Chin. Ann. Math.
30B(4), 2009, 333–378
DOI: 10.1007/s11401-008-0280-x

Chinese Annals of Mathematics, Series B

Carleman Estimates for Parabolic Equations with Nonhomogeneous Boundary Conditions****

Oleg Yu IMANUVILOV* Jean Pierre PUEL**
Masahiro YAMAMOTO***

Abstract The authors prove a new Carleman estimate for general linear second order parabolic equation with nonhomogeneous boundary conditions. On the basis of this estimate, improved Carleman estimates for the Stokes system and for a system of parabolic equations with a penalty term are obtained. This system can be viewed as an approximation of the Stokes system.

Keywords Controllability, Parabolic equations, Carleman estimates
2000 MR Subject Classification 35K20, 93B05, 93B07

1 Introduction

Local Carleman estimates for elliptic and parabolic equations have been known since [1, 7] and among other examples of applications they turned out to be essential for proving unique continuation properties. Global Carleman estimates for parabolic equations with homogeneous boundary conditions have been obtained by several authors in the recent years (see for example [9] for $L^2(0,T;L^2(\Omega))$ right-hand sides and [13] for $L^2(0,T;H^{-1}(\Omega))$ right-hand sides). They have been extensively used for obtaining observability inequalities in controllability theory and stability results for some inverse problems.

In the case of elliptic equations with nonhomogeneous boundary conditions and $H^{-1}(\Omega)$ right-hand sides, sharp Carleman estimates have been obtained in [12] and this result turned out to be essential for obtaining estimates on the pressure in the context of controllability for the Navier-Stokes equations (see [5]).

The main object of the present article is to obtain a similar result of global Carleman estimates for general parabolic equations with nonhomogeneous boundary conditions and right-hand sides in $L^2(0,T;H^{-1}(\Omega))$. To this aim, after localization and a change of coordinates, we use a factorization of the operator and successive estimates for first order pseudodifferential operators in order to obtain the Carleman estimate. The article is organized as follows. The main result together with its complete proof is precisely given in Section 2. In Section 3, this result is applied (using also the estimate for elliptic equations) to the Stokes operator. In

Manuscript received November 12, 2008. Published online May 12, 2009.
*Department of Mathematics, Colorado State University, 101 Weber Building, Fort Collins CO 80523, USA. E-mail: oleg@math.colostate.edu
**Laboratoire de Mathematiques de Versailles, Université de Versailles-St. Quentin, 45 Avenue des Etats Unis, 78035 Versailles, France. E-mail: jppuel@math.uvsq.fr
***Department of Mathematical Sciences, University of Tokyo, Komaba, Meguro, Tokyo 153, Japan. E-mail: myama@ms.u-tokyo.ac.jp
****Project supported by NSF grant DMS (No. 0808130) and ANR Project (No. C-QUID 06-BLAN-0052).

Section 4 it is applied to a compressible Stokes operator where the incompressibility condition is approximated by penalization. In Appendix, we give some useful technical results on calculus for pseudodifferential operators depending on a parameter.

First of all, we need some notations which are introduced below.

Set $B(\widetilde{x}_0,\delta)=\{\widetilde{x}\mid \widetilde{x}\in\mathbb{R}^m,\ |\widetilde{x}-\widetilde{x}_0|\le\delta\}$. ν is the outward unit normal vector to $\partial\Omega=\Gamma$, and $\frac{\partial}{\partial\nu_A}=\sum\limits_{i,j=1}^{n}a_{ij}\nu_i\frac{\partial}{\partial x_j}$. $Q=(0,T)\times\Omega$, $\Sigma=(0,T)\times\partial\Omega$, $Q_\omega=(0,T)\times\omega$, where ω is a subdomain of Ω. $\mathcal{L}(X,Y)$ is the space of linear continuous operators acting from a Banach space X into a Banach space Y, and $[L,A]=LA-AL$ is the commutator of operators L and A. Let

$$
\begin{aligned}
&D=\Big(\frac{1}{\mathrm{i}}\frac{\partial}{\partial x_0},\cdots,\frac{1}{\mathrm{i}}\frac{\partial}{\partial x_n}\Big),\quad D'=\Big(\frac{1}{\mathrm{i}}\frac{\partial}{\partial x_0},\cdots,\frac{1}{\mathrm{i}}\frac{\partial}{\partial x_{n-1}}\Big),\\
&\widetilde{D}=\Big(\frac{1}{\mathrm{i}}\frac{\partial}{\partial x_1},\cdots,\frac{1}{\mathrm{i}}\frac{\partial}{\partial x_n}\Big),\quad D_0=\frac{1}{\mathrm{i}}\frac{\partial}{\partial x_0},\\
&\nabla=\Big(\frac{\partial}{\partial x_0},\cdots,\frac{\partial}{\partial x_n}\Big),\quad \nabla'=\Big(\frac{\partial}{\partial x_0},\cdots,\frac{\partial}{\partial x_{n-1}}\Big),\quad \nabla_{\widetilde{x}}=\Big(\frac{\partial}{\partial x_1},\cdots,\frac{\partial}{\partial x_n}\Big),\\
&\xi'=(\xi_0,\cdots,\xi_{n-1}),\quad \widetilde{\xi}=(\xi_1,\cdots,\xi_n),\quad \xi=(\xi_0,\cdots,\xi_n),\\
&\widetilde{x}=(x_1,\cdots,x_n),\quad x=(x_0,\cdots,x_n),\quad x'=(x_0,\cdots,x_{n-1}).
\end{aligned}
$$

By $\widehat{u}(\xi')$ we denote the Fourier transform of the function $u(x')$:

$$\widehat{u}(\xi')=\frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}u(x')\mathrm{e}^{-\mathrm{i}\langle x',\xi'\rangle}\mathrm{d}x'.$$

We use the following functional spaces:

$$
\begin{aligned}
H^{1,2}(Q)&=\Big\{y\,\Big|\,\frac{\partial y}{\partial x_0},\frac{\partial^2 y}{\partial x_k\partial x_j},\frac{\partial y}{\partial x_k},y\in L^2(Q),\ \forall\,k,j\in\{1,\cdots,n\}\Big\},\\
W(Q)&=\Big\{y\,\Big|\,y,\frac{\partial y}{\partial x_i}\in L^2(Q),\ \forall\,i\in\{1,\cdots,n\},\ \frac{\partial y}{\partial x_0}\in L^2(0,T;H^{-1}(\Omega))\Big\},\\
H^{\frac14,\frac12}(\mathbb{R}^n)&=\Big\{y(x_1,\cdots,x_n)\,\Big|\,\|y\|^2_{H^{\frac14,\frac12}(\mathbb{R}^n)}=\int_{\mathbb{R}^n}\Big(1+|\xi_1|^{\frac12}+\sum_{i=2}^{n}|\xi_i|\Big)|\widehat{y}|^2\mathrm{d}\xi<+\infty\Big\}.
\end{aligned}
$$

The space $H^{\frac14,\frac12,s}(\mathbb{R}^n)$ is the space $H^{\frac14,\frac12}(\mathbb{R}^n)$ equipped with the norm

$$\|y\|_{H^{\frac14,\frac12,s}(\mathbb{R}^n)}=(\|y\|^2_{H^{\frac14,\frac12}(\mathbb{R}^n)}+|s|\|y\|^2_{L^2(\mathbb{R}^n)})^{\frac12}.$$

2 Statement of the Result

Let Ω be a bounded open set of $\mathbb{R}^n$ of class C^2 and let Γ be the boundary of Ω. We consider a solution $y\in W(Q)$ of the following linear second order parabolic equation:

$$
\begin{aligned}
L(x,D)y&=\frac{\partial y}{\partial x_0}-\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial y}{\partial x_j}\Big)+\sum_{j=1}^{n}b_j(x)\frac{\partial y}{\partial x_j}+\sum_{i=1}^{n}\frac{\partial}{\partial x_i}(c_i(x)y)+d(x)y\\
&=f+\sum_{j=1}^{n}\frac{\partial f_j}{\partial x_j},\quad \text{in } (0,T)\times\Omega, \qquad (2.1)\\
y&=g,\quad \text{on } (0,T)\times\Gamma, \qquad (2.2)
\end{aligned}
$$

where

$$a_{ij} \in C^2(\overline{Q}), \quad b_j, c_i, d \in L^\infty(Q) \quad \text{for } i,j \in \{1,\cdots,n\}, \tag{2.3}$$

$$a_{ij} = a_{ji} \quad \text{for } i,j \in \{1,\cdots,n\}, \tag{2.4}$$

and the coefficients a_{ij} satisfy the standard ellipticity condition

$$\exists\, \beta > 0,\ \forall\, \eta \in \mathbb{R}^n,\ \forall\, x \in \overline{Q}, \quad \sum_{i,j=1}^{n} a_{ij}(x)\eta_i\eta_j \geq \beta|\eta|^2. \tag{2.5}$$

On the other hand, we assume that

$$f \in L^2(Q), \quad f_j \in L^2(Q), \quad \forall\, j = 1,\cdots,n, \tag{2.6}$$

and g is the boundary value of a function in $W(Q)$. For the sake of simplicity, we will assume

$$g \in H^{\frac{1}{4},\frac{1}{2}}((0,T)\times\Gamma) = H^{\frac{1}{4},\frac{1}{2}}(\Sigma). \tag{2.7}$$

Our goal is to obtain a sharp global Carleman inequality for solutions of (2.1)–(2.2). In order to formulate our main result, we first have to introduce a suitable weight function.

Lemma 2.1 *Let ω be an arbitrary nonempty open set such that $\omega \subset \Omega$. Then there exists a function $\psi \in C^2(\overline{\Omega})$ such that*

$$\psi = 0, \quad \textit{on } \Gamma, \tag{2.8}$$

$$\psi(\widetilde{x}) > 0, \quad \forall\, \widetilde{x} \in \Omega, \tag{2.9}$$

$$|\nabla\psi(\widetilde{x})| > 0, \quad \forall\, \widetilde{x} \in \overline{\Omega \setminus \omega}. \tag{2.10}$$

Proof Let us consider a function $\theta \in C^2(\mathbb{R}^n)$ such that

$$\Omega = \{\widetilde{x} \mid \theta(\widetilde{x}) < 0\}, \quad |\nabla\theta(\widetilde{x})| \neq 0, \quad \forall\, \widetilde{x} \in \partial\Omega. \tag{2.11}$$

By the theorem on density of Morse functions (see [2]), there exists a sequence of Morse functions $\{\theta_k(\widetilde{x})\}_{k=1}^{\infty}$ such that

$$\theta_k \to \theta, \quad \text{in } C^2(\overline{\Omega}), \qquad \text{as } k \to +\infty. \tag{2.12}$$

We construct a Morse function $\mu \in C^2(\overline{\Omega})$ such that

$$\mu|_{\partial\Omega} = 0, \quad |\nabla\mu(\widetilde{x})| > 0, \quad \forall\, \widetilde{x} \in \partial\Omega. \tag{2.13}$$

We denote by $\mathcal{B} = \{\widetilde{x} \in \mathbb{R}^n \mid \nabla\theta(\widetilde{x}) = 0\}$ the set of critical points of functions θ. Since $|\nabla\theta||_{\partial\Omega} > 0$, there exists an open set $\Theta \subset \mathbb{R}^n$ such that

$$\overline{\Theta} \cap \mathcal{B} = \emptyset, \quad \partial\Omega \subset \Theta. \tag{2.14}$$

Let $e \in C_0^\infty(\Theta)$, $e|_{\partial\Omega} \equiv 1$. We set $\mu_k(\widetilde{x}) = \theta_k(\widetilde{x}) + e(\theta - \theta_k)(\widetilde{x})$. It is obvious that

$$\mu_k|_{\partial\Omega} = 0. \tag{2.15}$$

By the definition of function $e(\widetilde{x})$, we have

$$\nabla\mu_k(\widetilde{x}) = \nabla\theta_k(\widetilde{x}), \quad \forall\, \widetilde{x} \in \overline{\Omega \setminus \Theta}. \tag{2.16}$$

For all $\widetilde{x}$ from the set $\Theta \cap \Omega$, there holds

$$\nabla \mu_k(\widetilde{x}) = \nabla \theta_k(\widetilde{x}) + e(\widetilde{x})(\nabla \theta - \nabla \theta_k)(\widetilde{x}) + \nabla e(\widetilde{x})(\theta - \theta_k)(\widetilde{x}). \tag{2.17}$$

By (2.12) and (2.17), for all $\epsilon > 0$, there exists $k_0(\epsilon)$ such that on the set $\Theta \cap \Omega$, we have

$$|\nabla \mu_k(\widetilde{x})| \geq |\nabla \theta_k(\widetilde{x})| - \|e\|_{C^1(\overline{\Omega})} |(\nabla \theta - \nabla \theta_k)(\widetilde{x})| - \|e\|_{C^1(\overline{\Omega})} |(\theta - \theta_k)(\widetilde{x})| \geq |\nabla \theta_k(\widetilde{x})| - \epsilon, \tag{2.18}$$

where $k > k_0$.

From (2.12), (2.14), (2.16) and the last inequalities, there exist $\epsilon > 0$ and $\widehat{k}$ such that

$$|\nabla \mu_{\widehat{k}}(\widetilde{x})| > 0, \quad \text{in } \Theta \cap \Omega. \tag{2.19}$$

Set $\mu(\widetilde{x}) = \mu_{\widehat{k}}(\widetilde{x})$. By (2.15), (2.16) and (2.19), the Morse function $\mu_{\widehat{k}}(\widetilde{x})$ satisfies (2.13).

We denote by $\mathfrak{M}$ the set of critical points of function μ:

$$\mathfrak{M} = \{\widehat{x}_i \in \mathbb{R}^n \mid i = 1, \cdots, r\}.$$

Consider the sequence of functions $\{l_i\}_{i=1}^r \subset C^\infty([0,1]; \mathbb{R}^n)$, such that

$$l_i(t) \in \Omega, \ \ \forall\, t \in [0,1], \quad l_i(t_1) \neq l_i(t_2), \ \ \forall\, t_1, t_2 \in [0,1], \ t_1 \neq t_2, \ i = 1, \cdots, r, \tag{2.20}$$

$$l_i(1) = \widehat{x}_i, \quad l_i(0) \in \omega_0, \quad i = 1, \cdots, r, \tag{2.21}$$

$$l_i(t_1) \neq l_j(t_2), \quad \forall\, i \neq j, \ \forall\, t_1, t_2 \in [0,1], \tag{2.22}$$

where ω_0 is a nonempty open set such that $\overline{\omega}_0 \subset \omega$. By (2.20)–(2.22), there exists a sequence of functions $\{w^{(i)}\}_{i=1}^r \subset C^2(\mathbb{R}^n, \mathbb{R}^n)$ and $\{e_i\}_{i=1}^r \subset C_0^\infty(\Omega)$ such that

$$\frac{\mathrm{d} l_i(t)}{\mathrm{d}t} = w^{(i)}(l_i(t)), \quad \forall\, t \in [0,1], \ i = 1, \cdots, r, \tag{2.23}$$

$$\operatorname{supp} e_i \subset \Omega, \quad i = 1, \cdots, r, \tag{2.24}$$

$$\operatorname{supp} e_i \cap \operatorname{supp} e_j = \emptyset, \quad \forall\, i \neq j, \tag{2.25}$$

$$e_i(l_i(t)) = 1, \quad \forall\, t \in [0,1], \ i = 1, \cdots, r. \tag{2.26}$$

We set

$$V^{(i)}(\widetilde{x}) = e_i(\widetilde{x}) w^{(i)}(\widetilde{x}).$$

Consider the system of the ordinary differential equations

$$\frac{\mathrm{d}\widetilde{x}}{\mathrm{d}t} = V^{(i)}(\widetilde{x}), \quad \widetilde{x}(0) = \widetilde{x}_0. \tag{2.27}$$

We denote by $S_t^{(i)} : \mathbb{R}^n \to \mathbb{R}^n$ the operator such that $S_t^{(i)}(\widetilde{x}_0) = \widetilde{x}(t)$, where $\widetilde{x}(t)$ is the solution of problem (2.27).

By (2.21), (2.23) and (2.26), we have

$$S_1^{(i)}(l_i(0)) = \widehat{x}_i, \quad i = 1, \cdots, r.$$

Set

$$\psi(\widetilde{x}) = \mu(g_r(\widetilde{x})), \quad g_r(\widetilde{x}) = S_1^{(1)} \circ S_1^{(2)} \circ \cdots \circ S_1^{(r)}(\widetilde{x}). \tag{2.28}$$

By (2.24), there exists a domain $\mathfrak{S} \subset \mathbb{R}^n$ such that $\partial\Omega \subset \mathfrak{S}$ and

$$S_1^{(i)}(\widetilde{x}) = \widetilde{x}, \quad \forall\, \widetilde{x} \in \mathfrak{S}, \ i = 1, \cdots, r. \tag{2.29}$$

By (2.29), the mappings $S_1^{(i)}$ are diffeomorphisms on the domain Ω. Therefore g_r is a diffeomorphism on the domain Ω. By (2.29), $\psi(\widetilde{x}) = \mu(\widetilde{x})$, $\forall\, \widetilde{x} \in \mathfrak{S}$. Hence

$$\psi|_{\partial\Omega} = 0. \tag{2.30}$$

We denote by Ψ the set of critical points of function ψ. Since the mapping $g_r : \Omega \to \Omega$ is a diffeomorphism, we have

$$\Psi = \{\widetilde{x} \in \Omega \mid g_r(\widetilde{x}) \in \mathfrak{M}\}. \tag{2.31}$$

By (2.25) and (2.29), we have

$$g_r(l_i(0)) = \widehat{x}_i, \quad i = 1, \cdots, r. \tag{2.32}$$

It follows from (2.31) and (2.32) that $\Psi \subset \omega_0$. The proof of the lemma is complete.

We say that the function

$$L_2(x,\xi) = \mathrm{i}\xi_0 + \sum_{j,k=1}^{n} a_{jk}(x)\xi_j\xi_k$$

is the principal symbol of the operator $L(x,D)$.

For functions $f(x,\xi)$ and $g(x,\xi)$, we introduce the Poisson bracket

$$\{f,g\} = \sum_{j=0}^{n}\Big(\frac{\partial f}{\partial \xi_j}\frac{\partial g}{\partial x_j} - \frac{\partial g}{\partial \xi_j}\frac{\partial f}{\partial x_j}\Big).$$

Henceforth, Im and Re denote the imaginary and the real parts of a complex number, respectively.

Definition 2.1 *We say that the function $\alpha(x)$ is pseudoconvex with respect to the symbol $L_2(x,\xi)$ if there exists a constant $\widehat{C} > 0$ such that*

$$\frac{\mathrm{Im}\,\{\overline{L}_2(x,\xi_0,\overline{\widetilde{\zeta}}), L_2(x,\xi_0,\widetilde{\zeta})\}}{|s|} > 0, \quad \forall\,(x,\xi,s) \in \overline{Q\setminus Q_\omega} \times \mathcal{S},$$

where

$$\mathcal{S} = \{(x,\xi,s) \mid x \in \overline{Q\setminus Q_\omega},\ M(\xi,s) = 1,\ L_2(x,\xi_0,\widetilde{\zeta}) = 0\},$$
$$\widetilde{\zeta} = \Big(\xi_1 + \mathrm{i}|s|\frac{\partial\alpha}{\partial x_1}, \cdots, \xi_n + \mathrm{i}|s|\frac{\partial\alpha}{\partial x_n}\Big),$$
$$M(\xi,s) = \Big(\xi_0^2 + \sum_{i=1}^{n}\xi_i^4 + s^4\Big)^{\frac14}.$$

Now, using this function ψ, we construct two weight functions

$$\varphi(x) = \frac{\mathrm{e}^{\lambda\psi(\widetilde{x})}}{\ell^\kappa(x_0)}, \quad \alpha(x) = \frac{\mathrm{e}^{\lambda\psi(\widetilde{x})} - \mathrm{e}^{2\lambda\|\psi\|_{C^0(\overline{\Omega})}}}{\ell^\kappa(x_0)}, \tag{2.33}$$

where $\kappa \ge 2$, $\lambda \in \mathbb{R}$, $\lambda \ge 1$ will be chosen later on large enough, and

$$\ell \in C^\infty[0,T], \quad \ell(x_0) > 0, \quad \forall\, x_0 \in (0,T),$$
$$\ell(x_0) = x_0, \quad \forall\, x_0 \in \Big[0, \frac{T}{4}\Big], \quad \ell(x_0) = T - x_0, \quad x_0 \in \Big[\frac{3T}{4}, T\Big].$$

Proposition 2.1 *Let the function α be given by (2.33). Then there exist $\widehat{\lambda}$ and $\widehat{C}$ such that*

$$\mathrm{Im}\,\{\overline{L}_2(x,\xi_0,\overline{\widetilde{\zeta}}), L_2(x,\xi_0,\widetilde{\zeta})\} \ge \widehat{C}|s|\lambda^4 \frac{\mathrm{e}^{\lambda\psi(\widetilde{x})}}{\ell^\kappa(x_0)} M^2\Big(\xi, s\frac{\mathrm{e}^{\lambda\psi(\widetilde{x})}}{\ell^\kappa(x_0)}\Big) \tag{2.34}$$

for all $(x,\xi,s)\in\mathcal{S}$ and $\lambda\ge\widehat{\lambda}$. Here $\widehat{\lambda}$ is independent of s and $\widehat{C}$ is independent of s and λ.

Proof We introduce the following notations:

$$p^{(j)}(x,\xi) = \partial_{\xi_j}p(x,\xi), \quad p^{(j,i)}(x,\xi) = \frac{\partial^2 p(x,\xi)}{\partial\xi_i\partial\xi_j}, \quad p_{(j)}(x,\xi) = \frac{\partial p(x,\xi)}{\partial x_j},$$

$$\widetilde{\nabla}\alpha = \Big(0, \frac{\partial\alpha}{\partial x_1}, \cdots, \frac{\partial\alpha}{\partial x_n}\Big), \quad a(x,\widetilde{\xi},\widetilde{\eta}) = \sum_{i,j=1}^n a_{ij}\xi_i\eta_j.$$

After short computations, we obtain

$$\frac{\partial}{\partial\xi_0}L_2(x,\xi_0,\widetilde{\zeta}) = \mathrm{i}, \quad \frac{\partial}{\partial\xi_0}\overline{L}_2(x,\xi_0,\widetilde{\zeta}) = -\mathrm{i},$$

$$\frac{\partial}{\partial x_m}L_2(x,\xi_0,\eta) = L_{2,(m)}(x,\xi_0,\widetilde{\zeta}) + \mathrm{i}|s|\sum_{k=1}^n L_2^{(k)}(x,\xi_0,\widetilde{\zeta})\frac{\partial^2\alpha}{\partial x_k\partial x_m},$$

$$\frac{\partial}{\partial x_m}\overline{L}_2(x,\xi_0,\overline{\widetilde{\zeta}}) = L_{2,(m)}(x,\xi_0,\overline{\widetilde{\zeta}}) - \mathrm{i}|s|\sum_{k=1}^n L_2^{(k)}(x,\xi_0,\overline{\widetilde{\zeta}})\frac{\partial^2\alpha}{\partial x_k\partial x_m}.$$

Then

$$\begin{aligned}&\mathrm{Im}\,\{\overline{L}_2(x,\xi_0,\overline{\widetilde{\zeta}}), L_2(x,\xi_0,\widetilde{\zeta})\}\\ &= \mathrm{Im}\Big(\sum_{k=0}^n \overline{L}_2^{(k)}(x,\xi_0,\overline{\widetilde{\zeta}})L_{2,(k)}(x,\xi_0,\widetilde{\zeta}) - \overline{L}_{2,(k)}(x,\xi_0,\overline{\widetilde{\zeta}})L_2^{(k)}(x,\xi_0,\widetilde{\zeta})\Big).\end{aligned}$$

Simple computations provide the following formulas:

$$\begin{aligned}&\mathrm{Im}\Big(\frac{\partial}{\partial\xi_0}\overline{L}_2(x,\xi_0,\overline{\widetilde{\zeta}})\frac{\partial}{\partial x_0}L_2(x,\xi_0,\widetilde{\zeta}) - \frac{\partial}{\partial x_0}\overline{L}_2(x,\xi_0,\overline{\widetilde{\zeta}})\frac{\partial}{\partial\xi_0}L_2(x,\xi_0,\widetilde{\zeta})\Big)\\ &= \mathrm{Im}\Big(-\mathrm{i}\Big(L_{2,(0)}(x,\xi_0,\widetilde{\zeta}) + \mathrm{i}|s|\sum_{m=1}^n L_2^{(m)}(x,\xi_0,\widetilde{\zeta})\frac{\partial^2\alpha}{\partial x_m\partial x_0}\Big)\\ &\quad - \mathrm{i}\Big(L_{2,(0)}(x,\xi_0,\overline{\widetilde{\zeta}}) - \mathrm{i}|s|\sum_{m=1}^n L_2^{(m)}(x,\xi_0,\overline{\widetilde{\zeta}})\frac{\partial^2\alpha}{\partial x_m\partial x_0}\Big)\Big)\\ &= -2L_{2,(0)}(x,\xi) + 2s^2\sum_{m=1}^n L_2^{(m)}(x,\widetilde{\nabla}\alpha)\frac{\partial^2\alpha}{\partial x_m\partial x_0} + 2s^2 a_{x_0}(x,\nabla_{\widetilde{x}}\alpha,\nabla_{\widetilde{x}}\alpha),\end{aligned}$$

where $a_{x_0}(x,\widetilde{\eta},\widetilde{\eta}) = \sum_{i,j=1}^n \frac{\partial a_{ij}}{\partial x_0}\eta_i\eta_j$ and

$$\begin{aligned}&\mathrm{Im}\Big(\frac{\partial}{\partial\xi_k}\overline{L}_2(x,\xi_0,\overline{\widetilde{\zeta}})\frac{\partial}{\partial x_k}L_2(x,\xi_0,\widetilde{\zeta}) - \frac{\partial}{\partial x_k}\overline{L}_2(x,\xi_0,\overline{\widetilde{\zeta}})\frac{\partial}{\partial\xi_k}L_2(x,\xi_0,\widetilde{\zeta})\Big)\\ &= \mathrm{Im}\Big(\overline{L}_2^{(k)}(x,\xi_0,\overline{\widetilde{\zeta}})\Big(L_{2,(k)}(x,\xi_0,\widetilde{\zeta}) + \mathrm{i}|s|\sum_{m=1}^n L_2^{(m)}(x,\xi_0,\widetilde{\zeta})\frac{\partial^2\alpha}{\partial x_k\partial x_m}\Big)\\ &\quad - L_2^{(k)}(x,\xi_0,\widetilde{\zeta})\Big(L_{2,(k)}(x,\xi_0,\overline{\widetilde{\zeta}}) - \mathrm{i}|s|\sum_{m=1}^n L_2^{(k)}(x,\xi_0,\overline{\widetilde{\zeta}})\frac{\partial^2\alpha}{\partial x_k\partial x_m}\Big)\Big)\end{aligned}$$

$$
\begin{aligned}
&= -L_2^{(k)}(x,|s|\widetilde{\nabla}\alpha)(L_{2,(k)}(x,\xi)-L_{2,(k)}(x,|s|\widetilde{\nabla}\alpha))+L_2^{(k)}(x,\xi)\mathrm{Im}\,L_{2,(k)}(x,\xi_0,\widetilde{\zeta})\\
&\quad+|s|L_2^{(k)}(x,\xi)\sum_{m=1}^{n}L_2^{(m)}(x,\xi)\frac{\partial^2\alpha}{\partial x_k\partial x_m}+|s|L_2^{(k)}(x,|s|\widetilde{\nabla}\alpha)\sum_{m=1}^{n}L_2^{(m)}(x,|s|\widetilde{\nabla}\alpha)\frac{\partial^2\alpha}{\partial x_k\partial x_m}\\
&\quad-L_2^{(k)}(x,|s|\widetilde{\nabla}\alpha)(L_{2,(k)}(x,\xi)-L_{2,(k)}(x,|s|\widetilde{\nabla}\alpha))+L_2^{(k)}(x,\xi)\mathrm{Im}\,L_{2,(k)}(x,\xi_0,\widetilde{\zeta})\\
&\quad+|s|L_2^{(k)}(x,\xi)\sum_{m=1}^{n}L_2^{(m)}(x,\xi)\frac{\partial^2\alpha}{\partial x_k\partial x_m}+|s|L_2^{(k)}(x,|s|\widetilde{\nabla}\alpha)\sum_{m=1}^{n}L_2^{(m)}(x,|s|\widetilde{\nabla}\alpha)\frac{\partial^2\alpha}{\partial x_k\partial x_m}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\frac{1}{2}\mathrm{Im}\,\{\overline{L}_2(x,\xi_0,\overline{\widetilde{\zeta}}),L_2(x,\xi_0,\widetilde{\zeta})\}\\
&=\frac{1}{2}\mathrm{Im}\Big(\sum_{k=0}^{n}\frac{\partial}{\partial\xi_k}\overline{L}_2(x,\xi_0,\overline{\widetilde{\zeta}})\frac{\partial}{\partial x_k}L_{2,(k)}(x,\xi_0,\widetilde{\zeta})-\frac{\partial}{\partial x_k}\overline{L}_2(x,\xi_0,\overline{\widetilde{\zeta}})\frac{\partial}{\partial\xi_k}L_2(x,\xi_0,\widetilde{\zeta})\Big)\\
&=-L_{2,(0)}(x,\xi)+s^2\sum_{k=1}^{n}L_2^{(k)}(x,\widetilde{\nabla}\alpha)\frac{\partial^2\alpha}{\partial x_k\partial x_0}+2s^2a_{x_0}(x,\nabla_{\widetilde{x}}\alpha,\nabla_{\widetilde{x}}\alpha)\\
&\quad-L_2^{(k)}(x,|s|\widetilde{\nabla}\alpha)(L_{2,(k)}(x,\xi)-L_2^{(k)}(x,|s|\widetilde{\nabla}\alpha))+L_2^{(k)}(x,|s|\widetilde{\nabla}\alpha)\mathrm{Im}\,L_{2,(k)}(x,\xi_0,\widetilde{\zeta})\\
&\quad+\sum_{m,k=1}^{n}(|s|L_2^{(k)}(x,\xi)L_2^{(m)}(x,\xi)+|s|L_2^{(k)}(x,|s|\widetilde{\nabla}\alpha)L_2^{(m)}(x,|s|\widetilde{\nabla}\alpha))\frac{\partial^2\alpha}{\partial x_k\partial x_m}.
\end{aligned}
$$

Observing that $\frac{\partial^2\alpha}{\partial x_k\partial x_m}=(\lambda^2\frac{\partial\psi}{\partial x_k}\frac{\partial\psi}{\partial x_m}+\lambda\frac{\partial^2\psi}{\partial x_k\partial x_m})\frac{\mathrm{e}^{\lambda\psi}}{\ell^\kappa}$, we have

$$
\begin{aligned}
\mathrm{I}&=\sum_{m,k=1}^{n}(|s|L_2^{(k)}(x,\xi)L_2^{(m)}(x,\xi)+|s|L_2^{(k)}(x,|s|\widetilde{\nabla}\alpha)L_2^{(m)}(x,|s|\widetilde{\nabla}\alpha))\frac{\partial^2\alpha}{\partial x_k\partial x_m}\\
&=\lambda^2|s|\Big(a(x,\widetilde{\xi},\nabla_{\widetilde{x}}\psi)^2+s^2\frac{\mathrm{e}^{2\lambda\psi(x)}}{\ell^{2\kappa}}a(x,\nabla_{\widetilde{x}}\psi,\nabla_{\widetilde{x}}\psi)^2\Big)\frac{\mathrm{e}^{\lambda\psi}}{\ell^\kappa}\\
&\quad+\sum_{m,k=1}^{n}(|s|L_2^{(k)}(x,\xi)L_2^{(m)}(x,\xi)+|s|L_2^{(k)}(x,|s|\widetilde{\nabla}\alpha)L_2^{(m)}(x,|s|\widetilde{\nabla}\alpha))\lambda\psi_{x_kx_m}\frac{\mathrm{e}^{\lambda\psi}}{\ell^\kappa}.
\end{aligned}
$$

Since $(x,\xi,s)\in\mathcal{S}$, the following inequality holds:

$$
a(x,\widetilde{\xi},\nabla_{\widetilde{x}}\alpha)^2=s^2a(x,\nabla_{\widetilde{x}}\alpha,\nabla_{\widetilde{x}}\alpha)^2\geq\widehat{C}\Big|\Big(\widetilde{\xi},s\frac{\mathrm{e}^{\lambda\psi}}{\ell^\kappa}\Big)\Big|^2.
$$

Taking $\widehat{\lambda}$ sufficiently large, for all $\lambda\geq\widehat{\lambda}$, we have

$$
\mathrm{I}\geq\frac{\lambda^4}{2}\widehat{C}|s|\frac{\mathrm{e}^{\lambda\psi}}{\ell^\kappa}\Big|\Big(\widetilde{\xi},s\frac{\mathrm{e}^{\lambda\psi}}{\ell^\kappa}\Big)\Big|^2,\quad\forall\,(x,\xi,s)\in\mathcal{S},\tag{2.35}
$$

where $\widehat{C}$ is independent of (λ,x,ξ,s).

Finally, observing

$$
|\xi_0|\leq|a(x,\widetilde{\xi},|s|\nabla_{\widetilde{x}}\alpha)|,\quad\forall\,(x,\xi,s)\in\mathcal{S},
$$

from (2.35) we find

$$
\mathrm{I}\geq\frac{\lambda^4}{2}\widehat{C}\frac{\mathrm{e}^{\lambda\psi(\widetilde{x})}}{\ell^\kappa(x_0)}|s|M^2\Big(\xi,s\frac{\mathrm{e}^{\lambda\psi(\widetilde{x})}}{\ell^\kappa(x_0)}\Big),\quad\forall\,(x,\xi,s)\in\mathcal{S},\tag{2.36}
$$

where $\widehat{C}$ is independent of (λ, x, ξ, s). On the other hand,

$$\Big| - L_{2,(0)}(x,\xi) + 2s^2 a_{x_0}(x, \nabla_{\widetilde{x}}\alpha, \nabla_{\widetilde{x}}\alpha) + s^2 \sum_{k=1}^{n} L_2^{(k)}(x, \widetilde{\nabla}\alpha)\frac{\partial^2\alpha}{\partial x_k \partial x_0}$$
$$- L_2^{(m)}(x,|s|\widetilde{\nabla}\alpha)(L_{2,(m)}(x,\xi) - L_{2,(m)}(x,|s|\widetilde{\nabla}\alpha)) + L_2^{(m)}(x,|s|\widetilde{\nabla}\alpha)\mathrm{Im}\, L_{2,(m)}(x,\xi_0,\widetilde{\zeta})\Big|$$
$$\le C|s|\lambda^2 \frac{\mathrm{e}^{\lambda\psi}}{\ell^\kappa} M^2\Big(\xi, s\frac{\mathrm{e}^{\lambda\psi}}{\ell^\kappa}\Big). \tag{2.37}$$

Inequalities (2.37) and (2.36) imply (2.34).

We formulate our main Carleman estimate for the parabolic equation.

Theorem 2.1 *Assume that* (2.3)–(2.6) *hold. Let $y \in W(Q)$ be a solution of* (2.1)–(2.2). *Then there exists a constant $\widehat{\lambda}$, such that for any $\lambda > \widehat{\lambda}$ there exist constants $C > 0$ independent of s and $s_0(\lambda)$, such that*

$$\frac{1}{s}\int_Q \frac{1}{\varphi}\sum_{j=1}^n \Big|\frac{\partial y}{\partial x_j}\Big|^2 \mathrm{e}^{2s\alpha}\mathrm{d}x + s\int_Q \varphi|y|^2\mathrm{e}^{2s\alpha}\mathrm{d}x$$
$$\le C\Big(s^{-\frac12}\|\varphi^{-\frac14} g\mathrm{e}^{s\alpha}\|^2_{H^{\frac14,\frac12}(\Sigma)} + s^{-\frac12}\|\varphi^{-\frac14+\frac1\kappa} g\mathrm{e}^{s\alpha}\|^2_{L^2(\Sigma)} + \frac{1}{s^2}\int_Q \frac{|f|^2}{\varphi^2}\mathrm{e}^{2s\alpha}\mathrm{d}x$$
$$+ \sum_{j=1}^n \int_Q |f_j|^2 \mathrm{e}^{2s\alpha}\mathrm{d}x + \int_{Q_\omega} s\varphi|y|^2\mathrm{e}^{2s\alpha}\mathrm{d}x\Big), \quad \forall\, s \ge s_0 > 0. \tag{2.38}$$

Remark 2.1 (1) By a density argument, it suffices to prove the result when the solution y is assumed to be more regular, namely $y \in H^{1,2}(Q)$ and the right-hand side has a compact support. Actually, there exists a sequence of $\{f^k, f_1^k, \cdots, f_n^k, g^k\} \in (C_0^\infty(Q))^{n+1} \times C_0^\infty(\Sigma)$ converging to $\{f, f_1, \cdots, f_n, g\}$ in $L^2(Q)^{n+1} \times H^{\frac14,\frac12}(\Sigma)$, such that the corresponding solution y^k to problem (2.1)–(2.2) with right-hand side $f^k + \sum\limits_{j=1}^n \frac{\partial f_j^k}{\partial x_j}$ and boundary condition g^k satisfies $y^k \in H^{1,2}(Q)$ and

$$y^k \to y, \quad \text{in } L^2(0,T;H^1(\Omega)).$$

Hence, it suffices to prove the estimate (2.38) for solutions $y \in H^{1,2}(Q)$ and the terms of the right-hand side which have compact supports in Q and Σ.

(2) Without loss of generality, it is sufficient to consider the case where $b_j = 0$, $c_i = 0$ and $d = 0$ as the first and zero order terms in (2.1) can be added to the right-hand side and the corresponding terms in (2.38) can be absorbed by the terms on the left-hand side by choosing $\widehat{s}$ and $\widehat{\lambda}$ large enough.

The proof of Theorem 2.1 is divided into several steps and will be the content of the next subsections.

2.1 Localization in space and time

For every $\delta > 0$, we can consider a covering of $\overline{Q} = [0,T] \times \overline{\Omega}$ as follows:

$$\overline{Q} \subset \overline{Q}_0 \cup \Big(\bigcup_{k=1}^{I} B(\widehat{x}_k, \delta)\Big), \tag{2.39}$$

where $Q_0 = (0,T)\times\Omega_0$, $\overline{\Omega}_0 \subset \Omega$, $\widehat{x}_k \in (0,T)\times\partial\Omega$.

Let $(e_k)_{k=0}^{I}$ be a corresponding partition of unity, i.e.,

$$e_0 \in C_0^\infty(Q_0), \quad e_k \in C_0^\infty(B(\widehat{x}_k,\delta)), \quad k = 1,\cdots,I,$$
$$e_k(x) \ge 0, \quad k = 1,\cdots,I, \quad \sum_{k=0}^{I} e_k(x) = 1, \quad \forall\, x \in \overline{Q}.$$

We now define

$$y_k(x) = y(x)e_k(x), \quad k = 0,\cdots,I.$$

Then if $L(x,D)y = \frac{\partial y}{\partial x_0} - \sum\limits_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{ij}\frac{\partial y}{\partial x_j})$ (recall that from the previous remark this corresponds to the general case), we have

$$L(x,D)y_k = [L(x,D),e_k]y + e_k f + \sum_{j=1}^{n}\frac{\partial(e_k f_j)}{\partial x_j} - \sum_{j=1}^{n} f_j\frac{\partial e_k}{\partial x_j}, \quad \text{in } Q, \tag{2.40}$$
$$y_k = ge_k, \quad \text{on } (0,T)\times\Gamma \tag{2.41}$$

for each $k = 0,\cdots,I$, and

$$\operatorname{supp} y_0 \subset Q_0, \quad \operatorname{supp} y_k \subset B(\widehat{x}_k,\delta), \quad k = 1,\cdots,I. \tag{2.42}$$

Notice that the commutator $[L,e_k]$ is a first order operator and that $e_k f_j$ and $f_j\frac{\partial e_k}{\partial x_j}$ have compact supports in $B(\widehat{x}_k,\delta)$ for all $k = 1,\cdots,I$.

Assume that the assertion of Theorem 2.1 is true with the additional assumption that

$$\operatorname{supp} y \subset Q_0 \quad \text{or} \quad \operatorname{supp} y \subset B(\widehat{x},\delta), \quad \widehat{x} \in (0,T)\times\partial\Omega.$$

Then, as $y = \sum\limits_{k=0}^{I} y_k$, we have (the letter C denotes various constants independent of s)

$$\begin{aligned}
&\int_Q \Big(\frac{1}{s\varphi}\sum_{j=1}^{n}\Big|\frac{\partial y}{\partial x_j}\Big|^2 + s\varphi|y|^2\Big)\mathrm{e}^{2s\alpha}\mathrm{d}x \le C\sum_{k=0}^{I}\int_Q\Big(\frac{1}{s\varphi}\sum_{j=1}^{n}\Big|\frac{\partial y_k}{\partial x_j}\Big|^2 + s\varphi|y_k|^2\Big)\mathrm{e}^{2s\alpha}\mathrm{d}x\\
&\le C\sum_{k=0}^{I}\Big(s^{-\frac12}\|\varphi^{-\frac14}ge_k\mathrm{e}^{s\alpha}\|^2_{H^{\frac14,\frac12}(\Sigma)} + s^{-\frac12}\|\varphi^{-\frac14+\frac1\kappa}ge_k\mathrm{e}^{s\alpha}\|^2_{L^2(\Sigma)} + \int_Q\frac{|fe_k|^2}{s^2\varphi^2}\mathrm{e}^{2s\alpha}\mathrm{d}x\\
&\quad + \sum_{j=1}^{n}\int_Q|f_je_k|^2\mathrm{e}^{2s\alpha}\mathrm{d}x + \frac{1}{s^2}\sum_{j=1}^{n}\int_Q\frac{|f_j|^2}{\varphi^2}\mathrm{e}^{2s\alpha}\mathrm{d}x + \int_Q|y|^2\mathrm{e}^{2s\alpha}\mathrm{d}x + \int_{Q_\omega}s\varphi|y_k|^2\mathrm{e}^{2s\alpha}\mathrm{d}x\Big)\\
&\le C\Big(s^{-\frac12}\|\varphi^{-\frac14}g\mathrm{e}^{s\alpha}\|^2_{H^{\frac14,\frac12}(\Sigma)} + s^{-\frac12}\|\varphi^{-\frac14+\frac1\kappa}g\mathrm{e}^{s\alpha}\|^2_{L^2(\Sigma)} + \frac{1}{s^2}\int_Q\frac{|f|^2}{\varphi^2}\mathrm{e}^{2s\alpha}\mathrm{d}x\\
&\quad + \sum_{j=1}^{n}\int_Q|f_j|^2\mathrm{e}^{2s\alpha}\mathrm{d}x + \int_Q|y|^2\mathrm{e}^{2s\alpha}\mathrm{d}x + \int_{Q_\omega}s\varphi|y|^2\mathrm{e}^{2s\alpha}\mathrm{d}x\Big).
\end{aligned}\tag{2.43}$$

Now taking $\widehat{s}$ sufficiently large, we obtain (2.38) for $s \ge \widehat{s}$. Therefore, it is enough to prove Theorem 2.1 in two cases:

(i) $\operatorname{supp} y \subset Q_0$,

(ii) $\operatorname{supp} y \subset B(\widehat{x},\delta)$, $\widehat{x} \in (0,T)\times\partial\Omega$.

Case (i) immediately follows from [13]. So, we will concentrate on Case (ii).

2.2 Change of coordinates

Let us take $\widehat{x} \in (0,T)\times\partial\Omega$, $\delta > 0$ and a solution y of (2.1)–(2.2) such that $\operatorname{supp} y \subset B(\widehat{x},\delta)$. By (2.10), there exists a $\delta > 0$ sufficiently small, such that for some index $i_0 \in \{1,\cdots,n\}$,

$$\frac{\partial\psi}{\partial x_{i_0}}(\widetilde{x}) \neq 0, \quad \forall\,\widetilde{x} \quad \text{such that } (x_0,\widetilde{x}) \in B(\widehat{x},\delta).$$

After renumbering, we can assume that $i_0 = n$ and without loss of generality, we can assume that

$$\frac{\partial\psi}{\partial x_n}(\widetilde{x}) \neq 0, \quad \forall\,\widetilde{x} \quad \text{such that } (x_0,\widetilde{x}) \in B(\widehat{x},\delta). \tag{2.44}$$

We now take the new coordinate system

$$\widehat{x}_n = \psi(x_1,\cdots,x_n), \quad \widehat{x}_i = x_i, \quad i = 0,\cdots,n-1. \tag{2.45}$$

Since $\psi = 0$ on Γ, in the new coordinate system $((0,T)\times\partial\Omega)\cap B(\widehat{x},\delta)$ corresponds to $\widehat{x}_n = 0$. Writing

$$\widehat{y}(\widehat{x}_0,\widehat{x}_1,\cdots,\widehat{x}_n) = y(x_0,x_1,\cdots,x_n),$$

from (2.1)–(2.2), we find

$$\begin{aligned}\widehat{L}(\widehat{x},D)\widehat{y} &= a_0\frac{\partial\widehat{y}}{\partial x_0} - \frac{\partial^2\widehat{y}}{\partial\widehat{x}_n^2} - \sum_{j=1}^{n-1}\widehat{a}_{nj}\frac{\partial^2\widehat{y}}{\partial\widehat{x}_n\partial\widehat{x}_j} - \widehat{A}\widehat{y} + \widehat{B}\widehat{y}\\ &= \widehat{f} + \sum_{j=1}^{n}\frac{\partial\widehat{f}_j}{\partial\widehat{x}_j}, \quad \text{in } [0,T]\times\mathbb{R}^{n-1}\times(0,\gamma),\end{aligned} \tag{2.46}$$

$$\widehat{y}(\widehat{x}',0) = \widehat{g}(\widehat{x}'), \quad \text{on } [0,T]\times\mathbb{R}^{n-1}, \quad \widehat{x}' = (\widehat{x}_0,\cdots,\widehat{x}_{n-1}), \tag{2.47}$$

and $\widehat{y}$ vanishes in a neighborhood of the set $(\partial B'(0,\delta)\times[0,\gamma])\cup(B'(0,\delta)\times\{\gamma\})$ with $B'(0,\delta) = \{\widehat{x}' \in \mathbb{R}^n \mid |\widehat{x}'| \le \delta\}$ and $\widehat{x}' = (\widehat{x}_0,\cdots,\widehat{x}_{n-1})$.

Now we want to show an inequality analogous to (2.38) corresponding to the weight function

$$\widehat{\varphi}(\widehat{x}) = \mathrm{e}^{\lambda\widehat{x}_n}. \tag{2.48}$$

More precisely, we want to show that there exists a constant $\widehat{\lambda}$ such that for any $\lambda > \widehat{\lambda}$, there exist s_0 and $\widehat{C} > 0$ independent of s such that for all $s \ge s_0$,

$$\begin{aligned}&\int_{[0,T]\times\mathbb{R}^{n-1}\times[0,\gamma]}\Big(\frac{1}{s\widehat{\varphi}}\sum_{j=1}^{n}\Big|\frac{\partial\widehat{y}}{\partial x_j}\Big|^2 + s\widehat{\varphi}|\widehat{y}|^2\Big)\mathrm{e}^{2s\widehat{\alpha}}\mathrm{d}\widehat{x}\\ \le{}& \widehat{C}\Big(s^{-\frac12}\|\widehat{\varphi}^{-\frac14}\widehat{g}\mathrm{e}^{s\widehat{\alpha}}\|^2_{H^{\frac14,\frac12}([0,T]\times\mathbb{R}^{n-1})} + s^{-\frac12}\|\widehat{\varphi}^{-\frac14+\frac1\kappa}\widehat{g}\mathrm{e}^{s\widehat{\alpha}}\|^2_{L^2([0,T]\times\mathbb{R}^{n-1})}\\ &+ \int_{[0,T]\times\mathbb{R}^{n-1}\times[0,\gamma]}\Big(\frac{|\widehat{f}|^2}{s^2\widehat{\varphi}^2} + \sum_{j=1}^{n}|\widehat{f}_j|^2\Big)\mathrm{e}^{2s\widehat{\alpha}}\mathrm{d}\widehat{x}\Big).\end{aligned} \tag{2.49}$$

In (2.46), the operator $\widehat{A}$ has the form

$$\widehat{A}\widehat{y} = \sum_{i,j=1}^{n-1}\widehat{a}_{ij}(\widehat{x})\frac{\partial^2\widehat{y}}{\partial\widehat{x}_i\partial\widehat{x}_j}, \tag{2.50}$$

and the operator $\widehat{B}$ is a first order differential operator with L^∞ coefficients. We have already seen that we can ignore the first order terms. We also omit from now on the notation $\widehat{\ }$. Then we can write

$$
\begin{aligned}
L(x,D)y &= a_0\frac{\partial y}{\partial x_0} - \frac{\partial^2 y}{\partial x_n^2} - \sum_{j=1}^{n-1} a_{nj}\frac{\partial^2 y}{\partial x_n \partial x_j} - Ay \\
&= f + \sum_{j=1}^{n}\frac{\partial f_j}{\partial x_j}, \quad \text{in } [0,T]\times\mathbb{R}^{n-1}\times(0,\gamma), \qquad (2.51)\\
y(x',0) &= g(x'), \quad \text{on } [0,T]\times\mathbb{R}^{n-1}, \qquad (2.52)
\end{aligned}
$$

and

$$
y \text{ vanishes in the neigborhood of the set } (\partial B'(0,\delta)\times[0,\gamma])\cup(B'(0,\delta)\times\{\gamma\}). \qquad (2.53)
$$

Here $a_0 \in C^1(\overline{Q})$ is a strictly positive function. Notice that f, f_j also have compact supports in $B'(0,\delta)\times[0,\gamma)$ and that g has a compact support in $B'(0,\delta)$. Moreover, if we write

$$
\widetilde{a}(x,\xi^1,\xi^2) = \sum_{i,j=1}^{n-1} a_{ij}(x)\xi_i^1\xi_j^2 \qquad (2.54)
$$

for $x \in \mathbb{R}^{n+1}$ and $\xi^1, \xi^2 \in \mathbb{R}^{n-1}$, we have an ellipticity condition corresponding to (2.5), namely, there exists $\beta > 0$ such that for all $\widetilde{\xi} \in \mathbb{R}^n$,

$$
\xi_n^2 + \sum_{j=1}^{n-1} a_{nj}(x)\xi_n\xi_j + \widetilde{a}(x,\xi^*,\xi^*) \ge \beta|\widetilde{\xi}|^2, \quad \forall\, x \in \Pi_{\delta,\gamma} = B'(0,\delta)\times[0,\gamma], \qquad (2.55)
$$

where $\xi^* = (\xi_1,\cdots,\xi_{n-1}) \in \mathbb{R}^{n-1}$. This shows that

$$
\exists\,\widehat{\gamma} > 0,\ \forall\,\xi^* \in \mathbb{R}^{n-1}, \quad |\xi^*| = 1, \quad 4\widetilde{a}(x,\xi^*,\xi^*) - \Big(\sum_{j=1}^{n-1} a_{nj}(x)\xi_j\Big)^2 \ge \widehat{\gamma}. \qquad (2.56)
$$

2.3 Localization in time

From now on, it is convenient for us to work with the function $w(x) = \mathrm{e}^{|s|\alpha}y(x)$ instead of y. The function w verifies the equation

$$
\begin{aligned}
&L(x,D_0,\widetilde{D} + \mathrm{i}|s|\nabla_{\widetilde{x}}\alpha)w = F, \quad \text{in } [0,T]\times\mathbb{R}^{n-1}\times(0,\gamma), \qquad (2.57)\\
&w(x',0) = g\mathrm{e}^{|s|\alpha}, \qquad (2.58)\\
&\operatorname{supp} w \subset \Pi_{\delta,\gamma} = B'(0,\delta)\times[0,\gamma), \qquad (2.59)
\end{aligned}
$$

where $F(x) = F_0 + \sum\limits_{k=1}^{n}\frac{\partial F_k}{\partial x_k}$, $F_0 = \mathrm{e}^{|s|\alpha}f - \sum\limits_{j=1}^{n}|s|\frac{\partial\alpha}{\partial x_j}f_j\mathrm{e}^{|s|\alpha} - |s|\frac{\partial\alpha}{\partial x_0}w$, $F_k = f_k\mathrm{e}^{|s|\alpha}$. Since the function α has singularities at $x_0 = 0$ and $x_0 = T$, it is the same for some coefficients of equation (2.57). We overcome this difficulty by using a localization in time.

Let $\widetilde{\psi} \in C_0^\infty(\frac{1}{2},2)$ be a nonnegative function such that $\sum\limits_{j=-\infty}^{\infty}\widetilde{\psi}(2^{-j}x_0) = 1$ (we may take $\widetilde{\psi}(x_0) = \widetilde{\beta}(x_0) - \widetilde{\beta}(2x_0)$ where $\widetilde{\beta} \in C_0^\infty(-2,2)$ and $\widetilde{\beta}(x_0)$ equals 1 for $|x_0| \le 1$). Denote $\widetilde{\psi}_j(x_0) =$

$\widetilde{\psi}(2^{-j}x_0)$, $\mu_j(x_0)=\widetilde{\psi}(\frac{2^{-j}}{\ell^\kappa(x_0)})$, $\widetilde{\mu}_j(x_0)=\widetilde{\psi}'(\frac{2^{-j}}{\ell^\kappa(x_0)})$, $\Psi_j(x_0)=\mu_{j+1}(x_0)+\mu_j(x_0)+\mu_{j-1}(x_0)$ and $w_j(x)=\mu_j(x_0)w(x)$, $F_{k,j}(x)=\mu_j(x_0)F_k(x)$, $g_j(x)=\mu_j(x_0)g(x)\mathrm{e}^{|s|\alpha}$.

The function w_j satisfies the equation

$$L(x,D_0,\widetilde{D}+\mathrm{i}|s|\nabla_{\widetilde{x}}\alpha)w_j=\frac{\partial\mu_j}{\partial x_0}w+F_{0,j}+\sum_{k=1}^n\frac{\partial F_{k,j}}{\partial x_k},\quad \text{in } G=\mathbb{R}^n\times(0,\gamma), \tag{2.60}$$

$$w_j(x_0,\cdots,x_{n-1},0)=g_j, \tag{2.61}$$

$$\operatorname{supp} w_j\subset\Pi_{\delta,\gamma}. \tag{2.62}$$

Suppose that for a function w_j, the following Carleman estimate is already established:

$$\begin{aligned}&\sum_{k=1}^n\Big\|s^{-\frac12}\varphi^{-\frac12}\frac{\partial w_j}{\partial x_k}\Big\|_{L^2(G)}+\|s^{\frac12}\varphi^{\frac12}w_j\|_{L^2(G)}\\ &\le C\Big(\|s^{-\frac14}\varphi^{-\frac14}g_j\|_{H^{\frac14,\frac12}(\mathbb{R}^n)}+\|s^{-\frac14}\varphi^{-\frac14+\frac1\kappa}g_j\|_{L^2(\mathbb{R}^n)}\\ &\quad+\Big\|\frac{\frac{\partial\mu_j}{\partial x_0}w+F_{0,j}}{s\varphi}\Big\|_{L^2(G)}+\sum_{k=1}^n\|F_{k,j}\|_{L^2(G)}\Big),\quad\forall\, s>s_0>1,\end{aligned} \tag{2.63}$$

where C is independent of s and j, and s_0 is independent of j.

Observe that

$$\operatorname{supp}\mu_j\cap\operatorname{supp}\mu_k\neq\emptyset,\quad \operatorname{supp}\widetilde{\mu}_j\cap\operatorname{supp}\widetilde{\mu}_k\neq\emptyset,\quad\text{only if } |k-j|\le 1. \tag{2.64}$$

Let $\widetilde{f},\widetilde{h}\in L^2(G)$ be arbitrary functions and $\widetilde{f}_j=\mu_j\widetilde{f}$, $\widetilde{h}_j=\widetilde{\mu}_j\widetilde{h}$. By (2.64), we have

$$\sum_{j=-\infty}^{\infty}|\widetilde{f}_j(x)|^2\le C|\widetilde{f}(x)|^2,\qquad \sum_{j=-\infty}^{\infty}|\widetilde{h}_j(x)|^2\le C|\widetilde{h}(x)|^2,\quad\forall\, x\in G. \tag{2.65}$$

Therefore

$$\sum_{j=-\infty}^{\infty}\Big(\Big\|\frac{F_{0,j}}{s\varphi}\Big\|^2_{L^2(G)}+\sum_{k=1}^n\|F_{k,j}\|^2_{L^2(G)}\Big)\le 3\Big(\Big\|\frac{F_0}{s\varphi}\Big\|^2_{L^2(G)}+\sum_{k=1}^n\|F_k\|^2_{L^2(G)}\Big). \tag{2.66}$$

Observing that $\frac{\partial\mu_j}{\partial x_0}=-\kappa\widetilde{\mu}_j\frac{2^{-j}}{\ell^{\kappa+1}}\ell'$, we have

$$\sum_{j=-\infty}^{\infty}\Big\|\frac{\partial\mu_j}{\partial x_0}\frac{w}{s\varphi}\Big\|^2_{L^2(G)}=\sum_{j=-\infty}^{\infty}\Big\|\kappa\widetilde{\mu}_j\frac{2^{-j}}{\ell^{\kappa+1}}\ell'\frac{w}{s\varphi}\Big\|^2_{L^2(G)}\le C\sum_{j=-\infty}^{\infty}\Big\|\widetilde{\mu}_j\frac1\ell\frac{w}{s\varphi}\Big\|^2_{L^2(G)}.$$

Using (2.65) again, we find

$$\sum_{j=-\infty}^{\infty}\Big\|\frac{\partial\mu_j}{\partial x_0}\frac{w}{s\varphi}\Big\|^2_{L^2(G)}\le C\Big\|\frac{w}{s\varphi^{1-\frac1\kappa}}\Big\|^2_{L^2(G)}. \tag{2.67}$$

Since the restriction of φ on the hyperplane $\{x\mid x_n=0\}$ is independent of $(x_1,\cdots,x_{n-1})$, we have

$$\begin{aligned}\sum_{j=-\infty}^{\infty}\int_{-\infty}^{+\infty}\varphi^{-\frac12}\|g_j(x_0,\cdot)\|^2_{H^{\frac12}(\mathbb{R}^{n-1})}\mathrm{d}x_0&=\sum_{j=-\infty}^{\infty}\int_{-\infty}^{+\infty}\varphi^{-\frac12}\mu_j^2\|g(x_0,\cdot)\|^2_{H^{\frac12}(\mathbb{R}^{n-1})}\mathrm{d}x_0\\ &\le C\int_{-\infty}^{+\infty}\varphi^{-\frac12}\|g(x_0,\cdot)\|^2_{H^{\frac12}(\mathbb{R}^{n-1})}\mathrm{d}x_0\\ &\le C\|\varphi^{-\frac14}g\|^2_{H^{0,\frac12}(\mathbb{R}^n)}.\end{aligned} \tag{2.68}$$

Next we observe

$$\sum_{j=-\infty}^{\infty}\|\varphi^{-\frac14}g_j\|^2_{H^{\frac14,0}(\mathbb{R}^n)}=\sum_{j=-\infty}^{\infty}\int_{\mathbb{R}^{n-1}}\|\varphi^{-\frac14}g_j(\,\cdot\,,x_1,\cdots,x_{n-1})\|^2_{H^{\frac14}(\mathbb{R})}\mathrm{d}x_1\cdots\mathrm{d}x_{n-1}.$$

Denote $\mathcal{G}_j=\operatorname{supp}\mu_j$, $h_j=\varphi^{-\frac14}g_j$, $h=\varphi^{-\frac14}g$. According to the definition of the norm in the space $H^{\frac14}$, we have

$$\begin{aligned}\|h_j(\,\cdot\,,x_1,\cdots,x_{n-1})\|^2_{H^{\frac14}(\mathbb{R})}&=C\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{|(\mu_jh)(y_0,\,\cdot\,)-(\mu_jh)(x_0,\,\cdot\,)|^2}{|y_0-x_0|^{\frac32}}\mathrm{d}x_0\mathrm{d}y_0\\&\le C\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{|\mu_j(y_0)-\mu_j(x_0)|^2|h(y_0,\,\cdot\,)|^2}{|y_0-x_0|^{\frac32}}\mathrm{d}x_0\mathrm{d}y_0\\&\quad+C\int_{\mathbb{R}}\int_{\mathcal{G}_j}\frac{|h(y_0,\,\cdot\,)-h(x_0,\,\cdot\,)|^2|\mu_j(x_0)|^2}{|y_0-x_0|^{\frac32}}\mathrm{d}x_0\mathrm{d}y_0\\&=C(\mathrm{I}_1(j)+\mathrm{I}_2(j)).\end{aligned}\tag{2.69}$$

We estimate the terms I_1 and I_2 separately. By Young's inequality, we have

$$\begin{aligned}\mathrm{I}_1&=\int_{\mathcal{G}_j}\int_{[-\delta,\delta]}|\mu_j'(\zeta)|^2|h(y_0,x_1,\cdots,x_{n-1})|^2|y_0-x_0|^{\frac12}\mathrm{d}x_0\mathrm{d}y_0\\&\le\int_{\mathcal{G}_j}\int_{[-\delta,\delta]}|h(y_0,x_1,\cdots,x_{n-1})|^2|y_0-x_0|^{\frac12}\mathrm{d}x_0\mathrm{d}y_0\|\mu_j\|^2_{C^1(\mathbb{R})}\\&\le C\|\mu_j\|^2_{C^1(\mathbb{R})}\|h(\,\cdot\,,x_1,\cdots,x_{n-1})\|^2_{L^2(\mathcal{G}_j)}.\end{aligned}$$

Here ζ is some number between x_0 and y_0. Obviously,

$$\|\mu_j\|_{C^1(\mathbb{R})}\le C(1+\|\varphi^{\frac1\kappa}(\,\cdot\,,x_1,\cdots,x_{n-1},0)\|_{C^0(\mathcal{G}_j)}).$$

Next we notice

$$\frac{2^{-j}}{\ell^\kappa(x_0)}\in\Big[\frac12,2\Big],\quad\forall\,x_0\in\mathcal{G}_j.$$

Since $\varphi(\,\cdot\,,x_1,\cdots,x_{n-1},0)=\frac{1}{\ell^\kappa(x_0)}$, this implies

$$\varphi(\,\cdot\,,x_1,\cdots,x_{n-1},0)\in[2^{j-1},2^j],\quad\forall\,x_0\in\mathcal{G}_j$$

and

$$\mathrm{I}_1\le C\|\varphi^{-\frac14+\frac1\kappa}(\,\cdot\,,x_1,\cdots,x_{n-1},0)g(\,\cdot\,,x_1,\cdots,x_{n-1})\|^2_{L^2(\mathcal{G}_j)},\tag{2.70}$$

$$\begin{aligned}\mathrm{I}_2&\le\|\mu_j\|^2_{C^0(\mathbb{R})}\int_{\mathbb{R}}\int_{\mathcal{G}_j}\frac{|h(y_0,x_1,\cdots,x_{n-1})-h(x_0,x_1,\cdots,x_{n-1})|^2}{|y_0-x_0|^{\frac32}}\mathrm{d}x_0\mathrm{d}y_0\\&\le\|\mu_j\|^2_{C^0(\mathbb{R})}\Big(\|h(\,\cdot\,,x_1,\cdots,x_{n-1})\|^2_{H^{\frac14}(\bigcup_{l=j-2}^{j+2}\mathcal{G}_l)}+\int_{\mathbb{R}\setminus\bigcup_{l=j-1}^{j+1}\mathcal{G}_l}\int_{\mathcal{G}_j}\frac{|h(x_0,x_1,\cdots,x_{n-1})|^2}{|y_0-x_0|^{\frac32}}\mathrm{d}x_0\mathrm{d}y_0\Big)\\&\le C(\|h(\,\cdot\,,x_1,\cdots,x_{n-1})\|^2_{H^{\frac14}(\bigcup_{l=j-2}^{j+2}\mathcal{G}_l)}+\|\varphi^{-\frac14+\frac1\kappa}g(\,\cdot\,,x_1,\cdots,x_{n-1})\|^2_{L^2(\mathcal{G}_j)}).\end{aligned}\tag{2.71}$$

Here we used the fact that

$$\begin{aligned}\int_{\mathbb{R}\setminus\bigcup_{l=j-1}^{j+1}\mathcal{G}_l}\frac{1}{|y_0-x_0|^{\frac32}}\mathrm{d}y_0&\le C\int_{\{|y_0|^\kappa\ge2^{-j-2}\}}\frac{1}{|y_0|^{\frac32}}\mathrm{d}y_0\\&\le C\|\varphi^{-\frac14}(\,\cdot\,,x_1,\cdots,x_{n-1},0)\|^2_{C^0(\mathcal{G}_j)},\quad\forall\,x_0\in\mathcal{G}_j.\end{aligned}$$

By (2.64),

$$
\begin{aligned}
&\sum_{j=-\infty}^{\infty} \|h(\,\cdot\,,x_1,\cdots,x_{n-1})\|^2_{H^{\frac14}(\mathcal{G}_j)}\\
&\le \sum_{j=-\infty}^{\infty} \|h(\,\cdot\,,x_1,\cdots,x_{n-1})\|^2_{H^{\frac14}(\mathcal{G}_{2j})} + \sum_{j=-\infty}^{\infty} \|h(\,\cdot\,,x_1,\cdots,x_{n-1})\|^2_{H^{\frac14}(\mathcal{G}_{2j+1})}\\
&\le 2\|h(\,\cdot\,,x_1,\cdots,x_{n-1})\|^2_{H^{\frac14}(\mathbb{R})}.
\end{aligned}
\tag{2.72}
$$

By (2.70)–(2.72),

$$
\begin{aligned}
\sum_{j=-\infty}^{\infty} \|s^{-\frac14}\varphi^{-\frac14}g_j\|^2_{H^{\frac14,\frac12}(\mathbb{R}^n)} &= \sum_{j=-\infty}^{\infty} \|s^{-\frac14}\varphi^{-\frac14}g_j\|^2_{H^{0,\frac12}(\mathbb{R}^n)} + \sum_{j=-\infty}^{\infty} \|s^{-\frac14}\varphi^{-\frac14}g_j\|^2_{H^{\frac14,0}(\mathbb{R}^n)}\\
&\le C\Big(\|s^{-\frac14}\varphi^{-\frac14}g\|^2_{H^{0,\frac12}(\mathbb{R}^n)} + s^{-\frac12}\sum_{j=-\infty}^{\infty}(\mathrm{I}_1(j)+\mathrm{I}_2(j))\Big)\\
&\le C(\|s^{-\frac14}\varphi^{-\frac14}g\|^2_{H^{\frac14,\frac12}(\mathbb{R}^n)} + \|s^{-\frac14}\varphi^{-\frac14+\frac1\kappa}g\|^2_{L^2(\mathbb{R}^n)}).
\end{aligned}
\tag{2.73}
$$

By (2.63), for any $s \ge s_0$, we have

$$
\begin{aligned}
&\sum_{k=1}^{n}\Big\|s^{-\frac12}\varphi^{-\frac12}\frac{\partial w}{\partial x_k}\Big\|^2_{L^2(G)} + \|s^{\frac12}\varphi^{\frac12}w\|^2_{L^2(G)}\\
&\le \sum_{j=-\infty}^{\infty}\Big(\sum_{k=1}^{n}\Big\|s^{-\frac12}\varphi^{-\frac12}\frac{\partial w_j}{\partial x_k}\Big\|^2_{L^2(G)} + \|s^{\frac12}\varphi^{\frac12}w_j\|^2_{L^2(G)}\Big)\\
&\le C\sum_{j=-\infty}^{\infty}\Big(\|s^{-\frac14}\varphi^{-\frac14}g_j\|^2_{H^{\frac14,\frac12}(\mathbb{R}^n)} + \|s^{-\frac14}\varphi^{-\frac14+\frac1\kappa}g_j\|^2_{L^2(\mathbb{R}^n)}\\
&\quad + \Big\|\frac{\frac{\partial\mu_j}{\partial x_0}w + F_{0,j}}{s\varphi}\Big\|^2_{L^2(G)} + \sum_{k=1}^{n}\|F_{k,j}\|^2_{L^2(G)}\Big).
\end{aligned}
\tag{2.74}
$$

Finally, we estimate the right-hand side of (2.74) by using (2.66), (2.67) and (2.73),

$$
\begin{aligned}
&\sum_{k=1}^{n}\Big\|s^{-\frac12}\varphi^{-\frac12}\frac{\partial w}{\partial x_k}\Big\|^2_{L^2(G)} + \|s^{\frac12}\varphi^{\frac12}w\|^2_{L^2(G)}\\
&\le C\Big(\|s^{-\frac14}\varphi^{-\frac14}g\|^2_{H^{\frac14,\frac12}(\mathbb{R}^n)} + \|s^{-\frac14}\varphi^{-\frac14+\frac1\kappa}g\|^2_{L^2(\mathbb{R}^n)}\\
&\quad + \Big\|\frac{w}{s\varphi^{1-\frac1\kappa}}\Big\|^2_{L^2(G)} + \Big\|\frac{F_0}{s\varphi}\Big\|^2_{L^2(G)} + \sum_{k=1}^{n}\|F_k\|^2_{L^2(G)}\Big), \quad \forall\, s > s_0.
\end{aligned}
\tag{2.75}
$$

Using the definition of the functions F_k and increasing the parameter s_0 if necessary, we obtain (2.49) from (2.75). Thus, in order to prove (2.49), it suffices to prove (2.63). We concentrate on proving this estimate below.

2.4 Auxiliary problem

In the previous subsection, we showed that in order to prove the Carleman estimate (2.38), it suffices to establish a countable number of Carleman estimates for slightly simpler problems.

We put all these problems in the following general framework. Consider the following partial differential equation:

$$L(x, D_0, \widetilde{D} + \mathrm{i}|\tau|\nabla_{\widetilde{x}}\beta)w = \widetilde{f}, \quad \text{in } G, \tag{2.76}$$

$$w(x', 0) = \widetilde{g}, \tag{2.77}$$

$$\operatorname{supp} w \subset \Pi_{\delta,\gamma}. \tag{2.78}$$

Here $\widetilde{f} \in H^{-\frac{1}{2},-1,\tau}(G)$, $\operatorname{supp}\widetilde{f} \subset\subset \Pi_{\delta,\gamma}$, where the space $H_0^{\frac{1}{2},1,\tau}(G)$ can be defined similarly to $H^{\frac{1}{4},\frac{1}{2},s}(\mathbb{R}^n)$ and

$$H^{-\frac{1}{2},-1,\tau}(G) = (H_0^{\frac{1}{2},1,\tau}(G))^*, \quad \|\widetilde{f}\|_{H^{-\frac{1}{2},-1,\tau}(G)} = \sup_{w \in H_0^{\frac{1}{2},1,\tau}(G)} \frac{|\langle \widetilde{f}, w\rangle|}{\|w\|_{H_0^{\frac{1}{2},1,\tau}(G)}}.$$

We take a function β such that

$$\beta \in \mathcal{U}, \tag{2.79}$$

where the set $\mathcal{U}$ is constructed in the following way. First, we extend a function $\psi(x_1, \cdots, x_n)$ on the set $\mathbb{R}^{n-1} \times [0,\gamma]$ up to a C^2 function in such a way that ψ is constant outside a ball with a sufficiently large radius and $\psi(x) < 2\|\psi\|_{C^0(\overline{\Omega})}$ on $\mathbb{R}^{n-1} \times [0,\gamma]$. (Here $\|\psi\|_{C^0(\overline{\Omega})}$ is the norm of function ψ in the original coordinates.) We fix a sequence $\{x_{0,j}\}$ such that $x_{0,j} \in \operatorname{supp}\mu_j$. The set $\mathcal{U}$ consists of the functions of the form

$$(\mathrm{e}^{\lambda\psi} - \mathrm{e}^{2\lambda\|\psi\|_{C^0(\overline{\Omega})}})\frac{\ell^\kappa(x_{0,j})}{\widetilde{\ell}_j^\kappa(x_0)}.$$

The sequence of functions $\{\widetilde{\ell}_j\}$ is constructed in the following way. We fix sufficiently large $\widehat{j}$ such that for all $j \ge \widehat{j}$, $\operatorname{supp}\mu_j \subset [0, \frac{T}{8}] \cup [\frac{7T}{8}, T]$. There exist $0 < T_0(j) < T_1(j) < T$ such that $\operatorname{supp}\mu_j \subset [T_0(j), T_1(j)]$. Hence, for $j \le \widehat{j}$, we define $\widetilde{\ell}_j$ to be a smooth, strictly positive function on $[0,T]$, which coincides with ℓ on the segment $[T_0(j), T_1(j)]$ and is equal to some constants on $[T,+\infty)$ and $(-\infty, 0]$. If $j \ge \widehat{j}$, then $\operatorname{supp}\mu_j \subset [2^{-\frac{j+1}{\kappa}}, 2^{-\frac{j}{\kappa}}] \cup [T - 2^{-\frac{j+1}{\kappa}}, T - 2^{-\frac{j}{\kappa}}]$. Set $\widetilde{\ell} = 2^{-\frac{j+2}{\kappa}}$ on the segment $[0, 2^{-\frac{j+2}{\kappa}}]$ and on the segment $[2^{-\frac{j+2}{\kappa}}, 2^{-\frac{j+1}{\kappa}}]$. We extend $\widetilde{\ell}$ as a linear function in such a way that the resulting function is continuous. Similarly, on the segment $[T - 2^{-\frac{j-1}{\kappa}}, T]$ we set $\widetilde{\ell}(x_0) = T - 2^{-\frac{j-1}{\kappa}}$, and on the segment $[T - 2^{-\frac{j}{\kappa}}, T - 2^{-\frac{j-1}{\kappa}}]$ we let $\widetilde{\ell}$ be a linear function such that the resulting function is continuous. Finally, on the segment $[2^{-\frac{j}{\kappa}}, T - 2^{-\frac{j+1}{\kappa}}]$ the function $\widetilde{\ell}$ coincides with ℓ. It is not difficult to establish the following properties for functions of the set $\mathcal{U}$.

Proposition 2.2 (1) *There exists a positive constant C such that, for all $\beta \in \mathcal{U}$,*

$$\frac{\partial\beta}{\partial x_n}(x) \ge C > 0, \quad \forall\, x \in \Pi_{\delta,\gamma}, \quad \frac{\partial\beta}{\partial x_i} = 0, \quad \forall\, i \in \{1, \cdots, n-1\}, \quad \forall\, x \in \{x_n = 0\}. \tag{2.80}$$

(2) *There exists a positive constant $\widehat{C}$ such that*

$$\operatorname{Im}\{\overline{L}_2(x, \xi_0, \widetilde{\xi} - \mathrm{i}|\tau|\nabla_{\widetilde{x}}\beta), L_2(x, \xi_0, \widetilde{\xi} + \mathrm{i}|\tau|\nabla_{\widetilde{x}}\beta)\} \ge \widehat{C}|\tau|M^2(\xi,\tau) \tag{2.81}$$

for all $(x,\xi,\tau) \in \{(x,\xi,\tau) \mid x \in \Pi_{\delta_0,\gamma},\ L_2(x,\xi_0,\widetilde{\xi} + \mathrm{i}|\tau|\nabla_{\widetilde{x}}\beta) = 0\}$, where $\delta_0 > \delta$ is some constant independent of β.

Let us show that problem (2.60)–(2.62) can be reduced to problem (2.76)–(2.78). Set $w = w_j$, $\widetilde{f} = \frac{\partial\mu_j}{\partial x_0}w + F_{0,j} + \sum\limits_{k=1}^{n}\frac{\partial F_{k,j}}{\partial x_k}$, $\widetilde{g} = g_j$, and

$$\tau = \frac{s}{\ell^\kappa(x_{0,j})}, \quad \beta(x) = \alpha(x)\ell^\kappa(x_{0,j}), \quad \forall\, x_0 \in \operatorname{supp}\mu_j,\ x_{0,j} \in \operatorname{supp}\mu_j. \tag{2.82}$$

Since

$$\beta(x) = \alpha(x)\ell^\kappa(x_{0,j}), \quad \forall\, x_0 \in \operatorname{supp}\mu_j,$$

we have

$$\begin{aligned} L(x, D_0, \widetilde{D} + \mathrm{i}|\tau|\nabla_{\widetilde{x}}\beta)w &= L(x, D_0, \widetilde{D} + \mathrm{i}|\tau|\nabla_{\widetilde{x}}\beta)w_j = L(x, D_0, \widetilde{D} + \mathrm{i}|\tau|(\nabla_{\widetilde{x}}\alpha)\ell^\kappa(x_{0,j}))w_j \\ &= L(x, D_0, \widetilde{D} + \mathrm{i}|s|\nabla_{\widetilde{x}}\alpha)w_j = \frac{\partial\mu_j}{\partial x_0}w + F_{0,j} + \sum_{k=1}^{n}\frac{\partial F_{k,j}}{\partial x_k}. \end{aligned}$$

Next, we claim that the solutions of system (2.76)–(2.78) satisfy the following Carleman estimate: there exist constants C and τ_0, both independent of $\beta \in \mathcal{U}$, such that

$$\begin{aligned} &\sum_{j=1}^{n}\Big\|\tau^{-\frac{1}{2}}\frac{\partial w}{\partial x_j}\Big\|_{L^2(G)} + \|\tau^{\frac{1}{2}}w\|_{L^2(G)} \\ &\le C(\tau^{-\frac{1}{4}}\|\widetilde{g}\|_{H^{\frac{1}{4},\frac{1}{2}}(\mathbb{R}^n)} + \tau^{-\frac{1}{4}+\frac{1}{\kappa}}\|\widetilde{g}\|_{L^2(\mathbb{R}^n)} + \|\widetilde{f}\|_{H^{-\frac{1}{2},-1,\tau}(G)}), \quad \forall\, \tau \ge \tau_0. \end{aligned} \tag{2.83}$$

Suppose for the moment that this estimate is proved already. Let us show that it implies (2.63). We know already that functions $\Big(w_j, g_j, \frac{\partial\mu_j}{\partial x_0}w + F_{0,j} + \sum\limits_{k=1}^{n}\frac{\partial F_{k,j}}{\partial x_k}\Big)$ satisfy (2.76)–(2.78) with τ and β defined in (2.82). Making the change of unknown in (2.83), we arrive at the inequality

$$\begin{aligned} &\sum_{k=1}^{n}\Big\|\Big(\frac{s}{\ell^\kappa(x_{0,j})}\Big)^{-\frac{1}{2}}\frac{\partial w_j}{\partial x_k}\Big\|_{L^2(G)} + \Big\|\Big(\frac{|s|}{\ell^\kappa(x_{0,j})}\Big)^{\frac{1}{2}}w_j\Big\|_{L^2(G)} \\ &\le C\Big(\Big(\frac{|s|}{\ell^\kappa(x_{0,j})}\Big)^{-\frac{1}{4}}\|g_j\|_{H^{\frac{1}{4},\frac{1}{2}}(\mathbb{R}^n)} + \Big(\frac{|s|}{\ell^\kappa(x_{0,j})}\Big)^{-\frac{1}{4}+\frac{1}{\kappa}}\|g_j\|_{L^2(\mathbb{R}^n)} \\ &\quad + \Big\|\partial_{x_0}\mu_j w + F_{0,j} + \sum_{k=1}^{n}\frac{\partial F_{k,j}}{\partial x_k}\Big\|_{H^{-\frac{1}{2},-1,(\frac{s}{\ell^\kappa(x_{0,j})})}(G)}\Big), \end{aligned} \tag{2.84}$$

which holds for all $\tau \ge \tau_0$.

Note that there exist two constants $C_1 > 0$ and $C_2 > 0$ independent of s, j, such that

$$C_1\frac{1}{\varphi(x_0, \cdots, x_{n-1}, 0)} \le \ell^k(x_{0,j}) \le C_2\frac{1}{\varphi(x_0, \cdots, x_{n-1}, 0)}, \quad \forall\, x_0 \in \operatorname{supp}\mu_j. \tag{2.85}$$

Then the previous inequality can be written in the form

$$\begin{aligned} &\sum_{k=1}^{n}\Big\||s|^{-\frac{1}{2}}\varphi^{-\frac{1}{2}}\frac{\partial w_j}{\partial x_k}\Big\|_{L^2(G)} + \||s|^{\frac{1}{2}}\varphi^{\frac{1}{2}}w_j\|_{L^2(G)} \\ &\le C\Big(\Big(\frac{|s|}{\ell^\kappa(x_{0,j})}\Big)^{-\frac{1}{4}}\|g_j\|_{H^{\frac{1}{4},\frac{1}{2}}(\mathbb{R}^n)} + \|(|s|\varphi)^{-\frac{1}{4}+\frac{1}{\kappa}}g_j\|_{L^2(\mathbb{R}^n)} \\ &\quad + \Big\|\frac{\frac{\partial\mu_j}{\partial x_0}w + F_{0,j}}{s\varphi}\Big\|_{L^2(G)} + \sum_{k=1}^{n}\|F_{k,j}\|_{L^2(G)}\Big), \quad \forall\, s \ge s_0. \end{aligned} \tag{2.86}$$

What remains to be shown is the existence of a constant $C > 0$ such that

$$\Big(\frac{|s|}{\ell^k(x_{0,j})}\Big)^{-\frac{1}{4}}\|g_j\|_{H^{\frac{1}{4},\frac{1}{2}}(\mathbb{R}^n)} \le C(\|(|s|\varphi)^{-\frac{1}{4}}g_j\|_{H^{\frac{1}{4},\frac{1}{2}}(\mathbb{R}^n)} + \|(|s|\varphi)^{-\frac{1}{4}+\frac{1}{\kappa}}g_j\|_{L^2(\mathbb{R}^n)}).$$

Since (2.85) immediately implies the inequality

$$\Big(\frac{|s|}{\ell^k(x_{0,j})}\Big)^{-\frac{1}{4}}\|g_j\|_{H^{0,\frac{1}{2}}(\mathbb{R}^n)} \le C\|(|s|\varphi)^{-\frac{1}{4}}g_j\|_{H^{0,\frac{1}{2}}(\mathbb{R}^n)},$$

it suffices to prove

$$\Big(\frac{|s|}{\ell^k(x_{0,j})}\Big)^{-\frac{1}{4}}\|g_j\|_{H^{\frac{1}{4},0}(\mathbb{R}^n)} \le C(\|(|s|\varphi)^{-\frac{1}{4}}g_j\|_{H^{\frac{1}{4},0}(\mathbb{R}^n)} + \|(|s|\varphi)^{-\frac{1}{4}+\frac{1}{\kappa}}g_j\|_{L^2(\mathbb{R}^n)}). \tag{2.87}$$

Then, elementary computations provide that for any $x_0 \in \mathcal{G}_j$,

$$\begin{aligned}
&|h(x_0)\varphi^{-\frac{1}{4}}(x_0,\cdots,x_{n-1},0) - h(y_0)\varphi^{-\frac{1}{4}}(y_0,\cdots,y_{n-1},0)|^2\\
=&|\varphi^{-\frac{1}{4}}(x_0,\cdots,x_{n-1},0)(h(x_0)-h(y_0)) + h(y_0)(\varphi^{-\frac{1}{4}}(x_0\cdots,x_{n-1},0) - \varphi^{-\frac{1}{4}}(y_0\cdots,y_{n-1},0))|^2\\
\ge&|\varphi^{-\frac{1}{4}}(x_0,\cdots,x_{n-1},0)|^2|h(x_0)-h(y_0)|^2\\
&-4|h(y_0)|^2|\varphi^{-\frac{1}{4}}(x_0,\cdots,x_{n-1},0) - \varphi^{-\frac{1}{4}}(y_0,\cdots,y_{n-1},0)|^2\\
\ge&C\Big(\frac{1}{\ell^k(x_{0,j})}\Big)^{-\frac{1}{4}}|h(x_0)-h(y_0)|^2\\
&-4|h(y_0)|^2|\varphi^{-\frac{1}{4}}(x_0,\cdots,x_{n-1},0) - \varphi^{-\frac{1}{4}}(y_0,\cdots,y_{n-1},0)|^2.
\end{aligned} \tag{2.88}$$

Using (2.88) and the definition of $H^{\frac{1}{4},0}$-norm, we have

$$\begin{aligned}
&\Big(\frac{|s|}{\ell^k(x_{0,j})}\Big)^{-\frac{1}{4}}\|g_j\|_{H^{\frac{1}{4},0}(\mathbb{R}^n)} \le C\Big(\|(|s|\varphi)^{-\frac{1}{4}}g_j\|_{H^{\frac{1}{4},0}(\mathbb{R}^n)}\\
&\quad+\sqrt{\int_{\mathbb{R}^{n-1}}\int_{\mathcal{G}_j}\int_{\mathcal{G}_j}\frac{|\varphi^{-\frac{1}{4}}(x_0,\cdots,x_{n-1},0)-\varphi^{-\frac{1}{4}}(y_0,\cdots,y_{n-1},0)|^2}{|s|^{\frac{1}{4}}|h(y_0)|^{-2}|x_0-y_0|^{\frac{3}{2}}}\mathrm{d}x_0\mathrm{d}y_0\mathrm{d}x_1\cdots\mathrm{d}x_{n-1}}\Big)\\
&\le C\Big(\|(|s|\varphi)^{-\frac{1}{4}}g_j\|_{H^{\frac{1}{4},0}(\mathbb{R}^n)}\\
&\quad+\sqrt{\int_{\mathbb{R}^{n-1}}\int_{\mathcal{G}_j}\int_{\mathcal{G}_j}|s|^{-\frac{1}{4}}|h(y_0)|^2|(\ell^{\frac{\kappa}{4}})'(\zeta)|^2|x_0-y_0|^{\frac{1}{2}}\mathrm{d}x_0\mathrm{d}y_0\mathrm{d}x_1\cdots\mathrm{d}x_{n-1}}\Big)\\
&\le C(\|(|s|\varphi)^{-\frac{1}{4}}g_j\|_{H^{\frac{1}{4},0}(\mathbb{R}^n)} + \||s|^{-\frac{1}{4}}\varphi^{-\frac{1}{4}+\frac{1}{\kappa}}g_j\|_{L^2(\mathbb{R}^n)}).
\end{aligned}$$

Thus estimate (2.87) is established.

In (2.76), without loss of generality, one can assume $a_{nn} \equiv 1$. The principal symbol of operator $L(x, D_0, \widetilde{D} + \mathrm{i}|\tau|\nabla_{\widetilde{x}}\beta)$ can be written in the form

$$L_2(x,\xi,\tau) = \mathrm{i}a_0(x)\xi_0 + \sum_{k,j=1}^{n} a_{kj}(x)\xi_k\xi_j,$$

where $\zeta = \xi + \mathrm{i}|\tau|\nabla\beta$. Consider the equation $L_2(x,\xi,\tau) = 0$. The two roots in ξ_n of this equation are

$$-\mathrm{i}|\tau|\frac{\partial\beta(x)}{\partial x_n} + \lambda^{\pm}(x,\xi',\tau),$$

where

$$\lambda^{\pm}(x,\xi',\tau) = -\sum_{j=1}^{n-1} a_{nj}(x)\zeta_j \pm \sqrt{-(a(x,\zeta',\zeta') + \mathrm{i}a_0(x)\xi_0) + \Big(\sum_{j=1}^{n-1} a_{nj}(x)\zeta_j\Big)^2},$$

and $\zeta' = \xi' + \mathrm{i}|\tau|\nabla'\beta$.

Let $\mathcal{M} = \big\{(\xi',\tau) \mid \xi_0^2 + \tau^4 + \sum_{i=1}^{n-1}\xi_i^4 = 1\big\}$. If

$$(x,\xi',\tau) \in \Phi = \left\{(x,\xi',\tau) \in \Pi_{\delta,\gamma} \times \mathcal{M} \,\middle|\, -(a(x,\zeta',\zeta') + \mathrm{i}a_0(x)\xi_0) + \Big(\sum_{j=1}^{n-1} a_{nj}(x)\zeta_j\Big)^2 \in \mathbb{R}^1_+\right\},$$

we assume that

$$\operatorname{Im}\sqrt{-(a(x,\zeta',\zeta') + \mathrm{i}a_0(x)\xi_0) + \Big(\sum_{j=1}^{n-1} a_{nj}(x)\zeta_j\Big)^2}$$

is positive. Therefore, outside the set Φ, the functions $\lambda^{\pm}$ are smooth. In order to regularize this expression for $\lambda^{\pm}(x,\xi',\tau)$ near $(\xi',\tau) = 0$, we consider $\upsilon \in C^\infty(\mathbb{R}^+)$ such that

$$\begin{aligned} &\upsilon(t) = 0 \quad \text{for } t \in \Big[0, \frac{1}{2}\Big], \\ &\upsilon(t) = 1 \quad \text{for } t > 1, \\ &0 \le \upsilon(t) \le 1, \quad \forall\, t \in \mathbb{R}^+, \end{aligned}$$

and determine $\widetilde{\lambda}^{\pm}(x,\xi',\tau)$ as

$$\begin{aligned} &\widetilde{\lambda}^{\pm}(x,\xi',\tau) = -\sum_{j=1}^{n-1} a_{nj}(x)\zeta_j \pm \upsilon(M(\xi',\tau))\sqrt{-(a(x,\zeta',\zeta') + \mathrm{i}a_0(x)\xi_0) + \Big(\sum_{j=1}^{n-1} a_{nj}(x)\zeta_j\Big)^2}, \\ &M(\xi',\tau) = \Big(\xi_0^2 + \sum_{j=1}^{n-1}\xi_j^4 + \tau^4\Big)^{\frac{1}{4}}. \end{aligned}$$

We set

$$r_{\pm}(x,\xi',\tau) = |\tau|\frac{\partial\beta}{\partial x_n}(x) + \mathrm{i}\widetilde{\lambda}^{\pm}(x,\xi',\tau), \tag{2.89}$$

and introduce the following sets

$$\begin{aligned} &\Upsilon = \{(\xi',\tau) \mid \exists\, x \in \Pi_{\delta,\gamma} \text{ such that } (x,\xi',\tau) \in \Phi\}, \\ &\Upsilon_\epsilon = \{(\xi',\tau) \mid \operatorname{dist}((\xi',\tau),\Upsilon) \le \epsilon\}, \quad \Upsilon^1_\epsilon = \mathcal{M} \setminus \Upsilon_{\frac{\epsilon}{2}}. \end{aligned}$$

We claim that one can take a parameter $\gamma > 0$ small enough such that there exists a positive $\epsilon(\gamma)$, which can be taken arbitrarily small, and a pair of functions $\{\chi_0, \chi_1\}$ independent of $\beta \in \mathcal{U}$ such that

$$\chi_0, \chi_1 \in C^\infty(\mathcal{M}), \tag{2.90}$$

$$\operatorname{supp}\chi_0 \subset \Upsilon_\epsilon, \quad \operatorname{supp}\chi_1 \subset \Upsilon^1_\epsilon, \tag{2.91}$$

$$\chi_0 \ge 0, \quad \chi_1 \ge 0, \quad \chi_0 + \chi_1 \ge 1, \quad \text{on } \mathcal{M}, \tag{2.92}$$

$$\operatorname{dist}(\Upsilon, (0,\cdots,0,1)) \to +0, \quad \text{as } \gamma \to +0. \tag{2.93}$$

Really, we observe that $\frac{\partial\beta}{\partial x_i}(x',0)=0$ for $i\in\{1,\cdots,n-1\}$ and

$$\min_{x\in\{x|x_n=0,\ |x'|<\delta\}}\min\{\operatorname{Re}Z,-|\operatorname{Im}Z|\}<0,$$

where $Z=-4(a(x,\zeta',\zeta')+\mathrm{i}a_0(x)\xi_0)+\Big(\sum_{j=1}^{n-1}a_{nj}(x)\zeta_j\Big)^2$. Therefore, if a point (ξ',τ) belongs to Υ and $M(\xi',0)>0$, the quantity $\sum_{j=1}^{n-1}|\tau\frac{\partial\beta}{\partial x_j}|$ can not be closed to zero. Indeed, if $\tau=0$, then ζ' is a real vector. Since a_0 is a strictly positive function, $\xi_0=0$. However, in this case, the inequality (2.56) implies that Z is a negative number. This contradicts the fact $(\xi,\tau)\in\Upsilon$. On the other hand,

$$\max_{x\in\Pi_{\delta,\gamma}}\sum_{j=1}^{n-1}\Big|\frac{\partial\beta}{\partial x_j}\Big|\to+0,\quad \text{as } \gamma\to+0.$$

Therefore, (2.93) holds true, and for sufficiently small positive ϵ the choice of the functions χ_i is possible.

Next, we extend χ_μ to the set $\{(\xi',\tau)\mid M(\xi',\tau)>1\}$ by the formula

$$\chi_\mu(\xi',\tau)=\chi_\mu\Big(\frac{\xi_0}{M^2(\xi',\tau)},\frac{\xi_1}{M(\xi',\tau)},\cdots,\frac{\xi_n}{M(\xi',\tau)},\frac{\tau}{M(\xi',\tau)}\Big).$$

Then we extend functions χ_μ up to C^∞ function on $\{(\xi',\tau)\mid M(\xi',\tau)<1\}$. Let $\chi_\mu(D',\tau)$ be the pseudodifferential operator with symbol $\chi_\mu(\xi',\tau)$.

Applying the operator $\chi_\mu(D',\tau)$ to the both sides of equation (2.76), we find

$$L(x,D_0,\widetilde{D}+\mathrm{i}|\tau|\nabla_{\widetilde{x}}\beta)w_\mu=\chi_\mu\widetilde{f}-[\chi_\mu,L(x,D_0,\widetilde{D}+\mathrm{i}|\tau|\nabla_{\widetilde{x}}\beta)]w=\widetilde{f}_\mu,\quad \text{in } G, \tag{2.94}$$

$$w_\mu(x',0)=(\chi_\mu\widetilde{g})(x'),\quad x'\in\mathbb{R}^n, \tag{2.95}$$

where $w_\mu=\chi_\mu(D',\tau)w$.

By Lemma A.4, we observe

$$\|[\chi_\mu,L(x,D_0,\widetilde{D}+\mathrm{i}|\tau|\nabla_{\widetilde{x}}\beta)]w\|_{H^{-\frac12,-1,\tau}(G)}\le\frac{C\|w\|_{H^{\frac12,1,\tau}(G)}}{(1+|\tau|)^{1-\frac1\kappa}},\quad \mu\in\{0,1\}. \tag{2.96}$$

2.5 Proof of the main estimate

First, we obtain an a priori estimate for the function $w_0=\chi_0(D',\tau)w$. We claim that there exists a constant $C>0$ such that

$$\|w_0\|_{H^{\frac12,1,\tau}(G)}\le C\Big(\|\widetilde{f}\|_{H^{-\frac12,-1,\tau}(G)}+\frac{\|w\|_{H^{\frac12,1,\tau}(G)}}{(1+|\tau|)^{1-\frac1\kappa}}\Big). \tag{2.97}$$

Really, using the notations $W=(W_1,W_2)$ and $F=(0,\mathrm{i}\widetilde{f}_0)$, where $W_1=\widetilde{\Lambda}(D',\tau)w_0$, $W_2=\frac{\partial w_0}{\partial x_n}-|\tau|\frac{\partial\beta}{\partial x_n}w_0$ and $\widetilde{\Lambda}(D',\tau)w_0=\int_{\mathbb{R}^n}(1+M(\xi',\tau))\widehat{w}_0\mathrm{e}^{\mathrm{i}\langle\xi',x'\rangle}\mathrm{d}\xi'$, we rewrite system (2.94)–(2.95) in the form

$$\frac{\partial W}{\partial x_n}=K(x,D',\tau)W+F,\quad \text{in } G, \tag{2.98}$$

$$W(x',\gamma)=\frac{\partial}{\partial x_n}W(x',\gamma)=0. \tag{2.99}$$

Here we set

$$K(x,D',\tau)=|\tau|\frac{\partial\beta}{\partial x_n}I+\begin{pmatrix}0 & \widetilde{\Lambda}(D',\tau)+\Big[\widetilde{\Lambda},|\tau|\dfrac{\partial\beta}{\partial x_n}\Big]\\ K_{12}(x,D',\tau) & K_{22}(x,D',\tau)\end{pmatrix},$$

where

$$K_{12}(x,D',\tau)=\sum_{j,k=1}^{n-1}a_{kj}(x)\Big(D_j+\mathrm{i}|\tau|\frac{\partial\beta}{\partial x_j}\Big)\Big(D_k+\mathrm{i}|\tau|\frac{\partial\beta}{\partial x_k}\Big)\widetilde{\Lambda}^{-1}(D',\tau)+\mathrm{i}a_0D_0\widetilde{\Lambda}^{-1}(D',\tau),$$

$$K_{22}(x,D',\tau)=-\mathrm{i}\sum_{j=1}^{n-1}a_{jn}(x)\Big(D_j+\mathrm{i}|\tau|\frac{\partial\beta}{\partial x_j}\Big).$$

The eigenvalues of the matrix $K(x,\xi',\tau)$ are $r_\pm(x,\xi',\tau)$. Therefore, by (2.93), there exists a positive constant C independent of $\beta\in\mathcal{U}$ such that

$$\operatorname{Spec}K(x,\xi',\tau)\subset\Big\{z\in\mathbb{C}:\operatorname{Re}z\geq C\Big(\xi_0^2+\sum_{j=1}^{n-1}\xi_j^4+\tau^4\Big)^{\frac14}\Big\}\tag{2.100}$$

for all $(x,\xi',\tau)\in\Pi_{\delta,\gamma}\times(\Upsilon_\epsilon\cap\{(\xi',s)\mid M(\xi',\tau)\geq 1\})$. Here $\operatorname{Spec}K$ means the spectrum of K. We extend the symbol $K(x,\xi',\tau)$ from $\Pi_{\delta,\gamma}\times\Upsilon_\epsilon$ on $G\times\mathbb{R}^{n+1}$ in such a way that the new symbol $\widetilde{K}(x,\xi',\tau)\in C^1_{\mathrm{cl}}S^{\frac12,1,\tau}(G)$ and inequality (2.100) holds true on $G\times\mathbb{R}^{n+1}$ with the constant $\frac{C}{2}$. Here and henceforth, we understand that $C^k_{\mathrm{cl}}S^{\frac{\kappa}{2},\kappa,s}(\mathcal{O})$ is specified in Definition A.1 in Appendix. Moreover, the symbol $\widetilde{K}$ is independent of x' if $|x'|$ is sufficiently large. The function W verifies

$$\frac{\partial W}{\partial x_n}=\widetilde{K}(x,D',\tau)W+\widetilde{F},\quad\text{in }G,\tag{2.101}$$

$$W(x',\gamma)=\frac{\partial}{\partial x_n}W(x',\gamma)=0.\tag{2.102}$$

Applying Lemma A.7 and using (2.96), we obtain (2.97).

Next, we obtain an estimate for the function $w_1=\chi_1(D',\tau)w$. The symbols $r_\pm(x,\xi',\tau)$ are smooth on $\overline{\Pi_{\delta,\gamma}\times\Upsilon^1_\epsilon}$. Our goal is to extend symbols $r_\pm$ on the set $G\times\mathbb{R}^{n+1}$. First, for some positive constant C, we observe

$$\operatorname{Re}r_-(x,\xi',\tau)\geq CM(\xi',\tau),\quad\forall\,(x,\xi',\tau)\in\Pi_{\delta,\gamma}\times\Upsilon^1_\epsilon.$$

Therefore, we extend the symbol r_- to the set $G\times\mathbb{R}^{n+1}$ in such a way that $r_-\in C^1_{\mathrm{cl}}S^{\frac12,1}(G)$, the previous inequality holds true with a constant $\frac{C}{2}$ on $G\times\mathbb{R}^{n+1}$, and r_- is independent of x for all $|x'|\geq\widehat{K}$,

$$\operatorname{Re}r_-(x,\xi',\tau)>\frac{C}{2}M(\xi',\tau),\quad\forall\,(x,\xi',\tau)\in G\times\mathbb{R}^{n+1}.\tag{2.103}$$

Next we extend the symbol r_+. Note that

$$-\operatorname{Re}r_+(x,\xi',\tau)\geq CM(\xi',\tau),\quad\forall\,(x,\xi',\tau)\in\{(x,\xi',\tau)\mid x\in\Pi_{\delta,\gamma},\ (\xi',\tau)\in\partial\Upsilon^1_\epsilon\}.$$

Therefore, we extend r_+ on $\Pi_{\delta',\gamma}\times(\mathbb{R}^{n+1}\setminus\Upsilon^1_\epsilon)$ in such a way that

$$-\operatorname{Re}r_+(x,\xi',\tau)\geq CM(\xi',\tau),\quad\forall\,(x,\xi',\tau)\in\{(x,\xi',\tau)\mid x\in\Pi_{\delta',\gamma},\ (\xi',\tau)\in\mathbb{R}^{n+1}\setminus\Upsilon^1_\epsilon\}.\tag{2.104}$$

This is possible if the difference $\delta' - \delta > 0$ is small. Then in the definition of the symbol of operator λ^+, we substitute the function β by $\beta\chi_{-1}$, where $\chi_{-1} \in C_0^\infty(B(0,\delta'))$ and $\chi_{-1}|_{B(0,\delta)} = 1$. Finally, we extend r_+ from $\Pi_{\delta',\gamma} \times \mathbb{R}^{n+1}$ on $G \times \mathbb{R}^{n+1}$ up to a symbol of a class $C^1_{\mathrm{cl}} S^{\frac{1}{2},1,\tau}(G)$ in such a way that the symbol r_+ is independent of x for all x' such that $|x'| > \widehat{C}$ and

$$-\mathrm{Re}\, r_+(x,\xi',\tau) \geq CM(\xi',\tau), \quad \forall\, (x,\xi',\tau) \in \{(x,\xi',\tau) \mid |x'| > \widehat{C},\ (\xi',\tau) \in \mathbb{R}^{n+1}\}.$$

We denote by $R_\pm(x,D',\tau)$ the pseudodifferential operator with symbol $r_\pm(x,\xi',\tau)$, namely,

$$R_\pm(x,D',\tau)u(x) = \int_{\mathbb{R}^n} r_\pm(x,\xi',\tau)\widehat{u}(\xi',x_n)\mathrm{e}^{\mathrm{i}\langle x',\xi'\rangle}\mathrm{d}\xi'. \tag{2.105}$$

Since $r_\pm(x,\xi',\tau) \in C^1_{\mathrm{cl}} S^{\frac{1}{2},1,s}(G)$, by Lemma A.1, we have $R_\pm \in \mathcal{L}(H^{\frac{1}{2},1,\tau}(G); L^2(G))$. Using these pseudodifferential operators $R_\pm(x,D',\tau)$, we construct two operators

$$L_-(x,D,\tau) = \frac{\partial}{\partial x_n} - R_-(x,D',\tau), \quad L_+(x,D,\tau) = \frac{\partial}{\partial x_n} - R_+(x,D',\tau).$$

Symbols of the operators $L_-(x,D,\tau)$ and $L_+(x,D,\tau)$ are

$$L_-(x,\xi,\tau) = \mathrm{i}\xi_n - r_-(x,\xi',\tau), \quad L_+(x,\xi,\tau) = \mathrm{i}\xi_n - r_+(x,\xi',\tau). \tag{2.106}$$

If $\mathrm{supp}\, \widehat{w}(\xi',x_n,\tau) \subset \Upsilon^1_\epsilon$ for any $x_n \in [0,\gamma]$, the operator $L(x,D,\tau)$ can be represented as

$$\begin{aligned}
&L(x,D_0,\widetilde{D} + \mathrm{i}|\tau|\nabla_{\widetilde{x}}\beta)w \\
&= \Big(\frac{\partial}{\partial x_n} - R_-(x,D',\tau)\Big)\Big(\frac{\partial}{\partial x_n} - R_+(x,D',\tau)\Big)w + K(x_n)w, \quad x_n \in [0,\gamma],
\end{aligned} \tag{2.107}$$

where

$$Kw \in L^\infty(0,\gamma;\mathcal{L}(H^{\frac{1}{2},1,\tau}(\mathbb{R}^n), L^2(\mathbb{R}^n))), \quad \|K(x_n)\|_{\mathcal{L}(H^{\frac{1}{2},1,\tau},L^2)} \leq C(1+|\tau|)^{\frac{1}{\kappa}} \tag{2.108}$$

for all $x_n \in [0,\gamma]$. Indeed, note that

$$\begin{aligned}
&\Big(\frac{\partial}{\partial x_n} - R_-(x,D',\tau)\Big)\Big(\frac{\partial}{\partial x_n} - R_+(x,D',\tau)\Big) \\
&= \frac{\partial^2}{\partial x_n^2} + R_-(x,D',\tau)R_+(x,D',\tau) \\
&\quad - (R_+(x,D',\tau) + R_-(x,D',\tau))\frac{\partial}{\partial x_n} - R_{+,(x_n)}(x,D',\tau).
\end{aligned} \tag{2.109}$$

According to Lemma A.3, the operator $R_-R_+ = R + K_1$, where R is the pseudodifferential operator of the form (2.105) with symbol $r_+(x,\xi',\tau)r_-(x,\xi',\tau)$, and the operator $K_1(x_n) \in \mathcal{L}(H^{\frac{1}{2},1,\tau}(G), L^2(G))$ is such that $\sup\limits_{x_n\in[0,\gamma]} \|K_1(x_n)\| \leq C(\pi_{C^1(r_+)}, \pi_{C^1(r_-)}) \leq C(1+|\tau|)^{\frac{1}{\kappa}}$. The operator $R_{+,(x_n)}$ is the pseudodifferential operator with symbol $\frac{\partial}{\partial x_n} r_+(x,\xi',\tau) \in C^0_{\mathrm{cl}} S^{\frac{1}{2},1,\tau}(\mathbb{R}^n)$ for any x_n in $[0,\gamma]$. By Lemma A.1, it belongs to $L^\infty(0,\gamma;\mathcal{L}(H^{\frac{1}{2},1,\tau}(\mathbb{R}^n), L^2(\mathbb{R}^n)))$. Next, observe $(R_+(x,D',\tau) + R_-(x,D',\tau))\frac{\partial}{\partial x_n} = \sum\limits_{j=1}^{n-1} a_{nj}(D_j + \mathrm{i}|\tau|\frac{\partial \beta}{\partial x_j})\frac{\partial}{\partial x_n}$. By Lemma A.3, we have

$$R_-(x,D',\tau)R_+(x,D',\tau) = a(x,D' + \mathrm{i}|\tau|\nabla'\beta, D' + \mathrm{i}|\tau|\nabla'\beta) - \mathrm{i}D_0 + K_2(x_n),$$

where the operator $K_2(x_n) \in \mathcal{L}(H^{\frac{1}{2},1,\tau}(G), L^2(G))$ is such that $\sup_{x_n \in [0,\gamma]} \|K_2(x_n)\| \le C(\pi_{C^1(r_+)}, \pi_{C^1(r_-)}) \le C(1+|\tau|)^{\frac{1}{\kappa}}$. Here and henceforth, we understand that the seminorm $\pi_{C^k(a)}$ is defined after Definition A.1 in Appendix. This proves (2.107) and (2.108).

Denote the function $L_+(x,D,\tau)w_1$ as z,

$$L_+(x,D,\tau)w_1 = z, \quad \text{in } G. \tag{2.110}$$

Consider the initial value problem

$$L_-(x,D,\tau)z = \widetilde{f}_1 - Kw_1, \quad \text{in } G, \quad z(\,\cdot\,,\gamma) = 0. \tag{2.111}$$

By (2.103) and Lemma A.7, there exists a constant C independent of β such that

$$\|z\|_{L^2(G)} \le C(\|\widetilde{f}_1\|_{H^{-\frac{1}{2},-1,\tau}(G)} + \|Kw_1\|_{H^{-\frac{1}{2},-1,\tau}(G)}). \tag{2.112}$$

Now we concentrate on obtaining an a priori estimate for equation (2.110). We introduce the operators

$$Q(x,D',\tau) = \frac{1}{2}(L_+(x,D,\tau) + L_+(x,D,\tau)^*), \tag{2.113}$$

$$\begin{aligned} P(x,D,\tau) &= \frac{1}{2}(L_+(x,D,\tau) - L_+(x,D,\tau)^*) \\ &= \frac{\partial}{\partial x_n} - \frac{1}{2}(R_+(x,D',\tau) - R_+(x,D',\tau)^*). \end{aligned} \tag{2.114}$$

Therefore

$$Q(x,D',\tau) = Q(x,D',\tau)^*, \quad P(x,D,\tau)^* = -P(x,D',\tau).$$

Then equation (2.110) can be written in the form

$$Q(x,D',\tau)w_1 + P(x,D,\tau)w_1 = z, \quad \text{in } G.$$

Taking the L^2-norms of the left- and right-hand sides of this equation, we have

$$\|Qw_1\|^2_{L^2(G)} + \|Pw_1\|^2_{L^2(G)} + \mathrm{Re}\,(Qw_1, Pw_1)_{L^2(G)} = \|z\|^2_{L^2(G)}.$$

Observe that

$$\begin{aligned} \mathrm{Re}\,(Qw_1, Pw_1)_{L^2(G)} &= (Qw_1, Pw_1)_{L^2(G)} + (Pw_1, Qw_1)_{L^2(G)} \\ &= ([Q,P]w_1, w_1)_{L^2(G)} - (Q(x',0,D',\tau)w_1(\,\cdot\,,0), w_1(\,\cdot\,,0))_{L^2(\mathbb{R}^n)}. \end{aligned} \tag{2.115}$$

Therefore

$$\begin{aligned} \|z\|^2_{L^2(G)} = {}& \|Qw_1\|^2_{L^2(G)} + \|Pw_1\|^2_{L^2(G)} + ([Q,P]w_1, w_1)_{L^2(G)} \\ & - (Q(x',0,D',\tau)w_1(\,\cdot\,,0), w_1(\,\cdot\,,0))_{L^2(\mathbb{R}^n)}. \end{aligned} \tag{2.116}$$

By (2.81), there exists a positive constant $\widehat{C}$ such that

$$\mathrm{Re}\,\{Q,P\}(x,\xi',\tau) > \widehat{C}M(\xi',\tau) \tag{2.117}$$

for all $(x,\xi,\tau) \in \{(x,\xi,\tau) \mid x \in \Pi_{\delta',\gamma},\ Q(x,\xi',\tau) = 0,\ |\tau| > 1\}$, where $\delta' > \delta$.

Proposition 2.3 *Suppose that* (2.117) *holds true. Let* $w \in H^{\frac{1}{2},1}(G)$ *and* $\operatorname{supp} w \subset \Pi_{\beta',\gamma}$. *Then there exist positive constants* C_0 *and* C_1*, such that*

$$\|Qw\|^2_{L^2(G)} + \|Pw\|^2_{L^2(G)} + \operatorname{Re}([Q,P]w,w)_{L^2(G)} \geq C_0\|w\|^2_{H^{\frac{1}{4},\frac{1}{2},\tau}(G)} - C_1\|w\|^2_{L^2(G)}.$$

Proof The pseudodifferential operators P and Q have the symbols with C^1-smoothness in variable x. We approximate these operators by pseudodifferential operators with smooth symbols. The approximations are constructed in the following way:

$$Q_\tau = \frac{1}{2}(L_{+,\tau}(x,D,\tau) + L_{+,\tau}(x,D,\tau)^*), \quad P_\tau = \frac{1}{2}(L_{+,\tau}(x,D,\tau) - L_{+,\tau}(x,D,\tau)^*),$$

where $L_{+,\tau} = \frac{\partial}{\partial x_n} - R_{+,\tau}(x,D',\tau)$. The symbol of the operator $R_{+,\tau}$ is given by the formula

$$r_{+,\tau}(x,\xi',\tau) = |\tau|\frac{\partial \beta_\tau}{\partial x_n} + \mathrm{i}\lambda^+_\tau(x,\xi',\tau),$$

where

$$\lambda^+_\tau(x,\xi',\tau) = \sum_{j=1}^{n-1} a_{\tau,nj}(x)\widetilde{\zeta}_j \pm \upsilon(M(\xi',\tau))\sqrt{-(a_\tau(x,\widetilde{\zeta}',\widetilde{\zeta}') + \mathrm{i}a_{\tau,0}(x)\xi_0) + \Big(\sum_{j=1}^{n-1} a_{\tau,nj}(x)\widetilde{\zeta}_j\Big)^2}.$$

Here

$$a_{\tau,0} = a_0 * \eta_{\tau^{\frac{1}{2}+\epsilon}}, \quad a_{\tau,ij} = a_{ij} * \eta_{\tau^{\frac{1}{2}+\epsilon}}, \quad \beta_\tau = \beta * \eta_{\tau^{\frac{1}{2}+\epsilon}}, \quad \widetilde{\zeta}_j = \xi_j + \mathrm{i}|\tau|\frac{\partial \beta_\tau}{\partial x_j}, \tag{2.118}$$
$$\widetilde{\zeta}' = (\widetilde{\zeta}_0, \cdots, \widetilde{\zeta}_{n-1}),$$

where the function $\eta_{\widetilde{\varepsilon}}$ is the standard mollifier (see e.g. [3, p. 629]), and ϵ is a positive parameter.

Using the properties of mollifiers, we obtain

$$\begin{aligned} &\|Q - Q_\tau\|_{\mathcal{L}(H^{\frac{1}{2},1,\tau}(G);L^2(G))} + \|P - P_\tau\|_{\mathcal{L}(H^{\frac{1}{2},1,\tau}(G);L^2(G))} \\ &\leq C(\pi_{C^0(Q-O_\tau)} + \pi_{C^0(P-P_\tau)}) \leq \frac{C_\epsilon}{(1+|\tau|)^{\frac{1}{2}+\epsilon}}, \end{aligned} \tag{2.119}$$

$$(1+|\tau|)\|\widetilde{\Lambda}^{-\frac{1}{2}}Q_\tau w\|^2_{L^2(G)} \leq \frac{\widehat{C}}{2}\|Q_\tau w\|^2_{L^2(G)}, \tag{2.120}$$

where

$$\widetilde{\Lambda}^{-\frac{1}{2}} w = \int_{\mathbb{R}^n} \Big(1 + \xi_0^2 + \tau^4 + \sum_{i=1}^{n-1}\xi_i^4\Big)^{-\frac{1}{4}} \widehat{w}\, \mathrm{e}^{\mathrm{i}\langle x',\xi'\rangle}\mathrm{d}\xi'.$$

By (2.117) and (2.118), there exists a $\tau_0 > 0$ such that

$$\frac{\tau_0|Q_\tau(x,\xi',\tau)|^2}{M(\xi',\tau)} + \operatorname{Re}\{Q_\tau, P_\tau\}(x,\xi,\tau) > CM(\xi',\tau). \tag{2.121}$$

Some short computations provide

$$\begin{aligned} &\tau_0\|Q_\tau\widetilde{\Lambda}^{-\frac{1}{2}}w\|^2_{L^2(G)} + ([Q_\tau,P_\tau]w,w)_{L^2(G)} \\ &= (\tau_0(Q_\tau\widetilde{\Lambda}^{-\frac{1}{2}})^*Q_\tau\widetilde{\Lambda}^{-\frac{1}{2}}w,w)_{L^2(G)} + ([Q_\tau,P_\tau]w,w)_{L^2(G)} \\ &= \operatorname{Re}((\tau_0(Q_\tau\widetilde{\Lambda}^{-\frac{1}{2}})^*Q_\tau\widetilde{\Lambda}^{-\frac{1}{2}} + [Q_\tau,P_\tau])w,w)_{L^2(G)}. \end{aligned}$$

By (2.121) and applying Gårding's inequality, we have

$$\begin{aligned}&\tau_0\|Q_\tau\widetilde{\Lambda}^{-\frac{1}{2}}w\|^2_{L^2(G)}+\mathrm{Re}\,([Q_\tau,P_\tau]w,w)_{L^2(G)}\\ &\ge C\int_0^\gamma\|w(\,\cdot\,,x_n)\|^2_{H^{\frac14,\frac12,\tau}(\mathbb{R}^n)}\mathrm{d}x_n-C_1\|w\|^2_{L^2(G)}.\end{aligned}\tag{2.122}$$

Next, we observe

$$\begin{aligned}\|Q_\tau\widetilde{\Lambda}^{-\frac12}w\|_{L^2(G)}&=\|\widetilde{\Lambda}^{-\frac12}Q_\tau w+[Q_\tau,\widetilde{\Lambda}^{-\frac12}]w\|_{L^2(G)}\\ &\le C_0(\|\widetilde{\Lambda}^{-\frac12}Q_\tau w\|_{L^2(G)}+\|w\|_{L^2(G)})\\ &\le C\Big(\frac{\|Q_\tau w\|_{L^2(G)}}{1+|\tau|}+\|w\|_{L^2(G)}\Big),\end{aligned}\tag{2.123}$$

where in the last inequality we used the estimate (2.120). Combining (2.123) and (2.122), we obtain

$$\|Q_\tau w\|^2_{L^2(G)}+\mathrm{Re}\,([Q_\tau,P_\tau]w,w)_{L^2(G)}\ge C\int_0^\gamma\|w(\,\cdot\,,x_n)\|^2_{H^{\frac14,\frac12,\tau}(\mathbb{R}^n)}\mathrm{d}x_n-C_1\|w\|^2_{L^2(G)}.\tag{2.124}$$

Since there exists a constant $\widetilde{C}$ independent of τ such that

$$|Q_\tau(x,\xi',\tau)|^2+\widetilde{C}|\tau|^2\ge M^2(\xi',\tau),$$

Gårding's inequality yields

$$\|Q_\tau w\|^2_{L^2(G)}+\tau^2\|w\|^2_{L^2(G)}\ge C\|w\|^2_{H^{\frac12,1,\tau}(G)}.$$

This inequality and (2.124) imply

$$\|Q_\tau w\|^2_{L^2(G)}+\mathrm{Re}\,([Q_\tau,P_\tau]w,w)_{L^2(G)}\ge C\int_0^\gamma\frac{\|w(\,\cdot\,,x_n)\|^2_{H^{\frac12,1,\tau}(\mathbb{R}^n)}}{1+|\tau|}\mathrm{d}x_n-C_1\|w\|^2_{L^2(G)}.\tag{2.125}$$

On the other hand,

$$\Big\|\frac{\partial w}{\partial x_n}\Big\|_{L^2(G)}\le\|P_\tau w\|_{L^2(G)}+C\|w\|_{H^{\frac12,1,\tau}(G)}.$$

Hence (2.125) can be transformed into the following estimate:

$$\|P_\tau w\|^2_{L^2(G)}+\|Q_\tau w\|^2_{L^2(G)}+\mathrm{Re}\,([Q_\tau,P_\tau]w,w)_{L^2(G)}\ge C\|w\|^2_{H^{\frac12,1,\tau}(G)}-C_1\|w\|^2_{L^2(G)}.\tag{2.126}$$

By (2.119), we can put in the left-hand side of (2.126) the L^2-norms of the functions Pw and Qw instead of $P_\tau w$ and $Q_\tau w$ respectively. The proof of the proposition is finished.

By (2.116) and Proposition 2.3, there exist two positive constants C and C_1 independent of β,τ such that

$$\begin{aligned}\|z\|^2_{L^2(G)}\ge\frac12\|Qw_1\|^2_{L^2(G)}+\frac12\|Pw_1\|^2_{L^2(G)}+C\|w_1\|^2_{H^{\frac14,\frac12,\tau}(G)}-C_1\|w_1\|^2_{L^2(G)}\\ -\mathrm{Re}\,(Q(x',0,D',\tau)w_1(\,\cdot\,,0),w_1(\,\cdot\,,0))_{L^2(\mathbb{R}^n)}.\end{aligned}\tag{2.127}$$

Depending on the sign of the fifth term on the right-hand side of (2.127), we consider two cases.

Case 1 Assume that $\mathrm{Re}\,(Q(x',0,D',\tau)w_1(\,\cdot\,,0), w_1(\,\cdot\,,0))_{L^2(\mathbb{R}^n)} \le 0$. Then (2.127) implies

$$\|w_1\|_{H^{\frac14,\frac12,\tau}(G)} \le C(\|\widetilde{f}_1\|_{H^{-\frac12,-1,\tau}(G)} + \|w_1\|_{L^2(G)}). \tag{2.128}$$

Case 2 Assume that $\mathrm{Re}\,(Q(x',0,D',\tau)w_1(\,\cdot\,,0), w_1(\,\cdot\,,0))_{L^2(\mathbb{R}^n)} \ge 0$. By (2.80), (2.89) and (2.106), there exists a constant $C > 0$ independent of τ, β such that

$$\sqrt{1+|\tau|}\,\|w_1(\,\cdot\,,0)\|_{L^2(\mathbb{R}^n)} \le C\|w_1(\,\cdot\,,0)\|_{H^{\frac14,\frac12}(\mathbb{R}^n)}. \tag{2.129}$$

Let us consider the following (adjoint) problem:

$$L_+(x,D,\tau)^*p = \Big(-\frac{\partial}{\partial x_n} - R_+(x,D',\tau)^*\Big)p = (1+|\tau|)w_1 + v, \quad \text{in } G. \tag{2.130}$$

Set

$$m_\tau(x) = 1, \quad x \in \Pi_{\delta_0,\gamma}, \quad m_1(x) = (1+|\tau|)^{-1}, \quad x \in \mathbb{R}^{n+1} \setminus \Pi_{\delta_0,\gamma},$$

where $\delta_0 \in (\delta, \delta')$. We have

Lemma 2.2 *There exist a constant $C > 0$ independent of τ and a pair (p, v) satisfying* (2.130) *such that*

$$\sqrt{1+|\tau|}\int_G (|p|^2 + m_\tau^2|v|^2)\mathrm{d}x + \int_{\mathbb{R}^n} m_\tau^2|p(x',0)|^2\mathrm{d}x' \le C(1+|\tau|)^{\frac32}\int_G |w_1|^2\mathrm{d}x. \tag{2.131}$$

Proof For $\epsilon > 0$, let us consider the functional

$$J_\epsilon(p,v) = \frac12\|p\|^2_{L^2(G)} + \frac12\|m_\tau v\|^2_{L^2(G)} + \frac{1}{2\epsilon}\Big\|\frac{\partial p}{\partial x_n} + R_+(x,D',\tau)^*p + (1+|\tau|)w_1 + v\Big\|^2_{L^2(G)}. \tag{2.132}$$

Note that there exists a pair (p, v) such that $J_\epsilon(p, v)$ is finite, for example $(p, v) = 0$. We consider the minimization problem

$$\min_{(p,v)\in U} J_\epsilon(p,v),$$

where

$$U = \Big\{(p,v) \in L^2(G)\times L^2(G)\,\Big|\, \frac{\partial p}{\partial x_n} + R_+(x,D',\tau)^*p + (1+|\tau|)w_1 + v \in L^2(G)\Big\}.$$

We will follow a typical argument for the minimization problem (e.g., [14]). There exists a minimizing sequence $\{(p_k, v_k)\}_{k=1}^\infty$ such that $(p_k, v_k) \in U$ and

$$J_\epsilon(p_k,v_k) \to \inf_{(p,v)\in U} J_\epsilon(p,v).$$

Then $\|(p_k, v_k)\|_{L^2(G)}$ and $\|\frac{\partial p_k}{\partial x_n} + R_+(x,D',\tau)^*p_k + (1+|\tau|)w_1 + v_k\|_{L^2(G)}$ are bounded. Therefore $R_+(x,D',\tau)^*p_k$ and $\frac{\partial p_k}{\partial x_n}$ are both bounded in $L^2(0,\gamma; H^{-\frac12,-1,\tau}(\mathbb{R}^n))$.

We can then extract a subsequence, still denoted by $\{(p_k, v_k)\}_{k=1}^\infty$, such that

$$\begin{aligned}
&(p_k, v_k) \rightharpoonup (p_\epsilon, v_\epsilon), \quad \text{weakly in } L^2(G)\times L^2(G),\\
&\frac{\partial p_k}{\partial x_n} \rightharpoonup \frac{\partial p_\epsilon}{\partial x_n}, \quad \text{weakly in } L^2(0,\gamma;H^{-1}(\mathbb{R}^n)),\\
&\frac{\partial p_k}{\partial x_n} + R_+(x,D',\tau)^*p_k + (1+|\tau|)w_1 + v_k \rightharpoonup \frac{\partial p_\epsilon}{\partial x_n} + R_+(x,D',\tau)^*p_\epsilon + (1+|\tau|)w_1 + v_\epsilon,\\
&\hspace{20em} \text{weakly in } L^2(0,\gamma;H^{-\frac12,-1}(\mathbb{R}^n)).
\end{aligned}$$

However, since $\|\frac{\partial p_k}{\partial x_n} + R_+(x, D', \tau)^* p_k + (1+|\tau|)w_1 + v_k\|^2_{L^2(G)}$ stays bounded, we have

$$\frac{\partial p_k}{\partial x_n} + R_+(x, D', \tau)^* p_k + (1+|\tau|)w_1 + v_k \rightharpoonup \frac{\partial p_\epsilon}{\partial x_n} + R_+(x, D', \tau)^* p_\epsilon + (1+|\tau|)w_1 + v_\epsilon,$$

weakly in $L^2(G)$. Then (p_ϵ, v_ϵ) is a minimizer of J_ϵ, i.e., $(p_\epsilon, v_\epsilon) \in U$ and

$$J_\epsilon(p_\epsilon, v_\epsilon) = \min_{(p,v)\in U} J_\epsilon(p, v). \tag{2.133}$$

Writing the first order optimality conditions, we have

$$\langle \partial_p J_\epsilon(p_\epsilon, v_\epsilon), r\rangle = 0, \quad \langle \partial_v J_\epsilon(p_\epsilon, v_\epsilon), \widetilde{r}\rangle = 0, \quad \forall\, r \in H^{\frac{1}{2},1}(G),\ \forall\, \widetilde{r} \in L^2(G). \tag{2.134}$$

Let us define q_ϵ by

$$q_\epsilon = \frac{1}{\epsilon}\Big(\frac{\partial p_\epsilon}{\partial x_n} + R_+(x, D', \tau)^* p_\epsilon + (1+|\tau|)w_1 + v_\epsilon\Big). \tag{2.135}$$

We obtain from (2.134) that for every $r \in H^{\frac{1}{2},1}(G)$, there holds

$$\int_G p_\epsilon \overline{r}\mathrm{d}x + \int_G q_\epsilon \overline{\Big(\frac{\partial r}{\partial x_n} + R_+(x, D', \tau)^* r\Big)}\,\mathrm{d}x = 0 \tag{2.136}$$

and for every $\widetilde{r} \in L^2(G)$, there holds

$$\int_G v_\epsilon \overline{\widetilde{r}}\mathrm{d}x + \int_G q_\epsilon \overline{\widetilde{r}}\,\mathrm{d}x = 0. \tag{2.137}$$

Then q_ϵ satisfies the following problem:

$$L_+(x, D, \tau)q_\epsilon = \frac{\partial q_\epsilon}{\partial x_n} - R_+(x, D', \tau)q_\epsilon = p_\epsilon, \quad \text{in } G, \tag{2.138}$$

$$q_\epsilon = -m_\tau^2 v_\epsilon, \quad \text{in } G, \tag{2.139}$$

$$q_\epsilon(x', 0) = 0, \quad q_\epsilon(x', \gamma) = 0, \quad x' \in \mathbb{R}^n. \tag{2.140}$$

We can also write (2.138)–(2.140) in the form

$$L_+(x, D, \tau)q_\epsilon = (P+Q)(x, D, \tau)q_\epsilon = p_\epsilon, \quad \text{in } G, \tag{2.141}$$

$$q_\epsilon = -m_\tau^2 v_\epsilon, \quad \text{in } G, \tag{2.142}$$

$$q_\epsilon(x', 0) = 0, \quad q_\epsilon(x', \gamma) = 0, \quad x' \in \mathbb{R}^n. \tag{2.143}$$

Using Proposition 2.3 and (2.142), we obtain that there exists a constant $C > 0$ such that

$$\|Qq_\epsilon\|_{L^2(G)} + \|Pq_\epsilon\|_{L^2(G)} + \|q_\epsilon\|_{H^{\frac{1}{4},\frac{1}{2},\tau}(G)} \le C\|(p_\epsilon, m_\tau v_\epsilon)\|_{L^2(G)}. \tag{2.144}$$

Notice that $Qq_\epsilon \in L^2(G)$ implies $q_\epsilon \in L^2(0, \gamma; H^{\frac{1}{2},1}(\mathbb{R}^n))$, which implies $\frac{\partial q_\epsilon}{\partial x_n} \in L^2(G)$ from (2.138). Now, from the definition (2.135) of q_ϵ, we see that p_ϵ satisfies

$$\frac{\partial p_\epsilon}{\partial x_n} + R_+(x, D', \tau)^* p_\epsilon = \epsilon q_\epsilon - (1+|\tau|)w_1 - v_\epsilon,$$

which can be written as

$$(P-Q)(x, D, \tau)p_\epsilon = \frac{\partial p_\epsilon}{\partial x_n} + R_+(x, D', \tau)^* p_\epsilon = \epsilon q_\epsilon - (1+|\tau|)w_1 - v_\epsilon. \tag{2.145}$$

Multiplying (2.145) by q_ϵ in $L^2(G)$ and using the boundary conditions on q_ϵ, we obtain

$$-\int_G p_\epsilon \overline{L_+(x,D,\tau)q_\epsilon}\mathrm{d}x = \epsilon\int_G |q_\epsilon|^2\mathrm{d}x - \int_G (1+|\tau|)w_1\overline{q_\epsilon}\mathrm{d}x + \int_G m_\tau^2 v_\epsilon^2\mathrm{d}x,$$

and so

$$\int_G |p_\epsilon|^2\mathrm{d}x + \int_G m_\tau^2|v_\epsilon|^2\mathrm{d}x + \epsilon\int_G |q_\epsilon|^2\mathrm{d}x = \int_G (1+|\tau|)w_1\overline{q_\epsilon}\mathrm{d}x.$$

And by (2.144), we have

$$\begin{aligned}\|(p_\epsilon, m_\tau v_\epsilon)\|^2_{L^2(G)} &\le \Big(\int_G (1+|\tau|)|w_1|^2\mathrm{d}x\Big)^{\frac12}\Big(\int_G (1+|\tau|)|q_\epsilon|^2\mathrm{d}x\Big)^{\frac12}\\ &\le C\Big(\int_G (1+|\tau|)|w_1|^2\mathrm{d}x\Big)^{\frac12}\|(p_\epsilon, m_\tau v_\epsilon)\|_{L^2(G)}.\end{aligned}$$

Thus, we obtain the first estimate on (p_ϵ, v_ϵ),

$$\|(p_\epsilon, m_\tau v_\epsilon)\|_{L^2(G)} \le C\sqrt{1+|\tau|}\,\|w_1\|_{L^2(G)}. \tag{2.146}$$

By (2.146), there exists a subsequence $\{(p_{\epsilon_m}, v_{\epsilon_m})\}_{m=1}^\infty$ such that

$$\begin{aligned}&(p_{\epsilon_m}, v_{\epsilon_m}) \to (p, v), \quad \text{in } L^2(G)\times L^2(G),\\ &x_n^2 p_{\epsilon_m} \to x_n^2 p, \quad \text{in } H^{\frac12,1,\tau}(G),\\ &q_\epsilon \to q, \quad \text{in } H^{\frac12,1}(G).\end{aligned} \tag{2.147}$$

Using the above relations, we pass to the limit in (2.141)–(2.143). The pair $(p, v, q) \in L^2(G)\times L^2(G)\times L^2(0,\gamma; H^{\frac12,1,\tau}(\mathbb{R}^n))$ satisfies the optimality system

$$L_+(x,D,\tau)^*p = (1+|\tau|)w_1 + v, \quad \text{in } G, \tag{2.148}$$

$$L_+(x,D,\tau)q = p, \quad \text{in } G, \tag{2.149}$$

$$q = -m_\tau^2 v, \quad \text{in } G, \tag{2.150}$$

$$q(\,\cdot\,,0) = q(\,\cdot\,,\gamma) = 0. \tag{2.151}$$

Using Proposition 2.3 and (2.150), we have

$$\|Qq\|_{L^2(G)} + \|Pq\|_{L^2(G)} + \|q\|_{H^{\frac14,\frac12,\tau}(G)} \le C\|(p, m_\tau v)\|_{L^2(G)}. \tag{2.152}$$

Inequalities (2.146) and (2.147) imply

$$\|(p, m_\tau v)\|_{L^2(G)} \le C\sqrt{1+|\tau|}\,\|w_1\|_{L^2(G)}. \tag{2.153}$$

Observe that

$$p(\,\cdot\,,0) = \frac{\partial}{\partial x_n}q(\,\cdot\,,0).$$

By (2.148) and (2.149), we have

$$\mathfrak{G}(x,D,\tau)q = L_+(x,D,\tau)^*L_+(x,D,\tau)q = (Q^2 - P^2 + [Q,P])q = (1+|\tau|)w_1 + v, \quad \text{in } G.$$

Let $\theta \in C^\infty[0,\gamma]$, $\theta(0) = 1$, $\theta \equiv 0$ in a neighborhood of $x_n = \gamma$, and $\chi(x_0,\cdots,x_{n-1}) \in C_0^\infty(B(0,\delta'))$ such that $\chi|_{B(0,\delta_0)} \equiv 1$. Denote $\widetilde{\chi} = \chi\theta$. Then function $\widetilde{\chi}q$ verifies

$$\mathfrak{G}(x,D,\tau)(\widetilde{\chi}q) = (1+|\tau|)\widetilde{\chi}w_1 + \widetilde{\chi}v - [\widetilde{\chi},\mathfrak{G}]q, \quad \text{in } G.$$

Multiplying this equation by $P(x,D,\tau)(\widetilde{\chi}q)$, we have

$$\begin{aligned}\mathrm{Re}\,(\mathfrak{G}(\widetilde{\chi}q),P(\widetilde{\chi}q))_{L^2(G)} &= (Q^2(\widetilde{\chi}q),P(\widetilde{\chi}q))_{L^2(G)}+(P(\widetilde{\chi}q),Q^2(\widetilde{\chi}q))_{L^2(G)}-(P^2(\widetilde{\chi}q),P(\widetilde{\chi}q))_{L^2(G)}\\ &\quad-(P(\widetilde{\chi}q),P^2(\widetilde{\chi}q))_{L^2(G)}+\mathrm{Re}\,([Q,P](\widetilde{\chi}q),P(\widetilde{\chi}q))_{L^2(G)}\\ &=(1+|\tau|)(w_1,P(\widetilde{\chi}q))_{L^2(G)}+(\widetilde{\chi}v-[\widetilde{\chi},\mathfrak{G}]q,P(\widetilde{\chi}q))_{L^2(G)}.\end{aligned}\tag{2.154}$$

By (2.152) and (2.150), we obtain

$$\|Q(\widetilde{\chi}q)\|_{L^2(G)}+\|P(\widetilde{\chi}q)\|_{L^2(G)}+\frac{\|q\|_{H^{\frac12,1,\tau}(G)}}{\sqrt{1+|\tau|}}\le C\|(p,m_\tau v)\|_{L^2(G)}.\tag{2.155}$$

Using (2.155) and Lemma A.4, we have

$$\begin{aligned}|([Q,P](\widetilde{\chi}q),P(\widetilde{\chi}q))_{L^2(G)}| &\le \|[Q,P](\widetilde{\chi}q)\|_{L^2(G)}\|P(\widetilde{\chi}q)\|_{L^2(G)}\\ &\le C\|q\|_{H^{\frac12,1,\tau}(G)}\|P(\widetilde{\chi}q)\|_{L^2(G)}\\ &\le C\|q\|_{H^{\frac12,1,\tau}(G)}\|(p,m_\tau v)\|_{L^2(G)}\\ &\le C\sqrt{1+|\tau|}\|(p,m_\tau v)\|^2_{L^2(G)}.\end{aligned}\tag{2.156}$$

Next

$$\begin{aligned}&(Q^2(\widetilde{\chi}q),P(\widetilde{\chi}q))_{L^2(G)}+(P(\widetilde{\chi}q),Q^2(\widetilde{\chi}q))_{L^2(G)}\\ &=-(PQ^2(\widetilde{\chi}q),\widetilde{\chi}q)_{L^2(G)}+(Q^2P(\widetilde{\chi}q),\widetilde{\chi}q)_{L^2(G)}\\ &=-(QPQ(\widetilde{\chi}q),\widetilde{\chi}q)_{L^2(G)}+(Q^2P(\widetilde{\chi}q),\widetilde{\chi}q)_{L^2(G)}+([Q,P]Q(\widetilde{\chi}q),\widetilde{\chi}q)_{L^2(G)}\\ &=([Q,P](\widetilde{\chi}q),Q(\widetilde{\chi}q))_{L^2(G)}+(Q(\widetilde{\chi}q),[Q,P]^*(\widetilde{\chi}q))_{L^2(G)}.\end{aligned}\tag{2.157}$$

Hence, by (2.155) and Lemma A.4, we have

$$\begin{aligned}|\mathrm{Re}\,(Q^2(\widetilde{\chi}q),P(\widetilde{\chi}q))_{L^2(G)}| &\le C\|q\|_{H^{\frac12,1,\tau}(G)}\|(p,m_\tau v)\|_{L^2(G)}\\ &\le C\sqrt{1+|\tau|}\,\|(p,m_\tau v)\|^2_{L^2(G)}.\end{aligned}\tag{2.158}$$

Finally

$$\begin{aligned}\mathrm{Re}\,(P^2(\widetilde{\chi}q),P(\widetilde{\chi}q))_{L^2(G)} &= (P^2(\widetilde{\chi}q),P(\widetilde{\chi}q))_{L^2(G)}+(P(\widetilde{\chi}q),P^2(\widetilde{\chi}q))_{L^2(G)}\\ &=-(P^3(\widetilde{\chi}q),\widetilde{\chi}q)_{L^2(G)}+(P^3(\widetilde{\chi}q),\widetilde{\chi}q)_{L^2(G)}-\|\chi p(\,\cdot\,,0)\|^2_{L^2(\mathbb{R}^n)}\\ &=-\|\chi p(\,\cdot\,,0)\|^2_{L^2(\mathbb{R}^n)}.\end{aligned}\tag{2.159}$$

Note that

$$\begin{aligned}|(\widetilde{\chi}v-[\widetilde{\chi},\mathfrak{G}]q,P(\widetilde{\chi}q))_{L^2(G)}| &\le C(\|v\|_{L^2(G)}+\|Pq\|_{L^2(G)}+\|Qq\|_{L^2(G)})\|P(\widetilde{\chi}q)\|_{L^2(G)}\\ &\le C\|(p,m_\tau v)\|^2_{L^2(G)}.\end{aligned}\tag{2.160}$$

By (2.154)–(2.160), we have

$$\sqrt{1+|\tau|}\,\|\chi p(\,\cdot\,,0)\|^2_{L^2(\mathbb{R}^n)}\le C(1+|\tau|)\|(p,m_\tau v)\|^2_{L^2(G)}.$$

By (2.153), the right-hand side of this inequality can be estimated as follows:

$$\|\chi p(\,\cdot\,,0)\|^2_{L^2(\mathbb{R}^n)}\le C\sqrt{1+|\tau|}\,\|(p,m_\tau v)\|^2_{L^2(G)}\le C(1+|\tau|)^{\frac32}\|w_1\|^2_{L^2(G)}.\tag{2.161}$$

The proof of Lemma 2.2 is complete.

We recall that

$$\operatorname{supp} w(\,\cdot\,, x_n) \subset B(0,\delta), \quad \operatorname{supp} g(\,\cdot\,, x_n) \subset B(0,\delta), \quad \forall\, x_n \in [0,\gamma]. \tag{2.162}$$

Then by Lemma A.5, for any $\delta_2 > \delta$,

$$\|\chi_1 w(\,\cdot\,, x_n)\|_{L^2(\mathbb{R}^n\setminus B(0,\delta_2))} \le \frac{C(\delta_2)}{1+|\tau|}\|w(\,\cdot\,, x_n)\|_{L^2(\mathbb{R}^n)}, \tag{2.163}$$

$$\|\chi_1 g(\,\cdot\,, x_n)\|_{L^2(\mathbb{R}^n\setminus B(0,\delta_2))} \le \frac{C(\delta_2)}{1+|\tau|}\|g(\,\cdot\,, x_n)\|_{L^2(\mathbb{R}^n)}. \tag{2.164}$$

Taking the scalar products in $L^2(G)$ of (2.130) by w_1, integrating by parts and using (2.163), we have

$$\begin{aligned}
&(1+|\tau|)\|w_1\|^2_{L^2(G)} \\
&= (w_1, L_+(x,D,\tau)^* p - v)_{L^2(G)} \\
&= (L_+(x,D,\tau)w_1, p)_{L^2(G)} - (w_1, v)_{L^2(G)} + (\chi_1\widetilde{g}, p(\,\cdot\,,0))_{L^2(\mathbb{R}^n)} \\
&= -(w_1, v)_{L^2(G)} + (\chi_1\widetilde{g}, p(\,\cdot\,,0))_{L^2(\mathbb{R}^n)} + (\chi_1 z, p)_{L^2(G)} + ([L_+,\chi_1]w, p)_{L^2(G)} \\
&\le C(\|z\|_{L^2(G)}\|p\|_{L^2(G)} + \|\widetilde{g}\|_{L^2(\mathbb{R}^n)}\|m_\tau p(\,\cdot\,,0)\|_{L^2(\mathbb{R}^n)} \\
&\quad + \|w\|_{L^2(G)}\|m_\tau v\|_{L^2(G)} + \|w\|_{L^2(G)}\|p\|_{L^2(G)}).
\end{aligned}$$

By (2.131), from this inequality, we obtain

$$\begin{aligned}
(1+|\tau|)\|w_1\|^2_{L^2(G)} \le{}& C(\|z\|_{L^2(G)}\|p\|_{L^2(G)} + \|\widetilde{g}\|_{L^2(\mathbb{R}^n)}(1+|\tau|)^{\frac34}\|w_1\|_{L^2(G)} \\
&+ \|w\|_{L^2(G)}\|m_\tau v\|_{L^2(G)}) \\
\le{}& C(\|z\|_{L^2(G)}\sqrt{1+|\tau|}\,\|w_1\|_{L^2(G)} + \|\widetilde{g}\|_{L^2(\mathbb{R}^n)}(1+|\tau|)^{\frac34}\|w_1\|_{L^2(G)} \\
&+ \sqrt{1+|\tau|}\,\|w_1\|^2_{L^2(G)} + \sqrt{1+|\tau|}\,\|w\|^2_{L^2(G)}).
\end{aligned}$$

This inequality and (2.112) imply

$$\begin{aligned}
\|w_1\|_{L^2(G)} &\le C\Big(\frac{\|\chi_1\widetilde{g}\|_{L^2(\mathbb{R}^n)}}{(1+|\tau|)^{\frac14}} + \frac{\|z\|_{L^2(G)}}{\sqrt{1+|\tau|}} + \frac{\|w\|_{L^2(G)}}{\sqrt{1+|\tau|}}\Big) \\
&\le C\Big(\frac{\|\chi_1\widetilde{g}\|_{L^2(\mathbb{R}^n)}}{(1+|\tau|)^{\frac14}} + \frac{\|\widetilde{f}_1\|_{H^{-\frac12,-1,\tau}(G)}}{\sqrt{1+|\tau|}} + \frac{\|w\|_{L^2(G)}}{\sqrt{1+|\tau|}}\Big).
\end{aligned} \tag{2.165}$$

By (2.129) and (2.165), we have

$$\sqrt{1+|\tau|}\,\|w_1\|_{L^2(G)} \le C\Big(\frac{\|\chi_1\widetilde{g}\|_{H^{\frac14,\frac12}(\mathbb{R}^n)}}{(1+|\tau|)^{\frac14}} + \|\widetilde{f}_1\|_{H^{-\frac12,-1,\tau}(G)} + \|w\|_{L^2(G)}\Big). \tag{2.166}$$

Taking into account (2.166), (2.128) and (2.97), we obtain (2.83). Really,

$$\begin{aligned}
\sqrt{1+|\tau|}\,\|w\|_{L^2(G)} &= \sqrt{1+|\tau|}\sum_{\mu=0}^{1}\|w_\mu\|_{L^2(G)} \\
&\le C\sum_{\mu=0}^{1}(\|\chi_\mu\widetilde{f} + [\chi_\mu, L(x,D,\tau)]w\|_{H^{-\frac12,-1,\tau}(G)} + (1+|\tau|)^{-\frac14}\|\chi_\mu\widetilde{g}\|_{H^{\frac14,\frac12}(\mathbb{R}^n)})
\end{aligned}$$

$$\leq C\Big(\|\widetilde{f}\|_{H^{-\frac{1}{2},-1,\tau}(G)} + (1+|\tau|)^{-\frac{1}{4}}\|\widetilde{g}\|_{H^{\frac{1}{4},\frac{1}{2}}(\mathbb{R}^n)} + \frac{\|w\|_{H^{\frac{1}{2},1,\tau}(G)}}{(1+|\tau|)^{1-\frac{1}{\kappa}}}\Big). \tag{2.167}$$

Then, from the energy estimate for solutions of problem (2.76)–(2.78), we have

$$\|w\|_{H^{\frac{1}{2},1}(G)} \leq C(\|g\|_{H^{\frac{1}{4},\frac{1}{2}}(\mathbb{R}^n)} + (1+|\tau|)\|w\|_{L^2(G)} + \|\widetilde{f}\|_{H^{-\frac{1}{2},-1,\tau}(G)}). \tag{2.168}$$

By (2.167) and (2.168), we obtain (2.83). The proof of Theorem 2.1 is complete.

3 Carleman Estimate for the Stokes System

Consider the Stokes system

$$P(D)y = \frac{\partial y}{\partial x_0} - \Delta y = \nabla p + f, \quad \text{in } Q, \tag{3.1}$$

$$\operatorname{div} y = 0, \quad \text{in } Q,$$

$$y = 0, \quad \text{on } (0,T)\times\Gamma,$$

$$y(0,x) = y_0. \tag{3.2}$$

We introduce the following spaces

$$H = \{u = (u_1, \cdots, u_n) \in (L^2(\Omega))^n \mid \operatorname{div} u = 0, (u,\nu)|_{\partial\Omega} = 0\}, \quad n = 2, 3,$$
$$V = \{u = (u_1, \cdots, u_n) \in (H_0^1(\Omega))^n \mid u \in H\}.$$

The proof of the following proposition can be found in the classical book [18].

Proposition 3.1 (1) *Let $y_0 \in H$ and $f \in L^2(0,T;V')$. Then there exists a unique solution y to problem* (3.1)–(3.2) *with $y \in L^2(0,T;V) \cap C(0,T;H)$, $\frac{\partial y}{\partial x_0} \in L^2(0,T;V')$.*

(2) *Let $y_0 \in V$, $f \in L^2(0,T;H)$. Then there exists a solution (y,p) to problem* (3.1)–(3.2) *with $(y,p) \in C(0,T;V) \cap H^{1,2}(Q) \times L^2(0,T;H^1(\Omega))$, and the following a priori estimate holds:*

$$\|(y,p)\|_{H^{1,2}(Q)\times L^2(Q)} \leq C(\|y_0\|_V + \|f\|_{L^2(0,T;H)}). \tag{3.3}$$

The goal of this section is to prove the following Carleman estimate for solutions of problem (3.1)–(3.2).

Theorem 3.1 *Let $\kappa = 8$, $f \in L^2(0,T;H)$, $y_0 \in V$ and $y \in L^2(0,T,V) \cap H^{1,2}(Q)$ be a solution of* (3.1)–(3.2). *Then there exists a constant $\widehat{\lambda}$ such that for any $\lambda > \widehat{\lambda}$, there exist constants $C > 0$ and $\widehat{s}$ independent of s, such that*

$$\begin{aligned}&\|s^{\frac{1}{2}}\varphi^{\frac{1}{2}}(\operatorname{rot} y)\mathrm{e}^{s\alpha}\|_{L^2(Q)} + \|s\varphi y \mathrm{e}^{s\alpha}\|_{L^2(Q)}\\ \leq\,& C(\|f\mathrm{e}^{s\alpha}\|_{L^2(Q)} + \|s^{\frac{1}{2}}\varphi^{\frac{1}{2}}(\operatorname{rot} y)\mathrm{e}^{s\alpha}\|_{L^2(Q_\omega)} + \|s\varphi y\mathrm{e}^{s\alpha}\|_{L^2(Q_\omega)}), \quad \forall\, s \geq \widehat{s}.\end{aligned} \tag{3.4}$$

The Carleman estimates for the Stokes system are applicable to the exact controllability problem. See [4, 6, 10, 11] as related works.

First, we prove the following simple proposition.

Proposition 3.2 *Let $u \in H^{1,2}(Q) \cap L^2(0,T;H_0^1(\Omega))$. Then $\frac{\partial u}{\partial \nu} \in H^{\frac{1}{4},\frac{1}{2}}(\Sigma)$.*

Proof Let $\vec{n}(x_1,\cdots,x_n)\in C^1(\overline{\Omega})$ be a smooth vector field such that $\vec{n}=\nu$ on $\partial\Omega$. Since the function $\sum\limits_{i=1}^{n} n_i\partial_{x_i}u\in L^2(0,T;H^1(\Omega))$, we have $\frac{\partial y}{\partial\nu}\in L^2(0,T;H^{\frac12}(\partial\Omega))$. In order to show that $\frac{\partial u}{\partial\nu}\in H^{\frac14}(0,T;L^2(\partial\Omega))$, we observe that by using a partition of unity and a local change of variables, it suffices to consider a situation when $\Omega=\{(x_1,\cdots,x_n)\mid x_n>0\}$ and the function u has a support in $B(0,\delta)\cap\{(x_1,\cdots,x_n)\mid x_n\ge 0\}$. Denote by $\widehat{u}$ the Fourier transform with respect to the variables $x_0,\cdots,x_{n-1}$. Then

$$\begin{aligned}\|u\|^2_{H^{\frac14}(0,T;L^2(\partial\Omega))}&=-\int_{\mathbb{R}^{n+1}_+}\frac{\partial}{\partial x_n}\Big|\frac{\partial\widehat{u}}{\partial x_n}\Big|^2\sqrt{1+|\xi_0|}\,\mathrm{d}\xi_0\cdots\mathrm{d}\xi_{n-1}\mathrm{d}x_n\\&=-\int_{\mathbb{R}^{n+1}_+}(\partial^2_{x_nx_n}\widehat{u}\overline{\partial_{x_n}\widehat{u}}+\partial_{x_n}\widehat{u}\partial^2_{x_nx_n}\overline{\widehat{u}})\sqrt{1+|\xi_0|}\,\mathrm{d}\xi_0\cdots\mathrm{d}\xi_{n-1}\mathrm{d}x_n\\&\le\int_{\mathbb{R}^{n+1}_+}(|\partial^2_{x_nx_n}\widehat{u}|^2+(1+|\xi_0|)|\partial_{x_n}\widehat{u}|^2)\mathrm{d}\xi_0\cdots\mathrm{d}\xi_{n-1}\mathrm{d}x_n.\end{aligned}\tag{3.5}$$

Integrating by parts and taking into account the Dirichlet boundary conditions we have

$$\int_{\mathbb{R}^{n+1}_+}(1+|\xi_0|)|\partial_{x_n}\widehat{u}|^2\mathrm{d}\xi_0\cdots\mathrm{d}\xi_{n-1}\mathrm{d}x_n=\int_{\mathbb{R}^{n+1}_+}(1+|\xi_0|)\widehat{u}\overline{\partial^2_{x_nx_n}\widehat{u}}\mathrm{d}\xi_0\cdots\mathrm{d}\xi_{n-1}\mathrm{d}x_n.$$

Applying the Cauchy-Bynakovskii inequality, we obtain

$$\begin{aligned}&\int_{\mathbb{R}^{n+1}_+}(1+|\xi_0|)|\partial_{x_n}\widehat{u}|^2\mathrm{d}\xi_0\cdots\mathrm{d}\xi_{n-1}\mathrm{d}x_n\\\le&\int_{\mathbb{R}^{n+1}_+}(|\partial^2_{x_nx_n}\widehat{u}|^2+(1+|\xi_0|)^2|\widehat{u}|^2)\mathrm{d}\xi_0\cdots\mathrm{d}\xi_{n-1}\mathrm{d}x_n.\end{aligned}\tag{3.6}$$

Combining (3.5) and (3.6), we have

$$\|u\|_{H^{\frac14}(0,T;L^2(\partial\Omega))}\le C\|u\|_{H^{1,2}(Q)}.$$

The proof of the proposition is finished.

Proof of Theorem 3.1 Applying the operator rot to equation (3.1), we have

$$\frac{\partial\mathrm{rot}\,y}{\partial x_0}-\Delta\mathrm{rot}\,y=\mathrm{rot}\,f,\quad\text{in } Q.\tag{3.7}$$

Next, we apply the Carleman estimate (2.38) to (3.7). There exists an $s_0>0$ such that

$$\begin{aligned}s\int_Q\varphi|\mathrm{rot}\,y|^2\mathrm{e}^{2s\alpha}\mathrm{d}x\le&\,C\Big(s^{-\frac12}\|\varphi^{-\frac14}\mathrm{rot}\,y\mathrm{e}^{s\widehat{\alpha}}\|^2_{H^{\frac14,\frac12}(\Sigma)}+s^{-\frac12}\|\varphi^{-\frac14+\frac1\kappa}\mathrm{rot}\,y\mathrm{e}^{s\widehat{\alpha}}\|^2_{L^2(\Sigma)}\\&+\int_Q|f|^2\mathrm{e}^{2s\alpha}\mathrm{d}x+\int_{Q_\omega}s\varphi|\mathrm{rot}\,y|^2\mathrm{e}^{2s\alpha}\mathrm{d}x\Big),\quad\forall\,s\ge s_0,\end{aligned}\tag{3.8}$$

where we set $\widehat{\alpha}(x_0)=\alpha(x)|_{\partial\Omega}$ by (2.8). Since $\mathrm{div}\,y=0$, we have $-\Delta y=\mathrm{rot}\,\mathrm{rot}\,y$. Setting $u=y\mathrm{e}^{s\alpha}$, we obtain

$$-\mathrm{e}^{s\alpha}\Delta\mathrm{e}^{-s\alpha}u=\mathrm{e}^{s\alpha}\mathrm{rot}\,\mathrm{rot}\,y=\mathrm{rot}(\mathrm{e}^{s\alpha}\mathrm{rot}\,y)+[\mathrm{e}^{s\alpha},\mathrm{rot}]\mathrm{rot}\,y.$$

Notice that

$$[\mathrm{e}^{s\alpha},\mathrm{rot}]\mathrm{rot}\,y(x)=s\frac{c(\widetilde{x})}{\ell(x_0)^\kappa}(\mathrm{e}^{s\alpha}\mathrm{rot}\,y),\tag{3.9}$$

where $c \in (C^1(\overline{\Omega}))^3$ is some function.

Applying the Carleman estimate for elliptic equations obtained in [12] and using (3.9), we have

$$\begin{aligned}&\sqrt{\frac{s}{\ell^\kappa(x_0)}}\,\|y\mathrm{e}^{s\alpha}\|_{L^2(\Omega)}\\ &\le C\Big(\|\mathrm{rot}\, y\mathrm{e}^{s\alpha}\|_{L^2(\Omega)} + \sqrt{\frac{s}{\ell^\kappa(x_0)}}\,\|y\mathrm{e}^{s\alpha}\|_{L^2(\omega)}\Big), \quad \forall\, s \ge s_1,\ \forall\, x_0 \in [0,T], \end{aligned}\tag{3.10}$$

where the constants C and s_0 are independent of s, x_0. Combining (3.8) and (3.10), we obtain

$$\begin{aligned}&\int_Q s\varphi|\mathrm{rot}\, y|^2\mathrm{e}^{2s\alpha}\mathrm{d}x + s^2\|\varphi y\mathrm{e}^{s\alpha}\|^2_{L^2(Q)}\\ &\le C\Big(s^{-\frac12}\Big\|\varphi^{-\frac14}\frac{\partial y}{\partial\nu}\mathrm{e}^{s\widehat{\alpha}}\Big\|^2_{H^{\frac14,\frac12}(\Sigma)} + s^{-\frac12}\Big\|\varphi^{-\frac14+\frac1\kappa}\frac{\partial y}{\partial\nu}\mathrm{e}^{s\widehat{\alpha}}\Big\|^2_{L^2(\Sigma)} + \int_Q |f|^2\mathrm{e}^{2s\alpha}\mathrm{d}x\\ &\quad + \int_{Q_\omega} s\varphi|\mathrm{rot}\, y|^2\mathrm{e}^{2s\alpha}\mathrm{d}x + s^2\|\varphi y\mathrm{e}^{s\alpha}\|^2_{L^2(Q_\omega)}\Big), \quad \forall\, s \ge \max\{s_0, s_1\}.\end{aligned}\tag{3.11}$$

We need to estimate the first term on the right-hand side of (3.11). Denote

$$(w,q) = \ell(y\mathrm{e}^{s\widehat{\alpha}(x_0)}, p\mathrm{e}^{s\widehat{\alpha}(x_0)}).$$

The pair (w,q) solves the following initial/boundary value problem:

$$P(D)w = \frac{\partial w}{\partial x_0} - \Delta w = \nabla q + f\ell\mathrm{e}^{s\widehat{\alpha}} + s\widehat{\alpha}'\ell w + y\mathrm{e}^{s\widehat{\alpha}}\ell'\ell, \quad \text{in } Q, \tag{3.12}$$

$$\begin{aligned}&\mathrm{div}\, w = 0, \quad \text{in } Q,\\ &w = 0, \quad \text{on } (0,T)\times\Gamma,\\ &w(0,\,\cdot\,) = 0.\end{aligned}\tag{3.13}$$

By Proposition 3.1 and the fact that $\widehat{\alpha}(x_0) \le \alpha(x)$ for all $x \in Q$, for solutions to problem (3.12)–(3.13), we have

$$\begin{aligned}\|(w,q)\|_{H^{1,2}(Q)\times L^2(0,T;H^1(\Omega))} &\le C(\|f\mathrm{e}^{s\widehat{\alpha}}\|_{L^2(Q)} + \|sy\mathrm{e}^{s\widehat{\alpha}}\|_{L^2(Q)})\\ &\le C(\|f\mathrm{e}^{s\alpha}\|_{L^2(Q)} + \|sy\mathrm{e}^{s\alpha}\|_{L^2(Q)}).\end{aligned}\tag{3.14}$$

Using Proposition 3.2, we observe that there exists a constant $C > 0$ such that

$$\Big\|\frac{\partial w}{\partial\nu}\Big\|_{H^{\frac14,\frac12}(\Sigma)} \le C\|w\|_{H^{1,2}(Q)}. \tag{3.15}$$

Fix $\kappa = 8$. Then

$$\|s\varphi^{\frac34+\frac{3}{2\kappa}}y\mathrm{e}^{s\alpha}\|_{L^2(Q)} \le C\|s\varphi y\mathrm{e}^{s\alpha}\|_{L^2(Q)}.$$

Combining (3.15) with (3.14), we have

$$\Big\|\frac{\partial w}{\partial\nu}\Big\|_{H^{\frac14,\frac12}(\Sigma)} \le C(\|f\mathrm{e}^{s\alpha}\|_{L^2(Q)} + \|s\varphi y\mathrm{e}^{s\alpha}\|_{L^2(Q)}). \tag{3.16}$$

Hence

$$\begin{aligned}&s^{-\frac12}\Big\|\varphi^{-\frac14}\frac{\partial y}{\partial\nu}\mathrm{e}^{s\widehat{\alpha}}\Big\|^2_{H^{\frac14,\frac12}(\Sigma)} + s^{-\frac12}\Big\|\varphi^{-\frac14+\frac1\kappa}\frac{\partial y}{\partial\nu}\mathrm{e}^{s\widehat{\alpha}}\Big\|^2_{L^2(\Sigma)}\\ &\le Cs^{-\frac12}\Big\|\varphi^{-\frac14+\frac1\kappa}\frac{\partial y}{\partial\nu}\mathrm{e}^{s\widehat{\alpha}}\Big\|^2_{H^{\frac14,\frac12}(\Sigma)} \le Cs^{-\frac12}\|w\|^2_{H^{1,2}(Q)}\\ &\le Cs^{-\frac12}(\|f\mathrm{e}^{s\alpha}\|^2_{L^2(Q)} + \|s\varphi y\mathrm{e}^{s\alpha}\|^2_{L^2(Q)}).\end{aligned}\tag{3.17}$$

The first two terms on the right-hand side of (3.11) can be estimated by the right-hand side of (3.17). Observe that the term $Cs^{-\frac{1}{2}}\|s\varphi y e^{s\alpha}\|^2_{L^2(Q)}$ can be absorbed by the left-hand side of (3.11) for all sufficiently large s. This proves the statement of the theorem.

4 Observability Estimate for a Parabolic System with Parameter

Consider the system of parabolic equations

$$P(D)y = \frac{\partial y}{\partial x_0} - \Delta y - \frac{1}{\varepsilon}\nabla \mathrm{div}\, y = f, \quad \text{in } Q, \tag{4.1}$$

$$y = 0, \quad \text{on } (0,T)\times\Gamma, \tag{4.2}$$

$$y(0,x) = y_0. \tag{4.3}$$

Here ε is a positive parameter. The goal of this section is to obtain an observability estimate for system (4.1)–(4.3), which is uniform with respect to the small parameter ϵ. We have

Theorem 4.1 *Let $\kappa = 8$, $f \in L^2(Q)$, $y_0 \in H_0^1(\Omega)$, and $y \in L^2(0,T;H_0^1(\Omega)) \cap H^{1,2}(Q)$ be a solution of (4.1)–(4.3). Then there exists a constant $\widehat{\lambda}$ such that for any $\lambda > \widehat{\lambda}$, there exist constants $C>0$ and $\widehat{s}$ independent of s, such that*

$$\begin{aligned}
&\|s^{\frac{1}{2}}\varphi^{\frac{1}{2}}(\mathrm{div}\, y)e^{s\alpha}\|_{L^2(Q)} + \|s^{\frac{1}{2}}\varphi^{\frac{1}{2}}(\mathrm{rot}\, y)e^{s\alpha}\|_{L^2(Q)} + \|s\varphi y e^{s\alpha}\|_{L^2(Q)} \\
&\le C(\|f e^{s\alpha}\|_{L^2(Q)} + \|s^{\frac{1}{2}}\varphi^{\frac{1}{2}}(\mathrm{div}\, y)e^{s\alpha}\|_{L^2(Q_\omega)} \\
&\quad + \|s^{\frac{1}{2}}\varphi^{\frac{1}{2}}(\mathrm{rot}\, y)e^{s\alpha}\|_{L^2(Q_\omega)} + \|s\varphi y e^{s\alpha}\|_{L^2(Q_\omega)}), \quad \forall s \ge \widehat{s}.
\end{aligned} \tag{4.4}$$

Proof Applying the operators rot and div to equation (4.1), we have

$$\frac{\partial \mathrm{rot}\, y}{\partial x_0} - \Delta \mathrm{rot}\, y = \mathrm{rot}\, f, \quad \text{in } Q, \tag{4.5}$$

$$\frac{\partial \mathrm{div}\, y}{\partial x_0} - \Big(1 + \frac{1}{\epsilon}\Big)\Delta \mathrm{div}\, y = \mathrm{div}\, f, \quad \text{in } Q. \tag{4.6}$$

Next, we apply the Carleman estimate (2.38) to (4.5) and (4.6). There exists an $s_0 > 0$ such that

$$\begin{aligned}
s\int_Q \varphi|\mathrm{rot}\, y|^2 e^{2s\alpha}dx \le\, & C\Big(s^{-\frac{1}{2}}\|\varphi^{-\frac{1}{4}}\mathrm{rot}\, y e^{s\widehat{\alpha}}\|^2_{H^{\frac{1}{4},\frac{1}{2}}(\Sigma)} + s^{-\frac{1}{2}}\|\varphi^{-\frac{1}{4}+\frac{1}{\kappa}}\mathrm{rot}\, y e^{s\widehat{\alpha}}\|^2_{L^2(\Sigma)} \\
& + \int_Q |f|^2 e^{2s\alpha}dx + \int_{Q_\omega} s\varphi|\mathrm{rot}\, y|^2 e^{2s\alpha}dx\Big), \quad \forall s \ge s_0,
\end{aligned} \tag{4.7}$$

$$\begin{aligned}
s\int_Q \varphi|\mathrm{div}\, y|^2 e^{2s\alpha}dx \le\, & C\Big(s^{-\frac{1}{2}}\|\varphi^{-\frac{1}{4}}\mathrm{div}\, y e^{s\widehat{\alpha}}\|^2_{H^{\frac{1}{4},\frac{1}{2}}(\Sigma)} + s^{-\frac{1}{2}}\|\varphi^{-\frac{1}{4}+\frac{1}{\kappa}}\mathrm{rot}\, y e^{s\widehat{\alpha}}\|^2_{L^2(\Sigma)} \\
& + \int_Q |f|^2 e^{2s\alpha}dx + \int_{Q_\omega} s\varphi|\mathrm{div}\, y|^2 e^{2s\alpha}dx\Big), \quad \forall s \ge s_0,
\end{aligned} \tag{4.8}$$

where $\widehat{\alpha}(x_0) = \alpha(x)|_{\partial\Omega}$. Using the formula $\Delta y = -\mathrm{rot}\,\mathrm{rot}\, y + \nabla \mathrm{div}\, y$ and setting $u = ye^{s\alpha}$, we obtain

$$\begin{aligned}
e^{s\alpha}\Delta e^{-s\alpha}u &= e^{s\alpha}(-\mathrm{rot}\,\mathrm{rot}\, y + \nabla \mathrm{div}\, y) \\
&= -\mathrm{rot}(e^{s\alpha}\mathrm{rot}\, y) + \nabla(e^{s\alpha}\mathrm{div}\, y) - [e^{s\alpha}, \mathrm{rot}]\mathrm{rot}\, y + [e^{s\alpha}, \nabla]\mathrm{div}\, y.
\end{aligned}$$

Note that

$$[\mathrm{e}^{s\alpha}, \mathrm{rot}]\mathrm{rot}\, y(x) = s\frac{c(\widetilde{x})}{\ell(x_0)^\kappa}(\mathrm{e}^{s\alpha}\mathrm{rot}\, y), \tag{4.9}$$

where $c \in (C^1(\overline{\Omega}))^3$ is some function.

Applying the Carleman estimate for elliptic equations obtained in [12] and using (4.9), we have

$$\sqrt{\frac{s}{\ell^\kappa(x_0)}}\,\|y\mathrm{e}^{s\alpha}\|_{L^2(\Omega)} \le C\Big(\|\mathrm{rot}\, y\mathrm{e}^{s\alpha}\|_{L^2(\Omega)} + \|\mathrm{div}\, y\mathrm{e}^{s\alpha}\|_{L^2(\Omega)} + \sqrt{\frac{s}{\ell^\kappa(x_0)}}\,\|y\mathrm{e}^{s\alpha}\|_{L^2(\omega)}\Big) \tag{4.10}$$

for all $s \ge s_1$ and $x_0 \in [0,T]$, where the constants C and s_0 are independent of s, x_0. Therefore, combining (4.7), (4.8) and (4.10), we have

$$\begin{aligned}
&\int_Q s\varphi(|\mathrm{rot}\, y|^2 + |\mathrm{div}\, y|^2)\mathrm{e}^{2s\alpha}\mathrm{d}x + s^2\|\varphi y\mathrm{e}^{s\alpha}\|^2_{L^2(Q)} \\
&\le C\Big(s^{-\frac12}\Big\|\varphi^{-\frac14}\frac{\partial y}{\partial\nu}\mathrm{e}^{s\widehat{\alpha}}\Big\|^2_{H^{\frac14,\frac12}(\Sigma)} + s^{-\frac12}\Big\|\varphi^{-\frac14+\frac1\kappa}\frac{\partial y}{\partial\nu}\mathrm{e}^{s\widehat{\alpha}}\Big\|^2_{L^2(\Sigma)} + \int_Q |f|^2\mathrm{e}^{2s\alpha}\mathrm{d}x \\
&\quad + \int_{Q_\omega} s\varphi|\mathrm{rot}\, y|^2\mathrm{e}^{2s\alpha}\mathrm{d}x + \int_{Q_\omega} s\varphi|\mathrm{div}\, y|^2\mathrm{e}^{2s\alpha}\mathrm{d}x + s^2\|\varphi y\mathrm{e}^{s\alpha}\|^2_{L^2(Q_\omega)}\Big)
\end{aligned} \tag{4.11}$$

for all $s \ge \max\{s_0, s_1\}$. We need to estimate the first term on the right-hand side of (4.11). Denote $w = \ell y\mathrm{e}^{s\widehat{\alpha}(x_0)}$. The function w solves the following initial/boundary value problem:

$$P(D)w = \frac{\partial w}{\partial x_0} - \Delta w - \frac{1}{\varepsilon}\nabla\mathrm{div}\, w = f\ell\mathrm{e}^{s\widehat{\alpha}} + s\widehat{\alpha}'\ell w + y\mathrm{e}^{s\widehat{\alpha}}\ell'\ell, \quad \text{in } Q, \tag{4.12}$$

$$\begin{aligned}
&\mathrm{div}\, w = 0, \quad \text{in } Q, \\
&w = 0, \quad \text{on } (0,T)\times\Gamma, \\
&w(0,\cdot) = 0.
\end{aligned} \tag{4.13}$$

Using standard a priori estimates for a parabolic equations and the fact $\widehat{\alpha}(x_0) \le \alpha(x)$ for all $x \in Q$, we have the following estimate for solutions of (4.12)–(4.13):

$$\begin{aligned}
\|w\|_{H^{1,2}(Q)} &\le C(\|f\mathrm{e}^{s\widehat{\alpha}}\|_{L^2(Q)} + \|s\varphi^{\frac34+\frac{3}{2\kappa}}y\mathrm{e}^{s\widehat{\alpha}}\|_{L^2(Q)}) \\
&\le C(\|f\mathrm{e}^{s\alpha}\|_{L^2(Q)} + \|s\varphi^{\frac34+\frac{3}{2\kappa}}y\mathrm{e}^{s\alpha}\|_{L^2(Q)}).
\end{aligned} \tag{4.14}$$

Using Proposition 3.2, we observe that there exists a constant $C > 0$ such that

$$\Big\|\frac{\partial w}{\partial\nu}\Big\|_{H^{\frac14,\frac12}(\Sigma)} \le C\|w\|_{H^{1,2}(Q)}. \tag{4.15}$$

Fix $\kappa = 8$. Then

$$\|s\varphi^{\frac34+\frac{3}{2\kappa}}y\mathrm{e}^{s\alpha}\|_{L^2(Q)} \le C\|s\varphi y\mathrm{e}^{s\alpha}\|_{L^2(Q)}.$$

Combining (4.15) with (4.14), we have

$$\Big\|\frac{\partial w}{\partial\nu}\Big\|_{H^{\frac14,\frac12}(\Sigma)} \le C(\|f\mathrm{e}^{s\alpha}\|_{L^2(Q)} + \|s\varphi y\mathrm{e}^{s\alpha}\|_{L^2(Q)}). \tag{4.16}$$

Hence

$$\begin{aligned}
&s^{-\frac12}\Big\|\varphi^{-\frac14}\frac{\partial y}{\partial\nu}\mathrm{e}^{s\widehat{\alpha}}\Big\|^2_{H^{\frac14,\frac12}(\Sigma)} + s^{-\frac12}\Big\|\varphi^{-\frac14+\frac1\kappa}\frac{\partial y}{\partial\nu}\mathrm{e}^{s\widehat{\alpha}}\Big\|^2_{L^2(\Sigma)} \\
&\le Cs^{-\frac12}\Big\|\varphi^{-\frac14+\frac1\kappa}\frac{\partial y}{\partial\nu}\mathrm{e}^{s\widehat{\alpha}}\Big\|^2_{H^{\frac14,\frac12}(\Sigma)} \le Cs^{-\frac12}\|w\|^2_{H^{1,2}(Q)} \\
&\le Cs^{-\frac12}(\|f\mathrm{e}^{s\alpha}\|^2_{L^2(Q)} + \|s\varphi y\mathrm{e}^{s\alpha}\|^2_{L^2(Q)}).
\end{aligned} \tag{4.17}$$

The first two terms on the right-hand side of (4.11) can be estimated by the right-hand side of (4.17). Observe that the term $Cs^{-\frac{1}{2}}\|s\varphi y e^{s\alpha}\|^2_{L^2(Q)}$ can be absorbed by the left-hand side of (4.11) for all sufficiently large s. This proves the statement of the theorem.

Appendix Calculus for Pseudodifferential Operators with a Parameter

Let $\mathcal{O}$ be a domain in $\mathbb{R}^n$.

Definition A.1 *We say that the symbol $a(x',\xi',s) \in C^0(\overline{\mathcal{O}} \times \mathbb{R}^{n+1})$ belongs to the class $C^k_{\mathrm{cl}}S^{\frac{\kappa}{2},\kappa,s}(\mathcal{O})$, if*

(1) *There exists a compact set $K \subset\subset \mathcal{O}$, such that $a(x',\xi',s)|_{\mathcal{O}\setminus K} = 0$,*

(2) *For any $\beta = (\beta_0,\cdots,\beta_n)$, there exists a constant C_β such that*

$$\Big\|\frac{\partial^{\beta_0}}{\partial\xi_0^{\beta_0}}\cdots\frac{\partial^{\beta_{n-1}}}{\partial\xi_{n-1}^{\beta_{n-1}}}\frac{\partial^{\beta_n}}{\partial s^{\beta_n}}a(\,\cdot\,,\xi',s)\Big\|_{C^k(\overline{\mathcal{O}})} \le C_\beta\Big(|\xi_0| + s^2 + \sum_{i=1}^{n-1}\xi_i^2\Big)^{\frac{\kappa-|\beta|}{2}},$$

where $|\beta| = 2\beta_0 + \sum\limits_{j=1}^{n}\beta_j$ and $M(\xi',s) \ge 1$,

(3) *For any $N \in \mathbb{N}_+$, the symbol a can be represented as*

$$a(x',\xi',s) = \sum_{j=1}^{N} a_j(x',\xi',s) + R_N(x',\xi',s),$$

where the functions a_j have the following properties:

$$a_j(x',\tau^2\xi_0,\tau\xi_1,\cdots,\tau\xi_{n-1},\tau s) = \tau^{\kappa-j}a_j(x',\xi',s)$$

for all $\tau > 1$ and $(x',\xi',s) \in \{(x',\xi',s) \mid x' \in K,\ M(\xi',s) > 1\}$, and

$$\Big\|\frac{\partial^{\beta_0}}{\partial\xi_0^{\beta_0}}\cdots\frac{\partial^{\beta_{n-1}}}{\partial\xi_{n-1}^{\beta_{n-1}}}\frac{\partial^{\beta_n}}{\partial s^{\beta_n}}a_j(\,\cdot\,,\xi',s)\Big\|_{C^k(\overline{\mathcal{O}})} \le C_\beta\Big(|\xi_0| + s^2 + \sum_{i=1}^{n-1}\xi_i^2\Big)^{\frac{\kappa-j-|\beta|}{2}}$$

for all β and (ξ',s) such that $M(\xi',s) \ge 1$, and the term R_N satisfies the estimate

$$\|R_N(\,\cdot\,,\xi',s)\|_{C^k(\overline{\mathcal{O}})} \le C_N\Big(|\xi_0| + s^2 + \sum_{i=1}^{n-1}\xi_i^2\Big)^{\frac{\kappa-N}{2}}$$

for all (ξ',s) such that $M(\xi',s) \ge 1$.

For the symbol a, we introduce the seminorm

$$\begin{aligned}\pi_{C^k}(a) = \sum_{j=1}^{\widehat{N}} \sup_{|\beta|\le\widehat{N}} \sup_{M(\xi',s)\ge 1} \Big\|\frac{\partial^{\beta_0}}{\partial\xi_0^{\beta_0}}\cdots\frac{\partial^{\beta_{n-1}}}{\partial\xi_{n-1}^{\beta_{n-1}}}\frac{\partial^{\beta_n}}{\partial s^{\beta_n}}a_j(\,\cdot\,,\xi',s)\Big\|_{C^k(\overline{\mathcal{O}})}(1+M(\xi',s))^{-\kappa+j+|\beta|}\\ + \sup_{M(\xi',s)\le 1}\|a(\,\cdot\,,\xi',s)\|_{C^\kappa(\overline{\mathcal{O}})}.\end{aligned}$$

Let $\{\omega_j\}_{j=1}^\infty$ be a sequence of eigenfunctions of the operator Δ on $\mathcal{M} = \{(\xi_0,\cdots,\xi_n) \mid \xi_0^2 + \sum\limits_{j=1}^{n}\xi_i^4 = 1\}$, and let $\{\lambda_j\}_{j=1}^\infty$ be a sequence of corresponding eigenvalues. Assume that

$$(\omega_i,\omega_j)_{L^2(\mathcal{M})} = \delta_{i,j}.$$

The following asymptotic formula is established in [8]:

$$\lambda_j = cj^{\frac{2}{n}} + o(j^{\frac{2}{n}}), \quad \text{as } j \to +\infty.$$

For each k, thanks to the standard elliptic estimate for the Laplace operator, we have

$$\|\omega_j\|_{H^{2k}(\mathcal{M})} \le C_k \lambda_j^k. \tag{A.1}$$

Therefore, by the Sobolev embedding theorem, we have

$$\|\omega_j\|_{C^0(\mathcal{M})} \le C\lambda_j^n, \quad \forall\, j \in \{1, \cdots, \infty\}. \tag{A.2}$$

We extend the function ω_j on the set $\Big\{\xi \mid \xi_0^2 + \sum\limits_{i=1}^n \xi_i^4 \le 1\Big\}$ as a smooth function and set

$$\omega_j(\xi) = \omega_j\Big(\frac{\xi_0}{M^2(\xi)}, \frac{\xi_1}{M(\xi)}, \cdots, \frac{\xi_n}{M(\xi)}\Big), \quad M(\xi) = \Big(\xi_0^2 + \sum_{k=1}^n \xi_k^4\Big)^{\frac{1}{4}}.$$

We introduce the pseudodifferential operator

$$\widetilde{\omega}_j(D)w = \int_{\mathbb{R}^{n+1}} \omega_j(\xi) F_{(x',s)\to\xi} w \mathrm{e}^{\mathrm{i}\langle (x',s),\xi\rangle} \mathrm{d}\xi.$$

Here, in order to distinguish the Fourier transforms with respect to different variables, we will use the following notations:

$$F_{(x',s)\to\xi} u = \frac{1}{(2\pi)^{\frac{n+1}{2}}} \int_{\mathbb{R}^{n+1}} \mathrm{e}^{-\mathrm{i}\Big(\sum\limits_{j=0}^{n-1} x_j\xi_j + s\xi_n\Big)} u(x',s)\mathrm{d}x'\mathrm{d}s,$$
$$F_{x_n\to\xi_n} u = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}x_n\xi_n} u(x_n)\mathrm{d}x_n.$$

First, we define the operator $A(x', D', s)$ for functions in $C_0^\infty(\mathcal{O})$:

$$A(x', D', s)u = \int_{\mathbb{R}^n} a(x', \xi', s)\widehat{u}(\xi')\mathrm{d}x'.$$

The following lemma allows us to extend the definition of the operator A on Sobolev spaces.

Lemma A.1 *Let $a(x', \xi_0, \cdots, \xi_{n-1}, s) \in C^0_{\mathrm{cl}} S^{\frac{1}{2},1,s}(\mathcal{O})$. Then $A \in \mathcal{L}(H_0^{\frac{1}{2},1,s}(\mathcal{O}); L^2(\mathcal{O}))$ and $\|A\|_{\mathcal{L}(H_0^{\frac{1}{2},1,s}(\mathcal{O});L^2(\mathcal{O}))} \le C(\pi_{C^0}(a))$.*

Proof Thanks to Definition A.1(3), it suffices to consider the case where

$$a(x', \tau^2\xi_0, \tau\xi_1, \cdots, \tau\xi_{n-1}, \tau s) = \tau a(x', \xi_0, \cdots, \xi_{n-1}, s), \quad \forall\, \tau > 1. \tag{A.3}$$

The operator

$$\widetilde{A}(x', D)v = \int_{\Big\{\xi_0^2 + \sum\limits_{i=1}^n \xi_i^4 \le 1\Big\}} a(x', \xi_0, \cdots, \xi_n) F_{(x',s)\to\xi} v(\xi) \mathrm{e}^{\mathrm{i}\langle x,\xi\rangle} \mathrm{d}\xi$$

is a continuous operator from $L^2(\mathcal{O}\times\mathbb{R})$ into $L^2(\mathcal{O}\times\mathbb{R})$ with the norm estimated as

$$\|\widetilde{A}(x', D)\|_{\mathcal{L}(L^2(\mathcal{O}\times\mathbb{R}), L^2(\mathcal{O}\times\mathbb{R}))} \le C(\pi_{C^0}(a)).$$

Consider the symbol $b(x',\xi_0,\cdots,\xi_n)=a(x',\xi_0,\cdots,\xi_n)\Big(\xi_0^2+\sum\limits_{j=1}^n\xi_i^4\Big)^{-\frac14}$. Then by (A.3),

$$b(x',\tau^2\xi_0,\tau\xi_1,\cdots,\tau\xi_n)=b(x',\xi_0,\xi_1,\cdots,\xi_n),\quad \forall\,\tau\geq 1.$$

We can represent the symbol b as

$$b(x',\xi)=\sum_{j=1}^\infty b_j(x')\omega_j\Big(\frac{\xi_0}{M^2(\xi)},\frac{\xi_1}{M(\xi)},\cdots,\frac{\xi_n}{M(\xi)}\Big),\quad b_j(x')=(b(x',\xi),\omega_j(\xi))_{L^2(\mathcal{M})}.$$

Observe $b_j(x')=(\Delta_\xi^k b(x',\xi),\omega_j(\xi))_{L^2(\mathcal{M})}\lambda_j^{-k}$. So

$$\|b_j\|_{C^0(\overline{\mathcal{O}})}\leq C_m\lambda_j^{-m},\quad \forall\,m\in\{1,\cdots,\infty\}. \tag{A.4}$$

By (A.2) and (A.4),

$$\|B(x',D)v\|_{L^2(\mathcal{O}\times\mathbb{R})}\leq\sum_{j=1}^\infty\|b_j\|_{C^0(\overline{\mathcal{O}})}\|\widetilde{\omega}_j(D)\|\|v\|_{L^2(\mathcal{O}\times\mathbb{R})}\leq\sum_{j=1}^\infty C_m\lambda_j^{-m}\lambda_j^n\|v\|_{L^2(\mathcal{O}\times\mathbb{R})}.$$

Taking $m=3n$, we have

$$\|B(x',D)v\|_{L^2(\mathcal{O}\times\mathbb{R})}\leq\sum_{j=1}^\infty C_m\lambda_j^{-2n}\|v\|_{L^2(\mathcal{O}\times\mathbb{R})}.$$

Therefore, the operator

$$A^b(x',D)v=\int_{\Big\{\xi_0^2+\sum\limits_{i=1}^n\xi_i^4\geq 1\Big\}}a(x',\xi)F_{(x',s)\to\xi}v\mathrm{e}^{\mathrm{i}\langle x,\xi\rangle}\mathrm{d}\xi$$

is a continuous operator from $H_0^{\frac12,1}(\mathcal{O}\times\mathbb{R})$ into $L^2(\mathcal{O}\times\mathbb{R})$ with the norm satisfying the estimate

$$\|A^b\|_{\mathcal{L}(H_0^{\frac12,1}(\mathcal{O}\times\mathbb{R}),L^2(\mathcal{O}\times\mathbb{R}))}\leq\sum_{j=1}^\infty C_m\lambda_j^{-2n}.$$

Next, we observe that for the function $v(x)=u(x_0,\cdots,x_{n-1})w(x_n)$, there holds

$$\begin{aligned}\|\widetilde{A}(x',D)v\|_{L^2(\mathcal{O}\times\mathbb{R})}&=\sqrt{2\pi}\,\|A(x',D_0,\cdots,D_{n-1},\xi_n)uF_{x_n\to\xi_n}w\|_{L^2(\mathcal{O}\times\mathbb{R})}\\&\leq C(\pi_{C^0}(a))\Big(\int_{-\infty}^\infty\|u\|^2_{H^{\frac12,1,\xi_n}(\mathcal{O})}|F_{x_n\to\xi_n}w|^2\mathrm{d}\xi_n\Big)^{\frac12}.\end{aligned} \tag{A.5}$$

We take a sequence $\{w_j(x_n)\}_1^\infty$ such that $F_{x_n\to\xi_n}w_j(\xi_n)$ has a compact support and $|F_{x_n\to\xi_n}w_j|^2\to\ \delta(\xi_n-s)$, where $s\in\mathbb{R}$ is an arbitrary point. Since the function $\xi_n\to\|A(x',D',\xi_n)u\|_{L^2(\mathcal{O})}$ is continuous, we have

$$\begin{aligned}\|A(x',D_0,\cdots,D_{n-1},\xi_n)u\widehat{w}\|^2_{L^2(\mathcal{O}\times\mathbb{R})}&=\int_{\mathbb{R}}\|A(x',D_0,\cdots,D_{n-1},\xi_n)u\|^2_{L^2(\mathcal{O})}|F_{x_n\to\xi_n}w_j|^2\mathrm{d}\xi_n\\&\to\|A(x',D_0,\cdots,D_{n-1},s)u\|^2_{L^2(\mathcal{O})}.\end{aligned}$$

This fact and (A.5) imply

$$\|A(x',D',s)u\|_{L^2(\mathcal{O})}\leq C(\pi_{C^0}(a))\|u\|_{H^{\frac12,1,s}(\mathcal{O})}$$

for almost all s. Since the norm of the operator A is a continuous function of s, we have this inequality for all s.

The following theorem provides an estimate for a commutator of a Lipschitz function and the pseudodifferential operator $\widetilde{\omega}_j$.

Theorem A.1 *Let $f \in W^1_\infty(\mathcal{O})$ be a function with compact support. Then*

$$\|[f, \widetilde{\omega}_j]\|_{\mathcal{L}(L^2(\mathcal{O}), H^{\frac{1}{2},1,s}(\mathcal{O}))} \le C\|f\|_{W^1_\infty(\mathcal{O})}\lambda_j^{4n},$$

where the constant C is independent of j.

The proof of this theorem is similar to the proof of the corollary in [15, p. 309].

Lemma A.2 *Let $a(x', \xi_0, \cdots, \xi_{n-1}, s) \in C^1_{\mathrm{cl}} S^{\frac{1}{2},1,s}(\mathcal{O})$. Then $A(x', D', s)^* = A^*(x', D', s) + R$, where A^* is the pseudodifferential operator with symbol $\overline{a(x', \xi_0, \cdots, \xi_{n-1}, s)}$, and $R \in \mathcal{L}(L^2(\mathcal{O}), L^2(\mathcal{O}))$ satisfies*

$$\|R\|_{\mathcal{L}(L^2(\mathcal{O}), L^2(\mathcal{O}))} \le C\pi_{C^1}(a).$$

Proof Thanks to Definition A.1(3), it suffices to consider the case when

$$a(x', \tau^2\xi_0, \tau\xi_1, \cdots, \tau\xi_{n-1}, \tau s) = \tau a(x', \xi_0, \cdots, \xi_{n-1}, s), \quad \forall \tau > 1.$$

The symbol $a(x', \xi)$ can be represented as

$$a(x', \xi) = \sum_{j=1}^{\infty} a_j(x') M(\xi) \widetilde{\omega}_j(\xi).$$

Consider the operator

$$\widetilde{A}(x', D) = \sum_{j=1}^{\infty} a_j(x') M(D) \widetilde{\omega}_j(D), \quad M(D)w = \int_{\mathbb{R}^{n+1}} M(\xi)\widehat{w}\mathrm{e}^{\mathrm{i}\langle x, \xi\rangle}\mathrm{d}\xi.$$

Then

$$\begin{aligned}\widetilde{A}(x', D)^* &= \sum_{j=1}^{\infty} (a_j(x') M(D) \widetilde{\omega}_j(D))^* = \sum_{j=1}^{\infty} M(D) \widetilde{\omega}_j(D) \overline{a_j(x')} \\ &= \sum_{j=1}^{\infty} \overline{a_j(x')} M(D) \widetilde{\omega}_j(D) + \sum_{j=1}^{\infty} [M(D) \widetilde{\omega}_j(D), \overline{a_j(x')}].\end{aligned}$$

Observe that $\sum\limits_{j=1}^{\infty} \overline{a_j(x')} M(D) \widetilde{\omega}_j(D)$ is the operator with symbol $\overline{a(x', \xi_0, \cdots, \xi_n)} \in C^1_{\mathrm{cl}} S^{\frac{1}{2},1,s}(\mathcal{O})$. Let us estimate the norm of the operator $\sum\limits_{j=1}^{\infty} [M(D) \widetilde{\omega}_j(D), \overline{a_j(x')}]$. By Theorem A.1,

$$\begin{aligned}\Big\| \sum_{j=1}^{\infty} [\overline{a_j(x')}, M(D) \widetilde{\omega}_j(D)] \Big\|_{\mathcal{L}(L^2(\mathcal{O}), L^2(\mathcal{O}))} &\le C_m \sum_{j=1}^{\infty} \|a_j\|_{C^1(\overline{\mathcal{O}})} \lambda_j^{\widetilde{\kappa}(n)} \le C_m \sum_{j=1}^{\infty} \lambda_j^{\widetilde{\kappa}(n)} \pi_{C^1}(a) \lambda_j^{-m} \\ &\le C\pi_{C^1}(a).\end{aligned}$$

Denote $v = u(x_0, \cdots, x_{n-1}) w(x_n)$, $\widetilde{v} = \widetilde{u}(x_0, \cdots, x_{n-1}) \widetilde{w}(x_n)$. We have

$$(\widetilde{A}(x', D)v, \widetilde{v})_{L^2(\mathcal{O}\times\mathbb{R})} = (v, \widetilde{A}(x', D)^* \widetilde{v})_{L^2(\mathcal{O}\times\mathbb{R})} = (v, \widetilde{A}^*(x', D)\widetilde{v})_{L^2(\mathcal{O}\times\mathbb{R})} + (v, R\widetilde{v})_{L^2(\mathcal{O}\times\mathbb{R})}.$$

On the other hand,

$$\begin{aligned}(\widetilde{A}(x',D)v,\widetilde{v})_{L^2(\mathcal{O}\times\mathbb{R})} &= 2\pi\int_{\mathbb{R}}(A(x',D',\xi_n)u,\widetilde{u})_{L^2(\mathcal{O})}w\overline{\widetilde{w}}\mathrm{d}\xi_n\\ &= 2\pi\int_{\mathbb{R}}(u,A(x',D',\xi_n)^*\widetilde{u})_{L^2(\mathcal{O})}w\overline{\widetilde{w}}\mathrm{d}\xi_n.\end{aligned}$$

Taking into account that $(v,A^*(x',D)\widetilde{v})_{L^2(\mathcal{O}\times\mathbb{R})}=\int_{\mathbb{R}}(u,A^*(x',D',\xi_n)\widetilde{u})_{L^2(\mathcal{O})}w\overline{\widetilde{w}}\mathrm{d}\xi_n$, we have

$$\begin{aligned}\Big|\int_{\mathbb{R}}(u,(A(x',D',\xi_n)^*-A^*(x',D',\xi_n))\widetilde{u})_{L^2(\mathcal{O})}w\overline{\widetilde{w}}\mathrm{d}\xi_n\Big| &= |(v,R\widetilde{v})_{L^2(\mathcal{O}\times\mathbb{R})}|\\ &\le C\|v\|_{L^2(\mathcal{O}\times\mathbb{R})}\|\widetilde{v}\|_{L^2(\mathcal{O}\times\mathbb{R})}.\end{aligned}$$

We take a sequence $\{w_j\}_{j=1}^\infty$ such that $F_{x_n\to\xi_n}w_j$ has a compact support and $|F_{x_n\to\xi_n}w_j|^2\to\delta(\xi_n-s)$, where $s\in\mathbb{R}$ is an arbitrary point. Since the function $\xi_n\to\|A(x',D',\xi_n)u\|_{L^2(\mathcal{O})}$ is continuous, we have

$$\begin{aligned}&\Big|\int_{\mathbb{R}}(u,(A(x',D',\xi_n)^*-A^*(x',D',\xi_n))\widetilde{u})_{L^2(\mathcal{O})}|w_j|^2\mathrm{d}\xi_n\Big|\\ &\to|(u,(A(x',D',s)^*-A^*(x',D',s))\widetilde{u})_{L^2(\mathcal{O})}|.\end{aligned}$$

Since

$$|(u,(A(x',D',s)^*-A^*(x',D',s))\widetilde{u})_{L^2(\mathcal{O})}|\le C\|u\|_{L^2(\mathcal{O})}\|\widetilde{u}\|_{L^2(\mathcal{O})},$$

the statement of the lemma is proved.

Lemma A.3 *Let $a(x',\xi_0,\cdots,\xi_{n-1},s)\in C^1_{\mathrm{cl}}S^{\frac{\widehat{j}}{2},\widehat{j},s}(\mathcal{O})$, where $\widehat{j}\in\{0,1\}$, and $b(x',\xi_0,\cdots,\xi_{n-1},s)\in C^1_{\mathrm{cl}}S^{\frac{\mu}{2},\mu,s}(\mathcal{O})$. Then $A(x',D',s)B(x',D',s)=C(x',D',s)+R_0$, where $C(x',D',s)$ is the operator with symbol $a(x',\xi_0,\cdots,\xi_{n-1},s)b(x',\xi_0,\cdots,\xi_{n-1},s)$ and $R_0\in\mathcal{L}(H_0^{\frac{\mu+\tau}{2},\mu+\tau,s}(\mathcal{O}),H^{\frac{\tau+1}{2},\tau+1,s}(\mathcal{O}))$ for any $\tau\in[-1,0]$ if $\widehat{j}=0$, and $R_0\in\mathcal{L}(H_0^{\frac{\mu}{2},\mu,s}(\mathcal{O}),L^2(\mathcal{O}))$ if $\widehat{j}=1$. Moreover,*

$$\begin{aligned}&\|R_0\|_{\mathcal{L}(H_0^{\frac{\mu}{2},\mu,s}(\mathcal{O}),L^2(\mathcal{O}))}\le C\pi_{C^1}(\pi_{C^1}(a)\pi_{C^1}(b))\quad \textit{for } \widehat{j}=1,\\ &\|R_0\|_{\mathcal{L}(H_0^{\frac{\mu+\tau}{2},\mu+\tau,s}(\mathcal{O}),H^{\frac{\tau+1}{2},\tau+1,s}(\mathcal{O}))}\le C(\pi_{C^1}(a)\pi_{C^1}(b))\quad \textit{for } \widehat{j}=0.\end{aligned}$$

Proof We set

$$A(x',D)=\sum_{j=1}^\infty a_j(x')M^{\widehat{j}}(D)\widetilde{\omega}_j(D),\quad B(x',D)=\sum_{j=1}^\infty b_j(x')M^{\mu}(D)\widetilde{\omega}_j(D).$$

Observe

$$\begin{aligned}A(x',D)B(x',D)=&\sum_{m,k=1}^\infty a_m(x')b_k(x')M^{\widehat{j}+\mu}(D)\widetilde{\omega}_m(D)\widetilde{\omega}_k(D)\\ &+\sum_{m,k=1}^\infty a_m(x')[M^{\widehat{j}}\widetilde{\omega}_m,b_k]M^\mu(D)\widetilde{\omega}_k(D).\end{aligned}$$

Since $C(x',D)=\sum\limits_{m,k=1}^{\infty}a_m(x')b_k(x')M^{\widehat{j}+\mu}(D)\widetilde{\omega}_m(D)\widetilde{\omega}_k(D)$, and for $\widehat{j}=1$,

$$\begin{aligned}\|R_0\|_{\mathcal{L}(H_0^{\frac{\mu}{2},\mu}(\mathcal{O}),L^2(\mathcal{O}))}&=\Big\|\sum_{m,k=1}^{\infty}a_m(x')[M\widetilde{\omega}_m,b_k]M^{\mu}(D)\widetilde{\omega}_k(D)\Big\|_{\mathcal{L}(H_0^{\frac{\mu}{2},\mu}(\mathcal{O}),L^2(\mathcal{O}))}\\&\le\sum_{m,k=1}^{\infty}\|a_m\|_{C^1(\overline{\mathcal{O}})}\|[M\widetilde{\omega}_m,b_k]\|_{\mathcal{L}(L^2(\mathcal{O}),L^2(\mathcal{O}))}\|\widetilde{\omega}_k(D)\|_{\mathcal{L}(L^2(\mathcal{O}),L^2(\mathcal{O}))}\\&\le C_l\sum_{m,k=1}^{\infty}\lambda_m^{-l}\|[M\widetilde{\omega}_m,b_k]\|_{\mathcal{L}(L^2(\mathcal{O}),L^2(\mathcal{O}))}\lambda_k^{\widetilde{\kappa}(n)},\end{aligned}$$

by applying Theorem A.1, we obtain

$$\begin{aligned}\|R_0\|_{\mathcal{L}(H_0^{\frac{\mu}{2},\mu}(\mathcal{O}),L^2(\mathcal{O}))}&\le\sum_{m,k=1}^{\infty}\|a_m\|_{C^1(\overline{\mathcal{O}})}\|[M\widetilde{\omega}_m,b_k]\|_{\mathcal{L}(L^2,L^2)}\|\widetilde{\omega}_k(D)\|_{\mathcal{L}(L^2(\mathcal{O}),L^2(\mathcal{O}))}\\&\le C_l\sum_{m,k=1}^{\infty}\lambda_m^{-l}\|b_k\|_{C^1(\overline{\mathcal{O}})}\lambda_m^{\widetilde{\kappa}_1(n)}\lambda_k^{\widetilde{\kappa}(n)}\le C_{l,l_1}\sum_{m,k=1}^{\infty}\lambda_m^{-l}\lambda_k^{-l_1}\lambda_m^{\widetilde{\kappa}_1(n)}\lambda_k^{\widetilde{\kappa}(n)}\\&\le C_{l,l_1}\sum_{k=1}^{\infty}\lambda_m^{-l}\lambda_m^{\widetilde{\kappa}_1(n)}\sum_{m=1}^{\infty}\lambda_k^{-l_1}\lambda_k^{\widetilde{\kappa}(n)}<\infty.\end{aligned}\tag{A.6}$$

Let $v=v_j=u(x_0,\cdots,x_{n-1})w_j(x_n)$. We take a sequence $\{w_j\}_{j=1}^{\infty}$ such that $F_{x_n\to\xi_n}w_j$ has a compact support and $|F_{x_n\to\xi_n}w_j|^2\to\delta(\xi_n-s)$, where $s\in\mathbb{R}$ is arbitrary. Then for any $u\in H_0^{\frac{1}{2}+\frac{\mu}{2},1+\mu}(\mathcal{O})$,

$$\begin{aligned}&\|A(x',D)B(x',D)v_j-C(x',D)v_j\|^2_{L^2(\mathcal{O}\times\mathbb{R})}\\&=2\pi\int_{\mathbb{R}}\|(A(x',D',\xi_n)B(x',D',\xi_n)-C(x',D',\xi_n))u\|^2_{L^2(\mathcal{O})}|F_{x_n\to\xi_n}w_j|^2\mathrm{d}\xi_n\\&\le C\|v_j\|^2_{H_0^{\frac{\mu}{2},\mu}(\mathbb{R}^{n+1})}.\end{aligned}\tag{A.7}$$

Passing to the limit in (A.7) as $j\to+\infty$, we obtain

$$\|(A(x',D',s)B(x',D',s)-C(x',D',s))u\|^2_{L^2(\mathcal{O})}\le C\|u\|^2_{H_0^{\frac{\mu}{2},\mu,s}(\mathcal{O})}.$$

Let $\widehat{j}=0$. Then

$$\begin{aligned}&\|R_0\|_{\mathcal{L}(H_0^{\frac{\mu+\tau}{2},\mu+\tau}(\mathcal{O}),H_0^{\frac{\tau+1}{2},\tau+1,s}(\mathcal{O}))}\\&\le\sum_{m,k=1}^{\infty}\|a_m[\widetilde{\omega}_m,b_k]M^{-\tau}\|_{\mathcal{L}(L^2(\mathcal{O}),H_0^{\frac{\tau+1}{2},\tau+1}(\mathcal{O}))}\|\widetilde{\omega}_k(D)\|_{\mathcal{L}(L^2(\mathcal{O}),L^2(\mathcal{O}))}.\end{aligned}\tag{A.8}$$

In order to estimate the norm $\|a_m[\widetilde{\omega}_m,b_k]M^{-\tau}\|_{\mathcal{L}(L^2(\mathcal{O}),H_0^{\frac{\tau+1}{2},\tau+1}(\mathcal{O}))}$, we observe

$$M^{\tau+1}a_m[\widetilde{\omega}_m,b_k]M^{-\tau}=a_mM^{\tau+1}[\widetilde{\omega}_m,b_k]M^{-\tau}+[M^{\tau+1},a_m][\widetilde{\omega}_m,b_k]M^{-\tau}.$$

For the second term in this equality, we have

$$\begin{aligned}\|[\widetilde{\omega}_m,b_k]M^{-\tau}\|_{\mathcal{L}(L^2(\mathcal{O}),L^2(\mathcal{O}))}&\le\|b_k\|_{C^1(\overline{\mathcal{O}})}\|\widetilde{\omega}_m\|_{\mathcal{L}(L^2(\mathcal{O}),L^2(\mathcal{O}))},\\\|[M^{\tau+1},a_m]\|_{\mathcal{L}(L^2(\mathcal{O}),L^2(\mathcal{O}))}&\le C\|a_m\|_{C^1(\overline{\mathcal{O}})}.\end{aligned}\tag{A.9}$$

In order to estimate the first term, we observe $[\widetilde{\omega}_m, b_k]^* = -[\widetilde{\omega}_m, b_k]$. Then

$$[\widetilde{\omega}_m, b_k] \in \mathcal{L}(L^2(\mathbb{R}^n), H^{\frac{1}{2},1}(\mathbb{R}^n)), \quad [\widetilde{\omega}_m, b_k] \in \mathcal{L}(H^{-\frac{1}{2},-1}(\mathbb{R}^n), L^2(\mathbb{R}^n)).$$

Using an interpolation argument, we get

$$[\widetilde{\omega}_m, b_k] \in \mathcal{L}(H^{-\frac{\gamma}{2},-\gamma}(\mathbb{R}^n), H^{\frac{1-\gamma}{2},1-\gamma}(\mathbb{R}^n)), \quad \forall\, \gamma \in [0,1], \tag{A.10}$$

$$\|[\widetilde{\omega}_m, b_k]\|_{\mathcal{L}(H^{-\frac{\gamma}{2},-\gamma}(\mathbb{R}^n), H^{\frac{1-\gamma}{2},1-\gamma}(\mathbb{R}^n))} \le \|[\widetilde{\omega}_m, b_k]\|_{\mathcal{L}(L^2(\mathbb{R}^n), H^{\frac{1}{2},1}(\mathbb{R}^n))}, \quad \forall\, \gamma \in [0,1]. \tag{A.11}$$

Applying (A.9)–(A.11) to (A.8), we obtain

$$\begin{aligned}
\|R_0\|_{\mathcal{L}(H_0^{\frac{\mu+\tau}{2},\mu+\tau}(\mathcal{O}), H_0^{\frac{\tau+1}{2},\tau+1}(\mathcal{O}))} &\le C_l \sum_{m,k=1}^{\infty} \lambda_m^{-l} \|b_k\|_{C^1(\overline{\mathcal{O}})} \lambda_m^{\widetilde{\kappa}_1(n)} \lambda_k^{\widetilde{\kappa}(n)} \\
&\le C_{l,l_1} \sum_{m,k=1}^{\infty} \lambda_m^{-l} \lambda_k^{-l_1} \lambda_m^{\widetilde{\kappa}_1(n)} \lambda_k^{\widetilde{\kappa}(n)} \\
&\le C_{l,l_1} \sum_{k=1}^{\infty} \lambda_m^{-l} \lambda_m^{\widetilde{\kappa}_1(n)} \sum_{m=1}^{\infty} \lambda_k^{-l_1} \lambda_k^{\widetilde{\kappa}(n)} < \infty.
\end{aligned} \tag{A.12}$$

We finish the proof of the lemma by using similar arguments as in case $\widehat{j} = 1$.

The direct consequence of Lemma A.3 is the following commutator estimate.

Lemma A.4 *Let $a(x', \xi', s) \in C^1_{\rm cl} S^{\frac{1}{2},1,s}(\mathcal{O})$ and $b(x', \xi', s) \in C^1_{\rm cl} S^{0,0,s}(\mathcal{O})$. Then $[A, B] \in \mathcal{L}(L^2(\mathcal{O}); L^2(\mathcal{O}))$, and*

$$\|[A, B]\|_{\mathcal{L}(L^2(\mathcal{O}); L^2(\mathcal{O}))} \le C(\pi_{C^0}(a)\pi_{C^0}(b) + \pi_{C^0}(a)\pi_{C^1}(b) + \pi_{C^1}(a)\pi_{C^0}(b)).$$

Proof By Lemma A.3, we have

$$A(x', D', s)B(x', D', s) = C(x', D', s) + R_0, \quad B(x', D', s)A(x', D', s) = C(x', D', s) + \widetilde{R}_0,$$

where $R_0, \widetilde{R}_0 \in \mathcal{L}(L^2(\mathcal{O}), L^2(\mathcal{O}))$. Since $[A, B] = R_0 - \widetilde{R}_0$, we immediately obtain the statement of the lemma.

Lemma A.5 *Let $a(x', \xi', s) \in C^1_{\rm cl} S^{\frac{1}{2},1,s}(\mathcal{O})$ be a symbol with compact support in $\mathcal{O}$. Let $u \in H^{\frac{1}{2},1,s}(\mathcal{O})$, $\operatorname{supp} u \subset B(0, \delta)$, and $\delta' > \delta$. Then there exists a constant $C(\delta', \delta, \pi_{C^1}(a))$ such that*

$$\|A(x', D', s)u\|_{H^{\frac{1}{2},1,s}(\mathcal{O}\setminus B(0,\delta'))} \le C\|u\|_{H^{\frac{1}{2},1,s}(\mathcal{O})}.$$

Proof Consider the operator

$$\widetilde{A}(x', D) = \sum_{j=1}^{\infty} a_j(x') M(D) \widetilde{\omega}_j(D).$$

By (A.1), for any multiindex β, we have

$$\Big|\frac{\partial^\beta}{\partial \xi^\beta} \omega_j(\xi)\Big| \le C_\beta \lambda_j^{\widetilde{\kappa}_2 |\beta|}.$$

Hence, by Lemma 2.2 (see [17, Chapter II]), we have

$$\|M(D)\widetilde{\omega}_j(D)v\|_{H^{\frac{1}{2},1}(\mathcal{O}\setminus B(0,\delta')\times[-1,1]\setminus[-\frac{1}{2},\frac{1}{2}])} \le C\lambda_j^{\widehat{\kappa}} \|v\|_{H^{\frac{1}{2},1}(B(0,\delta)\times[-\frac{1}{4},\frac{1}{4}])}$$

for every function w such that $\operatorname{supp} w \subset B(0,\delta)\times[-\frac{1}{4},\frac{1}{4}]$. Therefore

$$\begin{aligned}
&\|\widetilde{A}(x',D)v\|_{H^1(\mathcal{O}\setminus B(0,\delta')\times[-1,1]\setminus[-\frac{1}{2},\frac{1}{2}])}\\
&\le \sum_{j=1}^{\infty}\|a_j(x')M(D)\widetilde{\omega}_j(D)v\|_{H^1(\mathcal{O}\setminus B(0,\delta')\times[-1,1]\setminus[-\frac{1}{2},\frac{1}{2}])}\\
&\le \sum_{j=1}^{\infty}\|a_j\|_{C^1(\overline{\mathcal{O}})}\|M(D)\widetilde{\omega}_j(D)v\|_{H^1(\mathcal{O}\setminus B(0,\delta')\times[-1,1]\setminus[-\frac{1}{2},\frac{1}{2}])}\\
&\le C\sum_{j=1}^{\infty}\lambda_j^{-m}\lambda_j^{\widehat{\kappa}}\|v\|_{H^{\frac{1}{2},1}(B(0,\delta)\times[-\frac{1}{4},\frac{1}{4}])}. \qquad (A.13)
\end{aligned}$$

Let $v=v_j=u(x_0,\cdots,x_{n-1})w_j(x_n)$. We take a sequence $\{w_j\}_{j=1}^{\infty}$ such that $F_{x_n\to\xi_n}w_j$ has a compact support and $|F_{x_n\to\xi_n}w_j|^2\to\delta(\xi_n-s_0)$, where $s_0\in\mathbb{R}$ is an arbitrary point. Then

$$\begin{aligned}
\|\widetilde{A}(x',D)v\|^2_{H^1(\mathcal{O}\setminus B(0,\delta')\times[-1,1]\setminus[-\frac{1}{2},\frac{1}{2}])} &= \int_{-\frac{1}{4}}^{\frac{1}{4}}\|A(x',D',s)u\|^2_{H^{1,s}(\mathcal{O}\setminus B(0,\delta'))}|F_{x_n\to\xi_n}w_j|^2\mathrm{d}\xi_n\\
&\to \|A(x,D',s_0)u\|^2_{H^{1,s_0}(\mathcal{O}\setminus B(0,\delta'))} \qquad (A.14)
\end{aligned}$$

and

$$\|v_j\|_{H^{\frac{1}{2},1}(B(0,\delta)\times[-\frac{1}{4},\frac{1}{4}])}\to\|u\|_{H^{\frac{1}{2},1,s_0}(B(0,\delta))},\quad \text{as } j\to+\infty.$$

These relations and (A.13) prove the statement of the lemma.

We shall use the following variant of Gårding's inequality.

Lemma A.6 *Let $p(x',\xi',s)\in C^1_{\rm cl}S^{\frac{1}{2},1,s}(\mathcal{O})$ be a symbol with compact support in $\mathcal{O}$. Let $u\in H^{\frac{1}{2},1,s}(\mathcal{O})$, $\operatorname{supp} u\subset B(0,\delta)$, and $\delta'>\delta$ be such that $\overline{B(0,\delta')}\subset\mathcal{O}$ and $\operatorname{Re} p(x',\xi',s)>\widehat{C}|s|M(\xi',s)$ for any $x\in B(0,\delta')$. Then then there exist positive constants C and C_1 independent of s, such that*

$$\operatorname{Re}(Pu,u)\ge C|s|\|u\|^2_{H^{\frac{1}{4},\frac{1}{2},s}(\mathcal{O})}-C_1\|u\|^2_{L^2(\mathcal{O})}.$$

Proof Let $\chi\in C_0^{\infty}(B(0,\delta'))$ be a function such that $\chi|_{B(0,\delta)}=1$. Consider the pseudo-differential operator $A(x',D',s)$ with symbol $A(x',\xi',s)=(\operatorname{Re} p(x',\xi',s)-\chi\frac{\widehat{C}}{2}|s|M(\xi',s))^{\frac{1}{2}}\in C^1_{\rm cl}S^{\frac{1}{4},\frac{1}{2},s}(\mathcal{O})$. Then, according to Lemma A.3, we have

$$A(x',D',s)^*A(x',D',s)=\operatorname{Re} p(x,\xi',s)-\chi\frac{\widehat{C}}{2}|s|M(\xi',s)+R,$$

where $R\in\mathcal{L}(L^2(\mathcal{O});L^2(\mathcal{O}))$. Therefore

$$\operatorname{Re}(Pu,u)_{L^2(\mathcal{O})}=\|A(x',D,s)u\|^2_{L^2(\mathcal{O})}-((1-\chi)M(D)u,u)_{L^2(\mathbb{R}^n)}+\frac{\widehat{C}}{2}\|u\|^2_{H^{\frac{1}{4},\frac{1}{2},s}(\mathcal{O})}+(Ru,u)_{L^2(\mathcal{O})}.$$

Observing that $|(Ru,u)_{L^2(\mathcal{O})}|\le C(\pi_{C^1}(p))\|u\|^2_{L^2(\mathcal{O})}$, and by Lemma 2.2 (see also [17, Chapter II]), we have $|((1-\chi)M(D)u,u)_{L^2(\mathbb{R}^n)}|\le C\|u\|^2_{L^2(\mathcal{O})}$. We obtain the statement of the lemma.

Consider the following system of equations:

$$\frac{\partial W}{\partial x_n}+K(x,D',s)W=F,\quad \text{in } G, \qquad (A.15)$$

$$W(x',0)=g, \qquad (A.16)$$

where $W = (w_1, \cdots, w_m)$, $F = (f_1, \cdots, f_m)$, $g = (g_1, \cdots, g_m)$. Let $K(x, D', s)$ be an $m \times m$ matrix pseudodifferential operator such that

$$K_{ij}(x, \xi', s) \in C^1_{\rm cl} S^{\frac{1}{2},1,s}(G). \tag{A.17}$$

Assume that there exists a constant $C > 0$ such that

$$\operatorname{Spec} K(x, \xi', s) \subset \Big\{ z \in \mathbb{C} : \operatorname{Re} z \geq C\Big(\xi_0^2 + \sum_{j=1}^{n-1} \xi_j^4 + s^4\Big)^{\frac{1}{4}} \Big\} \tag{A.18}$$

for all $(x, \xi', s) \in G \times \mathbb{R}^{n+1} \cap \{(\xi', s) \mid M(\xi', s) \geq 1\}$, and

$$\text{the matrix } K(x, \xi', s) \text{ is independent of } x \text{ outside a ball } B(0, \delta). \tag{A.19}$$

Lemma A.7 *Suppose that assumptions* (A.17)–(A.19) *hold true. Then*

(1) *For each $g \in H^{\frac{1}{4},\frac{1}{2},s}(\mathbb{R}^n)$ and $F \in L^2(G)$, there exists a unique solution $W \in H^{\frac{1}{2},1,s}(G)$ to problem* (A.15)–(A.16)*, and the following a priori estimate holds true:*

$$\|W\|_{H^{\frac{1}{2},1,s}(G)} + \|W\|_{L^\infty(0,\gamma;H^{\frac{1}{4},\frac{1}{2},s}(\mathbb{R}^n))} \leq C(\|g\|_{H^{\frac{1}{4},\frac{1}{2},s}(\mathbb{R}^n)} + \|F\|_{L^2(G)}). \tag{A.20}$$

(2) *For each $g \in L^2(\mathbb{R}^n)$ and $F \in H^{-\frac{1}{2},-1,s}(G)$ with $\operatorname{supp} F \subset\subset G$, there exists a unique solution $W \in L^2(G)$ to problem* (A.15)–(A.16)*, and the following a priori estimate holds true:*

$$\|W\|_{L^2(G)} + \|W\|_{L^\infty(0,\gamma;H^{-\frac{1}{4},-\frac{1}{2},s}(\mathbb{R}^n))} \leq C(\|g\|_{H^{-\frac{1}{4},-\frac{1}{2},s}(\mathbb{R}^n)} + \|F\|_{H^{-\frac{1}{2},-1,s}(G)}). \tag{A.21}$$

Remark A.1 For the initial data $g \in L^2(\mathbb{R}^n)$ and $F \in H^{-\frac{1}{2},-1,s}(G)$ with $\operatorname{supp} F \subset\subset G$, we understand the solution of problem (A.15)–(A.16) in the following way:

$$\Big(W, \Big(-\frac{\partial}{\partial x_n} + K(x, D', s)^*\Big)\Phi\Big)_{L^2(G)} = (g, \Phi(\,\cdot\,, 0))_{L^2(\mathbb{R}^n)} + (F, \Phi)_{L^2(G)}$$

for any function $\Phi \in H^{\frac{1}{2},1,s}(G)$, $\Phi(\,\cdot\,, \gamma) = 0$.

Proof Set

$$\widetilde{\Lambda}^{\frac{1}{2}}(D', s)w = \int_{\mathbb{R}^n} \Big(1 + \xi_0^2 + \sum_{j=1}^{n-1} \xi_j^4 + s^4\Big)^{\frac{1}{8}} \widehat{w} \mathrm{e}^{\langle x', \xi' \rangle} \mathrm{d}\xi'.$$

Let $K_1(x, \xi', s)$ be the principal symbol of the operator K. Consider the matrix

$$P(x, \xi', s) = \int_0^\infty \mathrm{e}^{-tK_1^*} \mathrm{e}^{-tK_1} \mathrm{d}t. \tag{A.22}$$

By (A.18), the integral on the right-hand side of (A.22) is convergent. Then for the principal symbol of the operator K, we have

$$PK_1 + K_1^* P = I.$$

For some positive constant $C_1 > 0$, we also observe

$$(P\vec{v}, \vec{v}) \geq C_1 |\vec{v}|^2, \quad \forall\, \vec{v} \in \mathbb{R}^m. \tag{A.23}$$

In order to show the solvability of (A.15)–(A.16), we will first consider regularized problems. For $\epsilon \in]0,1[$, let us consider a family of the Friedrichs mollifiers $(\mathcal{J}_\epsilon)_\epsilon$ with $\mathcal{J}_\epsilon \in \mathcal{S}^{-\infty}(\mathbb{R}^n)$ (see [16]). ($\mathcal{S}^p(\mathbb{R}^n)$ is the class of pseudodifferential operators of order p on $\mathbb{R}^n$.) It is known that

$$\begin{aligned}
&\mathcal{J}_\epsilon u \to u \quad \text{in } L^2(\mathbb{R}^n) \quad \text{as } \epsilon \to +0,\\
&\sup_{\epsilon\in[0,1]} \|\mathcal{J}_\epsilon\|_{\mathcal{L}(L^2(\mathbb{R}^n),L^2(\mathbb{R}^n))} \le C,\\
&\{[A,\mathcal{J}_\epsilon] : 0<\epsilon<1\} \text{ is a bounded set in } \mathcal{L}(L^2(\mathbb{R}^n),L^2(\mathbb{R}^n))
\end{aligned}$$

for any $A(x,D,s) \in C^0([0,1];C^1_{\mathrm{cl}}S^{\frac12,1,s}(\mathbb{R}^n))$. Let $\epsilon \in (0,1)$. We consider the following Cauchy problem for an ordinary differential equation in a Banach space:

$$\frac{\partial W_\epsilon}{\partial x_n} + \mathcal{J}_\epsilon^* K(x,D',s)\mathcal{J}_\epsilon W_\epsilon = \mathcal{J}_\epsilon^* F, \quad \text{in } G, \tag{A.24}$$

$$W_\epsilon(x',0) = g. \tag{A.25}$$

For any $g \in H^{\frac14,\frac12}(\mathbb{R}^n)$ and $F \in L^2(G)$, there exists a unique solution W_ϵ to problem (A.24)–(A.25) with $W_\epsilon \in L^\infty(0,\gamma;H^{\frac14,\frac12,s}(\mathbb{R}^n))$ and $\frac{\partial W_\epsilon}{\partial x_n} \in L^2(G)$.

Simple computations provide

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}x_n}(P\widetilde{\Lambda}^{\frac12}W_\epsilon,\widetilde{\Lambda}^{\frac12}W_\epsilon)_{L^2(\mathbb{R}^n)}\\
&=\Big(P\widetilde{\Lambda}^{\frac12}\frac{\partial W_\epsilon}{\partial x_n},\widetilde{\Lambda}^{\frac12}W_\epsilon\Big)_{L^2(\mathbb{R}^n)} + \Big(P\widetilde{\Lambda}^{\frac12}W_\epsilon,\widetilde{\Lambda}^{\frac12}\frac{\partial W_\epsilon}{\partial x_n}\Big)_{L^2(\mathbb{R}^n)} + (P'\widetilde{\Lambda}^{\frac12}W_\epsilon,\widetilde{\Lambda}^{\frac12}W_\epsilon)_{L^2(\mathbb{R}^n)}\\
&=(P\widetilde{\Lambda}^{\frac12}(-\mathcal{J}_\epsilon^*K\mathcal{J}_\epsilon W_\epsilon + \mathcal{J}_\epsilon^*F),\widetilde{\Lambda}^{\frac12}W_\epsilon)_{L^2(\mathbb{R}^n)} + (P\widetilde{\Lambda}^{\frac12}W_\epsilon,\widetilde{\Lambda}^{\frac12}(-\mathcal{J}_\epsilon^*K\mathcal{J}_\epsilon W_\epsilon + \mathcal{J}_\epsilon^*F))_{L^2(\mathbb{R}^n)}\\
&\quad + (P'\widetilde{\Lambda}^{\frac12}W_\epsilon,\widetilde{\Lambda}^{\frac12}W_\epsilon)_{L^2(\mathbb{R}^n)}\\
&= -(P\mathcal{J}_\epsilon^*K\mathcal{J}_\epsilon\widetilde{\Lambda}^{\frac12}W_\epsilon,\widetilde{\Lambda}^{\frac12}W_\epsilon)_{L^2(\mathbb{R}^n)} + (P\widetilde{\Lambda}^{\frac12}\mathcal{J}_\epsilon^*F,\widetilde{\Lambda}^{\frac12}W_\epsilon)_{L^2(\mathbb{R}^n)}\\
&\quad - (P\widetilde{\Lambda}^{\frac12}W_\epsilon,\mathcal{J}_\epsilon^*K\mathcal{J}_\epsilon\widetilde{\Lambda}^{\frac12}W_\epsilon)_{L^2(\mathbb{R}^n)} + (P\widetilde{\Lambda}^{\frac12}W_\epsilon,\widetilde{\Lambda}^{\frac12}\mathcal{J}_\epsilon^*F)_{L^2(\mathbb{R}^n)} + (P'\widetilde{\Lambda}^{\frac12}W_\epsilon,\widetilde{\Lambda}^{\frac12}W_\epsilon)_{L^2(\mathbb{R}^n)}\\
&= -((P\mathcal{J}_\epsilon^*K\mathcal{J}_\epsilon + \mathcal{J}_\epsilon^*K^*\mathcal{J}_\epsilon P)\widetilde{\Lambda}^{\frac12}W_\epsilon,\widetilde{\Lambda}^{\frac12}W_\epsilon)_{L^2(\mathbb{R}^n)} + (P\widetilde{\Lambda}^{\frac12}\mathcal{J}_\epsilon^*F,\widetilde{\Lambda}^{\frac12}W_\epsilon)_{L^2(\mathbb{R}^n)}\\
&\quad + (P\widetilde{\Lambda}^{\frac12}W_\epsilon,\widetilde{\Lambda}^{\frac12}\mathcal{J}_\epsilon^*F)_{L^2(\mathbb{R}^n)} + (P'\widetilde{\Lambda}^{\frac12}W_\epsilon,\widetilde{\Lambda}^{\frac12}W_\epsilon)_{L^2(\mathbb{R}^n)}\\
&= -((PK+K^*P)\mathcal{J}_\epsilon\widetilde{\Lambda}^{\frac12}W_\epsilon,\mathcal{J}_\epsilon\widetilde{\Lambda}^{\frac12}W_\epsilon)_{L^2(\mathbb{R}^n)} + (P\widetilde{\Lambda}^{\frac12}\mathcal{J}_\epsilon^*F,\widetilde{\Lambda}^{\frac12}W_\epsilon)_{L^2(\mathbb{R}^n)}\\
&\quad - (\mathcal{J}_\epsilon K^*[\mathcal{J}_\epsilon,P]\widetilde{\Lambda}^{\frac12}W_\epsilon,\widetilde{\Lambda}^{\frac12}W_\epsilon)_{L^2(\mathbb{R}^n)} + (P\widetilde{\Lambda}^{\frac12}W_\epsilon,\widetilde{\Lambda}^{\frac12}\mathcal{J}_\epsilon^*F)_{L^2(\mathbb{R}^n)} + (P'\widetilde{\Lambda}^{\frac12}W_\epsilon,\widetilde{\Lambda}^{\frac12}W_\epsilon)_{L^2(\mathbb{R}^n)}.
\end{aligned}$$

Here P' is the pseudodifferential operator with symbol $\frac{\partial}{\partial x_n}p(x,\xi',s)$. Note that

$$|(\mathcal{J}_\epsilon K^*[\mathcal{J}_\epsilon,P]\widetilde{\Lambda}^{\frac12}W_\epsilon,\widetilde{\Lambda}^{\frac12}W_\epsilon)_{L^2(\mathbb{R}^n)}| \le C\|W_\epsilon\|_{H^{\frac14,\frac12,s}(\mathbb{R}^n)}, \tag{A.26}$$

$$|(P'\widetilde{\Lambda}^{\frac12}W_\epsilon,\widetilde{\Lambda}^{\frac12}W_\epsilon)_{L^2(\mathbb{R}^n)}| \le C\|W_\epsilon\|_{H^{\frac14,\frac12,s}(\mathbb{R}^n)}. \tag{A.27}$$

Here and below, all constants C are independent of $\epsilon \in (0,1)$. After simple computations, we obtain

$$\begin{aligned}
&(P\widetilde{\Lambda}^{\frac12}\mathcal{J}_\epsilon^*F,\widetilde{\Lambda}^{\frac12}W_\epsilon)_{L^2(\mathbb{R}^n)} = ([P\widetilde{\Lambda}^{\frac12},\mathcal{J}_\epsilon^*]F,\widetilde{\Lambda}^{\frac12}W_\epsilon)_{L^2(\mathbb{R}^n)} + (P\widetilde{\Lambda}^{\frac12}F,\mathcal{J}_\epsilon\widetilde{\Lambda}^{\frac12}W_\epsilon)_{L^2(\mathbb{R}^n)}\\
&=([P\widetilde{\Lambda}^{\frac12},\mathcal{J}_\epsilon^*]F,\widetilde{\Lambda}^{\frac12}W_\epsilon)_{L^2(\mathbb{R}^n)} + (F,\widetilde{\Lambda}^{\frac12}P^*\mathcal{J}_\epsilon\widetilde{\Lambda}^{\frac12}W_\epsilon)_{L^2(\mathbb{R}^n)}\\
&=([P\widetilde{\Lambda}^{\frac12},\mathcal{J}_\epsilon^*]F,\widetilde{\Lambda}^{\frac12}W_\epsilon)_{L^2(\mathbb{R}^n)} + (F,\widetilde{\Lambda}^{\frac12}P^*\widetilde{\Lambda}^{\frac12}\mathcal{J}_\epsilon W_\epsilon)_{L^2(\mathbb{R}^n)} + (F,\widetilde{\Lambda}^{\frac12}P^*[\mathcal{J}_\epsilon,\widetilde{\Lambda}^{\frac12}]W_\epsilon)_{L^2(\mathbb{R}^n)}.
\end{aligned}$$

Therefore, for any positive δ, we have

$$\begin{aligned}&|(P\widetilde{\Lambda}^{\frac12}\mathcal{J}_\epsilon F,\widetilde{\Lambda}^{\frac12}W_\epsilon)_{L^2(\mathbb{R}^n)}|\\&\le C(\delta)\|F(\,\cdot\,,x_n)\|^2_{L^2(\mathbb{R}^n)}+C\|\widetilde{\Lambda}^{\frac12}W_\epsilon\|^2_{L^2(\mathbb{R}^n)}+\delta\|\mathcal{J}_\epsilon W_\epsilon\|^2_{H^{\frac12,1,s}(\mathbb{R}^n)}.\end{aligned}\tag{A.28}$$

Similarly, we have

$$\begin{aligned}&|(P\widetilde{\Lambda}^{\frac12}W_\epsilon,\widetilde{\Lambda}^{\frac12}\mathcal{J}^*_\epsilon F)_{L^2(\mathbb{R}^n)}|\\&\le C(\delta)\|F(\,\cdot\,,x_n)\|^2_{L^2(\mathbb{R}^n)}+C\|\widetilde{\Lambda}^{\frac12}W_\epsilon\|^2_{L^2(\mathbb{R}^n)}+\delta\|\mathcal{J}_\epsilon W_\epsilon\|^2_{H^{\frac12,1,s}(\mathbb{R}^n)}.\end{aligned}\tag{A.29}$$

Using (A.26)–(A.29), we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}x_n}(P\widetilde{\Lambda}^{\frac12}W_\epsilon,\widetilde{\Lambda}^{\frac12}W_\epsilon)_{L^2(\mathbb{R}^n)}+C_1\|\mathcal{J}_\epsilon W_\epsilon\|^2_{H^{\frac12,1,s}(\mathbb{R}^n)}\\&\le C(\|\widetilde{\Lambda}^{\frac12}W_\epsilon\|^2_{L^2(\mathbb{R}^n)}+\|F(\,\cdot\,,x_n)\|^2_{L^2(\mathbb{R}^n)}).\end{aligned}\tag{A.30}$$

Applying Gronwall's inequality, we obtain

$$\|W_\epsilon\|_{L^\infty(0,\gamma;H^{\frac14,\frac12,s}(\mathbb{R}^n))}+\|\mathcal{J}_\epsilon W_\epsilon\|_{L^2(0,\gamma;H^{\frac14,\frac12,s}(\mathbb{R}^n))}\le C(\|g\|_{H^{\frac14,\frac12,s}(\mathbb{R}^n)}+\|F\|_{L^2(G)}).\tag{A.31}$$

Using (A.31) from (A.15), we obtain the estimate for $\frac{\partial W_\epsilon}{\partial x_n}$:

$$\Big\|\frac{\partial W_\epsilon}{\partial x_n}\Big\|_{L^2(G)}\le C(\|g\|_{H^{\frac14,\frac12,s}(\mathbb{R}^n)}+\|F\|_{L^2(G)}).\tag{A.32}$$

Inequalities (A.31) and (A.32) imply

$$\|W_\epsilon\|_{H^{\frac12,1,s}(G)}+\|W_\epsilon\|_{L^\infty(0,\gamma;H^{\frac14,\frac12,s}(\mathbb{R}^n))}\le C(\|g\|_{H^{\frac14,\frac12,s}(\mathbb{R}^n)}+\|F\|_{L^2(G)}).\tag{A.33}$$

We can now extract a subsequence still denoted by ϵ, such that

$$W_\epsilon\rightharpoonup W,\qquad \text{weakly-}*\ \text{ in } L^\infty(0,\gamma;L^2(\mathbb{R}^n)),\tag{A.34}$$

$$\frac{\partial W_\epsilon}{\partial x_n}\rightharpoonup\frac{\partial W}{\partial x_n},\quad \text{weakly in } L^2(G),\tag{A.35}$$

$$\mathcal{J}_\epsilon W_\epsilon\rightharpoonup W,\qquad \text{weakly in } L^2(0,\gamma;H^{\frac12,1}(\mathbb{R}^n)),\tag{A.36}$$

$$\mathcal{J}_\epsilon W_\epsilon\rightharpoonup W,\qquad \text{weakly-}*\ \text{ in } L^\infty(0,\gamma;H^{\frac14,\frac12}(\mathbb{R}^n)).\tag{A.37}$$

Of course, $W\in H^{\frac12,1}(G)$ and W is a solution of (A.15)–(A.16). From (A.33) we obtain (A.20).

Now we prove (2) of this lemma. Since the space $L^2(G)$ is dense in the space $\{F\in H^{-\frac12,-1,s}(G)\mid \operatorname{supp}F\subset\subset G\}$, in order to prove (2), it suffices to establish the a priori estimate (A.21). Let Φ be a solution to the following boundary value problem:

$$\Big(-\frac{\partial}{\partial x_n}+K^*(x,D',s)\Big)\Phi=W,\ \text{ in } G,\quad \Phi(\,\cdot\,,\gamma)=0.\tag{A.38}$$

By the definition of a weak solution, we have

$$\|W\|^2_{L^2(G)}=(g,\Phi(\,\cdot\,,\gamma))_{L^2(\mathbb{R}^n)}+(F,\Phi)_{L^2(G)}.\tag{A.39}$$

By (1) of this lemma, the solution to (A.38) satisfies the estimate

$$\|\Phi\|_{H^{\frac12,1,s}(G)}+\|\Phi\|_{L^\infty(0,\gamma;H^{\frac14,\frac12,s}(\mathbb{R}^n))}\le C\|W\|_{L^2(G)}.\tag{A.40}$$

Using in equality (A.39) estimate (A.40), we obtain (A.21).

References

[1] Carleman, T., Sur un problème d'unicité pour les systèmes d'équations aux dérivées partielles à deux variables indépendentes, *Ark. Mat. Astr. Fys.*, **26B**, 1939, 1–9.

[2] Dubrovin, B. A., Fomenko, A. T. and Novikov, S. P., Modern Geometry: Methods and Applications, Part II, The Geometry and Topology of Manifolds, Springer-Verlag, Berlin, 1985.

[3] Evans, L., Partial Differential Equations, Graduate Studies in Mathematics, Vol. 19, A. M. S., Providence, RI, 1998.

[4] Fernandez-Cara, E., Guerrero, S., Imanuvilov, O. and Puel, J. P., Some controllability results for the N-dimensional Navier-Stokes and Boussinesq system, *SIAM J. Cont. Optim.*, **45**, 2006, 146–173.

[5] Fernandez-Cara, E., Guerrero, S., Imanuvilov, O. and Puel, J. P., Local exact controllability of the Navier-Stokes system, *J. Math. Pures Appl.*, **83**(12), 2005, 1501–1542.

[6] Fabre, C. and Lebeau, G., Prolongement unique des solutions de l'équation de Stokes, *Comm. Part. Diff. Eqs.*, **21**, 1996, 573–596.

[7] Hörmander, L., Linear Partial Differential Operators, Springer-Verlag, Berlin, 1963.

[8] Hörmander, L., The spectral function of elliptic operators, *Acta Math.*, **121**, 1968, 193–218.

[9] Imanuvilov, O., Controllability of parabolic equations, *Mat. Sb.*, **186**(6), 1995, 109–132.

[10] Imanuvilov, O., On exact controllability for the Navier-Stokes equations, *ESAIM Control Optim. Calc. Var.*, **3**, 1998, 97–131.

[11] Imanuvilov, O., Remarks on exact controllability for Navier-Stokes equations, *ESAIM Control Optim. Calc. Var.*, **6**, 2001, 39–72.

[12] Imanuvilov, O. and Puel, J. P., Global Carleman estimates for weak solutions of elliptic nonhomogeneous Dirichlet problems, *Int. Math. Res. Not.*, **16**, 2003, 883–913.

[13] Imanuvilov, O. and Yamamoto, M., Carleman inequalities for parabolic equations in Sobolev spaces of negative order and exact controllability for semilinear parabolic equations, *Publ. Res. Inst. Math. Sci.*, **39**, 2003, 227–274.

[14] Lions, J. L., Optimal Control of Systems Governed by Partial Differential Equations, Springer-Verlag, Berlin, 1971.

[15] Stein, E. M., Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals, Princeton University Press, New Jersey, 1993.

[16] Taylor, M., Pseudodifferential Operators and Nonlinear PDE, Birkhäuser, Berlin, 1991.

[17] Taylor, M., Pseudodifferential Operators, Princeton University Press, New Jersey, 1981.

[18] Temam, R., Navier-Stokes Equatons, A. M. S., Providence, RI, 2001.