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Ideals in the Roe Algebras of Discrete Metric Spaces with Coefficients in $\mathcal{B}(H)$***

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Abstract The notion of an ideal family of weighted subspaces of a discrete metric space X with bounded geometry is introduced. It is shown that, if X has Yu's property A, the ideal structure of the Roe algebra of X with coefficients in $\mathcal{B}(H)$ is completely characterized by the ideal families of weighted subspaces of X, where $\mathcal{B}(H)$ denotes the C^*-algebra of bounded linear operators on a separable Hilbert space H.

Keywords Roe algebra, Ideal, Metric space, Coarse geometry, Band-dominated operator
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1 Introduction

Let (X, d) be a discrete metric space with bounded geometry, i.e., for any $R > 0$ there exists $N > 0$ such that any ball in X of radius R contains at most N elements. Let H be a separable Hilbert space over the complex field $\mathbb{C}$. Denote by $\mathcal{B}(H)$ the C^*-algebra of all bounded linear operators on H and by $\mathcal{K}(H)$ the ideal of the compact operators on H. Throughout this paper, by an ideal in a C^*-algebra we mean a closed two-sided ideal.

Denote $H_X := \ell^2(X) \otimes H = \bigoplus\limits_{x\in X} H$. Any operator $T \in \mathcal{B}(H_X)$ admits a canonical form of $X \times X$ matrix

$$T = [T(x, y)]_{(x,y)\in X\times X}$$

with entries $T(x, y) \in \mathcal{B}(H)$. The support of T is defined as

$$\mathrm{Supp}\,(T) := \{(x, y) \in X \times X \mid T(x, y) \neq 0\}.$$

The propagation of T is defined to be

$$\mathrm{Prg}\,(T) := \sup\{d(x, y) \mid (x, y) \in \mathrm{Supp}\,(T)\}.$$

Denote by $\mathrm{FP}\,(X, \mathcal{B}(H))$ the collection of all finite propagation operators in $\mathcal{B}(H_X)$. It is a $*$-subalgebra of $\mathcal{B}(H_X)$.

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Definition 1.1 *The operator norm closure $C^*(X,\mathcal{B}(H))$ of $\mathrm{FP}(X,\mathcal{B}(H))$ is a C^*-subalgebra of $\mathcal{B}(H_X)$, called the Roe algebra of X with coefficients in $\mathcal{B}(H)$.*

Recall that the original "Roe algebra" $C^*(X) := C^*(X,\mathcal{K}(H))$ and the "uniform Roe algebra" $C_u^*(X) := C^*(X,\mathbb{C})$ are defined in a similar way by using finite propagation operators with coefficients in $\mathcal{K}(H)$ and $\mathbb{C}$ respectively (see [7, 8]). These C^*-algebras have great importance in geometry, topology and analysis. In the context of integral equations and operator theory, finite propagation operators and operators in these "Roe algebras" are called band operators and band-dominated operators, respectively, the investigation to which has a long and interesting history (see [6]). In applications, ideals of these Roe algebras play an important role. In [1–4], the ideal structure of the Roe algebra $C^*(X)$ and the uniform Roe algebra $C_u^*(X)$ are studied. The purpose of the present paper is to generalize the main idea and techniques in [2, 4] for $C^*(X)$ and $C_u^*(X)$ to the case of $C^*(X,\mathcal{B}(H))$. We introduce the notion of an ideal family of weighted subspaces of a discrete metric space X with bounded geometry and show that, if X has Yu's property A, any ideal in $C^*(X,\mathcal{B}(H))$ can be constructed from a unique ideal family of weighted subspaces of X.

Definition 1.2 *Let (X,d) be a discrete metric space with bounded geometry. An ideal family of weighted subspaces of X is a collection $\mathcal{L}$ of functions*

$$l : X \to \mathbb{Z}^+ \cup \{\infty\} := \{0,1,2,\cdots,\infty\}$$

satisfying the following conditions:

(1) *For any function $k : X \to \mathbb{Z}^+ \cup \{\infty\}$, if $k \le l$ for some $l \in \mathcal{L}$, then $k \in \mathcal{L}$,*

(2) *For any functions $k, l \in \mathcal{L}$, we have $k + l \in \mathcal{L}$, where $(k+l)(x) := k(x) + l(x)$ for all $x \in X$,*

(3) *For any $R > 0$ and $l \in \mathcal{L}$, we have $R * l \in \mathcal{L}$, where for all $x \in X$,*

$$(R * l)(x) := \sum_{y \in X : d(y,x) \le R} l(y).$$

Note that if $\mathcal{L}_1$ and $\mathcal{L}_2$ are ideal families of weighted subspaces of X, then $\mathcal{L}_1 \wedge \mathcal{L}_2 := \mathcal{L}_1 \cap \mathcal{L}_2$ and $\mathcal{L}_1 \vee \mathcal{L}_2 := \{l_1 + l_2 \mid l_1 \in \mathcal{L}_1, l_2 \in \mathcal{L}_2\}$ are also ideal families of weighted subspaces of X, so that the collection of all ideal families of weighted subspaces of X forms a lattice with respect to inclusion.

Recall that the rank of an operator $A \in \mathcal{B}(H)$ is defined to be the dimension of the range of A. In particular, $\mathrm{rank}(A) = \infty$ if A is not compact.

Definition 1.3 *An operator $T = [T(x,y)] \in \mathcal{B}(\ell^2(X) \otimes H)$ is said to be dominated by a function $l : X \to \mathbb{Z}^+ \cup \{\infty\}$, and denoted by $T \preceq l$, if $\mathrm{rank}(T(x,y)) \le \min\{l(x), l(y)\}$ for all $x, y \in X$.*

Let $\mathcal{L}$ be an ideal family of weighted subspaces of X. Set

$$\mathrm{FP}(\mathcal{L};X) = \bigcup_{l \in \mathcal{L}} \{T \in \mathrm{FP}(X,\mathcal{B}(H)) \mid T \preceq l\}.$$

For any $S \in \mathrm{FP}(X,\mathcal{B}(H))$ and $T \in \mathrm{FP}(\mathcal{L};X)$ with $T \preceq l \in \mathcal{L}$, it is easy to verify that $ST \preceq R * l \in \mathcal{L}$ and $TS \preceq R * l \in \mathcal{L}$, where $R = \max\{\mathrm{Prg}(S), \mathrm{Prg}(T)\}$, so that $\mathrm{FP}(\mathcal{L};X)$ is an (algebraic) ideal of $\mathrm{FP}(X,\mathcal{B}(H))$.

Definition 1.4 *Define $C^*(\mathcal{L};X)$ to be the operator norm closure of $\mathrm{FP}(\mathcal{L};X)$ in $C^*(X, \mathcal{B}(H))$.*

Thus, $C^*(\mathcal{L}; X)$ is an ideal of the Roe algebra $C^*(X, \mathcal{B}(H))$.

Recall that a discrete metric space X is said to have Yu's property A (see [9]), if for any $R > 0$ and $\varepsilon > 0$, there exists a family of nonempty finite subsets A_x of $X \times \mathbb{N}$ indexed by $x \in X$, such that (1) if $d(x, y) < R$, then $\frac{\#(A_x - A_y) + \#(A_y - A_x)}{\#(A_x \cap A_y)} < \varepsilon$, (2) there exists $S > 0$, such that $A_x \subseteq \mathrm{Ball}(x, S) \times \mathbb{N}$ for all $x \in X$. The class of metric spaces with property A includes all finitely generated exact groups and discrete metric spaces with finite asymptotic dimension.

The main result of this paper is the following characterization of ideal structure.

Theorem 1.1 *Let X be a discrete metric space with bounded geometry and Yu's property* A*. Then for any ideal I of the Roe algebra $C^*(X, \mathcal{B}(H))$, there exists a unique ideal family $\mathcal{L}$ of weighted subspaces of X such that $I = C^*(\mathcal{L}; X)$.*

Thus, the lattice of ideals of the Roe algebra $C^*(X, \mathcal{B}(H))$ is isomorphic to the lattice of ideal families of weighted subspaces of X.

2 Proof of Main Result

The main tool to prove Theorem 1.1 is a technique of controlled truncations (see [2, 4]). For any operator $A \in \mathcal{B}(H)$, by polar decomposition, there is a unique partial isometry $V \in \mathcal{B}(H)$ such that $A = V|A|$ and $\ker(A) = \ker(V)$, where $|A| = (A^*A)^{\frac{1}{2}}$. Let

$$|A| = \int x \mathrm{d} E_x$$

be the spectral resolution of $|A|$. For any $\varepsilon > 0$, the ε-truncation A_ε of A is defined by

$$A_\varepsilon := V|A|_\varepsilon := \begin{cases} V \displaystyle\int_{[\varepsilon, \|A\|]} x \mathrm{d} E_x, & \text{if } \varepsilon \leq \|A\|, \\ 0, & \text{if } \varepsilon > \|A\|. \end{cases}$$

Note that A_ε is either of finite rank or noncompact and $A = \lim\limits_{\varepsilon \to 0} A_\varepsilon$.

Throughout this section, let (X, d) be a discrete metric space with bounded geometry. Let $r, s : X \times X \to X$ be the two projections defined by

$$r(x, y) = x, \quad s(x, y) = y, \quad \forall\, (x, y) \in X \times X.$$

A subset $E \subseteq X \times X$ is called an entourage if there is $R > 0$ such that $d(x, y) \leq R$ for all $(x, y) \in E$. Denote by $\mathcal{E}$ the collection of all entourages in $X \times X$. An entourage F is called a partial translation if both r and s are injective on F.

Definition 2.1 *Let $T = [T(x, y)] \in \mathcal{B}(\ell^2(X) \otimes H)$, $E \in \mathcal{E}$ and $\varepsilon > 0$.*

(a) *The ε-truncation of T is the operator $T_\varepsilon = [T_\varepsilon(x, y)]$, where $T_\varepsilon(x, y) = (T(x, y))_\varepsilon$ is the ε-truncation of the entry $T(x, y) \in \mathcal{B}(H)$ for all $(x, y) \in X \times X$.*

(b) *The E-truncation of T is defined to be the operator $T_E = [T_E(x, y)]$, where $T_E(x, y) = T(x, y)$ if $(x, y) \in E$ and 0 otherwise.*

(c) *The (E, ε)-truncation of T is defined to be $T_{(E,\varepsilon)} = (T_E)_\varepsilon = (T_\varepsilon)_E$.*

Let $C^*(X, \mathcal{B}(H))$ be the Roe algebra of X with coefficients in $\mathcal{B}(H)$.

Lemma 2.1 *For any $T \in C^*(X, \mathcal{B}(H))$, $E \in \mathcal{E}$ and $\varepsilon > 0$, the truncations T_E, T_ε and $T_{(E,\varepsilon)}$ belong to $\langle T \rangle$, the principal ideal in $C^*(X, \mathcal{B}(H))$ generated by T.*

Proof The proof is similar to the proofs of [2, Theorem 3.5] and [4, Theorem 3.2] and therefore is omitted.

Note that the truncations T_E, T_ε and $T_{(E,\varepsilon)}$ are finite propagation operators and $T_\varepsilon(x,y)$ is either of finite rank or noncompact for $(x,y) \in X \times X$. For any $T = [T(x,y)] \in C^*(X, \mathcal{B}(H))$ and $\varepsilon > 0$, define a function

$$\lambda(T,\varepsilon) : X \to \mathbb{Z}^+ \cup \{\infty\}$$

by

$$\lambda(T,\varepsilon)(x) = \operatorname{rank}(T_\varepsilon(x,x)), \quad \forall\, x \in X.$$

Now, for an ideal I of the Roe algebra $C^*(X, \mathcal{B}(H))$, define

$$\Lambda(I) := \{\lambda(T,\varepsilon) \mid T \in I, \varepsilon > 0\}.$$

Lemma 2.2 *Let $l \in \Lambda(I)$ for an ideal I of $C^*(X, \mathcal{B}(H))$. For any operator $Q = \bigoplus_{x\in X} Q(x,x)$ in $\mathcal{B}(\ell^2(X) \otimes H)$ (i.e. $Q(x,y) = 0$ for $x \neq y$) such that $Q(x,x)$ is an orthogonal projection on H with*

$$\operatorname{rank}(Q(x,x)) \leq l(x)$$

for all $x \in X$, we have $Q \in I$.

Proof Since $l \in \Lambda(I)$, there exist $T = [T(x,y)] \in I$ and $\varepsilon > 0$ such that $l(x) = \operatorname{rank}(T_\varepsilon(x,x))$. Denote by Δ the diagonal of $X \times X$. Then it follows from Lemma 2.1 that the (Δ,ε)-truncation $T_{(\Delta,\varepsilon)} = \bigoplus_{x\in X} T_\varepsilon(x,x)$ belongs to I. For each $x \in X$, there exists a generalized inverse $T_\varepsilon(x,x)^\dagger$ such that $P(x,x) := T_\varepsilon(x,x)(T_\varepsilon(x,x)^\dagger)$ is the orthogonal projection of H on the range of $T_\varepsilon(x,x)$. It follows that $P := \bigoplus_{x\in X} P(x,x)$ belongs to I. Now, we have

$$\operatorname{rank}(Q(x,x)) \leq l(x) \leq \operatorname{rank}(T_\varepsilon(x,x)) = \operatorname{rank} P(x,x).$$

Take a partial isometry $W(x,x) \in \mathcal{B}(H)$ such that $W(x,x)^*W(x,x) = Q(x,x)$ and $W(x,x)W(x,x)^* \leq P(x,x)$ for each $x \in X$, and let $W = \bigoplus_{x\in X} W(x,x)$. Then we have $Q = W^*WW^*W = W^*(WW^*P)W \in I$ as desired.

Lemma 2.3 *$\Lambda(I)$ is an ideal family of weighted subspaces of X for any ideal I of $C^*(X, \mathcal{B}(H))$.*

Proof Let $R > 0$ and $l \in \Lambda(I)$. By Lemma 2.2 there is a projection $Q = \bigoplus_{x\in X} Q(x,x)$ such that $\operatorname{rank}(Q(x,x)) = l(x)$ for each $x \in X$. Let $X' := \{x \in X \mid l(x) < \infty\}$. Since X has bounded geometry, $X' \subseteq X$ is countable, say, $X' = \{x_1, x_2, \cdots, x_n, \cdots\}$. Let $\{e(1), e(2), \cdots, e(j), \cdots\}$ be an orthonormal basis of the Hilbert space H. For each integer $n > 0$ and $x_n \in X'$, define $P(x_n, x_n)$ to be the orthogonal projection of H onto the subspace spanned by

$$e\Big(\sum_{j=1}^{n-1} \operatorname{rank}(Q(x_j,x_j)) + 1\Big), e\Big(\sum_{j=1}^{n-1} \operatorname{rank}(Q(x_j,x_j)) + 2\Big), \cdots, e\Big(\sum_{j=1}^{n} \operatorname{rank}(Q(x_j,x_j))\Big).$$

Then $\operatorname{rank}(P(x_n,x_n)) = \operatorname{rank}(Q(x_n,x_n))$ and the projections $\{P(x_n,x_n)\}_n$ are mutually orthogonal. Denote by id_H the identity operator on H. Define $R := \bigoplus_{x\in X} R(x,x)$ by

$$R(x,x) = \begin{cases} P(x_n,x_n), & \text{if } x = x_n \in X', \\ \mathrm{id}_H, & \text{if } x \in X - X'. \end{cases}$$

Then $R \in I$ by Lemma 2.2. Define another operator $W = [W(x,y)]$ on $\ell^2(X) \otimes H$ by

$$W(x,y) = \begin{cases} \mathrm{id}_H, & \text{if } d(x,y) \le R, \\ 0, & \text{if } d(x,y) > R. \end{cases}$$

Then $W \in \mathrm{FP}(X, \mathcal{B}(H))$ so that $WRW^* \in I$. It is easy to verify that

$$R * l = \lambda(WRW^*, 1) \in \Lambda(I),$$

i.e., $\Lambda(I)$ satisfies condition (3) in Definition 1.2. Conditions (1) and (2) can be verified similarly. So $\Lambda(I)$ is an ideal family of weighted subspaces of X.

Lemma 2.4 *For any ideal I of $C^*(X, \mathcal{B}(H))$, we have $C^*(\Lambda(I); X) \subseteq I$.*

Proof It suffices to show that $\mathrm{FP}(\Lambda(I); X) \subseteq I$. Suppose $T \in \mathrm{FP}(X, \mathcal{B}(H))$ is such that $T \preceq l$ for some $l \in \Lambda(I)$. Since X has bounded geometry, $\mathrm{Supp}(T)$ can be decomposed into the union of finitely many disjoint partial translations, that is,

$$\mathrm{Supp}(T) = F_1 \sqcup F_2 \sqcup \cdots \sqcup F_m.$$

Let $T_j = T_{F_j}$ be the F_j-truncation of T for $j = 1, 2, \cdots, m$. For each j, define an operator $Q = \bigoplus_{x \in X} Q_j(x,x)$ such that, for each $x \in X$, $Q_j(x,x)$ is the orthogonal projection of H onto the range of $T_j(x,y)$ if $(x,y) \in F_j$ for some unique $y \in X$, and $Q_j(x,x) = 0$ otherwise. Then $T_j = Q_j T_j$ and

$$\mathrm{rank}(Q_j(x,x)) = \mathrm{rank}(T_j(x,y)) \le l(x).$$

By Lemma 2.2, we have $Q_j \in I$ for all $j = 1, 2, \cdots, m$ so that

$$T = \sum_{j=1}^{m} T_j = \sum_{j=1}^{m} Q_j T_j \in I.$$

Hence, we have $\mathrm{FP}(\Lambda(I); X) \subseteq I$ and $C^*(\Lambda(I); X) \subseteq I$.

Lemma 2.5 *If X has Yu's property* A*, then for any ideal I of $C^*(X, \mathcal{B}(H))$, we have $I \cap \mathrm{FP}(X, \mathcal{B}(H))$ is dense in I.*

Proof This follows from an approximation argument by using Schur multipliers of positive definite kernels with finite propagation. The proof is similar to the proof of Lemma 4.3 in [3] and therefore is omitted.

Lemma 2.6 *Suppose that X has Yu's property* A*. Then for any ideal I of $C^*(X, \mathcal{B}(H))$, we have $I \subseteq C^*(\Lambda(I); X)$.*

Proof By Lemma 2.5, it suffices to show that $I \cap \mathrm{FP}(X, \mathcal{B}(H)) \subseteq C^*(\Lambda(I); X)$. Suppose $T \in I \cap \mathrm{FP}(X, \mathcal{B}(H))$. Since X has bounded geometry, there exist finitely many partial translations $\{F_j\}_{j=1}^m$ such that $T = \lim_{\varepsilon \to 0} \sum_{j=1}^{m} T_{(F_j, \varepsilon)}$. Hence, to complete the proof, it suffices to show that $T_{(F,\varepsilon)} \in \mathrm{FP}(\Lambda(I); X)$ for any partial translation F and $\varepsilon > 0$. Note that $T_{(F,\varepsilon)} \in I$ by Lemma 2.1. Define an operator $S = [S(x,y)] \in \mathcal{B}(\ell^2(X) \otimes H)$ such that

$$S(x,y) = \begin{cases} \mathrm{id}_H, & \text{if } (y,x) \in E, \\ 0, & \text{otherwise.} \end{cases}$$

Then $T_{(F,\varepsilon)}S \in I$ and its support lies in the diagonal of $X \times X$. Hence, we have

$$T_{(F,\varepsilon)} \preceq \lambda(T_{(F,\varepsilon)}S, \varepsilon) \in \Lambda(I),$$

that is, $T_{(F,\varepsilon)} \in \mathrm{FP}(\Lambda(I); X)$ as desired. This completes the proof.

Lemma 2.7 *For any ideal family $\mathcal{L}$ of weighted subspaces of X, we have $\mathcal{L} = \Lambda(C^*(\mathcal{L}; X))$.*

Proof For any $l \in \mathcal{L}$, take an operator $Q = \bigoplus\limits_{x \in X} Q(x,x) \in \mathcal{B}(\ell^2(X) \otimes H)$ such that $Q(x,x)$ is an orthogonal projection in $\mathcal{B}(H)$ with $\mathrm{rank}(Q(x,x)) = l(x)$ for all $x \in X$. Then, by the definition of $C^*(\mathcal{L}; X)$, $Q \in C^*(\mathcal{L}; X)$ since $Q \preceq l$. On the other hand, $l = \lambda(Q, 1) \in \Lambda(C^*(\mathcal{L}; X))$. Hence, $\mathcal{L} \subseteq \Lambda(C^*(\mathcal{L}; X))$.

Conversely, suppose $l \in \Lambda(C^*(\mathcal{L}; X))$. By Lemma 2.2, there exists an operator $Q = \bigoplus\limits_{x \in X} Q(x,x) \in C^*(\mathcal{L}; X)$ such that $Q(x,x)$ are orthogonal projections on H with $\mathrm{rank}(Q(x,x)) = l(x)$. Thus, there exist $T \in \mathrm{FP}(\mathcal{L}; X)$ and $k \in \mathcal{L}$ such that

$$T \preceq k \quad \text{and} \quad \|Q - T\| < 1.$$

It follows that

$$l(x) = \mathrm{rank}(Q(x,x)) \leq \mathrm{rank}(T(x,x)) \leq k(x)$$

for all $x \in X$, namely, $l \leq k \in \mathcal{L}$. Hence, $l \in \mathcal{L}$ and $\Lambda(C^*(\mathcal{L}; X)) \subseteq \mathcal{L}$. The proof is complete.

Proof of Theorem 1.1 The existence is given by Lemmas 2.4 and 2.6. The uniqueness is given by Lemma 2.7.

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