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Error Bounds for Uniform Asymptotic Expansions—Modified Bessel Function of Purely Imaginary Order

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(*Dedicated to Professor Roger Temam on the Occasion of his* 70*th Birthday*)

Abstract The authors modify a method of Olde Daalhuis and Temme for representing the remainder and coefficients in Airy-type expansions of integrals. By using a class of rational functions, they express these quantities in terms of Cauchy-type integrals; these expressions are natural generalizations of integral representations of the coefficients and the remainders in the Taylor expansions of analytic functions. By using the new representation, a computable error bound for the remainder in the uniform asymptotic expansion of the modified Bessel function of purely imaginary order is derived.

Keywords Modified Bessel function of purely imaginary order, Airy function, Uniform asymptotic expansion, Error bound
2000 MR Subject Classification 41A60, 33E20, 30E20

1 Introduction

In recent years, many integrals arising in practical applications are given in the form

$$I(\lambda,\alpha)=\int_{\mathcal{C}}\mathrm{e}^{\lambda f(z,\alpha)}g(z)\mathrm{d}z, \tag{1.1}$$

where $\mathcal{C}$ is a contour in the complex plane, λ is a large asymptotic variable, $f(z,\alpha)$ and $g(z)$ are analytic functions of z in a neighborhood of $\mathcal{C}$, and α is an auxiliary parameter. For instance, in the study of Kelvin's ship wave pattern, Ursell [14] considered the asymptotic behavior of the contour integral

$$\int_{\infty\exp(-\frac{7\pi}{8}\mathrm{i})}^{\infty\exp(\frac{\pi}{8}\mathrm{i})}(1+u^2)\exp\{\mathrm{i}N[(\cos\theta-u\sin\theta)\sqrt{(1+u^2)}]\}\mathrm{d}u,$$

which represents the elevation of the free surface at the point in polar coordinates (r,θ) behind the ship. Here, $N=\frac{gr}{c^2}$, c is the velocity of the ship and g is the acceleration due to gravity.

Another example occurs in an investigation of the tsunami wave profiles (see [2]). The integral studied there is

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\overline{\zeta}(\gamma)\exp\{\mathrm{i}\Delta(\gamma\xi-\sqrt{|\gamma\tanh\gamma|}\,)\}\mathrm{d}\gamma,$$

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where Δ is a scaled time, ξ is a scaled distance, and $\overline{\zeta}(\gamma)$ is a scaled Fourier transform of the initial elevation.

A third example is the integral

$$\frac{1}{2\pi}\int_{\mathrm{i}a-\infty}^{\mathrm{i}a+\infty}\widehat{f}(\omega)\exp\Big[\frac{z}{c}\phi(\omega,\theta)\Big]\mathrm{d}\omega,$$

appearing in pulse propagation (see [11]), where $\phi(\omega,\theta)=\mathrm{i}\omega[n(\omega)-\theta]$, $\theta=\frac{ct}{z}$, $n(\omega)=\big(1-\frac{\omega_p^2}{\omega^2-\omega_0^2+2\mathrm{i}\delta\omega}\big)^{\frac{1}{2}}$, ω_p is the plasma frequency of the medium, ω_0 is a resonance frequency, δ is a damping constant, c is the vacuum speed of light, and t and z represent respectively time and distance.

If the phase function $f(z,\alpha)$ in (1.1) has exactly two simple saddle points, and if they coalesce into a double saddle when α runs into a critical value, then Chester, Friedman and Ursell [4] have shown that the cubic change of variable

$$f(z,\alpha)=\frac{1}{3}w^3-b^2w+c \tag{1.2}$$

transforms $I(\lambda,\alpha)$ into an equivalent but simpler integral of the form

$$I(\lambda,\alpha)=\mathrm{e}^{\lambda c}\int_{\mathcal{L}}\mathrm{e}^{\lambda(\frac{1}{3}w^3-b^2w)}h_0(w)\mathrm{d}w, \tag{1.3}$$

where $\mathcal{L}$ is the image of $\mathcal{C}$ under the transformation $z\to w$ in (1.2). The new parameters b and c in (1.2) are analytic functions of α.

To derive an asymptotic expansion which holds uniformly for α in a neighborhood of a critical value α_0, one of the two common approaches is to apply Bleistein's method of repeated integration by parts (see [3]). The other approach is to use a two-point expansion as given in [4], and to carry out integration term by term (see also [9, p. 351]). The technique of repeated integration by parts has two advantages over the method of two-point expansion, and the advantages are: (i) it provides a recursive formula for calculating the coefficients in the expansion, (ii) it leads to an explicit expression for the remainder term. The final result is

$$I(\lambda,\alpha)=2\pi\mathrm{i}\mathrm{e}^{\lambda c}\Big[\frac{\mathrm{Ai}(\lambda^{\frac{2}{3}}b^2)}{\lambda^{\frac{1}{3}}}\sum_{m=0}^{n-1}(-1)^m\frac{a_m}{\lambda^m}-\frac{\mathrm{Ai}'(\lambda^{\frac{2}{3}}b^2)}{\lambda^{\frac{2}{3}}}\sum_{m=0}^{n-1}(-1)^m\frac{b_m}{\lambda^m}\Big]+\mathrm{e}^{\lambda c}I_n(\lambda,\alpha). \tag{1.4}$$

The remainder term $I_n(\lambda,\alpha)$ in (1.4) is given by

$$I_n(\lambda,\alpha)=\frac{1}{\lambda^n}\int_{\mathcal{L}}\mathrm{e}^{\lambda(\frac{1}{3}w^3-b^2w)}h_n(w)\mathrm{d}w, \tag{1.5}$$

where the sequence $\{h_n(w):n=0,1,2,\cdots\}$ is defined recursively by

$$\begin{aligned}h_n(w)&=a_n+b_nw+(w^2-b^2)\phi_n(w),\\ h_{n+1}(w)&=\phi_n'(w),\quad n=0,1,2,\cdots.\end{aligned} \tag{1.6}$$

The problem of constructing a bound for the error term in (1.4) was proposed in a survey article of Wong [15]. But from (1.6), it is obvious that it would be very difficult to estimate the function $h_n(w)$ in the integral (1.5). If the integration path $\mathcal{L}$ in (1.5) is part of the real line, then it may be possible to express $h_n(w)$ in terms of $h_0^{(3n)}(\zeta)$ for some real number ζ (see [15, p. 431]). However, it is still difficult to evaluate these quantities since the function $h_0(w)$ in (1.3) is related to the original functions $f(z,\alpha)$ and $g(z)$ in (1.1) through a cubic transformation.

Recently, Olde Daalhuis and Temme [10] have presented a new method to express the remainder and the coefficients in the expansion (1.4). Their method is based on a class of rational functions with which the remainder can be represented in a manner analogous to the Cauchy-type integral for remainder in the Taylor expansion of an analytic function. There are two special features in their approach, with respect to the auxiliary parameter α. They are (i) it allows the domain of validity in $b = b(\alpha)$ to be unbounded; (ii) it exhibits the double asymptotic nature of the expansion. However, their method still provides only order estimates for the remainder.

By modifying the method of Olde Daalhuis and Temme [10], we show that it may be possible to derive a computable error bound for the Airy-type expansion in (1.4). In this paper, we illustrate this with the modified Bessel function of the third kind of purely imaginary order $K_{\mathrm{i}\nu}(\nu a)$. This function plays an important role in several problems in applied mathematics and physics. For instance, it was used in a study of diffraction theory of pulse by a circular cylinder [5], in a Dirichlet problem with boundary condition on a wedge (see [7, p. 150–153]), and in the time-independent radial solution to a one-dimensional Schrödinger equation [6]. It is also the kernel of the Lebedev transform (see [16, Chapter 6]). This transform has recently been used in problems concerning electrowetting with two immiscible electrolytic solutions (see [8]) and electrodynamic Casimir effect in medium-field wedge (see [13]). An interesting feature of the function $K_{\mathrm{i}\nu}(\nu a)$ is that there are infinitely many saddle points on the steepest descent path of the phase function in its integral representation

$$K_{\mathrm{i}\nu}(\nu a) = \frac{1}{2}\int_{-\infty}^{\infty} \mathrm{e}^{-\nu a \cosh t + \mathrm{i}\nu t}\mathrm{d}t, \tag{1.7}$$

when $0 < a \leq 1$, where we assume $a > 0$ and $\nu > 0$ (see Section 3 below). Note that in 1967, Balogh [1] had already given uniform asymptotic expansions of $K_{\mathrm{i}\nu}(\nu a)$. The method that he used is based on the modified Bessel equation and the turning-point theory developed for second-order linear differential equations. Here we shall start with the integral representation (1.7), and try to obtain a error bound for this uniform asymptotic expansion.

The material in this paper is arranged as follows. In Section 2, we briefly review the method of Olde Daaluis and Temme. In Section 3, we first show how to extend the validity of the uniform asymptotic expansion as the auxiliary parameter b in (1.3) tends to infinity. Then, by modifying their method, we give a new representation, and construct a computable error bound, for the remainder. In the final section, we use the algorithm developed by Vidunas and Temme [12] to calculate the coefficients a_m and b_m in (1.4), and give some numerical results for the error bound.

2 The Method of Olde Daalhuis and Temme

We start with the integral

$$F(z, b) = \frac{1}{2\pi\mathrm{i}}\int_{\mathcal{L}} \mathrm{e}^{z(\frac{1}{3}w^3 - b^2 w)} h_0(w)\mathrm{d}w, \tag{2.1}$$

where $h_0(w)$ is an analytic function in a neighborhood of $\mathcal{L}$, $\mathcal{L}$ being the steepest-descent path through $\pm b$. Analytically, $\mathcal{L}$ is given by $\mathcal{L} = \{w = x + \mathrm{i}y \in \mathbb{C} \mid y^2 = 3x^2 - 3b^2\}$ when $b \in [0, \infty)$, and $\mathcal{L} = \{w = x + \mathrm{i}y \in \mathbb{C} \mid 3yx^2 = (y \pm \mathrm{i}b)^2(y \mp 2\mathrm{i}b)\}$ when $b \in [0, \mathrm{i}\infty)$.

To derive an asymptotic expansion for this integral which holds uniformly for b near 0, Bleistein [3] introduced a clever technique of integration by parts. His method proceeds as follows. Define

$$h_n(w) = a_n + b_n w + (w^2 - b^2) g_n(w) \tag{2.2}$$

and

$$h_{n+1}(w) = \frac{\mathrm{d}}{\mathrm{d}w} g_n(w) \tag{2.3}$$

for $n = 0, 1, 2, \cdots$. The coefficients a_n and b_n are easily determined to be

$$a_n = \frac{1}{2}[h_n(b) + h_n(-b)], \quad b_n = \frac{1}{2b}[h_n(b) - h_n(-b)]. \tag{2.4}$$

Now one inserts (2.2) in (2.1) with $n = 0$ and integrates the last term by parts. Repeating the procedure n times leads to

$$F(z, b) = \mathrm{Ai}(z^{\frac{2}{3}} b^2) \sum_{k=0}^{n-1} \frac{(-1)^k a_k}{z^{k+\frac{1}{3}}} - \mathrm{Ai}'(z^{\frac{2}{3}} b^2) \sum_{k=0}^{n-1} \frac{(-1)^k b_k}{z^{k+\frac{2}{3}}} + \varepsilon_n, \tag{2.5}$$

where

$$\varepsilon_n = (-1)^n z^{-n} \frac{1}{2\pi \mathrm{i}} \int_{\mathcal{L}} \mathrm{e}^{z(\frac{1}{3}w^3 - b^2 w)} h_n(w) \mathrm{d}w. \tag{2.6}$$

Note that $h_n(w)$ has the same domain of analyticity as $h_0(w)$.

For large values of z and bounded $|b|$, the estimates of ε_n existing in the literature are usually of the form

$$|\varepsilon_n| \le \frac{M_n}{z^{n+\frac{1}{3}}} |\mathrm{Ai}(z^{\frac{2}{3}} b^2)| + \frac{N_n}{z^{n+\frac{2}{3}}} |\mathrm{Ai}'(z^{\frac{2}{3}} b^2)|, \tag{2.7}$$

where M_n and N_n depend on n but independent of b. The proof of this result is standard; but, as observed by Olde Daalhuis and Temme, the influence of large $|b|$ in (2.7) is not clear. For bounded $|b|$, an estimate like (2.7) will hold under rather mild conditions on $h_0(w)$. To obtain estimates for ε_n which hold in unbounded b-interval, they first introduced a new class of rational functions

$$\begin{aligned} R_0(u, w, b) &= \frac{1}{u - w}, \\ R_{n+1}(u, w, b) &= \frac{-1}{u^2 - b^2} \frac{\mathrm{d}}{\mathrm{d}u} R_n(u, w, b), \quad n = 0, 1, 2, \cdots, \end{aligned} \tag{2.8}$$

where $u, w, b \in \mathbb{C},\ u \ne w,$ and $u^2 \ne b^2$. Then they showed that the function $h_n(w)$ defined by the recursive formula (2.2) has the representation

$$h_n(w) = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma} R_n(u, w, b) h_0(u) \mathrm{d}u, \tag{2.9}$$

where Γ is any simple closed contour in the domain of analyticity of $h_0(w)$, enclosing the points w and $\pm b$.

Furthermore, by using induction on n, they proved that

$$R_n(u, w, b) = \sum_{i=0}^{n-1} \sum_{j=0}^{k_{n,i}} \frac{C_{ij} u^{i-j}}{(u - w)^{n+1-i-j} (u^2 - b^2)^{n+i}}, \quad n = 1, 2, \cdots, \tag{2.10}$$

where $k_{n,i} = \min(i, n-1-i)$ and the C_{ij}'s do not depend on u, w and b. Note that the rational functions defined in (2.8) are independent of the function $h_0(w)$. Hence, (2.9) can be considered as an analogue of the Cauchy integral representation for the remainder in a Taylor expansion. From (2.10), they deduced the following result.

Lemma 2.1 (i) *Let $w \in \mathbb{C}$ be such that $|w-b| = O(b)$ as $b \to \infty$, and Γ be a simple closed contour that encircles b and w. Then for $n = 1,\ 2,\ \cdots$,*

$$\frac{1}{2\pi \mathrm{i}} \int_\Gamma R_n(u,w,b)\mathrm{d}u = O(b^{-3n}), \quad \text{as } b \to \infty. \tag{2.11}$$

(ii) *Let $b \in \mathbb{C}$ and $\Omega(b) = \{(u,w) \in \mathbb{C}^2 : |u-b| = \rho(b), |w-b| \le \frac{1}{2}\rho(b)\}$, such that $\rho(b) = O(|b|^\theta)$ as $b \to \infty$, where $-\frac{1}{2} < \theta \le 1$. Then we can assign numbers A_n independent of b, such that*

$$\sup_{(u,w)\in\Omega(b)} |R_n(u,w,b)| \le A_n |b|^{-(1+2\theta)n-\theta}, \quad \text{as } b \to \infty. \tag{2.12}$$

Finally, they estimated $h_n(w)$ by formulating conditions on $h_0(w)$ in discs with centers at $\pm b$. These discs have radius $\rho(b)$, which is controlled by the nearest singularity of $h_0(w)$. Put

$$\rho_0(b) := \min\{|w \pm b| : w \text{ is a singularity of } h_0(w)\}, \tag{2.13}$$

and choose $\rho(b) \le \rho_0(b)$ such that $\rho(b) \sim \delta|b^\theta|$ as $b \to \infty$. With

$$\widetilde{h}_n := \sup_{|w\pm b|\le\frac{1}{2}\rho(b)} |h_n(w)|, \tag{2.14}$$

let Γ be a circle around $\pm b$ with radius $\rho(b)$, and ensure that not both saddle points are inside Γ. By Lemma 2.1, they proved

$$\begin{aligned} h_n(w) &= \frac{1}{2\pi\mathrm{i}} \int_\Gamma R_0(u,w,b)h_n(u)\mathrm{d}u \\ &= \frac{1}{2\pi\mathrm{i}} \int_\Gamma R_1(u,w,b)h_{n-1}(u)\mathrm{d}u - \frac{1}{2\pi\mathrm{i}} \int_\Gamma R_1(u,w,b)(\alpha_{n-1} + \beta_{n-1}u)\mathrm{d}u \\ &= \frac{1}{2\pi\mathrm{i}} \int_\Gamma R_1(u,w,b)h_{n-1}(u)\mathrm{d}u + \widetilde{h}_{n-1}O(b^{-3}) \\ &= \frac{1}{2\pi\mathrm{i}} \int_\Gamma R_n(u,w,b)h_0(u)\mathrm{d}u + \widetilde{h}_{n-1}O(b^{-3}) + \cdots + \widetilde{h}_0 O(b^{-3n}), \end{aligned} \tag{2.15}$$

as $b \to \infty$. Using induction, they have also proved the following theorem.

Theorem 2.1 *Let $\widetilde{h}_n$ $(n = 0, 1, 2, \cdots)$ be the upper bound of $h_n(w)$ defined by* (2.14)*. We have the estimate*

$$\widetilde{h}_n \le C_n |b|^{-(1+2\theta)n} \widetilde{h}_0, \quad \text{as } b \to \infty, \tag{2.16}$$

where C_n does not depend on b.

Note that because of the induction method used in the above argument, it is almost impossible to get an explicit expression for the constant C_n.

Next they splitted the contour $\mathcal{L}$ in (2.6) into $\mathcal{L}'$ and $\mathcal{L}''$. Then they estimated $\varepsilon_n|_{\mathcal{L}'}$ and $\varepsilon_n|_{\mathcal{L}''}$, respectively, and proved that $\varepsilon_n|_{\mathcal{L}''}$ is exponentially small in comparison with the estimate of $\varepsilon_n|_{\mathcal{L}'}$ as $z \to \infty$. Thus, they obtained the following theorem.

Theorem 2.2 *Let $F(z,b)$ be of the form* (2.1)*. The error term in* (2.6) *satisfies*

$$|\varepsilon_n| \le C_n(|b|+1)^{-(1+2\theta)n}\widetilde{h}_0 z^{-n-\frac{1}{3}}\widetilde{\mathrm{A}}\mathrm{i}(z^{\frac{2}{3}}b^2), \tag{2.17}$$

as $z\to\infty$ uniformly with respect to $b\in[0,\infty)\cup[0,\mathrm{i}\infty)$, where $\widetilde{h}_0$ is given in (2.14)*.*

In [10], the authors concentrated on proving that the Airy-type expansion (2.5) is valid for the auxiliary parameter b in an unbounded domain. Their new estimate gives the behavior of the remainder as b tends to infinity, and exhibits a double asymptotic nature in the sense that the role of the large variable z and the auxiliary parameter b can be interchanged. But, their estimates in Lemma 2.1 are not valid for b in the neighborhood of the origin, which is the domain of our major concern. In the next section, we shall use the modified Bessel function as an example to show that by modifying the method of Olde Daalhuis and Temme, one can actually prove that the Airy-type expansion (2.5) is valid for all b, and to present a computable bound for the error term.

3 The Airy-Type Expansion of $K_{\mathrm{i}\nu}(\nu a)$

Returning to (1.7)

$$K_{\mathrm{i}\nu}(\nu a) = \frac{1}{2}\int_{-\infty}^{\infty} \mathrm{e}^{-\nu a\cosh t+\mathrm{i}\nu t}\mathrm{d}t, \tag{3.1}$$

we assume $a>0$ and $\nu>0$. For convenience, we consider two different cases: (1) the monotonic case $a\ge 1$ (2) the oscillatory case $0<a\le 1$.

3.1 The monotonic case $a\ge 1$

For the monotonic case, we write $\frac{1}{a}=\cos\theta$, where $\theta\in[0,\frac{\pi}{2}]$. Then (3.1) becomes

$$K_{\mathrm{i}\nu}(\nu a) = \frac{1}{2}\int_{-\infty}^{\infty} \mathrm{e}^{\nu f(t,\theta)}\mathrm{d}t, \tag{3.2}$$

where

$$f(t,\theta) = -\frac{\cosh t}{\cos\theta}+\mathrm{i}t. \tag{3.3}$$

The saddle points are zeros of the equation $f'(t,\theta)=0$, they are at the points $t_k^{\pm}=\frac{\pi\mathrm{i}}{2}\mp\theta\mathrm{i}+2k\pi\mathrm{i}$. The only relevant saddle point is at $t_0^+=\frac{\pi\mathrm{i}}{2}-\theta\mathrm{i}$. The steepest descent path $\mathcal{C}$ is defined by the equation $\Im f(t,\theta)=\Im f(t_0^+,\theta)$ (see Figure 1). This is typically the case for the use of the cubic transformation, and we have

$$f(t,\theta) = -\frac{\cosh t}{\cos\theta}+\mathrm{i}t = \frac{1}{3}w^3 - b^2 w + c. \tag{3.4}$$

Since t_0^+ is the relevant saddle, we let $t_0^{\pm}$ correspond to $\pm b$. So we set

$$\begin{cases} -\dfrac{\sin\theta}{\cos\theta}+\theta-\dfrac{\pi}{2} = -\dfrac{2}{3}b^3+c, \\ \dfrac{\sin\theta}{\cos\theta}-\theta-\dfrac{\pi}{2} = \dfrac{2}{3}b^3+c. \end{cases}$$

This leads to

$$b^3 = \frac{3}{2}\Big(\frac{\sin\theta}{\cos\theta}-\theta\Big), \qquad c=-\frac{\pi}{2}. \tag{3.5}$$

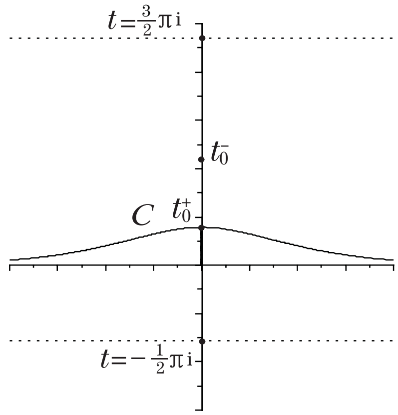

Figure 1 Steepest descent path $\mathcal{C}$ in t-plane, $a \geq 1$

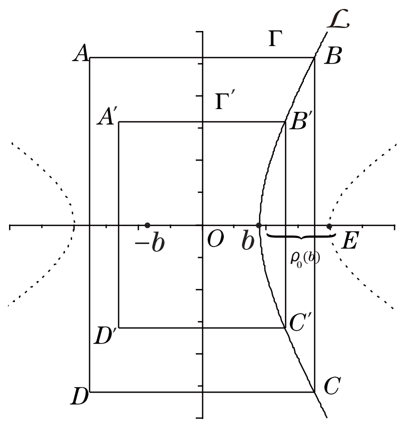

Figure 2 Steepest descent path $\mathcal{L}$ in w-plane, $a \geq 1$

We note that the transformation (3.4) maps the strip $-\frac{\pi}{2} \leq \Im t \leq \frac{3\pi}{2}$ conformally onto a region in the w-plane, and the mapping is one-to-one; see Figures 1 and 2. From (3.2) and (3.4), we obtain

$$K_{i\nu}(\nu a) = \frac{1}{2}\int_{\mathcal{L}} e^{\nu(\frac{1}{3}w^3 - b^2 w + c)} h_0(w) dw, \tag{3.6}$$

$$h_0(w) = \frac{dt}{dw} = \frac{(w^2 - b^2)\cos\theta}{-\sinh t + i\cos\theta}, \tag{3.7}$$

where $\mathcal{L}$ is the image of $\mathcal{C}$ in the w-plane.

To derive an asymptotic expansion for the integral in (3.6) which holds uniformly for b near 0, we use the Bleistein's method mentioned in Section 2, and obtain

$$K_{i\nu}(\nu a) = \pi i e^{-\frac{\nu\pi}{2}} \Big\{ \frac{\mathrm{Ai}(\nu^{\frac{2}{3}} b^2)}{\nu^{\frac{1}{3}}} \sum_{k=0}^{n-1} \frac{(-1)^k a_k}{\nu^k} - \frac{\mathrm{Ai}'(\nu^{\frac{2}{3}} b^2)}{\nu^{\frac{2}{3}}} \sum_{k=0}^{n-1} \frac{(-1)^k b_k}{\nu^k} + \varepsilon_n \Big\}, \tag{3.8}$$

where

$$\varepsilon_n = (-1)^n \nu^{-n} \frac{1}{2\pi \mathrm{i}} \int_{\mathcal{L}} \mathrm{e}^{\nu(\frac{1}{3}w^3 - b^2 w)} h_n(w) \mathrm{d}w. \tag{3.9}$$

The function $h_n(w)$ in (3.9) is defined recursively by (2.2) and (2.3), and it has the same domain of analyticity as $h_0(w)$.

There is a standard procedure to prove that (3.8) is an asymptotic expansion. But, because of the complicate form of $h_n(w)$, this procedure often works only when the parameter b is bounded and will not yield a computable error bound. Here, we shall modify the Olde Daalhuis-Temme method to prove that (3.8) is a uniform asymptotic expansion respect to b even when b is unbounded, and try to provide an error bound which is computable.

To extend the validity of the asymptotic expansion given in (3.8) to unbounded values of b, we define $\rho_0(b)$ similar to that in Section 2, but our $\rho_0(b)$ is not controlled by the nearest singularity. Since the mapping defined in (3.4) is one-to-one and analytic in the infinite strip $\Pi = \{t \in \mathbb{C}, -\frac{\pi}{2} < \Im t < \frac{3\pi}{2}\}$, all singular points of $h_0(w)$ lie outside the domain $\Pi^* = w(\Pi)$, i.e., the image of the strip in the w-plane (see Figure 2). Let Σ denote the infinite line $\Im t = -\frac{1}{2}\pi$, and Σ^* its image in the w-plane. With $w = x + y\mathrm{i}$, the equation of the curve Σ^* is given by

$$x^3 - 3xy^2 - 3b^2 x = 3\pi. \tag{3.10}$$

So here our $\rho_0(b)$ is controlled by the boundary of the conformal mapping. Let E denote the point where Σ^* cross the x-axis and O the origin (see Figure 2). Then we define $\rho_0(b) = |EO| - b$.

Below, we give a detailed discussion of $\rho_0(b)$. From (3.10), we know that the value $|EO| = x_0$ is the solution of the equation

$$x^3 - 3b^2 x = 3\pi, \tag{3.11}$$

and we have

$$x_0^3 - 3b^2 x_0 + 2b^3 = 3\pi + 2b^3. \tag{3.12}$$

Since $\rho_0(b) = x_0 - b$, it follows that

$$\rho_0(b)^3 + 3b\rho_0(b)^2 = 3\pi + 2b^3. \tag{3.13}$$

Differentiating both sides with respect to b gives

$$\rho_0'(b)(\rho_0(b)^2 + 2b\rho_0(b)) = 2b^2 - \rho_0(b)^2. \tag{3.14}$$

Clearly when $\rho_0(b) = \sqrt{2}\, b$, $\rho_0(b)$ attains its minimum. Substituting $\rho_0(b) = \sqrt{2}\, b$ into (3.13) and solving the equation, we obtain $b = (\frac{3\pi}{2\sqrt{2}+4})^{\frac{1}{3}}$. Thus, we have $\rho_0(b) \geq (\frac{3\pi}{1+\sqrt{2}})^{\frac{1}{3}}$. When $b \to \infty$, we also have

$$x_0 - \sqrt{3}\, b = \frac{3\pi}{x_0(x_0 + \sqrt{3}\, b)}, \tag{3.15}$$

which implies that $x_0 \sim \sqrt{3}\, b$ as $b \to \infty$, and $\rho_0(b) \sim (\sqrt{3} - 1)b$ as $b \to \infty$.

Next, we estimate $h_0(w)$. We know that $h_0(w)$ is analytic in the domain Π^*, and we have the following proposition.

Proposition 3.1 *There exists a positive number $\widetilde{h}_0$ independent of w and b, such that*

$$|h_0(w)| \le \widetilde{h}_0, \quad w \in \Pi^*. \tag{3.16}$$

More precisely, we have $\widetilde{h}_0 = 8.82$.

Proof Since $h_0(w)$ is analytic in the domain Π^*, we only need to estimate it on the boundary of Π^*. First, we estimate $h_0(w)$ on the line $\Im t = -\frac{\pi}{2}$.

If we write $t = \tau - \frac{\pi}{2}\mathrm{i}$, then we have

$$h_0(w) = \frac{(w^2 - b^2)\cos\theta}{\mathrm{i}\cosh\tau + \mathrm{i}\cos\theta}. \tag{3.17}$$

By the cubic transformation (3.4), we obtain

$$\begin{aligned} |h_0(w)| &= \frac{|w^2 - 3b^2 + 2b^2|\cos\theta}{|\cosh\tau + \cos\theta|} = \frac{|2b^2\cos\theta + 3(-\cosh t + \mathrm{i}t\cos\theta - c\cos\theta)\frac{1}{w}|}{|\cosh\tau + \cos\theta|} \\ &= \frac{|2b^2\cos\theta + 3(\mathrm{i}\sinh\tau + \mathrm{i}\tau\cos\theta + \pi\cos\theta)\frac{1}{w}|}{|\cosh\tau + \cos\theta|} \\ &\le 2b^2\cos\theta + \frac{3 + 3\pi}{|w|} + \frac{3|\tau\cos\theta|}{|w||\cosh\tau + \cos\theta|}. \end{aligned} \tag{3.18}$$

From (3.5), it follows that

$$2b^2\cos\theta = 2\Big(\frac{3}{2}\Big)^{\frac{2}{3}}\Big(\frac{\sin\theta}{\cos\theta} - \theta\Big)^{\frac{2}{3}}\cos\theta = 2\Big(\frac{3}{2}\Big)^{\frac{2}{3}}(\sin\theta - \theta\cos\theta)^{\frac{2}{3}}\cos^{\frac{1}{3}}\theta.$$

Since $h(\theta) = \sin\theta - \theta\cos\theta$ is a increasing function of θ for $0 \le \theta \le \frac{\pi}{2}$, we have $2b^2\cos\theta \le 2$.

Now we estimate the second term on the right-hand side of (3.18). From (3.10), we know that the value $|EO| = x_0$ is the solution of the equation

$$x^3 - 3b^2x = 3\pi. \tag{3.19}$$

Since x_0 is a function of b, differentiating both sides of this equation with respect to b gives

$$3x_0^2x_0' - 3b^2x_0' - 6bx_0 = 0. \tag{3.20}$$

We note that $x_0 > b \ge 0$. Hence, we have $x_0' \ge 0$. This means that x_0 is an increasing function of b. So, when $b = 0$, x_0 attain its minimum $(3\pi)^{\frac{1}{3}}$. Therefore, we obtain $|w| \ge (3\pi)^{\frac{1}{3}}$, and

$$\frac{3 + 3\pi}{|w|} \le \frac{3 + 3\pi}{(3\pi)^{\frac{1}{3}}}.$$

Next, we consider the third term, and let $h(\tau) = \frac{\tau}{\cosh\tau}$. Since $h'(\tau) = \frac{\cosh\tau - \tau\sinh\tau}{\cosh^2\tau}$, it is evident that $h(\tau)$ attains its maximum when $\tau = \tau_0$, which is the positive root of the equation $\cosh\tau - \tau\sinh\tau = 0$. One can solve this equation numerically and obtain $\tau_0 = 1.2$. Therefore, we have

$$\frac{3|\tau\cos\theta|}{|w||\cosh\tau + \cos\theta|} \le \frac{3\tau_0}{|w|\cosh\tau_0} \le \frac{1.99}{|w|} \le 0.94.$$

Now we have proved that there exists a constant $\widetilde{h}_0 = 8.82$ independent of b such that $|h_0(w)| \le \widetilde{h}_0$ on the portion $\Im t = -\frac{1}{2}\pi$. On the line $\Im t = \frac{3}{2}\pi$ the same estimate holds, thus Proposition 3.1 is proved.

Before modifying the method outlined in Section 2, let us first use the above result to extend the validity of (3.8) to unbounded values of b. With the estimate of $h_0(w)$, we let $\rho(b) = (\sqrt{3}-1)b$. Since $\rho_0(b) = x_0 - b$ and $x_0 > \sqrt{3}\,b$ by (3.15), it follows that $\rho(b) < \rho_0(b)$. Using the results in Section 2, we obtain from Theorem 2.1

$$\widetilde{h}_n \leq C_n |b|^{-3n} \widetilde{h}_0, \quad \text{as } b \to \infty. \tag{3.21}$$

Here and thereafter, C_n is used as a generic symbol for constants independent of b, whose values may change from place to place. By Theorem 2.2, asymptotic expansion (3.8) is valid for unbounded value of b, and

$$|\varepsilon_n| \leq C_n (|b|+1)^{-3n} \widetilde{h}_0 \nu^{-n-\frac{1}{3}} \widetilde{\mathrm{A}}\mathrm{i}(\nu^{\frac{2}{3}} b^2), \tag{3.22}$$

as $\nu \to \infty$ uniformly with respect to large value of b.

We note that in the proof presented in Section 2, Olde Daalhuis and Temme require the contour Γ in (2.15) to encircle just one saddle. When $\theta \to 0$, $\rho_0(b) = |EO| - b$ tends to a constant, but the two saddle points $\pm b$ approach each other. That is, we can not ensure that not both saddles are inside the circle Γ in (2.15). Recall the expression (2.9); this expression motivates us to consider a contour Γ that is larger than just a circle enclosing one saddle. So we choose Γ to be a contour that encircles both saddles $\pm b$ (see Figure 2). For the present example, this is more natural than the method mentioned in Section 2. Then we use (2.9) to estimate the remainder and avoid the O-terms in (2.15), thus the estimation is easier.

Now we use the modified method to prove that the expansion (3.8) is uniform for all b, and use the new estimate of the remainder to derive a computable error bound. Since the situations when $b \to 0$ and $b \to \infty$ are different, we divide our discussion into two cases: (i) $b \leq 4$, (ii) $b > 4$.

For case (i), we first determine the contour Γ. Since the choice of Γ is quite arbitrary as long as it encircles the two saddles $\pm b$, we simply choose it to be a rectangle, whose boundary is controlled by the point E. We divide the contour Γ into four parts, AB, BC, CD and DA (see Figure 2). Since $\rho_0(b)$ has a lower bound (see the line preceding (3.15)), we let the line segment BC be defined by the equation $\Re u = b + \rho_0(b)_{\min}$. Next, we determine the portion AB. Here we use the equation of steepest descent path $\mathcal{L} = \{w = x + \mathrm{i}y \in \mathbb{C} \mid y^2 = 3x^2 - 3b^2\}$ to locate point B. Since $b \leq 4$, when $b = 4$ we put $x = 4 + \rho_0(b)_{\min}$ into this equation, and obtain $y = 6.7$.

Now we choose Γ to be the rectangle $ABCD$ whose boundary lies on

$$\Re w = \pm(b + 1.574), \quad \Im w = \pm 6.7. \tag{3.23}$$

Similarly, we define a smaller rectangle Γ', with it boundary lying on the lines

$$\Re w = \pm\Big(b + \frac{1.574}{2}\Big), \quad \Im w = \pm 4.6. \tag{3.24}$$

As with Γ, the boundary lines $\Im w = \pm 4.6$ are determined by the equation of the steepest descent path.

For case (ii), we recall that $x_0 > \sqrt{3}\,b$ (see the sentence preceding (3.21)). So we choose the boundary Γ of the rectangle $ABCD$ to lie on the lines

$$\Re w = \pm\sqrt{3}\,b, \quad \Im w = \pm\sqrt{6}b. \tag{3.25}$$

Furthermore, the boundary Γ' of the smaller rectangle lies on the lines

$$\Re w = \pm\frac{\sqrt{3}+1}{2}b, \quad \Im w = \pm\Big(\frac{27}{4}\Big)^{\frac{1}{4}}b. \tag{3.26}$$

Note that, in case (ii), we could have still chosen Γ and Γ' as in case (i), but the error bound obtained would be much less accurate.

Now we define $\widetilde{h}_n$ as in Section 2. Let Ω' denote the domain bounded by Γ' and put

$$\widetilde{h}_n := \sup_{w\in\Omega'} |h_n(w)|. \tag{3.27}$$

We have following theorem.

Theorem 3.1 *There exists a positive constant C_n such that for each nonnegative integer n, we have*

$$\widetilde{h}_n \le C_n\widetilde{h}_0 \quad \textit{for } 0 \le b \le 4 \tag{3.28}$$

and

$$\widetilde{h}_n \le C_n b^{-3n}\widetilde{h}_0 \quad \textit{for } b > 4. \tag{3.29}$$

Proof We first consider the case $b > 4$. From (2.9), we have

$$h_n(w) = \frac{1}{2\pi\mathrm{i}}\int_\Gamma R_n(u,w,b)h_0(u)\mathrm{d}u. \tag{3.30}$$

We shall estimate the integral on the four parts of Γ separately. First we estimate it on the horizontal line segment AB. From (2.10), we have

$$R_n(u,w,b) = \sum_{i=0}^{n-1}\sum_{j=0}^{k_{n,i}} \frac{C_{ij}u^{i-j}}{(u-w)^{n+1-i-j}(u^2-b^2)^{n+i}}, \quad n = 1, 2, \cdots .$$

Because of the definition of AB, the points on AB satisfy $|u| \le 3b$, $|u-b| \ge \sqrt{6}\,b$ and $|u+b| \ge \sqrt{6}\,b$. Since $w \in \Omega'$, we also have $|u-w| \ge [\sqrt{6} - (\frac{27}{4})^{\frac{1}{4}}]b$. Thus,

$$|R_n(u,w,b)| \le \sum_{i=0}^{n-1}\sum_{j=0}^{k_{n,i}} \widetilde{C}_{ij}b^{-3n-1}.$$

Furthermore, we obtain

$$\Big|\frac{1}{2\pi\mathrm{i}}\int_{AB} R_n(u,w,b)h_0(u)\mathrm{d}u\Big| \le C_n b^{-3n}\widetilde{h}_0.$$

Next, we estimate it on BC. The points on BC satisfy $|u| \le 3b$, $|u-b| \ge (\sqrt{3}-1)b$ and $|u+b| \ge (\sqrt{3}+1)b$. So we easily have

$$\Big|\frac{1}{2\pi\mathrm{i}}\int_{BC} R_n(u,w,b)h_0(u)\mathrm{d}u\Big| \le C_n b^{-3n}\widetilde{h}_0.$$

Since the estimates on CD and DA are the same, (3.29) is proved.

Next, we consider the case $b \leq 4$. Since the points on Γ and Γ' satisfy (3.23) and (3.24), the quantities $|u|$, $|u^2 - b^2|$ and $|u - w|$ are all bounded, and we can easily obtain

$$\Big|\frac{1}{2\pi \mathrm{i}} \int_\Gamma R_n(u, w, b) h_0(u) \mathrm{d}u\Big| \leq C_n \widetilde{h}_0. \tag{3.31}$$

We note that C_n depends only on n; if n is fixed, we can compute C_n explicitly.

Now we estimate the error term ε_n. As in [10], we split the contour $\mathcal{L}$ into $\mathcal{L}'$ and $\mathcal{L}''$, and define

$$\varepsilon_n^{(1)} = (-1)^n \nu^{-n} \frac{1}{2\pi \mathrm{i}} \int_{\mathcal{L}'} h_n(w) \mathrm{e}^{\nu(\frac{1}{3}w^3 - b^2 w)} \mathrm{d}w \tag{3.32}$$

and

$$\varepsilon_n^{(2)} = (-1)^n \nu^{-n} \frac{1}{2\pi \mathrm{i}} \int_{\mathcal{L}''} h_n(w) \mathrm{e}^{\nu(\frac{1}{3}w^3 - b^2 w)} \mathrm{d}w, \tag{3.33}$$

where $\mathcal{L}' = \{w \in \mathcal{L} : w \in \Omega',\ \partial\Omega' = \Gamma'\}$ and $\mathcal{L}'' = \mathcal{L} - \mathcal{L}'$. By Theorem 3.1, we have

$$|\varepsilon_n^{(1)}| \leq \widetilde{h}_n \nu^{-n} \Big|\frac{1}{2\pi \mathrm{i}} \int_{\mathcal{L}'} \mathrm{e}^{\nu(\frac{1}{3}w^3 - b^2 w)} \mathrm{d}w\Big| \leq C_n \nu^{-n-\frac{1}{3}} \widetilde{h}_0 \mathrm{Ai}(\nu^{\frac{2}{3}} b^2) \tag{3.34}$$

when $0 \leq b \leq 4$, and

$$|\varepsilon_n^{(1)}| \leq C_n \nu^{-n-\frac{1}{3}} b^{-3n} \widetilde{h}_0 \mathrm{Ai}(\nu^{\frac{2}{3}} b^2), \tag{3.35}$$

when $b > 4$.

Finally, we estimate $\varepsilon_n^{(2)}$. Since the result given in [10] is too rough often due to over-estimation, to obtain a computable error bound, we shall try to derive a much more accurate estimate for $\varepsilon_n^{(2)}$. Recall that $h_0(w)$ is an analytic function in Π^*, which is the image of the infinite strip $\Pi = \{t \in \mathbb{C} :\ -\frac{\pi}{2} < \Im t < \frac{3\pi}{2}\}$. Hence, in particular it is analytic in a neighborhood Ω_0 containing $\mathcal{L}''$ (see the statement preceding (3.10)). Since the distance between the steepest descent path and the boundary of Π^* is increasing as $|w| \to \infty$ (see Figure 2), we can make sure that the disc with center $w \in \mathcal{L}''$ and radius $R = \frac{1}{4}\rho(b)$ is contained in Ω_0. Now we just consider the case $b > 4$, since the estimate of $\varepsilon_n^{(2)}$ when $b \leq 4$ can easily be obtained. Let $w \in \mathcal{L}''$ and Γ be the circle with center w and radius R. We will show that

$$\Big|\frac{1}{2\pi \mathrm{i}} \int_\Gamma R_n(u, w, b) \mathrm{d}u\Big| \leq C_n b^{-3n}. \tag{3.36}$$

From (2.10), it follows that

$$|R_n(u, w, b)| \leq \sum_{i=0}^{n-1} \sum_{j=0}^{k_{n,i}} \frac{C_{i,j}}{|u-w|^{n+1-i-j} |u-b|^{n+j} |u+b|^{n+j}} \Big[\frac{|u-b| + b}{|u-b|^2}\Big]^{i-j} \tag{3.37}$$

for all b, where we have made use of the fact that $|u + b| \geq |u - b|$. For w on $\mathcal{L}''$ moving away from the point B' to infinity along the steepest descent path, the quantity $|u - w|$ remains fixed, but the values of $|u - b|$ and $|u + b|$ will increase. Moreover, for the quantity inside the square brackets, we have

$$\frac{|u-b| + b}{|u-b|^2} \leq \frac{|u-b|_{\min} + b + 2R}{|u-b|^2_{\min}}. \tag{3.38}$$

Obviously, the right-hand side of (3.38) decreases when w (hence u) moves away from B'. So we just consider the estimate of $R_n(u,w,b)$ at the point $w = B' = \big(\frac{\sqrt{3}+1}{2}b, (\frac{27}{4})^{\frac{1}{4}}b\big)$, and obtain

$$\begin{aligned}|u-b|_{\min} &= \sqrt{\Big(\frac{\sqrt{3}-1}{2}b\Big)^2 + \Big(\frac{27}{4}\Big)^{\frac{1}{2}}b^2} - R\\ &= (1+\sqrt{3})^{\frac{1}{2}}b - R = (1+\sqrt{3})^{\frac{1}{2}}b - \frac{\sqrt{3}}{4}b;\end{aligned}$$

thus,

$$|R_n(u,w,b)| \le C_n b^{-3n-1}.$$

Since the length of Γ is $2\pi R$, we have

$$\Big|\frac{1}{2\pi\mathrm{i}}\int_\Gamma R_n(u,w,b)\mathrm{d}u\Big| \le C_n b^{-3n}.$$

Similarly, we have the estimate

$$\Big|\frac{1}{2\pi\mathrm{i}}\int_\Gamma uR_n(u,w,b)\mathrm{d}u\Big| \le C_n b^{-3n+1}. \tag{3.39}$$

Now, we can estimate $h_n(w)$ for w in the domain Ω_0 defined above, i.e., the envelop of all circles with center $w \in \mathcal{L}''$ and radius R. As in (2.15), we obtain by using (3.29)

$$\begin{aligned}h_n(w) &= \frac{1}{2\pi\mathrm{i}}\int_\Gamma R_0(u,w,b)h_n(u)\mathrm{d}u\\ &= \frac{1}{2\pi\mathrm{i}}\int_\Gamma R_1(u,w,b)h_{n-1}(u)\mathrm{d}u - \frac{1}{2\pi\mathrm{i}}\int_\Gamma R_1(u,w,b)(\alpha_{n-1}+\beta_{n-1}u)\mathrm{d}u\\ &= \frac{1}{2\pi\mathrm{i}}\int_\Gamma R_1(u,w,b)h_{n-1}(u)\mathrm{d}u + \widetilde{h}_{n-1}O(b^{-3})\\ &\ \ \vdots\\ &= \widetilde{h}_0 O(b^{-3n});\end{aligned} \tag{3.40}$$

that is, we have

$$|h_n(w)| \le C_n \widetilde{h}_0 b^{-3n}. \tag{3.41}$$

Since $\mathcal{L}''$ has the parametric representation

$$\Big\{\sqrt{\frac{1}{3}y^2+b^2} + \mathrm{i}y:\ |y| \ge y_0 = \Big(\frac{27}{4}\Big)^{\frac{1}{4}}b\Big\},$$

the integral in (3.33) can be written as

$$\varepsilon_n^{(2)} = (-1)^n\nu^{-n}\frac{1}{2\pi\mathrm{i}}\int_{y_0}^\infty \mathrm{e}^{-\nu\phi(y)}[H_n(y)+H_n(-y)]\mathrm{d}y, \tag{3.42}$$

where

$$\phi(y) = \Big(\frac{8}{9}y^2+\frac{2}{3}b^2\Big)\sqrt{\frac{1}{3}y^2+b^2}, \quad H_n(y) = h_n(x+\mathrm{i}y)\Big[\frac{\mathrm{d}x}{\mathrm{d}y}+\mathrm{i}\Big], \tag{3.43}$$

and

$$\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{\frac{1}{3}y}{\sqrt{\frac{1}{3}y^2 + b^2}}. \tag{3.44}$$

With (3.41), we obtain

$$\begin{aligned}|\varepsilon_n^{(2)}| &\le C_n \widetilde{h}_0 b^{-3n} \nu^{-n} \int_{y_0}^{\infty} \mathrm{e}^{-\nu\phi(y)} \mathrm{d}y \\ &\le C_n \widetilde{h}_0 b^{-3n} \mathrm{e}^{-\frac{2}{3}\nu b^3} \nu^{-n} \int_{y_0}^{\infty} \mathrm{e}^{-\frac{8}{9}\nu b y^2} \mathrm{d}y \\ &\le C_n \widetilde{h}_0 b^{-3n} \mathrm{e}^{-\frac{2}{3}\nu b^3} \nu^{-n} \frac{9\mathrm{e}^{-\frac{8}{9}\nu b y_0^2}}{16\nu b y_0} \\ &= C_n \widetilde{h}_0 b^{-3n} \nu^{-n} \mathrm{e}^{-\frac{2}{3}\nu b^3} \frac{\mathrm{e}^{-\frac{4}{3}\sqrt{3}\nu b^3}}{b^2 \nu}.\end{aligned} \tag{3.45}$$

When $b \to \infty$, the above estimate can also be expressed as

$$|\varepsilon_n^{(2)}| \le C_n \widetilde{h}_0 b^{-3n} \nu^{-n-\frac{1}{3}} \mathrm{Ai}(\nu^{\frac{2}{3}} b^2) \mathrm{e}^{-\frac{4}{3}\sqrt{3}\nu b^3}, \tag{3.46}$$

which is clearly exponentially small in comparison with $|\varepsilon_n^{(1)}|$. Note that the estimate of remainder ε_n involves the saddle-point parameter b, and that it exhibits the double asymptotic property mentioned earlier. If $b \le 4$, the procedure is the same, and it is obvious that we can derive a computable estimate for $\varepsilon_n^{(2)}$

$$|\varepsilon_n^{(2)}| \le C_n \widetilde{h}_0 \nu^{-n-1} \mathrm{e}^{-\frac{2}{3}\nu b^3 - 1.3\nu} \le C_n \widetilde{h}_0 \nu^{-n-\frac{1}{3}} \mathrm{Ai}(\nu^{\frac{2}{3}} b^2) \mathrm{e}^{-1.3\nu}. \tag{3.47}$$

Combining (3.34)–(3.35) and (3.46)–(3.47) together, we conclude that (3.8) is a uniform asymptotic expansion for all b. We note that when n is fixed, we can compute the constants α_n and β_n in (3.40) explicitly, and hence all the bounds C_n in estimates of $|\varepsilon_n^{(1)}|$ and $|\varepsilon_n^{(2)}|$. Thus we obtain a computable error bound. For an illustration, see Section 4.

3.2 The oscillatory case: $0 < a \le 1$

For the oscillatory case, we write $\frac{1}{a} = \cosh\alpha$. Then (3.1) becomes

$$K_{\mathrm{i}\nu}(\nu a) = \frac{1}{2}\int_{-\infty}^{\infty} \mathrm{e}^{\nu f(t,\alpha)} \mathrm{d}t, \tag{3.48}$$

where

$$f(t,\alpha) = -\frac{\cosh t}{\cosh\alpha} + \mathrm{i}t. \tag{3.49}$$

The saddle points $t_k^{\pm}$ can be obtained by solving the equation

$$f'(t,\alpha) = -\sinh t + \mathrm{i}\cosh\alpha = 0, \tag{3.50}$$

and we have

$$t_k^{\pm} = \pm\alpha + \frac{1}{2}\pi\mathrm{i} + 2k\pi\mathrm{i}, \quad k \in \mathbb{Z}. \tag{3.51}$$

The steepest descent paths through the saddle points $t_k^{\pm}$ are defined by the equation

$$\Im f(t,\alpha) = \pm\Big(\frac{\sinh\alpha}{\cosh\alpha} - \alpha\Big),$$

that is,

$$\sin\sigma = \cosh\alpha\frac{\tau}{\sinh\tau} \pm \frac{\sinh\alpha - \alpha\cosh\alpha}{\sinh\tau},$$

where $t = \tau + \mathrm{i}\sigma$. These pathes are shown in Figure 3, where we have used arrows to indicate

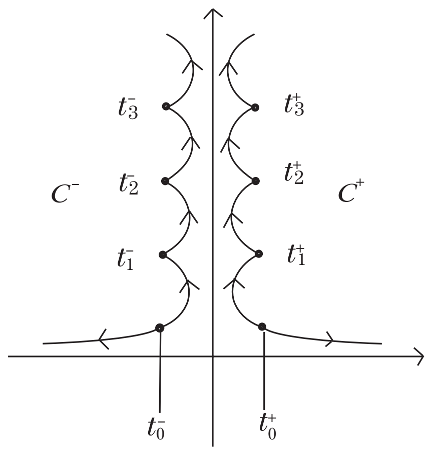

Figure 3 Steepest descent path $\mathcal{C}^+$ and $\mathcal{C}^-$, $0 < a \le 1$

directions of descent. We note that these curves do not intersect the imaginary axis. In fact, the imaginary axis divides these pathes into two parts, one denoted by $\mathcal{C}^+$ goes through all t_k^+, $k = 0, 1, 2, \cdots$, and the other denoted by $\mathcal{C}^-$ goes through all t_k^-, $k = 0, 1, 2, \cdots$. On each branch $\mathcal{C}^+$ or $\mathcal{C}^-$, there are infinitely many saddle points. Furthermore, these saddle points are in pairs, and each pair will coalescence when $\alpha \to 0$. This phenomenon is very interesting, and certainly is different from most of the cases that have been studied before.

To derive a uniform asymptotic expansion for $K_{\mathrm{i}\nu}(\nu a)$, we use as before the standard method mentioned in Section 2. Thus, we deform the integration path into steepest descent paths $\mathcal{C}^+$ and $\mathcal{C}^-$, so that (3.48) becomes

$$K_{\mathrm{i}\nu}(\nu a) = \frac{1}{2}\int_{\mathcal{C}^- + \mathcal{C}^+} \mathrm{e}^{\nu f(t,\alpha)}\mathrm{d}t. \tag{3.52}$$

We note that $t_0^{\pm} = \pm\alpha + \frac{1}{2}\pi\mathrm{i}$ are two relevant saddle points, and that they coalescence when $\alpha \to 0$. This is still typically the case for the use of the cubic transformation, and we have

$$f(t,\alpha) = -\frac{\cosh t}{\cosh\alpha} + \mathrm{i}t = \frac{1}{3}w^3 - b^2 w + c. \tag{3.53}$$

Since $t_0^{\pm}$ are the relevant saddles, they should correspond to $\pm b$. So we set

$$\begin{cases} -\mathrm{i}\dfrac{\sinh\alpha}{\cosh\alpha} + \mathrm{i}\alpha - \dfrac{\pi}{2} = -\dfrac{2}{3}b^3 + c, \\ \mathrm{i}\dfrac{\sinh\alpha}{\cosh\alpha} - \mathrm{i}\alpha - \dfrac{\pi}{2} = \dfrac{2}{3}b^3 + c. \end{cases}$$

This leads to

$$b^3 = \frac{3}{2}\mathrm{i}\Big(\frac{\sinh\alpha}{\cosh\alpha} - \alpha\Big), \quad c = -\frac{\pi}{2}. \tag{3.54}$$

We need the transformation $t \to w$ to be one-to-one and analytic, but the saddle points $t_k^{\pm}$ ($k \neq 0$) are all singular points of the mapping. So we should deform the steepest descent path to the curve $\mathcal{C}$ to avoid these saddle points which are not relevant (see Figure 4). The choice of $\mathcal{C}$ is some what arbitrary; we just need it to be in the strip $-\frac{1}{2}\pi \leq \Im t \leq \frac{3}{2}\pi$. By Cauchy's theorem, we can do so without changing the value of the integral in (3.52). Since

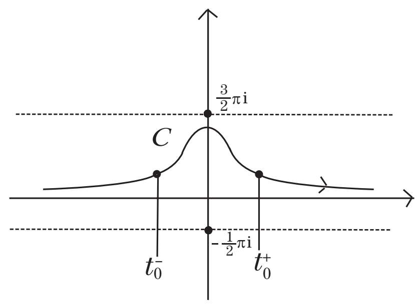

Figure 4 Integration path $\mathcal{C}$ in the t-plane, $0 < a \leq 1$

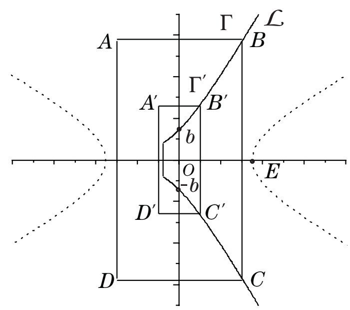

Figure 5 Steepest descent paths $\mathcal{L}$ in the w-plane, $0 < a \leq 1$

the transformation $t \to w$ maps the strip $-\frac{1}{2}\pi \leq \Im t \leq \frac{3}{2}\pi$ conformally into the w-plane (see Figures 4 and 5), we obtain from (3.52) and (3.53)

$$K_{\mathrm{i}\nu}(\nu a) = \frac{1}{2}\int_{\mathcal{L}} \mathrm{e}^{\nu(\frac{1}{3}w^3 - b^2 w + c)} h_0(w)\mathrm{d}w, \tag{3.55}$$

$$h_0(w) = \frac{\mathrm{d}t}{\mathrm{d}w} = \frac{(w^2 - b^2)\cosh\alpha}{-\sinh t + \mathrm{i}\cosh\alpha}, \tag{3.56}$$

where $\mathcal{L}$ is the image of $\mathcal{C}$ in the w-plane, and we still have the asymptotic expansion

$$K_{\mathrm{i}\nu}(\nu a) = \pi\mathrm{i}\mathrm{e}^{-\frac{\pi\nu}{2}}\Big\{\frac{\mathrm{Ai}(\nu^{\frac{2}{3}}b^2)}{\nu^{\frac{1}{3}}}\sum_{k=0}^{n-1}\frac{(-1)^k a_k}{\nu^k} - \frac{\mathrm{Ai}'(\nu^{\frac{2}{3}}b^2)}{\nu^{\frac{2}{3}}}\sum_{k=0}^{n-1}\frac{(-1)^k b_k}{\nu^k} + \varepsilon_n\Big\}, \tag{3.57}$$

where

$$\varepsilon_n = (-1)^n \nu^{-n}\frac{1}{2\pi\mathrm{i}}\int_{\mathcal{L}} \mathrm{e}^{\nu(\frac{1}{3}w^3 - b^2 w)} h_n(w)\mathrm{d}w. \tag{3.58}$$

In the oscillatory case, we let $0 \le \alpha \le 3$. We just consider this α-domain; since $\frac{1}{a} = \cosh \alpha$, we have almost covered the a-interval $0.099 < a \le 1$. As in the monotonic case, we estimate $h_0(w)$ first.

Proposition 3.2 *There exists a positive number $\widetilde{h}_0$ independent of w and b, such that*

$$|h_0(w)| \le \widetilde{h}_0, \quad w \in \Pi^*, \tag{3.59}$$

where Π^* is defined as before. More precisely, we have $\widetilde{h}_0 = 30.7$.

Proof Since $h_0(w)$ is analytic in the domain Π^*, we just need to estimate it on the boundary. First, we estimate $h_0(w)$ on the portion $\Im t = -\frac{\pi}{2}$.

For $t = \tau - \frac{\pi}{2}\mathrm{i}$, we have

$$h_0(w) = \frac{(w^2 - b^2)\cosh \alpha}{\mathrm{i}\cosh \tau + \mathrm{i}\cosh \alpha}. \tag{3.60}$$

From the cubic transformation (3.53), we have

$$\begin{aligned} |h_0(w)| &= \frac{|w^2 - 3b^2 + 2b^2|\cosh \alpha}{|\cosh \tau + \cosh \alpha|} \\ &= \frac{|2b^2 \cosh \alpha + 3(-\cosh t + \mathrm{i}t \cosh \alpha - c\cosh \alpha)\frac{1}{w}|}{|\cosh \tau + \cosh \alpha|} \\ &= \frac{|2b^2 \cosh \alpha + 3(\mathrm{i}\sinh \tau + \mathrm{i}\tau \cosh \alpha + \pi \cosh \alpha)\frac{1}{w}|}{|\cosh \tau + \cosh \alpha|} \\ &\le 2|b|^2 + \frac{3 + 3\pi}{|w|} + \frac{3|\tau \cosh \alpha|}{|w||\cosh \tau + \cosh \alpha|}. \end{aligned} \tag{3.61}$$

Since we just consider the case $\alpha \le 3$, the first term is obviously bounded. As we did in the monotonic case, we can prove that $|w| \ge 1.22$. Indeed, since $|w|$ attain its minimum at point E, and the value of $|EO|$ is the real root of the equation

$$x^3 + |b|^2 x = 3\pi. \tag{3.62}$$

Differentiation of both sides with respect to $|b|$ gives

$$(x^2 + |b|^2)x' + 2|b|x = 0; \tag{3.63}$$

thus, $x' \le 0$. So when $\alpha = 3$, $|w|$ attains its minimum 1.22, and we obtain a bound for the second term.

Now we estimate the last term. Since $\alpha \le 3$,

$$\frac{3|\tau \cosh \alpha|}{|w||\cosh \tau + \cosh \alpha|} \le \frac{3|\tau|\cosh 3}{|w|\cosh \tau}. \tag{3.64}$$

In the proof of Proposition 3.1, we have already show that $\frac{\tau}{\cosh \tau}$ attains its maximal value when $\tau = 1.2$. Hence,

$$\frac{3|\tau \cosh \alpha|}{|w||\cosh \tau + \cosh \alpha|} \le \frac{3\cosh 3}{|w|}\frac{1.2}{\cosh 1.2} \le 16.4,$$

thus proving that there exists a constant $\widetilde{h}_0 = 30.7$ independent of b such that $|h_0(w)| \le \widetilde{h}_0$. On the portion $\Im t = \frac{3}{2}\pi$, we can show that the same estimate holds; this completes the proof of Proposition 3.2.

Next we determine the contour Γ. We still choose the contour to be a rectangle, whose boundary is controlled by the length $|EO|$. Since we have proved in Proposition 3.2 that w is a decreasing function of b, we choose $|EO|$ to be the minimal; i.e., $|EO| = 1.21$. Then the boundary Γ of the rectangle $ABCD$ is given by

$$\Re w = \pm 1.21, \quad \Im w = \pm 2.93. \tag{3.65}$$

The corresponding boundary Γ' of the small rectangle $A'B'C'D'$ is given by

$$\Re w = \pm \frac{1.21}{3}, \quad \Im w = \pm 1.88. \tag{3.66}$$

With Γ and Γ' so chosen, we can estimate the quantity $\widetilde{h}_n$ defined in (3.27).

Theorem 3.2 *There exists a positive constant C_n such that for each nonnegative integer n, we have*

$$\widetilde{h}_n \le C_n \widetilde{h}_0 \quad \textit{for } 0 \le \alpha \le 3. \tag{3.67}$$

We are now ready to estimate $\varepsilon_n^{(1)}$ and $\varepsilon_n^{(2)}$ defined in (3.32) and (3.33), respectively. To estimate $\varepsilon_n^{(1)}$, we first divide $\mathcal{L}'$ into three parts; with $w = x + \mathrm{i}y$, define

$$\mathcal{L}'_+ = \{y > 0 \mid w \in \mathcal{L}' \text{ and on the steepest decent path through } b\}, \tag{3.68}$$

$$\mathcal{L}'_- = \{y < 0 \mid w \in \mathcal{L}' \text{ and on the steepest decent path through } -b\}, \tag{3.69}$$

$$\mathcal{L}'_1 = \Big\{ w \in \mathcal{L}' : x = -\frac{1.21}{4} \Big\}. \tag{3.70}$$

Clearly,

$$\frac{1}{2\pi \mathrm{i}} \int_{\mathcal{L}'} \mathrm{e}^{\nu(\frac{1}{3}w^3 - b^2 w)} h_n(w) \mathrm{d}w = \frac{1}{2\pi \mathrm{i}} \int_{\mathcal{L}'_+ + \mathcal{L}'_- + \mathcal{L}'_1} \mathrm{e}^{\nu(\frac{1}{3}w^3 - b^2 w)} h_n(w) \mathrm{d}w. \tag{3.71}$$

On $\mathcal{L}'_+$ and $\mathcal{L}'_-$, we have

$$\begin{aligned} &\frac{1}{2\pi \mathrm{i}} \int_{\mathcal{L}'_+ + \mathcal{L}'_-} \mathrm{e}^{\nu(\frac{1}{3}w^3 - b^2 w)} h_n(w) \mathrm{d}w \\ &= \frac{1}{2\pi \mathrm{i}} \int_{\mathcal{L}'_+} \mathrm{e}^{-\nu(y+\mathrm{i}b)^2 f(y)} g(y) [\mathrm{e}^{-\frac{2}{3}\nu b^3} h_n(w) - \mathrm{e}^{\frac{2}{3}\nu b^3} h_n(\overline{w})] \mathrm{d}y \\ &\quad + \frac{1}{2\pi} \int_{\mathcal{L}'_+} \mathrm{e}^{-\nu(y+\mathrm{i}b)^2 f(y)} [\mathrm{e}^{-\frac{2}{3}\nu b^3} h_n(w) + \mathrm{e}^{\frac{2}{3}\nu b^3} h_n(\overline{w})] \mathrm{d}y, \end{aligned}$$

where

$$f(y) = \frac{2(2y - \mathrm{i}b)^2}{9y} \sqrt{\frac{y - 2\mathrm{i}b}{3y}}, \quad g(y) = \sqrt{\frac{y - 2\mathrm{i}b}{3y}} + \frac{\mathrm{i}b(y + \mathrm{i}b)}{3y^2} \sqrt{\frac{3y}{y - 2\mathrm{i}b}}. \tag{3.72}$$

Since b is purely imaginary, we have $g(y) \ge 0$. Hence, it follows that

$$\begin{aligned} &\Big| \frac{1}{2\pi \mathrm{i}} \int_{\mathcal{L}'_+ + \mathcal{L}'_-} \mathrm{e}^{\nu(\frac{1}{3}w^3 - b^2 w)} h_n(w) \mathrm{d}w \Big| \\ &\le \frac{1}{2\pi} \int_{\mathcal{L}'_+} \mathrm{e}^{-\nu(y+\mathrm{i}b)^2 f(y)} (1 + g(y)) (|h_n(w)| + |h_n(\overline{w})|) \mathrm{d}y \\ &\le C_n \widetilde{h}_0 \frac{1}{\pi} \int_0^\infty \mathrm{e}^{-\nu(y+\mathrm{i}b)^2 f(y)} (1 + g(y)) \mathrm{d}y. \end{aligned}$$

The last integral has been evaluated exactly in [7, p. 311], and we obtain

$$\Big|\frac{1}{2\pi\mathrm{i}}\int_{\mathcal{L}'_{+}+\mathcal{L}'_{-}} \mathrm{e}^{\nu(\frac{1}{3}w^3-b^2w)}h_n(w)\mathrm{d}w\Big| \le C_n\widetilde{h}_0\sqrt{\frac{\mathrm{i}}{\nu b}}. \tag{3.73}$$

On $\mathcal{L}'_1$, we have

$$\Big|\frac{1}{2\pi\mathrm{i}}\int_{\mathcal{L}'_1} \mathrm{e}^{\nu(\frac{1}{3}w^3-b^2w)}h_n(w)\mathrm{d}w\Big| \le C_n\widetilde{h}_0\int_{\mathcal{L}'_1} \mathrm{e}^{\nu(\frac{1}{3}x^3-xy^2+x^2y\mathrm{i}-\frac{1}{3}y^3\mathrm{i}-b^2x-b^2y\mathrm{i})}\mathrm{d}y.$$

Since we have $x=-\frac{1.21}{4}$ on $\mathcal{L}'_1$, we can easily compute this integral, and obtain

$$\begin{aligned}\Big|\frac{1}{2\pi\mathrm{i}}\int_{\mathcal{L}'_1} \mathrm{e}^{\nu(\frac{1}{3}w^3-b^2w)}h_n(w)\mathrm{d}w\Big| &\le C_n\widetilde{h}_0\mathrm{e}^{\nu(\frac{1}{3}x^3-b^2x)}\int_0^T \mathrm{e}^{-\nu xy^2}\mathrm{d}y\\ &\le C_n\widetilde{h}_0\mathrm{e}^{\nu(\frac{1}{3}x^3-b^2x)}\frac{\mathrm{e}^{-\nu xT^2}}{\nu xT},\end{aligned}$$

where T is a constant depending on b and can be computed exactly by using the equation of the steepest descent path. It is obvious that $T<|b|$. Since $x=-\frac{1.21}{4}$, we have

$$\Big|\frac{1}{2\pi\mathrm{i}}\int_{\mathcal{L}'_1} \mathrm{e}^{\nu(\frac{1}{3}w^3-b^2w)}h_n(w)\mathrm{d}w\Big| \le C_n\widetilde{h}_0\frac{\mathrm{e}^{\frac{1}{3}\nu x^3}}{\nu}, \tag{3.74}$$

which is exponentially small in comparison with the estimate in (3.73) when $\nu\to\infty$.

Next we estimate $\varepsilon_n^{(2)}$. For $b\in[0,1.44\mathrm{i}]$, it is not difficult to show that

$$\mathcal{L}''=\left\{\sqrt{\frac{(y+\mathrm{i}b)^2(y-2\mathrm{i}b)}{3y}}+\mathrm{i}y \;\Big|\; |y|\ge y_0\right\},$$

where y_0 is a constant depending on b and $y_0\ge 1.88$. Since b is bounded, as in the monotonic case there is a neighborhood Ω_0 of $\mathcal{L}''$, in which we have

$$|h_n(w)|\le C_n\widetilde{h}_0.$$

Similarly to the representation for $\varepsilon_n^{(1)}$, we have

$$\varepsilon_n^{(2)}=(-1)^n\nu^{-n}\frac{1}{2\pi\mathrm{i}}\int_{y_0}^{\infty}\mathrm{e}^{-\nu(y+\mathrm{i}b)^2f(y)}[1+g(y)](h_n(w)+h_n(\overline{w}))\mathrm{d}y, \tag{3.75}$$

where $f(y)$ and $g(y)$ are given in (3.72). Since ν is large, we have

$$\begin{aligned}|\varepsilon_n^{(2)}| &\le C_n\widetilde{h}_0\nu^{-n}\int_{y_0}^{\infty}\mathrm{e}^{-\nu(y+\mathrm{i}b)^2f(y)}\mathrm{d}y\\ &\le C_n\widetilde{h}_0\nu^{-n}\int_{y_0}^{\infty}\mathrm{e}^{-\frac{8}{9\sqrt{3}}\nu(y+\mathrm{i}b)^2y}\mathrm{d}y\\ &\le C_n\widetilde{h}_0\nu^{-n-1}\mathrm{e}^{-\frac{8}{9\sqrt{3}}\times 0.19\nu y_0}\\ &\le C_n\widetilde{h}_0\nu^{-n-1}\mathrm{e}^{-0.18\nu},\end{aligned}$$

where we have used the fact that $y+\mathrm{i}b$ attains its minimum at $b=1.44\mathrm{i}$. Combining (3.73), (3.74) and (3.75) together, we obtain a computable error bound for the remainder.

Finally, we note that in this case, we can not let $b\to\infty$, since the modified Bessel function $K_{\mathrm{i}\nu}(\nu a)$ has no meaning when $a=0$. So, if b is bound but large, the estimate given above will be quite inaccurate.

4 Error Bound

Next, we give some numerical results. First we will evaluate the coefficients in the asymptotic expansion (3.8). Since this expansions is obtained by using an integral method, it is not possible to derive a recurrence relation for the coefficients. In most of the literature on Airy-type expansions, the authors gave just the first few coefficients, since the coefficients expressed in (2.4) are difficult to calculate. Now we can use the algorithm given in [12] to construct an analytic expression for the coefficients (see Appendix). The results in the following two tables are obtained with the aid of Maple.

Table 1 Values of coefficients ($a \geq 1$)

θ	a_0	b_0	a_1	b_1	a_2	b_2
0.2	1.25275	0	0	0.00205	−0.01298	0
0.785	1.17089	0	0	0.01669	−0.04545	0
1.4	0.80477	0	0	0.00295	−0.00056	0

Table 2 Values of coefficients ($0 < a \leq 1$)

α	a_0	b_0	a_1	b_1	a_2	b_2
0.1	1.26118i	0	0	0.02152i	−0.00252i	0
0.9	1.34793i	0	0	0.016012i	−0.00659i	0
2	1.55016i	0	0	0.04796i	−0.00551i	0

To illustrate how to compute the error bound, we take $n = 3$. In order to estimate ε_3, we first estimate $h_3(w)$. Since

$$h_3(w) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma R_3(u, w, b) h_0(w) \mathrm{d}w, \tag{4.1}$$

and

$$\begin{aligned} R_3(u, w, b) = &-\frac{2}{(u^2 - b^2)^4 (u - w)^2} + \frac{12u^2}{(u^2 - b^2)^5 (u - w)^2} \\ &+ \frac{4u}{(u^2 - b^2)^4 (u - w)^3} + \frac{3}{(u^2 - b^2)^3 (u - w)^4}, \end{aligned} \tag{4.2}$$

we first calculate the bounds of $R_3(u, w, b)$ and $h_0(w)$ for $\theta = 0.2,\ 0.785,\ 1.4$, $\alpha = 0.1,\ 0.9,\ 2$, and $\nu = 20$. Once we have the values of the coefficients and the estimate of $h_3(w)$, we can obtain an error bound for ε_3. The numerical results are presented in Table 3 and Table 4.

Table 3 Error bounds ($a \geq 1$)

θ	exact	appoximation	error	error bound
0.2	$1.01174 * 10^{-14}$	$1.01289 * 10^{-14}$	$1.15 * 10^{-17}$	$6.62 * 10^{-16}$
0.785	$8.64612 * 10^{-17}$	$8.65034 * 10^{-17}$	$4.22 * 10^{-20}$	$8.54 * 10^{-20}$
1.4	$1.66773 * 10^{-53}$	$1.66835 * 10^{-53}$	$5.72 * 10^{-57}$	$2.21 * 10^{-55}$

Table 4 Error bounds ($0 < a \leq 1$)

α	exact	appoximation	error	error bound
0.1	$1.21636 * 10^{-14}$	$1.21663 * 10^{-14}$	$2.8 * 10^{-18}$	$2.19 * 10^{-17}$
0.9	$-1.44545 * 10^{-14}$	$-1.44567 * 10^{-14}$	$2.2 * 10^{-18}$	$2.8 * 10^{-17}$
2	$6.11773 * 10^{-15}$	$6.09708 * 10^{-15}$	$2.07 * 10^{-17}$	$5.6 * 10^{-16}$

Appendix Algorithm for Computing the Coefficients

The algorithm in [12] was built on the basis of a two-point Taylor expansion. Assuming that the coefficients $p_k^{(1)}$ and $p_k^{(2)}$ of the expansions

$$h(w) = \sum_{k=0}^{\infty} p_k^{(1)} (w-b)^k, \quad h(-w) = \sum_{k=0}^{\infty} p_k^{(2)} (w-b)^k \tag{A.1}$$

are available, the authors established the following theorem [12, pp. 322–323]:

Theorem A.1 *Let the coefficients f_k^e and f_k^o be defined by*

$$f_k^e = \frac{1}{2}[p_k^{(1)} + p_k^{(2)}], \quad f_k^o = \frac{1}{2}[p_k^{(1)} - p_k^{(2)}], \quad k = 0, 1, 2, \cdots, \tag{A.2}$$

and the coefficients $f_k^{o,e}$ by the recursion

$$b f_k^{o,e} = f_k^o - f_{k-1}^{o,e}, \quad k \geq 0, \tag{A.3}$$

with $f_{-1}^{o,e} = 0$. Next, define the coefficients γ_k and δ_k by

$$\gamma_0 = f_0^e, \quad \delta_0 = f_0^{o,e} \tag{A.4}$$

and for $k \geq 1$

$$\begin{aligned} \gamma_k &= \sum_{j=1}^{k} \frac{(-1)^{k-j} j (2k-j-1)!}{(2b)^{2k-j} k! (k-j)!} f_j^e, \\ \delta_k &= \sum_{j=1}^{k} \frac{(-1)^{k-j} j (2k-j-1)!}{(2b)^{2k-j} k! (k-j)!} f_j^{o,e}. \end{aligned} \tag{A.5}$$

Finally, for $n \geq 0$ define the coefficients $\gamma_k^{(n)}$ and $\delta_k^{(n)}$ by the recursion

$$\begin{aligned} \gamma_k^{(n+1)} &= (2k+1)\delta_{k+1}^{(n)} + 2b^2(k+1)\delta_{k+2}^{(n)}, \\ \delta_k^{(n+1)} &= 2(k+1)\gamma_{k+2}^{(n)}, \quad k = 0, 1, 2, \cdots, \end{aligned} \tag{A.6}$$

with $\gamma_k^{(0)} = \gamma_k$, $\delta_k^{(0)} = \delta_k$. Then the coefficients a_n and b_n in the expansion (3.8) (*or* (3.57)) *are given by*

$$a_n = \gamma_0^{(n)}, \quad b_n = \delta_0^{(n)}, \quad n \geq 0. \tag{A.7}$$

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