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Generalized Tricomi Problem for a Quasilinear Mixed Type Equation**

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(*Dedicated to Professor Andrew Majda on the Occasion of his* 60*th Birthday*)

Abstract In this paper, the Tricomi problem and the generalized Tricomi problem for a quasilinear mixed type equation are studied. The coefficients of the mixed type equation are discontinuous on the line, where the equation changes its type. The existence of solution to these problems is proved. The method developed in this paper can be used to study more difficult problems for nonlinear mixed type equations arising in gas dynamics.

Keywords Mixed type equation, Tricomi problem, Mach configuration
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1 Introduction

In this paper, we are concerned with the boundary value problems of a nonlinear mixed type equation of second order. The motivation comes from the study of the stability of Mach configuration in gas dynamics. It is known that Mach configuration is a wave configuration frequently appearing in shock reflections. For instance, consider an incident plane shock front attacks a plane wall, if the incident angle is smaller than a critical value, then the regular reflection occurs, while if the incident angle is larger than the critical value, then the Mach reflection occurs. In the latter case the intersection of the incident shock and the reflected shock will not meet at the rigid wall, but at a point away from the wall. The intersect is connected with the rigid wall by another shock front, called Mach stem. Meanwhile, there is a slip line issuing from the intersection. These three shock fronts and a slip line near the intersection form a Mach configuration, which was first found by von Neumann (see [15]).

A crucial problem in studying Mach configuration is its stability under perturbation, because only a stable wave structure is physics and can actually occur. For a Mach configuration in a compressible flow, a part of the upstream flow first passes across the incident shock and then passes across the reflected shock, while another part of the upstream flow simply passes across the Mach stem. After passing shock fronts these two parts of the flow meet again at the downstream part, where they are separated by a stream line bearing a contact discontinuity.

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Generally, the flow behind the Mach stem is always subsonic, but the flow passing across both the incident shock and the reflected shock can be subsonic or supersonic. Therefore, referring to the flow in the downstream part one has two different cases: one is two subsonic regions separated by a streamline, the other is a subsonic region and a supersonic region adjacent to each other separated by a streamline. If one neglect the characteristics describing the effect of transport, then in the first case the flow in the downstream part should be described by an elliptic equation with discontinuous coefficients, while in the second case the wave configuration should be described by a mixed type equation. Generally, the first case is called E-E type Mach configuration, and the second case is called E-H type Mach configuration. The stability of E-E type Mach configuration has been verified in [3, 4], while the stability of E-H type Mach configuration is still an open problem so far.

Since the downstream flow behind the reflected shock front and the Mach stem for E-H Mach configuration is supersonic-subsonic flow separated by a contact discontinuity, so that the flow should be described by a nonlinear mixed type equation with discontinuous coefficients. As we know that the study on mixed type equations is much more difficult than the study on the purely elliptic equations and the purely hyperbolic equations, so that the results on mixed type equations are also much less than the latter. It turns out that the mixed type equation introduced in the study on E-H type Mach configurations has some similarity to Lavrentiev-Bitsadze mixed type equation (see [13]). Correspondingly, the stability of E-H type Mach configuration will be reduced to a generalized Tricomi problem of such a nonlinear mixed type equation.

As early in 1923, Tricomi initiated the study of the mixed type equation with the form

$$y\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{1.1}$$

and a special boundary value problem for it in [19]. They are called Tricomi equation and Tricomi problem by his successors respectively. Later, another mixed type equation, called Keldysh equation, with the form

$$y\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial x^2} = 0 \tag{1.2}$$

is also studied. These two equations have continuous coefficients. Moreover, both are elliptic on the upper half plane and are hyperbolic on the lower half plane. The difference is: when the characteristics on the lower half plane approaches the line $y = 0$, the characteristics of Tricomi equation is perpendicular to $y = 0$, while the characteristics of Keldysh equation is tangential to $y = 0$. Such a difference causes great divergence of the setting of the boundary value problems for these two equations as well as the properties of solutions for them. They stand for two basic models of mixed type equations.

In 1950, Lavrentiev and Bitsadze [13] introduced a new mixed type equation with the form

$$\frac{\partial^2 u}{\partial y^2} + \operatorname{sgn} y \frac{\partial^2 u}{\partial x^2} = 0. \tag{1.3}$$

The equation has discontinuous coefficients on the line $y = 0$, while the solution is required to be continuous and have continuous derivatives on $y = 0$. Equation (1.3) was considered as a simplest model of mixed type equations, but recent research found its new applications. As we mentioned above, in the study of E-H type Mach reflection of shock fronts the Euler system under consideration can be reduced to a nonlinear mixed type equation, which is defined in a neighborhood of triple intersection with coefficients discontinuous on the slip line (see [6]).

In this paper, we will study the following model quasilinear mixed type equation

$$(1+u)\frac{\partial^2 u}{\partial y^2} + \operatorname{sgn} y \frac{\partial^2 u}{\partial x^2} = 0 \tag{1.4}$$

near $u = 0$. We hope that such a study will give a good deal of enlightenment to study more general case. Let δ be a number in $(0, \frac{1}{2})$, and require $|u| \le \delta$ in all later discussions. Under such an assumption, equation (1.4) is elliptic in the upper half plane and is hyperbolic in the lower half plane.

The equation of characteristic curves of (1.4) in $y < 0$ is

$$(1+u)\mathrm{d}x^2 - \mathrm{d}y^2 = 0,$$

or

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \pm\sqrt{1+u}\,. \tag{1.5}$$

Hence the rightward characteristics starting from the origin is

$$\Gamma_1: \quad \begin{cases} \dfrac{\mathrm{d}y}{\mathrm{d}x} = -\sqrt{1+u}\,, \\ y|_{x=0} = 0, \end{cases} \tag{1.6}$$

while the leftward characteristics starting from $(1,0)$ is

$$\Gamma_2: \quad \begin{cases} \dfrac{\mathrm{d}y}{\mathrm{d}x} = \sqrt{1+u}\,, \\ y|_{x=1} = 0. \end{cases} \tag{1.7}$$

We emphasize that Γ_1 and Γ_2 depend on the solution u.

Let $\Gamma_0 : y = \gamma_0(x)$ be a C^2 curve satisfying $\gamma_0(x) > 0$ in $0 < x < 1$, and

$$\gamma_0(0) = \gamma_0(1) = 0, \quad \gamma_0'(0) > 0, \quad \gamma_0'(1) < 0. \tag{1.8}$$

Then we can set the following Tricomi problem

$$(\mathrm{P}_1): \quad \begin{cases} \text{equation (1.4)}, & (x,y) \in \Omega_1, \\ u(x,y) = \psi(x), & (x,y) \in \Gamma_0, \\ u(x,y) = \phi(x), & (x,y) \in \Gamma_1, \end{cases} \tag{1.9}$$

where Ω_1 is the domain bounded by Γ_0, Γ_1 and Γ_2. $|\phi|, |\psi| \le \delta$ and

$$\gamma_1'(x) = -\sqrt{1+\phi(x)}\,. \tag{1.10}$$

Next we are going to prove the following theorem.

Theorem 1.1 *Assume that* $\gamma_0(x) \in C^{1,\alpha_0}[0,1]$, $|\gamma'(0)|, |\gamma'(1)| < 1$; $\psi(x), \phi(x) \in C^{1,\alpha_0}[0,1]$ *satisfy* $|\psi|, |\phi| \le \delta$, $\phi(0) = \psi(0)$; $\gamma_1(x) \in C^{1,\alpha_0}[0,1]$ *satisfies* (1.10). *Then there exists a unique solution* $u(x,y) \in C^{1,\alpha}(\Omega_1)$ *of* (P_1), *where* α *is a number in* $(0, \alpha_0)$ *depending on* γ_0 *and* γ_1.

Remark 1.1 Since the curve Γ_2 also depends on u, the domain Ω_1 should be determined together with the solution u. Therefore, the data on the curve Γ_1 will generally be assigned on a longer segment than actually required. For instance, $\gamma_1(x)$ and $\phi(x)$ can be given on $[0,1]$ and condition (1.10) is satisfied on this interval. Furthermore, Γ_2 is on the left side of the straight line $\ell_2 : y = \sqrt{1+\delta}\,(x-1)$ because $|u| \le \delta$. Denote by $A(\overline{x}, \overline{y})$ the intersection $\Gamma_1 \cap \ell_2$. Then in order to determine the solution of (1.9), it is enough to have $\gamma_1(x)$ and $\phi(x)$ on $0 \le x \le \overline{x}$.

A more general boundary value problem for (1.4), which is called generalized Tricomi problem here, is discussed. Assume that Σ_1 is a C^2 curve $y = \zeta(x)$ satisfying

$$\zeta(0) = 0, \quad 0 < \sqrt{1-\delta} + \zeta'(x) < \eta, \tag{1.11}$$

where η is a small number. Denote by Ω_2 the domain bounded by Γ_0, Σ_1 and Γ_2, by Ω_+ the domain $\Omega_2 \cap \{y = 0\}$, by $(\frac{\partial u}{\partial y})_\pm$ the upper limit and the lower limit on $y = 0$. Then we can also set up the following boundary value problem:

$$(\mathrm{P}_2): \quad \begin{cases} \text{equation (1.4)}, & (x,y) \in \Omega_2, \\ u(x,y) = \psi(x), & (x,y) \in \Gamma_0, \\ u(x,y) = \phi(x), & (x,y) \in \Sigma_1, \\ u \text{ and } \nabla u \text{ are continuous}, & \text{on } y = 0. \end{cases} \tag{1.12}$$

For the problem (P_2), we are going to prove

Theorem 1.2 *Under the assumptions of Theorem* 1.1 *on ϕ, ψ and Γ_0 and the assumption that η is sufficiently small, there exists a unique solution $u(x,y) \in C^{0,\alpha}(\Omega_2)$ of* (P_2) *which satisfies $u(x,y) \in C^{1,\alpha}(\overline{\Omega}_\pm)$ and all conditions in* (1.11)*, where α is a number in $(0, \alpha_0)$ depending on γ_0 and γ_1.*

In the next sections, we will give the proof of the existence of solutions to the Tricomi problem (P_1) and the generalized Tricomi problem (P_2). Section 2 will give the outline of our method and the form of the linearized problems, which are Tricomi problem and generalized Tricomi problem for Lavrentiev-Bitsadze equation (1.3). The Tricomi problem for linear mixed type equation is solved in Section 3. There we first introduce an operator from the value of u on x-axis to the value of $\frac{\partial u}{\partial y}$. Combining the condition on the characteristics Γ_1, we establish a boundary value problem of Laplace equation on Ω_+ with oblique derivative boundary condition on $y = 0$. Section 4 treats the generalized Tricomi equation. In this case, the relation on x-axis is more general. Similar argument as did in Section 3 leads us to a boundary value problem of Laplace equation on Ω_+ with a nonlocal boundary condition on $y = 0$. Based on the results on the Tricomi problem and the generalized Tricomi problem for Lavrentiev-Bitsadze equation, the nonlinear problems (P_1) and (P_2) for equation (1.4) are solved by using implicit function theorem.

2 Linearization

First let us give the outline of our method to seek the solution to the problems (P_1) and (P_2). Without loss of any generality, we may assume $\psi(x) = 0$. Next we denote the value of the solution u on x-axis by $f(x)$. Combining it with the boundary condition on the curve γ_0, we obtain a solution $u(x,y)$ of (1.4) inside Ω_+, which is the restriction of the domain Ω_1 or Ω_2 on the upper half plane. Then the derivative $\frac{\partial u}{\partial y}$ on x-axis can be determined correspondingly, which is denoted by $h(x)$, i.e. $h(x) = \frac{\partial u}{\partial y}(x,0)$.

For definiteness we first consider the problem (P_1). Using the value of $f(x)$ and $h(x)$ as the initial data, we can obtain the solution $u(x,y)$ to (1.4) in $\{y<0\}$ by solving a Cauchy problem. The solution is defined in the domain bounded by x-axis and two characteristics issuing from the points $(0,0)$ and $(1,0)$. Denote the rightward characteristics issuing from $(0,0)$ and the leftward characteristics issuing from $(1,0)$ by Γ_1^f and Γ_2^f respectively. Denote the intersection of them by $P_a : (x_a, y_a)$, and denote the value of u on Γ_1^f by $u^f(x)$. We obviously should have $u^f(x) = \phi(x)$ on $(0, x_a)$. Hence if we define an operator Φ by

$$\Phi[f,\phi] = u_f\Big(\frac{x}{x_a}\Big) - \phi\Big(\frac{x}{x_a}\Big), \tag{2.1}$$

then to solve the problem (P_1) is equivalent to finding a function $f(x)$ satisfying

$$\Phi[f,\phi] = 0. \tag{2.2}$$

Let H be the set $C^{1,\alpha}[0,1]$, and F be the set of all function in H satisfying $f(0) = f(1) = 0$. Next we are going to show that the map $(f,\phi) \mapsto \Phi[f,\phi]$ is a nonlinear continuous map from $F \times H$ to H.

Lemma 2.1 *Let $u(x,y)$ be the solution to the boundary value problem*

$$\begin{cases} (1+u)\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0, \\ u|_{\Gamma_0} = 0, \\ u|_{y=0} = f(x) \end{cases} \tag{2.3}$$

in the domain Ω_+, and $h(x) = \frac{\partial u}{\partial y}(0,x)$. Then $f \mapsto h$ is a continuous map from $C^{1,\alpha}(0,1)$ to $C^{1,\alpha}(0,1)$.

Proof The existence of the solution to the Dirichlet problem can be obtained by the classical theory of nonlinear elliptic equations of second order (see [11]). We indicate here that it is impossible to expect the $C^{2,\alpha}$ solution to (2.3) even for $f(x) \in C^{2,\alpha}$, because of the appearance of the singular points $(0,0)$ and $(1,0)$ on the boundary. On the other hand, since both the angles formed by the boundary curve at these two points are less than π, the solution can be in $C^{1,\alpha}$ (see [10]), where $\alpha < \alpha_0$ is determined by the angles formed by the boundary curve Γ_0 and x-axis at $(0,0)$ and $(1,0)$. The estimate of the weak solution of Laplace equation indicates

$$\|u\|_{C^{1,\alpha}} \le C(\|f\|_{C^{1,\alpha}} + \|u\|_{L^\infty}). \tag{2.4}$$

In view of $\|u\|_{L^\infty} \le \max|f(x)| \le \|f\|_{C^{1,\alpha}}$, (2.4) can be simply written as

$$\|u\|_{C^{1,\alpha}} \le C\|f\|_{C^{1,\alpha}} \tag{2.5}$$

by possibly replacing the constant C. Then the estimate

$$\|h(x)\|_{C^\alpha(0,1)} \le \|u\|_{C^{1,\alpha}} \le C\|f\|_{C^{1,\alpha}} \tag{2.6}$$

follows, which gives the conclusion of the lemma.

Lemma 2.2 *Let $u(x,y)$ be the solution to the boundary value problem*

$$\begin{cases}(1+u)\dfrac{\partial^2 u}{\partial x^2}-\dfrac{\partial^2 u}{\partial y^2}=0,\\ u(x,0)=f(x), \quad \dfrac{\partial u}{\partial y}(x,0)=h(x)\end{cases} \tag{2.7}$$

in $y<0$, where $f(x)\in C^{1,\alpha}(0,1)$, $f(0)=f(1)=0$ and $h(x)\in C^{\alpha}(0,1)$. Denote by Ω_- the domain bounded by x-axis, the rightward characteristics $\Gamma_1: y=\gamma_1(x)$ issuing from $(0,0)$ and the leftward characteristics $\Gamma_2: y=\gamma_2(x)$ issuing from $(1,0)$. Denote the equations of Γ_i by $y=\gamma_i(x)$ for $i=1,2$, and $P_a(x_a,y_a)=\Gamma_1\cap\Gamma_2$, $\phi(x)=u(x,\gamma_1(x))$. Then

$$\begin{aligned}&u(x,y)\in C^{1,\alpha}(\Omega_-), \quad \phi(x)\in C^{1,\alpha}(0,x_a),\\ &\|\phi(x)\|_{C^{1,\alpha}}\le C(\|f\|_{C^{1,\alpha}}+\|h\|_{C^{\alpha}}).\end{aligned} \tag{2.8}$$

Proof Let $v_1=\frac{\partial u}{\partial x}$ and $v_2=\frac{\partial u}{\partial y}$. Then the equation in (2.7) can be written as

$$\begin{cases}\dfrac{\partial v_1}{\partial y}-\dfrac{\partial v_2}{\partial x}=0,\\ \dfrac{\partial v_2}{\partial y}-(1+u)\dfrac{\partial v_1}{\partial x}=0,\\ v_1(x,0)=f'(x), \quad v_2(x)=h(x).\end{cases} \tag{2.9}$$

The functions $f'(x)$ and $h(x)$ can be extended to $(-\infty,\infty)$ so that the extended functions are compactly supported, while their C^{α} norm are unchanged. Keeping in mind that $u=\int_{-\infty}^{x} v_1(x)\mathrm{d}x$, we are going to solve problem (2.9) in the lower half plane.

Denoting by $\partial_\pm$ the operators $\frac{\partial}{\partial y}\pm\sqrt{1+u}\,\frac{\partial}{\partial x}$, from (2.9) we have

$$\sqrt{1+u}\,\partial_\pm v_1\mp\partial_\pm v_2=0, \tag{2.10}$$

which can be written as

$$\partial_\pm(\sqrt{1+u}\,v_1\mp v_2)-\frac{v_1}{2\sqrt{1+u}}\partial_\pm u=0. \tag{2.11}$$

In view of

$$\begin{aligned}\partial_\pm u&=(\partial_y\pm\sqrt{1+u}\,\partial_x)\int_{-\infty}^{x} v_1\,\mathrm{d}x\\ &=\int_{-\infty}^{x} v_{2x}\,\mathrm{d}x\pm\sqrt{1+u}\,v_1=v_2\pm\sqrt{1+u}\,v_1,\end{aligned} \tag{2.12}$$

we have

$$\partial_\pm(\sqrt{1+u}\,v_1\mp v_2)-\frac{v_1v_2}{2\sqrt{1+u}}\pm\frac{v_1^2}{2}=0, \tag{2.13}$$

which gives a system of integral equations:

$$\begin{cases}\sqrt{1+u}\,v_1+v_2=\displaystyle\int_{\ell_-}\frac{v_1v_2}{2\sqrt{1+u}}+\frac{v_1^2}{2}\,\mathrm{d}s,\\ \sqrt{1+u}\,v_1-v_2=\displaystyle\int_{\ell_+}\frac{v_1v_2}{2\sqrt{1+u}}-\frac{v_1^2}{2}\,\mathrm{d}s.\end{cases} \tag{2.14}$$

Obviously, system (2.14) can be solved in C^α by using Picard iteration process. Denote the solution by (v_1, v_2), one can obtain $u \in C^{1,\alpha}$ by using $\partial_\pm u = v_2 \pm \sqrt{1+u}\, v_1$. Hence (2.9) is weakly satisfied. In accordance, (2.7) is also weakly satisfied.

From the approximate process of solving the integral system (2.14), we also know that the characteristics determined by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \pm \frac{1}{\sqrt{1+u}} \tag{2.15}$$

are $C^{1,\alpha}$ curves. It is evident that

$$\begin{aligned}
&\gamma_1(x) \in C^{1,\alpha}(0, x_a), \quad \gamma_2(x) \in C^{1,\alpha}(x_a, 1),\\
&\|\gamma_1(x)\|_{C^{1,\alpha}(0,x_a)} + \|\gamma_2(x)\|_{C^{1,\alpha}(x_a,1)} \le C(\|f\|_{C^{1,\alpha}} + \|h\|_{C^{0,\alpha}}),\\
&u(x,y) \in C^{1,\alpha}(\Omega_-), \quad \|u(x,y)\|_{C^{1,\alpha}(\Omega_-)} \le C(\|f\|_{C^{1,\alpha}} + \|h\|_{C^{0,\alpha}}).
\end{aligned} \tag{2.16}$$

Noticing $\phi(x) = u(x, \gamma_1(x))$, we obtain (2.8) immediately.

Since $h(x)$ is determined by $f(x)$ according to Lemma 2.1, we obtain that $\Phi[f, \phi]$ is a continuous map from $F \times H$ to H.

Obviously, $\Phi[0, 0] = 0$. Therefore, by using of the implicit function theorem, the solvability of (2.2) near $\phi = 0$ can be obtained from the reversibility of the linearized operator Φ' at $f = \phi = 0$. Notice that the operator Φ' at $f = \phi = 0$ is an operator defined similarly to the operator Φ, but one has to replace problem (P_1) in the definition of Φ by the Tricomi problem of Lavrentiev-Bitsadze equation as

$$\begin{cases}
\dfrac{\partial^2 u}{\partial x^2} + \operatorname{sgn} y \dfrac{\partial^2 u}{\partial y^2} = 0, & \\
u = 0, & \text{on } \Gamma_0,\\
u = \phi(x), & \text{on } x + y = 0.
\end{cases} \tag{2.17}$$

The linearization of problem (P_2) is similar. Notice that the curve Σ_1 in (1.12) is independent of the solution u. The above linearization procedure will lead us to a boundary value problem

$$\begin{cases}
\dfrac{\partial^2 u}{\partial x^2} + \operatorname{sgn} y \dfrac{\partial^2 u}{\partial y^2} = 0, & \\
u = 0, & \text{on } \Gamma_0,\\
u = \phi(x), & \text{on } \Sigma_1.
\end{cases} \tag{2.18}$$

In the next section, we will first treat (2.17). Then the solvability of (2.17) and the corresponding estimates ensure the solvability of the nonlinear problem (P_1).

3 Tricomi Problem for Linearized Mixed Type Equation

Problem (2.17) was studied by Bitsadze and Lavrentiev [1, 13], Frankle [9], Hua [12] and others.

Assume $f(x) \in C^{1,\alpha}(0,1)$ with $0 < \alpha < 1$. Similarly to the argument in Lemma 2.1, we know that the Dirichlet problem

$$\begin{cases}
\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0,\\
u|_{\Gamma_0 = 0} = 0,\\
u|_{y=0} = f(x)
\end{cases} \tag{3.1}$$

has a unique $C^{1,\alpha}(\Omega_+)$ solution satisfying

$$\|u(x,y)\|_{C^{1,\alpha}(\Omega_+)} \le C\|f\|_{C^{1,\alpha}(0,1)}. \tag{3.2}$$

Turn to the lower part of the domain Ω. Consider the initial value problem

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial^2 u}{\partial y^2} = 0, \\ u|_{y=0} = f(x), \\ u_y|_{y=0} = h(x). \end{cases} \tag{3.3}$$

The solution can be given by d'Alembert formula

$$u(x,y) = \frac{1}{2}(f(x-y) + f(x+y)) + \frac{1}{2}\int_{x-y}^{x+y} h(\xi)\mathrm{d}\xi. \tag{3.4}$$

By taking $y = -x$, we obtain

$$\phi(x) = \frac{1}{2}f(2x) + \frac{1}{2}\int_{2x}^{0} h(\xi)\mathrm{d}\xi. \tag{3.5}$$

Differentiating both sides gives

$$f'(x) - h(x) = \phi'\Big(\frac{x}{2}\Big), \tag{3.6}$$

which can be regarded as a boundary condition for $u(x,y)$ on x-axis. Then the solution $u(x,y)$ of (2.17) on Ω_+ satisfies

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0, & \\ u = 0, & \text{on } \Gamma_0, \\ \dfrac{\partial u}{\partial n} + \dfrac{\partial u}{\partial x} = -\phi'\Big(\dfrac{x}{2}\Big), & \text{on } y=0,\ 0<x<1, \end{cases} \tag{3.7}$$

where n is the outer normal direction of the domain Ω_+ on $y = 0$. According to the theory of elliptic equations in a polygonal domain (see [10] and the notations therein), the solution of problem (3.7) can be written as a sum of a regular part u_r and a singular part u_s, such that $u_r \in C^{2,\sigma}$ and $u_s = \sum c_{j,m}S_{j,m}$ with

$$S_{j,m} = r_j^{\lambda_{j,m}}\phi_{j,m}(\theta_j), \tag{3.8}$$

where (r_j, θ_j) is the local system coordinates in the neighborhood of the j-th vertex of the polygonal domain, $\phi_{j,m}$ are $C^{2,\sigma}$ functions of θ_j, and $\lambda_{j,m}$ are determined by the shape of the polygonal domain and the coefficients of the boundary conditions as

$$\lambda_{j,m} = \frac{\Phi_j - \Phi_{j+1} - m\pi}{\omega_j}. \tag{3.9}$$

Here $\lambda_{j,m}$ is assumed not to be integer, ω_j is the angle of the polygon formed by the sides Γ_j and Γ_{j+1}, m is an arbitrary integer such that $\lambda_{j,m} > 0$, and Φ_j is a number related to the boundary condition on the boundary Γ_j defined as

$$\Phi_j = \begin{cases} \dfrac{\pi}{2}, & \text{if a Dirichlet condition is given on } \Gamma_j, \\ \arctan\beta_j, & \text{if an oblique derivative } \dfrac{\partial u}{\partial n} + \beta_j\dfrac{\partial u}{\partial s} \text{ is given on } \Gamma_j. \end{cases}$$

In problem (3.7), the domain Ω_+ has two vertices $(0,0)$ and $(1,0)$, then $j=2$. Consider the vertex $(0,0)$, we have $j=0$, $\Phi_0=\frac{\pi}{2}$, $\Phi_1=\frac{\pi}{4}$. Therefore, if $\omega_0<\frac{\pi}{4}$, then the exponent $\lambda_{0,m}>1$ and the singular part (3.8) is in $C^{1,\alpha}$ for some $\alpha\in(0,1)$. Similarly, if the angle formed by Γ_0 and x-axis at $(1,0)$ is less than $\frac{\pi}{4}$, then the singular part is also in $C^{1,\alpha}$ for some $\alpha\in(0,1)$. Therefore, under the assumption that the angles between Γ_0 and x-axis at both $(0,0)$ and $(1,0)$ are less than $\frac{\pi}{4}$, problem (3.7) admits a $C^{1,\alpha}$ solution $u(x,y)$ satisfying

$$\|u(x,y)\|_{C^{1,\alpha}(\Omega_+)}\le C\|\phi\|_{C^{1,\alpha}(0,\frac{1}{2})}. \tag{3.10}$$

This is the basic fact required for applying the implicit function theorem as shown in Section 2. Correspondingly, the proof of Theorem 1.1 is complete.

4 Generalized Tricomi Problem ($\mathbf{P_2}$)

In this section, we are going to prove the existence of the solution to the generalized Tricomi problem (P_2). The outline of the method is similar to that for problem (P_1). Hence we will pay main attention to the new ingredients and omit similar arguments.

The linearized problem for (P_2) is

$$\begin{cases}\dfrac{\partial^2u}{\partial x^2}+\operatorname{sgn}y\dfrac{\partial^2u}{\partial y^2}=0,\\ u=0, & \text{on } \Gamma_0,\\ u=\phi(x), & \text{on } \Sigma_1,\\ u \text{ and } \nabla u \text{ are continuous}, & \text{on } y=0,\end{cases} \tag{4.1}$$

where Σ_1 is the curve $y=\zeta(x)$. In Ω_+ the treatment for problem (4.1) is the same as that for problem (2.17). In Ω_-, the equation in (4.1) becomes

$$\frac{\partial^2u}{\partial x^2}-\frac{\partial^2u}{\partial y^2}=0. \tag{4.2}$$

The d'Alembert formula gives

$$\phi(x)=\frac{1}{2}(f(x-\zeta(x))+f(x+\zeta(x)))+\frac{1}{2}\int_{x-\zeta(x)}^{x+\zeta(x)}h(\xi)\mathrm{d}\xi, \tag{4.3}$$

where $f(x)$ and $h(x)$ are the initial value of $u(x,y)$ and its derivative on $x=0$. Differentiating (4.3) gives

$$2\phi'(x)=(1-\zeta'(x))f'(x-\zeta(x))-h(x-\zeta(x))+(1+\zeta'(x))f'(x+\zeta(x))+h(x+\zeta(x)). \tag{4.4}$$

Denote by $m(x_1)$ the inverse of $x_1=x-\zeta(x)$. Let $\ell(x_1)=m(x_1)+\zeta(m(x_1))$. From (4.4) we have

$$\phi_1(x_1)=f'(x_1)-h(x_1)+\rho(x_1)(f'(\ell(x_1))+h(\ell(x_1))), \tag{4.5}$$

where $\phi_1(x_1)=\frac{2\phi'(m(x_1))}{1-\zeta'(m(x_1))}$, $\rho(x_1)=\frac{1+\zeta'(m(x_1))}{1-\zeta'(m(x_1))}$. Since $\zeta(0)=0$ and

$$0<\sqrt{1-\delta}+\zeta'(x)<\eta\quad\text{for all } x, \tag{4.6}$$

then

$$
\begin{aligned}
&-\sqrt{1-\delta}\,x < \zeta(x) < (-\sqrt{1-\delta}+\eta)x,\\
&\frac{x_1}{1+\sqrt{1-\delta}-\eta} > m(x_1) > \frac{x_1}{1+\sqrt{1-\delta}},\\
&0 < \rho(x_1) < \frac{1-\sqrt{1-\delta}+\eta}{1+\sqrt{1-\delta}-\eta},\\
&0 < \ell(x_1) < \frac{(1-\sqrt{1-\delta}+\eta)x_1}{1+\sqrt{1-\delta}-\eta}.
\end{aligned}
$$

Writing x_1 by x again, then equality (4.5) becomes

$$
\frac{\partial u}{\partial y}(x,0) - \frac{\partial u}{\partial x}(x,0) = \rho(x)\Big(\frac{\partial u}{\partial y}(\ell(x),0) + \frac{\partial u}{\partial x}(\ell(x),0)\Big) - \phi_1(x). \tag{4.7}
$$

Hence we obtain a boundary value problem

$$
\begin{cases}
\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0,\\
u = 0, \quad \text{on } \Gamma_0,\\
\dfrac{\partial u}{\partial y}(x,0) - \dfrac{\partial u}{\partial x}(x,0) = \rho(x)\Big(\dfrac{\partial u}{\partial y}(\ell(x),0) + \dfrac{\partial u}{\partial x}(\ell(x),0)\Big) - \phi_1(x), \quad \text{on } y=0,
\end{cases} \tag{4.8}
$$

which is essentially equivalent to (4.1). We emphasize here that condition (4.7) is a nonlocal condition, so that the classical existence theorem on elliptic boundary value problems does not work in this case.

In order to solve problem (4.8), we define an operator $\mathcal{L} : u \mapsto U$ as follows:

$$
\begin{cases}
\dfrac{\partial^2 U}{\partial x^2} + \dfrac{\partial^2 U}{\partial y^2} = 0,\\
U = 0, \quad \text{on } \Gamma_0,\\
\dfrac{\partial U}{\partial y}(x,0) - \dfrac{\partial U}{\partial x}(x,0) = \rho(x)\Big(\dfrac{\partial u}{\partial y}(\ell(x),0) + \dfrac{\partial u}{\partial x}(\ell(x),0)\Big) - \phi_1(x), \quad \text{on } y=0.
\end{cases} \tag{4.9}
$$

Lemma 4.1 *The operator $\mathcal{L} : u \mapsto U$ defined by* (4.9) *is an inner and contractive map in $C^{1,\alpha}(\Omega^+)$.*

Proof We notice that $\ell(x) \in C^2$, and $\ell(x)$ satisfies

$$
\ell'(x) < \frac{1-\sqrt{1-\delta}+\eta}{1+\sqrt{1-\delta}-\eta}.
$$

Then for any $f(x) \in C^{1,\alpha}(0,1)$, one has $f(\ell(x)) \in C^{1,\alpha}(0,1)$ and

$$
\|f(\ell(x))\|_{C^{k,\alpha}(0,1)} \le \|f(x)\|_{C^{k,\alpha}(0,1)},
$$

as $k = 0, 1$. Therefore,

$$
\begin{aligned}
\Big\|\rho(x)\Big(\frac{\partial u}{\partial y}(\ell(x),0) + \frac{\partial u}{\partial x}(\ell(x),0)\Big) - \phi_1(x)\Big\|_{C^\alpha(0,1)} &\le C(\|\rho(x)\|_{C^\alpha} \cdot \|u\|_{C^{1,\alpha}(\Omega_+)} + \|\phi_1\|_{C^\alpha})\\
&\le C((\delta+\eta)\|u\|_{C^{1,\alpha}(\Omega_+)} + \|\phi_1\|_{C^\alpha}).
\end{aligned} \tag{4.10}
$$

It turns out that as the solution of the oblique derivative problem (4.8), $U(x,y)$ is well defined in $C^{1,\alpha}(\Omega_+)$.

Furthermore, to prove the contraction of the map $\mathcal{L}$, we assume that U_1 and U_2 are the solutions of (4.8) with u replaced by u_1 and u_2. Then $W=U_1-U_2$ satisfies

$$\begin{cases} \dfrac{\partial^2 W}{\partial x^2}+\dfrac{\partial^2 W}{\partial y^2}=0, \\ W=0, \quad \text{on } \Gamma_0, \\ \dfrac{\partial W}{\partial y}(x,0)-\dfrac{\partial W}{\partial x}(x,0)=\rho(x)\Big(\dfrac{\partial(u_1-u_2)}{\partial y}(\ell(x),0)+\dfrac{\partial(u_1-u_2)}{\partial y}(\ell(x),0)\Big), \quad \text{on } y=0. \end{cases} \tag{4.11}$$

Then we have the estimate

$$\|W(x,y)\|_{C^{1,\alpha}(\Omega_+)}\le C(\delta+\eta)\|(u_1-u_2)(x,y)\|_{C^{1,\alpha}(\Omega_+)}.$$

By taking δ and η sufficiently small, we obtain the contraction of the operator $\mathcal{L}$ in $C^{1,\alpha}$.

Lemma 4.1 indicates that the operator $\mathcal{L}$ has a unique fixed point. It means that problem (4.8) admits a unique $C^{1,\alpha}$ solution. According to the equivalence of problems (4.8) and (4.1), we obtain the solution of the generalized Tricomi problem (4.1). Furthermore, problem (P_2) can also be solved by using implicit function theorem for small δ as indicated in Section 2. Hence Theorem 1.2 is also proved.

Remark 4.1 For more general nonlinear mixed type equation the line, where the equation changes its type, could also be unknown. It should be determined together with the solution. We will study such a case in the future.

Remark 4.2 Condition (1.11) means that the curve, where the data on the hyperbolic region is assigned, is near to the characteristics issuing from $(0,0)$. It is expected to relieve such a restriction.

Remark 4.3 In the problem arisen in the stability of Mach configuration, we may only have the continuity of part of flow parameters on the line where the equation changes its type. Corresponding to such a situation in our model problem, we could also require the unknown function is continuous, while the continuity condition for derivatives will be replaced by a consistence condition like

$$\alpha_+\Big(\frac{\partial u}{\partial y}\Big)_{++}\beta_+\Big(\frac{\partial u}{\partial y}\Big)_+ +\gamma_+u=\alpha_-\Big(\frac{\partial u}{\partial y}\Big)_- +\beta_-\Big(\frac{\partial u}{\partial y}\Big)_- +\gamma_-u, \quad \text{on } y=0. \tag{4.12}$$

It turns out that in this case we can still establish a similar boundary value problem like (4.7) with nonlocal boundary condition. Meanwhile, the above approach is also available to the new boundary value problem.

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