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The Continuity of Barrier Function with Respect to the Parameter**

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Abstract The authors study the continuity of barrier function $B_c(x)$ with respect to the parameter. A sufficient condition which makes $B_c(x)$ be continuous with respect to c is obtained, and an example of discontinuity when the condition is not satisfied is also constructed.

Keywords Lagrangian system, Average action, Barrier function, Minimal measure, Uniquely ergodic
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1 Introduction

In 1991 and 1993 respectively, Mather published two papers [1] and [2], which formed the framework of Mather theory. Associated to the Lagrangian $L - \eta_c$, where η_c is a closed 1-form such that $[\eta_c] = c \in H^1(M, \mathbb{R})$, he mainly considered the following sets:

Mather set $\widetilde{\mathcal{M}}_c$: the union of the supports of invariant minimal measure $\mathfrak{M}_c$,

Aubry set $\widetilde{\mathcal{A}}_c$: the union of the global c-static orbits,

Mañé set $\widetilde{\mathcal{N}}_c$: the union of the global c-semi-static orbits.

We use $\mathcal{M}_c$, $\mathcal{A}_c$, and $\mathcal{N}_c$ to denote the standard projection of $\widetilde{\mathcal{M}}_c$, $\widetilde{\mathcal{A}}_c$, and $\widetilde{\mathcal{N}}_c$ from $TM \times \mathbb{T}$ to $M \times \mathbb{T}$, respectively. Mather proved the following:

(1) $\widetilde{\mathcal{M}}_c \subset \widetilde{\mathcal{A}}_c \subset \widetilde{\mathcal{N}}_c$.

(2) Aubry set has the Lipschitz graph property. The mapping $\pi : \widetilde{\mathcal{A}}_c \to M \times \mathbb{T}$ is injective. Its inverse is Lipschitz, i.e., there exists a constant C, such that for each $x, y \in \widetilde{\mathcal{A}}_c$, we have

$$\mathrm{dist}(\pi^{-1}(x), \pi^{-1}(y)) \le C \mathrm{dist}(x, y).$$

(3) The mapping $c \mapsto \widetilde{\mathcal{N}}_c$ is upper semi-continuous.

Generally speaking, the action variables of the orbits in the different Mañé set are different. Then we may find the orbits whose action variables change sufficiently large by finding the orbits which connect different Mañé sets associated to different c. In general case, we do not know whether the connecting orbit exists, while Mather thought that if there is a c-equivalent curve which connects cohomology classes c_1 and c_2, then there exists an orbit which connects $\widetilde{\mathcal{N}}_{c_1}$ and $\widetilde{\mathcal{N}}_{c_2}$ (see [2]).

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It is quite important to find c-equivalent property in the proof of the existence of the connecting orbits. In addition, the c-equivalent property is determined by the topological structure of Mañé set. The continuity of barrier function with respect to c plays a very important role in studying the topological property of Mañé sets. In this paper, we present a sufficient condition which guarantees the continuity of the barrier function with respect to c. On the other hand, we also give an example whose barrier function is discontinuous with respect to c when it does not satisfy the given condition.

2 The Settings and Preliminary Results

Let M be a C^∞ compact manifold and TM denote the tangent bundle of M. $L : TM \times \mathbb{R} \to \mathbb{R}$ is 1-period C^r $(r \geq 2)$ Lagrangian and satisfies the following hypotheses:

(1) **Positive Definiteness** For each $m \in M$, $\theta \in \mathbb{T}$, the restriction of L to $TM_m \times \theta$ is strictly convex in the sense that its Hessian matrix is everywhere positive definite.

(2) **Superlinear Growth** Let $\|\cdot\|$ denote the norm associated to a Riemannian metric on M. Then

$$\frac{L(v,\theta)}{\|v\|} \to +\infty, \quad \text{as } \|v\| \to +\infty,$$

where v ranges over TM and $\theta \in \mathbb{T}$. This equals that, for every C_1, there exists C_2, such that $L(v,\theta) \geq C_1\|v\| - C_2$.

In other words, for every $C_1 > 0$, there exists $C_2 > 0$, such that $\|v\| \geq C_2$ implies $L(v,\theta) \geq C_1\|v\|$.

Since M is compact, this condition is independent of the choice of the Riemannian metric.

(3) **Completeness of the Euler-Lagrange Flow** Every maximal trajectory of Euler-Lagrange vector field E_L corresponding to L is defined for all time $\mathbb{R}$. For every E_L-invariant probability measure μ on $TM \times \mathbb{T}$, the average action is defined by

$$A(\mu) = \int L\mathrm{d}\mu,$$

and the rotation vector $\rho(\mu) \in H_1(M,\mathbb{R})$ is defined by the following equation:

$$\langle c, \rho(\mu)\rangle = \int \lambda_c \mathrm{d}\mu, \quad \forall\, c \in H^1(M,\mathbb{R}),\ [\lambda_c] = c,$$

where the bracket side is the canonical pairing of $H^1(M,\mathbb{R})$ and $H_1(M,\mathbb{R})$.

Define

$$A_c(\mu) = A(\mu) - \langle c, \rho(\mu)\rangle,$$
$$h_c(m,m') = \min \int_0^1 (L - \eta_c)(\mathrm{d}\gamma(t), t)\mathrm{d}t - \alpha(c),$$

where γ ranges over the set of absolutely continuous curves $\gamma : [0,1] \to M$, such that

$$\gamma(0) = m, \quad \gamma(1) = m',$$

and η_c is a smooth closed 1-form whose de Rham cohomology is $c \in H^1(M,\mathbb{R})$.

Let

$$-\alpha(c) = \min\{A_c(\mu) \mid \mu \in \mathfrak{M},\ \mathfrak{M} \text{ is the collection of } \phi_L\text{-invariant probability measures}\},$$

$$h_c^n(\xi,\eta) = \min\Big\{\sum_{i=0}^{n-1} h_c(m_i, m_{i+1}) : m_0 = \xi,\ m_n = \eta,\ \text{and } m_i \in M \text{ for } 0 \le i \le n\Big\},$$

$$h_c^\infty(\xi,\eta) = \liminf_{n\to\infty} h_c^n(\xi,\eta), \quad \forall\, \xi,\eta \in M.$$

It is easy to see that

$$h_c(\xi,\eta) \le h_c(\xi,m) + h_c(m,\eta).$$

$B_c(\xi) \triangleq h_c^\infty(\xi,\xi)$ is called the barrier function of the Lagrange system. It has the following properties:

Barrier function $B_c(m)$ is a non-negative Lipschitz function on M and vanishes identically on $\pi(\widetilde{\mathcal{M}}_c) \cap (M \times 0)$, where $\widetilde{\mathcal{M}}_c = \operatorname{supp} \mathfrak{M}_c$ (see [2–5]).

3 Main Results and Their Proofs

Theorem 3.1 *If the minimal probability measures $\mathfrak{M}_{c_0}$ is uniquely ergodic, then for each $m \in M$, $m' \in M$, the functions $h_c^\infty(m,m')$ and $B_c(m)$ are continuous at c_0 with respect to the parameter c.*

To prove this theorem, we need some lemmas.

Lemma 3.1 *Let $(\cdots, m_i, \cdots)$, $m_i \in M$ be a c-minimal configuration, and ω_1, ω_2 be two ω-limit points of the configuration. Then*

$$d_c(\omega_1,\omega_2) \triangleq h_c^\infty(\omega_1,\omega_2) + h_c^\infty(\omega_2,\omega_1) = 0.$$

For details of the proof of the lemma, please refer to [2].

Corollary 3.1 *For any two points x, y in Mather set $\mathcal{M}_c$, we have $d_c(x,y) = 0$, when c-minimal probability measure set $\mathfrak{M}_c$ is uniquely ergodic.*

This is a direct consequence of Lemma 3.1.

Lemma 3.2 (A priori Compactness) (see [4, 5]) *Consider a compact set $Q \subset H^1(M,\mathbb{R})$. For any given $c \in Q$ and every c-minimal curve $\gamma : [a,b] \to M$ of $L + \eta_c$, there exists a constant K such that, if $b \ge a+1$, then $\|\mathrm{d}\gamma(t)\| \le K$, $\forall\, t \in [a,b]$.*

Lemma 3.3 *If Q is a compact subset of $H^1(M,\mathbb{R})$, then for any $c \in Q$, $h_c(m,\xi)$ is a Lipschitz continuous function with a uniform Lipschitz constant.*

For details of the proof of the lemma, please refer to [5].

Similarly, it is easy to show that $h_c^n(\xi,\eta)$ and $h_c^\infty(\xi,\eta)$ are both Lipschitz continuous, and they have the same Lipschitz constant as $h_c(\xi,\eta)$ (see [2]).

Proof of Theorem 3.1 Choose η_c, which is a smooth 1-form on TM, such that $[\eta_c] = c \in H^1(M,R)$ and it is continuous with respect to c. Let $c_n \to c_0$, $\eta_{c_n} \to \eta_{c_0}$, and $\gamma_n : [0,\infty) \to M$ be a c_n-minimal curve, such that $\gamma_n(0) = m \in M$. $\omega(\gamma_n)$ stands for the ω-limit set of γ_n. Then

$$d_H(\omega(\gamma_n), \mathcal{M}_{c_0}) = \inf\{\operatorname{dist}(x,y) : x \in \omega(\gamma_n), y \in \mathcal{M}_{c_0}\} \to 0, \quad \text{as } n \to \infty,$$

where $\operatorname{dist}(x,y)$ denotes the distance between x and y induced by Riemann metric. Otherwise, there exists a subsequence $n_k \to \infty$ and a constant $\delta > 0$, such that

$$d_H(\omega(\gamma_{n_k}), \mathcal{M}_{c_0}) > \delta, \quad \forall\, n.$$

Suppose that μ_{n_k} is the limit measure of $\gamma_{n_k}(t)$. Then $\operatorname{supp}\mu_{n_k} \subset \omega(\gamma_{n_k})$. Let μ_0 be an accumulate point of μ_{n_k}, and further suppose that $\lim\limits_{k\to\infty} \mu_{n_k} = \mu_0$, in the sense of the vague topology. It follows that μ_0 is a c_0-minimal measure and

$$d_H(\operatorname{supp}\mu_0, \mathcal{M}_{c_0}) \geq \delta > 0.$$

We then obtain a contradiction with the assumption that $\mathfrak{M}_{c_0}$ is uniquely ergodic. Hence,

$$d_H(\omega_{\gamma_n}, \mathcal{M}_{c_0}) \to 0, \quad \text{as } n \to \infty.$$

By Lemma 3.2, it follows that $\dot{\gamma}_n(0)$ is contained in a compact set. Suppose that υ is a point of accumulation of $\dot{\gamma}_n(0)$. Obviously, the solution curve $\gamma_\upsilon(t)$ which has the initial velocity of υ is a unidirectional c_0-minimal curve. Hence, we can choose a subsequence $\{c_{n_k}\}_{k=1}^{\infty}$ ($c_{n_k} \to c_0$ as $k \to \infty$) of $\{c_n\}_{n=1}^{\infty}$ and a unidirectional c_{n_k}-minimal curve $\gamma_{n_k}(t)$ such that $\dot{\gamma}_{n_k}(0) \to \upsilon$.

Let ξ, ξ_{n_k} be points of accumulation of $\gamma_\upsilon(t)$ and $\gamma_{n_k}(t)$, respectively, that is to say,

$$\forall\, \delta > 0,\ \exists\, T > 0,\ \text{s.t. } \operatorname{dist}(\gamma_\upsilon(T), \xi) < \delta.$$

By the uniquely ergodic property of $\mathfrak{M}_c$, we have

$$\forall\, \eta > 0,\ \exists\, K,\ \text{s.t. } \operatorname{dist}(\xi_{n_k}, \xi) < \delta, \quad \forall\, k > K.$$

By the continuous dependence of solution on initial value, when $k > K$, we have

$$\begin{aligned}
&\operatorname{dist}(\gamma_{n_k}(t), \gamma_\upsilon(t)) < \eta, \quad \forall\, t \in [0,T],\\
&\operatorname{dist}(\dot{\gamma}_{n_k}(t), \dot{\gamma}_\upsilon(t)) < \eta, \quad \forall\, t \in [0,T].
\end{aligned}$$

Since α is continuous, when $k > K$, we also get

$$|\alpha(c_{n_k}) - \alpha(c_0)| < \eta.$$

As 1-form η_c is chosen to be continuous with respect to c, it follows that

$$|\eta_{c_{n_k}} - \eta_{c_0}| < \eta.$$

From Lemma 3.3, we obtain

$$|h^{\infty}_{c_{n_k}}(m, \xi) - h^{\infty}_{c_{n_k}}(m, \xi_{n_k})| \leq L\|\xi - \xi_{n_k}\| \leq L \cdot \delta. \tag{3.1}$$

In addition,

$$h^{\infty}_{c_{n_k}}(m, \xi_{n_k}) = h^{T}_{c_{n_k}}(m, \gamma_{n_k}(T)) + h^{\infty}_{c_{n_k}}(\gamma_{n_k}(T), \xi_{n_k}).$$

As ξ_{n_k} and ξ are ω-limit points of $\gamma_{n_k}(t)$ and $\gamma_\upsilon(t)$ respectively, by Lemma 3.1, we have

$$h^{\infty}_{c_{n_k}}(\xi_{n_k}, \xi_{n_k}) = \frac{1}{2} d_{c_{n_k}}(\xi_{n_k}, \xi_{n_k}) = 0$$

and

$$h_{c_0}^{\infty}(\xi,\xi)=\frac{1}{2}d_{c_0}(\xi,\xi)=0.$$

Thus, we obtain

$$|h_{c_{n_k}}^{\infty}(\gamma_{n_k}(T),\xi_{n_k})|=|h_{c_{n_k}}^{\infty}(\gamma_{n_k}(T),\xi_{n_k})-h_{c_{n_k}}^{\infty}(\xi_{n_k},\xi_{n_k})|\le L\cdot\|\gamma_{n_k}(T)-\xi_{n_k}\|\le L\cdot\delta, \tag{3.2}$$

$$|h_{c_0}^{\infty}(\gamma_{\upsilon}(T),\xi)|=|h_{c_0}^{\infty}(\gamma_{\upsilon}(T),\xi)-h_{c_0}^{\infty}(\xi,\xi)|\le L\cdot\|\gamma_{\upsilon}(T)-\xi\|\le L\cdot\delta. \tag{3.3}$$

On the other side, we get

$$\begin{aligned}
&|h_{c_{n_k}}^{T}(x,\gamma_{n_k}(T))-h_{c_0}^{T}(x,\gamma_{\upsilon}(T))|\\
=&\Big|\int_0^T(L-\eta_{c_{n_k}}+\alpha(c_{n_k}))(\mathrm{d}\gamma_{c_{n_k}}(t))\mathrm{d}t-\int_0^T(L-\eta_{c_0}+\alpha(c))(\mathrm{d}\gamma_{\upsilon}(t),t)\mathrm{d}t\Big|\\
\le&\int_0^T|L(\mathrm{d}\gamma_{\upsilon}(t),t)-L(\mathrm{d}\gamma_{n_k}(t),t)|\mathrm{d}t+\int_0^T|\eta_{c_0}(\mathrm{d}\gamma_{\upsilon}(t),t)-\eta_{c_{n_k}}(\mathrm{d}\gamma_{n_k}(t),t)|\mathrm{d}t\\
&+|\alpha(c_0)-\alpha(c_{n_k})|\cdot T\\
\le&\int_0^T|L(\mathrm{d}\gamma_{\upsilon}(t),t)-L(\mathrm{d}\gamma_{n_k}(t),t)|\mathrm{d}t+\int_0^T|\eta_{c_0}(\mathrm{d}\gamma_{\upsilon}(t),t)-\eta_{c_0}(\mathrm{d}\gamma_{n_k}(t),t)|\mathrm{d}t\\
&+\int_0^T|\eta_{c_{n_k}}(\mathrm{d}\gamma_{n_k}(t),t)-\eta_{c_0}(\mathrm{d}\gamma_{n_k}(t),t)|\mathrm{d}t+|\alpha(c_{n_k})-\alpha(c_0)|\cdot T\\
\le&\, AT\eta,
\end{aligned}\tag{3.4}$$

where $A=\{\sup(\dot{L}(x,\upsilon))+\sup(\dot{\eta}_c)(x,\upsilon):x\in M,\|\upsilon\|<K'\}+2$, and K' is some real constant. The last inequality above is ensured by Lemma 3.2, and $\eta\to 0$ as $k\to\infty$.

For each $\epsilon>0$, let δ and η be small enough, such that $AT\delta<\frac{\epsilon}{4}$ and $L(\delta+\eta)<\frac{\epsilon}{4}$. Combining (3.1)–(3.4) together, we have

$$\begin{aligned}
|h_{c_{n_k}}^{\infty}(m,\xi)-h_{c_0}^{\infty}(m,\xi)|&=|h_{c_{n_k}}^{\infty}(m,\xi_{n_k})-h_{c_0}^{\infty}(m,\xi)+h_{c_{n_k}}^{\infty}(m,\xi)-h_{c_{n_k}}^{\infty}(m,\xi_{n_k})|\\
&\le|h_{c_{n_k}}^{\infty}(m,\xi_{n_k})-h_{c_0}^{\infty}(m,\xi)|+|h_{c_{n_k}}^{\infty}(m,\xi)-h_{c_{n_k}}^{\infty}(m,\xi_{n_k})|\\
&\le|h_{c_{n_k}}^{T}(m,\gamma_{n_k}(T))-h_{c_0}^{T}(m,\gamma_{\upsilon}(T))|+|h_{c_{n_k}}^{\infty}(\gamma_{n_k}(T),\xi_{n_k})|\\
&\quad+|h_{c_0}^{\infty}(\gamma_{\upsilon}(T),\xi)|+\frac{\epsilon}{4}\\
&\le\frac{\epsilon}{4}+\frac{\epsilon}{4}+\frac{\epsilon}{4}+\frac{\epsilon}{4}\\
&=\epsilon.
\end{aligned}$$

Then we conclude that

$$|h_{c_n}^{\infty}(m,\xi)-h_{c_0}^{\infty}(m,\xi)|\to 0,\quad \text{as } c_n\to c_0.$$

This completes the proof of the theorem.

Corollary 3.2 *If the c-minimal probability measure set $\mathfrak{M}_{c_0}$ is uniquely ergodic, then the barrier function $B_c(m)$ is continuous at c_0 with respect to c.*

This is a direct consequence of Theorem 3.1.

Theorem 3.2 *For any given two points $m_1, m_2 \in \mathcal{M}_{c_0}$, it involves that $d_{c_0}(m_1, m_2) = 0$. Then the barrier function $B_c(m)$ is continuous at point c_0 with respect to c, without the assumption that $\mathfrak{M}_c$ is uniquely ergodic.*

The proof of this theorem is just the same as the proof of Theorem 3.1.

4 A Counterexample when Minimal Measures are not Uniquely Ergodic

In this section, we present an example whose barrier function is discontinuous with respect to c.

Consider the Lagrange system on $\mathbb{T} \times \mathbb{R}$:

$$L(q_1, q_2; \dot{q}_1, \dot{q}_2, t) = \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2) - \epsilon\lambda(q_1)\Big(\dot{q}_2 - 1 - \frac{\epsilon}{2}\Big) - \cos(2q_1),$$

where $\lambda(\,\cdot\,)$ is a smooth function satisfying the following condition:

$$\lambda|_{O_1} = 1, \quad \lambda|_{O_2} = 0,$$

in which O_1 and O_2 are two open neighborhoods of 0 and π, respectively, and $O_1 \cap O_2 = \emptyset$. If $\epsilon = 0$, the system

$$L_0 = \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2) - \cos(2q_1)$$

is an integrable system. For every $c = (0, c_2) \in H^1(M, \mathbb{R})$, where $\frac{1}{2} \leq c_2 \leq 2$, the support sets of the c-minimal invariant probability measures $\mathfrak{M}_c$ are included in the following two invariant hyperbolic cylinders:

$$\begin{aligned}
\Gamma^1 &: \{(q_1, q_2; \dot{q}_1, \dot{q}_2, t) \mid q_1 = 0 \ (\mathrm{mod}\, 2\pi),\ \dot{q}_1 = 0,\ \dot{q}_2 = c_2\},\\
\Gamma^2 &: \{(q_1, q_2; \dot{q}_1, \dot{q}_2, t) \mid q_1 = \pi \ (\mathrm{mod}\, 2\pi),\ \dot{q}_1 = 0,\ \dot{q}_2 = c_2\}.
\end{aligned}$$

By the structural stability of hyperbolic invariant manifold, there exists a $\delta_1 > 0$ such that if $|\epsilon| < \delta_1$, then the Lagrange system L has two invariant cylinders

$$\begin{aligned}
\Gamma^1_\epsilon &: \{(q_1, q_2; \dot{q}_1, \dot{q}_2, t) \mid q_1 = 0 \ (\mathrm{mod}\, 2\pi),\ \dot{q}_1 = 0,\ \dot{q}_2 = c_2 + \epsilon\},\\
\Gamma^2_\epsilon &: \{(q_1, q_2; \dot{q}_1, \dot{q}_2, t) \mid q_1 = \pi \ (\mathrm{mod}\, 2\pi),\ \dot{q}_1 = 0,\ \dot{q}_2 = c_2\}.
\end{aligned}$$

Lemma 4.1 *There exists a $\delta > 0$ such that, when $|\varepsilon| < \delta$, we have*

$$\widetilde{\mathcal{M}}_c \subset \Gamma^1_\varepsilon \cup \Gamma^2_\varepsilon$$

for every $c = (0, c_2) \in H^1(M, \mathbb{R})$, where $\frac{1}{2} \leq c_2 \leq 2$.

Proof Since Γ^1 and Γ^2 are two hyperbolic invariant cylinders, there exist two neighborhoods $\overline{\Omega}_1$ and $\overline{\Omega}_2$ of Γ^1 and Γ^2, respectively. The flow $\phi^t_{L_0}$ does not have invariant sets on $\overline{\Omega}_1 \setminus \Gamma^1$ and $\overline{\Omega}_2 \setminus \Gamma^2$. For the same reason, the flow ϕ^t_L does not have any invariant sets on $\overline{\Omega}_1 \setminus \Gamma^1_\varepsilon$ and $\overline{\Omega}_2 \setminus \Gamma^2_\varepsilon$, when δ is small enough.

Let $\Omega_1 = \pi\overline{\Omega}_1$ and $\Omega_2 = \pi\overline{\Omega}_2$, where $\pi : T\mathbb{T}^2 \times \mathbb{R} \to \mathbb{T}^2$ is the projection map. Since L_0 is an integrable system for any $x \in \mathbb{T}^2 \setminus (\Omega_1 \cup \Omega_2)$ and $c = (0, c_2) \in H^1(M, \mathbb{R})$ ($\frac{1}{2} \leq c_2 \leq 2$), we then deduce that

$$\lim_{\delta \to 0} B^c_L(x) \to B^c_{L_0}(x) > 0.$$

Consequently, we have $\mathcal{A}_c \cap (\mathbb{T}^2 \setminus (\Omega_1 \cup \Omega_2)) = \emptyset$. Since $\mathcal{M}_c \subset \mathcal{A}_c$, we also get $\mathcal{M}_c \cap (\mathbb{T}^2 \setminus (\Omega_1 \cup \Omega_2)) = \emptyset$ and $\widetilde{\mathcal{M}}_c \subset (\overline{\Omega}_1 \cup \overline{\Omega}_2)$. Moreover, there are no other invariant sets except Γ_ε^1 and Γ_ε^2 in $\overline{\Omega}_1 \cup \overline{\Omega}_2$. So $\widetilde{\mathcal{M}}_c \subset (\Gamma_\varepsilon^1 \cup \Gamma_\varepsilon^2)$, namely, the support sets of the c-minimal invariant measure $\mathfrak{M}_c$ of the Lagrange system L are included in the invariant cylinder $\Gamma_\varepsilon^1 \cup \Gamma_\varepsilon^2$. Thus we complete the proof.

Suppose that the subsystems are L_1 and L_2 respectively, when the Lagrange system L is restricted on the invariant subsets Γ_ε^1 and Γ_ε^2. Let

$$-\alpha^1(c_2) = \inf\Big\{\int (L_1 - \eta_c)\mathrm{d}\mu,\ \mu \in \mathfrak{M},\ \operatorname{supp}\mu \subset \Gamma_\varepsilon^1,\ [\eta_c] = (0, c_2)\Big\},$$
$$-\alpha^2(c_2) = \inf\Big\{\int (L_1 - \eta_c)\mathrm{d}\mu,\ \mu \in \mathfrak{M},\ \operatorname{supp}\mu \subset \Gamma_\varepsilon^2,\ [\eta_c] = (0, c_2)\Big\}.$$

By Lemma 4.1, we have $\mathcal{M}_c \subset \Gamma_\varepsilon^1 \cup \Gamma_\varepsilon^2$. Let $\widetilde{\mathcal{M}}_c = \widetilde{\mathcal{M}}_c^1 \cup \widetilde{\mathcal{M}}_c^2$, where $\widetilde{\mathcal{M}}_c^1 \subset \Gamma_\varepsilon^1$, $\widetilde{\mathcal{M}}_c^2 \subset \Gamma_\varepsilon^2$, and $\widetilde{\mathcal{M}}_c^i$ $(i = 1, 2)$ can be empty set. Then

$$\alpha(c) = \max\{\alpha^1(c_2), \alpha^2(c_2)\}.$$

However, it is easy to see that

$$\alpha^1(c_2) = \frac{1}{2}c_2^2 + \varepsilon(c_2 - 1) + 1,$$
$$\alpha^2(c_2) = \frac{1}{2}c_2^2 + 1.$$

Obviously, there hold

$$\alpha^1(c_2)\begin{cases} < \alpha^2(c_2), & \dfrac{1}{2} \le c_2 < 1, \\ = \alpha^2(c_2), & c_2 = 1, \\ > \alpha^2(c_2), & 1 < c_2 \le 2. \end{cases}$$

The above relationships have three probabilities. For $c = (0, 1)$, Mather set $\widetilde{\mathcal{M}}_c$ has two ergodic components $\widetilde{\mathcal{M}}_c^1 \subset \Gamma_\varepsilon^1$ and $\widetilde{\mathcal{M}}_c^2 \subset \Gamma_\varepsilon^2$. In this situation, the system is not uniquely ergodic.

For $c = (0, c_2)$, where $\frac{1}{2} \le c_2 < 1$, we have $\widetilde{\mathcal{M}}_c^1 = \emptyset$. So Mather set $\widetilde{M}_c$ has only one ergodic component $\widetilde{\mathcal{M}}_c^2 \subset \Gamma_\varepsilon^2$. Namely, it is uniquely ergodic.

For $c = (0, c_2)$, where $1 < c_2 \le 2$, we have $\widetilde{\mathcal{M}}_c^2 = \emptyset$. So Mather set $\widetilde{\mathcal{M}}_c$ has one ergodic component $\widetilde{\mathcal{M}}_c^1 \subset \Gamma_\varepsilon^1$, and it is uniquely ergodic, too.

We claim that, $\forall\, x \in \mathcal{M}_c^1 = \pi\widetilde{\mathcal{M}}_c^1$, the barrier function $B_c(x)$ is discontinuous at the point $c = (0, 1)$ with respect to c.

We are going to show the claim in the following two parts.

Part 1 In this part, we verify that

$$d_{(0,1)}(x, y) = h_{(0,1)}^\infty(x, y) + h_{(0,1)}^\infty(y, x) > 0, \quad \forall\, x \in \mathcal{M}_c^1,\ y \in \mathcal{M}_c^2,\ c = (0, 1).$$

Let Ω_1 and Ω_2 be neighborhoods of $\pi\Gamma_\varepsilon^1 \subset \mathbb{T}^2$ and $\pi\Gamma_\varepsilon^2 \subset \mathbb{T}^2$, respectively, and $d(\Omega_1, \Omega_2) = a > 0$. By Lemma 3.2, for every minimal curve $\gamma(t)$ which connects points x and y, its velocity $\dot\gamma(t)$ has a uniform upper bound K. Consequently, there exists at least time $T = \frac{a}{K}$ that makes $\gamma(t)$ stay in $\mathbb{T}^2 \setminus (\Omega_1 \cup \Omega_2)$. In addition,

$$L|_{\mathbb{T}^2 \setminus (\Omega_1 \cup \Omega_2) \times [-K, K] \times \mathbb{R}} > -\Big(\frac{1}{2}c^2 + 1\Big)$$

and $\mathbb{T}^2\backslash(\Omega_1\cup\Omega_2)$ is compact. Therefore, there exists a $\delta>0$, such that $(L+\alpha(c))|_{\mathbb{T}^2\backslash(\Omega_1\cup\Omega_2)}>\delta$, and

$$h_c^n(x,\,y)=\int_0^n (L+\alpha(c))(\gamma(t),\dot\gamma(t),t)\mathrm{d}t\geq\delta\cdot T>0,\quad \forall\, n.$$

Consequently,

$$h_c^\infty(x,y)=\liminf_{n\to\infty} h_c^n(x,y)\geq\delta\cdot T>0.$$

For the same reason, we also have $h_c^\infty(y,x)>0$. So we obtain

$$d_{(0,1)}(x,y)=h^\infty_{(0,1)}(x,y)+h^\infty_{(0,1)}(y,x)>2\delta\cdot T>0.$$

Part 2 For each $x\in\mathcal{M}^1_{(0,1)}$, we want to prove that $B_c(x)\to d_{(0,1)}(x,y)$ as $c\to(0,1-)$.

As $\mathcal{M}^2_{(0,1)}$ is an ergodic component of Lagrange system L, we have $d_{(0,1)}(y_1,\,y_2)=0$ for any $y_1,y_2\in\mathcal{M}^2_{(0,1)}$. Moreover, when $\frac12\leq c_2<1$, the c-minimal invariant measure set $\mathfrak{M}_c$ is uniquely ergodic with $\mathcal{M}_c=\mathcal{M}_c^2\subset\pi\Gamma^2_\varepsilon\subset\Omega_2$.

When $c\to1$, for each c-minimal curve γ, its ω-limit set is close enough to $\mathcal{M}_c^2$. Using the same method as the proof of Theorem 3.1, we see that

$$B_c(x)\to d_{(0,\,1)}(x,y),\quad c\to(0,1-).$$

On the other hand, one can get $x\in\mathcal{M}^1_{(0,1)}\subset\mathcal{M}_{(0,1)}$. Consequently, $B_{(0,1)}(x)=0$.

To sum up, we have

$$0<\lim_{c\to(0,1-)} B_c(x)=d_{(0,1)}(x,y)\neq B_{(0,1)}(x)=0.$$

Namely, we have proved that the barrier function $B_c(x)$ is discontinuous at $c_0=(0,1)$ with respect to c.

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