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Affine Structures on a Ringed Space and Schemes

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Abstract The author first introduces the notion of affine structures on a ringed space and then obtains several related properties. Affine structures on a ringed space, arising from complex analytical spaces of algebraic schemes, behave like differential structures on a smooth manifold.

As one does for differential manifolds, pseudogroups of affine transformations are used to define affine atlases on a ringed space. An atlas on a space is said to be an affine structure if it is maximal. An affine structure is said to be admissible if there is a sheaf on the underlying space such that they are coincide on all affine charts, which are in deed affine open sets of a scheme. In a rigour manner, a scheme is defined to be a ringed space with a specified affine structure if the affine structures make a contribution to the cases such as analytical spaces of algebraic schemes. Particularly, by the whole of affine structures on a space, two necessary and sufficient conditions, that two spaces are homeomorphic and that two schemes are isomorphic, coming from the main theorems of the paper, are obtained respectively. A conclusion is drawn that the whole of affine structures on a space and a scheme, as local data, encode and reflect the global properties of the space and the scheme, respectively.

Keywords Affine structure, Pseudogroup of affine transformations, Ringed space, Scheme
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1 Introduction

1.1 Background and motivation

As one studies differential structures on a manifold such as Milnor [11], affine structures on a scheme, taken as counterparts, will be introduced and discussed in this paper. Here, we will obtain several properties on affine structures in a rigour and systematic manner. These results in fact can be applied to the complex analytical space of an algebraic scheme over a number field.

1.1.1 An affine covering of a scheme

As well-known, a scheme (or a projective scheme, respectively) is defined to be a ringed space that can be covered by a family of affine (or projective, respectively) schemes, namely an affine covering of the scheme. An affine scheme is the spectrum of a commutative ring equipped with the sheaf (see [4, 7]). In the paper, it will be seen that such a family of affine schemes determines a unique affine structure on the underlying space of the scheme.

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1.1.2 Each affine covering produces a complex analytical space

Fixed an algebraic scheme X over a number field K. Let $\{A_\alpha\}_{\alpha\in\Gamma}$ be a family of finitely generated algebras over K such that their spectra $\mathrm{Spec}A_\alpha$ cover X, i.e.,

$$\bigcup_\alpha \mathrm{Spec}A_\alpha \supseteq X$$

holds. Here, each A_α is isomorphic to the quotient of some polynomial ring $K[t_1, t_2, \cdots, t_{n_\alpha}]$ by a finite number of polynomials

$$f_1, f_2, \cdots, f_{r_\alpha}$$

over K in the variables.

By Serre's GAGA (see [5, 12]), each open subscheme $\mathrm{Spec}A_\alpha$ has an analytical space X_α^{an} that is defined by the common zeros of the polynomials

$$f_1, f_2, \cdots, f_{r_\alpha}$$

mentioned above. Gluing these analytical spaces X_α^{an}, we will obtain an analytical space X^{an}, called the complex analytical space of X. Likewise, we have a complex space for a projective scheme over a number field of finite type. It has been seen that such a process has several functorial properties with respect to X (see [5, 12]).

Hence, every affine covering of X produces a complex analytical space X^{an} of X.

However, in general, an algebraic scheme X can have many affine coverings. Then what about the complex analytical spaces of X produced by different affine coverings of X?

1.1.3 Different affine coverings can produce different complex analytical spaces

Let us take an example raised by Serre [13]:

Let V be the nonsingular projective variety over a number field K as defined in [13]. Suppose that V_ϕ is a conjugate variety of V defined by an isomorphism ϕ. Then there is such an isomorphism ϕ that the complex analytic spaces V^{an} and V_ϕ^{an} are not of the same homotopy type.

Denote by A and A_ϕ the homogeneous coordinate rings of V and V_ϕ, respectively. Then we have two isomorphic projective schemes

$$X = \mathrm{Proj}\, A \quad \text{and} \quad X_\phi = \mathrm{Proj}\, A_\phi$$

with two different complex analytical spaces

$$X^{an} = V^{an} \quad \text{and} \quad X_\phi^{an} = V_\phi^{an}$$

respectively.

This shows that different affine coverings of the same projective scheme can produce different complex analytical spaces.

For this phenomenon, there are also some more examples arising from abelian varieties and Shimura varieties (see [2, 10]).

Now we come to a conclusion that there do exist evidences, such as related examples in [2, 5, 10, 13], that different affine coverings can produce different complex analytical spaces for a fixed algebraic scheme. For instance, for the case of algebraic schemes, consider the cones of the complex spaces, contained in the complex projective space, defined by the homogeneous rings mentioned above.

Why does there exist such a phenomena? It is an interesting problem. Such related topics will be discussed in our subsequent papers.

1.1.4 Several problems

In the above, it has been seen that different affine coverings of an algebraic scheme can produce different complex analytical spaces. Then it is natural for one to have several questions (similarly for projective schemes) such as the following:

(1) How can we give a restrictive definition that a family of affine schemes patch a scheme?

(2) Does there exist another family of affine schemes covering the fixed scheme and making it into a scheme?

(3) Given another family of affine schemes which cover the fixed scheme. Will we obtain the same scheme?

(4) Given a ringed space. How many families of affine schemes do patch it? How many schemes do there exist on the same underlying topological space?

(5) In particular, given an algebraic scheme X over a number field. There can be many families of affine schemes covering X. Each such a family produces an analytical space X^{an} of X. When are these analytical spaces X^{an} either diffeomorphic to each other or of the same homotopy types?

Several questions related to the above are in part discussed in the paper.

At the same time, affine structures have also been encountered by us during the discussions on a type of Galois covers of algebraic and arithmetic schemes, where such a scheme is said to be Galois closed if it has only one affine structure.

The Galois closed schemes have several nice properties with applications to class fields, for example, their Galois groups of rational fields are isomorphic to their groups of automorphisms (for instance, see [1]).

1.2 Techniques

As a counterpart, an affine structure on a scheme behaves exactly like a differential structure on a manifold.

1.2.1 A smooth manifold can have many differential structures

In a classical way, a differential manifold is a topological space covered by a family of open subsets in some Euclidean space, which is obtained by glueing such a family of open sets as patches. Under some technical conditions, such a family of open sets is called a differential structure on the manifold.

Nowadays, there have been many well-known facts about manifolds and their differential structures:

(1) There exist differential manifolds which have many differential structures on them and the differential structures produce many manifolds that are not diffeomorphic to each other, respectively (see [11]).

(2) There exist topological spaces that have no differential structures on them.

(3) There exist differential manifolds such that they will be of different properties if we establish different differential structures on the (same) underlying spaces.

1.2.2 Affine structure v.s. differential structure

Many approaches and skills in topology can be applied here to schemes. The techniques in the present paper, which we borrowed from differential topology, are thus not new to some certain degree.

1.2.3 Known related results on affine coverings

For affine coverings, there are several informal discussions, for example, see [7, 14], on how to patch a scheme, that is, how to glue a given family of affine schemes into a scheme; for a more abstract case of categories, there are fibered categories and groupoids (see [3, 5]) which can be used to discuss such coverings.

However, all those discussions involved in [3, 5, 7, 14] deal only with coverings, that is, a family of abstract objects over a fixed object.

1.2.4 Further results obtained in the paper

In the paper, we will have further discussions on such affine coverings and several related results will be obtained. We will introduce and discuss affine structures in a rigor and systematic manner, where an affine structure is an affine covering that is taken as maximal families of objects covering a given object and satisfying the certain properties.

In fact, the affine structures afford a platform to us to discuss the problems mentioned in Section 1.1. In particular, an algebraic scheme over $\mathbb{C}$ can have a unique associative analytical space if there exists only one affine structure on its underlying space.

Furthermore, in a rigor manner, a scheme is a locally ringed space with a specified affine structure on it. It follows that in such a case, an algebraic scheme over a number field can be associated exactly with a unique complex analytical space.

The main results obtained in the paper are that by the whole of affine structures on a space, it will be seen whether two spaces are homeomorphic and whether two schemes are isomorphic. In other words, the whole of affine structures on a space and a scheme, as local data, encode and reflect the global properties of the space and the scheme, respectively.

Such results can be applied to complex analytical spaces of algebraic schemes and arithmetic schemes.

1.3 Outline of the paper

At last, we give an outline of the paper. In Section 2, we use pseudogroups Γ of affine transformations to define an affine Γ-atlas on a topological space, which consists of a family of affine charts. An affine Γ-structure on a space is an affine Γ-atlas which is maximal. Our discussion can be regarded as an algebraic version of differential structures (see [6, 8]).

In Section 3, an affine Γ-structure on a space is said to be admissible if there is a sheaf on the space such that they are coincide with each other on each affine chart. Here, such a sheaf is called an extension of the given affine structure.

An affine structure which is not admissible will be of no practical use.

Given a scheme $(X, \mathcal{O}_X)$ in the usual manner (see [4, 7]). In Section 4, we discuss the special types of affine structures on the space X, called the canonical and the relative canonical affine structures in the scheme $(X, \mathcal{O}_X)$ respectively. Their extensions are called the associate schemes of $(X, \mathcal{O}_X)$.

Every scheme has an associate scheme. In particular, a scheme itself is an associate scheme of it. As schemes, a fixed scheme and their associate schemes are isomorphic with each other.

Now put

(1) $\mathbb{A}(X) \triangleq$ the set of all admissible affine structures on a topological space X;

(2) $\mathbb{A}_0(X, \mathcal{O}_X) \triangleq$ the set of all the relative canonical affine structures in a scheme $(X, \mathcal{O}_X)$.

Using the set of affine structures on a space, in Section 5, we give the statements of the two main theorems, that is, Theorems 5.1 and 5.2.

The two theorems are proved in Sections 7 and 8, respectively.

As a conclusion, in Section 6, we give several concluding remarks. Particularly, we come to a conclusion that to be precisely defined, a scheme should be a locally ringed space together with a given admissible affine structure on it if the affine structures in deed make a contribution to a particular case.

2 Definitions for Affine Structures

In this Section, we introduce affine structures on a space in a evident manner as one does for differential structures on a space (for instance, see [6, 8]).

2.1 Pseudogroup of affine transformations

Let $\mathfrak{Comm}$ be the category of commutative rings with identities, and $\mathfrak{Comm}/k$ the category of finitely generated algebras over a field k. Here, a pseudogroup (or groupoid) is a small category in which every morphism is invertible (see [9]).

Definition 2.1 *A pseudogroup Γ of affine transformations, as a subcategory of $\mathfrak{Comm}$, is a pseudogroup of isomorphisms between commutative rings satisfying the conditions* (1)–(5) :

(1) *Each $\sigma \in \Gamma$ is an isomorphism from a ring* $\mathrm{dom}(\sigma)$ *onto a ring* $\mathrm{rang}(\sigma)$ *contained in* Γ, *called the domain and range of σ, respectively.*

(2) *If $\sigma \in \Gamma$, the inverse σ^{-1} is contained in Γ.*

(3) *The identity map* id_A *on A is contained in Γ if there is some $\delta \in \Gamma$ with* $\mathrm{dom}(\delta) = A$.

(4) *If $\sigma \in \Gamma$, the isomorphism induced by σ defined on the localization* $\mathrm{dom}(\sigma)_f$ *of the ring* $\mathrm{dom}(\sigma)$ *at any* $0 \neq f \in \mathrm{dom}(\sigma)$ *is contained in* Γ.

(5) *Given any $\sigma, \delta \in \Gamma$. Then the isomorphism factorized by* $\mathrm{dom}(\tau)$ *from* $\mathrm{dom}(\sigma)_f$ *onto* $\mathrm{rang}(\delta)_g$ *is contained in Γ if for some $\tau \in \Gamma$ there are isomorphisms* $\mathrm{dom}(\tau) \cong \mathrm{dom}(\sigma)_f$ *and* $\mathrm{dom}(\tau) \cong \mathrm{rang}(\delta)_g$ *with* $0 \neq f \in \mathrm{dom}(\sigma)$ *and* $0 \neq g \in \mathrm{rang}(\delta)$.

Such a pseudogroup Γ is said to be a pseudogroup of k-affine transformation if Γ is contained in the category $\mathfrak{Comm}/k$, or equivalently, if each isomorphism in Γ is an isomorphism of finitely generated algebras over a field k.

2.2 Affine charts and affine atlas

For a topological space, we give the notions of affine charts and affine atlas.

Definition 2.2 *Let X be a topological space and Γ a pseudogroup of affine transformations. Then an affine Γ-atlas $\mathcal{A}(X,\Gamma)$ on X is a collection of pairs (U_j,φ_j) with $j\in\Delta$, called affine charts, satisfying the conditions* (1)–(3) :

(1) *For every pair $(U_j,\varphi_j)\in\mathcal{A}(X,\Gamma)$, U_j is an open subset of X and φ_j is a homeomorphism of U_j onto $\mathrm{Spec}(A_j)$, where A_j is a commutative ring contained in Γ.*

(2) $\bigcup_{j\in\Delta} U_j\supseteq X$ *is an open covering of X.*

(3) *Given any $(U_i,\varphi_i),(U_j,\varphi_j)\in\mathcal{A}(X,\Gamma)$ with $U_i\cap U_j\neq\varnothing$. There exists a pair $(W_{ij},\varphi_{ij})\in\mathcal{A}(X,\Gamma)$ such that $W_{ij}\subseteq U_i\cap U_j$ and that the isomorphism from the localization $(A_j)_{f_j}$ onto the localization $(A_i)_{f_i}$ that is induced by the restriction*

$$\varphi_j\circ\varphi_i^{-1}\mid_{W_{ij}}:\varphi_i(W_{ij})\to\varphi_j(W_{ij})$$

is also contained in Γ. Here A_i and A_j are commutative rings contained in Γ such that

$$\varphi_i(U_i)=\mathrm{Spec}A_i\quad\textit{and}\quad\varphi_j(U_j)=\mathrm{Spec}A_j$$

hold and that there are homeomorphisms

$$\varphi_i(W_{ij})\cong\mathrm{Spec}(A_i)_{f_i}\quad\textit{and}\quad\varphi_j(W_{ij})\cong\mathrm{Spec}(A_j)_{f_j}$$

for some $f_i\in A_i$ and $f_j\in A_j$.

Moreover, $\mathcal{A}(X,\Gamma)$ is said to be a k-affine Γ-atlas on X if Γ is a subcategory of $\mathfrak{Comm}/k$.

An affine Γ-atlas $\mathcal{A}(X,\Gamma)$ on X is said to be complete (*or maximal*) *if it can not be contained properly in any other affine Γ-atlas of X.*

Remark 2.1 The above construction in Definition 2.2 is well-defined since the open covering $\{U_j\}$ such that $(U_j,\varphi_j)\in\mathcal{A}(X,\Gamma)$ is a base for the topology on X.

Let $\mathcal{A}(X,\Gamma)$ and $\mathcal{A}(X,\Gamma')$ be atlases on a space X. Then $\Gamma\supseteq\Gamma'$ holds if $\mathcal{A}(X,\Gamma)\supseteq\mathcal{A}(X,\Gamma')$.

2.3 Affine structures

As one has differential structures on a manifold, here we have affine structures on a space such as the following.

Definition 2.3 *Let X be a topological space and Γ a pseudogroup of affine transformations. Then two affine Γ-atlases $\mathcal{A}$ and $\mathcal{A}'$ on X are said to be Γ-compatible if the condition below is satisfied:*

For any $(U,\varphi)\in\mathcal{A}$ and $(U',\varphi')\in\mathcal{A}'$ with $U\cap U'\neq\varnothing$ there exists an affine chart $(W,\varphi'')\in\mathcal{A}\cap\mathcal{A}'$ such that $W\subseteq U\cap U'$ and that the isomorphism from the localization $(A)_f$ onto the localization $(A')_{f'}$ induced by the restriction $\varphi'\circ\varphi^{-1}\mid_W$ is also contained in Γ. Here A and A' are commutative rings contained in Γ such that $\varphi(U)=\mathrm{Spec}A$ and $\varphi'(U')=\mathrm{Spec}A'$ hold and that there are homeomorphisms $\varphi(W)\cong\mathrm{Spec}(A)_f$ and $\varphi'(W)\cong\mathrm{Spec}(A')_{f'}$ for some $f\in A$ and $f'\in A'$.

Proposition 2.1 *Let X be a topological space and Γ be a pseudogroup of affine transformations. Then for any given affine Γ-atlas $\mathcal{A}$ on X, there is a unique complete affine Γ-atlas $\mathcal{A}_m$ on X such that*

(1) $\mathcal{A}\subseteq\mathcal{A}_m$;

(2) *$\mathcal{A}$ and $\mathcal{A}_m$ are Γ-compatible.*

In such a case, we say that $\mathcal{A}$ is a base for $\mathcal{A}_m$ and $\mathcal{A}_m$ is the complete affine Γ-atlas determined by $\mathcal{A}$.

Proof Prove the existence. Let Σ be the collection of affine Γ-atlases $\mathcal{A}_\alpha$ on X such that $\mathcal{A} \subseteq \mathcal{A}_\alpha$ and that $\mathcal{A}$ and $\mathcal{A}_\alpha$ are Γ-compatible.

Then Σ is a partially ordered set together with the inclusions of sets $\mathcal{A}_\alpha \subseteq \mathcal{A}_\beta$ for any $\mathcal{A}_\alpha, \mathcal{A}_\beta \in \Sigma$. It is clear that every totally ordered subset of Σ has a upper bound in Σ. By Zorn's Lemma, Σ has maximal elements.

Prove the uniqueness. Let $\mathcal{A}_m$ and $\mathcal{A}'_m$ be two maximal elements of Σ. Then we must have $\mathcal{A}_m = \mathcal{A}'_m$. Otherwise, hypothesize $\mathcal{A}_m \neq \mathcal{A}'_m$. It is seen that $\mathcal{A}_m$ and $\mathcal{A}'_m$ are Γ-compatible since they are Γ-compatible respectively with $\mathcal{A}$. Then the union $\mathcal{A}_m \cup \mathcal{A}'_m$ is contained in Σ, where we obtain a contradiction.

Definition 2.4 *Let X be a topological space. An affine Γ-structure on X is a complete affine Γ-atlas $\mathcal{A}(\Gamma)$ on X, where Γ is a given pseudogroup of affine transformations.*

Likewise, we define a k-affine Γ-structure if $\Gamma \subseteq \mathfrak{Comm}/k$.

3 Admissible Affine Structures

By Proposition 2.1, it is seen that an affine atlas on a topological space X determines a unique affine structure on it. From this view of point, we sometimes identify an affine atlas on X with its determined complete affine structure on X. In this section, we discuss admissible affine structures on a space. On a given space, only admissible affine structures are interesting and are of the practical uses.

Definition 3.1 *Let $\mathcal{A}(\Gamma)$ be an affine Γ-structure on a topological space X. Suppose that there exists a locally ringed space $(X, \mathcal{F})$ such that $\varphi_{\alpha *}\mathcal{F} \mid_{U_\alpha} (\mathrm{Spec} A_\alpha) = A_\alpha$ holds for each $(U_\alpha, \varphi_\alpha) \in \mathcal{A}(\Gamma)$, where A_α is a commutative ring contained in Γ with $\varphi_\alpha(U_\alpha) = \mathrm{Spec} A_\alpha$.*

Then $\mathcal{A}(\Gamma)$ is said to be an admissible affine structure on X and $(X, \mathcal{F})$ is said to be an extension of the affine Γ-structure $\mathcal{A}(\Gamma)$.

Proposition 3.1 *All extensions of an admissible affine structure on a topological space are schemes which are isomorphic with each other.*

Proof Let $\mathcal{A}$ be an admissible affine structure on a topological space X. It is evident that each extension of $\mathcal{A}$ on X is a scheme.

Now fixed any extensions $(X, \mathcal{F})$ and $(X, \mathcal{G})$ of $\mathcal{A}$ on X. We prove $\mathcal{F} \cong \mathcal{G}$.

In deed, let U_α be an open subset of X contained in $\mathcal{A}$. From the assumption, we have

$$\Gamma(\mathcal{F}, U_\alpha) = \Gamma(\mathcal{G}, U_\alpha).$$

Take any open subset U of X. We have

$$U = \bigcup_\alpha U_\alpha$$

with $U_\alpha \in \mathcal{A}$. Define a map

$$\phi : \Gamma(\mathcal{F}, U) = \Gamma(\mathcal{G}, U), \quad t \mapsto \phi(t),$$

where $\phi(t) \in \Gamma(\mathcal{G}, U)$ is the section on U determined by

$$t \mid_{U_\alpha} = \phi(t) \mid_{U_\alpha}.$$

Then ϕ is an isomorphism for every open subset U of X.

By ϕ, we obtain an isomorphism $\mathcal{F}_x \cong \mathcal{G}_x$ at every $x \in X$, and hence $\mathcal{F} \cong \mathcal{G}$ holds.

Corollary 3.1 *For affine structures, there are the following statements:*

(1) *Let $(X, \mathcal{F})$ be an extension of the affine Γ-structure $\mathcal{A}(\Gamma)$ on a space X. Then we have*

$$(U, \mathcal{F} \mid_U) \cong (\mathrm{Spec} A, \widetilde{A}) \quad \textit{and} \quad \mathcal{F} \mid_U = \varphi_*^{-1} \widetilde{A}$$

for every affine chart $(U, \varphi) \in \mathcal{A}(\Gamma)$ with $\varphi(U) = \mathrm{Spec}(A)$, where A is a commutative ring contained in Γ.

(2) *An affine structure $\mathcal{A}$ on a space X is admissible if and only if $\mathcal{A}$ can be extended to be a sheaf $\mathcal{F}$ on X such that $(X, \mathcal{F})$ is a locally ringed space.*

Proof It is immediate from Definition 3.1 and Proposition 3.1.

4 Canonical Affine Structures

In this section, it is seen that for a given scheme there can be many different admissible affine structures on the underlying topological space of the scheme. That is, a given scheme can have many associate schemes. All associate schemes of a given scheme are isomorphic as schemes but have different affine structures so that their complex analytical spaces can be very different for the case of algebraic schemes over a number field.

4.1 Canonical pseudogroups of affine transformations

To start with, let us consider an example.

Example 4.1 (Different Affine Structures) Let k be a field.

(1) Put

$$\begin{aligned}
\Gamma_1 &= \{\text{the identity } 1_k : k \rightarrow k\}, \\
\Gamma_2 &= \{\text{the identity } 1_k : k \rightarrow k\} \cup \{\text{a field isomorphism } \sigma : k \rightarrow k'\} \\
&\quad \cup \{\text{the inverse } \sigma^{-1} : k' \rightarrow k\}.
\end{aligned}$$

Then Γ_1 and Γ_2 are both pseudogroups of affine transformations.

(2) Let

$$\mathcal{A}(\Gamma_1) = \{(U, \varphi)\}, \quad \mathcal{A}(\Gamma_2) = \{(U, \varphi), (V, \eta)\},$$

where $U = V = \mathrm{Spec}\,(k)$, $\varphi(U) = \mathrm{Spec}\,(k)$ and $\eta(V) = \mathrm{Spec}\,(k')$. Then $\mathrm{Spec}\,(k)$ is an extension of the affine structure $\mathcal{A}(\Gamma_1)$. In general, it is not true that Spec(k) is an extension of $\mathcal{A}(\Gamma_2)$. For example, let $\sqrt[3]{2}, \xi, \overline{\xi}$ be the roots of the equation $X^3 - 2 = 0$ in $\mathbb{C}$. Consider $k = \mathbb{Q}(\sqrt[3]{2})$, $k' = \mathbb{Q}(\xi)$.

Definition 4.1 *Let $(X, \mathcal{O}_X)$ be a scheme. Denote by Γ_0 (respectively, $\Gamma^{\max}$) the union of the set of some (respectively, all) identities of commutative rings*

$$\mathrm{id}_{A_\alpha} : A_\alpha \to A_\alpha$$

and the set of some (respectively, all) isomorphisms of commutative rings

$$\sigma_{\alpha\beta} : (A_\alpha)_{f_\alpha} \to (A_\beta)_{f_\beta},$$

satisfying the conditions (1)–(2) :

(1) *Each $A_\alpha, A_\beta, A_\gamma \in \mathfrak{Comm}$ are commutative rings such that there are affine open subsets*

$$U_\alpha, U_\beta \quad \textit{and} \quad U_\gamma \subseteq U_\alpha \cap U_\beta$$

of X satisfying the conditions

$$\varphi_\alpha(U_\alpha) = \mathrm{Spec} A_\alpha, \quad \varphi_\beta(U_\beta) = \mathrm{Spec} A_\beta \quad \textit{and} \quad \varphi_\gamma(U_\gamma) = \mathrm{Spec} A_\gamma.$$

(2) *Each $\sigma_{\alpha\beta} : (A_\alpha)_{f_\alpha} \to (A_\beta)_{f_\beta}$ is induced from the homeomorphism*

$$\varphi_\alpha \circ \varphi_\beta^{-1} |_{U_\gamma} : \varphi_\beta(U_\gamma) \to \varphi_\alpha(U_\gamma)$$

such that

$$\varphi_\alpha(U_\gamma) \cong \mathrm{Spec}(A_\alpha)_{f_\alpha} \quad \textit{and} \quad \varphi_\beta(U_\gamma) \cong \mathrm{Spec}(A_\beta)_{f_\beta}$$

hold for some $f_\alpha \in A_\alpha$ and $f_\beta \in A_\beta$.

Then the pseudogroup generated by Γ_0 in $\mathfrak{Comm}$, denoted by $\Gamma_{X,\mathcal{O}_X}$, which is the smallest pseudogroup containing Γ_0 in $\mathfrak{Comm}$, is called a pseudogroup of affine transformations in $(X, \mathcal{O}_X)$.

The pseudogroup generated by $\Gamma^{\max}$ in $\mathfrak{Comm}$, denoted by $\Gamma^{\max}_{X,\mathcal{O}_X}$, is called the maximal pseudogroup of affine transformations in $(X, \mathcal{O}_X)$.

For any given $\Gamma_{X,\mathcal{O}_X}$, define

$$\mathcal{A}^*(\Gamma_{X,\mathcal{O}_X}) = \{(U_\alpha, \varphi_\alpha) : \varphi_\alpha(U_\alpha) = \mathrm{Spec} A_\alpha \textit{ and } A_\alpha \in \Gamma_{X,\mathcal{O}_X}\},$$

where each U_α is an affine open subset in the scheme X.

Definition 4.2 *Let $(X, \mathcal{O}_X)$ be a scheme. Given such a pseudogroup $\Gamma_{X,\mathcal{O}_X}$ in $(X, \mathcal{O}_X)$. Suppose that $\mathcal{A}^*(\Gamma_{X,\mathcal{O}_X})$ is an affine $\Gamma_{X,\mathcal{O}_X}$-atlas on the space X. Then $\Gamma_{X,\mathcal{O}_X}$ is said to be a canonical pseudogroup of affine transformations in the scheme $(X, \mathcal{O}_X)$ and $\mathcal{A}^*(\Gamma_{X,\mathcal{O}_X})$ is said to be an affine atlas in the scheme $(X, \mathcal{O}_X)$.*

It is immediate that $\Gamma_{X,\mathcal{O}_X}$ is a sub-pseudogroup of $\Gamma^{\max}_{X,\mathcal{O}_X}$. There can be many canonical pseudogroups of affine transformations in the scheme $(X, \mathcal{O}_X)$. By Zorn's Lemma, it is seen that $\Gamma^{\max}_{X,\mathcal{O}_X}$ is maximal among these pseudogroups.

Take an example. Let $X = \mathrm{Spec}\,(\mathbb{Z})$ and Y be the disjoint union of X. Then there are three canonical pseudogroups of affine transformations in the scheme Y, which are generated respectively by $\mathbb{Z}$ and its localisations, by $\mathbb{Z} \oplus \mathbb{Z}$ and its localisations, and by $\mathbb{Z}$ and $\mathbb{Z} \oplus \mathbb{Z}$ and their localisations.

4.2 Canonical affine structures

In general, the underlying space of a scheme can have many affine structures on it.

Definition 4.3 *Let Γ be a canonical pseudogroup of affine transformations in a scheme $(X, \mathcal{O}_X)$.*

An affine Γ-atlas $\mathcal{A}$ on the space X is said to be a canonical affine structure in the scheme $(X, \mathcal{O}_X)$ if $\mathcal{A}$ is the affine Γ-structure on X determined by the affine Γ-atlas $\mathcal{A}^*(\Gamma)$.

An affine Γ-atlas $\mathcal{A}$ on the space X is said to be a relative canonical affine structure in the scheme $(X, \mathcal{O}_X)$ if $\mathcal{A}$ is maximal among all the affine Γ-atlases in $(X, \mathcal{O}_X)$ which contain the affine Γ-atlas $\mathcal{A}^*(\Gamma)$ and are Γ-compatible.

A scheme is said to have a unique (respectively, relative) canonical affine structure if there exists only one (respectively, relative) canonical affine structure in it.

Proposition 4.1 *Let Γ be the maximal pseudogroup of affine transformations in a scheme $(X, \mathcal{O}_X)$. Then $\mathcal{A}^*(\Gamma)$ is a relative canonical affine Γ-structure in $(X, \mathcal{O}_X)$.*

Proof Prove that $\mathcal{A}^*(\Gamma)$ is a Γ-atlas of the space X. In fact, it is clear that $\mathcal{A}^*(\Gamma)$ is Γ-compatible. Hence, it suffices to prove that $\mathcal{A}^*(\Gamma)$ affords us a base for the topology on the space X.

Fixed any point $x \in X$. Take any affine open subset $U \ni x$ in $(X, \mathcal{O}_X)$ such that there is an isomorphism

$$(\varphi, \widetilde{\varphi}) : (U, \mathcal{O}_X \mid_U) \cong (\mathrm{Spec}A, \widetilde{A}),$$

where $A \in \mathfrak{Comm}$.

There is some $f \in A$ such that A_f is contained in Γ. Hypothesize that $A_f \notin \Gamma$ holds for any $f \in A$. Then $\{(U, \varphi)\}$ and $\mathcal{A}^*(\Gamma)$ are Γ-compatible; it follows that $\{\mathrm{id}_A\} \cup \Gamma$ is a pseudogroup of affine transformations in $(X, \mathcal{O}_X)$, where $\mathrm{id}_A : A \to A$ is the identity map; hence, we have

$$\Gamma \subsetneqq \{\mathrm{id}_A\} \cup \Gamma,$$

which is in contradiction with the assumption.

Now take any $f \in A$ such that $A_f \in \Gamma$. Let $\mathrm{Spec}A$ be irreducible without loss of generality. We have

$$\mathrm{Spec}(A_f) \cong D(f) \subseteq \mathrm{Spec}A.$$

Then $\{(U, \varphi)\}$ and $\mathcal{A}^*(\Gamma)$ are Γ-compatible.

As U is an affine open subset in X, we have $(U, \varphi) \in \mathcal{A}^*(\Gamma)$; as Γ is maximal in $(X, \mathcal{O}_X)$, it is seen that A is contained in Γ.

This proves that for any $x \in X$ there is an affine chart $(U, \varphi) \in \mathcal{A}^*(\Gamma)$ such that $x \in U$.

Remark 4.1 Let $(X, \mathcal{O}_X)$ be a scheme. Then $\mathcal{A}^*(\Gamma_{X,\mathcal{O}_X}) \subseteq \mathcal{A}^*(\Gamma^m_{X,\mathcal{O}_X})$. In particular, each relative canonical affine $\Gamma_{X,\mathcal{O}_X}$-structure in $(X, \mathcal{O}_X)$ is contained in $\mathcal{A}^*(\Gamma^m_{X,\mathcal{O}_X})$. In general, there can be different relative canonical affine structures in a scheme.

Furthermore, we have the following conclusions:

Proposition 4.2 *Let $(X, \mathcal{O}_X)$ be a scheme. There are the following statements:*

(1) *Let Γ be a canonical pseudogroup of affine transformations in $(X, \mathcal{O}_X)$. Then there is a unique (respectively, relative) canonical affine Γ-structure in $(X, \mathcal{O}_X)$.*

Furthermore, given any affine open subset U in $(X, \mathcal{O}_X)$. Then U is contained in the canonical affine Γ-structure in $(X, \mathcal{O}_X)$ if and only if U is contained in the relative canonical affine Γ-structure in $(X, \mathcal{O}_X)$.

(2) *The scheme $(X, \mathcal{O}_X)$ has a unique affine structure if and only if $(X, \mathcal{O}_X)$ has a unique relative affine structure.*

Proof (1) It is immediate from definition.

(2) Assume that $(X, \mathcal{O}_X)$ has a unique affine structure. Hypothesize that $\mathcal{A}(\Gamma_1)$ and $\mathcal{A}(\Gamma_2)$ are two distinct relative canonical affine structures in $(X, \mathcal{O}_X)$ together with the canonical pseudogroups Γ_1 and Γ_2 respectively.

From Γ_1 and Γ_2, we obtain two canonical affine structures $\mathcal{B}(\Gamma_1)$ and $\mathcal{B}(\Gamma_2)$ in $(X, \mathcal{O}_X)$. Then $\mathcal{B}(\Gamma_1)$ and $\mathcal{B}(\Gamma_2)$ are neither Γ_1-compatible nor Γ_2-compatible. Otherwise, if they are Γ_1-compatible, by (1) it will be seen that $\mathcal{A}(\Gamma_1)$ and $\mathcal{A}(\Gamma_2)$ are Γ_1-compatible.

Hence, there are two distinct canonical affine structures in $(X, \mathcal{O}_X)$, which is a contradiction to the assumption.

Conversely, assume that $(X, \mathcal{O}_X)$ has a unique relative affine structure. If $(X, \mathcal{O}_X)$ has two distinct canonical affine structures $\mathcal{B}(\Gamma_1)$ and $\mathcal{B}(\Gamma_2)$, by (1) we obtain two relative canonical affine structures in $(X, \mathcal{O}_X)$ which are neither Γ_1-compatible nor Γ_2-compatible in virtue of the property of a base for the topology on X, which is a contradiction.

4.3 Associate schemes

In the following, it is seen that a scheme can have many associate schemes.

Proposition 4.3 *All (respectively, relative) canonical affine structures in a scheme $(X, \mathcal{O}_X)$ are admissible; moreover, their extensions are all isomorphic to $(X, \mathcal{O}_X)$ as schemes.*

Proof Let $\mathcal{A}^*(X, \mathcal{O}_X)$ be a (respectively, relative) canonical affine structure on X. Take any $(U_\alpha, \varphi_\alpha) \in \mathcal{A}^*(X, \mathcal{O}_X)$. There is the isomorphism

$$(\tau_\alpha, \widetilde{\tau}_\alpha) : (U_\alpha, \mathcal{O}_X \mid_{U_\alpha}) \cong (\mathrm{Spec} A_\alpha, \widetilde{A}_\alpha),$$

where $\varphi_\alpha(U_\alpha) = \tau_\alpha(U_\alpha),\ \widetilde{\tau}_\alpha(\widetilde{A}_\alpha) = \tau_{\alpha *}\mathcal{O}_X \mid_{U_\alpha}$.

This proves that the scheme $(X, \mathcal{O}_X)$ is at least an extension of $\mathcal{A}^*(X, \mathcal{O}_X)$. It follows that $\mathcal{A}^*(X, \mathcal{O}_X)$ is admissible.

Take any extension $(X, \mathcal{F})$ of $\mathcal{A}^*(X, \mathcal{O}_X)$. By considering sections, it is seen that $(X, \mathcal{F})$ and $(X, \mathcal{O}_X)$ are isomorphic schemes.

Definition 4.4 *An associate scheme of a given scheme $(X, \mathcal{O}_X)$ is an extension on the space X of a canonical affine structure or a relative canonical affine structure in $(X, \mathcal{O}_X)$.*

Remark 4.2 By Proposition 4.3, we have the following statements:

(1) Every scheme has an associate scheme. In particular, a scheme is an associate scheme of itself.

(2) All associate schemes of a given scheme are isomorphic as schemes. However, their complex analytical spaces can be very different for the case of algebraic schemes.

5 Statements of the Main Theorems

In this section, we give the statements of the main theorems in the paper.

5.1 Definitions and notations

Let us fix notations and terminology.

For a topological space X, put

(1) $\mathbb{A}(X) \triangleq$ the set of all admissible affine structures on the space X.

For a scheme $(X, \mathcal{O}_X)$, set

(2) $\mathbb{A}_0(X, \mathcal{O}_X) \triangleq$ the set of all the relative canonical affine structures in the scheme $(X, \mathcal{O}_X)$.

Likewise, we can define $\mathbb{A}(X; k)$ and $\mathbb{A}_0(X, \mathcal{O}_X; k)$ for k-affine structures.

Definition 5.1 *For two spaces X and Y, we say*

$$\mathbb{A}(X) \subseteq \mathbb{A}(Y)$$

if the below condition is satisfied:

Given any affine chart $(U_\alpha, \varphi_\alpha)$ contained in an affine structure $\mathcal{A}(X)$ belonging to $\mathbb{A}(X)$. There is an affine chart (V_α, ψ_α) contained in an affine structure $\mathcal{A}(Y)$ belonging to $\mathbb{A}(Y)$ such that $B_\alpha = A_\alpha$. Here $A_\alpha, B_\alpha \in \mathfrak{Comm}$, $\varphi_a(U_\alpha) = \mathrm{Spec}A_\alpha$, and $\psi_\alpha(V_\alpha) = \mathrm{Spec}B_\alpha$.

Definition 5.2 *For two spaces X and Y, we say*

$$\mathbb{A}(X) = \mathbb{A}(Y)$$

if there are relations $\mathbb{A}(X) \subseteq \mathbb{A}(Y)$ and $\mathbb{A}(X) \supseteq \mathbb{A}(Y)$.

Likewise, replacing admissible by relative canonical, we define

$$\mathbb{A}_0(X, \mathcal{O}_X) = \mathbb{A}_0(Y, \mathcal{O}_Y)$$

for two schemes $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$.

Definition 5.3 *For two spaces X and Y, we say*

$$\mathbb{A}(X) \trianglelefteq \mathbb{A}(Y)$$

if the below conditions (1)–(3) are satisfied:

(1) (*Local Isomorphism*) *Given any affine chart $(U_\alpha, \varphi_\alpha)$ contained in an affine structure $\mathcal{A}(X)$ belonging to $\mathbb{A}(X)$.*

Then there is an affine chart (V_α, ψ_α) contained in an affine structure $\mathcal{A}(Y)$ belonging to $\mathbb{A}(Y)$ such that A_α and B_α are isomorphic rings. Here $A_\alpha, B_\alpha \in \mathfrak{Comm}$, $\varphi_a(U_\alpha) = \mathrm{Spec}A_\alpha$, and $\psi_\alpha(V_\alpha) = \mathrm{Spec}B_\alpha$.

(2) (*Covering*) *Let $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in\Gamma}$ be a family of affine charts $(U_\alpha, \varphi_\alpha)$ contained in some affine structures $\mathcal{A}(\Gamma_\alpha)$ belonging to $\mathbb{A}(X)$ such that $\varphi_a(U_\alpha) = \mathrm{Spec}A_\alpha$ and $\bigcup_{\alpha\in\Gamma} U_\alpha \supseteq X$.*

Then $\bigcup_{i,\alpha} V_{i,\alpha} \supseteq Y$ holds, where $V_{i,\alpha}$ runs through all the affine charts $(V_{i,\alpha}, \psi_{i,\alpha})$ contained in any affine structures $\mathcal{A}(\Gamma_{i,\alpha})$ belonging to $\mathbb{A}(Y)$ such that $B_{i,\alpha} \cong A_\alpha$ and $\psi_{i,\alpha}(B_{i,\alpha}) = \mathrm{Spec}B_{i,\alpha}$.

(3) (*Filtering*) *Let* $(U_\alpha, \varphi_\alpha)$ *and* (U_β, φ_β) *be two affine charts contained in some affine structures* $\mathcal{A}(\Gamma_\alpha)$ *and* $\mathcal{A}(\Gamma_\beta)$ *belonging to* $\mathbb{A}(X)$ *respectively. Given any* $x_\alpha \in \mathrm{Spec}A_\alpha$ *and* $x_\beta \in \mathrm{Spec}A_\beta$ *with* $\varphi_\alpha^{-1}(x_\alpha) = \varphi_\beta^{-1}(x_\beta)$, *where* $\varphi_\alpha(U_\alpha) = \mathrm{Spec}A_\alpha$ *and* $\varphi_\beta(U_\beta) = \mathrm{Spec}A_\beta$.

Then there exist affine charts (V_α, ψ_α) *and* (V_β, ψ_β) *respectively contained in some affine structures* $\mathcal{A}'(\Gamma'_\alpha)$ *and* $\mathcal{A}'(\Gamma'_\beta)$ *belonging to* $\mathbb{A}(Y)$ *such that* $\psi_\alpha^{-1} \circ \sigma_\alpha(x_\alpha) = \psi_\beta^{-1} \circ \sigma_\beta(x_\beta)$ *holds and that there are ring isomorphisms* $\delta_\alpha : B_\alpha \cong A_\alpha$ *and* $\delta_\beta : B_\beta \cong A_\beta$, *where* $\psi_\alpha(V_\alpha) = \mathrm{Spec}B_\alpha$, $\psi_\beta(V_\beta) = \mathrm{Spec}B_\beta$, *and* $\sigma_\alpha : \mathrm{Spec}A_\alpha \to \mathrm{Spec}B_\alpha$ *and* $\sigma_\beta : \mathrm{Spec}A_\beta \to \mathrm{Spec}B_\beta$ *are the isomorphisms induced from* δ_α *and* δ_β, *respectively.*

Likewise, replacing admissible by relative canonical, we define

$$\mathbb{A}_0(X, \mathcal{O}_X) \trianglelefteq \mathbb{A}_0(Y, \mathcal{O}_Y)$$

for two schemes $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$.

Such an isomorphism $\delta_\alpha : A_\alpha \cong B_\alpha$ is called a deck transformation from X into Y.

Definition 5.4 *For two spaces* X *and* Y, *we say*

$$\mathbb{A}(X) \cong \mathbb{A}(Y)$$

if there are relations $\mathbb{A}(X) \trianglelefteq \mathbb{A}(Y)$ *and* $\mathbb{A}(X) \trianglerighteq \mathbb{A}(Y)$.

Likewise, replacing admissible by relative canonical, we define

$$\mathbb{A}_0(X, \mathcal{O}_X) \cong \mathbb{A}_0(Y, \mathcal{O}_Y)$$

for two schemes $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$.

5.2 Statements of the main theorems

Here there are the main theorems of the present paper.

Theorem 5.1 *Let* X *and* Y *be two topological spaces such that either* $\mathbb{A}(X) \neq \emptyset$ *or* $\mathbb{A}(Y) \neq \emptyset$ *holds. Then* X *and* Y *are homeomorphic if and only if there is*

$$\mathbb{A}(X) = \mathbb{A}(Y).$$

Theorem 5.2 *Any two schemes* $(X, \mathcal{O}_X)$ *and* $(Y, \mathcal{O}_Y)$ *are isomorphic if and only if we have*

$$\mathbb{A}_0(X, \mathcal{O}_X) \cong \mathbb{A}_0(Y, \mathcal{O}_Y).$$

We will prove Theorems 5.1 and 5.2 in Sections 7 and 8, respectively.

Remark 5.1 From the two main theorems above, it is seen that the whole of affine structures on a space and the underlying space of a scheme, as local data of the space, encode the global data of the space and the scheme, in particular, the global topology of the space and the scheme, respectively.

Remark 5.2 In Theorem 5.2, the condition

$$\mathbb{A}_0(X, \mathcal{O}_X) \cong \mathbb{A}_0(Y, \mathcal{O}_Y)$$

can not be replaced by

$$\mathbb{A}(X, \mathcal{O}_X) = \mathbb{A}(Y, \mathcal{O}_Y).$$

For example, consider $X = \text{Spec}\,(\mathbb{Q})$ and $Y = \text{Spec}\,(\mathbb{Q}(\sqrt{2}))$.

6 Concluding Remarks

From Remark 4.2 and Theorem 5.2, we have the following proposition, a comparison between two schemes of the same underlying space.

Proposition 6.1 *Let $(X, \mathcal{O}_X)$ and $(X, \mathcal{O}'_X)$ be two schemes. The following statements are equivalent:*

(1) $\mathbb{A}_0(X, \mathcal{O}_X) \cong \mathbb{A}_0(X, \mathcal{O}'_X)$ *holds,*

(2) $(X, \mathcal{O}_X)$ *and* $(X, \mathcal{O}'_X)$ *are isomorphic schemes,*

(3) *There is an isomorphism* $(X, \widehat{\mathcal{O}_X}) \cong (X, \widehat{\mathcal{O}'_X})$ *for any associate schemes* $(X, \widehat{\mathcal{O}_X})$ *of* $(X, \mathcal{O}_X)$ *and* $(X, \widehat{\mathcal{O}'_X})$ *of* $(X, \mathcal{O}'_X)$.

Example 6.1 Let K/k be a Galois extension. Then Spec$K$ has a unique associate scheme and there exists a unique admissible k-affine structure in the scheme Spec$K$.

Definition 6.1 *A scheme $(X, \mathcal{O}_X)$ is said to have a property P for an admissible affine structures $\mathcal{A}$ on X if as a scheme any extension $(X, \mathcal{O}_{\mathcal{A}(\Gamma)})$ of $\mathcal{A}$ has that property P.*

Remark 6.1 There are the following conclusions.

(1) Fixed an associate scheme $(X, \mathcal{O}_{\mathcal{A}})$ of a scheme $(X, \mathcal{O}_X)$. In general, it is not true that $(X, \mathcal{O}_{\mathcal{A}}) = (X, \mathcal{O}_X)$ although they are isomorphic.

(2) There exists a scheme $(X, \mathcal{O}_X)$ and an admissible affine structure $\mathcal{A}$ on the space X such that there is some property P that $(X, \mathcal{O}_X)$ holds but an extension $(X, \mathcal{O}_{\mathcal{A}})$ of $\mathcal{A}$ does not hold.

(3) In general, one says that a scheme $(X, \mathcal{O}_X)$ has a property P. But it is not specified that the property P holds for some certain or all the admissible affine structures on the space X.

This situation is very similar to that in differential topology. As usual, a differential manifold is said to have some property if the property holds for all the differential structures until such one is especially specified.

It has been known that there is some property P on some manifold X which does not hold for any other differential structures on X.

Remark 6.2 In a precise and rigor manner, a scheme is defined to be a ringed space together with a specified admissible affine structure on it if the affine structures make indeed a particular contribution to the case.

Remark 6.3 Let $(X, \mathcal{O}_X)$ be a scheme. For any $\mathcal{A}, \mathcal{B} \in \mathbb{A}(X)$, we say

$$\mathcal{A} \sim \mathcal{B}$$

if and only if there is an isomorphism $(X, \mathcal{O}_{\mathcal{A}}) \cong (X, \mathcal{O}_{\mathcal{B}})$.

Then the quotient set

$$\mathbb{A}(X)/\sim$$

is the whole of the schemes on the space X upon isomorphisms.

7 Proof of the First Main Theorem

In this section, we give the proof of Theorem 5.1.

Proof of Theorem 5.1 As $\mathbb{A}(X) \neq \emptyset$ and $\mathbb{A}(Y) \neq \emptyset$, we can choose two admissible affine structures on X and Y respectively. Fixed their extensions $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$, which form respectively two schemes.

"$\Rightarrow$" Prove that $\mathbb{A}(X) \subseteq \mathbb{A}(Y)$ holds.

In deed, take any affine chart $(U_\alpha, \varphi_\alpha)$ contained in an affine structure $\mathcal{A}(X)$ belonging to $\mathbb{A}(X)$, where $\varphi_a(U_\alpha) = \mathrm{Spec}A_\alpha$ and $A_\alpha \in \mathfrak{Comm}$.

Then $(\tau(U), \varphi_\alpha \circ \tau^{-1})$ is an affine chart contained in an affine structure

$$\mathcal{A}(\tau(X)) = \mathcal{A}(Y)$$

belonging to $\mathbb{A}(Y)$. Hence, we have

$$\mathbb{A}(X) \subseteq \mathbb{A}(Y).$$

Similarly, we have

$$\mathbb{A}(X) \supseteq \mathbb{A}(Y).$$

So

$$\mathbb{A}(X) = \mathbb{A}(Y).$$

"$\Leftarrow$" Let $\mathbb{A}(X) = \mathbb{A}(Y)$. We prove that there exists a homeomorphism

$$\tau : X \to Y.$$

We proceed in several steps.

(i) Take an affine chart $(U_\alpha, \varphi_\alpha)$ contained in an affine structure $\mathcal{A}(X)$ belonging to $\mathbb{A}(X)$, where $A_\alpha \in \mathfrak{Comm}$ is a commutative ring and

$$\varphi_a(U_\alpha) = \mathrm{Spec}A_\alpha.$$

Then there is an affine chart (V_α, ψ_α) contained in an affine structure $\mathcal{A}(Y)$ belonging to $\mathbb{A}(Y)$ such that

$$\psi_\alpha(V_\alpha) = \mathrm{Spec}A_\alpha.$$

The converse is true since we have $\mathbb{A}(X) = \mathbb{A}(Y)$.

Let Σ be the disjoint union of all such open sets $\mathrm{Spec}A_\alpha$. Take any points $x, y \in \Sigma$.

We say $x \sim_X y$ if there exist admissible affine structures $\mathcal{A}(\Gamma_\alpha)$ and $\mathcal{A}(\Gamma_\beta)$ contained in $\mathbb{A}(X)$ satisfying the condition:

There are affine charts $(U_\alpha, \varphi_\alpha) \in \mathcal{A}(\Gamma_\alpha)$, $(U_\beta, \varphi_\beta) \in \mathcal{A}(\Gamma_\beta)$ such that $\varphi_\alpha^{-1}(x) = \varphi_\beta^{-1}(y)$, where $x \in \mathrm{Spec}A_\alpha = \varphi_\alpha(U_\alpha)$ and $y \in \mathrm{Spec}A_\beta = \varphi_\beta(U_\beta)$.

Likewise, we say $x \sim_Y y$ if there exist admissible affine structures $\mathcal{A}(\Gamma_\alpha)$ and $\mathcal{A}(\Gamma_\beta)$ contained in $\mathbb{A}(Y)$ satisfying the condition:

There are affine charts $(V_\alpha, \psi_\alpha) \in \mathcal{A}(\Gamma_\alpha)$, $(V_\beta, \psi_\beta) \in \mathcal{A}(\Gamma_\beta)$ such that $\psi_\alpha^{-1}(x) = \psi_\beta^{-1}(y)$, where $x \in \mathrm{Spec}A_\alpha = \psi_\alpha(V_\alpha)$ and $y \in \mathrm{Spec}A_\beta = \psi_\beta(V_\beta)$.

(ii) Let Σ_X be the quotient of the set Σ by relation $\sim_X$, and

$$\pi_X : \Sigma \to \Sigma_X$$

be the canonical map.

Prove that there is a bijection ρ_X from the set Σ_X onto the set X.

In deed, we have a mapping

$$\rho : \Sigma \to X$$

given by

$$z \mapsto \varphi_\alpha^{-1}(z),$$

where $(U_\alpha, \varphi_\alpha)$ is the affine chart contained in an affine structure belonging to $\mathbb{A}(X)$ such that

$$z \in \mathrm{Spec} A_\alpha = \varphi_\alpha(U_\alpha).$$

Then we have a map

$$\rho_X : \Sigma_X \to X, \quad \pi_X(z) \mapsto \rho(z).$$

Evidently, ρ_X is a surjection. From the definition for $\sim_X$, it is easily seen that ρ_X is an injection. This proves that ρ_X is a bijection.

Hence, Σ_X is a topological space together with the topology on the space X.

Similarly, let Σ_Y be the quotient of the set Σ by relation $\sim_Y$, and

$$\pi_Y : \Sigma \to \Sigma_Y$$

be the canonical map. As

$$\mathbb{A}(X) = \mathbb{A}(Y),$$

it is clear that there is a bijection ρ_Y from the set Σ_Y onto the set Y. Then Σ_Y is a topological space together with the topology on the space Y.

(iii) Take any $x, y \in \Sigma$. Prove that $x \sim_X y$ holds if and only if $x \sim_Y y$ holds.

In deed, let $x \sim_X y$, that is, we have

$$\varphi_\alpha^{-1}(x) = \varphi_\beta^{-1}(y)$$

for some affine charts

$$(U_\alpha, \varphi_\alpha) \in \mathcal{A}(\Gamma_\alpha) \quad \text{and} \quad (U_\beta, \varphi_\beta) \in \mathcal{A}(\Gamma_\beta)$$

such that

$$x \in \mathrm{Spec} A_\alpha = \varphi_\alpha(U_\alpha) \quad \text{and} \quad y \in \mathrm{Spec} A_\beta = \varphi_\beta(U_\beta).$$

Let $\Gamma_{X,\mathcal{O}_X}^{\max}$ be the maximal pseudogroup of affine transformations in the scheme $(X, \mathcal{O}_X)$.

We choose the above open sets U_α and U_β to be affine open subsets of the scheme $(X, \mathcal{O}_X)$. That is, U_α and U_β are contained in the pseudogroup $\Gamma_{X,\mathcal{O}_X}^{\max}$.

Then there is an affine open subset $U_{\alpha\beta}$ of $(X, \mathcal{O}_X)$ contained in $\Gamma_{X,\mathcal{O}_X}^{\max}$ such that

$$\varphi_\alpha^{-1}(x) \in U_{\alpha\beta} \subseteq U_\alpha \cap U_\beta, \quad \varphi_\alpha(U_{\alpha\beta}) = \mathrm{Spec}\,(A_\alpha)_{f_\alpha}, \quad \varphi_\beta(U_{\alpha\beta}) = \mathrm{Spec}\,(A_\beta)_{f_\beta}$$

for some $f_\alpha \in A_\alpha$ and $f_\beta \in A_\beta$, where the isomorphism $\sigma_{\alpha\beta}$ from $(A_\alpha)_{f_\alpha}$ onto $(A_\beta)_{f_\beta}$ is contained in $\Gamma^{\max}_{X,\mathcal{O}_X}$.

As

$$\mathbb{A}(X) = \mathbb{A}(Y),$$

we have affine charts (V_α, ψ_α) and (V_β, ψ_β) respectively contained in some affine structures belonging to $\mathbb{A}(Y)$, where $V_\alpha = \psi_\alpha^{-1}(\mathrm{Spec}A_\alpha)$ and $V_\beta = \psi_\beta^{-1}(\mathrm{Spec}A_\beta)$.

Set

$$V_{\alpha\beta} = \psi_\alpha^{-1}(\mathrm{Spec}\,(A_\alpha)_{f_\alpha}), \quad V_{\beta\alpha} = \psi_\beta^{-1}(\mathrm{Spec}\,(A_\beta)_{f_\beta}).$$

Denote by $\psi_{\beta\alpha}$ the homeomorphism of $V_{\beta\alpha}$ onto $V_{\alpha\beta}$ which is induced from $\sigma_{\alpha\beta}$.

It is easily seen that $(V_{\alpha\beta}, \psi_\beta \circ \psi_{\beta\alpha}^{-1})$ is an affine chart contained in some admissible affine structure belonging to $\mathbb{A}(Y)$. In fact, fixed any admissible affine Γ_0-structure $\mathcal{A}(\Gamma_0)$ on the space Y which contains the affine chart (V_α, ψ_α). Let Γ_1 be the pseudogroup of affine transformations in $\mathfrak{Comm}$ generated by the union of Γ_0 and the set of the identity on $(A_\beta)_{f_\beta}$ and all the possible isomorphisms between the localisations of the rings. Then $\{(V_{\alpha\beta}, \psi_\beta \circ \psi_{\beta\alpha}^{-1})\}$ and $\mathcal{A}(\Gamma_1)$ are Γ_1-compatible. Hence,

$$(V_{\alpha\beta}, \psi_\beta \circ \psi_{\beta\alpha}^{-1}) \in \mathcal{A}(\Gamma_1).$$

Consider

$$y \in V_{\beta\alpha}, \quad x \in \mathrm{Spec}\,(A_\alpha)_{f_\alpha} \subseteq \mathrm{Spec}A_\alpha.$$

It is evident that $\psi_\alpha^{-1}(x) = \psi_{\beta\alpha} \circ \psi_\beta^{-1}(y)$ holds since $\psi_\alpha^{-1}(\mathrm{Spec}\,(A_\alpha)_{f_\alpha}) = \psi_{\beta\alpha} \circ \psi_\beta^{-1}(\mathrm{Spec}\,(A_\beta)_{f_\beta})$ $= V_{\alpha\beta}$. This proves that $x \sim_Y y$ holds.

In a similar manner, it is seen that the converse is true.

(iv) The map from Σ_X into Σ_Y defined by

$$\pi_X(z) \longmapsto \pi_Y(z)$$

for $z \in \Sigma$ gives us a bijection

$$\tau : X \longrightarrow Y,$$

which is well-defined from (iii).

All the open sets $\mathrm{Spec}A_\alpha$ determine a topology on the set Σ in such a manner:

A subset W of Σ is open if and only if $\pi_X(W)$ is open in Σ_X.

It follows that Σ_X is the quotient space of Σ by π_X. As $\mathbb{A}(X) = \mathbb{A}(Y)$, Σ_Y is the quotient space of Σ by π_Y. Hence,

$$\tau : X \to Y$$

is a homeomorphism.

This completes the proof.

8 Proof of the Second Main Theorem

In this section, we give the proof of Theorem 5.2.

Proof of Theorem 5.2 "$\Rightarrow$" Let

$$\tau : (X, \mathcal{O}_X) \cong (Y, \mathcal{O}_Y)$$

be an isomorphism.

As $\tau_*\mathcal{O}_X \cong \mathcal{O}_Y$, we have

$$\begin{aligned}(\varphi_\alpha^{-1}(\mathrm{Spec}A_\alpha), (\varphi_\alpha^{-1})_*\widetilde{A}_\alpha) &\cong (U_\alpha, \mathcal{O}_X\mid_{U_\alpha}) \\ &\cong (\tau(U_\alpha), \tau_*\mathcal{O}_X\mid_{U_\alpha}) \\ &\cong (\tau(U_\alpha), \mathcal{O}_Y\mid_{\tau(U_\alpha)}) \\ &\cong (\psi_\alpha^{-1}(\mathrm{Spec}B_\alpha), (\psi_\alpha^{-1})_*\widetilde{B}_\alpha)\end{aligned}$$

for any affine open set U_α of X such that

$$\varphi_\alpha(U_\alpha) = \mathrm{Spec}A_\alpha$$

and $(U_\alpha, \varphi_\alpha)$ is contained in some canonical affine structure belonging to $\mathbb{A}_0(X, \mathcal{O}_X)$.

Then $(\tau(U_\alpha), \varphi_\alpha)$ is an affine chart with

$$\psi_\alpha(\tau(U_\alpha)) = \mathrm{Spec}B_\alpha,$$

which is contained in some canonical affine structure belonging to $\mathbb{A}_0(Y, \mathcal{O}_Y)$.

By the isomorphism τ, it is easily seen that the conditions (1)–(2) in Definition 5.3 are satisfied.

Now let Γ_X and Γ_Y be the maximal pseudogroups of affine transformations in the schemes $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$, respectively. Via the isomorphism τ, every affine chart in $\mathcal{A}^*(\Gamma_X)$ is an affine chart in the $\mathcal{A}^*(\Gamma_Y)$; the converse is true.

Using the same procedure in proving Theorem 5.1, we can prove such a claim that "$x \sim_X y$" for X holds if and only if "$x \sim_Y y$" for Y holds.

It follows that the condition (3) in Definition 5.3 is satisfied. Hence, we have

$$\mathbb{A}_0(X, \mathcal{O}_X) \trianglelefteq \mathbb{A}_0(Y, \mathcal{O}_Y).$$

Similarly, we prove

$$\mathbb{A}_0(X, \mathcal{O}_X) \trianglerighteq \mathbb{A}_0(Y, \mathcal{O}_Y).$$

This proves

$$\mathbb{A}_0(X, \mathcal{O}_X) \cong \mathbb{A}_0(Y, \mathcal{O}_Y).$$

"$\Rightarrow$" Put

$$\mathbb{A}_0(X, \mathcal{O}_X) \cong \mathbb{A}_0(Y, \mathcal{O}_Y).$$

We prove that there exists an isomorphism from $(X, \mathcal{O}_X)$ onto $(Y, \mathcal{O}_Y)$.

In the following, we proceed in several steps in a manner similar to the procedure in proving Theorem 5.1.

(i) Let Γ_Y be the maximal pseudogroup of affine transformations in the scheme $(Y, \mathcal{O}_Y)$. We obtain a relative canonical affine Γ_Y-structure

$$\mathcal{A}(\Gamma_Y) \triangleq \mathcal{A}^*(\Gamma_Y), \quad \text{on } Y.$$

Then V_α is an affine open set in the scheme $(Y, \mathcal{O}_Y)$ for every (V_α, ψ_α) contained in $\mathcal{A}(\Gamma_Y)$.

For each $(V_\alpha, \psi_\alpha) \in \mathcal{A}(\Gamma_Y)$, we put

$$\psi_\alpha(V_\alpha) = \mathrm{Spec} B_\alpha,$$

where B_α is a commutative ring contained in the pseudogroup Γ_Y.

From Definition 4.1, we have

$$\mathcal{A}(\Gamma_Y) \supseteq \mathcal{A}^*(\Gamma_Y).$$

Then

$$\bigcup_{(V_\alpha, \psi_\alpha) \in \mathcal{A}(\Gamma_Y)} V_\alpha \supseteq Y.$$

Let Σ be the disjoint union of all the open sets $\mathrm{Spec} B_\alpha$ such that

$$\psi_\alpha(V_\alpha) = \mathrm{Spec} B_\alpha \quad \text{and} \quad (V_\alpha, \psi_\alpha) \in \mathcal{A}(\Gamma_Y).$$

Take any points $x, y \in \Sigma$.

We say $x \sim_Y y$ if there are affine charts $(V_\alpha, \psi_\alpha), (V_\beta, \psi_\beta) \in \mathcal{A}(\Gamma_Y)$ such that

$$\psi_\alpha^{-1}(x) = \psi_\beta^{-1}(y),$$

where

$$x \in \mathrm{Spec} B_\alpha = \psi_\alpha(V_\alpha), \quad y \in \mathrm{Spec} B_\beta = \psi_\beta(V_\beta).$$

(ii) For each $(V_\alpha, \psi_\alpha) \in \mathcal{A}(\Gamma_Y)$, define $\{(U_{i,\alpha}, \varphi_{i,\alpha})\}_{i \in I_\alpha}$ to be the set of all the affine charts contained in each relative canonical affine structures in the scheme $(X, \mathcal{O}_X)$ such that

$$\varphi_{i,\alpha}(U_{i,\alpha}) = \mathrm{Spec} A_{i,\alpha}$$

and that there is an isomorphism

$$\delta_{i,\alpha} : A_{i,\alpha} \cong B_\alpha.$$

Denote by Δ_X the set of all such affine charts $(U_{i,\alpha}, \varphi_{i,\alpha})$, where $i \in I_\alpha$ and $(V_\alpha, \psi_\alpha) \in \mathcal{A}(\Gamma_Y)$.

As $\mathbb{A}_0(Y, \mathcal{O}_Y) \trianglelefteq \mathbb{A}_0(X, \mathcal{O}_X)$, we have

$$\bigcup_{(U_{i,\alpha}, \varphi_{i,\alpha}) \in \Delta_X} U_{i,\alpha} \supseteq X.$$

Let Σ^* be the disjoint union of all the open sets $\mathrm{Spec} A_{i,\alpha}$ such that

$$A_{i,\alpha} \cong B_\alpha \quad \text{and} \quad (U_{i,\alpha}, \varphi_{i,\alpha}) \in \Delta_X.$$

Take any $x, y \in \Sigma^*$.

We say $x \sim_X y$ if there are affine charts $(U_{i,\alpha}, \varphi_{i,\alpha}), (U_{j,\beta}, \varphi_{j,\beta}) \in \Delta_X$ such that

$$\varphi_{i,\alpha}^{-1}(x) = \varphi_{j,\beta}^{-1}(y)$$

holds, where

$$x \in \mathrm{Spec} A_{i,\alpha} = \varphi_{i,\alpha}(U_{i,\alpha}), \quad y \in \mathrm{Spec} A_{j,\beta} = \varphi_{j,\beta}(U_{j,\beta}).$$

We say $x \sim_\Sigma y$ if there are affine charts $(U_{i,\alpha}, \varphi_{i,\alpha}), (U_{j,\beta}, \varphi_{j,\beta}) \in \Delta_X$ such that

$$\sigma_{i,\alpha}^{-1}(x) = \sigma_{j,\beta}^{-1}(y)$$

holds, where

$$x \in \mathrm{Spec} A_{i,\alpha} = \varphi_{i,\alpha}(U_{i,\alpha}), \quad y \in \mathrm{Spec} A_{j,\beta} = \varphi_{j,\beta}(U_{j,\beta}),$$

and

$$\sigma_{i,\alpha} : \mathrm{Spec} B_\alpha \to \mathrm{Spec} A_{i,\alpha} \quad \text{and} \quad \sigma_{j,\beta} : \mathrm{Spec} B_\beta \to \mathrm{Spec} A_{j,\beta}$$

are the scheme isomorphisms induced from the ring isomorphisms

$$\delta_{i,\alpha} : A_{i,\alpha} \cong B_\alpha \quad \text{and} \quad \delta_{j,\beta} : A_{j,\beta} \cong B_\beta$$

respectively.

(iii) Let Σ_X^* be the quotient of the set Σ^* by $\sim_X$, and

$$\pi_X : \Sigma^* \to \Sigma_X^*$$

be the canonical map. We get schemes Σ^* and Σ_X^* in an evident manner. It is clear that π_X is a morphism of the schemes.

Prove that there is an isomorphism ρ_X from the scheme Σ_X^* onto the scheme X.

In deed, we have a mapping

$$\rho : \Sigma^* \to X$$

given by

$$z \longmapsto \varphi_{i,\alpha}^{-1}(z),$$

where $(U_{i,\alpha}, \varphi_{i,\alpha}) \in \Delta_X$ such that

$$z \in \mathrm{Spec} A_{i,\alpha} = \varphi_{i,\alpha}(U_{i,\alpha}).$$

Then we have a mapping

$$\rho_X : \Sigma_X^* \to X, \quad \pi_X(z) \mapsto \rho(z).$$

Evidently, ρ_X is a surjection. From the definition for $\sim_X$, it is seen that ρ_X is an injection. Hence, ρ_X is a homeomorphism from the space Σ_X^* onto the space X. By the construction, it is seen that ρ_X is an isomorphism of the schemes.

Similarly, let Σ_Y be the quotient of the set Σ by $\sim_Y$, and

$$\pi_Y : \Sigma \to \Sigma_Y$$

be the canonical map.

Then Σ and Σ_Y are schemes, and π_Y is a scheme morphism. There is an isomorphism ρ_Y from the scheme Σ_Y onto the scheme Y.

Let Σ_Σ^* be the quotient of the set Σ^* by $\sim_\Sigma$, and

$$\pi_\Sigma : \Sigma^* \to \Sigma_\Sigma^*$$

be the canonical map.

Then Σ^*_Σ is a scheme and π_Σ is a morphism. There is an isomorphism ρ_Σ from the scheme Σ^*_Σ onto the scheme Σ.

(iv) Take any $x, y \in \Sigma^*$. We prove

$$\rho_X \circ \pi_X(x) = \rho_X \circ \pi_X(y)$$

if and only if

$$\rho_Y \circ \pi_Y \circ \rho_\Sigma \circ \pi_\Sigma(x) = \rho_Y \circ \pi_Y \circ \rho_\Sigma \circ \pi_\Sigma(y).$$

In deed, let

$$\rho_X \circ \pi_X(x) = \rho_X \circ \pi_X(y).$$

We have

$$\varphi^{-1}_{i,\alpha}(x) = \varphi^{-1}_{j,\beta}(y)$$

for some affine charts $(U_{i,\alpha}, \varphi_{i,\alpha}), (U_{j,\beta}, \varphi_{j,\beta}) \in \Delta_X$ such that

$$x \in \mathrm{Spec}A_{i,\alpha} = \varphi_{i,\alpha}(U_{i,\alpha}), \quad y \in \mathrm{Spec}A_{j,\beta} = \varphi_{j,\beta}(U_{j,\beta}).$$

As $\mathbb{A}_0(X, \mathcal{O}_X) \trianglelefteq \mathbb{A}_0(Y, \mathcal{O}_Y)$, we have affine charts $(V_{i,\alpha}, \psi_{i,\alpha})$ and $(V_{j,\beta}, \psi_{j,\beta})$ contained in $\mathcal{A}(\Gamma_Y)$ such that

$$\psi^{-1}_{i,\alpha} \circ \sigma^{-1}_{i,\alpha}(x) = \psi^{-1}_{j,\beta} \circ \sigma^{-1}_{j,\beta}(y),$$

where

$$\psi_{i,\alpha}(V_{i,\alpha}) = \mathrm{Spec}B_{i,\alpha}, \quad \psi_{j,\beta}(V_{j,\beta}) = \mathrm{Spec}B_{j,\beta}$$

and

$$\sigma_{i,\alpha} : \mathrm{Spec}B_{i,\alpha} \to \mathrm{Spec}A_{i,\alpha} \quad \text{and} \quad \sigma_{j,\beta} : \mathrm{Spec}B_{j,\beta} \to \mathrm{Spec}A_{j,\beta}$$

are the isomorphisms induced from the ring isomorphisms

$$\delta_{i,\alpha} : A_{i,\alpha} \cong B_{i,\alpha} \quad \text{and} \quad \delta_{j,\beta} : A_{j,\beta} \cong B_{j,\beta}$$

respectively.

Hence, we have

$$\rho_Y \circ \pi_Y \circ \rho_\Sigma \circ \pi_\Sigma(x) = \rho_Y \circ \pi_Y \circ \rho_\Sigma \circ \pi_\Sigma(y).$$

In a similar manner, it is seen that the converse is true.

(v) Define a map $\tau : X \to Y$ by

$$\tau(\rho_X \circ \pi_X(z)) = \rho_Y \circ \pi_Y \circ \rho_\Sigma \circ \pi_\Sigma(z)$$

for every $z \in \Sigma^*$.

By (iv), it is seen that τ is well-defined. It is immediate that τ is the desired scheme isomorphism from $(X, \mathcal{O}_X)$ onto $(Y, \mathcal{O}_Y)$.

This completes the proof.

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