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A Type of General Forward-Backward Stochastic Differential Equations and Applications*

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Abstract The authors discuss one type of general forward-backward stochastic differential equations (FBSDEs) with Itô's stochastic delayed equations as the forward equations and anticipated backward stochastic differential equations as the backward equations. The existence and uniqueness results of the general FBSDEs are obtained. In the framework of the general FBSDEs in this paper, the explicit form of the optimal control for linear-quadratic stochastic optimal control problem with delay and the Nash equilibrium point for nonzero sum differential games problem with delay are obtained.

Keywords Stochastic delayed differential equations, Anticipated backward stochastic differential equations, Forward-backward stochastic differential equations, Linear-quadratic stochastic optimal control with delay, Nonzero sum stochastic differential game with delay

2000 MR Subject Classification 60H10, 93E20

1 Introduction

1.1 Background of the problem

It is well-known that linear-quadratic (LQ) problem is one of the most important class of optimal control and game problems. And the LQ problem was studied by linking to a linear fully coupled forward-backward stochastic differential equation (FBSDE). The FBSDE can also be encountered in the optimization problem when we apply stochastic maximum principle and in mathematic finance when we consider large investor (see [2]).

Hu, Peng [3] and Peng, Wu [7] obtained the existence and uniqueness results for FBSDE under some monotone conditions. Yong [11] let the method in [3, 7] be systematic and called it the "continuation method". Then Wu [9, 10] discussed the application of FBSDEs in LQ problem and maximum principle for optimal control problem of FBSDE systems. Yu and Ji [12] used the results of FBSDE to study one kind of LQ nonzero-sum stochastic differential

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game problem. In the previous results, the FBSDE is of the form

$$\begin{cases} \mathrm{d}x_t = b(t, x_t, y_t, z_t)\mathrm{d}t + \sigma(t, x_t, y_t, z_t)\mathrm{d}B_t, \\ -\mathrm{d}y_t = f(t, x_t, y_t, z_t)\mathrm{d}t - z_t\mathrm{d}B_t, \\ x_0 = a, \quad y_T = \Phi(x_T) \end{cases}$$

with the classical Itô's stochastic differential equation (SDE) as the state equation and backward stochastic differential equation (BSDE) as a dual equation in the control system which is also called Hamilton system in optimal control theory.

However, the study of various natural and social phenomena shows that the future development of many processes depends not only on their present state but also on their previous history. Such processes can be described mathematically by using the stochastic delayed differential equations (SDDEs), such as the following form:

$$\begin{cases} \mathrm{d}x_t = \ b(t, x_t, x_{t-\delta})\mathrm{d}t + \sigma(t, x_t, x_{t-\delta})\mathrm{d}B_t, & t \in [0, T], \\ x_t = \ \varphi_t, & t \in [-\delta, 0], \end{cases}$$

where $\varphi : [-\delta, 0] \to \mathbb{R}^n$ is the initial path of x and $\delta \geq 0$ is the time delay. This kind of SDDE can be encountered in population growth problem, economics, biology, automatics and other areas of human activity. More examples and applications in the related fields can be found in [1, 4, 5].

A stochastic control system whose state function is described by the solution of SDDE is called delayed system. Øksendal and Sulem [6] discussed a certain class of stochastic control systems with delay and they gave the sufficient conditions for the stochastic maximum principle of this kind of control system. However, to study the delayed optimal control system, it is natural to introduce the adjoint equation which is a new type of BSDE as following:

$$\begin{cases} -\mathrm{d}y_t = f(t, y_t, z_t, y_{t+\delta}, z_{t+\delta})\mathrm{d}t - z_t\mathrm{d}B_t, & t \in [0, T], \\ y_t = \xi_t, \quad z_t = \eta_t, & t \in [T, T + \delta]. \end{cases}$$

Here $\xi(\,\cdot\,), \eta(\,\cdot\,)$ are two integral functions as the terminal conditions of y, z respectively. Recently, Peng and Yang [8] discussed this kind of BSDE which is called anticipated BSDE. Under some proper assumptions, they obtained the existence and uniqueness of solution for anticipated BSDE.

So it is necessary to explore the following general FBSDE:

$$\begin{cases} \mathrm{d}x_t = b(t, x_t, y_t, z_t, x_{t-\delta})\mathrm{d}t + \sigma(t, x_t, y_t, z_t, x_{t-\delta})\mathrm{d}B_t, & t \in [0, T], \\ -\mathrm{d}y_t = f(t, x_t, y_t, z_t, y_{t+\delta}, z_{t+\delta})\mathrm{d}t - z_t\mathrm{d}B_t, & t \in [0, T], \\ x_t = \varphi_t, & t \in [-\delta, 0], \\ y_t = \xi_t, \quad z_t = \eta_t, & t \in [T, T + \delta]. \end{cases}$$

Here forward SDE is with delay, BSDE is anticipated, and they form a type of general fully coupled FBSDE.

In the next section, we deal with this kind of general FBSDE and get the existence and uniqueness results of the solution. And then, in Section 3, we study an LQ stochastic optimal control problem with delay. Using the solution of the general FBSDE, we give the explicit unique optimal control for this problem. In the last section, we continue to study the LQ nonzero sum stochastic differential game problem with delay which is more complicated than the control problem. Using our general FBSDE, we get the Nash equilibrium point for the game problem. Our paper is the first attempt to study this type of general FBSDE and apply it to stochastic optimal control and game problem with delay to the authors' knowledge.

1.2 Notations

Let $\{B_t\}_{t\geq 0}$ be a d-dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$. And let $\{\mathcal{F}_t\}_{t\geq 0}$ be the natural filtration of $\{B_t\}$, where $\mathcal{F}_0$ contains all P-null sets of $\mathcal{F}$. $0 < T < +\infty$ is the time horizon. If x belongs to $\mathbb{R}^n$, $|x|$ denotes its Euclidean norm. We will denote by $\langle \cdot , \cdot \rangle$ the inner product. And the following notations will be used throughout our paper:

$$\begin{aligned}
C[-\delta,0]^n &:= \Big\{\varphi_t : [-\delta,0] \to \mathbb{R}^n \text{ is continuous and } \sup_{-\delta\leq t\leq 0} |\varphi_t| < +\infty\Big\},\\
L^2(\mathcal{F}_T;\mathbb{R}^m) &:= \{\varphi \text{ is } \mathbb{R}^m\text{-valued } \mathcal{F}_T\text{-measurable random variable s.t. } \mathbb{E}[\varphi^2] < +\infty\},\\
L^2_{\mathcal{F}}(0,T;\mathbb{R}^n) &:= \Big\{\varphi_t,\ 0\leq t\leq T, \text{ is an } \mathcal{F}_t\text{-adapted stochastic process s.t.}\\
&\qquad\qquad \mathbb{E}\int_0^T |\varphi_t|^2 \mathrm{d}t < +\infty\Big\}.
\end{aligned}$$

2 The General Forward-Backward Differential Equations

Assume that for all $t \in [0,T]$,

$$\begin{aligned}
&b : \Omega \times [0,T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \times L^2(\mathcal{F}_s;\mathbb{R}^n) \to L^2(\mathcal{F}_t;\mathbb{R}^n),\\
&\sigma : \Omega \times [0,T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \times L^2(\mathcal{F}_s;\mathbb{R}^n) \to L^2(\mathcal{F}_t;\mathbb{R}^{n\times d}),\\
&f : \Omega \times [0,T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \times L^2(\mathcal{F}_r;\mathbb{R}^m) \times L^2(\mathcal{F}_{r'};\mathbb{R}^{m\times d}) \to L^2(\mathcal{F}_t;\mathbb{R}^m),
\end{aligned}$$

where $s \in [-\delta, t]$ and $r, r' \in [t, T+\delta]$.

Then we consider the following general forward-backward stochastic differential equation (FBSDE) with forward equation being SDDE and backward equation being anticipated BSDE:

$$\begin{cases}
\mathrm{d}x_t = b(t,x_t,y_t,z_t,x_{t-\delta})\mathrm{d}t + \sigma(t,x_t,y_t,z_t,x_{t-\delta})\mathrm{d}B_t, & t\in[0,T],\\
-\mathrm{d}y_t = f(t,x_t,y_t,z_t,y_{t+\delta},z_{t+\delta})\mathrm{d}t - z_t\mathrm{d}B_t, & t\in[0,T],\\
x_t = \varphi_t, & t\in[-\delta,0],\\
y_T = \Phi(x_T), \quad y_t = \xi_t, & t\in(T,T+\delta],\\
z_t = \eta_t, & t\in[T,T+\delta]
\end{cases} \tag{2.1}$$

with $\Phi : \Omega\times\mathbb{R}^n \to \mathbb{R}^m$ and $\varphi_t \in C[-\delta,0]^n$, $\xi_t \in L^2_{\mathcal{F}}(T,T+\delta;\mathbb{R}^m)$, $\eta_t \in L^2_{\mathcal{F}}(T,T+\delta;\mathbb{R}^{m\times d})$.

Given an $m\times n$ full-rank matrix G, we will use the notations

$$u = \begin{pmatrix} x\\ y\\ z\end{pmatrix}, \quad \begin{pmatrix} x_{\cdot-\delta}\\ y_{\cdot+\delta}\\ z_{\cdot+\delta}\end{pmatrix} = \begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix}, \quad A(t,u,\alpha,\beta,\gamma) = \begin{pmatrix} -G^\tau f(t,u,\beta,\gamma)\\ Gb(t,u,\alpha)\\ G\sigma(t,u,\alpha)\end{pmatrix},$$

where $G\sigma = (G\sigma_1,\cdots,G\sigma_d)$.

Definition 2.1 *A triple of process* $(X,Y,Z) : \Omega\times[-\delta,T]\times[0,T+\delta]\times[0,T+\delta] \to \mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^{m\times d}$ *is called an adapted solution to FBSDE* (2.1) *if* $(X,Y,Z) \in L^2_{\mathcal{F}}(-\delta,T;\mathbb{R}^n)\times L^2_{\mathcal{F}}(0,T+\delta;\mathbb{R}^m)\times L^2_{\mathcal{F}}(0,T+\delta;\mathbb{R}^{m\times d})$ *and satisfies FBSDE* (2.1).

We assume that

$$
\text{(H2.1)}\begin{cases}
\text{(i) There exists a constant } C>0, \text{ s.t.}\\
|A(t,u,\alpha,\beta,\gamma)-A(t,u',\alpha',\beta',\gamma')|\\
\quad \le C(|u-u'|+|\alpha-\alpha'|+\mathbb{E}^{\mathcal{F}_t}[|\beta-\beta'|+|\gamma-\gamma'|]) \text{ for all } u,u',\alpha,\alpha',\beta,\beta',\gamma,\gamma';\\
\text{(ii) for each } u,\alpha,\beta,\gamma,\ A(\cdot,u,\alpha,\beta,\gamma) \text{ is in } L^2_{\mathcal{F}}(0,T);\\
\text{(iii) } \Phi(x) \text{ is in } L^2(\mathcal{F}_T;\mathbb{R}^m) \text{ and it is uniformly Lipschitz with respect to } x\in\mathbb{R}^n.
\end{cases}
$$

$$
\text{(H2.2)}\begin{cases}
\displaystyle\int_0^T \langle A(t,u_t,\alpha_t,\beta_t,\gamma_t)-A(t,\overline{u}_t,\overline{\alpha}_t,\overline{\beta}_t,\overline{\gamma}_t),u-\overline{u}\rangle \mathrm{d}t\\
\quad \displaystyle\le \int_0^T[-\beta_1|G\widehat{x}_t|^2-\beta_2(|G^\tau\widehat{y}_t|^2+|G^\tau\widehat{z}_t|^2)]\mathrm{d}t,\\
\langle \Phi(x)-\Phi(\overline{x}),G(x-\overline{x})\rangle \ge \mu_1|G\widehat{x}|^2
\end{cases}
$$

for all $u=(x,y,z)$, $\overline{u}=(\overline{x},\overline{y},\overline{z})$, $\widehat{x}=x-\overline{x}$, $\widehat{y}=y-\overline{y}$, $\widehat{z}=z-\overline{z}$, where β_1,β_2 and μ_1 are given nonnegative constants with $\beta_1+\beta_2>0$, $\beta_2+\mu_1>0$. Moreover, we have $\beta_1>0$, $\mu_1>0$ (resp. $\beta_2>0$) when $m>n$ (resp. $n>m$).

Theorem 2.1 *Let* (H2.1) *and* (H2.2) *hold. Then there exists a unique adapted solution* (X,Y,Z) *to the general FBSDE* (2.1).

Proof Since the initial path of x in $[-\delta,0]$ and the terminal conditions and trajectories of y,z in $[T,T+\delta]$ are given in advance, we only need to consider (x_t,y_t,z_t), $0\le t\le T$.

Uniqueness First we prove the uniqueness. For $0\le t\le T$, let $u_t=(x_t,y_t,z_t)$ and $\overline{u}_t=(\overline{x}_t,\overline{y}_t,\overline{z}_t)$ be two solutions of (2.1). We set $\widehat{u}_t=(x_t-\overline{x}_t,y_t-\overline{y}_t,z_t-\overline{z}_t)=(\widehat{x}_t,\widehat{y}_t,\widehat{z}_t)$.

Applying the Itô's formula to $\langle G\widehat{x}_t,\widehat{y}_t\rangle$, we have

$$
\begin{aligned}
&\mathbb{E}\langle \Phi(x_T)-\Phi(\overline{x}_T),G\widehat{x}_T\rangle\\
&=\mathbb{E}\int_0^T\langle A(t,u_t,\alpha_t,\beta_t,\gamma_t)-A(t,\overline{u}_t,\overline{\alpha}_t,\overline{\beta}_t,\overline{\gamma}_t),\widehat{u}_t\rangle\mathrm{d}t\\
&\le -\beta_1\mathbb{E}\int_0^T\langle G\widehat{x}_t,G\widehat{x}_t\rangle\mathrm{d}t-\beta_2\mathbb{E}\int_0^T(\langle G^\tau\widehat{y}_t,G^\tau\widehat{y}_t\rangle+\langle G^\tau\widehat{z}_t,G^\tau\widehat{z}_t,\rangle)\mathrm{d}t.
\end{aligned}
$$

By (H2.2), we also have

$$
\beta_1\mathbb{E}\int_0^T\langle G\widehat{x}_t,G\widehat{x}_t\rangle\mathrm{d}t+\beta_2\mathbb{E}\int_0^T(\langle G^\tau\widehat{y}_t,G^\tau\widehat{y}_t\rangle+\langle G^\tau\widehat{z}_t,G^\tau\widehat{z}_t,\rangle)\mathrm{d}t+\mu_1\mathbb{E}\langle G\widehat{x}_T,G\widehat{x}_T\rangle\le 0.
$$

We first treat the case where $m>n$. Then $\beta_1>0$, $\mu_1>0$, $\langle G\widehat{x}_t,G\widehat{x}_t\rangle\equiv 0$. In this case, we have $\widehat{x}_t\equiv 0$. Thus $x_t\equiv\overline{x}_t$. Then from the uniqueness of anticipated BSDE (see [8]), it follows that $y_t\equiv\overline{y}_t$, $z_t\equiv\overline{z}_t$.

Now we discuss the second case where $m<n$, then $\beta_2>0$, $\langle G^\tau\widehat{y}_t,G^\tau\widehat{y}_t\rangle\equiv 0$. In this case, we have $y_t\equiv\overline{y}_t$. Applying the Itô's formula to $\widehat{y}_t\equiv 0$, it follows that $\int_0^T|\widehat{z}_t|\mathrm{d}t=0$ which implies that $z_t\equiv\overline{z}_t$. Finally, from the uniqueness of SDDE (see [4, 5]), it follows that $x_t\equiv\overline{x}_t$.

Similarly to the above two cases, the result can be obtained easily in the case $m=n$.

Remark 2.1 From the proof process of uniqueness, (H2.2) can be relaxed as

$$
(\mathrm{H2.2})' \quad \begin{cases} \displaystyle\int_0^T \langle A(t,u_t,\alpha_t,\beta_t,\gamma_t) - A(t,\overline{u}_t,\overline{\alpha}_t,\overline{\beta}_t,\overline{\gamma}_t), u-\overline{u}\rangle \mathrm{d}t \\ \displaystyle\quad \le \int_0^T [-\beta_1|G\widehat{x}_t|^2 - \beta_2|G^\tau \widehat{y}_t|^2]\mathrm{d}t, \\ \langle \Phi(x)-\Phi(\overline{x}), G(x-\overline{x})\rangle \ge 0, \end{cases}
$$

where β_1, β_2 are given nonnegative constants with $\beta_1+\beta_2>0$. Moreover, we require that $\beta_1>0$ (resp. $\beta_2>0$), when $m>n$ (resp. $n>m$).

The proof of existence is more complicated. We will analyze different cases according to different dimensions of x and y respectively.

First case If $m>n$, then $\beta_1>0$, $\mu_1>0$. We consider the following family of generalized FBSDEs with parameter $\epsilon\in[0,1]$:

$$
\begin{cases} \mathrm{d}x_t^\epsilon = [\epsilon b(t,u_t^\epsilon,x_{t-\delta}^\epsilon)+\phi_t]\mathrm{d}t + [\epsilon\sigma(t,u_t^\epsilon,x_{t-\delta}^\epsilon)+\psi_t]\mathrm{d}B_t, & t\in[0,T], \\ -\mathrm{d}y_t^\epsilon = [(1-\epsilon)\beta_1 G x_t^\epsilon + \epsilon f(t,u_t^\epsilon,y_{t+\delta}^\epsilon,z_{t+\delta}^\epsilon)+\rho_t]\mathrm{d}t - z_t^\epsilon \mathrm{d}B_t, & t\in[0,T], \\ x_t^\epsilon = \varphi_t, & t\in[-\delta,0], \\ y_T^\epsilon = \epsilon\Phi(x_T^\epsilon)+(1-\epsilon)Gx_T^\epsilon+\theta, \quad y_t^\epsilon=\xi_t, & t\in(T,T+\delta], \\ z_t^\epsilon = \eta_t, & t\in[T,T+\delta], \end{cases} \tag{2.2}
$$

where $\phi,\psi,\rho\in L^2_{\mathcal{F}}(0,T)$ with values in $\mathbb{R}^n, \mathbb{R}^{n\times d}$ and $\mathbb{R}^m$ respectively, and $\theta\in L^2(\mathcal{F}_T;\mathbb{R}^m)$. Obviously, when $\epsilon=1$ the existence of solution for equation (2.2) implies that for equation (2.1). From the existence and uniqueness of solutions for SDDE and anticipated BSDE, when $\epsilon=0$, equation (2.2) has a uniqueness solution. In order to prove Theorem 2.1, we give the following priori estimate for the existence interval of (2.2) with respect to $\epsilon\in[0,1]$.

Lemma 2.1 *We assume $m>n$,* (H2.1) *and* (H2.2)*. If for some $\epsilon_0\in[0,1)$ there exists a solution $(x^{\epsilon_0},y^{\epsilon_0},z^{\epsilon_0})$ to* (2.2)*, then there exists a positive constant $c_0>0$ independent of ϵ_0, such that for each $c\in[0,c_0]$ there exists a solution $(x^{\epsilon_0+c},y^{\epsilon_0+c},z^{\epsilon_0+c})$ to generalized FBSDE* (2.2) *for $\epsilon=\epsilon_0+c$.*

Proof Since for each $\varphi_t\in C[-\delta,0]^n$, $\theta\in L^2(\mathcal{F}_T;\mathbb{R}^m)$ and $(\phi,\psi,\rho)\in L^2_{\mathcal{F}}(0,T;\mathbb{R}^n\times\mathbb{R}^{n\times d}\times\mathbb{R}^m)$, $\epsilon_0\in[0,1)$, there exists a (unique) solution to (2.2), thus for each triple $u_s=(x_s,y_s,z_s)\in L^2_{\mathcal{F}}(-\delta,T;\mathbb{R}^n)\times L^2_{\mathcal{F}}(0,T+\delta;\mathbb{R}^m)\times L^2_{\mathcal{F}}(0,T+\delta;\mathbb{R}^{m\times d})$, there exists a unique triple $U_s=(X_s,Y_s,Z_s)\in L^2_{\mathcal{F}}(-\delta,T;\mathbb{R}^n)\times L^2_{\mathcal{F}}(0,T+\delta;\mathbb{R}^m)\times L^2_{\mathcal{F}}(0,T+\delta;\mathbb{R}^{m\times d})$ satisfying the following FBSDE:

$$
\begin{cases} \mathrm{d}X_t = [\epsilon_0 b(t,U_t,X_{t-\delta}) + cb(t,u_t,x_{t-\delta})+\phi_t]\mathrm{d}t \\ \qquad\qquad +[\epsilon_0\sigma(t,U_t,X_{t-\delta})+c\sigma(t,u_t,x_{t-\delta})+\psi_t]\mathrm{d}B_t, & t\in[0,T], \\ -\mathrm{d}Y_t = [(1-\epsilon_0)\beta_1 GX_t + \epsilon_0 f(t,U_t,Y_{t+\delta},Z_{t+\delta}) \\ \qquad\qquad +c(-\beta_1 Gx_t + f(t,u_t,y_{t+\delta},z_{s+\delta}))+\rho_t]\mathrm{d}t - Z_t\mathrm{d}B_t, & t\in[0,T], \\ X_t=\varphi_t, & t\in[-\delta,0], \\ Y_T = \epsilon_0\Phi(X_T)+c\Phi(x_T)+(1-\epsilon_0)GX_T - cGx_T+\theta, \quad Y_t=\xi_t, & t\in(T,T+\delta], \\ Z_t=\eta_t, & t\in[T,T+\delta]. \end{cases}
$$

We desire to prove that the mapping defined by

$$\begin{aligned}I_{\epsilon_0+c}(u)=U: L^2_{\mathcal{F}}(-\delta,T;\mathbb{R}^n)\times L^2_{\mathcal{F}}(0,T+\delta;\mathbb{R}^m)\times L^2_{\mathcal{F}}(0,T+\delta;\mathbb{R}^{m\times d})\\ \to L^2_{\mathcal{F}}(-\delta,T;\mathbb{R}^n)\times L^2_{\mathcal{F}}(0,T+\delta;\mathbb{R}^m)\times L^2_{\mathcal{F}}(0,T+\delta;\mathbb{R}^{m\times d})\end{aligned}$$

is a contraction.

Let $\overline{u}=(\overline{x},\overline{y},\overline{z})\in L^2_{\mathcal{F}}(-\delta,T;\mathbb{R}^n)\times L^2_{\mathcal{F}}(0,T+\delta;\mathbb{R}^m)\times L^2_{\mathcal{F}}(0,T+\delta;\mathbb{R}^{m\times d})$ and $\overline{U}=(\overline{X},\overline{Y},\overline{Z})=I_{\epsilon_0+c}(\overline{u})$. We set $\widehat{u}=(\widehat{x},\widehat{y},\widehat{z})=(x-\overline{x},y-\overline{y},z-\overline{z})$, $\widehat{U}=(\widehat{X},\widehat{Y},\widehat{Z})=(X-\overline{X},Y-\overline{Y},Z-\overline{Z})$. Applying the Itô's formula to $\langle G\widehat{X}_t,\widehat{Y}_t\rangle$ yields

$$\begin{aligned}&[\mu_1\epsilon_0+(1-\epsilon_0)]\mathbb{E}|G\widehat{X}_T|^2+\beta_1\mathbb{E}\int_0^T|G\widehat{X}_s|^2\mathrm{d}s+\epsilon_0\beta_2\mathbb{E}\int_0^T(|G^\tau\widehat{Y}_s|^2+|G^\tau\widehat{Z}_s|^2)\mathrm{d}s\\&\le C_1c\mathbb{E}|\widehat{X}_T|^2+C_1c\mathbb{E}|\widehat{x}_T|^2+C_1c\Big[\mathbb{E}\int_0^T(|\widehat{x}_t|^2+|\widehat{y}_t|^2+|\widehat{z}_t|^2)\mathrm{d}t\\&\quad+\mathbb{E}\int_0^T(|\widehat{x}_{t-\delta}|^2+|\widehat{y}_{t+\delta}|^2+|\widehat{z}_{t+\delta}|^2)\mathrm{d}t\Big]+C_1c\mathbb{E}\int_0^T(|\widehat{X}_t|^2+|\widehat{Y}_t|^2+|\widehat{Z}_t|^2)\mathrm{d}t\\&\le C_1c\mathbb{E}|\widehat{X}_T|^2+C_1c\mathbb{E}|\widehat{x}_T|^2+C_1c\Big[\mathbb{E}\int_0^T|\widehat{x}_t|^2\mathrm{d}t+\mathbb{E}\int_0^{T+\delta}(|\widehat{y}_t|^2+|\widehat{z}_t|^2)\mathrm{d}t\Big]\\&\quad+C_1c\mathbb{E}\int_0^T(|\widehat{X}_t|^2+|\widehat{Y}_t|^2+|\widehat{Z}_t|^2)\mathrm{d}t. \qquad (2.3)\end{aligned}$$

For $(\widehat{Y},\widehat{Z})$, in virtue of the estimate of anticipated BSDE, we can derive

$$\begin{aligned}\mathbb{E}\int_0^T(|\widehat{Y}_t|^2+|\widehat{Z}_t|^2)\mathrm{d}t\le C_1c\Big[\mathbb{E}\int_0^T|x_t|^2\mathrm{d}t+\mathbb{E}\int_0^{T+\delta}(|\widehat{y}_t|^2+|\widehat{z}_t|^2)\mathrm{d}t\Big]\\+C_1c\mathbb{E}|\widehat{x}_T|^2+C_1\mathbb{E}\int_0^T|\widehat{X}_t|^2\mathrm{d}t+C_1\mathbb{E}|\widehat{X}_T|^2\mathrm{d}t, \qquad (2.4)\end{aligned}$$

where C_1 is a constant depending on G, β_1, T and Lipschitz's constant and can be changed line by line. If $\mu_1>0$, then $\mu_1\epsilon_0+(1-\epsilon_0)\ge\mu$, $\mu=\min(1,\mu_1)>0$. Combining the estimates (2.3) and (2.4), we have

$$\begin{aligned}&\mathbb{E}\int_{-\delta}^T|\widehat{X}_t|^2\mathrm{d}t+\mathbb{E}\int_0^{T+\delta}(|\widehat{Y}_t|^2+|\widehat{Z}_t|^2)\mathrm{d}t+\mathbb{E}|\widehat{X}_T|^2\\&\le Kc\Big(\mathbb{E}\int_{-\delta}^T|\widehat{x}_t|^2\mathrm{d}t+\mathbb{E}\int_0^{T+\delta}(|\widehat{y}_t|^2+|\widehat{z}_t|^2)\mathrm{d}t+\mathbb{E}|\widehat{x}_T|^2\Big),\end{aligned}$$

where constant K depends only on β_1,μ,C_1,T. Now we choose $c_0=\frac{1}{2K}$. It is clear that, for each fixed $c\in[0,c_0]$, the mapping I_{ϵ_0+c} is a contraction in the sense that

$$\begin{aligned}&\mathbb{E}\int_{-\delta}^T|\widehat{X}_t|^2\mathrm{d}t+\mathbb{E}\int_0^{T+\delta}(|\widehat{Y}_t|^2+|\widehat{Z}_t|^2)\mathrm{d}t+\mathbb{E}|\widehat{X}_T|^2\\&\le\frac{1}{2}\Big(\mathbb{E}\int_{-\delta}^T|\widehat{x}_t|^2\mathrm{d}t+\mathbb{E}\int_0^{T+\delta}(|\widehat{y}_t|^2+|\widehat{z}_t|^2)\mathrm{d}t+\mathbb{E}|\widehat{x}_T|^2\Big).\end{aligned}$$

It follows immediately that this mapping has a unique fixed point, $U^{\epsilon_0+c}=(X^{\epsilon_0+c},Y^{\epsilon_0+c},Z^{\epsilon_0+c})$, which is the solution to (2.2) for $\epsilon=\epsilon_0+c$. The proof is completed.

Second Case If $m < n$, then $\beta_2 > 0$. We consider the following FBSDE:

$$\begin{cases} \mathrm{d}x_t^\epsilon = [(1-\epsilon)\beta_2(-G^\tau y_t^\epsilon) + \epsilon b(t, u_t^\epsilon, x_{t-\delta}^\epsilon) + \phi_t]\mathrm{d}t \\ \qquad\quad +[(1-\epsilon)\beta_2(-G^\tau z_t^\epsilon) + \epsilon\sigma(t, u_t^\epsilon, x_{t-\delta}^\epsilon) + \psi_t]\mathrm{d}B_t, & t \in [0,T], \\ -\mathrm{d}y_t^\epsilon = [\epsilon f(t, u_t^\epsilon, y_{t+\delta}^\epsilon, z_{t+\delta}^\epsilon) + \rho_t]\mathrm{d}t - z_t^\epsilon \mathrm{d}B_t, & t \in [0,T], \\ x_t^\epsilon = \varphi_t, & t \in [-\delta, 0], \\ y_T^\epsilon = \epsilon\Phi(x_T^\epsilon) + \theta, \quad y_t^\epsilon = \xi_t, & t \in (T, T+\delta], \\ z_t^\epsilon = \eta_t, & t \in [T, T+\delta]. \end{cases} \tag{2.5}$$

When $\epsilon = 0$, we know that equation (2.5) has a unique solution. Our purpose is to derive that there exists a solution to (2.5) when $\epsilon = 1$, and then we can get the existence of solution to (2.1). Similarly to Lemma 2.1, we have the following lemma.

Lemma 2.2 *Suppose that $m < n$, and* (H2.1), (H2.2) *hold. If there exists some $\epsilon_0 \in [0,1)$ such that* (2.5) *has a solution $(x^{\epsilon_0}, y^{\epsilon_0}, z^{\epsilon_0})$, then there is a positive constant c_0 independent of ϵ_0 such that for each $c \in [0, c_0]$, $(x^{\epsilon_0+c}, y^{\epsilon_0+c}, z^{\epsilon_0+c})$ also satisfies the equation* (2.5).

Third Case $m = n$. By $(H2.2)$, we only need to consider

(i) If $\beta_1 > 0,\ \beta_2 \geq 0,\ \mu_1 > 0$, we can have the same result like Lemma 2.1;

(ii) If $\beta_1 \geq 0,\ \beta_2 > 0,\ \mu_1 \geq 0$, the same result as Lemma 2.2 can be obtained.

Now we can proceed to give the proof of the existence for Theorem 2.1.

Existence We first treat the case where $m > n$. When $\epsilon = 0$, equation (2.2) has a unique solution. It then follows from Lemma 2.1 that there exists a positive constant c_0 depending on Lipschitz constants, β_1, G and T, such that, for each $c \in [0, c_0]$, equation (2.2) has a unique solution for $\epsilon = c$. We can repeat this process for N times with $1 \leq Nc_0 < 1 + c_0$. It then follows that FBSDE (2.2) has a unique solution for $\epsilon = 1$.

In the case where $m < n$ and $m = n$, our desired result can be obtained similarly. The proof of Theorem 2.1 is complete.

Remark 2.2 If b and σ in FBSDE (2.1) are independent of z_t and $\beta_2 > 0$ in (H2.2)$'$, then, similarly, we can prove that FBSDE (2.1) has a unique solution (x_t, y_t, z_t) under (H2.1) and (H2.2)$'$.

Moreover, for the application in optimal control and game problem, we consider the following kind of general FBSDE:

$$\begin{cases} \mathrm{d}x_t = b(t, x_t, By_t, Dz_t, x_{t-\delta})\mathrm{d}t + \sigma(t, x_t, By_t, Dz_t, x_{t-\delta})\mathrm{d}B_t, & t \in [0,T], \\ -\mathrm{d}y_t = f(t, x_t, y_t, z_t, y_{t+\delta}, z_{t+\delta})\mathrm{d}t - z_t \mathrm{d}B_t, & t \in [0,T], \\ x_t = \varphi_t, & t \in [-\delta, 0], \\ y_T = \Phi(x_T), \quad y_t = \xi_t, & t \in (T, T+\delta], \\ z_t = \eta_t, & t \in [T, T+\delta], \end{cases} \tag{2.6}$$

where B, D are $k \times n$ matrixes. For notational simplification, we assume the dimension of Brownian motion $d = 1$. $(x, y, z) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$, and b, f, σ have appropriate dimensions. We also use the notations

$$u = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \begin{pmatrix} x_{\cdot-\delta} \\ y_{\cdot+\delta} \\ z_{\cdot+\delta} \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}, \quad A(t, u, \alpha, \beta, \gamma) = \begin{pmatrix} -f(t, u, \beta, \gamma) \\ b(t, u, \alpha) \\ \sigma(t, u, \alpha) \end{pmatrix},$$

and impose the following monotone conditions:

$$\text{(H2.3)}\quad \begin{cases} \displaystyle\int_0^T \langle A(t,u_t,\alpha_t,\beta_t,\gamma_t) - A(t,\overline{u}_t,\overline{\alpha}_t,\overline{\beta}_t,\overline{\gamma}_t), u-\overline{u}\rangle \mathrm{d}t \\ \quad \displaystyle\leq \int_0^T [-\nu_1|\widehat{x}_t|^2 - \nu_2|B\widehat{y}_t + D\widehat{z}_t|^2]\mathrm{d}t, \\ \langle \Phi(x)-\Phi(\overline{x}), G(x-\overline{x})\rangle \geq 0 \end{cases}$$

for all $u=(x,y,z),\ \overline{u}=(\overline{x},\overline{y},\overline{z}),\ \widehat{x}=x-\overline{x},\ \widehat{y}=y-\overline{y},\ \widehat{z}=z-\overline{z}$, where $\nu_1\geq 0$, $\nu_2>0$. We also assume that

$$\text{(H2.4)}\ \begin{cases} \text{(i)}\ \ \text{There exists a constant } C>0,\ \text{s.t.} \\ |A(t,u,\alpha,\beta,\gamma)-A(t,u',\alpha',\beta',\gamma')| \\ \quad \leq C(|u-u'|+|\alpha-\alpha'|+\mathbb{E}^{\mathcal{F}_t}[|\beta-\beta'|+|\gamma-\gamma'|]) \text{ for all } u,u',\alpha,\alpha',\beta,\beta',\gamma,\gamma'; \\ \text{(ii)}\ \ \text{for each } u,\alpha,\beta,\gamma,\ A(\cdot,u,\alpha,\beta,\gamma) \text{ is in } L^2_{\mathcal{F}}(0,T); \\ \text{(iii)}\ \ \Phi(x) \text{ is in } L^2(\mathcal{F}_T;\mathbb{R}^m) \text{ and it is uniformly Lipschitz with respect to } x\in\mathbb{R}^n; \\ \text{(iv)}\ \ \forall x,\alpha,\ |l(t,x,By,Dz,\alpha)-l(t,x,By',Dz',\alpha)| \leq K[|B(y-y')+D(z-z')|], \\ \hfill l=b,\sigma,\ K>0. \end{cases}$$

Theorem 2.2 *Let* (H2.3) *and* (H2.4) *hold. Then there exists a unique solution* (x_t,y_t,z_t) *satisfying the general FBSDE* (2.6).

The proof method of Theorem 2.2 is combined that of Theorem 2.1 with that of Theorem 3.1 in [7]. We omit it.

3 Linear Quadratic Stochastic Optimal Control Problem with Delay

In this section, we consider the following linear control system with delay:

$$\begin{cases} \mathrm{d}x_t = (A_t x_t + B_t x_{t-\delta} + C_t v_t)\mathrm{d}t + (D_t x_t + E_t x_{t-\delta} + F_t v_t)\mathrm{d}B_t, & t\in[0,T], \\ x_t = \varphi_t, & t\in[-\delta,0], \end{cases} \tag{3.1}$$

where $\varphi\in C[-\delta,0]^n$ is deterministic function; v_t $(t\in[0,T])$ is an $\mathcal{F}_t$-adapted square-integrable process taking values in $U\subset\mathbb{R}^k$ (we call such v_t admissible control and the set of admissible control will be denoted by $\mathcal{U}_{\mathrm{ad}}$); A_t,B_t,C_t,D_t,E_t,F_t are bounded progressively measurable matrix-valued processes with appropriate dimensions.

We introduce the classical quadratic optimal control cost function:

$$J(v(\cdot)) = \frac{1}{2}\mathbb{E}\Big[\int_0^T (\langle R_t x_t, x_t\rangle + \langle N_t v_t, v_t\rangle)\mathrm{d}t + \langle Q x_T, x_T\rangle\Big], \tag{3.2}$$

where Q is the $\mathcal{F}_T$-measurable nonnegative symmetric bounded matrix and R_t is $n\times n$ nonnegative symmetric bounded progressively measurable matrix-valued process, N_t is positive symmetric bounded progressively measurable matrix-valued process with the dimension $k\times k$ and the inverse N_t^{-1} is also bounded.

Problem 3.1 Our problem is to find admissible control $u(\cdot)$ such that

$$J(u(\cdot)) = \inf_{v(\cdot)} J(v(\cdot)).$$

For the above LQ optimal control problem with delay, we can get the explicit form of the optimal control by virtue of the solution of the general FBSDE.

Theorem 3.1 *The process*

$$u_t = -N_t^{-1}(C_t^\tau y_t + F_t^\tau z_t), \quad t \in [0, T] \tag{3.3}$$

is the unique optimal control of Problem 3.1*, where* (x_t, y_t, z_t) *is the solution of the following general FBSDE:*

$$\begin{cases} \mathrm{d}x_t = [A_t x_t + B_t x_{t-\delta} - C_t N_t^{-1}(C_t^\tau y_t + F_t^\tau z_t)]\mathrm{d}t \\ \qquad\quad +[D_t x_t + E_t x_{t-\delta} - F_t N_t^{-1}(C_t^\tau y_t + F_t^\tau z_t)]\mathrm{d}B_t, & t \in [0, T], \\ -\mathrm{d}y_t = [A_t^\tau y_t + D_t^\tau z_t + \mathbb{E}^{\mathcal{F}_t}(B_{t+\delta}^\tau y_{t+\delta} + E_{t+\delta}^\tau z_{t+\delta}) \\ \qquad\quad +R_t x_t]\mathrm{d}t - z_t \mathrm{d}B_t, & t \in [0, T], \\ x_t = \varphi_t, & t \in [-\delta, 0], \\ y_T = Q x_T, \quad y_t = 0, & t \in (T, T+\delta], \\ z_t = 0, & t \in [T, T+\delta]. \end{cases} \tag{3.4}$$

Proof It is easy to verify that (3.4) satisfies assumptions (H2.3) and (H2.4). Then according to Theorem 2.2, we know that the general FBSDE (3.4) has a unique solution (x_t, y_t, z_t).

For each $v(\cdot) \in \mathcal{U}_{\mathrm{ad}}$, we denote x_t^v the corresponding trajectory of system (3.1). Then

$$\begin{aligned} & J(v(\cdot)) - J(u(\cdot)) \\ = & \frac{1}{2}\mathbb{E}\Big[\int_0^T (\langle R_t x_t^v, x_t^v\rangle - \langle R_t x_t, x_t\rangle + \langle N_t v_t, v_t\rangle - \langle N_t u_t, u_t\rangle)\mathrm{d}t + \langle Q x_T^v, x_T^v\rangle - \langle Q x_T, x_T\rangle\Big] \\ = & \frac{1}{2}\mathbb{E}\Big[\int_0^T (\langle R_t(x_t^v - x_t), x_t^v - x_t\rangle + \langle N_t(v_t - u_t), v_t - u_t\rangle + 2\langle R_t x_t, x_t^v - x_t\rangle \\ & + 2\langle N_t u_t, v_t - u_t\rangle)\mathrm{d}t + \langle Q(x_T^v - x_T), x_T^v - x_T\rangle + 2\langle Q x_T, x_T^v - x_T\rangle\Big]. \end{aligned}$$

Applying the Itô's formula to $\langle x_T^v - x_T, y_T\rangle$ and noticing the initial and terminal conditions, we get

$$\mathbb{E}\langle x_T^v - x_T, y_T\rangle = \mathbb{E}\int_0^T (\langle -R_t x_t, x_t^v - x_t\rangle + \langle C_t(v_t - u_t), y_t\rangle + \langle F_t(v_t - u_t), z_t\rangle)\mathrm{d}t.$$

In fact, the above result is due to the following virtue of the initial and terminal conditions:

$$\begin{aligned} & \mathbb{E}\int_0^T [\langle B_t(x_{t-\delta}^v - x_{t-\delta}), y_t\rangle - \langle \mathbb{E}^{\mathcal{F}_t}(B_{t+\delta}^\tau y_{t+\delta}), x_t^v - x_t\rangle]\mathrm{d}t \\ = & \mathbb{E}\int_0^T \langle B_t(x_{t-\delta}^v - x_{t-\delta}), y_t\rangle \mathrm{d}t - \mathbb{E}\int_\delta^{T+\delta} \langle B_t(x_{t-\delta}^v - x_{t-\delta}), y_t\rangle \mathrm{d}t \\ = & \mathbb{E}\int_0^\delta \langle B_t(x_{t-\delta}^v - x_{t-\delta}), y_t\rangle \mathrm{d}t - \mathbb{E}\int_T^{T+\delta} \langle B_t(x_{t-\delta}^v - x_{t-\delta}), y_t\rangle \mathrm{d}t \\ = & 0. \end{aligned}$$

Since R_t and Q are nonnegative, N_t is positive, we have

$$J(v(\cdot)) - J(u(\cdot)) \geq \mathbb{E}\int_0^T (\langle N_t u_t, v_t - u_t\rangle + \langle C_t(v_t - u_t), y_t\rangle + \langle F(v_t - u_t), z_t\rangle)\mathrm{d}t = 0.$$

So $u_t=-N_t^{-1}(C_t^\tau y_t+F_t^\tau z_t)$ is an optimal control.

The method to prove the uniqueness of the optimal control is classical, which can also be seen in [10]. For completeness and convenience of the readers, we give the details as follows.

We assume that $u^1(\cdot)$ and $u^2(\cdot)$ are both optimal controls, and the corresponding trajectories are $x^1(\cdot)$ and $x^2(\cdot)$. It is easy to know the trajectories corresponding to $\frac{u^1(\cdot)+u^2(\cdot)}{2}$ are $\frac{x^1(\cdot)+x^2(\cdot)}{2}$. Since N_t is positive, R_t and Q are nonnegative, we know that $J(u^1(\cdot))=J(u^2(\cdot))=\lambda\geq 0$, and from the parallelogram rule, we have

$$
\begin{aligned}
2\lambda&=J(u^1(\cdot))+J(u^2(\cdot))\\
&=2J\Big(\frac{u^1(\cdot)+u^2(\cdot)}{2}\Big)+\mathbb{E}\Big[\int_0^T\Big(\Big\langle R_t\frac{x_t^1-x_t^2}{2},\frac{x_t^1-x_t^2}{2}\Big\rangle+\Big\langle N_t\frac{u_t^1-u_t^2}{2},\frac{u_t^1-u_t^2}{2}\Big\rangle\Big)\mathrm{d}t\\
&\quad+\Big\langle Q\frac{x_T^1-x_T^2}{2},\frac{x_T^1-x_T^2}{2}\Big\rangle\Big]\\
&\geq 2\lambda+\mathbb{E}\int_0^T\Big\langle N_t\frac{u_t^1-u_t^2}{2},\frac{u_t^1-u_t^2}{2}\Big\rangle\mathrm{d}t.
\end{aligned}
$$

Because of N_t being positive, we have $u^1(\cdot)=u^2(\cdot)$.

4 Linear Quadratic Nonzero Sum Stochastic Differential Games with Delay

In this section, we study the linear-quadratic nonzero sum stochastic differential game problem with delay which is more complicated. For notational simplification, we assume the dimension of Brownian motion $d=1$ and only consider the case of two players, which is similar to that of n players.

The controlled system is

$$
\begin{cases}
\mathrm{d}x_t^v=(A_tx_t^v+A_t^1x_{t-\delta}^v+B_t^1v_t^1+B_t^2v_t^2+\phi_t)\mathrm{d}t\\
\qquad\qquad+(C_tx_t^v+C_t^1x_{t-\delta}^v+D_t^1v_t^1+D_t^2v_t^2+\psi_t)\mathrm{d}B_t, & t\in[0,T],\\
x_t=\varphi_t, & t\in[-\delta,0],
\end{cases}
\tag{4.1}
$$

where $\varphi\in C[-\delta,0]^n$, $\phi,\psi\in L_{\mathcal{F}}^2(0,T;\mathbb{R}^n)$; v_t^1 and v_t^2 are admissible control process; $A_t, A_t^1, B_t^1, B_t^2, C_t, C_t^1, D_t^1, D_t^2$ are $\mathcal{F}_t$-adapted matrix-valued bounded processes with appropriate dimensions.

We denote $J^1(v(\cdot)), J^2(v(\cdot)),\ v(\cdot)=(v^1(\cdot),v^2(\cdot))$, which are the cost functionals corresponding to the players 1 and 2:

$$
\begin{aligned}
J^1(v(\cdot))&=\frac{1}{2}\mathbb{E}\Big[\int_0^T(\langle R_t^1x_t^v,x_t^v\rangle+\langle N_t^1v_t^1,v_t^1\rangle)\mathrm{d}t+\langle Q^1x_T^v,x_T^v\rangle\Big],\\
J^2(v(\cdot))&=\frac{1}{2}\mathbb{E}\Big[\int_0^T(\langle R_t^2x_t^v,x_t^v\rangle+\langle N_t^2v_t^2,v_t^2\rangle)\mathrm{d}t+\langle Q^2x_T^v,x_T^v\rangle\Big],
\end{aligned}
$$

where Q^i is $\mathcal{F}_T$-measurable nonnegative symmetric bounded matrix; R_t^i is $\mathcal{F}_t$-adapted nonnegative symmetric bounded matrix-valued process; N_t^i is $\mathcal{F}_t$-adapted positive symmetric bounded matrix-valued process and the inverse $(N_t^i)^{-1}$ is also bounded $(i=1,2)$.

Problem 4.1 The problem is to look for admissible control $(u^1(\cdot),u^2(\cdot))$ which is called the Nash equilibrium point of the delayed game, such that

$$
J^1(u^1(\cdot),u^2(\cdot))\leq J^1(v^1(\cdot),u^2(\cdot)),\quad J^2(u^1(\cdot),u^2(\cdot))\leq J^2(u^1(\cdot),v^2(\cdot)).
$$

Theorem 4.1 *$(u^1(\cdot),u^2(\cdot))$ is a Nash equilibrium point for the above game Problem 4.1, if and only if $(u^1(\cdot),u^2(\cdot))$ has the form*

$$(u_t^1,u_t^2)=(-(N_t^1)^{-1}[(B_t^1)^\tau y_t^1+(D_t^1)^\tau z_t^1],-(N_t^2)^{-1}[(B_t^2)^\tau y_t^2+(D_t^2)^\tau z_t^2]),\quad t\in[0,T],$$

with $(x_t,y_t^1,y_t^2,z_t^1,z_t^2)$ being the solution of the following general FBSDE

$$\begin{cases}\mathrm{d}x_t=\{A_tx_t+A_t^1x_{t-\delta}-B_t^1(N_t^1)^{-1}[(B_t^1)^\tau y_t^1+(D_t^1)^\tau z_t^1]\\ \qquad -B_t^2(N_t^2)^{-1}[(B_t^2)^\tau y_t^2+(D_t^2)^\tau z_t^2]+\phi_t\}\mathrm{d}t\\ \qquad +\{C_tx_t+C_t^1x_{t-\delta}-D_t^1(N_t^1)^{-1}[(B_t^1)^\tau y_t^1+(D_t^1)^\tau z_t^1]\\ \qquad -D_t^2(N_t^2)^{-1}[(B_t^2)^\tau y_t^2+(D_t^2)^\tau z_t^2]+\psi_t\}\mathrm{d}B_t, & t\in[0,T],\\ -\mathrm{d}y_t^1=[A_t^\tau y_t^1C_t^\tau z_t^1+\mathbb{E}^{\mathcal{F}_t}[(A_{t+\delta}^1)^\tau y_{t+\delta}^1+(C_{t+\delta}^1)^\tau z_{t+\delta}^1]\\ \qquad +R_t^1x_t]\mathrm{d}t-z_t^1\mathrm{d}B_t, & t\in[0,T],\\ -\mathrm{d}y_t^2=[A_t^\tau y_t^2C_t^\tau z_t^2+\mathbb{E}^{\mathcal{F}_t}[(A_{t+\delta}^2)^\tau y_{t+\delta}^1+(C_{t+\delta}^2)^\tau z_{t+\delta}^2]\\ \qquad +R_t^2x_t]\mathrm{d}t-z_t^2\mathrm{d}B_t, & t\in[0,T],\\ x_t=\varphi_t, & t\in[-\delta,0],\\ y_T^1=Q^1x_T,\quad y_T^1=Q^2x_T,\quad y_t^1=y_t^2=0, & t\in(T,T+\delta],\\ z_t^1=z_t^2=0, & t\in[T,T+\delta].\end{cases}\tag{4.2}$$

Proof From the definition of Nash equilibrium point and Theorem 3.1, we know that the following three statements are equivalent:

(i) $(u^1(\cdot),u^2(\cdot))$ is a Nash equilibrium point for our game problem;

(ii) $u^i(\cdot)$ is an optimal control for the following control problem with delay $(i=1,2)$

$$\begin{cases}\mathrm{d}x_t=[A_tx_t+A_t^1x_{t-\delta}+B_t^iv_t^i+B_t^ju_t^j+\phi_t]\mathrm{d}t\\ \qquad +[C_tx_t+C_t^1x_{t-\delta}+D_t^iv_t^i+D_t^ju_t^j+\psi_t]\mathrm{d}B_t, & t\in[0,T],\\ x_t=\varphi_t, & t\in[-\delta,0],\end{cases}\tag{4.3}$$

with the cost functional

$$J^{u^j}(v^i(\cdot))=\frac{1}{2}\mathbb{E}\Big[\int_0^T(\langle R_t^ix_t,x_t\rangle+\langle N_t^iv_t^i,v_t^i\rangle)\mathrm{d}t+\langle Q^ix_T,x_T\rangle\Big],$$

where $j=1,2$, but $j\neq i$.

(iii) $u_t^i=-(N_t^i)^{-1}[(B_t^i)^\tau y_t^i+(D_t^i)^\tau z_t^i]$, $t\in[0,T]$, and (x_t,y_t^i,z_t^i) satisfies the following general FBSDE $(i=1,2)$:

$$\begin{cases}\mathrm{d}x_t=[A_tx_t+A_t^1x_{t-\delta}+B_t^iu_t^i+B_t^ju_t^j+\phi_t]\mathrm{d}t\\ \qquad +[C_tx_t+C_t^1x_{t-\delta}+D_t^iu_t^i+D_t^ju_t^j+\psi_t]\mathrm{d}B_t, & t\in[0,T],\\ -\mathrm{d}y_t^i=\{A_t^\tau y_t^i+C_t^\tau z_t^i+\mathbb{E}^{\mathcal{F}_s}[(A_{t+\delta}^1)^\tau y_{t+\delta}^i+(C_{t+\delta}^1)^\tau z_{t+\delta}^i]\\ \qquad +R_t^ix_t\}\mathrm{d}t-z_t^i\mathrm{d}B_t, & t\in[0,T],\\ x_t=\varphi_t, & t\in[-\delta,0],\\ y_T^i=Q^ix_T,\quad y_t^i=0, & t\in(T,T+\delta],\\ z_t^i=0, & t\in[T,T+\delta].\end{cases}\tag{4.4}$$

Combining cases for $i=1$ and $i=2$, we can rewrite (4.4) as (4.2). Our desired result is proved.

Remark 4.1 FBSDE (4.2) seems very complicated and it is not easy to get the existence and uniqueness of its solution. However, if it has a unique solution, our game problem with delay also has a unique Nash equilibrium point. For some particular cases, we can derive the following results.

Theorem 4.2 (a) *For the case that $D_t^1 \equiv D_t^2 \equiv 0$ in system* (4.1) *and for $i=1,2$, the matricial process $B_t^i(N_t^i)^{-1}(B_t^i)^\tau$ is independent of t satisfying*

$$B_t^i(N_t^i)^{-1}(B_t^i)^\tau S = SB_t^i(N_t^i)^{-1}(B_t^i)^\tau, \quad S = A_t^\tau, C_t^\tau, (A_t^1)^\tau, (C_t^1)^\tau.$$

Then

$$(u_t^1, u_t^2) = (-(N_t^1)^{-1}(B_t^1)^\tau y_t^1, -(N_t^2)^{-1}(B_t^2)^\tau y_t^2), \quad t \in [0,T]$$

is the unique Nash equilibrium point for the game Problem 4.1.

(b) *For the case that $B_t^1 \equiv B_t^2 \equiv 0$ in system* (4.1) *and for $i=1,2$, the matricial process $D_t^i(N_t^i)^{-1}(D_t^i)^\tau$ is independent of t satisfying*

$$D_t^i(N_t^i)^{-1}(D_t^i)^\tau S = SD_t^i(N_t^i)^{-1}(D_t^i)^\tau, \quad S = A_t^\tau, C_t^\tau, (A_t^1)^\tau, (C_t^1)^\tau.$$

Then

$$(u_t^1, u_t^2) = (-(N_t^1)^{-1}(D_t^1)^\tau z_t^1, -(N_t^2)^{-1}(D_t^2)^\tau z_t^2), \quad t \in [0,T]$$

is the unique Nash equilibrium point for the game Problem 4.1.

Proof We only prove (a). (b) can be proved by the similar method. Under the assumption of (a), FBSDE (4.2) becomes

$$\begin{cases} \mathrm{d}x_t = [A_t x_t + A_t^1 x_{t-\delta} - B_t^1(N_t^1)^{-1}(B_t^1)^\tau y_t^1 \\ \qquad -B_t^2(N_t^2)^{-1}\cdot(B_t^2)^\tau y_t^2 + \phi_t]\mathrm{d}t + (C_t x_t + C_t^1 x_{t-\delta} + \psi_t)\mathrm{d}B_t, & t \in [0,T], \\ -\mathrm{d}y_t^1 = \{A_t^\tau y_t^1 + C_t^\tau z_t^1 + \mathbb{E}^{\mathcal{F}_t}[(A_{t+\delta}^1)^\tau y_{t+\delta}^1 + (C_{t+\delta}^1)^\tau z_{t+\delta}^1] + R_t^1 x_t\}\mathrm{d}t \\ \qquad -z_t^1\mathrm{d}B_t, & t \in [0,T], \\ -\mathrm{d}y_t^2 = \{A_t^\tau y_t^2 + C_t^\tau z_t^2 + \mathbb{E}^{\mathcal{F}_t}[(A_{t+\delta}^2)^\tau y_{t+\delta}^2 + (C_{t+\delta}^2)^\tau z_{t+\delta}^2] + R_t^2 x_t\}\mathrm{d}t \\ \qquad -z_t^2\mathrm{d}B_t, & t \in [0,T], \\ x_t = \varphi_t, & t \in [-\delta, 0], \\ y_T^1 = Q^1 x_T, \quad y_T^1 = Q^2 x_T, \quad y_t^1 = y_t^2 = 0, & t \in (T, T+\delta], \\ z_t^1 = z_t^2 = 0, & t \in [T, T+\delta]. \end{cases} \tag{4.5}$$

We set

$$\begin{aligned} X_t &= x_t, \\ Y_t &= B_t^1(N_t^1)^{-1}(B_t^1)^\tau y_t^1 + B_t^2(N_t^2)^{-1}(B_t^2)^\tau y_t^2, \\ Z_t &= B_t^1(N_t^1)^{-1}(B_t^1)^\tau z_t^1 + B_t^2(N_t^2)^{-1}(B_t^2)^\tau z_t^2, \end{aligned}$$

where $(x_t, y_t^1, y_t^2, z_t^1, z_t^2)$ satisfies (4.5). By the commutation relation between matrix, we get

$$\begin{cases} \mathrm{d}X_t = [A_t X_t + A_t^1 X_{t-\delta} - Y_t + \phi_t]\mathrm{d}t + [C_t X_t + C_t^1 X_{t-\delta} + \psi_t]\mathrm{d}B_t, & t \in [0,T], \\ -\mathrm{d}Y_t = \{A_t^\tau Y_t + C_t^\tau Z_t + \mathbb{E}^{\mathcal{F}_t}[(A_{t+\delta}^1)^\tau Y_{t+\delta} + (C_{t+\delta}^1)^\tau Z_{t+\delta}] & \\ \qquad\qquad +[B_t^1 (N_t^1)^{-1} (B_t^1)^\tau R_t^1 + B_t^2 (N_t^2)^{-1} (B_t^2)^\tau R_t^2] X_t\}\mathrm{d}t - Z_t \mathrm{d}B_t, & t \in [0,T], \\ X_t = \varphi_t, & t \in [-\delta, 0], \\ Y_T = [B_t^1 (N_t^1)^{-1} (B_t^1)^\tau Q^1 + B_t^2 (N_t^2)^{-1} (B_t^2)^\tau Q^2] X_T, \quad Y_t = 0, & t \in (T, T+\delta], \\ Z_t = 0, & t \in [T, T+\delta]. \end{cases} \tag{4.6}$$

On the other hand, if (X_t, Y_t, Z_t) is the solution of the above equations, we can let $x_t = X_t$ in (4.5). Then we get $(y_t^1, y_t^2, z_t^1, z_t^2)$ from the following anticipated BSDE:

$$\begin{cases} -\mathrm{d}y_t^1 = \{A_t^\tau y_t^1 + C_t^\tau z_t^1 + \mathbb{E}^{\mathcal{F}_t}[(A_{t+\delta}^1)^\tau y_{t+\delta}^1 + (C_{t+\delta}^1)^\tau z_{t+\delta}^1] + R_t^1 x_t\}\mathrm{d}t & \\ \qquad\qquad -z_t^1 \mathrm{d}B_t, & t \in [0,T], \\ -\mathrm{d}y_t^2 = \{A_t^\tau y_t^2 + C_t^\tau z_t^2 + \mathbb{E}^{\mathcal{F}_t}[(A_{t+\delta}^2)^\tau y_{t+\delta}^2 + (C_{t+\delta}^2)^\tau z_{t+\delta}^2] + R_t^2 x_t\}\mathrm{d}t & \\ \qquad\qquad -z_t^2 \mathrm{d}B_t, & t \in [0,T], \\ y_T^1 = Q^1 x_T, \quad y_T^1 = Q^2 x_T, \quad y_t^1 = y_t^2 = 0, & t \in (T, T+\delta], \\ z_t^1 = z_t^2 = 0, & t \in [T, T+\delta]. \end{cases} \tag{4.7}$$

If we can prove that (Y_t, Z_t) is in the form

$$Y_t = B_t^1 (N_t^1)^{-1} (B_t^1)^\tau y_t^1 + B_t^2 (N_t^2)^{-1} (B_t^2)^\tau y_t^2,$$
$$Z_t = B_t^1 (N_t^1)^{-1} (B_t^1)^\tau z_t^1 + B_t^2 (N_t^2)^{-1} (B_t^2)^\tau z_t^2,$$

then we can assert that (4.5) has a solution. In order to prove that, we let

$$\overline{Y}_t = B_t^1 (N_t^1)^{-1} (B_t^1)^\tau y_t^1 + B_t^2 (N_t^2)^{-1} (B_t^2)^\tau y_t^2,$$
$$\overline{Z}_t = B_t^1 (N_t^1)^{-1} (B_t^1)^\tau z_t^1 + B_t^2 (N_t^2)^{-1} (B_t^2)^\tau z_t^2.$$

Hence, we obtain

$$\begin{cases} -d\overline{Y}_t = \{A_t^\tau \overline{Y}_t + C_t^\tau \overline{Z}_t + \mathbb{E}^{\mathcal{F}_t}[(A_{t+\delta}^1)^\tau \overline{Y}_{t+\delta} + (C_{t+\delta}^1)^\tau \overline{Z}_{t+\delta}] & \\ \qquad\qquad +[B_t^1 (N_t^1)^{-1} (B_t^1)^\tau R_t^1 + B_t^2 (N_t^2)^{-1} (B_t^2)^\tau R_t^2] X_t\}\mathrm{d}t - \overline{Z}_t \mathrm{d}B_t, & t \in [0,T], \\ \overline{Y}_T = [B_t^1 (N_t^1)^{-1} (B_t^1)^\tau Q^1 + B_t^2 (N_t^2)^{-1} (B_t^2)^\tau Q^2] X_T, \quad \overline{Y}_t = 0, & t \in (T, T+\delta], \\ \overline{Z}_t = 0, & t \in [T, T+\delta]. \end{cases} \tag{4.8}$$

As a result of the uniqueness of the solution of the anticipated BSDE (see [8]), we have

$$Y_t = \overline{Y}_t = B_t^1 (N_t^1)^{-1} (B_t^1)^\tau y_t^1 + B_t^2 (N_t^2)^{-1} (B_t^2)^\tau y_t^2,$$
$$Z_t = \overline{Z}_t = B_t^1 (N_t^1)^{-1} (B_t^1)^\tau z_t^1 + B_t^2 (N_t^2)^{-1} (B_t^2)^\tau z_t^2.$$

This implies that the existence and uniqueness of (4.5) is equivalent to the existence and uniqueness of (4.6). It is easy to check that FBSDE (4.6) satisfies (H2.1) and (H2.2)$'$. From Remark 2.2 we know that the general FBSDE (4.6) has a unique solution. So the general FBSDE (4.5) has a unique solution. Combining Theorem 4.1, we prove that (a) holds.

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