Chin. Ann. Math.
30B(3), 2009, 293–310
DOI: 10.1007/s11401-007-0492-5

Chinese Annals of Mathematics, Series B

Toeplitz and Hankel Products on Bergman Spaces of the Unit Ball**

Yufeng LU* Chaomei LIU*

Abstract The authors give some new necessary conditions for the boundedness of Toeplitz products $T_f^\alpha T_{\overline{g}}^\alpha$ on the weighted Bergman space A_α^2 of the unit ball, where f and g are analytic on the unit ball. Hankel products $H_f H_g^*$ on the weighted Bergman space of the unit ball are studied, and the results analogous to those Stroethoff and Zheng obtained in the setting of unit disk are proved.

Keywords Weighted Bergman space, Unit ball, Toeplitz operator, Hankel operator, Berezin transform
2000 MR Subject Classification 47B35, 47B47

1 Introduction

Throughout this paper, let $n \geq 2$ be a fixed integer. Denote the unit ball in $\mathbb{C}^n$ by B_n, and let v be the normalized Lebesgue volume measure on B_n. For $-1 < \alpha < \infty$, we denote by v_α the measure on B_n defined by $\mathrm{d}v_\alpha(z) = c_\alpha(1-|z|^2)^\alpha \mathrm{d}v(z)$, where $c_\alpha = \frac{\Gamma(n+\alpha+1)}{n!\Gamma(\alpha+1)}$ is a normalizing constant such that $v_\alpha(B_n) = 1$. For $1 \leq p < \infty$, we write $\|\cdot\|_{\alpha,p}$ for the norm on $L^p(B_n, \mathrm{d}v_\alpha)$ and $\langle\,\cdot\,,\,\cdot\,\rangle_\alpha$ for the inner product on $L^2(B_n, \mathrm{d}v_\alpha)$. The weighted Bergman space A_α^2 is the space of analytic functions on B_n that are also in $L^2(B_n, \mathrm{d}v_\alpha)$. Reproducing kernels $K_w^{(\alpha)}$ and normalized reproducing kernels $k_w^{(\alpha)}$ in A_α^2 are given by, respectively,

$$K_w^{(\alpha)}(z) = \frac{1}{(1-\langle z,w\rangle)^{n+\alpha+1}} \quad \text{and} \quad k_w^{(\alpha)}(z) = \frac{(1-|w|^2)^{\frac{n+\alpha+1}{2}}}{(1-\langle z,w\rangle)^{n+\alpha+1}}$$

for $z, w \in B_n$. For every $h \in A_\alpha^2$, we have $\langle h, K_w^{(\alpha)}\rangle_\alpha = h(w)$ for all $w \in B_n$. The orthogonal projection P_α of $L^2(B_n, \mathrm{d}v_\alpha)$ onto A_α^2 is given by

$$(P_\alpha g)(w) = \langle g, K_w^{(\alpha)}\rangle_\alpha = \int_{B_n} g(z)\frac{1}{(1-\langle w,z\rangle)^{n+\alpha+1}}\mathrm{d}v_\alpha(z)$$

for $g \in L^2(B_n, \mathrm{d}v_\alpha)$ and $w \in B_n$.

Given $f \in L^\infty(B_n)$, the Toeplitz operator T_f^α is defined on A_α^2 by $T_f^\alpha h = P_\alpha(fh)$. We have

$$(T_f^\alpha h)(w) = \langle T_f^\alpha h, K_w^{(\alpha)}\rangle_\alpha = \langle fh, K_w^{(\alpha)}\rangle_\alpha = \int_{B_n} \frac{f(z)h(z)}{(1-\langle w,z\rangle)^{n+\alpha+1}}\mathrm{d}v_\alpha(z)$$

Manuscript received November 26, 2007. Published online April 16, 2009.
*School of Mathematical Sciences, Dalian University of Technology, Dalian 116024, Liaoning, China.
E-mail: lyfdlut1@yahoo.com.cn liuchaomeidjtu@126.com
**Project supported by the National Natural Science Foundation of China (No. 10671028).

for $h \in A^2_\alpha$ and $w \in B_n$. Note that the above integral formula makes sense, and defines a function analytic on B_n, if $f \in L^2(B_n, \mathrm{d}v_\alpha)$. So, if $g \in A^2_\alpha$, we define $T^\alpha_{\overline{g}}$ by the formula

$$(T^\alpha_{\overline{g}} h)(w) = \int_{B_n} \frac{\overline{g(z)} h(z)}{(1 - \langle w, z\rangle)^{n+\alpha+1}} \mathrm{d}v_\alpha(z)$$

for all $h \in A^2_\alpha$ and $w \in B_n$. If also $f \in A^2_\alpha$, then $T^\alpha_f T^\alpha_{\overline{g}} h$ is the analytic function $f T^\alpha_{\overline{g}} h$ for $h \in H^\infty(B_n)$.

Given $f \in L^\infty(B_n)$, the Hankel operator H_f is defined on A^2_α by $H_f h = (I - P_\alpha)(fh)$. Then

$$(H_f h)(w) = f(w)h(w) - P_\alpha(fh)(w) = \int_{B_n} \frac{(f(w) - f(z))h(z)}{(1 - \langle w, z\rangle)^{n+\alpha+1}} \mathrm{d}v_\alpha(z)$$

for $h \in A^2_\alpha$ and $w \in B_n$. The latter formula will be used to define H_f densely on A^2_α if $f \in L^2(B_n, \mathrm{d}v_\alpha)$. If $g \in L^\infty(B_n)$ and $u \in (A^2_\alpha)^\perp$, then

$$H^*_g u(w) = \langle H^*_g u, K^{(\alpha)}_w \rangle_\alpha = \langle u, H_g K^{(\alpha)}_w \rangle_\alpha = \langle u, g K^{(\alpha)}_w \rangle_\alpha$$

for $w \in B_n$. Since $K^{(\alpha)}_w$ is bounded, the latter formula makes sense for all $g \in L^2(B_n, \mathrm{d}v_\alpha)$, and we use it to define the operator H^*_g densely on $(A^2_\alpha)^\perp$. Note that the star need no longer be the adjoint (but would of course coincide with the adjoint in case the operator H_g is itself bounded).

By [1, Theorem 3.14], $C_c(B_n)$, the set of all continuous functions with compact support in B_n, is dense in $L^2(B_n, \mathrm{d}v_\alpha)$, so certainly $C_c(B_n) \cap (A^2_\alpha)^\perp$ is dense in $(A^2_\alpha)^\perp$. If $f, g \in L^2(B_n, \mathrm{d}v_\alpha)$ and $u \in C_c(B_n) \cap (A^2_\alpha)^\perp$, then $H^*_g u$ is bounded, and the meaning of $H_f H^*_g u$ is clear: it is the function $H_f(H^*_g u)$. This defines the Hankel product $H_f H^*_g$ on a dense subset of $(A^2_\alpha)^\perp$, namely, $C_c(B_n) \cap (A^2_\alpha)^\perp$.

It is well-known that Toeplitz operator, Hankel operator and dual Toeplitz operator are closely related to each other. Under the decomposition $L^2(B_n, \mathrm{d}v_\alpha) = A^2_\alpha \oplus (A^2_\alpha)^\perp$, for $f \in L^\infty(B_n)$, the multiplication operator M_f is represented as

$$M_f = \begin{pmatrix} T^\alpha_f & H^*_{\overline{f}} \\ H_f & S_f \end{pmatrix}.$$

The operator S_f is an operator on $(A^2_\alpha)^\perp$, which is called the dual Toeplitz operator with symbol f. The identity $M_{fg} = M_f M_g$ implies the following basic algebraic relation between these operators

$$H_{fg} = H_f T^\alpha_g + S_f H_g.$$

Suppose $\varphi \in H^\infty(B_n)$ and $\psi \in L^\infty(B_n)$. Then we have

$$H_\psi T^\alpha_\varphi = S_\varphi H \psi, \tag{1.1}$$

and, by taking adjoints, we have

$$T^\alpha_{\overline{\varphi}} H^*_\psi = H^*_\psi S_{\overline{\varphi}}. \tag{1.2}$$

It is easy to prove that identities (1.1) and (1.2) also hold if $\varphi \in H^\infty(B_n)$ and $\psi \in L^2(B_n, \mathrm{d}v_\alpha)$.

In this paper, we shall consider questions of when, for analytic functions f and g, the product $T_f^\alpha T_{\overline{g}}^\alpha$ extends to a bounded linear operator on A_α^2, and when, for square integrable functions f and g, the product $H_f H_g^*$ extends to a bounded linear operator on $(A_\alpha^2)^\perp$.

On the Hardy space $H^2(\mathbb{T})$, bounded Toeplitz operators arise only from bounded symbols. In [2], Sarason posed the problem for which f and g in $H^2(\mathbb{T})$ the densely defined operator $T_f T_{\overline{g}}$ is bounded on $H^2(\mathbb{T})$, and he conjectured that a necessary condition founded by S. Treil, namely,

$$\sup_{w\in\mathbb{D}} \langle |f|^2 \widetilde{k}_w, \widetilde{k}_w\rangle \langle |g|^2 \widetilde{k}_w, \widetilde{k}_w\rangle < \infty,$$

where $\widetilde{k}_w = (1-|w|^2)^{\frac{1}{2}} \frac{1}{1-\overline{w}z}$ denotes the normalized reproducing kernels of $H^2(\mathbb{T})$, is also sufficient. This question turned out to have close links with the question of boundedness of the two-weight Hilbert transform on $L^2(\mathbb{T})$ (see [3]). In [4], Cruz-Uribe characterized the outer functions f and g for which the Toeplitz product $T_f T_{\overline{g}}$ is bounded and invertible on $H^2(\mathbb{T})$, providing support for Sarason's conjecture. In [5], Zheng obtained a partial answer to Sarason's problem by showing that a condition slightly stronger than the one in Sarason's conjecture is sufficient for boundedness of these Toeplitz products on the Hardy space. Unfortunately, Sarason's conjecture on the Hardy space was answered in the negative by Nazarov [6].

On the Bergman space of the unit disk, there are unbounded symbols that induce bounded Toeplitz operators. A Toeplitz operator with analytic symbol is, however, bounded if and only if its symbol is bounded on the unit disk. In [2], Sarason also asked for which analytic functions f and g the densely defined product $T_f^0 T_{\overline{g}}^0$ is bounded on $A_0^2(\mathbb{D})$. In [7], Stroethoff and Zheng found necessary conditions on the unit disk $\mathbb{D}$ and they also proved that the necessary condition is very close to being sufficient, as shown for Toeplitz products on the Hardy space of the unit circle in [5]:

(1) If $f, g \in A_0^2(\mathbb{D})$ and $T_f^0 T_{\overline{g}}^0$ is bounded, then

$$\sup_{w\in\mathbb{D}} \langle |f|^2 k_w^{(0)}, k_w^{(0)}\rangle_0 \langle |g|^2 k_w^{(0)}, k_w^{(0)}\rangle_0 = \sup_{w\in\mathbb{D}} B_0(|f|^2) B_0(|g|^2) < \infty;$$

(2) If $f, g \in A_0^2(\mathbb{D})$ and there exists an $\varepsilon > 0$ such that

$$\sup_{w\in\mathbb{D}} B_0(|f|^{2+\varepsilon}) B_0(|g|^{2+\varepsilon}) < \infty,$$

then $T_f^0 T_{\overline{g}}^0$ is bounded.

Stroethoff and Zheng showed the analogous result on the Bergman space of the polydisk in [8] and on the weighted Bergman space of the unit disk in [9] and the unit ball in [10]. In [11], Park gave the analogous result for Toeplitz products on the Bergman space of the unit ball. In [12], Pott and Strouse also obtained a sufficient and a necessary condition for boundedness of the Toeplitz products on the weighted Bergman space of the unit disk. But Sarason's problem is still open on various settings.

On the Bergman space, little is known concerning the products $H_f^* H_g$ or $H_f H_g^*$ for $f, g \in L^2(\mathbb{D}, \mathrm{d}A)$. Many interesting questions concerning Hankel products still remain open. In [7], Stroethoff and Zheng obtained a necessary condition on boundedness of Hankel products $H_f H_g^*$

and proved that the necessary condition is very close to being sufficient, as shown for Toeplitz products on the Bergman space of the unit disk. In [15], Lu and Shang proved a similar result for Hankel products on the Bergman space of the polydisk.

In this paper, we continue to investigate conditions for boundedness of the Toeplitz products on the weighted Bergman space of the unit ball and obtain new necessary conditions to guarantee the boundedness of the Toeplitz products on the weighted Bergman space of the unit ball. Meanwhile, we study Hankel products $H_f H_g^*$ on the weighted Bergman space of the unit ball and prove results analogous to those Stroethoff and Zheng [7] obtained in the setting of unit disk.

2 Some Lemmas and Basic Inequalities

For $w \in B_n$, let φ_w be the automorphism of B_n, which is described in [13, Section 2.2]. It has real Jacobian equal to

$$|\varphi_w'|^2 = \frac{(1-|w|^2)^{n+1}}{|1-\langle z, w\rangle|^{2n+2}},$$

and it also has properties as follows:

$$1-\langle \varphi_w(z), w\rangle = \frac{1-|w|^2}{1-\langle z, w\rangle} \quad \text{and} \quad k_w^{(\alpha)}(\varphi_w(z)) = \frac{1}{k_w^{(\alpha)}(z)}$$

for $z, w \in B_n$. Thus we have the following change-of-variable formula

$$\int_{B_n} h\circ\varphi_w(z)\mathrm{d}v_\alpha(z) = \int_{B_n} h(z)|k_w^{(\alpha)}(z)|^2\mathrm{d}v_\alpha(z) \tag{2.1}$$

for every $h \in L^1(B_n, \mathrm{d}v_\alpha)$ (see [14] for the proof).

For $w \in B_n$, the operator $U_w^{(\alpha)}$ on A_α^2 is defined by

$$U_w^{(\alpha)}h = (h\circ\varphi_w)k_w^{(\alpha)}.$$

It is easy to see that $U_w^{(\alpha)}$ is a unitary operator and $(U_w^{(\alpha)})^{-1} = U_w^{(\alpha)}$. In particular,

$$T^\alpha_{f\circ\varphi_w}U_w^{(\alpha)} = U_w^{(\alpha)}T_f^\alpha \tag{2.2}$$

holds for $f \in L^\infty(B_n)$ (see [10] for the proof).

For a function $u \in L^1(B_n, \mathrm{d}v_\alpha)$, the Berezin transform $B_\alpha[u]$ is the function on B_n defined by

$$B_\alpha[u](w) = \int_{B_n} u(z)\frac{(1-|w|^2)^{n+\alpha+1}}{|1-\langle z, w\rangle|^{2n+2\alpha+2}}\mathrm{d}v_\alpha(z).$$

Suppose $f, g \in A_\alpha^2$. Consider the operator $f\otimes g$ on A_α^2 defined by

$$(f\otimes g)h = \langle h, g\rangle_\alpha f$$

for $h \in A_\alpha^2$. It is easily proved that $f\otimes g$ is bounded on A_α^2 with norm equal to $\|f\otimes g\| = \|f\|_{\alpha,2}\|g\|_{\alpha,2}$.

We observe that the Taylor expansion of the function $(1-z)^{n+\alpha+1}$ around 0, i.e.,

$$(1-z)^{n+\alpha+1} = \sum_{k=0}^{\infty} C_{n,\alpha,k} z^k,$$

where $C_{n,\alpha,k} = (-1)^k \frac{(n+\alpha+1)(n+\alpha)\cdots(n+\alpha+2-k)}{k!}$, $k = 1, 2, \cdots$, $C_{n,\alpha,0} = 1$, is absolutely convergent on the closed unit disk in $\mathbb{C}$ for $\alpha > -1$.

The term multi-index refers to an ordered n-tuple

$$m = (m_1, \cdots, m_n)$$

of nonnegative integer m_i. The following abbreviated notations will be used:

$$z^m = z_1^{m_1} \cdots z_n^{m_n}, \quad |m| = m_1 + \cdots + m_n, \quad m! = m_1! \cdots m_n!.$$

We have the multinomial formula

$$(z_1 + \cdots + z_n)^N = \sum_{|m|=N} \frac{N!}{m!} z^m.$$

In this paper, the letter C denotes a positive constant, possibly different on each occurrence.

Lemma 2.1 *On A_α^2, we have*

$$k_w^{(\alpha)} \otimes k_w^{(\alpha)} = \sum_{k=0}^{\infty} C_{n,\alpha,k} \sum_{|\gamma|=k} \frac{k!}{\gamma!} T^\alpha_{\varphi_w^\gamma} T^\alpha_{\overline{\varphi_w^\gamma}} \tag{2.3}$$

for $w \in B_n$.

Proof For $f \in A_\alpha^2$, by the mean value property, we have

$$f(0) = (1 \otimes 1) f = \int_{B_n} f(w) \mathrm{d}v_\alpha(w) = \int_{B_n} (K_w^{(\alpha)}(z))^{-1} K_w^{(\alpha)}(z) f(w) \mathrm{d}v_\alpha(w).$$

By the multinomial formula, we have

$$(K_w^{(\alpha)}(z))^{-1} = (1 - \langle z, w\rangle)^{n+\alpha+1} = \sum_{k=0}^{\infty} C_{n,\alpha,k} \sum_{|\gamma|=k} \frac{k!}{\gamma!} \overline{w}^\gamma z^\gamma.$$

Since the series $\sum\limits_{k=0}^{\infty} |C_{n,\alpha,k}|$ is convergent and $T^\alpha_{\overline{w}^\gamma} f(z) = \int_{B_n} \overline{w}^\gamma K_w^{(\alpha)}(z) f(w) \mathrm{d}v_\alpha(w)$, we have

$$\begin{aligned} f(0) = (1 \otimes 1) f &= \int_{B_n} f(w) \mathrm{d}v_\alpha(w) \\ &= \sum_{k=0}^{\infty} C_{n,\alpha,k} \sum_{|\gamma|=k} \frac{k!}{\gamma!} z^\gamma \int_{B_n} \overline{w}^\gamma K_w^{(\alpha)}(z) f(w) \mathrm{d}v_\alpha(w) \\ &= \sum_{k=0}^{\infty} C_{n,\alpha,k} \sum_{|\gamma|=k} \frac{k!}{\gamma!} T^\alpha_{z^\gamma} T^\alpha_{\overline{z}^\gamma} f. \end{aligned}$$

Then it follows that

$$1\otimes 1=\sum_{k=0}^{\infty}C_{n,\alpha,k}\sum_{|\gamma|=k}\frac{k!}{\gamma!}T^{\alpha}_{z^{\gamma}}T^{\alpha}_{\overline{z}^{\gamma}}.$$

For $w\in B_n$, we use the unitary operator $U_w^{(\alpha)}$ to obtain

$$\begin{aligned}k_w^{(\alpha)}\otimes k_w^{(\alpha)}&=(U_w^{(\alpha)}1)\otimes(U_w^{(\alpha)}1)=U_w^{(\alpha)}(1\otimes 1)U_w^{(\alpha)}\\&=U_w^{(\alpha)}\Big(\sum_{k=0}^{\infty}C_{n,\alpha,k}\sum_{|\gamma|=k}\frac{k!}{\gamma!}T^{\alpha}_{z^{\gamma}}T^{\alpha}_{\overline{z}^{\gamma}}\Big)U_w^{(\alpha)}\\&=\sum_{k=0}^{\infty}C_{n,\alpha,k}\sum_{|\gamma|=k}\frac{k!}{\gamma!}U_w^{(\alpha)}T^{\alpha}_{z^{\gamma}}T^{\alpha}_{\overline{z}^{\gamma}}U_w^{(\alpha)}\\&=\sum_{k=0}^{\infty}C_{n,\alpha,k}\sum_{|\gamma|=k}\frac{k!}{\gamma!}T^{\alpha}_{\varphi_w^{\gamma}}T^{\alpha}_{\overline{\varphi}_w^{\gamma}}.\end{aligned}$$

The following inner product formula in A^2_α will play an important role in this paper, which was proved in [10].

Lemma 2.2 (see [10]) *Let $-1<\alpha<\infty$, and m be a positive integer. Then there exist constants $a_1,a_2,\cdots,a_{2m-1}$ and $b_1,b_2,\cdots,b_m$ such that, for any $F,G\in A^2_\alpha$,*

$$\begin{aligned}\langle F,G\rangle_\alpha=&\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+2m+1)}\sum_{|\gamma|=m}\int_{B_n}D^\gamma F(z)\overline{D^\gamma G(z)}(1-|z|^2)^{2m}\mathrm{d}v_\alpha(z)\\&+\sum_{j=1}^{2m-1}a_j\sum_{|\gamma|=m}\int_{B_n}D^\gamma F(z)\overline{D^\gamma G(z)}(1-|z|^2)^{2m+j}\mathrm{d}v_\alpha(z)\\&+\sum_{j=1}^{m}b_j\int_{B_n}F(z)\overline{G(z)}(1-|z|^2)^{2m+j-1}\mathrm{d}v_\alpha(z).\end{aligned}\tag{2.4}$$

The following lemma will be frequently used in the following calculations (see [14]).

Lemma 2.3 (see [14]) *Fix two real parameters a and b, and define two integral operators $T_{a,b}$ and $Q_{a,b}$ as follows:*

$$T_{a,b}f(z)=(1-|z|^2)^a\int_{B_n}\frac{(1-|w|^2)^b}{(1-\langle z,w\rangle)^{n+1+a+b}}f(w)\mathrm{d}v(w)$$

and

$$Q_{a,b}f(z)=(1-|z|^2)^a\int_{B_n}\frac{(1-|w|^2)^b}{|1-\langle z,w\rangle|^{n+1+a+b}}f(w)\mathrm{d}v(w).$$

Then, for $-1<t<\infty$ and $1\le p<\infty$, the following conditions are equivalent:

(a) *$T_{a,b}$ is bounded on $L^p(B_n,\mathrm{d}v_t)$,*

(b) *$Q_{a,b}$ is bounded on $L^p(B_n,\mathrm{d}v_t)$,*

(c) $-pa<t+1<p(b+1)$.

Lemma 2.4 *Let $-1<\gamma<\alpha<\infty$. For $f\in L^2(B_n,\mathrm{d}v_\gamma)$ and $h\in H^\infty(B_n)$, we have*

$$|(T^{\alpha}_{\overline{f}}h)(w)|\le\frac{B_\alpha[|f|^2](w)^{\frac12}}{(1-|w|^2)^{\frac{n+1+\alpha}{2}}}\|h\|_{\gamma,2}$$

and

$$|(T^{\alpha}_{\overline{f}}h)(w)| \le C\frac{B_\gamma[|f|^2](w)^{\frac{1}{2}}}{(1-|w|^2)^{\frac{n+1+\alpha}{2}}}\|h\|_{\alpha,2}$$

for all $w \in B_n$.

Proof Suppose $h \in H^\infty(B_n)$. Using Hölder's inequality, we have

$$|(T^{\alpha}_{\overline{f}}h)(w)| = |\langle \overline{f}h, K_w^{(\alpha)}\rangle_\alpha| = |\langle h, fK_w^{(\alpha)}\rangle_\alpha| \le \|h\|_{\alpha,2}\|fK_w^{(\alpha)}\|_{\alpha,2}.$$

Since

$$B_\alpha[|f|^2](w) = \Big\|f\frac{K_w^{(\alpha)}}{\|K_w^{(\alpha)}\|_{\alpha,2}}\Big\|^2_{\alpha,2} = (1-|w|^2)^{n+\alpha+1}\|fK_w^{(\alpha)}\|^2_{\alpha,2},$$

we see that

$$|(T^{\alpha}_{\overline{f}}h)(w)| \le \frac{B_\alpha[|f|^2](w)^{\frac{1}{2}}}{(1-|w|^2)^{\frac{n+\alpha+1}{2}}}\|h\|_{\alpha,2}.$$

Since $\gamma \le \alpha$ implies $\|h\|_{\alpha,2} \le \|h\|_{\gamma,2}$, the first inequality follows. Since

$$\begin{aligned}
B_\alpha[|f|^2](w) &= \int_{B_n}|f|^2(z)\frac{(1-|w|^2)^{n+\alpha+1}}{|1-\langle z,w\rangle|^{2n+2\alpha+2}}\mathrm{d}v_\alpha(z)\\
&= \int_{B_n}|f|^2(z)\frac{(1-|w|^2)^{n+\gamma+1}(1-|w|^2)^{\alpha-\gamma}(1-|z|^2)^{\alpha-\gamma}}{|1-\langle z,w\rangle|^{2n+2\gamma+2}|1-\langle z,w\rangle|^{2\alpha-2\gamma}}\mathrm{d}v_\gamma(z)\\
&\le 4^{\alpha-\gamma}\int_{B_n}|f|^2(z)\frac{(1-|w|^2)^{n+\gamma+1}}{|1-\langle z,w\rangle|^{2n+2\gamma+2}}\mathrm{d}v_\gamma(z)\\
&= C^2B_\gamma[|f|^2](w),
\end{aligned}$$

the second inequality follows.

Lemma 2.5 *Let $-1 < \gamma < \alpha < \infty$. For $f \in L^2(B_n, \mathrm{d}v_\gamma)$, $h \in H^\infty(B_n)$ and multi-index s with $|s| = m \ge \frac{n+\alpha+1}{2}$, we have*

$$|(D^sT^{\alpha}_{\overline{f}}h)(w)| \le C\frac{B_\gamma[|f|^2](w)^{\frac{1}{2}}}{(1-|w|^2)^m}(Q_{0,2\alpha-\gamma}(|h|^2)(w))^{\frac{1}{2}}.$$

Proof For $f \in L^2(B_n, \mathrm{d}v_\gamma)$ and $h \in H^\infty(B_n)$, we have

$$(T^{\alpha}_{\overline{f}}h)(w) = \langle T^{\alpha}_{\overline{f}}h, K_w^{(\alpha)}\rangle_\alpha = \int_{B_n}\frac{\overline{f(z)}h(z)}{(1-\langle w,z\rangle)^{n+\alpha+1}}\mathrm{d}v_\alpha(z).$$

Thus

$$(D^sT^{\alpha}_{\overline{f}}h)(w) = \frac{\Gamma(n+\alpha+m+1)}{\Gamma(n+\alpha+1)}\int_{B_n}\frac{\overline{z^sf(z)}h(z)}{(1-\langle w,z\rangle)^{n+\alpha+m+1}}\mathrm{d}v_\alpha(z)$$

for every multi-index s with $|s| = m$. Applying Hölder's inequality, we get

$$\begin{aligned}
|(D^sT^{\alpha}_{\overline{f}}h)(w)| &\le C\int_{B_n}\frac{|f(z)||h(z)|}{|1-\langle w,z\rangle|^{n+\alpha+m+1}}\mathrm{d}v_\alpha(z)\\
&= C\int_{B_n}\frac{|f(z)|}{|1-\langle w,z\rangle|^{n+\gamma+1}}\frac{|h(z)|(1-|z|^2)^{\alpha-\gamma}}{|1-\langle w,z\rangle|^{m+\alpha-\gamma}}\mathrm{d}v_\gamma(z)\\
&\le C\Big(\int_{B_n}\frac{|f(z)|^2}{|1-\langle w,z\rangle|^{2n+2\gamma+2}}\mathrm{d}v_\gamma(z)\Big)^{\frac{1}{2}}\Big(\int_{B_n}\frac{|h(z)|^2(1-|z|^2)^{2\alpha-2\gamma}}{|1-\langle w,z\rangle|^{2m+2\alpha-2\gamma}}\mathrm{d}v_\gamma(z)\Big)^{\frac{1}{2}}
\end{aligned}$$

$$= C\frac{B_\gamma[|f|^2](w)^{\frac12}}{(1-|w|^2)^{\frac{n+\gamma+1}{2}}}\Big(\int_{B_n}\frac{|h(z)|^2(1-|z|^2)^{2\alpha-2\gamma}}{|1-\langle w,z\rangle|^{2m+2\alpha-2\gamma}}\mathrm{d}v_\gamma(z)\Big)^{\frac12}.$$

Since $2m\ge n+\alpha+1$ and $|1-\langle w,z\rangle|\ge 2^{-1}(1-|w|^2)$, we have

$$\Big(\int_{B_n}\frac{|h(z)|^2(1-|z|^2)^{2\alpha-2\gamma}}{|1-\langle w,z\rangle|^{2m+2\alpha-2\gamma}}\mathrm{d}v_\gamma(z)\Big)^{\frac12}\le\frac{2^{m-\frac{n+\gamma+1}{2}}}{(1-|w|^2)^{m-\frac{n+\gamma+1}{2}}}\Big(\int_{B_n}\frac{|h(z)|^2(1-|z|^2)^{2\alpha-\gamma}}{|1-\langle z,w\rangle|^{n+2\alpha-\gamma+1}}\mathrm{d}v(z)\Big)^{\frac12}.$$

Hence

$$\begin{aligned}|(D^sT_{\overline{f}}^{\alpha}h)(w)|&\le C\frac{B_\gamma[|f|^2](w)^{\frac12}}{(1-|w|^2)^{\frac{n+\gamma+1}{2}}}\frac{2^{m-\frac{n+\gamma+1}{2}}}{(1-|w|^2)^{m-\frac{n+\gamma+1}{2}}}\Big(\int_{B_n}\frac{|h(z)|^2(1-|z|^2)^{2\alpha-\gamma}}{|1-\langle z,w\rangle|^{n+2\alpha-\gamma+1}}\mathrm{d}v(z)\Big)^{\frac12}\\&=C\frac{B_\gamma[|f|^2](w)^{\frac12}}{(1-|w|^2)^{m}}\Big(\int_{B_n}\frac{|h(z)|^2(1-|z|^2)^{2\alpha-\gamma}}{|1-\langle z,w\rangle|^{n+2\alpha-\gamma+1}}\mathrm{d}v(z)\Big)^{\frac12}\\&=C\frac{B_\gamma[|f|^2](w)^{\frac12}}{(1-|w|^2)^{m}}(Q_{0,2\alpha-\gamma}(|h|^2)(w))^{\frac12}.\end{aligned}$$

This completes the proof.

Lemma 2.6 *Suppose $\beta>-1$. For $f\in L^2(B_n,\mathrm{d}v_\beta)$, $h\in H^\infty(B_n)$ and multi-index s with $|s|=m\ge\frac{n+\beta+1}{2}$, we have*

$$|(D^sT_{\overline{f}}^{\beta}h)(w)|\le C\frac{B_\beta[|f|^2](w)^{\frac12}}{(1-|w|^2)^{m}}(Q_{0,\beta}(|h|^2)(w))^{\frac12}.$$

Proof Suppose $h\in H^\infty(B_n)$. We proceed as the proof of Lemma 2.5 to see that

$$|(D^sT_{\overline{f}}^{\beta}h)(w)|\le C\int_{B_n}\frac{|f(z)||h(z)|}{|1-\langle w,z\rangle|^{n+\beta+m+1}}\mathrm{d}v_\beta(z)$$

for every multi-index s with $|s|=m$. Applying Hölder's inequality, we get

$$\begin{aligned}|(D^sT_{\overline{f}}^{\beta}h)(w)|&\le C\frac{B_\beta[|f|^2](w)^{\frac12}}{(1-|w|^2)^{\frac{n+\beta+1}{2}}}\Big(\int_{B_n}\frac{|h(z)|^2}{|1-\langle w,z\rangle|^{2m}}\mathrm{d}v_\beta(z)\Big)^{\frac12}\\&\le C\frac{B_\beta[|f|^2](w)^{\frac12}}{(1-|w|^2)^{m}}\Big(\int_{B_n}\frac{|h(z)|^2}{|1-\langle w,z\rangle|^{n+\beta+1}}\mathrm{d}v_\beta(z)\Big)^{\frac12}\\&=C\frac{B_\beta[|f|^2](w)^{\frac12}}{(1-|w|^2)^{m}}(Q_{0,\beta}(|h|^2)(w))^{\frac12},\end{aligned}$$

since $2m\ge n+\beta+1$ and $|1-\langle w,z\rangle|\ge 2^{-1}(1-|w|^2)$.

This proves the stated inequality.

Lemma 2.7 *Let $-1<\alpha<\infty$ and $f\in L^2(B_n,\mathrm{d}v_\alpha)$. Then*

$$|(H_f^*u)(w)|\le\frac{1}{(1-|w|^2)^{\frac{n+\alpha+1}{2}}}\|f\circ\varphi_w-P_\alpha(f\circ\varphi_w)\|_{\alpha,2}\|u\|_{\alpha,2}$$

for all $u\in(A_\alpha^2)^\perp$ and $w\in B_n$.

Proof It is easy to see that $H_f k_w^{(\alpha)} = (f - P_\alpha(f \circ \varphi_w) \circ \varphi_w) k_w^{(\alpha)}$. We have

$$H_f^* u(w) = \frac{1}{(1-|w|^2)^{\frac{n+\alpha+1}{2}}} \langle u, H_f k_w^{(\alpha)} \rangle_\alpha = \frac{1}{(1-|w|^2)^{\frac{n+\alpha+1}{2}}} \langle u, (f - P_\alpha(f \circ \varphi_w) \circ \varphi_w) k_w^{(\alpha)} \rangle_\alpha.$$

By change-of-variable formula (2.1), we have

$$\|(f - P_\alpha(f \circ \varphi_w) \circ \varphi_w) k_w^{(\alpha)}\|_{\alpha,2} = \|f \circ \varphi_w - P_\alpha(f \circ \varphi_w)\|_{\alpha,2}.$$

Therefore, applying Cauchy-Schwartz's inequality, we get

$$|\langle u, (f - P_\alpha(f \circ \varphi_w) \circ \varphi_w) k_w^{(\alpha)} \rangle_\alpha| \le \|u\|_{\alpha,2} \|f \circ \varphi_w - P_\alpha(f \circ \varphi_w)\|_{\alpha,2}.$$

Lemma 2.8 *Let $-1 < \alpha < \infty$ and $\varepsilon > 0$. For $g \in L^2(B_n, \mathrm{d}v_\alpha)$, $u \in (A_\alpha^2)^\perp$ and multi-index γ with $|\gamma| = m \ge \frac{n+\alpha+1}{2}$, we have*

$$|(D^\gamma H_g^* u)(w)| \le C \frac{1}{(1-|w|^2)^m} \|g \circ \varphi_w - P_\alpha(g \circ \varphi_w)\|_{\alpha,2+\varepsilon} (Q_{0,\alpha}(|h|^\delta)(w))^{\frac{1}{\delta}}$$

for all $w \in B_n$, where $\delta = \frac{2+\varepsilon}{1+\varepsilon}$.

Proof For $u \in (A_\alpha^2)^\perp$, we have

$$(H_g^* u)(w) = \langle H_g^* u, K_w^{(\alpha)} \rangle_\alpha = \langle u, H_g K_w^{(\alpha)} \rangle_\alpha = \int_{B_n} \frac{u(z)\overline{g(z)}}{(1-\langle w, z\rangle)^{n+\alpha+1}} \mathrm{d}v_\alpha(z).$$

Thus

$$(D^\gamma H_g^* u)(w) = \frac{\Gamma(n+\alpha+m+1)}{\Gamma(n+\alpha+1)} \int_{B_n} \frac{u(z)\overline{z^\gamma g(z)}}{(1-\langle w, z\rangle)^{n+\alpha+m+1}} \mathrm{d}v_\alpha(z)$$

for every multi-index γ with $|\gamma| = m$.

Let G_w denote $P_\alpha(g \circ \varphi_w) \circ \varphi_w$. The function $z \to \frac{z^\gamma G_w(z)}{(1-\langle z, w\rangle)^{n+\alpha+m+1}}$ is in A_α^2, and since $u \in (A_\alpha^2)^\perp$, we get

$$\int_{B_n} \frac{u(z)\overline{z^\gamma G_w(z)}}{(1-\langle w, z\rangle)^{n+\alpha+m+1}} \mathrm{d}v_\alpha(z) = 0.$$

Thus

$$(D^\gamma H_g^* u)(w) = \frac{\Gamma(n+\alpha+m+1)}{\Gamma(n+\alpha+1)} \int_{B_n} \frac{u(z)\overline{z^\gamma (g(z) - G_w(z))}}{(1-\langle w, z\rangle)^{n+\alpha+m+1}} \mathrm{d}v_\alpha(z).$$

Since

$$1 - \langle \varphi_w(z), w\rangle = \frac{1-|w|^2}{1-\langle z, w\rangle} \quad \text{and} \quad |k_w^{(\alpha)}(z)|^2 = \frac{(1-|w|^2)^{n+\alpha+1}}{|1-\langle z, w\rangle|^{2(n+\alpha+1)}},$$

applying change-of-variable formula (2.1) and Hölder's inequality, we have

$$\begin{aligned} |(D^\gamma H_g^* u)(w)| &\le C \int_{B_n} \frac{|u(z)||g(z) - G_w(z)|}{|1-\langle w, z\rangle|^{n+\alpha+m+1}} \mathrm{d}v_\alpha(z) \\ &= C \int_{B_n} \frac{|u \circ \varphi_w(z)||g \circ \varphi_w(z) - P_\alpha(g \circ \varphi_w)(z)|}{|1-\langle w, \varphi_w(z)\rangle|^{n+\alpha+m+1}} |k_w^{(\alpha)}(z)|^2 \mathrm{d}v_\alpha(z) \\ &= C \frac{1}{(1-|w|^2)^m} \int_{B_n} \frac{|u \circ \varphi_w(z)||g \circ \varphi_w(z) - P_\alpha(g \circ \varphi_w)(z)|}{|1-\langle z, w\rangle|^{(n+\alpha+1)-m}} \mathrm{d}v_\alpha(z) \end{aligned}$$

$$
\begin{aligned}
&\le C\frac{\|g\circ\varphi_w-P_\alpha(g\circ\varphi_w)\|_{\alpha,2+\varepsilon}}{(1-|w|^2)^m}\Big(\int_{B_n}\frac{|u\circ\varphi_w(z)|^\delta}{|1-\langle z,w\rangle|^{[(n+\alpha+1)-m]\delta}}\mathrm{d}v_\alpha(z)\Big)^{\frac{1}{\delta}}\\
&=C\frac{\|g\circ\varphi_w-P_\alpha(g\circ\varphi_w)\|_{\alpha,2+\varepsilon}}{(1-|w|^2)^m}\Big(\int_{B_n}\frac{|u(z)|^\delta|k_w^{(\alpha)}(z)|^2}{|1-\langle \varphi_w(z),w\rangle|^{[(n+\alpha+1)-m]\delta}}\mathrm{d}v_\alpha(z)\Big)^{\frac{1}{\delta}}\\
&=C\frac{\|g\circ\varphi_w-P_\alpha(g\circ\varphi_w)\|_{\alpha,2+\varepsilon}}{(1-|w|^2)^m}\Big(\int_{B_n}\frac{|u(z)|^\delta(1-|w|^2)^\beta}{|1-\langle z,w\rangle|^{(n+\alpha+1)+\beta}}\mathrm{d}v_\alpha(z)\Big)^{\frac{1}{\delta}},
\end{aligned}
$$

where $\beta=m\delta+(n+\alpha+1)(1-\delta)$. Since $m\ge\frac{n+\alpha+1}{2}$ and $\delta=\frac{2+\varepsilon}{1+\varepsilon}$, we have $\beta>0$ and

$$
\begin{aligned}
|1-\langle z,w\rangle|^{(n+\alpha+1)+\beta}&\ge|1-\langle z,w\rangle|^{n+\alpha+1}(1-|w|)^\beta\\
&\ge 2^{-\beta}(1-|w|^2)^\beta|1-\langle z,w\rangle|^{n+\alpha+1}.
\end{aligned}
$$

Thus

$$
\Big(\int_{B_n}\frac{|u(z)|^\delta(1-|w|^2)^\beta}{|1-\langle z,w\rangle|^{(n+\alpha+1)+\beta}}\mathrm{d}v_\alpha(z)\Big)^{\frac{1}{\delta}}\le 2^{\frac{\beta}{\delta}}\Big(\int_{B_n}\frac{|u(z)|^\delta}{|1-\langle z,w\rangle|^{n+\alpha+1}}\mathrm{d}v_\alpha(z)\Big)^{\frac{1}{\delta}}.
$$

Hence

$$
\begin{aligned}
|(D^\gamma H_g^*u)(w)|&\le C\frac{\|g\circ\varphi_w-P_\alpha(g\circ\varphi_w)\|_{\alpha,2+\varepsilon}}{(1-|w|^2)^m}2^{\frac{\beta}{\delta}}\Big(\int_{B_n}\frac{|u(z)|^\delta}{|1-\langle z,w\rangle|^{n+\alpha+1}}\mathrm{d}v_\alpha(z)\Big)^{\frac{1}{\delta}}\\
&\le C\frac{\|g\circ\varphi_w-P_\alpha(g\circ\varphi_w)\|_{\alpha,2+\varepsilon}}{(1-|w|^2)^m}\Big(\int_{B_n}\frac{|u(z)|^\delta}{|1-\langle z,w\rangle|^{n+\alpha+1}}\mathrm{d}v_\alpha(z)\Big)^{\frac{1}{\delta}}\\
&=C\frac{\|g\circ\varphi_w-P_\alpha(g\circ\varphi_w)\|_{\alpha,2+\varepsilon}}{(1-|w|^2)^m}(Q_{0,\alpha}(|h|^\delta)(w))^{\frac{1}{\delta}}.
\end{aligned}
$$

This proves the stated inequality.

3 Bounded Toeplitz Products and Hankel Products

We now prove our main results on boundedness of Toeplitz products.

Theorem 3.1 *Let $-1<\gamma<\infty$ and $f,g\in A^2_\gamma$. If*

$$
\sup_{w\in B_n}B_\gamma[|f|^2](w)B_\gamma[|g|^2](w)<\infty,
$$

then for each $\alpha>\gamma$, $T_f^\alpha T_{\overline{g}}^\alpha$ determines a bounded linear operator $A^2_\alpha\to A^2_\alpha$.

Proof Assume that M is a positive constant such that

$$
B_\gamma[|f|^2](w)B_\gamma[|g|^2](w)\le M^2
$$

for all $w\in B_n$.

Let h and k be bounded analytic functions on B_n. It follows from Lemma 2.4 that

$$
|(T_{\overline{f}}^\alpha h)(w)(T_{\overline{g}}^\alpha k)(w)|\le\frac{C}{(1-|w|^2)^{n+\alpha+1}}\|h\|_{\alpha,2}\|k\|_{\alpha,2}.
$$

Thus

$$
\int_{B_n}|(T_{\overline{f}}^\alpha h)(w)(T_{\overline{g}}^\alpha k)(w)|(1-|w|^2)^q\mathrm{d}v_\alpha(w)\le C\|h\|_{\alpha,2}\|k\|_{\alpha,2}
$$

for all $q \geq n+\alpha+1$. So if we choose a large m such that $2m \geq n+\alpha+1$, then we have

$$\int_{B_n} |(T^{\alpha}_{\overline{f}} h)(w)(T^{\alpha}_{\overline{g}} k)(w)|(1-|w|^2)^{2m+j-1}\mathrm{d}v_\alpha(w) \leq C\|h\|_{\alpha,2}\|k\|_{\alpha,2}$$

for $j = 1, \cdots, m$.

By Lemma 2.5 for a multi-index s with $|s| = m \geq \frac{n+\alpha+1}{2}$, we get

$$|(D^s T^{\alpha}_{\overline{g}} k)(w)\overline{(D^s T^{\alpha}_{\overline{f}} h)(w)}| \leq \frac{C}{(1-|w|^2)^{2m}}(Q_{0,2\alpha-\gamma}|h|^2(w))^{\frac{1}{2}}(Q_{0,2\alpha-\gamma}|k|^2(w))^{\frac{1}{2}}$$

for all $w \in B_n$. Since $Q_{0,2\alpha-\gamma}$ is bounded on $L^1(B_n, \mathrm{d}v_\alpha)$ by Lemma 2.3, we have

$$\int_{B_n}(Q_{0,2\alpha-\gamma}|h|^2)(w)\mathrm{d}v_\alpha(w) \leq \|Q_{0,2\alpha-\gamma}\|\int_{B_n}|h|^2(w)\mathrm{d}v_\alpha(w) = \|Q_{0,2\alpha-\gamma}\|\|h\|^2_{\alpha,2},$$

and, likewise,

$$\int_{B_n}(Q_{0,2\alpha-\gamma}|k|^2)(w)\mathrm{d}v_\alpha(w) \leq \|Q_{0,2\alpha-\gamma}\|\|k\|^2_{\alpha,2}.$$

By Cauchy-Schwartz's inequality, we have

$$\int_{B_n}(Q_{0,2\alpha-\gamma}|h|^2(w))^{\frac{1}{2}}(Q_{0,2\alpha-\gamma}|k|^2(w))^{\frac{1}{2}}\mathrm{d}v_\alpha(w) \leq \|Q_{0,2\alpha-\gamma}\|\|h\|_{\alpha,2}\|k\|_{\alpha,2}.$$

We conclude that

$$\Big|\int_{B_n} D^s T^{\alpha}_{\overline{g}} k(z)\overline{D^s T^{\alpha}_{\overline{f}} h(z)}(1-|z|^2)^{2m+j}\mathrm{d}v_\alpha(z)\Big| \leq C\|Q_{0,2\alpha-\gamma}\|\|h\|_{\alpha,2}\|k\|_{\alpha,2}$$

for $j = 0, 1, \cdots, 2m-1$. Using the inner product formula (2.4) in Lemma 2.2 with $F = T^{\alpha}_{\overline{g}}k$ and $G = T^{\alpha}_{\overline{f}}h$, we see that there is a finite constant C such that

$$|\langle T^{\alpha}_{f} T^{\alpha}_{\overline{g}} k, h\rangle_\alpha| \leq C\|h\|_{\alpha,2}\|k\|_{\alpha,2}$$

for all bounded analytic functions h and k on B_n. Hence, the operator $T^{\alpha}_{f} T^{\alpha}_{\overline{g}}$ is bounded on A^2_α.

Theorem 3.2 *Let $-1 < \gamma < \infty$ and $f, g \in A^2_\gamma$. If*

$$\sup_{w \in B_n} B_\gamma[|f|^2](w)B_\gamma[|g|^2](w) < \infty,$$

then $T^{\gamma}_{f} T^{\gamma}_{\overline{g}} : A^2_\alpha \to A^2_\alpha$ is a bounded operator for $-1 < \alpha < \gamma$.

Proof Let

$$M = \sup_{w \in B_n} B_\gamma[|f|^2](w)B_\gamma[|g|^2](w) < \infty.$$

Precisely as in the proof of Theorem 3.1, it suffices to show that there exists a positive constant C, such that for any $h, k \in H^\infty(B_n)$, we have

$$|\langle T^{\gamma}_{f} T^{\gamma}_{\overline{g}} h, k\rangle_\alpha| \leq C\|h\|_{\alpha,2}\|k\|_{\alpha,2}.$$

By Lemma 2.4, we see that if we choose a large m, such that $2m \geq n+\gamma+1$, then

$$\int_{B_n} |(T_f^\gamma h)(w)(T_{\overline{g}}^\gamma k)(w)|(1-|w|^2)^{2m+j-1}\mathrm{d}v_\alpha(w) \leq M^{\frac{1}{2}}\|h\|_{\alpha,2}\|k\|_{\alpha,2}$$

for $j=1,\cdots,m$.

Applying Lemma 2.6 and Hölder's inequality, we obtain, for $j=0,1,\cdots,2m-1$,

$$\begin{aligned}
&\Big|\int_{B_n} D^s T_{\overline{g}}^\gamma k(w)\overline{D^s T_{\overline{f}}^\gamma h(w)}(1-|w|^2)^{2m+j}\mathrm{d}v_\alpha(w)\Big| \\
\leq &C\int_{B_n}(Q_{0,\gamma}|k|^2(w))^{\frac{1}{2}}(Q_{0,\gamma}|h|^2(w))^{\frac{1}{2}}\mathrm{d}v_\alpha(w) \\
\leq &C\Big(\int_{B_n} Q_{0,\gamma}|k|^2(w)\mathrm{d}v_\alpha(w)\Big)^{\frac{1}{2}}\Big(\int_{B_n} Q_{0,\gamma}|h|^2(w)\mathrm{d}v_\alpha(w)\Big)^{\frac{1}{2}} \\
\leq &C\Big(\int_{B_n} |k|^2(w)\mathrm{d}v_\alpha(w)\Big)^{\frac{1}{2}}\Big(\int_{B_n} |h|^2(w)\mathrm{d}v_\alpha(w)\Big)^{\frac{1}{2}} \\
= &C\|k\|_{\alpha,2}\|h\|_{\alpha,2},
\end{aligned}$$

since $Q_{0,\gamma}$ is bounded on $L^1(B_n,\mathrm{d}v_\alpha)$ by Lemma 2.3. By the inner product formula (2.4) in Lemma 2.2, we see that there exists a constant C, such that

$$|\langle T_f^\gamma T_{\overline{g}}^\gamma h, k\rangle_\alpha| \leq C\|h\|_{\alpha,2}\|k\|_{\alpha,2}$$

for all bounded analytic functions h and k on B_n. Hence $T_f^\gamma T_{\overline{g}}^\gamma$ is bounded on A_α^2.

Remark 3.1 Suppose that $f,g\in A_\gamma^2$ satisfy the conditions in the above theorem. Since for any $h\in A_\alpha^2$ and $\beta\geq\alpha$, $\|T_f^\gamma T_{\overline{g}}^\gamma h\|_{\beta,2}\leq\|T_f^\gamma T_{\overline{g}}^\gamma h\|_{\alpha,2}$, it follows that $T_f^\gamma T_{\overline{g}}^\gamma : A_\alpha^2\to A_\beta^2$ is also a bounded operator for $-1<\alpha<\gamma$.

Using exactly the same argument as in the proof of Lemma 3.3 in [10], we have the following lemma.

Lemma 3.1 *Let $-1<\alpha<\infty$. If S is a bounded linear operator on $(A_\alpha^2)^\perp$, then*

$$\Big\|\sum_{|\gamma|=m}\frac{m!}{\gamma!}S_{\varphi_w^\gamma}SS_{\overline{\varphi}_w^\gamma}\Big\| \leq \|S\|$$

for every positive integer m and $w\in B_n$.

The following Theorems 3.3 and 3.4 are analogous to those Stroethoff and Zheng [7] obtained in the setting of unit disk. While our method is partially adapted from [7], a substantial amount of extra work is necessary for the setting of the unit ball.

Theorem 3.3 *Let $-1<\alpha<\infty$ and $f,g\in L^2(B_n,\mathrm{d}v_\alpha)$. If $H_fH_g^*$ is bounded, then*

$$\sup_{w\in B_n}\|f\circ\varphi_w-P_\alpha(f\circ\varphi_w)\|_{\alpha,2}\|g\circ\varphi_w-P_\alpha(g\circ\varphi_w)\|_{\alpha,2}<\infty.$$

Proof Using identities (1.1), (1.2) and (2.3), we have

$$\begin{aligned} H_f(k_w^{(\alpha)} \otimes k_w^{(\alpha)})H_g^* &= H_f\Big(\sum_{k=0}^{\infty} C_{n,\alpha,k} \sum_{|\gamma|=k} \frac{k!}{\gamma!} T^{\alpha}_{\varphi_w^\gamma} T^{\alpha}_{\overline{\varphi}_w^\gamma}\Big)H_g^* \\ &= \sum_{k=0}^{\infty} C_{n,\alpha,k} \sum_{|\gamma|=k} \frac{k!}{\gamma!} H_f T^{\alpha}_{\varphi_w^\gamma} T^{\alpha}_{\overline{\varphi}_w^\gamma} H_g^* \\ &= \sum_{k=0}^{\infty} C_{n,\alpha,k} \sum_{|\gamma|=k} \frac{k!}{\gamma!} S_{\varphi_w^\gamma} H_f H_g^* S_{\overline{\varphi}_w^\gamma}, \end{aligned}$$

and since $H_f(k_w^{(\alpha)} \otimes k_w^{(\alpha)})H_g^* = (H_f k_w^{(\alpha)}) \otimes (H_g k_w^{(\alpha)})$, we have

$$\begin{aligned} \|(H_f k_w^{(\alpha)}) \otimes (H_g k_w^{(\alpha)})\| &= \|H_f k_w^{(\alpha)}\|_{\alpha,2} \|H_g k_w^{(\alpha)})\|_{\alpha,2} \\ &= \|f \circ \varphi_w - P_\alpha(f \circ \varphi_w)\|_{\alpha,2} \|g \circ \varphi_w - P_\alpha(g \circ \varphi_w)\|_{\alpha,2}. \end{aligned}$$

Thus, by Lemma 3.1, we have

$$\begin{aligned} &\|f \circ \varphi_w - P_\alpha(f \circ \varphi_w)\|_{\alpha,2} \|g \circ \varphi_w - P_\alpha(g \circ \varphi_w)\|_{\alpha,2} \\ &= \Big\| \sum_{k=0}^{\infty} C_{n,\alpha,k} \sum_{|\gamma|=k} \frac{k!}{\gamma!} S_{\varphi_w^\gamma} H_f H_g^* S_{\overline{\varphi}_w^\gamma} \Big\| \\ &\le \sum_{k=0}^{\infty} |C_{n,\alpha,k}| \Big\| \sum_{|\gamma|=k} \frac{k!}{\gamma!} S_{\varphi_w^\gamma} H_f H_g^* S_{\overline{\varphi}_w^\gamma} \Big\| \\ &\le \sum_{k=0}^{\infty} |C_{n,\alpha,k}| \|H_f H_g^*\| \\ &\le C\|H_f H_g^*\| < \infty, \end{aligned}$$

since $\sum\limits_{k=0}^{\infty} |C_{n,\alpha,k}|$ is convergent.

Theorem 3.4 *Let $-1 < \alpha < \infty$ and $f, g \in L^2(B_n, \mathrm{d}v_\alpha)$. If there exists a positive constant $\varepsilon > 0$ such that*

$$\sup_{w \in B_n} \|f \circ \varphi_w - P_\alpha(f \circ \varphi_w)\|_{\alpha,2+\varepsilon} \|g \circ \varphi_w - P_\alpha(g \circ \varphi_w)\|_{\alpha,2+\varepsilon} < \infty,$$

then the operator $H_f H_g^*$ is bounded.

Proof Let $u, v \in C_c(B_n) \cap (A_\alpha^2)^\perp$. Using the definitions of $H_g^* u$ and $H_f^* v$ and Fubini's theorem, we have

$$\begin{aligned} \langle H_g^* u, H_f^* v\rangle_\alpha &= \int_{B_n} \Big\{ \int_{B_n} \frac{u(z)\overline{g(z)}}{(1 - \langle w, z\rangle)^{n+\alpha+1}} \mathrm{d}v_\alpha(z) \Big\} \Big\{ \int_{B_n} \frac{f(\lambda)\overline{v(\lambda)}}{(1 - \langle \lambda, w\rangle)^{n+\alpha+1}} \mathrm{d}v_\alpha(\lambda) \Big\} \mathrm{d}v_\alpha(w) \\ &= \int_{B_n} f(\lambda) H_g^* u(\lambda) \overline{v(\lambda)} \mathrm{d}v_\alpha(\lambda) = \langle f H_g^* u, v\rangle_\alpha = \langle H_f H_g^* u, v\rangle_\alpha. \end{aligned}$$

Thus, by Lemma 2.2, we have

$$\langle H_f H_g^* u, v\rangle_\alpha = \langle H_g^* u, H_f^* v\rangle_\alpha = \mathrm{I} + \mathrm{II} + \mathrm{III}$$

for $m \geq \frac{n+\alpha+1}{2}$, where

$$\begin{aligned}
\mathrm{I} &= \sum_{j=1}^{m} b_j \int_{B_n} (H_g^* u)(z)\overline{(H_f^* u)(z)}(1-|z|^2)^{2m+j-1} \mathrm{d}v_\alpha(z), \\
\mathrm{II} &= \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+2m+1)} \sum_{|\gamma|=m} \int_{B_n} D^\gamma (H_g^* u)(z)\overline{D^\gamma (H_f^* u)(z)}(1-|z|^2)^{2m} \mathrm{d}v_\alpha(z), \\
\mathrm{III} &= \sum_{j=1}^{2m-1} \sum_{|\gamma|=m} a_j \int_{B_n} D^\gamma (H_g^* u)(z)\overline{D^\gamma (H_f^* u)(z)}(1-|z|^2)^{2m+j} \mathrm{d}v_\alpha(z).
\end{aligned}$$

It follows from Lemma 2.7 that

$$|\mathrm{I}| \leq C_1 \sup_{w\in B_n} \|f\circ\varphi_w - P_\alpha(f\circ\varphi_w)\|_{\alpha,2}\|g\circ\varphi_w - P_\alpha(g\circ\varphi_w)\|_{\alpha,2}\|u\|_{\alpha,2}\|v\|_{\alpha,2}.$$

Note $p = \frac{2}{\delta} > 1$. Using Lemma 2.8 and since $Q_{0,\alpha}$ is bounded on $L^p(B_n, \mathrm{d}v_\alpha)$ by Lemma 2.3, we have

$$|\mathrm{II}| \leq C_2 \sup_{w\in B_n} \|f\circ\varphi_w - P_\alpha(f\circ\varphi_w)\|_{\alpha,2+\varepsilon}\|g\circ\varphi_w - P_\alpha(g\circ\varphi_w)\|_{\alpha,2+\varepsilon}\|u\|_{\alpha,2}\|v\|_{\alpha,2}.$$

The estimate of III is similar to that of II, and combining the estimates, we get

$$|\langle H_f H_g^* u, v\rangle_\alpha| \leq M \sup_{w\in B_n} \|f\circ\varphi_w - P_\alpha(f\circ\varphi_w)\|_{\alpha,2+\varepsilon}\|g\circ\varphi_w - P_\alpha(g\circ\varphi_w)\|_{\alpha,2+\varepsilon}\|u\|_{\alpha,2}\|v\|_{\alpha,2}$$

for some constant $M > 0$. So the product $H_f H_g^*$ is bounded as desired.

4 Compact Hankel Products

In this section, we discuss the condition for compactness of the Hankel products.

Lemma 4.1 *For any $z \in B_n$ and multi-index γ, we have $w^\gamma - \varphi_w^\gamma \to 0$ as $w \in B_n$ tends to $\xi \in \partial B_n$.*

Proof By definition,

$$\varphi_w(z) = \frac{w - P_w z - sQ_w z}{1-\langle z, w\rangle},$$

where $P_w z = \frac{\langle z,w\rangle}{|w|^2} w$, $Q_w z = (I - P_w)z$, $s = (1-|w|^2)^{\frac{1}{2}}$. Hence we have

$$\varphi_w(z) = \frac{|w|^2 - \langle z,w\rangle}{1-\langle z,w\rangle}\frac{1}{|w|^2}w + \frac{\langle z,w\rangle}{1-\langle z,w\rangle}\frac{s}{|w|^2}w - \frac{s}{1-\langle z,w\rangle}z.$$

Set $w = (w_1, \cdots, w_n)$, $z = (z_1, \cdots, z_n)$, $\gamma = (\gamma_1, \cdots, \gamma_n)$, and let

$$P_1(w) = \frac{|w|^2 - \langle z,w\rangle}{1-\langle z,w\rangle}\frac{1}{|w|^2}, \quad P_2(w) = \frac{\langle z,w\rangle}{1-\langle z,w\rangle}\frac{s}{|w|^2}, \quad P_3(w) = \frac{s}{1-\langle z,w\rangle}.$$

Then

$$\begin{aligned}
\varphi_w(z) = &((P_1(w)+P_2(w))w_1 - P_3(w)z_1, (P_1(w)+P_2(w))w_2 - P_3(w)z_2, \\
&\cdots, (P_1(w)+P_2(w))w_n - P_3(w)z_n).
\end{aligned}$$

Hence

$$\begin{aligned}\varphi_w^\gamma(z) = &((P_1(w)+P_2(w))w_1 - P_3(w)z_1)^{\gamma_1}((P_1(w)+P_2(w))w_2 - P_3(w)z_2)^{\gamma_2}\\ &\times\cdots\times((P_1(w)+P_2(w))w_n - P_3(w)z_n)^{\gamma_n}.\end{aligned}$$

If $w \in B_n \to \xi = (\xi_1, \cdots, \xi_n) \in \partial B_n$, then $w_i \to \xi_i$, $i = 1, 2, \cdots, n$. Clearly, if $w \in B_n \to \xi$, then $P_1(w) \to 1$, $P_2(w) \to 0$, $P_3(w) \to 0$. We get

$$\varphi_w^\gamma(z) \to \xi^\gamma = \xi_1^{\gamma_1}\cdots\xi_n^{\gamma}.$$

The following lemma gives a necessary condition for compactness of operators on $(A_\alpha^2)^\perp$.

Lemma 4.2 *Let T be a compact operator on $(A_\alpha^2)^\perp$. Then*

$$\lim_{|w|\to 1^-}\Big\|\sum_{k=0}^\infty C_{n,\alpha,k}\sum_{|\gamma|=k}\frac{k!}{\gamma!}S_{\varphi_w^\gamma}TS_{\overline{\varphi}_w^\gamma}\Big\| = 0. \tag{4.1}$$

Proof If H_1 and H_2 are Hilbert spaces and $T : H_1 \to H_2$ is a compact operator, since operators of finite rank are dense in the set of compact operators, given $\varepsilon > 0$, there exist $f_1, \cdots, f_n \in H_1$ and $g_1, \cdots, g_n \in H_1$, such that

$$\Big\|T - \sum_{i=1}^n f_i \otimes g_i\Big\| < \varepsilon.$$

Thus the lemma follows, once we show (4.1) for operators of rank one.

If $f \in L^2(B_n, \mathrm{d}v_\alpha)$ as $|w| \to 1^-$, then for every $z \in B_n$ and multi-index γ, we have $w^\gamma - \varphi_w^\gamma(z) \to 0$ by Lemma 4.1. So by Lebesgue's dominated convergence theorem, we get

$$\|w^\gamma f - \varphi_w^\gamma f\|_{\alpha,2}^2 = \int_{B_n}|w^\gamma f(z) - \varphi_w^\gamma(z)f(z)|^2\mathrm{d}v_\alpha(z) \to 0,$$

as $|w| \to 1^-$. It follows that $\|\xi^\gamma f - \varphi_w^\gamma f\|_{\alpha,2} \to 0$, as $w \in B_n$ tends to $\xi \in \partial B_n$.

Suppose $f \in (A_\alpha^2)^\perp$. Then

$$(I-P)(\xi^\gamma f) = \xi^\gamma f,$$

and consequently

$$\|\xi^\gamma f - S_{\varphi_w^\gamma}f\|_{\alpha,2} = \|(I-P)(\xi^\gamma f - \varphi_w^\gamma f)\|_{\alpha,2} \to 0,$$

as $w \in B_n$ tends to $\xi \in \partial B_n$. If $f, g \in (A_\alpha^2)^\perp$, then

$$\begin{aligned}\|\xi^\gamma(f\otimes g)\overline{\xi}^\gamma - S_{\varphi_w^\gamma}(f\otimes g)S_{\overline{\varphi}_w^\gamma}\| &= \|(\xi^\gamma f)\otimes(\xi^\gamma g) - (S_{\varphi_w^\gamma}f)\otimes(S_{\varphi_w^\gamma}g)\|\\ &\le \|(\xi^\gamma f - S_{\varphi_w^\gamma}f)\otimes(\xi^\gamma g)\| + \|(S_{\varphi_w^\gamma}f)\otimes(\xi^\gamma g - S_{\varphi_w^\gamma}g)\|\\ &\le \|\xi^\gamma f - S_{\varphi_w^\gamma}f\|_{\alpha,2}\|g\|_{\alpha,2} + \|f\|_{\alpha,2}\|\xi^\gamma g - S_{\varphi_w^\gamma}g\|_{\alpha,2}.\end{aligned}$$

We get

$$\|\xi^\gamma(f\otimes g)\overline{\xi}^\gamma - S_{\varphi_w^\gamma}(f\otimes g)S_{\overline{\varphi}_w^\gamma}\| \to 0,$$

as $w \in B_n$ tends to $\xi \in \partial B_n$.

Hence, for any nonnegative integer k, we get

$$\Big\|\sum_{|\gamma|=k}\frac{k!}{\gamma!}(\xi^\gamma(f\otimes g)\overline{\xi}^\gamma - S_{\varphi_w^\gamma}(f\otimes g)S_{\overline{\varphi}_w^\gamma})\Big\| \to 0,$$

as $w\in B_n$ tends to $\xi\in\partial B_n$. Since

$$\begin{aligned}\Big\|\sum_{k=0}^\infty C_{n,\alpha,k}\sum_{|\gamma|=k}\frac{k!}{\gamma!}S_{\varphi_w^\gamma}(f\otimes g)S_{\overline{\varphi}_w^\gamma}\Big\| &= \Big\|\sum_{k=0}^\infty C_{n,\alpha,k}\sum_{|\gamma|=k}\frac{k!}{\gamma!}(S_{\varphi_w^\gamma}(f\otimes g)S_{\overline{\varphi}_w^\gamma} - \xi^\gamma(f\otimes g)\overline{\xi}^\gamma)\Big\|\\ &\le \sum_{k=0}^\infty |C_{n,\alpha,k}|\Big\|\sum_{|\gamma|=k}\frac{k!}{\gamma!}(\xi^\gamma(f\otimes g)\overline{\xi}^\gamma - S_{\varphi_w^\gamma}(f\otimes g)S_{\overline{\varphi}_w^\gamma})\Big\|,\end{aligned}$$

and the series $\sum\limits_{k=0}^\infty |C_{n,\alpha,k}|$ is convergent, by Lemma 3.1, we have

$$\Big\|\sum_{k=0}^\infty C_{n,\alpha,k}\sum_{|\gamma|=k}\frac{k!}{\gamma!}S_{\varphi_w^\gamma}(f\otimes g)S_{\overline{\varphi}_w^\gamma}\Big\| \to 0,$$

as $w\in B_n$ tends to $\xi\in\partial B_n$.

Theorem 4.1 *Let f and g be in $L^\infty(B_n, \mathrm{d}v_\alpha)$. Then $H_fH_g^*$ is compact if and only if*

$$\lim_{|w|\to 1^-}\|f\circ\varphi_w - P_\alpha(f\circ\varphi_w)\|_{\alpha,2}\|g\circ\varphi_w - P_\alpha(g\circ\varphi_w)\|_{\alpha,2} = 0.$$

Proof First, we show the "if" part. If $H_fH_g^*$ is compact, then by Lemma 4.2, we have

$$\lim_{|w|\to 1^-}\|f\circ\varphi_w - P_\alpha(f\circ\varphi_w)\|_{\alpha,2}\|g\circ\varphi_w - P_\alpha(g\circ\varphi_w)\|_{\alpha,2} = 0,$$

since

$$\|f\circ\varphi_w - P_\alpha(f\circ\varphi_w)\|_{\alpha,2}\|g\circ\varphi_w - P_\alpha(g\circ\varphi_w)\|_{\alpha,2} = \|(H_fk_w^{(\alpha)})\otimes(H_gk_w^{(\alpha)})\|$$

and

$$H_f(k_w^{(\alpha)}\otimes k_w^{(\alpha)})H_g^* = (H_fk_w^{(\alpha)})\otimes(H_gk_w^{(\alpha)}) = \sum_{k=0}^\infty C_{n,\alpha,k}\sum_{|\gamma|=k}\frac{k!}{\gamma!}S_{\varphi_w^\gamma}H_fH_g^*S_{\overline{\varphi}_w^\gamma}.$$

Now we turn to the "only if" part. For $u,v\in C_c(B_n)\cap(A_\alpha^2)^\perp$ and $m\ge\frac{n+\alpha+1}{2}$, we have

$$\langle H_fH_g^*u, v\rangle_\alpha = \langle H_g^*u, H_f^*v\rangle_\alpha = \mathrm{I}+\mathrm{II}+\mathrm{III},$$

where I, II and III are as those in the proof of Theorem 3.4. For $0<s<1$, we write $\mathrm{I} = \mathrm{I}_s + \mathrm{I}_s'$, $\mathrm{II} = \mathrm{II}_s + \mathrm{II}_s'$ and $\mathrm{III} = \mathrm{III}_s + \mathrm{III}_s'$, where

$$\begin{aligned}\mathrm{I}_s &= \sum_{j=1}^m b_j\int_{s<|z|<1}(H_g^*u)(z)\overline{(H_f^*u)(z)}(1-|z|^2)^{2m+j-1}\mathrm{d}v_\alpha(z),\\ \mathrm{II}_s &= \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+2m+1)}\sum_{|\gamma|=m}\int_{s<|z|<1}D^\gamma(H_g^*u)(z)\overline{D^\gamma(H_f^*u)(z)}(1-|z|^2)^{2m}\mathrm{d}v_\alpha(z),\\ \mathrm{III}_s &= \sum_{j=1}^{2m-1}\sum_{|\gamma|=m}a_j\int_{s<|z|<1}D^\gamma(H_g^*u)(z)\overline{D^\gamma(H_f^*u)(z)}(1-|z|^2)^{2m+j}\mathrm{d}v_\alpha(z).\end{aligned}$$

It is easy to see that there exists a compact operator C_s such that $\langle (H_f H_g^* - C_s)u, v\rangle_\alpha = \mathrm{I}_s + \mathrm{II}_s + \mathrm{III}_s$. By Lemma 2.7, we get

$$|\mathrm{I}_s| \le C \sup_{s<|w|<1} \|f\circ\varphi_w - P_\alpha(f\circ\varphi_w)\|_{\alpha,2}\|g\circ\varphi_w - P_\alpha(g\circ\varphi_w)\|_{\alpha,2}\|u\|_{\alpha,2}\|v\|_{\alpha,2}.$$

Using Lemma 2.8 and since $Q_{0,\alpha}$ is bounded on $L^p(B_n, \mathrm{d}v_\alpha)$ by Lemma 2.3, we have

$$|\mathrm{II}_s| \le C \sup_{s<|w|<1} \|f\circ\varphi_w - P_\alpha(f\circ\varphi_w)\|_{\alpha,2+\varepsilon}\|g\circ\varphi_w - P_\alpha(g\circ\varphi_w)\|_{\alpha,2+\varepsilon}\|u\|_{\alpha,2}\|v\|_{\alpha,2}.$$

The estimate of III_s is similar to that of II_s. Then we obtain

$$\begin{aligned}|\langle (H_f H_g^* - C_s)u, v\rangle_\alpha| \le{}& C \sup_{s<|w|<1} \|f\circ\varphi_w - P_\alpha(f\circ\varphi_w)\|_{\alpha,2+\varepsilon}\\ &\times \|g\circ\varphi_w - P_\alpha(g\circ\varphi_w)\|_{\alpha,2+\varepsilon}\|u\|_{\alpha,2}\|v\|_{\alpha,2}\end{aligned}$$

for some constant $C > 0$. Since P_α is bounded on $L^{2+2\varepsilon}(B_n, \mathrm{d}v_\alpha)$, there exists a constant C such that

$$\|f\circ\varphi_w - P_\alpha(f\circ\varphi_w)\|_{\alpha,2+\varepsilon} \le C\|f\|_\infty^{\frac{1+\varepsilon}{2+\varepsilon}}\|f\circ\varphi_w - P_\alpha(f\circ\varphi_w)\|_{\alpha,2}^{\frac{1}{2+\varepsilon}}.$$

A similar inequality holds for $\|g\circ\varphi_w - P_\alpha(g\circ\varphi_w)\|_{\alpha,2+\varepsilon}$. Thus there exists a constant C such that

$$\begin{aligned}|\langle (H_f H_g^* - C_s)u, v\rangle_\alpha| \le{}& C \sup_{s<|w|<1} (\|f\circ\varphi_w - P_\alpha(f\circ\varphi_w)\|_{\alpha,2}\\ &\times \|g\circ\varphi_w - P_\alpha(g\circ\varphi_w)\|_{\alpha,2})^{\frac{1}{2+\varepsilon}}\|u\|_{\alpha,2}\|v\|_{\alpha,2},\end{aligned}$$

from which we conclude that

$$\|H_f H_g^* - C_s\| \le C \sup_{s<|w|<1} (\|f\circ\varphi_w - P_\alpha(f\circ\varphi_w)\|_{\alpha,2}\|g\circ\varphi_w - P_\alpha(g\circ\varphi_w)\|_{\alpha,2})^{\frac{1}{2+\varepsilon}}.$$

So if

$$\lim_{|w|\to 1^-} \|f\circ\varphi_w - P_\alpha(f\circ\varphi_w)\|_{\alpha,2}\|g\circ\varphi_w - P_\alpha(g\circ\varphi_w)\|_{\alpha,2} = 0,$$

it follows from the above inequality that C_s converges to $H_f H_g^*$ in operator norm as $s \to 1^-$, and since each of the C_s is compact, we conclude that the operator $H_f H_g^*$ is compact.

References

[1] Rudin, W., Real and Complex Analysis, McGraw-Hill, New York, 1966.

[2] Sarason, D., Products of Toeplitz operators, Linear and Complex Analysis Problem Book 3, V. P. Havin and N. K. Nikolski (eds.), Part I, Lecture Notes in Math., **1573**, Springer-Verlag, Berlin, 1994, 318–319.

[3] Treil, S., Volberg, A. and Zheng, D., Hilbert transform, Toeplitz operators and Hankel operators, and invariant A_∞ weights, *Rev. Mat. Iberoamericana*, **13**(2), 1997, 319–360.

[4] Cruz-Uribe, D., The invertibility of the product of unbounded Toeplitz operators, *Integr. Eqs. Oper. Th.*, **20**(2), 1994, 231–237.

[5] Zheng, D., The distribution function inequality and products of Toeplitz operators and Hankel operators, *J. Funct. Anal.*, **138**(2), 1996, 477–501.

[6] Nazarov, F., A counterexample to Sarason's conjuecture, preprint. www.math.msu.edu/fedja/perp.html

[7] Stroethoff, K. and Zheng, D., Products of Hankel and Toeplitz operators on the Bergman space, *J. Funct. Anal.*, **169**(1), 1999, 289–313.

[8] Stroethoff, K. and Zheng, D., Bounded Toeplitz products on the Bergman space of the polydisk, *J. Math. Anal. Appl.*, **278**(1), 2003, 125–135.

[9] Stroethoff, K. and Zheng, D., Bounded Toeplitz products on Weighted Bergman spaces, *J. Oper. Th.*, **59**(2), 2008, 277–308.

[10] Stroethoff, K. and Zheng, D., Bounded Toeplitz products on Bergman spaces of the unit ball, *J. Math. Anal. Appl.*, **325**(1), 2007, 114–129.

[11] Park, J. D., Bounded Toeplitz products on the Bergman space of the unit ball in $\mathbb{C}^n$, *Integr. Eqs. Oper. Th.*, **54**(4), 2006, 571–584.

[12] Pott, S. and Strouse, E., Products of Toeplitz operators on the Bergman spaces A^2_α, *St. Petersburg Math. J.*, **18**(1), 2007, 105–118.

[13] Rudin, W., Function Theory in the Unit Ball in $\mathbb{C}^n$, Springer-Verlag, New York, 1980.

[14] Zhu, K., Spaces of Holomorphic Functions in the Unit Ball, Graduate Texts in Mathematics, **226**, Springer-Verlag, New York, 2005.

[15] Lu, Y. F. and Shang, S. X., Bounded Hankel products on the Bergman space of the polydisk, *Can. J. Math.*, **60**(1), 2009, 190–204.