Chin. Ann. Math.
29B(3), 2008, 239–246
DOI: 10.1007/s11401-007-0179-y

Chinese Annals of Mathematics, Series B

Real Renormings on Complex Banach Spaces

Francisco J GARCÍA-PACHECO* Alejandro MIRALLES**

Abstract The authors prove that every complex Banach space admits an equivalent real norm that is far away from being a complex norm. Furthermore, this real norm can be chosen to share many properties with complex norms, but it is still not a complex norm.

Keywords Complex norms, Real norms, Hilbert space
2000 MR Subject Classification 46B03, 46B20, 46B07

1 Introduction

As we all know, any complex Banach space is topologically identic to its underlying real space. However, they differ algebraically and geometrically. The aim of this paper is to show two ways to obtain real Banach spaces whose norm is not complex.

Given a complex Banach space X, we let $X_{\mathbb{R}}$ denote the underlying real Banach space of X. It is obvious that if $\|\cdot\|$ is an equivalent complex norm on X then this norm induces on $X_{\mathbb{R}}$ an equivalent real norm $\|\cdot\|_{\mathbb{R}}$. Therefore, given an equivalent real norm $\|\cdot\|_r$ on $X_{\mathbb{R}}$, we will say that this norm comes from a complex norm on X if there exists an equivalent norm $\|\cdot\|$ on X such that $\|\cdot\|_{\mathbb{R}} = \|\cdot\|_r$. Obviously, $\|\cdot\|_r$ comes from a complex norm if and only if $\|\lambda x\|_r = |\lambda|\|x\|_r$ for all $\lambda \in \mathbb{C}$ and $x \in X$.

In this paper we will show, in two different ways (algebraically and geometrically), that there always exists an equivalent real norm $\|\cdot\|_r$ on $X_{\mathbb{R}}$ that cannot come from a complex norm.

To finish the introduction, we will introduce the notation that we will make use of. Let X denote a normed space. Then B_X and S_X will respectively denote the closed unit ball of X and the unit sphere of X. Whenever we refer to a closed ball or sphere of center x and radius ε we will add (x, ε) next to the right of the above expressions.

2 Geometric Renorming

If we look at any 1-dimensional complex Banach space as a real space, we realize that its unit ball is strictly convex. In particular, this unit ball is free of convex sets with non-empty interior relative to the unit sphere. We can easily find many examples of complex spaces whose unit sphere does contain non-trivial segments. Nevertheless, the next theorem shows that this

Manuscript received April 22, 2007. Published online April 23, 2008.
*Department of Mathematical Sciences, Kent State University, Kent 44242, Ohio, USA.
E-mail: fgarcia@math.kent.edu
**Departamento de Análisis Matemático, Universidad de Valencia, Burjassot, 46100, Valencia, Spain.

cannot happen if we substitute "non-trivial segment" with "convex set with non-empty interior relative to the unit sphere" in the sentence right above.

Theorem 2.1 *Let X be a complex Banach space. If $C \subset \mathsf{S}_X$ is a convex set, then* $\mathrm{int}_{\mathsf{S}_X}(C) = \emptyset$.

Proof Assume that there are $c \in C$ and $r > 0$ such that $\mathsf{B}_X(c, r) \cap \mathsf{S}_X \subseteq C$. Now, choose $\lambda \in \mathbb{C} \setminus \{1\}$ with $|\lambda| = 1$ and $|\lambda - 1| < r$. Obviously, $\lambda c \in \mathsf{B}_X(c, r) \cap \mathsf{S}_X$, therefore $\lambda c \in C$. Since C is convex, we have $\frac{\lambda c + c}{2} \in C$ and hence $\|\frac{\lambda c + c}{2}\| = 1$, but this means that $\lambda = 1$, which is a contradiction.

We will take advantage of Theorem 2.1 to show that many real norms can be obtained that do not come from complex norms. To do this, the following lemmas will be very helpful.

Lemma 2.1 *Let X be a real Banach space. Let $f \in \mathsf{S}_{X^*}$ be a norm-attaining functional. Let $t \in (0, 1]$ and $(s_n)_{n\in\mathbb{N}} \subset [t, 1]$ converging to 1. Then, there exists a subsequence $(s_{n_k})_{k\in\mathbb{N}}$ of $(s_n)_{n\in\mathbb{N}}$ such that for every $k \in \mathbb{N}$ there are $u_k, v_k \in \mathsf{B}_X \cap f^{-1}(\{t\})$ with $\|u_k\| = \|v_k\| = s_{n_k}$ and* $\mathrm{diam}(\mathsf{B}_X \cap f^{-1}(\{t\})) - \|u_k - v_k\| \to 0$ *as $k \to \infty$.*

Proof Let us fix $k \in \mathbb{N}$ and consider $a_k, b_k \in \mathsf{S}_X \cap f^{-1}(\{t\})$ such that

$$\mathrm{diam}(\mathsf{B}_X \cap f^{-1}(\{t\})) - \|a_k - b_k\| < \frac{1}{k}.$$

Since the segment $[a_k, b_k]$ is a compact set, there exits $\lambda_k \in (0, 1)$ such that the norm of $c_k := \lambda_k a_k + (1 - \lambda_k) b_k$ is minimum on the above segment. Observe that

$$t = f(c_k) \leq \|c_k\| \leq 1.$$

Thus we can find $n_k \in \mathbb{N}$ big enough and use the Intermediate Value Theorem to assure the existence of $u_k \in [a_k, c_k]$ and $v_k \in [c_k, b_k]$ such that $\|u_k\| = \|v_k\| = s_{n_k}$ and $\|a_k - u_k\|, \|b_k - v_k\| \leq \frac{1}{k}$. Obviously, we can choose the sequence $(n_k)_{k\in\mathbb{N}}$ to be strictly increasing. As a consequence, $\mathrm{diam}(\mathsf{B}_X \cap f^{-1}(\{t\})) - \|u_k - v_k\| \to 0$ as $k \to \infty$.

Lemma 2.2 *Let X be a real Banach space. Let $f \in \mathsf{S}_{X^*}$ be a norm-attaining functional and consider the function*

$$\begin{aligned} \phi : [0, 1] &\to \mathbb{R} \\ t &\mapsto \phi(t) = \mathrm{diam}(\mathsf{B}_X \cap f^{-1}(\{t\})). \end{aligned}$$

Then

(1) *For every $0 < s \leq t \leq 1$ we have*

$$\frac{\phi(t)}{t} \leq \frac{\phi(s)}{s}.$$

(2) *The function ϕ is continuous from the right.*

(3) *The space X is strongly rotund if and only if X is rotund and the function ϕ is continuous at 1 for every norm-attaining $f \in \mathsf{S}_{X^*}$.*

Proof For our purposes in this proof, let us pick an element $z \in \mathsf{S}_X$ and $f(z) = 1$. We will also assume that $\dim(X) > 1$, because otherwise $\phi = 0$. Observe that with this assumption $\phi(0) = 2$.

(1) Let $x, y \in \mathsf{B}_X \cap f^{-1}(\{t\})$. Then $(\frac{s}{t})x, (\frac{s}{t})y \in \mathsf{B}_X \cap f^{-1}(\{s\})$. Thus

$$\|x - y\| = \frac{t}{s}\left\|\frac{s}{t}x - \frac{s}{t}y\right\| \leq \frac{t}{s}\phi(s).$$

Since x and y were arbitrarily chosen in $\mathsf{B}_X \cap f^{-1}(\{t\})$, we have

$$\frac{\phi(t)}{t} \leq \frac{\phi(s)}{s}.$$

(2) First, let us see that ϕ is continuous at 0. Let $(t_n)_{n\in\mathbb{N}} \subset [0, 1]$ be a sequence converging to 0. For every $n \in \mathbb{N}$, let us choose $x_n \in \ker(f_n)$ with $\|x_n\| = 1 - t_n$, $v_n = t_n z + x_n$, and $w_n = t_n z - x_n$. Then, we have $f(v_n) = f(w_n) = t_n$, $\|v_n\|, \|w_n\| \leq 1$, and $\|v_n - w_n\| = 2(1 - t_n)$ for every $n \in \mathbb{N}$. Since

$$2 \geq \phi(t_n) \geq 2(1 - t_n)$$

for every $n \in \mathbb{N}$, we deduce that $(\phi(t))_{n\in\mathbb{N}}$ converges to $2 = \phi(0)$. Finally, let us see that ϕ is continuous from the right at every $t \in (0, 1]$. Let $(t_n)_{n\in\mathbb{N}} \subset [t, 1]$ be a sequence converging to t. Then

$$\frac{\phi(t_n)}{t_n} \leq \frac{\phi(t)}{t}$$

for all $n \in \mathbb{N}$, which means that $\limsup_{n\to\infty} \phi(t_n) \leq \phi(t)$. Now, in accordance to Lemma 2.1, there exists a subsequence $(t_{n_k})_{k\in\mathbb{N}}$ of $(t_n)_{n\in\mathbb{N}}$ such that for every $k \in \mathbb{N}$ we can choose $u_k, v_k \in \mathsf{B}_X \cap f^{-1}(\{t\})$ with $\|u_k\| = \|v_k\| = 1 - (t_{n_k} - t)$ and $\phi(t) - \|u_k - v_k\| \to 0$ as $k \to \infty$. For every $k \in \mathbb{N}$, let $x_k = (t_{n_k} - t)z + u_k$ and $y_k = (t_{n_k} - t)z + v_k$. Then, we have $f(x_k) = f(y_k) = t_{n_k}$, $\|x_k\|, \|y_k\| \leq 1$, and $\|x_k - y_k\| = \|u_k - v_k\|$ for every $k \in \mathbb{N}$. Therefore, $\liminf_{k\to\infty} \phi(t_{n_k}) \geq \phi(t)$. This proves that $(\phi(t_{n_k}))_{k\in\mathbb{N}}$ converges to $\phi(t)$, and hence ϕ is continuous from the right at t.

(3) Assume first that X is strongly rotund (see [2] for all definitions). We clearly have that X is rotund. Now, let $(t_n)_{n\in\mathbb{N}} \subset [0, 1]$ be a sequence converging to 1. For every $n \in \mathbb{N}$ we can choose $x_n, y_n \in \mathsf{B}_X \cap f^{-1}(\{t_n\})$ such that $\phi(t_n) - \|x_n - y_n\| \to 0$ as $n \to \infty$. Now, both sequences $(f(x_n))_{n\in\mathbb{N}}$ and $(f(y_n))_{n\in\mathbb{N}}$ converge to 1, so by the strong rotundity of X we have that both $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ converge to z. Next, for all $n \in \mathbb{N}$,

$$\begin{aligned}\phi(t_n) &= \phi(t_n) - \|x_n - y_n\| + \|x_n - y_n\| \\ &\leq \phi(t_n) - \|x_n - y_n\| + \|x_n - z\| + \|z - y_n\|,\end{aligned}$$

that is, $(\phi(t_n))_{n\in\mathbb{N}}$ converges to 0. Observe that

$$\phi(1) = \operatorname{diam}(\mathsf{B}_X \cap f^{-1}(\{1\})) = \operatorname{diam}(\{z\}) = 0$$

because X is rotund. Conversely, let $(x_n)_{n\in\mathbb{N}} \subset \mathsf{B}_X$ be a sequence such that $(f(x_n))_{n\in\mathbb{N}}$ converges to 1. By hypothesis, $\phi(f(x_n)) \to \phi(1) = 0$ as $n \to \infty$. Now, $x_n, f(x_n)z \in \mathsf{B}_X \cap f^{-1}(\{f(x_n)\})$ for all $n \in \mathbb{N}$. Thus

$$\begin{aligned}\|x_n - z\| &\leq \|x_n - f(x_n)z\| + \|f(x_n)z - z\| \\ &\leq \phi(f(x_n)) + |f(x_n) - 1|\end{aligned}$$

for all $n \in \mathbb{N}$, that is, the sequence $(x_n)_{n\in\mathbb{N}}$ converges to z.

Lemma 2.3 *Let X be a real Banach space with $\dim(X) > 1$. Then, X can be equivalently renormed to have a convex subset in its unit ball with non-empty interior relative to the unit sphere and of diameter greater than or equal to δ, for any previously fixed $\delta \in (0, 2)$.*

Proof Let $f \in \mathsf{S}_{X^*}$ be a norm-attaining functional and consider the function

$$\begin{aligned} \phi : [0,1] &\to \mathbb{R} \\ t &\mapsto \phi(t) = \operatorname{diam}(\mathsf{B}_X \cap f^{-1}(\{t\})). \end{aligned}$$

According to Lemma 2.2, ϕ is continuous at 0. Therefore we can find $0 < t < 1$ closed enough to 0 such that $\phi(t) \geq \delta$. Finally, let us consider the new norm on X whose unit ball is given by $\mathcal{B} := \mathsf{B}_X \cap f^{-1}([-t,t])$. We obviously have that $\mathcal{B} \subset \mathsf{B}_X$, therefore $\|\cdot\| \leq \|\cdot\|_{\mathcal{B}}$, and $\operatorname{diam}_{\mathcal{B}}(\mathsf{B}_X \cap f^{-1}(\{t\})) \geq \operatorname{diam}(\mathsf{B}_X \cap f^{-1}(\{t\})) \geq \delta$.

Now, we can state and prove the main theorem in this section.

Theorem 2.2 *Let X be a complex Banach space. Then, $X_{\mathbb{R}}$ can be equivalently renormed so that the new real norm on $X_{\mathbb{R}}$ cannot come from a complex norm.*

Proof According to Lemma 2.3, $X_{\mathbb{R}}$ can be equivalently renormed to have a convex subset with non-empty interior relative to the unit sphere. Now, by applying Lemma 2.1, we deduce that this new norm cannot come from a complex norm.

Notice that we can improve Lemma 2.3 by showing that the convex subset with non-empty interior relative to the unit sphere actually can be chosen to have diameter equal to 2. In order to prove this, we will make use of the following result, which can be found in [3].

Theorem 2.3 (see [3]) *Let X and Y be real Banach spaces. The unit ball of $X \oplus_\infty Y$ has a convex subset with non-empty interior relative to the unit sphere if and only if either the unit ball of X or the unit ball of Y has a convex subset with non-empty interior relative to the unit sphere.*

Theorem 2.4 *Let X be a real Banach space with $\dim(X) > 1$. Then, X can be equivalently renormed to have a convex subset in its unit ball with non-empty interior relative to the unit sphere and of diameter equal to 2.*

Proof Let us take $x \in \mathsf{S}_X$ and $x^* \in \mathsf{S}_{X^*}$ such that $x^*(x) = 1$. Then, X is isomorphic to $Y := \mathbb{R}x \oplus_\infty \ker(x^*)$. Finally, $\mathbb{R}x$ possesses convex subsets in this unit ball with non-empty interior relative to the unit sphere (for instance, $\{x\}$ and $\{-x\}$). Therefore, according to Theorem 2.3, Y possesses such subsets (for instance, $\{x\} \times \mathsf{B}_{\ker(x^*)}$ and $\{-x\} \times \mathsf{B}_{\ker(x^*)}$).

The final results of this section show that the norm 1 does also serve our purposes.

Lemma 2.4 *Let X and Y be real Banach spaces. Let $(x^*, y^*) \in \mathsf{S}_{X^* \oplus_\infty Y^*}$ and $(x, y) \in \mathsf{S}_{X \oplus_1 Y}$. Then, $(x^*, y^*)(x, y) = 1$ if and only if $x^*(x) = \|x\|$ and $y^*(y) = \|y\|$.*

Proof Suppose that either $x^*(x) < \|x^*\|\|x\|$ or $y^*(y) < \|y^*\|\|y\|$. Then

$$1 = (x^*, y^*)(x, y) = x^*(x) + y^*(y) < \|x^*\|\|x\| + \|y^*\|\|y\| \leq \|x\| + \|y\| = 1,$$

which is a contradiction. Conversely, if $x^*(x) = \|x\|$ and $y^*(y) = \|y\|$, then

$$(x^*, y^*)(x, y) = x^*(x) + y^*(y) = \|x\| + \|y\| = 1.$$

Note that the next lemma follows from the previous one without making use of any proof. Remember that a point in the unit sphere of a real Banach space is said to be a smooth point of the unit ball if there is only one functional in the unit sphere of the dual attaining its norm at the point.

Lemma 2.5 *Let X and Y be real Banach spaces. Then, an element $(x, y) \in \mathsf{S}_{X\oplus_1 Y}$ is a smooth point of $\mathsf{B}_{X\oplus_1 Y}$ if and only if $x, y \neq 0$ and $\frac{x}{\|x\|}$ and $\frac{y}{\|y\|}$ are respectively smooth points of B_X and B_Y.*

Remark 2.1 Let X be a real Banach space. Then, for every $a, b \in X \setminus \{0\}$ we have

$$\left\| \frac{a}{\|a\|} - \frac{b}{\|b\|} \right\| \leq \frac{2}{\|b\|} \|a - b\|.$$

Theorem 2.5 *Let X and Y be real Banach spaces. The unit ball of $X \oplus_1 Y$ has a convex subset with non-empty interior relative to the unit sphere if and only if both the unit ball of X and the unit ball of Y have a convex subset with non-empty interior relative to the unit sphere.*

Proof Let $C \subset \mathsf{S}_{X\oplus_1 Y}$ be a convex subset with non-empty interior relative to $\mathsf{S}_{X\oplus_1 Y}$. By Hahn-Banach, there exists $(x^*, y^*) \in \mathsf{S}_{X^*\oplus_\infty Y^*}$ such that $(x^*, y^*)(C) = \{1\}$. Actually, since $\mathrm{int}_{\mathsf{S}_{X\oplus_1 Y}}(C) \neq \emptyset$ we deduce that (x^*, y^*) is the only element in $\mathsf{S}_{X^*\oplus_\infty Y^*}$ such that $(x^*, y^*)(C) = \{1\}$ and all elements in $\mathrm{int}_{\mathsf{S}_{X\oplus_1 Y}}(C)$ are smooth points (see, for instance, [1, 4], or [3].) There also are $\varepsilon > 0$ and $(x, y) \in C$ such that $\mathsf{B}_{X\oplus_1 Y}((x, y), \varepsilon) \cap \mathsf{S}_{X\oplus_1 Y} \subseteq C$. By Lemma 2.5, we have that $x, y \neq 0$. Now, we will show that $\mathsf{B}_Y(\frac{y}{\|y\|}, \varepsilon) \cap \mathsf{S}_Y \subseteq (y^*)^{-1}(\{1\}) \cap \mathsf{B}_Y$. So, let $v \in \mathsf{B}_Y(\frac{y}{\|y\|}, \varepsilon) \cap \mathsf{S}_Y$. Then, $(x, \|y\|v) \in \mathsf{S}_{X\oplus_1 Y}$ and

$$\|(x, y) - (x, \|y\|v)\|_1 = \|y\| \left\| \frac{y}{\|y\|} - v \right\| \leq \varepsilon.$$

As a consequence,

$$(x, \|y\|v) \in C \quad \text{and} \quad (x^*, y^*)(x, \|y\|v) = 1.$$

By applying Lemma 2.4, we have $y^*(v) = 1$. Similarly, $\mathrm{int}_{\mathsf{S}_X}((x^*)^{-1}(\{1\}) \cap \mathsf{B}_X) \neq \emptyset$. Conversely, let $x^* \in \mathsf{S}_{X^*}$ and $y^* \in \mathsf{S}_{Y^*}$ such that

$$\mathrm{int}_{\mathsf{S}_X}((x^*)^{-1}(\{1\}) \cap \mathsf{B}_X),\ \mathrm{int}_{\mathsf{S}_Y}((y^*)^{-1}(\{1\}) \cap \mathsf{B}_Y) \neq \emptyset.$$

Let $x \in \mathsf{S}_X$, $y \in \mathsf{S}_Y$, and $\varepsilon > 0$ such that

$$\mathsf{B}_X(x, \varepsilon) \cap \mathsf{S}_X \subseteq (x^*)^{-1}(\{1\}) \cap \mathsf{B}_X,$$
$$\mathsf{B}_X(y, \varepsilon) \cap \mathsf{S}_X \subseteq (y^*)^{-1}(\{1\}) \cap \mathsf{B}_Y.$$

We will show that

$$\mathsf{B}_{X\oplus_1 Y}\left(\left(\frac{x}{2}, \frac{y}{2}\right), \frac{\varepsilon}{4}\right) \cap \mathsf{S}_{X\oplus_1 Y} \subseteq (x^*, y^*)^{-1}(\{1\}) \cap \mathsf{B}_{X\oplus_1 Y}.$$

So, let $(u,v) \in \mathsf{B}_{X\oplus_1 Y}((\frac{x}{2},\frac{y}{2}),\frac{\varepsilon}{4}) \cap \mathsf{S}_{X\oplus_1 Y}$. Then, by Remark 2.1, we have

$$\left\|\frac{u}{\|u\|} - x\right\| = \left\|\frac{u}{\|u\|} - \frac{\frac{x}{2}}{\|\frac{x}{2}\|}\right\| \leq \frac{2}{\|\frac{x}{2}\|}\left\|u - \frac{x}{2}\right\| \leq \varepsilon.$$

This means that

$$\frac{u}{\|u\|} \in \mathsf{B}_X(x,\varepsilon) \cap \mathsf{S}_X \subseteq (x^*)^{-1}(\{1\}) \cap \mathsf{B}_X$$

and thus

$$x^*(u) = \|u\|.$$

Likewise, $y^*(v) = \|v\|$. Therefore, by applying Lemma 2.4, we have

$$(u,v) \in (x^*,y^*)^{-1}(\{1\}) \cap \mathsf{B}_{X\oplus_1 Y}.$$

3 Algebraic Renorming

In this section, we will take advantage of the algebraical properties of complex spaces to find real norms that cannot come from complex norms. In concrete terms, we have the following remark that clarifies quite well the real structure of a complex Banach space.

Remark 3.1 Let X be a complex vector space and let $\|\cdot\|_r$ be a real norm on $X_{\mathbb{R}}$. The following conditions are equivalent:

(1) The norm $\|\cdot\|_r$ comes from a complex norm on X.

(2) For every $x \in X$, we have $\|x\|_r = \|\mathrm{i}x\|_r$ and $\mathrm{span}_{\mathbb{R}}\{x, \mathrm{i}x\} = \mathbb{R}x \oplus_2 \mathbb{R}(\mathrm{i}x)$.

In particular, if X is a complex Banach space then every $x \in X_{\mathbb{R}}$ is contained in a 2-dimensional Hilbert subspace of $X_{\mathbb{R}}$.

The conclusion that we infer from the last remark is that, given any complex Banach space X, $X_{\mathbb{R}}$ has many 2-dimensional Hilbert subspaces. This gives us the hint to find the appropriate real renorming that we are looking for. Nevertheless, we want to present first the following theorem, which shows that Theorem 2.1 can be generalized to real spaces that have many 2-dimensional Hilbert subspaces.

Theorem 3.1 *Let X be a real Banach space such that every $x \in X$ is contained in a 2-dimensional Hilbert subspace. Then, the unit ball B_X of X is free of convex subsets with non-empty interior relative to the unit sphere.*

Proof Let $C \subset \mathsf{S}_X$ be convex and have non-empty interior relative to S_X. Let $c \in \mathrm{int}_{\mathsf{S}_X}(C)$. By hypothesis, there is $d \in X$ such that $Y := \mathrm{span}\{c,d\}$ is a Hilbert space of dimension 2. Now, we have $c \in \mathrm{int}_{\mathsf{S}_Y}(C \cap Y)$ and this is impossible because the unit ball B_Y of Y is free of convex subsets with non-empty interior relative to the unit sphere.

The next remark shows, as expected, that the converse to Theorem 3.1 does not hold in general.

Remark 3.2 If we let X stand for ℓ_p^2 then the unit ball of X is free of convex subsets with non-empty interior relative to the unit sphere, but no element $x \in X$ is contained in a 2-dimensional Hilbert subspace.

The next theorem that we present is an improvement of Remark 3.1 and suggests some hints to prove the main theorem in this section.

Theorem 3.2 *Let X be a complex vector space and let $\|\cdot\|_r$ be a real norm on $X_{\mathbb{R}}$. The following conditions are equivalent:*

(1) *The norm $\|\cdot\|_r$ comes from a complex norm on X.*

(2) *For every $x \in X$ we have* $\operatorname{span}_{\mathbb{R}}\{x, \mathrm{i}x\} = \mathbb{R}x \oplus_2 \mathbb{R}(\mathrm{i}x)$.

Proof Note that, according to Remark 3.1, we only need to show that $\|x\|_r = \|\mathrm{i}x\|_r$ for every $x \in X$. So, let $x \in X$. By hypothesis,

$$\operatorname{span}_{\mathbb{R}}\{x + \mathrm{i}x, -x + \mathrm{i}x\} = \mathbb{R}(x + \mathrm{i}x) \oplus_2 \mathbb{R}(-x + \mathrm{i}x),$$

in other words,

$$4\|\mathrm{i}x\|_r^2 = \|(x + \mathrm{i}x) + (-x + \mathrm{i}x)\|_r^2 = \|x + \mathrm{i}x\|_r^2 + \|-x + \mathrm{i}x\|_r^2 = 2\|x\|_r^2 + 2\|\mathrm{i}x\|_r^2,$$

which implies that $\|x\|_r = \|\mathrm{i}x\|_r$.

To finish, we state and prove the main theorem in this section, which shows, among other things, that Theorem 3.1 is really a generalization of Theorem 2.1.

Theorem 3.3 *Let X be a complex Banach space. Then, $X_{\mathbb{R}}$ can be equivalently renormed so that the new real norm on $X_{\mathbb{R}}$ cannot come from a complex norm but verifies that every $x \in X_{\mathbb{R}}$ is contained in a 2-dimensional Hilbert subspace.*

Proof Let us fix an element $x \in X_{\mathbb{R}}$ with $\|x\| = 1$. We can consider on $N := \operatorname{span}_{\mathbb{R}}\{x, \mathrm{i}x\}$ the following equivalent norm:

$$\|\alpha x + \beta(x + \mathrm{i}x)\|_s := \sqrt{|\alpha|^2 + |\beta|^2}$$

for all $\alpha, \beta \in \mathbb{R}$. Observe that $\operatorname{span}_{\mathbb{R}}\{x, \mathrm{i}x\}$ endowed with the norm $\|\cdot\|_s$ is a Hilbert space (it is, in fact, $\mathbb{R}x \oplus_2 \mathbb{R}(x + \mathrm{i}x)$). Furthermore,

$$\|x\|_s = 1 \quad \text{and} \quad \|\mathrm{i}x\|_s = \|(-x) + (x + \mathrm{i}x)\|_s = \sqrt{2}.$$

Now, let M be a topological complement for $\mathbb{C}x$ in X. We can consider on $X_{\mathbb{R}} = N \oplus M_{\mathbb{R}}$ the following equivalent norm:

$$\|y\|_r := \sqrt{\|n\|_s^2 + \|m\|^2}$$

for all $y \in X_{\mathbb{R}}$, where $y = n + m$ with $n \in N$ and $m \in M_{\mathbb{R}}$. Finally, we will show that the norm $\|\cdot\|_r$ verifies the required properties:

(1) The real norm $\|\cdot\|_r$ does not come from a complex norm on X. Indeed,

$$\|\mathrm{i}x\|_r = \|\mathrm{i}x\|_s = \sqrt{2} \neq 1 = |\mathrm{i}|\|x\|_s = |\mathrm{i}|\|x\|_r.$$

(2) Every $y \in X_{\mathbb{R}}$ is contained in a 2-dimensional Hilbert subspace. Indeed, let us write $y = n + m$ with $n \in N$ and $m \in M_{\mathbb{R}}$. Let n' be an element in the orthogonal $(\mathbb{R}n)^{\perp}$ of $\mathbb{R}n$ in N endowed with the norm $\|\cdot\|_s$. We will show that

$$\operatorname{span}_{\mathbb{R}}\{y, y'\} = \mathbb{R}y \oplus_2 \mathbb{R}(y'),$$

where $y' = n' + \mathrm{i}m$. Let us take any $\lambda, \gamma \in \mathbb{R}$. Then

$$\begin{aligned}
\|\lambda y + \gamma y'\|_r^2 &= \|(\lambda n + \gamma n') + (\lambda m + \gamma \mathrm{i}m)\|_r^2 \\
&= \|\lambda n + \gamma n'\|_s^2 + \|(\lambda + \mathrm{i}\gamma)m\|^2 \\
&= \|\lambda n\|_s^2 + \|\gamma n'\|_s^2 + \|\lambda m\|^2 + \|\gamma \mathrm{i}m\|^2 \\
&= \|\lambda n + \lambda m\|_r^2 + \|\gamma n' + \gamma \mathrm{i}m\|_r^2 \\
&= \|\lambda y\|_r^2 + \|\gamma y'\|_r^2.
\end{aligned}$$

Acknowledgement The authors would like to thank the Math Department at Kent State University for the nice environment during the preparation of this manuscript.

References

[1] Diestel, J., Geometry of Banach Spaces — Selected Topics, Lecture Notes in Mathematics, **485**, Springer-Verlag, Berlin-New York, 1975.

[2] Megginson, R. E., An Introduction to Banach Space Theory, Graduate Texts in Mathematics, **183**, Springer-Verlag, New York, 1998.

[3] Acosta, M. D., Aizpuru, A., Aron, R. M. and García-Pacheco, F. J., Functionals that do not attain their norm, *Bull. Belg. Math. Soc. Simon Stevin*, **14**, 2007, 407–418.

[4] Aizpuru, A. and García-Pacheco, F. J., Some questions about rotundity and renormings in Banach spaces, *J. Aust. Math. Soc.*, **79**, 2005, 131–140.