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Global Well-Posedness of the BCL System with Viscosity**

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Abstract The BCL system, a kind of equations governing the motion of the free surface of water waves in $\mathbb{R}^3$, is studied. Some results on the global existence, uniqueness and regularity of solutions to such system with small initial data are obtained.

Keywords Pseudo-differential operator, Water waves, Global well-posedness
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1 Introduction

We will consider a movement of the free-surface of water waves which is an inviscid fluid and whose depth is finite. Generally speaking, the motion of water waves is described by the Euler equations. The well-posedness of the Euler equations has been obtained by Sijue Wu [16, 17] for infinite depth and D. Lannes [10] for finite depth. With assumptions of the long waves approximation and shallow water, J. L. Bona et al. [2] derived a system from the water waves equations equivalent to the Euler equations in 2-dimension

$$\begin{aligned} \eta_t + V_x + (\eta V)_x &= aV_{xxx} + b\eta_{xxt}, \\ V_t + \eta_x + VV_x &= c\eta_{xxx} + dV_{xxt}. \end{aligned} \tag{1.1}$$

Here η is the deviation of the free-surface with respect to the vertical direction, V is the velocity of surface in the horizontal direction, and a, b, c, d (see [2, 3]) are constants, which depend on the nature of the water. It has attracted much attentions of many mathematicians to study equations (1.1) with special a, b, c, d respectively. For example, C. J. Amick [1] and M. E. Schonbek [13] have showed the existence of the solutions to equations (1.1) in the case of $a = b = c = 0$ and $d > 0$. There are many studies (see [6, 12]) of (1.1) in the soliton theory.

J. L. Bona et al. [3] classified equations (1.1) with the values a, b, c and d. It should be pointed out that linearized equations of (1.1) are well-posed in Sobolev spaces only when $ac \geq 0$ and $b, d > 0$ (see [2]). With the restriction of $b = d > 0$, the global well-posedness of equations (1.1) in the cases of $a, c > 0$ and $a = 0$, $c > 0$ are respectively obtained, since there exists a

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Hamiltonian to (1.1). In [4], J. L. Bona, T. Colin and D. Lannes derived the following equations

$$\begin{aligned} &\eta_t + \nabla \cdot V + \nabla \cdot (\eta V) = a\Delta\nabla \cdot V + b\Delta\eta_t, \\ &V_t + \nabla\eta + \frac{1}{2}\nabla|V|^2 = c\Delta\nabla\eta + d\Delta V_t \end{aligned} \tag{1.2}$$

from the Euler equations in 3-dimension and gave the error estimate for the difference between the solutions to equations (1.2) and the ones to Euler equations, and also pointed out that there exists a conservation law to (1.2) in the case of $a, c > 0$ and $b = d > 0$.

The question one concerns is whether there exists global well-posedness of (1.2) for general values a, b, c, d. Firstly, we will consider the Cauchy problem of equations (1.2) under the condition $b = d > 0$, which is of some conservation law to (1.2). For general case, by adding partial viscosity to the second equation in (1.2), we will consider the Cauchy problem

$$\begin{cases} \eta_t + \nabla \cdot V + \nabla \cdot (\eta V) = a\Delta\nabla \cdot V + b\Delta\eta_t, \\ V_t + \nabla\eta - \mu\Delta V + \frac{1}{2}\nabla|V|^2 = c\Delta\nabla\eta + d\Delta V_t, \quad \forall\, t \in \overline{\mathbb{R}}_+,\ \forall\, x \in \mathbb{R}^2. \\ \eta(0,x) = \eta_0(x), \quad V(0,x) = V_0(x), \end{cases} \tag{1.3}$$

Throughout the present paper, we call (1.3) the BCL system with viscosity.

The space of continuous functions in $t \in \overline{\mathbb{R}}_+$ valued in $H^M(\mathbb{R}^2)$ with bounded norms in $L^\infty(\overline{\mathbb{R}}_+, H^M(\mathbb{R}^2))$ is denoted by $C_b(\overline{\mathbb{R}}_+, H^M(\mathbb{R}^2))$.

Our main results in the present article are as follows.

Theorem 1.1 *Assume that a, b, c, d are positive, and $b = d$. If $H(\eta_0, V_0) < \frac{\sqrt{a}}{2}$ and $\|\eta_0\|_{L^2} < \sqrt{a}$ (or $H(\eta_0, V_0) < \frac{\sqrt{a}}{2}$, $\|V_0\|_{L^2} < \sqrt{a}$), where $H(\eta, V)$ is the Hamiltonian of* (1.2) *defined by*

$$H(\eta, V) = \frac{1}{2}\int_{\mathbb{R}^2} (|\eta|^2 + |V|^2 + \eta|V|^2 + c|\nabla\eta|^2 + a|\nabla V|^2)\mathrm{d}x\mathrm{d}y,$$

then the Cauchy problem (1.2) *with initial data (η_0, V_0) admits unique global solutions $(\eta, V) \in C([0,\infty), (H^1(\mathbb{R}^2))^3)$. Moreover, $(\eta_t, V_t) \in C([0,\infty), (L^2(\mathbb{R}^2))^3)$.*

Theorem 1.2 *Assume that a, b, c, d, μ are positive numbers, and $q = \frac{2+2\delta}{2+\delta}$, $p = \frac{2+2\delta}{3\delta}$ with $\delta > 0$. Then there exist two sufficiently small positive numbers δ_0 and ε, such that there exists a unique solution*

$$(\eta, V) \in L^\infty(\overline{\mathbb{R}}_+, W^{1,p}(\mathbb{R}^2) \times W^{1,p}(\mathbb{R}^2) \times W^{1,p}(\mathbb{R}^2)) \cap C_b(\overline{\mathbb{R}}_+, H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2))$$

to the Cauchy problem (1.3) *with initial data $(\eta_0, V_0) \in (W^{4,q}(\mathbb{R}^2))^3$, if $\|\eta_0\|_{W^{4,q}} + \|V_0\|_{W^{4,q}} \le \varepsilon$ and $0 < \delta < \delta_0$.*

In the case of $a = 0$, $c > 0$, a similar theorem (see Remark 3.2) is also obtained.

In Section 2, the conservation law of equations (1.2) is given under the condition $b = d > 0$. Then local solutions to the Cauchy problem of equations (1.2) are obtained by Banach contraction principle. In terms of the conservation law, the global solutions of equations (1.2) are obtained. However, since the conservation law is not positive-definite and the embedding $H^1(\mathbb{R}^2) \subset L^\infty(\mathbb{R}^2)$ is not valid, the Ladyzhenskaya inequality is useful for us to bypass these obstacles.

In Section 3, the Cauchy problem (1.3) is considered since the conservation law ceases to hold in general case. In the preceding article (see [7]), we consider the interaction of the wave operator $\mathrm{e}^{\mathrm{i}t|D|}$, which is the effect of the dispersion and the heat operator $\mathrm{e}^{-t|D|^2}$ caused by the viscosity. Such estimates of the interactions mentioned above are the key to the global well-posedness of the nonlinear equations. Fortunately, the decay rate is not too bad to make the iteration divergence. The global existence, uniqueness and regularity of the solutions to the nonlinear equations with small initial data are given. Another key lemma is Lemma 3.2, which claims that the L^2-norm decays in the diffusion equations. Generally speaking, the L^2-norm is only bounded. On the other hand, the loss of derivatives which arises in high frequency occurs in the nonlinear iteration. We attack the effect of the loss of derivatives in nonlinear iteration by separating the estimates (see Theorem 3.1) into high frequency part and low frequency part. Our scheme of nonlinear iteration is essentially standard. It also works for some kind of the system in [14].

2 Global Well-Posedness of the BCL System with Conservation Law

2.1 Conservation law of the BCL system

In this section, we will consider the global well-posedness of the following system:

$$\begin{aligned} &\eta_t + \nabla\cdot V + \nabla\cdot(\eta V) = a\Delta\nabla\cdot V + b\Delta\eta_t,\\ &V_t + \nabla\eta + \frac{1}{2}\nabla|V|^2 = c\Delta\nabla\eta + d\Delta V_t. \end{aligned} \tag{2.1}$$

In order to obtain the well-posedness of the equations, we must derive an energy estimate for the equations. To this end, we firstly show that the BCL system has a conservation law. We rewrite equations (2.1) as follows:

$$\begin{aligned} &\eta_t = -(I-b\Delta)^{-1}\nabla\cdot(V+\eta V - a\Delta V),\\ &V_t = -(I-d\Delta)^{-1}\nabla\Big(\eta + \frac{1}{2}|V|^2 - c\Delta\eta\Big). \end{aligned}$$

Here $(I-b\Delta)^{-1}$ is defined by

$$(I-b\Delta)^{-1}\phi(x) = \int \frac{\widehat{\phi}(x)}{(1+b|\xi|^2)}\mathrm{e}^{-\mathrm{i}x\cdot\xi}\mathrm{d}\xi.$$

If we introduce a quantity $H(\eta,V)$ as

$$H(\eta,V) = \frac{1}{2}\int_{\mathbb{R}^2}(|\eta|^2 + |V|^2 + \eta|V|^2 + c|\nabla\eta|^2 + a|\nabla V|^2)\mathrm{d}x\mathrm{d}y, \tag{2.2}$$

it is easy to verify that the variation gradient of H is just the right-hand side of (2.1), i.e.,

$$\begin{pmatrix} \dfrac{\delta H}{\delta\eta} \\ \dfrac{\delta H}{\delta V} \end{pmatrix} = \begin{pmatrix} \eta + \dfrac{1}{2}|V|^2 - c\Delta\eta \\ V + \eta V - a\Delta V \end{pmatrix}.$$

If we introduce an operator

$$J = -\begin{pmatrix} 0 & (I - b\Delta)^{-1}\nabla \\ ((I - d\Delta)^{-1}\nabla)^{\mathrm{T}} & 0 \end{pmatrix},$$

where the notation $^{\mathrm{T}}$ is the transpose operator of the vector, then system (2.1) is equivalent to the following form:

$$\partial_t \begin{pmatrix} \eta \\ V \end{pmatrix} = J \begin{pmatrix} \dfrac{\delta H}{\delta \eta} \\ \dfrac{\delta H}{\delta V} \end{pmatrix}. \tag{2.3}$$

In terms of the Hamiltonian theory in infinite dimension, if the operator J is skew-symmetric, i.e., $b = d > 0$, then we have the following theorem.

Theorem 2.1 *Assume that $b = d > 0$. Then $H(\eta, V)$ defined in* (2.2) *is the Hamiltonian action of the BCL system* (2.1)*, and the operator J is skew-symmetric. System* (2.1) *is written equivalently as the Hamiltonian form* (2.3)*.*

This theorem shows that $H(\eta, V)$ is a Hamiltonian, that is, $\frac{\mathrm{d}H}{\mathrm{d}t} = 0$. In fact, by formal computation, there holds

$$\frac{\mathrm{d}H}{\mathrm{d}t} = 2(b - d)\int_{\mathbb{R}^2} \eta_t \nabla \cdot V_t \mathrm{d}x\mathrm{d}y.$$

It is obvious that $H(\eta, V)$ is the Hamiltonian of (2.1) when $b = d$ is valid. Here we will give a proof of conservation law.

Proof of Conservation Law By using (2.3), it implies that

$$\begin{aligned}
\frac{\mathrm{d}H}{\mathrm{d}t} &= \int_{\mathbb{R}^2} \Big(\frac{\delta H}{\delta \eta}\partial_t \eta + \frac{\delta H}{\delta V} \cdot \partial_t V\Big)\mathrm{d}x\mathrm{d}y \\
&= -\int_{\mathbb{R}^2} \sum_{j=1}^{2} \Big(\frac{\delta H}{\delta \eta}\frac{\nabla_j}{1 + b|D|^2}\frac{\delta H}{\delta V_j} + \frac{\delta H}{\delta V_j}\frac{\nabla_j}{1 + b|D|^2}\frac{\delta H}{\delta \eta}\Big)\mathrm{d}x\mathrm{d}y \\
&= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \sum_{j=1}^{2} \Big(\frac{\delta H}{\delta \eta}(x, y)K_j(x, y, w, z)\frac{\delta H}{\delta V_j}(w, z) \\
&\quad + \frac{\delta H}{\delta \eta}(w, z)K_j(x, y, w, z)\frac{\delta H}{\delta V_j}(x, y)\Big)\mathrm{d}x\mathrm{d}y\mathrm{d}w\mathrm{d}z \\
&= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \sum_{j=1}^{2} \frac{\delta H}{\delta \eta}(x, y)(K_j(x, y, w, z) + K_j(w, z, x, y))\frac{\delta H}{\delta V_j}(w, z)\mathrm{d}x\mathrm{d}y\mathrm{d}w\mathrm{d}z,
\end{aligned}$$

where the integral kernel is defined by the oscillatory integral

$$K_j(x, y, w, z) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} \mathrm{e}^{\mathrm{i}(\langle (x-w, y-z), \xi\rangle)}\frac{\mathrm{i}\xi_j}{1 + b(|\xi_1|^2 + |\xi_2|^2)}\mathrm{d}\xi_1\mathrm{d}\xi_2,$$

where $j = 1, 2$, $\langle (x - w, y - z), \xi\rangle$ is the inner product of the vectors $(x - w, y - z)$ and $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$. If we notice that the amplitude function in the oscillatory integral is odd, it implies that $K_j(x, y, w, z) = -K_j(w, z, x, y)$ for $j = 1, 2$. Hence

$$\frac{\mathrm{d}H}{\mathrm{d}t} = 0.$$

It indicates that $H(\eta, V)$ is the conservation quantity for the BCL system.

It is easy to see that (2.2) is not positive-definite. It will lead to some restriction of the initial data in order to get the global solutions.

2.2 Local solutions to equations (2.1)

Consider the local solutions to (2.1). Firstly, the equivalent form of (2.1) is given by Fourier analysis. Taking Fourier transform of both sides of (2.1), we have

$$\begin{aligned} &\widehat{\eta}_t + \frac{1+a|\xi|^2}{1+b|\xi|^2}(\mathrm{i}\xi)\cdot\widehat{V} = -\frac{\mathrm{i}\xi\cdot(\eta V)^\wedge}{1+b|\xi|^2}, \\ &\widehat{V}_t + \frac{1+c|\xi|^2}{1+d|\xi|^2}(\mathrm{i}\xi)\widehat{\eta} = -\frac{\mathrm{i}\xi(|V|^2)^\wedge}{2(1+d|\xi|^2)}. \end{aligned} \tag{2.4}$$

In order to diagnose equations (2.4), it is convenient to introduce an auxiliary function Z as follows:

$$\widehat{Z} = \frac{\xi^\perp}{|\xi|}\cdot\widehat{V}, \quad \xi^\perp = (\xi_2, -\xi_1). \tag{2.5}$$

By taking the inner-product of the vector $\xi^\perp$ and the second equation in (2.4), there holds

$$\partial_t Z = 0. \tag{2.6}$$

We introduce two other auxiliary functions $W^\pm$ defined by

$$\widehat{W^\pm} = \widehat{\eta} \pm \sqrt{\frac{\omega_1}{\omega_2}}\frac{\xi}{|\xi|}\cdot\widehat{V}, \tag{2.7}$$

where $\omega_1 = \frac{1+a|\xi|^2}{1+b|\xi|^2}$, $\omega_2 = \frac{1+c|\xi|^2}{1+d|\xi|^2}$. Hence by taking the inner-product of $\pm\sqrt{\frac{\omega_1}{\omega_2}}\frac{\xi}{|\xi|}$ and the second equation in (2.4), and adding to the first equation in (2.4), we obtain

$$\partial_t W^\pm \pm \mathrm{i}|D|\sqrt{\omega_1\omega_2}\,(D)W^\pm = -f^\pm(\eta, V), \tag{2.8}$$

where the nonlinear terms $f^\pm(\eta, V)$ are

$$f^\pm(\eta, V) = \frac{D}{1+b|D|^2}\cdot(\eta V) \pm \sqrt{\frac{\omega_1}{\omega_2}}\,(D)\frac{|D|}{2(1+d|D|^2)}(|V|^2). \tag{2.9}$$

In terms of Duhamel principal, the solutions to (2.6) and (2.8) are written as

$$\begin{aligned} Z(t,x,y) &= Z_0(x,y), \\ W^\pm(t,x,y) &= \mathrm{e}^{\mp\mathrm{i}t|D|\sqrt{\omega_1\omega_2}\,(D)}W_0^\pm - \int_0^t \mathrm{e}^{\mp\mathrm{i}(t-s)|D|\sqrt{\omega_1\omega_2}\,(D)}f^\pm(\eta, V)(s,x,y)\mathrm{d}s. \end{aligned}$$

Here the initial data Z_0, W_0 are respectively of forms

$$Z_0 = \frac{D^\perp}{|D|}\cdot V_0, \quad W_0^\pm = \eta_0 \pm \sqrt{\frac{\omega_1}{\omega_2}}\,(D)\frac{D}{|D|}\cdot V_0,$$

and the operator $\frac{D^\perp}{|D|}$ is defined as the Calderon-Zygmund singular integral operator. By definitions (2.5) and (2.7) of Z and $W^\pm$, the solutions (η, V) are given as follows:

$$\begin{aligned}\eta(t,x,y) &= \frac{W^+ + W^-}{2},\\ V &= \frac{1}{2}\sqrt{\frac{\omega_2}{\omega_1}}(D)\frac{D}{|D|}(W^+ - W^-) + \frac{D^\perp}{|D|}Z.\end{aligned}$$

At last, by substituting the solutions Z, $W^\pm$ into the above expressions, we have

$$\begin{aligned}\eta(t,x,y) &= A(t,D)\eta_0(x,y) + B(t,D)\cdot V_0(x,y) + \int_0^t A(t-s,D)\frac{D}{1+b|D|^2}\cdot(\eta V)\mathrm{d}s\\ &\quad + \int_0^t B(t-s,D)\cdot\frac{D}{2(1+d|D|^2)}(|V|^2)\mathrm{d}s,\\ V(t,x,y) &= B(t,D)\eta_0(x,y) + F(t,D)\cdot V_0(x,y) + \int_0^t B(t-s,D)\frac{D}{1+b|D|^2}\cdot(\eta V)\mathrm{d}s\\ &\quad + \frac{D^\perp}{|D|}\Big(\frac{D^\perp}{|D|}\cdot V_0(x,y)\Big) + \int_0^t F(t-s,D)\cdot\frac{D}{2(1+d|D|^2)}(|V|^2)\mathrm{d}s.\end{aligned} \tag{2.10}$$

Here the symbols of the operators $A(t,D)$, $B(t,D)$, $F(t,D)$ are respectively defined by

$$\begin{aligned}A(t,\xi) &= \frac{1}{2}(\mathrm{e}^{\mathrm{i}t|\xi|\sqrt{\omega_1\omega_2}(\xi)} + \mathrm{e}^{-\mathrm{i}t|\xi|\sqrt{\omega_1\omega_2}(\xi)}),\\ B(t,\xi) &= \frac{1}{2}\sqrt{\frac{\omega_1}{\omega_2}}(\xi)\frac{\xi}{|\xi|}(\mathrm{e}^{-\mathrm{i}t|\xi|\sqrt{\omega_1\omega_2}(\xi)} - \mathrm{e}^{\mathrm{i}t|\xi|\sqrt{\omega_1\omega_2}(\xi)}),\\ F(t,\xi) &= \frac{1}{2}\frac{\omega_1}{\omega_2}(\xi)\frac{\xi^2}{|\xi|^2}(\mathrm{e}^{-\mathrm{i}t|\xi|\sqrt{\omega_1\omega_2}(\xi)} + \mathrm{e}^{\mathrm{i}t|\xi|\sqrt{\omega_1\omega_2}(\xi)}).\end{aligned}$$

It should be pointed out that some of these operators are of smooth symbols of Hörmander pseudo-differential operator and some are of singular symbols of Calderon-Zygmund singular integral operator.

Lemma 2.1 *Assume that a, b, c, d are positive, and s is an arbitrary number. Then there exists a positive constant $C = C(a,b,c,d,s)$, such that there hold, for $t > 0$,*

$$\begin{aligned}\|A(t,D)\phi\|_{H^s(\mathbb{R}^2)} &\le C\|\phi\|_{H^s(\mathbb{R}^2)},\\ \|B(t,D)\phi\|_{H^s(\mathbb{R}^2)} &\le C\|\phi\|_{H^s(\mathbb{R}^2)},\\ \|F(t,D)\phi\|_{H^s(\mathbb{R}^2)} &\le C\|\phi\|_{H^s(\mathbb{R}^2)}.\end{aligned}$$

Moreover, these operators are strongly continuous operators in t with respect to $H^s(\mathbb{R}^2)$-norm.

Proof The desired estimates are actually the L^2- and L^p-boundedness of the pseudo-differential operators and the singular integral operators. We will give direct proofs by Plancherel's theorem. Firstly, noting that $S(\mathbb{R}^2)$ is dense in $H^s(\mathbb{R}^2)$, we just need to prove the above estimates in $S(\mathbb{R}^2)$. It is clear that

$$\|A(t,D)\phi\|_{H^s(\mathbb{R}^2)} = \int_{\mathbb{R}^2}(1+|\xi|^2)^s|A(t,\xi)\widehat{\phi}(\xi)|^2\mathrm{d}\xi.$$

If we notice that $|A(t,\xi)| \le 1$, we obtain

$$\|A(t,D)\phi\|_{H^s(\mathbb{R}^2)} \le \int_{\mathbb{R}^2} (1+|\xi|^2)^s |\widehat{\phi}(\xi)|^2 \mathrm{d}\xi \le \|\phi\|_{H^s(\mathbb{R}^2)}.$$

For the operators $B(t,D)$ and $F(t,D)$, the similar proof is available because of the boundedness of the corresponding symbols $B(t,\xi)$ and $F(t,\xi)$. In order to show the strong continuities of the operators, firstly assume that $\phi \in S(\mathbb{R}^2)$ ($S(\mathbb{R}^2)$ is Schwartz space), and t_n is a sequence of non-negative numbers, t_0 is some non-negative number, such that $t_n \to t_0$ as $n \to \infty$. Note that $A(t,\xi)$, $B(t,\xi)$, $F(t,\xi)$ are continuous in t and uniformly bounded in ξ. By Plancherel's theorem, there holds

$$\|(A(t_n,D) - A(t_0,D))\phi\|_{H^s(\mathbb{R}^2)} = \int_{\mathbb{R}^2} (1+|\xi|^2)^s |A(t_n,\xi) - A(t_0,\xi)|^2 |\widehat{\phi}(\xi)|^2 \mathrm{d}\xi.$$

In terms of Lebesgue dominated convergence, we obtain

$$\|(A(t_n,D) - A(t_0,D))\phi\|_{H^s(\mathbb{R}^2)} \to 0, \quad t_n \to 0.$$

Remark 2.1 There exists a positive constant C, such that

$$\Big\|\frac{D^\perp}{|D|}\Big(\frac{D^\perp}{|D|}\cdot\phi\Big)\Big\|_{H^s(\mathbb{R}^2)} \le C\|\phi\|_{H^s(\mathbb{R}^2)}.$$

This remark easily follows from the proof of the preceding theorem.

Lemma 2.2 *Assume that $\phi, \psi \in H^1(\mathbb{R}^2)$. Then there holds Ladyzhenskaya inequality*

$$\|\phi\|^2_{L^4(\mathbb{R}^2)} \le \|\phi\|_{L^2(\mathbb{R}^2)} \|D\phi\|_{L^2(\mathbb{R}^2)}.$$

Furthermore,

$$\|\phi\psi\|^2_{L^2(\mathbb{R}^2)} \le \|\phi\|_{L^2(\mathbb{R}^2)} \|D\phi\|_{L^2(\mathbb{R}^2)} \|\psi\|_{L^2(\mathbb{R}^2)} \|D\psi\|_{L^2(\mathbb{R}^2)}.$$

Proof By the Sobolev inequality in $\mathbb{R}^2$, we have

$$\|\psi\|_{L^q(\mathbb{R}^2)} \le C(p,q)\|D\psi\|_{L^p(\mathbb{R}^2)},$$

where $\frac{1}{q} + \frac{1}{2} = \frac{1}{p}$, $C(p,q) = \frac{1}{2}\frac{p}{2-p}$. Take $p = 1$, $q = 2$. Then there holds

$$\|\psi\|_{L^2(\mathbb{R}^2)} \le \frac{1}{2}\|D\psi\|_{L^1(\mathbb{R}^2)}.$$

If we substitute $\psi = \phi^2$ into the above inequality, it implies the Ladyzhenskaya inequality by the Cauchy-Schwarz inequality. Note

$$\|\phi\psi\|^2_{L^2(\mathbb{R}^2)} \le \|\phi\|_{L^4(\mathbb{R}^2)} \|\psi\|_{L^4(\mathbb{R}^2)}.$$

Then the second inequality is an immediate conclusion of Ladyzhenskaya inequality.

In order to show the existence of local solutions to (2.1), we will construct a studied function space firstly. For any given initial data $\eta_0 \in H^1(\mathbb{R}^2)$, $V_0 \in (H^1(\mathbb{R}^2))^2$, the function space is defined by

$$\begin{aligned} B_R = \{&(\eta(t,\cdot), V(t,\cdot)) \in C([0,T], (H^1(\mathbb{R}^2))^3) : \\ &\|\eta(t,\cdot)\|_{H^1(\mathbb{R}^2)} \le R,\ \|V(t,\cdot)\|_{H^1(\mathbb{R}^2)} \le R,\ t \in [0,T]\}, \end{aligned}$$

where R is a sufficiently large number to be determined later, and T is a sufficiently small positive number depending on R. Of course, the standard metric ρ in B_R, defined by

$$\rho((\eta_1, V_1), (\eta_2, V_2)) = \sup_{t\in[0,T]} (\|\eta_1(t,\cdot) - \eta_2(t,\cdot)\|_{H^1(\mathbb{R}^2)} + \|V_1(t,\cdot) - V_2(t,\cdot)\|_{H^1(\mathbb{R}^2)}),$$

is given. It is clear that (B_R, ρ) is a complete metric space. We denote the right-hand side of (2.10) by the maps $S(\eta, V) = (S_1(\eta, V), S_2(\eta, V))$. Before using the Banach contraction theorem, we need some lemmas.

Lemma 2.3 *Assume that a, b, c, d are positive numbers, $\eta_0 \in H^1(\mathbb{R}^2)$ and $V_0 \in (H^1(\mathbb{R}^2))^2$. If R is taken sufficiently large, such that $R \geq 2C(\|\eta_0\|_{H^1(\mathbb{R}^2)} + \|V_0\|_{H^1(\mathbb{R}^2)})$, and T is taken sufficient small, such that $T \leq \frac{1}{4CR}$, then the map S is from B_R to B_R. Here the constant C is a positive number only depending on a, b, c, d.*

Proof Let $(\eta(t,\cdot), V(t,\cdot)) \in B_R$. By Lemma 2.1 and Remark 2.1, there holds

$$\begin{aligned}\|S_1(\eta, V)\|_{H^1(\mathbb{R}^2)} &\leq C(\|\eta_0\|_{H^1(\mathbb{R}^2)} + \|V_0\|_{H^1(\mathbb{R}^2)}) + \int_0^t \Big\|\frac{D}{1+b|D|^2}\cdot(\eta V)(\tau)\Big\|_{H^1(\mathbb{R}^2)}\mathrm{d}\tau \\ &\quad + \int_0^t \Big\|\frac{D}{2(1+d|D|^2)}(|V|^2)(\tau)\Big\|_{H^1(\mathbb{R}^2)}\mathrm{d}\tau \\ &\leq C(\|\eta_0\|_{H^1(\mathbb{R}^2)} + \|V_0\|_{H^1(\mathbb{R}^2)}) + \int_0^t C\|(\eta V)(\tau)\|_{L^2(\mathbb{R}^2)}\mathrm{d}\tau \\ &\quad + \int_0^t C\|(|V|^2)(\tau)\|_{L^2(\mathbb{R}^2)}\mathrm{d}\tau,\end{aligned}$$

where we have used the L^2-boundedness of the pseudo-differential operator of order -1. To estimate the integrand, by using the inequalities in Lemma 2.2, we have

$$\begin{aligned}\|S_1(\eta, V)\|_{H^1(\mathbb{R}^2)} &\leq C(\|\eta_0\|_{H^1(\mathbb{R}^2)} + \|V_0\|_{H^1(\mathbb{R}^2)}) + \int_0^t C\|V(\tau)\|_{L^2(\mathbb{R}^2)}\|DV(\tau)\|_{L^2(\mathbb{R}^2)}\mathrm{d}\tau \\ &\quad + \int_0^t C\|\eta(\tau)\|^{\frac12}_{L^2(\mathbb{R}^2)}\|D\eta(\tau)\|^{\frac12}_{L^2(\mathbb{R}^2)}\|V(\tau)\|^{\frac12}_{L^2(\mathbb{R}^2)}\|DV(\tau)\|^{\frac12}_{L^2(\mathbb{R}^2)}\mathrm{d}\tau \\ &\leq C(\|\eta_0\|_{H^1(\mathbb{R}^2)} + \|V_0\|_{H^1(\mathbb{R}^2)}) + 2CTR^2\end{aligned}$$

for $t \in [0,T]$. Hence we take R sufficiently large and T sufficiently small, so that $R > 2C(\|\eta_0\|_{H^1(\mathbb{R}^2)} + \|V_0\|_{H^1(\mathbb{R}^2)})$, $T \leq \frac{1}{4CR}$. The same proof works for S_2. Thus, the map S is from B_R to B_R.

The following lemma will show that the map S is a contraction from B_R to B_R if T is taken much smaller.

Lemma 2.4 *Assume that a, b, c, d are positive numbers, $\eta_0 \in H^1(\mathbb{R}^2)$ and $V_0 \in (H^1(\mathbb{R}^2))^2$. If R is taken sufficiently large, such that $R \geq 2C(\|\eta_0\|_{H^1(\mathbb{R}^2)} + \|V_0\|_{H^1(\mathbb{R}^2)})$, and T is taken sufficiently small, such that $T < \frac{1}{8CR}$, then the map S is a contraction from B_R to B_R. Here the constant C is a positive number only depending on a, b, c, d.*

Proof Let $(\eta_1(t,\cdot),V_1(t,\cdot)),(\eta_2(t,\cdot),V_2(t,\cdot))\in B_R$. By Lemma 2.1, Remark 2.1 and Lemma 2.2, there holds

$$\begin{aligned}\|S_1(\eta_1,V_1)-S_1(\eta_2,V_2)\|_{H^1(\mathbb{R}^2)}&\le\int_0^t\Big\|\frac{D}{1+b|D|^2}\cdot((\eta_1V_1)(\tau)-(\eta_2V_2)(\tau))\Big\|_{H^1(\mathbb{R}^2)}\mathrm{d}\tau\\&\quad+\int_0^t\Big\|\frac{D}{2(1+d|D|^2)}((|V|_1^2)(\tau)-(|V|_2^2)(\tau))\Big\|_{H^1(\mathbb{R}^2)}\mathrm{d}\tau\\&\le\int_0^t C(\|((\eta_1-\eta_2)V_1)(\tau)\|_{L^2(\mathbb{R}^2)}+\|(\eta_2(V_1-V_2))(\tau)\|_{L^2(\mathbb{R}^2)})\mathrm{d}\tau\\&\quad+\int_0^t C\|((V_1+V_2)\cdot(V_1-V_2))(\tau)\|_{L^2(\mathbb{R}^2)}\mathrm{d}\tau\\&\le 4CRT\rho((\eta_1,V_1),(\eta_2,V_2))\end{aligned}$$

for $t\in[0,T]$. The other estimates are similarly proved. Here the constant C is a positive number only depending on a, b, c, d. If T is taken much smaller so that $T<\frac{1}{8CR}$, the map S is a contraction map.

In terms of Banach contraction theorem, it implies the existence of local solutions.

Theorem 2.2 *Assume that a, b, c, d are positive numbers, $\eta_0\in H^1(\mathbb{R}^2)$ and $V_0\in(H^1(\mathbb{R}^2))^2$. If $T<\frac{1}{16C^2(\|\eta_0\|_{H^1(\mathbb{R}^2)}+\|V_0\|_{H^1(\mathbb{R}^2)})}$ is valid, then the Cauchy problem* (2.1) *with initial data (η_0,V_0) admits a unique solution*

$$(\eta,V)\in C([0,T],(H^1(\mathbb{R}^2))^3).$$

Moreover, there also holds $(\eta_t,V_t)\in C([0,T],(L^2(\mathbb{R}^2))^3)$.

Proof The existence and uniqueness of the solution to the Cauchy problem are derived from Lemmas 2.3 and 2.4. To show the strong continuities of solutions, we only make use of the proof of Lemma 2.1 and the strong continuity of Bochner integral of (2.10). In order to prove the regularities of (η_t,V_t), we rewrite (2.1) as follows:

$$\begin{aligned}\eta_t&=(I-b\Delta)^{-1}(-\nabla\cdot V-\nabla\cdot(\eta V)+a\Delta\nabla\cdot V),\\V_t&=(I-d\Delta)^{-1}\Big(-\nabla\eta-\frac{1}{2}\nabla|V|^2+c\Delta\nabla\eta\Big).\end{aligned}$$

By the property of the Bessel potential, i.e., $(I-b\Delta)^{-1}$ is a bounded operator from $H^s(\mathbb{R}^2)$ to $H^{s+2}(\mathbb{R}^2)$, and by Lemma 2.1, the desired regularities of solutions are obtained.

2.3 Global solutions to the BCL system

To show the global existence of solutions, we need the following theorem.

Theorem 2.3 *Assume that a, b, c, d are positive, $\eta_0\in H^1(\mathbb{R}^2)$ and $V_0\in(H^1(\mathbb{R}^2))^2$. Provided that $T^*>0$ is the maximum of the existence interval of the Cauchy problem* (2.1) *with initial data (η_0,V_0), that is, $0<T<T^*$, the Cauchy problem* (2.1) *with initial data (η_0,V_0) admits a unique solution $(\eta,V)\in C([0,T],(H^1(\mathbb{R}^2))^3)$. If $T^*<\infty$, then*

$$\liminf_{t\to T^*}(\|\eta(t,\cdot)\|_{H^1(\mathbb{R}^2)}+\|V(t,\cdot)\|_{H^1(\mathbb{R}^2)})=+\infty.$$

Proof We shall prove the above theorem by contradiction. Assume that

$$\liminf_{t\to T^*}(\|\eta(t,\cdot)\|_{H^1(\mathbb{R}^2)}+\|V(t,\cdot)\|_{H^1(\mathbb{R}^2)})<+\infty,$$

when $T^*<\infty$. Then there exists a positive constant M, such that, for each $0<t<T^*$,

$$\|\eta(t,\cdot)\|_{H^1(\mathbb{R}^2)}+\|V(t,\cdot)\|_{H^1(\mathbb{R}^2)}\le M.$$

By the definition of T^*, we can take $T_0=T^*-\frac{1}{32C^2M}<T^*$, such that the Cauchy problem (2.1) with initial data (η_0,V_0) admits a unique solution

$$(\eta,V)\in C([0,T_0],(H^1(\mathbb{R}^2))^3).$$

Now let $(\eta(T_0),V(T_0))$ be the new initial data. By Theorem 2.2, the Cauchy problem with initial data $(\eta(T_0),V(T_0))$ admits a unique solution, such that the length of the existence interval of solutions is greater than $\frac{1}{32C^2M}$. Thus, it implies a contradiction to the definition of T^*.

Theorem 2.4 *Assume that a, b, c, d are positive, and $b=d$. If $H(\eta_0,V_0)<\frac{\sqrt{a}}{2}$ and $\|\eta_0\|_{L^2}<\sqrt{a}$ (or $H(\eta_0,V_0)<\frac{\sqrt{a}}{2}$, $\|V_0\|_{L^2}<\sqrt{a}$), then the Cauchy problem* (2.1) *with initial data (η_0,V_0) admits a unique global solution $(\eta,V)\in C([0,\infty),(H^1(\mathbb{R}^2))^3)$. Moreover, $(\eta_t,V_t)\in C([0,\infty),(L^2(\mathbb{R}^2))^3)$.*

Proof By Theorem 2.2, if $T>0$ is sufficiently small, the Cauchy problem (2.1) with initial data (η_0,V_0) always admits a unique solution $(\eta,V)\in C([0,T],(H^1(\mathbb{R}^2))^3)$. Since the solution is continuous in t, and $\|\eta_0\|_{L^2}<\sqrt{a}$, there exists a positive number $0<T_0<T$, such that $\|\eta(t,\cdot)\|_{L^2}\le\sqrt{a}$ is valid for $t\in[0,T_0]$. By Lemma 2.2, for $t\in[0,T_0]$, there holds

$$\begin{aligned}\Big|\int_{\mathbb{R}^2}\eta(t,x,y)|V(t,x,y)|^2\mathrm{d}x\mathrm{d}y\Big| &\le\|\eta(t)\|_{L^2(\mathbb{R}^2)}\|V^2(t)\|_{L^2(\mathbb{R}^2)}\\ &\le\|\eta(t)\|_{L^2(\mathbb{R}^2)}\|V(t)\|_{L^2(\mathbb{R}^2)}\|\nabla V(t)\|_{L^2(\mathbb{R}^2)}\\ &\le\|\eta(t)\|_{L^2(\mathbb{R}^2)}\Big(\frac{1}{\sqrt{a}}\|V(t)\|^2_{L^2(\mathbb{R}^2)}+\frac{\sqrt{a}}{2}\|\nabla V(t)\|^2_{L^2(\mathbb{R}^2)}\Big).\end{aligned}$$

For the Hamiltonian $H(\eta(t),V(t))$, if $t\in[0,T_0]$, there holds

$$\begin{aligned}H(\eta(t),V(t))\ge\frac{1}{2}\Big(&\|\eta(t)\|^2_{L^2(\mathbb{R}^2)}+c\|\nabla\eta(t)\|^2_{L^2(\mathbb{R}^2)}+\Big(1-\frac{\|\eta(t)\|_{L^2(\mathbb{R}^2)}}{2\sqrt{a}}\Big)\|V(t)\|^2_{L^2(\mathbb{R}^2)}\\ &+\Big(a-\frac{\sqrt{a}}{2}\|\eta(t)\|_{L^2(\mathbb{R}^2)}\Big)\|\nabla V(t)\|^2_{L^2(\mathbb{R}^2)}\Big).\end{aligned}$$

It implies that $H(\eta(t),V(t))$ is positive-definite for $t\in[0,T_0]$. And by conservation law, there holds

$$\begin{aligned}&\|\eta(t,\cdot)\|_{L^2(\mathbb{R}^2)}\le 2H(\eta(t),V(t))<\sqrt{a},\\ &\|\eta(t,\cdot)\|_{H^1(\mathbb{R}^2)}+\|V(t,\cdot)\|_{H^1(\mathbb{R}^2)}\le\beta\sqrt{a}\end{aligned}$$

for $t\in[0,T_0]$, where $\beta=\max\{4,\frac{2}{c},\frac{4}{a}\}$. By Theorem 2.3, the solutions can always be extended till $T^*=\infty$.

Remark 2.2 Assume that a, b, c, d are positive, and $b=d$. If $H(\eta_0,V_0)<\frac{\sqrt{a}}{2}$ and $\|\eta_0\|_{L^2}<\sqrt{a}$ (or $H(\eta_0,V_0)<\frac{\sqrt{a}}{2}$, $\|V_0\|_{L^2}<\sqrt{a}$), then the Cauchy problem (2.1) with initial data (η_0,V_0) admits a unique global solution $(\eta,V)\in C^1([0,\infty),(C^\infty(\mathbb{R}^2))^3)$, if $(\eta_0,V_0)\in(C^\infty(\mathbb{R}^2))^3$.

3 Global Well-Posedness of the BCL System with Viscosity

3.1 The statement of the problem

Consider the following equations:

$$\begin{aligned}&\eta_t+\nabla\cdot V+\nabla\cdot(\eta V)=a\Delta\nabla\cdot V+b\Delta\eta_t,\\&V_t+\nabla\eta-\mu\Delta V+\frac{1}{2}\nabla|V|^2=c\Delta\nabla\eta+d\Delta V_t.\end{aligned}\tag{3.1}$$

Without the restriction of $b=d$, we want to give the global existence of the solutions to (3.1) with initial data $(\eta_0,V_0)\in\overline{\mathbb{R}}_+\times\mathbb{R}^2$. The linearized equations of (3.1) at $(\eta,V)=(0,0)$

$$\begin{aligned}&\eta_t+\nabla\cdot V=a\Delta\nabla\cdot V+b\Delta\eta_t,\\&V_t+\nabla\eta=c\Delta\nabla\eta+d\Delta V+\mu\Delta V_t\end{aligned}\tag{3.2}$$

should be considered firstly. By using the similar procedure in [7], the nonlinear equations (3.1) are transformed into

$$\begin{aligned}\eta(t,x)&=A(t,D)\eta_0(x)+B(t,D)\cdot V_0(x)\\&\quad-\int_0^t A(t-s,D)\frac{\nabla}{1+b|D|^2}\cdot(\eta V)(s,x)\mathrm{d}s\\&\quad-\int_0^t B(t-s,D)\cdot\frac{\nabla}{2(1+d|D|^2)}(|V|^2)(s,x)\mathrm{d}s,\\V(t,x)&=F(t,D)\eta_0(x)+G(t,D)\cdot V_0(x)-\frac{D^\perp}{|D|}\mathrm{e}^{t\sigma(D)}\frac{D^\perp}{|D|}\cdot V_0(x)\\&\quad-\int_0^t F(t-s,D)\frac{\nabla}{1+b|D|^2}\cdot(\eta V)(s,x)\mathrm{d}s\\&\quad-\int_0^t G(t-s,D)\cdot\frac{\nabla}{2(1+d|D|^2)}(|V|^2)(s,x)\mathrm{d}s,\end{aligned}\tag{3.3}$$

where the operators $\mathrm{e}^{t\sigma(D)}$, $A(t,D)$, $B(t,D)$, $F(t,D)$ and $G(t,D)$ are defined as in [7]. Next we mainly consider the global well-posedness of equations (3.3)

3.2 The main estimates

For the later needs, some a priori estimates for the operators $A(t,D)$, $B(t,D)$, etc. without proofs are listed as follows. For the details, one can refer to [7, Remark 3.4 and Theorem 3.9].

Theorem 3.1 *Assume that a, b, c, d, μ are positive numbers, M is an arbitrary number and $q=\frac{2+2\delta}{2+\delta}$, $p=\frac{2+2\delta}{3\delta}$ with $0<\delta<\frac{1}{4}$, p' is the conjugate exponent of p. Let $\chi_2(\xi)\in C_0^\infty(\mathbb{R}^2)$ with $\chi_2(\xi)=1$ for $|\xi|<R$, where R is sufficiently large, and s, N be non-negative integers. For $t>0$, there hold*

$$\begin{aligned}\|D^sA(t,D)\varphi\|_{H^N}&\le\frac{C}{(1+t)^{\frac{s}{2}}}\|\varphi\|_{H^{N+s}},\\\|D^sA(t,D)(1-\chi_2(D))\varphi\|_{W^{N,p}}&\le C\mathrm{e}^{-C_1t}\|\varphi\|_{W^{N+s+1,p}},\\\|D^sA(t,D)\chi_2(D)\varphi\|_{W^{N,p}}&\le C(1+t)^{\frac{5\delta}{2(1+\delta)}-\frac{2+s}{2}}\|\varphi\|_{W^{M,q}},\\\|D^sA(t,D)\varphi\|_{W^{N,p}}&\le C(1+t)^{\frac{5\delta}{2(1+\delta)}-\frac{2+s}{2}}\|\varphi\|_{W^{N+s+3,q}}.\end{aligned}$$

There also holds

$$\|D^s A(t,D)\varphi\|_{W^{N,p'}} \le C(1+t)^{\frac{5}{4(1+\delta)}-\frac{2+s}{2}}\|\varphi\|_{W^{N+s+3,q}}.$$

Here $C_1 = C_1(a,b,c,d,\mu)$ and $C = C(s,N,M,a,b,c,d,\mu,\delta)$ are positive constants only depending on s, N, M, a, b, c, d, μ, δ. If $A(t,D)$ in this theorem is replaced by the operators $\mathrm{e}^{t\sigma(D)}$, $B(t,D)$, $F(t,D)$ and $G(t,D)$, the same estimates are also valid.

Theorem 3.2 *Assume that a, b, c, d, μ are positive numbers, and $q=\frac{2+2\delta}{2+\delta}$, $p=\frac{2+2\delta}{3\delta}$ with $0<\delta<\frac14$, p' is the conjugate exponent of p. If the initial data*

$$(\eta_0,V_0)\in (W^{N+3,q}(\mathbb{R}^2)\times W^{N+3,q}(\mathbb{R}^2)\times W^{N+3,q}(\mathbb{R}^2))\cap(H^N(\mathbb{R}^2)\times H^N(\mathbb{R}^2)\times H^N(\mathbb{R}^2))$$

for every non-negative integer N, then the linearized problem of (1.3) admits a solution

$$\begin{aligned}(\eta,V)\in\, & L^\infty(\overline{\mathbb{R}}_+, W^{N,p}(\mathbb{R}^2)\times W^{N,p}(\mathbb{R}^2)\times W^{N,p}(\mathbb{R}^2))\\ & \cap C_b(\overline{\mathbb{R}}_+, H^N(\mathbb{R}^2)\times H^N(\mathbb{R}^2)\times H^N(\mathbb{R}^2)).\end{aligned}$$

Moreover, there exists a positive constant $C=C(N,a,b,c,d,\mu,\delta)$, such that

$$\begin{aligned}\|D^s\eta(t,\cdot)\|_{L^p} &\le C(1+t)^{\frac{5\delta}{2(1+\delta)}-\frac{2+s}{2}}(\|\eta_0\|_{W^{s+3,q}}+\|V_0\|_{W^{s+3,q}}),\\ \|D^sV(t,\cdot)\|_{L^p} &\le C(1+t)^{\frac{5\delta}{2(1+\delta)}-\frac{2+s}{2}}(\|\eta_0\|_{W^{s+3,q}}+\|V_0\|_{W^{s+3,q}}),\\ \|D^s\eta(t,\cdot)\|_{L^2} &\le C(1+t)^{-\frac{s}{2}}(\|\eta_0\|_{H^s}+\|V_0\|_{H^s}),\\ \|D^sV(t,\cdot)\|_{L^2} &\le C(1+t)^{-\frac{s}{2}}(\|\eta_0\|_{H^s}+\|V_0\|_{H^s})\end{aligned}$$

for a non-negative integer $s\le N$. There also hold

$$\begin{aligned}\|D^s\eta(t,\cdot)\|_{L^{p'}} &\le C(1+t)^{\frac{5}{4(1+\delta)}-\frac{2+s}{2}}(\|\eta_0\|_{W^{s+3,q}}+\|V_0\|_{W^{s+3,q}}),\\ \|D^sV(t,\cdot)\|_{L^{p'}} &\le C(1+t)^{\frac{5}{4(1+\delta)}-\frac{2+s}{2}}(\|\eta_0\|_{W^{s+3,q}}+\|V_0\|_{W^{s+3,q}}).\end{aligned}$$

3.3 The global well-posedness of nonlinear equations (3.1)

In the case of $a>0$, $c>0$, we introduce a suitable function space as follows:

$$\begin{aligned}Y_E=\{&\eta(t,\cdot)\in S'(\mathbb{R}^2);\ (1+t)^{1-\frac{5\delta}{2(1+\delta)}}\|\eta(t,\cdot)\|_{L^p}+(1+t)^{1+\beta}\|D\eta(t,\cdot)\|_{L^p}\\ &+(1+t)^{\nu}\|\eta(t,\cdot)\|_{L^2}+(1+t)^{\nu+\frac12}\|D\eta(t,\cdot)\|_{L^2}\le E,\ \forall\, t\in\overline{\mathbb{R}}_+\}.\end{aligned}$$

Here $\nu=1-\frac58\frac{1+2\delta}{1+\delta}$ for the same δ as the one in $q=\frac{2+2\delta}{2+\delta}$, $p=\frac{2+2\delta}{3\delta}$ such that $\frac{5\delta}{2(1+\delta)}<\nu$, and $\beta>0$ is chosen such that

$$\theta\Big(\frac{5\delta}{2(1+\delta)}-1\Big)-\theta(1+\beta)-2(1-\theta)\nu-\frac{1-\theta}{2}<-1-\beta,$$

where θ satisfies $\frac{1}{2q}=\frac{1-\theta}{2}+\frac{\theta}{p}$.

Lemma 3.1 *Y_E is nontrivial. If the distance $\rho(\eta_1,\eta_2)$ is defined by*

$$\begin{aligned}\rho(\eta_1,\eta_2)=\sup_{t\in\overline{\mathbb{R}}_+}\{&(1+t)^{1-\frac{5\delta}{2(1+\delta)}}\|(\eta_1-\eta_2)(t,\cdot)\|_{L^p}+(1+t)^{1+\beta}\|D(\eta_1-\eta_2)(t,\cdot)\|_{L^p}\\ &+(1+t)^{\nu}\|(\eta_1-\eta_2)(t,\cdot)\|_{L^2}+(1+t)^{\nu+\frac12}\|D(\eta_1-\eta_2)(t,\cdot)\|_{L^2}\},\end{aligned}$$

then (Y_E,ρ) is a complete metric space.

The proof is obvious. With no confusion, we also denote the distance in $Y_E \times Y_E \times Y_E$ by ρ.

We denote the right-hand side of equations (3.3) by a map $T(\eta, V) = (T_1(\eta, V), T_2(\eta, V))$. If we assign $S(\eta_0, V_0) = (S_1(\eta_0, V_0), S_2(\eta_0, V_0))$ to the solution $T(0,0)$ in the linearized equations (3.2) with initial data (η_0, V_0), we have

Lemma 3.2 *Assume that a, b, c, d, μ are positive numbers, N is a non-negative integer, and $q = \frac{2+2\delta}{2+\delta}$, $p = \frac{2+2\delta}{3\delta}$ with $0 < \delta < \frac{1}{4}$, p' is the conjugate exponent of p. If initial data*

$$(\eta_0, V_0) \in (W^{N+3,q} \times W^{N+3,q} \times W^{N+3,q}),$$

then there holds

$$\sum_{l=0}^{N} (1+t)^{\nu+\frac{l}{2}} \|D^l S(\eta_0, V_0)\|_{L^2} \le C(\|\eta_0\|_{W^{N+3,q}} + \|V_0\|_{W^{N+3,q}}).$$

Here $C = C(N, a, b, c, d, \mu, \delta)$ is a positive constant only depending on s, N, a, b, c, d, μ, δ.

Proof If we observe these estimates for the linearized equations in Theorem 3.2, we have

$$\begin{aligned}
\|S_1(\eta_0, V_0)\|_{L^2} &\le \|S_1(\eta_0, V_0)\|_{L^p}^{\frac{1}{2}} \|S_1(\eta_0, V_0)\|_{L^{p'}}^{\frac{1}{2}} \\
&\le C(1+t)^{\frac{5\delta}{4(1+\delta)} - \frac{1}{2}} (\|\eta_0\|_{W^{3,q}} + \|V_0\|_{W^{3,q}})^{\frac{1}{2}} \\
&\quad \cdot (1+t)^{\frac{5}{8(1+\delta)} - \frac{1}{2}} (\|\eta_0\|_{W^{3,q}} + \|V_0\|_{W^{3,q}})^{\frac{1}{2}} \\
&\le C(1+t)^{-\nu} (\|\eta_0\|_{W^{3,q}} + \|V_0\|_{W^{3,q}}).
\end{aligned}$$

The other proofs are all the same.

Lemma 3.3 *Let a, b be positive numbers. Then there hold*

$$\begin{aligned}
\int_0^t \frac{1}{(1+t-s)^a} \frac{1}{(1+s)^b} \mathrm{d}s &\le \frac{C}{(1+t)^{\min(a,b)}}, \\
\int_0^t \mathrm{e}^{-C_1(1+t-s)} \frac{1}{(1+s)^b} \mathrm{d}s &\le \frac{C}{(1+t)^b},
\end{aligned}$$

if $\max(a,b) > 1$. Here $C = C(a, b, C_1)$ is only dependent on a, b, C_1.

The proof is obvious. For the details, one can refer to [11].

The following interpolation formulae are also needed later.

Lemma 3.4 *It is true that $\|f^2\|_{L^q} \le \|f\|_{L^2}^{2(1-\theta)} \|f\|_{L^p}^{2\theta}$ is valid, where $\frac{1}{2q} = \frac{1-\theta}{2} + \frac{\theta}{p}$. More generally, we have $\|fg\|_{L^q} \le \|f\|_{L^2}^{1-\theta} \|f\|_{L^p}^{\theta} \|g\|_{L^2}^{1-\theta} \|g\|_{L^p}^{\theta}$.*

Now we can prove that the map T is from $Y_E \times Y_E \times Y_E$ into $Y_E \times Y_E \times Y_E$.

Theorem 3.3 *If $a, b, c, d, \mu > 0$, $q = \frac{2+2\delta}{2+\delta}$, $p = \frac{2+2\delta}{3\delta}$ with $\delta > 0$, and $(\eta_0, V_0) \in W^{4,q} \times W^{4,q} \times W^{4,q}$, then there exist two sufficiently small positive numbers δ_0 and E, such that the map T is from $Y_E \times Y_E \times Y_E$ into $Y_E \times Y_E \times Y_E$, if $0 < \delta < \delta_0$ and $\|\eta_0\|_{W^{4,q}} + \|V_0\|_{W^{4,q}} \le E^2$.*

Proof If $(\eta, V) \in Y_E \times Y_E \times Y_E$ is valid and the initial data (η_0, V_0) satisfy the conditions listed in the present theorem, then by the definition of the map T_1 and Theorem 3.2, we have

$$\begin{aligned}\|T_1(\eta,V)\|_{L^p} \leq & CE^2(1+t)^{\frac{5\delta}{2(1+\delta)}-1} \\ & + \int_0^t \Big\| A(t-s,D)\chi_2(D)\frac{\nabla}{1+b|D|^2}\cdot(\eta V)(s,x)\Big\|_{L^p}\mathrm{d}s \\ & + \int_0^t \Big\| A(t-s,D)(1-\chi_2(D))\frac{\nabla}{1+b|D|^2}\cdot(\eta V)(s,x)\Big\|_{L^p}\mathrm{d}s \\ & + \int_0^t \Big\| B(t-s,D)\chi_2(D)\cdot\frac{\nabla}{2(1+d|D|^2)}(|V|^2)(s,x)\Big\|_{L^p}\mathrm{d}s \\ & + \int_0^t \Big\| B(t-s,D)(1-\chi_2(D))\cdot\frac{\nabla}{2(1+d|D|^2)}(|V|^2)(s,x)\Big\|_{L^p}\mathrm{d}s \\ = & CE^2(1+t)^{\frac{5\delta}{2(1+\delta)}-1} + \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3 + \mathrm{I}_4.\end{aligned}$$

By Theorem 3.1, it follows that

$$\begin{aligned}\mathrm{I}_1 &\leq C\int_0^t (1+t-s)^{\frac{5\delta}{2(1+\delta)}-1}\|\nabla\cdot(\eta V)(s,x)\|_{L^q}\mathrm{d}s, \\ \mathrm{I}_2 &\leq C\int_0^t \mathrm{e}^{-C_1(t-s)}\Big\|\frac{1}{1+b|D|^2}(\eta V)(s,x)\Big\|_{W^{2,p}}\mathrm{d}s, \\ \mathrm{I}_3 &\leq C\int_0^t (1+t-s)^{\frac{5\delta}{2(1+\delta)}-1}\|\nabla(|V|^2)(s,x)\|_{L^q}\mathrm{d}s, \\ \mathrm{I}_4 &\leq C\int_0^t \mathrm{e}^{-C_1(t-s)}\Big\|\frac{1}{1+d|D|^2}(|V|^2)(s,x)\Big\|_{W^{2,p}}\mathrm{d}s.\end{aligned}$$

To obtain the boundedness of I_1, I_3, noting that θ satisfying $\frac{1}{2q} = \frac{1-\theta}{2} + \frac{\theta}{p}$,

$$\|\eta\nabla\cdot V\|_{L^q} \leq \|\nabla\eta(s,x)\|_{L^p}^{\theta}\|\nabla\eta(s,x)\|_{L^2}^{1-\theta}\|V(s,x)\|_{L^p}^{\theta}\|V(s,x)\|_{L^2}^{1-\theta}$$

is valid by Lemma 3.4 as well as the inequalities for $\|V\cdot\nabla\eta\|_{L^q}$ and $\|\nabla(|V|^2)\|_{L^q}$, we have

$$\mathrm{I}_1 + \mathrm{I}_3 \leq 2CE^2\int_0^t (1+t-s)^{\frac{5\delta}{2(1+\delta)}-1}(1+s)^{\theta(\frac{5\delta}{2(1+\delta)}-1)-\theta(1+\beta)-2(1-\theta)\nu-\frac{1-\theta}{2}}\mathrm{d}s.$$

Since it is clear that

$$\theta\Big(\frac{5\delta}{2(1+\delta)}-1\Big)-\theta-2(1-\theta)\nu-\frac{1-\theta}{2} \to -\frac{5}{4}, \quad \text{as } \delta\to 0,$$

it implies that

$$\theta\Big(\frac{5\delta}{2(1+\delta)}-1\Big)-\theta(1+\beta)-2(1-\theta)\nu-\frac{1-\theta}{2} < -1,$$

if we take a sufficient small $\delta_0 > 0$, such that $0 < \delta < \delta_0$. Hence there holds

$$\mathrm{I}_1 + \mathrm{I}_3 \leq 2CE^2(1+t)^{\frac{5\delta}{2(1+\delta)}-1}$$

by Lemma 3.3. Noting the L^p-boundedness of pseudo-differential operator and the Sobolev embedding theorem in $\mathbb{R}^2$, that is, $\|u\|_{L^\infty} \le C\|u\|_{W^{1,p}}$ for large p, we have

$$\begin{aligned}
\mathrm{I}_2 + \mathrm{I}_4 &\le C\int_0^t \mathrm{e}^{-C_1(t-s)}(\|\eta V\|_{L^p} + \||V|^2\|_{L^p})\mathrm{d}s \\
&\le C\int_0^t \mathrm{e}^{-C_1(t-s)}\|V\|_{L^p}(\|\eta\|_{L^\infty} + \|V\|_{L^\infty})\mathrm{d}s \\
&\le C\int_0^t \mathrm{e}^{-C_1(t-s)}\|V\|_{L^p}(\|\eta\|_{W^{1,p}} + \|V\|_{W^{1,p}})\mathrm{d}s \\
&\le 2CE^2\int_0^t \mathrm{e}^{-C_1(t-s)}(1+s)^{2(\frac{5\delta}{2(1+\delta)}-1)}\mathrm{d}s.
\end{aligned}$$

By Lemma 3.3, there also holds

$$\mathrm{I}_2 + \mathrm{I}_4 \le 2CE^2(1+t)^{\frac{5\delta}{2(1+\delta)}-1}.$$

Hence it implies

$$\|T_1(\eta, V)\|_{L^p} \le 5CE^2(1+t)^{\frac{5\delta}{2(1+\delta)}-1}. \tag{3.4}$$

By Theorem 3.2, it is clear that

$$\begin{aligned}
\|DT_1(\eta, V)\|_{L^p} \le{}& CE^2(1+t)^{\frac{5\delta}{2(1+\delta)}-\frac{3}{2}} \\
&+ \int_0^t \Big\|DA(t-s,D)\chi_2(D)\frac{\nabla}{1+b|D|^2}\cdot(\eta V)(s,x)\Big\|_{L^p}\mathrm{d}s \\
&+ \int_0^t \Big\|DA(t-s,D)(1-\chi_2(D))\frac{\nabla}{1+b|D|^2}\cdot(\eta V)(s,x)\Big\|_{L^p}\mathrm{d}s \\
&+ \int_0^t \Big\|DB(t-s,D)\chi_2(D)\cdot\frac{\nabla}{2(1+d|D|^2)}(|V|^2)(s,x)\Big\|_{L^p}\mathrm{d}s \\
&+ \int_0^t \Big\|DB(t-s,D)(1-\chi_2(D))\cdot\frac{\nabla}{2(1+d|D|^2)}(|V|^2)(s,x)\Big\|_{L^p}\mathrm{d}s \\
={}& CE^2(1+t)^{\frac{5\delta}{2(1+\delta)}-\frac{3}{2}} + \mathrm{I}_5 + \mathrm{I}_6 + \mathrm{I}_7 + \mathrm{I}_8.
\end{aligned}$$

By the same reason for the above estimates of I_1 and I_3, there holds

$$\mathrm{I}_5 + \mathrm{I}_7 \le 2CE^2\int_0^t (1+t-s)^{\frac{5\delta}{2(1+\delta)}-\frac{3}{2}}(1+s)^{\theta(\frac{5\delta}{2(1+\delta)}-1)-\theta(1+\beta)-2(1-\theta)\nu-\frac{1-\theta}{2}}\mathrm{d}s.$$

Here $\beta > 0$ is taken sufficiently small, such that

$$\theta\Big(\frac{5\delta}{2(1+\delta)}-1\Big) - \theta(1+\beta) - 2(1-\theta)\nu - \frac{1-\theta}{2} < -1-\beta,$$

since it is valid that

$$\theta\Big(\frac{5\delta}{2(1+\delta)}-1\Big) - \theta - 2(1-\theta)\nu - \frac{1-\theta}{2} < -1$$

for the same δ since $0 < \theta < 1$. By Lemma 3.3, it implies

$$\mathrm{I}_5 + \mathrm{I}_7 \le 2CE^2(1+t)^{-1-\beta},$$

since $\frac{5\delta}{2(1+\delta)}-\frac{3}{2}<-1-\beta$ for $0<\delta<\delta_0$. Similar to I_2 and I_4, there holds

$$\begin{aligned}\mathrm{I}_6+\mathrm{I}_8&\le C\int_0^t \mathrm{e}^{-C_1(t-s)}(\|(\eta V)(s,x)\|_{W^{1,p}}+\|(|V|^2)(s,x)\|_{W^{1,p}})\mathrm{d}s\\&\le C\int_0^t \mathrm{e}^{-C_1(t-s)}\|V(s,x)\|_{W^{1,p}}(\|\eta(s,x)\|_{W^{1,p}}+\|V(s,x)\|_{W^{1,p}})\mathrm{d}s\\&\le 2CE^2\int_0^t \mathrm{e}^{-C_1(t-s)}(1+s)^{2(\frac{5\delta}{2(1+\delta)}-1)}\mathrm{d}s.\end{aligned}$$

By Lemma 3.3, it implies

$$\mathrm{I}_6+\mathrm{I}_8\le 2CE^2(1+t)^{-1-\beta}.$$

Since it is valid that $\frac{5\delta}{2(1+\delta)}-\frac{3}{2}<-1-\beta$ if β and δ_0 are taken sufficiently small, it implies

$$\|DT_1(\eta,V)\|_{L^p}\le 5CE^2(1+t)^{-1-\beta}. \tag{3.5}$$

By Lemma 3.2, we have

$$\begin{aligned}\|T_1(\eta,V)\|_{L^2}\le CE^2(1+t)^{-\nu}&+\int_0^t\Big\|A(t-s,D)\frac{\nabla}{1+b|D|^2}\cdot(\eta V)(s,x)\Big\|_{L^2}\mathrm{d}s\\&+\int_0^t\Big\|B(t-s,D)\cdot\frac{\nabla}{2(1+d|D|^2)}(|V|^2)(s,x)\Big\|_{L^2}\mathrm{d}s.\end{aligned}$$

By Theorem 3.1, L^2-boundedness of pseudo-differential operator and the Sobolev embedding theorem in $\mathbb{R}^2$, there holds

$$\begin{aligned}\|T_1(\eta,V)\|_{L^2}&\le CE^2(1+t)^{-\nu}+C\int_0^t(1+t-s)^{-\frac{1}{2}}\|(\eta V)(s,x)\|_{L^2}\mathrm{d}s\\&\quad+C\int_0^t(1+t-s)^{-\frac{1}{2}}\|(|V|^2)(s,x)\|_{L^2}\mathrm{d}s\\&\le CE^2(1+t)^{-\nu}+C\int_0^t(1+t-s)^{-\frac{1}{2}}\|\eta(s,x)\|_{L^2}\|V(s,x)\|_{W^{1,p}}\mathrm{d}s\\&\quad+C\int_0^t(1+t-s)^{-\frac{1}{2}}\|V(s,x)\|_{L^2}\|V(s,x)\|_{W^{1,p}}\mathrm{d}s\\&\le CE^2(1+t)^{-\nu}+2CE^2\int_0^t(1+t-s)^{-\frac{1}{2}}(1+s)^{(\frac{5\delta}{2(1+\delta)}-1)-\nu}\mathrm{d}s.\end{aligned}$$

Since $\nu\to-\frac{3}{8}$, and $(\frac{5\delta}{2(1+\delta)}-1)-\nu\to-\frac{11}{8}$ as $\delta\to 0$, by Lemma 3.3, it implies

$$\|T_1(\eta,V)\|_{L^2}\le 3CE^2(1+t)^{-\nu}, \tag{3.6}$$

if δ is taken sufficiently small. Similarly, there holds

$$\begin{aligned}\|DT_1(\eta,V)\|_{L^2}&\le CE^2(1+t)^{-\nu-\frac{1}{2}}+\int_0^t\Big\|DA(t-s,D)\frac{\nabla}{1+b|D|^2}\cdot(\eta V)(s,x)\Big\|_{L^2}\mathrm{d}s\\&\quad+\int_0^t\Big\|DB(t-s,D)\cdot\frac{\nabla}{2(1+d|D|^2)}(|V|^2)(s,x)\Big\|_{L^2}\mathrm{d}s\\&\le CE^2(1+t)^{-\nu-\frac{1}{2}}+C\int_0^t(1+t-s)^{-1}\|(\eta V)(s,x)\|_{L^2}\mathrm{d}s\\&\quad+C\int_0^t(1+t-s)^{-1}\|(|V|^2)(s,x)\|_{L^2}\mathrm{d}s\end{aligned}$$

$$
\begin{aligned}
&\le CE^2(1+t)^{-\nu-\frac{1}{2}} + C\int_0^t (1+t-s)^{-1}\|\eta(s,x)\|_{L^2}\|V(s,x)\|_{L^\infty}\mathrm{d}s\\
&\quad + C\int_0^t (1+t-s)^{-1}\|V(s,x)\|_{L^2}\|V(s,x)\|_{L^\infty}\mathrm{d}s\\
&\le CE^2(1+t)^{-\nu-\frac{1}{2}} + 2CE^2\int_0^t (1+t-s)^{-1}(1+s)^{(\frac{5\delta}{2(1+\delta)}-1)-\nu}\mathrm{d}s,
\end{aligned}
$$

that is,

$$
\|DT_1(\eta,V)\|_{L^2} \le 3CE^2(1+t)^{-\nu-\frac{1}{2}}. \tag{3.7}
$$

The other estimates for the map T_2 are similarly obtained, and the constant C in these estimates only depends on a, b, c, d, μ and δ. Together with (3.4), (3.5), (3.6) and (3.7), the proof is complete, if we choose E so small that $16CE^2 \le E$ holds.

Theorem 3.4 *Assume that $a,b,c,d>0$, $q=\frac{2+2\delta}{2+\delta}$, $p=\frac{2+2\delta}{3\delta}$ with $\delta>0$, and $(\eta_0,V_0)\in W^{4,q}\times W^{4,q}\times W^{4,q}$. Then there exist two sufficiently small positive numbers δ_0 and E, such that the map T is a contraction map from $Y_E\times Y_E\times Y_E$ into $Y_E\times Y_E\times Y_E$, if $0<\delta<\delta_0$ and $\|\eta_0\|_{W^{4,q}}+\|V_0\|_{W^{4,q}}<E^2$.*

Proof If $(\eta_1,V_1),(\eta_2,V_2)\in Y_E\times Y_E\times Y_E$, then we have

$$
\begin{aligned}
&\|T_1(\eta_1,V_1)-T_1(\eta_2,V_2)\|_{L^p}\\
&\le \int_0^t \Big\|A(t-s,D)\frac{\nabla}{1+b|D|^2}\cdot((\eta_1V_1)-(\eta_2V_2))(s,x)\Big\|_{L^p}\mathrm{d}s\\
&\quad + \int_0^t \Big\|B(t-s,D)\cdot\frac{\nabla}{2(1+d|D|^2)}(|V_1|^2-|V_2|^2)(s,x)\Big\|_{L^p}\mathrm{d}s\\
&\le C\int_0^t (1+t-s)^{\frac{5\delta}{2(1+\delta)}-1}(\|(\eta_1V_1-\eta_2V_2)(s,x)\|_{L^q}+\|(|V_1|^2-|V_2|^2)(s,x)\|_{L^q})\mathrm{d}s\\
&\quad + C\int_0^t \mathrm{e}^{-C_1(t-s)}(\|(\eta_1V_1-\eta_2V_2)(s,x)\|_{L^p}+\|(|V_1|^2-|V_1|^2)(s,x)\|_{L^p})\mathrm{d}s\\
&\le 8CE(1+t)^{\frac{5\delta}{2(1+\delta)}-1}\rho((\eta_1,V_1),(\eta_2,V_2)),
\end{aligned}
$$

where the constant C is only dependent on a, b, c, d, μ and δ. The other estimates are similarly obtained. The proof is complete, if E is taken so small that $24CE<1$ is valid.

Theorem 3.5 *Assume that $a,b,c,d,\mu>0$, $q=\frac{2+2\delta}{2+\delta}$, $p=\frac{2+2\delta}{3\delta}$ with $\delta>0$, and $(\eta_0,V_0)\in W^{4,q}\times W^{4,q}\times W^{4,q}$. Then there exist two sufficiently small positive numbers δ_0 and E, such that there exists a unique solution $(\eta,V)\in Y_E\times Y_E\times Y_E$ to equations* (3.1) *with initial data (η_0,V_0), if $0<\delta<\delta_0$ and $\|\eta_0\|_{W^{4,q}}+\|V_0\|_{W^{4,q}}\le E^2$. Moreover, for every $T\in(0,\infty)$, there hold*

$$
\begin{aligned}
(\eta,V)&\in L^\infty([0,T],W^{1,p}(\mathbb{R}^2)\times W^{1,p}(\mathbb{R}^2)\times W^{1,p}(\mathbb{R}^2))\\
&\quad\cap C([0,T],H^1(\mathbb{R}^2)\times H^1(\mathbb{R}^2)\times H^1(\mathbb{R}^2)),\\
(\eta_t,V_t)&\in L^\infty([0,T],L^p(\mathbb{R}^2)\times L^p(\mathbb{R}^2)\times L^p(\mathbb{R}^2))\\
&\quad\cap C([0,T],L^2(\mathbb{R}^2)\times L^2(\mathbb{R}^2)\times L^2(\mathbb{R}^2)).
\end{aligned}
$$

Proof The existence of the unique solution $(\eta, V) \in Y_E \times Y_E \times Y_E$ to the equations (3.1) with initial data (η_0, V_0) follows directly from Banach's contraction principle. It is clear that

$$\begin{aligned}(\eta, V) \in\, & L^\infty((0,\infty), W^{1,p}(\mathbb{R}^2) \times W^{1,p}(\mathbb{R}^2) \times W^{1,p}(\mathbb{R}^2)) \\ & \cap L^\infty((0,\infty), H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2)),\end{aligned}$$

since $(\eta, V) \in Y_E \times Y_E \times Y_E$ is valid. From the proof of the continuity in t of the solution to the linearized equations (3.2) in Theorem 3.2 (see [7]), we know that the operators $A(t, D)$, $B(t, D)$, etc. are continuous in t with respect to the corresponding norms. If we observe

$$\begin{aligned}\eta(t, x) &= A(t, D)\eta_0(x) + B(t, D) \cdot V_0(x) \\ &\quad - \int_0^t A(t-s, D) \frac{\nabla}{1 + b|D|^2} \cdot (\eta V)(s, x) \mathrm{d}s \\ &\quad - \int_0^t B(t-s, D) \cdot \frac{\nabla}{2(1 + d|D|^2)} (|V|^2)(s, x) \mathrm{d}s,\end{aligned}$$

by the continuity of Bochner integral and the operators $A(t, D)$, $B(t, D)$, we have

$$\eta \in C([0, T], H^1(\mathbb{R}^2))$$

for every $T \in (0, \infty)$. If (3.1) is transformed into

$$\eta_t = (I - b\Delta)^{-1}(a\Delta\nabla \cdot V - \nabla \cdot V - \nabla \cdot (\eta V)),$$

by the property of Bessel potential $(I - b\Delta)^{-1}$, it implies that

$$\eta_t \in L^\infty([0, T], L^p(\mathbb{R}^2)) \cap C([0, T], L^2(\mathbb{R}^2))$$

for every $T \in (0, \infty)$. The remainders are similarly obtained.

Remark 3.1 Assume that $a, b, c, d, \mu > 0$, $q = \frac{2+2\delta}{2+\delta}$, $p = \frac{2+2\delta}{3\delta}$ with $\delta > 0$, and $(\eta_0, V_0) \in H^\infty(\mathbb{R}^2) \times H^\infty(\mathbb{R}^2) \times H^\infty(\mathbb{R}^2)$. Then there exist two sufficiently small positive numbers δ_0 and E, such that there exists a unique solution $(\eta, V) \in C^1(\overline{\mathbb{R}}_+, C^\infty(\mathbb{R}^2) \times C^\infty(\mathbb{R}^2) \times C^\infty(\mathbb{R}^2))$ to equations (3.1) with initial data (η_0, V_0), if $0 < \delta < \delta_0$ and $\|\eta_0\|_{W^{4,q}} + \|V_0\|_{W^{4,q}} \le E^2$.

Proof By Theorem 3.5, the existence of the solutions

$$(\eta, V) \in L^\infty(\mathbb{R}_+, W^{1,p}(\mathbb{R}^2) \times W^{1,p}(\mathbb{R}^2) \times W^{1,p}(\mathbb{R}^2)) \cap C(\overline{\mathbb{R}}_+, H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2))$$

to equations (3.1) has been obtained. Since the proof for (η_t, V_t) is analogous to the one for (η, V), we have to show that

$$\begin{aligned}(\eta, V) \in\, & L^\infty(\overline{\mathbb{R}}_+, W^{N+1,p}(\mathbb{R}^2) \times W^{N+1,p}(\mathbb{R}^2) \times W^{N+1,p}(\mathbb{R}^2)) \\ & \cap C_b(\overline{\mathbb{R}}_+, H^{N+1}(\mathbb{R}^2) \times H^{N+1}(\mathbb{R}^2) \times H^{N+1}(\mathbb{R}^2))\end{aligned}$$

to equations (3.1), if $(\eta_0, V_0) \in W^{N+4,q} \times W^{N+4,q} \times W^{N+4,q}$ for every non-negative integer N satisfies $\|\eta_0\|_{W^{4,q}} + \|V_0\|_{W^{4,q}} \le E^2$. We only prove the case $N = 1$, since the case for $N \ge 1$

could be proved by induction on N. For every fixed positive number $T > 0$, from (3.3) we have

$$
\begin{aligned}
\nabla_j^h \eta(t,x) &= A(t,D)\nabla_j^h \eta_0(x) + B(t,D)\cdot \nabla_j^h V_0(x) \\
&\quad - \int_0^t A(t-s,D)\frac{\nabla}{1+b|D|^2}\cdot ((\nabla_j^h \eta(s,x))(V(s,x+h)) + (\eta(s,x))(\nabla_j^h V(s,x)))\mathrm{d}s \\
&\quad - \int_0^t B(t-s,D)\cdot \frac{\nabla}{2(1+d|D|^2)}((\nabla_j^h V(s,x))(V(s,x+h)+V(s,x)))(s,x)\mathrm{d}s.
\end{aligned}
$$

Here the difference quotient of u is defined by $\nabla_j^h u = \frac{u(x+he_j)-u(x)}{h}$, where e_j is the unit vector of the j-th axis. With the expression of $\nabla_j^h \eta(s,x)$ and Gronwall's inequality, we have

$$
\begin{aligned}
\|\nabla_j^h \eta(s,\cdot)\|_{W^{1,p}} &\le C(T), \\
\|\nabla_j^h V(s,\cdot)\|_{W^{1,p}} &\le C(T),
\end{aligned}
$$

where $C(T)$ is a constant depending on T, and independent of j and h. By the weak convergence of L^p, we have $(\eta(t,\cdot),V(t,\cdot)) \in W^{2,p}\times W^{2,p}\times W^{2,p}$ for every $t\in[0,T]$.

By means of Bootstrap arguments and using the standard inequality, we can easily finish the proof.

In the case of $a = 0$, $c > 0$, we omit all the details of the proofs and list the main results, since the method used in the case of $a > 0$, $c > 0$ also works in this case.

Remark 3.2 Assume that $a = 0$, $b,c,d,\mu > 0$, and $q = \frac{2+2\delta}{2+\delta}$, $p = \frac{2+2\delta}{3\delta}$ with $\delta > 0$, and $(\eta_0,V_0) \in W^{4,q}\times W^{3,q}\times W^{3,q}$. Then there exist two sufficiently small positive numbers δ_0 and E, such that there exists a unique solution (η,V) to equations (3.1) with initial data (η_0,V_0), if $0<\delta<\delta_0$ and $\|\eta_0\|_{W^{4,q}} + \|V_0\|_{W^{3,q}} \le E^2$. Moreover, for every $T\in(0,\infty)$, there hold

$$
\begin{aligned}
(\eta,V) &\in L^\infty([0,T],W^{2,p}(\mathbb{R}^2)\times W^{1,p}(\mathbb{R}^2)\times W^{1,p}(\mathbb{R}^2)) \\
&\quad \cap C([0,T],H^2(\mathbb{R}^2)\times H^1(\mathbb{R}^2)\times H^1(\mathbb{R}^2)), \\
(\eta_t,V_t) &\in L^\infty([0,T],W^{1,p}(\mathbb{R}^2)\times L^p(\mathbb{R}^2)\times L^p(\mathbb{R}^2)) \\
&\quad \cap C([0,T],H^1(\mathbb{R}^2)\times L^2(\mathbb{R}^2)\times L^2(\mathbb{R}^2)).
\end{aligned}
$$

Remark 3.3 Assume that $a = 0$, $b,c,d,\mu > 0$, and $q = \frac{2+2\delta}{2+\delta}$, $p = \frac{2+2\delta}{3\delta}$ with $\delta > 0$, and $(\eta_0,V_0)\in H^\infty(\mathbb{R}^2)\times H^\infty(\mathbb{R}^2)\times H^\infty(\mathbb{R}^2)$. Then there exist two sufficiently small positive numbers δ_0 and E, such that there exists a unique solution $(\eta,V)\in C^1(\overline{\mathbb{R}}_+,C^\infty(\mathbb{R}^2)\times C^\infty(\mathbb{R}^2)\times C^\infty(\mathbb{R}^2))$ to equations (2.1) with initial data (η_0,V_0), if $0<\delta<\delta_0$ and $\|\eta_0\|_{W^{4,q}} + \|V_0\|_{W^{3,q}} \le E^2$.

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