The Laplacian eigenvalues of graphs: a survey*

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Abstract

The Laplacian matrix of a simple graph is the difference of the diagonal matrix of vertex degree and the (0,1) adjacency matrix. In the past decades, the Laplacian spectrum has received much more and more attention, since it has been applied to several fields, such as randomized algorithms, combinatorial optimization problems and machine learning. This paper is primarily a survey of various aspects of the eigenvalues of the Laplacian matrix of a graph for the past teens. In addition, some new unpublished results and questions are concluded. Emphasis is given on classifications of the upper and lower bounds for the Laplacian eigenvalues of graphs (including some special graphs, such as trees, bipartite graphs, triangular-free graphs, cubic graphs, etc.) as a function of other graph invariants, such as degree sequence, the average 2-degree, diameter, the maximal independence number, the maximal matching number, vertex connectivity, the domination number, the number of the spanning trees, etc.

Key words: Laplacian matrix, Laplacian eigenvalue, graph, tree, upper bound, lower bound, degree sequence, the independence number, majorization.

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1

1 Introduction

The Laplacian matrix has a long history. The first celebrated result is attributable to Kirchhoff [50] in an 1847 paper concerned with electrical networks. However, it did not receive much attention until the work of Fiedler, which appeared in 1973 [29] and 1975 [30]. Mohar in his survey [72] argued that, because of its importance in various physical and chemical theories, the spectrum of the Laplacian matrix is more natural and important than the more widely studied adjacency spectrum. In [2], Alon used the smallest positive eigenvalue of the Laplacian matrix to estimate the expander and magnifying coefficients of graphs.

There are several books and survey papers concerned with the Laplacian matrix of a graph. For example, in 1997, Chung [12] published his book entitled "Spectral graph theory" which investigated the theory of the Laplacian matrix with aid of the ideas and methods of differential manifold. In 1991 and 1992, Mohar [72], [74] surveyed a detailed introduction to the Laplacian matrix. Further, in 1997, he surveyed several applications of eigenvalues of the Laplacian matrices of graphs in graph theory and in combinatorial optimization. In 1994, Merris [66] surveyed the properties of the Laplacian matrix from the view of linear algebra and graph theory. Further, in 1995, he [68] surveyed the relations between the parameters and the spectrum of the Laplacian matrix and some applications which was not appeared in [66]. In 1991, Grone [32] surveyed the geometry properties of the Laplacian matrix. Recently, Abreu [1] surveyed the old and new results of the second smallest Laplacian eigenvalue. For the more background and motivation on research of the Laplacian matrix, the reader may be referred to the above books, surveys and their references in there.

This paper is a survey of recent new results and questions on the spectrum of the Laplacian matrix. The present content is biased by the viewpoint and the interests of the authors and can not be complete. Therefore we apologize to all those who feel that their work is missing in the references or has not been emphasized sufficiently in this survey.

Let $G = (V, \ E)$ be a simple graph (no loops or multiple edges) with vertex set $V(G) = \{v_1, \cdots, v_n\}$ and edge set $E(G)$. Denote by $d(v_i)$ or $d_G(v_i)$ the *degree* of vertex v_i. If $D(G) = diag(d(u), u \in V)$ is the diagonal matrix of vertex degrees of G and $A(G)$ is the $(0, 1)$ *adjacency matrix* of G, then the matrix $L(G) = D(G) - A(G)$ is called the *Laplacian matrix* of a graph G. It is obvious that $L(G)$ is positive semidefinite and singular $M-$matrix. Thus the all eigenvalues of $L(G)$ are called the *Laplacian*

eigenvalues (or sometimes just eigenvalues) of G and arranged in nonincreasing order:

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} \geq \lambda_n = 0.$$

When more than one graph is under discussion, we may write $\lambda_i(G)$ instead of λ_i. From the matrix-tree theorem, $\lambda_{n-1} > 0$ if and only if G is connected. This observation led Fiedler to define the *algebraic connectivity* of G by $\alpha(G) = \lambda_{n-1}(G)$, which may be considered a quantitative measure of connectivity.

Let $G = (V, E)$ be a simple graph with vertex set $V(G) = \{v_1, \cdots, v_n\}$ and edge set $E(G) = (e_1, e_2, \cdots, e_m)$. For each edge $e_k = (v_i, v_j)$, choose one of v_i or v_j to be the positive end of e_k and the other to be the negative end. We refer to this procedure by saying G has been given an *orientation*. For an arbitrary given orientation of G, the *oriented vertex-edge incidence matrix* is the $n \times m$ matrix $Q = Q(G) = (q_{ij})$, where

$$q_{ij} = \begin{cases} +1, & \text{if } v_i \text{ is the positive end of } e_j \\ -1, & \text{if } v_i \text{ is the negative end of } e_j \\ 0, & \text{otherwise.} \end{cases}$$

While Q depends on the orientation of G, QQ^T does not. In fact, for any orientation of G, it is easy to see that

$$Q(G)Q(G)^T = D(G) - A(G) = L(G).$$

Thus one may also describe $L(G)$ by means of its quadratic form

$$x^T L(G)x = (Q(G)^T x)^T (Q(G)^T x) = \sum (x_i - x_j)^2,$$

where $x = (x_1, \cdots, x_n)^T$ is $n-$dimension real vector and the sum is taken over all pairs $i < j$ for which $(v_i, v_j) \in E(G)$.

The first appearance of $L(G)$ may occur in Kirchhoff's *matrix-tree theorem* [50]:

Theorem 1.1 *([50]) Let $L(i|j)$ be the $(n-1) \times (n-1)$ submatrix of $L(G)$, which is obtained by deleting its $i-$th row and $j-$column. Denote by $\tau(G)$ the number of spanning trees in G. Then*

$$\tau(G) = (-1)^{i+j} \det L(i|j) = \frac{1}{n} \prod_{i=1}^{n-1} \lambda_i.$$

In view of this result, $L(G)$ is sometimes called the *Kirchhoff matrix* or *matrix of admittance* (admitance=conductivity, the reciprocal of impedance). However, we will

refer to $L(G)$ as a Laplacian matrix because it is a discrete analogue of the Laplace differential operator. The Laplacian matrix of a graph and its eigenvalues can be used in several areas of mathematical research and have a physical interpretation in various physical and chemical theories. The adjacency matrix of a graph and its eigenvalues have been much investigated in the monographs [14] and [15]. The normalized Laplacian matrix $\mathcal{L}(G) = D^{-1/2}L(G)D^{-1/2}$ of a graph and its eigenvalues has studied in the monographs [12].

In this paper, we survey the Laplacian eigenvalues of a graph. In section 2, some basic and important properties of the Laplacian eigenvalues are reviewed. In section 3, the largest Laplacian eigenvalue is heavily investigated. Many upper and lower bounds for the largest Laplacian eigenvalues of graphs and special graphs (including tree, cubic graphs, triangular free graphs, etc.) are presented. Proofs of part important results are also given. In section 4, the second Laplacian eigenvalue is studied and a question is proposed. In section 5, the bounds for the $k-$largest Laplacian eigenvalue are discussed. In section 6, the upper and lower bounds for the second smallest Laplacian eigenvalue, i.e., algebraic connectivity, are studied. Moreover, the relations between algebraic connectivity and graph parameters are obtained. In section 7, the sum of the Laplacian eigenvalues are investigated with emphasizing on two conjectures of Grone and Merris in [35].

2   Preliminary

Let $G = (V(G), E(G))$ be a simple graph. The line graph of G, written G^l, is the graph whose vertex set is the edge set $E(G)$ of G and whose two vertices are adjacent if and only if they have one common vertex in G. Denoted by $D(G) = diag(d(u), u \in V)$ and $A(G)$ the diagonal matrix of vertex degrees of G and the $(0, 1)$ *adjacency matrix* of G respectively. The matrix $K(G) = D(G) + A(G)$ is called the *unoriented Laplacian matrix* of G. Moreover, denote by $Q(G)$ the oriented vertex-edge incidence matrix. Let $X = (x_{ij})$ be an $(n \times n)$ matrix. Denote by $|X| = (|x_{ij}|)$ the matrix whose entries are absolute values of the entries of X. Denote by $\rho(X)$ the largest modulus of eigenvalues of X. Then we sum up some preliminary results from [58], [65], [66] [85] as follows:

Lemma 2.1 *Let G be a simple graph. Then*

$$K(G) = D(G) + A(G) = |Q(G)Q(G)^T| = |Q(G)||Q(G)^T|. \tag{1}$$

4

$$|Q^T(G)Q(G)| = 2I + A(G^l),\tag{2}$$

where I is the identity matrix.

$$\lambda_1(G) \le \rho(K(G)) = 2 + \rho(A(G^l))\tag{3}$$

with equality if and only G is bipartite.

A *semiregular graph* $G = (V, E)$ is a graph with bipartition (V_1, V_2) of V such that all vertices in V_i have the same degree k_i for $i = 1, 2$.

Lemma 2.2 *([101] Let G be a simple connected graph. Then the line graph G^l of G is regular or semiregular if and only if G is regular or semiregular or a path of order 4.*

Proof. since sufficiency is obvious, we only consider necessity. If G^l is k-regular, then for each edge $e_{uv} = (u, v) \in E(G)$, the degree of vertex e_{uv} in G^l is equal to $d_{G^l}(e_{uv}) = d_G(u) + d(v) - 2$. Hence if two vertices of G share a common vertex, then they have the same degree. Since G is connected, this implies that there are at most two different degrees. If two adjacent vertices have same degree, it is easy to show that G is regular by means of induction argument. If G contains a cycle of odd length, then it must have two adjacent vertices with the same degree. Therefore, if G is not regular, then it does not contain any cycle of odd length, which implies that G is bipartite. So G is semiregular. ∎

Lemma 2.3 *([66]) Let G be a simple graph on n vertices and G^c be the complement graph of G in the complement graph. Then*

$$\lambda_1(G) \le n.\tag{4}$$

$$\lambda_i(G^c) = n - \lambda_{n-i}(G) \text{ for } i = 1, \cdots, n - 1.\tag{5}$$

Proof. Since

$$L(G) + L(G^c) = nI - J,$$

where J is the $n \times n$ matrix each of whose entries is 1. It follows that the Laplacian spectrum of G^c is

$$n - \lambda_{n-1}(G) \ge n - \lambda_{n-2}(G) \ge \cdots \ge n - \lambda_1(G) \ge 0.$$

Therefore the assertion holds. ■

There are several useful min-max formulas for the expression of eigenvalues of a symmetric matrix and their sums. If M is a real symmetric matrix of order $n \times n$ and $\mathbf{R^n}$ is the n real dimension vector space, then Rayleigh-Ritz ration (see p.176 in [47]) may be expressed as follows.

$$\lambda_1(M) = \max\{x^T M x \mid \; \| x \| = 1, x \in \mathbf{R^n} \; \} \tag{6}$$

and

$$\lambda_1(M) = \min\{x^T M x \mid \; \| x \| = 1, x \in \mathbf{R^n} \; \}. \tag{7}$$

In general, the min-max characterization of $\lambda_k(M)$ is called *Courant-Fischer "min-max theorem"*(see p.179 in [47])

$$\lambda_k(M) = \max_U \min_x \{x^T M x \mid \; \| x \| = 1, x \in U \; \}, \tag{8}$$

where the first minimum is over all $k-$dimensional subspaces U of R^n.

3 The Largest Laplacian eigenvalue

In this section, we will discuss the upper and lower bounds for the largest Laplacian eigenvalue for graphs and several kinds of special graphs, including trees, triangular-free graphs, cubic graphs. There are a lot of papers focus on this topic.

3.1 The upper bound versus degree sequences

In 1985, Anderson and Morley [3] may first obtain the upper bound for the largest Laplacian eigenvalue. They showed the following:

Theorem 3.1 *([3]) Let G be a simple graph. Then*

$$\lambda_1 \leq \max\{d(u) + d(v) | (u,v) \in E(G)\}, \tag{9}$$

where $d(u)$ is the degree of vertex u.

In 1997, this result was improved by Li and Zhang [58]. Their main result is as follows:

Theorem 3.2 *([58])Let G be a simple graph. Denote by $r = \max\{d(u) + d(v)|(u, v) \in E(G)\}$ and $s = \max\{d(u) + d(v)|(u, v) \in E(G) - (x, y)\}$ with $(x, y) \in E(G)$ such that $d(x) + d(y) = r$. Then*

$$\lambda(G) \leq 2 + \sqrt{(r-2)(s-2)}, \tag{10}$$

Pan in [78] gave the necessary and sufficient conditions for the holding of equality in (10) . In fact this result may further be improved. We can state as follows:

Theorem 3.3 *Let G be a simple connected graph. Then*

$$\lambda(G) \leq 2 + \max\left\{\sqrt{(d(u) + d(v) - 2)(d(u) + d(w) - 2)}\right\}, \tag{11}$$

where the maximum is taken over all pairs $(u, v), (u, w) \in E(G)$. Moreover, equality holds in (11) if and only if G is regular bipartite graph or a semiregular graph, or a path of order four.

Proof. For each edge $e_{uv} = (u, v) \in E(G)$, the degree $d(e_{uv})$ of vertex e_{uv} in G^l is equal to $d_G(u) + d_G(v) - 2$. By Lemma 2.1 in [7],

$$\rho(G^l) \leq \max\left\{\sqrt{(d(u) + d(v) - 2)(d(u) + d(w) - 2)}\right\},$$

where the maximum is taken over all pairs $(u, v), (u, w) \in E(G)$. Hence it follows from (3) in Lemma 2.1 that (11) holds. Clearly, if G is regular bipartite graph or a semiregular graph, or a path of order four, by some calculations, it is easy to argue that equality in (11) holds. Conversely, if equality in (11) holds, then

$$\rho(G^l) = \max\left\{\sqrt{(d(u) + d(v) - 2)(d(u) + d(w) - 2)}\right\},$$

where the maximum is taken over all pairs $(u, v), (u, w) \in E(G)$. By Lemma 2.1 in [7], G^l is regular or semiregular. Consequently it follows from Lemma 2.2 that G is regular bipartite graph or a semiregular graph, or a path of order four. ∎

We notice that Theorem 3.3 is a new result and better than Theorems 3.1 and 3.2. In 2002, Shu, Hong and Wen [85] gave an upper bound in terms of degree sequences.

Theorem 3.4 *([85]) Let G be a simple graph. Assume that the degree sequence of G is $d_1 \geq d_2 \geq \cdots \geq d_n$. Then*

$$\lambda_1(G) \leq d_n + \frac{1}{2} + \sqrt{(d_n - \frac{1}{2})^2 + \sum_{i=1}^{n} d_i(d_i - d_n)} \tag{12}$$

with equality if and only if G is a regular bipartite graph.

7

Sketch of Proof. With the aid of the result in [45] and Lemma 2.1, it not difficult to argue with some calculations that (12) holds. ■

Das in [19] also gave several related upper bounds for the largest Laplacian eigenvalue in terms of degree sequence.

3.2 The upper bounds versus the average 2-degree

Let G be a simple graph. Denote by $m(v)$ the average of the degrees of the vertices adjacent to v. Then $d(v)m(v)$ is the "$2 - degree$" of vertex v. In 1998, Merris [70] used another approach method to provide another upper bound:

Theorem 3.5 *([70]) Let G be a simple graph. Then*

$$\lambda_1(G) \leq \max\{d(v) + m(v) \mid v \in V(G)\}. \tag{13}$$

We observed that Merris' bound (13) was only involved in one vertex, while Li and Zhang's bound (10) was involved in the adjacent vertices. It was natural to stimulate us to consider whether there was an better upper bound than Merris' upper bound for graphs with the adjacent relations. Li and Zhang in [59] followed this idea and obtained an better upper bound. Later Pan in [78] characterized equality situation.

Theorem 3.6 *([59],[78]) Let G be a simple graph. Then*

$$\lambda_1(G) \leq \max\left\{\frac{d(u)(d(u) + m(u)) + d(v)(d(v) + m(v))}{d(u) + d(v)} : (u, v) \in E(G)\right\}. \tag{14}$$

If G is connected, then equality in (14) holds if and only if G is regular bipartite or semiregular.

Sketch of Proof. Let P be sum of the degree diagonal matrix of the line graph G^l of a graph G and two multiple of the identity matrix. Let

$$N = P^{-1}(2I + A(G^l))P^{-1}.$$

If $e_{uv} = (u, v)$ is an edge of G, then e_{uv} is an vertex of G^l and the corresponding row sum of N is equal to

$$\frac{\sum_{x \sim u}(d(x) + d(u)) + \sum_{y \sim v}(d(y) + d(v))}{d(u) + d(v)} = \frac{d(u)(d(u) + m(u)) + d(v)(d(v) + m(v))}{d(u) + d(v)},$$

where $u \sim v$ mean that u and v in G are adjacent. Hence

$$\rho(N) \leq \max \left\{ \frac{d(u)(d(u)+m(u))+d(v)(d(v)+m(v))}{d(u)+d(v)} \ : \ (u,v) \in E(G) \right\}.$$

On the other hand, by Lemma 2.1, we have

$$\lambda_1(G) \leq \rho(2I + A(G^l)) = \rho(N).$$

Therefore (14) holds. For the equality situation, the proof is omitted. ■

Denote by $t = \max\{d(v)+m(v) \mid v \in V(G)\}$. It is obvious that (14) is better than (13), since

$$\max \left\{ \frac{d(u)(d(u)+m(u))+d(v)(d(v)+m(v))}{d(u)+d(v)} : (u,v) \in E(G) \right\} \leq \frac{d(u)t+d(v)t}{d(u)+d(v)} = t.$$

By a similar method, we could get another two upper bounds

Theorem 3.7 *([96]) Let G be a simple connected graph. Denote by $t(u) = d(u) + m(u)$. Then*

$$\lambda_1(G) \leq \max \left\{ 2 + \sqrt{(d(u)(t(u)-4)+d(v)(t(v)-4)+4} \right\} \tag{15}$$

and

$$\lambda_1(G) \leq \max \left\{ \sqrt{d(u)t(u)+d(v)t(v)} \right\}, \tag{16}$$

where the maximum is taken over all pairs $(u,v) \in E(G)$. Moreover, equality in (15) holds if and only if G is bipartite regular or semi-regular, or a path of order four. Equality in (16) holds if and only if G is bipartite regular or semi-regular.

3.3 The upper bound versus eigenvectors

In this subsection, we use the relationships between eigenvalues and eigenvectors to investigate the largest Laplacian eigenvalue. Li and Pan in [56] showed the following result.

Theorem 3.8 *([56]) Let G be a simple connected graph. Then*

$$\lambda_1(G) \leq \max\{\sqrt{2d(u)(d(u)+m(u))} \mid u \in V(G)\} \tag{17}$$

with equality if and only if G regular bipartite.

Zhang in [96] followed Li and Pan's method and improved the above result.

Theorem 3.9 *([96]) Let G be a simple connected graph. Then*

$$\lambda_1(G) \leq \max\{d(u) + \sqrt{d(u)m(u)} \mid u \in V(G)\} \tag{18}$$

with equality if and only if G is bipartite regular or semiregular.

Proof. Let $x = (x_v, v \in V(G))^T$ be an eigenvector with $||x||_2 = 1$ corresponding to $\lambda(G)$. Thus $L(G)x = \lambda_1(G)x$. Hence for any $u \in V(G)$,

$$\lambda_1(G)x_u = d(u)x_u - \sum_{v \in V(G)} a_{uv}x_v = \sum_{(u,v) \in E(G)} (x_u - x_v).$$

By the Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
\lambda_1(G)^2 x_u^2 &\leq \Big(\sum_{(u,v) \in E(G)} 1^2 \Big)\Big(\sum_{(u,v) \in E(G)} (x_u - x_v)^2 \Big) \\
&= d(u)^2 x_u^2 + 2d(u)x_u^2(\lambda_1(G) - d(u)) + d(u) \sum_{(u,v) \in E(G)} x_v^2.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\sum_{u \in V(G)} \lambda_1(G)^2 x_u^2 &\leq \sum_{u \in V(G)} (2d(u)\lambda_1(G) - d(u)^2)x_u^2 + \sum_{u \in V(G)} d(u) \sum_{(u,v) \in E(G)} x_v^2 \\
&= \sum_{u \in V(G)} (2d(u)\lambda_1(G) - d(u)^2)x_u^2 + \sum_{u \in V(G)} d(u)m(u)x_u^2.
\end{aligned}
$$

Therefore, we have

$$\sum_{u \in V(G)} (\lambda_1(G)^2 - 2d(u)\lambda_1(G) + d(u)^2 - d(u)m(u))x_u^2 \leq 0.$$

Then there must exist a vertex u such that

$$\lambda_1(G)^2 - 2d(u)\lambda_1(G) + d(u)^2 - d(u)m(u) \leq 0,$$

which implies $\lambda_1(G) \leq d(u) + \sqrt{d(u)m(u)}$. it follows that (18) holds.

If G is bipartite regular or semi-regular, it is easy to see that equality in (18) holds by a simply calculation.

Conversely, if equality in (18) holds, it follows from the above proof that for each $u \in V(G), (u,v) \in E(G), (u,w) \in E(G)$, we have $x_u - x_v = x_u - x_w$, which implies that all x_v are equal for all vertices adjacent to vertex u. Fixed a vertex $w \in V(G)$,

10

we may define that $V_1(G) = \{v \in V(G)|$ the distance between v and w is even $\}$ and $V_2(G) = \{v \in V(G)|$ the distance between v and w is odd $\}$. Clearly, V_1 and V_2 are a partition of $V(G)$. Since G is connected, it is not difficult to see that all x_v are equal for any $v \in V_1$ and denoted by a, and that all x_v are equal for any $v \in V_2$ and denoted by b. We claim that G is bipartite. In fact, if there exists an edge $(u_1, u_2) \in E(G)$, where $u_1, u_2 \in V_1$ or $u_1, u_2 \in V_2$, then $a = b$. Hence $\lambda_1(G)x_w = \sum_{(v,w)\in E(G)}(x_w - x_v)) = 0$ which implies $x_w = 0$. Therefore $x = 0$ and it is a contradiction. For any $u \in V_1$, we have $\lambda_1(G)x_u = \sum_{(v,u)\in E(G)}(x_u - x_v)) = (a - b)d(u)$, which result in $d(u) = \frac{a\lambda_1(G)}{a-b}$ for any $u \in V_1$. Similarly, $d(u) = \frac{-b\lambda_1(G)}{a-b}$ for any $u \in V_2$. Hence we conclude that G is regular or semi-regular. ■

Since $(d(u) + \sqrt{d(u)m(u)}) \leq 2d(u)(d(u) + m(u))$, for any $u \in V(G)$, we have that (18) is always better than (17).

On the other hand, if the common neighbors of two adjacent vertices are involved, (17) can be also improved. Das in [16] and [17] showed the following

Theorem 3.10 *([16]) Let G be a simple connected graph. Denote by*

$$m'(u) = \frac{\sum_{v \sim u}(d(u) - |N(u) \cap N(v)|)}{d(u)},$$

where $v \sim u$ means that v and u are adjacent and $N(u)$ is the set of all neighbor vertices of u. Then

$$\lambda_1(G) \leq \max\{\sqrt{2d(u)(d(u) + m'(u))} \mid u \in V(G)\} \tag{19}$$

with equality if and only if G bipartite regular.

With aid of the relationships between the eigenvalues and eigenvectors, we improved and generalized some equalities and inequalities for the largest Laplacian eigenvalue. For example, in 2002, Zhang and Li [101] generalized the result for the largest eigenvalue of mixed graphs. In 2003, Zhang and Luo in [104] were able to get the new upper bounds for the Largest Laplacian eigenvalues of mixed graphs (including simple graphs), while in 2004, Das in [17] also obtained the same result for simple graphs.

Theorem 3.11 *([17], [104]) Let G be a simple connected graph of order n. Denote by $d(u)$ and $m(u)$ the degree and average 2-degree of the vertex $u \in V(G)$, respectively. Then*

$$\lambda_1(G) \leq \max\left\{\frac{d(u) + d(v) + \sqrt{(d(u) - d(v))^2 + 4m(u)m(v)}}{2} \mid (u, v) \in E(G)\right\} \tag{20}$$

with equality if and only G is bipartite regular or semiregular.

3.4   The upper bounds versus related matrices

In this subsection, we introduced another approach to obtain the upper bound for the largest Laplacian eigenvalue. Li and Pan in [57] used the relationships of the eigenvalues of between the matrix $K(G) = D(G) + A(G)$ and $L(G)$, and nonnegative matrix theory to present some upper bounds for the largest Laplacian eigenvalue of G.

Lemma 3.12 *([57], [61]) Let G be a simple connected graph and let $f(x)$ be a polynomial on x. Denote by $\rho(K)$ the spectral radius of the matrix $K = D(G) + A(G)$. Let $R_v(f(K))$ be the corresponding $v-$th row sum of $f(K)$. Then*

$$\min\{R_v(f(K)) \mid v \in V(G)\} \le f(\rho(K)) \le \max\{R_v(f(K)) \mid v \in V(G)\}. \qquad (21)$$

Moreover, if the row sums of $f(K))$ are not all equal, then both inequalities in (21) are strict.

Proof. Let $x = (x_v, v \in V(G))^T$ be a positive eigenvector of K with $sum_{v \in V(G)} x_v = 1$. Then by

$$f(K)x = f(\rho(K))x,$$

we have

$$f(\rho(K)) = f(\rho(K)) \sum_{x \in V(G)} x_v = \sum_{v \in V(G)} (f(K)x)_v = \sum_{v \in V(G)} x_v R_v(f(K)).$$

Therefore the desired result holds since the entries of x are positive and their sum is equal to 1. ∎

Theorem 3.13 *([57]) Let G be a simple connected graph with n vertices and m edges. Denote by Δ and δ the maximum and minimum degrees of G, respectively. Then*

$$\lambda_1(G) \le \frac{\delta - 1 + \sqrt{(\delta - 1)^2 + 8(\Delta^2 + 2m - (n-1)\delta)}}{2} \qquad (22)$$

with equality if and only if G is bipartite and regular.

Proof. Let $K = D(G) + A(G)$. Then $K^2 = D(G)^2 + D(G)A(G) + A(G)D(G) + A(G)^2$. Then the $u-$row sum of K^2 is

$$
\begin{aligned}
R_u(K^2) &= 2d(u) + 2\sum_{v \sim u} d(v) = 2d(u)^2 + 4m - 2d(u) - 2\sum_{v \nsim u, v \neq u} d(v) \\
&\leq 2\Delta^2 + 4m - 2d(u) - 2(n - 1 - d(u))\delta \\
&= 2\Delta^2 + 4m + 2(\delta - 1)d(u) - 2(n - 1)\delta.
\end{aligned}
$$

Let $f(x) = x^2 - (\delta - 1)x$. It follows from Lemma 3.12 that

$$
\rho(K)^2 - (\delta - 1)\rho(K) \leq 2\Delta^2 + 4m - 2(n - 1)\delta.
$$

Combining the above inequality and (3), we are able to obtain (22). ∎

Using the similar method, Li et.al in [57] and Liu et.al in [61] gave the following:

Theorem 3.14 *([57], [61]) Let G be a simple connected graph with n vertices and m edges. Denote by Δ and δ the maximum and minimum degrees of G, respectively. Then*

$$
\lambda_1(G) \leq \frac{\Delta + \delta - 1 + \sqrt{(\Delta + \delta - 1)^2 + 8(2m - (n - 1)\delta)}}{2} \tag{23}
$$

with equality if and only if G is bipartite and regular.

3.5 Always nontrivial upper bounds

In the above subsections, several kind upper bounds for the largest Laplacian eigenvalue are presented. However, sometime these bounds exceed the number of vertices in G, which becomes an trivial upper bounds. Rojo et.al. in [83] obtained an always nontrivial upper bound. Their result is

Theorem 3.15 *([83]) Let G be a simple graph. Denote by $N(u)$ the set of all neighbor vertices of vertex u in G. Then*

$$
\lambda_1(G) \leq \max\left\{d(u) + d(v) - |N(u)\bigcap N(v)| \mid u, v \in V(G)\right\}. \tag{24}
$$

Before giving an proof, we need the following Lemma

Lemma 3.16 *([4]) Let $B = (b_{ij})$ be an $n \times n$ nonnegative matrix. Denote by $\xi(B)$ the second largest modulus of the eigenvalues of B. If $w = (w_1, \cdots, w_n)^T$ is a positive eigenvector of B corresponding to the spectral radius $\rho(B)$, then*

$$
\xi(B) \leq \frac{1}{2}\max\left\{\sum_{k=1}^{n} w_k|\frac{b_{ik}}{w_i} - \frac{b_{jk}}{w_j}| \mid 1 \leq i, j \leq n\right\}. \tag{25}
$$

Now we use the lemma to prove Theorem 3.15. Let $B = L(G) + ee^T$, where e is the all ones $n-$dimensional column vector. Thus M has a positive eigenvector e corresponding to $\rho(B) = n$ and $\xi(B) = \lambda_1(G)$. Then from Lemma 3.16 and some calculations, it is not difficult to get the desired result. Clearly this upper bound is always nontrivial. But we notice that the vertices u and v in Theorem 3.15 may be or not adjacent. It stimulated researcher to consider whether this result may be improved by the adjacent relationships. In 2003, Das [16] improved this upper bound. Further, Das in [17] considered when the upper bound is attained and proposed a conjecture. Yu at.el in [93] confirmed the conjecture. Before stating this theorem, we need the following notation. Let $F = (V, E)$ be a semiregular with bipartition $V = V_1 \bigcup V_2$ and let $F^+ = (V, E^+)$ be a *super graph* of F constructed by joining those pairs of vertices of V_1 (or V_2) which have same set of neighbors in the other set V_2 (or V_1), if such pairs exist, where E^+ is equal to E with some new edges (if new edges were constructed).

Theorem 3.17 *([16], [17], [93]) Let G be a simple connected graph. Then*

$$\lambda_1(G) \leq \max \left\{ d(u) + d(v) - |N(u) \bigcap N(v)| \mid (u, v) \in E(G) \right\} \tag{26}$$

with equality if and only if G is a super graph of a semiregular graph.

3.6   The lower bounds for the largest Laplacian eigenvalue

The first lower bound for the largest Laplacian eigenvalue may be contributed to Fiedler [29]. He showed the following result

Theorem 3.18 *([29]) Let G be a graph with on n vertices and the maximum degree Δ. Then*

$$\lambda_1(G) \geq \frac{n}{n-1}\Delta \tag{27}$$

Grone and Merris in [35] improved (27). Moreover, Zhang and Luo in [103] gave a new proof of this lower bound and characterized equality situation.

Theorem 3.19 *([35], [103]) Let G be a simple connected graph with at least one edge and the maximum degree Δ. Then*

$$\lambda_1(G) \geq \Delta + 1 \tag{28}$$

with equality if and only if there exists a vertex is adjacent all other vertices in G.

Proof. It is easy to see that G contains a star graph H with $\Delta + 1$ vertices. By a simple calculation, the largest Laplacian eigenvalue of H is $\Delta + 1$. Hence the result follows from Theorem 4.1 in [36].

If there exists a vertex is adjacent all other vertices in G, then $\Delta = n - 1$, where n is the number of vertices in G. By (28) and Lemma 2.1, equality in (28) holds. Conversely, if $\Delta < n - 1$, then let $d(z) = \Delta$ and there exist vertices y_1 and y_2 such that $(z, y_1) \in E(G), (z, y_2) \notin E(G)$ and $(y_1, y_2) \in E(G)$. Let H_1 be a subgraph of G obtained from a star graph with $\Delta + 1$ vertices and joining a new vertex and new edge. By a simple calculation and Theorem 4.1 in [36], $\lambda_1(G) \geq \lambda_1(H_1) > \Delta + 1$. ■

Another lower bound for the largest Laplacian eigenvalue in terms of the number of vertices and edges was given in [99].

Theorem 3.20 *([99]) Let G be a simple graph with n vertices and m edges. Then*

$$\lambda_1(G) \geq \frac{1}{n-1} \left(2m + \sqrt{\frac{2m(n(n-1) - 2m)}{n(n-2)}} \right) \tag{29}$$

with equality if and only if G is the complete graph.

Proof. Clearly,

$$((n-1)\lambda_1 - Tr(L(G)))^2 \geq \sum_{i=1}^{n-1} (\lambda_1 - \lambda_i)^2,$$

while

$$\sum_{i=1}^{n-1} (\lambda_1 - \lambda_i)^2 = Tr(L(G)^2) - 2\lambda_1 Tr(L(G)) + (n-1)\lambda_1^2.$$

Since $Tr(L(G)) = 2m$ and $Tr(L(G)^2) \geq 2m + \frac{(2m)^2}{n}$, we have

$$((n-1)\lambda_1 - 2m)^2 \geq (2m + \frac{(2m)^2}{n}) - 4m\lambda_1 + (n-1)\lambda_1^2.$$

By solving this quadratic form, it is easy to obtain (29). ■

Das in [18] considered the largest Laplacian eigenvalues of special subgraphs of a graph and obtained a lower bound for the largest Laplacian eigenvalue of graphs in term of degree sequence and their neighbor sets.

Theorem 3.21 *([18]) Let G be a simple graph with at least one edge. Denote by $c_{uv} = d(u) - |N(u) \cap N(v)| - 1$, $t_u = d(u)^2 + 2d(u)$, Then*

$$\lambda_1(G) \geq \max \left\{ \sqrt{\frac{1}{2} \left(t_u - 2d(v) - 2 + \sqrt{(t_u + 2d(v) + 4)^2 + 4c_{uv}c_{vu}} \right)} \right\}, \tag{30}$$

where the maximum is taken over all pairs $(u, v) \in E(G)$.

3.7 The upper and lower bounds for special graphs

Now we turn to consider the upper and lower bounds for the largest Laplacian eigenvalue of special graphs. Zhang and Luo in [103] provided the following lower bound for the largest Laplacian eigenvalue of triangle-free graphs.

Theorem 3.22 *([103]) Let $G = (V, E)$ be a triangle-free graph. If d_u and m_u are the degree and the average 2-degree of a vertex u, respectively, then*

$$\lambda_1(G) \geq \max\{\frac{1}{2}(d(u) + m(u) + \sqrt{(d(u) - m(u))^2 + 4d(u)}, \quad u \in V\}. \quad (31)$$

Proof. Let $L(U)$ be the principal submatrix of $L(G)$ corresponding to U, where $U = \{u, v_1, \cdots, v_k\}$ is the closed neighborhood of a vertex u and $d(u) = k$. Obviously, $\lambda_1(L(G)) \geq \lambda_1(L(U))$. Since G is triangle-free, we may assume that

$$L(U) = \begin{pmatrix} d_u & -1 & -1 & \cdots & -1 \\ -1 & d_{v_1} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -1 & 0 & 0 & \cdots & d_{v_k} \end{pmatrix}.$$

With elementary calculations, we have that the characteristic polynomial of $L(U)$ is

$$det(\lambda I - L(U)) = (\lambda - d(u) - \sum_{i=1}^{k} \frac{1}{\lambda - d(v_i)}) \prod_{i=1}^{k} (\lambda - d(v_i)).$$

Note that $\lambda_1(L(G)) \geq \lambda_1(L(U)) > d(v_i)$ for each $i = 1, \cdots, k$. Hence $\lambda_1(L(G))$ satisfies

$$\lambda_1(L(G)) - d(u) \geq \sum_{i=1}^{k} \frac{1}{\lambda_1(L(G)) - d(v_i)}.$$

By Cauchy-Schwarz inequality, we have

$$\sum_{i=1}^{k} (\lambda_1(L(G)) - d(v_i)) \sum_{i=1}^{k} \frac{1}{\lambda_1(L(G)) - d(v_i)} \geq \left(\sum_{i=1}^{k} \frac{\sqrt{\lambda_1(L(G)) - d(v_i)}}{\sqrt{\lambda_1(L(G)) - d(v_i)}} \right)^2 = k^2.$$

Hence

$$\lambda_1(L(G)) - d(u) \geq \frac{k^2}{\sum_{i=1}^{k}(\lambda_1(L(G)) - d(v_i))} = \frac{d(u)}{\lambda_1(L(G)) - m(u)},$$

since $m(u) = \frac{1}{k} \sum_{i=1}^{k} d(v_i)$. This inequality yields the desired result. ■

Yu et al. in [92] used the 2-degree vertex to present a lower bound for the Laplacian eigenvalue of bipartite graphs.

Theorem 3.23 *([92]) Let G be a simple connected bipartite graph. Then*

$$\lambda_1(G) \geq \sqrt{\frac{\sum_{v \in V(G)} d(v)^2(d(v) + m(v))^2}{\sum_{v \in V(G)} d(v)^2}} \tag{32}$$

with equality if and only if G is regular or semiregular.

Hong and Zhang in [46] gave another lower bound for the largest Laplacian eigenvalue of bipartite graphs.

Theorem 3.24 *([46]) Let G be a simple connected bipartite graph. Then*

$$\lambda_1(G) \geq 2 + \sqrt{\frac{1}{m} \sum_{u \sim v} (d(u) + d(v) - 2)^2} \ , \tag{33}$$

where m is the edge number of G. Moreover, equality in (33) holds if and only if G is either a regular connected bipartite graph, or a semiregular connected bipartite graph, or the path with four vertices.

If we consider tree, what are about upper and lower bounds for the largest Laplacian eigenvalue? Stevanović in [87] presented an upper bound for the largest Laplacian eigenvalue of a tree in terms of the largest vertex degree.

Theorem 3.25 *([87]) Let T be a tree with the largest vertex degree Δ. Then*

$$\lambda(T) < \Delta + 2\sqrt{\Delta - 1}. \tag{34}$$

In 2005, Rojo [82] improved Stevanović's result.

Theorem 3.26 *([82]) Let T be a tree with the largest vertex degree Δ. Let u be a vertex of T with $d(u) = \Delta$. Denote by $k - 1$ the largest distance from u to any other vertex of tree. For $j = 1, \cdots, k - 1$, let $\delta_j = \max\{d(v) : dist(v, u) = j\}$. Then*

$$\lambda(G) < \max\{\max_{2 \leq j \leq k-2}\{\sqrt{\delta_j - 1} + \delta_j + \sqrt{\delta_{j-1} - 1}\}, \sqrt{\delta_1 - 1} + \delta_1 + \sqrt{\Delta}, \Delta + \sqrt{\Delta}\}. \tag{35}$$

From the proofs of [87] and [82], we are able to this upper bound is not achieved. It is natural to ask what is the best upper bound for trees. Thus we may propose the following question:

Question 3.27 *Let T be a tree with the largest vertex degree Δ. What is the best upper bound for the largest Laplacian eigenvalue of T?*

3.8 The bounds in terms of graph parameters

In the above several sections, we have mainly investigated some upper and lower bounds for the largest eigenvalue of graphs in terms of the following basic invariants of G, including, the vertex number, the edge number, the maximum and minimum degrees, 2-average degree, degree sequence. In this subsection, we just focus on relations between the largest Laplacian eigenvalue and other graphs parameters.

A subset U of vertex set V of a graph $G = (V, E)$ is called *an independent set* of G if no two vertices of U are adjacent in G. The *independence number* $\alpha(G)$ of G is the maximum size of independent sets of G. In 2004, Zhang [97] proved two conjectures on the Laplacian eigenvalue and the independence number.

Theorem 3.28 *([97]) Let G be a graph of order n with at least one edge and the independence number $\alpha(G)$. Then*

$$\lambda_1(G) \geq \frac{n}{\alpha(G)} \tag{36}$$

with equality if and only if $\alpha(G)$ is a factor and G has $\alpha(G)$ components each of which is the complete graph $K_{\frac{n}{\alpha(G)}}$.

In 2005, Lu et al. [62] also obtained the same result for connected graphs. Recently, Nikiforov in [77] gave a slight improvement and showed that $\lambda_1(G) \geq \left\lceil \frac{n}{\alpha(G)} \right\rceil$, where $\lceil x \rceil$ the smallest integer no less than x. Let $K_{1,m}$ denote the star on $m + 1$ vertices. If $\frac{n-1}{2} < m \leq n - 1$, then $T_{n,m}$ is the tree created from $K_{1,m}$ by adding a pendent edge to $n - m - 1$ of the pendent vertices of $K_{1,m}$.

Theorem 3.29 *([97]) Let T be a tree of order n and the independence number $\alpha(T)$. Denote by a the largest root of the equation $x^3 - (\alpha(T) + 4)x^2 + (3\alpha(T) + 4)x - n = 0$. Then*

$$\lambda_1(T) \leq a \tag{37}$$

with equality if and only if T is $T_{n,\alpha(T)}$.

A *matching* in a simple graph G is a set of edges with no shared common vertex The *matching number* of G is the maximum size among all matching in G. Guo in [38] showed that the largest Laplacian eigenvalue of a tree in terms of the matching number.

Theorem 3.30 *([38]) Let T be a tree of order n with the matching number $\beta(T)$. Denote by a the largest root of the equation $x^3 - (n - \beta(T) + 4)x^2 + (3n - 3\beta(T) + 4)x - n = 0$. Then*

$$\lambda_1(T) \leq a \tag{38}$$

with equality if and only if T is $T_{n,n-\beta(T)}$.

Let G be a simple graph and let H be any bipartite subgraph of G with the maximum edges. Thus

$$b(G) = \frac{|E(H)|}{|E(G)|}$$

is called *the bipartite density* of G. Berman and Zhang in [8] gave a lower bound for the largest Laplacian eigenvalue of cubic graphs in terms of their bipartite density. Stevanović in [88] characterized all exemtral graphs which attain the lower bound.

Theorem 3.31 *([8], [88]) Let G be a connected cubic graph of order n with the bipartite density $b(G)$. Then*

$$\lambda_1(G) \geq \frac{10b(G) - 4}{b(G)} \tag{39}$$

with equality if and only if G is bipartite graph, or the complete graph K_4, or the Petersen graph, or the four special graphs of order 10.

4   The second largest Laplacian eigenvalue

Since there are a lot of upper and lower bounds for the largest Laplacian eigenvalues of graphs, On upper and lower bounds for the second largest Laplacian eigenvalue of graphs, what can we say? Up to now, there are just a few results on it. Firstly, Zhang and Li in [98] investigated the second largest Laplacian eigenvalue of a tree. They obtained the upper bound in terms of the number of vertices and characterized all extremal graphs which attained the upper bound.

Theorem 4.1 *([98]) Let T be a tree of order n. Denote by $\lceil x \rceil$ the smallest integer no less than x. Then*

$$\lambda_2(T) \leq \left\lceil \frac{n}{2} \right\rceil \tag{40}$$

with equality if and only if n is even and T is obtained joining one edge from any one vertex to another vertex between the two copies star graphs $K_{1,\frac{n}{2}-1}$.

Using the relations between graph partition and the Laplacian eigenvalue and Cauch-Poincare separation theorem, Li and Pan in [55] showed the the second largest Laplacian eigenvalue of a graph is at least its second largest degree.

Theorem 4.2 *([55]) Let G be a simple connected graph with $n \geq 3$ vertices. Denote by d_2 the second largest degree of G. Then*

$$\lambda_2(G) \geq d_2 \tag{41}$$

with equality if G is a complete bipartite graph.

Das in [18] studied the Laplacian eigenvalues of induced subgraph of a graph obtained from the vertices of two vertices with the largest two degrees and their neighbors. Basing these properties and Cauch-Poincare separation theorem, He improved Li and Pan's lower bound.

Theorem 4.3 *([18]) Let G be a simple connected graph with at least three vertices. Denote by $d_1 = d(u)$ and $d_2 = d(v)$ the largest and second largest degree of G, respectively, and $c_{uv} = |N(u) \cap N(v)|$. Then*

$$\lambda_2(G) \geq \begin{cases} \frac{d_2+2+\sqrt{(d_2-2)^2+4c_{uv}}}{2}, & \text{if } (u,v) \in E \\ \frac{d_2+1+\sqrt{(d_2+1)^2-4c_{uv}}}{2}, & \text{if } (u,v) \notin E. \end{cases} \tag{42}$$

For most upper and lower bounds for the largest Laplacian eigenvalues, we are able to characterize all extremal graphs which attain their bounds. For the same season, we also expect to characterize all extremal graphs which achieve this lower bounds. Although (42) is better than (41), it is still not able to help us to find all extremal graphs which attain the lower bound (41). Pan and Hou in [79] gave the two necessary conditions for graphs with the second largest Laplacian eigenvalue equal to the second largest degree.

Theorem 4.4 *([79]) Let G be a simple connected graph of order $n \geq 3$ other than the star graph. Denote by $d_1 = d(u)$ and $d_2 = d(v)$ the largest and second largest degree of G, respectively. Assume that $\lambda_2(G) = d_2$.*
  (1) If $(u,v) \in E(G)$, then $N(u) = N(v)$.
  (2) If $(u,v) \notin E(G)$, then $N(u) \cap N(v) = d_1 = d_2 = \frac{n}{2}$.

On the other hand, there are many graphs whose second largest Laplacian eigenvalue is equal to its second largest degree, for example, double star graphs which is obtained from joining a new edge from the centers of two star graphs, etc. Basing the above situation, Li *et al.* in [60] proposed the following question:

Question 4.5 *([60]) Characterize all extremal graphs such that its second largest Laplacian eigenvalue is equal to its second largest degree.*

5 The $k-$th largest Laplacian eigenvalue

In this section, we consider some upper and lower bounds for the $k-$th largest Laplacian eigenvalues of graphs or trees. Zhang and Li in [99] gave the upper and lower bounds for the $k-$th largest Laplacian eigenvalues of graphs in terms of the number of vertices, edges and the number of spanning trees.

Theorem 5.1 *([99]) Let G be a simple connected graph of order n with m edges. Denote by $M(G) = \min\{m((n-4)m + 2(n-1)), 2m(n(n-1) - 2m)\}$. Then for $k = 1, \cdots, n-1$,*

$$\lambda_k(G) \leq \frac{1}{n-1}\left\{2m + \sqrt{\frac{n-k-1}{k}M(G)}\right\} \tag{43}$$

with equality in (43) for some $1 \leq k_0 \leq n-1$ if and only if G is the complete graph or star graph.

Proof. Clearly,

$$Tr(L(G)^2) = \sum_{i=1}^{k}\lambda_i^2 + \sum_{i=k+1}^{n-1}\lambda_i \geq \frac{(\sum_{i=1}^{k}\lambda_i)^2}{k} + \frac{(\sum_{i=k+1}^{n-1}\lambda_i)^2}{n-k-1}.$$

Let $\varphi_k = \sum_{i=1}^{k}\lambda_i$. Then

$$Tr(L(G)^2) \geq \frac{\varphi_k^2}{k} + \frac{(2m-\varphi_k)^2}{n-k-1}$$

which implies

$$\lambda_k \leq \frac{\varphi_k}{k} \leq \frac{1}{n-1}\left\{2m + \sqrt{\frac{n-k-1}{k}[(n-1)Tr(L(G)^2) - 4m^2]}\right\}.$$

21

We observe that

$$(n-1)Tr(L(G)^2)-4m^2 = (n-1)\sum_{v\in V} d(v)^2+2m(n-1)-4m^2 \leq m((n-4)m+2(n-1))$$

and

$$(n-1)Tr(L(G)^2)-4m^2 \leq (n-1)\sum_{v\in V} d(v)(n-1)+2m(n-1)-4m^2 = 2m(n(n-1)-2m),$$

since $d(v) \leq n-1$. Hence (43) holds. ∎

Next, Zhang and Li in [99] used the number of spanning trees and edges to obtain the lower bounds for the $k-$th largest Laplacian eigenvalues of graphs.

Theorem 5.2 ([99]) *Let G be a simple connected graph of order n with m edges. Denote by τ the number of spanning trees of G. Then*

$$\lambda_k \geq \frac{1}{n-k}\left\{(n-1)(2^{n-k}n\tau)^{\frac{1}{n-1}} - 2m\right\}. \tag{44}$$

If G is a strongly regular graph on the parameters $(a^2, 2(a-1), a-2, 2)$, equality in (44) holds for $k = (a-1)^2 + 1$.

From [35] and [55], we have that $\lambda_1(G) \geq d_1 + 1$ and $\lambda_2(G) \geq d_2$ if $d_1 \geq d_2 \geq \cdots \geq d_n$ are degree sequence of G. These results arise a question what about the relations between the k-th largest Laplacian eigenvalue and the $k-$th largest degree. Guo in [40] found that in general $\lambda_k \geq d_k$ does not hold. But he showed that the following inequality.

Theorem 5.3 ([40]) *Let G be a simple connected graph with at least four vertices. Denote by d_3 the third largest degree of G. Then*

$$\lambda_3(G) \geq d_3 - 1. \tag{45}$$

Basing his result and observing, he proposed the following conjecture:

Conjecture 5.4 ([40]) *Let G be a simple connected graph of order n. Denote by d_k the $k-$th largest Laplacian eigenvalue of G. Then*

$$\lambda_k(G) \geq d_k - k + 2, \quad \text{for all } k = 1, \cdots, n-1. \tag{46}$$

22

Recently, Wang et al. in [91] confirmed this conjecture and characterized all extremal graphs which attain the lower bounds.

From (40), we may obtain $\lambda_k(G) \leq \lceil \frac{n}{k} \rceil$ for a tree of order if $k = 1, 2$. It is natural to expect whether the result is able to generalize for any k. Recently, Guo [39] followed this idea and showed the following:

Theorem 5.5 *([39]) Let T be a tree of order n. Then*

$$\lambda_k(T) \leq \left\lceil \frac{n}{k} \right\rceil \quad \text{for } 1 \leq k \leq n - 1 \tag{47}$$

with equality if and only if $k | n$ and T is spanned by k vertices disjoint copies of the star graph $K_{1, \frac{n}{k} - 1}$.

6 The second smallest Laplacian eigenvalue

In 1973, Fiedler in [29] called the second smallest Laplacian eigenvalue the algebraic connectivity of a graph, since it is a good parameter to measure, to a certain extent, how well a graph is connected. For example, The second smallest eigenvalue is positive if and only if G is connected. Moreover, the eigenvectors corresponding to the algebraic connectivity are called *Fiedler vectors* (see,[30], [53], [54]). Recently, there is an excellent survey on algebraic connectivity of graphs written by de Abreu [1]. One of the earliest result may be is due to Fielder [29]

Theorem 6.1 *([29]) Let G be a simple graph of order n other than a complete graph with vertex connectivity $\kappa(G)$ and edge connectivity $\kappa'(G)$. Then*

$$2\kappa'(G)(1 - \cos(\pi/n) \leq \lambda_{n-1}(G) \leq \kappa(G) \leq \kappa'(G). \tag{48}$$

It is natural to investigate all extremal graphs which attain the bound in (48). In order to characterize all extremal graphs with $\lambda_{n-1}(G) \leq \kappa(G) \leq \kappa'(G)$, we recall the definitions of the union and join of graphs. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two disjoint graphs. The *union* of G_1 and G_2 is $G_1 + G_1 = (V_1 \cup V_2, E_1 \cup E_2)$ and the *join* $G_1 \vee G_2$ of G_1 and G_2 is a graph from $G_1 + G_2$ by adding new edges from each vertex in G_1 to every vertex of G_2. Kirkland et al. [53] obtained the necessary and sufficient conditions for the second smallest eigenvalue equal to the vertex connectivity.

23

Theorem 6.2 *([53]) Let G be a simple connected graph or order n rather than a complete graph. Then $\lambda_{n-1}(G) = \kappa(G)$ if and only if G can be written as $G_1 \vee G_2$, where G_1 is a disconnected graph of order $n - \kappa(G)$ and G_2 is a graph of order $\kappa(G)$ with $\lambda_{\kappa(G)-1}(G_2) \geq 2\kappa(G) - n$.*

Now we present some new results which are not appeared in [1]. A *dominating set* in G is a subset U of $V(G)$ such that each vertex in $V(G) - U$ is adjacent to at least one vertex of U. The *domination number* $\gamma(G)$ is the minimum size of a dominating set in G. Lu et al. in [62] gave an upper bound for the second smallest Laplacian eigenvalue in terms of the domination number.

Theorem 6.3 *([62]) Let G be a simple connected graph of order $n \geq 2$. Then*

$$\lambda_{n-1}(G) \leq \frac{n(n - 2\gamma(G) + 1)}{n - \gamma(G)} \tag{49}$$

with equality if and only if G is the complete bipartite graph $K_{2,2}$.

Recently, Nikiforov in [77] gave another upper bound.

Theorem 6.4 *([77])(Let G be a simple connected graph other than a complete graph. Then*

$$\lambda_{n-1}(G) \leq n - \gamma(G). \tag{50}$$

We notice that (49) and (50) are not comparable. Another important graph parameter is diameter. There are several results on the upper and lower bounds for the second smallest Laplacian eigenvalue in terms of diameter of G. The reader may refer to [1]. In here, we only present an up-to-date result by Lu et al. [63].

Theorem 6.5 *([63]) Let G be a simple connected graph of order n with m edges and diameter $diam(G)$. Then*

$$\lambda_{n-1}(G) \geq \frac{2n}{2 + n(n - 1)(diam(G)) - 2m(diam(G))} \tag{51}$$

with equality if and only if G is a path of order 3 or a complete graph.

For trees, we gave an upper bound for the second smallest Laplacian eigenvalue in terms of the independence number $\alpha(G)$. Zhang in [97] proved the following:

Theorem 6.6 ([97]) Let T be a tree of order n with the independence number $\alpha(T)$. If T is not the star graph $K_{1,n-1}$ or $T_{n,n-2}$, then

$$\lambda_{n-1}(T) \leq \frac{3 - \sqrt{5}}{2} \tag{52}$$

with if and only if T is $T_{n,\alpha(T)}$

By a simple calculation, we have following corollary due to Grone et al. [36]

Corollary 6.7 ([36]) Let T be a tree of order $n \geq 6$ other than the star graph $K_{1,n-1}$. Then $\lambda_{n-1}(T) < 0.49$.

Merris in [69] introduced the *doubly stochastic matrix* of a graph which is defined to be $\Omega(G) = (\omega_{ij}) = (I + L(G))^{-1}$. Denote by $\omega(G) = \min\{\omega_{ij} \mid 1 \leq i, j \leq n\}$. In the study of relations between smallest entry of this doubly stochastic matrix and the algebraic connectivity. In 1998, Merris [71] proposed the following two conjectures.

Conjecture 6.8 ([71]) Let G be a graph on n vertices. Then

$$\lambda_{n-1}(G) \geq 2(n+1)\omega(G). \tag{53}$$

Conjecture 6.9 ([71]) Let E_n be the degree anti-regular graph, that is, the unique connected graph whose vertex degrees attain all values between 1 and $n - 1$. Then

$$\omega(E_n) = \frac{1}{2(n+1)}. \tag{54}$$

In 2000, Berman and Zhang [6] confirmed Conjecture 6.9. Recently, Zhang and Wu in [105] firstly obtained sharp upper and lower bounds for the smallest entries of doubly stochastic matrices of trees, which is used to disprove Conjecture 6.8. Hence we may propose the following question:

Question 6.10 *What is the best lower bound for the algebraic connectivity in terms of the vertex number and the smallest entry of the doubly stochastic matrix of a graph?*

7   The sum of the Laplacian eigenvalues

Before presenting some results, we need to recall some notations. If $(a) = (a_1, a_2, \cdots, a_r)$ and $(b) = (b_1, b_2, \cdots, b_s)$ are nonincreasing sequences of real number, then (a) *majorizes* (b), denoted by $(a) \succeq (b)$, if

$$\sum_{i=1}^{k} a_i \geq \sum_{i=1}^{k} b_i, \quad \text{for } k = 1, 2, \cdots, \min\{r, s\}$$

and
$$\sum_{i=1}^{r} a_i = \sum_{i=1}^{s} b_i.$$

Moreover, if (a) is a integer nonincreasing sequence, denote by $(a)^* = (a_1^*, a_2^*, \cdots, a_t^*)$ the *conjugate* sequence of (a), where a_i is the cardinality of the set $\{j \mid a_j \geq i\}$.

Since $L(G)$ is positive semidefinite, it follows from Schur's theorem (see [64]) that the Laplacian eigenvalues of a graph majorizes the degree sequence (when both are arranged in nonincreasing order). It is not surprising that such a result should be, to some extent, improved upon restriction to the class of the Laplacian matrices. Grone and Merris in [35] proposed the following two conjectures on the Laplacian eigenvalues.

Conjecture 7.1 *([35]) Let G be a connected graph of order $n \geq 2$ with nonincreasing degree sequence $(d_1, d_2, \cdots, d_n)$. Then*

$$(\lambda_1(G), \lambda_2(G), \cdots, \lambda_{n-1}(G)) \succeq (d_1 + 1, d_2, \cdots, d_n - 1). \tag{55}$$

Conjecture 7.2 *([35]) Let G be a connected graph of order $n \geq 2$ with nonincreasing degree sequence $(d_1, d_2, \cdots, d_n)$. Then*

$$(\lambda_1(G), \lambda_2(G), \cdots, \lambda_{n-1}(G)) \preceq (d_1^*, d_2^*, \cdots, d_n^*). \tag{56}$$

On Conjecture 7.1, Grone and Merris in [35] showed the part result on this conjecture.

Theorem 7.3 *([35]) Let $G = (V, E)$ be a connected graph of order $n > 2$. If the induced subgraph by subset U of V with $|U| = k$ contains r pair disjoint edges, then*

$$\sum_{i=1}^{k} \lambda_i(G) \geq \sum_{u \in U} d(u) + k - r \tag{57}$$

Further, using M-matrix theory and graph structure, Grone in [33] confirmed Conjecture 7.1. However, it seems to be difficult to prove Conjecture 7.2. In [35], Grone and Merris only showed that $\lambda_{n-1}(G) \geq d_{n-1}^*$, in other words, the first and last inequalities in the majorization inequality hold. In 2002, Duval and Reiner [20] investigated the combinatorial Laplace operators associated to the boundary maps in a shifted simplicial complex. They proposed a generalization of Conjecture 7.2 and only proved the following :

26

Theorem 7.4 *([20]) Let G be a connected graph with the nonincreasing degree sequence $(d_1^*, \cdots, d_n^*)$. Then*

$$\lambda_1(G) + \lambda_2(G) \leq d_1^* + d_2^* \tag{58}$$

Moreover, there are more and more evidence to indicate that Conjecture 7.2 may hold. For example, Merris in [67] studied the relations between spectra and structure for a class of graphs which are called *degree maximal graphs* and found that $(\lambda_1(G), \cdots, \lambda_n(G)) = (d_1^*, \cdots, d_n^*)$. In other words, equality in Conjecture 7.2 holds. Hammer and Klemans in [43] investigated the question of which graphs have integer spectra and found that the *threshold graphs* are Laplacian integer. In fact, the degree maximal graphs are exactly the threshold graphs. It is known that Conjecture 7.2 holds for regular graphs and nearly regular graphs whose vertices have degree either k or $k-1$. In 2004, Stephen [86] showed that Conjecture 7.2 holds for trees. However, up to now, this Conjecture has still not been proved or disproved.

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