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Deleting Vertices and Interlacing Laplacian Eigenvalues****

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Abstract The authors obtain an interlacing relation between the Laplacian spectra of a graph G and its subgraph $G - U$, which is obtained from G by deleting all the vertices in the vertex subset U together with their incident edges. Also, some applications of this interlacing property are explored and this interlacing property is extended to the edge weighted graphs.

Keywords Interlacing inequality, Eigenvalue, Spectrum, Laplacian matrix
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1 Introduction

In this work, we are primarily interested in undirected graphs without loops or multiple edges, i.e., simple graphs. Let $G = (V, E)$ be a simple graph with vertex set $V = V(G) = \{v_1, v_2, \cdots, v_n\}$ and edge set $E = E(G) = \{e_1, e_2, \cdots, e_m\}$. Each edge in E can be represented by its endpoints such as (u, v). If $(u, v) \in E$, we use the notation $u \sim v$. Denote the set of neighbors of v by $N_G(v)$ and the degree of v by $d_G(v) = |N_G(v)|$. We say v is a pendant vertex if $d_G(v) = 1$.

The Laplacian matrix of G is defined as $L(G) = D(G) - A(G)$, where $D(G)$ is the diagonal matrix of vertex degrees and $A(G)$ is the $(0, 1)$ adjacency matrix of G. It is well-known that $L(G)$ is a real symmetric, positive semidefinite matrix and its eigenvalues are all real and nonnegative.

The eigenvalues of an $n \times n$ real symmetric matrix M are denoted by $\lambda_i(M)$ $(i = 1, 2, \cdots, n)$, where we always assume the eigenvalues to be arranged in nonincreasing order: $\lambda_1(M) \geq \lambda_2(M) \geq \cdots \geq \lambda_n(M)$. While for a graph G, we use $\lambda_i(G)$ instead of $\lambda_i(L(G))$ $(i = 1, 2, \cdots, n)$. The Laplacian spectrum of G is defined as $\mathrm{LSpec}(G) = (\lambda_1(G), \lambda_2(G), \cdots, \lambda_n(G))$. It turns out that $\lambda_n(G) = 0$ and $\lambda_1(G) \leq n$ with equality if and only if the complement of G is not connected.

Let e be an edge of G, and $G - e$ be the subgraph obtained from G by deleting e. In [4, 9], Grone and Mohar first studied the relation between the Laplacian spectra of G and $G - e$.

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Using different methods they both showed an interlacing property as

$$\lambda_i(G) \geq \lambda_i(G-e) \geq \lambda_{i+1}(G), \quad i = 1, 2, \cdots, n-1. \tag{1.1}$$

Let u be a vertex of G, and $G-u$ be the subgraph obtained from G by deleting u together with its incident edges. In [8], Lotker studied the relation between the Laplacian spectra of G and $G-u$, and showed

$$\lambda_i(G) \geq \lambda_i(G-u) \geq \lambda_{i+1}(G) - 1, \quad i = 1, 2, \cdots, n-1. \tag{1.2}$$

Let U be a subset of $V(G)$ with $|U| = k$ $(0 \leq k \leq n)$. Then $G-U$ is the subgraph obtained from G by deleting all the vertices in U together with their incident edges. In this work, we will investigate the relation between the Laplacian spectra of G and $G-U$, and will present an interlacing property (see Theorem 2.1 in Section 2) as follows

$$\lambda_i(G) - w_1 \geq \lambda_i(G-U) \geq \lambda_{i+k}(G) - w_2, \quad i = 1, 2, \cdots, n-k, \tag{1.3}$$

where $w_1 = \min\limits_{v \in V \setminus U} |N_G(v) \cap U|$ and $w_2 = \max\limits_{v \in V \setminus U} |N_G(v) \cap U|$.

This interlacing inequality (1.3) generalizes the interlacing inequality (1.2). In addition, it has some interesting applications (see Corollaries 3.1–3.6) on the algebraic connectivity, vertex connectivity, bounds of $\lambda_i(G)$, etc., which will be shown in Section 3. Moreover, we will give a generalization of (1.3) for edge weighted graphs in Section 4.

2 Relation Between the Laplacian Spectra of G and $G-U$

In this section, we first give some preliminary results on real symmetric matrices. Then we prove the interlacing inequality (1.3) of the Laplacian spectra of G and $G-U$.

Lemma 2.1 (see [7]) *Let A be an $n \times n$ real symmetric matrix, m be an integer with $1 \leq m \leq n$, and A_m be an $m \times m$ principal submatrix of A. Then*

$$\lambda_i(A) \geq \lambda_i(A_m) \geq \lambda_{i+n-m}(A), \quad i = 1, 2, \cdots, m.$$

Lemma 2.2 (see [7]) *Let A, B be two $n \times n$ real symmetric matrices. Then for each integer i with $1 \leq i \leq n$ we have*

$$\min_{r+s=i+1}\{\lambda_r(A) + \lambda_s(B)\} \geq \lambda_i(A+B) \geq \max_{r+s=n+i}\{\lambda_r(A) + \lambda_s(B)\}.$$

Theorem 2.1 *Let G be a graph on the vertex set V with $|V| = n$, and $U \subseteq V$ with $|U| = k$. Then*

$$\lambda_i(G) - w_1 \geq \lambda_i(G-U) \geq \lambda_{i+k}(G) - w_2, \quad i = 1, 2, \cdots, n-k,$$

where $w_1 = \min\limits_{v \in V \setminus U} |N_G(v) \cap U|$ and $w_2 = \max\limits_{v \in V \setminus U} |N_G(v) \cap U|$.

Proof Remove the rows and columns of $L(G)$ that correspond to the vertices in U, and the resulting principal submatrix of $L(G)$ is denoted by $L_U(G)$. Applying Cauchy-Poincaré's theorem (see Lemma 2.1), we have $\lambda_i(G) \geq \lambda_i(L_U(G)) \geq \lambda_{i+k}(G)$ $(i = 1, 2, \cdots, n-k)$.

Let $D_U(G) = L_U(G) - L(G-U)$. Then $D_U(G)$ is a diagonal matrix whose diagonal entry corresponding to v is $|N_G(v) \cap U|$. Thus by the Weyl's inequalities (see Lemma 2.2), for each $i = 1, 2, \cdots, n-k$, we have

$$\begin{aligned}
\lambda_i(G-U) &= \lambda_i(L(G-U)) = \lambda_i(L_U(G) - D_U(G)) \\
&\leq \min_{r+s=i+1} \{\lambda_r(L_U(G)) + \lambda_s(-D_U(G))\} \\
&\leq \lambda_i(L_U(G)) + \lambda_1(-D_U(G)) \\
&= \lambda_i(L_U(G)) - \lambda_{n-k}(D_U(G)) \\
&\leq \lambda_i(G) - \min_{v \in V\setminus U} |N_G(v) \cap U|, \qquad (2.1) \\
\lambda_i(G-U) &\geq \max_{r+s=n-k+i} \{\lambda_r(L_U(G)) + \lambda_s(-D_U(G))\} \\
&\geq \lambda_i(L_U(G)) + \lambda_{n-k}(-D_U(G)) \\
&= \lambda_i(L_U(G)) - \lambda_1(D_U(G)) \\
&\geq \lambda_{i+k}(G) - \max_{v \in V\setminus U} |N_G(v) \cap U|. \qquad (2.2)
\end{aligned}$$

Combining (2.1) and (2.2), we complete the proof.

The following Corollaries 2.1–2.2 immediately follow from Theorem 2.1.

Corollary 2.1 *Let G be a graph on the vertex set V with $|V| = n$, and $U \subseteq V$ with $|U| = k$. Then*

$$\lambda_i(G) \geq \lambda_i(G-U) \geq \lambda_{i+k}(G) - k, \quad i = 1, 2, \cdots, n-k. \qquad (2.3)$$

Moreover, if every vertex of U is adjacent to all the vertices of $V\setminus U$, then

$$\lambda_i(G) - k \geq \lambda_i(G-U) \geq \lambda_{i+k}(G) - k, \quad i = 1, 2, \cdots, n-k. \qquad (2.4)$$

Corollary 2.2 *Let G be a graph on n vertices, and u be a vertex of G. Then*

$$\lambda_i(G) \geq \lambda_i(G-u) \geq \lambda_{i+1}(G) - 1, \quad i = 1, 2, \cdots, n-1. \qquad (2.5)$$

Moreover, if u is adjacent to all the other vertices of G, then

$$\lambda_i(G) - 1 \geq \lambda_i(G-u) \geq \lambda_{i+1}(G) - 1, \quad i = 1, 2, \cdots, n-1. \qquad (2.6)$$

The interlacing inequality (2.5) is the main result of [8]. While it can be improved as (2.6) provided that the vertex u is the common neighbor of all the other vertices of G.

Now we study the tightness of the inequality (2.6). Firstly, $\lambda_i(G-u) = \lambda_{i+1}(G) - 1$ will never occur when $i = n-1$ since $\lambda_{n-1}(G-u) = \lambda_n(G) = 0$. Secondly, as we will see in the following, there exist graphs such that both the equalities in (2.6) hold for some i.

Example 2.1 Consider the complete graph K_n. Delete a vertex u from K_n and the resulting graph is K_{n-1}. Recall

$$\mathrm{LSpec}(K_n) = (\underbrace{n, \cdots, n}_{n-1}, 0) \quad \text{and} \quad \mathrm{LSpec}(K_{n-1}) = (\underbrace{n-1, \cdots, n-1}_{n-2}, 0).$$

Thus, $\lambda_i(G) - 1 = \lambda_i(G-u) = \lambda_{i+1}(G) - 1 = n-1$ $(1 \leq i \leq n-2)$.

Example 2.2 Consider the star $K_{1,n-1}$. Delete the central vertex u from $K_{1,n-1}$ and the resulting graph consists of $n-1$ isolated vertices, whose eigenvalues are all 0 with multiplicity $n-1$. Noting

$$\mathrm{LSpec}(K_{1,n-1}) = (n, \underbrace{1, \cdots, 1}_{n-2}, 0),$$

we have $\lambda_i(G) - 1 = \lambda_i(G-u) = \lambda_{i+1}(G) - 1 = 0$ $(2 \le i \le n-2)$.

3 Applications

In this section, we give some applications of our previous interlacing results on the algebraic connectivity, vertex connectivity, bounds of $\lambda_i(G)$, etc., which can be obtained from Theorem 2.1 and its consequences Corollaries 2.1–2.2.

For a graph G, the second smallest eigenvalue $\lambda_{n-1}(G) > 0$ if and only if G is connected. This observation led M. Fiedler [2] to define the algebraic connectivity of G by $a(G) = \lambda_{n-1}(G)$. The following Corollary 3.1 reflects the relation between the algebraic connectivity of G and that of $G - U$.

Corollary 3.1 *Let G be a graph on the vertex set V with $|V| = n$, and $U \subseteq V$ with $|U| \le n-2$. Then*

$$a(G) \le a(G-U) + \max_{v \in V \setminus U} |N_G(v) \cap U|. \tag{3.1}$$

Proof Let $k = |U|$ and take $i = n-k-1 (\ge 1)$ in (1.3). Then we have

$$\lambda_{n-k-1}(G-U) \ge \lambda_{n-1}(G) - \max_{v \in V \setminus U} |N_G(v) \cap U|,$$

which implies the result.

Consequently, when $|U| \le n-2$ one can immediately obtain a known bound (see [3, p. 288]) from (3.1) as

$$a(G) \le a(G-U) + |U|.$$

In addition, the equality in (3.1) can occur. For instance, delete $k (\le n-2)$ vertices of the complete graph K_n and the resulting graph is K_{n-k}. As we see, $a(K_n) = a(K_{n-k}) + k = n$.

Let $\kappa(G)$ be the vertex connectivity of G. It is well-known that $a(G) \le \kappa(G)$ if $G \ne K_n$ (see [2]). In the following, we show that this result, together with a necessary condition for the equality case, can also be derived from Theorem 2.1.

Corollary 3.2 *Let G be a connected graph on the vertex set V with $|V| = n$. If $G \ne K_n$, then*

$$a(G) \le \kappa(G).$$

Moreover, if equality holds, then for any minimum vertex cut U there exists a vertex of $V \setminus U$ adjacent to all the vertices of U.

Proof Let U be a minimum vertex cut of G, and $k = |U| = \kappa(G)$. Then $k \le n-2$ since $G \ne K_n$. Take $i = n-k-1 (\ge 1)$ in (1.3). Then we have

$$\lambda_{n-1}(G) \le \lambda_{n-k-1}(G-U) + \max_{v \in V \setminus U} |N_G(v) \cap U| \le a(G-U) + k = k,$$

i.e.,

$$a(G) \le \kappa(G).$$

Moreover, if $a(G) = \kappa(G)$, then $\max_{v \in V\setminus U} |N_G(v) \cap U| = k = |U|$. The proof is completed.

Corollary 3.3 *Let G be a connected graph on n vertices, and $\beta(G)$ be the vertex cover number of G. Then*

$$\begin{cases} \lambda_i(G) \ge 1, & i = 1, \cdots, n - \beta(G), \\ \lambda_j(G) \le \beta(G), & j = \beta(G) + 1, \cdots, n. \end{cases}$$

Proof Let U be a minimum vertex cover of G, and $k = |U| = \beta(G)$. By Theorem 2.1, we have

$$\lambda_i(G) - \min_{v \in V\setminus U} |N_G(v) \cap U| \ge \lambda_i(G - U) \ge \lambda_{i+k}(G) - \max_{v \in V\setminus U} |N_G(v) \cap U|, \quad i = 1, \cdots, n - k.$$

Note $\lambda_i(G - U) = 0$ $(i = 1, \cdots, n - k)$. Thus, for each $i = 1, \cdots, n - k$, we have

$$\lambda_i(G) \ge \min_{v \in V\setminus U} |N_G(v) \cap U| \ge 1 \quad \text{and} \quad \lambda_{i+k}(G) \le \max_{v \in V\setminus U} |N_G(v) \cap U| \le k,$$

which implies the result.

Let $d_1 \ge \cdots \ge d_n$ be the degrees of G. In [6], Guo conjectured that $\lambda_i(G) \ge d_i - i + 2$ $(0 \le i \le n - 1)$, where G is a connected graph on n vertices. Subsequently, Brouwer and Haemers proved it in [1]. Here we can give a proof of a weak form of this conjecture, which can be obtained from Theorem 2.1.

Corollary 3.4 *Let G be a graph on n vertices. Then $\lambda_i(G) \ge d_i - i + 1$ $(1 \le i \le n)$.*

Proof It is obvious that $\lambda_n(G) \ge d_n - n + 1$. Hence in the following we may assume $1 \le i \le n - 1$. Let $V = \{v_1, \cdots, v_n\}$ be the vertex set of G, where the degree of v_j is d_j $(1 \le j \le n)$. Set $U = V\setminus\{v_1, \cdots, v_i\}$. Then $V\setminus U = \{v_1, \cdots, v_i\}$. According to Theorem 2.1, we have

$$\begin{aligned} \lambda_i(G) &\ge \lambda_i(G - U) + \min_{v \in V\setminus U} |N_G(v) \cap U| \ge \min_{v \in V\setminus U} |N_G(v) \cap U| \\ &= \min_{v \in \{v_1, \cdots, v_i\}} (|N_G(v)| - |N_G(v) \cap \{v_1, \cdots, v_i\}|) \ge d_i - i + 1. \end{aligned}$$

The following upper bounds for the second largest eigenvalue $\lambda_2(G)$ can also be directly obtained from the interlacing property.

Corollary 3.5 (see [5]) *Let G be a connected graph with a cut vertex u. If the largest component of $G - u$ contains k vertices, then $\lambda_2(G) \le k + 1$.*

Proof Taking $i = 1$ in (2.5), we have $\lambda_1(G - u) \ge \lambda_2(G) - 1$. Thus $\lambda_2(G) \le \lambda_1(G - u) + 1 \le k + 1$.

Corollary 3.6 (see [5]) *Let G be a connected graph on $n > 2$ vertices. Suppose that u is a vertex of G adjacent to k pendant vertices, then $\lambda_2(G) \le n - k$.*

Proof If $G = K_{1,n-1}$, then $k = n - 1$ and $\lambda_2(G) = 1 = n - k$. Otherwise, the largest component of $G - u$ contains at most $n - k - 1$ vertices. Also taking $i = 1$ in (2.5), we have $\lambda_2(G) \le \lambda_1(G - u) + 1 \le n - k$.

4 Extension to Edge Weighted Graphs

In this section, we establish an improved version of Theorem 2.1 which extend the interlacing property to a broader range of graphs, i.e., edge weighted graphs.

An edge weighted graph, $\widehat{G}$, consists of a vertex set $V = V(\widehat{G})$, an edge set $E = E(\widehat{G})$ and a positive-valued weight function w on E. For convenience, we simply write the weight of the edge (u, v) as $w(u, v)$. Define the adjacency matrix of $\widehat{G}$ by $A(\widehat{G}) = (a_{ij})$, where $a_{ij} = w(v_i, v_j)$ if $v_i \sim v_j$ and $a_{ij} = 0$ otherwise. Let $d_{\widehat{G}}(v) = \sum\limits_{u\sim v} w(u, v)$ and $D(\widehat{G})$ be the diagonal matrix whose diagonal entry corresponding to v is $d_{\widehat{G}}(v)$. Then the Laplacian matrix of $\widehat{G}$, as before, is defined as $L(\widehat{G}) = D(\widehat{G}) - A(\widehat{G})$.

An unweighted graph G can be regarded as the special case of a weighted graph by taking the weight function as $w \equiv 1$ on E.

If U is a subset of $V(\widehat{G})$ with $|U| = k$ $(0 \le k \le n)$, then $\widehat{G} - U$ is the subgraph which is obtained from $\widehat{G}$ by deleting all the vertices in U together with their incident edges and whose weight function w is restricted on the set of the remaining edges.

In the process of the proof of Theorem 2.1, substituting $|N_G(v) \cap U|$ by $\sum\limits_{\substack{u\in U\\ u\sim v}} w(u, v)$, we have the following Theorem 4.1.

Theorem 4.1 *Let $\widehat{G}$ be an edge weighted graph on the vertex set V with $|V| = n$, w be the weight function of $\widehat{G}$ and $U \subseteq V$ with $|U| = k$. Then*

$$\lambda_i(\widehat{G}) - w_1 \ge \lambda_i(\widehat{G} - U) \ge \lambda_{i+k}(\widehat{G}) - w_2, \quad i = 1, 2, \cdots, n-k.$$

where $w_1 = \min\limits_{v\in V\setminus U} \Big(\sum\limits_{\substack{u\in U\\ u\sim v}} w(u, v)\Big)$ *and* $w_2 = \max\limits_{v\in V\setminus U} \Big(\sum\limits_{\substack{u\in U\\ u\sim v}} w(u, v)\Big)$.

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