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Álgebra del operador nabla

Operadores de primer orden

Gradiente: $$\nabla\phi = \frac{\partial\phi}{\partial x}\mathbf{u}_x + \frac{\partial\phi}{\partial y}\mathbf{u}_y + \frac{\partial\phi}{\partial z}\mathbf{u}_z$$

Divergencia: $$\nabla\cdot\mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

Rotacional: $$\nabla\wedge\mathbf{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\mathbf{u}_x + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\mathbf{u}_y + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\mathbf{u}_z$$

Aplicación sobre productos

$$\begin{aligned}
\nabla(\phi\psi) &= \psi\,\nabla\phi + \phi\,\nabla\psi \\
\nabla\cdot(\phi\mathbf{A}) &= \nabla\phi\cdot\mathbf{A} + \phi\,\nabla\cdot\mathbf{A} \\
\nabla\wedge(\phi\mathbf{A}) &= \nabla\phi\wedge\mathbf{A} + \phi\,\nabla\wedge\mathbf{A} \\
\nabla\cdot(\mathbf{A}\wedge\mathbf{B}) &= (\nabla\wedge\mathbf{A})\cdot\mathbf{B} - (\nabla\wedge\mathbf{B})\cdot\mathbf{A} \\
\nabla\wedge(\mathbf{A}\wedge\mathbf{B}) &= \mathbf{A}(\nabla\cdot\mathbf{B}) + (\mathbf{B}\cdot\nabla)\mathbf{A} - \mathbf{B}(\nabla\cdot\mathbf{A}) - (\mathbf{A}\cdot\nabla)\mathbf{B} \\
\nabla(\mathbf{A}\cdot\mathbf{B}) &= \mathbf{A}\wedge(\nabla\wedge\mathbf{B}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{B}\wedge(\nabla\wedge\mathbf{A}) + (\mathbf{B}\cdot\nabla)\mathbf{A}
\end{aligned}$$

Operadores de segundo orden

$$\nabla\cdot(\nabla\phi) = \nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} \qquad (\textbf{Laplaciano})$$

$$\begin{aligned}
\nabla\wedge(\nabla\phi) &= 0 \\
\nabla\cdot(\nabla\wedge\mathbf{A}) &= 0 \\
\nabla\wedge(\nabla\wedge\mathbf{A}) &= \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}
\end{aligned}$$

Identidades de Green

1ª identidad: $\nabla\cdot(\phi\nabla\psi) = \nabla\phi\cdot\nabla\psi + \phi\nabla^2\psi$

2ª identidad: $\nabla\cdot(\phi\nabla\psi - \psi\nabla\phi) = \phi\nabla^2\psi - \psi\nabla^2\phi$

Resumen de fórmulas de coordenadas curvilíneas ortogonales

$x = \rho\cos\varphi = r\,\mathrm{sen}\,\theta\cos\varphi$	$\sqrt{x^2+y^2} = \rho = r\,\mathrm{sen}\,\theta$	$\sqrt{x^2+y^2+z^2} = \sqrt{\rho^2+z^2} = r$
$y = \rho\,\mathrm{sen}\,\varphi = r\,\mathrm{sen}\,\theta\,\mathrm{sen}\,\varphi$	$\mathrm{arctg}\,\dfrac{y}{x} = \varphi = \varphi$	$\mathrm{arctg}\,\dfrac{\sqrt{x^2+y^2}}{z} = \mathrm{arctg}\,\dfrac{\rho}{z} = \theta$
$z = z = r\cos\theta$	$z = z = r\cos\theta$	$\mathrm{arctg}\,\dfrac{y}{x} = \varphi = \varphi$

Vector de posición:
$\mathbf{r} = x\mathbf{u}_x + y\,\mathbf{u}_y + z\mathbf{u}_z$
$\mathbf{r} = \rho\,\mathbf{u}_\rho + z\mathbf{u}_z$
$\mathbf{r} = r\mathbf{u}_r$

Factores de escala:			
$h_i = \left\lvert \dfrac{\partial \mathbf{r}}{\partial q_i} \right\rvert$	$h_x = 1$	$h_y = 1$	$h_z = 1$
	$h_\rho = 1$	$h_\varphi = \rho$	$h_z = 1$
	$h_r = 1$	$h_\theta = r$	$h_\varphi = r\,\mathrm{sen}\,\theta$

Diferencial de longitud:	**Diferencial de volumen:**
$d\mathbf{r} = h_1\,dq_1\,\mathbf{u}_1 + h_2\,dq_2\,\mathbf{u}_2 + h_3\,dq_3\,\mathbf{u}_3$	$d\tau = h_1h_2h_3\,dq_1\,dq_2\,dq_3$
$d\mathbf{r} = dx\mathbf{u}_x + dy\,\mathbf{u}_y + dz\mathbf{u}_z$	$d\tau = dx\,dy\,dz$
$d\mathbf{r} = d\rho\,\mathbf{u}_\rho + \rho\,d\varphi\,\mathbf{u}_\varphi + dz\mathbf{u}_z$	$d\tau = \rho\,d\rho\,d\varphi\,dz$
$d\mathbf{r} = dr\mathbf{u}_r + r\,d\theta\,\mathbf{u}_\theta + r\,\mathrm{sen}\,\theta\,d\varphi\,\mathbf{u}_\varphi$	$d\tau = r^2\,\mathrm{sen}\,\theta\,dr\,d\theta\,d\varphi$

Delta de Dirac:

$$\delta(\mathbf{r}-\mathbf{r}') = \frac{\delta(q_1-q_1')\delta(q_2-q_2')\delta(q_3-q_3')}{h_1h_2h_3}$$

$$\delta(\mathbf{r}-\mathbf{r}') = \delta(x-x')\delta(y-y')\delta(z-z')$$

$$\delta(\mathbf{r}-\mathbf{r}') = \frac{\delta(\rho-\rho')\delta(\varphi-\varphi')\delta(z-z')}{\rho}$$

$$\delta(\mathbf{r}-\mathbf{r}') = \frac{\delta(r-r')\delta(\theta-\theta')\delta(\varphi-\varphi')}{r^2\,\mathrm{sen}\,\theta}$$

Gradiente:

$$\nabla\phi = \frac{1}{h_1}\frac{\partial\phi}{\partial q_1}\mathbf{u}_1 + \frac{1}{h_2}\frac{\partial\phi}{\partial q_2}\mathbf{u}_2 + \frac{1}{h_3}\frac{\partial\phi}{\partial q_3}\mathbf{u}_3$$

$$\nabla\phi = \frac{\partial\phi}{\partial x}\mathbf{u}_x + \frac{\partial\phi}{\partial y}\mathbf{u}_y + \frac{\partial\phi}{\partial z}\mathbf{u}_z$$

$$\nabla\phi = \frac{\partial\phi}{\partial\rho}\mathbf{u}_\rho + \frac{1}{\rho}\frac{\partial\phi}{\partial\varphi}\mathbf{u}_\varphi + \frac{\partial\phi}{\partial z}\mathbf{u}_z$$

$$\nabla\phi = \frac{\partial\phi}{\partial r}\mathbf{u}_r + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\mathbf{u}_\theta + \frac{1}{r\,\mathrm{sen}\,\theta}\frac{\partial\phi}{\partial\varphi}\mathbf{u}_\varphi$$

Vectores unitarios:	
$\mathbf{u}_i = \dfrac{1}{h_i}\dfrac{\partial\mathbf{r}}{\partial q_i}$	$\mathbf{u}_x = \cos\varphi\mathbf{u}_\rho - \mathrm{sen}\,\varphi\mathbf{u}_\varphi = \mathrm{sen}\,\theta\cos\varphi\mathbf{u}_r + \cos\theta\cos\varphi\mathbf{u}_\theta - \mathrm{sen}\,\varphi\mathbf{u}_\varphi$ $\mathbf{u}_y = \mathrm{sen}\,\varphi\mathbf{u}_\rho + \cos\varphi\mathbf{u}_\varphi = \mathrm{sen}\,\theta\,\mathrm{sen}\,\varphi\mathbf{u}_r + \cos\theta\,\mathrm{sen}\,\varphi\mathbf{u}_\theta + \cos\varphi\mathbf{u}_\varphi$ $\mathbf{u}_z = \mathbf{u}_z = \cos\theta\mathbf{u}_r - \mathrm{sen}\,\theta\mathbf{u}_\theta$
$\cos\varphi\mathbf{u}_x + \mathrm{sen}\,\varphi\mathbf{u}_y = \mathbf{u}_\rho = \mathrm{sen}\,\theta\mathbf{u}_r + \cos\theta\mathbf{u}_\theta$ $-\mathrm{sen}\,\varphi\mathbf{u}_x + \cos\varphi\mathbf{u}_y = \mathbf{u}_\varphi = \mathbf{u}_\varphi$ $\mathbf{u}_z = \mathbf{u}_z = \cos\theta\mathbf{u}_r - \mathrm{sen}\,\theta\mathbf{u}_\theta$	$\mathrm{sen}\,\theta\cos\varphi\mathbf{u}_x + \mathrm{sen}\,\theta\,\mathrm{sen}\,\varphi\mathbf{u}_y + \cos\theta\mathbf{u}_z = \mathrm{sen}\,\theta\mathbf{u}_\rho + \cos\theta\mathbf{u}_z = \mathbf{u}_r$ $\cos\theta\cos\varphi\mathbf{u}_x + \cos\theta\,\mathrm{sen}\,\varphi\mathbf{u}_y - \mathrm{sen}\,\theta\mathbf{u}_z = \cos\theta\mathbf{u}_\rho - \mathrm{sen}\,\theta\mathbf{u}_z = \mathbf{u}_\theta$ $-\mathrm{sen}\,\varphi\mathbf{u}_x + \cos\varphi\mathbf{u}_y = \mathbf{u}_\varphi = \mathbf{u}_\varphi$

Rotacional:

$$\nabla\wedge\mathbf{A} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\mathbf{u}_1 & h_2\mathbf{u}_2 & h_3\mathbf{u}_3 \\ \dfrac{\partial}{\partial q_1} & \dfrac{\partial}{\partial q_2} & \dfrac{\partial}{\partial q_3} \\ h_1A_1 & h_2A_2 & h_3A_3 \end{vmatrix}$$

$$\nabla\wedge\mathbf{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\mathbf{u}_x + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\mathbf{u}_y + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\mathbf{u}_z$$

$$\nabla\wedge\mathbf{A} = \left(\frac{1}{\rho}\frac{\partial A_z}{\partial\varphi} - \frac{\partial A_\varphi}{\partial z}\right)\mathbf{u}_\rho + \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial\rho}\right)\mathbf{u}_\varphi + \frac{1}{\rho}\left(\frac{\partial(\rho A_\varphi)}{\partial\rho} - \frac{\partial A_\rho}{\partial\varphi}\right)\mathbf{u}_z$$

$$\nabla\wedge\mathbf{A} = \frac{1}{r\,\mathrm{sen}\,\theta}\left(\frac{\partial(\mathrm{sen}\,\theta A_\varphi)}{\partial\theta} - \frac{\partial A_\theta}{\partial\varphi}\right)\mathbf{u}_r + \frac{1}{r}\left(\frac{1}{\mathrm{sen}\,\theta}\frac{\partial A_r}{\partial\varphi} - \frac{\partial(rA_\varphi)}{\partial r}\right)\mathbf{u}_\theta + \frac{1}{r}\left(\frac{\partial(rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial\theta}\right)\mathbf{u}_\varphi$$

Divergencia:

$$\nabla\cdot\mathbf{A} = \frac{1}{h_1h_2h_3}\left(\frac{\partial(A_1h_2h_3)}{\partial q_1} + \frac{\partial(h_1A_2h_3)}{\partial q_2} + \frac{\partial(h_1h_2A_3)}{\partial q_3}\right)$$

$$\nabla\cdot\mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla\cdot\mathbf{A} = \frac{1}{\rho}\frac{\partial(\rho A_\rho)}{\partial\rho} + \frac{1}{\rho}\frac{\partial A_\varphi}{\partial\varphi} + \frac{\partial A_z}{\partial z}$$

$$\nabla\cdot\mathbf{A} = \frac{1}{r^2}\frac{\partial(r^2A_r)}{\partial r} + \frac{1}{r\,\mathrm{sen}\,\theta}\frac{\partial(\mathrm{sen}\,\theta A_\theta)}{\partial\theta} + \frac{1}{r\,\mathrm{sen}\,\theta}\frac{\partial A_\varphi}{\partial\varphi}$$

Laplaciano:

$$\nabla^2\phi = \frac{1}{h_1h_2h_3}\left(\frac{\partial}{\partial q_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\phi}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{h_1h_3}{h_2}\frac{\partial\phi}{\partial q_2}\right) + \frac{\partial}{\partial q_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\phi}{\partial q_3}\right)\right)$$

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

$$\nabla^2\phi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\phi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\phi}{\partial\varphi^2} + \frac{\partial^2\phi}{\partial z^2}$$

$$\nabla^2\phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2\,\mathrm{sen}\,\theta}\frac{\partial}{\partial\theta}\left(\mathrm{sen}\,\theta\frac{\partial\phi}{\partial\theta}\right) + \frac{1}{r^2\,\mathrm{sen}^2\,\theta}\frac{\partial^2\phi}{\partial\varphi^2}$$