Eigen Calculations Using the Jacobi Iteration Method

How are eigenvalues and eigenvectors calculated in practice?
Look at simple case when A is symmetric.
Jacobi method circa 1845.

Orthogonal matrices are essential. Q is orthogonal if $Q^T = Q^{-1}$

Examples: $Q = \begin{bmatrix} c & s \\ -s & c \end{bmatrix}$ when $c^2 + s^s = 1$

$$Q_1 = \begin{bmatrix} c & s & 0 \\ -s & c & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} Q & 0 \\ 0 & 1 \end{bmatrix}$$ use block matrix calculations

Suppose Q is orthogonal and P is a permutation matrix
Then P^TQP is an orthogonal matrix.

$$Q_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c & s \\ 0 & -s & c \end{bmatrix}$$ obtained from Q_1 using a permutation matrix

Basic ideas from earlier.
If A is symmetric then Q^TAQ is also.
If Q is orthogonal, then Q^TAQ and A have the same eigenvalues.

The basic idea is to choose special orthogonal matrices that zero out specified off-diagonal elements. Givens rotations are one choice(see page 104, problem 23).

$$A = \begin{bmatrix} 5 & -2 & 0 \\ -2 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

and $Q_1 = \begin{bmatrix} c & s & 0 \\ -s & c & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is a Givens rotation designed to zero out the (2,1) element.

$$A_1 = Q_1^T A Q_1 = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

The columns of Q are eigenvectors and the eigenvalues are on the diagonal.

Rotations that zero out a specified element for this note are called Givens rotations after the scientist at Oak Ridge who pioneerred their use more than fifty years ago.

Related orthogonal transformations that zero out a portion of a column are called Householder transformations.

Start with $A = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 4 \end{bmatrix}$ and construct a Givens rotation Q_1that zeros out the (2,1) element in then sense that the (2,1) element of $A_1 = Q_1^T A Q_1$ is zero.

Then find a Givens rotation Q_2 that zeros out the (3,1) element of A_1 in the same manner $A_2 = Q_2^T A_1 Q_2$. Proceed with another rotation Q_3 that zeros out the (3,2) element of A_2 and set $A_3 = Q_3^T A_2 Q_3$. Start over again with the (2,1) location for the latest iterate and continue the process.

The off-diagonal elements go to zero after several iterations leaving an essentially diagonal matrix of eigenvalues.
After p iterations, we have

$$A_p = Q_p^T A_{p-1} Q_p \text{ or } A_p = Q^T A Q$$

for the orthogonal matrix $Q = Q_1 Q_2 \cdots Q_{p-1} Q_p$.

After several sweeps, A_p is almost diagonal. This means that A is orthogonally similar to a diagonal matrix A_p which holds the eigenvalues of A. Q holds the corresponding eigenvectors of A.

```
The symmetric matrix A is

      3      1      2
      1      2      1
      2      1      4

Begin the sweep process.
Zero out spot (2,1)
    Givens rotation
0.8507 -0.5257  0.0000
0.5257  0.8507  0.0000
0.0000  0.0000  1.0000
*** *** *** *** *** ***
         Q'*A*Q           |            Q
3.6180  0.0000  2.2270  |  0.8507 -0.5257  0.0000
0.0000  1.3820 -0.2008  |  0.5257  0.8507  0.0000
2.2270 -0.2008  4.0000  |  0.0000  0.0000  1.0000
*** ***


Zero out spot (3,1)
     Givens rotation
 0.7367  0.0000  0.6762         current product
 0.0000  1.0000  0.0000        of Givens rotations
-0.6762  0.0000  0.7367                |
*** *** *** *** *** ***               \|/
         Q'*A*Q           |             Q
1.5738  0.1358  0.0000  |   0.6267 -0.5257  0.5752
0.1358  1.3820 -0.1479  |   0.3873  0.8507  0.3555
0.0000 -0.1479  6.0442  |  -0.6762  0.0000  0.7367
*** ***

Notice how pushing an element to zero lets a
previously zeroed element to reappear.
```

```
Zero out spot (3,2)
   Givens rotation
1.0000   0.0000   0.0000
0.0000   0.9995  -0.0317
0.0000   0.0317   0.9995
*** *** *** *** *** ***
        Q'*A*Q              |              Q
 1.5738  0.1357 -0.0043  |   0.6267 -0.5072  0.5916
 0.1357  1.3773  0.0000  |   0.3873  0.8615  0.3284
-0.0043  0.0000  6.0489  |  -0.6762  0.0233  0.7363
*** ***

One sweep is complete.
Absolute sum of off-diagonal elements = 2.800552e-01

Another sweep?y/n-->y
Zero out spot (2,1)
   Givens rotation
0.8906  -0.4547   0.0000
0.4547   0.8906   0.0000
0.0000   0.0000   1.0000
*** *** *** *** *** ***
        Q'*A*Q              |              Q
 1.6431  0.0000 -0.0038  |   0.3275 -0.7367  0.5916
 0.0000  1.3080  0.0020  |   0.7367  0.5911  0.3284
-0.0038  0.0020  6.0489  |  -0.5917  0.3283  0.7363
*** ***

Zero out spot (3,1)
   Givens rotation
1.0000   0.0000  -0.0009
0.0000   1.0000   0.0000
0.0009   0.0000   1.0000
*** *** *** *** *** ***
        Q'*A*Q             |              Q
1.6431  0.0000  0.0000  |   0.3280 -0.7367  0.5913
0.0000  1.3080  0.0020  |   0.7370  0.5911  0.3277
0.0000  0.0020  6.0489  |  -0.5910  0.3283  0.7368
*** ***
```

```
Zero out spot (3,2)
   Givens rotation
1.0000  0.0000  0.0000
0.0000  1.0000  0.0004
0.0000 -0.0004  1.0000
*** *** *** *** *** ***
       Q'*A*Q          |          Q
1.6431  0.0000  0.0000  |   0.3280 -0.7370  0.5910
0.0000  1.3080  0.0000  |   0.7370  0.5910  0.3280
0.0000  0.0000  6.0489  |  -0.5910  0.3280  0.7370
*** ***

2 sweeps are complete.
Absolute sum of off-diagonal elements = 3.404415e-06

Another sweep?y/n-->y
Zero out spot (2,1)
   Givens rotation
1.0000 -0.0000  0.0000
0.0000  1.0000  0.0000
0.0000  0.0000  1.0000
*** *** *** *** *** ***
       Q'*A*Q          |          Q
1.6431  0.0000  0.0000  |   0.3280 -0.7370  0.5910
0.0000  1.3080 -0.0000  |   0.7370  0.5910  0.3280
0.0000 -0.0000  6.0489  |  -0.5910  0.3280  0.7370
*** ***


Zero out spot (3,1)
    Givens rotation
 1.0000  0.0000  0.0000
 0.0000  1.0000  0.0000
-0.0000  0.0000  1.0000
*** *** *** *** *** ***
       Q'*A*Q          |          Q
1.6431  0.0000  0.0000  |   0.3280 -0.7370  0.5910
0.0000  1.3080 -0.0000  |   0.7370  0.5910  0.3280
0.0000 -0.0000  6.0489  |  -0.5910  0.3280  0.7370
*** ***
```

```
Zero out spot (3,2)
   Givens rotation
1.0000  0.0000  0.0000
0.0000  1.0000 -0.0000
0.0000  0.0000  1.0000
*** *** *** *** *** ***
        Q'*A*Q            |             Q
 1.6431  0.0000 -0.0000   |    0.3280 -0.7370  0.5910
 0.0000  1.3080  0.0000   |    0.7370  0.5910  0.3280
-0.0000  0.0000  6.0489   |   -0.5910  0.3280  0.7370
*** ***

3 sweeps are complete.
Absolute sum of off-diagonal elements = 1.136271e-24

Another sweep?y/n-->n
Diagonal matrix D of approximate eigenvalues

    1.6431    0.0000   -0.0000
    0.0000    1.3080         0
   -0.0000         0    6.0489

Matrix Q of corresponding approximate eigenvectors

    0.3280   -0.7370    0.5910
    0.7370    0.5910    0.3280
   -0.5910    0.3280    0.7370
```

```
*****************************************************
*** *** ***      Summary Information      *** *** ***
*****************************************************
```

```
At termination, 3 sweeps were completed.
D should be diagonal and it is
    1.6431    0.0000   -0.0000
    0.0000    1.3080         0
   -0.0000         0    6.0489

Orthogonal Q is
    0.3280   -0.7370    0.5910
    0.7370    0.5910    0.3280
   -0.5910    0.3280    0.7370
```

What about MATLAB?

```
»[Q,D] = eig(A);
»disp(Q)
    0.7370   -0.3280    0.5910
   -0.5910   -0.7370    0.3280
   -0.3280    0.5910    0.7370

»disp(D)
    1.3080         0         0
         0    1.6431         0
         0         0    6.0489
```

The results are basically the same but the Q's are different. This is acceptable because MATLAB used a somewhat different algorithm.