Matrix Factorization

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date: 7/27/2019

1. About the Data

The data is from a part of “EachMovie” dataset (http://www.cs.cmu.edu/~lebanon/IR-lab/data.html (http://www.cs.cmu.edu/~lebanon/IR-lab/data.html))

Analyze the rating data of 1623 movies by 1000 users.

Make a recommendation system to offer best movies for each user by matrix factorization.

The data is like bellow:

In [1]:

```
import pandas as pd
import matplotlib.pyplot as plt
import numpy as np
import warnings
warnings.simplefilter(action='ignore', category=FutureWarning)
```

In [2]:

```
#import data
df=pd.read_table("/Users/daigofujiwara/Documents/授業資料/論理生命学/MatrixFactorization/mov
print("(row,col)=",df.shape)
df.head()
```

(row,col)= (1000, 1623)

Out[2]:

	0	1	2	3	4	5	6	7	8	9	...	1613	1614	1615	1616	1617	1618	1619	1620	1621	16
0	0	0	0	0	0	0	4	0	0	0	...	0	0	0	0	0	0	0	0	0	
1	2	0	0	0	0	6	5	0	0	0	...	0	0	0	0	0	0	0	0	0	
2	0	0	0	0	0	0	5	0	0	0	...	0	0	0	0	0	0	0	0	0	
3	0	0	4	0	0	0	6	0	0	0	...	0	0	0	0	0	0	0	0	0	
4	0	2	0	0	0	0	0	0	4	0	...	0	0	0	0	0	0	0	0	0	

5 rows × 1623 columns

```

```

2. Formulation of Method1

I will explain a formulation of the recommendation problem and a solution using matrix factorization, using the following notations: r_{ij} is the rating score of item i given by user j. Ω is the set of the indices of observed ratings, i.e. r_{ij} is observed if $(i, j) \in \Omega$

First, in recomendation, usually the size of dimension and sample data is large, and rating score matrix is sparse, so we need feature reduction.Feature reduction model is expressed as bellow:

$$\boldsymbol{R} = \boldsymbol{U}^T V$$

$$\{\boldsymbol{R}\}_{ij} = r_{ij}, \boldsymbol{R} \in \mathbb{R}^{m \times n}, \boldsymbol{U} \in \mathbb{R}^{k \times m}, \boldsymbol{V} \in \mathbb{R}^{k \times n}, \min(m, n) \geq k$$

Then naively it is come up with the idea of interpretting un-observed value to zero, and by SVD, reducing dimension of rating score matrix. However, in this idea, we thought un-observed value of rating score as zero, but in fact, un-obseved value is just ”un-observed” and not neessarily zero, so this model is not better.

Alternartively, there is some way filling un-observed value by regression or minimizing error. In this method, U,V is determined based on only observed score, minimizing this loss function:

$$\min_{U,V} L = \min_{U,V} \left(\sum_{(i,j) \in \Omega} \left(r_{ij} - \boldsymbol{u}_i^\top \boldsymbol{v}_j \right)^2 + \lambda_1 \|\boldsymbol{U}\|_F^2 + \lambda_2 \|\boldsymbol{V}\|_F^2 \right)$$

where,

$$\boldsymbol{U} = \left[\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_m \right]$$
$$\boldsymbol{V} = \left[\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n \right]$$

Minimizing this loss is done by these alternative update.

$$\boldsymbol{u}_i = \left(\sum_{(i,j) \in \Omega} \boldsymbol{v}_j \boldsymbol{v}_j^\top + \lambda_1 \boldsymbol{I} \right)^{-1} \sum_{(i,j) \in \Omega} r_{ij} \boldsymbol{v}_j \tag{1}$$

$$\boldsymbol{v}_j = \left(\sum_{(i,j) \in \Omega} \boldsymbol{u}_i \boldsymbol{u}_i^\top + \lambda_2 \boldsymbol{I} \right)^{-1} \sum_{(i,j) \in \Omega} r_{ij} \boldsymbol{u}_i \tag{2}$$

Each update formula is derived from Loss's partial differential.

$$L = \sum_{(i,j) \in \Omega} \left(r_{ij} - \boldsymbol{u}_i^\top \boldsymbol{v}_j \right)^2 + \lambda_1 \|\boldsymbol{U}\|_F^2 + \lambda_2 \|\boldsymbol{V}\|_F^2$$

$$\frac{\partial L}{\partial \boldsymbol{u}_i} = 2 \sum_{(i,j) \in \Omega} (\boldsymbol{u}_i^T \boldsymbol{v}_j - r_{ij}) \boldsymbol{v}_j + 2\lambda_1 \boldsymbol{u}_i = 0$$

$$\therefore \left(\sum_{(i,j) \in \Omega} \boldsymbol{v}_j \boldsymbol{v}_j^T + \lambda_1 \boldsymbol{I} \right) \boldsymbol{u}_i = \sum_{(i,j) \in \Omega} r_{ij} \boldsymbol{v}_j$$

$$\therefore \boldsymbol{u}_i = \left(\sum_{(i,j) \in \Omega} \boldsymbol{v}_j \boldsymbol{v}_j^\top + \lambda_1 \boldsymbol{I} \right)^{-1} \sum_{(i,j) \in \Omega} r_{ij} \boldsymbol{v}_j \tag{1}$$

And as well, from $\frac{\partial L}{\partial \boldsymbol{v}_j}$, formula (2) is derived.

In [4]:

```
Rmasked = df.values.copy()
Rmasked = Rmasked.T
Rmasked[:100,:100]=np.zeros((100,100))

class MatrixFactorization:
    def __init__(self,R):
        self.R=R.copy()
        #parameter
        self.M=R.shape[0]
        self.N=R.shape[1]
        ##initializing matrix
        self.R_exist = np.where(R!=0)
        self.R_exist = np.array([self.R_exist[0],self.R_exist[1]])

    def fit(self,K,noprint=1):
        ##initializing matrix
        U=np.random.rand(self.M,K)
        V=np.random.rand(self.N,K)
        #calculate loss
        loss=np.inf
        ##regularization coefficient
        lambda0=1
        lambda1=lambda0
        lambda2=self.M/self.N*lambda0

        self.Loss=[]

        k=0
        if not noprint:
            print("loss=")
        while 1:
            for i in range(self.M):
                #extract observed area in i row
                arr=self.R_exist[1,(self.R_exist==i)[0]]
                temp1=np.zeros((K,K))
                temp2=np.zeros(K)
                for j in arr:
                    temp1=temp1+np.dot(V[j].reshape(V[j].shape[0],1),V[j].reshape(1,V[j].shape[0]))
                    temp2=temp2+self.R[i,j]*V[j]
                inv=np.linalg.inv(temp1+lambda1*np.eye(K))
                U[i]=np.dot(temp2,inv)

            temp3=0

            for j in range(self.N):
                #extract observed area in j colm
                arr=self.R_exist[0,(self.R_exist==j)[1]]
                temp1=np.zeros((K,K))
                temp2=np.zeros(K)
                for i in arr:
                    temp1=temp1+np.dot(U[i].reshape(U[i].shape[0],1),U[i].reshape(1,U[i].shape[0]))
                    temp2=temp2+self.R[i,j]*U[i]
                    #calculate loss
                    temp3+=np.square(self.R[i,j]-np.dot(U[i],V[j]))
                inv=np.linalg.inv(temp1+lambda2*np.eye(K))
                V[j]=np.dot(temp2,inv)

            pre=loss

```

```
            loss=temp3+lambda1*np.linalg.norm(U,ord=2)*np.linalg.norm(U,ord=2) \
            +lambda2*np.linalg.norm(V,ord=2)*lambda2*np.linalg.norm(V,ord=2)

            self.Loss.append([loss,k])

            if pre-loss<0.1 or k>1000:
                break
            if not noprint:
                if not (k%10):
                    print(loss,", iter=",k)
            k+=1

        if not noprint:
            print("finished !")
        self.U=U
        self.V=V
        self.Loss=np.array(self.Loss)
        self.Loss=self.Loss.T
        self.Rest=np.dot(U,V.T)
        return self.Rest
```

3. Implementation

3.1 Confirmation the decrease of loss

Confirming the decrease of loss, using reduced dimension k=5 model and showing the graph of loss decrease for each iteration step.

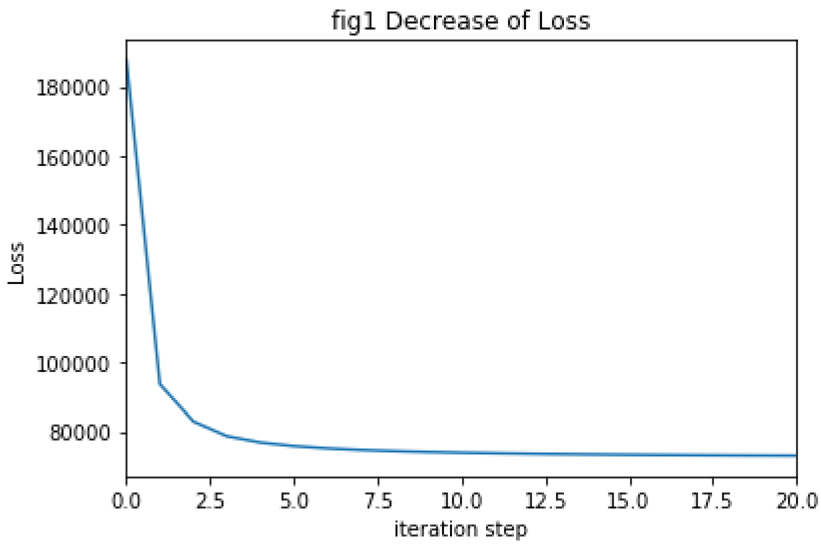
In [5]:

```
mf=MatrixFactorization(Rmasked)
Rest=mf.fit(5,noprint=0)
```

```
loss=
187724.70798117723 , iter= 0
73837.51096647854 , iter= 10
72980.55351907827 , iter= 20
72805.70560859605 , iter= 30
72780.94787997485 , iter= 40
finished !
```

In [6]:

```
plt.title("fig1 Decrease of Loss")
plt.plot(mf.Loss[1],mf.Loss[0])
plt.xlabel("iteration step")
plt.xlim(0,20)
plt.ylabel("Loss")
plt.show()
```



3.2 Evaluation

Hiding upper left 100x100 rating data, I estimate the rating matrix by ALS. After that, revealing the real value, I compare estimation value and real value. I use RMSE as the evaluation criteria.

$$RMSE = \sqrt{\frac{1}{N_\Omega} \sum_{(i,j)\in\Omega} \left(r_{ij} - u_i^\top v_j\right)^2}$$

N_Ω is the number of elements of Ω. Ω is now limited to upper left 100x100 area.

And I compare the RMSEs of our model and Ramdom model.

(model dimension) k=5 method1 model

In thi part, we use our matrix factorization model as k=5 and calculate RMSE.

In [7]:

```
RealR=df.values.copy()
RealR=RealR.T
RealR=RealR[:100,:100]
R_estimate=mf.Rest[:100,:100]

R_exist100 = np.where(RealR!=0)
R_exist100 = np.array([R_exist100[0],R_exist100[1]])

err=0

for i in range(100):
    arr=R_exist100[1,(R_exist100==i)[0]]
    for j in arr:
        err+=np.square(RealR[i,j]-R_estimate[i,j])

MSE=err/R_exist100.shape[1]
RMSE=np.sqrt(MSE)
RMSE
```

Out[7]:

```
1.1292267112130494
```

RMSE is about 1.129

Random model

In this model,it is used uniform distribution in the section $[1, 6]$ for each value of estimation matrix.

In [8]:

```
R_uni=1+5*np.random.rand(100,100)
err=0
for i in range(100):
    arr=R_exist100[1,(R_exist100==i)[0]]
    for j in arr:
        err+=np.square(RealR[i,j]-R_uni[i,j])

MSE_uni=err/R_exist100.shape[1]
RMSE_uni=np.sqrt(MSE_uni)
RMSE_uni
```

Out[8]:

```
2.193987378539665
```

RMSE is about 2.194

small conlusion

It is concluded that this method is superior to random estimation.

3.3 Transition of each K

I made some models where model dimesion k is deifferent, observed the transition for each k, and estimte the best model.

In [9]:

```
RealR=df.values.copy()
RealR=RealR.T
RealR=RealR[:100,:100]
R_exist100 = np.where(RealR!=0)
R_exist100 = np.array([R_exist100[0],R_exist100[1]])
Ks=list([1,2,3,4,5,6,7,8,9,10])
Rests=[]
```

In [10]:

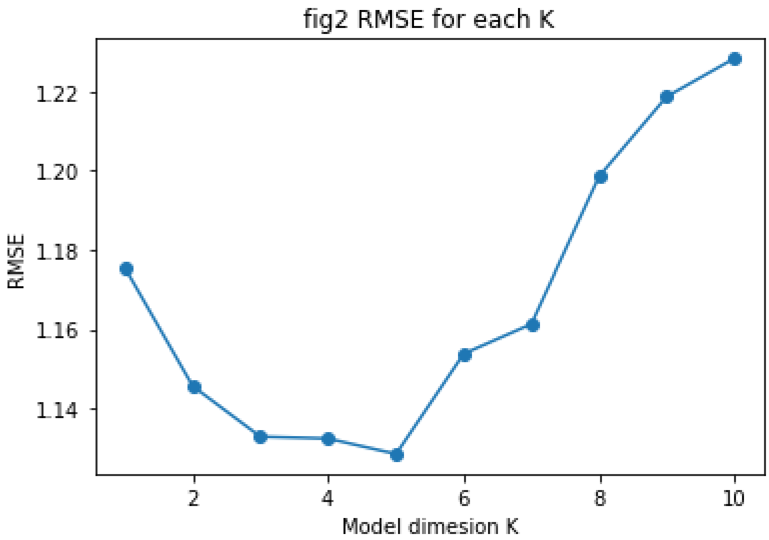
```
for k in Ks:
    Rests.append(mf.fit(k))
```

In [11]:

```
MSEs=np.array([])

for k in range(len(Rests)):
    err=0

    for i in range(100):
        arr=R_exist100[1,(R_exist100==i)[0]]
        for j in arr:
            err+=np.square(RealR[i,j]-Rests[k][i,j])

    MSEs=np.append(MSEs,err/R_exist100.shape[1])

RMSEs=np.sqrt(MSEs)
```

In [12]:

```
plt.title("fig2 RMSE for each K")
plt.plot(Ks,RMSEs,"o-")
plt.xlabel("Model dimesion K")
plt.ylabel("RMSE")
plt.show()
```



small conclusion

The best model is $k = 3$ or $k = 5$ model. And too large k makes the acuracy of model worse. It is very suprising because naively we think that more degree of freedom can make RMSE smaller.

Then, I will also show the acuracy (RMSE) for the trainnig data. In other words,

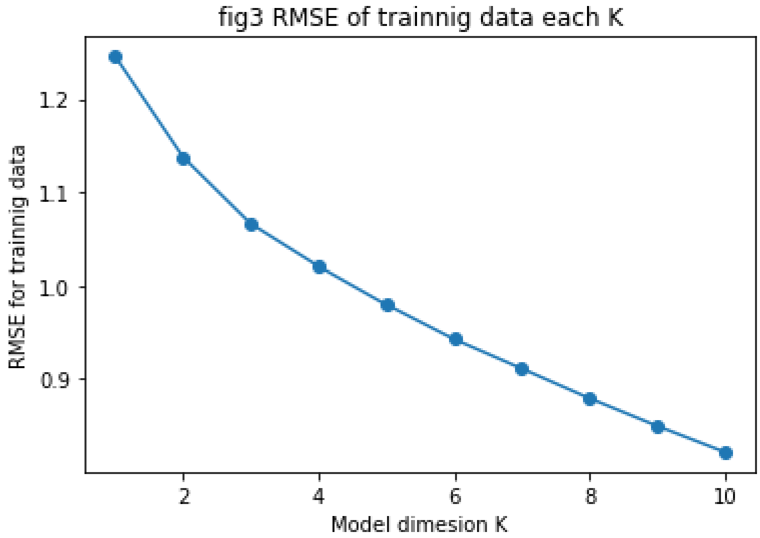
$$\overline{RMSE} = \sqrt{\frac{1}{N_{\overline{\Omega}}} \sum_{(i,j)\in\overline{\Omega}} \left(r_{ij} - \boldsymbol{u}_i^\top \boldsymbol{v}_j\right)^2}$$

In [13]:

```
RealR_=df.values.copy()
RealR_=RealR_.T
MSEs_=np.array([])

for k in range(len(Rests)):
    err=0

    for i in range(mf.M):
        arr=mf.R_exist[1,(mf.R_exist==i)[0]]
        for j in arr:
            err+=np.square(RealR_[i,j]-Rests[k][i,j])

    MSEs_=np.append(MSEs_,err/mf.R_exist.shape[1])

MSEs_=(MSEs_*mf.R_exist.shape[1]-MSEs*R_exist100.shape[1])/(mf.R_exist.shape[1]-R_exist1

RMSEs_=np.sqrt(MSEs_)
```

In [16]:

```
plt.title("fig3 RMSE of trainnig data each K")
plt.plot(Ks,RMSEs_,"o-")
plt.xlabel("Model dimesion K")
plt.ylabel("RMSE for trainnig data")
plt.show()
```



Seeing fig3, this RMSE for trainnnig data is smaller when the model dimension k is larger, but RMSE for test data is not. It is supposed to be because too much degree of freedom cause overfitting.