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Compressible Limit of the Nonlinear Schrödinger Equation with Different-Degree Small Parameter Nonlinearities*

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Abstract The authors study the compressible limit of the nonlinear Schrödinger equation with different-degree small parameter nonlinearities in small time for initial data with Sobolev regularity before the formation of singularities in the limit system. On the one hand, the existence and uniqueness of the classical solution are proved for the dispersive perturbation of the quasi-linear symmetric system corresponding to the initial value problem of the above nonlinear Schrödinger equation. On the other hand, in the limit system, it is shown that the density converges to the solution of the compressible Euler equation and the validity of the WKB expansion is justified.

Keywords Nonlinear Schrödinger equation, Compressible limit, Compressible Euler equation, WKB expansion
2000 MR Subject Classification 35Q55, 35C20

1 Introduction

In this paper, we study the compressible limit of the Cauchy problem with rapidly oscillating initial data for the nonlinear Schrödinger equation with different-degree small parameter nonlinearities:

$$\mathrm{i}\epsilon\partial_t\psi^\epsilon + \frac{\epsilon^2}{2}\Delta_x\psi^\epsilon + a\epsilon^2\psi^\epsilon|\psi^\epsilon|^2 = f(|\psi^\epsilon|^2)\psi^\epsilon, \quad t\in\mathbb{R}^+,\ x\in\mathbb{R}^m, \tag{1.1}$$

$$\psi^\epsilon(0,x) = \psi_0^\epsilon(x) = A_0^\epsilon(x)\exp\Big(\frac{\mathrm{i}}{\epsilon}S_0(x)\Big), \tag{1.2}$$

where ψ^ϵ denotes the condensate wave function in the quantum mechanics, ϵ denotes the Planck constant, i is the imaginary unit, Δ_x denotes the Laplace operator, $a\in R$, $f\in C^\infty(\mathbb{R}^+,\mathbb{R})$, $S_0(x)$ is a function of $H^s(\mathbb{R}^m)$ for s large enough, and $A_0^\epsilon(x)$ is a function, polynomial in ϵ, with coefficients of Sobolev regularity in x. The study of compressible limit is realized by studying the behaviour of solutions to the Cauchy problem (1.1)–(1.2) as $\epsilon\to 0$, $x\in\mathbb{R}^m$ and $0\le t\le T$, i.e., within an arbitrary finite time T.

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When $a = 0$, equation (1.1) reduces to the classical nonlinear Schrödinger equation

$$\mathrm{i}\epsilon\partial_t\psi^\epsilon + \frac{\epsilon^2}{2}\Delta_x\psi^\epsilon = f(|\psi^\epsilon|^2)\psi^\epsilon, \quad t \in \mathbb{R}^+,\ x \in \mathbb{R}^m. \tag{1.3}$$

For equation (1.3), the compressible limit for initial data with Sobolev regularity in short time was obtained by Grenier [1, 2]. Especially, for one space dimension and $f(\rho) = \rho$, Jin, Levermore and Mc Laughlin [3] used the integrability of the cubic nonlinear Schrödinger equation to establish the compressible limit of the one-dimensional defocusing cubic nonlinear Schrödinger equation. In addition, with particular assumptions on m and f, Ginibre and Velo [4] proved that equation (1.3) without small parameter ϵ has global smooth solutions.

Since (1.1) has different-order small parameter nonlinearities which causes that the singular perturbation can also create energy, we must make some energy estimates for this term since it does not vanish in the energy estimates. According to Schrödinger's original idea on reformulation of the quantum mechanics in terms of the pair time reflection invariant diffusion equations (see [5]), the Schrödinger type equations can represent as a dispersive perturbation of a symmetric quasilinear hyperbolic system (see [1, 2, 6]).

Like the usual WKB method, we introduce the complex-valued wave function

$$\psi^\epsilon(t,x) = A^\epsilon(t,x)\exp\Big(\frac{\mathrm{i}}{\epsilon}S(t,x)\Big). \tag{1.4}$$

Here, A^ϵ and S are real-valued functions, A^ϵ is called the amplitude and S the classical action (phase). The motivation of this transformation comes from the semiclassical limit of the nonlinear Schrödinger equation where a short wave limit is considered. Plugging (1.4) into (1.1), we obtain

$$\begin{aligned} &-\mathrm{i}\epsilon\partial_t A^\epsilon + A^\epsilon\partial_t S - \frac{\epsilon^2}{2}\triangle_x A^\epsilon - \mathrm{i}\epsilon\nabla_x S\nabla_x A^\epsilon - \frac{\mathrm{i}\epsilon}{2}A^\epsilon\triangle_x S \\ &+ \frac{1}{2}A^\epsilon|\nabla_x S|^2 - a\epsilon^2|A^\epsilon|^2A^\epsilon + f(|A^\epsilon|^2)A^\epsilon = 0. \end{aligned} \tag{1.5}$$

Taking real part and imaginary part in (1.5), respectively, we obtain

$$\partial_t A^\epsilon + \nabla_x S\nabla_x A^\epsilon + \frac{1}{2}A^\epsilon\triangle_x S = 0, \tag{1.6}$$

$$\partial_t S + \frac{1}{2}|\nabla_x S|^2 + f(|A^\epsilon|^2) = a\epsilon^2|A^\epsilon|^2 + \frac{\epsilon^2}{2}\frac{\triangle_x A^\epsilon}{A^\epsilon}. \tag{1.7}$$

Let

$$\rho = |A^\epsilon|^2 = |\psi^\epsilon|^2, \quad u = \nabla_x S. \tag{1.8}$$

Then multiplying (1.6) by $2A^\epsilon$ and differentiating (1.7) with respect to the space variable x, we have

$$\partial_t\rho + \nabla_x\cdot(\rho u) = 0, \tag{1.9}$$

$$\partial_t u + \nabla_x\Big(\frac{|u|^2}{2} + f(\rho)\Big) = a\epsilon^2\nabla_x\rho + \frac{\epsilon^2}{2}\nabla_x\Big(\frac{\triangle_x\sqrt{\rho}}{\sqrt{\rho}}\Big). \tag{1.10}$$

System (1.9)–(1.10) is a perturbation of the Euler equations of compressible isentropic fluid mechanics

$$\partial_t\rho + \nabla_x\cdot(\rho u) = 0, \tag{1.11}$$

$$\partial_t u + \nabla_x\Big(\frac{|u|^2}{2} + f(\rho)\Big) = 0. \tag{1.12}$$

For system (1.11)–(1.12), Majda [7] proved that if $f' > 0$, the system has smooth solutions on a time interval $[0, T]$ for initial data with Sobolev regularity. The case of initial data A_0^ϵ and S_0 with analytic regularity, Gérard [8] proved the existence of smooth solutions ψ^ϵ of (1.3) on a time interval $[0, T]$ independent of ϵ, and justified the WKB expansion on the same interval of time T being linked to the existence time of smooth solution to (1.11)–(1.12). System (1.9)–(1.10) comprise a closed system governing ρ and u which has the form of a perturbation of the modified Euler equations. Therefore, we can conclude as follows.

Proposition 1.1. *Equation* (1.1) *is equivalent to the dispersive perturbation of the hyperbolic system* (1.9)–(1.10)*. The density ρ is the conservative quantities of the equation* (1.1)*, that is,*

$$\int_{\mathbb{R}^m} \rho \mathrm{d}x = C = \int_{\mathbb{R}^m} |\psi^\epsilon|^2 \mathrm{d}x = \int_{\mathbb{R}^m} |A^\epsilon|^2 \mathrm{d}x.$$

This paper proceeds as follows. In Section 2, the local existence and uniqueness of the smooth solutions of equation (1.1) are obtained by employing the classical quasilinear hyperbolic system theory. Section 3 is devoted to studying the semiclassical limit of equation (1.1) with ϵ small enough. In Section 4, the validity of the WKB expansion for equation (1.1) is justified in detail.

2 Existence of Smooth Solutions to the Nonlinear Schrödinger Equation

In order to study the semiclassical limit of equation (1.1), we must show the existence of the smooth solutions ψ^ϵ to (1.1) on a finite time interval $[0, T]$ independent of ε, for initial data A_0^ϵ and S_0 with Sobolev regularity. According to the idea of Grenier [1, 2] and Schochet-Weinstein [6], we transform equation (1.1) into a dispersive perturbation of a symmetric hyperbolic system. As suggested by Grenier [1, 2], instead of looking as usual at solutions ψ^ϵ of the form

$$\psi^\epsilon(t, x) = A^\epsilon(t, x) \exp\Big(\frac{\mathrm{i}}{\epsilon} S(t, x)\Big),$$

where S is independent of ϵ, we allow S to depend on ϵ in order to get better equations for A^ϵ and S^ϵ. We search for solutions ψ^ϵ of the form

$$\psi^\epsilon(t, x) = A^\epsilon(t, x) \exp\Big(\frac{\mathrm{i}}{\epsilon} S^\epsilon(t, x)\Big), \tag{2.1}$$

where complex-valued function $A^\epsilon = A_1^\epsilon + \mathrm{i} A_2^\epsilon$ ($A_1^\epsilon, A_2^\epsilon \in \mathbb{R}$) represents the amplitude and real-valued function S^ϵ represents the phase.

Putting (2.1) into equation (1.1), we obtain

$$\begin{aligned} &-\mathrm{i}\epsilon\partial_t A^\epsilon + A^\epsilon \partial_t S^\epsilon - \frac{\epsilon^2}{2} \triangle_x A^\epsilon - \mathrm{i}\epsilon \nabla_x S^\epsilon \nabla_x A^\epsilon - \frac{\mathrm{i}\epsilon}{2} A^\epsilon \triangle_x S^\epsilon \\ &+ \frac{1}{2} A^\epsilon |\nabla_x S^\epsilon|^2 - a\epsilon^2 |A^\epsilon|^2 A^\epsilon + f(|A^\epsilon|^2) A^\epsilon = 0, \end{aligned}$$

which can be split into

$$\begin{aligned} &\partial_t A^\epsilon + \nabla_x S^\epsilon \nabla_x A^\epsilon + \frac{1}{2} A^\epsilon \triangle_x S^\epsilon = \mathrm{i} a\epsilon |A^\epsilon|^2 A^\epsilon + \frac{\mathrm{i}\epsilon}{2} \triangle_x A^\epsilon, \\ &\partial_t S^\epsilon + \frac{1}{2} |\nabla_x S^\epsilon|^2 + f(|A^\epsilon|^2) = 0. \end{aligned}$$

Let $w^\epsilon = \nabla_x S^\epsilon$. Then one has

$$\partial_t A^\epsilon + (w^\epsilon \cdot \nabla_x) A^\epsilon + \frac{1}{2} A^\epsilon \nabla_x \cdot w^\epsilon = \mathrm{i} a\epsilon |A^\epsilon|^2 A^\epsilon + \frac{\mathrm{i}\epsilon}{2} \triangle_x A^\epsilon, \tag{2.2}$$

$$\partial_t w^\epsilon + (w^\epsilon \cdot \nabla_x) w^\epsilon + f'(|A^\epsilon|^2) \nabla_x |A^\epsilon|^2 = 0. \tag{2.3}$$

Using the fact that $A^\epsilon = A_1^\epsilon + \mathrm{i} A_2^\epsilon$, taking real part and imaginary part in (2.2), respectively, we have the equivalent form of (2.2)–(2.3)

$$\partial_t A_1^\epsilon + \sum_{j=1}^m w_j^\epsilon \partial_j A_1^\epsilon + \frac{1}{2} A_1^\epsilon \sum_{j=1}^m \partial_j w_j^\epsilon = -\frac{\epsilon}{2} \triangle_x A_2^\epsilon - a\epsilon (|A_1^\epsilon|^2 + |A_2^\epsilon|^2) A_2^\epsilon, \tag{2.4}$$

$$\partial_t A_2^\epsilon + \sum_{j=1}^m w_j^\epsilon \partial_j A_2^\epsilon + \frac{1}{2} A_2^\epsilon \sum_{j=1}^m \partial_j w_j^\epsilon = \frac{\epsilon}{2} \triangle_x A_1^\epsilon + a\epsilon (|A_1^\epsilon|^2 + |A_2^\epsilon|^2) A_1^\epsilon, \tag{2.5}$$

$$\partial_t w_i^\epsilon + f'(|A_1^\epsilon|^2 + |A_2^\epsilon|^2)(2A_1^\epsilon \partial_i A_1^\epsilon + 2A_2^\epsilon \partial_i A_2^\epsilon) + \sum_{j=1}^m w_j^\epsilon \partial_j w_i^\epsilon = 0 \tag{2.6}$$

with initial data

$$A_1^\epsilon(0,x) = A_{10}^\epsilon(x), \quad A_2^\epsilon(0,x) = A_{20}^\epsilon(x), \quad w^\epsilon(0,x) = w_0^\epsilon(x) \tag{2.7}$$

satisfying

$$|A_{10}^\epsilon(x)|^2 + |A_{20}^\epsilon(x)|^2 = |A_0^\epsilon(x)|^2, \quad w_0^\epsilon(x) = \nabla_x S_0^\epsilon(x), \tag{2.8}$$

where $i = 1, 2, \cdots, m$, w_i^ϵ is the i^{th} component of w^ϵ and $\partial_i = \frac{\partial}{\partial x_i}$. The Cauchy problem (2.4)–(2.7) can be written in the form

$$\partial_t U^\epsilon + \sum_{i=1}^m B^i(U^\epsilon) \partial_i U^\epsilon = \epsilon L(U^\epsilon) + a\epsilon N(U^\epsilon), \tag{2.9}$$

$$\begin{aligned} &U^\epsilon = (A_1^\epsilon, A_2^\epsilon, w_1^\epsilon, \cdots, w_m^\epsilon)^{\mathrm{T}}, \\ &U_0^\epsilon = U^\epsilon(0,x) = (A_{10}^\epsilon, A_{20}^\epsilon, w_{10}^\epsilon, \cdots, w_{m0}^\epsilon)^{\mathrm{T}}, \end{aligned} \tag{2.10}$$

where

$$B(U^\epsilon, \eta) = \sum_{j=1}^m \eta_j B^j(U^\epsilon) = \begin{pmatrix} \sum\limits_{i=1}^m \eta_i w_i^\epsilon & 0 & \frac{\eta_1 A_1^\epsilon}{2} & \frac{\eta_2 A_1^\epsilon}{2} & \cdots \\ 0 & \sum\limits_{i=1}^m \eta_i w_i^\epsilon & \frac{\eta_1 A_2^\epsilon}{2} & \frac{\eta_2 A_2^\epsilon}{2} & \cdots \\ 2\eta_1 A_1^\epsilon f' & 2\eta_1 A_2^\epsilon f' & \sum\limits_{i=1}^m \eta_i w_i^\epsilon & 0 & \cdots \\ 2\eta_2 A_1^\epsilon f' & 2\eta_2 A_2^\epsilon f' & 0 & \sum\limits_{i=1}^m \eta_i w_i^\epsilon & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and

$$\epsilon L(U^\epsilon) + a\epsilon N(U^\epsilon) = \epsilon \begin{pmatrix} 0 & -\frac{1}{2}\triangle_x & 0 & \cdots \\ \frac{1}{2}\triangle_x & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} A_1^\epsilon \\ A_2^\epsilon \\ w_1^\epsilon \\ \vdots \\ w_m^\epsilon \end{pmatrix}$$

$$
+ a\epsilon \begin{pmatrix} 0 & -(|A_1^\epsilon|^2 + |A_2^\epsilon|^2) & 0 & \cdots \\ (|A_1^\epsilon|^2 + |A_2^\epsilon|^2) & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} A_1^\epsilon \\ A_2^\epsilon \\ w_1^\epsilon \\ \vdots \\ w_m^\epsilon \end{pmatrix}
$$

$$
= \frac{\epsilon}{2} \begin{pmatrix} -\triangle_x A_2^\epsilon \\ \triangle_x A_1^\epsilon \\ 0 \\ \vdots \\ 0 \end{pmatrix} + a\epsilon \begin{pmatrix} -\left(|A_1^\epsilon|^2 + |A_2^\epsilon|^2\right) A_2^\epsilon \\ \left(|A_1^\epsilon|^2 + |A_2^\epsilon|^2\right) A_1^\epsilon \\ 0 \\ \vdots \\ 0 \end{pmatrix}.
$$

The matrix $B(U^\epsilon, \eta)$ can be symmetrized by

$$
S(U^\epsilon) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{4f'} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{4f'} \end{pmatrix},
$$

which is symmetric and positive since $f' > 0$. Thus we write (1.1) as a dispersive perturbation of a quasilinear symmetric hyperbolic system

$$
S(U^\epsilon)\partial_t U^\epsilon + S(U^\epsilon)\sum_{i=1}^m B^i(U^\epsilon)\partial_i U^\epsilon = \epsilon L^*(U^\epsilon) + a\epsilon N^*(U^\epsilon), \tag{2.11}
$$

where $L^*(U^\epsilon) = S(U^\epsilon)L(U^\epsilon),\ N^*(U^\epsilon) = S(U^\epsilon)N(U^\epsilon)$. The importance of symmetry is that it leads to simple L^2 and more generally H^s estimates which are often related to physical quantities like energy or entropy. The antisymmetric operator $L^* = SL$ reflects the dispersive nature of the equations. In the following, we prove the local existence in time of (2.9) by using the iteration scheme.

Define

$$
\begin{aligned}
U^\epsilon(0,x) &= (A_1^\epsilon(0,x), A_2^\epsilon(0,x), w_1^\epsilon(0,x), \cdots, w_m^\epsilon(0,x)) \\
&= U_0^\epsilon(x) = (A_{10}^\epsilon(x), A_{20}^\epsilon(x), w_{10}^\epsilon(x), \cdots, w_{m0}^\epsilon(x)),
\end{aligned}
$$

where $U_0^\epsilon(x)$ denotes the initial data and define $U_{p+1}^\epsilon(t,x)$ inductively as the solution of the linear equation $(p = 0, 1, 2, 3, \cdots)$.

$$
\begin{aligned}
&S(U_p^\epsilon)\partial_t U_{p+1}^\epsilon + S(U_p^\epsilon)\sum_{i=1}^m B^i(U_p^\epsilon)\partial_i U_{p+1}^\epsilon = \epsilon L^*(U_{p+1}^\epsilon) + a\epsilon N^*(U_{p+1}^\epsilon), \\
&U_{p+1}^\epsilon(0,x) = U_0^\epsilon(x).
\end{aligned} \tag{2.12}
$$

For further reference, we ignore the subscripts p and consider $U^\epsilon \in C^\infty$, $V^\epsilon \in C^\infty$ satisfying

$$
\begin{aligned}
&S(V^\epsilon)\partial_t U^\epsilon + S(V^\epsilon)\sum_{i=1}^m B^i(V^\epsilon)\partial_i U^\epsilon = \epsilon L^*(U^\epsilon) + a\epsilon N^*(U^\epsilon), \\
&U^\epsilon(0,x) = U_0^\epsilon(x).
\end{aligned} \tag{2.13}
$$

For a certain T, let the function $U^\epsilon(t,x)$ be a solution to (2.13) of class $C^2([0,T]\times\Omega)$ which is of compact support in x for each $t\in[0,T]$. The canonical energy associated with the dispersive perturbation of the symmetric hyperbolic system (2.13) is defined by the scalar product

$$\|U^\epsilon(t)\|_E^2=\int_{\mathbb{R}^m}\langle S\partial_x^\alpha U^\epsilon,\partial_x^\alpha U^\epsilon\rangle\mathrm{d}x, \tag{2.14}$$

where α is a multi-index of length $|\alpha|\le s$ and $\alpha=0,1,\cdots,s$. By (2.11), using the symmetry of S and integrating by parts, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|U^\epsilon(t)\|_E^2&=\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^m}\langle S\partial_x^\alpha U^\epsilon,\partial_x^\alpha U^\epsilon\rangle\mathrm{d}x\\&=\int_{\mathbb{R}^m}\langle\partial_t S\partial_x^\alpha U^\epsilon,\partial_x^\alpha U^\epsilon\rangle\mathrm{d}x+2\int_{\mathbb{R}^m}\langle S\partial_t\partial_x^\alpha U^\epsilon,\partial_x^\alpha U^\epsilon\rangle\mathrm{d}x\\&=\int_{\mathbb{R}^m}\langle\partial_t S\partial_x^\alpha U^\epsilon,\partial_x^\alpha U^\epsilon\rangle\mathrm{d}x+2\epsilon\int_{\mathbb{R}^m}\langle S\partial_x^\alpha L(U^\epsilon),\partial_x^\alpha U^\epsilon\rangle\mathrm{d}x\\&\quad+2a\epsilon\int_{\mathbb{R}^m}\langle S\partial_x^\alpha N(U^\epsilon),\partial_x^\alpha U^\epsilon\rangle\mathrm{d}x\\&\quad-2\int_{\mathbb{R}^m}\Big\langle S\partial_x^\alpha\Big(\sum_{i=1}^m B^i(U^\epsilon)\partial_i U^\epsilon\Big),\partial_x^\alpha U^\epsilon\Big\rangle\mathrm{d}x.\end{aligned} \tag{2.15}$$

Unlike the usual singular perturbation which does not create energy by the choice of S, the singular perturbation $N^*=SN$ can create energy except the case $|\alpha|=0$ in which $N^*=SN$ does not produce energy.

The first term in (2.15) can be bounded by

$$\int_{\mathbb{R}^m}\langle\partial_t S\partial_x^\alpha U^\epsilon,\partial_x^\alpha U^\epsilon\rangle\mathrm{d}x\le|\partial_t S|_{L^\infty}\|\partial_x^\alpha U^\epsilon\|_{L^2}^2.$$

From Sobolev injections and equation (2.9), it follows that

$$\begin{aligned}|\partial_t S|_{L^\infty}&\le C\Big|\frac{f''}{f'^2}(A_1^\epsilon\partial_t A_1^\epsilon+A_2^\epsilon\partial_t A_2^\epsilon)\Big|_{L^\infty}\\&\le C(|U^\epsilon|_{L^\infty})|\partial_t U^\epsilon|_{L^\infty}\\&\le C(|U^\epsilon|_{L^\infty})\Big(|\epsilon L(U^\epsilon)|_{L^\infty}+|a\epsilon N(U^\epsilon)|_{L^\infty}+\Big|\sum_{i=1}^m B^i(U^\epsilon)\partial_i U^\epsilon\Big|_{L^\infty}\Big)\\&\le C(\|U^\epsilon\|_{H^s})\|U^\epsilon\|_{H^s},\end{aligned}$$

where $s>\frac{m}{2}+2$ and

$$\|U^\epsilon\|_{H^s}^2=\sum_{|\alpha|\le s}\|\partial_x^\alpha U^\epsilon\|_{L^2}^2.$$

Thus

$$\sum_{|\alpha|\le s}\int_{\mathbb{R}^m}\langle\partial_t S\partial_x^\alpha U^\epsilon,\partial_x^\alpha U^\epsilon\rangle\mathrm{d}x\le C(\|U^\epsilon\|_{H^s})\|U^\epsilon\|_{H^s}^2. \tag{2.16}$$

The second term and the third term in (2.15) can be bounded by

$$\begin{aligned}
&2\epsilon\int_{\mathbb{R}^m}\langle S\partial_x^\alpha L(U^\epsilon),\partial_x^\alpha U^\epsilon\rangle \mathrm{d}x\\
=&2\epsilon\int_{\mathbb{R}^m}\langle SL(\partial_x^\alpha U^\epsilon),\partial_x^\alpha U^\epsilon\rangle \mathrm{d}x\\
=&\epsilon\int_{\mathbb{R}^m}(-\partial_x^\alpha(\triangle_x A_2^\epsilon)\cdot\partial_x^\alpha A_1^\epsilon+\partial_x^\alpha(\triangle_x A_1^\epsilon)\cdot\partial_x^\alpha A_2^\epsilon)\mathrm{d}x\\
=&0, \qquad (2.17)\\
&2a\epsilon\int_{\mathbb{R}^m}\langle S\partial_x^\alpha N(U^\epsilon),\partial_x^\alpha U^\epsilon\rangle \mathrm{d}x\\
=&2a\epsilon\int_{\mathbb{R}^m}\{-\partial_x^\alpha[(|A_1^\epsilon|^2+|A_2^\epsilon|^2)A_2^\epsilon]\cdot\partial_x^\alpha A_1^\epsilon+\partial_x^\alpha[(|A_1^\epsilon|^2+|A_2^\epsilon|^2)A_1^\epsilon]\cdot\partial_x^\alpha A_2^\epsilon\}\mathrm{d}x\\
=&\begin{cases}0, \quad |\alpha|=0,\\ 2\epsilon\int_{\mathbb{R}^m}\{-\partial_x^\alpha[(|A_1^\epsilon|^2+|A_2^\epsilon|^2)A_2^\epsilon]\cdot\partial_x^\alpha A_1^\epsilon+\partial_x^\alpha[(|A_1^\epsilon|^2+|A_2^\epsilon|^2)A_1^\epsilon]\cdot\partial_x^\alpha A_2^\epsilon\}\mathrm{d}x, \quad 0<|\alpha|\le s\end{cases}\\
\le& C(\|\partial_x^\alpha((|A_1^\epsilon|^2+|A_2^\epsilon|^2)A_2^\epsilon)\|_{L^2}\cdot\|\partial_x^\alpha A_1^\epsilon\|_{L^2}+\|\partial_x^\alpha((|A_1^\epsilon|^2+|A_2^\epsilon|^2)A_1^\epsilon)\|_{L^2}\cdot\|\partial_x^\alpha A_2^\epsilon\|_{L^2})\\
\le& C\|\partial_x^\alpha(|A_1^\epsilon|^2+|A_2^\epsilon|^2)\|_{L^2}\cdot\|\partial_x^\alpha A_1^\epsilon\|_{L^2}\cdot\|\partial_x^\alpha A_2^\epsilon\|_{L^2}.
\end{aligned}$$

Thus

$$\sum_{|\alpha|\le s}\int_{\mathbb{R}^m}\langle S\partial_x^\alpha N(U^\epsilon),\partial_x^\alpha U^\epsilon\rangle \mathrm{d}x\le C(\|U^\epsilon\|_{H^s})\|U^\epsilon\|_{H^s}^2. \tag{2.18}$$

The fourth term in (2.15) can be bounded by

$$\begin{aligned}
&-2\int_{\mathbb{R}^m}\Big\langle S\partial_x^\alpha\Big(\sum_{i=1}^m B^i(U^\epsilon)\partial_i U^\epsilon\Big),\partial_x^\alpha U^\epsilon\Big\rangle \mathrm{d}x\\
=&-2\int_{\mathbb{R}^m}\Big\langle S\sum_{i=1}^m B^i(U^\epsilon)\partial_i(\partial_x^\alpha U^\epsilon),\partial_x^\alpha U^\epsilon\Big\rangle \mathrm{d}x\\
=&-2\int_{\mathbb{R}^m}\Big\langle S\Big[\partial_x^\alpha\Big(\sum_{i=1}^m B^i(U^\epsilon)\partial_i U^\epsilon\Big)-\sum_{i=1}^m B^i(U^\epsilon)\partial_i(\partial_x^\alpha U^\epsilon)\Big],\partial_x^\alpha U^\epsilon\Big\rangle \mathrm{d}x. \qquad (2.19)
\end{aligned}$$

Since $SB^i(U^\epsilon)$ is a symmetric matrix, we get by integration by parts

$$\begin{aligned}
&-2\int_{\mathbb{R}^m}\Big\langle S\sum_{i=1}^m B^i(U^\epsilon)\partial_i(\partial_x^\alpha U^\epsilon),\partial_x^\alpha U^\epsilon\Big\rangle \mathrm{d}x-2\sum_{i=1}^m\int_{\mathbb{R}^m}\langle SB^i(U^\epsilon)\partial_i(\partial_x^\alpha U^\epsilon),\partial_x^\alpha U^\epsilon\rangle \mathrm{d}x\\
=&2\sum_{i=1}^m\int_{\mathbb{R}^m}\langle \partial_i(SB^i(U^\epsilon))\partial_x^\alpha U^\epsilon,\partial_x^\alpha U^\epsilon\rangle \mathrm{d}x+2\sum_{i=1}^m\int_{\mathbb{R}^m}\langle SB^i(U^\epsilon)\partial_i(\partial_x^\alpha U^\epsilon),\partial_x^\alpha U^\epsilon\rangle \mathrm{d}x.
\end{aligned}$$

So

$$\begin{aligned}
-2\int_{\mathbb{R}^m}\Big\langle S\sum_{i=1}^m B^i(U^\epsilon)\partial_i(\partial_x^\alpha U^\epsilon),\partial_x^\alpha U^\epsilon\Big\rangle \mathrm{d}x&=\sum_{i=1}^m\int_{\mathbb{R}^m}\langle \partial_i(SB^i(U^\epsilon))\partial_x^\alpha U^\epsilon,\partial_x^\alpha U^\epsilon\rangle \mathrm{d}x\\
&\le C\Big|\sum_{i=1}^m\partial_i(SB^i(U^\epsilon))\Big|_{L^\infty}\cdot\|\partial_x^\alpha U^\epsilon\|_{L^2}^2\\
&\le C(|U^\epsilon|_{L^\infty})\cdot|\nabla_x U^\epsilon|_{L^\infty}\cdot\|\partial_x^\alpha U^\epsilon\|_{L^2}^2.
\end{aligned}$$

Thus from the Sobolev embedding and $s > \frac{m}{2} + 2$, we get

$$\sum_{|\alpha| \le s} \Big(-2 \int_{\mathbb{R}^m} \Big\langle S \sum_{i=1}^{m} B^i(U^\epsilon) \partial_i (\partial_x^\alpha U^\epsilon), \partial_x^\alpha U^\epsilon \Big\rangle \mathrm{d}x \Big) \le C(\|U^\epsilon\|_{H^s}) \|U^\epsilon\|_{H^s}^2. \tag{2.20}$$

For the second term in (2.19), using the symmetry of $SB^i(U^\epsilon)$ and the usual estimates on commutators, we obtain

$$\begin{aligned} & -2 \int_{\mathbb{R}^m} \Big\langle S \Big[\partial_x^\alpha \Big(\sum_{i=1}^{m} B^i(U^\epsilon) \partial_i U^\epsilon \Big) - \sum_{i=1}^{m} B^i(U^\epsilon) \partial_i (\partial_x^\alpha U^\epsilon) \Big], \partial_x^\alpha U^\epsilon \Big\rangle \mathrm{d}x \\ &= -2 \int_{\mathbb{R}^m} \Big\langle S \sum_{i=1}^{m} (\partial_x^\alpha (B^i(U^\epsilon) \partial_i U^\epsilon) - B^i(U^\epsilon) \partial_x^\alpha \partial_i U^\epsilon), \partial_x^\alpha U^\epsilon \Big\rangle \mathrm{d}x \\ &= -2 \int_{\mathbb{R}^m} \Big\langle S \sum_{i=1}^{m} [\partial_x^\alpha, B^i(U^\epsilon)] \partial_i U^\epsilon, \partial_x^\alpha U^\epsilon \Big\rangle \mathrm{d}x \\ &\le C(|\partial_x^\alpha U^\epsilon|_{L^\infty}) \cdot |\nabla_x U^\epsilon|_{L^\infty} \cdot \|\partial_x^\alpha U^\epsilon\|_{L^2}^2, \end{aligned}$$

thus

$$\begin{aligned} & \sum_{|\alpha| \le s} \Big(-2 \int_{\mathbb{R}^m} \Big\langle S \Big[\partial_x^\alpha \Big(\sum_{i=1}^{m} B^i(U^\epsilon) \partial_i U^\epsilon \Big) - \sum_{i=1}^{m} B^i(U^\epsilon) \partial_i (\partial_x^\alpha U^\epsilon) \Big], \partial_x^\alpha U^\epsilon \Big\rangle \mathrm{d}x \Big) \\ & \le C \|U^\epsilon\|_{H^s}) \|U^\epsilon\|_{H^s}^2. \end{aligned} \tag{2.21}$$

So from (2.15)–(2.21), we get for $s > \frac{m}{2} + 2$,

$$\frac{\mathrm{d}}{\mathrm{d}t} \sum_{|\alpha| \le s} \int_{\mathbb{R}^m} \langle S \partial_x^\alpha U^\epsilon, \partial_x^\alpha U^\epsilon \rangle \mathrm{d}x \le C(\|U^\epsilon\|_{H^s}) \|U^\epsilon\|_{H^s}^2. \tag{2.22}$$

This energy estimate is independent of ϵ. According to Gronwall Lemma along with a continuity argument and (2.22), we get

$$\|U^\epsilon\|_{H^s}^2 \le C(T) \|U_0^\epsilon\|_{H^s}^2.$$

Thus we obtain

$$\|U_p^\epsilon\|_{H^s}^2 \le C \tag{2.23}$$

as soon as $U_0^\epsilon \in H^s$. It implies the convergence of the iteration $\{U_p^\epsilon\}_{p=1}^\infty$ to a unique classical solution of system (2.9) or (2.11). It follows from (2.11) and (2.23) that

$$\|\partial_t A_1^\epsilon\|_{H^{s-2}} \le C, \quad \|\partial_t A_2^\epsilon\|_{H^{s-2}} \le C, \quad \|\partial_t w_i^\epsilon\|_{H^{s-1}} \le C.$$

From the Sobolev embedding $H^{s-1} \hookrightarrow H^{s-2}$, we get

$$\|U_p^\epsilon\|_{H^{s-2}} \le C. \tag{2.24}$$

This ensures that for any fixed ϵ, we can construct a sequence $\{U_p^\epsilon\}_{p=0}^\infty$ belonging to $C([0,T]; H^s)$ $\cap C^1([0,T]; H^{s-2})$ satisfying (2.12) as well as the uniform estimates

$$\max_{0 \le t \le T} (\|\partial_t U_p^\epsilon(t)\|_{H^{s-2}} + \|U_p^\epsilon(t)\|_{H^s}) \le C,$$

where the constant C is independent of t. It follows by Ascoli-Arzelà Theorem (see [9]) that there exists

$$U^{\epsilon} \in L^{\infty}([0,T]; H^{s}) \wedge \mathrm{Lip}([0,T]; H^{s-2}) \tag{2.25}$$

such that

$$\max_{0 \le t \le T} \|U_p^{\epsilon} - U^{\epsilon}\|_{H^{s-2}} \to 0, \quad \text{as } p \to \infty.$$

Therefore, by the standard interpolation inequality, we have the convergence

$$U_p^{\epsilon} \to U^{\epsilon}, \quad \text{in } C([0,T]; H^{s-\theta}) \tag{2.26}$$

for an appropriate θ with $0 < \theta < 2$. Choose s such that $s - \theta - 2 > \frac{1}{2}$. Then the space H^s becomes an algebra, thus we can overcome the difficulty of the nonlinearity. From [7, 10–12], we can prove that

$$U^{\epsilon} \in C([0,T]; H^{s}) \wedge C^1([0,T]; H^{s-2}) \tag{2.27}$$

and U^{ϵ} is a solution of (2.9)–(2.10). The transition from estimates for integral of square to pointwise estimates is furnished by one of the Sobolev inequalities. By Sobolev's embedding theorem, we have

$$C([0,T]; H^{s}) \wedge C^1([0,T]; H^{s-2}) \hookrightarrow C^1([0,T] \times \Omega) \tag{2.28}$$

and hence the constructed solutions are classical.

In the following, we prove the uniqueness of solution of (2.4)–(2.8) or (2.9)–(2.10). Suppose that there exist two solutions $U_1^{\epsilon}(t,x)$ and $U_2^{\epsilon}(t,x)$ to the initial value problem (2.9)–(2.10) on $[0,T]$. Let $V^{\epsilon} = U_1^{\epsilon} - U_2^{\epsilon}$. Then $V^{\epsilon}(0) = 0$,

$$\begin{aligned} \partial_t V^{\epsilon} &= \partial_t U_1^{\epsilon} - \partial_t U_2^{\epsilon} \\ &= \epsilon L(U_1^{\epsilon}) + \epsilon N(U_1^{\epsilon}) - \sum_{i=1}^{m} B^i(U_1^{\epsilon}) \partial_i U_1^{\epsilon} - \Big(\epsilon L(U_2^{\epsilon}) + \epsilon N(U_2^{\epsilon}) - \sum_{i=1}^{m} B^i(U_2^{\epsilon}) \partial_i U_2^{\epsilon} \Big) \\ &= \epsilon L(U_1^{\epsilon} - U_2^{\epsilon}) + \epsilon N(U_1^{\epsilon}) - \epsilon N(U_2^{\epsilon}) - \sum_{i=1}^{m} (B^i(U_1^{\epsilon}) \partial_i U_1^{\epsilon} - B^i(U_2^{\epsilon}) \partial_i U_2^{\epsilon}). \end{aligned} \tag{2.29}$$

Define

$$\|V^{\epsilon}(t)\|_E^2 = \int_{R^m} \langle S \partial_x^{\alpha} V^{\epsilon}, \partial_x^{\alpha} V^{\epsilon} \rangle \mathrm{d}x, \tag{2.30}$$

where α is a multi-index of $|\alpha| \le s$. Thus making the same energy estimates as before, we obtain

$$\|V^{\epsilon}\|_{H^s}^2 \le C(T) \, \|V^{\epsilon}(0)\|_{H^s}^2 \, .$$

Therefore, from $V^{\epsilon}(0) = 0$, we get $U_1^{\epsilon} = U_2^{\epsilon}$ on $t \in [0,T]$. Namely, the solution of the initial value problem (2.9)–(2.10) is unique. Using the above argument, we prove the existence and uniqueness of the classical solution to the dispersive perturbation of the quasi-linear symmetric system (2.4)–(2.8). That is as follows.

Theorem 2.1 *Let $f \in C^\infty(\mathbb{R}^+, \mathbb{R})$ with $f' > 0$ and $s > \frac{m}{2} + 2$. Assume that the initial data $U_0^\epsilon = (A_{10}^\epsilon, A_{20}^\epsilon, w_{10}^\epsilon, \cdots, w_{m0}^\epsilon)^{\mathrm{T}} \in H^s(\mathbb{R}^m) \times H^s(\mathbb{R}^m) \times H^s(\mathbb{R}^m) \times \cdots \times H^s(\mathbb{R}^m)$ satisfies the uniform bound*

$$\|U_0^\epsilon\|_{H^s(\mathbb{R}^m)} = \|A_{10}^\epsilon\|_{H^s(\mathbb{R}^m)} + \|A_{20}^\epsilon\|_{H^s(\mathbb{R}^m)} + \|w_{10}^\epsilon\|_{H^s(\mathbb{R}^m)} + \cdots + \|w_{m0}^\epsilon\|_{H^s(\mathbb{R}^m)} < C_1. \quad (2.31)$$

Then there exists a time interval $[0, T]$ with $T > 0$, so that the initial value problem (2.4)–(2.8) has a unique classical solution $U^\epsilon = (A_1^\epsilon, A_2^\epsilon, w_1^\epsilon, \cdots, w_m^\epsilon)^{\mathrm{T}}$,

$$\begin{aligned} (A_1^\epsilon(t,x), A_2^\epsilon(t,x)) &\in (C^1([0,T] \times \Omega) \wedge C^1([0,T]; C^2))^2 \\ w_i^\epsilon(t,x) &\in C^1([0,T] \times \Omega). \end{aligned} \quad (2.32)$$

Furthermore,

$$U^\epsilon \in C([0,T]; H^s) \wedge C^1([0,T]; H^{s-2}) \quad (2.33)$$

and T depends on the bound C_1 in (2.31) and in particular not on ϵ. In addition, the solution $U^\epsilon = (A_1^\epsilon, A_2^\epsilon, w_1^\epsilon, \cdots, w_m^\epsilon)^{\mathrm{T}}$ satisfies the estimate

$$\|U^\epsilon\|_{H^s} = \|A_1^\epsilon\|_{H^s} + \|A_2^\epsilon\|_{H^s} + \|w_1^\epsilon\|_{H^s} + \cdots + \|w_m^\epsilon\|_{H^s} < C_2 \quad (2.34)$$

for all $t \in [0, T]$, and the constant C_2 is also independent of ϵ.

For equation (1.1), we have the following equivalent result.

Theorem 2.2 *Assume that Theorem 2.1 holds. In addition, suppose $(A_0^\epsilon, S_0^\epsilon) \in H^s \times H^{s+1}$. Then the initial value problem (1.1)–(1.2) has a unique classical solution in $C^1([0,T] \times \Omega) \wedge C^1([0,T]; C^2)$ of the form $\psi^\epsilon(t,x) = A^\epsilon(t,x) \exp(\frac{\mathrm{i}}{\epsilon} S^\epsilon(t,x))$ on the time interval $[0,T]$. Moreover, A^ϵ and $\nabla_x S^\epsilon$ are bounded in $L^\infty([0,T]; H^s)$.*

Proof Since $A^\epsilon = A_1^\epsilon + \mathrm{i} A_2^\epsilon$ and $w^\epsilon = \nabla_x S^\epsilon$, it follows from (2.32)–(2.34) that

$$\begin{aligned} &A^\epsilon \in C([0,T]; H^s) \wedge C^1([0,T]; H^{s-2}), \\ &S^\epsilon \in C([0,T]; H^{s+1}) \wedge C^1([0,T]; H^s). \end{aligned}$$

Thus by Sobolev embedding theorem, we get

$$\begin{aligned} &A^\epsilon \in C^1([0,T] \times \Omega) \wedge C^1([0,T]; C^2), \\ &S^\epsilon \in C^1([0,T]; C^2). \end{aligned}$$

Due to the expression of ψ^ϵ in the short wave form (2.1), ψ^ϵ has the same regularity as A^ϵ, and

$$\psi^\epsilon \in C([0,T]; H^s) \wedge C^1([0,T]; H^{s-2}),$$

thus

$$\psi^\epsilon \in C^1([0,T] \times \Omega) \wedge C^1([0,T]; C^2).$$

For classical solution, equation (1.1) is equivalent to the dispersive quasi-linear hyperbolic system (2.9). Using this equivalent relation and Theorem 2.1, we can get the result of Theorem 2.2.

3 Existence of Solution with ϵ Small Enough

In this section, we link T in Theorem 2.1 and Theorem 2.2 to the existence time of a smooth solution to (1.11)–(1.12).

Theorem 3.1 *Let $f \in C^\infty(\mathbb{R}^+, \mathbb{R})$ with $f' > 0$, $s > \frac{m}{2} + 2$, $S_0(x) \in H^s(\mathbb{R}^m)$, $A_0^\epsilon(x)$ be a sequence of functions uniformly bounded in $H^s(\mathbb{R}^m)$ and $A_0^\epsilon(x)$ converges to A_0 in $H^s(\mathbb{R}^m)$ as ϵ goes to 0. Moreover, if system (1.11)–(1.12) with initial data $\rho(0, x) = |A_0(x)|^2$ and $v(0, x) = \nabla_x S_0(x)$ has a solution in $L^\infty([0, T], H^{s+2}(\mathbb{R}^m))$, then for ϵ small enough, there exist solutions to equation (1.1) of the form $\psi^\epsilon(t, x) = A^\epsilon(t, x) \exp(\frac{i}{\epsilon} S^\epsilon(t, x))$ on $[0, T]$, where A^ϵ and $\nabla_x S^\epsilon$ are bounded in $L^\infty([0, T], H^s(\mathbb{R}^m))$ uniformly in ϵ.*

Proof According to the assumptions of this theorem, there exists a solution (ρ, v) in $L^\infty([0, T], H^{s+2}(\mathbb{R}^m))$ to system (1.11)–(1.12) on a time interval $[0, T]$ with $s > \frac{m}{2} + 2$ for the initial data

$$\rho = \Big|\lim_{\epsilon\to 0} A_0^\epsilon\Big|^2, \quad v = \nabla_x S_0. \tag{3.1}$$

Let $U = (A_1, A_2, w)$. The limit system of (2.9) is

$$\partial_t U + \sum_{i=1}^m B^i(U)\partial_i U = 0, \tag{3.2}$$

which admits a solution on a maximal time interval $[0, T']$. In the following, we prove that $T' > T$ by contradiction. Assume that $T' \le T$. Let $\rho = |A_1|^2 + |A_2|^2$ and $v = w$. Then (ρ, v) satisfy (1.11)–(1.12) with initial data (3.1) and ρ and v are in $L^\infty([0, T'], H^s(\mathbb{R}^m))$. Thus $w \in L^\infty([0, T'], H^s(\mathbb{R}^m))$. Using (2.4) and (2.5), we get that A_1 and A_2 are in $L^\infty([0,T'], H^{s-1}(\mathbb{R}^m))$, which is impossible since T' is assumed to be the maximal existence time. Therefore, $T' > T$ and system (3.2) has a smooth solution on the time interval $[0, T]$.

Setting $V^\epsilon = U^\epsilon - U$, we obtain from (2.9) and (3.2) that

$$\begin{aligned}\partial_t V^\epsilon &= \partial_t U^\epsilon - \partial_t U\\ &= \epsilon L(U^\epsilon) + \epsilon N(U^\epsilon) - \sum_{i=1}^m B^i(U^\epsilon)\partial_i U^\epsilon - \Big(-\sum_{i=1}^m B^i(U)\partial_i U\Big)\\ &= \epsilon L(U^\epsilon) + \epsilon N(U^\epsilon) - \sum_{i=1}^m B^i(U + V^\epsilon)\partial_i V^\epsilon - \sum_{i=1}^m (B^i(U + V^\epsilon) - B^i(U))\partial_i U,\end{aligned}$$

namely,

$$\begin{aligned}&\partial_t V^\epsilon + \sum_{i=1}^m B^i(U + V^\epsilon)\partial_i V^\epsilon + \sum_{i=1}^m (B^i(U + V^\epsilon) - B^i(U))\partial_i U\\ &= \epsilon L(U) + \epsilon L(V^\epsilon) + \epsilon N(U + V^\epsilon).\end{aligned} \tag{3.3}$$

The matrix $\sum\limits_{i=1}^m B^i(U + V^\epsilon)\eta_i$ is symmetrisable. We can make the same energy estimates as those in Section 2. Because S is symmetric, we get for $|\alpha| \le s$ and $s > \frac{m}{2} + 2$,

$$\partial_t \int_{\mathbb{R}^m} \langle S\partial_x^\alpha V^\epsilon, \partial_x^\alpha V^\epsilon\rangle \mathrm{d}x = \int_{\mathbb{R}^m} \langle \partial_t S\partial_x^\alpha V^\epsilon, \partial_x^\alpha V^\epsilon\rangle \mathrm{d}x + 2\int_{\mathbb{R}^m} \langle S\partial_t\partial_x^\alpha V^\epsilon, \partial_x^\alpha V^\epsilon\rangle \mathrm{d}x. \tag{3.4}$$

The first term can be bounded by

$$\sum_{|\alpha|\le s}\int_{\mathbb{R}^m}\langle \partial_t S\partial_x^\alpha V^\epsilon, \partial_x^\alpha V^\epsilon\rangle \mathrm{d}x \le C(\|V^\epsilon\|_{H^s}, \|U\|_{H^{s+2}}^2)\|V^\epsilon\|_{H^s}^2. \tag{3.5}$$

For the second term of (3.4), from (3.3) we get

$$\begin{aligned}\int_{\mathbb{R}^m}\langle S\partial_t\partial_x^\alpha V^\epsilon, \partial_x^\alpha V^\epsilon\rangle \mathrm{d}x &= \epsilon\int_{\mathbb{R}^m}\langle SL(\partial_x^\alpha V^\epsilon) + SL(\partial_x^\alpha U), \partial_x^\alpha V^\epsilon\rangle \mathrm{d}x\\
&\quad + \epsilon\int_{\mathbb{R}^m}\langle S\partial_x^\alpha N(U+V^\epsilon), \partial_x^\alpha V^\epsilon\rangle \mathrm{d}x\\
&\quad - \int_{\mathbb{R}^m}\Big\langle S\partial_x^\alpha\Big(\sum_{i=1}^m B^i(U+V^\epsilon)\partial_i V^\epsilon\Big), \partial_x^\alpha V^\epsilon\Big\rangle \mathrm{d}x\\
&\quad - \int_{\mathbb{R}^m}\Big\langle S\partial_x^\alpha\Big(\sum_{i=1}^m (B^i(U+V^\epsilon) - B^i(U))\partial_i U\Big), \partial_x^\alpha V^\epsilon\Big\rangle \mathrm{d}x.\end{aligned} \tag{3.6}$$

For the first term in (3.6), using integration by parts we obtain

$$\begin{aligned}&\epsilon\int_{\mathbb{R}^m}\langle SL(\partial_x^\alpha V^\epsilon) + SL(\partial_x^\alpha U), \partial_x^\alpha V^\epsilon\rangle \mathrm{d}x\\
&= \epsilon\int_{\mathbb{R}^m}\langle SL(\partial_x^\alpha V^\epsilon), \partial_x^\alpha V^\epsilon\rangle \mathrm{d}x + \epsilon\int_{\mathbb{R}^m}\langle SL(\partial_x^\alpha U), \partial_x^\alpha V^\epsilon\rangle \mathrm{d}x\\
&= \epsilon\int_{\mathbb{R}^m}\langle L(\partial_x^\alpha V^\epsilon), \partial_x^\alpha V^\epsilon\rangle \mathrm{d}x + \epsilon\int_{\mathbb{R}^m}\langle L(\partial_x^\alpha U), \partial_x^\alpha V^\epsilon\rangle \mathrm{d}x\\
&= 0 + \epsilon\int_{\mathbb{R}^m}\langle L(\partial_x^\alpha U), \partial_x^\alpha V^\epsilon\rangle \mathrm{d}x\\
&= \epsilon\int_{\mathbb{R}^m}\Big(-\frac{1}{2}\triangle_x\partial_x^\alpha A_2\cdot\partial_x^\alpha V_1^\epsilon + \frac{1}{2}\triangle_x\partial_x^\alpha A_1\cdot\partial_x^\alpha V_2^\epsilon\Big)\mathrm{d}x\\
&\le \epsilon C\|\partial_x^\alpha U\|_{L^2}\|\partial_x^\alpha V^\epsilon\|_{L^2}.\end{aligned}$$

Thus

$$\sum_{|\alpha|\le s}\epsilon\int_{\mathbb{R}^m}\langle SL(\partial_x^\alpha V^\epsilon) + SL(\partial_x^\alpha U), \partial_x^\alpha V^\epsilon\rangle \mathrm{d}x \le \epsilon C\|U\|_{H^{s+2}}\|V^\epsilon\|_{H^s}. \tag{3.7}$$

For the second term in (3.6), we can make the following estimates:

$$\begin{aligned}\epsilon\int_{\mathbb{R}^m}\langle S\partial_x^\alpha N(U+V^\epsilon), \partial_x^\alpha V^\epsilon\rangle \mathrm{d}x &= \epsilon\int_{\mathbb{R}^m}(-\partial_x^\alpha((|A_1+V_1^\epsilon|^2 + |A_2+V_2^\epsilon|^2)(A_2+V_2^\epsilon))\partial_x^\alpha V_1^\epsilon\\
&\quad + \partial_x^\alpha((|A_1+V_1^\epsilon|^2 + |A_2+V_2^\epsilon|^2)(A_1+V_1^\epsilon))\partial_x^\alpha V_2^\epsilon)\mathrm{d}x\\
&\le \epsilon C(\|\partial_x^\alpha(U^2U)\|_{L^2} + \|\partial_x^\alpha({V^\epsilon}^2U)\|_{L^2})\|\partial_x^\alpha V^\epsilon\|_{L^2}.\end{aligned} \tag{3.8}$$

Thus

$$\sum_{|\alpha|\le s}\epsilon\int_{\mathbb{R}^m}\langle S\partial_x^\alpha N(U+V^\epsilon), \partial_x^\alpha V^\epsilon\rangle \mathrm{d}x \le \epsilon C(\|U\|_{H^s}^3 + \|V^\epsilon\|_{H^s})\|V^\epsilon\|_{H^s}. \tag{3.9}$$

For the third term in (3.6), since $SB^i(U+V^\epsilon)$ is a symmetric matrix, we obtain

$$\begin{aligned}
&-\int_{\mathbb{R}^m}\Big\langle S\partial_x^\alpha\Big(\sum_{i=1}^m B^i(U+V^\epsilon)\partial_i V^\epsilon\Big),\partial_x^\alpha V^\epsilon\Big\rangle \mathrm{d}x\\
=&-\int_{\mathbb{R}^m}\Big\langle S\sum_{i=1}^m B^i(U+V^\epsilon)\partial_i\partial_x^\alpha V^\epsilon,\partial_x^\alpha V^\epsilon\Big\rangle \mathrm{d}x\\
&-\int_{\mathbb{R}^m}\Big\langle S\Big(\partial_x^\alpha\Big(\sum_{i=1}^m B^i(U+V^\epsilon)\partial_i V^\epsilon\Big)-\sum_{i=1}^m B^i(U+V^\epsilon)\partial_i\partial_x^\alpha V^\epsilon\Big),\partial_x^\alpha V^\epsilon\Big\rangle \mathrm{d}x. \qquad (3.10)
\end{aligned}$$

But

$$\begin{aligned}
&-\int_{\mathbb{R}^m}\Big\langle S\sum_{i=1}^m B^i(U+V^\epsilon)\partial_i\partial_x^\alpha V^\epsilon,\partial_x^\alpha V^\epsilon\Big\rangle \mathrm{d}x\\
=&-\sum_{i=1}^m\int_{\mathbb{R}^m}\langle SB^i(U+V^\epsilon)\partial_i\partial_x^\alpha V^\epsilon,\partial_x^\alpha V^\epsilon\rangle \mathrm{d}x\\
=&\sum_{i=1}^m\int_{\mathbb{R}^m}\langle \partial_i(SB^i(U+V^\epsilon))\partial_x^\alpha V^\epsilon,\partial_x^\alpha V^\epsilon\rangle \mathrm{d}x+\sum_{i=1}^m\int_{\mathbb{R}^m}\langle SB^i(U+V^\epsilon)\partial_i\partial_x^\alpha V^\epsilon,\partial_x^\alpha V^\epsilon\rangle \mathrm{d}x,
\end{aligned}$$

thus

$$\begin{aligned}
&-\int_{\mathbb{R}^m}\Big\langle S\sum_{i=1}^m B^i(U+V^\epsilon)\partial_i\partial_x^\alpha V^\epsilon,\partial_x^\alpha V^\epsilon\Big\rangle \mathrm{d}x\\
=&\frac{1}{2}\sum_{i=1}^m\int_{\mathbb{R}^m}\langle \partial_i(SB^i(U+V^\epsilon))\partial_x^\alpha V^\epsilon,\partial_x^\alpha V^\epsilon\rangle \mathrm{d}x\\
\le& C(|U+V^\epsilon|_{L^\infty})\|\partial_x^\alpha V^\epsilon\|_{L^2}^2|\nabla_x(U+V^\epsilon)|_{L^\infty}. \qquad (3.11)
\end{aligned}$$

Therefore by $s>\frac{m}{2}+2$, we get

$$\sum_{|\alpha|\le s}-\int_{\mathbb{R}^m}\Big\langle S\sum_{i=1}^m B^i(U+V^\epsilon)\partial_i\partial_x^\alpha V^\epsilon,\partial_x^\alpha V^\epsilon\Big\rangle \mathrm{d}x\le C(\|U\|_{H^{s+2}},\|V^\epsilon\|_{H^s})\|V^\epsilon\|_{H^s}^2. \qquad (3.12)$$

For the last term in (3.10), applying (3.11) and the usual estimates on communicators, we obtain

$$\begin{aligned}
&-\int_{\mathbb{R}^m}\Big\langle S\Big(\partial_x^\alpha\Big(\sum_{i=1}^m B^i(U+V^\epsilon)\partial_i V^\epsilon\Big)-\sum_{i=1}^m B^i(U+V^\epsilon)\partial_i\partial_x^\alpha V^\epsilon\Big),\partial_x^\alpha V^\epsilon\Big\rangle \mathrm{d}x\\
=&-\int_{\mathbb{R}^m}\Big\langle S\sum_{i=1}^m(\partial_x^\alpha(B^i(U+V^\epsilon)\partial_i V^\epsilon)-B^i(U+V^\epsilon)\partial_x^\alpha(\partial_i V^\epsilon)),\partial_x^\alpha V^\epsilon\Big\rangle \mathrm{d}x\\
=&-\int_{\mathbb{R}^m}\Big\langle S\sum_{i=1}^m(\partial_x^\alpha(B^i(U+V^\epsilon))\partial_i V^\epsilon),\partial_x^\alpha V^\epsilon\Big\rangle \mathrm{d}x\\
\le& C(\|\partial_x^\alpha U\|_{L^2},\|\partial_x^\alpha V^\epsilon\|_{L^2})\|\partial_x^\alpha V^\epsilon\|_{L^2}\|\nabla_x V^\epsilon\|_{L^2},
\end{aligned}$$

so

$$\begin{aligned}
&\sum_{|\alpha|\le s}-\int_{\mathbb{R}^m}\Big\langle S\Big(\partial_x^\alpha\Big(\sum_{i=1}^m B^i(U+V^\epsilon)\partial_i V^\epsilon\Big)-\sum_{i=1}^m B^i(U+V^\epsilon)\partial_i\partial_x^\alpha V^\epsilon\Big),\partial_x^\alpha V^\epsilon\Big\rangle \mathrm{d}x\\
\le& C(\|U\|_{H^s},\|V^\epsilon\|_{H^s})\|V^\epsilon\|_{H^s}^2. \qquad (3.13)
\end{aligned}$$

For the last term in (3.6), we get

$$
\begin{aligned}
&-\int_{\mathbb{R}^m} \Big\langle S\partial_x^\alpha\Big(\sum_{i=1}^m (B^i(U+V^\epsilon)-B^i(U))\partial_i U\Big), \partial_x^\alpha V^\epsilon\Big\rangle \mathrm{d}x \\
&\le C(\|\partial_x^\alpha U\|_{L^2}, \|\partial_x^\alpha V^\epsilon\|_{L^2})\|\partial_x^\alpha V^\epsilon\|_{L^2}^2,
\end{aligned}
$$

so

$$
\begin{aligned}
&\sum_{|\alpha|\le s} -\int_{\mathbb{R}^m} \Big\langle S\partial_x^\alpha\Big(\sum_{i=1}^m (B^i(U+V^\epsilon)-B^i(U))\partial_i U\Big), \partial_x^\alpha V^\epsilon\Big\rangle \mathrm{d}x \\
&\le C(\|U\|_{H^{s+1}}, \|V^\epsilon\|_{H^s})\|V^\epsilon\|_{H^s}^2. \qquad (3.14)
\end{aligned}
$$

Thus from (3.4)–(3.14), we get

$$
\begin{aligned}
&\partial_t \sum_{|\alpha|\le s}\int_{\mathbb{R}^m} \langle S\partial_x^\alpha V^\epsilon, \partial_x^\alpha V^\epsilon\rangle \mathrm{d}x \\
&\le \epsilon C\|U\|_{H^{s+2}}\|V^\epsilon\|_{H^s} + \epsilon C(\|U\|_{H^s}^3 + \|V^\epsilon\|_{H^s})\|V^\epsilon\|_{H^s} + C(\|U\|_{H^{s+2}}, \|V^\epsilon\|_{H^s})\|V^\epsilon\|_{H^s}^2 \qquad (3.15)
\end{aligned}
$$

for $s > \frac{m}{2}+2$. Moreover,

$$
\lim_{\epsilon\to 0}\|V^\epsilon(t=0)\|_{H^s} = \lim_{\epsilon\to 0}\|U^\epsilon(t=0)-U(t=0)\|_{H^s} = 0. \qquad (3.16)
$$

Therefore Gronwall's lemma along with a continuity argument and (3.15) show that for ϵ small, there exists a constant $C(\epsilon)$ such that

$$
\|V^\epsilon\|_{H^s} \le C(\epsilon), \quad \text{on } [0,T]
$$

with $C(\epsilon)\to 0$ as $\epsilon\to 0$. That is,

$$
\lim_{\epsilon\to 0}\|U^\epsilon - U\|_{H^s} = 0.
$$

Since $U=(A_1,A_2,w)$ and $w=\nabla_x S$ exist on $[0,T]$, U^ϵ also exists on $[0,T]$. Thus for ϵ small enough, there exist solutions to equation (1.1) of the form

$$
\psi^\epsilon(t,x) = A^\epsilon(t,x)\exp\Big(\frac{\mathrm{i}S^\epsilon(t,x)}{\epsilon}\Big), \quad \text{on } [0,T].
$$

Making the same estimates as those in Theorem 2.2, we can obtain that A^ϵ and $\nabla_x S^\epsilon$ are bounded in $L^\infty([0,T],H^s)$ uniformly in ϵ.

4 WKB Expansion (Approximation)

In this section, we justify the WKB expansion.

Theorem 4.1 *Under the assumptions of Theorem* 3.1, *suppose that the initial amplitude* $A_0^\epsilon(x)$ *admits the following expansion:*

$$
A_0^\epsilon(x) = \sum_{k=0}^N A_0^{(k)}(x)\epsilon^k + \epsilon^N R_N(x,\epsilon), \qquad (4.1)
$$

where

$$\lim_{\epsilon\to 0}\|R_N(x,\epsilon)\|_{H^s}=0 \tag{4.2}$$

for $N\in\mathbf{N}$ *and* $s>2N+2+\frac{m}{2}$. *Thus, on the time interval* $[0,T]$ *given by Theorem* 3.1, *equation* (1.1) *can be represented as*

$$\psi^\epsilon(t,x)=A^\epsilon(t,x)\exp\Big(\frac{\mathrm{i}S^\epsilon(t,x)}{\epsilon}\Big)=\sum_{k=0}^{N}A^{(k)}(t,x)\epsilon^k\exp\Big(\frac{\mathrm{i}S(t,x)}{\epsilon}\Big)+\epsilon^N R_N(t,x,\epsilon) \tag{4.3}$$

as ϵ *goes to zero, where* $S(t,x)$ *and* $A^{(k)}(t,x)$ *are given by WKB method, and*

$$\lim_{\epsilon\to 0}\|R_N(t,x,\epsilon)\|_{C([0,T];H^{s-2N-2-\frac{m}{2}}(\mathbb{R}^m))}=0. \tag{4.4}$$

Proof We look for formally asymptotic solutions of (2.9) in the form

$$U^\epsilon=U_0'+\epsilon U_1'+\epsilon^2U_2'+\cdots+\epsilon^N U_N'+\cdots, \tag{4.5}$$

where $U^\epsilon=(A_1^\epsilon,A_2^\epsilon,w_1^\epsilon,\cdots,w_m^\epsilon)^{\mathrm{T}}$, $A^\epsilon=A_1^\epsilon+\mathrm{i}A_2^\epsilon$ and $w^\epsilon=(w_1^\epsilon,\cdots,w_m^\epsilon)^{\mathrm{T}}=\nabla_x S^\epsilon$. We first consider the zeroth order.

4.1 Zeroth order approximation

From Theorem 3.1, we know that A^ϵ and w^ϵ are bounded in $L^\infty([0,T],H^s(\mathbb{R}^m))$. Then from (2.2) and (2.3) it follows that ∂_tA^ϵ and ∂_tw^ϵ are bounded in $L^\infty([0,T],H^{s-2}(\mathbb{R}^m))$. Therefore, by using the classical compactness arguments, Ascoli-Arzelà Theorem (applied in the time variable) and the Rellich Lemma (applied in the space variable), we deduce from (2.2) and (2.3) that there exist subsequences of A^ϵ and w^ϵ (always denoted by A^ϵ and w^ϵ) such that A^ϵ and w^ϵ converge uniformly in $L^\infty([0,T],H^{s-2}(\mathbb{R}^m))$ to the functions A_0' and w_0', where A_0' and w_0' satisfy the quasilinear hyperbolic system

$$\partial_tA_0'+(w_0'\cdot\nabla_x)A_0'+\frac{1}{2}A_0'\nabla_x\cdot w_0'=0, \tag{4.6}$$

$$\partial_tw_0'+(w_0'\cdot\nabla_x)w_0'+f'(|A_0'|^2)\nabla_x|A_0'|^2=0 \tag{4.7}$$

with the initial data complemented by

$$A_0'(0,x)=\lim_{\epsilon\to 0}A_0^\epsilon(x),\quad w_0'(0,x)=\nabla_xS_0(x).$$

This system admits a unique solution, which implies in fact that all the sequences (A^ϵ,w^ϵ) converge.

4.2 First order approximation

For convenience, we set $V_1^\epsilon\equiv U^\epsilon-U_0'$, where $U_0'=(A_{10}',A_{20}',w_{10}',\cdots,w_{m0}')$, $A_0'=A_{10}'+\mathrm{i}A_{20}'$, $w_0'=(w_{10}',\cdots,w_{m0}')$. Then by using the same energy estimate as those in Section 3, we obtain

$$\|V_1^\epsilon\|_{H^{s-2}(R^m)}\le\epsilon C(\|U_0'\|_{L^\infty([0,T],H^s(\mathbb{R}^m))})$$

for all $t \leq T$. Let $\widetilde{V}_1^\epsilon = \frac{V_1^\epsilon}{\epsilon}$. Thus $\widetilde{V}_1^\epsilon$ is bounded in $L^\infty([0,T], H^{s-2}(\mathbb{R}^m))$ and $\partial_t \widetilde{V}_1^\epsilon$ is bounded in $L^\infty([0,T], H^{s-4}(\mathbb{R}^m))$. Thus for a subsequence, $\widetilde{V}_1^\epsilon$ converges strongly in $L^\infty([0,T], H^{s-4}(\mathbb{R}^m))$ to a function U_1' and

$$\begin{aligned}
\partial_t \widetilde{V}_1^\epsilon &= \frac{1}{\epsilon}(\partial_t U^\epsilon - \partial_t U_0') \\
&= \frac{1}{\epsilon}\Big[\epsilon L(U^\epsilon) + \epsilon N(U^\epsilon) - \sum_{i=1}^m B^i(U^\epsilon)\partial_i U^\epsilon - \Big(- \sum_{i=1}^m B^i(U_0')\partial_i U_0'\Big)\Big] \\
&= L(U^\epsilon) + N(U^\epsilon) - \frac{1}{\epsilon}\sum_{i=1}^m (B^i(U^\epsilon)\partial_i U^\epsilon - B^i(U_0')\partial_i U_0') \\
&= L(U^\epsilon) + N(U^\epsilon) - \sum_{i=1}^m B^i(U_0' + \epsilon \widetilde{V}_1^\epsilon)\partial_i \widetilde{V}_1^\epsilon - \frac{1}{\epsilon}\sum_{i=1}^m [B^i(U_0' + \epsilon \widetilde{V}_1^\epsilon) - B^i(U_0')]\partial_i U_0' \\
&= L(U_0' + \epsilon \widetilde{V}_1^\epsilon) + N(U_0' + \epsilon \widetilde{V}_1^\epsilon) - \sum_{i=1}^m B^i(U_0' + \epsilon \widetilde{V}_1^\epsilon)\partial_i \widetilde{V}_1^\epsilon \\
&\quad - \frac{1}{\epsilon}\sum_{i=1}^m [B^i(U_0' + \epsilon \widetilde{V}_1^\epsilon) - B^i(U_0')]\partial_i U_0'.
\end{aligned} \tag{4.8}$$

Taking the limit of the equation (4.8), we obtain

$$\begin{aligned}
&\lim_{\epsilon\to 0}\partial_t \widetilde{V}_1^\epsilon + \lim_{\epsilon\to 0}\sum_{i=1}^m B^i(U_0' + \epsilon \widetilde{V}_1^\epsilon)\partial_i \widetilde{V}_1^\epsilon + \lim_{\epsilon\to 0}\sum_{i=1}^m \frac{1}{\epsilon}[B^i(U_0' + \epsilon \widetilde{V}_1^\epsilon) - B^i(U_0')]\partial_i U_0' \\
&= \lim_{\epsilon\to 0}[L(U_0' + \epsilon \widetilde{V}_1^\epsilon) + N(U_0' + \epsilon \widetilde{V}_1^\epsilon)].
\end{aligned} \tag{4.9}$$

From the definition of Frechét derivative, we get

$$\lim_{\epsilon\to 0}\sum_{i=1}^m \frac{1}{\epsilon}[B^i(U_0' + \epsilon \widetilde{V}_1^\epsilon) - B^i(U_0')]\partial_i U_0' = \sum_{i=1}^m (\nabla B^i(U_0')U_1')\partial_i U_0'.$$

Thus we obtain from (4.9) that

$$\partial_t U_1' + \sum_{i=1}^m B^i(U_0'\partial_i U_1') + \sum_{i=1}^m (\nabla B^i(U_0')U_1')\partial_i U_0' = L(U_0') + N(U_0') \tag{4.10}$$

with initial data

$$U_1'(0,x) = \lim_{\epsilon\to 0}\frac{1}{\epsilon}(U^\epsilon(0,x) - U_0'(0,x)). \tag{4.11}$$

As the solution to problem (4.10)–(4.11) is unique, we get that the whole sequence $\widetilde{V}_1^\epsilon$ converges to U_1'.

Similarly, the higher order approximation can be obtained as follows.

4.3 Higher order approximation

Suppose that we have already obtained an asymptotic expansion to the order N

$$U^\epsilon = \sum_{j=0}^N U_j'\epsilon^j + o(\epsilon^N), \tag{4.12}$$

where

$$f(\epsilon^N)=o(\epsilon^N)\Longleftrightarrow\Big|\frac{f(\epsilon^N)}{\epsilon^N}\Big|\to 0,\quad \text{as } \epsilon^N\to 0.$$

Firstly, let

$$\overline{U}^\epsilon=\sum_{j=0}^{N}U_j'\epsilon^j \quad\text{and}\quad V_{N+1}^\epsilon=U^\epsilon-\sum_{j=0}^{N}U_j'\epsilon^j, \tag{4.13}$$

the functions U_j' being in $L^\infty([0,T],H^{s-2j}(\mathbb{R}^m))$, $j=0,1,2,\cdots,N$.

Then, put

$$\widetilde{V}_{N+1}^\epsilon=\frac{1}{\epsilon^{N+1}}\Big[U^\epsilon-\sum_{j=0}^{N}U_j'\epsilon^j\Big]. \tag{4.14}$$

Since $U_k'\in L^\infty([0,T],H^{s-2k}(\mathbb{R}^m))$, we get $\partial_t U_k'\in L^\infty([0,T],H^{s-2k-2}(\mathbb{R}^m))$. Moreover, by the assumption on the initial data, $\widetilde{V}_{N+1}^\epsilon(0)$ is bounded in $H^s(\mathbb{R}^m)$. Making the similar energy estimates to the previous section, we obtain $\widetilde{V}_{N+1}^\epsilon$ is bounded in $L^\infty([0,T],H^{s-2N-2}(\mathbb{R}^m))$ and $\partial_t\widetilde{V}_{N+1}^\epsilon$ is bounded in $L^\infty([0,T],H^{s-2N-4}(\mathbb{R}^m))$. Thus, for a subsequence, $\widetilde{V}_{N+1}^\epsilon$ converges strongly in $L^\infty([0,T],H^{s-2N-4}(\mathbb{R}^m))$ to a function U_{N+1}' as ϵ goes to 0.

From (4.12) and (4.13), in view of equation (2.9) for U^ϵ, we get

$$\partial_t V_{N+1}^\epsilon=\partial_t U^\epsilon-\partial_t\overline{U}^\epsilon=\epsilon L(U^\epsilon)+\epsilon N(U^\epsilon)-\sum_{i=1}^{m}B^i(U^\epsilon)\partial_i U^\epsilon-\partial_t\overline{U}^\epsilon. \tag{4.15}$$

Then, making a Taylor expansion on B^i we obtain

$$\begin{aligned}&\partial_t V_{N+1}^\epsilon+\sum_{i=1}^{m}B^i(\overline{U}^\epsilon+V_{N+1}^\epsilon)\partial_i V_{N+1}^\epsilon-\sum_{i=1}^{m}(B^i(\overline{U}^\epsilon)-B^i(\overline{U}^\epsilon+V_{N+1}^\epsilon))\partial_i\overline{U}^\epsilon\\&=\epsilon L(V_{N+1}^\epsilon)+\epsilon N(V_{N+1}^\epsilon)+\epsilon^{2N+1}C_N^\epsilon+\epsilon^{3N+1}D_N^\epsilon+\epsilon^{N+1}B_N^\epsilon,\end{aligned}$$

where B_N^ϵ is a function depending on $\overline{U}^\epsilon$ and bounded in $L^\infty\big([0,T],H^{s-2N-2}(\mathbb{R}^m)\big)$ uniformly in ϵ, C_N^ϵ is a function depending on $\overline{U}^\epsilon$ and V_{N+1}^ϵ, D_N^ϵ is a function depending on $\overline{U}^\epsilon$.

To find the equation for U_{N+1}', write the term of order ϵ^{N+1} as

$$\begin{aligned}&\partial_t(\overline{U}^\epsilon+\epsilon^{N+1}\widetilde{V}_{N+1}^\epsilon)+\sum_{i=1}^{m}B^i(\overline{U}^\epsilon+\epsilon^{N+1}\widetilde{V}_{N+1}^\epsilon)\partial_i(\overline{U}^\epsilon+\epsilon^{N+1}\widetilde{V}_{N+1}^\epsilon)\\&-\epsilon L(\overline{U}^\epsilon+\epsilon^{N+1}\widetilde{V}_{N+1}^\epsilon)-\epsilon N(\overline{U}^\epsilon+\epsilon^{N+1}\widetilde{V}_{N+1}^\epsilon)=0\end{aligned}$$

and take limit with initial data as

$$U_{N+1}'(0,x)=\lim_{\epsilon\to 0}\frac{1}{\epsilon^{N+1}}(U^\epsilon(0,x)-U_0'(0,x)-\cdots-\epsilon^N U_N'(0,x)).$$

Since we have obtained the formal approximation expansion of A^ϵ and of S^ϵ to an arbitrarily high order. Now we get back to the usual WKB expansion. The usual WKB method is to look for solutions of (1.1) of the form

$$\psi^\epsilon(t,x)=A(t,x,\epsilon)\exp\Big(\frac{\mathrm{i}}{\epsilon}S(t,x)\Big),$$

where

$$A(t,x,\epsilon)=\sum_{j=0}^{\infty}A_j(t,x)\epsilon^j.$$

In order to obtain A_j and S, we only write the identity of the two following formal series:

$$\sum_{j=0}^{\infty}A_j(t,x)\epsilon^j\exp\Big(\frac{\mathrm{i}}{\epsilon}S(t,x)\Big)=\Big(\sum_{k=0}^{\infty}A'_k\epsilon^k\Big)\exp\Big(\frac{\mathrm{i}}{\epsilon}\sum_{k=0}^{\infty}S'_k\epsilon^k\Big),$$

where

$$A'_k=A'_{1k}+\mathrm{i}A'_{2k},\quad U'_k=(A'_{1k},A'_{2k},w'_{1k},\cdots,w'_{mk}),\quad w'_k=\nabla_x S'_k=(w'_{1k},\cdots,w'_{mk}).$$

For instance, we can get

$$S=S'_0,\quad A_0=A'_0\mathrm{e}^{\mathrm{i}S'_1},\quad A_1=\mathrm{e}^{\mathrm{i}S'_1}(A'_1+\mathrm{i}S'_2A'_0),$$
$$A_2=\mathrm{e}^{\mathrm{i}S'_1}\Big(A'_2+\mathrm{i}S'_2A'_1+\Big(\mathrm{i}S'_3-\frac{{S'_2}^2}{2}\Big)A'_0\Big),\ \cdots.$$

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