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A Poincaré Inequality in a Sobolev Space with a Variable Exponent

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Abstract Let Ω be a domain in $\mathbb{R}^N$. It is shown that a generalized Poincaré inequality holds in cones contained in the Sobolev space $W^{1,p(\cdot)}(\Omega)$, where $p(\cdot):\overline{\Omega}\to[1,\infty[$ is a variable exponent. This inequality is itself a corollary to a more general result about equivalent norms over such cones. The approach in this paper avoids the difficulty arising from the possible lack of density of the space $\mathcal{D}(\Omega)$ in the space $\{v\in W^{1,p(\cdot)}(\Omega); \operatorname{tr} v = 0 \text{ on } \partial\Omega\}$. Two applications are also discussed.

Keywords Poincaré inequality, Sobolev spaces with variable exponent
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1 Introduction

All notions and definitions not defined in this introduction are defined in the next section.

Let Ω be a domain in $\mathbb{R}^N$ and let $\Gamma := \partial\Omega$. The usual Lebesgue and Sobolev spaces, i.e., with a constant exponent $p \geq 1$ are denoted $L^p(\Omega)$ and $W^{1,p}(\Omega)$, while the Lebesgue and Sobolev spaces with a variable exponent are denoted $L^{p(\cdot)}(\Omega)$ and $W^{1,p(\cdot)}(\Omega)$, the variable exponent $p(\cdot)$ being now any function in the space $L^\infty(\Omega)$ that satisfies $p(x)\geq 1$ for almost all $x\in\Omega$ (the main properties of these spaces are reviewed in Section 2).

The classical Poincaré inequality in Sobolev spaces with a constant exponent asserts that, given any real number $p\geq 1$, there exists a constant $C=C(\Omega,p)$ such that

$$\int_\Omega |v(x)|^p \,\mathrm{d}x \leq C\sum_{i=1}^N \int_\Omega |\partial_i v(x)|^p \,\mathrm{d}x \quad \text{for all } v\in W_0^{1,p}(\Omega),$$

where

$$W_0^{1,p}(\Omega) := \{v\in W^{1,p}(\Omega); \operatorname{tr} v = 0 \text{ on } \Gamma\}.$$

However, as shown in [10], this inequality does not necessarily hold when the constant exponent p is replaced by a variable exponent $p(\cdot):\Omega\to[1,\infty[$ (even if $p\in\mathcal{C}(\overline{\Omega})$ and $N=1$; see also the counter-example given in [7]). As shown by Kováčik and Rákosník [9], this inequality does hold for all $v\in\mathcal{D}(\Omega)$ if $p\in\mathcal{C}(\overline{\Omega})$ and if the norm $\|\cdot\|_{L^p(\Omega)}$ is replaced by the Luxemburg norm (the appropriate norm when the exponent is variable (see Section 2)).

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Still, this does not provide a satisfactory generalization of the classical Poincaré inequality, because the equality which holds for a constant exponent p:

$$\overline{\mathcal{D}(\Omega)}^{\|\cdot\|_{W^{1,p}(\Omega)}} = \{v \in W^{1,p}(\Omega); \operatorname{tr} v = 0 \text{ on } \Gamma\}$$

is replaced, in the case of a variable exponent $p(\cdot)$, by the inclusion

$$\overline{\mathcal{D}(\Omega)}^{\|\cdot\|_{W^{1,p(\cdot)}(\Omega)}} \subset \{v \in W^{1,p(\cdot)}(\Omega); \operatorname{tr} v = 0 \text{ on } \Gamma\},$$

which may be strict, unless additional assumptions are imposed on the function $p(\cdot)$. Consequently, even if such a generalized Poincaré inequality can be established for all functions in the space $\mathcal{D}(\Omega)$, it cannot be extended in this case by means of a density argument to all functions in the space $W^{1,p(\cdot)}(\Omega)$ whose traces vanish on Γ.

The purpose of this paper is to provide a way, new to the best of our knowledge, to circumvent such shortcomings. Instead of proving the Poincaré inequality for functions in $\mathcal{D}(\Omega)$ and then extending it by a density argument (as is often done for a constant exponent), we obtain the Poincaré inequality as an immediate corollary of an equivalence of norms in ad hoc cones of the space $W^{1,p(\cdot)}(\Omega)$ (see Theorems 3.1 and 3.2).

This approach has several advantages. Firstly, it can be carried out in the space $\{v \in W^{1,p(\cdot)}(\Omega); \operatorname{tr} v = 0 \text{ on } \Gamma\}$, thus avoiding the difficulty arising from the possible lack of density of the space $\mathcal{D}(\Omega)$ in this space; secondly, it can be extended at no extra cost to the space $\{v \in W^{1,p(\cdot)}(\Omega); \operatorname{tr} v = 0 \text{ on } \Gamma_0\}$, where Γ_0 is a subset of Γ with $\mathrm{d}\Gamma\text{-meas}\,\Gamma_0 > 0$ (see Theorem 4.1); thirdly, it allows to establish a Poincaré inequality in ad hoc cones in $W^{1,p(\cdot)}(\Omega)$, and thus in subsets (associated for instance with a Nemytsky operator (see Theorem 4.2)) that need not be subspaces.

2 Notations and Preliminaries

All vector and function spaces considered in this paper are real.

This section gathers various definitions and basic properties related to Lebesgue and Sobolev spaces with variable exponents. For proofs and references, see [6, 7].

The Lebesgue measure in $\mathbb{R}^N$ is denoted $\mathrm{d}x$. Throughout this paper, Ω designates a domain in $\mathbb{R}^N$, i.e., a bounded and connected open subset of $\mathbb{R}^N$ whose boundary Γ is Lipschitz-continuous, the set Ω being locally on the same side of Γ. A measure, denoted $\mathrm{d}\Gamma$, can then be defined on Γ. For details, see, e.g., [1] or [11].

No distinction will be made between $\mathrm{d}x$-measurable, resp. $\mathrm{d}\Gamma$-measurable, functions and their equivalence classes modulo the relation of $\mathrm{d}x$-almost everywhere, resp. $\mathrm{d}\Gamma$-almost everywhere, equality.

A cone (with vertex at the origin) in a vector space V is a subset U of V with the property that $\lambda \geq 0$ and $v \in V$ implies $\lambda v \in U$.

Unless a specific notation is used, $\|\cdot\|_V$ denotes the norm in a normed vector space V and $\overline{A}^{\|\cdot\|_V}$ designates the closure in V of a subset A of V with respect to the norm $\|\cdot\|_V$.

Given two normed vector spaces V and W, the notation $V \hookrightarrow W$, resp. $V \Subset W$, means that $V \subset W$ and the canonical injection from V into W is continuous, resp. compact.

The notation $\mathcal{D}(\Omega)$ denotes the space of functions that are infinitely differentiable in Ω and whose support is a compact subset of Ω. Given a real number $p \geq 1$, the notations $L^p(\Omega)$, $W^{1,p}(\Omega)$, and

$$W_0^{1,p}(\Omega) := \overline{\mathcal{D}(\Omega)}^{\|\cdot\|_{W^{1,p}(\Omega)}} = \{v \in W^{1,p}(\Omega); \operatorname{tr} v = 0 \text{ on } \Gamma\},$$

designate the usual Lebesgue and Sobolev spaces; "usual" means here that the exponent $p \geq 1$ is a constant.

Given a function $p(\,\cdot\,) \in L^\infty(\Omega)$ that satisfies

$$1 \leq p^- := \operatorname*{essinf}_{x \in \Omega} p(x),$$

the Lebesgue space $L^{p(\,\cdot\,)}(\Omega)$ with a variable exponent $p(\,\cdot\,)$ is defined as

$$L^{p(\,\cdot\,)}(\Omega) := \Big\{ v : \Omega \to \mathbb{R};\ v \text{ is } \mathrm{d}x\text{-measurable and } \int_\Omega |v(x)|^{p(x)}\,\mathrm{d}x < \infty \Big\}.$$

Likewise, given a function $q(\,\cdot\,) \in L^\infty(\Gamma)$ that satisfies

$$1 \leq \operatorname*{essinf}_{y \in \Gamma} q(y),$$

the Lebesgue space $L^{q(\,\cdot\,)}(\Gamma)$ with a variable exponent $q(\,\cdot\,)$ is defined as

$$L^{q(\,\cdot\,)}(\Gamma) := \Big\{ v : \Gamma \to \mathbb{R};\ v \text{ is } \mathrm{d}\Gamma\text{-measurable and } \int_\Gamma |v(y)|^{q(y)}\,\mathrm{d}y < \infty \Big\}.$$

Theorem 2.1 *Let Ω be a domain in $\mathbb{R}^N$.*

(a) *Let $p(\cdot) \in L^\infty(\Omega)$ be such that $p^- \geq 1$. Equipped with the norm*

$$v \in L^{p(\,\cdot\,)}(\Omega) \to \|v\|_{0,p(\,\cdot\,)} := \inf \Big\{ \lambda \geq 0;\ \int_\Omega \Big| \frac{v(x)}{\lambda} \Big|^{p(x)} \mathrm{d}x \leq 1 \Big\},$$

the space $L^{p(\,\cdot\,)}(\Omega)$ is a separable Banach space. If $p^- > 1$, the space $L^{p(\,\cdot\,)}(\Omega)$ is uniformly convex, hence reflexive.

(b) *Let $p_1(\,\cdot\,) \in L^\infty(\Omega)$ and $p_2(\,\cdot\,) \in L^\infty(\Omega)$ be such that $p_1^- \geq 1$ and $p_2^- \geq 1$. Then*

$$L^{p_2(\,\cdot\,)}(\Omega) \hookrightarrow L^{p_1(\,\cdot\,)}(\Omega)$$

if and only if

$$p_1(x) \leq p_2(x) \quad \text{for almost all } x \in \Omega.$$

(c) *Given $p(\,\cdot\,) \in L^\infty(\Omega)$ such that $p^- > 1$, let $p'(\,\cdot\,) \in L^\infty(\Omega)$ be defined by*

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1 \quad \text{for almost all } x \in \Omega.$$

Then, given any function $u \in L^{p'(\,\cdot\,)}(\Omega)$, the linear functional

$$\ell : v \in L^{p(\,\cdot\,)}(\Omega) \to \int_\Omega u(x)v(x)\,\mathrm{d}x \in \mathbb{R}$$

is continuous; conversely, given any continuous linear functional $\ell : L^{p(\,\cdot\,)}(\Omega) \to \mathbb{R}$, there exists one, and only one, function $u_\ell \in L^{p'(\,\cdot\,)}(\Omega)$ such that

$$\ell(v) = \int_\Omega u_\ell(x) v(x)\,\mathrm{d}x \quad \text{for all } v \in L^{p(\,\cdot\,)}(\Omega).$$

Remark 2.1 The norm $\|\cdot\|_{0,p(\,\cdot\,)}$, which is often called the Luxemburg norm, reduces to the norm $\|\cdot\|_{L^p(\Omega)}$ if the function $p(\,\cdot\,)$ is constant and equal to p.

Given a function $p(\,\cdot\,) \in L^\infty(\Omega)$ that satisfies $p^- \geq 1$, the Sobolev space $W^{1,p(\,\cdot\,)}(\Omega)$ with a variable exponent $p(\,\cdot\,)$ is defined as

$$W^{1,p(\,\cdot\,)}(\Omega) := \{v \in L^{p(\,\cdot\,)}(\Omega);\ \partial_i v \in L^{p(\,\cdot\,)}(\Omega),\ 1 \leq i \leq N\},$$

where, for each $1 \leq i \leq N$, ∂_i denotes the distributional derivative operator with respect to the i-th variable.

Theorem 2.2 *Let Ω be a domain in $\mathbb{R}^N$.*

(a) *Let $p(\,\cdot\,) \in L^\infty(\Omega)$ be such that $p^- \geq 1$. Equipped with the norm*

$$v \in W^{1,p(\,\cdot\,)}(\Omega) \to \|v\|_{1,p(\,\cdot\,)} := \|v\|_{0,p(\,\cdot\,)} + \sum_{i=1}^N \|\partial_i v\|_{0,p(\,\cdot\,)},$$

the space $W^{1,p(\,\cdot\,)}(\Omega)$ is a separable Banach space. If $p^- > 1$, the space $W^{1,p(\,\cdot\,)}(\Omega)$ is reflexive.

(b) *Let $p_1(\,\cdot\,) \in L^\infty(\Omega)$ with $p_1^- \geq 1$ and $p_2(\,\cdot\,) \in L^\infty(\Omega)$ with $p_2^- \geq 1$ be such that*

$$p_1(x) \leq p_2(x) \quad \text{for almost all } x \in \Omega.$$

Then

$$W^{1,p_2(\,\cdot\,)}(\Omega) \hookrightarrow W^{1,p_1(\,\cdot\,)}(\Omega).$$

(c) *Let $p(\,\cdot\,) \in \mathcal{C}(\overline{\Omega})$ be such that $p^- \geq 1$. Given any $x \in \overline{\Omega}$, let*

$$p^*(x) := \frac{Np(x)}{N - p(x)}, \quad \text{if } p(x) < N \quad \text{and} \quad p^*(x) := \infty, \quad \text{if } p(x) \geq N,$$

and let there be a function $q(\,\cdot\,) \in \mathcal{C}(\overline{\Omega})$ that satisfies

$$1 \leq q(x) < p^*(x) \quad \text{for each } x \in \overline{\Omega}.$$

Then the following compact injection holds:

$$W^{1,p(\,\cdot\,)}(\Omega) \Subset L^{q(\,\cdot\,)}(\Omega).$$

Thus, in particular,

$$W^{1,p(\,\cdot\,)}(\Omega) \Subset L^{p(\,\cdot\,)}(\Omega).$$

Finally, we state several properties of traces of functions in the space $W^{1,p(\,\cdot\,)}(\Omega)$.

Theorem 2.3 *Let Ω be a domain in $\mathbb{R}^N$.*

(a) *Let $p(\,\cdot\,) \in L^\infty(\Omega)$ be such that $p^- \geq 1$. Since $W^{1,p(\,\cdot\,)}(\Omega) \hookrightarrow W^{1,1}(\Omega)$ (see Theorem 2.2(b)), the trace on Γ of any function $v \in W^{1,p(\,\cdot\,)}(\Omega)$ is well-defined as a function, denoted $\operatorname{tr} v$, in the space $L^1(\Gamma)$.*

(b) *Let there be a function $p(\,\cdot\,) \in \mathcal{C}(\overline{\Omega})$ that satisfies $p^- > 1$. Given any $x \in \Gamma$, we let*

$$p^\partial(x) := \frac{(N-1)p(x)}{N-p(x)}, \quad \text{if } p(x) < N \quad \text{and} \quad p^\partial(x) := \infty, \quad \text{if } p(x) \geq N,$$

and let there be a function $q(\,\cdot\,) \in \mathcal{C}(\Gamma)$ that satisfies

$$1 \leq q(x) < p^\partial(x) \quad \text{for each } x \in \Gamma.$$

Then, given any function $v \in W^{1,p(\,\cdot\,)}(\Omega)$, $\operatorname{tr} v \in L^{q(\,\cdot\,)}(\Gamma)$, the trace operator

$$\operatorname{tr} : W^{1,p(\,\cdot\,)}(\Omega) \to L^{q(\,\cdot\,)}(\Gamma)$$

defined in this fashion is compact. Thus, in particular, the trace operator

$$\operatorname{tr} : W^{1,p(\,\cdot\,)}(\Omega) \to L^{p(\,\cdot\,)}(\Gamma)$$

is compact.

3 Equivalence of Norms and Poincaré Inequality in a Cone in $W^{1,p(\,\cdot\,)}(\Omega)$

We now establish the main results of this paper, which extend to Sobolev spaces with variable exponents established by Jebelean and Precup [8] for the usual Sobolev spaces.

Recall that, as a domain in $\mathbb{R}^N$, the open set Ω is in particular connected. So, a function $v \in W^{1,1}(\Omega)$ that satisfies $\partial_i v = 0$ a.e. in Ω, $1 \leq i \leq N$, is a constant function.

Theorem 3.1 *Let Ω be a domain in $\mathbb{R}^N$. Let $p(\,\cdot\,) \in \mathcal{C}(\overline{\Omega})$ be such that $p^- > 1$, and let $U \neq \{0\}$ be a cone in the space $W^{1,p(\,\cdot\,)}(\Omega)$ that is sequentially weakly closed and that does not contain nonzero constant functions. In other words,*

$$v_k \in U,\ k \geq 1, \text{ and } v_k \rightharpoonup v \text{ weakly in } W^{1,p(\,\cdot\,)}(\Omega) \text{ as } k \to \infty \text{ imply } v \in U,$$
$$v \in U \text{ and } \partial_i v = 0 \text{ a.e. in } \Omega,\ 1 \leq i \leq N \text{ imply } v = 0.$$

Then there exists a constant $C = C(U)$ such that

$$\|v\|_{1,p(\,\cdot\,)} \leq C|v|_{1,p(\,\cdot\,)} \quad \text{for all } v \in U,$$

where the semi-norm $|\cdot|_{1,p(\,\cdot\,)} : W^{1,p(\,\cdot\,)}(\Omega) \to \mathbb{R}$ is defined by

$$|v|_{1,p(\,\cdot\,)} := \sum_{i=1}^N \|\partial_i v\|_{0,p(\,\cdot\,)} \quad \text{for all } v \in W^{1,p(\,\cdot\,)}(\Omega).$$

Proof Assume that the property is false. Then, for each integer $k \geq 1$, there exists a function $w_k \in U$ such that

$$\|w_k\|_{1,p(\cdot)} > k|w_k|_{1,p(\cdot)}.$$

Therefore, the functions $v_k := \frac{w_k}{\|w_k\|}$, $k \geq 1$, which belong to the cone U since $\frac{1}{\|w_k\|} > 0$, satisfy

$$\|v_k\|_{1,p(\cdot)} = 1 \text{ for all } k \geq 1 \quad \text{and} \quad |v_k|_{1,p(\cdot)} \to 0 \text{ as } k \to \infty.$$

The sequence $(v_k)_{k=1}^{\infty}$ is thus bounded in the reflexive Banach space $W^{1,p(\cdot)}(\Omega)$ (see Theorem 2.2(a)). Therefore, by the Banach-Eberlein-Shmulyan theorem (see, e.g., [12, Chapter 5]), there exists a subsequence, still denoted $(v_k)_{k=1}^{\infty}$ for convenience, that weakly converges in $W^{1,p(\cdot)}(\Omega)$ as $k \to \infty$ to a limit v that belongs to U, since U is sequentially weakly closed by assumption.

But $W^{1,p(\cdot)}(\Omega) \Subset L^{p(\cdot)}(\Omega)$ (see Theorem 2.2(b)). Hence, $(v_k)_{k=1}^{\infty}$ strongly converges in $L^{p(\cdot)}(\Omega)$ as $k \to \infty$. On the other hand, $|v_k|_{1,p(\cdot)} \to 0$ as $k \to \infty$. Hence, the sequence $(v_k)_{k=1}^{\infty}$ is a Cauchy sequence in $W^{1,p(\cdot)}(\Omega)$, which strongly converges to $v \in U$.

Since the semi-norm $|\cdot|_{1,p(\cdot)} : W^{1,p(\cdot)}(\Omega) \to \mathbb{R}$ is strongly continuous, it further follows that

$$|v|_{1,p(\cdot)} = \lim_{k\to\infty} |v_k|_{1,p(\cdot)} = 0.$$

Hence, $\partial_i v = 0$ a.e. in Ω, $1 \leq i \leq N$, and thus $v = 0$ by assumption. But this contradicts the relation $\|v\|_{1,p(\cdot)} = \lim\limits_{k\to\infty} \|v_k\|_{1,p(\cdot)} = 1$. This completes the proof.

We immediately infer from Theorem 3.1 that, if its assumptions are satisfied, there exists a constant $C = C(U)$ such that

$$\|v\|_{0,p(\cdot)} \leq C|v|_{1,p(\cdot)} \quad \text{for all } v \in U.$$

This relation constitutes the Poincaré inequality in a Sobolev space with a variable exponent announced in the title of this paper.

Actually, with a little further ado, we can even characterize the "best" (i.e., the smallest) constant C appearing in this relation.

Theorem 3.2 *Let the assumptions about the set Ω, the function $p(\cdot)$, and the cone U be those of Theorem* 3.1*, and let*

$$\mu(U) := \inf\{|v|_{1,p(\cdot)};\ v \in U,\ \|v\|_{0,p(\cdot)} = 1\}.$$

Then

$$\mu(U) > 0,$$

and there exists $v \in U$ such that

$$\|v\|_{0,p(\cdot)} = 1 \quad \textit{and} \quad \mu(U) = |v|_{1,p(\cdot)}.$$

Hence

$$\|v\|_{0,p(\cdot)} \leq \frac{1}{\mu(U)}|v|_{1,p(\cdot)} \quad \textit{for all } v \in U,$$

and $\frac{1}{\mu(U)}$ is the best possible constant in this Poincaré inequality.

Proof Some arguments are the same as in the previous proof and for this reason, will not be repeated. By definition, there exists a sequence $(v_k)_{k=1}^{\infty}$ of elements $v_k \in U$ such that

$$\|v_k\|_{0,p(\,\cdot\,)} = 1 \quad \text{for all } k \geq 1 \quad \text{and} \quad |v_k|_{1,p(\,\cdot\,)} \to \mu(U) \quad \text{as } k \to \infty.$$

Since $(v_k)_{k=1}^{\infty}$ is then a bounded sequence in $W^{1,p(\cdot)}(\Omega)$, there exists a subsequence, still denoted $(v_k)_{k=1}^{\infty}$ for convenience, and an element $v \in U$ such that v_k converges weakly to v in $W^{1,p(\,\cdot\,)}(\Omega)$ as $k \to \infty$.

Besides, $v_k \to v$ in $L^{p(\,\cdot\,)}(\Omega)$ as $k \to \infty$, so that

$$\|v\|_{0,p(\,\cdot\,)} = \lim_{k\to\infty} \|v_k\|_{0,p(\,\cdot\,)} = 1.$$

The definition of $\mu(U)$ then implies

$$\mu(U) \leq |v|_{1,p(\,\cdot\,)}$$

on the one hand. Since a semi-norm is sequentially weakly lower semi-continuous,

$$|v|_{1,p(\,\cdot\,)} \leq \liminf_{k\to\infty} |v_k|_{1,p(\,\cdot\,)} = \mu(U),$$

on the other hand. Hence $\mu(U) = |v|_{1,p(\,\cdot\,)}$ and thus $\mu(U) > 0$, for otherwise v would vanish. But this is impossible since $\|v\|_{0,p(\,\cdot\,)} = 1$. This completes the proof.

4 Applications

Our first application is to the case where the cone U in $W^{1,p(\,\cdot\,)}(\Omega)$ is a subspace, associated with a homogeneous Dirichlet condition.

Theorem 4.1 *Let Ω be a domain in $\mathbb{R}^N$, let Γ_0 be a $\mathrm{d}\Gamma$-measurable subset of $\Gamma = \partial\Omega$ that satisfies $\mathrm{d}\Gamma\text{-meas}\,\Gamma_0 > 0$, let $p(\,\cdot\,) \in \mathcal{C}(\overline{\Omega})$ be such that $p^- > 1$, and let*

$$U := \{v \in W^{1,p(\,\cdot\,)}(\Omega);\ \mathrm{tr}\, v = 0 \text{ on } \Gamma_0\},$$

where the trace operator tr *is defined as in Theorem* 2.3*. Then there exists a constant $C = C(U)$ such that*

$$\|v\|_{1,p(\,\cdot\,)} \leq C|v|_{1,p(\,\cdot\,)} \quad \textit{for all } v \in U.$$

Proof It suffices to verify that the above set U satisfies the assumptions of Theorem 3.1. To show that U is sequentially weakly closed in $W^{1,p(\,\cdot\,)}(\Omega)$, it suffices to show that U is strongly closed since U is a subspace.

So, let $v_k \in U$, $k \geq 1$, and $v \in W^{1,p(\,\cdot\,)}(\Omega)$ be such that $v_k \to v$ in $W^{1,p(\,\cdot\,)}(\Omega)$ as $k \to \infty$. Then $v_k \to v$ in $W^{1,1}(\Omega)$ since $W^{1,p(\,\cdot\,)}(\Omega) \hookrightarrow W^{1,1}(\Omega)$ (see Theorem 2.2(b)). Consequently, $\mathrm{tr}\, v_k \to \mathrm{tr}\, v$ in $L^1(\Gamma)$ as $k \to \infty$, which in turn implies that $\mathrm{tr}\, v_k|_{\Gamma_0} \to \mathrm{tr}\, v|_{\Gamma_0}$ in $L^1(\Gamma_0)$. But $\mathrm{tr}\, v_k|_{\Gamma_0} = 0$ for all $k \geq 1$ and the limit of a sequence in a normed vector space is unique, which shows that $\mathrm{tr}\, v|_{\Gamma_0} = 0$. Hence $v \in U$, which shows that U is closed.

It remains to show that $|\cdot|_{1,p(\,\cdot\,)}$ is a norm over the space U. So, let $v \in U$ be such that $|v|_{1,p(\,\cdot\,)} = 0$. Then v is a constant function by virtue of the connectedness of the set Ω.

Therefore its trace on Γ is a constant function that takes the same value, and this value is zero since the trace vanishes on Γ_0 and $d\Gamma$-meas $\Gamma_0 > 0$. This completes the proof.

When $\Gamma_0 = \Gamma$, Theorem 4.1 thus shows that the following Poincaré inequality holds: There exists a constant C such that

$$\|v\|_{0,p(\cdot)} \le C|v|_{1,p(\cdot)} \quad \text{for all } v \in \mathring{W}^{1,p(\cdot)}(\Omega),$$

where

$$\mathring{W}^{1,p(\cdot)}(\Omega) := \{v \in W^{1,p(\cdot)}(\Omega);\ \operatorname{tr} v = 0 \text{ on } \Gamma\}.$$

But, as indicated in the introduction, the space

$$W_0^{1,p(\cdot)}(\Omega) := \overline{\mathcal{D}(\Omega)}^{\|\cdot\|_{1,p(\cdot)}}$$

may be only strictly contained in the space $\mathring{W}^{1,p(\cdot)}(\Omega)$. Hence in this case, establishing the Poincaré inequality for all functions in $\mathcal{D}(\Omega)$ would yield the same inequality for all functions in $W_0^{1,p(\cdot)}(\Omega)$, but not for all functions in $\mathring{W}^{1,p(\cdot)}(\Omega)$.

Remark 4.1 For a constant exponent p, Theorem 4.1 is well-known; see, e.g., [3, Theorem 1.2.1] or [2, Theorem 2.15] (in both cases, $p = 2$, but the proof is similar for any $p \ge 1$).

Remark 4.2 The spaces $W_0^{1,p(\cdot)}(\Omega)$ and $\mathring{W}_0^{1,p(\cdot)}(\Omega)$ coincide if the function $p(\cdot) \in L^\infty(\Omega)$ with $p^- \ge 1$ satisfies the following Diening-Fan-Zhao-Zhikov condition (so named after Diening [4], Fan and Zhao [7], and Zhikov [13]): There exists a constant C such that

$$|p(x) - p(y)| \le \frac{C}{|\ln \|x-y\||} \quad \text{for all } x, y \in \overline{\Omega} \quad \text{such that } \|x - y\| < \frac{1}{2},$$

where $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^N$.

Our second application is to "genuine" cones U (i.e., which are not subspaces) of a specific form.

Theorem 4.2 *Let Ω be a domain in $\mathbb{R}^N$, let $p(\cdot) \in \mathcal{C}(\overline{\Omega})$ with $p^- > 1$, let $q(\cdot) \in \mathcal{C}(\overline{\Omega})$ with $q^- > 1$ be such that*

$$q(x) < p^*(x) \quad \textit{for all } x \in \overline{\Omega},$$

where the function $p^* : \overline{\Omega} \to \mathbb{R}$ is defined in Theorem 2.2(c), and let the function $q'(\cdot) \in \mathcal{C}(\overline{\Omega})$ be defined by

$$\frac{1}{q(x)} + \frac{1}{q'(x)} = 1 \quad \textit{for all } x \in \overline{\Omega}.$$

Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function (i.e., such that $f(x,\cdot) \in \mathcal{C}(\mathbb{R})$ for almost all $x \in \Omega$ and $f(\cdot, s) : \Omega \to \mathbb{R}$ is measurable for all $s \in \mathbb{R}$) such that, for each $\lambda \ge 0$, there exist two constants C_λ^- and C_λ^+ with the property that

$$C_\lambda^- f(x,s) \le f(x, \lambda s) \le C_\lambda^+ f(x,s) \quad \textit{for almost all } x \in \Omega \textit{ and all } s \in \mathbb{R},$$

and such that there exist a non-negative function $a \in L^{q'(\cdot)}(\Omega)$ and a constant $b \ge 0$ such that

$$|f(x,s)| \le a(x) + b|s|^{q(x)-1} \quad \textit{for almost all } x \in \Omega \textit{ and all } s \in \mathbb{R}.$$

Finally, assume that the set

$$U := \left\{ v \in W^{1,p(\,\cdot\,)}(\Omega);\ \int_\Omega f(x, v(x))\,\mathrm{d}x = 0 \right\}$$

does not contain nonzero constant functions. Then there exists a constant $C = C(U)$ such that

$$\|v\|_{1,p(\,\cdot\,)} \le C|v|_{1,p(\,\cdot\,)} \quad \textit{for all } v \in U.$$

Proof Again it suffices to verify that the above set U satisfies the assumption of Theorem 3.1. First, the existence of the constants C_λ^- and C_λ^+ for each $\lambda \ge 0$ and the definition of the set U together imply that U is a cone. It thus remains to verify that U is sequentially weakly closed. Because

$$|f(x,s)| \le a(x) + b|s|^{q(x)/q'(x)} \quad \text{for almost all } x \in \Omega \text{ and all } s \in \mathbb{R},$$

the associated Nemytsky operator N_f, defined for any function $v \in L^{q(\,\cdot\,)}(\Omega)$ by

$$N_f v(x) = f(x, v(x)) \quad \text{for almost all } x \in \Omega,$$

maps $L^{q(\,\cdot\,)}(\Omega)$ into $L^{q'(\,\cdot\,)}(\Omega)$, and is continuous between these two spaces (see [7, Theorem 1.16]).

The mapping

$$v \in L^{q(\,\cdot\,)}(\Omega) \to \int_\Omega f(x, v(x))\,\mathrm{d}x \in \mathbb{R}$$

is thus continuous, as composed of the continuous mappings $N_f : L^{q(\,\cdot\,)}(\Omega) \to L^{q'(\,\cdot\,)}(\Omega)$ and $g \in L^{q'(\,\cdot\,)}(\Omega) \to \int_\Omega g(x)\,\mathrm{d}x \in \mathbb{R}$ (while the latter mapping is continuous following Theorem 2.1(c) and the observation that the constant function equalling to one belongs to $L^{q(\,\cdot\,)}(\Omega)$).

Let then $v_k \in U$, $k \ge 1$, and $v \in W^{1,p(\,\cdot\,)}(\Omega)$ be such that v_k converges weakly to v in $W^{1,p(\,\cdot\,)}(\Omega)$ as $k \to \infty$. Consequently, $v_k \to v$ in $L^{q(\,\cdot\,)}(\Omega)$ since $W^{1,p(\,\cdot\,)}(\Omega) \Subset L^{q(\,\cdot\,)}(\Omega)$ (see Theorem 2.2(c)), which in turn implies that

$$\int_\Omega f(x, v(x))\,\mathrm{d}x = \lim_{k\to\infty} \int_\Omega f(x, v_k(x))\,\mathrm{d}x = 0.$$

Therefore $v \in U$, which shows that U is sequentially weakly closed. This completes the proof.

Remark 4.3 An example of a function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ that satisfies the assumptions of Theorem 4.2 is given by $(x,s) \in \Omega \times \mathbb{R} \to f(x,s) := |s|^{q(x)-2}s$, since in this case the existence of the constants C_λ^- and C_λ^+ for each $\lambda \ge 0$ follows the inequalities $\lambda^{q^+} \le \lambda^{q(x)} \le \lambda^{q^-}$ for all $x \in \overline{\Omega}$ if $0 \le \lambda \le 1$ and $\lambda^{q^-} \le \lambda^{q(x)} \le \lambda^{q^+}$ for all $x \in \overline{\Omega}$ if $1 < \lambda$, where $q^+ := \sup\limits_{x\in\overline{\Omega}} |q(x)|$, and $|f(x,s)| = |s|^{q(x)-1}$ for almost all $x \in \Omega$ and all $s \in \mathbb{R}$. For a constant exponent p, functions of this type naturally appear in the analysis of the p-Laplace operator (see, e.g., [5, 8]).

Remark 4.4 Theorem 2.1(c) shows that the same conclusion holds if the set U is of the more general form $U = \left\{ v \in W^{1,p(\,\cdot\,)}(\Omega);\ \int_\Omega h(x) f(x, v(x))\,\mathrm{d}x = 0 \right\}$, where h is a given non-negative function in the space $L^{q(\,\cdot\,)}(\Omega)$.

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