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Comparisons of Metrics on Teichmüller Space***

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Abstract For a Riemann surface X of conformally finite type (g,n), let d_T, d_L and d_{P_i} $(i=1,2)$ be the Teichmüller metric, the length spectrum metric and Thurston's pseudo-metrics on the Teichmüller space $T(X)$, respectively. The authors get a description of the Teichmüller distance in terms of the Jenkins-Strebel differential lengths of simple closed curves. Using this result, by relatively short arguments, some comparisons between d_T and d_L, d_{P_i} $(i=1,2)$ on $T_\varepsilon(X)$ and $T(X)$ are obtained, respectively. These comparisons improve a corresponding result of Li a little. As applications, the authors first get an alternative proof of the topological equivalence of d_T to any one of d_L, d_{P_1} and d_{P_2} on $T(X)$. Second, a new proof of the completeness of the length spectrum metric from the viewpoint of Finsler geometry is given. Third, a simple proof of the following result of Liu-Papadopoulos is given: a sequence goes to infinity in $T(X)$ with respect to d_T if and only if it goes to infinity with respect to d_L (as well as d_{P_i} $(i=1,2)$).

Keywords Length spectrum metric, Teichmüller metric, Thurston's pseudo-metrics
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1 Introduction

In this paper, X is a non-elementary Riemann surface of conformally finite type (g,n). For any quasi-conformal mapping $f: X \to X_0$, we denote by the pair (X_0, f) a marked Riemann surface. Two marked Riemann surfaces (X_1, f_1) and (X_2, f_2) are equivalent if there is a conformal mapping $c: X_1 \to X_2$ which is homotopic to $f_2 \circ f_1^{-1}$. Denote $[X,f]$ to be the equivalence class of (X,f). The Teichmüller space $T(X)$ is the set of the equivalence classes $[X,f]$.

As we know, Teichmüller gave a metric on $T(X)$,

$$d_T([X_1,f_1],[X_2,f_2]) = \log\{\inf K(f_0)\},$$

where the infimum is taken over all $f_0: X_1 \to X_2$ in the homotopy class of $f_2 \circ f_1^{-1}$, and $K(f_0)$ is its dilatation.

For any non-trivial simple closed curve $\gamma \subset X$, let $l_X(\gamma)$ be the shortest length under the Poincaré metric (hyperbolic metric) of closed curves in the free homotopy class of γ. $l_X(\gamma)$ is called the Poincaré length or hyperbolic length of γ. Let Σ_X'' be the set of homotopy classes of essential curves on X; that is, Σ_X'' is the set of homotopy classes of simple closed curves which

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are non-trivial and not homotopic to a puncture. The length spectrum metric d_L is defined as (see [1])

$$d_L([X_1, f_1], [X_2, f_2]) = \log \rho([X_1, f_1], [X_2, f_2]),$$

where

$$\rho([X_1, f_1], [X_2, f_2]) = \sup_{\gamma \in \Sigma''_{X_1}} \Big\{ \frac{l_{X_2}(f(\gamma))}{l_{X_1}(\gamma)}, \frac{l_{X_1}(\gamma)}{l_{X_2}(f(\gamma))} \Big\},$$

and $f = f_2 \circ f_1^{-1}$.

Thurston's pseudo-metrics d_{P_1} and d_{P_2} are defined as follows (see [2]):

$$d_{P_1}([X_1, f_1], [X_2, f_2]) = \log \sup_{\gamma \in \Sigma''_{X_1}} \frac{l_{X_2}(f(\gamma))}{l_{X_1}(\gamma)},$$

$$d_{P_2}([X_1, f_1], [X_2, f_2]) = \log \sup_{\gamma \in \Sigma''_{X_1}} \frac{l_{X_1}(\gamma)}{l_{X_2}(f(\gamma))},$$

where $f = f_2 \circ f_1^{-1}$. In [3, 4], Papadopoulos called d_{P_i} $(i = 1, 2)$ Thurston's asymmetric metrics. Thurston [2] showed that the equalities

$$d_{P_i}([X_1, f_1], [X_2, f_2]) = d_{P_i}([X_2, f_2], [X_1, f_1]), \quad i = 1, 2$$

are not true generally. Therefore, d_{P_1} and d_{P_2} are pseudo-metrics on $T(X)$. We know from the definitions that

$$d_{P_i}([X_1, f_1], [X_2, f_2]) \le d_L([X_1, f_1], [X_2, f_2]), \quad i = 1, 2.$$

The following result of Wolpert [5] is well-known.

Proposition 1.1 *Let $f : X_1 \to X_2$ be a quasi-conformal mapping between hyperbolic Riemann surfaces. Then*

$$\frac{l_{X_2}(f(\alpha))}{l_{X_1}(\alpha)} \le K(f)$$

holds for all non-trivial simple closed curves $\alpha \subset X_1$.

From this result, we immediately get the following lemma.

Lemma 1.1

$$d_L \le d_T, \quad d_{P_i} \le d_T, \quad i = 1, 2.$$

Now, we state some terminologies. Let d_1 and d_2 be two (pseudo-)metrics on a set F.

(1) We say that d_1 is topologically equivalent to d_2, if for any sequence $\{t_n\}_{n=0}^{\infty} \subset F$, we have $\lim_{n\to\infty} d_1(t_n, t_0) = 0$ if and only if $\lim_{n\to\infty} d_2(t_n, t_0) = 0$.

(2) We say that d_1 is quasi-isometric to d_2 if there exists a $K > 0$ such that

$$\frac{1}{K} d_1(x, y) \le d_2(x, y) \le K d_1(x, y)$$

for any $x, y \in F$.

The study of the relations of various metrics or pseudo-metrics on $T(X)$ is very interesting. In 1972, Sorvali [1] defined and studied the length spectrum metric and asked the following problem: is the Teichmüller metric d_T topologically equivalent to the length spectrum metric d_L for Teichmüller space of topologically finite Riemann surface? In 1975, Sorvali [6] solved this problem for tori. In 1986, Li [7] gave a positive answer to this question for the Teichmüller spaces of compact Riemann surfaces. In 1999, Liu [8] proved that the Teichmüller metric d_T is topologically equivalent to d_L for the Teichmüller spaces of topologically finite Riemann surfaces. This result gave an affirmative answer to Sorvali's problem and he asked the problem whether d_T is topologically equivalent to d_L in the Teichmüller spaces of Riemann surfaces of infinite topological type (see [8]). In 2003, Shiga [9] gave a negative answer to Liu's question by constructing a counter-example, and he gave a sufficient condition for the topological equivalence of d_T and d_L on $T(X)$. Most recently, Kinjo [10] showed that Shiga's condition is not a necessary one. Liu [11] also showed that the metrics d_T, d_L and the pseudo-metrics d_{P_i}, $i = 1, 2$ are topologically equivalent to each other in the Teichmüller spaces of topologically finite Riemann surfaces. Recently, Papadopoulos-Théret [3, 4] proved the same result. Actually, they have obtained many results about Thurston's pseudo-metrics. In 2008, Liu-Sun-Wei [12] gave a new proof of Shiga's result. They provided a class of Riemann surfaces X of topologically infinite type, such that d_L, d_{P_1} and d_{P_2} are not topologically equivalent to d_T on $T(X)$. They also gave a necessary condition for the topological equivalence of d_T to any one of d_L, d_{P_1} and d_{P_2} on $T(X)$.

On the other hand, many authors studied the quasi-isometric equivalence of the above metrics and pseudo-metrics. Thurston [2] (see also [13]) showed that the Thurston's pseudo-metrics are asymmetry, that is, $d_1 \neq d_2$. Liu [13] proved that d_{P_1} is not quasi-isometric to d_{P_2}. This also implies that d_L is not quasi-isometric to d_{P_i}, $i = 1, 2$. Liu [14] also showed that d_T is not quasi-isometric to d_{P_i}, $i = 1, 2$. In 2003, Li [15] proved that d_T is not quasi-isometric to d_L. Actually, he proved that there exist two sequences of points $\{\tau_n\}_{n=1}^{\infty}$ and $\{\tau'_n\}_{n=1}^{\infty}$ in $T(X)$ (X is a compact Riemann surface), such that $\lim\limits_{n\to\infty} d_L(\tau_n, \tau'_n) = 0$ while $\lim\limits_{n\to\infty} d_T(\tau_n, \tau'_n) > d_0$, where d_0 is a positive constant. In [12], Liu-Sun-Wei gave a generalization of Li's above result; that is, they showed that in the Teichmüller spaces of Riemann surfaces of topologically finite or infinite type, there exist two sequences $\{\widetilde{\tau}\}_{n=1}^{\infty}$ and $\{\widehat{\tau}\}_{n=1}^{\infty}$, such that as $n \to \infty$, $d_L(\widetilde{\tau}_n, \widehat{\tau}_n) \to 0$, $d_{P_1}(\widetilde{\tau}_n, \widehat{\tau}_n) \to 0$, $d_{P_2}(\widetilde{\tau}_n, \widehat{\tau}_n) \to 0$, while $d_T(\widetilde{\tau}_n, \widehat{\tau}_n) \to \infty$.

The rest of this paper is organized as follows. In Section 2, we will give some comparisons of the hyperbolic length, the extremal length and the quadratic differential length. Section 3 contains our main results. Theorem 3.1 gives a description of the Teichmüller distance in terms of the Jenkins-Strebel differential lengths of simple closed curves. By this result and a comparison between the hyperbolic length and the quadratic differential length, we will give, in Theorem 3.2 and Theorem 3.3, comparisons on $T(X)$ of the Teichmüller distance with the corresponding length spectrum distance and Thurston's pseudo-distances. These results improve Li's results in [7] a little, and our proofs of Theorem 3.2 and Theorem 3.3 are relatively short. In Theorem 3.4 and Theorem 3.5, we will give comparisons of d_T with d_L and d_{P_i}, $i = 1, 2$ on the thick parts $T_\varepsilon(X)$, respectively. In Section 4, we will give some applications to Theorems 3.1 and 3.2. As the first application, following Li [7], we will give an alternative proof of the topological equivalence of d_T to d_L, d_{P_1} and d_{P_2}, respectively. Second, we will give a new proof of the completeness of the length spectrum metric from the viewpoint of Finsler geometry. Third, we will give a simple proof of the following result in Liu-Papadopoulos [17, Theorem 2.25]: a sequence goes to infinity in $T(X)$ with respect to d_T if and only if it goes to infinity with respect to d_L (as well as d_{P_i}, $i = 1, 2$).

2 Preliminaries

In this section, we will give some comparisons of the hyperbolic length, the extremal length and the quadratic differential length. These comparisons will be used in the next section to give estimations of the Teichmüller distance, the length spectrum distance and Thurston's pseudo-distances on $T(X)$ and $T_\varepsilon(X)$.

First, for later use, we summarize some results in Teichmüller's theory and quadratic differential theory. References are [18–22].

The extremal length $\mathrm{ext}_X(\alpha)$ of a simple closed curve $\alpha \subset X$ is defined as

$$\mathrm{ext}_X(\alpha) = \sup_{\rho} \frac{(l_\rho(\alpha))^2}{A_\rho},$$

where ρ ranges over all conformal metrics on X with area $0 < A_\rho < \infty$, and $l_\rho(\alpha)$ denotes the infimum of the ρ-lengths of all the simple closed curves which are homotopic to α.

In his remarkable paper [19], Kerckhoff gave the following description of the Teichmüller distance in terms of the extremal lengths.

Lemma 2.1 (Kerckhoff)

$$d_T([X_1, f_1], [X_2, f_2]) = \log \sup_{\gamma \in \Sigma''_X} \Big\{ \frac{\mathrm{ext}_{X_1}(f_1(\gamma))}{\mathrm{ext}_{X_2}(f_2(\gamma))} \Big\}.$$

Let $QD(X)$ be the space of holomorphic quadratic differentials on X. Let $PQD(X)$ be the space of projective equivalence classes of elements in $QD(X)$, where $q_1 \in QD(X)$ and $q_2 \in QD(X)$ are projectively equivalent if they differ by a positive multiplier. We may endow $QD(X)$ with the L_1 norm

$$\|q\|_1 = \int_X |q(z)||\mathrm{d}z|^2.$$

The complex dimensions of $QD(X)$ and $PQD(X)$ are $3g-3+n$ and $3g-4+n$, respectively. $PQD(X)$ may be viewed as the unit sphere in $QD(X)$, thus it is a compact set. A quadratic differential $q \in QD(X)$ induces a metric whose line element can be written locally as $|q(z)|^{\frac{1}{2}}|\mathrm{d}z|$. We call this metric the q-metric, or the quadratic differential metric for short. This metric gives a measure of lengths as follows. For any simple closed curve $\gamma \subset X$, let

$$l_q(\gamma) = \inf_{\alpha \sim \gamma} \Big\{ \int_\alpha |q(z)|^{\frac{1}{2}}|\mathrm{d}z| \Big\}.$$

Then $l_q(\gamma)$ is called the quadratic differential length of γ. The quantity

$$\inf_{\alpha \sim \gamma} \Big\{ \int_\alpha |\Im\{q(z)^{\frac{1}{2}}\mathrm{d}z\}| \Big\}$$

is called the height of γ in the q-metric, which is denoted by $h_q(\gamma)$.

There is a special class of holomorphic quadratic differentials $q \in QD(X)$ with prescribed trajectory structures. Given m disjoint simple loops $\alpha_1, \alpha_2, \cdots, \alpha_m$ $(1 \le m \le 3g-3+n)$ on X which are not pair-wisely homotopic, and m positive numbers $h_1, h_2, \cdots, h_m$, there exists a unique holomorphic quadratic differential $q \in QD(X)$ such that (1) the complement of the critical trajectories of q is the union of cylinders A_j, $j = 1, 2, \cdots, m$, which are homotopic to α_j, $j = 1, 2, \cdots, m$, respectively; (2) the height of A_j is equal to h_j, $j = 1, 2, \cdots, m$. Such quadratic differentials are called the Jenkins-Strebel differentials determined by $\alpha_1, \alpha_2, \cdots, \alpha_m$ and $h_1, h_2, \cdots, h_m$. The cylinders A_j, $j = 1, 2, \cdots, m$ are called the characteristic cylinders of q.

A special case is $m = 1$. Given a simple closed curve $\alpha \subset S$, we will use the notation $\phi[\alpha]$ to denote the unique Jenkins-Strebel differential on S determined by α with the height of its characteristic cylinder equal to 1. This kind of differentials are of great importance to us. In the rest of this paper, we will also call this kind of differentials the simple differentials.

Let $M(X)$ be the moduli space of X. For any $\varepsilon > 0$, let $M_\varepsilon \subset M(X)$ be the set of Riemann surfaces with the property that the hyperbolic length of any essential curve is not less than ε. By Mumford's compactness theorem, we know that M_ε is a compact subset of the moduli space. Let $T_\varepsilon(X) \subset T(X)$ be the set of $[X_1, f_1]$ where X_1 satisfies the property that the hyperbolic length of any essential curve is not less than ε. We call $T_\varepsilon(X)$ the ε-thick part of $T(X)$, and $T(X) - T_\varepsilon(X)$ the ε-thin part of $T(X)$.

Let $\mathcal{MF}$ be the space of measured foliations on a topological surface X (here, we need not the complex structure on X), and $\mathcal{PMF}$ be the set of its projective classes. The complex dimensions of $\mathcal{MF}$ and $\mathcal{PMF}$ are $3g-3+n$ and $3g-4+n$, respectively. $\mathcal{PMF}$ may be viewed as the unit sphere in $\mathcal{MF}$. Thus $\mathcal{PMF}$ is a compact subset of $\mathcal{MF}$. Therefore, $M_\varepsilon \times \mathcal{PMF}$ and $M_\varepsilon \times \mathcal{PMF} \times PQD(X)$ are compact subsets.

Note that the spaces $QD(X)$ and $PQD(X)$ are defined with respect to the complex structure on X, while $\mathcal{MF}$ and $\mathcal{PMF}$ are only topological objects which do not depend on the complex structure on X.

In [22] (see also [19, 23]), Hubbard-Masur proved that given any Riemann surface $S \in T(X)$ and any measured foliation $\mathcal{F} \in \mathcal{MF}$, there exists a unique quadratic differential $q \in QD(S)$ whose horizontal measured foliation is measured equivalent to $\mathcal{F}$. Hubbard-Masur's result generalized the existence of the Jenkins-Strebel differentials.

In the rest of this section, we will devote ourselves to the comparisons of the hyperbolic length, the extremal length and the quadratic differential length. We will work on $T(X)$ and sometimes on $T_\varepsilon(X)$.

We have the following relation between the extremal length and the simple differential length.

Lemma 2.2 *Let $\phi[\gamma]$ be the simple differential determined by a simple closed curve $\gamma \subset X$. Then, the $\phi[\gamma]$-metric is the metric that realizes the supremum in $\sup\limits_\rho \frac{(l_\rho([\alpha]))^2}{A_\rho}$. Consequently,*

$$\mathrm{ext}_X(\gamma) = l_{\phi[\gamma]}(\gamma) = \|\phi[\gamma]\|_1,$$

where $l_{\phi[\gamma]}(\gamma)$ denotes the length of γ in the metric induced by $\phi[\gamma]$.

Proof The first half of the lemma is well-known. See [20], and see also [19, Theorem 3.1].

Now we prove the second half. According to a remark after [19, Proposition 3], $\mathrm{ext}_X(\gamma)$ is equal to the area of X in the metric induced $\phi[\gamma]$. Namely,

$$\mathrm{ext}_X(\gamma) = \int_X |\phi[\gamma]||\mathrm{d}z|^2 = \|\phi[\gamma]\|_1.$$

On the other hand, we have

$$\mathrm{ext}_X(\gamma) = \frac{l^2_{\phi[\gamma]}(\gamma)}{\|\phi[\gamma]\|_1}.$$

Combining the above two equalities, we get

$$\mathrm{ext}_X(\gamma) = \|\phi[\gamma]\|_1 = \frac{l^2_{\phi[\gamma]}(\gamma)}{\|\phi[\gamma]\|_1}.$$

Now, the lemma follows trivially.

As to the relation between the hyperbolic length and the extremal length, we have the following result.

Lemma 2.3 (see [24])

$$\frac{l_X(\gamma)}{\pi} \leq \mathrm{ext}_X(\gamma) \leq \frac{l_X(\gamma)}{2} \mathrm{e}^{\frac{l_X(\gamma)}{2}}.$$

For any $q \in QD(X)$, we have two norms, the L_1 norm $\|q\|_1$ and Bers' sup-norm

$$\|q\|_B = \sup_{z\in X} \frac{|q(z)|}{\delta^2(z)},$$

where $\delta^2(z)|\mathrm{d}z|^2$ is the Poincaré area density. Both of the two norms are of great importance in quadratic differential theory and Teichmüller theory: $(QD(X), \|\cdot\|_B)$ is the model in Bers embedding which gives $T(X)$ a natural complex manifold structure, and $(QD(S), \|\cdot\|_1)$ is the complex cotangent space of $T(X)$ at a point $S \in T(X)$.

Bers' sup-norm and the L_1 norm are actually equivalent.

Lemma 2.4 (see [25]) *We have*

$$\|q\|_1 < \infty \quad \textit{if and only if} \quad \|q\|_B < \infty.$$

Furthermore, the two norms $\|\cdot\|_B$ and $\|\cdot\|_1$ are equivalent: for any $S \in T(X)$, there exists a constant $C = C_S(g, n) > 0$ which depends only on S, g and n, such that

$$\frac{\|q\|_1}{C} \leq \|q\|_B \leq C\|q\|_1$$

holds for any $q \in QD(S)$.

For general relations between the hyperbolic length and the quadratic differential length, we have the following result.

Lemma 2.5 *For any $q = q(z)\mathrm{d}z^2 \in QD(X)$, we have that*

$$l_q(\gamma) \leq l_X(\gamma)\Big(\sup_{z\in[\gamma]} \frac{|q(z)|}{\delta^2(z)}\Big)^{\frac{1}{2}} \leq l_X(\gamma)\Big(\sup_{z\in X} \frac{|q(z)|}{\delta^2(z)}\Big)^{\frac{1}{2}}$$

holds for any simple closed curve $\gamma \subset X$, where $[\gamma]$ denotes the set of all the simple closed curves in the homotopy class of γ, and $\delta^2(z)|\mathrm{d}z|^2$ is the Poincaré area density on X. Note that the constant $C_q = \sup\limits_{z\in X} \frac{|q(z)|}{\delta^2(z)}$ is Bers' sup-norm of $q \in QD(X)$.

Proof Since X is of type (g, n), any $q \in QD(X)$ is of finite L_1 norm, i.e.,

$$\|q\|_1 < \infty.$$

By Lemma 2.4, this is equivalent to

$$\|q\|_B = \sup_{z\in X} \frac{|q(z)|}{\delta^2(z)} < \infty.$$

Thus, for any simple closed curve $\alpha \in [\gamma]$, we have

$$\int_\alpha \sqrt{|q(z)|}\,|\mathrm{d}z| = \int_\alpha \frac{\sqrt{|q(z)|}}{\delta(z)} \delta(z)|\mathrm{d}z| \leq \int_\alpha \delta(z)|\mathrm{d}z| \sup_{z\in[\gamma]} \frac{\sqrt{|q(z)|}}{\delta(z)}.$$

Take the infimum over all simple closed curves α in the homotopy class of γ, we get the desired inequality.

Particularly, when considering the simple differential lengths, we have the following result.

Corollary 2.1 *For any $S \in T(X)$, there exists a constant $C = C_S(g,n) > 0$ (which is exactly the same as the one in Lemma 2.4), such that*

$$\mathrm{ext}_S(\gamma) = l_{\phi[\gamma]}(\gamma) \leq C\, l_S^2(\gamma)$$

holds for any simple closed curve $\gamma \subset S$, and the corresponding simple differential $\phi[\gamma] \in QD(S)$ determined by γ.

Proof For simplicity, denote $q = \phi[\gamma]$. From Lemma 2.5, we know

$$l_q(\gamma) \leq l_S(\gamma)\Big(\sup_{z\in S} \frac{|q(z)|}{\delta^2(z)}\Big)^{\frac{1}{2}}.$$

By Lemma 2.4, there exists a constant $C = C_S(g,n) > 0$ such that

$$\sup_{z\in S} \frac{|q(z)|}{\delta^2(z)} \leq C\|q\|_1$$

holds for any holomorphic quadratic differential in $QD(S)$. Thus, the above two inequalities give

$$l_q(\gamma) \leq C^{\frac{1}{2}} l_S(\gamma)\|q\|_1^{\frac{1}{2}}.$$

At the same time, Lemma 2.2 tells us that

$$\mathrm{ext}_S(\gamma) = l_q(\gamma) = \|q\|_1.$$

Therefore,

$$\mathrm{ext}_S(\gamma) = l_q(\gamma) \leq C^{\frac{1}{2}} l_S(\gamma)\mathrm{ext}_S^{\frac{1}{2}}(\gamma).$$

This implies

$$\mathrm{ext}_S(\gamma) = l_q(\gamma) \leq C\, l_S^2(\gamma).$$

A converse inequality to the one in Corollary 2.1 can be easily obtained from the definition of the extremal lengths and the Gauss-Bonnet theorem. Given any $S \in T(X)$ and any $\gamma \subset S$, we have

$$\mathrm{ext}_S(\gamma) = l_{\phi[\gamma]}(\gamma) \geq \frac{l_S^2(\gamma)}{2\pi(2g-2+n)}.$$

This inequality together with Corollary 2.1 gives the following lemma.

Lemma 2.6 *For any $S \in T(X)$, there exists a constant $C = C_S(g,n) > 0$, such that*

$$\frac{l_S^2(\gamma)}{C} \leq \mathrm{ext}_S(\gamma) = l_{\phi[\gamma]}(\gamma) \leq C\, l_S^2(\gamma)$$

holds for any simple closed curve $\gamma \subset S$, and the corresponding simple differential $\phi[\gamma] \in QD(S)$ determined by γ.

When $S \in T_\varepsilon(X)$, we have the following lemma in correspondence with Lemma 2.5 and Lemma 2.6.

Lemma 2.7 *There exists a universal constant $\mathfrak{C} = \mathfrak{C}(g, n, \varepsilon)$ such that*

$$\frac{l_q(\gamma)}{\mathfrak{C}} \leq l_S(\gamma) \leq \mathfrak{C}\, l_q(\gamma)$$

holds for any $S \in T_\varepsilon(X)$, any essential simple closed curve $\gamma \subset S$, and any $q \in PQD(S)$. Especially, there exists a universal constant $\mathfrak{C} = \mathfrak{C}(g, n, \varepsilon)$ (which is exactly the same as the above one), such that

$$\frac{l_{\phi[\gamma]}^{\frac{1}{2}}(\gamma)}{\mathfrak{C}} \leq l_S(\gamma) \leq \mathfrak{C}\, l_{\phi[\gamma]}^{\frac{1}{2}}(\gamma)$$

holds for any $S \in T_\varepsilon(X)$, any $\gamma \subset S$, and the corresponding simple differential $\phi[\gamma] \in QD(S)$.

Proof For any $S \in T_\varepsilon(X)$, any $\gamma \subset S$, and any quadratic differential $q \in PQD(S)$ (which is viewed as the unit sphere in $QD(S)$), set

$$\mathcal{J} = \mathcal{J}(S, \gamma, q) = \frac{l_S(\gamma)}{l_q(\gamma)}.$$

Then $\mathcal{J}$ is a well-defined, positive and continuous function on the compact set $M_\varepsilon \times \mathcal{PMF} \times \bigcup\limits_{S \in T_\varepsilon(X)} PQD(S)$ $\Big($note that $\bigcup\limits_{S \in T_\varepsilon(X)} PQD(S)$ is a finite union of compact sets $PQD(S)$ with its cardinality non-greater than $3g - 3 + n\Big)$. Let c_1 and c_2 be the maximum and minimum values of $\mathcal{J}$, respectively. Then the first result follows by setting $\mathfrak{C} = \max\{c_1, \frac{1}{c_2}\}$.

From Lemma 2.2, we know $l_{\phi[\gamma]}(\gamma) = \|\phi[\gamma]\|_1$. Thus, by a similar argument, we get the second result by considering those differentials $\frac{\phi[\gamma]}{\|\phi[\gamma]\|_1}$ with unit norms.

To end this section, we make the following remark.

Remark 2.1 Lemma 2.6 holds for any $S \in T(X)$, but its disadvantage is that the constant $C = C_S(g, n)$ depends on $S \in T(X)$.

In the second inequality of Lemma 2.7, the constant $\mathfrak{C} = \mathfrak{C}(g, n, \varepsilon)$ is a universal constant which does not depend on $S \in T_\varepsilon(X)$ or $\gamma \subset S$. A similar result is given in [12, Theorem 1].

3 Main Results

In Theorem 3.1, we will give a description of the Teichmüller distance in terms of the simple differential lengths of all the simple closed curves.

Theorem 3.1 *For any two points $[X_1, f_1], [X_2, f_2] \in T(X)$, we have*

$$d_T([X_1, f_1], [X_2, f_2]) = \log \sup_{\gamma \in \Sigma''_X} \Big\{ \frac{l_{\phi[f_1(\gamma)]}(f_1(\gamma))}{l_{\phi[f_2(\gamma)]}(f_2(\gamma))} \Big\}.$$

Proof By Lemma 2.1, we have

$$d_T([X_1, f_1], [X_2, f_2]) = \log \sup_{\gamma \in \Sigma''_X} \Big\{ \frac{\mathrm{ext}_{X_1}(f_1(\gamma))}{\mathrm{ext}_{X_2}(f_2(\gamma))} \Big\}. \tag{3.1}$$

According to Lemma 2.2, replacing the extremal lengths in (3.1) by the corresponding simple differential lengths, we get

$$d_T([X_1, f_1], [X_2, f_2]) = \log \sup_{\gamma \in \Sigma''_X} \Big\{ \frac{l_{\phi[f_1(\gamma)]}(f_1(\gamma))}{l_{\phi[f_2(\gamma)]}(f_2(\gamma))} \Big\}.$$

In the following Theorems 3.2–3.5, we will use Theorem 3.1 and some related lemmas in the preceding section to give comparisons of the Teichmüller distance, the length spectrum distance and Thurston's pseudo-distances on $T(X)$ and $T_\varepsilon(X)$, respectively.

First, we give comparisons of these distances on the whole of $T(X)$.

Theorem 3.2 *For any two points $[X_1, f_1], [X_2, f_2] \in T(X)$, we have*

$$d_T([X_1, f_1], [X_2, f_2]) \le d_{P_1}([X_1, f_1], [X_2, f_2]) + C_B([X_2, f_2]), \tag{3.2}$$

$$d_T([X_1, f_1], [X_2, f_2]) \le d_{P_2}([X_1, f_1], [X_2, f_2]) + C_B([X_1, f_1]), \tag{3.3}$$

$$d_T([X_1, f_1], [X_2, f_2]) \le d_L([X_1, f_1], [X_2, f_2]) + C_B([X_i, f_i]), \quad i = 1, 2. \tag{3.4}$$

Here $C_B = C_B([X_i, f_i])$ is a constant which depends only on $[X_i, f_i]$, $i = 1, 2$.

Proof By Theorem 3.1, we have

$$d_T([X_1, f_1], [X_2, f_2]) = \log \sup_{\gamma \in \Sigma''_X} \Big\{ \frac{l_{q_1}(f_1(\gamma))}{l_{q_2}(f_2(\gamma))} \Big\}. \tag{3.5}$$

Here, for simplicity, we denote the simple differentials $\phi[f_1(\gamma)]$ and $\phi[f_2(\gamma)]$ by q_1 and q_2, respectively.

Now, we estimate the right-hand side of (3.5). From Lemmas 2.2 and 2.3, we have

$$\frac{l_{X_2}(f_2(\gamma))}{\pi} \le \mathrm{ext}_{X_2}(f_2(\gamma)) = l_{q_2}(f_2(\gamma)). \tag{3.6}$$

On the other hand, by Lemma 2.5, we get

$$l_{q_1}(f_1(\gamma)) \le l_{X_1}(f_1(\gamma)) \Big(\sup_{z \in [f_1(\gamma)]} \frac{|q_1(z)|}{\delta^2(z)} \Big)^{\frac{1}{2}}. \tag{3.7}$$

Combining (3.6) and (3.7), we obtain

$$\frac{l_{q_1}(f_1(\gamma))}{l_{q_2}(f_2(\gamma))} \le \pi \Big(\sup_{z \in [f_1(\gamma)]} \frac{|q_1(z)|}{\delta^2(z)} \Big)^{\frac{1}{2}} \frac{l_{X_1}(f_1(\gamma))}{l_{X_2}(f_2(\gamma))}. \tag{3.8}$$

Taking the supremum in (3.8) over all $\gamma \in \Sigma''_X$, we have

$$\sup_{\gamma \in \Sigma''_X} \Big\{ \frac{l_{q_1}(f_1(\gamma))}{l_{q_2}(f_2(\gamma))} \Big\} \le \pi \sup_{\gamma \in \Sigma''_X} \Big\{ \Big(\sup_{z \in [f_1(\gamma)]} \frac{|q_1(z)|}{\delta^2(z)} \Big)^{\frac{1}{2}} \Big\} \sup_{\gamma \in \Sigma''_X} \Big\{ \frac{l_{X_1}(f_1(\gamma))}{l_{X_2}(f_2(\gamma))} \Big\}. \tag{3.9}$$

Therefore, from (3.5) and (3.9), we get the desired inequality (3.3),

$$d_T([X_1, f_1], [X_2, f_2]) \le d_{P_2}([X_1, f_1], [X_2, f_2]) + C_B([X_1, f_1]),$$

where

$$C_B([X_1, f_1]) = \frac{1}{2} \log \sup_{\gamma \in \Sigma''_{X_1}} \Big\{ \sup_{z \in [\gamma]} \frac{|q_1(z)|}{\delta^2(z)} \Big\} + \log \pi.$$

Note

$$C_B([X_1, f_1]) \le \frac{1}{2} \log \sup_{z \in X_1} \frac{|q_1(z)|}{\delta^2(z)} + \log \pi < \infty.$$

Similarly, we get the desired inequality (3.2),

$$d_T([X_1, f_1], [X_2, f_2]) \le d_{P_1}([X_1, f_1], [X_2, f_2]) + C_B([X_2, f_2]),$$

where

$$C_B([X_2, f_2]) = \frac{1}{2} \log \sup_{\gamma \in \Sigma''_{X_2}} \Big\{ \sup_{z \in [\gamma]} \frac{|q_2(z)|}{\delta^2(z)} \Big\} + \log \pi.$$

By definitions,

$$d_{P_i}([X_1, f_1], [X_2, f_2]) \leq d_L([X_1, f_1], [X_2, f_2]), \quad i = 1, 2.$$

Thus, from (3.2) and (3.3), we obtain the desired inequality (3.4),

$$d_T([X_1, f_1], [X_2, f_2]) \leq d_L([X_1, f_1], [X_2, f_2]) + C_B([X_i, f_i]), \quad i = 1, 2.$$

Correspondingly, from the proof of Theorem 3.2, we get the following result.

Theorem 3.3 *For any two points $[X_1, f_1], [X_2, f_2] \in T(X)$, let $f : X_1 \to X_2$ be the Teichmüller mapping in the homotopy class of $f_2 \circ f_1^{-1}$. Then we have*

$$K(f) \leq \mathfrak{C}_B([X_2, f_2]) \sup_{\gamma \in \Sigma''_X} \Big\{ \frac{l_{X_2}(f_2(\gamma))}{l_{X_1}(f_1(\gamma))} \Big\},$$

$$K(f) \leq \mathfrak{C}_B([X_1, f_1]) \sup_{\gamma \in \Sigma''_X} \Big\{ \frac{l_{X_1}(f_1(\gamma))}{l_{X_2}(f_2(\gamma))} \Big\},$$

$$K(f) \leq \mathfrak{C}_B([X_i, f_i]) \sup_{\gamma \in \Sigma''_X} \Big\{ \frac{l_{X_1}(f_1(\gamma))}{l_{X_2}(f_2(\gamma))}, \frac{l_{X_2}(f_2(\gamma))}{l_{X_1}(f_1(\gamma))} \Big\}, \quad i = 1, 2.$$

Here $\mathfrak{C}_B = \mathfrak{C}_B([X_i, f_i])$ is a constant which depends only on $[X_i, f_i]$, $i = 1, 2$.

Proof Set

$$\mathfrak{C}_B([X_1, f_1]) = \pi \sup_{\gamma \in \Sigma''_{X_1}} \Big\{ \Big(\sup_{z \in [\gamma]} \frac{|q_1(z)|}{\delta^2(z)} \Big)^{\frac{1}{2}} \Big\},$$

$$\mathfrak{C}_B([X_2, f_2]) = \pi \sup_{\gamma \in \Sigma''_{X_2}} \Big\{ \Big(\sup_{z \in [\gamma]} \frac{|q_2(z)|}{\delta^2(z)} \Big)^{\frac{1}{2}} \Big\}.$$

Then, from the definitions of d_T, d_L and d_{P_i}, $i = 1, 2$, we get the desired inequalities by (3.9) and its similarities.

Remark 3.1 For Teichmüller spaces of compact Riemann surfaces, Li [7] proved the following inequality

$$K(f) \leq m([X_1, f_1]) \sup_{\gamma \in \Sigma''_X} \Big\{ \Big(\frac{l_{X_1}(f_1(\gamma))}{l_{X_2}(f_2(\gamma))} \Big)^2, \Big(\frac{l_{X_2}(f_2(\gamma))}{l_{X_1}(f_1(\gamma))} \Big)^2 \Big\}, \tag{3.10}$$

where $m([X_1, f_1])$ is a constant which depends only on $[X_1, f_1]$.

In the same paper [7], by using inequality (3.10), Li obtained the following important inequality from which he further showed the topological equivalence of the Teichmüller metric and the length spectrum metric on Teichmüller spaces of compact Riemann surfaces. Li's inequality is as follows:

$$d_L(\tau_1, \tau_2) \leq d_T(\tau_1, \tau_2) \leq 2 d_L(\tau_1, \tau_2) + C(\tau_1), \tag{3.11}$$

where $\tau_1, \tau_2 \in T(S_0)$, $C(\tau_1)$ is a constant depending on τ_1 and S_0 is a compact Riemann surface.

Theorems 3.2 and 3.3 improve Li's above results (3.10) and (3.11) a little. Theorems 3.2 and 3.3 hold for Riemann surfaces of conformally finite type, whereas (3.10) and (3.11) only hold for compact Riemann surfaces (this is because, in Li's [7] proof of (3.10) and (3.11), a compactness argument is used). And we remark that the constants $C_B([X_i, f_i])$, $i = 1, 2$ in Theorem 3.2 and $\mathfrak{C}_B([X_i, f_i])$, $i = 1, 2$ in Theorem 3.3 are essentially related to Bers' sup-norm of simple differentials, whereas the constants in (3.10) and (3.11) are obtained by a compactness argument.

In Theorems 3.4 and 3.5, we will give comparisons of the Teichmüller distance, the length spectrum distance and Thurston's pseudo-distances on $T_\varepsilon(X)$. These comparisons are a little different from those given in Theorems 3.2 and 3.3. Note that the constants in the following Theorems 3.4 and 3.5 are universal in the sense that they depend only on g, n and ε.

Theorem 3.4 *There exist universal constants $\mathscr{C}_i = \mathscr{C}_i(g, n, \varepsilon)$, $i = 1, 2$ which depend only on g, n and ε, such that for any two points $[X_1, f_1], [X_2, f_2] \in T_\varepsilon(X)$, we have*

$$d_T([X_1, f_1], [X_2, f_2]) \leq 2d_{P_1}([X_1, f_1], [X_2, f_2]) + \mathscr{C}_1, \tag{3.12}$$
$$d_T([X_1, f_1], [X_2, f_2]) \leq 2d_{P_2}([X_1, f_1], [X_2, f_2]) + \mathscr{C}_2, \tag{3.13}$$
$$d_T([X_1, f_1], [X_2, f_2]) \leq 2d_L([X_1, f_1], [X_2, f_2]) + \mathscr{C}_i, \quad i = 1, 2. \tag{3.14}$$

Proof By Theorem 3.1, we have

$$d_T([X_1, f_1], [X_2, f_2]) = \log \sup_{\gamma \in \Sigma''_X} \Big\{ \frac{l_{q_1}(f_1(\gamma))}{l_{q_2}(f_2(\gamma))} \Big\}. \tag{3.15}$$

Here, for simplicity, we denote the simple differentials $\phi[f_1(\gamma)]$ and $\phi[f_2(\gamma)]$ by q_1 and q_2, respectively.

By Lemma 2.7, the right-hand side of (3.15) can be estimated by

$$\sup_{\gamma \in \Sigma''_X} \Big\{ \frac{l_{q_1}(f_1(\gamma))}{l_{q_2}(f_2(\gamma))} \Big\} \leq \mathfrak{C}_2 \sup_{\gamma \in \Sigma''_X} \Big\{ \frac{l^2_{X_1}(f_1(\gamma))}{l^2_{X_2}(f_2(\gamma))} \Big\}, \tag{3.16}$$

where $\mathfrak{C}_2$ comes from Lemma 2.7 which depends only on g, n and ε.

Thus, from (3.15) and (3.16), we get the desired inequality (3.13) by setting $\mathscr{C}_2 = \log \mathfrak{C}_2$,

$$d_T([X_1, f_1], [X_2, f_2]) \leq 2d_{P_2}([X_1, f_1], [X_2, f_2]) + \log \mathfrak{C}_2.$$

Similarly, we get inequality (3.12).

By definitions,

$$d_{P_i}([X_1, f_1], [X_2, f_2]) \leq d_L([X_1, f_1], [X_2, f_2]), \quad i = 1, 2.$$

Thus, we get inequality (3.14) from (3.12) and (3.13).

Correspondingly, we get the following theorem from the definitions of d_T, d_L, d_{P_i} $(i = 1, 2)$ and the proof of Theorem 3.4, especially from (3.16) and its similarities.

Theorem 3.5 *There exist universal constants $\mathfrak{C}_i = \mathfrak{C}_i(g, n, \varepsilon)$, $i = 1, 2$ (which are given in (3.16) and its similarities), such that for any two points $[X_1, f_1], [X_2, f_2] \in T_\varepsilon(X)$ and the*

Teichmüller mapping $f : X_1 \to X_2$ in the homotopy class of $f_2 \circ f_1^{-1}$, we have

$$K(f) \le \mathfrak{C}_1 \sup_{\gamma \in \Sigma''_X} \Big\{ \frac{l^2_{X_2}(f_2(\gamma))}{l^2_{X_1}(f_1(\gamma))} \Big\}, \tag{3.17}$$

$$K(f) \le \mathfrak{C}_2 \sup_{\gamma \in \Sigma''_X} \Big\{ \frac{l^2_{X_1}(f_1(\gamma))}{l^2_{X_2}(f_2(\gamma))} \Big\}, \tag{3.18}$$

$$K(f) \le \mathfrak{C}_i \sup_{\gamma \in \Sigma''_X} \Big\{ \frac{l^2_{X_2}(f_2(\gamma))}{l^2_{X_1}(f_1(\gamma))}, \frac{l^2_{X_1}(f_1(\gamma))}{l^2_{X_2}(f_2(\gamma))} \Big\}, \quad i = 1, 2. \tag{3.19}$$

Remark 3.2 (i) In [16, Theorem B], Choi-Rafi proved that there is a constant c depending on g, n and ε such that, for any $\sigma, \tau \in T_\varepsilon(X)$, we have

$$d_T(\sigma, \tau) \le 2d_L(\sigma, \tau) + c.$$

(Note that the Teichmüller distance in [16] is one half of ours). In [12, Theorem 2], similar inequalities are obtained with additional multiplicative constants on the right-hand side of the inequalities.

(ii) As stated in the introduction, Liu [13] proved that the length spectrum metric is not quasi-isometric to Thurston's pseudo-metrics. Thus, inequalities (3.12) and (3.13) of Theorem 3.4 may not be obtained from inequality (3.14) trivially. Similarly, inequalities (3.17) and (3.18) of Theorem 3.5 may not be obtained from inequality (3.19) trivially.

4 Applications

First, following Li [7], we can immediately get the following result from Theorem 3.2.

Theorem 4.1 *Teichmüller metric d_T is topologically equivalent to d_L, d_{P_1} and d_{P_2}, respectively.*

Proof Recall that Li [7] has got the topological equivalence between the Teichmüller metric and the length spectrum metric from inequality (3.11). Since Theorem 3.2 is a slight improvement of (3.11), we will get the topological equivalences by following Li's idea in proving his Theorem 1 in [7].

Second, we show the completeness of the length spectrum metric d_L from the viewpoint of Finsler geometry.

Theorem 4.2 *The length spectrum metric d_L is a complete Finsler metric on $T(X)$.*

Proof Thurston [2] showed that d_L is a Finsler metric.

Recall the following version of the Hopf-Rinow theorem (see [26]) for Finsler metrics on connected manifolds: a Finsler metric is compete if and only if every bounded closed subset is compact. Since d_L and d_T are both Finsler metrics, we will use this criterion to show the completeness of d_L.

Let $V \subset T(X)$ be a bounded closed subset with respect to d_L. Then, from Theorems 3.2 and 4.1, V is also a bounded closed subset with respect to d_T. But d_T is complete as a Finsler metric, so from the above Hopf-Rinow theorem we know that V is compact. Again from the Hopf-Rinow theorem, it follows that the length spectrum metric d_L is complete.

Theorem 4.1 tells us that d_T and each of d_L, d_{P_i}, $i = 1, 2$ go to zero simultaneously. As the last application to Theorem 3.2, we will give a simple proof of the following theorem (see [17, Theorem 2.25]).

Theorem 4.3 *Let X be a non-elementary Riemann surface of conformally finite type (g,n). Let $\{x_n\}_{n=0}^{\infty}$ be a sequence in $T(X)$. Then*

$$\lim_{n\to\infty} d_T(x_n,x_0)=\infty \Longleftrightarrow \lim_{n\to\infty} d_L(x_n,x_0)=\infty,$$
$$\lim_{n\to\infty} d_T(x_n,x_0)=\infty \Longleftrightarrow \lim_{n\to\infty} d_{P_i}(x_n,x_0)=\infty, \quad i=1,2.$$

Proof We will prove the first equivalence in this theorem. The second one can be proved similarly.

The implication
$$\lim_{n\to\infty} d_L(x_n,x_0)=\infty \Longrightarrow \lim_{n\to\infty} d_T(x_n,x_0)=\infty$$
follows directly from Lemma 1.1.

For the other implication, Theorem 3.2 gives

$$d_T(x_0,x_n)\le d_L(x_0,x_n)+C_B(x_0).$$

This indicates that
$$\lim_{n\to\infty} d_T(x_n,x_0)=\infty \Longrightarrow \lim_{n\to\infty} d_L(x_n,x_0)=\infty.$$

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