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A Note on Residue Formulas for the Euler Class of Sphere Fibrations*

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Abstract This paper presents a definition of residue formulas for the Euler class of cohomology-oriented sphere fibrations ξ. If the base of ξ is a topological manifold, a Hopf index theorem can be obtained and, for the smooth category, a generalization of a residue formula is derived for real vector bundles given in [2].

Keywords Euler class, Sphere fibration, Hopf index theorem, Residue formulas
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1 Introduction

Let $\xi : E \xrightarrow{\pi} B$ be a fibration whose fibre has the R-cohomology of an $(r-1)$-sphere, where R is a commutative ring with a unit, π is surjective and B is a Hausdorff, path-connected, paracompact space.

We suppose that ξ is endowed with an R-orientation, i.e., a Thom class $t(\xi) \in H^r(Z_\pi, E; R) \cong R$ is given such that the natural inclusion $j_x : (C(\pi^{-1}(x)), \pi^{-1}(x)) \subset (Z_\pi, E)$ induces isomorphisms in cohomology for every $x \in B$, where $C(\pi^{-1}(x))$ is the cone over $\pi^{-1}(x)$ and Z_π is the cylinder of π, i.e., Z_π is the quotient of $E \times [0,1]$ by the equivalent relation identifying $(z,0)$ to $(z',0)$ whenever $\pi(z) = \pi(z')$.

Finally, we assume that the Thom homomorphism $H^p(B;R) \to H^{p+r}(Z_\pi, E; R)$, given by $\alpha \to \rho^*(\alpha)t(\xi)$, is an isomorphism. Here $\rho : Z_\pi \to B$ is given by $\rho([z,t]) = \pi(z)$.

Define then the Euler class $e(\xi) \in H^r(B;R)$ by the relation $j^*(t(\xi)) = \rho^*(e(\xi))$, where $j : Z_\pi \to (Z_\pi, E)$ is inclusion.

It is clear that the Euler class of ξ depends only on the equivalence class of the R-oriented fibration ξ.

The long exact sequence for the pair (Z_π, E), together with Thom isomorphism, gives the Gysin sequence

$$\cdots \to H^{p-1}(E;R) \to H^{p-r}(B;R) \to H^p(B;R) \xrightarrow{\pi^*} H^p(E;R) \to \cdots,$$

where the central map above is given by $\alpha \to \alpha.e(\xi)$.

The main example for an R-oriented fibration is given by the R-oriented sphere bundle associated to an R-oriented real vector bundle.

In this paper, we specify the definition of residue formulas for the Euler class associated to the given sections, find a Hopf index theorem, and by restricting to the smooth category and to real vector bundles, obtain a generalization of a result by Feng Huitao and Guo Enli [2].

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2 Residue Formulas

Let X be a closed subset of B. We say that a cross section σ of ξ over $U - X$ is equivalent to a cross section σ' of ξ over $U' - X$, where U and U' are open neighbourhoods of X, if and only if σ and σ' agree on $V - X$ for some open neighbourhood V of X.

The class $[\sigma]$ of a cross section σ defined on $U - X$ is called the germ of σ at X, and $\mathrm{Sec}(\xi, X)$ will denote the set of those germs of ξ at X.

A residue formula for the Euler class of ξ consists of giving a map $\mathrm{Res}_X : \mathrm{Sec}(\xi, X) \to H^r(B, B - X; R)$ for every closed subset X with $\mathrm{Sec}(\xi, X) \neq \emptyset$, such that the following two conditions hold:

(i) $e(\xi) = j_X^*(\mathrm{Res}_X([\sigma]))$ for all $[\sigma] \in \mathrm{Sec}(\xi, X)$, where $j_X^* : H^r(B, B - X; R) \to H^r(B; R)$ is induced by inclusion.

(ii) Let $X \subset Y$ be closed subsets of B, U a neighborhood of Y in B, σ_X a cross section of ξ over $U - X$ and σ_Y the restriction of σ_X to $U - Y$. Then

$$j_{XY}^*(\mathrm{Res}_X[\sigma_X]) = \mathrm{Res}_Y[\sigma_Y],$$

where $j_{XY}^* : H^r(B, B - X; R) \to H^r(B, B - Y; R)$ is induced by inclusion.

Observe that given a cross section σ of ξ over $B - X$, X closed in B, $e(\xi_{|B-X}) = 0$. So there is a class $\alpha_X \in H^r(B, B - X; R)$ such that $j_X^*(\alpha_X) = e(\xi)$. Thus, without condition (ii) above, the definition of a residue formula would be uninteresting.

The following theorem proves existence of residue formulas.

Theorem 2.1 *Let B be metrizable and π proper, i.e., $\pi^{-1}(K)$ is compact for any compact K of B. There exist then residue formulas for the Euler class of ξ.*

Proof Let σ be a cross section of ξ over $U - X$, where U is an open neighborhood of a closed subset X of B. We may choose a nonnegative real continuous function $f : B \to \mathbb{R}$ such that $f^{-1}(0) = X$, because B is metrizable and so perfectly normal.

Define $\sigma_f : U \to \rho^{-1}(U) \subset Z_\pi$ by

$$\sigma_f(x) = \begin{cases} [\sigma(x), f(x)], & \text{if } x \in U - X, \\ \sigma_0(x), & \text{if } x \in X, \end{cases}$$

where [ ] denotes the equivalence class and $\sigma_0(x) = [z, 0]$ for any $z \in \pi^{-1}(x)$.

It is clear that $\rho\sigma_f$ is the identity on U and σ_f is continuous because π is proper.

Define then $\mathrm{Res}_X([\sigma]) \in H^r(B, B - X; R)$ by the formula

$$i_{XC}^*(\mathrm{Res}_X([\sigma])) = \sigma_f^*(\bar{t}(\xi)),$$

where C is a closed neighborhood of X in U, i_{XC}^* is the excision isomorphism for the inclusion $(C, C - X) \subset (B, B - X)$, and $\bar{t}(\xi) \in H^r(Z_\pi, Z_\pi - \sigma_0(B); R)$ is the class applied to the Thom class $t(\xi)$ by the isomorphism associated to inclusion $(Z_\pi, E) \subset (Z_\pi, Z_\pi - \sigma_0(B))$.

If we replace f by another function f', we get a homotopy from σ_f to $\sigma_{f'}$ by $H(x, t) = [\sigma(x), (1 - t)f(x) + tf'(x)]$, which shows that the definition of Res_X is correct.

Finally, it is easy to check that we obtain in this way a residue formula.

Suppose now that X is a compact homologically locally connected subspace of B (see [3, Chapter 6]). This implies that the connected and path connected components of X coincide and they are open and closed in X. In particular, X has a finite number of connected components and we can choose an open neighborhood V_F for each connected component F of X, such that $\overline{V}_F \cap \overline{V}_{F'} = \emptyset$ whenever F and F' are distinct connected components of X.

Lemma 2.1 *The following diagram commutes:*

$$\begin{array}{ccccc} \mathrm{Sec}(\xi, X) & \xrightarrow{\mathrm{Res}_X} & H^r(B, B-X; R) & \searrow & \\ \downarrow & & \uparrow \sum\limits_F j^*_{FX} & & \oplus_F H^r(\overline{V}_F, \overline{V}_F - F; R) \\ \prod_F \mathrm{Sec}(\xi, F) & \xrightarrow{\oplus_F \mathrm{Res}_F} & \oplus_F H^r(B, B-F; R) & \nearrow & \end{array}$$

where F runs through the connected components of X, and $j_{FX} : (B, B-X) \subset (B, B-F)$.

Suppose now that B is a compact R-oriented topological manifold of dimension n and X is a closed homologically locally connected subspace of B, so that we have Alexander duality isomorphism $D_X : H^i(B, B-X; R) \to H_{n-i}(X; R)$ (see [3, Chapter 6]). Then we have the following Hopf index theorem for a given cross section σ of ξ over $B-X$.

Theorem 2.2

$$De(\xi) = \sum_F (i_F)_* D_F(\mathrm{Res}_F([\sigma])),$$

where F runs through the connected components of X, D represents Poincaré duality for B, D_F is Alexander duality for F, and $i_F : F \subset B$ inclusion.

Proof In fact, the previous lemma gives

$$\mathrm{Res}_X[\sigma] = \sum_F j^*_{FX}(\mathrm{Res}_F[\sigma]),$$

and so

$$D_X(\mathrm{Res}_X[\sigma]) = \sum_F (i_{FX})_*(D_F(\mathrm{Res}_F[\sigma]))$$

with $i_{FX} : F \subset X$, and then

$$De(\xi) = Dj^*_X(\mathrm{Res}_X([\sigma])) = (i_X)_* D_X \mathrm{Res}_X([\sigma]) = \sum_F (i_F)_* D_F(\mathrm{Res}_F[\sigma]),$$

where $i_x : X \subset B$.

Suppose now that each connected component F of X is an R-oriented closed topological manifold, $t_{FB} \in H^{n-n(F)}(B, B-F; R)$ the Thom class for the inclusion $F \subset B$, i.e., $D_F(t_{FB}) = [F]$, where $[F] \in H_{n(F)}(F; R)$ is the homology fundamental orientation class of F and $n(F)$ the dimension of F.

We define $\theta_F : H^i(F; R) \to H^{i+n-n(F)}(B, B-F; R)$ by the formula $D_F(\theta_F(\gamma)) = D(\gamma)$, where $D : H^i(F; R) \to H_{n(F)-i}(F; R)$ is Poincaré duality. In particular, $\theta_F(1) = t_{FB}$.

We also define the normal class of F in B by $e_{FB} = j^*_F(t_{FB})$, where $j_F : F \subset (B, B-F)$. Therefore the composition $i^*_F \circ \theta_F : H^i(F; R) \to H^{i+n-n(F)}(F; R)$ consists of right multiplication by e_{FB}.

If we further assume that B is a smooth manifold and F are smooth submanifolds with a normal bundle ν_F, then e_{FB} coincides with the Euler class $e(\nu_F)$, and in this case, composition $i^*_F \circ \theta_F : H^i(F; R) \to H^{i+n-n(F)}(F; R)$ is the map appearing in the Gysin sequence for ν_F and consists of right multiplication by $e(\nu_F)$.

Finally, suppose that $\xi : E \xrightarrow{\pi} B$ is an r-dimensional smooth real vector bundle, X a compact subset of B such that each connected component of X is a smooth submanifold, and σ is a cross section of ξ without zeros on $B-X$.

Define then $\mathcal{L}_\sigma(x) : T_x(B) \to \pi^{-1}(x)$ for all $x \in X$ such that $\sigma(x) = 0$ by $\mathcal{L}_\sigma(x)v = \sum\limits_{i=1}^r vf_i.e_i(x)$, where $e_1, \cdots, e_r$ is a basis of cross sections of ξ in a neighborhood of x such that $\sigma(x) = 0$.

It is clear that the above definition is correct because if ∇ is a linear connection on ξ, the linear map $(\nabla\sigma)_x : T_xB \to \pi^{-1}(x)$, given by $v \to \nabla_v\sigma$, extends $\mathcal{L}_\sigma(x)$ and $T_xF \subset \ker\mathcal{L}_\sigma(x)$ for all $x \in F$. Also assume that $T_xF = \ker\mathcal{L}_\sigma(x)$ for all $x \in F$, i.e., σ is nondegenerate in the sense of Bott [1].

We have then the following result.

Theorem 2.3 *The residue of σ at F is given by*

$$De(\xi_F/\mathcal{L}_\sigma(\nu_F)) = D_F(\mathrm{Res}_F[\sigma]),$$

and we have

$$De(\xi) = \sum_F (i_F)_* De(\xi_F/\mathcal{L}_\sigma(\nu_F)).$$

Proof In fact, we have the exact sequence of vector bundles over F

$$0 \to \nu_F \xrightarrow{\mathcal{L}_\sigma} \xi_F \to \xi_F/\mathcal{L}_\sigma(\nu_F) \to 0.$$

Therefore,

$$e(\xi_F) = e(\xi_F/\mathcal{L}_\sigma(\nu_F))e(\nu_F).$$

Consider then the commutative diagram

$$\begin{array}{ccccc} H^r(\pi^{-1}(F)) & \overset{\sigma^*}{\to} & H^r(F) & \overset{j^*}{\leftarrow} & H^r(\overline{V}_F) \\ \uparrow & & & & \uparrow \\ H^r(\pi^{-1}F, \pi^{-1}F - \sigma_0 F) & \overset{\lambda^*}{\leftarrow} & H^r(\pi^{-1}\overline{V}_F, \pi^{-1}\overline{V}_F - \sigma_0\overline{V}_F) & \overset{\sigma^*}{\to} & H^r(\overline{V}_F, \overline{V}_F - F) \end{array}$$

and so we have

$$e(\xi_F) = \sigma^*\pi^*e(\xi_F) = \sigma^*i^*(t(\xi_F)) = \sigma^*i^*\lambda^*(t(\xi_{\overline{V}_F})) = j^*i^*\sigma^*(t(\xi_{\overline{V}_F})) = j^*i^*i^*_{F\overline{V}_F}(\mathrm{Res}_F[\sigma]).$$

Let $\alpha_F(\sigma) \in H^{r-n+n(F)}(F)$ be given by $\theta_F(\alpha_F(\sigma)) = \mathrm{Res}_F[\sigma]$. So we get

$$e(\xi_F) = i^*_F\theta_F(\alpha_F(\sigma)) = \alpha_F(\sigma).e_{FB} = \alpha_F(\sigma).e(\nu_F).$$

Whence

$$\alpha_F(\sigma) = e(\xi_F/\mathcal{L}_\sigma(\nu_F)),$$

and so

$$De(\xi_F/\mathcal{L}_\sigma(\nu_F)) = D_F(\mathrm{Res}_F[\sigma]).$$

Therefore

$$D_X(\mathrm{Res}_X[\sigma]) = \sum_F (i_{FX})_* D_F(\mathrm{Res}_F[\sigma]) = \sum_F (i_{FX})_* De(\xi_F/\mathcal{L}_\sigma(\nu_F)),$$

and so

$$De(\xi) = \sum_F (i_F)_* De(\xi_F/\mathcal{L}_\sigma(\nu_F)).$$

Observe that all we needed was that ξ_F contains a subbundle isomorphic to ν_F.

Remark 2.1 The theorem above holds for any coefficient ring R, in particular, for $\mathbb{Z}$ or $\mathbb{Z}/(2)$, and generalizes a formula given in [2].

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