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Bloch Constant of Holomorphic Mappings on the Unit Ball of $\mathbb{C}^n$***

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Abstract In this paper, the authors establish distortion theorems for various subfamilies $H_k(\mathbb{B})$ of holomorphic mappings defined in the unit ball in $\mathbb{C}^n$ with critical points, where k is any positive integer. In particular, the distortion theorem for locally biholomorphic mappings is obtained when k tends to $+\infty$. These distortion theorems give lower bounds on $|\det f'(z)|$ and $\operatorname{Re}\det f'(z)$. As an application of these distortion theorems, the authors give lower and upper bounds of Bloch constants for the subfamilies $\beta_k(M)$ of holomorphic mappings. Moreover, these distortion theorems are sharp. When $\mathbb{B}$ is the unit disk in $\mathbb{C}$, these theorems reduce to the results of Liu and Minda. A new distortion result of $\operatorname{Re}\det f'(z)$ for locally biholomorphic mappings is also obtained.

Keywords Bloch constant, Holomorphic mappings, Locally biholomorphic mappings, Critical points
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1 Introduction

In the case of one complex variable, the following Bloch theorem is well known.

Theorem 1.1 (Cf. [2]) *There exists an absolute constant $r > 0$ such that if $f \in H(D)$ and $f'(0) = 1$, then some open subset of $U \subset D$ is mapped biholomorphically by f onto some disk with radius r.*

The Bloch constant B is defined as the greatest lower bound of r for all above f. Since de Branges solved the Bieberbach conjecture, finding the exact value of the Bloch constant is the number one important problem in the geometric function theory of one complex variable. Though the precise value of the Bloch constant is still unsolved, Ahlfors [1] and Heins [10] showed that $B \geq \frac{\sqrt{3}}{4}$. In 1988, Bonk [4] established the following distortion theorem

$$\operatorname{Re} f'(z) \geq \frac{\sqrt{3} - |z|}{(1 - \frac{|z|}{\sqrt{3}})^3}, \quad |z| \leq \frac{1}{\sqrt{3}}.$$

By making use of the above inequality, Bonk gave a simple new proof to $B \geq \frac{\sqrt{3}}{4}$ and improved the lower bound of the Bloch constant to $B \geq \frac{\sqrt{3}}{4} + 10^{-14}$.

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In the case of several complex variables, however, Wu [15] pointed out that the Bloch theorem fails unless we have some restrictive assumption on holomorphic mapping f. For example, for positive integer n, set $f_n(z_1, z_2) = (nz_1, \frac{1}{n}z_2)$. Then $f_n \in H(\mathbb{B})$ and $|\det J_{f_n}(0)| = 1$, where $\mathbb{B}$ is the unit ball of $\mathbb{C}^2$. But the image of f_n does not contain the univalent ball of radius $\frac{1}{n}$. In order that there is a positive Bloch constant, it is necessary to restrict the class of holomorphic mappings in higher dimensions. Bochner [3], Hahn [7], Harris [9] and Takahashi [14] obtained the Bloch theorem on holomorphic quasiregular mappings and established the corresponding lower bounds of Bloch constants on the unit ball of $\mathbb{C}^n$. While the lower bounds of Bloch constants cannot reduce to that of Ahlfors in one complex variable.

Liu [12], however, discussed some subfamilies of Bloch mappings defined on the unit ball $\mathbb{B}$ in $\mathbb{C}^n$. Moreover, lower and upper bounds of the corresponding Bloch constants were obtained. In particular, the result can reduce to that of Ahlfors when $\mathbb{B}$ is the unit disk in $\mathbb{C}$. FitzGerald and Gong [5] gave the estimates of Bloch constants on the domains of classical type in the sense of Hua [11]. More general results have been obtained by Gong [8] and Gong and Yan [16].

In this paper, we will establish distortion theorems (see Theorem 1.2) for subfamilies $H_k(\mathbb{B})$ of holomorphic mappings with critical points defined in the unit ball of $\mathbb{C}^n$. By making use of the distortion theorems, we obtain lower and upper bounds of Bloch constants $B_k(M)$ (see Theorem 1.3). As a corollary, we obtain a distortion theorem and estimates of Bloch constant for locally biholomorphic mappings if we let k tend to $+\infty$. Our theorems reduce to the results of Liu and Minda [13] when $\mathbb{B}$ is the unit disk. When we take $k = 1$ and let k tend to $+\infty$, our results reduce to the results of Liu [12]. Moreover, we obtain an interesting distortion theorem of $\operatorname{Re}\det f'(z)$ about locally biholomorphic mappings (see Corollary 1.1(b)).

In all what follows, we first introduce some notation. Let D be the unit disk in the complex plane $\mathbb{C}$. Denote by $\mathbb{C}^n$ as the n-dimensional complex Hilbert space with the inner product and the absolute value given by

$$\langle z, w\rangle = \sum_{j=1}^{n} z_j \overline{w}_j, \quad |z| = (\langle z, z\rangle)^{\frac{1}{2}},$$

where $z,\ w \in \mathbb{C}^n$. Let $\mathbb{B}$ be the open unit ball in $\mathbb{C}^n$, i.e., $\mathbb{B} = \{z \in \mathbb{C}^n : |z| < 1\}$. The unit sphere of $\mathbb{C}^n$ is denoted by $\partial\mathbb{B} = \{z \in \mathbb{C}^n : |z| = 1\}$. Denote by $\mathbb{B}(x, r)$ as the ball of radius r with the center x. Let $H(\mathbb{B})$ be the set of all holomorphic mappings from $\mathbb{B}$ to $\mathbb{C}^n$. Throughout the paper, we will write a point $z \in \mathbb{C}^n$ as a column vector in the following $n \times 1$ matrix form

$$z = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix}.$$

For a holomorphic mapping $f \in H(\mathbb{B})$, we also write f as the $n \times 1$ matrix form

$$f = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix},$$

where all of f_i are holomorphic functions from $\mathbb{B}$ to $\mathbb{C}$. The derivative of a mapping $f \in H(\mathbb{B})$ at a point $a \in \mathbb{B}$ is the complex Jacobian of f given by

$$f'(a) = \Big(\frac{\partial f_i}{\partial z_j}\Big)_{z=a}.$$

Then $f'(a)$ is a linear mapping from $\mathbb{C}^n$ to $\mathbb{C}^n$. Denote by $\|f'(a)\|$ the norm of complex Jacobian matrix $f'(a)$.

A holomorphic mapping $f \in H(\mathbb{B})$ is said to be locally biholomorphic if $\det f'(z) \neq 0$ for every $z \in \mathbb{B}$. A point z_0 is called a critical point of f if $\det f'(z_0) = 0$ (in one complex variable, such a point is called a branch point).

Definition 1.1 (Cf. [12]) *A holomorphic mapping $f \in H(\mathbb{B})$ is called a Bloch mapping if the family*

$$F_f = \{g : g(z) = f(\varphi(z)) - f(\varphi(0)),\ \varphi \in \mathrm{Aut}(\mathbb{B})\}$$

is a normal family. The Bloch semi-norm of the Bloch mapping $f(z)$ is defined as

$$\|f\|_{\mathcal{B}} = \sup\Big\{\Big\|\frac{\partial(f\circ\varphi)}{\partial z}(0)\Big\| : \varphi \in \mathrm{Aut}(\mathbb{B})\Big\}.$$

Definition 1.2 (Cf. [12]) *Suppose $f \in H(\mathbb{B})$. We define the prenorm $\|f\|_0$ of f by*

$$\|f\|_0 = \sup\{|\det(g'(0))|^{\frac{1}{n}} : g \in F_f\} = \sup\{(1-|z|^2)^{\frac{n+1}{2n}}|\det(f'(z))|^{\frac{1}{n}} : z \in \mathbb{B}\}.$$

It is clear that $\|f\|_0$ is invariant under the group of holomorphic automorphisms $\mathrm{Aut}(\mathbb{B})$.

Definition 1.3 *For any positive integer k, we define $H_k(\mathbb{B})$ by*

$$H_k(\mathbb{B}) = \{f \in H(\mathbb{B}) : \forall\, a \in \mathbb{B},\ \text{if}\ \det f'(a) = 0,\ \text{then}\ \det f'(z) = O(|z-a|^k)\}.$$

It is clear that $H_1(\mathbb{B}) = H(\mathbb{B})$ and $H_{k+1}(\mathbb{B}) \subset H_k(\mathbb{B})$. Let $H_{\mathrm{loc}}(\mathbb{B})$ be the class of locally biholomorphic mappings from $\mathbb{B}$ to $\mathbb{C}^n$. Then we have $H_{\mathrm{loc}}(\mathbb{B}) \subset H_k(\mathbb{B})$.

In this paper, we will consider the following subfamilies of $H(\mathbb{B})$:

$$\beta_k(M) = \{f \in H_k(\mathbb{B}) : \|f\|_{\mathcal{B}} \le M\}, \qquad 1 \le M < +\infty.$$
$$\beta_{\mathrm{loc}}(M) = \{f \in H_{\mathrm{loc}}(\mathbb{B}) : \|f\|_{\mathcal{B}} \le M\}, \quad 1 \le M < +\infty.$$

Given a holomorphic mapping $f \in H(\mathbb{B})$, we denote by $r(a,f)$ the radius of the biggest univalent ball of f centered at $f(a)$ (a univalent ball $\mathbb{B}(f(a),r) \subset f(\mathbb{B})$ means that f maps an open subset of $\mathbb{B}$ containing the point a biholomorphically onto this ball). Let $r(f) = \sup\{r(a,f) : a \in \mathbb{B}\}$. Denote by $B_k(M)$ the Bloch constant of the family $\beta_k(M)$, i.e.,

$$B_k(M) = \inf\{r(f) : f \in \beta_k(M),\ \det f'(0) = 1\}.$$

We also write $B_{\mathrm{loc}}(M)$ as the Bloch constant of the family $\beta_{\mathrm{loc}}(M)$, i.e.,

$$B_{\mathrm{loc}}(M) = \inf\{r(f) : f \in \beta_{\mathrm{loc}}(M),\ \det f'(0) = 1\}.$$

Then we have $B_{\mathrm{loc}}(M) = \lim\limits_{k\to+\infty} B_k(M)$.

Now we are ready to present our theorems and corollaries.

Theorem 1.2 *If $f \in H_k(\mathbb{B})$, $\|f\|_0 = 1$ and $\det f'(0) = 1$, then*
(a)

$$|\det f'(z)| \geq \frac{\left(1 - \sqrt{\frac{n+k+1}{k}}|z|\right)^k}{\left(1 - \sqrt{\frac{k}{n+k+1}}|z|\right)^{n+k+1}}$$

for $|z| \leq \sqrt{\frac{k}{n+k+1}}$. Moreover, the above inequality cannot be improved.
(b)

$$\mathrm{Re}\det f'(z) \geq \frac{\left(1 - \sqrt{\frac{n+k+1}{k}}|z|\right)^k}{\left(1 - \sqrt{\frac{k}{n+k+1}}|z|\right)^{n+k+1}}$$

for $|z| \leq \frac{2\sqrt{k(n+k+1)}}{(n+1)(k+1)+2k}$. Moreover, the above inequality cannot be improved.

When $n = 1$ and $k = 1$, Theorem 1.2 reduces to the Bonk's distortion theorem. Moreover, if we let k tend to $+\infty$, then we can obtain the following Corollary 1.1.

Corollary 1.1 *If $f \in H_{\mathrm{loc}}(\mathbb{B})$, $\|f\|_0 = 1$ and $\det f'(0) = 1$, then*
(a)

$$|\det f'(z)| \geq (1 - |z|)^{-(n+1)} \exp\left\{\frac{-(n+1)|z|}{1 - |z|}\right\}$$

for all $z \in \mathbb{B}$. Moreover, the above inequality cannot be improved.
(b)

$$\mathrm{Re}\det f'(z) \geq (1 - |z|)^{-(n+1)} \exp\left\{\frac{-(n+1)|z|}{1 - |z|}\right\}$$

for $|z| \leq \frac{2}{n+3}$. Moreover, the above inequality cannot be improved.

Remark 1.1 When $n = 1$, Corollary 1.1 reduces to Theorem 1 in [13]. By making use of $H_{\mathrm{loc}}(\mathbb{B}) \subset H_k(\mathbb{B})$, we get the above distortion theorem of $|\det f'(z)|$ which is a new proof of the result in [12]. We also obtain a new distortion theorem of $\mathrm{Re}\det f'(z)$ for $H_{\mathrm{loc}}(\mathbb{B})$.

Theorem 1.2 leads to the following estimates of Bloch constants for $\beta_k(M)$.

Theorem 1.3 *If $f \in \beta_k(M)$, $\|f\|_0 = 1$ and $\det f'(0) = 1$, then the Bloch constant $B_k(M)$ of $\beta_k(M)$ satisfies the following inequality:*

$$M^{1-n} \geq B_k(M) \geq \int r(0, f) \geq M^{1-n} \int_0^{\sqrt{\frac{k}{n+k+1}}} \frac{(1-t^2)^{n-1}\left(1 - \sqrt{\frac{n+k+1}{k}}t\right)^k}{\left(1 - \sqrt{\frac{k}{n+k+1}}t\right)^{n+k+1}} dt.$$

Letting $k \to +\infty$, we can easily obtain the following Corollary 1.2 which is established by Liu [12].

Corollary 1.2 *If $f \in \beta_{\mathrm{loc}}(M)$, $\|f\|_0 = 1$ and $\det f'(0) = 1$, then the Bloch constant $B_{\mathrm{loc}}(M)$ of $\beta_{\mathrm{loc}}(M)$ satisfies the following inequality:*

$$M^{1-n} \geq B_{\mathrm{loc}}(M) \geq \int r(0, f) \geq M^{1-n} \int_0^1 \frac{(1+t)^{n-1}}{(1-t)^2} \exp\left\{\frac{-(n+1)t}{1-t}\right\} dt \geq \frac{M^{1-n}}{n+1}.$$

When $n = 1$ and $M = 1$, $B_{\mathrm{loc}}(1)$ reduces to the well-known Landau constant L (see [1]).

2 Some Lemmas

In order to prove the desired theorems, we need the following lemmas.

Lemma 2.1 (Cf. [13]) *Let g be a holomorphic function on $D\cup\{1\}$. $g(D)\subset D$, $g(1)=1$ and all zeros of g have multiplicity at least k. If $g'(1)=k$, then*

(a) $|g(x)|\ge x^k$ *for $x\in[0,1)$ with equality for some x if and only if $g(z)=z^k$.*

(b) $\operatorname{Re} g(x)\ge x^k$ *for $\frac{k-1}{k+1}\le x<1$ with equality for some x if and only if $g(z)=z^k$.*

Lemma 2.2 *If $f\in H(\mathbb{B})$, $\|f\|_0=1$ and $\det f'(0)=1$, then*

$$|\det f'(z)|=1+o(|z|).$$

It can be showed by the inequality

$$|\det f'(z)|\le\frac{1}{(1-|z|^2)^{\frac{n+1}{2}}},$$

which is obtained by the definition of $\|f\|_0=1$.

Lemma 2.3 (Cf. [12]) *Suppose that f is a Bloch mapping on the unit ball $\mathbb{B}$, then we have*

$$\|f'(z)\|\le\frac{\|f\|_{\mathcal{B}}}{1-|z|^2}.$$

Lemma 2.4 (Cf. [6]) *Suppose that $A=(a_{ij})$ is an $n\times n$ complex matrix. If $\|A\|>0$, then for any unit vector $\xi\in\partial\mathcal{B}^n$, the following inequality holds:*

$$|A\xi|\ge\frac{|\det A|}{\|A\|^{n-1}}.$$

3 Proofs of the Theorems

Proof of Theorem 1.2 (a) For any $\xi\in\partial\mathbb{B}$, we set

$$g(u)=\Big(\frac{1-a^2}{1-a^2u}\Big)^{n+1}\det f'\Big(\frac{a-au}{1-a^2u}\xi\Big),$$

where $a=a(n,k)=\sqrt{\frac{k}{n+k+1}}$.

Then g is a holomorphic function in the closure of D and $g(1)=1$. From Lemma 2.2, we get $g'(1)=k$. It is clear that all zeros of g have multiplicity at least k from the definition of $H_k(\mathbb{B})$. For any $u\in D$, since $\|f\|_0=1$, we obtain that

$$\begin{aligned}|g(u)|&=\Big|\frac{1-a^2}{1-a^2u}\Big|^{n+1}\Big|\det f'\Big(\frac{a-au}{1-a^2u}\xi\Big)\Big|\\&\le\Big|\frac{1-a^2}{1-a^2u}\Big|^{n+1}\frac{1}{\big(1-\big|\frac{a-au}{1-a^2u}\big|^2\big)^{\frac{n+1}{2}}}\\&=\Big|\frac{1-a^2}{1-a^2u}\Big|^{n+1}\Big(\frac{|1-a^2u|^2}{(1-a^2)(1-|au|^2)}\Big)^{\frac{n+1}{2}}\\&=\Big(\frac{1-a^2}{1-|au|^2}\Big)^{\frac{n+1}{2}}\le 1.\end{aligned}$$

Hence $g(D) \subset D$. Lemma 2.1 shows that $|g(u)| \geq u^k$ for $u \in [0,1]$. Hence we have

$$|\det f'(v\xi)| \geq \frac{1}{(1-av)^{n+1}} \left(\frac{1}{a} \frac{a-v}{1-av} \right)^k = \frac{(1-\frac{1}{a}v)^k}{(1-av)^{n+k+1}},$$

where $v = \frac{a-au}{1-a^2u}$, i.e., $u = \frac{1}{a}\frac{a-v}{1-av}$ and $\frac{1-a^2}{1-a^2u} = 1-av$.

When $u \in [0,1]$, we have $v \in \left[0, \sqrt{\frac{k}{n+k+1}}\right]$.

Take $z \in \mathbb{B}$ such that $|z| \leq \sqrt{\frac{k}{n+k+1}}$. Let $v = |z|$, $\xi = \frac{z}{|z|}$. Then the above inequality implies that

$$|\det f'(z)| \geq \frac{\left(1 - \sqrt{\frac{n+k+1}{k}}|z|\right)^k}{\left(1 - \sqrt{\frac{k}{n+k+1}}|z|\right)^{n+k+1}}.$$

This completes the proof of part (a).

For the proof of part (b), we need only to show that $|v| \leq \frac{2\sqrt{k(n+k+1)}}{(n+1)(k+1)+2k}$. This can be calculated by $v = \frac{a-au}{1-a^2u}$ and $u \in [\frac{k-1}{k+1}, 1]$.

This inequality cannot be improved because we can find a holomorphic mapping $F \in H_k(\mathbb{B})$ satisfying $F(0) = 0$ and

$$F'(z) = \begin{pmatrix} \frac{\left(1-\sqrt{\frac{n+k+1}{k}}z_1\right)^k}{\left(1-\sqrt{\frac{k}{n+k+1}}z_1\right)^{n+k+1}} & 0 \\ 0 & I_{n-1} \end{pmatrix}.$$

Obviously, $\det F'(0) = 1$. Hence we will only show that $\|F\|_0 = 1$. By making use of the definition of $\|F\|_0 = \sup\{(1-|z|^2)^{\frac{n+1}{2n}} |\det F'(z)|^{\frac{1}{n}} : z \in \mathbb{B}\}$, we need only prove that $(1-|z|^2)^{\frac{n+1}{2}} |\det F'(z)| \leq 1$ for every $z \in \mathbb{B}$.

Let $T(z_1) = \frac{\sqrt{\frac{k}{n+k+1}} - z_1}{1-\sqrt{\frac{k}{n+k+1}}z_1}$. Then $T(D) \in \mathrm{Aut}(D)$ and $1 - |T(z_1)|^2 = \frac{n+1}{n+k+1} \frac{1-|z_1|^2}{\left|1-\sqrt{\frac{k}{n+k+1}}z_1\right|^2}$. Hence

$$\begin{aligned}
(1-|z|^2)^{\frac{n+1}{2}} |\det F'(z)| &= \frac{(1-|z|^2)^{\frac{n+1}{2}} \left|1 - \sqrt{\frac{n+k+1}{k}}z_1\right|^k}{\left|1 - \sqrt{\frac{k}{n+k+1}}z_1\right|^{n+k+1}} \\
&\leq \left(\frac{n+k+1}{k}\right)^{\frac{k}{2}} \left(\frac{1-|z_1|^2}{\left|1 - \sqrt{\frac{k}{n+k+1}}z_1\right|^2} \right)^{\frac{n+1}{2}} |T(z_1)|^k \\
&\leq \left(\frac{n+k+1}{k}\right)^{\frac{k}{2}} \left(\frac{n+k+1}{n+1}\right)^{\frac{n+1}{2}} (1-|T(z_1)|^2)^{\frac{n+1}{2}} |T(z_1)|^k \\
&\leq 1.
\end{aligned}$$

The last inequality is obtained by the inequality

$$(1-x^2)^{\frac{n+1}{2}} x^k \leq \left(\frac{n+1}{n+k+1}\right)^{\frac{n+1}{2}} \left(\frac{k}{n+k+1}\right)^{\frac{k}{2}} \quad \text{for } 0 \leq x \leq 1.$$

Thus $\|F\|_0 = 1$ and the inequality of Theorem 1.2 cannot be improved.

Proof of Theorem 1.3 If we take

$$f(z)=\begin{pmatrix} M^{1-n}z_1 \\ Mz_2 \\ \vdots \\ Mz_n \end{pmatrix},$$

then we can obtain $M^{1-n}\geq B_k(M)$. Hence, we will only prove

$$B_k(M)\geq\int r(0,f)\geq M^{1-n}\int_0^{\sqrt{\frac{k}{n+k+1}}}\frac{(1-t^2)^{n-1}\Big(1-\sqrt{\frac{n+k+1}{k}}t\Big)^k}{\Big(1-\sqrt{\frac{k}{n+k+1}}t\Big)^{n+k+1}}dt.$$

Since $\det f'(0)=1$, there is a small ball centered at the origin such that the mapping f is biholomorphic on the small ball. When the ball in the range expands, the preimage reaches a point of the unit sphere $\partial\mathbb{B}$ or reaches a point at which $\det f'(z)=0$. Otherwise, we can enlarge the ball in range $f(\mathbb{B})$ by the estimate of $\det f'(z)$ in Theorem 1.2. So we only consider the case when $\det f'(z)$ is non-zero to obtain the estimate of $B_k(M)$. Take a straight line interval $\Gamma\subset f(\mathbb{B})$. The interval Γ starts at the origin and goes as far as it can with its preimage not running through the boundary of $\mathbb{B}$ or $\det f'(z)=0$. Note that $r(0,f)$ is the largest nonnegative number r such that there exists a domain $V\subset\mathbb{B}$, and f maps biholomorphically the V onto a ball centered at the origin with radius r, therefore

$$r(0,f)\geq\Big|\int_\Gamma dw\Big|=\int_\Gamma|dw|=\int_\gamma\Big|\frac{\partial f}{\partial z}\frac{dz}{|dz|}\Big|\cdot|dz|,$$

where $\gamma=f^{-1}(\Gamma)$.

By making use of Lemma 2.4, we have

$$\int_\gamma\Big|\frac{\partial f}{\partial z}\frac{dz}{|dz|}\Big|\cdot|dz|\geq\int_\gamma\frac{|\det f'(z)|}{\|\frac{\partial f}{\partial z}\|^{n-1}}d|z|.$$

In view of Theorem 1.2(a) and Lemma 2.3, we obtain the right-hand side of the preceding inequality is greater than or equal to

$$\int_0^{\sqrt{\frac{k}{n+k+1}}}\frac{(1-|z|^2)^{n-1}\Big(1-\sqrt{\frac{n+k+1}{k}}|z|\Big)^k}{M^{n-1}\Big(1-\sqrt{\frac{k}{n+k+1}}|z|\Big)^{n+k+1}}d|z|.$$

Thus we have

$$M^{1-n}\geq B_k(M)\geq\int r(0,f)\geq M^{1-n}\int_0^{\sqrt{\frac{k}{n+k+1}}}\frac{(1-t^2)^{n-1}\Big(1-\sqrt{\frac{n+k+1}{k}}t\Big)^k}{\Big(1-\sqrt{\frac{k}{n+k+1}}t\Big)^{n+k+1}}dt.$$

The proof of Theorem 1.3 is completed.

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