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The Cauchy Integral Operator on Weighted Hardy Space***

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Abstract The authors show that the Cauchy integral operator is bounded from $H^p_\omega(R^1)$ to $h^p_\omega(R^1)$ (the weighted local Hardy space). To prove the results, a kind of generalized atoms is introduced and a variant of weighted "Tb theorem" is considered.

Keywords Cauchy integral, Calderón-Zygmund operator, Weighted Hardy space, Weighted local Hardy space
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1 Introduction

Let R^n be the n-dimensional Euclidean space. The Cauchy integral operator is defined by

$$C_Af(x) = \text{p.v.} \int_{R^1} \frac{1}{x-y+\mathrm{i}(A(x)-A(y))} f(y)\mathrm{d}y, \tag{1.1}$$

where $A(x)$ is a real valued function. This operator is very important in real and complex analysis, and has attracted many mathematicians to investigate it (see, for example, [2–4, 11]).

It is well-known that C_A is bounded on $L^p(R^1)$, but few results are known on the Hardy space $H^p(R^1)$. Recently, Komori [14] showed that C_A is bounded from $H^p(R^1)$ to $h^p(R^1)$ (the local Hardy space). In this paper, we consider the weighted version of Hardy space and show that C_A is bounded from $H^p_w(R^1)$ to $h^p_w(R^1)$. To prove the theorem, we introduce a kind of generalized atoms and consider a variant of weighted "Tb theorem".

2 Definitions and Notations

Throughout this paper, we always use the letter C to denote positive constants that may vary at each occurrence, but is independent of the essential variables. And we assume that, unless otherwise stated, all given functions are complex valued.

We denote the Euclidean ball with center x of radius r by $B(x,r)$, and the Lebesgue measure of a measurable set E by $|E|$.

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Let φ be a fixed real valued Schwartz function in $S(R^n)$ such that $\mathrm{supp}(\varphi) \subset B(0,1)$ and $\int \varphi(x)\mathrm{d}x = 1$. We denote

$$f^{++}(x) = \sup_{t>0} |\varphi_t * f(x)|, \quad f^{+}(x) = \sup_{0<t<1} |\varphi_t * f(x)|,$$

where $\varphi_t(x) = \frac{1}{t^n}\varphi(\frac{x}{t})$.

Definition 2.1 *Let $0 < p \leq 1$, the Hardy space $H^p(R^n)$ (see [6]) and the local Hardy space $h^p(R^n)$ (see [9]) are defined respectively by*

$$H^p(R^n) = \{f \in S'(R^n) : \|f\|_{H^p} = \|f^{++}\|_{L^p} < \infty\}$$

and

$$h^p(R^n) = \{f \in S'(R^n) : \|f\|_{h^p} = \|f^{+}\|_{L^p} < \infty\}.$$

We remark that $H^p(R^n) \subset h^p(R^n)$, $h^1(R^n) \subset L^1(R^n)$ and all the inclusions are proper.

Definition 2.2 *For $0 < \alpha \leq 1$, the Lipschitz space $\Lambda^\alpha(R^n)$ and the local Lipschitz space $\Lambda^\alpha_{\mathrm{loc}}(R^n)$ are the sets of all functions f satisfying the following conditions respectively*

$$\|f\|_{\Lambda^\alpha} = \sup_{0<|x-y|} \frac{|f(x)-f(y)|}{|x-y|} < \infty,$$
$$\|f\|_{\Lambda^\alpha_{\mathrm{loc}}} = \sup_{0<|x-y|<2} \frac{|f(x)-f(y)|}{|x-y|} < \infty.$$

It is easy to see that $\Lambda^1(R^n) = \Lambda^1_{\mathrm{loc}}(R^n)$ and $\Lambda^\alpha(R^n) \subset \Lambda^\alpha_{\mathrm{loc}}(R^n)$ $(0 < \alpha < 1)$, where the inclusion is proper. Furthermore, we know that the dual space of $H^p(R^n)$ is $\Lambda^{n(\frac{1}{p}-1)}(R^n)$, i.e. $(H^p(R^n))^* = \Lambda^{n(\frac{1}{p}-1)}(R^n)$, where $\frac{n}{n+1} < p < 1$ (see [6]).

Definition 2.3 *A locally integrable function f is in BMO if*

$$\sup_B \frac{1}{|B|}\int_B |f - m_B f|\mathrm{d}x < \infty,$$

where $m_B f = \frac{1}{|B|}\int_B f(x)\mathrm{d}x$ and the supremum is taken over all balls B. We denote the supremum by $\|f\|_{\mathrm{BMO}}$.

A weight is a nonnegative, locally integrable function. We consider weights satisfying the following conditions.

Definition 2.4 *Let $1 < q < \infty$. We say that a weight w satisfies the A_q condition if there exists a positive constant C such that for all balls B,*

$$\Big(\frac{1}{|B|}\int_B w(x)\mathrm{d}x\Big)\Big(\frac{1}{|B|}\int_B w(x)^{-\frac{1}{q-1}}\mathrm{d}x\Big)^{q-1} \leq C.$$

We also say that a weight w satisfies the A_1 condition if there is a constant $C > 0$ such that for all balls B,

$$\frac{1}{|B|}\int_B w(x)\mathrm{d}x \leq C \operatorname*{ess\,inf}_{x\in B} w(x).$$

Finally, we define $A_\infty = \bigcup_{q\geq 1} A_q$.

Observe that $A_p \subset A_q$ if $1 \le p < q$.

Definition 2.5 (see [17]) *Let $w \in A_\infty$ and $0 < p \le 1$. We define the weighted Hardy space $H^p_w(R^n)$ and the weighted local Hardy space $h^p_w(R^n)$ as follows:*

$$H^p_w(R^n) = \{f \in S'(R^n) : \|f\|_{H^p_w} = \|f^{++}\|_{L^p_w} < \infty\}$$

and

$$h^p_w(R^n) = \{f \in S'(R^n) : \|f\|_{h^p_w} = \|f^{+}\|_{L^p_w} < \infty\}.$$

Next we define Calderón-Zygmund operators. One may refere to [10, 16]. But since we are interested in the Cauchy integral operator, our definitions will be presented as follows (see [14]).

Definition 2.6 *Let $0 < \delta \le 1$. A locally integrable function $K(x, y)$ defined on $\{(x, y) \in R^n \times R^n : x \neq y\}$ is called a Calderón-Zygmund kernel if it satisfies the following conditions:*

$$|K(x, y)| \le \frac{C}{|x - y|^n}, \tag{2.1}$$

$$|K(x, y) - K(x, z)| + |K(y, x) - K(z, x)| \le C\frac{|y - z|^\delta}{|x - z|^{n+\delta}}, \quad 2|y - z| < |x - z|. \tag{2.2}$$

Definition 2.7 *We say that an operator T is a δ-Calderón-Zygmund operator associated with a Calderón-Zygmund kernel $K(x, y)$ if for every $f \in L^2(R^n)$,*

$$Tf(x) = \lim_{\epsilon \to 0} \int_{|x-y|>\epsilon} K(x, y)f(y)\mathrm{d}y$$

exists almost everywhere in R^n and T is bounded on $L^2(R^n)$, i.e. $\|Tf\|_{L^2} \le C\|f\|_{L^2}$.

Remark 2.1 If T is a δ-Calderón-Zygmund operator and $w \in A_q$, then T is bounded on $L^q_w(R^n)$, $q > 1$ (see [7], [11, p. 52] and [15]).

Definition 2.8 *The transpose of an operator T is denoted by*

$${}^{\mathrm{t}}Tf(x) = \lim_{\epsilon \to 0} \int_{|x-y|>\epsilon} K(y, x)f(y)\mathrm{d}y.$$

Definition 2.9 *For a bounded function b, we define*

$$\widetilde{{}^{\mathrm{t}}T}b(x) = \lim_{\epsilon \to 0} \int_{|x-y|>\epsilon} \{K(y, x) - K(y, 0)\chi_{|y|\ge 1}(y)\}b(y)\mathrm{d}y.$$

Note that if $b \in L^2(R^n) \cap L^\infty(R^n)$, then $\widetilde{{}^{\mathrm{t}}T}b(x) = {}^{\mathrm{t}}Tb(x) + C_b$ a.e., where C_b is a constant.

Definition 2.10 *Let $\beta > 0$. A bounded function b is said to be β-accretive if $\mathrm{Re}\, b(x) \ge \beta$ for almost all x.*

3 Theorems

First we recall some known results. The L^p boundedness of C_A is well-known, and the following theorem is the most essential (see [3] and [10, p. 647]).

Theorem 3.1 *If $A' \in L^\infty(R^1)$, then the Cauchy integral operator C_A defined by* (1.1) *is a* 1-*Calderón-Zygmund operator.*

Recently, Komori [14] showed that C_A is bounded from $H^p(R^1)$ to $h^p(R^1)$, i.e., the following theorem.

Theorem 3.2 *Let $0 < \alpha < 1$ and $\frac{1}{1+\alpha} \le p \le 1$. If $A' \in L^\infty(R^1) \cap \wedge^\alpha(R^1)$, then C_A is bounded from $H^p(R^1)$ to $h^p(R^1)$.*

In order to prove Theorem 3.2, he considered a variant of "Tb theorem" as follows.

Theorem 3.3 *Let $0 < \alpha < 1$, $\frac{n}{n+\delta} < p \le 1$, $\frac{n}{n+\alpha} \le p$ and T be a δ-Calderón-Zygmund operator. If there exists a β-accretive function b such that $b, \widetilde{{}^tT}b \in \wedge^\alpha(R^n)$, then T is a bounded operator from $H^p(R^n)$ to $h^p(R^n)$ and*

$$\|Tf\|_{h^p} \le C\|f\|_{H^p}.$$

As a corollary, he got Theorem 3.2 by Theorem 3.3.

Remark 3.1 In [14], the author obtained the $h^p(R^n)$ $(0 < p < 1)$ estimate by $(H^p(R^n))^* = \Lambda^{n(\frac{1}{p}-1)}(R^n)$ and the $h^1(R^n)$ estimate by interpolation. In this paper, we use a different method to prove our theorems. The details are presented in Section 5.

Now we turn to the weighted Hardy space case. Our main results are the following.

Theorem 3.4 *Let $0 < \alpha \le 1 \le q$, $\frac{nq}{n+\delta} < p \le 1$ and $\frac{nq}{n+\alpha} \le p < q$. Assume that $w \in A_q$ and T is a δ-Calderón-Zygmund operator. If there exists a β-accretive function b such that b, $\widetilde{{}^tT}b \in \wedge^\alpha_{\mathrm{loc}}(R^n)$, then T is a bounded operator from $H^p_w(R^n)$ to $h^p_w(R^n)$ and*

$$\|Tf\|_{h^p_w} \le C\|f\|_{H^p_w}.$$

At the end point $p = 1$, if we strengthen the weight condition, we have

Theorem 3.5 *If $w \in A_1$ and T is a δ-Calderón-Zygmund operator, then T is a bounded operator from $H^1_w(R^n)$ to $h^1_w(R^n)$ and*

$$\|Tf\|_{h^1_w} \le C\|f\|_{H^1_w}.$$

Remark 3.2 For a δ-Calderón-Zygmund operator T and $w \in A_1$, it is easy to check that $\|Tf\|_{L^1_w} \le C\|f\|_{H^1_w}$ by some standard argument. Furthermore, Quek and Yang [15] obtained that $\|Tf\|_{H^1_w} \le C\|f\|_{H^1_w}$ if $\widetilde{{}^tT}1 = C$ and Komori [13] obtained that $\|Tf\|_{h^1_w} \le C\|f\|_{H^1_w}$ if $\widetilde{{}^tT}1 \in \wedge^\alpha(R^n)$. Since $H^1_w(R^n) \subset h^1_w(R^n) \subset L^1_w(R^n)$, Theorem 3.5 extends these results. Especially, taking $w = 1$, our results are also new.

As a corollary of these theorems, we obtain the boundedness of the Cauchy integral operator.

Theorem 3.6 *Let $0 < \alpha \le 1 \le q$, $\frac{q}{2} < p \le 1$ and $\frac{q}{1+\alpha} \le p < q$. If $w \in A_q$ and $A' \in L^\infty(R^1) \cap \wedge^\alpha_{\mathrm{loc}}(R^1)$, then C_A is bounded from $H^p_w(R^1)$ to $h^p_w(R^1)$.*

Theorem 3.7 *If $A' \in L^\infty(R^1)$ and $w \in A_1$, then C_A is bounded from $H^1_w(R^1)$ to $h^1_w(R^1)$.*

4 Some Lemmas

First we present two elementary lemmas on weight functions without proof (see [8] or [19, p. 226]).

Lemma 4.1 *If* $w \in A_q$, $q \geq 1$, *then there exists a positive constant* C *such that*

$$\frac{w(B(x_0,r))}{w(B(x_0,s))} \leq C\Big(\frac{|B(x_0,r)|}{|B(x_0,s)|}\Big)^q$$

for all $r > s$ *and* $x_0 \in R^n$. *Especially*

$$w(B(x_0,2^j r)) \leq C2^{nqj} w(B(x_0,r)).$$

Lemma 4.2 *Let* f *be a nonnegative locally integrable function. If* $w \in A_q$, $q \geq 1$, *then*

$$\frac{1}{|B(x_0,r)|}\int_{B(x_0,r)} f(x)\mathrm{d}x \leq C\Big(\frac{1}{w(B(x_0,r))}\int_{B(x_0,r)} f(x)^q w(x)\mathrm{d}x\Big)^{\frac{1}{q}}.$$

Next we define atoms and molecules in $H^p_w(R^n)$ and $h^p_w(R^n)$ and obtain some properties of Hardy spaces.

Definition 4.1 *Let* $1 \leq q \leq \infty$. *A function* $a(x)$ *is called an* (H^p_w, q)*-atom centered at* x_0 *if there exists a ball* $B(x_0,r)$ *such that the following conditions are satisfied*

$$\mathrm{supp}(a) \subset B(x_0,r), \tag{4.1}$$

$$\|a\|_{L^q_w} \leq w(B(x_0,r))^{\frac{1}{q}-\frac{1}{p}}, \tag{4.2}$$

$$\int_{R^n} a(x)\mathrm{d}x = 0. \tag{4.3}$$

Lemma 4.3 (see [7] and [17, p. 111]) *Let* $\frac{nq}{n+1} < p \leq 1$ *and* $p < q$. *If* $w \in A_q$ *and* $a(x)$ *is an* (H^p_w, q)*-atom, then* $a \in H^p_w(R^n)$ *and*

$$\|a(x)\|_{H^p_w} \leq C.$$

Furthermore, we have the atomic decomposition of $H^p_w(R^n)$ (see [7, 17]) as follows.

Lemma 4.4 *Let* $\frac{nq}{n+1} < p \leq 1$ *and* $p < q$. *If* $w \in A_q$ *and* $f \in H^p_w(R^n)$, *then* f *can be written as*

$$f = \sum_{j=1}^{\infty} \lambda_j a_j,$$

where a_j*'s are* (H^p_w, q)*-atoms and*

$$\sum_{j=1}^{\infty} |\lambda_j|^p \approx \|f\|^p_{H^p_w}.$$

Definition 4.2 *Let* $1 \leq q \leq \infty$. *A function* $a(x)$ *is called a large* (h^p_w, q)*-atom centered at* x_0 *if there exists a ball* $B(x_0,r)$ *of radius* $r \geq 1$ *such that conditions* (4.1) *and* (4.2) *are satisfied.*

Lemma 4.5 (see [13]) *Let* $\frac{n}{n+1} < p \leq 1 \leq q$ *and* $p < q$. *If* $w \in A_q$ *and* $a(x)$ *is a large* (h^p_w, q)*-atom, then* $a \in h^p_w(R^n)$ *and*

$$\|a\|_{h^p_w} \leq C.$$

Definition 4.3 *Let $1 \le q \le \infty$. A function $a(x)$ is a small (h_w^p, q)-atom centered at x_0 if there exists a ball $B(x_0, r)$ of radius $r < 1$, which satisfies conditions* (4.1), (4.2) *and*

$$\Big|\int_{R^n} a(x)\mathrm{d}x\Big| \le r^{n\frac{q-1}{p}}\Big(\frac{|B(x_0,r)|}{w(B(x_0,r))}\Big)^{\frac{1}{p}}.$$

Lemma 4.6 (see [13]) *Let $\frac{nq}{n+1} < p \le 1$ and $p < q$. If $w \in A_q$ and $a(x)$ is a small (h_w^p, q)-atom, then $a \in h_w^p(R^n)$ and*

$$\|a\|_{h_w^p} \le C.$$

Definition 4.4 *Let b be β-accretive and $1 \le q \le \infty$. A function $a(x)$ is a small (h_w^p, q, b)-atom centered at x_0 if there exists a ball $B(x_0, r)$ of radius $r < 1$, which satisfies conditions* (4.1), (4.2) *and*

$$\Big|\int_{R^n} a(x)b(x)\mathrm{d}x\Big| \le r^{n\frac{q-1}{p}}\Big(\frac{|B(x_0,r)|}{w(B(x_0,r))}\Big)^{\frac{1}{p}}. \tag{4.4}$$

Lemma 4.7 *Let b be β-accretive and $b \in \wedge_{\mathrm{loc}}^{\alpha}(R^n)$ $(0 < \alpha \le 1)$. Assume that $\frac{nq}{n+1} < p \le 1 \le q$ and $\frac{nq}{n+\alpha} \le p < q$. If $w \in A_q$ and $a(x)$ is a small (h_w^p, q, b)-atom, then $a \in h_w^p(R^n)$ and*

$$\|a\|_{h_w^p} \le C.$$

Proof According to Lemma 4.6, we only need to show that a is a small (h_w^p, q)-atom.

$$\begin{aligned}
\Big|\int_{B(x_0,r)} a(x)\mathrm{d}x\Big| &\le \Big|\frac{1}{b(x_0)}\int_{B(x_0,r)} a(x)(b(x)-b(x_0))\mathrm{d}x\Big| + \Big|\frac{1}{b(x_0)}\int_{B(x_0,r)} a(x)b(x)\mathrm{d}x\Big| \\
&\le C\frac{r^{\alpha}}{\beta}\Big(\int_{B(x_0,r)} |a(x)|^q w(x)\mathrm{d}x\Big)^{\frac{1}{q}}\frac{|B(x_0,r)|}{w(B(x_0,r))^{\frac{1}{q}}} + \frac{1}{\beta}r^{n\frac{q-1}{p}}\Big(\frac{|B(x_0,r)|}{w(B(x_0,r))}\Big)^{\frac{1}{p}} \\
&\le Cr^{\alpha+n-\frac{n}{p}}\Big(\frac{|B(x_0,r)|}{w(B(x_0,r))}\Big)^{\frac{1}{p}} + \frac{1}{\beta}r^{n\frac{q-1}{p}}\Big(\frac{|B(x_0,r)|}{w(B(x_0,r))}\Big)^{\frac{1}{p}} \\
&\le Cr^{n\frac{q-1}{p}}\Big(\frac{|B(x_0,r)|}{w(B(x_0,r))}\Big)^{\frac{1}{p}}.
\end{aligned}$$

Note that we have used the fact $r < 1$ and $\frac{nq}{n+\alpha} \le p$ in the last inequality.

Definition 4.5 *Let $w \in A_q$. A function $M(x)$ is called a large (h_w^p, q, δ)-molecule centered at x_0 if there exists $r \ge 1$ such that the following conditions are satisfied:*

$$\Big(\int_{|x-x_0|\le 2r} |M(x)|^q w(x)\mathrm{d}x\Big)^{\frac{1}{q}} \le Cw(B(x_0,r))^{\frac{1}{q}-\frac{1}{p}}, \tag{4.5}$$

$$|M(x)| \le \frac{r^{n+\delta}w(B(x_0,r))^{-\frac{1}{p}}}{|x-x_0|^{n+\delta}}, \quad |x-x_0| \ge 2r. \tag{4.6}$$

Definition 4.6 *Let $w \in A_q$ and b be β-accretive. A function $M(x)$ is called a small (h_w^p, q, δ, b)-molecule centered at x_0 if there exists $r < 1$ such that* (4.5), (4.6) *and the following condition*

$$\Big|\int_{R^n} M(x)b(x)\mathrm{d}x\Big| \le Cr^{n\frac{q-1}{p}}\Big(\frac{|B(x_0,r)|}{w(B(x_0,r))}\Big)^{\frac{1}{p}}. \tag{4.7}$$

Lemma 4.8 *Let $\frac{nq}{n+\delta} < p \le 1 \le q$ and $p < q$. If $w \in A_q$ and a function $M(x)$ is a large (h^p_w, q, δ)-molecule, then $M \in h^p_w(R^n)$ and*

$$\|M\|_{h^p_w} \le C.$$

Lemma 4.9 *Let $0 < \alpha \le 1 \le q$, $\frac{nq}{n+\delta} < p < q$ and $\frac{nq}{n+\alpha} \le p \le 1$. Suppose that b is β-accretive and $b \in \wedge^\alpha_{\mathrm{loc}}(R^n)$. If a function $M(x)$ is a small (h^p_w, q, δ, b)-molecule, then $M \in h^p_w(R^n)$ and*

$$\|M\|_{h^p_w} \le C.$$

Lemmas 4.8 and 4.9 are the key lemmas to prove our theorems. The proofs of the two lemmas are similar in nature. So we shall only prove Lemma 4.9 below. The idea of our proof comes from [12–14, 18].

Proof of Lemma 4.9 Let $E_0 = \{x : |x - x_0| < 2r\}$ and $E_i = \{x : 2^i r \le |x - x_0| < 2^{i+1} r\}$, $i = 1, 2, 3, \cdots$, $b(E_i) = \int_{E_i} b(x)\mathrm{d}x$. Since $b(E_i) \ne 0$, we denote $\chi_i = \chi_{E_i}(x)$, $\widetilde{\chi}_i = \frac{\chi_i}{b(E_i)}$, $m_i = \frac{1}{b(E_i)} \int_{E_i} b(x)M(x)\mathrm{d}x$ and $\widetilde{m}_i = \int_{E_i} b(x)M(x)\mathrm{d}x$.

We write

$$M(x) = \sum_{i=0}^{\infty}(M(x) - m_i)\chi_i(x) + \sum_{i=0}^{\infty} m_i\chi_i(x) = \sum_{i=0}^{\infty} M_i(x) + \sum_{i=0}^{\infty} \widetilde{m}_i\widetilde{\chi}_i(x),$$

where $M_i(x) = (M(x) - m_i)\chi_i(x)$. Letting $N_j = \sum\limits_{i=j}^{\infty} \widetilde{m}_i$, we have

$$\begin{aligned} M(x) &= \sum_{i=0}^{\infty} M_i(x) + \sum_{i=1}^{\infty} N_i(\widetilde{\chi}_i(x) - \widetilde{\chi}_{i-1}(x)) + N_0\widetilde{\chi}_0(x) \\ &= \mathrm{I} + \mathrm{II} + \mathrm{III}. \end{aligned}$$

Next we shall estimate the above three terms.

(a) It is clear that $\mathrm{supp}(M_i) \subset B(x_0, 2^{i+1}r)$ and $\int M_i(x)b(x)\mathrm{d}x = 0$. So

$$\begin{aligned} \Big(\int |M_0(x)|^q w(x)\mathrm{d}x\Big)^{\frac{1}{q}} &\le \Big(\int_{E_0} |M(x)|^q w(x)\mathrm{d}x\Big)^{\frac{1}{q}} + |m_0|\Big|\int_{E_0} w(x)\mathrm{d}x\Big|^{\frac{1}{q}} \\ &\le Cw(B(x_0, r))^{\frac{1}{q}-\frac{1}{p}} + |m_0|w(E_0)^{\frac{1}{q}}. \end{aligned}$$

By the definition of m_0 and Lemma 4.2, we have

$$\begin{aligned} |m_0| &= \frac{1}{|b(E_0)|}\Big|\int_{E_0} M(y)b(y)\mathrm{d}y\Big| \\ &\le C\frac{\|b\|_{L^\infty}}{\beta|E_0|}\int_{E_0} |M(y)|\mathrm{d}y \\ &\le C\Big(\frac{1}{w(E_0)}\int_{E_0} |M(y)|^q w(y)\mathrm{d}y\Big)^{\frac{1}{q}} \\ &\le Cw(B(x_0, r))^{\frac{1}{q}-\frac{1}{p}} w(E_0)^{-\frac{1}{q}}. \end{aligned}$$

Therefore, we get

$$\Big(\int |M_0(x)|^q w(x)\mathrm{d}x\Big)^{\frac{1}{q}} \le Cw(B(x_0, r))^{\frac{1}{q}-\frac{1}{p}}.$$

So by Lemma 4.5 or 4.7 we have

$$\|M_0\|_{h^p_w} \leq C.$$

When $i \geq 1$, we have

$$\begin{aligned}\Big(\int |M_i(x)|^q w(x)\mathrm{d}x\Big)^{\frac{1}{q}} &\leq \Big(\int_{E_i} |M(x)|^q w(x)\mathrm{d}x\Big)^{\frac{1}{q}} + |m_i| w(E_i)^{\frac{1}{q}} \\ &= \widetilde{\mathrm{I}} + \widetilde{\mathrm{II}}.\end{aligned}$$

By condition (4.6), we have

$$\begin{aligned}\widetilde{\mathrm{I}} &\leq r^{n+\delta} w(B(x_0,r))^{-\frac{1}{p}} \Big(\int_{E_i} \frac{w(x)}{|x-x_0|^{q(n+\delta)}}\mathrm{d}x\Big)^{\frac{1}{q}} \\ &\leq \frac{r^{n+\delta}}{(2^i r)^{n+\delta}} w(B(x_0,r))^{-\frac{1}{p}} w(B(x_0,2^{i+1}r))^{\frac{1}{q}} \\ &\leq C2^{-i\delta} w(B(x_0,r))^{\frac{1}{q}-\frac{1}{p}}.\end{aligned}$$

The last inequality was obtained by Lemma 4.1.

Using the condition (4.6) again and the fact $|E_i| \approx |B(x_0,2^{i+1}r)|$, we have

$$\begin{aligned}\widetilde{\mathrm{II}} &\leq w(E_i)^{\frac{1}{q}} \frac{\|b\|_{L^\infty}}{\beta|E_i|} \int_{E_i} |M(y)|\mathrm{d}y \\ &\leq C\frac{w(E_i)^{\frac{1}{q}}}{|E_i|} \int_{E_i} \frac{r^{n+\delta} w(B(x_0,r))^{-\frac{1}{p}}}{|y-x_0|^{n+\delta}}\mathrm{d}y \\ &\leq C2^{-i\delta} w(B(x_0,r))^{\frac{1}{q}-\frac{1}{p}}.\end{aligned}$$

According to Lemma 4.1, we reach

$$\begin{aligned}\Big(\int |M_i|^q w(x)\mathrm{d}x\Big)^{\frac{1}{q}} &\leq C2^{-i\delta} w(B(x_0,r))^{\frac{1}{q}-\frac{1}{p}} \\ &\leq C2^{-i(\delta+nq(\frac{1}{q}-\frac{1}{p}))} w(B(x_0,2^{i+1}r))^{\frac{1}{q}-\frac{1}{p}}.\end{aligned}$$

By Lemma 4.5 or 4.7, we get

$$\|M_i\|_{h^p_w} \leq C2^{-i(\delta+nq(\frac{1}{q}-\frac{1}{p}))}.$$

Since $\frac{nq}{n+\delta} < p$, we finally conclude

$$\sum_{i=0}^{\infty} \|M_i\|^p_{h^p_w} \leq C \quad \text{and} \quad \|\mathrm{I}\|_{h^p_w} \leq C.$$

(b) Let $A_i = N_i(\widetilde{\chi}_i(x) - \widetilde{\chi}_{i-1}(x))$. It is easy to see that $\mathrm{supp}(A_i) \subset B(x_0,2^{i+1}r)$ and $\int_{R^n} A_i(x)b(x)\mathrm{d}x = 0$, $i = 1,2,3\cdots$.

Using condition (4.6) and Lemma 4.1, we have

$$\begin{aligned}\Big(\int |A_i(x)|^q w(x)\mathrm{d}x\Big)^{\frac{1}{q}} &\le |N_i|\Big(\int |\widetilde{\chi}_i(x)-\widetilde{\chi}_{i-1}(x)|^q w(x)\mathrm{d}x\Big)^{\frac{1}{q}}\\ &\le \Big(\frac{|w(E_i)|^{\frac{1}{q}}}{|b(E_i)|}+\frac{|w(E_{i-1})|^{\frac{1}{q}}}{|b(E_{i-1})|}\Big)\sum_{k=i}^{\infty}\Big|\int_{E_k} M(y)b(y)\mathrm{d}y\Big|\\ &\le C\frac{w(B(x_0,2^{i+1}r))^{\frac{1}{q}}}{\beta(2^i r)^n}\|b\|_{L^\infty}\sum_{k=i}^{\infty}\int_{E_k}|M(y)|\mathrm{d}y\\ &\le C\frac{w(B(x_0,r))^{\frac{1}{q}}}{r^n}\sum_{k=i}^{\infty}\int_{E_k}\frac{r^{n+\delta}w(B(x_0,r))^{-\frac{1}{p}}}{|y-x_0|^{n+\delta}}\mathrm{d}y\\ &\le Cr^{\delta}w(B(x_0,r))^{\frac{1}{q}-\frac{1}{p}}\int_{|y-x_0|\ge 2^i r}\frac{1}{|y-x_0|^{n+\delta}}\mathrm{d}y\\ &\le C2^{-i\delta}w(B(x_0,r))^{\frac{1}{q}-\frac{1}{p}}\\ &\le C2^{-i(\delta+nq(\frac{1}{q}-\frac{1}{p}))}w(B(x_0,2^{i+1}r))^{\frac{1}{q}-\frac{1}{p}}.\end{aligned}$$

So by Lemma 4.5 or 4.7, we have

$$\sum_{i=0}^{\infty}\|A_i\|_{h_w^p}^p\le C \quad \text{and} \quad \|\mathrm{II}\|_{h_w^p}\le C.$$

(c) Conditions (4.5), (4.6) together with the fact $\operatorname{supp}(N_0\widetilde{\chi}_0(x))\subset B(x_0,2r)$ imply

$$\begin{aligned}\Big(\int |N_0\widetilde{\chi}_0(x)|^q w(x)\mathrm{d}x\Big)^{\frac{1}{q}} &\le |N_0|\frac{1}{|b(E_0)|}\Big(\int_{E_0} w(x)\mathrm{d}x\Big)^{\frac{1}{q}}\\ &\le C\frac{w(E_0)^{\frac{1}{q}}}{\beta r^n}\Big|\int M(x)b(x)\mathrm{d}x\Big|\\ &\le C\frac{w(E_0)^{\frac{1}{q}}}{r^n}\|b\|_{L^\infty}\Big(\int_{|x-x_0|<2r}|M(x)|\mathrm{d}x+\int_{|x-x_0|\ge 2r}|M(x)|\mathrm{d}x\Big)\\ &\le Cw(B(x_0,2r))^{\frac{1}{q}-\frac{1}{p}}.\end{aligned}$$

By condition (4.7), we have

$$\Big|\int_{R^n} N_0\widetilde{\chi}_0(x)b(x)\mathrm{d}x\Big| = \Big|\int_{R^n} M(x)b(x)\mathrm{d}x\Big|\le Cr^{n\frac{q-1}{p}}\Big(\frac{|B(x_0,r)|}{w(B(x_0,r))}\Big)^{\frac{1}{p}}.$$

By Lemma 4.5 or 4.7, we have

$$\|N_0\widetilde{\chi}_0\|_{h_w^p}\le C.$$

Finally combining (a), (b) and (c), we complete the proof of Lemma 4.9.

Lemma 4.10 *Let $w\in A_1$ and $M(x)$ be a function on R^n. If there exists a ball $B(x_0,r)$, $r>0$, satisfies the following conditions*

$$\int_{|x-x_0|\le 2r}|M(x)|w(x)\mathrm{d}x\le C, \tag{4.8}$$

$$|M(x)|\le C\frac{r^{n+\delta}w(B(x_0,r))^{-1}}{|x-x_0|^{n+\delta}}, \quad |x-x_0|\ge 2r; \tag{4.9}$$

and furthermore, if $0<r<1$, $M(x)$ *also satisfies*

$$\Big|\int_{R^n} M(x)\mathrm{d}x\Big| \le C\frac{|B(x_0,r)|}{w(B(x_0,r))}, \tag{4.10}$$

then $M(x)\in h^1_w(R^n)$ *and*

$$\|M(x)\|_{h^1_w}\le C.$$

The proof of Lemma 4.10 is almost similar to that of Lemma 7 in [13], so we omit the details.

5 Proof of the Theorems

Proof of Theorem 3.4 By Lemma 4.4 and Theorem 7.2 in [1], it suffices to show that there is a constant $C>0$ such that $\|Ta\|_{h^p_w}\le C$ for every $(H^p_w,2q)$-atom $a(x)$.

Assuming (H^p_w,∞)-atom $a(x)$ supported in $B(x_0,r)$, we show that $Ta(x)$ is a constant multiple of a large (h^p_w,q,δ)-molecule with $r\ge 1$ or a constant multiple of a small (h^p_w,q,δ,b)-molecule with $r<1$.

Since T is bounded on $L^{2q}_w(R^n)$ (see [11, p. 52]), we have

$$\begin{aligned}\Big(\int_{|x-x_0|\le 2r}|Ta(x)|^q w(x)\mathrm{d}x\Big)^{\frac1q} &\le C\Big(\int_{|x-x_0|\le 2r}|Ta(x)|^{2q} w(x)\mathrm{d}x\Big)^{\frac1{2q}}\Big(\int_{|x-x_0|\le 2r} w(x)\mathrm{d}x\Big)^{\frac1{2q}}\\ &\le C\|a\|_{L^{2q}_w} w(B(x_0,r))^{\frac1{2q}}\\ &\le C w(B(x_0,r))^{\frac1q-\frac1p}.\end{aligned}$$

If $|x-x_0|\ge 2r$, then

$$\begin{aligned}|Ta(x)| &= \Big|\int_{B(x_0,r)}[K(x,y)-K(x,x_0)]a(y)\mathrm{d}y\Big|\\ &\le C\frac{r^\delta}{|x-x_0|^{n+\delta}}\int_{B(x_0,r)}|a(y)|\mathrm{d}y\\ &\le C\frac{r^{n+\delta}}{|x-x_0|^{n+\delta}} w(B(x_0,r))^{-\frac1p}.\end{aligned}$$

If $r\ge 1$, by Lemma 4.8, we have

$$\|Ta(x)\|_{h^p_w}\le C.$$

If $r<1$, by (4.2) and (4.3), we have

$$\begin{aligned}\Big|\int Ta(x)b(x)\mathrm{d}x\Big| &= |\langle a, {}^{\mathrm t}\widetilde{T}b\rangle| = \Big|\int_{B(x_0,r)} a(x)[{}^{\mathrm t}\widetilde{T}b(x)-{}^{\mathrm t}\widetilde{T}b(x_0)]\mathrm{d}x\Big|\\ &\le \|{}^{\mathrm t}\widetilde{T}b\|_{\Lambda^\alpha_{\mathrm{loc}}} r^\alpha\int_{B(x_0,r)}|a(x)|\mathrm{d}x\\ &\le C r^\alpha w(B(x_0,r))^{-\frac1p}|B(x_0,r)|\\ &\le C r^{\alpha+n\frac{p-1}{p}}\Big(\frac{|B(x_0,r)|}{w(B(x_0,r))}\Big)^{\frac1p}\\ &\le C r^{n\frac{q-1}{p}}\Big(\frac{|B(x_0,r)|}{w(B(x_0,r))}\Big)^{\frac1p},\end{aligned}$$

where the last inequality is obtained for $r < 1$ and $\frac{nq}{n+\alpha} \le p$.

By Lemma 4.9 and the above argument, we obtain the desired result

$$\|Ta\|_{h_w^p} \le C.$$

Proof of Theorem 3.5 Similar to the argument of Theorem 3.4, it suffices to show that there is a constant $C > 0$ such that $\|Ta\|_{h_w^1} \le C$ for all $(H_w^1, 2)$-atoms $a(x)$ supported in $B(x_0, r)$. It is easy to see

$$\int_{|x-x_0|\le 2r} |Ta(x)|w(x)\mathrm{d}x \le C.$$

And if $|x - x_0| \ge 2r$,

$$|Ta(x)| \le C\frac{r^{n+\delta}w(B(x_0,r))^{-1}}{|x-x_0|^{n+\delta}}.$$

If T is a δ-Calderón-Zygmund operator, then $\widetilde{{}^{\mathrm{t}}T}1 \in \mathrm{BMO}$ and $\|\widetilde{{}^{\mathrm{t}}T}1\|_{\mathrm{BMO}} \le C$ by the famous "T1 theorem" (see [5]). So we have

$$\begin{aligned}\Big|\int_{R^n} Ta(x)\mathrm{d}x\Big| &= |\langle a, \widetilde{{}^{\mathrm{t}}T}1\rangle| = \Big|\int_{B(x_0,r)} a(x)[\widetilde{{}^{\mathrm{t}}T}1(x) - m_B\widetilde{{}^{\mathrm{t}}T}1]\mathrm{d}x\Big| \\ &\le \|\widetilde{{}^{\mathrm{t}}T}1\|_{\mathrm{BMO}}|B(x_0,r)|w(B(x_0,r))^{-1} \\ &\le C\frac{|B(x_0,r)|}{w(B(x_0,r))},\end{aligned}$$

where $m_B f = \frac{1}{|B(x_0,r)|}\int_{B(x_0,r)} f(x)\mathrm{d}x$. According to Lemma 4.10 and the above argument, we get the desired conclusion immediately.

Proof of Theorem 3.6 Note that C_A is a 1-Calderón-Zygmund operator by Theorem 3.1. Let $b(x) = 1 + \mathrm{i}A'(x)$. Then b is a 1-accretive and $b \in \wedge_{\mathrm{loc}}^{\alpha}(R^1)$. By the calculus of complex analysis (refer to calculation in [14] or [19, p. 407]),

$$\begin{aligned}\widetilde{{}^{\mathrm{t}}C_A}b(x) &= \lim_{\epsilon\to 0}\int_{|x-y|>\epsilon}\Big\{\frac{1+\mathrm{i}A'(y)}{y-x+\mathrm{i}(A(y)-A(x))} - \frac{1+\mathrm{i}A'(y)}{y+\mathrm{i}(A(y)-A(0))}\chi_{|y|\ge 1}(y)\Big\}\mathrm{d}y \\ &= \text{constant},\end{aligned}$$

which implies $\widetilde{{}^{\mathrm{t}}C_A}b(x) \in \wedge_{\mathrm{loc}}^{\alpha}(R^1)$. Therefore the theorem is proved by Theorem 3.4.

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