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Global Smooth Solutions to the 2-D Inhomogeneous Navier-Stokes Equations with Variable Viscosity**

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(*Dedicated to Professor Andrew Majda on the Occasion of his* 60*th Birthday*)

Abstract Under the assumptions that the initial density ρ_0 is close enough to 1 and $\rho_0 - 1 \in H^{s+1}(\mathbb{R}^2)$, $u_0 \in H^s(\mathbb{R}^2) \cap \dot{H}^{-\varepsilon}(\mathbb{R}^2)$ for $s > 2$ and $0 < \varepsilon < 1$, the authors prove the global existence and uniqueness of smooth solutions to the 2-D inhomogeneous Navier-Stokes equations with the viscous coefficient depending on the density of the fluid. Furthermore, the L^2 decay rate of the velocity field is obtained.

Keywords Inhomogeneous Navier-Stokes equations, Littlewood-Paley theory, Global smooth solutions
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1 Introduction

In this paper, we consider the following 2-D inhomogeneous incompressible Navier-Stokes equations (INS for short)

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^2, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \operatorname{div}(\mu \mathcal{M}) + \nabla p = 0, \\ \operatorname{div} u = 0, \end{cases} \tag{1.1}$$

where $\rho, u = (u_1, u_2)$ stand for the density and velocity of the fluid respectively, $\mathcal{M} = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$, p is a scalar pressure function, and the viscosity coefficient $\mu(\rho)$ is a smooth, positive function on $[0, \infty)$. Such system describes a fluid which is obtained by mixing two miscible fluids that are incompressible and that have different densities. It may also describe a fluid containing a melted substance. One may check [14] for the detailed derivation.

Let ρ_0, m_0 satisfy

$$\begin{cases} \rho_0 \geq 0 \quad \text{a.e. in } \mathbb{R}^N, \quad \rho_0 \in L^\infty(\mathbb{R}^N), \\ m_0 \in L^2(\mathbb{R}^N)^N, \quad m_0 = 0 \quad \text{a.e. on } \{\rho_0 = 0\}, \quad \frac{|m_0|^2}{\rho_0} \in L^1(\mathbb{R}^N), \end{cases}$$

where we agree that $\frac{|m_0|^2}{\rho_0} = 0$ a.e. on $\{\rho_0 = 0\}$, and we impose

$$\rho|_{t=0} = \rho_0, \quad \rho u|_{t=0} = m_0. \tag{1.2}$$

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Concerning (1.1)–(1.2), DiPerna and Lions [11, 14] proved the following celebrated theorem in N space dimensions.

Theorem 1.1 *There exists a global weak solution* (ρ, u) *of* (1.1)–(1.2) *such that the following energy inequality holds:*

$$\int_{\mathbb{R}^N} \rho|u|^2 \,\mathrm{d}x + \frac{1}{2}\int_0^t\int_{\mathbb{R}^N} \mu(\rho)(\partial_i u_j + \partial_j u_i)^2 \,\mathrm{d}x\,\mathrm{d}t' \le \int_{\mathbb{R}^N} \frac{|m_0|^2}{\rho_0}\,\mathrm{d}x \quad a.e.\ t\in(0,\infty).$$

Furthermore, for all $0\le\alpha\le\beta<\infty$,

$$\mathrm{meas}\{x\in\mathbb{R}^N : \alpha\le\rho(t,x)\le\beta\} \ \textit{is independent of}\ t\ge 0.$$

One may check [14] for the detailed proof. However, the uniqueness and regularities of thus obtained weak solutions are big open questions even in two space dimension, as was mentioned by Lions in [14].

On the other hand, when $\mu(\rho)$ is independent of ρ, i.e. μ is a positive constant, and ρ_0 is bounded away from 0, it was shown by Kazhikov [12] (see also [3, 4]) that (1.1) has a unique local smooth solution with regular initial data. In addition, they proved the global existence and uniqueness for small enough data in any space dimensions and for all data in two-dimensional case. Similar results were proved by Ladyženskaja and Solonnikov [13] for the initial boundary value problem of (1.1). While when $\rho_0\ge 0$, Simon [16] proved the global existence of weak solutions to (1.1).

According to the statement in [14, p. 31], even when $N=2$, further regularities of the weak solutions obtained in Theorem 1.1 does not seem to be available when μ depends on ρ. Except under the assumptions that

$$\inf_{c>0}\Big\|\frac{\mu(\rho_0)}{c}-1\Big\|_{L^\infty(\mathbb{T}^2)} < \epsilon \quad \text{and} \quad u_0\in H^1(\mathbb{T}^2).$$

Desjardins [10] proved that $u\in L^\infty([0,T];H^1(\mathbb{T}^2))$ and $\rho\in L^\infty([0,T]\times\mathbb{T}^2)$ for the weak solution (ρ,u) constructed in [14]. Moreover, with additional assumptions, he could also prove that $u\in L^2([0,\tau];H^2(\mathbb{T}^2))$ for some short time τ. To understand this problem further, the second author of this paper proved in [19] the global wellposedness to the following model problem with large regular initial data:

$$\begin{cases} \dfrac{\partial\rho}{\partial t} + \mathrm{div}(\rho u) = 0, \quad (t,x)\in\mathbb{R}^+\times\mathbb{R}^2, \\ \dfrac{\partial(\rho u)}{\partial t} + \mathrm{div}(\rho u\otimes u) - \nabla^\perp(\mu(\rho)\omega) + \nabla p = 0, \\ \mathrm{div}\, u = 0, \end{cases} \tag{1.3}$$

where $\omega = \partial_1 u_2 - \partial_2 u_1$ is the vorticity of the fluid, and $\nabla^\perp f = (-\partial_2 f, \partial_1 f)$. Notice that as $\mathrm{div}\,u=0$, $\nabla^\perp(\mu\omega) = \mu\Delta u$ when μ is independent of ρ, so (1.3) coincides with the classical inhomogeneous, incompressible Navier-Stokes equation in this case.

On the other hand, denoting $a \stackrel{\text{def}}{=} \frac{1}{\rho}-1$, and $b\stackrel{\text{def}}{=} a+1 = \frac{1}{\rho}$, $\widetilde{\mu}(a)\stackrel{\text{def}}{=}\mu(\rho)$, we can see that system (1.1) can be reformulated as

$$\text{(INS)} \quad \begin{cases} \partial_t a + u \cdot \nabla a = 0, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^2, \\ \partial_t u + u \cdot \nabla u + b(\nabla p - \operatorname{div}(\widetilde{\mu}(a)\mathcal{M})) = 0, \\ \operatorname{div} u = 0, \\ (a,u)|_{t=0} = (a_0, u_0). \end{cases}$$

In [1], Abidi proved in general space dimension N that if $1 < p < 2N$, $0 < \underline{\mu} < \mu(\rho)$, $u_0 \in \dot{B}^{\frac{N}{p}-1}_{p,1}(\mathbb{R}^N)$ and $a_0 = \frac{1}{\rho_0} - 1 \in \dot{B}^{\frac{N}{p}}_{p,1}(\mathbb{R}^N)$, then (INS) has a global solution provided that $\|a_0\|_{\dot{B}^{\frac{N}{p}}_{p,1}} + \|u_0\|_{\dot{B}^{\frac{N}{p}-1}_{p,1}} \le c_0$ for c_0 sufficiently small. Furthermore, the obtained solution is unique if $1 < p \le N$. This result generalized the corresponding result of [9] for the constant viscosity case. Very recently, Abidi and Paicu [2] improved the wellposedness results in [1, 9] for more general p when $\widetilde{\mu}(a) = \mu$.

We shall prove in this paper that when $N = 2$, (INS) has a unique global smooth solution with smooth initial data provided that a_0 is small enough, while we do not need any restriction for the size of the initial velocity field u_0. More precisely, we have

Theorem 1.2 *Let $s > 2$, $\varepsilon \in (0,1)$. Let $a_0 \in H^{s+1}(\mathbb{R}^2)$ and $u_0 \in \dot{H}^{-\varepsilon}(\mathbb{R}^2) \cap \dot{H}^s(\mathbb{R}^2)$ with* div $u_0 = 0$. *We assume that $\mu(\rho)$ is a smooth and positive function on $[0,\infty)$. Then* (INS) *has a unique global solution (a,u) such that $a \in C([0,\infty), H^{s+1}(\mathbb{R}^2))$ and $u \in C([0,\infty), \dot{H}^{-\varepsilon}(\mathbb{R}^2) \cap \dot{H}^s(\mathbb{R}^2)) \cap \widetilde{L}^1((0,\infty), \dot{H}^{s+2}(\mathbb{R}^2) \cap \dot{H}^{2-\varepsilon}(\mathbb{R}^2))$ provided that*

$$\|a_0\|_{H^{s+1}} < c_0$$

for some small enough constant c_0. Furthermore, there exists some $C > 0$ such that

$$\|u(t)\|_{L^2} \le C(1+t)^{-\min(\varepsilon, \frac{1}{2})}. \tag{1.4}$$

Remark 1.1 (i) One may check the definition of the function spaces $\widetilde{L}^r_T(\dot{H}^s(\mathbb{R}^2))$ in Definition 2.2, which was first introduced by Chemin and Lerner [8].

(ii) The main difficulty of the proof of Theorem 1.2 lies in the estimate of $\|a(t)\|_{H^{s+1}}$. As a satisfies a free transport equation, it is easy to get any L^p control of $\|a(t)\|_{L^p}$ in terms of $\|a_0\|_{L^p}$ via characteristic method. However, in general the derivative estimate of $a(t)$ can not be obtained in such a trivial way. In fact, to control the size of $\|a(t)\|_{H^{s+1}}$, we need to use $\|u\|_{\widetilde{L}^1_\infty(\dot{H}^{s+2})}$ and $\int_0^\infty \|\nabla u(t)\|_{L^\infty} \mathrm{d}t$ (see (4.2)). The assumption that $u_0 \in \dot{H}^{-\varepsilon}(\mathbb{R}^2)$ is related to the decay of $u(t)$ at ∞. In general, if we assume that $\|\mathrm{e}^{\frac{1}{2}\widetilde{\mu}(0)t\Delta} u_0\|_{L^2} \le C(1+t)^{-\nu}$ for $0 < \varepsilon \le \nu$, then one may improve (1.4) to

$$\|u(t)\|_{L^2} \le C(1+t)^{-\min(\nu, \frac{1}{2})}.$$

Compared with the L^2 decay estimates for the homogeneous case in [15, 18], the decay rate in (1.4) might not be the optimal. However as we are mainly concerned with the global well-posedness of (INS) here, we may not pursue this point here.

Notations Let A, B be two operators. We denote $[A; B] = AB - BA$, the commutator between A and B. By $a \lesssim b$, we mean that there is a uniform constant C, which may be different on different lines, such that $a \leq Cb$. We denote $(c_j)_{j\in\mathbf{N}}$ (or $(c_j(t))_{j\in\mathbf{N}}$) to be a sequence in ℓ^2 with norm 1, $(a \mid b)_{L^2} = \int_{\mathbb{R}^2} ab\,\mathrm{d}x$ the standard L^2 inner product of a and b, and $\|u\|_{L^2_T(L^p)} \stackrel{\text{def}}{=} \|u\|_{L^2([0,T];L^p(\mathbb{R}^2))}$.

2 Littlewood-Paley Analysis

For the convenience of the reader, we shall recall some basic facts on Littlewood-Paley Theory. One may check [5–7, 17] for more details.

Proposition 2.1 (Littlewood-Paley Decomposition) *Let $\mathcal{C} \stackrel{def}{=} \{\xi \in \mathbb{R}^d,\ \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}$. There exists a radial function $\varphi \in C_c^\infty(\mathcal{C})$ such that*

$$\sum_{j\in\mathbb{Z}} \varphi(2^{-j}\xi) = 1, \quad \forall\, \xi \in \mathbb{R}^d \setminus \{0\},$$
$$|j - j'| \geq 2 \Rightarrow \mathrm{Supp}\ \varphi(2^{-j}\cdot) \cap \mathrm{Supp}\ \varphi(2^{-j'}\cdot) = \emptyset.$$

Let $h \stackrel{\text{def}}{=} \mathcal{F}^{-1}\varphi$. Then the dyadic operators $\dot{\Delta}_j$, $\dot{S}_j$ with $j \in \mathbb{Z}$ can be defined as follows:

$$\begin{aligned} &\dot{\Delta}_j f \stackrel{\text{def}}{=} \varphi(2^{-j}D)f = 2^{jd}\int_{\mathbb{R}^d} h(2^j y) f(x-y)\mathrm{d}y, \\ &\dot{S}_j f \stackrel{\text{def}}{=} \sum_{j'\leq j-1} \dot{\Delta}_{j'} f. \end{aligned} \tag{2.1}$$

Lemma 2.1 (Bernstein's Inequality) *Let $\mathcal{B}$ be a given ball with center 0. A constant C exists so that, for any positive real number λ, any nonnegative integer k, any smooth homogeneous function σ of degree m, and any couple of real numbers (a, b) with $b \geq a \geq 1$, there hold*

$$\begin{aligned} &\mathrm{Supp}\ \widehat{u} \subset \lambda\mathcal{B} \Rightarrow \sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^b} \leq C^{k+1}\lambda^{k+d(\frac{1}{a}-\frac{1}{b})}\|u\|_{L^a}, \\ &\mathrm{Supp}\ \widehat{u} \subset \lambda\mathcal{C} \Rightarrow C^{-1-k}\lambda^k\|u\|_{L^a} \leq \sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^a} \leq C^{1+k}\lambda^k\|u\|_{L^a}, \\ &\mathrm{Supp}\ \widehat{u} \subset \lambda\mathcal{C} \Rightarrow \|\sigma(D)u\|_{L^b} \leq C_{\sigma,m}\lambda^{m+d(\frac{1}{a}-\frac{1}{b})}\|u\|_{L^a} \end{aligned} \tag{2.2}$$

for any function $u \in L^a$.

With the introduction to $\dot{\Delta}_j$ and $\dot{S}_j$, we recall the definition of the homogeneous Besov space from [17].

Definition 2.1 (Besov Spaces) *Let $s \in \mathbb{R}, 1 \leq p, r \leq \infty$. The homogenous Besov space $\dot{B}^s_{p,r}(\mathbb{R}^d)$ is defined by*

$$\dot{B}^s_{p,r}(\mathbb{R}^d) \stackrel{def}{=} \{f \in \mathcal{S}'_h(\mathbb{R}^d);\ \|f\|_{\dot{B}^s_{p,r}} < \infty\},$$

where

$$\|f\|_{\dot{B}^s_{p,r}} \stackrel{def}{=} \begin{cases} \Big(\sum_{j\in\mathbb{Z}} 2^{jsr}\|\dot{\Delta}_j f\|^r_{L^p}\Big)^{\frac{1}{r}} & \textit{for } r < \infty, \\ \sup_{j\in\mathbb{Z}} 2^{js}\|\dot{\Delta}_j f\|_{L^p} & \textit{for } r = \infty \end{cases}$$

and

$$\mathcal{S}'_h(\mathbb{R}^d) \stackrel{def}{=} \{f \in \mathcal{S}'(\mathbb{R}^d);\ \lim_{j\to-\infty} \dot{S}_j f = 0 \textit{ in } \mathcal{S}'(\mathbb{R}^d)\}.$$

Remark 2.1 It is easy to verify that the homogeneous Besov space $\dot{B}^s_{2,2}(\mathbb{R}^d)$ coincides with the classical homogeneous Sobolev space $\dot{H}^s(\mathbb{R}^d)$ and $\dot{B}^s_{\infty,\infty}(\mathbb{R}^d)$ coincides with the classical homogeneous Hölder space $\dot{C}^s(\mathbb{R}^d)$ when s is not a positive integer. In case s is a nonnegative integer, $\dot{B}^s_{\infty,\infty}(\mathbb{R}^d)$ coincides with the classical homogeneous Zygmund space $\dot{C}^s_*(\mathbb{R}^d)$.

An immediate corollary of Definition 2.1 is

Corollary 2.1 *Let $s \in \mathbb{R}, 1 \le p, r \le \infty$, and $u \in \mathcal{S}'_h$. Then u belongs to $\dot{B}^s_{p,r}$ if and only if there exists $\{c_j\}_{j\in\mathbb{Z}}$ such that $\|c_j\|_{\ell^r} = 1$ and*

$$\|\dot{\Delta}_j u\|_{L^p} \le C c_j 2^{-js} \|u\|_{\dot{B}^s_{p,r}}.$$

In order to study the global wellposedness of (INS) with initial data in Sobolev space, we need also the following spaces from [8].

Definition 2.2 *For $r \in [0, +\infty]$, $s \in \mathbb{R}$ and $T \in [0, +\infty]$, we define*

$$\widetilde{L}^r_T(\dot{H}^s(\mathbb{R}^d)) \stackrel{def}{=} \{u \in \mathcal{S}'([0,T] \times \mathbb{R}^d) : \|u\|_{\widetilde{L}^r_T(\dot{H}^s)} < \infty\},$$

where

$$\|u\|_{\widetilde{L}^r_T(\dot{H}^s)} \stackrel{def}{=} \Big(\sum_{j\in\mathbb{Z}} 2^{2js}\Big(\int_0^T \|\dot{\Delta}_j u(t)\|^r_{L^2}\Big)^{\frac{2}{r}}\Big)^{\frac{1}{2}}.$$

Remark 2.2 Similarly to Definitions 2.1 and 2.2, one may also define inhomogeneous Besov space $B^s_{p,r}(\mathbb{R}^d)$ and $\widetilde{L}^r_T(H^s)$. One may check [8, 17] for more details.

Thanks to Minkowski's inequality and an standard argument of interpolations, we have

Proposition 2.2 *Let $u \in C([0,T]; \mathcal{S}(\mathbb{R}^d))$. There hold*

$$\|u\|_{\widetilde{L}^r_T(\dot{H}^s)} \le \|u\|_{L^r_T(\dot{H}^s)}, \quad \text{if } r \le 2, \quad \|u\|_{L^r_T(\dot{H}^s)} \le \|u\|_{\widetilde{L}^r_T(\dot{H}^s)}, \quad \text{if } r \ge 2 \tag{2.3}$$

and

$$\|u\|_{\widetilde{L}^r_T(\dot{H}^s)} \lesssim \|u\|^{\theta}_{\widetilde{L}^{r_1}_T(\dot{H}^{s_1})} \|u\|^{1-\theta}_{\widetilde{L}^{r_2}_T(\dot{H}^{s_2})} \tag{2.4}$$

with $0 \le \theta \le 1$, $\frac{1}{r} = \frac{\theta}{r_1} + \frac{1-\theta}{r_2}$ and $s = \theta s_1 + (1-\theta)s_2$.

In what follows, we shall constantly use the homogeneous Bony's decomposition (see [5]):

$$uv = \dot{T}_u v + \dot{R}'(u,v) = \dot{T}_u v + \dot{T}_v u + \dot{R}(u,v), \tag{2.5}$$

where

$$\begin{aligned} &\dot{T}_u v \stackrel{\text{def}}{=} \sum_{j\in\mathbb{Z}} \dot{S}_{j-1} u \dot{\Delta}_j v, \quad \dot{R}'(u,v) \stackrel{\text{def}}{=} \sum_{j\in\mathbb{Z}} \dot{\Delta}_j u \dot{S}_{j+2} v, \\ &\dot{R}(u,v) \stackrel{\text{def}}{=} \sum_{|j-j'|\le 1} \dot{\Delta}_j u \dot{\Delta}_{j'} v = \sum_j \dot{\Delta}_j u \widetilde{\dot{\Delta}}_j v, \quad \widetilde{\dot{\Delta}}_j v \stackrel{\text{def}}{=} \sum_{|j'-j|\le 1} \dot{\Delta}_{j'} v. \end{aligned}$$

By using the Bony's decomposition (see [1]), one has

Proposition 2.3 *Let $s > 0$, $G \in W^{[s]+1,\infty}_{\text{loc}}$, $G(0) = 0$, $T > 0$, and $u \in \widetilde{L}^\rho_T(\dot{H}^s) \cap L^\infty_T(L^\infty)$. Then*

$$\|G(u)\|_{\widetilde{L}^\rho_T(\dot{H}^s)} \le C(1 + \|u\|_{L^\infty_T(L^\infty)})^{[s]+1} \|u\|_{\widetilde{L}^\rho_T(\dot{H}^s)}. \tag{2.6}$$

3 Preliminaries

In this section, we shall apply Littlewood-Paley analysis to study some commutator and product estimates, which will be used in the subsequent sections.

Lemma 3.1 *Let $s>0$, $f,g^1,g^2\in C([0,T];\mathcal{S}(\mathbb{R}^2))$ and $g=(g^1,g^2)$. Then there holds*

$$\begin{aligned}&\Big(\sum_{j\in\mathbb{Z}}2^{2js}\|\dot\Delta_j(f\,\mathrm{div}\,g)-\mathrm{div}\,(f\,\dot\Delta_jg)\|^2_{L^1_T(L^2)}\Big)^{\frac12}\\&\lesssim\|\nabla f\|_{L^\infty_T(L^\infty)}\|g\|_{\widetilde{L}^1_T(\dot H^s)}+\|f\|_{\widetilde{L}^\infty_T(\dot H^s)}\|\nabla g\|_{L^1_T(L^\infty)}.\end{aligned}\tag{3.1}$$

Proof We first get by using Bony's decomposition (2.5) that

$$\begin{aligned}&\dot\Delta_j(f\,\mathrm{div}\,g)-\mathrm{div}\,(f\,\dot\Delta_jg)\\&=\dot\Delta_j(\dot T_f\partial_ig^i+\dot R'(f,\partial_ig^i))-\partial_i(\dot T_f\dot\Delta_jg^i)-\partial_i\dot R'(f,\dot\Delta_jg^i)\\&=\partial_i([\dot\Delta_j;\ \dot T_f]g^i)-\dot\Delta_j(T_{\partial_if}g^i)+\dot\Delta_jR'(f,\partial_ig^i)-\partial_i\dot R'(f,\dot\Delta_jg^i)\overset{\text{def}}{=}\sum_{\ell=1}^4\mathcal{A}_j^\ell.\end{aligned}\tag{3.2}$$

Here and in what follows, we use Einstein convention of summations. Thanks to (2.1), one has

$$[\dot\Delta_j;\dot S_{j'-1}f]\dot\Delta_{j'}g^i(x)=2^{2j}\int_{\mathbb{R}^2}h(2^j(x-y))[\dot S_{j'-1}f(y)-\dot S_{j'-1}f(x)]\dot\Delta_{j'}g^i(y)\,\mathrm{d}y.\tag{3.3}$$

Then we get

$$\begin{aligned}\|\mathcal{A}_j^1\|_{L^1_T(L^2)}&=\Big\|\sum_{|j-j'|\le4}\partial_i[\dot\Delta_j;\ \dot S_{j'-1}f]\dot\Delta_{j'}g^i\Big\|_{L^1_T(L^2)}\\&\lesssim\sum_{|j-j'|\le4}2^{j'-j}\|\nabla\dot S_{j'-1}f\|_{L^\infty_T(L^\infty)}\|\dot\Delta_{j'}g^i\|_{L^1_T(L^2)}\\&\lesssim\|\nabla f\|_{L^\infty_T(L^\infty)}\sum_{|j-j'|\le4}\|\dot\Delta_{j'}g^i\|_{L^1_T(L^2)},\end{aligned}$$

from which we deduce that

$$\begin{aligned}\Big(\sum_{j\in\mathbb{Z}}2^{2js}\|\mathcal{A}_j^1\|^2_{L^1_T(L^2)}\Big)^{\frac12}&\lesssim\|\nabla f\|_{L^\infty_T(L^\infty)}\Big[\sum_{j\in\mathbb{Z}}\Big(2^{js}\sum_{|j-j'|\le4}\|\dot\Delta_{j'}g^i\|_{L^1_T(L^2)}\Big)^2\Big]^{\frac12}\\&\lesssim\|\nabla f\|_{L^\infty_T(L^\infty)}\|g\|_{\widetilde{L}^1_T(\dot H^s)}.\end{aligned}\tag{3.4}$$

A similar estimate holds for $\mathcal{A}_j^2$. While as

$$\begin{aligned}\|\mathcal{A}_j^3\|_{L^2}&=\Big\|\sum_{j'\ge j-2}\dot\Delta_j(\dot\Delta_{j'}f\dot S_{j'+2}\partial_ig^i)\Big\|_{L^2}\\&\lesssim\sum_{j'\ge j-2}\|\dot\Delta_{j'}f\|_{L^2}\|\dot S_{j'+2}\partial_ig^i\|_{L^\infty}\lesssim\|\partial_ig^i\|_{L^\infty}\sum_{j'\ge j-2}\|\dot\Delta_{j'}f\|_{L^2},\end{aligned}$$

we obtain

$$\begin{aligned}\Big(\sum_{j\in\mathbb{Z}}2^{2js}\|\mathcal{A}_j^3\|^2_{L^1_T(L^2)}\Big)^{\frac12}&\lesssim\Big[\sum_{j\in\mathbb{Z}}\Big(\sum_{j'\ge j-2}2^{(j-j')s}\int_0^T2^{j's}\|\dot\Delta_{j'}f\|_{L^2}\|\nabla g\|_{L^\infty}\mathrm{d}t\Big)^2\Big]^{\frac12}\\&\lesssim\Big[\sum_{j\in\mathbb{Z}}\sum_{j'\ge j-2}2^{(j-j')s}\Big(\int_0^T2^{j's}\|\dot\Delta_{j'}f\|_{L^2}\|\nabla g\|_{L^\infty}\mathrm{d}t\Big)^2\Big]^{\frac12}\\&\lesssim\|f\|_{\widetilde{L}^\infty_T(\dot H^s)}\|\nabla g\|_{L^1_T(L^\infty)}.\end{aligned}\tag{3.5}$$

A similar estimate holds for $\mathcal{A}_j^4$. Combining (3.2), (3.4) with (3.5), we get the inequality (3.1).

A similar proof of Lemma 3.1 ensures

Lemma 3.2 (Communicator Estimate) *Let $s>0$ and $f,g\in C([0,T];\mathcal{S}(\mathbb{R}^2))$. There hold*

$$\Big(\sum_{j\in\mathbb{Z}}2^{2js}\|[f;\ \dot{\Delta}_j]\nabla g\|_{L^1_T(L^2)}^2\Big)^{\frac12}\lesssim\|\nabla f\|_{L^\infty_T(L^\infty)}\|g\|_{\widetilde{L}^1_T(\dot{H}^s)}+\|f\|_{\widetilde{L}^\infty_T(\dot{H}^s)}\|\nabla g\|_{L^1_T(L^\infty)},\tag{3.6}$$

$$\Big(\sum_{j\in\mathbb{Z}}2^{2js}\|[f;\ \dot{\Delta}_j]\nabla g\|_{L^1_T(L^2)}^2\Big)^{\frac12}\lesssim\|f\|_{L^\infty_T(L^\infty)}\|g\|_{\widetilde{L}^1_T(\dot{H}^{s+1})}+\|f\|_{\widetilde{L}^\infty_T(\dot{H}^{s+1})}\|\nabla g\|_{L^1_T(L^\infty)}.\tag{3.7}$$

Proof The main idea in proving (3.6) is to use Bony's decomposition (2.5) to get

$$[f;\ \dot{\Delta}_j]\nabla g=[\dot{T}_f;\ \dot{\Delta}_j]\nabla g+\dot{R}'(f,\dot{\Delta}_j\nabla g)-\dot{\Delta}_j\dot{R}'(f,\nabla g).$$

Then a similar proof of Lemma 3.1 ensures (3.6). To prove (3.7), we only need to notice from (3.3) that

$$\|[\dot{\Delta}_j;\dot{S}_{j'-1}f]\dot{\Delta}_{j'}g\|_{L^1_T(L^2)}\lesssim\|f\|_{L^\infty_T(L^\infty)}\|\dot{\Delta}_{j'}g\|_{L^1_T(L^2)}.$$

The other details are omitted here.

However, when $f=u=(u_1,u_2)$ with $\operatorname{div}u=0$, we can improve (3.6) as follows.

Lemma 3.3 *Let $s>-1$, and $u\in C([0,T];\mathcal{S}(\mathbb{R}^d))$ be a solenoidal vector field. Then for any $\delta>0$, there hold*

(i)

$$\Big(\sum_{j\in\mathbb{Z}}2^{2js}\|[u,\dot{\Delta}_j]\cdot\nabla u\|_{L^1_T(L^2)}^2\Big)^{\frac12}\le\delta\|u\|_{\widetilde{L}^1_T(\dot{H}^{s+2})}+C\delta^{-1}\int_0^T\|u\|_{\dot{C}^0_*}^2\|u\|_{\dot{H}^s}\mathrm{d}t,\tag{3.8}$$

(ii)

$$\|u\cdot\nabla u\|_{\widetilde{L}^1_T(\dot{H}^s)}\le\delta\|u\|_{\widetilde{L}^1_T(\dot{H}^{s+2})}+C\delta^{-1}\int_0^T\|u\|_{\dot{C}^0_*}^2\|u\|_{\dot{H}^s}\mathrm{d}t.\tag{3.9}$$

Proof (i) Thanks to (2.5), we first split $[u;\ \dot{\Delta}_j]\cdot\nabla u$ as

$$[u;\ \dot{\Delta}_j]\cdot\nabla u=[\dot{T}_u;\ \dot{\Delta}_j]\cdot\nabla u+\dot{R}'(u,\dot{\Delta}_j\nabla u)-\dot{\Delta}_j\dot{T}_{\nabla u}u-\dot{\Delta}_j\nabla\cdot\dot{R}(u,u)\stackrel{\text{def}}{=}\sum_{\ell=1}^4\mathcal{B}_j^\ell.\tag{3.10}$$

However, thanks to (3.3), we have

$$\|\mathcal{B}_j^1\|_{L^2}\lesssim\sum_{|j-j'|\le4}2^{-j}\|\nabla\dot{S}_{j'-1}u\|_{L^\infty}\|\dot{\Delta}_{j'}\nabla u\|_{L^2},$$

and it is easy to observe that

$$\|\nabla\dot{S}_{j'-1}u\|_{L^\infty}\lesssim\|u\|_{\dot{C}^0_*}\sum_{\ell\le j'-2}2^\ell\lesssim2^{j'}\|u\|_{\dot{C}^0_*},\tag{3.11}$$

which ensures

$$\|\mathcal{B}_j^1\|_{L^2}\lesssim\|u\|_{\dot{C}^0_*}\Big(\sum_{|j-j'|\le4}\|\dot{\Delta}_{j'}\nabla u\|_{L^2}\Big).$$

Therefore, we obtain

$$\begin{aligned}&\Big(\sum_{j\in\mathbb{Z}}2^{2js}\|\mathcal{B}_j^1\|_{L^1_T(L^2)}^2\Big)^{\frac12}\\ &\lesssim\Big[\sum_{j\in\mathbb{Z}}\Big(\int_0^T(2^{j(s+1)}\|\dot\Delta_j\nabla u\|_{L^2})\,\mathrm{d}t\Big)^2\Big]^{\frac14}\Big[\sum_{j\in\mathbb{Z}}\Big(\int_0^T(2^{j(s-1)}\|\dot\Delta_j\nabla u\|_{L^2})\|u\|_{\dot C^0_*}^2\mathrm{d}t\Big)^2\Big]^{\frac14}.\end{aligned}$$

Applying Lemma 2.1 and Minkowski inequality gives

$$\begin{aligned}&\Big[\sum_{j\in\mathbb{Z}}\Big(\int_0^T(2^{j(s-1)}\|\dot\Delta_j\nabla u\|_{L^2})\|u\|_{\dot C^0_*}^2\mathrm{d}t\Big)^2\Big]^{\frac14}\\ &\lesssim\Big[\int_0^T\Big(\sum_{j\in\mathbb{Z}}2^{2js}\|\dot\Delta_j u\|_{L^2}^2\|u\|_{\dot C^0_*}^4\Big)^{\frac12}\mathrm{d}t\Big]^{\frac12}\lesssim\Big[\int_0^T\|u\|_{\dot C^0_*}^2\|u\|_{\dot H^s}\mathrm{d}t\Big]^{\frac12},\end{aligned}$$

from which we deduce that for any $\delta>0$,

$$\Big(\sum_{j\in\mathbb{Z}}2^{2js}\|\mathcal{B}_j^1\|_{L^1_T(L^2)}^2\Big)^{\frac12}\le\delta\|u\|_{\widetilde L^1_T(\dot H^{s+2})}+C\delta^{-1}\int_0^T\|u\|_{\dot C^0_*}^2\|u\|_{\dot H^s}\mathrm{d}t.\tag{3.12}$$

A similar but easier argument gives the same estimate for $\mathcal{B}_j^3$. While thanks to (3.11), we have

$$\|\mathcal{B}_j^2\|_{L^2}\lesssim\sum_{j'\ge j-2}\|\dot S_{j'+2}\dot\Delta_j\nabla u\|_{L^\infty}\|\dot\Delta_{j'}u\|_{L^2}\lesssim 2^j\|u\|_{\dot C^0_*}\sum_{j'\ge j-2}\|\dot\Delta_{j'}u\|_{L^2},$$

from which we deduce that

$$\begin{aligned}\Big(\sum_{j\in\mathbb{Z}}2^{2js}\|\mathcal{B}_j^2\|_{L^1_T(L^2)}^2\Big)^{\frac12}\lesssim\Big[\sum_{j\in\mathbb{Z}}\Big(\int_0^T\sum_{j'\ge j-2}&\|u\|_{\dot C^0_*}(2^{j's}\|\dot\Delta_{j'}u\|_{L^2})^{\frac12}2^{\frac12(j-j')(s+1)}\\ &\times(2^{j'(s+2)}\|\dot\Delta_{j'}u\|_{L^2})^{\frac12}2^{\frac12(j-j')(s+1)}\mathrm{d}t\Big)^2\Big]^{\frac12}.\end{aligned}$$

Note that

$$\begin{aligned}&\Big[\sum_{j\in\mathbb{Z}}\Big(\sum_{j'\ge j-2}\int_0^T2^{j'(s+2)}\|\dot\Delta_{j'}u\|_{L^2}2^{(j-j')(s+1)}\mathrm{d}t\Big)^2\Big]^{\frac12}\\ &\lesssim\Big[\sum_{j\in\mathbb{Z}}\sum_{j'\ge j-2}\Big(\int_0^T2^{j'(s+2)}\|\dot\Delta_{j'}u\|_{L^2}\mathrm{d}t\Big)^2 2^{(j-j')(s+1)}\Big]^{\frac12}\lesssim\|u\|_{\widetilde L^1_T(\dot H^{s+2})},\end{aligned}$$

and by Minkowski inequality we have

$$\begin{aligned}&\Big[\sum_{j\in\mathbb{Z}}\Big(\sum_{j'\ge j-2}\int_0^T\|u\|_{\dot C^0_*}^2 2^{j's}\|\dot\Delta_{j'}u\|_{L^2}2^{(j-j')(s+1)}\,\mathrm{d}t\Big)^2\Big]^{\frac12}\\ &\lesssim\int_0^T\|u\|_{\dot C^0_*}^2\Big[\sum_{j\in\mathbb{Z}}\Big(\sum_{j'\ge j-2}2^{j's}\|\dot\Delta_{j'}u\|_{L^2}2^{(j-j')(s+1)}\Big)^2\Big]^{\frac12}\mathrm{d}t\\ &\lesssim\int_0^T\|u\|_{\dot C^0_*}^2\|u\|_{\dot H^s}\mathrm{d}t.\end{aligned}$$

As a consequence, we obtain

$$\Big(\sum_{j\in\mathbb{Z}} 2^{2js}\|\mathcal{B}_2^2\|^2_{L^1_T(L^2)}\Big)^{\frac{1}{2}} \le \delta\|u\|_{\widetilde{L}^1_T(\dot{H}^{s+2})} + C\delta^{-1}\int_0^T \|u\|^2_{\dot{C}^0_*}\|u\|_{\dot{H}^s}\mathrm{d}t. \tag{3.13}$$

Finally, as

$$\|\mathcal{B}_j^4\|_{L^2} \lesssim 2^j \sum_{j'\ge j-4} \|\dot{\Delta}_{j'}u\|_{L^2}\|\dot{\Delta}_{j'}u\|_{L^\infty} \lesssim 2^j\|u\|_{\dot{C}^0_*}\sum_{j'\ge j-2}\|\dot{\Delta}_{j'}u\|_{L^2},$$

from the proof of (3.13), $\mathcal{B}_j^4$ satisfies (3.13) as well. Summing up (3.10)–(3.13), we conclude the proof of (3.8).

(ii) As $\operatorname{div} u = 0$, we get by using Bony's decomposition (2.5) that

$$u\cdot\nabla u = 2\dot{T}_{\nabla u}u + \nabla\cdot\dot{R}(u,u).$$

Noting that

$$\dot{\Delta}_j(\dot{T}_{\nabla u}u) = \sum_{|j'-j|\le 4}\dot{\Delta}_j(\dot{S}_{j'-1}\nabla u\cdot\dot{\Delta}_{j'}u),$$

from (3.11), we obtain

$$\begin{aligned}\|\dot{T}_{\nabla u}u\|_{\widetilde{L}^1_T(\dot{H}^s)} &\lesssim \Big[\sum_{j\in\mathbb{Z}}\Big(\int_0^T 2^{js}\sum_{|j'-j|\le 4}\|\dot{S}_{j'-1}\nabla u\|_{L^\infty}\|\dot{\Delta}_{j'}u\|_{L^2}\mathrm{d}t\Big)^2\Big]^{\frac{1}{2}}\\ &\lesssim \Big[\sum_{j\in\mathbb{Z}}\Big(\int_0^T \|u\|_{\dot{C}^0_*}(2^{j(s+2)}\|\dot{\Delta}_j u\|_{L^2})^{\frac{1}{2}}(2^{js}\|\dot{\Delta}_j u\|_{L^2})^{\frac{1}{2}}\mathrm{d}t\Big)^2\Big]^{\frac{1}{2}},\end{aligned}$$

which together with the proof of (3.12) ensures

$$\|\dot{T}_{\nabla u}u\|_{\widetilde{L}^1_T(\dot{H}^s)} \lesssim \delta\|u\|_{\widetilde{L}^1_T(\dot{H}^{s+2})} + C\delta^{-1}\int_0^T \|u\|^2_{\dot{C}^0_*}\|u\|_{\dot{H}^s}\mathrm{d}t. \tag{3.14}$$

On the other hand, as

$$\dot{\Delta}_j\nabla\cdot\dot{R}(u,u) = \nabla\cdot\sum_{j'\ge j-2}\dot{\Delta}_j(\widetilde{\dot{\Delta}}_{j'}u\otimes\dot{\Delta}_{j'}u),$$

we get by using Lemma 2.1 that

$$\begin{aligned}\|\nabla\cdot\dot{R}(u,u)\|_{\widetilde{L}^1_T(\dot{H}^s)} &\lesssim \Big[\sum_{j\in\mathbb{Z}}\Big(\int_0^T 2^{j(s+1)}\sum_{j'\ge j-2}\|\widetilde{\dot{\Delta}}_{j'}u\|_{L^\infty}\|\dot{\Delta}_{j'}u\|_{L^2}\mathrm{d}t\Big)^2\Big]^{\frac{1}{2}}\\ &\lesssim \Big[\sum_{j\in\mathbb{Z}}\Big(\int_0^T\sum_{j'\ge j-2}2^{(j-j')(s+1)}(2^{j'(s+2)}\|\dot{\Delta}_{j'}u\|_{L^2})^{\frac{1}{2}}\\ &\qquad\times(2^{j's}\|\dot{\Delta}_{j'}u\|_{L^2})^{\frac{1}{2}}\|\widetilde{\dot{\Delta}}_{j'}u\|_{L^\infty}\mathrm{d}t\Big)^2\Big]^{\frac{1}{2}},\end{aligned}$$

from which we deduce by a similar proof of (3.13) that

$$\|\nabla\cdot\dot{R}(u,u)\|_{\widetilde{L}^1_T(\dot{H}^s)} \lesssim \delta\|u\|_{\widetilde{L}^1_T(\dot{H}^{s+2})} + C\delta^{-1}\int_0^T \|u\|^2_{\dot{C}^0_*}\|u\|_{\dot{H}^s}\mathrm{d}t. \tag{3.15}$$

Summing (3.14) and (3.15), we obtain (3.9). This completes the proof of Lemma 3.3.

To study the propagation of regularities of the velocity field in the negative Sobolev spaces, we also need the following forms of commutator estimate and product law.

Lemma 3.4 *Let $f,g\in C([0,T];\mathcal{S}(\mathbb{R}^2))$. Then for any $0<\varepsilon<1$, there holds*

$$\Big\{\sum_{q\in\mathbb{Z}}2^{-2q\varepsilon}\|[\dot\Delta_q;f]g\|^2_{L^1_T(L^2)}\Big\}^{\frac12}\lesssim(\|f\|_{L^\infty_T(L^\infty)}+\|f\|_{\widetilde L^\infty_T(\dot H^1)})\|g\|_{\widetilde L^1_T(\dot H^{-\varepsilon})}. \tag{3.16}$$

Proof Firstly, similarly to (3.10), we decompose $[\dot\Delta_q;f]g$ as

$$[\dot\Delta_q;f]g=[\dot\Delta_q;\dot T_f]g+\dot\Delta_q(\dot T_gf)+\dot\Delta_q\dot R(f,g)-\dot R'(f,\dot\Delta_qg).$$

Applying (3.3) and Corollary 2.1, we have

$$\begin{aligned}\|[\dot\Delta_q;\dot T_f]g\|_{L^1_T(L^2)}&\lesssim\sum_{|q-\ell|\le4}\|[\dot\Delta_q;\dot S_{\ell-1}f]\dot\Delta_\ell g\|_{L^1_T(L^2)}\lesssim\|f\|_{L^\infty_T(L^\infty)}\|\dot\Delta_qg\|_{L^1_T(L^2)}\\&\lesssim c_q2^{q\varepsilon}\|f\|_{L^\infty_T(L^\infty)}\|g\|_{\widetilde L^1_T(\dot H^{-\varepsilon})}.\end{aligned}\tag{3.17}$$

Similarly

$$\|\dot\Delta_q(\dot T_gf)\|_{L^1_T(L^2)}\lesssim\sum_{|q-\ell|\le4}\|\dot S_{\ell-1}g\|_{L^1_T(L^2)}\|\dot\Delta_\ell f\|_{L^\infty_T(L^\infty)}\lesssim c_q2^{q\varepsilon}\|f\|_{L^\infty_T(L^\infty)}\|g\|_{\widetilde L^1_T(\dot H^{-\varepsilon})}. \tag{3.18}$$

Thanks to Corollary 2.1 and Lemma 2.1, we have

$$\begin{aligned}\|\dot\Delta_q\dot R(f,g)\|_{L^1_T(L^2)}&\lesssim2^q\sum_{\ell\ge q-2}\|\widetilde{\dot\Delta}_\ell f\|_{L^1_T(L^2)}\|\dot\Delta_\ell g\|_{L^\infty_T(L^2)}\\&\lesssim2^{q\varepsilon}\sum_{\ell\ge q-2}c_\ell2^{(q-\ell)(1-\varepsilon)}\|f\|_{\widetilde L^\infty_T(\dot H^1)}\|g\|_{\widetilde L^1_T(\dot H^{-\varepsilon})}\\&\lesssim c_q2^{q\varepsilon}\|f\|_{\widetilde L^\infty_T(\dot H^1)}\|g\|_{\widetilde L^1_T(\dot H^{-\varepsilon})}.\end{aligned}\tag{3.19}$$

Finally, again thanks to Corollary 2.1, we obtain

$$\begin{aligned}\|R'(f,\dot\Delta_qg)\|_{L^1_T(L^2)}&=\Big\|\sum_{\ell\ge q-2}\dot S_{\ell+2}\dot\Delta_qg\dot\Delta_\ell f\Big\|_{L^1_T(L^2)}\\&\lesssim\|\dot\Delta_qg\|_{L^1_T(L^\infty)}\sum_{\ell\ge q-2}\|\dot\Delta_\ell f\|_{L^\infty_T(L^2)}\\&\lesssim2^{q\varepsilon}c_q\Big(\sum_{\ell\ge q-2}c_\ell2^{q-\ell}\Big)\|f\|_{\widetilde L^\infty_T(\dot H^1)}\|g\|_{\widetilde L^1_T(\dot H^{-\varepsilon})}\\&\lesssim c_q2^{q\varepsilon}\|f\|_{\widetilde L^\infty_T(\dot H^1)}\|g\|_{\widetilde L^1_T(\dot H^{-\varepsilon})},\end{aligned}$$

which together with (3.17)–(3.19) implies (3.16).

Lemma 3.5 *Let $-1<s<1$ and $0<\varepsilon<1$. Let $f,g\in C([0,T];\mathcal{S}(\mathbb{R}^2))$. Then there holds*

$$\|fg\|_{\widetilde L^1_T(\dot H^s)}\lesssim(\|f\|_{L^\infty_T(L^\infty)}+\|f\|_{\widetilde L^\infty_T(\dot H^1)})\|g\|_{\widetilde L^1_T(\dot H^s)}. \tag{3.20}$$

Proof By using Bony's decomposition (2.5), we first get

$$fg = \dot{T}_f g + \dot{T}_g f + \dot{R}(f,g).$$

Applying Corollary 2.1, we have

$$\begin{aligned}
\|\dot{\Delta}_q(\dot{T}_f g)\|_{L^1_T(L^2)} &\lesssim \sum_{|q-\ell|\le 4} \|\dot{S}_{\ell-1} f\|_{L^\infty_T(L^\infty)} \|\dot{\Delta}_\ell g\|_{L^1_T(L^2)} \lesssim c_q 2^{-qs} \|f\|_{L^\infty_T(L^\infty)} \|g\|_{\widetilde{L}^1_T(\dot{H}^s)},\\
\|\dot{\Delta}_q(\dot{T}_g f)\|_{L^1_T(L^2)} &\lesssim \sum_{|q-\ell|\le 4} \|\dot{S}_{\ell-1} g\|_{L^1_T(L^\infty)} \|\dot{\Delta}_\ell f\|_{L^\infty_T(L^2)} \lesssim c_q 2^{-qs} \|f\|_{\widetilde{L}^\infty_T(\dot{H}^1)} \|g\|_{\widetilde{L}^1_T(\dot{H}^s)},
\end{aligned}$$

where we use (2.2) and $s<1$, so that

$$\|\dot{S}_{\ell-1} g\|_{L^1_T(L^\infty)} \lesssim c_\ell 2^{(1-s)\ell} \|g\|_{\widetilde{L}^1_T(\dot{H}^s)}.$$

On the other hand, as $s>-1$, by using Lemma 2.1, one gets

$$\begin{aligned}
\|\dot{\Delta}_q(\dot{R}(f,g)\|_{L^1_T(L^2)} &\lesssim 2^q \sum_{\ell\ge q-2} \|\dot{\Delta}_\ell f\|_{L^\infty_T(L^2)} \|\widetilde{\dot{\Delta}}_\ell g\|_{L^1_T(L^2)}\\
&\lesssim 2^{-qs} \sum_{\ell\ge q-2} c_\ell 2^{-(\ell-q)(1+s)} \|f\|_{\widetilde{L}^\infty_T(\dot{H}^1)} \|g\|_{\widetilde{L}^1_T(\dot{H}^s)}\\
&\lesssim c_q 2^{-qs} \|f\|_{\widetilde{L}^\infty_T(\dot{H}^1)} \|g\|_{\widetilde{L}^1_T(\dot{H}^s)}.
\end{aligned}$$

This proves (3.20).

4 Proof of Theorem 1.2

We shall first provide all the necessary a priori estimates for the existence proof of Theorem 1.2.

4.1 The transport equation

We first deal with the continuity equation in (INS):

$$\begin{cases} \partial_t a + u\cdot\nabla a = 0, & (t,x)\in\mathbb{R}^+\times\mathbb{R}^2,\\ a|_{t=0} = a_0. \end{cases} \tag{4.1}$$

Lemma 4.1 *Let $\alpha>2$ and $a_0\in H^\alpha(\mathbb{R}^2)$. Let u be a solenoidal vector field with $u\in \widetilde{L}^1([0,T];\dot{H}^{\alpha+1}(\mathbb{R}^2))$ and $\nabla u\in L^\infty([0,T];L^\infty(\mathbb{R}^2))$. Then (4.1) has a unique solution $a\in \widetilde{L}^\infty_T(H^\alpha(\mathbb{R}^2))\cap C([0,T];H^\alpha(\mathbb{R}^2))$, which satisfies*

$$\begin{aligned}
&\|a\|_{L^\infty_T(L^p)} = \|a_0\|_{L^p}, \quad \forall\, 1\le p\le\infty,\\
&\|a\|_{\widetilde{L}^\infty_T(H^\alpha)} \le C_\alpha \|a_0\|_{H^\alpha}(1+\|u\|_{\widetilde{L}^1_T(\dot{H}^{\alpha+1})}) \exp\Big(\int_0^T \|\nabla u(t)\|_{L^\infty}\mathrm{d}t\Big).
\end{aligned} \tag{4.2}$$

Proof As the existence of solutions to (4.1) essentially follows from the a priori estimates, for simplicity, we just present the detailed proof to (4.2). The first part of (4.2) follows from the standard characteristic method and $\operatorname{div} u = 0$. On the other hand, again as $\operatorname{div} u=0$, by acting $\dot{\Delta}_j$ to (4.1) and taking the L^2 inner product of the resulting equation with $\dot{\Delta}_j a$, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\dot{\Delta}_j a\|_{L^2}^2 + ([\dot{\Delta}_j;\ u]\cdot\nabla a \mid \dot{\Delta}_j a)_{L^2} = 0. \tag{4.3}$$

While thanks to (2.5), we have

$$[\dot{\Delta}_j;\ u]\cdot\nabla a = [\dot{\Delta}_j;\ \dot{T}_u]\nabla a + \dot{\Delta}_j(\dot{R}'(u,\nabla a)) - \dot{R}'(u,\nabla\dot{\Delta}_j a).$$

It is easy to observe

$$\|[\dot{\Delta}_j;\ \dot{T}_u]\nabla a\|_{L^2} \lesssim \sum_{|j-\ell|\leq 4}\|\nabla\dot{S}_{\ell-1}u\|_{L^\infty}\|\dot{\Delta}_\ell a\|_{L^2} \lesssim \|\nabla u\|_{L^\infty}\sum_{|j-\ell|\leq 4}\|\dot{\Delta}_\ell a\|_{L^2},$$

$$\|\dot{\Delta}_j(R'(u,\nabla a))\|_{L^2} \lesssim \sum_{\ell\geq j-N}\|\nabla\dot{S}_{\ell+2}a\|_{L^\infty}2^{-\ell}\|\dot{\Delta}_\ell\nabla u\|_{L^2} \lesssim \|a\|_{L^\infty}\sum_{\ell\geq j-N}\|\dot{\Delta}_\ell\nabla u\|_{L^2}.$$

Similarly,

$$\|\dot{R}'(u,\nabla\Delta_j a)\|_{L^2} \lesssim \|a\|_{L^\infty}\sum_{\ell\geq j-N}\|\dot{\Delta}_\ell\nabla u\|_{L^2}.$$

Then integrating (4.3) over $[0,t]$ for $t\leq T$, we have

$$\begin{aligned}\|\dot{\Delta}_j a\|_{L^\infty_t(L^2)} \lesssim \|\dot{\Delta}_j a_0\|_{L^2} &+ \sum_{|j-\ell|\leq 4}\int_0^t\|\nabla u(\tau)\|_{L^\infty}\|\dot{\Delta}_\ell a\|_{L^\infty_\tau(L^2)}\mathrm{d}\tau\\ &+ \|a_0\|_{L^\infty}\sum_{\ell\geq j-N}\int_0^t\|\dot{\Delta}_\ell\nabla u\|_{L^2}\,\mathrm{d}\tau.\end{aligned}$$

But as

$$\sum_{\ell\geq j-N}\int_0^t\|\dot{\Delta}_\ell\nabla u\|_{L^2}\,\mathrm{d}\tau \lesssim \Big(\sum_{\ell\geq j-N}c_\ell 2^{-\ell\alpha}\Big)\|u\|_{\widetilde{L}^1_t(\dot{H}^{\alpha+1})} \lesssim c_j 2^{-j\alpha}\|u\|_{\widetilde{L}^1_T(\dot{H}^{\alpha+1})}$$

and

$$\sum_{|j-\ell|\leq 4}\int_0^t\|\nabla u(\tau)\|_{L^\infty}\|\dot{\Delta}_\ell a\|_{L^\infty_\tau(L^2)}\,\mathrm{d}\tau \lesssim 2^{-j\alpha}\int_0^t c_j(\tau)\|\nabla u(\tau)\|_{L^\infty}\|a\|_{\widetilde{L}^\infty_\tau(\dot{H}^\alpha)}\,\mathrm{d}\tau,$$

by using Minkowski inequality, we get

$$\|a\|_{\widetilde{L}^\infty_t(\dot{H}^\alpha)} \lesssim \|a_0\|_{\dot{H}^\alpha} + \|a_0\|_{L^\infty}\|u\|_{\widetilde{L}^1_T(\dot{H}^{\alpha+1})} + \int_0^t\|\nabla u(\tau)\|_{L^\infty}\|a\|_{\widetilde{L}^\infty_\tau(\dot{H}^\alpha)}\,\mathrm{d}\tau.$$

Gronwall inequality together with the first part of (4.2) gives (4.2). Since $a\in\widetilde{L}^\infty_T(H^\alpha(\mathbb{R}^2))$ and a satisfies (4.1), it is standard to prove that $a\in C([0,T];H^\alpha(\mathbb{R}^2))$. We omit the details here.

4.2 Elliptic estimates

To deal with the pressure term in (INS), we need to handle the following type of elliptic equation:

$$\text{(E)}\quad \mathrm{div}\,(b\nabla p) = \mathrm{div}\,\vec{F}.$$

Lemma 4.2 *Let $s>2$, $\vec{F}=(F_1,F_2)$, $b\overset{def}{=}1+a$, with $\underline{b}\overset{def}{=}\inf_{(t,x)\in[0,T]\times\mathbb{R}^2}b(t,x)>0$ and $\nabla a\in\widetilde{L}^\infty_T(H^{s-1}(\mathbb{R}^2))$. Then up to a constant,* (E) *has a unique solution p such that*

$$\underline{b}\|\nabla p\|_{\widetilde{L}^1_T(H^s)} \leq C(\mathcal{A}_T)^s\|\vec{F}\|_{\widetilde{L}^1_T}(H^s), \tag{4.4}$$

where

$$\mathcal{A}_T \overset{def}{=} 1+\underline{b}^{-1}\|\nabla a\|_{\widetilde{L}^\infty_T(H^{s-1})}.$$

Proof Again for simplicity, we only present the detailed proof to the a priori estimate (4.4). We first take the L^2 inner product of (E) with p and use integration by parts to obtain

$$(b\nabla p \,|\, \nabla p)_{L^2} = (\vec{F} \,|\, \nabla p)_{L^2},$$

which ensures

$$\|\nabla p\|_{L^1(L^2)} \leq \underline{b}^{-1}\|\vec{F}\|_{L^1(L^2)}. \tag{4.5}$$

Furthermore, applying $\dot{\Delta}_q$ to (E) and taking the L^2 inner product of the resulting equation with $\dot{\Delta}_q p$, we get

$$(\dot{\Delta}_q(b\nabla p) \mid \dot{\Delta}_q\nabla p)_{L^2} = (\dot{\Delta}_q\vec{F} \mid \dot{\Delta}_q\nabla p)_{L^2},$$

which gives

$$(b\dot{\Delta}_q\nabla p \mid \dot{\Delta}_q\nabla p)_{L^2} = (\dot{\Delta}_q\vec{F} \mid \dot{\Delta}_q\nabla p)_{L^2} + ([b;\dot{\Delta}_q]\nabla p \mid \dot{\Delta}_q\nabla p)_{L^2}.$$

Then, by integrating the above inequality over $[0,T]$, we get

$$\underline{b}\|\dot{\Delta}_q\nabla p\|_{L^1(L^2)} \leq \|\dot{\Delta}_q\vec{F}\|_{L^1_T(L^2)} + \|[a;\ \dot{\Delta}_q]\nabla p\|_{L^1_T(L^2)}.$$

Applying (3.7), we have

$$\begin{aligned}\underline{b}\|\nabla p\|_{\widetilde{L}^1_T(\dot{H}^s)} &\leq \|\vec{F}\|_{\widetilde{L}^1_T(\dot{H}^s)} + \Big(\sum_{q\in\mathbf{Z}} 2^{2qs}\|[b;\ \dot{\Delta}_q]\nabla p\|^2_{L^1_T(L^2)}\Big)^{\frac{1}{2}}\\ &\leq \|F\|_{\widetilde{L}^1_T(\dot{H}^s)} + C\|\nabla a\|_{\widetilde{L}^\infty_T(H^{s-1})}\|\nabla p\|_{\widetilde{L}^1_T(H^{s-1})},\end{aligned} \tag{4.6}$$

while thanks to Proposition 2.2, we have

$$\begin{aligned}&\|\nabla p\|_{\widetilde{L}^1_T(H^{s-1})} \leq C\|\nabla p\|^{\frac{1}{s}}_{\widetilde{L}^1_T(H^0)}\|\nabla p\|^{\frac{s-1}{s}}_{\widetilde{L}^1_T(H^s)},\\ &\|\nabla p\|_{\widetilde{L}^1_T(H^0)} \lesssim \|\nabla p\|_{L^1_T(L^2)} \lesssim \underline{b}^{-1}\|F\|_{L^1_T(L^2)} \lesssim \underline{b}^{-1}\|F\|_{\widetilde{L}^1_T(H^s)},\end{aligned}$$

which together with (4.5) and (4.6) gives (4.4).

4.3 The momentum equation

The goal of this section is to study the momentum equation in the system (INS), which is the key part in the existence proof of Theorem 1.2.

$$\text{(M)}\quad \partial_t u + u\cdot\nabla u + b(\nabla p - \mathrm{div}\,(\widetilde{\mu}(a)\mathcal{M}) = 0.$$

Lemma 4.3 *Let $0<\varepsilon<1$, $s>0$, and $u\in C([0,T];\mathcal{S}(\mathbb{R}^2))$. Then there hold*

$$\begin{aligned}&\|u\|_{\widetilde{L}^\infty_T(H^s)} \lesssim \|u\|_{\widetilde{L}^\infty_T(\dot{H}^{-\varepsilon})} + \|u\|_{\widetilde{L}^\infty_T(\dot{H}^s)},\\ &\|\nabla u\|_{L^1_T(L^\infty)} + \|u\|_{L^1_T(\dot{H}^2)} \lesssim \|u\|_{\widetilde{L}^1_T(\dot{H}^{2-\varepsilon})} + \|u\|_{\widetilde{L}^1_T(\dot{H}^{s+2})}.\end{aligned} \tag{4.7}$$

Proof Thanks to Definition 2.2, to prove the first part of (4.7), we only need to show that

$$\|\dot{S}_{-1}u\|_{L^\infty_T(L^2)} \lesssim \Big(\sum_{j\leq -2} 2^{-2j\varepsilon}\|\dot{\Delta}_j u\|^2_{L^\infty_T(L^2)}\Big)^{\frac{1}{2}},$$

which is a consequence of

$$\begin{aligned}\|\dot S_{-1}u\|_{L^\infty_T(L^2)} &\le \sum_{j\le -2}\|\dot\Delta_j u\|_{L^\infty_T(L^2)} \le \Big(\sum_{j\le -2}2^{-2j\varepsilon}\|\dot\Delta_j u\|^2_{L^\infty_T(L^2)}\Big)^{\frac12}\Big(\sum_{j\le -2}2^{2j\varepsilon}\Big)^{\frac12}\\ &\lesssim\Big(\sum_{j\le -2}2^{-2j\varepsilon}\|\dot\Delta_j u\|^2_{L^\infty_T(L^2)}\Big)^{\frac12}.\end{aligned}$$

On the other hand, thanks to Littlewood-Paley decomposition, we have

$$\begin{aligned}\int_0^T\|\nabla u(t)\|_{L^\infty}\,\mathrm{d}t &\lesssim \sum_{j\le 0}\|\dot\Delta_j\nabla u\|_{L^1_T(L^\infty)}+\sum_{j\ge 0}\|\dot\Delta_j\nabla u\|_{L^1_T(L^\infty)}\\ &\lesssim \sum_{j\le 0}2^{2j}c_j2^{-j(2-\varepsilon)}\|u\|_{\widetilde L^1_T(\dot H^{2-\varepsilon})}+\sum_{j\ge 0}2^{2j}c_j2^{-j(2+s)}\|u\|_{\widetilde L^1_T(\dot H^{s+2})}\\ &\lesssim \|u\|_{\widetilde L^1_T(\dot H^{2-\varepsilon})}+\|u\|_{\widetilde L^1_T(\dot H^{s+2})}. \end{aligned}\tag{4.8}$$

A similar argument gives the same estimate for $\|u\|_{L^1_T(\dot H^2)}$.

Proposition 4.1 *Let $s>2$, $\varepsilon\in(0,1)$. Let (a,u) be a given smooth enough solution of* (INS) *on $[0,T]$. We assume that $\underline{b}\stackrel{def}{=}\inf_{x\in\mathbb{R}^2}b_0>0$ (resp. $\overline{b}\stackrel{def}{=}\|b_0\|_{L^\infty}$), and $\underline{\mu}\stackrel{def}{=}\inf_{b\in[\underline{b},\overline{b}]}\mu(\frac1b)>0$. Then there exist positive constants c and C such that for all $\delta>0$, there holds*

$$\begin{aligned}&\|u\|_{\widetilde L^\infty_T(\dot H^s)}+c\underline{b}\ \underline{\mu}\|u\|_{\widetilde L^1_T(\dot H^{s+2})}\\ \le{}&\|u_0\|_{\dot H^s}+C\Big[(\delta+\|\nabla a\|_{\widetilde L^\infty_T(H^s)}(1+\|a\|_{\widetilde L^\infty_T(H^{s+1})})^{s+2})[\|u\|_{\widetilde L^1_T(\dot H^{2-\varepsilon})}\\ &+\|u\|_{\widetilde L^1_T(\dot H^{s+2})}]+\delta^{-1}\int_0^T\|u\|^2_{\dot C^0_*}\|u\|_{H^s}\,\mathrm{d}t\Big].\end{aligned}\tag{4.9}$$

Proof We first apply $\dot\Delta_q$ to the momentum equation (M) to yield

$$(\mathrm{M}_q)\quad \partial_t\dot\Delta_q u+u\cdot\nabla\dot\Delta_q u+\dot\Delta_q\nabla p-\mathrm{div}\,(b\widetilde\mu(a)\dot\Delta_q\mathcal{M})=-\dot\Delta_q(a\nabla p)+[u;\ \dot\Delta_q]\cdot\nabla u+R_q$$

with

$$R_q\stackrel{\text{def}}{=}\dot\Delta_q(b\,\mathrm{div}\,(\widetilde\mu(a)\mathcal{M}))-\mathrm{div}\,(b\,\widetilde\mu(a)\dot\Delta_q\mathcal{M}).$$

We split R_q as follows:

$$\begin{aligned}R_q&=-\mathrm{div}\,\{b[\widetilde\mu(a)-\widetilde\mu(0);\ \dot\Delta_q]\mathcal{M}\}+\widetilde\mu(0)\{\dot\Delta_q(a\,\mathrm{div}\,\mathcal{M})-\mathrm{div}\,(a\,\dot\Delta_q\mathcal{M})\}\\ &\quad+\dot\Delta_q\{a\,\mathrm{div}\,[(\widetilde\mu(a)-\widetilde\mu(0))\mathcal{M}]\}-\mathrm{div}\,\{a\,\dot\Delta_q[(\widetilde\mu(a)-\widetilde\mu(0))\mathcal{M}]\}\\ &=-\mathrm{div}\,\{b[\widetilde\mu(a)-\widetilde\mu(0);\ \dot\Delta_q]\mathcal{M}\}+R_q^1+R_q^2,\end{aligned}\tag{4.10}$$

where

$$\begin{aligned}&R_q^1\stackrel{\text{def}}{=}\widetilde\mu(0)\{\dot\Delta_q(a\,\mathrm{div}\,\mathcal{M})-\mathrm{div}\,(a\,\dot\Delta_q\mathcal{M})\},\\ &R_q^2\stackrel{\text{def}}{=}\dot\Delta_q\{a\,\mathrm{div}\,[(\widetilde\mu(a)-\widetilde\mu(0))\mathcal{M}]\}-\mathrm{div}\,\{a\,\dot\Delta_q[(\widetilde\mu(a)-\widetilde\mu(0))\mathcal{M}]\}.\end{aligned}$$

Notice from (4.1) that $\inf_{(t,x)\in[0,T]\times\mathbb{R}^2}b(t,x)=\underline{b}$ and $\|b\|_{L^\infty([0,T]\times\mathbb{R}^2)}=\overline{b}$. Then thanks to Lemma 2.1, we get by taking L^2 inner product of (M_q) with $\dot\Delta_q u$ that

$$\begin{aligned}\frac12\frac{\mathrm{d}}{\mathrm{d}t}\|\dot\Delta_q u\|^2_{L^2}+c\underline{b}\underline{\mu}2^{2q}\|\dot\Delta_q u\|^2_{L^2}&\le\|\dot\Delta_q u\|_{L^2}(\|R_q^1\|_{L^2}+\|R_q^2\|_{L^2}+\|[u;\dot\Delta_q]\cdot\nabla u\|_{L^2}\\ &\quad+C\,\overline{b}2^q\|[\widetilde\mu(a)-\widetilde\mu(0);\dot\Delta_q]\mathcal{M}\|_{L^2})-(\dot\Delta_q(a\nabla p)\,|\,\dot\Delta_q u)_{L^2}\end{aligned}\tag{4.11}$$

for some constants $c, C > 0$. However thanks to (2.5) and using the fact that $\operatorname{div} u = 0$, we have

$$\begin{aligned}(\dot\Delta_q(a\nabla p) \mid \dot\Delta_q u)_{L^2} &= (\dot\Delta_q(T_a\nabla p) \mid \dot\Delta_q u)_{L^2} + (\dot\Delta_q R'(a,\nabla p) \mid \dot\Delta_q u)_{L^2}\\ &= -(\dot\Delta_q(T_{\nabla a} p) \mid \dot\Delta_q u)_{L^2} + (\dot\Delta_q R'(a,\nabla p) \mid \dot\Delta_q u)_{L^2},\end{aligned}$$

which together with (4.11) ensures that

$$\begin{aligned}&\|\dot\Delta_q u\|_{L^\infty_T(L^2)} + c\underline{b}\ \underline{\mu} 2^{2q}\|\dot\Delta_q u\|_{L^1_T(L^2)}\\ &\le \|\dot\Delta_q u_0\|_{L^2} + \|R^1_q\|_{L^1_T(L^2)} + \|R^2_q\|_{L^1_T(L^2)} + \|[u;\dot\Delta_q]\cdot\nabla u\|_{L^1_T(L^2)}\\ &\quad + \|\dot\Delta_q(\dot T_{\nabla a} p)\|_{L^1_T(L^2)} + \|\dot\Delta_q \dot R'(a,\nabla p)\|_{L^1_T(L^2)} + C\,\overline{b} 2^q \|[\widetilde\mu(a)-\widetilde\mu(0);\dot\Delta_q]\mathcal{M}\|_{L^1_T(L^2)},\end{aligned}$$

from which we deduce that

$$\begin{aligned}&\|u\|_{\widetilde L^\infty_T(\dot H^s)} + c\underline{b}\ \underline{\mu}\|u\|_{\widetilde L^1_T(\dot H^{s+2})}\\ &\le \|u_0\|_{\dot H^s} + \Big\{\sum_{q\in\mathbb{Z}} 2^{2qs}\|R^1_q\|^2_{L^1_T(L^2)}\Big\}^{\frac12} + \Big\{\sum_{q\in\mathbb{Z}} 2^{2qs}\|R^2_q\|^2_{L^1_T(L^2)}\Big\}^{\frac12}\\ &\quad + \|\dot T_{\nabla a} p\|_{\widetilde L^1_T(\dot H^s)} + \|\dot R'(a,\nabla p)\|_{\widetilde L^1_T(\dot H^s)} + \Big(\sum_{q\in\mathbb{Z}} 2^{2qs}\|[u;\dot\Delta_q]\cdot\nabla u\|^2_{L^1_T(L^2)}\Big)^{\frac12}\\ &\quad + C\,\overline{b}\Big(\sum_{q\in\mathbb{Z}} 2^{2q(s+1)}\|[\widetilde\mu(a)-\widetilde\mu(0);\dot\Delta_q]\mathcal{M}\|^2_{L^1_T(L^2)}\Big)^{\frac12}. \end{aligned}\tag{4.12}$$

By Lemma 3.1, we have

$$\Big\{\sum_{q\in\mathbb{Z}} 2^{2qs}\|R^1_q\|^2_{L^1_T(L^2)}\Big\}^{\frac12} \lesssim \|\nabla a\|_{L^\infty_T(L^\infty)}\|\nabla u\|_{\widetilde L^1_T(\dot H^s)} + \|a\|_{\widetilde L^\infty_T(\dot H^s)}\|\nabla^2 u\|_{L^1_T(L^\infty)},$$

but a similar argument as (4.8) implies that

$$\|\nabla^2 u\|_{L^1_T(L^\infty)} \lesssim \|u\|_{\widetilde L^1_T(\dot H^{2-\varepsilon})} + \|u\|_{\widetilde L^1_T(\dot H^{s+2})},$$

from which, by (2.4), we obtain

$$\Big\{\sum_{q\in\mathbb{Z}} 2^{2qs}\|R^1_q\|^2_{L^1_T(L^2)}\Big\}^{\frac12} \lesssim \|\nabla a\|_{\widetilde L^\infty_T(H^{s-1})}[\|u\|_{\widetilde L^1_T(\dot H^{2-\varepsilon})} + \|u\|_{\widetilde L^1_T(\dot H^{s+2})}]. \tag{4.13}$$

Applying Proposition 2.3 and Lemma 3.1 once again gives

$$\begin{aligned}\Big\{\sum_{q\in\mathbb{Z}} 2^{2qs}\|R^2_q\|^2_{L^1_T(L^2)}\Big\}^{\frac12} &\lesssim \|\nabla a\|_{L^\infty_T(L^\infty)}\|(\widetilde\mu(a)-\widetilde\mu(0))\nabla u\|_{\widetilde L^1_T(\dot H^s)}\\ &\quad + \|\nabla a\|_{\widetilde L^\infty_T(\dot H^s)}\|(\widetilde\mu(a)-\widetilde\mu(0))\nabla u\|_{L^1_T(L^\infty)}\\ &\lesssim \|\nabla a\|_{\widetilde L^\infty_T(H^s)}(1+\|a\|_{\widetilde L^\infty_T(H^s)})[\|\nabla u\|_{L^1_T(L^\infty)} + \|\nabla u\|_{\widetilde L^1_T(\dot H^s)}],\end{aligned}$$

which together with (2.4) and (4.7) implies that

$$\Big\{\sum_{q\in\mathbb{Z}} 2^{2qs}\|R^2_q\|^2_{L^1_T(L^2)}\Big\}^{\frac12} \lesssim \|\nabla a\|_{\widetilde L^\infty_T(H^s)}(1+\|a\|_{\widetilde L^\infty_T(H^s)})[\|u\|_{\widetilde L^1_T(\dot H^{2-\varepsilon})} + \|u\|_{\widetilde L^1_T(\dot H^{s+2})}]. \tag{4.14}$$

Similarly, applying (3.6) gives

$$
\begin{aligned}
&\Big(\sum_{q\in\mathbb{Z}} 2^{2q(s+1)}\|[\widetilde{\mu}(a)-\widetilde{\mu}(0);\dot{\Delta}_q]\mathcal{M}\|^2_{L^1_T(L^2)}\Big)^{\frac{1}{2}}\\
&\lesssim \|\nabla a\|_{L^\infty_T(L^\infty)}\|u\|_{\widetilde{L}^1_T(\dot{H}^{s+1})} + \|a\|_{\widetilde{L}^\infty_T(\dot{H}^{s+1})}\|\nabla u\|_{L^1_T(L^\infty)}\\
&\lesssim \|\nabla a\|_{\widetilde{L}^\infty_T(H^s)}[\|u\|_{\widetilde{L}^1_T(\dot{H}^{2-\varepsilon})} + \|u\|_{\widetilde{L}^1_T(\dot{H}^{s+2})}]. \qquad (4.15)
\end{aligned}
$$

On the other hand, by taking divergence to (M), we get

$$
\operatorname{div}(b\nabla p) = \operatorname{div}\vec{F}, \qquad (4.16)
$$

with

$$
\begin{aligned}
\vec{F} &= b\operatorname{div}(\widetilde{\mu}(a)\mathcal{M}) - u\cdot\nabla u - \widetilde{\mu}(0)\Delta u\\
&= (1+a)\operatorname{div}[(\widetilde{\mu}(a)-\widetilde{\mu}(0)\mathcal{M})] + a\widetilde{\mu}(0)\Delta u - u\cdot\nabla u.
\end{aligned}
$$

Then Proposition 2.3 together with (2.4), (3.9) and (4.7) gives

$$
\begin{aligned}
\|\vec{F}\|_{L^1_T(L^2)} &\le (\delta + \|a\|_{L^\infty})\|u\|_{\widetilde{L}^1_T(\dot{H}^2)} + C\Big(\|a\|_{\widetilde{L}^\infty_T(\dot{H}^1)}\|\nabla u\|_{L^1_T(L^\infty)} + \delta^{-1}\int_0^T \|u\|^2_{\dot{C}^0_*}\|u\|_{L^2}\,\mathrm{d}t\Big)\\
&\le C(\delta + \|a\|_{L^\infty} + \|a\|_{\widetilde{L}^\infty_T(\dot{H}^1)})[\|u\|_{\widetilde{L}^1_T(\dot{H}^{2-\varepsilon})} + \|u\|_{\widetilde{L}^1_T(\dot{H}^{s+2})}]\\
&\quad + C\delta^{-1}\int_0^T \|u\|^2_{\dot{C}^0_*}\|u\|_{L^2}\,\mathrm{d}t,\\
\|\vec{F}\|_{\widetilde{L}^1_T(\dot{H}^s)} &\le C\|a\|_{\widetilde{L}^\infty_T(H^{s+1})}(1+\|a\|_{\widetilde{L}^\infty_T(H^s)})[\|\nabla u\|_{L^1_T(L^\infty)} + \|u\|_{\widetilde{L}^1_T(\dot{H}^{s+2})}] + \|u\cdot\nabla u\|_{\widetilde{L}^1_T(\dot{H}^s)}\\
&\le C[\delta + \|a\|_{\widetilde{L}^\infty_T(H^{s+1})}(1+\|a\|_{\widetilde{L}^\infty_T(H^s)})][\|u\|_{\widetilde{L}^1_T(\dot{H}^{2-\varepsilon})} + \|u\|_{\widetilde{L}^1_T(\dot{H}^{s+2})}]\\
&\quad + C\delta^{-1}\int_0^T \|u\|^2_{\dot{C}^0_*}\|u\|_{\dot{H}^s}\,\mathrm{d}t.
\end{aligned}
$$

Then applying Lemma 4.2 gives

$$
\begin{aligned}
\underline{b}\|\nabla p\|_{\widetilde{L}^1_T(H^s)} &\le C(\mathcal{A}_T)^s[\|F\|_{L^1_T(L^2)} + \|F\|_{\widetilde{L}^1_T(\dot{H}^s)}]\\
&\le C(\mathcal{A}_T)^s\Big[[\delta + \|a\|_{\widetilde{L}^\infty_T(H^{s+1})}(1+\|a\|_{\widetilde{L}^\infty_T(H^s)})][\|u\|_{\widetilde{L}^1_T(\dot{H}^{2-\varepsilon})}\\
&\quad + \|u\|_{\widetilde{L}^1_T(\dot{H}^{s+2})}] + \delta^{-1}\int_0^T \|u\|^2_{\dot{C}^0_*}\|u\|_{H^s}\,\mathrm{d}t\Big]. \qquad (4.17)
\end{aligned}
$$

One the other hand, a similar proof of Lemma 3.1 yields

$$
\begin{aligned}
\|\dot{T}_{\nabla a}p\|_{\widetilde{L}^1_T(\dot{H}^s)} + \|\dot{R}'(a,\nabla p)\|_{\widetilde{L}^1_T(\dot{H}^s)} &\lesssim \|\nabla a\|_{L^\infty_T(L^\infty)}\|\nabla p\|_{\widetilde{L}^1_T(\dot{H}^{s-1})} + \|a\|_{\widetilde{L}^\infty_T(\dot{H}^s)}\|\nabla p\|_{L^1_T(L^\infty)}\\
&\lesssim \|\nabla a\|_{\widetilde{L}^\infty_T(H^{s-1})}\|\nabla p\|_{\widetilde{L}^1_T(H^{s-1})}. \qquad (4.18)
\end{aligned}
$$

Plugging (3.8), (4.13)–(4.18) into (4.12), we obtain (4.9), which completes the proof of Proposition 4.1.

Now let us turn to the estimate of $\|u\|_{\widetilde{L}^\infty_T(\dot{H}^{-\varepsilon})} + \|u\|_{\widetilde{L}^1_T(\dot{H}^{2-\varepsilon})}$.

Proposition 4.2 *Let* (a,u) *be a smooth enough solution to* (INS). *Then under the assumptions of Proposition* 4.1*, we have*

$$\begin{aligned}
&\|u\|_{\widetilde{L}^\infty_T(\dot{H}^{-\varepsilon})} + c\underline{b}\ \underline{\mu}\|u\|_{\widetilde{L}^1_T(\dot{H}^{2-\varepsilon})}\\
&\le \|u_0\|_{\dot{H}^{-\varepsilon}} + C\Big\{\delta + \|a\|_{\widetilde{L}^\infty_T(H^s)}(1+\|a\|_{\widetilde{L}^\infty_T(H^s)})\Big(1+\frac{\|a\|_{\widetilde{L}^\infty_T(H^s)}}{\underline{b}-C\|a\|_{\widetilde{L}^\infty_T(H^s)}}\Big)\Big\}\\
&\quad\times[\|u\|_{\widetilde{L}^1_T(\dot{H}^{2-\varepsilon})} + \|u\|_{\widetilde{L}^1_T(\dot{H}^{s+2})}] + C\Big(\delta^{-1}+\frac{\|a\|_{\widetilde{L}^\infty_T(H^s)}}{\underline{b}-C\|a\|_{\widetilde{L}^\infty_T(H^s)}}\Big)\int_0^T \|u\|^2_{\dot{C}^0_*}\|u\|_{\dot{H}^{-\varepsilon}}\mathrm{d}t. \quad (4.19)
\end{aligned}$$

Proof Firstly, by applying $\dot{\Delta}_q$ to (M), we get

$$\begin{aligned}
&\partial_t\dot{\Delta}_q u + \dot{\Delta}_q(u\cdot\nabla u) + \dot{\Delta}_q\nabla p - \operatorname{div}(b\widetilde{\mu}(a)\dot{\Delta}_q\mathcal{M})\\
&= -\dot{\Delta}_q(a\nabla p) + R^3_q + R^4_q + \operatorname{div}(b[\dot{\Delta}_q;\widetilde{\mu}(a)]\mathcal{M}),
\end{aligned}$$

where

$$R^3_q \stackrel{\text{def}}{=} [\dot{\Delta}_q;\ a]\operatorname{div}(\widetilde{\mu}(a)\mathcal{M}), \quad R^4_q \stackrel{\text{def}}{=} -\nabla a\cdot\dot{\Delta}_q(\widetilde{\mu}(a)\mathcal{M}),$$

from which we deduce that

$$\begin{aligned}
&\|\dot{\Delta}_q u\|_{L^\infty_T(L^2)} + c\underline{b}\ \underline{\mu}2^{2q}\|\dot{\Delta}_q u\|_{L^1_T(L^2)}\\
&\le \|\dot{\Delta}_q u_0\|_{L^2} + \|\dot{\Delta}_q(u\cdot\nabla u)\|_{L^1_T(L^2)} + \|\dot{\Delta}_q(a\nabla p)\|_{L^1_T(L^2)} + \|R^3_q\|_{L^1_T(L^2)}\\
&\quad + \|R^4_q\|_{L^1_T(L^2)} + C\overline{b}2^q\|[\dot{\Delta}_q;\widetilde{\mu}(a)-\widetilde{\mu}(0)]\mathcal{M}\|_{L^1_T(L^2)}.
\end{aligned}$$

Multiplying the above inequality by $2^{-q\varepsilon}$ and taking the ℓ^2 norm to the resulting inequality, we obtain

$$\begin{aligned}
&\|u\|_{\widetilde{L}^\infty_T(\dot{H}^{-\varepsilon})} + c\underline{b}\ \underline{\mu}\|u\|_{\widetilde{L}^1_T(\dot{H}^{2-\varepsilon})}\\
&\le \|u_0\|_{\dot{H}^{-\varepsilon}} + \|a\nabla p\|_{\widetilde{L}^1_T(\dot{H}^{-\varepsilon})} + \|u\cdot\nabla u\|_{\widetilde{L}^1_T(\dot{H}^{-\varepsilon})} + \Big\{\sum_{q\in\mathbb{Z}}2^{-2q\varepsilon}\|R^3_q\|^2_{L^1_T(L^2)}\Big\}^{\frac12}\\
&\quad + \Big\{\sum_{q\in\mathbb{Z}}2^{-2q\varepsilon}\|R^4_q\|^2_{L^1_T(L^2)}\Big\}^{\frac12} + \overline{b}\Big\{\sum_{q\in\mathbb{Z}}2^{2q(1-\varepsilon)}\|[\dot{\Delta}_q;\widetilde{\mu}(a)-\widetilde{\mu}(0)]\mathcal{M}\|^2_{L^1_T(L^2)}\Big\}^{\frac12}. \quad (4.20)
\end{aligned}$$

Firstly, applying Lemma 3.4 and (3.20), we have

$$\begin{aligned}
&\Big\{\sum_{q\in\mathbb{Z}}2^{-2q\varepsilon}\|R^3_q\|^2_{L^1_T(L^2)}\Big\}^{\frac12}\\
&\lesssim (\|a\|_{L^\infty_T(L^\infty)} + \|a\|_{\widetilde{L}^1_T(\dot{H}^1)})[\|(\widetilde{\mu}(a)-\widetilde{\mu}(0))\mathcal{M}\|_{\widetilde{L}^1_T(\dot{H}^{1-\varepsilon})} + \widetilde{\mu}(0)\|u\|_{\widetilde{L}^1_T(\dot{H}^{2-\varepsilon})}]\\
&\lesssim (\|a\|_{L^\infty_T(L^\infty)} + \|a\|_{\widetilde{L}^1_T(\dot{H}^1)})(1+\|a\|_{L^\infty_T(L^\infty)} + \|a\|_{\widetilde{L}^1_T(\dot{H}^1)})\|u\|_{\widetilde{L}^1_T(\dot{H}^{2-\varepsilon})}. \quad (4.21)
\end{aligned}$$

And applying (3.7) and (4.7), we have

$$\begin{aligned}
&\Big\{\sum_{q\in\mathbb{Z}}2^{2q(1-\varepsilon)}\|[\dot{\Delta}_q;\widetilde{\mu}(a)-\widetilde{\mu}(0)]\mathcal{M}\|^2_{L^1_T(L^2)}\Big\}^{\frac12}\\
&\lesssim \|\widetilde{\mu}(a)-\widetilde{\mu}(0)\|_{L^\infty_T(L^\infty)}\|u\|_{\widetilde{L}^1_T(\dot{H}^{2-\varepsilon})} + \|\nabla\widetilde{\mu}(a)\|_{\widetilde{L}^\infty_T(\dot{H}^{1-\varepsilon})}\|\nabla u\|_{L^1_T(L^\infty)}\\
&\lesssim (\|a\|_{L^\infty_T(L^\infty)} + \|a\|_{\widetilde{L}^\infty_T(\dot{H}^{2-\varepsilon})})(\|u\|_{\widetilde{L}^1_T(\dot{H}^{2-\varepsilon})} + \|u\|_{\widetilde{L}^1_T(\dot{H}^{s+2})}). \quad (4.22)
\end{aligned}$$

On the other hand, thanks to Bony's decomposition (2.5), we have

$$\widetilde{\mu}(a)\mathcal{M} = \dot{T}_{\widetilde{\mu}(a)}\mathcal{M} + \dot{T}_{\mathcal{M}}\widetilde{\mu}(a) + \dot{R}(\widetilde{\mu}(a), \mathcal{M}).$$

Note that

$$\|\dot{\Delta}_q \dot{T}_{\widetilde{\mu}(a)}\mathcal{M}\|_{L^1_T(L^\infty)} \lesssim \sum_{|q-\ell|\le 4} \|\dot{S}_{\ell-1}(\widetilde{\mu}(a))\|_{L^\infty_T(L^\infty)} \|\dot{\Delta}_\ell(\nabla u)\|_{L^1_T(L^\infty)} \lesssim c_q 2^{q\varepsilon} \|u\|_{\widetilde{L}^1_T(\dot{H}^{2-\varepsilon})}.$$

So, a similar estimate holds for $\dot{T}_{\mathcal{M}}\widetilde{\mu}(a)$. While thanks to Lemma 2.1, we have

$$\|\dot{\Delta}_q(\dot{R}(\widetilde{\mu}(a), \mathcal{M}))\|_{L^1_T(L^\infty)} \lesssim 2^q \sum_{\ell\ge q-1} \|\dot{\Delta}_\ell(\widetilde{\mu}(a))\|_{L^\infty_T(L^\infty)} \|\widetilde{\dot{\Delta}}_\ell(\mathcal{M})\|_{L^1_T(L^2)} \lesssim c_q 2^{q\varepsilon} \|u\|_{\widetilde{L}^1_T(\dot{H}^{2-\varepsilon})}.$$

Therefore, we obtain

$$\Big\{ \sum_{q\in\mathbb{Z}} 2^{-2q\varepsilon} \|R^4_q\|^2_{L^1_T(L^2)} \Big\}^{\frac{1}{2}} \lesssim \|\nabla a\|_{L^\infty_T(L^2)} \|u\|_{\widetilde{L}^1_T(\dot{H}^{2-\varepsilon})}. \tag{4.23}$$

Finally, to handle $\|a\nabla p\|_{\widetilde{L}^1_T(\dot{H}^{-\varepsilon})}$, we first deduce from (4.16) that

$$\underline{b}\|\dot{\Delta}_q \nabla p\|_{L^1(L^2)} \le \|\dot{\Delta}_q \vec{F}\|_{L^1_T(L^2)} + \|[a; \dot{\Delta}_q]\nabla p\|_{L^1_T(L^2)},$$

which gives

$$\begin{aligned} \underline{b}\|\nabla p\|_{\widetilde{L}^1_T(\dot{H}^{-\varepsilon})} &\le \|\vec{F}\|_{\widetilde{L}^1_T(\dot{H}^{-\varepsilon})} + \Big\{ \sum_{q\in\mathbb{Z}} 2^{-2q\varepsilon} \|[a; \dot{\Delta}_q]\nabla p\|^2_{L^1_T(L^2)} \Big\}^{\frac{1}{2}} \\ &\lesssim \|\vec{F}\|_{\widetilde{L}^1_T(\dot{H}^{-\varepsilon})} + (\|a\|_{\widetilde{L}^\infty_T(\dot{H}^1)} + \|a\|_{L^\infty_T(L^\infty)}) \|\nabla p\|_{\widetilde{L}^1_T(\dot{H}^{-\varepsilon})}. \end{aligned} \tag{4.24}$$

However, by applying (3.20), we get

$$\|\operatorname{div}[(\widetilde{\mu}(a) - \widetilde{\mu}(0))\mathcal{M}]\|_{\widetilde{L}^1_T(\dot{H}^{-\varepsilon})} \lesssim (\|a\|_{L^\infty_T(L^\infty)} + \|a\|_{\widetilde{L}^\infty_T(\dot{H}^1)}) \|u\|_{\widetilde{L}^1_T(\dot{H}^{2-\varepsilon})},$$

while (3.20) gives

$$\|a \operatorname{div}[(\widetilde{\mu}(a) - \widetilde{\mu}(0))\mathcal{M}]\|_{\widetilde{L}^1_T(\dot{H}^{-\varepsilon})} \lesssim (\|a\|_{L^\infty_T(L^\infty)} + \|a\|_{\widetilde{L}^\infty_T(\dot{H}^1)}) \|\operatorname{div}[(\widetilde{\mu}(a) - \widetilde{\mu}(0))\mathcal{M}]\|_{\widetilde{L}^1_T(\dot{H}^{-\varepsilon})}.$$

Therefore, it follows from (3.9) and (4.24) that

$$\begin{aligned} &(\underline{b} - C(\|a\|_{L^\infty_T(L^\infty)} + \|a\|_{\widetilde{L}^\infty_T(\dot{H}^1)})) \|\nabla p\|_{\widetilde{L}^1_T(\dot{H}^{-\varepsilon})} \le C\|\vec{F}\|_{\widetilde{L}^1_T(\dot{H}^{-\varepsilon})} \\ &\le C(1 + \|a\|_{L^\infty_T(L^\infty)} + \|a\|_{\widetilde{L}^\infty_T(\dot{H}^1)})^2 \|u\|_{\widetilde{L}^1_T(\dot{H}^{2-\varepsilon})} + C\int_0^T \|u\|^2_{\dot{C}^0_*} \|u\|_{\dot{H}^{-\varepsilon}} \mathrm{d}t. \end{aligned} \tag{4.25}$$

On the other hand, thanks to (3.20), we have

$$\|a\nabla p\|_{\widetilde{L}^\infty_T(\dot{H}^{-\varepsilon})} \lesssim (\|a\|_{L^\infty_T(L^\infty)} + \|a\|_{\widetilde{L}^\infty_T(\dot{H}^1)}) \|\nabla p\|_{\widetilde{L}^1_T(\dot{H}^{-\varepsilon})},$$

which together with (4.25) gives

$$\|a\nabla p\|_{\widetilde{L}^1_T(\dot{H}^{-\varepsilon})} \le \frac{C\|a\|_{\widetilde{L}^\infty_T(H^s)}}{\underline{b} - C\|a\|_{\widetilde{L}^\infty_T(H^s)}} \Big((1 + \|a\|_{\widetilde{L}^\infty_T(H^s)})^2 \|u\|_{\widetilde{L}^1_T(\dot{H}^{2-\varepsilon})} + \int_0^T \|u\|^2_{\dot{C}^0_*} \|u\|_{\dot{H}^{-\varepsilon}} \mathrm{d}t \Big). \tag{4.26}$$

Plunging (3.8), (4.21)–(4.23) and (4.26) into (4.20), we get (4.19). This completes the proof of Proposition 4.2.

4.4 Proof of Theorem 1.2 (existence part)

Now we are in a position to complete the proof of the existence part of Theorem 1.2.

Proof of Theorem 1.2 (Existence Part) Firstly, as $u_0 \in \dot{H}^s(\mathbb{R}^2) \cap \dot{H}^{-\varepsilon}(\mathbb{R}^2)$, a similar proof of (4.7) ensures that $u_0 \in L^2(\mathbb{R}^2)$, and consequently $u_0 \in H^s(\mathbb{R}^2)$. Given $(a_0, u_0) \in H^{s+1}(\mathbb{R}^2) \times H^s(\mathbb{R}^2)$, it is standard (see e.g. [1]) to prove that (INS) has a unique solution (a, u) on $[0, T]$ for some $T > 0$, and the following energy equality holds on $[0, T]$:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2} \frac{1}{b}|u|^2 \,\mathrm{d}x + \int_{\mathbb{R}^2} \widetilde{\mu}(a)|\mathcal{M}|^2 \,\mathrm{d}x = 0. \tag{4.27}$$

Thanks to (4.2), as long as $|a_0| < 1$,

$$0 < \underline{b} \le 1 + a = b < 2,$$

which together with (4.27) gives

$$\frac{1}{4}\int_{\mathbb{R}^2} |u|^2 \,\mathrm{d}x + \underline{\mu}\int_0^t\int_{\mathbb{R}^2} |\nabla u|^2 \,\mathrm{d}x\,\mathrm{d}t \le \frac{1}{\underline{b}}\int_{\mathbb{R}^2} |u_0|^2 \,\mathrm{d}x. \tag{4.28}$$

Furthermore, thanks to (4.9) and (4.19), we have

$$\begin{aligned}
&\|u\|_{\widetilde{L}^\infty_t(\dot{H}^s)} + \|u\|_{\widetilde{L}^\infty_t(\dot{H}^{-\varepsilon})} + \Big[c\underline{b}\,\underline{\mu} - C\Big[\delta + \|a\|_{\widetilde{L}^\infty_t(H^{s+1})}(1 + \|a\|_{\widetilde{L}^\infty_t(H^{s+1})})^{s+2} \\
&\times\Big(1 + \frac{\|a\|_{\widetilde{L}^\infty_t(H^s)}}{\underline{b} - C\|a\|_{\widetilde{L}^\infty_t(H^s)}}\Big)\Big]\Big][\|u\|_{\widetilde{L}^1_t(\dot{H}^{2-\varepsilon})} + \|u\|_{\widetilde{L}^1_t(\dot{H}^{s+2})}] \\
\le{}& \|u_0\|_{\dot{H}^s} + \|u_0\|_{\dot{H}^{-\varepsilon}} + C\delta^{-1}\|u_0\|^3_{L^2} \\
&+ C\Big[\delta^{-1} + \frac{\|a\|_{\widetilde{L}^\infty_t(H^s)}}{\underline{b} - C\|a\|_{\widetilde{L}^\infty_T(H^s)}}\Big]\int_0^t \|u(t')\|^2_{\dot{C}^0_*}(\|u\|_{\widetilde{L}^\infty_{t'}(\dot{H}^s)} + \|u\|_{\widetilde{L}^\infty_{t'}(\dot{H}^{-\varepsilon})})\,\mathrm{d}t'
\end{aligned} \tag{4.29}$$

for $t \le T$, where we use (4.28) and $\|u\|_{\dot{C}^0_*} \le C\|\nabla u\|_{L^2}$, which follows from Lemma 2.1, so that

$$\int_0^t \|u(t')\|^2_{\dot{C}^0_*}\|u(t')\|_{L^2}\,\mathrm{d}t' \le C\|u_0\|^3_{L^2}.$$

Now let us define

$$T^* \overset{\text{def}}{=} \sup\Big\{T > 0 : \|a\|_{\widetilde{L}^\infty_T(H^{s+1})} \le \min\Big(\frac{\underline{b}}{2C},\ \frac{c\underline{b}\,\underline{\mu}}{4(1+\frac{1}{C})(1+\frac{\underline{b}}{2C})^{s+2}}\Big) \overset{\text{def}}{=} \zeta_0\Big\}. \tag{4.30}$$

If $T^* < \infty$, by taking $\delta \le \frac{c\underline{b}\,\underline{\mu}}{4C}$ in (4.29), we get

$$\begin{aligned}
&\|u\|_{\widetilde{L}^\infty_t(\dot{H}^{-\varepsilon})} + \|u\|_{\widetilde{L}^\infty_t(\dot{H}^s)} + \frac{c\underline{b}\,\underline{\mu}}{2}[\|u\|_{\widetilde{L}^1_t(\dot{H}^{2-\varepsilon})} + \|u\|_{\widetilde{L}^1_t(\dot{H}^{s+2})}] \\
\le{}& \|u_0\|_{\dot{H}^s} + \|u_0\|_{\dot{H}^{-\varepsilon}} + \frac{4C}{c\underline{b}\,\underline{\mu}}\|u_0\|^3_{L^2} \\
&+ C(\delta^{-1} + C^{-1})\int_0^t \|u\|^2_{\dot{C}^0_*}(\|u\|_{\widetilde{L}^\infty_{t'}(\dot{H}^s)} + \|u\|_{\widetilde{L}^\infty_{t'}(\dot{H}^{-\varepsilon})})\,\mathrm{d}t'.
\end{aligned} \tag{4.31}$$

Then, by Gronwall inequality and (4.28), we get

$$\begin{aligned}&\|u\|_{\widetilde{L}^\infty_t(\dot{H}^{-\varepsilon})}+\|u\|_{\widetilde{L}^\infty_t(\dot{H}^{s})}+\frac{c\underline{b}\,\underline{\mu}}{2}[\|u\|_{\widetilde{L}^1_t(\dot{H}^{2-\varepsilon})}+\|u\|_{\widetilde{L}^1_t(\dot{H}^{s+2})}]\\&\leq C(\|u_0\|_{\dot{H}^s}+\|u_0\|_{\dot{H}^{-\varepsilon}})(1+(\|u_0\|_{\dot{H}^s}+\|u_0\|_{\dot{H}^{-\varepsilon}}))^2\\&\quad\times\exp\Big(C(\delta^{-1}+C^{-1})\frac{\|u_0\|_{L^2}^2}{\underline{b}\,\underline{\mu}}\Big)\overset{\text{def}}{=}\eta_0\end{aligned}\tag{4.32}$$

for $t<T^*$, from which, together with (4.2) and (4.7), we deduce that

$$\|a\|_{\widetilde{L}^\infty_T(H^{s+1})}\leq C\|a_0\|_{H^{s+1}}\Big(1+\frac{2\eta_0}{c\underline{b}\,\underline{\mu}}\Big)\exp\Big(\frac{2C\eta_0}{c\underline{b}\,\underline{\mu}}\Big),\quad T<T^*.\tag{4.33}$$

Therefore, if we take a_0 small enough so that

$$\|a_0\|_{H^{s+1}}\leq\frac{\zeta_0}{\max\big(4,2C\big(1+\frac{2\eta_0}{c\underline{b}\,\underline{\mu}}\big)\big)}\exp\Big(-\frac{2C\eta_0}{c\underline{b}\,\underline{\mu}}\Big),$$

then (4.33) implies that

$$\|a\|_{\widetilde{L}^\infty_T(H^{s+1})}\leq\frac{\zeta_0}{2},\quad\forall\,T<T^*,$$

which contradicts (4.30), and therefore $T^*=\infty$. This completes the existence proof of Theorem 1.2.

4.5 Proof of Theorem 1.2 (L^2 decay part)

The main goal of this subsection is to prove (1.4). Motivated by [18], we shall first focus on a logarithmic-type decay estimate of $\|u(t)\|_{L^2}$.

Lemma 4.4 *Under the assumptions of Theorem* 1.2*, there holds*

$$\|u(t)\|_{L^2}\lesssim\ln^{-1}(\mathrm{e}+t).\tag{4.34}$$

Proof Firstly, thanks to (4.27), one has

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\sqrt{\rho}u\|_{L^2}^2+\underline{\mu}\|\nabla u\|_{L^2}^2\leq 0.$$

Applying Schonbek's strategy in [15], by splitting the phase-space $\mathbb{R}^2$ into two time-dependent parts, we get

$$\|\nabla u(t)\|_{L^2}^2=\int_{S(t)}|\xi|^2|\widehat{u}(t,\xi)|^2\mathrm{d}\xi+\int_{S(t)^c}|\xi|^2|\widehat{u}(t,\xi)|^2\mathrm{d}\xi,$$

where $S(t)\overset{\text{def}}{=}\big\{\xi:|\xi|\leq\sqrt{\frac{\overline{\rho}}{\underline{\mu}}}\;g(t)\big\}$, $\overline{\rho}\overset{\text{def}}{=}\sup\limits_{(t,x)\in\mathbb{R}^+\times\mathbb{R}^2}\rho(t,x)=\sup\limits_{x\in\mathbb{R}^2}\rho_0(x)$, and $g(t)$ satisfies $g(t)\lesssim(1+t)^{-\frac12}$, which will be chosen later on. Then we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\sqrt{\rho}u(t)\|_{L^2}^2+g^2(t)\|\sqrt{\rho}\,u(t)\|_{L^2}^2\leq\overline{\rho}g^2(t)\int_{S(t)}|\widehat{u}(t,\xi)|^2\mathrm{d}\xi.\tag{4.35}$$

To deal with the low frequency part of u on the right-hand side of (4.35), noting that $\operatorname{div}\mathcal{M} = \frac{1}{2}\Delta u$, we rewrite the momentum equations in (INS) as

$$
\begin{aligned}
&\partial_t u - \mu_0\Delta u + u\cdot\nabla u + \nabla p + a\nabla p - \mu_0 a\Delta u\\
&\quad - \operatorname{div}[(1+a)(\widetilde{\mu}(a) - \widetilde{\mu}(0))\mathcal{M}] + (\widetilde{\mu}(a) - \widetilde{\mu}(0))\nabla a\mathcal{M} = 0,
\end{aligned}
$$

where $\mu_0 \overset{\text{def}}{=} \frac{1}{2}\widetilde{\mu}(0)$. Denoting $\mathbb{P}$ to be the Leray projection operator, by using Duhamel's principle, we get

$$
\begin{aligned}
u = \mathrm{e}^{\mu_0 t\Delta}u_0 + \int_0^t &\mathrm{e}^{\mu_0(t-t')\Delta}\mathbb{P}(\nabla\cdot(-u\otimes u) + \operatorname{div}[(1+a)(\widetilde{\mu}(a) - \widetilde{\mu}(0))\mathcal{M}]\\
&+ \mu_0 a\Delta u - a\nabla p - (\widetilde{\mu}(a) - \widetilde{\mu}(0))\nabla a\mathcal{M})\,\mathrm{d}t'.
\end{aligned}
$$

Taking Fourier transform with respect to x variables gives rise to

$$
\begin{aligned}
|\widehat{u}(t,\xi)| \lesssim \mathrm{e}^{-\mu_0 t|\xi|^2}|\widehat{u}_0(\xi)| + \int_0^t &\mathrm{e}^{-\mu_0(t-t')|\xi|^2}[|\xi|(|\mathcal{F}_x(u\otimes u)| + |\mathcal{F}_x[(1+a)(\widetilde{\mu}(a) - \widetilde{\mu}(0))\mathcal{M}]|)\\
&+ |\mathcal{F}_x(a\Delta u)| + |\mathcal{F}_x(a\nabla p)| + |\mathcal{F}_x[(\widetilde{\mu}(a) - \widetilde{\mu}(0))\nabla a\mathcal{M}]|]\mathrm{d}t',
\end{aligned}
$$

so that

$$
\begin{aligned}
\int_{S(t)}|\widehat{u}(t,\xi)|^2\mathrm{d}\xi \lesssim \int_{S(t)}&\mathrm{e}^{-2\mu_0 t|\xi|^2}|\widehat{u}_0(\xi)|^2\mathrm{d}\xi + g^4(t)\Big[\int_0^t(\|\mathcal{F}_x(u\otimes u)\|_{L^\infty_\xi}\\
&+ \|\mathcal{F}_x[(1+a)(\widetilde{\mu}(a) - \widetilde{\mu}(0))\mathcal{M}]\|_{L^\infty_\xi})\,\mathrm{d}t'\Big]^2 + g^2(t)\Big[\int_0^t(\|\mathcal{F}_x(a\Delta u)\|_{L^\infty_\xi}\\
&+ \|\mathcal{F}_x(a\nabla p)\|_{L^\infty_\xi} + \|\mathcal{F}_x[(\widetilde{\mu}(a) - \widetilde{\mu}(0))\nabla a\mathcal{M}]\|_{L^\infty_\xi})\,\mathrm{d}t'\Big]^2.
\end{aligned}
\tag{4.36}
$$

Applying (4.28) and (4.33) gives

$$
\begin{aligned}
\Big(\int_0^t\|\mathcal{F}_x[(1+a)(\widetilde{\mu}(a) - \widetilde{\mu}(0))\mathcal{M}]\|_{L^\infty_\xi}\,\mathrm{d}t'\Big)^2 &\le \Big(\int_0^t\|(1+a)(\widetilde{\mu}(a) - \widetilde{\mu}(0))\mathcal{M}\|_{L^1}\,\mathrm{d}t'\Big)^2\\
&\lesssim \|\widetilde{\mu}(a) - \widetilde{\mu}(0)\|^2_{L^\infty_t(L^2)}\|\nabla u\|^2_{L^1_t(L^2)}\\
&\lesssim (1+t)\|\nabla u\|_{L^2_t(L^2)} \lesssim (1+t)
\end{aligned}
\tag{4.37}
$$

and

$$
\Big(\int_0^t\|\mathcal{F}_x(a\nabla p)\|_{L^\infty_\xi}\,\mathrm{d}t'\Big)^2 \le \Big(\int_0^t\|a\nabla p\|_{L^1}\,\mathrm{d}t'\Big)^2 \lesssim \|a\|^2_{L^\infty_t(L^2)}\|\nabla p\|^2_{L^1_t(L^2)} \le C.
$$

Furthermore, thanks to (4.7) and (4.32), we have

$$
\begin{aligned}
&\Big(\int_0^t\|\mathcal{F}_x(a\Delta u)\|_{L^\infty_\xi}\,\mathrm{d}t'\Big)^2 \le \|a\|_{L^\infty_t(L^2)}\|\Delta u\|_{L^1_t(L^2)} \le C,\\
&\Big(\int_0^t\|\mathcal{F}_x[(\widetilde{\mu}(a) - \widetilde{\mu}(0))\nabla a\mathcal{M}]\|_{L^\infty_\xi}\mathrm{d}t'\Big)^2 \lesssim \|(\widetilde{\mu}(a) - \widetilde{\mu}(0))\|^2_{L^\infty_t(L^2)}\|\nabla a\|^2_{L^\infty_t(L^2)}\|\nabla u\|^2_{L^1_t(L^\infty)} \lesssim C.
\end{aligned}
$$

Finally, it is easy to observe that

$$
\Big(\int_0^t\|\mathcal{F}_x(u\otimes u)\|_{L^\infty_\xi}\,\mathrm{d}t'\Big)^2 \le \Big(\int_0^t\|u\otimes u\|_{L^1}\,\mathrm{d}t'\Big)^2 = \Big(\int_0^t\|u\|^2_{L^2}\,\mathrm{d}t'\Big)^2.
$$

Noting that $u_0 \in \dot{H}^s \cap \dot{H}^{-\varepsilon}$, one has

$$\int_{S(t)} \mathrm{e}^{-2\mu_0 t|\xi|^2} |\widehat{u}_0(\xi)|^2 \mathrm{d}\xi \lesssim (t+1)^{-2\varepsilon}.$$

Then as $g(t) \lesssim (1+t)^{-\frac{1}{2}}$, one deduces from (4.36) that

$$\int_{S(t)} |\widehat{u}(t,\xi)|^2 \mathrm{d}\xi \lesssim g^4(t) \Big(\int_0^t \|u(t')\|_{L^2}^2 \,\mathrm{d}t' \Big)^2 + (1+t)^{-2\kappa} \tag{4.38}$$

with $\kappa \stackrel{\text{def}}{=} \min\{\frac{1}{2}, \varepsilon\}$. Substituting (4.38) to (4.35) results in

$$\frac{\mathrm{d}}{\mathrm{d}t} \|\sqrt{\rho} u\|_{L^2}^2 + g^2(t) \|\sqrt{\rho}\, u\|_{L^2}^2 \lesssim g^2(t)(t+1)^{-2\kappa} + g^6(t) \Big(\int_0^t \|u(t')\|_{L^2}^2 \,\mathrm{d}t' \Big)^2, \tag{4.39}$$

which gives

$$\begin{aligned} &\mathrm{e}^{\int_0^t g^2(t')\,\mathrm{d}t'} \|\sqrt{\rho}\, u(t)\|_{L^2}^2 \\ &\lesssim \|\sqrt{\rho_0}\, u_0\|_{L^2}^2 + \int_0^t \mathrm{e}^{\int_0^{t'} g^2(r)\mathrm{d}r} \Big(g^2(t')(1+t')^{-2\kappa} + g^6(t') \Big(\int_0^{t'} \|u(\tau)\|_{L^2}^2 \mathrm{d}\tau \Big)^2 \Big) \,\mathrm{d}t'. \end{aligned} \tag{4.40}$$

Now taking $g^2(t) = \frac{3}{(\mathrm{e}+t)\ln(\mathrm{e}+t)}$ in (4.40), we deduce from (4.28) and (4.40) that

$$\begin{aligned} \|\sqrt{\rho} u(t)\|_{L^2}^2 \ln^3(\mathrm{e}+t) &\lesssim 1 + \int_0^t \Big(\frac{\ln^2(\mathrm{e}+t')}{(\mathrm{e}+t')^{1+2\kappa}} + \frac{1}{(\mathrm{e}+t')^3} \Big(\int_0^{t'} \|u(\tau)\|_{L^2}^2 \mathrm{d}\tau \Big)^2 \Big) \,\mathrm{d}t' \\ &\lesssim 1 + \int_0^t \frac{1}{(\mathrm{e}+t')} \,\mathrm{d}t' \lesssim \ln(\mathrm{e}+t), \end{aligned}$$

which gives (4.34), and this completes the proof of Lemma 4.4.

With the fundamental estimate (4.34), we can follow the main ideas of [18] to prove (1.4).

Proof of Theorem 1.2 (Decay Part) We shall essentially follow the ideas of [18]. For completeness, we shall present most of the details here. First, by applying (4.34), we get

$$\int_0^t \|u(t')\|_{L^2}^2 \,\mathrm{d}t' \leq C(\mathrm{e}+t) \ln^{-2}(\mathrm{e}+t). \tag{4.41}$$

Taking $g^2(t) = \frac{\alpha}{t+\mathrm{e}}$ in (4.39) for some $\alpha \in [2\kappa, 1]$ and thanks to (4.41), we obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} ((\mathrm{e}+t)^\alpha \|\sqrt{\rho} u(t)\|_{L^2}^2) &\lesssim (\mathrm{e}+t)^{\alpha-1-2\kappa} + (\mathrm{e}+t)^{\alpha-3} \Big(\int_0^t \|u(t')\|_{L^2}^2 \,\mathrm{d}t' \Big)^2 \\ &\lesssim (\mathrm{e}+t)^{\alpha-1-2\kappa} + (e+t)^{\alpha-2} \ln^{-2}(\mathrm{e}+t) \int_0^t \|u(t')\|_{L^2}^2 \,\mathrm{d}t', \end{aligned}$$

which gives

$$(\mathrm{e}+t)^\alpha \|\sqrt{\rho} u(t)\|_{L^2}^2 \lesssim (\mathrm{e}+t)^{\alpha-2\kappa} + \int_0^t (\mathrm{e}+t')^{\alpha-2} \ln^{-2}(\mathrm{e}+t') \Big(\int_0^{t'} \|u(\tau)\|_{L^2}^2 \mathrm{d}\tau \Big) \,\mathrm{d}t'. \tag{4.42}$$

For $t \geq 1$, we define

$$y(t) \stackrel{\text{def}}{=} \int_0^t (\mathrm{e}+t')^\alpha \|\sqrt{\rho}\, u(t')\|_{L^2}^2 \,\mathrm{d}t' \quad \text{and} \quad Y(t) \stackrel{\text{def}}{=} \max\{y(s); 1 \leq s \leq t\}.$$

Let $I(t) \stackrel{\text{def}}{=} \int_0^t \|u(t')\|_{L^2}^2 \, \mathrm{d}t'$. Then one has

$$
\begin{aligned}
I(t) &= \int_0^{t-[t]} \|u(t')\|_{L^2}^2 \, \mathrm{d}t' + \int_{t-[t]}^{t} \|u(t')\|_{L^2}^2 \, \mathrm{d}t' \\
&\le C_0 + \sum_{j=0}^{[t]-1} \int_{t-j-1}^{t-j} (\mathrm{e}+t')^{\alpha} \|u(t')\|_{L^2}^2 (\mathrm{e}+t')^{-\alpha} \, \mathrm{d}t' \\
&\le C_0 + Y(s) \sum_{j=0}^{[t]-1} (t-j)^{-\alpha} \le C_0 + Y(t) \frac{(\mathrm{e}+t)^{1-\alpha}}{1-\alpha},
\end{aligned}
\tag{4.43}
$$

from which, by integrating (4.42) from $t-1$ to t, we get

$$
y(t) \lesssim (\mathrm{e}+t)^{\alpha-2\kappa} + \int_0^t (\mathrm{e}+t')^{-1} \ln^{-2}(\mathrm{e}+t') Y(t') \, \mathrm{d}t'.
$$

Then, applying Gronwall's inequality, we have

$$
Y(t) \lesssim (\mathrm{e}+t)^{\alpha-2\kappa} + \int_0^t (\mathrm{e}+t')^{\alpha-2\kappa-1} \ln^{-2}(\mathrm{e}+t') \, \mathrm{d}t' \lesssim (\mathrm{e}+t)^{\alpha-2\kappa}. \tag{4.44}
$$

Plunging (4.44) into (4.43) gives rise to $I(t) \lesssim (\mathrm{e}+t)^{1-2\kappa}$. Then it follows from (4.42) that

$$
(\mathrm{e}+t)^{\alpha} \|\sqrt{\rho} u(t)\|_{L^2}^2 \lesssim (\mathrm{e}+t)^{\alpha-2\kappa} + \int_0^t (\mathrm{e}+t')^{\alpha-1-2\kappa} \ln^{-2}(\mathrm{e}+t') \, \mathrm{d}t' \lesssim (\mathrm{e}+t)^{\alpha-2\kappa},
$$

which gives (1.4), and this completes the proof of Theorem 1.2.

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