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A Note on the Completeness of an Exponential Type Sequence*

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Abstract For any given coprime integers p and q greater than 1, in 1959, B. J. Birch proved that all sufficiently large integers can be expressed as a sum of pairwise distinct terms of the form $p^a q^b$. As Davenport observed, Birch's proof can be modified to show that the exponent b can be bounded in terms of p and q. In 2000, N. Hegyvari gave an effective version of this bound. The author improves this bound.

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1 Introduction

A positive integer set A is called complete if all sufficiently large integers can be expressed as the sum of distinct terms taken from A. Denote by $\mathbb{N}_0$ the set of non-negative integers. In 1959, B. J. Birch [1] proved that for given integers p and q greater than 1, the set $Y = \{p^a q^b : a, b \in \mathbb{N}_0\}$ is complete if and only if $(p, q) = 1$, which verifies the conjecture of P. Erdős.

Theorem 1.1 (see [1]) *Given any positive coprime integers p, q greater than 1, there exists a number $N(p, q)$ such that every $n > N(p, q)$ is expressible as a sum of the form $n = p^{a_1} q^{b_1} + p^{a_2} q^{b_2} + \cdots$, where (a_i, b_i) are distinct pairs of positive integers.*

As Davenport observed, Birch's proof can be modified to show that for every coprime integers p and q greater than 1, there exists an integer $K = K(p, q)$ such that the sequence $Y_K = \{p^a q^b : \ a, b \in \mathbb{N}_0, \ 0 \le b \le K\}$ is complete.

For such K, Erdős mentioned that, "of course the exact value of $K(p, q)$ is not known and no doubt will be very difficult to determine". In 2000, Hegyvari [2] obtained an effective upper bound for $K(p, q)$.

Theorem 1.2 (see [2]) *For every coprime integers p and q greater than 1, there exists an integer $K = K(p, q)$ such that the set*

$$Y_K = \{p^a q^b : \ a, b \in \mathbb{N}_0, \ 0 \le b \le K\}$$

is complete. Furthermore, we have

$$K(p, q) \le 2p^{2c^{2^{2q^{4p+3}}}},$$

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where $c = 1152\log_2 p \log_2 q$.

In this paper, we improve this upper bound. The basic idea is similar to that in [2]. What is more, we add in proof a nice result of V. H. Vu on subset sums, which greatly reduces the upper bound obtained by Hegyvari. For more details, see Lemma 2.6 in Section 2.

Theorem 1.3

$$K(p,q) \le p^{c^{2^{q^{2p+3}}}},$$

where $c = 1152\log_2 p \log_2 q$.

2 Lemmas

Before the proof of the lemmas, we introduce the following notation and definitions. Let $\mathbb{N}$ be the set of positive integers, and $A = \{a_1 < a_2 < \cdots < a_n < \cdots\}$ be a sequence of positive integers. Denote $P(A)$ as

$$P(A) = \Big\{\sum \varepsilon_i a_i : \varepsilon_i = 0 \text{ or } 1, \ \sum \varepsilon_i < \infty\Big\}.$$

We call (x, y) disjoint if there exist $X, Y \subseteq \mathbb{N}$, $X \cap Y = \emptyset$, such that $x = \sum\limits_{i\in X} a_i$, $y = \sum\limits_{j\in Y} a_j$. The sets X, Y are disjoint if for every $x \in X$, $y \in Y$, x and y are disjoint. Denote $Z \subseteq P(A)$ as a d-set if all elements of Z are pairwise disjoint.

Lemma 2.1 *Let* $A = \{a_1 < a_2 < \cdots < a_n < \cdots\}$ *be a sequence of positive integers. Assume that there exists an integer* n_0 *such that for every* $n > n_0$, $a_n < a_1 + a_2 + \cdots + a_{n-1}$. *Then* $P(A)$ *has bounded gaps, i.e., if* $P(A) = \{x_1 < x_2 < \cdots\}$, *then for every* k *we have* $x_{k+1} - x_k \le a_{n_0}$.

Proof Assume that $A_k = \{a_1 < a_2 < \cdots < a_k\}$ and $P(A_k) = \{x_{k_1} < x_{k_2} < \cdots\}$. We will take induction on k to prove that for any l, $x_{k_{l+1}} - x_{k_l} \le a_{n_0}$.

If $k \le n_0$, then for any l, there exists an integer $i < n_0$, such that $a_1 + a_2 + \cdots + a_i \le x_{k_l} \le a_1 + a_2 + \cdots + a_i + a_{i+1}$ and $a_1 + a_2 + \cdots + a_i \le x_{k_{l+1}} \le a_1 + a_2 + \cdots + a_i + a_{i+1}$. Hence $x_{k_{l+1}} - x_{k_l} \le a_{i+1} \le a_{n_0}$.

Now assume that the proposition holds for $k (\ge n_0)$. Namely, for any l, $x_{k_{l+1}} - x_{k_l} \le a_{n_0}$. Assume $P(A_{k+1}) = \{y_1 < y_2 < \cdots\}$ for convenience. Since $k \ge n_0$, by the precondition of Lemma 2.1, we have $a_{k+1} < a_1 + a_2 + \cdots + a_k$. Let n_1 be the largest number no larger than $a_1 + a_2 + \cdots + a_k$ with the form $a_{k+1} + \sum\limits_{1\le i\le k} \varepsilon_i a_i$, and n_2 be the least number larger than $a_1 + a_2 + \cdots + a_k$ with the same form as above.

Then for any m, we have the following three possibilities:

Case 1 $y_m < y_{m+1} \le a_1 + a_2 + \cdots + a_k$. Then by the induction hypothesis, we have $y_{m+1} - y_m \le a_{n_0}$.

Case 2 $y_m = a_1 + \cdots + a_k$, $y_{m+1} = n_2$. Then

$$y_{m+1} - y_m \le y_{m+1} - n_1 = n_2 - n_1.$$

By the choice of n_1, n_2 and the induction hypothesis, we have $y_{m+1} - y_m \le a_{n_0}$.

Case 3 $n_2 \le y_m < y_{m+1} \le a_1 + a_2 + \cdots + a_{k+1}$. Then we assume that $y_m = a_{k+1} + y'_m$ and $y_{m+1} = a_{k+1} + y'_{m+1}$. We can find that the elements y'_m and y'_{m+1} are adjacent in $P(A_k)$. By the induction hypothesis, we have $y_{m+1} - y_m = y'_{m+1} - y'_m \le a_{n_0}$.

Collecting the above discussion, we know that for any m, $y_{m+1} - y_m \le a_{n_0}$. This completes the proof of Lemma 2.1.

Lemma 2.2 *Let p, q be positive integers greater than 1. Let $Y_{2p,2} = \{p^k q^{2m} : k \ge 0, 1 \le m \le 2p\}$ and assume $P(Y_{2p,2}) = \{x_1 < x_2 < \cdots\}$. Then for every n, we have $x_{n+1} - x_n < \triangle$, where*

$$\triangle \le q^{2p+2}.$$

Proof Assume that x is the number larger than q^{2p+2} with the form $p^k q^{2m}$. Then

$$\sum_{p^t q^{2s} < x} p^t q^{2s} = \sum_{s=1}^{[\frac{1}{2}\log_q x]} q^{2s} \cdot \sum_{p^t < \frac{x}{q^{2s}}} p^t = \sum_{s=1}^{[\frac{1}{2}\log_q x]} q^{2s} \cdot \frac{p^{T+1} - 1}{p - 1},$$

where $p^T < \frac{x}{q^{2s}} \le p^{T+1}$.

Since

$$x > q^{2p+2},$$

by direct calculation, we have

$$\sum_{p^t q^{2s} < x} p^t q^{2s} = \sum_{s=1}^{[\frac{1}{2}\log_q x]} q^{2s} \cdot \frac{p^{T+1} - 1}{p - 1} \ge \sum_{s=1}^{[\frac{1}{2}\log_q x]} \frac{x - q^{2s}}{p - 1} > x.$$

Hence, by Lemma 2.1, we have $\triangle \le q^{2p+2}$. This completes the proof of Lemma 2.2.

Lemma 2.3 (see [2]) *Let $c, d \ge 2$ with $(c, d) = 1$. Let $x \ge d^{4\mathrm{A}}$ and*

$$Y_{\mathrm{A}} = \{c^a d^b : a \in N, 1 \le b \le \mathrm{A} = [5 \log_2 c] + 1\}.$$

Then there exists a number n with $1 \le n \le x$, which has at least two representations $n = \sum_{y \in Y_{\mathrm{A}}} \varepsilon_y y = \sum_{y \in Y_{\mathrm{A}}} \varepsilon'_y y$, where $\varepsilon_y, \varepsilon'_y \in \{0, 1\}$ and $\sum_{y \in Y_{\mathrm{A}}} \varepsilon_y \varepsilon'_y = 0$ (i.e., the representations are disjoint).

Lemma 2.4 (see [2]) *Let p, q be integers greater than 1, $(p, q) = 1$ and let $g = q^2$. Let $a_1 = b_1 = 1$, and for $i > 0$, let*

$$a_{i+1} = [24 a_i b_i \log_2 g], \quad b_{i+1} = [24 a_i b_i \log_2 p], \quad p_i = p^{a_i}, \quad q_i = g^{b_i},$$

and $\mathrm{A}_i = [5 \log_2 p_i] + 1$. Then, for every n, there exist sets

$$U_n = \{u_1 < u_2 < \cdots < u_n\}, \quad V_n = \{v_1 < v_2 < \cdots < v_n\}$$

for which

$$\begin{aligned} &u_i, v_i \in P(Y_{\mathrm{A}_i}) = P(\{p_i^k q_i^m : k \in N, 1 \le m \le \mathrm{A}_i\}), \\ &v_i - u_i = p^{k_i} g^{m_i}, \quad u_i, v_i \text{ are disjoint}, \quad i = 1, 2, \cdots, n \end{aligned}$$

and

$$\{p^{k_j-k_i}g^{m_j-m_i}u_i,\ p^{k_j-k_i}g^{m_j-m_i}v_i,\ u_j,\ v_j\}$$

is a d-set for any $1 \le i < j \le n$.

Corollary 2.1 (see [2]) *Let*

$$c_1 = 48\log_2 q, \quad c_2 = 24\log_2 p, \quad c = c_1c_2.$$

Then, for every n, *there exists a d-set*

$$D = \{x_1, y_1, x_2, y_2, \cdots, x_n, y_n\}$$

for which

$$y_1 - x_1 = y_2 - x_2 = \cdots = y_n - x_n = p^{k_n}q^{2m_n}, \quad D \subseteq P(Y_{L_n}),$$

where $L_n \le 2b_{n+1}$. *Furthermore, for* $k > 1$, *we have*

$$a_k \le \frac{1}{c_2}c^{2^{k-1}} \quad \text{and} \quad b_k \le \frac{1}{c_1}c^{2^{k-1}}.$$

Lemma 2.5 (see [2]) *Let* $A = \{a_1 < a_2 < \cdots < a_n < \cdots\}$ *be a sequence of positive integers. Assume*

$$U = \{x_1, x_2, \cdots, x_k, y_1, y_2, \cdots, y_k\} \subseteq P(A),$$

where U *is a d-set and for every* j *with* $1 \le j \le k$, $y_j - x_j = d > 0$ *for some fixed* d. *Then* $P(A)$ *contains an arithmetic progression of length* $k+1$.

Lemma 2.6 *Let* p, q, a, b *be positive integers with* $(p,q) = 1$ *and let* $T = p^a$. *Let*

$$R_T = \{p^r, q^s, r \in \mathbb{N}, 1 \le s \le T\}.$$

Then for every r *with* $1 \le r \le p^aq^b$, *there exists an* $x_r \in P(R_T)$ *such that* $r \equiv x_r \pmod{p^aq^b}$.

The conclusion of Lemma 2.6 is an application of Lemma 2.1 in [3].

Lemma 2.7 (see [3, Lemma 2.1]) *Let* n *be a positive integer and* A *be a multi-set of* n *integers coprime to* n. *Then* $P(A)$ *contains every residue modulo* n.

Proof of Lemma 2.6 Assume that $n = p^a$ and $A = \{q, q^2, \cdots, q^{p^a}\}$. Then, by Lemma 2.7, $P(A)$ contains every residue modulo p^a. Hence, for any integer r with $1 \le r \le p^aq^b$, we have

$$r \equiv \sum_i q^i \pmod{p^a},$$

where $i \le p^a$. Then, we assume that $r = \sum\limits_i q^i + Mp^a$.

Since

$$M \equiv \sum_j p^{j\phi(q^b)} \pmod{q^b},$$

where ϕ is the Euler's totient function, we can assume that $M = \sum_j p^{j\phi(q^b)} + q^b N$. Combining the above equalities, we have

$$r \equiv \sum_i q^i + \sum_j p^{a+j\phi(q^b)} \pmod{p^a q^b}.$$

By the definition of R_T and the fact that $i \le p^a$, we know that $\sum_i q^i + \sum_j p^{a+j\phi(q^b)} \in P(R_T)$. This completes the proof of Lemma 2.6.

3 Proof of Theorem 1.3

Let $n = q^{2p+3}$. By Corollary 2.1 and Lemma 2.5, there is an arithmetic progression of length n and difference $d = p^{k_n} q^{2m_n}$. Furthermore, $H = \{h_0 + kd : k = 0, 1, \cdots, n-1\} \subseteq P(Y_{L_n})$, where $L_n \le c^{2^n}$. If $p^k q^s$ is a term of any element of H, then s is even and $k_n \le a_{n+1}$, and $m_n \le b_{n+1}$.

Let $Y^* = dqY_{2q,2}$. Assume that $P(Y^*) = \{x_1 < x_2 < \cdots < x_n \cdots\}$. Then, by Lemma 2.2, we know that the biggest gap in $P(Y^*)$ is at most $dq \cdot q^{2p+2}$. If $p^k q^s$ is a term of any element of Y^*, then s is odd. Hence, $P(Y^*)$ and H are disjoint.

Now we will prove that $P(Y^*)+H$ contains an infinite arithmetic progression with difference d, i.e., $\{x_1 + h_0 + kd : k \in \mathbb{N}_0\} \subseteq P(Y^*) + H$. For any t, there exists an integer s, such that $x_s \le x_1 + td < x_{s+1}$. Hence

$$dq \cdot q^{2p+2} > x_{s+1} - x_s > x_1 + td - x_s = \left(t - \frac{x_s - x_1}{d}\right) \cdot d.$$

Since

$$0 \le t - \frac{x_s - x_1}{d} < q^{2p+3} = n,$$

there exists an integer $z = t - \frac{x_s - x_1}{d}$ such that $h_0 + zd \in H$. Hence

$$x_1 + h_0 + td = h_0 + \left(t - \frac{x_s - x_1}{d}\right) \cdot d + x_s = h_0 + zd + x_s \in H + P(Y^*).$$

Let $a = k_n$, $b = 2m_n$. By Lemma 2.6, there exists a set $P(R_T)$, such that for any r with $1 \le r \le p^{k_n} q^{2m_n}$, there exists an $x_r \in P(R_T)$ such that $r \equiv x_r \pmod{p^a q^b}$.

By the definition of R_T, we know that $P(R_T)$, $P(Y^*)$ and H are disjoint. It is easy to see that $P(R_T) + P(Y^*) + H$ contains every sufficiently large number. So $R_T \cup Y^* \cup Y_{L_n}$ is complete.

Now we only need to give an upper bound for $K(p,q)$. Denote by $K_1 = K_1(p,q)$, $K_2 = K_2(p,q)$ and $K_3 = K_3(p,q)$ the greatest s for which $p^k q^s$ is a term of an element of $P(Y^*)$, H and $P(R_T)$ respectively. Following the same discussion as in [2], we have

(1) An upper bound for $K_1 = K_1(p,q)$. Since $Y^* = dqY_{2q,2}$, we have that if $p^k q^s \in Y^*$ then

$$K_1 \le 2m_n + 1 + 2p \le 2b_{n+1} + 2p + 1 < 3c^{2^n}.$$

(2) An upper bound for $K_2 = K_2(p,q)$. By Corollary 2.1, $K_2 \le 2b_{n+1} \le 2c^{2^n}$.

(3) An upper bound for $K_3 = K_3(p,q)$. By Lemma 2.6 and the definition of R_T, we have

$$K_3 \le p^{k_n} < p^{c^{2^n}}.$$

It is easy to find that the last upper bound is the biggest one. Hence, we have

$$K(p,q) \le p^{c^{2^n}} = p^{c^{2^{q^{2p+3}}}},$$

where $c = 1152 \log_2 p \log_2 q$. This completes the proof of Theorem 1.3.

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