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Gradient Estimates for the Heat Kernels in Higher Dimensional Heisenberg Groups**

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Abstract The author obtains sharp gradient estimates for the heat kernels in two kinds of higher dimensional Heisenberg groups — the non-isotropic Heisenberg group and the Heisenberg type group $\mathbb{H}_{n,m}$. The method used here relies on the positive property of the Bakry-Émery curvature Γ_2 on the radial functions and some associated semigroup technics.

Keywords Gradient estimates, Γ_2 curvature, Heat kernels, Sublaplace, Heisenberg group
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1 Introduction

The study of properties of second-order self-adjoint differential operators often depends on Gaussian bounds for the corresponding heat kernels (gradient estimates or not), that is, the kernel of the semigroup generated by these operators. In the past few years, kinds of estimates for the heat kernels were obtained, see [3, 5, 7–9, 13–16, 18].

In the study of the long (or small) time behavior (e.g., gradient estimates, ergodicity, etc.) of simple linear parabolic evolution equations, one often uses lower bounds on the Ricci curvature associated to the generator of the heat kernel (see, for example, [1, 12, 18] and the references therein). But this method fails in general hypoelliptic evolution equations, since the Ricci (Γ_2-) curvature in even the simplest example of the Heisenberg group can not be bounded below as explained, e.g., in [2, 11]. Nevertheless, in the Heisenberg group (type, or nonisotropic) case, many properties of the elliptic case remain true. In this paper, we focus on the gradient estimates for the heat kernels in higher dimensional Heisenberg groups.

Let us recall first some basic facts.

The elliptic case Let M be a complete Riemannian manifold of dimension n and let $\mathcal{L} := \Delta + \nabla h$, where Δ is the Laplace-Beltrami operator. For $t \geq 0$, denote by P_t the heat semigroup generated by $\mathcal{L}$ (that is formally $P_t = \exp(t\mathcal{L})$), and let $H(t, x, y)$ be the heat kernel.

In [8], Engoulatov obtained the following gradient estimates for the heat kernels on Riemannian manifolds.

Theorem 1.1 *Let M be a complete Riemannian manifold of dimension n with Ricci curvature bounded from below, and* $\mathrm{Ric}(M) \geq -\rho$, $\rho \geq 0$.

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(i) *Suppose a non-collapsing condition is satisfied on M, namely, there exist $t_0 > 0$ and $\nu_0 > 0$, such that for any $x \in M$, the volume of the geodesic ball of radius t_0 centered at x is not too small, i.e.* $\mathrm{Vol}(B_x(t_0)) \geq \nu_0$. *Then there exist two constants $C(\rho, n, \nu_0, t_0)$ and $\overline{C}(t_0) > 0$, such that*

$$|\nabla \log H(t,x,y)| \leq C(\rho, n, \nu_0, t_0)\Big(\frac{d(x,y)}{t} + \frac{1}{\sqrt{t}}\Big)$$

uniformly on $(0, \overline{C}(t_0)] \times M \times M$, where $d(x,y)$ is the Riemannian distance between x and y.

(ii) *Suppose that M has a diameter bounded by D. Then there exists a constant $C(\rho, n)$, such that*

$$|\nabla \log H(t,x,y)| \leq C(\rho, n)\Big(\frac{D}{t} + \frac{1}{\sqrt{t}} + \rho\sqrt{t}\Big)$$

uniformly on $(0, \infty) \times M \times M$.

Heisenbeg group case The Heisenberg group can be seen as the Euclidean space $\mathbb{R}^3$ with a group structure $\circ$, which is defined, for $\vec{x} = (x,y,z)$ and $\vec{y} = (x', y', z') \in \mathbb{R}^3$, by

$$\vec{x} \circ \vec{y} = \Big(x + x', y + y', z + z' + \frac{1}{2}(xy' - x'y)\Big).$$

The left invariant vector fields which are given by

$$\begin{aligned} X(f) &= \lim_{\varepsilon \to 0} \frac{f(\vec{x} \circ (\varepsilon, 0, 0)) - f(\vec{x})}{\varepsilon} = \Big(\partial_x - \frac{y}{2}\partial_z\Big)f, \\ Y(f) &= \lim_{\varepsilon \to 0} \frac{f(\vec{x} \circ (0, \varepsilon, 0)) - f(\vec{x})}{\varepsilon} = \Big(\partial_y + \frac{x}{2}\partial_z\Big)f, \\ Z(f) &= \lim_{\varepsilon \to 0} \frac{f(\vec{x} \circ (0, 0, \varepsilon)) - f(\vec{x})}{\varepsilon} = \partial_z f. \end{aligned}$$

The sublaplace is defined by

$$\Delta_{\mathbb{H}} = X^2 + Y^2.$$

Let P_t be the associated semigroup generated by $\Delta_{\mathbb{H}}$, and $p_t(\vec{x}, \vec{y})$ be the heat kernel of P_t. Though the Γ_2 curvature can not be bounded below, explained in [2, 11]. Recently, Li has shown the following sharp gradient estimates for the heat kernel in the Heisenberg group (see [13]).

Theorem 1.2 *There exists a constant $C > 0$ such that for $t > 0$ and $g = (x,y,z) \in \mathbb{H}$,*

$$|\nabla \log p_t(g)| \leq \frac{Cd(g)}{t}, \tag{1.1}$$

where $d(g)$ denotes the Carnot-Carathéodory distance between the origin $o = (0,0,0)$ and g, and $\nabla f := (Xf, Yf)$, $|\nabla f|^2 := (Xf)^2 + (Yf)^2$.

Note that in [13–15], Li obtained the precisely upper and lower bounds for the heat kernel p_t in the Heisenberg (type) group. And the expansion for the heat kernels, Li inequality (i.e. $|\nabla P_t f| \leq C P_t |\nabla f|$ for some $C > 0$), were also obtained. In the present paper, we will use another method to proof the sharp gradient estimates for the heat kernels. This method relies on the positive property of the Bakry-Émery curvature and some basic facts on groups (see [3, 4]). Our method is more applicable in variant models than the one in [13–15].

In the following, we will study two high dimensional Heisenberg groups — non-isotropic Heisenberg group $\mathbb{G}$, and Heisenberg type group $\mathbb{H}_{n,m}$. We derive the sharp gradient estimates for the associated heat kernels (see Propositions 2.2 and 3.1).

2 Non-isotropic Heisenberg Group

We study the $(2n+1)$-dimensional Heisenberg group $\mathbb{G}$ (here we use the same symbol as in [5, Subsection 3.4]), normalized as follows. We equip $\mathbb{R}^{2n}\times\mathbb{R}$ with the group law

$$(x,z)\circ(x',z')=\Big(x+x',z+z'+2\sum_{j=1}^{n}a_j[x_{2j}x'_{2j-1}-x_{2j-1}x'_{2j}]\Big),\quad x,x'\in\mathbb{R}^{2n},\ z,z'\in\mathbb{R},$$

where

$$0<a_1\le a_2\le\cdots\le a_l\le a_{l+1}=\cdots=a_n.$$

If a_i is equal to each other ($a_1=a_n$), it reduces to the isotropic model. In the isotropic model, the optimal gradient estimates for the heat kernels were obtained by Li [14, 15]. In the non-isotropic case ($a_1\ne a_n$), although the method used there is not applicable, we will derive the sharp gradient estimate for the heat kernel in this context.

2.1 Some basic facts

The left invariant vector fields in $\mathbb{G}$ are

$$X_{2j-1}=\partial_{2j-1}+2a_jx_{2j}\partial_z,\quad X_{2j}=\partial_{2j}-2a_jx_{2j-1}\partial_z,\quad Z=\partial_z,$$

where $\partial_i=\partial_{x_i}$. We have the following properties:

(i) For $1\le j\le n$, $[X_{2j-1},X_{2j}]=-4a_jZ$, $[X_{2j-1},Z]=[X_{2j},Z]=0$;

(ii) For $1\le i,j\le n$, $i\ne j$, $[X_{2i-1},X_{2j-1}]=[X_{2i-1},X_{2j}]=[X_{2i},X_{2j-1}]=[X_{2i},X_{2j}]=0$.

The sublaplace is defined by

$$\Delta=\sum_{j=1}^{2n}X_j^2.$$

Denote the semigroup $P_t=\mathrm{e}^{t\Delta}$. The heat kernel $p_t(x,z)$ (density of the semigroup P_t) is given by

$$p_t(x,z)=\frac{1}{2(4\pi t)^{n+1}}\int_{-\infty}^{\infty}\mathrm{e}^{-\frac{f(x,z,\tau)}{4t}}V(\tau)\,\mathrm{d}\tau,$$

where

$$f(x,z,\tau)=-\mathrm{i}\,\tau z+\sum_{j=1}^{n}a_j\tau\coth(a_j\tau)r_j^2,\quad V(\tau)=\prod_{j=1}^{n}\frac{a_j\tau}{\sinh(a_j\tau)},$$

where $\mathrm{i}=\sqrt{-1}$, $r_j^2=x_{2j-1}^2+x_{2j}^2$ (see [5]). We have the time scaling property $p_t(x,z)=t^{-n-1}p_1(\frac{x}{\sqrt{t}},\frac{z}{t})$. So it is enough to get the estimates for p_1.

Set $x=(\widetilde{x},\widehat{x})$ and $\widehat{x}=(x_{2l+1},x_{2l},\cdots,x_{2n})$, then the Carnot-Carathéodory distance $d(x,z)$ from (x,z) to the origin $(0,0)$ is

$$d^2(x,z)=\begin{cases}\displaystyle\sum_{j=1}^{n}\frac{(2a_j\theta_c)^2r_j^2}{2\sin^2(2a_j\theta_c)}, & \text{if } \widehat{x}\ne0,\\ \displaystyle\sum_{j=1}^{n}\frac{(2a_j\theta_c)^2r_j^2}{2\sin^2(2a_j\theta_c)}, & \text{if } |z|<\displaystyle\sum_{j=1}^{n}a_j\mu\Big(\frac{a_j\pi}{a_n}\Big)r_j^2,\ \text{and } \widetilde{x}\ne0,\ \widehat{x}=0,\\ \displaystyle\frac{\pi}{a_n}\Big(|z|+\sum_{j=1}^{n}a_j\cot\Big(\frac{a_j\pi}{a_n}\Big)\Big)r_j^2, & \text{if } |z|\ge\displaystyle\sum_{j=1}^{n}a_j\mu\Big(\frac{a_j\pi}{a_n}\Big)r_j^2,\ \text{and } \widetilde{x}\ne0,\ \widehat{x}=0,\end{cases}$$

where $\mu(\phi) = \frac{\phi}{\sin^2\phi} - \cot\phi$ and θ_c is the unique solution of $|z| = \sum_{j=1}^{n} a_j\mu(2a_j\theta)r_j^2$ in the interval $[0, \frac{\pi}{2a_n})$ (see [5, Theorems 3.18, 3.24 and 3.52]). In the first two cases, we clearly have $d(x,z) \geq |x|$ $\left(|x|^2 := \sum_{j=1}^{2n} x_j^2\right)$. In the third case, since $d(x,z) \simeq |x| + |z|^{\frac{1}{2}}$ (see [5, Theorem 3.52]). (Because $d(x,z)$ is continuous in this case and $d(x,z)(|x|+|z|^{\frac{1}{2}})^{-1}$ is homogeneous of degree 0.) In all, we have

$$d(x,z) \geq c|x| \tag{2.1}$$

for some positive constant c.

Denote

$$\Gamma(f,g) = \frac{1}{2}(\Delta(fg) - g\Delta f - f\Delta g), \quad \Gamma_2(f,f) := \frac{1}{2}(\Delta\Gamma(f,f) - 2\Gamma(f,\Delta f)).$$

Through calculation, we have

$$\begin{aligned}
\Gamma(f,g) &= \sum_{i=1}^{2n}(X_i f)(X_i g),\\
\Gamma_2(f,f) &= \sum_{i,j=1}^{2n}(X_j X_i f)^2 + 8\sum_{k=1}^{n} a_k(X_{2k-1}f(X_{2k}Zf) - X_{2k}f(X_{2k-1}Zf)).
\end{aligned}$$

The mixed term $\sum_{k=1}^{n} a_k(X_{2k-1}f(X_{2k}Zf) - X_{2k}f(X_{2k-1}Zf))$ prevents the existence of any constant ρ such that the curvature dimensional condition $CD(\rho,\infty)$ holds. Thus the method used in the elliptic context is not applicable.

For any radial function $f = f(r_1, r_2, \cdots, r_n, z)$, the sublaplace Δ has the following form

$$\begin{aligned}
\Delta f &= \sum_{i,j=1}^{n} f_{ij}\Gamma(r_i,r_j) + f_{zz}\Gamma(z,z) + \sum_{i=1}^{n} f_{iz}\Gamma(r_i,z) + \sum_{i=1}^{n} f_i\Delta r_i + f_z\Delta z\\
&= \sum_{i=1}^{n} f_{ii} + 4\sum_{i=1}^{n} a_i^2 r_i^2 f_{zz} + \sum_{i=1}^{n}\frac{1}{r_i}f_i\\
&:= \mathcal{L}f,
\end{aligned}$$

where $\mathcal{L}$ is defined by

$$\mathcal{L} = \sum_{i=1}^{n}\partial_{ii} + 4\sum_{i=1}^{n} a_i^2 r_i^2\partial_{zz} + \sum_{i=1}^{n}\frac{1}{r_i}\partial_i.$$

Hence, for radial functions $f = f(r_1, r_2, \cdots, r_n, z)$ and $g = g(r_1, r_2, \cdots, r_n, z)$,

$$\begin{aligned}
\Gamma(f,g) &:= \frac{1}{2}(\Delta(fg) - f\Delta g - g\Delta f)\\
&= \frac{1}{2}(\mathcal{L}(fg) - f\mathcal{L}g - g\mathcal{L}f)\\
&= \Gamma^{\mathcal{L}}(f,g)\\
&= \sum_{i=1}^{n}(f_i g_i + 4a_i^2 r_i^2 f_z g_z).
\end{aligned}$$

Moreover, through calculation, we have

$$\begin{aligned}\Gamma_2(f,f) &:= \frac{1}{2}(\Delta\Gamma(f,f) - 2\Gamma(f,\Delta f))\\ &= \frac{1}{2}(\mathcal{L}\Gamma^{\mathcal{L}}(f,f) - 2\Gamma^{\mathcal{L}}(f,\mathcal{L}f))\\ &= \Gamma_2^{\mathcal{L}}(f,f)\\ &= \sum_{i=1}^n f_{ii}^2 + \Big(4a_i^2 r_i^2 f_{zz} - \frac{f_z}{r_i}\Big)^2 + 8a_i^2(f_z + r_i f_{zi})^2\\ &\quad + \sum_{i\neq j} f_{ij}^2 + 4a_i^2 r_i^2 f_{jz}^2 + 4a_j^2 r_j^2 f_{iz}^2 + 16 a_i^2 a_j^2 r_i^2 r_j^2 f_{zz}^2\\ &\geq 0. \end{aligned} \tag{2.2}$$

Note that the heat kernel p_t depends only on $r_1, r_2, \cdots, r_n, z$. So, in particular, we have

$$\Gamma_2(p_t, p_t) \geq 0, \quad \forall\, t \geq 0.$$

2.2 Gradient estimates for the heat kernels

Let us firstly state the Li-Yau type inequality in the non-isotropic Heisenberg group $\mathbb{G}$.

Proposition 2.1 *There exist positive constants C_1, C_2, C_3 such that for any positive function $f : \mathbb{G} \to \mathbb{R}^+$, if $u = \log P_t f$, we have*

$$\partial_t u \geq C_1 \Gamma(u) + C_2 t |Zu|^2 - \frac{C_3}{t}. \tag{2.3}$$

Proof Following [3], for fixed $t > 0$, let $u_s = P_{t-s} f$, $g_s = \log u_s$, and

$$\Phi_1(s) = P_s(u_s \Gamma(g_s)), \quad \Phi_2(s) = P_s(u_s (Z g_s)^2).$$

It follows that

$$\Phi_1'(s) = 2P_s(u_s \Gamma_2(g_s, g_s)), \quad \Phi_2'(s) = 2P_s(u_s \Gamma(Z g_s)).$$

Note that

$$\begin{aligned}\Gamma_2(f,f) &= \sum_{i,j=1}^{2n} (X_j X_i f)^2 + 8\sum_{k=1}^n a_k((X_{2k-1}f)(X_{2k}Zf) - (X_{2k}f)(X_{2k-1}Zf))\\ &\overset{(*)}{\geq} \frac{1}{2n}(\Delta f)^2 + 8\sum_{k=1}^n a_k^2 (Zf)^2 + 8\sum_{k=1}^n a_k((X_{2k-1}f)(X_{2k}Zf) - (X_{2k}f)(X_{2k-1}Zf))\\ &\geq \frac{1}{2n}(\Delta f)^2 + 8\sum_{k=1}^n a_k^2 (Zf)^2 - 4\sum_{k=1}^n a_k\Big(\frac{1}{\lambda}|\nabla_k f|^2 + \lambda|\nabla_k(Zf)|^2\Big)\\ &\geq \frac{1}{2n}(\Delta f)^2 + 8\sum_{k=1}^n a_k^2 (Zf)^2 - 4a_n\Big(\frac{1}{\lambda}\Gamma(f) + \lambda\Gamma(Zf)\Big),\end{aligned}$$

where $\lambda > 0$ and $|\nabla_k(f)|^2 := (X_{2k-1}f)^2 + (X_{2k}f)^2$. Here we use the assumption that for

$1 \le k \le n$, $0 < a_k \le a_n$, and inequality $(*)$ follows from

$$\begin{aligned}\sum_{i,j=1}^{2n}(X_jX_if)^2 &= \sum_{i=1}^{2n}(X_i^2f)^2 + \sum_{1\le i<j\le 2n}((X_jX_if)^2+(X_jX_if)^2)\\ &\ge \frac{1}{2n}(\Delta f)^2 + \frac{1}{2}\sum_{1\le i<j\le 2n}(X_iX_jf - X_jX_if)^2\\ &= \frac{1}{2n}(\Delta f)^2 + 8\sum_{k=1}^{n}a_k^2(Zf)^2.\end{aligned}$$

Here we use the basic property (i) and (ii) in Subsection 2.1. Since for any $\gamma\in\mathbb{R}$,

$$(\Delta u_s)^2 \ge 2\gamma\Delta u_s - \gamma^2, \quad \Delta g_s = \frac{\Delta u_s}{u_s} - \Gamma(g_s),$$

it yields

$$\Phi_1'(s) \ge -\Big(\frac{2\gamma}{n}+\frac{8a_n}{\lambda}\Big)\Phi_1(s) + 16\sum_{k=1}^{n}a_k^2\Phi_2(s) - 4\lambda a_n\Phi_2'(s) + \frac{2\gamma}{n}\Delta P_tf - \frac{\gamma^2}{n}P_tf.$$

Let a, b be two positive functions defined on $[0,t)$, with b decreasing. We have

$$\begin{aligned}(a(s)\Phi_1(s)+b(s)\Phi_2(s))' \ge{}& \Big(a' - \frac{8a_na}{\lambda} - \frac{2a\gamma}{n}\Big)\Phi_1(s) + \Big(16\sum_{k=1}^{n}a_k^2a + b'\Big)\Phi_2(s)\\ &+ (b-4\lambda a_na)\Phi_2'(s) + \frac{2\gamma a}{n}\Delta P_tf - \frac{\gamma^2a}{n}P_tf.\end{aligned}$$

Choose

$$a = -\frac{b'}{16\sum\limits_{k=1}^{n}a_k^2}, \quad \lambda = -\frac{4\sum\limits_{k=1}^{n}a_k^2b}{a_nb'}, \quad \gamma = \frac{nb''}{2b'} + \frac{na_n^2b'}{\sum\limits_{k=1}^{n}a_k^2b},$$

such that the coefficients of $\Phi_1(s)$, $\Phi_2(s)$ and $\Phi_2'(s)$ in the right-hand side reduce to zero. Then choose $b(s) = (t-s)^\alpha$ for some $\alpha > 2$. Integrating the above differential inequality from 0 to t, the desired result follows.

As a consequence, we have the following Harnack inequality. There exist positive constants A_1 and A_2, such that for $t_2 > t_1 > 0$ and $g_1, g_2 \in \mathbb{G}$,

$$\frac{p_{t_1}(g_1)}{p_{t_2}(g_2)} \le \Big(\frac{t_2}{t_1}\Big)^{A_1} \mathrm{e}^{A_2\frac{d^2(g_1,g_2)}{t_2-t_1}} \tag{2.4}$$

(see [16]).

Proposition 2.2 *There exists a constant $C > 0$ such that for $t > 0$ and $g = (x,z) \in \mathbb{G}$,*

$$\sqrt{\Gamma(\log p_t)(g)} \le \frac{Cd(g)}{t}, \tag{2.5}$$

where $d(g)$ denotes the Carnot-Carathéodory distance between 0 and g.

Proof Following [4], for $0 < s < t$, let $\Phi(s) = P_s(p_{t-s}\log p_{t-s})$. We have

$$\Phi'(s) = P_s(p_{t-s}\Gamma(\log p_{t-s})), \quad \Phi''(s) = 2P_s(p_{t-s}\Gamma_2(\log p_{t-s})).$$

By the fact that $\Gamma_2(p_t, p_t) \geq 0$ (here we use the positive property of Γ_2 on the radial functions), we have that for all $t \geq 0$, Φ'' is positive, whence Φ' is non-deceasing. Thus

$$\int_0^{\frac{t}{2}} \Phi'(s)\,\mathrm{d}s \geq \frac{t}{2}\Phi'(0).$$

That is

$$p_t\Gamma(\log p_t) \leq \frac{2}{t}(P_{\frac{t}{2}}(p_{\frac{t}{2}}\log p_{\frac{t}{2}}) - p_t\log p_t).$$

The right-hand side can be bounded by applying the above Harnack inequality (2.4) and the basic fact $p_{\frac{t}{2}}(g) \leq p_{\frac{t}{2}}(0)$ for all $g = (x, z) \in \mathbb{G}$. We have

$$\sqrt{\Gamma(\log p_t)(g)} \leq C\Big(\frac{d(g)}{t} + \frac{1}{\sqrt{t}}\Big).$$

In particular,

$$\sqrt{\Gamma(\log p_1)(g)} \leq C(d(g) + 1).$$

If $d(g) \geq 1$, it is trivial to get the desired result. If $d(g) \leq 1$, by (2.1), we have $|x| \leq \frac{1}{c}$, $g = (x, z)$.

As in [14], denote

$$W_{1,j} = \int_{\mathbb{R}} \frac{a_j\tau\cosh(a_j\tau)}{\sinh(a_j\tau)} \cdot \prod_{k=1}^{n} \frac{a_k\tau}{\sinh(a_k\tau)}\,\mathrm{d}\tau, \quad W_{2,j} = \int_{\mathbb{R}} a_j\tau \cdot \prod_{k=1}^{n} \frac{a_k\tau}{\sinh(a_k\tau)}\,\mathrm{d}\tau.$$

We have

$$\Gamma(p_1) \leq C\sum_{j=1}^{n} r_j^2(|W_{1,j}| + |W_{2,j}|)^2,$$

where C depends on the dimension n. From the expression of $W_{1,j}$ and $W_{2,j}$, we can easily obtain that $|W_{i,j}|_{i=1,2,\ j=1,\cdots,n}$ are bounded. So $\Gamma(p_1) \leq C_1|x|^2 \leq C_2 d(g)$ for some positive constants C_1 and C_2. By the classic estimates on the heat kernel, $p_1 \simeq 1$, on $|d(g)| \leq 1$ (see, for example, [18, Theorems IV 4.2 and IV 4.3]). Thus

$$\Gamma(\log p_1)(g) \leq Cd(g).$$

Therefore, we complete the proof by the time scaling property.

3 Heisenberg-Type Group $\mathbb{H}_{n,m}$

Heisenberg type group $\mathbb{H}_{n,m}$ is a nature generalization of Heisenberg group $\mathbb{H}$. Let us firstly recall the definition of Heisenberg type groups. It can be seen as $\mathbb{R}^{n+m}$ with a multiplying law $\circ$ (refer to [6, 10, 15] and references therein). The multiplier $\circ$ is defined by

$$(x,t)\circ(\xi,\tau) = \begin{pmatrix} x_j + \xi_j, & j = 1,\cdots,n, \\ t_j + \tau_j + \frac{1}{2}\langle x, U^{(j)}\xi\rangle, & j = 1,\cdots,m, \end{pmatrix}, \quad x,\xi \in \mathbb{R}^n,\ t,\tau \in \mathbb{R}^m,$$

where the matrices $U^{(1)}, \cdots, U^{(m)}$ have the following properties:

(1) $U^{(j)}$ is an $n \times n$ anti-symmetric and orthogonal matrix (skew matrix) for every $j = 1, 2, \cdots, n$;

(2) $U^{(i)}U^{(j)} + U^{(j)}U^{(i)} = 0$ for every $i, j \in \{1, \cdots, m\}$ with $i \neq j$,

and m, n satisfy $m < \rho(n) = 8p + 2^q$, where $n = (2a+1)2^{4p+q}$ for $a, p \in \mathbb{N}$ and $0 \leq q < 3$. Note that n is even, and clearly we have $n \geq \rho(n) \geq m+1$. The left invariant vector fields are

$$X_j = \frac{\partial}{\partial x_j} + \frac{1}{2}\sum_{k=1}^{m}\sum_{i=1}^{n} x_i U_{i,j}^{(k)} \frac{\partial}{\partial t_k}. \tag{3.1}$$

By using the anti-symmetric of $U^{(j)}$, we obtain

$$[X_i, X_j] = \sum_{k=1}^{m} U_{i,j}^{(k)} \frac{\partial}{\partial t_k} := Z_{ij} \quad \text{for every } i, j \in \{1, \cdots, n\}.$$

Also, we have

$$Z_{ij} + Z_{ji} = 0.$$

This yields the commutative relations

$$[X_i, Z_{jk}] = 0 \quad \text{for every } i, j, k \in \{i, \cdots, n\}.$$

Note that for the case $n = 2$ and $m = 1$, it reduces to the Heisenberg groups case. Let us introduce the canonical sub-Laplacian operator in the Heisenberg-type groups

$$\Delta := \sum_{j=1}^{n} X_j^2 = \Delta_x + \frac{1}{4}|x|^2\Delta_t + \sum_{k=1}^{m}\langle x, U^{(k)}\nabla_x\rangle \frac{\partial}{\partial t_k},$$

where the $U^{(k)}$ are as (3.1). Here, we use the notation $\Delta_x = \sum\limits_{j=1}^{n}(\frac{\partial}{\partial x_j})^2$, $\Delta_t = \sum\limits_{k=1}^{m}(\frac{\partial}{\partial t_k})^2$ and $\nabla_x = (\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n})^{\mathrm{T}}$. In this model, for the Carnot-Carathéodory distance $d(g)$ between 0 and $g = (x, t) \in \mathbb{H}_{n,m}$, we have $d(g) \geq |x|$.

Moreover, for any radial function $f(r, t)$, $r^2 := \sum\limits_{i=1}^{n} x_i^2$, as done in the non-isotropic case, Δ has the following form

$$\Delta f = \Delta_x f + \frac{1}{4}|x|^2\Delta_t f = \frac{\partial^2 f}{\partial r^2} + \frac{n-1}{r}\frac{\partial f}{\partial r} + \frac{1}{4}r^2\Delta_t f. \tag{3.2}$$

Hence for any radial functions $f = f(r, t)$ and $g = g(r, t)$, we have

$$\Gamma(f, g) = \frac{\partial f}{\partial r}\frac{\partial g}{\partial r} + \frac{1}{4}r^2\nabla_t f \nabla_t g \tag{3.3}$$

and

$$\begin{aligned}\Gamma_2(f,f) &= \Big(\frac{\partial^2 f}{\partial r^2}\Big)^2 + \frac{n-1}{4}|\nabla_t f|^2 + \frac{1}{2}\Big|\nabla_t f + r\nabla_t \frac{\partial f}{\partial r}\Big|^2 + \frac{r^4}{16}\Big(\sum_{i,j=1}^m \frac{\partial^2 f}{\partial t_i \partial t_j}\Big)^2 \\ &\quad - \frac{r}{2}\frac{\partial f}{\partial r}\Delta_t f + \frac{n-1}{r^2}\Big(\frac{\partial f}{\partial r}\Big)^2 \\ &\geq \Big(\frac{\partial^2 f}{\partial r^2}\Big)^2 + \frac{n-1}{4}|\nabla_t f|^2 + \frac{1}{2}\Big|\nabla_t f + r\nabla_t \frac{\partial f}{\partial r}\Big|^2 + \frac{r^4}{16m}(\Delta_t f)^2 \\ &\quad - \frac{r}{2}\frac{\partial f}{\partial r}\Delta_t f + \frac{n-1}{r^2}\Big(\frac{\partial f}{\partial r}\Big)^2 \\ &= \Big(\frac{\partial^2 f}{\partial r^2}\Big)^2 + \frac{n-1}{4}|\nabla_t f|^2 + \frac{1}{2}\Big|\nabla_t f + r\nabla_t \frac{\partial f}{\partial r}\Big|^2 \\ &\quad + \Big(\frac{r^2}{4\sqrt{m}}\Delta_t f - \frac{\sqrt{m}}{r}\frac{\partial f}{\partial r}\Big)^2 + \frac{n-1-m}{r^2}\Big(\frac{\partial f}{\partial r}\Big)^2 \\ &\geq 0,\end{aligned}$$

where the last inequality follows from the fact that $n \geq 1+m$.

Let p_s be the heat kernel of the semi-group P_s starting from the origin o. Then we have the following form (see [15, 17]):

$$\begin{aligned}p_s(x,t) &= (2\pi)^{-m}(4\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^m} \Big(\frac{|\lambda|}{\sinh s|\lambda|}\Big)^{\frac{n}{2}} \exp\Big(-t\cdot\lambda\,\mathrm{i} - \frac{|x|^2|\lambda|}{4}\coth s|\lambda|\Big)\mathrm{d}\lambda \\ &= (2\pi)^{-m}(4\pi)^{-\frac{n}{2}} s^{-\frac{n}{2}-m} \int_{\mathbb{R}^m} \Big(\frac{|\lambda|}{\sinh|\lambda|}\Big)^{\frac{n}{2}} \exp\frac{1}{s}\Big(-t\cdot\lambda\,\mathrm{i} - \frac{|x|^2|\lambda|}{4}\coth|\lambda|\Big)\mathrm{d}\lambda. \end{aligned} \tag{3.4}$$

As done above, we can also get the Li-Yau type inequality holds in the Heisenberg type group $\mathbb{H}_{n,m}$, hence the Harnack type inequality follows. Also, we have the following sharp gradient estimates for the heat kernel. This proposition was obtained by Li [15] by a different method.

Proposition 3.1 *There exists a constant $C>0$ such that for $t>0$ and $g=(x,t)\in\mathbb{H}_{n,m}$,*

$$\sqrt{\Gamma(\log p_t)(g)} \leq \frac{Cd(g)}{t}, \tag{3.5}$$

where $d(g)$ denotes the Carnot-Carathéodory distance between 0 and g.

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