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Strong (Weak) Exact Controllability and Strong (Weak) Exact Observability for Quasilinear Hyperbolic Systems***

Tatsien LI* Bopeng RAO**

(*Dedicated to Professor Roger Temam on the Occasion of his* 70*th Birthday*)

Abstract In this paper, the authors define the strong (weak) exact boundary controllability and the strong (weak) exact boundary observability for first order quasilinear hyperbolic systems, and study their properties and the relationship between them.

Keywords Strong (weak) exact boundary controllability, Strong (weak) exact boundary observability, First order quasilinear hyperbolic system
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1 Introduction and Preliminaries

A complete theory on the exact controllability and the exact observability for linear hyperbolic systems and for 1-D quasilinear hyperbolic systems can be found in [1, 2] and [3], respectively. In this paper, we give the definition of the strong (resp. weak) exact controllability and the strong (resp. weak) exact observability, and study their relationship for 1-D first order quasilinear hyperbolic systems. Since all the discussions are made in the framework of classical solutions, the controllability and the observability in the quasilinear hyperbolic case mean the local controllability and the local observability in a neighborhood of the equilibrium $u = 0$. However, in the linear hyperbolic case, the controllability and the observability should be the global controllability and the global observability.

Consider the following 1-D first order quasilinear hyperbolic system

$$\frac{\partial u}{\partial t} + A(u)\frac{\partial u}{\partial x} = F(u), \tag{1.1}$$

where $u = (u_1, \cdots, u_n)^{\mathrm{T}}$ is the unknown vector function of (t, x), $A(u)$ is an $n \times n$ matrix with C^1 entries $a_{ij}(u)$ $(i, j = 1, \cdots, n)$, and $F(u) = (f_1(u), \cdots, f_n(u))^{\mathrm{T}}$ is a C^1 vector function with

$$F(0) = 0. \tag{1.2}$$

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*School of Mathematical Sciences, Fudan University, Shanghai 200433, China; Shanghai Key Laboratory for Contemporary Applied Mathematics; Nonlinear Mathematical Modeling and Methods Laboratory. E-mail: dqli@fudan.edu.cn
**Institut de Recherche Mathématique Avancée, Université de Strasbourg, 67084 Strasbourg, France. E-mail: bopeng.rao@math.unistra.fr
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By (1.2), $u = 0$ is an equilibrium of system (1.1).

By hyperbolicity, for any given u on the domain under consideration, the matrix $A(u)$ possesses n real eigenvalues

$$\lambda_1(u), \lambda_2(u), \cdots, \lambda_n(u) \tag{1.3}$$

and a complete set of left eigenvectors $l_i(u) = (l_{i1}(u), \cdots, l_{in}(u))$ $(i = 1, \cdots, n)$ and right eigenvectors $r_i(u) = (r_{1i}(u), \cdots, r_{ni}(u))^{\mathrm{T}}$ $(i = 1, \cdots, n)$:

$$l_i(u)A(u) = \lambda_i(u)l_i(u) \tag{1.4}$$

and

$$A(u)r_i(u) = \lambda_i(u)r_i(u). \tag{1.5}$$

Without loss of generality, we assume that

$$l_i(u)r_j(u) \equiv \delta_{ij}, \quad i, j = 1, \cdots, n, \tag{1.6}$$

where δ_{ij} denotes the Kronecher symbol.

We suppose that all $\lambda_i(u)$, $l_i(u)$ and $r_i(u)$ $(i = 1, \cdots, n)$ have the same regularity as $A(u) = (a_{ij}(u))$.

Suppose that there are no zero eigenvalues:

$$\lambda_r(u) < 0 < \lambda_s(u), \quad r = 1, \cdots, m; \; s = m + 1, \cdots, n. \tag{1.7}$$

Let

$$v_i = l_i(u)u, \quad i = 1, \cdots, n. \tag{1.8}$$

v_i is called to be the diagonal variable corresponding to $\lambda_i(u)$ $(i = 1, \cdots, n)$. In a neighborhood of $u = 0$, $v = (v_1, \cdots, v_n)^{\mathrm{T}}$ is a C^1 diffeomorphism of $u = (u_1, \cdots, u_n)^{\mathrm{T}}$.

Under assumption (1.7), the most general boundary conditions which guarantee the well-posedness of the forward problem on the domain $\{(t, x) \mid t \geq 0, \; 0 \leq x \leq L\}$ can be written as (see [4])

$$x = 0 : v_s = G_s(t, v_1, \cdots, v_m) + H_s(t), \qquad s = m + 1, \cdots, n, \tag{1.9}$$

$$x = L : v_r = G_r(t, v_{m+1}, \cdots, v_n) + H_r(t), \quad r = 1, \cdots, m, \tag{1.10}$$

where G_i and H_i $(i = 1, \cdots, n)$ are C^1 functions and, without loss of generality, we suppose that

$$G_i(t, 0, \cdots, 0) \equiv 0, \quad i = 1, \cdots, n. \tag{1.11}$$

By (1.7), all the characteristics $\frac{\mathrm{d}x}{\mathrm{d}t} = \lambda_s(u)$ $(s = m + 1, \cdots, n)$ (resp. $\frac{\mathrm{d}x}{\mathrm{d}t} = \lambda_r(u)$ $(r = 1, \cdots, m)$) corresponding to the positive (resp. negative) eigenvalues are called to be the coming (resp. departing) characteristics on $x = 0$, since they reach (resp. leave) the boundary $x = 0$ from (resp. to) the interior of the domain $\{(t, x) \mid t \geq 0, \; 0 \leq x \leq L\}$. Similarly, all the characteristics $\frac{\mathrm{d}x}{\mathrm{d}t} = \lambda_r(u)$ $(r = 1, \cdots, m)$ (resp. $\frac{\mathrm{d}x}{\mathrm{d}t} = \lambda_s(u)$ $(s = m+1, \cdots, n)$) corresponding to the negative (resp. positive) eigenvalues are called to be the coming (resp. departing) characteristics on $x = L$.

Thus, the characters of boundary conditions (1.9)–(1.10) are as follows.

(1) On $x = 0$, the number of the boundary conditions = the number of the coming characteristics = $n - m$, while, on $x = L$, the number of the boundary conditions = the number of the coming characteristics = m.

(2) The boundary conditions on $x = 0$ are written in the form that the diagonal variables v_s ($s = m+1, \cdots, n$) corresponding to the coming characteristics on $x = 0$ are explicitly expressed by the diagonal variables v_r ($r = 1, \cdots, m$) corresponding to other characteristics, while, the boundary conditions on $x = L$ are written in the form that all the diagonal variables v_r ($r = 1, \cdots, m$) corresponding to the coming characteristics on $x = L$ are explicitly expressed by the diagonal variables v_s ($s = m+1, \cdots, n$) corresponding to other characteristics.

For the forward mixed initial-boundary value problem (1.1) and (1.9)–(1.10) with the initial condition

$$t = 0 : u = \varphi(x), \quad 0 \le x \le L, \tag{1.12}$$

according to the theory on the semi-global C^1 solution (see [5]), we have the following lemma.

Lemma 1.1 *Suppose that* (1.2), (1.7) *and* (1.11) *hold. Suppose furthermore that the conditions of C^1 compatibility are satisfied at the points $(t, x) = (0, 0)$ and $(0, L)$, respectively. For any given and possibly quite large $T_0 > 0$, if $\|\varphi\|_{C^1[0,L]}$ and $\|H\|_{C^1[0,T_0]}$ (in which $H = (H_1, \cdots, H_n)^{\mathrm{T}}$) are sufficiently small (depending on T_0), then the forward mixed problem* (1.1), (1.9)–(1.10) *and* (1.12) *admits a unique semi-global C^1 solution $u = u(t, x)$ on the domain $R(T_0) = \{(t, x) \mid 0 \le t \le T_0,\ 0 \le x \le L\}$. Moreover, under the additional hypothesis that $\frac{\partial G_i}{\partial t}$ $(i = 1, \cdots, n)$ satisfy the local Lipschitz condition with respect to the variable $v = (v_1, \cdots, v_n)^{\mathrm{T}}$ or can be controlled by $\|H\|_{C^1[0,T_0]}$ for the variable v in any given bounded set, we have*

$$\|u\|_{C^1[R(T_0)]} \le C(\|\varphi\|_{C^1[0,L]} + \|H\|_{C^1[0,T_0]}), \tag{1.13}$$

where C is a positive constant depending on T_0.

Remark 1.1 Suppose that (1.2) holds. If $\|\varphi\|_{C^1[0,L]}$ is sufficiently small, then Cauchy problem (1.1) and (1.12) admits a unique global C^1 solution $u = u(t, x)$ on the whole maximum determinate domain D and

$$\|u\|_{C^1[D]} \le C\|\varphi\|_{C^1[0,L]}, \tag{1.14}$$

where C is a positive constant and

$$D = \{(t, x) \mid t \ge 0,\ x_1(t) \le x \le x_2(t)\}, \tag{1.15}$$

in which $x = x_1(t)$ is the maximum characteristic passing through the point $(t, x) = (0, 0)$:

$$\begin{cases} \dfrac{\mathrm{d}x_1(t)}{\mathrm{d}t} = \max\limits_{i=1,\cdots,n} \lambda_i(u(t, x_1(t))), \\ x_1(0) = 0, \end{cases} \tag{1.16}$$

while, $x = x_2(t)$ is the minimum characteristic passing through the point $(t, x) = (0, L)$:

$$\begin{cases} \dfrac{\mathrm{d}x_2(t)}{\mathrm{d}t} = \min\limits_{i=1,\cdots,n} \lambda_i(u(t, x_2(t))), \\ x_2(0) = L. \end{cases} \tag{1.17}$$

Lemma 1.1 and Remark 1.1 are the basis of the nonlinear method given in [3] for establishing the exact controllability and the exact observability of 1-D quasilinear hyperbolic systems by means of a simple constructive method. They are also a basis in this paper.

2 Strong Exact Controllability and Weak Exact Controllability

For any given initial data $\varphi(x)$ and final data $\psi(x)$ with small $C^1[0,L]$ norm, if there exists a $T>0$ such that, taking $H_i(t)$ $(i=1,\cdots,n)$ or a part of $H_i(t)$ $(i=1,\cdots,n)$ with small $C^1[0,T]$ norm as boundary controls, the corresponding mixed initial-boundary value problem (1.1), (1.9)–(1.10) and (1.12) admits a unique semi-global C^1 solution $u=u(t,x)$ with small C^1 norm on the domain $R(T)=\{(t,x)\mid 0\le t\le T,\ 0\le x\le L\}$, which satisfies exactly the final condition

$$t=T: u=\psi(x),\quad 0\le x\le L, \tag{2.1}$$

then we say that there is a local exact boundary controllability. In this situation, by means of boundary controls, the system under consideration can drive any given initial data $\varphi(x)$ at $t=0$ exactly to any given final data $\psi(x)$ at $t=T$. This kind of controllability is called to be the strong exact boundary controllability in this paper.

On the other hand, there is another kind of controllability, namely, the so-called zero controllability (see [1, 2, 6]), for which the final data are specially taken as

$$t=T: u=0,\quad 0\le x\le L. \tag{2.2}$$

In this situation, by means of boundary controls, the system under consideration can drive any given initial data $\varphi(x)$ at $t=0$ exactly to the equilibrium $u=0$ at $t=T$. This kind of controllability is called to be the weak exact boundary controllability in this paper.

Obviously, the strong exact controllability implies the weak exact controllability, hence, when the strong exact controllability can be realized, it is not necessary to consider the corresponding weak exact controllability. However, when we do not know if the strong exact controllability can be realized or not, it is quite natural to ask if the corresponding weak exact controllability can be realized and if it is possible to get the strong exact controllability from the weak exact controllability under certain additional hypotheses.

By [7–10], for the strong exact boundary controllability, we have the following results.

Theorem 2.1 (Two-Sided Strong Exact Boundary Controllability) *Let*

$$T>L\max_{\substack{r=1,\cdots,m\\ s=m+1,\cdots,n}}\Big(\frac{1}{|\lambda_r(0)|},\frac{1}{\lambda_s(0)}\Big). \tag{2.3}$$

There exist boundary controls $H_i(t)$ $(i=1,\cdots,n)$ with small $C^1[0,T]$ norm, such that the corresponding mixed initial-boundary value problem (1.1), (1.9)–(1.10) *and* (1.12) *admits a unique semi-global C^1 solution $u=u(t,x)$ with small C^1 norm on the domain $R(T)=\{(t,x)\mid 0\le t\le T,\ 0\le x\le L\}$, which satisfies exactly the final condition* (2.1).

Theorem 2.2 (One-Sided Strong Exact Boundary Controllability) *Suppose that the number of positive eigenvalues is not greater than that of negative ones:*

$$\overline{m}\overset{\text{def.}}{=}n-m\le m,\quad i.e.,\quad n\le 2m. \tag{2.4}$$

Suppose furthermore that in a neighborhood of $u=0$, the boundary conditions (1.9) *on $x=0$* (*the side with less coming characteristics*) *can be equivalently rewritten as*

$$x=0: v_{\overline{r}}=\overline{G}_{\overline{r}}(t,v_{\overline{m}+1},\cdots,v_m,v_{m+1},\cdots,v_n)+\overline{H}_{\overline{r}}(t),\quad \overline{r}=1,\cdots,\overline{m} \tag{2.5}$$

with

$$\overline{G}_{\overline{r}}(t,0,\cdots,0)\equiv 0,\quad \overline{r}=1,\cdots,\overline{m}. \tag{2.6}$$

Let

$$T > L\Big(\max_{r=1,\cdots,m}\frac{1}{|\lambda_r(0)|}+\max_{s=m+1,\cdots,n}\frac{1}{\lambda_s(0)}\Big). \tag{2.7}$$

For any given $H_s(t)$ $(s=m+1,\cdots,n)$ with small $C^1[0,T]$ norm, satisfying the conditions of C^1 compatibility at the points $(t,x)=(0,0)$ and $(T,0)$, respectively, there exist boundary controls $H_r(t)$ $(r=1,\cdots,m)$ with small $C^1[0,T]$ norm on $x=L$ (the side with more coming characteristics), such that the conclusion of Theorem 2.1 *holds.*

Remark 2.1 In the linear case with the hypotheses that the number of positive eigenvalues is equal to that of negative ones ($n=2m$) and $H_s(t)\equiv 0$ $(s=m+1,\cdots,n)$, the corresponding result can be found in [1].

Theorem 2.3 (Two-Sided Strong Exact Boundary Controllability with Less Controls) *Suppose that the number of positive eigenvalues is less than that of negative ones:*

$$\overline{m}\overset{\text{def.}}{=} n-m<m,\quad i.e.,\quad n<2m. \tag{2.8}$$

Suppose furthermore that, in a neighborhood of $u=0$, without loss of generality, the first $\overline{m}$ boundary conditions in (1.10) *on $x=L$ (the side with more coming characteristics)*

$$x=L: v_{\overline{r}}=G_{\overline{r}}(t,v_{m+1},\cdots,v_n)+H_{\overline{r}}(t),\quad \overline{r}=1,\cdots,\overline{m} \tag{2.9}$$

can be equivalently rewritten as

$$x=L: v_s=\overline{G}_s(t,v_1,\cdots,v_{\overline{m}})+\overline{H}_s(t),\quad s=m+1,\cdots,n \tag{2.10}$$

with

$$\overline{G}_s(t,0,\cdots,0)\equiv 0,\quad s=m+1,\cdots,n. \tag{2.11}$$

Let $T>0$ satisfy (2.7). *For any given $H_{\overline{r}}(t)$ $(\overline{r}=1,\cdots,\overline{m})$ with small $C^1[0,T]$ norm, satisfying the corresponding conditions of C^1 compatibility at the points $(t,x)=(0,L)$ and (T,L), respectively, there exist boundary controls $H_s(t)$ $(s=m+1,\cdots,n)$ on $x=0$ (the side with less coming characteristics) and boundary controls $H_{\overline{\overline{r}}}(t)$ $(\overline{\overline{r}}=\overline{m}+1,\cdots,m)$ on $x=L$ (the side with more coming characteristics) with small $C^1[0,T]$ norm, such that the conclusion of Theorem* 2.1 *holds.*

In Theorems 2.1–2.3, estimates (2.3) and (2.7) on the controllability time are all sharp. Moreover, in order to realize the strong exact boundary controllability on the interval $[0,T]$, the number of boundary controls can not be reduced generically.

By Theorem 2.2, in order to realize the one-sided strong exact boundary controllability, we should utilize all the boundary functions $H_r(t)$ $(r=1,\cdots,m)$ as boundary controls on $x=L$ (the side with more coming characteristics) and we should suppose that the boundary condition (1.9) on the non-control side $x=0$ satisfies hypothesis (2.5). However, for getting the one-sided weak exact boundary controllability, boundary controls can be acted on any given side and the boundary conditions on the non-control side are not asked to satisfy a hypothesis of form (2.5), but all the boundary functions should be identically equal to zero in the boundary conditions on the non-control side. Precisely speaking, we have the following theorem.

Theorem 2.4 (One-Sided Weak Exact Boundary Controllability) *Let $T > 0$ satisfy* (2.7).

(1) *Suppose that the boundary condition* (1.10) *on $x = L$ is specially taken as*

$$x = L : v_r = G_r(t, v_{m+1}, \cdots, v_n), \quad r = 1, \cdots, m, \tag{2.12}$$

namely, $H_r(t) \equiv 0$ $(r = 1, \cdots, m)$. For any given initial data $\varphi(x)$ with small $C^1[0, L]$ norm, such that the conditions of C^1 compatibility are satisfied at the point $(t, x) = (0, L)$, there exist boundary controls $H_s(t)$ $(s = m+1, \cdots, n)$ with small $C^1[0, T]$ norm on $x = 0$, such that the corresponding mixed problem (1.1), (1.9), (2.12) *and* (1.12) *admits a unique semi-global C^1 solution $u = u(t, x)$ with small C^1 norm on the domain $R(T) = \{(t, x) \mid 0 \le t \le T,\ 0 \le x \le L\}$, which verifies exactly the zero final condition* (2.2).

(2) *Suppose that the boundary condition* (1.9) *on $x = 0$ is specially taken as*

$$x = 0 : v_s = G_s(t, v_1, \cdots, v_m), \quad s = m+1, \cdots, n, \tag{2.13}$$

namely, $H_s(t) \equiv 0$ $(s = m+1, \cdots, n)$. For any given initial data $\varphi(x)$ with small $C^1[0, L]$ norm, such that the conditions of C^1 compatibility are satisfied at the point $(t, x) = (0, 0)$, there exist boundary controls $H_r(t)$ $(r = 1, \cdots, m)$ with small $C^1[0, T]$ norm on $x = L$, such that the corresponding mixed problem (1.1), (2.13), (1.10) *and* (1.12) *admits a unique semi-global C^1 solution $u = u(t, x)$ with small C^1 norm on the domain $R(T)$, which verifies exactly the zero final condition* (2.2).

Remark 2.2 Noting (1.2) and (1.11), it is easy to see that the conditions of C^1 compatibility are satisfied at the point $(t, x) = (T, L)$ (resp. $(T, 0)$) for system (1.1), boundary condition (2.12) (resp. (2.13)) on $x = L$ (resp. $x = 0$) and the zero final condition (2.2).

Proof of Theorem 2.4 We only prove the first part of Theorem 2.4. The proof of the second part is similar.

Similarly to the proof of Theorem 2.2 (see [9]), it suffices to construct a C^1 solution $u = u(t, x)$ to system (1.1) on $R(T)$, such that it satisfies simultaneously the initial condition (1.12), the zero final condition (2.2) and the boundary condition (2.12) on $x = L$.

By (2.7), there exists an $\varepsilon_0 > 0$ so small that

$$T > L\Big(\sup_{|u| \le \varepsilon_0} \max_{r=1,\cdots,m} \frac{1}{|\lambda_r(u)|} + \sup_{|u| \le \varepsilon_0} \max_{s=m+1,\cdots,n} \frac{1}{\lambda_s(u)}\Big). \tag{2.14}$$

Let

$$T_1 = L \sup_{|u| \le \varepsilon_0} \max_{r=1,\cdots,m} \frac{1}{|\lambda_r(u)|} \tag{2.15}$$

and

$$T_2 = L \sup_{|u| \le \varepsilon_0} \max_{s=m+1,\cdots,n} \frac{1}{\lambda_s(u)}. \tag{2.16}$$

(i) On the domain

$$R_f = \{(t, x) \mid 0 \le t \le T_1,\ 0 \le x \le L\} \tag{2.17}$$

we consider the following forward mixed initial-boundary value problem for system (1.1) with the initial condition (1.12), the boundary condition (2.12) on $x = L$ and the following artificial boundary condition on $x = 0$:

$$x = 0 : v_s = f_s(t), \quad s = m+1, \cdots, n, \tag{2.18}$$

where $f_s(t)$ $(s = m+1, \cdots, n)$ are any given functions of t with small $C^1[0, T_1]$ norm, satisfying the conditions of C^1 compatibility at the point $(t, x) = (0, 0)$. By Lemma 1.1, this forward problem admits a unique semi-global C^1 solution $u = u_f(t, x)$ with small C^1 norm on the domain R_f. In particular, we have

$$|u_f(t,x)| \leq \varepsilon_0, \quad \forall\, (t,x) \in R_f. \tag{2.19}$$

Thus, we can determine the value of $u = u_f(t, x)$ on $x = L$ as

$$x = L : u_f = a(t), \quad 0 \leq t \leq T_1. \tag{2.20}$$

The $C^1[0, T_1]$ norm of $a(t)$ is small and $a(t)$ satisfies the boundary condition (2.12) on the interval $[0, T_1]$.

(ii) Next, on the domain

$$R_b = \{(t,x) \mid T - T_2 \leq t \leq T,\ 0 \leq x \leq L\}, \tag{2.21}$$

we want to get a C^1 solution which satisfies system (1.1), the zero final condition (2.2) and the boundary condition (2.12) on $x = L$. Different from the proof of Theorem 2.2, this C^1 solution can not be obtained by solving the corresponding backward mixed initial-boundary value problem generically, since in the special case that (2.8) holds, the boundary condition (2.12) on $x = L$ does not fit the requirement of well-posedness presented in Section 1. However, noting (2.2) and the special form of boundary condition (2.12), it is easy to see that $u = u_b(t, x) \equiv 0$ is just a desired C^1 solution.

(iii) Noting (2.14)–(2.16), there exists a $c(t)$ with small $C^1[0, T]$ norm, such that

$$c(t) = \begin{cases} a(t), & 0 \leq t \leq T_1, \\ 0, & T - T_2 \leq t \leq T, \end{cases} \tag{2.22}$$

and $c(t)$ satisfies the boundary condition (2.12) on the whole interval $[0, T]$.

Changing the role of t and x, we solve the following leftward mixed initial-boundary value problem for system (1.1) with the initial condition

$$x = L : u = c(t), \quad 0 \leq t \leq T \tag{2.23}$$

and the following boundary conditions reduced from the original initial data (1.12) and the final data (2.2):

$$t = 0 : v_r = l_r(\varphi(x))\varphi(x), \quad r = 1, \cdots, m, \tag{2.24}$$

$$t = T : v_s = 0, \qquad s = m+1, \cdots, n. \tag{2.25}$$

By Lemma 1.1, this mixed problem admits a unique semi-global C^1 solution $u = u(t, x)$ on the domain $R(T)$.

By means of the uniqueness of C^1 solution to the one-sided mixed initial-boundary value problem (see [4]), similarly to the proof of Theorem 2.2, it is easy to get that $u = u(t, x)$ also satisfies the initial condition (1.12) and the final condition (2.2). This finishes the proof.

Remark 2.3 A treatment similar to (ii) in the proof of Theorem 2.4 can be found in [1] for the linear hyperbolic case.

Remark 2.4 The estimate (2.7) on the weak controllability time is still sharp.

Similarly to Theorem 2.4, we have the following result related to Theorem 2.3.

Theorem 2.5 (Two-Sided Weak Exact Boundary Controllability with Less Controls) *Let $T > 0$ satisfy* (2.7).

(1) *Suppose that, without loss of generality, the first h $(0 < h \le m)$ boundary conditions in* (1.10) *on $x = L$ take the following form:*

$$x = L : v_a = G_a(t, v_{m+1}, \cdots, v_n), \quad a = 1, \cdots, h, \tag{2.26}$$

namely, $H_a(t) \equiv 0$ $(a = 1, \cdots, h)$. Then there exist boundary controls $H_s(t)$ $(s = m+1, \cdots, n)$ on $x = 0$ and boundary controls $H_b(t)$ $(b = h+1, \cdots, m)$ on $x = L$ with small $C^1[0, T]$ norm, such that the corresponding forward problem (1.1), (1.9)–(1.10) *and* (1.12) *admits a unique semi-global C^1 solution $u = u(t, x)$ with small C^1 norm on the domain $R(T)$, which verifies exactly the zero final condition* (2.2).

(2) *Suppose that, without loss of generality, the first g $(0 < g \le n-m)$ boundary conditions in* (1.9) *on $x = 0$ take the following form:*

$$x = 0 : v_c = G_c(t, v_1, \cdots, v_m), \quad c = m+1, \cdots, m+g, \tag{2.27}$$

namely, $H_c(t) \equiv 0$ $(c = m+1, \cdots, m+g)$. Then there exist boundary controls $H_d(t)$ $(d = m+g+1, \cdots, n)$ on $x = 0$ and boundary controls $H_r(t)$ $(r = 1, \cdots, m)$ on $x = L$ with small $C^1[0, T]$ norm, such that we have the same conclusion as in (1).

Remark 2.5 When $h = m$ or $g = n - m$, Theorem 2.5 implies Theorem 2.4.

Remark 2.6 In order to get the strong controllability from the corresponding weak controllability, we should suppose that

(a) The number of positive eigenvalues is equal to that of negative ones:

$$n - m = m, \quad \text{i.e.,} \quad n = 2m. \tag{2.28}$$

(b) In a neighborhood of $u = 0$, the boundary conditions (1.9) and (1.10) can be equivalently rewritten, respectively, as

$$x = 0 : v_r = \overline{G}_r(t, v_{m+1}, \cdots, v_n) + \overline{H}_r(t), \quad r = 1, \cdots, m \tag{2.29}$$

and

$$x = L : v_s = \overline{G}_s(t, v_1, \cdots, v_m) + \overline{H}_s(t), \quad s = m+1, \cdots, n \tag{2.30}$$

with

$$\overline{G}_i(t, 0, \cdots, 0) \equiv 0, \quad i = 1, \cdots, n. \tag{2.31}$$

Under these hypotheses, for system (1.1) and boundary conditions (1.9)–(1.10), both the forward mixed problem and the backward mixed problem are all well-posed. Therefore, if suitable boundary controls on the interval $[0, T]$ can be chosen to realize the weak exact boundary controllability for any given system (1.1) and any given boundary conditions (1.9)–(1.10), namely, to drive any given initial data $\varphi(x)$ with small $C^1[0, L]$ norm at $t = 0$ to the zero final data at $t = T$, then it is easily shown that suitable boundary controls on the interval $[0, T]$ can be used to realize the strong exact boundary controllability for driving the zero initial data at $t=0$ to any given final data $\psi(x)$ with small $C^1[0,L]$ norm at $t=T$.

In fact, consider the following quasilinear hyperbolic system

$$\frac{\partial u}{\partial \bar{t}} - A(u)\frac{\partial u}{\partial x} = -F(u) \tag{2.32}$$

and boundary conditions

$$x=0: v_r = \overline{G}_r(T-\bar{t}, v_{m+1}, \cdots, v_n) + \overline{H}_r(T-\bar{t}), \quad r=1,\cdots,m, \tag{2.33}$$

$$x=L: v_s = \overline{G}_s(T-\bar{t}, v_1, \cdots, v_m) + \overline{H}_s(T-\bar{t}), \qquad s=m+1,\cdots,n, \tag{2.34}$$

in which $\overline{G}_i$ and $\overline{H}_i$ $(i=1,\cdots,n)$ are given by (2.29)–(2.30) and

$$\bar{t} = T - t. \tag{2.35}$$

Noting (2.31), we have

$$\overline{G}_i(T-\bar{t}, 0, \cdots, 0) \equiv 0, \quad i=1,\cdots,n. \tag{2.36}$$

By the weak exact boundary controllability, suitable boundary controls on the interval $[0,T]$ can be used to drive any given initial data $\psi(x)$ with small $C^1[0,L]$ norm at $\bar{t}=0$ to the zero final data at $\bar{t}=T$. Hence, noting (2.35), system (1.1) with the corresponding boundary conditions (2.29)–(2.30) (i.e., (1.9)–(1.10)) can drive the zero initial data at $t=0$ to any given final data $\psi(x)$ with small $C^1[0,L]$ norm at $t=T$.

Thus, using the weak exact boundary controllability again, we get that suitable boundary controls on the interval $[0,2T]$ can be used to drive any given initial data $\varphi(x)$ with small $C^1[0,L]$ norm at $t=0$ to any given final data $\psi(x)$ with small $C^1[0,L]$ norm at $t=2T$. That is to say, the strong exact boundary controllability follows from the weak exact boundary controllability, however, the controllability time should be doubled in this indirect way.

3 Strong Exact Observability and Weak Exact Observability

For the mixed initial-boundary value problem (1.1), (1.9)–(1.10) and (1.12), if there exists a $T>0$ such that, under the assumption that the $C^1[0,L]$ norm of the initial data $\varphi(x)$ and the $C^1[0,T]$ norm of the boundary functions $H(t)=(H_1(t),\cdots,H_n(t))^{\mathrm{T}}$ are sufficiently small, and the conditions of C^1 compatibility are satisfied at the points $(t,x)=(0,0)$ and $(0,L)$ respectively, suitable boundary observations together with boundary functions $H(t)=(H_1(t),\cdots,H_n(t))^{\mathrm{T}}$ on the interval $[0,T]$ can uniquely determine the initial data $\varphi(x)$ $(0\le x\le L)$ and then the C^1 solution $u=u(t,x)$ on the domain $R(T)=\{(t,x)\mid 0\le t\le T,\ 0\le x\le L\}$, hence we have the (local) exact boundary observability which will be called to be the strong exact boundary observability in this paper.

On the other hand, in the linear hyperbolic case, D. Russell has introduced another kind of observability in [1], in which, for the backward problem, it requires to uniquely determine the initial data from boundary observations. Correspondingly, for the forward problem, it requires to uniquely determine the final data form boundary observations. Since, under assumptions (a) and (b) in Section 2 (see (2.28)–(2.31)), both the forward mixed problem (1.1), (1.9)–(1.10) and (1.12) and the backward mixed problem (1.1), (1.9)–(1.10) and (2.1) are well-posed, it is easy to see that these two kinds of observablilities are actually equivalent. However, in the general situation without assumptions (a) and (b), from the final data uniquely determined by boundary observations one can not uniquely determine the whole solution $u = u(t, x)$ on the domain $R(T)$ for the forward problem. This kind of observability is called to be the weak exact boundary observability in this paper.

Obviously, the strong exact observability implies the weak exact observability. Hence, when the strong exact observability can be realized, it is not necessary to consider the corresponding weak exact observability. However, when we do not know if the strong exact observability can be realized or not, it is quite natural to ask if the corresponding weak exact observability can be realized.

By [11, 12], for the strong exact boundary observability, we have the following results, in which the boundary functions $H(t) = (H_1(t), \cdots, H_n(t))^T$ with small C^1 norm are given.

Theorem 3.1 (Two-Sided Strong Exact Boundary Observability) *Suppose that* $T > 0$ *satisfies* (2.3). *For any given initial data* $\varphi(x)$ *with small* $C^1[0, L]$ *norm, satisfying the conditions of* C^1 *compatibility at the points* $(t, x) = (0, 0)$ *and* $(0, L)$, *respectively, the boundary observations* $v_r = \overline{v}_r(t)$ $(r = 1, \cdots, m)$ *corresponding to the departing characteristics on* $x = 0$ *and* $v_s = \overline{\overline{v}}_s(t)$ $(s = m + 1, \cdots, n)$ *corresponding to the departing characteristics on* $x = L$ *on the interval* $[0, T]$ *can be used to uniquely determine the initial data* $\varphi(x)$, *and the following strong observability inequality holds:*

$$\|\varphi\|_{C^1[0,L]} \le C\Big(\sum_{r=1}^{m} \|\overline{v}_r\|_{C^1[0,T]} + \sum_{s=m+1}^{n} \|\overline{\overline{v}}_s\|_{C^1[0,T]} + \|H\|_{C^1[0,T]}\Big), \tag{3.1}$$

where C *is a positive constant.*

Theorem 3.2 (One-Sided Strong Exact Boundary Observability) *Suppose that* (2.4) *holds and* $T > 0$ *satisfies* (2.7). *Suppose furthermore that in a neighborhood of* $u = 0$, *boundary condition* (1.10) *on* $x = L$ (*the side with more coming characteristics*) *implies*

$$x = L : v_s = \overline{G}_s(t, v_1, \cdots, v_m) + \overline{H}_s(t), \quad s = m + 1, \cdots, n \tag{3.2}$$

with

$$\overline{G}_s(t, 0, \cdots, 0) \equiv 0, \quad s = m + 1, \cdots, n. \tag{3.3}$$

Suppose finally that $\frac{\partial \overline{G}_s}{\partial t}$ $(s = m + 1, \cdots, n)$ *satisfy the local Lipschitz condition with respect to the variable* $v = (v_1, \cdots, v_n)^{\mathrm{T}}$ *or can be controlled by* $\|\overline{H}_s\|_{C^1[0,T]}$ $(s = m + 1, \cdots, n)$ *for the variable* v *in any given bounded set. For any given initial data* $\varphi(x)$ *satisfying the same properties as in Theorem* 3.1, *the boundary observations* $v_r = \overline{v}_r(t)$ $(r = 1, \cdots, m)$ *corresponding to the departing characteristics on* $x = 0$ (*the side with less coming characteristics*) *can be used to uniquely determine the initial data* $\varphi(x)$, *and the following strong observability inequality holds:*

$$\|\varphi\|_{C^1[0,L]} \le C\Big(\sum_{r=1}^{m} \|\overline{v}_r\|_{C^1[0,T]} + \|H\|_{C^1[0,T]}\Big), \tag{3.4}$$

where C *is a positive constant.*

Theorem 3.3 (Two-Sided Strong Exact Boundary Observability with Less Observations) *Suppose that* (2.8) *holds and* $T > 0$ *satisfies* (2.7). *Suppose furthermore that, without loss of* *generality, in a neighborhood of $u = 0$, boundary condition* (1.9) *on $x = 0$* (*the side with less coming characteristics*) *can be equivalently rewritten as*

$$x = 0 : v_{\overline{r}} = \overline{G}_{\overline{r}}(t, v_{\overline{m}+1}, \cdots, v_m, v_{m+1}, \cdots, v_n) + \overline{H}_{\overline{r}}(t), \quad \overline{r} = 1, \cdots, \overline{m} \tag{3.5}$$

with

$$\overline{G}_{\overline{r}}(t, 0, \cdots, 0) \equiv 0, \quad \overline{r} = 1, \cdots, \overline{m}. \tag{3.6}$$

Suppose finally that $\frac{\partial \overline{G}_{\overline{r}}}{\partial t}$ ($\overline{r} = 1, \cdots, \overline{m}$) satisfy the local Lipschitz condition with respect to the variable $v = (v_1, \cdots, v_n)^{\mathrm{T}}$ or can be controlled by $\|\overline{H}_{\overline{r}}\|_{C^1[0,T]}$ ($\overline{r} = 1, \cdots, \overline{m}$) for the variable v in any given bounded set. For any given initial data $\varphi(x)$ satisfying the same properties as in Theorem 3.1*, the boundary observations $v_{\overline{s}} = \overline{v}_{\overline{s}}(t)$ ($\overline{s} = \overline{m} + 1, \cdots, m$) corresponding to a part of the departing characteristics on $x = 0$ and $v_s = \overline{\overline{v}}_s(t)$ ($s = m + 1, \cdots, n$) corresponding to all the departing characteristics on $x = L$ can be used to uniquely determine the initial data $\varphi(x)$, and the following strong observability inequality holds:*

$$\|\varphi\|_{C^1[0,L]} \le C\Big(\sum_{\overline{s}=\overline{m}+1}^{m} \|\overline{v}_{\overline{s}}\|_{C^1[0,T]} + \sum_{s=m+1}^{n} \|\overline{\overline{v}}_s\|_{C^1[0,T]} + \|H\|_{C^1[0,T]}\Big), \tag{3.7}$$

where C is a positive constant.

In Theorems 3.1–3.3, the estimates (2.3) and (2.7) on the observability time, which coincide with the corresponding estimates on the controllability time, are both sharp. Moreover, in order to realize the strong exact boundary observability on the interval $[0, T]$, the number of boundary observations can not be reduced generically.

By Theorem 3.2, in order to realize the one-sided strong exact boundary observability, we should utilize all the boundary observations $v_r = \overline{v}_r(t)$ ($r = 1, \cdots, m$) on $x = 0$ (the side with less coming characteristics) and we should suppose that the boundary condition (1.10) on the non-observation side $x = L$ satisfies hypothesis (3.2). However, for getting the one-sided weak exact boundary observability, boundary observations can be taken on any given side and the boundary conditions on the non-observation side are not asked to satisfy a hypothesis of form (3.2). Precisely speaking, we have

Theorem 3.4 (One-Sided Weak Exact Boundary Observability) *Suppose that $T > 0$ satisfies* (2.7)*. For any given initial data $\varphi(x)$ satisfying the same properties as in Theorem* 3.1*, we have*

(1) *Suppose that $\frac{\partial G_s}{\partial t}$ ($s = m + 1, \cdots, n$) satisfy the local Lipschitz condition with respect to the variable $v = (v_1, \cdots, v_n)^{\mathrm{T}}$ or can be controlled by $\|H_s\|_{C^1[0,T]}$ ($s = m + 1, \cdots, n$) for the variable v in any given bounded set. Then, boundary observations $v_s = \overline{\overline{v}}_s(t)$ ($s = m+1, \cdots, n$) corresponding to all the departing characteristics on $x = L$ on the interval $[0, T]$ can be used to uniquely determine the final data $\psi(x)$ on $t = T$, and the following weak observability inequality holds:*

$$\|\psi\|_{C^1[0,L]} \le C\Big(\sum_{s=m+1}^{n} \|\overline{\overline{v}}_s\|_{C^1[0,T]} + \|H\|_{C^1[0,T]}\Big), \tag{3.8}$$

where C is a positive constant.

(2) *Suppose that $\frac{\partial G_r}{\partial t}$ ($r = 1, \cdots, m$) satisfy the local Lipschitz condition with respect to the variable $v = (v_1, \cdots, v_n)^{\mathrm{T}}$ or can be controlled by $\|H_r\|_{C^1[0,T]}$ ($r = 1, \cdots, m$) for the*

variable v in any given bounded set. Then, boundary observations $v_r = \overline{v}_r(t)$ $(r = 1, \cdots, m)$ corresponding to all the departing characteristics on $x = 0$ on the interval $[0, T]$ can be used to uniquely determine the final data $\psi(x)$ on $t = T$, and the following weak observability inequality holds:

$$\|\psi\|_{C^1[0,L]} \le C\Big(\sum_{r=1}^{m} \|\overline{v}_r\|_{C^1[0,T]} + \|H\|_{C^1[0,T]} \Big), \tag{3.9}$$

where C is a positive constant.

Proof We only prove the first part of Theorem 3.4. The proof of the second part is similar.

Similarly to the proof of Theorem 3.2 (see [12]), using the boundary observations $v_s = \overline{\overline{v}}_s(t)$ $(s = m+1, \cdots, n)$ on $x = L$ and the boundary condition (1.10) on $x = L$, we get the values $v_r = \overline{\overline{v}}_r(t)$ $(r = 1, \cdots, m)$ on $x = L$, where

$$\overline{\overline{v}}_r(t) = G_r(t, \overline{\overline{v}}_{m+1}(t), \cdots, \overline{\overline{v}}_n(t)) + H_r(t), \quad r = 1, \cdots, m. \tag{3.10}$$

Then, noting (1.11), we have

$$\sum_{r=1}^{m} \|\overline{\overline{v}}_r\|_{C^1[0,T]} \le C\Big(\sum_{s=m+1}^{n} \|\overline{\overline{v}}_s\|_{C^1[0,T]} + \sum_{r=1}^{m} \|H_r\|_{C^1[0,T]} \Big). \tag{3.11}$$

Here and hereafter C denotes a positive constant. Thus, the value $\overline{\overline{u}}(t)$ of the solution $u = u(t, x)$ on $x = L$ can be uniquely determined by the boundary observations $\overline{\overline{v}}_s(t)$ $(s = m+1, \cdots, n)$ and the boundary functions $H_r(t)$ $(r = 1, \cdots, m)$, and

$$\|\overline{\overline{u}}\|_{C^1[0,T]} \le C\Big(\sum_{s=m+1}^{n} \|\overline{\overline{v}}_s\|_{C^1[0,T]} + \sum_{r=1}^{m} \|H_r\|_{C^1[0,T]} \Big). \tag{3.12}$$

By Remark 1.1, the leftward Cauchy problem for system (1.1) with the initial condition

$$x = L : u = \overline{\overline{u}}(t), \quad 0 \le t \le T \tag{3.13}$$

admits a unique global C^1 solution $u = \widetilde{\widetilde{u}}(t, x)$ on the corresponding maximum determinate domain and

$$\|\widetilde{\widetilde{u}}\|_{C^1} \le C\Big(\sum_{s=m+1}^{n} \|\overline{\overline{v}}_s\|_{C^1[0,T]} + \sum_{r=1}^{m} \|H_r\|_{C^1[0,T]} \Big). \tag{3.14}$$

$u = \widetilde{\widetilde{u}}(t, x)$ is the restriction of the C^1 solution $u = u(t, x)$ to the original mixed problem (1.1), (1.9)–(1.10) and (1.12) on the intersection of this maximum determinate domain with $R(T)$.

Noting (2.7) and the smallness of the data, this maximum determinate domain must intersect $x = 0$. Hence, there exists a T_0 $(0 < T_0 < T)$ such that the value $\widehat{u}(x)$ of $u = u(t, x)$ at $t = T_0$ can be uniquely determined by $u = \widetilde{\widetilde{u}}(t, x)$, and

$$\|\widehat{u}\|_{C^1[0,L]} \le C\Big(\sum_{s=m+1}^{n} \|\overline{\overline{v}}_s\|_{C^1[0,T]} + \sum_{r=1}^{m} \|H_r\|_{C^1[0,T]} \Big). \tag{3.15}$$

We now solve the forward mixed problem for system (1.1) with the initial condition

$$t = T_0 : u = \widehat{u}(x), \quad 0 \le x \le L, \tag{3.16}$$

the boundary condition (1.9) on $x = 0$ and the following boundary condition

$$x = L : v_r = \overline{\overline{v}}_r(t), \quad r = 1, \cdots, m. \tag{3.17}$$

By Lemma 1.1 and noting (3.11) and (3.15), $u = u(t, x)$ as its unique C^1 solution on the domain $\{(t, x) \mid T_0 \le t \le T,\ 0 \le x \le L\}$ satisfies

$$\|u\|_{C^1} \le C\Big(\sum_{s=m+1}^{n} \|\overline{\overline{v}}_s\|_{C^1[0,T]} + \|H\|_{C^1[0,T]} \Big). \tag{3.18}$$

In particular, the final data $\psi(x)$ can be uniquely determined and (3.8) holds.

Remark 3.1 In Theorem 3.4, the estimate (2.7) on the weak observability time is sharp.

For the purpose in what follows, we rewrite Theorems 3.2 and 3.4 in the corresponding form for the backward problem.

Noting that the coming (resp. departing) characteristics for the forward problem become the departing (resp. coming) characteristics for the backward problem, for system (1.1) with hypothesis (1.7), the boundary conditions which guarantee the well-posedness for the backward problem should be

$$x = 0 : v_r = \widetilde{G}_r(t, v_{m+1}, \cdots, v_n) + \widetilde{H}_r(t), \quad r = 1, \cdots, m \tag{3.19}$$

and

$$x = L : v_s = \widetilde{G}_s(t, v_1, \cdots, v_m) + \widetilde{H}_s(t), \quad s = m+1, \cdots, n \tag{3.20}$$

with

$$\widetilde{G}_i(t, 0, \cdots, 0) \equiv 0, \quad i = 1, \cdots, n, \tag{3.21}$$

and the final condition is supposed to be

$$t = T : u = \Phi(x), \quad 0 \le x \le L. \tag{3.22}$$

For the backward problem, Theorem 3.2 can be written as follows.

Theorem 3.2′ (One-Sided Strong Exact Boundary Observability) *Suppose that* (2.4) *holds and* $T > 0$ *satisfies* (2.7). *Suppose furthermore that in a neighborhood of* $u = 0$, *boundary condition* (3.19) *on* $x = 0$ (*the side with more coming characteristics for the backward problem*) *implies*

$$x = 0 : v_s = \overline{\overline{\widetilde{G}}}_s(t, v_1, \cdots, v_m) + \overline{\overline{\widetilde{H}}}_s(t), \quad s = m+1, \cdots, n \tag{3.23}$$

with

$$\overline{\overline{\widetilde{G}}}_s(t, 0, \cdots, 0) \equiv 0, \quad s = m+1, \cdots, n, \tag{3.24}$$

and $\frac{\partial \overline{\overline{\widetilde{G}}}_s}{\partial t}$ ($s = m+1, \cdots, n$) *satisfy the local Lipschitz condition with respect to the variable* $v = (v_1, \cdots, v_n)^{\mathrm{T}}$ *or can be controlled by* $\|\overline{\overline{\widetilde{H}}}_s\|_{C^1[0,T]}$ ($s = m+1, \cdots, n$) *for the variable* v *in any given bounded set. For any given final data* $\Phi(x)$ *with small* $C^1[0, L]$ *norm, such that the conditions of* C^1 *compatibility are satisfied at the points* $(t, x) = (T, 0)$ *and* (T, L) *respectively, the boundary observations* $v_r = \overline{\overline{v}}_r(t)$ ($r = 1, \cdots, m$) *on* $x = L$ (*the side with less* *coming characteristics for the backward problem) on the interval* $[0,T]$ *can be used to uniquely determine the final data* $\Phi(x)$*, and the following strong observability inequality holds:*

$$\|\Phi\|_{C^1[0,L]} \leq C\Big(\sum_{r=1}^{m}\|\overline{\overline{v}}_r\|_{C^1[0,T]} + \|\widetilde{H}\|_{C^1[0,T]}\Big), \tag{3.25}$$

where C *is a positive constant.*

The corresponding form of Theorem 3.4 for the backward problem is as follows.

Theorem 3.4$'$ (One-Sided Weak Exact Boundary Observability) *Suppose that* $T > 0$ *satisfies* (2.7)*. For any given final data* $\Phi(x)$ *with the same properties as shown in Theorem* 3.2$'$*, we have*

(1) *Suppose that* $\frac{\partial \widetilde{G}_s}{\partial t}$ $(s = m+1,\cdots,n)$ *satisfy the local Lipschitz condition with respect to the variable* $v = (v_1,\cdots,v_n)^{\mathrm{T}}$ *or can be controlled by* $\|\widetilde{H}_s\|_{C^1[0,T]}$ $(s = m+1,\cdots,n)$ *for the variable* v *in any given bounded set. The boundary observations* $v_s = \overline{v}_s(t)$ $(s = m+1,\cdots,n)$ *on* $x = 0$ *on the interval* $[0,T]$ *can be used to uniquely determine the initial data* $\Psi(x)$ *of the solution* $u = u(t,x)$ *at* $t = 0$*, and the following weak observability inequality holds:*

$$\|\Psi\|_{C^1[0,L]} \leq C\Big(\sum_{s=m+1}^{n}\|\overline{v}_s\|_{C^1[0,T]} + \|\widetilde{H}\|_{C^1[0,T]}\Big), \tag{3.26}$$

where C *is a positive constant.*

(2) *Suppose that* $\frac{\partial \widetilde{G}_r}{\partial t}$ $(r = 1,\cdots,m)$ *satisfy the local Lipschitz condition with respect to the variable* $v = (v_1,\cdots,v_n)^{\mathrm{T}}$ *or can be controlled by* $\|\widetilde{H}_r\|_{C^1[0,T]}$ $(r = 1,\cdots,m)$ *for the variable* v *in any given bounded set. The boundary observations* $v_r = \overline{\overline{v}}_r(t)$ $(r = 1,\cdots,m)$ *on* $x = L$ *on the interval* $[0,T]$ *can be used to uniquely determine the initial data* $\Psi(x)$ *of the solution* $u = u(t,x)$ *at* $t = 0$*, and the following weak observability inequality holds:*

$$\|\Psi\|_{C^1[0,L]} \leq C\Big(\sum_{r=1}^{m}\|\overline{\overline{v}}_r\|_{C^1[0,T]} + \|\widetilde{H}\|_{C^1[0,T]}\Big), \tag{3.27}$$

where C *is a positive constant.*

4 Relationship Between the Strong (Weak) Controllability and the Strong (Weak) Observability — Non-observability Implies Non-controllability

In this section and the next section, we illustrate the relationship between the strong (weak) controllability and the strong (weak) observability for the following linear hyperbolic system

$$\frac{\partial u_i}{\partial t} + \lambda_i \frac{\partial u_i}{\partial x} = 0, \quad i = 1,\cdots,n, \tag{4.1}$$

in which λ_i $(i = 1\cdots,n)$ are constants and

$$\lambda_r < 0 < \lambda_s, \quad r = 1,\cdots,m;\ s = m+1,\cdots,n. \tag{4.2}$$

The linear boundary conditions are prescribed as follows:

$$x = 0 : u_s = \sum_{r=1}^{m} a_{sr} u_r, \quad s = m+1,\cdots,n \tag{4.3}$$

and

$$x = L : u_r = \sum_{s=m+1}^{n} a_{rs} u_s + H_r(t), \quad r = 1, \cdots, m, \tag{4.4}$$

where a_{sr} and a_{rs} ($r = 1, \cdots, m$; $s = m+1, \cdots, n$) are constants, and H_r ($r = 1, \cdots, m$) are C^1 functions of t.

We consider the forward mixed initial-boundary value problem for system (4.1) with boundary conditions (4.3)–(4.4) and the initial condition

$$t = 0 : u = \varphi(x), \quad 0 \le x \le L. \tag{4.5}$$

Correspondingly, we consider the following backward mixed problem for the adjoint system:

$$\frac{\partial w_i}{\partial t} + \lambda_i \frac{\partial w_i}{\partial x} = 0, \qquad i = 1, \cdots, n, \tag{4.6}$$

$$x = 0 : w_r = -\sum_{s=m+1}^{n} \frac{\lambda_s}{\lambda_r} a_{sr} w_s, \quad r = 1, \cdots, m, \tag{4.7}$$

$$x = L : w_s = -\sum_{r=1}^{m} \frac{\lambda_r}{\lambda_s} a_{rs} w_r, \qquad s = m+1, \cdots, n, \tag{4.8}$$

$$t = T : w = \Phi(x), \qquad 0 \le x \le L. \tag{4.9}$$

Multiplying the i-th equation in (4.1) by w_i, integrating it with respect to t and x on the domain $R(T) = \{(t,x) \mid 0 \le t \le T,\ 0 \le x \le L\}$ and summing up with respect to i from 1 to n, by integration by parts it is easy to get the following dual integral formula:

$$\sum_{i=1}^{n} \int_0^L u_i w_i(T,x)\,\mathrm{d}x - \sum_{i=1}^{n} \int_0^L u_i w_i(0,x)\,\mathrm{d}x + \sum_{r=1}^{m} \int_0^T \lambda_r H_r(t) w_r(t,L)\,\mathrm{d}t = 0. \tag{4.10}$$

Using (4.10), under the assumption that the number of boundary observations can not be reduced in the one-sided strong (resp. weak) exact boundary observability for the backward problem (4.6)–(4.9), we can reach the conclusion that the number of boundary controls can not be reduced generically in the one-sided strong (resp. weak) exact boundary controllability for the forward problem (4.1) and (4.3)–(4.5).

(1) We first consider the relationship between the weak exact observability and the weak exact controllability.

Suppose that the boundary observations $w_r = \overline{\overline{w}}_r(t)$ ($r = 1, \cdots, m$) on $x = L$ on an interval $[0, T]$ can be used to uniquely determine the initial data $w(0,x) \triangleq \Psi(x)$, $0 \le x \le L$, for the backward problem (4.6)–(4.9) (see the second part of Theorem 3.4′). In particular, when

$$\overline{\overline{w}}_r(t) \equiv 0, \quad r = 1, \cdots, m,\ 0 \le t \le T, \tag{4.11}$$

we have

$$\Psi(x) \equiv 0, \quad 0 \le x \le L. \tag{4.12}$$

Suppose furthermore that in order to realize the one-sided weak exact boundary observability, the number of boundary observations can not be reduced. For instance, even though the boundary observations

$$(w_2(t,L), \cdots, w_m(t,L)) = (\overline{\overline{w}}_2(t), \cdots, \overline{\overline{w}}_m(t)) \equiv 0, \quad 0 \le t \le T, \tag{4.13}$$

we still have

$$\Psi(x) \not\equiv 0, \quad 0 \le x \le L. \tag{4.14}$$

Then, we want to prove that the number of boundary controls can not be reduced generically in the one-sided weak exact boundary controllability for the forward problem (4.1) and (4.3)–(4.5).

For this purpose, we specially take the initial condition for the forward problem as

$$t = 0 : u = \Psi(x), \quad 0 \le x \le L, \tag{4.15}$$

where $\Psi(x)$ is given by (4.14). Suppose that boundary controls $H_r(t)$ $(r = 1, \cdots, m)$ on $x = L$ on the interval $[0, T]$ can be used to realize the weak controllability:

$$u(T, x) \equiv 0, \quad 0 \le x \le L \tag{4.16}$$

(see the second part of Theorem 2.4). We now prove that in order to realize the one-sided weak exact boundary controllability, the number of boundary controls can not be reduced generically. For this purpose, supposing that only boundary controls $H_2(t), \cdots, H_m(t)$ on the interval $[0, T]$ are enough to get (4.16), we may specially take

$$H_1(t) \equiv 0, \quad 0 \le t \le T. \tag{4.17}$$

Substituting (4.13) and (4.15)–(4.17) into (4.10) yields

$$\int_0^L |\Psi(x)|^2 \mathrm{d}x = 0, \tag{4.18}$$

which contradicts (4.14).

(2) We next consider the relationship between the strong exact observability and the strong exact controllability.

Suppose that (2.4) holds. Suppose furthermore that the boundary observations $w_r = \overline{\overline{w}}_r(t)$ $(r = 1, \cdots, m)$ on $x = L$ on an interval $[0, T]$ can be used to uniquely determine the final data $w(T, x) = \Phi(x)$, $0 \le x \le L$, for the backward problem (4.6)–(4.9) (see Theorem 3.2$'$). In particular, when (4.11) holds, we have

$$\Phi(x) \equiv 0, \quad 0 \le x \le L. \tag{4.19}$$

Suppose finally that in order to realize the one-sided strong exact boundary observability, the number of boundary observations can not be reduced. For instance, even though (4.13) holds, we still have

$$\Phi(x) \not\equiv 0, \quad 0 \le x \le L. \tag{4.20}$$

Then, we want to prove that the number of boundary controls can not be reduced generically in the one-sided strong exact boundary controllability for the forward problem (4.1) and (4.3)–(4.5).

In fact, suppose that boundary controls $H_r(t)$ $(r = 1, \cdots, m)$ on $x = L$ on the interval $[0, T]$ can be used to realize the strong controllability for the forward problem (4.1) and (4.3)–(4.5) (see Theorem 2.2). We now show that in order to realize the one-sided strong exact boundary controllability on $[0, T]$, the number of boundary controls can not be reduced generically. For this purpose, we specially take the initial condition and the final condition for the forward problem as

$$t = 0 : u = 0, \quad 0 \le x \le L \tag{4.21}$$

and

$$t = 0 : u = \Phi(x), \quad 0 \le x \le L, \tag{4.22}$$

respectively, where $\Phi(x)$ is given by (4.20). Supposing that only boundary controls $H_2(t), \cdots, H_m(t)$ on the interval $[0, T]$ are enough to drive $u = 0$ at $t = 0$ exactly to $\Phi(x)$ at $t = T$, we may assume that (4.17) holds. Thus, noting (4.13), (4.17) and (4.21)–(4.22), it follows form (4.10) that

$$\int_0^L |\Phi(x)|^2 \mathrm{d}x = 0, \tag{4.23}$$

which is a contradiction to (4.20).

From the previous discussions, we reach the conclusion that in order to show that the number of boundary controls can not be reduced generically in the one-sided strong (resp. weak) exact boundary controllability for the forward problem, it suffices to show that the number of boundary observations can not be reduced generically in the one-sided strong (resp. weak) exact boundary observability for the corresponding backward problem.

Remark 4.1 For the backward problem (4.6)–(4.9), if we take the boundary observations on $x = L$ not only for $(w_2, \cdots, w_m) = (\overline{\overline{w}}_2(t), \cdots, \overline{\overline{w}}_m(t))$ on the interval $[0, T]$, but also for $w_1 = \overline{\overline{w}}_1(t)$ on a smaller interval $[\overline{T}, T]$ with $0 < \overline{T} < T$, such that under the assumptions (4.13) and

$$\overline{\overline{w}}_1(t) \equiv 0, \quad \overline{T} \le t \le T, \tag{4.24}$$

we still have (4.14) for the weak exact boundary observability and (4.20) for the strong exact boundary observability. Then, for the forward problem (4.1) and (4.3)–(4.5), the one-sided weak (resp. strong) exact boundary controllability can still not be realized generically by means of both the boundary controls $H_2(t), \cdots, H_m(t)$ on the interval $[0, T]$ and the boundary control $H_1(t)$ on an interval smaller than $[0, T]$. In fact, if it is not the case, supposing that $H_1(t)$ as a boundary control is given on the interval $[\widetilde{T}, T]$ with $\widetilde{T} > \overline{T}$ and specially taking

$$H_1(t) \equiv 0, \quad 0 \le t \le \overline{T}, \tag{4.25}$$

similarly to the previous discussion, noting (4.24)–(4.25), we can still get (4.18) and (4.23) which reach a contradiction.

We now take some examples to show that the number of boundary observations can not be reduced in the one-sided weak (resp. strong) exact boundary observability for the backward problem.

Consider the backward problem for the following system:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{\partial u}{\partial x} = 0, \\ \dfrac{\partial v}{\partial t} + \dfrac{\partial v}{\partial x} = 0 \end{cases} \tag{4.26}$$

and the following boundary conditions:

$$x = 0 : u = v, \tag{4.27}$$

$$x = L : v = u. \tag{4.28}$$

By the second part of Theorem 3.4′, if $T \geq 2L$ and we observe $u = \overline{\overline{u}}(t)$ $(0 \leq t \leq T)$ on $x = L$, then we have the one-sided weak exact boundary observability. However, if we do not take any observation on $x = L$, then we can not expect to have the one-sided weak exact boundary observability, since, generically speaking, the initial value at $t = 0$ of the solution to the backward problem is not identically equal to zero. This shows that the number of boundary observations can not be reduced generically in the one-sided weak exact boundary observability.

In order to show that the number of boundary observations can not be reduced generically in the one-sided strong exact boundary observability for the backward problem, we still consider system (4.26) and boundary conditions (4.27)–(4.28). By Theorem 3.2′, if $T \geq 2L$ and we observe $u = \overline{\overline{u}}(t)$ $(0 \leq t \leq T)$ on $x = L$, then we have the one-sided strong exact boundary observability. However, if we do not take any observation on $x = L$, then we can not expect to have the one-sided strong exact boundary observability, since the final data at $t = T$ of the solution to the backward problem is arbitrarily given.

Remark 4.2 We now observe $u = \overline{\overline{u}}(t)$ for $\overline{T} \leq t \leq T$ on $x = L$, but the observed time $T - \overline{T}$ is less than $2L$. Without loss of generality we suppose

$$L < T - \overline{T} < 2L. \tag{4.29}$$

Suppose that

$$\overline{\overline{u}}(t) \equiv 0, \quad \overline{T} \leq t \leq T. \tag{4.30}$$

When $\overline{T} < L$, noting boundary conditions (4.27)–(4.28), it is easy to see that

$$u(0, x) \equiv 0, \quad 0 \leq x \leq L \tag{4.31}$$

and

$$v(0, x) \equiv 0, \quad 0 \leq x \leq L - \overline{T}, \tag{4.32}$$

but in general $v(0, x)$ on $L - \overline{T} \leq x \leq L$ can not be identically equal to zero. Similarly, when $\overline{T} \geq L$, in general $v(0, x)$ on $0 \leq x \leq L$ can not be identically equal to zero. It shows that the corresponding one-sided weak exact boundary observability can not be realized in this case.

We now show that the corresponding one-sided strong exact boundary observability can not be realized in this case, too. In fact, noting (4.29) and using boundary conditions (4.27)–(4.28), it is easy to see that

$$u(T, x) \equiv 0, \quad 0 \leq x \leq L \tag{4.33}$$

and

$$v(T, x) \equiv 0, \quad 0 \leq x \leq T - \overline{T} - L, \tag{4.34}$$

but $v(T, x)$ on $T - \overline{T} - L \leq x \leq L$ should be arbitrarily given.

By Remark 4.1, corresponding conclusions on the one-sided weak (resp. strong) exact boundary controllability can be obtained for the related forward problem.

5 Relationship Between the Strong Controllability and the Strong Observability — Non-controllability implies Non-observability

In this section, we consider the forward problem for system (4.1) with the boundary condition (4.3) on $x = 0$, the boundary condition

$$x = L : u_r = H_r(t), \quad r = 1, \cdots, m \tag{5.1}$$

and the initial condition

$$t = 0 : u = 0, \quad 0 \le x \le L. \tag{5.2}$$

By Theorem 2.2, if

$$n \le 2m \tag{5.3}$$

and

$$T > L\Big(\max_{r=1,\cdots,m} \frac{1}{|\lambda_r|} + \max_{s=m+1,\cdots,n} \frac{1}{\lambda_s}\Big), \tag{5.4}$$

boundary controls $H_r(t)$ $(r = 1, \cdots, m)$ on $x = L$ on the interval $[0, T]$ can be used to realize the one-sided strong exact boundary controllability for driving the initial data (5.2) exactly to any given final data $\psi(x)$ on $t = T$:

$$t = T : u = \psi(x), \quad 0 \le x \le L. \tag{5.5}$$

Moreover, the number of boundary controls can not be reduced in this situation. In fact, in the lack of boundary controls, without loss of generality we may suppose that

$$H_1(t) \equiv 0, \quad 0 \le t \le T. \tag{5.6}$$

Then it is easy to see that

$$u_1(T, x) \equiv 0, \quad 0 \le x \le L. \tag{5.7}$$

Hence, for any given final data (5.5) in which $\psi(x) = \Phi(x) \triangleq (\varphi(x), 0, \cdots, 0)^{\mathrm{T}}$ and $\varphi(x)$ is a non-trivial C^1 function with compact support, there is no way to reach it at $t = T$, no matter how to choose boundary controls $H_2(t), \cdots, H_m(t)$.

We now consider the following adjoint backward problem:

$$\begin{cases} \dfrac{\partial w_i}{\partial t} + \lambda_i \dfrac{\partial w_i}{\partial x} = 0, & i = 1, \cdots, n, \qquad (5.8) \\ x = 0 : w_r = -\displaystyle\sum_{s=m+1}^{n} \frac{\lambda_s}{\lambda_r} a_{sr} w_s, & r = 1, \cdots, m, \qquad (5.9) \\ x = L : w_s = 0, & s = m+1, \cdots, n, \qquad (5.10) \\ t = T : w = \Phi(x), & \qquad (5.11) \end{cases}$$

in which $\Phi(x)$ is given in the previous way.

We still have (4.10). Using (5.2) and (5.6) and specially taking

$$H_{\widetilde{r}}(t) = w_{\widetilde{r}}(t, L), \quad \widetilde{r} = 2, \cdots, m, \tag{5.12}$$

we get

$$\int_0^L u^{\mathrm{T}}(T, x)\Phi(x)\,\mathrm{d}x + \sum_{\widetilde{r}=2}^{m} \int_0^T \lambda_{\widetilde{r}} w_{\widetilde{r}}^2(t, L)\,\mathrm{d}t = 0. \tag{5.13}$$

Noting (5.7) and the special choice of $\Phi(x)$, we have

$$\int_0^L u^{\mathrm{T}}(T,x)\Phi(x)\,\mathrm{d}x = 0. \tag{5.14}$$

Then, noting (4.2), it follows from (5.13) that

$$w_{\widetilde{r}}(t,L) \equiv 0, \quad \widetilde{r} = 2,\cdots,m,\ 0 \le t \le T. \tag{5.15}$$

Thus, for the solution $w = w(t,x)$ to the backward problem (5.8)–(5.11), we have the zero boundary observation with lack of observations (see Theorem 3.2′), but the final data are not identically equal to zero. This shows that the number of boundary observations can not be reduced in order to realize the one-sided strong exact boundary observability for the backward problem.

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