Chin. Ann. Math.
31B(1), 2010, 133–144
DOI: 10.1007/s11401-008-0028-7

Chinese Annals of Mathematics, Series B

Koszul Differential Graded Algebras and BGG Correspondence II***

Jiwei HE*　　Quanshui WU**

Abstract The concept of Koszul differential graded (DG for short) algebra is introduced in [8]. Let A be a Koszul DG algebra. If the Ext-algebra of A is finite-dimensional, i.e., the trivial module ${}_Ak$ is a compact object in the derived category of DG A-modules, then it is shown in [8] that A has many nice properties. However, if the Ext-algebra is infinite-dimensional, little is known about A. As shown in [15] (see also Proposition 2.2), ${}_Ak$ is not compact if $H(A)$ is finite-dimensional. In this paper, it is proved that the Koszul duality theorem also holds when $H(A)$ is finite-dimensional by using Foxby duality. A DG version of the BGG correspondence is deduced from the Koszul duality theorem.

Keywords Koszul differential graded algebra, Koszul duality, BGG correspondence
2000 MR Subject Classification 16E05, 16E40, 16W50

1 Introduction

In [8], we introduced the concept of Koszul DG algebras. Let k be a field. By a connected DG algebra we mean a cochain k-algebra A such that $A = \bigoplus_{n\geq 0} A^n$ and $A^0 = k$. A connected DG algebra A is said to be Koszul if the trivial DG module ${}_Ak$ has a semifree resolution with a semifree basis concentrated in degree 0. Let A be a connected DG algebra. We write $D(A)$ to be the derived category of A, and $D^c(A)$ to be the full triangulated subcategory of $D(A)$ generated by ${}_AA$, that is, the smallest triangulated subcategory of $D(A)$ containing ${}_AA$ as an object and closed under isomorphisms. We say that a DG module ${}_AM$ is compact if ${}_AM \in D^c(A)$. Let A be a Koszul DG algebra, and let $E = \mathrm{Ext}^*_A({}_Ak, {}_Ak)$ be its Ext-algebra. If the trivial DG module ${}_Ak$ is compact, then the Koszul DG algebra A has nice properties such as (i) the Yoneda algebra of E is isomorphic to the cohomology algebra $H(A)$; (ii) there is a duality of triangulated categories between the bounded derived category of finite modules over E and $D^c(A)$. However, if ${}_Ak$ is not compact, little is known about E and A.

Examples of DG algebra from differential geometry and algebraic topology usually have the property that the cohomology algebra is finite-dimensional. A DG algebra with this property was said to be compact by Kontsevich and Soibelman [10]. In this case, the trivial module ${}_Ak$ must not be compact (see Proposition 2.2). Hence the results obtained in [8] can not be applied

Manuscript received January 18, 2008. Published online December 11, 2009.
*Department of Mathematics, Shaoxing College of Arts and Sciences, Shaoxing 312000, Zhejiang, China.
E-mail: jwhe@zscas.edu.cn
**School of Mathematical Sciences, Fudan University, Shanghai 200433, China.
E-mail: qswu@fudan.edu.cn
***Project supported by the National Natural Science Foundation of China (Nos. 10801099, 10731070) and the Doctoral Program Foundation of the Ministry of Education of China (No. 20060246003).

to these DG algebras. In this paper, by using Foxby duality, we show that a Koszul DG algebra A such that $H(A)$ is finite dimensional still has the same properties (see Theorems 3.1, 3.2 and 4.1) as in the case that ${}_Ak$ is compact. The method of this paper is different from that of [8].

Throughout, k is a fixed field, unadorned $\otimes$ means $\otimes_k$. Let B be a graded algebra, M and N be graded B-modules. We write $\underline{\mathrm{Hom}}_B(M,N) = \bigoplus\limits_{n\in\mathbb{Z}} \mathrm{Hom}_B(M, N(n))$ to be the set of all graded B-module homomorphisms and $\underline{\mathrm{Ext}}_B(-,-)$ to be the derived functor of $\underline{\mathrm{Hom}}_B(-,-)$. For the notations and properties of DG algebras we refer to the references [1, 4, 8, 11].

2 Some Basic Properties of Koszul DG Algebras

Let A be a connected DG algebra. For convenience, we use $A^\natural$ to denote the underlying connected graded algebra of A. Similarly, if M is a DG A-module, we use $M^\natural$ to denote the underlying graded $A^\natural$-module. We use $M^\sharp$ to denote the graded vector space dual.

As in [4], let $R = B(A)$ be its bar construction for the augmented DG algebra A.

Lemma 2.1 (see [4]) (i) *The augmentation map* $B(A;A) = A\otimes R \xrightarrow{\epsilon\otimes\epsilon} {}_Ak$ *is a quasi-isomorphism, and hence* $A\otimes R$ *is a semifree resolution of* ${}_Ak$.

(ii) *The map* $\varphi : R^\sharp \longrightarrow \mathrm{End}_A(A\otimes R)$ *defined by*

$$\varphi(f)(1[a_1|\cdots|a_n]) = \sum_{i=0}^{n} (-1)^{|f|\omega_i} 1[a_1|\cdots|a_i] f([a_{i+1}|\cdots|a_n])$$

is a quasi-isomorphism of DG algebras, where $R^\sharp = \underline{\mathrm{Hom}}_k(R,k) = \bigoplus\limits_{i\in\mathbb{Z}} \mathrm{Hom}_k(R^i,k)$ *and* $\omega_i = |a_1| + \cdots + |a_i| + i$.

Proposition 2.1 *Let A be a Koszul DG algebra. If* $\mathrm{gl.dim}\, A^\natural < \infty$ *and* $\dim A^1 < \infty$, *then* ${}_Ak$ *is compact.*

Proof By the previous lemma, $\mathrm{Ext}^n_A(k,k) \cong H^n(R^\sharp)$. Let us inspect the cohomology of R. Consider the following second quadrant double complex $\mathbf{P}$:

$$\begin{tikzcd}
 & \vdots \arrow[d, "d^v"] & & & & \\
\cdots & \sum\limits_{i+j+k=4} A^i\otimes A^j\otimes A^k \arrow[l, "d^h"'] \arrow[d, "d^v"] & A^1\otimes A^1\otimes A^1 \arrow[l, "d^h"'] \arrow[d, "d^v"] & & & \\
\cdots & \sum\limits_{i+j=4} A^i\otimes A^j \arrow[l, "d^h"'] \arrow[d, "d^v"] & \sum\limits_{i+j=3} A^i\otimes A^j \arrow[l, "d^h"'] \arrow[d, "d^v"] & A^1\otimes A^1 \arrow[l, "d^h"'] \arrow[d, "d^v"] & & \\
\cdots & A^4 \arrow[l, "d^h"'] \arrow[d] & A^3 \arrow[l, "d^h"'] \arrow[d] & A^2 \arrow[l, "d^h"'] \arrow[d] & A^1 \arrow[l, "d^h"'] \arrow[d] & \\
\cdots & 0 \arrow[l] & 0 \arrow[l] & 0 \arrow[l] & 0 \arrow[l] & k \arrow[l]
\end{tikzcd}$$

where d^v and d^h are defined as follows:

$$d^h(a_1 \otimes \cdots \otimes a_n) = \sum_{i=1}^{n} (-1)^{\omega_i + 1} a_1 \otimes \cdots \otimes d(a_i) \otimes \cdots \otimes a_n,$$

$$d^v(a_1 \otimes \cdots \otimes a_n) = \sum_{i=2}^{n} (-1)^{\omega_i} a_1 \otimes \cdots \otimes a_{i-1}a_i \otimes \cdots \otimes a_n,$$

in which $\omega_i = \sum\limits_{j<i} (|a_j| - 1)$.

It is clear that the bar complex $R = \mathrm{Tot}^{\oplus}\mathbf{P}$. Let $\widehat{R} = \mathrm{Tot}^{\Pi}\mathbf{P}$. Then R is a subcomplex of $\widehat{R}$. Choose a filtration on $\widehat{R}$ as follows:

$$F_0\widehat{R} = \widehat{R}, \quad F_n\widehat{R} = \mathrm{Tot}^{\Pi}\mathbf{P}(n) \ \text{ for } n < 0,$$

where $\mathbf{P}(n)$ is the double subcomplex of $\mathbf{P}$ by deleting the right n columns. Then this filtration is exhaustive and complete. The E_0-level of the spectral sequence induced by the filtration $\{F_n\widehat{R}\}$ is the following diagram:

$$\begin{tikzcd}
 & \vdots \arrow[d, "d^v"] & & & & \\
\cdots & \sum\limits_{i+j+k=4} A^i \otimes A^j \otimes A^k \arrow[d, "d^v"] & A^1 \otimes A^1 \otimes A^1 \arrow[d, "d^v"] & & & \\
\cdots & \sum\limits_{i+j=4} A^i \otimes A^j \arrow[d, "d^v"] & \sum\limits_{i+j=3} A^i \otimes A^j \arrow[d, "d^v"] & A^1 \otimes A^1 \arrow[d, "d^v"] & & \\
\cdots & A^4 \arrow[d] & A^3 \arrow[d] & A^2 \arrow[d] & A^1 \arrow[d] & \\
\cdots & 0 & 0 & 0 & 0 & k
\end{tikzcd}$$

It follows from $\mathrm{gl.dim}\, A^{\natural} < \infty$ that the spectral sequence is bounded. By the complete convergence theorem (see [20]), the spectral sequence converges to $H(\widehat{R})$. Since A^1 is finite-dimensional and $\mathrm{gl.dim}\, A^{\natural} < \infty$, it follows that $\dim H^0(\widehat{R}) < \infty$. Since R is a subcomplex of $\widehat{R}$, $H^0(R) = Z^0(R) \subseteq Z^0(\widehat{R}) = H^0(\widehat{R})$. Hence $\dim \mathrm{Ext}_A^0(k,k) = \dim(H^0(R^{\sharp})) = \dim H^0(R) \leq \dim H^0(\widehat{R}) < \infty$.

We do not know when ${}_Ak$ is compact in general. However, we have the following proposition which is proved in [15], as a corollary of some homological identities over DG algebras. For completeness we give a direct proof here.

Proposition 2.2 *Let A be a nontrivial connected DG algebra (that is, $H(A) \neq k$). If $\dim H(A) < \infty$, then the trivial module ${}_Ak$ is not compact.*

Proof Suppose that ${}_Ak$ is compact. Let P be a minimal semifree resolution of ${}_Ak$ with a semifree filtration $P(0) \subseteq P(1) \subseteq \cdots \subseteq P(i) \subseteq \cdots$ and a finite set of semifree basis. By adjusting the filtration, we may get a new semifree filtration $F(0) \subseteq F(1) \subseteq \cdots \subseteq F(i) \subseteq \cdots$ of P satisfying the following conditions: (i) there are graded vector spaces $0 \neq U(i)$ for $i = 0, \cdots, t$ such that $F(i)/F(i-1) = A \otimes U(i)$ for $i = 0, \cdots, t$ and $F(i)/F(i-1) = 0$ for $i > t$; (ii) the graded vector space $U(i)$ is concentrated in degree j_i for each i and $j_0 \leq j_1 \leq \cdots \leq j_t$. Since $H(A)$ is finite-dimensional, there is an integer n such that $H^n(A) \neq 0$ and $H^i(A) = 0$ for all $i > n$. By the truncation of A at the n-th position, we have a bounded DG module N such that $N^i = 0$ for $i > n$ or $i < 0$ and a quasi-isomorphism of DG modules $A \longrightarrow N$. Then we get a quasi-isomorphism of DG modules $P = A \otimes_A P \longrightarrow N \otimes_A P$. Write $M = N \otimes_A P$. There is a natural filtration $M(0) \subseteq M(1) \subseteq \cdots \subseteq M(i) \subseteq \cdots$ inheriting from P. Clearly, we have $M(i)/M(i-1) = N \otimes U(i)$ for $i = 0, \cdots, t$ and $M(i)/M(i-1) = 0$ for $i > t$. By inspecting the $(j_t + n)$-th cohomology of M, one can easily see that $H^{j_t+n}(M) \neq 0$, hence a contradiction. It follows that ${}_Ak$ is not compact.

Proposition 2.3 *Let A be a Koszul DG algebra and $E = \mathrm{Ext}^0_A(k,k)$ be its Ext-algebra. Then E is a local algebra with residue field k.*

Proof If ${}_Ak$ is compact the result was proved in [8]. We now assume that ${}_Ak$ is not compact, hence E is infinite dimensional. By [8, Theorem 3.1], $E = k \oplus \Big(\prod\limits_{i\geq 1} E_i\Big)$ for some vector spaces E_i, such that each $F_n = \prod\limits_{i\geq n} E_i$ $(n \geq 1)$ is an ideal of E and E is a filtered algebra with the filtration $E = F_0 \supset F_1 \supset \cdots$. The filtration defines a topology on E so that E is a complete topological algebra. Next, we show that $J = F_1$ is the Jacobson radical of E. To this end, it suffices to show that for any $0 \neq x \in J$, $1 + x$ has a left inverse. Since E is complete, we may write $x = x_1 + x_2 + \cdots$ with $x_i \in E_i$. Set $x_1^{(1)} = x_1$. We have $(1 - x_1^{(1)})(1 + x) = 1 + x_1 + \sum\limits_{i\geq 2} x_i - x_1^{(1)} - x_1^{(1)}x = 1 + \sum\limits_{i\geq 2} x_i - x_1^{(1)}x$. Since E is filtered, $x^{(2)} = \sum\limits_{i\geq 2} x_i - x_1^{(1)}x \in F_2$. Hence, we may write $x^{(2)} = x_2^{(2)} + x_3^{(2)} + \cdots$ with $x_i^{(2)} \in E_i$ for $i \geq 2$. We have

$$\begin{aligned}(1 - x_2^{(2)})(1 - x_1^{(1)})(1 - x) &= (1 - x_2^{(2)})(1 + x^{(2)}) \\ &= 1 + x_2^{(2)} + \sum_{i\geq 3} x_i^{(2)} - x_2^{(2)} - x_2^{(2)}x^{(2)} \\ &= 1 + \Big(\sum_{i\geq 3} x_i^{(2)} - x_2^{(2)}x^{(2)}\Big).\end{aligned}$$

Now $x^{(3)} = \sum\limits_{i\geq 3} x_i^{(2)} - x_2^{(2)}x^{(2)} \in F_3$. Similarly to the previous procedure, we may write $x^{(3)} = x_3^{(3)} + x_4^{(3)} + \cdots$ with $x_i^{(3)} \in E_i$ for $i \geq 3$.

Inductively, we have a sequence of elements $\{x_n^{(n)} \mid n \geq 1\}$ such that

$$(1 - x_n^{(n)})(1 - x_{n-1}^{(n-1)}) \cdots (1 - x_1^{(1)})(1 + x) = 1 + x_{n+1}^{(n+1)} + x_{n+2}^{(n+1)} + \cdots \tag{2.1}$$

with $x_i^{(n+1)} \in E_i$ for $i \geq n+1$. Set $y_n = \prod\limits_{i=0}^{n-1}(1 - x_{n-i}^{(n-i)})$ and $z_n = 1 + \sum\limits_{i\geq n+1} x_i^{(i)}$. We get two sequences of elements $\{y_n \mid n \geq 1\}$ and $\{z_n \mid n \geq 1\}$. Since $y_n - y_{n-1} = (1 - x_n^n)y_{n-1} - y_{n-1} = -x_n^n y_{n-1} \in F_n$, it follows that $\{y_n \mid n \geq 1\}$ is a Cauchy sequence. Hence $\{y_n \mid n \geq 1\}$ converges in E. Set $y = \lim\limits_{n\to\infty} y_n$. Taking limit on both sides of (2.1), we have

$$\lim_{n\to\infty}(y_n(1+x)) = \lim_{n\to\infty} z_n.$$

It follows

$$y(1+x) = \Big(\lim_{n\to\infty} y_n\Big)(1+x) = \lim_{n\to\infty} z_n = 1,$$

that is, $1+x$ has a left inverse. Hence E is local.

3 Koszul Duality

Examples of DG algebra from differential geometry and algebraic topology usually have the property that the cohomology algebra is finite-dimensional. For example, let X be a finite CW complex, and let $B = C^*(X;k)$ be the singular cochain algebra on X. Then $H(B)$ is finite-dimensional. There is a connected DG algebra A which is weakly equivalent to B (see [4]), that is, there are finitely many DG algebras $D_1, \cdots, D_n$ which are connected by quasi-isomorphisms:

$$B \xrightarrow{\simeq} D_1 \xleftarrow{\simeq} \cdots \xrightarrow{\simeq} D_n \xleftarrow{\simeq} A.$$

Of course, $H(A)$ is finite-dimensional. By Proposition 2.2, the trivial module ${}_Ak$ is not compact. Hence, the Koszul duality established in [8] is not applied to this class of DG algebras. In this section, we prove a version of Koszul duality theorem for Koszul DG algebras with finite-dimensional cohomology algebra by using Foxby duality (see [5]).

If A is a Koszul DG algebra, Proposition 2.3 says that $E = \operatorname{Ext}^0_A(k,k)$ is a local algebra with residue field k. Hence k is a left E-module by the trivial action. If we view E as a DG algebra concentrated in degree 0, then $A \otimes E$ is an augmented DG algebra.

For any DG A-modules M and N, let $\underline{\mathrm{Hom}}_A(M,N) = \underline{\mathrm{Hom}}_{A^\natural}(M^\natural, N^\natural)$. Let $\mathrm{R}\underline{\mathrm{Hom}}_A(-,N)$ be the right derived functor of $\underline{\mathrm{Hom}}_A(-,N)$.

Let $\mathcal{I}$ be a K-injective resolution of ${}_Ak$, and let $B = \underline{\mathrm{Hom}}_A(\mathcal{I},\mathcal{I})$. Then B is a DG algebra and $\mathcal{I}$ is a DG $A\otimes B$-module. Since A is Koszul, $H^i(B) = 0$ for $i \neq 0$. We have the following truncation:

$$B' := \cdots B^{-n} \longrightarrow B^{-n+1} \longrightarrow \cdots \longrightarrow B^{-1} \longrightarrow Z^0(B) \longrightarrow 0,$$

where $Z^0(B)$ is the 0-th cocycles of B. Then one can easily check that B' is a DG subalgebra of B, and the inclusion map $B' \hookrightarrow B$ is a quasi-isomorphism. Hence $\mathcal{I}$ is a DG $A\otimes B'$-module. Write

$$\begin{aligned} F &= \mathrm{R}\underline{\mathrm{Hom}}_A\,(-,\mathcal{I}):\ D(A) \longrightarrow D(B'),\\ G &= \mathrm{R}\underline{\mathrm{Hom}}_{B'}(-,\mathcal{I}): D(B') \longrightarrow D(A). \end{aligned}$$

Then F and G is a pair of adjoint contravariant functors.

Let

$$\mathcal{A}(A) = \{M \in D(A) \mid \text{the adjunction map } M \longrightarrow GFM \text{ is isomorphic}\}$$

be the Auslander class, and

$$\mathcal{B}(B') = \{N \in D(B') \mid \text{the adjunction map } N \longrightarrow FGN \text{ is isomorphic}\}$$

be the Bass class.

Lemma 3.1 (Foxby Duality) *If (F, G) is a pair of adjoint contravariant triangulated functors between triangulated categories $\mathcal{C}$ and $\mathcal{D}$, then*

(i) *the Auslander class and the Bass class are full triangulated subcategories,*

(ii) *F and G induce a pair of dualities between the Auslander class and the Bass class.*

By the Foxby duality, $F : \mathcal{A}(A) \longrightarrow \mathcal{B}(B')$ and $G : \mathcal{B}(B') \longrightarrow \mathcal{A}(A)$ is a pair of duality functors between triangulated categories.

Lemma 3.2 *The regular DG module ${}_{B'}B'$ is in the Bass class $\mathcal{B}(B')$.*

Proof In $D(B')$, we have

$$\begin{aligned} FG(B') &= \mathrm{R}\underline{\mathrm{Hom}}_A(\mathrm{R}\underline{\mathrm{Hom}}_{B'}(B', \mathcal{I}), \mathcal{I}) \\ &= \underline{\mathrm{Hom}}_A(\underline{\mathrm{Hom}}_{B'}(B', \mathcal{I}), \mathcal{I}) \\ &= \underline{\mathrm{Hom}}_A(\mathcal{I}, \mathcal{I}) \\ &\cong {}_{B'}B'. \end{aligned}$$

Hence ${}_{B'}B'$ is in the Bass class $\mathcal{B}(B')$.

Temporarily, we write $\langle {}_A k\rangle$ and $\langle {}_{B'}B'\rangle$ to be the full triangulated categories of $D(A)$ and $D(B')$ generated by ${}_A k$ and ${}_{B'}B'$ respectively.

Lemma 3.3 *The trivial DG module ${}_A k$ is in the Auslander class $\mathcal{A}(A)$, and hence F and G induce a pair of duality functors between $\langle {}_A k\rangle$ and $\langle {}_{B'}B'\rangle$.*

Proof In $D(A)$, since we have $G({}_{B'}B') = \mathrm{R}\underline{\mathrm{Hom}}_{B'}(B', \mathcal{I}) = \underline{\mathrm{Hom}}_{B'}(B', \mathcal{I}) = \mathcal{I} \cong {}_A k$, by Foxby duality, ${}_A k$ is in the Auslander class $\mathcal{A}(A)$. Hence we have a pair of duality functors

$$\begin{aligned} F :&\ \langle {}_A k\rangle \longrightarrow \langle {}_{B'}B'\rangle, \\ G :&\ \langle {}_{B'}B'\rangle \longrightarrow \langle {}_A k\rangle. \end{aligned}$$

Since A is Koszul, $E = \mathrm{Ext}_A^0(k, k) = H^0(B) = H^0(B')$. Since B' is concentrated in negative degrees, there is a quasi-isomorphism of DG algebras

$$\varphi : B' \longrightarrow E.$$

Lemma 3.4 (see [11]) *Let D and D' be DG algebras. If there is a quasi-isomorphism of DG algebras $\psi : D \longrightarrow D'$, then the restriction of ψ induces an equivalence of triangulated categories $\psi^* : D(D') \longrightarrow D(D)$ with the inverse functor $D' \otimes_D^L -$.*

By the lemma above, we have a pair of quasi-inverse equivalences of triangulated categories $\varphi^* : D(E) \longrightarrow D(B')$ and $E \otimes_{B'}^L - : D(B') \longrightarrow D(E)$. Clearly $\varphi^*({}_E E) = {}_{B'}B'$. By the restriction of φ^* and $E \otimes_{B'}^L -$, we get a pair of quasi-inverse equivalences

$$\begin{aligned} \zeta :&\ \langle {}_E E\rangle \longrightarrow \langle {}_{B'}B'\rangle, \\ \xi :&\ \langle {}_{B'}B'\rangle \longrightarrow \langle {}_E E\rangle, \end{aligned}$$

such that $\zeta({}_E E) = {}_{B'}B'$.

Proposition 3.1 *Let A be a Koszul DG algebra. There is a pair of duality functors between triangulated categories*

$$\theta : D_{fd}(A) \longrightarrow \langle {}_E E\rangle,$$
$$\phi : \quad \langle {}_E E\rangle \longrightarrow D_{fd}(A),$$

where $D_{fd}(A)$ is the full triangulated subcategory of $D(A)$ consisting of DG modules M such that $\dim H(M) < \infty$.

Proof Let $\theta = \xi \circ F$ and $\phi = G \circ \zeta$. Then we have a pair of duality functors

$$\theta : \langle {}_A k\rangle \longrightarrow \langle {}_E E\rangle,$$
$$\phi : \langle {}_E E\rangle \longrightarrow \langle {}_A k\rangle.$$

By [8, Lemma 5.5], we have $\langle {}_A k\rangle = D_{fd}(A)$. Hence the result follows.

Theorem 3.1 (Koszul Duality on Ext-Algebra) *Let A be a Koszul DG algebra, E be its* Ext-*algebra. If* $\dim H(A) < \infty$, *then* $\mathrm{Ext}^*_E(k,k) \cong H(A)$.

Proof Since $\dim H(A) < \infty$, ${}_A A \in D_{fd}(A)$. By Proposition 3.1, we have

$$\begin{aligned}\theta({}_A A) &= \xi \circ F({}_A A)\\ &= E \otimes^L_{B'} \mathrm{R}\underline{\mathrm{Hom}}_A({}_A A, \mathcal{I})\\ &= E \otimes^L_{B'} \mathcal{I}\\ &\overset{(a)}{\cong} {}_E k,\end{aligned} \tag{3.1}$$

where the isomorphism (a) holds for the following reason. Let $X = E \otimes^L_{B'} \mathcal{I}$. Since E and B' are quasi-isomorphic, $B'_{B'}$ is a semifree resolution of $E_{B'}$. Hence $X \cong B' \otimes^L_{B'} \mathcal{I} = \mathcal{I}$ as the complex of vector spaces. Hence $H^0(X) = k$ and $H^i(X) = 0$ for $i \neq 0$. Since E is concentrated in degree zero, the DG E-module X is exactly a cochain complex of E-modules. Hence, by suitable truncations, X is isomorphic to a simple E-module in $D(E)$. While Proposition 2.3 says that E is a local algebra with residue field k, hence there is a unique simple module in the category of E-modules. Thus X is isomorphic to ${}_E k$ in $D(E)$.

The isomorphisms in (3.1) say that the trivial module ${}_E k$ is in $\langle {}_E E\rangle$. We have the following equalities:

$$\begin{aligned}\mathrm{Ext}^*_E(k,k) &= \bigoplus_{i\geq 0} \mathrm{Hom}_{D(E)}({}_E k, {}_E k[i])\\ &\cong \Big(\bigoplus_{i\geq 0} \mathrm{Hom}_{D(A)}(\phi({}_E k), \phi({}_E k)[i])\Big)^{\mathrm{op}}\\ &\cong \Big(\bigoplus_{i\geq 0} \mathrm{Hom}_{D(A)}({}_A A, {}_A A[i])\Big)^{\mathrm{op}}\\ &\cong ((H(A))^{\mathrm{op}})^{\mathrm{op}}\\ &= H(A).\end{aligned}$$

Corollary 3.1 *Let A be a Koszul DG algebra, E be its* Ext-*algebra. If* $\dim H(A) < \infty$ *and E is Noetherian, then* $\mathrm{gl.dim}\, E < \infty$.

Proof Since E is local and Noetherian, $\mathrm{gl.dim}\, E = \mathrm{pd}\,_E k$. By Theorem 3.1, there is an integer n such that $\mathrm{Ext}^n_E(k,k) = 0$ since $\dim H(A) < \infty$.

Similarly to [8, Corollaries 3.8 and 3.9], we have the following results.

Corollary 3.2 *Let A be a Koszul DG algebra, E be its* Ext*-algebra. If $\dim H(A) < \infty$, then*

(i) *the local algebra E is quasi-Koszul* (*see* [6]) *if and only if $H(A)$ is generated by $H^1(A)$;*

(ii) *E is strongly quasi-Koszul* (*see* [6]) *if and only if $H(A)$ is a Koszul algebra.*

Theorem 3.2 (Koszul Duality) *Let A be a Koszul DG algebra, E be its Ext-algebra. If $\dim H(A) < \infty$ and E is Noetherian, then we have a pair of duality functors between triangulated categories*

$$\theta : D_{fd}(A) \longrightarrow D^b(\mathrm{mod}\, E), \quad \phi : D^b(\mathrm{mod}\, E) \longrightarrow D_{fd}(A),$$

where $\mathrm{mod}\, E$ *is the category of finitely generated left E-modules and* $D^b(\mathrm{mod}\, E)$ *is the bounded derived category of* $\mathrm{mod}\, E$.

Moreover, under these dualities $\theta({}_A A) = {}_E k$ and $\theta({}_A k) = {}_E E$.

Proof By Proposition 3.1, it suffices to show $\langle {}_E E\rangle = D^b(\mathrm{mod}\, E)$. This follows from Corollary 3.1 and the hypothesis that E is Noetherian.

For the rest of this section, assume that A is a compact Koszul DG algebra, E is its Ext-algebra and E is Noetherian. We next show that there is a natural t-structure on the triangulated category $D_{fd}(A)$ and the heart of the t-structure is a category of modules.

Define subclasses

$$D^{\geq n} = \{X \in D_{fd}(A) \mid \mathrm{Ext}^i_A(X,k) = 0 \text{ for all } i > -n\},$$
$$D^{\leq n} = \{X \in D_{fd}(A) \mid \mathrm{Ext}^i_A(X,k) = 0 \text{ for all } i < -n\}.$$

Proposition 3.2 *$(D^{\geq 0}, D^{\leq 0})$ is a t-structure on $D_{fd}(A)$.*

Proof Let

$$\mathcal{T}^{\geq n} = \{U \in D^b(\mathrm{mod}\, E) \mid H^i(U) = 0 \text{ for all } i < n\},$$
$$\mathcal{T}^{\leq n} = \{U \in D^b(\mathrm{mod}\, E) \mid H^i(U) = 0 \text{ for all } i > n\}.$$

Then $(\mathcal{T}^{\geq n}, \mathcal{T}^{\leq n})$ is a t-structure on $D^b(\mathrm{mod}\, E)$. By Theorem 3.2, there is an anti-equivalence

$$\phi : D^b(\mathrm{mod}\, E) \longrightarrow D_{fd}(A).$$

The anti-equivalence induces a t-structure $(\phi(\mathcal{T}^{\leq n}), \phi(\mathcal{T}^{\geq n}))$ on $D_{fd}(A)$. We next show $\phi(\mathcal{T}^{\geq n}) = D^{\leq -n}$ and $\phi(\mathcal{T}^{\leq n}) = D^{\geq -n}$. We check the following steps:

$$\begin{aligned} U \in \mathcal{T}^{\geq n} &\Longleftrightarrow H^i(U) = 0 \quad \text{for all } i < n \\ &\Longleftrightarrow \mathrm{Hom}_{D^b(\mathrm{mod} E)}(E, U[i]) = 0 \quad \text{for all } i < n \\ &\Longleftrightarrow \mathrm{Hom}_{D_{fd}(A)}(\phi(U), \phi(E)[i]) = 0 \quad \text{for all } i < n \\ &\overset{(a)}{\Longleftrightarrow} \mathrm{Hom}_{D_{fd}(A)}(\phi(U), k[i]) = 0 \quad \text{for all } i < n \\ &\Longleftrightarrow \mathrm{Ext}^i_A(\phi(U), k_A) = 0 \quad \text{for all } i < n \\ &\Longleftrightarrow \phi(U) \in D^{\leq -n}, \end{aligned}$$

where (a) holds by Theorem 3.2. Similarly, we have $\phi(\mathcal{T}^{\leq n}) = D^{\geq -n}$. Hence the proposition follows.

Let $\mathcal{K}$ be the heart of the t-structure $(D^{\geq 0}, D^{\leq 0})$. If a DG module X is an object of $\mathcal{K}$, then $\mathrm{Ext}^i_A(X, k) = 0$ for all $i \neq 0$. We call such a DG module a Koszul DG module.

Theorem 3.3 $\mathrm{Ext}^0_A(-, k) : \mathcal{K} \longrightarrow \mathrm{mod}\, E$ *is an anti-equivalence of Abelian categories.*

Proof By Proposition 3.2, $\phi^{-1} : \mathcal{K} \longrightarrow \mathrm{mod}\, E$ is an anti-equivalence of Abelian categories. We have natural isomorphisms

$$\begin{aligned}
\phi^{-1}(X) &= \mathrm{Hom}_E(E, \phi^{-1}(X)) \\
&= \mathrm{Hom}_{D^b(\mathrm{mod} E)}(E, \phi^{-1}(X)) \\
&\cong \mathrm{Hom}_{D_{fd}(A)}(X, \phi(E)) \\
&\cong \mathrm{Hom}_{D_{fd}(A)}(X, k) \\
&= \mathrm{Ext}^0_A(X, k).
\end{aligned}$$

Hence $\phi^{-1} = \mathrm{Ext}^0_A(-, k)$.

4 BGG Correspondence

In this section, we form a correspondence of triangulated categories similar to the BGG correspondence (see [3, 8, 9, 16, 18]) for Koszul DG algebras with finite-dimensional cohomology algebra.

Throughout this section, A is a Koszul DG algebra with $H(A)$ finite-dimensional, and E is its Ext-algebra.

First of all, we recall some terminologies.

Definition 4.1 (see [7]) *Let R be a Noetherian local algebra with residue field k. R is said to be Gorenstein if there is an integer $d \geq 0$ such that*

$$\mathrm{Ext}^n_R(k, R) = \begin{cases} 0, & n \neq d, \\ k, & n = d. \end{cases}$$

Definition 4.2 (see [13]) *A connected DG algebra B is Frobenius if there is a quasi-isomorphism ${}_BB \longrightarrow {}_BB^\sharp[l]$ for some integer l.*

Proposition 4.1 *E is Gorenstein if and only if A is Frobenius.*

Proof By Proposition 3.1 and the isomorphisms in (3.1), we have the following equalities:

$$\begin{aligned}
\mathrm{Ext}^n_E(k, E) &= \mathrm{Hom}_{D(E)}({}_Ek, {}_EE[n]) \\
&\cong \mathrm{Hom}_{D(A)}(\phi({}_EE), \phi({}_Ek)[n]) \\
&= \mathrm{Hom}_{D(A)}({}_Ak, {}_AA[n]).
\end{aligned}$$

It follows that $\mathrm{Ext}^n_E(k, E) = 0$ for $n \neq d$ if and only if $\mathrm{Hom}_{D(A)}({}_Ak, {}_AA)[n] = 0$ for $n \neq d$, and $\mathrm{Ext}^d_E(k, E) = k$ if and only if $\mathrm{Hom}_{D(A)}({}_Ak, {}_AA)[d] = k$. We claim that

$$\mathrm{Hom}_{D(A)}({}_Ak, {}_AA[n]) = \begin{cases} 0, & n \neq d, \\ k, & n = d, \end{cases}$$

if and only if A is Frobenius. Since $H(A)$ is finite dimensional, the DG A-module $A^\sharp_A$ has minimal semifree resolution P with a set of semibasis $\{e_\alpha \mid \alpha \in \Lambda\}$. Then $P^\sharp$ is a K-injective resolution of ${}_AA$. If

$$\operatorname{Hom}_{D(A)}({}_Ak, {}_AA[n]) = \operatorname{Hom}_A({}_Ak, P^\sharp[n]) = \begin{cases} 0, & n \neq d, \\ k, & n = d, \end{cases}$$

then the index set Λ has only one element. Moreover $P^\sharp = A^\sharp[-d]$, that is, ${}_AA \cong {}_AA^\sharp[-d]$. Hence A is Frobenius. The other direction of the claim is clear.

Lemma 4.1 *A connected DG algebra B is Frobenius if and only if $H(B)$ is graded Frobenius.*

Proof If B is Frobenius, then there is a quasi-isomorphism of DG modules ${}_BB \longrightarrow {}_BB^\sharp[l]$, which implies a graded module isomorphism ${}_{H(B)}H(B) \longrightarrow {}_{H(B)}H(B^\sharp)[l]$. Hence $H(B)$ is graded Frobenius. Conversely, if $H(B)$ is graded Frobenius, then

$${}_{H(B)}H(B) \longrightarrow {}_{H(B)}H(B^\sharp)[l] \tag{4.1}$$

is a free resolution of $H(B^\sharp)$. The Eilenberg-Moore resolution (see [4]) of ${}_BB^\sharp[l]$ defined by (4.1) is

$${}_BB \longrightarrow {}_BB^\sharp[l].$$

Hence the DG algebra B is Frobenius.

Corollary 4.1 *E is Gorenstein if and only if $H(A)$ is a graded Frobenius algebra.*

Proof The proof is directly from Proposition 4.1 and Lemma 4.1.

Now suppose that E is Noetherian. Let J be its Jacobson radical. An E-module M is called a J-torsion module if for any element $m \in M$ there is an integer n such that $J^n m = 0$. Let $\operatorname{tor} E$ be the full subcategory of $\operatorname{mod} E$ consisting of all the J-torsion modules. Since E is Noetherian, $\operatorname{tor} E$ is a thick Abelian subcategory of $\operatorname{mod} E$. Write $\operatorname{qmod} E$ to be the quotient category $\operatorname{mod} E/\operatorname{tor} E$. Since E is Noetherian, $\operatorname{tor} E$ is exactly the category of all finite dimensional E-modules.

Theorem 4.1 (BGG Correspondence) *If E is Noetherian, then we have an anti-equivalence of triangulated categories*

$$D^b(\operatorname{qmod} E) \longrightarrow D_{fd}(A)/D^c(A).$$

Proof By Theorem 3.2, there is an anti-equivalence

$$D^b(\operatorname{mod} E) \longrightarrow D_{fd}(A).$$

Under this anti-equivalence, ${}_Ek$ is corresponding to ${}_AA$. Hence we have an anti-equivalence

$$D^b(\operatorname{mod} E)/\langle {}_Ek\rangle \longrightarrow D_{fd}(A)/D^c(A).$$

It is clear $\langle {}_Ek\rangle = D^b_{\operatorname{tor} E}(\operatorname{mod} E)$, the full triangulated subcategory of $D^b(\operatorname{mod} E)$ consisting of complexes M such that each cohomology $H^\bullet(M)$ is a J-torsion module. By [17],

$$D^b(\operatorname{mod} E)/D^b_{\operatorname{tor} E}(\operatorname{mod} E) \cong D^b(\operatorname{qmod} E).$$

Hence the result follows.

Remark 4.1 The BGG correspondence established in Theorem 4.1 also implies the classical one in [3].

In fact, as we see in [8], if A is a Koszul Adams connected DG algebra, then its Ext-algebra E is a connected graded algebra. Following the notations in [8], we write $AD_{dg}(A)$ to be the derived category of left DG A-modules, $AD^c(A)$ to be the full triangulated subcategory of $AD_{dg}(A)$ generated by ${}_AA$, and $AD_{fd}(A)$ to be the full triangulated subcategory of $AD_{dg}(A)$ consisting of objects with finite-dimensional cohomologies. Now the Koszul duality is of the following form:

$$D^b(\operatorname{gr} E) \xrightarrow{\cong} AD_{fd}(A), \tag{4.2}$$

and the BGG correspondence is of the following form:

$$D^b(\operatorname{qgr} E) \xrightarrow{\cong} AD_{fd}(A)/AD^c(A). \tag{4.3}$$

Now let R be a Noetherian Koszul AS-regular algebra. Then its Yoneda algebra $S = R^!$ is a graded Frobenius algebra. Let A be the Adams connected DG algebra given as follows: $A_i^i = S_i$ for $i \geq 0$ and $A_j^i = 0$ for $i \neq j$. Then A is a Koszul Adams connected DG algebra. The Ext-algebra of A is R. Since $AD_{dg}(A) = D(S)$ and S is finite dimensional, it follows that $AD_{fd}(A) = D^b(\operatorname{gr} S)$ and $AD^c(A) = D^b(\operatorname{proj} S)$, where $\operatorname{proj} S$ is the category of finitely generated graded projective S-modules. Hence the Koszul duality (4.2) can be written as

$$D^b(\operatorname{gr} R) \xrightarrow{\cong} D^b(S),$$

and the BGG correspondence (4.3) as

$$D^b(\operatorname{qgr} R) \xrightarrow{\cong} D^b(S)/D^b(\operatorname{proj} S).$$

By [19], $D^b(S)/D^b(\operatorname{proj} S) \cong \underline{\operatorname{gr}}\, S$ since S is graded Frobenius. Hence the BGG correspondence is of the form

$$D^b(\operatorname{qgr} R) \xrightarrow{\cong} \underline{\operatorname{gr}}\, S,$$

which was proved in [3, 9, 18].

References

[1] Avramov, L. L., Foxby, H. B. and Halperin, S., Differetial Graded Homological Algebra, preprint.

[2] Bezrukavnikov, R., Koszul DG-algebras arising from configuration spaces, *Geom. Funct. Anal.*, **4**(2), 1994, 119–135.

[3] Bernstein, I. N., Gelfand, I. M. and Gelfand, S. I., Algebraic bundles over $\mathbb{P}^n$ and problems in linear algebra, *Funct. Anal. Appl.*, **12**, 1979, 212–214.

[4] Félix, Y., Halperin, S. and Thomas, J. C., Rational Homotopy Theory, Graduate Texts in Mathematics, Vol. 205, Springer-Verlag, New York, 2001.

[5] Frankild, A. and Jørgenson, P., Foxby equivalence, complete modules, and torsion modules, *J. Pure Appl. Algebra*, **174**(2), 2002, 135–147.

[6] Green, E. L. and Martínez-Villa, R., Koszul and Yoneda algebras, Representation Theory of Algebras, CMS Conference Proceedings, Vol. 18, 1996, 247–297.

[7] Hartshorne, R., Residues and Duality, Lecture Notes in Mathematics, Vol. 20, Springer-Verlag, New York, 1966.

[8] He, J. W. and Wu, Q. S., Koszul differential graded algebras and BGG correspondence, *J. Algebra*, **320**(7), 2008, 2934–2962.

[9] Jørgensen, P., A noncommutative BGG correspondence, *Pacific J. Math.*, **218**(2), 2005, 357–378.

[10] Kontsevich, M. and Soibelman, Y., Notes on A_∞-algebras, A_∞-categories and non-commutative geometry I, 2006. arXiv:math.RA/0606241

[11] Kříž, I. and May, J. P., Operads, Algebras, Modules and Motives, Astérisque, Vol. 233, Société Mathématique de France, Paris, 1995.

[12] Löfwall, C., On the subalgebra generated by the one-dimensional elements in the Yoneda Ext-algebra, Algebra, Algebraic Topology and Their Interactions, Lecture Notes in Mathematics, Vol. 1183, Springer-Verlag, New York, 1986, 291–338.

[13] Lu, D. M., Palmieri, J. H., Wu, Q. S. and Zhang, J. J., A_∞-algebras for ring theorists, *Algebra Colloq.*, **11**(1), 2004, 91–128.

[14] Mao, X. F. and Wu, Q. S., Homological identities of DG algebras, *Comm. Algebra*, **36**(8), 2008, 3050–3072.

[15] Mao, X. F. and Wu, Q. S., Compact DG modules and Gorenstein DG algebras, *Sci. China Ser. A*, **52**(4), 2009, 648–676.

[16] Martínez-Villa, R. and Saorín, M., Koszul equivalences and dualities, *Pacific J. Math.*, **214**(2), 2004, 359–378.

[17] Miyachi, J. I., Localization of triangulated categories and derived categories, *J. Algebra*, **141**(2), 1991, 463–483.

[18] Mori, I., Riemann-Roch like theorem for triangulated categories, *J. Pure Appl. Algebra*, **193**(1–3), 2004, 263–285.

[19] Orlov, D., Derived categories of coherent sheaves and triangulated categories of singularities, 2005. arXiv: math.AG/0503632

[20] Weibel, C. A., An Introduction to Homological Algebra, Cambridge University Press, Cambridge, 1994.