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Notes on the Incompressible Euler and Related Equations on $\mathbb{R}^N$**

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(*Dedicated to Professor Andrew Majda with Profound Admiration on the Occasion of his* 60*th Birthday*)

Abstract The author reviews briefly some of the recent results on the blow-up problem for the incompressible Euler equations on $\mathbb{R}^N$, and also presents Liouville type theorems for the incompressible and compressible fluid equations.

Keywords Euler equations, Navier-Stokes equations, Liouville theorem
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1 Introduction

The motion of homogeneous ideal incompressible fluids is governed by the Navier-Stokes (Euler for $\nu = 0$) equations:

$$(\mathrm{NS,E})\ \begin{cases} \dfrac{\partial v}{\partial t} + (v \cdot \nabla)v = -\nabla p + \nu\Delta v, \\ \operatorname{div} v = 0, \end{cases}$$

where $v = v(x,t) = (v^1, \cdots, v^N)$ is the velocity of the fluid, $p = p(x,t)$ is the pressure, and $\nu \geq 0$ is the viscosity of the fluid. We denote the system (NS, E) as (NS) if $\nu > 0$, and (E) if $\nu = 0$ respectively. Below our main concern is on the Euler system (E). The local in time existence of classical solution of (E) for $v_0 \in H^m(\mathbb{R}^3)$ $(m > \frac{5}{2})$ is established by Kato [28]. The questions of the finite time blow-up of the local smooth solution for (E) is a wide open problem (see e.g. [1, 20, 14, 31] for graduate level text and survey articles on the status of the problem). In this study, there is celebrated result by Beale, Kato and Majda (BKM) in [2], which says that the blow-up of solution is controlled by the temporal accumulation of the maximum of the vorticity, $\omega = \operatorname{curl} v$, namely

$$\limsup_{t\to T_*} \|v(t)\|_{H^m} = \infty \quad \text{if and only if} \quad \int_0^{T_*} \|\omega(s)\|_{L^\infty} \mathrm{d}s = \infty.$$

The blow-up criterion incorporating the direction of vorticity has been derived by Constantin, Fefferman and Majda (CFM) in [23], which says that smooth change of vorticity directions

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implies regularity of the solution. This is based on the structure of nonlinear depletion in the vortex stretching term of the vorticity equation of the Euler and the Navier-Stokes equations, which was observed previously in [22]. The interpolation between the BKM and the CFM criterion, using the Triebel-Lizorkin space for the direction field, ξ has been recently done in [18], which generalizes the argument of [17] (for related results see also [3, 4] for the Navier-Stokes equations and [26] for the Euler equations). As for another direction of the study of the finite time blow-up problem of the Euler system, the exclusion of plausible scenarios have been investigated. The scenario of vortex tube collapse was excluded by Córdoba-Fefferman [24, 25] (under the assumption of $\int_0^T \|v(t)\|_{L^\infty}\mathrm{d}t < \infty$). Later the exclusion of self-similar blow-up scenario was derived in [16], proving that there exists no solution (v,p) to (E) of the form

$$\begin{cases} v(x,t) = \dfrac{1}{(T_*-t)^{\frac{\alpha}{\alpha+1}}} V\Big(\dfrac{x}{(T_*-t)^{\frac{1}{\alpha+1}}}\Big), \\ p(x,t) = \dfrac{1}{(T_*-t)^{\frac{2\alpha}{\alpha+1}}} P\Big(\dfrac{x}{(T_*-t)^{\frac{1}{\alpha+1}}}\Big) \end{cases} \tag{1.1}$$

for some (V,P), for which $\Omega = \operatorname{curl} V$ has suitable decay condition near infinity. Note that, compared with Leray's ansatz for the self-similar blow-up solution in [30], the parameter $\alpha \neq -1$ in (1.1) is arbitrary. Thus exclusion of such form of solution is more difficult since we need to exclude infinite family of solutions. Moreover, the argument of nonexistence of the Leray's self-similar solution for (NS), as proved in [33, 34, 36], cannot be adapted to the case of Euler system. The result in [16] was refined later in [15], showing that there exists no asymptotically self-similar blow-up scenario. More precisely, if

$$\lim_{t\to T}(T-t)\Big\| v(\,\cdot\,,t) - \frac{1}{(T_*-t)^{\frac{\alpha}{\alpha+1}}}\overline{V}\Big(\frac{\cdot}{(T-t)^{\frac{1}{\alpha+1}}}\Big)\Big\|_{\mathrm{Lip}(\mathbb{R}^3)} = 0$$

for $\overline{V}$ satisfying the above mentioned decay condition near the infinity and $\alpha > -1$, then the solution is shown to be regular beyond $t = T$. Generalizing these results one can also prove the nonexistence of (asymptotically) self-similar singularity for the MHD system (see [12, 13]).

2 Similarity Transforms and New a priori Estimates

Given a classical solution (v,p) of (E) and $\gamma \geq 1$, $\alpha > -1$, we consider the "similarity transform" $(v,p) \to (V,P)$ introduced in [10], which is defined by

$$\begin{aligned} v(x,t) &= V(y,s)\exp\Big[\frac{\gamma\alpha}{\alpha+1}\int_0^t \|\nabla v(\tau)\|_{L^\infty}\mathrm{d}\tau\Big], \\ p(x,t) &= P(y,s)\exp\Big[\frac{2\gamma\alpha}{\alpha+1}\int_0^t \|\nabla v(\tau)\|_{L^\infty}\mathrm{d}\tau\Big] \end{aligned}$$

with

$$\begin{aligned} y &= \exp\Big[\frac{\gamma}{\alpha+1}\int_0^t \|\nabla v(\tau)\|_{L^\infty}\mathrm{d}\tau\Big]x, \\ s &= \int_0^t \exp\Big[\gamma\int_0^\tau \|\nabla v(\sigma)\|_{L^\infty}\mathrm{d}\sigma\Big]\mathrm{d}\tau. \end{aligned}$$

Substituting (v,p) into (E), we find that (V,P) satisfies

$$(\mathrm{E}_*)\quad\begin{cases}-\gamma\|\nabla V(s)\|_{L^\infty}\Big[\dfrac{\alpha}{\alpha+1}V+\dfrac{1}{\alpha+1}(y\cdot\nabla)V\Big]=V_s+(V\cdot\nabla)V+\nabla P,\\ \operatorname{div}V=0,\\ V(y,0)=V_0(y)=v_0(y).\end{cases}$$

The global well-posedness of the system (E_*) follows from the local well-posedness of the Euler equations. Indeed, if T_* is the maximal time for (E), then the maximal time of the existence of classical solution S_* for (E_*) is

$$S_*=\int_0^{T_*}\exp\Big[\gamma\int_0^\tau\|\nabla v(\sigma)\|_{L^\infty}\mathrm{d}\sigma\Big]\mathrm{d}\tau\ge\frac{1}{\|\omega_0\|_{L^\infty}}\int_0^{T_*}\|\omega(\tau)\|_{L^\infty}\mathrm{d}\tau=\infty$$

thanks to the BKM criterion. We observe the integral invariant of the transform

$$\int_0^t\|\nabla v(t)\|_{L^\infty}\mathrm{d}t=\int_0^s\|\nabla V(s)\|_{L^\infty}\mathrm{d}s.$$

From the estimates for V we can transform back to the original physical variables in order to obtain the following estimates for the solutions to the Euler equations.

Theorem 2.1 *Let* $\omega=\operatorname{curl}v$ *be the vorticity of a classical solution* v *of* (E). *Then we have an upper estimate*

$$\|\omega(t)\|_{L^\infty}\le\frac{\|\omega_0\|_{L^\infty}\exp[\gamma\int_0^t\|\nabla v(\tau)\|_{L^\infty}\mathrm{d}\tau]}{1+(\gamma-1)\|\omega_0\|_{L^\infty}\int_0^t\exp[\gamma\int_0^\tau\|\nabla v(\sigma)\|_{L^\infty}\mathrm{d}\sigma]\mathrm{d}\tau}$$

and a lower one

$$\|\omega(t)\|_{L^\infty}\ge\frac{\|\omega_0\|_{L^\infty}\exp[-\gamma\int_0^t\|\nabla v(\tau)\|_{L^\infty}\mathrm{d}\tau]}{1-(\gamma-1)\|\omega_0\|_{L^\infty}\int_0^t\exp[-\gamma\int_0^\tau\|\nabla v(\sigma)\|_{L^\infty}\mathrm{d}\sigma]\mathrm{d}\tau}$$

for all $\gamma\ge1$, *the denominator of the right-hand side of which can be estimated from below as*

$$1-(\gamma-1)\|\omega_0\|_{L^\infty}\int_0^t\exp\Big[-\gamma\int_0^\tau\|\nabla v(\sigma)\|_{L^\infty}\mathrm{d}\sigma\Big]\mathrm{d}\tau\ge\frac{1}{(1+\|\omega_0\|_{L^\infty}t)^{\gamma-1}},$$

which shows that the finite time blow-up does not follow from the above inequality.

For $\gamma=1$, the above estimates reduce to the well-known ones. We briefly describe the proof of the above theorem. For more details, we refer to [10].

Outline of the Proof The vorticity equation for (E_*) is

$$-\gamma\|\nabla V\|_{L^\infty}\Big[\Omega-\frac{1}{\alpha+1}(y\cdot\nabla)\Omega\Big]=\Omega_s+(V\cdot\nabla)\Omega-(\Omega\cdot\nabla)V.$$

Multiplying $\Xi=\frac{\Omega}{|\Omega|}$ on the both sides of the above, we have

$$\begin{aligned}|\Omega|_s+(V\cdot\nabla)|\Omega|-\frac{\|\nabla V(s)\|_{L^\infty}}{\alpha+1}(y\cdot\nabla)|\Omega|&=(\Xi\cdot\nabla V\cdot\Xi-\|\nabla V\|_{L^\infty})|\Omega|-(\gamma-1)\|\nabla V\|_{L^\infty}|\Omega|\\&\le-(\gamma-1)|\Omega|^2.\end{aligned}$$

Introducing the particle trajectories $\{Y(a,s)\}$ defined by the ordinary differential equations

$$\frac{\partial Y(a,s)}{\partial s} = V(Y(a,s),s) - \frac{\|\nabla V(s)\|_{L^\infty}}{\alpha+1} Y(a,s), \quad Y(a,0)=a,$$

we have the differential inequality

$$\frac{\mathrm{D}}{\mathrm{D}s}|\Omega| \le -(\gamma-1)|\Omega|^2.$$

Solving the above differential inequality along the particle trajectories, we obtain

$$|\Omega(Y(a,s),s)| \le \frac{|\Omega_0(a)|}{1+(\gamma-1)s|\Omega_0(a)|}.$$

Hence

$$\|\Omega(s)\|_{L^\infty} \le \frac{\|\Omega_0\|_{L^\infty}}{1+(\gamma-1)s\|\Omega_0\|_{L^\infty}}.$$

Going back to the original physical variables, we have the upper estimate part of Theorem 2.1. The lower estimate is similar.

In the case of the Navier-Stokes equations, we consider the "similarity transform" $(v,p) \to (V,P)$ given by

$$v(x,t) = \exp\left[\frac{\gamma}{2}\int_0^t \|v(\tau)\|_{L^p}^{\frac{2p}{p-3}}\mathrm{d}\tau\right] V(y,s),$$
$$p(x,t) = \exp\left[\gamma\int_0^t \|v(\tau)\|_{L^p}^{\frac{2p}{p-3}}\mathrm{d}\tau\right] P(y,s)$$

with

$$y = \exp\left[\frac{\gamma}{2}\int_0^t \|v(\tau)\|_{L^p}^{\frac{2p}{p-3}}\mathrm{d}\tau\right] x,$$
$$s = \int_0^t \exp\left[\gamma\int_0^\tau \|v(\sigma)\|_{L^p}^{\frac{2p}{p-3}}\mathrm{d}\sigma\right]\mathrm{d}\tau.$$

In this case, we have the following invariant of the transform

$$\|v(t)\|_{L^3} = \|V(s)\|_{L^3}, \quad \int_0^t \|v(\tau)\|_{L^p}^{\frac{2p}{p-3}}\mathrm{d}\tau = \int_0^s \|V(\sigma)\|_{L^p}^{\frac{2p}{p-3}}\mathrm{d}\sigma.$$

Substituting (v,p) into (NS), we obtain an equivalent system of equations for (V,P):

$$(\mathrm{NS})_* \begin{cases} -\frac{\gamma}{2}\|V(s)\|_{L^p}^{\frac{2p}{p-3}}[V+(y\cdot\nabla)V] = V_s + (V\cdot\nabla)V + \nabla P - \nu\Delta V, \\ \operatorname{div} V = 0, \\ V(y,0) = V_0(y) = v_0(y). \end{cases}$$

By the similar method to the Euler equations case we can obtain the following estimate.

Theorem 2.2 *Let $p \in (3,\infty)$. Then there exists an absolute constant $C_0 = C_0(\nu,p)$ such that for all $\gamma \ge C_0$ the following inequality holds true:*

$$\|v(t)\|_{L^p} \le \frac{\|v_0\|_{L^p}\exp[\frac{(p-3)\gamma}{2p}\int_0^t \|v(\tau)\|_{L^p}^{\frac{2p}{p-3}}\mathrm{d}\tau]}{\{1+(\gamma-C_0)\|v_0\|_{L^p}^{\frac{2p}{p-3}}\int_0^t \exp[\gamma\int_0^\tau \|v(\sigma)\|_{L^p}^{\frac{2p}{p-3}}\mathrm{d}\sigma]\mathrm{d}\tau\}^{\frac{p-3}{2p}}},$$

with an upper estimate of the denominator

$$1+(\gamma-C_0)\|v_0\|_{L^p}^{\frac{2p}{p-3}}\int_0^t \exp\Big[\gamma\int_0^\tau \|v(\sigma)\|_{L^p}^{\frac{2p}{p-3}}\mathrm{d}\sigma\Big]\mathrm{d}\tau \le \frac{1}{(1-C_0\|v_0\|_{L^p}^{\frac{2p}{p-3}}t)^{\frac{\gamma}{C_0}-1}}.$$

For the proof of the above theorem we refer to [10]. Generalizing the arguments of this section, one can also deduce various new a priori estimates different from Theorems 2.1 and 2.2 for the Euler, Navier-Stokes and the surface quasi-geostrophic equations (see [11]).

3 Liouville Type Theorems in the Fluid Mechanics

3.1 The case of incompressible fluids

Definition 3.1 *We say that the pair* $(v,p)\in L^2(0,T;L^2_\sigma(\mathbb{R}^N))\times L^1(0,T;\mathcal{D}(\mathbb{R}^N))$ *is a weak solution of* $(\mathrm{NS})_\nu$ *(Euler for* $\nu=0$*) on* $\mathbb{R}^N\times(0,T)$ *if*

$$\begin{aligned}&-\int_0^T\int_{\mathbb{R}^N} v(x,t)\cdot\phi(x)\xi'(t)\mathrm{d}x\mathrm{d}t-\int_0^T\int_{\mathbb{R}^N} v(x,t)\otimes v(x,t):\nabla\phi(x)\xi(t)\mathrm{d}x\mathrm{d}t\\ &=\int_0^T\langle p(\,\cdot\,,t),\operatorname{div}\phi\rangle\xi(t)\mathrm{d}x\mathrm{d}t+\nu\int_0^T\int_{\mathbb{R}^N} v(x,t)\cdot\Delta\phi(x)\xi(t)\mathrm{d}x\mathrm{d}t\end{aligned}$$

for all $\xi\in C_0^1(0,T)$ *and* $\phi=[C_0^\infty(\mathbb{R}^N)]^N$*.*

Given $q\in(0,1]$, the Hardy space $\mathcal{H}^q(\mathbb{R}^N)$ is defined as follows. Let $\varphi_t(x)=t^{-N}\varphi(t^{-1}x)$, $t>0$, where $\varphi\in\mathcal{S}(\mathbb{R}^N)$ (the Schwartz class) with $\int_{\mathbb{R}^N}\varphi(x)\mathrm{d}x\neq 0$. Then

$$\mathcal{M}_\varphi f(x):=\sup_{t>0}|f*\varphi_t(x)|\in L^q(\mathbb{R}^N)\Rightarrow f\in\mathcal{H}^q(\mathbb{R}^N).$$

Theorem 3.1 *Suppose* (v,p) *is a weak solution to* $(\mathrm{NS})_\nu$ $(\nu\ge 0)$*.*

(i) (Energy Equipartition) *If*

$$|v|^2+|p|\in L^1(0,T;L^1(\mathbb{R}^N)),$$

then

$$\int_{\mathbb{R}^N}(v^j(x,t))^2\mathrm{d}x=-\int_{\mathbb{R}^N}p(x,t)\mathrm{d}x,\quad j=1,\cdots,N$$

for almost every $t\in(0,T)$*.*

(ii) (Liouville Type of Property) *If* $v\in L^2(0,T;L^2_\sigma(\mathbb{R}^N))$ *and*

$$p\in L^1(0,T;\mathcal{H}^q(\mathbb{R}^N))$$

for some $q\in(0,1]$*, then* $v(x,t)=0$ *and* $p(x,t)=0$ *almost everywhere in* $\mathbb{R}^N\times(0,T)$*.*

Remark 3.1 Professor D. Serre informed me that part (i) of the above theorem was obtained previously by L. Brandolese.

Remark 3.2 Note that the condition

$$p\in L^1(0,T;\mathcal{H}^1(\mathbb{R}^N))$$

is far beyond the natural scaling of the usual regularity criterion on the pressure for weak solutions of the Navier-Stokes equations

$$p \in L^q(0,T;L^r(\mathbb{R}^N)), \quad \frac{2}{q}+\frac{N}{r} \le 2$$

(see e.g. [5, 19]), and our conclusion is not only that the solution is regular, but also that it is trivially zero.

The pressure for the Leray weak solutions v of the N-dimensional ($N=2,3$) Navier-Stokes equations satisfies

$$p \in L^1(0,T;L^q(\mathbb{R}^N)), \quad \forall\, q \in \Big(1, \frac{N}{N-2}\Big].$$

In [29], Kock, Nadirashvili, Seregin and Šverák also studied the Liouville properties for the 3D axisymmetric Navier-Stokes equations: A bounded axisymmetric weak solution $v(x,t)$ to (NS) such that

$$|v(x,t)| \le \frac{C}{\sqrt{x_1^2+x_2^2}}, \quad \text{in } \mathbb{R}^3 \times (-\infty,0)$$

is zero, which is completely different to the above theorem. Below we outline the proof of Theorem 3.1 (see [6] for more details).

Outline of the Proof Let us consider a cut-off function $\sigma \in C_0^\infty(\mathbb{R}^N)$ such that

$$\sigma(x)=\sigma(|x|)=\begin{cases} 1, & \text{if } |x|<1,\\ 0, & \text{if } |x|>2,\end{cases}$$

and $0 \le \sigma(x) \le 1$ for $1<|x|<2$. Given $\xi \in C_0^1(0,T)$, $j \in \{1,\cdots,N\}$, we choose the vector test function

$$\phi = \nabla\Big(x_j^2 \sigma\Big(\frac{x}{R}\Big)\Big)$$

in the definition of weak solutions. Then we obtain

$$\int_0^T\int_{\mathbb{R}^N} (v^j(x,t))^2 \sigma_R(x)\xi(t)\mathrm{d}x\mathrm{d}t + \int_0^T\int_{\mathbb{R}^N} p(x,t)\sigma_R(x)\xi(t)\mathrm{d}x\mathrm{d}t = o(1),$$

where $o(1)\to 0$ as $R\to\infty$. The estimate of typical terms containing velocity in $o(1)$ is

$$\begin{aligned}
&\Big|\int_0^T\int_{\mathbb{R}^N} \xi(t)v^k(x,t)v^l(x,t)x_j^m D^n\sigma_R(x)\mathrm{d}x\mathrm{d}t\Big|\\
&\le \frac{1}{R^n}\sup_{1<s<2}|\sigma^{(n)}(s)|\int_0^T\int_{R<|x|<2R}|\xi(t)||v(x,t)|^2|x|^m\mathrm{d}x\mathrm{d}t\\
&\le R^{m-n}2^m \sup_{1<s<2}|\sigma^{(n)}(s)|\sup_{0<t<T}|\xi(t)|\int_0^T\int_{R<|x|<2R}|v(x,t)|^2\mathrm{d}x\mathrm{d}t\\
&\to 0
\end{aligned}$$

for $n \ge m$ by the dominated convergence theorem. On the other hand, the estimate of typical terms containing pressure $p(x,t)$ in $o(1)$ is

$$\begin{aligned}&\Big|\int_0^T\int_{\mathbb{R}^N}\xi(t)p(x,t)x_j^m D^n\sigma_R(x)\mathrm{d}x\mathrm{d}t\Big|\\ &\le \frac{1}{R^n}\sup_{1<s<2}|\sigma^{(n)}(s)|\int_0^T\int_{R<|x|<2R}|\xi(t)||p(x,t)||x|^m\mathrm{d}x\mathrm{d}t\\ &\le R^{m-n}2^m\sup_{1<s<2}|\sigma^{(n)}(s)|\sup_{0<t<T}|\xi(t)|\int_0^T\int_{R<|x|<2R}|p(x,t)|\mathrm{d}x\mathrm{d}t\\ &\to 0\end{aligned}$$

for $n\ge m$ by the dominated convergence theorem. Thus, passing $R\to\infty$, we have

$$\int_{\mathbb{R}^N}(v^j(x,t))^2\mathrm{d}x=-\int_{\mathbb{R}^N}p(x,t)\mathrm{d}x,\quad \text{a.e. } t\in(0,T)$$

for all $j=1,2,\cdots$. Similarly, if we choose

$$\phi=\nabla\Big(x_jx_k\sigma\Big(\frac{x}{R}\Big)\Big),\quad j\ne k,$$

we can derive the orthogonality

$$\int_0^T\int_{\mathbb{R}^N}v^j(x,t)v^k(x,t)\mathrm{d}x\mathrm{d}t=o(1).$$

In the case $p\in L^1(0,T;\mathcal{H}^q(\mathbb{R}^N))$, $0<q<1$, the pressure term can be estimated as ($\phi=\nabla(x_j^2\sigma)$ below)

$$\begin{aligned}\Big|\int_0^T\xi(t)\langle p(\,\cdot\,,t),\operatorname{div}\phi\rangle\mathrm{d}t\Big|&\le\sup_{0\le t\le T}\int_0^T|\xi(t)|\|p(t)\|_{\mathcal{H}^q}\mathrm{d}t\,\|\operatorname{div}\phi\|_{C^\gamma}\\ &\le\frac{C}{R^\gamma}\|\xi\|_{L^\infty(0,T)}\|p(t)\|_{L^1(0,T;\mathcal{H}^q)}\\ &\to 0,\end{aligned}$$

where $\gamma=N(\frac{1}{q}-1)>0$.

One can generalize Theorem 3.1 to more complicated system of magnetohydrodynamic (MHD) equations. The system of incompressible MHD equations in $\mathbb{R}^N$ ($N\ge 2$) is

$$(\mathrm{MHD})_{\mu,\nu}\ \begin{cases}\dfrac{\partial v}{\partial t}+(v\cdot\nabla)v=(b\cdot\nabla)b-\nabla\Big(p+\dfrac12|b|^2\Big)+\nu\Delta v+f,\\ \dfrac{\partial b}{\partial t}+(v\cdot\nabla)b=(b\cdot\nabla)v+\mu\Delta b+g,\\ \operatorname{div}v=\operatorname{div}b=0,\\ v(x,0)=v_0(x),\quad b(x,0)=b_0(x),\end{cases}$$

where $v=(v_1,\cdots,v_N)$, $v_j=v_j(x,t)$ ($j=1,\cdots,N$) is the velocity of the flow, $p=p(x,t)$ is the scalar pressure, $b=(b_1,\cdots,b_N)$ ($b_j=b_j(x,t)$) is the magnetic field, and v_0 and b_0 are the given initial velocity and magnetic field satisfying $\operatorname{div}v_0=\operatorname{div}b_0=0$, respectively. The forcing terms satisfy $\operatorname{div}f=\operatorname{div}g=0$. If $b=0$, $(\mathrm{MHD})_{\mu,\nu}\Rightarrow(\mathrm{NS})_\nu$. Our generalization of Theorem 3.1 is the following theorem.

Theorem 3.2 *Suppose that* (v, b, p) *is a weak solution to* $(\mathrm{MHD})_{\mu,\nu}$ *with* $\mu, \nu \geq 0$ *on* $\mathbb{R}^N \times (0, T)$, $N \geq 3$. *Suppose*

$$\int_0^T (|v(x,t)|^2 + |b(x,t)|^2 + |p(x,t)|)\mathrm{d}x\mathrm{d}t < \infty$$

and either

$$\int_0^T \int_{\mathbb{R}^N} p(x,t)\mathrm{d}x\mathrm{d}t \geq 0$$

or

$$p \in L^1(0, T; \mathcal{H}^q(\mathbb{R}^N))$$

for some $q \in (0, 1]$. *Then* $v(x,t) = b(x,t) = 0$ *and* $p(x,t) = 0$ *a.e. in* $\mathbb{R}^N \times (0, T)$.

Below we outline the proof of the above theorem. See [6] for more details.

Outline of the Proof We choose the vector test function

$$\phi = \nabla\Big(\frac{|x|^2}{2}\sigma\Big), \quad j \in \{1, \cdots, N\}.$$

Then, we obtain

$$\begin{aligned} &-\int_0^T \int_{\mathbb{R}^N} |v(x,t)|^2 \xi(t)\mathrm{d}x\mathrm{d}t - \frac{N-2}{2}\int_0^T \int_{\mathbb{R}^N} |b(x,t)|^2 \xi(t)\mathrm{d}x\mathrm{d}t \\ &= N\int_0^T \int_{\mathbb{R}^N} p(x,t)\xi(t)\mathrm{d}x\mathrm{d}t + o(1). \end{aligned}$$

Hence, passing $R \to \infty$, we find

$$-\int_{\mathbb{R}^N} |v(x,t)|^2\mathrm{d}x - \frac{N-2}{2}\int_{\mathbb{R}^N} |b(x,t)|^2\mathrm{d}x = N\int_{\mathbb{R}^N} p(x,t)\mathrm{d}x$$

almost everywhere in $(0, T)$.

One can also generalize Theorem 3.1 by introducing suitable weight functions as follows.

Theorem 3.3 *Let* $w \in L^1_{\mathrm{loc}}([0, \infty))$ *be given, which is positive almost everywhere on* $[0, \infty)$. *Suppose that* (v, p) *is a weak solution to* $(\mathrm{NS})_\nu$, $\nu \geq 0$ *on* $\mathbb{R}^N \times (0, T)$ *such that*

$$\int_0^T \int_{\mathbb{R}^N} (|v(x,t)|^2 + |p(x,t)|)\Big[w(|x|) + \frac{1}{|x|}\int_0^{|x|} w(s)\mathrm{d}s + \frac{1}{|x|^2}\int_0^{|x|}\int_0^r w(s)\mathrm{d}s\mathrm{d}r\Big]\mathrm{d}x\mathrm{d}t < \infty$$

and

$$\int_{\mathbb{R}^N} p(x,t)\Big[w(|x|) + \frac{N-1}{|x|}\int_0^{|x|} w(s)\mathrm{d}s\Big]\mathrm{d}x \geq 0 \quad \textit{for } t \in (0,T).$$

Then $v(x,t) = 0$ *and* $p(x,t) = 0$ *almost everywhere on* $\mathbb{R}^N \times (0, T)$.

If we choose $w(s) \equiv 1$ on $[0, \infty)$, then we recover Liouville part of the results of Theorem 3.1. For the proof of Theorem 3.3, we just choose the vector test function

$$\phi = \nabla\Big[\sigma\Big(\frac{|x|}{R}\Big)\int_0^{|x|}\int_0^r w(s)\mathrm{d}s\mathrm{d}r\Big]$$

and follow the argument of Theorem 3.1 (see [7] for details).

Corollary 3.1 *Let* (v,p) *be a weak solution to* $(\mathrm{NS})_\nu$ *with* $\nu \geq 0$ *on* $\mathbb{R}^N \times (0,T)$ *such that either*

$$\int_0^T \int_{\mathbb{R}^N} \frac{(|v(x,t)|^2 + |p(x,t)|)}{1+|x|} \mathrm{d}x\mathrm{d}t < \infty$$

or

$$p(x,t) \to 0 \quad \text{as } |x| \to \infty \quad \text{for almost every } t \in (0,T), \text{ and}$$
$$v \in L^2(0,T;L^q(\mathbb{R}^N)) \quad \text{for some } q \text{ with } 2 < q < \frac{2N}{N-1}.$$

Suppose that there exists a $w \in L^1(0,\infty)$ *such that* $0 < w(r) \leq \frac{C}{1+r}$ *and*

$$\int_{\mathbb{R}^N} p(x,t)\Big[w(|x|) + \frac{N-1}{|x|}\int_0^{|x|} w(s)\mathrm{d}s\Big]\mathrm{d}x \geq 0 \quad \text{for almost every } t \in (0,T).$$

Then $v(x,t) = 0$ *and* $p(x,t) = 0$ *almost everywhere on* $\mathbb{R}^N \times (0,T)$.

For the proof of the above corollary from Theorem 3.3, we just observe that if $w \in L^1(0,\infty)$ such that $0 < w(r) \leq \frac{C}{1+r}$, then

$$w(r) + \frac{1}{r}\int_0^r w(s)\mathrm{d}s + \frac{1}{r^2}\int_0^r\int_0^s w(\rho)\mathrm{d}\rho\mathrm{d}s \leq \frac{C}{1+r}.$$

As an application of Corollary 3.1 to Leray's weak solution (see [30]), we have the following corollary.

Corollary 3.2 *Let* $v \in L^\infty(0,T;L^2_\sigma(\mathbb{R}^N)) \cap L^2(0,T;H^1_\sigma(\mathbb{R}^N))$ *be an Leray's weak solution to* $(\mathrm{NS})_\nu$ *with* $\nu > 0$. *Suppose that the pressure* $p(x,t)$ *satisfies the conditions of Corollary* 3.1. *Then* $v(x,t) = 0$ *almost everywhere on* $\mathbb{R}^N \times (0,T)$.

Similar to the case of the Euler/Navier-Stokes system, the generalization of Theorem 3.3 to the MHD case is the following theorem.

Theorem 3.4 *Fix* $\mu, \nu \geq 0$ *and* $N \geq 3$. *Let* $w \in L^1_{\mathrm{loc}}([0,\infty))$ *be given, which is positive, non-increasing on* $[0,\infty)$. *Suppose that* (v,b,p) *is a weak solution to* $(\mathrm{MHD})_{\mu,\nu}$ *such that*

$$\int_0^T \int_{\mathbb{R}^N} (|v(x,t)|^2 + |b(x,t)|^2 + |p(x,t)|)$$
$$\times \Big[w(|x|) + \frac{1}{|x|}\int_0^{|x|} w(s)\mathrm{d}s + \frac{1}{|x|^2}\int_0^{|x|}\int_0^r w(s)\mathrm{d}s\mathrm{d}r\Big]\mathrm{d}x\mathrm{d}t < \infty$$

and

$$\int_{\mathbb{R}^N} p(x,t)\Big[w(|x|) + \frac{N-1}{|x|}\int_0^{|x|} w(s)\mathrm{d}s\Big]\mathrm{d}x \geq 0 \quad \text{for } t \in (0,T).$$

Then $b(x,t) = v(x,t) = 0$ *and* $p(x,t) = 0$.

Similar to the case of Euler equations, if we choose $w(s) \equiv 1$ on $[0,\infty)$, then we recover a part of Liouville type result in Theorem 3.2.

Corollary 3.3 *Let (v,b,p) be a weak solution to $(\mathrm{MHD})_{\mu,\nu}$ with $\mu,\nu \geq 0$ on $\mathbb{R}^N \times (0,T)$, $N \geq 3$, such that either*

$$\int_0^T \int_{\mathbb{R}^N} \frac{|v(x,t)|^2 + |b(x,t)|^2 + |p(x,t)|}{1+|x|} \mathrm{d}x\mathrm{d}t < \infty$$

or

$$|p(x,t)| \to 0 \quad \text{as } |x| \to \infty \quad \text{for almost every } t \in (0,T), \text{ and}$$
$$|v| + |b| \in L^2(0,T;L^q(\mathbb{R}^N)) \quad \text{for some } q \text{ with } 2 < q < \frac{2N}{N-1}.$$

Suppose that there exists a $w \in L^1(0,\infty)$ such that $0 < w(r) \leq \frac{C}{1+r}$, non-increasing on $[0,\infty)$, such that

$$\int_{\mathbb{R}^N} p(x,t)\Big[w(|x|) + \frac{N-1}{|x|}\int_0^{|x|} w(s)\mathrm{d}s\Big]\mathrm{d}x \geq 0.$$

Then $v(x,t) = b(x,t) = 0$ and $p(x,t) = 0$.

The above corollary can be applied to the weak solutions of the fully viscous weak solution of MHD constructed by Sermange and Temam [35] as follows.

Corollary 3.4 *Let*

$$(v,b,p) \in [L^\infty(0,T;L^2_\sigma(\mathbb{R}^N)) \cap L^2(0,T;H^1_\sigma(\mathbb{R}^N))]^2 \times L^1(0,T;L^1_{\mathrm{loc}}(\mathbb{R}^N))$$

be a weak solution to $(\mathrm{MHD})_{\mu,\nu}$ with $\mu,\nu > 0$. Suppose that the pressure $p(x,t)$ satisfies the above conditions of Corollary 3.3. Then $v = b = 0$ and $p = 0$.

3.2 Compressible fluids

The Navier-Stokes (Euler for $\mu = \lambda = 0$) equations of the compressible gas are

$$(\mathrm{CNS}) \quad \begin{cases} \partial_t \rho + \mathrm{div}(\rho v) = 0, \\ \partial_t(\rho v) + \mathrm{div}(\rho v \otimes v) = -\nabla p + \mu\Delta v + (\mu+\lambda)\nabla \mathrm{div}\, v + f, \\ \partial_t S + (v\cdot\nabla)S = 0, \\ p = \kappa \mathrm{e}^S \rho^\gamma, \quad \kappa > 0,\ \gamma > 1, \\ (\rho, v, S)(x,0) = (\rho_0, v_0, S_0)(x). \end{cases}$$

The system (CNS) describes compressible gas flows with the adiabatic exponent γ, and ρ, v, S, p and f denote the density, velocity, specific entropy, pressure and the external force respectively (see e.g. [31] for general introduction to compressible fluid equations). In the Liouville type theorems for the incompressible fluid equations in Subsection 3.1, we need to assume a sign condition for a pressure integral. For the case of compressible fluids, however, the pressure is nonnegative definite pointwisely and the convection term has a structure similar to that in the incompressible case. Thus one can expect a strengthened version of Liouville type theorems for the compressible fluid equations. In this case, however, the treatment of the time derivative term is not straightforward as in the incompressible case. For the stationary equations there exists no such problem.

Theorem 3.5 *Let $N \geq 1$, and let the external force $f \in [L^1_{\mathrm{loc}}(\mathbb{R}^N)]^N$ satisfy $\operatorname{div} f = 0$ in the sense of distribution. Suppose that (ρ, v, S) is a stationary weak solution to* (CNS) *satisfying one of the following conditions depending on μ and λ.*

(i) *In the inviscid case ($\mu = \lambda = 0$),*

$$\int_{\mathbb{R}^N} (\rho|v|^2 + \rho^\gamma \mathrm{e}^S) \frac{\mathrm{d}x}{1+|x|} < \infty.$$

(ii) *In the viscous case ($\mu > 0$),*

(a) *if $2\mu + \lambda = 0$,*

$$\int_{\mathbb{R}^N} (\rho|v|^2 + \rho^\gamma \mathrm{e}^S) \mathrm{d}x < \infty.$$

(b) *if $2\mu + \lambda \neq 0$,*

$$\int_{\mathbb{R}^N} (\rho|v|^2 + |v|^{\frac{N}{N-1}} + \rho^\gamma \mathrm{e}^S) \mathrm{d}x < \infty.$$

Then $\rho(x) = 0$ for almost every $x \in \mathbb{R}^N$.

In the special case of $N \geq 3$, $\mu > 0$, $\mu + \lambda > 0$ and $f = \rho \nabla \Phi$, where the connected component of $\{\Phi(x) > -c\}$ is unbounded, P. L. Lions showed the nonexistence of stationary solutions under appropriate integrability condition for (ρ, v), which is completely different from the above. Below we outline the proof of the above theorem. See [8] for the full proof.

Outline of the Proof Let us choose the vector test function as previously,

$$\phi = \nabla(|x|^2 \sigma_R(x)).$$

The proof for the inviscid case is similar to the proof of Theorem 3.1. In the viscous case, it suffices to show that the viscosity term vanishes, namely

$$\mu \int_{\mathbb{R}^N} v \cdot \Delta\phi \mathrm{d}x + (\mu + \lambda) \int_{\mathbb{R}^N} v \cdot \nabla \operatorname{div} \phi \mathrm{d}x = o(1),$$

as $R \to \infty$. If $2\mu + \lambda = 0$, then

$$\begin{aligned} J &:= \mu \int_{\mathbb{R}^N} v \cdot \Delta \nabla(|x|^2 \sigma_R) \mathrm{d}x + (\mu + \lambda) \int_{\mathbb{R}^N} v \cdot \nabla[\operatorname{div} \nabla(|x|^2 \sigma_R)] \mathrm{d}x \\ &= (2\mu + \lambda) \int_{\mathbb{R}^N} v \cdot \nabla \Delta\Big(|x|^2 \sigma\Big(\frac{|x|}{R}\Big)\Big) \mathrm{d}x = 0, \end{aligned}$$

while if $2\mu + \lambda \neq 0$, then

$$\begin{aligned} |J| &= 2|\mu + \lambda| \Big| \int_{\mathbb{R}^N} v \cdot \nabla \Delta\Big(|x|^2 \sigma\Big(\frac{|x|}{R}\Big)\Big) \mathrm{d}x \Big| \\ &\leq |2\mu + \lambda| \Big| \int_{\mathbb{R}^N} (N+5) \Big[\frac{(v \cdot x)}{R|x|} \sigma'\Big(\frac{|x|}{R}\Big) + \frac{(v \cdot x)}{R^2} \sigma''\Big(\frac{|x|}{R}\Big)\Big] \mathrm{d}x \Big| \\ &\quad + |2\mu + \lambda| \Big| \int_{\mathbb{R}^N} \frac{|x|(v \cdot x)}{R^3} \sigma'''\Big(\frac{|x|}{R}\Big) \mathrm{d}x \Big| \\ &\leq \frac{C}{R} \int_{R \leq |x| \leq 2R} |v(x)| \mathrm{d}x \leq C\Big(\int_{R \leq |x| \leq 2R} |v(x)|^{\frac{N}{N-1}} \mathrm{d}x\Big)^{\frac{N-1}{N}} \to 0, \end{aligned}$$

as $R \to \infty$.

The system of compressible magnetohydrodynamic equations on $\mathbb{R}^N$ with magnetic field H is

$$
\text{(CMHD)}\quad \begin{cases}
\partial_t \rho + \operatorname{div}(\rho v) = 0, \\
\partial_t(\rho v) + \operatorname{div}(\rho v \otimes v - H \otimes H) = -\nabla\Big(p + \dfrac{1}{2}|H|^2\Big) + \mu \Delta v + (\mu + \lambda)\nabla \operatorname{div} v + f, \\
\partial_t H - \operatorname{curl}(v \times H) = 0, \\
\operatorname{div} H = 0, \\
\partial_t S + (v \cdot \nabla) S = 0, \\
p = \kappa \mathrm{e}^S \rho^\gamma, \quad \kappa > 0,\ \gamma \geq 1,\ \rho \geq 0, \\
(\rho, v, H, S)(x, 0) = (\rho_0, v_0, H_0, S_0)(x).
\end{cases}
$$

In the stationary case, we can extend Theorem 3.5 to the MHD as follows (see [8]).

Theorem 3.6 *Let $N \geq 2$, and let the external force $f \in [L^1_{\mathrm{loc}}(\mathbb{R}^N)]^N$ satisfy* $\operatorname{div} f = 0$ *in the sense of distribution. Suppose that (ρ, v, H, S) is a stationary weak solution to* (CMHD) *satisfying the following conditions depending on μ and λ:*

(i) *In the inviscid case ($\mu = \lambda = 0$),*

$$\int_{\mathbb{R}^N} (\rho |v|^2 + |H|^2 + \rho^\gamma \mathrm{e}^S) \frac{\mathrm{d}x}{1 + |x|} < \infty.$$

(ii) *In the viscous case ($\mu > 0$),*

(a) *if $2\mu + \lambda = 0$,*

$$\int_{\mathbb{R}^N} (\rho |v|^2 + |H|^2 + \rho^\gamma \mathrm{e}^S) \mathrm{d}x < \infty.$$

(b) *if $2\mu + \lambda \neq 0$,*

$$\int_{\mathbb{R}^N} (\rho |v|^2 + |v|^{\frac{N}{N-1}} + |H|^2 + \rho^\gamma \mathrm{e}^S) \mathrm{d}x < \infty.$$

Then $\rho(x) = 0$ and $H(x) = 0$ for almost every $x \in \mathbb{R}^N$.

In order to generalize the Liouville theorems for the stationary equations to the time dependent case, we need to impose a temporal decay condition of the second moment of the density. More precisely, we have the following theorem.

Theorem 3.7 *Let ρ_0 satisfy $\rho_0 v_0 |x| \in L^1(\mathbb{R}^N)$ and*

$$\int_{\mathbb{R}^N} \rho_0 v_0 \cdot x \mathrm{d}x \geq 0.$$

Then any finite energy global weak solution to the time-dependent compressible Euler equations, satisfying

$$\limsup_{\tau \to \infty} \int_{\tau \leq t \leq 2\tau} \int_{\mathbb{R}^2} \frac{\rho(x,t)|x|^2}{1 + t^2} \mathrm{d}x \mathrm{d}t = 0, \tag{3.1}$$

corresponds to the vacuum $\rho = 0$. If

$$\int_{\mathbb{R}^N} \rho_0 v_0 \cdot x \mathrm{d}x > 0,$$

then there exists no finite energy global weak solution to the time-dependent compressible Euler equations satisfying (3.1).

See [9] for the proof of the above theorem. Similar to the case of incompressible fluid equations, extensions to the inviscid, compressible (MHD) can be obtained. In the viscous case (Navier-Stokes, viscous MHD), we need extra integrability condition

$$\int_0^T \int_{\mathbb{R}^N} |v(x,t)|^{\frac{N}{N-1}} \mathrm{d}x \mathrm{d}t < \infty$$

for all $T > 0$, besides the finite energy condition (see [9]). One crucial observation obtained in [27] is that the condition (3.1) is implied by the energy inequality for the compressible viscous fluids.

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