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Remark on the Regularities of Kato's Solutions to Navier-Stokes Equations with Initial Data in $L^d(\mathbb{R}^d)$**

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Abstract Motivated by the results of J. Y. Chemin in "J. Anal. Math., **77**, 1999, 27–50" and G. Furioli et al in "Revista Mat. Iberoamer., **16**, 2002, 605–667", the author considers further regularities of the mild solutions to Navier-Stokes equation with initial data $u_0 \in L^d(\mathbb{R}^d)$. In particular, it is proved that if $u \in C([0,T^*);L^d(\mathbb{R}^d))$ is a mild solution of (NS_ν), then $u(t,x)-\mathrm{e}^{\nu t\Delta}u_0 \in \widetilde{L}^\infty((0,T);\dot{B}^1_{\frac{d}{2},\infty}) \cap \widetilde{L}^1((0,T);\dot{B}^3_{\frac{d}{2},\infty})$ for any $T<T^*$.

Keywords Navier-Stokes equations, Kato's solutions, Para-differential decomposition
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1 Introduction

In this article, we consider Navier-Stokes system for incompressible fluids in the whole space:

$$(NS_\nu) \quad \begin{cases} \partial_t u + u\cdot\nabla u - \nu\Delta u = -\nabla p, \quad (t,x)\in(0,\infty)\times\mathbb{R}^d, \\ \operatorname{div} u = 0, \\ u|_{t=0} = u_0, \end{cases}$$

where $u(t,x)$ denotes the fluid velocity and $p(t,x)$ the pressure.

In 1964, Fujita and Kato proved the local well-posedness of (NS_ν) with initial data in $\dot{H}^{\frac12}(\mathbb{R}^3)$, which is the space of distributions u with Fourier transform satisfying

$$\|u\|^2_{\dot{H}^{\frac12}} \stackrel{\text{def}}{=} \int_{\mathbb{R}^3} |\xi||\widehat{u}(\xi)|^2 \mathrm{d}\xi < \infty.$$

The reason why they consider (NS_ν) with initial data in $\dot{H}^{\frac12}(\mathbb{R}^3)$ is motivated by the following observation: let $u(t,x)$ be a solution to (NS_ν) on a time interval $[0,T]$ with initial data $u_0(x)$, then the vector field u_λ defined by

$$u_\lambda(t,x) \stackrel{\text{def}}{=} \lambda u(\lambda^2 t, \lambda x)$$

is also a solution of (NS_ν) on the time interval $[0,\lambda^{-2}T]$ with initial data $\lambda u_0(\lambda x)$. It is easy to check that the norm $\|u_0\|_{\dot{H}^{\frac12}}$ is scaling invariant under the transformation: $u_0(x)\to\lambda u_0(\lambda x)$

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for all $\lambda > 0$. The other typical examples are given by the Lebesgue space $L^3(\mathbb{R}^3)$ in [9, 3], the homogeneous Besov space $\dot{B}^{-1+\frac{3}{p}}_{p,\infty}(\mathbb{R}^3)$ for $p > 3$ in [2].

The standard procedure used in the study of the well-posedness to (NS_ν) is: one first transforms (NS_ν) to the integral form

$$u(t) = S(t)u_0 + B(u,u)(t), \tag{1.1}$$

where

$$B(v,u)(t) \stackrel{\text{def}}{=} -\int_0^t \mathbb{P}S(t-s)\nabla\cdot(v\otimes u)(s)\mathrm{d}s,$$

$\mathbb{P} = I - \nabla\Delta^{-1}\operatorname{div}$ is the projection operator onto divergence free vector fields and $S(t) \stackrel{\text{def}}{=} \mathrm{e}^{\nu t\Delta}$ is the heat semigroup; then one uses fixed point theorem for (1.1) in an appropriate Banach space.

However, as noticed by Oru in [10], the operator $B(v,u)$ is not continuous on $C([0,T^*); L^d(\mathbb{R}^d))$. Therefore, given initial data $u_0 \in L^d(\mathbb{R}^d)$, one can not use fixed point theorem for (1.1) in $C([0,T^*); L^d(\mathbb{R}^d))$. Instead, one has to search for an appropriate smaller subspace of $C([0,T^*); L^d(\mathbb{R}^d))$, on which one can use fixed theorem to prove both the existence as well as the uniqueness of solutions to (1.1) in this subspace.

It was not until 1998 that Furioli, Lemarié-Rieusset and Terraneo [8] proved the uniqueness of solution to (NS_ν) in the class of $C([0,T^*); L^d(\mathbb{R}^d))$. One key observation in [8] is that: when $u \in C([0,T^*); L^d(\mathbb{R}^d))$, $B(u,u) \in L^\infty((0,T^*); B^{\frac{d}{p}-1}_{p,\infty})$ for every $p \in [\frac{d}{2}, \infty)$. In this text, we are going to use the function space introduced by Chemin in [6] to improve the regularity of $B(u,u)$ for Kato's solution $u \in C([0,T^*); L^d(\mathbb{R}^d))$ of (NS_ν) (see [9]).

Before we present the main result of this article, let us start with the space we are going to work. As it requires the dyadic decomposition of the Fourier space, let us first recall the following operators of localization in Fourier space:

$$\Delta_j a = \mathcal{F}^{-1}(\varphi(2^{-j}|\xi|)\widehat{a}) \quad \text{and} \quad S_j a = \sum_{j'\le j-1}\Delta_{j'}a, \tag{1.2}$$

where $\mathcal{F}a$ and $\widehat{a}$ denote the Fourier transform of the distribution a, $\varphi(\tau)$ is a smooth function with $\operatorname{supp}\varphi \subset \{\tau :\ \frac{3}{4} \le \tau \le \frac{8}{3}\}$, and

$$\sum_{j\in\mathbf{Z}}\varphi(2^{-j}\tau) = 1, \quad \forall\,\tau > 0.$$

Motivated by [6, 8], to study further regularities of the mild solution $u \in C([0,T^*); L^d(\mathbb{R}^d))$ to (NS_ν), we recall the following time dependent space, $E_{\frac{d}{2}}(T)$, from [6].

Definition 1.1 *Let $T > 0$. $E_{\frac{d}{2}}(T)$ is the space of tempered distributions $u \in L^\infty\big([0,T]; \dot{B}^1_{\frac{d}{2},\infty}\big)$ such that*

$$\|u\|_{E_{\frac{d}{2}}(T)} \stackrel{\text{def}}{=} \|u\|_{\widetilde{L}^\infty_T\left(\dot{B}^1_{\frac{d}{2},\infty}\right)} + \nu\|u\|_{\widetilde{L}^1_T\left(\dot{B}^3_{\frac{d}{2},\infty}\right)} < \infty,$$

with

$$\|u\|_{\widetilde{L}^\infty_T\left(\dot{B}^1_{\frac{d}{2},\infty}\right)} \stackrel{\text{def}}{=} \sup_{j\in\mathbf{Z}} 2^j\|\Delta_j u\|_{L^\infty_T(L^{\frac{d}{2}})} \quad \textit{and} \quad \|u\|_{\widetilde{L}^1_T\left(\dot{B}^3_{\frac{d}{2},\infty}\right)} \stackrel{\text{def}}{=} \sup_{j\in\mathbf{Z}} 2^{3j}\|\Delta_j u\|_{L^1_T(L^{\frac{d}{2}})}.$$

Now, we present the main result of this article.

Theorem 1.1 *Let $u \in C([0,T^*); L^d(\mathbb{R}^d))$ be the unique mild solution of (NS_ν) with initial data $u_0 \in L^d(\mathbb{R}^d)$. Then*

$$u - \mathrm{e}^{\nu t\Delta} u_0 \in E_{\frac{d}{2}}(T) \quad \text{for any } T < T^*.$$

We conclude this section by recalling the para-differential decomposition from [1]: let $a, b \in \mathcal{S}'(\mathbb{R}^3)$,

$$ab = T_a b + R(a,b), \tag{1.3}$$

where

$$T_a b = \sum_{j\in\mathbf{Z}} S_{j-1} a \Delta_j b \quad \text{and} \quad R(a,b) = \sum_{j\in\mathbf{Z}} \Delta_j a S_{j+2} b.$$

Notations By $a \lesssim b$, we mean that there is a uniform constant C, which may be different on different lines, such that $a \le Cb$; and we denote by $L^r_T(L^p)$ the space $L^r([0,T]; L^p(\mathbb{R}^d))$.

2 Proof of Theorem 1.1

As we will constantly use the Littlewood-Paley theory in the subsequence, for convenience of the reader, we recite it in the following, and one may refer to [4] or [5] for further details.

Lemma 2.1 *Let $\mathcal{B}$ be a ball of $\mathbb{R}^d$, and $\mathcal{C}$ a ring of $\mathbb{R}^d$; let $1 \le p_2 \le p_1 \le \infty$. Then there holds:*

If the support of $\widehat{a}$ is included in $2^k\mathcal{B}$, then

$$\|\partial^\alpha a\|_{L^{p_1}(\mathbb{R}^d))} \lesssim 2^{k(|\alpha| + d(\frac{1}{p_2} - \frac{1}{p_1}))} \|a\|_{L^{p_2}(\mathbb{R}^d)}.$$

If the support of $\widehat{a}$ is included in $2^k\mathcal{C}$, then

$$\|a\|_{L^{p_1}(\mathbb{R}^d)} \lesssim 2^{-kN} \sup_{|\alpha|=N} \|\partial^\alpha a\|_{L^{p_1}(\mathbb{R}^d)}.$$

Lemma 2.2 *Let $\mathcal{C}$ be a ring of $\mathbb{R}^d$. Then there exist two positive constants c and C such that, for any $r > 1$ and any couple (t, λ) of positive real numbers, there holds*

$$\operatorname{Supp} \widehat{u} \subset \lambda\mathcal{C} \Longrightarrow \|\mathrm{e}^{t\Delta} u\|_{L^r} \le C\mathrm{e}^{-ct\lambda^2} \|u\|_{L^r}.$$

The proof of Theorem 1.1 is essentially based on the following three Lemmas.

Lemma 2.3 *Let $w_1, w_2 \in L^\infty((0,T); L^d(\mathbb{R}^d)) \cap E_{\frac{d}{2}}(T)$. Then there holds*

$$\|B(w_1,w_2)\|_{E_{\frac{d}{2}}(T)} \lesssim \frac{1}{\nu} (\|w_1\|_{L^\infty_T(L^d)} \|w_2\|_{E_{\frac{d}{2}}(T)} + \|w_2\|_{L^\infty_T(L^d)} \|w_1\|_{E_{\frac{d}{2}}(T)}).$$

Proof Thanks to Bony's decomposition (1.3), we write

$$w_1 w_2 = T_{w_1} w_2 + R(w_1, w_2).$$

Considering the support to the Fourier transform to terms above, we have

$$\begin{aligned}\Delta_j(T_{w_1}w_2) &= \sum_{|j'-j|\le 4}\Delta_j(S_{j'-1}w_1\Delta_{j'}w_2),\\ \Delta_j(R(w_1,w_2)) &= \sum_{j'\ge j-N_0}\Delta_j(\Delta_{j'}w_1S_{j'+2}w_2)\end{aligned}\tag{2.1}$$

for some positive integer N_0.

Then we get by applying Hölder inequality

$$\|\Delta_j(R(w_1,w_2))\|_{L^1_T(L^{\frac{d}{2}})}\lesssim\sum_{j'\ge j-N_0}\|\Delta_{j'}w_1\|_{L^1_T(L^{\frac{d}{2}})}\|S_{j'+2}w_2\|_{L^\infty_T(L^\infty)}.\tag{2.2}$$

However, thanks to Lemma 2.1, we have

$$\|S_{j'+2}w_2\|_{L^\infty_T(L^\infty)}\lesssim 2^{j'}\|S_{j'+2}w_2\|_{L^\infty_T(L^d)}\lesssim 2^{j'}\|w_2\|_{L^\infty_T(L^d)}.$$

As a consequence, we obtain

$$\begin{aligned}\|\Delta_j(R(w_1,w_2))\|_{L^1_T(L^{\frac{d}{2}})}&\lesssim\frac{1}{\nu}\|w_1\|_{E_{\frac{d}{2}}(T)}\|w_2\|_{L^\infty_T(L^d)}\sum_{j'\ge j-N_0}2^{-2j'}\\ &\lesssim\frac{2^{-2j}}{\nu}\|w_1\|_{E_{\frac{d}{2}}(T)}\|w_2\|_{L^\infty_T(L^d)}.\end{aligned}$$

A similar but easier proof gives

$$\|\Delta_j(T_{w_1}w_2)\|_{L^1_T(L^{\frac{d}{2}})}\lesssim\frac{2^{-2j}}{\nu}\|w_1\|_{L^\infty_T(L^d)}\|w_2\|_{E_{\frac{d}{2}}(T)}.$$

Therefore,

$$\|\Delta_j(w_1w_2)\|_{L^1_T(L^{\frac{d}{2}})}\lesssim\frac{2^{-2j}}{\nu}(\|w_1\|_{L^\infty_T(L^d)}\|w_2\|_{E_{\frac{d}{2}}(T)}+\|w_1\|_{E_{\frac{d}{2}}(T)}\|w_2\|_{L^\infty_T(L^d)}),$$

from which, and thanks to Lemma 2.2, we get by applying Young's inequality

$$\begin{aligned}\|\Delta_j(B(w_1,w_2))\|_{L^\infty_T(L^{\frac{d}{2}})}&\lesssim 2^j\Big\|\int_0^t e^{-c\nu(t-s)2^{2j}}\|\Delta_j(w_1w_2)(s)\|_{L^{\frac{d}{2}}}\,\mathrm{d}s\Big\|_{L^\infty_T}\\ &\lesssim 2^j\|\Delta_j(w_1w_2)\|_{L^1_T(L^{\frac{d}{2}})}\\ &\lesssim\frac{2^{-j}}{\nu}(\|w_1\|_{L^\infty_T(L^d)}\|w_2\|_{E_{\frac{d}{2}}(T)}+\|w_1\|_{E_{\frac{d}{2}}(T)}\|w_2\|_{L^\infty_T(L^d)}).\end{aligned}\tag{2.3}$$

Similarly, we get by applying Young's inequality once again

$$\begin{aligned}\nu 2^{2j}\|\Delta_j(B(w_1,w_2))\|_{L^1_T(L^{\frac{d}{2}})}&\lesssim 2^j\|\Delta_j(w_1w_2)\|_{L^1_T(L^{\frac{d}{2}})}\\ &\lesssim\frac{2^{-j}}{\nu}(\|w_1\|_{L^\infty_T(L^d)}\|w_2\|_{E_{\frac{d}{2}}(T)}+\|w_1\|_{E_{\frac{d}{2}}(T)}\|w_2\|_{L^\infty_T(L^d)}).\end{aligned}\tag{2.4}$$

This together with (2.3) completes the proof of this lemma.

Lemma 2.4 *Let $u_0\in L^d(\mathbb{R}^d)$, $u_L\stackrel{\text{def}}{=}e^{\nu t\Delta}u_0$ and $w\in L^\infty((0,T);L^d(\mathbb{R}^d))\cap E_{\frac{d}{2}}(T)$. Then for every $\varepsilon>0$, there exists $C_\varepsilon>0$ such that*

$$\|B(w,u_L)\|_{E_{\frac{d}{2}}(T)}\lesssim\frac{1}{\nu}(\|u_0\|_{L^d}\|w\|_{L^\infty_T(L^d)}+\min(\|u_0\|_{L^d},\varepsilon+C_\varepsilon(\nu T)^{\frac{1}{2}})\|w\|_{E_{\frac{d}{2}}(T)}).$$

Proof Again thanks to Bony's decomposition (1.3), we write

$$wu_L = T_w u_L + R(w, u_L),$$

with $\Delta_j(T_w u_L)$ and $\Delta_j(R(w, u_L))$ having the similar expressions as the correspondences in (2.1). Then we get by applying Hölder inequality

$$\|\Delta_j(T_w u_L)\|_{L^1_T(L^{\frac{d}{2}})} \lesssim \sum_{|j'-j|\le 4} \|S_{j'-1}w\|_{L^\infty_T(L^d)} \|\Delta_{j'}(u_L)\|_{L^1_T(L^d)}.$$

But thanks to (1.2),

$$\|S_{j'-1}w\|_{L^\infty_T(L^d)} \lesssim \|w\|_{L^\infty_T(L^d)},$$

and thanks to Lemma 2.2, we have

$$\|\Delta_j(u_L)\|_{L^1_T(L^d)} \lesssim \|\mathrm{e}^{-c\nu t 2^{2j}} \Delta_j(u_0)\|_{L^1_T(L^d)} \lesssim \frac{2^{-2j}}{\nu} \|\Delta_j(u_0)\|_{L^d} \lesssim \frac{2^{-2j}}{\nu} \|u_0\|_{L^d}. \tag{2.5}$$

Hence

$$\|\Delta_j(T_w u_L)\|_{L^1_T(L^{\frac{d}{2}})} \lesssim \frac{2^{-2j}}{\nu} \|u_0\|_{L^d} \|w\|_{L^\infty_T(L^d)}, \tag{2.6}$$

whereas we get by applying Hölder inequality to $\Delta_j(R(w, u_L))$

$$\|\Delta_j(R(w, u_L))\|_{L^1_T(L^{\frac{d}{2}})} \lesssim \sum_{j'\ge j-N_0} \|\Delta_{j'}w\|_{L^1_T(L^d)} \|S_{j'+2}(u_L)\|_{L^\infty_T(L^d)}.$$

Thanks to Lemma 2.1, we have

$$\|\Delta_{j'}w\|_{L^1_T(L^d)} \lesssim 2^{j'} \|\Delta_{j'}w\|_{L^1_T(L^{\frac{d}{2}})} \lesssim \frac{2^{-2j'}}{\nu} \|w\|_{E_{\frac{d}{2}}(T)},$$

and trivially

$$\|S_{j'+2}(u_L)\|_{L^\infty_T(L^d)} \lesssim \|u_L\|_{L^\infty_T(L^d)} \le \|u_0\|_{L^d},$$

from which we deduce

$$\|\Delta_j(R(w, u_L))\|_{L^1_T(L^{\frac{d}{2}})} \lesssim \frac{2^{-2j'}}{\nu} \|u_0\|_{L^d} \|w\|_{E_{\frac{d}{2}}(T)}. \tag{2.7}$$

When $\|u_0\|_{L^d}$ is large, for every $\varepsilon > 0$, we can find some $u_1 \in C_c^\infty(\mathbb{R}^d)$ such that $\|u_0 - u_1\|_{L^d} \le \varepsilon$. We denote $u_2 \overset{\text{def}}{=} u_0 - u_1$. Then similar proof of (2.7) gives

$$\|\Delta_j(R(w, \mathrm{e}^{\nu t\Delta} u_2))\|_{L^1_T(L^{\frac{d}{2}})} \lesssim \frac{\varepsilon}{\nu} 2^{-2j'} \|w\|_{E_{\frac{d}{2}}(T)}. \tag{2.8}$$

While thanks to (2.1), we get by applying Hölder inequality

$$\|\Delta_j(R(w, \mathrm{e}^{\nu t\Delta} u_1))\|_{L^1_T(L^{\frac{d}{2}})} \lesssim \sum_{j'\ge j-N_0} \|\Delta_{j'}w\|_{L^2_T(L^{\frac{d}{2}})} \|S_{j'+2}(\mathrm{e}^{\nu t\Delta} u_1)\|_{L^2_T(L^\infty)}.$$

However, using Hölder inequality once again, we have

$$\|\Delta_{j'}w\|_{L^2_T(L^{\frac{d}{2}})} \lesssim \|\Delta_{j'}w\|^{\frac{1}{2}}_{L^1_T(L^{\frac{d}{2}})} \|\Delta_{j'}w\|^{\frac{1}{2}}_{L^\infty_T(L^{\frac{d}{2}})} \lesssim \frac{2^{-2j}}{\nu^{\frac{1}{2}}} \|w\|_{E_{\frac{d}{2}}(T)},$$

$$\|S_{j'+2}(\mathrm{e}^{\nu t\Delta} u_1)\|_{L^2_T(L^\infty)} \lesssim T^{\frac{1}{2}} \|S_{j'+2}(\mathrm{e}^{\nu t\Delta} u_1)\|_{L^\infty_T(L^\infty)} \lesssim T^{\frac{1}{2}} \|u_1\|_{L^\infty} \le C_\varepsilon T^{\frac{1}{2}}.$$

This reaches the conclusion that

$$\|\Delta_j(R(w, \mathrm{e}^{\nu t\Delta}u_1))\|_{L^1_T(L^{\frac{d}{2}})} \lesssim \frac{C_\varepsilon}{\sqrt{\nu}} 2^{-2j} T^{\frac{1}{2}} \|w\|_{E_{\frac{d}{2}}(T)}. \tag{2.9}$$

By summing up (2.6)–(2.9), we complete the proof of Lemma 2.4 by repeating the argument in (2.3) and (2.4).

Lemma 2.5 *Let $u_0 \in L^d(\mathbb{R}^d)$, $u_L \stackrel{\text{def}}{=} \mathrm{e}^{\nu t\Delta}u_0$. Then, there holds*

$$\|B(u_L, u_L)\|_{E_{\frac{d}{2}}(T)} \lesssim \frac{1}{\nu}\|u_0\|^2_{L^d}.$$

Proof We use Bony's decomposition (1.3) to write

$$\Delta_j(u_L \otimes u_L) = \Delta_j(T_{u_L} \otimes u_L) + \Delta_j(R(u_L, u_L)).$$

Then thanks to (2.1), we get by applying Hölder inequality

$$\|\Delta_j(R(u_L, u_L))\|_{L^1_T(L^{\frac{d}{2}})} \lesssim \sum_{j' \geq j - N_0} \|\Delta_{j'}(u_L)\|_{L^1_T(L^d)} \|S_{j'+2}(u_L)\|_{L^\infty_T(L^d)},$$

which together with (2.5) gives

$$\|\Delta_j(R(u_L, u_L))\|_{L^1_T(L^{\frac{d}{2}})} \lesssim \frac{2^{-2j}}{\nu}\|u_0\|^2_{L^d}.$$

Similar inequality holds for $\|\Delta_j(T_{u_L} \otimes u_L)\|_{L^1_T(L^{\frac{d}{2}})}$. Then a proof similar to (2.3) and (2.4) concludes the proof of this lemma.

Now we are in a position to complete the proof of Theorem 1.1.

Proof of Theorem 1.1 Thanks to the uniqueness result in [8] that: given $u_0 \in L^d(\mathbb{R}^d)$, (NS_ν) has a unique solution $u \in C([0, T^*); L^d(\mathbb{R}^d))$. To get further regularities of this unique solution, we only need to do the a priori estimate for the approximate solutions, like what constructed by Kato in [9]. For simplicity, we just present the a priori estimate for the exact solution of (NS_ν).

We denote $u_L \stackrel{\text{def}}{=} \mathrm{e}^{\nu t\Delta}u_0$ and $w \stackrel{\text{def}}{=} u - u_L$. Then by substituting $u = u_L + w$ in (1.1) we get

$$w = B(w, w) + B(w, u_L) + B(u_L, w) + B(u_L, u_L), \tag{2.10}$$

from which we get by applying Lemma 2.3 to Lemma 2.5

$$\|w\|_{E_{\frac{d}{2}}(T)} \leq \frac{C}{\nu}[(\|w\|_{L^\infty_T(L^d)} + \varepsilon + C_\varepsilon(\nu T)^{\frac{1}{2}})\|w\|_{E_{\frac{d}{2}}(T)} + \|u_0\|_{L^d}(\|u_0\|_{L^d} + \|w\|_{L^\infty_T(L^d)})]. \tag{2.11}$$

Note from the assumption that $u \in C([0, T^*); L^d(\mathbb{R}^d))$ and $\mathrm{e}^{\nu t\Delta}$ is a continuous semi-group on $L^d(\mathbb{R}^d)$. Therefore, from its definition we get

$$\lim_{t \to 0^+} \|w\|_{L^\infty([0,t]; L^d)} = 0,$$

from which we deduce that there is a positive constant τ_1 such that

$$\|w\|_{L^\infty_{\tau_1}(L^d)} \leq \frac{\nu}{4C}.$$

And then we choose ε sufficiently small so that

$$\varepsilon \le \frac{\nu}{8C} \quad \text{and} \quad \tau_2 = \frac{\nu}{(8CC_\varepsilon)^2}.$$

We denote

$$T_1 \stackrel{\text{def}}{=} \min(\tau_1, \tau_2).$$

Then thanks to (2.11), we obtain

$$\|w\|_{E_{\frac{d}{2}}(T_1)} \le \frac{2C}{\nu}\|u_0\|_{L^d}(2\|u_0\|_{L^d} + \|u\|_{L^\infty_{T_1}(L^d)}). \tag{2.12}$$

If $T_1 < T^*$, as $u \in C([0, T^*); L^d(\mathbb{R}^d))$, we write

$$u(t) = S(t - T_1)u(T_1) - \int_{T_1}^t \mathbb{P}S(t-s)\nabla\cdot(u \otimes u)(s)\mathrm{d}s, \quad t > T_1, \tag{2.13}$$

and set

$$\widetilde{w} \stackrel{\text{def}}{=} u(t) - \mathrm{e}^{\nu(t-T_1)\Delta}u(T_1),$$

from which we repeat the argument used in the proof of (2.12) to find some $T_2 > T_1$ such that

$$\begin{aligned} \|\widetilde{w}\|_{E_{\frac{d}{2}}(T_1,T_2)} &\stackrel{\text{def}}{=} \sup_{j\in\mathbf{Z}} 2^j\|\Delta_j\widetilde{w}\|_{L^\infty((T_1,T_2);L^{\frac{d}{2}})} + \nu\sup_{j\in\mathbf{Z}} 2^{3j}\|\Delta_j\widetilde{w}\|_{L^1((T_1,T_2);L^{\frac{d}{2}})} \\ &\le \frac{2C}{\nu}\|u(T_1)\|_{L^d}(2\|u(T_1)\|_{L^d} + \|u\|_{L^\infty((T_1,T_2);L^d)}). \end{aligned} \tag{2.14}$$

On the other hand, as $w \in E_{\frac{d}{2}}(T_1)$, $w(T_1) \in \dot{B}^1_{\frac{d}{2},\infty}$. Then we get by using Lemma 2.2

$$\|\mathrm{e}^{\nu(t-T_1)\Delta}w(T_1)\|_{E_{\frac{d}{2}}(T_1,T_2)} \le C\|w(T_1)\|_{\dot{B}^1_{\frac{d}{2},\infty}},$$

which together with (2.14) ensures

$$\|w\|_{E_{\frac{d}{2}}(T_1,T_2)} \le \|\widetilde{w}\|_{E_{\frac{d}{2}}(T_1,T_2)} + \|\mathrm{e}^{\nu(t-T_1)\Delta}w(T_1)\|_{E_{\frac{d}{2}}(T_1,T_2)} \le C(\|u\|_{L^\infty_{T_2}(L^d)}), \tag{2.15}$$

where we have used the fact that $u(T_1) = \mathrm{e}^{\nu T_1\Delta}u_0 + w(T_1)$. Thanks to (2.12) and (2.15), we obtain

$$\|w\|_{E_{\frac{d}{2}}(T_2)} \le C(\|u\|_{L^\infty_{T_2}(L^d)}). \tag{2.16}$$

If $T_2 < T^*$, we can repeat the argument from (2.13) to (2.14) so that we can find some $T_3 > T_2$ such that (2.14) holds with T_1, T_2 there being replaced by T_2 and T_3 respectively. In this way, if $T^* < \infty$, we can find a maximal time τ^* such that (2.16) holds for any $T < \tau^*$. We claim that $\tau^* = T^*$. Otherwise, if $\tau^* < T^*$, as $u \in C([0,T^*); L^d(\mathbb{R}^d))$, we can replace T_1 in (2.13) by τ^*. Then repeating the argument from (2.13) to (2.16), we can find a $\tau^* < T' < T^*$ such that (2.16) holds with T_2 there being replaced by T', which contradicts the definition of τ^*. When $T^* = \infty$, a similar argument deduces that for any $T < \infty$, (2.16) holds with T_2 there being replaced by T. This completes the proof of Theorem 1.1.

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