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On JB-Rings

Huanyin CHEN*

Abstract A ring R is a QB-ring provided that $aR + bR = R$ with $a, b \in R$ implies that there exists a $y \in R$ such that $a + by \in R_q^{-1}$. It is said that a ring R is a JB-ring provided that $R/J(R)$ is a QB-ring, where $J(R)$ is the Jacobson radical of R. In this paper, various necessary and sufficient conditions, under which a ring is a JB-ring, are established. It is proved that JB-rings can be characterized by pseudo-similarity. Furthermore, the author proves that R is a JB-ring iff so is $R/J(R)^2$.

Keywords JB-Rings, Exchange rings, Subdirect product
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1 Introduction

A ring R is a B-ring (i.e., ring having stable range one) provided that $aR + bR = R$ with $a, b \in R$ implies that there exists a $y \in R$ such that $a + by \in U(R)$, where $U(R)$ is the set of all units in R. It is well known that every strongly π-regular ring is a B-ring. Many authors have studied B-rings from different view points such as [11] and [13]. So as to study directly infinite rings, Ara et al. discovered a new class of rings, the QB-rings. We say that $x, y \in R$ are centrally orthogonal, in symbols $x \perp y$, if $xRy = 0$ and $yRx = 0$. A ring R is said to be a QB-ring if $aR + bR = R$ with $a, b \in R$ implies that $a + by \in R_q^{-1}$ for a $y \in R$, where

$$R_q^{-1} = \{u \in R \mid \exists\, a, b \in R \text{ such that } (1 - ua) \perp (1 - bu)\}.$$

The class of QB-rings is very large. For example, all exchange rings satisfying related comparability are QB-rings (cf. [2, Example 8.8]). Let $\mathbb{F}$ be a field, and let $\mathbb{B}(\mathbb{F})$ denote the algebra of all row- and column-finite matrices over $\mathbb{F}$. Then $\mathbb{B}(\mathbb{F})$ is a QB-ring (cf. [2, Example 8.8]). Very recently, Ara proved that every purely infinite simple ring is an exchange QB-ring (see [1, Theorem 1.1]).

We say that a ring R is a JB-ring provided that $R/J(R)$ is a QB-ring, where $J(R)$ is the Jacobson radical of R. Clearly, every QB-ring is a JB-ring, but the converse is not true (see Section 5). The examples below point out that the class of JB-ring is much larger than the class of QB-ring. We say that R is a local ring provided that $R/J(R)$ is a division. A ring R is a left perfect ring iff $R/J(R)$ is artinian and $J(R)$ is left T-nilpotent, that is, for any $a_1, a_2, \cdots \in J(R)$, $a_1a_2 \cdots a_n = 0$ for some n. We say that R is a semilocal ring provided that $R/J(R)$ is an artinian ring. A ring R is a semiperfect ring iff $R/J(R)$ is an artinian ring and

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*Department of Mathematics, Hunan Normal University, Changsha 410006, China.
E-mail: chyzxl@hunnu.edu.cn

idempotents lift modulo $J(R)$. We say that R is a semiregular ring provided that $R/J(R)$ is a regular ring and idempotents lift modulo $J(R)$. Also we see that a ring R is a P-exchange ring iff $R/J(R)$ is a regular ring and $J(R)$ is T-nilpotent. From these, we see that the difference between R and $R/J(R)$ is very large from ring point of view. Furthermore, we note that the difference between QB-ring and JB-ring is very large. For example, R/I is a JB-ring iff so is R/I^2 for any ideal I of R. But we can construct an ideal I of a ring R such that R/I is a QB-ring, while R/I^2 is not (see Corollary 5.5).

We establish, in this paper, various necessary and sufficient conditions under which a ring is a JB-ring. These results show that JB-rings behave like QB-ring in several aspects, though one cannot obtain these results by applying QB properties to the residue ring $R/J(R)$. A ring R is an exchange ring if for every right R-module A and any two decompositions $A = M \oplus N = \bigoplus_{i\in I} A_i$, where $M_R \cong R$ and the index set I is finite, there exist submodules $A_i' \subseteq A_i$ such that $A = M \oplus \left(\bigoplus_{i\in I} A_i'\right)$ (cf. [1, 4, 12]). We will prove that exchange JB-ring can be characterized by pseudo-similarity. As an application, we give a new example of QB-ring (see Example 5.8).

Throughout, all rings are associative with identity. $M_n(R)$ denotes the ring of all $n \times n$ matrices and $\mathbb{N}$ denotes the set of all natural numbers. An element $a \in R$ is regular if there exists an $x \in R$ such that $a = axa$.

2 JB-Rings

Recall that $x, y \in R$ are J-orthogonal, in symbols $x\sharp y$, if $xRy, yRx \subseteq J(R)$. Let $R_J^{-1} = \{u \in R \mid \exists\, a, b \in R \text{ such that } (1-ua)\sharp(1-bu)\}$. The main purpose of this section is to give some elementary properties of JB-rings.

Theorem 2.1 *Let R be a ring. Then the following are equivalent:*

(1) *R is a JB-ring;*

(2) *$aR + bR = R$ with $a, b \in R$ implies that there exists a $y \in R$ such that $a + by \in R_J^{-1}$;*

(3) *$Ra + Rb = R$ with $a, b \in R$ implies that there exists a $z \in R$ such that $a + zb \in R_J^{-1}$.*

Proof (1) $\Rightarrow$ (2) Given $aR + bR = R$ with $a, b \in R$, we have $\overline{a}\overline{R} + \overline{b}\,\overline{R} = \overline{R}$. Since $R/J(R)$ is a QB-ring, we can find a $y \in R$ such that $\overline{a} + \overline{b}\,\overline{y} \in \overline{R}_q^{-1}$. Thus, $a + by \in R_J^{-1}$, as required.

(2) $\Rightarrow$ (1) Given $\overline{a}\overline{R} + \overline{b}\,\overline{R} = \overline{R}$, we see that there are $x, y \in R$ and $r \in J(R)$ such that $ax + by + r = 1$. As $r \in J(R)$, $1 - r \in U(R)$. Hence, $ax(1-r)^{-1} + by(1-r)^{-1} = 1$, and then $aR + bR = R$. By the assumption, we can find a $z \in R$ such that $a + bz \in R_J^{-1}$. This implies that $\overline{a} + \overline{b}\,\overline{y} \in \overline{R}_q^{-1}$. This infers that $R/J(R)$ is a QB-ring.

(1) $\Leftrightarrow$ (3) is proved by symmetry.

Corollary 2.1 *Let R be a ring. Then the following are equivalent:*

(1) *R is a JB-ring;*

(2) *$aR + bR = dR$ with $a, b, d \in R$ implies that there exist $y \in R$, $u \in R_J^{-1}$ such that $a + by = du$;*

(3) *$Ra + Rb = Rd$ with $a, b \in R$ implies that there exist $z \in R$, $u \in R_J^{-1}$ such that $a + zb = ud$.*

Proof (1) $\Rightarrow$ (2) Given $aR + bR = dR$ with $a, b, d \in R$, we see that there are $x, y, s, t \in R$ such that $ax + by = d, a = ds$ and $b = dt$. Thus, $dsx + dty = d$. As $sx + ty + (1 - sx - ty) = 1$, there exists $z \in R$ such that $u := s + tyz + (1 - sx - ty)z \in R_J^{-1}$. As a result, we deduce that $du = ds + dtyz = a + byz$, as required.

(2) $\Rightarrow$ (1) is trivial by Theorem 2.1.

(1) $\Leftrightarrow$ (3) is proved by symmetry.

As a consequence of Corollary 2.1, we deduce that if R is a JB-ring then $aR = bR$ with $a, b \in R$ implies that $a = bu$ for a $u \in R_J^{-1}$.

Lemma 2.1 *Let R be a JB-ring. If $x = xyx$, then there exists a $u \in R_J^{-1}$ such that $x = xyu = uyx$.*

Proof Assume that $x = xyx$. Then $x = xzx, z = zxz$, where $z = yxy$. Since $xz+(1-xz) = 1$, it follows from Theorem 2.1 that there exists a $t \in R$ such that $v := x + (1 - xz)t \in R_J^{-1}$. Hence, $z = zvz$. Let $u = (1 - xz - vz)v(1 - zx - zv)$. One easily checks that $(1 - xz - vz)^2 = 1 = (1 - zx - zv)^2$. Hence $u \in R_J^{-1}$. It is easy to see that

$$xzu = -xzv(1 - zx - zv) = -xzv + xzx + xzv = xzx = x,$$

$$uzx = (1 - xz - vz)v(-zvzx) = -(1 - xz - vz)vzx = -vzx + xzx + vzx = xzx = x.$$

Thus, $x = xzu = x(yxy)u = xyu$ and $x = uzx = u(yxy)x = uyx$.

Let R be a ring and $a, b \in R$. We say that a and b are pseudo-similar, denoted by $a \overset{\sim}{-} b$, if there exist $x, y \in R$ such that $a = xby, b = yax, x = xyx$ and $y = yxy$ (cf. [6]).

Theorem 2.2 *If R is a JB-ring. Then $a \overset{\sim}{-} b$ with $a, b \in R$ implies that there exists a $u \in R_J^{-1}$ such that $au = ub$.*

Proof Suppose that $a \overset{\sim}{-} b$ with $a, b \in R$. Then we have $x, y \in R$ such that $a = xby, b = yax, x = xyx$ and $y = yxy$. By virtue of Lemma 2.1, there exists a $u \in R_J^{-1}$ such that $x = xyu = uyx$. One easily checks that $ax = a(xyu) = (xby)xyu = (xby)u = au$ and $xb = (uyx)b = (uyx)(yax) = (uyxy)ax = u(yax) = ub$. In addition, $ax = (xby)x = x(yax)yx = x(yax) = xb$. Thus, $au = xb = ub$, as asserted.

Corollary 2.2 *Let R be a JB-ring. Then for any idempotents $e, f \in R$, $eR \cong fR$ implies that there exists a $u \in R_J^{-1}$ such that $eu = uf$.*

Proof For any idempotents $e, f \in R$, $eR \cong fR$ implies that there exist $a \in eRf$ and $b \in fRe$ such that $e = ab$ and $f = ba$. Hence, $e = afb, f = bea, a = aba, b = bab$. That is, $e \overset{\sim}{-} f$. By virtue of Theorem 2.2, we have a $u \in R_J^{-1}$ such that $eu = uf$.

3 Exchange Rings

As is well known, a ring R is an exchange ring if and only if for any $a \in R$, there exists an idempotent $e \in aR$ such that $1 - e \in (1 - a)R$. In this section, we investigate necessary and sufficient conditions under which an exchange ring R is a JB-ring.

Theorem 3.1 *Let R be an exchange ring. Then the following are equivalent:*

(1) *R is a JB-ring;*

(2) *Every regular element is a product of an idempotent in R and an element in R_J^{-1}.*

Proof (1) $\Rightarrow$ (2) Given any regular $x \in R$, we see that there exists a $y \in R$ such that $x = xyx$. From $xy + (1 - xy) = 1$, we have $z \in R$ such that $x + (1 - xy)z = u \in R_J^{-1}$ such that $x = xy(x + (1 - xy)z) = (xy)u$. Let $e = xy$. Then x is the product of the idempotent $e \in R$ and $u \in R_J^{-1}$.

(2) $\Rightarrow$ (1) Given $ax + b = 1$ in R, by [12, Proposition 28.5] we see that there exists an idempotent $e \in bR$ such that $1 - e \in (1 - b)R$. Thus, $axt + e = 1$ for some $t \in R$. Hence, $(1 - e)axt + e = 1$, and so $(1 - e)axt(1 - e)a = (1 - e)a$. By virtue of Lemma 2.1, there exist $f = f^2 \in R$ and $u \in R_J^{-1}$ such that $(1 - e)a = fu$. As a result, $fuxt + e = 1$, and so $fuxt + e(1 - f) = 1 - f$. This implies that $f + e(1 - f) = 1 - fuxt(1 - f) \in U(R)$. As a result, $(1 - e)a + e(1 - f)u = (1 - fuxt(1 - f))u \in R_J^{-1}$. Thus, $a + e((1 - f)u - a) \in R_J^{-1}$. Clearly, there exists a $y \in R$ such that $e = by$. Therefore $a + by((1 - f)u - a) \in R_J^{-1}$. According to Theorem 2.1, R is a JB-ring.

Corollary 3.1 *Let R be an exchange ring. Then the following are equivalent:*

(1) *R is a JB-ring;*

(2) *For any regular $x \in R$, there exists $u \in R_J^{-1}$ such that $x = xux$;*

(3) *For any regular $x \in R$, there exists $u \in R_J^{-1}$ such that ux is an idempotent.*

Proof (1) $\Rightarrow$ (2) For any regular $x \in R$, there exists a $y \in R$ such that $x = xyx$ and $y = yxy$. Since $yx + (1 - yx) = 1$, there exists a $z \in R$ such that $u := y + (1 - yx)z \in R_J^{-1}$; hence, $x = x(y + (1 - yx)z)x = xux$.

(2) $\Rightarrow$ (3) is obvious.

(3) $\Rightarrow$ (1) For any regular $x \in R$, there exists a $y \in R$ such that $x = xyx$ and $y = yxy$. By the hypothesis, we have a $u \in R_J^{-1}$ such that ux is an idempotent. From $xy + (1 - xy) = 1$, we get $uxy + u(1 - xy) = u$. Let $e = ux$. Then $e(y + u(1 - xy)) + (1 - e)u(1 - xy) = u$. Clearly, $(1 - e)u(1 - xy) = (1 - e)u$. As $u \in R_J^{-1}$, we see that $\overline{u} \in (R/J(R))_q^{-1}$. In view of [2, Proposition 2.2], there is a $\overline{v} \in R/J(R)$ such that $\overline{u} = \overline{uvu}$. Thus, $\overline{(1 - e)uv(1 - e)u} = \overline{(1 - e)uv(1 - ux)u} = \overline{(1 - e)(u - uxu)} = (1 - e)u$. Let $g = (1 - e)uv(1 - e)$. Then $(1 - e)u = gu + r$ for some $r \in J(R)$. Hence, $(e + (1 - e)u(1 - xy)v(1 - e))u = (e + g)u = eu + gu = u + r'$ for a $r' \in J(R)$. This implies that

$$
\begin{aligned}
&u(x + (1 - xy)v(1 - e)(1 + eu(1 - xy)v(1 - e)))(1 - eu(1 - xy)v(1 - e))u \\
&= (e + u(1 - xy)v(1 - e)(1 + eu(1 - xy)v(1 - e)))(1 - eu(1 - xy)v(1 - e))u \\
&= (e(1 - eu(1 - xy)v(1 - e)) + u(1 - xy)v(1 - e))u \\
&= u + r'.
\end{aligned}
$$

Clearly, we see that

$$
\overline{(1 - eu(1 - xy)v(1 - e))u} = \overline{(1 - eu(1 - xy)v(1 - e))(u + r')} \in (R/J(R))_q^{-1},
$$

and then $\overline{x+(1-xy)z} \in (R/J(R))_q^{-1}$ for a $z \in R$. This implies that $w := x+(1-xy)z \in R_J^{-1}$. Therefore $x = xyx = xy(= x+(1-xy)z) = xyw$. According to Theorem 3.1, we complete the proof.

Lemma 3.1 *Let R be a ring and $x \in R$. Then the following are equivalent:*

(1) *There exists a $v \in R_J^{-1}$ such that $x = xvx$;*

(2) $x = xyx = xyu$*, where* $y \in R, u \in R_J^{-1}$;

(3) $x = xyx = uyx$*, where* $y \in R, u \in R_J^{-1}$.

Proof (1) $\Rightarrow$ (2) Since $xv + (1-xv) = 1$ with $v \in R_J^{-1}$, we have that $\overline{xv+(1-xv)} = \overline{1}$ with $\overline{v} \in (R/J(R))_q^{-1}$. In view of [2, Lemma 4.4], we see that $\overline{x+(1-xy)z} \in (R/J(R))_q^{-1}$ for a $z \in R$. Hence $u := x+(1-xy)z \in R_J^{-1}$. Furthermore, we get $x = xy(x+(1-xy)z) = xyu$, as required.

(2) $\Rightarrow$ (1) Suppose that $x = xyx = xyu$, where $y \in R$, $u \in R_J^{-1}$. Let $e = xy$. Then $e \in R$ is an idempotent. Since $xy + (1-xy) = 1$, we have that $euy + (1-xy) = 1$, and so $euy(1-e)+(1-xy)(1-e) = 1-e$. This implies that $e+(1-xy)(1-e) = 1-euy(1-e) \in U(R)$. Therefore we get $x+(1-xy)(1-e) = (1-euy(1-e))u \in R_J^{-1}$. Applying [2, Lemma 4.4] to $R/J(R)$, we can find a $z \in R$ such that $w := y+z(1-xy) \in R_J^{-1}$. Thus, $x = x(y+z(1-xy))x = xwx$.

(1) $\Leftrightarrow$ (3) is symmetric.

Theorem 3.2 *Let R be an exchange ring. Then the following are equivalent:*

(1) *R is a JB-ring;*

(2) *Whenever $x = xyx$, there exists a $u \in R_J^{-1}$ such that $x = xyu$;*

(3) *Whenever $x = xyx$, there exists a $u \in R_J^{-1}$ such that $x = uyx$.*

Proof (1) $\Rightarrow$ (2) Given $x = xyx$, by Corollary 3.1 we have $x = xvx$ for a $v \in R_J^{-1}$. In view of Lemma 3.1, there exists a $u \in R_J^{-1}$ such that $x = xvx = xvu$. Let $e = xv$. Then $e = e^2 \in R$. Since $xy + (1-xy) = 1$, we have that $euy + (1-xy) = 1$; hence, $euy(1-e)+(1-xy)(1-e) = 1-e$. This implies that $e+(1-xy)(1-e) = 1-euy(1-e) \in U(R)$, and so $x+(1-xy)(1-e) = (1-euy(1-e))u \in R_J^{-1}$. Let $w = (1-euy(1-e))u$. Then $x = xyx = xy(x+(1-xy)(1-e)) = xyw$.

(2) $\Rightarrow$ (1) is obvious from Theorem 3.1.

(1) $\Leftrightarrow$ (3) is proved in the same manner.

Corollary 3.2 *Let R be an exchange ring. Then the following are equivalent:*

(1) *R is a JB-ring;*

(2) *$a \overline{\sim} b$ with $a, b \in R$ implies that there exists a $u \in R_J^{-1}$ such that $au = ub$;*

(3) *For any idempotents $e, f \in R$, $eR \cong fR$ implies that there exists a $u \in R_J^{-1}$ such that $eu = uf$.*

Proof (1) $\Rightarrow$ (2) is clear by Theorem 2.2.

(2) $\Rightarrow$ (3) For any idempotents $e, f \in R$, $eR \cong fR$ implies that $e \overline{\sim} f$. Hence, we have a $u \in R_J^{-1}$ such that $eu = uf$.

(3) $\Rightarrow$ (1) Given $x = xyx$, we have $x = xzx$ and $z = zxz$, where $z = yxy$. Clearly, $\varphi : xR = xzR \cong zxR$ is given by $\varphi(xr) = zxr$ for any $r \in R$. By the hypothesis, there exists a $u \in R_J^{-1}$ such that $zxu = uxz$. Thus, we have $a, b \in R$ such that $(1-au)\sharp(1-ub)$. Let $v = a + b - aub$. Then $1 - uv = (1-ua)(1-ub)$ and $1 - vu = (1-au)(1-bu)$. This implies that $(1-uv)\sharp(1-vu)$. Let $s = z + u(1-xz)$ and $t = x + (1-xz)v(1-zx)$. Then

$$\begin{aligned} 1 - st &= 1 - zx - u(1-xz)v(1-zx) \\ &= 1 - zx - (1-zx)uv(1-zx) \\ &= (1-zx)(1-uv)(1-zx). \end{aligned}$$

Likewise,

$$\begin{aligned} 1 - ts &= 1 - xz - xu(1-xz) - (1-xz)v(1-zx)u(1-xz) \\ &= 1 - xz - x(1-zx)u - (1-xz)vu(1-xz) \\ &= (1-xz)(1-vu)(1-xz). \end{aligned}$$

This implies that $(1-st)\sharp(1-ts)$. Hence, $s \in R_J^{-1}$. It is easy to see that $x = xzx = xz(x + (1-xz)v(1-zx)) = xzt = xyt$. According to Theorem 3.2, R is a JB-ring.

4 Extensions

A ring R is the subdirect product of rings A_i $(i \in I)$ provided that there exist ring epimorphisms $\phi_i : R \twoheadrightarrow A_i$ such that $\bigcap\limits_{i \in I} \operatorname{Ker}\phi_i = 0$. Let $\mathrm{cl}(R_J^{-1}) = \{a \in R \mid Ra + Rb = R \Longrightarrow$ there exists a $z \in R$ such that $a + zb \in R_J^{-1}\}$.

Lemma 4.1 *Let R be a ring, and let $u \in R$. Then the following are equivalent:*

(1) $u \in R_J^{-1}$;

(2) *There exists a $v \in R$ such that $(1-uv)\sharp(1-vu)$ and $u \equiv uvu, v \equiv vuv \pmod{J(R)}$.*

Proof (2) $\Rightarrow$ (1) is trivial.

(1) $\Rightarrow$ (2) If $u \in R_J^{-1}$, then there are $a, b \in R$ such that $(1-au)\sharp(1-ub)$. Let $w = a + b - aub$. Then $1 - uw = (1-ua)(1-ub)$ and $1 - wu = (1-au)(1-bu)$. This implies that $(1-uw)\sharp(1-wu)$. In addition, $u \equiv uwu \pmod{J(R)}$. Let $v = wuw$. Then $(1-uv)\sharp(1-vu)$ and $u \equiv uvu, v \equiv vuv \pmod{J(R)}$, as required.

Let I be an ideal of a ring R, and let $\pi : R \to R/I$ be the natural map. Set $Q(I) = \{x \in I \mid \pi(x) \in J(R/I)\}$.

Lemma 4.2 *Let I be an ideal of a JB-ring R. Then*

$$(Q(I) + R_J^{-1})/I = (R/I)_J^{-1}.$$

Proof Given any $u \in R_J^{-1}$, $r \in Q(I)$, we can find a $v \in R$ such that $(1-uv)R(1-vu), (1-vu)R(1-uv) \subseteq J(R)$. Thus, $(\overline{1} - \overline{(u+r)}\,\overline{v})\overline{R}(\overline{1} - \overline{v}\,\overline{(u+r)}) \subseteq J(R/I)$. Likewise, $(\overline{1} - \overline{v}\,\overline{(u+r)})\overline{R}(\overline{1} - \overline{(u+r)}\,\overline{v}) \subseteq J(R/I)$. Therefore we get $\overline{u+r} \in (R/I)_J^{-1}$, and so $(Q(I) + R_J^{-1})/I \subseteq (R/I)_J^{-1}$.

Let $\pi(a)\in(R/I)_J^{-1}$. By virtue of Lemma 4.1, we have a $\pi(b)\in R/I$ such that

$$\begin{aligned}&\pi(a)\equiv\pi(a)\pi(b)\pi(a),\quad \pi(b)\equiv\pi(b)\pi(a)\pi(b)\pmod{J(R/I)},\\&(1-\pi(a)\pi(b))\pi(R)(1-\pi(b)\pi(a)),\ (1-\pi(b)\pi(a))\pi(R)(1-\pi(a)\pi(b))\subseteq J(R/I).\end{aligned}$$

Since $ab+(1-ab)=1$, by the hypothesis, there exists a $y\in R$ such that $v=b+y(1-ab)\in R_J^{-1}$. Thus, we have a $v\in R$ such that

$$\begin{aligned}&u\equiv uvu,\quad v\equiv vuv\pmod{J(R)},\\&(1-uv)R(1-vu),\ (1-vu)R(1-uv)\subseteq J(R).\end{aligned}$$

Choose $w=u+a(1-vu)+(1-uv)a$. Then $1-wv\equiv(1-uv)(1-av),\ 1-vw\equiv(1-va)(1-vu)\pmod{J(R)}$. This implies that

$$(1-wv)R(1-vw),\ (1-vw)R(1-wv)\subseteq J(R).$$

That is, $w\in R_J^{-1}$. As $(1-\pi(b)\pi(a))\pi(R)(1-\pi(a)\pi(b))\subseteq J(R/I)$, we get

$$\begin{aligned}\pi(v)\pi(a)\pi(v)&=\pi(b+y(1-ab))\pi(a)\pi(b+y(1-ab))\\&\equiv\pi(ba)\pi(b+y(1-ab))\\&\equiv\pi(b+bay(1-ab))\\&\equiv\pi(b+y(1-ab))\\&=\pi(v).\end{aligned}$$

It follows from $(1-uv)a(1-vu)\in J(R)$ that

$$\pi(a)\equiv\pi(u)+\pi(a)(1-\pi(v)\pi(u))+(1-\pi(u)\pi(v))\pi(a)=\pi(w)\pmod{J(R/I)}.$$

Therefore, we can find an $\overline{r}\in J(R/I)$ such that $\pi(a)=\pi(w+r)$. This implies that $(Q(I)+R_J^{-1})/I\supseteq(R/I)_J^{-1}$, as required.

Theorem 4.1 *Let I be an ideal of a ring R. Then R is a JB-ring if and only if the following hold:*

(1) *R/I is a JB-ring,*
(2) $(Q(I)+R_J^{-1})/I=(R/I)_J^{-1}$,
(3) $Q(I)+R_J^{-1}\subseteq \mathrm{cl}(R_J^{-1})$.

Proof Assume that R is a JB-ring. It follows from $(I+J(R))/I\subseteq J(R/I)$ that R/I is a JB-ring. Obviously, $Q(I)+R_J^{-1}\subseteq\mathrm{cl}(R_J^{-1})$. By Lemma 4.2, $(Q(I)+R_J^{-1})/I+J(R/I)=(R/I)_J^{-1}$.

Conversely, assume that (1)–(3) hold. Let $\pi:R\to R/I$ be the quotient map. Suppose that $ax+b=1$ in R. Then $\pi(a)\pi(x)+\pi(b)=\pi(1)$ in R/I, and so we have a $y\in R$ such that $\pi(a)+\pi(b)\pi(y)\in(R/I)_J^{-1}$. As $(Q(I)+R_J^{-1})/I=(R/I)_J^{-1}$, there exist $w\in R_J^{-1}$ and $r\in R$ such that $\pi(a)+\pi(b)\pi(y)=\pi(w+r)$ and $\pi(r)\in J(R/I)$. Hence $a+by-w-r\in I$, and then $a+by\in Q(I)+R_J^{-1}$. Since $(a+by)x+b(1-yx)=1$, we can find $z\in R$ such that $a+b(y+(1-yx)z)=a+by+b(1-yx)z\in R_J^{-1}$, as required.

Recall that an ideal I of a ring R is a B-ideal provided that $aR+bR=R$ with $a\in 1+I, b\in R$ implies that there exists a $y\in R$ such that $a+by\in U(R)$. As is well known, I is a B-ideal of a ring R if and only if $Ra+Rb=R$ with $a\in 1+I, b\in R$ implies that there exists a $z\in R$ such that $a+zb\in U(R)$.

Corollary 4.1 *Let I be a B-ideal of a ring R. Then R is a JB-ring if and only if the following hold:*

(1) *R/I is a JB-ring,*

(2) *$(Q(I)+R_J^{-1})/I=(R/I)_J^{-1}$.*

Proof One direction is obvious.

Conversely, assume that (1)–(2) hold. Take $u\in R_J^{-1}$ and $t\in Q(I)$ and assume that $x(u-t)+b=1$ with $x,b\in R$. In view of Lemma 4.1, there exists a $v\in R$ such that $(1-uv)\sharp(1-vu)$ and $u\equiv uvu,\ v\equiv vuv \pmod{J(R)}$. It is easy to see that

$$1=xu(1-vt)-x(1-uv)t+b.$$

As $t\in Q(I)$, $\overline{vt}\in J(R/I)$; hence, $\overline{1-vt}\in U(R/I)$. Since I is a B-ideal, it is easy to find a $w\in U(R)$ such that $1-vt\equiv w \pmod{I}$. That is, $1-vt=w+r$ for an $r\in I$. As a result, we deduce that $xu(w+r)+b-x(1-uv)t=1$. This implies that

$$wxu(1+rw^{-1})+w(b-x(1-uv)t)w^{-1}=1.$$

Since $1+rw^{-1}\in 1+I$, we have a $z\in R$ such that

$$1+rw^{-1}+zw(b-x(1-uv)t)w^{-1}\in U(R).$$

Then $w+r+zw(b-x(1-uv)t)\in U(R)$. That is,

$$w_1:=1-vt+zw(b-x(1-uv)t)\in U(R).$$

Clearly,

$$\begin{aligned} uw_1 &= u-uvt+uzw(b-x(1-uv)t)\\ &\equiv u-t+uzwb+(1-uzwx(1-uv))(1-uv)t \pmod{J(R)}. \end{aligned}$$

As $(uzwx(1-uv))^2\in J(R)$, we see that $1-(uzwx(1-uv))^2\in U(R)$, and so $1-uzwx(1-uv)\in U(R)$. Let $w_2=(1-uzwx(1-uv))^{-1}$. Then $w_2\equiv 1+uzwx(1-uv) \pmod{J(R)}$, and so $w_2u\equiv u \pmod{J(R)}$. This implies that

$$w_2(u-t+uzwb)w_1^{-1}\equiv w_2u-(1-uv)tw_1^{-1}\equiv u-(1-uv)tw_1^{-1} \pmod{J(R)}.$$

Thus, we have some $s\in J(R)$ such that $w_2(u-t+uzwb)w_1^{-1}=u-(1-uv)tw_1^{-1}+s$. Let $u'=u-(1-uv)tw_1^{-1}+s$. Then $1-u'v=(1-uv)(1+tw_1^{-1})-u'r\equiv(1-uv)(1+tw_1^{-1}) \pmod{J(R)}$ and $1-vu'=1-vu-(v-vuv)tw_1^{-1}-vs\equiv 1-vu \pmod{J(R)}$. As $(1-uv)\sharp(1-vu)$, we deduce that $(1-u'v)\sharp(1-vu')$, whence, $u'\in R_J^{-1}$. Therefore $u-t+uzwb\in R_J^{-1}$. According to Theorem 4.1, R is a JB-ring.

Corollary 4.2 *Let I be a B-ideal of a ring R, and S be a JB-subring of R containing* 1. *If $R = I + S$ and $S_J^{-1} \subseteq R_J^{-1}$, then R is a JB-ring.*

Proof Clearly, $R/I = (I+S)/I \cong S/(I \cap S)$. As S is a JB-ring, so is $S/(I \cap S)$ by Theorem 4.1. Thus, R/I is a JB-ring. Obviously, $(Q(I) + R_J^{-1})/I \subseteq (R/I)_J^{-1}$. Given any $\overline{u} \in (R/I)_J^{-1}$, we see that there exist $a \in I$ and $b \in S$ such that $u = a + b$. Thus, $u + I = b + I$. It is easy to verify that $\overline{b} \in (S/(I \cap S))_J^{-1}$. In view of Theorem 4.1, $\overline{b} \in (Q(I \cap S) + S_J^{-1})/(I \cap S)$. Thus, we have a $c \in S_J^{-1}$ such that $(b - c) + I \cap S \subseteq J(S/(I \cap S))$. This implies that $b - c \in I + J(R/I)$. As $S_J^{-1} \subseteq R_J^{-1}$, we see that $c \in R_J^{-1}$, and then $(R/I)_J^{-1} \subseteq (Q(I) + R_J^{-1})/I$. By Corollary 4.1, we complete the proof.

5 Related JB-Rings

Lemma 5.1 *Let R be a JB-ring, and let $e \in R$ be an idempotent. Then eRe is a JB-ring.*

Proof Given $(eae)(exe)+ebe = e$ with $a, x, b \in R$, we have $(eae+1-e)(exe+1-e)+ebe = 1$. Since R is a JB-ring, we have a $y \in R$ such that $eae + 1 - e + ebey \in R_J^{-1}$. By Lemma 4.1, there is a $u \in R$ such that $R(1 - (eae + 1 - e + ebey)u)R(1 - u(eae + 1 - e + ebey))R \subseteq J(R)$. Hence

$$R((1-e)ue)(eRe)(e - (eue)(eae + (ebe)(eye))R,$$
$$R(e - (eae + ebey)ue)(eRe)(e - (eue)(eae + (ebe)(eye))R \subseteq J(R).$$

As a result, we deduce that

$$\begin{aligned} &(eRe)(e - (eae + (ebe)(eye))(eue))(eRe)(e - (eue)(eae + (ebe)(eye)))(eRe) \\ \subseteq\ & eR((1-e)ue)(eRe)(e - (eue)(eae + (ebe)(eye)))Re \\ &+ eR(e - (eae + ebey)ue)(eRe)(e - (eue)(eae + (ebe)(eye)))Re \\ \subseteq\ & eJ(R)e = J(eRe). \end{aligned}$$

Likewise, we show that

$$(eRe)(e - (eue)(eae + (ebe)(eye)))(eRe)(e - (eae + (ebe)(eye))(eue))(eRe) \subseteq J(eRe).$$

Therefore $eae + ebe(eye) \in (eRe)_J^{-1}$, as desired.

Theorem 5.1 *Let A be a finitely generated projective right module over a JB-ring R. Then* $\mathrm{End}_R(A)$ *is a JB-ring.*

Proof Since A is a finitely generated projective right R-module, there exists an idempotent $E \in M_n(R)$ such that $A \cong E(nR)$. Hence, $\mathrm{End}_R(A) \cong EM_n(R)E$. As $M_n(R)/J(M_n(R)) \cong M_n(R/J(R))$, it follows from [2, Theorem 6.4] that $M_n(R/J(R))$ is a QB-ring, and so $M_n(R)$ is a JB-ring. Therefore the result follows from Lemma 5.1.

Corollary 5.1 *Let R be a JB-ring, and let $A \in M_n(R)$ be regular. Then there exist an idempotent $E \in M_n(R)$ and a $U \in M_n(R)_J^{-1}$ such that $A = EU$.*

Proof In view of Theorem 5.1, $M_n(R)$ is a JB-ring. Since $A \in M_n(R)$ is regular, there exists a $B \in M_n(R)$ such that $A = ABA$. In view of Lemma 2.1, there exists a $U \in M_n(R)_J^{-1}$ such that $A = ABU$. Take $E = AB$. Then $E = E^2 \in M_n(R)$, as required.

Let $TM_n(R)$ be the ring of all $n \times n$ upper triangular matrices over a ring R. If R is a JB-ring, we claim that $TM_n(R)$ is a JB-ring for all $n \in \mathbb{N}$. One easily checks that

$$J(TM_n(R)) = \begin{pmatrix} J(R) & R & \cdots & R \\ 0 & J(R) & \cdots & R \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J(R) \end{pmatrix};$$

hence,

$$TM_n(R)/J(TM_n(R)) = \begin{pmatrix} R/J(R) & 0 & \cdots & 0 \\ 0 & R/J(R) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & R/J(R) \end{pmatrix}.$$

As R is a JB-ring, $R/J(R)$ is a QB-ring. Hence, $TM_n(R)/J(TM_n(R))$ is a QB-ring. Therefore $TM_n(R)$ is a JB-ring, as desired.

A ring is the subproduct of the rings R_i $(i \in I)$ provided that there are surjective ring homomorphisms $\varphi_i : R \to R_i$ such that $\bigcap\limits_{i \in I} \operatorname{Ker} \varphi_i = 0$.

Theorem 5.2 *Every finite subdirect product of JB-rings is a JB-ring.*

Proof Let R be the subdirect product of two JB-rings A and B. It will suffice to show that R is a JB-ring. Suppose that there exist two surjective ring homomorphisms $\varphi : R \to A$ and $\psi : R \to B$ such that $\operatorname{Ker} \varphi \cap \operatorname{Ker} \psi = 0$. Clearly, there exist two corresponding surjective ring homomorphisms $\varphi^* : R/J(R) \to A/J(A)$ and $\psi^* : R/J(R) \to B/J(B)$. Given any $\overline{x} \in \operatorname{Ker} \varphi^* \cap \operatorname{Ker} \psi^*$, we have $\overline{xR} \subseteq \operatorname{Ker} \varphi^* \cap \operatorname{Ker} \psi^*$. Assume that $xR + M = R$ for a right R-module M. Then $\varphi(xR) + \varphi(M) = \varphi(R) = A$. As $\varphi(xR) \subseteq J(A)$, $J(A) + \varphi(M) = A$. It follows, by Nakayama's lemma, that $\varphi(M) = A$. Likewise, $\psi(M) = B$. Thus, we can find some $x, y \in M$ such that $\varphi(1_R) = \varphi(x)$ and $\psi(1_R) = \psi(y)$. Thus, $1_R - x \in \operatorname{Ker} \varphi$ and $1_R - y \in \operatorname{Ker} \psi$; hence, $(1_R - x)(1_R - y) \in \operatorname{Ker} \varphi \cap \operatorname{Ker} \psi$. Consequently, $1_R - y - x + xy = 0$. This infers that $R = M$. That is, xR is a superfluous submodule of R. So $xR \subseteq J(R)$, whence, $\overline{x} = 0$. This implies that $R/J(R)$ is the subdirect product of $A/J(A)$ and $B/J(B)$. Since $A/J(A)$ and $B/J(B)$ are both QB-rings, it follows by [3, Corollary 2.4] that $R/J(R)$ is a QB-ring. Therefore R is a JB-ring, as required.

Corollary 5.2 *Let I and J be ideals of a ring R. Then the following are equivalent:*

(1) *R/I and R/J are JB-rings;*

(2) *$R/(IJ)$ is a JB-ring;*

(3) *$R/(I \cap J)$ is a JB-ring.*

Proof (1) $\Rightarrow$ (3) Let $\varphi : R/(I \cap J) \twoheadrightarrow (R/(I \cap J))/(I/(I \cap J))$ and $\psi : R/(I \cap J) \twoheadrightarrow (R/(I \cap J))/(J/(I \cap J))$ be quotient maps. Then $\operatorname{Ker} \varphi \cap \operatorname{Ker} \psi = 0$. Clearly,

$$(R/(I \cap J))/(I/(I \cap J)) \cong R/I \quad \text{and} \quad (R/(I \cap J))/(J/(I \cap J)) \cong R/J.$$

Thus, $R/(I\cap J)$ is a JB-ring by Theorem 5.2.

(3) $\Rightarrow$ (2) Let $K=(I\cap J)/(I\cap J)^2$. By assumption, $R/(I\cap J)^2/K$ is a JB-ring. As $K^2=0$, one easily checks that $K\subseteq J(R/(I\cap J)^2)$; hence, $R/(I\cap J)^2$ is a JB-ring. As $R/(IJ)\cong(R/(I\cap J)^2)/((IJ)/(I\cap J)^2)$, we see that $R/(IJ)$ is a JB-ring.

(2) $\Rightarrow$ (1) As $R/I\cong(R/(IJ))/(I/(IJ))$, it follows from Theorem 4.1 that R/I is a JB-ring. Likewise, R/J is a JB-ring, as asserted.

Theorem 5.3 *Let I be an ideal of a ring R, and let $I\subseteq J(R)$. Then the following are equivalent:*

(1) *R is a JB-ring;*

(2) *R/I is a JB-ring;*

(3) *$R/J(R)^2$ is a JB-ring.*

Proof (1) $\Rightarrow$ (2) is clear from Theorem 4.1.

(2) $\Rightarrow$ (1) Since $R/J(R)\cong(R/I)/(J(R)/I)$, it follows from Theorem 4.1 that $R/J(R)$ is a JB-ring. As $J(R/J(R))=0$, $R/J(R)$ is a QB-ring.

(1) $\Leftrightarrow$ (3) Clearly, $J(R)^2\subseteq J(R)$. Applying "(1) $\Leftrightarrow$ (2)" to $J(R)^2$, we complete the proof.

Obviously, $\{B\text{-Rings}\}\subsetneqq\{QB\text{-rings}\}\subsetneqq\{JB\text{-rings}\}$. Let I be an ideal of a ring R with $I\subseteq J(R)$. As well known, R has stable range one if and only if so has R/I. Corollary 5.3 shows that R is a JB-ring if and only if so is R/I. We note that "R is a QB-ring"$\not\Leftrightarrow$"R/I is a QB-ring". Let V be an infinite-dimensional vector space over a division ring D, and let

$$R=\begin{pmatrix}\mathrm{End}_D(V) & \mathrm{End}_D(V)\\ 0 & \mathrm{End}_D(V)\end{pmatrix}.$$

Take

$$I=\begin{pmatrix}0 & \mathrm{End}_D(V)\\ 0 & 0\end{pmatrix}.$$

Then R/I is a QB-ring with $I\subseteq J(R)$, while R is not a QB-ring by [7, Example 3.4]. Let $R[[x]]$ be the ring of all formal power series in one variable over a ring R. We now derive the following corollary.

Corollary 5.3 *A ring R is a JB-ring if and only if so is $R[[x]]$.*

Proof Let $\varphi: R[[x]]\to R$ be given by $f(x)\mapsto f(0)$. Then $R\cong R[[x]]/\mathrm{Ker}\,\varphi$ with $\mathrm{Ker}\,\varphi\subseteq J(R[[x]])$. By virtue of Theorem 5.3, the proof is completed.

Example 5.1 Let V be an infinite dimensional vector space over a field $\mathbb{F}$, let $Q=\mathrm{End}_{\mathbb{F}}(V)$, and $J=\{x\in Q\mid \dim_{\mathbb{F}}(xV)<\infty\}$. Set $R=\{(x,y)\in Q\times Q\mid x-y\in J\}$. Then R is a QB-ring.

Proof Clearly, R is a subring of $Q\times Q$. Since $J\times J$ is an ideal of R and $R/(J\times J)\cong Q/J$ is regular. Thus, R is regular by [10, Lemma 1.3].

Set $P_1=J\times 0$ and $P_2=0\times J$. Since $R/P_1\cong R/P_2\cong Q$, R/P_1 and R/P_2 are QB-rings. Hence, R/P_1 and R/P_2 are both JB-rings. In addition, $P_1P_2=0$. According to Corollary 5.2, R is a JB-ring. As $J(R)=0$, we conclude that R is a QB-ring.

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