle on C, we can speak of stable pure dimension one sheaves of $\mathcal{O}_C$-modules. We let $\mathcal{M}_C$ be the moduli of such sheaves $\mathcal{E}$ that have Euler characteristic $\chi(\mathcal{E}) = 1$ and have determinants $\operatorname{Det}(\jmath_*\mathcal{E}) = C$. Equivalently, the multiplicity of $\mathcal{E}$ along any irreducible component D of C is equal to the multiplicity of C along D. The moduli space $\mathcal{M}_C$ is projective, admits a universal family and is acted on by G via the tensor product:

$$(\mathcal{E}, \mathcal{L}) \in \mathcal{M}_C \times G \longmapsto \mathcal{E} \otimes \mathcal{L} \in \mathcal{M}_C.$$

The spectral decomposition to be constructed will present sheaves $\mathcal{E}$ in the fixed loci $\mathcal{M}_C^G$ as the direct image sheaf $\iota_*\widetilde{\mathcal{E}}$ of a pair of a scheme $\iota\colon \widetilde{C} \to C$ and a sheaf of $\mathcal{O}_{\widetilde{C}}$-modules $\widetilde{\mathcal{E}}$. For this, we need to specify a finite sheaf of $\mathcal{O}_C$-subalgebras $\mathcal{A} \subset \mathcal{E}nd(\mathcal{E})$ and define ι be the tautological projection

$$\iota : \widetilde{C} \triangleq \operatorname{Spec}_C \mathcal{A} \longrightarrow C.$$

To make this construction funtorial, we shall work out its relative version in details here. Let $W \subset \mathcal{M}_C^G$ be a connected affine Zariski open subset, endowed with reduced scheme structure, and let $\mathcal{E}$ over $C \times W$ be the restriction to W of the universal family of $\mathcal{M}_C$. Because W is fixed by G, there is an invertible sheaf $\mathcal{K}$ on $W \times G$ such that as sheaves of $\mathcal{O}_{C\times W\times G}$-modules,

$$\pi_{12}^*\mathcal{E} \otimes \pi_{13}^*\mathcal{L} \cong \pi_{12}^*\mathcal{E} \otimes \pi_{23}^*\mathcal{K}.$$

By shrinking W if necessary, we can assume that $\mathcal{K} = \mathcal{O}_{W\times G}$. Thus the above isomorphism reduces to

$$\Phi : \pi_{12}^*\mathcal{E} \otimes \pi_{13}^*\mathcal{L} \xrightarrow{\cong} \pi_{12}^*\mathcal{E}. \tag{4.9}$$

Combining this isomorphism with the dual multiplication homomorphism $\mathfrak{m}^*$, we obtain two sequences of sheaf isomorphisms over $C \times M_1 \times G \times G$:

$$\pi_{12}^*\mathcal{E} \otimes \pi_{13}^*\mathcal{L} \otimes \pi_{14}^*\mathcal{L} \xrightarrow{\pi_{123}^*(\Phi)} \pi_{12}^*\mathcal{E} \otimes \pi_{14}^*\mathcal{L} \xrightarrow{\pi_{124}^*(\Phi)} \pi_{12}^*\mathcal{E}$$

and

$$\pi_{12}^*\mathcal{E} \otimes \pi_{13}^*\mathcal{L} \otimes \pi_{14}^*\mathcal{L} \xrightarrow{\pi_{134}^*(\mathfrak{m}^*)} \pi_{12}^*\mathcal{E} \otimes (1_C \times 1_W \times \mathfrak{m})^*\mathcal{L} \xrightarrow{(1_C\times 1_W\times\mathfrak{m})^*(\Phi)} \pi_{12}^*\mathcal{E}.$$

Because W is reduced and because all families involved are families of stable sheaves, the individual composition of each of the above two sequences differ by a nowhere vanishing regular function ρ on $W \times G \times G$:

$$(1_C \times 1_W \times \mathfrak{m})^*(\Phi) \circ \pi_{134}^*(\mathfrak{m}^*) = \rho \cdot \pi_{124}^*(\Phi) \circ \pi_{123}^*(\Phi). \tag{4.10}$$

Lemma 4.2 *We can choose* Φ *so that* (4.10) *holds for* $\rho \equiv 1$.

Proof By shrinking W if necessary, we can assume that there is a closed point $x \in C - \Lambda$ so that $\mathcal{E}_x \triangleq \mathcal{E}|_{x\times W}$, viewed as a sheaf of $\mathcal{O}_W$-modules, is locally free. For convenience, we let E be the vector bundle on W so that $\mathcal{O}_W(E) \cong \mathcal{E}_x$. Then the restriction to x of Φ, coupled with the trivialization η_U, defines a canonical isomorphism of vector bundles over $W \times G$:

$$\Phi_x : \pi_W^* E \longrightarrow \pi_W^* E,$$

which in turns defines a morphism

$$\psi : G \longrightarrow \Gamma(\mathrm{PGL}_W(E)), \ ^2$$

that, thanks to (4.10), must be a group homomorphism.

We now argue that ψ can be lifted to a homomorphism

$$\tilde{\psi} : G \longrightarrow \Gamma(\mathrm{GL}_W(E)).$$

Indeed, because G is Abelian, E decomposes into a direct sum of vector bundles $E \cong \oplus_\nu E_\nu$ such that each $\mathbf{P}_W(E_\nu) \subset \mathbf{P}_W(E)$ is fixed by the G action on $\mathbf{P}_W(E)$ induced by the homomorphism ψ. We then pick a nowhere vanishing section of one of E_ν, say $v \in \Gamma(E_\nu)$. Using $\mathcal{O}_W(E) \cong \mathcal{E}_x$, it associates to a unique section of $\mathcal{E}_x$, still denoted by v. Then necessarily, $\Phi_x(u) = \rho^{-1} \cdot v$

[2] $\mathrm{PGL}_W(E)$ is the vector bundle over W whose fiber over each $w \in W$ is the group $\mathrm{PGL}(E_w)$.

for a $\rho \in \Gamma(\mathcal{O}^*_{W\times G})$. In other words, if we replace Φ by $\rho\Phi$, then the new Φ_x will keep v fixed. Therefore, the difference of the two sides of (4.10), which is

$$(1_C\times 1_W\times \mathfrak{m})^*(\Phi)\circ\pi^*_{134}(\mathfrak{m}^*) - \pi^*_{124}(\Phi)\circ\pi^*_{123}(\Phi),$$

fails to be an isomorphism along each slice $C\times w\times\sigma\times\sigma'$. This is possible only if the above difference is identically zero because $\mathcal{E}$ is a family of stable sheaves while $\mathcal{L}$ is a family of invertible sheaves in $\mathrm{Pic}^0(C)$. This proves the lemma.

Restricting the homomorphism Φ in (4.9) to $C\times W\times\sigma$, we obtain sheaf homomorphisms

$$\Phi_\sigma : \pi^*_C\mathcal{L}(\sigma) \longrightarrow \mathcal{E}nd_{\mathcal{O}_{C\times W}}(\mathcal{E}) \tag{4.11}$$

parameterized by $\sigma\in G$. In the next section, we will show that the image of all Φ_σ generates a sheaf of commutative $\mathcal{O}_{C\times W}$-subalgebras in $\mathcal{E}nd_{\mathcal{O}_{C\times W}}(\mathcal{E})$. Hence, if we denote this sheaf of subalgebra by $\mathcal{A}$, then $Z = \mathrm{Spec}_{C\times W}\,\mathcal{A}$ becomes a scheme over $C\times W$; $\mathcal{E}$ becomes a sheaf of $\mathcal{A}$-modules; and the direct image sheaf of $\mathcal{E}$ under $Z\to C\times W$ is $\mathcal{E}$ itself.

5 Spectral Triples

Our strategy of studying the sheaf $\mathcal{A}$, and thus Z, is to apply the spectral decomposition of the representation of G induced by (4.11).

Before plunging into the details, let us first fix a few more notations that will be used here. First, we let Π be the space of maps from Λ to $\mathbb{Z}$, which is isomorphic to the space of characters of G. For $\sigma = (\sigma_\alpha)_\alpha$ and $\lambda\in\Pi$, we agree that

$$\sigma^\lambda = \prod_{\alpha\in\Lambda}\sigma_\alpha^{\lambda(\alpha)}.$$

Also, in case a vector space (or a sheaf) comes with a direct sum decomposition $\mathcal{F} = \oplus\mathcal{F}_i$, we will use $1_{\mathcal{F}_i}$ to mean both the identity map of $\mathcal{F}_i$ and the endomorphism of $\mathcal{F}$ that is the composites of the projection $\mathcal{F}\to\mathcal{F}_i$, the $1_{\mathcal{F}_i}:\mathcal{F}_i\to\mathcal{F}_i$ and the inclusion $\mathcal{F}_i\to\mathcal{F}$. Note that when we view $1_{\mathcal{F}_i}$ as an endomorphism of $\mathcal{F}$, we have $1_{\mathcal{F}_i}\cdot 1_{\mathcal{F}_i} = 1_{\mathcal{F}_i}$ and $1_{\mathcal{F}_i}\cdot 1_{\mathcal{F}_j} = 0$ for $i\neq j$.

We are now ready to construct and investigate the representations promised. We begin with the part of $\mathcal{A}$ over U. For this, we compose the trivialization $\eta_U:\mathcal{O}_U\to\mathcal{L}(\sigma)|_U$ with the homomorphism Φ_σ to obtain the homomorphism

$$\psi(\sigma) \triangleq \Phi_\sigma\circ\pi^*_U\eta_U : \mathcal{O}_{U\times W}\longrightarrow \mathcal{E}nd_{\mathcal{O}_{U\times W}}(\mathcal{E}), \tag{5.1}$$

a homomorphism that defines an automorphism of the vector space

$$\psi(\sigma) : \Gamma(U\times W,\mathcal{E}) \longrightarrow \Gamma(U\times W,\mathcal{E}). \tag{5.2}$$

Because of Lemma 4.2 and the remark immediately after (4.6), this assignment defines a group homomorphism

$$\psi : G\longrightarrow \mathrm{GL}(\Gamma(U\times W,\mathcal{E})).$$

Because G is commutative, the vector space $\Gamma(U\times W,\mathcal{E})$ decomposes into the direct sum of common eigenspaces of ψ, that is

$$\Gamma(U\times W,\mathcal{E})=\bigoplus_{\lambda\in\Pi}F_\lambda \quad\text{and}\quad \psi(\sigma)=\sum_{\lambda\in\Pi}\sigma^\lambda\cdot 1_{F_\lambda}. \tag{5.3}$$

At the sheaf level, we let $\mathcal{F}_\lambda\subset\mathcal{E}|_{U\times W}$ be the subsheaf generated by the sections $F_\lambda\subset\Gamma(\mathcal{E}|_{U\times W})$. Because each $\psi(\sigma)\in\mathrm{GL}(\Gamma(\mathcal{E}|_{U\times W}))$ is induced by a global sheaf endomorphism (5.1), the naturality of the Raynold operator implies that

$$\mathcal{E}|_{U\times W}=\bigoplus_{\lambda\in\Pi}\mathcal{F}_\lambda \quad\text{and}\quad \Phi_\sigma|_{U\times W}=\sum_{\lambda\in\Pi}\sigma^\lambda\cdot 1_{\mathcal{F}_\lambda}.$$

This, combined with $\mathcal{E}|_{U\times W}$ being coherent, implies that only finitely many $\mathcal{F}_\lambda$ are non-zero, and hence only finitely many $1_{\mathcal{F}_\lambda}:\mathcal{F}_\lambda\to\mathcal{F}_\lambda$ are non-trivial. We let Π_U be those λ so that $\mathcal{F}_\lambda\neq 0$; it is finite. Further, because $1_{\mathcal{F}_\lambda}\cdot 1_{\mathcal{F}_{\lambda'}}$ is zero in case $\lambda\neq\lambda'$, and is $1_{\mathcal{F}_\lambda}$ in case $\lambda=\lambda'$, the subsheaf

$$\mathcal{O}_{U\times W}1_{\mathcal{F}_\lambda}\subset\mathcal{E}nd_{\mathcal{O}_{U\times W}}(\mathcal{E})$$

generated by the section $1_{\mathcal{F}_\lambda}$ is a sheaf of subalgebras, and the direct sum

$$\bigoplus_{\lambda\in\Pi_U}\mathcal{O}_{U\times W}1_{\mathcal{F}_\lambda}\subset\mathcal{E}nd_{\mathcal{O}_{U\times W}}(\mathcal{E})$$

is the sheaf $\mathcal{A}\otimes\mathcal{O}_{U\times W}$. In particular, it is a finite and commutative sheaf of $\mathcal{O}_{U\times W}$-algebras.

We summarize these into the following lemma.

Lemma 5.1 *The sheaf of algebras $\mathcal{A}\otimes\mathcal{O}_{U\times W}$ is a finite and commutative sheaf of $\mathcal{O}_{U\times W}$-algebras. Further, for any irreducible component D of U and for Π_D the set of those $\lambda\in\Pi$ of which $\mathcal{F}_\lambda\otimes\mathcal{O}_{D\times W}\neq 0$, we have*

$$Z\times_{C\times W}(D_{\mathrm{red}}\times W)=\coprod_{\Pi_D}D_{\mathrm{red}}\times W,$$

a disjoint union of Π_D copies of $D_{\mathrm{red}}\times W$.

Proof Only the last statement remains to be checked. For this, we need to show that the kernel of the tautological homomorphism $\mathcal{O}_{U\times W}\to\mathcal{O}_{U\times W}1_{\mathcal{F}_\lambda}$ are nilponent elements of $\mathcal{O}_{D\times W}$. Because $\mathcal{O}_{D\times W}$ is locally free, this will be true if for general point $\xi\in D\times W$ the sheaf $\mathcal{O}_{D\times W}\cdot 1_{\mathcal{F}_\lambda}\otimes k(w)\neq 0$; but this is true since $\lambda\in\Pi_D$. This completes the proof of the lemma.

We next turn our attention to the fiber of Z over nodes of C. Let α be one of the node of C; let V_α be the analytic neighborhood of $\alpha\in C$ mentioned in (4.7). Then like the previous case, the trivialization $\eta_{V_\alpha}:\mathcal{O}_{V_\alpha}\to\mathcal{L}(\sigma)|_{V_\alpha}$, together with the homomorphism Φ_σ, gives rise to homomorphism

$$\psi_\alpha(\sigma):\mathcal{E}|_{V_\alpha}\longrightarrow\mathcal{E}|_{V_\alpha} \tag{5.4}$$

and its accompanied group representation

$$\psi_\alpha:G\longrightarrow\mathrm{GL}(\Gamma(V_\alpha\times W,\mathcal{E})). \tag{5.5}$$

Just as before, this provides us a unique spectral decomposition

$$\mathcal{E}|_{V_\alpha\times W}=\bigoplus_{\lambda\in\Pi}\mathcal{F}_{\alpha,\lambda}\quad\text{and}\quad\psi_\alpha(\sigma)=\sum_{\lambda\in\Pi}\sigma^\lambda\cdot 1_{\mathcal{F}_{\alpha,\lambda}}. \tag{5.6}$$

Based on these, we immediately conclude, as in the previous case, that

$$\mathcal{A}\otimes\mathcal{O}_{V_\alpha\times W}=\bigoplus_{\lambda\in\Pi_\alpha}\mathcal{O}_{V_\alpha\times W}1_{\mathcal{F}_{\alpha,\lambda}},\quad \Pi_\alpha=\{\lambda\mid\mathcal{F}_{\alpha,\lambda}\neq 0\},$$

is a finite and commutative sheaf of $\mathcal{O}_{V_\alpha\times W}$-algebras.

To proceed, we need to investigate how $\mathcal{F}_\lambda$ relates to $\mathcal{F}_{\alpha,\lambda}$. First, we write $V_\alpha\cap U$ as the union of two irreducible open subschemes $V_{\alpha,1}$ and $V_{\alpha,2}$, as defined immediately after (4.7). Because restricting to $V_{\alpha,1}$ the trivializations η_U and η_{V_α} coincide,

$$\psi(\sigma)|_{V_{\alpha,1}\times W}=\psi_\alpha(\sigma)|_{V_{\alpha,1}\times W},$$

and hence

$$\mathcal{F}_\lambda\otimes\mathcal{O}_{V_{\alpha,1}\times W}=\mathcal{F}_{\alpha,\lambda}\otimes\mathcal{O}_{V_{\alpha,1}\times W}.$$

On the other hand, restricting to $V_{\alpha,2}$ the two trivializations differ by σ_α, as shown in (4.8). Therefore

$$\psi(\sigma)|_{V_{\alpha,1}\times W}=\sigma^{c_\alpha}\cdot\psi_\alpha(\sigma)|_{V_{\alpha,1}\times W}$$

for $c_\alpha\in\Pi$ the map $c_\alpha(\beta)=\delta_{\alpha\beta}$, and hence

$$\mathcal{F}_\lambda\otimes\mathcal{O}_{V_{\alpha,1}\times W}=\mathcal{F}_{\alpha,\lambda+c_\alpha}\otimes\mathcal{O}_{V_{\alpha,1}\times W}.$$

We let Π_α be the set of those $\lambda\in\Pi$ so that $\mathcal{F}_{\alpha,\lambda}\neq 0$.

Lemma 5.2 *The sheaf of algebras $\mathcal{A}\otimes\mathcal{O}_{V_\alpha\times W}$ is a finite and commutative sheaf of $\mathcal{O}_{V_\alpha\times W}$-algebras. Further, for any $\lambda\in\Pi_\alpha$, the scheme $\operatorname{Spec}(\mathcal{O}_{V_\alpha\times W}1_{\mathcal{F}_{\alpha,\lambda}})_{\rm red}$ is either isomorphic to $V_{\alpha,\rm red}\times W$ or the product of W with one of the irreducible component of V_α with reduced scheme structure.*

Proof We shall skip the proof since it is parallel to that of the previous lemma, except commenting on the explicit isomorphisms stated in the later part of the lemma.

Since V_α has two irreducible components, one containing $V_{\alpha,1}$ and the other containing $V_{\alpha,2}$, when $\mathcal{F}_{\alpha,\lambda}\neq 0$ we have three possibilities: both $\mathcal{F}_{\alpha,\lambda}\otimes\mathcal{O}_{V_{\alpha,1}\times W}$ and $\mathcal{F}_{\alpha,\lambda}\otimes\mathcal{O}_{V_{\alpha,2}\times W}$ are non-zero; the first is non-zero; and the second is non-zero. In the first case, $\operatorname{Spec}(\mathcal{O}_{V_\alpha\times W}1_{\mathcal{F}_{\alpha,\lambda}})_{\rm red}$ is isomorphic to $V_{\alpha,\rm red}\times W$; in the second case it is isomorphic to $\{|z_1|<1\}\times W$ and in the last case isomorphic to $\{|z_2|<1\}\times W$.

With these two lemmas, we see that the scheme $Z_{\rm red}$ is a constant family of nodal curves over W. As to the scheme Z, we are less fortunate to hope that it is also a constant family of curves over Z. Indeed, it might not be flat over W. On the other hand, since our focus is on the sheaf of $\mathcal{O}_Z$-modules that push forward to $\mathcal{E}$, we can and shall enlarge Z so to make it a constant family over W. For this purpose, we shall pause and study $\mathcal{E}$ for now.

Because $\mathcal{E}$ is a sheaf of $\mathcal{E}nd(\mathcal{E})$-modules and $\mathcal{A}$ is a submodule of $\mathcal{E}nd(\mathcal{E})$, $\mathcal{E}$ becomes a sheaf of $\mathcal{A}$-modules, thus a sheaf of $\mathcal{O}_Z$-modules. To avoid possible confusion, we denote the so defined sheaf of $\mathcal{O}_Z$-modules by $\widetilde{\mathcal{E}}$. Then for the projection $\iota\colon Z \to C \times W$, $\iota_*\widetilde{\mathcal{E}} = \mathcal{E}$.

Lemma 5.3 *The sheaf $\widetilde{\mathcal{E}}$ is flat over W, and with respect to the pull back polarization on Z, is a W-family of stable sheaves on Z.*

Proof This follows immediately from that $Z \to C \times W$ is finite.

Now let $D \subset U$ be an irreducible open subscheme, let $Z_i \subset \iota^{-1}(D \times W)$ be a connected component, as an open subscheme of Z, and let $\iota_i : Z_i \to D \times W$ be the restriction of ι to $Z_i \subset Z$. Associated to them are the the multiplicity of D, denoted by m, and the multiplicity m_i of $\widetilde{\mathcal{E}}$ along Z_i, defined as the multiplicity of $\widetilde{\mathcal{E}}|_{\iota_i^{-1}(D\times w)}$ for any closed $w \in W$. Note that for any nilpotent element $f \in \mathcal{O}_{D\times W}$, the subsheaf $\iota_i^* f^{m_i} \cdot \widetilde{\mathcal{E}} \otimes \mathcal{O}_{Z_i} \subset \widetilde{\mathcal{E}} \otimes \mathcal{O}_{Z_i}$ is the zero subsheaf, a statement that follows immediately from that the multiplicity of $\widetilde{\mathcal{E}}$ along Z_i is m_i. Hence because $Z_i = \operatorname{Spec} \mathcal{O}_{D\times W} 1_{\mathcal{F}_\lambda}$ for some λ, $\iota_i^* f^{m_i} = 0$.

Another feature of the morphism $Z \to C \times W$ is that it is an analytic immersion.

Definition 5.4 *We say a morphism $f\colon Y \to Y'$ between two schemes an analytic immersion if for any closed point $y \in Y$, the induced homomorphism $T_yY \to T_{f(y)}Y'$ is injective.*

Our next step is to modify the nilpotent structure of Z so to make it a constant family of curves while contain Z as its closed subscheme.

Lemma 5.5 *We can extend $\iota\colon Z \to C\times W$ uniquely to an analytic immersion $\tilde\iota\colon \widetilde{Z} \to C\times W$ so that*

(1) *$Z \subset \widetilde{Z}$ and $\widetilde{Z}$ with the reduced scheme structure is identical to Z_{red};*

(2) *$\widetilde{Z} \to W$ is a flat family of planar curves; thus without embedded points;*

(3) *For any irreducible component $D \subset U$, any connected component $\widetilde{Z}_i \subset \tilde\iota^{-1}(D \times W)$ and $Z_i = \widetilde{Z}_i \cap Z$, the multiplicity of $\widetilde{Z}_i$ is exactly the multiplicity of $\widetilde{\mathcal{E}}$ along Z_i.*

Further more, there is a planar curve $\widetilde{C}$ and a projection $\widetilde{C} \to C$ so that $\widetilde{Z} = \widetilde{C} \times W$.

Proof We only need to show that the multiplicity of D is greater than or equal to the multiplicity of $\widetilde{\mathcal{E}}$ along Z_i; but this follows immediately from $\operatorname{Det}(\mathcal{E}) = D$ and $\iota_*\widetilde{\mathcal{E}} = \mathcal{E}$.

Because $Z \subset \widetilde{Z}$, we can and shall view $\widetilde{\mathcal{E}}$ as a sheaf of $\mathcal{O}_{\widetilde{Z}}$-modules. And because $\widetilde{\mathcal{E}}$ is flat over W, we can view it as flat family of stable sheaves on $\widetilde{C}$. In the following, we call the triple $(\widetilde{Z}, \iota, \widetilde{\mathcal{E}})$ the spectral triple of the pair $(C \times W, \mathcal{E})$, which leads us to the notion of admissible triples.

Definition 5.6 *We call a triple $(\widetilde{C}, \iota, \widetilde{\mathcal{E}}\,)$ of an analytic immersion $\iota\colon \widetilde{C} \to C$ and a sheaf of $\mathcal{O}_{\widetilde{C}}$-modules $\widetilde{\mathcal{E}}$ admissible if $\widetilde{\mathcal{E}}$ is stable, $\chi(\widetilde{\mathcal{E}}\,) = 1$, and the determinant divisor* $\operatorname{Det}(\iota_*\widetilde{\mathcal{E}}\,) = C$ *and* $\operatorname{Det}(\widetilde{\mathcal{E}}\,) = \widetilde{C}$, *which is the same as for any irreducible component $D \subset \widetilde{C}$ the multiplicity* $\operatorname{mult}_D\widetilde{C} = \operatorname{mult}_D\widetilde{\mathcal{E}}$.

Because the possible curves $\iota : \widetilde{C} \to C$ that can appear as part of an admissible triple is finite, the collection of all admissible triples is bounded. Therefore a standard argument shows that there is a moduli space of admissible triples with a universal family. For simplicity, we will take this moduli with the reduced scheme structure, and denote it by $\mathcal{M}$ with $\mathcal{U}$ over $\mathcal{C}$ its universal family.

Now we look at the set $\mathcal{M}_C^G$. For each of its connected component W (with reduced scheme structure), the spectral decomposition provides us a family of admissible triples over W, a family of sheaves over a single curves $\widetilde{C} \to C$. Hence this family induces an injective morphism $W \to \mathcal{M}$, and an injective

$$\Psi_1 : \mathcal{M}_C^G \longrightarrow \mathcal{M},$$

after running this over all connected components of $\mathcal{M}_C^G$. We let $\mathcal{M}_2$ be the image scheme $\Psi_1(\mathcal{M}_C^G)$. Then

$$e(\mathcal{M}_C) = e(\mathcal{M}_C^G) = e(\mathcal{M}_2).$$

We repeat this procedure again to $\mathcal{M}_2$. Suppose $W \subset \mathcal{M}_2$ is a connected component lying in a connected component W' of $\mathcal{M}$. Since all triples in W' consists of sheaves over a single, say, $\iota\colon \widetilde{C} \to C$, the group $G(\widetilde{C})$ acts on W' via tensoring sheaves by invertible sheaves in $G(\widetilde{C})$. It is easy to check that W is invariant under $G(\widetilde{C})$. We let W^{fix} be the fixed loci $W^{G(\widetilde{C})}$. Applying the spectral decomposition again to the $G(\widetilde{C})$-invariant family over W^{fix}, we derive a new family of admissible triples over the same W^{fix}, which in turns induces a canonical injective morphism

$$\Psi_W : W^{\text{fix}} \to \mathcal{M}.$$

We let $\mathcal{M}_2^{\text{fix}}$ be the union of W^{fix} over all connected components of $\mathcal{M}_2$, and let $\Psi_2 : \mathcal{M}_2^{\text{fix}} \longrightarrow \mathcal{M}$ be induced by the individual Ψ_W. Again, Ψ_2 is injective and its image $\mathcal{M}_3 = \Psi_2(\mathcal{M}_2^{\text{fix}})$ satisfies

$$e(\mathcal{M}_2) = e(\mathcal{M}_2^{\text{fix}}) = e(\mathcal{M}_3).$$

Repeating this procedure, we obtain a sequence of subschemes $\mathcal{M}_j \subset \mathcal{M}$, injectives $\Psi_j : \mathcal{M}_j^{\text{fix}} \to \mathcal{M}$ that obeys $\mathcal{M}_j = \operatorname{Im} \Psi_{j-1}$ and

$$e(\mathcal{M}_C) = e(\mathcal{M}_2) = \cdots = e(\mathcal{M}_k) = \cdots.$$

The sequence $\mathcal{M}_i$ eventually stabilize.

Proposition 5.7 *Let $\mathcal{M}_\infty$ be the subset of triples $(\widetilde{C}, \iota, \widetilde{\mathcal{E}})$ of which the base curve $\widetilde{C}$ is simply connected. Then for large i, $\mathcal{M}_i = \mathcal{M}_\infty$.*

Proof It is easy to see that $\mathcal{M}_k \subset \mathcal{M}_\infty$ for any integer k exceeding the total multiplicity of C. Indeed, let $(C, \mathcal{E})$ be any element in $\mathcal{M}_C^G$ and let $(C_1, \iota_1, \mathcal{E}_1) \in \mathcal{M}_1$ be its image under Ψ_1. Then, each irreducible component of C_1 with reduced scheme structure must be smooth. Afterwards, in case $(C_1, \mathcal{E}_1)$ is fixed by $G(C_1)$ and $(C_2, \iota_2, \mathcal{E}_2)$ is its image under Ψ_2, then the number of irreducible components of C_2 is bigger than that of C_1 when C_1 is not simply connected. Continuing this process, we see that any triple $(C_k, \iota_k, \mathcal{E}_k)$ in $\mathcal{M}_k$ for k exceeding the total multiplicity of D must have C_k simply connected. This proves the first half of the proposition.

We now show that $\mathcal{M}_\infty \subset \mathcal{M}_i$ for sufficiently large i. Let $(\widetilde{C}, \iota, \widetilde{\mathcal{E}})$ be an element in $\mathcal{M}_\infty$. To show that it belongs to $\mathcal{M}_i$ for some i we need to construct a sequence of nodal planar curves

$$\widetilde{C} = C_k \xrightarrow{\iota_k} C_{k-1} \xrightarrow{\iota_{k-1}} \cdots \xrightarrow{\iota_1} C_0 = C$$

and sheaves $\mathcal{E}_i$ over C_i so that $\mathcal{E}_0 = \iota_*\widetilde{\mathcal{E}}$ and $\mathcal{E}_k = \widetilde{\mathcal{E}}$, and that each $(C_i, \iota_i, \mathcal{E}_i)$ is the spectral triple of $(C_{i-1}, \mathcal{E}_{i-1})$.

The logical first step is to construct representation Φ_σ as in (4.9) for all $\sigma \in G(C)$. For this, we first pick a point $w_0 \in C - \Lambda$, and for each $\sigma \in G(C)$ we pick the isomorphism

$$\varphi_\sigma : \widetilde{\mathcal{E}} \longrightarrow \widetilde{\mathcal{E}} \otimes \iota^*\mathcal{L}(\sigma) \tag{5.7}$$

so that its restriction to w_0, that is

$$\varphi_\sigma(w_0) : \widetilde{\mathcal{E}} \otimes k(w_0) \longrightarrow \widetilde{\mathcal{E}} \otimes \iota^*\mathcal{L}(\sigma) \otimes k(w_0),$$

is the identity map after replacing $\iota^*\mathcal{L}(\sigma) \otimes k(w_0)$ by $k(w_0)$ using the trivialization η_U of $\mathcal{L}(\sigma)|_U$. Note that such φ_σ exists and is unique because $\iota^*\mathcal{L}(\sigma) \cong \mathcal{O}_{\widetilde{C}}$ and because $\widetilde{\mathcal{E}}$ is stable. Taking the direct image under ι, we obtain isomorphisms

$$\Phi_\sigma = \iota_*\varphi_\sigma : \iota_*\widetilde{\mathcal{E}} \longrightarrow \iota_*(\widetilde{\mathcal{E}} \otimes \iota^*\mathcal{L}(\sigma)) = \iota_*\widetilde{\mathcal{E}} \otimes \mathcal{L}(\sigma) \tag{5.8}$$

that satisfies the requirement of Lemma 4.2. Such Φ allows us to construct the spectral triple $(C_1, \iota_1, \mathcal{E}_1)$ as we did.

To reiterate this procedure, we need to find a morphism $\rho_1 : \widetilde{C} \to C_1$ so that in addition to that the composite $\iota_1 \circ \rho_1$ from $\widetilde{C}$ to C_1 and then to C is identical to ι we have $\mathcal{E}_1 = \rho_{1*}\widetilde{\mathcal{E}}$. For once the ρ_1 is constructed, we can produce yet another spectral triple $(C_2, \iota_2, \mathcal{E}_2)$ out of $\rho_{1*}\widetilde{\mathcal{E}} = \mathcal{E}_1$; and so on.

To construct ρ_1, we shall use the dual graph of $\widetilde{C}$ with additional orientation on its edges and marking on its vertices. As is known, the dual graph of $\widetilde{C}$ is constructed by assigning to each irreducible component D of $\widetilde{C}$ a vertex $[D]$, and assigning to each node ξ in $\widetilde{C}$ an edge $[\xi]$ connecting the two vertices whose associated irreducible components intersect at ξ. Because $\widetilde{C}$ is connected and simply connected, the resulting graph is a tree. To assign an orientation to an edge $[\xi]$, we need to specify the initial vertex $[D_{\xi,1}]$ and the terminal vertex $[D_{\xi,2}]$ of $[\xi]$; for this, we require that the germ of $D_{\xi,1}$ at ξ is mapped onto the branch $V_{\iota(\xi),1}$ of the germ of C at the node $\iota(\xi)$ according to the convention adopted before. We further assign each vertex $[D]$ a character $\lambda_D \in \Pi = \mathrm{Map}(\Lambda, \mathbb{Z})$ according to the following rule: to the irreducible component D_0 containing the w_0 specified before, we assign to it the zero map; to the initial vertex $[D_{\xi,1}]$ and the terminal vertex $[D_{\xi,2}]$ bordering the edge $[\xi]$, we require

$$\lambda_{D_{\xi,2}} = \lambda_{D_{\xi,1}} + c_{\iota(\xi)}.$$

Because the graph is a connected tree, this rule will complete a map from the vertices of this graph to the set Π.

With this assignment, it is easy to describe the spectral decomposition of the module $\Gamma(U, \iota_*\widetilde{\mathcal{E}})$. Specifically, for any irreducible component $D \subset \widetilde{C}$ and $D^0 = D \cap \iota^{-1}(U)$, the submodule $\Gamma(\iota_*(\widetilde{\mathcal{E}}|_{D^0})) \subset \Gamma(U, \mathcal{E})$ lies in the common eigenspace of eigenvalue λ_D. Alternatively, for any $\lambda \in \Pi$,

$$V_\lambda = \bigoplus_{D:\lambda_D = \lambda} \Gamma(\iota_*(\widetilde{\mathcal{E}}|_{D^0})). \tag{5.9}$$

Then the explicit construction of C_1 spelled out in the previous section shows that as set C_1 is the quotient of $\widetilde{C}$ by the equivalence relation $\sim$ as follows. Two points z_1 and z_2 are equivalent only if $\iota(z_1)=\iota(z_2)$; when they do and the common image $\iota(z_1)$ is not a node of C then $z_1\sim z_2$ if and only if the two irreducible components containing z_1 and z_2 are assigned identical characters in Π, and when the common image $\iota(z_1)$ is a node of C then $z_1\sim z_2$ if the markings of the two components containing z_1 are identical to the markings of the two irreducible components containing z_2. It follows from the explicit construction of C_1 that $C/\sim = C_1$ as sets; hence $\iota\colon \widetilde{C}\to C$ factor through $\rho_1\colon \widetilde{C}\to C_1$, and that the sheaf $\mathcal{E}_1=\rho_{1*}\widetilde{\mathcal{E}}$. Since the checking is direct, we shall omit the details here.

Once we have $\rho_1:\widetilde{C}\to C_1$ with $\mathcal{E}_1=\rho_{1*}\widetilde{\mathcal{E}}$, we can repeat this procedure to produce a sequence of curves C_i and sheaves $\mathcal{E}_i$ on C_i with specified properties. Then because the total multiplicity of C_i is the same as the total multiplicity of C, this sequence must stabilize. Hence $C_k=\widetilde{C}$ and $\mathcal{E}_k=\widetilde{\mathcal{E}}$ for some k.

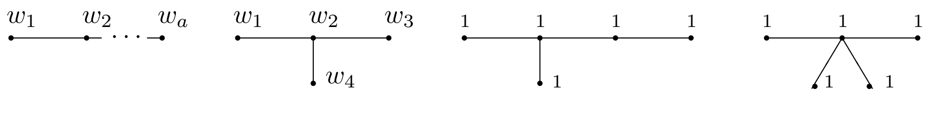
To complete the proof of the Proposition we still have to show that $\mathcal{E}=\iota_*\widetilde{\mathcal{E}}$ is stable. Suppose not, then because the maximal destabilizing subsheaf $\mathcal{F}\subset\mathcal{E}$ is unique, $\mathcal{F}$ must be the direct image of a subsheaf $\mathcal{F}_1\subset\mathcal{E}_1$, and be the direct image sheaf of a subsheaf $\mathcal{F}_2\subset\mathcal{E}_2$, and so on. In the end we obtain a subsheaf $\widetilde{\mathcal{F}}\subset\widetilde{\mathcal{E}}$ so that $\iota_*\widetilde{\mathcal{F}}=\mathcal{F}$, hence it will destabilize $\widetilde{\mathcal{E}}$, a contradiction. This completes the proof of the proposition.

6 Stable Sheaves over Simply Connected Curves

We now classify the set $\mathcal{M}_\infty$ for a planar nodal rational curve C inside a K3 surface S. In this case, if $\iota\colon\widetilde{C}\to C$ is an analytic immersion, $\widetilde{C}$ is simply connected, and $D\subset\widetilde{C}$ is an irreducible component with non-reduced scheme structure, then the normal bundle to D in $\widetilde{C}$, which is defined as the dual of $I_{D_{\mathrm{red}}\subset\widetilde{C}}\otimes_{\mathcal{O}_{\widetilde{C}}}\mathcal{O}_{D_{\mathrm{red}}}$, has degree -2.

As we mentioned, the topology of such curves $\widetilde{C}$ are encoded in their dual graphes. To account for the possibility of non-reduced scheme structure of $\widetilde{C}$, we shall assign to each vertex $[D]$ the integer that is the multiplicity of $\widetilde{C}$ along D. For such a graph, we define its total weight to be the sum of the weights of all its vertices.

For instance, the set of trees of total weights five consists of trees of various lengths and three additional graphes, as shown:



Lemma 6.1 *Let $\widetilde{C}$ be a simply connected planar nodal rational curve embedable to an analytic* K3 *surface and let $\mathcal{E}$ be a pure dimension one sheaf of $\mathcal{O}_{\widetilde{C}}$-modules with $\chi(\mathcal{E})=1$. Suppose $\chi(\mathcal{O}_D)\ge 1$ for all one dimensional subscheme $D\subset\widetilde{C}$. Then in case $\chi(\mathcal{O}_{\widetilde{C}})=1$, $\mathcal{E}$ is stable if and only if $\mathcal{E}\cong\mathcal{O}_{\widetilde{C}}$; and in case $\chi(\mathcal{O}_{\widetilde{C}})>1$, $\mathcal{E}$ is not stable.*

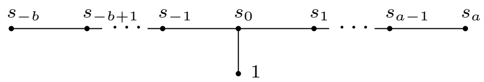
Proof Because $\chi(\mathcal{E})=1$, $\mathcal{E}$ has non-trivial global sections, say, $s:\mathcal{O}_{\widetilde{C}}\to\mathcal{E}$. We let $\mathcal{F}$ be the image of s. Then the kernel of $s\colon\mathcal{O}_{\widetilde{C}}\to\mathcal{F}$ defines a closed subscheme $D\subset C$. Then $\mathcal{O}_D\cong\mathcal{F}$, and by assumption, $\chi(\mathcal{F})\ge 1$. Therefore in case $\mathcal{E}$ is stable, we must have $\mathcal{F}=\mathcal{E}$, and hence $\mathcal{E} \cong \mathcal{O}_D$ for some closed subscheme $D \subset \widetilde{C}$. But then $\mathrm{Det}(\mathcal{E}) = \widetilde{C}$, this is possible only if $D = \widetilde{C}$ and therefore $\mathcal{E} \cong \mathcal{O}_{\widetilde{C}}$ and $\chi(\mathcal{O}_{\widetilde{C}}) = 1$. In case $\chi(\mathcal{O}_{\widetilde{C}}) = 1$, it is easy to check that $\mathcal{O}_{\widetilde{C}}$ is stable. This proves the lemma.

Using this lemma, we can classify certain elements in $\mathcal{M}_\infty$.

Lemma 6.2 *In case $\widetilde{C}$ is a curve consisting of a chain of rational curves: Then $\chi(\mathcal{O}_{\widetilde{C}}) \geq 1$ for every closed subscheme $D \subset \widetilde{C}$; and $\chi(\mathcal{O}_{\widetilde{C}}) = 1$ if and only if $\widetilde{C}$ is reduced. In case $\widetilde{C}$ is a curve consisting of a chain of rational curves $D_{-b}, \cdots, D_a$ with multiplicities s_i plus an additional D_* intersecting D_0: Then $\chi(\mathcal{O}_{\widetilde{C}}) \geq 1$ for every closed subscheme $D \subset \widetilde{C}$; and $\chi(\mathcal{O}_{\widetilde{C}}) = 1$ if and only if $|s_{i+1} - s_i| \leq 1$ and the two sequences $\{s_0, s_1, \cdots\}$ and $\{s_0, s_{-1}, \cdots\}$ are decreasing.*

Proof We shall prove the second part; the first is easier.

We first note that the dual graphes of the curves in the second part of the lemma are



with s_i the multiplicity of D_i in $\widetilde{C}$. It is easy to see that

$$\chi(\mathcal{O}_{\widetilde{C}}) = 1 + \sum_{i=-b}^{a} {s_i}^2 - \sum_{i=-b}^{a-1} s_i s_{i+1} - s_0 = 1 + \sum_{i=-b}^{a-1} \frac{1}{2}(s_i - s_{i+1})^2 + \frac{1}{2}s_{-b}^2 + \frac{1}{2}s_a^2 - s_0.$$

Since $x^2 \geq x$ for integers x, we obtain

$$\sum_{i=-b}^{-1} \frac{1}{2}(s_i - s_{i+1})^2 + \frac{1}{2}s_{-b}^2 \geq \sum_{i=-b}^{-1} \frac{1}{2}|s_i - s_{i+1}| + \frac{1}{2}|s_{-b}| \geq \frac{1}{2}s_0.$$

In the same way,

$$\sum_{i=0}^{a-1} \frac{1}{2}(s_i - s_{i+1})^2 + \frac{1}{2}s_a^2 \geq \frac{1}{2}s_0.$$

Therefore $\chi(\mathcal{O}_{\widetilde{C}}) \geq 1$ and equality holds if and only if $\{s_n\}$ is one-admissible in the sense that $|s_{i+1} - s_i| \leq 1$ and the two sequences $\{s_0, s_1, \cdots\}$ and $\{s_0, s_{-1}, \cdots\}$ are decreasing (see [4]). Since every subscheme $D \subset \widetilde{C}$ has dual graph either a chain or similar to $\widetilde{C}$, we get $\chi(\mathcal{O}_D) \geq 1$. This proves the lemma.

It is direct to check that all weighted trees with total weight not greater than 5 satisfy the condition $\chi(\mathcal{O}_D) \geq 1$ for all closed subschemes $D \subset C$. Therefore

Corollary 6.3 *Let $(\widetilde{C}, \iota, \widetilde{\mathcal{E}})$ be a triple in $\mathcal{M}_\infty$ such that the total weight of $\widetilde{C}$ is no more than 5. Then $\chi(\mathcal{O}_{\widetilde{C}}) = 1$ and $\widetilde{\mathcal{E}} \cong \mathcal{O}_{\widetilde{C}}$.*

Let X be a K3 fibred local Calabi-Yau three-fold and let β_0 be a fiberwise rigid curve class in X_0. We now state and prove the main result of this note:

Theorem 6.4 *Suppose $\mathrm{Pic}(X_0) = \mathbb{Z}$ and is generated by β_0. Suppose further that for an integer $d \leq 5$ all (reduced) rational curves in the linear series $|k\beta_0|$ for $k \leq d$ are nodal. Then* *the conjecture* 1.2 *holds up to* $q^{d\beta_0}$:

$$\mathcal{F}_{X,0}(q) \equiv \sum_{\beta\in H_2(X,\mathbb{Z})} \Big(\sum_{k|\beta} \frac{1}{k^3}\, n_0(\beta/k)\Big) q^{\beta} \mod q^{(d+1)\beta_0}.$$

In the sequel of this paper, we shall remove the technical condition on the nodal curves. For now, we are content to prove this partial result.

Proof As we have discussed, since rational curves in X_0 are rigid, we can localize the invariants along rational curves and reduce the degree zero Gopakumar-Vafa conjecture to its local version.

For any rational C in $|\beta|$, those $\Sigma \to X$ that factor through $C \subset X$ form a closed and open substack $\mathcal{M}_0(X,\beta)_C$ of $\mathcal{M}_0(X,\beta)$ that decomposes further into disjoint union of open and closed substacks according to their liftings to a simply connected analytic immersion $\iota\colon \widetilde{C} \to C$. In other words, for any simply connected analytic immersion $\iota\colon \widetilde{C} \to C$ such that $\iota_*[\widetilde{C}] = [C]$, we let $\mathcal{M}_0(X,\beta)_{\widetilde{C}}$ be the substack of those $f\colon \Sigma \to X$ that lift to $\tilde{f}\colon \Sigma \to \widetilde{C}$ so that $\tilde{f}_*[\Sigma] = [\widetilde{C}]$. Obviously,

$$\mathcal{M}_0(X,\beta) = \coprod_{\iota\colon \widetilde{C}\to C} \mathcal{M}_0(X,\beta)_{\widetilde{C}}$$

is a decomposition into disjoint open and closed substacks. Hence we can assign each $\mathcal{M}_0(X,\beta)_{\widetilde{C}}$ its own obstruction theory and its own virtual cycle.

Similarly, for each analytic immersion $\iota\colon \widetilde{C} \to C$ with $\widetilde{C}$ simply connected and $\iota_*[\widetilde{C}] = [C]$, we let $\mathcal{M}_{\widetilde{C}}$ be the space of all stable sheaves $\mathcal{E}$ of $\mathcal{O}_{\widetilde{C}}$-modules of $\chi(\mathcal{E}) = 1$ and $\mathrm{mult}_D\mathcal{E} = \mathrm{mult}_D\widetilde{C}$ for all irreducible component $D \subset \widetilde{C}$.

Now we confine ourselves to $C \in |d\beta_0|$ with $d \le 5$. We first look at the case where $\widetilde{C}$ is a chain of rational curves. Then by the result of [9], $\deg[\mathcal{M}_{\widetilde{C}}]^{\mathrm{vir}} = d^{-3}$ when the multiplicities of all irreducible components of $\widetilde{C}$ are equal to d; and is zero otherwise.

In case $\widetilde{C}$ is reduced, then $\deg[\mathcal{M}_{\widetilde{C}}]^{\mathrm{vir}} = 1$ because the moduli space is a single smooth point. In this case $e(\mathcal{M}_{\widetilde{C}}) = 1$ as well.

The only remaining case is when $\widetilde{C}$ is a chain of rational curves $D_{-b}, \cdots, D_a$ plus an additional D_* intersecting D_0. Then according to [4], $\deg[\mathcal{M}_{\widetilde{C}}]^{\mathrm{vir}} = 1$ if the multiplicities s_i of D_i is one-admissible; and is zero otherwise.

Combined, this proves the identity

$$\mathcal{F}_{X,0}(q) \equiv \sum_{\beta\in H_2(X,\mathbb{Z})} \Big(\sum_{k|\beta} \frac{1}{k^3}\, n_0(\beta/k)\Big) q^{\beta} \mod q^{(d+1)\beta_0}.$$

An application of the technique developed in this section allows us to enumerate the expected irreducible nodal curves in K3 surfaces. For the case of two multiples of a primitive class, please see [21].

The technique developed here allows us to conjecture on the Gromov-Witten invariants of rigid rational curves in a local Calabi-Yau threefold. For instance, the local Gopakumar-Vafa conjecture holds in case C is a rational curve with single node in a Calabi-Yau with normal bundle $(-1,-1)$. For more general nodal rational curves in a local Calabi-Yau manifold, it turns out that the moduli $\mathcal{M}_\infty$ can have positive dimensional moduli space, thus resisting to offer a simple combinatoric expression in its Euler numbers.

We also remark that in his recent preprint [10], Katz checked the validity of this conjecture for the case of contractible $\mathbf{P}^1$ in a Calabi-Yau threefold.

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