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The Classification of Proper Holomorphic Mappings Between Special Hartogs Triangles of Different Dimensions***

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Abstract The authors discuss the proper holomorphic mappings between special Hartogs triangles of different dimensions and obtain a corresponding classification theorem.

Keywords Proper holomorphic mapping, Hartogs triangle, Automorphism group
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1 Introduction

Proper holomorphic mapping theory dates from 1950s, and there are many good results on it (see [1–9]). The classification of proper holomorphic mappings (see the definition in [1]) is an important and difficult problem, especially between bounded domains of different dimensions (see [2–4]). Assume that $g, f : D_1 \to D_2$ are proper holomorphic mappings, D_1 and D_2 are bounded domains in $\mathbb{C}^n$ and $\mathbb{C}^N$ respectively. If there exist $h_1 \in \mathrm{Aut}(D_1)$ and $h_2 \in \mathrm{Aut}(D_2)$ such that $f = h_2 \circ g \circ h_1$, then f and g are equivalent. Let B^n be the unit ball in $\mathbb{C}^n$. By a classical result of Alexander [5], every proper holomorphic self-mapping of B^n with $n \geq 2$ is equivalent to the identity mapping.

For $1 < n < N$, denote by $\mathrm{Rat}(B^n, B^N)$ the collection of all rational proper holomorphic mappings from B^n to B^N. For $n > 2$, the authors of [2] proved that there are only two equivalence classes in $\mathrm{Rat}(B^n, B^N)$. In [3], the authors got a new gap phenomenon for proper holomorphic mappings from B^n to B^N when $N \leq 3n-4$. When $N < 2n-1$ Huang Xiaojun gave the following classical theorem on the classification of proper holomorphic mappings from B^n to B^N.

Theorem A (see [4]) *Let B^n, B^m $(n > 1, n < m < 2n-1)$ be the unit balls in $\mathbb{C}^n, \mathbb{C}^m$ respectively. Let $f : B^n \to B^m$ be a holomorphic proper mapping that is twice continuously differentiable up to the boundary. Then there exist $\sigma \in \mathrm{Aut}(B^n), \tau \in \mathrm{Aut}(B^m)$ such that*

$$\tau \circ f \circ \sigma(z_1, z_2, \cdots, z_n) = (z_1, z_2, \cdots, z_n, 0, \cdots, 0).$$

From the above theorem, we know that the holomorphic proper mapping from B^n to B^m is unique up the holomorphic automorphisms of B^n and B^m.

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In this paper, we will discuss the proper holomorphic mappings between special Hartogs triangles of different dimensions and obtain a corresponding classification theorem. The main idea is from [6], in which the authors gave the classification of proper holomorphic mappings between generalized Hartogs triangles of same dimensions.

Firstly, we give the definition of the special Hartogs triangles:

$$\Omega(n_1,m_1)=\Big\{(z,w)\in\mathbb{C}^{n_1+m_1}:0<\sum_{i=1}^{n_1}|z_i|^2<\sum_{j=1}^{m_1}|w_j|^2<1\Big\},$$
$$\Omega(n_2,m_2)=\Big\{(z',w')\in\mathbb{C}^{n_2+m_2}:0<\sum_{i=1}^{n_2}|z_i'|^2<\sum_{j=1}^{m_2}|w_j'|^2<1\Big\},$$

where

$$1<n_1<n_2<\min\{n_1+m_1-1,2n_1-1\},\quad 1<m_1<m_2<2m_1-1, \tag{1.1}$$

and we use the notations

$$|z|^2:=\sum_{i=1}^{n_1}|z_i|^2,\quad |w|^2:=\sum_{j=1}^{m_1}|w_j|^2,\quad |z'|^2:=\sum_{i=1}^{n_2}|z_i'|^2,\quad |w'|^2:=\sum_{j=1}^{m_2}|w_j'|^2.$$

The main result is as follows.

Theorem 1.1 *Let $\Omega(n_1,m_1)$ and $\Omega(n_2,m_2)$ be Hartogs triangles with the dimensional assumption* (1.1). *Let $F:\Omega(n_1,m_1)\to\Omega(n_2,m_2)$ be a proper holomorphic mapping that is twice continuously differentiable up to the boundary. Then there exist $\sigma\in\mathrm{Aut}(\Omega(n_1,m_1))$ and $\tau\in\mathrm{Aut}(\Omega(n_2,m_2))$, such that*

$$\tau\circ F\circ\sigma(z,w)=(z_1,\cdots,z_{n_1},\underbrace{0,\cdots,0}_{n_2-n_1},w_1,\cdots,w_{m_1},\underbrace{0,\cdots,0}_{m_2-m_1}).$$

2 Main Lemmas

Let $F=(F_1,F_2):\Omega(n_1,m_1)\to\Omega(n_2,m_2)$ be a proper holomorphic mapping, where $F_1=(f_1,\cdots,f_{n_2})$, $F_2=(f_{n_2+1},\cdots,f_{n_2+m_2})$.

Let $\partial\Omega(n_1,m_1)=A\cup B\cup C$, where

$$\begin{aligned}A&=\{\,(z,w)\in\mathbb{C}^{n_1+m_1}\mid |z|^2-|w|^2=0,|z|^2\neq0,|w|^2\neq1\},\\ B&=\{\,(z,w)\in\mathbb{C}^{n_1+m_1}\mid |w|^2=1\},\\ C&=\{\,0\in\mathbb{C}^{n_1+m_1}\}.\end{aligned}$$

It is obvious that $A\cap B=B\cap C=A\cap C=\emptyset$.

Similarly, $\partial\Omega(n_2,m_2)=A'\cup B'\cup C'$, where

$$\begin{aligned}A'&=\{\,(z',w')\in\mathbb{C}^{n_2+m_2}\mid |z'|^2-|w'|^2=0,|z'|^2\neq0,|w'|^2\neq1\},\\ B'&=\{\,(z',w')\in\mathbb{C}^{n_2+m_2}\mid |w'|^2=1\},\\ C'&=\{\,0\in\mathbb{C}^{n_2+m_2}\}.\end{aligned}$$

We also have $A'\cap B'=B'\cap C'=A'\cap C'=\emptyset$.

Lemma 2.1 *$F = (F_1, F_2) : \Omega(n_1, m_1) \to \Omega(n_2, m_2)$ is a proper holomorphic mapping that is twice continuously differentiable up to the boundary. Then $F(B) \subset B'$.*

Proof Since F is proper, and twice continuously differentiable up to the boundary, we have $F(B) \subset \partial\Omega(n_2, m_2)$.

If there exists $x_0 \in B$, such that $F(x_0) \in A'$, then by the continuity of F, there exist an open set U of x_0 in $\mathbb{C}^{n_1+m_1}$ and an open set V of $F(x_0)$ in $\mathbb{C}^{n_2+m_2}$, such that $F(U) \subset V$.

Let

$$S = \{(z, w) \in \partial\Omega(n_1, m_1) : \mathrm{rank}(F') < n_1 + m_1\},$$

where F' is the Jacobian matrix of F. Select $x_1 \in B \backslash S$. Then we can find a suitable open set U_1 of x_1, such that $F|U_1 : U_1 \to F(U_1)$ has maximum rank. Then $(|z'|^2 - |w'|^2) \circ F$ and $|w|^2 - 1$ are local defining functions of $U_1 \cap B$.

The coefficient matrices of the Levi-forms of $|w|^2 - 1$ and $(|z'|^2 - |w'|^2) \circ F$ are respectively

$$\begin{pmatrix} 0 & 0 \\ 0 & I_{m_1} \end{pmatrix} \quad \text{and} \quad (F')^t \begin{pmatrix} I_{n_1} & 0 \\ 0 & -I_{m_2} \end{pmatrix} (\overline{F'}), \tag{2.1}$$

where $(F')^t$ is an $(n_1 + m_1) \times (n_2 + m_2)$ matrix which defines on $U_1 \cap B$ with maximum rank. Then $(F')^t$ can be expressed as the following

$$(F')^t = R(0, I_{m_1+n_1})V, \tag{2.2}$$

where R and V are $(n_1 + m_1) \times (n_1 + m_1)$ and $(n_2 + m_2) \times (n_2 + m_2)$ nonsingular matrices respectively, $I_{n_1+m_1}$ is the $(n_1 + m_1) \times (n_1 + m_1)$ unit matrix.

Let

$$V = \begin{pmatrix} V_1 & V_2 \\ V_3 & V_4 \end{pmatrix},$$

where V_1 is an $(n_2 + m_2 - n_1 - m_1) \times n_2$ matrix, V_4 is an $(n_1 + m_1) \times (m_2)$ matrix.

Setting

$$\begin{aligned} \eta &= (0, \cdots, 0, b_1, \cdots, b_{n_1+m_1})_{1\times(n_2+m_2)}, \\ \eta_1 &= (b_1, \cdots, b_{n_1+m_1})_{1\times(n_1+m_1)}, \end{aligned}$$

we consider the equation system:

$$\begin{cases} \eta_1 V_3 = (\underbrace{0, \cdots, 0}_{n_2}), \\ \mathrm{grad}(|w|^2 - 1)(R^{-1})^t(0 \quad I)_{(n_1+m_1)\times(n_2+m_2)}\eta^t = 0, \end{cases} \tag{2.3}$$

where

$$\mathrm{grad}(|w|^2 - 1) = (0, \cdots, 0, \overline{w}_1, \cdots, \overline{w}_{m_1}).$$

(2.3) has $n_1 + m_1$ variables and $n_2 + 1$ equations. By the assumption condition (1.1), there exist nontrivial solutions of (2.3). Since η_1 is nontrivial and $\eta_1 V_3 = 0$, we have $\eta_1 V_4 = (d_1, \cdots, d_{m_2}) \neq 0$.

Set

$$\xi = \eta \begin{pmatrix} 0 \\ I \end{pmatrix}_{(n_2+m_2)\times(n_1+m_1)} R^{-1} = (c_1, \cdots, c_{n_1+m_1}).$$

From (2.3), we get $\operatorname{grad}(|w|^2-1)\xi^t=0$. On the other hand, by the coefficient matrices of Levi-forms of $|w|^2-1$, we have

$$L(|w|^2-1)(\xi,\xi)=(c_1,\cdots,c_{n_1+m_1})\begin{pmatrix}0 & 0\\ 0 & I_{m_1}\end{pmatrix}(\overline{c}_1,\cdots,\overline{c}_{n_1+m_1})^T\geq 0. \tag{2.4}$$

From the expression of ξ,

$$\xi(F')^t=\eta\begin{pmatrix}0 & 0\\ V_3 & V_4\end{pmatrix}=(\underbrace{0,\cdots,0,d_1,\cdots,d_{m_2}}_{n_2+m_2}), \tag{2.5}$$

then

$$L((|z'|^2-|w'|^2)\circ F)(\xi,\xi)=\xi(F')^t\begin{pmatrix}I_{n_2} & 0\\ 0 & -I_{m_2}\end{pmatrix}(\overline{F'})\overline{\xi^t}=-\sum_{j=1}^{m_2}|d_j|^2<0. \tag{2.6}$$

But it is impossible. Because $|w|^2-1$ and $(|z'|^2-|w'|^2)\circ F$ are local defining functions of B, their Levi-forms $L(|w|^2-1), L((|z'|^2-|w'|^2)\circ F)$ considered as Hermitian quadratic form on $T^{(1,0)}(B)$ are only different by a positive factor, so (2.4), (2.6) deduce contradiction. Therefore the assumption $F(x_0)\in A'$ is impossible.

We now only need to verify that when $x_0\in B$, $F(x_0)\in C'$ is impossible. If $x_0\in B$, $F(x_0)=0$, and there exists an open set U of x_0, such that $F(B\cap U)\equiv 0$. Since $B\cap U$ is a real manifold of dimension $2(n_1+m_1)-1$ in $\Omega(n_1,m_1)$, we have $F\equiv 0$ on $\Omega(n_1,m_1)$, which is impossible by the definition of F. Otherwise there exists an open set U of x_0. By the continuity of F, $F((B\backslash S)\cap U)\cap A'\neq\emptyset$ is impossible, so we have proved that if $x_0\in B$, $F(x_0)\notin C'$. Thus, we complete the proof of Lemma 2.1.

Lemma 2.2 *$F=(F_1,F_2)$ is a proper holomorphic mapping as in Lemma* 2.1. *Then F_2 is independent of $z=(z_1,\cdots,z_{n_1})$.*

Proof Let $w=(w_1,\cdots,w_{m_1})\in B$. From Lemma 2.1, we have $F(B)\subset B'$, i.e.,

$$\sum_{j=1}^{m_2}|\, F_{n_2+j}(z,w)\,|^2=1. \tag{2.7}$$

Operating $\sum\limits_{k=1}^{n_1}\frac{\partial^2}{\partial z_k\partial\overline{z}_k}$ to (2.7), we have

$$\sum_{k=1}^{n_1}\sum_{j=1}^{m_2}\Big|\frac{\partial F_{n_2+j}(z,w)}{\partial z_k}\Big|^2=0.$$

Therefore

$$\frac{\partial F_{n_2+j}(z,w)}{\partial z_k}\equiv 0,\quad 1\leq k\leq n_1,\ 1\leq j\leq m_2$$

on B. Since B is a real manifold of dimension $2(n_1+m_1)-1$ in $\Omega(n_1,m_1)$, and $\frac{\partial F_{n_2+j}}{\partial z_k}$ is a holomorphic function, we have $\frac{\partial F_{n_2+j}}{\partial z_k}\equiv 0$, $1\leq k\leq n_1$, $1\leq j\leq m_2$, on $\Omega(n_1,m_1)$, which means that F_2 is independent of z.

3 Proof of Main Results

Proof of Theorem 1.1 Step 1 Fix w_0 such that $|w_0|^2 = 1$. From Lemmas 2.1 and 2.2, we can get $|F_2(w_0)|^2 = 1$. Set

$$F_{w_0} : \{z \in \mathbb{C}^{n_1} : 0 < |z|^2 < |w_0|^2 = 1\} \to \{z' \in \mathbb{C}^{n_2} : 0 < |z'|^2 < |F_2(w_0)|^2 = 1\},$$

which is a proper holomorphic mapping. Since $F_{w_0} : B \to B'$, we have $\forall z_n \to 0$, where $z_n \in \{z \in \mathbb{C}^{n_1} : 0 < |z|^2 < |w_0|^2 = 1\}$, $F_{w_0}(z_n) \to 0$. Otherwise, $F_{w_0}(z_n) \to B'$. By Hartogs extension theorem, we can extend F_{w_0} and we will still use F_{w_0} to denote this extended mapping:

$$F_{w_0} : \{z \in \mathbb{C}^{n_1} : |z|^2 < |w_0|^2 = 1\} \to \{z' \in \mathbb{C}^{n_2} : |z'|^2 \le |F_2(w_0)|^2 = 1\}.$$

If $F_{w_0}(z_n) \to B'$, then $|F_{w_0}(0)| = 1$, which contradicts the maximum principle. So $|F_{w_0}(0)| = 0$.

By Theorem A, we have

$$F_{w_0} = \theta_2(\underbrace{\theta_1 z}_{n_1}, \underbrace{0, \cdots, 0}_{n_2-n_1}), \tag{3.1}$$

where $\theta_1 \in \mathrm{Aut}(B_{n_1}), \theta_2 \in \mathrm{Aut}(B_{n_2})$. Using the representation of automorphism of the unit ball, and $\theta_1(z^0) = 0, \theta_2(u^0) = 0, z^0 \in \mathbb{C}^{n_1}, u^0 \in \mathbb{C}^{n_2}$, we have

$$\theta_1 : (z_1, \cdots, z_{n_1}) \to (u_1, \cdots, u_{n_1}) \quad \text{and} \quad u_j = \frac{\sum\limits_{k=1}^{n_1} q_{jk}(z_k - z_k^0)}{\Big(1 - \sum\limits_{k=1}^{n_1} \overline{z_k^0} z_k\Big) R_1},$$

where

$$\begin{aligned} &z^0 = (z_1^0, \cdots, z_{n_1}^0) \in \mathbb{C}^{n_1}, \quad Q = (q_{jk})_{1 \le j,k \le n_1}, \\ &\overline{Q}(I - \overline{z^0}^t z^0) Q^t = I_{n_1}, \quad \overline{R}_1(1 - z^0 \overline{z^0}^t) R_1 = 1; \\ &\theta_2 : (u_1, \cdots, u_{n_1}, 0, \cdots, 0) \to (f_1, \cdots, f_{n_2}) \quad \text{and} \quad f_j = \frac{\sum\limits_{k=1}^{n_1} q^*_{jk}(u_k - u_k^0)}{\Big(1 - \sum\limits_{k=1}^{n_1} \overline{u_k^0} u_k\Big) R_2}, \end{aligned} \tag{3.2}$$

where

$$\begin{aligned} &u^0 = (u_1^0, \cdots, u_{n_2}^0) \in \mathbb{C}^{n_2}, \quad Q^* = (q^*_{jk})_{1 \le j,k \le n_2}, \\ &\overline{Q^*}(I - \overline{u^0}^t u^0) Q^{*t} = I_{n_2}, \quad \overline{R}_2(1 - u^0 \overline{u^0}^t) R_2 = 1. \end{aligned} \tag{3.3}$$

Set

$$\lambda_1 = \Big(1 - \sum_{k=1}^{n_1} \overline{z_k^0} z_k\Big) R_1, \quad \lambda_2 = \Big(1 - \sum_{k=1}^{n_1} \overline{u_k^0} u_k\Big) R_2.$$

By the properties of $F_{w_0}(0, w_0) = 0$, we have

$$u^0 = \Big(\frac{-z^0 Q^t}{R_1}, 0\Big). \tag{3.4}$$

Then from (3.1), (3.2), $F^t_{w_0}$ can be expressed as follows:

$$\begin{aligned} F^t_{w_0} &= \frac{1}{\lambda_1\lambda_2} Q^* \begin{pmatrix} Q(z^t - {z^0}^t) + Q{z^0}^t(1-\overline{z^0}z^t) \\ 0 \end{pmatrix} \\ &= \frac{1}{\lambda_1\lambda_2} Q^* \begin{pmatrix} (Qz^t - Q{z^0}^t\overline{z^0}z^t) \\ 0 \end{pmatrix} = \frac{1}{\lambda_1\lambda_2} Q^* \begin{pmatrix} Q(I - {z^0}^t\overline{z^0})z^t \\ 0 \end{pmatrix} \\ &= \frac{1}{\lambda_1\lambda_2} Q^* \begin{pmatrix} \overline{Q^t}^{-1} z^t \\ 0 \end{pmatrix}. \end{aligned} \tag{3.5}$$

By (3.3), (3.4),

$$\begin{aligned} \lambda_1\lambda_2 &= (1-\overline{u^0}u^t)R_2(1-\overline{z^0}z^t)R_1 \\ &= (1-\overline{z^0}z^t)R_1R_2 + \frac{\overline{z^0}\,\overline{Q^t}Q(z^t - {z^0}^t)R_2}{\overline{R}_1} \\ &= \frac{R_2}{\overline{R}_1}(\overline{R}_1(1-\overline{z^0}z^t)R_1 + \overline{z^0}\,\overline{Q^t}Q(z^t - {z^0}^t)) \end{aligned} \tag{3.6}$$

and

$$I - \overline{z^0}^t z^0 = \overline{Q}^{-1}{Q^t}^{-1}.$$

Then

$$F^t_{w_0} = \frac{\overline{R}_1}{R_2} \frac{Q^*\begin{pmatrix} \overline{Q^t}^{-1} z^t \\ 0 \end{pmatrix}}{\overline{R}_1(1-\overline{z^0}z^t)R_1 + \overline{z^0}\,\overline{Q^t}Q(z^t - {z^0}^t)}. \tag{3.7}$$

Without loss of generality, let

$$z^0 = (z^0, 0, \cdots, 0), \quad z = (\mathrm{e}^{\mathrm{i}\theta}, 0, \cdots, 0), \tag{3.8}$$

since (U, I_{m_1}) is an automorphism of $\Omega(n_1, m_1)$, where U is a unitary matrix.

Using (3.2) again, we have

$$\begin{aligned} &|R_1|^2 = \frac{1}{1-|z^0|^2}, \\ &\overline{Q}\begin{pmatrix} 1-|z^0|^2 & 0 \\ 0 & I_{n_1-1} \end{pmatrix} Q^t = I, \\ &Q^{-1}\overline{{Q^t}^{-1}} = (\overline{Q^t}Q)^{-1} = \begin{pmatrix} 1-|z^0|^2 & 0 \\ 0 & I_{n_1-1} \end{pmatrix}. \end{aligned} \tag{3.9}$$

Then

$$\overline{Q^t}Q = \begin{pmatrix} \frac{1}{1-|z^0|^2} & 0 \\ 0 & I_{n_1-1} \end{pmatrix}. \tag{3.10}$$

Similarly, using (3.3) and (3.4) again, we have

$$\begin{aligned} &|R_2|^2 = \frac{1}{1-|u^0|^2} = \frac{1}{1-\frac{\overline{z^0}\,\overline{Q^t}Q{z^0}^t}{|R_1|^2}} = \frac{|R_1|^2}{|R_1|^2 - |R_1|^2|z^0|^2} = |R_1|^2, \\ &\overline{Q^*}\,(I - \overline{u^0}^t u^0){Q^*}^t = \overline{Q^*}\left(I - \begin{pmatrix} \frac{-\overline{Q}\,\overline{{z^0}^t}}{\overline{R}_1} \\ 0 \end{pmatrix}\left(\frac{-z^0Q^t}{R_1}, 0\right)\right){Q^*}^t = I, \\ &({Q^*}^t\overline{Q^*})^{-1} = \overline{{Q^*}^{-1}}{{Q^*}^t}^{-1} = \begin{pmatrix} I_{n_1} - \frac{\overline{Q}\,\overline{{z^0}^t}z^0Q^t}{|R_1|^2} & 0 \\ 0 & I_{n_2-n_1} \end{pmatrix}. \end{aligned} \tag{3.11}$$

Then

$$\overline{Q^{*t}}Q^* = \begin{pmatrix} I_{n_1} - \frac{Qz^{0t}\overline{z^0}\,\overline{Q^t}}{|R_1|^2} & 0 \\ 0 & I_{n_2-n_1} \end{pmatrix}^{-1}. \tag{3.12}$$

By (3.2),

$$Q\overline{Q^t} - Qz^{0t}\overline{z^0}\,\overline{Q^t} = I_{n_1} \quad \text{and} \quad Qz^{0t}\overline{z^0}\,\overline{Q^t} = Q\overline{Q^t} - I_{n_1}.$$

Then from (3.12), we have

$$\overline{Q^{*t}}Q^* = \begin{pmatrix} I_{n_1} - \frac{Q\overline{Q^t}-I_{n_1}}{|R_1|^2} & 0 \\ 0 & I_{n_2-n_1} \end{pmatrix}^{-1}. \tag{3.13}$$

On the other hand, by (3.8)–(3.10)

$$\begin{aligned} \overline{R}_1(1-\overline{z^0}z^t)R_1 + \overline{z^0}\,\overline{Q^t}Q(z^t - z^{0t}) &= |R_1|^2(1-\overline{z^0}z^t) + |R_1|^2(\overline{z^0}z^t - \overline{z^0}z^{0t}) \\ &= |R_1|^2(1-\overline{z^0}z^{0t}) = 1. \end{aligned} \tag{3.14}$$

Now we can rewrite (3.7) as follows:

$$F_{w_0}{}^t = \frac{\overline{R}_1}{R_2}Q^* \begin{pmatrix} \overline{Q^t}^{-1}z^t \\ 0 \end{pmatrix}. \tag{3.15}$$

So

$$\begin{aligned} |F_{w_0}|^2 &= (\overline{z}Q^{-1},0)\overline{Q^{*t}}Q^* \begin{pmatrix} \overline{Q^{t-1}}z^t \\ 0 \end{pmatrix} \\ &= (\overline{z}Q^{-1},0) \begin{pmatrix} I_{n_1} - \frac{Q\overline{Q^t}-I_{n_1}}{|R_1|^2} & 0 \\ 0 & I_{n_2-n_1} \end{pmatrix}^{-1} \begin{pmatrix} \overline{Q^{t-1}}z^t \\ 0 \end{pmatrix} \\ &= \overline{z}Q^{-1}\Big(I_{n_1} - \frac{Q\overline{Q^t}-I_{n_1}}{|R_1|^2}\Big)^{-1}\overline{Q^{t-1}}z^t, \end{aligned} \tag{3.16}$$

where

$$\begin{aligned} &Q^{-1}\Big(I_{n_1} - \frac{Q\overline{Q^t}-I_{n_1}}{|R_1|^2}\Big)^{-1}\overline{Q^{t-1}} \\ &= \Big(\overline{Q^t}\Big(I_{n_1} - \frac{Q\overline{Q^t}-I_{n_1}}{|R_1|^2}\Big)Q\Big)^{-1} = \Big(\overline{Q^t}Q + \frac{\overline{Q^t}Q}{|R_1|^2} - \frac{\overline{Q^t}Q\overline{Q^t}Q}{|R_1|^2}\Big)^{-1} \\ &= \left(1 + \frac{1}{|R_1|^2}\begin{pmatrix} |R_1|^2 & 0 \\ 0 & I_{n_1-1} \end{pmatrix} - \frac{1}{|R_1|^2}\begin{pmatrix} |R_1|^4 & 0 \\ 0 & I_{n_1-1} \end{pmatrix}\right)^{-1} \\ &= \left(\frac{1}{|R_1|^2}\begin{pmatrix} |R_1|^2 & 0 \\ 0 & |R_1|^2 I_{n_1-1} \end{pmatrix}\right)^{-1} = I_{n_1}. \end{aligned} \tag{3.17}$$

Let

$$\gamma^* = \begin{pmatrix} \frac{1}{|R_1|} & 0 & 0 \\ 0 & \sqrt{1-\frac{1}{|R_1|^2}}\,I_{n_1-1} & 0 \\ 0 & 0 & I_{n_2-n_1} \end{pmatrix} \quad \text{and} \quad \gamma = \begin{pmatrix} \frac{1}{|R_1|} & 0 \\ 0 & \sqrt{1-\frac{1}{|R_1|^2}}\,I_{n_1-1} \end{pmatrix}. \tag{3.18}$$

From (3.2), (3.10), (3.13),

$$\begin{aligned}\overline{Q^{*t}}Q^* &= \begin{pmatrix} I_{n_1} - \frac{Q\overline{Q^t}-I_{n_1}}{|R_1|^2} & 0 \\ 0 & I_{n_2-n_1} \end{pmatrix}^{-1} \\ &= \begin{pmatrix} \frac{1}{|R_1|^2} & 0 & 0 \\ 0 & 1-\frac{1}{|R_1|^2}I_{n_1-1} & 0 \\ 0 & 0 & I_{n_2-n_1} \end{pmatrix}^{-1} \\ &= (\gamma^*\overline{\gamma^*}^t)^{-1} = \overline{\gamma^*}^{t\,-1}\gamma^{*\,-1}. \end{aligned} \tag{3.19}$$

Now, let $\overline{\mathbf{B}^t(w_0)} = Q^*\gamma^*$. Then

$$\mathbf{B}(w_0)\overline{\mathbf{B}^t(w_0)} = \overline{\gamma^*}^t\overline{Q^*}^tQ^*\gamma^* = I_{n_2},$$

which means that $\mathbf{B}(w_0) \in \mathbf{U}(n_2)$, where $\mathbf{U}(n_2)$ is the unitary group of degree n_2.

Since $|R_1|^2 = |R_2|^2$, we have $\frac{R_1}{R_2} = \mathrm{e}^{\mathrm{i}\alpha}, \alpha \in \mathbb{R}$. Let $\overline{\mathbf{A}^t(w_0)} = \mathrm{e}^{\mathrm{i}\alpha}\gamma^{-1}\overline{Q^{t\,-1}}$. Then from (3.17) and the definition of γ, we have

$$\mathbf{A}^t(w_0)\overline{\mathbf{A}(w_0)} = I_{n_1},$$

which means that $\mathbf{A}(w_0) \in \mathbf{U}(n_1)$, where $\mathbf{U}(n_1)$ is the unitary group of degree n_1.

Now we can rewrite (3.15) as

$$F_{w_0}^t = \overline{\mathbf{B}^t(w_0)}\begin{pmatrix} \overline{\mathbf{A}^t(w_0)}z^t \\ 0 \end{pmatrix}, \quad \text{i.e.,} \quad F_{w_0} = (z\mathbf{A}(w_0), 0)\mathbf{B}(w_0). \tag{3.20}$$

Step 2 From Lemma 2.2, F_2 is independent of z.

$$F_2 : \{w \in \mathbb{C}^{m_1} : 0 < |w|^2 < 1\} \to \{w' \in \mathbb{C}^{m_2} : 0 < |w'|^2 < 1\}$$

is a proper holomorphic mapping. Then by Hartogs extension theorem, we can extend F_2 so that

$$F_2 : \{w \in \mathbb{C}^{m_1} : |w|^2 < 1\} \to \{w' \in \mathbb{C}^{m_2} : |w'|^2 < 1\}$$

with $F_2(0) = 0$. Use Theorem A again,

$$F_2 = \theta_2'(\underbrace{\theta_1' w}_{m_1}, \underbrace{0, \cdots, 0}_{m_2-m_1}).$$

By the proper properties of F_2, for every $w : |w| = 1, |F_2| = 1$. With the same argument used in Step 1, we have that θ_1', θ_2' are unitary transformations.

Now we can assume

$$\theta'(z) = z\mathbf{A}',\ \mathbf{A}' \in \mathbf{U}(m_1), \quad \theta_1'(w) = w\mathbf{B}',\ \mathbf{B}' \in \mathbf{U}(m_2),$$

where $z \in \mathbb{C}^{m_1},\ w \in \mathbb{C}^{m_2}$, $\mathbf{U}(m_1),\ \mathbf{U}(m_2)$ are unitary groups of degree m_1 and m_2 respectively. Then

$$F_2(w) = (w\mathbf{A}', 0)\mathbf{B}'. \tag{3.21}$$

Step 3 From the above expression of F_2, we have $|F_2(w)| = |w|, \forall\, w : |w| = l \le 1$. Now for a given $w : |w| = l \le 1$, set

$$F_1(z, w) : \{z \in \mathbb{C}^{n_1} : 0 < |z|^2 < |w|^2 = l^2\} \to \{z' \in \mathbb{C}^{n_2} : 0 < |z'|^2 < |F_2(w)|^2 = l^2\},$$

which is a proper holomorphic mapping. By Hartogs extension theorem, we can extend F_1 so that

$$F_1(z,w):\{z\in\mathbb{C}^{n_1}:|z|^2<|w|^2=l^2\}\to\{z'\in\mathbb{C}^{n_2}:|z'|^2<|F_2(w)|^2=l^2\}$$

is a proper holomorphic mapping with $F_1(0,w)=0$. It is easy to see that

$$F_1(z,w)=\frac{|F_2(w)|}{|w|}(z\mathbf{A}(w),0)\mathbf{B}(w)=\frac{l}{l}(z\mathbf{A}(w),0)\mathbf{B}(w)=(z\mathbf{A}(w),0)\mathbf{B}(w),$$

where $\mathbf{A}(w)\in\mathbf{U}(n_1)$, $\mathbf{B}(w)\in\mathbf{U}(n_2)$, $\mathbf{U}(n_1),\mathbf{U}(n_2)$ are unitary groups of degree n_1 and n_2 respectively. Since $F_1(z,w)$ is holomorphic on z and w, we have

$$\frac{\mathrm{d}}{\mathrm{d}\overline{w}}(\mathbf{A}(w),0)\mathbf{B}(w)=0.$$

As we know, $\mathbf{A}(w)\overline{\mathbf{A}^t(w)}=I_{n_1}$, $\mathbf{B}(w)\overline{\mathbf{B}^t(w)}=I_{n_2}$. Set

$$\mathbf{B}(w)=\begin{pmatrix}\mathbf{B}_1(w) & \mathbf{B}_2(w)\\ \mathbf{B}_3(w) & \mathbf{B}_4(w)\end{pmatrix},$$

where $\mathbf{B}_1(w)$ is an $n_1\times n_1$ matrix, $\mathbf{B}_2(w)$ is an $n_1\times(n_2-n_1)$ matrix. Then $(\mathbf{A}(w),0)\mathbf{B}(w)=(\mathbf{A}(w)\mathbf{B}_1(w),\mathbf{A}(w)\mathbf{B}_2(w))$, and

$$\begin{aligned}&(\mathbf{A}(w)\mathbf{B}_1(w),\mathbf{A}(w)\mathbf{B}_2(w))\overline{(\mathbf{A}(w)\mathbf{B}_1(w),\mathbf{A}(w)\mathbf{B}_2(w))}^t\\ &=\mathbf{A}(w)\mathbf{B}_1(w)\overline{(\mathbf{A}(w)\mathbf{B}_1(w))}^t+\mathbf{A}(w)\mathbf{B}_2(w)\overline{(\mathbf{A}(w)\mathbf{B}_2(w))}^t\\ &=\mathbf{A}(w)(\mathbf{B}_1(w)\overline{\mathbf{B}_1(w)}^t+\mathbf{B}_2(w)\overline{\mathbf{B}_2(w)}^t)\overline{\mathbf{A}(w)}^t=\mathbf{A}(w)\overline{\mathbf{A}(w)}^t=I_{n_1}.\end{aligned}$$

Set $(\mathbf{A}(w)\mathbf{B}_1(w))=(\varphi_{ij})_{1\le i,j\le n_1}$, and $(\mathbf{A}(w)\mathbf{B}_2(w))=(\psi_{i\alpha})_{1\le i\le n_1,1\le\alpha\le n_2-n_1}$, where φ_{ij}, $\psi_{i\alpha}$ are holomorphic dependent on $w_1,\cdots,w_{m_1}$, and

$$\begin{aligned}&\sum_{1\le i,j\le n_1}|\varphi_{ij}|^2+\sum_{1\le i\le n_1,1\le\alpha\le n_2-n_1}|\psi_{i\alpha}|^2\\ &=\mathrm{tr}[(\mathbf{A}(w)\mathbf{B}_1(w),\mathbf{A}(w)\mathbf{B}_2(w))\overline{(\mathbf{A}(w)\mathbf{B}_1(w),\mathbf{A}(w)\mathbf{B}_2(w))}^t]\\ &=n_1.\end{aligned}$$

Operating $\sum\limits_{k=1}^{m_1}\frac{\partial^2}{\partial w_k\partial\overline{w}_k}$ on the above equation, we have

$$\sum_{\substack{1\le i,j\le n_1\\1\le k\le m_1}}\Big|\frac{\partial\varphi_{ij}}{\partial w_k}\Big|^2+\sum_{\substack{1\le i\le n_1\\1\le\alpha\le n_2-n_1\\1\le k\le m_1}}\Big|\frac{\partial\psi_{i\alpha}}{\partial w_k}\Big|^2=0,$$

i.e., $\varphi_{ij},\psi_{i\alpha}$ are all independent of $w_1,\cdots,w_{m_1}$, so that $\mathbf{A}(w)\mathbf{B}_1(w)$ and $\mathbf{A}(w)\mathbf{B}_2(w)$ are all independent of $w_1,\cdots,w_{m_1}$. Hence, there exist constant matrices $A\in\mathbf{U}(n_1)$ and $B\in\mathbf{U}(n_2)$ such that $(\mathbf{A}(w),0)\mathbf{B}(w)=(\mathbf{A},0)\mathbf{B}$. Thus

$$F_1(z,w)=(z\mathbf{A},0)\mathbf{B}.\tag{3.22}$$

Step 4 Now it is obvious from (3.21), (3.22) that

$$F=(F_1,F_2):(z,w)\to(\underbrace{(z\mathbf{A},0)\mathbf{B}}_{n_2},\underbrace{(w\mathbf{A}',0)\mathbf{B}'}_{m_2}).\tag{3.23}$$

By the main theorem in [7], any proper holomorphic self-mapping of $\Omega(n_1, m_1)$ or $\Omega(n_2, m_2)$ is automorphism. Without loss of generality, let $\sigma \in \mathrm{Aut}(\Omega(n_1, m_1))$, $\tau \in \mathrm{Aut}(\Omega(n_2, m_2))$, such that

$$\sigma : (z, w) \to (z\mathbf{A}, w\mathbf{A}'),$$
$$\tau : (z', w') \to (z'\mathbf{B}, w'\mathbf{B}'),$$

where $\mathbf{A}, \mathbf{A}', \mathbf{B}, \mathbf{B}'$ are unitary matrices of degree n_1, m_1, n_2, m_2 respectively. Then from (3.23), for the proper holomorphic mapping: $F : \Omega(n_1, m_1) \to \Omega(n_2, m_2)$, that is twice continuously differentiable up to the boundary, there exist σ, τ which are automorphisms of $\Omega(n_1, m_1)$ and $\Omega(n_2, m_2)$ respectively, such that

$$\tau \circ F \circ \sigma(z, w) = (z_1, \cdots, z_{n_1}, \underbrace{0, \cdots, 0}_{n_2 - n_1}, w_1, \cdots, w_{m_1}, \underbrace{0, \cdots, 0}_{m_2 - m_1}).$$

The proof of Theorem 1.1 is completed.

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