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Regions of Applicability of Aubry-Mather Theory for Non-convex Hamiltonian*

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Abstract Herman constructed an autonomous system of two degrees of freedom which says that in non-convex situations, oscillations do happen and Aubry-Mather Theory cannot apply (see the results due to W. F. Chen in 1992). In this paper, it is shown that although the orbits could visit a region far away from the initial point in phase space, they can only exist in some fixed regions in $I = (I_1, I_2)$ plane. Moreover, Aubry-Mather Theory can be applied outside the regions.

Keywords Twist map, Aubry-Mather Theory, Non-convex Hamiltonian
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1 Introduction

By far, the most celebrated theory of small perturbations of completely integrable Hamiltonian systems is the so-called KAM-Theory which was discovered by A. Kolmogorov, V. Arnold and J. Moser during the 1950s to 1970s. The theory guarantees the existence of invariant tori with Diophantine rotation numbers. It is natural to ask after perturbation what happens to the rest of the invariant tori of the unperturbed completely integrable system?

For the so-called twist maps S. Aubry [2–4] and J. Mather [17] (see [5, 18] for complete survey) established the existence of special invariant sets which are projected injectively to the circle and carry motions with any given rotation number. Moreover, the map preserves the cyclic order of points on any of those invariant sets. This theory is now well-known as Aubry-Mather Theory which is based on two key ingredients: (1) the variational principle for finding desired motions; (2) the regularity of projection of any order-preserving orbit to the circle. The variational method could be substituted by other topological methods (see [6, 14]). The regularity of the projection allows us to take the limits with respect to rotation numbers. On the other hand, solutions representing global minima in various variational problems associated to a twist map and posed without assuming preservation of order turn out to be order-preserving too.

Any attempt to apply Aubry-Mather Approach to the case of $n(\geq 2)$ degrees of freedom faces the obvious problem that the order-preserving property is no longer available. However, there are some results about the preservation of periodic orbits, and the earliest one of those results is Birkhoff-Lewis Theorem (see [8]), the accurate proof of which was given by Moser

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[20]. Conley and Zehnder [11] improved the results, and they discovered a remarkable global method for finding periodic orbits for symplectic maps and Hamiltonian systems under non-degenerate conditions. Bernstein and Katok [7] proved the existence of periodic orbits with any rational rotation vector with the assumption of positive (or negative) definiteness. Katok [15] also showed that there are infinitely many rotation vectors for which KAM tori do not exist.

For autonomous, nearly integrable, and positively (or negatively) definite systems with two-degree of freedom, one can always reduce them to area-preserving twist maps of the annulus or the cylinder [13]. But to indefinite systems, it is much more complicated. In this paper, we will show that for any fixed energy surface with energy not too small comparing with perturbation, one can still reduce the indefinite systems to exact area-preserving twist maps. Actually, the two lines $I_i = 0$ $(i = 1, 2)$ divide the whole energy surface to different connected components, and every connected component determines a two-dimensional, half-infinite cylinder. Therefore, Aubry-Mather Theory can be applied here, which asserts the existence of periodic and quasi-periodic orbits, while KAM-Theory guarantees these orbits cannot wander in a large region on this energy surface.

Theorem 1.1 *Suppose that a C^r $(r > 1)$ function $g(\theta_1, \theta_2) \neq 0$ for any $\theta = (\theta_1, \theta_2) \in \mathbb{T}^2$ and the C^1 norm of $g(\theta_1, \theta_2)$ is bounded from below and above. Then there exists a constant $C > 0$, such that on every energy surface S_E with $E \leq -C\epsilon$ or $E \geq C\epsilon$, there are periodic and quasi-periodic orbits for the Hamiltonian system*

$$H(I_1, I_2, \theta_1, \theta_2) = \frac{1}{2} I_1^2 - \frac{1}{2} I_2^2 + \epsilon g(\theta_1, \theta_2), \quad \theta = (\theta_1, \theta_2) \in \mathbb{T}^2.$$

We know from [7] that periodic orbits for positively (or negatively) definite system have the regularity property, and their action variable cannot change too much, i.e., they have the uniform bound. However, this is not true for indefinite systems. Herman constructed an example (see [10]) which shows that if the period of a periodic orbit is large enough, then this orbit can visit a large region in the energy surface, i.e., its action variable does not have a uniform bound. In the last section of this paper, we will show that this kind of orbits can only exist in the energy surface with small energy in an absolute sense, i.e., they can only exist in a small neighborhood of the resonant region. We will also prove that these orbits do not have the ordering property.

Theorem 1.2 *For any small enough $\epsilon > 0$ and a constant C given in Theorem 1.1, if $q > \frac{1}{C}\epsilon^{-\frac{1}{2}}$, the q-periodic orbits constructed by Herman can only exist in an energy surface with energy $-C\epsilon < E < C\epsilon$. Moreover, if $q > \frac{2\pi}{\epsilon}$, these periodic orbits have no ordering property.*

2 Existence of Aubry-Mather Sets

2.1 Settings

In this paper, we consider a system with Hamiltonian

$$H(I_1, I_2, \theta_1, \theta_2) = \frac{I_1^2}{2} - \frac{I_2^2}{2} + \varepsilon g(\theta_1, \theta_2), \quad \theta = (\theta_1, \theta_2) \in \mathbb{T}^2. \tag{2.1}$$

To make the computations simple, we assume that the function g is a C^r $(r > 1)$ function of θ, and $0 < \sup_{\theta} |g(\theta)| < C_1$ for some constant $C_1 > 0$.

We restrict the Hamiltonian $H = \frac{I_1^2}{2} - \frac{I_2^2}{2} + \varepsilon g(\theta_1, \theta_2)$ to some fixed energy surface S_E, i.e., we set

$$E = \frac{I_1^2}{2} - \frac{I_2^2}{2} + \varepsilon g(\theta_1, \theta_2). \tag{2.2}$$

Case (1) If $E \le -C_1\epsilon$, we eliminate I_2 from the above equation (2.2). Thus we get two functions, and denote them by K_1 and K_2 respectively, i.e.,

$$\begin{cases} K_1(I_1, \theta_1, \theta_2) = \sqrt{I_1^2 - 2E + 2\epsilon g(\theta_1, \theta_2)}, \\ K_2(I_1, \theta_1, \theta_2) = -\sqrt{I_1^2 - 2E + 2\epsilon g(\theta_1, \theta_2)}. \end{cases}$$

Setting $T = -\theta_2$ as the new time variable and using (I_1, θ_1, E) as local coordinates on the energy surface S_E, one can easily check the formulas below

$$\begin{cases} \dfrac{\mathrm{d}I_1}{\mathrm{d}T} = -\dfrac{\dot{I}_1}{\dot{\theta}_2} = -\dfrac{\varepsilon \frac{\partial g}{\partial \theta_1}}{\sqrt{I_1^2 - 2E + 2\varepsilon g}} = -\dfrac{\partial K_1}{\partial \theta_1}, \\ \dfrac{\mathrm{d}\theta_1}{\mathrm{d}T} = -\dfrac{\dot{\theta}_1}{\dot{\theta}_2} = \dfrac{I_1}{\sqrt{I_1^2 - 2E + 2\varepsilon g}} = \dfrac{\partial K_1}{\partial I_1}, \end{cases} \tag{2.3}$$

where $\dot{I}_1, \dot{\theta}_1, \dot{\theta}_2$ denote the derivatives of I_1, θ_1, θ_2 with respect to time t. Formula (2.3) implies that function $K_1(I_1, \theta_1, T)$ is a new Hamiltonian function on the energy surface S_E, and similarly, function $K_2(I_1, \theta_1, T)$ defines a new Hamiltonian function on the energy surface S_E too.

Case (2) If $E \ge C_1\epsilon$, we can eliminate I_1 from equation (2.2). We also get two functions given by the following equations:

$$\begin{cases} K_3(I_2, \theta_2, \theta_1) = \sqrt{I_2^2 + 2E - 2\epsilon g(\theta_2, \theta_1)}, \\ K_4(I_2, \theta_2, \theta_1) = -\sqrt{I_2^2 + 2E - 2\epsilon g(\theta_2, \theta_1)}. \end{cases}$$

Set $T = -\theta_1$ and use (I_2, θ_2, E) as local coordinates on the energy surface S_E. It is easy to check that functions $K_3(I_2, \theta_2, T)$ and $K_4(I_2, \theta_2, T)$ are also Hamiltonian functions defined on S_E.

From the above discussions, we know that if the energy $E \ge C_1\epsilon$ or $E \le -C_1\epsilon$, the two lines $I_i = 0$ $(i = 1, 2)$ divide the energy surface into different connected open regions and K_1, K_2 are well defined in the region where $I_2 \ne 0$, while K_3, K_4 are well defined in the region where $I_1 \ne 0$. In the following, we will show that K_i $(i = 1, 2, 3, 4)$ is a convex or concave function with respect to the action variable in the region where it is well defined. Obviously, it is enough for us to prove that K_1 is convex with respect to the action variable I_1.

Since $E \le -C_1\varepsilon$ and $0 < |g(\theta_1, -T)| < C_1$, it is easy to know that

$$\frac{\partial^2 K_1}{\partial I_1^2}(I_1, \theta_1, T) = \frac{2\varepsilon g - 2E}{(I_1^2 - 2E + 2\varepsilon g)^{\frac{3}{2}}} > 0,$$

which implies that Hamiltonian function K_1 is convex with respect to the variable I_1. Consider the time-2π map f defined by the new Hamiltonian function K_1 on the energy surface S_E. It is clear that f is a Poincaré return map and the Mean Value Theorem of Integral implies that f has the following form:

$$f(I_1, \theta_1) = \Big(I_1 + \varepsilon g_1(I_1, \theta_1),\ \theta_1 + \frac{2\pi I_1}{\sqrt{I_1^2 - 2E + 2\varepsilon g}}\Big), \tag{2.4}$$

where g_1 arises from the perturbation term $g \neq 0$.

There is a standard way to define a time-periodic Hamiltonian $K_1(I_1, \theta_1, T)$ of the time period 2π, so that after the identification $-\theta_2 = T$, on the energy surface S_E, trajectories of K_1 and H locally coincide up to time reparametrization (see, e.g., [9, Section 4.1]). It implies that the Poincaré return map f of K_1 coincides with the time-2π map of H. Since the time-2π map of K_1 is symplectic, it preserves the canonical 2-form $\omega^2 = \mathrm{d}I_1 \wedge \mathrm{d}\theta_1$. Restriction of ω^2 onto S_E is an area form, which implies that f preserves a smooth area form on S_E in its local domain of definition. Moreover, from (2.4), one could check it easily that f is a twist map on S_E. In the same way, one can check that K_2 is concave with respect to I_1, and K_3 (K_4) is convex (concave) with respect to I_2 respectively, so that the time-2π maps of them are also twist maps.

So far, we have checked that f is an orientation-preserving (due to the fact that f is a Poincaré return map for a Hamiltonian system), area-preserving, and twist map. To use Aubry-Mather Theory, we still need to check that f has the property of exactness, which means that, given any loop γ which goes once around S_E, the area between γ and its image $\gamma' = f(\gamma)$ is zero, i.e.,

$$\int_{\gamma'-\gamma} I\mathrm{d}\theta = 0.$$

2.2 Proof of exactness

Because f defined by (2.4) is a smooth area-preserving twist map, so f can be defined by a so-called generating function $h : \mathbb{R}^2 \to \mathbb{R}^2$ in the following way. Let F: $\mathbb{R}^2 \to \mathbb{R}^2$ be a lift of f, given by $F(x + 2\pi, x') = F(x, x') + (2\pi, 0)$. Then $F(x, I) = (x', I')$ can be implicitly defined by the following equations:

$$\begin{cases} I = -\partial_1 h(x, x'), \\ I' = \partial_2 h(x, x'). \end{cases} \tag{2.5}$$

Here ∂_i ($i = 1, 2$) is the partial derivative with respect to the ith component, $h \in C^2$, $h(x + 2\pi, x' + 2\pi) = h(x, x')$, and $\partial_2\partial_1 h \leq -c < 0$ for some $c > 0$.

The value of $h(x, x')$ equals the minimal action getting from x to x' in time 2π, where action arises from the time-periodic Lagrangian system associated to F by Moser's Theorem (see [21]). In our case, $\frac{\partial^2 K_1}{\partial I_1^2}(I_1, \theta_1, T) > 0$ implies that our Hamiltonian is positive definite in the twist region. We can thus apply the Legendre transformation to K_1 and get a Lagrangian L, since our Poincaré return map is a time-2π map for K_1. Then the generating function h is given by the following formula (see, e.g., [1]):

$$h(\theta, \theta') = \inf_{\substack{\gamma(0)=\theta \\ \gamma(2\pi)=\theta'}} \int_{\theta}^{\theta'} L(\gamma(t), \gamma'(t), t)\mathrm{d}t,$$

where the infimum is taken over all absolutely continuous curves γ that start at $\gamma(0) = \theta$ and arrive at $\gamma(2\pi) = \theta'$ in time 2π. Since K_1 is periodic in θ, so is L. Therefore, h is also periodic in θ.

From (2.5), we know $I = -\partial_1 h(\theta, \theta')$, and from (2.4), we have $\theta_1' = \theta_1 + \frac{2\pi I_1}{\sqrt{I_1^2 - 2E + 2\varepsilon g}}$. Thus

$$\theta_1' = \theta_1 - \frac{2\pi\partial_1 h}{\sqrt{(\partial_1 h)^2 - 2E + 2\varepsilon g}}.$$

Differentiating the above equation with respect to θ_1' and rearranging, we have

$$\partial_2\partial_1 h = \Big(\frac{4\pi(E-\varepsilon g)}{(I_1^2-2E+2\varepsilon g)^{\frac{3}{2}}}\Big)^{-1} < 0. \tag{2.6}$$

The property of exactness for the map f defined by (2.4) comes from the following theorem.

Theorem 2.1 (Exactness Theorem (see [16, p. 1])) *Any smooth twist cylinder map f satisfying the monotone twist condition $\frac{\partial\theta'}{\partial I} > 0$ possesses a generating function h, such that the map is given by* (2.5) *implicitly. Moreover, the map is exact: $\int_{f(\gamma)-\gamma} I\mathrm{d}\theta = 0$, where γ is an arbitrarily smooth, noncontractible circle on the cylinder if and only if $h(\theta+1,\theta'+1) = h(\theta,\theta')$ and $\partial_2\partial_1 h < 0$.*

2.3 Introduction to Aubry-Mather Theorem

Aubry-Mather Theory studies the orbit structure of exact area-preserving twist (EAPT) maps by projecting orbits into their first components, which form the configuration space. Consider the space of configurations $\mathbb{R}^{\mathbb{Z}} = \{\Theta \mid \Theta : \mathbb{Z} \to \mathbb{R}\}$, that is, the space of bi-infinite sequences of real numbers with product topology. Given a function $h : \mathbb{R}^2 \to \mathbb{R}$, we extend h to arbitrary finite segments $(x_j, \cdots, x_k)$ $(j < k)$ of configurations $\Theta \in \mathbb{R}^{\mathbb{Z}}$ by

$$h(x_j, \cdots, x_k) = \sum_{i=j}^{k-1} h(x_i, x_{i+1}).$$

Say that segment is minimal or action-minimizing with respect to h, if

$$h(x_j, \cdots, x_k) \le h(x_j^*, \cdots, x_k^*)$$

for all $(x_j^*, \cdots, x_k^*)$ with $x_j^* = x_j$ and $x_k^* = x_k$.

A configuration $x \in \mathbb{R}^{\mathbb{Z}}$ is called minimal or action-minimizing with respect to h if every finite segment of x is minimal or action-minimizing with respect to h. The set of all action-minimizing trajectories is denoted by $\widetilde{\Sigma} = \widetilde{\Sigma}(h) \subset \mathbb{R}^{\mathbb{Z}}$.

A configuration $x \in \mathbb{R}^{\mathbb{Z}}$ is called stationary, if

$$\partial_2 h(x_{k-1}, x_k) + \partial_1 h(x_k, x_{k+1}) = 0 \tag{2.7}$$

for all $k \in \mathbb{Z}$.

This equation is an analogue of the Euler-Lagrange equation. Indeed, this equation says that the sum $\sum\limits_k h(x_k, x_{k+1})$ is extremized with respect to each x_k, because the formal derivative of the sum with respect to each x_k is zero. In particular, each minimal configuration is stationary.

By direct calculation using (2.6), we have the following lemma.

Lemma 2.1 *Suppose that h is a C^2 smooth function. Then there is one-to-one correspondence between stationary configurations and orbits of an EAPT Φ: $\mathbb{A} \to \mathbb{A}$, given by the following relation: Let $0 \le x_0 = \theta_0 < 1$. Then*

$$\begin{aligned}&\{x_k\}_{k\in\mathbb{Z}} \to \Phi^k(\theta_0, I_0) = (x_k \mod 2\pi, \partial_2 h(x_{k-1}, x_k)),\\ &\{\Phi^k(\theta_0, I_0)\}_{k\in\mathbb{Z}} \to \widetilde{\Phi}^k(x_k, I_k), \quad \{x_k\}_{k\in\mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}}.\end{aligned}$$

Here, $\widetilde{\Phi}$ is a lift of Φ.

Action-minimizing configurations have the following properties.

Theorem 2.2 (Aubry-Mather) *Every minimal configuration $\Theta \in \widetilde{\Sigma}$ has a rotation number $\rho(\Theta)$, and for every rotation number $\omega \in \mathbb{R}$, there is a minimal configuration $\Theta \in \widetilde{\Sigma}$ with $\rho(\Sigma) = \omega$. Moreover, there exists a circle homeomorphism ψ with rotation number ω, such that Θ is the orbit of the lift $\widetilde{\psi}$ of ψ.*

2.4 Existence of Aubry-Mather sets

In this section, we will prove that on the energy surface S_E there are Aubry-Mather sets existing in regions where the Hamiltonian functions K_i $(i = 1, 2, 3, 4)$ are well defined. More exactly, we will define a region $(\mathbb{S})$ in $I = (I_1, I_2)$-plane, and we will show if the action variable I is outside of the region $(\mathbb{S})$, then there are Aubry-Mather sets existing for some Hamiltonian function K_i $(i = 1, 2, 3, 4)$. To prove such a result, we need the following Theorem 2.3. Let us first explain the notations used in the theorem.

Consider an integrable symplectic diffeomorphism $f_0 : f_0(x, r) = (x + a(r), r)$, where $a : U \subset \mathbb{R}^n \to \mathbb{R}^n$ is a regular injective map. Add a small and periodic perturbation to f_0 and denote the new map by f. By saying the rotation vector v, we mean that if for some lift $F^m(x_0, r_0) = (x_m, r_m),\ m \in \mathbb{Z}$, the formula below holds:

$$\lim_{m\to\pm\infty} \frac{x_m - x_0}{m} = v.$$

Theorem 2.3 (see [15, Proposition 2]) *Suppose that C^1 norm of perturbation term is bounded from below and above. If a minimal orbit has the rotation vector v, then for all integers $n = 0, 1, 2, 3, \cdots$, there exists a constant $C_2 > 0$, such that the following holds:*

$$|r_n - a^{-1}(v)| < C_2 \varepsilon^{\frac{1}{2}}.$$

Proposition 2.1 *For any small enough ε, define region $(\mathbb{S})$ in the $I = (I_1, I_2)$ plane below:*

$$\begin{cases} I_1 - C_0\varepsilon^{\frac{1}{2}} < I_2 < I_1 + C_0\varepsilon^{\frac{1}{2}}, \\ -I_1 - C_0\varepsilon^{\frac{1}{2}} < I_2 < -I_1 + C_0\varepsilon^{\frac{1}{2}}, \\ -C_0\varepsilon^{\frac{1}{2}} < I_1 < C_0\varepsilon^{\frac{1}{2}}, \\ -C_0\varepsilon^{\frac{1}{2}} < I_2 < C_0\varepsilon^{\frac{1}{2}}, \end{cases} \tag{2.8}$$

where $C_0 = C_2 + 2\sqrt{C_1}$. Then after perturbation outside region $(\mathbb{S})$, there are Aubry-Mather sets for Hamiltonian function K_i $(i = 1, 2, 3, 4)$ with the same rotation number of unperturbed orbits.

Proof In Section 2.1, we have shown that outside region satisfies

$$-C_1\varepsilon < \frac{I_1^2}{2} - \frac{I_2^2}{2} + \varepsilon g < C_1\varepsilon.$$

We can always find a new Hamiltonian function whose time-2π map is a twist map. Therefore, Aubry-Mather Theory guarantees the existence of minimal orbits with appropriate rotation numbers. On the other hand, Theorem 2.3 implies that after ϵ-perturbation, the action variable of any minimal orbits changes less than $C_2\epsilon^{\frac{1}{2}}$.

Proof of Theorem 1.1 Proposition 2.1 directly shows $C = \frac{1}{2}C_0^2$. There are regions on the energy surface S_E, and these regions are homeomorphic to half-infinity cylinders, while on cylinders there are periodic and quasi-periodic orbits with a rotation number ω satisfying $|\omega| \geq |I_1| \geq C\epsilon^{\frac{1}{2}}$ or $|\omega| \geq |I_2| \geq C\epsilon^{\frac{1}{2}}$.

2.5 KAM result

Recall the basic KAM Theorem. Let $U \subset \mathbb{R}^n$ be an open-bounded set in $\mathbb{R}^n$, $\mathcal{M} := U \times \mathbb{T}^n$ be the phase space, and suppose that a Hamiltonian function has the form

$$H_\varepsilon(I, \theta) := H_0(I) + \varepsilon H_1(I, \theta)$$

with real-analytic functions H_0, H_1 and ε being a small real number. The variables (I, θ) are the standard symplectic "action-angle" variables, and the symplectic form is given by the formula below

$$\mathrm{d}I \wedge \mathrm{d}\theta := \sum_{i=1}^{n} \mathrm{d}I_i \wedge \mathrm{d}\theta_i.$$

Theorem 2.4 (Kolmogorov) *In any neighborhood of any torus $I_0 \times \mathbb{T}^n \subset \mathcal{M}$ such that*

$$\det \partial_{II}^2 H_0(I_0) = \det\Big(\frac{\partial^2 H_0}{\partial I_i \partial I_j}(I_0)\Big) \neq 0, \tag{2.9}$$

there exists a positive measure set of points in the phase space $\mathcal{M}$ which belongs to the analytic KAM tori for H_ε, provided that ε is small enough.

To the Hamiltonian function defined by equation (2.1), $H_0 = \frac{I_1^2}{2} - \frac{I_2^2}{2}$, a simple computation shows that $\det \partial_{II}^2 H_0(I_0) = -1 \neq 0$, so Theorem 2.4 implies that after ϵ-perturbation, for any $I = (I_1, I_2)$, there are KAM tori in the neighborhood of it. Especially, in any neighborhood of the lines

$$\pm I_1 - C_0\varepsilon^{\frac{1}{2}} = I_2 = \pm I_1 + C_0\varepsilon^{\frac{1}{2}}, \quad I_1 = \pm C_0\varepsilon^{\frac{1}{2}}, \quad I_2 = \pm C_0\varepsilon^{\frac{1}{2}},$$

there are KAM tori, and those KAM tori separate the whole energy surface in disjoint parts.

3 Regions Where the Orbits Constructed by Herman Exist

3.1 Herman's example

Consider the integrable map $f : T^*(\mathbb{T}^2) \to T^*(\mathbb{T}^2)$,

$$f(\theta, r) = (\theta + rB, r),$$

where $B = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $\theta = (\theta_1, \theta_2)$ and $r = (r_1, r_2)$.

For small ε, add an ε-size perburbation to f, such that the perturbed map in the covering space has the following form:

$$F(x, r) = (x_1 + r_2, x_2 + r_1, r_1 + \varepsilon \cos(2\pi(x_1 + r_2)), r_2). \tag{3.1}$$

For $j \geq 2$, the jth iterates of F is

$$\begin{aligned} F^j(x,r) &= (x_1^j, x_2^j, r_1^j, r_2^j) \\ &= \Big(x_1 + jr_2, x_2 + jr_1 + \varepsilon \sum_{k=1}^{j-1} (j-k)\cos\left(2\pi(x_1 + kr_2)\right), \\ &\quad r_1 + \varepsilon \sum_{k=1}^{j} \cos\left(2\pi(x_1 + kr_2)\right), r_2\Big). \end{aligned}$$

Take any integer $q \geq 2$ and an integer vector $\omega = (1,1)$. Then the action variables of an unperturbed tori with the rotation vector $\frac{\omega}{q}$ are $r = r_{\omega,q} = (\frac{1}{q}, \frac{1}{q})$. With this rotation vector and choosing x_2 arbitrarily, one can check it easily that $\{F^i(-\frac{1}{2q}, x_2, \frac{1}{q}, \frac{1}{q}), i \in \mathbb{Z}\}$ is a Birkhoff periodic orbit. $\Big($Let h be the generating function of F, and define $L_{\omega,q}(x) = \sum_{i=0}^{q-1} h(x^i, x^{i+1})$ as an action of a periodic orbit x with rotation number ω. x is called a Birkhoff periodic orbit if it satisfies $\frac{\partial L_{\omega,q}}{\partial x^i} = 0,\ \forall\, i \in \mathbb{Z}.\Big)$ For $1 \leq j \leq q$,

$$r_1^j - r_1 = \varepsilon \sum_{k=1}^{j} \cos\Big(2\pi\Big(k - \frac{1}{2}\Big)r_2\Big) = \varepsilon \frac{\sin(2\pi j r_2)}{2\sin(\pi r_2)},$$

where $r_2 = \frac{1}{q}$. Taking $j = [\frac{q}{4}]$ the integer part of $\frac{q}{4}$, for large q, we have

$$|r_1^j - r_1| \approx \frac{\varepsilon q}{2\pi}. \tag{3.2}$$

To this example, it is easy to check that the map defined in (3.1) has the following properties:

(1) F is an area-preserving map;

(2) F is a symplectic map;

(3) $\det(DF^q - Id) = 0$, i.e., the periodic orbits of F are all degenerate;

(4) For $q > \frac{2\pi}{\varepsilon}$, the action variables of those orbits will deviate more than 1.

To coincide with Section 2, we do the following transformations:

$$\begin{cases} I_1 = \dfrac{r_1 + r_2}{\sqrt{2}}, \quad I_2 = \dfrac{r_1 - r_2}{\sqrt{2}}, \\ \theta_1 = \dfrac{x_1 + x_2}{\sqrt{2}}, \quad \theta_2 = \dfrac{x_1 - x_2}{\sqrt{2}}. \end{cases} \tag{3.3}$$

It is easy to check that $\mathrm{d}r_1 \wedge \mathrm{d}x_1 + \mathrm{d}r_2 \wedge \mathrm{d}x_2 = \mathrm{d}I_1 \wedge \mathrm{d}\theta_1 + \mathrm{d}I_2 \wedge \mathrm{d}\theta_2$, so this is a symplectic transformation, and the map in new coordinates has the following form:

$$\begin{aligned} F(\theta_1, \theta_2, I_1, I_2) &= (\theta_1', \theta_2', I_1', I_2') \\ &= \Big(\theta_1 + I_1, \theta_2 - I_2, I_1 + \frac{\varepsilon}{\sqrt{2}} \cos\left(\sqrt{2}\,\pi(I_1 - I_2 + \theta_1 + \theta_2)\right), \\ &\quad I_2 + \frac{\varepsilon}{\sqrt{2}} \cos\left(\sqrt{2}\,\pi(I_1 - I_2 + \theta_1 + \theta_2)\right)\Big). \end{aligned} \tag{3.4}$$

The generating function of (3.4) is

$$h(\theta, \theta') = \frac{1}{2}(\theta_1' - \theta_1)^2 - \frac{1}{2}(\theta_2' - \theta_2)^2 + \frac{\varepsilon}{2\pi} \sin\sqrt{2}\,\pi(\theta_1' + \theta_2'),$$

where $\theta = (\theta_1, \theta_2), \theta' = (\theta_1', \theta_2') \in \mathbb{T}^2$.

Using the so-called generating function method (see, e.g., [19, Proposition 9.18] and [12, Theorem 58.9]), we know that the Hamiltonian function of (3.4) has the following form:

$$H(I_1, I_2, \theta_1, \theta_2) = \frac{I_1^2}{2} - \frac{I_2^2}{2} + \varepsilon g(I_1, I_2, \theta_1, \theta_2),$$

where g is a trigonometric function, and so it is bounded from below and above.

Proposition 3.1 *For $q > \frac{\sqrt{2}}{C_0}\epsilon^{-\frac{1}{2}}$, these periodic orbits given by* (3.4) *are all in region* ($\mathbb{S}$)*, i.e., these orbits are all on the energy surface S_E with energy $|E| < C\epsilon$.*

Proof From (3.3), we know action variables of any two points on the periodic orbits $\{F^i(-\frac{1}{2q}, x_2, \frac{1}{q}, \frac{1}{q}), i \in \mathbb{Z}\}$ all satisfy $I_1 - I_2 = \sqrt{2}r_2 = \frac{\sqrt{2}}{q}$, so if $q > \frac{\sqrt{2}}{C_0}\epsilon^{-\frac{1}{2}}$, we have $0 < I_1 - I_2 < C_0\epsilon^{\frac{1}{2}}$. This relation implies $|E| < \frac{1}{2}C_0^2\epsilon = C\epsilon$.

Project orbits defined in (3.1) into their angular components, and consider the "state space" $(\mathbb{R}^2)^{\mathbb{Z}} = \{\Theta \,|\, \Theta : \mathbb{Z} \to \mathbb{R}^2\}$, which is composed of bi-infinite sequences $y = (\cdots, y_{-1}, y_0, y_1, \cdots)$ of vectors in $\mathbb{R}^2$ with product topology and satisfies the following periodicity condition:

$$y_{i+q} = y_i + (1, 1).$$

We defined the "Aubry-graph" to be the union of the line segments in $\mathbb{R}^3$ joining (i, y_i) and $(i+1, y_{i+1})$ for any integer $i \in \mathbb{Z}$. Without loss of generality let $y_0 = (\frac{1}{2q}, x_2)$ for any fixed x_2, and we define the translation to be $(T_{(a,b)}y)_i = y_{i-a} + (b, b)$ for $a, b \in \mathbb{Z}$.

Proposition 3.2 *Assume $q > \frac{2\pi}{\epsilon}$. Let $\{F^i(-\frac{1}{2q}, x_2, \frac{1}{q}, \frac{1}{q}), i \in \mathbb{Z}\}$ be a periodic orbit with the rotation vector $\omega = (\frac{1}{q}, \frac{1}{q})$. Let y be the corresponding state with $y_0 = (-\frac{1}{2q}, x_2)$. There exists an $r \in \mathbb{Z}$, such that $(T_{r,0}y)$ intersects with y.*

Proof If $\frac{q}{4} \in \mathbb{Z}$, take $r = \frac{q}{4}$; or else take $r = \frac{q+1}{4}$. It is clear that if $q > \frac{2\pi}{\epsilon}$, the deviation of y_r is more than $\frac{3}{4}$, so the following computations are obvious:

$$(T_{r,0}y)_0 = y_{-r} > \frac{3}{4} - \frac{1}{2} = \frac{1}{4} > 0 = y_0,$$
$$(T_{r,0}y)_r = y_0 = 0 < \frac{3}{4} < y_r.$$

These two relation imply that $(T_{r,0}y)$ intersects with y.

Remark 3.1 In this example, one can check it easily that $\{F^i(-\frac{1}{2q} + \frac{1}{2}, x_2, \frac{1}{q}, \frac{1}{q}), i \in \mathbb{Z}\}$ is another family of periodic orbits with the rotation vector $\omega = (\frac{1}{q}, \frac{1}{q})$. Actually, $\{F^i(-\frac{1}{2q}, x_2, \pm\frac{1}{q}, \frac{1}{q}), i \in \mathbb{Z}\}$ and $\{F^i(-\frac{1}{2q} + \frac{1}{2}, x_2, \pm\frac{1}{q}, \frac{1}{q}), i \in \mathbb{Z}\}$ are periodic orbits with the rotation vector $\omega = (\pm\frac{1}{q}, \frac{1}{q})$; $\{F^i(\frac{1}{2q}, x_2, \pm\frac{1}{q}, -\frac{1}{q}), i \in \mathbb{Z}\}$ and $\{F^i(\frac{1}{2q} - \frac{1}{2}, x_2, \pm\frac{1}{q}, -\frac{1}{q}), i \in \mathbb{Z}\}$ are periodic orbits with the rotation vector $\omega = (\pm\frac{1}{q}, -\frac{1}{q})$. Almost in the same way, one can check that Propositions 3.1 and 3.2 are valid for all these periodic orbits.

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