Chin. Ann. Math.
30B(1), 2009, 17–26
DOI: 10.1007/s11401-008-0039-4

Chinese Annals of Mathematics, Series B

The Incompressible Limits of Compressible Navier-Stokes Equations in the Whole Space with General Initial Data****

Ling HSIAO* Qiangchang JU** Fucai LI***

Abstract It is showed that, as the Mach number goes to zero, the weak solution of the compressible Navier-Stokes equations in the whole space with general initial data converges to the strong solution of the incompressible Navier-Stokes equations as long as the later exists. The proof of the result relies on the new modulated energy functional and the Strichartz's estimate of linear wave equation.

Keywords Compressible Navier-Stokes equations, Incompressible Navier-Stokes equations, Low Mach number limit, Modulated energy functional, Strichartz's estimate

2000 MR Subject Classification 35Q30, 35B40

1 Introduction

The motion of an ideal fluid can be described by the following compressible Navier-Stokes equations

$$\partial_t\widetilde{\rho} + \operatorname{div}(\widetilde{\rho}\,\widetilde{u}) = 0, \tag{1.1}$$

$$\partial_t(\widetilde{\rho}\,\widetilde{u}) + \operatorname{div}(\widetilde{\rho}\,\widetilde{u}\otimes\widetilde{u}) + \nabla\widetilde{P}(\widetilde{\rho}) = \widetilde{\mu}\Delta\widetilde{u} + (\widetilde{\mu}+\widetilde{\xi})\nabla\operatorname{div}\widetilde{u}. \tag{1.2}$$

Here $x\in\mathbb{R}^d$, $d=2$ or 3, $t>0$, the unknowns $\widetilde{\rho}$ and $\widetilde{u}$ denote the fluid density and velocity, respectively. $\widetilde{\mu}$ and $\widetilde{\xi}$ are constant viscous coefficients satisfying $\widetilde{\mu}>0$ and $2\widetilde{\mu}+d\widetilde{\xi}\ge 0$. $\widetilde{P}(\widetilde{\rho})$ is the pressure-density function and here we consider the isentropic case

$$\widetilde{P}(\widetilde{\rho}) = a\widetilde{\rho}^{\,\gamma}, \quad a>0, \gamma>1. \tag{1.3}$$

The purpose of this paper is to derive the incompressible Navier-Stokes equations from the compressible one (1.1)–(1.2). For this end, we first give some formal analysis. We introduce

Manuscript received January 23, 2008. Revised July 2, 2008. Published online January 9, 2009.
*Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100080, China. E-mail: hsiaol@amss.ac.cn
**Institute of Applied Physics and Computational Mathematics, PO Box 8009-28, Beijing 100088, China. E-mail: qiangchang_ju@yahoo.com
***Corresponding author. Department of Mathematics, Nanjing University, Nanjing 210093, China. E-mail: fli@nju.edu.cn
****Project supported by the National Natural Science Foundation of China (Nos. 10431060, 10701011, 10501047) and the Nanjing University Talent Development Foundation.

the incompressible limit scaling as follows:

$$\rho^\epsilon(x,t)=\widetilde{\rho}\Big(x,\frac{t}{\epsilon}\Big),\quad u^\epsilon(x,t)=\frac{1}{\epsilon}\widetilde{u}\Big(x,\frac{t}{\epsilon}\Big),$$

and we assume that the viscosity coefficients $\widetilde{\mu}$ and $\widetilde{\xi}$ are small constants and scale like

$$\widetilde{\mu}=\epsilon\mu,\quad \widetilde{\xi}=\epsilon\xi,$$

where $\epsilon\in(0,1)$ is a small parameter. For simplicity we assume here that μ and ξ are positive constants independent of ϵ. The case when μ and ξ are dependent on ϵ and converge to some positive constants as ϵ goes to 0 can be treated similarly.

With the preceding scalings and using the expression of the pressure (1.3), the compressible Navier-Stokes equations (1.1)–(1.2) take the form

$$\partial_t\rho^\epsilon+\operatorname{div}(\rho^\epsilon u^\epsilon)=0, \tag{1.4}$$

$$\partial_t(\rho^\epsilon u^\epsilon)+\operatorname{div}(\rho^\epsilon u^\epsilon\otimes u^\epsilon)+\frac{a\nabla(\rho^\epsilon)^\gamma}{\epsilon^2}=\mu\Delta u^\epsilon+(\mu+\xi)\nabla\operatorname{div}u^\epsilon. \tag{1.5}$$

Replacing ϵ by $\sqrt{a\gamma}\,\epsilon$, we can always assume $a=\frac{1}{\gamma}$. Formally if we let $\epsilon\to0$, we obtain from the momentum equation (1.5) that ρ^ϵ converges to some function $\overline{\rho}(t)\geq0$. If we further assume that the initial data ρ_0^ϵ is of order $1+O(\epsilon)$ (this can be guaranteed by the initial energy bound (1.11) below), then we can expect that $\overline{\rho}=1$. Thus the continuity equation (1.4) gives $\operatorname{div}u=0$. Therefore, we obtain the following incompressible Navier-Stokes equations

$$\nabla\cdot u=0,\quad x\in\mathbb{R}^d,\ t>0, \tag{1.6}$$

$$\partial_t u+(u\cdot\nabla)u-\mu\Delta u+\nabla p=0,\quad x\in\mathbb{R}^d,\ t>0. \tag{1.7}$$

In the present paper, we will give a rigorous proof that the weak solution to compressible Navier-Stokes equations (1.4)–(1.5) with general initial data converges to, as the small parameter ϵ goes to 0, the strong solution of the incompressible Navier-Stokes equations (1.6)–(1.7).

Before stating our results, we give some notations used in the sequel. We denote the space $L_2^q(\mathbb{R}^d)$ by

$$L_2^q(\mathbb{R}^d)=\{f\in L_{\rm loc}(\mathbb{R}^d): f1_{\{|f|\geq1\}}\in L^q, f1_{\{|f|\leq1\}}\in L^2\}.$$

C or C_T denotes various positive constants independent of ϵ and C_T may depend on T. For convenience we denote $\int f=\int_{\mathbb{R}^d}f\mathrm{d}x$ and $W^{s,r}$ for the standard Sobolev space. For any vector field v, we denote by Pv and Qv respectively the divergence-free part of v and the gradient part of v, namely, $Qv=\nabla\Delta^{-1}(\operatorname{div}v)$ and $Pv=v-Qv$.

Firstly, we recall the local existence of strong solution to the incompressible Navier-Stokes equations (1.6)–(1.7).

Proposition 1.1 (see [8, 13]) *Assume that the initial data $u(x,t=0)=u_0(x)$ satisfies $u_0\in H^s$, $s>1+\frac{d}{2}$, $\nabla\cdot u_0=0$. Then there exist a $T^*\in(0,\infty)$ ($T^*=+\infty$ if $d=2$) and a unique solution $u\in L^\infty_{\rm loc}([0,T^*),H^s)$ to the incompressible Navier-Stokes equations* (1.6)–(1.7) *satisfying, for any $0<T<T^*$, $\nabla\cdot u=0$ and*

$$\sup_{0\leq t\leq T}(\|u\|_{H^s}+\|\partial_t u\|_{H^{s-1}}+\|\nabla p\|_{H^s}+\|\partial_t\nabla p\|_{H^{s-1}})\leq C(T) \tag{1.8}$$

for some positive constant $C(T)$, depending only upon T.

We prescribe the initial conditions for (1.4)–(1.5)

$$\rho^\epsilon|_{t=0} = \rho_0^\epsilon(x), \quad \rho^\epsilon u^\epsilon|_{t=0} = \rho_0^\epsilon(x) u_0^\epsilon(x) \equiv m_0^\epsilon(x), \tag{1.9}$$

and assume that

$$\rho_0^\epsilon \geq 0, \quad \rho_0^\epsilon - 1 \in L_2^\gamma, \quad \rho_0^\epsilon |u_0^\epsilon|^2 \in L^1, \quad m_0^\epsilon = 0 \quad \text{for a.e. } \rho_0^\epsilon = 0. \tag{1.10}$$

The initial conditions also satisfy the following uniform bound

$$\int \Big[\frac{1}{2}\rho_0^\epsilon |u_0^\epsilon|^2 + \frac{a}{\epsilon^2(\gamma-1)}((\rho_0^\epsilon)^\gamma - 1 - \gamma(\rho_0^\epsilon - 1))\Big] \mathrm{d}x \leq C. \tag{1.11}$$

We also need to impose the following conditions on the solution $(\rho^\epsilon, u^\epsilon)$ at infinity

$$\rho^\epsilon \to 1 \quad \text{as } |x| \to +\infty, \quad u^\epsilon \to 0 \quad \text{as } |x| \to +\infty.$$

Under the above assumptions, we have the following result on the global existence of weak solutions to the compressible Navier-Stokes equations (1.4)–(1.5).

Proposition 1.2 (see [4, 9, 10]) *Let $\gamma > \frac{d}{2}$. Suppose that the initial data $(\rho_0^\epsilon, m_0^\epsilon)$ satisfy the assumptions* (1.10)–(1.11). *Then the compressible Navier-Stokes equations* (1.4)–(1.5) *with initial condition* (1.9) *enjoy at least one global weak solution $(\rho^\epsilon, u^\epsilon)$ satisfying*

(1) $\rho^\epsilon - 1 \in L^\infty(0,\infty; L_2^\gamma) \cap C([0,\infty), L_2^r)$ *for all* $1 \leq r < \gamma$, $u^\epsilon \in (0,T; H^1)$ *for all* $T \in (0,\infty)$, $\rho^\epsilon |u^\epsilon|^2 \in L^\infty(0,\infty; L^1)$;

(2) *the energy inequality*

$$\mathcal{E}^\epsilon(t) + \mu \int_0^t\!\!\int |\nabla u^\epsilon(x,s)|^2 + (\mu + \xi) \int_0^t\!\!\int |\mathrm{div}\, u^\epsilon(x,s)|^2 \leq \mathcal{E}^\epsilon(0) \tag{1.12}$$

holds with the finite total energy

$$\mathcal{E}^\epsilon(t) \equiv \int \Big[\frac{1}{2}\rho^\epsilon |u^\epsilon|^2 + \frac{a}{\epsilon^2(\gamma-1)}((\rho^\epsilon)^\gamma - 1 - \gamma(\rho^\epsilon - 1))\Big] \mathrm{d}x; \tag{1.13}$$

(3) *the continuity equation is satisfied in the sense of renormalized solutions, i.e.,*

$$\partial_t b(\rho^\epsilon) + \mathrm{div}\,(b(\rho^\epsilon) u^\epsilon) + (b'(\rho^\epsilon)\rho^\epsilon - b(\rho^\epsilon))\mathrm{div}\, u^\epsilon = 0 \tag{1.14}$$

for any $b \in C^1(\mathbb{R})$ such that $b'(z)$ is a constant for z large enough;

(4) *the equations* (1.4)–(1.5) *hold in $\mathcal{D}'((0,\infty) \times \mathbb{R}^d)$.*

The initial energy bound (1.11) implies that ρ_0^ϵ is of order $1 + O(\epsilon)$. We write $\rho^\epsilon = 1 + \epsilon \varphi^\epsilon$ and denote

$$\Pi^\epsilon(x,t) = \frac{1}{\epsilon}\sqrt{\frac{2a}{\gamma-1}((\rho^\epsilon)^\gamma - 1 - \gamma(\rho^\epsilon - 1))}\,.$$

We use the above approximation because we can not obtain any bound for φ^ϵ in $L^\infty(0,T; L^2)$ directly if $\gamma < 2$. The main results of this paper can be stated as follows.

Theorem 1.1 *Suppose that the conditions in Proposition* 1.2 *hold. Moreover, we assume that* $\sqrt{\rho_0^\epsilon}\,u_0^\epsilon$ *converges strongly in* L^2 *to* $\widetilde{u}_0$*, and* $\Pi^\epsilon(x,t=0)=\Pi_0^\epsilon$ *converges strongly in* L^2 *to some* φ_0*. Let* u *be the smooth solution to the incompressible Navier-Stokes equations defined on* $[0,T^*)$ *with* $u_0=P\widetilde{u}_0$*. Then, for any* $0<T<T^*$*, the global weak solution* $(\rho^\epsilon,u^\epsilon)$ *of the compressible Navier-Stokes equations* (1.4)–(1.5) *established in Proposition* 1.2 *satisfies*

(1) ρ^ϵ *converges strongly to* 1 *in* $C([0,T];L^\gamma(\mathbb{R}^d))$;

(2) ∇u^ϵ *converges strongly to* ∇u *in* $L^2(0,T;L^2(\mathbb{R}^d))$;

(3) $P(\sqrt{\rho^\epsilon}\,u^\epsilon)$ *converges strongly to* u *in* $L^\infty(0,T;L^2(\mathbb{R}^d))$;

(4) $\sqrt{\rho^\epsilon}\,u^\epsilon$ *converges strongly to* u *in* $L^r(0,T;L^2_{\mathrm{loc}}(\mathbb{R}^d))$ *for all* $1\le r<+\infty$.

Remark 1.1 The assumption that Π_0^ϵ converges strongly in L^2 to some φ_0 in fact implies that φ_0^ϵ converges strongly to φ_0 in L_2^γ.

The proof of above result is based on the modulated energy method, motivated by Y. Brenier [1], Strichartz's estimate of linear wave equation and the weak convergence method. The idea of modulated energy method is to modulate the energy of the equations by test functions which are solutions of the limiting equations. Compared with the results in [6, 7, 11, 15], where the limiting equations are incompressible Euler equations and their analyses depend on the smallness of μ and ξ, here we must deal with the diffusion term very carefully. Our new ingredient of this paper is that we will modulate the partial energy instead of the total energy as usual case. The dissipative effect of viscous term is also exploited carefully. On the other hand, because there is an initial layer, oscillation appears and propagates with the solution. We will use Strichartz's estimate of linear wave equation to deal with such oscillation. Finally, the weak convergence method and refined energy analysis are used to obtain the desired convergence results.

There are many results on the low Mach number limit (incompressible limit) of compressible Navier-Stokes equations, for which we just mention a few. The smooth solution case was investigated by D. Hoff [5]. P. L. Lions and N. Masmoudi [9] studied the limit of the weak solution of compressible Navier-Stokes equations to the weak solution of incompressible Navier-Stokes equations in the whole space and bounded domain cases. N. Masmoudi [11] considered the incompressible, inviscid convergence of weak solution of compressible Navier-Stokes equations to the strong solution of the Euler equations in the whole space and the torus. See also B. Desjardins and E. Grenier [3] for the weak solution in the whole space case and R. Danchin [2] in the critical space case. The interested reader can refer to the survey paper [12] for more relative results.

2 Proof of Theorem 1.1

We shall prove our convergence results by the combining of the modulated energy method, Strichartz's estimate of linear wave equation and the weak convergence method.

We divide the proof into four steps.

Step 1 Basic energy estimates and compact arguments

By the assumptions on the initial data we obtain, from the energy inequality (1.12), that the total energy $\mathcal{E}^\epsilon(t)$ has a uniform upper bound for a.e. $t \in [0,T]$, $T > 0$. This uniform bound of $\mathcal{E}^\epsilon(t)$ implies that $\rho^\epsilon|u^\epsilon|^2$ and $\frac{1}{\epsilon^2}((\rho^\epsilon)^\gamma - 1 - \gamma(\rho^\epsilon - 1))$ are bounded in $L^\infty(0,T;L^1)$ and ∇u^ϵ is bounded in $L^2(0,T;L^2)$. From these facts we can obtain that

$$\rho^\epsilon \to 1 \quad \text{strongly in } C([0,T],L^\gamma), \tag{2.1}$$

and u^ϵ is bounded in $L^2(0,T;L^2)$ by following the same arguments as that in [3, 9], and we will not repeat it here.

Notice the fact that $\rho^\epsilon|u^\epsilon|^2$ is bounded in $L^\infty(0,T;L^1)$ implies the following convergence (up to the extraction of a subsequence ϵ_n):

$$\sqrt{\rho^\epsilon}\, u^\epsilon \quad \text{converges weakly-} * \text{ to some } J \text{ in } L^\infty(0,T;L^2(\mathbb{R}^d)).$$

Our main task in this section is to show that $J = u$ in some sense, where u is the strong solution to the incompressible Navier-Stokes equations (1.6)–(1.7).

Step 2 Description and cancelation of oscillations

In order to describe the oscillation involved by the initial data, we use some ideas introduced in [11, 14] and the dispersion property of the linear wave equation (see [3, 11]).

We introduce the following group defined by $\mathscr{L}(\tau) = \mathrm{e}^{\tau L}$, $\tau \in \mathbb{R}$, where L is the operator defined on $\mathcal{D} \times (\mathcal{D}')^d$ by

$$L \begin{pmatrix} \phi \\ v \end{pmatrix} = \begin{pmatrix} -\mathrm{div}\, v \\ -\nabla\phi \end{pmatrix}.$$

Then, it is easy to check that $\mathrm{e}^{\tau L}$ is an isometry on each $H^s \times (H^s)^d$ for all $s \in \mathbb{R}$ and for all $\tau \in \mathbb{R}$. Denote

$$\begin{pmatrix} \overline{\phi}(\tau) \\ \overline{v}(\tau) \end{pmatrix} = \mathrm{e}^{\tau L} \begin{pmatrix} \phi \\ v \end{pmatrix}.$$

Then we have

$$\frac{\partial\overline{\phi}}{\partial\tau} = -\mathrm{div}\overline{v}, \quad \frac{\partial\overline{v}}{\partial\tau} = -\nabla\overline{\phi}.$$

Thus, $\frac{\partial^2\overline{\phi}}{\partial\tau^2} - \Delta\overline{\phi} = 0$.

Let $(\phi^\epsilon, g^\epsilon = \nabla q^\epsilon)$ be the solution of the following system

$$\frac{\partial\phi^\epsilon}{\partial t} = -\frac{1}{\epsilon}\mathrm{div}\, g^\epsilon, \quad \phi^\epsilon|_{t=0} = \Pi_0^\epsilon, \tag{2.2}$$

$$\frac{\partial g^\epsilon}{\partial t} = -\frac{1}{\epsilon}\nabla\phi^\epsilon, \quad g^\epsilon|_{t=0} = Q(\sqrt{\rho_0^\epsilon}\, u_0^\epsilon). \tag{2.3}$$

Our main idea is to use ϕ^ϵ and g^ϵ as test functions and plug them into the total energy $\mathcal{E}^\epsilon(t)$ to cancel the oscillation. In order to make the computation go smoothly, we introduce the following regularization for the initial data, $\Pi_0^{\epsilon,\delta} = \Pi_0^\epsilon * \chi^\delta$, $Q^\delta(\sqrt{\rho_0^\epsilon}\, u_0^\epsilon) = Q(\sqrt{\rho_0^\epsilon}\, u_0^\epsilon) * \chi^\delta$, and denote by $(\phi^{\epsilon,\delta}, g^{\epsilon,\delta} = \nabla q^{\epsilon,\delta})$ the corresponding solution to the equations (2.2)–(2.3) with initial data $\phi^{\epsilon,\delta}|_{t=0} = \Pi_0^{\epsilon,\delta}$, $g^{\epsilon,\delta}|_{t=0} = Q^\delta(\sqrt{\rho_0^\epsilon}\, u_0^\epsilon)$. Here $\chi \in C_0^\infty(\mathbb{R}^d)$ is the Friedrich's mollifier, i.e., $\int \chi = 1$ and $\chi^\delta(x) = \frac{1}{\delta^d}\chi(\frac{x}{\delta})$. Since the equations (2.2)–(2.3) are linear, it is easy to verify that $\phi^{\epsilon,\delta} = \phi^\epsilon * \chi^\delta$, $g^{\epsilon,\delta} = g^\epsilon * \chi^\delta$.

Using the Strichartz estimate of linear wave equation (see [3, 11]), we have

$$\left\| \begin{pmatrix} \phi^{\epsilon,\delta} \\ \nabla q^{\epsilon,\delta} \end{pmatrix} \right\|_{L^l(\mathbb{R},W^{s,q}(\mathbb{R}^d))} \le C\epsilon^{\frac{1}{l}} \left\| \begin{pmatrix} \Pi_0^{\epsilon,\delta} \\ Q^\delta(\sqrt{\rho_0^\epsilon}\,u_0^\epsilon) \end{pmatrix} \right\|_{H^{s+\sigma}(\mathbb{R}^d)} \tag{2.4}$$

for all $l, q > 2$ and $\sigma > 0$ such that

$$\frac{2}{q} = (d-1)\Big(\frac{1}{2} - \frac{1}{l}\Big), \quad \sigma = \frac{d+1}{d-1}.$$

The estimate (2.4) implies that for all fixed δ and for all $s \in \mathbb{R}$, we have

$$\phi^{\epsilon,\delta}, g^{\epsilon,\delta} \to 0, \quad \text{in } L^l(\mathbb{R}, W^{s,q}(\mathbb{R}^d)), \text{ as } \epsilon \to 0. \tag{2.5}$$

Step 3 The modulated energy functional and uniform estimate

We first recall the energy inequality to the compressible Navier-Stokes equations (1.4)–(1.5) for almost all t,

$$\begin{aligned} &\frac{1}{2}\int [\rho^\epsilon(t)|u^\epsilon|^2(t) + (\Pi^\epsilon(x,t))^2] + \mu \int_0^t\!\!\int |\nabla u^\epsilon(x,s)|^2 + (\mu+\xi)\int_0^t\!\!\int |\operatorname{div} u^\epsilon(x,s)|^2 \\ \le\ &\frac{1}{2}\int [\rho_0^\epsilon |u_0^\epsilon|^2 + (\Pi_0^\epsilon)^2]. \end{aligned} \tag{2.6}$$

The conservation of energy for the incompressible Navier-Stokes equations (1.6)–(1.7) reads

$$\frac{1}{2}\int |u|^2(t) + \mu\int_0^t\!\!\int |\nabla u|^2 = \frac{1}{2}\int |u_0|^2. \tag{2.7}$$

From system (2.2)–(2.3), we obtain

$$\frac{1}{2}\int [|\phi^{\epsilon,\delta}|^2 + |g^{\epsilon,\delta}|^2](t) = \frac{1}{2}\int [|\phi^{\epsilon,\delta}|^2 + |g^{\epsilon,\delta}|^2](0) \tag{2.8}$$

for all t.

Using $\phi^{\epsilon,\delta}$ as a test function and noticing $\rho^\epsilon = 1 + \epsilon\varphi^\epsilon$, we obtain the following weak formulation of the continuity equation (1.4):

$$\int \phi^{\epsilon,\delta}(t)\varphi^\epsilon(t) + \frac{1}{\epsilon}\int_0^t\!\!\int [\operatorname{div}(\nabla q^{\epsilon,\delta})\varphi^\epsilon - \rho^\epsilon u^\epsilon \cdot \nabla\phi^{\epsilon,\delta}] = \int \phi^{\epsilon,\delta}(0)\varphi_0^\epsilon. \tag{2.9}$$

Similarly, using u and $g^{\epsilon,\delta} = \nabla q^{\epsilon,\delta}$ as a test function to the momentum equation (1.5) respectively, we get

$$\begin{aligned} &\int (\rho^\epsilon u^\epsilon \cdot u)(t) + \int_0^t\!\!\int [\rho^\epsilon u^\epsilon \cdot (u\cdot\nabla u - \mu\Delta u + \nabla p)] \\ &- \int_0^t\!\!\int (\rho^\epsilon u^\epsilon \otimes u^\epsilon)\cdot\nabla u + \mu\int_0^t\!\!\int \nabla u^\epsilon \cdot \nabla u \\ =\ &\int \rho_0^\epsilon u_0^\epsilon \cdot u_0 \end{aligned} \tag{2.10}$$

and

$$\begin{aligned}
&\int(\rho^\epsilon u^\epsilon\cdot\nabla q^{\epsilon,\delta})(t)+\int_0^t\!\!\int\rho^\epsilon u^\epsilon\Big(\frac{1}{\epsilon}\nabla\phi^{\epsilon,\delta}\Big)-\int_0^t\!\!\int(\rho^\epsilon u^\epsilon\otimes u^\epsilon)\cdot\nabla g^{\epsilon,\delta}\\
&+\int_0^t\!\!\int[\mu\nabla u^\epsilon\cdot\nabla g^{\epsilon,\delta}+(\mu+\xi)\mathrm{div}\,u^\epsilon\mathrm{div}\,g^{\epsilon,\delta}]-\int_0^t\!\!\int\Big(\frac{1}{\epsilon}\varphi^\epsilon+\frac{\gamma-1}{2}(\Pi^\epsilon)^2\Big)\mathrm{div}\,g^{\epsilon,\delta}\\
=&\int\rho_0^\epsilon u_0^\epsilon\cdot g^{\epsilon,\delta}(0).
\end{aligned}\tag{2.11}$$

Summing up (2.6)–(2.8), plugging (2.9)–(2.11) into the result, and using the fact $\mathrm{div}\,u=0$, we can deduce the following inequality by the straightforward computations:

$$\begin{aligned}
&\frac{1}{2}\int\{|\sqrt{\rho^\epsilon}\,u^\epsilon-u-g^{\epsilon,\delta}|^2(t)+(\Pi^\epsilon-\phi^{\epsilon,\delta})^2(t)\}\\
&+\frac{\mu}{2}\int_0^t\!\!\int|\nabla u^\epsilon-\nabla u|^2+\frac{\mu}{2}\int_0^t\!\!\int|\nabla u^\epsilon|^2\\
&+\frac{\mu}{2}\int_0^t\!\!\int|\nabla u|^2+(\mu+\xi)\int_0^t\!\!\int|\mathrm{div}\,u^\epsilon(x,t)|^2\\
\le&\int[(\sqrt{\rho^\epsilon}-1)\sqrt{\rho^\epsilon}\,u^\epsilon\cdot(u+g^{\epsilon,\delta})](t)-\int[(\Pi^\epsilon-\varphi^\epsilon)\phi^{\epsilon,\delta}](t)\\
&+\int_0^t\!\!\int[\rho^\epsilon u^\epsilon\cdot(u\cdot\nabla u-\mu\Delta u+\nabla p)]\\
&-\int_0^t\!\!\int(\rho^\epsilon u^\epsilon\otimes u^\epsilon)\cdot\nabla u-\int_0^t\!\!\int(\rho^\epsilon u^\epsilon\otimes u^\epsilon)\cdot\nabla g^{\epsilon,\delta}\\
&+\int_0^t\!\!\int[\mu\nabla u^\epsilon\cdot\nabla g^{\epsilon,\delta}+(\mu+\xi)\mathrm{div}\,u^\epsilon\mathrm{div}\,g^{\epsilon,\delta}]\\
&-\frac{\gamma-1}{2}\int_0^t\!\!\int(\Pi^\epsilon)^2\mathrm{div}\,g^{\epsilon,\delta}-\int[(\sqrt{\rho^\epsilon}-1)\sqrt{\rho^\epsilon}\,u^\epsilon\cdot(u+g^{\epsilon,\delta})](0)\\
&+\int(\Pi_0^\epsilon-\varphi_0^\epsilon)\phi^{\epsilon,\delta}(0)+\frac{1}{2}\int\{|\sqrt{\rho^\epsilon}\,u^\epsilon-u-g^{\epsilon,\delta}|^2(0)+(\Pi_0^\epsilon-\phi^{\epsilon,\delta}(0))^2\}.
\end{aligned}\tag{2.12}$$

We first deal with the right-hand side of the inequality (2.12). Denoting $z^{\epsilon,\delta}=\sqrt{\rho^\epsilon}\,u^\epsilon-u-g^{\epsilon,\delta}$, we have

$$\begin{aligned}
&\int_0^t\!\!\int[\rho^\epsilon u^\epsilon\cdot(u\cdot\nabla u-\mu\Delta u+\nabla p)]-\int_0^t\!\!\int(\rho^\epsilon u^\epsilon\otimes u^\epsilon)\cdot\nabla u\\
=&-\int_0^t\!\!\int(z^{\epsilon,\delta}\otimes z^{\epsilon,\delta})\cdot\nabla u+\mu\int_0^t\!\!\int\nabla u^\epsilon\cdot\nabla u\\
&+\int_0^t\!\!\int(\rho^\epsilon-\sqrt{\rho^\epsilon}\,)u^\epsilon\cdot(u\cdot\nabla u)-\int_0^t\!\!\int g^{\epsilon,\delta}\cdot\nabla u\,z^{\epsilon,\delta}\\
&+\int_0^t\!\!\int(\sqrt{\rho^\epsilon}\,u^\epsilon-u)\cdot\nabla u\cdot z^{\epsilon,\delta}-\mu\int_0^t\!\!\int(\rho^\epsilon-1)u^\epsilon\Delta u\\
&+\int_0^t\!\!\int\rho^\epsilon u^\epsilon\cdot\nabla p-\int_0^t\!\!\int(\sqrt{\rho^\epsilon}\,u^\epsilon-u)\cdot\nabla\Big(\frac{u^2}{2}\Big).
\end{aligned}\tag{2.13}$$

Plugging (2.13) into inequality (2.12), we obtain

$$\|z^{\epsilon,\delta}(t)\|_{L^2}^2+\|\Pi^\epsilon(x,t)-\phi^{\epsilon,\delta}(t)\|_{L^2}^2+\mu\int_0^t\!\!\int|\nabla u^\epsilon-\nabla u|^2+2(\mu+\xi)\int_0^t\!\!\int|\mathrm{div}\,u^\epsilon(x,t)|^2$$

$$\le \|z^{\epsilon,\delta}(0)\|_{L^2}^2 + \|\Pi_0^\epsilon - \phi^{\epsilon,\delta}(0)\|_{L^2}^2 + 2C\int_0^t \|z^{\epsilon,\delta}(s)\|_{L^2}^2 \|\nabla u(s)\|_{L^\infty} \mathrm{d}s$$
$$+2R_1^{\epsilon,\delta}(t) + 2R_2^\epsilon(t) + 2R_3^\epsilon(t), \tag{2.14}$$

where

$$\begin{aligned}
R_1^{\epsilon,\delta}(t) = & \int [(\sqrt{\rho^\epsilon}-1)\sqrt{\rho^\epsilon}\, u^\epsilon \cdot (u+g^{\epsilon,\delta})](t) - \int [(\Pi^\epsilon - \varphi^\epsilon)\phi^{\epsilon,\delta}](t) \\
& - \int_0^t\!\!\int (\rho^\epsilon u^\epsilon \otimes u^\epsilon)\cdot \nabla g^{\epsilon,\delta} - \frac{\gamma-1}{2}\int_0^t\!\!\int (\Pi^\epsilon)^2 \mathrm{div}\, g^{\epsilon,\delta} \\
& - \int_0^t\!\!\int [\mu \nabla u^\epsilon \cdot \nabla g^{\epsilon,\delta} + (\mu+\xi)\mathrm{div}\, u^\epsilon \mathrm{div}\, g^{\epsilon,\delta}] \\
& - \int [(\sqrt{\rho^\epsilon}-1)\sqrt{\rho^\epsilon}\, u^\epsilon \cdot (u+g^{\epsilon,\delta})](0) + \int [(\Pi^\epsilon - \varphi^\epsilon)\phi^{\epsilon,\delta}](0) \\
& + \int_0^t\!\!\int (\rho^\epsilon - \sqrt{\rho^\epsilon}\,) u^\epsilon \cdot (u\cdot\nabla u) \\
& - \int_0^t\!\!\int g^{\epsilon,\delta}\cdot \nabla u\, z^{\epsilon,\delta} + \int_0^t\!\!\int (\sqrt{\rho^\epsilon}\, u^\epsilon - u)\cdot \nabla u \cdot z^{\epsilon,\delta}, \\
R_2^\epsilon(t) = & -\mu \int_0^t\!\!\int (\rho^\epsilon - 1) u^\epsilon \Delta u + \int_0^t\!\!\int \rho^\epsilon u^\epsilon \cdot \nabla p, \\
R_3^\epsilon(t) = & -\int_0^t\!\!\int (\sqrt{\rho^\epsilon}\, u^\epsilon - u)\cdot \nabla\Big(\frac{u^2}{2}\Big).
\end{aligned}$$

Step 4 Convergence of the modulated energy functional

To show the convergence of the modulated energy functional and to finish our proof, we need to deal with the reminders $R_1^{\epsilon,\delta}(t)$, $R_2^\epsilon(t)$ and $R_3^\epsilon(t)$. First, by using inequality (2.5), the assumptions on the initial data, the strong convergence of ρ^ϵ and the estimate on $\sqrt{\rho^\epsilon}\, u^\epsilon$, it is easy to know that $R_1^{\epsilon,\delta}(t)$ converges to 0 for almost all t, uniformly in t when ϵ goes to 0.

Next, the term $R_2^\epsilon(t)$ can be treated as follows:

$$\begin{aligned}
R_2^\epsilon(t) &= -\mu \int_0^t\!\!\int \epsilon\varphi^\epsilon u^\epsilon \Delta u + \int_0^t\!\!\int \rho^\epsilon u^\epsilon \cdot \nabla p \\
&= -\epsilon\mu \int_0^t\!\!\int \varphi^\epsilon u^\epsilon \Delta u + \epsilon \int [(\varphi^\epsilon p)(t) - (\varphi^\epsilon p)(0)] - \epsilon \int_0^t\!\!\int \varphi^\epsilon \partial_t p \\
&\le C_T \epsilon,
\end{aligned}$$

where we have used the inequality (1.8) and (2.1).

From the fact $\mathrm{div}\, u = 0$, the strong convergence of ρ^ϵ, the estimate on u^ϵ and the inequality (1.8), the term $R_3^\epsilon(t)$ enjoys the following estimate:

$$\begin{aligned}
R_3^\epsilon(t) &= -\int_0^t\!\!\int (\sqrt{\rho^\epsilon}\, u^\epsilon - u)\cdot \nabla\Big(\frac{u^2}{2}\Big) \\
&= \int_0^t\!\!\int \sqrt{\rho^\epsilon}\,(\sqrt{\rho^\epsilon}-1) u^\epsilon \cdot \nabla\Big(\frac{u^2}{2}\Big) - \int_0^t\!\!\int \rho^\epsilon u^\epsilon \cdot \nabla\Big(\frac{u^2}{2}\Big)
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^t\!\!\int \sqrt{\rho^\epsilon}\,(\sqrt{\rho^\epsilon}-1)u^\epsilon\cdot\nabla\Big(\frac{u^2}{2}\Big) + \epsilon\int_0^t\!\!\int \varphi^\epsilon\partial_t\Big(\frac{u^2}{2}\Big)\\
&\quad - \epsilon\int\Big[\Big(\varphi^\epsilon\Big(\frac{u^2}{2}\Big)\Big)(t) - \Big(\varphi^\epsilon\Big(\frac{u^2}{2}\Big)\Big)(0)\Big]\\
&\le C_T\epsilon.
\end{aligned}
$$

Thus, by Gronwall inequality, we deduce that, for almost all $t\in(0,T)$,

$$
\begin{aligned}
&\|z^{\epsilon,\delta}(t)\|_{L^2}^2 + \|\Pi^\epsilon(t)-\phi^{\epsilon,\delta}(t)\|_{L^2}^2\\
&\le \overline{C}\Big[\|z^{\epsilon,\delta}(0)\|_{L^2}^2 + \|\Pi_0^\epsilon - \phi^{\epsilon,\delta}(0)\|_{L^2}^2 + C_T\epsilon + \sup_{0\le s\le t} R_1^{\epsilon,\delta}(t)\Big],
\end{aligned}
\tag{2.15}
$$

where $\overline{C}=\exp\{C\int_0^t\|\nabla u\|_{L^\infty}^2\mathrm{d}s\}$. Then, letting ϵ go to 0, we obtain

$$
\begin{aligned}
\|J-u\|_{L^\infty(0,T;L^2)}^2 &\le C\overline{\lim_{\epsilon\to0}}[\|z^{\epsilon,\delta}(t)\|_{L^\infty(0,T;L^2)} + \|\Pi^\epsilon(t)-\phi^{\epsilon,\delta}(t)\|_{L^\infty(0,T;L^2)}^2]\\
&\le C\overline{C}[\|J_0-u_0-Q(J_0)*\chi_\delta\|_{L^2}^2 + \|\varphi_0-\varphi_0*\chi_\delta\|_{L^2}^2].
\end{aligned}
\tag{2.16}
$$

Hence we deduce $J=u$ in $L^\infty(0,T;L^2)$ by letting δ go to 0. Moreover, inequalities (2.14)–(2.16) imply directly that ∇u^ϵ converges strongly to ∇u in $L^2(0,T;L^2(\mathbb{R}^d))$ as ϵ goes to 0.

Noticing

$$
\overline{\lim_{\epsilon\to0}}\|P(\sqrt{\rho^\epsilon}\,u^\epsilon)-u\|_{L^\infty(0,T;L^2)} \le C\overline{C}\lim_{\delta\to0}[\|J_0-u_0-Q(J_0)*\chi_\delta\|_{L^2}^2 + \|\varphi_0-\varphi_0*\chi_\delta\|_{L^2}^2]=0,
$$

we have the uniform convergence (in t) of $P(\sqrt{\rho^\epsilon}\,u^\epsilon)$ to u in $L^2(\mathbb{R}^d)$.

Finally, we show the local strong convergence of $\sqrt{\rho^\epsilon}\,u^\epsilon$ to u in $L^r(0,T;L^2(\Omega))$ for all $1\le r<+\infty$ on any bounded domain $\Omega\subset\mathbb{R}^d$. In fact, for all t, we have

$$
\begin{aligned}
\|\sqrt{\rho^\epsilon}\,u^\epsilon-u\|_{L^2(\Omega)} &\le \|\sqrt{\rho^\epsilon}\,u^\epsilon-u-g^{\epsilon,\delta}\|_{L^2(\Omega)} + \|g^{\epsilon,\delta}\|_{L^2(\Omega)}\\
&\le \|\sqrt{\rho^\epsilon}\,u^\epsilon-u-g^{\epsilon,\delta}\|_{L^2(\Omega)} + \|g^{\epsilon,\delta}\|_{L^q(\Omega)}
\end{aligned}
$$

for any $q>2$. Using the estimate (2.5), we can take the limit on ϵ and then on δ as above to obtain that $\sqrt{\rho^\epsilon}\,u^\epsilon$ converges to u in $L^r(0,T;L^2(\Omega))$. Thus we complete our proof.

References

[1] Brenier, Y., Convergence of the Vlasov-Poisson system to the incompressible Euler equations, *Comm. Part. Diff. Eqs.*, **25**(3–4), 2000, 737–754.

[2] Danchin, R., Zero Mach number limit in critical spaces for compressible Navier-Stokes equations, *Ann. Sci. École Norm. Sup.* (4), **35**(1), 2002, 27–75.

[3] Desjardins, B. and Grenier, E., Low Mach number limit of viscous compressible flows in the whole space, *R. Soc. Lond. Proc., Ser. A, Math. Phys. Eng. Sci.*, **455**(1986), 1999, 2271–2279.

[4] Feireisl, E., Novotný, A. and Petzeltová, H., On the existence of globally defined weak solutions to the Navier-Stokes equations, *J. Math. Fluid Mech.*, **3**(4), 2001, 358–392.

[5] Hoff, D., The zero-Mach limit of compressible flows, *Comm. Math. Phys.*, **192**(3), 1998, 543–554.

[6] Hsiao, L., Li, F. C. and Wang, S., Convergence of the Vlasov-Poisson-Fokker-Planck system to the incompressible Euler equations, *Sci. in Chin., Ser. A*, **49**(2), 2006, 255–266.

[7] Hsiao, L., Li, F. C. and Wang, S., Coupled quasineutral and inviscid limit of the Vlasov-Poisson-Fokker-Planck system, *Comm. Pure Appl. Anal.*, **7**(3), 2008, 579–589.

[8] Kato, T., Nonstationary flows of viscous and ideal fluids in $\mathbf{R}^3$, *J. Func. Anal.*, **9**(3), 1972, 296–305.

[9] Lions, P. L. and Masmoudi, N., Incompressible limit for a viscous compressible fluid, *J. Math. Pures Appl.* (99), **77**(6), 1998, 585–627.

[10] Lions, P. L., Mathematical Topics in Fluid Mechanics, Compressible Models, Oxford Lecture Series in Mathematics and its Applications 10, Vol. 2, The Clarendon Press and Oxford University Press, New York, 1998.

[11] Masmoudi, N., Incompressible, inviscid limit of the compressible Navier-Stokes system, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **18**(2), 2001, 199–224.

[12] Masmoudi, N., Examples of singular limits in hydrodynamics, Handbook of Differential Equations, Evolutionary Equations, Vol. 3, North-Holland, Amsterdam, 2007, 195–275.

[13] McGrath, F. J., Nonstationary plane flow of viscous and ideal fluids, *Arch. Rat. Mech. Anal.*, **27**, 1967, 329–348.

[14] Schochet, S., Fast singular limits of hyperbolic PDEs, *J. Diff. Eqs.*, **114**(2), 1994, 476–512.

[15] Wang, S. and Jiang, S., The convergence of the Navier-Stokes-Poisson system to the incompressible Euler equations, *Comm. Part. Diff. Eqs.*, **31**(4–6), 2006, 571–591.