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# A General Law of Precise Asymptotics for the Complete Moment Convergence***

Yong ZHANG* Xiaoyun YANG* Zhishan DONG**

**Abstract** The authors achieve a general law of precise asymptotics for a new kind of complete moment convergence of i.i.d. random variables, which includes complete convergence as a special case. It can describe the relations among the boundary function, weighted function, convergence rate and limit value in studies of complete convergence. This extends and generalizes the corresponding results of Liu and Lin in 2006.

**Keywords** Complete moment convergence, General law, Precise asymptotics
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## 1 Introduction

Throughout this paper, let $\{X, X_n, n \geq 1\}$ be a sequence of i.i.d. random variables, $S_n = \sum\limits_{i=1}^{n} X_i$ and $N$ be the standard normal random variable. $C$ denotes positive constants, possibly varying from place to place, and $[x]$ denotes the largest integer $\leq x$.

Since P. L. Hsu and H. Robbins [6] introduced the concept of complete convergence, there have been extensions in several directions. One of them is to discuss the precise rate and limit value of $\sum\limits_{n=1}^{\infty} \varphi(n) P\{|S_n| \geq \varepsilon g(n)\}$ as $\varepsilon \downarrow a$, $a \geq 0$, where $\varphi(x)$ and $g(x)$ are the positive functions defined on $[0, \infty)$. We call $\varphi(x)$ and $g(x)$ weighted function and boundary function respectively. The first result in this direction was due to C. C. Heyde [5], who proved that

$$\lim_{\varepsilon \downarrow 0} \varepsilon^2 \sum_{n=1}^{\infty} P\{|S_n| \geq \varepsilon n\} = EX^2, \tag{1.1}$$

where $EX = 0$ and $EX^2 < \infty$. For analogous results in more general case, see [2, 3, 9, 11], etc. The research in this field is called the precise asymptotics. W. D. Liu and Z. Y. Lin [7] studied the precise asymptotics for a new kind of complete moment convergence of i.i.d. random variables.

W. D. Liu and Z. Y. Lin [7] achieved the following three results.

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*School of Mathematics, Jilin University (Qianwei Campus), Changchun 130012, China.
E-mail: zyong2661@sina.com
**Corresponding author. School of Mathematics, Jilin University (Qianwei Campus), Changchun 130012, China. E-mail: Dongzs@mail.jlu.edu.cn
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**Theorem 1.1** *Suppose that*

$$EX = 0, \quad EX^2 = \sigma^2 \quad \text{and} \quad EX^2 \log^+ |X| < \infty. \tag{1.2}$$

*Then we have*

$$\lim_{\varepsilon \downarrow 0} \frac{1}{-\log \varepsilon} \sum_{n=1}^{\infty} \frac{1}{n^2} ES_n^2 I\{|S_n| \geq \varepsilon n\} = 2\sigma^2. \tag{1.3}$$

*Conversely, if* (1.3) *is true, then* (1.2) *holds.*

**Theorem 1.2** *Suppose that*

$$EX = 0, \quad EX^2 = \sigma^2 < \infty. \tag{1.4}$$

*Then we have*

$$\lim_{\varepsilon \downarrow 0} \varepsilon^{2-p} \sum_{n=1}^{\infty} \frac{1}{n^p} E|S_n|^p I\{|S_n| \geq \varepsilon n\} = \frac{2}{2-p} \sigma^2 \tag{1.5}$$

*for* $0 \leq p < 2$*. Conversely, if* (1.5) *is true for some* $0 \leq p < 2$*, then* (1.4) *holds.*

**Theorem 1.3** *Suppose that*

$$EX = 0, \quad EX^2 = \sigma^2 \quad \text{and} \quad EX^2 (\log^+ |X|)^\delta < \infty, \tag{1.6}$$

*where* $0 < \delta \leq 1$*. Then we have*

$$\lim_{\varepsilon \downarrow 0} \varepsilon^{2\delta} \sum_{n=1}^{\infty} \frac{(\log n)^{\delta-1}}{n^2} ES_n^2 I\{|S_n| \geq \varepsilon \sqrt{n \log n}\} = \frac{\sigma^{2\delta+2}}{\delta} E|N|^{2\delta+2}. \tag{1.7}$$

*Conversely, if* (1.7) *is true, then* (1.6) *holds.*

In this paper, we will extend the scope of the weighted functions and boundary functions, and give a general law of precise asymptotics of i.i.d. random variables, which extend and generalize the direct part of [7, Theorems 1.1–1.3].

## 2 Main Results

We will make some appropriate limitations to $\varphi(x)$ and $g(x)$ in the following theorems, and then get some corollaries according to the kind of $g(x)$. From these corollaries we can conclude a series of interesting results, which contain the direct part of [7, Theorems 1.1–1.3].

**Theorem 2.1** *Let* $g(x)$ *be a positive and differentiable function defined on* $[n_0, \infty)$*, which is strictly increasing to* $\infty$*.* $g'(x)$ *is monotone. If* $g'(x)$ *is monotone nondecreasing, we assume* $\lim\limits_{n\to\infty} \frac{g'(n+1)}{g'(n)} = 1$*. Assume that* $\varphi(x) = \frac{g'(x)}{g(x)}$ *is monotone, and if* $\varphi(x)$ *is monotone nondecreasing, we assume* $\lim\limits_{n\to\infty} \frac{\varphi(n+1)}{\varphi(n)} = 1$*. And assume that the following condition is satisfied:*

$$\limsup_{n\to\infty} n\varphi(n) < \infty. \tag{2.1}$$

*Finally suppose that the following conditions of i.i.d. random variables* $\{X, X_n, n \geq 1\}$ *are satisfied:*

$$EX = 0, \quad EX^2 = \sigma^2 \quad \text{and} \quad EX^2 \log^+ |X| < \infty. \tag{2.2}$$

*Then we have*

$$\lim_{\varepsilon\downarrow 0}\frac{1}{-\log\varepsilon}\sum_{n=n_0}^{\infty}\varphi(n)E\Big|\frac{S_n}{\sqrt{n}}\Big|^2 I\{|S_n|\ge\varepsilon\sqrt{ng(n)}\}=2\sigma^2. \tag{2.3}$$

**Theorem 2.2** *Let $g(x)$ be a positive and differentiable function defined on $[n_0,\infty)$, which is strictly increasing to $\infty$. $g'(x)$ is monotone. If $g'(x)$ is monotone nondecreasing, we assume $\lim_{n\to\infty}\frac{g'(n+1)}{g'(n)}=1$. Assume that $\varphi(x)=\frac{g'(x)}{g^{ps}(x)}$ is monotone, where $\frac{1}{s}>p\ge 0$. If $\varphi(x)$ is monotone nondecreasing, we assume $\lim_{n\to\infty}\frac{\varphi(n+1)}{\varphi(n)}=1$. And assume that the following condition is satisfied:*

$$\limsup_{n\to\infty} n^{\frac{p}{2}}\varphi(n)<\infty. \tag{2.4}$$

*Finally suppose that the following conditions of i.i.d. random variables $\{X,X_n,n\ge 1\}$ are satisfied:*

$$EX=0,\ EX^2=\sigma^2<\infty\quad \textit{for } 0\le p<2; \tag{2.5}$$

$$EX=0,\ EX^2=\sigma^2,\ EX^2\log^+|X|<\infty\quad \textit{for } p=2; \tag{2.6}$$

$$EX=0,\ EX^2=\sigma^2,\ E|X|^p<\infty\quad \textit{for } p>2. \tag{2.7}$$

*Then we have*

$$\lim_{\varepsilon\downarrow 0}\varepsilon^{\frac{1}{s}-p}\sum_{n=n_0}^{\infty}\varphi(n)E\Big|\frac{S_n}{\sqrt{n}}\Big|^p I\{|S_n|\ge\varepsilon\sqrt{n}g^s(n)\}=\frac{\sigma^{\frac{1}{s}}}{1-ps}E|N|^{\frac{1}{s}}. \tag{2.8}$$

**Remark 2.1** In Theorem 2.1 or Theorem 2.2, $\lim_{n\to\infty}\frac{\varphi(n+1)}{\varphi(n)}=1$, (2.1) and (2.4) are all mild conditions. $g(x)=x^\alpha$, $(\log x)^\beta$, $(\log\log x)^\gamma$ with some suitable conditions of $\alpha>0$, $\beta>0$, $\gamma>0$ and some others all satisfy these conditions.

**Remark 2.2** Letting $g(x)=x$ in Theorem 2.1, we can get the direct part of [7, Theorem 1].

**Remark 2.3** If $s=\frac{1}{2}$, $0\le p<2$ and $g(x)=x$, we can get the direct part of [7, Theorem 2].

**Remark 2.4** If $s=\frac{1}{2\delta+2}$, $p=2$ and $g(x)=(\log x)^{\delta+1}$, we can get the direct part of [7, Theorem 3] with $EX^2\log^+|X|<\infty$ instead of $EX^2(\log^+|X|)^\delta<\infty$. W. D. Liu and Z. Y. Lin [7] restricts $0<\delta\le 1$, but in our theorem we only restrict $\delta>0$.

**Remark 2.5** If $p=0$, the conditions of Theorem 2.2 still hold, then we can get [1, Theorem 1] with $\alpha=2$.

**Remark 2.6** If $p=0$, $s=\frac{1}{2}$, $g(x)=\log\log x$, we can get [3, Theorem 2].

**Remark 2.7** If $p=0$, $s=\frac{2-t}{2(r-t)}$, $g(x)=x^{\frac{r}{t}-1}$ for $0<t<2$, $r>t$, we can get [2, Corollary 1].

## 3 Several Lemmas

To prove our theorems, we need some lemmas as follows.

**Lemma 3.1** (see [1]) *Suppose that $0<\alpha\le 2$, $\{X, X_n, n\ge 1\}$ is a sequence of i.i.d. random variables which belongs to the normal domain of attraction of a nondegenerate stable distribution $G_\alpha$ with characteristic exponent $\alpha$ and $EX=0$ when $1<\alpha\le 2$. Let $g(x)$ be a positive and differentiable function defined on $[n_0,\infty)$, which is strictly increasing to $\infty$. $g'(x)$ is monotone. If $g'(x)$ is monotone nondecreasing, we assume $\lim\limits_{n\to\infty}\frac{g'(n+1)}{g'(n)}=1$. Then, $\forall\, s>\frac{1}{\alpha}$, we have*

$$\lim_{\varepsilon\downarrow 0}\varepsilon^{\frac{1}{s}}\sum_{n=n_0}^{\infty}g'(n)P\{|S_n-a_n|\ge\varepsilon b_n g^s(n)\}=E|Z_\alpha|^{\frac{1}{s}},$$

*where $Z_\alpha$ is a random variable having the distribution $G_\alpha$, $a_n$ and $b_n$ are the centralizing and normalizing constants respectively.*

**Lemma 3.2** *Suppose that $\{X, X_n, n\ge 1\}$ is a sequence of i.i.d. random variables with $E|X|^\beta<\infty$, where $1<\beta\le 2$. Let $S_n=\sum\limits_{i=1}^{n}X_i$. Then $\forall\, x,y>0$, we have*

$$P\{|S_n|\ge x\}\le nP\{|X|\ge y\}+2n^{\frac{x}{y}}\Big(\frac{\mathrm{e}E|X|^\beta}{xy^{\beta-1}}\Big)^{\frac{x}{y}}. \tag{3.1}$$

**Proof** To prove (3.1), it suffices to show that

$$P\{|S_n|\ge x\}\le nP\{|X|\ge y\}+2\mathrm{e}^{\frac{x}{y}}\Big(\frac{nE|X|^\beta}{nE|X|^\beta+xy^{\beta-1}}\Big)^{\frac{x}{y}}. \tag{3.2}$$

However, (3.2) is [11, Lemma 2], which, in turn, is based on [8, Theorem 1.2].

**Lemma 3.3** (see [10]) *Suppose that $\{X, X_n, n\ge 1\}$ is a sequence of i.i.d. random variables, $S_n=\sum\limits_{i=1}^{n}X_i$. We have*

$$C_1\lambda^{-2}EX^2I\{|X|\ge\lambda\}\le\sum_{n=1}^{\infty}P\{|S_n|\ge n\lambda\}\le C_2\lambda^{-2}EX^2I\{|X|\ge\lambda\} \tag{3.3}$$

*for any $\lambda>0$, where $C_1$ and $C_2$ are positive absolute constants.*

## 4 Proof of Theorem 2.1

Set $b(\varepsilon)=[g^{-1}(\varepsilon^{-2})]$, where $g^{-1}(x)$ is the inverse function of $g(x)$. Without loss of generality, we assume that $\sigma^2=1$.

**Proposition 4.1**

$$\lim_{\varepsilon\downarrow 0}\frac{1}{-\log\varepsilon}\sum_{n=n_0}^{\infty}\varphi(n)\int_{\varepsilon\sqrt{g(n)}}^{\infty}2xP\{|N|\ge x\}\mathrm{d}x=2.$$

**Proof** At first, we discuss the relations between the integral and the series. If $\varphi(y)$ is nonincreasing, then $\varphi(y)\int_{\varepsilon\sqrt{g(y)}}^{\infty}2xP\{|N|\ge x\}\mathrm{d}x$ is nonincreasing. Hence we have

$$\begin{aligned}\int_{n_0+1}^{\infty}\varphi(y)\int_{\varepsilon\sqrt{g(y)}}^{\infty}2xP\{|N|\ge x\}\mathrm{d}x\mathrm{d}y&\le\sum_{n=n_0+1}^{\infty}\varphi(n)\int_{\varepsilon\sqrt{g(n)}}^{\infty}2xP\{|N|\ge x\}\mathrm{d}x\\&\le\int_{n_0}^{\infty}\varphi(y)\int_{\varepsilon\sqrt{g(y)}}^{\infty}2xP\{|N|\ge x\}\mathrm{d}x\mathrm{d}y.\end{aligned}$$

Then we have

$$\begin{aligned}
&\lim_{\varepsilon\downarrow 0}\frac{1}{-\log\varepsilon}\sum_{n=n_0}^{\infty}\varphi(n)\int_{\varepsilon\sqrt{g(n)}}^{\infty}2xP\{|N|\ge x\}\mathrm{d}x\\
&=\lim_{\varepsilon\downarrow 0}\frac{1}{-\log\varepsilon}\int_{n_0}^{\infty}\frac{g'(y)}{g(y)}\int_{\varepsilon\sqrt{g(y)}}^{\infty}2xP\{|N|\ge x\}\mathrm{d}x\mathrm{d}y\\
&=\lim_{\varepsilon\downarrow 0}\frac{1}{-\log\varepsilon}\int_{g(n_0)}^{\infty}\frac{1}{t}\int_{\varepsilon\sqrt{t}}^{\infty}2xP\{|N|\ge x\}\mathrm{d}x\mathrm{d}t\\
&=\lim_{\varepsilon\downarrow 0}\frac{2}{-\log\varepsilon}\int_{\varepsilon\sqrt{g(n_0)}}^{\infty}\frac{1}{y}\int_{y}^{\infty}2xP\{|N|\ge x\}\mathrm{d}x\mathrm{d}y\\
&=\lim_{\varepsilon\downarrow 0}2\int_{\varepsilon\sqrt{g(n_0)}}^{\infty}2xP\{|N|\ge x\}\mathrm{d}x\\
&=2. \qquad (4.1)
\end{aligned}$$

If $\varphi(y)$ is nondecreasing, then by $\lim_{n\to\infty}\frac{\varphi(n+1)}{\varphi(n)}=1$ for any $0<\delta<1$, there exists $n_1=n_1(\delta)$, such that $\frac{\varphi(n+1)}{\varphi(n)}<1+\delta$ and $\frac{\varphi(n)}{\varphi(n+1)}>1-\delta$ for $n\ge n_1$. Thus we have

$$\begin{aligned}
&(1+\delta)^{-1}\int_{n_1+1}^{\infty}\varphi(y)\int_{\varepsilon\sqrt{g(y)}}^{\infty}2xP\{|N|\ge x\}\mathrm{d}x\mathrm{d}y\\
&\le\sum_{n=n_1+1}^{\infty}\varphi(n)\int_{\varepsilon\sqrt{g(n)}}^{\infty}2xP\{|N|\ge x\}\mathrm{d}x\\
&\le(1-\delta)^{-1}\int_{n_1}^{\infty}\varphi(y)\int_{\varepsilon\sqrt{g(y)}}^{\infty}2xP\{|N|\ge x\}\mathrm{d}x\mathrm{d}y.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\lim_{\varepsilon\downarrow 0}\frac{1}{-\log\varepsilon}(1+\delta)^{-1}\int_{n_1+1}^{\infty}\varphi(y)\int_{\varepsilon\sqrt{g(y)}}^{\infty}2xP\{|N|\ge x\}\mathrm{d}x\mathrm{d}y\\
&\le\lim_{\varepsilon\downarrow 0}\frac{1}{-\log\varepsilon}\sum_{n=n_0}^{\infty}\varphi(n)\int_{\varepsilon\sqrt{g(n)}}^{\infty}2xP\{|N|\ge x\}\mathrm{d}x\\
&\le\lim_{\varepsilon\downarrow 0}\frac{1}{-\log\varepsilon}(1-\delta)^{-1}\int_{n_1}^{\infty}\varphi(y)\int_{\varepsilon\sqrt{g(y)}}^{\infty}2xP\{|N|\ge x\}\mathrm{d}x\mathrm{d}y.
\end{aligned}$$

And then by (4.1), we know

$$2(1+\delta)^{-1}\le\lim_{\varepsilon\downarrow 0}\frac{1}{-\log\varepsilon}\sum_{n=n_0}^{\infty}\varphi(n)\int_{\varepsilon\sqrt{g(n)}}^{\infty}2xP\{|N|\ge x\}\mathrm{d}x\le 2(1-\delta)^{-1}.$$

Let $\delta\downarrow 0$. Then we can conclude

$$\lim_{\varepsilon\downarrow 0}\frac{1}{-\log\varepsilon}\sum_{n=n_0}^{\infty}\varphi(n)\int_{\varepsilon\sqrt{g(n)}}^{\infty}2xP\{|N|\ge x\}\mathrm{d}x=2.$$

**Remark 4.1** In the following, for simplicity, we will omit the discuss of $\varphi(x)$, but the process is similar to the discussion of Proposition 4.1.

**Proposition 4.2**

$$\lim_{\varepsilon\downarrow 0}\frac{1}{-\log\varepsilon}\sum_{n=n_0}^{b(\varepsilon)}\varphi(n)\Big|\int_{\varepsilon\sqrt{g(n)}}^{\infty}2xP\{|S_n|\geq\sqrt{n}x\}\mathrm{d}x-\int_{\varepsilon\sqrt{g(n)}}^{\infty}2xP\{|N|\geq x\}\mathrm{d}x\Big|=0. \tag{4.2}$$

**Proof** Obviously,

$$\begin{aligned}&\sum_{n=n_0}^{b(\varepsilon)}\varphi(n)\Big|\int_{\varepsilon\sqrt{g(n)}}^{\infty}2xP\{|S_n|\geq\sqrt{n}x\}\mathrm{d}x-\int_{\varepsilon\sqrt{g(n)}}^{\infty}2xP\{|N|\geq x\}\mathrm{d}x\Big|\\&=\sum_{n=n_0}^{b(\varepsilon)}\frac{g'(n)}{g(n)}\Big|\int_0^{\infty}2(x+\varepsilon\sqrt{g(n)}\,)P\{|S_n|\geq\sqrt{n}(x+\varepsilon\sqrt{g(n)}\,)\}\mathrm{d}x\\&\quad-\int_0^{\infty}2(x+\varepsilon\sqrt{g(n)}\,)P\{|N|\geq(x+\varepsilon\sqrt{g(n)}\,)\}\mathrm{d}x\Big|\\&\leq\sum_{n=n_0}^{b(\varepsilon)}\frac{g'(n)}{g(n)}(\Delta_{n1}+\Delta_{n2}+\Delta_{n3}),\end{aligned}$$

where

$$\begin{aligned}\Delta_n&=\sup_x|P\{|S_n|\geq\sqrt{n}x\}-P\{|N|\geq x\}|,\\\Delta_{n1}&=\int_0^{\Delta_n^{-\frac14}}2(x+\varepsilon\sqrt{g(n)}\,)|P\{|S_n|\geq\sqrt{n}(x+\varepsilon\sqrt{g(n)}\,)\}-P\{|N|\geq(x+\varepsilon\sqrt{g(n)}\,)\}|\mathrm{d}x,\\\Delta_{n2}&=\int_{\Delta_n^{-\frac14}}^{\infty}2(x+\varepsilon\sqrt{g(n)}\,)P\{|S_n|\geq\sqrt{n}(x+\varepsilon\sqrt{g(n)}\,)\}\mathrm{d}x,\\\Delta_{n3}&=\int_{\Delta_n^{-\frac14}}^{\infty}2(x+\varepsilon\sqrt{g(n)}\,)P\{|N|\geq(x+\varepsilon\sqrt{g(n)}\,)\}\mathrm{d}x.\end{aligned}$$

Since $n\leq b(\varepsilon)$ implies $\varepsilon\sqrt{g(n)}\leq 1$, we have

$$\Delta_{n1}\leq\int_0^{\Delta_n^{-\frac14}}2(x+\varepsilon\sqrt{g(n)}\,)\Delta_n\mathrm{d}x\leq\Delta_n(\Delta_n^{-\frac14}+\varepsilon\sqrt{g(n)}\,)^2\leq(\Delta_n^{\frac14}+\Delta_n^{\frac12})^2. \tag{4.3}$$

For $\Delta_{n3}$, by Markov inequality, we have

$$\Delta_{n3}\leq C\int_{\Delta_n^{-\frac14}}^{\infty}\frac{1}{(x+\varepsilon\sqrt{g(n)}\,)^3}\mathrm{d}x\leq C\Delta_n^{\frac12}. \tag{4.4}$$

Now we estimate $\Delta_{n2}$. By Lemma 3.2, we have

$$\begin{aligned}\Delta_{n2}&\leq\int_{\Delta_n^{-\frac14}}^{\infty}2(x+\varepsilon\sqrt{g(n)}\,)nP\Big\{|X|\geq\frac{\sqrt{n}}{2}(x+\varepsilon\sqrt{g(n)}\,)\Big\}\mathrm{d}x\\&\quad+C\int_{\Delta_n^{-\frac14}}^{\infty}2(x+\varepsilon\sqrt{g(n)}\,)n^2\frac{1}{n^2(x+\varepsilon\sqrt{g(n)}\,)^4}\mathrm{d}x\\&\triangleq \mathrm{I}_1+\mathrm{I}_2.\end{aligned}$$

By Fubini Theorem and the fact $0 \le \Delta_n \le 1$, we have

$$\begin{aligned}
\mathrm{I}_1 &= E\int_{\Delta_n^{-\frac{1}{4}}}^{\infty} 2(x+\varepsilon\sqrt{g(n)}\,)nI\Big\{|X| \ge \frac{\sqrt{n}}{2}(x+\varepsilon\sqrt{g(n)}\,)\Big\}\mathrm{d}x \\
&\le EI\{4|X| \ge \sqrt{n}\,\}\int_0^{\infty} 2(x+\varepsilon\sqrt{g(n)}\,)nI\Big\{|X| \ge \frac{\sqrt{n}}{2}(x+\varepsilon\sqrt{g(n)}\,)\Big\}\mathrm{d}x \\
&\le EI\{4|X| \ge \sqrt{n}\,\}\int_0^{\frac{2|X|}{\sqrt{n}}-\varepsilon\sqrt{g(n)}} 2n(x+\varepsilon\sqrt{g(n)}\,)\mathrm{d}x \\
&\le 4E|X|^2 I\{4|X| \ge \sqrt{n}\,\}
\end{aligned} \tag{4.5}$$

and

$$\mathrm{I}_2 \le C\int_{\Delta_n^{-\frac{1}{4}}}^{\infty} \frac{1}{(x+\varepsilon\sqrt{g(n)}\,)^3}\mathrm{d}x \le C\Delta_n^{\frac{1}{2}}. \tag{4.6}$$

From (4.3)–(4.6) and the fact $\Delta_n \to 0$ as $n \to \infty$, we can get

$$\Delta_{n1} + \Delta_{n2} + \Delta_{n3} \to 0, \quad \text{as } n \to \infty. \tag{4.7}$$

Then by (4.7), the monotonicity of $\varphi(x)$ and Toeplitz Lemma (see [4]), we get (4.2). The proposition is now proved.

**Proposition 4.3**

$$\lim_{\varepsilon\downarrow 0} \frac{1}{-\log\varepsilon} \sum_{n=b(\varepsilon)+1}^{\infty} \varphi(n)\int_{\varepsilon\sqrt{g(n)}}^{\infty} 2xP\{|N| \ge x\}\mathrm{d}x = 0. \tag{4.8}$$

**Proof** By the monotonicity of $\varphi(x)$ and Markov inequality, we have

$$\begin{aligned}
&\lim_{\varepsilon\downarrow 0} \frac{1}{-\log\varepsilon} \sum_{n=b(\varepsilon)+1}^{\infty} \varphi(n)\int_{\varepsilon\sqrt{g(n)}}^{\infty} 2xP\{|N| \ge x\}\mathrm{d}x \\
\le &\lim_{\varepsilon\downarrow 0} \frac{1}{-\log\varepsilon} \sum_{n=b(\varepsilon)+1}^{\infty} \varphi(n)\int_{\varepsilon\sqrt{g(n)}}^{\infty} 2x\frac{E|N|^4}{x^4}\mathrm{d}x \\
\le &\lim_{\varepsilon\downarrow 0} \frac{C}{-\varepsilon^2\log\varepsilon} \sum_{n=b(\varepsilon)+1}^{\infty} \frac{g'(n)}{g^2(n)} \\
\le &\lim_{\varepsilon\downarrow 0} \frac{C}{-\varepsilon^2\log\varepsilon}\int_{\varepsilon^{-2}}^{\infty} \frac{1}{y^2}\mathrm{d}y \le \lim_{\varepsilon\downarrow 0}\frac{C}{-\log\varepsilon} = 0.
\end{aligned}$$

**Proposition 4.4**

$$\lim_{\varepsilon\downarrow 0} \frac{1}{-\log\varepsilon} \sum_{n=b(\varepsilon)+1}^{\infty} \varphi(n)\int_{\varepsilon\sqrt{g(n)}}^{\infty} 2xP\{|S_n| \ge \sqrt{n}x\}\mathrm{d}x = 0. \tag{4.9}$$

**Proof** Set

$$\begin{aligned}
&X_i' = X_i I\{|X_i| \le \lambda\}, \quad X_i'' = X_i I\{|X_i| > \lambda\}, \\
&S_n' = \sum_{i=1}^{n}(X_i' - EX_i'), \quad S_n'' = \sum_{i=1}^{n}(X_i'' - EX_i''),
\end{aligned}$$

where $\lambda > 1$. Clearly, $S_n = S'_n + S''_n$. Therefore,

$$\begin{aligned}
&\sum_{n=b(\varepsilon)+1}^{\infty} \varphi(n) \int_{\varepsilon\sqrt{g(n)}}^{\infty} 2xP\{|S_n| \geq \sqrt{n}x\}\mathrm{d}x \\
&\leq \sum_{n=b(\varepsilon)+1}^{\infty} \varphi(n) \int_{\varepsilon\sqrt{g(n)}}^{\infty} 2xP\Big\{|S'_n| \geq \frac{\sqrt{n}x}{2}\Big\}\mathrm{d}x + \sum_{n=b(\varepsilon)+1}^{\infty} \varphi(n) \int_{\varepsilon\sqrt{g(n)}}^{\infty} 2xP\Big\{|S''_n| \geq \frac{\sqrt{n}x}{2}\Big\}\mathrm{d}x \\
&\triangleq \mathrm{I}_3 + \mathrm{I}_4.
\end{aligned}$$

For $\mathrm{I}_3$, by the monotonicity of $\varphi(x)$, Markov inequality and $E|S'_n|^4 \leq Cn^2\lambda^4$, we have

$$\begin{aligned}
\lim_{\varepsilon\downarrow 0} \frac{1}{-\log\varepsilon}\mathrm{I}_3 &\leq \lim_{\varepsilon\downarrow 0} \frac{1}{-\log\varepsilon} \sum_{n=b(\varepsilon)+1}^{\infty} \varphi(n) \int_{\varepsilon\sqrt{g(n)}}^{\infty} 2x\frac{4E|S'_n|^4}{n^2x^4}\mathrm{d}x \\
&\leq \lim_{\varepsilon\downarrow 0} \frac{C}{-\varepsilon^2\log\varepsilon} \sum_{n=b(\varepsilon)+1}^{\infty} \frac{g'(n)}{g^2(n)} \\
&\leq \lim_{\varepsilon\downarrow 0} \frac{C}{-\varepsilon^2\log\varepsilon} \int_{\varepsilon^{-2}}^{\infty} \frac{1}{y^2}\mathrm{d}y \leq \lim_{\varepsilon\downarrow 0} \frac{C}{-\log\varepsilon} = 0. \qquad (4.10)
\end{aligned}$$

For $\mathrm{I}_4$, by condition (2.1), Fubini Theorem and Lemma 3.3, we get

$$\begin{aligned}
\mathrm{I}_4 &= \sum_{n=b(\varepsilon)+1}^{\infty} \varphi(n) \int_{\varepsilon\sqrt{g(n)}}^{\infty} 2xP\Big\{|S''_n| \geq \frac{\sqrt{n}x}{2}\Big\}\mathrm{d}x \\
&= \sum_{n=b(\varepsilon)+1}^{\infty} n\varphi(n) \int_{0}^{\infty} 2(t+\varepsilon)P\Big\{|S''_n| \geq \frac{n(t+\varepsilon)}{2}\Big\}\mathrm{d}t \\
&\leq C\int_{0}^{\infty} (t+\varepsilon) \sum_{n=1}^{\infty} P\Big\{|S''_n| \geq \frac{n(t+\varepsilon)}{2}\Big\}\mathrm{d}t \\
&\leq C\int_{0}^{\infty} (t+\varepsilon) \frac{E|Y|^2 I\{|Y| \geq \frac{t+\varepsilon}{2}\}}{(t+\varepsilon)^2}\mathrm{d}t \\
&= CE\Big(|Y|^2 \int_{0}^{\infty} \frac{I\{|Y| \geq \frac{t+\varepsilon}{2}\}}{(t+\varepsilon)}\mathrm{d}t\Big) \\
&\leq CE|Y|^2\log^{+} 2|Y| - C(\log\varepsilon)E|Y|^2,
\end{aligned}$$

where $Y$ denotes $X''_1 - EX''_1$. Then by (2.2) we have

$$\lim_{\lambda\to\infty} \limsup_{\varepsilon\downarrow 0} \frac{1}{-\log\varepsilon}\mathrm{I}_4 = 0. \qquad (4.11)$$

From (4.10) and (4.11), we complete the proof of Proposition 4.4.

**Proof of Theorem 2.1** Note that

$$\begin{aligned}
&\sum_{n=n_0}^{\infty} \varphi(n)E\Big|\frac{S_n}{\sqrt{n}}\Big|^2 I\{|S_n| \geq \varepsilon\sqrt{ng(n)}\} \\
&= \varepsilon^2 \sum_{n=n_0}^{\infty} \varphi(n)g(n)P\{|S_n| \geq \varepsilon\sqrt{ng(n)}\} + \sum_{n=n_0}^{\infty} \varphi(n) \int_{\varepsilon\sqrt{g(n)}}^{\infty} 2xP\{|S_n| \geq \sqrt{n}x\}\mathrm{d}x.
\end{aligned}$$

In order to prove (2.3), we just need to show that

$$\lim_{\varepsilon\downarrow 0}\frac{\varepsilon^2}{-\log\varepsilon}\sum_{n=n_0}^{\infty}g'(n)P\{|S_n|\geq\varepsilon\sqrt{ng(n)}\,\}=0 \tag{4.12}$$

and

$$\lim_{\varepsilon\downarrow 0}\frac{1}{-\log\varepsilon}\sum_{n=n_0}^{\infty}\varphi(n)\int_{\varepsilon\sqrt{g(n)}}^{\infty}2xP\{|S_n|\geq\sqrt{n}x\}\mathrm{d}x=2. \tag{4.13}$$

By the same argument of Lemma 3.1, we know that for any $\frac{1}{s}>p$,

$$\lim_{\varepsilon\downarrow 0}\varepsilon^{\frac{1}{s}}\sum_{n=n_0}^{\infty}g'(n)P\{|S_n|\geq\varepsilon\sqrt{n}g^s(n)\}=E|N|^{\frac{1}{s}}. \tag{4.14}$$

Let $s=\frac{1}{2}$ in (4.14). Then we can obtain (4.12). (4.13) can be proved by Propositions 4.1–4.4 and the triangular inequality.

## 5 Proof of Theorem 2.2

Set $d(\varepsilon)=[g^{-1}(M\varepsilon^{-\frac{1}{s}})]$, where $g^{-1}(x)$ is the inverse function of $g(x)$, $M\geq 1$. Without loss of generality, we assume that $\sigma^2=1$.

**Proposition 5.1** *For $p>0$, we have*

$$\lim_{\varepsilon\downarrow 0}\varepsilon^{\frac{1}{s}-p}\sum_{n=n_0}^{\infty}\varphi(n)\int_{\varepsilon g^s(n)}^{\infty}px^{p-1}P\{|N|\geq x\}\mathrm{d}x=\frac{ps}{1-ps}E|N|^{\frac{1}{s}}. \tag{5.1}$$

**Proof** At first we discuss the relations between the integral and the series. If $\varphi(y)$ is nonincreasing, then $\varphi(y)\int_{\varepsilon g^s(y)}^{\infty}px^{p-1}P\{|N|\geq x\}\mathrm{d}x$ is nonincreasing. Hence we have

$$\begin{aligned}\int_{n_0+1}^{\infty}\varphi(y)\int_{\varepsilon g^s(y)}^{\infty}px^{p-1}P\{|N|\geq x\}\mathrm{d}x\mathrm{d}y &\leq\sum_{n=n_0+1}^{\infty}\varphi(n)\int_{\varepsilon g^s(n)}^{\infty}px^{p-1}P\{|N|\geq x\}\mathrm{d}x\\ &\leq\int_{n_0}^{\infty}\varphi(y)\int_{\varepsilon g^s(y)}^{\infty}px^{p-1}P\{|N|\geq x\}\mathrm{d}x\mathrm{d}y.\end{aligned}$$

Then we have

$$\begin{aligned}&\lim_{\varepsilon\downarrow 0}\varepsilon^{\frac{1}{s}-p}\sum_{n=n_0}^{\infty}\varphi(n)\int_{\varepsilon g^s(n)}^{\infty}px^{p-1}P\{|N|\geq x\}\mathrm{d}x\\ =&\lim_{\varepsilon\downarrow 0}\varepsilon^{\frac{1}{s}-p}\int_{n_0}^{\infty}\varphi(y)\int_{\varepsilon g^s(y)}^{\infty}px^{p-1}P\{|N|\geq x\}\mathrm{d}x\mathrm{d}y\\ =&\lim_{\varepsilon\downarrow 0}\varepsilon^{\frac{1}{s}-p}\int_{g(n_0)}^{\infty}\frac{1}{t^{ps}}\int_{\varepsilon t^s}^{\infty}px^{p-1}P\{|N|\geq x\}\mathrm{d}x\mathrm{d}t\end{aligned}$$

$$
\begin{aligned}
&= \lim_{\varepsilon\downarrow 0}\frac{p}{s}\int_{\varepsilon g^s(n_0)}^{\infty}\frac{1}{t^{p+1-\frac{1}{s}}}\int_t^{\infty}x^{p-1}P\{|N|\geq x\}\mathrm{d}x\mathrm{d}t\\
&= \frac{p}{s}\int_0^{\infty}\frac{1}{t^{p+1-\frac{1}{s}}}\int_t^{\infty}x^{p-1}P\{|N|\geq x\}\mathrm{d}x\mathrm{d}t\\
&= \frac{p}{s}\int_0^{\infty}x^{p-1}P\{|N|\geq x\}\int_0^{x}\frac{1}{t^{p+1-\frac{1}{s}}}\mathrm{d}t\mathrm{d}x\\
&= \frac{ps}{1-ps}\int_0^{\infty}\frac{1}{s}x^{\frac{1}{s}-1}P\{|N|\geq x\}\mathrm{d}x\\
&= \frac{ps}{1-ps}E|X|^{\frac{1}{s}}.
\end{aligned}
$$

If $\varphi(y)$ is nondecreasing, then by $\lim\limits_{n\to\infty}\frac{\varphi(n+1)}{\varphi(n)}=1$, the proof is similar to that of Proposition 4.1. Thus we can get Proposition 5.1 by the above steps.

**Remark 5.1** In the following, for simplicity, we will omit the discussion of $\varphi(x)$, but the process is similar to the discussion of Proposition 5.1.

**Proposition 5.2** *For $p>0$, we have*

$$
\lim_{\varepsilon\downarrow 0}\varepsilon^{\frac{1}{s}-p}\sum_{n=n_0}^{d(\varepsilon)}\varphi(n)\Big|\int_{\varepsilon g^s(n)}^{\infty}px^{p-1}P\{|S_n|\geq\sqrt{n}x\}\mathrm{d}x-\int_{\varepsilon g^s(n)}^{\infty}px^{p-1}P\{|N|\geq x\}\mathrm{d}x\Big|=0. \tag{5.2}
$$

**Proof** It is easy to see that

$$
\begin{aligned}
&\varepsilon^{\frac{1}{s}-p}\sum_{n=n_0}^{d(\varepsilon)}\varphi(n)\Big|\int_{\varepsilon g^s(n)}^{\infty}px^{p-1}P\{|S_n|\geq\sqrt{n}x\}\mathrm{d}x-\int_{\varepsilon g^s(n)}^{\infty}px^{p-1}P\{|N|\geq x\}\mathrm{d}x\Big|\\
\leq&\sum_{n=n_0}^{d(\varepsilon)}\varphi(n)\int_0^{\infty}p(x+\varepsilon g^s(n))^{p-1}|P\{|S_n|\geq\sqrt{n}(x+\varepsilon g^s(n))\}-P\{|N|\geq(x+\varepsilon g^s(n))\}|\mathrm{d}x\\
\leq&\sum_{n=n_0}^{d(\varepsilon)}\varphi(n)(\Delta'_{n1}+\Delta'_{n2}+\Delta'_{n3}),
\end{aligned}
$$

where

$$
\begin{aligned}
\Delta'_{n1}&=\int_0^{\Delta_n^{-\frac{1}{2p}}}p(x+\varepsilon g^s(n))^{p-1}|P\{|S_n|\geq\sqrt{n}(x+\varepsilon g^s(n))\}-P\{|N|\geq(x+\varepsilon g^s(n))\}|\mathrm{d}x,\\
\Delta'_{n2}&=\int_{\Delta_n^{-\frac{1}{2p}}}^{\infty}p(x+\varepsilon g^s(n))^{p-1}P\{|S_n|\geq\sqrt{n}(x+\varepsilon g^s(n))\}\mathrm{d}x,\\
\Delta'_{n3}&=\int_{\Delta_n^{-\frac{1}{2p}}}^{\infty}p(x+\varepsilon g^s(n))^{p-1}P\{|N|\geq(x+\varepsilon g^s(n))\}\mathrm{d}x.
\end{aligned}
$$

Since $n\leq d(\varepsilon)$ implies $\varepsilon g^s(n)\leq M^s$, we have

$$
\Delta'_{n1}\leq\int_0^{\Delta_n^{-\frac{1}{2p}}}p(x+\varepsilon g^s(n))^{p-1}\Delta_n\mathrm{d}x\leq\Delta_n(\Delta_n^{-\frac{1}{2p}}+\varepsilon g^s(n))^p\leq(\Delta_n^{\frac{1}{2p}}+M^s\Delta_n^{\frac{1}{p}})^p. \tag{5.3}
$$

For $\Delta'_{n3}$, by Markov inequality, we have

$$
\Delta'_{n3}\leq C\int_{\Delta_n^{-\frac{1}{2p}}}^{\infty}\frac{1}{(x+\varepsilon g^s(n))^3}\mathrm{d}x\leq C\Delta_n^{\frac{1}{p}}. \tag{5.4}
$$

Now we estimate $\Delta'_{n2}$. First we consider $0 < p < 2$. By Markov inequality, we have

$$\Delta'_{n2} \le C \int_{\Delta_n^{-\frac{1}{2p}}}^{\infty} \frac{1}{(x+\varepsilon g^s(n))^{3-p}} \mathrm{d}x \le C \Delta_n^{\frac{1}{p}-\frac{1}{2}}. \tag{5.5}$$

For $p \ge 2$, by Lemma 3.2, choosing $\beta = 2$ and $x = py$, we know

$$\begin{aligned} \Delta'_{n2} &\le \int_{\Delta_n^{-\frac{1}{2p}}}^{\infty} p(x+\varepsilon g^s(n))^{p-1} n P\Big\{|X| \ge \frac{\sqrt{n}}{p}(x+\varepsilon g^s(n))\Big\} \mathrm{d}x \\ &\quad + C \int_{\Delta_n^{-\frac{1}{2p}}}^{\infty} p(x+\varepsilon g^s(n))^{p-1} n^p \frac{1}{n^p (x+\varepsilon g^s(n))^{2p}} \mathrm{d}x \\ &\triangleq \mathrm{J}_1 + \mathrm{J}_2. \end{aligned}$$

By Fubini Theorem and the fact $0 \le \Delta_n \le 1$, we have

$$\begin{aligned} \mathrm{J}_1 &= E \int_{\Delta_n^{-\frac{1}{2p}}}^{\infty} p(x+\varepsilon g^s(n))^{p-1} n I\Big\{|X| \ge \frac{\sqrt{n}}{p}(x+\varepsilon g^s(n))\Big\} \mathrm{d}x \\ &\le E I\{2p|X| \ge \sqrt{n}\} \int_0^{\infty} p(x+\varepsilon g^s(n))^{p-1} n I\Big\{|X| \ge \frac{\sqrt{n}}{p}(x+\varepsilon g^s(n))\Big\} \mathrm{d}x \\ &\le E I\{2p|X| \ge \sqrt{n}\} \int_0^{\frac{p|X|}{\sqrt{n}}-\varepsilon g^s(n)} p(x+\varepsilon g^s(n))^{p-1} n \mathrm{d}x \\ &\le C E I\{2p|X| \ge \sqrt{n}\} n \frac{|X|^p}{n^{\frac{p}{2}}} \\ &\le C E|X|^p I\{2p|X| \ge \sqrt{n}\} \end{aligned} \tag{5.6}$$

and

$$\mathrm{J}_2 \le C \int_{\Delta_n^{-\frac{1}{2p}}}^{\infty} \frac{1}{(x+\varepsilon g^s(n))^{p+1}} \mathrm{d}x \le C \Delta_n^{\frac{1}{2}}. \tag{5.7}$$

From (5.3)–(5.7) and the fact $\Delta_n \to 0$ as $n \to \infty$, we can get

$$\Delta'_{n1} + \Delta'_{n2} + \Delta'_{n3} \to 0 \quad \text{as } n \to \infty. \tag{5.8}$$

Then by (5.8), the monotonicity of $\varphi(x)$ and Toeplitz Lemma (see [4]), we get (5.2). The proposition is now proved.

**Proposition 5.3** *For $p > 0$, we have*

$$\lim_{M\to\infty} \limsup_{\varepsilon\downarrow 0} \varepsilon^{\frac{1}{s}-p} \sum_{n=d(\varepsilon)+1}^{\infty} \varphi(n) \int_{\varepsilon g^s(n)}^{\infty} p x^{p-1} P\{|N| \ge x\} \mathrm{d}x = 0. \tag{5.9}$$

**Proof** By the monotonicity of $\varphi(x)$ and Markov inequality, we have

$$\begin{aligned} &\lim_{M\to\infty} \limsup_{\varepsilon\downarrow 0} \varepsilon^{\frac{1}{s}-p} \sum_{n=d(\varepsilon)+1}^{\infty} \varphi(n) \int_{\varepsilon g^s(n)}^{\infty} p x^{p-1} P\{|N| \ge x\} \mathrm{d}x \\ \le &\lim_{M\to\infty} \limsup_{\varepsilon\downarrow 0} C \varepsilon^{\frac{1}{s}-p} \sum_{n=d(\varepsilon)+1}^{\infty} \frac{g'(n)}{g^{ps}(n)} \int_{\varepsilon g^s(n)}^{\infty} p x^{p-1} \frac{E|N|^{\frac{2}{s}}}{x^{\frac{2}{s}}} \mathrm{d}x \end{aligned}$$

$$\begin{aligned}
&\le \lim_{M\to\infty}\limsup_{\varepsilon\downarrow 0} C\varepsilon^{-\frac{1}{s}}\sum_{n=d(\varepsilon)+1}^{\infty}\frac{g'(n)}{g^2(n)}\\
&\le \lim_{M\to\infty}\limsup_{\varepsilon\downarrow 0} C\varepsilon^{-\frac{1}{s}}\int_{M\varepsilon^{-\frac{1}{s}}}^{\infty}\frac{1}{y^2}\mathrm{d}y\\
&\le \lim_{M\to\infty}\limsup_{\varepsilon\downarrow 0} C\frac{1}{M}=0.
\end{aligned}$$

**Proposition 5.4** *For $p>0$, we have*

$$\lim_{M\to\infty}\lim_{\varepsilon\downarrow 0}\varepsilon^{\frac{1}{s}-p}\sum_{n=d(\varepsilon)+1}^{\infty}\varphi(n)\int_{\varepsilon g^s(n)}^{\infty}px^{p-1}P\{|S_n|\ge\sqrt{n}x\}\mathrm{d}x=0. \tag{5.10}$$

**Proof** Define $X_i'$, $X_i''$, $S_n'$ and $S_n''$ as in the proof of Proposition 4.4. Therefore

$$\begin{aligned}
&\sum_{n=d(\varepsilon)+1}^{\infty}\varphi(n)\int_{\varepsilon g^s(n)}^{\infty}px^{p-1}P\{|S_n|\ge\sqrt{n}x\}\mathrm{d}x\\
&\le\sum_{n=d(\varepsilon)+1}^{\infty}\varphi(n)\Big[\int_{\varepsilon g^s(n)}^{\infty}px^{p-1}P\Big\{|S_n'|\ge\frac{\sqrt{n}x}{2}\Big\}\mathrm{d}x+\int_{\varepsilon g^s(n)}^{\infty}px^{p-1}P\Big\{|S_n''|\ge\frac{\sqrt{n}x}{2}\Big\}\mathrm{d}x\Big]\\
&\triangleq \mathrm{J}_3+\mathrm{J}_4.
\end{aligned}$$

For $\mathrm{J}_3$, by the monotonicity of $\varphi(x)$, Markov inequality and $E|S_n'|^{\frac{2}{s}}\le Cn^{\frac{1}{s}}\lambda^{\frac{2}{s}}$, we have

$$\begin{aligned}
\lim_{M\to\infty}\limsup_{\varepsilon\downarrow 0}\varepsilon^{\frac{1}{s}-p}\mathrm{J}_3 &\le \lim_{M\to\infty}\limsup_{\varepsilon\downarrow 0} C\varepsilon^{\frac{1}{s}-p}\sum_{n=d(\varepsilon)+1}^{\infty}\varphi(n)\int_{\varepsilon g^s(n)}^{\infty}px^{p-1}\frac{E|S_n'|^{\frac{2}{s}}}{n^{\frac{1}{s}}x^{\frac{2}{s}}}\mathrm{d}x\\
&\le \lim_{M\to\infty}\limsup_{\varepsilon\downarrow 0} C\varepsilon^{\frac{1}{s}-p}\sum_{n=d(\varepsilon)+1}^{\infty}\varphi(n)\int_{\varepsilon g^s(n)}^{\infty}\frac{1}{x^{\frac{2}{s}-p+1}}\mathrm{d}x\\
&\le \lim_{M\to\infty}\limsup_{\varepsilon\downarrow 0} C\varepsilon^{-\frac{1}{s}}\sum_{n=d(\varepsilon)+1}^{\infty}\frac{g'(n)}{g^2(n)}\\
&\le \lim_{M\to\infty}\limsup_{\varepsilon\downarrow 0} C\varepsilon^{-\frac{1}{s}}\int_{M\varepsilon^{-\frac{1}{s}}}^{\infty}\frac{1}{y^2}\mathrm{d}y\\
&\le \lim_{M\to\infty}\limsup_{\varepsilon\downarrow 0} C\frac{1}{M}=0.
\end{aligned}$$

For $\mathrm{J}_4$, by condition (2.4), Fubini Theorem and Lemma 3.3, we get

$$\begin{aligned}
\mathrm{J}_4 &= \sum_{n=d(\varepsilon)+1}^{\infty}\varphi(n)\int_{\varepsilon g^s(n)}^{\infty}px^{p-1}P\Big\{|S_n''|\ge\frac{\sqrt{n}x}{2}\Big\}\mathrm{d}x\\
&= \sum_{n=d(\varepsilon)+1}^{\infty}n^{\frac{p}{2}}\varphi(n)\int_0^{\infty}p(x+\varepsilon)^{p-1}P\Big\{|S_n''|\ge\frac{n(x+\varepsilon)}{2}\Big\}\mathrm{d}x\\
&\le C\int_0^{\infty}p(x+\varepsilon)^{p-1}\sum_{n=1}^{\infty}P\Big\{|S_n''|\ge\frac{n(x+\varepsilon)}{2}\Big\}\mathrm{d}x
\end{aligned}$$

$$\leq C\int_0^\infty p(x+\varepsilon)^{p-1}\frac{E|Y|^2 I\{|Y|\geq \frac{x+\varepsilon}{2}\}}{(x+\varepsilon)^2}\mathrm{d}x$$
$$= CpEY^2\int_0^\infty \frac{I\{|Y|\geq \frac{x+\varepsilon}{2}\}}{(x+\varepsilon)^{3-p}}\mathrm{d}x,$$

where $Y$ denotes $X_1'' - EX_1''$.

Note that

$$\mathrm{J}_4 \leq CEY^2\log^+|Y| - C(\log\varepsilon)EY^2$$

for $p=2$, and

$$\mathrm{J}_4 \leq CE|Y|^P - C(\varepsilon^{p-2})EY^2$$

for $p\neq 2$, and note that $\frac{1}{s}>p$, so we know

$$\lim_{\lambda\to\infty}\limsup_{\varepsilon\downarrow 0}\varepsilon^{\frac{1}{s}-p}\mathrm{J}_4 = 0.$$

From the above steps, we complete the proof.

**Proof of Theorem 2.2** Since

$$E\Big|\frac{S_n}{\sqrt{n}}\Big|^p I\{|S_n|\geq \varepsilon\sqrt{n}\,g^s(n)\} = P\{|S_n|\geq \varepsilon\sqrt{n}\,g^s(n)\},$$

where $p=0$, by (4.14) we can get Theorem 2.2. Therefore we just need to discuss the case $\frac{1}{s}>p>0$. Note that

$$\sum_{n=n_0}^\infty \varphi(n)E\Big|\frac{S_n}{\sqrt{n}}\Big|^p I\{|S_n|\geq \varepsilon\sqrt{n}g^s(n)\}$$
$$= \varepsilon^p\sum_{n=n_0}^\infty \varphi(n)g^{ps}(n)P\{|S_n|\geq \varepsilon\sqrt{n}g^s(n)\} + \sum_{n=n_0}^\infty \varphi(n)\int_{\varepsilon g^s(n)}^\infty px^{p-1}P\{|S_n|\geq \sqrt{n}x\}\mathrm{d}x.$$

In order to prove (2.8), we just need to show that

$$\lim_{\varepsilon\downarrow 0}\varepsilon^{\frac{1}{s}}\sum_{n=n_0}^\infty g'(n)P\{|S_n|\geq \varepsilon\sqrt{n}g^s(n)\} = E|N|^{\frac{1}{s}}, \tag{5.11}$$

$$\lim_{\varepsilon\downarrow 0}\varepsilon^{\frac{1}{s}-p}\sum_{n=n_0}^\infty \varphi(n)\int_{\varepsilon g^s(n)}^\infty px^{p-1}P\{|S_n|\geq \sqrt{n}x\}\mathrm{d}x = \frac{ps}{1-ps}E|N|^{\frac{1}{s}}. \tag{5.12}$$

By (4.14), we can get (5.11). (5.12) can be proved by Propositions 5.1–5.4 and the triangular inequality.

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