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Remarks on Thurston's Construction of Pseudo-Anosov Maps of Riemann Surfaces

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Abstract It is well known that certain isotopy classes of pseudo-Anosov maps on a Riemann surface $\widetilde{S}$ of non-excluded type can be defined through Dehn twists $t_{\widetilde{\alpha}}$ and $t_{\widetilde{\beta}}$ along simple closed geodesics $\widetilde{\alpha}$ and $\widetilde{\beta}$ on $\widetilde{S}$, respectively. Let G be the corresponding Fuchsian group acting on the hyperbolic plane $\mathbb{H}$ so that $\mathbb{H}/G \cong \widetilde{S}$. For any point $a \in \widetilde{S}$, define $S = \widetilde{S}\backslash\{a\}$. In this article, the author gives explicit parabolic elements of G from which he constructs pseudo-Anosov classes on S that can be projected to a given pseudo-Anosov class on $\widetilde{S}$ obtained from Thurston's construction.

Keywords Quasiconformal mappings, Riemann surfaces, Teichmüller spaces, Mapping classes, Dehn twists, Pseudo-Anosov, Bers fiber spaces
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1 Introduction and Statement of Results

Let $\widetilde{S}$ be a Riemann surface of genus p and n punctures. We assume that $3p-3+n>0$ and $n \geq 1$. Let $a \in \widetilde{S}$ be an arbitrary point and let $S = \widetilde{S}\backslash\{a\}$. The relationship between the mapping class groups Mod_S and $\mathrm{Mod}_{\widetilde{S}}$ of S and $\widetilde{S}$ was extensively studied in [5, 8] and literature given there.

In [3], Bers developed a complex analytic method to investigate the two groups and their relationships. We denote by Mod_S^a the subgroup of Mod_S consisting of mapping classes fixing a. There is a natural group epimorphism $i : \mathrm{Mod}_S^a \to \mathrm{Mod}_{\widetilde{S}}$ defined by filling in the puncture a. Bers [3] used his method to show that the kernel of i is isomorphic to a Fuchsian group G that uniformizes $\widetilde{S}$ (the result is also known in [5]). Later in [9], Kra classified all pseudo-Anosov mapping classes that project to the trivial class on $\widetilde{S}$ as a is filled in. Kra also considered more general questions of finding pseudo-Anosov classes on the Riemann surface S that fix a and can be projected to classes with given types, and obtained important results by using methods of [3, 4].

In this paper, we continue to study this problem. We consider a pair $\{\widetilde{\alpha}, \widetilde{\beta}\}$ of simple closed geodesics on $\widetilde{S}$ that fills $\widetilde{S}$ in the sense that $\widetilde{S}\backslash\{\widetilde{\alpha}, \widetilde{\beta}\}$ consists of disks and once punctured disks. It was shown by Thurston in [13] (see also [7, 8]) that $t_{\widetilde{\beta}}^n \circ t_{\widetilde{\alpha}}^{-m}$ represents a pseudo-Anosov class for any positive integers n and m. More generally, the following theorem is well known:

Theorem 1.1 (see [10, 12, 13]) *Let $\widetilde{S}$ be a Riemann surface of type (p,n) with $3p-3+n>0$. Let $(\widetilde{\alpha}, \widetilde{\beta})$ be a pair of simple closed geodesics that fills $\widetilde{S}$, and let $\{(n_i, m_i)\}$ be a finite number*

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of pairs of non-negative integers such that at least one n_i *and one* m_i *are non-zero. Then the finite product*

$$\prod_i (t_{\widetilde{\beta}}^{n_i} \circ t_{\widetilde{\alpha}}^{-m_i}) \tag{1.1}$$

represents a pseudo-Anosov class on $\widetilde{S}$.

For general information of constructions of pseudo-Anosov class, see, for example, [7, 8, 10, 12, 13].

A question arises as to whether there is a pseudo-Anosov map on S that projects to a map isotopic to (1.1).

If $\widetilde{S}$ is compact, then every map of S that projects to a map isotopic to (1.1) is pseudo-Anosov (see [14]). Let us consider the case that $\widetilde{S}$ is non-compact, that is, S contains some punctures. Note that $\widetilde{\alpha}$ and $\widetilde{\beta}$ can be thought of as closed curves α and β on S. If there is a disk component D of $\widetilde{S}\backslash\{\widetilde{\alpha},\widetilde{\beta}\}$ and a is positioned in D, then $D\backslash\{a\}$ is a newly created once punctured disk among the components of $S\backslash\{\alpha,\beta\}$, and it is obvious that $\{\alpha,\beta\}$ still fills S. Thus (1.1) intimately represents a pseudo-Anosov class on S that projects to the class represented by (1.1) in a trivial way when a is filled in. But this case does not always occur. For example, if $\widetilde{S}\backslash\{\widetilde{\alpha},\widetilde{\beta}\}$ only consists of once punctured disks, then any positioning of a that is off the curves will provide a twice punctured disk. In this case, the naturally defined map (1.1) represents a reducible class on S yet it still projects to the pseudo-Anosov class on $\widetilde{S}$ represented by (1.1).

Another way of producing pseudo-Anosov classes on S based on an existing pseudo-Anosov class on $\widetilde{S}$ is due to Kra [9]. Since (1.1) represents a pseudo-Anosov class, we can find a surface $\widetilde{S}$ so that $\widetilde{S}$ is minimal for the class (see [4] for the definition and terminology). Let $w : \widetilde{S} \to \widetilde{S}$ be an extremal Teichmüller mapping that is isotopic to a map of form (1.1).

Let ϕ be the corresponding quadratic differential. If $p \geq 2$, ϕ must have zeros invariant under w. We assume, by taking a suitable power if necessary, that w fixes a zero on $\widetilde{S}$ denoted by a. It turns out that the restriction $w|S = w|\widetilde{S}\backslash\{\alpha\}$ is pseudo-Anosov and has a required property. This construction is, however, contingent upon the assumption that ϕ has a zero on $\widetilde{S}$ fixed by w. It was shown in [17] that if $\widetilde{S}\backslash\{\widetilde{\alpha},\widetilde{\beta}\}$ only consists of once punctured disks, then all zeros of ϕ are punctures of $\widetilde{S}$. Thus the construction of Kra [9] does not apply in the current situation.

The question remains as to whether or not there exist pseudo-Anosov classes on S projecting to a class represented by (1.1) under any circumstances. In this paper, we answer this question in a positive way by proving the following result:

Theorem 1.2 *Let* $\widetilde{S}$ *be a Riemann surface of type* (p, n) *with* $3p - 3 + n > 0$. *There are* (*infinitely many*) *pseudo-Anosov classes on* S *that project to the pseudo-Anosov class represented by* (1.1).

The idea of the proof is the following. We first lift $\widetilde{\alpha}$ and $\widetilde{\beta}$ in a natural way to curves α and β on S. Note that the pair (α,β) may not fill S. But based on β one constructs a non-trivial curve δ that becomes trivial on $\widetilde{S}$ so that (α,δ) fills S. Furthermore, we show that the image curve β' of β under the positive Dehn twist along δ has the properties: (1) (α,β') fills S; (2) β' is isotopic to β on $\widetilde{S}$. Finally we prove the theorem by invoking Theorem 1.1 on the Riemann surface S.

The same method can also be used to study pseudo-Anosov classes on S projecting to a given Dehn twist. Some results are discussed in Section 4.

2 Preliminaries

In this section, we review some definitions and basic facts on Teichmüller theory. For general information about the structure of Teichmüller space, the reader is referred to [1, 3, 4, 9].

Let R be a compact Riemann surface of genus p with $n \geq 1$ points removed. Assume that $3p-3+n>0$. The Teichmüller space $T(R)$ is defined as a space of conformal structures μ on R modulo an equivalence relation, where two structures μ, ν are called equivalent, if there is an isometry h between them such that h is isotopic to the identity as a self-map of the underlying surface. The equivalence class of μ is denoted by $[\mu]$.

Let $\mathbb{H} = \{z \in \mathbb{C};\ \mathrm{Im}\ z > 0\}$ denote the hyperbolic plane and $\varrho : \mathbb{H} \to \widetilde{S}$ the universal covering with covering group G. Associate to each μ is a quasiconformal map w^{μ} of $\mathbb{C}$ (see [2]) that fixes $0, 1$; is conformal on $\mathbb{H}^* = \{z \in \mathbb{C},\ \mathrm{Im}\ z < 0\}$; and has Beltrami coefficient μ on $\mathbb{H}$. It was shown in [3] that the domain $w^{\mu}(\mathbb{H})$ depends only on $[\mu]$. Therefore, we can form a total space $F(R) = \{([\mu], z),\ [\mu] \in T(R),\ \text{and}\ z \in w^{\mu}(\mathbb{H})\}$, which is called a Bers fiber space over $T(R)$.

The group of isotopy classes of self-maps (mapping classes) of R forms a mapping class group and is denoted by Mod_R. An element χ in Mod_R acts on $T(R)$ in the form $[\mu] \mapsto [\mu \cdot w^{-1}]$, where $w : R \to R$ is a representative of χ. w is said to represent a pseudo-Anosov class χ if there is a point $x_0 \in T(R)$ such that $\langle x_0,\ \chi(x_0)\rangle$ is a positive minimum for $\langle x,\ \chi(x)\rangle$ when x ranges over all points in $T(R)$, where $\langle\, ,\, \rangle$ denotes the Teichmüller metric on $T(R)$.

Denote by $i(c, c')$ the infimum of the geometric intersections of two simple closed curves c and c' on R running over their homotopy classes. Let t_c denote the positive Dehn twist along the geodesic representative homotopic to c. Assume that $w : R \to R$ represents a pseudo-Anosov class. It was shown in [10] that $t_c^n \circ w$ represents a pseudo-Anosov class except for only a finite number of values of n. This result was refined by Fathi [6]. To be more precise, we first note that the set $\{c, w(c), \cdots, w^k(c), \cdots\}$ of closed curves always fills R. Let $k > 0$ be an integer such that $\mu_k := i(c, w^k(c)) > 0$ and $i(c, w^l(c)) = 0$ for $l = 0, 1, \cdots, k-1$. From [6, Proposition 5.1], there are real numbers z_j and $y \geq 0$ such that for any real number λ,

$$i(t_c^{\lambda} \circ w^k(c),\ w^{-k}(c)) = y + \sum_{j=1}^{\mu_k} |z_j - \lambda \mu_k|,$$

where t_c^{λ} denotes the interpolated twist along c by the factor λ. See [6] for more information. Let

$$\lambda_0 = \Big(\sum_{j=1}^{\mu_k} z_j\Big)(\mu_k \mu_{-k})^{-1},$$

and let I be the interval with center λ_0 and length $2 + \frac{4}{\mu_k}$. [6, Theorem 5.1] tells us that if $n \in \mathbb{Z}\backslash I$, then the map $t_c^n \circ w$ also represents a pseudo-Anosov class. In particular, since the length of I is ≤ 6, $t_c^n \circ w$ represents a pseudo-Anosov class except for 7 consecutive values of n.

Let $\widetilde{S}$ and S be defined as in the introduction. Let $\widehat{w}$, $\widehat{w}' : \mathbb{H} \to \mathbb{H}$ be two maps that project to self-maps w and w' of $\widetilde{S}$, respectively. $\widehat{w}$ and $\widehat{w}'$ are called equivalent (the equivalence class is denoted by $[\widehat{w}]$) if

$$\widehat{w} \circ g \circ (\widehat{w})^{-1} = \widehat{w}' \circ g \circ (\widehat{w}')^{-1} \quad \text{for every } g \in G.$$

The group mod $\widetilde{S}$ is the collection of $[\widehat{w}]$ for all $w : \widetilde{S} \to \widetilde{S}$. Elements in the set of restrictions $\widehat{w}|_{\mathbb{H}}$ are one-to-one corespondent with elements in mod $\widetilde{S}$. Every element $[\widehat{w}]$ of mod $\widetilde{S}$ acts on $F(\widetilde{S})$ by the formula:

$$[\widehat{w}]\ ([\mu], z) = ([\nu], w^\nu \circ \widehat{w} \circ (w^\mu)^{-1}(z)),$$

where ν is the Beltrami coefficient of $w^\mu \circ \widehat{w}^{-1}$. We see that $[\widehat{w}]$ acts on $F(\widetilde{S})$ as a fiber-preserving biholomorphism. By [3, Theorem 9], there is a biholomorphic map $\varphi : F(\widetilde{S}) \to T(S)$ that is called a Bers isomorphism in the literature. From [3, Theorem 10], the isomorphism φ induces an isomorphism sending elements $[\widehat{w}] \in \text{mod}\ \widetilde{S}$ onto the elements $[\widehat{w}]^* \in \text{Mod}_S^a$.

Example 2.1 Take a simple closed geodesic $\widetilde{\alpha} \subset \widetilde{S}$ that avoids a. The Dehn twist $t_{\widetilde{\alpha}}$ can be lifted to a map $\widetilde{t_{\widehat{\alpha}}}$ of $\mathbb{H}$ along a geodesic $\widehat{\alpha} \subset \mathbb{H}$ with $\varrho(\widehat{\alpha}) = \widetilde{\alpha}$. It was shown in [16] that $[\widetilde{t_{\widehat{\alpha}}}]^*$ is represented by a Dehn twist t_α, where α is a simple closed curve on S that descends to $\widetilde{\alpha}$ as a is filled in. Among the lifts of $t_{\widetilde{\alpha}}$ there is a lift (also called $\widetilde{t_{\widehat{\alpha}}}$) such that $[\widetilde{t_{\widehat{\alpha}}}]^*$ is represented by t_α that is obtained just from $t_{\widetilde{\alpha}}$ by deleting a. In what follows, we use the same notation to denote a self-map of a Riemann surface as well as its mapping class. In this convention, we simply write $[\widetilde{t_{\widehat{\alpha}}}]^* = t_\alpha$.

Every element $g \in G$ naturally acts on $F(\widetilde{S})$ by

$$g([\mu], z) = ([\mu], w^\mu \circ g \circ (w^\mu)^{-1}(z)).$$

In this way, $G \cong \pi_1(\widetilde{S}, a)$ can be regarded as a normal subgroup of mod $\widetilde{S}$. We use the letter G to denote the Fuchsian group as well as the normal subgroup of mod $\widetilde{S}$. Note that g^* is an element of Mod_S^a that projects to the trivial class of $\text{Mod}_{\widetilde{S}}$. Theorem 2 of [9, 11] states that g^* is represented by a Dehn twist along a boundary curve c of a twice punctured disk on S enclosing a (write as $g^* = t_c$) if and only if $g \in G$ is parabolic.

To proceed, we let $b_1, \cdots, b_n$ denote the punctures on $\widetilde{S}$. Then, of course, S has punctures $a, b_1, \cdots, b_n$.

Lemma 2.1 *Let D_i be an arbitrary twice punctured disk on S enclosing a and b_i and let $c = \partial D_i$. Let $\{T_i\}$ denote the conjugacy class in G corresponding to the puncture b_i. Then there is a parabolic element $g_i \in G$ in $\{T_i\}$ such that g_i^* is represented by t_c.*

Proof Since c becomes a trivial loop as a is filled in. $i(t_c)$ is a trivial mapping class. This means that t_c lies in the kernel of the homomorphism $i : \text{Mod}_S^a \to \text{Mod}_{\widetilde{S}}$. By [3, Theorem 10], there is an element $g \in G$ such that $g^* = t_c$. It follows from Theorem 2 of [9, 11] that g is a parabolic element. Since c shrinks to the puncture b_i as a is filled in, g is in $\{T_i\}$.

3 Filling Curves and Pseudo-Anosov Maps

Let S^* be the compactification $\widetilde{S} \cup \{b_1, \cdots, b_n\}$ with marked points $b_1, \cdots, b_n$, and let S_a^* be S^* with one more marked point a added. Let $\mathcal{P}_a$ denote the set of equivalence classes of paths P_i on S_a^* connecting from a to another marked point b_i and no any other marked points are on P_i, where two paths P_i and P_i' are considered equivalent if both paths connect a and b_i and P_i is homotopic to P_i' by a homotopy fixing a and b_i without interfering with any other b_j, $j \neq i$.

Let $\alpha \subset S$ be defined as in the example in Section 2. Let $b = b_i$ be an arbitrary puncture. We have

Lemma 3.1 *Assume that there is a simple closed geodesic $\widetilde{\beta}$ such that $\{\widetilde{\alpha},\widetilde{\beta}\}$ fills $\widetilde{S}$. There is an element Γ in $\mathcal{P}_a$ connecting a and b such that each component of $S_a^*\backslash\{\Gamma,\widetilde{\alpha}\}$ is either a disk or a disk with only one marked point enclosed.*

Proof By hypothesis, $\{\widetilde{\alpha},\widetilde{\beta}\}$ fills $S^*=\widetilde{S}\cup\{b_1,\cdots,b_n\}$. For any marked point $b=b_i$, let Ω_1 be the disk component of $S^*\backslash\{\widetilde{\alpha},\widetilde{\beta}\}$ that includes b.

There is a subarc Γ_0 of $\widetilde{\beta}$ that serves as a boundary portion of Ω_1. Let $\{A,B\}$ be the endpoints of Γ_0, i.e., $\Gamma_0\cap\widetilde{\alpha}=\{A,B\}$.

Cut off Γ_0 in the middle point, and denote by C,D the endpoints resulted from the cutting. Γ_0 breaks down to two smaller arcs, one of which connects from D to B, denoted by Γ_1; and the other connects from A to C and is denoted by Γ_3. We parametrize Γ_1, Γ_3 by $\Gamma_1=\Gamma_1(s)$ and $\Gamma_3(s)$, respectively, where $0\le s\le 1$. Similarly, the subarc $\Gamma_2=\widetilde{\beta}\backslash\Gamma_0$ can be parametrized by $\Gamma_2(s)$, $0\le s\le 1$, such that $\Gamma_2(0)=B$, and $\Gamma_2(1)=A$. Finally, we draw a small arc Γ_4 that connects C and b without leaving Ω_1. Γ_4 can be parametrized as $\Gamma_4(s)$, $0\le s\le 1$.

Now we construct a path Γ originating at D and terminating at b by the formula:

$$\Gamma(s)=\begin{cases}\Gamma_1(4s), & 0\le s\le \dfrac{1}{4},\\ \Gamma_2(4s-1), & \dfrac{1}{4}<s\le\dfrac{1}{2},\\ \Gamma_3(4s-2), & \dfrac{1}{2}<s\le\dfrac{3}{4},\\ \Gamma_4(4s-3), & \dfrac{3}{4}<s\le 1.\end{cases}$$

Obviously, $\Gamma=\Gamma(s)$ is a path on S^* connecting from D to b, and no marked points other than b lie on Γ. If necessary, a quasiconformal mapping of S^* onto S_a^* can be made to send b_i to b_i and D to a. One may assume that $D=a$. Since $\Gamma(0)=a$ and $\Gamma(1)=b$, we obtain a path Γ in $\mathcal{P}_a$.

To see that $S_a^*\backslash\{\Gamma,\widetilde{\alpha}\}$ consists of disks and disks with only one marked point enclosed, we note that Γ_0 is a common boundary segment of two components Ω_1 and Ω_2 of $S_a^*\backslash\{\widetilde{\alpha},\widetilde{\beta}\}$, and no other components of $S_a^*\backslash\{\widetilde{\alpha},\widetilde{\beta}\}$ take Γ_0 as a boundary segment. Observe that Ω_2 is either a disk or a disk with one marked point enclosed. Let Ω denote the interior of $\overline{\Omega_1\cup\Omega_2}$. If Ω_2 is a disk with one marked point, then $\Omega\backslash\Gamma$ is also a topological disk with the same marked point. If Ω_2 is a disk, so is $\Omega\backslash\Gamma$.

Finally, let Ω' be any component of $S_a^*\backslash\{\widetilde{\alpha},\widetilde{\beta}\}$ other than Ω_1 and Ω_2. In this case, all boundary segments of Ω' are portions of $\widetilde{\alpha}$ and $\widetilde{\beta}$, which means that Ω' is either a disk or a disk with one marked point enclosed. Since components of $S_a^*\backslash\{\widetilde{\alpha},\widetilde{\beta}\}$ other than Ω_1 and Ω_2 are one-to-one correspondent with components of $S_a^*\backslash\{\Gamma,\widetilde{\alpha}\}$ away from Ω, we conclude that every component of $S_a^*\backslash\{\Gamma,\widetilde{\alpha}\}$ is either a disk or a disk with one marked point enclosed. So Γ is as required.

Let $\mathcal{E}_a$ denote the set of equivalence classes of twice punctured disks on S enclosing a and another puncture, where two such disks are considered equivalent if both enclose the same punctures and both boundaries are homotopic to each other. There is a bijection

$$j:\mathcal{P}_a\to\mathcal{E}_a$$

defined as follows. For any path $P_i \in \mathcal{P}_a$ connecting a and b_i, fattening P_i and then deleting a and all b_i, $1 \leq i \leq n$. We thus obtain a twice punctured disk on S. Conversely, any twice punctured disk $\Delta_i \in \mathcal{E}_a$ defines a disk $\widetilde{\Delta}_i$ on S_a^* with two marked points a and b_i enclosed if all points a and b_i, $1 \leq i \leq n$, are filled in. One obtains a path P_i in $\mathcal{P}_a$ defined to be a deformation retract of $\widetilde{\Delta}_i$ on S_a^*.

Let α, $\beta \subset S$ be natural lifts of $\widetilde{\alpha}$ and $\widetilde{\beta}$ that are obtained from deleting the point a.

Lemma 3.2 *For each puncture $b = b_i$ of S, there is a twice punctured disk Δ on S that satisfies:*

(1) Δ *encloses a and b,*

(2) $\delta = \partial\Delta$ *together with α fills S.*

Proof Let Γ be given as in Lemma 3.1, and let $\Delta = j(\Gamma)$. Then $\Delta \subset S$ is a twice punctured disk enclosing a and b. A schematic diagram is drawn in Figure 1.

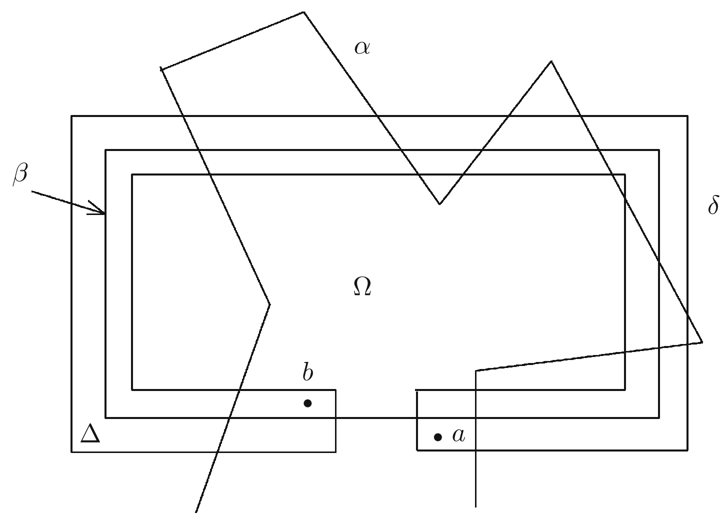

Figure 1

Clearly, $\delta \subset S$ is a non-peripheral simple closed curve. We need to show that $\alpha \cup \delta$ fills S. By definition of Δ, $\widetilde{\alpha}$ crosses Γ at a point x if and only if α cuts Δ transversally along a segment passing through x. Note that α cuts Δ into several regions in order that they are all quadrilaterals except for the first one H_1, which is an a-punctured disk; and the last one H_2, which is a b-punctured disk. Let Ω, Ω' be defined as in Lemma 3.1. Each component of $S\backslash\{\alpha, \delta\}$ falls into one of the following collections:

(I) H_1 and H_2;

(II) components of $\Delta\backslash\{\alpha\}$ that are not H_1 and H_2;

(III) components of $S\backslash\{\alpha, \Delta\}$ corresponding to Ω';

(IV) the (unique) component of $S\backslash\{\alpha, \Delta\}$ that corresponds to the interior of $\Omega\backslash\{H_1, H_2\}$.

As noted above, a type I component is a once punctured disk, and a type II component is a quadrilateral that is isotopic to a disk of course. Every type III component can be deformed into an Ω', which is a disk or a once punctured disk. Finally, let $\Omega^* \subset S$ denote the (unique) component of type IV. Then Ω^* can be deformed to $\Omega\backslash\Gamma$ as defined in Lemma 3.1. It follows from Lemma 3.1 that Ω^* is either a disk or a once punctured disk. Overall, we conclude that each component of $S\backslash\{\alpha, \delta\}$ is either a disk or a once punctured disk, which means that $\{\alpha, \delta\}$ fills S.

Lemma 3.3 *Let δ and Δ be given as in Lemma 3.2, and let $t_\delta(\beta)$ be the image loop of β under the Dehn twist along the boundary curve δ of Δ. Then $t_\delta(\beta)$ is a non-peripheral simple closed curve with the properties:* (1) *$t_\delta(\beta)$ is homotopic to β on $\widetilde{S}$;* (2) *$\{t_\delta(\beta), \alpha\}$ fills S.*

Proof Without loss of generality, we assume that a and b lie in different sides of β as shown in Figure 1. The curve $t_\delta(\beta)$ is drawn in Figure 2. Following the same notations as in Lemmas 3.1 and 3.2, let $S' = S\backslash\Omega$. It is clear that $S' \cap \{t_\delta(\beta)\}$ is a disjoint union of simple arcs. These subarcs are almost parallel and placed in Δ in a nice position after a suitable homotopy is performed.

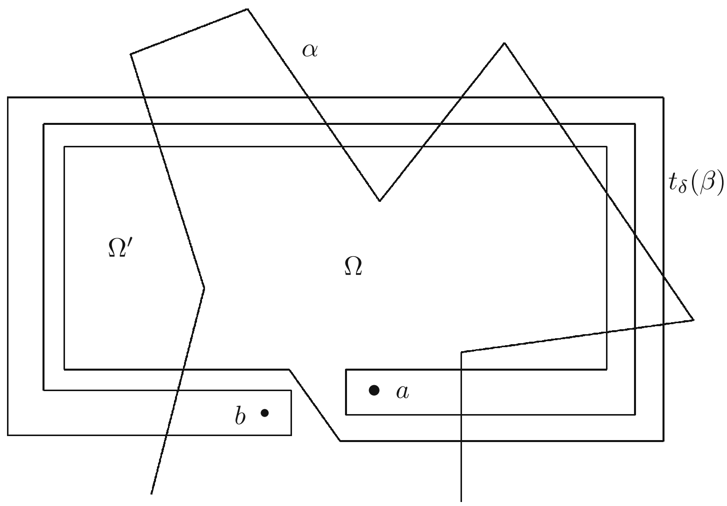

Figure 2

We need to examine each component of $S\backslash\{\alpha, t_\delta(\beta)\}$. Any type III component (defined in Lemma 3.2) can be deformed to a component of $S\backslash\{\alpha, t_\delta(\beta)\}$. So these components are either disks or once punctured disks. Any type II component is divided by $S' \cap \{t_\delta(\beta)\}$ into two smaller quadrilaterals, each of which is still a topological disk.

Finally, observe that $\{t_\delta(\beta)\} \cap \Omega$ divides Ω into four regions, among which one is an a-punctured disk; one is a b-punctured disk; and the rest two components are homeomorphic to disks (see Figure 2).

Since $t_\delta(\beta)$ is an image of a simple non-peripheral closed curve under a homeomorphism, it is simple and non-peripheral as well. Finally, notice that δ is peripheral on $\widetilde{S}$, t_δ is trivial on $\widetilde{S}$, and thus $t_\delta(\beta)$ is homotopic to β on $\widetilde{S}$.

Proof of Theorem 1.2 By Lemma 3.3, the two simple closed curves $\sigma = t_\delta(\beta)$ and α fills S. Therefore by Theorem 1.1, the finite product

$$\prod_i (t_\sigma^{n_i} \circ t_\alpha^{-m_i}) \tag{3.1}$$

represent a pseudo-Anosov class. But $t_\sigma = t_\delta \circ t_\beta \circ t_\delta^{-1}$. By Lemma 2.1, there is a parabolic element $g \in G$ such that $g^* = t_\delta$. Together with (3.1), we see that

$$\prod_i (g^* \circ t_\beta^{n_i} \circ (g^*)^{-1} \circ t_\alpha^{-m_i}) \tag{3.2}$$

represents a pseudo-Anosov class on S. By [3, Theorem 10], $i(g^*) = \mathrm{id}$. Hence the class (3.2) projects to the class represented by (1.1) under the epimorphism $i : \mathrm{Mod}_S^a \to \mathrm{Mod}_{\widetilde{S}}$. This completes the proof of Theorem 1.2.

4 Pseudo-Anosov Maps Obtained from Parabolic Elements

In this section, we use the parabolic elements g of G obtained from Section 3 to study pseudo-Anosov classes in Mod_S^a that project to a simple Dehn twist. We also discuss some properties of these classes. For convenience, $\widetilde{\alpha} \subset \widetilde{S}$ is called to pair with a geodesic $\widetilde{\beta} \subset \widetilde{S}$ if $\{\widetilde{\alpha}, \widetilde{\beta}\}$ fills $\widetilde{S}$.

Associate to each simple geodesic $\alpha \subset S$ with $i(t_\alpha) = t_{\widetilde{\alpha}}$ is a primary simple hyperbolic element g_α of G such that g_α^* commutes with t_α. We denote by A_g the axis of a hyperbolic element g of G in $\mathbb{H}$. It was shown in [15, 16] that if g has the property that $\varrho(A_g)$ fills $\widetilde{S}$, then $t_\alpha \circ g^* \in \mathrm{Mod}_S^a$ represents a pseudo-Anosov class. If g is simple hyperbolic and there is a fundamental region $F \subset \mathbb{H}$ of G such that F catches portions of A_g and A_{g_α}, then $t_\alpha \circ g^*$ is pseudo-Anosov if and only if $\{\widetilde{\alpha}, \varrho(A_g)\}$ fills $\widetilde{S}$. On the other hand, if g is parabolic with fixed point being a vertex of F that catches a portion of A_{g_α}, then t_α can be defined so that $t_\alpha \circ g^*$ is not pseudo-Anosov.

Theorem 4.1 *Let $\widetilde{S}$ be a Riemann surface of type (p, n) with $3p - 3 + n > 0$ and $n \geq 1$. Let $\widetilde{\alpha}$ be a simple closed geodesic on $\widetilde{S}$ that pairs with a simple closed geodesic. Then for each puncture b_i, there are elements $g_i \in G$ in the conjugacy class of b_i such that $t_\alpha \circ g_i^* \in \mathrm{Mod}_S^a$ are pseudo-Anosov and project to the class represented by the Dehn twist $t_{\widetilde{\alpha}}$.*

Proof By Lemma 3.2, there is a non-peripheral closed curve δ on S that satisfies the conditions: (1) δ becomes a trivial loop around b_i on $\widetilde{S}$ if a is filled in; (2) δ together with α fills S. By Thurston's construction of pseudo-Anosov maps (see also [7]), the product $t_\alpha \circ t_\delta^{-1}$ represents a pseudo-Anosov class in Mod_S^a. By Lemma 2.1, we can find a parabolic element g of G so that g^* is represented by t_δ^{-1}, which tells us that $t_\alpha \circ g^*$ represents a pseudo-Anosov class in Mod_S^a, while it still projects to the Dehn twist $t_{\widetilde{\alpha}}$. This completes the proof of Theorem 4.1.

Theorem 4.1 can be used to discuss some properties of mapping classes in Mod_S^a.

Theorem 4.2 *There are infinitely many pseudo-Anosov classes $\theta_n \in \mathrm{Mod}_S^a$ and a fixed parabolic element g of G such that $i(\theta_n \circ g^*) \in \mathrm{Mod}_{\widetilde{S}}$ and $\theta_n \circ g^* \in \mathrm{Mod}_S^a$ are all distinct pseudo-Anosov classes.*

Proof Let $\widetilde{\alpha} \subset \widetilde{S}$ be a curve that pairs with a curve $\widetilde{\beta}$. Let α and $\beta \subset S$ be natural lifts of $\widetilde{\alpha}$ and $\widetilde{\beta}$, respectively. By Theorem 4.1, we can find a parabolic element $g \in G$ so that for any positive integer m, $t_\alpha^{-m} \circ g^*$ is a pseudo-Anosov class in Mod_S^a. Since β is a non-peripheral closed curve on S, from [6, Theorem 5.1] (see also Section 2 for an exposition), $t_\alpha^{-m} \circ g^* \circ t_\beta^n$ is pseudo-Anosov except for at most 7 consecutive positive values of n. This particularly implies that $t_\alpha^{-m} \circ g^* \circ t_\beta^n = (t_\alpha^{-m} \circ g^* \circ t_\beta^n) \circ (g^*)^{-1} \circ g^*$ is pseudo-Anosov for infinitely many values of n.

Let $\theta_n = t_\alpha^{-m} \circ g^* \circ t_\beta^n \circ (g^*)^{-1}$. By Lemma 3.3 and the same argument of Theorem 1.2, we see that θ_n are pseudo-Anosov for infinitely many values of n. It is also easy to see that $i(\theta_n \circ g^*) = t_{\widetilde{\alpha}}^{-m} \circ t_{\widetilde{\beta}}^n$. By Theorem 1.1, $i(\theta_n \circ g^*)$ is pseudo-Anosov. Note also that for $n \neq n'$, $t_{\widetilde{\alpha}}^{-m} \circ t_{\widetilde{\beta}}^n$ is not isotopic to $t_{\widetilde{\alpha}}^{-m} \circ t_{\widetilde{\beta}}^{n'}$ on $\widetilde{S}$, that is, $i(\theta_n \circ g^*)$ is not isotopic to $i(\theta_{n'} \circ g^*)$. It follows that $\theta_n \circ g^*$ and $\theta_{n'} \circ g^*$ lie in different fibers. In particular, this implies that $\theta_n \circ g^* \neq \theta_{n'} \circ g^*$. We conclude that all $\theta_n \circ g^*$ for different values of n are distinct. This completes the proof of Theorem 4.2.

As another application of Theorem 4.1, we intend to represent some pseudo-Anosov maps by virtue of products of Dehn twists along a pair of filling geodesics on S. Let $\Sigma = \{(\alpha_1, \beta_1), \cdots, (\alpha_k, \beta_k)\}$ denote a non-empty finite set of pairs of simple closed geodesics on S such that each pair fills S. From Theorem 4.1, there is an integer $N > 0$ (depending on a fixed integer m) such that for any $n > N$, the map

$$f_n = t_\beta^n \circ t_\alpha^{-m} \circ g^* \tag{4.1}$$

represents a pseudo-Anosov class in Mod_S^a, where g^* is defined in Theorem 4.2. We ask if there is such a non-empty finite set Σ so that for each $n > N$, f_n is isotopic to

$$t_{\beta_i}^{c_n} \circ t_{\alpha_i}^{d_n} \tag{4.2}$$

for $(\alpha_i, \beta_i) \in \Sigma$ and certain integers $c_n, d_n \in \mathbb{Z}$. We will give a negative answer to this question by proving:

Theorem 4.3 *There does not exist any non-empty finite set Σ of pairs of simple closed geodesics on S such that any pseudo-Anosov map f_n in* (4.1) *are isotopic to a map of form* (4.2).

Proof By taking a subsequence if needed, we may assume that

$$f_n = t_{\beta'}^{c_n} \circ t_{\alpha'}^{d_n}, \tag{4.3}$$

where (α', β') is a fixed pair in Σ. Since f_n projects to $t_{\widetilde{\beta}}^n \circ t_{\widetilde{\alpha}}^{-m}$, α' and β' are isotopic on $\widetilde{S}$ to $\widetilde{\alpha}$ and $\widetilde{\beta}$, respectively. It follows that $c_n = n$, $d_n = -m$, and there are maps f_1 and f_2 of S, isotopic to the identity on $\widetilde{S}$, such that $f_1(\beta) = \beta'$ and $f_2(\alpha) = \alpha'$. We write $f_1 = u^*$ and $f_2 = v^*$ for some elements $u, v \in G$. Then

$$u^*(\beta) = \beta' \quad \text{and} \quad v^*(\alpha) = \alpha'. \tag{4.4}$$

Hence (4.3) can be written as $f_n = t_{u^*(\beta)}^n \circ t_{v^*(\alpha)}^{-m}$, or

$$f_n = (u^* \circ t_\beta^n \circ (u^*)^{-1}) \circ (v^* \circ t_\alpha^{-m} \circ (v^*)^{-1}). \tag{4.5}$$

As we remarked in Section 2, geodesics $\widehat{\alpha}, \widehat{\beta} \subset \mathbb{H}$ can be chosen with the properties: (1) $\varrho(\widehat{\alpha}) = \widetilde{\alpha}$ and $\varrho(\widehat{\beta}) = \widetilde{\beta}$; (2) there are lifts $\widetilde{t}_{\widehat{\alpha}}$ and $\widetilde{t}_{\widehat{\beta}}$ along $\widehat{\alpha}$ and $\widehat{\beta}$, respectively, such that $t_\alpha = [\widetilde{t}_{\widehat{\alpha}}]^*$ and $t_\beta = [\widetilde{t}_{\widehat{\beta}}]^*$. Therefore, using Bers isomorphism, together with (4.3) and (4.1) we obtain

$$[\widetilde{t}_{\widehat{\beta}}]^n \circ [\widetilde{t}_{\widehat{\alpha}}]^{-m} \circ g = (u \circ [\widetilde{t}_{\widehat{\beta}}]^n \circ u^{-1}) \circ (v \circ [\widetilde{t}_{\widehat{\alpha}}]^{-m} \circ v^{-1}), \tag{4.6}$$

which is equivalent to

$$\widetilde{t}_{\widehat{\beta}}^{-n} \circ u^{-1} \circ \widetilde{t}_{\widehat{\beta}}^{n} = (u^{-1} \circ v) \circ (\widetilde{t}_{\widehat{\alpha}}^{-m} \circ v^{-1} \circ g^{-1} \circ \widetilde{t}_{\widehat{\alpha}}^{m}) \quad \text{on } \mathbb{R}. \tag{4.7}$$

Note that $u_n = \widetilde{t}_{\widehat{\beta}}^{-n} \circ u^{-1} \circ \widetilde{t}_{\widehat{\beta}}^{n}$, $w = u^{-1} \circ v$, and $w' = \widetilde{t}_{\widehat{\alpha}}^{-m} \circ v^{-1} \circ g^{-1} \circ \widetilde{t}_{\widehat{\alpha}}^{m}$ are all elements of G. From (4.7), we get $u_n = w \circ w'$. Denote $u' = w \circ w'$. This says that u_n is a fixed element u' of G that is independent of n.

Let x be a fixed point of u. Then for all n, $\widetilde{t}_{\widehat{\beta}}^{-n}(x)$ is fixed by $\widetilde{t}_{\widehat{\beta}}^{-n} \circ u^{-1} \circ \widetilde{t}_{\widehat{\beta}}^{n}$, and thus it is fixed by u' also. But this is impossible unless $\widetilde{t}_{\widehat{\beta}}^{-n}(x) = x$. This implies that u commutes with $\widetilde{t}_{\widehat{\beta}}$. It follows that $u^*(\beta) = \beta$. By (4.4) we get $\beta = \beta'$.

Now (4.1) together with (4.3) yields $t_{\alpha}^{-m} \circ g^* = t_{\alpha'}^{-m}$. By Theorem 4.1, $t_{\alpha}^{-m} \circ g^*$ is pseudo-Anosov, while $t_{\alpha'}^{-m}$ is a power of a Dehn twist. This contradiction proves Theorem 4.3.

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