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Classification of Quasifinite Modules with Nonzero Central Charges for EALAs of Type A with Coordinates in Quantum Torus**

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Abstract The author first constructs a Lie algebra $\mathfrak{L} := \mathfrak{L}(q, w_d)$ from rank 3 quantum torus, which is isomorphic to the core of EALAs of type A_{d-1} with coordinates in quantum torus C_{q^d}, and then gives the necessary and sufficient conditions for the highest weight modules to be quasifinite. Finally the irreducible $\mathbb{Z}$-graded quasifinite $\mathfrak{L}$-modules with nonzero central charges are classified.

Keywords Core of EALAs, Graded modules, Quasifinite module, Highest weight module, Quantum torus
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1 Introduction

Extended affine Lie algebras (EALAs) are generalizations of affine Kac-Moody Lie algebras introduced in [1]. They were constructed and studied in many papers, for more details we refer the reader to, for example, [2–7]. The structure theory of the EALAs of type A_{d-1} is tied up with Lie algebra $\mathrm{gl}_d(\mathbb{C}) \otimes C_q$, where C_q is the quantum torus. Quantum torus defined in [8] are noncommutative analogue of Laurent polynomial algebras. The universal central extension of the derivation Lie algebra of rank 2 quantum torus is known as the q-analog Virasoro-like algebra (see [9]), which is studied in many papers (see [15–18]). Representations for Lie algebras coordinated by certain quantum tori have been studied by many people (see [10, 14]).

In this paper, we first construct a Lie algebra $\mathfrak{L}$ from rank 3 quantum torus, which is isomorphic to the core of EALAs of type A_{d-1} with coordinates in quantum torus C_{q^d}, and then give the necessary and sufficient conditions for the highest weight modules to be quasifinite. Finally, the irreducible $\mathbb{Z}$-graded quasifinite modules of $\mathfrak{L}$ with nonzero central charges are classified. The results generalize those in [23] from $d = 2$ to arbitrary $d \geq 2$.

This paper is organized as follows. In Section 2, we first recall some concepts and results about quantum torus in [2, 20, 23], then introduce the $\mathbb{Z}^2$-extragraded Lie algebras $\widetilde{L}$ and give the necessary and sufficient conditions for the highest weight modules to be quasifinite. And the necessary and sufficient conditions for the $\mathbb{Z}$-graded highest weight $\mathfrak{L}$-modules to be quasifinite

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are obtained in Theorem 2.3. In Section 3, we give the classification of irreducible $\mathbb{Z}$-graded quasifinite $\mathfrak{L}$-modules with nonzero central charges in Theorem 3.2.

2 Highest Weight Quasifinite Modules over $\mathfrak{L}(q, w_d)$

Through this paper, we use $\mathbb{C}$, $\mathbb{Z}$, $\mathbb{Z}_+$ and $\mathbb{N}$ to denote the sets of complex numbers, integers, nonnegative integers and positive integers respectively. For any additive group G, we denote $G^* := G \setminus \{0\}$. For any Lie algebra L, denote its center by $Z(L)$, and $L' := [L, L]$. For any set S, we define the Kronecker delta

$$\delta_{x,S} = \begin{cases} 1, & \text{if } x \in S, \\ 0, & \text{otherwise.} \end{cases} \tag{2.1}$$

For any $m, n \in \mathbb{N}$, denote the maximal common factor of m, n by $\langle m, n\rangle$.

2.1 Basics

Let $\mathcal{Q} = (q_{i,j})_{i,j=1}^n$ be an $n \times n$ matrix over $\mathbb{C}$ satisfying

$$q_{i,i} = 1, \quad q_{i,j} = q_{j,i}^{-1}, \tag{2.2}$$

where n is a positive integer. The $\mathcal{Q}$-quantum torus $C_{\mathcal{Q}}$ is the unital associative algebra over $\mathbb{C}$ generated by $t_1^{\pm 1}, \cdots, t_n^{\pm 1}$ and subject to the defining relations

$$t_i t_i^{-1} = t_i^{-1} t_i = 1, \quad t_i t_j = q_{i,j} t_j t_i. \tag{2.3}$$

For any $\mathbf{m} \in \mathbb{Z}^n$, we always write

$$\mathbf{m} = (m_1, \cdots, m_n), \quad t^a = t_1^{m_1} \cdots t_n^{m_n}. \tag{2.4}$$

For any $\mathbf{m}, \mathbf{n} \in \mathbb{Z}^n$, we define the functions $\sigma_{\mathcal{Q}}(\mathbf{m}, \mathbf{n})$ and $f_{\mathcal{Q}}(\mathbf{m}, \mathbf{n})$ by

$$t^{\mathbf{m}} t^{\mathbf{n}} = \sigma_{\mathcal{Q}}(\mathbf{m}, \mathbf{n}) t^{\mathbf{m}+\mathbf{n}}, \quad t^{\mathbf{m}} t^{\mathbf{n}} = f_{\mathcal{Q}}(\mathbf{m}, \mathbf{n}) t^{\mathbf{n}} t^{\mathbf{m}}. \tag{2.5}$$

Then

$$\sigma_{\mathcal{Q}}(\mathbf{m}, \mathbf{n}) = \prod_{1 \le i < j \le n} q_{j,i}^{m_j n_i}, \quad f_{\mathcal{Q}}(\mathbf{m}, \mathbf{n}) = \prod_{i,j=1}^{n} q_{j,i}^{m_j n_i}, \tag{2.6}$$

and $f_{\mathcal{Q}}(\mathbf{m}, \mathbf{n}) = \sigma_{\mathcal{Q}}(\mathbf{m}, \mathbf{n}) \sigma_{\mathcal{Q}}(\mathbf{n}, \mathbf{m})^{-1}$. We define

$$\mathrm{rad} f_{\mathcal{Q}} = \{\mathbf{m} \in \mathbb{Z}^n \mid f_{\mathcal{Q}}(\mathbf{m}, \mathbb{Z}^n) = 1\}. \tag{2.7}$$

For the properties of $C_{\mathcal{Q}}$, $f_{\mathcal{Q}}$ and $\sigma_{\mathcal{Q}}$, please refer to [2].

In the case $\mathcal{Q} = \begin{pmatrix} 1 & q^{-1} \\ q & 1 \end{pmatrix}$, we will simply denote $C_{\mathcal{Q}}$, $f_{\mathcal{Q}}$ and $\sigma_{\mathcal{Q}}$ by C_q, f_q and σ_q respectively.

Let w_d be a d-th primitive root of unity with $d \ge 2$, q a fixed generic complex number, and

$$\mathcal{Q}_d := \begin{pmatrix} 1 & q^{-1} & 1 \\ q & 1 & w_d^{-1} \\ 1 & w_d & 1 \end{pmatrix}. \tag{2.8}$$

Then $Z(C_{\mathcal{Q}_d}) = \mathbb{C}[t_3^d, t_3^{-d}]$. Let J be the idea of the associative algebra $C_{\mathcal{Q}_d}$ generated by $t_3^d - 1$. Denote $\mathbb{Z}_d = \mathbb{Z}/d\mathbb{Z}$. Define

$$\overline{C}_{\mathcal{Q}_d} = C_{\mathcal{Q}_d}/J = \mathrm{span}_{\mathbb{C}}\{t_1^i t_2^j t_3^k \mid i, j \in \mathbb{Z}, k \in \mathbb{Z}_d\} \tag{2.9}$$

to be the quotient of $C_{\mathcal{Q}_d}$ and identify t_3^k with its image in $\overline{C}_{\mathcal{Q}_d}$.

The derived Lie subalgebra of $\overline{C}_{\mathcal{Q}_d}$ is

$$\overline{C}'_{\mathcal{Q}_d} = \mathrm{span}_{\mathbb{C}}\{t_1^i t_2^j t_3^k \mid (i, j, k) \in (\mathbb{Z}^2 \times \mathbb{Z}_d)^*\}. \tag{2.10}$$

Let $M_d(C_{q^d})$ be the set of $d \times d$ matrices over C_{q^d}, and E_{ij} be the $d \times d$ matrix whose entry is 1 for the (i, j)-entry and 0 elsewhere.

We have the following proposition from a manuscript of K. M. Zhao.

Proposition 2.1 (a) *$\overline{C}_{\mathcal{Q}_d}$ is a simple associative algebra, and $\overline{C}'_{\mathcal{Q}_d}$ is a simple Lie algebra.*
(b) *$\overline{C}_{\mathcal{Q}_d} \cong M_d(C_{q^d})$ as associative algebras.*

Proof (a) Suppose that H is a nonzero associative ideal of $\overline{C}_{\mathcal{Q}_d}$. We want to show that $H = \overline{C}_{\mathcal{Q}_d}$. Take a nonzero element

$$x = \sum_{i=1}^{r} x_i t_1^{a_i} t_2^{b_i} t_3^{c_i} \in H,$$

where $x_i \in \mathbb{C}^*$ and $(a_i, b_i, c_i) \in \mathbb{Z}^2 \times \mathbb{Z}_d$ are pairwise distinct for $i = 1, \cdots, r$.

We may assume that r is minimal. If $r = 1$, clearly $H = \overline{C}_{\mathcal{Q}_d}$. Now assume that $r > 1$. Without loss of generality, we may also assume that $(a_r, b_r, c_r) = (0, 0, \overline{0})$. Then there exists $(a, b, c) \in \mathbb{Z}^2 \times \mathbb{Z}_d$ such that $0 \neq [t_1^a t_2^b t_3^c, t_1^{a_1} t_2^{b_1} t_3^{c_1}] \in \overline{C}_{\mathcal{Q}_d}$. Now

$$0 \neq [t_1^a t_2^b t_3^c, x] = \sum_{i=1}^{r-1} x_i [t_1^a t_2^b t_3^c, t_1^{a_i} t_2^{b_i} t_3^{c_i}] \in H.$$

This is in contradiction with the choice of x. Thus $\overline{C}_{\mathcal{Q}_d}$ is simple.

Now by using Herstein's theorem in [19] and $\overline{C}'_{\mathcal{Q}_d} \cap Z(C_{\mathcal{Q}_d}) = \{0\}$, we see that $\overline{C}'_{\mathcal{Q}_d}$ is a simple Lie algebra.

(b) Define an associative algebra embedding $\varphi_1 : \overline{C}_{\mathcal{Q}_d} \to M_d(C_q)$ by

$$\varphi_1(t_1^i t_2^j t_3^k) = t_1^i t_2^j F^j E^k, \tag{2.11}$$

where $E = \sum\limits_{i=1}^{d} w_d^{-i} E_{i,i}$, $F = E_{d,1} + \sum\limits_{i=1}^{d-1} E_{i,i+1} \in M_d(\mathbb{C})$. Then $\varphi_1(\overline{C}_{\mathcal{Q}_d})$ is spanned by

$$E_{i,j}(t_1^m t_2^{dk+j-i}), \quad 1 \leq i, j \leq d, k, m \in \mathbb{Z}. \tag{2.12}$$

Now, we define the associative algebra isomorphism

$$\varphi_2 : \varphi_1(\overline{C}_{\mathcal{Q}_d}) \to M_d(\mathbb{C}[t_1^{\pm 1}, t_2^{\pm d}]) \subset M_d(C_q), \quad E_{i,j}(t_1^m t_2^{dk+j-i}) \mapsto q^{im} E_{i,j}(t_1^m t_2^{dk}).$$

All the verifications are straightforward.

Now we construct our Lie algebra as a central extension of $\overline{C}'_{\mathcal{Q}_d}$, which will be denoted by $\mathfrak{L} := \mathfrak{L}(q, w_d) = \overline{C}'_{\mathcal{Q}_d} + \mathrm{span}_{\mathbb{C}}\{c_1, c_2\}$, with the following Lie bracket

$$[t^{\mathbf{m}}t_3^i, t^{\mathbf{n}}t_3^j] = (w_d^{n_2 i}q^{m_2 n_1} - w_d^{m_2 j}q^{m_1 n_2})t^{\mathbf{m}+\mathbf{n}}t_3^{i+j} + \delta_{\mathbf{m}+\mathbf{n},0}\delta_{i+j,\overline{0}}w_d^{n_2 i}q^{m_2 n_1}(m_1c_1 + m_2c_2), \quad (2.13)$$

where $\mathbf{m}, \mathbf{n} \in \mathbb{Z}^2$, $i, j \in \mathbb{Z}_d$.

From Proposition 2.1(b), we may easily deduce that $\mathfrak{L}$ is isomorphic to the core of the EALAs of type A_{d-1} with coordinates in C_{q^d}.

Next, we will recall some concepts about the $\mathbb{Z}$-graded $\mathfrak{L}$-modules in [23].

Fix a $\mathbb{Z}$-basis

$$\mathbf{m}_1 = (m_{11}, m_{12}), \quad \mathbf{m}_2 = (m_{21}, m_{22}) \in \mathbb{Z}^2. \quad (2.14)$$

If we define the degree of the nonzero elements in $\mathrm{span}_{\mathbb{C}}\{t^{i\mathbf{m}_1 + j\mathbf{m}_2}t_3^k \in \mathfrak{L} \mid i, j \in \mathbb{Z}, k \in \mathbb{Z}_d\}$ to be i and the degree of the nonzero elements in $\mathbb{C}c_1 + \mathbb{C}c_2$ to be zero, then $\mathfrak{L}$ can be regarded as a $\mathbb{Z}$-graded Lie algebra

$$\mathfrak{L}_i = \mathrm{span}_{\mathbb{C}}\{t^{i\mathbf{m}_1 + j\mathbf{m}_2}t_3^k \in \mathfrak{L} \mid j \in \mathbb{Z}, k \in \mathbb{Z}_d\} + \delta_{i,0}(\mathbb{C}c_1 + \mathbb{C}c_2). \quad (2.15)$$

Set

$$\mathfrak{L}_+ = \bigoplus_{i>0}\mathfrak{L}_i, \quad \mathfrak{L}_- = \bigoplus_{i<0}\mathfrak{L}_i.$$

Then $\mathfrak{L} = \bigoplus_{i\in\mathbb{Z}}\mathfrak{L}_i$ and $\mathfrak{L}$ has the following triangular decomposition

$$\mathfrak{L} = \mathfrak{L}_- \oplus \mathfrak{L}_0 \oplus \mathfrak{L}_+.$$

Definition 2.1 *For any $\mathfrak{L}$-module V, if $V = \bigoplus_{m\in\mathbb{Z}} V_x$ with $\mathfrak{L}_i V_m = V_{m+i}$, $\forall\, i, m \in \mathbb{Z}$, then V is called a $\mathbb{Z}$-graded $\mathfrak{L}$-module (w.r.t $(\mathbf{m}_1, \mathbf{m}_2)$) and V_m is called a homogeneous subspace of V with degree m. The $\mathfrak{L}$-module V is called*

(i) *a quasi-finite $\mathbb{Z}$-graded module, if $\dim V_m < \infty$, $\forall\, m \in \mathbb{Z}$,*

(ii) *a uniformly bounded module, if there exists some $N \in \mathbb{N}$, such that $\dim V_m < N$, $\forall\, m \in \mathbb{Z}$,*

(iii) *a highest (resp. lowest) weight module, if there exists a nonzero homogeneous vector $v \in V_m$, such that V is generated by v and $\mathfrak{L}_+ v = 0$ (resp. $\mathfrak{L}_- v = 0$),*

(iv) *a generalized highest weight module with highest degree m, if there exist a $\mathbb{Z}$-basis $\mathbf{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ of $\mathbb{Z}^2$ and a nonzero vector $v \in V_m$, such that V is generated by v and $t^{\mathbf{m}}t_3^i v = 0$, $\forall\, \mathbf{m} \in \mathbb{Z}_+\mathbf{b}_1 + \mathbb{Z}_+\mathbf{b}_2, i \in \mathbb{Z}_d$,*

(v) *an irreducible $\mathbb{Z}$-graded module, if V does not have any nontrivial $\mathbb{Z}$-graded submodule.*

We denote the set of quasi-finite irreducible $\mathbb{Z}$-graded $\mathfrak{L}$-modules by $\mathscr{O}_{\mathbb{Z}}(\mathbf{m}_1, \mathbf{m}_2)$.

From the definition, one sees that the generalized highest weight modules contain the highest weight modules and the lowest weight modules as their special cases. As the central elements c_1, c_2 of $\mathfrak{L}$ act on irreducible graded modules V as scalars, we shall use the same symbols to denote these scalars.

2.2 Finite dimensional irreducible modules over $\mathfrak{L}_0$

Now we study the finite dimensional irreducible modules over $\mathfrak{L}_0$. Note that by the theory of Verma modules, the irreducible $\mathbb{Z}$-graded highest (or lowest) weight $\mathfrak{L}$-modules are classified by the irreducible modules of $\mathfrak{L}_0$.

Recall $\mathfrak{L}_0 = \mathrm{span}_{\mathbb{C}}\{t^{j\mathbf{m}_2}t_3^k, c_1, c_2 \mid (j,k) \in (\mathbb{Z}\times\mathbb{Z}_d)^*\}$.

Clearly

$$\mathfrak{L}_0' = \mathrm{span}_{\mathbb{C}}\Big\{t^{k\mathbf{m}_2}t_3^i, m_{21}c_1 + m_{22}c_2 \,\Big|\, (k,i) \notin \frac{d}{\langle d, m_{22}\rangle}(\mathbb{Z}\times\mathbb{Z}_d)\Big\}. \tag{2.16}$$

Denote

$$H_0 = \mathrm{span}_{\mathbb{C}}\Big\{t^{k\mathbf{m}_2}t_3^i, c_1, c_2 \,\Big|\, (k,i) \in \frac{d}{\langle d, m_{22}\rangle}(\mathbb{Z}\times\mathbb{Z}_d)\Big\}. \tag{2.17}$$

Clearly H_0 is an ideal of $\mathfrak{L}_0$, and we have

$$\mathfrak{L}_0 = H_0 + \mathfrak{L}_0'. \tag{2.18}$$

Suppose that A is an arbitrary finite dimensional irreducible module over $\mathfrak{L}_0$. Then c_1, c_2 act as scalars on A, and considering the action of the three dimensional Heisenberg Lie subalgebra $\mathrm{span}_{\mathbb{C}}\{t^{\mathbf{m}_2}, t^{-\mathbf{m}_2}, m_{21}c_1+m_{22}c_2\}$, we have $(m_{21}c_1+m_{22}c_2)A = 0$. Hence, we can regard A as a module over $\mathfrak{L}_0/(\mathbb{C}(m_{21}c_1+m_{22}c_2))$. Noting that $H_0/(\mathbb{C}(m_{21}c_1+m_{22}c_2)) = Z(\mathfrak{L}_0/(\mathbb{C}(m_{21}c_1+m_{22}c_2))$, we see that elements in H_0 act as scalars on A and A is an irreducible $\mathfrak{L}_0'/(\mathbb{C}(m_{21}c_1+m_{22}c_2))$ module.

Define

$$\phi_A : H_0 \to \mathbb{C} \tag{2.19}$$

by $\phi(x)v = xv$ for all $x \in H_0$ and $v \in A$.

Make A to be an $\mathfrak{L}_0 + \mathfrak{L}_+$-module by defining $\mathfrak{L}_+A = 0$. Then we have the highest Verma $\mathfrak{L}$-module

$$\widetilde{M}^+(A; \mathbf{m}_1, \mathbf{m}_2) = \mathrm{Ind}_{\mathfrak{L}_0+\mathfrak{L}_+}^{\mathfrak{L}} A. \tag{2.20}$$

$\widetilde{M}^+(A; \mathbf{m}_1, \mathbf{m}_2)$ has a unique maximal $\mathbb{Z}$-graded $\mathfrak{L}$ proper submodule $\widetilde{M}^{+'}(A; \mathbf{m}_1, \mathbf{m}_2)$, and the unique irreducible $\mathbb{Z}$-graded quotient module

$$M^+(A; \mathbf{m}_1, \mathbf{m}_2) := \widetilde{M}^+(A; \mathbf{m}_1, \mathbf{m}_2)/\widetilde{M}^{+'}(A; \mathbf{m}_1, \mathbf{m}_2). \tag{2.21}$$

Similarly, we have the lowest Verma module $\widetilde{M}^-(A; \mathbf{m}_1, \mathbf{m}_2)$ and the irreducible $\mathbb{Z}$-graded quotient module $M^-(A; \mathbf{m}_1, \mathbf{m}_2)$.

Let us recall the properties of finite dimensional irreducible modules in [20].

Theorem 2.1 (see [20]) (a) *Let $F(X) = (X-x_1)\cdots(X-x_r)$, $G(X) = (X-y_1)\cdots(X-y_s) \in \mathbb{C}^* + X\mathbb{C}[X]$, and*

$$I'(F,G) := \mathrm{span}_{\mathbb{C}}\{t^{\mathbf{n}}F(t_1^m), t^{\mathbf{n}}G(t_2^m) \mid \mathbf{n} \in \mathbb{Z}^2 \setminus (m\mathbb{Z}^2)\} \subset C_{w_m}, \tag{2.22}$$

where w_m is an m-th primitive root of unity with $m > 1$. Then the quotient Lie algebra $C'_{w_m}/I'(F,G) \cong \oplus^{rs}\mathrm{sl}_m(\mathbb{C})$ if and only if $(F,F') = (G,G') = 1$.

(b) *Let A be an irreducible finite dimensional module over the Lie algebra C_{w_m} with $\dim A > 1$. Then there exist nonzero polynomials $F(X), G(X) \in \mathbb{C}^* + X\mathbb{C}[X]$ with $(F,F') = (G,G') = 1$ such that $I'(F,G) \subset \mathrm{Ann}\, A$, and*

(1) *A is an irreducible module over $C'_{w_m}/I'(F,G)$,*

(2) *elements in $Z(C_{w_m})$ act as scalars on A.*

Proof (a) and (b) are Lemma 2.5 and Theorem 3.2 in [20] respectively.

Corollary 2.1 *Suppose that $\langle d, m_{22}\rangle \neq d$, and A is an irreducible finite dimensional module over $\mathfrak{L}_0$ with $\dim A > 1$. Then there exists a nonconstant polynomial $F(X) \in \mathbb{C}^* + X\mathbb{C}[X]$ with $(F, F') = 1$ such that*

(1) $(m_{21}c_1 + m_{22}c_2)A = 0$,

(2) *A is an irreducible module over the semisimple finite dimensional algebra $\mathfrak{L}'_0/(I'(F) + \mathbb{C}(m_{21}c_1 + m_{22}c_2))$ (which is isomorphic to some direct sums of $\mathrm{sl}_{\frac{d}{\langle d, m_{22}\rangle}}(\mathbb{C})$), where*

$$I'(F) := \mathrm{span}_{\mathbb{C}}\Big\{F((t^{\mathbf{m}_2})^{\frac{d}{\langle d,m_{22}\rangle}})t^{k\mathbf{m}_2}t_3^i \,\Big|\, (k,i) \notin \frac{d}{\langle d, m_{22}\rangle}(\mathbb{Z}\times\mathbb{Z}_d)\Big\}, \tag{2.23}$$

(3) *elements in H_0 act on A as scalars.*

2.3 Highest weight modules over $\mathbb{Z}^2$-extragraded Lie algebra $\widetilde{L}$

Let us recall a $\mathbb{Z}^2$-extragraded Lie algebra in [22] for some special case.

Denote

$$\mathcal{Q}' = (q'_{i,j}) := \begin{pmatrix} 1 & q^{m_{12}m_{21}-m_{11}m_{22}} & w_d^{-m_{12}} \\ q^{-m_{12}m_{21}+m_{11}m_{22}} & 1 & w_d^{-m_{22}} \\ w_d^{m_{12}} & w_d^{m_{22}} & 1 \end{pmatrix},$$

where $\mathbf{m}_1, \mathbf{m}_2$ are defined in (2.14) and q is generic.

We have an associative algebra isomorphism

$$\rho : C_{\mathcal{Q}'} \to C_{\mathcal{Q}_d} \tag{2.24}$$

with $\rho(t_3) = t_3$ and $\rho(t_i) = t^{\mathbf{m}_i}$ for $i = 1, 2$. Further, we have

$$\rho(t_1^i t_2^j t_3^k) = q^{\frac{m_{11}m_{12}i(i-1)+m_{21}m_{22}j(j-1)}{2}+ijm_{12}m_{21}}t^{i\mathbf{m}_1+j\mathbf{m}_2}t_3^k. \tag{2.25}$$

Define the Lie algebra $\widetilde{L} := \widetilde{L}_{\mathcal{Q}'}$ with $\widetilde{L} = C_{\mathcal{Q}'}$ as vector space and the relations

$$\begin{aligned}[t^{\mathbf{a}}, t^{\mathbf{b}}] &= t^{\mathbf{a}}t^{\mathbf{b}} - t^{\mathbf{b}}t^{\mathbf{a}} + \delta_{a_1+b_1,0}\delta_{\mathbf{a}+\mathbf{b},\mathrm{rad} f_{\mathcal{Q}'}}a_1 t^{\mathbf{a}}t^{\mathbf{b}} \\ &= \sigma_{\mathcal{Q}'}(\mathbf{a},\mathbf{b})(1 - f_{\mathcal{Q}'}(\mathbf{b},\mathbf{a}) + \delta_{a_1+b_1,0}\delta_{\mathbf{a}+\mathbf{b},\mathrm{rad} f_{\mathcal{Q}'}}a_1)t^{\mathbf{a}+\mathbf{b}}, \quad \forall\, \mathbf{a},\mathbf{b}\in\mathbb{Z}^3.\end{aligned} \tag{2.26}$$

Now we need to recall some notations in [21].

Definition 2.2 (a) *The algebra of exp-polynomial functions in r' variables $m_1, m_2, \cdots, m_{r'}$ is the algebra of functions $f(m_1, \cdots, m_{r'}) : \mathbb{Z}^{r'} \to \mathbb{C}$ generated as an algebra by functions m_j and a^{m_j} for various constants $a \in \mathbb{C}^* = \mathbb{C}\setminus\{0\}$, $j = 1, \cdots, r'$.*

(b) *Let $G = \bigoplus_{(i,\mathbf{a})\in\mathbb{Z}^{n+1}} G_{i,\mathbf{a}}$ be any $\mathbb{Z}^{n+1}$-graded Lie algebra, $\mathcal{K} = \{K_i \mid i \in \mathbb{Z}\}$ be a family of finite sets and*

$$\mathcal{B} = \{g_i^{(k_i)}(\mathbf{a}) \mid k_i \in K_i, (i,\mathbf{a}) \in \mathbb{Z}^{n+1}\} \tag{2.27}$$

be any homogenous spanning set of G with $g_i^{(k_i)}(\mathbf{a}) \in G_{i,\mathbf{a}}$. Then G is said to be a $\mathbb{Z}^n$-extragraded Lie algebra with respect to $\mathcal{K}$ and $\mathcal{B}$, if there exists a family of exp-polynomial *functions* $\{f_{i,j,i+j}^{k_i,k_j,k_{i+j}}(\mathbf{a},\mathbf{b})\}$ *in the* $2n$ *variables* a_l, b_l, $l = 1, 2, \cdots, n$, *where* $k_i \in K_i$, $\forall\, i \in \mathbb{Z}$, *such that*

$$[g_i^{(k_i)}(\mathbf{a}), g_j^{(k_j)}(\mathbf{b})] = \sum_{k_{i+j}\in K_{i+j}} f_{i,j,i+j}^{k_i,k_j,k_{i+j}}(\mathbf{a},\mathbf{b}) g_{i+j}^{(k_{i+j})}(\mathbf{a}+\mathbf{b}). \tag{2.28}$$

(c) *Let* G *be a* $\mathbb{Z}^n$*-extragraded Lie algebra w.r.t* $\mathcal{B}$ *and* $\mathcal{K}$ *as defined in* (b), *and* $G_0 := \bigoplus_{\mathbf{a}\in\mathbb{Z}^n} G_{0,\mathbf{a}}$. *A finite dimensional nonzero module* A *over* G_0 *is called an exp-polynomial* G_0*-module, if there exists a basis* $\{v_i\}_{i\in J}$, *and there exists a family of* n*-variable exp-polynomial functions* $h_{s,j}^k(\mathbf{a})$ *for* $k \in K_0$, $j, s \in J$, *such that*

$$g_0^k(\mathbf{a}) v_j = \sum_{j\in J} h_{s,j}^k(\mathbf{a}) v_s.$$

Lemma 2.1 (see [22]) *Suppose that* $\psi : \mathbb{Z}^n \to \mathbb{C}$ *is a function,* $h_i(t) = \sum_{j=0}^{m_i} x_{i,j} t^j = \prod_{j=1}^{l_i} (t - y_{i,j})^{s_{i,j}}$, $i = 1, \cdots, n$, *are polynomials in* $\mathbb{C}[t]$, *where* $s_{i,j}, m_i \in \mathbb{N}$, *and* $x_{i,j}, y_{i,j} \in \mathbb{C}$ *with* $x_{i,0} x_{i,m_i} \neq 0$. *For* $k = 1, 2, \cdots, n$, *let*

$$\begin{aligned}\mathcal{F}_k &= \{f_{k,0}(r), f_{k,1}(r), \cdots, f_{k,m_k-1}(r)\}\\ &:= \{y_{k,1}^r, r y_{k,1}^r, \cdots, r^{s_{k,1}-1} y_{k,1}^r; y_{k,2}^r, \cdots, r^{s_{k,2}-1} y_{k,2}^r; \cdots; y_{k,l_k}^r, r y_{k,l_k}^r, \cdots, r^{s_{k,l_k}-1} y_{k,l_k}^r\}\end{aligned}$$

be a set of functions in $r \in \mathbb{Z}$. *Then*

$$\sum_{j=0}^{m_i} x_{i,j} \psi(\mathbf{a} + j\overline{\varepsilon_i}) = 0, \quad \forall\, \mathbf{a} \in \mathbb{Z}^n,\ i = 1, 2, \cdots, n, \tag{2.29}$$

if and only if there exist $\prod_{i=1}^n m_i$ *complex numbers* $z_{(b_1,\cdots,b_n)}$, $0 \le b_i \le m_i - 1$, $i = 1, \cdots, n$, *such that*

$$\psi(\mathbf{a}) = \sum_{b_1=0}^{m_1-1} \cdots \sum_{b_n=0}^{m_n-1} z_{(b_1,\cdots,b_n)} \prod_{i=1}^n f_{i,b_i}(a_i), \quad \forall\, \mathbf{a} \in \mathbb{Z}^n. \tag{2.30}$$

Lemma 2.2 (see [22]) $\widetilde{L}_{\mathcal{Q}'}$ *is a* $\mathbb{Z}^2$*-extragraded Lie algebra with respect to* $\mathcal{K}$ *and* $\mathcal{B}$, *where*

$$\mathcal{K} = \{K_i \mid i \in \mathbb{Z}\}, \quad \mathcal{B} = \{g_i^{(k_i)}(\mathbf{a}) \mid k_i \in K_i, (i, \mathbf{a}) \in \mathbb{Z}^3\}, \tag{2.31}$$

$$K_0 = \{1, 2\} \text{ and } K_i = \{1\}, \quad \forall\, i \neq 0, \tag{2.32}$$

$$g_0^{(1)}(\mathbf{a}) = \delta_{\mathbf{a}, \frac{d}{\langle d, m_{22}\rangle}\mathbb{Z}^2} (1 - q^{(m_{11}m_{22} - m_{12}m_{21})a_1} w_d^{m_{12}a_2} + \delta_{a_1,0}\delta_{a_2,d\mathbb{Z}}) t^{(0,\mathbf{a})}, \tag{2.33}$$

$$g_0^{(2)}(\mathbf{a}) = (1 - \delta_{\mathbf{a}, \frac{d}{\langle d, m_{22}\rangle}\mathbb{Z}^2}) t^{(0,\mathbf{a})}, \tag{2.34}$$

$$g_i^{(1)}(\mathbf{a}) = t^{(i,\mathbf{a})}, \quad \forall\, i \neq 0. \tag{2.35}$$

$\widetilde{L}$ has a natural $\mathbb{Z}$-gradation with $\widetilde{L}_i = \mathrm{span}_{\mathbb{C}}\{t_1^i t_2^j t_3^k \mid (j,k) \in \mathbb{Z}^2\}$. Similarly, we have the notations of $\mathbb{Z}$-graded modules and quasi-finite modules over $\widetilde{L}$. And for any irreducible module A over $\widetilde{L}_0$, make A to be an $\widetilde{L}_0 + \widetilde{L}_+$-module by defining $\widetilde{L}_+ A = 0$. Then we have the Verma module

$$\widetilde{M}^+_{\widetilde{L}_{\mathcal{Q}'}}(A) = \mathrm{Ind}_{\widetilde{L}_0 + \widetilde{L}_+}^{\widetilde{L}} A \tag{2.36}$$

and the unique irreducible $\mathbb{Z}$-graded quotient module $M^+_{\widetilde{L}_{\mathcal{Q}'}}(A)$. Similarly, we have $M^-_{\widetilde{L}_{\mathcal{Q}'}}(A)$.

Theorem 2.2 *Let A be any finite dimensional irreducible $\widetilde{L}_0$ module. Then $M^{\pm}_{\widetilde{L}_{\mathcal{Q}'}}(A)$ is quasifinite, if and only if there exists some 2-variable exp-polynomial function $\psi:\mathbb{Z}^2\to\mathbb{C}$, such that*

$$(t_2^{\frac{d}{\langle d,m_{22}\rangle}i}t_3^{\frac{d}{\langle d,m_{22}\rangle}j})v=\frac{\psi((i,j))}{1-q^{\frac{(m_{11}m_{22}-m_{12}m_{21})di}{\langle d,m_{22}\rangle}}w_d^{\frac{m_{12}dj}{\langle d,m_{22}\rangle}}+\delta_{i,0}\delta_{\frac{j}{\langle d,m_{22}\rangle},\mathbb{Z}}}v \tag{2.37}$$

for all $(i,j)\in\mathbb{Z}^2$ and $v\in A$.

Proof If $\dim A=1$, then the theorem follows from [22, Theorem 2.11]. So we may assume that $\dim A>1$. Suppose that $M^{\pm}_{\widetilde{L}_{\mathcal{Q}'}}(A)$ is quasifinite. Fix $0\neq v\in A$. Let $\widetilde{H}:=\mathbb{C}[t_1^{\pm1},t_2^{\pm\frac{d}{\langle d,m_{22}\rangle}},t_3^{\pm\frac{d}{\langle d,m_{22}\rangle}}]\subset\widetilde{L}_{\mathcal{Q}'}$. Then $U(\widetilde{H})v$ is a quasifinite $\mathbb{Z}$-graded $\widetilde{H}$ module. And from [22, Theorem 2.11], we get (2.37).

On the other hand, suppose that (2.37) holds. Note that

$$\widetilde{L}_0\cong C_{w_d^{m_{22}}},\quad \widetilde{L}_0=\widetilde{L}_0'\oplus Z(\widetilde{L}_0). \tag{2.38}$$

From Theorem 2.1(a), we have a nonconstant polynomial $F_1(X),G_1(X)\in\mathbb{C}^*+X\mathbb{C}[X]$, such that

$$\Big\{t_2^it_3^jF_1(t_2^{\frac{d}{\langle d,m_{22}\rangle}}),t_2^it_3^jG_1(t_3^{\frac{d}{\langle d,m_{22}\rangle}})\;\Big|\;(i,j)\notin\frac{d}{\langle d,m_{22}\rangle}\mathbb{Z}^2\Big\}\subset\mathrm{Ann}A. \tag{2.39}$$

Now, by Lemma 2.1, it is direct to check that A is an exp-polynomial module (see Definition 2.2(c)). Hence the theorem follows from [21, Theorem 1.7].

2.4 Irreducible quasifinite highest weight modules over $\mathfrak{L}(q,w_d)$

Define a Lie algebra surjective homomorphism $\varrho:\widetilde{L}_{\mathcal{Q}'}\to\mathfrak{L}/\mathbb{C}(m_{21}c_1+m_{22}c_2)$ by

$$\varrho(t_1^it_2^jt_3^k)=\begin{cases}q^{\frac{m_{11}m_{12}i(i-1)+m_{21}m_{22}j(j-1)+2ijm_{12}m_{21}}{2}}t^{i\mathbf{m}_1+j\mathbf{m}_2}t_3^k, & (i,j,k)\notin(0,0,d\mathbb{Z}),\\ m_{11}c_1+m_{12}c_2, & (i,j,k)\in(0,0,d\mathbb{Z}).\end{cases} \tag{2.40}$$

Theorem 2.3 *Let A be an irreducible finite dimensional $\mathfrak{L}_0$ module. Then the highest weight $\mathfrak{L}$-module $M^{\pm}(A;\mathbf{m}_1;\mathbf{m}_2)$ is quasifinite, if and only if there exist 1-variable exp-polynomial functions $\psi_0,\psi_2,\cdots,\psi_{\langle d,m_{22}\rangle-1}:\mathbb{Z}\to\mathbb{C}$, such that*

$$\phi_A(m_{11}c_1+m_{12}c_2)=\psi_0(0),\quad \phi_A(m_{21}c_1+m_{22}c_2)=0, \tag{2.41}$$

$$\phi_A(t^{\frac{id\mathbf{m}_2}{\langle d,m_{22}\rangle}}t_3^{\frac{kd}{\langle d,m_{22}\rangle}})=\frac{\psi_k(i)}{(1-q^{\frac{(m_{11}m_{22}-m_{12}m_{21})di}{\langle d,m_{22}\rangle}}w_d^{\frac{m_{12}dk}{\langle d,m_{22}\rangle}})q^{\frac{m_{21}m_{22}id(id-\langle d,m_{22}\rangle)}{2\langle d,m_{22}\rangle^2}}} \tag{2.42}$$

for all $(i,k)\notin(0,\langle d,m_{22}\rangle\mathbb{Z}_d)$, $k=0,\cdots,\langle d,m_{22}\rangle-1$, where ϕ_A is defined in (2.19).

Proof Note that we may regard $\mathfrak{L}$-module $M^{\pm}(A;\mathbf{m}_1;\mathbf{m}_2)$ as $M^{\pm}_{\widetilde{L}_{\mathcal{Q}'}}(A)$ via the surjective homomorphism ϱ defined in (2.41). The theorem follows from Theorem 2.2 and Lemma 2.1.

3 Classification of Irreducible Quasifinite $\mathbb{Z}$-graded Modules with Nonzero Central Charges for $\mathfrak{L}(q,w_d)$

In this section, we will give the classification of irreducible quasifinite $\mathbb{Z}$-graded modules with nonzero charges for $\mathfrak{L}(q,w_d)$.

We will omit the details of the proof in this section, since they are almost the same as in [23].

We need to point out that "t_3" in this paper is corresponding to "t_0" in [23].

Proposition 3.1 *If $V \in \mathscr{O}_{\mathbb{Z}}(\mathbf{m}_1, \mathbf{m}_2)$, then V is a generalized highest weight module or a uniformly bounded module.*

Proof If V is not a generalized highest weight module, then one may deduce that $\dim V_m \leq d(\dim V_0 + \dim V_1)$ in the same way as in the proof of [23, Proposition 2.7].

Lemma 3.1 *If V is a nontrivial irreducible generalized highest weight $\mathbb{Z}$-graded $\mathfrak{L}$-module corresponding to a $\mathbb{Z}$-basis $B = \{\mathbf{b}_1, \mathbf{b}_2\}$ of $\mathbb{Z}^2$, then*

(a) *For any $0 \neq v \in V$, there is some $p \in N$ such that $t^{m_1\mathbf{b}_1+m_2\mathbf{b}_2}t_3^i \cdot v = 0$ for all $m_1, m_2 \geq p$ and $i \in \mathbb{Z}_d$.*

(b) *For any $0 \neq v \in V$ and $m_1, m_2 > 0$, $i \in \mathbb{Z}_d$, we have $t^{-m_1\mathbf{b}_1-m_2\mathbf{b}_2}t_3^i \cdot v \neq 0$.*

Proof The proof is the same as [23, Lemma 4.1].

Lemma 3.2 *If $V \in \mathscr{O}_{\mathbb{Z}}(\mathbf{m}_1, \mathbf{m}_2)$ is a generalized highest weight $\mathfrak{L}(q, w_d)$-module, then V must be a highest or lowest weight module.*

Proof The proof is the same as [23, Lemma 4.2].

From the above lemma and the results in Section 2, we have the following theorem.

Theorem 3.1 *If V is a quasifinite irreducible $\mathbb{Z}$-graded $\mathfrak{L}$-module w.r.t $(\mathbf{m}_1, \mathbf{m}_2)$, then V is either $M^+(A; \mathbf{m}_1, \mathbf{m}_2)$, $M^-(A; \mathbf{m}_1, \mathbf{m}_2)$ with ϕ_A satisfying* (2.41) *and* (2.42)*, or a uniformly bounded module.*

Then we have the same result as [23, Theorem 4.4].

Theorem 3.2 *If $V \in \mathcal{O}_{\mathbb{Z}}(\mathbf{m}_1, \mathbf{m}_2)$ is an irreducible $\mathfrak{L}(q, w_d)$-module with nontrivial central charges, then there exists some finite dimensional irreducible $\mathfrak{L}_0$ module A with ϕ_A satisfying* (2.41) *and* (2.42)*, such that $V \cong M^+(A; \mathbf{m}_1, \mathbf{m}_2)$ or $V \cong M^-(A; \mathbf{m}_1, \mathbf{m}_2)$.*

One can also construct a class of highest weight $\mathbb{Z}^2$-graded $\mathfrak{L}(q, w_d)$-modules $V_{\mathbb{Z}^2} = V \otimes \mathbb{C}[x^{\pm 1}]$ from the $\mathbb{Z}$-graded module V w.r.t $(\mathbf{m}_1, \mathbf{m}_2)$ as follows:

$$t^{i\mathbf{m}_1+j\mathbf{m}_2}t_3^k \cdot (v \otimes x^r) = (t^{i\mathbf{m}_1+j\mathbf{m}_2}t_3^k \cdot v) \otimes x^{r+j}.$$

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