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Stochastic Wave Equations with Memory**

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Abstract The authors show the existence and uniqueness of solution for a class of stochastic wave equations with memory. The decay estimate of the energy function of the solution is obtained as well.

Keywords Stochastic wave equations with memory, Resolvent, Infinite dimensional Wiener process, Energy function
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1 Introduction

The wave equation with memory of the following form

$$\begin{cases} u_{tt} - \Delta u + \displaystyle\int_0^t \beta(t-s)\mathrm{div}\{a(x) \bigtriangledown u\}\mathrm{d}s + f(u) + b(x)h(u_t) = 0, \\ u(t,x) = 0, \quad \text{on } \partial D, \\ u(0) = u_0, \\ u_t(0) = u_1, \quad (x,t) \in D \times [0,T] \end{cases} \tag{1.1}$$

describes the model of materials consisting of an elastic part (without memory) and a viscoelastic part (memory) with $u(t,x)$ giving the position of material particle x at time t. The term β is the relaxation function, f denotes the body force and h is the damping term. The properties of the solution to (1.1) has been studied by many authors. For the case that the damping term is zero, Dafermos [10] proved that the solution of the viscoelastic system decays to zero as time goes to infinity. Unfortunately, the explicit rate was not obtained. Rivera [14] obtained the uniform rates of decay for the solution of a linear viscoelastic system with memory, based on second-order estimates. For the partially viscoelastic case, Rivera and Salvatierra [13] showed that the energy of the solution decays exponentially when β decays exponentially. On the other hand, Cavalcanti and Oquendo [6] studied the nonlinear equation with a nonlinear and localized frictional damping and proved the exponential and polynomial decay rates of the energy. Recently, Alalau-Boussouira et al [1] developed a unified method for the decay estimates of the energy of the general second order integro-differential equations.

In fact, the driving force may be affected by the environment randomly. In view of this, we

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consider the following stochastic wave equation with memory:

$$\begin{cases} u_{tt} - \Delta u + 2\alpha u_t + \displaystyle\int_0^t \beta(t-s)\Delta u \,\mathrm{d}s + f(u) = \partial_t W(t,x), \\ u(t,x) = 0, \quad \text{on } \partial D, \\ u(0) = u_0, \\ u_t(0) = u_1 \end{cases} \tag{1.2}$$

for $(x,t) \in D \times [0,T]$, where Δ denotes the Laplace operator on D with Dirichlet boundary condition, $\alpha \geq 0$ is a constant, and W is an infinite dimensional Wiener process which may be treated as the random force. $D \subset \mathbb{R}^d$ is a bounded open domain with some smooth boundary ∂D for $d \geq 3$, and $T > 0$ is a constant.

Before solving the equation above, we first mention some important studies on the general stochastic wave equations. Using estimates on the energy function, Chow [7] proved the existence of a global solution to stochastic wave equations with a polynomial nonlinearity. With the similar method, Chow [8, 9] studied further properties of the solution such as asymptotic stability and invariant measure. On the other hand, Barbu et al [2] obtained the existence and uniqueness of the invariant measure of the solution without the computation of the energy. Moreover, using the energy inequality, Bo et al [3] proposed sufficient conditions that the solutions of a class of stochastic wave equations blow up with a positive probability or in L^2 sense. However, for the current equation (1.2), the memory part makes it difficult to estimate the energy by using these methods. Hence, we solve it in another way. We use the definition of solutions in [4] and extend them to the stochastic cases. Then we first prove the existence and uniqueness of a local mild solution. Following from the arguments for the decay estimate on the energy function in [1, 6, 13, 14], we prove the global existence of the solution.

The remaining part of this article is organized as follows. Definitions of the solution to Equation (1.2) are given in Section 2. In Section 3, we show the local existence and uniqueness of the mild solution. In Section 4, the decay estimate of the energy function is obtained and the global existence of the solution is proved.

2 Preliminaries

According to the arguments in [12] and [4], we solve equation (1.1) as an integro-differential equation. More precisely, consider the following integral-differential equation

$$\begin{cases} u_{tt} + Au + \displaystyle\int_0^t B(t-s)u \,\mathrm{d}s + f(u) + h(u_t) = 0, \\ u(t) = 0, \quad \text{on } \partial D, \\ u(0) = u_0, \\ u_t(0) = u_1 \end{cases} \tag{2.1}$$

with $u \in L^1([0,T];X)$, where X is a real Hilbert space, A and B satisfy the conditions in [4], i.e., A and $B(\,\cdot\,)$ are linear unbounded self-adjoint operators with domains $D(A)$ and $D(B(\,\cdot\,))$ respectively, satisfying that

(A1) $D(A) \subset D(B(t))$ for any $t \geq 0$ and $D(A)$ is dense in X.

(A2) $\langle Ay, y\rangle \geq a_0\|y\|^2$ for any $y \in D(A)$ and some constant $a_0 > 0$.

(A3) $B(\,\cdot\,)y \in W^{1,1}_{\mathrm{loc}}(0,+\infty;X)$ for any $y \in D(A)$.

(A4) $B(t)$ commutes with A, that is

$$B(t)D(A^2) \subset D(A) \quad \text{and} \quad AB(t)y = B(t)Ay, \quad y \in D(A^2),\ t \geq 0.$$

Here, f becomes the operator defined by $f(u(t))(x) = f(u(t,x))$, and the similar change is made on h.

Definition 2.1 *A family of bounded linear operators $\{S(t)\}_{t\geq 0}$ in X is called a resolvent for equation* (2.1) *with $f = 0$ and $h = 0$, if the following conditions are satisfied:*

(S1) *$S(0) = I$ and $S(t)$ is strong continuous on $[0,\infty)$. That is, for all $x \in X$, $S(\cdot)x$ is continuous on $[0,\infty)$.*

(S2) *$S(t)$ commutes A, which means that $S(t)D(A) \subset D(A)$ and $AS(t)y = S(t)Ay$ for all $y \in D(A)$ and $t \geq 0$.*

(S3) *For any $y \in D(A)$, $S(\cdot)y$ is twice continuously differentiable in X on $[0,\infty)$ and $\dot{S}(0) = 0$.*

(S4) *For any $y \in D(A)$ and $t \geq 0$, the resolvent equation is*

$$\ddot{S}(t)y + AS(t)y + \int_0^t B(t-r)S(r)y\,\mathrm{d}r = 0. \tag{2.2}$$

It is easy to check that when $A = -\Delta$ and $B(t) = \beta(t)\Delta$, with Δ being the Laplace operator on D with Dirichlet boundary condition and $\beta(\cdot)$ being continuous differentiable function on t, A and B satisfy the conditions (A1)–(A4).

Let $\langle\cdot,\cdot\rangle$ be the inner product and $\|\cdot\|$ be the norm on the Hilbert space $L^2(D)$, and we have the following theorem, which is a consequence of Theorem 2 in [4].

Theorem 2.1 *Assume that $A = -\Delta$ and $B(t) = \beta(t)\Delta$, with Δ being the Laplace operator on D with Dirichlet boundary condition and $\beta(\cdot)$ being continuous differentiable function on t, $f = 0$ and $h = 0$. Then there exists a unique resolvent $\{S(t)\}_{t\geq 0}$ for equation* (2.1)*. Furthermore, the resolvent satisfies the following properties:*

(i) *The operators $S(t)$ are self-adjoint.*

(ii) *$S(t)$ commutes with $\sqrt{A}$, that is $S(t)D(\sqrt{A}) \subset D(\sqrt{A})$ and $\sqrt{A}\,S(t)x = S(t)\sqrt{A}\,x$ for all $x \in D(\sqrt{A})$ and $t \geq 0$.*

(iii) *For any $x \in L^2(D)$, the function $t \to \int_0^t S(r)x\,\mathrm{d}r$ belongs to $C([0,\infty); D(\sqrt{A}))$ and for any $T > 0$, there exists a constant C_T such that*

$$\|S(t)x\| + \left\|\sqrt{A}\int_0^t S(r)x\,\mathrm{d}r\right\| \leq C_T\|x\| \quad \text{for any } t \in [0,T]. \tag{2.3}$$

(iv) *For any $x \in D(\sqrt{A})$, the function $t \to \int_0^t S(r)x\,\mathrm{d}r$ belongs to $C([0,\infty); D(\sqrt{A}))$ and for any $T > 0$,*

$$\left\|A\int_0^t S(r)x\,\mathrm{d}r\right\| \leq C_T\|\sqrt{A}\,x\|, \qquad t \in [0,T], \tag{2.4}$$

$$\|\dot{S}x\| \leq C_T(\|x\| + \|\sqrt{A}\,x\|), \qquad t \in [0,T], \tag{2.5}$$

$$\dot{S}x + A\int_0^t S(r)x\,\mathrm{d}r + B * 1 * S(t)x = 0, \quad t \geq 0, \tag{2.6}$$

where $*$ stands for the convolution of two functions.

(v) *For any $x \in D(\sqrt{A})$ the function $\dot{S}(\cdot)x$ belongs to $C([0,\infty); D(\sqrt{A}))$.*

With the resolvent, we can define the solutions of the stochastic wave equation (1.2). We rewrite the equation as

$$\begin{cases} u_{tt} + Au + \int_0^t B(t-s)u\,\mathrm{d}s + f(u) + 2\alpha u_t = \dfrac{\mathrm{d}}{\mathrm{d}t}W(t), \\ u(t,x) = 0, \quad \text{on } \partial D, \\ u(0) = u_0, \\ u_t(0) = u_1, \end{cases} \tag{2.7}$$

where $A = -\Delta$, $B(t) = \beta(t)\Delta$, and W is a Q-Wiener process in X on some probability space $(\Omega, P, \mathcal{F})$ with the variance operator Q satisfying $\mathbf{Tr}\,Q < \infty$ and $\{\mathcal{F}_t, t \geq 0\}$ as its natural filtration satisfying the usual conditions. (Here, we use form $\frac{\mathrm{d}}{\mathrm{d}t}W(t)$ instead of $\dot{W}(t)$ to denote the white noise in time since we will use the Itô formula with respect to the infinite dimensional Wiener process.) Moreover, we can assume that Q has the following form

$$Qe_i = \lambda_i e_i, \quad i = 1, 2, \cdots,$$

where $\{\lambda_i\}$ are eigenvalues of Q satisfying $\sum\limits_{i=1}^{\infty} \lambda_i < \infty$ and $\{e_i\}$ are the corresponding eigenfunctions which form an orthonormal base of X. In this case,

$$W(t) = \sum_{i=1}^{\infty} \sqrt{\lambda_i}\, B_i(t) e_i,$$

where $\{B_i(t)\}$ is a sequence of independent copies of standard Brownian motions in one dimension. Let $\mathcal{H}$ be the set of $L_2^0 = L^2(Q^{\frac{1}{2}}X, X)$-valued processes with the norm

$$\begin{aligned} \|\Psi(t)\|_{\mathcal{H}} &= \Big[\mathbf{E}\int_0^t \|\Psi(s)\|_{L_2^0}^2\,\mathrm{d}s\Big]^{\frac{1}{2}} \\ &= \Big[\mathbf{E}\int_0^t \mathbf{Tr}((\Psi(s)Q^{\frac{1}{2}})(\Psi(s)Q^{\frac{1}{2}})^*)\,\mathrm{d}s\Big]^{\frac{1}{2}} < \infty, \end{aligned}$$

where $(\Psi(s)Q^{\frac{1}{2}})^*$ denotes the adjoint operator of $\Psi(s)Q^{\frac{1}{2}}$. Let $\{t_k\}_{k=1}^n$ be a partition on $[0,T]$ such that $0 = t_0 < t_1 < \cdots < t_n = T$. For a process $\Psi \in \mathcal{H}$, define the stochastic integral with respect to the Q-Wiener process as

$$\int_0^t \Psi(s)\,\mathrm{d}W(s) = \lim_{n\to\infty} \sum_{k=0}^{n-1} \Psi(t_k)(W(t_{k+1} \wedge t) - W(t_k \wedge t)),$$

where the sequence converges in $\mathcal{H}$-sense. It is not difficult to check that the integral process $\int_0^t \Psi(s)\,\mathrm{d}W(s)$ is a martingale for any $\Psi \in \mathcal{H}$, and the quadratic variation process is given by

$$\Big\langle\!\Big\langle \int_0^t \Psi(s)\,\mathrm{d}W(s) \Big\rangle\!\Big\rangle = \int_0^t \mathbf{Tr}((\Psi(s)Q^{\frac{1}{2}})(\Psi(s)Q^{\frac{1}{2}})^*)\,\mathrm{d}s.$$

In particular, if we take $\Psi \equiv 1$, then the equation above becomes

$$\langle\!\langle W(t) \rangle\!\rangle = \int_0^t \mathbf{Tr}((Q^{\frac{1}{2}})(Q^{\frac{1}{2}})^*)\,\mathrm{d}s = t\,\mathbf{Tr}\,Q.$$

For more details about the infinite dimension Wiener process and the stochastic integral, we refer to [11].

Definition 2.2 *Let $f \in C([0,T] \times L^2(D); L^2(D))$.*

(i) *We say that u is a strong solution to* (1.2) *if u is $\{\mathcal{F}_t\}_{t\geq 0}$-adapted that belongs to $C^2([0,T]\times\Omega; X) \cap C([0,T]\times\Omega; D(A))$ and satisfies* (1.2).

(ii) *An $\{\mathcal{F}_t\}_{t\geq 0}$-adapted X valued stochastic process u is said to be a weak solution to* (1.2) *if $u \in C^1([0,T]\times\Omega; X)\cap C([0,T]\times\Omega; D(\sqrt{A}))$, and for any $\psi \in D(\sqrt{A})$, the following equation holds:*

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\langle u_t, \psi\rangle + \langle\sqrt{A}\,u(t), \sqrt{A}\,\psi\rangle - \Big\langle \int_0^t \beta(t-s)\sqrt{A}\,u(s)\,\mathrm{d}s, \sqrt{A}\,\psi\Big\rangle\\ =\ & -2\alpha\langle u_t, \psi\rangle - \langle f, \psi\rangle + \Big\langle \psi, \frac{\mathrm{d}}{\mathrm{d}t}W(t)\Big\rangle. \end{aligned} \tag{2.8}$$

(iii) *An $\{\mathcal{F}_t\}_{t\geq 0}$-adapted X valued stochastic process u is said to be a mild solution to* (1.2) *if $u \in C^1([0,T]\times\Omega; X)\cap C([0,T]\times\Omega; D(\sqrt{A}))$ and the following equation holds:*

$$\begin{aligned} u(t) &= S(t)u_0 + \int_0^t S(r)u_1\,\mathrm{d}r - 2\alpha\int_0^t 1 * S(t-r)u_r\,\mathrm{d}r\\ &\quad - \int_0^t 1*S(t-r)f(r,u(r))\,\mathrm{d}r + \int_0^t 1*S(t-r)\,\mathrm{d}W(r)\\ &= S(t)u_0 + 2\alpha\int_0^t S(r)u_0\,\mathrm{d}r + \int_0^t S(r)u_1\,\mathrm{d}r - 2\alpha\int_0^t S(t-r)u(r)\,\mathrm{d}r\\ &\quad - \int_0^t 1*S(t-r)f(u(r))\,\mathrm{d}r + \int_0^t 1*S(t-r)\,\mathrm{d}W(r), \end{aligned} \tag{2.9}$$

where S is the resolve operator of (2.1) *with $f = 0$ and $g = 0$.*

Remark 2.1 By the definitions above and Theorem 2.1, we have the following facts.

(i) The stochastic integral in (2.9) is well-defined.

(ii) A strong solution of (1.2) is also a weak solution and a mild solution.

(iii) If u is a mild solution to (1.2) satisfying $u_0 \in D(A)$ and $u_1 \in D(\sqrt{A})$, then u is a strong solution.

(iv) A weak solution may be not equivalent to a mild solution, because the resolve G may be not a semigroup. When the operator B in (2.7) equals 0, (2.7) degenerates to the general stochastic wave equation and the mild solution is equivalent to the weak solution then.

Here, we focus on the mild solution to equation (1.2) and give the existence and uniqueness, as well as the decay estimate of the energy.

3 Existence and Uniqueness of Local Solution

In this section, we give the main theorem below that states the existence and uniqueness of the mild solution. The drift coefficient f is assumed to be non-Lipschitzian which was proposed in [6]. Hence, we could not conclude the global solution directly. In another way, the proof is split into two steps. First, the local solution is obtained in a completed metric space in this section. Then, following the estimation on the energy function in next section, the global existence is proved.

Theorem 3.1 *Let f satisfy the following hypotheses.*

(i) *For any $x \in \mathbb{R}$, there exists a constant $C > 0$ such that*

$$|f(x)| \leq C(1 + |x|^{\rho-1})|x|. \tag{3.1}$$

(ii) *For any* $x, y \in \mathbb{R}$,

$$|f(x) - f(y)| \leq C(1 + |x|^{\rho-1} + |y|^{\rho-1})|x - y|. \tag{3.2}$$

(iii) *For any* $x \in \mathbb{R}$,

$$f(x)x \geq 0, \quad (\rho+1)F(x) \leq f(x)x, \quad F(x) := \int_0^x f(y)\,\mathrm{d}y \tag{3.3}$$

for some positive constant $\rho \geq 1$, *and* $(d-2)\rho \leq d$. *Assume* $u_0 \in D(\sqrt{A})$, $u_1 \in L^2(D)$, $\beta(t) = \mathrm{e}^{-\sigma t}$ *such that* $\int_0^\infty \beta(t)\,\mathrm{d}t < 1$. *Then there exists a unique mild solution* u *to equation* (1.2) *which belongs to* $C([0,T] \times \Omega, D(\sqrt{A}))$.

Proof We first solve the truncated equation with f satisfying the global Lipschitz condition. For every $n \geq 1$, define $\Pi_n : [0,\infty) \to [0,1]$ as a C^1 function such that

$$\Pi_n(x) = \begin{cases} 1, & \text{if } x \leq n, \\ 0, & \text{if } x > n+1 \end{cases} \tag{3.4}$$

satisfies that $|\Pi_n| \leq 1$ and $|\Pi_n'| \leq 2$. We will prove the existence and uniqueness of the solution to the following stochastic integral equation

$$\begin{aligned} u_n(t) = {} & S(t)u_0 + 2\alpha \int_0^t S(r)u_0\,\mathrm{d}r + \int_0^t S(r)u_1\,\mathrm{d}r - 2\alpha \int_0^t S(t-r)u_n(r)\,\mathrm{d}r \\ & - \int_0^t 1 * S(t-r)\Pi_n(\|u_n(r)\|)f(u_n(r))\,\mathrm{d}r + \int_0^t 1 * S(t-r)\,\mathrm{d}W(r). \end{aligned} \tag{3.5}$$

Set

$$u_n^{(0)} = S(t)u_0, \tag{3.6}$$

$$\begin{aligned} u_n^{(k)} = {} & S(t)u_0 + 2\alpha \int_0^t S(r)u_0\,\mathrm{d}r + \int_0^t S(r)u_1\,\mathrm{d}r - 2\alpha \int_0^t S(t-r)u_n^{(k-1)}(r)\,\mathrm{d}r \\ & - \int_0^t 1 * S(t-r)\Pi_n(\|u_n(r)^{(k-1)}\|)f(u_n(r)^{(k-1)})\,\mathrm{d}r + \int_0^t 1 * S(t-r)\,\mathrm{d}W(r). \end{aligned} \tag{3.7}$$

Denote

$$\Lambda := \Big\{ v \in C([0,T] \times \Omega; D(\sqrt{A})); \ \sup_{0 \leq t \leq T} \mathbf{E}\|\sqrt{A}\,v(t)\| < \infty \Big\}$$

equipped with the distance generated by $C([0,T] \times \Omega; D(\sqrt{A}))$, i.e.,

$$d(v_1, v_2) = \sup_{0 \leq s \leq T} \mathbf{E}\|\sqrt{A}\,v_1(s) - \sqrt{A}\,v_2(s)\|.$$

Then (Λ, d) is a complete metric space. We will show that $\{u_n^{(k)}\}$ is a Cauchy sequence in Λ for each n and the limit exists and is a solution to (3.5). Note that $A = -\Delta$ is nonnegative definite and uniformly elliptic. By the Sobolev inequality, there exists a constant $\eta > 0$ such that

$$\|v\| \leq \eta \|\sqrt{A}\,v\|$$

for any $v \in D(\sqrt{A})$. From Theorem 2.1,

$$\begin{aligned} \mathbf{E}\|\sqrt{A}\,(u_n^{(1)}(t) - u_n^{(0)}(t))\| & \leq C_T\Big[\|u_1\| + 2\,\mathbf{E}\,\alpha\|u_n^{(0)}\| + C_n \int_0^t \mathbf{E}\|u_n^{(0)}\|\,\mathrm{d}r + \Big(t\sum_{i=0}^\infty \lambda_i\Big)^{\frac{1}{2}}\Big] \\ & \leq C_{n,T}\Big[\|\sqrt{A}\,u_0\| + \|u_1\| + \Big(T\sum_{i=0}^\infty \lambda_i\Big)^{\frac{1}{2}}\Big] < \infty. \end{aligned} \tag{3.8}$$

According to (2.3) in Theorem 2.1, we have for any $v_1, v_2 \in \Lambda$ that

$$\begin{aligned}
&\Big\|\sqrt{A}\Big[\int_0^t 1 * S(t-r)(\Pi_n(\|\sqrt{A}\,v_1(r)\|)f(v_1(r)) - \Pi_n(\|\sqrt{A}\,v_2(r)\|)f(v_2(r)))\,\mathrm{d}r\Big]\Big\| \\
&\le \Big\|\sqrt{A}\Big[\int_0^t 1 * S(t-r)\Pi_n(\|\sqrt{A}\,v_2(r)\|)(f(v_1(r)) - f(v_2(r)))\,\mathrm{d}r\Big]\Big\| \\
&\quad + \Big\|\sqrt{A}\Big[\int_0^t 1 * S(t-r)(\Pi_n(\|\sqrt{A}\,v_1(r)\|) - \Pi_n(\|\sqrt{A}\,v_2(r)\|))f(v_1(r))\,\mathrm{d}r\Big]\Big\| \\
&\le \Big\|\int_0^t \Pi_n(\|\sqrt{A}\,v_2(r)\|)(f(v_1(r)) - f(v_2(r)))\,\mathrm{d}r\Big\| \\
&\quad + \Big\|\int_0^t (\Pi_n(\|\sqrt{A}\,v_1(r)\|) - \Pi_n(\|\sqrt{A}\,v_2(r)\|))f(v_1(r))\,\mathrm{d}r\Big\|.
\end{aligned}$$

From the argument in [5, p. 785] and the definition of Π_n, we have

$$\begin{aligned}
\Big\|\int_0^t \Pi_n(\|\sqrt{A}\,v_2(r)\|)(f(v_1(r)) - f(v_2(r)))\,\mathrm{d}r\Big\| &\le C(1+(n+1)^{\rho-1})\int_0^t \|v_1(r) - v_2(r)\|\,\mathrm{d}r \\
&\le C_n \int_0^t \|\sqrt{A}\,(v_1(r) - v_2(r))\|\,\mathrm{d}r
\end{aligned}$$

and

$$\begin{aligned}
&\Big\|\int_0^t (\Pi_n(\|\sqrt{A}\,v_1(r)\|) - \Pi_n(\|\sqrt{A}\,v_2(r)\|))f(v_1(r))\,\mathrm{d}r\Big\| \\
&\le C(n+1+(n+1)^{\rho})\int_0^t |\|v_1(r)\| - \|v_2(r)\||\,\mathrm{d}r \\
&\le C_n \int_0^t \|v_1(r) - v_2(r)\|\,\mathrm{d}r \\
&\le C_n \int_0^t \|\sqrt{A}\,(v_1(r) - v_2(r))\|\,\mathrm{d}r,
\end{aligned}$$

where we use the mean value theorem to get

$$\begin{aligned}
|\Pi_n(\|\sqrt{A}\,v_1(r)\|) - \Pi_n(\|\sqrt{A}\,v_2(r)\|)| &\le |\Pi_n'|\,|\|v_1(r)\| - \|v_2(r)\|| \\
&\le 2|\|v_1(r)\| - \|v_2(r)\||.
\end{aligned}$$

Combining the three inequalities above, we conclude

$$\begin{aligned}
&\Big\|\sqrt{A}\Big[\int_0^t 1 * S(t-r)(\Pi_n(\|\sqrt{A}\,v_1(r)\|)f(v_1(r)) - \Pi_n(\|\sqrt{A}\,v_2(r)\|)f(v_2(r)))\,\mathrm{d}r\Big]\Big\| \\
&\le C_n \int_0^t \|\sqrt{A}\,(v_1(r) - v_2(r))\|\,\mathrm{d}r. \qquad (3.9)
\end{aligned}$$

Then

$$\begin{aligned}
\mathbf{E}\|\sqrt{A}\,(u_n^{(k+1)}(t) - u_n^{(k)}(t))\| &\le \alpha\,\mathbf{E}\Big\|\sqrt{A}\int_0^t S(t-r)(u_n^{(k)}(r) - u_n^{(k-1)}(r))\,\mathrm{d}r\Big\| \\
&\quad + \mathbf{E}\int_0^t \|\sqrt{A}\,1 * S(t-r)(\Pi_n(\|\sqrt{A}\,u_n^{(k)}(r)\|)f(u_n^{(k)}(r)) \\
&\quad - \Pi_n(\|\sqrt{A}\,u_n^{(k-1)}(r)\|)f(u_n^{(k-1)}(r)))\|\,\mathrm{d}r \\
&\le C_{T,n}\int_0^t \|\sqrt{A}\,(u_n^{(k)}(r) - u_n^{(k-1)}(r))\|\,\mathrm{d}r.
\end{aligned}$$

Iterating this inequality, we get

$$\sup_{0\le t\le T}\mathbf{E}\|\sqrt{A}\,(u_n^{(k+1)}(t)-u_n^{(k)}(t))\|\le\frac{C_{T,n}^k}{k!}\sup_{0\le t\le T}\mathbf{E}\|\sqrt{A}\,(u_n^{(1)}(t)-u_n^{(0)}(t))\|. \tag{3.10}$$

Therefore,

$$\sum_{k=1}^{\infty}\sup_{0\le t\le T}\mathbf{E}\|\sqrt{A}\,(u_n^{(k+1)}(t)-u_n^{(k)}(t))\|<\infty, \tag{3.11}$$

and there exists $u_n\in\Lambda$ such that $\lim\limits_{k\to\infty}u_n^{(k)}(t)=u_n(t)$ uniformly. It is easy to check that u_n is a solution to equation (3.5).

For the uniqueness, let u_n and v_n be two solutions to (3.5). Then by the similar arguments as above,

$$\sup_{0\le t\le T}\mathbf{E}\|\sqrt{A}\,(u_n(t)-v_n(t))\|\le C_{T,n}\int_0^T\sup_{0\le s\le t}\mathbf{E}\|\sqrt{A}\,(u_n(s)-v_n(s))\|\,\mathrm{d}s. \tag{3.12}$$

Using Gronwall inequality, we have

$$\sup_{0\le t\le T}\mathbf{E}\|\sqrt{A}\,(u_n(t)-v_n(t))\|=0. \tag{3.13}$$

Finally, the continuity of u_n follows from the continuity of S and the integrals.

For each n, define the stopping time τ_n by

$$\tau_n=\inf\{t\ge 0;\ \|\sqrt{A}\,u_n(t)\|\ge n\}.$$

By the uniqueness of the solution, for $m>n$, $u_m(t)=u_n(t)$ on $[0,\tau_n]$. So we can define a local solution u of (1.2) by $u(t)=u_n(t)$ on $[0,T\wedge\tau_n]$. Let $\tau_\infty=\lim\limits_{n\to\infty}\tau_n$. Hence, we construct a unique continuous local solution to (1.2) on $[0,T\wedge\tau_\infty)$. In the next section, using estimate (4.12) of the energy function defined as (4.7), we will prove $\tau_\infty=\infty$ and finish the proof of the global existence.

4 Exponential Decay of the Energy

In this section, we will prove that the energy function of the mild solution to equation (1.2) exponentially decays to a positive constant with more general conditions. With the estimates of the energy, we continue to prove the global existence of the solution. Note that we could only use Itô formula on a strong solution to equation (1.2). However, $D(\sqrt{A}\,)$ is dense in $D(A)$ and a strong solution is also a mild solution. So we can approximate the energy function of a mild solution u by a sequence of energy functions such that the corresponding strong solution sequence $\{u_n\}$ converges to u. Hence, the following arguments that should be derived for a strong solution can be easily extended to a mild solution.

As well-known, equation (1.2) is equivalent to the following Itô system

$$\begin{cases}\mathrm{d}u_t=v_t\,\mathrm{d}t,\\ \mathrm{d}v_t=\Big[-Au_t-2\alpha v_t+\displaystyle\int_0^t\beta(t-s)Au_s\,\mathrm{d}s-f(u_t)\Big]\mathrm{d}t+\mathrm{d}W_t,\\ u_t=0,\quad\text{on }\partial D,\\ u(0)=u_0,\\ v(0)=u_1.\end{cases} \tag{4.1}$$

We will prove a more general result with assumptions on β, of which $\beta(t) = \mathrm{e}^{-\sigma t}$ is a particular case, that

$$\int_0^\infty \beta(t)\,\mathrm{d}t < 1, \tag{4.2}$$

$$\beta'(t) \leq -C\beta(t), \quad \beta''(t) \leq C\beta(t), \quad \beta'''(t) \geq C\beta'(t), \quad \forall\, t \geq 0 \tag{4.3}$$

for some constant $C > 0$. Furthermore, we assume that the initial data satisfies

$$\|\sqrt{A}\,u_0\|^2 + \|u_1\|^2 \leq \lambda \tag{4.4}$$

for some $\lambda > 0$.

Set

$$(\beta\Box\omega)(t) := \int_0^t \beta(t-s)\|\omega(s) - \omega(t)\|^2 \mathrm{d}s \tag{4.5}$$

and

$$k(t) = 1 - \int_0^t \beta(s)\,\mathrm{d}s. \tag{4.6}$$

Let u be the unique mild solution to equation (4.1). Then the energy function of u is defined by

$$e(u_t, v_t) := \frac{1}{2}[k(t)\|\sqrt{A}\,u_t\|^2 + \|v_t\|^2 + (\beta\Box\sqrt{A}\,u)(t)] + \int_D F(u_t)\,\mathrm{d}x \tag{4.7}$$

with $e(u_0, v_0) = \frac{1}{2}(\|\sqrt{A}\,u_0\|^2 + \|v_0\|^2) = \frac{1}{2}\lambda$.

Remark 4.1 Under conditions (3.3) and (4.2), it is easy to check that $e(u_t, v_t) \geq 0$.

Before we give the estimation of the energy function, we first introduce a useful lemma as follows.

Lemma 4.1 *For any function $h \in C^1([0,T]; D(\sqrt{A}\,))$, we have*

$$\begin{aligned}\int_0^t \beta(t-s)\langle\sqrt{A}\,h'(t), \sqrt{A}\,h(s)\,\mathrm{d}s\rangle &= -\frac{1}{2}\beta(t)\|\sqrt{A}\,h(t)\|^2 + \frac{1}{2}(\beta'\Box\sqrt{A}\,h)(t) \\ &\quad - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big[(\beta\Box\sqrt{A}\,h)(t) - \Big(\int_0^t \beta(s)\,\mathrm{d}s\Big)\|\sqrt{A}\,h(t)\|^2\Big]. \end{aligned} \tag{4.8}$$

Proof This is a direct consequence of [14, Lemma 2.1] by setting $a(x) = 1$.

Proposition 4.1 *The energy function* (4.7) *satisfies the following equation:*

$$\begin{aligned} e(u_t, v_t) &= e(u_0, v_0) - \frac{1}{2}\Big[\int_0^t \beta(s)\|\sqrt{A}\,u_s\|^2 \mathrm{d}s - \int_0^t (\beta'\Box\sqrt{A}\,u)(s)\,\mathrm{d}s\Big] \\ &\quad - \alpha\int_0^t \|v_s\|^2 \mathrm{d}s + \int_0^t \langle v_s, \mathrm{d}W_s\rangle + \frac{1}{2}t\cdot \mathbf{Tr}\,Q. \end{aligned} \tag{4.9}$$

Moreover, we have

$$\begin{aligned} \mathbf{E}\,e(u_t, v_t) &= e(u_0, v_0) - \frac{1}{2}\Big[\int_0^t \beta(s)\mathbf{E}\|\sqrt{A}\,u_s\|^2 \mathrm{d}s - \int_0^t \mathbf{E}(\beta'\Box\sqrt{A}\,u)(s)\,\mathrm{d}s\Big] \\ &\quad - \alpha\int_0^t \mathbf{E}\|v_s\|^2 \mathrm{d}s + \frac{1}{2}t\cdot\mathbf{Tr}\,Q \end{aligned} \tag{4.10}$$

and

$$\frac{\mathrm{d}\,\mathbf{E}\,e(u_t,v_t)}{\mathrm{d}t}=-\frac{1}{2}\beta(t)\mathbf{E}\|\sqrt{A}\,u_t\|^2+\frac{1}{2}\mathbf{E}(\beta'\Box\sqrt{u}\,)(t)-\alpha\,\mathbf{E}\|v_t\|^2+\frac{1}{2}\mathbf{Tr}\,Q. \tag{4.11}$$

Proof Using Itô formula in the Hilbert space to $\|v_t\|^2$ and by Lemma 4.1, we get

$$\begin{aligned}\|v_t\|^2&=\|v_0\|^2+2\int_0^t\langle v_s,\mathrm{d}v_s\rangle+\int_0^t\mathrm{d}\langle v_s,v_s\rangle\\&=\|v_0\|^2+2\int_0^t\Big\langle v_s,-Au_s-2\alpha v_s+\int_0^s\beta(s-r)Au_r\,\mathrm{d}r-f(u_s)\Big\rangle\mathrm{d}s\\&\quad+\int_0^t\langle v_s,\mathrm{d}W(s)\rangle+t\cdot\mathbf{Tr}\,Q\\&=\|v_0\|^2+\|\sqrt{A}\,u_0\|^2-\|\sqrt{A}\,u_t\|^2-2\alpha\int_0^t\|v_s\|^2\mathrm{d}s-\int_0^t\beta(s)\|\sqrt{A}\,u_s\|^2\mathrm{d}s\\&\quad+\int_0^t(\beta'\Box\sqrt{A}\,u)(s)\,\mathrm{d}s-(\beta\Box\sqrt{A}\,u)(t)+\Big(\int_0^t\beta(s)\,\mathrm{d}s\Big)\|\sqrt{A}\,u(t)\|^2\\&\quad-2F(u_t)+\int_0^t\langle v_s,\mathrm{d}W(s)\rangle+t\cdot\mathbf{Tr}\,Q.\end{aligned}$$

Hence, (4.9) follows. Taking the expectation for (4.9), we get (4.10). Finally, (4.11) follows from (4.10) by taking derivative.

Next, we will give our main estimation of the energy function.

Theorem 4.1 *Under the assumptions* (3.1)–(3.3) *and* (4.2)–(4.4)*, there exist constants* $c_1,c_2,c_3>0$ *such that*

$$\mathbf{E}\,e(u_t,v_t)\le c_1\mathrm{e}^{-c_2t}+c_3\mathbf{Tr}\,Q \tag{4.12}$$

for any $t\in[0,T]$.

We prove the theorem by the following lemmas.

Consider a nonnegative bounded function $\phi\in C^1(\overline{D}\,)$ such that

$$\phi(x)\ge\frac{\delta}{2}>0,\quad x\in\overline{D}\,. \tag{4.13}$$

Set

$$R_1(t):=-\int_D\phi(x)\Big[v_t(\beta*u)_t'-\frac{1}{2}|\beta*\sqrt{A}\,u|^2-\frac{1}{2}\beta(t)'|u|^2+\frac{1}{2}(\beta''\diamond u)(t)\Big](x)\,\mathrm{d}x, \tag{4.14}$$

where

$$\begin{aligned}(\beta*u)_t(x)&:=\int_0^t\beta(t-s)u(s,x)\,\mathrm{d}s,\\(\beta''\diamond u)_t(x)&:=\int_0^t\beta''(t-s)|u(t,x)-u(s,x)|^2\mathrm{d}s\end{aligned}$$

for each $x\in D$.

Lemma 4.2 *For any given* $\epsilon>0$*, there exists a constant* $C>0$ *such that*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}\,R_1(t)\le\mathbf{E}\Big[&-\frac{\delta}{2}\beta(0)\|v_t\|^2+C\epsilon((\beta\Box\sqrt{A}\,u)_t+k(t)\|\sqrt{A}\,u_t\|^2+2\alpha\|v_t\|^2)\\&+\frac{C}{\epsilon}(\beta(t)\|\sqrt{A}\,u_t\|^2-(\beta'\Box\sqrt{A}\,u)_t)\Big].\end{aligned} \tag{4.15}$$

Proof Using the Itô formula, we have

$$\begin{aligned}
\mathrm{d}\int_D v_t(x)\phi(x)(\beta*u)_t'(x)\,\mathrm{d}x &:= \mathrm{d}\langle v_t, \phi(\beta*u)_t'\rangle \\
&= [-\langle\sqrt{A}\,u_t, (\sqrt{A}\,\phi)(\beta*u)_t'\rangle - \langle\sqrt{A}\,u_t, \phi(\beta*\sqrt{A}\,u_t)_t'\rangle \\
&\quad + \langle(\beta*\sqrt{A}\,u)_t, (\sqrt{A}\,\phi)(\beta*u)_t'\rangle + \langle\beta*\sqrt{A}\,u_t, \phi(\beta*\sqrt{A}\,u)_t'\rangle \\
&\quad - \langle f(u_t), \phi(\beta*u)_t'\rangle - 2\alpha\langle v_t, \phi(\beta*u)_t'\rangle \\
&\quad + \langle v_t, \phi(\beta*u)_t''\rangle]\,\mathrm{d}t + \langle\phi(\beta*u)_t', \mathrm{d}W_t\rangle. \qquad (4.16)
\end{aligned}$$

From (4.6), it follows that

$$\begin{aligned}
&-\langle\sqrt{A}\,u_t, (\sqrt{A}\,\phi)(\beta*u)_t'\rangle + \langle(\beta*\sqrt{A}\,u)_t, (\sqrt{A}\,\phi)(\beta*u)_t'\rangle \\
&= -\langle(\beta\circ\sqrt{A}\,u)_t + k(t)\sqrt{A}\,u_t, (\sqrt{A}\,\phi)(\beta*u)_t'\rangle, \qquad (4.17)
\end{aligned}$$

where

$$(\beta\circ\sqrt{A}\,u)_t := \int_0^t \beta(t-s)(\sqrt{A}\,u_t - \sqrt{A}\,u_s)\,\mathrm{d}s.$$

By Lemma 4.1, we get

$$\begin{aligned}
\langle v_t, \phi(\beta*u)_t''\rangle &= \langle v_t, \phi(\beta(0)v_t + \beta'(0)u_t + (\beta''*u)_t)\rangle \\
&= \beta(0)\|\sqrt{\phi}\,v_t\|^2 + \frac{1}{2}\beta'(0)\frac{\mathrm{d}}{\mathrm{d}t}\|\sqrt{\phi}\,u_t\|^2 + \frac{1}{2}\langle\beta'''\diamond u, \phi\rangle - \frac{1}{2}\beta''(t)\|\sqrt{\phi}\,u_t\|^2 \\
&\quad - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\langle\beta''\diamond u, \phi\rangle + (\beta'(0)-\beta'(t))\|\sqrt{\phi}\,u_t\|^2) \\
&= \beta(0)\|\sqrt{\phi}\,v_t\|^2 - \frac{1}{2}\beta''(t)\|\sqrt{\phi}\,u_t\|^2 + \frac{1}{2}\langle\beta'''\diamond u, \phi\rangle \\
&\quad + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\beta'(t)\|\sqrt{\phi}\,u_t\|^2 - \langle(\beta''\diamond u)_t, \phi\rangle). \qquad (4.18)
\end{aligned}$$

Then, the assumption (3.2) on f, (4.3), (4.16)–(4.18), Fubini theorem and the Young inequality imply

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}\,R_1(t) &= \mathbf{E}\Big[\langle(\beta\circ\sqrt{A}\,u)_t + k(t)\sqrt{A}\,u_t, (\sqrt{A}\,\phi)(\beta*u)_t'\rangle \\
&\quad + \langle\sqrt{A}\,u_t, \phi(\beta*\sqrt{A}\,u)_t'\rangle - \beta(0)\|\sqrt{\phi}\,v_t\|^2 + \frac{1}{2}\beta''(t)\|\sqrt{\phi}\,u_t\|^2 \\
&\quad - \frac{1}{2}\langle\beta'''\diamond u, \phi\rangle + 2\alpha\langle v_t, \phi(\beta*u)_t'\rangle + \langle f(u_t), \phi(\beta*u)_t'\rangle\Big] \\
&\le \mathbf{E}\Big[-\beta(0)\|\sqrt{\phi}v_t\|^2 + C\beta(t)\|\sqrt{\phi}\,u_t\|^2 - C\langle\beta'\diamond u, \phi\rangle \\
&\quad + C_1\epsilon(\|(\beta\circ\sqrt{A}\,u)_t + k(t)\sqrt{A}\,u_t\|^2 + \|\sqrt{A}\,u_t\|^2 + \|f(u_t)\|^2 + 2\alpha\|v_t\|^2) \\
&\quad + \frac{C_1}{\epsilon}(\|(\sqrt{A}\,\phi)(\beta(t)u_t - (\beta'\circ u)_t)\|^2 + \|\phi(\beta*\sqrt{A}\,u)_t'\|^2 \\
&\quad + \|\phi(\beta*u)_t\|^2 + \|\phi(\beta*u)_t'\|^2)\Big] \\
&\le \mathbf{E}\Big[-\frac{\delta}{2}\beta(0)\|v_t\|^2 + C\epsilon((\beta\Box\sqrt{A}\,u)_t + k(t)\|\sqrt{A}\,u\|^2 + 2\alpha\|v_t\|^2) \\
&\quad + \frac{C}{\epsilon}(\beta(t)\|\sqrt{A}\,u_t\|^2 - (\beta'\Box\sqrt{A}\,u)_t)\Big].
\end{aligned}$$

Here we use the fact $L^{2\rho}(D) \supset D(\sqrt{A})$, which together with assumption (3.1) indicates

$$\|f(u)\|^2 \le C\|\sqrt{A}\,u\|^2.$$

Now, we introduce another functional to estimate the energy function.

Define

$$R_2(t) := \langle u_t, v_t \rangle. \tag{4.19}$$

Then we have the following lemma.

Lemma 4.3 *There exists some constant $C > 0$ such that*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}\,R_2(t) \le \mathbf{E}\Big[&\|v_t\|^2 - \frac{1}{2}k(t)\|\sqrt{A}\,u_t\|^2 + C(\beta\Box\sqrt{A}\,u)_t \\ &+ 2C\alpha\|v_t\|^2 - (\rho+1)\int_D F(u_t(x))\,\mathrm{d}x\Big].\end{aligned} \tag{4.20}$$

Proof Using Itô formula again to $\langle u_t, v_t\rangle$, we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}R_2(t) &= -\|\sqrt{A}\,u_t\|^2 + \langle (\beta * \sqrt{A}\,u)_t, \sqrt{A}\,u_t\rangle - \langle f(u_t), u_t\rangle \\ &\quad - 2\alpha\langle v_t, u_t\rangle + \langle u_t, \mathrm{d}W_t\rangle + \|v_t\|^2 \\ &= \|v_t\|^2 - k(t)\|\sqrt{A}\,u_t\|^2 - \langle (\beta\circ\sqrt{A}\,u)_t, \sqrt{A}\,u_t\rangle \\ &\quad - \langle f(u_t), u_t\rangle - 2\alpha\langle v_t, u_t\rangle + \langle u_t, \mathrm{d}W_t\rangle.\end{aligned} \tag{4.21}$$

Fubini theorem and Young inequality yield

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}\,R_2(t) \le \mathbf{E}\Big[&\|v_t\|^2 - k(t)\|\sqrt{A}\,u_t\|^2 + \frac{\epsilon}{2}\|\sqrt{A}\,u_t\|^2 + \frac{2}{\epsilon}(\beta\Box\sqrt{A}\,u)_t \\ &+ \frac{\epsilon}{2}\|\sqrt{A}\,u_t\|^2 + \frac{4\alpha}{\epsilon}\|v_t\|^2 - (\rho+1)\int_D F(u_t(x))\,\mathrm{d}x\Big].\end{aligned} \tag{4.22}$$

According to (4.2), we can take $0 < \epsilon \le \frac{1}{2} - \frac{1}{2}\int_0^\infty \beta(t)\,\mathrm{d}t$ in (4.22) and conclude that (4.20) holds for some constant C associated to ϵ.

Define

$$R(t) = R_1(t) + \frac{\delta\beta(0)}{4}R_2(t), \tag{4.23}$$

and this functional gives a control to the energy by the following lemma.

Lemma 4.4 *There exist two positive constants C and θ such that*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}\,R(t) \le \mathbf{E}[-\theta e(u_t, v_t) + C(\beta(t)\|\sqrt{A}\,u\|^2 + (\beta\Box\sqrt{A}\,u)_t - (\beta'\Box\sqrt{A}\,u)_t + 2\alpha\|v_t\|^2)]. \tag{4.24}$$

Proof The conclusion follows directly from combining Lemmas 4.2 and 4.3 by taking $\epsilon = \frac{1}{16}\delta\beta(0)$ in (4.15).

Proof of Theorem 4.1 By (4.3), we have

$$(\beta\Box\sqrt{A}\,u)_t \le -C(\beta'\Box\sqrt{A}\,u)_t,$$

which implies that (4.24) becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}\,R(t) &\le \mathbf{E}[-\theta e(u_t, v_t) + C(\beta(t)\|\sqrt{A}\,u\|^2 - (\beta'\Box\sqrt{A}\,u)_t + 2\alpha\|v_t\|^2)] \\ &\le -\theta\Big(\mathbf{E}\,e(u_t, v_t) - \frac{1}{2}t\cdot\mathbf{Tr}\,Q\Big) - 2C\frac{\mathrm{d}}{\mathrm{d}t}\Big(\mathbf{E}\,e(u_t, v_t) - \frac{1}{2}t\cdot\mathbf{Tr}\,Q\Big).\end{aligned} \tag{4.25}$$

From (4.1) in Proposition 4.1, we know that $\mathbf{E}\, e(u_t, v_t) - \frac{1}{2}t \cdot \mathbf{Tr}\, Q$ decreases as t increases.

Define

$$\Phi(t) := 2N\, \mathbf{E}\, e(u_t, v_t) + \mathbf{E}\, R(t).$$

There exists a positive constant N large enough such that

$$\Phi(t) \geq N\Big(\mathbf{E}\, e(u_t, v_t) - \frac{1}{2}t \cdot \mathbf{Tr}\, Q\Big) - C\lambda \tag{4.26}$$

and

$$\Phi(t) \leq 3N\Big(\mathbf{E}\, e(u_t, v_t) + \frac{1}{2}t \cdot \mathbf{Tr}\, Q\Big) + C\lambda. \tag{4.27}$$

On the other hand, (4.25) yields that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi(t) \leq -\theta\, \mathbf{E}\, e(u_t, v_t) + \frac{1}{2}N\, \mathbf{Tr}\, Q,$$

which indicates that, in view of inequalities (4.26) and (4.27),

$$\mathbf{E}\, e(u_t, v_t) \leq -\frac{\theta}{N}\int_0^t \mathbf{E}\, e(u_s, v_s)\, \mathrm{d}s + t \cdot \mathbf{Tr}\, Q + \frac{F(0) + C\lambda}{N}.$$

By Gronwall inequality, we obtain

$$\begin{aligned}
\mathbf{E}\, e(u_t, v_t) &\leq -\frac{\theta}{N}\int_0^t \Big(s \cdot \mathbf{Tr}\, Q + \frac{F(0) + C\lambda}{N}\Big)\mathrm{e}^{-\frac{\theta}{N}(t-s)}\mathrm{d}s + t \cdot \mathbf{Tr}\, Q + \frac{F(0) + C\lambda}{N} \\
&\leq \frac{F(0) + C\lambda}{N}\mathrm{e}^{-\frac{\theta}{N}t} + \frac{N}{\theta}\mathbf{Tr}\, Q.
\end{aligned} \tag{4.28}$$

The proof is completed.

Proof of Theorem 3.1 (Continuity) Since $\beta(t) = \mathrm{e}^{-\sigma t}$, it satisfies condition (4.3). By similar arguments, there exists a positive constant C such that, for each stopping time τ_n,

$$\mathbf{E}\, e(u_{T\wedge\tau_n}, v_{T\wedge\tau_n}) \leq C. \tag{4.29}$$

On the other hand,

$$e(u_{T\wedge\tau_n}, v_{T\wedge\tau_n}) \geq C' \|\sqrt{A}\, u_{T\wedge\tau_n}\|^2$$

for some positive constant C'. Then Chebyshev's inequality indicates

$$\begin{aligned}
\mathbf{E}\, e(u_{T\wedge\tau_n}, v_{T\wedge\tau_n}) &\geq \mathbf{E}[I_{\{\tau_n \leq T\}} e(u_{\tau_n}, v_{\tau_n})] \\
&\geq C'\mathbf{E}[I_{\{\tau_n \leq T\}} \|\sqrt{A}\, u_{\tau_n}\|^2] \\
&\geq C' n^2 P(\tau_n \leq T).
\end{aligned} \tag{4.30}$$

Combining (4.29) and (4.30), we get

$$P(\tau_n \leq T) \leq \frac{CC'}{n^2}. \tag{4.31}$$

Finally, Borel-Cantelli lemma implies that

$$p(\tau_\infty \leq T) = 0$$

for any $T > 0$, or equivalent, $\lim\limits_{n\to\infty} \tau_n = \infty$ a.s. This completes the proof of Theorem 3.1.

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