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Global Exact Boundary Controllability for Cubic Semi-linear Wave Equations and Klein-Gordon Equations***

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Abstract The authors prove the global exact boundary controllability for the cubic semi-linear wave equation in three space dimensions, subject to Dirichlet, Neumann, or any other kind of boundary controls which result in the well-posedness of the corresponding initial-boundary value problem. The exponential decay of energy is first established for the cubic semi-linear wave equation with some boundary condition by the multiplier method, which reduces the global exact boundary controllability problem to a local one. The proof is carried out in line with [2, 15]. Then a constructive method that has been developed in [13] is used to study the local problem. Especially when the region is star-complemented, it is obtained that the control function only need to be applied on a relatively open subset of the boundary. For the cubic Klein-Gordon equation, similar results of the global exact boundary controllability are proved by such an idea.

Keywords Global exact boundary controllability, Cubic semi-linear wave equations, The exponential decay, Star-shaped, Star-complemented, Cubic Klein-Gordon equations
2000 MR Subject Classification 35B37, 35L05

1 Introduction

In this paper, we continue to study the exact boundary controllability problem for nonlinear wave equations. The local exact boundary controllability for semi-linear and quasi-linear wave equations was built in [13]. The aim of this paper is to study the global exact boundary controllability for semi-linear wave equations. Here by local we mean that the initial and final data are small in some suitable Sobolev spaces, while by global we mean there is no smallness restriction on the initial and final data. We first prove the dissipative energy estimate for the semi-linear wave equation, which reduces the problem of the global exact boundary controllability to a local one. Then we apply the constructive method introduced in [13] to establish the local exact boundary controllability.

To best illustrate our idea, we take the cubic semi-linear wave equation in three space

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dimensions for example:

$$\Box u + \lambda u^3 = 0, \quad 0 < t < T,\ x \in \Omega, \tag{1.1}$$

where $\Box = \partial_t^2 - \triangle$, $\triangle = \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2}$, λ is a positive constant and Ω is a bounded open subset of $\mathbb{R}^3$.

Consider the initial state

$$u(0,x) = f_0(x), \quad u_t(0,x) = f_1(x), \quad x \in \Omega \tag{1.2}$$

and the final state

$$u(T,x) = g_0(x), \quad u_t(T,x) = g_1(x), \quad x \in \Omega. \tag{1.3}$$

Let $s \geq 2$, $f_0, g_0 \in H^s(\Omega)$ and $f_1, g_1 \in H^{s-1}(\Omega)$, where $H^s(\Omega)$ is the standard Sobolev space of order s. For $0 \leq t \leq T$ and $x \in \partial\Omega$, we impose any of the following boundary conditions on equation (1.1):

$$\begin{cases} u = h(t,x) & \text{of Dirichlet type,} \\ \dfrac{\partial u}{\partial n} = h(t,x) & \text{of Neumann type,} \\ \dfrac{\partial u}{\partial n} + bu = h(t,x) & \text{of the third type,} \\ \dfrac{\partial u}{\partial n} + \overline{b} u_t = h(t,x) & \text{of the dissipative type,} \end{cases} \tag{1.4}$$

where b and $\overline{b}$ are given positive constants. Here we can use any other kind of boundary condition as long as the corresponding initial-boundary value problem is well-posed.

Then the problem of the exact boundary controllability for the equation (1.1) is stated as follows: Given $T > 0$, is it possible to find a corresponding boundary control $h(t,x)$ driving the equation (1.1) with the initial state (f_0, f_1) to the desired state (g_0, g_1) at time T?

In this paper, we will establish the global exact boundary controllability for the cubic semi-linear wave equation.

Precisely we prove the following theorem.

Theorem 1.1 *Suppose $f_0, g_0 \in H^s(\Omega)$, $f_1, g_1 \in H^{s-1}(\Omega)$, $s \geq 2$. There exists a sufficiently large positive constant T_0 depending only on the Sobolev norm of the data $\|f_0\|_{H^s(\Omega)}$, $\|f_1\|_{H^{s-1}(\Omega)}$, $\|g_0\|_{H^s(\Omega)}$, $\|g_1\|_{H^{s-1}(\Omega)}$ and a boundary control function h, such that the cubic semi-linear wave equation* (1.1) *with the initial state* (1.2) *and one of the boundary conditions* (1.4) *admits a unique solution on the domain* $(0,T) \times \Omega$ *which verifies the desired state* (1.3)*, provided that the time $T > T_0$.*

B. Dehman, G. Lebeau and E. Zuazua obtained similar results on the exact internal controllability problem in [2]. Note that the controllability time T depends on the initial and final data. Whether T may be independent of the initial and final data is certainly one of the main open problems in the context of controllability of nonlinear PDE.

There are an extremely large number of publications on the exact boundary controllability problem. Some classical references can be found in [7,9]. For semi-linear hyperbolic equations, E. Zuazua [14] introduced a variant of the Hilbert uniqueness method to study the control problem for semi-linear wave equations $y'' - \triangle y + f(y) = 0$ in n space dimensions with both

Dirichlet and Neumann boundary conditions, where $f \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R})$ is a locally Lipschitz function. The author got the exact controllability when the nonlinearity is asymptotically linear and local controllability results for a large class of nonlinearities under some natural growth assumptions on the nonlinearities in [14]. E. Zuazua also studied the exact controllability for semi-linear wave equations $y_{tt} - y_{xx} + f(y) = h$ with Dirichlet boundary condition in one space dimension in [16], where the author established the exact controllability in $H^1_0(\Omega) \times L^2(\Omega)$ with controls $h \in L^2(\Omega \times (0,T))$ supported in any open and non-empty subset of Ω if $\frac{|f(s)|}{|s| \log^2 |s|} \to 0$ as $s \to \infty$ by HUM and a fixed point technique, which can be also applied to the wave equation with Neumann type boundary condition. It was also shown in [16] that if f behaves like $-s \log^p(1+|s|)$ with $p > 2$ as $|s| \to \infty$, the system is not exactly controllable in any time T. In [4], X. Fu, J. Yong and X. Zhang obtained a global exact controllability result for a class of multidimensional semi-linear hyperbolic equations with super-linear nonlinearity and variable coefficients, via an observability estimate for the linear hyperbolic equation with an unbounded potential, which is obtained by a point-wise estimate and a global Carleman estimate for the hyperbolic differential operators and analysis on the regularity of the optimal solution to an auxiliary optimal control problem. I. Lasiecka and R. Triggiani [6] studied the (global) exact controllability for the semi-linear wave equation $u_{tt} - \triangle u = f(u)$, where f is an absolutely continuous function with first derivative f' a.e. being uniformly bounded $|f'| \le C$ a.e., and obtained the exact controllability results on any state space $H = H^\gamma_0(\Omega) \times H^{\gamma-1}(\Omega)$ using the control space $H^\gamma_0([0,T], L_2(\partial\Omega))$, $0 \le \gamma \le 1$, $\gamma \ne \frac{1}{2}$, as well as the special case $\gamma = \frac{1}{2}$ in [6].

Then let us show our strategy of establishing the global exact controllability for the cubic semi-linear wave equation.

Without loss of generality, we assume $\Omega \subset\subset \mathbb{B}_1$, where $\mathbb{B}_1$ is the unit ball centered at the origin with the boundary $\partial\mathbb{B}_1$. We can always extend the functions f_0, f_1, g_0, g_1 to $\widetilde{f}_0, \widetilde{f}_1, \widetilde{g}_0, \widetilde{g}_1$ such that

$$\operatorname{supp}(\widetilde{f}_0, \widetilde{f}_1, \widetilde{g}_0, \widetilde{g}_1) \subset\subset \mathbb{B}_1 \tag{1.5}$$

and

$$\begin{aligned} &\|\widetilde{f}_0\|_{H^s(\mathbb{B}_1)} \le C_s \|f_0\|_{H^s(\Omega)}, \quad \|\widetilde{f}_1\|_{H^{s-1}(\mathbb{B}_1)} \le C_s \|f_1\|_{H^{s-1}(\Omega)}, \\ &\|\widetilde{g}_0\|_{H^s(\mathbb{B}_1)} \le C_s \|g_0\|_{H^s(\Omega)}, \quad \|\widetilde{g}_1\|_{H^{s-1}(\mathbb{B}_1)} \le C_s \|g_1\|_{H^{s-1}(\Omega)} \end{aligned} \tag{1.6}$$

for some constant $C_s > 0$ and all $s \ge 0$ (for example, see [3]). In what follows, we will use the same extension for several times and always denote the extension operator by $\sim: f \to \widetilde{f}$.

We shall construct a solution of (1.1) with initial data $\widetilde{f}_0, \widetilde{f}_1$ and final data $\widetilde{g}_0, \widetilde{g}_1$ on the domain $(0,T) \times \mathbb{B}_1$ for sufficiently large T. Then the restriction of the solution to $\partial\Omega$ yields the desired boundary control function. To this end, we first evolve the equation (1.1) with the initial data $\widetilde{f}_0, \widetilde{f}_1$ and the boundary condition $\frac{\partial u}{\partial r} + \frac{\partial u}{\partial t} + u = 0$ on the domain $[0, T_1] \times \mathbb{B}_1$ and prove the global existence and an exponential decay of energy estimate for the solution of this problem. Consequently, $u(T_1, x)$ and $u_t(T_1, x)$ will be sufficiently small in appropriate Sobolev spaces if T_1 is large enough. Similarly, we evolve equation (1.1) with the final data $\widetilde{g}_0, \widetilde{g}_1$ and backward with the boundary condition $\frac{\partial u}{\partial r} - \frac{\partial u}{\partial t} + u = 0$ on the domain $[T - T_2, T] \times \mathbb{B}_1$. Then $u(T - T_2, x)$ and $u_t(T - T_2, x)$ will be small enough in suitable Sobolev spaces provided that T_2 is large enough. By taking $T > T_1 + T_2$, we find that it suffices to construct a solution on the domain $[T_1, T - T_2] \times \Omega$ with initial condition $u(T_1, x)$, $u_t(T_1, x)$ and final condition $u(T - T_2, x)$, $u_t(T - T_2, x)$. Noting that $u(T_1, x)$, $u_t(T_1, x)$ and $u(T - T_2, x)$, $u_t(T - T_2, x)$ are small, we have reduced the global exact controllability problem to a local one, which was studied by Zhou and Lei [13].

Now we take a look at the control problem of equation (1.1) in the star-complemented region. For this problem, we attempt to obtain a control function only applied on a relatively open subset of the boundary. D. Russell [10] and G. Chen [1] studied such problem for the linear wave equation with the help of the decay estimate for the solution of the wave equation on an exterior domain due to C. S. Morawetz.

First we give some useful definitions.

Definition 1.1 (see [3]) *Let Ω^* be an open subset of $\mathbb{R}^3$. We say Ω^* is star-shaped if there exists a point $x^* \in \Omega^*$ such that for all $x \in \overline{\Omega^*}$ the line segment $\{\xi \mid \xi = (1-t)x^* + tx,\ \forall\, 0 \leq t \leq 1\}$ lies in $\overline{\Omega^*}$. We also call it star-shaped with respect to x^*.*

Definition 1.2 (see [10]) *The pair (Ω, Γ) is star-complemented if Γ is a relatively open subset of $\partial\Omega$ and there is a point $x^* \in \overline{\Omega}^c$ with the property that each point $x \in \partial\Omega - \Gamma$ can be connected to x^* by a line segment which, except for x itself, lies entirely outside $\overline{\Omega}$.*

Here $\overline{\Omega}$ means the closure of Ω and Ω^c means the complement of Ω.

Assume that there exists a star-complemented pair (Ω, Γ) for the region Ω and $\partial\Omega$ is a regular, piecewise C^∞ manifold of dimension two. What the word "regular" means will be given in Section 4. Let $\Gamma_1 = \partial\Omega - \Gamma$.

Now we introduce the boundary control conditions, any of which we impose on the equation (1.1):

$$u(t,x) = 0, \quad 0 \leq t \leq T,\ x \in \Gamma_1 \tag{1.7}$$

and

$$\begin{cases} u = h(t,x) & \text{of Dirichlet type,} \\ \dfrac{\partial u}{\partial n} = h(t,x) & \text{of Neumann type,} \\ \dfrac{\partial u}{\partial n} + cu = h(t,x) & \text{of the third type,} \\ \dfrac{\partial u}{\partial n} + \overline{c}u_t = h(t,x) & \text{of the dissipative type,} \end{cases} \tag{1.8}$$

where c and $\overline{c}$ are given positive constants, for $0 \leq t \leq T$ and $x \in \Gamma$.

Briefly we establish the following theorem for the control problem of the cubic semi-linear wave equation (1.1) in the star-complemented region.

Theorem 1.2 *Assume that the bounded region Ω is star-complemented. For any $f_0, g_0 \in H^s(\Omega)$, $f_1, g_1 \in H^{s-1}(\Omega)$, $s \geq 2$ with the property that $\partial_x^\alpha f_0 = \partial_x^\alpha g_0 = 0$ on Γ_1 for $|\alpha| \leq s-1$ and $\partial_x^\beta f_1 = \partial_x^\beta g_1 = 0$ on Γ_1 for $|\beta| \leq s-2$, there exists a sufficiently large constant $T_0 > 0$ depending only on the Sobolev norm of the data $\|f_0\|_{H^s(\Omega)}$, $\|f_1\|_{H^{s-1}(\Omega)}$, $\|g_0\|_{H^s(\Omega)}$, $\|g_1\|_{H^{s-1}(\Omega)}$ and a boundary control h only applied on Γ such that the cubic semi-linear wave equation* (1.1) *with the initial data* (1.2)*, the boundary condition* (1.7) *and one of the conditions* (1.8) *admits a unique solution on the domain $(0,T)\times\Omega$ satisfying the desired data* (1.3)*, provided that $T > T_0$.*

Remark 1.1 The solution in Theorem 1.1 or 1.2 belongs to $\bigcap\limits_{j=0}^{s} C^j([0,T], H^{s-j}(\Omega))$ and the boundary control obtained by the way of our construction is not unique.

The proof of Theorem 1.2 is similar to that of Theorem 1.1 to a large extent, however it is more complicated.

The rest of this paper is organized as follows. In Section 2, we prove the global existence and an exponentially dissipative energy estimate for the solution of the equation (1.1) with the initial data $\widetilde{f}_0, \widetilde{f}_1$ and boundary condition $\frac{\partial u}{\partial t} + \frac{\partial u}{\partial r} + u = 0$ on the domain $[0,+\infty) \times \mathbb{B}_1$. Then we prove Theorem 1.1 in Section 3 and Theorem 1.2 in Section 4 respectively. In Section 5, we prove similar results of the global exact boundary controllability for the cubic Klein-Gordon equation.

2 The Global Existence and Exponentially Dissipative Energy Estimates for the Cubic Semi-linear Wave Equation

In this section, we will study the global existence of the strong solution to the mixed initial-boundary value problem:

$$\begin{cases} \Box u + \lambda u^3 = 0, & t \geq 0,\ x \in \mathbb{B}_1, \\ t = 0 : u = \widetilde{f}_0,\ u_t = \widetilde{f}_1, & x \in \mathbb{B}_1, \\ \dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial r} + u = 0, & t \geq 0,\ x \in \partial\mathbb{B}_1. \end{cases} \tag{2.1}$$

The proof relies on a local existence theory and an exponentially dissipative a priori energy estimate.

We first establish the following local existence for system (2.1).

Lemma 2.1 *Assume $s \geq 2$, $\widetilde{f}_0 \in H^s(\mathbb{B}_1)$, $\widetilde{f}_1 \in H^{s-1}(\mathbb{B}_1)$ and $\mathrm{supp}(\widetilde{f}_0, \widetilde{f}_1) \subset\subset \mathbb{B}_1$. There exist positive constants T^* and M depending only on $\|\widetilde{f}_0\|_{H^s(\mathbb{B}_1)}$ and $\|\widetilde{f}_1\|_{H^{s-1}(\mathbb{B}_1)}$ such that system (2.1) admits a unique solution $u(t,x)$ on the domain $[0,T^*] \times \mathbb{B}_1$ satisfying*

$$\|\partial_t^s u(t,\cdot)\|^2_{L^2(\mathbb{B}_1)} + \sum_{l=0}^{s-1} \|\partial_t^l u(t,\cdot)\|^2_{H^1(\mathbb{B}_1)} \leq M^2, \quad \forall\, 0 \leq t \leq T^*. \tag{2.2}$$

Proof In what follows, we will use C to denote a generic positive constant which may vary from line to line (unless otherwise stated).

The condition $\mathrm{supp}(\widetilde{f}_0, \widetilde{f}_1) \subset\subset \mathbb{B}_1$ implies $\mathrm{supp}(\partial_t^l u(0,\cdot)) \subset\subset \mathbb{B}_1$ for any $l \geq 0$. Then even if we apply ∂_t^l $(0 \leq l \leq s-1)$ to system (2.1), the compatible condition still holds.

Now we prove the local existence of solution to system (2.1) by energy estimates and the standard contraction mapping theorem.

For any $v \in D_M^T$, where

$$\begin{aligned} D_M^T = \Big\{ v : [0,T] \times \mathbb{B}_1 \to \mathbb{R} \,\Big|\, & v(0,\cdot) = \widetilde{f}_0,\ v_t(0,\cdot) = \widetilde{f}_1, \\ & \sup_{0 \leq t \leq T} \Big(\|\partial_t^s v(t,\cdot)\|^2_{L^2(\mathbb{B}_1)} + \sum_{l=0}^{s-1} \|\partial_t^l v(t,\cdot)\|^2_{H^1(\mathbb{B}_1)} \Big) \leq M^2 \Big\}, \end{aligned}$$

we define a map $\Pi : v \longrightarrow u$, where u satisfies the mixed initial-boundary value problem:

$$\begin{cases} \Box u + \lambda v^3 = 0, & t \geq 0,\ x \in \mathbb{B}_1, \\ t = 0 : u = \widetilde{f}_0,\ u_t = \widetilde{f}_1, & x \in \mathbb{B}_1, \\ \dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial r} + u = 0, & t \geq 0,\ x \in \partial\mathbb{B}_1. \end{cases} \tag{2.3}$$

Applying ∂_t^l $(0 \le l \le s-1)$ to system (2.3) and taking the L^2 inner product of the resulting equation with $\partial_t^{l+1}u$, we obtain the energy estimate:

$$\begin{aligned}&\sum_{l=0}^{s-1}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\partial_t^{l+1}u\|_{L^2(\mathbb{B}_1)}^2+\|\nabla\partial_t^l u\|_{L^2(\mathbb{B}_1)}^2+\|\partial_t^l u\|_{L^2(\partial\mathbb{B}_1)}^2)\\&+\sum_{l=0}^{s-1}\|\partial_t^{l+1}u\|_{L^2(\partial\mathbb{B}_1)}^2+\lambda\sum_{l=0}^{s-1}\int_{\mathbb{B}_1}\partial_t^l(v^3)\partial_t^{l+1}u\,\mathrm{d}x=0,\end{aligned}\tag{2.4}$$

where we used the boundary conditions $\partial_t^{l+1}u+\partial_r\partial_t^l u+\partial_t^l u=0$ for all $0\le l\le s-1$. Consequently, we get

$$\sum_{l=0}^{s-1}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\partial_t^{l+1}u\|_{L^2(\mathbb{B}_1)}^2+\|\nabla\partial_t^l u\|_{L^2(\mathbb{B}_1)}^2+\|\partial_t^l u\|_{L^2(\partial\mathbb{B}_1)}^2)\le\lambda\sum_{l=0}^{s-1}\|\partial_t^l(v^3)\partial_t^{l+1}u\|_{L^1(\mathbb{B}_1)}.\tag{2.5}$$

By Hölder inequality, for any $l_1+l_2+l_3=l$, we have

$$\|\partial_t^{l_1}v\partial_t^{l_2}v\partial_t^{l_3}v\partial_t^{l+1}u\|_{L^1(\mathbb{B}_1)}\le\|\partial_t^{l_1}v\|_{L^6(\mathbb{B}_1)}\|\partial_t^{l_2}v\|_{L^6(\mathbb{B}_1)}\|\partial_t^{l_3}v\|_{L^6(\mathbb{B}_1)}\|\partial_t^{l+1}u\|_{L^2(\mathbb{B}_1)}.$$

By Sobolev embedding theorem that $H^1(\mathbb{B}_1)\hookrightarrow L^6(\mathbb{B}_1)$ in $\mathbb{R}^3$ and the inequality (2.5), we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|\partial_t^s u\|_{L^2(\mathbb{B}_1)}^2+\sum_{l=1}^{s-1}\|\partial_t^l u\|_{H^1(\mathbb{B}_1)}^2+\|\nabla u\|_{L^2(\mathbb{B}_1)}^2+\sum_{l=0}^{s-1}\|\partial_t^l u\|_{L^2(\partial\mathbb{B}_1)}^2\Big)^{\frac{1}{2}}\le CM^3.\tag{2.6}$$

Let

$$M=4\Big(\|\partial_t^s u(0,\cdot)\|_{L^2(\mathbb{B}_1)}^2+\sum_{l=0}^{s-1}(\|\partial_t^l u(0,\cdot)\|_{H^1(\mathbb{B}_1)}^2+\|\partial_t^l u(0,\cdot)\|_{L^2(\partial\mathbb{B}_1)}^2)\Big)^{\frac{1}{2}}$$

and integrate the inequality (2.6) with respect to time t over $[0,t_0]$. Hence there exists a positive constant T_1^* depending only on M such that for any $0\le t_0\le T_1^*$,

$$\Big(\|\partial_t^s u(t_0,\cdot)\|_{L^2(\mathbb{B}_1)}^2+\sum_{l=1}^{s-1}\|\partial_t^l u(t_0,\cdot)\|_{H^1(\mathbb{B}_1)}^2+\|\nabla u(t_0,\cdot)\|_{L^2(\mathbb{B}_1)}^2\Big)^{\frac{1}{2}}\le\frac{M}{2}.\tag{2.7}$$

From the equality $u(t_0,\cdot)=\int_0^{t_0}u_t(t,\cdot)\mathrm{d}t-u(0,\cdot)$ for any $t_0\ge 0$, we know that there exists a constant $T_2^*<T_1^*$ such that $\|u(t_0,\cdot)\|_{L^2(\mathbb{B}_1)}\le\frac{M}{2}$ for any $0\le t_0\le T_2^*$, which implies

$$\sup_{0\le t\le T_2^*}\Big(\|\partial_t^s u(t,\cdot)\|_{L^2(\mathbb{B}_1)}^2+\sum_{l=0}^{s-1}\|\partial_t^l u(t,\cdot)\|_{H^1(\mathbb{B}_1)}^2\Big)\le M^2$$

by the inequality (2.7). Therefore, the map Π is $D_M^{T_2^*}$ to $D_M^{T_2^*}$.

According to system (2.3), we have $\partial_t^{l+1}u=-\lambda\partial_t^{l-1}(v^3)+\Delta\partial_t^{l-1}u$ for all $1\le l\le s-1$. Then M is controlled by a constant depending only on $\|\widetilde{f}_0\|_{H^s(\mathbb{B}_1)}$ and $\|\widetilde{f}_1\|_{H^{s-1}(\mathbb{B}_1)}$.

For any $v_1,v_2\in D_M^{T_2^*}$, define $u_1=\Pi v_1$ and $u_2=\Pi v_2$. Similarly, there exists a positive constant $T^*\le T_2^*$ depending only on $\|\widetilde{f}_0\|_{H^s(\mathbb{B}_1)}$ and $\|\widetilde{f}_1\|_{H^{s-1}(\mathbb{B}_1)}$ such that Π is a strict contraction from $D_M^{T^*}$ to $D_M^{T^*}$. By the standard contraction mapping theorem, there exists a unique fixed point $u\in D_M^{T^*}$ such that $\Pi u=u$.

Therefore u solves system (2.1) and the inequality (2.2) holds.

Remark 2.1 The local solution in Lemma 2.1 belongs to $\bigcap_{j=0}^{s} C^j([0,T^*], H^{s-j}(\mathbb{B}_1))$.

To conclude Remark 2.1, we recall the following elliptic estimates involving the boundary condition of Neumann type (see [11]).

Lemma 2.2 *Suppose that Ω is a bounded domain with the smooth boundary $\partial\Omega$. Consider the Neumann system:*

$$\begin{cases} -\Delta h = f, & in\ \Omega, \\ \dfrac{\partial h}{\partial n} = g, & on\ \partial\Omega. \end{cases} \tag{2.8}$$

For any given $f \in H^k(\Omega)$ and $g \in H^{k+\frac{1}{2}}(\partial\Omega)$ for $k = 0, 1, \cdots$, any solution h to system (2.8) *satisfies*

$$\|h\|^2_{H^{k+2}(\Omega)} \le C_k(\|f\|^2_{H^k(\Omega)} + \|g\|^2_{H^{k+\frac{1}{2}}(\partial\Omega)} + \|h\|^2_{L^2(\Omega)}). \tag{2.9}$$

Lemma 2.3 *Suppose that u solves the mixed initial-boundary value problem* (2.1)*. We have*

$$\sum_{j=0}^{s} \|\partial_t^j u\|^2_{H^{s-j}(\mathbb{B}_1)} \le C_1\Big(\|\partial_t^s u\|^2_{L^2(\mathbb{B}_1)} + \sum_{l=0}^{s-1} \|\partial_t^l u\|^2_{H^1(\mathbb{B}_1)}\Big). \tag{2.10}$$

Here $C_1(\,\cdot\,)$ is a function in the form $C_1(z) = \sum_{j=1}^{k_s} c_{1j} z^j$, where k_s is an integer depending only on s and c_{1j} is a constant depending only on the integer j.

Proof First we regard system (2.1) as the Neumann boundary problem of elliptic equations:

$$\begin{cases} \triangle u = \partial_{tt} u + \lambda u^3, & x \in \mathbb{B}_1, \\ \dfrac{\partial u}{\partial r} = -\dfrac{\partial u}{\partial t} - u, & x \in \partial\mathbb{B}_1. \end{cases} \tag{2.11}$$

Applying ∂_t^l $(0 \le l \le s-2)$ to system (2.11) and using the inequality (2.9), we conclude

$$\|\partial_t^l u\|^2_{H^2(\mathbb{B}_1)} \le C\|\partial_t^{l+2} u + \lambda\partial_t^l(u^3)\|^2_{L^2(\mathbb{B}_1)} + C\|\partial_t^{l+1} u + \partial_t^l u\|^2_{H^{\frac{1}{2}}(\partial\mathbb{B}_1)} + C\|\partial_t^l u\|^2_{L^2(\mathbb{B}_1)}. \tag{2.12}$$

By Hölder inequality and Sobolev embedding theorem, for any $l_1 + l_2 + l_3 = l$, we obtain

$$\begin{aligned} \|\partial_t^{l_1} u \partial_t^{l_2} u \partial_t^{l_3} u\|^2_{L^2(\mathbb{B}_1)} &\le \|\partial_t^{l_1} u\|^2_{L^6(\mathbb{B}_1)} \|\partial_t^{l_2} u\|^2_{L^6(\mathbb{B}_1)} \|\partial_t^{l_3} u\|^2_{L^6(\mathbb{B}_1)} \\ &\le C\Big(\sum_{k\le l} \|\partial_t^k u\|^2_{H^1(\mathbb{B}_1)}\Big)^3. \end{aligned} \tag{2.13}$$

By the trace theorem that $\|\phi\|_{H^{\frac{1}{2}}(\partial\mathbb{B}_1)} \le C\|\phi\|_{H^1(\mathbb{B}_1)}$ for any ϕ with the right part bounded and the inequalities (2.12)–(2.13), we deduce

$$\|\partial_t^l u\|^2_{H^2(\mathbb{B}_1)} \le C\Big(\|\partial_t^s u\|^2_{L^2(\mathbb{B}_1)} + \sum_{l=0}^{s-1} \|\partial_t^l u\|^2_{H^1(\mathbb{B}_1)}\Big) + C\Big(\|\partial_t^s u\|^2_{L^2(\mathbb{B}_1)} + \sum_{l=0}^{s-1} \|\partial_t^l u\|^2_{H^1(\mathbb{B}_1)}\Big)^3 \tag{2.14}$$

for all $0 \le l \le s-2$.

Using (2.14), we similarly prove that $\|\partial_t^l u\|^2_{H^3(\mathbb{B}_1)}$ is controlled by the right part of the inequality (2.10) for all $0 \le l \le s-3$. Then we can get the inequality (2.10) by induction.

Obviously the inequality (2.10) shows that Remark 2.1 holds.

Next we establish the global existence for system (2.1).

Theorem 2.1 *Assume $s \geq 2$, $\widetilde{f}_0 \in H^s(\mathbb{B}_1)$, $\widetilde{f}_1 \in H^{s-1}(\mathbb{B}_1)$ and $\operatorname{supp}(\widetilde{f}_0, \widetilde{f}_1) \subset\subset \mathbb{B}_1$. Then the mixed initial-boundary value problem (2.1) admits a global solution $u(t,x)$. And for any constant $\alpha \in (0, \frac{1}{2})$, there exists a constant $C_2 > 0$ such that*

$$\|\partial_t^s u(t,\cdot)\|^2_{L^2(\mathbb{B}_1)} + \sum_{l=0}^{s-1} \|\partial_t^l u(t,\cdot)\|^2_{H^1(\mathbb{B}_1)} \leq C_2\, \mathrm{e}^{-\alpha t}, \quad \forall\, t \geq 0, \tag{2.15}$$

where C_2 depends only on λ, α, $\|\widetilde{f}_0\|_{H^s(\mathbb{B}_1)}$ and $\|\widetilde{f}_1\|_{H^{s-1}(\mathbb{B}_1)}$.

Proof According to the local existence for system (2.1), it suffices to establish the uniform a priori estimates on the H^s norm of $u(t,\cdot)$ and H^{s-1} norm of $u_t(t,\cdot)$. The combination of the inequalities (2.10) and (2.15) shows that there exist a constant $\gamma > 0$ and a positive constant C_3 depending only on λ, γ, $\|\widetilde{f}_0\|_{H^s(\mathbb{B}_1)}$ and $\|\widetilde{f}_1\|_{H^{s-1}(\mathbb{B}_1)}$ such that

$$\sum_{j=0}^{s} \|\partial_t^j u(t,\cdot)\|^2_{H^{s-j}(\mathbb{B}_1)} \leq C_3\, \mathrm{e}^{-\gamma t}, \tag{2.16}$$

which implies that $\|u(t,\cdot)\|_{H^s(\mathbb{B}_1)}$ and $\|u_t(t,\cdot)\|_{H^{s-1}(\mathbb{B}_1)}$ are uniformly bounded on the interval $[0,+\infty)$. Then we extend the local solution to a global one assuming (2.15) holds.

Now let us prove (2.15) by induction. The proof is close to [5] and [13] but a little different. However, for reader's convenience, we give the detail. The proof is divided into two steps.

Step 1 First let us consider $\|u_t(t,\cdot)\|^2_{L^2(\mathbb{B}_1)} + \|u(t,\cdot)\|^2_{H^1(\mathbb{B}_1)}$. By taking the L^2 inner product of the equation in system (2.1) with u_t and using the boundary condition, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|u_t\|^2_{L^2(\mathbb{B}_1)} + \|\nabla u\|^2_{L^2(\mathbb{B}_1)} + \|u\|^2_{L^2(\partial\mathbb{B}_1)}) + \|u_t\|^2_{L^2(\partial\mathbb{B}_1)} + \frac{\lambda}{4}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^4_{L^4(\mathbb{B}_1)} = 0. \tag{2.17}$$

We do the energy estimate of Morawetz type by taking the L^2 inner product of the equation in system (2.1) with $x\cdot\nabla u$:

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{B}_1}(x\cdot\nabla u)u_t\,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{B}_1} x\cdot\nabla(u_t^2)\mathrm{d}x + \frac{\lambda}{4}\int_{\mathbb{B}_1} x\cdot\nabla(u^4)\mathrm{d}x \\ &= \int_{\mathbb{B}_1}\nabla_k(\nabla_k u x\cdot\nabla u)\mathrm{d}x - \int_{\mathbb{B}_1}|\nabla u|^2\mathrm{d}x - \frac{1}{2}\int_{\mathbb{B}_1} x\cdot\nabla|\nabla u|^2\mathrm{d}x. \end{aligned} \tag{2.18}$$

By integration by parts and using the boundary condition, we have

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{B}_1}(x\cdot\nabla u)u_t\,\mathrm{d}x + \frac{1}{2}(\|u_t\|^2_{L^2(\mathbb{B}_1)} + \|\nabla u\|^2_{L^2(\mathbb{B}_1)}) - \frac{1}{2}\|u_t\|^2_{L^2(\partial\mathbb{B}_1)} \\ &+ \frac{\lambda}{4}\|u\|^4_{L^4(\partial\mathbb{B}_1)} - \frac{3\lambda}{4}\|u\|^4_{L^4(\mathbb{B}_1)} + \frac{1}{2}\|\nabla u\|^2_{L^2(\partial\mathbb{B}_1)} \\ &= -\int_{\partial\mathbb{B}_1}(u_t+u)\frac{\partial u}{\partial r}\mathrm{d}\Gamma + (\|\nabla u\|^2_{L^2(\mathbb{B}_1)} - \|u_t\|^2_{L^2(\mathbb{B}_1)}). \end{aligned} \tag{2.19}$$

By taking L^2 inner product of the equation in system (2.1) with u, we get

$$\begin{aligned} \int_{\mathbb{B}_1}(u_t^2 - |\nabla u|^2)\mathrm{d}x &= \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{B}_1} uu_t\,\mathrm{d}x - \int_{\mathbb{B}_1}(u(\triangle u - \lambda u^3) + |\nabla u|^2)\mathrm{d}x \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{B}_1} uu_t\,\mathrm{d}x - \int_{\partial\mathbb{B}_1} uu_r\,\mathrm{d}\Gamma + \lambda\|u\|^4_{L^4(\mathbb{B}_1)} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{B}_1} uu_t\,\mathrm{d}x + \lambda\|u\|^4_{L^4(\mathbb{B}_1)} + \int_{\partial\mathbb{B}_1} u(u+u_t)\mathrm{d}\Gamma, \end{aligned} \tag{2.20}$$

where we used the boundary condition.

Let

$$E(t) = \frac{1}{2}\|u_t\|^2_{L^2(\mathbb{B}_1)} + \frac{1}{2}\|\nabla u\|^2_{L^2(\mathbb{B}_1)} + \frac{1}{2}\|u\|^2_{L^2(\partial\mathbb{B}_1)} + \frac{\lambda}{4}\|u\|^4_{L^4(\mathbb{B}_1)}. \tag{2.21}$$

Then the equality (2.17) implies that for any $0 \le S \le T$,

$$E(S) - E(T) = \int_S^T \|u_t(t,\cdot)\|^2_{L^2(\partial\mathbb{B}_1)}\mathrm{d}t. \tag{2.22}$$

The combination of the equalities (2.19) and (2.20) shows

$$\begin{aligned}
&\frac{1}{2}(\|u_t\|^2_{L^2(\mathbb{B}_1)} + \|\nabla u\|^2_{L^2(\mathbb{B}_1)}) + \frac{\lambda}{4}\|u\|^4_{L^4(\mathbb{B}_1)} \\
&\le -\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{B}_1} u_t(u + x\cdot\nabla u)\mathrm{d}x + \frac{1}{2}\|u_t\|^2_{L^2(\partial\mathbb{B}_1)} - \frac{1}{2}\|\nabla u\|^2_{L^2(\partial\mathbb{B}_1)} \\
&\quad - \int_{\partial\mathbb{B}_1} u(u+u_t)\mathrm{d}\Gamma - \int_{\partial\mathbb{B}_1}(u_t+u)\frac{\partial u}{\partial r}\mathrm{d}\Gamma.
\end{aligned} \tag{2.23}$$

From $|(u_t+u)\frac{\partial u}{\partial r}| \le \frac{1}{2}((u_t+u)^2 + u_r^2)$ and the inequality (2.23), we have

$$E(t) \le -\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{B}_1} u_t(u + x\cdot\nabla u)\mathrm{d}x + \|u_t\|^2_{L^2(\partial\mathbb{B}_1)}. \tag{2.24}$$

On the other hand, a straightforward calculation shows

$$\begin{aligned}
\Big|\int_{\mathbb{B}_1}(x\cdot\nabla u + u)u_t\,\mathrm{d}x\Big| &\le \frac{1}{2}\int_{\mathbb{B}_1}(x\cdot\nabla u+u)^2\mathrm{d}x + \frac{1}{2}\|u_t\|^2_{L^2(\mathbb{B}_1)} \\
&\le \frac{1}{2}\|u_t\|^2_{L^2(\mathbb{B}_1)} + \frac{1}{2}\|\nabla u\|^2_{L^2(\mathbb{B}_1)} + \frac{1}{2}\|u\|^2_{L^2(\mathbb{B}_1)} + \frac{1}{2}\int_{\mathbb{B}_1} x\cdot\nabla(u^2)\mathrm{d}x \\
&= \frac{1}{2}\|u_t\|^2_{L^2(\mathbb{B}_1)} + \frac{1}{2}\|\nabla u\|^2_{L^2(\mathbb{B}_1)} - \|u\|^2_{L^2(\mathbb{B}_1)} + \frac{1}{2}\|u\|^2_{L^2(\partial\mathbb{B}_1)},
\end{aligned} \tag{2.25}$$

which implies

$$\|u\|^2_{L^2(\mathbb{B}_1)} \le \frac{1}{2}\|u_t\|^2_{L^2(\mathbb{B}_1)} + \frac{1}{2}\|\nabla u\|^2_{L^2(\mathbb{B}_1)} + \frac{1}{2}\|u\|^2_{L^2(\partial\mathbb{B}_1)}. \tag{2.26}$$

Integrating the inequality (2.24) from S to T and by the inequality (2.25), we have

$$\int_S^T E(t)\mathrm{d}t \le (E(S) + E(T)) + \int_S^T \|u_t(t,\cdot)\|^2_{L^2(\partial\mathbb{B}_1)}\mathrm{d}t, \tag{2.27}$$

which implies

$$\int_S^T E(t)\mathrm{d}t \le 2E(S), \tag{2.28}$$

with the help of the equality (2.22).

For any $S \ge 0$, let $T \to \infty$. Then we have

$$\int_S^\infty E(t)\mathrm{d}t \le 2E(S). \tag{2.29}$$

Let $M(S) = \mathrm{e}^{\frac{S}{2}}\int_S^\infty E(t)\mathrm{d}t$. So the inequality (2.29) implies $M'(S) \le 0$ for any $S \ge 0$. Hence

$$\mathrm{e}^{\frac{S}{2}}\int_S^\infty E(t)\mathrm{d}t \le \int_0^\infty E(t)\mathrm{d}t \le 2E(0), \tag{2.30}$$

which implies that for any $S \geq 0$,

$$E(S+1) \leq \int_S^{S+1} E(t)\mathrm{d}t \leq \int_S^{\infty} E(t)\mathrm{d}t \leq 2E(0)\,\mathrm{e}^{-\frac{S}{2}}. \tag{2.31}$$

So for any $S \geq 1$, we have

$$E(S) \leq 2E(0)\,\mathrm{e}^{-\frac{S-1}{2}}. \tag{2.32}$$

By the inequality (2.26), for any $t \geq 1$ we have

$$\|u(t,\cdot)\|^2_{L^2(\mathbb{B}_1)} \leq E(t) \leq 2E(0)\,\mathrm{e}^{-\frac{t-1}{2}}, \tag{2.33}$$

which implies that $\|u_t(t,\cdot)\|^2_{L^2(\mathbb{B}_1)} + \|u(t,\cdot)\|^2_{H^1(\mathbb{B}_1)} \leq C\,\mathrm{e}^{-\frac{t}{2}}$ holds for any $t \geq 1$. Therefore, there exists a positive constant C_1 depending only on $\|\widetilde{f}_0\|_{H^1(\mathbb{B}_1)}$ and $\|\widetilde{f}_1\|_{L^2(\mathbb{B}_1)}$ such that for any $t \geq 0$,

$$\|u_t(t,\cdot)\|^2_{L^2(\mathbb{B}_1)} + \|u(t,\cdot)\|^2_{H^1(\mathbb{B}_1)} \leq C_1\,\mathrm{e}^{-\frac{t}{2}}. \tag{2.34}$$

Step 2 Assume that for any $2 \leq k \leq s$, there exists a constant $\alpha_{k-1} > 0$ and a positive constant C_{k-1} depending only on λ, α_{k-1}, $\|\widetilde{f}_0\|_{H^{k-1}(\mathbb{B}_1)}$ and $\|\widetilde{f}_1\|_{H^{k-2}(\mathbb{B}_1)}$ such that for any $t \geq 0$,

$$\|\partial_t^{k-1}u(t,\cdot)\|^2_{L^2(\mathbb{B}_1)} + \sum_{l=0}^{k-2}\|\partial_t^l u(t,\cdot)\|^2_{H^1(\mathbb{B}_1)} \leq C_{k-1}\,\mathrm{e}^{-\alpha_{k-1}t}. \tag{2.35}$$

Applying ∂_t^l $(0 \leq l \leq k-1)$ to system (2.1) and taking the L^2 inner product of the resulting equations with $\partial_t^{l+1}u$, we obtain the higher-order energy estimates:

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\partial_t^{l+1}u\|^2_{L^2(\mathbb{B}_1)} + \|\nabla\partial_t^l u\|^2_{L^2(\mathbb{B}_1)} + \|\partial_t^l u\|^2_{L^2(\partial\mathbb{B}_1)})\\ &+ \|\partial_t^{l+1}u\|^2_{L^2(\partial\mathbb{B}_1)} + \lambda\int_{\mathbb{B}_1}\partial_t^l(u^3)\partial_t^{l+1}u\,\mathrm{d}x = 0,\end{aligned} \tag{2.36}$$

where we used the boundary conditions $\partial_t^{l+1}u + \partial_t^l u + \partial_r\partial_t^l u = 0$, $x \in \partial\mathbb{B}_1$.

Applying ∂_t^l $(0 \leq l \leq k-1)$ to system (2.1) and taking the L^2 inner product of the resulting equations with $x\cdot\nabla\partial_t^l u$, we get higher-order Morawetz's energy estimates:

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{B}_1}(x\cdot\nabla\partial_t^l u)\partial_t^{l+1}u\,\mathrm{d}x - \frac{1}{2}\|\partial_t^{l+1}u\|^2_{L^2(\partial\mathbb{B}_1)} + \frac{3}{2}\|\partial_t^{l+1}u\|^2_{L^2(\mathbb{B}_1)}\\ &+ \frac{1}{2}\|\nabla\partial_t^l u\|^2_{L^2(\partial\mathbb{B}_1)} + \lambda\int_{\mathbb{B}_1}\partial_t^l(u^3)x\cdot\nabla\partial_t^l u\,\mathrm{d}x\\ =& -\int_{\partial\mathbb{B}_1}(\partial_t^{l+1}u + \partial_t^l u)\frac{\partial}{\partial r}\partial_t^l u\,\mathrm{d}\Gamma + \frac{1}{2}\|\nabla\partial_t^l u\|^2_{L^2(\mathbb{B}_1)},\end{aligned} \tag{2.37}$$

where we used the boundary conditions $\partial_t^{l+1}u + \partial_t^l u + \partial_r\partial_t^l u = 0$, $x \in \partial\mathbb{B}_1$.

Applying ∂_t^l $(0 \leq l \leq k-1)$ to system (2.1) and taking the L^2 inner product of the resulting equations with $\partial_t^l u$, we find the following estimates:

$$\begin{aligned}\|\nabla\partial_t^l u\|^2_{L^2(\mathbb{B}_1)} - \|\partial_t^{l+1}u\|^2_{L^2(\mathbb{B}_1)} =& -\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{B}_1}\partial_t^l u\partial_t^{l+1}u\,\mathrm{d}x - \int_{\partial\mathbb{B}_1}\partial_t^l u(\partial_t^l u + \partial_t^{l+1}u)\mathrm{d}\Gamma\\ &- \lambda\int_{\mathbb{B}_1}\partial_t^l(u^3)\partial_t^l u\,\mathrm{d}x,\end{aligned} \tag{2.38}$$

where we used the boundary conditions $\partial_t^{l+1}u + \partial_t^l u + \partial_r \partial_t^l u = 0$, $x \in \partial\mathbb{B}_1$.

The combination of the inequalities (2.37) and (2.38) shows that

$$\begin{aligned}
&\frac{1}{2}(\|\nabla\partial_t^l u\|^2_{L^2(\mathbb{B}_1)} + \|\partial_t^{l+1}u\|^2_{L^2(\mathbb{B}_1)}) \\
=& -\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{B}_1}(\partial_t^l u + x\cdot\nabla\partial_t^l u)\partial_t^{l+1}u\,\mathrm{d}x - \lambda\int_{\mathbb{B}_1}\partial_t^l(u^3)(\partial_t^l u + x\cdot\nabla\partial_t^l u)\mathrm{d}x \\
&-\int_{\partial\mathbb{B}_1}(\partial_t^{l+1}u + \partial_t^l u)\frac{\partial}{\partial r}\partial_t^l u\,\mathrm{d}\Gamma - \int_{\partial\mathbb{B}_1}\partial_t^l u(\partial_t^l u + \partial_t^{l+1}u)\mathrm{d}\Gamma \\
&+\frac{1}{2}\|\partial_t^{l+1}u\|^2_{L^2(\partial\mathbb{B}_1)} - \frac{1}{2}\|\nabla\partial_t^l u\|^2_{L^2(\partial\mathbb{B}_1)},
\end{aligned} \tag{2.39}$$

which implies

$$\begin{aligned}
&\sum_{l=0}^{k-1}\frac{1}{2}(\|\nabla\partial_t^l u\|^2_{L^2(\mathbb{B}_1)} + \|\partial_t^{l+1}u\|^2_{L^2(\mathbb{B}_1)} + \|\partial_t^l u\|^2_{L^2(\partial\mathbb{B}_1)}) \\
\le& -\sum_{l=0}^{k-1}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{B}_1}(\partial_t^l u + x\cdot\nabla\partial_t^l u)\partial_t^{l+1}u\,\mathrm{d}x \\
&-\lambda\sum_{l=0}^{k-1}\int_{\mathbb{B}_1}\partial_t^l(u^3)(\partial_t^l u + x\cdot\nabla\partial_t^l u)\mathrm{d}x + \sum_{l=0}^{k-1}\|\partial_t^{l+1}u\|^2_{L^2(\partial\mathbb{B}_1)} \\
=& -\sum_{l=0}^{k-1}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{B}_1}(\partial_t^l u + x\cdot\nabla\partial_t^l u)\partial_t^{l+1}u\,\mathrm{d}x \\
&-\frac{1}{2}\sum_{l=0}^{k-1}\frac{\mathrm{d}}{\mathrm{d}t}(\|\partial_t^{l+1}u\|^2_{L^2(\mathbb{B}_1)} + \|\nabla\partial_t^l u\|^2_{L^2(\mathbb{B}_1)} + \|\partial_t^l u\|^2_{L^2(\partial\mathbb{B}_1)}) \\
&-\lambda\sum_{l=0}^{k-1}\int_{\mathbb{B}_1}\partial_t^l(u^3)(\partial_t^l u + x\cdot\nabla\partial_t^l u + \partial_t^{l+1}u)\mathrm{d}x
\end{aligned} \tag{2.40}$$

by using the same argument as the case $k = 1$ and with the help of the inequality (2.36).

Let

$$A_k(t) = \frac{1}{2}\sum_{l=0}^{k-1}(\|\partial_t^{l+1}u\|^2_{L^2(\mathbb{B}_1)} + \|\nabla\partial_t^l u\|^2_{L^2(\mathbb{B}_1)} + \|\partial_t^l u\|^2_{L^2(\partial\mathbb{B}_1)}), \tag{2.41}$$

$$B_k(t) = \|\partial_t^k u\|^2_{L^2(\mathbb{B}_1)} + \sum_{l=0}^{k-1}\|\partial_t^l u\|^2_{H^1(\mathbb{B}_1)}. \tag{2.42}$$

By the trace theorem, we know that there exists a constant $C > 0$ such that $A_k(t) \le CB_k(t)$ for any $t \ge 0$. The inequality (2.26) shows that there exists a constant $C > 0$ such that $B_k(t) \le CA_k(t)$ for any $t \ge 0$.

By Hölder inequality and Sobolev embedding theorem, for $0 \le l \le k-1$, we deduce

$$\begin{aligned}
\int_{\mathbb{B}_1}|\partial_t^l(u^3)\partial_t^{l+1}u|\mathrm{d}x \le& \sum_{l_1+l_2+l_3=l}\|\partial_t^{l+1}u\|_{L^2(\mathbb{B}_1)}\|\partial_t^{l_1}u\|_{L^6(\mathbb{B}_1)}\|\partial_t^{l_2}u\|_{L^6(\mathbb{B}_1)}\|\partial_t^{l_3}u\|_{L^6(\mathbb{B}_1)} \\
\le& C\|\partial_t^{l+1}u\|_{L^2(\mathbb{B}_1)}\sum_{j=0}^{l-1}\|\partial_t^j u\|^2_{H^1(\mathbb{B}_1)}\sum_{l_3=0}^{l}\|\partial_t^{l_3}u\|_{H^1(\mathbb{B}_1)} \\
\le& C\,\mathrm{e}^{-\alpha_{k-1}t}A_k(t).
\end{aligned} \tag{2.43}$$

The same argument is used to deal with the rest of the last term of the inequality (2.40).

For $0 \le l \le k-1$, the inequalities (2.25) and (2.26) still hold if u is substituted by $\partial_t^l u$. Then the equality (2.40) implies that for any $0 \le S \le T$,

$$\int_S^T A_k(t)\mathrm{d}t \le 2A_k(S) + \int_S^T C\,\mathrm{e}^{-\alpha_{k-1}t} A_k(t)\mathrm{d}t. \tag{2.44}$$

In view of the equality (2.36) and (2.43), we know that for any $t \ge 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t} A_k(t) \le C\,\mathrm{e}^{-\alpha_{k-1}t} A_k(t), \tag{2.45}$$

which implies that there exists a constant $M_1 > 0$ depending only on λ, α_{k-1}, $\|\widetilde{f}_0\|_{H^k(\mathbb{B}_1)}$ and $\|\widetilde{f}_1\|_{H^{k-1}(\mathbb{B}_1)}$ such that the inequality $A_k(t) \le M_1$ holds for all $t \ge 0$.

Let $T \to \infty$. By the inequality (2.44), we have

$$\int_S^\infty A_k(t)\mathrm{d}t \le 2A_k(S) + C\,\mathrm{e}^{-\alpha_{k-1}S}. \tag{2.46}$$

Obviously, there exists a positive constant α_k such that $\alpha_k < \alpha_{k-1}$ and $\frac{1}{\alpha_k} > 2$. Therefore, the inequality (2.46) implies

$$\int_S^\infty A_k(t)\mathrm{d}t \le \frac{1}{\alpha_k} A_k(S) + C\,\mathrm{e}^{-\alpha_{k-1}S}. \tag{2.47}$$

Let $M_k(S) = \mathrm{e}^{\alpha_k S} \int_S^\infty A_k(t)\mathrm{d}t$. By the inequality (2.47), we have $M_k'(S) \le C\alpha_k\,\mathrm{e}^{(\alpha_k - \alpha_{k-1})S}$, which implies

$$\int_S^{S+1} A_k(t)\mathrm{d}t \le \int_S^\infty A_k(t)\mathrm{d}t \le C\,\mathrm{e}^{-\alpha_k S} \tag{2.48}$$

by using the inequality (2.47) when $S = 0$.

By the inequality (2.45), we have $\frac{\mathrm{d}}{\mathrm{d}t} A_k(t) \le C\,\mathrm{e}^{-\alpha_{k-1}t}$, which implies that for any $0 \le \beta \le 1$,

$$A_k(S+1) - A_k(S+\beta) \le C\,\mathrm{e}^{-\alpha_{k-1}S}. \tag{2.49}$$

By the inequalities (2.48) and (2.49), we have $A_k(S+1) \le C\,\mathrm{e}^{-\alpha_k S}$ for any $S \ge 0$. Then $A_k(t) \le C\,\mathrm{e}^{-\alpha_k t}$ for any $t \ge 1$, which implies $B_k(t) \le C\,\mathrm{e}^{-\alpha_k t}$ for any $t \ge 1$.

So there exists a positive constant C_k only depending on λ, α_k, $\|\widetilde{f}_0\|_{H^k(\mathbb{B}_1)}$ and $\|\widetilde{f}_1\|_{H^{k-1}(\mathbb{B}_1)}$ such that

$$\|\partial_t^k u\|_{L^2(\mathbb{B}_1)}^2 + \sum_{l=0}^{k-1} \|\partial_t^l u\|_{H^1(\mathbb{B}_1)}^2 \le C_k\,\mathrm{e}^{-\alpha_k t}$$

for any $t \ge 0$.

Therefore, we prove the inequality (2.15) inductively.

Now let us take a look at the inverted initial-boundary value problem:

$$\begin{cases} \Box u + \lambda u^3 = 0, & 0 \le t \le T,\ x \in \mathbb{B}_1, \\ t = T : u = \widetilde{g}_0,\ \partial_t u = \widetilde{g}_1, & x \in \mathbb{B}_1, \\ -\dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial r} + u = 0, & 0 \le t \le T,\ x \in \partial\mathbb{B}_1. \end{cases} \tag{2.50}$$

We make a change of variable $t \to T - t$ and then system (2.50) converts into the problem (2.1). According to Theorem 2.1, we get the following theorem for system (2.50).

Theorem 2.2 *Let $s \geq 2$, $\widetilde{g}_0 \in H^s(\mathbb{B}_1)$, $\widetilde{g}_1 \in H^{s-1}(\mathbb{B}_1)$ and $\operatorname{supp}(\widetilde{g}_0, \widetilde{g}_1) \subset\subset \mathbb{B}_1$. Then the problem (2.50) admits a unique solution $u(t,x)$ on the domain $[0,T] \times \mathbb{B}_1$ and there exists a positive constant C_4 depending only on λ, α $\|\widetilde{g}_0\|_{H^s(\mathbb{B}_1)}$ and $\|\widetilde{g}_1\|_{H^{s-1}(\mathbb{B}_1)}$ such that*

$$\|\partial_t^s u(t,\cdot)\|^2_{L^2(\mathbb{B}_1)} + \sum_{l=0}^{s-1} \|\partial_t^l u(t,\cdot)\|^2_{H^1(\mathbb{B}_1)} \leq C_4\, \mathrm{e}^{-\alpha(T-t)}, \quad \forall\, 0 \leq t \leq T. \tag{2.51}$$

3 Global Exact Boundary Controllability for the Cubic Semi-linear Wave Equation in the Ordinary Region

In this section, we will give the proof of Theorem 1.1, which is divided into several steps.

Step 1 By Lemma 2.3 and in view of Theorems 2.1 and 2.2, for any T_1 and T_2 satisfying $0 < T_1, T_2 < T$, system (2.1) admits a unique solution u_1 on the domain $[0,T_1] \times \mathbb{B}_1$ and there exists a unique solution u_2 to system (2.50) on the domain $[T-T_2, T] \times \mathbb{B}_1$. In addition, there exists a positive constant $\widetilde{C}_2$ depending on λ, α, $\|\widetilde{f}_0\|_{H^s(\mathbb{B}_1)}$ and $\|\widetilde{f}_1\|_{H^{s-1}(\mathbb{B}_1)}$ such that $\sum\limits_{j=0}^{s} \|\partial_t^j u_1(T_1,\cdot)\|^2_{H^{s-j}(\Omega)} \leq \widetilde{C}_2\, \mathrm{e}^{-\alpha T_1}$ and a positive constant $\widetilde{C}_4$ depending on λ, α, $\|\widetilde{g}_0\|_{H^s(\mathbb{B}_1)}$ and $\|\widetilde{g}_1\|_{H^{s-1}(\mathbb{B}_1)}$ such that $\sum\limits_{j=0}^{s} \|\partial_t^j u_2(T-T_2,\cdot)\|^2_{H^{s-j}(\Omega)} \leq \widetilde{C}_4\, \mathrm{e}^{-\alpha T_2}$. Take $T > T_1 + T_2$.

To best illustrate our proof, let $T_1 = T_2 = \frac{1}{4}T$. Then it suffices to study the exact boundary controllability problem

$$\begin{cases} \Box u + \lambda u^3 = 0, & \dfrac{1}{4}T \leq t \leq \dfrac{3}{4}T,\ x \in \Omega, \\ t = \dfrac{1}{4}T : u = u_1,\ \partial_t u = \partial_t u_1, & x \in \Omega, \\ t = \dfrac{3}{4}T : u = u_2,\ \partial_t u = \partial_t u_2, & x \in \Omega. \end{cases} \tag{3.1}$$

Define

$$D_\Lambda(\psi) = \sup_{\frac{1}{4}T \leq t \leq \frac{3}{4}T} \Big(\sum_{j=0}^{s} \|\partial_t^j \psi(t,\cdot)\|^2_{H^{s-j}(\Omega)} \Big)^{\frac{1}{2}} \tag{3.2}$$

and

$$\begin{aligned} \Lambda_\theta = \Big\{ \psi : \Big[\frac{1}{4}T, \frac{3}{4}T\Big] \times \Omega \to \mathbb{R} \,\Big|\, & \psi\Big(\frac{1}{4}T,\cdot\Big) = u_1\Big(\frac{1}{4}T,\cdot\Big),\ \psi_t\Big(\frac{1}{4}T,\cdot\Big) = u_{1t}\Big(\frac{1}{4}T,\cdot\Big), \\ & \psi\Big(\frac{3}{4}T,\cdot\Big) = u_2\Big(\frac{3}{4}T,\cdot\Big),\ \psi_t\Big(\frac{3}{4}T,\cdot\Big) = u_{2t}\Big(\frac{3}{4}T,\cdot\Big),\ D_\Lambda(\psi) \leq \theta \Big\}. \end{aligned} \tag{3.3}$$

For any $\phi \in \Lambda_\theta$, we define a map $\Pi : \phi \to v + w$, where v satisfies the mixed initial-boundary value problem

$$\begin{cases} \Box v + \lambda \phi^3 = 0, & \dfrac{1}{4}T \leq t \leq \dfrac{3}{4}T,\ x \in \Omega, \\ t = \dfrac{1}{4}T : v = 0,\ \partial_t v = 0, & x \in \Omega, \\ v = 0, & \dfrac{1}{4}T \leq t \leq \dfrac{3}{4}T,\ x \in \partial\Omega, \end{cases} \tag{3.4}$$

and w is defined via the exact boundary controllability problem

$$\begin{cases} \Box w = 0, & \frac{1}{4}T \le t \le \frac{3}{4}T,\ x \in \Omega, \\ t = \frac{1}{4}T : w = u_1,\ \partial_t w = u_{1t}, & x \in \Omega, \\ t = \frac{3}{4}T : w = u_2 - v,\ \partial_t w = u_{2t} - v_t, & x \in \Omega. \end{cases} \tag{3.5}$$

Step 2 As to system (3.4), we get the energy estimate

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|\partial_t^s v\|^2_{L^2(\Omega)} + \sum_{l=1}^{s-1}\|\partial_t^l v\|^2_{H^1(\Omega)} + \|\nabla v\|^2_{L^2(\Omega)}\Big) + \sum_{l=0}^{s-1}\lambda\int_\Omega \partial_t^l(\phi^3)\partial_t^{l+1}v\,\mathrm{d}x = 0. \tag{3.6}$$

There is no boundary term since $\partial_t^{l+1}v(t,x) \equiv 0$ for $l = 0, 1, \cdots, s-1$ while $x \in \partial\Omega$. Consequently, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|\partial_t^s v\|^2_{L^2(\Omega)} + \sum_{l=1}^{s-1}\|\partial_t^l v\|^2_{H^1(\Omega)} + \|\nabla v\|^2_{L^2(\Omega)}\Big) \le \lambda\int_\Omega \sum_{l=0}^{s-1}|\partial_t^l(\phi^3)\partial_t^{l+1}v|\mathrm{d}x. \tag{3.7}$$

By Hölder inequality and Sobolev embedding theorem, we deduce

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|\partial_t^s v\|^2_{L^2(\Omega)} + \sum_{l=1}^{s-1}\|\partial_t^l v\|^2_{H^1(\Omega)} + \|\nabla v\|^2_{L^2(\Omega)}\Big)^{\frac{1}{2}} \le C\theta^3. \tag{3.8}$$

For any $\frac{1}{4}T \le t_0 \le \frac{3}{4}T$, integrating the inequality (3.8) with respect to time t over $[\frac{1}{4}T, t_0]$, we arrive at

$$\begin{aligned} &\Big(\|\partial_t^s v(t_0,\cdot)\|^2_{L^2(\Omega)} + \sum_{l=1}^{s-1}\|\partial_t^l v(t_0,\cdot)\|^2_{H^1(\Omega)} + \|\nabla v(t_0,\cdot)\|^2_{L^2(\Omega)}\Big)^{\frac{1}{2}} \\ &\le \Big(\Big\|\partial_t^s v\Big(\frac{T}{4},\cdot\Big)\Big\|^2_{L^2(\Omega)} + \sum_{l=1}^{s-1}\Big\|\partial_t^l v\Big(\frac{T}{4},\cdot\Big)\Big\|^2_{H^1(\Omega)} + \Big\|\nabla v\Big(\frac{T}{4},\cdot\Big)\Big\|^2_{L^2(\Omega)}\Big)^{\frac{1}{2}} + C\theta^3\frac{T}{2}. \end{aligned} \tag{3.9}$$

System (3.4) shows

$$\Big\|\partial_t^s v\Big(\frac{T}{4},\cdot\Big)\Big\|^2_{L^2(\Omega)} + \sum_{l=1}^{s-1}\Big\|\partial_t^l v\Big(\frac{T}{4},\cdot\Big)\Big\|^2_{H^1(\Omega)} \le C\theta^6$$

by induction. Then for $\frac{1}{4}T \le t \le \frac{3}{4}T$, we have

$$\Big(\|\partial_t^s v(t,\cdot)\|^2_{L^2(\Omega)} + \sum_{l=1}^{s-1}\|\partial_t^l v(t,\cdot)\|^2_{H^1(\Omega)} + \|\nabla v(t,\cdot)\|^2_{L^2(\Omega)}\Big)^{\frac{1}{2}} \le C\theta^3(T+1).$$

Now we use Poincaré's lemma to deal with $\|v(t,\cdot)\|_{L^2(\Omega)}$, because of the null Dirichlet's boundary condition of system (3.4). Then for any $\frac{1}{4}T \le t \le \frac{3}{4}T$, we have

$$\Big(\|\partial_t^s v(t,\cdot)\|^2_{L^2(\Omega)} + \sum_{l=0}^{s-1}\|\partial_t^l v(t,\cdot)\|^2_{H^1(\Omega)}\Big)^{\frac{1}{2}} \le C\theta^3(T+1). \tag{3.10}$$

Next we regard system (3.4) as the Dirichlet boundary value problem of elliptic equations and use the elliptic estimates (4.7), then we have $D_\Lambda(v) \leq C\theta^3(T+1)$, assuming θ is sufficiently small.

Step 3 Now we prove that w is well-defined. (Because wave equations are time invertible, the exact controllability is equivalent to the null controllability. So, here, we can use the Huygens principle to obtain the null controllability, which implies that w is well-defined. But when the region is star-complemented, we want to get the null control of part of the boundary and Huygens principle can not assure this. Here, for convenience, we prove that w is well-defined by the constructive method introduced in [13], which can be also applied to the case that the region is star-complemented.)

By using the extension operator $\sim$, we define the series $\varphi^{(i)}$ and $\psi^{(i)}$ as follows.

Let $\varphi^{(1)}$ be the solution of the initial-boundary value problem

$$\begin{cases} \Box\varphi^{(1)} = 0, & \frac{1}{4}T \leq t \leq \frac{3}{4}T,\ x \in \mathbb{B}_1, \\ \varphi_t^{(1)} + \varphi_r^{(1)} + \varphi^{(1)} = 0, & \frac{1}{4}T \leq t \leq \frac{3}{4}T,\ x \in \partial\mathbb{B}_1, \\ t = \frac{1}{4}T : \varphi^{(1)} = [\chi u_1]^\sim,\ \partial_t\varphi^{(1)} = [\chi u_{1t}]^\sim, & x \in \mathbb{B}_1, \end{cases} \tag{3.11}$$

and let $\psi^{(1)}$ be the solution of the inverted initial-boundary value problem

$$\begin{cases} \Box\psi^{(1)} = 0, & \frac{1}{4}T \leq t \leq \frac{3}{4}T,\ x \in \mathbb{B}_1, \\ -\psi_t^{(1)} + \psi_r^{(1)} + \psi^{(1)} = 0, & \frac{1}{4}T \leq t \leq \frac{3}{4}T,\ x \in \partial\mathbb{B}_1, \\ t = \frac{3}{4}T : \psi^{(1)} = [\chi(u_2 - v)]^\sim,\ \partial_t\psi^{(1)} = [\chi(u_{2t} - v_t)]^\sim, & x \in \mathbb{B}_1. \end{cases} \tag{3.12}$$

For $j \geq 2$, $\varphi^{(j)}$ is defined inductively as the solution of the initial-boundary value problem

$$\begin{cases} \Box\varphi^{(j)} = 0, & \frac{1}{4}T \leq t \leq \frac{3}{4}T,\ x \in \mathbb{B}_1, \\ \varphi_t^{(j)} + \varphi_r^{(j)} + \varphi^{(j)} = 0, & \frac{1}{4}T \leq t \leq \frac{3}{4}T,\ x \in \partial\mathbb{B}_1, \\ t = \frac{1}{4}T : \varphi^{(j)} = [\chi\psi^{(j-1)}]^\sim,\ \varphi_t^{(j)} = [\chi\psi_t^{(j-1)}]^\sim, & x \in \mathbb{B}_1, \end{cases} \tag{3.13}$$

and $\psi^{(j)}$ is defined as the solution of the inverted initial-boundary value problem

$$\begin{cases} \Box\psi^{(j)} = 0, & \frac{1}{4}T \leq t \leq \frac{3}{4}T,\ x \in \mathbb{B}_1, \\ -\psi_t^{(j)} + \psi_r^{(j)} + \psi^{(j)} = 0, & \frac{1}{4}T \leq t \leq \frac{3}{4}T,\ x \in \partial\mathbb{B}_1, \\ t = \frac{3}{4}T : \psi^{(j)} = [\chi\varphi^{(j-1)}]^\sim,\ \partial_t\psi^{(j)} = [\chi\varphi_t^{(j-1)}]^\sim, & x \in \mathbb{B}_1, \end{cases} \tag{3.14}$$

where χ is the characteristic function defined as

$$\chi = \begin{cases} 1, & x \in \Omega, \\ 0, & x \in \mathbb{B}_1 \setminus \Omega. \end{cases}$$

We define

$$w^m=\sum_{j=1}^{m}(-1)^{j-1}(\varphi^{(j)}+\psi^{(j)}). \tag{3.15}$$

First of all, we observe that for any $m\ge 1$,

$$\Box w^m=0,\quad \frac14 T\le t\le \frac34 T,\ x\in\Omega \tag{3.16}$$

and

$$\begin{cases} w^m\Big(\dfrac14 T,x\Big)=u_1\Big(\dfrac14 T,x\Big)+(-1)^{m-1}\psi^{(m)}\Big(\dfrac14 T,x\Big), & x\in\Omega,\\ w_t^m\Big(\dfrac14 T,x\Big)=u_{1t}\Big(\dfrac14 T,x\Big)+(-1)^{m-1}\psi_t^{(m)}\Big(\dfrac14 T,x\Big), & x\in\Omega,\\ w^m\Big(\dfrac34 T,x\Big)=(u_2-v)\Big(\dfrac34 T,x\Big)+(-1)^{m-1}\varphi^{(m)}\Big(\dfrac34 T,x\Big), & x\in\Omega,\\ w_t^m\Big(\dfrac34 T,x\Big)=(u_{2t}-v_t)\Big(\dfrac34 T,x\Big)+(-1)^{m-1}\varphi_t^{(m)}\Big(\dfrac34 T,x\Big), & x\in\Omega. \end{cases} \tag{3.17}$$

To show that the sequence $\{w^m\}$ defined in the equality (3.15) is convergent, let us take a look at the mixed initial-boundary value problem

$$\begin{cases} \Box u=0, & t\ge 0,\ x\in\mathbb{B}_1,\\ t=0: u=f,\ u_t=g, & x\in\mathbb{B}_1,\\ u_t+u_r+u=0, & t\ge 0,\ x\in\partial\mathbb{B}_1. \end{cases} \tag{3.18}$$

Theorem 3.1 *If $s\ge 2$, $f\in H^s(\mathbb{B}_1)$, $g\in H^{s-1}(\mathbb{B}_1)$ and $\operatorname{supp}(f,g)\subset\subset\mathbb{B}_1$, then there exists a global solution $u(t,x)$ to system (3.18) and constants $C_5,\beta>0$ such that for any $t\ge 0$,*

$$\|\partial_t^s u(t,\cdot)\|_{L^2(\mathbb{B}_1)}^2+\sum_{l=0}^{s-1}\|\partial_t^l u(t,\cdot)\|_{H^1(\mathbb{B}_1)}^2\le C_5(\|f\|_{H^s(\mathbb{B}_1)}^2+\|g\|_{H^{s-1}(\mathbb{B}_1)}^2)\mathrm{e}^{-\beta t}. \tag{3.19}$$

Proof The proof of Theorem 3.1 is similar to that of Theorem 2.1 to a large extent.

By the inequality (3.19) and using the same argument as in Lemma 2.3, there exists a positive constant C_6 such that

$$\begin{aligned} &\Big(\Big\|\partial_t^s\varphi^{(j)}\Big(\frac34 T,\cdot\Big)\Big\|_{L^2(\mathbb{B}_1)}^2+\sum_{l=0}^{s-1}\Big\|\partial_t^l\varphi^{(j)}\Big(\frac34 T,\cdot\Big)\Big\|_{H^1(\mathbb{B}_1)}^2\Big)\\ &\le C_5\,\mathrm{e}^{-\frac{\beta T}{2}}\Big(\Big\|[\chi\psi^{(j-1)}]^\sim\Big(\frac14 T,\cdot\Big)\Big\|_{H^s(\mathbb{B}_1)}^2+\Big\|[\chi\psi_t^{(j-1)}]^\sim\Big(\frac14 T,\cdot\Big)\Big\|_{H^{s-1}(\mathbb{B}_1)}^2\Big)\\ &\le C_6\,\mathrm{e}^{-\frac{\beta T}{2}}\Big(\Big\|\partial_t^s\psi^{(j-1)}\Big(\frac14 T,\cdot\Big)\Big\|_{L^2(\mathbb{B}_1)}^2+\sum_{l=0}^{s-1}\Big\|\partial_t^l\psi^{(j-1)}\Big(\frac14 T,\cdot\Big)\Big\|_{H^1(\mathbb{B}_1)}^2\Big) \end{aligned} \tag{3.20}$$

and

$$\begin{aligned} &\Big\|\partial_t^s\psi^{(j)}\Big(\frac14 T,\cdot\Big)\Big\|_{L^2(\mathbb{B}_1)}^2+\sum_{l=0}^{s-1}\Big\|\partial_t^l\psi^{(j)}\Big(\frac14 T,\cdot\Big)\Big\|_{H^1(\mathbb{B}_1)}^2\\ &\le C_6\,\mathrm{e}^{-\frac{\beta T}{2}}\Big(\Big\|\partial_t^s\varphi^{(j-1)}\Big(\frac34 T,\cdot\Big)\Big\|_{L^2(\mathbb{B}_1)}^2+\sum_{l=0}^{s-1}\Big\|\partial_t^l\varphi^{(j-1)}\Big(\frac34 T,\cdot\Big)\Big\|_{H^1(\mathbb{B}_1)}^2\Big). \end{aligned} \tag{3.21}$$

Combining (3.20) and (3.21), we arrive at

$$\begin{aligned}
&\Big\|\partial_t^s\psi^{(m)}\Big(\frac{1}{4}T,\,\cdot\Big)\Big\|^2_{L^2(\mathbb{B}_1)}+\sum_{l=0}^{s-1}\Big\|\partial_t^l\psi^{(m)}\Big(\frac{1}{4}T,\,\cdot\Big)\Big\|^2_{H^1(\mathbb{B}_1)}\\
&\quad+\Big\|\partial_t^s\varphi^{(m)}\Big(\frac{3}{4}T,\,\cdot\Big)\Big\|^2_{L^2(\mathbb{B}_1)}+\sum_{l=0}^{s-1}\Big\|\partial_t^l\varphi^{(m)}\Big(\frac{3}{4}T,\,\cdot\Big)\Big\|^2_{H^1(\mathbb{B}_1)}\\
&\le (C_6\,\mathrm{e}^{-\frac{\beta T}{2}})^m\Big(\Big\|[\chi u_1]^{\sim}\Big(\frac{1}{4}T,\,\cdot\Big)\Big\|^2_{H^s(\mathbb{B}_1)}+\Big\|[\chi u_{1t}]^{\sim}\Big(\frac{1}{4}T,\,\cdot\Big)\Big\|^2_{H^{s-1}(\mathbb{B}_1)}\\
&\quad+\Big\|[\chi(u_2-v)]^{\sim}\Big(\frac{3}{4}T,\,\cdot\Big)\Big\|^2_{H^s(\mathbb{B}_1)}+\Big\|[\chi(u_{2t}-v_t)]^{\sim}\Big(\frac{3}{4}T,\,\cdot\Big)\Big\|^2_{H^{s-1}(\mathbb{B}_1)}\Big).
\end{aligned}\tag{3.22}$$

Take a sufficiently large positive constant T depending only on λ, $\|f_0\|_{H^s(\Omega)}$, $\|f_1\|_{H^{s-1}(\Omega)}$, $\|g_0\|_{H^s(\Omega)}$, $\|g_1\|_{H^{s-1}(\Omega)}$ and θ satisfying $\theta T=1$ such that $C_6\,\mathrm{e}^{\frac{-\beta T}{2}}<\frac{1}{2}$, $\widetilde{C}_2\,\mathrm{e}^{-\frac{\alpha T}{4}}\le\theta^4$ and $\widetilde{C}_4\,\mathrm{e}^{-\frac{\alpha T}{4}}\le\theta^4$. Therefore, we know that $D_\Lambda(v)\le C\theta^2$ and $\sum\limits_{l=0}^{s}\|\partial_t^l u_1(\frac{1}{4}T,\,\cdot)\|^2_{H^{s-l}(\mathbb{B}_1)}\le\theta^4$ and $\sum\limits_{l=0}^{s}\|\partial_t^l u_2(\frac{3}{4}T,\,\cdot)\|^2_{H^{s-l}(\mathbb{B}_1)}\le\theta^4$.

By the inequality (3.22) and similarly as in Lemma 2.3, we deduce the conclusion that when $m\to\infty$, $w^m(\frac{1}{4}T,\,\cdot)\to u_1(\frac{1}{4}T,\,\cdot)$, $w^m(\frac{3}{4}T,\,\cdot)\to(u_2-v)(\frac{3}{4}T,\,\cdot)$ in $H^s(\Omega)$ and $w_t^m(\frac{1}{4}T,\,\cdot)\to u_{1t}(\frac{1}{4}T,\,\cdot)$, $w_t^m(\frac{3}{4}T,\,\cdot)\to(u_{2t}-v_t)(\frac{3}{4}T,\,\cdot)$ in $H^{s-1}(\Omega)$.

What is more, for any $\frac{1}{4}T\le t\le\frac{3}{4}T$, we conclude that the inequality (3.22) still holds even if we substitute t for $\frac{1}{4}T$, $\frac{3}{4}T$ in the left part of the inequality (3.22) and $m-1$ for m in the right part of the inequality (3.22) at the same time. Using the same argument as in Lemma 2.3, we deduce that w^m is a Cauchy sequence in $\bigcap\limits_{j=0}^{s}C^j([\frac{1}{4}T,\frac{3}{4}T],H^{s-j}(\Omega))$. Denote $w^m\to w$. Hence w satisfies system (3.5). For any $\frac{1}{4}T\le t\le\frac{3}{4}T$, it follows that

$$\begin{aligned}
&\sum_{l=0}^{s-1}\|\partial_t^l w(t,\,\cdot)\|^2_{H^{s-1}(\Omega)}\\
&\le C\Big(\Big\|u_1\Big(\frac{1}{4}T,\,\cdot\Big)\Big\|^2_{H^s(\Omega)}+\Big\|u_{1t}\Big(\frac{1}{4}T,\,\cdot\Big)\Big\|^2_{H^{s-1}(\Omega)}\\
&\quad+\Big\|(u_2-v)\Big(\frac{3}{4}T,\,\cdot\Big)\Big\|^2_{H^s(\Omega)}+\Big\|(u_{2t}-v_t)\Big(\frac{3}{4}T,\,\cdot\Big)\Big\|^2_{H^{s-1}(\Omega)}\Big),
\end{aligned}\tag{3.23}$$

which implies $D_\Lambda(w)\le C\theta^2$.

Therefore $D_\Lambda(v+w)\le D_\Lambda(v)+D_\Lambda(w)\le C\theta^2$. Taking T sufficiently large and θ sufficiently small, we have $D_\Lambda(v+w)\le\theta$. So $v+w\in\Lambda_\theta$, which says the map $\Pi:\Lambda_\theta\to\Lambda_\theta$.

Step 4 In the end, we prove that Π is a strict contraction. For any $\phi_1,\phi_2\in\Lambda_\theta$, define $\Pi\phi_1=v_1+w_1$, $\Pi\phi_2=v_2+w_2$, $\overline{\phi}=\phi_1-\phi_2$, $\overline{v}=v_1-v_2$ and $\overline{w}=w_1-w_2$.

Hence $\overline{v}$ solves the initial-boundary value problem

$$\begin{cases}
\Box\overline{v}+\lambda(\phi_1^3-\phi_2^3)=0, & \dfrac{1}{4}T\le t\le\dfrac{3}{4}T,\ x\in\Omega,\\
t=\dfrac{1}{4}T:\overline{v}=0,\ \partial_t\overline{v}=0, & x\in\Omega,\\
\overline{v}=0, & \dfrac{1}{4}T\le t\le\dfrac{3}{4}T,\ x\in\partial\Omega,
\end{cases}\tag{3.24}$$

and $\overline{w}$ solves the exact boundary controllability problem

$$\begin{cases} \Box\overline{w} = 0, & \frac{1}{4}T \le t \le \frac{3}{4}T,\ x \in \Omega, \\ t = \frac{1}{4}T : \overline{w} = 0,\ \partial_t\overline{w} = 0, & x \in \Omega, \\ t = \frac{3}{4}T : \overline{w} = -\overline{v},\ \partial_t\overline{w} = -\overline{v}_t, & x \in \Omega. \end{cases} \tag{3.25}$$

Similarly, we have $D_\Lambda(\overline{w}) \le CD_\Lambda(\overline{v})$ and $D_\Lambda(\overline{v}) \le C\theta^2(T+1)D_\Lambda(\overline{\phi})$. Choosing T large enough and θ small enough, we know $D_\Lambda(\Pi\phi_1 - \Pi\phi_2) \le D_\Lambda(\overline{v}) + D_\Lambda(\overline{w}) \le C\theta D_\Lambda(\overline{\phi}) \le \frac{1}{2}D_\Lambda(\overline{\phi})$. Therefore Π is a strict contraction from Λ_θ to Λ_θ.

By the standard contraction mapping theorem, there exists a fixed point $u_0 \in \Lambda_\theta$ such that $\Pi u_0 = u_0$. It follows that u_0 solves system (3.1).

To prove Theorem 1.1, it suffices to take a Dirichlet boundary condition. The other boundary condition can be obtained in a similar way. For $(t,x) \in [\frac{1}{4}T, \frac{3}{4}T] \times \partial\Omega$, let $h(t,x) = u_0(t,x)$; for $(t,x) \in [0, \frac{1}{4}T] \times \partial\Omega$, let $h(t,x) = u_1(t,x)$; for $(t,x) \in [\frac{3}{4}T, T] \times \partial\Omega$, let $h(t,x) = u_2(t,x)$. Therefore h is the desired boundary control function. We easily know the existence of the solution on time interval $[0,T]$ to the initial-boundary value problem (1.1)–(1.2) and the Dirichlet boundary condition (1.4) from the proof above. And we obtain the uniqueness of the initial-boundary value problem (1.1), (1.2) and (1.4) by doing energy estimates and applying Gronwall's inequality. Then the proof of Theorem 1.1 is completed.

4 Global Exact Boundary Controllability for the Cubic Semi-linear Wave Equation in the Star-Complemented Region

In this section, we prove Theorem 1.2.

We say that $\partial\Omega$ is regular if given any $x \in \partial\Omega$, there is a neighborhood $\mathbb{U}$ of x in $\mathbb{R}^3$ and C^∞ functions $g_1, \cdots, g_{n(x)}$ defined on $\mathbb{U}$, with the gradient vector functions $\operatorname{grad} g_1, \cdots, \operatorname{grad} g_{n(x)}$ being everywhere linearly independent in $\mathbb{U}$, such that $\Omega \cap \mathbb{U} = \{y \in \mathbb{U} \mid y \in M_1 * (M_2 * \cdots (M_{n(x)-1} * M_{n(x)}))\}$, where $*$ is one of the operators $\cap$ or $\cup$ and $M_i = \{y \in \mathbb{U} \mid g_i(y) \ge 0\}$. There are more general definitions of this term but the present one suffices for our purpose.

There exists a star-shaped region Ω^* with a piecewise smooth boundary $\partial\Omega^*$ such that $\overline{\Omega \cup \Omega^*} \subseteq \mathbb{B}_1$, $\partial\Omega - \Gamma \subseteq \partial\Omega^*$, $\Omega \subseteq \overline{\Omega^*}^c$ (see Figure 1), and we can employ the reflection method described in [7] to extend f_0, g_0 to $H^s(\mathbb{R}^3)$ and f_1, g_1 to $H^{s-1}(\mathbb{R}^3)$, which we still call f_0, g_0, f_1, g_1, such that

$$f_0(x) = f_1(x) = g_0(x) = g_1(x) = 0, \quad x \in \Omega^*, \tag{4.1}$$

$$\operatorname{supp}(f_0, g_0, f_1, g_1) \subset\subset \mathbb{B}_1, \tag{4.2}$$

$$\begin{aligned} &\|f_0\|_{H^s(\mathbb{R}^3)} \le K_s\|f_0\|_{H^s(\Omega)}, \quad \|f_1\|_{H^{s-1}(\mathbb{R}^3)} \le K_{s-1}\|f_1\|_{H^{s-1}(\Omega)}, \\ &\|g_0\|_{H^s(\mathbb{R}^3)} \le K_s\|g_0\|_{H^s(\Omega)}, \quad \|g_1\|_{H^{s-1}(\mathbb{R}^3)} \le K_{s-1}\|g_1\|_{H^{s-1}(\Omega)}, \end{aligned} \tag{4.3}$$

where K_s and K_{s-1} are positive constants. (For example, see [9,10].)

Without loss of generality, we assume that Ω^* is star-shaped with respect to 0.

Let $\Omega_1 = \mathbb{B}_1 - \overline{\Omega^*}$. Then it suffices to construct the solution of the equation (1.1) with initial data f_0, f_1 and final data g_0, g_1 on the domain $[0,T] \times \Omega_1$ for large enough time T. By

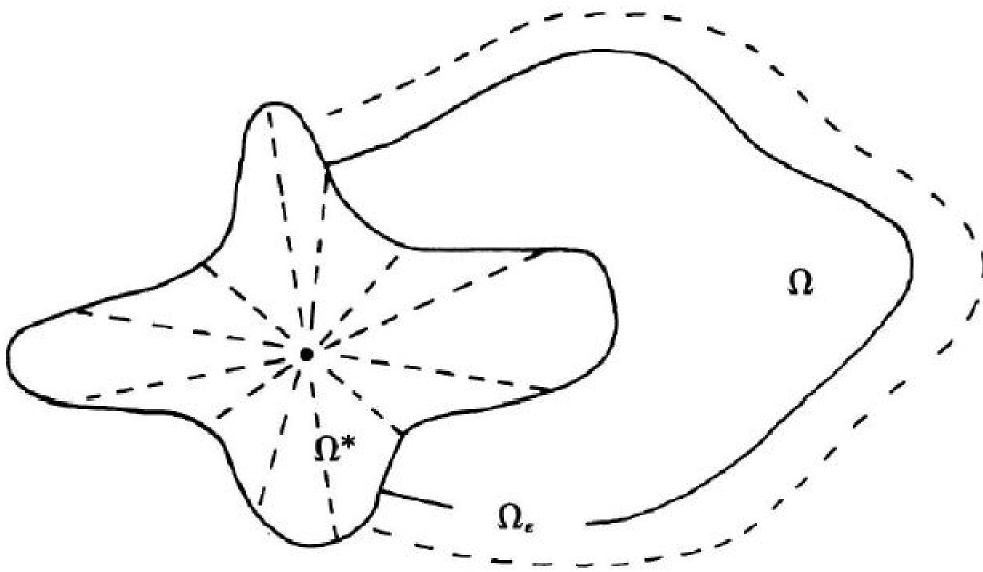

Figure 1 The domains Ω, Ω^* and Ω_ε

restricting the solution to Γ, we obtain the desired boundary control function. To this end, we evolve the equation (1.1) on the domain $[0,T]\times\Omega_1$ with the initial data (f_0,f_1), the boundary condition $\frac{\partial u}{\partial t}+\frac{\partial u}{\partial r}+u=0$ on $[0,T]\times\partial\mathbb{B}_1$ and the Dirichlet boundary condition $u=0$ on $[0,T]\times\partial\Omega^*$, and prove the global existence and an exponential decay of energy for the solution of this problem. Then we reduce the global control problem to a local one which is solved by a constructive method developed in [13].

First we study the global existence of the strong solution to the following mixed initial-boundary value problem:

$$\begin{cases}\Box u+\lambda u^3=0, & t\geq 0,\ x\in\Omega_1,\\ t=0: u=f_0,\ u_t=f_1, & x\in\Omega_1,\\ \dfrac{\partial u}{\partial t}+\dfrac{\partial u}{\partial r}+u=0, & t\geq 0,\ x\in\partial\mathbb{B}_1,\\ u=0, & t\geq 0,\ x\in\partial\Omega^*.\end{cases}\tag{4.4}$$

Even if we apply ∂_t^l $(0\leq l\leq s-1)$ to system (4.4), the compatible condition for the resulting system still holds according to (4.1) and (4.2). Similarly to Lemma 2.1, we establish the following local existence for system (4.4).

Lemma 4.1 *There exist positive constants T^* and M such that system* (4.4) *admits a unique solution $u(t,x)$ on the domain $[0,T^*]\times\Omega_1$ satisfying*

$$\|\partial_t^s u(t,\cdot)\|^2_{L^2(\Omega_1)}+\sum_{l=0}^{s-1}\|\partial_t^l u(t,\cdot)\|^2_{H^1(\Omega_1)}\leq M^2,\quad \forall\, 0\leq t\leq T^*,\tag{4.5}$$

where T^* and M depend only on $\|f_0\|_{H^s(\Omega_1)}$ and $\|f_1\|_{H^{s-1}(\Omega_1)}$.

Remark 4.1 The local solution in Lemma 4.1 belongs to $\bigcap\limits_{j=0}^{s} C^j([0,T^*],H^{s-j}(\Omega_1))$.

To conclude Remark 4.1, we recall the elliptic estimates involving the null Dirichlet boundary condition presented in [11].

Lemma 4.2 *Suppose that Ω is a bounded domain with the smooth boundary $\partial\Omega$. Consider the following system:*

$$\begin{cases} -\Delta h = f, & \text{in } \Omega, \\ h = 0, & \text{on } \partial\Omega. \end{cases} \tag{4.6}$$

Given $f \in H^{k-1}(\Omega)$ $(k = 0, 1, \cdots)$, a solution h to system (4.6) belongs to $H^{k+1}(\Omega)$ and we have the estimate

$$\|h\|^2_{H^{k+1}(\Omega)} \leq C_k(\|f\|^2_{H^{k-1}(\Omega)} + \|h\|^2_{H^k(\Omega)}). \tag{4.7}$$

Then the combination of Lemmas 2.2 and 4.2 implies the following lemma.

Lemma 4.3 *Consider the following system:*

$$\begin{cases} -\Delta h = f, & \text{in } \Omega_1, \\ \dfrac{\partial h}{\partial r} = g, & \text{on } \partial\mathbb{B}_1, \\ h = 0, & \text{on } \partial\Omega^*. \end{cases} \tag{4.8}$$

Given $f \in H^k(\Omega_1)$ and $g \in H^{k+\frac{1}{2}}(\partial\mathbb{B}_1)$ $(k = 0, 1, \cdots)$, a solution h to system (4.8) belongs to $H^{k+2}(\Omega_1)$ and we have

$$\|h\|^2_{H^{k+2}(\Omega_1)} \leq C_k(\|f\|^2_{H^k(\Omega_1)} + \|g\|^2_{H^{k+\frac{1}{2}}(\partial\mathbb{B}_1)} + \|h\|^2_{L^2(\Omega_1)}). \tag{4.9}$$

Proof Lemma 4.3 holds according to the proof of Lemmas 2.2 and 4.2.

Using the same argument as that in Lemma 2.3 and in view of Lemma 4.3, we get the following lemma.

Lemma 4.4 *Assume that u solves the initial-boundary value problem (4.4). We have*

$$\sum_{j=0}^{s} \|\partial_t^j u\|^2_{H^{s-j}(\Omega_1)} \leq C_2\Big(\|\partial_t^s u\|^2_{L^2(\Omega_1)} + \sum_{l=0}^{s-1} \|\partial_t^l u\|^2_{H^1(\Omega_1)}\Big), \tag{4.10}$$

where $C_2(\cdot)$ is a function in the form $C_2(z) = \sum\limits_{j=1}^{k_s} c_{2j} z^j$ with k_s an integer depending only on s and c_{2j} a constant depending only on j.

Obviously, (4.10) implies that Remark 4.1 holds.

In the discussion later, we need the following lemma on the star-shaped region (see [3]).

Lemma 4.5 *Assume that Ω^* is star-shaped with respect to 0 and $\partial\Omega^*$ is $\mathbb{C}^1$. Then $x \cdot \boldsymbol{v}(x) \geq 0$ for all $x \in \partial\Omega^*$, where $\boldsymbol{v}$ denotes the unit outward normal.*

Now we are ready to establish the global existence and an exponential decay of the energy for system (4.4).

Precisely, we prove the following theorem.

Theorem 4.1 *The initial-boundary value problem* (4.4) *admits a global solution* $u(t,x)$, *and for any* $0<\delta<\frac{1}{2}$, *there exists a positive constant* C_8 *such that*

$$\|\partial_t^s u(t,\cdot)\|_{L^2(\Omega_1)}^2+\sum_{l=0}^{s-1}\|\partial_t^l u(t,\cdot)\|_{H^1(\Omega_1)}^2\le C_7\,\mathrm{e}^{-\delta t},\quad \forall\, t\ge 0,\tag{4.11}$$

where C_7 *depends only on* λ, δ, $\|f_0\|_{H^s(\Omega_1)}$ *and* $\|f_1\|_{H^{s-1}(\Omega_1)}$.

Proof The proof is similar to that of Theorem 2.1. First we consider the following estimates. We get the standard energy estimate by taking the $L^2(\Omega_1)$ inner product of the equation in system (4.4) with u_t,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|u_t\|_{L^2(\Omega_1)}^2+\|\nabla u\|_{L^2(\Omega_1)}^2+\|u\|_{L^2(\partial\mathbb{B}_1)}^2)+\|u_t\|_{L^2(\partial\mathbb{B}_1)}^2+\frac{\lambda}{4}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{L^4(\Omega_1)}^4=0,\tag{4.12}$$

where we used the boundary condition $u_t=0$, $x\in\partial\Omega^*$ and $u_t+u_r+u=0$, $x\in\partial\mathbb{B}_1$.

We do the energy estimate of Morawetz type by taking the $L^2(\Omega_1)$ inner product of the equation in system (4.4) with $x\cdot\nabla u$:

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_1}(x\cdot\nabla u)u_t\,\mathrm{d}x-\frac{1}{2}\int_{\Omega_1}x\cdot\nabla(u_t^2)\mathrm{d}x+\frac{\lambda}{4}\int_{\Omega_1}x\cdot\nabla(u^4)\mathrm{d}x\\&=\int_{\Omega_1}\nabla_k(\nabla_k ux\cdot\nabla u)\mathrm{d}x-\int_{\Omega_1}|\nabla u|^2\mathrm{d}x-\frac{1}{2}\int_{\Omega_1}x\cdot\nabla|\nabla u|^2\mathrm{d}x.\end{aligned}\tag{4.13}$$

Let $\boldsymbol{v}=(v_1,v_2,v_3)^{\mathrm{T}}$ be the unit exterior normal to the boundary $\partial\Omega^*$ of the region Ω^*.

By integration by parts and using the boundary condition $u=0$, $u_t=0$, $x\in\partial\Omega^*$ and $u_t+u_r+u=0$, $x\in\partial\mathbb{B}_1$, we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_1}(x\cdot\nabla u)u_t\,\mathrm{d}x+\frac{1}{2}(\|u_t\|_{L^2(\Omega_1)}^2+\|\nabla u\|_{L^2(\Omega_1)}^2)\\&-\frac{1}{2}\|u_t\|_{L^2(\partial\mathbb{B}_1)}^2+\frac{\lambda}{4}\|u\|_{L^4(\partial\mathbb{B}_1)}^4-\frac{3\lambda}{4}\|u\|_{L^4(\Omega_1)}^4\\&=-\int_{\partial\Omega^*}\frac{\partial u}{\partial v}(x\cdot\nabla u)\mathrm{d}\sigma-\int_{\partial\mathbb{B}_1}(u_t+u)\frac{\partial u}{\partial r}\mathrm{d}\Gamma-\frac{1}{2}\|\nabla u\|_{L^2(\partial\mathbb{B}_1)}^2\\&\quad+\frac{1}{2}\int_{\partial\Omega^*}\sum_{i=1}^3 x_iv_i|\nabla u|^2\mathrm{d}\sigma+(\|\nabla u\|_{L^2(\Omega_1)}^2-\|u_t\|_{L^2(\Omega_1)}^2).\end{aligned}\tag{4.14}$$

By taking $L^2(\Omega_1)$ inner product of the equation in system (4.4) with u and using the boundary condition $u=0$, $x\in\partial\Omega^*$ and $u_t+u_r+u=0$, $x\in\partial\mathbb{B}_1$, we get

$$\begin{aligned}\int_{\Omega_1}(u_t^2-|\nabla u|^2)\mathrm{d}x&=\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_1}uu_t\,\mathrm{d}x-\int_{\Omega_1}u(\triangle u-\lambda u^3)+|\nabla u|^2\mathrm{d}x\\&=\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_1}uu_t\,\mathrm{d}x-\int_{\partial\mathbb{B}_1}uu_r\,\mathrm{d}x+\int_{\partial\Omega^*}u\frac{\partial u}{\partial v}\mathrm{d}\sigma+\lambda\|u\|_{L^4(\Omega_1)}^4\\&=\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_1}uu_t\,\mathrm{d}x+\lambda\|u\|_{L^4(\Omega_1)}^4+\int_{\partial\mathbb{B}_1}u(u_t+u)\mathrm{d}x.\end{aligned}\tag{4.15}$$

Next we prove

$$-\int_{\partial\Omega^*}\frac{\partial u}{\partial v}(x\cdot\nabla u)\mathrm{d}\sigma+\frac{1}{2}\int_{\partial\Omega^*}\sum_{i=1}^3 x_iv_i|\nabla u|^2\mathrm{d}\sigma\le 0.\tag{4.16}$$

Without loss of generality, we assume $v_1 \neq 0$. Therefore $\tau_1 = (-v_2, v_1, 0)^{\mathrm{T}}$ and $\tau_2 = (-v_3, 0, v_1)^{\mathrm{T}}$ are the tangents to the boundary $\partial\Omega^*$. Then we know that $\boldsymbol{v}$, τ_1 and τ_2 are linearly independent. Since $u(t,x) = 0$ for $t \geq 0$ and $x \in \partial\Omega^*$, we easily get

$$\frac{\partial u}{\partial \tau_1}(t,x) = 0, \quad \frac{\partial u}{\partial \tau_2}(t,x) = 0, \quad t \geq 0,\ x \in \partial\Omega^*. \tag{4.17}$$

Consequently, for $t \geq 0$ and $x \in \partial\Omega^*$, we deduce

$$\frac{\partial u}{\partial x_1} = v_1 \frac{\partial u}{\partial v}, \quad \frac{\partial u}{\partial x_2} = v_2 \frac{\partial u}{\partial v}, \quad \frac{\partial u}{\partial x_3} = v_3 \frac{\partial u}{\partial v}, \quad t \geq 0,\ x \in \partial\Omega^*. \tag{4.18}$$

From the equality (4.18), we obtain

$$-\frac{\partial u}{\partial v}(x \cdot \nabla u) + \frac{1}{2}\sum_{i=1}^{3} x_i v_i |\nabla u|^2 = -\frac{1}{2}\sum_{i=1}^{3} x_i v_i \Big|\frac{\partial u}{\partial v}\Big|^2 \leq 0, \tag{4.19}$$

where we used Lemma 4.5. Then the inequality (4.16) holds.

The rest proof of Theorem 4.1 has a great deal in common with that of Theorem 2.1.

Now we consider the inverted initial-boundary value problem

$$\begin{cases} \Box u + \lambda u^3 = 0, & 0 \leq t \leq T,\ x \in \Omega_1, \\ t = T : u = g_0,\ u_t = g_1, & x \in \Omega_1, \\ -\dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial r} + u = 0, & 0 \leq t \leq T,\ x \in \partial\mathbb{B}_1, \\ u = 0, & 0 \leq t \leq T,\ x \in \partial\Omega^*. \end{cases} \tag{4.20}$$

We only need to make a change of variable $t \to T - t$. Then we get the following theorem for system (4.20) according to Theorem 4.1.

Theorem 4.2 *The problem* (4.20) *admits a unique solution* $u(t,x)$ *on the domain* $[0,T] \times \Omega_1$ *and there exists a constant* $C_8 > 0$ *depending only on* λ, δ, $\|g_0\|_{H^s(\Omega_1)}$ *and* $\|g_1\|_{H^{s-1}(\Omega_1)}$ *such that*

$$\|\partial_t^s u(t,\cdot)\|^2_{L^2(\Omega_1)} + \sum_{l=0}^{s-1} \|\partial_t^l u(t,\cdot)\|^2_{H^1(\Omega_1)} \leq C_9\, \mathrm{e}^{-\delta(T-t)}, \quad \forall\, 0 \leq t \leq T. \tag{4.21}$$

The rest proof of Theorem 1.2 is word for word the same as that in Section 3.

5 Global Exact Boundary Controllability for Cubic Klein-Gordon Equations

In this section, we shall be concerned with the cubic Klein-Gordon equation

$$\Box u + u^3 + u = 0, \quad 0 < t < T,\ x \in \Omega, \tag{5.1}$$

where Ω is a bounded open subset of $\mathbb{R}^3$.

By a similar proof of the global exact controllability for the cubic semi-linear wave equation, we obtain the following theorem for the cubic Klein-Gordon equation in the ordinary region.

Theorem 5.1 *Suppose $f_0, g_0 \in H^s(\Omega)$, $f_1, g_1 \in H^{s-1}(\Omega)$, $s \geq 2$. There exists a large enough positive constant T_0 depending only on the Sobolev norm of the data $\|f_0\|_{H^s(\Omega)}$, $\|f_1\|_{H^{s-1}(\Omega)}$, $\|g_0\|_{H^s(\Omega)}$, $\|g_1\|_{H^{s-1}(\Omega)}$ and a boundary control function h such that the cubic Klein-Gordon equation* (5.1) *with the initial state* (1.2) *and one of the boundary conditions* (1.4) *admits a unique solution on the domain $(0,T)\times\Omega$ which verifies the desired state* (1.3)*, provided that $T > T_0$.*

Especially when the region Ω is star-complemented, we similarly obtain the following theorem for the cubic Klein-Gordon equation.

Theorem 5.2 *Assume that the bounded region Ω is star-complemented. For any $f_0, g_0 \in H^s(\Omega)$, $f_1, g_1 \in H^{s-1}(\Omega)$, $s \geq 2$ with the property that $\partial_x^\alpha f_0 = \partial_x^\alpha g_0 = 0$ on Γ_1 for $|\alpha| \leq s-1$ and $\partial_x^\beta f_1 = \partial_x^\beta g_1 = 0$ on Γ_1 for $|\beta| \leq s-2$, there exists a sufficiently large constant $T_0 > 0$ depending only on the Sobolev norm of the data $\|f_0\|_{H^s(\Omega)}$, $\|f_1\|_{H^{s-1}(\Omega)}$, $\|g_0\|_{H^s(\Omega)}$, $\|g_1\|_{H^{s-1}(\Omega)}$ and a boundary control h only applied on Γ such that the cubic Klein-Gordon equation* (5.1) *with the initial data* (1.2)*, the boundary condition* (1.7) *and one of the conditions* (1.8) *admits a unique solution on the domain $(0,T)\times\Omega$ satisfying the desired data* (1.3)*, provided that $T > T_0$.*

Remark 5.1 The solution in Theorem 5.1 or 5.2 belongs to $\bigcap_{j=0}^{s} C^j([0,T], H^{s-j}(\Omega))$.

Remark 5.2 For the equation $\Box u + f(u) = 0$ with the nonlinear function f satisfying "good-sign" growth conditions (see [2, 12]), we obtain similar global exact boundary controllability results. If $f(0) = 0$ and $f'(0) = 0$, the semi-linear wave equation and the cubic semi-linear wave equation (1.1) is its classical example. If $f(0) = 0$ and $f'(0) > 0$, Klein-Gordon equations and the cubic Klein-Gordon equation (5.1) is its classical example.

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