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A Note on Schwarz-Pick Estimate***

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Abstract A Schwarz-Pick estimate of higher order derivative for holomorphic functions with positive real part on $\mathbf{B}_n$ is presented. This improves the earlier work on Schwarz-Pick estimate of higher order derivatives for holomorphic functions with positive real part on the unit disk in $\mathbb{C}$.

Keywords Schwarz-Pick estimate, Holomorphic function, Taylor expansion
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1 Introduction

For notation, let $\mathbf{D}$ be the unit disk in $\mathbb{C}$, $\mathbf{B}_n$ be the unit ball in $\mathbb{C}^n$. A multi-index $v = (v_1, \cdots, v_n)$ consists of n nonnegative integers v_i $(1 \le i \le n)$, and the degree of the multi-index v is the sum $|v| = \sum\limits_{i=1}^{n} v_i$. For vectors $z = (z_1, \cdots, z_n) \in \mathbb{C}^n$, $|z| = \Big(\sum\limits_{i=1}^{n} |z_i|^2\Big)^{\frac{1}{2}}$, and the multi-indices can be used as exponents in a product $z^v = \prod\limits_{i=1}^{n} z_i^{v_i}$; similarly, a_v represents the coefficient $a_{v_1,\cdots,v_n}$ of z^v in the Taylor expansion of a holomorphic function.

In this paper, we denote $\mathbf{\Omega}$ as the complete Reinhardt domain. Let $\mathfrak{B}(\mathbf{\Omega})$ and $\mathfrak{R}(\mathbf{\Omega})$ be the sets of all holomorphic functions $\varphi(z)$ in $\mathbf{\Omega}$ with $|\varphi(z)| < 1$ and the real part $\Re\varphi(z) > 0$ for each $z \in \mathbf{\Omega}$ respectively. Obviously, $\mathbf{B}_n$ is a special complete Reinhardt domain.

The classical Schwarz-Pick estimate is the inequality $|\varphi'(z)| \le \frac{1-|\varphi(z)|^2}{1-|z|^2}$, $|z| < 1$, for a holomorphic function $\varphi(z)$ satisfying $|\varphi(z)| < 1$ on the unit disk of the complex plane.

On the other hand, holomorphic function with positive real part is also an important part in function theory. There were some results on it (see [5, 6, 8, 9, 11]).

If $\varphi(z) \in \mathfrak{R}(\mathbf{D})$, in 2008, Dai and Pan had the following estimate of higher order derivatives for positive real part holomorphic functions on $\mathbf{D}$.

Theorem A (see [5]) *Let $\varphi(z)$ be a holomorphic function in $\mathbf{D}$ and $\Re\varphi > 0$ for each $z \in \mathbf{D}$. Then* $|\varphi^{(m)}(z)| \le \frac{2m!\Re\varphi(z)}{(1-|z|^2)^m}(1+|z|)^{m-1}$.

Recently, Liu and Chen considered generalized Schwarz-Pick estimate for positive real part holomorphic functions on the unit ball of $\mathbb{C}^n$ and they had the following result which would deduce Theorem A when $n = 1$.

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Theorem B (see [9]) *Let $\varphi(z) \in \mathfrak{R}(\mathbf{B}_n)$. Then for multi-index $m = (m_1, \cdots, m_n)$,*

$$|\partial^m \varphi(z)| \leq \binom{n+|m|-1}{n-1}^{n+2} \sqrt{\frac{|m|^{|m|}}{(\frac{|m|}{n})^{\frac{|m|}{n}}}} \frac{2|m|!\Re\varphi(z)}{(1-|z|^2)^{|m|}}(1+|z|)^{|m|-1}, \tag{1.1}$$

where $\partial^m \varphi(z) = \frac{\partial^{|m|}\varphi(z)}{\partial z_1^{m_1}\cdots\partial z_n^{m_n}}$.

In [4], the authors proved a high order Schwarz-Pick lemma for mappings between unit balls in complex spaces in terms of the Bergman metric, and Schwarz-Pick estimates for partial derivatives of arbitrary order of mappings were deduced. Motivated by [2, 4], in this paper we obtain the coefficient inequality on bounded holomorphic functions in complete Reinhardt domains; furthermore, estimates of higher order derivatives for all the positive real part holomorphic functions on $\mathbf{B}_n$ are given.

The following theorems are the main results of this paper.

Theorem 1.1 *Let $\varphi(z) \in \mathfrak{R}(\mathbf{B}_n)$. Then for multi-index $m = (m_1, \cdots, m_n)$ and $v = (v_1, \cdots, v_n) \neq 0$,*

$$\sum_{|\alpha|=|m|} \Big|\frac{|m|!}{\alpha!}\partial^m\varphi(z)\Big|^2 \frac{v^\alpha}{|v|^{|\alpha|}} \leq \Big[\frac{2|m|!\Re\varphi(z)}{(1-|z|^2)^{|m|}}(1+|z|)^{|m|-1}\Big]^2, \tag{1.2}$$

where $\partial^m \varphi(z) = \frac{\partial^{|m|}\varphi(z)}{\partial z_1^{\alpha_1}\cdots\partial z_n^{\alpha_n}}$.

Theorem 1.2 *Let $\varphi(z) \in \mathfrak{R}(\mathbf{B}_n)$. Then for multi-index $m = (m_1, \cdots, m_n)$,*

$$|\partial^m \varphi(z)| \leq \sqrt{\frac{|m|^{|m|}}{m^m}} \frac{2m!\Re\varphi(z)}{(1-|z|^2)^{|m|}}(1+|z|)^{|m|-1}, \tag{1.3}$$

where $\partial^m \varphi(z) = \frac{\partial^{|m|}\varphi(z)}{\partial z_1^{m_1}\cdots\partial z_n^{m_n}}$.

Remark 1.1 Theorem 1.2 gives a better estimate than Theorem B. Since the factor

$$\binom{n+|m|-1}{n-1}^{n+2}$$

is canceled, $m! \leq |m|!$ and $\sqrt{\frac{|m|^{|m|}}{m^m}} \leq \sqrt{\frac{|m|^{|m|}}{(\frac{|m|}{n})^{\frac{|m|}{n}}}}$.

Remark 1.2 When $n = 1$, our results can deduce Theorem A.

2 Main Lemma

Lemma 2.1 *Let $\varphi(z) \in \mathfrak{B}(\mathbf{\Omega})$ and $\varphi(z) = \sum_\alpha a_\alpha z^\alpha$. Then we have*

$$\sum_\alpha |a_\alpha|^2 |\beta^{2\alpha}| \leq 1 \quad \textit{for any } \beta \in \partial\Omega. \tag{2.1}$$

Proof For any $\zeta = (\zeta_1, \cdots, \zeta_n) \in \mathbf{\Omega}$, $\zeta\theta := (\zeta_1 e^{i\theta_1}, \cdots, \zeta_n e^{i\theta_n}) \in \mathbf{\Omega}$, where $\theta_i \in \mathbb{R}$ $(i = 1, \cdots, n)$. Considering $\varphi(\zeta\theta)$, when $|\alpha| > 0$, by the orthogonality we have

$$\begin{aligned} 1 &\geq \frac{1}{(2\pi)^n}\int_0^{2\pi}\cdots\int_0^{2\pi} |\varphi(\zeta_1 e^{i\theta_1}, \cdots, \zeta_n e^{i\theta_n})|^2 d\theta_1\cdots d\theta_n \\ &= \sum_\alpha |a_\alpha|^2 |\zeta_1^{\alpha_1}\cdots\zeta_n^{\alpha_n}|^2 = \sum_\alpha |a_\alpha|^2|\zeta^\alpha|^2. \end{aligned} \tag{2.2}$$

Let $\zeta \to \beta$. The desired result is concluded.

Remark 2.1 From Lemma 2.1, when $\varphi(z) \in \mathfrak{B}(\mathbf{B}_n)$, for multi-index $v = (v_1, \cdots, v_n) \neq 0$, let $\beta = \left(\sqrt{\frac{v_1}{|v|}}, \cdots, \sqrt{\frac{v_n}{|v|}}\right)$. Then

$$\sum_{\alpha} |a_\alpha|^2 \frac{v^\alpha}{|v|^{|\alpha|}} \leq 1, \tag{2.3}$$

especially,

$$|a_\alpha| \leq \sqrt{\frac{|v|^{|\alpha|}}{v^\alpha}}. \tag{2.4}$$

In fact, we can also deduce the above remark from [4].

3 Proofs of Theorems 1.1 and 1.2

Motivated by [4], now we give the proof of Theorem 1.1.

Proof of Theorem 1.1 For multi-index $m = (m_1, \cdots, m_n)$ and $v = (v_1, \cdots, v_n) \neq 0$, let $|m| \geq 1$, $\beta = (\beta_1, \cdots, \beta_n) \in \partial\mathbf{B}_n$ and $\xi = (\xi_1, \cdots, \xi_n) \in \mathbf{B}_n$ be given. Now we consider the disk $\Delta = \{\zeta \in \mathbb{C} : |\xi + \zeta\beta|^2 = |\xi_1 + \zeta\beta_1|^2 + \cdots + |\xi_n + \zeta\beta_n|^2 < 1\}$. For simplicity, let U be a unitary matrix such that $U\beta = (1, 0, \cdots, 0)^{\mathrm{T}}$. Denote $U\xi = \eta = (\eta_1, \cdots, \eta_n)^{\mathrm{T}}$. Then $|\xi + \zeta\beta|^2 = |U(\xi + \zeta\beta)|^2 = |\eta_1 + \zeta|^2 + |\eta_2|^2 + \cdots + |\eta_n|^2$. Rewrite Δ as $\Delta = \{\zeta \in \mathbb{C} : |\eta_1 + \zeta|^2 < 1 - |\eta_2|^2 - \cdots - |\eta_n|^2\}$. Now we set $\sigma = (1 - |\eta_2|^2 - \cdots - |\eta_n|^2)^{\frac{1}{2}}$, $\gamma = \sigma\beta$, $\zeta = \sigma\omega - \eta_1$, $z = L(\omega) = \xi + \omega\gamma - \eta_1\beta$.

Let $g(\omega) := \varphi(L(\omega)) \in \mathfrak{R}(\mathbf{D})$. Using Theorem A to $g(\omega)$ at the point $\omega = \omega' = \frac{\eta_1}{\sigma}$, we have

$$|g^{(|m|)}(\omega')| \leq \frac{2|m|!\Re g(\omega')}{(1 - |\omega'|^2)^{|m|}}(1 + |\omega'|)^{|m|-1}. \tag{3.1}$$

On the other hand, $g(\omega') = \varphi(\xi)$, $|\eta| = |U\xi| = |\xi|$, $\eta_1 = \langle U\xi, U\beta\rangle = \langle \xi, \beta\rangle$, and $\sigma^2 = 1 - |\eta|^2 + |\eta_1|^2 = 1 - |\xi|^2 + |\langle \xi, \beta\rangle|^2 \leq 1$, $|\omega'| = \frac{|\langle \xi, \beta\rangle|}{(1 - |\xi|^2 + |\langle \xi, \beta\rangle|^2)^{\frac{1}{2}}}$, $1 - |\omega'|^2 = \frac{1 - |\xi|^2}{\sigma^2}$. By the chain rule,

$$g^{(k)}(\omega') = \sum_{|\alpha|=k} \frac{k!}{\alpha!} \frac{\partial^k \varphi(\xi)}{\partial z_1^{\alpha_1} \cdots \partial z_1^{\alpha_1}} \gamma^\alpha = \sigma^k \sum_{|\alpha|=k} \frac{k!}{\alpha!} \frac{\partial^k \varphi(\xi)}{\partial z_1^{\alpha_1} \cdots \partial z_1^{\alpha_1}} \beta^\alpha.$$

Thus, (3.1) can be written as

$$\Big|\sigma^{|m|} \sum_{|\alpha|=|m|} \frac{|m|!}{\alpha!} \frac{\partial^{|m|}\varphi(\xi)}{\partial z_1^{\alpha_1} \cdots \partial z_1^{\alpha_1}} \beta^\alpha\Big| \leq \frac{2\sigma^{2|m|}|m|!\Re\varphi(\xi)}{(1 - |\xi|^2)^{|m|}} \Big(1 + \frac{|\langle \xi, \beta\rangle|}{(1 - |\xi|^2 + |\langle \xi, \beta\rangle|^2)^{\frac{1}{2}}}\Big)^{|m|-1}, \tag{3.2}$$

i.e.,

$$\Big|\sum_{|\alpha|=|m|} \frac{|m|!}{\alpha!} \frac{\partial^{|m|}\varphi(\xi)}{\partial z_1^{\alpha_1} \cdots \partial z_1^{\alpha_1}} \beta^\alpha\Big| \leq \frac{2\sigma^{|m|}|m|!\Re\varphi(\xi)}{(1 - |\xi|^2)^{|m|}} \Big(1 + \frac{|\langle \xi, \beta\rangle|}{(1 - |\xi|^2 + |\langle \xi, \beta\rangle|^2)^{\frac{1}{2}}}\Big)^{|m|-1}.$$

Since $\sigma^{|m|}\Big(1 + \frac{|\langle \xi, \beta\rangle|}{(1 - |\xi|^2 + |\langle \xi, \beta\rangle|^2)^{\frac{1}{2}}}\Big)^{|m|-1} = \sigma(\sigma + |\langle \xi, \beta\rangle|) \leq 1 + |\xi|$, we have

$$\Big|\sum_{|\alpha|=|m|} \frac{|m|!}{\alpha!} \frac{\partial^{|m|}\varphi(\xi)}{\partial z_1^{\alpha_1} \cdots \partial z_1^{\alpha_1}} \beta^\alpha\Big| \leq \frac{2|m|!\Re\varphi(\xi)}{(1 - |\xi|^2)^{|m|}}(1 + |\xi|)^{|m|-1}. \tag{3.3}$$

Let $A := \frac{2|m|!\Re\varphi(\xi)}{(1-|\xi|^2)^{|m|}}(1+|\xi|)^{|m|-1}$, and let $z = \rho\beta \in \mathbf{B}_n$ for $0 \le \rho < 1$. Define $h(z) := \frac{1}{A}\sum\limits_{|\alpha|=|m|} \frac{|m|!}{\alpha!}\frac{\partial^{|m|}\varphi(\xi)}{\partial z_1^{\alpha_1}\cdots\partial z_1^{\alpha_1}} z^\alpha$. From (3.3), we have

$$
\begin{aligned}
|h(z)| &= \Big|\frac{1}{A}\sum_{|\alpha|=|m|}\frac{|m|!}{\alpha!}\frac{\partial^{|m|}\varphi(\xi)}{\partial z_1^{\alpha_1}\cdots\partial z_1^{\alpha_1}} z^\alpha\Big| = \Big|\frac{1}{A}\sum_{|\alpha|=|m|}\frac{|m|!}{\alpha!}\frac{\partial^{|m|}\varphi(\xi)}{\partial z_1^{\alpha_1}\cdots\partial z_1^{\alpha_1}} (\rho\beta)^\alpha\Big| \\
&= \rho^{|m|}\Big|\frac{1}{A}\sum_{|\alpha|=|m|}\frac{|m|!}{\alpha!}\frac{\partial^{|m|}\varphi(\xi)}{\partial z_1^{\alpha_1}\cdots\partial z_1^{\alpha_1}} \beta^\alpha\Big| < 1. \qquad (3.4)
\end{aligned}
$$

Then $h(z) \in \mathfrak{B}(\mathbf{B}_n)$. From Remark 2.1, we have

$$
\sum_{|\alpha|=|m|}\Big|\frac{|m|!}{\alpha!}\frac{\partial^{|m|}\varphi(\xi)}{\partial z_1^{\alpha_1}\cdots\partial z_1^{\alpha_1}}\Big|^2\frac{v^\alpha}{|v|^{|\alpha|}} \le A^2. \qquad (3.5)
$$

Theorem 1.1 is proved for $z = \xi$.

In particular, from (3.5), letting $v = m$, we have

$$
\Big|\frac{|m|!}{m!}\frac{\partial^{|m|}\varphi(\xi)}{\partial z_1^{m_1}\cdots\partial z_1^{m_1}}\Big|\sqrt{\frac{m^m}{|m|^{|m|}}} \le A = \frac{2|m|!\Re\varphi(\xi)}{(1-|\xi|^2)^{|m|}}(1+|\xi|)^{|m|-1}. \qquad (3.6)
$$

Then

$$
\Big|\frac{\partial^{|m|}\varphi(\xi)}{\partial z_1^{m_1}\cdots\partial z_1^{m_1}}\Big| \le \sqrt{\frac{|m|^{|m|}}{m^m}}\frac{2m!\Re\varphi(\xi)}{(1-|\xi|^2)^{|m|}}(1+|\xi|)^{|m|-1}. \qquad (3.7)
$$

By replacing ξ with z, Theorem 1.2 is proved.

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