Chin. Ann. Math.
32B(1), 2011, 45–58
DOI: 10.1007/s11401-010-0625-0

Chinese Annals of Mathematics, Series B

On a Class of Infinite-Dimensional Hamiltonian Systems with Asymptotically Periodic Nonlinearities*

Minbo YANG[1] Zifei SHEN[2] Yanheng DING[2]

Abstract The authors study the existence of homoclinic type solutions for the following system of diffusion equations on $\mathbb{R}\times\mathbb{R}^N$:

$$\begin{cases}\partial_t u-\Delta_x u+b\cdot\nabla_x u+au+V(t,x)v=H_v(t,x,u,v),\\ -\partial_t v-\Delta_x v-b\cdot\nabla_x v+av+V(t,x)u=H_u(t,x,u,v),\end{cases}$$

where $z=(u,v):\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}^m\times\mathbb{R}^m$, $a>0$, $b=(b_1,\cdots,b_N)$ is a constant vector and $V\in C(\mathbb{R}\times\mathbb{R}^N,\mathbb{R})$, $H\in C^1(\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^{2m},\mathbb{R})$. Under suitable conditions on $V(t,x)$ and the nonlinearity for $H(t,x,z)$, at least one non-stationary homoclinic solution with least energy is obtained.

Keywords Variational methods, Least energy solution, Hamiltonian system
2000 MR Subject Classification 35R30, 49J40, 58D30

1 Introduction

Recently, the following system of diffusion equations has been wildly considered:

$$\begin{cases}\partial_t u-\Delta_x u+b(t,x)\cdot\nabla_x u+V(x)u=H_v(t,x,u,v),\\ -\partial_t v-\Delta_x v-b(t,x)\cdot\nabla_x v+V(x)v=H_u(t,x,u,v),\end{cases}\qquad (t,x)\in\mathbb{R}\times\Omega,\tag{1.1}$$

where Ω is a domain of $\mathbb{R}^N$, $z=(u,v):\mathbb{R}\times\Omega\to\mathbb{R}^m\times\mathbb{R}^m$, $b=(b_1,\cdots,b_N)$ is a vector, $V\in\mathbb{C}(\Omega,\mathbb{R})$ and $H\in\mathbb{C}^1(\mathbb{R}\times\Omega\times\mathbb{R}^{2m},\mathbb{R})$. Such problems arise from optimal control of systems governed by partial differential equations (cf. [22]), and are related to the Schrödinger equations (cf. [25]).

When the existence of stationary solutions is involved. Many authors have devoted to the research of the Hamiltonian type elliptic systems. For example: de Figueiredo and Jianfu Yang [18] considered

$$\begin{cases}-\Delta\varphi+\varphi=g(x,\psi),\quad \text{in } \mathbb{R}^N,\\ -\Delta\psi+\psi=f(x,\varphi),\quad \text{in } \mathbb{R}^N,\end{cases}\tag{1.2}$$

Manuscript received October 21, 2009. Revised April 29, 2010. Published online December 28, 2010.
[1]Department of Mathematics, Zhejiang Normal University, Jinhua 321004, Zhejiang, China.
E-mail: mbyang@zjnu.cn
[2]Institute of Mathematics, AMSS, Chinese Academy of Sciences, Beijing 100190, China.
*Project supported by the National Natural Science Foundation of China (No. 10971194), the Zhejiang Provincial Natural Science Foundation of China (Nos. Y7080008, R6090109) and the Zhejiang Innovation Project (No. T200905).

and showed the existence of a strong radial solution pair. There are also some results about the existence of solutions for (1.2) in bounded domain. Hulshof-Van der Vorst [20] and de Figueiredo, Felmer [17] considered the elliptic systems by using the Sobolev spaces of fractional order.

As far as we know, there is not so much work on the existence of nonstationary solutions for systems like (1.1). In the case $b(t,x)\equiv 0, V(x)\equiv 0$, Brézis and Nirenberg [6] considered the system

$$\begin{cases}\partial_t u-\Delta_x u=-v^5+f,\\ -\partial_t v-\Delta_x v=u^3+g,\end{cases}\quad \text{in } (0,T)\times\Omega, \tag{1.3}$$

where Ω is a bounded domain, $f,g\in L^\infty(\Omega)$, subject to the boundary conditions $u=v=0$ on $(0,T)\times\partial\Omega$ and $u(0,x)=v(T,x)=0$ on Ω. Using Schauder's fixed point theorem, they obtained a (generalized) solution (u,v) with $u\in L^4$ and $v\in L^6$. Clément, Felmer and Mitidieri [7] considered

$$\begin{cases}\partial_t u-\Delta_x u=|v|^{q-2}v,\\ -\partial_t v-\Delta_x v=|u|^{p-2}u,\end{cases}\quad \text{in } (-T,\ T)\times\Omega, \tag{1.4}$$

where p,q satisfy

$$\frac{N}{N+2}<\frac{1}{p}+\frac{1}{q}<1.$$

By using the mountain pass theorem, they proved that there exists $T_0>0$ such that for each $T>T_0$, problem (1.4) has at least one positive solution.

In the case $b(t,x)\equiv 0, V(x)\neq 0$, Bartsch and Ding [4] investigated the following infinite-dimensional Hamiltonian system:

$$\begin{cases}\partial_t u-\Delta_x u+V(x)u=H_v(t,x,u,v),\\ -\partial_t v-\Delta_x v+V(x)v=H_u(t,x,u,v).\end{cases} \tag{1.5}$$

They established the existence and multiplicity of solutions of homoclinic type under the assumptions that $V(x)$ and $H(t,x,u,v)$ are periodic in t, x, and $H(t,x,u,v)$ is superlinear at infinity.

For the case $b(t,x)\neq 0, V(x)\neq 0$, the diffusion equations with periodic potential and nonlinearities was recently considered by Ding, Luan and Willem in [14]. They assumed that $H(t,x,0)\equiv 0$ and $H(t,x,z)$ is asymptotically quadratic or super-quadratic as $|z|\to\infty$. By establishing a proper variational setting based on some recent critical point theorems, they obtained that at least one nontrivial solution and infinitely many solutions provided moreover H are symmetric in z.

For other results concerning Hamiltonian system, we refer readers to [2, 8, 10–13, 19, 21, 27].

The aim of this paper is to investigate the existence of homoclinic type solutions for the following system of diffusion equations:

$$\text{(H.S.)}\quad \begin{cases}\partial_t u-\Delta_x u+b\cdot\nabla_x u+au+V(t,x)v=H_v(t,x,u,v),\\ -\partial_t v-\Delta_x v-b\cdot\nabla_x v+av+V(t,x)u=H_u(t,x,u,v),\end{cases} \tag{1.6}$$

where $z = (u, v) : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^m \times \mathbb{R}^m$, $a > 0$, $b = (b_1, \cdots, b_N)$ is a constant vector, $V \in C(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$, and $H \in C^1(\mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^{2m}, \mathbb{R})$ is asymptotically periodic.

To solve the problem by variational methods, we need to overcome some difficulties. First, there is no compactness for the Sobolev imbedding. Second, the energy functional is strongly indefinite, the classical critical point can not be applied directly. Third, the potential $V(t, x)$ and the nonlinearity $H(t, x, z)$ are both nonperiodic in variables t , x; moreover, the nonlinearity $H(t, x, z)$ is super-quadratic as $|z| \to \infty$. We can not use the the periodicity property to obtain the existence of nontrivial solutions. Inspired by recent works of Ding and Wei [15] and Li and Yang [24], we are going to investigate the existence of nontrivial homoclinic type solutions for (H.S.). In [15], the authors considered a class of nonlinear Dirac equations with general potential and special nonlinearities (satisfying Ambrosetti-Rabinowitz condition and $(\mathrm{H}_1), (\mathrm{H}_4)$, see below) by a reduction discussion, where they also gave the exponential decaying proposition for the solutions.

To simplify the notation, we denote

$$\mathcal{J} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}, \quad \mathcal{J}_0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \mathcal{S} = -\Delta_x + a, \quad A = \mathcal{J}_0 \mathcal{S} + \mathcal{J} b \cdot \nabla_x. \tag{1.7}$$

Then system (H.S.) can be rewritten in the form of $\mathcal{J} \frac{\mathrm{d}}{\mathrm{d}t} z + (A + V(t, x))z = H_z(t, x, z)$. It was called an unbounded Hamiltonian system (cf. [3]), or an infinite dimensional Hamiltonian system (cf. [4]). Indeed, it has the representation

$$\mathcal{J} \frac{\mathrm{d}}{\mathrm{d}t} z = \mathrm{grad}_z \mathcal{H}(t, z)$$

with the Hamiltonian

$$\mathcal{H}(t, z) := -\int_{\mathbb{R}^N} \Big(\nabla_x u \nabla_x v + b \cdot \nabla_x u v + a u v + \frac{1}{2} V(t, x) |z|^2 - H(t, x, z) \Big) \mathrm{d}x$$

in $L^2(\mathbb{R}, \mathbb{R}^{2m})$, where grad_z denotes the gradient operator in $L^2(\mathbb{R}, \mathbb{R}^{2m})$ and

$$\nabla_x u \nabla_x v = \sum_{j=1}^{m} \sum_{i=1}^{N} \partial_{x_i} u_j \partial_{x_i} v_j \,, \quad b \cdot \nabla_x u v = \sum_{j=1}^{m} \sum_{i=1}^{N} b_i \partial_{x_i} u_j v_j$$

for $u = (u_1, \cdots, u_m)$ and $v = (v_1 \cdots, v_m)$.

In order to state our main results, we introduce for $r \geq 1$ the Banach space

$$B_r = B_r(\mathbb{R} \times \mathbb{R}^N, \mathbb{R}^{2m}) := W^{1,r}(\mathbb{R}, L^r(\mathbb{R}^N, \mathbb{R}^{2m})) \cap L^r(\mathbb{R}, W^{2,r} \cap W^{1,r}(\mathbb{R}^N, \mathbb{R}^{2m}))$$

equipped with norm

$$\|z\|_{B_r} = \Big(\int_{\mathbb{R} \times \mathbb{R}^N} \Big(|z|^r + |\partial_t z|^r + \sum_{j=1}^{N} |\partial^2_{x_j} z|^r \Big) \Big)^{\frac{1}{r}}.$$

Clearly, B_r is the completion of $\mathcal{C}_0^\infty(\mathbb{R} \times \mathbb{R}^N, \mathbb{R}^{2m})$ with respect to the norm $\| \cdot \|_{B_r}$. If $r = 2$, B_2 is a Hilbert space.

Let $N^* := \infty$ if $N = 1$, and $N^* := \frac{2(N+2)}{N}$ if $N \geq 2$, and define

$$\begin{aligned} V(\infty) &:= \lim_{|t|+|x|\to\infty} V(t,x), \quad W(t,x) := V(\infty) - V(t,x). \\ V_{\sup} &:= \sup_{(t,x)\in\mathbb{R}\times\mathbb{R}^N} |V(t,x)|, \quad W_{\sup} := \sup_{(t,x)\in\mathbb{R}\times\mathbb{R}^N} W(t,x). \end{aligned} \tag{1.8}$$

We make the following assumptions on the potential and nonlinearities:

(V_1) $V_{\sup} < a$, $W_{\sup} < a - V(\infty)$, $W(t,x) > 0$ for all t, x,

(H_1) $H_z(t,x,z) = h(t,x,|z|)z$, $h(t,x,s) \geq 0$, $h(t,x,s) = o(s)$ as $s \to 0$,

(H_2) $h(t,x,|z|) \to \infty$ uniformly in t, x, as $|z| \to \infty$,

(H_3) there is $r > 0$ and $1 < q < \frac{2}{\sigma-1}$ with $\sigma > 1$ if $N = 1$, $\sigma > 1 + \frac{N}{2}$ if $N \geq 2$, such that

$$\begin{aligned} H(t,x,z) &\geq C_0|z|^{q+1}, \\ |h(t,x,|z|)|^\sigma &\leq C_1\widetilde{H}(t,x,z), \quad \text{if } |z| \geq r, \end{aligned}$$

where $\widetilde{H}(t,x,z) = \dfrac{1}{2}h(t,x,|z|)|z|^2 - H(t,x,z) > 0$, if $z \neq 0$,

(H_4) there is $h_\infty \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ with $h'_\infty(s) > 0$ for $s > 0$ such that $h(t,x,s) \to h_\infty(s)$ as $|t| + |x| \to \infty$ uniformly on bounded sets of s, and $h_\infty(s) \leq h(t,x,s)$ for all (t,x,s). Moreover $\widetilde{H}_\infty(z) = \dfrac{1}{2}h_\infty(|z|)|z|^2 - H_\infty(z) > 0$, if $z \neq 0$.

For a solution z of (H.S.), we denote the associated action functional by

$$\Phi(z) := \int_{\mathbb{R}\times\mathbb{R}^N} \Big(\frac{1}{2}\mathcal{J}\dot{z}\cdot z - \mathcal{H}(t,x,z)\Big)\,\mathrm{d}t.$$

Set

$$c_{\min} := \inf\big\{\Phi(z) : z \neq 0 \,\text{is a solution to (H.S.)}\big\},$$

a solution $z_0 \neq 0$ with $\Phi(z_0) = c_{\min}$ called a least energy solution. Let $S_{\min}$ denote the set of all least energy solutions to (H.S.).

The main result of this paper is the following theorem.

Theorem 1.1 *Let* (V_1) *and* (H_1) $-$ (H_4) *be satisfied. Then*

(i) *system* (H.S.) *has at least one least energy solution;*

(ii) $S_{\min}$ *is compact in* B_2.

Theorem 1.1 can be applied to the following special case:

$$\begin{cases} \partial_t u - \Delta_x u + b\cdot\nabla_x u + au + V(t,x)v = h(t,x)|z|^{p-2}v, \\ -\partial_t v - \Delta_x v - b\cdot\nabla_x v + av + V(t,x)u = h(t,x)|z|^{p-2}u, \end{cases} \tag{1.9}$$

with $2 < p < N^*$ and $h(t,x)$ satisfying (h_0): $h \in C(\mathbb{R}\times\mathbb{R}^N, \mathbb{R})$, $h(t,x) \geq h_0 > 0$ for all t, x. Here $h_0 := \lim\limits_{|t|+|x|\to\infty} h(t,x)$.

Corollary 1.1 *Let* (V_1) *and* (h_0) *be satisfied. Then*

(i) (1.9) *has at least one least energy solution;*

(ii) $S_{\min}$ *is compact in* B_2.

This paper is organized as follows. In Section 2, we formulate the variational setting and recall some critical point theorems required. In Section 3, we discuss the least action solutions of the associated limit equation. And finally, in Section 4, we complete the proof of the main results.

2 The Variational Setting and Critical Point Theorem

Let $A_0 := \mathcal{J}\partial_t + A$, $L_0 := \mathcal{J}\partial_t + (A + V(\infty))$ and $L = \mathcal{J}\partial_t + (A + V(t,x))$, where $\mathcal{J}$ and A are given by (1.7). Since b is a constant, A_0 is a selfadjoint operator acting in $L^2(\mathbb{R}\times\mathbb{R}^N, \mathbb{R}^{2m})$ with domain $\mathcal{D}(A_0) = B_2(\mathbb{R}\times\mathbb{R}^N, \mathbb{R}^{2m})$ (cf. [14]). Let $\sigma(\cdot)$ and $\sigma_c(\cdot)$ denote the spectrum and continuous spectrum respectively. Recall that the operator $\mathcal{S}$ is self-adjoint on $L^2(\mathbb{R}^N, \mathbb{R})$ and $\sigma(\mathcal{S}) \subset [a, \infty)$. By [14, Lemma 2.1], we know $\sigma(A_0) = \sigma_c(A_0) \subset \mathbb{R}\setminus(-a,\ a)$.

Lemma 2.1 *Under the assumptions on V, we have the following*

(1) *L_0 are self-adjoint operators and $\sigma(L_0) \subset \mathbb{R}\setminus(V(\infty)-a,\ a-V(\infty))$;*

(2) *L are self-adjoint operators and $\sigma(L) \subset \mathbb{R}\setminus(V_{\sup}-a,\ a-V_{\sup})$.*

Proof We check (2) only, the proof of L_0 is similar. Since $V_{\sup} < a$, it follows from the Kato-Rellich theorem that L is selfadjoint. Furthermore,

$$|Lz|_2 = |(A_0 + V(t,x))z|_2 \ge |A_0 z|_2 - |V(t,x)z|_2 \ge a|z|_2 - V_{\sup}|z|_2 = (a - V_{\sup})|z|_2,$$

thus, $\sigma(L) \subset \mathbb{R}\setminus(V_{\sup}-a,\ a-V_{\sup})$.

It follows from Lemma 2.1 that the space $L^2(\mathbb{R}\times\mathbb{R}^N)$ possesses the orthogonal decomposition:

$$L^2 = L^- \oplus L^+, \quad z = z^- + z^+,$$

so that L_0 is negative definite (resp. positive definite) in L^- (resp. L^+). Let $|L_0|$ denote the absolute, $|L_0|^{\frac{1}{2}}$ the squared root, and take $E = \mathcal{D}(|L_0|^{\frac{1}{2}})$. E is a Hilbert space equipped with the inner product

$$(z, w) = (|L_0|^{\frac{1}{2}} z, |L_0|^{\frac{1}{2}} w)_2$$

and the induced norm $\|z\| = (z,z)^{\frac{1}{2}}$. E possesses the following decomposition

$$E = E^- \oplus E^+ \quad \text{with } E^{\pm} = E \cap L^{\pm},$$

orthogonal with respect to both $(\cdot,\cdot)_2$ and $(\cdot,\cdot)$ inner products. It is clear that $\|z\|^2 \ge (a - V(\infty))|z|_2^2$ for all $z \in E$. Since L_0 is periodic, the following results can be obtained similarly to [14].

Lemma 2.2 *There exist c_1, c_2 such that*

$$c_1\|z\|_{B_2}^2 \le |L_0 z|_2^2 \le c_2\|z\|_{B_2}^2$$

for all $z \in B_2$.

Lemma 2.3 *E is continuously embedded in L^r for any $r \geq 2$ if $N = 1$, and for $r \in [2, N^*]$ if $N \geq 2$. E is compactly embedded in L^r_{loc} for all $r \in [1, N^*)$.*

Assuming that (V_1) and $(\mathrm{H}_1) - (\mathrm{H}_4)$ are satisfied, we define on E the following functional:

$$\Phi(z) = \frac{1}{2}\left(\|z^+\|^2 - \|z^-\|^2\right) - \frac{1}{2}\int_{\mathbb{R}\times\mathbb{R}^N} W(t,x)|z|^2 - \Psi(z)$$

for all $z = z^- + z^+ \in E$, where

$$\Psi(z) := \int_{\mathbb{R}\times\mathbb{R}^N} H(t,x,z).$$

Then $\Phi \in C^1(E,\mathbb{R})$ and a standard argument show that critical points of Φ are solutions to (H.S.).

Using the spectrum decomposition of L, one may give Φ another representation as follows. By Lemma 2.1, we have $E = \mathcal{D}(|L|^{\frac{1}{2}})$ with the equivalent inner product

$$(z,v)_L := (|L|^{\frac{1}{2}}z,\, |L|^{\frac{1}{2}}v)_2$$

and norm $\|z\|_L := (u,u)_L^{\frac{1}{2}}$. Then as above, there is a decomposition

$$E = E_L^- \oplus E_L^+$$

with

$$\|z^{\pm}\|_L^2 \geq (a - V_{\sup})|z^{\pm}|_2^2 \quad \text{for all } z^{\pm} \in E_L^{\pm}.$$

Now Φ can be represented as

$$\Phi(u) = \frac{1}{2}\left(\|z^+\|_L^2 - \|z^-\|_L^2\right) - \Psi(z)$$

for all $z = z^- + z^+ \in E_L^- \oplus E_L^+$.

In order to find critical points of Φ, we use the following abstract theorem which is taken from [4, 10].

Let E be a Banach space with direct sum decomposition $E = X \oplus Y$, $z = x + y$ and corresponding projections P_X, P_Y onto X, Y, respectively. For a functional $\Phi \in C^1(E,\mathbb{R})$, we write $\Phi_a = \{z \in E : \Phi(z) \geq a\}$. Recall that a sequence $(z_n) \subset E$ is said to be a $(\mathrm{C})_c$-sequence if $\Phi(z_n) \to c$ and $(1+\|z_n\|)\Phi'(z_n) \to 0$. Φ is said to satisfy the $(\mathrm{C})_c$-condition if any $(\mathrm{C})_c$-sequence has a convergent subsequence.

Now we assume that X is separable and reflexive, and we fix a countable dense subset $\mathcal{S} \subset X^*$. For each $s \in \mathcal{S}$, there is a semi-norm on E defined by

$$p_s : E \to \mathbb{R}, \quad p_s(z) = |s(x)| + \|y\| \quad \text{for } z = x + y \in X \oplus Y.$$

We denote by $\mathcal{T}_{\mathcal{S}}$ the induced topology. Let w^* denote the weak*-topology on E^*. Suppose that

(Φ_0) There exists $\zeta > 0$ such that $\|z\| < \zeta\|P_Y z\|$ for all $z \in \Phi_0$;

(Φ_1) For any $c \in \mathbb{R}$, Φ_c is $\mathcal{T}_{\mathcal{S}}$-closed, and $\Phi' : (\Phi_c, \mathcal{T}_{\mathcal{S}}) \to (E^*, w^*)$ is continuous;

(Φ_2) There exists $\rho > 0$ with $\kappa := \inf \Phi(S_\rho Y) > 0$ where $S_\rho Y := \{z \in Y : \|z\| = \rho\}$.

The following theorem is taken from [4] (also cf. [10]).

Theorem 2.1 *Let* (Φ_0)–(Φ_2) *be satisfied and suppose that there are* $R > \rho > 0$ *and* $e \in Y$ *with* $\|e\| = 1$ *such that* $\sup \Phi(\partial Q) \le \kappa$ *where* $Q = \{z = x + te : x \in X,\ t \ge 0,\ \|z\| < R\}$. *Then* Φ *has a* $(\mathrm{C})_c$*-sequence with* $\kappa \le c \le \sup \Phi(Q)$.

The following lemma is useful to verify (Φ_1) (cf. [4, 10]).

Lemma 2.4 *Suppose that* $\Phi \in C^1(E, \mathbb{R})$ *is of the form*

$$\Phi(z) = \frac{1}{2}\big(\|y\|^2 - \|x\|^2\big) - \Psi(z) \quad \textit{for } z = x + y \in E = X \oplus Y,$$

such that

(i) $\Psi \in C^1(E, \mathbb{R})$ *is bounded from below;*

(ii) $\Psi : (E, \mathcal{T}_w) \to \mathbb{R}$ *is sequentially lower semicontinuous, that is,* $z_n \rightharpoonup z$ *in* E *implies* $\Psi(z) \le \liminf \Psi(u_n)$;

(iii) $\Psi' : (E, \mathcal{T}_w) \to (E^*, \mathcal{T}_{w^*})$ *is sequentially continuous*;

(iv) $\nu : E \to \mathbb{R}$, $\nu(z) = \|z\|^2$, *is* C^1 *and* $\nu' : (E, \mathcal{T}_w) \to (E^*, \mathcal{T}_{w^*})$ *is sequentially continuous. Then* Φ *satisfies* (Φ_1).

3 The Autonomous Problem

In this section, we study the following limit equation related to (H.S.):

$$\begin{cases} \mathcal{J}\dfrac{\mathrm{d}}{\mathrm{d}t}z + (A + V(\infty))z = h_\infty(|z|)z, \\ z(t,x) \to 0, \quad \text{as } |t| + |x| \to \infty, \end{cases} \tag{3.1}$$

where h_∞ is the function from assumption (H_4).

From the assumptions $(\mathrm{H}_3), (\mathrm{H}_4)$, we know that there are C_0', C_1' such that

$$\begin{aligned} &H_\infty(z) \ge C_0'|z|^{q+1}, \\ &|h_\infty(|z|)|^\sigma \le C_1'\widetilde{H}_\infty(z), \quad \text{if } |z| \ge r, \end{aligned} \tag{3.2}$$

which imply

$$|h_\infty(|z|)z| \le C_2|z|^{\frac{\sigma+1}{\sigma-1}}, \quad \text{if } |z| \ge r. \tag{3.3}$$

Choosing $\frac{2\sigma}{\sigma-1} \le p < N^*$ and by (H_1), we know that for any $\varepsilon > 0$, there exists a $C_\varepsilon > 0$ such that

$$H_\infty(z) \le \varepsilon|z|^2 + C_\varepsilon|z|^p \quad \text{for all } z \in \mathbb{R}^{2m}. \tag{3.4}$$

Set

$$\Psi_\infty(z) := \int_{\mathbb{R}\times\mathbb{R}^N} H_\infty(z)$$

and define the functional

$$F(z) := \frac{1}{2}\|z^+\|^2 - \frac{1}{2}\|z^-\|^2 - \Psi_\infty(z)$$

for $z = z^- + z^+ \in E^- \oplus E^+$. It follows from the assumption on h_∞ that $F \in C^1(E, \mathbb{R})$ and its critical points are solutions to (3.1).

Lemma 3.1 *F possesses the following properties:*

(1) *Ψ_∞ is weakly sequentially lower semicontinuous and F' is weakly sequentially continuous.*

(2) *For any finite dimensional subspace $Z \subset E^+$,*

$$F(z) \to -\infty, \quad \text{as } z \in E^- \oplus Z,\ \|z\| \to \infty\,.$$

(3) *There are $\rho > 0$ and $\kappa > 0$ such that*

$$F|_{B_\rho \cap E^+} \geq 0 \quad \text{and} \quad F|_{\partial B_\rho \cap E^+} \geq \kappa.$$

(4) *Let (z_j) be a $(\mathrm{C})_c$ sequence for F. Then it is bounded and $c \geq 0$.*

Proof (1) The first conclusion follows easily because of Lemma 2.3.

(2) for all $z \in E^- \oplus Z$,

$$F(z) = \frac{1}{2}\|z^+\|^2 - \frac{1}{2}\|z^-\|^2 - \Psi_\infty(z) \leq \left(\frac{1}{2}\|z^+\|^2 - C_0' \int_{\mathbb{R}\times\mathbb{R}^N} |z^+|^{q+1}\right) - \frac{1}{2}\|z^-\|^2.$$

Since all norms in Z are equivalent and $q > 1$, one obtains easily the desired conclusion.

(3) From (3.4), for ε there exists a C_ε such that $H_\infty(z) \leq \varepsilon|z|^2 + C_\varepsilon|z|^p$. Thus for $z \in E_+$,

$$F(z) \quad = \tfrac{1}{2}\|z\|^2 - \Psi_\infty(z) \geq \tfrac{1}{2}\|z\|^2 - \varepsilon\|z\|^2 - \lambda C_\varepsilon\|z\|^p.$$

Consequently, the conclusion follows since $p > 2$.

(4) The arguments in [14] show that (z_j) is bounded.

Let $\widehat{\mathcal{K}} := \{z \in E :\ F'(z) = 0\}$ be the critical set of F.

Lemma 3.2 *$\widehat{\mathcal{K}} \setminus \{0\} \neq \emptyset$, $\widehat{\mathcal{K}} \subset \bigcap_{r\geq 2} B_r$,*

$$\widehat{C} := \inf\{F(z) :\ z \in \widehat{\mathcal{K}} \setminus \{0\}\}\ >\ 0$$

and is attained.

Proof Setting $X = E^-$ and $Y = E^+$, one has $E = X \oplus Y$. From Lemma 3.1, it is easy to see that all the assumptions of Lemma 2.4 are satisfied. Thus there is a $(\mathrm{C})_c$ sequence (z_j) for F with $\kappa \leq c \leq \sup F(Q)$. By Lemma 3.1(4), the $(\mathrm{C})_c$ sequence (z_j) is also bounded in E. From a standard concentration compactness argument in [23], there exist $\gamma, \eta > 0$ and $(a_j) \subset \mathbb{R}^{1+N}$ such that $\limsup_{j\to\infty} \int_{B(a_j,\gamma)} |z_j|^2 \geq \eta$. Set $v_j := a_j * z_j$ by

$$(a_j * z)(t,x) := z(t + a_j^0, x_1 + a_j^1, \cdots, x_N + a_j^N) \quad \text{for all } (t,x) \in \mathbb{R}^{1+N}.$$

We know $\|v_j\| = \|u_j\| \leq C$ and $F(v_j) \to c \geq \kappa,\ F'(v_j) \to 0$. Therefore $v_j \rightharpoonup v$ in E with $v \neq 0$ and $F'(v) = 0$, that is, v is a nontrivial solution of (3.1), therefore

$$\widehat{\mathcal{K}} \setminus \{0\} \neq \emptyset.$$

By bootstrap argument (cf. e.g., [4, 16]) we know $\widehat{\mathcal{K}} \subset \bigcap_{r\geq 2} B_r$.

To show that there is a $z \in \widehat{\mathcal{K}}$ with $F(z) = \widehat{C}$. Let $z_j \in \widehat{\mathcal{K}}\backslash\{0\}$ be such that $F(z_j) \to \widehat{C}$. Then (z_j) is bounded. By applying the concentration principle, one may assume $z_j \rightharpoonup z \in \widehat{\mathcal{K}} \setminus \{0\}$. Now

$$\widehat{C} = \lim_{j\to\infty} F(u_j) = \lim_{j\to\infty} \int_{\mathbb{R}\times\mathbb{R}^N} \widetilde{H}_\infty(z_j) \geq \int_{\mathbb{R}\times\mathbb{R}^N} \widetilde{H}_\infty(z) = F(z) \geq \widehat{C},$$

that is, $F(z) = \widehat{C}$. We prove $\widehat{C} > 0$ by contradiction. Assume $\widehat{C} = 0$. If $z \in \widehat{\mathcal{K}}$, one has

$$F(z) = F(z) - \frac{1}{2}F'(z)z = \int_{\mathbb{R}\times\mathbb{R}^N} \frac{1}{2}h_\infty(|z|)|z|^2 - H_\infty(z) \geq 0,$$

which means $C \geq 0$. If $\widehat{C} = 0$, let $z_j \in \widehat{\mathcal{K}} \setminus \{0\}$ be such that $F(z_j) \to 0$. Then (z_j) is a $(\mathrm{C})_0$-sequence, hence is bounded by Lemma 3.1. We can suppose $z_j \rightharpoonup z \in \widehat{\mathcal{K}}$. Then

$$F(z_j) = \int_{\mathbb{R}\times\mathbb{R}^N} \frac{1}{2}h_\infty(|z_j|)|z_j|^2 - H_\infty(z_j) = \int_{\mathbb{R}\times\mathbb{R}^N} \widetilde{H}_\infty(z_j) \to 0.$$

By Hölder inequality $\frac{1}{\sigma} + \frac{1}{\sigma'} = 1$, we have

$$\begin{aligned}
\|z_j\|^2 &= \int_{\mathbb{R}\times\mathbb{R}^N} h_\infty(|z_j|)z_j(z_j^+ - z_j^-) \\
&\leq \varepsilon|u_j|_2^2 + c_\varepsilon \int_{\mathbb{R}\times\mathbb{R}^N} \widetilde{H}_\infty(z_j)^{\frac{1}{\nu}}|z_j||z_j^+ - z_j^-| \\
&\leq \varepsilon|z_j|_2^2 + c_1c_\varepsilon\Big(\int_{\mathbb{R}\times\mathbb{R}^N} \widetilde{H}_\infty(z_j)\Big)^{\frac{1}{\sigma}}|z_j|_{\sigma'}^2, \\
&\leq c_2\varepsilon\|z_j\|^2 + c_3c_\varepsilon F(z_j)^{\frac{1}{\sigma}}\|z_j\|^2.
\end{aligned}$$

Hence $1 \leq c_2\varepsilon + o(1)$, a contradiction.

The following two lemmas are important to prove the main results of this paper.

Lemma 3.3 *If $z_0 \neq 0$ is a critical point of F, then $F''(z_0)$ is negative definite on $\widetilde{E} \equiv E^- \oplus \mathbb{R}z_0 = E^- \oplus \mathbb{R}z_0^+$. More generally, if $\widetilde{E} \equiv \widetilde{E}^- \oplus \mathbb{R}z_0 = \widetilde{E}^- \oplus \mathbb{R}z_0^+$, where $\widetilde{E}^- \subset E^-$, and if $z_0 \neq 0$ is a critical point of $F\,|_{\widetilde{E}}$, then $F''(z_0)$ is negative definite on $\widetilde{E}$.*

Proof It suffices to prove the second statement. We denote $J = F\,|_{\widetilde{E}}$ and suppose that $z \neq 0$ is a critical point of J. For $z \in \widetilde{E}$ we write $z = tz_0 + v$ where $v \in \widetilde{E}^-$. Since $z_0 \neq 0$ is a critical point of J, from assumption (H_4), we get

$$\begin{aligned}
J''(z_0)[tz_0 + v, tz_0 + v] &= J''(z_0)[tz_0 + v, tz_0 + v] - t\big(J'(z_0), tz_0 + 2v\big) \\
&= t^2\|z_0^+\|^2 - t^2\|z_0^-\|^2 - 2t(z_0^-, v) - \|v\|^2 \\
&\quad - \int_{\mathbb{R}\times\mathbb{R}^N} \frac{h'_\infty(|z_0|)}{|z_0|}(z_0(tz_0 + v))^2 - \int_{\mathbb{R}\times\mathbb{R}^N} h_\infty(|z_0|)|tz_0 + v|^2 - t^2\|z_0^+\|^2 \\
&\quad + t^2\|z_0^-\|^2 + 2t\big(z_0^-, v\big) + t\int_{\mathbb{R}\times\mathbb{R}^N} h_\infty(|z_0|)z_0(tz_0 + 2v) \\
&= -\int_{\mathbb{R}\times\mathbb{R}^N} \frac{h'_\infty(|z_0|)}{|z_0|}(z_0(tz_0 + v))^2 - \int_{\mathbb{R}\times\mathbb{R}^N} h_\infty(|z_0|)|v|^2 - \|v\|^2 < 0,
\end{aligned}$$

which means that $F''(z_0)$ is negative definite on $\widetilde{E}$.

Now we select $z_0 \in \widehat{\mathcal{K}}$ with $F(z_0) = \widehat{C}$ and set $\widetilde{E} \equiv E^- \oplus \mathbb{R}z_0 = E^- \oplus \mathbb{R}z_0^+$, $Q := \{z = z^- + sz_0^+ : z^- \in E^-, s \geq 0, \|z\| < R\}$ where R be large enough. We have the lemma below.

Lemma 3.4 $\sup\limits_{u \in Q} F(u) = F(z_0) = \widehat{C}$.

Proof The proof is contained in [23]. We sketch the proof here for the convenience of the readers and the completeness of the paper.

Let $z = tz_0^+ + z^- \in Q$. We have

$$F(z) := \tfrac{1}{2}t^2\|z_0^+\|^2 - \tfrac{1}{2}\|z^-\|^2 - \Psi_\infty(z).$$

F takes its maximum on Q at some v_0 and $F(v_0) > 0$. We shall show that z_0 is the only critical point of $F|_{Q\setminus\{0\}}$. It will then follow that $v_0 = z_0$ and $\sup\limits_{u \in Q} F(u) = F(z_0)$.

Suppose that F has another critical point on $Q\setminus\{0\}$ and let $\widetilde{E} \equiv \widetilde{E}^- \oplus \mathbb{R}z_0 = \widetilde{E}^- \oplus \mathbb{R}z_0^+$ be a finite-dimensional space such that $\widetilde{E}^- \subset E^-$ contains z_0 and the second critical point. Let $\widetilde{Q} := (Q\setminus B(0,\varepsilon)) \cap \widetilde{E}$ and $J := F|_{\widetilde{E}}$. It follows from the assumptions on H_∞ that J has no critical points on $\partial\widetilde{Q}$ provide ε is small enough.

Define $\Pi : \widetilde{Q} \times [0,1] \to \widetilde{Q}$ by

$$\begin{aligned} \Pi(z,s) : &= (1-s)(-tz_0^+ + z^- + \Psi'_\infty(z)) + s((t-1)z_0^+ + z^-) \\ &= -(1-s)J'(z) + s(z - z_0^+). \end{aligned} \tag{3.5}$$

It is easy to prove that Π is an admissible homotopy. Since $\Pi(z,0) = -J'(z)$, $\Pi(z,1) = z - z_0^+$ and $z_0^+ \in \widetilde{Q}$, we get

$$\deg(-J', \widetilde{Q}, 0) = \deg(I, \widetilde{Q}, z_0^+) = 1,$$

where I is the identity mapping.

However, by Lemma 3.3, if $z_0 \neq 0$ is a critical point of $J = F\,|_{\widetilde{E}}$, then $-J''(z_0) = -F''(z_0)$ is positive definite on $\widetilde{E}$. Hence v_0 is an isolated zero of $-J'$ and the local degree at v_0 $\deg(-J', v_0, 0) = 1$, and each $z \in \widetilde{Q}$ with $-J'(z) = 0$ is isolated with local degree 1.

It then follows from the additivity property of the degree that no second critical point can exist in $\widetilde{Q}$. Therefore $v_0 = z_0$.

4 Proof of the Main Result

In this section, we complete the proof of the main results.

Lemma 4.1 *The functional Φ possesses the following properties:*

(1) *Ψ is weakly sequentially lower semicontinuous and Φ' is weakly sequentially continuous.*

(2) *There exist $r > 0$ and $\rho > 0$ such that $\Phi|_{B_r^+}(z) \geq 0$ and $\Phi|_{S_r^+} \geq \rho$, where $B_r^+ = \{z \in E^+ : \|z\| \leq r\}$ and $S_r^+ = \{z \in E^+ : \|z\| = r\}$.*

(3) *There is $R > 0$ such that, for any $e \in E^+$ with $\|e\| = 1$ and $E_e = E^- \oplus \mathbb{R}\,e$,*

$$\Phi(u) < 0 \quad \text{for all } u \in E_e \setminus B_R.$$

(4) *Any $(\mathrm{C})_c$-sequence for Φ is bounded.*

Proof (1) The proof is standard.

(2) Observe that, for $z \in E^+$,

$$\begin{aligned}\Phi(z) &= \frac{1}{2}\|z\|^2 - \frac{1}{2}\int_{\mathbb{R}\times\mathbb{R}^N} W(t,x)|z|^2 - \int_{\mathbb{R}\times\mathbb{R}^N} H(t,x,z)\\ &\geq \frac{1}{2}\|z\|^2 - \frac{1}{2}W_{\sup}|z|_2^2 - \int_{\mathbb{R}\times\mathbb{R}^N} H(t,x,z)\\ &\geq \frac{1}{2}\Big(1 - \frac{W_{\sup}}{a - V(\infty)}\Big)\|z\|^2 - \int_{\mathbb{R}\times\mathbb{R}^N} H(t,x,z).\end{aligned}$$

Since for any $\varepsilon > 0$, there exists a $C_\varepsilon > 0$ such that

$$H(t,x,z) \leq \varepsilon|z|^2 + C_\varepsilon|z|^p$$

for all $z \in \mathbb{R}^{2m}$, the result follows by the embedding theorem.

(3) This follows from the following facts:

$$\begin{aligned}\Phi(z) &= \frac{1}{2}\big(\|z^+\|^2 - \|z^-\|^2\big) - \frac{1}{2}\int_{\mathbb{R}\times\mathbb{R}^N} W(t,x)|z|^2 - \int_{\mathbb{R}\times\mathbb{R}^N} H(t,x,z)\\ &\leq \frac{1}{2}\big(\|z^+\|^2 - \|z^-\|^2\big) - \int_{\mathbb{R}\times\mathbb{R}^N} H_\infty(z) = F(z),\end{aligned}$$

and F satisfies the conclusion by Lemma 3.1.

(4) By conditions $(\mathrm{H}_3), (\mathrm{H}_4)$, a careful observation of the argument in [14] shows that (z_j) is bounded.

In particular, let $z_0 \in \widehat{\mathcal{K}}$ with $F(z_0) = \widehat{C}$. Setting $e \equiv z_0^+$ and $\widetilde{E} \equiv E^- \oplus \mathbb{R}e$ and $Q := \{z = z^- + sz_0^+ :\ z^- \in E^-, s \geq 0, \|z\| < R\}$, we have the following lemma.

Lemma 4.2 $d := \sup\{\Phi(z) :\ z \in \widetilde{E}\} = \sup \Phi(Q) < \widehat{C}$.

Proof By Lemma 4.1 and the linking property, we have $d \geq \rho$. From Lemma 3.4 and $\Phi(z) \leq F(z)$ for all $z = v + sz_0^+$, we know $\Phi(z) \leq \widehat{C}$.

If $d = \widehat{C}$, let $w_j = v + s_j z_0^+ \in \widetilde{E}$ be such that $d - \frac{1}{j} \leq \Phi(w_j) \to d$. It follows from Lemma 4.1 that w_j is bounded and we can assume $w_j \rightharpoonup w$ in E with $v_j \rightharpoonup v \in E^-$ and $s_j \to s$. It is clear that $s > 0$ (otherwise there should appear the contradiction that $d = 0$). Then

$$d - \frac{1}{j} \quad \leq \Phi(w_j) \leq F(w_j) - \frac{1}{2}\int_{\mathbb{R}\times\mathbb{R}^N} W(t,x)w_j w_j \leq \widehat{C} - \frac{1}{2}\int_{\mathbb{R}\times\mathbb{R}^N} W(t,x)w_j w_j.$$

Taking the limit yields $\widehat{C} \leq \widehat{C} - \frac{1}{2}\int_{\mathbb{R}\times\mathbb{R}^N} W(t,x)ww$ which implies that $w = 0$, a contradiction.

By Lemma 4.1 any $(\mathrm{C})_c$-sequence is bounded, hence it is a $(\mathrm{PS})_c$-sequence. The following lemma is an representation of $(\mathrm{PS})_c$-sequence and its proof is well-known, see for example [1], hence the details are omitted here.

Lemma 4.3 *Let* (z_j) *be a* $(\mathrm{C})_c$*-sequence for* Φ*. Then either*

(i) $z_j \to 0$ (*and hence* $c = 0$), *or*

(ii) *$c \geq \rho$ and there exist a critical point z_0 of Φ, a positive integer $\ell \leq \left[\frac{c}{\rho}\right]$, points $\overline{z}_1, \cdots, \overline{z}_\ell \in \widehat{\mathcal{K}} \setminus \{0\}$, a subsequence denoted again by (z_j), and sequences $(a_j^i) \subset \mathbb{R}$, such that*

$$\left\|z_j - z_0 - \sum_{i=1}^{\ell}(a_j^i * \overline{z}_i)\right\| \to 0, \quad \textit{as } j \to \infty,$$
$$|a_j^i - a_j^k| \to \infty \quad \textit{for } i \neq k, \quad \textit{as } j \to \infty$$

and

$$\Phi(z_0) + \sum_{i=1}^{\ell} F(\overline{z}_i) = c.$$

As a straight consequence of Lemma 4.3, we have the following result.

Lemma 4.4 Φ *satisfies the* $(\mathrm{C})_c$*-condition for all* $c < \widetilde{C}$.

Let $\mathcal{K} := \{z \in E : \ \Phi'(z) = 0\}$ be the critical set of Φ. Recall that

$$c_{\min} := \inf\{\Phi(z) : \ u \in \mathcal{K} \setminus \{0\}\},$$
$$S_{\min} := \{z \in \mathcal{K} : \ \Phi(z) = c_{\min}\}.$$

We now in a position to complete the proof of Theorem 1.1.

Proof of Theorem 1.1 It is easy to see that all the linking conditions of Theorem 2.1 are satisfied. Combining with Lemma 4.2 yields a $(\mathrm{C})_c$-sequence (z_j) with $c < \widehat{C}$ for Φ. Lemma 4.4 shows $z_j \to z$ and so $\Phi'(z) = 0$ and $\Phi(z) \geq \rho$, which imply $\mathcal{K} \setminus \{0\} \neq \emptyset$. By a bootstrap argument, we have for any $q \geq 2$, $z \in B_q$.

It is easy to see $c_{\min} \geq 0$ by (H_3). Let $z_j \in \mathcal{K} \setminus \{0\}$ be such that $\Phi(z_j) \to c_{\min}$. Then (z_j) is a $(\mathrm{C})_{c_{\min}}$ sequence, and since $c_{\min} < \widehat{C}$, Lemma 4.4 implies $z_j \to z \in S_{\min}$. Note that $\Phi(z_j) = \int_{\mathbb{R}^N} \widetilde{H}(t, x, z_j) \to c_{\min}$. As before, we obtain

$$\begin{aligned}
\|z_j\|_L^2 &= \int_{\mathbb{R}\times\mathbb{R}^N} h(t, x, |z_j|)z_j(z_j^+ - z_j^-) \\
&\leq \varepsilon|z_j|_2^2 + c_\varepsilon \int_{\mathbb{R}\times\mathbb{R}^N} \widetilde{H}(t, x, z_j)^{\frac{1}{\sigma}}|z_j||z_j^+ - z_j^-| \\
&\leq \varepsilon|z_j|_2^2 + c_1c_\varepsilon(\widetilde{H}(t, x, z_j))^{\frac{1}{\nu}}|z_j|_\sigma^2 \\
&\leq c_2\varepsilon\|z_j\|_L^2 + c_3c_\varepsilon\Phi(z_j)^{\frac{1}{\sigma}}\|z_j\|_L^2.
\end{aligned}$$

Hence $1 \leq c_2\varepsilon + c_3c_\varepsilon c_{\min}^{\frac{1}{\nu}}$, consequently $c_{\min} > 0$.

We now prove that $S_{\min}$ is compact in B_2. $S_{\min}$ is bounded. Let $z_j \in S_{\min}$. We have $z_j \to z \in S_{\min}$ alone a subsequence and $|z_j|_2 \leq C_2$ for some $C_2 > 0$. The bootstrap argument also tells that for each $q \in [2, \infty)$, there is $\Lambda_q > 0$ such that

$$\|z_j\|_{B_q} \leq \Lambda_q,$$

which implies that for some Λ_∞, $|z_j|_\infty \leq \Lambda_\infty$.

By

$$L_0 z = W(t,x)z + h(t,x,|z|)z,$$

one has

$$\begin{aligned}|L_0(z_j - z)|_2 &\le |(W(t,x)(z_j - z)|_2 + |h(\,\cdot\,,\,\cdot\,,|z_j|)z_j - h(\,\cdot\,,\,\cdot\,,|z|)z|_2 \\ &\le o(1) + |h(\,\cdot\,,\,\cdot\,,|z_j|)(z_j - z)|_2 + |(h(\,\cdot\,,\,\cdot\,,|z_j|) - h(\,\cdot\,,\,\cdot\,,|z|))z|_2.\end{aligned}$$

Since $|z_j|_\infty \le \Lambda_\infty$ and $z_j \to z$ in E, we get

$$\int_{\mathbb{R}\times\mathbb{R}^N} |h(t,x,|z_j|)^2 |z_j - z|^2 \le C|z_j - z|_2^2 \to 0,$$

and since $|z(t,x)| \to 0$ as $|t| + |x| \to \infty$, we get

$$\begin{aligned}\int_{\mathbb{R}\times\mathbb{R}^N} |(h(t,x,|z_j|) - h(t,x,|z|))z|^2 &= \Big(\int_{|t|+|x|<M} + \int_{|t|+|x|\ge M}\Big)|(h(t,x,|z_j|) - h(t,x,|z|))z|^2 \\ &\to 0,\end{aligned}$$

Therefore, one can see that $|L_0(z_j - z)|_2 \to 0$, i.e., $z_j \to z$ in B_2.

Acknowledgements The second author thanks Professor Yanheng Ding for inviting him to visit Institute of Mathematics, AMSS. The authors would also like to thank the referees for their valuable suggestions.

References

[1] Alama, A. and Li, Y. Y., On "multibump" bound states for certain semilinear elliptic equations, *Indiana Univ. Math. J.*, **41**, 1992, 983–1026.

[2] Arioli, G. and Szulkin, A., Homoclinic solutions of Hamiltonian systems with symmetry, *J. Diff. Eqs.*, **158**, 1999, 291–313.

[3] Barbu, V., Periodic solutions to unbounded Hamiltonian system, *Discr. Contin. Dynam. Syst.*, **1**, 1995, 277–283.

[4] Bartsch, T. and Ding, Y. H., Homoclinic solutions of an infinite-dimensional Hamiltonian system, *Math. Z.*, **240**, 2002, 289–310.

[5] Bartsch, T. and Ding, Y. H., Deformation theorems on non-metrizable vector spaces and applications to critical point theory, *Math. Nachr.*, **279**, 2006, 1267–1288

[6] Brézis, H. and Nirenberg, L., Characterization of the ranges of some nonlinear operators and applications to boundary value problems, *Ann. Scuola Norm. Sup. Pisa*, **5**, 1978, 225–326.

[7] Clément, P., Felmer, P. and Mitidieri, E., Homoclinic orbits for a class of infinite dimensional Hamiltonian systems, *Ann. Scuola Norm. Sup. Pisa*, **24**, 1997, 367–393.

[8] Coti-Zelati, V. and Rabinowitz, P., Homoclinic orbits for second order Hamiltonian systems possessing superquadratic potentials, *J. Amer. Math. Soc.*, **4**, 1991, 693–727.

[9] Dautray, R. and Lions, J. L., Mathematical Analysis and Numerical Methods for Science and Technology, Vol. 3, Springer-Verlag, Berlin, 1990.

[10] Ding, Y. H., Variational Methods for Strongly Indefinite Problems, Interdisciplinary Math. Sci., Vol. 7, World Scientific Publishing, Singapore, 2007.

[11] Ding, Y. H., Multiple homoclinics in a Hamiltonian system with asymptotically or super linear terms, *Commun. Contemp. Math.*, **8**, 2006, 453–480.

[12] Ding, Y. H. and Jeanjean, L., Homoclinic orbits for a nonperiodic Hamiltonian system, *J. Diff. Eqs.*, **237**, 2007, 473–490.

[13] Ding, Y. H. and Willem, M., Homoclinic orbits of a Hamiltonian system, *Z. Angew. Math. Phys.*, **50**, 1999, 759–778.

[14] Ding, Y. H., Luan, S. and Willem, M., Solutions of a system of diffusion equations, *J. Fixed Point Theory Appl.*, **2**, 2007, 117–139.

[15] Ding, Y. H. and Wei, J. C., Stationary states of nonlinear Dirac equations with general potentials, *Rev. Math. Phys.*, **20**, 2008, 1007–1032.

[16] Esteban, M. and Séré, E., Stationary ststes of the nonlinear Dirac equation: a variational approach, *Commun. Math. Phys.*, **171**, 1995, 323–250.

[17] de Figueiredo, D. and Felmer, P., On superquadratic elliptic systems, *Trans. Amer. Math. Soc.*, **343**, 1994, 99–116.

[18] de Figueiredo, D. G. and Yang, J. F., Decay, symmetry and existence of solutions of semilinear elliptic systems, *Nonlinear Anal., T.M.A.*, **33**, 1998, 211–234.

[19] Hofer, H. and Wysocki, K., First order elliptic systems and the existence of homoclinic orbits in Hamiltonian systems, *Math. Ann.*, **288**, 1990, 483–503.

[20] Hulshof, J. and van der Vorst, R., Differential systems with strongly indefinite variational structure, *J. Funct. Anal.*, **114**, 1993, 32–58.

[21] Kryszewki, W. and Szulkin, A., Generalized linking theorem with an application to semilinear Schrödinger equation, *Adv. Diff. Equations*, **3**, 1998, 441–472.

[22] Lions, J. L., Optimal Control of Systems Governed by Partial Differential Equations, Springer-Verlag, New York, 1971.

[23] Lions, P. L., The concentration-compactness principle in the calculus of variations: The locally compact cases, Part II, *Ann. Inst. H. Poincaré, Anal. Nonlinéaire*, **1**, 1984, 223–283.

[24] Li, G. B. and Yang, J. F., Asymptotically linear elliptic systems, *Comm. Part. Diff. Eqs.*, **29**, 2004, 925–954.

[25] Nagasawa, M, Schrödinger Equations and Diffusion Theory, Birkhäuser, Basel, 1993.

[26] Reed, M. and Simon, B., Methods of Mathematical Physics, Vol. I–IV, Academic Press, New York, 1978.

[27] Séré, E., Existence of infinitely many homoclinic orbits in Hamiltonian systems, *Math. Z.*, **209**, 1992, 27–42.