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On the Cauchy Problem Describing an Electron-Phonon Interaction*

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Abstract In this paper, a model is derived to describe a quartic anharmonic interatomic interaction with an external potential involving a pair electron-phonon. The authors study the corresponding Cauchy Problem in the semilinear and quasilinear cases.

Keywords Schrödinger-like equations, Cauchy problem, Blow-up, Phonon-electron interaction
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1 Introduction

Electron-phonon interactions play a crucial role in the determination of the physical properties of many mixed cristals (see [16]).

In the present paper, we study the well-posedness of a nonlinear dispersive system arising in the framework of the electron-phonon interaction in a one-dimensional lattice. In [8], V. Konotop treated the temporal dynamics of such a system in the presence of resonant interactions between the electron and phonon subsystems. The Hamiltonian H for such a one-dimensional chain of particles is given by $H = H_{\text{el}} + H_{\text{ph}} + H_{\text{el-ph}}$, where, denoting by a dot the time derivative, the Hamiltonians for each subsystem and their interaction read in braket notation

$$\begin{aligned}
H_{\text{el}} &= -J\sum_n (|n >< n+1| + |n >< n-1|),\\
H_{\text{ph}} &= \frac{M}{2}\sum_n \dot\rho_n^2 + \frac{U}{2}\sum_n (\rho_{n+1}-\rho_n)^2,\\
H_{\text{el-ph}} &= \chi\sum_n |n >< n|(\rho_{n+1}-\rho_{n-1}).
\end{aligned}$$

Here, ρ_n denotes the distance to the equilibrium position of the nth atom of mass M, J is the energetical constant determined by the overlapping of the electronic orbitals, U is a force constant and χ represents the strenght of the electron-phonon interaction.

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In the continuum limit, the above Hamiltonians become

$$H_{\text{el}} = -J\int |u_x|^2 \mathrm{d}x, \quad H_{\text{ph}} = \frac{M}{2}\int \rho_t^2 \mathrm{d}x + \frac{U}{2}\int \rho_x^2 \mathrm{d}x, \quad H_{\text{el-ph}} = \chi \int |u|^2 \rho_x \mathrm{d}x,$$

where u is the electronic wave-function.

Putting $q = \rho$, $p = M\rho_t$, we obtain the Hamilton evolution set of equations

$$\begin{cases} \dot{q}_{\text{ph}} = \dfrac{\partial (H_{\text{ph}} + H_{\text{el-ph}})}{\partial p_{\text{ph}}}, \\ \dot{p}_{\text{ph}} = -\dfrac{\partial (H_{\text{ph}} + H_{\text{el-ph}})}{\partial q_{\text{ph}}}, \\ \mathrm{i}\hbar u_t = \dfrac{\partial (H_{\text{el}} + H_{\text{el-ph}})}{\partial u}. \end{cases} \tag{1.1}$$

In the present paper, we will treat the Cauchy problem associated with this evolution system. We will replace the Hamiltonian of the electronic and phonon subsystems respectively by

$$H_{\text{el}} = -J\int |u_x|^2 \mathrm{d}x + \frac{\alpha}{4}\int |u|^4 \mathrm{d}x, \quad \alpha \in \mathbb{R} \tag{1.2}$$

and

$$H_{\text{ph}} = \frac{M}{2}\int \rho_t^2 \mathrm{d}x + \frac{U}{2}\int \rho_x^2 \mathrm{d}x - \frac{\beta}{4}\int \rho^4 \mathrm{d}x, \quad \beta \in \mathbb{R}, \tag{1.3}$$

allowing the possibility of nonlinear cubic potentials for the evolution of u and ρ. Also, we will incorporate in $H_{\text{el-ph}}$ a term to account for the anharmonic interatomic interactions (see [1]):

$$H_{\text{el-ph}} = \chi \int |u|^2 \rho_x \mathrm{d}x + \lambda \int (\rho_x)^4 \mathrm{d}x, \quad \lambda \geq 0. \tag{1.4}$$

By replacing (1.2)–(1.4) in (1.1), we obtain the system

$$\begin{cases} \mathrm{i}\hbar u_t + J u_{xx} = 2\chi u \rho_x + \alpha |u|^2 u, \quad x \in \mathbb{R},\ t \geq 0, \\ M\rho_{tt} - [U\rho_x + \lambda \rho_x^3]_x = \chi(|u|^2)_x + \beta \rho^3. \end{cases} \tag{1.5}$$

Finally, after putting all physical constants equal to the unity and scaling out the remaining coefficient of the term $u\rho_x$ by the transformation $\widetilde{\rho} = 2\rho$ and $\widetilde{u} = \sqrt{2}\,u$, we obtain the initial value problem

$$\begin{cases} \mathrm{i}u_t + u_{xx} = u\rho_x + \alpha |u|^2 u, \quad x \in \mathbb{R},\ t \geq 0, \\ \rho_{tt} - [\rho_x + \lambda \rho_x^3]_x = (|u|^2)_x + \beta \rho^3, \\ u(0,x) = u_0(x), \quad \rho(0,x) = \rho_0(x), \quad \rho_t(0,x) = \rho_1(x). \end{cases} \tag{1.6}$$

For $\alpha = \beta = \lambda = 0$, by putting $n = \rho_x$, we obtain the classical Zakharov system

$$\begin{cases} \mathrm{i}u_t + u_{xx} = un, \\ n_{tt} - n_{xx} = (|u|^2)_{xx}. \end{cases} \tag{1.7}$$

The initial value problem for (1.7) is studied in [6, 12]. Also, in the case where $\beta = \lambda = 0$, $\alpha \neq 0$, (1.6) falls in the scope of the Zakharov-Rubenchik equation studied in [10, 9] for the global well-posedness and stability of solitary waves and in [11] for the adiabatic limit to the cubic nonlinear Schrödinger equation.

The rest of this paper is organized as follows.

In Section 2, we treat the local well-posedness of (1.6). The main difficulty of this system is the presence of the strongly nonlinear term with derivative-loss $\rho_x^2\rho_{xx}$. In order to overcome this problem, we translate (1.6) in terms of its Riemann invariants. Next, we perform a change of functions technique developed in [15, 10, 5] which takes care of the derivative-loss and use a variant of a result derived by Kato [7] to prove the existence and uniqueness of strong local solutions to (1.6) for initial data $(u_0, \rho_0, \rho_{t0}) \in H^3(\mathbb{R}) \times H^3(\mathbb{R}) \times H^2(\mathbb{R})$.

In Section 3, we derive some conservation laws for (1.6) and prove the existence of solutions which blow-up in L^2 in finite time (provided that $\beta > 0$) by adapting a result due to Reed and Simon [13]. Also, for $\beta \le 0$ and $\lambda = 0$, we prove that the solutions obtained in the previous section are in fact global in time.

Finally, if $\lambda > 0$ and $\beta \le 0$, we establish in Section 4 the global existence of weak solutions for (1.6) by applying a compensated-compactness method developed in [14] by Serre and Shearer (see also [2]). The adaptation of this method to a Schrödinger-nonlinear elasticity system was made in [4]. The technique of using this compensated-compactness result in order to prove the existence of global weak solutions was introduced in [3] in the framework of a Schrödinger-conservation law system.

2 Local Existence of Strong Solutions

In this section, we address the local well-posedness of the initial value problem (1.6).

Let $u_0 \in H^3(\mathbb{R})$, $\rho_0 \in H^3(\mathbb{R})$ and $\rho_1 \in H^2(\mathbb{R})$.

By setting $v = \rho_x$, $w = \rho_t$ and $\sigma(v) = v + \lambda v^3$, the Cauchy problem (1.6) is equivalent to

$$\begin{cases} \mathrm{i}u_t + u_{xx} = uv + \alpha|u|^2 u, \\ \rho_t = w, \\ v_t - w_x = 0, \\ w_t - (\sigma(v))_x = (|u|^2)_x + \beta\rho^3 \end{cases} \tag{2.1}$$

with initial data

$$\begin{aligned} &u(\,\cdot\,,0) = u_0 \in H^3(\mathbb{R}), & &\rho(\,\cdot\,,0) = \rho_0 \in H^3(\mathbb{R}), \\ &v(\,\cdot\,,0) = v_0 := \rho_{0x} \in H^2(\mathbb{R}), & &w(\,\cdot\,,0) = w_0 := \rho_1 \in H^2(\mathbb{R}). \end{aligned} \tag{2.2}$$

Let $\lambda \ge 0$. By introducing the Riemann invariants

$$l = w + \int_0^v \sqrt{1+3\lambda\xi^2}\,\mathrm{d}\xi \quad \text{and} \quad r = w - \int_0^v \sqrt{1+3\lambda\xi^2}\,\mathrm{d}\xi,$$

we derive

$$l - r = 2\int_0^v \sqrt{1+3\lambda\xi^2}\,\mathrm{d}\xi = v\sqrt{1+3\lambda v^2} + \frac{1}{\sqrt{3\lambda}}\operatorname{arcsinh}(\sqrt{3\lambda}\,v), \quad w = \frac{l+r}{2}.$$

Noticing that

$$f(v) = v\sqrt{1+3\lambda v^2} + \frac{1}{\sqrt{3\lambda}}\operatorname{arcsinh}(\sqrt{3\lambda}\,v)$$

is one-one and smooth, we put $v = f^{-1}(l-r) = v(l,r)$. For classical solutions, the Cauchy problem (2.1)–(2.2) is equivalent to

$$\begin{cases} \mathrm{i}u_t + u_{xx} = uv + \alpha|u|^2u, \\ \rho_t = \dfrac{1}{2}(l+r), \\ l_t - \sqrt{1+3\lambda v^2}\, l_x = (|u|^2)_x + \beta\rho^3, \\ r_t + \sqrt{1+3\lambda v^2}\, r_x = (|u|^2)_x + \beta\rho^3 \end{cases} \tag{2.3}$$

with initial data

$$\begin{aligned} &u(\,\cdot\,,0) = u_0 \in H^3(\mathbb{R}), \quad \rho(\,\cdot\,,0) = \rho_0 \in H^3(\mathbb{R}), \\ &l(\,\cdot\,,0) = l_0 \in H^2(\mathbb{R}), \quad\ \ r(\,\cdot\,,0) = r_0 \in H^2(\mathbb{R}), \end{aligned} \tag{2.4}$$

where

$$l_0 = w_0 + \int_0^{v_0} \sqrt{1+3\lambda\xi^2}\,\mathrm{d}\xi \quad \text{and} \quad r_0 = w_0 - \int_0^{v_0} \sqrt{1+3\lambda\xi^2}\,\mathrm{d}\xi. \tag{2.5}$$

In order to obtain a local strong solution to the Cauchy problem (2.3)–(2.4) for a fixed $\lambda \geq 0$, we will follow the technique employed in [10, 5].

We consider the auxiliary system with non-local source terms

$$\begin{cases} \mathrm{i}F_t + F_{xx} = 2\alpha|u|^2F + \alpha u^2\overline{F} + Fv + \dfrac{1}{2}u(l_x + r_x), \\ \rho_t = \dfrac{1}{2}(l+r), \\ l_t - \sqrt{1+3\lambda v^2}\, l_x = (|\widetilde{u}|^2)_x + \beta\rho^3, \\ r_t + \sqrt{1+3\lambda v^2}\, r_x = (|\widetilde{u}|^2)_x + \beta\rho^3, \end{cases} \tag{2.6}$$

where $\overline{F}$ is the complex conjugate of F and

$$\begin{aligned} &u(x,t) = u_0(x) + \int_0^t F(x,s)\mathrm{d}s, \\ &\widetilde{u}(x,t) = (\Delta - 1)^{-1}(\alpha|u|^2u + u(v-1) - \mathrm{i}F), \end{aligned} \tag{2.7}$$

with initial data

$$\begin{aligned} &F(\,\cdot\,,0) = F_0 \in H^1(\mathbb{R}), \quad \rho(\,\cdot\,,0) = \rho_0 \in H^3(\mathbb{R}), \quad l(\,\cdot\,,0) = l_0 \in H^2(\mathbb{R}), \\ &r(\,\cdot\,,0) = r_0 \in H^2(\mathbb{R}), \quad l_0 \text{ and } r_0 \text{ given by (2.5)}. \end{aligned} \tag{2.8}$$

We will prove the following result.

Theorem 2.1 *Let* $(F_0, \rho_0, l_0, r_0) \in H^1 \times H^3 \times H^2 \times H^2$, $l_0 - r_0 = f(\rho_{x0})$. *There exists a* $T^* = T^*(F_0, \rho_0, l_0, r_0) > 0$, *such that for all* $T < T^*$ *there exists a unique solution* (F, ρ, l, r) *to the Cauchy problem* (2.6)–(2.8) *with*

$$\begin{aligned} (F,\rho,l,r) \in\ & C^j([0,T]; H^{1-2j}) \times C^j([0,T]; H^{3-j}) \\ & \times C^j([0,T]; H^{2-j}) \times C^j([0,T]; H^{2-j}), \quad j = 0, 1. \end{aligned}$$

From this result, we will prove the following theorem.

Theorem 2.2 *Let* $(u_0, \rho_0, \rho_1) \in H^3 \times H^3 \times H^2$. *There exists a* $T^* = T^*(u_0, \rho_0, \rho_1) > 0$ *such that for all* $T < T^*$ *there exists a unique solution* (u, ρ) *to the Cauchy problem* (1.6) *with*

$$(u,\rho) \in C^j([0,T]; H^{3-2j}) \times (C^j([0,T]; H^{3-j}) \cap C^{j+1}([0,T]; H^{2-j})), \quad j = 0, 1.$$

Proof of Theorem 2.1 We want to apply a variant of [7, Theorem 6]. Hence, we need to put the Cauchy problem in the framework of real spaces. Introduce the new variables $F_1 = \mathrm{Re}(F)$, $F_2 = \mathrm{Im}(F)$, $u_1 = \mathrm{Re}(u)$, $u_2 = \mathrm{Im}(u)$.

By setting $U = (F_1, F_2, \rho, l, r)$, $F_{10} = \mathrm{Re}(F_0)$ and $F_{20} = \mathrm{Im}(F_0)$, the initial value problem (2.6) and (2.8) can be written in the form

$$\begin{cases} \dfrac{\partial}{\partial t}U + A(U)U = g(t, U), \\ U(\,\cdot\,, 0) = U_0, \end{cases} \tag{2.9}$$

where

$$A(U) = \begin{bmatrix} 0 & \Delta & 0 & 0 & 0 \\ -\Delta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sqrt{1+3\lambda v^2} & 0 \\ 0 & 0 & 0 & 0 & \sqrt{1+3\lambda v^2} \end{bmatrix},$$

$$g(t, U) = \begin{bmatrix} 2\alpha|u|^2F_2 - \alpha(u_1^2 - u_2^2)F_2 + 2\alpha u_1u_2F_1 + F_2v + \frac{1}{2}u_2(l_x + r_x) \\ 2\alpha|u|^2F_1 - \alpha(u_1^2 - u_2^2)F_1 - 2\alpha u_1u_2F_2 - F_1v - \frac{1}{2}u_2(l_x + r_x) \\ \frac{1}{2}(l + r) \\ (|\widetilde{u}|^2)_x + \beta\rho^3 \\ (|\widetilde{u}|^2)_x + \beta\rho^3 \end{bmatrix},$$

$$U_0 = (F_{10}, F_{20}, \rho_0, l_0, r_0) \in Y = (H^1(\mathbb{R}))^2 \times (H^2(\mathbb{R}))^3$$

(The condition $\rho_0 \in H^3(\mathbb{R})$ will be used later).

Note that the source term $g(t, U)$ is non-local.

We now set $X = (H^{-1}(\mathbb{R}))^2 \times (L^2(\mathbb{R}))^3$ and $S = (1 - \Delta)I$, which is an isomorphism $S : Y \to X$.

Furthermore, we denote by W_R the open ball in Y of radius R centered at the origin and by $G(X, 1, \omega)$ the set of linear operators $\Lambda : D(\Lambda) \subset X \to X$, such that

(1) $-\Lambda$ generates a C_0-semigroup $\{\mathrm{e}^{-t\Lambda}\}_{t\in\mathbb{R}_+}$;

(2) for all $t \geq 0$, $\|\mathrm{e}^{-t\Lambda}\| \leq \mathrm{e}^{\omega t}$, where for all $U \in W_R$,

$$\omega = \frac{1}{2}\sup_{x\in\mathbb{R}}\left\|\frac{\partial}{\partial x}a(\rho, l, r)\right\| \leq c(R), \quad c : [0, +\infty[\to [0, +\infty[\ \text{ is continuous},$$

$$a(\rho, l, r) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\sqrt{1+3\lambda v^2} & 0 \\ 0 & 0 & \sqrt{1+3\lambda v^2} \end{bmatrix}.$$

Following from [7, Paragraph 12], we get $A : U = (F_1, F_2, \rho, l, r) \in W_R \to G(X, 1, \omega)$. It is easy to see that g verifies for a fixed $T > 0$, $\|g(t, U(t))\|_Y \leq \theta_R$, $t \in [0, T]$, $U \in C([0, T]; W_R)$. For (ρ, l, r) in a ball $\widetilde{W}$ in $(H^2(\mathbb{R}))^3$, we set

$$B_0(\rho, l, r) = [(1 - \Delta), a(\rho, l, r)](1 - \Delta)^{-1} \in \mathcal{L}((L^2(\mathbb{R}))^3)$$

(see [7, 12.6]). We now introduce the operator $B(U) \in \mathcal{L}(X)$, $U = (F_1, F_2, \rho, l, r) \in W_R$ by

$$B(U) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & & & \\ 0 & 0 & & B_0(\rho, l, r) & \\ 0 & 0 & & & \end{bmatrix}.$$

In [7, Paragraph 12], Kato proved that for $(\rho, l, r) \in \widetilde{W}$, we have

$$(1 - \Delta)a(\rho, l, r)(1 - \Delta)^{-1} = a(\rho, l, r) + B_0(\rho, l, r).$$

Hence, we easily derive for $U \in W_R$, $SA(U)S^{-1} = A(U) + B(U)$.

Now, for each pair $U, U^* \in C([0, T]; W_R)$, $U = (F_1, F_2, \rho, l, r)$, $U^* = (F_1^*, F_2^*, \rho^*, l^*, r^*)$, we claim that

$$\|g(\,\cdot\,, U) - g(\,\cdot\,, U^*)\|_{L^1(0,T';X)} \leq c(T') \sup_{0 \leq t \leq T'} \|U(t) - U^*(t)\|_X, \tag{2.10}$$

where $0 \leq T' \leq T$ and $c(T')$ is a non-decreasing continuous function such that $c(0) = 0$.

Indeed, let us point out that for $h \in L^2(\mathbb{R})$ and $w \in H^1(\mathbb{R})$, $\|hw\|_{H^{-1}} \leq \|h\|_{H^{-1}}\|w\|_{H^1}$. Hence, for example,

$$\|F_1 u_1(u_1^* - u_1)\|_{H^{-1}} \leq \|F_1\|_{H^1}\|u_1\|_{H^1}\|u_1^* - u_1\|_{H^{-1}}$$

and for $t \leq T'$,

$$\begin{aligned} \Big\|(l_x + r_x)\Big(\int_0^t F_2 \mathrm{d}s - \int_0^t F_2^* \mathrm{d}s\Big)\Big\|_{H^{-1}} &\leq \|l_x + r_x\|_{H^1} \int_0^t \|F - F*\|_{H^{-1}} \mathrm{d}\tau \\ &\leq c(T') \sup_{0 \leq t \leq T'} \|U(t) - U^*(t)\|_X. \end{aligned}$$

Finally, applying [7, Theorem 6] and replacing the local condition (7.7) in [7] by (2.10), we obtain the result described in Theorem 2.1, but with $\rho \in C^j([0, T]; H^{2-j}), j = 0, 1$. To obtain $\rho \in C^j([0, T]; H^{3-j})$, it is enough to remark that, since $\rho_t = w$, $\rho_0 \in H^3$, $v_0 = \rho_{0x} \in H^2$, $w_0 \in \rho_1 \in H^2$, we derive $\rho_x = v \in C^j([0, T], H^{2-j})$.

Proof of Theorem 2.2 We will follow here the ideas in [5].

If (F, ρ, l, r) is a solution to (2.6) and (2.8), by differenciating (2.7) with respect to t, we obtain $u_t = F$. Applying it to the first equation of (2.6), we obtain

$$(\mathrm{i}u_t + u_{xx})_t = 2\alpha|u|^2 F + \alpha u^2 \overline{F} + Fv + \frac{1}{2}u(l_x + r_x) = 2\alpha|u|^2 u_t + \alpha u^2 \overline{u}_t + u_t v + u v_t.$$

Hence, $(\mathrm{i}u_t + u_{xx} - \alpha|u|^2 u - uv)_t = 0$ and $\mathrm{i}u_t + u_x x - \alpha|u|^2 u - uv = \phi_0(x)$, where $\phi_0(x) = \mathrm{i}F_0 + u_0'' - \alpha|u_0|^2 u_0 - u_0 v_0$. By choosing $F_0 = \mathrm{i}(u_0'' - \alpha|u_0|^2 u_0 - u_0 v_0)$, we obtain that $\phi_0 = 0$ and (u, v) satisfy the first equation in (2.3).

Furthermore, from this equation, we derive

$$u = (\Delta - 1)^{-1}(\alpha|u|^2 u + u(v - 1) - \mathrm{i}u_t). \tag{2.11}$$

Therefore $u = \widetilde{u}$ and (u, ρ, l, r) satisfy (2.3)–(2.4). Note that $u_t = F \in C([0, T]; H^1)$. Moreover,

$$u(x, t) = u_0(x) + \int_0^t F(x, s)\mathrm{d}s \in C([0, T]; H^1).$$

But from (2.11), we have in fact $u \in C([0, T]; H^3)$.

3 Global Well-Posedness for $\lambda = 0$ and Blow-Up Results

In this section, we prove that the local solutions obtained in Theorem 2.2 are in fact global in time in the case where $\beta \le 0$ and $\lambda = 0$. Conversely, if $\beta > 0$, we show the blow-up of the local solutions in finite time under some conditions on the initial data.

We consider the initial data $(u_0, \rho_0, \rho_1) \in H^3 \times H^3 \times H^2$. Let

$$(u, \rho) \in C^j([0,T]; H^{3-2j}) \times (C^j([0,T]; H^{3-j}) \cap C^{j+1}([0,T]; H^{2-j})), \quad j = 0, 1$$

be the unique corresponding maximal solution to the Cauchy problem (1.6). We begin the proof by deriving the following conservation laws:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int |u|^2 \mathrm{d}x = 0, \quad t \in [0, T[, \tag{3.1}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} E(t) = 0, \quad t \in [0, T[, \tag{3.2}$$

where the energy $E(t)$ is given by

$$\begin{aligned} E(t) = & \frac{1}{2}\int (\rho_t)^2 \mathrm{d}x + \frac{1}{2}\int (\rho_x)^2 \mathrm{d}x + \frac{\lambda}{4}\int (\rho_x)^4 \mathrm{d}x - \frac{\beta}{4}\int \rho^4 \mathrm{d}x + \int \rho_x |u|^2 \mathrm{d}x \\ & + \int |u_x|^2 \mathrm{d}x + \frac{\alpha}{2}\int |u|^4 \mathrm{d}x. \end{aligned}$$

For the first one we multiply the first equation in (1.6) by $\overline{u}$, and integrate the imaginary part. To obtain the conservation of energy, we derive from (1.6) that

$$\begin{aligned} \mathrm{Re}\int \mathrm{i} u_t \overline{u}_t \mathrm{d}x + \mathrm{Re}\int u_{xx}\overline{u}_t \mathrm{d}x &= \mathrm{Re}\int \rho_x u \overline{u}_t \mathrm{d}x + \alpha \mathrm{Re}\int |u|^2 u \overline{u}_t \mathrm{d}x, \\ -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |u_x|^2 \mathrm{d}x &= \frac{1}{2}\int \rho_x \frac{\partial}{\partial t}|u|^2 \mathrm{d}x + \frac{\alpha}{4}\frac{\mathrm{d}}{\mathrm{d}t}\int |u|^4 \mathrm{d}x \\ &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int \rho_x |u|^2 \mathrm{d}x - \frac{1}{2}\int \frac{\partial}{\partial t}\rho_x |u|^2 \mathrm{d}x + \frac{\alpha}{4}\frac{\mathrm{d}}{\mathrm{d}t}\int |u|^4 \mathrm{d}x. \end{aligned}$$

Finally,

$$\begin{aligned} -\frac{1}{2}\int \frac{\partial^2 \rho}{\partial x \partial t}|u|^2 \mathrm{d}x &= \frac{1}{2}\int \frac{\partial \rho}{\partial t}(|u|^2)_x \mathrm{d}x = \frac{1}{2}\int \frac{\partial \rho}{\partial t}\Big\{\frac{\partial^2 \rho}{\partial t^2} - \frac{\partial}{\partial x}[\rho_x + \lambda(\rho_x)^3] - \beta\rho^3\Big\}\mathrm{d}x \\ &= \frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}\int (\rho_t)^2 \mathrm{d}x + \frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}\int (\rho_x)^2 \mathrm{d}x + \frac{\lambda}{8}\frac{\mathrm{d}}{\mathrm{d}t}\int (\rho_x)^4 - \frac{\beta}{8}\frac{\mathrm{d}}{\mathrm{d}t}\int \rho^4 \mathrm{d}x, \end{aligned}$$

and (3.2) is proved.

Next, we will prove the following result.

Theorem 3.1 *Let $\beta \le 0$ and $\lambda = 0$. Then Theorem 2.2 holds for $T^* = +\infty$.*

Proof In order to prove this result, it is sufficient to derive a priori bounds for the norms $\|u\|_{H^3}$, $\|\rho\|_{H^3}$, $\|\rho_t\|_{H^2}$ and $\|\rho_{tt}\|_{H^1}$.

Let us begin the proof by noticing that $|\int \rho_x |u|^2 \mathrm{d}x| \le \frac{1}{4}\int (\rho_x)^2 \mathrm{d}x + \int |u|^4 \mathrm{d}x$. By the Gagliardo-Nirenberg inequality and (3.1), we have $\|u\|_{L^4}^4 \le c_0 \|u\|_{L^2}^3 \|u_x\|_{L^2} \le c_0 \|u_0\|_{L^2}^3 \|u_x\|_{L^2} \le c\|u_0\|_{L^2}^6 + \frac{1}{2}\|u_x\|_{L^2}^2$. Since $\beta \le 0$, we obtain from (3.2) that

$$\int (\rho_t)^2 \mathrm{d}x + \int [(\rho_x)^2 + \lambda(\rho_x)^4] \mathrm{d}x + \int |u_x|^2 \mathrm{d}x \le c \tag{3.3}$$

with c depending only on $(\|u_0\|_{H^1}, \|\rho_0\|_{H^2}, \|\rho_1\|_{H^1})$.

Moreover, since $\rho(t) = \rho_0 + \int_0^t \rho_t(\tau)\mathrm{d}\tau$, we have $\|\rho(t)\|_{L^2} \le \|\rho_0\|_{L^2} + \int_0^t \|\rho_t(\tau)\|_{L^2}\mathrm{d}\tau$. Hence, since $\beta \le 0$, we have

$$\int (\rho_t)^2 \mathrm{d}x + \int (\rho)^2 \mathrm{d}x + \int (\rho_x)^2 \mathrm{d}x + \int |u|^2 \mathrm{d}x + \int |u_x|^2 \mathrm{d}x \le C(1+t) \tag{3.4}$$

with C depending exclusively on the initial data.

Next, we estimate $\|u_{xx}\|_{L^2}$, $\|\rho_{xt}\|_{L^2}$ and $\|\rho_{xx}\|_{L^2}$. For $\lambda = 0$, the system (2.3) reads

$$\begin{cases} \mathrm{i}u_t + u_{xx} = uv + \alpha |u|^2 u, \\ \rho_t = \dfrac{1}{2}(l+r), \\ l_t - l_x = (|u|^2)_x + \beta\rho^3, \\ r_t + r_x = (|u|^2)_x + \beta\rho^3. \end{cases} \tag{3.5}$$

We put

$$\gamma(t) = \int (r_x)^2 \mathrm{d}x + \int (l_x)^2 \mathrm{d}x + \int |u_t|^2 \mathrm{d}x.$$

In what follows, we will denote by $A(t)$ a generic positive continuous function $A : \mathbb{R}_+ \to \mathbb{R}_+$, which can change from line to line.

By differentiating with respect to x the last equation in (3.5), multiplying by r_x and integrating, we get

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int (r_x)^2 \mathrm{d}x \\ &\le 2\int |u u_{xx} r_x| \mathrm{d}x + 2\int |u_x^2 r_x| \mathrm{d}x + 3|\beta| \int \rho^2 |\rho_x r_x| \mathrm{d}x \\ &\le A(t)\Big[\Big(\int r_x^2 \mathrm{d}x\Big)^{\frac{1}{2}}\Big(\int |u_{xx}|^2 \mathrm{d}x\Big)^{\frac{1}{2}} + \|u\|_\infty \Big(\int |u_x|^2 \mathrm{d}x\Big)^{\frac{1}{2}}\Big(\int r_x^2 \mathrm{d}x\Big)^{\frac{1}{2}} + \Big(\int r_x^2 \mathrm{d}x\Big)^{\frac{1}{2}}\Big] \\ &\le A(t)\Big[\Big(\int r_x^2 \mathrm{d}x\Big)^{\frac{1}{2}}\Big(\int |u_{xx}|^2 \mathrm{d}x\Big)^{\frac{1}{2}} + \Big(\int r_x^2 \mathrm{d}x\Big)^{\frac{1}{2}}\Big], \end{aligned}$$

where the Sobolev injection $\|u_x\|_\infty \le c\|u_x\|_{H^1}$ and (3.4) are used.

By a similar estimate for l_x, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int ((r_x)^2 + (l_x)^2)\mathrm{d}x \le A(t)\Big[\gamma^{\frac{1}{2}}(t) + \gamma^{\frac{1}{2}}(t)\Big(\int |u_{xx}|^2 \mathrm{d}x\Big)^{\frac{1}{2}}\Big]. \tag{3.6}$$

From the first equation in (3.5), we have

$$\|u_{xx}\|_{L^2} \le \|u_t\|_{L^2} + A(t) \le \gamma^{\frac{1}{2}}(t) + A(t). \tag{3.7}$$

By using it in (3.6), we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int ((r_x)^2 + (l_x)^2)\mathrm{d}x \le A(t)[\gamma^{\frac{1}{2}}(t) + \gamma(t)]. \tag{3.8}$$

Moreover, since $\rho_t = \frac{1}{2}(l+r)$, we have

$$\|\rho_{xt}\|_{L^2} \le c\gamma^{\frac{1}{2}}(t). \tag{3.9}$$

Now, multiplying the first equation in (3.5) by $\overline{u}_t$, integrating the imaginary part and using the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int|u_t|^2\mathrm{d}x&=\int\rho_{xt}\mathrm{Im}(u\overline{u}_t)\mathrm{d}x+\alpha\int(|u|^2)_t\mathrm{Im}(u\overline{u}_t)\mathrm{d}x\\&\le\|u\|_\infty\|\rho_{xt}\|_{L^2}\Big(\int|u_t|^2\mathrm{d}x\Big)^{\frac{1}{2}}+2|\alpha|\|u\|_\infty^2\int|u_t|^2\mathrm{d}x\le c\gamma(t).\end{aligned}$$

Finally, using (3.8), we get

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\gamma(t)\le A(t)[\gamma^{\frac{1}{2}}(t)+\gamma(t)]\le A(t)[1+\gamma(t)],\\&\gamma(t)\le(1+\gamma(0))\mathrm{e}^{\int_0^t A(\tau)\mathrm{d}\tau}-1.\end{aligned}$$

Hence, by (3.7) and (3.9), we have $\|u_{xx}\|_{L^2}+\|\rho_{xt}\|_{L^2}\le A(t)$. By the second and the third equations in (3.5), we have $\|l_t\|_{L^2}+\|r_t\|_{L^2}\le A(t)$. Therefore

$$\begin{aligned}\|\rho_{tt}\|_{L^2}&=\frac{1}{2}\|l_t+r_t\|_{L^2}\le A(t),\\\|\rho_{xx}\|_{L^2}&=\|\rho_{tt}-(|u|^2)_x-\beta\rho^3\|_{L^2}\le A(t).\end{aligned}$$

To obtain a continuous bound on $\|\rho_{xxx}\|_{L^2}$, $\|u_{xxx}\|_{L^2}$, $\|\rho_{txx}\|_{L^2}$ and $\|\rho_{ttx}\|_{L^2}$, the exact same method can be used by setting

$$\gamma(t)=\int(r_{xx})^2\mathrm{d}x+\int(l_{xx})^2\mathrm{d}x+\int|u_{xt}|^2\mathrm{d}x$$

and differentiating system (3.5) with respect to x.

We now assume $\beta>0$. In what follows, we will consider the following conditions on the initial data:

$$\int\rho_0\rho_1\mathrm{d}x>0,\tag{3.10}$$

$$E(0)\le-\frac{1}{64}\Big(\frac{9}{4}+2|\alpha|\Big)^2\|u_0\|_{L^2}^6.\tag{3.11}$$

We will prove the following blow-up result.

Theorem 3.2 *Let $\beta>0$ and $\lambda\ge0$. Under the conditions of Theorem 2.2, by assuming that the initial data (u_0,ρ_0,ρ_1) satisfy conditions (3.10) and (3.11), there exists a time $0<T^*\le T_0:=(\int\rho_0^2\mathrm{d}x)(\int\rho_0\rho_1\mathrm{d}x)^{-1}$, such that, if the solution exists in $[0,T^*]$, then*

$$\lim_{t\to T^{*-}}\int\rho^2\mathrm{d}x=+\infty.$$

Proof Following [13, Chapter 10, Paragraph 13], we put

$$G(t)=\int\rho^2\mathrm{d}x\quad\text{and}\quad F(t)=(G(t))^{-\frac{1}{2}}.\tag{3.12}$$

We have $F'(t)=-\frac{1}{2}G(t)^{-\frac{3}{2}}G'(t)=-G(t)^{-\frac{3}{2}}\int\rho\rho_t\mathrm{d}x$, and from (3.10), $F'(0)<0$.

Furthermore, we set $Q(t)=-2G(t)^{\frac{5}{2}}F''(t)=G''(t)G(t)-\frac{3}{2}G'(t)^2$ with

$$G''(t)=6\int(\rho_t)^2\mathrm{d}x+2H(t)\quad\text{and}\quad H(t)=\int[\rho\rho_{tt}-2(\rho_t)^2]\mathrm{d}x.$$

We have

$$Q(t) = 6\Big[\Big(\int \rho^2 \mathrm{d}x\Big)\Big(\int (\rho_t)^2 \mathrm{d}x\Big) - \Big(\int (\rho\rho_t)^2 \mathrm{d}x\Big)\Big] + 2G(t)H(t).$$

By the Cauchy-Schwarz inequality, we obtain $Q(t) \geq 0$, and consequently $F''(t) \leq 0$ provided $H(t) \geq 0$.

The last fact is easy to check. From (1.6) and (3.2), we have

$$\begin{aligned} H(t) &= -4E(t) + 4\Big[\int |u_x|^2 \mathrm{d}x + \frac{\alpha}{2}\int |u|^4 \mathrm{d}x + \frac{3}{4}\int \rho_x |u|^2 \mathrm{d}x\Big] + \int (\rho_x)^2 \mathrm{d}x \\ &= -4E(0) + 4\Big[\int |u_x|^2 \mathrm{d}x + \frac{\alpha}{2}\int |u|^4 \mathrm{d}x + \frac{3}{4}\int \rho_x |u|^2 \mathrm{d}x\Big] + \int (\rho_x)^2 \mathrm{d}x. \end{aligned}$$

Then

$$3\int \rho_x |u|^2 \mathrm{d}x \leq \int (\rho_x)^2 \mathrm{d}x + \frac{9}{4}\int |u|^4 \mathrm{d}x.$$

By the Gagliardo-Nirenberg inequality and (3.1), we have

$$\Big(\frac{9}{4} + 2|\alpha|\Big)\int |u|^4 \mathrm{d}x \leq \Big(\frac{9}{4} + 2|\alpha|\Big)\|u_0\|_{L^2}^3 \|u_x\|_{L^2} \leq 4\int |u_x|^2 \mathrm{d}x + \frac{1}{16}\Big(\frac{9}{4} + 2|\alpha|\Big)^2 \|u_0\|_{L^2}^6.$$

From condition (3.11), we have $H(t) \geq -4E(0) - \frac{1}{16}(\frac{9}{4} + 2|\alpha|)^2 \|u_0\|_{L^2}^6 \geq 0$.

Hence, we have shown that for all $t \in [0, T[$, $F''(t) \leq 0$, which implies Theorem 3.2.

4 Global Existence of Weak Solutions to the Quasilinear System

For the study of the existence of a global weak solution to the Cauchy problem (1.6), we consider for $\epsilon > 0$, the regularized problem (see [4] for the case $\beta = 0$)

$$\begin{cases} \mathrm{i}u_t + u_{xx} = u\rho_x + \alpha|u|^2 u, \\ \rho_t = w, \\ w_t - \epsilon w_{xx} = \beta\rho^3 + (\sigma(\rho_x))_x + (|u|^2)_x \end{cases} \tag{4.1}$$

with the initial data (we have dropped the ϵ parameter on u, w and ρ)

$$\begin{aligned} &u(0,x) = u_0(x) \in H^1(\mathbb{R}), \quad \rho(0,x) = \rho_0(x) \in H^2(\mathbb{R}), \\ &w(x,0) = \rho_t(0,x) = \rho_1(x) \in H^1(\mathbb{R}). \end{aligned} \tag{4.2}$$

Here, $\sigma(v) = v + \lambda v^3$ and $\lambda > 0$ (hence, $\sigma'(v) = 1 + 3\lambda v^2 \geq 1$).

For a smooth solution to (4.1)–(4.2), the energy identity (3.2) takes the form

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\Big\{\frac{1}{2}\int w^2 \mathrm{d}x + \frac{1}{2}\int v^2 \mathrm{d}x + \frac{\lambda}{4}\int v^4 \mathrm{d}x - \frac{\beta}{4}\int \rho^4 \mathrm{d}x \\ &\quad + \int v|u|^2 \mathrm{d}x + \int |u_x|^2 \mathrm{d}x + \frac{\alpha}{2}\int |u|^4 \mathrm{d}x\Big\} \\ &= -\epsilon\int (w_x)^2 \mathrm{d}x, \end{aligned} \tag{4.3}$$

where we have put $v = \rho_x$. On the other hand, the conservation law

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\int |u|^2 \mathrm{d}x\Big) = 0 \tag{4.4}$$

still holds. Also, we deduce (see [4] and following [14])

$$\begin{aligned}\int[w_t v_x-\sigma'(v)(v_x)^2]\mathrm{d}x&=\int(|u|^2)_x v_x\mathrm{d}x+\beta\int\rho^3 v_x\mathrm{d}x+\epsilon\int w_{xx}v_x\mathrm{d}x,\\ &-\frac{\mathrm{d}}{\mathrm{d}t}\int w_x v\mathrm{d}x+\int(w_x)^2\mathrm{d}x-\int\sigma'(v)(v_x)^2\mathrm{d}x\\ &=\int(|u|^2)_x v_x\mathrm{d}x+\beta\int\rho^3\rho_{xx}\mathrm{d}x+\frac{\epsilon}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int(v_x)^2\mathrm{d}x,\end{aligned}$$

since

$$\begin{aligned}-\frac{\mathrm{d}}{\mathrm{d}t}\int w_x v\mathrm{d}x&=-\int w_{xt}v\mathrm{d}x-\int w_x v_t\mathrm{d}x=\int w_t v_x\mathrm{d}x-\int w_x v_t\mathrm{d}x,\\ v_t&=\rho_{xt}=w_x.\end{aligned}$$

Integrating this identity over the time interval $[0,t]$, we obtain with $v_0(x)=v(x,0)$,

$$\begin{aligned}&-\int w_x v\mathrm{d}x+\int\rho_{1x}v_0\mathrm{d}x+\int_0^t\int(w_x)^2\mathrm{d}x\mathrm{d}\tau-\int_0^t\int\sigma'(v)(v_x)^2\mathrm{d}x\mathrm{d}\tau\\ &=\int_0^t\int(|u|^2)_x v_x\mathrm{d}x\mathrm{d}\tau-3\beta\int_0^t\int\rho^2(\rho_x)^2\mathrm{d}x\mathrm{d}\tau+\frac{\epsilon}{2}\int(v_x)^2\mathrm{d}x-\frac{\epsilon}{2}\int(v_{0x})^2\mathrm{d}x.\end{aligned}$$

Since $-\int w_x v\mathrm{d}x=\int w v_x\mathrm{d}x$, we get

$$\begin{aligned}&\int_0^t\int\sigma'(v)(v_x)^2\mathrm{d}x\mathrm{d}\tau+\frac{\epsilon}{2}\int(v_x)^2\mathrm{d}x\\ &\le\frac{\epsilon}{4}\int(v_x)^2\mathrm{d}x+\frac{1}{\epsilon}\int w^2\mathrm{d}x+\int|v_0\rho_{1x}|\mathrm{d}x\\ &\quad+\frac{\epsilon}{2}\int(v_{0x})^2\mathrm{d}x+3\beta\int_0^t\int\rho^2v^2\mathrm{d}x\mathrm{d}\tau+\epsilon\int_0^t\int(w_x)^2\mathrm{d}x\mathrm{d}\tau+2\int_0^t\int|uu_xv_x|\mathrm{d}x\mathrm{d}\tau\end{aligned}\tag{4.5}$$

and

$$2\int_0^t\int|uu_xv_x|\mathrm{d}x\mathrm{d}\tau\le2\int_0^t\int|uu_x|^2\mathrm{d}x\mathrm{d}\tau+\frac{1}{2}\int_0^t\int(v_x)^2\mathrm{d}x\mathrm{d}\tau.\tag{4.6}$$

Now, we assume $\beta\le0$. Since $\epsilon>0$, we can derive from (4.3), as in (3.3),

$$\int w^2\mathrm{d}x+\int(v^2+\lambda v^4)\mathrm{d}x+\int|u_x|^2\mathrm{d}x+\epsilon\int_0^t\int(w_x)^2\mathrm{d}x\mathrm{d}\tau\le C,\tag{4.7}$$

where C only depends on $(\|u_0\|_{H^1},\|\rho_0\|_{H^2},\|\rho_1\|_{H^1})$.

Hence, from (4.4), (4.6) and (4.7), we have

$$2\int_0^t\int|uu_xv_x|\mathrm{d}x\mathrm{d}\tau\le Ct+\frac{1}{2}\int_0^t\int\sigma'(v)(v_x)^2\mathrm{d}x\mathrm{d}\tau.\tag{4.8}$$

Taking $\epsilon\le1$, we deduce from (4.5)–(4.8) that

$$\epsilon\int_0^t\int\sigma'(v)(v_x)^2\mathrm{d}x\mathrm{d}\tau+\epsilon^2\int(v_x)^2\mathrm{d}x+\epsilon|\beta|\int_0^t\int\rho^2v^2\mathrm{d}x\mathrm{d}\tau\le C(1+t).\tag{4.9}$$

Let us now analyse the problem of the existence and uniqueness of a solution

$$(u,\rho,w)\in C([0,+\infty[;H^1)\times C([0,+\infty[;H^2)\times C([0,+\infty[;H^1)$$

to the Cauchy problem (4.1)–(4.2). Without loss of generality, we may assume $\epsilon = 1$. We start with the existence and uniqueness of a local (in time) solution. We fix $0 < T < +\infty$ and introduce the Banach spaces $X_T = C([0,T];H^1)$ (complex) and $Y_T = C([0,T];H^2)$ (real) endowed with the usual norms. Furthermore, we consider the product space $\widetilde{B}_R^T \times B_R^T$, where

$$\widetilde{B}_R^T = \{u \in X_T : \|u\|_{X_T} \le R\} \quad \text{and} \quad B_R^T = \{u \in Y_T : \|u\|_{Y_T} \le R\}.$$

Finally, we consider the application $\Phi : (\widetilde{u},\widetilde{\rho}) \in \widetilde{B}_R^T \times B_R^T \to (u,\rho) \in X_T \times Y_T$. Here, u denotes the solution to the linear problem

$$\begin{cases} \mathrm{i}u_t + u_{xx} = \widetilde{\rho}_x\widetilde{u} + \alpha|\widetilde{u}|^2\widetilde{u}, \\ u(\,\cdot\,,0) = u_0 \in H^1 \end{cases} \tag{4.10}$$

and

$$\rho(t) = \rho_0 + \int_0^t w\,\mathrm{d}\tau, \quad \rho(\,\cdot\,,0) = \rho_0 \in H^2, \tag{4.11}$$

where w is the unique solution to

$$\begin{cases} w_t - w_{xx} = \beta\widetilde{\rho}^3 + (\sigma(\widetilde{\rho}_x))_x + (|\widetilde{u}|^2)_x, \\ w(\,\cdot\,,0) = w_0(x) \in H^1, \end{cases} \tag{4.12}$$

verifying $w \in L^2(0,T;H^2)$, $w_t \in L^2(0,T;L^2)$. We have

$$u(t) = \mathrm{e}^{\mathrm{i}t\partial_{xx}}u_0 - \mathrm{i}\int_0^t \mathrm{e}^{\mathrm{i}(t-s)\partial_{xx}}(\widetilde{\rho}_x\widetilde{u} + \alpha|\widetilde{u}|^2\widetilde{u})(s)\mathrm{d}s$$

and $\beta\widetilde{\rho}^3 + (\sigma(\widetilde{\rho}_x))_x + (|\widetilde{u}|^2)_x \in C([0,T];L^2)$.

The existence and uniqueness of a local solution is a consequence of the Banach fixed-point theorem for a convenient choice of R and T, $R > \max(\|u_0\|_{H^1}, \|\rho_0\|_{H^2})$. We have $w_t - w_{xx} = \beta\rho^3 + (\sigma(\rho_x))_x + (|u|^2)_x$. From (4.3), (4.4), (4.7) and (4.9)–(4.11), we derive the a priori estimate $|w_t - w_{xx}|_{L^2(0,T;L^2)} \le C(T)$, $C \in C([0,+\infty[;\mathbb{R}_+)$, which implies $w \in L^2(0,T;H^2)$ and a similar a priori estimate for $\|w\|_{L^2(0,T;H^2)}$ and so for $\|w_t\|_{L^2(0,T;L^2)}$ and $\|w\|_{C([0,T];H^1)}$.

We conclude that $\rho \in Y_T$ and $u \in X_T$ with similar estimates for $\|\rho\|_{Y_T}$ and $\|u\|_{X_T}$. Hence, we can extend the solution to $[0,+\infty[$.

Hence, if we write

$$\rho_\epsilon(t) = \rho_0 + \int_0^t w_\epsilon\,\mathrm{d}\tau, \quad \rho_0 \in H^2(\mathbb{R}),\ 0 < \epsilon \le 1, \tag{4.13}$$

we get, with

$$u_\epsilon(0,x) = u_0(x) \in H^1, \quad v_\epsilon(0,x) = v_0(x) \in H^1, \quad w_\epsilon(0,x) = \rho_t(0,x) = \rho_1(x) \in H^1, \tag{4.14}$$

a unique solution

$$(u_\epsilon, v_\epsilon, w_\epsilon) \in (C([0,+\infty[;H^1))^3 \tag{4.15}$$

to the Cauchy problem

$$\begin{cases} \mathrm{i}u_{\epsilon t} + u_{\epsilon xx} = u_\epsilon v_\epsilon + \alpha |u_\epsilon|^2 u_\epsilon, \\ v_{\epsilon t} = w_{\epsilon x}, \\ w_{\epsilon t} = (\sigma(v))_x + (|u_\epsilon|^2)_x + \beta {\rho_\epsilon}^3 + \epsilon w_{\epsilon xx} \end{cases} \tag{4.16}$$

with the initial data (4.14).

Moreover, for each $T > 0$, by (4.4), (4.7) and the first equation in (4.1), we have

$$\{u_\epsilon\}_\epsilon \text{ bounded in } L^\infty(0, +\infty; H^1),$$
$$\{u_{\epsilon t}\}_\epsilon \text{ bounded in } L^\infty(0, +\infty; H^{-1}).$$

Hence, $\{u_\epsilon\}_\epsilon$ belongs to a compact set of $L^2(0, T; L^2(I_R))$ for each interval $I_R = [-R, R]$, $R \geq 0$. By applying a standard diagonalization method, we conclude that there exists a $u \in L^\infty(0, +\infty; H^1)$ and a subsequence, still denoted by $\{u_\epsilon\}_\epsilon$, such that

$$u_\epsilon \to u, \quad \text{in } L^\infty(0, +\infty; H^1) \text{ weak* and in } L^1_{\text{loc}}(\mathbb{R} \times [0, \infty[).$$

By (4.7), we also have $\{w_\epsilon\}_\epsilon$ bounded in $L^2_{\text{loc}}(\mathbb{R} \times [0, \infty[)$. With $\sum(v) = \frac{1}{2}v^2 + \frac{\lambda}{4}v^4$, we have $\{v_\epsilon\}_\epsilon$ bounded in $L^\Sigma_{\text{loc}}(\mathbb{R} \times [0, \infty[)$, where $v \in L^\Sigma_{\text{loc}}(\mathbb{R} \times [0, \infty[)$ means $\int_K \sum(v) \mathrm{d}x\mathrm{d}t < +\infty$ for each compact $K \subset \mathbb{R} \times [0, +\infty[$. Finally, by (4.13), we have $\{\rho_\epsilon\}_\epsilon$ bounded in $L^2_{\text{loc}}(\mathbb{R} \times [0, +\infty[)$.

By (4.7) and (4.9), we derive for $\epsilon \leq 1$,

$$\epsilon \int_0^t \int [(w_{\epsilon x})^2 + \sigma'(v_\epsilon)(v_{\epsilon x})^2] \mathrm{d}x\mathrm{d}\tau \leq C(1 + t), \tag{4.17}$$

where C only depends on $(\|u_0\|_{H^1}, \|\rho_0\|_{H^2}, \|\rho_1\|_{H^1})$.

Now we consider the quasilinear hyperbolic system

$$\begin{cases} v_t = w_x, \\ w_t = (\sigma(v))_x, \end{cases} \tag{4.18}$$

and let $(\eta(v, w), q(v, w))$ $((v, w) \in \mathbb{R}^2)$ be a pair of smooth convex entropy-entropy flux for (4.18), such that η_w, η_{ww} and $\frac{\eta_{vw}}{\sqrt{\sigma'}}$ are bounded in $\mathbb{R}^2$.

From (4.4) and the estimates (4.7) and (4.17), we can deduce that (see [14, 2, 4])

$$\frac{\partial}{\partial t}\eta(v_\epsilon, w_\epsilon) + \frac{\partial}{\partial x}q(v_\epsilon, w_\epsilon)$$

belongs to a compact subset of $W^{-1,2}_{\text{loc}}(\mathbb{R} \times [0, +\infty[)$.

Hence, we can use a result on compensated compactness of Serre and Shearer [14] to conclude that $\{(v_\epsilon, w_\epsilon)\}_\epsilon$ is pre-compact in $(L^1_{\text{loc}}(\mathbb{R} \times [0, +\infty[))^2$. Hence, there exist a subsequence $\{(u_\epsilon, v_\epsilon, w_\epsilon)\}_\epsilon$ and a $(u, v, w) \in L^\infty(]0, +\infty[; H^1) \times L^\Sigma_{\text{loc}}(\mathbb{R} \times [0, +\infty[) \times L^2_{\text{loc}}(\mathbb{R} \times [0, +\infty[)$, such that

$$(u_\epsilon, v_\epsilon, w_\epsilon) \to (u, v, w), \quad \text{in } (L^1_{\text{loc}}(\mathbb{R} \times [0, +\infty[))^3,$$
$$\rho_\epsilon = \rho_0 + \int_0^t w_\epsilon \mathrm{d}\tau \to \rho = \rho_0 + \int_0^t w \mathrm{d}\tau, \quad \text{in } L^1_{\text{loc}}(\mathbb{R} \times [0, +\infty[).$$

Hence, we obtain from (4.16) the following result.

Theorem 4.1 *Assume* $(u_0, \rho_0, \rho_1) \in H^1 \times H^2 \times H^1$, $\lambda > 0$ *and* $\beta \leq 0$. *Then, there exists a* $(u, v, w) \in L^\infty(0, +\infty; H^1) \times L^\Sigma_{\text{loc}}(\mathbb{R} \times [0, +\infty[) \times L^2_{\text{loc}}(\mathbb{R} \times [0, +\infty[)$, *such that, with* $\rho(x, t) = \rho_0(x) + \int_0^t w(x, \tau)\mathrm{d}\tau$, *we have*

$$\begin{aligned} &-\mathrm{i}\int_0^{+\infty}\int u\theta_t \mathrm{d}x\mathrm{d}t - \int_0^{+\infty}\int u_x\theta_x \mathrm{d}x\mathrm{d}t + \int u_0(x)\theta(x,0)\mathrm{d}x \\ &= \int_0^{+\infty}\int vu\theta \mathrm{d}x\mathrm{d}t + \alpha\int_0^{+\infty}\int |u|^2 u\theta \mathrm{d}x\mathrm{d}t \end{aligned}$$

for all $\theta \in C_0^1(\mathbb{R} \times [0, +\infty[)$ (*complex-valued*), *and*

$$\begin{aligned} &\int_0^{+\infty}\int (v\phi_t - w\phi_x)\mathrm{d}x\mathrm{d}t + \int \rho_{0x}\phi(x,0)\mathrm{d}x + \int_0^{+\infty}\int (w\psi_t - \sigma(v)\psi_x + \beta\rho^3\psi)\mathrm{d}x\mathrm{d}t \\ &+ \int \rho_1\psi(x,0)\mathrm{d}x + \int_0^{+\infty}\int (|u|^2)_x\psi \mathrm{d}x\mathrm{d}t = 0 \end{aligned}$$

for all $\phi, \psi \in C_0^1(\mathbb{R} \times [0, +\infty[)$ (*real-valued*).

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