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REGULAR ARTICLE

Aggregation of heterogeneous interacting agents: the variant representative agent framework

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Abstract In this paper we have presented a variant of the stochastic aggregation approach which basically consists in exploring the evolution over time of higher moments of the economic units' distribution. In a sense therefore, we propose to focus on the behavior of a Variant Representative Agent. An application to a classical growth model shows that changes in aggregate output usually associated with *Total Factor Productivity* in the aggregative interpretation of the framework may be due to changes in the distribution of agents in terms of capital intensity. The application to a model by Gatti et al. (Interaction and market structure, Springer, 2000) shows the efficacy of the method in capturing the evolution over time of the distribution of firms in terms of financial solidity (equity ratio). The method seems general enough to cover a wide range of economic situations in which heterogeneity is relevant and persistent. It seems also simple enough to deserve the attention of the macroeconomist dissatisfied with the RA who wants to derive meaningful and microfounded macroeconomic results.

1 Introduction

In the last few years, considerable attention has been devoted to the interaction of heterogeneous agents and the role of the distribution of their characteristics

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in shaping macro-economic outcomes. The literature developed on this issue, however, has not had much impact on standard macro theory, which is still cast in terms of a representative agent, claiming, for example, that the main source of business fluctuations is a technological shock to a representative firm or that aggregate consumption depends on aggregate income and wealth, neglecting distributional effect by construction.

Even when heterogeneity is explicitly taken into account, interaction is generally ignored. Of course, also the distributional effects of economic policy cannot be properly evaluated because of the particular framework mainstream macroeconomics adopts, i.e. the representative agent (RA).

The RA framework has a long tradition in economics (Hartley 1997). Despite the stringency of the logical requirements for consistent aggregation (more on this in Sect. 2), the RA has been one of the most successful tools in economics. It is the cornerstone of microfoundations in macroeconomics because it allows to extend individual behavior to the aggregate in the most straightforward way: The analysis of the aggregate in fact can be reduced to the analysis of a single, representative, individual, ignoring by construction any form of heterogeneity and interaction.

Heterogeneity, however, is a persistent feature in many fields. Empirical investigations, for instance, have repeatedly shown that the distribution of firms' size is described by a power law. By itself this fact falsifies the RA hypothesis and the related myth of the *optimal size* of the firm. The RA is also far from being a neutral assumption in econometrics. For instance, the results of the econometric analysis of the relation between aggregate consumption and aggregate income depend on the assumption of linearity and absence of heterogeneity, as Forni and Lippi (1997) showed.

With the passing of time, therefore, economists have become more and more dissatisfied with the RA device (Kirman (1992); Malinvaud (1993); Grandmont (1993); Chavas (1993); and the proceedings of the WEHIA) and have tried to put forward a theory of aggregation in the presence of persistent heterogeneity. The set of assumptions necessary to reach exact aggregation in this case, however, is impressive: The general equilibrium theorist may not feel at ease with the RA assumption because some of the building blocks of general equilibrium theory do not hold in the presence of a representative agent (e.g. the "Weak Axiom of Revealed Preferences" or "Arrow's Impossibility Theorem", Kirman (1992) p. 122). From a different theoretical perspective, the very idea of asymmetric information of the New Keynesian Economics is inconsistent with the RA hypothesis (Stiglitz 1992).

Moreover, the adoption of a RA framework ignores the problem of coordination [which is of a crucial importance when informational imperfections are taken into account: see Leijonhufvud (1981)].

Since the empirical evidence does not corroborate the RA assumption (Stoker 1993) and theoretical investigations show also its analytical inconsistencies (Kirman 1992), one could ask why it is still the standard framework in economics. Several answers may be given to this question but the most fundamental reason in our opinion is that if individuals are homogeneous there is not room for *interaction*. In a RA framework *aggregation* is made simple and the connection between micro and macro-behaviour is immediate. By construction, the RA behaves like Robinson Crusoe: problems arise when he meets Friday.

In the literature we can find several attempts to produce adequate tools for aggregation in the presence of heterogeneity (see Sect. 2 for a review). From the realization of the impossibility of exact aggregation, we moved to investigate an alternative aggregation procedure which allows to deal with a dynamic heterogeneous interacting agents framework. We illustrate the methodology in Sect. 3, where its main advantages and shortcomings are also emphasized. Before drawing some conclusion, Sect. 4 illustrates the application of our framework by means of simulations.

2 Aggregation procedures

Macroeconomists assess the implications a given theory in terms of relationships between aggregate quantities. This is exactly the key point of the *aggregation problem*: having the *micro-equation*s describing/representing the (optimal) choices of the economic units, what can we say about the *macro-equations*. Do they have the same functional form of the micro-equations (the *analogy* principle in Theil's wording, see Theil 1954)? If not, how to derive the macro-theory?

The aggregation problem in macroeconomics has a long history. Every survey on the subject starts with the work of Gorman (1953) in which the conditions are derived that preferences must satisfy in order to have an aggregate demand with the same functional form of the individual demand. As is known, these conditions include *quasi-homothetic preferences* that imply linear demand functions known in the literature as *Gorman's polar forms*, i.e.

$$x_{it}^{j}\left(p_t, w_t\right) = \alpha_i^{j}\left(p_t\right) + \beta^{j}\left(p_t\right)^{\mathrm{T}} w_{it}^{j}, \tag{1}$$

where x^j is the demand of good j and w^j is the vector collecting all the relevant variables regarding the choice of agent i. Note that α may differ across agents whereas β (the individual response to changes in w) *has to be the same for all agents*.

When the above conditions are satisfied, it is possible to use an agent that represent exactly the *community preference fields* (the *representative agent hypothesis*).

One year after Gorman's work, Theil (1954) analyzed the aggregation problem from an econometric perspective assuming linear and static specifications. He investigated the conditions under which it is possible to apply the *analogy principle* and estimate correctly the micro-parameters showing that, from an econometric point of view, unless for very particular conditions, omitting the micro-variables in the macro-relations implies a specification error of the *omitted variables* type. In other terms, assuming the *analogy principle* as in the RAH, estimates are generally biased both in finite samples and asymptotically. Furthermore, within his framework, it is possible to show that dynamic aggregate relationships are present even if the micro-relationships are all static. This is due to the fact that the aggregate error term inherits the probabilistic properties of the micro-variables that are often auto-correlated.

To summarize, Theil's message is the same as Gorman's: *In order to have an exact aggregation (i.e. no aggregation bias)micro-relationships must be linear with the same slope parameters.*

The exact aggregation theory was subsequently generalized by Jorgenson et al. (1982) to the situation in which the economist has additional information (statistics) about the distribution of w.

A statistics may be defined as the expected value of a transformation of w, say $h_k\left(w_{it}^j\right)$. Having $k = 1, \ldots, m$ such statistics, $E_t\left[h_k\left(w_{it}^j\right)\right]$, they show that the necessary and sufficient condition to have an exact aggregation is that the demand functions have the following form:

$$x_{it}^j\left(p_t, w_t\right) = \alpha_i^j\left(p_t\right) + \beta_1^j\left(p_t\right)^{\mathrm{T}} h_1\left(w_{it}^j\right) + \cdots + \beta_m^j\left(p_t\right)^{\mathrm{T}} h_m\left(w_{it}^j\right),$$

where the variables involved have the same meaning as in Eq. 1.

This implies that it is possible to relax Gorman's stringent conditions if and only if appropriate additional information on the agents' cross-sectional distribution is available.

The aggregation problem becomes even more complex if one takes into account the fact that agents' distribution changes over time in an unknown way. Lewbel (1989), studied conditions under which changes of the distribution do not matter in the class of log–linear relationships that are often used in theoretical and applied analysis. Consider the following micro-relation:

$$\ln y_{ijt} = \sum_{j=1}^{J} b_j \ln x_{ijt} + r_{ijt}, \tag{2}$$

with x_{ijt} positive, r_{ijt} is the individual (heterogeneous) component and b_j are the elasticity parameters with respect to x, uniform across agents. Non-linearity yields in general an aggregation problem when we use the analogy principle to build the aggregate relationship

$$\ln Y_t = \sum_{j=1}^{J} b_j \ln X_{jt} + R_t,$$

where $X_{jt} = E\left[x_{ijt}\right]$ and $Y_{jt} = E\left[y_{ijt}\right]$ with respect to the agents' distribution.

Let $F\left(x|X_t, \theta_t\right) = G\left(x|X_{1t}z_{1t}, \ldots, X_{jt}z_{jt}, \ldots, X_{Jt}z_{Jt}, \theta_t\right)$ be the distribution of the x's with $z_{ijt} = x_{ijt}/X_{ijt}$ and call $G\left(z|X_t, \theta_t\right)$ the distribution of the z's. Lewbel (1992) shows that it is possible to use the analogy principle if the distribution of z_t is independent from the mean X_t (*mean-scaled*), $G\left(z|X_t, \theta_t\right) = G^*\left(z|\theta_t\right)$.

In other terms, if a test on the mean-scaled property is not rejected, we can use the analogy principle for log–linear models.

Granger (1980) studied the dynamics of aggregate economic variables when individual relationships are linear, heterogeneous and dynamic. He found that aggregating short-memory (i.e., the auto-correlation functions decay exponentially) individual stochastic processes, like linear autoregressive processes with heterogeneous parameters, the resulting aggregate relationship may belong to the *fractional integrated processes* family with the auto-correlation function falling in an hyperbolic way. In other terms, the aggregate process shows a long memory property even though individual processes do not.

In the same line, recent developments of the exact aggregation approach show a huge number of negative results concerning the aggregation problem. For instance, properties such as the *random walk property* implied by the *life cycle/permanent income theory of consumption*, the *Granger causality* (i.e., time anticipation of some variables compared to others) or the *cointegration* (i.e., long run equilibrium) between variables are generally destroyed by the aggregation (see Forni and Lippi 1997). But, they may be also a mere property of the aggregation of heterogeneous agents not holding at the individual level as in the Granger (1980) analysis. These results make it really difficult the elaboration of statistical tests able to reject an economic theory.

2.1 The stochastic aggregation approach

In the previous section we have shown that exact aggregation is too restrictive. The conditions that must hold in order to use the analogy principle or the RAH, are rarely fulfilled in real situations. As said above, when an economist rejects a theory built on the RAH it is difficult to say whether he/she is really rejecting the theory, the RAH or both.

The approach suggested by Kalejan (1980) and Stoker (1984) puts forward less restrictive conditions in order to build aggregate relationships. They do not require the aggregate model to have the same functional form of the micro-relationship. Instead they look for the relationship between the first moments of the macro-variables given the micro-theory.

To present their ideas, suppose the in the reduced form the micro-relationship is

$$y_{it} = f_i\left(x_{it}, u_{it}, \theta_i\right) \quad i = 1, 2, \ldots, N \;\; t = 1, 2, \ldots, T, \tag{3}$$

where y_{it} is the strategic choice of the economic unit i that depends on the vector x_{it}, whereas u_{it} is a vector of unobservable characteristics in individual choice and θ_i a vector of parameters.

Call $P\left(x_{it}, u_{it}, \theta_i; \phi_t\right)$ the joint distribution of all variables characterized by a time-variant vector ϕ_t. Then, using a continuous notation for the sake of simplicity:

$$E\left[y_t\right] = \mu_y(t) = \Psi_y\left(\phi_t\right) = \int f\left(x_t, u_t, \theta_i\right) P\left(x_t, u_t, \theta_i; \phi_t\right) \mathrm{d}x_t \mathrm{d}u_t \mathrm{d}\theta \tag{4}$$

and the following equation gives the expected (aggregate) variables:

$$E\left[x_t\right] = \mu_x(t) = \Psi_x\left(\phi_t\right) = \int x_t P\left(x_t, u_t, \theta_i; \phi_t\right) \mathrm{d}x_t \mathrm{d}u_t \mathrm{d}\theta, \tag{5}$$

where the two integrals are computed on the support of P.

The idea behind this line of research is to divide ϕ in two sub-vectors, $\phi_t = (\phi_{1t}, \phi_{2t})$, such that:

(1) ϕ_{2t} has the same dimensions of x_{it} and
(2) an invertible function exists relating ϕ_{2t} and $\mu_x(t)$, say

$$\phi_{2t} = \Psi_x^{-1}\left(\phi_{1t}, \mu_x(t)\right).$$

From conditions (1) and (2) follows that we can write

$$\mu_y(t) = \Psi_y\left[\phi_{1t}, \Psi_x^{-1}\left(\phi_{1t}, \mu_x(t)\right)\right] = F\left(\mu_x(t), \phi_{1t}\right)$$

that is the (aggregate) relation between the first moments of the economic variables.

Two remarks are important at this point. First, the aggregate relationship F between $\mu_y(t)$ and $\mu_x(t)$ depends on a set of parameters ϕ_{1t} of the joint distribution. Second, in general, the aggregate relationship F differs from the micro-one f. In other words, if one allows for heterogeneity, it is impossible to use the analogy principle even if we want an aggregate model that holds only on average.

This novel and interesting approach has rarely been used in the literature probably because of its complexity (above all in dynamical settings) and the necessity to gather a lot of information about distributions of economic units (see Pesaran 2000).

Masanao Aoki has developed a stochastic method for modeling aggregate behaviour and fluctuations (Aoki 1998, 2001). His approach is fully dynamic and uses the formalism of Fokker–Planck partial differential equations as well as Langevin stochastic differential equations.

Therefore, in principle, Aoki's analysis is able to overcome the problem of dynamic stochastic aggregation, present in the method developed by Stocker. The mathematical subtleties of the formalism may have so far prevented its widespread use among economists.

3 The variant representative agent (VRA) approach

The stochastic aggregation framework is a very important step in the analysis of aggregation but, in order to be applied, it needs a huge amount of prior information about distribution of economic units. Therefore, it is important to investigate the effect of the distribution of agents according to some relevant characteristics onto the aggregate macro variables. In this part we suggest to approximate Eqs. (4) and (5) along the line of Keller (1980) and extend such methodology to a dynamical context.

In other terms, when possible we can expand the micro-relation (3) in Taylor series around the mean up to a certain order k (usually 2) and take the expectation operator with respect to agents' distribution. To simplify the analysis, suppose (as it is the case in many economic models) that the random components u_{it} is additive, the function f_i and the parameters $\theta_i$are the same across agents ($f_i = f; \theta_i = \theta$) and to simplify notation let us neglect the dependence of f on θ

$$y_{it} = f(x_{it}) + u_{it}.$$

The deterministic part of the above equation can be expanded in Taylor series up to order k provided that $f(x)$ has the first $k+1$ derivatives. In the following, to avoid complications we analyze the case $k = 2$, but the reader can easily generalize the idea to values higher than 2. Therefore

$$f(x_{it}) = f(\mu_{xt} + \varepsilon_{it}) = f(\mu_{xt}) + f'(\mu_{xt})\,\varepsilon_{it} + \tfrac{1}{2}\varepsilon_{it}^{\mathrm{T}} f''(\mu_{xt})\,\varepsilon_{it} + o(||\varepsilon_{it}||),$$

where μ_{xt} is the expected vector of x_{it}, ε_{it} the vector spread from the mean and f' and f'' are, respectively, the gradient and the hessian matrix of $f(x)$.

Taking the expected value we get

$$E\left[y_{it}\right] = f(\mu_{xt}) + \tfrac{1}{2}E\left[tr\left(\varepsilon_{it}\varepsilon_{it}^{\mathrm{T}} f''(\mu_{xt})\right)\right] + E\left[o(||\varepsilon_{it}||)\right],$$

where E is the expectation operator and tr is the trace operator. Using the linearity property of E, we obtain:

$$\mu_{yt} = E\left[y_{it}\right] = f(\mu_{xt}) + \tfrac{1}{2}tr\left(\Sigma_t f''(\mu_{xt})\right) + E\left[o(||\varepsilon_{it}||)\right], \tag{6}$$

where Σ_t is the variance-covariance matrix of x. The above equation is the exact aggregate relationship between y and x. When there is no dispersion among agents (Σ_t converges to the null matrix) it boils down to the representative agent equation in which the *analogy principle* holds true between first moments of agents' distribution

$$\mu_{yt} = f(\mu_{xt}).$$

In other terms, we can interpret the two terms

$$\tfrac{1}{2}tr\left(\Sigma_t f''(\mu_{xt})\right) + E\left[o(||\varepsilon_{it}||)\right]$$

as the error we made using the RAH when there is important heterogeneity among agents/firms[1].

When we have prior information about agents' distribution we can invert the relationships between first moments and the parameters of the distribution as described in the previous section. In the case of scalar micro-equations (i.e, x is a scalar variable Equation (6) becomes[2]

$$\mu_{yt} = f(\mu_{xt}) + \tfrac{1}{2} f''(\mu_{xt})\, \sigma_{xt}^2 + E[o(||\varepsilon_{it}||)]. \tag{7}$$

If, for example, we know that the empirical distribution of x can be approximated by an exponential probability density function with parameter b,

$$\frac{1}{b}\exp\left(-\frac{x}{b}\right)$$

then the mean, μ, is equal to b and the variance, σ^2, to b^2. This implies that $\sigma^2 = \mu^2$ and the exact aggregate equation becomes

$$\mu_{yt} = f(\mu_{xt}) + \tfrac{1}{2} f''(\mu_{xt})\, \mu_{xt}^2 + E[o(||\varepsilon_{it}||)] = h_1(\mu_{xt}) + E[o(||\varepsilon_{it}||)].$$

Expanding the micro-relationship setting $k = 2$ suggests to use the approximate equation

$$\mu_{yt} \approx h_1(\mu_{xt})$$

as a *second order approximation* (in general, an approximation of order k) to the aggregation problem.

Alternatively, when we do not have prior information on the distribution we can use the empirical/theoretical information we have about the dynamic evolution of x[3]. For example, let's assume that the law of motion of the micro-variable

[1] In probability theory, those terms represent the error made linearizing a non-linear transformation of random variables. In fact, as discussed in the previous part, there is no aggregation error in an economic theory only when the relationships between micro-variables are all linear or affine.

[2] This method may be applied in principle even when second moments of x do not exist (but the first moment of y exists) using an appropriate transformation of the microvariable x. In such cases, the approach can give only qualitative insights.

[3] This type of information can be easily collected in the industrial dynamics literature in which firm's size evolve according to a Gibrat's law (1931) in the sense that the rate of growth is stochastically independent (or mean independent as it seem from recent works; see Stanley et al. 1996) from firm's size. In this literature the asymptotic distribution of the process can be derived that, according to the structure of the model may be log-normal or power law (Axtell 2001; Gaffeo et al. 2003). In this last case we have to take into account the fact that higher moments do not exist. To analyze approximated aggregate relationships we have to use a non-linear transformation of the variables of interest as said in note 2. We stress again that in such situations the approximate method gives only qualitative insights of the effect of the distribution of agents on the macro-relationships.

x is described by the following first order dynamical relation in discrete time,

$$x_{it} = g\left(x_{it-1}\right) + z_{it}, \tag{8}$$

where g is a function of the variable x and z_{it} is some idiosyncratic component with zero mean and variance δ^2. Assuming, as before, that g may be differentiated three times and computing the expected value of x_{it}, we can write it as an approximate function of the mean and the variance at time $t-1$. That is,

$$\mu_{x,t} \cong h_x\left(\mu_{x,t-1}, \sigma^2_{x,t-1}\right). \tag{9}$$

Now calling h_y the function relating the first moment of y with the first and second moment of x we can write

$$\mu_{y,t} \cong h_y\left(\mu_{x,t}, \sigma^2_{x,t}\right). \tag{10}$$

Finally, expanding Eq. (8) in Taylor's series and applying the variance operator to both sides it is possible to compute the approximate relation between second moments at time t and first and second moments at time $t-1$

$$\sigma^2_{xt} \approx g'\left(\mu_{xt-1}\right)^2 \sigma^2_{xt-1} + \delta^2 = v\left(\mu_{xt-1}, \sigma^2_{xt-1}, \delta^2\right) \tag{11}$$

that can be substituted in the above equation

$$\mu_{y,t} \cong h_y\left(\mu_{x,t}, v\left(\mu_{xt-1}, \sigma^2_{xt-1}, \delta^2\right)\right) = h_{yy}\left(\mu_{x,t}, \mu_{xt-1}, \sigma^2_{xt-1}, \delta^2\right). \tag{12}$$

Then, by inverting Eq. (9) with respect to $\sigma^2_{x,t-1}$, we get

$$\sigma^2_{x,t-1} \cong l_x\left(\mu_{X,t}, \mu_{x,t-1}\right) \tag{13}$$

and by replacing Eq. (13) in Eq. (12) we get a second (approximate) aggregate relationship

$$\mu_{y,t} \cong h_2\left(\mu_{x,t}, \mu_{xt-1}, \delta^2\right)$$

relating the mean (aggregate) of the macro-variable y at time t with the mean (aggregate) of the macro-variable x at time t and $t-1$ and depending on prior information on the conditional variance of the stochastic process x.

We have to take care of the presence of direct interaction in the individual micro-equations. In other terms, the micro-relation may be the following

$$y_{it} = f\left(x_{it}, \mu^e_{yit}\right) + u_{it}$$

in which μ^e_{yit} is the agent-i expected value of y. In analyzing the macro-relationship in equilibrium we have to impose the *consistency principle*; *i.e.*, individual expectations have to be equal to the mean of y

$$\mu^e_{yit} = \mu_{yt}$$

otherwise some agents make errors and revise their expectations. To make an example, assume the following micro-specification

$$y_{it} = f(x_{it}) + b\mu^e_{yit} + u_{it}$$

then the second order approximation of the aggregate relationship, applying the consistency principle, is

$$\mu_{yt} = f(\mu_{xt}) + \tfrac{1}{2}f''(\mu_{xt})\sigma^2_{xt} + b\mu_{yt} + E[o(||\varepsilon_{it}||)]$$

In situation in which we can neglect the $E[o(||\varepsilon_{it}||)]$ term, the approximate relation implies that

$$\mu_{yt} \approx \frac{1}{1-b}\left[f(\mu_{xt}) + \tfrac{1}{2}f''(\mu_{xt})\sigma^2_{xt}\right],$$

where $(1-b)^{-1}$ may be interpreted as the (approximate) social multiplier generated by the interaction term. When the interaction term enters non-linearly in the micro-relation this may cause a multiplicity of equilibria and the well-know *coordination problem* (Cooper – John 1988).

In summary, when we study some theory describing the stochastic process on a micro-variable x, we suggest to approximate the dynamic evolution of the first two moments (in general the first k moments) by the following map:

$$\begin{cases} \mu_{xt+1} \approx g(\mu_{xt}) + \tfrac{1}{2}g''(\mu_{xt})\,\sigma^2_{xt} \\ \sigma^2_{xt+1} \approx g'(\mu_{xt})^2\,\sigma^2_{xt} + \delta^2 \end{cases}$$

Therefore, it is important to investigate how this approximation describes distributional evolution. This issue is analyzed in Sect. 4, in the framework of Delli Gatti et al. (2000) model for firms' growth. In the next sub-section we present a simple economic application of this approach.

4 A simple application: the theory of growth and distributional changes

The starting point of the theory of growth (see Romer 2000) is the equality between aggregate investment and aggregate savings. In a simplified version, that assumption may be written as follows,

$$\sum_{i=1}^{N} \dot{k}_{it} + \delta \sum_{i=1}^{N} k_{it} = \sum_{i=1}^{N} sf\left(k_{it}\right).$$

In the above equation, without loss of generality, we assume homogeneity in the saving rate (s) and in the rate of depreciation of capital (δ). The only form of heterogeneity is the amount of capital (k_{it}). The production function $f\left(k_{it}\right)$ use only capital[4]. In the theory of perfect aggregation of production functions (see Fisher 1992) a necessary condition in order to aggregate without bias is the hypothesis of constant (and identical) return to scale technologies. But what changes the analysis if this hypothesis does not hold exactly? First, note that we assume no technology growth, so by construction Solow's residual (or total factor productivity) is zero. In this situation, changes of aggregate output $y_t = \sum_{i=1}^{N} f\left(k_{it}\right)$ can be explained by means of variations in aggregate capital.

Multiplying both sides of the dynamical equation by N^{-1} we get

$$N^{-1} \sum_{i=1}^{N} \dot{k}_{it} + \delta N^{-1} \sum_{i=1}^{N} k_{it} = N^{-1} \sum_{i=1}^{N} sf\left(k_{it}\right).$$

Now, writing $\dot{\bar{k}}_t = N^{-1} \sum_{i=1}^{N} \dot{k}_{it}, \bar{k}_t = N^{-1} \sum_{i=1}^{N} k_{it}$ and assuming that the law of large numbers applies, we can use the above approximation to obtain the aggregate dynamic equation

$$\dot{\bar{k}}_t \approx sf\left(\bar{k}_t\right) + \tfrac{1}{2} f''\left(\bar{k}_t\right) \sigma_{kt}^2 - \delta \bar{k}_t,$$

which is Solow's equation modified with the new term $\frac{1}{2} f''\left(\bar{k}_t\right) \sigma_{kt}^2$ implying that aggregate equilibrium depends on the distribution. For example, in Fig. 1, a change in the aggregate equilibrium is plotted, assuming a Cobb–Douglas production function, when the variance of k goes from 0 to 3.

Movements of distribution may be erroneously attributed to changes in Solow's residual, a quantity that in this example is zero by construction.

5 How the VRA actually performs: a simulation case

In this section we apply the approximated approach described in the previous section to the framework of Delli Gatti et al. (2000).

In that model, the economy is characterized by a large number of firms. Each one of them produces a homogeneous good by means of a constant return to scale technology where capital is the only input. As in Greenwald and Stiglitz (1993), firms cannot raise funds on the Stock market because of equity rationing, but they have unlimited access to credit. This means that firms do not issue new

[4] This hypothesis is made for the sake of simplicity. It avoids to take trace of the heterogeneity of labor and its relation with the heterogeneity of capital.

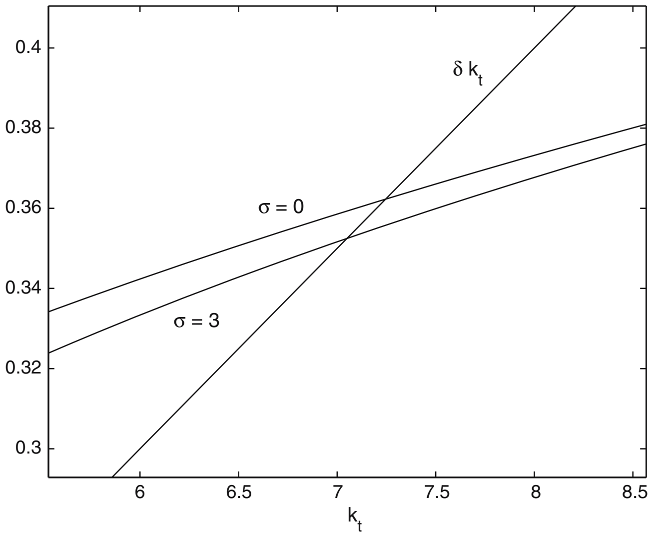

Fig. 1 A change of the variance affects the aggregate equilibrium. The figure is obtained using a Cobb–Douglas production function $f(k) = k^a$ with a = 0.3, s = 0.2 and $\delta = 0.05$

equities but can obtain from banks all the credit they need to finance production at the (exogenous) rate of interest, r.

The model implies the following micro-dynamics for the equity ratio a_{it}, the ratio of equity to total capital:

$$a_{it} = \Gamma_1 a_{it-1} - \Gamma_2 a_{it-1}^2 + \Gamma_{0i} \tag{14}$$

with

$$\begin{aligned} \Gamma_1 &= 1 - \beta_0(1 + \gamma\beta_1) \\ \Gamma_2 &= \beta_1 \left(1 + \frac{\gamma}{2}\beta_1\right) \\ \Gamma_{0i} &= u_{it}\phi - r_t q - \frac{\gamma}{2}(\beta_0)^2, \end{aligned}$$

where q is the cost of capital, γ is the capital adjustment cost parameter, $\beta_{0,1}$ are parameters of the relation between capital accumulation and the equity ratio. Finally, u_{it} is the relative price, for the good of firm i at time t, that is supposed to follow a uniform IIU(0,2) stochastic process.

Since the Eq. (14) is quadratic, it is reasonable to use a second order Taylor expansion for the mean and a first order for the variance as described in the end of the previous part.

$$\begin{cases} a_t \approx \Gamma_1 a_{t-1} - \Gamma_2 a_{t-1}^2 + \Gamma_0 - \Gamma_2 V_{t-1} \\ V_t \approx \left(\Gamma_1 - 2\Gamma_2 a_{t-1}\right)^2 V_{t-1} + V\left(\Gamma_{0i}\right) \end{cases} \tag{15}$$

that is

$$\begin{cases} a_t \approx \Gamma_1 a_{t-1} - \Gamma_2 a_{t-1}^2 + \Gamma_0 - \Gamma_2 V_{t-1} \\ V_t \approx \left(\Gamma_1 - 2\Gamma_2 a_{t-1}\right)^2 V_{t-1} + \frac{\phi^2}{3} \end{cases} \tag{16}$$

In Figs. 2 and 3, we present the results of a simulation of the equity ratio stochastic process described by Eq. (14). A set of 10,000 firms has been simulated for 20

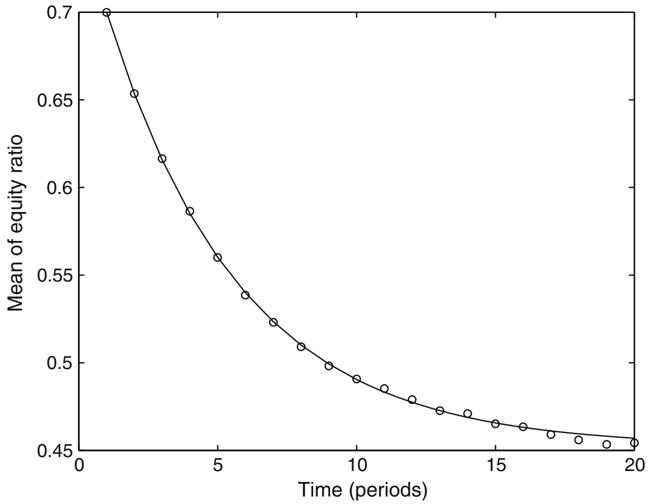

Fig. 2 Mean of equity ratio. The *dots* represent the results of the stochastic simulation (see text for details), whereas the *solid line* is the solution of the quadratic map (16)

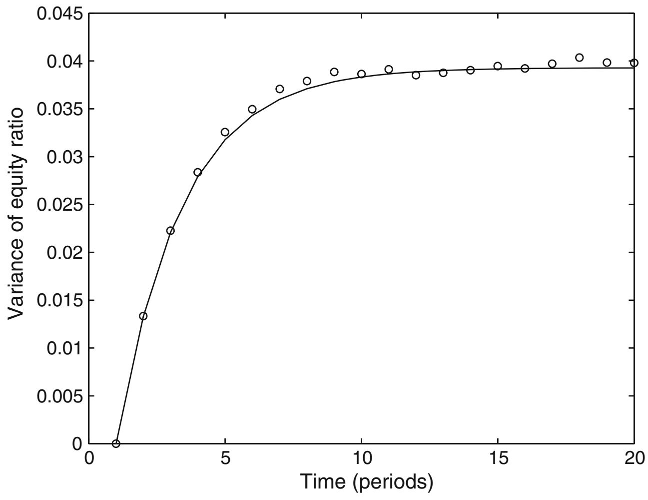

Fig. 3 Variance of equity ratio. The *dots* represent the results of the stochastic simulation (see text for details), whereas the *solid line* is the solution of the quadratic map (16)

periods. In the initial condition, all the firms share the same equity ratio set at a value of 70% and the initial variance is 0. For each period, the average and the variance of the firm equity ratio distribution has been recorded. Both quantities are respectively plotted in Figs. 2 and 3 as open circles. In both figures, the solid line represents the approximation given by Eqs. (15) and (16). The parameters of the simulation are $\phi = 0.2; \beta_0 = 0.19; \beta_1 = 0.0012; \gamma = 5; r = 0.03; q = 1$. With this choice, in the quadratic map (16), both the average and the variance converge to a stable fixed point.

6 Conclusive remarks

The profession at large seems increasingly aware of the shortcomings of the RA methodology in macroeconomics and econometrics. Intellectual consistency would require the widespread acceptance of microfoundations as the only scientific way to build sensible macromodels to be coupled with the recognition that agents are heterogeneous and generally interact, sometimes in complex ways. Microfoundations based on the RA assumption, in fact, amount to no more than a tautology, void of meaningful contents in the presence of persistent heterogeneity.

Consistent microfoundations, therefore, must be based on aggregation procedures which emphasize heterogeneity from the beginning. There have been some attempts to put forward methods to solve the aggregation problem in this case – for instance the stochastic aggregation procedure proposed by Stocker or Aoki's combinatorial method – but they are seldom adopted in the literature, perhaps because of their complexity or the fact that they do not fit easily with the conceptual framework of mainstream macroeconomics.

In this paper we have presented a variant of the stochastic aggregation approach which basically consists in exploring the evolution over time of the first and second moments of the distribution. In a sense therefore, we propose to focus on the behaviour of a Variant Representative Agent. An application to Solow growth model shows that changes aggregate output usually associated with total factor productivity in the aggregative interpretation of the framework may be due to changes in the distribution of agents in terms of capital intensity. The application to a model by Delli Gatti et al. (2000) shows the efficacy of the method in capturing the evolution over time of the distribution of firms in terms of financial solidity (equity ratio). The method seems general enough to cover a wide range of economic situations in which heterogeneity is relevant and persistent. It seems also simple enough to deserve the attention of the macroeconomist dissatisfied with the RA who wants to derive meaningful and microfounded macroeconomic results.

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