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Topological Representations of Distributive Hypercontinuous Lattices**

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Abstract The concept of locally strong compactness on domains is generalized to general topological spaces. It is proved that for each distributive hypercontinuous lattice L, the space Spec L of nonunit prime elements endowed with the hull-kernel topology is locally strongly compact, and for each locally strongly compact space X, the complete lattice of all open sets $\mathcal{O}(X)$ is distributive hypercontinuous. For the case of distributive hyperalgebraic lattices, the similar result is given. For a sober space X, it is shown that there is an order reversing isomorphism between the set of upper-open filters of the lattice $\mathcal{O}(X)$ of open subsets of X and the set of strongly compact saturated subsets of X, which is analogous to the well-known Hofmann-Mislove Theorem.

Keywords Hypercontinuous lattice, Locally strongly compact space, Hull-kernel topology, Hyperalgebraic lattice, Strongly locally compact space
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1 Introduction

The representation of lattices by topologies, especially by the hull-kernel topologies, goes back to Stone's famous work on the topological representation of Boolean algebras and distributive lattices (see [6, 8]). In [4], Hofmann and Lawson proved that for every distributive continuous lattice L, the space Spec L of nonunit prime elements endowed with the hull-kernel topology is always a locally compact sober space and conversely that the lattice of open sets of a locally compact sober space is a continuous lattice. Furthermore, the correspondence between distributive continuous lattices and locally compact sober spaces is functorial and thus a dual equivalence between them is established.

In this paper, the concept of locally strongly compactness, which was posed by Heckmann in [3] for the purpose of defining multi-continuous domains, is generalized to general topological spaces. It is proved that for a distributive hypercontinuous lattice L, the space Spec L with the hull-kernel topology is locally strongly compact, and conversely that the lattice of open sets of a locally strongly compact space is hypercontinuous. Also in the paper, the concept of strongly locally compact sober spaces is introduced, which is proved to be a correspondence to distributive hyperalgebraic lattices. A result analogous to the well-known Hofmann-Mislove

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Theorem is obtained, that is, there is an order reversing isomorphism between the set of upper-open filters of the lattice $\mathcal{O}(X)$ of open subsets of X and the set of strongly compact saturated subsets of X.

In this paper, spaces are assumed to be T_0. For a set X, let $X^{(<\omega)} = \{F \subseteq X : F \text{ is finite}\}$. A domain D is a partially ordered set such that every directed set E of D has a least upper bound $\vee E$ in D. Let $\uparrow x = \{y \in D : x \leq y\}$ and $\uparrow A = \bigcup_{a\in A} \uparrow a$; $\downarrow x$ and $\downarrow A$ are defined dually. The topology generated by the collection of sets $D\setminus \downarrow x$ (as a subbase) is called the upper topology and denoted by $\upsilon(D)$; the lower topology $\omega(D)$ on D is defined dually. The topology $\theta(D) = \upsilon(D) \vee \omega(D)$ is called the interval topology on D. A subset U of a domain D is called Scott open, provided that $U =\uparrow U$ and $E \cap U \neq \emptyset$ for all directed set $E \subseteq D$ with $\vee E \in U$. The topology formed by all the Scott open sets of D is called the Scott topology on D, written as $\sigma(D)$. The topology $\lambda(D) = \sigma(D) \vee \omega(D)$ is called the Lawson topology on D. For a poset $(X, \leq)$ and a topology τ on X, define $\tau^{\uparrow} = \{U \in \tau : U =\uparrow U\}$ and $\tau^{\downarrow} = \{U \in \tau : U =\downarrow U\}$. It is easy to check that $\theta(L)^{\uparrow} = \upsilon(L)$ and $\theta(L)^{\downarrow} = \omega(L)$ for a complete lattice L.

For a domain D, we define a binary relation on D, called the way below relation, by $x \ll y$ if and only if for each directed set $E \subseteq D$ with $y \leq \vee E$, there exists $e \in E$ such that $x \leq e$. A domain D is called a continuous domain, if the set $\{a \in D : a \ll x\}$ is directed and its supremum is x for all $x \in D$. A complete lattice L is called a continuous lattice if it is a continuous domain. A complete lattice L is called an algebraic lattice if $x = \vee\{b \in L : b \ll b \leq x\}$ for all $x \in L$.

An element p of a lattice L is called prime if $a \wedge b \leq p$ always implies $a \leq p$ or $b \leq p$. The set of all nonunit prime elements of L is denoted by $\mathrm{Spec}\,L$. Let $\triangle_L\,(a) = \mathrm{Spec}\,L\setminus \uparrow a$. For a complete lattice L, it is easy to check that the family of all sets $\triangle_L$ forms a topology on $\mathrm{Spec}\,L$, called the hull-kernel topology on $\mathrm{Spec}\,L$. In this paper, $\mathrm{Spec}\,L$ is always endowed with this topology, i.e., $\mathcal{O}(\mathrm{Spec}\,L) = \{\triangle_L\,(a) : a \in L\}$.

Definition 1.1 (see [1]) *For a complete lattice L, define a relation $\prec$ on L by $x \prec y \Leftrightarrow y \in \mathrm{int}_{\upsilon(L)} \uparrow x$. Let $i(x) = \{u \in L : u \prec x\}$. L is called hypercontinuous if $x = \vee i(x)$ for all $x \in L$.*

It is easy to get the following

Proposition 1.1 *For a complete lattice L, the following conditions are equivalent:*

(1) *L is hypercontinuous,*

(2) *L satisfies the following two conditions:*

(i) *$\prec$ satisfies (INT), i.e., $x \prec y \Rightarrow \exists\, z \in L,\ x \prec z \prec y$,*

(ii) *$x \neq y \Rightarrow i(x) \neq i(y)$.*

Theorem 1.1 (see [1]) *Let L be a complete lattice. Then the following conditions are equivalent:*

(1) *L is hypercontinuous,*

(2) *L is continuous and $\ll = \prec$,*

(3) *L is continuous and $\sigma(L) = \upsilon(L)$,*

(4) *L is continuous and $\lambda(L) = \theta(L)$,*

(5) *L is continuous and $(L, \theta(L))$ is Hausdorff.*

For a poset P, let $\mathcal{D}(P) = \{E : E \subseteq P \text{ and } E \text{ is direct}\}$, $\mathbf{Up}\,P = \{A \subseteq P : A =\uparrow A\}$ and $\mathbf{Fin}\,P = \{\uparrow A : A \in P^{(<\omega)}\}$. Define a mapping $\mathbf{min} : \mathbf{Fin}\,P \to 2^P$ by $\mathbf{min}(F) = \{x \in F : x \text{ is a minimal element of } F\}$. For a family of sets $\mathcal{M}$, the poset $(\mathcal{M}, \supseteq)$ always means that the order on $\mathcal{M}$ is the inverse inclusion order of sets. $\mathcal{M}$ is called down-directed if $\forall\, A, B \in \mathcal{M}$, $\exists\, C \in \mathcal{M}$ such that $C \subseteq A \cap B$, i.e., $(\mathcal{M}, \supseteq)$ is directed. When $\mathbf{Fin}\,P$ is regarded as a poset, we always mean the poset $(\mathbf{Fin}\,P, \supseteq)$.

In [7], Rudin got the following well-known result.

Lemma 1.1 (Rudin's Lemma) *Let P be a poset and $E \in \mathbf{Up}\,P$. Suppose that $\mathcal{G} \subseteq \mathbf{Fin}\,P$ is down-directed (i.e., $\mathcal{G} \in \mathcal{D}(\mathbf{Fin}\,P)$), $\emptyset \notin \mathcal{G}$ and $\cap\mathcal{G} \subseteq E$. Then $\exists\, K \subseteq \cup\{\mathbf{min}(G) : G \in \mathcal{G}\}$ such that*

(i) $\forall\, G \in \mathcal{G}$, $K \cap \mathbf{min}(G) \neq \emptyset$,

(ii) $K \in \mathcal{D}(P)$,

(iii) $\cap\{\uparrow k : k \in K\} \subseteq E$,

(iv) $\forall\, G, H \in \mathcal{G}$, $G \subseteq H \Rightarrow K \cap \mathbf{min}(G) \subseteq \uparrow (K \cap \mathbf{min}(H))$.

Rudin's lemma has many important applications (see [1–3]). The following result is one of them.

Corollary 1.1 (see [3]) *Let D be a domain. Suppose that $\mathcal{G} \subseteq \mathbf{Fin}\,D$ is down-directed (i.e., $\mathcal{G} \in \mathcal{D}(\mathbf{Fin}\,P)$) and $U \in \sigma(D)$. If $\cap\mathcal{G} \subseteq U$, then $\exists \uparrow G \in \mathcal{G}$ such that $\uparrow G \subseteq U$.*

2 Topological Representations for Distributive Hypercontinuous Lattices

In this section, we give an intrinsic characterization of hypercontinuous lattice and investigate the topological representations for distributive hypercontinuous lattices.

For a topological space (X, τ), we define a binary relation $\leq_\tau$ as follows: $x \leq_\tau y \Leftrightarrow x \in \mathrm{cl}_\tau\{y\}$. Then $\leq_\tau$ is a partial order on X since (X, τ) is T_0. Let $\uparrow_\tau x = \{y \in X : y \leq_\tau x\}$ and $\uparrow_\tau A = \bigcup\limits_{a \in A} \uparrow_\tau a$; $\downarrow_\tau x$ and $\downarrow_\tau A$ are defined dually. For the sake of no confusion, we let $\uparrow\{U\} = \{V \in \tau : U \subseteq V\}$ and $\downarrow\{U\} = \{W \in \tau : W \subseteq U\}$ for $U \in \tau$.

For a poset $(P, \leq)$ and a topology τ on P, the triple $(P, \leq, \tau)$ is called a pospace if the relation $\leq$ is closed in the product space $(P, \tau) \times (P, \tau)$.

Let L be a hypercontinuous lattice and $x, y \in L$ with $x \not\leq y$. Then there exists $u \in L$ such that $u \prec x$ and $u \not\leq y$, which implies $(x, y) \in \mathrm{int}_{\upsilon(L)} \uparrow u \times (L \setminus \uparrow u) \subseteq X \times X \setminus \leq$. Thus $(L, \leq, \theta(L))$ is a pospace.

Theorem 2.1 *For a complete lattice L, the following conditions are equivalent:*

(1) *L is hypercontinuous,*

(2) *For each $U \in \upsilon(L)$, $x \in U$, there exists $y \in U$ such that $x \in \mathrm{int}_{\upsilon(L)} \uparrow y \subseteq \uparrow y \subseteq U$,*

(3) *If $x, y \in L$ with $x \not\leq y$, then there exist $F \in L^{(<\omega)}$ and $u \in L$ satisfying the following conditions:*

(i) *$x \notin \downarrow F$, $u \notin \uparrow y$,*

(ii) *For each $z \in L$, either $z \in \downarrow F$ or $z \in \uparrow u$.*

Proof (1) $\Rightarrow$ (2) Let $U \in \upsilon(L)$ and $x \in U$. Since $\{y \in L : y \prec x\}$ is directed and $\vee\{y \in L : y \prec x\} = x \in U$, there exists $y \prec x$ such that $y \in U$. Then $x \in \mathrm{int}_{\upsilon(L)} \uparrow y \subseteq \uparrow y \subseteq U$.

(2) $\Rightarrow$ (3) Let $x, y \in L$ with $x \not\leq y$. Then $x \in L \setminus \downarrow y \in \upsilon(L)$. By (2), there exists $u \in L \setminus \downarrow y$ such that $x \in \mathrm{int}_{\upsilon(L)} \uparrow u \subseteq \uparrow u \subseteq L \setminus \downarrow y$. Choose $F \in L^{(<\omega)}$ such that $x \in L \setminus \downarrow F \subseteq \mathrm{int}_{\upsilon(L)} \uparrow u \subseteq \uparrow u$. Then F and u satisfy the conditions (i) and (ii).

(3) $\Rightarrow$ (1) For each $x \in L$, let $y = \{a \in L : a \prec x\}$. Then $y \leq x$. If $x \not\leq y$, then there exist $F \in L^{(<\omega)}$ and $u \in L$ satisfying the conditions (i) and (ii) in (3). Then $u \prec x$ and $u \not\leq y$, which is a contradiction. Therefore $x = y = \{a \in L : a \prec x\}$.

The condition (3) in the above theorem is called the intrinsic characterization of hypercontinuous lattices, which is first obtained in [9] (see also [10]).

Definition 2.1 (see [3]) *Let (X, τ) be a topological space and $S \subseteq X$. S is called strongly compact provided for all $U \in \tau$ with $S \subseteq U$, there is $F \in X^{(<\omega)}$ such that $S \subseteq \uparrow_\tau F \subseteq U$.*

Definition 2.2 *A topological space (X, τ) is called locally strongly compact if for each $U \in \tau$, $x \in U$, there exists $F \in X^{(<\omega)}$ such that $x \in \mathrm{int}_\tau \uparrow_\tau F \subseteq \uparrow_\tau F \subseteq U$.*

Remark 2.1 The concept of locally strong compactness was first posed on domains by Heckmann for the purpose of defining multi-continuous domains. A domain D is called multi-continuous if the topological space $(D, \sigma(D))$ is locally strongly compact (see [3]), i.e., for every $U \in \sigma(D)$, $x \in U$, there exists $F \in D^{(<\omega)}$ such that $x \in \mathrm{int}_{\sigma(D)} \uparrow F \subseteq \uparrow F \subseteq U$. It can be checked that multi-continuous domains are exactly quasicontinuous domains defined in [2]. Here we carry the concept of locally strong compactness to general topological spaces.

Lemma 2.1 *Let (X, τ) be a topological space and $U, V \in \tau$. If $U \subseteq \uparrow_\tau F \subseteq V$ for some $F \in X^{(<\omega)}$, then $U \prec V$. The converse is true if (X, τ) is locally strongly compact.*

Proof Let $F = \{x_1, x_2, \cdots, x_n\}$ and $\mathcal{H} = \{W \in \tau : W \not\subseteq X \setminus \mathrm{cl}_\tau\{x_i\}$ for all $i = 1, 2, \cdots, n\} = \tau \setminus \downarrow \{X \setminus \mathrm{cl}_\tau\{x_1\}, X \setminus \mathrm{cl}_\tau\{x_2\}, \cdots, X \setminus \mathrm{cl}_\tau\{x_n\}\}$. Then $\mathcal{H} \in \upsilon(\tau)$ and $V \in \mathcal{H} \subseteq \uparrow \{U\}$. By the definition of $\prec$, we have $U \prec V$. Conversely, if (X, τ) is locally strongly compact, then for each $v \in V$, there exists $F_v \in X^{(<\omega)}$ such that $x \in \mathrm{int}_\tau \uparrow_\tau F_v \subseteq \uparrow_\tau F_v \subseteq V$. Hence $V = \bigcup\limits_{v \in V} \mathrm{int} \uparrow_\tau F_v$. Notice that $U \prec V$ implies $U \ll V$. Therefore there exists $\{v_1, v_2, \cdots, v_n\} \subseteq V$ such that

$$U \subseteq \bigcup_{i=1}^{n} \mathrm{int}_\tau \uparrow_\tau F_{v_i} \subseteq \bigcup_{i=1}^{n} \uparrow_\tau F_{v_i} \subseteq V.$$

Let $F = \bigcup\limits_{i=1}^{n} F_{v_i}$. Then $U \subseteq \uparrow_\tau F \subseteq V$.

Lemma 2.2 *For a topological space (X, τ), the following conditions are equivalent:*

(1) *(X, τ) is locally strongly compact,*

(2) *For every $U \in \tau$, $x \in U$, there exists $V \in \tau$ such that $x \in V \prec U$,*

(3) *$(\tau, \subseteq)$ is a hypercontinuous lattice,*

(4) *For every $U \in \tau$, $x \in U$, there exists $\mathcal{H} \in \upsilon(\tau)$ such that $U \in \mathcal{H}$ and $\bigcap\limits_{V \in \mathcal{H}} V$ is a neighborhood of x in (X, τ).*

Proof (1) $\Rightarrow$ (2) Let $U \in \tau$ and $x \in U$. Then there exists $F \in X^{(<\omega)}$ such that $x \in \mathrm{int}_\tau \uparrow_\tau F \subseteq \uparrow_\tau F \subseteq U$. By Lemma 2.1, we have $\mathrm{int}_\tau \uparrow_\tau F \prec U$.

(2) $\Leftrightarrow$ (3) Trivial.

(3) $\Rightarrow$ (1) Let $U \in \tau$, $x \in U$. Then $U \not\subseteq X \setminus \mathrm{cl}_\tau\{x\}$. By Theorem 2.1, there exist $V, V_1, V_2, \cdots, V_n \in \tau$ satisfying the following two conditions:

(i) $V \not\subseteq X \setminus \mathrm{cl}_\tau\{x\}$ and $U \not\subseteq V_i$ for all $i = 1, 2, \cdots, n$,

(ii) $\forall W \in \tau$, either $V \subseteq W$ or $W \subseteq V_i$ for some $i \in \{1, 2, \cdots, n\}$.

For each $i \in \{1, 2, \cdots, n\}$, choose $x_i \in U \setminus V_i$ and let $F = \{x_1, x_2, \cdots, x_n\}$. Then $\uparrow_\tau F \subseteq U$ since U is open. Now we show that $V \subseteq \uparrow_\tau F$. Suppose that $V \not\subseteq \uparrow_\tau F$. Then $\exists\, v \in V \setminus \uparrow_\tau F$. Let $W_0 = X \setminus \mathrm{cl}_\tau\{v\}$. Then $F \subseteq W_0$. By $F \not\subseteq V_i$ $(i = 1, 2, \cdots, n)$, we know $W_0 \not\subseteq V_i$ for all $i \in \{1, 2, \cdots, n\}$. By (ii), we have $V \subseteq W_0$, which contradicts $v \in V$. Since $V \not\subseteq X \setminus \mathrm{cl}_\tau\{x\} \Leftrightarrow x \in V$, it follows that $x \in V \subseteq \mathrm{int}_\tau \uparrow_\tau F \subseteq \uparrow_\tau F \subseteq U$.

(3) $\Rightarrow$ (4) Let $U \in \tau$ and $x \in U$. Then there exists $W \in \tau$ such that $x \in W \prec U$. By the definition of $\prec$, there exists $\mathcal{H} \in \upsilon(\tau)$ such that $U \in \mathcal{H} \subseteq \uparrow \{W\}$. Then $x \in W \subseteq \bigcap\limits_{V \in \mathcal{H}} V$.

(4) $\Rightarrow$ (2) For every $U \in \tau$, $x \in U$, let $\mathcal{H} \in \upsilon(\tau)$ satisfy the condition (4). Then there exists $V \in \tau$ such that $x \in V \subseteq \bigcap\limits_{V \in \mathcal{H}} V$. Then $U \in \mathcal{H} \subseteq \uparrow \{V\}$. Hence $x \in V \prec U$.

Corollary 2.1 (see [2]) *For a domain D, D is quasicontinuous if and only if the lattice $\sigma(D)$ of all Scott open sets of D is hypercontinuous.*

Lemma 2.3 *For a distributive hypercontinuous lattice L, the topological space* $\mathrm{Spec}\,L$ *is a locally strongly compact sober space.*

Proof Since L is a distributive hypercontinuous lattice, $\mathrm{Spec}\,L$ is order generating. Hence $\mathrm{Spec}\,L$ is sober. Now we prove that $(\mathcal{O}(\mathrm{Spec}\,L), \subseteq)$ is hypercontinuous. Let $\triangle_L(a)$ and $\triangle_L(b)$ be two open sets in the hull-kernel topology with $\triangle_L(a) \not\subseteq \triangle_L(b)$. Then $a \not\leq b$ since $\mathrm{Spec}\,L$ is order generating. By the hypercontinuity of L, there exist $u \in L$ and $F = \{x_1, x_2, \cdots, x_n\} \subseteq L$ such that

(i) $a \notin \downarrow F$, $b \notin \uparrow u$,

(ii) For all $x \in L$, either $x \in \downarrow F$ or $x \in \uparrow u$.

Since $\mathrm{Spec}\,L$ is order generating, $\triangle_L(u)$ and $\{\triangle_L(x_1), \triangle_L(x_2), \cdots, \triangle_L(x_n)\}$ satisfy the following two conditions:

(i)$'$ $\triangle_L(u) \not\subseteq \triangle_L(b)$ and $\triangle_L(a) \not\subseteq \triangle_L(x_i)$ for all $i = 1, 2, \cdots, n$,

(ii)$'$ For each $\triangle_L(c) \in \mathcal{O}(\mathrm{Spec}\,L)$, either $\triangle_L(u) \subseteq \triangle_L(c)$ or $\triangle_L(c) \subseteq \triangle_L(x_i)$ for some $i \in \{1, 2, \cdots, n\}$.

By Theorem 2.1, $(\mathcal{O}(\mathrm{Spec}\,L), \subseteq)$ is a hypercontinuous lattice. Therefore, by Lemma 2.2, $\mathrm{Spec}\,L$ is locally strongly compact.

Theorem 2.2 (Topological Representations for Distributive Hypercontinuous Lattices)

(1) *For a distributive hypercontinuous lattice L, the topological space* $\mathrm{Spec}\,L$ *is a locally strongly compact space and L is order isomorphic to* $\mathcal{O}(\mathrm{Spec}\,L)$.

(2) *For a locally strongly compact space X, $\mathcal{O}(X)$ is a distributive hypercontinuous lattice and X is homeomorphic to* $\mathrm{Spec}\,\mathcal{O}(X)$ *if, in addition, X is a sober space.*

Let **SOB** be the category of all sober spaces and all continuous maps, and $\mathbf{FRM}_0$ be the category of all complete lattices in which the prime elements are order generating and the maps between them preserve arbitrary sups and finite infs. It is well-known that the categories **SOB** and $\mathbf{FRM}_0$ are dual equivalent through the functors $\mathcal{O}$ and Spec (see [1, V-5] for details). Let **LSCSOB** denote the full subcategory of **SOB** whose objects are the locally strongly compact sober spaces, and **DHCL** the full subcategory of $\mathbf{FRM}_0$ whose objects are distributive hypercontinuous lattices. Then the categories **LSCSOB** and **DHCL** are dual equivalent. Unfortunately, the morphisms preserving arbitrary sups and finite infs seem to be of no particular significance for hypercontinuous lattices. So it is worthwhile finding appropriate morphisms for hypercontinuous lattices.

3 Topological Representations for Distributive Hyperalgebraic Lattices

In this section, we apply the developments of the preceding section to hyperalgebraic lattices, i.e., algebraic hypercontinuous lattices.

Definition 3.1 *A topological space (X, τ) is called strongly locally compact if it has a base consisting of fintary open upper sets, i.e., for all $U \in \tau$, $x \in U$, there exists $F \in X^{(<\omega)}$ such that $x \in \mathrm{int}_\tau \uparrow_\tau F = \uparrow_\tau F \subseteq U$.*

Definition 3.2 *A complete lattice L is called hyperalgebraic provided $x = \vee\{y \in L : y \prec y \leq x\}$ for all $x \in L$.*

Lemma 3.1 *For a complete lattice L, the following conditions are equivalent:*

(1) *L is hyperalgebraic,*

(2) *L is both hypercontinuous and algebraic,*

(3) *For all $U \in \upsilon(L)$, $x \in U$, there exists $y \in U$ such that $x \in \mathrm{int}_{\upsilon(L)} \uparrow y = \uparrow y \subseteq U$,*

(4) *For all $x, y \in L$ with $x \not\leq y$, there exist $F \in L^{(<\omega)}$ and $u \in L$ satisfying the following two conditions:*

(i) *$x \notin \downarrow F$, $y \notin \uparrow u$, $\downarrow F \cap \uparrow u = \emptyset$,*

(ii) *For every $z \in L$, either $z \in \downarrow F$ or $z \in \uparrow u$.*

Proof (1) $\Rightarrow$ (2) Trivial.

(2) $\Rightarrow$ (3) Let $U \in \upsilon(L)$, $x \in U$. By Theorem 1.1, $\{u \in L : u \prec u \leq x\} = \{u \in L : u \ll u \leq x\}$ is a sup-semilattice of L. So by $\vee\{u \in L : u \prec u \leq x\} = x \in U \in \upsilon(L) = \sigma(L)$, there exists $y \in \{u \in L : u \prec u \leq x\}$ with $y \in U$. Therefore $x \in \mathrm{int}_{\upsilon(L)} \uparrow y = \uparrow y \subseteq U$.

(3) $\Rightarrow$ (4) Let $x, y \in L$ with $x \not\leq y$. Then $x \in L \setminus \downarrow y \in \upsilon(L)$. By (3), there exists $u \in L \setminus \downarrow y$ with $x \in \mathrm{int}_{\upsilon(L)} \uparrow u = \uparrow u \subseteq L \setminus \downarrow y$. By the definition of the upper topology, there exists a family $\{F_i \in L^{(<\omega)} : i \in I\}$ such that $\bigcup_{i\in I}(L \setminus \downarrow F_i) = L \setminus \bigcap_{i \in I} \downarrow F_i = \mathrm{int}_{\upsilon(L)} \uparrow u = \uparrow u$. Hence $\bigcap_{i\in L} \downarrow F_i = L \setminus \uparrow u \in \omega(L)$. By Corollary 1.1 (applying to the dual of L), there exists a finite $I_0 \subseteq I$ such that $\bigcap_{i\in I_0} \downarrow F_i = L \setminus \uparrow u \in \omega(L)$. Let $F = \left\{\bigwedge_{i\in I_0} \varphi(i) : \varphi \in \prod_{i \in I_0} F_i\right\}$. Then $F \in L^{(<\omega)}$ and $x \in L \setminus \downarrow F = L \setminus \bigcap_{i\in I_0} \downarrow F_i = \uparrow u \subseteq L \setminus \downarrow y$. Hence F and u satisfy the conditions (i) and (ii).

(4) $\Rightarrow$ (1) For $x \in L$, let $y = \vee\{y \in L : y \prec y \leq x\}$. If $x \not\leq y$, then $x \in L \setminus \downarrow y \in \upsilon(L)$. By (4), there exist $F \in L^{(<\omega)}$ and $u \in L$ such that the conditions (i) and (ii) are satisfied. That is, $x \in L \setminus \downarrow F = \uparrow u \subseteq L \setminus \downarrow y$. Hence $\uparrow u \in \upsilon(L)$. It follows that $u \prec u$, which is in contradiction with $\uparrow u \subseteq L \setminus \downarrow y$ and $y = \vee\{y \in L : y \prec y \leq x\}$. So $x = y = \vee\{y \in L : y \prec y \leq x\}$. Therefore L is hyperalgebraic.

The condition (4) in the above theorem is called the intrinsic characterization of hyperalgebraic lattices, which is first obtained in [11].

Lemma 3.2 *Let (X, τ) be a topological space and $U \in \tau$. If $U = \mathrm{int}_\tau \uparrow_\tau F_u = \uparrow_\tau F$ for some $F \in X^{(<\omega)}$, then $U \prec U$. The converse is true if (X, τ) is strongly locally compact.*

Proof Let $U \in \tau$. If $U = \mathrm{int}_\tau \uparrow_\tau F_u = \uparrow_\tau F$ for some $F \in X^{(<\omega)}$, then $U \prec U$ by Lemma 2.1. Conversely, if (X, τ) is strongly locally compact, then for all $u \in U$, there exists $F_u \in X^{(<\omega)}$ such that $x \in \mathrm{int}_\tau \uparrow_\tau F_u = \uparrow_\tau F_u \subseteq U$. Hence $U = \bigcup_{u\in U} \mathrm{int} \uparrow_\tau F_u = \bigcup_{u \in U} \uparrow_\tau F_u$. Notice that $U \prec U$ implies $U \ll U$. Therefore there exists $\{u_1, u_2, \cdots, u_k\} \subseteq U$ such that $U \subseteq \bigcup_{i=1}^{k} \mathrm{int}_\tau \uparrow_\tau F_{u_i} = \bigcup_{i=1}^{k} \uparrow_\tau F_{u_i}$. Let $F = \bigcup_{i=1}^{k} F_{u_i}$. Then $F \in X^{(<\omega)}$ and $U = \mathrm{int}_\tau \uparrow_\tau F_u = \uparrow_\tau F$.

Lemma 3.3 *For a topological space (X, τ), the following two conditions are equivalent:*

(1) *(X, τ) is strongly locally compact,*

(2) *$(\tau, \subseteq)$ is a hyperalgebraic lattice.*

Proof (1) $\Rightarrow$ (2) Let $U \in \tau$. For $x \in U$, by the strongly local compactness of (X, τ), there exists $F \in X^{(<\omega)}$ such that $x \in \mathrm{int}_\tau \uparrow_\tau F = \uparrow_\tau F \subseteq U$. Let $V = \mathrm{int}_\tau \uparrow_\tau F$. Then by

Lemma 3.2, we have $x \in V \prec V \subseteq U$. Whence $U = \cup\{V \in \tau : V \prec V \subseteq U\}$. Thus $(\tau, \subseteq)$ is hyperalgebraic.

(2) $\Rightarrow$ (1) Let $U \in \tau$ and $x \in U$. Then by (2), there exists $V \in \tau$ such that $x \in V \prec V \subseteq U$. Hence there exists $\{V_1, V_2, \cdots, V_n\} \in \tau^{(<\omega)}$ with $x \in V \in \tau \setminus \downarrow \{V_1, V_2, \cdots, V_n\} \subseteq \uparrow \{V\}$. For each $i \in \{1, 2, \cdots, n\}$, choose $x_i \in V \setminus V_i$. Let $F = \{x_1, x_2, \cdots, x_n\}$. Then $x \in \uparrow_\tau F$. If not, then $F \subseteq X \setminus \mathrm{cl}_\tau\{x\}$; hence $X \setminus \mathrm{cl}_\tau\{x\} \in \tau \setminus \downarrow \{V_1, V_2, \cdots, V_n\} \subseteq \uparrow \{V\}$, which is in contradiction with $x \in V$. So $x \in \uparrow_\tau F$. Now we show that $V = \uparrow_\tau F$. Clearly, $\uparrow_\tau F \subseteq V$ since V is open. On the other hand, if there is a $y \in V \setminus \uparrow_\tau F$, then $\{x_1, x_2, \cdots, x_n\} \subseteq X \setminus \mathrm{cl}_\tau\{y\}$. It follows that $X \setminus \mathrm{cl}_\tau\{y\} \in \tau \setminus \downarrow \{V_1, V_2, \cdots, V_n\} \subseteq \uparrow \{V\}$, which is in contradiction with $y \in V$. Therefore $V = \uparrow_\tau F$. Hence $x \in \mathrm{int}_\tau \uparrow_\tau F = \uparrow_\tau F \subseteq U$. Thus (X, τ) is strongly locally compact.

Lemma 3.4 *For a distributive hyperalgebraic lattice L, the topological space* $\mathrm{Spec}\, L$ *is a strongly locally compact sober space.*

Proof By Lemma 3.1, L is distributive, hypercontinuous and algebraic, so $\mathrm{Spec}\, L$ is order generating. Hence $\mathrm{Spec}\, L$ is sober. Now we prove that $(\mathcal{O}(\mathrm{Spec}\, L), \subseteq)$ is hyperalgebraic. Let $\Delta_L(a)$ and $\Delta_L(b)$ be two open sets in the hull-kernel topology with $\Delta_L(a) \not\subseteq \Delta_L(b)$. Then $a \not\leq b$ since $\mathrm{Spec}\, L$ is order generating. Since L is hyperalgebraic, there exist $F = \{x_1, x_2, \cdots, x_n\} \subseteq L$ and $u \in L$ such that

(i) $a \notin \downarrow F$, $b \notin \uparrow u$, $\downarrow F \cap \uparrow u = \emptyset$,

(ii) For all $x \in L$, either $x \in \downarrow F$ or $x \in \uparrow u$.

Since $\mathrm{Spec}\, L$ is order generating, $\Delta_L(u)$ and $\{\Delta_L(x_1), \Delta_L(x_2), \cdots, \Delta_L(x_n)\}$ satisfy the following two conditions:

(i)$'$ $\Delta_L(u) \not\subseteq \Delta_L(b)$, $\Delta_L(a) \not\subseteq \Delta_L(x_i)$ for all $i = 1, 2, \cdots, n$, and $\downarrow \{\Delta_L(x_1), \Delta_L(x_2), \cdots, \Delta_L(x_n)\} \cap \uparrow \{\Delta_L(u)\} = \emptyset$,

(ii)$'$ For each $\Delta_L(c) \in \mathcal{O}(\mathrm{Spec}\, L)$, either $\Delta_L(u) \subseteq \Delta_L(c)$ or $\Delta_L(c) \subseteq \Delta_L(x_i)$ for some $i \in \{1, 2, \cdots, n\}$.

By Lemma 3.1, $(\mathcal{O}(\mathrm{Spec}\, L), \subseteq)$ is hyperalgebraic. Therefore, by Lemma 3.3, $\mathrm{Spec}\, L$ is strongly locally compact.

By Lemmas 3.3 and 3.4, we get the following

Theorem 3.1 (Topological Representations for Distributive Hyperalgebraic Lattices)

(1) *For a distributive hyperalgebraic lattice L, the topological space* $\mathrm{Spec}\, L$ *is a strongly locally compact space and L is order isomorphic to* $\mathcal{O}(\mathrm{Spec}\, L)$.

(2) *For a strongly locally compact space X, $\mathcal{O}(X)$ is a hyperalgebraic lattice and X is homeomorphic to* $\mathrm{Spec}\,\mathcal{O}(X)$ *if, in addition, X is a sober space.*

For a topological space (X, τ), let $L = (\tau, \subseteq)$. We call $\mathcal{F} \subseteq L$ an upper-open filter in L if $\mathcal{F}$ is upper-open, i.e., $\mathcal{F} \in \upsilon(L)$, and a filter in L. The set of upper-open filters in L is denoted by $\mathrm{Filt}\, \upsilon(L)$. A subset $A \subseteq X$ is called saturated if $A = \cap\{U \in \tau : A \subseteq U\}$. It is easy to check that A is saturated $\Leftrightarrow A = \uparrow_\tau A$. The set of strongly compact saturated subsets of X is written as $S(X)$.

We will end this paper by showing that $\mathrm{Filt}\, \upsilon(L)$ is order-isomorphic to $S(X)$, i.e., there is a result analogous to the well-known Hofmann-Mislove Theorem.

The following important lemma is due to Keimel and Paseka [5].

Lemma 3.5 *Let X be a sober space and $\mathcal{F}$ a Scott-open filter in $\mathcal{O}(X)$. Then every open set U containing $K = \cap\mathcal{F}$ is already a member of $\mathcal{F}$.*

Lemma 3.6 *Let (X, τ) be a sober space. If $\mathcal{F}$ is an upper-open filter in $\mathcal{O}(X)$, then $K = \cap\mathcal{F}$ is strongly compact; if, in addition, all the member of $\mathcal{F}$ are nonempty, then $K = \cap\mathcal{F}$ is* *nonempty, too.*

Proof Let $U \in \tau$ and $K = \cap\mathcal{F}$. If $K \subseteq U$, then $U \in \mathcal{F}$ by Lemma 3.5. As $\mathcal{F}$ is upper-open, there exist $V_1, V_2, \cdots, V_n \in \tau$ such that $U \in \tau \setminus \downarrow \{V_1, V_2, \cdots, V_n\} \subseteq \mathcal{F}$. Thus there exists $x_i \in U \setminus V_i$ for each $i \in \{1, 2, \cdots, n\}$. Let $F = \{x_1, x_2, \cdots, x_n\}$. Then $\uparrow_\tau F \subseteq U$ since U is open. Now we show that $K \subseteq \uparrow_\tau F$. If not, then there exists $k \in K \setminus \uparrow_\tau F$. Hence $F \subseteq X \setminus \mathrm{cl}_\tau\{k\} \in \tau$. Since $x_i \in X \setminus \mathrm{cl}_\tau\{k\}$ and $x_i \notin V_i$, it follows that $X \setminus \mathrm{cl}_\tau\{k\} \not\subseteq V_i$ for each $i \in \{1, 2, \cdots, n\}$. Thus $X \setminus \mathrm{cl}_\tau\{k\} \in \tau \setminus \downarrow \{V_1, V_2, \cdots, V_n\} \subseteq \mathcal{F}$. It implies $K = \cap\mathcal{F} \subseteq X \setminus \mathrm{cl}_\tau\{k\}$, which is a contradiction. Therefore $K \subseteq \uparrow_\tau F \subseteq U$. Thus $K = \cap\mathcal{F}$ is strongly compact. If $K = \emptyset$, then $\emptyset \in \mathcal{F}$ again by Lemma 3.5.

Lemma 3.7 *Let (X, τ) be a topological space and S a strongly compact set in X. Then $\mathcal{U}(S) = \{U \in \tau : S \subseteq U\}$ is an upper-open filter in τ.*

Proof Obviously, $\mathcal{U}(S)$ is a filter in τ. Let $U \in \mathcal{U}(S)$. Then by the strong compactness of S, there exists $F = \{x_1, \cdots, x_n\} \subseteq X$ such that $S \subseteq \uparrow_\tau F \subseteq U$. Let $V_i = X \setminus \mathrm{cl}_\tau\{x_i\}$ and $\mathcal{H} = \{V \in \tau : V \not\subseteq V_i \text{ for all } i = 1, 2, \cdots, n\} = \tau \setminus \downarrow \{V_1, V_2, \cdots, V_n\} \in \upsilon(\tau)$. Then $U \in \mathcal{H} \subseteq \mathcal{U}(S)$. Therefore, $\mathcal{U}(S)$ is upper-open in τ.

By Lemmas 3.6 and 3.7, we get the following

Theorem 3.2 *Let (X, τ) be a sober space and $L = (\tau, \subseteq)$. Then $(S(X), \supseteq)$ is order-isomorphic to $(\mathrm{Filt}\,\upsilon(L), \subseteq)$, the isomorphisms being*

$$S(X) \to \mathrm{Filt}\,\upsilon(L), \quad S \mapsto \mathcal{U}(S) \quad \textit{and} \quad \mathrm{Filt}\,\upsilon(L) \to S(X), \quad \mathcal{F} \mapsto \cap\mathcal{F}.$$

Corollary 3.1 *Let X be a sober space and $(S_i)_{i \in I}$ a filtered family of nonempty strongly compact saturated subsets of X. Then $\bigcap\limits_{i \in I} S_i$ is nonempty, strongly compact and saturated.*

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