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REGULAR ARTICLE

Market clearing by maximum entropy in agent models of stock markets

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Abstract We replace the conventional market clearing process by maximizing the information entropy. At fixed return agents optimize their demands using an utility with a statistical tolerance against deviations from the deterministic maximum of the utility which leads to information entropies for the agents. Interactions are described by coupling the agents to a large system, called 'market'. The main problem in economic markets is the absence of the analogue of the first law of thermodynamics (energy conservation in physics). We solve this problem by restricting the utilities to be at most quadratic in the demand (ideal gas approximation). Maximizing the sum of agent and market entropy serves to eliminate the unknown properties of the latter. Assuming a stochastic volatility decomposition for the return we derive an expression for the pdf of the return which is in excellent agreement with the daily DAX data. The pdf exhibits a power law behaviour with an integer tail index equal to the number of agent groups. With the assumption of an Ornstein Uhlenbeck model for the risk aversity parameters of the agents the autocorrelation for the absolute return is well described up to time lags of 2.5 years.

1 Introduction

Deviations of the behaviour of the return in financial markets from an iid Gaussian are referred to as stylized facts (Lux and Ausloos 2002). Among these the power law in the pdf at large values of the return with a universal tail index and the persistent autocorrelation of the absolute return are particularly difficult to explain. We want to

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show that maximizing the entropy within a simple model of heterogeneous agents can account also for these stylized facts.

We start with the observation that determination of the return of stocks in heterogeneous agent models (Gaunersdorfer 2000; Chiarella and He 2002; Alfarano et al. 2005; Brock and Hommes 1998) involves usually three processes on very different time scales. The first is the market clearing condition at time scale Δt_m, where the agents chose their demand by optimization of an utility function at fixed return r. Δt_m must occur on a very short scale, otherwise quotation for the DAX every 30 s does not make any sense. A Walrasian market used in Kirman type models (Alfarano et al. 2005; Kirman 1993; Kirman and Teyssière 2002) or bounded heterogeneous agent models (Chiarella et al. 2009; LeBaron 2006; Hommes 2006) corresponds to $\Delta t_m = 0$. The second process must ensure that the return at observation time Δt_o is given by the product of an iid Gaussian noise (Wagner et al. 2010) and a volatility factor changing on a much longer scale Δt_v as seen from the autocorrelation. The noise can be due to fluctuation of the fundamental price (Lux and Marchesi 2000) or fast change between optimistic and pessimistic chartists (Alfarano et al. 2005). Since the volatility factor will depend on the parameters describing the behaviour of the agents, a third process must occur changing these parameters on a scale larger than Δt_v.

In this paper we perform market clearing by maximizing the entropy on the various time scales. At scales Δt_m the agents optimize their utilities allowing for statistical uncertainty in their demands. At larger times the demands are equilibrium distributed and we can define an information entropy (Jaynes 1957) as the expectation value of the log of the pdf. Trying to treat this entropy in analogy to thermodynamics in physics one encounters the following problem. By definition we have the second law (existence of entropy), but there is no analogue of the first law (existence of energy). A natural choice for the negative energy would be the total utility, but there is no reason why utility should be a quantity describing the system independent of history. However, in thermodynamics the ideal gas as a reference substance plays an important role. For this hypothetical system the second law can be derived from the first, and for us more important vice versa.[1] In statistical physics the ideal gas is described by an energy functional quadratic in the dynamical variables. Therefore we make the following ideal gas approximation, that the utility of agent i has to be quadratic in the demand π_i. This solves another problem in applying thermodynamical methods to economics. In the latter each agent optimizes his own utility u_i, whereas in statistical physics the total utility $\sum u_i$ should be optimized. For statistically independent agents both methods are equivalent. However, this does not allow interaction. A minimal interaction can be build in by introducing a large system. Agents exchange information with this system called the market. In analogy to a heat bath in physics we maximize at time scales larger than Δt_m the sum of the agent entropy and an unknown market entropy. This total entropy will depend on the return and the constants describing the behaviour of the agents, which both vary on scales larger than Δt_m. For the return we assume a stochastic volatility decomposition (Cont 2001) as a product of an iid noise factor varying with observation scale Δt_o and a volatility factor v depending on the stochastic agent

[1] See any textbook on statistical physics and thermodynamics.

parameters varying with scale Δt_m. As shown by Wagner et al. (2010) this decomposition with Gaussian noise is supported by empirical data. We have a situation similar to a spin glass system (Mezard et al. 1987) in physics, where magnetic moments with slowly varying couplings reach equilibrium very fast (analogue to the optimization of utilities). At later times observables (f. e. entropy) have to be averaged over the couplings. Since we have explicit expressions for the latter the nuisance of replica method will not be needed. Maximizing the averaged entropy with respect to the volatility factor leads to an equation of state describing the dependence of the return on the agent parameters. Maximizing the entropy with respect to the latter a considerable reduction of the number of independent parameters may be achieved.

The importance of a conservation law analogue to the first law of thermodynamics has been pointed out by Yakovenco and Rosser (2009). In their work (Dragulescu and Yakovenco 2000) utility is restricted to the conserved money, which leads to an exponential distribution of wealth or income. An alternative consists in the introduction of a new quantity. Examples are a welfare function (Negishi 1960), a hypothetical good used by Smith and Foley (2008) or surplus utility (Saslow 1999). In these cases application of thermodynamics leads to an interpretation of the new quantity with little predictive power. Restricting oneself to a two variable macro economic model (Mimke 2010) an unknown integrating factor can replace the missing conservation law.

We organize the paper in the following way. In Sect. 2 we describe the optimization of the utility of a representative agent by a Langevin equation leading the equilibrium information entropy for the ideal gas model. Section 3 contains the market clearing by the stationary properties of the entropy. Despite the elimination of the market entropy is different from physical systems the resulting total entropy is the analogue of the Gibbs potential in thermodynamics. In Sect. 4 we discuss the averaging over the slowly varying stochastic quantities, which leads to the equation of state and determination of agent parameters. Finally we apply in Sect. 5 our model to daily DAX data showing the stylized facts are successfully accounted for. Section 6 contains our conclusions.

2 Optimization of single agent utility

We consider a simplified market with only one stock with price P_t (index). We assume n representative agents with a demand π_i equal to the number of traded stocks. The agents are described by an utility based on risk aversity, which is optimized with respect to π_i at fixed values of the return $r = \ln P_t/P_{t-1}$. An Arrow–Pratt (Arrow 1971; Pratt 1964) utility $U(W)$ depends on the gain (good) $W = r\pi$ by trading π_i stocks at the expected return r. It should be an increasing and concave function of W (Neumann and Morgenstern 1953). For stock markets one uses frequently the exponential utility corresponding to a constant risk aversity γ_i ($CARA$)

$$U_{CARA}(W) = 1 - \exp(-\gamma_i W), \tag{1}$$

Following Chiarella et al. (2009) an uncertainty about W described by an Gaussian noise in r leads to an additional $\Delta W = \pi_i \eta$ with $\eta \in N(0, 1)$. The optimal value D of the demand π_i is found from the maximum of the mean value $E[U_{CARA}(W + \Delta W)]$. This leads to a linear relation between D_i and r

$$D_i = \frac{r}{\gamma_i}, \tag{2}$$

and a value of $E[U_{CARA}]$ at optimum

$$E[U_{CARA}(W + \Delta W)] = 1 - \exp(-r^2/2). \tag{3}$$

The size of ΔW at optimum is given by

$$E[(\Delta W)^2] = \frac{r^2}{\gamma_i^2}. \tag{4}$$

In this approach the demand is a deterministic quantity and no entropy exists. A non trivial information entropy can be achieved by placing the uncertainty into a stochastic optimization of π_i instead of in r. If the agents are characterized by an utility $u_i(\pi_i, r)$ (different from U_{CARA}), we use for the optimization a Langevin equation

$$d\pi = \nu_m \frac{\partial u_i}{\partial \pi} dt + \sqrt{2 \nu_m dt/\beta_i} \cdot \eta_t. \tag{5}$$

The first term corresponds to a deterministic maximization and the second to a statistical uncertainty described by a Brownian motion of size $1/\beta_i$. If we chose the inverse rate constant $1/\nu_m$ in the order of the market clearing time Δt_m, we need to consider at time scales larger than Δt_m only equilibrium properties. In equilibrium the demand π possesses the distribution (for details see "Appendix A")

$$w_i(\pi) \propto \exp(\beta_i u_i(\pi)) \tag{6}$$

which allows the computation of a relative information entropy S_i (Jaynes 1957):

$$S_i = E[-\ln(w_i/w_0)]. \tag{7}$$

The a priori distribution w_0 is needed in case of a continuous variable π. To treat the maximum entropy principle in analogy to physics one needs the entropy as function of the mean utility $U_i = E[u_i]$ with the property

$$\frac{\partial S_i}{\partial U_i} = -\beta_i. \tag{8}$$

This condition is met only for utilities $u_i(\pi)$ which are a quadratic form in π corresponding to an ideal gas. Therefore we add for $u_i(\pi)$ to the gain $r\pi$ a negative quadratic term

$$u_i(\pi, r) = r\pi - \frac{\gamma_i'}{2}\pi^2. \tag{9}$$

With this utility π is Gaussian distributed with a mean demand $D_i = E[\pi]$ given by

$$D_i = \frac{r}{\gamma_i'} \tag{10}$$

and a variance

$$\Delta D^2 = \frac{1}{\beta\gamma_i'} \tag{11}$$

To see the meaning of the coefficient γ_i' we compute $E[U_{CARA}(W+\Delta W)]$ with mean gain $W = D_i r$ and a noise from π as

$$\Delta W = \frac{\sqrt{\beta\gamma_i'}}{\gamma_i}(\pi - D_i) \tag{12}$$

The factor in front of the noise has to be chosen such that $E[(\Delta W)^2]$ agrees with the size of the fluctuation in the CARA mechanism from Eq. (4). Carrying out the Gaussian integration we get

$$E[U_{CARA}(W + \Delta W)] = 1 - \exp\left(-r^2\frac{2\gamma_i - \gamma_i'}{2\gamma_i'}\right). \tag{13}$$

For $\gamma_i' = \gamma_i$ both the value of $E[U_{CARA}]$ and the linear relation between D_i and r coincide for the conventional treatment and the stochastic optimization. Therefore γ_i' can be identified with the risk aversity parameter. The same result is obtained whether one optimizes $E[U_{CARA}(W+\Delta W)]$ with a noise placed in r with respect to π or uses a stochastic optimization of a quadratic utility and calculates $E[U_{CARA}(W+\Delta W)]$. However, only the latter method leads to an information entropy given by

$$S_i(\beta_i, r) = \frac{1}{2}\left(\ln\frac{2\pi}{\beta_i\gamma_i} + 1\right) + E[-\ln w_0]. \tag{14}$$

S_i from Eq. (14) has the disadvantage well known in physics that the first term becomes negative for large β_i and the contribution from w_0 cannot be neglected. In physical problems this is solved by quantum mechanics leading to discrete values of u_i and $E[-\ln w_0]$ can be put to zero. In analogy one could use integer values of π. This, however prevents any analytical calculations. Later on we will need only the limit of small β_i where this problem does not occur. Therefore we neglect the contribution of $E[-\ln w_0]$ in the following.

To prove the consistency relation (8) we consider $S_i(\beta_i, r)$ as function of the demand D_i and the mean utility U_i given by

$$U_i = E[u_i] = -\frac{1}{2\beta_i} + \frac{r^2}{2\gamma_i}. \tag{15}$$

By differentiation (see "Appendix A") we get Eq. (8) and

$$\frac{\partial S_i}{\partial D_i} = r\beta_i. \tag{16}$$

These relations are specific to the ideal gas model. For general $u(\pi)$ such relations exist only for the aggregated quantities $\sum U_i$ or $\sum D_i$ taking the number n of agents to ∞ (thermodynamic limit [see footnote 1]).

The representative agents are characterized by their risk aversity γ_i and their uncertainty β_i in the Langevin equation. The return $r(t)$ and $\gamma_i(t)$ can still be stochastic, time dependent variables provided they do not change over a time scale Δt_m. In the following sections we will use the maximum entropy principle to restrict the possible values of the parameters. In analogy to physics we will need the so called Gibbs function $G_i(\beta_i, r) = S_i + \beta_i\ U_i\ -\ \beta_i\ r D_i$ which is given by

$$G_i(\beta_i, r) = \frac{1}{2}\ln\frac{2\pi}{\beta_i\gamma_i}\ -\ \frac{\beta_i r^2}{2\gamma_i}\ . \tag{17}$$

3 Market clearing conditions from maximum entropy

The maximum entropy principle applies to closed systems. To obtain such a system, we assume the existence of a large system (market) with an equilibrium information entropy S_M. A minimal interaction between the agents can be build in analoguous to a system of free atoms or magnetic moments in thermal equilibrium. This is performed by taking the sum of the market entropy and the agent's entropy

$$S_{tot} = \sum_i S_i(U_i, D_i)\ +\ S_M(U, D_M). \tag{18}$$

The market demand D_M may differ from $-\sum D_i$ by a possible constant unsatisfied demand. S_M depends on U_i due to the following argument. As one sees f.e. from the German XETRA market (1997) the return offered by the market is found by an iterative process during which the market learns the demand D_i and its spread ΔD_i. Specific to the ideal gas model U_i can be expressed as

$$U_i = \frac{rD_i}{2}\left(1 - \frac{\Delta^2 D_i}{D_i^2}\right). \tag{19}$$

Therefore U_i is also known to the market. We use the maximization of S_{tot} in two ways. In equilibrium S_{tot} is stationary for any change δU_i or δD_i, which allows the determination of the properties of the unknown S_M. Then we use the maximum entropy principle in the next section for the derivation of an equation of state and for reducing the number of agent parameters.

In equilibrium S_{tot} is stationary for any change δU_i or δD_i

$$0 = \left(\frac{\partial S_i}{\partial U_i} + \frac{\partial S_M}{\partial U_i} \right) \delta U_i \; + \; \left(\frac{\partial S_i}{\partial D_i} + \frac{\partial S_M}{\partial D_i} \right) \delta D_i . \tag{20}$$

Since stocks are conserved we have $dD_i = -dD_M$ and therefore

$$\frac{\partial S_M}{\partial D_i} = -\frac{\partial S_M}{\partial D_M} = -p_M . \tag{21}$$

The market reacts on change of D_i with a constant p_M independent of the agents. Inserting Eq. (16) for $\partial S_i/\partial D_i$ we obtain that the coefficients β_i

$$\beta_i r = p_M \tag{22}$$

have the same value β and the market parameter is given by $p_M = \beta r$. Inserting Eq. (8) for $\partial S_i/\partial U_i$ we get

$$\frac{\partial S_M}{\partial U_i} = \beta . \tag{23}$$

Now we consider a change of the agent parameters implying a change of δU_i or δD_i which will increase the total entropy to a new equilibrium value. Using Eqs. (21) and (23) for the derivatives of S_M this change of entropy is given by

$$\delta S_{tot} = \sum_i \left(\frac{\partial S_i}{\partial U_i} + \beta \right) \delta U_i + \left(\frac{\partial S_i}{\partial D_i} - \beta \, r \right) \delta D_i . \tag{24}$$

In Eq. (24) the market entropy has disappeared. Its properties are taken into account only by the constants β and βr. By adding terms which vanish upon differentiation we write

$$\left(\frac{\partial S_i}{\partial U_i} \right)_{D_i = \text{const}} + \beta = \frac{\partial}{\partial U_i} \left(S_i + \beta U_i - \beta \, r D_i \right) \tag{25}$$

$$\left(\frac{\partial S_i}{\partial D_i} \right)_{U_i = \text{const}} - r \, \beta = \frac{\partial}{\partial D_i} \left(S_i + \beta U_i - \beta r D_i \right) . \tag{26}$$

Inserting (25) and (26) into Eq. (24) we get

$$\delta S_{tot} = \delta G \tag{27}$$

with the total Gibbs function G

$$G = \sum_i \left(S_i + \beta U_i - \beta r D_i\right). \tag{28}$$

Maximizing the total entropy is equivalent to maximize G depending only on the agent properties. In physics $-G/\beta$ corresponds to the Gibbs potential. With Eq. (17) we get for the ideal gas model

$$G = \frac{1}{2} \sum_i \left(\ln \frac{2\pi}{\beta \gamma_i} - \frac{r^2 \beta}{\gamma_i} \right). \tag{29}$$

The return r and the risk aversities can still be stochastic time dependent quantities as long they can be considered as constant on the time scale Δt_m of market clearing.

4 Determination of volatility and risk aversity parameters

In the entropy (29) we can insert any stochastic expression for the returns and risk aversity as long they do not change over time Δt_m. A frequently used parametrization of the return consists in the decomposition in a product of a slowly varying volatility factor and a Gaussian noise (Cont 2001)

$$r(t) = \sqrt{v(t)}\, \eta_t, \tag{30}$$

where η changes with observation scale Δt_o. γ_i and v vary on a much longer scale Δt_v. With this assumption we can average G over η in analogy to the spin glass systems in physics:

$$E[G]_\eta = \frac{1}{2} \sum_i \left(\ln \frac{2\pi}{\beta \gamma_i} - \frac{v\, \beta}{\gamma_i} \right). \tag{31}$$

Since in the limit $v \to 0$ there is no need for risk aversity, we assume that $1/\gamma_i$ vanishes with a power $(1-q)/q$ for $v \to 0$ which leads to following parametrization

$$1/\gamma_i = \alpha^2\, v^{1/q-1} \cdot \Phi_i^2. \tag{32}$$

The stochastic variables Φ_i expressing the risk aversity of the agents will change over the time scale Δt_v and are described by an Ornstein Uhlenbeck process:

$$d\Phi_i = -\nu_i \Phi_i dt + \sigma_i \sqrt{2\nu_i dt}\, \epsilon_i \tag{33}$$

with Gaussian iid distributed ϵ_i (Brownian noise). The rate constants ν_i are of order $1/\Delta t_v$. The main motivation for this choice is mathematical convenience to allow analytic computations. As long the Φ_i are iid distributed other choices will lead qualitatively to the same result.

Equation (33) can be integrated and there is no difference between the continuous time and a discrete time version of (33) (see "Appendix B"). In equilibrium the Φ_i are normal distributed with mean zero and variance σ_i^2. Since a common scale in σ_i^2 can be absorbed into α, we can normalize the risk aversity parameters to

$$\sum_i E[\Phi_i^2] = \sum_i \sigma_i^2 = n. \tag{34}$$

Inserting Eq. (32) into Eq. (31) we get for the G averaged over η

$$E[G]_\eta = G_0 + \frac{1}{2}\sum_i \ln \Phi_i^2 \; + \; \frac{n(1-q)}{2}\left(\ln(\frac{v}{v_0})^{1/q} \; - \; (\frac{v}{v_0})^{1/q} \cdot \sum_i \Phi_i^2 \right) \tag{35}$$

with a constant $G_0 = (n/2)\ln(2\pi\alpha^2/\beta)$ and a normalization constant

$$v_0 = \left(\frac{n(1-q)}{\beta\alpha^2} \right)^q . \tag{36}$$

Maximizing $E[G]_\eta$ as function of v yields the equation of state

$$v = \frac{v_0}{(\sum_i \Phi_i^2)^q} \tag{37}$$

At the maximum $E[G]_\eta$ the entropy can be written as

$$E[G]_\eta = G_0 + \frac{1}{2}\sum_i \ln \Phi_i^2 + \frac{n(1-q)}{2q}\left(\ln v_0 - q - q \ln \sum_i \Phi_i^2 \right). \tag{38}$$

If we average G over the remaining time dependence on scale Δt_v we calculate this average by taking the expectation value of Φ_i in equilibrium. It is easy to see that the maximum of $E[g_\Phi]$ conditional under Eq. (34) is obtained for agent independent σ_i

$$\sigma_i^2 = 1. \tag{39}$$

After these optimizations the time independent entropy $E[G]_{\eta,v}$ reads as

$$E[G]_{\eta,v} = G_0(\alpha, \beta) + \frac{n}{2}L_1 + \frac{n(1-q)}{2q}(\ln v_0 - q - qL_n) \, , \tag{40}$$

where the expectation values $L_n = E[\ln \sum_i \Phi_i^2]$ can be expressed by Euler's ψ function.

$$L_n = \psi\left(\frac{n}{2}\right) + \ln 2. \tag{41}$$

The entropy $E[G]_{\eta,v}$ in Eq. (40) increases as function of q, provided we use a value for v_0 (see "Appendix B") given by the empirical value of $E[r^2]$:

$$v_0 = \frac{2^q \Gamma(n/2)}{\Gamma(n/2-q)} E[r^2]. \tag{42}$$

Since $q < 1$, maximum entropy requires a value close to 1. To avoid a vanishing v_0 in Eq. (36) the product $\alpha^2\beta$ has to compensate the factor $1-q$ by a small β.

5 Comparison with empirical data (DAX)

Our ideal gas model contains for each agent as free parameters the tolerance $1/\beta_i$ against the deviations from a deterministic optimum, the relative size of the risk aversity σ_i and a rate parameter ν_i. Together with the exponent q this corresponds to $3n+1$ free parameters. The maximum entropy principle reduces this number to agent independent values $\beta_i = \beta$ and $\sigma_i = 1$ and will fix q to a value near to 1. The agents can differ only by their rate constants ν_i. So we are left with a normalization factor v_0, the number n of representative agents and n rate parameters ν_i in the Ornstein Uhlenbeck process for Φ_i.

To obtain the pdf of the return we insert Eq. (37) into Eq. (30)

$$r(t) = \sqrt{v_0}\left(\sum_i \Phi_i^2\right)^{-q/2} \cdot \eta_t. \tag{43}$$

In equilibrium the Φ_i are iid unit Gaussian distributed. For the pdf for r we get an expression differing slightly from a general class of stochastic volatility models discussed by Anteneodo and Riera (2005). Only in the limiting case of $q = 1$ one recovers a special case of the so called 3/2 model developed by Ahn and Gao (1999). Therefore this 3/2 model can be interpreted as an agent model with little dependence of the risk parameter γ on the volatility. Another interpretation of the 3/2 model in terms of an agent model with Kirman type herding has been given in Wagner (2006). For $q = 1$ the pdf of r corresponds a Student't distribution:

$$f(r) = \frac{\Gamma((n+1)/2)}{\Gamma(n/2)}\left(2^{n+1}\pi v_0\right)^{-1/2} \cdot \left(1+\frac{r^2}{v_0}\right)^{-(n+1)/2}. \tag{44}$$

We can eliminate the normalization factor v_0 in favor of the observed $E[r^2]$ using the moment relation

$$v_0 = (n-2)\ E[r^2] \tag{45}$$

and considering the normalized return

$$\rho = \frac{|r|}{\sqrt{E[r^2]}}. \tag{46}$$

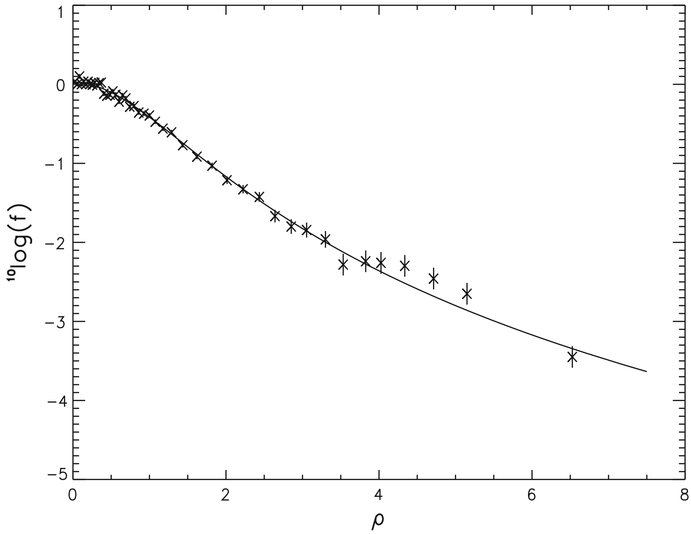

Fig. 1 The pdf for the DAX on a log scale as function of the normalized return ρ. The *solid line* corresponds to the prediction of the ideal gas model Eq. (48)

At large r the pdf exhibits a power law behaviour with integer tail index equal to the number of representative agent groups. This is similar to the universality found in critical systems in physics where the exponents observed in power laws of observables depend only on the dimensions of the system, but not on details of the interaction. We compare the pdf from Eq. (44) with empirical normalized return from the German stock index DAX[2] in the years 1980–2008. As for other stocks or indices the observed tail index is in the order 3–4. A more accurate quantity to determine n is to use the moment ratio

$$E^2[\rho] = \frac{\Gamma^2((n-1)/2)}{\pi\,\Gamma(n/2)\Gamma(n/2-1)}, \tag{47}$$

which excludes $n = 3$. Its observed value 0.51 agrees well with 1/2 resulting from $n = 4$. With this choice we find the pdf of ρ as

$$f(\rho) = 6\left(2+\rho^2\right)^{-5/2}. \tag{48}$$

Figure 1 shows that this zero parameter expression is in very good agreement with the empirical data of the DAX.

After having a successful explanation of a universal power law we turn now to the second hard stylized fact, the persistent autocorrelation function for the absolute

[2] Data provided by the Deutsche Bank Research.

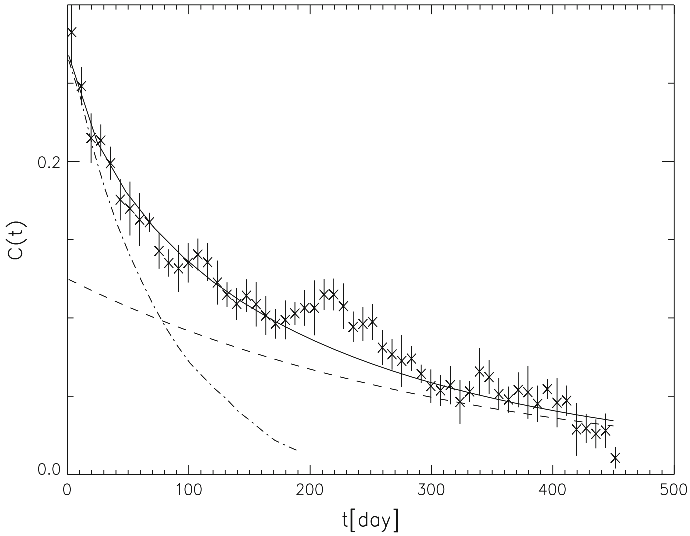

Fig. 2 Correlation function for $|r|$ from the DAX as function the time lag in [days]. The *solid line* gives the prediction of the ideal gas model from Eq. (73). The deviation around 1 year cannot be described by a pure statistical model. *Dashed line* gives the asymptotic behaviour and the *dashed-dotted line* the prediction of the GARCH(1,1) model

return. This is defined as

$$C(t) = \frac{E[|r(t)r(0)|] - E^2[|r|]}{\mathrm{var}[|r|]}. \tag{49}$$

It depends only on the rate constants ν_i. For arbitrary ν_i calculation of $C(t)$ involves an $2n$ dimensional integration over Φ_i at $t = 0$ and Φ_i at time lag t with a complicated integrand. Therefore estimate of ν_i with maximum likelihood or chisq methods seems not to be feasible. To show that the model can explain the correlation we simplify the problem by assuming equal decay constants $\nu_i = \nu_0$ for the dynamic of the Φ_i. For $n = 4$ six integrations can be carried out (see "Appendix B"). The remaining two angle integrations in C are given by

$$C(t) = \frac{4}{\pi^2} \int_0^\pi d\varphi \sin^2(\varphi) \int_0^\pi (\sin(\theta))^2 \left[\left(\frac{1 - \cos(\theta)}{1 + (2a^{2t} - 1)\cos(\theta) + 2a^t \sqrt{1 - a^{2t}} \cos(\varphi) \sin(\theta)} \right)^{1/2} - 1 \right]. \tag{50}$$

The rate enters by the constant $a = \exp(-\nu_0)$. Adjusting ν_0 to the observed correlation function we get $\nu_0 = 0.00155$ [day]$^{-1}$ indicating a very slow dynamic of Φ_i. In Fig. 2 we compare the daily autocorrelation with empirical data for the DAX. The ideal gas model (solid line) describes the correlation very well up to a time lag of 2 years. The characteristic time of $\Delta t_v \sim 120$ days, for which the correlation drops by a factor e^{-1}

is in good agreement with the requirement $\Delta t_v >> \Delta t_o$. The autocorrelation poses a serious problem for models based on a simple Markov process leading usually to an exponential decay. As an example we show (dashed-dotted line) the result from the GARCH(1,1) model (Bollerslev 1986) considered generally as a bench mark model. Fitting the free parameters with the maximum likelihood method[3] the exponential decay of $C(t)$ is adjusted to the behaviour at time lags below 50 days and can not describe larger times. Only the multifractal of Calvet and Fisher (2001) can account (Lux 2004) for the persistent autocorrelation. It leads, however, to a less satisfactory description of the pdf for the return compared to our model.

The reason for the agreement of our model is that it exhibits an exponential decay only asymptotically

$$C(t) = \frac{1}{8}\exp(-2\nu_0 t) \qquad \text{for } \nu_0 t \text{ large}\ , \tag{51}$$

whereas the value near $t = 0$ given by

$$C(0) = \frac{4}{\pi} - 1 \tag{52}$$

is more than a factor two larger than the value obtained by extrapolating the asymptotic behaviour (dashed line in Fig. 2) from Eq. (51) to $t = 0$. The large value of $1/\nu_0 \sim 2.5$ years can be interpreted in the following way. If all Φ_i are small we observe a volatility cluster. At these values the Ornstein–Uhlenbeck process corresponds to a random walk with large activity in the return. The switch of one or more of the Φ_i to large values with normal returns will take a long time. Therefore $1/\nu_0$ corresponds to average waiting time for a volatility cluster. Due to the small value of ν_0 the time window for determining the empirical pdf for the return must be substantially larger than 2.5 years. This explains why fitting the DAX in 5 year periods within the 3/2 model (Wagner 2006) leads to large variations of the shape parameter around the mean observed for all times.

6 Conclusions

In our model for heterogeneous agents Gaussian approximations are used for the stochastic processes at various time scales. At the shortest scale an utility u_i quadratic in the demands of the agents is essential in order to avoid the assumption of a conserved quantity as total utility or welfare function. The free parameter in u_i can be interpreted as risk aversity in the sense that the resulting expectation value for the exponential utility function and the linear relation between demand and return agree with the conventional treatment. At observation time scale we use a stochastic volatility decomposition with Gaussian noise which is supported by empirical evidence. At the longest time scale the change of the agent's risk aversity parameters is described

[3] GARCH tool in program package MATLAB.

by an Ornstein–Uhlenbeck process. This could be replaced by other processes depending in equilibrium only on a scale parameter. Maximization of the information entropy at the various time scales reduces the number of independent parameters substantially. We find an equilibrium pdf for the normalized return which depends only on the number n of involved agent groups equal to the tail index. This pdf agrees very well with the daily DAX for $n = 4$. The agent groups can differ only on the rate the risk aversity parameter are changing with. Already equal rate constants leads to a satisfactory description of the autocorrelation of the absolute return.

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Appendix A: Langevin equation

The continuum Langevin equation (5) (suppressing the agent index i) corresponds to the following Fokker-Planck equation

$$\frac{\partial w}{\partial t} = \frac{\partial}{\partial \pi}\left[-\frac{\partial u}{\partial \pi}\, w \;+\; \frac{1}{\beta}\frac{\partial w}{\partial \pi}\right] \tag{53}$$

for the time dependent distribution $w(t, \pi)$. In equilibrium with natural boundary conditions at $\pi \to \pm\infty$ the bracket [ ] has to vanish leading to the equilibrium pdf given in Eq. (6). For the quadratic utility (9) with $\gamma' = \gamma$ we obtain a Gaussian pdf given by

$$w(\pi) = \left(\frac{\beta\gamma}{2\pi}\right)^{1/2} \exp\left(-\frac{\beta\gamma}{2}(\pi - \frac{r}{\gamma})^2\right). \tag{54}$$

Mean and variance of π prove Eqs. (10) and (11). $u_i(\pi)$ and $\ln w(\pi)$ are both quadratic in π which leads to Eqs. (15) and (14). Using Eqs. (10) and (15) we consider S as a function of D and U instead of r and β

$$S(U, D) = \frac{1}{2}\ln\frac{2\pi e}{\gamma} \;+\; \frac{1}{2}\ln(\gamma D^2 - 2U) \tag{55}$$

Differentiation leads to

$$\left(\frac{\partial S}{\partial U}\right)_{D=\text{constant}} = -\frac{1}{\gamma D^2 - 2U} \tag{56}$$

$$\left(\frac{\partial S}{\partial D}\right)_{U=\text{constant}} = \frac{\gamma D}{\gamma D^2 - 2U}. \tag{57}$$

Replacing U and D by the variables r and β proves the derivative rules in Eqs. (8) and (16).

Appendix B: Ornstein Uhlenbeck model

The discrete Ornstein–Uhlenbeck model is characterized by a linear stochastic difference equation

$$\Delta\Phi = -\nu_i \Phi \Delta t \ + \ \sigma_i \sqrt{\Delta t \nu_i (2 - \nu_i \Delta t)}\, \epsilon_t. \tag{58}$$

The factor in the noise term guarantees that the variance of Φ_i is given by σ_i^2. At integer time steps $\tau = t/\Delta t$ and with the abbreviation $c_i = 1 - \nu_i \ \Delta t$ Eq. (58) reads as

$$\Phi_i(\tau + 1) = c_i \ \Phi_i(\tau) + \ \sigma_i \sqrt{1 - c_i^2}\, \epsilon_\tau \tag{59}$$

For a given set of the Gaussian noise factors ϵ_τ this linear difference equation can be summed with an initial value $\Phi_i(0)$

$$\Phi_i(\tau) = c_i^\tau \ \Phi(0) \ + \sigma_i \sqrt{1 - c_i^2} \sum_{\tau'=0}^{\tau-1} c_i^{\tau'} \ \epsilon_{\tau-\tau'-1}. \tag{60}$$

A sum of iid Gaussians is again Gaussian leading to

$$\Phi_i(\tau) = c_i^\tau \ \Phi(0) \ + \sigma_i \sqrt{1 - c_i^{2\tau}}\, \epsilon_\tau. \tag{61}$$

The conditional distribution for Φ_i at time t can be expressed as

$$w(\Phi_i, t|\Phi_{0i}) = \left(2\pi(1 - a_{i,t}^2)\sigma_i^2\right)^{1/2} \exp\left(-\frac{(\Phi_i - a_{i,t}\Phi_{0i})^2}{2(1 - a_{i,t}^2)\sigma_i^2}\right) \tag{62}$$

with the time dependent constants

$$a_{i,t} = c_i^{t/\Delta t}. \tag{63}$$

The continuum limes $\Delta t \to 0$ differs from the discrete case only by a rsescaling of the rate parameters in the latter as

$$a_{i,t} = \exp(-\nu_{i,c} t) \tag{64}$$

with

$$\nu_{i,c} = -\frac{\ln(1 - \nu_i \Delta t)}{\Delta t}. \tag{65}$$

In equilibrium $a_{i,t}$ vanishes and the Φ_i are normal distributed with variance σ_i^2. For equal variances $\sigma_i^2 = 1$ the pdf ot r follows from Eq. (30) and the equation of state (37)

$$f(r)=\left(\frac{1}{(2\pi)^{(n+1)}v_0}\right)^{1/2}\int d^n\Phi\left(\Phi^2\right)^{q/2}\exp\left(-\frac{1}{2}(r^2(\Phi^2)^q/v_0+\Phi^2)\right) \quad (66)$$

with $\Phi^2=\sum\Phi_i^2$. Introducing polar coordinates for Φ_i the angle integrations can be carried out and one is left with an integration over $t=\Phi^2$

$$f(r)=\left(2v_0\pi^n\right)^{-1/2}\frac{1}{\Gamma(n/2)}\int\limits_0^\infty dt t^{(n-2+q)/2}\exp\left(-\frac{1}{2}(t+t^q r^2/v_0)\right). \quad (67)$$

For large r we get a power law in Eq. (67) with tail index n/q. Also moments of r can be determined as

$$E[(r^2)^\alpha]=2^{\alpha(1-q)}\frac{\Gamma(n/2-\alpha q)}{\Gamma(n/2)}.\frac{\Gamma(\alpha+1/2)}{\sqrt{\pi}} \quad (68)$$

In the special case of $q=1$ the remaining integration can be performed which proves Eq. (44) for the return.

For the autocorrelation function we need the common distribution for r at t and r_0 at $t=0$. Assuming an equilibrium distribution for Φ_0 at $t=0$ it is given for $q=1$ by

$$f(r,t,r_0)=\frac{1}{(2\pi)^{1+n/2}v_0}\int d^n\Phi\int d^n\Phi_0\left(\Phi^2\Phi_0^2\right)^{1/2}$$
$$\exp\left(-\frac{r^2\Phi^2}{2v_0}+\frac{r_0^2\Phi_0^2}{2v_0}\right).\prod_i w(\Phi_i,t|\Phi_{0i})\exp\left(-\frac{\Phi_0^2}{2}\right) \quad (69)$$

with the the conditional distribution Eq. (62). The integration in (69) can be simplified with agent independent rate constants $a_{i,t}=a^t$. Taking moments $|r(t)|^\alpha$ and $|r(0)|^\alpha$ we calculate from Eq. (69)

$$R(t)=\frac{E[|r(t)|^\alpha|r(0)|^\alpha]}{E^2[|r|^\alpha]}-1. \quad (70)$$

In the special case $n=4$ we obtain

$$R(t)=\frac{2^\alpha}{(2\pi)^4\Gamma^2(2-\alpha/2)(1-a^{2t})^2}\int d^4\Phi\int d^4\Phi_0\left(\Phi_0^2\right)^{-\alpha/2}$$
$$\exp\left(-\frac{\Phi_0^2}{2}\right)\cdot A(\Phi,\Phi_0). \quad (71)$$

with the kernel

$$A(\Phi,\Phi_0)=\frac{(\Phi)^{-\alpha/2}}{(1-a^{2t})^2}\left[\exp\left(-\frac{(\Phi-a^t\Phi_0)^2}{2(1-a^{2t})}\right)-\exp\left(-\frac{\Phi^2}{2}\right)\right].$$

Substituting for Φ_i in the first term $(\Phi - a^t \Phi_0)_i/\sqrt{1-a^{2t}}$ we get

$$A(\Phi, \Phi_0) = \exp\left(-\frac{\Phi^2}{2}\right) \cdot \left(\Phi^2\right)^{-\alpha/2} \left[\left(\frac{\Phi^2}{(\sqrt{1-a^{2t}}\Phi + a^t \Phi_0)^2}\right)^{-\alpha/2} - 1\right].$$

Using rotation invariance and introducing angle coordinates we write the integration as

$$\int d^4\Phi \int d^4\Phi_0 = \frac{\pi^3}{2} \int_0^\infty d\rho \rho^7 \int_0^\pi d\varphi \int_0^\pi d\theta \cdot \sin^2(\varphi) \sin^3(\theta)$$

with $2\Phi_0^2 = \rho^2(1+\cos(\theta), 2\Phi^2 = \rho^2(1-\cos(\theta)$ and $2\Phi\Phi_0 = \rho^2 \sin(\theta)\cos(\varphi)$. The ρ integration can be carried out:

$$R(t) = \frac{2^\alpha \Gamma(4-\alpha)}{4\pi \Gamma^2(2-\alpha/2)} \int_0^\pi d\varphi \sin^2(\varphi) \int_0^\pi (\sin(\theta))^{3-\alpha}$$
$$\left[\left(\frac{1-\cos(\theta)}{1+(2a^{2t}-1)\cos(\theta)+2a^t\sqrt{1-a^{2t}}\cos(\varphi)\sin(\theta)}\right)^{\alpha/2} - 1\right]. \quad (72)$$

The remaining integration in Eq. (72) has to be carried out numerically. With the normalization for the auto correlation we have finally

$$C(t) = \frac{E^2[|r|^\alpha]}{\mathrm{var}[|r|^\alpha]} \cdot R(t). \quad (73)$$

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