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The Partial Positivity of the Curvature in Riemannian Symmetric Spaces

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Abstract In this paper, the partial positivity (resp., negativity) of the curvature of all irreducible Riemannian symmetric spaces is determined. From the classifications of abstract root systems and maximal subsystems, the author gives the calculations for symmetric spaces both in classical types and in exceptional types.

Keywords Partial positivity, Symmetric space, Semi-simple Lie algebra
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1 Introduction

We recall the definition of s-positive curvature from [13]. A Riemannian manifold M has s-positive (resp., s-negative) curvature if and only if for each $x \in M$ and for any $(s+1)$ orthonormal vectors $\{e_0, e_1, \cdots, e_s\}$ in M_x, $\sum\limits_{i=1}^{s} K(e_0, e_i) > 0$ (resp., < 0), where $K(e_0, e_i)$ denotes the sectional curvature of the plane spanned by e_0 and e_i. The 1-positive curvature is equivalent to positive sectional curvature, and $(n-1)$-positive curvature is equivalent to positive Ricci curvature.

The manifolds which have s-positive (or negative) curvature have some topological implications as well as their geometric properties. These manifolds were studied by Wu [13], Shen [12], Kenmotsu and Xia [4, 5].

Among them it was shown in [12] that if a proper open manifold M has s-positive curvature then M has the homotopy type of a CW complex with cells each of dimension $\leq s-1$. In particular, $H_i(M, \mathbb{Z}) = 0$, for $i \geq s$.

Frankel [2] showed that two compact totally geodesic submanifolds in an n-dimensional complete Riemannian manifold N of positive sectional curvature must intersect if the sum of their dimension is greater than or equal to n; and he proved [3] that if V is an r-dimensional compact totally geodesic immersed submanifold of N with $2r > n$, then the homomorphism of fundamental group $\pi_1(V) \to \pi_1(N)$ is surjective.

These results had been generalized in the case of manifolds with partially positive curvature by Kenmotsu and Xia [5]. They showed that in an n-dimensional complete connected Riemannian manifold with k-nonnegative curvature, let V and W be two complete immersed totally geodesic submanifolds of dimensions r and s, let one of V and W be compact and suppose N has k-positive curvature either at all points of V or at all points of W, if $r + s \geq n + k - 1$

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then V and W must intersect. At the same time they also proved that if V is an r-dimensional totally geodesic submanifold with $2r \geq n+k-1$, then the homomorphism of fundamental group $\pi_1(V) \to \pi_1(N)$ is surjective. These show that there exist topological obstructions for the existence of higher dimensional totally geodesic submanifolds in the Riemannian manifolds with partially positive curvature.

Those results show that the notion of s-positivity (or negativity) is a subtler curvature condition. Naturally, determining all the s-values for irreducible Riemannian symmetric spaces of compact type is an interesting problem. This has been started by Lee [10]. She gave the s-values for the symmetric spaces of classical types by using the matrix representation.

What about the exceptional cases? Considering that more and more physicists pay attention to the exceptional geometry, we calculate the partial positivity of the curvature of all irreducible Riemannian symmetric spaces of compact type. If M has s-positive curvature, the dual of M is a simply connected irreducible symmetric space of noncompact type which has s-negative curvature.

Our method is different from hers and we can deal with both classical types as well as the exceptional types. For the classical types we recover Lee's result in [10]. Our main results are in the following table.

Table 1.1

Type	compact type	rank	dimension	s
AI	$\mathrm{SU}(n)/\mathrm{SO}(n)$	$n-1$	$\frac{1}{2}(n-1)(n+2)$	$\frac{n(n-1)}{2}$
AII	$\mathrm{SU}(2n)/\mathrm{Sp}(n)$	$n-1$	$(n-1)(2n+1)$	$(n-1)(2n-3)$
AIII	$\mathrm{SU}(p+q)/\mathrm{S}(\mathrm{U}_p \times \mathrm{U}_q)$	$\min(p,q)$	$2pq$	$1+2(p-1)(q-1)$
BDI	$\mathrm{SO}(p+q)/\mathrm{SO}(p)\times\mathrm{SO}(q)$	$\min(p,q)$	pq	$1+(p-1)(q-1)$
DIII	$\mathrm{SO}(2n)/\mathrm{U}(n)$	$[\frac{1}{2}n]$	$n(n-1)$	$1+(n-2)(n-3)$
CI	$\mathrm{Sp}(n)/\mathrm{U}(n)$	n	$n(n+1)$	$1+n(n-1)$
CII	$\mathrm{Sp}(p+q)/\mathrm{Sp}(p)\times\mathrm{Sp}(q)$	$\min(p,q)$	$4pq$	$1+4(p-1)(q-1)$
EI	$(\mathfrak{e}_{6(-78)}, \mathfrak{sp}(4))$	6	42	26
EII	$(\mathfrak{e}_{6(-78)}, \mathfrak{su}(6)+\mathfrak{su}(2))$	4	40	19
EIII	$(\mathfrak{e}_{6(-78)}, \mathfrak{so}(10)+\mathbb{R})$	2	32	11
EIV	$(\mathfrak{e}_{6(-78)}, \mathfrak{f}(4))$	2	26	10
EV	$(\mathfrak{e}_{7(-133)}, \mathfrak{su}(8))$	7	70	43
EVI	$(\mathfrak{e}_{7(-133)}, \mathfrak{so}(12)+\mathfrak{su}(2))$	4	64	31
EVII	$(\mathfrak{e}_{7(-133)}, \mathfrak{e}_6+\mathbb{R})$	3	54	27
EVIII	$(\mathfrak{e}_{8(-248)}, \mathfrak{so}(16))$	8	128	71
EIX	$(\mathfrak{e}_{8(-248)}, \mathfrak{e}_7+\mathfrak{su}(2))$	4	112	55
FI	$(\mathfrak{f}_{4(-52)}, \mathfrak{sp}(3)+\mathfrak{su}(2))$	4	28	13
FII	$(\mathfrak{f}_{4(-52)}, \mathfrak{so}(9))$	1	16	1
G	$(\mathfrak{g}_{2(-14)}, \mathfrak{su}(2)+\mathfrak{su}(2))$	2	8	3

Remark 1.1 We note some exceptional cases:

(i) if $r = \mathrm{rank}(M) = 1$, then $s = r = 1$;

(ii) for AIII, $p = q = 2,\ s = 4$;

(iii) for BDI, $p = q = 2,\ s = 3;\ p = q = 3,\ s = 6$.

2 Abstract Root System and Subsystem

An abstract root system in a finite dimensional real inner product space V with inner product $\langle\,,\,\rangle$ is a finite set Δ (whose element is called a root) of $V-\{0\}$ such that (i) Δ spans V; (ii) for $\alpha\in\Delta$, the root reflection $s_\alpha(h)=h-a_{h,\alpha}\alpha$ carries Δ to itself, where $h\in V,\ a_{h,\alpha}=\frac{2\langle h,\alpha\rangle}{\langle\alpha,\alpha\rangle}$; (iii) $a_{\beta,\alpha}$ is an integer whenever $\alpha,\beta\in\Delta$.

$l=\dim V$ is called the rank of Δ. The Weyl group $W(\Delta)$ is the subgroup of orthogonal group of V generated by the reflection s_α for $\alpha\in\Delta$. An abstract root system is said to be reduced if $\alpha\in\Delta$ implies $2\alpha\notin\Delta$. If α is a root and $\frac{1}{2}\alpha$ is not a root, we say that α is reduced. An abstract root system Δ is said to be reducible if Δ admits an nontrivial disjoint decomposition $\Delta=\Delta'\cup\Delta''$ with every member of Δ' orthogonal to every member of Δ''. We say Δ is irreducible if it admits no such nontrivial decomposition.

For $\alpha,\beta\in\Delta$, the α string containing β is the set of all members of $\Delta\cup\{0\}$ of the form $\beta+k\alpha, k\in\mathbb{Z}$. In fact, there are no gaps, $-p\le k\le q$, $p\ge 0$, $q\ge 0$, and $a_{\beta,\alpha}=p-q$. The α string containing β contains at most four roots.

We can choose a lexicographic ordering so that $\Delta=\Delta^+\cup\Delta^-$ as disjoint sum of the set of positive roots and the set of negative roots. A root α is called simple if $\alpha>0$ and α does not decompose as the sum of two positive roots. A simple root is necessarily reduced. We can choose l simple roots $\alpha_1,\dots,\alpha_l$ which are linearly independent, such that every root α has the form $\alpha=\sum\limits_{i=1}^{l}m_i(\alpha)\alpha_i$ with all $m_i(\alpha)$ nonnegative or nonpositive. We call $\Pi=\{\alpha_1,\cdots,\alpha_l\}$ a simple system or a fundamental system.

For an abstract root system we associate a Dynkin diagram.

Lemma 2.1 (Classifications of Root Systems) (see [8, 6]) *Up to an isomorphism the irreducible reduced abstract root system are $A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4$ and G_2. For a nonreduced abstract root system, the reduced roots form an abstract reduced root system Δ_s, and the roots $\alpha\in\Delta$ satisfying $2\alpha\notin\Delta$ form a reduced abstract root system Δ_l. The Weyl groups of Δ,Δ_s,Δ_l coincide. Up to an isomorphism the only irreducible abstract root system that are not reduced is of the form $(BC)_n$.*

A subset Δ_1 of Δ is called a subsystem of Δ if it is closed, i.e., (i) if $\alpha\in\Delta_1$, then $-\alpha\in\Delta_1$; (ii) if $\alpha,\beta\in\Delta_{1,}\alpha+\beta\in\Delta$, then $\alpha+\beta\in\Delta_1$. A subsystem Δ_1 is called maximal if Δ_1 is a proper subset of Δ and Δ_1 is not properly contained in any properly subsystem of Δ. A subsystem Δ_1 is called $l-1$ maximal if it is maximal and $\mathrm{rank}(\Delta_1)=l-1$, where $l=\mathrm{rank}(\Delta)$.

Lemma 2.2 (Classifications of $l-1$ Maximal Subsystems) (see [7])

(1) *Let Δ be an irreducible reduced root system. Then the maximal* (*properly*) *subsystem is l maximal or $l-1$ maximal.*

(2) *Let Δ_1 be an $l-1$ maximal subsystem, Π_1 be a fundamental system of Δ_1. Then there exists a fundamental system Π of Δ such that $\Pi_1=\Pi\cap\Delta_1$.*

(3) *Let $\mu=\sum\mu_i\alpha_i$ be the highest root of Δ, $\Delta_1=\{\alpha\in\Delta\mid\alpha=\sum m_i\alpha_i, m_1=0\}$. Then Δ_1 is $l-1$ maximal if and only if $\mu_1=1$. Every $l-1$ maximal subsystem can be obtained in this manner.*

3 Real Semi-simple Lie Algebras and Restricted Root System

We adopt the notation as in [8]. Let $M = G/K$ be an irreducible Riemannian symmetric space of noncompact type with the Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. We can extend the Cartan involution θ to be complex linear on the complex simple Lie algebra $\mathfrak{g}^{\mathbb{C}}$:

$$\theta|_{\mathfrak{k}^{\mathbb{C}}} = \mathrm{id}, \quad \theta|_{\mathfrak{p}^{\mathbb{C}}} = -\mathrm{id}. \tag{3.1}$$

Up to isomorphism the real simple Lie algebra $\mathfrak{g}$ is uniquely determined by $\mathfrak{g}^{\mathbb{C}}$ and θ.

Let $\mathfrak{h} = \mathfrak{h}_{\mathfrak{k}} + \mathfrak{a}$ be a Cartan subalgebra of $\mathfrak{g}$ with $\mathfrak{a}$ being maximal in $\mathfrak{p}$.

$$\dim \mathfrak{a} = \mathrm{rank}(M) = r.$$

Let $\Delta = \Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}})$ be the corresponding root system of $\mathfrak{g}^{\mathbb{C}}$.

Every root takes real value on $\mathfrak{h}_o = i\mathfrak{h}_{\mathfrak{k}} + \mathfrak{a}$. Through the Killing form of $\mathfrak{g}^{\mathbb{C}}$ we can embed the root into $\mathfrak{h}_0$ by

$$\alpha(h) = B(\alpha, h) \quad \text{for } h \in \mathfrak{h}^{\mathbb{C}}.$$

The restricted roots are the elements in $\mathfrak{a}$ of the form

$$\Big\{\lambda = \alpha' = \frac{1}{2}(\alpha - \theta\alpha) \,\Big|\, a \in \Delta\Big\}.$$

All the restricted roots form an abstract root system Σ which can be nonreduced. For $\lambda \in \Sigma$, we denote $\mathfrak{g}_\lambda$ the corresponding root space with multiplicity

$$m_\lambda = \dim \mathfrak{g}_\lambda.$$

We can choose the ordering in Σ which is compatible to the ordering of Δ.

The restricted roots and the restricted root spaces have the following properties:

(i) $\mathfrak{g}_o = \mathfrak{a} \oplus \mathfrak{m}, \mathfrak{m} = Z_{\mathfrak{k}}(\mathfrak{a}) = \{Z \in \mathfrak{k} \mid [Z, A] = 0 \text{ for all } A \in \mathfrak{a}\}$,

(ii) $[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subset \mathfrak{g}_{\lambda+\mu}$,

(iii) $\theta\ (\mathfrak{g}_\lambda) = \mathfrak{g}_{-\lambda}$, and $\lambda \in \Sigma$ implies $-\lambda \in \Sigma$,

(iv) $\mathfrak{g} = \mathfrak{g}_0 + \sum_{\lambda \in \Sigma} \mathfrak{g}_\lambda$ as direct sum.

For any $X \in \mathfrak{g}$, there exist $H \in \mathfrak{a}, X_0 \in \mathfrak{m}, X_\lambda \in \mathfrak{g}_\lambda$ such that

$$X = H + X_0 + \sum_{\lambda \in \Sigma} X_\lambda = \Big(X_0 + \sum_{\lambda \in \Sigma^+} (X_{-\lambda} + \theta X_{-\lambda})\Big) + H + \sum_{\lambda \in \Sigma^+} (X_\lambda - \theta X_{-\lambda}).$$

We have Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \sum_{\lambda \in \Sigma^+} \mathfrak{g}_\lambda$.

For $h \in \mathfrak{a}$, the centralizer of h in $\mathfrak{p}$ is

$$Z_{\mathfrak{p}}(h) = \mathfrak{a} + \sum_{\substack{\lambda \in \Sigma^+ \\ \lambda(h)=0}} \mathfrak{g}_\lambda. \tag{3.2}$$

Our objective is to get $s = \max_{h \in \mathfrak{a}} \{\dim Z_{\mathfrak{p}}(h)\}$.

$h \in \mathfrak{a}$ is called regular if $\lambda(h) \neq 0$ for all $\lambda \in \Sigma$, otherwise singular. Let $C^+ = \{h \in \mathfrak{a} \mid \lambda(h) > 0 \text{ for } \lambda \in \Sigma^+\}$ be a restricted Weyl chamber, whoes closure is the closed Weyl chamber

$\overline{C^+}$. Let $W(\Sigma)$ be the Weyl group of Σ. We know that for any $h \in \mathfrak{a}$, there exists $w \in W(\Sigma)$ such that $w(h) \in \overline{C^+}$.

Let $r = \dim \mathfrak{a}$ denote the rank of M. If h is regular then $\dim Z_{\mathfrak{p}}(h) = \dim(\mathfrak{a}) = r$. Obviously if $\dim Z_{\mathfrak{p}}(h)$ takes maximal value, then h must be a singular element. Now let h be singular. Set $\Sigma(h) = \{\lambda \in \Sigma \mid \lambda(h) = B(\lambda, h) = 0\}$. It is obviously a subsystem of Σ. On the other hand, let Σ_1 be a subsystem of Σ with $\mathrm{rank}(\Sigma_1) < r$, and the span of Σ_1 be V_1. Then $\dim V_1 < r$. Let h_0 be any nonzero vector in the orthogonal complement of V_1 in $\mathfrak{a}$. Then $\Sigma(h_0)$ is a subsystem containing Σ_1. If $\mathrm{rank}(\Sigma_1) = r$, and the orthogonal complement of Σ_1 in $\mathfrak{a}$ is 0, then there exists no nonzero vector h in $\mathfrak{a}$ such that $\Sigma(h)$ contains Σ_1. So if $\dim \Sigma(h_0)$ takes the maximum s, $\Sigma(h_0)$ should be a maximal $l-1$ subsystem.

Let $\Pi' = \{\lambda_{1,} \cdots, \lambda_r\}$ be the fundamental system of Σ. From the lemma of maximal subsystem we see that $l-1$ maximal subsystem must be of the form of $\Sigma_k = \{\lambda \in \Sigma \mid \lambda = \sum m'_i \lambda_i,\ m'_k = 0\}$, where $1 \le k \le r$.

Since s is the maximum of $\dim Z_{\mathfrak{p}}(h)$ as h varies in $\mathfrak{a}$, from (3.2) we have

$$s = r + \max_{1 \le k \le r} \sum_{\lambda \in \Sigma_k^+} m_\lambda. \tag{3.3}$$

We note that $s = r$ if $r = 1$. The restricted root systems and the Stake diagrams are listed in [8, pp. 532–534].

4 The Partial Positivity for Riemannian Symmetric Spaces

Let $M = U/K$ be a simply connected irreducible symmetric spaces of compact type. Let $\mathfrak{u} = \mathfrak{k} + \mathfrak{p}_*$ be its Cartan decomposition. We identify $\mathfrak{p}_*$ with the tangent space of M at $o = eK$. From the theory of symmetric spaces, up to constants there exists uniquely U-invariant Riemannian metric on M. We fix such one invariant metric. We know that the curvature tensor at o is

$$R(X,Y)Z = -[[X,Y],Z] \quad \text{for all } X, Y, Z \in \mathfrak{p}_*.$$

If $X, Y \in \mathfrak{p}_*$ are orthonormal vectors, then the sectional curvature of the plane spanned by X and Y is

$$K(X,Y) = \|[X,Y]\|^2.$$

Thus $K(X,Y) = 0$ if and only if $[X,Y] = 0$. Suppose that the M has s-positive curvature. From definition we have

(1) For any $s+1$ orthonormal vectors $\{X_0, X_1, \cdots, X_s\}$ in $\mathfrak{p}_*$,

$$\sum_{i=1}^{s} K(X_0, X_i) = \sum_{i=1}^{s} \|[X_0, X_i]\|^2 > 0.$$

(2) There exist s orthonormal vectors $\{X_0, X_1, \cdots, X_{s-1}\}$ in $\mathfrak{p}_*$ such that

$$\sum_{i=1}^{s-1} K(X_0, X_i) = \sum_{i=1}^{s-1} \|[X_0, X_i]\|^2 = 0.$$

This is equivalent to

(3) for any $X \in \mathfrak{p}_*$, $\dim Z_{\mathfrak{p}_*}(X) < s+1$, where

$$Z_{\mathfrak{p}_*}(X) = \{Y \in \mathfrak{p}_* \mid [Y,X] = 0\}$$

is the centralizer of X in $\mathfrak{p}_*$; and

(4) there exists at least an $X_0 \in \mathfrak{p}_*$ such that $\dim Z_{\mathfrak{p}_*}(X_0) = s$.

So we have

$$s = \max_{X \in \mathfrak{p}_*}\{\dim Z_{\mathfrak{p}_*}(X)\}.$$

We calculate the s in the following program.

Let $\Pi = \{\alpha_1, \cdots, \alpha_l\}$ be the simple system of the complex simple Lie algebra $\mathfrak{g}^{\mathbb{C}}$. Every root can be given as the integral linear combination

$$\alpha = m_1(\alpha)\alpha_1 + m_2(\alpha)\alpha_2 + \cdots + m_l(\alpha)\alpha_l. \tag{4.1}$$

For $\alpha \in \Delta$, α' is its restriction (or projection) to $\mathfrak{a}$. We have

$$\alpha_i' = \lambda_{i^\theta}, \quad 1 \le i^\theta \le r \quad \text{for } 1 \le i \le l.$$

Then

$$\alpha' = \sum_{i=1}^{l} m_i \lambda_{i^\theta} = \sum_{j=1}^{r} m_j'(\alpha)\lambda_j.$$

From this we can get all restricted roots and their multiplicities.

However, in our calculations, we need not know the multiplicities explicitly. From (3.3), for $1 \le k \le r$, let

$$\Delta_k = \{\alpha \in \Delta \mid \alpha' \neq 0, m_k'(\alpha) = 0\}.$$

We denote the number of positive roots in $\Delta(k)$ by s_k'. Let $s_k = r + s_k'$. Then we have

$$s = \max_{1 \le k \le r}\{s_k\}.$$

In the following calculation, we suppose that $r > 1$ since $s = r = 1$ if $r = 1$. For four classical complex simple Lie algebras we imbed the root system into Euclidean spaces. We adopt the Dynkin diagrams of complex simple Lie algebras as in [8, p. 476]. The root system $\Delta = \Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}})$ is reduced while the restricted root system $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{a})$ may not be reduced. We denote the rank of Δ and Σ by l and r respectively. Note that our l, r are r, l in the notation of [8, pp. 532–534] respectively. For simple Lie algebra of exception type, we can get all the roots as listed in [1].

The following formulas are trivial

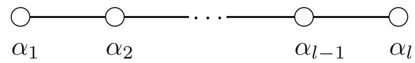
$$(l-k)(l-1-k) = l(l-1) + k^2 - k(2l-1), \tag{4.2}$$

$$(l-k)(l-1-k) - (l-r)(l-1-r) = (r-k)(2l-1-r-k). \tag{4.3}$$

Lemma 4.1 *Let* $f(t) = t(T-t)$, $T > 0$, *where* t *takes integer values* $1, 2, \cdots, r$, $r \le T$. *If* $T - r \ge 1$, *then*

$$\min_{1 \le t \le r} f(t) = f(1).$$

The Dynkin diagram of $\mathfrak{a}_l = \mathfrak{sl}(l+1, \mathbb{C})$ is



Let $\varepsilon_j,\ 1 \le j \le l+1$, be an orthogonal base of $\mathbb{R}^{l+1}$ with $|\varepsilon_j|^2 = \frac{1}{2(l+1)}$. The simple root system of $\mathfrak{a}_l$ is

$$\Pi = \{\alpha_j = \varepsilon_j - \varepsilon_{j+1}, j = 1, 2, \cdots, l\}, \quad |\alpha_j|^2 = \frac{1}{l+1}.$$

The positive roots are

$$\varepsilon_i - \varepsilon_j = \alpha_i + \cdots + \alpha_{j-1}, \quad 1 \le i < j \le l+1.$$

(1) AI, $M = \mathrm{SU}(n)/\mathrm{SO}(n),\ \mathfrak{g}^{\mathbb{C}} = \mathfrak{a}_l = \mathfrak{sl}(n, \mathbb{C}),\ r = l = n-1$.

In this case, the restricted root system $\Sigma = \Delta$.

For $1 \le k \le r = l,\ \lambda_k = \alpha'_k$. Then $\alpha' = (\sum m_i \alpha_i)' = \sum m_i \lambda_i$.

We have

$$\Delta_k^+ = \{\alpha > 0 \mid m_k(\alpha) = 0\}.$$

Let $\alpha = \varepsilon_i - \varepsilon_j = \alpha_i + \cdots + \alpha_j,\ 1 \le i < j \le l+1$, and $m_k(\alpha) = 0$ if and only if

$$i, j \le k \quad \text{or} \quad i, j > k.$$

Then

$$s_k = r + \frac{k(k-1)}{2} + \frac{(l+1-k)(l-k)}{2} = -k(l+1-k) + r + \frac{(l+1)l}{2}.$$

From Lemma 4.1 we get

$$s = \max_{1 \le k \le r} s_k = s_1 = l + \frac{l(l-1)}{2} = \frac{l(l+1)}{2} = \frac{n(n-1)}{2}.$$

(2) AII, $M = \mathrm{SU}(2n)/\mathrm{Sp}(n),\ \mathfrak{g}^{\mathbb{C}} = \mathfrak{a}_l,\ l = 2n-1,\ r = n-1,\ l = 2r+1$.

We have

$$\alpha'_1 = \alpha'_3 = \cdots = \alpha'_{2r-1} = \alpha'_{2r+1} = 0, \quad \alpha'_2 = \lambda_1, \quad \alpha'_4 = \lambda_2, \quad \cdots, \quad \alpha'_{2r} = \lambda_r.$$

Let $\alpha = \varepsilon_i - \varepsilon_j,\ 1 \le i < j \le l+1 = 2n$, and $\alpha' = 0$ if and only if $\alpha = \alpha_1, \alpha_3, \cdots, \alpha_{2n-1}$.

For $1 \le k \le r = n-1,\ \Delta_k = \{\alpha \in \Delta \mid m'_k(\alpha) = 0\} = \{\alpha \in \Delta \mid m'_{2k}(\alpha) = 0\}$, and

$$m_{2k}(\alpha) = 0 \quad \text{if and only if} \quad i, j \le 2k \text{ or } j > i > 2k.$$

We see that

$$\begin{aligned} s_k &= r + \frac{2k(2k-1)}{2} + \frac{(l+1-2k)(l-2k)}{2} - (r+1) \\ &= -2k(l+1-2k) + \frac{(l+1)l}{2} - 1. \end{aligned}$$

From Lemma 4.1 we get

$$s = s_1 = \frac{(l-1)(l-2)}{2} = (n-1)(2n-3).$$

(3) AIII, we suppose that $p \le q$, $M = \mathrm{SU}(p+q)/\mathrm{S}(\mathrm{U}_p \times \mathrm{U}_q)$, $\mathfrak{g} = \mathfrak{a}_l$, $l = p+q-1$, $r = \min(p,q) = p$, $r \le \frac{l+1}{2}$.

We have

$$\alpha'_i = \alpha'_{l+1-i} = \lambda_i,\ 1 \le i \le r; \quad \alpha'_j = 0,\ r < j \le l-r.$$

Let $\alpha = \varepsilon_i - \varepsilon_j,\ 1 \le i < j \le l+1,$ and $\alpha' = 0$ if and only if $r < i < j \le l+1-r$.

$m'_k(\alpha) = 0$ if $m_k(\alpha) = 0$ or $m_{l+1-k}(\alpha) = 0$.

Then $m'_k(\alpha) = 0$ implies

$$1 \le i < j \le k \quad \text{or} \quad l+1-k < i < j \le l+1 \quad \text{or} \quad k \le i < j \le l+1-k.$$

We get

$$\begin{aligned} s_k &= r + 2\frac{k(k-1)}{2} + \frac{(l+1-2k)(l-2k)}{2} - \frac{(l+1-2r)(l-2r)}{2} \\ &= k(k-1) + 2k^2 - k(2l+1) + \frac{(l+1)l}{2} + r - \frac{(l+1-2r)(l-2r)}{2} \\ &= -k(2l+2-3k) + \frac{(l+1)l}{2} + r - \frac{(l+1-2r)(l-2r)}{2}. \end{aligned}$$

If $\frac{2l+2}{3} - r \ge 1$, i.e., $r \le \frac{2l-1}{3}$, then $\max\{s_k\} = s_1$.

As $r \le \frac{l+1}{2}$, we have $r \le \frac{2l-1}{3}$ when $l \ge 5$.

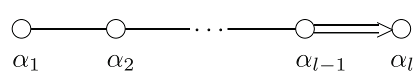
we have for $l \ge 5$, i.e., $q \ge 3$,

$$\begin{aligned} s = s_1 &= r + \frac{(l+1-2)(l-2)}{2} - \frac{(l+1-2r)(l-2r)}{2} \\ &= r + \frac{1}{2}(2r-2)(2l+1-2r-2) \\ &= 1 + (r-1) + (r-1)(2l-1-2r) \\ &= 1 + 2(r-1)(l-r) \\ &= 1 + 2(p-1)(q-1). \end{aligned}$$

If $l = 4,\ r \le \frac{5}{2},\ r = 2$ (we suppose $r \ge 2$), as $1 \cdot (10-3) < 2 \cdot (10-6)$, then $s_1 > s_2,\ s = s_2$. In this case $p = 2,\ q = 3$.

If $l = 3,\ r \le 2,\ r = 2$, as $1 \cdot (8-3) > 2 \cdot (8-6)$, then $s_1 < s_2,\ s = s_2 = 4$. In this case $p = q = 2$.

The Dynkin diagram of $\mathfrak{b}_l = \mathfrak{so}(2l+1, \mathbb{C})$ is



Let $\varepsilon_j, 1 \le j \le l$, be an orthogonal base of $\mathbb{R}^l$, and $|\varepsilon_j|^2 = \frac{1}{2(2l-1)}$. The simple root system of $\mathfrak{b}_l$ is

$$\Pi = \{\alpha_j = \varepsilon_j - \varepsilon_{j+1},\ j = 1, 2, \cdots, l-1,\ \alpha_l = \varepsilon_l\}.$$

The positive roots are

$$\varepsilon_i,\ \varepsilon_i \pm \varepsilon_j, \quad 1 \le i < j \le l,$$

where

$$\varepsilon_i = \alpha_i + \cdots + \alpha_l, \quad \varepsilon_i - \varepsilon_j = \alpha_i + \cdots + \alpha_j, \quad \varepsilon_i + \varepsilon_j = \alpha_i + \cdots + \alpha_{j-1} + 2(\alpha_j + \cdots + \alpha_l).$$

(4) BI, $M = \mathrm{SO}(p+q)/\mathrm{SO}(p) \times \mathrm{SO}(q),\ p+q = 2l+1,\ \mathfrak{g} = \mathfrak{b}_l,\ r = p \leq l.$

We have

$$\alpha_i' = \lambda_i,\ 1 \leq i \leq r; \quad \alpha_j' = 0,\ r < j \leq l.$$

If $\alpha \in \Delta^+,\ \alpha' = 0$, then α is one of the following roots:

$$\varepsilon_i,\ i > r; \quad \varepsilon_i \pm \varepsilon_j,\ j > i > r.$$

If $\alpha > 0,\ m_k'(\alpha) = 0$, then α is one of the following roots:

$$\varepsilon_i,\ i > k; \quad \varepsilon_i - \varepsilon_j,\ i < j \leq k \text{ or } j > i > k; \quad \varepsilon_i + \varepsilon_j,\ j > i > k.$$

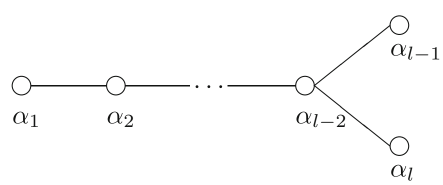
Then

$$\begin{aligned} s_k &= r + [(l-k) - (l-r)] + \frac{k(k-1)}{2} + 2\Big[\frac{(l-k)(l-1-k)}{2} - \frac{(l-r)(l-1-r)}{2}\Big] \\ &= 2r - k + \frac{k(k-1)}{2} + k^2 - k(2l-1) + l(l-1) - (l-r)(l-1-r) \\ &= -\frac{k}{2}(4l+1-3k) + 2r + l(l-1) - (l-r)(l-1-r). \end{aligned}$$

Since $\frac{4l+1}{3} - r \geq \frac{4l+1}{3} - l = \frac{l+1}{3} \geq 1$, we have

$$\begin{aligned} s = s_1 &= 2r - 1 + (l-1)(l-2) - (l-r)(l-1-r) \\ &= 1 + 2(r-1) + (r-1)(2l-1-r-1) \\ &= 1 + (r-1)(2l-r) \\ &= 1 + (p-1)(q-1). \end{aligned}$$

The Dynkin diagram of $\mathfrak{d}_l = \mathfrak{so}(2l, \mathbb{C})$ is



Let $\varepsilon_j,\ 1 \leq j \leq l$, be an orthogonal base of $\mathbb{R}^l$, and $|\varepsilon_j|^2 = \frac{1}{4(l-1)}$. The simple root system is

$$\Pi = \{\alpha_j = \varepsilon_j - \varepsilon_{j+1},\ j = 1, 2, \cdots, l-1,\ \alpha_l = \varepsilon_{l-1} + \varepsilon_l\}.$$

The positive roots are

$$\varepsilon_i \pm \varepsilon_j, \quad i < j,$$

where

$$\begin{aligned} &\varepsilon_i - \varepsilon_j = \alpha_i + \cdots + \alpha_{j-1}, \\ &\varepsilon_i + \varepsilon_j = \alpha_i + \cdots + \alpha_{l-2} + \alpha_j + \cdots + \alpha_l = \alpha_i + \cdots + \alpha_{j-1} + 2(\alpha_j + \cdots + \alpha_{l-2}) + \alpha_{l-1} + \alpha_l. \end{aligned}$$

(5) DI, $M = \mathrm{SO}(p+q)/\mathrm{SO}(p)\times\mathrm{SO}(q),\ p+q=2l,\ \mathfrak{g}=\mathfrak{d}_l,\ r=p\le l.$

There are three cases.

(i) $2\le r\le l-2$. We have

$$\alpha_i' = \lambda_i,\ 1\le i\le r;\quad \alpha_j'=0,\ r<j\le l.$$

If $\alpha\in\Delta^+,\ \alpha'=0$, then α is one of the following roots:

$$\varepsilon_i,\ i>r;\quad \varepsilon_i\pm\varepsilon_j,\ j>i>r.$$

If $\alpha>0,\ m_k'(\alpha)=0$, then α is one of the following roots:

$$\varepsilon_i,\ i>k;\quad \varepsilon_i-\varepsilon_j,\ i<j\le k \text{ or } j>i>k;\quad \varepsilon_i+\varepsilon_j,\ j>i>k.$$

Then

$$\begin{aligned} s_k &= r+\frac{k(k-1)}{2}+2\Big[\frac{(l-k)(l-1-k)}{2}-\frac{(l-r)(l-1-r)}{2}\Big]\\ &= \frac{k(k-1)}{2}+k^2-k(2l-1)+l(l-1)-(l-r)(l-1-r)+r\\ &= -\frac{k}{2}(4l-1-3k)+l(l-1)-(l-r)(l-1-r)+r. \end{aligned}$$

Since $\frac{4l-1}{3}-r\ge\frac{l+5}{3}\ge 1$, we have

$$\begin{aligned} s=s_1 &= r+(l-1)(l-2)-(l-r)(l-1-r)\\ &= 1+(r-1)+(r-1)(2l-1-r-1)\\ &= 1+(r-1)(2l-1-r)\\ &= 1+(p-1)(q-1). \end{aligned}$$

(ii) $r=p=l-1,\ q=l+1$. We have

$$\alpha_i'=\lambda_i,\ 1\le i\le l-2;\quad \alpha_{l-1}'=\alpha_l'=\lambda_{l-1}=\lambda_r.$$

$\alpha'>0$ implies $\alpha=0$.

If $\alpha>0$ and $m_k'(\alpha)=0$, then α is one of the following roots:

$$\varepsilon_i,\ i>k;\quad \varepsilon_i-\varepsilon_j,\ i<j\le k \text{ or } j>i>k;\quad \varepsilon_i+\varepsilon_j,\ j>i>k.$$

Then

$$\begin{aligned} s_k &= r+\frac{k(k-1)}{2}+2\frac{(l-k)(l-1-k)}{2}\\ &= \frac{k(k-1)}{2}+k^2-k(2l-1)+l(l-1)+r\\ &= -\frac{k}{2}(4l-1-3k)+l(l-1)+r. \end{aligned}$$

Since $\frac{4l-1}{3}-r=\frac{l+2}{3}\ge 1$, we have

$$s=s_1=r+(l-1)(l-2)=(l-1)^2=1+l(l-2)=1+(p-1)(q-1).$$

(iii) $r = p = l,\ q = l$. We have

$$\alpha'_i = \lambda_i, \quad 1 \leq i \leq l.$$

$\alpha' > 0$ implies $\alpha = 0$.

If $\alpha > 0$, and $m'_k(\alpha) = 0$, then α is one of the following roots:

$$\varepsilon_i,\ i > k; \quad \varepsilon_i - \varepsilon_j,\ i < j \leq k \text{ or } j > i > k; \quad \varepsilon_i + \varepsilon_j,\ j > i > k.$$

Then

$$\begin{aligned} s_k &= r + \frac{k(k-1)}{2} + 2\frac{(l-k)(l-1-k)}{2} \\ &= \frac{k(k-1)}{2} + k^2 - k(2l-1) + l(l-1) + r \\ &= -\frac{k}{2}(4l-1-3k) + l(l-1) + r. \end{aligned}$$

Since $\frac{4l-1}{3} - r = \frac{l-1}{3}$, we have for $l \geq 4$, i.e., $q \geq 4$,

$$s = s_1 = r + (l-1)(l-2) = 1 + (l-1) + (l-1)(l-2) = 1 + (l-1)^2 = 1 + (p-1)(q-1).$$

When $r \geq 2,\ l \geq r = 2$. For $l = 2$, $r = p = 2$, $q = 2$, $s_1 < s_2,\ s = s_2 = 3$.

For $l = 3$, $r = p = 2$, $q = 4$, $s_1 > s_2,\ s = s_1$; $r = p = 3$, $q = 3$, $s_3 > s_1 > s_2,\ s = s_3 = 6$.

(6) DIII, $M = \mathrm{SO}(2n)/\mathrm{U}(n),\ \mathfrak{g}^{\mathbb{C}} = \mathfrak{g}_l,\ l = n,\ r = [\frac{l}{2}]$.

There are two cases.

(i) l is even. We have

$$\alpha'_{2i} = \lambda_i,\ 1 \leq i \leq r = \frac{l}{2}; \quad \alpha'_{2i-1} = 0,\ 1 \leq i \leq r.$$

If $\alpha > 0$, then $\alpha' > 0$ implies $\alpha = \alpha_1, \alpha_3, \cdots, \alpha_{l-1}$.

If $\alpha > 0$ and $m'_k(\alpha) = 0$, then α is one of the following roots:

$$\varepsilon_i - \varepsilon_j,\ i < j \leq 2k \text{ or } j > i > 2k; \quad \varepsilon_i + \varepsilon_j,\ j > i > 2k.$$

Then

$$\begin{aligned} s_k &= r + \Big[\frac{2k(2k-1)}{2} + \frac{(l-2k)(l-1-2k)}{2} - \frac{l}{2}\Big] + \frac{(l-2k)(l-1-2k)}{2} \\ &= \frac{k(2k-1)}{2} + (l-2k)(l-1-2k) \\ &= -k(4l-1-6k) + l(l-1). \end{aligned}$$

For $\frac{4l-1}{6} - \frac{l}{2} = \frac{l-1}{6} \geq 1$, i.e., $l \geq 7$, we have

$$s = s_1 = 1 + (l-2)(l-3) = 1 + (n-2)(n-3).$$

For $l = 4$, $r = 2$, $s_1 < s_2,\ s = s_2 = 6$.

For $l = 6$, $r = 3$, $s_2 < s_1 < s_3,\ s = s_3 = 15$.

(ii) l is odd. We have

$$\alpha'_{2i} = \lambda_i,\ 1 \le i \le \frac{l-3}{2}; \quad \alpha'_{l-1} = \alpha'_l = \lambda_r,\ r = \frac{l-1}{2}; \quad \alpha'_{2i-1} = 0,\ 1 \le i \le r.$$

If $\alpha > 0$, then $\alpha' > 0$ implies $\alpha = \alpha_1, \alpha_3, \cdots, \alpha_{l-2}$.

If $\alpha > 0$ and $m'_k(\alpha) = 0$, then α is one of the following roots:

$$\varepsilon_i - \varepsilon_j,\ i < j \le 2k \text{ or } j > i > 2k; \quad \varepsilon_i + \varepsilon_j,\ j > i > 2k.$$

Then

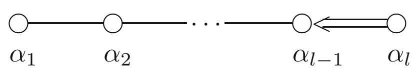
$$\begin{aligned} s_k &= r + \Big[\frac{2k(2k-1)}{2} + \frac{(l-2k)(l-1-2k)}{2} - r\Big] + \frac{(l-2k)(l-1-2k)}{2} \\ &= \frac{k(2k-1)}{2} + (l-2k)(l-1-2k) \\ &= -k(4l-1-6k) + l(l-1). \end{aligned}$$

For $\frac{4l-1}{6} - \frac{l-1}{2} = \frac{l+2}{6} \ge 1$, i.e., $l \ge 4$, we have

$$s = s_1 = 1 + (l-2)(l-3) = 1 + (n-2)(n-3).$$

For $l = 3$, $r = 1,\ s = r = 1$.

The Dynkin diagram of $\mathfrak{c}_l = \mathfrak{sp}(l, \mathbb{C})$ is



Let $\varepsilon_j,\ 1 \le j \le l$ be an orthogonal base of $\mathbb{R}^l$, and $|\varepsilon_j|^2 = \frac{1}{4(l+1)}$. The simple root system is

$$\Pi = \{\alpha_j = \varepsilon_j - \varepsilon_{j+1}, j = 1, 2, \cdots, l-1, \alpha_l = 2\varepsilon_l\}.$$

The positive roots are

$$2\varepsilon_i,\ \varepsilon_i \pm \varepsilon_j, \quad i < j,$$

where

$$\begin{aligned} &2\varepsilon_i = 2(\alpha_i + \cdots + \alpha_{l-1}) + \alpha_l, \quad \varepsilon_i - \varepsilon_j = \alpha_i + \cdots + \alpha_{j-1}, \\ &\varepsilon_i + \varepsilon_j = \alpha_i + \cdots + \alpha_{j-2} + 2(\alpha_j + \cdots + \alpha_{l-1}) + \alpha_l. \end{aligned}$$

(7) CI, $M = \mathrm{Sp}(n)/\mathrm{U}(n),\ \mathfrak{g}^{\mathbb{C}} = \mathfrak{c}_l,\ l = n$.
We have

$$\alpha'_i = \lambda_i, \quad 1 \le i \le l.$$

$\alpha' > 0$ implies $\alpha = 0$.

If $\alpha > 0$ and $m'_k(\alpha) = 0$, then α is one of the following roots:

$$2\varepsilon_i,\ i > k; \quad \varepsilon_i - \varepsilon_j,\ i < j \le 2k \text{ or } j > i > 2k; \quad \varepsilon_i + \varepsilon_j,\ j > i > 2k.$$

Then

$$\begin{aligned} s_k &= r + (l-k) + \frac{k(k-1)}{2} + 2\frac{(l-k)(l-1-k)}{2} \\ &= 2l - k + \frac{k(k-1)}{2} + (l-k)(l-1-k) \\ &= -k(4l+1-3k) + l(l-1) + 2l. \end{aligned}$$

For $\frac{4l+1}{3} - l = \frac{l+1}{3} \geq 1$, i.e., $l \geq 2$, we have

$$s = s_1 = 2l - 1 + (l-1)(l-2) = 1 + l(l-1) = 1 + n(n-1).$$

(8) CII, $M = \mathrm{Sp}(p+q)/\mathrm{SP}(p) \times \mathrm{SP}(q),\ \mathfrak{g}^{\mathbb{C}} = \mathfrak{c}_l,\ l = p+q,\ r = p \leq [\frac{l}{2}]$.
We have

$$\alpha'_{2i} = \lambda_i,\ 1 \leq i \leq r; \quad \alpha'_j = 0,\ j = 1, 3, \cdots, 2r-1,\ \text{or } j > 2r.$$

If $\alpha > 0$, then $\alpha' > 0$ implies $\alpha = \alpha_1, \alpha_3, \cdots, \alpha_{2r-1};$ or $\alpha = \varepsilon_i - \varepsilon_j,\ j > i > 2r$.
If $\alpha > 0$ and $m'_k(\alpha) = 0$, then α is one of the following roots:

$$2\varepsilon_i,\ i > 2k; \quad \varepsilon_i - \varepsilon_j,\ i < j \leq 2k \text{ or } j > i > 2k; \quad \varepsilon_i + \varepsilon_j,\ j > i > 2k.$$

Then

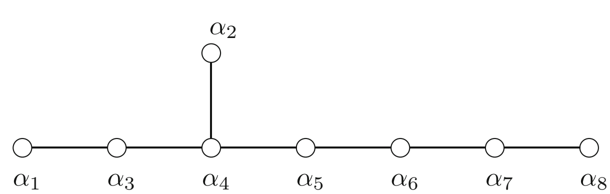
$$\begin{aligned} s_k &= r + (r-k) + \Big[\frac{2k(2k-1)}{2} - k\Big] + 2\Big[\frac{(l-2k)(l-1-2k)}{2} - \frac{(l-2r)(l-1-2r)}{2}\Big] \\ &= 2r + k(2k-3) + (l-2k)(l-1-2k) - (l-2r)(l-1-2r) \\ &= -k(4l+1-6k) + 2r + l(l-1) - (l-2r)(l-1-2r). \end{aligned}$$

For $\frac{4l+1}{6} - \frac{l}{2} = \frac{l+1}{6} \geq 1$, i.e., $l \geq 5$, we have

$$\begin{aligned} s = s_1 &= 2r - 1 + (l-2)(l-3) - (l-2r)(l-1-2r) \\ &= 1 + 4(r-1)(l-1-r) \\ &= 1 + 4(p-1)(q-1). \end{aligned}$$

For $l = 4$, $r = p = 2$, $q = 2$, $s_1 < s_2,\ s = s_2 = 6$.

Now we consider the cases of exception type. Since we can calculate the s-values by using the same method, we only give the full detail in the case of type EVIII. We draw the Dynkin diagram of $\mathfrak{e}_8$ as follows:



(9) EVIII, $\mathfrak{g}^{\mathbb{C}} = \mathfrak{e}_8,\ l = 8,\ r = 8$.
We have

$$\alpha'_i = \lambda_i, \quad 1 \leq i \leq 8.$$

$\alpha' = 0$ implies $\alpha = 0$.

We list Δ_k^+ as follows:

$$\begin{aligned}
\Delta_1^+ = \{&\alpha_3, \quad \alpha_4, \quad \alpha_5, \quad \alpha_6, \quad \alpha_7, \quad \alpha_8, \quad \alpha_2,\\
&\alpha_3+\alpha_4, \quad \alpha_4+\alpha_2, \quad \alpha_4+\alpha_5, \quad \alpha_5+\alpha_6, \quad \alpha_6+\alpha_7, \quad \alpha_7+\alpha_8,\\
&\alpha_3+\alpha_4+\alpha_2, \quad \alpha_3+\alpha_4+\alpha_5, \quad \alpha_4+\alpha_5+\alpha_2,\\
&\alpha_4+\alpha_5+\alpha_6, \quad \alpha_5+\alpha_6+\alpha_7, \quad \alpha_6+\alpha_7+\alpha_8,\\
&\alpha_3+\alpha_4+\alpha_5+\alpha_2, \quad \alpha_3+\alpha_4+\alpha_5+\alpha_6, \quad \alpha_4+\alpha_5+\alpha_6+\alpha_2,\\
&\alpha_4+\alpha_5+\alpha_6+\alpha_7, \quad \alpha_5+\alpha_6+\alpha_7+\alpha_8, \quad \alpha_3+2\alpha_4+\alpha_5+\alpha_2,\\
&\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_2, \quad \alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7, \quad \alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_2,\\
&\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8, \quad \alpha_3+2\alpha_4+\alpha_5+\alpha_6+\alpha_2, \quad \alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_2,\\
&\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8, \quad \alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8+\alpha_2,\\
&\alpha_3+2\alpha_4+2\alpha_5+\alpha_6+\alpha_2, \quad \alpha_3+2\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_2,\\
&\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8+\alpha_2, \quad \alpha_3+2\alpha_4+2\alpha_5+\alpha_6+\alpha_7+\alpha_2,\\
&\alpha_3+2\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8+\alpha_2, \quad \alpha_3+2\alpha_4+2\alpha_5+2\alpha_6+\alpha_7+\alpha_2,\\
&\alpha_3+2\alpha_4+2\alpha_5+\alpha_6+\alpha_7+\alpha_8+\alpha_2, \quad \alpha_3+2\alpha_4+2\alpha_5+2\alpha_6+\alpha_7+\alpha_8+\alpha_2,\\
&\alpha_3+2\alpha_4+2\alpha_5+2\alpha_6+2\alpha_7+\alpha_8+\alpha_2\},
\end{aligned}$$

$$\begin{aligned}
\Delta_2^+ = \{&\alpha_1, \quad \alpha_3, \quad \alpha_4, \quad \alpha_5, \quad \alpha_6, \quad \alpha_7, \quad \alpha_8,\\
&\alpha_1+\alpha_3, \quad \alpha_3+\alpha_4, \quad \alpha_4+\alpha_5, \quad \alpha_5+\alpha_6, \quad \alpha_6+\alpha_7, \quad \alpha_7+\alpha_8,\\
&\alpha_1+\alpha_3+\alpha_4, \quad \alpha_3+\alpha_4+\alpha_5, \quad \alpha_4+\alpha_5+\alpha_6,\\
&\alpha_5+\alpha_6+\alpha_7, \quad \alpha_6+\alpha_7+\alpha_8, \quad \alpha_1+\alpha_3+\alpha_4+\alpha_5,\\
&\alpha_3+\alpha_4+\alpha_5+\alpha_6, \quad \alpha_4+\alpha_5+\alpha_6+\alpha_7, \quad \alpha_5+\alpha_6+\alpha_7+\alpha_8,\\
&\alpha_1+\alpha_3+\alpha_4+\alpha_5+\alpha_6, \quad \alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7, \quad \alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8,\\
&\alpha_1+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7, \quad \alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8,\\
&\alpha_1+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8\},
\end{aligned}$$

$$\begin{aligned}
\Delta_3^+ = \{&\alpha_1, \quad \alpha_4, \quad \alpha_5, \quad \alpha_6, \quad \alpha_7, \quad \alpha_8, \quad \alpha_2,\\
&\alpha_4+\alpha_2, \quad \alpha_4+\alpha_5, \quad \alpha_5+\alpha_6, \quad \alpha_6+\alpha_7, \quad \alpha_7+\alpha_8,\\
&\alpha_4+\alpha_5+\alpha_2, \quad \alpha_4+\alpha_5+\alpha_6, \quad \alpha_5+\alpha_6+\alpha_7, \quad \alpha_6+\alpha_7+\alpha_8,\\
&\alpha_4+\alpha_5+\alpha_6+\alpha_2, \quad \alpha_4+\alpha_5+\alpha_6+\alpha_7, \quad \alpha_5+\alpha_6+\alpha_7+\alpha_8,\\
&\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_2, \quad \alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8, \quad \alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8+\alpha_2\},
\end{aligned}$$

$$\begin{aligned}
\Delta_4^+ = \{&\alpha_1, \quad \alpha_3, \quad \alpha_5, \quad \alpha_6, \quad \alpha_7, \quad \alpha_8, \quad \alpha_2,\\
&\alpha_1+\alpha_3, \quad \alpha_5+\alpha_6, \quad \alpha_6+\alpha_7, \quad \alpha_7+\alpha_8,\\
&\alpha_5+\alpha_6+\alpha_7, \quad \alpha_6+\alpha_7+\alpha_8, \quad \alpha_5+\alpha_6+\alpha_7+\alpha_8\},
\end{aligned}$$

$$\begin{aligned}
\Delta_5^+ = \{&\alpha_1, \quad \alpha_3, \quad \alpha_4, \quad \alpha_6, \quad \alpha_7, \quad \alpha_8, \quad \alpha_2,\\
&\alpha_1+\alpha_3, \quad \alpha_3+\alpha_4, \quad \alpha_4+\alpha_2, \quad \alpha_6+\alpha_7, \quad \alpha_7+\alpha_8,\\
&\alpha_1+\alpha_3+\alpha_4, \quad \alpha_3+\alpha_4+\alpha_2, \quad \alpha_6+\alpha_7+\alpha_8,\\
&\alpha_1+\alpha_3+\alpha_4+\alpha_2\},
\end{aligned}$$

$$\begin{aligned}\Delta_6^+ = \{&\alpha_1, \quad \alpha_3, \quad \alpha_4, \quad \alpha_5, \quad \alpha_7, \quad \alpha_8, \quad \alpha_2,\\
&\alpha_1+\alpha_3, \quad \alpha_3+\alpha_4, \quad \alpha_4+\alpha_2, \quad \alpha_4+\alpha_5, \quad \alpha_7+\alpha_8,\\
&\alpha_1+\alpha_3+\alpha_4, \quad \alpha_3+\alpha_4+\alpha_2, \quad \alpha_3+\alpha_4+\alpha_5, \quad \alpha_4+\alpha_5+\alpha_2,\\
&\alpha_1+\alpha_3+\alpha_4+\alpha_5, \quad \alpha_1+\alpha_3+\alpha_4+\alpha_2, \quad \alpha_3+\alpha_4+\alpha_5+\alpha_2,\\
&\alpha_1+\alpha_3+\alpha_4+\alpha_5+\alpha_2, \quad \alpha_3+2\alpha_4+\alpha_5+\alpha_2, \quad \alpha_1+\alpha_3+2\alpha_4+\alpha_5+\alpha_2,\\
&\alpha_1+2\alpha_3+2\alpha_4+\alpha_5+\alpha_2\},\end{aligned}$$

$$\begin{aligned}\Delta_7^+ = \{&\alpha_1, \quad \alpha_3, \quad \alpha_4, \quad \alpha_5, \quad \alpha_6, \quad \alpha_8, \quad \alpha_2,\\
&\alpha_1+\alpha_3, \quad \alpha_3+\alpha_4, \quad \alpha_4+\alpha_2, \quad \alpha_4+\alpha_5, \quad \alpha_5+\alpha_6,\\
&\alpha_1+\alpha_3+\alpha_4, \quad \alpha_3+\alpha_4+\alpha_2, \quad \alpha_3+\alpha_4+\alpha_5,\\
&\alpha_4+\alpha_5+\alpha_2, \quad \alpha_4+\alpha_5+\alpha_6,\\
&\alpha_1+\alpha_3+\alpha_4+\alpha_5, \quad \alpha_1+\alpha_3+\alpha_4+\alpha_2, \quad \alpha_3+\alpha_4+\alpha_5+\alpha_2,\\
&\alpha_3+\alpha_4+\alpha_5+\alpha_6, \quad \alpha_4+\alpha_5+\alpha_6+\alpha_2,\\
&\alpha_1+\alpha_3+\alpha_4+\alpha_5+\alpha_2, \quad \alpha_3+2\alpha_4+\alpha_5+\alpha_2, \quad \alpha_1+\alpha_3+\alpha_4+\alpha_5+\alpha_6,\\
&\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_2, \quad \alpha_1+\alpha_3+2\alpha_4+\alpha_5+\alpha_2,\\
&\alpha_1+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_2, \quad \alpha_3+2\alpha_4+\alpha_5+\alpha_6+\alpha_2,\\
&\alpha_1+2\alpha_3+2\alpha_4+\alpha_5+\alpha_2, \quad \alpha_1+\alpha_3+2\alpha_4+\alpha_5+\alpha_6+\alpha_2,\\
&\alpha_3+2\alpha_4+2\alpha_5+\alpha_6+\alpha_2, \quad \alpha_1+2\alpha_3+2\alpha_4+\alpha_5+\alpha_6+\alpha_2,\\
&\alpha_1+\alpha_3+2\alpha_4+2\alpha_5+\alpha_6+\alpha_2, \quad \alpha_1+2\alpha_3+2\alpha_4+2\alpha_5+\alpha_6+\alpha_2,\\
&\alpha_1+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6+\alpha_2, \quad \alpha_1+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6+2\alpha_2\},\end{aligned}$$

$$\begin{aligned}\Delta_8^+ = \{&\alpha_1, \quad \alpha_3, \quad \alpha_4, \quad \alpha_5, \quad \alpha_6, \quad \alpha_7, \quad \alpha_2,\\
&\alpha_1+\alpha_3, \quad \alpha_3+\alpha_4, \quad \alpha_4+\alpha_2, \quad \alpha_4+\alpha_5, \quad \alpha_5+\alpha_6, \quad \alpha_6+\alpha_7,\\
&\alpha_1+\alpha_3+\alpha_4, \quad \alpha_3+\alpha_4+\alpha_2, \quad \alpha_3+\alpha_4+\alpha_5,\\
&\alpha_4+\alpha_5+\alpha_2, \quad \alpha_4+\alpha_5+\alpha_6, \quad \alpha_5+\alpha_6+\alpha_7,\\
&\alpha_1+\alpha_3+\alpha_4+\alpha_5, \quad \alpha_1+\alpha_3+\alpha_4+\alpha_2, \quad \alpha_3+\alpha_4+\alpha_5+\alpha_2,\\
&\alpha_3+\alpha_4+\alpha_5+\alpha_6, \quad \alpha_4+\alpha_5+\alpha_6+\alpha_2, \quad \alpha_4+\alpha_5+\alpha_6+\alpha_7,\\
&\alpha_1+\alpha_3+\alpha_4+\alpha_5+\alpha_2, \quad \alpha_3+2\alpha_4+\alpha_5+\alpha_2, \quad \alpha_1+\alpha_3+\alpha_4+\alpha_5+\alpha_6,\\
&\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_2, \quad \alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7, \quad \alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_2,\\
&\alpha_1+\alpha_3+2\alpha_4+\alpha_5+\alpha_2, \quad \alpha_1+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_2,\\
&\alpha_3+2\alpha_4+\alpha_5+\alpha_6+\alpha_2, \quad \alpha_1+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7,\\
&\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_2, \quad \alpha_1+2\alpha_3+2\alpha_4+\alpha_5+\alpha_2,\\
&\alpha_1+\alpha_3+2\alpha_4+\alpha_5+\alpha_6+\alpha_2, \quad \alpha_3+2\alpha_4+2\alpha_5+\alpha_6+\alpha_2,\\
&\alpha_1+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_2, \quad \alpha_3+2\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_2,\\
&\alpha_1+2\alpha_3+2\alpha_4+\alpha_5+\alpha_6+\alpha_2, \quad \alpha_1+\alpha_3+2\alpha_4+2\alpha_5+\alpha_6+\alpha_2,\\
&\alpha_1+\alpha_3+2\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_2, \quad \alpha_3+2\alpha_4+2\alpha_5+\alpha_6+\alpha_7+\alpha_2,\\
&\alpha_1+2\alpha_3+2\alpha_4+2\alpha_5+\alpha_6+\alpha_2, \quad \alpha_1+2\alpha_3+2\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_2,\end{aligned}$$

$$\alpha_1+\alpha_3+2\alpha_4+2\alpha_5+\alpha_6+\alpha_7+\alpha_2,\quad \alpha_3+2\alpha_4+2\alpha_5+2\alpha_6+\alpha_7+\alpha_2,$$
$$\alpha_1+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6+\alpha_2,\quad \alpha_1+2\alpha_3+2\alpha_4+2\alpha_5+\alpha_6+\alpha_7+\alpha_2,$$
$$\alpha_1+\alpha_3+2\alpha_4+2\alpha_5+2\alpha_6+\alpha_7+\alpha_2,\quad \alpha_1+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6+2\alpha_2,$$
$$\alpha_1+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6+\alpha_7+\alpha_2,\quad \alpha_1+2\alpha_3+2\alpha_4+2\alpha_5+2\alpha_6+\alpha_7+\alpha_2,$$
$$\alpha_1+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6+\alpha_7+2\alpha_2,\quad \alpha_1+2\alpha_3+3\alpha_4+2\alpha_5+2\alpha_6+\alpha_7+\alpha_2,$$
$$\alpha_1+2\alpha_3+3\alpha_4+2\alpha_5+2\alpha_6+\alpha_7+2\alpha_2,\quad \alpha_1+2\alpha_3+3\alpha_4+3\alpha_5+2\alpha_6+\alpha_7+\alpha_2,$$
$$\alpha_1+2\alpha_3+3\alpha_4+3\alpha_5+2\alpha_6+\alpha_7+2\alpha_2,\quad \alpha_1+2\alpha_3+4\alpha_4+3\alpha_5+2\alpha_6+\alpha_7+2\alpha_2,$$
$$\alpha_1+3\alpha_3+4\alpha_4+3\alpha_5+2\alpha_6+\alpha_7+2\alpha_2,\quad 2\alpha_1+3\alpha_3+4\alpha_4+3\alpha_5+2\alpha_6+\alpha_7+2\alpha_2\}.$$

We get s_k as follows:

$$s_1=50,\quad s_2=36,\quad s_3=30,\quad s_4=22,\quad s_5=24,\quad s_6=31,\quad s_7=45,\quad s_8=71.$$

Then $s=71$.

Now we get all s-values for irreducible Riemannian symmetric spaces of compact type. In summary, we have Table 1.1 in Section 1 of the present paper.

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