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Global Solutions of Shock Reflection by Wedges for the Nonlinear Wave Equation*

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Abstract When a plane shock hits a wedge head on, it experiences a reflection-diffraction process and then a self-similar reflected shock moves outward as the original shock moves forward in time. In this paper, shock reflection by large-angle wedges for compressible flow modeled by the nonlinear wave equation is studied and a global theory of existence, stability and regularity is established. Moreover, $C^{0,1}$ is the optimal regularity for the solutions across the degenerate sonic boundary.

Keywords Compressible flow, Conservation laws, Nonlinear wave system, Regular reflection
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1 Introduction

We are concerned with the problems of shock reflection by wedges, which are modeled by the nonlinear wave equation. When a plane shock hits a wedge head on, it experiences a reflection-diffraction process and then a self-similar reflected shock moves outward as the original shock moves forward in time. In [5], G.-Q. Chen and Feldman analyzed these phenomena of shock reflection by large-angle wedges for potential flow, which is the first global theory for this problem.

The compressible isentropic gas dynamics, neglecting the inertial terms, become

$$\begin{aligned} \rho_t + m_x + n_y &= 0, \\ m_t + p_x &= 0, \\ n_t + p_y &= 0, \end{aligned} \tag{1.1}$$

for $(t, x, y) \in [0, \infty) \times \mathbb{R}^2$, where ρ, p and (m, n) stand for density, pressure and momenta in x and y directions respectively. We denote $c^2(\rho) := p'(\rho) = \rho^\gamma$, with $\gamma > 0$, and remark that $c^2(\rho)$ is a positive and increasing function for all $\rho > 0$.

For smooth solutions or in regions where a solution $U = (\rho, m, n)$ is smooth, eliminating m

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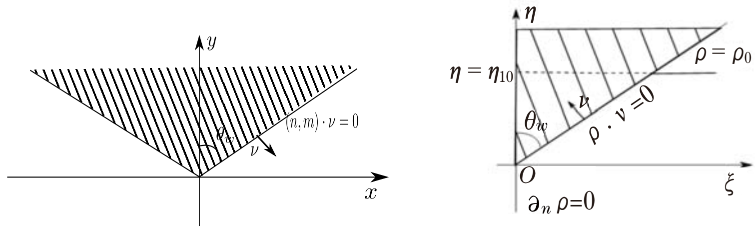


(a) Initial-boundary value problem (b) Boundary value problem

Figure 1 Initial-boundary value problem and boundary value problem

and n in (1.1), we obtain a second order equation for ρ,

$$\rho_{tt} = -m_{tx} - n_{ty} = p_{xx} + p_{yy} = \operatorname{div}(c^2(\rho)\nabla\rho). \tag{1.2}$$

For more details for the derivation of (1.2), please refer to [3], in which the equation was first studied systematically. When a plane shock with the lower state $U_1 = (\rho_1, m_1, 0)$ and the upper state $U_0 = (\rho_0, 0, 0)$, where $m_1 = \sqrt{(p(\rho_1) - p(\rho_0))(\rho_1 - \rho_0)} > 0$ and $\rho_0 < \rho_1$, hits a symmetric wedge $W := \{y > |x| \cot\theta_w\}$ head on, it experiences a reflection-diffraction process, and the reflection problem can be formulated as follows.

Problem 1.1 (Initial-Boundary Value Problem) (see Figure 1(a)) Seek a solution to (1.1), with the initial condition at $t = 0$

$$U|_{t=0} = \begin{cases} U_0, & \text{for } |x| > y\tan\theta_w,\ y > 0, \\ U_1, & \text{for } y < 0, \end{cases} \tag{1.3}$$

and the momenta (m, n) parallel to the wall (see Figure 1(a))

$$m = n\tan\theta_w. \tag{1.4}$$

Notice that the initial-boundary value problem (1.1) with (1.3)–(1.4) is invariant under the self-similar scaling: $(x, y, t) \to (\alpha x, \alpha y, \alpha t)$ for $\alpha \neq 0$. Thus we seek self-similar solutions with the form $(\rho, m, n)(x, y, t) = (\rho, m, n)(\xi, \eta)$ for $(\xi, \eta) = (\frac{x}{t}, \frac{y}{t})$. Write system (1.1) in self-similar coordinates,

$$\begin{aligned} -\xi\rho_\xi - \eta\rho_\eta + m_\xi + n_\eta &= 0, \\ -\xi m_\xi - \eta m_\eta + c^2(\rho)\rho_\xi &= 0, \\ -\xi n_\xi - \eta n_\eta + c^2(\rho)\rho_\eta &= 0. \end{aligned} \tag{1.5}$$

If the solutions are smooth, ρ satisfies

$$((c^2 - \xi^2)\rho_\xi - \xi\eta\rho_\eta)_\xi + ((c^2 - \eta^2)\rho_\eta - \xi\eta\rho_\xi)_\eta + \xi\rho_\xi + \eta\rho_\eta = 0. \tag{1.6}$$

The eigenvalues of the coefficient matrix of the second order terms of (1.6) are $c^2(\rho)$ and $c^2(\rho) - \xi^2 - \eta^2$.

The plane incident shock in the (ξ, η)-coordinates satisfies $U = (\rho_0, 0, 0)$ for $\eta > \eta_{10}$, and $U = (\rho_1, 0, n_1)$ for $\eta < \eta_{10}$, where $\eta_{10} = \sqrt{\frac{p(\rho_1) - p(\rho_0)}{\rho_1 - \rho_0}}$ is the location of the incident shock, uniquely determined by ρ_0 and ρ_1.

Since the problem is symmetric with respect to the axis $\xi = 0$, it suffices to consider the problem in the half-plane $\xi \geq 0$ outside the half-wedge

$$\Lambda := \{\xi \geq 0,\ \eta < 0\} \cup \{\xi \geq \eta \tan\theta_w,\ \eta > 0\}.$$

Then the initial-boundary value problem (1.1) and (1.3)–(1.4) in the (x, y, t)-coordinates can be formulated as the following boundary value problem in the (ξ, η)-coordinates.

Problem 1.2 (Boundary Value Problem) (see Figure 1(b)) Seek a solution to (1.6) in the self-domain Λ with the slip boundary condition on the wedge boundary $\partial\Lambda$

$$D\rho \cdot \nu = 0 \tag{1.7}$$

and the asymptotic boundary condition at infinity

$$\rho \to \widetilde{\rho} = \begin{cases} \rho_0 & \text{for } \eta > \eta_{10},\ \xi > \eta\tan\theta_w, \\ \rho_1 & \text{for } \eta < \eta_{10},\ \xi > 0, \end{cases} \quad \text{when } \xi^2 + \eta^2 \to \infty, \tag{1.8}$$

in the sense that $\lim\limits_{R\to\infty} \|\rho - \widetilde{\rho}\|_{C(\Lambda\setminus B_R)(0)} = 0$, where ν denotes the exterior unit normal to Ω on the wedge.

Remark 1.1 On the wedge, the boundary condition $m = n\tan\theta_w$ becomes $\partial_\nu\rho = 0$. The last two equations in (1.5) are used for determining (m, n) once ρ is obtained. Thus Problem 1.1 is equivalent to Problem 1.2.

Since the momenta $(0, n_1)$ does not parallel the wall, the solution must differ from ρ_1 in $\{\eta < \eta_{10}\} \cap \Lambda$. Thus a shock diffraction by the wedge occurs. In this paper, we first follow the von Neumann criterion to establish a local existence of regular shock reflection near the reflection point and show that the structure of the solutions is as in Figure 2, when the wedge angle is large and close to $\frac{\pi}{2}$, in which the horizontal line is the incident shock $S = \{\eta = \eta_{10}\}$ that hits the wedge at the point $P_0 = (\eta_{10}\tan\theta_w, \eta_{10})$, and the state (0) and the state (1) ahead of and behind S are given by ρ_0 and ρ_1 respectively. The solution ρ differs from ρ_1 in the domain $P_0P_1P_2O$ because of the shock diffraction by the wedge vertex, where the curve $P_0P_1P_2$ is the reflected shock with the straight segment P_0P_1. State (2) is behind P_0P_1.

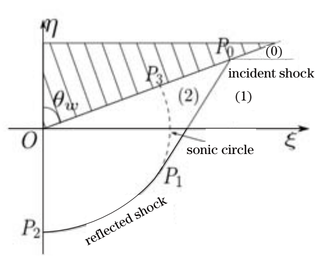


Figure 2 Regular reflection

Theorem 1.1 *There exists a $\theta_0 = \theta_0(\rho_0, \rho_1) \in (0, \frac{\pi}{2})$ such that, for any $\theta_w \in [\theta_0, \frac{\pi}{2})$, there exists a global self-similar solution to Problem* 1.2 (*equivalently, Problem* 1.1)*, which satisfies* *that, for* $(\xi,\eta)=(\frac{x}{t},\frac{y}{t})$, $\rho\in C^\infty$ *in the open domain* $OP_2P_1P_3$, *with*

$$\rho=\begin{cases}\rho_0 & \textit{for } \eta>\eta_{10} \textit{ and } \xi>\eta\tan\theta_w,\\ \rho_1 & \textit{for } \eta<\eta_{10} \textit{ below the reflection shock } P_0P_1P_2,\\ \rho_2 & \textit{in } P_0P_1P_3,\end{cases} \tag{1.9}$$

and ρ *is* $C^{0,1}$, *which is the optimal regularity across the degenerate sonic boundary* P_1P_3, *and the reflected shock* $P_0P_1P_2$ *is* $C^{1,1}$ *at* P_1 *and* C^∞ *except* P_1. *Moreover, the solutions tend to the normal reflection when* $\theta_w\to\frac{\pi}{2}$.

There are two main difficulties to get the global existence. First, the ellipticity degenerates at the sonic circle P_1P_3 (the boundary of the subsonic flow). Second, the oblique boundary degenerates at P_2. The techniques used here to prove the global existence of the solutions rely on the Perron method developed in [10], which is to show the global existence of the solutions to the linearized fixed boundary value problem; and on the application of the Schauder fixed point theorem for the nonlinear free boundary value problem, which is based on [5] and [3]. In this paper, we cannot get the estimates of $\rho_2-\rho$ directly in the process of proving the existence of the solutions, when θ_w tends to $\frac{\pi}{2}$, since it is not easy to find a global supersolution to the boundary value problem about $\phi=c^2(\rho_2)-c^2(\rho)$, because of the nonlinearity of the coefficients of the governing equation and the equation for the oblique boundary condition. In addition, we use the self-similar coordinates and polar coordinates simultaneously. The reason is that it is hard to show $\widetilde{\eta}\widetilde{\eta}'(\xi)+\xi>0$ for $\xi>0$, and is then hard to get the obliqueness condition on the shock $(\xi,\widetilde{\eta}(\xi))$ during the iteration, which is an obstacle to proving the existence of the solutions to the fixed boundary problem. But, it is easy to prove the obliqueness condition in the polar coordinates, thus to get the global existence of the solutions for the regularized free boundary value problem. However, the position of the reflected shock can be described more precisely in self-similar coordinates (ξ,η) than in polar coordinates (r,θ), namely convexity. Moreover, if there exists a solution to the regularized nonlinear free boundary problem in polar coordinates, we can show that it is also a solution in self-similar coordinates.

In order to show the regularity near the sonic boundary, we write (1.6) in terms of the function $\psi=c^2(\rho_2)-c^2(\rho)$ in the new (x,y)-coordinates, which will be specified in Section 5, defined near P_1P_3 such that P_1P_3 becomes a segment on $\{x=0\}$, of the form

$$(2c_2x-\psi)\psi_{xx}+c_2\psi_x-(\psi_x)^2+\psi_{yy}-\frac{1}{\gamma c_2^2}\psi_y^2=0,\quad \text{in } x>0 \text{ and near } x=0, \tag{1.10}$$

plus "small" terms, since ρ and ψ have the same regularity in Ω. For the solution ψ, (1.10) is elliptic in $\{x>0\}$; also $\psi>0$ in $\{x>0\}$ and $\psi=0$ on $\{x=0\}$. The proof of the regularity is exactly the same as that in [4], so we just list the results about the optimal regularity in Section 5.

As we know, much effort has been devoted to the study of the phenomena of shock reflection. Čanić, Keyfitz and Kim [2] got the existence of regular transonic shock reflection for the UTSD. And Čanić, Keyfitz and Kim [3] established the existence results of Mach stem for the nonlinear wave system. Zheng [15] studied the existence of the global solutions of two dimensional regular shock reflection for the pressure system.

The organization of this paper is as the following. In Section 2, we derive the second-order operator and the boundary conditions for the nonlinear wave system (1.1) in self-similar coordinates and in the polar coordinates, as in [3] and in [9]; and give the mathematical statement of our results, Theorem 2.1. In Section 3, by using a regularized differential operator, with $\epsilon\triangle\rho$ added, we prove the existence of the solutions for the uniformly elliptic free boundary problem in polar coordinates, as well as in self-similar coordinates. In Section 4, we proceed to the limit as $\epsilon \to 0$ to get the global existence of the solutions to the original problem. In Section 5, we establish the optimal regularity $C^{0,1}$ of the solutions ρ across the degenerate sonic boundary.

2 The von Neumann Criterion and Local Theory for Shock Reflection

In this section, we first discuss the normal reflection solution, then follow the von Neumann criterion to derive the necessary condition for the existence of the regular reflection and show that the shock reflection is regular locally when the wedge angle is large, that is, when θ_w is close to $\frac{\pi}{2}$ or, equivalently, the angle between the incident shock and the wedge

$$\sigma = \frac{\pi}{2} - \theta_w \tag{2.1}$$

tends to zero.

To find the reflected shock and the state between the wedge and it, denoted by state (2), we need the Rankine-Hugoniot relation. Rewrite system (1.5) in the conservation form

$$\partial_\xi \begin{pmatrix} m-\xi\rho \\ p-\xi m \\ -\xi n \end{pmatrix} + \partial_\eta \begin{pmatrix} n-\eta\rho \\ -\eta m \\ p-\eta n \end{pmatrix} = -2 \begin{pmatrix} \rho \\ m \\ n \end{pmatrix}.$$

Let $\eta = \eta(\xi)$ with slope $\sigma' = \eta'(\xi)$ being a shock. Then

$$\begin{aligned} &(\eta - \sigma'\xi)[m] + \sigma'[p] = 0, \\ &(\eta - \sigma'\xi)[n] - [p] = 0, \\ &(\eta - \sigma'\xi)[\rho] + \sigma'[m] - [n] = 0, \end{aligned} \tag{2.2}$$

where $[f] = f - f_1$ denotes the jump of f across the shock wave. For $[\rho] \neq 0$, we can solve them to obtain

$$\begin{aligned} &\frac{\mathrm{d}\eta}{\mathrm{d}\xi} = \sigma' = \frac{\xi\eta \pm \bar{c}\sqrt{\xi^2+\eta^2-\bar{c}^2}}{\xi^2-\bar{c}^2}, \\ &[p] = \xi[m] + \eta[n], \\ &[p][\rho] = [m]^2 + [n]^2, \end{aligned} \tag{2.3}$$

where

$$\bar{c}^2(\rho, \rho_1) = \frac{p(\rho) - p(\rho_1)}{\rho - \rho_1}.$$

A useful and equivalent form for the Rankine-Hugoniot relation is

$$\begin{aligned} \frac{\mathrm{d}\eta}{\mathrm{d}\xi} &= \sigma' = \frac{\xi\eta \pm \bar{c}\sqrt{\xi^2+\eta^2-\bar{c}^2}}{\xi^2-\bar{c}^2}, \\ [m] &= \frac{\bar{c}^2\xi \pm \bar{c}\eta\sqrt{\xi^2+\eta^2-\bar{c}^2}}{\bar{c}^2(\xi^2+\eta^2)}[p], \\ [n] &= \frac{\bar{c}^2\eta \mp \bar{c}\xi\sqrt{\xi^2+\eta^2-\bar{c}^2}}{\bar{c}^2(\xi^2+\eta^2)}[p]. \end{aligned} \tag{2.4}$$

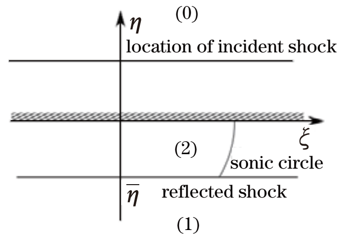


Figure 3 Normal reflection

Use the plus branch for the reflected shock, which gives the shock evolution equation

$$\frac{\mathrm{d}\eta}{\mathrm{d}\xi} = f(\xi,\eta,\rho) = \frac{\xi\eta + \overline{c}\sqrt{\xi^2+\eta^2-\overline{c}^2}}{\xi^2-\overline{c}^2} = \frac{\eta^2-\overline{c}^2}{\xi\eta - \overline{c}\sqrt{\xi^2+\eta^2-\overline{c}^2}}. \tag{2.5}$$

The second expression is equivalent to the first one, and both are well defined if $\overline{c}^2(\rho) \le \xi^2+\eta^2$. Denote $P_2 = (0,\eta(0))$, the point at the foot of the shock, and observe that we need $\eta'(0) = \sqrt{\frac{\eta^2-\overline{c}^2}{\overline{c}^2}}$ to be zero by symmetry. Thus

$$\eta(0) = -\overline{c}(\rho,\rho_1) = -\sqrt{\frac{p(\rho)-p(\rho_1)}{\rho-\rho_1}}. \tag{2.6}$$

This can be interpreted as a condition which determines $\rho(P_2)$ in the subsonic region at the foot of the shock.

2.1 Normal shock reflection

In this case, the wedge angle is $\frac{\pi}{2}$, i.e., $\sigma = 0$, and the incident shock normally reflects (see Figure 3). The reflected shock is also a plane at $\eta = \overline{\eta} < 0$, which will be defined below. Then $\overline{m}_2 = \overline{n}_2 = 0$, and it follows from the Rankine-Hugoniot relation (2.2) that

$$\overline{\eta} = -\sqrt{\frac{p(\overline{\rho}_2)-p(\rho_1)}{\overline{\rho}_2-\rho_1}}. \tag{2.7}$$

At the reflected shock $\eta = \overline{\eta} < 0$, the Rankine-Hugoniot relation (2.2) implies

$$-n_1 = \overline{\eta}(\overline{\rho}_2 - \rho_1). \tag{2.8}$$

Thus

$$(p(\rho_1)-p(\rho_0))(\rho_1-\rho_0) = (p(\overline{\rho}_2)-p(\rho_1))(\overline{\rho}_2-\rho_1). \tag{2.9}$$

It can be shown that there is a unique solution $\overline{\rho}_2$ to (2.9) such that $\overline{\rho}_2 > \rho_1$. Indeed, for fixed ρ_1 and ρ_0, and denoting by $F(\overline{\rho}_2)$ the right-hand side of (2.9), we have

$$F(\rho_1) = 0, \quad F(\infty) = \infty,$$
$$F'(s) = \Big(p'(s) + \int_0^1 p(\rho_1+\theta(s-\rho_1))\mathrm{d}\theta\Big)(s-\rho_1) > 0 \quad \text{for } s > \rho_1.$$

Thus there exists a unique $\overline{\rho}_2 \in (\rho_1,\infty)$ satisfying $F(\overline{\rho}_2) = n_1^2$, i.e., (2.9) holds. Then the position of the reflected shock $\eta = \overline{\eta} < 0$ is uniquely determined by (2.7).

Moreover, for the sonic speed $c(\overline{\rho}_2) = \sqrt{p'(\overline{\rho}_2)}$ of state (2), we have

$$|\overline{\eta}| < c(\overline{\rho}_2). \tag{2.10}$$

2.2 The von Neumann criterion and local theory for regular reflection

In this subsection, we first follow the von Neumann criterion to derive the necessary condition for the existence of regular reflection and show that, when the wedge angle is large, there exists a unique state (2) with a two-shock structure at the reflected point, which is close to the solution $(\overline{\rho}_2, \overline{m}_2, \overline{n}_2) = (\overline{\rho}_2, 0, 0)$ of the normal reflection.

For a possible two-shock configuration satisfying the corresponding boundary condition on the wedge $\eta = \xi \cot\theta_w$, we set the reflected point $P_0 = (\eta_{10}\tan\theta_w, \eta_{10})$ and assume that the line that coincides with the reflected shock in state (2) will intersect with the axis $\eta = 0$ at the point $(\widetilde{\eta}, 0)$ with the angle θ_s between the line and $\xi = 0$. It is easy to check that

$$\widetilde{\eta} = \eta_{10}\tan\theta_w - \eta_{10}\tan\theta_s. \tag{2.11}$$

In addition, the momenta (m_2, n_2) should be parallel to the wall, i.e.,

$$m_2 = n_2 \tan\theta_w. \tag{2.12}$$

This requirement and the Rankine-Hugoniot relation determine the state (2).

Proposition 2.1 (Regular Reflection of the Algebraic Portion) *There exists a $\theta_c \in (0, \frac{\pi}{2})$, depending only on ρ_0 and ρ_1, such that if $\theta_w > \theta_c$, there exists a constant state (ρ_2, m_2, n_2) with $\rho_2 > \rho_1$, satisfying* (2.12) *and the Rankine-Hugoniot condition.*

Proof It follows from the second equation of (2.3) that,

$$p_2 - p_1 = \eta_{10}(1 + \tan^2\theta_w)n_2 - \eta_{10}n_1. \tag{2.13}$$

Denoting $\widetilde{n}_2 := (1 + \tan^2\theta_w)n_2$, we have

$$\widetilde{n}_2 - n_1 = (p(\rho_2) - p(\rho_1))\sqrt{\frac{\rho_1 - \rho_0}{p(\rho_1) - p(\rho_0)}}. \tag{2.14}$$

Manipulating the third equation of (2.3), we obtain

$$(p(\rho_2) - p(\rho_1))(\rho_2 - \rho_1) - \sin^2\theta_w n_1^2 = \cos^2\theta_w(\widetilde{n}_2 - n_1)^2. \tag{2.15}$$

It follows from (2.14) and (2.15) that

$$\begin{aligned}(p(\rho_2) - p(\rho_1))^2\frac{\rho_1 - \rho_0}{p(\rho_1) - p(\rho_0)} &= (1 + \tan^2\theta_w)(p(\rho_2) - p(\rho_1))(\rho_2 - \rho_1) \\ &\quad - \tan^2\theta_w(p(\rho_1) - p(\rho_0))(\rho_1 - \rho_0).\end{aligned} \tag{2.16}$$

Consider

$$\begin{aligned}f(\rho) = {} & (1 + \tan^2\theta_w)(p(\rho) - p(\rho_1))(p(\rho_1) - p(\rho_0))(\rho - \rho_1) \\ & - (p(\rho) - p(\rho_1))^2(p(\rho_1) - p(\rho_0)) - \tan^2\theta_w(p(\rho_1) - p(\rho_0))(\rho_1 - \rho_0).\end{aligned} \tag{2.17}$$

We need to show that there exists a $\rho_2 > \rho_1$, such that $f(\rho_2) = 0$. In fact,

$$\begin{aligned}f'(\rho) = {} & (1 + \tan^2\theta_w)(p(\rho_1) - p(\rho_0))(\rho - \rho_1)\Big[p'(\rho) + \frac{p(\rho) - p(\rho_1)}{\rho - \rho_1}\Big] \\ & - 2(p(\rho) - p(\rho_1))(\rho_1 - \rho_0)p'(\rho).\end{aligned}$$

By the convexity of $p(\rho)$, it is easy to show that $f'(\rho) > 0$ for σ sufficiently small. Moreover, $f(\rho_1) = -\tan^2\theta_w(p(\rho_1) - p(\rho_0))(\rho_1 - \rho_0) < 0$, $f(\rho) \to \infty$ if $\rho \to \infty$, and by the continuity of $f(\rho)$, there exists a $\rho_2 > \rho_1$, such that $f(\rho_2) = 0$. Define $\theta_c = \inf\{\theta_w \mid f'(\rho) > 0 \text{ and } \frac{\pi}{2} - \theta_w > 0\}$. We obtain (ρ_2, m_2, n_2) satisfying Rankine-Hugoniot relation for $\theta_w > \theta_c$, where (m_2, n_2) could be obtained from (2.4).

This finishes the proof of the proposition.

Moreover, for $\sigma = \frac{\pi}{2} - \theta_w \in (0, \sigma_1)$, where σ_1 is sufficiently small, depending only on ρ_0, ρ_1 and γ, we have

$$|\rho_2 - \overline{\rho}_2| + \left|\frac{\pi}{2} - \theta_s\right| + |\widetilde{\eta} - \overline{\eta}| + |c_2 - c(\overline{\rho}_2)| \le C_1\sigma, \tag{2.18}$$

where $c_2 = \rho_2^{\frac{\gamma}{2}}$ is the sonic speed of state (2). It follows from (2.10) and (2.18) that, if $\sigma_1 > 0$ is small, then

$$|\widetilde{\eta}| < c_2. \tag{2.19}$$

Thus we have established the local existence of the two-shock configuration near the reflected point, so that behind the straight reflected shock emanating from the reflection point, state (2) is pseudo-supersonic up to the sonic circle of state (2). Furthermore, this local structure is stable in the limit $\theta_w \to \frac{\pi}{2}$, i.e., $\sigma \to 0$.

2.3 The oblique derivative boundary conditions

Following [3] and [9], since vorticity is confined to the lines of discontinuity of the Riemann data, and these lines lie above the shock, that means $m_\eta - n_\xi = 0$. Using this equation and (1.5),

$$\begin{aligned} n_\xi = m_\eta &= \frac{1}{\xi^2+\eta^2}\big(\eta(c^2-\xi^2)\rho_\xi + \xi(c^2-\eta^2)\rho_\eta\big),\\ m_\xi &= \frac{1}{\xi^2+\eta^2}\big(\xi(c^2+\eta^2)\rho_\xi - \eta(c^2-\eta^2)\rho_\eta\big),\\ n_\eta &= \frac{1}{\xi^2+\eta^2}\big(\xi(-c^2+\xi^2)\rho_\xi + \eta(c^2+\xi^2)\rho_\eta\big). \end{aligned} \tag{2.20}$$

Differentiating the third equation of (2.3) along $\Gamma_{\text{shock}} = \{\xi, \eta(\xi)\}$, we get

$$(c^2(\rho)[\rho] + [p])(\rho_\xi + \eta'\rho_\eta) = 2[n](-\eta'(m_\xi + (1-(\eta')^2)m_\eta) + \eta' n_\eta), \tag{2.21}$$

where $[m] = -\eta'[n]$ are used. Replacing derivatives $\mathrm{D}m$ and $\mathrm{D}n$ by $\mathrm{D}\rho$, and using (2.21) and $[n] = \frac{[p]}{-\eta'\xi+\eta}$, we get

$$\beta^{(1)}\cdot\nabla\rho = \beta_1^{(1)}\rho_\xi + \beta_2^{(1)}\rho_\eta = 0, \tag{2.22}$$

where $\beta^{(1)}$ is given by

$$\begin{aligned} \beta_1^{(1)}(\rho) &= (\xi^2+\eta^2)(-\eta'\xi+\eta)(c^2(\rho) + \overline{c}^2(\rho))\\ &\quad - 2\overline{c}^2(\rho)\{-\eta'\xi(c^2+\eta^2) + (1-(\eta')^2)\eta(c^2-\xi^2) + \eta'\xi(-c^2+\xi^2)\} \end{aligned} \tag{2.23}$$

and

$$\beta_2^{(1)}(\rho) = \eta'(\xi^2+\eta^2)(-\eta'\xi+\eta)(c^2(\rho)+\overline{c}^2(\rho)) \\ - 2\overline{c}^2(\rho)\{\eta'\eta(c^2-\eta^2)+(1-(\eta')^2)\xi(c^2-\eta^2)+\eta'\eta(c^2+\xi^2)\}. \tag{2.24}$$

Thus the obliqueness becomes

$$\beta^{(1)}\cdot\nu = 2\overline{c}^2(\rho)(\eta\eta'+\xi)\{(c^2-\xi^2)^2(\eta')^2+2\xi\eta\eta'+c^2-\eta^2\}, \tag{2.25}$$

where $\nu=(\eta',-1)$ is the outward normal to Ω at Γ_{shock}. It is easy to check that

$$(c^2-\xi^2)(\eta')^2+2\xi\eta\eta'+c^2-\eta^2\neq 0,$$

if $\xi^2+\eta^2<c^2$. In fact, let $g(y)=(c^2-\xi^2)^2y^2+2\xi\eta y+c^2-\eta^2$, which is a quadratic polynomial with coefficients depending smoothly on (ξ,η) and ρ. Notice $\Delta=-4c^2(\rho)(c^2(\rho)-\xi^2-\eta^2)$ and $c^2(\rho)-\xi^2>0$, so $g(y)>0$.

Thus the obliqueness depends only on whether $\eta\eta'+\xi$ equals zero.

Hereafter, we let $\eta=l(\xi)$ denote the location of the reflected shock of state (2), the straight part, that is,

$$l(\xi)=\xi\cot\theta_s+\widetilde{\eta}\quad\text{with } \widetilde{\eta}=\sqrt{\frac{p(\rho_1)-p(\rho_0)}{\rho_1-\rho_0}}\Big(1-\frac{\tan\theta_w}{\tan\theta_s}\Big)<0, \tag{2.26}$$

where θ_s is the angle between $l(\xi)$ and the axis $\xi=0$.

Another condition on the free boundary $\eta(\xi)$ comes from the fact that the curved part and the straight part of the reflected shock should match at least up to the first order. Denote by $P_1=(\xi_1,\eta_1)$ with $\xi_1>0$ and $\eta_1<0$, the intersection point of the line $\eta=l(\xi)$ and the sonic circle $\xi^2+\eta^2=c_2^2$, i.e., (ξ_1,η_1) is the unique point for a small $\sigma>0$ satisfying $l(\xi_1)^2+\xi_1^2=c_2^2$, $\eta_1=l(\xi_1)$, $\xi_1>0$. The existence and uniqueness of such a point (ξ_1,η_1) follow from $-c_2<\widetilde{\eta}<0$. Then at P_1, $\eta(\xi)$ satisfies

$$\eta(\xi_1)=l(\xi_1),\quad \eta'(\xi_1)=l'(\xi_1)=\frac{\xi_1\eta_1+\overline{c}(\rho_2)\sqrt{\xi_1^2+\eta_1^2-\overline{c}^2(\rho_2)}}{\xi_1^2-\overline{c}^2(\rho_2)}. \tag{2.27}$$

2.4 The free boundary problem in polar coordinates

We discuss the problem in polar coordinates first for the technical reason. Let $(\xi,\eta)=(r\cos\theta,r\sin\theta)$, and rewrite (1.6) as

$$((c^2-r^2)\rho_r)_r+\frac{c^2}{r}\rho_r+\Big(\frac{c^2}{r^2}\rho_\theta\Big)_\theta=0. \tag{2.28}$$

As in self-similar coordinates, using the Rankine-Hugoniot relation in polar coordinates correspondingly, we have

$$\beta_i^{(2)}D_i\rho=\beta_1^{(2)}\rho_r+\beta_2^{(2)}\rho_\theta=0, \tag{2.29}$$

along $\{(r(\theta),\theta)\}$ in (r,θ)-coordinates, and

$$\beta_1^{(2)}=r'(c^2(r^2-\overline{c}^2)-3\overline{c}^2(c^2-r^2)),\quad \beta_2^{(2)}=3c^2(r^2-\overline{c}^2)-\overline{c}^2(c^2-r^2).$$

Thus the obliqueness becomes

$$\beta^{(2)} \cdot (1, -r') = -2r'(c^2 - \overline{c}^2)r^2 \equiv \mu,$$

where $(1, -r'(\theta))$ is the outward normal to Ω at Γ_{shock}. Note that μ becomes zero when $r'(\theta) = 0$, that is, $r = \overline{c}(\rho)$. When the obliqueness fails, we have $\beta_1^{(2)} = 0$ and $\beta_2^{(2)} = -\overline{c}^2(c^2 - r^2) < 0$ in a subsonic region.

Next define Q to be the governing second-order quasi-linear operator in the subsonic domain

$$Q\rho = ((c^2 - r^2)\rho_r)_r + \frac{c^2}{r}\rho_r + \Big(\frac{c^2}{r^2}\rho_\theta\Big)_\theta = 0, \tag{2.30}$$

and M to be the derivative boundary operator

$$M\rho = \beta_1^{(2)}\rho_r + \beta_2^{(2)}\rho_\theta = 0, \quad \text{on } \Gamma_{\text{shock}} = \{(r(\theta), \theta)\}. \tag{2.31}$$

Here $\beta^{(2)} = (\beta_1^{(2)}, \beta_2^{(2)})$ is a vector field. The second condition on Γ_{shock} is the shock evolution equation

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = r\frac{\sqrt{r^2 - \overline{c}^2(\rho)}}{\overline{c}(\rho)} := g(r, \theta, \rho(r, \theta)) \quad \text{with } r(\theta_1) = r_1, \tag{2.32}$$

where (r_1, θ_1) is the polar coordinates of (ξ_1, η_1).

The boundary conditions on the other parts of $\partial\Omega$ are

$$\rho = \rho_2, \quad \text{on } \Gamma_{\text{sonic}} = \partial\Omega \cap \partial B_{c_2}(0), \tag{2.33}$$

$$\rho_\nu = 0, \quad \text{on } \Gamma_{\text{wedge}} = \partial\Omega \cap \{\theta = \theta_w\}, \tag{2.34}$$

$$\rho_\nu = 0, \quad \text{on } \Sigma_0 = \partial\Omega \cap \Big\{\theta = -\frac{\pi}{2}\Big\}. \tag{2.35}$$

At the Dirichlet boundary Γ_{sonic}, the ellipticity of the operator Q degenerates. At the point P_2, $r'(-\frac{\pi}{2}) = 0$, M fails to be oblique. We may alternatively express this as a one-point Dirichlet condition by solving $r(-\frac{\pi}{2}) = \overline{c}(\rho(-\frac{\pi}{2}, r(-\frac{\pi}{2})), \rho_1)$. In order to deal with this equation, we introduce the notation

$$a = \overline{c}_b^{-1}(r), \quad \text{when } \overline{c}(a, b) = r \tag{2.36}$$

for a fixed b. Thus,

$$\overline{\rho} = \rho(P_2) = \overline{c}_{\rho_1}^{-1}\Big(r\Big(-\frac{\pi}{2}\Big)\Big). \tag{2.37}$$

In this paper, we will establish the following theorem.

Theorem 2.1 *There exists a $\theta_0 \in [\theta_c, \frac{\pi}{2})$ such that if $\theta_w \in [\theta_0, \frac{\pi}{2})$, there exists a solution $\rho \in C^{2+\alpha}(\Omega) \cap C^{0,1}(\overline{\Omega})$ for the initial data* (1.8)*, to the free boundary value problem* (2.30)–(2.35) *and* (2.37)*. Lipschitz continuity is the optimal regularity for ρ across Γ_{sonic}. Moreover, ρ tends to $\overline{\rho}_2$ as $\theta_w \to \frac{\pi}{2}$.*

The existence part of Theorem 2.1 is proved in two stages. First, we solve the regularized free boundary value problem for $Q^\epsilon = Q + \epsilon\triangle$ ($\triangle$ is the Laplace operator) in Section 3. Second, we consider the limit $\epsilon \to 0$ and show that this limit yields a solution to (2.30)–(2.35) and (2.37) in Section 4.

3 The Regularized Problem

For a fixed $\epsilon \in (0,1)$, we solve the free boundary value problem defined in Subsection 2.4. But with Q replaced by the regularized operator Q^ϵ, the equation for ρ in the subsonic region is now

$$Q^\epsilon \rho = ((c^2 - r^2 + \epsilon)\rho_r)_r + \frac{c^2+\epsilon}{r}\rho_r + \Big(\frac{c^2+\epsilon}{r^2}\rho_\theta\Big)_\theta = 0. \tag{3.1}$$

The shock evolution equation remains the same

$$r' = g(r,\theta,\rho), \quad r(\theta_1) = r_1, \tag{3.2}$$

and the boundary conditions are as before

$$M\rho = \beta^{(2)} \cdot \nabla\rho, \quad \text{on } \Gamma_{\text{shock}} = \Big\{(r,\theta) : -\frac{\pi}{2} < \theta < \theta_1\Big\}, \tag{3.3}$$

$$\rho = \rho_2, \quad \text{on } \Gamma_{\text{sonic}}, \quad \rho_\nu = 0, \quad \text{on } \Gamma_{\text{wedge}} \cup \Sigma_0, \tag{3.4}$$

where ν is the outward normal to Ω at $\Gamma_{\text{wedge}} \cup \Sigma_0$, and

$$\rho(P_2) = \overline{\rho} = \overline{c}_{\rho_1}^{-1}\Big(r\Big(-\frac{\pi}{2}\Big)\Big). \tag{3.5}$$

We will focus on the proof of the existence theorem in this section as follows.

Theorem 3.1 *There exists a $\theta_0 \in [\theta_c, \frac{\pi}{2})$ such that if $\theta_w \in [\theta_0, \frac{\pi}{2})$, then for each $\epsilon \in (0,\epsilon_0)$ with some $\epsilon_0 > 0$, there exists a solution $(\rho^\epsilon, r^\epsilon) \in C_{2+\alpha}^{-\gamma_1}(\Omega^\epsilon) \times C^{1+1}([-\frac{\pi}{2},\theta_1])$ to the regularized free boundary problem* (3.1)–(3.5) *such that*

$$\rho_1 < \overline{\rho}^\epsilon \le \rho^\epsilon < \rho_2 \quad \text{and} \quad c^2(\rho^\epsilon) > r^2, \quad \text{in } \overline{\Omega}^\epsilon \backslash \Gamma_{\text{shock}} \tag{3.6}$$

for some $\alpha, \gamma \in (0,1)$ depending on ϵ, ρ_0, ρ_1 and θ_w. The function $r^\epsilon(\theta)$, defining the position of the free boundary $\Gamma^\epsilon_{\text{shock}}$, is in $\mathcal{K}^\epsilon$, which will be defined later. Here Ω^ϵ is bounded by $\Gamma^\epsilon_{\text{shock}}$, Σ_0, Γ_{wedge} and Γ_{sonic}.

We prove Theorem 3.1 in the following steps (which take up four subsections of this section).

Step 1 Since the governing equation (3.1) is nonlinear, and the ellipticity is not known a priori, we introduce a cut-off function into the equation $Q^\epsilon\rho = 0$, which is a smooth increasing function $f \in C^\infty$, such that

$$f(s) = \begin{cases} s, & \text{if } s \ge 0, \\ -\dfrac{1}{2}\epsilon, & \text{if } s < -\epsilon \end{cases} \tag{3.7}$$

and $|f'(s)| \le 1$. Consider the following modified equation:

$$\begin{aligned} Q^{\epsilon,+}\rho &= [(f(c^2 - r^2) + \epsilon)\rho_r]_r + \Big[\frac{1}{r}(f(c^2-r^2)+\epsilon) + r\Big]\rho_r + \Big(\frac{c^2+\epsilon}{r^2}\rho_\theta\Big)_\theta \\ &= D_i(a^\epsilon_{ii}(r,\theta,\rho)D_i\rho) + b^\epsilon(r,\rho)D_r\rho = 0, \quad \text{in } \Omega. \end{aligned} \tag{3.8}$$

Step 2 We show the existence of the solutions to the linear problem with fixed boundary Γ_{shock} defined by $r(\theta) \in \mathcal{K}^{\epsilon,\delta}$ and establish the Schauder estimates at Γ_{shock}, particularly near the point where obliqueness loses, and the Schauder estimates are near the corners and are locally in the rest of the domain. For these elliptic estimates, we introduce some notations first.

Let $V = \{P_1, P_2, O, P_3\}$ denote the corners of Ω, $V' = V\backslash\{P_2\}$. Set $\Omega' = \overline{\Omega}\backslash(V \cup \Gamma_{\mathrm{shock}})$. For $\Xi \in V$, define the corner region

$$\Omega_\Xi(\delta) = \{x \in \Omega : \mathrm{dist}(x, \Xi) \le \sigma\}$$

and

$$\Gamma'(\sigma) = \{\Xi \in \Gamma_{\mathrm{shock}} \mid \mathrm{dist}(\Xi, P_1) > \sigma\},$$
$$\Gamma(\delta) = \Big\{x \in \Omega \cap \bigcup_{\Xi \in \Gamma'(\sigma)} B_\sigma(\Xi)\Big\},$$

where $B_\delta(\Xi)$ is a ball of radius δ centered at Ξ. So we define a region that is close to Γ_{shock} but does not contain the corner P_1. We then define the weighted space

$$C_a^b \equiv \Big\{u : \|u\|_a^b \equiv \sup_{\delta>0} \delta^{a+b} |u|_{a, \overline{\Omega}\backslash(\Gamma(\delta) \cup \Omega_{V'}(\delta))} < \infty\Big\}. \tag{3.9}$$

In this paper, we cannot use the results in [11]–[14] directly to show the existence of the solutions to the fixed boundary value problem. Instead, by using the Hölder gradient bounds to the linear problem, we establish the existence result to the nonlinear fixed boundary problem via the Perron method developed in [10].

Step 3 We apply the Schauder fixed point theorem to prove the existence of the solutions to the nonlinear fixed boundary problem and then to the free boundary problem. Here we will remove the cut-off function and prove that the shock evolution equation can always be well-defined. In order to use the Schauder fixed point theorem, we now define $\mathcal{K} = \mathcal{K}^{\epsilon,\delta}$, a closed, convex subset of a Hölder space $C^{1+\alpha_1}([-\frac{\pi}{2}, \theta_1]) \cap C^{2+\alpha_1}([-\frac{\pi}{2}, \tau_1))$ (τ_1 may depend on $\widetilde{r}$, which will be specified later), where α_1 depends on ϵ and will be specified later, and the mapping on it is $\widetilde{r}(\theta) = Jr$, where Jr will be defined in Subsection 3.4. The functions in $\mathcal{K}$ satisfy the following properties:

(K_1) $r(\theta_1) = r_1$ and $r'(\theta_1) = r_1 \frac{\sqrt{r_2^2 - \overline{c}^2(\rho_2)}}{\overline{c}(\rho_2)}$;
(K_2) $r'(-\frac{\pi}{2}) = 0$ and $r''(-\frac{\pi}{2}) = 0$;
(K_3) $c(\rho_1) + \delta \le r(-\frac{\pi}{2})$;
(K_4) $0 \le r'(\theta) \le \frac{r_1^2}{\overline{c}(\rho_1)}$ for $-\frac{\pi}{2} \le \theta \le \theta_1$.

Note that (K_3) guarantees that $r(\theta)$ does not touch the sonic circle $r = \overline{\rho_1}$.

3.1 The regularized linear fixed boundary problem

Replace ρ in the coefficients a_{ii}, b of (3.1) and $\beta_i^{(2)}$ of (3.3) by a function w in a set $\mathcal{W}$ defined in a bounded domain Ω^ϵ, depending on given values ρ_2 and ρ_1 as follows.

Definition 3.1 *The elements in $\mathcal{W} \in C_2^{-\gamma}$ satisfy*
(W1) $\rho_1 < \overline{\rho}^\epsilon \le w \le \rho_2$, $w(P_2) = \overline{\rho}^\epsilon$, $w = \rho_2$ *on* Γ_{sonic}, $w_\nu = 0$ *on* $\Sigma_0 \cup \Gamma_{\mathrm{wedge}}$;
(W2) $\|w\|_2^{-\gamma_1} \le K$;
(W3) $|w|_{\alpha_0, \Omega'_{\mathrm{loc}}} \le K_0$ *and* $|w|_{1+\mu, \Gamma(d)} \le K_0$.

The weighted Sobolev space is defined by (3.9). The values of $\gamma_1, \alpha_0 \in (0, 1)$, and K, K_0 will be specified latter. Obviously, $\mathcal{W}$ is closed, bounded and convex.

The quasilinear equation (3.8) and boundary condition (3.3) are now replaced by the linear problem (repeated indices are summed up)

$$
\begin{aligned}
&L^{\epsilon,+}u = D_i(a^{\epsilon}_{ii}(\Xi,w)D_i u) + b^{\epsilon}(\Xi,w)D_1 u = 0, \quad \text{in } \Omega,\\
&\quad Mu = \beta^{(2)}_1(\Xi,w)D_r u + \beta^{(2)}_2(\Xi,w)D_\theta u = 0, \quad \text{on } \Gamma^{\epsilon}_{\text{shock}} = \Big\{(r(\theta),\theta)\,\Big|\, -\frac{\pi}{2} \le \theta \le \theta_1\Big\},
\end{aligned} \tag{3.10}
$$

with the remaining boundary conditions

$$
u = \rho_2 \quad \text{on } \Gamma_{\text{shock}}, \quad u_\theta = 0 \quad \text{on } \Sigma_0, \quad u_\nu = 0 \quad \text{on } \Gamma_{\text{wedge}}, \quad u(P_2) = \overline{\rho}^{\epsilon}, \tag{3.11}
$$

where $r(\theta) \in \mathcal{K}^{\epsilon,\delta} \subset C^{1+\alpha_1}([\theta_w,\theta_1]) \cap C^2((-\frac{\pi}{2},\theta_1))$ are given and $w \in \mathcal{W}$. Because of the cut-off function f, $L^{\epsilon,+}$ is uniformly elliptic in Ω^ϵ. In this subsection, we demonstrate that the solutions u to the linear problems (3.10) and (3.11) satisfy Hölder and Schauder estimates in Ω', especially a uniform $C^{1+\mu}(\Gamma(d_0))$ estimate near Γ_{shock} for any $\mu < \min\{\gamma_1,\alpha_1\}$. This bound gives the good enough compactness to establish the existence of a solution to the nonlinear problem by applying the Schauder fixed point theorem.

First, we state the Schauder estimates including the Dirichlet and fixed Neumann boundaries, Γ_{sonic} and $\Sigma_0 \cup \Gamma_{\text{wedge}}$, and the Hölder estimates at the corners V'.

Lemma 3.1 *Assume that Γ_{shock} is given by $\{(r(\theta),\theta)\}$ with $r(\theta) \in \mathcal{K}^{\epsilon,\delta}$ for some α_1 and that $w \in \mathcal{W}$ for given K, K_0, α_0 and γ. Then there exist $\gamma_V, \alpha_\Omega \in (0,1)$ such that the solution $u \in C^{2+\alpha_\Omega}_{\text{loc}}(\Omega') \cap C^{\gamma_V}(\Omega_{V'}(d_0))$ to the linear problems* (3.10) *and* (3.11) *satisfies*

$$
|u|_{\gamma,\Omega_{V'}(d_0)} \le C_1 |u|_0 \tag{3.12}
$$

for any $\gamma \le \gamma_V$ and

$$
|u|_{2+\alpha,\Omega'_{\text{loc}}} \le C_2 |u|_0 \tag{3.13}
$$

for any $\alpha \le \alpha_\Omega$. The exponent γ_V depends on the Riemann data ρ_0, ρ_1, θ_w, and both α_Ω and γ_V depend on ϵ but are independent of α_1 and γ_1. The constant C_2 is independent of K but depends on K_0.

Proof We refer to [14, Theorem 1] for the corner estimates at P_1 and P_3. Near the origin, since the governing equation can be written in self-similar coordinates in the form of (1.6), we refer to [13] to get the corner estimate at O. Here γ_V is a fixed value that depends on the Riemann data ρ_0, ρ_1 and θ_w, as well as the ellipticity ratio ϵ, but not on γ_1, α_1, K or K_0. Next we can use standard interior and boundary Schauder estimates to get the local estimate (3.13). The constant C_2 depends on ϵ, the C^α-norm of the coefficients a_{ij} and the domain.

Because the interior Schauder estimates can be applied once more, a solution in $C^{2+\alpha}_{\text{loc}}(\Omega')$ is actually in $C^3_{\text{loc}}(\Omega')$.

We next state the Hölder gradient estimates at Γ_{shock}, especially at the point P_2 where the boundary operator M is not oblique.

Lemma 3.2 *Assume that Γ_{shock} is given by $\{(r(\theta),\theta)\}$ with $r(\theta) \in \mathcal{K}^{\epsilon,\delta}$ for some α_1 and that $w \in \mathcal{W}$ for given K, K_0, α_0 and γ_1. Then there exists a positive constant d_0 such that* *for any $d \le d_0$, the solution $u \in C^1_{\mathrm{loc}}(\Omega \cup \Gamma_{\mathrm{shock}}) \cup C^3_{\mathrm{loc}}(\Omega)$ to the linear problem* (3.10)–(3.11) *satisfies*

$$|u|_{1+\mu,\Gamma(d)\backslash B_d(P_1)} \le C(\epsilon,\delta,\alpha_1,\gamma_1,K,d_0)|u|_0 \tag{3.14}$$

for any $\mu < \min\{\gamma_1, \alpha_1\}$.

We omit the long proof here since it is the same as the one for Theorem 3.5 in [3].

The next lemma will be used in the proof of the local existence of the solutions to the fixed boundary value problem near P_2.

Lemma 3.3 *There exists a neighborhood of P_2 on Γ_{shock}, such that $\eta'(\xi) > 0$.*

Proof By the Implicit Theorem, the shock wave can be described by $\eta = \eta(\xi)$ locally. Thus, $rr' = \xi(\frac{\partial \xi}{\partial \theta} + r'(\theta)\frac{\partial \xi}{\partial r}) + \eta\eta'(\frac{\partial \xi}{\partial \theta} + r'(\theta)\frac{\partial \xi}{\partial r}) = r\cos\theta(-r\sin\theta + r'(\theta)\cos\theta) + 2r\sin\theta\eta'(-r\sin\theta + r'\cos\theta)$, so $\eta' = \frac{r\cos\theta + r'\sin\theta}{-r\sin\theta + r'\cos\theta}$. We claim that there exists a $d_0 > 0$, such that $\eta'(\xi) > 0$ for $(\xi, \eta(\xi)) \in B_{d_0}(P_2)$. In fact, let $f(\theta) = r\cos\theta + r'\sin\theta$, then $f(-\frac{\pi}{2}) = 0$. By using the fact $r''(-\frac{\pi}{2}) = 0$ in Property (K$_2$), we have $f'(-\frac{\pi}{2}) = r(-\frac{\pi}{2}) > 0$. So $f(\theta) > 0$, and thus $\eta'(\xi) > 0$ for $(r(\theta), \theta) \in B_{d_0}(P_2)$ with some $d_0 > 0$ and $\xi > 0$.

Now, we will focus on the proof of the existence of the solutions.

Before giving the existence of the solutions, we introduce two definitions with some modification compared to [10]. We call (3.10)–(3.11) is locally solvable, if for each $y \in \overline{\Omega}$, there is a neighborhood $O(y)$ and let $N = O(y) \cap \{\overline{\Omega}\backslash(\{P_2\} \cup \Gamma_{\mathrm{sonic}})\}$ such that for any $h \in C(\overline{N})$, there is a solution $v \in C^2(N) \cap C(\overline{N})$ to the problem

$$L^{\epsilon,+}v = 0 \quad \text{in } N \cap \Omega, \quad Mv = 0 \quad \text{on } N \cap \partial\Omega, \quad v = h \quad \text{on } \partial' N,$$

when $P_2 \notin N(y)$; or

$$L^{\epsilon,+}v = 0 \quad \text{in } N \cap \Omega, \quad Mv = 0 \quad \text{on } N \cap \partial\Omega, \quad v = h \quad \text{on } \partial' N, \quad v|_{P_2} = \overline{\rho},$$

when $P_2 \in N(y)$. Here $\partial' N = \partial N \cap \Omega$. We denote this function v by $(h)_y$ to emphasize its dependence on h and y.

A subsolution (supersolution) to (3.10)–(3.11) is a function $w \in C(\overline{\Omega})$, $v(r(\theta_w), \theta_w) = \overline{\rho}$ such that for any $y \in \overline{\Omega}$, if $h \ge w$ ($h \le w$) on $\partial' N$, then $(h)_y \ge w$ ($(h)_y \le w$) in N. The set of all subsolutions (supersolutions) is denoted by S^- (S^+).

We now establish the existence of the solutions to (3.10) and (3.11).

Lemma 3.4 *Assume that Γ_{shock} is given by $\{(r(\theta), \theta)\}$ with $r(\theta) \in \mathcal{K}^{\epsilon,\delta}$ for some α_1 and that $w \in \mathcal{W}$ for given K, K_0, α_0 and γ_1. Then there exist $\gamma_V, \alpha_\Omega \in (0,1)$ and $d_0 > 0$, where γ_V, α_Ω and d_0 are independent of γ_1 and α_1, such that the solution in $C^{1+\mu}(\Gamma(d_0)\backslash B_{d_0}(P_1)) \cap C^{2+\alpha}_{\mathrm{loc}}(\Omega') \cap C^\gamma(\Omega_{V'}(d_0))$ to the linear problems* (3.10) *and* (3.11) *exists for any $\alpha \le \alpha_\Omega$, $\mu < \min\{\gamma_1, \alpha_1\}$, $\gamma \le \gamma_V$ and $d \le d_0$ and satisfies* (3.12), (3.13) *and* (3.14).

Proof For fixed $\epsilon > 0$ and $\delta > 0$, without confusion, let $u^{\epsilon,\delta} = u$.

We use the Perron method to show the existence of a solution to (3.10) and (3.11).

Compared to [10], the local existence at P_2 is the only new case we need to show. In fact, let B_2 be a neighborhood of P_2 with smooth boundary. B_2 is sufficiently small such that $O \notin B_2$,

$\beta_1^{(2)} \le 0$ and $\beta_2^{(2)} < 0$. Thus, we can study the local existence in (ξ,η)-coordinates in B_2. Introduce the coordinate transformation near P_2

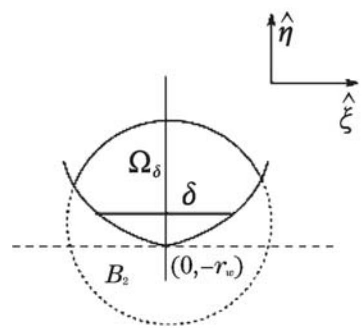
$$\begin{cases} \widehat{\xi} = \widehat{\xi}(\theta), \\ \widehat{\eta} = \widehat{\eta}(r,\theta) \end{cases} \tag{3.15}$$

such that $\widehat{\xi}(\overline{c}(\overline{\rho}), -\frac{\pi}{2}) = 0$, $\widehat{\eta}(\overline{c}(\overline{\rho}, -\frac{\pi}{2})) = -\overline{c}(\overline{\rho})$, $\frac{\partial \widehat{\xi}}{\partial r} = 0$, $\frac{\partial \widehat{\xi}}{\partial \theta} = -\frac{1}{\beta_2^{(2)}} > 0$, $\frac{\partial \widehat{\eta}}{\partial r} = -1$ and $\frac{\partial \widehat{\eta}}{\partial \theta} = -\frac{\beta_1^{(2)}}{\beta_2^{(2)}} \ge 0$. Moreover, $\widehat{\eta}(r,\theta) = \widehat{\eta}(\widehat{\xi}(r(\theta),\theta))$ along $\Gamma_{\mathrm{shock}} \cap B_2$. Thus,

$$\widehat{\eta}'(\widehat{\xi}) = \frac{\frac{\partial \widehat{\eta}}{\partial r} r'(\theta) + \frac{\partial \widehat{\eta}}{\partial \theta}}{\frac{\partial \widehat{\xi}}{\partial r} r'(\theta) + \frac{\partial \widehat{\xi}}{\partial \theta}} = -(\beta_1^{(2)} - \beta_2^{(2)} r'(\theta)) \ge 0.$$

So $\widehat{\eta}(\widehat{\xi})$ is an increasing function on $\Gamma_{\mathrm{shock}} \cap B_2$. From $\frac{\partial \widehat{\xi}}{\partial \theta} = -\frac{1}{\beta_2^{(2)}} > 0$ and $\frac{\partial \widehat{\xi}}{\partial r} = 0$, we know that $\widehat{\eta}(\widehat{\xi}) \ge -\overline{c}(\overline{\rho})$. Reflect the region B_2 across $\widehat{\xi} = 0$ to obtain a new region, still denoted by B_2. Furthermore, we replace Ω by Ω_σ which is σ-distance from the point P_2 upward (see Figure 4). On the bottom straight boundary of Ω_σ, impose

$$u = \overline{\rho}, \quad \text{on bottom of } \Omega_\sigma.$$



(a)

Figure 4 Domain with tip P_2 removed

Now, we study the following boundary value problem:

$$\begin{cases} \widehat{L}^{\epsilon,\delta} u = \widehat{a}_{ij} D_j u + \widehat{b}_i D_i u = 0, & \text{in } \Omega_\sigma, \\ \widehat{M} u = \partial_{\widehat{\xi}} u = 0, & \text{on } \partial\Omega_\sigma \cap \Gamma_{\mathrm{shock}}, \\ u = h, & \text{on } \partial B_2 \cap \Omega, \\ u = \overline{\rho}, & \text{on } \Sigma_\sigma, \end{cases} \tag{3.16}$$

where

$$\widetilde{a}_{11}^\epsilon = \frac{\widehat{a}_{11}^\epsilon}{\widehat{\beta}_2^2}, \quad \widetilde{a}_{12}^\epsilon = \widetilde{a}_{21}^\epsilon = -\frac{\widehat{\beta}_1}{\widehat{\beta}_2^2} \widehat{a}_{22}^\epsilon, \quad \widetilde{a}_{22}^\epsilon = \widehat{a}_{11}^\epsilon + \Big(\frac{\widehat{\beta}_1}{\widehat{\beta}_2}\Big)^2 \widehat{a}_{22},$$

$$\widetilde{b}_1^\epsilon = \frac{\partial \widehat{a}_{11}^\epsilon}{\partial \widehat{\eta}} - \frac{\widehat{a}_{22}^\epsilon}{\widehat{\beta}_2^2} \frac{\partial \widehat{\beta}_1}{\partial \widehat{\xi}} + \frac{\widehat{\beta}_1 \widehat{a}_{22}^\epsilon}{\widehat{\beta}_2^2} \frac{\partial \widehat{\beta}_2}{\partial \widehat{\eta}} + \frac{\widehat{\beta}_1 \widehat{a}_{22}^\epsilon}{\widehat{\beta}_2^3} \frac{\partial \widehat{\beta}_2}{\partial \widehat{\xi}} - \frac{\widehat{\beta}_1^2 \widehat{a}_{22}^\epsilon}{\widehat{\beta}_2^3} \frac{\partial \widehat{\beta}_2}{\partial \widehat{\eta}} + \Big(\frac{\widehat{\beta}_1}{\widehat{\beta}_2}\Big)^2 \frac{\partial \widehat{a}_{22}^\epsilon}{\partial \widehat{\eta}} - \widehat{b}^\epsilon,$$

$$\widetilde{b}_2^\epsilon = -\frac{\widehat{a}_{22}^\epsilon}{\widehat{\beta}_2^3} \frac{\partial \widehat{\beta}_2}{\partial \widehat{\xi}} + \frac{\widehat{a}_{22}^\epsilon \widehat{\beta}_1}{\widehat{\beta}_2^3} + \frac{1}{\widehat{\beta}^2} \frac{\partial \widehat{a}_{22}^\epsilon}{\partial \widehat{\xi}} - \frac{\widehat{\beta}_1}{\widehat{\beta}_2^2} \frac{\partial \widehat{a}_{22}^\epsilon}{\partial \widehat{\eta}}.$$

Here $\widehat{a}_{ii}^{\epsilon}$, $\widehat{b}^{\epsilon}$ and $\widehat{\beta}_i$ $(i=1,2)$ are the coefficients of (3.10) and (3.11) in $(\widehat{\xi},\widehat{\eta})$-coordinate, and h is a continuous function satisfying $\overline{\rho}<h\leq\rho_2$.

From now on, the proceeder is the same as [4]. Roughly speaking, we get the solutions u_σ to this problem, and then let $\sigma\to 0$ with the barrier function $v=\overline{\rho}+c(1-\mathrm{e}^{-l(\widehat{\eta}+r_w)})$ to deduce that the limiting function solves the problem locally. Please refer to [4] for more details.

3.2 The regularized nonlinear fixed boundary problem

This subsection is devoted to proving the existence of the solutions to the nonlinear problem (3.1) with a fixed boundary.

We have the following existence lemma for the fixed boundary.

Lemma 3.5 *For $\epsilon\in(0,\epsilon_0)$ and $\delta\in(0,\delta_0)$, given $r(\theta)\in\mathcal{K}^{\epsilon,\delta}\subset C^{1+\alpha_1}$, there exists a solution $\rho^{\epsilon,\delta}\in C_{2+\alpha}^{-\gamma_1}(\Omega^{\epsilon,\delta})$ to (3.1) and (3.3)–(3.5) such that*

$$\rho_1<\overline{\rho}^{\epsilon,\delta}\leq\rho^{\epsilon,\delta}\leq\rho_2 \tag{3.17}$$

for some $\alpha(\epsilon,\delta),\gamma(\epsilon,\delta)\in(0,1)$. Moreover, for some $d_0>0$, the solution $\rho^{\epsilon,\delta}$ satisfies

$$|\rho^{\epsilon,\delta}|_{\gamma,\Gamma(d_0)\cup B_{d_0}(P_1)}\leq K_0, \tag{3.18}$$

where γ and K_1 depend on δ, ϵ, γ_V and K, but both are independent of α_1.

The proof based on Schauder fixed point theorem is the same as in [4] or in [3], so we omit the details.

3.3 Three important properties for nonlinear problems

In this subsection, we will show three properties of the solutions to the nonlinear problems (3.1) and (3.3)–(3.5). First, we show $c^2(\rho^{\epsilon,\delta})-r^2\geq 0$ in $\overline{\Omega}^{\epsilon,\delta}$, which guarantees the ellipticity of the nonlinear equations. Thus the cut-off function can be removed.

Lemma 3.6 *There exist positive constants ϵ_0 and δ_0, such that for $0<\epsilon\leq\epsilon_0$ and $0<\delta\leq\delta_0$, the solution $\rho^{\epsilon,\delta}\in C(\overline{\Omega})\cap C^2(\Omega)\cap C^1(\Omega\backslash\Gamma_{\rm sonic})$ to (3.1), (3.3)–(3.5) satisfies*

$$c^2(\rho^{\epsilon,\delta})\geq r^2,\quad \text{in } \overline{\Omega}^{\epsilon,\delta}. \tag{3.19}$$

Proof For the notational simplicity, throughout the proof, we write $\rho=\rho^{\epsilon,\delta}$.

We show the lemma by contradiction arguments. More precisely, assume that there exists a nonempty set $D=\{(\xi,\eta)\in\overline{\Omega}:c^2(\rho)-r^2<0\}$ and let $X_{\min}\in D$ be the minimum point. First, it is easy to check that $P_2\notin D$. Also $O\notin D$, thus $D\subset\Omega_s$, where $\Omega_s=\{X\in\overline{\Omega}\backslash V:r^2>\overline{c}^2(\rho)\}$ and V is the set of all the corner points of Ω. Hence there are three possible locations of $X_{\min}$.

First, if $X_{\min}$ is the inner point of Ω. For notational simplicity, denote $c^2(\rho)=\rho^\gamma=u$ from now on. Then multiplying $\gamma\rho^{\gamma-1}$ over the equation $Q^{\epsilon,+}\rho=0$, we have

$$\begin{aligned}Lu&=\gamma\rho^{\gamma-1}\cdot Q^{\epsilon,+}\rho\\&=a_{ii}^{\epsilon}\Big(D_{ii}u-\frac{\gamma-1}{\gamma}\frac{1}{\rho^\gamma}|D_iu|^2\Big)+f'(c^2-r^2)(c^2-r^2)_ru_r+\frac{1}{r^2}u_\theta^2+b^\epsilon u_r\\&=0.\end{aligned} \tag{3.20}$$

Note that $\frac{\epsilon}{2} \leq a_{11}^{\epsilon} \leq \epsilon$ due to the cut-off function f in D. We evaluate Lr^2 in D

$$\begin{aligned} Lr^2 &\geq -2\epsilon\Big|1 - 2\frac{\gamma-1}{\gamma}\frac{1}{\rho^{\gamma}}r^2\Big| + f'(c^2-r^2)(c^2-r^2)_r u_r + 2c^2 \\ &\geq 2\overline{c}^2(\overline{\rho}) - 2\epsilon\Big|1 - 2\frac{|\gamma-1|}{\gamma}\frac{1}{\rho_1^{\gamma}}\overline{c}^2(\overline{\rho})\Big| > 0 \end{aligned} \tag{3.21}$$

with any small $\epsilon < \epsilon_0$ where $\epsilon_0 = \frac{\gamma c^2(\rho_1)\overline{c}^2(\overline{\rho})}{|\gamma c^2(\rho_1) - 2|\gamma-1|\overline{c}^2(\overline{\rho})|}$. Then using the fact that $(c^2 - r^2)_r(X_{\min}) = 0$, we obtain

$$\begin{aligned} 0 &> Lu - Lr^2 \\ &= a_{ii}^{\epsilon}D_{ii}(u-r^2) - \frac{\gamma-1}{\gamma\rho^{\gamma}}a_{ii}^{\epsilon}D_i(u+r^2)D_i(u-r^2) \\ &\quad + f'(c^2-r^2)(c^2-r^2)_r(u-r^2)_r + \frac{1}{r^2}(u+r^2)_\theta(u-r^2)_\theta + b(u-r^2)_r. \end{aligned} \tag{3.22}$$

Since $X_{\min}$ is an interior minimum point, we have $D_i(u-r^2)(X_{\min}) = 0$, and $a_{ii}^{\epsilon}D_{ii}(u - r^2)(X_{\min}) \geq 0$, which contradicts the inequality $Lu - Lr^2 < 0$ in $D \cap \Omega$.

Second, if $X_{\min}$ is located on $\Gamma_{\text{shock}} \cap D$, multiplying $\gamma\rho^{\gamma-1}$ over the equation $M\rho = 0$, we have $0 = \gamma\rho^{\gamma-1}M\rho = \widetilde{M}u = \beta_i D_i u$. On the one hand,

$$\widetilde{M}r^2 = 2r\beta_1^{(2)} = 2rr'(c^2(r^2-\overline{c}^2) - 3\overline{c}^2(c^2-r^2)) > 0 \tag{3.23}$$

in $\Gamma_{\text{shock}} \cap D$, where we use the fact $r^2 \geq c^2 \geq \overline{c}^2$ in Ω_s. On the other hand, at $X_{\min}$, the outward normal derivative of $u - r^2$ becomes non-positive (that is, $\nabla(u-r^2)(1,-r') \leq 0$) and the tangential derivative becomes zero (that is, $\nabla(u-r^2)(r',1) = 0$), so $(1+(r')^2)(u-r^2)_r \leq 0$ at $X_{\min}$, which implies $(u-r^2)_r \leq 0$. Thus we have

$$0 > \widetilde{M}(u-r^2) = (\beta_1^{(2)} - r'\beta_2^{(2)})(u-r^2)_r = \mu(u-r^2)_r \geq 0,$$

which is a contradiction.

Finally, if $X_{\min}$ is located on $\{\Sigma_0 \cup \Gamma_{\text{wedge}}\} \cap D$, then

$$\gamma\rho^{\gamma-1}\frac{\partial\rho}{\partial n} - \frac{\partial r^2}{\partial n} = \gamma\rho^{\gamma-1}\rho_\nu = 0,$$

which is a contradiction due to Hopf Lemma, i.e., $\frac{\partial(u-r^2)}{\partial n(X_{\min})} < 0$. Therefore, there is no minimum point and thus the set $D = \emptyset$, which completes the proof.

Based on Lemma 3.6, as Lemma 3.2 in [9], we can show that the solutions to the fixed boundary value problems (3.1) and (3.3)–(3.5) satisfy $r - \overline{c}(\rho) \geq 0$ on Γ_{shock}.

Lemma 3.7 *Let* $0 < \epsilon \leq \epsilon_0$ *and* $0 < \delta \leq \delta_0$, *and* $\rho^{\epsilon,\delta} \in C(\overline{\Omega}) \cap C^2(\Omega) \cap C^1(\Omega\backslash\Gamma_{\text{sonic}})$ *is a solution to the boundary value problems* (3.1) *and* (3.3)–(3.5). *Then* $\overline{c}(\rho^{\varepsilon,\delta}) - r \leq 0$ *on* $\Gamma_{\text{shock}}^{\epsilon,\delta}$.

We omit the long proof here. One could refer to [4] or [9] for details. With this lemma, the integration in (3.24) in the next section is always well defined on Γ_{shock}. Next, as in [3] and [15], we have the monotonicity of ρ along Γ_{shock}, which will be used to describe the convexity of the shock wave in (ξ,η)-coordinate.

Lemma 3.8 *Suppose that* $\rho^{\epsilon,\delta} \in C^1(\Omega \cup \Gamma_{\text{shock}} \cup \Gamma_{\text{wedge}} \cup \Sigma_0) \cap C^{\alpha}(\overline{\Omega})$ *is a solution to the boundary value problems* (3.1) *and* (3.3)–(3.5). *Then* $\rho^{\epsilon,\delta}$ *is monotonic on* Γ_{shock}.

3.4 The regularized nonlinear free boundary problem

We will show the existence of the solutions to the regularized free boundary problems.

Lemma 3.9 *For each $\epsilon \in (0,\epsilon_0)$ and $\delta \in (0,\delta_0)$ with some $\delta_0 > 0$, there exists a solution $(\rho^{\epsilon,\delta}, r^{\epsilon,\delta}) \in C^{-\gamma}_{2+\alpha}(\Omega^{\epsilon,\delta}) \times C^{1+\alpha}([-\frac{\pi}{2},\theta_1))$ to the regularized free boundary problems* (3.1), (3.3)–(3.5) *and* (3.2) *at the points of $\Gamma^{\epsilon,\delta}_{\rm shock}$ where $r^{\epsilon,\delta} \geq c(\rho_0) + 2\delta$.*

Proof For notational simplicity, we suppress the ϵ and δ dependence.

For each $r(\theta) \in \mathcal{K}^{\epsilon,\delta} \subset C^{1+\alpha}([-\frac{\pi}{2},\theta_1]) \cap C^2([-\frac{\pi}{2},\overline{\delta}))$, using the solution ρ to the nonlinear fixed boundary problems (3.1) and (3.3)–(3.5) given by Lemma 3.4, we first define the map J on $\mathcal{K}$, $\widetilde{r} = Jr$, as

$$\widetilde{r}(\theta) = r_2 + \int_{\theta_1}^{\theta} g(r(s), s, \rho(r(s), s))\mathrm{d}s. \tag{3.24}$$

There are two cases for the approximate shock position $\widetilde{r}(\theta)$.

Case 1 $\widetilde{r}(-\frac{\pi}{2}) \geq c(\rho_1) + \delta$ (see Figure 5(a)). We check that J maps $\mathcal{K}$ into itself. It is easy to check that $\widetilde{r}(\theta) \in C^{1+\alpha}([\theta_w,\theta_1]) \cap C^2([\theta_w,\theta))$. Property ($\mathrm{K}_1$) follows from (3.24). By the definition of g and $\rho(P_2) = \overline{\rho}$, $\widetilde{r}'(\theta) = 0$ holds. From the oblique boundary condition, we have

$$\frac{\rho_\theta(\theta) - \rho_\theta(-\frac{\pi}{2})}{|\theta + \frac{\pi}{2}|} = -\frac{\beta_1^{(2)}\rho_r}{\beta_2^{(2)} r'} \cdot \frac{r'(\theta) - r'(-\frac{\pi}{2})}{|\theta + \frac{\pi}{2}|} \to 0$$

as θ tends to $-\frac{\pi}{2}$, since $\rho \in C^{1+\mu}(\Gamma_{d_0} \backslash B_{d_0}(P_1))$ and $r''(-\frac{\pi}{2}) = 0$ for the older one. Thus, $\rho(\theta) - \rho(-\frac{\pi}{2}) = o(1)|\theta + \frac{\pi}{2}|^2$ for θ close to $-\frac{\pi}{2}$. This implies that $\overline{c}(\rho) = r(-\frac{\pi}{2}) + o(1)|\theta + \frac{\pi}{2}|^2$ for θ close to $-\frac{\pi}{2}$. Moreover, since $r''(-\frac{\pi}{2}) = 0$, we have $r - r(-\frac{\pi}{2}) = o(1)|\theta + \frac{\pi}{2}|^2$ for θ close to $-\frac{\pi}{2}$. Thus $\widetilde{r}' = \sqrt{\frac{r^2(r^2-\overline{c}^2)}{\overline{c}^2}} = o(1)|\theta + \frac{\pi}{2}|$ for θ close to $-\frac{\pi}{2}$, which implies that $\widetilde{r}''(-\frac{\pi}{2}) = 0$, and we get Property ($\mathrm{K}_2$). The only thing left is to show that Property (K_4) holds. In fact, it comes from the expression of $g(r(\theta),\theta,\rho(r(\theta),\theta))$, the upper and lower bounds of ρ, Lemma 3.5 and the bound of r in Lemma 3.6.

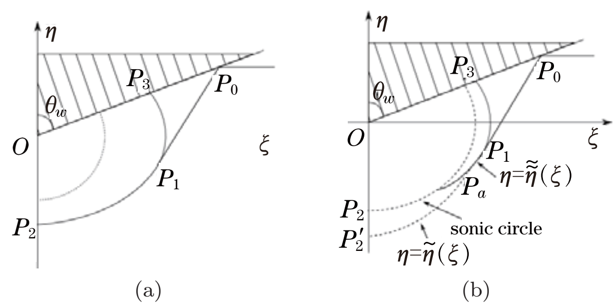


Figure 5 Approximate shock position

Case 2 $\widetilde{r}(-\frac{\pi}{2}) < c(\rho_1) + \delta$. Since $\widetilde{r}'(\theta) > 0$ for $\theta \in (-\frac{\pi}{2},\theta_1)$ and $r_2 = c(\rho_2) > c(\rho_1) + \delta$, there exists a unique $\theta_a \in (\theta_w,\theta_1)$ such that $\widetilde{r}(\theta_a) = c(\rho_1) + \delta$ (see Figure 5(b)) . Now choosing τ which will be determined later such that $\widetilde{r}(\theta_a + \tau) \leq c(\rho_1) + 2\delta$ and letting $x_1 = \theta_a + \tau + \frac{\pi}{2}$, we modify the approximate shock position on $-\frac{\pi}{2} \leq \theta \leq \theta_a + \tau$ by defining

$$\widetilde{\widetilde{r}}(\theta) = c(\rho_1) + \delta + A\Big(\theta + \frac{\pi}{2}\Big)^3 + B\Big(\theta + \frac{\pi}{2}\Big)^n$$

with $a = \widetilde{r}(\theta_a + \tau) - c(\rho_1) - \delta$, $b = \widetilde{r}'(\theta_a + \tau)$, $A = \frac{na - bx_1}{(n-3)x_1^3}$ and $B = \frac{bx_1 - 3a}{(n-3)x_1^n}$. Let τ be small enough such that $bx_1 - 3a > 0$. Then choose n sufficiently large such that $na - bx_1 > 0$, where n depends on δ but not on the iteration. In fact, it is easy to see that $|b| \le \sqrt{\overline{c}^2(\rho_1)\frac{c^2(\rho_2) - c^2(\rho_1)}{c^2(\rho_1)}}$, so there exists a constant $C(\rho_1, \rho_2)$, such that $|bx_1| \le C(\rho_1, \rho_2)$. If $3\delta \le bx_1$, we choose τ such that $a = \delta$ and $n_1 = \frac{2C(\rho_0, \rho_1)}{\delta}$, which depend only on ρ_0 and ρ_1. If $3\delta > bx_1$, we can let τ be small enough, such that we can get new a and b such that $3a = bx_1$ since $bx_1 > 0$ and $\widetilde{r}(\theta_a) = c(\rho_0) + \delta$. Thus, choosing $n_2 = 4$, we have $A > 0$ and $B = 0$. Let $n = \max(n_1, n_2)$, which is independent of the iterative process, and thus $\widetilde{\widetilde{r}}(\theta)$ is a strictly increasing function on $[-\frac{\pi}{2}, \theta_a + \tau]$. Furthermore, $0 = \widetilde{\widetilde{r}}'(-\frac{\pi}{2}) \le \widetilde{\widetilde{r}}'(\theta) \le \widetilde{\widetilde{r}}'(\theta_a + \tau) = \widetilde{r}'(\theta_a + \tau)$. We define

$$\widetilde{r}(\theta) = \begin{cases} \widetilde{r}(\theta) & \text{for } \theta \in [\theta_a + \tau, \theta_1], \\ \widetilde{\widetilde{r}}(\theta) & \text{for } \theta \in [-\frac{\pi}{2}, \theta_a + \tau]. \end{cases}$$

From the definition, it is easy to show that $\widetilde{r}(\theta)$, $\theta \in [-\frac{\pi}{2}, \theta_1]$ satisfies properties (K_1)–(K_4). We only need to show that $\widetilde{r}(\theta) \in C^{1+\mu}([-\frac{\pi}{2}, \theta_1]) \cap C^2([-\frac{\pi}{2}, \overline{\delta}))$. In fact, $\widetilde{r}(\theta) \in C^{1+\alpha_1}([\theta_a + \tau, \theta_1])$, $\widetilde{r}(\theta) \in C^{+\infty}([-\frac{\pi}{2}, \theta_1])$, and $\widetilde{r}'(\theta) \in C([-\frac{\pi}{2}, \theta_1])$, so we have $\widetilde{r}(\theta) \in C^{1+\alpha_1}([-\frac{\pi}{2}, \theta_1])$. Thus $\|\widetilde{r}\|_{C^{1+\alpha}([-\frac{\pi}{2}, \theta_1])} \le C(\rho_1, \rho_2, \epsilon, \delta)$, and then ($K_1$)–($K_4$) hold.

As in [4], we could easily prove that the map is continuous and compact since n is uniquely determined. Thus, we get the existence of the solution $(\rho^{\epsilon,\delta}, r^{\epsilon,\delta})$ to the free boundary problem by Schauder fixed point argument, and $r^{\epsilon,\delta} \in C^{1+\mu}([-\frac{\pi}{2}, \theta_1]) \cap C^2([-\frac{\pi}{2}, \theta_1))$ for $\mu \le \alpha_1$. This completes the proof of the lemma.

Remark 3.1 There may be two cases for the solution pair $(\rho^{\epsilon,\delta}, r^{\epsilon,\delta})$ as follows:

Case I If $r^{\epsilon,\delta} > c(\rho_1) + 2\delta$ for all $\theta \in (-\frac{\pi}{2}, \theta_1)$, then $r^{\epsilon,\delta} \in C^{2+\alpha}((-\frac{\pi}{2}, \theta_1))$ and $\frac{\mathrm{d}r^{\epsilon,\delta}}{\mathrm{d}\theta} = r^{\epsilon,\delta}\frac{\sqrt{(r^{\epsilon,\delta})^2 - \overline{c}^2(\rho^{\epsilon,\delta})}}{\overline{c}(\rho^{\epsilon,\delta})}$.

Case II If $r^{\epsilon,\delta} < c(\rho_1) + 2\delta$ for some point, then there exists a $\theta^* \in (-\frac{\pi}{2}, \theta_1)$, such that

(1) for each $\theta \in (-\frac{\pi}{2}, \theta^*)$, $r(\theta) = c(\rho_1) + \delta + A(\theta + \frac{\pi}{2})^3 + B(\theta + \frac{\pi}{2})^n$;

(2) for any $\theta \in (\theta^*, \theta_1)$, $\frac{\mathrm{d}r^{\epsilon,\delta}}{\mathrm{d}\theta} = r^{\epsilon,\delta}\frac{\sqrt{(r^{\epsilon,\delta})^2 - \overline{c}^2(\rho^{\epsilon,\delta})}}{\overline{c}(\rho^{\epsilon,\delta})}$.

In the following, we consider Case I first and give the precise description of the shock wave in (ξ, η)-coordinates.

Lemma 3.10 *For the solutions to* (3.1)–(3.5)*, the free boundary can be described as* $\Gamma_{\text{shock}} = \{(\xi, \eta(\xi)) \mid 0 < \xi < \xi_1\}$ *with* $\eta(\xi) \in C^2_{\text{loc}}((0, \xi_1))$, $\eta'(\xi) > 0$ *and* $\eta'' \ge 0$. *In addition,* $\eta(\xi) > l(\xi)$ *for* $0 < \xi < \xi_1$.

Proof We define

$$F(\xi, \eta) = \xi^2 + \eta^2 - r^2(\theta(\xi, \eta)) = 0, \quad \text{on } \Gamma_{\text{shock}}. \tag{3.25}$$

It is easy to check that $F_\eta = (2\eta - 2rr'\theta_\eta)|_{\xi=0} = 2\eta(0) \ne 0$. By the Implicit Theorem, there exists an $\eta = \eta(\xi)$ such that (3.25) holds locally on Γ_{shock} near $\xi = 0$, and $\frac{\partial \eta}{\partial \xi}\big|_{\xi=0} = 0$, that is, there exists a $\overline{\xi} > 0$, such that $(\xi, \eta(\xi)) \in \Gamma_{\text{shock}}$ for $0 < \xi \le \overline{\xi}$.

Recall that $\eta' = f(\xi, \eta(\xi), \rho(\xi, \eta(\xi)))$, and we calculate $\eta'' = f_\xi + f_\eta\eta' + f_\rho\rho'$ for $\xi \in (0, \overline{\xi})$. Observing that if ρ is a constant the shock would be a straight line, we get $f_\xi + f_\eta\eta' = 0$. Therefore, the sign of η'' is determined entirely by the sign of f_ρ and ρ'. Since $\rho' > 0$ by ρ

increasing, $\frac{\mathrm{d}\overline{c}^2}{\mathrm{d}\rho} > 0$ and

$$\frac{\partial f}{\partial \overline{c}^2} = \frac{(\xi\overline{c} - \eta\sqrt{\xi^2+\eta^2-\overline{c}^2})^2}{\overline{c}(\xi\eta - \overline{c}\sqrt{\xi^2+\eta^2-\overline{c}^2})^2\sqrt{\xi^2+\eta^2-\overline{c}^2}} \geq 0,$$

this finishes the proof of the convexity. Then $0 \leq \eta'(\xi) < l'(\xi_1) \leq C\sigma$ for $0 < \xi < \xi_1$. Thus, we have $\eta(\xi) = \eta_1 + \int_{\xi_1}^{\xi} \eta'(s)\mathrm{d}s > \xi_1 + \int_{\xi_1}^{\xi} l'(\xi_1)\mathrm{d}s = l(\xi)$, for $\xi < \xi_1$, which finishes the proof of the lemma.

Next, we will demonstrate that Case II will not happen if the angle of the wedge is large and δ_0 is small only depending on ρ_1, ρ_0 in the following remark.

Remark 3.2 Let θ be close to θ^* from the right-hand side. We have

$$\frac{\mathrm{d}r^{\epsilon,\delta}(\theta)}{\mathrm{d}\theta} = r^{\epsilon,\delta}\frac{\sqrt{(r^{\epsilon,\delta})^2 - \overline{c}^2(\rho^{\epsilon,\delta})}}{\overline{c}(\rho^{\epsilon,\delta})}.$$

As the proof of Lemma 3.10, the shock reflection boundary can be expressed locally in (ξ,η)-coordinates, which satisfies

$$(\eta^{\epsilon,\delta}(\xi))' = \frac{\overline{c}^2(\rho^{\epsilon,\delta}) - (\eta^{\epsilon,\delta})^2}{\overline{c}(\rho^{\epsilon,\delta})\sqrt{\xi^2+(\eta^{\epsilon,\delta})^2-\overline{c}^2(\rho^{\epsilon,\delta})} - \xi\eta^{\epsilon,\delta}}.$$

As $\delta \to 0$, we divide it into two cases. First, if $|\theta^* + \frac{\pi}{2}| < C \ll 1$, where C is independent of δ and will be specified later, as in Lemma 3.3, $\eta' = \frac{r\cos\theta + r'\sin\theta}{-r\sin\theta + r'\cos\theta}$. Let $(r^{\epsilon,\delta}(\theta^*),\theta^*) = (\xi^*,\eta^*)$. We have that if $\xi < \xi^*$, because $A = \frac{na-bx_1}{(n-3)x_1^3} < \frac{na}{(n-3)x_1^3} \leq C(\rho_1,\rho_2)\delta$, then

$$\eta' = \frac{c(\rho_1) + \delta + (O(1)\xi^*)r \times |O(1)|(\theta^*+\frac{\pi}{2})^2 + \delta^* O(1)(\theta^*+\frac{\pi}{2})^2}{-(c(\rho_1)+\delta)\sin\theta + O(1)\theta^*} > 0,$$

when $\delta, |\theta^* + \frac{\pi}{2}|$ are small enough depending only on ρ_1. Then from the C^1-regularity, we obtain that $\eta' > 0$ for $0 < \xi - \xi^* \ll 1$. We can show that Lemma 3.10 holds for $\xi \in (\xi^*, \xi_1)$. Thus, from the fact that $0 \leq \eta'(\xi) \leq \eta'(\xi_1) \leq C\sigma$, there exists a $\theta_0 \in [\theta_c, \frac{\pi}{2})$ and $\tau^* > 0$, such that $\xi^2 + (\eta^{\epsilon,\delta}(\xi))^2 > c(\rho_1) + \tau^*$ for $\theta \in (\theta^*, \theta_1)$, if $\theta_w \in [\theta_0, \frac{\pi}{2})$, where θ_0 is independent of δ, which contradicts the continuity of $r^{\epsilon,\delta}$ at θ^*.

For another case, i.e., $\theta^* + \frac{\pi}{2} > C > 0$ only depending on ρ_1, let $(r(\theta^*),\theta^*) = (\xi^*,\eta^*)$. Then $\eta^* > -c(\rho_1) + O(1)\xi^*$ and $\rho^{\epsilon,\delta}(\xi^*,\eta^*) = \rho_1 + O(1)\delta$. Thus $\overline{c}(\rho^{\epsilon,\delta}) > (\eta^{\epsilon,\delta})^2$ for $0 < \xi - \xi^* \ll 1$, if δ is small enough, which implies that $(\eta^{\epsilon,\delta}(\xi))' > 0$. Thus as in the first case, we could obtain the contradiction to deduce that Case II does not happen for our regular shock reflection if the angle of wedge is large.

Now, we focus on the proof of Theorem 3.1. Here $\eta'(\xi) > 0$ for $\xi \in (0,\xi_1)$.

Proof of Theorem 3.1 We note that Remark 3.2 and Lemma 3.10 imply that there exists a constant $\delta^* > 0$ independent of ϵ, such that $r^{\epsilon,\delta} \geq \rho_1 + 2\delta^*$. By choosing $\delta_0 < \delta^*$, the solution pair $(\rho^{\epsilon,\delta}, r^{\epsilon,\delta})$ is independent of δ, and then we discard the note of δ. Thus, we have $c^2(P_2) > r^2(P_2)$, which implies that $\beta_2^{(2)} \leq -\overline{\delta} < 0$ for some $\overline{\delta} > 0$. So the estimates obtained in Lemma 3.2 and Lemma 3.3 do not depend on δ, and ρ^ϵ satisfies all the estimates in Theorem 3.1.

Remark 3.3 Now, it is easy to show that the free boundary value problem (3.1)–(3.5) is equivalent to the following problem in self-similar coordinates:

$$L^\epsilon\rho = D_i(a_{ij}(\Xi,\rho)D_j\rho) + \epsilon\triangle\rho + b_i(\Xi,\rho)D_i\rho = 0, \quad \text{in } \Omega, \tag{3.26}$$

where $a_{11}(\xi,\eta) = c^2(\rho) - \xi^2 + \epsilon$, $a_{22}(\xi,\eta) = c^2(\rho) - \eta^2 + \epsilon$, $a_{12}(\xi,\eta) = a_{21}(\xi,\eta) = -\xi\eta$, $b_1(\xi,\eta) = \xi$ and $b_2(\xi,\eta) = \eta$, and the shock evolution equation

$$\frac{\mathrm{d}\eta}{\mathrm{d}\xi} = f(\xi,\eta,\rho) \quad \text{with } \eta(\xi_1) = \eta_1,$$

and with the boundary condition on Γ_{shock},

$$Nu = \beta_i D_i\rho = \beta(\Xi,\rho)D_i\rho = 0, \quad \text{on } \Gamma_{\text{shock}} = \{\eta = \eta(\xi) \mid 0 \le \xi \le \xi_1\}, \tag{3.27}$$

where β_i is a function of (ξ,η), ρ and η' are defined in (2.23) and (2.24), with the remaining boundary conditions

$$u = \rho_2 \quad \text{on } \Gamma_{\text{sonic}}, \quad u_\xi = 0 \quad \text{on } \Sigma_0, \quad u_\nu = 0 \quad \text{on } \Gamma_{\text{wedge}}, \quad u(P_2) = \overline{\rho}, \tag{3.28}$$

where ν is the outward normal to Ω at $\eta = \xi\cot\theta_w$. We remark that from the expression of η', it is easy to show that (2.25) implies that (3.27) is oblique on Γ_{shock}.

4 The Limiting Solution

In this section, we study the limiting solutions, as the elliptic regularization parameter ϵ tends to zero. We start with the regularized solutions (3.26)–(3.28) in (ξ,η)-coordinates, whose existence is guaranteed by Theorem 3.1. Denote by ρ^ϵ a sequence of regularized solutions of the free boundary value problem.

Following [3], we could find a uniform lower barrier to obtain the uniform ellipticity in any compact domain contained by $\overline{\Omega}\backslash\Gamma_{\text{sonic}}$ for the solutions to the regularized problems.

Lemma 4.1 *There exists a positive function φ, which is independent of ϵ, such that $c^2(\rho^\epsilon) - (\xi^2+\eta^2) \ge \varphi$ in $\overline{\Omega}\backslash\Gamma_{\text{sonic}}$, and φ tends to zero, as dist$((\xi,\eta),\Gamma_{\text{sonic}}) \to 0$.*

Lemma 4.1 implies that we can get the uniform ellipticity of (3.1) which is independent of ϵ in $B_{3R_{X_0}/4}(X_0) \cap \overline{\Omega_\epsilon}$.

The existence of a uniform lower bound of $c^2 - \xi^2 - \eta^2$ independent of ϵ implies that the governing equation (3.1) is locally uniform elliptic independent of ϵ, which allows us to use standard local compactness arguments to get a limit ρ locally in the interior of the domain. We next show that the sequence of domain Ω^ϵ converges to a domain Ω, as ϵ tends to zero.

Lemma 4.2 *The sequence η^ϵ has a convergent subsequence, whose limit η belongs to $C^\gamma([0,\xi_1])$ for all $\gamma \in (0,1)$. The limiting curve η is convex.*

Proof Theorem 3.1 gives the existence of a sequence $(\rho^\epsilon,\eta^\epsilon)$ of the solutions to the regularized free boundary problems for which η^ϵ belongs to the set $\mathcal{K}^\epsilon$ for each ϵ. Now $\rho_1 < \overline{\rho} \le \rho^\epsilon \le \rho_2$, and the definition of $\mathcal{K}^\epsilon$ immediately gives a C^1 bound for η^ϵ, uniformly in ϵ. Thus by the Arzela-Ascoli theorem, η^ϵ has a convergent subsequence, and the limit $\eta \in C^\gamma([0,\xi_1])$ for all $\gamma \in (0,1)$. In addition, as we know, η^ϵ is convex for each $\epsilon > 0$, so is the limiting function.

We remark that away from point P_1, from Lemma 3.2 and Lemma 4.1, $\|\eta^\epsilon\|_{1+\alpha}$ is uniformly bounded. By using Arzela-Ascoli Theorem again, the limit function $\eta(\xi)$ is in fact in $C^{1+\alpha}([0,\xi_1))$. The limit value $\eta(0)=\lim\limits_{\epsilon\to 0}\eta^\epsilon(0)$ is also established, and the corresponding subsequence of domains Ω^ϵ also has a limit, Ω.

In the remaining lemmas, without further comments, we carry out the limiting arguments using the convergent subsequence of η^ϵ, which is still written as η^ϵ.

Lemma 4.3 *The sequence ρ^ϵ has a limit $\rho\in C^{2+\alpha'}_{\rm loc}$ for some $\alpha'>0$. The limit ρ satisfies the quasi-linear degenerate elliptic equation* (3.26)*. Moreover, $\rho_1+\delta^*\le\overline{\rho}^\epsilon\le\rho^\epsilon<\rho_2$ in Ω.*

Proof The proof is based on local compactness arguments and on uniform L^∞ bounds for ρ^ϵ: $\rho_1+\delta^*<\overline{\rho}^\epsilon<\rho^\epsilon<\rho_2$. The main ideas follow those used in Lemma 4.2 of [3]. Fix $\Omega_1\Subset\Omega$. There exists an ϵ' (which depends on Ω_1), such that $\Omega_1\subsetneq\Omega^\epsilon$ for $\epsilon\le\epsilon'$, and then for $\Omega_2\subsetneq\Omega_1$, $|\rho^\epsilon|_{C^{\overline{\alpha}}(\overline{\Omega_2})}\le C$, where $\overline{\alpha}\in(0,1)$ and C is independent of ϵ. With these estimates of the coefficients of Q^ϵ, and the boundness of ρ^ϵ, we get from the standard estimates in [7] (Theorem 8.32 and Theorem 6.2 for the interior, and Theorem 8.33 and Lemma 6.5 for the boundary Ω_2) that $|\rho^\epsilon|_{C^{2,\overline{\alpha}}(\overline{\Omega_2})}\le C$. By the Arzela-Ascoli theorem, there exists a $C^{2,\alpha'}_{\rm loc}(\overline{\Omega_2})$-convergent subsequence for $\alpha'<\overline{\alpha}$. Now let Ω_1 vary in Ω and use a diagonalization argument to obtain a subsequence of ρ^ϵ which converges in $C^{2,\alpha'}_{\rm loc}(\Omega)$ to a limit $C^{2,\alpha'}_{\rm loc}(\Omega)$ which satisfies $Q\rho=0$ in Ω. From the uniform L^∞ bounds for ρ^ϵ, we get $\rho_1<\overline{\rho}\le\rho<\rho_2$ in Ω.

In the next lemma, we prove the Lipschitz continuity of the solutions near the degenerate sonic boundary.

Lemma 4.4 *The solution ρ to the free boundary value problem* (3.26)–(3.28) *is Lipschitz continuous up to the boundary $\Gamma_{\rm sonic}$.*

Proof On the one hand, since $\rho\le\rho_2$ in Ω, $c^2(\rho)-\xi^2-\eta^2<c^2(\rho_2)-\xi^2-\eta^2$.

On the other hand, it follows from Lemma 3.1 that $c^2(\rho)-\xi^2-\eta^2>\xi^2+\eta^2-c^2(\rho_2)$ in Ω. Letting $r_2^2=c^2(\rho_2)$, we have

$$\begin{aligned}|c^2(\rho)-c^2(\rho_2)|&\le|c^2(\rho)-\xi^2-\eta^2|+|c^2(\rho_2)-\xi^2-\eta^2|\\&\le 2|c^2(\rho_2)-\xi^2-\eta^2|\\&\le 4r_2|r_2-\sqrt{\xi^2+\eta^2}|,\end{aligned}$$

which implies that ρ is Lipschitz continuous up to the degenerate boundary $\Gamma_{\rm sonic}$.

The next task is to show that ρ and η satisfy both the shock evolution equation (2.5) and the oblique derivative boundary condition (3.27) on $\Gamma_{\rm shock}$.

Lemma 4.5 *The limits η and ρ satisfy*

$$\eta'=f(\xi,\eta,\rho)\quad\text{and}\quad N\rho=\beta(\xi,\eta(\xi))\cdot\nabla\rho=0,\quad\text{on }\Gamma_{\rm shock}.$$

Furthermore, $\eta\in C^{2+\alpha'}([0,\xi_1))\cap C^1([0,\xi_1])$ and $\rho\in C^{2+\alpha'}_{\rm loc}(\Omega\cup\Gamma_{\rm shock}\cup\Sigma_0\cup\Gamma_{\rm wedge}\backslash B_\delta(V))\cap C(\Omega\cup\Gamma_{\rm shock}\cup\Sigma_0\cup\Gamma_{\rm wedge})$ for some $\alpha'>0$. In addition, $\rho=\overline{\rho}$ at $P_2=(0,\eta(0))$, where $\overline{\rho}=\overline{c}_{\rho_1}^{-1}(-\eta(0))$.

Proof As in [3], we just focus on dealing with the behavior of the solutions near P_2.

Since $\eta^\epsilon \to \eta(\xi)$ in $C^{2+\alpha'}_{\rm loc}$ for $\xi \neq 0$, and $\rho^\epsilon \to \rho$ in $C^{1+\alpha'}_{\rm loc}$, we have $(\eta^\epsilon)' = f(\xi, \eta^\epsilon, \rho^\epsilon) \to f(\xi, \eta, \rho)$, $\forall \xi \neq 0$, and thus $\eta' = f(\xi, \eta, \rho)$ for $\xi \neq 0$. Furthermore,

$$0 = N\rho = \beta(\eta^\epsilon(\xi), \rho^\epsilon) \cdot \nabla \rho^\epsilon(\xi, \eta^\epsilon(\xi)) \to \beta(\eta(\xi), \rho) \cdot \nabla \rho(\xi, \eta(\xi)), \quad \forall \xi \neq 0,$$

where we use the continuity of β and ρ. Then $\beta(\eta, \rho) \cdot \nabla \rho = 0$ on $\Gamma_{\rm shock} \backslash \{(0, \eta(0))\}$.

We now focus on the behavior of the solutions at P_2. By Lemma 4.2, $\eta^\epsilon \to \eta$ in $C^\gamma([0, \xi_1])$ for any $0 < \gamma < 1$. Furthermore $\overline{c}^2(\overline{\rho}^\epsilon, \rho_1) = (\eta^\epsilon(0))^2$, where $\overline{\rho}^\epsilon = \rho(0, \eta^\epsilon(0))$ for fixed $\epsilon > 0$. Therefore, as $\epsilon \to 0$, the right-hand side converges to $\eta^2(0)$. Hence $\overline{c}^2(\overline{\rho}^\epsilon, \rho_1) \to \eta^2(0)$. By the continuity and monotonicity of $\overline{c}$, the sequence $\overline{\rho}^\epsilon$ has a limit $\overline{\overline{\rho}}$. Moreover, $\overline{c}(\overline{\overline{\rho}}, \rho_1) = -\eta(0)$, which defines $\overline{\rho}$, therefore $\overline{\rho} = \overline{\overline{\rho}}$, and the sequence of traces of the functions ρ^ϵ at $(0, \eta^\epsilon(0))$ converges to $\overline{\rho}$. We still have to show that ρ is continuous at P_2, i.e., $\lim\limits_{\xi \to 0} \rho(\xi, \eta(\xi)) = \overline{\rho}$. In fact, η'_ϵ has a limit $\eta' = f(\xi, \eta(\xi), \rho(\xi, \eta))$ in $C^{1+\alpha}$ for $\xi \neq 0$, and $\eta'_\epsilon(0) = 0$ for each $\epsilon > 0$, then for any $\delta > 0$, there exists an $h_0 \neq 0$ such that $|\eta'(h)| \le |\eta'(h) - \eta'_\epsilon(h)| + |\eta'_\epsilon(h)| \le \delta$ for $0 < h < h_0$, which implies the continuity of η' at $\xi = 0$ and $\eta'(0) = 0$. Thus

$$f(h, \eta(h), \rho(h, \eta(h))) = \eta'(h) \to \eta'(0) = 0 = f(0, \eta(0), \overline{\rho}), \quad \text{as } h \to 0.$$

This implies that $\rho(h, \eta(h)) \to \overline{\rho}$ and so ρ is continuous at P_2. Moreover $\rho(P_2) = \overline{\rho}$ with $\overline{\rho} = \overline{c}^{-1}_{\rho_1}(-\eta(0))$.

This finishes the proof of the lemma.

Proof of the Existence Part of Theorem 2.1 The above four lemmas, i.e., Lemmas 4.2–4.5 show that there exists a solution pair $(\rho, \eta) \in C^{2+\alpha}(\Omega) \cap C(\overline{\Omega} \backslash \Gamma_{\rm sonic}) \cap C^{0,1}(\Omega \cup \Gamma_{\rm sonic}) \times C^{2+\alpha'}(0, \xi_1)$ satisfying (3.26)–(3.28). This finishes the proof of the existence part of Theorem 2.1.

Finally, we show that the solution ρ obtained in Theorem 2.1 tends to the normal reflection solution $\overline{\rho}_2$, as $\theta_w \to \frac{\pi}{2}$.

Lemma 4.6 *Assume that ρ is the solution to the free boundary value problem* (3.26)–(3.28)*. Then ρ tends to $\overline{\rho}_2$, as $\theta_w \to \frac{\pi}{2}$.*

Proof It is easy to see that $\eta_c \le \eta(\xi) \le \eta_1$, where $\eta_c = \eta_1 - \eta'(\xi_1)\xi_1$. Moreover, it follows from the definition of $\eta'(\xi_1)$ that $\eta'(\xi_1) \to 0$, as $\theta_w \to \frac{\pi}{2}$. Thus $|\eta(\xi) - \eta_1| \le |\eta_1 - \eta_c| \le \int_0^{\xi_1} |l'(s)| {\rm d}s = |\eta'(\xi_1)|\xi_1$. This implies that $\eta(\xi) \to \overline{\eta}$, since $\eta_1 \to \overline{\eta}$, as $\theta_w \to \frac{\pi}{2}$. It is easy to see that $\eta_c \le \overline{c}(\overline{\rho}) \le \eta_1$. By the Squeeze Theorem, $\overline{c}(\overline{\rho})$ tends to $\overline{c}(\overline{\rho}_2)$ as $\theta_w \to \frac{\pi}{2}$. Thus $\rho \to \overline{\rho}_2$, as $\theta_w \to \frac{\pi}{2}$.

This finishes the proof of the lemma.

5 Optimal Regularity near the Sonic Boundary

In this section, we will prove that Lipschitz continuity is the optimal regularity for ρ across the sonic boundary $\Gamma_{\rm sonic}$, since we have proven that the solution ρ to the free boundary value problem (3.26)–(3.28) is Lipschitz continuous in Ω up to the degenerate boundary $\Gamma_{\rm sonic}$. We will study the behaviors of ρ near $r = r_2 := c(\rho_2)$, where $(r, \theta) = (\sqrt{\xi^2 + \eta^2}, \arctan(\frac{\eta}{\xi}))$ are polar coordinates with respect to self-similar coordinates (ξ, η). The proof of Theorem 1.1 is long and exactly the same as that in Section 5 of [4]. So we just sketch them in this section.

For $\varepsilon \in (0, \frac{c_2}{2})$, denote $\Omega_\varepsilon := \Omega \cap \{(r,\theta) : 0 < c_2 - r < \varepsilon\}$, the ε-neighborhood of the sonic circle Γ_{sonic} within Ω. In Ω_ε, introduce the coordinates

$$x = c_2 - r, \quad y = \theta_w - \theta. \tag{5.1}$$

It is convenient to study the regularity in terms of the difference between $c^2(\rho_2)$ and $c^2(\rho)$, since ψ and ρ have the same regularity in Ω_ε. Thus we introduce

$$\psi = c^2(\rho_2) - c^2(\rho). \tag{5.2}$$

It follows from (2.28) that ψ satisfies

$$\begin{aligned} \mathcal{L}_1\psi := (2c_2x - \psi + O_1)\psi_{xx} + (c_2 + O_2)\psi_x - (1 + O_3)\psi_x^2 \\ + (1 + O_4)\psi_{yy} - \Big(\frac{1}{\gamma c_2^2} + O_5\Big)\psi_y^2 = 0, \quad \text{in } Q^+_{r,R} \end{aligned} \tag{5.3}$$

in the (x,y)-coordinates, where

$$\begin{cases} O_1(x,\psi) = -x^2, \\ O_2(x,\psi) = -3x + \dfrac{\psi}{c_2}, \\ O_3(x,\psi) = -\dfrac{\gamma-1}{\gamma}(2c_2x - \psi - x^2), \\ O_4(x,\psi) = \dfrac{(c_2)^2 - \psi}{(c_2 - x)^2} - 1, \\ O_5(x,\psi) = \dfrac{1}{(c_2-x)^2} - \dfrac{1}{(c_2)^2}. \end{cases} \tag{5.4}$$

Moreover, ψ satisfies the following conditions:

$$\psi > 0, \quad \text{in } Q^+_{r,R}, \tag{5.5}$$

$$\psi = 0, \quad \text{on } \partial Q^+_{r,R} \cap \{x = 0\}, \tag{5.6}$$

where $Q^+_{r,R} := \{(x,y) : x \in (0,\epsilon), |y| < R\} \subset \mathbb{R}^2$, with $R = \theta_w - \arctan(\frac{\eta_1}{\xi_1})$, since we can extend $\psi(x,y)$ from Ω_ε by defining $\psi(x,y) = \psi(x,-y)$ for $(x,y) \in \Omega_\varepsilon$, and also the domain Ω_ε with respect to y. Thus, without further comments, we study the behaviors of ψ in $Q^+_{r,R}$. It is easy to see that the terms $O_i(x,y),\ i = 1,\cdots,5$, are continuously differentiable and

$$\frac{|O_1(x,y)|}{x^2}, \ \frac{|O_k(x,y)|}{x} \le N \quad \text{for } k = 2,\cdots,5, \tag{5.7}$$

$$\frac{|DO_1(x,y)|}{x}, \ |DO_k(x,y)| \le N \quad \text{for } k = 2,\cdots,5 \tag{5.8}$$

in $\{x > 0\}$ for some constant N depending only on c_2 and γ. Inequalities (5.7) and (5.8) imply that the terms $O_i(x,y),\ i = 1,\cdots,5$, are “small”. Thus, the main terms of (5.3) form the following equation:

$$(2c_2 - \psi)\psi_{xx} + c_2\psi_x - \psi_x^2 + \psi_{yy} - \frac{1}{\gamma c_2^2}\psi_y^2 = 0, \quad \text{in } Q^+_{r,R}. \tag{5.9}$$

It follows from Lemma 4.1 and Lemma 4.5 that

$$0 \le \psi \le 2(c_2 - \vartheta)x, \tag{5.10}$$

where ϑ depends only on ρ_2 and γ. Then (5.9) is uniformly elliptic in every subdomain $\{x > \delta\}$ with $\delta > 0$. It is the same to (5.3) in $Q^+_{r,R}$ if r is sufficiently small. Then we have the following two theorems, the proofs of which are quite long and similar to [1]. One can refer to [4] for details.

Theorem 5.1 *Let $\rho \in C^{2+\alpha}(\Omega) \cap C(\overline{\Omega})$ be the solution to the free boundary value problem* (3.26)–(3.28) *obtained in Section* 4*. Then ρ cannot be C^1 across the degenerate sonic boundary $\Gamma_{\rm sonic}$.*

In the following theorem, we study more detailed regularity of ρ near the sonic circle in the case of $C^{0,1}$ interacting transonic shock solutions.

We use a localized version of Ω_ε: For a given neighborhood $\mathcal{N}(\Gamma_{\rm sonic})$ of $\Gamma_{\rm sonic}$ and $\varepsilon > 0$, define $\Omega_\varepsilon := \Omega \cap \mathcal{N}(\Gamma_{\rm sonic}) \cap \{x < \varepsilon\}$. Since $\mathcal{N}(\Gamma_{\rm sonic})$ will be fixed in the following theorem, we do not specify the dependence of Ω_ε on $\mathcal{N}(\Gamma_{\rm sonic})$.

Theorem 5.2 *Let $\psi = c_2^2 - c^2(\rho)$, where ρ is the solution to the free boundary value problem* (3.26)–(3.28) *obtained in Section* 4*, and satisfies the following properties:*

There exists a neighborhood $\mathcal{N}(\Gamma_{\rm sonic})$ of $\Gamma_{\rm sonic}$ such that

(a) *ψ is $C^{0,1}$ across the part $\Gamma_{\rm sonic}$ of the degenerate sonic boundary;*

(b) *there exists a $\vartheta_0 > 0$ so that, in the coordinates introduced by* (5.1),

$$|\psi| \le (2c_2 - \vartheta_0)x, \quad \text{in } \Omega \cap \mathcal{N}(\Gamma_{\rm sonic}). \tag{5.11}$$

Then we have

(i) *there exists an $\varepsilon_0 > 0$, such that ψ is $C^{1,\alpha}$ in Ω up to $\Gamma_{\rm sonic}$ away from the point P_1 for any $\alpha \in (0,1)$, that is, for any $\alpha \in (0,1)$ and any given $(\xi_0, \eta_0) \in \overline{\Gamma_{\rm sonic}} \backslash P_1$, there exists a $K < \infty$ depending only on ρ_0, ρ_1, γ, ε_0, α, $\|\psi\|_{C^{0,1}}$ and $d = dist((\xi_0, \eta_0), \Gamma_{\rm shock})$ so that*

$$\|\psi\|_{1,\alpha;\overline{B_{\frac{d}{2}}(\xi_0,\eta_0)}\cap\Omega_{\frac{\varepsilon_0}{2}}} \le K;$$

(ii) *for any $(\xi_0, \eta_0) \in \Gamma_{\rm sonic} \backslash P_1$,*

$$\lim_{\substack{(\xi,\eta)\to(\xi_0,\eta_0)\\(\xi,\eta)\in\Omega}} D_r\psi = c_2;$$

(iii) *the limit $\lim\limits_{\substack{(\xi,\eta)\to P_1\\(\xi,\eta)\in\Omega}} D_r\psi$ does not exist.*

6 Conclusions

A solution ρ has been constructed by Theorem 2.1 to the differential equation (3.26) in Ω, and combining this function with $\rho = \rho_i$ in state (i), i.e., we have obtained a solution which is piecewise constant in the supersonic region, which is Lipschitz continuous across the degenerate sonic boundary $\Gamma_{\rm sonic}$ from Ω to state (2). To recover the momentum components, m and n, we could in principle integrate the second and the third equation in (1.5), which can be written as transport equations in the radial variable r,

$$\frac{\partial m}{\partial r} = \frac{1}{r} c^2(\rho)\rho_\xi, \quad \frac{\partial n}{\partial r} = \frac{1}{r} c^2(\rho)\rho_\eta, \tag{6.1}$$

and integrated from the boundary of the subsonic region toward the origin. We note that the sonic boundary can be written as $r = r_2$ for $\theta \in [\arctan(\frac{\eta_1}{\xi_1}), \theta_w]$, and the boundary conditions for m and n are of the form $m(r_2, \theta) = m_2$ and $n(r_2, \theta) = n_2$ respectively, since (m, n) have the same regularity as ρ in $P_0P_1P_2O$. Note that we have proven that $D\rho$ does not converge in Ω as (ξ, η) tends to (ξ_1, η_1), thus (6.1) may not be meaningful. In addition, the behavior $c^2(\rho)\frac{\rho_\eta}{r}$ in (6.1) at the origin causes a logarithmic singularity in n (but not in m, since $c^2(\rho)\frac{\rho_\xi}{r}$ remains bounded since $\rho_\xi(0, 0) = 0$).

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References

[1] Bae, M., Chen, G.-Q. and Feldman, M., Regularity of solutions to regular shock reflection for potential flow, *Ivent. Math.*, **175**, 2009, 505–543.

[2] Čanić, S., Keyfitz, B. L. and Kim, E. H., A free boundary problem for a quasilinear degenerate elliptic equation: transonic regular reflection of weak shocks, *Comm. Pure Appl. Math.*, **55**, 2002, 71–92.

[3] Čanić, S., Keyfitz, B. L. and Kim, E. H., Free boundary problems for nonlinear wave systems: mach stems for interacting shocks, *SIAM J. Math. Anal.*, **37**, 2006, 1947–1977.

[4] Chen, G.-Q., Deng, X. M. and Xiang, W., The global existence and optimal regularity of solutions for shock diffraction problem to the nonlinear wave systems, preprint.

[5] Chen, G.-Q. and Feldman, M., Global solutions of shock reflection by large-angle wedges for potential flow, *Ann. of Math.*, **171**(2), 2010, 1067–1182.

[6] Gilbarg, D. and Hörmander, L., Intermediate Schauder estimates, *Arch. Ration. Mech. Anal.*, **74**, 1980, 297–318.

[7] Gilbarg, D. and Trudinger, N., Elliptic Partial Differential Equations of Second Order, 2nd ed., Springer-Verlag, Berlin, 1983.

[8] Guan, P. and Sawyer, E., Regularity estimates for the oblique derivative problem on non-smooth domains (I), *Chin. Ann. Math.*, **16B**(3), 1985, 299–324.

[9] Kim, E. H., A global sub-sonic solution to an interacting transonic shock of the self-similar nonlinear wave equation, *J. Differ. Equ.*, **248**, 2010, 2906–2930.

[10] Lieberman, G., The Perron process applied to oblique derivative problems, *Adv. Math.*, **55**, 1985, 161–172.

[11] Lieberman, G., Regularized distance and its applications, *Pacific J. Math.*, **117**, 1985, 329–352.

[12] Lieberman, G., Mixed boundary value problems for elliptic and parabolic differential equations of second order, *J. Math. Anal. Appl.*, **113**, 1986, 422–440.

[13] Lieberman, G., Oblique derivative problems in Lipschitz domains, II, discontinuous boundary data, *J. Reine Angew. Math.*, **389**, 1988, 1–21.

[14] Lieberman, G., Optimal Hölder regularity for mixed boundary value problems, *J. Math. Anal. Appl.*, **143**, 1989, 572–586.

[15] Zheng, Y., Two-dimensional regular shock reflection for the pressure gradient system of conservation laws, *Acta Math. Sin.*, **22**, 2006, 177–210.