Chin. Ann. Math.
32B(5), 2011, 699–710
DOI: 10.1007/s11401-011-0671-2

Chinese Annals of Mathematics, Series B

A Criterion of Normality Concerning Holomorphic Functions Whose Derivative Omits a Function*

Xiaojun LIU[1] Yasheng YE[1]

Abstract The authors discuss the normality concerning holomorphic functions and get the following result. Let $\mathcal{F}$ be a family of holomorphic functions on a domain $D \subset \mathbb{C}$, all of whose zeros have multiplicity at least k, where $k \geq 2$ is an integer. And let $h(z) \not\equiv 0$ be a holomorphic function on D. Assume also that the following two conditions hold for every $f \in \mathcal{F}$: (a) $f(z) = 0 \Longrightarrow |f^{(k)}(z)| < |h(z)|$; (b) $f^{(k)}(z) \neq h(z)$. Then $\mathcal{F}$ is normal on D.

Keywords Normal family, Holomorphic functions, Omitted function
2000 MR Subject Classification 30D35

1 Introduction

In [5], X. C. Pang, D. G. Yang and L. Zalcman proved the following theorem.

Theorem 1.1 (see [5]) *Let $\mathcal{F}$ be a family of meromorphic functions on a domain $D \subset \mathbb{C}$, all of whose zeros have multiplicity at least $k+3$, where $k \geq 1$ is an integer, and let $h(z)$ ($\not\equiv 0$) be a holomorphic function on D. Suppose that for every $f \in \mathcal{F}$, $f^{(k)}(z) \neq h(z)$, $z \in D$, then $\mathcal{F}$ is a normal family on D.*

Also in [5], they considered reducing the multiplicity for the zeros of f and proved the following result.

Theorem 1.2 (see [5]) *Let $\mathcal{F}$ be a family of meromorphic functions on a domain $D \subset \mathbb{C}$, all of whose zeros have multiplicity at least $k+2$, where $k \geq 1$ is an integer. Let $h(z)$ ($\not\equiv 0$) be a holomorphic function on D, all of whose zeros have multiplicity at least 2. Suppose that for every $f \in \mathcal{F}$, $f^{(k)}(z) \neq h(z)$, $z \in D$, then $\mathcal{F}$ is a normal family on D.*

The question is that can the restriction for the zeros of $f(z)$ with multiplicity at least $k+2$ be reduced to k? In this paper, we continue to study the above problem and get the confirmed result.

Theorem 1.3 *Let $\mathcal{F}$ be a family of functions holomorphic on a domain $D \subset \mathbb{C}$, all of whose zeros have multiplicity at least k, where $k \geq 2$ is an integer. And let $h(z) \not\equiv 0$ be a holomorphic*

Manuscript received February 28, 2011. Revised June 4, 2011.
[1]Department of Mathematics, College of Science, University of Shanghai for Science and Technology, Shanghai 200093, China. E-mail: xiaojunliu2007@hotmail.com yashengye@yahoo.com.cn
*Project supported by the National Natural Science Foundation of China (No. 11071074) and the Outstanding Youth Foundation of Shanghai (No. slg10015).

function on D. Assume also that the following two conditions hold for every $f \in \mathcal{F}$:

(a) $f(z) = 0 \Longrightarrow |f^{(k)}(z)| < |h(z)|$;

(b) $f^{(k)}(z) \neq h(z)$.

Then $\mathcal{F}$ is normal on D.

The following counterexample shows that Theorem 1.3 does not hold for meromorphic functions when $k = 2$.

Example 1.1 *Let $D = \Delta = \{z : |z| < 1\}$ be a unit disc,*

$$f_n(z) = \frac{\left(z + \frac{1}{n}\right)^2 \left(z + \frac{2}{n}\right)^2}{6\left(z + \frac{6}{n}\right)} \quad \textit{and} \quad h(z) = z.$$

It is easy to check that f_n are meromorphic on Δ and have only two zeros $z_1^{(n)} = -\frac{1}{n}$ and $z_2^{(n)} = -\frac{2}{n}$ with multiplicity 2. By calculation, we have

$$f_n''(z) = z + \frac{400}{3n^4\left(z + \frac{6}{n}\right)^3}.$$

So

$$\begin{aligned} f_n = 0 &\Longrightarrow z_1^{(n)} = -\frac{1}{n}, \quad z_2^{(n)} = -\frac{2}{n} \\ &\Longrightarrow |f_n''(z_{1,2}^{(n)})| = |z_{1,2}^{(n)}| \left| 1 + \frac{400}{3n^4 z_{1,2}^{(n)} \left(z_{1,2}^{(n)} + \frac{6}{n}\right)^3} \right| < |z_{1,2}^{(n)}| = |h(z_{1,2}^{(n)})| \\ &\Longrightarrow |f_n''| < |h| \quad \textit{and} \quad f_n''(z) = z + \frac{400}{3n^4\left(z + \frac{6}{n}\right)^3} \neq z = h(z). \end{aligned}$$

But, $\mathcal{F} = \{f_n\}$ is not normal on Δ.

So, the question is what about the case $k \geq 3$?

Question 1.1 *Let $\mathcal{F}$ be a family of functions meromorphic on a domain $D \subset \mathbb{C}$, all of whose zeros have multiplicity at least k, where $k \geq 3$ is an integer. And let $h(z) \not\equiv 0$ be a holomorphic function on D. Assume also that the following two conditions hold for every $f \in \mathcal{F}$:*

(a) $f(z) = 0 \Longrightarrow |f^{(k)}(z)| < |h(z)|$;

(b) $f^{(k)}(z) \neq h(z)$.

Then is $\mathcal{F}$ normal on D?

Let us set some notations. Throughout this paper, D is a domain in $\mathbb{C}$. For $z_0 \in \mathbb{C}$ and $r > 0$, $\Delta(z_0, r) = \{z : |z - z_0| < r\}$ and $\Delta'(z_0, r) = \{z : 0 < |z - z_0| < r\}$. The unit disc is denoted by Δ and $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. We write $f_n(z) \overset{\chi}{\Longrightarrow} f(z)$ on D to indicate that the sequence $\{f_n\}$ converges to f in the spherical metric, uniformly on compact subsets of D, and $f_n \Longrightarrow f$ on D if the convergence is in the Euclidean metric. The spherical derivative of the meromorphic function f at the point z is denoted by $f^\sharp(z)$.

Frequently, given a sequence $\{f_n\}_1^\infty$ of functions, we need to extract an appropriate subsequence. This necessity may recur within a single proof. To avoid the awkwardness of multiple indices, we again denote the extracted subsequence by $\{f_n\}$ (rather than, say, $\{f_{n_k}\}$) and signal this operation by writing "taking a subsequence and renumbering" or simply "renumbering". The same convention applies to the sequences of constants.

The plan of this paper is as follows. In Section 2, we state a number of preliminary results. Then, in Section 3, we prove Theorem 1.3.

2 Preliminary Results

The following lemma is taken from [2, p. 145], [5, p. 259] and [10, pp. 216–217].

Lemma 2.1 *Let $\mathcal{F}$ be a family of functions meromorphic on a domain D, all of whose zeros have multiplicity at least k, and suppose that there exists an $A \geq 1$, such that $|f^{(k)}(z)| \leq A$ whenever $f(z) = 0$. Then if $\mathcal{F}$ is not normal at $z_0 \in D$, for each $0 \leq \alpha \leq k$, there exist*

(a) *points $z_n \to z_0$;*

(b) *functions $f_n \in \mathcal{F}$;*

(c) *positive numbers $\rho_n \to 0^+$,*

such that $g_n(\zeta) := \rho_n^{-\alpha} f_n(z_n + f_n\zeta) \overset{\chi}{\Longrightarrow} g(\zeta)$ on $\mathbb{C}$, where g is a nonconstant meromorphic function on $\mathbb{C}$, such that for every $\zeta \in \mathbb{C}$, $g^{\#}(\zeta) \leq g^{\#}(0) = kA + 1$.

Lemma 2.2 (see [1, pp. 118–119, 122–123]) *Let f be a meromorphic function on $\mathbb{C}$. If $f^{\#}$ is uniformly bounded on $\mathbb{C}$, then the order of f is at most 2. If f is an entire function, then the order of f is at most 1.*

Lemma 2.3 *Let f be an entire function of finite order $\rho(f)$ on $\mathbb{C}$, all of whose zeros have multiplicity at least k, where $k \geq 2$ is an integer and $a \neq 0$ is a constant. Suppose that $\rho(f) \leq 1$ and $f(z)$ satisfies the following two conditions:*

(a) $f(z) = 0 \Longrightarrow |f^{(k)}(z)| < |a|$;

(b) $f^{(k)}(z) \neq a$.

Then

$$f(z) = \frac{b(z - z_0)^k}{k!},$$

where $b \neq a$ and z_0 are constants.

Proof We separate it into two cases.

Case 1 f is a transcendental entire function on $\mathbb{C}$.

By $\rho(f^{(k)}) = \rho(f) \leq 1$ and $f^{(k)} \neq a$, we have $f^{(k)}(z) = a + B\exp(A\zeta)$, where $A, B \in \mathbb{C}^*$ are two constants.

By calculation,

$$f(z) = \frac{az^k}{k!} + a_{k-1}z^{k-1} + \cdots + a_0 + BA^{-k}\exp(A\zeta),$$

where $a_{k-1}, \cdots, a_0$ are constants.

So there exist z_m, $z_m \to \infty$, such that $f(z_m) = 0$, $m = 1, 2, \cdots$. By the condition that all zeros of f have multiplicity at least k (≥ 2), we have $f'(z_m) = 0$. Set

$$P(z) = A^{-1}f'(z) - f(z).$$

It is obvious to see that P is a polynomial and $P(z_m) = 0$, $m = 1, 2, \cdots$. Then we have $P(z) \equiv 0$, $f(z) = C\exp(Az)$, where $C \neq 0$ is a constant, a contradiction.

Case 2 f is a polynomial.

Then by $f^{(k)} \neq a$, we have $f^{(k)}(z) = b$, where $b \neq a$ is a constant. Since all zeros of f have multiplicity at least k (≥ 2),

$$f(z) = \frac{b(z - z_0)^k}{k!},$$

where z_0 is a constant.

Lemma 2.4 *Let $\{f_n\}$ be a sequence of functions holomorphic on a domain $D \subset \mathbb{C}$, all of whose zeros have multiplicity at least k and $\{h_n\}$ be a sequence of functions analytic on D such that $h_n(z) \Longrightarrow h(z)$ on D, where $h(z) \neq 0$ for $z \in D$ and $k \geq 2$ is an integer. Suppose that, for each n, $f_n(z) = 0 \Longrightarrow |f_n^{(k)}(z)| < |h_n(z)|$ and $f_n^{(k)}(z) \neq h_n(z)$. Then $\{f_n\}$ is normal on D.*

Proof Suppose to the contrary that there exists a $z_0 \in D$ such that $\{f_n\}$ is not normal at z_0. The convergence of $\{h_n\}$ to h implies that, in some neighborhood of z_0, we have $f_n(z) = 0 \Longrightarrow |f_n^{(k)}(z)| \leq |h(z_0)| + 1$ (for large enough n). Thus we can apply Lemma 2.1 with $\alpha = k$ and $A = |h(z_0)| + 1$. So we can take an appropriate subsequence of $\{f_n\}$ (denoted also by $\{f_n\}$ after renumbering), together with points $z_n \to z_0$ and positive numbers $\rho_n \to 0^+$ such that

$$g_n(\zeta) = \frac{f_n(z_n + \rho_n\zeta)}{\rho_n^k} \overset{\chi}{\Longrightarrow} g(\zeta), \quad \text{on } \mathbb{C},$$

where g is a nonconstant entire function, all of whose zeros have multiplicity at least k and $g^\sharp(\zeta) \leq g^\sharp(0) = k(|h(z_0)| + 1) + 1$.

We claim that $g = 0 \Longrightarrow |g^{(k)}| \leq |h(z_0)|$ and $g^{(k)} \neq h(z_0)$.

In fact, if there exists a $\zeta_0 \in \mathbb{C}$, such that $g(\zeta_0) = 0$, then since $g(\zeta) \not\equiv 0$, there exist ζ_n, $\zeta_n \to \zeta_0$, such that if n is sufficiently large,

$$g_n(\zeta_n) = \frac{f_n(z_n + \rho_n\zeta_n)}{\rho_n^k} = 0.$$

Thus $f_n(z_n + \rho_n\zeta_n) = 0$, so that $|f_n^{(k)}(z_n + \rho_n\zeta_n)| < |h_n(z_n + \rho_n\zeta_n)|$, i.e., $|g_n^{(k)}(\zeta_n)| < |h_n(z_n + \rho_n\zeta_n)|$. Since $|g^{(k)}(\zeta_0)| = \lim\limits_{n\to\infty} |g_n^{(k)}(\zeta_n)| \leq |h(z_0)|$, we have established the first part of the claim.

Now, suppose that there exists a $\zeta_0 \in \mathbb{C}$, such that $g^{(k)}(\zeta_0) = h(z_0)$. If $g^{(k)}(\zeta) \equiv h(z_0)$, then we have $g^\sharp(0) \leq k|h(z_0)|$, which contradicts $g^\sharp(0) = k(|h(z_0)| + 1) + 1$. Thus $g^{(k)}$ is not constant. So by Hurwitz's theorem, there exist ζ_n, $\zeta_n \to \zeta_0$, such that

$$f_n^{(k)}(z_n + \rho_n\zeta_n) - h_n(z_n + \rho_n\zeta_n) = g_n^{(k)}(\zeta_n) - h_n(z_n + \rho_n\zeta_n) = 0,$$

which contradicts $f_n^{(k)} \neq h_n$. This completes the proof of the claim.

By Lemma 2.3,

$$g(\zeta) = \frac{b}{k!}(\zeta - \zeta_0)^k,$$

where $\zeta_0 \in \mathbb{C}$ and $b \neq h(z_0)$ are constants. Since $g(\zeta_0) = 0$, $|g^{(k)}(\zeta_0)| = |b| \leq |h(z_0)|$. We have $g^\sharp(0) \leq k|b| \leq k|h(z_0)|$, a contradiction. The lemma is proved.

Lemma 2.5 *Let h be a holomorphic function on D with a zero of order ℓ (≥ 1) at $z_0 \in D$, $\{f_n\}_1^\infty$ be a sequence of functions such that $\{f_n\}$ and h satisfy conditions* (a) *and* (b) *of Theorem* 1.3. *Let $\{\alpha_n\}_{n=1}^\infty$ be a sequence of nonzero numbers such that $\alpha_n \to 0$ as $n \to \infty$. Then*

(a) $\left\{\frac{f_n(z_0+\alpha_n\zeta)}{\alpha_n^{k+\ell}}\right\}_{n=1}^\infty$ *is normal in $\mathbb{C}^*$.*

In addition, if

$$\frac{f_n(z_0+\alpha_n\zeta)}{\alpha_n^{k+\ell}} \Longrightarrow G(\zeta), \quad \text{on } \mathbb{C}^* \ (\text{or on } \mathbb{C}),$$

where $G(\zeta) \not\equiv 0$, then

(b)

(i) $G(\zeta_0) = 0 \Longrightarrow |G^{(k)}(\zeta_0)| \leq |\zeta_0^\ell|$ *for every $\zeta_0 \in \mathbb{C}^*$ (or for every $\zeta_0 \in \mathbb{C}$);*

(ii) *If $G^{(k)}(\zeta) \not\equiv \zeta^\ell$, then $G^{(k)}(\zeta) \neq \zeta^\ell$.*

Proof Without loss of generality, we may assume that $z_0 = 0$. In a neighborhood of the origin, we have $h(z) = z^\ell b(z)$, where $b(z)$ is analytic, $b(0) \neq 0$. Without loss of generality, we can assume that $b(0) = 1$. Define $r_n(\zeta) = \zeta^\ell b(\alpha_n\zeta)$. We will show that the assumptions of Lemma 2.4 hold in $\mathbb{C}^*$ for the sequence $\{G_n(\zeta)\}_{n=1}^\infty$, $G_n(\zeta) := \frac{f_n(\alpha_n\zeta)}{\alpha_n^{k+\ell}}$ and $\{r_n(\zeta)\}_{n=1}^\infty$. First, we have that $r_n(\zeta) \Longrightarrow \zeta^\ell$ on $\mathbb{C}$ and $\zeta^\ell \neq 0$ in $\mathbb{C}^*$. Assume that $G_n(\zeta) = 0$. Hence $f_n(\alpha_n\zeta) = 0$ and $|f_n^{(k)}(\alpha_n\zeta)| < |(\alpha_n\zeta)^\ell b(\alpha_n\zeta)|$, and we get $|G_n^{(k)}(\zeta)| < |r_n(\zeta)|$. Obviously, we have

$$G_n^{(k)}(\zeta) = \frac{f_n^{(k)}(\alpha_n\zeta)}{\alpha_n^\ell} \neq \frac{h(\alpha_n\zeta)}{\alpha_n^\ell} = r_n(\zeta),$$

which means that the assumptions of Lemma 2.4 hold. Hence we deduce that $\{G_n(\zeta)\}$ is normal in $\mathbb{C}^*$, and (a) is proved.

Suppose now that $G(\zeta_0) = 0$. Then there exist $\zeta_n \to \zeta_0$ such that $G_n(\zeta_n) = 0$, i.e., $f_n(\alpha_n\zeta_n) = 0$. It then follows that $|f_n^{(k)}(\alpha_n\zeta_n)| < |\alpha_n^\ell\zeta_n^\ell b(\alpha_n\zeta_n)|$, and this implies $|G_n^{(k)}(\zeta_n)| < |\zeta_n^\ell b(\alpha_n\zeta_n)|$. Letting $n \to \infty$, $|G^{(k)}(\zeta_0)| \leq |\zeta_0^\ell|$, so (i) of (b) is proved.

For the proof of (ii), observe first that

$$\frac{f_n^{(k)}(\alpha_n\zeta)}{\alpha_n^\ell b(\alpha_n\zeta)} = G_n^{(k)}(\alpha_n\zeta)b(\alpha_n\zeta) \overset{\chi}{\Longrightarrow} G^{(k)}(\zeta), \quad \text{on } \mathbb{C}. \tag{2.1}$$

If $G^{(k)}(\zeta_0) = \zeta_0^\ell$, then by (2.1) we have $\zeta_n \to \zeta_0$ such that

$$f_n^{(k)}(\alpha_n\zeta_n) = \left[\alpha_n^\ell b(\alpha_n\zeta_n)\right]\zeta_n^\ell = h(\alpha_n\zeta_n),$$

which contradicts the condition (b) of Theorem 1.3. This completes the proof of the lemma.

3 Proof of Theorem 1.3

By Lemma 2.4, $\mathcal{F}$ is normal at every point $z_0 \in D$ at which $h(z_0) \neq 0$ (so that $\mathcal{F}$ is quasinormal in D). Consider $z_0 \in D$ such that $h(z_0) = 0$. Without loss of generality, we can assume $z_0 = 0$. Then

$$h(z) = z^\ell b(z), \tag{3.1}$$

where ℓ (≥ 1) is an integer, $b(z) \neq 0$ is an analytic function in $\Delta(0,\delta)$ and we can assume also that $b(0) = 1$. We take a subsequence $\{f_n\}_1^\infty \subset \mathcal{F}$, and we want to prove that $\{f_n\}$ is normal at $z = 0$. Suppose by negation that $\{f_n\}$ is not normal at $z = 0$. Since $\{f_n\}$ is normal in $\Delta'(0,\delta)$, we can assume (after renumbering) that $f_n \Longrightarrow F$ on $\Delta'(0,\delta)$. If $F(z) \not\equiv \infty$, then it is a holomorphic function. Hence by the maximum principle, F extends to be analytic also at $z = 0$. So $f_n \Longrightarrow F$ on $\Delta(0,\delta)$, and we are done. Hence we assume that

$$f_n(z) \Longrightarrow \infty, \quad \text{on } \Delta'(0,\delta). \tag{3.2}$$

Define $\mathcal{F}_1 = \left\{F_n = \frac{f_n}{h} : n \in \mathbb{N}\right\}$. It is enough to prove that $\mathcal{F}_1$ is normal in $\Delta(0,\delta)$. Indeed, if (after renumbering) $\frac{f_n(z)}{h} \Longrightarrow H(z)$ on $\Delta(0,\delta)$, then since $h \neq 0$ in $\Delta'(0,\delta)$, it follows from (3.2) that $H(z) \equiv \infty$ in $\Delta'(0,\delta)$, and thus $H(z) \equiv \infty$ also in $\Delta(0,\delta)$. In particular, $\frac{f_n}{h}(z) \neq 0$ on each compact subset of $\Delta(0,\delta)$ for large enough n. Since $h \neq 0$ on $\Delta'(0,\delta)$ and $f_n(0) \neq 0$ for every $n \geq 1$, by the assumptions of the theorem, we obtain $f_n(z) \neq 0$ on each compact subset of $\Delta(0,\delta)$ for large enough n. Then by the minimum principle, it follows from (3.2) that $f_n(z) \Longrightarrow \infty$ on $\Delta(0,\delta)$, and this implies the normality of $\mathcal{F}$. So suppose to the contrary that $\mathcal{F}_1$ is not normal at $z = 0$. By Lemma 2.1 and the assumptions of Theorem 1.3, there exist (after renumbering) points $z_n \to 0$, $\rho_n \to 0^+$ and a nonconstant meromorphic function on $\mathbb{C}$, $g(\zeta)$ such that

$$g_n(\zeta) = \frac{F_n(z_n + \rho_n\zeta)}{\rho_n^k} = \frac{f_n(z_n + \rho_n\zeta)}{\rho_n^k h(z_n + \rho_n\zeta)} \stackrel{\chi}{\Longrightarrow} g(\zeta), \quad \text{on } \mathbb{C}, \tag{3.3}$$

all of whose zeros have multiplicity at least k and

$$\text{for every } \zeta \in \mathbb{C}, \quad g^\sharp(\zeta) \leq g^\sharp(0) = kA + 1, \tag{3.4}$$

where $A > 1$ is a constant. Here we have used Lemma 2.1 with $\alpha = k$. Observe that $g_n(\zeta) = 0$ implies $|g_n^{(k)}(\zeta)| < 1$ and so A can be chosen to be any number such that $A \geq 1$. After renumbering, we can assume that $\left\{\frac{z_n}{\rho_n}\right\}_{n=1}^\infty$ converges. We separate it now into two cases.

Case 1

$$\frac{z_n}{\rho_n} \to \infty. \tag{3.5}$$

Claim (1) $g(\zeta) = 0 \Longrightarrow |g^{(k)}(\zeta)| \leq 1$; (2) $g^{(k)}(\zeta) \neq 1$.

Proof of the Claim From (3.3) and the fact that $h(z) \neq 0$ in $\Delta'(0,\delta)$, we have that g is an entire function. Suppose $g(\zeta_0) = 0$. Since $g(\zeta) \not\equiv 0$, there exist $\zeta_n \to \zeta_0$, such that $g_n(\zeta_n) = 0$. Thus $f_n(z_n + \rho_n\zeta_n) = 0$. By assumption, we then have $f_n^{(j)}(z_n + \rho_n\zeta_n) = 0$ and $|f_n^{(k)}(z_n + \rho_n\zeta_n)| < |h(z_n + \rho_n\zeta_n)|$, where $j = 2, 3, \cdots, k-1$. Thus $|g_n^{(k)}(\zeta_n)| < 1$. Letting $n \to \infty$, we obtain $|g^{(k)}(\zeta_0)| \leq 1$.

If there exists a $\zeta_0 \in \mathbb{C}$ such that $g^{(k)}(\zeta_0) = 1$, then there exists a neighborhood $U = U(\zeta_0)$ of ζ_0, such that the functions $g_n^{(j)}$ are analytic on U for sufficiently large n, $j = 0, 1, \cdots, k+1$.

Obviously,

$$\begin{aligned}g_n^{(k)}(\zeta) = F_n^{(k)}(z_n+\rho_n\zeta) &= \Big(\frac{f_n(z)}{h(z)}\Big)^{(k)}\Big|_{z=z_n+\rho_n\zeta}\\ &= \Big[\frac{f_n^{(k)}(z)}{h(z)} + \sum_{j=1}^{k}\binom{k}{j} f_n^{(k-j)}(z)\Big(\frac{1}{h(z)}\Big)^{(j)}\Big]\Big|_{z=z_n+\rho_n\zeta}.\end{aligned}$$

By Leibniz's formula, we have that

$$f_n^{(k-j)}(z) = [F_n(z)h(z)]^{(k-j)} = \sum_{s=0}^{k-j}\binom{k-j}{s}\rho_n^{j+s} g_n^{(k-j-s)}\Big(\frac{z-z_n}{\rho_n}\Big)h^{(s)}(z)$$

and

$$\Big(\frac{1}{h(z)}\Big)^{(j)} = [z^{-\ell}\widetilde{b}(z)]^{(j)} = z^{-\ell-j}[(-1)^j\ell(\ell+1)\cdots(\ell+j-1)\widetilde{b}(z)+P(z)],$$

where $\widetilde{b}(z) = \frac{1}{b(z)}$, and $P(z)$ is holomorphic on $\Delta(0,\delta)$ with $P(0)=0$.

Since

$$\begin{aligned}\rho_n^{j+s}h^{(s)}(z)z^{-\ell-j}|_{z=z_n+\rho_n\zeta} &= \rho_n^{j+s}z^{\ell-s}Q(z)z^{-\ell-j}|_{z=z_n+\rho_n\zeta}\\ &= \frac{\rho_n^{j+s}}{(z_n+\rho_n\zeta)^{j+s}}Q(z_n+\rho_n\zeta) \Longrightarrow 0\end{aligned}$$

on $\mathbb{C}$, where $Q(z)$ is holomorphic on $\Delta(0,\delta)$ and $Q(0) = \ell(\ell-1)\cdots(\ell-s+1)\neq 0$, we have

$$\begin{aligned}f_n^{(k-j)}(z)\Big(\frac{1}{h(z)}\Big)^{(j)}\Big|_{z=z_n+\rho_n\zeta} &= \sum_{s=0}^{k-j}\binom{k-j}{s}\rho_n^{j+s}g_n^{(k-j-s)}\Big(\frac{z-z_n}{\rho_n}\Big)h^{(s)}(z)\\ &\quad\times z^{-\ell-j}[(-1)^j\ell(\ell+1)\cdots(\ell+j-1)\widetilde{b}(z)+P(z)]\Big|_{z=z_n+\rho_n\zeta}\\ &\Longrightarrow 0, \quad \text{on } \mathbb{C}\setminus\{\text{the poles of } g\}.\end{aligned} \tag{3.6}$$

Now

$$\frac{f_n^{(k)}(z_n+\rho_n\zeta)}{h(z_n+\rho_n\zeta)} \Longrightarrow g^{(k)}(\zeta), \quad \text{on } \mathbb{C}\setminus\{\text{the poles of } g\}.$$

So $\frac{f_n^{(k)}(z_n+\rho_n\zeta)}{h(z_n+\rho_n\zeta)}$ converges locally uniformly to $g^{(k)}(\zeta)$ on U. By (3.4) we deduce that $g^{(k)}(\zeta)\not\equiv 1$. Thus there exist $\zeta_n\to\zeta_0$, such that $\frac{f_n^{(k)}(z_n+\rho_n\zeta_n)}{h(z_n+\rho_n\zeta_n)} = 1$. So

$$f_n^{(k)}(z_n+\rho_n\zeta_n) = h(z_n+\rho_n\zeta_n), \tag{3.7}$$

which contradicts the condition (b) of Theorem 1.3. Then the claim is proved.

Also by Lemma 2.3, we have

$$g(\zeta) = \frac{b}{k!}(\zeta-\zeta_0)^k,$$

where $\zeta_0\in\mathbb{C}$ and $b\neq 1$ are constants. Since $g(\zeta_0)=0$, $|g^{(k)}(\zeta_0)| = |b|\leq 1$. We have $g^\sharp(0)\leq k|b|\leq k$, a contradiction.

Case 2

$$\frac{z_n}{\rho_n}\to\alpha\in\mathbb{C}. \tag{3.8}$$

As before, we have $g(\zeta_0)=0 \Longrightarrow |g^{(k)}(\zeta_0)|\le 1$. Now set

$$G_n(\zeta)=\frac{f_n(\rho_n\zeta)}{\rho_n^{k+\ell}}.$$

From (3.3) and (3.8) we have

$$G_n(\zeta)\Longrightarrow G(\zeta)=g(\zeta-\alpha)\zeta^{\ell}, \quad \text{on } \mathbb{C}.$$

Indeed,

$$\frac{f_n(\rho_n\zeta)}{\rho_n^{k+\ell}}=\frac{f_n(\rho_n\zeta)}{\rho_n^k h(\rho_n\zeta)}\cdot\frac{h(\rho_n\zeta)}{\rho_n^{\ell}}=\frac{f_n(z_n+\rho_n(\zeta-\frac{z_n}{\rho_n}))}{\rho_n^k h\big(z_n+\rho_n(\zeta-\frac{z_n}{\rho_n})\big)}\frac{(\rho_n\zeta)^{\ell}b(\rho_n\zeta)}{\rho_n^{\ell}}$$

(see [10, p. 7]). Since g has a pole of order ℓ at $\zeta=-\alpha$,

$$G(0)\ne 0,\ \infty. \tag{3.9}$$

We now consider several subcases, depending on the nature of G.

Case 2.1 G is a polynomial.

Since $\{f_n\}$ is not normal at $z=0$, there exists (after renumbering) a sequence $z_n^*\to 0$ such that

$$f_n(z_n^*)=0. \tag{3.10}$$

Otherwise, there is some δ', $0<\delta'<\delta$ such that (before renumbering) $f_n(z)\ne 0$ in $\Delta(0,\delta')$. Since $f_n(z)\Longrightarrow\infty$ on $\Delta'(0,\delta)$, by the minimum principle, we would have that $f_n(z)\Longrightarrow\infty$ on $\Delta(0,\delta)$, a contradiction to the non-normality of $\{f_n\}$ at $z=0$. If G is a polynomial of degree $\ell\ge 1$, then by Lemma 2.5 and (3.9), all zeros of $G(\zeta)$ have multiplicity exactly k. We consider now two kinds of possibilities.

Case 2.1.1 $G^{(k)}\equiv\zeta^{\ell}$.

Since $k\ge 2$, we have $G^{(k-1)}(\zeta)=\frac{\zeta^{\ell+1}}{\ell+1}+C$ and $G^{(k-2)}(\zeta)=\frac{\zeta^{\ell+2}}{(\ell+1)(\ell+2)}+C\zeta+D$, where C and D are two constants. Since all zeros of G have multiplicity at least k, for any zero ζ_j of G, we have $G^{(k-2)}(\zeta_j)=G^{(k-1)}(\zeta_j)=0$. So

$$\frac{\zeta_j^{\ell+1}}{\ell+1}+C=0 \quad\text{and}\quad \frac{\zeta_j^{\ell+2}}{(\ell+1)(\ell+2)}+C\zeta_j+D=0. \tag{3.11}$$

By calculation, we have $\frac{(\ell+1)C}{\ell+2}\zeta_j=-D$. If $CD=0$, then by (3.11), $\zeta_j=0$, a contradiction. So $CD\ne 0$ and $\zeta_j=-\frac{(\ell+2)D}{(\ell+1)C}$, which implies that G has only one zero ζ_0. Thus

$$G=\frac{\ell!(\zeta-\zeta_0)^{k+\ell}}{(k+\ell)!}. \tag{3.12}$$

Since $G^{(k)}\equiv\zeta^{\ell}$, $\zeta_0=0$, a contradiction.

Case 2.1.2 $G^{(k)}\not\equiv\zeta^{\ell}$.

By Lemma 2.5, we have $G(\zeta)=0\Longrightarrow|G^{(k)}(\zeta)|\le|\zeta^{\ell}|$ and $G^{(k)}\ne\zeta^{\ell}$. So G is a nonconstant polynomial and $G^{(k)}=\zeta^{\ell}+B$, where $B\ne 0$ is a constant. Since all zeros of G have multiplicity at least k, for any zero ζ_j of G, we have $G^{(k-2)}(\zeta_j) = G^{(k-1)}(\zeta_j) = 0$. So

$$\frac{\zeta_j^{\ell+1}}{\ell+1} + B\zeta_j + C = 0 \quad \text{and} \quad \frac{\zeta_j^{\ell+2}}{(\ell+1)(\ell+2)} + \frac{B\zeta_j^2}{2} + C\zeta_j + D = 0. \tag{3.13}$$

By calculation, we have $\frac{\ell B}{2(\ell+2)}\zeta_j^2 + \frac{C(\ell+1)}{\ell+2}\zeta_j + D = 0$, which implies that G has at most two zeros ζ_1, ζ_2. Then we divide it into two subcases.

Case 2.1.2(a) G has only one zero ζ_1.

Set

$$G(\zeta) = \frac{\ell!}{(k+\ell)!}(\zeta - \zeta_1)^{k+\ell}. \tag{3.14}$$

Since $G^{(k)} = \zeta^\ell + B$, we have $\ell = 1$ and $\zeta_1 = -B$. So

$$G(\zeta) = \frac{(\zeta + B)^{k+1}}{(k+1)!}. \tag{3.15}$$

By Hurwitz's theorem, there exists a sequence $\zeta_{n,0} \to -B$, such that $G_n(\zeta_{n,0}) = 0$. If there exists a δ', $0 < \delta' < \delta$, such that for every n (after renumbering), $f_n(z)$ has only one zero $z_{n,0} = \rho_n \zeta_{n,0}$ in $\Delta(0, \delta')$.

Set

$$H_n(z) = \frac{f_n(z)}{(z - z_{n,0})^{k+1}}.$$

Since $H_n(z)$ is a nonvanishing holomorphic function in $\Delta(0, \delta')$ and $H_n(z) \Longrightarrow \infty$ on $\Delta'(0, \delta)$, we can deduce as before by the minimum principle that $H_n(z) \Longrightarrow \infty$ on $\Delta(0, \delta')$. But

$$H_n(2z_{n,0}) = \frac{f_n(2z_{n,0})}{z_{n,0}^{k+1}} = \frac{G_n(2\zeta_{n,0})}{\zeta_{n,0}^{k+1}} \to \frac{1}{(k+1)!}, \tag{3.16}$$

a contradiction. Thus, we can assume, after renumbering, that for every $\delta' > 0$, f_n has at least two zeros in $\Delta(0, \delta')$ for large enough n. Thus, there exists another sequence of points $z_{n,1} = \rho_n \zeta_{n,1}$, tending to zero, where $z_{n,1}$ is also a zero of $f_n(z)$ and $\zeta_{n,1} \to \infty$, as $n \to \infty$. We can also assume that $z_{n,1}$ is the closest zero to the origin of f_n, except $z_{n,0}$. Now set $c_n = \frac{z_{n,0}}{z_{n,1}}$ and define $K_n(\zeta) = \frac{f_n(z_{n,1}\zeta)}{z_{n,1}^{k+1}}$. By Lemma 2.5, $\{K_n(\zeta)\}$ is normal in $\mathbb{C}^*$. Now, if $\{K_n\}$ is normal at $\zeta = 0$, then after renumbering we can assume that

$$K_n(\zeta) \Longrightarrow K(\zeta), \quad \text{on } \mathbb{C}.$$

Since $K_n(c_n) = 0$ and $c_n \to \infty$, letting $n \to \infty$, we obtain $K(0) = 0$. Also we have $K^{(k)}(\zeta) \equiv \zeta$ or $K^{(k)}(\zeta) \neq \zeta$, by $K_n^{(k)}(\zeta) = \frac{f_n^{(k)}(z_{n,1}\zeta)}{z_{n,1}} \neq \zeta b(z_{n,1}\zeta)$.

If $K^{(k)}(\zeta) \equiv \zeta$, by $K(0) = 0$, we have $K(\zeta) = \frac{z^{k+1}}{(k+1)!}$, which contradicts $K(1) = 0$.

If $K^{(k)}(\zeta) \neq \zeta$, by Lemma 2.5, we have $K(\zeta) = 0 \Longrightarrow |K^{(k)}(\zeta)| \leq |\zeta|$ and then $K^{(k)}(0) = 0$, a contradiction.

Hence we can deduce that $\{K_n\}$ is not normal at $\zeta = 0$. Since $K_n(\zeta)$ is holomorphic in Δ, we have

$$K_n(\zeta) \Longrightarrow \infty, \quad \text{on } \mathbb{C}^*.$$

But $K_n(1) = 0$, a contradiction.

Case 2.1.2(b) G has exactly two distinct zeros ζ_1, ζ_2.

By $G^{(k+1)} = \ell\zeta^{\ell-1}$, we have that none of the two zeros of G has multiplicity at least $k+2$.

If both of the two zeros of G has multiplicity exactly $k+1$, then we may assume that

$$G(\zeta) = \frac{\ell!}{(k+\ell)!}(\zeta-\zeta_1)^{k+1}(\zeta-\zeta_2)^{k+1}. \tag{3.17}$$

Since $G^{(k)}(\zeta) = \zeta^\ell + B$, by calculation, we have $\ell = k+2$ and $\zeta_1+\zeta_2 = 0$, $\zeta_1\zeta_2 = 0$, a contradiction.

If only one of the two zeros of G have multiplicity exactly $k+1$, then we may assume that

$$G(\zeta) = \frac{\ell!}{(k+\ell)!}(\zeta-\zeta_1)^{k+1}(\zeta-\zeta_2)^{k}. \tag{3.18}$$

By (3.18),

$$G(\zeta) = \frac{\ell!}{(k+\ell)!}(\zeta-\zeta_1)\Big[\zeta^{2k} - k(\zeta_1+\zeta_2)\zeta^{2k-1} + \Big(k\zeta_1\zeta_2 + \binom{k}{2}(\zeta_1+\zeta_2)^2\Big)\zeta^{2k-2} + \cdots\Big].$$

Since $G^{(k)}(\zeta) = \zeta^\ell + B$, by calculation, we have $\ell = k+1$ and

$$k(\zeta_1+\zeta_2)+\zeta_1 = 0, \quad k(\zeta_1+\zeta_2)\zeta_1 + k\zeta_1\zeta_2 + \binom{k}{2}(\zeta_1+\zeta_2)^2 = 0, \tag{3.19}$$

which means $\zeta_1 = 0$, a contradiction.

If both of the two zeros of G have multiplicity exactly k, then we may assume that

$$G(\zeta) = \frac{\ell!}{(k+\ell)!}(\zeta-\zeta_1)^{k}(\zeta-\zeta_2)^{k}. \tag{3.20}$$

Since $G^{(k)}(\zeta) = \zeta^\ell + B$, by calculation, we have $\ell = k$ and $\zeta_1+\zeta_2 = 0$.

For $k \geq 3$, we also have $\zeta_1\zeta_2 = 0$, a contradiction.

For $k = 2$, we have

$$G(\zeta) = \frac{1}{12}(\zeta-\zeta_1)^2(\zeta+\zeta_1)^2. \tag{3.21}$$

By Hurwitz's theorem, there exist sequences $\zeta_{n,1} \to \zeta_1$, $\zeta_{n,2} \to -\zeta_1$, such that $G_n(\zeta_{n,j}) = 0$, $j = 1, 2$. If there exists a δ', $0 < \delta' < \delta$, such that for every n (after renumbering), $f_n(z)$ has only two zeros $z_{n,j} = \rho_n\zeta_{n,j}$, $j = 1, 2$ in $\Delta(0, \delta')$.

Set

$$H_n(z) = \frac{f_n(z)}{(z-z_{n,1})^2(z-z_{n,2})^2}.$$

Since $H_n(z)$ is a nonvanishing holomorphic function in $\Delta(0,\delta')$ and $H_n(z) \Longrightarrow \infty$ on $\Delta'(0,\delta)$, we can deduce as before by the minimum principle that $H_n(z) \Longrightarrow \infty$ on $\Delta(0,\delta')$. But

$$H_n(2z_{n,1}) = \frac{f_n(2z_{n,1})}{z_{n,1}^2(2z_{n,1}-z_{n,2})^2} = \frac{G_n(2\zeta_{n,1})}{\zeta_{n,1}^2(2\zeta_{n,1}-\zeta_{n,2})^2} \to \frac{1}{12}, \tag{3.22}$$

a contradiction. Thus, we can assume, after renumbering, that for every $\delta' > 0$, f_n has at least two zeros in $\Delta(0,\delta')$ for large enough n. Thus, there exists another sequence of points

$z_{n,3} = \rho_n \zeta_{n,3}$ tending to zero, where $z_{n,3}$ is also a zero of $f_n(z)$ and $\zeta_{n,3} \to \infty$, as $n \to \infty$. We can also assume that $z_{n,3}$ is the closest zero to the origin of f_n, except $z_{n,j}$, $j = 1, 2$. Now set $c_{n,j} = \frac{z_{n,j}}{z_{n,3}}$, $j = 1, 2$ and define $K_n(\zeta) = \frac{f_n(z_{n,3}\zeta)}{z_{n,3}^4}$. By Lemma 2.5, $\{K_n(\zeta)\}$ is normal in $\mathbb{C}^*$. Now, if $\{K_n\}$ is normal at $\zeta = 0$, then after renumbering we can assume that

$$K_n(\zeta) \Longrightarrow K(\zeta), \quad \text{on } \mathbb{C}.$$

Since $K_n(c_{n,j}) = 0$ and $c_{n,j} \to \infty$, $j = 1, 2$, letting $n \to \infty$, we obtain $K(0) = 0$. Also we have $K''(\zeta) \equiv \zeta^2$ or $K''(\zeta) \neq \zeta^2$, by $K_n''(\zeta) = \frac{f_n''(z_{n,3}\zeta)}{z_{n,3}^2} \neq \zeta^2 b(z_{n,3}\zeta)$.

If $K''(\zeta) \equiv \zeta^2$, by $K(0) = 0$, we have $K(\zeta) = \frac{\zeta^4}{12}$, which contradicts $K(1) = 0$.

If $K''(\zeta) \neq \zeta^2$, by Lemma 2.5, we have $K(\zeta) = 0 \Longrightarrow |K''(\zeta)| \leq |\zeta^2|$ and then $K''(0) = 0$, a contradiction.

Hence we can deduce that $\{K_n\}$ is not normal at $\zeta = 0$. Since $K_n(\zeta)$ is holomorphic in Δ, we have

$$K_n(\zeta) \Longrightarrow \infty, \quad \text{on } \mathbb{C}^*.$$

But $K_n(1) = 0$, a contradiction.

Case 2.2 $G(\zeta)$ is a transcendental entire function.

By Lemma 2.5, we have

$$G(\zeta) = 0 \Longrightarrow |G^{(k)}(\zeta)| \leq |\zeta^\ell| \quad \text{and} \quad G^{(k)}(\zeta) \neq \zeta^\ell. \tag{3.23}$$

Since G is a transcendental entire function with order at most 1, we have

$$G^{(k)}(\zeta) = \zeta^\ell + B\exp(A\zeta), \tag{3.24}$$

where $A \neq 0$, $B \neq 0$ are two constants. By calculation,

$$G(\zeta) = \frac{\ell!}{(k+\ell)!}\zeta^{k+\ell} + a_{k-1}\zeta^{k-1} + \cdots + a_0 + BA^{-k}\exp(A\zeta). \tag{3.25}$$

Obviously, G has infinitely many zeros ζ_m on $\mathbb{C}$, and $\zeta_m \to \infty$, $m \to \infty$. By (3.23), $|G^{(k)}(\zeta_m)| = |\zeta_m^\ell + B\exp(A\zeta_m)| \leq |\zeta_m^\ell|$, there exists an $M > 0$, such that, for every m,

$$\Big|\frac{\exp(A\zeta_m)}{\zeta_m^\ell}\Big| \leq M.$$

But

$$\Big|\frac{G(\zeta_m)}{\zeta_m^\ell}\Big| = \Big|\frac{\ell!}{(k+\ell)!}\zeta_n^k + a_{k-1}\zeta_m^{k-1-\ell} + \cdots + a_0\zeta_m^{-\ell} + \frac{BA^{-k}\exp(A\zeta_m)}{\zeta_m^\ell}\Big| \to \infty,$$

a contradiction. The theorem is proved.

Acknowledgement The authors are very grateful to the referee for his or her many valuable suggestions.

References

[1] Clunie, J. and Hayman, W. K., The spherical derivative of integral and meromorphic functions, *Comment. Math. Helv.*, **40**, 1966, 117–148.

[2] Liu, X. J. and Nevo, S., A criterion of normality based on a single holomorphic function, *Acta Math. Sinica,* **27**, 2011, 141–154.

[3] Pang, X. C., Bloch's principle and normal criterion, *Sci. China Ser. A*, **32**, 1989, 782–791.

[4] Pang, X. C., Shared values and normal families, *Analysis*, **22**, 2002, 175–182.

[5] Pang, X. C., Yang, D. G. and Zalcman, L., Normal families of meromorphic functions whose derivative omit a function, *Comput. Methods Funct. Theory*, **2**, 2002, 257–265.

[6] Pang, X. C. and Zalcman, L., Normal families and shared values, *Bull. London Math. Soc.*, **32**, 2000, 325–331.

[7] Pang, X. C. and Zalcman, L., Normal families of meromorphic functions with multiple zeros and poles, *Israel J. Math.*, **136**, 2003, 1–9.

[8] Wang, Y. F. and Fang, M. L., Picard values and normal families of meromorphic functions with multiple zeros, *Acta Math. Sinica* (*N. S.*), **14**, 1998, 17–26.

[9] Zalcman, L., A heuristic principle in complex function theory, *Amer. Math. Monthly*, **82**, 1975, 813–817.

[10] Zalcman, L., Normal families: new perspectives, *Bull. Amer. Math. Soc.* (*N. S.*), **35**, 1998, 215–230.