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Global Classical Solutions to Partially Dissipative Quasilinear Hyperbolic Systems*

Yi ZHOU[1]

Abstract The author considers the Cauchy problem for quasilinear inhomogeneous hyperbolic systems. Under the assumption that the system is weakly dissipative, Hanouzet and Natalini established the global existence of smooth solutions for small initial data (in Arch. Rational Mech. Anal., Vol. 169, 2003, pp. 89–117). The aim of this paper is to give a completely different proof of this result with slightly different assumptions.

Keywords Cauchy problem, Global classical solution, Partially dissipative quasilinear hyperbolic system
2000 MR Subject Classification 35L45, 35L60

1 Introduction and Main Results

In this paper, we consider the following first order quasilinear inhomogeneous hyperbolic systems

$$\frac{\partial u}{\partial t}+A(u)\frac{\partial u}{\partial x}+f(u)=0, \tag{1.1}$$

where $u=(u_1,\cdots,u_n)^{\mathrm{T}}$ is the unknown vector function of (t,x) and $f(u)=(f_1(u),\cdots,f_n(u))^{\mathrm{T}}$ with C^2 functions $f_i(u)$ $(i=1,\cdots,n)$. Moreover, $A(u)=(a_{ij}(u))$ is an $n\times n$ matrix with C^2 elements $a_{ij}(u)$ $(i,j=1,\cdots,n)$.

We assume strictly hyperbolicity of the system. For any given u on the domain under consideration, $A(u)$ has n distinct real eigenvalues

$$\lambda_1(u)<\lambda_2(u)<\cdots<\lambda_n(u). \tag{1.2}$$

Let $l_i(u)=(l_{i1}(u),\cdots,l_{in}(u))$ (resp. $r_i(u)=(r_{i1}(u),\cdots,r_{in}(u))^{\mathrm{T}}$) be a left (resp. right) eigenvector corresponding to $\lambda_i(u)$ $(i=1,\cdots,n)$:

$$l_i(u)A(u)=\lambda_i(u)l_i(u)\quad(\text{resp. } A(u)r_i(u)=\lambda_i(u)r_i(u)). \tag{1.3}$$

We have

$$\det|l_{ij}(u)|\neq 0\quad(\text{equivalently, } \det|r_{ij}(u)|\neq 0). \tag{1.4}$$

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[1]Key Laboratory of Mathematics for Nonlinear Sciences, Ministry of Education, China; Shanghai Key Laboratory for Contemporary Applied Mathematics; School of Mathematical Sciences, Fudan University, Shanghai 200433, China. E-mail: yizhou@fudan.ac.cn
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All $\lambda_i(u),\ l_{ij}(u)$ and $r_{ij}(u)$ have the C^2 regularity.

Without loss of generality, we may suppose that

$$l_i(u)r_j(u) \equiv \delta_{ij}, \quad i,j = 1,\cdots,n \tag{1.5}$$

and

$$r_i^{\mathrm{T}}(u)r_i(u) \equiv 1, \quad i = 1,\cdots,n, \tag{1.6}$$

where δ_{ij} stands for the Kronecker's symbol.

In this paper, we assume

$$f_i(0) = 0, \quad i = 1,\cdots,n. \tag{1.7}$$

Thus, $u = 0$ is a solution to the system.

For the Cauchy problem of system (1.1) with initial data

$$t = 0 : u = \phi(x), \tag{1.8}$$

where $\phi(x)$ is a "small" C^1 vector function, we shall investigate the global existence of classical solutions to (1.1) and (1.8).

In the general case of system (1.1), a well-known assumption for the global existence is given by the total dissipation condition (see [8, 6]):

$$g_{ii}(0) > \sum_{j\neq i} |g_{ij}(0)|, \quad i = 1,\cdots n, \tag{1.9}$$

where

$$g_{ij}(u) = \sum_{l,m=1}^{n} l_{il}(u)\frac{\partial f_l(u)}{\partial u_m} r_{mj}(u). \tag{1.10}$$

Then, In [3], for the special case of diagonal system

$$A(u) = \mathrm{diag}(\lambda_1(u),\cdots,\lambda_n(u)), \tag{1.11}$$

the global existence for small initial data is established by only assuming

$$g_{ii}(0) \geq \sum_{j\neq i} |g_{ij}(0)|, \quad i = 1,\cdots n \tag{1.12}$$

and

$$g_{ii}(0) > 0, \quad i = 1,\cdots n. \tag{1.13}$$

Finally, in [2, 11], for general systems of hyperbolic balance laws with a convex entropy, the global existence for small initial data is obtained by assuming a suitable version of the Kawashima condition (see [5]) and a strict entropy dissipative condition. The aim of this paper is to give a completely different proof of the result in [2] with slightly different assumptions on the system. Our advantage is that we only assume a weak entropy dissipative condition. The disadvantage is that we need strictly hyperbolic assumptions. Our key idea of the proof is to use the formula on the decomposition of waves and combine it with a technique in [3]. In [6], the formula on the decomposition of waves was first used to prove the global existence of classical solutions to quasilinear inhomogeneous hyperbolic systems.

The main assumption of the present paper consists of two parts. The first one is just (1.13), which is related to the so called Kawashima condition. The second one is the so called entropy condition. We assume that system (1.1) has a strictly convex entropy $S(u)$ with entropy flux $q(u)$, such that

$$\sum_{j=1}^{n} \frac{\partial S(u)}{\partial u_j} a_{jk}(u) = \frac{\partial q(u)}{\partial u_k}, \quad k = 1, \cdots n \tag{1.14}$$

and

$$\sum_{j=1}^{n} \frac{\partial S(u)}{\partial u_j} f_j(u) \geq 0. \tag{1.15}$$

Moreover, we assume

$$S(0) = 0, \quad S'(0) = 0, \quad S''(0) > 0. \tag{1.16}$$

It follows from our assumptions that we have the entropy inequality

$$S(u)_t + q(u)_x \leq 0. \tag{1.17}$$

Our main result is given in Theorem 1.1.

Theorem 1.1 *Suppose that* (1.2) *holds in a neighborhood of origin, and system* (1.1) *has a strictly convex entropy satisfying* (1.14)–(1.16)*. Suppose furthermore that* (1.13) *is satisfied. Let* $\phi(x)$ *be a* C^1 *vector function satisfying that*

$$\varepsilon \triangleq \|\phi\|_{C^1(\mathbb{R})} + \|\phi\|_{H^1(\mathbb{R})} < \infty, \tag{1.18}$$

where $\|\phi\|_{C^1(\mathbb{R})} = \sup\limits_{x\in\mathbb{R}}(|\phi(x)| + |\phi'(x)|)$ *and* $\|\phi\|_{H^1} = \|\phi\|_{L^2(\mathbb{R})} + \|\phi'\|_{L^2(\mathbb{R})}$*. Then there exists an* $\varepsilon_0 > 0$ *so small that for any given* $\varepsilon \in [0, \varepsilon_0]$*, Cauchy problem* (1.1) *and* (1.8) *admits a unique global* C^1 *solution* $u = u(t, x)$ *as* $t \geq 0$.

Finally, we refer to [1, 6–10, 12–14] for related results on the global existence of smooth solutions to quasilinear hyperbolic systems.

2 Formula on the Decomposition of Waves

The purpose of this section is to derive formula on the decomposition of waves for the inhomogeneous quasilinear hyperbolic system, and combine it with a technique of [3]. The formula on the decomposition of waves for the homogenous quasilinear hyperbolic system was first derived by John in [4]. The inhomogeneous case was first treated by Kong in [6] (see also [7]).

Without loss of generality, we assume that

$$A(0) = \mathrm{diag}(\lambda_1(0), \cdots, \lambda_n(0)). \tag{2.1}$$

Let

$$w_i = l_i(u)u_x, \tag{2.2}$$

where $l_i(u)$ denotes the ith left eigenvector.

By (1.5), it is easy to see that

$$u_x = \sum_{k=1}^{n} w_k r_k(u). \tag{2.3}$$

Differentiating the equation with respect to x gives

$$(u_x)_t + (A(u)u_x)_x + (u_x \cdot \nabla_u)f(u) = 0. \tag{2.4}$$

Substituting (2.3) into this equation, we get

$$\sum_{j=1}^{n}[(w_j r_j(u))_t + (\lambda_j(u)w_j r_j(u))_x + (r_j(u)\cdot\nabla_u f(u))w_j] = 0. \tag{2.5}$$

Taking inner product with respect to $l_i(u)$ gives

$$w_{it} + (\lambda_i(u)w_i)_x = -\sum_{j=1}^{n}[l_i(u)\cdot(r_j(u)_t + \lambda_j(u)r_j(u)_x) + l_i(u)\cdot(r_j(u)\cdot\nabla_u)f(u)]w_j. \tag{2.6}$$

We have

$$(\lambda_i(u)w_i)_x = \lambda_i(u)w_{it} + u_x\cdot\nabla\lambda_i(u)w_i = \lambda_i(u)w_{it} + \sum_{j=1}^{n} r_j(u)\cdot\nabla\lambda_i(u)w_j w_i \tag{2.7}$$

and

$$\begin{aligned} r_j(u)_t + \lambda_j(u)r_j(u)_x &= u_t\cdot\nabla_u r_j(u) + \lambda_j(u)u_x\cdot\nabla_u r_j(u) \\ &= -(A(u)u_x + f(u))\cdot\nabla_u r_j(u) + \lambda_j(u)u_x\cdot\nabla_u r_j(u) \\ &= (\lambda_j(u)-\lambda_k(u))r_k(u)\cdot\nabla_u r_j(u)w_k - f(u)\cdot\nabla_u r_j(u). \end{aligned} \tag{2.8}$$

Let

$$\frac{\mathrm{d}}{\mathrm{d}_i t} = \frac{\partial}{\partial t} + \lambda_i(u)\frac{\partial}{\partial x} \tag{2.9}$$

be the directional derivative along the ith characteristic. We finally get

$$\frac{\mathrm{d}w_i}{\mathrm{d}_i t} = \sum_{j,k=1}^{n}\gamma_{ijk}(u)w_j w_k - \sum_{j=1}^{n}\overline{g}_{ij}(u)w_j, \quad i = 1,\cdots,n, \tag{2.10}$$

where

$$\gamma_{ijk}(u) = (\lambda_j(u)-\lambda_k(u))l_i(u)\cdot(r_k(u)\cdot\nabla_u r_j(u)) - r_j(u)\cdot\nabla_u\lambda_k(u)\delta_{ik} \tag{2.11}$$

and

$$\overline{g}_{ij}(u) = g_{ij}(u) - l_i(u)\cdot(f(u)\cdot\nabla_u r_j(u)) \tag{2.12}$$

with $g_{ij}(u)$ being defined by (1.10). At this point, we use a technique similar to that of [3]. For any $j \neq i$, we have

$$\begin{aligned}\frac{\mathrm{d}u_j}{\mathrm{d}_i t} &= l_j(0)\cdot(u_t+\lambda_i(u)u_x)\\ &= l_j(0)\cdot(-A(u)u_x - f(u) + \lambda_i(u)u_x)\\ &= \sum_{k=1}^{n}(\lambda_i(u)-\lambda_k(u))l_j(0)\cdot r_k(u)w_k - f_j(u).\end{aligned} \tag{2.13}$$

Noting that $l_j(0)\cdot r_j(0) = 1$, for u sufficiently small, we have $l_j(0)\cdot r_j(u) \neq 0$. Therefore, for $j \neq i$, we get

$$w_j = \frac{\frac{\mathrm{d}u_j}{\mathrm{d}_i t} + f_j(u)}{(\lambda_i(u)-\lambda_j(u))l_j(0)\cdot r_j(u)} - \sum_{k\neq j}\frac{(\lambda_i(u)-\lambda_k(u))l_j(0)\cdot r_k(u)}{(\lambda_i(u)-\lambda_j(u))l_j(0)\cdot r_j(u)}w_k. \tag{2.14}$$

Thus, it follows that

$$\sum_{j\neq i}\overline{g}_{ij}(u)w_j = \sum_{j\neq i}\alpha_{ij}(u)\frac{\mathrm{d}u_j}{\mathrm{d}_i t} + \sum_{k=1}^{n} b_{ik}(u)w_k + \alpha_i(u), \tag{2.15}$$

where

$$\alpha_{ij}(u) = \frac{\overline{g}_{ij}(u)}{(\lambda_i(u)-\lambda_j(u))l_j(0)\cdot r_j(u)}, \tag{2.16}$$

$$b_{ik}(u) = \sum_{\substack{j\neq i\\ j\neq k}}\frac{\overline{g}_{ij}(u)(\lambda_i(u)-\lambda_k(u))l_j(0)\cdot r_k(u)}{(\lambda_i(u)-\lambda_j(u))l_j(0)\cdot r_j(u)}, \tag{2.17}$$

$$\alpha_i(u) = \sum_{j\neq i}\frac{\overline{g}_{ij}(u)f_j(u)}{(\lambda_i(u)-\lambda_j(u))l_j(0)\cdot r_j(u)}. \tag{2.18}$$

Noting that $j \neq k$, $l_j(0)\cdot r_k(0) = 0$, we get

$$b_{ik}(0) = 0. \tag{2.19}$$

Let

$$a_i(u) = \sum_{j\neq i}\alpha_{ij}(u)u_j. \tag{2.20}$$

We get

$$\begin{aligned}\sum_{j\neq i}\alpha_{ij}(u)\frac{\mathrm{d}u_j}{\mathrm{d}_i t} &= \frac{\mathrm{d}a_i(u)}{\mathrm{d}_i t} - \sum_{j\neq i}\frac{\mathrm{d}u}{\mathrm{d}_i t}\cdot\nabla_u\alpha_{ij}(u)u_j\\ &= \frac{\mathrm{d}a_i(u)}{\mathrm{d}_i t} - \sum_{j\neq i}(-A(u)u_x - f(u) + \lambda_i(u)u_x)\cdot\nabla_u\alpha_{ij}(u)u_j\\ &= \frac{\mathrm{d}a_i(u)}{\mathrm{d}_i t} - \sum_{j\neq i}\Big[-f(u)\cdot\nabla_u\alpha_{ij}(u)u_j\\ &\quad + \sum_{k=1}^{n}(\lambda_i(u)-\lambda_k(u))r_k(u)\cdot\nabla_u\alpha_{ij}(u)u_j w_k\Big].\end{aligned} \tag{2.21}$$

Thus, we get

$$\frac{\mathrm{d}w_i}{\mathrm{d}_i t} + \overline{g}_{ii}(u)w_i = -\frac{\mathrm{d}a_i(u)}{\mathrm{d}_i t} + \sum_{j,k=1}^{n} \gamma_{ijk}(u)w_j w_k + \sum_{k=1}^{n} \overline{b}_{ik}(u)w_k + g_i(u), \tag{2.22}$$

where

$$\overline{b}_{ik}(u) = b_{ik}(u) + \sum_{j\neq i}(\lambda_i(u) - \lambda_k(u))r_k(u) \cdot \nabla_u \alpha_{ij}(u)u_j, \tag{2.23}$$

$$g_i(u) = -\sum_{j\neq i} f(u) \cdot \nabla_u \alpha_{ij}(u)u_j + \alpha_i(u). \tag{2.24}$$

Finally, let

$$z_i = w_i + a_i(u). \tag{2.25}$$

We get

$$\frac{\mathrm{d}z_i}{d_i t} + \overline{g}_{ii}(u)z_i = \sum_{j,k=1}^{n} \gamma_{ijk}(u)w_j w_k + \sum_{k=1}^{n} \overline{b}_{ik}(u)w_k + \overline{g}_i(u), \tag{2.26}$$

where

$$\overline{g}_i(u) = g_i(u) + \overline{g}_{ii}(u)a_i(u). \tag{2.27}$$

We have

$$\overline{b}_{ik}(0) = 0, \quad \forall\, i, k = 1, \cdots, n, \tag{2.28}$$

$$\overline{g}_i(0) = 0, \quad \forall\, i = 1, \cdots, n. \tag{2.29}$$

3 Proof of Theorem 1.1

By the local existence and uniqueness of C^1 solutions, we only need to get a priori estimate on the C^1 norm of the solution in order to get the global existence. For that purpose, we use a bootstrap argument. We first assume that on the existence domain $D = \{(t, x) \mid 0 \le t \le T, x \in \mathbb{R}\}$, we have

$$\sup_{x\in\mathbb{R}}(|u(t, x)| + |u_x(t, x)|) \le \sqrt{\varepsilon}, \quad \forall\, 0 \le t \le T. \tag{3.1}$$

We shall use this to prove

$$\sup_{x\in\mathbb{R}}(|u(t, x)| + |u_x(t, x)|) \le \frac{1}{2}\sqrt{\varepsilon}, \quad \forall\, 0 \le t \le T. \tag{3.2}$$

Then, (3.1) will always be valid, and the proof will be done.

First of all, we use entropy inequality (1.17) to get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{+\infty} S(u(t, x))\mathrm{d}x \le 0. \tag{3.3}$$

Thus

$$\int_{-\infty}^{+\infty} S(u(t,x))\mathrm{d}x \le \int_{-\infty}^{+\infty} S(\phi(x))\mathrm{d}x. \tag{3.4}$$

By (1.16), there exists two positive constant λ and Λ, such that

$$\lambda|u|^2 \le S(u) \le \Lambda|u|^2. \tag{3.5}$$

Then it follows that

$$\|u(t)\|^2_{L^2(\mathbb{R})} \le \frac{\Lambda}{\lambda}\|\phi\|^2_{L^2(\mathbb{R})}. \tag{3.6}$$

Therefore, we get

$$\|u(t)\|_{L^2(\mathbb{R})} \le C\varepsilon, \quad \forall\, 0 \le t \le T. \tag{3.7}$$

Here and hereafter, C will denote a positive constant independent of ε, and its meaning may change from line to line.

Next, we estimate L^2 norm of the derivatives of the solutions. By (2.3), it is enough to estimate L^2 norm of $w = (w_1, \cdots, w_n)^{\mathrm{T}}$. For that purpose, we first rewrite (2.26) as

$$\frac{\partial z_i}{\partial t} + \lambda_i(u)\frac{\partial z_i}{\partial x} + \overline{g}_{ii}(u)z_i = \sum_{j,k=1}^{n} \gamma_{ijk}(u)w_j w_k + \sum_{k=1}^{n} \overline{b}_{ik}(u)w_k + \overline{g}_i(u). \tag{3.8}$$

We multiply the equation by z_i and integration by parts to get

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{+\infty} z_i^2\mathrm{d}x + \int_{-\infty}^{+\infty} \overline{g}_{ii}(u)z_i^2\mathrm{d}x\\
={}& \frac{1}{2}\int_{-\infty}^{+\infty} \lambda_i(u)_x z_i^2\mathrm{d}x + \int_{-\infty}^{+\infty} \sum_{j,k=1}^{n} \gamma_{ijk}(u)w_j w_k z_i\mathrm{d}x\\
&+ \int_{-\infty}^{+\infty} \sum_{k=1}^{n} \overline{b}_{ik}(u)w_k z_i\mathrm{d}x + \int_{-\infty}^{+\infty} \overline{g}_i(u)z_i\mathrm{d}x.
\end{aligned} \tag{3.9}$$

We have

$$\begin{aligned}
\int_{-\infty}^{+\infty} \lambda_i(u)_x z_i^2\mathrm{d}x &= \int_{-\infty}^{+\infty} u_x \cdot \nabla_u\lambda_i(u)z_i^2\mathrm{d}x\\
&\le C\|u_x\|_{C^0}\|z_i\|^2_{L^2(\mathbb{R})} \le C\sqrt{\varepsilon}\|z_i\|^2_{L^2(\mathbb{R})},
\end{aligned} \tag{3.10}$$

$$\begin{aligned}
\int_{-\infty}^{+\infty} \sum_{j,k=1}^{n} \gamma_{ijk}(u)w_j w_k z_i\mathrm{d}x &\le C\|z_i\|_{C^0}\|w\|^2_{L^2(\mathbb{R})}\\
&\le C\sqrt{\varepsilon}\|w\|^2_{L^2(\mathbb{R})}\\
&\le C\sqrt{\varepsilon}\Big(\sum_{j=1}^{n}\|z_j\|^2_{L^2(\mathbb{R})} + \|u\|^2_{L^2(\mathbb{R})}\Big)\\
&\le C\sqrt{\varepsilon}\sum_{j=1}^{n}\|z_j\|^2_{L^2(\mathbb{R})} + C\varepsilon^{\frac{5}{2}},
\end{aligned} \tag{3.11}$$

$$\begin{aligned}\int_{-\infty}^{+\infty}\sum_{k=1}^{n}\overline{b}_{ik}(u)w_k z_i\mathrm{d}x &\le C\|z_i\|_{C^0}\|w\|_{L^2}\|u\|_{L^2}\\ &\le C\sqrt{\varepsilon}(\|w\|^2_{L^2(\mathbb{R})}+\|u\|^2_{L^2(\mathbb{R})})\\ &\le C\sqrt{\varepsilon}\Big(\sum_{j=1}^{n}\|z_j\|^2_{L^2(\mathbb{R})}+\|u\|^2_{L^2(\mathbb{R})}\Big)\\ &\le C\sqrt{\varepsilon}\sum_{j=1}^{n}\|z_j\|^2_{L^2(\mathbb{R})}+C\varepsilon^{\frac{5}{2}}. \end{aligned}\tag{3.12}$$

Finally,

$$\begin{aligned}\int_{-\infty}^{+\infty}\overline{g}_i(u)z_i\mathrm{d}x &\le C\|u\|_{L^2(\mathbb{R})}\|z_i\|_{L^2(\mathbb{R})}\\ &\le \delta\|z_i\|^2_{L^2(\mathbb{R})}+\frac{C}{\delta}\|u\|^2_{L^2(\mathbb{R})}\\ &\le \delta\|z_i\|^2_{L^2(\mathbb{R})}+\frac{C}{\delta}\varepsilon^2, \end{aligned}\tag{3.13}$$

where δ is to be chosen later. By our assumption

$$\overline{g}_{ii}(0)=g_{ii}(0)>0,\tag{3.14}$$

there exists a positive constant μ, such that

$$\overline{g}_{ii}(u)\ge\mu,\quad \forall\, i=1,\cdots,n,\tag{3.15}$$

provided that $|u|$ is sufficiently small. Therefore, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|z_i(t,\cdot)\|^2_{L^2(\mathbb{R})}+\mu\|z_i(t,\cdot)\|^2_{L^2(\mathbb{R})}\le(C\sqrt{\varepsilon}+\delta)\sum_{i=1}^{n}\|z_i(t,\cdot)\|^2_{L^2(\mathbb{R})}+C(\delta)\varepsilon^2.\tag{3.16}$$

Summing up for i, we get

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^{n}\|z_i(t,\cdot)\|^2_{L^2(\mathbb{R})}+\mu\sum_{i=1}^{n}\|z_i(t,\cdot)\|^2_{L^2(\mathbb{R})}\\ &\le(C\sqrt{\varepsilon}+n\delta)\sum_{i=1}^{n}\|z_i(t,\cdot)\|^2_{L^2(\mathbb{R})}+C(\delta)\varepsilon^2. \end{aligned}\tag{3.17}$$

Taking

$$\delta=\frac{\mu}{4n},\tag{3.18}$$

$$C\sqrt{\varepsilon}\le\frac{\mu}{4},\tag{3.19}$$

we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^{n}\|z_i(t,\cdot)\|^2_{L^2(\mathbb{R})}+\mu\sum_{i=1}^{n}\|z_i(t,\cdot)\|^2_{L^2(\mathbb{R})}\le C\varepsilon^2.\tag{3.20}$$

Solving this differential inequality, we get

$$\sum_{i=1}^{n}\|z_i(t,\cdot)\|_{L^2(\mathbb{R})}^2\le C\varepsilon^2\mathrm{e}^{-\mu t}+C(1-\mathrm{e}^{-\mu t})\varepsilon^2\le C\varepsilon^2. \tag{3.21}$$

Noting

$$\|w_i\|_{L^2(\mathbb{R})}\le\|z_i\|_{L^2(\mathbb{R})}+\|a_i(u)\|_{L^2(\mathbb{R})}\le\|z_i\|_{L^2(\mathbb{R})}+C\varepsilon, \tag{3.22}$$

we finally get

$$\|w(t,\cdot)\|_{L^2(\mathbb{R})}\le C\varepsilon,\quad\forall\,0\le t\le T. \tag{3.23}$$

This implies

$$\|u_x(t,\cdot)\|_{L^2(\mathbb{R})}\le C\varepsilon,\quad\forall\,0\le t\le T. \tag{3.24}$$

By Sobolev embedding theorem, we have

$$\|u(t,\cdot)\|_{C^0(\mathbb{R})}\le C\varepsilon,\quad\forall\,0\le t\le T. \tag{3.25}$$

We now estimate the derivatives of u. For that purpose, we multiply (2.26) by $\mathrm{sgn}(z_i)$ to obtain

$$\begin{aligned}\frac{\mathrm{d}|z_i|}{\mathrm{d}_it}+\overline{g}_{ii}(u)|z_i|&=\mathrm{sgn}(z_i)\Big(\sum_{j,k=1}^{n}\gamma_{ijk}(u)w_jw_k+\sum_{k=1}^{n}\overline{b}_{ik}(u)w_k+\overline{g}_i(u)\Big)\\&\le C\|w(t,\cdot)\|_{C^0(\mathbb{R})}^2+C\|u(t,\cdot)\|_{C^0(\mathbb{R})}\|w(t,\cdot)\|_{C^0(\mathbb{R})}+C\|u(t,\cdot)\|_{C^0(\mathbb{R})}\\&\le C(\sqrt{\varepsilon})^2+C\varepsilon^{\frac{3}{2}}+C\varepsilon\\&\le C\varepsilon.\end{aligned} \tag{3.26}$$

Thus, we get

$$\frac{\mathrm{d}|z_i|}{\mathrm{d}_it}+\mu|z_i|\le C\varepsilon. \tag{3.27}$$

Integrating this inequality along characteristics yields

$$|z_i(t,x)|\le C\varepsilon\mathrm{e}^{-\mu t}+C(1-\mathrm{e}^{-\mu t})\varepsilon\le C\varepsilon. \tag{3.28}$$

Noting

$$|w_i(t,x)|\le|z_i(t,x)|+|a_i(u)(t,x)|\le|z_i(t,x)|+C\varepsilon, \tag{3.29}$$

we get

$$|w_i(t,x)|\le C\varepsilon. \tag{3.30}$$

Taking supreme for $(t,x)\in D$ and $i=1,\cdots,n$, we get

$$\sup_{0\le t\le T}\|w(t,\cdot)\|_{C^0(\mathbb{R})}\le C\varepsilon. \tag{3.31}$$

Combining this with (3.25), we finally get

$$\sup_{0\le t\le T}\|u(t,\cdot)\|_{C^1(\mathbb{R})}\le C\varepsilon. \tag{3.32}$$

This again implies (3.2) provided that ε is sufficiently small. This completes the proof of Theorem 1.1.

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