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On the Existence and Stability of a Global Subsonic Flow in a 3D Infinitely Long Cylindrical Nozzle***

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Abstract This paper is concerned with the problem on the global existence and stability of a subsonic flow in an infinitely long cylindrical nozzle for the 3D steady potential flow equation. Such a problem was indicated by Courant-Friedrichs in [8, p. 377]: A flow through a duct should be considered as a steady, isentropic, irrotational flow with cylindrical symmetry and should be determined by solving the 3D potential flow equations with appropriate boundary conditions. By introducing some suitably weighted Hölder spaces and establishing a priori estimates, the authors prove the global existence and stability of a subsonic potential flow in a 3D nozzle when the state of subsonic flow at negative infinity is given.

Keywords Subsonic flow, Potential flow equation, Bessel function, Weighted Hölder space, Global existence
2000 MR Subject Classification 35L70, 35L65, 35L67, 76N15

1 Introduction and Main Results

In this paper, we are concerned with the problem on the global existence and stability of a subsonic flow in an infinitely long cylindrical nozzle for the three-dimensional steady potential flow equation. Such a problem was indicated in [8, p. 377]: A flow through a duct should be considered as a steady, isentropic, irrotational flow with cylindrical symmetry and should be determined by solving the 3D potential flow equations with appropriate boundary conditions. With respect to the global existence of subsonic flows in 2D nozzles, there have been many results (see [15, 20–23] and the references therein). However, for the 3D case, due to the multidimensional reason (i.e., the standard stream function method which is extensively used in 2D case, does not work for the 3D case in general), there are few results except some asymptotic analysis and computational examples (see [14] and references therein). In the present paper, we will focus on the 3D subsonic flow problem in an infinitely long cylindrical nozzle (see Figure 1).

We use the potential flow equation to describe the motion of the subsonic gas in a 3D nozzle. Let $\varphi(x)$ be the potential of velocity $u = (u_1, u_2, u_3)$, i.e., $u_i = \partial_i \varphi$. Then it follows from the

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Bernoulli's law that

$$\frac{1}{2}|\nabla\varphi|^2 + h(\rho) = C_0, \tag{1.1}$$

where $\nabla = (\partial_1, \partial_2, \partial_3)$, $h(\rho) = \frac{c^2(\rho)}{\gamma-1}$ is the specific enthalpy for the polytropic gas with the state equation $P = A\rho^\gamma$ $(1 < \gamma < 3)$ and the sonic speed $c(\rho) = \sqrt{P'(\rho)}$, $C_0 = \frac{1}{2}q_0^2 + h(\rho_0)$ stands for the Bernoulli's constant, where the far velocity field $(q_0, 0, 0; \rho_0)$ at minus infinity of the nozzle is subsonic, i.e., $q_0 < c(\rho_0)$ holds true.

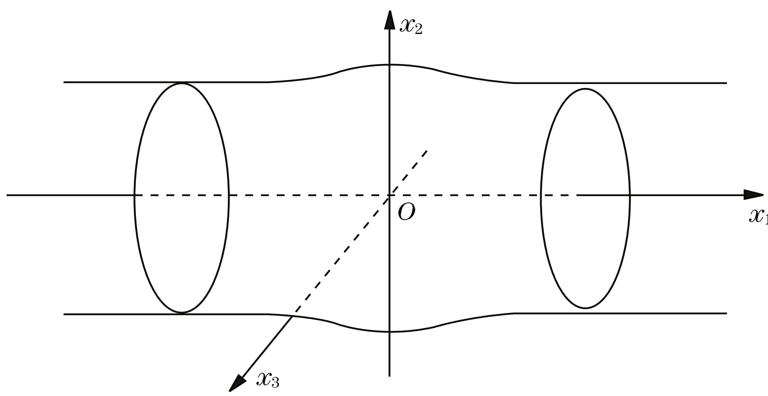

Figure 1 The 3D subsonic flow problem in an infinitely long cylindrical nozzle

By use of (1.1) and the implicit function theorem, the density function $\rho(x)$ of gas can be expressed as

$$\rho = h^{-1}\Big(C_0 - \frac{1}{2}|\nabla\varphi|^2\Big) \equiv H(\nabla\varphi). \tag{1.2}$$

Substituting (1.2) into the mass conservation equation $\sum\limits_{j=1}^{3} \partial_j(\rho u_j) = 0$ of gas yields

$$\sum_{i=1}^{3}((\partial_i\varphi)^2 - c^2)\partial_i^2\varphi + 2\sum_{1\le i<j\le 3} \partial_i\varphi\partial_j\varphi\partial_{ij}^2\varphi = 0, \tag{1.3}$$

where $c = c(H(\nabla\varphi))$.

Assume that the 3D infinitely long nozzle Ω is bounded by the wall Σ: $\sqrt{x_2^2 + x_3^2} = 1 + \varepsilon g(x_1, x_2, x_3)$, where $g(x_1, x_2, x_3) \in C_0^\infty((-X_0, X_0) \times (-\infty, +\infty) \times (-\infty, +\infty))$ for some fixed positive constant X_0, and $\varepsilon > 0$ is a suitably small constant.

Due to the fixed wall condition, one has

$$(\partial_1\varphi, \partial_2\varphi, \partial_3\varphi) \cdot \Big(- \varepsilon\partial_{x_1}g, \frac{x_2}{\sqrt{x_2^2 + x_3^2}} - \varepsilon\partial_{x_2}g, \frac{x_3}{\sqrt{x_2^2 + x_3^2}} - \varepsilon\partial_{x_3}g\Big) = 0, \quad \text{on } \Sigma. \tag{1.4}$$

In addition, suppose that the state of subsonic flow at minus infinity satisfies

$$\lim_{x_1\to-\infty}(\varphi(x) - q_0x_1) = 0 \quad \text{for } x \in \Omega. \tag{1.5}$$

On the other hand, from the physical point of view (see [2, 3, 6, 8–11, 16–18] and the references therein), when a subsonic flow in an unbounded domain is called to be stable, it should admit a determined state at infinity, namely,

$$\lim_{x_1\to+\infty} \nabla\varphi(x) \text{ exists for } x\in\Omega. \tag{1.6}$$

The main result in our paper can be stated as follows.

Theorem 1.1 *If the* $3D$ *infinitely long cylindrical nozzle* Ω *is bounded by* Σ: $\sqrt{x_2^2+x_3^2}=1+\varepsilon g(x_1,x_2,x_3)$, *where* $g(x_1,x_2,x_3)\in C_0^\infty((-X_0,X_0)\times(-\infty,+\infty)\times(-\infty,+\infty))$ *for some fixed positive constant* X_0, *then there exists a small constant* $\varepsilon_0>0$ *such that the problem* (1.3)–(1.6) *has a global smooth solution* $\varphi(x)$ *as* $\varepsilon<\varepsilon_0$, *which admits*

(i) $|\nabla\varphi|<c(H(\nabla\varphi))$, *namely, the flow is globally subsonic in the whole domain* Ω;

(ii) *For* $x_1<0$ *and* $x\in\Omega$, *there exist a suitable constant* $\delta_0>0$ *and a constant* $C_0>0$ *such that*

$$|\varphi(x)-q_0x_1|+|\nabla(\varphi(x)-q_0x_1)|\le C_0\varepsilon \mathrm{e}^{-\delta_0|x_1|};$$

(iii) *For* $x_1>0$ *and* $x\in\Omega$, *there exists a constant* $C_0>0$ *such that*

$$|\varphi(x)-q_0x_1|\le C_0\varepsilon(1+x_1);$$

(iv) $\lim\limits_{\substack{x_1\to+\infty\\ x\in\Omega}}\nabla\varphi(x)=(q_0,0,0)$ *holds true. Moreover, for* $x_1>0$ *and* $x\in\Omega$, *there exists a constant* $C_0>0$ *such that*

$$|\nabla_{x_2,x_3}\varphi(x)|\le C_0\varepsilon \mathrm{e}^{-\delta_0 x_1},$$

where $\delta_0>0$ *is given in* (ii).

Remark 1.1 We emphasize that the assumption on the compact support of $g(x)$ is only for the convenience to stating our problem. In fact, $g(x)$ can be permitted to suitably decay at infinity.

Remark 1.2 If the Mach number of the subsonic flow is sufficiently small, then the variation of the nozzle wall can be permitted to be suitably large. Since the proof procedure is completely analogous to that in Theorem 1.1, we omit the details.

Remark 1.3 Although there have been many results on the weighted $W^{2,p}(\Omega)$ $(1<p<\infty)$ estimates of solution to the second order linear elliptic equation in an unbounded strip domain Ω or half-space Ω (see [1, 13] and the references therein), it is difficult for us to use these results to treat the existence of solution to the quasilinear elliptic equation (1.3) as well as the asymptotic state and asymptotic behavior of solution at minus or positive infinity since the related weighted Sobolev spaces in [1] or [13] can not be imbedded into the suitable Hölder space $C^\delta(\overline{\Omega})$ with some positive constant $\delta>0$.

Remark 1.4 So far there have been many papers (see [27] and the references therein) to treat the global existence problems for the incompressible or compressible Navier-Stokes equations and Euler equations in suitable function spaces.

It is noted that there have been extensive works on the global subsonic flows or subsonic-sonic flows for the gas past bounded obstacles/curved infinite surfaces or through 2D nozzles (see [2–11, 15–18, 20–23] and the references therein). For examples, in [2, 4, 6, 9, 11, 17, 18], for the case of the gas past an obstacle, by use of the Kelvin transformation, the authors can reduce the exterior problem on the 2D or 3D potential flow equation into a boundary value problem in a bounded domain. From this, together with the maximum principle, some a priori estimates on the solutions to second order linear elliptic equations in bounded domain and Schauder fixed point theorem, the authors can show that the global subsonic flow fields exist uniquely outside the obstacles. In [10], by use of the "good" geometrical property of half-plane, the author established the global existence and uniqueness of the plane subsonic flow. In addition, with respect to the 2D subsonic nozzle flow case, the authors in [15, 23] used the stream function method to reduce the 2D potential flow equation or 2D full Euler system so that a second order quasilinear elliptic equation with a Dirichlet boundary value or a first order nonlinear elliptic system with suitable boundary values can be obtained. From this, by use of the maximum principle and the theory of second order linear elliptic equations, the authors established the global existence and stability of a global subsonic flow in a 2D nozzle. However, for the 3D subsonic nozzle flows, the streamline function method does not work (this is also illustrated in [8, Chapter VI]), we have to directly treat the 3D potential flow equation with the fixed nozzle wall condition, which is described by the Neumann boundary value condition. In this case, the crucial comparison principle on second order elliptic equations can not be used directly; consequently, the L^∞ norm and further $C^{1,\alpha}$-norm of $\varphi - q_0x_1$ can not be obtained correspondingly. This implies that we have to use some new ingredients to overcome such essential difficulties.

We now comment on the proof of the main result in this paper. By introducing some suitable coordinate transformation and linearizing the nonlinear equation (1.3), we can actually get a Laplacian equation $\Delta u = f$ in an unbounded cylindrical domain $\widetilde{\Omega} = \{(z_1, z_2, z_3) : -\infty < z_1 < \infty, z_2^2 + z_3^2 \le 1\}$ together with the Neumann boundary condition on $z_2^2 + z_3^2 = 1$, one Dirichlet boundary value condition at minus infinity (i.e., $z_1 \to -\infty$) and one restriction condition on the existence of $\lim\limits_{z_1 \to +\infty} \nabla_z u(z)$. In order to solve such a Laplacian equation in $\widetilde{\Omega}$, our ingredient is to use Sturm-Liouville theorem and the separation variable method to write out formal expression of $u(z)$. From this, together with some delicate analysis, we can show that this formal expression is actually a solution of $\Delta u = f$ and its derivatives will decay at the rate of $\mathrm{e}^{-\delta_0|z_1|}$ ($\delta_0 > 0$ is a suitable constant) for $z_1 < 0$; on the other hand, for $z_1 > 0$, the solution $u(z)$ increases at the rate of $(1 + z_1)$ meanwhile its partial derivative $\partial_{z_1} u$ is bounded and the partial derivatives $(\partial_{z_2} u, \partial_{z_3} u)$ decay at the rate of $\mathrm{e}^{-\delta_0 z_1}$. In terms of these properties, some inhomogeneous weighted Hölder spaces will be introduced by us and further be used to treat the regularity and existence of solution to the second order nonlinear elliptic problem in an unbounded cylindrical domain. In this procedure, some detailed analysis on the expression of solution will be required; moreover, a priori estimates with different weighted norms are required to be established. Subsequently, by use of the continuity method, we can complete the proof of Theorem 1.1. It should be mentioned that our approach is influenced by the work in [26], yet some analysis there (especially, the analysis on the linearized equation with different boundary conditions in cuboid domain which are treated in [26]) can not be applied here due to the different geometric properties of nozzles.

Our paper is organized as follows. In Section 2, first we reformulate the problem (1.3) with (1.4)–(1.6), and then give a more precise descriptions on Theorem 1.1 in some suitably weighted Hörder spaces. In Section 3, we linearize the nonlinear problem (1.3) with (1.4)–(1.6). By such a linearization, we essentially obtain the Laplacian equation $\Delta u = \widetilde{f}(z)$ in the cylindrical domain $\widetilde{\Omega} = \{z = (z_1, z_2, z_3) : -\infty < z_1 < \infty, z_2^2 + z_3^2 \le 1\}$ with Neumann boundary condition on $z_2^2 + z_3^2 = 1$, together with $\lim\limits_{z_1 \to -\infty} u(z) = 0$ and the requirement on the existence of $\lim\limits_{z_1 \to \infty} \nabla_z u(z)$. By use of Sturm-Liouville theorem and the separation variable method, we can derive the formal expression of $u(z)$ in $\widetilde{\Omega}$. Subsequently, it follows from some detailed estimates that we can obtain the existence and regularity of $u(z)$ in $\widetilde{\Omega}$. In Section 4, based on the crucial estimates and properties given in Section 3, by use of the suitable iteration scheme, we can complete the proof of Theorem 1.1 and further obtain the asymptotic behavior of $\nabla_x\varphi$ at negative and positive infinity in the nozzle domain Ω respectively.

2 The Reformulation on (1.3)–(1.6) and More Precise Descriptions on Theorem 1.1

In this section, we first introduce some notations and weighted Hölder norms as in [26] so that Theorem 1.1 can be given a more precise description.

Let $\Omega\subset\mathbb{R}^3$ be an open set including the origin $O=(0,0,0)$. If $u\in C^{m,\alpha}(\Omega)$ with $0\le\alpha<1$, then we define the following weighted Hölder norms for $x,y\in\Omega$, some positive constant $\delta>0$ and $m\in\mathbb{N}\cup\{0\}$:

$$
\begin{aligned}
&[u]^{(\delta)}_{m,0;\Omega}\equiv\sum_{|\beta|=m}\sup_{x\in\Omega}\mathrm{e}^{\delta|x_1|}|D^\beta u(x)|,\\
&[u]^{(\delta)}_{m,\alpha;\Omega}\equiv\sum_{|\beta|=m}\sup_{x,y\in\Omega}\mathrm{e}^{\delta d_{x,y}}\frac{|D^\beta u(x)-D^\beta u(y)|}{|x-y|^\alpha},\quad\text{where } d_{x,y}=\min(|x_1|,|y_1|),\\
&|u|^{(\delta)}_{m,\alpha;\Omega}\equiv\sum_{0\le k\le m}[u]^{(\delta)}_{k,0;\Omega}+[u]^{(\delta)}_{m,\alpha;\Omega},\\
&\|u\|^{(\delta)}_{m,\alpha;\Omega}\equiv\sup_{\substack{x\in\Omega\\x_1<0}}\mathrm{e}^{\delta|x_1|}|u(x)|+\sup_{\substack{x\in\Omega\\x_1>0}}(1+x_1)^{-1}|u(x)|+\sup_{\substack{x\in\Omega\\x_1<0}}\mathrm{e}^{\delta|x_1|}|\partial_{x_1}u(x)|\\
&\qquad\qquad+\sup_{\substack{x\in\Omega\\x_1>0}}|\partial_{x_1}u(x)|+\sup_{x\in\Omega}\mathrm{e}^{\delta|x_1|}(|\partial_{x_2}u(x)|+|\partial_{x_3}u(x)|)+\sum_{2\le k\le m}[u]^{(\delta)}_{k,0;\Omega}+[u]^{(\delta)}_{m,\alpha;\Omega}
\end{aligned}
$$

and the corresponding function spaces are defined as

$$
\begin{aligned}
&H^{(\delta)}_{m,\alpha}(\Omega)=\{u(x)\in C^{m,\alpha}(\Omega):|u|^{(\delta)}_{m,\alpha}<+\infty\},\\
&\mathbb{H}^{(\delta)}_{m,\alpha}(\Omega)=\{u(x)\in C^{m,\alpha}(\Omega):\|u\|^{(\delta)}_{m,\alpha}<+\infty\}.
\end{aligned}
$$

By the definitions of the above spaces and norms, we can arrive at the following lemma.

Lemma 2.1 *For $u(x)\in C^{m,\alpha}(\overline{\Omega})$, one has*
(i) $H^{(\delta)}_{m,\alpha}(\Omega)\subset\mathbb{H}^{(\delta)}_{m,\alpha}(\Omega)$,
(ii) $|\partial_{x_i}u|^{(\delta)}_{m-1,\alpha;\Omega}\le\|u\|^{(\delta)}_{m,\alpha;\Omega}$ *for $i=2,3$ and $m\ge1$,*
(iii) $|D^2u|^{(\delta)}_{m-2,\alpha;\Omega}\le\|u\|^{(\delta)}_{m,\alpha;\Omega}$ *for $m\ge2$.*

Proof Since these properties can be directly verified by use of the definitions of the norms $|\cdot|^{(\delta)}_{m,\alpha}$ and $\|\cdot\|^{(\delta)}_{m,\alpha}$, we omit them.

By use of the weighted Hölder norms introduced in the above, Theorem 1.1 can be stated more precisely as follows.

Theorem 2.1 *Under the assumptions of Theorem* 1.1, *in the nozzle Ω, the problem* (1.3)–(1.6) *has a unique solution $\varphi(x)\in C^{6,\alpha}(\Omega)$ (any fixed constant $0<\alpha<1$), which admits*
(i) $\|\varphi(x)-q_0x_1\|^{(\delta_0)}_{6,\alpha;\Omega}\le\widetilde{C}\varepsilon$, *where $\delta_0>0$ is some suitable constant,*
(ii) $\lim\limits_{\substack{x\in\Omega\\x_1\to+\infty}}\nabla\varphi(x)=(q_0,0,0)$.

Remark 2.1 By the results on the interior regularities and boundary regularities of solutions to second order elliptic equations (see [12, Chapter 6]), we know that $\varphi(x) \in C^\infty(\overline{\Omega})$ holds true in Theorem 2.1.

For the requirements to show Theorem 2.1, we intend to introduce the following transformation so that the domain Ω can be changed into a standard nozzle domain $\widetilde{\Omega} \equiv \{z = (z_1, z_2, z_3) : -\infty < z_1 < \infty, z_2^2 + z_3^2 \le 1\}$:

$$\begin{cases} z_1 = x_1, \\ z_2 = \dfrac{x_2}{1 + \varepsilon g(x_1, x_2, x_3)}, \\ z_3 = \dfrac{x_3}{1 + \varepsilon g(x_1, x_2, x_3)}. \end{cases} \tag{2.1}$$

In this case, for the notational convenience, we still denote the solution by $\varphi(z)$ instead of $\varphi(x)$ under the transformation (2.1). It follows from a direct computation that the problem (1.3)–(1.6) can be changed into

$$\begin{cases} \sum\limits_{i,j=1}^{3} A_{ij}(z, \nabla_z\varphi)\partial^2_{z_i z_j}\varphi + B(z, \nabla_z\varphi)\partial_{z_2}\varphi = 0, \quad \text{in } \widetilde{\Omega}, \\ b_1(z)\partial_{z_1}\varphi + b_2(z)\partial_{z_2}\varphi + b_3(z)\partial_{z_3}\varphi = 0, \quad \text{on } z_1^2 + z_2^2 = 1, \\ \lim\limits_{z_1 \to -\infty} (\varphi(z) - q_0 z_1) = 0, \\ \lim\limits_{\substack{z \in \widetilde{\Omega} \\ z_1 \to +\infty}} \nabla_z\varphi(z) \text{ exists}, \end{cases} \tag{2.2}$$

where

$$\begin{aligned}
A_{11}(z, \nabla_z\varphi) &= c^2(H(\nabla_x\varphi)) - (\partial_{x_1}\varphi)^2, \\
A_{kk}(z, \nabla_z\varphi) &= \sum_{i=1}^{3}(c^2(H(\nabla_x\varphi)) - (\partial_{x_i}\varphi)^2)\Big(\frac{\partial z_k}{\partial x_i}\Big)^2 - 2\sum_{1 \le i < j \le 3} \partial_{x_i}\varphi\partial_{x_j}\varphi\frac{\partial z_k}{\partial x_i}\frac{\partial z_k}{\partial x_j}, \quad k = 2, 3, \\
A_{1k}(z, \nabla_z\varphi) &= A_{k1}(z, \nabla_z\varphi) = (c^2(H(\nabla_x\varphi)) - (\partial_{x_1}\varphi)^2)\frac{\partial z_k}{\partial x_1} - \partial_{x_1}\varphi\partial_{x_2}\varphi\frac{\partial z_k}{\partial x_2} - \partial_{x_1}\varphi\partial_{x_3}\varphi\frac{\partial z_k}{\partial x_3}, \\
A_{23}(z, \nabla_z\varphi) &= A_{32}(z, \nabla_z\varphi) \\
&= \sum_{i=1}^{3}(c^2(H(\nabla_x\varphi)) - (\partial_{x_i}\varphi)^2)\frac{\partial z_2}{\partial x_i}\frac{\partial z_3}{\partial x_i} - \sum_{1 \le i < j \le 3} \partial_{x_i}\varphi\partial_{x_j}\varphi\Big(\frac{\partial z_2}{\partial x_i}\frac{\partial z_3}{\partial x_j} + \frac{\partial z_2}{\partial x_j}\frac{\partial z_3}{\partial x_i}\Big), \\
B_k(z, \nabla_z\varphi) &= \sum_{i=1}^{3}(c^2(H(\nabla_x\varphi)) - (\partial_{x_i}\varphi)^2)\frac{\partial^2 z_k}{\partial x_i^2} - 2\sum_{1 \le i < j \le 3} \partial_{x_i}\varphi\partial_{x_j}\varphi\frac{\partial^2 z_k}{\partial x_i \partial x_j}, \quad k = 2, 3, \\
b_1(z) &= -\varepsilon\partial_{x_1} g, \\
b_k(z) &= \frac{z_2}{\sqrt{z_2^2 + z_3^2}}\frac{\partial z_k}{\partial x_2} + \frac{z_3}{\sqrt{z_2^2 + z_3^2}}\frac{\partial z_k}{\partial x_3} - \varepsilon\sum_{i=1}^{3}\partial_{x_i} g\frac{\partial z_k}{\partial x_i}, \quad k = 2, 3,
\end{aligned}$$

with

$$\partial_{x_1}\varphi = \partial_{z_1}\varphi + \partial_{z_2}\varphi\frac{\partial z_2}{\partial x_1} + \partial_{z_3}\varphi\frac{\partial z_3}{\partial x_1}, \quad \partial_{x_i}\varphi = \partial_{z_2}\varphi\frac{\partial z_2}{\partial x_i} + \partial_{z_3}\varphi\frac{\partial z_3}{\partial x_i}, \quad i = 2, 3.$$

By the transformation (2.1), together with the properties of $g(x)$ and the definition of the norm $\|\cdot\|^{(\delta)}_{m,\alpha}$, in order to show Theorem 2.1, we only need to establish Theorem 2.2.

Theorem 2.2 *Under the assumptions in Theorem* 1.1, *the problem* (2.2) *has a unique solution* $\varphi(z) \in C^{6,\alpha}(\widetilde{\Omega})$ *which satisfies*

(i) $\|\varphi(z) - q_0 z_1\|^{(\delta_0)}_{6,\alpha;\widetilde{\Omega}} \le \widetilde{C}\varepsilon$,

(ii) $\lim\limits_{\substack{z\in\widetilde{\Omega}\\ z_1\to+\infty}} \nabla_z\varphi(z) = (q_0, 0, 0)$.

In next sections, we will focus on the proof of Theorem 2.2.

3 Solvability and a priori Estimates for the Linearized Problem of (2.2)

In order to solve the nonlinear problem (2.2), we first consider its linearized case, which corresponds to a mixed boundary value problem of a second order linear elliptic equation in an infinite nozzle domain $\widetilde{\Omega}$. In terms of the smallness of perturbed nozzle walls and by use of direct computations, the linearized problem of (2.2) can be essentially expressed as

$$\begin{cases} \overline{L}(v)\dot{u} \equiv \sum\limits_{i,j=1}^{3} a_{ij}(z,\nabla_z v)\partial^2_{z_i z_j}\dot{u} \\ \qquad\qquad \equiv \sum\limits_{i=1}^{3}(c^2(H(\nabla_z v)) - (\partial_{z_i}v)^2)\partial^2_{z_i}\dot{u} - 2\sum\limits_{1\le i<j\le 3}\partial_{z_i}v\partial_{z_j}v\partial^2_{z_i z_j}\dot{u} = \dot{f}, \quad \text{in } \widetilde{\Omega}, \\ \partial_n\dot{u} = \dot{g}, \quad \text{on } z_2^2+z_3^2=1, \\ \lim\limits_{\substack{z_1\to-\infty\\ z\in\widetilde{\Omega}}} \dot{u}(z) = 0, \\ \lim\limits_{\substack{z_1\to+\infty\\ z\in\widetilde{\Omega}}} \nabla_z\dot{u}(z) \text{ exists,} \end{cases} \tag{3.1}$$

where $v \in \mathbb{H}^{(\delta_0)}_{6,\alpha}(\widetilde{\Omega})$ with $\|v - q_0 z_1\|^{(\delta_0)}_{6,\alpha;\widetilde{\Omega}} < \varepsilon$ and $\delta_0(>0)$ is a suitably fixed constant.

It is easy to verify that the coefficients of the problem (3.1) satisfy the following uniformly elliptic condition in $\widetilde{\Omega}$:

$$\lambda|\xi|^2 \le \sum_{i,j=1}^{3} a_{ij}(z,\nabla_z v)\xi_i\xi_j \le \Lambda|\xi|^2 \quad \text{for all } \xi=(\xi_1,\xi_2,\xi_3)\in\mathbb{R}^3 \text{ and } z\in\widetilde{\Omega}, \tag{3.2}$$

where λ and Λ are two appropriate constants.

Next, we study the solvability of problem (3.1) as well as the regularity and a priori estimates of solution $\dot{u}(z)$ to (3.1). To this end, we first study the Laplacian equation in $\mathbb{R}^3$ with the following boundary conditions:

$$\begin{cases} L_0 u \equiv \Delta u = \widetilde{f}, \quad \text{in } \widetilde{\Omega}, \\ \partial_n u = \widetilde{g}, \quad \text{on } z_2^2+z_3^2=1, \\ \lim\limits_{z_1\to-\infty} u(z) = 0, \\ \lim\limits_{z_1\to+\infty} \nabla_z u(z) \text{ exists.} \end{cases} \tag{3.3}$$

In order to study the existence and regularity of solution to (3.3), we need to give some estimates on the eigenvalues and eigenfunctions to the following equation with $n \in \mathbb{N}\cup\{0\}$:

$$\begin{cases} R''(r) + \dfrac{1}{r}R'(r) + \Big(\nu - \dfrac{n^2}{r^2}\Big)R(r) = 0, \\ R'(1) = 0, \quad R(0) \text{ is bounded.} \end{cases} \tag{3.4}$$

First, we show that $\nu \geq 0$ holds true.

In fact, multiplying $rR(r)$ on two hand sides of (3.4) and integrating over $[0,1]$ yield

$$\int_0^1 \Big(\nu r R^2(r) - \frac{n^2}{r} R^2(r)\Big) \mathrm{d}r = \int_0^1 r(R'(r))^2 \mathrm{d}r \geq 0. \tag{3.5}$$

Thus,

$$\nu \geq 0.$$

For notational conveniences, we set $\nu = \lambda^2$ with $\lambda \geq 0$. Next we give a result on the estimates of the eigenvalues and eigenfunctions of (3.4).

Lemma 3.1 *For the following Bessel's equation with $n \in \mathbb{N} \cup \{0\}$:*

$$\begin{cases} R''(r) + \dfrac{1}{r} R'(r) + \Big(\lambda^2 - \dfrac{n^2}{r^2}\Big) R(r) = 0, \\ R'(1) = 0, \quad R(0) \textit{ is bounded}, \end{cases} \tag{3.6}$$

there exists a countable eigenvalues λ^2's for each fixed n $(n = 0, 1, 2, \cdots)$, which are denoted by $\lambda_{1n} < \lambda_{2n} < \cdots < \lambda_{mn} < \cdots$ for $n \geq 1$ and $\lambda_{00} < \lambda_{10} < \cdots < \lambda_{m0} < \cdots$ for $n = 0$, where $\lambda_{1n} > 0$, $\lambda_{00} = 0$, and $\lim\limits_{m \to \infty} \lambda_{mn} = +\infty$.

Additionally, $\lambda_{mn} > n$ holds true for $n \geq 1$.

Moreover, the corresponding eigenfunctions $\{R_{mn}(r)\}_{m=1}^{\infty}$ are orthogonal in the following weighted sense:

$$\begin{cases} \displaystyle\int_0^1 r R_{mn}(r) R_{kn}(r) \mathrm{d}r = \delta_{mk} \quad \textit{for } m, k = 1, 2, \cdots, \\ \displaystyle\int_0^1 r R_{m0}(r) R_{k0}(r) \mathrm{d}r = \delta_{mk} \quad \textit{for } m, k = 0, 1, 2, \cdots, \end{cases} \tag{3.7}$$

which admit the following estimates:

$$R_{00}(r) = \sqrt{2}, \tag{3.8}$$

$$|R_{mn}(r)| \leq \frac{C}{m} \lambda_{mn}^{\frac{3}{2}}, \quad m = 1, 2, \cdots, \tag{3.9}$$

$$|R_{mn}(1)| \leq \frac{C}{m} \lambda_{mn}, \tag{3.10}$$

$$\Big|\frac{1}{r} R_{mn}(r)\Big| \leq \frac{C}{m} \lambda_{mn}^{\frac{5}{2}}, \quad |R'_{mn}(r)| \leq \frac{C}{m} \lambda_{mn}^{\frac{5}{2}}, \quad |R''_{mn}(r)| \leq \frac{C}{m} \lambda_{mn}^{\frac{7}{2}} \quad \textit{for } n \geq 1, \tag{3.11}$$

$$\Big|\frac{1}{r} R'_{mn}(r)\Big| \leq \frac{C}{mn} \lambda_{mn}^{\frac{7}{2}}, \quad \Big|\frac{n^2}{r^2} R_{mn}(r)\Big| \leq \frac{C}{m} \lambda_{mn}^{\frac{7}{2}} \quad \textit{for } n \geq 2, \tag{3.12}$$

where $C > 0$ is a generic constant.

In order to prove Lemma 3.1, next we list some useful properties of Bessel functions, where the Bessel functions are defined by the ordinary differential equation $y''(x) + \frac{1}{x} y'(x) + (1 - \frac{n^2}{x^2}) y(x) = 0$.

Lemma 3.2 *Denote by $J_n(x)$ and $N_n(x)$ the Bessel function of the first kind and the Bessel function of the second kind of order n respectively. Then we have*

(i) $J_n(x) = \frac{1}{\pi} \int_0^{\pi} \cos(n\theta - x \sin\theta) \mathrm{d}\theta$;

(ii) $J_0'(x) = -J_1(x)$, $J_n'(x) = \frac{1}{2}(J_{n-1}(x) - J_{n+1}(x)) = J_{n-1}(x) - \frac{n}{x}J_n(x) = \frac{n}{x}J_n(x) - J_{n+1}(x)$;

(iii) $J_n(x) \sim \frac{x^n}{2^n\Gamma(1+n)}$ *and* $N_n(x) \sim -\frac{(n-1)!}{\pi}(\frac{x}{2})^{-n}$ *for* $n \geq 1$, $N_0(x) \sim \frac{2}{\pi}\ln\frac{x}{2}$, *as* $x \to 0$;

(iv) $J_n(x) \sim \sqrt{\frac{2}{\pi x}}\cos(x - \frac{1}{2}n\pi - \frac{1}{4}\pi)$, *as* $x \to +\infty$;

(v) $\int_0^x tJ_n(at)J_n(bt)\mathrm{d}t = \frac{x}{a^2-b^2}(J_n(ax)\frac{\mathrm{d}}{\mathrm{d}x}J_n(bx) - J_n(bx)\frac{\mathrm{d}}{\mathrm{d}x}J_n(ax))$, *here* $a^2 \neq b^2$;

(vi) $J_0''(x) = \frac{1}{2}(J_2(x) - J_0(x))$, $J_1''(x) = -J_1(x) + \frac{1}{2}J_3(x)$, $J_n''(x) = \frac{1}{4}(J_{n-2}(x) - 2J_n(x) + J_{n+2}(x))$, $n \geq 2$;

(vii) $\frac{1}{x}J_n'(x) = \frac{1}{4(n-1)}(J_{n-2}(x) + J_n(x)) - \frac{1}{4(n+1)}(J_n(x) - J_{n+2}(x))$, $n \geq 2$;

(viii) $\frac{n^2}{x^2}J_n(x) = J_n''(x) + J_n(x) + \frac{1}{4(n-1)}(J_{n-2}(x) + J_n(x)) - \frac{1}{4(n+1)}(J_n(x) - J_{n+2}(x))$, $n \geq 2$.

Let $\{\lambda_{mn}\}_{m=1}^{\infty}$ *stands for the countable zeroes of* $J_n'(x) = 0$ *with* $n \geq 1$. *Then*

(ix) *If* $J_n'(\lambda_{mn}) = 0$ *for* $n \geq 1$ *and* $\lambda_{mn}^2 \geq \frac{n^2}{4n^2-3}(4n^2 + 4 + \sqrt{48n^2 + 13})$, *then* $J_n(\lambda_{mn}) \geq \sqrt{\frac{2}{\pi\lambda_{mn}}}$;

(x) $\lambda_{mn} \geq \sqrt{n(n+2)}$ *for* $m, n \geq 1$.

In addition, if we denote by $J_\nu(x)$ *the Bessel function of the first kind of order* ν *which is a solution of* $y''(x) + \frac{1}{x}y'(x) + (1 - \frac{\nu^2}{x^2})y(x) = 0$ *with* $\nu > -1$ *and* $\nu \in \mathbb{R}$, *and write* $\{\widetilde{\lambda}_{m\nu}\}_{m=1}^{\infty}$ *as the countable zeroes of* $J_\nu(x) = 0$ *with* $\nu \geq 0$, *then*

(xi) *If* $J_\nu(\widetilde{\lambda}_{m,\nu}) = 0$ *for* $m \geq 1$ *and* $\nu > 0$, *then* $\frac{\mathrm{d}}{\mathrm{d}\nu}\widetilde{\lambda}_{m,\nu} = \frac{2\nu}{\widetilde{\lambda}_{m,\nu}J_{\nu+1}^2(\widetilde{\lambda}_{m,\nu})}\int_0^{\widetilde{\lambda}_{m,\nu}}\frac{1}{t}J_\nu^2(t)\mathrm{d}t$;

(xii) *For* $-\frac{1}{2} < \nu \leq \frac{1}{2}$, *the positive zeroes of* $J_\nu(x)$ *lie in the intervals* $(m\pi - \frac{1}{4}\pi + \frac{1}{2}\nu\pi, m\pi - \frac{1}{8}\pi + \frac{1}{4}\nu\pi)$;

For $\frac{1}{2} < \nu < \frac{5}{2}$, *all* $\widetilde{\lambda}_{m,\nu}$ *lie in the intervals* $(m\pi - \frac{1}{8}\pi + \frac{1}{4}\nu\pi, m\pi - \frac{1}{4}\pi + \frac{1}{2}\nu\pi)$;

(xiii) *The zeroes of* $J_\nu'(x)$ *are interlaced with those of* $J_{\nu+1}(x)$.

Proof (i)–(v) and (ix)–(xiii) can be found in [19].

(vi) follows from (ii) directly. We only need to show (vii)–(viii).

For $n \geq 2$, it follows from (ii) that

$$
\begin{aligned}
J_{n-1}(x) &= \frac{x}{n-1}(J_{n-2}(x) - J_{n-1}'(x)) = \frac{x}{2(n-1)}(J_{n-2}(x) + J_n(x)),\\
J_{n+1}(x) &= \frac{x}{n+1}(J_n(x) - J_{n+1}'(x)) = \frac{x}{2(n+1)}(J_n(x) + J_{n+2}(x)).
\end{aligned}
$$

Thus, one has for $n \geq 2$,

$$
\begin{aligned}
\frac{1}{x}J_n'(x) &= \frac{1}{2x}(J_{n-1} - J_{n+1}(x))\\
&= \frac{1}{4(n-1)}(J_{n-2}(x) + J_n(x)) - \frac{1}{4(n+1)}(J_n(x) - J_{n+2}(x)).
\end{aligned}
$$

Namely, (vii) is proved.

Next, we verify (viii).

It is noted that the first Bessel function $J_n(x)$ satisfies

$$
J_n''(x) + \frac{1}{x}J_n'(x) + \Big(1 - \frac{n^2}{x^2}\Big)J_n(x) = 0,
$$

which together with (vii) yields (viii).

Based on Lemma 3.2, we now start to prove Lemma 3.1.

Proof of Lemma 3.1 If $\lambda = 0$, then it follows from (3.5) that $n = 0$ holds when (3.6) has the nontrivial solution. In this case, it is easy to verify that $\lambda_{00} = 0$ is an eigenvalue, and the corresponding eigenfunction is $R_{00}(r) = \sqrt{2}$. Namely, (3.8) is proved.

Next, we consider the case of $\lambda > 0$.

Set $t = \lambda r$ and $y(t) = R(r)$. Then (3.6) is changed into the following Bessel equation of order n ($n \in \mathbb{N} \cup \{0\}$):

$$\begin{cases} y''(t) + \frac{1}{t} y'(t) + \Big(1 - \dfrac{n^2}{t^2}\Big) y(t) = 0, \\ y'(\lambda) = 0, \quad y(0) \text{ is bounded.} \end{cases} \tag{3.13}$$

Its general solutions can be expressed as

$$y_n(t) = C_1 J_n(t) + C_2 N_n(t), \tag{3.14}$$

where $J_n(t)$ and $N_n(x)$ are the Bessel functions of the first kind and the second kind of order n respectively.

From the properties (iii) of Lemma 3.2, one has $\lim\limits_{t \to 0} N_n(t) = \infty$. This, together with the boundary conditions in (3.13), yields $C_2 = 0$ in (3.14). In addition, it follows from $y_n'(\lambda) = 0$ that $J_n'(\lambda) = 0$ holds. By the properties of Bessel function in Lemma 3.2, $J_n'(\lambda)$ has countable zero points $\{\lambda_{mn}\}_{m=1}^{\infty}$, which satisfies $0 < \lambda_{1n} < \lambda_{2n} < \cdots < \lambda_{mn} < \cdots$, and $\lim\limits_{m \to \infty} \lambda_{mn} = +\infty$.

Thus, returning to $R(r)$ in (3.6), we obtain the corresponding the eigenfunction

$$R_{mn}(r) = \frac{J_n(\lambda_{mn} r)}{\big(\int_0^1 r J_n^2(\lambda_{mn} r) \mathrm{d}r\big)^{\frac{1}{2}}}. \tag{3.15}$$

This together with Lemma 3.2(v) yields

$$\begin{cases} \displaystyle\int_0^1 r R_{mn}(r) R_{kn}(r) \mathrm{d}r = \delta_{mk}, \quad m, k = 1, 2, \cdots, \\ \displaystyle\int_0^1 r R_{m0}(r) R_{k0}(r) \mathrm{d}r = \delta_{mk}, \quad m, k = 0, 1, 2, \cdots. \end{cases}$$

Next, we start to estimate the bound of eigenfunctions $R_{mn}(r)$.

Multiplying $2r^2 R_{mn}'(r)$ on two hand sides of (3.6) and integrating over $[0, 1]$ yield

$$\int_0^1 (2r R_{mn}'(r)(r R_{mn}'(r))' + 2(r^2 \lambda_{mn}^2 - n^2) R_{mn}(r) R_{mn}'(r)) \mathrm{d}r = 0.$$

This derives

$$\int_0^1 r R_{mn}^2(r) \mathrm{d}r = \frac{1}{2\lambda_{mn}^2} \big((\lambda_m n^2 - n^2) R_{mn}^2(1) + n^2 R_{mn}^2(0)\big) = \frac{(\lambda_{mn}^2 - n^2) R_{mn}^2(1)}{2\lambda_{mn}^2}. \tag{3.16}$$

Substituting (3.15) into (3.16) yields

$$\int_0^1 r J_n^2(\lambda_{mn} r) \mathrm{d}r = \frac{(\lambda_{mn}^2 - n^2) J_n^2(\lambda_{mn})}{2\lambda_{mn}^2}. \tag{3.17}$$

Combining (3.17) with (3.15), we arrive at

$$R_{mn}(r) = \frac{\sqrt{2}\, \lambda_{mn} J_n(\lambda_{mn} r)}{\sqrt{\lambda_{mn}^2 - n^2} J_n(\lambda_{mn})}. \tag{3.18}$$

By use of Lemma 3.2(ii) and (iv), for large λ and fixed n, we have

$$J_n'(\lambda) \sim \sqrt{\frac{1}{\pi\lambda}}\cos\Big(\lambda - \frac{1}{2}n\pi + \frac{1}{4}\pi\Big). \tag{3.19}$$

Thus, for sufficiently large m and for $n \le 7$, it follows from $J_n'(\lambda_{mn}) = 0$ that

$$\lambda_{mn} \sim (m + k(n))\pi + \frac{1}{2}n\pi + \frac{1}{4}, \quad n = 0, 1, \cdots, 7, \tag{3.20}$$

where $k(n)$ is some fixed positive integer depending only on n.

On the other hand, in terms of Lemma 3.2(iv), we have

$$J_n(\lambda_{mn}) \sim \sqrt{\frac{1}{\pi\lambda_{mn}}}\cos(m + k(n))\pi \quad \text{for large } m.$$

This yields

$$|J_n(\lambda_{mn})| \ge C\sqrt{\frac{1}{\lambda_{mn}}}, \quad m = 1, 2, \cdots, n \le 7.$$

When $n > 7$, by use of Lemma 3.2(x) and direct computation, one has

$$\lambda_{mn}^2 > n(n+2) > \frac{n^2}{4n^2 - 3}(4n^2 + 4 + \sqrt{48n^2 + 13}).$$

This, together with Lemma 3.2(ix) and the estimate on $|J_n(\lambda_{mn})|$ in the case of $n \le 7$, yields

$$|J_n(\lambda_{mn})| \ge C\sqrt{\frac{1}{\lambda_{mn}}}, \quad m = 1, 2, \cdots, \ n = 0, 1, \cdots. \tag{3.21}$$

We now estimate the more precise upper and lower bound of λ_{mn}.

Since $J_\nu(x)$ satisfies

$$J_\nu''(t) + \frac{1}{t}J_\nu'(t) + \Big(1 - \frac{\nu^2}{t^2}\Big)J_\nu(t) = 0, \tag{3.22}$$

multiplying $2t^2 J_\nu'(t)$ on two hand sides of (3.22) and integrating over $[0, \widetilde{\lambda}_{m,\nu}]$, where $\widetilde{\lambda}_{m,\nu}$ denotes the zero point of $J_\nu(x)$, we have

$$\int_0^{\widetilde{\lambda}_{m,\nu}} (2tJ_\nu'(tJ_\nu'(t))' + 2(t^2 - \nu^2)J_\nu(t)J_\nu'(t))\mathrm{d}t = 0.$$

This derives

$$\int_0^{\widetilde{\lambda}_{m,\nu}} tJ_\nu^2(t)\mathrm{d}t = \frac{1}{2}\widetilde{\lambda}_{m,\nu}^2 J_\nu'^2(\widetilde{\lambda}_{m,\nu}).$$

Due to $J_\nu'(\widetilde{\lambda}_{m,\nu}) = \frac{\nu}{\widetilde{\lambda}_{m,\nu}}J_\nu(\widetilde{\lambda}_{m,\nu}) - J_{\nu+1}(\widetilde{\lambda}_{m,\nu}) = -J_{\nu+1}(\widetilde{\lambda}_{m,\nu})$, one has

$$\int_0^{\widetilde{\lambda}_{m,\nu}} tJ_\nu^2(t)\mathrm{d}t = \frac{1}{2}\widetilde{\lambda}_{m,\nu}^2 J_{\nu+1}^2(\widetilde{\lambda}_{m,\nu}). \tag{3.23}$$

Combining (3.23) with Lemma 3.2(xi) yields

$$\frac{\mathrm{d}}{\mathrm{d}\nu}\widetilde{\lambda}_{m,\nu} = \frac{2\nu}{\widetilde{\lambda}_{m,\nu}J_{\nu+1}^2(\widetilde{\lambda}_{m,\nu})}\int_0^{\widetilde{\lambda}_{m,\nu}} \frac{1}{t}J_\nu^2(t)\mathrm{d}t = \nu\widetilde{\lambda}_{m,\nu}\frac{\int_0^{\widetilde{\lambda}_{m,\nu}} \frac{1}{t}J_\nu^2(t)\mathrm{d}t}{\int_0^{\widetilde{\lambda}_{m,\nu}} tJ_\nu^2(t)\mathrm{d}t}. \tag{3.24}$$

This implies

$$\frac{\mathrm{d}}{\mathrm{d}\nu}\widetilde{\lambda}_{m,\nu} = \frac{\nu}{\widetilde{\lambda}_{m,\nu}} \frac{\int_0^1 \frac{1}{s} J_\nu^2(\widetilde{\lambda}_{m,\nu} s)\mathrm{d}s}{\int_0^1 s J_\nu^2(\widetilde{\lambda}_{m,\nu} s)\mathrm{d}s} \geq \frac{\nu}{\widetilde{\lambda}_{m,\nu}}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}\nu^2}(\widetilde{\lambda}_{m,\nu})^2 \geq 1.$$

Thus, we have

$$\widetilde{\lambda}^2_{m,\nu} \geq \widetilde{\lambda}^2_{m,1} + \nu^2 - 1.$$

This together with Lemma 3.2(xii) yields

$$\widetilde{\lambda}_{m,n} \geq \sqrt{m^2\pi^2 + n^2 + C}. \tag{3.25}$$

Consequently, it follows from Lemma 3.2(xiii) and (3.25) that

$$\lambda_{mn} \geq \sqrt{m^2\pi^2 + n^2 + C}. \tag{3.26}$$

On the other hand, multiplying $tJ_\nu(t)$ on two hand sides of (3.22) and integrating over $[0, \widetilde{\lambda}_{m,\nu}]$ yield

$$\int_0^{\widetilde{\lambda}_{m,\nu}} \Big(t - \frac{\nu^2}{t}\Big) J_\nu(t)\mathrm{d}t = \int_0^{\widetilde{\lambda}_{mn}} t J_\nu'^2(x)\mathrm{d}x \geq 0.$$

Combining this with (3.24) yields

$$\frac{\mathrm{d}}{\mathrm{d}\nu}\widetilde{\lambda}_{m,\nu} = \frac{\widetilde{\lambda}_{m,\nu}}{\nu} \frac{\int_0^{\widetilde{\lambda}_{m,\nu}} \frac{\nu^2}{t} J_\nu^2(t)\mathrm{d}t}{\int_0^{\widetilde{\lambda}_{m,\nu}} t J_\nu^2(t)\mathrm{d}t} \leq \frac{\widetilde{\lambda}_{m,\nu}}{\nu}.$$

This implies

$$\widetilde{\lambda}_{mn} \leq \widetilde{\lambda}_{m1} n.$$

By use of Lemma 3.2(xii), we arrive at

$$\widetilde{\lambda}_{mn} \leq Cm(n+1).$$

Thus, together with Lemma 3.2(xiii), one has

$$\lambda_{mn} \leq Cm(n+1). \tag{3.27}$$

Based on (3.26)–(3.27), we now show (3.9)–(3.12).

Substituting (3.26) into (3.18), one has

$$|R_{mn}(1)| \leq \frac{C}{m}\lambda_{mn}. \tag{3.28}$$

On the other hand, by use of Lemma 3.2(i), we can deduce $|J_n(x)| \leq 1$. Combining (3.21) and (3.26) with (3.18) yields

$$|R_{mn}(r)| \leq \frac{C}{m}\lambda_{mn}^{\frac{3}{2}}. \tag{3.29}$$

In addition, it follows from (3.18), (3.21) and Lemma 3.2 that

$$\begin{aligned}
\Big|\frac{1}{r}R_{mn}(r)\Big| &= \Big|\frac{\sqrt{2}\,\lambda_{mn}^2}{\sqrt{\lambda_{mn}^2-n^2}J_n(\lambda_{mn})}\frac{1}{\lambda_{mn}r}J_n(\lambda_{mn}r)\Big| \\
&= \Big|\frac{\sqrt{2}\,\lambda_{mn}^2}{\sqrt{\lambda_{mn}^2-n^2}J_n(\lambda_{mn})}(J_n'(\lambda_{mn}r)+J_{n+1}(\lambda_{mn}r))\Big| \le \frac{C}{m}\lambda_{mn}^{\frac{5}{2}}, \\
|R_{mn}'(r)| &= \Big|\frac{\sqrt{2}\,\lambda_{mn}^2 J_n'(\lambda_{mn}r)}{\sqrt{\lambda_{mn}^2-n^2}J_n(\lambda_{mn})}\Big| \le \frac{C}{m}\lambda_{mn}^{\frac{5}{2}}, \\
|R_{mn}''(r)| &= \Big|\frac{\sqrt{2}\,\lambda_{mn}^3 J_n''(\lambda_{mn}r)}{\sqrt{\lambda_{mn}^2-n^2}J_n(\lambda_{mn})}\Big| \le \frac{C}{m}\lambda_{mn}^{\frac{7}{2}}.
\end{aligned}$$

In the case of $n\ge 2$, by (3.15), (3.21) and Lemma 3.2, we can also obtain

$$\begin{aligned}
\Big|\frac{1}{r}R_{mn}'(r)\Big| &= \Big|\frac{\sqrt{2}\,\lambda_{mn}^3}{\sqrt{\lambda_{mn}^2-n^2}J_n(\lambda_{mn})}\frac{1}{\lambda_{mn}r}J_n'(\lambda_{mn}r)\Big| \le \frac{C}{mn}\lambda_{mn}^{\frac{7}{2}}, \\
\Big|\frac{n^2}{r^2}R_{mn}(r)\Big| &= \Big|\frac{\sqrt{2}\,\lambda_{mn}^3}{\sqrt{\lambda_{mn}^2-n^2}J_n(\lambda_{mn})}\frac{n^2}{\lambda_{mn}^2r^2}J_n(\lambda_{mn}r)\Big| \le \frac{C}{m}\lambda_{mn}^{\frac{7}{2}}.
\end{aligned}$$

Thus, we complete the proof of Lemma 3.1.

In order to solve (3.3) and for the requirement later on, we now give a lemma on the function $f(z)\in H_{4,\alpha}^{(\delta_0)}(\widetilde{\Omega})$.

Lemma 3.3 *For $f\in H_{4,\alpha}^{(\delta_0)}(\widetilde{\Omega})$, if we set*

$$\begin{aligned}
f_{m0}(z_1) &= \frac{1}{2\pi}\int_0^1\int_0^{2\pi} rf(z_1,r,\theta)R_{m0}(r)\mathrm{d}\theta\mathrm{d}r, \\
f_{mn}^1(z_1) &= \frac{1}{2\pi}\int_0^1\int_0^{2\pi} rf(z_1,r,\theta)R_{mn}(r)\cos n\theta\mathrm{d}\theta\mathrm{d}r, \\
f_{mn}^2(z_1) &= \frac{1}{2\pi}\int_0^1\int_0^{2\pi} rf(z_1,r,\theta)R_{mn}(r)\sin n\theta\mathrm{d}\theta\mathrm{d}r
\end{aligned}$$

for $m,n\in\mathbb{N}$, then

$$\begin{aligned}
|f_{m0}(z_1)| &\le \frac{C}{\lambda_{m0}^2}|f|_{4,\alpha;\widetilde{\Omega}}^{(\delta_0)}\,\mathrm{e}^{-\delta_0|z_1|}, \\
|f_{mn}^i(z_1)| &\le \frac{C}{\lambda_{mn}mn^2}|f|_{4,\alpha;\widetilde{\Omega}}^{(\delta_0)}\,\mathrm{e}^{-\delta_0|z_1|}, \quad i=1,2.
\end{aligned}$$

Proof Multiplying r on both sides of (3.6) yields

$$(rR'(r))'+r\Big(\lambda^2-\frac{n^2}{r^2}\Big)R(r)=0.$$

This implies

$$rR(r)=\frac{1}{\lambda^2}\Big(-(rR'(r))'+\frac{n^2}{r}R(r)\Big). \tag{3.30}$$

In this case, we have

$$f_{m0}(z_1)=-\frac{1}{2\pi\lambda_{m0}^2}\int_0^1\int_0^{2\pi} f(z_1,r,\theta)(rR_{m0}'(r))'\mathrm{d}\theta\mathrm{d}r.$$

By integration by parts, we arrived at

$$\begin{aligned} f_{m0}(z_1) &= \frac{1}{2\pi\lambda_{m0}^2}\int_0^{2\pi}\Big(rR_{m0}(r)\partial_r f(z_1,r,\theta)\big|_0^1 - \int_0^1 R_{m0}(r)(\partial_r f + r\partial_r^2 f)\mathrm{d}r\Big)\mathrm{d}\theta \\ &= \frac{1}{2\pi\lambda_{m0}^2}\int_0^{2\pi}\Big(R_{m0}(1)\partial_r f(z_1,1,\theta) - \int_0^1 R_{m0}(r)(\partial_r f + r\partial_r^2 f)\mathrm{d}r\Big)\mathrm{d}\theta. \end{aligned} \tag{3.31}$$

Due to $f \in H_{4,\alpha}^{(\delta_0)}(\widetilde{\Omega})$ and Lemma 3.1, we have

$$\Big|\int_0^1 rR_{m0}(r)\partial_r^2 f\mathrm{d}r\Big| \le \Big(\int_0^1 rR_{m0}^2(r)\mathrm{d}r\Big)^{\frac12}\Big(\int_0^1 r(\partial_r^2 f)^2\mathrm{d}r\Big)^{\frac12} \le C\mathrm{e}^{-\delta_0|z_1|}|f|_{4,\alpha;\widetilde{\Omega}}^{(\delta_0)}. \tag{3.32}$$

Since

$$\begin{aligned} &\int_0^{2\pi}\int_0^1 R_{m0}(r)\partial_r f\mathrm{d}r\mathrm{d}\theta \\ &= \int_0^1 R_{m0}(r)\int_0^{2\pi}(\partial_{z_2} f\cos\theta + \partial_{z_3} f\sin\theta)\mathrm{d}\theta\mathrm{d}r \\ &= \int_0^1\int_0^{2\pi} rR_{m0}(r)(\partial_{z_2}^2 f\sin^2\theta - 2\sin\theta\cos\theta\partial_{z_2z_3}^2 f + \partial_{z_3}^2 f\cos^2\theta)\mathrm{d}\theta\mathrm{d}r \end{aligned} \tag{3.33}$$

and

$$\begin{aligned} &\Big|\int_0^1 rR_{m0}(r)(\partial_{z_2}^2 f\sin^2\theta - 2\sin\theta\cos\theta\partial_{z_2z_3}^2 f + \partial_{z_3}^2 f\cos^2\theta)\mathrm{d}r\Big| \\ &\le \Big(\int_0^1 rR_{m0}^2(r)\mathrm{d}r\Big)^{\frac12}\Big(\int_0^1 r(\partial_{z_2}^2 f\sin^2\theta - 2\sin\theta\cos\theta\partial_{z_2z_3}^2 f + \partial_{z_3}^2 f\cos^2\theta)^2\mathrm{d}r\Big)^{\frac12} \\ &\le C\mathrm{e}^{-\delta_0|z_1|}|f|_{4,\alpha;\widetilde{\Omega}}^{(\delta_0)}, \end{aligned} \tag{3.34}$$

it follows from (3.31)–(3.34) and Lemma 3.1 that

$$|f_{m0}(z_1)| \le \frac{C}{\lambda_{m0}^2}|f|_{4,\alpha;\widetilde{\Omega}}^{(\delta_0)}\mathrm{e}^{-\delta_0|z_1|}.$$

Next, we estimate $f_{mn}^i(z_1)$ for $i = 1, 2$.

It follows from (3.30) and integration by parts with respect to r that

$$\begin{aligned} f_{mn}^1(z_1) = &-\frac{1}{2\pi\lambda_{mn}^2}\int_0^{2\pi}\int_0^1 R_{mn}(r)(\partial_r f(z_1,r,\theta) + r\partial_r^2 f(z_1,r,\theta))\cos n\theta\mathrm{d}r\mathrm{d}\theta \\ &+ \frac{R_{mn}(1)}{2\pi\lambda_{mn}^2}\int_0^{2\pi}\partial_r f(z_1,1,\theta)\cos n\theta\mathrm{d}\theta \\ &+ \frac{n^2}{2\pi\lambda_{mn}^2}\int_0^{2\pi}\int_0^1 \frac{1}{r}f(z_1,r,\theta)R_{mn}(r)\cos n\theta\mathrm{d}r\mathrm{d}\theta. \end{aligned}$$

By integration by parts with respect to θ, we can obtain

$$\begin{aligned} f_{mn}^1(z_1) = &\frac{R_{mn}(1)}{2\pi\lambda_{mn}^2 n^2}\int_0^{2\pi}\partial_r\partial_\theta^2 f(z_1,1,\theta)\cos n\theta\mathrm{d}\theta \\ &+ \frac{1}{2\pi\lambda_{mn}^2 n^2}\int_0^{2\pi}\int_0^1 R_{mn}(r)\Big(\partial_\theta^2\Big(\partial_r + \frac{1}{r}\partial_\theta^2\Big)f(z_1,r,\theta) \\ &+ r\partial_r^2\partial_\theta^2 f(z_1,r,\theta)\Big)\cos n\theta\mathrm{d}r\mathrm{d}\theta. \end{aligned} \tag{3.35}$$

Note

$$\partial_r f + \frac{1}{r}\partial_\theta^2 f = r(\sin^2\theta\partial_{z_2}^2 f - 2\sin\theta\cos\theta\partial_{z_2z_3}^2 f + \cos^2\theta\partial_{z_3}^2 f). \tag{3.36}$$

Due to $f \in H_{4,\alpha}^{(\delta_0)}(\widetilde{\Omega})$ and Lemma 3.1, we have

$$\begin{aligned}&\Big|\int_0^1 R_{mn}(r)\Big(\partial_\theta^2\Big(\partial_r + \frac{1}{r}\partial_\theta^2\Big)f(z_1,r,\theta) + r\partial_r^2\partial_\theta^2 f(z_1,r,\theta)\Big)\Big|\\ &\le \Big(\int_0^1 rR_{mn}^2(r)\mathrm{d}r\Big)^{\frac{1}{2}}\Big(\int_0^1 r(\partial_\theta^2(\sin^2\theta\partial_{z_2}^2 f - 2\sin\theta\cos\theta\partial_{z_2z_3}^2 f + \cos^2\theta\partial_{z_3}^2 f) + \partial_r^2\partial_\theta^2 f)^2\mathrm{d}r\Big)^{\frac{1}{2}}\\ &\le C|f|_{4,\alpha;\widetilde{\Omega}}^{(\delta_0)}\mathrm{e}^{-\delta_0|z_1|}.\end{aligned} \tag{3.37}$$

Thus, it follows from (3.35)–(3.37) and Lemma 3.1 that

$$|f_{mn}^1(z_1)| \le \frac{C}{\lambda_{mn}mn^2}|f|_{4,\alpha;\widetilde{\Omega}}^{(\delta_0)}\mathrm{e}^{-\delta_0|z_1|}.$$

Analogously, $|f_{mn}^2(z_1)| \le \frac{C}{\lambda_{mn}mn^2}|f|_{4,\alpha;\widetilde{\Omega}}^{(\delta_0)}\mathrm{e}^{-\delta_0|z_1|}$ can be shown.

Lemma 3.4 *If $\widetilde{f} \in H_{4,\alpha}^{(\delta_0)}(\widetilde{\Omega})$ and $\widetilde{g} \in H_{5,\alpha}^{(\delta_0)}(\widetilde{\Omega})$ with $0 < \delta_0 < \min\{\lambda_{10}, 1\}$, then equation (3.3) has a solution $u \in C^2(\overline{\widetilde{\Omega}})$, which admits the following estimate:*

$$\|u\|_{2,0;\widetilde{\Omega}}^{(\delta_0)} \le C(|\widetilde{f}|_{4,\alpha;\widetilde{\Omega}}^{(\delta_0)} + |\widetilde{g}|_{5,\alpha;\widetilde{\Omega}}^{(\delta_0)}). \tag{3.38}$$

Proof We intend to use the method of separation variables to study the solvability and regularities of solution u to (3.3). To this end, we firstly introduce such cylindrical coordinate transformation

$$z_1 = z_1, \quad z_2 = r\cos\theta, \quad z_3 = r\sin\theta, \tag{3.39}$$

where $r = \sqrt{z_2^2 + z_3^2}$.

Thus, (3.3) is changed into the following equation:

$$\begin{cases} L_0 u = \partial_{z_1}^2 u + \partial_r^2 u + \frac{1}{r}\partial_r u + \frac{1}{r^2}\partial_\theta^2 u = \widetilde{f}, \quad \text{in } \widetilde{\Omega},\\ \partial_r u = \widetilde{g}, \quad \text{on } r = 1,\\ \lim\limits_{z_1\to-\infty} u = 0, \quad \lim\limits_{z_1\to+\infty}\nabla u \text{ exists.}\end{cases} \tag{3.40}$$

Let us consider the nontrivial solutions of the following problem:

$$\begin{cases} \partial_{z_1}^2 u + \partial_r^2 u + \frac{1}{r}\partial_r u + \frac{1}{r^2}\partial_\theta^2 u = 0, \quad \text{in } \widetilde{\Omega},\\ \partial_r u = 0, \quad \text{on } r = 1.\end{cases} \tag{3.41}$$

Set $u(z) = Z(z_1)R(r)\Theta(\theta)$. Then it follows from (3.41) that

$$\begin{cases} \Theta''(\theta) + \mu\Theta(\theta) = 0,\\ \Theta(\theta + 2\pi) = \Theta(\theta),\end{cases} \tag{3.42}$$

$$\begin{cases} R''(r) + \frac{1}{r}R'(r) + \Big(\nu - \frac{\mu}{r^2}\Big)R(r) = 0,\\ R'(1) = 0, \quad R(0) \text{ is bounded}\end{cases} \tag{3.43}$$

and

$$Z''(z_1) - \nu Z(z_1) = 0, \tag{3.44}$$

where $\nu, \mu \in \mathbb{R}$.

By a simple computation, we know that the eigenvalues of (3.42) are $\mu_n = n^2$ $(n = 0, 1, 2, \cdots)$, and the corresponding eigenfunctions are $\cos n\theta$ and $\sin n\theta$ respectively. In addition, by Lemma 3.1, we obtain that the eigenvalues of (3.43) are $\nu_{mn} = \lambda_{mn}^2$ $(m, n = 0, 1, 2, \cdots)$, and the corresponding eigenfunctions are $R_{mn}(r)$.

We now solve equation (3.40) by use of the eigenfunction expansion method in terms of the complete weighted orthogonal basis $\{R_{00}, R_{m0}(r), R_{mn}\cos n\theta, R_{mn}\sin n\theta\}|_{m,n=1}^{+\infty}$.

Set $h(z_1, r, \theta) = \frac{1}{2}r^2\widetilde{g}(z_1, \theta)$ and $v(z_1, r, \theta) = u(z_1, r, \theta) - h(z_1, r, \theta)$. Then it follows from (3.40) that $v(z_1, r, \theta)$ satisfies

$$\begin{cases} L_0 v = \partial_{z_1}^2 v + \partial_r^2 v + \dfrac{1}{r}\partial_r v + \dfrac{1}{r^2}\partial_\theta^2 v = \widetilde{f} - L_0 h \equiv f, \quad \text{in } \widetilde{\Omega}, \\ \partial_r v = 0, \quad \text{on } r = 1, \\ \lim\limits_{z_1 \to -\infty} v(z_1, r, \theta) = 0, \\ \lim\limits_{z_1 \to +\infty} \nabla v(z_1, r, \theta) \ \text{ exists.} \end{cases} \tag{3.45}$$

Let

$$\begin{aligned} v(z_1, r, \theta) = {} & Z_{00}(z_1) + \sum_{m=1}^{\infty} Z_{m0}(z_1)R_{m0}(r) \\ & + \sum_{m,n=1}^{\infty} (Z_{mn}^1(z_1)R_{mn}(r)\cos n\theta + Z_{mn}^2(z_1)R_{mn}(r)\sin n\theta) \end{aligned} \tag{3.46}$$

and

$$\begin{aligned} f(z) = {} & f_{00}(z_1) + \sum_{m=1}^{\infty} f_{m0}(z_1)R_{m0}(r) \\ & + \sum_{m,n=1}^{\infty} (f_{mn}^1(z_1)R_{mn}(r)\cos n\theta + f_{mn}^2(z_1)R_{mn}(r)\sin n\theta), \end{aligned}$$

where

$$\begin{cases} f_{00}(z_1) = \dfrac{1}{2\pi}\displaystyle\int_0^1\int_0^{2\pi} rf(z_1, r, \theta)\mathrm{d}\theta\mathrm{d}r, \\ f_{m0}(z_1) = \dfrac{1}{2\pi}\displaystyle\int_0^1\int_0^{2\pi} rf(z_1, r, \theta)R_{m0}(r)\mathrm{d}\theta\mathrm{d}r, \\ f_{mn}^1(z_1) = \dfrac{1}{2\pi}\displaystyle\int_0^1\int_0^{2\pi} rf(z_1, r, \theta)R_{mn}(r)\cos n\theta\mathrm{d}\theta\mathrm{d}r, \\ f_{mn}^2(z_1) = \dfrac{1}{2\pi}\displaystyle\int_0^1\int_0^{2\pi} rf(z_1, r, \theta)R_{mn}(r)\sin n\theta\mathrm{d}\theta\mathrm{d}r. \end{cases}$$

Next, we start to determine the terms $Z_{00}(z_1), Z_{m0}(z_1), Z_{0n}^i(z_1)$ and $Z_{mn}^i(z_1)$ $(i = 1, 2)$ in (3.46).

It follows from (3.45) and (3.46) that we can formally obtain

$$\begin{cases} Z''_{00}(z_1) = f_{00}(z_1), \\ \lim\limits_{z_1\to-\infty} Z_{00}(z_1) = 0, \quad \lim\limits_{z_1\to+\infty} Z'_{00}(z_1) \text{ exists}, \end{cases} \tag{3.47}$$

$$\begin{cases} Z''_{m0}(z_1) - \lambda_{m0}^2 Z_{m0}(z_1) = f_{m0}(z_1), \\ \lim\limits_{z_1\to-\infty} Z_{m0}(z_1) = 0, \quad \lim\limits_{z_1\to+\infty} Z'_{m0}(z_1) \text{ exists}, \end{cases} \tag{3.48}$$

$$\begin{cases} (Z^i_{mn})''(z_1) - \lambda_{mn}^2 Z^i_{mn}(z_1) = f^i_{mn}(z_1), \\ \lim\limits_{z_1\to-\infty} Z^i_{mn}(z_1) = 0, \quad \lim\limits_{z_1\to+\infty} (Z^i_{mn})'(z_1) \text{ exists}. \end{cases} \tag{3.49}$$

Solving these ordinary differential equations directly yields

$$Z_{00}(z_1) = \int_{-\infty}^{z_1}\int_{-\infty}^{t} f_{00}(\xi)\mathrm{d}\xi\mathrm{d}t, \tag{3.50}$$

$$Z_{m0}(z_1) = \mathrm{e}^{\lambda_{m0}z_1}\int_{+\infty}^{z_1}\mathrm{e}^{-2\lambda_{m0}t}\int_{-\infty}^{t}\mathrm{e}^{\lambda_{m0}\xi} f_{m0}(\xi)\mathrm{d}\xi\mathrm{d}t, \quad m \ge 1, \tag{3.51}$$

$$Z^i_{mn}(z_1) = \mathrm{e}^{\lambda_{mn}z_1}\int_{+\infty}^{z_1}\mathrm{e}^{-2\lambda_{mn}t}\int_{-\infty}^{t}\mathrm{e}^{\lambda_{mn}\xi} f^i_{mn}(\xi)\mathrm{d}\xi\mathrm{d}t, \quad m, n \ge 1. \tag{3.52}$$

As in [26], together with Lemmas 3.1–3.3, we can show the following properties:

(A)
$$\|Z_{00}(z_1)\|^{(\delta_0)}_{0,0} \le C|f|^{(\delta_0)}_{4,\alpha;\widetilde{\Omega}}, \quad \|Z'_{00}(z_1)\|^{(\delta_0)}_{0,0} \le C|f|^{(\delta_0)}_{4,\alpha;\widetilde{\Omega}} \tag{3.53}$$

and

$$\lim_{z_1\to+\infty} Z'_{00}(z_1) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_0^{2\pi}\int_0^1 rf(t,r,\theta)\mathrm{d}r\mathrm{d}\theta\mathrm{d}t. \tag{3.54}$$

(B)
$$|Z_{m0}(z_1)|^{(\delta_0)}_{0,0} \le \frac{C}{\lambda_{m0}^3(\lambda_{m0}-\delta_0)}|f|^{(\delta_0)}_{4,\alpha;\widetilde{\Omega}}, \tag{3.55}$$

$$|Z'_{m0}(z_1)|^{(\delta_0)}_{0,0} \le \frac{C}{\lambda_{m0}^2(\lambda_{m0}-\delta_0)}|f|^{(\delta_0)}_{3,\alpha;\widetilde{\Omega}} \tag{3.56}$$

and

$$\lim_{z_1\to\infty} Z'_{m0}(z_1) = 0. \tag{3.57}$$

(C)
$$|Z^i_{mn}(z_1)|^{(\delta_0)}_{0,0} \le \frac{C}{\lambda_{mn}^2 mn^2(\lambda_{mn}-\delta_0)}(|\widetilde{f}|^{(\delta_0)}_{4,\alpha;\widetilde{\Omega}} + |\widetilde{g}|^{(\delta_0)}_{5,\alpha;\widetilde{\Omega}}), \tag{3.58}$$

$$|(Z^i_{mn})'(z_1)|^{(\delta_0)}_{0,0} \le \frac{C}{\lambda_{mn} mn^2(\lambda_{mn}-\delta_0)}(|\widetilde{f}|^{(\delta_0)}_{4,\alpha;\widetilde{\Omega}} + |\widetilde{g}|^{(\delta_0)}_{5,\alpha;\widetilde{\Omega}}) \tag{3.59}$$

and

$$\lim_{z_1\to\infty} (Z^i_{mn})'(z_1) = 0. \tag{3.60}$$

Based on (A)–(C), we now show that the formal solution (3.46) is actually a classical solution of (3.45). For the convenience, we set

$$v(z) = Z_{00}(z_1) + \mathrm{I}(z), \tag{3.61}$$

where $\mathrm{I}(z)=\sum\limits_{k=1}^{3}\mathrm{I}_k(z)$ with

$$\begin{aligned}\mathrm{I}_1(z)&\equiv \mathrm{I}_1(z_1,r)=\sum_{m=1}^{\infty}Z_{m0}(z_1)R_{m0}(r),\\ \mathrm{I}_2(z)&=\sum_{m,n=1}^{\infty}Z^1_{mn}(z_1)R_{mn}(r)\cos n\theta,\\ \mathrm{I}_3(z)&=\sum_{m,n=1}^{\infty}Z^2_{mn}(z_1)R_{mn}(r)\sin n\theta.\end{aligned}$$

We now show that $\mathrm{I}_k(z)$ $(1\le k\le 3)$ is convergent for $(z_1,r,\theta)\in(-\infty,+\infty)\times[0,1]\times[0,2\pi]$. Indeed, by (A)–(C), we have

$$|\mathrm{I}_1(z)|\le\sum_{m=1}^{+\infty}\frac{C\sqrt{\lambda_{m0}}}{\lambda_{m0}^3(\lambda_{m0}-\delta_0)}|f|^{(\delta_0)}_{3,\alpha}\,\mathrm{e}^{-\delta_0 z_1}\le C|f|^{(\delta_0)}_{4,\alpha;\widetilde{\Omega}}\,\mathrm{e}^{-\delta_0 z_1},\tag{3.62}$$

$$\begin{aligned}|\mathrm{I}_2(z)|+|\mathrm{I}_3(z)|&\le\sum_{m,n=1}^{+\infty}\frac{C\sqrt{\lambda_{mn}}}{\lambda_{mn}m^2n^2(\lambda_{mn}-\delta_0)}(|\widetilde{f}|^{(\delta_0)}_{4,\alpha;\widetilde{\Omega}}+|\widetilde{g}|^{(\delta_0)}_{5,\alpha;\widetilde{\Omega}})\,\mathrm{e}^{-\delta_0|z_1|}\\&\le C(|\widetilde{f}|^{(\delta_0)}_{4,\alpha;\widetilde{\Omega}}+|\widetilde{g}|^{(\delta_0)}_{5,\alpha;\widetilde{\Omega}})\,\mathrm{e}^{-\delta_0|z_1|}.\end{aligned}\tag{3.63}$$

Then

$$|\mathrm{I}_k(z)|^{(\delta_0)}_{0,0}\le C(|\widetilde{f}|^{(\delta_0)}_{4,\alpha;\widetilde{\Omega}}+|\widetilde{g}|^{(\delta_0)}_{5,\alpha;\widetilde{\Omega}})\quad\text{for } k=2,3.\tag{3.64}$$

Thus, the series $\mathrm{I}(z)$ and further $v(z)$ are continuous due to the uniform convergence of $\mathrm{I}_k(z)$ in any compact subset of $\widetilde{\Omega}=(-\infty,+\infty)\times[0,1]\times(-\infty,+\infty)$.

Next, we show $\mathrm{I}(z)\in C^1(\widetilde{\Omega})$ and further $v(z)\in C^1(\widetilde{\Omega})$.

Note

$$|\partial_{z_1}\mathrm{I}_1(z)|\le\sum_{m=1}^{+\infty}|Z'_{m0}(z_1)|\le\sum_{m=1}^{+\infty}\frac{C\sqrt{\lambda_{m0}}}{\lambda_{m0}^2(\lambda_{m0}-\delta_0)}|f|^{(\delta_0)}_{4,\alpha;\widetilde{\Omega}}\,\mathrm{e}^{-\delta_0|z_1|}\le C|f|^{(\delta_0)}_{4,\alpha;\widetilde{\Omega}}\,\mathrm{e}^{-\delta_0|z_1|}.$$

It follows from Lemma 3.3 that

$$\begin{aligned}|\partial_{z_2}\mathrm{I}_1(z)|&=\Big|\partial_r\mathrm{I}_1(z)\cos\theta-\frac{1}{r}\partial_\theta\mathrm{I}_1(z)\sin\theta\Big|\le\sum_{m=1}^{+\infty}\lambda_{m0}^{\frac{3}{2}}|Z_{m0}(z_1)|\\&\le C\sum_{m=1}^{+\infty}\frac{C\sqrt{\lambda_{m0}}}{\lambda_{m0}^2(\lambda_{m0}-\delta_0)}|f|^{(\delta_0)}_{4,\alpha;\widetilde{\Omega}}\,\mathrm{e}^{-\delta_0|z_1|}\le C|f|^{(\delta_0)}_{4,\alpha;\widetilde{\Omega}}\,\mathrm{e}^{-\delta_0|z_1|}.\end{aligned}$$

Analogously,

$$|\partial_{z_3}\mathrm{I}_1(z)|\le C|f|^{(\delta_0)}_{4,\alpha;\widetilde{\Omega}}\,\mathrm{e}^{-\delta_0|z_1|}.$$

Therefore, $\mathrm{I}_1(z)\in C^1(\widetilde{\Omega})$ holds true, and admits the following estimate:

$$|\nabla_z\mathrm{I}_1(z)|^{(\delta_0)}_{0,0}\le C|f|^{(\delta_0)}_{4,\alpha;\widetilde{\Omega}}.$$

Analogously, $\mathrm{I}_k(z)\in C^1(\widetilde{\Omega})$ $(k=2,3)$ holds true. Moreover, we have

$$|\nabla_z\mathrm{I}(z)|\le C(|\widetilde{f}|^{(\delta_0)}_{4,\alpha;\widetilde{\Omega}}+|\widetilde{g}|^{(\delta_0)}_{5,\alpha;\widetilde{\Omega}})\,\mathrm{e}^{-\delta_0|z_1|}$$

and

$$|\nabla_z \mathrm{I}(z)|_{0,0}^{(\delta_0)} \le C(|\widetilde{f}|_{4,\alpha;\widetilde{\Omega}}^{(\delta_0)} + |\widetilde{g}|_{5,\alpha;\widetilde{\Omega}}^{(\delta_0)}). \tag{3.65}$$

Finally, we show $\mathrm{I}(z) \in C^2(\widetilde{\Omega})$ and further $v(z) \in C^2(\widetilde{\Omega})$.

By use of the expression of $\mathrm{I}_1(z)$ and (3.13), we have

$$\partial_{z_1}^2 \mathrm{I}_1(z) = \sum_{m=1}^{+\infty} Z''_{m0}(z_1) R_{m0}(r) = \sum_{m=1}^{+\infty} (\lambda_{m0}^2 Z_{m0}(z_1) + f_{m0}(z_1)) R_{m0}(r).$$

It follows from Lemma 3.3 and (B) that

$$\begin{aligned}|\partial_{z_1}^2 \mathrm{I}_1(z)| &\le \sum_{m=1}^{+\infty} \Big(\frac{C}{\sqrt{\lambda_{m0}}(\lambda_{m0}-\delta_0)} + \frac{C\sqrt{\lambda_{m0}}}{\lambda_{m0}^2} \Big) (|\widetilde{f}|_{4,\alpha;\widetilde{\Omega}}^{(\delta_0)} + |\widetilde{g}|_{5,\alpha;\widetilde{\Omega}}^{(\delta_0)})\, \mathrm{e}^{-\delta_0|z_1|} \\ &\le C(|\widetilde{f}|_{4,\alpha;\widetilde{\Omega}}^{(\delta_0)} + |\widetilde{g}|_{5,\alpha;\widetilde{\Omega}}^{(\delta_0)})\, \mathrm{e}^{-\delta_0|z_1|}.\end{aligned}$$

Analogously, one has

$$\begin{aligned}|\partial_{z_1}^2 \mathrm{I}_2(z)| + |\partial_{z_1}^2 \mathrm{I}_3(z)| &\le \sum_{m,n=1}^{+\infty} \Big(\frac{C\lambda_{mn}^{\frac{3}{2}}}{m^2n^2(\lambda_{mn}-\delta_0)} + \frac{C\sqrt{\lambda_{mn}}}{m^2n^2} \Big) (|\widetilde{f}|_{4,\alpha;\widetilde{\Omega}}^{(\delta_0)} + |\widetilde{g}|_{5,\alpha;\widetilde{\Omega}}^{(\delta_0)})\, \mathrm{e}^{-\delta_0|z_1|} \\ &\le C(|\widetilde{f}|_{4,\alpha;\widetilde{\Omega}}^{(\delta_0)} + |\widetilde{g}|_{5,\alpha;\widetilde{\Omega}}^{(\delta_0)})\, \mathrm{e}^{-\delta_0|z_1|}.\end{aligned}$$

This derives

$$|\partial_{z_1}^2 \mathrm{I}(z)|_{0,0}^{(\delta_0)} \le C(|\widetilde{f}|_{4,\alpha;\widetilde{\Omega}}^{(\delta_0)} + |\widetilde{g}|_{5,\alpha;\widetilde{\Omega}}^{(\delta_0)}).$$

Similarly, we can arrive at

$$|\partial_{z_1z_j}^2 \mathrm{I}(z)|_{0,0}^{(\delta_0)} = |\partial_{z_jz_1}^2 \mathrm{I}(z)|_{0,0}^{(\delta_0)} \le C(|\widetilde{f}|_{4,\alpha;\widetilde{\Omega}}^{(\delta_0)} + |\widetilde{g}|_{5,\alpha;\widetilde{\Omega}}^{(\delta_0)}) \quad \text{for } 1 < j \le 3. \tag{3.66}$$

Next, we treat $\partial_{z_iz_j}^2 \mathrm{I}(z)$ $(i,j=2,3)$.

Since

$$\partial_{z_2}^2 = \cos^2\theta\partial_r^2 - \frac{2}{r}\sin\theta\cos\theta\partial_{r\theta}^2 + \frac{1}{r^2}\sin^2\theta\partial_\theta^2 + \frac{2}{r^2}\sin\theta\cos\theta\partial_\theta + \frac{1}{r}\sin^2\theta\partial_r,$$

we have

$$\begin{aligned}|\partial_{z_2}^2 \mathrm{I}_1(z)| &= \Big| \sum_{m=1}^{+\infty} Z_{m0}(z_1)\Big(R''_{m0}(r)\cos^2\theta + \frac{1}{r}R'_{m0}(r)\sin^2\theta \Big) \Big| \\ &\le \sum_{m=1}^{+\infty} \frac{C}{\sqrt{\lambda_{m0}}(\lambda_{m0}-\delta_0)} (|\widetilde{f}|_{4,\alpha;\widetilde{\Omega}}^{(\delta_0)} + |\widetilde{g}|_{5,\alpha;\widetilde{\Omega}}^{(\delta_0)}) \le C(|\widetilde{f}|_{4,\alpha;\widetilde{\Omega}}^{(\delta_0)} + |\widetilde{g}|_{5,\alpha;\widetilde{\Omega}}^{(\delta_0)}).\end{aligned} \tag{3.67}$$

Next, we consider $\partial_{z_2}^2 \mathrm{I}_2(z)$.

For $n=1$, it follows from $\frac{1}{r}R'_{m1}(r) - \frac{1}{r^2}R_{m1}(r) = -R''_{m1}(r) - \lambda_{m1}^2 R_{m1}(r)$ that

$$\begin{aligned}|\partial_{z_2}^2 (R_{m1}(r)\cos\theta)| &= \Big| \cos^3\theta R''_{m1}(r) + \frac{3}{r}\sin^2\theta\cos\theta R'_{m1}(r) - \frac{3}{r^2}\sin^2\theta\cos\theta R_{m1}(r) \Big| \\ &= |\cos^3\theta R''_{m1}(r) - 3\sin^2\theta\cos\theta(R''_{m1}(r) - \lambda_{m1}^2 R_{m1}(r))| \le C\lambda_{m1}^{\frac{5}{2}}.\end{aligned} \tag{3.68}$$

Analogously,

$$|\partial_{z_2}^2(R_{m1}(r)\sin\theta)| \le C\lambda_{m1}^{\frac{5}{2}}. \tag{3.69}$$

For $n \ge 2$, it follows from Lemma 3.2 that

$$\begin{aligned}|\partial_{z_2}^2(R_{mn}(r)\cos n\theta)| = \Big|&\cos^2\theta\cos n\theta R''_{mn}(r) + n\sin 2\theta\Big(\sin n\theta\frac{1}{r}R'_{mn}(r)\\ &-\cos n\theta\frac{1}{r^2}R_{mn}(r)\Big) + \sin^2\theta\cos n\theta\Big(\frac{1}{r}R'_{mn}(r) - \frac{n^2}{r^2}R_{mn}\Big)\Big|\\ \le\ &\frac{C}{m}\lambda_{mn}^{\frac{7}{2}}.\end{aligned} \tag{3.70}$$

Analogously,

$$|\partial_{z_2}^2(R_{mn}(r)\sin n\theta)| \le \frac{C}{m}\lambda_{mn}^{\frac{7}{2}}. \tag{3.71}$$

Thus,

$$|\partial_{z_2}^2\mathrm{I}_2(r)| \le \sum_{m,n=1}^{+\infty}\frac{C\lambda_{mn}^{\frac{3}{2}}}{m^2n^2(\lambda_{mn}-\delta_0)}(|\widetilde{f}|_{4,\alpha;\widetilde{\Omega}}^{(\delta_0)} + |\widetilde{g}|_{5,\alpha;\widetilde{\Omega}}^{(\delta_0)}) \le C(|\widetilde{f}|_{4,\alpha;\widetilde{\Omega}}^{(\delta_0)} + |\widetilde{g}|_{5,\alpha;\widetilde{\Omega}}^{(\delta_0)}). \tag{3.72}$$

Analogously,

$$|\partial_{z_iz_j}^2\mathrm{I}_2(r)| + |\partial_{z_iz_j}^2\mathrm{I}_3(r)| \le C(|\widetilde{f}|_{4,\alpha;\widetilde{\Omega}}^{(\delta_0)} + |\widetilde{g}|_{5,\alpha;\widetilde{\Omega}}^{(\delta_0)}), \quad i,j=2,3. \tag{3.73}$$

This yields

$$|\partial_{z_iz_j}^2\mathrm{I}(z)|_{0,0}^{(\delta_0)} \le C(|\widetilde{f}|_{4,\alpha;\widetilde{\Omega}}^{(\delta_0)} + |\widetilde{g}|_{5,\alpha;\widetilde{\Omega}}^{(\delta_0)}) \quad \text{for } 1\le i,j\le 3. \tag{3.74}$$

Collecting the estimates above yield (3.38).

On the other hand, we have

$$\lim_{z_1\to+\infty}\partial_{z_1}u(z) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_0^1\int_0^{2\pi}\widetilde{r}f(t,r,\theta)\mathrm{d}\theta\mathrm{d}r\mathrm{d}t - \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_0^{2\pi}\widetilde{g}(t,\theta)\mathrm{d}\theta\mathrm{d}t \tag{3.75}$$

and

$$\lim_{z_1\to+\infty}\partial_{z_k}u(z) = 0 \quad \text{for } k=2,3.$$

Thus we completed the proof of Lemma 3.4.

Next, we intend to establish the higher regularities and higher order norm (i.e., $\|u\|_{6,\alpha}^{(\delta_0)}$) estimates of $u(z)$ to (3.3) and further treat the nonlinear problem (3.1) in the unbounded nozzle domain $\widetilde{\Omega}$. For notational convenience, we use a weighted Hölder norm which is introduced in [12, Chapter 6] and the references therein as follows.

Let $D \subset \mathbb{R}^3$ be an open set. For $x, y \in D$, we define $r_{x,y} = \min(|x|, |y|)$. For $m \in \mathbb{N}\cup\{0\}$,

$\alpha \in \mathbb{R}^+$, $\mu \in \mathbb{R}^+$, $\mu_1, \mu_2 \in \mathbb{R}$ and $v \in C^{m,\alpha}(D)$, we define

$$
\begin{aligned}
{[v]]}^{(\mu)}_{m,0;D} &\equiv \sum_{|\beta|=m} \sup_{x\in D} |x|^{m+\mu} |D^\beta v(x)|,\\
{[v]]}^{(\mu)}_{m,\alpha;D} &\equiv \sum_{|\beta|=m} \sup_{\substack{x,y\in D\\ x\neq y}} r_{x,y}^{m+\alpha+\mu} \frac{|D^\beta v(x) - D^\beta v(y)|}{|x-y|^\alpha},\\
|v||^{(\mu)}_{m,\alpha;D} &\equiv \sum_{0\le i\le m} [v]]^{(\mu)}_{i,0;D} + [v]]^{(\mu)}_{m,\alpha;D},\\
[[v]]^{(\mu_1,\mu_2)}_{m,0;D} &\equiv \max\Big\{ \sup_{|x|<1} \sum_{|\beta|=m} |x|^{m+\mu_1} |D^\beta v(x)|,\ \sup_{|x|>1} \sum_{|\beta|=m} |x|^{m+\mu_2} |D^\beta v(x)| \Big\},\\
[[v]]^{(\mu_1,\mu_2)}_{m,\alpha;D} &\equiv \max\Big\{ \sup_{0<r_{x,y}<1} \sum_{|\beta|=m} r_{x,y}^{\mu_1+m+\alpha} \frac{|D^\beta v(x) - D^\beta v(y)|}{|x-y|^\alpha},\\
&\qquad\qquad \sup_{r_{x,y}>1} \sum_{|\beta|=m} r_{x,y}^{\mu_2+m+\alpha} \frac{|D^\beta v(x) - D^\beta v(y)|}{|x-y|^\alpha} \Big\},\\
|||v|||^{(\mu_1,\mu_2)}_{m,\alpha;D} &\equiv \sum_{0\le i\le m} [[v]]^{(\mu_1,\mu_2)}_{i,0;D} + [[v]]^{(\mu_1,\mu_2)}_{m,\alpha;D}.
\end{aligned}
$$

Now let us consider the following equation:

$$
\begin{cases}
\partial^2_{z_1} w + \partial^2_r w + \dfrac{1}{r}\partial_r w + \dfrac{1}{r^2}\partial^2_\theta w = \widehat{f}, \quad \text{in } \widetilde{\Omega},\\
\partial_r w = 0, \quad \text{on } r = 1,\\
\lim\limits_{z_1\to-\infty} w(z) = \lim\limits_{z_1\to+\infty} w(z) = 0,
\end{cases}
\tag{3.76}
$$

where $\widehat{f} \in H^{(\delta_0)}_{2,\alpha}(\widetilde{\Omega})$.

Lemma 3.5 *If $w \in H^{(\delta_0)}_{4,\alpha}(\widetilde{\Omega})$ is a solution of* (3.76)*, which satisfies*

$$
\sup_{z\in\widetilde{\Omega}} (\mathrm{e}^{\delta_0|z_1|} |w(z)|) \le C|\widehat{f}|^{(\delta_0)}_{2,\alpha;\widetilde{\Omega}},
\tag{3.77}
$$

then we have

$$
|w|^{(\delta_0)}_{4,\alpha;\widetilde{\Omega}} \le C|\widehat{f}|^{(\delta_0)}_{2,\alpha;\widetilde{\Omega}}.
$$

Proof First, we introduce such a coordinate transformation:

$$
y_1 = \mathrm{e}^{z_1}\cos r, \quad y_2 = \mathrm{e}^{z_1}\cos\theta\sin r, \quad y_3 = \mathrm{e}^{z_1}\sin\theta\sin r.
\tag{3.78}
$$

In this case, the cylindrical domain $\widetilde{\Omega}$ is changed into an unbounded domain D which is bounded by infinite cone $\{y : y_1 = (\cot 1)\sqrt{y_2^2 + y_3^2}\,\}$.

It follows from the transformation (3.78) that (3.76) can be changed into the following problem:

$$
\begin{cases}
\sum\limits_{i,j=1}^{3} \widetilde{a}_{ij}(y)\partial_{ij} w + \sum\limits_{i=1}^{3} \widetilde{b}_i(y)\partial_i w = F(y), \quad \text{in } D \equiv \{y : 0 \le y_1 < (\cot 1)\sqrt{y_2^2 + y_3^2}\,\},\\
\partial_n w = 0, \quad \text{on } y_1 = (\cot 1)\sqrt{y_2^2 + y_3^2},\\
w(0,0,0) = \lim\limits_{r\to+\infty} w(z_1, z_2, z_3) = 0,
\end{cases}
\tag{3.79}
$$

where

$$\widetilde{a}_{22}=\frac{1}{|y|^2}\Big(y_2^2+\frac{y_1^2y_2^2}{y_2^2+y_3^2}+\frac{y_3^2}{\arctan^2\frac{\sqrt{y_2^2+y_3^2}}{y_1}}\Big),\quad \widetilde{a}_{33}=\frac{1}{|y|^2}\Big(y_3^2+\frac{y_1^2y_3^2}{y_2^2+y_3^2}+\frac{y_2^2}{\arctan^2\frac{\sqrt{y_2^2+y_3^2}}{y_1}}\Big),$$

$$\widetilde{a}_{11}=1,\quad \widetilde{a}_{12}=\widetilde{a}_{21}=\widetilde{a}_{13}=\widetilde{a}_{31}=0,\quad \widetilde{a}_{23}=\widetilde{a}_{32}=\frac{1}{|y|^2}\Big(y_2y_3+\frac{y_1^2y_2y_3}{y_2^2+y_3^2}-\frac{y_2y_3}{\arctan^2\frac{\sqrt{y_2^2+y_3^2}}{y_1}}\Big),$$

$$\widetilde{b}_1=-\frac{1}{|y|^2}\frac{\sqrt{y_2^2+y_3^2}}{\arctan\frac{\sqrt{y_2^2+y_3^2}}{y_1}},\quad \widetilde{b}_2=\frac{1}{|y|^2}\Big(\frac{y_1y_2}{\sqrt{y_2^2+y_3^2}\arctan\frac{\sqrt{y_2^2+y_3^2}}{y_1}}-\frac{y_2}{\arctan^2\frac{\sqrt{y_2^2+y_3^2}}{y_1}}\Big),$$

$$\widetilde{b}_3=\frac{1}{|y|^2}\Big(\frac{y_1y_3}{\sqrt{y_2^2+y_3^2}\arctan\frac{\sqrt{y_2^2+y_3^2}}{y_1}}-\frac{y_3}{\arctan^2\frac{\sqrt{y_2^2+y_3^2}}{y_1}}\Big),\quad F=\frac{1}{|y|^2}\widehat{f}.$$

In addition, it follows from (3.77) and the transformation (3.78) that

$$\sup|y|^{\delta_0}|w|\le C|\widehat{f}|^{(\delta_0)}_{2,\alpha;\widetilde{\Omega}}.\tag{3.80}$$

As in [25, 26], we can show the following estimate

$$|w\|^{(\delta_0)}_{4,\alpha;D}\le C|\widehat{f}|^{(\delta_0)}_{2,\alpha;\widetilde{\Omega}}.\tag{3.81}$$

Returning to the coordinate $z=(z_1,z_2,z_3)$ for $[D^4w]]^{(\delta_0)}_{0,\alpha;D}$, we can derive

$$\sum_{|\beta|\le 4}\sup_{z\in\widetilde{\Omega}}\mathrm{e}^{\delta_0z_1}|D^\beta w(z)|+\sum_{|\beta|=4}\sup_{\substack{z,\widetilde{z}\in\widetilde{\Omega}\\ z\ne\widetilde{z}}}\mathrm{e}^{\delta_0\min(z_1,\widetilde{z}_1)}\frac{|D_z^\beta w(z)-D_z^\beta w(\widetilde{z})|}{|z-\widetilde{z}|^\alpha}\le C|\widehat{f}|^{(\delta_0)}_{2,\alpha;\widetilde{\Omega}}.\tag{3.82}$$

On the other hand, if we introduce such a coordinate transformation:

$$y_1=\mathrm{e}^{-z_1}\cos r,\quad y_2=\mathrm{e}^{-z_1}\cos\theta\sin r,\quad y_3=\mathrm{e}^{-z_1}\sin\theta\sin r,\tag{3.83}$$

then by using the same method to deduce (3.82), we can arrive at

$$\sum_{|\beta|\le 4}\sup_{z\in\widetilde{\Omega}}\mathrm{e}^{\delta_0z_1}|D^\beta w(z)|+\sum_{|\beta|=4}\sup_{\substack{z,\widetilde{z}\in\widetilde{\Omega}\\ z\ne\widetilde{z}}}\mathrm{e}^{\delta_0\max(z_1,\widetilde{z}_1)}\frac{|D_z^\beta w(z)-D_z^\beta w(\widetilde{z})|}{|z-\widetilde{z}|^\alpha}\le C|\widehat{f}|^{(\delta_0)}_{2,\alpha;\widetilde{\Omega}}.\tag{3.84}$$

Combining (3.82) with (3.84) yields

$$[w]^{(\delta_0)}_{i,0;\widetilde{\Omega}}\le C|\widehat{f}|^{(\delta_0)}_{2,\alpha;\widetilde{\Omega}},\quad i=1,2,\cdots,4\tag{3.85}$$

and

$$\sup_{z_1\widetilde{z}_1>0}\mathrm{e}^{\delta_0\min\{|z_1|,|\widetilde{z}_1|\}}\frac{|D^4w(z)-D^4w(\widetilde{z})|}{|z-\widetilde{z}|^\alpha}\le C|\widehat{f}|^{(\delta_0)}_{2,\alpha;\widetilde{\Omega}}.\tag{3.86}$$

For the case of $z_1\widetilde{z}_1<0$ in (3.86), it follows from an analogous analysis that we can get

$$\mathrm{e}^{\delta_0\min\{-z_1,\widetilde{z}_1\}}\frac{|D^4w(z)-D^4w(\widetilde{z})|}{|z-\widetilde{z}|^\alpha}\le C|\widehat{f}|^{(\delta_0)}_{2,\alpha;\widetilde{\Omega}}.$$

Thus, it is proved that

$$[w]^{(\delta_0)}_{4,\alpha;\widetilde{\Omega}}\le C|\widehat{f}|^{(\delta_0)}_{2,\alpha;\widetilde{\Omega}}.\tag{3.87}$$

Namely, by use of (3.80) and (3.87), we complete the proof of Lemma 3.5.

Based on Lemmas 3.4 and 3.5, we now give the estimate of $\|u\|^{(\delta_0)}_{6,\alpha;\widetilde{\Omega}}$ to the solution of (3.3).

Lemma 3.6 *Under the assumptions of Lemma* 3.4*, the solution* $u(z)$ *of* (3.3) *satisfies the following estimate:*

$$\|u\|^{(\delta_0)}_{6,\alpha;\widetilde{\Omega}} \le C(|\widetilde{f}|^{(\delta_0)}_{4,\alpha;\widetilde{\Omega}} + |\widetilde{g}|^{(\delta_0)}_{5,\alpha;\widetilde{\Omega}}).$$

Proof In order to prove Lemma 3.6, by use of Lemma 3.4, it suffices to prove

$$\begin{cases} |\partial^2_{z_1} u|^{(\delta_0)}_{4,\alpha;\widetilde{\Omega}} \le C(|\widetilde{f}|^{(\delta_0)}_{4,\alpha;\widetilde{\Omega}} + |\widetilde{g}|^{(\delta_0)}_{5,\alpha;\widetilde{\Omega}}), \\ |\partial_{z_k} u|^{(\delta_0)}_{5,\alpha;\widetilde{\Omega}} \le C(|\widetilde{f}|^{(\delta_0)}_{4,\alpha;\widetilde{\Omega}} + |\widetilde{g}|^{(\delta_0)}_{5,\alpha;\widetilde{\Omega}}), \quad k = 2, 3. \end{cases} \tag{3.88}$$

Set $w(z) = \partial^2_{z_1} v$, where $v(z)$ is a solution of (3.45). Then it is easy to know that $w(z)$ satisfies the equation (3.76) with $\widehat{f}(z) = \partial^2_{z_1} f(z)$. Therefore, by Lemma 3.5, we have

$$|\partial^2_{z_1} v|^{(\delta_0)}_{4,\alpha;\widetilde{\Omega}} \le C|\widehat{f}|^{(\delta_0)}_{2,\alpha;\widetilde{\Omega}} \le C|f|^{(\delta_0)}_{4,\alpha;\widetilde{\Omega}}. \tag{3.89}$$

Due to $u(z) = v(z) + h(z)$ with $h(z) = r\widetilde{g}(z_1, \theta)$, then it follows from (3.89) and the interior estimate on the second order elliptic equation that

$$|\partial^2_{z_1} u|^{(\delta_0)}_{4,\alpha;\widetilde{\Omega}} \le C(|\widetilde{f}|^{(\delta_0)}_{4,\alpha;\widetilde{\Omega}} + |\widetilde{g}|^{(\delta_0)}_{5,\alpha;\widetilde{\Omega}}). \tag{3.90}$$

Next, we consider $|\partial_{z_k} u|^{(\delta_0)}_{5,\alpha;\widetilde{\Omega}}$ $(k = 2, 3)$. For this end, we will divide this procedure into three steps.

Step 1 Multiplying r on two hand sides of equation (3.45) and subsequently taking the partial derivative on r, we have

$$\partial^2_{z_1}(r\partial_r v) + \partial^2_r(r\partial_r v) + \frac{1}{r^2}\partial^2_\theta(r\partial_r v) + \partial^2_{z_1} v - \frac{1}{r^2}\partial^2_\theta v = f + r\partial_r f. \tag{3.91}$$

Set $w = r\partial_r v$. Then by use of (3.45) and (3.91), we have

$$\begin{cases} \partial^2_{z_1} w + \partial^2_r w + \dfrac{1}{r}\partial_r w + \dfrac{1}{r^2}\partial^2_\theta w = 2f + r\partial_r f - 2\partial^2_{z_1} v, \quad \text{in } \widetilde{\Omega}, \\ w = 0, \quad \text{on } r = 1, \\ \lim\limits_{z_1 \to -\infty} w = \lim\limits_{z_1 \to +\infty} w = 0. \end{cases} \tag{3.92}$$

By use of the similar method in Lemma 3.5, we can arrive at

$$|r\partial_r v|^{(\delta_0)}_{5,\alpha;\widetilde{\Omega}} \le C|2f + r\partial_r f - 2\partial^2_{z_1} v|^{(\delta_0)}_{3,\alpha;\widetilde{\Omega}} \le C|f|^{(\delta_0)}_{4,\alpha;\widetilde{\Omega}}$$

and further

$$|r\partial_r u|^{(\delta_0)}_{5,\alpha;\widetilde{\Omega}} \le C(|\widetilde{f}|^{(\delta_0)}_{4,\alpha;\widetilde{\Omega}} + |\widetilde{g}|^{(\delta_0)}_{5,\alpha;\widetilde{\Omega}}). \tag{3.93}$$

Step 2 Differentiating (3.45) with respect to θ and writing $\widetilde{w} = \partial_\theta v$, we have

$$\begin{cases} \partial^2_{z_1} \widetilde{w} + \partial^2_r \widetilde{w} + \dfrac{1}{r}\widetilde{w} + \dfrac{1}{r^2}\partial^2_\theta \widetilde{w} = \partial_\theta f, \quad \text{in } \widetilde{\Omega}, \\ \partial_r \widetilde{w} = 0, \quad \text{on } r = 1, \\ \lim\limits_{z_1 \to -\infty} \widetilde{w} = \lim\limits_{z_1 \to +\infty} \widetilde{w} = 0. \end{cases} \tag{3.94}$$

Analogous to the proof in Step 1, we can obtain

$$|\widetilde{w}|^{(\delta_0)}_{5,\alpha;\widetilde{\Omega}} \le C(|\widetilde{f}|^{(\delta_0)}_{4,\alpha;\widetilde{\Omega}} + |\widetilde{g}|^{(\delta_0)}_{5,\alpha;\widetilde{\Omega}}). \tag{3.95}$$

Thus, combining (3.93) with (3.95), we have for $\frac{1}{2} \le r \le 1$,

$$|\partial_{z_i} u|^{(\delta_0)}_{5,\alpha;\{z\in\widetilde{\Omega}:\frac{1}{2}\le\sqrt{z_2^2+z_3^2}\le 1\}} \le C(|\widetilde{f}|^{(\delta_0)}_{4,\alpha;\widetilde{\Omega}} + |\widetilde{g}|^{(\delta_0)}_{5,\alpha;\widetilde{\Omega}}). \tag{3.96}$$

Step 3 In order to treat the singularity of solution on $r = 0$, which is induced by the cylindrical coordinate transformation, we return to the original equation (3.45).

Differentiating (3.45) with respect to z_2 and writing $\overline{w} = \partial_{z_2} v$, we obtain

$$\begin{cases} \Delta\overline{w} = \partial_{z_2} f, \quad \text{in } \sqrt{z_2^2+z_3^2} \le \dfrac{3}{4}, \\ \overline{w}(z) = \partial_{z_2} v(z), \quad \text{on } \sqrt{z_2^2+z_3^2} = \dfrac{3}{4}, \\ \lim\limits_{z_1\to\pm\infty} \overline{w} = 0. \end{cases} \tag{3.97}$$

Similarly to the proof on Lemma 3.5, and by use of (3.97), one has

$$|\partial_{z_2} u|^{(\delta_0)}_{5,\alpha;\{z\in\widetilde{\Omega}:\sqrt{z_2^2+z_3^2}\le\frac{3}{4}\}} \le C(|\widetilde{f}|^{(\delta_0)}_{4,\alpha;\widetilde{\Omega}} + |\widetilde{g}|^{(\delta_0)}_{5,\alpha;\widetilde{\Omega}}). \tag{3.98}$$

Combining (3.96) with (3.98) yields

$$|\partial_{z_2} u|^{(\delta_0)}_{5,\alpha;\widetilde{\Omega}} \le C(|\widetilde{f}|^{(\delta_0)}_{4,\alpha;\widetilde{\Omega}} + |\widetilde{g}|^{(\delta_0)}_{5,\alpha;\widetilde{\Omega}}). \tag{3.99}$$

Analogously,

$$|\partial_{z_3} u|^{(\delta_0)}_{5,\alpha;\widetilde{\Omega}} \le C(|\widetilde{f}|^{(\delta_0)}_{4,\alpha;\widetilde{\Omega}} + |\widetilde{g}|^{(\delta_0)}_{5,\alpha;\widetilde{\Omega}}). \tag{3.100}$$

Thus, we complete the proof of Lemma 3.6.

Based on Lemma 3.6, we now derive the uniform estimates on the solution $\dot{u}(z)$ to problem (3.1).

Lemma 3.7 *Suppose that the assumption* (3.2) *holds true, and* $\dot{u} \in C^2(\overline{\widetilde{\Omega}})$ *is a solution of* (3.1). *Then there exists a positive constant* δ_0 *such that for any* $\dot{f} \in H^{(\delta_0)}_{4,\alpha}(\widetilde{\Omega})$, $\dot{g} \in H^{(\delta_0)}_{5,\alpha}(\widetilde{\Omega})$, *we have* $\dot{u} \in \mathbb{H}^{(\delta_0)}_{6,\alpha}(\widetilde{\Omega})$ *with*

$$\|\dot{u}\|^{(\delta_0)}_{6,\alpha;\widetilde{\Omega}} \le C(|\dot{f}|^{(\delta_0)}_{4,\alpha;\widetilde{\Omega}} + |\dot{g}|^{(\delta_0)}_{5,\alpha;\widetilde{\Omega}}), \tag{3.101}$$

where $C > 0$ *depends only on the constants* Λ *and* λ *in* (3.2).

Proof Firstly, we introduce a coordinate transformation as follows:

$$\widetilde{z}_1 = k_1 z_1, \quad \widetilde{z}_2 = k_2 z_2, \quad \widetilde{z}_3 = k_3 z_3 \tag{3.102}$$

with $k_1 = \frac{1}{\sqrt{c^2(\rho_0)-q_0^2}}$ and $k_2 = k_3 = \frac{1}{c(\rho_0)}$.

Under this transformation, the domain $\widetilde{\Omega}$ is changed into the domain $Q \equiv \big\{(\widetilde{z}) : \widetilde{z}_1 \in (-\infty,+\infty), \sqrt{\widetilde{z}_2^2+\widetilde{z}_3^2} \le \frac{1}{c(\rho_0)}\big\}$, and equation (3.1) can be rewritten as

$$\begin{cases} \Delta\dot{u} = \overline{f}, \quad \text{in } Q, \\ \partial_n \dot{u} = \overline{g}, \quad \text{on } \sqrt{z_2^2+z_3^2} = l, \\ \lim\limits_{\widetilde{z}_1\to-\infty} \dot{u} = 0, \quad \lim\limits_{\widetilde{z}_1\to+\infty} \nabla_{\widetilde{z}}\dot{u} \text{ exists}. \end{cases} \tag{3.103}$$

where $l = \frac{1}{c(\rho_0)}$ and $\overline{f} = \dot{f} + \sum_{i=1}^{3}(1 - k_i^2(c^2(\nabla v) - \partial_{z_1}^2 v))\partial_{\tilde{z}_i}^2 \dot{u} - 2\sum_{1\le i<j\le 3} k_i k_j \partial_{z_i} v \partial_{z_j} v \partial_{\tilde{z}_i\tilde{z}_j}\dot{u}$, $\overline{g} = c(\rho_0)\dot{g}$.

For simplicity and without loss of generality, we assume $l = 1$ in (3.103).

By the assumption $\|v - q_0 z_1\|_{6,\alpha;\widetilde{\Omega}}^{(\delta_0)} < \varepsilon$ and Lemma 2.1, we have

$$|\overline{f}|_{4,\alpha;\widetilde{\Omega}}^{(\delta_0)} \le O(\varepsilon)\|\dot{u}\|_{6,\alpha;\widetilde{\Omega}}^{(\delta_0)} + |\dot{f}|_{4,\alpha;\widetilde{\Omega}}^{(\delta_0)}. \tag{3.104}$$

On the other hand, by use of Lemma 3.6, one has

$$\|\dot{u}\|_{6,\alpha;\widetilde{\Omega}}^{(\delta_0)} \le C(|\overline{f}|_{4,\alpha;\widetilde{\Omega}}^{(\delta_0)} + |\dot{g}|_{5,\alpha;\widetilde{\Omega}}^{(\delta_0)}). \tag{3.105}$$

Substituting (3.105) into (3.104) yields (3.101).

Moreover, it follows from (3.75) and (3.76) that

$$\begin{cases} \lim\limits_{z_1\to+\infty} |\partial_{z_1}\dot{u}| \le C(|\dot{f}|_{4,\alpha;\widetilde{\Omega}}^{(\delta_0)} + |\dot{g}|_{5,\alpha;\widetilde{\Omega}}^{(\delta_0)}), \\ \lim\limits_{z_1\to+\infty} \partial_{z_i}\dot{u} = 0, \quad i = 2, 3. \end{cases} \tag{3.106}$$

Therefore, we complete the proof of Lemma 3.7.

Based on Lemma 3.4 and Lemma 3.7, the next theorem follows from the standard continuity method (one can see [12, Theorem 5.2]).

Theorem 3.1 *There exists a unique solution $\dot{u} \in \mathbb{H}_{6,\alpha}^{(\delta_0)}(\widetilde{\Omega})$ to problem* (3.1) *for some $\delta_0 > 0$, which admits the following estimate*

$$\|\dot{u}\|_{6,\alpha;\widetilde{\Omega}}^{(\delta_0)} \le C(|\dot{f}|_{4,\alpha;\widetilde{\Omega}}^{(\delta_0)} + |\dot{g}|_{5,\alpha;\widetilde{\Omega}}^{(\delta_0)}). \tag{3.107}$$

4 The Proofs of Theorems 2.3, 2.2 and 1.1

In this section, first we intend to use the contraction mapping principle to show Theorem 2.3. To this end, we define the space $K = \{\psi(z) : \psi(z) - \varphi_0(z) \in \mathbb{H}_{6,\alpha}^{(\delta_0)}(\widetilde{\Omega}), \|\psi(z) - \varphi_0(z)\|_{6,\alpha;\widetilde{\Omega}}^{(\delta_0)} \le \varepsilon\}$ with $\varphi_0(z) = q_0 z_1$.

Set $\varphi = \dot{\varphi} + \varphi_0$. Then $\dot{\varphi}$ satisfies

$$\begin{cases} L(\psi)\dot{\varphi} = \sum\limits_{i,j=1}^{3} a_{ij}(z, D\psi)\partial_{z_i z_j}^2\dot{\varphi} = \dot{f}(z, D\psi, D^2\psi), \quad \text{in } \widetilde{\Omega}, \\ G(\psi)\dot{\varphi} = \partial_n\dot{\varphi} = \dot{g}(z, D\psi), \quad \text{on } \sqrt{z_2^2 + z_3^2} = 1, \\ \lim\limits_{z_1\to-\infty} \dot{\varphi} = 0, \qquad \lim\limits_{z_1\to+\infty} \nabla_z\dot{\varphi} \ \text{ exists}, \end{cases} \tag{4.1}$$

where

$$\begin{aligned} \dot{f}(z, D\psi, D^2\psi) &= (L(\varphi_0)\varphi_0 - L(\psi)\varphi_0) + \sum_{i,j=1}^{3}(a_{ij}(z,\nabla\psi) - A_{ij}(z,\nabla\psi))\partial_{z_i z_j}\dot{\psi} \\ &\quad - \sum_{i=2}^{3} B_i(z,\nabla\psi)\partial_{z_i}\dot{\psi}, \\ \dot{g}(D\psi) &= -b_1(z)\partial_{z_1}\dot{\psi} + \Big(\frac{z_2}{\sqrt{z_2^2+z_3^2}} - b_2(z)\Big)\partial_{z_2}\dot{\psi} + \Big(\frac{z_3}{\sqrt{z_2^2+z_3^2}} - b_3(z)\Big)\partial_{z_3}\dot{\psi} \end{aligned}$$

with $\dot{\psi} = \psi - \varphi_0$.

Define the nonlinear mapping J by $J(\psi) = \varphi$. Then we have the following lemma.

Lemma 4.1 *Suppose that α and δ_0 are the positive constants given in Lemma* 3.7. *Then there exists an $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$, J is a mapping from K to itself.*

Proof By the definitions of $A_{ij}(z, \nabla\psi)$ and $B(z, \nabla\psi)$ in (2.2), we arrived at

$$|(a_{ij}(z, \nabla\psi) - A_{ij}(z, \nabla\psi))\partial_{z_i z_j}\dot{\psi}|^{(\delta_0)}_{4,\alpha;\widetilde{\Omega}} \le |a_{ij}(z, \nabla\psi) - A_{ij}(z, \nabla\psi)|^{(0)}_{4,\alpha;\widetilde{\Omega}} \|\dot{\psi}\|^{(\delta_0)}_{6,\alpha;\widetilde{\Omega}} \le C\varepsilon^2$$

and

$$|B_i(z, \nabla\psi)\partial_{z_i}\dot{\psi}|^{(\delta_0)}_{4,\alpha;\widetilde{\Omega}} \le |B_i(z, \nabla\psi)|^{(0)}_{4,\alpha;\widetilde{\Omega}} \|\dot{\psi}\|^{(\delta_0)}_{6,\alpha;\widetilde{\Omega}} \le C\varepsilon^2.$$

In addition, by use of $\varphi_0 = q_0 z_1$, we have

$$L(\varphi_0)\varphi_0 - L(\psi)\varphi_0 = 0.$$

Thus, we arrive at

$$|\dot{f}|^{(\delta_0)}_{4,\alpha;\widetilde{\Omega}} \le C\varepsilon^2. \tag{4.2}$$

Analogously, one has

$$|\dot{g}|^{(\delta_0)}_{5,\alpha;\widetilde{\Omega}} \le C\varepsilon^2. \tag{4.3}$$

It follows from Theorem 3.8 that

$$\|\dot{\varphi}\|^{(\delta_0)}_{6,\alpha;\widetilde{\Omega}} \le C(|\dot{f}|^{(\delta_0)}_{4,\alpha;\widetilde{\Omega}} + |\dot{g}|^{(\delta_0)}_{5,\alpha;\widetilde{\Omega}}) \le C\varepsilon^2, \tag{4.4}$$

where $C > 0$ depends only on Λ and λ.

Choosing $\varepsilon_0 = \frac{1}{2C}$, for any $0 < \sigma < \varepsilon < \varepsilon_0$, by (4.4), we obtain

$$\|\dot{\varphi}\|^{(\delta_0)}_{6,\alpha;\widetilde{\Omega}} < \varepsilon. \tag{4.5}$$

This means that the mapping J is from K into itself.

Next we show that the mapping J defined above is contractible.

Lemma 4.2 *Under the assumptions of Lemma* 4.1, *the mapping J is a contractible mapping from K to itself.*

Proof Take $\psi_1, \psi_2 \in K$. Let $\varphi_i = J\psi_i$ and $\dot{\varphi}_i = \varphi_i - \varphi_0$. Then we have

$$\begin{cases} L(\psi_2)(\varphi_2 - \varphi_1) = \dot{f}(z, D\psi_2, D^2\psi_2) - \dot{f}(z, D\psi_1, D^2\psi_1) - (L(\psi_2) - L(\psi_1))\dot{\varphi}_1, \quad \text{in } \widetilde{\Omega}, \\ \partial_n(\varphi_2 - \varphi_1) = \dot{g}(z, \psi_2) - \dot{g}(z, \psi_1), \quad \text{on } z_2^2 + z_3^2 = 1, \\ \lim\limits_{z_1 \to -\infty}(\varphi_2 - \varphi_1) = 0, \quad \lim\limits_{z_1 \to +\infty} \nabla_z(\varphi_2 - \varphi_1) \ \text{ exists.} \end{cases} \tag{4.6}$$

As in Lemma 4.1, a direct computation yields

$$\begin{aligned} &|\dot{f}(z, D\psi_2, D^2\psi_2) - \dot{f}(z, D\psi_1, D^2\psi_1)|^{(\delta_0)}_{4,\alpha;\widetilde{\Omega}} \le C\varepsilon\|\psi_2 - \psi_1\|^{(\delta_0)}_{6,\alpha;\widetilde{\Omega}}, \\ &|\dot{g}(z, \psi_2) - \dot{g}(z, \psi_1)|^{(\delta_0)}_{5,\alpha;\widetilde{\Omega}} \le C\varepsilon\|\psi_2 - \psi_1\|^{(\delta_0)}_{6,\alpha;\widetilde{\Omega}}, \\ &|(L(\psi_2) - L(\psi_1))\dot{\varphi}_1|^{(\delta_0)}_{4,\alpha;\widetilde{\Omega}} \le C\varepsilon\|\psi_2 - \psi_1\|^{(\delta_0)}_{6,\alpha;\widetilde{\Omega}}. \end{aligned}$$

It follows from Theorem 3.8 that

$$\|\varphi_2 - \varphi_1\|^{(\delta_0)}_{6,\alpha;\widetilde{\Omega}} \le C\varepsilon\|\psi_2 - \psi_1\|^{(\delta_0)}_{6,\alpha;\widetilde{\Omega}}.$$

Choosing appropriately small ε_0 and letting $0 < \varepsilon < \varepsilon_0$ yield

$$\|J(\psi_2) - J(\psi_1)\|_{6,\alpha;\widetilde{\Omega}}^{(\delta_0)} \leq \frac{1}{2}\|\psi_2 - \psi_1\|_{6,\alpha;\widetilde{\Omega}}^{(\delta_0)}.$$

This means that J is a contractible mapping.

Based on Lemmas 4.1 and 4.2, we now show Theorem 2.3.

Proof of Theorem 2.3 By Lemmas 4.1 and 4.2, we know that the mapping $J\psi = \varphi$ has a unique fixed point in $\mathbb{H}_{6,\alpha}^{(\delta_0)}(\widetilde{\Omega})$.

Next, we show $\lim\limits_{z_1\to+\infty} \nabla_z\varphi(z)$ exists as in [26].

Since for $Z_1 > Z_2 > 0$, we have $|\partial_{z_1}\varphi(Z_1, z_2, z_3) - \partial_{z_1}\varphi(Z_2, z_2, z_3)| = (Z_1 - Z_2)\left|\int_0^1 \partial_{z_1}^2\varphi(\theta Z_1 + (1-\theta)Z_2, z_2, z_3)\mathrm{d}\theta\right| \leq C\mathrm{e}^{-\delta_0 Z_2}$. This means that there exists a function $q(z_2, z_3)$ such that $\partial_{z_1}\varphi(z_1, z_2, z_3)$ converges to $q(z_2, z_3)$ uniformly as $z_1 \to +\infty$. On the other hand, $|\partial_{z_1 z_k}^2\varphi(z_1, z_2, z_3)| \leq C\mathrm{e}^{-\delta_0 z_1}$ for $k = 2, 3$. Thus, we can derive $q(z_2, z_3) \equiv q$, where q is a constant which will be determined later on. In addition, due to $|\partial_{z_k}\varphi(z)| \leq C\mathrm{e}^{-\delta_0|z_1|}$ $(k = 2, 3)$, then $\lim\limits_{z_1\to\pm\infty} \partial_{z_k}\varphi(z) = 0$. From the analysis above, we can also obtain under the x-coordinates

$$\lim_{x_1\to\infty} \partial_{x_1}\varphi = q \quad \text{and} \quad \lim_{x_1\to\pm\infty} \partial_{x_i}\varphi = 0 \quad \text{for } i = 2, 3. \tag{4.7}$$

We now show that $q = q_0$ holds true.

Integrating the mass conservation equation $\sum\limits_{j=1}^{3} \partial_{x_j}(\rho(|\nabla\varphi|)\partial_{x_j}\varphi) = 0$ in $\Omega_R = \Omega \cap \{x : -R \leq x_1 \leq R\}$ yields

$$0 = -\int_{x_1=-R} \rho(\nabla\varphi)\partial_{x_1}\varphi\mathrm{d}\sigma + \int_{x_1=R} \rho(\nabla\varphi)\partial_{x_1}\varphi\mathrm{d}\sigma. \tag{4.8}$$

Using (4.7) and letting $R \to +\infty$ in (4.8), we arrive at

$$\rho(q)q = \rho(q_0)q_0. \tag{4.9}$$

In addition, it is easy to verify that $\rho(q)q$ is an increasing function of q for $q < c(\rho_0)$, then we derive $q = q_0$.

From the analysis above, we complete the proof of Theorem 2.3.

Proofs of Theorems 2.2 and 1.1 Since the proofs come from Theorem 2.3 directly, we omit them.

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