Chin. Ann. Math.
31B(2), 2010, 145–162
DOI: 10.1007/s11401-009-0165-7

Chinese Annals of Mathematics, Series B

Determination of Unknown Boundary in the Composite Materials with Stefan-Boltzmann Conditions**

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Abstract The authors consider one specific kind of heat transfer problems in a three-dimensional layered domain, with nonlinear Stefan-Boltzmann conditions on the boundaries as well as on the interfaces. To determine the unknown part of the boundary (or corrosion) by the Cauchy data on the reachable part is an important inverse problem in engineering. The mathematical model of this problem is introduced, the well-posedness of the forward problems and the uniqueness of the inverse problems are obtained.

Keywords Inverse heat problem, Stefan-Boltzmann conditions, Uniqueness
2000 MR Subject Classification 35R30, 35K20

1 Introduction

The parabolic equations with nonlinear boundary conditions have been extensively studied in recent decades, and there are some fundamental results in [1, 15, 16, 25]. Such kind of problems, which involve the Stefan-Boltzmann interface condition, are usually derived from the modeling of heat transfer process that can be found in the fields such as materials science, geophysics, engineering and so on. The problems studied here arise from the procedure of the steel-making, where the steel is heated and melt in the container built with composite materials. Sometimes, corrosions will appear on the inside surface of the container, and the detection of them turns out to be an important objective for both the theory and the application. The additional information for detecting the corrosions is the thermal observation data on the outside surface of the container. Mathematically, the problems can be treated as boundary determination problems.

Similar problems have been studied by Banks, Kojima and Winfree [5, 6], where the non-destructive evaluation methods in thermal tomography are used to characterize structural flaws (e.g. corrosion, cracks, etc.) which may not be detectable by visual inspection. Besides, some interesting theoretical results have been developed, for instance, the uniqueness theory by Bryan and Caudill [9] and by Chapko, Kress and Yoon [10], the logarithmic stability by Vessella [24] and by Canuto, Rosset and Vessella [12]. Reconstruction methods for the heat equation have been proposed by Banks, Kojima and Winfree [5, 6] in the case where Ω is a rectangle, Chapko,

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**Project supported by the Shanghai Natural Science Foundation of China (No. 07JC14001), the Doctoral Program Foundation of the Ministry of Education of China, the Tianyuan Fund for Mathematics (No. 10826105), the National Basic Research Program (No. 2005CB321701) and the 111 Project (No. B08018).

Kress and Yoon [10] where domain Ω is the unit disk, and Bryan and Caudill [7, 8] where Ω is a strip. Recently, Bryan and Caudill also considered the inverse Cauchy problem in n dimensions and developed a Newton-like algorithm (see [4]).

Different from the works we mentioned above, here we consider the heat transfer problem with the Stefan-Boltzmann boundary conditions. The Stefan-Boltzmann law comes from the black-body radiation theory, which describes the nonlinear heat transfer phenomenon of the high temperature materials. The purpose for using the nonlinear conditions is to characterize the discontinuity of the temperature across the interfaces of the multi-layered domain. The nature of discontinuity is discovered by the experiments data from the thermal sensors in the applications, and has been verified by our numerical simulations. Yang, Yamamoto and Cheng have provided some theoretical results for such kind of heat transfer problem in one-dimensional case (see [26]), and we intend to extend the results into a multi-dimensional situation in this paper. Additionally, we will introduce the corresponding inverse problem in later sections, which is a boundary determination problem arising from the practical need in industry.

In our previous work of [20], we focused on the numerical methods of this boundary determination problem. We provided a stable numerical scheme for the forward problem, which is used to simulate the heat transfer process in composite materials with Stefan-Boltzmann conditions; the numerical results showed the accordance between the mathematical modeling and the practical observations. Besides, we also proposed an effective reconstruction method for determining the inside boundary (where the corrosions appear) with the Cauchy data on the outside surface; more importantly, the initial value was set to be unknown for the inverse problem which is from practical requirement. The numerical results showed that these algorithms are applicable. In [13], we proposed the Robin-Robin domain decomposition methods combining with the monotone method to decouple the nonlinear interface and boundary condition. The monotone properties are proven for both the multiplicative and the additive domain decomposition methods.

In this paper, we mainly study the mathematical theories for this kind of heat transfer problem. For the forward problem, which describes the heat transfer process in composite materials with Stefan-Boltzmann conditions in three dimensional case, we prove its well-posedness nature; especially, we focus on the existence and the uniqueness of the solution to the problem. Then for the corresponding inverse problem, which is described as a boundary determination problem here, we provide a uniqueness theorem for it. And we mention that our uniqueness result for the inverse problem does not require the knowledge of the initial value, which accords with the requirements of the industry applications and the numerical methods we developed in [20].

The paper is organized as follows. In Section 2, we introduce the mathematical formulations of both the forward problem and the inverse problem. In Section 3, the well-posedness of the forward problem is studied. In Section 4, the uniqueness of the inverse problem is obtained. The conclusions are given in Section 5.

2 Mathematical Formulation

The problem is discussed in the bounded domain $\Omega \in \mathbb{R}^3$, which is shown in Figure 1. The

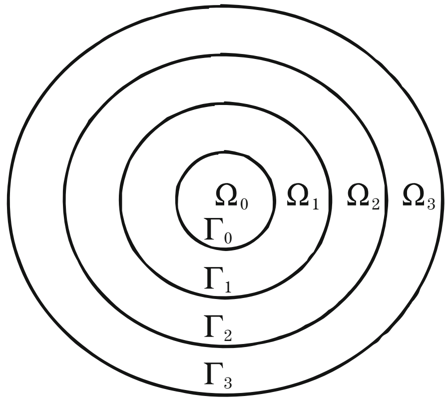

Figure 1 $\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3$

mathematical model is described as

$$\partial_t u_1(\mathbf{x}, t) = \alpha_1 \Delta u_1(\mathbf{x}, t), \quad \mathbf{x} \in \Omega_1,\ t \in (0, T), \tag{2.1}$$

$$\partial_t u_2(\mathbf{x}, t) = \alpha_2 \Delta u_2(\mathbf{x}, t), \quad \mathbf{x} \in \Omega_2,\ t \in (0, T), \tag{2.2}$$

$$\partial_t u_3(\mathbf{x}, t) = \alpha_3 \Delta u_3(\mathbf{x}, t), \quad \mathbf{x} \in \Omega_3,\ t \in (0, T), \tag{2.3}$$

where $\partial\Omega_i$ is C^1 continuous, $i = 1, 2, 3$, and ∂_t denotes the derivative $\frac{\partial}{\partial t}$. The boundary conditions and interface conditions are defined as follows

$$u_1(\mathbf{x}, t) = u_M, \qquad \mathbf{x} \in \Gamma_0, \tag{2.4}$$

$$-\lambda_1 \partial_{n_1} u_1(\mathbf{x}, t) = \sigma_1(u_1^4(\mathbf{x}, t) - u_2^4(\mathbf{x}, t)), \quad \mathbf{x} \in \Gamma_1, \tag{2.5}$$

$$\lambda_1 \partial_{n_1} u_1(\mathbf{x}, t) = -\lambda_2 \partial_{n_2} u_2(\mathbf{x}, t), \qquad \mathbf{x} \in \Gamma_1, \tag{2.6}$$

$$-\lambda_2 \partial_{n_2} u_2(\mathbf{x}, t) = \sigma_2(u_2^4(\mathbf{x}, t) - u_3^4(\mathbf{x}, t)), \quad \mathbf{x} \in \Gamma_2, \tag{2.7}$$

$$\lambda_2 \partial_{n_2} u_2(\mathbf{x}, t) = -\lambda_3 \partial_{n_3} u_3(\mathbf{x}, t), \qquad \mathbf{x} \in \Gamma_2, \tag{2.8}$$

$$-\lambda_3 \partial_{n_3} u_3(\mathbf{x}, t) = \sigma_3(u_3^4(\mathbf{x}, t) - u_A^4), \qquad \mathbf{x} \in \Gamma_3, \tag{2.9}$$

where n_j is the outward unit normal of Ω_j, ∂_{n_j} denotes the normal derivative $\frac{\partial}{\partial n_j}$, and $\alpha_j > 0$, $\lambda_j > 0$, $\sigma_j > 0$, $u_M, u_A > 0$ are constants for $j = 1, 2$ and 3. The initial value is set as

$$u(\mathbf{x}, 0) = a(\mathbf{x}), \quad \mathbf{x} \in \Omega_1 \cup \Omega_2 \cup \Omega_3, \tag{2.10}$$

and we assume the compatibility condition for $a(\mathbf{x})$,

$$\begin{cases} a(\mathbf{x}) \in C^{2+\kappa}(\Omega_1 \cup \Omega_2 \cup \Omega_3), & \kappa \in (0, 1), \\ a(\mathbf{x}) > 0, & \mathbf{x} \in (\Omega_1 \cup \Omega_2 \cup \Omega_3), \\ a(\mathbf{x}) = u_M, & \mathbf{x} \in \Gamma_0, \\ -\lambda_3 \partial_{n_3} a(\mathbf{x}) = \sigma_3(a^4(\mathbf{x}) - u_A^4), & \mathbf{x} \in \Gamma_3. \end{cases} \tag{2.11}$$

And all the following results are discussed in the class of

$$u_1 \in C^1([0, T], C^2(\overline{\Omega}_1)), \quad u_2 \in C^1([0, T], C^2(\overline{\Omega}_2)), \quad u_3 \in C^1([0, T], C^2(\overline{\Omega}_3)). \tag{2.12}$$

Now we introduce our forward problems and inverse problems.

Problem 2.1 (Forward Problems) Assume that (2.1)–(2.12) are satisfied. Determine $u(\mathbf{x},t)=(u_1,u_2,u_3)$, if $a(\mathbf{x})$, u_M, u_A, Ω_i, α_i, λ_i, σ_i, $i=1,2,3$ are given.

Problem 2.2 (Inverse Problems) Assume that (2.1)–(2.12) are satisfied. Determine $u(\mathbf{x},t) = (u_1,u_2,u_3)$ and Γ_0, if u_M, u_A, Ω_1, Ω_2, α_i, λ_i, σ_i, $i=1,2,3$ and the Dirichlet condition on Γ_3 are given,

$$u_3(\mathbf{x},t)=f(\mathbf{x},t),\quad \mathbf{x}\in\Gamma_3,\ t\in(0,T). \tag{2.13}$$

Actually, with (2.9) and (2.13), we have the Neumann condition on Γ_3 for Problem 2.2,

$$\partial_{n_3}u_3(\mathbf{x},t)=g(\mathbf{x},t)=-\frac{\sigma_3}{\lambda_3}(u_3^4(\mathbf{x},t)-u_A^4),\quad \mathbf{x}\in\Gamma_3.$$

Therefore, the inverse problem we consider here is to reconstruct the inside boundary Γ_0 by the Cauchy data on the outside boundary Γ_3, while the initial value $a(\mathbf{x})$ is not given.

We list our main results of the paper. For Problem 2.1 (forward problems), we have the following theorems.

Theorem 2.1 (Global Uniqueness and Existence in Time) *Assume that $a(\mathbf{x})$ satisfies* (2.11)*. Let T be arbitrarily given. Then there exists a unique solution $u(\mathbf{x},t)$ of the problem* (2.1)–(2.10) *within the class of* (2.12).

And for Problem 2.2 (inverse problems), we have the result as follows.

Theorem 2.2 (Uniqueness) *Assume that $T=\infty$, and there exist constants $M,\gamma_0>0$, such that $\|f\|,\|g\|\le M$, $\|f-u_M\|\ge\gamma_0$. If there exist $\{\Gamma_0,u(\mathbf{x},t)\}$ and $\{\widetilde{\Gamma}_0,\widetilde{u}(\mathbf{x},t)\}$ which satisfy Problem* 1.2 *respectively, then $\Gamma_0=\widetilde{\Gamma}_0$ and $u(\mathbf{x},t)=\widetilde{u}(\mathbf{x},t)$ must hold.*

3 Well-Posedness of the Mathematical Model

3.1 Green's functions

In subdomain Ω_1, the Green's function $G_1(\mathbf{x},t;\boldsymbol{\xi},\tau)$ is the solution of the following problem (see [14, 22]):

$$\begin{aligned}
&-\partial_\tau G_1-\alpha_1\Delta_{\boldsymbol{\xi}}G_1=\delta(\boldsymbol{\xi}-\mathbf{x})\delta(\tau-t),\\
&G_1(\mathbf{x},t;\boldsymbol{\xi},\tau)=0,\quad \boldsymbol{\xi}\in\Gamma_0,\\
&\partial_{n_1}G_1(\mathbf{x},t;\boldsymbol{\xi},\tau)=0,\quad \boldsymbol{\xi}\in\Gamma_1,\\
&G_1=0,\quad \tau>t.
\end{aligned}$$

By this Green's function, we have

$$u_1(\mathbf{x},t)=-\int_0^t\int_{\Omega_1}(G_1(\partial_\tau u_1-\alpha_1\Delta_{\boldsymbol{\xi}}u_1)+u_1(\partial_\tau G_1+\alpha_1\Delta_{\boldsymbol{\xi}}G_1))\mathrm{d}\boldsymbol{\xi}\mathrm{d}\tau. \tag{3.1}$$

On the other hand, we have

$$-\int_0^t\int_{\Omega_1}(G_1\partial_\tau u_1+u_1\partial_\tau G_1)\mathrm{d}\boldsymbol{\xi}\mathrm{d}\tau=\int_{\Omega_1}G_1(\mathbf{x},t;\boldsymbol{\xi},0)u_1(\boldsymbol{\xi},0)\mathrm{d}\boldsymbol{\xi}=\int_{\Omega_1}G_1(\mathbf{x},t;\boldsymbol{\xi},0)a(\boldsymbol{\xi})\mathrm{d}\boldsymbol{\xi},$$

as well as

$$
\begin{aligned}
&-\int_0^t\int_{\Omega_1}\alpha_1(-G_1\Delta_{\boldsymbol{\xi}}u_1+u_1\Delta_{\boldsymbol{\xi}}G_1)\mathrm{d}\boldsymbol{\xi}\mathrm{d}\tau\\
=&\,\alpha_1\int_0^t\int_{\partial\Omega_1}(G_1\partial_{n_1}u_1-u_1\partial_{n_1}G_1)\mathrm{d}\boldsymbol{\xi}\mathrm{d}\tau\\
=&\,\alpha_1\int_0^t\int_{\Gamma_1}(G_1\partial_{n_1}u_1-u_1\partial_{n_1}G_1)\mathrm{d}\boldsymbol{\xi}\mathrm{d}\tau+\alpha_1\int_0^t\int_{\Gamma_0}(G_1\partial_{n_1}u_1-u_1\partial_{n_1}G_1)\mathrm{d}\boldsymbol{\xi}\mathrm{d}\tau\\
=&\,\alpha_1\int_0^t\int_{\Gamma_1}G_1\partial_{n_1}u_1\mathrm{d}\boldsymbol{\xi}\mathrm{d}\tau-\alpha_1\int_0^t\int_{\Gamma_0}u_M\partial_{n_1}G_1\mathrm{d}\boldsymbol{\xi}\mathrm{d}\tau.
\end{aligned}
$$

Here Green's formula is used in the first equality. Combining with (3.1), we get the expression

$$
\begin{aligned}
u_1(\mathbf{x},t)=&\,\alpha_1\int_0^t\int_{\Gamma_1}G_1\partial_{n_1}u_1\mathrm{d}\boldsymbol{\xi}\mathrm{d}\tau-\alpha_1\int_0^t\int_{\Gamma_0}u_M\partial_{n_1}G_1\mathrm{d}\boldsymbol{\xi}\mathrm{d}\tau+\int_{\Omega_1}G_1(\mathbf{x},t;\boldsymbol{\xi},0)a(\boldsymbol{\xi})\mathrm{d}\boldsymbol{\xi}\\
\equiv&\,\widetilde{u}_1(\mathbf{x},t)+\alpha_1\int_0^t\int_{\Gamma_1}G_1(\mathbf{x},t;\boldsymbol{\xi},\tau)f_1(\boldsymbol{\xi},\tau)\mathrm{d}\boldsymbol{\xi}\mathrm{d}\tau, \qquad (3.2)
\end{aligned}
$$

where $\widetilde{u}_1$ is

$$
\widetilde{u}_1(\mathbf{x},t)=\int_{\Omega_1}G_1(\mathbf{x},t;\boldsymbol{\xi},0)a(\boldsymbol{\xi})\mathrm{d}\boldsymbol{\xi}-\alpha_1\int_0^t\int_{\Gamma_0}u_M\partial_{n_1}G_1\mathrm{d}\boldsymbol{\xi}\mathrm{d}\tau.
$$

Similarly, G_i $(i=2,3)$ are the Green's functions on Ω_i respectively,

$$
\begin{aligned}
&-\partial_\tau G_i-\alpha_i\Delta_{\boldsymbol{\xi}}G_i=\delta(\boldsymbol{\xi}-\mathbf{x})\delta(\tau-t),\\
&\partial_{n_i}G_i(\mathbf{x},t;\boldsymbol{\xi},\tau)=0,\quad \boldsymbol{\xi}\in\Gamma_{i-1},\\
&\partial_{n_i}G_i(\mathbf{x},t;\boldsymbol{\xi},\tau)=0,\quad \boldsymbol{\xi}\in\Gamma_{i},\\
&G_i=0,\quad \tau>t,
\end{aligned}
$$

and let

$$
\widetilde{u}_i(\mathbf{x},t)=\int_{\Omega_i}G_i(\mathbf{x},t;\boldsymbol{\xi},0)a(\boldsymbol{\xi})\mathrm{d}\boldsymbol{\xi}.
$$

Then $u_2(\mathbf{x},t)$ can be expressed as

$$
\begin{aligned}
u_2(\mathbf{x},t)=&\,\widetilde{u}_2(\mathbf{x},t)+\alpha_2\int_0^t\int_{\Gamma_2}G_2(\mathbf{x},t;\boldsymbol{\xi},\tau)f_2(\boldsymbol{\xi},\tau)\mathrm{d}\boldsymbol{\xi}\mathrm{d}\tau\\
&-\alpha_2\int_0^t\int_{\Gamma_1}G_2(\mathbf{x},t;\boldsymbol{\xi},\tau)\frac{\lambda_1}{\lambda_2}f_1(\boldsymbol{\xi},\tau)\mathrm{d}\boldsymbol{\xi}\mathrm{d}\tau, \qquad (3.3)\\
u_3(\mathbf{x},t)=&\,\widetilde{u}_3(\mathbf{x},t)+\alpha_3\int_0^t\int_{\Gamma_3}G_3(\mathbf{x},t;\boldsymbol{\xi},\tau)f_3(\boldsymbol{\xi},\tau)\mathrm{d}\boldsymbol{\xi}\mathrm{d}\tau\\
&-\alpha_3\int_0^t\int_{\Gamma_2}G_3(\mathbf{x},t;\boldsymbol{\xi},\tau)\frac{\lambda_2}{\lambda_3}f_1(\boldsymbol{\xi},\tau)\mathrm{d}\boldsymbol{\xi}\mathrm{d}\tau. \qquad (3.4)
\end{aligned}
$$

Now let f_i be the normal derivatives on the interfaces Γ_i respectively, i.e., $f_i(\mathbf{x},t)=\partial_{n_i}u_i(\mathbf{x},t)$, for $\mathbf{x}\in\Gamma_i$. According to the definition of $f_i(\mathbf{x},t)$ and the interface conditions (2.5), (2.7), (2.9), we obtain

$$
f_1=\frac{\sigma_1}{\lambda_1}(u_2^4(\mathbf{x},t)-u_1^4(\mathbf{x},t))
$$

$$
\begin{aligned}
&= \frac{\sigma_1}{\lambda_1}\Big\{\Big[\widetilde{u}_2 + \alpha_2\int_0^t\int_{\Gamma_2} G_2 f_2 \mathrm{d}\boldsymbol{\xi}\mathrm{d}\tau - \alpha_2\int_0^t\int_{\Gamma_1} G_2\frac{\lambda_1}{\lambda_2} f_1 \mathrm{d}\boldsymbol{\xi}\mathrm{d}\tau\Big]^4 \\
&\quad - \Big[\widetilde{u}_1 + \alpha_1\int_0^t\int_{\Gamma_1} G_1 f_1 \mathrm{d}\boldsymbol{\xi}\mathrm{d}\tau\Big]^4\Big\}, \qquad (3.5)\\
f_2 &= \frac{\sigma_2}{\lambda_2}(u_3^4(\mathbf{x},t) - u_2^4(\mathbf{x},t))\\
&= \frac{\sigma_2}{\lambda_2}\Big\{\Big[\widetilde{u}_3 + \alpha_3\int_0^t\int_{\Gamma_3} G_3 f_3 \mathrm{d}\boldsymbol{\xi}\mathrm{d}\tau - \alpha_3\int_0^t\int_{\Gamma_2} G_3\frac{\lambda_2}{\lambda_3} f_2 \mathrm{d}\boldsymbol{\xi}\mathrm{d}\tau\Big]^4 \\
&\quad - \Big[\widetilde{u}_2 + \alpha_2\int_0^t\int_{\Gamma_2} G_2 f_2 \mathrm{d}\boldsymbol{\xi}\mathrm{d}\tau - \alpha_2\int_0^t\int_{\Gamma_1} G_2\frac{\lambda_1}{\lambda_2} f_1 \mathrm{d}\boldsymbol{\xi}\mathrm{d}\tau\Big]^4\Big\}, \qquad (3.6)\\
f_3 &= \frac{\sigma_3}{\lambda_3}(u_A^4 - u_3^4(\mathbf{x},t))\\
&= \frac{\sigma_3}{\lambda_3}\Big\{u_A^4 - \Big[\widetilde{u}_3 + \alpha_3\int_0^t\int_{\Gamma_3} G_3 f_3 \mathrm{d}\boldsymbol{\xi}\mathrm{d}\tau - \alpha_3\int_0^t\int_{\Gamma_2} G_3\frac{\lambda_2}{\lambda_3} f_2 \mathrm{d}\boldsymbol{\xi}\mathrm{d}\tau\Big]^4\Big\}. \qquad (3.7)
\end{aligned}
$$

Based on the integral equations, the local uniqueness of the solution will be proved by fixed point theorem.

3.2 Local uniqueness and existence in time

Proposition 3.1 *Assume that $a(\mathbf{x})$ satisfies (2.11). Then there exists a sufficiently small $t_0 > 0$, such that problem (2.1)–(2.10) has a unique solution $u(\mathbf{x},t) = (u_1,u_2,u_3)$ with $u_1 \in C^1([0,t_0], C^2(\overline{\Omega}_1))$, $u_2 \in C^1([0,t_0], C^2(\overline{\Omega}_2))$, and $u_3 \in C^1([0,t_0], C^2(\overline{\Omega}_3))$.*

We need the following lemma to prove Proposition 3.1.

Lemma 3.1 *For all $\epsilon > 0$, there exists $t_0 \in (0,1)$ such that*

$$
\int_0^{t_0} G_i(\mathbf{x},t;\boldsymbol{\xi},\tau)\mathrm{d}\tau < \epsilon, \quad i = 1,2,3.
$$

Proof It follows from [2, 11] that the Green's function $G(\mathbf{x},\boldsymbol{\xi},\tau)$ satisfies the following inequality

$$
0 \le G(\mathbf{x},\boldsymbol{\xi},\tau) \le \frac{C}{\tau^{\frac{d}{2}}}\exp\Big(\frac{-|\mathbf{x}-\boldsymbol{\xi}|^2}{C\tau}\Big),
$$

where C is a constant, d represents the dimension, and $\mathbf{x},\boldsymbol{\xi} \in \mathbb{R}^d$. In our 3D case, $d = 3$, therefore, we have

$$
\begin{aligned}
\int_0^{t_0} G_i \mathrm{d}\tau &\le \int_0^{t_0}\frac{C}{\tau^{\frac{d}{2}}}\exp\Big(\frac{-|\mathbf{x}-\boldsymbol{\xi}|^2}{C\tau}\Big)\mathrm{d}\tau \le C_1\int_0^{t_0}\frac{C_2}{\tau^{\frac{3}{2}}}\exp\Big(-\frac{C_2}{\tau}\Big)\mathrm{d}\tau\\
&\le C_1\int_0^{t_0}\frac{C_2}{\tau^2}\exp\Big(-\frac{C_2}{\tau}\Big)\mathrm{d}\tau = C_1\exp\Big(-\frac{C_2}{\tau}\Big)\Big|_0^{t_0} = C_1\exp\Big(-\frac{C_2}{t_0}\Big),
\end{aligned}
$$

where $C_1 = \frac{C}{C_2}$, $C_2 = \min\frac{|\mathbf{x}-\boldsymbol{\xi}|^2}{C}$. It is clear that $\exp(-\frac{C_2}{t_0}) \to 0$ as $t_0 \to 0^+$. Then the proof is complete.

In order to use the fixed point theorem, we need one mapping relation. From (3.5)–(3.7), we define the mapping K by

$$
(f_1,f_2,f_3) = K(f_1,f_2,f_3),
$$

and define the set A as

$$A = \Big\{(f_1, f_2, f_3) \,\Big|\, f_i \in C(\Omega_i \times [0,t]),\ \Big\| f_1 - \frac{\sigma_1}{\lambda_1}(\widetilde{u}_2^4(\mathbf{x},t) - \widetilde{u}_1^4(\mathbf{x},t))\Big\| \le N_1,$$
$$\Big\| f_2 - \frac{\sigma_2}{\lambda_2}(\widetilde{u}_3^4(\mathbf{x},t) - \widetilde{u}_2^4(\mathbf{x},t))\Big\| \le N_2,\ \Big\| f_3 - \frac{\sigma_3}{\lambda_3}(u_A^4 - \widetilde{u}_3^4(\mathbf{x},t))\Big\| \le N_3\Big\},$$

where $N_i > 0$ are constants, and $\|\cdot\|$ denotes the norm in $C(\Omega_i \times [0,t])$. We introduce some notations to simplify the expressions

$$\begin{aligned}
X &= \widetilde{u}_2 + \alpha_2 \int_0^t \int_{\Gamma_2} G_2 f_2 \mathrm{d}\boldsymbol{\xi}\mathrm{d}\tau - \alpha_2 \int_0^t \int_{\Gamma_1} G_2 \frac{\lambda_1}{\lambda_2} f_1 \mathrm{d}\boldsymbol{\xi}\mathrm{d}\tau,\\
Y &= \widetilde{u}_1 + \alpha_1 \int_0^t \int_{\Gamma_1} G_1 f_1 \mathrm{d}\boldsymbol{\xi}\mathrm{d}\tau,\\
\widetilde{X} &= \widetilde{u}_2 + \alpha_2 \int_0^t \int_{\Gamma_2} G_2 \widetilde{f}_2 \mathrm{d}\boldsymbol{\xi}\mathrm{d}\tau - \alpha_2 \int_0^t \int_{\Gamma_1} G_2 \frac{\lambda_1}{\lambda_2} \widetilde{f}_1 \mathrm{d}\boldsymbol{\xi}\mathrm{d}\tau,\\
\widetilde{Y} &= \widetilde{u}_1 + \alpha_1 \int_0^t \int_{\Gamma_1} G_1 \widetilde{f}_1 \mathrm{d}\boldsymbol{\xi}\mathrm{d}\tau.
\end{aligned}$$

It can be shown that for arbitrarily small $\widetilde{M} > 0$, there exists $t_0 \in (0,1)$, such that

$$\max\Big\{\Big\|\alpha_2 \int_0^t \int_{\Gamma_2} G_2 f_2 \mathrm{d}\boldsymbol{\xi}\mathrm{d}\tau\Big\|, \Big\|\alpha_2 \int_0^t \int_{\Gamma_2} G_2 \widetilde{f}_2 \mathrm{d}\boldsymbol{\xi}\mathrm{d}\tau\Big\|, \Big\|\alpha_2 \int_0^t \int_{\Gamma_1} G_2 \frac{\lambda_1}{\lambda_2} f_1 \mathrm{d}\boldsymbol{\xi}\mathrm{d}\tau\Big\|,$$
$$\Big\|\alpha_2 \int_0^t \int_{\Gamma_1} G_2 \frac{\lambda_1}{\lambda_2} \widetilde{f}_1 \mathrm{d}\boldsymbol{\xi}\mathrm{d}\tau\Big\|, \Big\|\alpha_1 \int_0^t \int_{\Gamma_1} G_1 f_1 \mathrm{d}\boldsymbol{\xi}\mathrm{d}\tau\Big\|, \Big\|\alpha_1 \int_0^t \int_{\Gamma_1} G_1 \widetilde{f}_1 \mathrm{d}\boldsymbol{\xi}\mathrm{d}\tau\Big\|\Big\} < \widetilde{M}$$

for $0 < t < t_0$. Since $f_1, f_2, \widetilde{f}_1, \widetilde{f}_2$ are bounded due to the definition of set A, we can conclude the results above by Lemma 3.1. Therefore, we know that X, Y, $\widetilde{X}$, $\widetilde{Y}$ will be bounded by some constant $M > 0$ when $0 < t < t_0$,

$$\max\{\|X\|, \|Y\|, \|\widetilde{X}\|, \|\widetilde{Y}\|\} \le M.$$

Similarly, for arbitrarily small $\widetilde{M} > 0$, there exists $t_0 \in (0,1)$, such that for all $0 < t < t_0$ and $f_i, \widetilde{f}_i \in A$,

$$\max\Big\{\Big\|\alpha_2 \int_0^t \int_{\Gamma_2} G_2 \mathrm{d}\boldsymbol{\xi}\mathrm{d}\tau\Big\|, \Big\|\alpha_2 \int_0^t \int_{\Gamma_1} G_2 \frac{\lambda_1}{\lambda_2} \mathrm{d}\boldsymbol{\xi}\mathrm{d}\tau\Big\|, \Big\|\alpha_1 \int_0^t \int_{\Gamma_1} G_1 \mathrm{d}\boldsymbol{\xi}\mathrm{d}\tau\Big\|\Big\} < \widetilde{M}.$$

Lemma 3.2 *If $t \in (0,1)$ is sufficiently small, then*

(1) *$KA \subseteq A$,*

(2) *there exists an $h \in (0,1)$, such that for all $f_i, \widetilde{f}_i \in A$,*

$$\|K(f_1, f_2, f_3) - K(\widetilde{f}_1, \widetilde{f}_2, \widetilde{f}_3)\| \le h\|(f_1, f_2, f_3) - (\widetilde{f}_1, \widetilde{f}_2, \widetilde{f}_3)\|, \tag{3.8}$$

i.e., K is a contraction mapping.

Proof Let $0 < t < t_0$. Firstly, we prove $KA \subseteq A$, and we just consider $K(f_1)$ since the rest parts are similar. From the definition of K,

$$K(f_1) = \frac{\sigma_1}{\lambda_1}\Big\{\Big[\widetilde{u}_2 + \alpha_2 \int_0^t \int_{\Gamma_2} G_2 f_2 \mathrm{d}\boldsymbol{\xi}\mathrm{d}\tau - \alpha_2 \int_0^t \int_{\Gamma_1} G_2 \frac{\lambda_1}{\lambda_2} f_1 \mathrm{d}\boldsymbol{\xi}\mathrm{d}\tau\Big]^4$$

$$- \Big[\widetilde{u}_1 + \alpha_1 \int_0^t \int_{\Gamma_1} G_1 f_1 \mathrm{d}\boldsymbol{\xi}\mathrm{d}\tau\Big]^4\Big\}.$$

Then

$$\begin{aligned}
&\Big\|K(f_1) - \frac{\sigma_1}{\lambda_1}(\widetilde{u}_2^4(\mathbf{x},t) - \widetilde{u}_1^4(\mathbf{x},t))\Big\| \\
=& \frac{\sigma_1}{\lambda_1}\Big\|\Big[\widetilde{u}_2 + \alpha_2 \int_0^t \int_{\Gamma_2} G_2 f_2 \mathrm{d}\boldsymbol{\xi}\mathrm{d}\tau - \alpha_2 \int_0^t \int_{\Gamma_1} G_2 \frac{\lambda_1}{\lambda_2} f_1 \mathrm{d}\boldsymbol{\xi}\mathrm{d}\tau\Big]^4 \\
&- \Big[\widetilde{u}_1 + \alpha_1 \int_0^t \int_{\Gamma_1} G_1 f_1 \mathrm{d}\boldsymbol{\xi}\mathrm{d}\tau\Big]^4 - (\widetilde{u}_2^4 - \widetilde{u}_1^4)\Big\| \\
=& \frac{\sigma_1}{\lambda_1}\Big\|\Big[\alpha_2 \int_0^t \int_{\Gamma_2} G_2 f_2 \mathrm{d}\boldsymbol{\xi}\mathrm{d}\tau - \alpha_2 \int_0^t \int_{\Gamma_1} G_2 \frac{\lambda_1}{\lambda_2} f_1 \mathrm{d}\boldsymbol{\xi}\mathrm{d}\tau\Big](X + \widetilde{u}_2)(X^2 + \widetilde{u}_2^2) \\
&- \alpha_1 \int_0^t \int_{\Gamma_1} G_1 f_1 \mathrm{d}\boldsymbol{\xi}\mathrm{d}\tau (Y + \widetilde{u}_1)(Y^2 + \widetilde{u}_1^2)\Big\| \\
\le& \frac{\sigma_1}{\lambda_1}[2\widetilde{M}(M + \|\widetilde{u}_2\|)(M^2 + \|\widetilde{u}_2\|^2) + \widetilde{M}(M + \|\widetilde{u}_1\|)(M^2 + \|\widetilde{u}_1\|^2)] \\
=& \widetilde{M}\frac{\sigma_1}{\lambda_1}[2(M + \|\widetilde{u}_2\|)(M^2 + \|\widetilde{u}_2\|^2) + (M + \|\widetilde{u}_1\|)(M^2 + \|\widetilde{u}_1\|^2)].
\end{aligned}$$

Since $\widetilde{M}$ can be arbitrarily small, and all other parts are bounded, it is obvious that there exists a $t_0 \in (0,1)$ such that, for all $0 < t < t_0$,

$$\Big\|K(f_1) - \frac{\sigma_1}{\lambda_1}(\widetilde{u}_2^4(\mathbf{x},t) - \widetilde{u}_1^4(\mathbf{x},t))\Big\| \le N_1.$$

As for $K(f_2)$ and $K(f_3)$, we can make the similar conclusion, while it is easy to know that $K(f_i) \in C(\Omega_i \times [0,t])$. Thus the first part of Lemma 3.2 is proved.

Next, we prove the second part of Lemma 3.2.

$$\begin{aligned}
\|K(f_1) - K(\widetilde{f}_1)\| =& \Big\|\frac{\sigma_1}{\lambda_1}\Big\{\Big[\widetilde{u}_2 + \alpha_2 \int_0^t \int_{\Gamma_2} G_2 f_2 \mathrm{d}\boldsymbol{\xi}\mathrm{d}\tau - \alpha_2 \int_0^t \int_{\Gamma_1} G_2 \frac{\lambda_1}{\lambda_2} f_1 \mathrm{d}\boldsymbol{\xi}\mathrm{d}\tau\Big]^4 \\
&- \Big[\widetilde{u}_1 + \alpha_1 \int_0^t \int_{\Gamma_1} G_1 f_1 \mathrm{d}\boldsymbol{\xi}\mathrm{d}\tau\Big]^4\Big\} \\
&- \frac{\sigma_1}{\lambda_1}\Big\{\Big[\widetilde{u}_2 + \alpha_2 \int_0^t \int_{\Gamma_2} G_2 \widetilde{f}_2 \mathrm{d}\boldsymbol{\xi}\mathrm{d}\tau - \alpha_2 \int_0^t \int_{\Gamma_1} G_2 \frac{\lambda_1}{\lambda_2} \widetilde{f}_1 \mathrm{d}\boldsymbol{\xi}\mathrm{d}\tau\Big]^4 \\
&- \Big[\widetilde{u}_1 + \alpha_1 \int_0^t \int_{\Gamma_1} G_1 \widetilde{f}_1 \mathrm{d}\boldsymbol{\xi}\mathrm{d}\tau\Big]^4\Big\}\Big\| \\
=& \frac{\sigma_1}{\lambda_1}\Big\|\Big(\alpha_2 \int_0^t \int_{\Gamma_2} G_2 (f_2 - \widetilde{f}_2) \mathrm{d}\boldsymbol{\xi}\mathrm{d}\tau - \alpha_2 \int_0^t \int_{\Gamma_1} G_2 \frac{\lambda_1}{\lambda_2} (f_1 - \widetilde{f}_1) \mathrm{d}\boldsymbol{\xi}\mathrm{d}\tau\Big) \\
&(X + \widetilde{X})(X^2 + \widetilde{X}^2) - \alpha_1 \int_0^t \int_{\Gamma_1} G_1 (f_1 - \widetilde{f}_1) \mathrm{d}\boldsymbol{\xi}\mathrm{d}\tau (Y + \widetilde{Y})(Y^2 + \widetilde{Y}^2)\Big\| \\
\le& \frac{\sigma_1}{\lambda_1}\Big\{\|f_2 - \widetilde{f}_2\|\Big\|\alpha_2 \int_0^t \int_{\Gamma_2} G_2 \mathrm{d}\boldsymbol{\xi}\mathrm{d}\tau\Big\|\|X + \widetilde{X}\|\|X^2 + \widetilde{X}^2\| \\
&+ \|f_1 - \widetilde{f}_1\|\Big\|\alpha_2 \int_0^t \int_{\Gamma_1} G_2 \frac{\lambda_1}{\lambda_2} \mathrm{d}\boldsymbol{\xi}\mathrm{d}\tau\Big\|\|X + \widetilde{X}\|\|X^2 + \widetilde{X}^2\| \\
&+ \|f_1 - \widetilde{f}_1\|\Big\|\alpha_1 \int_0^t \int_{\Gamma_1} G_1 \mathrm{d}\boldsymbol{\xi}\mathrm{d}\tau\Big\|\|Y + \widetilde{Y}\|\|Y^2 + \widetilde{Y}^2\|\Big\}
\end{aligned}$$

$$\leq \frac{\sigma_1}{\lambda_1}(8\widetilde{M}M^3\|f_1 - \widetilde{f}_1\| + 4\widetilde{M}M^3\|f_2 - \widetilde{f}_2\|)$$
$$= C_1\|f_1 - \widetilde{f}_1\| + C_2\|f_2 - \widetilde{f}_2\|,$$

where $C_1 = 8\widetilde{M}M^3\frac{\sigma_1}{\lambda_1}$, and $C_2 = 4\widetilde{M}M^3\frac{\sigma_1}{\lambda_1}$. As for $\|K(f_2) - K(\widetilde{f}_2)\|$ and $\|K(f_3) - K(\widetilde{f}_3)\|$, the following results can be argued similarly:

$$\|K(f_2) - K(\widetilde{f}_2)\| \leq C_3\|f_1 - \widetilde{f}_1\| + C_4\|f_2 - \widetilde{f}_2\| + C_5\|f_3 - \widetilde{f}_3\|,$$
$$\|K(f_3) - K(\widetilde{f}_3)\| \leq C_6\|f_2 - \widetilde{f}_2\| + C_7\|f_3 - \widetilde{f}_3\|.$$

Therefore we can obtain

$$\|K(f_1, f_2, f_3) - K(\widetilde{f}_1, \widetilde{f}_2, \widetilde{f}_3)\| \leq h\|(f_1, f_2, f_3) - (\widetilde{f}_1, \widetilde{f}_2, \widetilde{f}_3)\|.$$

Then we can conclude that K is a contraction mapping (see [23]).

Now we can complete our proof of Proposition 3.1. With Lemma 3.2 and the Banach fixed point theorem, we know that there exists a unique $f = (f_1, f_2, f_3)$ such that $Kf = f$, $f_i \in C(\Omega_i \times [0, t_0])$. By the Volterra integral equations, we can improve the regularity of f_j,

$$f_j \in C^{1+\kappa}(\Omega_j \times [0, t_0]), \quad j = 1, 2, 3,\ \kappa \in (0, 1).$$

Then by Theorems 5.2 and 5.3 (see [21, pp. 317–323]), we can conclude that there exists a unique solution $u = (u_1, u_2, u_3)$ to problem (2.1)–(2.10) with $u_i \in C^1([0, t_0], C^2(\overline{\Omega}_i))$ for $i = 1, 2, 3$. The proof of Proposition 3.1 is completed.

By the local existence and uniqueness result of Proposition 3.1, we can directly obtain the global uniqueness.

Proposition 3.2 *Assume that* $a(\mathbf{x})$ *satisfies* (2.11). *Let* T *be arbitrarily given. If the solution* $u(\mathbf{x}, t)$ *to the problem* (2.1)–(2.10) *exists, then it should be unique within the class of* (2.12).

Proof Assume contrarily that the theorem is not right. Then we will have two different solutions $u(\mathbf{x}, t)$, $\widetilde{u}(\mathbf{x}, t)$ satisfying the problem (2.1)–(2.10). We can also make the assumption that there exists a $t_1 \geq 0$, such that $u(\mathbf{x}, t) \neq \widetilde{u}(\mathbf{x}, t)$ while $t > t_1$. Then we can set the initial value of the time interval to t_1, and make the following conclusion with Theorem 2.1: there exists a $t_0 \in (0, 1)$, such that the problem (2.1)–(2.10) has a unique solution in $t \in [t_1, t_1 + t_0]$, which is obviously a contradiction to our previous assumptions.

3.3 Global existence in time

By the maximum principle of the parabolic equation (see [22]), we have the following lemmas.

Lemma 3.3 *Let* u_i *satisfy* (2.1)–(2.10). *Then* u_i *attains its maximum or minimum value only on the parabolic boundary, i.e.,*

(1) for $u_1(\mathbf{x}, t) : \{\mathbf{x} \in \Gamma_0\}, \{\mathbf{x} \in \Gamma_1\}$, or $u_1|_{t=0} = a(\mathbf{x})$,
(2) for $u_2(\mathbf{x}, t) : \{\mathbf{x} \in \Gamma_1\}, \{\mathbf{x} \in \Gamma_2\}$, or $u_2|_{t=0} = a(\mathbf{x})$,
(3) for $u_3(\mathbf{x}, t) : \{\mathbf{x} \in \Gamma_2\}, \{\mathbf{x} \in \Gamma_3\}$, or $u_3|_{t=0} = a(\mathbf{x})$.

Lemma 3.4 *Let u_i satisfy* (2.1)–(2.10)*. Then*

$$u_1|_{\mathbf{x}\in\Gamma_0}, u_1|_{\mathbf{x}\in\Gamma_1}, u_2|_{\mathbf{x}\in\Gamma_1}, u_2|_{\mathbf{x}\in\Gamma_2}, u_3|_{\mathbf{x}\in\Gamma_2}, u_3|_{\mathbf{x}\in\Gamma_3} > 0.$$

Proof Assume contrarily that the lemma is not correct. From the continuity of u_1,u_2,u_3, at least one or more $u_i(\mathbf{x_j}, t_k)$ on the corresponding boundary would equal zero at some time t_k. Set $t_0 = \min\{t_k\}$. Without loss of generality, we consider the boundary Γ_1, and assume that there is one point $\mathbf{x} \in \Gamma_1$ which gives $u_i(\mathbf{x}, t_0) = 0$. Then we have the following three cases:

(1) $u_1(\mathbf{x}, t_0)$ equals zero while $u_2(\mathbf{x}, t_0)$ does not. Then by the equality from (2.5),

$$-\lambda_1\partial_{n_1} u_1(\mathbf{x}, t_0) = \sigma_1(u_1^4(\mathbf{x}, t_0) - u_2^4(\mathbf{x}, t_0)), \quad \mathbf{x} \in \Gamma_1.$$

We get $\partial_{n_1} u_1(\mathbf{x}, t_0) > 0$. By the continuity of u_1, there must exist $\mathbf{x_1} \in \Omega_1$, such that $u_1(\mathbf{x_1}, t_0) < 0$, which is a contradiction to Lemma 3.3.

(2) $u_2(\mathbf{x}, t_0)$ equals zero while $u_1(\mathbf{x}, t_0)$ does not. Similarly, as the first case, it is impossible.

(3) $u_1(\mathbf{x}, t_0)$ and $u_2(\mathbf{x}, t_0)$ both equal zero. From the equalities we used above, we can derive $\partial_{n_1} u_1(\mathbf{x}, t_0) = 0$, but by the Strong Maximum Principle for parabolic equation we know that it is impossible.

Therefore, $u_1|_{\mathbf{x}\in\Gamma_1}$ and $u_2|_{\mathbf{x}\in\Gamma_1}$ cannot equal zero. And we can similarly prove that $u_1|_{\mathbf{x}\in\Gamma_0}$, $u_2|_{\mathbf{x}\in\Gamma_2}$, $u_3|_{\mathbf{x}\in\Gamma_2}$, $u_3|_{\mathbf{x}\in\Gamma_3}$ cannot equal zero. Since $a(\mathbf{x}) > 0$ and u_i are continuous, we can conclude that they must be positive.

Lemmas 3.3 and 3.4 lead us to the following proposition, which is essential for the theorem of global existence.

Proposition 3.3 *Let $u = (u_1, u_2, u_3)$ satisfy* (2.1)–(2.10)*, and $T > 0$ be arbitrarily fixed. Then we have*

$$\begin{aligned}&\max\{\|u_1\|_{C(\overline{\Omega}_1\times[0,T])}, \|u_2\|_{C(\overline{\Omega}_2\times[0,T])}, \|u_3\|_{C(\overline{\Omega}_3\times[0,T])}\}\\ &\le \max\{\|a\|_{C(\Omega_1\cup\Omega_2\cup\Omega_3)}, |u_M|, |u_A|\}.\end{aligned}$$

Proof Set $M_1 = \max\left\{\max\limits_{\mathbf{x}\in\Omega_1\cup\Omega_2\cup\Omega_3} a(\mathbf{x}), u_M, u_A\right\}$. We first prove

$$u_j(\mathbf{x}, t) \le M_1, \quad \mathbf{x} \in \Omega_j,\ t \in [0, T],\ j = 1, 2, 3. \tag{3.9}$$

Set $\mu < 0$ and $v_j(\mathbf{x}, t) = \mathrm{e}^{\mu t}(u_j(\mathbf{x}, t) - M_1)$. Then for $i = 1, 2$ and 3, we have

$$\partial_t v_i(\mathbf{x}, t) = \alpha_i\Delta v_i(\mathbf{x}, t) + \mu v_i(\mathbf{x}, t),$$

and (v_1, v_2, v_3) remains in the class defined by (2.12). Assume that v_j attains the maximum at $\mathbf{x_0} \in \Omega_1 \cup \Omega_2 \cup \Omega_3$, $t_0 \in (0, T]$. By Lemma 3.3, we know

$$\max_{j=1,2,3} \max_{\substack{\mathbf{x}\in\Omega_j\\ t\in[0,T]}} v_j(\mathbf{x}, t) = \begin{cases} v_1(\mathbf{x_0}, t_0) \text{ or } v_2(\mathbf{x_0}, t_0), & \text{if } \mathbf{x_0} \in \Gamma_1,\\ v_2(\mathbf{x_0}, t_0) \text{ or } v_3(\mathbf{x_0}, t_0), & \text{if } \mathbf{x_0} \in \Gamma_2.\end{cases}$$

Define $M_0 = \max\limits_{j=1,2,3} \max\limits_{\substack{\mathbf{x}\in\Omega_j\\ t\in[0,T]}} v_j(\mathbf{x}, t)$. If $M_0 \le 0$, then $v_j(\mathbf{x}, t) = \mathrm{e}^{\mu t}(u_j(\mathbf{x}, t) - M_1) \le 0$, we have proved (3.9). If $M_0 > 0$, without loss of generality, we assume $\mathbf{x_0} \in \Gamma_1$, then

$$\begin{cases} -\lambda_1\partial_{n_1} v_1(\mathbf{x_0}, t_0) = \sigma_1\mathrm{e}^{\mu t_0}(u_1^4(\mathbf{x_0}, t_0) - u_2^4(\mathbf{x_0}, t_0)),\\ \lambda_1\partial_{n_1} v_1(\mathbf{x_0}, t_0) = -\lambda_2\partial_{n_2} v_2(\mathbf{x_0}, t_0).\end{cases} \tag{3.10}$$

If $M_0 = v_1(\mathbf{x_0}, t_0)$, then $\partial_{n_1} v_1(\mathbf{x_0}, t_0) \geq 0$ since $v_1(\,\cdot\,, t_0)$ attains maximum in Ω_1 at x_0. Now if $\partial_{n_1} v_1(\mathbf{x_0}, t_0) > 0$, then $u_1(\mathbf{x_0}, t_0) < u_2(\mathbf{x_0}, t_0)$ due to (3.10) and Lemma 3.4. This implies $v_1(\mathbf{x_0}, t_0) < v_2(\mathbf{x_0}, t_0)$, which is a contradiction since $v_1(\mathbf{x_0}, t_0)$ is the maximum. Otherwise, if $\partial_{n_1} v_1(\mathbf{x_0}, t_0) = 0$, we have $\partial_t v_1(\mathbf{x_0}, t_0) \geq 0$ and $\Delta v_2(\mathbf{x_0}, t_0) \leq 0$, moreover, $\mu v_1(\mathbf{x_0}, t_0) < 0$. It is also impossible since

$$0 \leq \partial_t v_1(\mathbf{x_0}, t_0) = \alpha_1 \Delta v_1(\mathbf{x_0}, t_0) + \mu v_1(\mathbf{x_0}, t_0) < 0.$$

Similarly, it is impossible that $M_0 = v_2(\mathbf{x_0}, t_0)$. We can get the same conclusion for $x_0 \in \Gamma_2$. Thus, M_0 cannot be positive, so that (3.9) is proved.

Next, set

$$m_1 = \min\Big\{ \min_{\mathbf{x} \in \Omega_1 \cup \Omega_2 \cup \Omega_3} a(\mathbf{x}), u_M, u_A \Big\},$$

with $\mu < 0$ and $w_j(\mathbf{x}, t) = \mathrm{e}^{\mu t}(u_j(\mathbf{x}, t) - m_1)$. We can argue similarly to see that

$$u_j(\mathbf{x}, t) \geq m_1, \quad \mathbf{x} \in \Omega_j,\ t \in [0, T],\ j = 1, 2, 3. \tag{3.11}$$

Combining this inequality with (3.9), the proof of proposition is complete.

Now we finish the proof of Theorem 2.1.

Proof of Theorem 2.1 With Propositions 3.1–3.3, by referring to [16, Theorem 3], we can conclude that there exists a unique solution to problem (2.1)–(2.10) within the class of (2.12), when $a(\mathbf{x})$ satisfies (2.11).

4 Uniqueness of Inverse Problem

4.1 Some lemmas

In this subsection, we introduce several important lemmas for the proof of our theorem. For the constant $\alpha > 0$, denote the operator $L \equiv \partial_t - \alpha\Delta$. In this section, the convergence $v \to u$ is in the sense of L^2 norm, i.e., $\|u - v\|_{L^2} = 0$.

Lemma 4.1 *For any bounded domain $\Omega \in \mathbb{R}^3$ with piecewise smooth boundary $\partial\Omega$, if function $u(\mathbf{x}, t)$ satisfies*

$$\begin{aligned} Lu &= 0, \quad \mathbf{x} \in \Omega,\ t > 0, \\ u(\mathbf{x}, t) &= C, \quad \mathbf{x} \in \partial\Omega,\ t > 0, \end{aligned}$$

where C is a constant, then we have

$$u(\mathbf{x}, t) \to C, \quad \mathbf{x} \in \Omega,\ t \to \infty.$$

The result of Lemma 4.1 can be derived by the theorem of eigenvalues and eigenfunctions of the operator, or one can refer to [15].

Next, we introduce a Carleman estimate for parabolic equations, obtained by Imanuvilov and Yamamoto (see [17, Theorem 2.1]).

Let $(t,\mathbf{x}) \in Q = (0,T) \times \Omega$. We assume that the boundary $\partial\Omega$ is sufficiently smooth. Let $\omega \subset \Omega$ be an arbitrarily fixed subdomain, $Q_\omega = (0,T) \times \omega$ and $\omega_0 \subset \omega$ be an arbitrary fixed subdomain of Ω such that $\overline{\omega}_0 \subset \omega$. Then there exists a function $\psi \in C^2(\overline{\Omega})$ such that $\psi(\mathbf{x}) > 0, \mathbf{x} \in \Omega$, $\psi|_{\partial\Omega} = 0$, and $|\nabla\psi(\mathbf{x})| > 0$ for $\mathbf{x} \in \overline{\Omega \setminus \omega_0}$. The weight functions $\varphi(t,\mathbf{x})$ and $\alpha(t,\mathbf{x})$ are

$$\varphi(t,\mathbf{x}) = \frac{\mathrm{e}^{\lambda\psi(\mathbf{x})}}{t(T-t)} \quad \text{and} \quad \alpha(t,\mathbf{x}) = \frac{\mathrm{e}^{\lambda\psi(\mathbf{x})} - \mathrm{e}^{2\lambda\|\psi\|_{C(\overline{\Omega})}}}{t(T-t)}, \tag{4.1}$$

where $\lambda > 0$ is a parameter. Consider the initial boundary value problem

$$Lu = 0, \quad (t,\mathbf{x}) \in Q, \tag{4.2}$$

$$u|_{\partial\Omega} = 0, \quad u(0,\cdot) = u_0. \tag{4.3}$$

Lemma 4.2 (Carleman Estimate) *Let functions φ and α be defined by (4.1). Then there exists a number $\widehat{\lambda} > 0$ such that for arbitrary $\lambda \geq \widehat{\lambda}$, we can choose $s_0(\lambda) > 0$ satisfying: there exists a constant $C_1 > 0$ such that for each $s \geq s_0(\lambda)$ the solution $u \in L^2(Q)$ to the problem (4.2) and (4.3) satisfies the following inequality:*

$$\int_Q ((s\varphi)^{1-2\ell}|\nabla u|^2 + (s\varphi)^{3-2\ell}u^2)\mathrm{e}^{2s\alpha}\mathrm{d}\mathbf{x}\mathrm{d}t \leq C_1 \int_{Q_\omega} (s\varphi)^{3-2\ell}u^2\mathrm{e}^{2s\alpha}\mathrm{d}\mathbf{x}\mathrm{d}t, \tag{4.4}$$

where $\ell \in [0,1]$, constant C_1 is dependent continuously on λ and independent of s.

Remark 4.1 We would like to mention that the functions φ, α defined by (4.1) satisfy the following for any $k \geq 0$,

$$\lim_{t\downarrow 0} \varphi^k(t,\mathbf{x})\mathrm{e}^{2s\alpha(t,\mathbf{x})} = \lim_{t\uparrow T} \varphi^k(t,\mathbf{x})\mathrm{e}^{2s\alpha(t,\mathbf{x})} = 0.$$

With Lemma 4.2, we can derive the following proposition.

Proposition 4.1 *Let domain $\omega \subset \Omega$, function $u(\mathbf{x},t)$ satisfy*

$$Lu = 0, \quad \mathbf{x} \in \Omega,\ t > 0.$$

If we have the following estimation in ω,

$$u(\mathbf{x},t) \to C, \quad \mathbf{x} \in \omega,\ t \to \infty,$$

where C is a constant, then we can conclude the following in Ω,

$$u(\mathbf{x},t) \to C, \quad \mathbf{x} \in \Omega,\ t \to \infty.$$

Proof Set $\upsilon(\mathbf{x},t) = u(\mathbf{x},t) - C$. Then $\upsilon(\mathbf{x},t)$ satisfies

$$\partial_t\upsilon(\mathbf{x},t) = \alpha\Delta\upsilon(\mathbf{x},t), \quad \mathbf{x} \in \Omega,\ t > 0.$$

By the assumptions, we know that for any $\varepsilon > 0$, there exists $T_0 > 0$ such that for any $t > T_0$ the function $u(\mathbf{x},t)$ satisfies

$$\|u(\mathbf{x},t) - C\|_{L^2(\omega)} < \varepsilon, \quad \mathbf{x} \in \omega,\ t > T_0,$$

which means

$$\|\upsilon(\mathbf{x},t)\|_{L^2(\omega)} < \varepsilon, \quad \mathbf{x}\in\omega,\ t > T_0.$$

For the simplicity of the proof, we set $\widehat{t} = t - T_0$ and have

$$\|\upsilon(\mathbf{x},\widehat{t}\,)\|_{L^2(\omega)} < \varepsilon, \quad \mathbf{x}\in\omega,\ \widehat{t} > 0.$$

By Lemma 4.2, we get

$$\int_Q ((s\varphi)^{1-2\ell}|\nabla\upsilon|^2 + (s\varphi)^{3-2\ell}\upsilon^2)\mathrm{e}^{2s\alpha}\mathrm{d}\mathbf{x}\mathrm{d}\widehat{t} \le C_1\int_{Q_\omega}(s\varphi)^{3-2\ell}\upsilon^2\mathrm{e}^{2s\alpha}\mathrm{d}\mathbf{x}\mathrm{d}\widehat{t},$$

where $Q = (0,T)\times\Omega$, $Q_\omega = (0,T)\times\omega$, $\ell\in[0,1]$ and $T > 0$ is an arbitrary constant. We can choose $\ell = 1$ above, and then obtain the following since $s,\varphi > 0$,

$$\int_Q s\varphi\upsilon^2\mathrm{e}^{2s\alpha}\mathrm{d}\mathbf{x}\mathrm{d}\widehat{t} \le C_1\int_{Q_\omega} s\varphi\upsilon^2\mathrm{e}^{2s\alpha}\mathrm{d}\mathbf{x}\mathrm{d}\widehat{t}. \tag{4.5}$$

Now we estimate both sides of the inequality (4.5) respectively, and we consider the right-hand side (RHS) first. Let $C_2 = C_1 s\mathrm{e}^{\lambda\|\psi\|_{C(\overline{\Omega})}}$, $-M = 2s(\mathrm{e}^{\lambda\|\psi\|_{C(\overline{\Omega})}} - \mathrm{e}^{2\lambda\|\psi\|_{C(\overline{\Omega})}})$. Then $M > 0$ and we get

$$\begin{aligned}
\text{RHS} &= C_1\int_{Q_\omega} s\varphi\upsilon^2\mathrm{e}^{2s\alpha}\mathrm{d}\mathbf{x}\mathrm{d}\widehat{t}\\
&= C_1\int_\omega\int_0^T s\frac{\mathrm{e}^{\lambda\psi(\mathbf{x})}}{\widehat{t}(T-\widehat{t}\,)}\upsilon^2(\mathbf{x},\widehat{t}\,)\exp\Big\{2s\frac{\mathrm{e}^{\lambda\psi(\mathbf{x})} - \mathrm{e}^{2\lambda\|\psi\|_{C(\overline{\Omega})}}}{\widehat{t}(T-\widehat{t}\,)}\Big\}\mathrm{d}\mathbf{x}\mathrm{d}\widehat{t}\\
&\le C_1 s\int_\omega\int_0^T \frac{\mathrm{e}^{\lambda\psi(\mathbf{x})}}{\widehat{t}(T-\widehat{t}\,)}\upsilon^2(\mathbf{x},\widehat{t}\,)\exp\Big\{2s\frac{\mathrm{e}^{\lambda\psi(\mathbf{x})} - \mathrm{e}^{2\lambda\|\psi\|_{C(\overline{\Omega})}}}{\widehat{t}(T-\widehat{t}\,)}\Big\}\mathrm{d}\mathbf{x}\mathrm{d}\widehat{t}\\
&\le C_1 s\mathrm{e}^{\lambda\|\psi\|_{C(\overline{\Omega})}}\int_\omega\int_0^T \frac{1}{\widehat{t}(T-\widehat{t}\,)}\upsilon^2(\mathbf{x},\widehat{t}\,)\exp\Big\{2s\frac{\mathrm{e}^{\lambda\|\psi\|_{C(\overline{\Omega})}} - \mathrm{e}^{2\lambda\|\psi\|_{C(\overline{\Omega})}}}{\widehat{t}(T-\widehat{t}\,)}\Big\}\mathrm{d}\mathbf{x}\mathrm{d}\widehat{t}\\
&= C_2\int_\omega\int_0^T \frac{1}{\widehat{t}(T-\widehat{t}\,)}\upsilon^2(\mathbf{x},\widehat{t}\,)\exp\Big\{-\frac{M}{\widehat{t}(T-\widehat{t}\,)}\Big\}\mathrm{d}\mathbf{x}\mathrm{d}\widehat{t}.
\end{aligned}$$

For the function $f(\widehat{t}\,) = \frac{1}{\widehat{t}(T-\widehat{t}\,)}\exp\big\{-\frac{M}{\widehat{t}(T-\widehat{t}\,)}\big\}$ in the integral part, it is clear that $f(\widehat{t}\,) > 0$ holds true for any $\widehat{t}\in(0,T)$. Besides, we know from Lemma 4.2 that

$$\lim_{\widehat{t}\downarrow 0} f(\widehat{t}\,) = \lim_{\widehat{t}\uparrow T} f(\widehat{t}\,) = 0.$$

And further analysis shows

$$f'(\widehat{t}\,) = \frac{(T-2\widehat{t}\,)(M+\widehat{t}(\widehat{t}-T))}{\widehat{t}^3(T-\widehat{t}\,)^3}\exp\Big\{-\frac{M}{\widehat{t}(T-\widehat{t}\,)}\Big\}.$$

Since $s \ge s_0(\lambda)$ in Lemma 4.2 can be arbitrarily chosen, we can find an $M > 0$ which is sufficiently large to ensure that $M + \widehat{t}(\widehat{t}-T) > 0$ holds true for any $\widehat{t}\in(0,T)$. Then we get

$$\begin{cases}
f'(\widehat{t}\,) > 0, & \widehat{t}\in\Big(0,\dfrac{T}{2}\Big),\\
f'(\widehat{t}\,) = 0, & \widehat{t} = \dfrac{T}{2},\\
f'(\widehat{t}\,) < 0, & \widehat{t}\in\Big(\dfrac{T}{2},T\Big).
\end{cases}$$

Therefore the range of function $f(\widehat{t})$ is $(0, \frac{4}{T^2}\mathrm{e}^{-\frac{4M}{T^2}})$ in $(0,T)$, where its maximum is reached when $\widehat{t} = \frac{T}{2}$. Thus,

$$\begin{aligned}\mathrm{RHS} &\le C_2 \int_\omega \int_0^T \frac{1}{\widehat{t}(T-\widehat{t}\,)} \upsilon^2(\mathbf{x},\widehat{t}\,) \exp\Big\{-\frac{M}{\widehat{t}(T-\widehat{t}\,)}\Big\} \mathrm{d}\mathbf{x}\mathrm{d}\widehat{t} \\ &= C_2 \int_\omega \int_0^T f(\widehat{t}\,)\upsilon^2(\mathbf{x},\widehat{t}\,)\mathrm{d}\mathbf{x}\mathrm{d}\widehat{t} \\ &\le \frac{4C_2}{T^2}\mathrm{e}^{-\frac{4M}{T^2}} \int_\omega \int_0^T \upsilon^2(\mathbf{x},\widehat{t}\,)\mathrm{d}\mathbf{x}\mathrm{d}\widehat{t}.\end{aligned}$$

Next, we consider the left-hand side (LHS) of inequality (4.5). Denote $N = -2s(1 - \mathrm{e}^{2\lambda\|\psi\|_{C(\overline{\Omega})}})$. Then $N > 0$ and we have

$$\begin{aligned}\mathrm{LHS} &= \int_Q s\varphi\upsilon^2\mathrm{e}^{2s\alpha}\mathrm{d}\mathbf{x}\mathrm{d}\widehat{t} \\ &= \int_\Omega \int_0^T s\frac{\mathrm{e}^{\lambda\psi(\mathbf{x})}}{\widehat{t}(T-\widehat{t}\,)}\upsilon^2(\mathbf{x},\widehat{t}\,)\exp\Big\{2s\frac{\mathrm{e}^{\lambda\psi(\mathbf{x})} - \mathrm{e}^{2\lambda\|\psi\|_{C(\overline{\Omega})}}}{\widehat{t}(T-\widehat{t}\,)}\Big\}\mathrm{d}\mathbf{x}\mathrm{d}\widehat{t} \\ &\ge \int_\Omega \int_0^T s\frac{1}{\widehat{t}(T-\widehat{t}\,)}\upsilon^2(\mathbf{x},\widehat{t}\,)\exp\Big\{2s\frac{1 - \mathrm{e}^{2\lambda\|\psi\|_{C(\overline{\Omega})}}}{\widehat{t}(T-\widehat{t}\,)}\Big\}\mathrm{d}\mathbf{x}\mathrm{d}\widehat{t} \\ &= s\int_\Omega \int_0^T \frac{1}{\widehat{t}(T-\widehat{t}\,)}\upsilon^2(\mathbf{x},\widehat{t}\,)\exp\Big\{-\frac{N}{\widehat{t}(T-\widehat{t}\,)}\Big\}\mathrm{d}\mathbf{x}\mathrm{d}\widehat{t} \\ &\ge s\int_\Omega \int_{\frac14 T}^{\frac34 T} \frac{1}{\widehat{t}(T-\widehat{t}\,)}\upsilon^2(\mathbf{x},\widehat{t}\,)\exp\Big\{-\frac{N}{\widehat{t}(T-\widehat{t}\,)}\Big\}\mathrm{d}\mathbf{x}\mathrm{d}\widehat{t}.\end{aligned}$$

Similarly, for the function $g(\widehat{t}) = \frac{1}{\widehat{t}(T-\widehat{t})}\exp\{-\frac{N}{\widehat{t}(T-\widehat{t})}\}$ in the integral part, $f(\widehat{t}) > 0$ holds true for any $\widehat{t} \in (0,T)$. We also have $\lim_{\widehat{t}\downarrow 0} g(\widehat{t}) = \lim_{\widehat{t}\uparrow T} g(\widehat{t}) = 0$ and

$$g'(\widehat{t}) = \frac{(T-2\widehat{t})(N+\widehat{t}(\widehat{t}-T))}{\widehat{t}^3(T-\widehat{t})^3}\exp\Big\{-\frac{N}{\widehat{t}(T-\widehat{t})}\Big\}.$$

In the same way, since $s \ge s_0(\lambda)$ in Lemma 4.2 can be arbitrarily chosen, we can find an $N > 0$ which is sufficiently large to ensure that $N + \widehat{t}(\widehat{t} - T) > 0$ holds true for any $\widehat{t} \in (0,T)$. Then we get

$$\begin{cases} g'(\widehat{t}\,) > 0, & \widehat{t} \in \Big(0, \dfrac{T}{2}\Big), \\ g'(\widehat{t}\,) = 0, & \widehat{t} = \dfrac{T}{2}, \\ g'(\widehat{t}\,) < 0, & \widehat{t} \in \Big(\dfrac{T}{2}, T\Big). \end{cases}$$

Therefore, the range of function $g(\widehat{t})$ is $\big[\frac{16}{3T^2}\mathrm{e}^{-\frac{16N}{3T^2}}, \frac{4}{T^2}\mathrm{e}^{-\frac{4N}{T^2}}\big]$ in $[\frac{T}{4}, \frac{3T}{4}]$, where its maximum is reached when $\widehat{t} = \frac{T}{2}$ and its minimum is reached when $\widehat{t} = \frac{T}{4}, \frac{3T}{4}$. Thus

$$\mathrm{LHS} \ge s\int_\Omega \int_{\frac14 T}^{\frac34 T} \frac{1}{\widehat{t}(T-\widehat{t})}\upsilon^2(\mathbf{x},\widehat{t})\exp\Big\{-\frac{N}{\widehat{t}(T-\widehat{t})}\Big\}\mathrm{d}\mathbf{x}\mathrm{d}\widehat{t}$$

$$
\begin{aligned}
&= s\int_\Omega\int_{\frac{1}{4}T}^{\frac{3}{4}T} g(\widehat{t})\upsilon^2(\mathbf{x},\widehat{t})\mathrm{d}\mathbf{x}\mathrm{d}\widehat{t} \\
&\geq \frac{16s}{3T^2}\mathrm{e}^{-\frac{16N}{3T^2}}\int_\Omega\int_{\frac{1}{4}T}^{\frac{3}{4}T} \upsilon^2(\mathbf{x},\widehat{t})\mathrm{d}\mathbf{x}\mathrm{d}\widehat{t}.
\end{aligned}
$$

Combining the estimations above for both sides of inequality (4.5), while choosing a sufficiently large $s \geq s_0(\lambda)$ in Lemma 4.2, we have

$$
\frac{16s}{3T^2}\mathrm{e}^{-\frac{16N}{3T^2}}\int_\Omega\int_{\frac{1}{4}T}^{\frac{3}{4}T} \upsilon^2(\mathbf{x},\widehat{t}\,)\mathrm{d}\mathbf{x}\mathrm{d}\widehat{t} \leq \frac{4C_2}{T^2}\mathrm{e}^{-\frac{4M}{T^2}}\int_\omega\int_0^T \upsilon^2(\mathbf{x},\widehat{t}\,)\mathrm{d}\mathbf{x}\mathrm{d}\widehat{t}.
$$

Let $C_3 = (\frac{3C_2}{4s}\mathrm{e}^{\frac{(16N-12M)}{3T^2}})^{\frac{1}{2}}$. By taking square root on both sides above we have

$$
\|\upsilon(\mathbf{x},\widehat{t}\,)\|_{L^2((\frac{T}{4},\frac{3T}{4})\times\Omega)} \leq C_3\|\upsilon(\mathbf{x},\widehat{t}\,)\|_{L^2((0,T)\times\omega)}.
$$

Since

$$
\|\upsilon(\mathbf{x},\widehat{t}\,)\|^2_{L^2((0,T)\times\omega)} = \int_0^T\int_\omega \upsilon^2\mathrm{d}\mathbf{x}\mathrm{d}\widehat{t} = \int_0^T \|\upsilon(\mathbf{x},\widehat{t}\,)\|^2_{L^2(\omega)}\mathrm{d}\widehat{t} \leq T\varepsilon^2,
$$

letting $C_4 = C_3T^{\frac{1}{2}}$, we get

$$
\|\upsilon(\mathbf{x},\widehat{t}\,)\|_{L^2((\frac{T}{4},\frac{3T}{4})\times\Omega)} \leq C_4\varepsilon. \tag{4.6}
$$

Recalling the variable replacement $\widehat{t} = t - T_0$ we have made in the beginning part, (4.6) tells us that, for any $\varepsilon > 0$, there exists $T_0 > 0$ such that $\|\upsilon(\mathbf{x},t)\|_{L^2((T_0+\frac{T}{4},T_0+\frac{3T}{4})\times\Omega)} \leq C_4\varepsilon$ when $t > T_0$. Because ε can be arbitrarily chosen, we can conclude

$$
\upsilon(\mathbf{x},t) \to 0, \quad \mathbf{x}\in\Omega,\ t\to\infty.
$$

By the definition of υ, we have

$$
u(\mathbf{x},t) \to C, \quad \mathbf{x}\in\Omega,\ t\to\infty.
$$

The proof of Proposition 4.1 is completed.

4.2 Uniqueness proof of inverse problems

With the lemmas and proposition in last subsection, we are now able to provide the uniqueness of the inverse problems.

Proof of Theorem 2.2 First, we have the equality on $\Gamma_3 \subset \partial\Omega_3$,

$$
u_3(\mathbf{x},t) = \widetilde{u}_3(\mathbf{x},t), \quad \partial_{n_3}u_3(\mathbf{x},t) = \partial_{n_3}\widetilde{u}_3(\mathbf{x},t), \quad \mathbf{x}\in\Gamma_3.
$$

By the uniqueness of the Cauchy problem (Holmgren theorem, refer to [18, 19]), we have the following results in Ω_3,

$$
u_3(\mathbf{x},t) = \widetilde{u}_3(\mathbf{x},t), \quad \mathbf{x}\in\Omega_3,
$$

which already prove the uniqueness of $u(\mathbf{x},t)$ in Ω_3. Moreover, we get

$$u_3(\mathbf{x},t)=\widetilde{u}_3(\mathbf{x},t),\quad \partial_{n_3}u_3(\mathbf{x},t)=\partial_{n_3}\widetilde{u}_3(\mathbf{x},t),\quad \mathbf{x}\in\Gamma_2.$$

With the boundary conditions (2.7) and (2.8) on interface Γ_2, we know

$$u_2(\mathbf{x},t)=\widetilde{u}_2(\mathbf{x},t),\quad \partial_{n_2}u_2(\mathbf{x},t)=\partial_{n_2}\widetilde{u}_2(\mathbf{x},t),\quad \mathbf{x}\in\Gamma_2.$$

Repeating the process above, we can prove the uniqueness of $u(\mathbf{x},t)$ in Ω_2, and we obtain the following on Γ_1,

$$u_1(\mathbf{x},t)=\widetilde{u}_1(\mathbf{x},t),\quad \partial_{n_1}u_1(\mathbf{x},t)=\partial_{n_1}\widetilde{u}_1(\mathbf{x},t),\quad \mathbf{x}\in\Gamma_1.$$

Suppose contrarily that boundary $\Gamma_0\neq\widetilde{\Gamma}_0$. Then by the uniqueness of Cauchy problem we have

$$u_1(\mathbf{x},t)=\widetilde{u}_1(\mathbf{x},t),\quad \mathbf{x}\in(\Omega_1\setminus\widetilde{\Omega}_0)\subset\Omega_1.$$

Since $u_1(\mathbf{x},t)$, $\widetilde{u}_1(\mathbf{x},t)$ satisfy (2.4) on Γ_0, $\widetilde{\Gamma}_0$ respectively, we know

$$u_1(\mathbf{x},t)=u_M,\quad \mathbf{x}\in\partial(\widetilde{\Omega}_0\setminus\Omega_0).$$

Then we can conclude with Lemma 4.1 that

$$u_1(\mathbf{x},t)\to u_M,\quad \mathbf{x}\in(\widetilde{\Omega}_0\setminus\Omega_0)\subset\Omega_1,\ t\to\infty.$$

And by Proposition 4.1, we derive

$$u_1(\mathbf{x},t)\to u_M,\quad \mathbf{x}\in\Omega_1,\ t\to\infty.$$

Thus, we obtain the following on Γ_1,

$$u_1(\mathbf{x},t)\to u_M,\quad \partial_{n_1}u_1(\mathbf{x},t)\to 0,\quad \mathbf{x}\in\Gamma_1,\ t\to\infty.$$

Again, with the boundary conditions (2.5) and (2.6) on interface Γ_1, we get

$$u_2(\mathbf{x},t)\to u_M,\quad \partial_{n_2}u_2(\mathbf{x},t)\to 0,\quad \mathbf{x}\in\Gamma_1,\ t\to\infty.$$

Assume that $\overline{u}_2(x,t)$ satisfies

$$\begin{aligned}
&\partial_t\overline{u}_2(\mathbf{x},t)=\alpha_2\Delta\overline{u}_2(\mathbf{x},t), && \mathbf{x}\in\Omega_2,\ t\in(0,T),\\
&\overline{u}_2(\mathbf{x},t)=u_M,\quad \partial_{n_2}\overline{u}_2(\mathbf{x},t)=0, && \mathbf{x}\in\Gamma_1,\ t\in(0,T).
\end{aligned}$$

Then by the uniqueness of Cauchy problem, we know that $\overline{u}_2$ must satisfy

$$\overline{u}_2=u_M,\quad \mathbf{x}\in\Omega_2,\ t\in(0,T).$$

Due to the work of Bell [3, p. 785], the similarity between u_2 and $\overline{u}_2$ is defined in the sense that

$$\varepsilon^2=\|u_2-u_M\|^2_{H^1(\Gamma_1)}+\|\partial_{n_2}u_2-0\|^2_{L^2(\Gamma_1)}+\|(\partial_t u_2-\alpha_2\Delta u_2)-0\|^2_{L^2(\Omega_2)},$$

and there exists a domain $D_\alpha \subset \Omega_2$ which is contiguous with Γ_1, such that u_2 and $\overline{u}_2$ satisfy the following estimate inequality:

$$\iint_{D_\alpha} (u_2 - \overline{u}_2)^2 \mathrm{d}\mathbf{x}\mathrm{d}t \leq c\Big\{\varepsilon^{2(1-\alpha)} M^{2\alpha} + \frac{2M^2}{\log(\frac{M^2}{\varepsilon^2})}\Big\} \frac{1}{\log(\frac{M^2}{\varepsilon^2})}, \quad 0 < \alpha \leq 1.$$

Since $\varepsilon \to 0$ as $t \to \infty$, it indicates that

$$u_2(\mathbf{x}, t) \to u_M, \quad \mathbf{x} \in D_\alpha \subset \Omega_2,\ t \to \infty.$$

Thus, by Proposition 4.1 we get

$$u_2(\mathbf{x}, t) \to u_M, \quad \mathbf{x} \in \Omega_2,\ t \to \infty.$$

Therefore we have the following results on Γ_2,

$$u_2(\mathbf{x}, t) \to u_M, \quad \partial_{n_2} u_2(\mathbf{x}, t) \to 0, \quad \mathbf{x} \in \Gamma_2,\ t \to \infty.$$

Similarly, repeating the analysis process in Ω_3, we can finally derive

$$u_3(\mathbf{x}, t) \to u_M, \quad \mathbf{x} \in \Gamma_3,\ t \to \infty,$$

which is clearly a contradiction to the assumption $\|f - u_M\| = \|u_3(\mathbf{x}, t) - u_M\| \geq \gamma_0 > 0$. $u(\mathbf{x}, t) = \widetilde{u}(\mathbf{x}, t)$ can be obtained by the uniqueness results of the Cauchy problems if the domain is uniquely determined.

5 Conclusion

In this paper, we mainly focus on the heat transfer problem with Stefan-Boltzmann conditions on the boundaries or interfaces, which is considered in the multi-dimensional case and in composite material situation. We provide the theorem of global existence and uniqueness in time for the forward problem, as well as the uniqueness theorem for the related inverse problem of boundary determination.

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