Chin. Ann. Math.
32B(5), 2011, 741–748
DOI: 10.1007/s11401-011-0668-x

Chinese Annals of Mathematics, Series B

A Characterization of Counterexamples to the Kodaira-Ramanujam Vanishing Theorem on Surfaces in Positive Characteristic*

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Abstract The author gives a characterization of counterexamples to the Kodaira-Ramanujam vanishing theorem on smooth projective surfaces in positive characteristic. More precisely, it is reproved that if there is a counterexample to the Kodaira-Ramanujam vanishing theorem on a smooth projective surface X in positive characteristic, then X is either a quasi-elliptic surface of Kodaira dimension 1 or a surface of general type. Furthermore, it is proved that up to blow-ups, X admits a fibration to a smooth projective curve, such that each fiber is a singular curve.

Keywords Characterization, Counterexample, Kodaira-Ramanujam vanishing
2000 MR Subject Classification 14F17, 14E30

1 Introduction

The Kodaira vanishing theorem claims that for an ample line bundle $\mathcal{L}$ on a smooth projective complex variety X, $H^i(X, \mathcal{L}^{-1}) = 0$ holds for any $i < \dim X$. However, this statement fails in positive characteristic. Raynaud [11] gave counterexamples to the Kodaira vanishing theorem on smooth projective surfaces over an algebraically closed field k of characteristic $p > 0$. More precisely, the counterexample constructed in [11] is either a quasi-elliptic surface of Kodaira dimension 1 or a surface of general type, which has a fibration to a smooth projective curve, such that each fiber is a singular rational curve.

In this paper, we shall prove the following main theorem, which is almost the converse to the above result, and in fact, gives a characterization of counterexamples to the Kodaira-Ramanujam vanishing theorem in [10] on smooth projective surfaces in positive characteristic.

Theorem 1.1 *Let X be a smooth projective surface over an algebraically closed field k of characteristic $p > 0$, $\mathcal{L}$ a nef and big line bundle on X. If $H^1(X, \mathcal{L}^{-1}) \neq 0$, then*

(i) *X is either a quasi-elliptic surface of Kodaira dimension 1 or a surface of general type,*

(ii) *up to blow-ups, X admits a fibration to a smooth projective curve, such that each fiber is a singular curve.*

Manuscript received September 12, 2010. Revised March 24, 2011.
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*Project supported by the National Natural Science Foundation of China (No. 10901037), the Doctoral Program Foundation of the Ministry of Education of China (No. 20090071120004) and the Scientific Research Foundation for the Returned Overseas Chinese Scholars, State Education Ministry.

We shall prove Theorem 1.1 by using two methods together. One is Tango's criterion, which is deduced from the Cartier isomorphism and was firstly used in [14] to treat vanishing problems. Another is Raynaud's criterion (see [3, Corollaire 2.8]), which claims that if a smooth projective surface X can be lifted over $W_2(k)$, then the Kodaira-Ramanujam vanishing theorem holds on X. The main idea of the proof of Theorem 1.1 consists of two key points. The first one is to prove that blow-ups of a liftable surface are also liftable over $W_2(k)$. The second one is to use Bombieri-Mumford's classification of algebraic surfaces in positive characteristic.

It should be mentioned that Theorem 1.1 is a slight improvement of one of the main results in [8], which dealt with the counterexamples to the Kodaira vanishing theorem on smooth projective surfaces. Although Theorem 1.1(i) was proved in [4, Theorem II.1.6] via the foliation argument and in [15, Theorem 1.6] via the d-very ampleness argument, the methods used here are more intuitive from the geometric point of view, while Theorem 1.1(ii) is fresh and seems more interesting. Furthermore, it should be mentioned that the Kodaira-Ramanujam vanishing theorem fails for the canonical sheaf on a minimal surface of general type in positive characteristic (see [4, Main Theorem] for more details).

We shall give the proof of Theorem 1.1 in Section 2 and some remarks in Section 3. In what follows, we always work over an algebraically closed field k of characteristic $p > 0$, unless otherwise stated.

2 Proof of the Main Theorem

Let X be a smooth projective surface throughout this section, and $F : X \to X$ the absolute Frobenius morphism. We have the absolute Cartier isomorphism, i.e., the following isomorphism of $\mathcal{O}_X$-modules for any $i \geq 0$ (see [13, Propositions 1–3]):

$$C : \mathcal{H}^i(F_*\Omega^\bullet_X) \xrightarrow{\sim} \Omega^i_X,$$

which induces the following commutative diagram with an exact row:

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}_X \arrow[r] & F_*\mathcal{O}_X \arrow[r] \arrow[dr, "F_*(d)"'] & \mathcal{B}^1_X \arrow[r] \arrow[d, hook] & 0 \\ & & & F_*\Omega^1_X & \end{tikzcd}$$

where $\mathcal{B}^1_X = \mathrm{im}(F_*(d) : F_*\mathcal{O}_X \to F_*\Omega^1_X)$.

Definition 2.1 *Let $\mathcal{L}$ be a line bundle on X. We say that X satisfies the T_1 condition for $\mathcal{L}$, if $H^0(X, \Omega^1_X \otimes \mathcal{L}^{-1}) = 0$ holds.*

Lemma 2.1 (i) *$H^0(X, \mathcal{B}^1_X(-L)) = \{dh \mid h \in K(X), (dh) \geq pL\}$, where L is a divisor on X, $K(X)$ is the rational function field of X and $(dh) = \sum_E v_E(dh)$, where the sum runs over all prime divisors E on X, and v_E is the associated discrete valuation.*

(ii) *If X satisfies the T_1 condition for any nef and big line bundle $\mathcal{L}$ on X, then $H^1(X, \mathcal{L}^{-1}) = 0$ holds for any nef and big line bundle $\mathcal{L}$ on X.*

Proof (i) Since $H^0(X, \mathcal{B}^1_X(-L)) \hookrightarrow H^0(X, F_*\Omega^1_X(-L)) = H^0(X, \Omega^1_X(-pL))$ and $\mathcal{B}^1_X$ is the image of $F_*(d) : F_*\mathcal{O}_X \to F_*\Omega^1_X$, we have $H^0(X, \mathcal{B}^1_X(-L)) = \{dh \in \Omega^1_X(-pL) \mid h \in K(X)\} = \{dh \mid h \in K(X), (dh) \geq pL\}$.

(ii) Since $\mathcal{L}$ is nef and big, we have the following exact sequences:

$$0 \to H^0(X, \mathcal{B}_X^1 \otimes \mathcal{L}^{-1}) \to H^1(X, \mathcal{L}^{-1}) \xrightarrow{F^*} H^1(X, \mathcal{L}^{-p}), \tag{2.1}$$

$$0 \to H^0(X, \mathcal{B}_X^1 \otimes \mathcal{L}^{-1}) \to H^0(X, F_*\Omega_X^1 \otimes \mathcal{L}^{-1}) = H^0(X, \Omega_X^1 \otimes \mathcal{L}^{-p}). \tag{2.2}$$

If X satisfies the T_1 condition for any nef and big line bundle $\mathcal{L}$, then $H^0(X, \mathcal{B}_X^1 \otimes \mathcal{L}^{-1}) = 0$ holds for any nef and big line bundle $\mathcal{L}$. Hence, $F^* : H^1(X, \mathcal{L}^{-1}) \to H^1(X, \mathcal{L}^{-p})$ is injective for any nef and big line bundle $\mathcal{L}$. For a given nef and big line bundle $\mathcal{L}$, we have $H^1(X, \mathcal{L}^{-n}) = 0$ for all $n \gg 0$ by [7, Proposition 2]. Hence, we obtain $H^1(X, \mathcal{L}^{-1}) = 0$ for any nef and big line bundle $\mathcal{L}$ on X.

Proposition 2.1 *Let $f : X \to C$ be a ruled or elliptic fibration to a smooth projective curve C, i.e., the general fiber of f is a smooth rational or smooth elliptic curve. Then the T_1 condition holds for any nef and big line bundle $\mathcal{L}$ on X.*

Proof Assume on the contrary that $H^0(X, \Omega_X^1 \otimes \mathcal{L}^{-1}) \neq 0$ holds for a nef and big line bundle $\mathcal{L}$. Then we can take a section $0 \neq s \in H^0(X, \Omega_X^1 \otimes \mathcal{L}^{-1})$ and a general fiber F of f, such that $0 \neq s|_F \in H^0(F, (\Omega_X^1 \otimes \mathcal{L}^{-1})|_F)$. Since F is smooth, we have the following exact sequence:

$$0 \to \mathcal{I}_F/\mathcal{I}_F^2 \to \Omega_X^1|_F \to \omega_F \to 0,$$

where $\mathcal{I}_F/\mathcal{I}_F^2 = \mathcal{O}_F(-F) \cong \mathcal{O}_F$. Tensoring the above exact sequence with $\mathcal{L}^{-1}$, we obtain the exact sequence:

$$0 \to \mathcal{L}^{-1}|_F \to (\Omega_X^1 \otimes \mathcal{L}^{-1})|_F \to \omega_F \otimes \mathcal{L}^{-1}|_F \to 0. \tag{2.3}$$

Since F is smooth rational or smooth elliptic and $\mathcal{L}$ is nef and big, we have $\deg(\mathcal{L}^{-1}|_F) < 0$ and $\deg(\omega_F \otimes \mathcal{L}^{-1}|_F) < 0$. Taking the exact sequence of cohomology groups of (2.3), we have $H^0(F, (\Omega_X^1 \otimes \mathcal{L}^{-1})|_F) = 0$, which yields a contradiction.

Definition 2.2 *Let $W_2(k)$ be the ring of Witt vectors of length two of k. Then $W_2(k)$ is flat over $\mathbb{Z}/p^2\mathbb{Z}$, and $W_2(k) \otimes_{\mathbb{Z}/p^2\mathbb{Z}} \mathbb{F}_p = k$. For the explicit construction and further properties of $W_2(k)$, we refer the reader to* [12, II.6]. *The following definition generalizes slightly the definition in* [3, 1.6] *of liftings of smooth schemes over $W_2(k)$.*

Let Z be a smooth scheme over k, and V a closed subscheme of Z smooth over k of codimension $s \geq 2$. A lifting of (Z, V) over $W_2(k)$ consists of a smooth scheme $\widetilde{Z}$ over $W_2(k)$ and a closed subscheme $\widetilde{V} \subset \widetilde{Z}$ smooth over $W_2(k)$, such that $Z = \widetilde{Z} \times_{\operatorname{Spec} W_2(k)} \operatorname{Spec} k$, and $V = \widetilde{V} \times_{\operatorname{Spec} W_2(k)} \operatorname{Spec} k$. We say that $(\widetilde{Z}, \widetilde{V})$ is a lifting of (Z, V) over $W_2(k)$, if no confusion is possible.

We need the following standard results.

Lemma 2.2 *Let Z be a smooth scheme over k, V a closed subscheme of Z smooth over k of codimension $s \geq 2$, and $\pi : Z' \to Z$ the blow-up of Z along V. Assume that (Z, V) admits a lifting over $W_2(k)$. Then Z' admits a lifting over $W_2(k)$.*

Proof Let $(\widetilde{Z}, \widetilde{V})$ be a lifting of (Z, V) over $W_2(k)$. Then $\widetilde{V} \subset \widetilde{Z}$ is a closed subscheme smooth over $W_2(k)$ of codimension $s \geq 2$. Let $\widetilde{I}$ be the ideal sheaf of $\widetilde{V}$ in $\widetilde{Z}$, $\widetilde{\pi} : \widetilde{Z}' \to \widetilde{Z}$ the blow-up of $\widetilde{Z}$ along $\widetilde{V}$. By [5, Corollary II.7.15], we have the following commutative diagram:

$$\begin{tikzcd} Z'' \arrow[r, hook] \arrow[d, "\pi'"'] & \widetilde{Z}' \arrow[d, "\widetilde{\pi}"] \\ Z \arrow[r, hook] & \widetilde{Z} \end{tikzcd}$$

where $\pi' : Z'' \to Z$ is the blow-up of Z with respect to the ideal sheaf $\widetilde{I} \otimes_{W_2(k)} k = I$, which is the ideal sheaf of V in Z. Hence we have $Z'' = Z'$ and $\pi' = \pi$. Since $\widetilde{Z}$ is smooth over $W_2(k)$, so is $\widetilde{Z}'$. Note that $\widetilde{Z}' \times_{\operatorname{Spec} W_2(k)} \operatorname{Spec} k = \operatorname{Proj}\Big(\bigoplus_i \widetilde{I}^i\Big) \times_{\operatorname{Spec} W_2(k)} \operatorname{Spec} k = \operatorname{Proj}\Big(\bigoplus_i \widetilde{I}^i \otimes_{W_2(k)} k\Big) = \operatorname{Proj}\Big(\bigoplus_i I^i\Big) = Z'$. So $\widetilde{Z}'$ is a lifting of Z' over $W_2(k)$.

Lemma 2.3 *Let Z be a smooth scheme over k, $P \in Z$ a closed point, and $\pi : Z' \to Z$ the blow-up of Z along P. If Z is liftable over $W_2(k)$, then so is Z'.*

Proof Let $\widetilde{Z}$ be a lifting of Z over $W_2(k)$, $Z \hookrightarrow \widetilde{Z}$ the induced closed immersion, and $\eta : \widetilde{Z} \to \operatorname{Spec} W_2(k)$ the structure morphism. Let $\operatorname{Spec} k \hookrightarrow Z$ be the closed immersion associated to the closed point $P \in Z$, and $\operatorname{Spec} k \hookrightarrow \operatorname{Spec} W_2(k)$ the natural closed immersion. We have the following commutative square:

$$\begin{tikzcd} \operatorname{Spec} k \arrow[r, hook] \arrow[d, hook] & Z \arrow[r, hook] & \widetilde{Z} \arrow[d, "\eta"] \\ \operatorname{Spec} W_2(k) \arrow[rru, dashed, "\xi"] \arrow[rr, equal] & & \operatorname{Spec} W_2(k) \end{tikzcd}$$

Since $\operatorname{Spec} k \hookrightarrow \operatorname{Spec} W_2(k)$ is a closed immersion of the ideal sheaf square zero and $\eta : \widetilde{Z} \to \operatorname{Spec} W_2(k)$ is smooth, there is a morphism $\xi : \operatorname{Spec} W_2(k) \to \widetilde{Z}$, such that the induced diagrams are commutative. Since ξ is a section of η, it defines a closed subscheme $\widetilde{P} \subset \widetilde{Z}$ smooth over $W_2(k)$. It follows from the upper commutativity that $P = \widetilde{P} \times_{\operatorname{Spec} W_2(k)} \operatorname{Spec} k$ holds. Therefore, $(\widetilde{Z}, \widetilde{P})$ is a lifting of (Z, P) over $W_2(k)$. By Lemma 2.2, Z' is liftable over $W_2(k)$.

Proposition 2.2 *Let X be a smooth projective surface, and $\pi : X \to Y$ a composition of Castelnuovo contractions of (-1)-curves. If Y is liftable over $W_2(k)$, then so is X.*

Proof It follows from Lemma 2.3.

Lemma 2.4 *We use the same notation as in Proposition 2.2, and assume further that $K_Y \equiv 0$, $\chi(\mathcal{O}_X) \geq 1$, and $\mathcal{L} = \mathcal{O}_X(L)$ is nef and big. Then we have $h^0(X, K_X + L) \geq 2$.*

Proof By Serre duality, $h^2(X, K_X + L) = h^0(X, -L) = 0$, we have $h^0(X, K_X + L) \geq \chi(X, K_X + L) = \frac{1}{2}(K_X + L) \cdot L + \chi(\mathcal{O}_X)$. Note that $K_X = \pi^* K_Y + E \equiv E$, where E is an effective divisor supported by the exceptional locus of f. Therefore, we have $h^0(X, K_X + L) \geq \frac{1}{2}L^2 + \chi(\mathcal{O}_X) > 1$, i.e., $h^0(X, K_X + L) \geq 2$.

We fix some notations. Let X be a smooth projective surface, $\mathcal{L} = \mathcal{O}_X(L)$ a nef and big line bundle on X, Y a relatively minimal model of X, and $\kappa = \kappa(X) = \kappa(Y)$ the Kodaira dimension of both X and Y.

Proof of Theorem 1.1 (i) We proceed to exclude all possibilities except for quasi-elliptic surfaces of Kodaira dimension 1 and surfaces of general type by means of Bombieri-Mumford's classification of algebraic surfaces in characteristic p (see [2, 1]).

First of all, it is easy to see that we can exclude the case $X \cong \mathbb{P}^2_k$. If $\kappa < 0$, then X admits a ruled fibration. If $\kappa = 1$ and Y is an elliptic surface, then X admits an elliptic fibration. If $\kappa = 0$ and Y is a hyperelliptic or quasi-hyperelliptic surface, then X also admits an elliptic fibration by [1, Theorem 4] and [2, Theorem 1]. These cases can be excluded due to Lemma 2.1(ii) and Proposition 2.1.

A theorem of Grothendieck [9] claims that any abelian variety is liftable over $W_2(k)$. If $\kappa = 0$ and Y is an abelian surface, then this case can be excluded by [3, Corollaire 2.8] and Proposition 2.2.

It remains to exclude the cases where Y is $K3$ or Enriques. Since $H^1(X, \mathcal{L}^{-1}) \neq 0$, after replacing $\mathcal{L}^{p^\nu}$ by $\mathcal{L}$, we may assume that $F^* : H^1(X, \mathcal{L}^{-1}) \to H^1(X, \mathcal{L}^{-p})$ is not injective. Hence we have $H^0(X, \mathcal{B}^1_X \otimes \mathcal{L}^{-1}) \neq 0$ by the exact sequence (2.1). Given a section $0 \neq \xi \in H^0(X, \mathcal{B}^1_X \otimes \mathcal{L}^{-1})$, the map $H^0(X, \mathcal{L} \otimes \omega_X) \to H^0(X, \mathcal{B}^1_X \otimes \omega_X)$, sending s to $s \cdot \xi$, is an injective linear map of k-vector spaces. As a consequence, we have $\dim H^0(X, \mathcal{B}^1_X \otimes \omega_X) \geq \dim H^0(X, \mathcal{L} \otimes \omega_X) \geq 2$ by Lemma 2.4.

Tensoring the exact sequence

$$0 \to \mathcal{O}_X \to F_*\mathcal{O}_X \to \mathcal{B}^1_X \to 0$$

with ω_X and taking the cohomology groups, we obtain the exact sequence:

$$0 \to H^0(X, \omega_X) \to H^0(X, \omega_X^p) \to H^0(X, \mathcal{B}^1_X \otimes \omega_X) \to H^1(X, \omega_X) \xrightarrow{F^*} H^1(X, \omega_X^p).$$

We have $h^0(X, \omega_X) = P_1(X) = P_1(Y)$, $h^0(X, \omega_X^p) = P_p(X) = P_p(Y)$, and that by Serre duality, $h^1(X, \omega_X) = h^1(X, \mathcal{O}_X)$ is a birational invariant. If Y is $K3$, then we have $P_1(X) = P_p(X) = 1$ and $h^1(X, \omega_X) = 0$, which contradict $\dim H^0(X, \mathcal{B}^1_X \otimes \omega_X) \geq 2$. If Y is a classical Enriques surface, then we have $P_1(X) = 0$, $P_p(X) \leq 1$ and $h^1(X, \omega_X) = 0$, which contradict $\dim H^0(X, \mathcal{B}^1_X \otimes \omega_X) \geq 2$. If Y is a non-classical Enriques surface (see [2, §3]), then we have $\omega_Y \cong \mathcal{O}_Y$ and $h^1(Y, \mathcal{O}_Y) = 1$. Hence $P_1(X) = P_p(X) = 1$ and $h^1(X, \omega_X) = 1$, which also contradict $\dim H^0(X, \mathcal{B}^1_X \otimes \omega_X) \geq 2$.

Therefore, X has to be a quasi-elliptic surface of Kodaira dimension 1 or a surface of general type, which completes the proof of Theorem 1.1(i).

(ii) Since $H^1(X, \mathcal{L}^{-1}) \neq 0$, after replacing $\mathcal{L}^{p^\nu}$ by $\mathcal{L}$, we may assume that $F^* : H^1(X, \mathcal{L}^{-1}) \to H^1(X, \mathcal{L}^{-p})$ is not injective. Hence we have $H^0(X, \mathcal{B}^1_X \otimes \mathcal{L}^{-1}) \neq 0$ by the exact sequence (2.1). By Lemma 2.1(i), we have

$$0 \neq H^0(X, \mathcal{B}^1_X(-L)) = \{dh \mid h \in K(X), (dh) \geq pL\},$$

where L is a nef and big divisor on X such that $\mathcal{L} = \mathcal{O}_X(L)$.

Let $h \in K(X)$ be a rational function such that $(dh) \geq pL$. Then we can define a rational map $g : X \dashrightarrow \mathbb{P}^1$ via the rational function h. Through some blow-ups $\sigma : X' \to X$, we get a morphism $g' : X' \to \mathbb{P}^1$. Taking the Stein factorization of g', we obtain a fibration $f : X' \to C$

to a smooth projective curve C with connected fibers and a finite morphism $\tau : C \to \mathbb{P}^1$.

$$\begin{tikzcd} X' \arrow[r, "f"] \arrow[d, "\sigma"'] \arrow[dr, "g'"] & C \arrow[d, "\tau"] \\ X \arrow[r, dashed, "g"] & \mathbb{P}^1 \end{tikzcd}$$

Let $L' = \sigma^* L$, $\mathcal{G} = \mathrm{im}(\mathcal{O}_{X'}(pL') \xrightarrow{dh} \Omega^1_{X'})$ and $\mathcal{F} = \mathrm{im}(f^*\Omega^1_C \to \Omega^1_{X'})$. Then we have

$$\Omega^1_{X'/C} = \Omega^1_{X'}/\mathcal{F} \supseteq \mathcal{T} := \mathcal{G}/\mathcal{G} \cap \mathcal{F}.$$

Since $\tau^*(dh)$ can be regarded as a section of the line bundle $\Omega^1_C \to C$ over a non-empty open subset of C, there is a non-empty open subset U of X', such that both $\mathcal{G}$ and $\mathcal{G} \cap \mathcal{F}$ are generated by dh on U. Hence $\mathcal{G}$ and $\mathcal{G} \cap \mathcal{F}$ coincide on U and $\mathcal{T}$ is a torsion sheaf on X'. Let $A = c_1(\mathcal{T}) = c_1(\mathcal{G}) - c_1(\mathcal{G} \cap \mathcal{F})$. Since $c_1(\mathcal{G}) = pL'$ is nef and big on X' and $c_1(\mathcal{G} \cap \mathcal{F}) \leq f^*K_C$, the divisor A must contain a certain irreducible component which is horizontal with respect to f, i.e., it is not contained in any fiber of f. Denote such an irreducible component of A by C'.

Lemma 2.5 *We use the same notation and assumptions as above. Let F be an irreducible and reduced fiber of $f : X' \to C$. Then we have $\Omega^1_F \cong \Omega^1_{X'/C}|_F$.*

Proof By [5, Proposition II.8.11], we have the following exact sequence:

$$f^*\Omega^1_C \to \Omega^1_{X'} \to \Omega^1_{X'/C} \to 0,$$

which, by restriction to F, gives rise to an exact sequence:

$$0 \to f^*\Omega^1_C|_F \xrightarrow{\xi} \Omega^1_{X'}|_F \xrightarrow{\eta} \Omega^1_{X'/C}|_F \to 0. \tag{2.4}$$

Since ξ is generically injective and $f^*\Omega^1_C|_F$ is locally free, ξ is injective. Let $\mathcal{I} = \mathcal{O}_{X'}(-F)$ be the ideal sheaf of F in X'. By [5, Proposition II.8.12], we have the following exact sequence:

$$0 \to \mathcal{I}/\mathcal{I}^2 \xrightarrow{\nu} \Omega^1_{X'}|_F \xrightarrow{\pi} \Omega^1_F \to 0. \tag{2.5}$$

Since ν is generically injective and $\mathcal{I}/\mathcal{I}^2 = \mathcal{O}_{X'}(-F)|_F$ is locally free, ν is injective.

It is easy to see that the composition $f^*\Omega^1_C|_F \xrightarrow{\xi} \Omega^1_{X'}|_F \xrightarrow{\pi} \Omega^1_F$ is zero, which induces natural homomorphisms $\varphi : f^*\Omega^1_C|_F \to \mathcal{O}_{X'}(-F)|_F$ and $\psi : \Omega^1_{X'/C}|_F \to \Omega^1_F$. Thus, we have the following commutative diagram with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & f^*\Omega^1_C|_F \arrow[r, "\xi"] \arrow[d, "\varphi"] & \Omega^1_{X'}|_F \arrow[r, "\eta"] \arrow[d, equal] & \Omega^1_{X'/C}|_F \arrow[r] \arrow[d, "\psi"] & 0 \\ 0 \arrow[r] & \mathcal{O}_{X'}(-F)|_F \arrow[r, "\nu"] & \Omega^1_{X'}|_F \arrow[r, "\pi"] & \Omega^1_F \arrow[r] & 0 \end{tikzcd}$$

Since $f^*\Omega^1_C|_F \cong \mathcal{O}_F$, $\mathcal{O}_{X'}(-F)|_F \cong \mathcal{O}_F$ and both the isomorphisms are induced by the restriction map, we have that $\varphi : f^*\Omega^1_C|_F \to \mathcal{O}_{X'}(-F)|_F$ is an isomorphism, and then so is $\psi : \Omega^1_{X'/C}|_F \to \Omega^1_F$ by the five-lemma.

Let F be a fiber of $f : X' \to C$. If F is not irreducible or not reduced, then F is singular. If F is irreducible and reduced, then by Lemma 2.5, $\Omega^1_F \cong \Omega^1_{X'/C}|_F$ always has a torsion part along the intersection $F \cap C' \neq \emptyset$. In particular, F is a singular curve, which completes the proof of Theorem 1.1(ii).

3 Some Remarks

The following definition is a generalization of Definition 2.1.

Definition 3.1 *Let X be a smooth projective variety, $\mathcal{L}$ a line bundle on X and k a positive integer. We say that X satisfies the T_k condition for $\mathcal{L}$, if $H^i(X, \Omega_X^j \otimes \mathcal{L}^{-1}) = 0$ holds for all $i \geq 0, j > 0$ and $i + j \leq k$.*

Remark 3.1 The T_k condition was first defined and used by Tango to give a criterion [14, Theorem 5] of the Kodaira vanishing theorem. We recall this criterion here for the convenience of the reader, while the idea of the proof is to use the exact sequences repeatedly associated to the Cartier isomorphism. Though such a Tango's criterion exists, it seems impossible to generalize Theorem 1.1 to the higher dimensional cases, since the minimal model program for higher dimensional varieties is much more complicated than that for the surfaces.

Theorem 3.1 *Let X be a smooth projective variety of dimension n, and let $0 < k < n$ be an integer. If X satisfies the T_k condition for all ample line bundles $\mathcal{L}$ on X, then $H^i(X, \mathcal{L}^{-1}) = 0$ holds for all $0 \leq i \leq k$ and all ample line bundles $\mathcal{L}$ on X.*

Remark 3.2 As is well-known, the Kawamata-Viehweg vanishing theorem is a generalization of the Kodaira-Ramanujam vansihing theorem. For simplicity, we recall here a smooth surface version of the Kawamata-Viehweg vanishing theorem for nef and big $\mathbb{Q}$-divisors (see [6, Theorem 1-2-3]).

($*$) Let X be a smooth projective surface over an algebraically closed field k with $\mathrm{char}(k) = 0$. Let L be a nef and big $\mathbb{Q}$-divisor on X, such that the fractional part of L has simple normal crossing support. Then $H^1(X, -\ulcorner L \urcorner) = 0$, where $\ulcorner L \urcorner$ is the round-up of L.

Assume $\mathrm{char}(k) > 0$. When L is integral, Theorem 1.1 shows that ($*$) only fails for quasi-elliptic surfaces of Kodaira dimension 1 and surfaces of general type. When L is a $\mathbb{Q}$-divisor, the situation becomes more delicate. For instance, the author has given counterexamples [16, Theorem 3.1] to ($*$) on certain ruled surfaces, and has also proved that ($*$) holds for any surface which is birational to a strongly liftable smooth projective surface (see [18, Theorem 1.4]). In particular, the Kawamata-Viehweg vanishing theorem holds for all rational surfaces [17, Theorem 1.4], and hence for all log del Pezzo surfaces [17, Corollary 1.6].

Acknowledgement The author would like to express his gratitude to the referees for their careful reading and useful comments.

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