Chin. Ann. Math.
31B(1), 2010, 119–132
DOI: 10.1007/s11401-008-0055-4

Chinese Annals of Mathematics, Series B

Weierstrass Representation for Surfaces in the Three-Dimensional Heisenberg Group***

Qun CHEN* Hongbing QIU**

Abstract The authors define the Gauss map of surfaces in the three-dimensional Heisenberg group and give a representation formula for surfaces of prescribed mean curvature. Furthermore, a second order partial differential equation for the Gauss map is obtained, and it is shown that this equation is the complete integrability condition of the representation.

Keywords Heisenberg group, Mean curvature, Weierstrass representation
2000 MR Subject Classification 53B25, 53C40

1 Introduction

It is well-known that the classical Weierstrass representation formula represents minimal surfaces in $\mathbb{R}^3$ via holomorphic functions. Since it is a fundamental and extremely useful tool in the theory of surfaces (see, e.g., [13]), many efforts have been made by geometers to extend it to general cases. For example, Kenmotsu [7] discovered a representation formula for surfaces of prescribed mean curvature in $\mathbb{R}^3$. In three-dimensional Minkowski space L^3, Kobayashi [8] proved the Weierstrass representation formula for maximal surfaces, and then Akutagawa and Nishikawa [1] generalized his results to the case of spacelike surfaces with prescribed mean curvature in L^3. In hyperbolic space H^3, Bryant [2] gave a representation formula for surfaces of constant mean curvature one. Later, Kokubu [9] obtained a formula for minimal surfaces in H^3. Generalizing these, Shi [11] proved the Weierstrass representation formula for surfaces of prescribed mean curvature in H^3.

On the other hand, the three-dimensional Heisenberg group Nil_3 equipped with the left invariant metric is one of the eight models in Thurston's geometries (see [12]), and it is interesting to consider surfaces in this space. In 2000, Inoguchi, Kumamoto, et al [5] derived a Weierstrass representation for minimal surfaces in Nil_3. Later, Inoguchi [6] obtained another integral representation formula for minimal surfaces in Nil_3 by making some improvement. Meanwhile, Mercuri, Montaldo and Piu [10] used a new method to get a representation formula for minimal surfaces in Nil_3. Also, Daniel [4] developed a method which is different from those of Inoguchi [6] and Mercuri, Montaldo and Piu [10] to give a representation for minimal surfaces in Nil_3.

Manuscript received February 14, 2008. Published online June 8, 2009.
*School of Mathematics and Statistics, Wuhan University, Wuhan 430072, China.
E-mail: qunchen@whu.edu.cn
**Corresponding author. School of Mathematical Sciences, Fudan University, Shanghai 200433, China.
E-mail: qhbing_2001@yahoo.com.cn
***Project supported by the National Natural Science Foundation of China (Nos. 10571068, 10871149) and the Research Fund for the Doctoral Program of Higher Education (No. 200804860046).

Recently, Berdinsky and Taimanov [3] obtained a generalized Weierstrass representation for surfaces in three-dimensional Lie groups including Nil_3 by using Dirac equations.

In this paper, we define the Gauss map of surfaces in the Heisenberg group Nil_3 and obtain a Weierstrass representation formula for surfaces of prescribed mean curvature, which is a generalization of the above mentioned results of Mercuri et al. We find that the complete integrability condition of this representation formula is exactly a second order partial differential equation for the Gauss map. Using our representation formula, we explicitly construct some examples of minimal surfaces as well as surface with constant mean curvature in Nil_3.

2 Surface Theory in Nil_3

The three-dimensional Heisenberg group Nil_3 can be viewed as $\mathbb{R}^3$ endowed with the left-invariant metric

$$g = \mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \Big(\mathrm{d}x_3 + \frac{1}{2}x_2\mathrm{d}x_1 - \frac{1}{2}x_1\mathrm{d}x_2\Big)^2.$$

It can also be represented in $\mathrm{GL}(3,\mathbb{R})$ by

$$\begin{bmatrix} 1 & x_1 & x_3 + \frac{1}{2}x_1x_2 \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{bmatrix}$$

with $x_i \in \mathbb{R}$, $i = 1, 2, 3$. The left-invariant orthonormal frame $\{E_1, E_2, E_3\}$ is given by

$$E_1 = \frac{\partial}{\partial x_1} - \frac{x_2}{2}\frac{\partial}{\partial x_3}, \quad E_2 = \frac{\partial}{\partial x_2} + \frac{x_1}{2}\frac{\partial}{\partial x_3}, \quad E_3 = \frac{\partial}{\partial x_3}.$$

Let $\overline{\nabla}$ be the Levi-Civita connection of Nil_3. The expression of $\overline{\nabla}$ in this frame is the following:

$$\begin{aligned}
&\overline{\nabla}_{E_1}E_1 = 0, && \overline{\nabla}_{E_2}E_1 = -\frac{1}{2}E_3, && \overline{\nabla}_{E_3}E_1 = -\frac{1}{2}E_2,\\
&\overline{\nabla}_{E_1}E_2 = \frac{1}{2}E_3, && \overline{\nabla}_{E_2}E_2 = 0, && \overline{\nabla}_{E_3}E_2 = \frac{1}{2}E_1,\\
&\overline{\nabla}_{E_1}E_3 = -\frac{1}{2}E_2, && \overline{\nabla}_{E_2}E_3 = \frac{1}{2}E_1, && \overline{\nabla}_{E_3}E_3 = 0.
\end{aligned}$$

Let $x = (x_1, x_2, x_3)$ be a point in Nil_3. For an arbitrary tangent vector X at x, if

$$X = a_1\frac{\partial}{\partial x_1} + a_2\frac{\partial}{\partial x_2} + a_3\frac{\partial}{\partial x_3},$$

then

$$X = a_1E_1 + a_2E_2 + \Big[a_3 + \frac{1}{2}(x_2a_1 - x_1a_2)\Big]E_3.$$

Let Σ be an oriented two-dimensional connected Riemannian manifold and $x : \Sigma \to \mathrm{Nil}_3$ an isometric immersion of Σ into Nil_3. In a neighborhood of any point of Σ, we choose an isothermal coordinate $z = \xi_1 + \mathrm{i}\xi_2$ and making use of it the metric of Σ can be written as $\mathrm{d}s^2 = \lambda^2|\mathrm{d}z|^2$ $(\lambda > 0)$. For $i = 1, 2$, let

$$e_i = \frac{1}{\lambda}\frac{\partial}{\partial \xi_i} = \frac{1}{\lambda}\sum_{k=1}^{3}\frac{\partial x_k}{\partial \xi_i}\frac{\partial}{\partial x_k} = \frac{1}{\lambda}\Big[\frac{\partial x_1}{\partial \xi_i}E_1 + \frac{\partial x_2}{\partial \xi_i}E_2 + \Big(\frac{\partial x_3}{\partial \xi_i} + \frac{1}{2}\Big(x_2\frac{\partial x_1}{\partial \xi_i} - x_1\frac{\partial x_2}{\partial \xi_i}\Big)\Big)E_3\Big].$$

Then $\{e_1, e_2\}$ defines an orthonomal tangent frame field on Σ compatible with the orientation. From $\langle e_i, e_j\rangle = \delta_{ij}$, we have

$$\frac{\partial x_1}{\partial \xi_i}\frac{\partial x_1}{\partial \xi_j}+\frac{\partial x_2}{\partial \xi_i}\frac{\partial x_2}{\partial \xi_j}+\Big[\frac{\partial x_3}{\partial \xi_i}+\frac{1}{2}\Big(x_2\frac{\partial x_1}{\partial \xi_i}-x_1\frac{\partial x_2}{\partial \xi_i}\Big)\Big]\Big[\frac{\partial x_3}{\partial \xi_j}+\frac{1}{2}\Big(x_2\frac{\partial x_1}{\partial \xi_j}-x_1\frac{\partial x_2}{\partial \xi_j}\Big)\Big]=\lambda^2\delta_{ij}, \quad (2.1)$$

where $i, j = 1, 2$.

Let n be a unit normal vector field of Σ, that is, $\langle n, n\rangle = 1$ and $\langle n, e_i\rangle = 0$ for $i = 1, 2$. In terms of the left-invariant orthonormal frame, n is given explicitly by $n = \sum_{i=1}^{3} e_{3i}E_i$, where

$$\begin{aligned}
e_{31} &= \frac{1}{\lambda^2}\Big[\frac{\partial x_2}{\partial \xi_1}\frac{\partial x_3}{\partial \xi_2}-\frac{\partial x_3}{\partial \xi_1}\frac{\partial x_2}{\partial \xi_2}+\frac{1}{2}x_2\Big(\frac{\partial x_2}{\partial \xi_1}\frac{\partial x_1}{\partial \xi_2}-\frac{\partial x_1}{\partial \xi_1}\frac{\partial x_2}{\partial \xi_2}\Big)\Big],\\
e_{32} &= \frac{1}{\lambda^2}\Big[\frac{\partial x_3}{\partial \xi_1}\frac{\partial x_1}{\partial \xi_2}-\frac{\partial x_1}{\partial \xi_1}\frac{\partial x_3}{\partial \xi_2}+\frac{1}{2}x_1\Big(\frac{\partial x_1}{\partial \xi_1}\frac{\partial x_2}{\partial \xi_2}-\frac{\partial x_2}{\partial \xi_1}\frac{\partial x_1}{\partial \xi_2}\Big)\Big],\\
e_{33} &= \frac{1}{\lambda^2}\Big(\frac{\partial x_1}{\partial \xi_1}\frac{\partial x_2}{\partial \xi_2}-\frac{\partial x_2}{\partial \xi_1}\frac{\partial x_1}{\partial \xi_2}\Big), \quad e_{31}^2+e_{32}^2+e_{33}^2=1.
\end{aligned} \quad (2.2)$$

We note that the dual coframe of $\{e_1, e_2\}$ on Σ is

$$w^i = \lambda \mathrm{d}\xi_i, \quad i = 1, 2,$$

and the connection 1-forms are

$$\omega_i^j = \frac{1}{\lambda^2}\frac{\partial \lambda}{\partial \xi_i}\omega^j - \frac{1}{\lambda^2}\frac{\partial \lambda}{\partial \xi_j}\omega^i, \quad i, j = 1, 2.$$

The formulas of Gauss and Weingarten for Σ in Nil_3 are the following:

$$\overline{\nabla}_{e_i} e_j = \sum_{k=1}^{2}\omega_j^k(e_i)e_k + h_{ij}n, \quad \overline{\nabla}_{e_i} n = -\sum_{k=1}^{2} h_{ik}e_k, \quad i, j = 1, 2,$$

where h_{ij} is the second fundamental form on Σ. By an elementary calculation, we see that the above formulas can be written in terms of local coordinates as follows:

$$\begin{aligned}
\frac{\partial^2 x_1}{\partial \xi_i \partial \xi_j} &= \frac{1}{\lambda}\Big(\frac{\partial \lambda}{\partial \xi_i}\frac{\partial x_1}{\partial \xi_j}+\frac{\partial \lambda}{\partial \xi_j}\frac{\partial x_1}{\partial \xi_i}-\delta_i^j\sum_{k=1}^{2}\frac{\partial \lambda}{\partial \xi_k}\frac{\partial x_1}{\partial \xi_k}\Big)-\frac{1}{2}\frac{\partial x_2}{\partial \xi_i}\Big[\frac{\partial x_3}{\partial \xi_j}+\frac{1}{2}\Big(x_2\frac{\partial x_1}{\partial \xi_j}\\
&\quad -x_1\frac{\partial x_2}{\partial \xi_j}\Big)\Big]-\frac{1}{2}\frac{\partial x_2}{\partial \xi_j}\Big[\frac{\partial x_3}{\partial \xi_i}+\frac{1}{2}\Big(x_2\frac{\partial x_1}{\partial \xi_i}-x_1\frac{\partial x_2}{\partial \xi_i}\Big)\Big]+\lambda^2 h_{ij}e_{31},\\
\frac{\partial^2 x_2}{\partial \xi_i \partial \xi_j} &= \frac{1}{\lambda}\Big(\frac{\partial \lambda}{\partial \xi_i}\frac{\partial x_2}{\partial \xi_j}+\frac{\partial \lambda}{\partial \xi_j}\frac{\partial x_2}{\partial \xi_i}-\delta_i^j\sum_{k=1}^{2}\frac{\partial \lambda}{\partial \xi_k}\frac{\partial x_2}{\partial \xi_k}\Big)+\frac{1}{2}\frac{\partial x_1}{\partial \xi_i}\Big[\frac{\partial x_3}{\partial \xi_j}+\frac{1}{2}\Big(x_2\frac{\partial x_1}{\partial \xi_j}\\
&\quad -x_1\frac{\partial x_2}{\partial \xi_j}\Big)\Big]+\frac{1}{2}\frac{\partial x_1}{\partial \xi_j}\Big[\frac{\partial x_3}{\partial \xi_i}+\frac{1}{2}\Big(x_2\frac{\partial x_1}{\partial \xi_i}-x_1\frac{\partial x_2}{\partial \xi_i}\Big)\Big]+\lambda^2 h_{ij}e_{32},\\
\frac{\partial^2 x_3}{\partial \xi_i \partial \xi_j} &= \frac{1}{\lambda}\Big[\frac{\partial \lambda}{\partial \xi_i}\frac{\partial x_3}{\partial \xi_j}+\frac{\partial \lambda}{\partial \xi_j}\frac{\partial x_3}{\partial \xi_i}+\frac{1}{2}\frac{\partial \lambda}{\partial \xi_i}\Big(x_2\frac{\partial x_1}{\partial \xi_j}-x_1\frac{\partial x_2}{\partial \xi_j}\Big)+\frac{1}{2}\frac{\partial \lambda}{\partial \xi_j}\Big(x_2\frac{\partial x_1}{\partial \xi_i}-x_1\frac{\partial x_2}{\partial \xi_i}\Big)\\
&\quad +\frac{x_1}{2}\frac{\partial \lambda}{\partial \xi_i}\frac{\partial x_2}{\partial \xi_j}+\frac{x_1}{2}\frac{\partial \lambda}{\partial \xi_j}\frac{\partial x_2}{\partial \xi_i}-\frac{x_2}{2}\frac{\partial \lambda}{\partial \xi_i}\frac{\partial x_1}{\partial \xi_j}-\frac{x_2}{2}\frac{\partial \lambda}{\partial \xi_j}\frac{\partial x_1}{\partial \xi_i}-\delta_i^j\sum_{k=1}^{2}\frac{\partial \lambda}{\partial \xi_k}\frac{\partial x_3}{\partial \xi_k}\\
&\quad -\delta_i^j\sum_{k=1}^{2}\frac{1}{2}\frac{\partial \lambda}{\partial \xi_k}\Big(x_2\frac{\partial x_1}{\partial \xi_k}-x_1\frac{\partial x_2}{\partial \xi_k}\Big)-\frac{x_1}{2}\delta_i^j\sum_{k=1}^{2}\frac{\partial \lambda}{\partial \xi_k}\frac{\partial x_2}{\partial \xi_k}+\frac{x_2}{2}\delta_i^j\sum_{k=1}^{2}\frac{\partial \lambda}{\partial \xi_k}\frac{\partial x_1}{\partial \xi_k}\Big]\\
&\quad +\frac{1}{4}x_1\frac{\partial x_1}{\partial \xi_i}\Big[\frac{\partial x_3}{\partial \xi_j}+\frac{1}{2}\Big(x_2\frac{\partial x_1}{\partial \xi_j}-x_1\frac{\partial x_2}{\partial \xi_j}\Big)\Big]+\frac{1}{4}x_1\frac{\partial x_1}{\partial \xi_j}\Big[\frac{\partial x_3}{\partial \xi_i}+\frac{1}{2}\Big(x_2\frac{\partial x_1}{\partial \xi_i}
\end{aligned} \quad (2.3)$$

$$- x_1\frac{\partial x_2}{\partial \xi_i}\Big)\Big] + \frac{1}{4}x_2\frac{\partial x_2}{\partial \xi_i}\Big[\frac{\partial x_3}{\partial \xi_j} + \frac{1}{2}\Big(x_2\frac{\partial x_1}{\partial \xi_j} - x_1\frac{\partial x_2}{\partial \xi_j}\Big)\Big] + \frac{1}{4}x_2\frac{\partial x_2}{\partial \xi_j}\Big[\frac{\partial x_3}{\partial \xi_i} + \frac{1}{2}\Big(x_2\frac{\partial x_1}{\partial \xi_i} - x_1\frac{\partial x_2}{\partial \xi_i}\Big)\Big] - \frac{\lambda^2}{2}x_2h_{ij}e_{31} + \frac{\lambda^2}{2}x_1h_{ij}e_{32} + \lambda^2 h_{ij}e_{33},$$

and

$$\begin{aligned}\frac{\partial e_{31}}{\partial \xi_i} &= -\frac{1}{2}\frac{\partial x_2}{\partial \xi_i}e_{33} - \frac{1}{2}\Big[\frac{\partial x_3}{\partial \xi_i} + \frac{1}{2}\Big(x_2\frac{\partial x_1}{\partial \xi_i} - x_1\frac{\partial x_2}{\partial \xi_i}\Big)\Big]e_{32} - \sum_{k=1}^{2} h_{ik}\frac{\partial x_1}{\partial \xi_k},\\ \frac{\partial e_{32}}{\partial \xi_i} &= \frac{1}{2}\frac{\partial x_1}{\partial \xi_i}e_{33} + \frac{1}{2}\Big[\frac{\partial x_3}{\partial \xi_i} + \frac{1}{2}\Big(x_2\frac{\partial x_1}{\partial \xi_i} - x_1\frac{\partial x_2}{\partial \xi_i}\Big)\Big]e_{31} - \sum_{k=1}^{2} h_{ik}\frac{\partial x_2}{\partial \xi_k},\\ \frac{\partial e_{33}}{\partial \xi_i} &= \frac{1}{2}\frac{\partial x_2}{\partial \xi_i}e_{31} - \frac{1}{2}\frac{\partial x_1}{\partial \xi_i}e_{32} - \sum_{k=1}^{2} h_{ik}\Big[\frac{\partial x_3}{\partial \xi_k} + \frac{1}{2}\Big(x_2\frac{\partial x_1}{\partial \xi_k} - x_1\frac{\partial x_2}{\partial \xi_k}\Big)\Big].\end{aligned} \tag{2.4}$$

The mean curvature is given by $H = \frac{1}{2}(h_{11} + h_{22})$. Let $\phi = \frac{1}{2}(h_{11} - h_{22}) - \mathrm{i}h_{12}$. The multiplication in Nil_3 is defined to be the multiplication of matrices, i.e., $x \cdot y = (x_1 + y_1, x_2 + y_2, x_3 + y_3 + \frac{1}{2}(x_1y_2 - x_2y_1))$ for $x = (x_1, x_2, x_3),\ y = (y_1, y_2, y_3) \in \mathrm{Nil}_3$. The unit element of Nil_3 is $e = (0, 0, 0)$, and $x^{-1} = (-x_1, -x_2, -x_3)$ for $x = (x_1, x_2, x_3) \in \mathrm{Nil}_3$.

As Nil_3 is a Lie group, by the left translation of n, we get

$$\widetilde{n} = L_{x^{-1}*}(n) = e_{31}\frac{\partial}{\partial x_1}(e) + e_{32}\frac{\partial}{\partial x_2}(e) + e_{33}\frac{\partial}{\partial x_3}(e) \in T_e(\mathrm{Nil}_3)$$

by using the stereographic projection with respect to the north pole and the south pole respectively, and we have two maps of Σ into $\mathbb{C} \cup \{\infty\}$ as follows:

$$\begin{aligned}G_1(x) &= \frac{e_{31} + \mathrm{i}e_{32}}{1 - e_{33}} \quad \text{for } \widetilde{n} \in U_1 = S^2(1) \setminus \{N\},\\ G_2(x) &= \frac{e_{31} - \mathrm{i}e_{32}}{1 + e_{33}} \quad \text{for } \widetilde{n} \in U_2 = S^2(1) \setminus \{S\},\end{aligned} \tag{2.5}$$

where $S^2(1)$ is the unit sphere of the Lie algebra of Nil_3, $N(S)$ is the north (south) pole of $S^2(1)$. The map $G = G_1$ (or G_2) is called the Gauss map of the surface $x(\Sigma)$.

3 Weierstrass Representation Formula

Let Σ be a surface immersed in Nil_3 by a mapping $x : \Sigma \to \mathrm{Nil}_3$, and G denote the Gauss map of Σ into $\mathbb{C} \cup \{\infty\}$ as in the above section. In this section, we shall give a Weierstrass representation formula for surfaces of prescribed mean curvature. First, we prove the following lemma.

Lemma 3.1 *If $x = (x_1, x_2, x_3) : \Sigma \to \mathrm{Nil}_3$ is an isometric immersion, then*

$$\frac{\partial x_1}{\partial z} + \mathrm{i}\frac{\partial x_2}{\partial z} = -G_1\Big[\frac{\partial x_3}{\partial z} + \frac{1}{2}\Big(x_2\frac{\partial x_1}{\partial z} - x_1\frac{\partial x_2}{\partial z}\Big)\Big], \tag{3.1}$$

$$\frac{\partial x_3}{\partial z} + \frac{1}{2}\Big(x_2\frac{\partial x_1}{\partial z} - x_1\frac{\partial x_2}{\partial z}\Big) = G_1\Big(\frac{\partial x_1}{\partial z} - \mathrm{i}\frac{\partial x_2}{\partial z}\Big), \tag{3.2}$$

$$\Big[\frac{\partial x_3}{\partial z} + \frac{1}{2}\Big(x_2\frac{\partial x_1}{\partial z} - x_1\frac{\partial x_2}{\partial z}\Big)\Big]\Big(\frac{\partial x_1}{\partial \overline{z}} + \mathrm{i}\frac{\partial x_2}{\partial \overline{z}}\Big) = \lambda^2\frac{G_1}{(1 + |G_1|^2)^2}. \tag{3.3}$$

Proof We see from (2.2) that

$$\begin{aligned} e_{31} &= \frac{2\mathrm{i}}{\lambda^2}\Big[\frac{\partial x_2}{\partial \overline{z}}\frac{\partial x_3}{\partial z} - \frac{\partial x_2}{\partial z}\frac{\partial x_3}{\partial \overline{z}} + \frac{1}{2}x_2\Big(\frac{\partial x_1}{\partial z}\frac{\partial x_2}{\partial \overline{z}} - \frac{\partial x_1}{\partial \overline{z}}\frac{\partial x_2}{\partial z}\Big)\Big], \\ e_{32} &= \frac{2\mathrm{i}}{\lambda^2}\Big[\frac{\partial x_1}{\partial z}\frac{\partial x_3}{\partial \overline{z}} - \frac{\partial x_1}{\partial \overline{z}}\frac{\partial x_3}{\partial z} + \frac{1}{2}x_1\Big(\frac{\partial x_1}{\partial \overline{z}}\frac{\partial x_2}{\partial z} - \frac{\partial x_1}{\partial z}\frac{\partial x_2}{\partial \overline{z}}\Big)\Big], \\ e_{33} &= \frac{2\mathrm{i}}{\lambda^2}\Big(\frac{\partial x_1}{\partial \overline{z}}\frac{\partial x_2}{\partial z} - \frac{\partial x_1}{\partial z}\frac{\partial x_2}{\partial \overline{z}}\Big). \end{aligned} \tag{3.4}$$

And from (2.5), we have

$$G_1 = \frac{e_{31} + \mathrm{i}e_{32}}{1 - e_{33}}, \tag{3.5}$$

$$(1 + |G_1|^2)(1 - e_{33}) = 2. \tag{3.6}$$

On the other hand, since $z = \xi_1 + \mathrm{i}\xi_2$, for which (ξ_1, ξ_2) is an isothermal coordinates on Σ, it follows from $\mathrm{d}s^2 = \lambda^2|\mathrm{d}z|^2$ that

$$\Big\langle\frac{\partial}{\partial z}, \frac{\partial}{\partial \overline{z}}\Big\rangle = \frac{\lambda^2}{2}, \quad \Big\langle\frac{\partial}{\partial z}, \frac{\partial}{\partial z}\Big\rangle = \Big\langle\frac{\partial}{\partial \overline{z}}, \frac{\partial}{\partial \overline{z}}\Big\rangle = 0,$$

i.e.,

$$\begin{aligned} &\frac{\partial x_1}{\partial z}\frac{\partial x_1}{\partial \overline{z}} + \frac{\partial x_2}{\partial z}\frac{\partial x_2}{\partial \overline{z}} + \Big[\frac{\partial x_3}{\partial z} + \frac{1}{2}\Big(x_2\frac{\partial x_1}{\partial z} - x_1\frac{\partial x_2}{\partial z}\Big)\Big]\Big[\frac{\partial x_3}{\partial \overline{z}} + \frac{1}{2}\Big(x_2\frac{\partial x_1}{\partial \overline{z}} - x_1\frac{\partial x_2}{\partial \overline{z}}\Big)\Big] = \frac{\lambda^2}{2}, \\ &\Big(\frac{\partial x_1}{\partial z}\Big)^2 + \Big(\frac{\partial x_2}{\partial z}\Big)^2 + \Big[\frac{\partial x_3}{\partial z} + \frac{1}{2}\Big(x_2\frac{\partial x_1}{\partial z} - x_1\frac{\partial x_2}{\partial z}\Big)\Big]^2 = 0, \\ &\Big(\frac{\partial x_1}{\partial \overline{z}}\Big)^2 + \Big(\frac{\partial x_2}{\partial \overline{z}}\Big)^2 + \Big[\frac{\partial x_3}{\partial \overline{z}} + \frac{1}{2}\Big(x_2\frac{\partial x_1}{\partial \overline{z}} - x_1\frac{\partial x_2}{\partial \overline{z}}\Big)\Big]^2 = 0. \end{aligned} \tag{3.7}$$

Substituting (3.4) into (3.5), and making use of (3.7) and (3.6), we can then obtain (3.1)–(3.3) through a straightforward calculation.

We shall now compute the derivatives of the Gauss map G. First we prove the following proposition.

Proposition 3.1

$$\frac{\partial G_1}{\partial \overline{z}} = -\frac{(1 + |G_1|^2)^2}{4}(\mathrm{i}e_{33}^2 + 2H)\Big(\frac{\partial x_1}{\partial \overline{z}} + \mathrm{i}\frac{\partial x_2}{\partial \overline{z}}\Big), \tag{3.8}$$

$$\frac{\partial G_1}{\partial z} = -\frac{(1 + |G_1|^2)^2}{2}\phi\Big(\frac{\partial x_1}{\partial \overline{z}} + \mathrm{i}\frac{\partial x_2}{\partial \overline{z}}\Big) - \mathrm{i}\Big(\frac{\partial x_1}{\partial z} + \mathrm{i}\frac{\partial x_2}{\partial z}\Big). \tag{3.9}$$

Proof By a simple calculation, we get

$$\frac{1}{2}\Big(h_{11}\frac{\partial x_i}{\partial \xi_1} + h_{12}\frac{\partial x_i}{\partial \xi_2} - \mathrm{i}h_{21}\frac{\partial x_i}{\partial \xi_1} - \mathrm{i}h_{22}\frac{\partial x_i}{\partial \xi_2}\Big) = H\frac{\partial x_i}{\partial z} + \phi\frac{\partial x_i}{\partial \overline{z}}.$$

Then from (2.4), we see

$$\begin{aligned} \frac{\partial e_{31}}{\partial z} &= -\frac{1}{2}e_{33}\frac{\partial x_2}{\partial z} - \frac{1}{2}e_{32}\Big(\frac{\partial x_3}{\partial z} + \frac{1}{2}x_2\frac{\partial x_1}{\partial z} - \frac{1}{2}x_1\frac{\partial x_2}{\partial z}\Big) - H\frac{\partial x_1}{\partial z} - \phi\frac{\partial x_1}{\partial \overline{z}}, \\ \frac{\partial e_{32}}{\partial z} &= \frac{1}{2}e_{33}\frac{\partial x_1}{\partial z} + \frac{1}{2}e_{31}\Big(\frac{\partial x_3}{\partial z} + \frac{1}{2}x_2\frac{\partial x_1}{\partial z} - \frac{1}{2}x_1\frac{\partial x_2}{\partial z}\Big) - H\frac{\partial x_2}{\partial z} - \phi\frac{\partial x_2}{\partial \overline{z}}, \\ \frac{\partial e_{33}}{\partial z} &= \frac{1}{2}e_{31}\frac{\partial x_2}{\partial z} - \frac{1}{2}e_{32}\frac{\partial x_1}{\partial z} - H\frac{\partial x_3}{\partial z} - \phi\frac{\partial x_3}{\partial \overline{z}} - \frac{1}{2}x_2\Big(H\frac{\partial x_1}{\partial z} + \phi\frac{\partial x_1}{\partial \overline{z}}\Big) \\ &\quad + \frac{1}{2}x_1\Big(H\frac{\partial x_2}{\partial z} + \phi\frac{\partial x_2}{\partial \overline{z}}\Big). \end{aligned} \tag{3.10}$$

Differentiating (3.5) with respect to $\overline{z}$ and applying (3.10), we get

$$\begin{aligned}\frac{\partial G_1}{\partial \overline{z}} &= \frac{\partial}{\partial \overline{z}}\Big(\frac{e_{31}+\mathrm{i}e_{32}}{1-e_{33}}\Big) = \frac{1}{1-e_{33}}\Big(\frac{\partial e_{31}}{\partial \overline{z}} + \mathrm{i}\frac{\partial e_{32}}{\partial \overline{z}} + G_1\frac{\partial e_{33}}{\partial \overline{z}}\Big)\\ &= \frac{1}{1-e_{33}}\Big[\frac{\mathrm{i}}{2}e_{33}\Big(\frac{\partial x_1}{\partial \overline{z}} + \mathrm{i}\frac{\partial x_2}{\partial \overline{z}}\Big) + \frac{\mathrm{i}}{2}\Big(\frac{\partial x_3}{\partial \overline{z}} + \frac{1}{2}x_2\frac{\partial x_1}{\partial \overline{z}} - \frac{1}{2}x_1\frac{\partial x_2}{\partial \overline{z}}\Big)G_1(1-e_{33})\\ &\quad - H\Big(\frac{\partial x_1}{\partial \overline{z}} + \mathrm{i}\frac{\partial x_2}{\partial \overline{z}}\Big) - \overline{\phi}\Big(\frac{\partial x_1}{\partial z} + \mathrm{i}\frac{\partial x_2}{\partial z}\Big) + G_1\Big(\frac{1}{2}e_{31}\frac{\partial x_2}{\partial \overline{z}} - \frac{1}{2}e_{32}\frac{\partial x_1}{\partial \overline{z}}\Big)\\ &\quad - G_1 H\Big(\frac{\partial x_3}{\partial \overline{z}} + \frac{1}{2}x_2\frac{\partial x_1}{\partial \overline{z}} - \frac{1}{2}x_1\frac{\partial x_2}{\partial \overline{z}}\Big) - G_1\overline{\phi}\Big(\frac{\partial x_3}{\partial z} + \frac{1}{2}x_2\frac{\partial x_1}{\partial z} - \frac{1}{2}x_1\frac{\partial x_2}{\partial z}\Big)\Big].\end{aligned}$$

Then by (3.1), (3.2) and (3.6), it is verified that

$$\frac{\partial G_1}{\partial \overline{z}} = -\frac{(1+|G_1|^2)^2}{4}(\mathrm{i}e_{33}^2 + 2H)\Big(\frac{\partial x_1}{\partial \overline{z}} + \mathrm{i}\frac{\partial x_2}{\partial \overline{z}}\Big).$$

Similarly, we obtain

$$\begin{aligned}\frac{\partial G_1}{\partial z} &= \frac{\partial}{\partial z}\Big(\frac{e_{31}+\mathrm{i}e_{32}}{1-e_{33}}\Big) = \frac{1}{1-e_{33}}\Big(\frac{\partial e_{31}}{\partial z} + \mathrm{i}\frac{\partial e_{32}}{\partial z} + G_1\frac{\partial e_{33}}{\partial z}\Big)\\ &= \frac{1}{1-e_{33}}\Big[\frac{\mathrm{i}}{2}e_{33}\Big(\frac{\partial x_1}{\partial z} + \mathrm{i}\frac{\partial x_2}{\partial z}\Big) + \frac{\mathrm{i}}{2}\Big(\frac{\partial x_3}{\partial z} + \frac{1}{2}x_2\frac{\partial x_1}{\partial z} - \frac{1}{2}x_1\frac{\partial x_2}{\partial z}\Big)G_1(1-e_{33})\\ &\quad - H\Big(\frac{\partial x_1}{\partial z} + \mathrm{i}\frac{\partial x_2}{\partial z}\Big) - \phi\Big(\frac{\partial x_1}{\partial \overline{z}} + \mathrm{i}\frac{\partial x_2}{\partial \overline{z}}\Big) + G_1\Big(\frac{1}{2}e_{31}\frac{\partial x_2}{\partial z} - \frac{1}{2}e_{32}\frac{\partial x_1}{\partial z}\Big)\\ &\quad - G_1 H\Big(\frac{\partial x_3}{\partial z} + \frac{1}{2}x_2\frac{\partial x_1}{\partial z} - \frac{1}{2}x_1\frac{\partial x_2}{\partial z}\Big) - G_1\phi\Big(\frac{\partial x_3}{\partial \overline{z}} + \frac{1}{2}x_2\frac{\partial x_1}{\partial \overline{z}} - \frac{1}{2}x_1\frac{\partial x_2}{\partial \overline{z}}\Big)\Big].\end{aligned}$$

By (3.1) and (3.2), it is verified that

$$\frac{\partial G_1}{\partial z} = -\frac{(1+|G_1|^2)^2}{2}\phi\Big(\frac{\partial x_1}{\partial \overline{z}} + \mathrm{i}\frac{\partial x_2}{\partial \overline{z}}\Big) - \mathrm{i}\Big(\frac{\partial x_1}{\partial z} + \mathrm{i}\frac{\partial x_2}{\partial z}\Big).$$

By the same argument, we can also prove the following

Proposition 3.2

$$\frac{\partial G_2}{\partial \overline{z}} = -\frac{(1+|G_2|^2)^2}{4}(\mathrm{i}e_{33}^2 + 2H)\Big(\frac{\partial x_1}{\partial \overline{z}} - \mathrm{i}\frac{\partial x_2}{\partial \overline{z}}\Big), \tag{3.11}$$

$$\frac{\partial G_2}{\partial z} = -\frac{(1+|G_2|^2)^2}{2}\phi\Big(\frac{\partial x_1}{\partial \overline{z}} - \mathrm{i}\frac{\partial x_2}{\partial \overline{z}}\Big) - \mathrm{i}\Big(\frac{\partial x_1}{\partial z} - \mathrm{i}\frac{\partial x_2}{\partial z}\Big). \tag{3.12}$$

We can calculate the norms of the complex vectors $\frac{\partial G_1}{\partial \overline{z}}$ and $\frac{\partial G_2}{\partial \overline{z}}$.

Corollary 3.1

$$\Big|\frac{\partial G_1}{\partial \overline{z}}\Big| = \frac{\lambda}{4}|\mathrm{i}e_{33}^2 + 2H|(1+|G_1|^2), \tag{3.13}$$

$$\Big|\frac{\partial G_2}{\partial \overline{z}}\Big| = \frac{\lambda}{4}|\mathrm{i}e_{33}^2 + 2H|(1+|G_2|^2). \tag{3.14}$$

Proof We shall prove only (3.13). By making use of (3.2) and (3.3), we have

$$\Big|\frac{\partial x_1}{\partial \overline{z}} + \mathrm{i}\frac{\partial x_2}{\partial \overline{z}}\Big| = \frac{\lambda}{1+|G_1|^2}.$$

Then from (3.8), we see

$$\Big|\frac{\partial G_1}{\partial \overline{z}}\Big| = \frac{\lambda}{4}|\mathrm{i}e_{33}^2 + 2H|(1+|G_1|^2).$$

Thus we have the following representation formula.

Theorem 3.1 *Let* $x = (x_1, x_2, x_3) : \Sigma \to \mathrm{Nil}_3$ *be an isometric immersion and* $G : \Sigma \to \mathbb{C} \cup \{\infty\}$ *be the Gauss map. Then we have*

$$\begin{aligned}
(2H - \mathrm{i}e_{33}^2)\frac{\partial x_1}{\partial z} &= \frac{2(G_1^2-1)}{(1+|G_1|^2)^2}\frac{\partial \overline{G}_1}{\partial z},\\
(2H - \mathrm{i}e_{33}^2)\frac{\partial x_2}{\partial z} &= -\frac{2\mathrm{i}(1+G_1^2)}{(1+|G_1|^2)^2}\frac{\partial \overline{G}_1}{\partial z},\\
(2H - \mathrm{i}e_{33}^2)\frac{\partial x_3}{\partial z} &= \Big[-\frac{4G_1}{(1+|G_1|^2)^2} - \frac{x_2(G_1^2-1)}{(1+|G_1|^2)^2} - \frac{\mathrm{i}x_1(1+G_1^2)}{(1+|G_1|^2)^2}\Big]\frac{\partial \overline{G}_1}{\partial z},
\end{aligned}\tag{3.15}$$

on U_1*, where* $e_{33} = \frac{|G_1|^2-1}{1+|G_1|^2}$*, and*

$$\begin{aligned}
(2H - \mathrm{i}e_{33}^2)\frac{\partial x_1}{\partial z} &= \frac{2(G_2^2-1)}{(1+|G_2|^2)^2}\frac{\partial \overline{G}_2}{\partial z},\\
(2H - \mathrm{i}e_{33}^2)\frac{\partial x_2}{\partial z} &= \frac{2\mathrm{i}(1+G_2^2)}{(1+|G_2|^2)^2}\frac{\partial \overline{G}_2}{\partial z},\\
(2H - \mathrm{i}e_{33}^2)\frac{\partial x_3}{\partial z} &= \Big[\frac{4G_2}{(1+|G_2|^2)^2} - \frac{x_2(G_2^2-1)}{(1+|G_2|^2)^2} + \frac{\mathrm{i}x_1(1+G_2^2)}{(1+|G_2|^2)^2}\Big]\frac{\partial \overline{G_2}}{\partial z},
\end{aligned}\tag{3.16}$$

on U_2*, where* $e_{33} = \frac{1-|G_2|^2}{1+|G_2|^2}$.

Proof By (3.8), we get

$$\frac{\partial \overline{G}_1}{\partial z} = \overline{\frac{\partial G_1}{\partial \overline{z}}} = -\frac{(1+|G_1|^2)^2}{4}(2H - \mathrm{i}e_{33}^2)\Big(\frac{\partial x_1}{\partial z} - \mathrm{i}\frac{\partial x_2}{\partial z}\Big), \quad \text{on } U_1. \tag{3.17}$$

Then from (3.1) and (3.2), we see

$$G_1^2\frac{\partial \overline{G}_1}{\partial z} = \frac{(1+|G_1|^2)^2}{4}(2H - \mathrm{i}e_{33}^2)\Big(\frac{\partial x_1}{\partial z} + \mathrm{i}\frac{\partial x_2}{\partial z}\Big). \tag{3.18}$$

Hence, by adding (3.17) to (3.18), we have

$$(1+G_1^2)\frac{\partial \overline{G}_1}{\partial z} = \mathrm{i}\frac{(1+|G_1|^2)^2}{2}(2H - \mathrm{i}e_{33}^2)\frac{\partial x_2}{\partial z},$$

and by subtracting (3.17) from (3.18), we have

$$(G_1^2-1)\frac{\partial \overline{G}_1}{\partial z} = \frac{(1+|G_1|^2)^2}{2}(2H - \mathrm{i}e_{33}^2)\frac{\partial x_1}{\partial z}.$$

Thus we get

$$(2H - \mathrm{i}e_{33}^2)\frac{\partial x_1}{\partial z} = \frac{2(G_1^2-1)}{(1+|G_1|^2)^2}\frac{\partial \overline{G}_1}{\partial z}, \tag{3.19}$$

$$(2H - \mathrm{i}e_{33}^2)\frac{\partial x_2}{\partial z} = -\frac{2\mathrm{i}(1+G_1^2)}{(1+|G_1|^2)^2}\frac{\partial \overline{G}_1}{\partial z}, \tag{3.20}$$

on U_1. Note that from (3.2) we also have

$$(2H-\mathrm{i}e_{33}^2)\Big[\frac{\partial x_3}{\partial z}+\frac{1}{2}\Big(x_2\frac{\partial x_1}{\partial z}-x_1\frac{\partial x_2}{\partial z}\Big)\Big]=(2H-\mathrm{i}e_{33}^2)G_1\Big(\frac{\partial x_1}{\partial z}-\mathrm{i}\frac{\partial x_2}{\partial z}\Big). \tag{3.21}$$

It then follows from (3.17) and (3.21) that on U_1,

$$(2H-\mathrm{i}e_{33}^2)\Big[\frac{\partial x_3}{\partial z}+\frac{1}{2}\Big(x_2\frac{\partial x_1}{\partial z}-x_1\frac{\partial x_2}{\partial z}\Big)\Big]=-\frac{4G_1}{(1+|G_1|^2)^2}\frac{\partial\overline{G}_1}{\partial z},$$

i.e.,

$$(2H-\mathrm{i}e_{33}^2)\frac{\partial x_3}{\partial z}=\Big[-\frac{4G_1}{(1+|G_1|^2)^2}-\frac{x_2(G_1^2-1)}{(1+|G_1|^2)^2}-\frac{\mathrm{i}x_1(1+G_1^2)}{(1+|G_1|^2)^2}\Big]\frac{\partial\overline{G}_1}{\partial z}. \tag{3.22}$$

(3.16) can be proved in a similar way.

Remark 3.1 By making use of equations (3.9) and (3.12) instead of (3.8) and (3.11), we obtain the following representation formula:

$$\begin{aligned}\overline{\phi}\frac{\partial x_1}{\partial z}&=\frac{G_1^2-1}{(1+|G_1|^2)^2}\overline{\frac{\partial(G_1-x_2+\mathrm{i}x_1)}{\partial z}},\\ \overline{\phi}\frac{\partial x_2}{\partial z}&=-\mathrm{i}\frac{1+G_1^2}{(1+|G_1|^2)^2}\overline{\frac{\partial(G_1-x_2+\mathrm{i}x_1)}{\partial z}},\\ \overline{\phi}\frac{\partial x_3}{\partial z}&=\Big[-\frac{2G_1}{(1+|G_1|^2)^2}-\frac{x_2}{2}\frac{G_1^2-1}{(1+|G_1|^2)^2}-\frac{\mathrm{i}x_1}{2}\frac{1+G_1^2}{(1+|G_1|^2)^2}\Big]\overline{\frac{\partial(G_1-x_2+\mathrm{i}x_1)}{\partial z}},\end{aligned} \tag{3.23}$$

on U_1, and

$$\begin{aligned}\overline{\phi}\frac{\partial x_1}{\partial z}&=\frac{G_2^2-1}{(1+|G_2|^2)^2}\overline{\frac{\partial(G_2+x_2+\mathrm{i}x_1)}{\partial z}},\\ \overline{\phi}\frac{\partial x_2}{\partial z}&=\mathrm{i}\frac{1+G_2^2}{(1+|G_2|^2)^2}\overline{\frac{\partial(G_2+x_2+\mathrm{i}x_1)}{\partial z}},\\ \overline{\phi}\frac{\partial x_3}{\partial z}&=\Big[\frac{2G_2}{(1+|G_2|^2)^2}-\frac{x_2}{2}\frac{G_2^2-1}{(1+|G_2|^2)^2}+\frac{\mathrm{i}x_1}{2}\frac{1+G_2^2}{(1+|G_2|^2)^2}\Big]\overline{\frac{\partial(G_2+x_2+\mathrm{i}x_1)}{\partial z}},\end{aligned} \tag{3.24}$$

on U_2.

4 Integrability Condition

In this section, we shall show that the Gauss map of an arbitrary surface in Nil_3 satisfies a second order differential equation, which is the complete integrability condition for the system (3.15).

Theorem 4.1 *Let $x:\Sigma\to\mathrm{Nil}_3$ be an isometric immersion. Then the Gauss map G must satisfy*

$$\begin{aligned}\frac{\partial^2G_1}{\partial z\partial\overline{z}}-\frac{2\overline{G}_1}{1+|G_1|^2}\frac{\partial G_1}{\partial z}\frac{\partial G_1}{\partial\overline{z}}&=\frac{1}{\mathrm{i}e_{33}^2+2H}\frac{\partial(\mathrm{i}e_{33}^2+2H)}{\partial z}\frac{\partial G_1}{\partial\overline{z}}\\&\quad+\Big[\frac{2}{1+|G_1|^2}-\frac{4H(\mathrm{i}e_{33}^2+2H)}{|\mathrm{i}e_{33}^2+2H|^2(1+|G_1|^2)}\\&\quad-\frac{2\mathrm{i}(1-|G_1|^2)(\mathrm{i}e_{33}^2+2H)}{|\mathrm{i}e_{33}^2+2H|^2(1+|G_1|^2)^2}\Big]G_1\Big|\frac{\partial G_1}{\partial\overline{z}}\Big|^2,\end{aligned} \tag{4.1}$$

where $e_{33} = \frac{|G_1|^2-1}{1+|G_1|^2}$, *and*

$$\begin{aligned}\frac{\partial^2 G_2}{\partial z\partial\overline{z}} - \frac{2\overline{G}_2}{1+|G_2|^2}\frac{\partial G_2}{\partial z}\frac{\partial G_2}{\partial\overline{z}} &= \frac{1}{\mathrm{i}e_{33}^2+2H}\frac{\partial(\mathrm{i}e_{33}^2+2H)}{\partial z}\frac{\partial G_2}{\partial\overline{z}} \\ &\quad + \Big[\frac{2}{1+|G_2|^2} - \frac{4H(\mathrm{i}e_{33}^2+2H)}{|\mathrm{i}e_{33}^2+2H|^2(1+|G_2|^2)} \\ &\quad - \frac{2\mathrm{i}(1-|G_2|^2)(\mathrm{i}e_{33}^2+2H)}{|\mathrm{i}e_{33}^2+2H|^2(1+|G_2|^2)^2}\Big]G_2\Big|\frac{\partial G_2}{\partial\overline{z}}\Big|^2, \end{aligned} \tag{4.2}$$

where $e_{33} = \frac{1-|G_2|^2}{1+|G_2|^2}$.

Proof We shall only prove (4.1) for G_1, since (4.2) can be proved in a similar way. From (2.3), we see

$$\begin{aligned}\frac{\partial^2 x_1}{\partial z\partial\overline{z}} + \mathrm{i}\frac{\partial^2 x_2}{\partial z\partial\overline{z}} &= \frac{1}{4}\Big(\frac{\partial^2 x_1}{\partial\xi_1^2} + \frac{\partial^2 x_1}{\partial\xi_2^2}\Big) + \frac{\mathrm{i}}{4}\Big(\frac{\partial^2 x_2}{\partial\xi_1^2} + \frac{\partial^2 x_2}{\partial\xi_2^2}\Big) \\ &= \frac{\mathrm{i}}{2}\Big[\frac{\partial x_3}{\partial z} + \frac{1}{2}\Big(x_2\frac{\partial x_1}{\partial z} - x_1\frac{\partial x_2}{\partial z}\Big)\Big]\Big(\frac{\partial x_1}{\partial\overline{z}} + \mathrm{i}\frac{\partial x_2}{\partial\overline{z}}\Big) \\ &\quad + \frac{\mathrm{i}}{2}\Big[\frac{\partial x_3}{\partial\overline{z}} + \frac{1}{2}\Big(x_2\frac{\partial x_1}{\partial\overline{z}} - x_1\frac{\partial x_2}{\partial\overline{z}}\Big)\Big]\Big(\frac{\partial x_1}{\partial z} + \mathrm{i}\frac{\partial x_2}{\partial z}\Big) + \frac{\lambda^2 H}{2}(e_{31} + \mathrm{i}e_{32}).\end{aligned}$$

By making use of (3.1)–(3.3), we have

$$\frac{\partial^2 x_1}{\partial z\partial\overline{z}} + \mathrm{i}\frac{\partial^2 x_2}{\partial z\partial\overline{z}} = \frac{\mathrm{i}\lambda^2 G_1}{2(1+|G_1|^2)^2} - \frac{\mathrm{i}\lambda^2 G_1|G_1|^2}{2(1+|G_1|^2)^2} + \frac{\lambda^2 HG_1}{1+|G_1|^2} = \frac{\mathrm{i}\lambda^2 G_1(1-|G_1|^2)}{2(1+|G_1|^2)^2} + \frac{\lambda^2 HG_1}{1+|G_1|^2}.$$

From (3.13), we have

$$\lambda^2 = \frac{16}{|\mathrm{i}e_{33}^2+2H|^2(1+|G_1|^2)^2}\Big|\frac{\partial G_1}{\partial\overline{z}}\Big|^2.$$

Then we have

$$\frac{\partial^2 x_1}{\partial z\partial\overline{z}} + \mathrm{i}\frac{\partial^2 x_2}{\partial z\partial\overline{z}} = \frac{8\mathrm{i}G_1(1-|G_1|^2)}{|\mathrm{i}e_{33}^2+2H|^2(1+|G_1|^2)^4}\Big|\frac{\partial G_1}{\partial\overline{z}}\Big|^2 + \frac{16HG_1}{|\mathrm{i}e_{33}^2+2H|^2(1+|G_1|^2)^3}\Big|\frac{\partial G_1}{\partial\overline{z}}\Big|^2. \tag{4.3}$$

From (3.8), we get

$$\begin{aligned}\frac{\partial^2 G_1}{\partial z\partial\overline{z}} &= \frac{\partial}{\partial z}\Big(\frac{\partial G_1}{\partial\overline{z}}\Big) = \frac{\partial}{\partial z}\Big\{-\frac{(1+|G_1|^2)^2}{4}(\mathrm{i}e_{33}^2+2H)\Big(\frac{\partial x_1}{\partial\overline{z}} + \mathrm{i}\frac{\partial x_2}{\partial\overline{z}}\Big)\Big\} \\ &= -\frac{(1+|G_1|^2)^2}{4}\Big(\frac{\partial x_1}{\partial\overline{z}} + \mathrm{i}\frac{\partial x_2}{\partial\overline{z}}\Big)\frac{\partial(\mathrm{i}e_{33}^2+2H)}{\partial z} - \frac{1}{4}(\mathrm{i}e_{33}^2+2H)\Big(\frac{\partial x_1}{\partial\overline{z}} + \mathrm{i}\frac{\partial x_2}{\partial\overline{z}}\Big) \\ &\quad\times 2(1+|G_1|^2)\frac{\partial(1+|G_1|^2)}{\partial z} - \frac{(1+|G_1|^2)^2}{4}(\mathrm{i}e_{33}^2+2H)\Big(\frac{\partial^2 x_1}{\partial z\partial\overline{z}} + \mathrm{i}\frac{\partial^2 x_2}{\partial z\partial\overline{z}}\Big).\end{aligned} \tag{4.4}$$

By (3.8), (4.3) and (4.4), it is verified that

$$\begin{aligned}\frac{\partial^2 G_1}{\partial z\partial\overline{z}} - \frac{2\overline{G}_1}{1+|G_1|^2}\frac{\partial G_1}{\partial z}\frac{\partial G_1}{\partial\overline{z}} &= \frac{1}{\mathrm{i}e_{33}^2+2H}\frac{\partial(\mathrm{i}e_{33}^2+2H)}{\partial z}\frac{\partial G_1}{\partial\overline{z}} \\ &\quad + \Big[\frac{2}{1+|G_1|^2} - \frac{4H(\mathrm{i}e_{33}^2+2H)}{|\mathrm{i}e_{33}^2+2H|^2(1+|G_1|^2)} \\ &\quad - \frac{2\mathrm{i}(1-|G_1|^2)(\mathrm{i}e_{33}^2+2H)}{|\mathrm{i}e_{33}^2+2H|^2(1+|G_1|^2)^2}\Big]G_1\Big|\frac{\partial G_1}{\partial\overline{z}}\Big|^2.\end{aligned}$$

Remark 4.1 When $H=0$, the Gauss map satisfies

$$\frac{\partial^2 G_1}{\partial z\partial\overline{z}}+\frac{2\overline{G}_1}{1-|G_1|^2}\frac{\partial G_1}{\partial z}\frac{\partial G_1}{\partial\overline{z}}=0.$$

It shows that the Gauss map of minimal surfaces in Nil_3 is harmonic (see [4]).

Theorem 4.2 *Equation* (4.1) *is the complete integrability condition of system* (3.15).

Proof From (3.15), we see

$$\begin{aligned}\frac{\partial x_1}{\partial z}&=\frac{1}{2H-\mathrm{i}e_{33}^2}\frac{2(G_1^2-1)}{(1+|G_1|^2)^2}\frac{\partial\overline{G}_1}{\partial z},\\ \frac{\partial x_2}{\partial z}&=-\frac{1}{2H-\mathrm{i}e_{33}^2}\frac{2\mathrm{i}(1+G_1^2)}{(1+|G_1|^2)^2}\frac{\partial\overline{G}_1}{\partial z},\\ \frac{\partial x_3}{\partial z}&=-\frac{1}{2H-\mathrm{i}e_{33}^2}\Big[\frac{4G_1}{(1+|G_1|^2)^2}\frac{\partial\overline{G}_1}{\partial z}+\frac{x_2(G_1^2-1)}{(1+|G_1|^2)^2}\frac{\partial\overline{G}_1}{\partial z}+\frac{\mathrm{i}x_1(1+G_1^2)}{(1+|G_1|^2)^2}\frac{\partial\overline{G}_1}{\partial z}\Big].\end{aligned}$$

Set

$$F=\frac{1}{2H-\mathrm{i}e_{33}^2}\frac{1}{(1+|G_1|^2)^2}\frac{\partial\overline{G}_1}{\partial z}.\tag{4.5}$$

Then

$$\begin{aligned}\frac{\partial x_1}{\partial z}&=2F(G_1^2-1),\\ \frac{\partial x_2}{\partial z}&=-2\mathrm{i}F(1+G_1^2),\\ \frac{\partial x_3}{\partial z}&=-F[4G_1+x_2(G_1^2-1)+\mathrm{i}x_1(1+G_1^2)].\end{aligned}$$

Set

$$P=(P_1,P_2,P_3)=(2F(G_1^2-1),-2\mathrm{i}F(1+G_1^2),-F[4G_1+x_2(G_1^2-1)+\mathrm{i}x_1(1+G_1^2)]).$$

Differentiating (4.5) with respect to $\overline{z}$, we get

$$\begin{aligned}\frac{\partial F}{\partial\overline{z}}=&\frac{1}{2H-\mathrm{i}e_{33}^2}\frac{1}{(1+|G_1|^2)^2}\Big(\frac{\partial^2\overline{G}_1}{\partial z\partial\overline{z}}-\frac{2G_1}{1+|G_1|^2}\frac{\partial\overline{G}_1}{\partial z}\frac{\partial\overline{G}_1}{\partial\overline{z}}\Big)\\ &-\frac{1}{(2H-\mathrm{i}e_{33}^2)^2}\frac{\partial(2H-\mathrm{i}e_{33}^2)}{\partial\overline{z}}\frac{1}{(1+|G_1|^2)^2}\frac{\partial\overline{G}_1}{\partial z}-\frac{2\overline{G}_1}{2H-\mathrm{i}e_{33}^2}\frac{1}{(1+|G_1|^2)^3}\frac{\partial\overline{G}_1}{\partial z}\frac{\partial G_1}{\partial\overline{z}}.\end{aligned}$$

By (4.1), we have

$$\frac{\partial F}{\partial\overline{z}}=-\frac{4H\overline{G}_1}{|\mathrm{i}e_{33}^2+2H|^2(1+|G_1|^2)^3}\Big|\frac{\partial G_1}{\partial\overline{z}}\Big|^2+\frac{2\mathrm{i}(1-|G_1|^2)\overline{G}_1}{|\mathrm{i}e_{33}^2+2H|^2(1+|G_1|^2)^4}\Big|\frac{\partial G_1}{\partial\overline{z}}\Big|^2.$$

Hence

$$\begin{aligned}\frac{\partial P_1}{\partial\overline{z}}=&2(G_1^2-1)\frac{\partial F}{\partial\overline{z}}+2F\cdot 2G_1\frac{\partial G_1}{\partial\overline{z}}\\ =&-\frac{8H\overline{G}_1(G_1^2-1)}{|\mathrm{i}e_{33}^2+2H|^2(1+|G_1|^2)^3}\Big|\frac{\partial G_1}{\partial\overline{z}}\Big|^2+\frac{4\mathrm{i}(1-|G_1|^2)\overline{G}_1(G_1^2-1)}{|\mathrm{i}e_{33}^2+2H|^2(1+|G_1|^2)^4}\Big|\frac{\partial G_1}{\partial\overline{z}}\Big|^2\\ &+\frac{4G_1}{(2H-\mathrm{i}e_{33}^2)(1+|G_1|^2)^2}\Big|\frac{\partial G_1}{\partial\overline{z}}\Big|^2\\ =&\frac{16H(1+|G_1|^2)\mathrm{Re}\,G_1-8(1-|G_1|^2)\mathrm{Im}\,G_1}{|\mathrm{i}e_{33}^2+2H|^2(1+|G_1|^2)^4}\Big|\frac{\partial G_1}{\partial\overline{z}}\Big|^2\in\mathbb{R}.\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}\frac{\partial P_2}{\partial \overline{z}} &= -2\mathrm{i}(1+G_1^2)\frac{\partial F}{\partial \overline{z}} - 2\mathrm{i}F\cdot 2G_1\frac{\partial G_1}{\partial \overline{z}}\\ &= \frac{16H(1+|G_1|^2)\mathrm{Im}\,G_1 + 8(1-|G_1|^2)\mathrm{Re}\,G_1}{|\mathrm{i}e_{33}^2+2H|^2(1+|G_1|^2)^4}\Big|\frac{\partial G_1}{\partial \overline{z}}\Big|^2 \in \mathbb{R}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial P_3}{\partial \overline{z}} =& -[4G_1 + x_2(G_1^2-1) + \mathrm{i}x_1(1+G_1^2)]\frac{\partial F}{\partial \overline{z}}\\ &- F\Big[4\frac{\partial G_1}{\partial \overline{z}} + (G_1^2-1)\frac{\partial x_2}{\partial \overline{z}} + x_2\cdot 2G_1\frac{\partial G_1}{\partial \overline{z}} + \mathrm{i}(1+G_1^2)\frac{\partial x_1}{\partial \overline{z}} + \mathrm{i}x_1\cdot 2G_1\frac{\partial G_1}{\partial \overline{z}}\Big]\\ =& \frac{1}{|\mathrm{i}e_{33}^2+2H|^2(1+|G_1|^2)^4}[8H(1+|G_1|^2)(|G_1|^2-1) - 8Hx_2(1+|G_1|^2)\mathrm{Re}\,G_1\\ &+ 8Hx_1(1+|G_1|^2)\mathrm{Im}\,G_1 + 4x_2(1-|G_1|^2)\mathrm{Im}\,G_1 + 4x_1(1-|G_1|^2)\mathrm{Re}\,G_1]\Big|\frac{\partial G_1}{\partial \overline{z}}\Big|^2 \in \mathbb{R},\end{aligned}$$

i.e.,

$$\frac{\partial P}{\partial \overline{z}} = \Big(\frac{\partial P_1}{\partial \overline{z}}, \frac{\partial P_2}{\partial \overline{z}}, \frac{\partial P_3}{\partial \overline{z}}\Big) \in \mathbb{R}^3.$$

So (4.1) is the complete integrability condition of (3.15).

Remark 4.2 By a similar argument, one can show that equation (4.2) is the complete integrability condition of system (3.16).

Therefore, we have the following representation formula.

Theorem 4.3 *Let* Σ *be a simply connected Riemann surface,* $H:\Sigma\to\mathbb{R}$ *be a* C^1*-function, and* $G:\Sigma\to\mathbb{C}\cup\{\infty\}$ *be a smooth mapping which is defined on* U_1 *(resp.* U_2*) by* G_1 *(resp.* G_2*). Assume that* G *satisfies the differential equations* (4.1) *and* (4.2) *for the above* H*. In the case of* $G(z)\in U_1$*, we set*

$$\begin{aligned}x_1 =& 2\mathrm{Re}\Big\{\int_{z_0}^{z}\frac{G_1^2-1}{(2H-\mathrm{i}e_{33}^2)(1+|G_1|^2)^2}\frac{\partial \overline{G}_1}{\partial z}\mathrm{d}z\Big\} + c_1,\\ x_2 =& 2\mathrm{Re}\Big\{\int_{z_0}^{z}\frac{-\mathrm{i}(1+G_1^2)}{(2H-\mathrm{i}e_{33}^2)(1+|G_1|^2)^2}\frac{\partial \overline{G}_1}{\partial z}\mathrm{d}z\Big\} + c_2,\\ x_3 =& 2\mathrm{Re}\Big\{\int_{z_0}^{z}\frac{1}{(2H-\mathrm{i}e_{33}^2)(1+|G_1|^2)^2}\Big[-2G_1 - \frac{1}{2}x_2(G_1^2-1)\\ &- \frac{\mathrm{i}}{2}x_1(1+G_1^2)\Big]\frac{\partial \overline{G_1}}{\partial z}\mathrm{d}z\Big\} + c_3,\end{aligned}\tag{4.6}$$

where $e_{33}=\frac{|G_1|^2-1}{1+|G_1|^2}$*. In the case of* $G(z)\in U_2$*, we set*

$$\begin{aligned}x_1 =& 2\mathrm{Re}\Big\{\int_{z_0}^{z}\frac{G_2^2-1}{(2H-\mathrm{i}e_{33}^2)(1+|G_2|^2)^2}\frac{\partial \overline{G}_2}{\partial z}\mathrm{d}z\Big\} + c_1,\\ x_2 =& 2\mathrm{Re}\Big\{\int_{z_0}^{z}\frac{\mathrm{i}(1+G_2^2)}{(2H-\mathrm{i}e_{33}^2)(1+|G_2|^2)^2}\frac{\partial \overline{G}_2}{\partial z}\mathrm{d}z\Big\} + c_2,\\ x_3 =& 2\mathrm{Re}\Big\{\int_{z_0}^{z}\frac{1}{(2H-\mathrm{i}e_{33}^2)(1+|G_2|^2)^2}\Big[2G_2 - \frac{1}{2}x_2(G_2^2-1)\\ &+ \frac{\mathrm{i}}{2}x_1(1+G_2^2)\Big]\frac{\partial \overline{G}_2}{\partial z}\mathrm{d}z\Big\} + c_3,\end{aligned}\tag{4.7}$$

where $e_{33} = \frac{1-|G_2|^2}{1+|G_2|^2}$. *Then* $x = (x_1, x_2, x_3)$ *is a branched surface such that the mean curvature is* H *and the Gauss map of* x *is* G. *Moreover, from Corollary* 3.1, *if* $G_{\overline{z}} \neq 0$ *on* Σ, *then* x *is a regular surface.*

Remark 4.3 We assume that $|G_1| \neq 1$. By (4.6), we get the Weierstrass representation formula for minimal surfaces in Nil_3 as follows:

$$\begin{aligned}(x_1, x_2, x_3) = \Big(&2\mathrm{Re}\Big\{\int_{z_0}^{z} F(1-G_1^2)\mathrm{d}z\Big\}, 2\mathrm{Re}\Big\{\int_{z_0}^{z} \mathrm{i}F(1+G_1^2)\mathrm{d}z\Big\},\\ &2\mathrm{Re}\Big\{\int_{z_0}^{z}\Big[2FG_1 - \frac{x_2}{2}F(1-G_1^2) + \frac{\mathrm{i}x_1}{2}F(1+G_1^2)\Big]\mathrm{d}z\Big\}\Big),\end{aligned}$$

where $F = -\frac{\mathrm{i}}{(1-|G_1|^2)^2}\frac{\partial \overline{G_1}}{\partial z}$.

This is the result of Mercuri, Montaldo and Piu [10].

Remark 4.4 We have found a correspondence from the set of solutions of the differential equations (4.1) and (4.2) to the set of surfaces in Nil_3 by Theorem 4.3.

Next we shall study the uniqueness of the correspondence.

Theorem 4.4 *Let* $G(z)$ *(resp.* $\widehat{G}(w)$*) be a smooth mapping satisfying* (4.1) *for some positive function* $H(z)$ *(resp.* $\widehat{H}(w)$*) on a simply connected two-dimensional manifold* Σ. *We define a branched immersion* $x(z)$ *(resp.* $\widehat{x}(w)$*) by Theorem* 4.3. *Then the following two conditions are equivalent:*

(1) *There exists a holomorphic mapping* $w = f(z)$ *with* $f'(z) \neq 0$ *on* Σ *and a motion* θ *of* Nil_3, *such that* $\widehat{x} \circ f(z) = \theta \circ x(z)$, $z \in \Sigma$.

(2) *There exists a holomorphic mapping* $w = f(z)$ *with* $f'(z) \neq 0$ *on* Σ, *such that it has relations* $G(z) = \widehat{G} \circ f(z)$, $H(z) = \widehat{H} \circ f(z)$, $z \in \Sigma$.

Proof (1) $\Rightarrow$ (2) We may assume $\theta =$identity. We have

$$\frac{\partial x}{\partial z} = \frac{\partial \widehat{x}}{\partial w} f'(z) \quad \text{and} \quad \frac{\partial x}{\partial \overline{z}} = \frac{\partial \widehat{x}}{\partial \overline{w}} \overline{f'(z)}.$$

Since

$$\mathrm{d}s^2 = \lambda^2|\mathrm{d}z|^2 = \widehat{\lambda}^2|\mathrm{d}w|^2 = \widehat{\lambda}^2|f'(z)\mathrm{d}z|^2 = \widehat{\lambda}^2|f'(z)|^2|\mathrm{d}z|^2,$$

we get

$$\lambda^2 = \widehat{\lambda}^2|f'(z)|.$$

Then we have

$$\begin{aligned}e_1 + \mathrm{i}e_2 &= \frac{2}{\lambda}\frac{\partial x}{\partial \overline{z}} = \frac{2}{\widehat{\lambda}|f'(z)|}\frac{\partial \widehat{x}}{\partial \overline{w}}\overline{f'(z)} = (\widehat{e}_1 + \mathrm{i}\widehat{e}_2)\frac{\overline{f'(z)}}{|f'(z)|},\\ e_1 - \mathrm{i}e_2 &= \frac{2}{\lambda}\frac{\partial x}{\partial z} = \frac{2}{\widehat{\lambda}|f'(z)|}\frac{\partial \widehat{x}}{\partial w}f'(z) = (\widehat{e}_1 - \mathrm{i}\widehat{e}_2)\frac{f'(z)}{|f'(z)|}.\end{aligned}$$

So

$$2n(z) = \mathrm{i}(e_1 + \mathrm{i}e_2) \times (e_1 - \mathrm{i}e_2) = \mathrm{i}(\widehat{e}_1 + \mathrm{i}\widehat{e}_2)\frac{\overline{f'(z)}}{|f'(z)|} \times (\widehat{e}_1 - \mathrm{i}\widehat{e}_2)\frac{f'(z)}{|f'(z)|} = 2\widehat{n}(w),$$

i.e., $n(z) = \widehat{n}(w)$. Hence

$$G(z) = \widehat{G}(f(z)).$$

Then by (3.8), we get

$$H(z) = \widehat{H}(f(z)).$$

(2) $\Rightarrow$ (1) By the assumption and Theorem 4.3, we have $\frac{\partial x_j}{\partial z} = \frac{\partial \widehat{x}_j}{\partial w} f'(z)$, $j = 1, 2$, i.e., $x_j = \widehat{x}_j + c_j$. Then we get

$$x_3 = \widehat{x}_3 + \frac{c_1}{2}\widehat{x}_2 - \frac{c_2}{2}\widehat{x}_1 + c_3,$$

where c_1, c_2, c_3 are constants.

5 Examples

Let us give some examples.

Example 5.1 Let $\Sigma = \mathbb{C}$, $H = 0$, and define $G : \mathbb{C} \to \mathbb{C} \cup \{\infty\}$ by $G(z) = \overline{z}$. Then G and H satisfy (4.1), and the immersion x defined by (4.6) is written as

$$x(z) = (x_1(z), x_2(z), x_3(z)) = \Big(\frac{2\mathrm{Im}\, z}{1 - |z|^2}, -\frac{2\mathrm{Re}\, z}{1 - |z|^2}, \frac{4\mathrm{Re}\, z \mathrm{Im}\, z}{(1 - |z|^2)^2}\Big), \quad |z| \neq 1,$$

i.e., $x_3 = -x_1 x_2$.

Example 5.2 Let $\Sigma = \mathbb{C} \setminus \{0\}$, $H = 0$, and define $G : \mathbb{C} \setminus \{0\} \to \mathbb{C} \cup \{\infty\}$ by $G(z) = \frac{1}{\overline{z}}$. Then G and H satisfy (4.1), and the immersion x defined by (4.6) is written as

$$x(z) = (x_1(z), x_2(z), x_3(z)) = \Big(\frac{2\mathrm{Im}\, z}{|z|^2(1 - |z|^2)}, \frac{2\mathrm{Re}\, z}{|z|^2(1 - |z|^2)}, \frac{-4\mathrm{Re}\, z \mathrm{Im}\, z}{|z|^2(1 - |z|^2)^2}\Big), \quad |z| \neq 1.$$

Example 5.3 Let $\Sigma = \mathbb{C}$, $H = 0$, and define $G : \mathbb{C} \to \mathbb{C} \cup \{\infty\}$ by $G(z) = \mathrm{e}^{\overline{z}}$. Then G and H satisfy (4.1), and the immersion x defined by (4.6) is written as

$$x(z) = (x_1(z), x_2(z), x_3(z)) = \Big(\frac{\mathrm{i}(\mathrm{e}^{\overline{z}} - \mathrm{e}^z)}{1 - |\mathrm{e}^z|^2}, \frac{\mathrm{e}^z + \mathrm{e}^{\overline{z}}}{1 - |\mathrm{e}^z|^2}, \frac{2|\mathrm{e}^z|^2 \mathrm{Im}\, z}{(1 - |\mathrm{e}^z|^2)^2}\Big), \quad \mathrm{Re}\, z \neq 0.$$

Example 5.4 Let $\Sigma = \mathbb{C}$, $H = \frac{1}{4}$, and define $G : \mathbb{C} \to \mathbb{C} \cup \{\infty\}$ by $G(z) = -\mathrm{e}^{\mathrm{i}\mathrm{Re}\, z}$. Then G and H satisfy (4.1), and the immersion x defined by (4.6) is written as

$$x(z) = (x_1(z), x_2(z), x_3(z)) = \Big(\cos(\mathrm{Re}\, z), \sin(\mathrm{Re}\, z), \frac{1}{2}\mathrm{Re}\, z - \mathrm{Im}\, z\Big),$$

i.e., $x_1^2 + x_2^2 = 1$. It is the unit circular cylinder in Nil_3.

References

[1] Akutagawa, K. and Nishikawa, S., The Gauss map and spacelike surfaces with prescribed mean curvature in Minkowski 3-space, *Tohoku Math. J.* (2), **42**(1), 1990, 67–82.

[2] Bryant, R. L., Surfaces of mean curvature one in hyperbolic space, *Astérisque*, **154**–**155**, 1987, 321–347.

[3] Berdinsky, D. A. and Taimanov, I. A., Surfaces in three-dimensional Lie groups, *Siberian Math. J.*, **46**(6), 2005, 1005–1019.

[4] Daniel, B., The Gauss map of minimal surfaces in the Heisenberg group, 2006. arXiv:math.DG/0606299

[5] Inoguchi, J., Kumamoto, T., Ohsugi, N. et al, Differential geometry of curves and surfaces in 3-dimensional homogeneous spaces II, *Fukuoka Univ. Sci. Rep.*, **30**, 2000, 17–47.

[6] Inoguchi, J., Minimal surfaces in the 3-dimensional Heisenberg group, *Differ. Geom. Dyn. Syst.*, **10**, 2008, 163–169.

[7] Kenmotsu, K., Weierstrass formula for surfaces of prescribed mean curvature, *Math. Ann.*, **245**(2), 1979, 89–99.

[8] Kobayashi, O., Maximal surfaces in the 3-dimensional Minkowski space L^3, *Tokyo J. Math.*, **6**, 1983, 297–309.

[9] Kokubu, M., Weierstrass representation for minimal surfaces in hyperbolic space, *Tohoku Math. J.*, **49**(3), 1997, 367–377.

[10] Mercuri, F., Montaldo, S. and Piu, P., A Weierstrass representation formula for minimal surfaces in $\mathbb{H}_3$ and $\mathbb{H}^2 \times \mathbb{R}$, *Acta Math. Sin., Engl. Ser.*, **22**(6), 2006, 1603–1612.

[11] Shi, S. G., Weierstrass representation for surfaces of prescribed mean curvature in the hyperbolic 3-dimensional space (in Chinese), *Chin. Ann. Math.*, **22A**(6), 2001, 691–700.

[12] Thurston, W. P., Three dimensional manifolds, Kleinian groups and hyperbolic geometry, *Bull. Amer. Math. Soc.*, **6**(3), 1982, 357–381.

[13] Xin, Y. L., Minimal Submanifolds and Related Topics, Nankai Tracts in Mathematics, **8**, World Scientific, Singapore, 2003.