Part I

Summaries

Chapter I)

Section 1) Explains fundamentals of set theory and basic set operations

Section 2) Explains set relations, set functions and some more advanced set operations

Section 3) Presents categories, and multiple examples of categories. Some are simple, some are advanced.

Section 4) Presents monomorphisms and epimorphisms in more detail, taking care to distinguish general morphisms from set functions and their accolytes (inj, surj, etc)

Section 5) Presents more advanced concepts from category theory, mostly some important universal properties

Chapter II)

Part II

Group Weekly Reports

Week 1 : Today we mostly talked about the first chapter first section's reading; going over the vocabulary term by term (see the lexicon on the github repo), and going more in depth over certain concepts (particularly relating to set relations). We also saw a bit of a "teaser" of how these notions are used. We did not go over the exercises since not everyone had done them.

Week 2 : Today we continued on discussing the first chapter, it was mostly freeform. We mostly talked about foundations of set theory (mostly stemming from the discussion of exercise 1 on russell's paradox), why we use function notation the way we do, and about some of the operators over sets themselves (including through some examples from linear algebra and things like the subobject classifier which is seen at the end of section 3).

Week 3 : Today we finished discussing the first chapter. We went over all exercises. We mostly spoke about equivalence relations and partitions. We also spoke about the geometry/topology of quotients of sets by equivalence relations. This was naturally related to exercises 1.2 to 1.7.

Week 4 : We went over monomorphisms and epimorphisms in more depth. We corrected exercises 2.1 to 2.3 (included)

Week 5 : We went in depth over the distinction between isomorphisms and bijections (foreshadowing a bunch of category theory while we were at it) and corrected exercises 2.4 and 2.5.

Week 6 : We went in depth over the notion of section. We corrected exercises 2.6 and 2.7. For the latter exercise, we understood Tristan's solution by ourselves ! (written by Amric)

Week 7 : We reviewed the notions of algebraic quotient and well-definition. We broached the notion of universal property. We used this to correct exercises 2.8 and 2.9.

Week 8 : We corrected exercises 2.10 and 2.11. We then did some preliminary explanations to present categories and help with the reading of section 3.

Week 9 : We spoke more in depth about category theory, concrete categories, local smallness, algebraic structures (and their vocabulary) and applied category theory.

Week 10 : We reviewed examples 3.2, 3.3, 3.4 and gave a bunch of disambiguation ideas for 3.5. Next week we'll go over 3.5 and 3.6 in a bit more detail, and start correcting the exercises for this section. We'll leave 3.7 and above for when we get to their respective exercises

Part III

Lexicon

Chapter 1

Section 1

- Set (not a multiset)
- : the empty set, containing no elements;
- N: the set of natural numbers (that is, nonnegative integers);
- Z: the set of integers;
- Q: the set of rational numbers;
- R: the set of real numbers;
- C: the set of complex numbers.
- Singleton:
- : existential quantifier, "there exists"
- : universal quantifier, "for all"
- inclusion:
- subset:
- cardinal:
- powerset:
- : the union:
- : the intersection:

- : the difference:
- $\coprod$: the disjoint union:
- $\times$: the (Cartesian) product:
- complement of a subset
- relation
- order relation
- equivalence relation
- reflexivity
- symmetry
- antisymmetry
- transitivity
- partition
- quotient by an equivalence relation

Section 2

- function
- graph
- (categorical, function) diagram
- identity function
- kernel (of a function)
- image (of a function)
- restriction (of a function to a subset)

- multiset
- composition
- commutative (diagram)
- injection
- surjection
- bijection
- isomorphism
- inverse
- pre-inverse, right-inverse
- post-inverse, left-inverse
- monomorphism
- epimorphism
- natural projection
- natural injection
- canonical decomposition (of a function)

Section 3

- category
- object
- morphism
- endomorphism
- operation
- discrete category

- small category
- locally small category
- slice category
- coslice category
- comma category (mentioned, undefined)
- pointed set
- $C^{A,B}$ category ?? (bislice, bicoslice, fibered bislice, fibered bicoslice)
- dual category

Section 4

- automorphism

Section 5

- universal property
- initial object
- terminal object
- (categorical) product
- (categorical) coproduct

Part IV

Exercise solutions

Chapter I)

Section 1)

1.1)

In a nutshell, Russell's paradox proves, by contradiction, that certain mathematical collections cannot be sets. It posits the existence of a "set of all sets that don't contain themselves". Such a set can neither contain itself (since in that case, it would be a "set that does contain itself", and should be excluded); nor can it exclude it itself (since in that case, it would be a "set that doesn't contain itself", and should be included).

1.2)

Prove that any equivalence relation over a set S defines a partition of $\mathcal{P}_S$.

a) $\mathcal{P}_S$ has no empty elements: any element in S is part of at least one equivalence class, the class containing at least that element itself. Since there is no equivalence class constructed independently from elements, there are no empty equivalence classes.

b) Elements of $\mathcal{P}_S$ are disjoint: suppose there is an element x that is part of A and B, two distinct equivalence classes. $\forall a \in A, x \sim a$ and $\forall b \in B, x \sim b$. By transivity through x, $\forall a \in A, \forall b \in B, a \sim b$. Therefore, A and B are the same equivalence class: $A = B$. Contradiction. Therefore all elements of $\mathcal{P}_S$ are disjoint subsets of S.

c) The union of all elements of $\mathcal{P}_S$ makes up S: suppose $\exists x \in S$ such that $x \notin \bigcup_{S_i \in \mathcal{P}_S} S_i$. From the argument made in (a), x exists in at least one equivalence class, the class which contains only x itself. This is one of ou S_i sets. Contradiction. Therefore, $\bigcup_{S_i \in \mathcal{P}_S} S_i = S$

1.3)

Given a partition $\mathcal{P}$ on a set S, show how to define a relation $\sim$ on S such that P is the corresponding partition.

The insight here is to build an equivalence relation such that two elements are equivalent if and only if they are part of the same subset of S, which is understood as their common equivalence class.

We define $\sim$ such that $\forall S_i, S_j \in \mathcal{P}, \forall x \in S_i, \forall y \in S_j, x \sim y \Leftrightarrow S_i = S_j$.

Let us prove that $\sim$ is an equivalence relation.

a) Reflexivity:

$$\forall A \in \mathcal{P}, \forall x \in A, A = A \Rightarrow x \sim x$$

b) Symmetry:

$$\forall S_i, S_j \in \mathcal{P}, \forall x \in S_i, \forall y \in S_j, x \sim y \Leftrightarrow S_i = S_j \Leftrightarrow S_j = S_i \Leftrightarrow y \sim x$$

c) Transitivity:

$$\begin{aligned}\forall S_i, S_j, S_k \in \mathcal{P}, \forall x \in S_i, \forall y \in S_j, \forall z \in S_k,\\ (x \sim y) \cap (y \sim z)\\ \Leftrightarrow\\ (S_i = S_j) \cap (S_j = S_k)\\ \Rightarrow\\ S_i = S_k\\ \Leftrightarrow\\ x \sim z\end{aligned}$$

Therefore, $\sim$ is indeed an equivalence relation, and is generated uniquely by the partition.

1.4)

How many different equivalence relations may be defined on the set $\{1, 2, 3\}$?

If we start with the 1 element set, we have only one possible partition, one possible equivalence class.

With the 2 element set, there are 2 partitions, $\{\{1, 2\}\}$ and $\{\{1\}, \{2\}\}$.

With the 3 element set, there is:

- 1 partition of type 1-1-1: $\{\{1\},\{2\},\{3\}\}$.
- 3 partitions of type 2-1: $\{\{1\},\{2,3\}\}$, $\{\{2\},\{1,3\}\}$, and $\{\{3\},\{1,2\}\}$.
- 1 partition of type 3: $\{\{1,2,3\}\}$.

Hence, there are five equivalence classes on the 3 element set.
See the Bell numbers: https://oeis.org/A000110

1.5)

Give an example of a relation that is reflexive and symmetric, but not transitive. What happens if you attempt to use this relation to define a partition on the set?

Let's imagine a "similarity relation" we can notate with $\simeq$. We can imagine it to work like a looser version of equality (say for example, if an integer is only 1 away, then it counts as similar).

- reflexive: $\forall a \in S, a \simeq a$ (an element is always "similar" to itself)
- symmetric: $\forall a, b \in S, a \simeq b \Rightarrow b \simeq a$ ("similarity" goes both ways)
- not transitive: $\exists a, b, c \in S, (a \simeq b) \wedge (b \simeq c) \wedge \neg(a \simeq c)$ (just because $a \simeq b$ and $b \simeq c$ are similar, that doesn't mean $a \simeq c$ works, because it is possible for the "similarity gap" to be too large to qualify as "similar". E.g.: $(a, b, c) = (1, 2, 3)$.).

If we use this to define a partition P on some set S: $S/\simeq := P_{\simeq}$, there is ambiguity as to which element should go into which equivalence class.

This idea deserves further discussion.

In terms of graph theory, if we express a set with an internal relation as a graph, we can represent elements as nodes and relationships as edges. Reflexivity means that every node has a loop (unary, self-edge). Symmetry means that the graph is not directed (since every relationship goes both ways). Transitivity means that every connected subset of nodes is a maximal clique (synonymously, every connected component is a complete subgraph).

In a relation which is reflexive and symmetric, but not transitive, you would have connected components of this graph which are not cliques. For these, there is ambiguity as to how you would group their nodes. Two obvious choices would be either:

- to remove the minimal number of edges to obtain n distinct cliques (thereby gaining the *transitive restriction* of the relation) from a given non-clique; or

- to complete the connected subgraph into a clique (thereby gaining the *transitive closure* of the relation).

1.6)

Define a relation $\sim$ on the set $\mathbb{R}$ of real numbers, by setting $a \sim b \Leftrightarrow b - a \in \mathbb{Z}$. Prove that this is an equivalence relation, and find a 'compelling' description for $\mathbb{R}/\sim$. Do the same for the relation $\approx$ on the plane $\mathbb{R} \times \mathbb{R}$ defined by declaring $(a_1, a_2) \approx (b_1, b_2) \Leftrightarrow b_1 - a_1 \in \mathbb{Z}$ and $b_2 - a_2 \in \mathbb{Z}$.

TODO: forgot to prove that it's an equivalence relation

$b - a \in \mathbb{Z}$ means that 2 real numbers differ by an integral amount. This means that the equivalence relation algebraically describes the idea that "with this relation, 2 real numbers are the same iff they have the same fractional component x (or $1 - x$ for negative numbers)". Eg, $4.76 \sim 1024.76 \sim -5.34$, since $-5.34 + 10 = 4.76$, etc.

To make an algebraic quotient of a set by an equivalence relation, we take the function which maps each element to its corresponding equivalence class, in the set (partition) containing these equivalence class. Intuitively, this is similar to keeping only one representative element per equivalence class. For the example class above, we can keep the representative 0.76. There is such an equivalence class for every fractional part possible, that is, one for every number in $[0, 1[$. The corresponding map is the "real remainder of division modulo 1". This map is well-defined because each real number has only one output for this map, and all real numbers that are equivalent through $\sim$ are mapped to the same value in the output set.

We should also notice that since $0 \sim 1$, this space loops around on itself. Intuitively, if you increase linearly in the input space $\mathbb{R}$, it goes back to 0 after 0.9999999... in the output space. This output space is thus a circle of perimeter 1.

Similarly, $b_1 - a_1 \in \mathbb{Z}$ and $b_2 - a_2 \in \mathbb{Z}$ means that 2 points in the 2D plane are the same iff they differ in each coordinate by an integral amount. This boils down to combining two such loops from the first part of the exercise: one in the x direction and one in the y direction: what this gives is the small square $[0,1[\times[0,1[$, which loops to $x=0$ (resp. $y=0$) when $x=1$ (resp. $y=1$) is reached. This space functions like a small torus, of area 1.

Section 2)

2.1)

How many different bijections are there between a set S with n elements and itself?

Any bijection is a choice of a pairs from 2 sets of the same size, where each element is used only once, and each pair has one element from each set. At first there are n choices in each set. We go through each possible input element in order (no choice), for each one, we pick one amongst n possibilities for an output.

There are then $(n-1)$ choice of output left, etc.

Ccl°: $\prod_{i=1}^{i=n} i = n!$

2.2)

Prove that a function has a right-inverse (pre-inverse) iff it is surjective (can use AC).

Let $f \in (A \to B)$.

2.2.a) $\Rightarrow$

Suppose that f has a right-inverse (pre-inverse). We have $\exists g \in (B \to A), f \circ g = id_B$

Suppose that f is not a surjection. This means $\exists b \in B, \nexists a \in A, b = f(a)$

$f(g(b)) = id_B(b) = b$ Necessarily, $g(b)$ is such an a, so $\exists a \in A, b = f(a)$. Contradiction.

Ccl°:: f is a surjection.

2.2.b) $\Leftarrow$

Suppose that f is a surjection.

$\forall b \in B, \exists a \in A, b = f(a)$

We will construct a pre-inverse for f.

The insight here is to realize that a surjection divides its input set into a partition, where each 2-by-2 disjoint subset corresponds to $f^{-1}(\{q\})$, for every q in the output set. More formally, each "fiber" (preimage of a singleton) is a disjoint subset of the input set, and the union of fibers is the input set itself. You can see this in the following diagram:

(add diagram) 1234 to ab 1a 2a (fiber from a) 3b 4b (fiber from b) https://tex.stackexchange.com/questions/157450/producing-a-diagram-showing-relations-between-sets https://tex.stackexchange.com/questions/79009/drawing-the-mapping-of-elements-for-sets-in-latex

Using AC, we select a single element from each such fiber. For each $q \in B$, we name $p_q \in f^{-1}(\{q\})$ the chosen element. We define g as $g \in (B \to A), g = (q \mapsto p_q)$. With this, $\forall b \in B, f \circ g(b) = b$, and so $f \circ g = id_A$. Thus, f has a preinverse.

A summary of this idea: all surjection preinverses are simply a choice of a representative for each fiber of the surjection as the output to the respective singleton.

2.3)

Prove that the inverse of a bijection is a bijection, and that the composition of two bijections is a bijection.

2.3.a)

Using the fact that a function is a bijection iff it has a two-sided inverse (Corollary 2.2) we can see from this defining fact, $f \in (A \to B)$ bijective $\Leftrightarrow \exists f^{-1} \in (B \to A), (f^{-1} \circ f = id_A \text{ and } f \circ f^{-1} = id_B)$ that f is naturally f^{-1}'s (unique) two-sided inverse, and so f^{-1} is also a bijection.

2.3.b)

Let be $f \in (A \to B), g \in (B \to C)$, both bijective (hence with inverses in the respective function spaces). Let $h \in (A \to C), h = g \circ f$ and $h^{-1} \in (C \to A), h^{-1} = f^{-1} \circ g^{-1}$. We have:

$$
\begin{aligned}
h^{-1} \circ h &= (f^{-1} \circ g^{-1}) \circ (g \circ f) \\
&= f^{-1} \circ g^{-1} \circ g \circ f \\
&= f^{-1} \circ id_B \circ f \\
&= f^{-1} \circ f \\
&= id_A
\end{aligned}
$$

$$
\begin{aligned}
h \circ h^{-1} &= (g \circ f) \circ (f^{-1} \circ g^{-1}) \\
&= g \circ f \circ f^{-1} \circ g^{-1} \\
&= g \circ id_B \circ g^{-1} \\
&= g \circ g^{-1} \\
&= id_C
\end{aligned}
$$

Therefore h and h^{-1} are two-sided inverses of each other, and thus bijections. From this we conclude that the composition of any two bijections is also a bijection.

2.4)

Prove that 'isomorphism' is an equivalence relation (on any set of sets).

2.4.a) Problem statement

Let $\mathcal{A}$ be a set of sets. We define the relation $\simeq$ between the elements of $\mathcal{A}$ as the following:

$$\forall X, Y \in \mathcal{A},\ X \simeq Y \Leftrightarrow \text{there exists a bijection between } X \text{ and } Y$$

Let us show that $\simeq$ is an equivalence relation.

2.4.b) Reflexivity

For any set $A \in \mathcal{A}$, the identity mapping on A is a bijection. This means that $\forall A \in \mathcal{A}, A \simeq A$, ie, $\simeq$ is reflexive.

2.4.c) Symmetry

$$\begin{aligned}\forall X, Y \in \mathcal{A},\ X \simeq Y &\Rightarrow \exists f \in (X \to Y) \text{ bijective} \\ &\Rightarrow \exists f^{-1} \in (Y \to X) \text{ bijective} \\ &\Rightarrow Y \simeq X\end{aligned}$$

Therefore, $\simeq$ is symmetric.

2.4.d) Transitivity

Let be $X, Y, Z \in \mathcal{A}$. Suppose that $X \simeq Y$ and $Y \simeq Z$. This means $\exists f \in (X \to Y), g \in (Y \to Z)$, both bijections. Let be $h \in (X \to Z), h = g \circ f$. h is also a bijection since the composition of two bijections is also a bijection (exercise 2.3).

The existence of h implies $X \simeq Z$.

Therefore $\simeq$ is transitive.

2.4.e) Conclusion

Isomorphism, $\simeq$, is a relation on an arbitrary set (of sets) which is always reflexive, symmetric and transitive. It is thus an equivalence relation.

2.5)

Formulate a notion of epimorphism and prove that epimorphisms and surjections are equivalent.

See "notes" file: section "Proofs of mono/inj and epi/surj equivalence".

2.6)

With notation as in Example 2.4, explain how any function $f \in (A \to B)$ determines a section of π_A.

A section is the preinverse of a surjection. Here, the surjection in question is π_A the projection of $A \times B$ onto A.

Let $f \in (A \to B)$.

We now consider the function which maps an input $a \in A$ of f to its "geometric representation" (its coordinates in the enclosing space $A \times B$, corresponding to a point of the graph Γ_f).

$$\hat{f} \in (A \to (A \times B)), \hat{f} = (\ a \mapsto (a, f(a))\ )$$

We notice that $\hat{f}(A) = \Gamma_f$.

Naturally, $\pi_A \circ \hat{f} = (a \mapsto a) = id_A$, therefore, $\hat{f}$ is a pre-inverse (section) of π_A.

This set of relationships can be expressed in the following commutative diagram:

$$\begin{tikzcd}
& \begin{matrix}\Gamma_f \subseteq A \times B \\ (a, f(a))\end{matrix} \arrow[dl, shift left, two heads, "\pi_A"] \arrow[dr, two heads, "\pi_B"] & \\
\begin{matrix}A \\ a\end{matrix} \arrow[ur, shift left, hook, "\hat{f}"] \arrow[rr, "f"'] & & \begin{matrix}B \\ f(a)\end{matrix}
\end{tikzcd}$$

PS: see "On sections and fibers" in the "notes" file for a worked example.

2.7)

Let $f \in (A \to B)$ be any function. Prove that the graph Γ_f of f is isomorphic to A.

Using the elements from the previous exercise, we know that $\hat{f}$ is injective from A into $A \times B$. This property is inherited to any restriction of the codomain $Z \subseteq A \times B$, and corresponding implied restriction of the domain to $Y = \hat{f}^{-1}(Z) \subseteq A$. In particular, here, $Y = A$ and $Z = \Gamma_f = \hat{f}(A)$. We now consider $\overline{f} \in (A \to \Gamma_f), \overline{f} = (a \mapsto \hat{f}(a))$. We can see that $\overline{f}$ is injective from being a restriction of an injective function to a smaller codomain. We also know that $\overline{f}$ is surjective, since its domain is its image. Therefore, $\overline{f}$ is a bijection. This means that $A \simeq \Gamma_f$.

2.8)

Describe as explicitly as you can all terms in the canonical decomposition of the function $f \in (\mathbb{R} \to \mathbb{C})$ defined by $f = (r \mapsto e^{2\pi i r})$. (This exercise matches one assigned previously, which one?)

Firstly, elements of $\mathbb{R}$ are equivalent by this map (they have the same output) if they vary by 1 from each other. This is a reference to the equivalence relation $\sim$ in exercise 1.6. Therefore, we will use $\mathbb{R}/\sim \simeq S^1$ in our decomposition. Obviously, the map from $(\mathbb{R} \to \mathbb{R}/\sim)$, which maps each element of $\mathbb{R}$ to respective their equivalence class is a surjection (since there's no empty equivalence class).

Secondly, as mentioned, we have a bijection $\tilde{f}$ between $\mathbb{R}/\sim$ and S^1, the circle group of unit complex numbers, namely $\tilde{f} = (x \mapsto e^{2\pi i x}$, where each element x of $\mathbb{R}/\sim$ can be understood to correspond to a (class representative) value in the interval $[0, 1[$.

Finally, we do the canonical injection of S^1 into its superset $\mathbb{C}$.

2.9)

Show that if $A \simeq A'$ and $B \simeq B'$, and further $A \cap B = \emptyset$ and $A' \cap B' = \emptyset$, then $A \cup B \simeq A' \cup B'$. Conclude that the operation $A \coprod B$ (as described in §1.4) is well-defined up to isomorphism.

We suppose the aforementioned.

Let f_A be a bijection from $A \to A'$, and f_B be a bijection from $B \to B'$.

We define the following:

$$f \in (A \cup B \to A' \cup B'), \text{ such that } \begin{cases} \forall a \in A, \ f(a) = f_A(a) \\ \forall b \in B, \ f(b) = f_B(b) \end{cases}$$

This function is a well-defined function, since $A \cap B = \emptyset$: every element of the domain has one, and only one, possible image.

Similarly, we define:

$$g \in (A' \cup B' \to A \cup B), \text{ such that } \begin{cases} \forall a \in A', \ g(a) = f_A^{-1}(a) \\ \forall b \in B', \ g(b) = f_B^{-1}(b) \end{cases}$$

Similarly, because $A' \cap B' = \emptyset$, g is well-defined.

Let us study $g \circ f$. We have:

$$\begin{cases} \forall a \in A, \ g(f(a)) = f_A^{-1}(f_A(a)) = a \\ \forall b \in B, \ g(f(b)) = f_B^{-1}(f_B(b)) = b \end{cases}$$

Hence, $g \circ f = id_{A \cup B}$. Similarly, $f \circ g = id_{A' \cup B'}$. Therefore, $g = f^{-1}$, f is a bijection, and $A \cup B \simeq A' \cup B'$.

We'll now do a shift in notation. Let be some arbitrary sets A and B. Let be A_1, A_2, B_1, B_2 such that $A_1 = \{1\} \times A$, $A_2 = \{2\} \times A$, $B_1 = \{1\} \times B$, and $B_2 = \{2\} \times B$. This means $A \simeq A_1$, $A \simeq A_2$, $B \simeq B_1$, and $B \simeq B_2$.

It also means $A_1 \cap A_2 = \emptyset$ and $B_1 \cap B_2 = \emptyset$. From the above, this implies $A_1 \cup B_1 \simeq A_2 \cup B_2$.

This means that the disjoint union of A and B is indeed well-defined, up to isomorphism: so long as 2 respective copies of A and B are made in a way that their intersection is empty, the 2 respective unions of 1 copy each will be isomorphic.

2.10)

Show that if A and B are finite sets, then $|B^A| = |B|^{|A|}$.

The number of $|B^A|$ functions in $B^A = (A \to B)$ can be counted in the following way.

For each element a of A, of which there are $|A|$, we can pick any element of B as the image; a total of $|B|$ choices per choice of a. This means $|B| \times ... \times |B|$, a total of $|A|$ times. Hence, $|B^A| = |B|^{|A|}$.

2.11)

In view of Exercise 2.10, it is not unreasonable to use 2^A to denote the set of functions from an arbitrary set A to a set with 2 elements (say $\mathbb{B} = \{0, 1\}$). Prove that there is a bijection between 2^A and the power set $\mathcal{P}(A)$ of A.

Simply put, every subset A_i of A is built through a series of $|A|$ choices: for each element a in A, do we keep the element a in our subset A_i (output 1) or do we remove it (output 0) ? It is then easy to see that such a series of choices can easily be encoded as a unique function in $A \to \mathbb{B}$. The totality of such series of choices thus corresponds both to the space $A \to \mathbb{B}$, and to the powerset $\mathcal{P}(A)$, and there is a bijection between the two.

Section 3)

3.1)

Let $\mathcal{C}$ be a category. Consider a structure $\mathcal{C}^{op}$ with:

- $Obj(\mathcal{C}^{op}) \coloneqq Obj(\mathcal{C})$;
- for A, B objects of $\mathcal{C}^{op}$ (hence, objects of $\mathcal{C}$), $Hom_{\mathcal{C}^{op}}(A, B) \coloneqq Hom_{\mathcal{C}}(B, A)$

Show how to make this into a category.

3.1.a) Composition

First, to make things clearer and more rigorous, let us distinguish composition in $\mathcal{C}$ as $\circ$ and composition in $\mathcal{C}^{op}$ as $\star$. We define $\star$ as:

$$\begin{aligned}
&\forall f \in Hom_{\mathcal{C}^{op}}(B, A) = Hom_{\mathcal{C}}(A, B),\\
&\forall g \in Hom_{\mathcal{C}^{op}}(C, B) = Hom_{\mathcal{C}}(B, C),\\
&\exists h \in Hom_{\mathcal{C}^{op}}(C, A) = Hom_{\mathcal{C}}(A, C),\\
&f \star g \coloneqq g \circ f = h
\end{aligned}$$

We will now show that $\mathcal{C}^{op}$ with $\star$ verifies the other axioms of a category (namely identity and associativity of composition).

3.1.b) Identity

Since $\mathcal{C}$ is a category, since $\mathcal{C}^{op}$ has the same objects, and since, by definition, for all object A, we have $Hom_{\mathcal{C}^{op}}(A, A) = Hom_{\mathcal{C}}(A, A)$, we can take every $id_A \in Hom_{\mathcal{C}}(A, A)$ as the same identity in $\mathcal{C}^{op}$. We can verify that this is compatible with $\star$:

$$\begin{aligned}
\forall A, B \in Obj(\mathcal{C}) &= Obj(\mathcal{C}^{op}),\\
\exists id_A \in Hom_{\mathcal{C}}(A, A) &= Hom_{\mathcal{C}^{op}}(A, A),\\
\exists id_B \in Hom_{\mathcal{C}}(B, B) &= Hom_{\mathcal{C}^{op}}(B, B),\\
\forall f \in Hom_{\mathcal{C}}(A, B) &= Hom_{\mathcal{C}^{op}}(B, A),\\
f = f \circ id_A &= id_A \star f,\\
f = id_B \circ f &= f \star id_B
\end{aligned}$$

3.1.c) Associativity

Using associativity in $\mathcal{C}$, we have:

$$\begin{aligned}
\forall A, B, C, D \in Obj(\mathcal{C}) &= Obj(\mathcal{C}^{op}),\\
\forall f \in Hom_{\mathcal{C}}(A, B) &= Hom_{\mathcal{C}^{op}}(B, A),\\
\forall g \in Hom_{\mathcal{C}}(B, C) &= Hom_{\mathcal{C}^{op}}(C, B),\\
\forall h \in Hom_{\mathcal{C}}(C, D) &= Hom_{\mathcal{C}^{op}}(D, C),
\end{aligned}$$

$$
\begin{aligned}
h \star (g \star f) = \quad & h \star (f \circ g) \\
= \quad & (f \circ g) \circ h \\
= \quad & f \circ (g \circ h) \\
= \quad & (g \circ h) \star f \\
= \quad & (h \star g) \star f
\end{aligned}
$$

Therefore, $\star$ is associative.

We conclude that $\mathcal{C}^{op}$ is a category.

3.2)

If A is a finite set, how large is $End_{\text{Set}}(A)$?

We know that, in Set, $End_{\text{Set}}(A) = (A \to A) = A^A$. From a previous exercise, we know that $|B^A| = |B|^{|A|}$, therefore $|End_{\text{Set}}(A)| = |A|^{|A|}$.

3.3)

Formulate precisely what it means to say that "1_a is an identity with respect to composition" in Example 3.3, and prove this assertion.

Example 3.3 is that of a category over a set S with a (reflexive, transitive) relation $\sim$, where the objects of the category are the elements of S, and the homset between two elements a and b is the singleton (a, b) if $a \sim b$, and $\emptyset$ otherwise. Composition $\circ$ is given by transitivity of $\sim$, where $(b, c) \circ (a, b) = (a, c)$. Reflexivity gives the identities ($id_a = (a, a)$ for any element a).

In this context, to say that "1_a is an identity with respect to composition" means that we can cancel out an element of the form (a, a) from a composition.

Formally, we have:

$$\forall a, b \in S, (b, b) \circ (a, b) = (a, b) = (a, b) \circ (a, a)$$

proving that (b, b) is indeed a post-identity, and (a, a) a pre-identity, in this context.

3.4)

Can we define a category in the style of Example 3.3, using the relation $<$ on the set $\mathbb{Z}$?

(Description of example 3.3 in the exercise 3.3 just above.)

Naively, saying like in example 3.3 "there is a singleton homset $\mathrm{Hom}(a, b)$ each time we have $a < b$", we cannot define such a category, since $<$ is not reflexive, and we would thus lack identity morphisms.

However, in a roundabout way, we can define a category over the *negation* of $<$: "there is a singleton homset $\mathrm{Hom}(a, b)$ each time we DO NOT have $a < b$". Namely this corresponds to the relation $\geq$, which is, itself, reflexive, transitive (and antisymmetric), and is a valid instance of the kind of category presented in example 3.3.

In fact, the pair $(\mathbb{Z}, \geq)$ is an instance of what is called a "totally ordered set", which is a more restrictive kind of "partially ordered set" (also called "poset" for short). Consequently, this kind of category is called a "poset category".

3.5)

Explain in what sense Example 3.4 is an instance of the categories considered in Example 3.3.

(Description of example 3.3 in the exercise 3.3 just above.)

Example 3.4 describes a category $\hat{S}$ where the objects are the subsets of a set S (equivalently: elements of the powerset $\mathcal{P}(S)$ of S), and morphisms between two subsets A and B of S are singleton (or empty) homsets based on whether the inclusion is true (or false).

Inclusion of sets, $\subset$, is also an order relation, this time between the elements of a set of sets (here, $\mathcal{P}(S)$). This means inclusion is reflexive, transitive, and antisymmetric. This makes $\hat{S}$ a poset category, and thus another instance of example 3.3.

3.6)

Define a category V by taking $Obj(V) = \mathbb{N}$, and $Hom_V(n, m) = Mat_{\mathbb{R}}(m, n)$, the set of $m \times n$ matrices with real entries, for all $n, m \in \mathbb{N}$. (I will leave the reader the task to make sense of a matrix with 0 rows or columns.) Use product of matrices to define composition. Does this category 'feel' familiar ?

The formulation of the exercise is strange. It says to use the product of matrices to define composition, and to have homsets be sets of matrices, but objects of the category are supposed to be integers. I don't know of any matrix with real entries that maps an integer to an integer in this way.

We thus infer that the meaning of the exercise can be one of two things.

Either we suppose the set of objects could rather be understood as "something isomorphic to $\mathbb{N}$", ie, the collection of real vector spaces with finite bases (ie, $\forall n \in \mathbb{N}, \mathbb{R}^n$). In which case, this is just the category of real vector spaces with finite basis (and linear maps as morphisms), which is a subcategory of the category real vector spaces (commonly called $Vect_{\mathbb{R}}$). In this context, any morphism starting from $0 \simeq \mathbb{R}^0 = \{0\}$ is just the injection of the origin into the codomain; and any morphism ending at 0 is the mapping of all elements to the origin.

Otherwise, we understand this as "yes, the objects of the category are integers: this means you should ignore the actual content of the matrices, and instead consider only their effect on the dimensionality of domains and codomains". In this case, this category is a complete directed graph over $\mathbb{N}$ where each edge corresponds to the change in dimension (from domain to codomain) caused by a given linear map.

3.7)

Define carefully objects and morphisms in Example 3.7, and draw the diagram corresponding to composition.

Example 3.7 (on coslice categories) refers to example 3.5 (on slice categories). Let's go over slice categories (since example 3.5 asks the reader to "check all [their various properties]").

3.7.1) Slice categories

Slice categories are categories made by singling out an object (say A) in some parent (larger) category (say $\mathcal{C}$), and studying all morphisms into that object. These morphisms become the objects of a new category (ie, for any Z of $\mathcal{C}$, $f \in (Z \to A)$ is an object of the slice category, called $\mathcal{C}_A$ in this context). In the slice category, morphisms are defined as those morphism in $\mathcal{C}$ that preserve composition between 2 morphisms into A.

Note that there exist pairs of morphisms $f_1 \in (Z_1 \to A)$ and $f_2 \in (Z_2 \to A)$ between which there is no morphism that exists in the slice category. One such example we can make is in $(Vect_{\mathbb{R}})_{\mathbb{R}^2}$ (see notes "On the morphisms of slice and coslice categories" for more details).

3.7.1.a) Identity

A generic identity morphism is expressed diagrammatically in $\mathcal{C}_A$ as:

$$\begin{tikzcd} Z \arrow[r, "id_Z"] \arrow[d, "f"'] & Z \arrow[dl, "f"] \\ A \arrow[loop below, "id_A"] & \end{tikzcd}$$

We can see that since $f = f \circ id_Z$ in $\mathcal{C}$, this is compatible with the definition of a (pre-/right-)unit morphism in $\mathcal{C}_A$. Also, since the only maps post-f are maps from $A \to A$, we have id_A as the (post-/left-)unit for every morphism f (ie, $f = id_A \circ f$.

3.7.1.b) Composition

Taking 3 objects of the slice category ($f_1 \in (Z_1 \to A)$, $f_2 \in (Z_2 \to A)$ and $f_3 \in (Z_3 \to A)$), and two morphisms (σ_A mapping f_1 to f_2 via a $\mathcal{C}$-morphism $\sigma \in (Z_1 \to Z_2)$, and τ_A mapping f_2 to f_3 via a $\mathcal{C}$-morphism $\tau \in (Z_2 \to Z_3)$), we have that $f_1 = f_2 \circ \sigma$ and $f_2 = f_3 \circ \tau$. This is expressed as the following commutative diagram.

$$\begin{tikzcd} Z_1 \arrow[r, "\sigma"] \arrow[dr, "f_1"'] & Z_2 \arrow[r, "\tau"] \arrow[d, "f_2"] & Z_3 \arrow[dl, "f_3"] \\ & A & \end{tikzcd}$$

Composition of morphisms is then defined as $\tau_A \circ_A \sigma_A$ as a mapping from f_1 to f_3, such that $f_1 = f_3 \circ (\tau \circ \sigma)$. This can be understood through the following commutative diagram:

$$\begin{tikzcd} Z_1 \arrow[rr, "\tau\circ\sigma"] \arrow[dr, "f_1"'] & & Z_3 \arrow[dl, "f_3"] \\ & A & \end{tikzcd}$$

Which commutes, because we have:

$$\begin{aligned} f_1 = & \quad f_2 \circ \sigma \\ = & \quad (f_3 \circ \tau) \circ \sigma \\ = & \quad f_3 \circ (\tau \circ \sigma) \end{aligned}$$

Thus, we have a working composition of morphisms.

3.7.1.c) Associativity

We take 4 objects of the slice category ($f_1 \in (Z_1 \to A)$, $f_2 \in (Z_2 \to A)$, $f_3 \in (Z_3 \to A)$ and $f_4 \in (Z_4 \to A)$), and three morphisms (σ_A mapping f_1

to f_2, τ_A mapping f_2 to f_3, and υ_A mapping f_3 to f_4). Using composition defined as above, we have

$$\begin{aligned} f_1 = &\quad f_4 \circ (\upsilon \circ (\tau \circ \sigma)) \\ = &\quad f_4 \circ ((\upsilon \circ \tau) \circ \sigma) \\ \Rightarrow & \\ &\quad \upsilon_A \circ (\tau_A \circ \sigma_A) \\ = &\quad (\upsilon_A \circ \tau_A) \circ \sigma_A \end{aligned}$$

Through associativity in $\mathcal{C}$.

3.7.2) Coslice categories

A coslice category $\mathcal{C}^A$ is similar, except it takes the morphisms coming *from* a chosen object A, rather than those going *to* this object A. Below is a commutative diagram in the style of the one of the textbook for slice categories.

$$\begin{tikzcd} & A \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & \\ Z_1 \arrow[rr, "\sigma"] & & Z_2 \end{tikzcd}$$

We can similarly show that this also defines a category.

3.7.2.a) Identity

A generic identity morphism is expressed diagrammatically in $\mathcal{C}^A$ as:

$$\begin{tikzcd} A \arrow[loop above, "id_A"] \arrow[d, "f"] \arrow[dr, "f"] & \\ Z \arrow[r, "id_Z"'] & Z \end{tikzcd}$$

We can see that since $f = id_Z \circ f$ in $\mathcal{C}$, this is compatible with the definition of a (post-/left-)unit morphism in $\mathcal{C}^A$. Also, since the only maps pre-f are maps from $A \to A$, we have id_A as the (pre-/right-)unit for every morphism f (ie, $f = f \circ id_A$.

3.7.2.b) Composition

Taking 3 objects of the slice category ($f_1 \in (A \to Z_1)$, $f_2 \in (A \to Z_2)$ and $f_3 \in (A \to Z_3)$), and two morphisms (σ^A mapping f_1 to f_2 via a $\mathcal{C}$-morphism

$\sigma \in (Z_1 \to Z_2)$, and τ^A mapping f_2 to f_3 via a $\mathcal{C}$-morphism $\tau \in (Z_2 \to Z_3)$), we have that $f_1 = \sigma \circ f_2$ and $f_2 = \tau \circ f_3$. This is expressed as the following commutative diagram.

$$\begin{tikzcd}
 & A \arrow[dl, "f_1"'] \arrow[d, "f_2"] \arrow[dr, "f_3"] & \\
Z_1 \arrow[r, "\sigma"] & Z_2 \arrow[r, "\tau"] & Z_3
\end{tikzcd}$$

Composition of morphisms is then defined as $\tau^A \circ^A \sigma^A$ as a mapping from f_1 to f_3, such that $f_3 = (\tau \circ \sigma) \circ f_1$. This can be understood through the following commutative diagram:

$$\begin{tikzcd}
 & A \arrow[dl, "f_1"'] \arrow[dr, "f_3"] & \\
Z_1 \arrow[rr, "\tau\circ\sigma"] & & Z_3
\end{tikzcd}$$

Which commutes, because we have:

$$\begin{aligned}
f_3 = & \quad \tau \circ f_2 \\
= & \quad \tau \circ (\sigma \circ f_1) \\
= & \quad (\tau \circ \sigma) \circ f_1
\end{aligned}$$

Thus, we have a working composition of morphisms.

3.7.2.c) Associativity

We take 4 objects of the slice category ($f_1 \in (A \to Z_1)$, $f_2 \in (A \to Z_2)$, $f_3 \in (A \to Z_3)$ and $f_4 \in (A \to Z_4)$), and three morphisms (σ^A mapping f_1 to f_2, τ^A mapping f_2 to f_3, and υ^A mapping f_3 to f_4). Using composition defined as above, we have

$$\begin{aligned}
f_4 = & \quad (\upsilon \circ (\tau \circ \sigma)) \circ f_1 \\
= & \quad ((\upsilon \circ \tau) \circ \sigma) \circ f_1 \\
\Rightarrow & \\
& \quad \upsilon^A \circ (\tau^A \circ \sigma^A) \\
= & \quad (\upsilon^A \circ \tau^A) \circ \sigma^A
\end{aligned}$$

Through associativity in $\mathcal{C}$.

3.8)

A subcategory $\mathcal{C}'$ of a category $\mathcal{C}$ consists of a collection of objects of $\mathcal{C}$, with morphisms $Hom_{\mathcal{C}'}(A, B) \subseteq Hom_{\mathcal{C}}(A, B)$ for all objects A, B in $Obj(\mathcal{C}')$, such that identities and compositions in $\mathcal{C}$ make $\mathcal{C}'$ into a category. A subcategory $\mathcal{C}'$ is *full* if $Hom_{\mathcal{C}'}(A, B) = Hom_{\mathcal{C}}(A, B)$ for all A, B in $Obj(\mathcal{C}')$. Construct a category of *infinite sets* and explain how it may be viewed as a full subcategory of **Set**.

To put it less technically, a "subcategory" $\mathcal{C}'$ is just "picking only certain items of a base category $\mathcal{C}$, and making sure that things stay closed uneder morphism composition". It is "full" if *all* morphisms between the objects that remain are also conserved.

We can construct a category **InfSet** of infinite sets by taking all the objects A of **Set** such that $\nexists n \in \mathbb{N}, |A| = n$, and only homsets between these objects. This is clearly a subcategory of **Set**, since it inherits all identity morphisms, composition works the same, and so does associativity; also, restricting the choice of homsets makes it so that the category is closed (you can't reach a finite set via a homset that went from an infinite to a finite set).

For this category to not be full, there would need to be some homset that loses a morphism, or fully disappears, in the ordeal. However, there is no restriction as to the kind of morphism that is conserved, so any homset that is kept is identical to its original version. Finally, homsets between infinite sets are also infinite sets, so they don't disappear in this operation.

Consequently **InfSet** defined as such is a full subcategory of **Set**.

3.9)

An alternative to the notion of multiset introduced in §2.2 is obtained by considering sets endowed with equivalence relations; equivalent elements are taken to be multiple instances of elements 'of the same kind'. Define a notion of morphism between such enhanced sets, obtaining a category **MSet** containing (a 'copy' of) **Set** as a full subcategory. (There may be more than one reasonable way to do this! This is intentionally an open-ended exercise.) Which objects in **MSet** determine ordinary multisets as defined in §2.2, and how? Spell out what a morphism of multisets would be from this point of view. (There are several natural notions of morphisms of multisets. Try to define morphisms in MSet so that the notion you obtain for ordinary multisets captures your intuitive understanding of these objects.) [§2.2, §3.2, 4.5]

Let us recall how multisets were defined in §2.2. Since duplicate elements do not exist in sets, multisets were instead defined as functions from a set S to $\mathbb{N}*$, the set of (nonzero) positive integers. This allows each element in S to have a "count", thereby encoding the intuitive notion of multiset. A similar, and equivalent (isomorphic), way of defining it is *via* pairs $(s, n) \in S \times \mathbb{N}*$, which is simpler to think about. We'll call this category **CMSet**, for "count multiset" (TODO: probably has a conventional and better name, but I don't know it). As for morphisms in **CMSet**, we can consider that for any multisets $A = S_A \times \mathbb{N}*$ and $B = S_B \times \mathbb{N}*$, the homset from A to B is simply the set functions from $S_A \times \mathbb{N}*$ to $S_B \times \mathbb{N}*$ as usual.

We first notice that if we restrict **CMSet** to only the objects for which all elements have a count of 1, and where morphisms only ever output to $\{1\}$ in the second coordinate (a subcategory we can call **C1MSet**, for example), we get a "copy" of **Set**: **C1MSet** and **Set** are isomorphic in **Cat**. This is a full subcategory because there are no morphisms that map counts to anything else than $\{1\}$ if we restrict our objects to this form; so all morphisms between the kept objects are also kept.

Now let us do a similar construction, but based on equivalence classes instead. We know that each equivalence class over a set corresponds uniquely to a partition of that set. By considering only these partitions (these "sets of sets") as objects, we can build a category **EMSet** (for "equivalence multiset"). The "count" corresponds simply to the cardinal of a top-level element in the partition. For example, the top-level elements of $M = \{S_1, S_2, S_3\} = \{\{a\}, \{b, c\}, \{d, e, f\}\}$ would be understood to have counts $|S_1| = 1$, $|S_2| = 2$ and $|S_3| = 3$ respectively.

As for morphisms in **EMSet**, they simply map each top-level element of the domain multiset (a distinct subset of the original set) to some other top-level elements in the codomain multiset. This has precisely the same effect as mapping pairs of "value and count" as seen in the previous **CMSet** construction.

In this example, any set itself, when "injected" (by a functor) into **EMSet** would just nest all of its elements into singletons. I.e., $S = \{a, b, c\}$ in **Set** would become $S = \{\{a\}, \{b\}, \{c\}\}$ in **EMSet**. This also shows how restricting **EMSet** to "only objects that are a set of (toplevel) singletons" makes **EMSet** have a "copy" of **Set** as a full subcategory (for similar arguments as above).

Yet another example could be something akin to polynomials with integer coefficients on freeform indeterminates of degree 1 (which would be our set elements); raising the operators one rank, a product of freeform variables with integer powers (multiplicities), etc.

3.10)

Since the objects of a category $\mathcal{C}$ are not (necessarily) sets, it is not clear how to make sense of a notion of 'subobject' in general. In some situations it does make sense to talk about subobjects, and the subobjects of any given object A in $\mathcal{C}$ are in one-to-one correspondence with the morphisms $A \to \Omega$ for a fixed, special object Ω of $\mathcal{C}$, called a subobject classifier. Show that **Set** has a subobject classifier.

We define the set $\mathbb{B} = \{0, 1\}$, aka the binary alphabet or booleans, as the subobject classifier of **Set**. For any subset A of B, there is a unique map $f : B \to \mathbb{B}$, such that $\forall b \in B, f(b) = 1 \Leftrightarrow b \in A$ (otherwise $f(b) = 0$, of course, as the equivalence and lack of alternatives to 0 as an output imply). The map f always fully describes A from its relationship with B.

3.11)

Draw the relevant diagrams and define composition and identities for the category $\mathcal{C}^{A,B}$ mentioned in Example 3.9. Do the same for the category $\mathcal{C}^{\alpha,\beta}$ mentioned in Example 3.10. [§5.5, 5.12]

For lack of a better term, we will refer to the categories of the form $\mathcal{C}_{A,B}$ represented by Example 3.9 as "bi-slice categories". The first part of the exercise is thus asking us to define and explain what "bi-coslice categories" (of the form $\mathcal{C}^{A,B}$) are.

Similarly, we will refer to the categories of the form $\mathcal{C}_{\alpha,\beta}$ represented by Example 3.10 as "fibered bi-slice categories". The second part of the exercise is thus asking us to define and explain what "fibered bi-coslice categories" (of the form $\mathcal{C}^{\alpha,\beta}$) are.

We will, of course, attempt to make more formal and pedagogical all definitions broached in the textbook's examples as well.

3.11.1) Bi-slice categories

3.11.1.a) Objects and morphisms

To make a bi-slice category $\mathcal{C}_{A,B}$, we pick 2 objects A and B of a base category $\mathcal{C}$, and consider for all other objects Z of $\mathcal{C}$, all pairs of morphisms $(f, g) \in (Z \to A) \times (Z \to B)$. These pairs of morphisms are the objects of the bi-slice category $\mathcal{C}_{A,B}$. Morphisms $\sigma_{A,B}$ are defined from an object $p_1 = (f_1, g_1) \in (Z_1 \to A) \times (Z_1 \to B)$ to an object $p_2 = (f_2, g_2) \in (Z_2 \to A) \times (Z_2 \to B)$ so that we have both $f_1 = f_2 \circ \sigma$ and $g_1 = g_2 \circ \sigma$, for some $\sigma \in (Z_1 \to Z_2)$.

A generic object in $\mathcal{C}_{A,B}$ is of the form:

$$\begin{tikzcd} & A \\ Z \arrow[ur, "f"'] \arrow[dr, "g"] & \\ & B \end{tikzcd}$$

3.11.1.b) Morphisms

Morphisms are defined between objects as

$$\begin{tikzcd} & A & & & A \\ Z_1 \arrow[ur, "f_1"'] \arrow[dr, "g_1"] & & \xrightarrow{\sigma_{A,B}} & Z_2 \arrow[ur, "f_2"] \arrow[dr, "g_2"] & \\ & B & & & B \end{tikzcd}$$

such that the following diagram commutes

$$\begin{tikzcd} & A \\ Z_1 \arrow[ur, "f_1"] \arrow[r, "\sigma"] \arrow[dr, "g_1"'] & Z_2 \arrow[u, "f_2"'] \arrow[d, "g_2"] \\ & B \end{tikzcd}$$

3.11.1.c) Identity

It is clear that identity morphisms exist for all objects, simply by taking

$Z = Z_1 = Z_2$, $f_1 = f_2$, $g_1 = g_2$ and $\sigma = id_Z$, in the diagram above.

3.11.1.d) Composition

Let be 3 objects of $\mathcal{C}_{A,B}$, which we will name p_1, p_2 and p_3 (and define with the respective (Z_n, f_n, g_n) triplet for p_n).

Composition $\tau_{A,B} \circ \sigma_{A,B} = p_1 \mapsto p_3$ of two morphisms $\sigma_{A,B} = p_1 \mapsto p_2$ and $\tau_{A,B} = p_2 \mapsto p_3$ is defined so that the following diagram commutes.

$$\begin{tikzcd}
 & A & \\
Z_1 \arrow[ur, "f_1"] \arrow[r, "\sigma"] \arrow[dr, "g_1"'] & Z_2 \arrow[u, "f_2"'] \arrow[r, "\tau"] \arrow[d, "g_2"] & Z_3 \arrow[ul, "f_3"'] \arrow[dl, "g_3"] \\
 & B &
\end{tikzcd}$$

3.11.1.e) Associativity

Associativity follows from associativity of morphisms in $\mathcal{C}$, similarly to what was done for slice categories in exercise 3.7 .

3.11.2) Bi-coslice categories

3.11.2.a) Objects and morphisms

To make a bi-coslice category $\mathcal{C}^{A,B}$, we similarly pick 2 objects A and B of our base category $\mathcal{C}$, but instead consider, for all other objects Z of $\mathcal{C}$, all pairs of morphisms $(f, g) \in (A \to Z) \times (B \to Z)$.

A generic object in $\mathcal{C}^{A,B}$ is of the form:

$$\begin{tikzcd}
A \arrow[dr, "f"] & \\
 & Z \\
B \arrow[ur, "g"'] &
\end{tikzcd}$$

3.11.2.b) Morphisms

Morphisms are defined between objects as

$$\begin{tikzcd}
A \arrow[dr, "f_1"] & & & A \arrow[dr, "f_2"] & \\
& Z_1 \arrow[r, "\sigma^{A,B}"] & \; & & Z_2 \\
B \arrow[ur, "g_1"'] & & & B \arrow[ur, "g_2"'] &
\end{tikzcd}$$

such that the following diagram commutes

$$\begin{tikzcd}
A \arrow[d, "f_1"'] \arrow[dr, "f_2"] & \\
Z_1 \arrow[r, "\sigma"] & Z_2 \\
B \arrow[u, "g_1"] \arrow[ur, "g_2"'] &
\end{tikzcd}$$

3.11.2.c) Identity

It is clear that identity morphisms exist for all objects, simply by taking $Z = Z_1 = Z_2$, $f_1 = f_2$, $g_1 = g_2$ and $\sigma = id_Z$, in the diagram above.

3.11.2.d) Composition

Let be 3 objects of $\mathcal{C}^{A,B}$, which we will name p_1, p_2 and p_3 (and define with the respective (Z_n, f_n, g_n) triplet for p_n).

Composition $\tau^{A,B} \circ \sigma^{A,B} = p_1 \mapsto p_3$ of two morphisms $\sigma^{A,B} = p_1 \mapsto p_2$ and $\tau^{A,B} = p_2 \mapsto p_3$ is defined so that the following diagram commutes.

$$\begin{tikzcd}
& A \arrow[dl, "f_1"'] \arrow[d, "f_2"] \arrow[dr, "f_3"] & \\
Z_1 \arrow[r, "\sigma"] & Z_2 \arrow[r, "\tau"] & Z_3 \\
& B \arrow[ul, "g_1"] \arrow[u, "g_2"'] \arrow[ur, "g_3"'] &
\end{tikzcd}$$

3.11.2.e) Associativity

Associativity follows from associativity of morphisms in $\mathcal{C}$, similarly to what was done for slice categories in exercise 3.7 .

3.11.3) Fibered bi-slice categories

3.11.3.a) Objects

To build a fibered bi-slice category $\mathcal{C}_{\alpha,\beta}$, one takes a base category $\mathcal{C}$, as well as a fixed pair of morphisms $\alpha : A \to C$ and $\beta : B \to C$ in $\mathcal{C}$, that point to a common object C of $\mathcal{C}$. Our basic "fixed construct" from $\mathcal{C}$ looks like so:

$$\begin{tikzcd} A \arrow[dr, "\alpha"] & \\ & C \\ B \arrow[ur, "\beta"'] & \end{tikzcd}$$

The role of the category $\mathcal{C}_{\alpha,\beta}$ is now to study the morphisms into this construct. A generic object from this new category looks like so:

$$\begin{tikzcd} & A \arrow[dr, "\alpha"] & \\ Z \arrow[ur, "f"] \arrow[dr, "g"'] & & C \\ & B \arrow[ur, "\beta"'] & \end{tikzcd}$$

such that the diagram commutes. This means that valid object in $\mathcal{C}_{\alpha,\beta}$ are triplets (Z, f, g), with $f : Z \to A$ and $g : Z \to B$, such that $\alpha \circ f = \beta \circ g$. In a caricatural way, this boils down to studying "the comparison of the different paths one can use to reach C, knowing that the last steps are on one hand, α, and on the other, β".

3.11.3.b) Morphisms

Morphisms are defined between objects as:

$$\begin{tikzcd} & A \arrow[dr, "\alpha"] & & & & A \arrow[dr, "\alpha"] & \\ Z_1 \arrow[ur, "f_1"] \arrow[dr, "g_1"'] & & C & \xrightarrow{\sigma_{\alpha,\beta}} & Z_2 \arrow[ur, "f_2"] \arrow[dr, "g_2"'] & & C \\ & B \arrow[ur, "\beta"'] & & & & B \arrow[ur, "\beta"'] & \end{tikzcd}$$

such that the following diagram commutes

$$\begin{tikzcd}
& & A \arrow[dr, "\alpha"] & \\
Z_1 \arrow[r, "\sigma"] \arrow[urr, "f_1"] \arrow[drr, "g_1"'] & Z_2 \arrow[ur, "f_2"'] \arrow[dr, "g_2"] & & C \\
& & B \arrow[ur, "\beta"'] &
\end{tikzcd}$$

<u>3.11.3.c) Identity</u>

Once again, it is clear that identity morphisms exist for all objects, simply by taking $Z = Z_1 = Z_2$, $f_1 = f_2$, $g_1 = g_2$ and $\sigma = id_Z$, in the diagram above.

<u>3.11.3.d) Composition</u>

Let be 3 objects of $\mathcal{C}_{\alpha,\beta}$, which we will name p_1, p_2 and p_3 (and define with the respective (Z_n, f_n, g_n) triplet for p_n).

Composition $\tau_{\alpha,\beta} \circ \sigma_{\alpha,\beta} = p_1 \mapsto p_3$ of two morphisms $\sigma_{\alpha,\beta} = p_1 \mapsto p_2$ and $\tau_{\alpha,\beta} = p_2 \mapsto p_3$ is defined so that the following diagram commutes.

$$\begin{tikzcd}
& & & A \arrow[dr, "\alpha"] & \\
Z_1 \arrow[r, "\sigma"] \arrow[urrr, "f_1"] \arrow[drrr, "g_1"'] & Z_2 \arrow[r, "\tau"] \arrow[urr, "f_2"] \arrow[drr, "g_2"'] & Z_3 \arrow[ur, "f_3"'] \arrow[dr, "g_3"] & & C \\
& & & B \arrow[ur, "\beta"'] &
\end{tikzcd}$$

<u>3.11.3.e) Associativity</u>

Associativity follows from associativity of morphisms in $\mathcal{C}$, similarly to what was done for slice categories in exercise 3.7 .

3.11.4) Fibered bi-coslice categories

<u>3.11.4.a) Objects</u>

To build a fibered bi-coslice category $\mathcal{C}^{\alpha,\beta}$, one takes a base category $\mathcal{C}$, as well as a fixed pair of morphisms $\alpha : C \to A$ and $\beta : C \to B$ in $\mathcal{C}$, that originate from a common object C of $\mathcal{C}$. Our basic "fixed construct" from $\mathcal{C}$ looks like so:

$$\begin{tikzcd}
 & A \\
C \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & \\
 & B
\end{tikzcd}$$

The role of the category $\mathcal{C}^{\alpha,\beta}$ is now to study the morphisms from this construct. A generic object from this new category looks like so:

$$\begin{tikzcd}
 & A \arrow[dr, "f"] & \\
C \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & & Z \\
 & B \arrow[ur, "g"'] &
\end{tikzcd}$$

such that the diagram commutes. This means that valid object in $\mathcal{C}^{\alpha,\beta}$ are triplets (Z, f, g), with $f : A \to Z$ and $g : B \to Z$, such that $f \circ \alpha = g \circ \beta$. In a caricatural way, this boils down to studying "the comparison of the different paths one can build by starting from C, knowing that the choice of first step is on one hand, α, and on the other, β".

3.11.4.b) Morphisms

Morphisms are defined between objects as:

$$\begin{tikzcd}
 & A \arrow[dr, "f_1"] & & & & A \arrow[dr, "f_2"] & \\
C \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & & Z_1 & \arrow[r, "\sigma^{\alpha,\beta}"] & C \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & & Z_2 \\
 & B \arrow[ur, "g_1"'] & & & & B \arrow[ur, "g_2"'] &
\end{tikzcd}$$

such that the following diagram commutes

$$\begin{tikzcd}
 & A \arrow[dr, "f_1"] \arrow[drr, "f_2"] & & \\
C \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & & Z_1 \arrow[r, dashed, "\sigma"] & Z_2 \\
 & B \arrow[ur, "g_1"'] \arrow[urr, "g_2"'] & &
\end{tikzcd}$$

3.11.4.c) Identity

Once again, it is clear that identity morphisms exist for all objects, simply by taking $Z = Z_1 = Z_2$, $f_1 = f_2$, $g_1 = g_2$ and $\sigma = id_Z$, in the diagram above.

3.11.4.d) Composition

Let be 3 objects of $\mathcal{C}^{\alpha,\beta}$, which we will name p_1, p_2 and p_3 (and define with the respective (Z_n, f_n, g_n) triplet for p_n).

Composition $\tau^{\alpha,\beta} \circ \sigma^{\alpha,\beta} = p_1 \mapsto p_3$ of two morphisms $\sigma^{\alpha,\beta} = p_1 \mapsto p_2$ and $\tau^{\alpha,\beta} = p_2 \mapsto p_3$ is defined so that the following diagram commutes.

$$\begin{tikzcd}
& A \arrow[dr, "f_1"] \arrow[drr, "f_2"] \arrow[drrr, "f_3"] & & & \\
C \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & & Z_1 \arrow[r, "\sigma"] & Z_2 \arrow[r, "\tau"] & Z_3 \\
& B \arrow[ur, "g_1"] \arrow[urr, "g_2"'] \arrow[urrr, "g_3"'] & & &
\end{tikzcd}$$

3.11.4.e) Associativity

Associativity follows from associativity of morphisms in $\mathcal{C}$, similarly to what was done for slice categories in exercise 3.7 .

Section 4)

4.1)

Composition is defined for *two* morphisms. If more than 2 morphisms are given, one may compose them in several ways, so that every step only consists in composing 2 morphisms. Prove that for any such valid sequence of morphisms, the order of parentheses doesn't matter.

This boils down to showing that associativity is a global property, that doesn't just make parentheses meaningless when there are 3 elements and 2 operators between them, but in general n elements with $(n-1)$ operators between them.

Note: A useful way of visualizing this is representing the order of operations as a binary tree, and noticing that applying associativity (forwards or backwards) is just a tree rotation (resp. right or left) at a given node. Then it is easy to show that one can always obtain a "left comb binary tree". Since every choice of parentheses is equal to this left comb choice, and equality is transitive, every choice of parentheses is equal to every other choice.

To be more rigorous, we will proceed by induction.

Hypothesis: $P(n) =$ "for a given n, for $f_n f_{n-1} \cdot f_1$ any valid, composable, ordered sequence of morphisms in our base category $\mathcal{C}$, any choice H of parentheses to compose elements of this sequence 2-by-2, giving a formula s_H, will lead to the same result, which can be seen by always having $s_H = (\cdot(f_n f_{n-1})\cdot)f_1$".

Initialization: We initialize at $n = 3$; the validity is immediate as it is precisely the definition of associativity.

Heredity: We suppose the hypothesis $P(n)$ true for a given $n \geq 3$; let us show that this implies that the hypothesis is true for $P(n+1)$.

What this means is that, no matter the composable ordered sequence $f_n f_{n-1} \cdot f_1$ of n functions, for a fixed n, the order of parentheses does not matter. Note that though n is chosen and fixed; the statement is true for EVERY (ordered, composable) sequence of functions. We add a new function g to this sequence. By a simple renaming of the functions, we deduce that it doesn't matter where we insert g, so we'll insert it at the very right to simplify our argument, giving us the sequence $f_n f_{n-1} \cdot f_1 g$.

Here, there are 3 cases. Either:

- g is part of the last composition (i.e., it's not in a semantically necessary parenthethical grouping; it can be made external to all parentheses),
- g is part of the first composition (i.e., the first operation is $(f_1 g)$)
- it isn't either (it's inside some non-removable parentheses, and needs to be composed earlier on, but not as the first operation).

If g is part of the last composition, then by applying the hypothesis $P(n)$ to the terms $f_n f_{n-1} \cdot f_1$, we immediately find that our new sequence can be made equal to $((\cdot(f_n f_{n-1})\cdot)f_1)g$, which is precisely what we wanted for $P(n+1)$.

If g is part of the first composition, we isolate it so that it isn't anymore. To do so, we apply "backwards" associativity on the grouping of terms $F_k(f_1 g)$ in order to obtain $(F_k f_1)g$, where F_k is the appropriate choice of $(f_k \cdot f_2)$ such that associativity can be applied (with $2 \leq k \leq n$). This makes it so that our problem is identical to our final case, solved just below.

If g is part of neither the first nor last composition, then we consider the innermost composition $(f_k f_{k-1})$ to be a single element h. We now have a sequence of only n terms. We apply our hypothesis $P(n)$. This makes g the outermost right term, part of the last composition. Unravelling h back into two members, we see that we are back at our initial case, with an arbitrary order of parentheses for the $f_n f_{n-1} \cdot f_1$ terms, and g outermost. We already saw that this implied $P(n+1)$.

Conclusion: since we have initialization and heredity of our hypothesis in all cases, we can conclude by induction that it is true for all $n \geq 3$.

4.2)

In Example 3.3 we have seen how to construct a category from a set endowed with a relation, provided the latter is reflexive and transitive. For what types of relations is the corresponding category a groupoid (cf. Example 4.6) ?

We remind example 4.6 : a groupoid is a category in which every morphism is an isomorphism. This means that every morphism needs to be 2-way invertible.

In this context, this means that for every morphism $a \sim b$, there should be a corresponding inverse morphism $b \sim a$. This property is precisely the symmetry of a relation.

This means that all sets with an equivalence relation can be reconstructed into a groupoid.

4.3)

Let A, B be objects of a category $\mathcal{C}$, and $f \in Hom_{\mathcal{C}}(A, B)$ a morphism. Prove that if f has a pre-inverse, then f is an epimorphism. Show that the converse does not hold, by giving an explicit example of a category and an epimorphism without a pre-inverse.

4.3.a)

f has a pre-inverse $\Rightarrow$ f is an epimorphism

Let $\mathcal{C}$ be a category. Let $f \in Hom_{\mathcal{C}}(A, B)$, having some pre-inverse which we'll call $g \in Hom_{\mathcal{C}}(B, A)$:

Let Z be an arbitrary object of $\mathcal{C}$, and $\beta', \beta" \in Hom_{\mathcal{C}}(B, Z)$:

$$\begin{aligned}\beta' \circ f = \beta'' \circ f \Rightarrow (\beta' \circ f) \circ g &= (\beta'' \circ f) \circ g \\ &= \beta' \circ (f \circ g) = \beta'' \circ (f \circ g) \\ &= \beta' \circ id_B = \beta'' \circ id_B \\ &= \beta' = \beta''\end{aligned}$$

This means that f is an epimorphism.

4.3.b)

f is an epimorphism $\not\Rightarrow$ f has a pre-inverse

As was mentioned in the text, "order" categories (poset categories) where there's only at most one morphism between any two objects makes it so that every morphism is trivially an epimorphism. However, only identities have any kind of inverse (since they are isomorphisms, they are their own inverse).

See also here and here.

4.4)

Prove that the composition of two monomorphisms is a monomorphism. Deduce that one can define a subcategory $\mathcal{C}_{mono}$ of a category $\mathcal{C}$ by taking the same objects as in $\mathcal{C}$, and defining $Hom_{\mathcal{C}_{mono}}(A, B)$ to be the subset of $Hom_{\mathcal{C}}(A, B)$ consisting of monomorphisms, for all objects A, B. (Cf. Exercise 3.8; of course, in general $\mathcal{C}_{mono}$ is not full in $\mathcal{C}$.) Do the same for epimorphisms. Can you define a subcategory $\mathcal{C}_{nonmono}$ of $\mathcal{C}$ by restricting to morphisms that are not monomorphisms?

4.4.a)

Mono

Let be $f \in Hom_{\mathcal{C}}(A, B)$ and $g \in Hom_{\mathcal{C}}(B, C)$ be monomorphisms. Let us show that $g \circ f$ is also a monomorphism.

Let Z be an arbitrary object of $\mathcal{C}$, and $\alpha', \alpha" \in Hom_{\mathcal{A}}(Z, A)$:

$$\begin{aligned}(g \circ f) \circ \alpha' = (g \circ f) \circ \alpha'' = g \circ (f \circ \alpha') &= g \circ (f \circ \alpha'') \\ &\Rightarrow f \circ \alpha' = f \circ \alpha'' \text{ because } g \text{ is mono} \\ &\Rightarrow \alpha' = \alpha'' \text{ because } f \text{ is mono}\end{aligned}$$

This means that the composition of 2 monomorphisms is always an monomorphism. We can thus make a subcategory. Taking all objects, properties, and homsets of $\mathcal{C}$, but restricting the homsets only to the monomorphisms, we know that this makes a new category $\mathcal{C}_{mono}$ since it is closed under composition, has identities (which are iso, and *a fortiori* mono) and associativity.

4.4.b)

Epi

Let be $f \in Hom_{\mathcal{C}}(A,B)$ and $g \in Hom_{\mathcal{C}}(B,C)$ be epimorphisms. Let us show that $g \circ f$ is also a epimorphism.

Let Z be an arbitrary object of $\mathcal{C}$, and $\beta', \beta" \in Hom_{\mathcal{C}}(C,Z)$:

$$\begin{aligned} \beta' \circ (g \circ f) = \beta'' \circ (g \circ f) &= (\beta' \circ g) \circ f = (\beta'' \circ g) \circ f \\ &\Rightarrow \beta' \circ g = \beta'' \circ g \text{ because } f \text{ is epi} \\ &\Rightarrow \beta' = \beta'' \text{ because } g \text{ is epi} \end{aligned}$$

This means that the composition of 2 epimorphisms is always an epimorphism. We can thus make a subcategory. Taking all objects, properties, and homsets of $\mathcal{C}$, but restricting the homsets only to the epimorphisms, we know that this makes a new category $\mathcal{C}_{epi}$ since it is closed under composition, has identities (which are iso, and *a fortiori* epi) and associativity.

4.4.c)

Nonmono and nonepi

We could consider the fact that (TODO prove lol) we can't obtain a monomorphism from the composition of two non-monomorphisms (you need at least one monomorphism in the mix). However, the real problem is identities. Identities are iso, and thus mono. You can't make a category without identities, so there is no such $\mathcal{C}_{nonmono}$. the same reasoning applies to $\mathcal{C}_{nonepi}$.

4.5)

Give a concrete description of monomorphisms and epimorphisms in the category **MSet** you constructed in Exercise 3.9. (Your answer will depend on the notion of morphism you defined in that exercise!)

We'll use our **CMSet** construction, where elements of multisets consisted of a pair of the set-element and its count in the multiset.

We recall that in the way we formulated this, morphisms were just simple set functions on "(element, count)" pairs (i.e., returning any other "(element, count)" pair of the codomain). Let be a morphism of multisets $f \in (A \to B)$. Labelling the elements of the domain A as a_i and of the codomain B as b_j with $i \in I$, $j \in J$, and I, J any two indexing sets such that $card(A) = card(I)$ and $card(B) = card(J)$, we can see that A and B now just look like "normal" sets.

We now simply recycle the notion of injections and surjections. These form our monomorphisms and epimorphisms respectively.

Section 5)

5.1)

Prove that a final object in a category $\mathcal{C}$ is initial in the opposite category $\mathcal{C}^{op}$

Let $\mathcal{C}$ be a category. Let $\mathcal{C}^{op}$ be the dual category on $\mathcal{C}$. Let F be a final object in $\mathcal{C}$. This means that for every object Z in $\mathcal{C}$, there is a single morphism from Z to F. We will call this morphism f_Z (respectively).

We remind how we defined composition in $\mathcal{C}^{op}$ as $\star$, respecting:

$$\begin{aligned}
&\forall f \in Hom_{\mathcal{C}^{op}}(B, A) = Hom_{\mathcal{C}}(A, B),\\
&\forall g \in Hom_{\mathcal{C}^{op}}(C, B) = Hom_{\mathcal{C}}(B, C),\\
&\exists h \in Hom_{\mathcal{C}^{op}}(C, A) = Hom_{\mathcal{C}}(A, C),\\
&f \star g \coloneqq g \circ f = h
\end{aligned}$$

In this case, we see that $\forall Z \in Obj(\mathcal{C}^{op}) = Obj(\mathcal{C}), f_Z \in Hom_{\mathcal{C}^{op}}(F, Z) = Hom_{\mathcal{C}}(Z, F)$. This implies that the homset $Hom_{\mathcal{C}^{op}}(F, Z)$ contains a single morphism, f_Z. This means that F is initial in $mathcalC^{op}$.

5.2)

Prove that $\emptyset$ is the *unique* initial object in **Set**.

First we will prove that it is initial, then that it is unique.

Initiality: we take an arbitrary set Z in **Set**. We wish to study $Hom_{\mathbf{Set}}(\emptyset, Z) = Z^{\emptyset}$. We recall that functions (in category theory) are defined as "applications" / "mappings" are in traditional set theory (i.e., as a relation between sets where every antecedent in the domain has a singleton image in the codomain; the key point being that "no input has no result when passed through the function"). Let I be an initial element in **Set**. We write $|I| = n$ and $|Z| = m$. We know that $|Z^I| = |Z|^{|I|} = m^n$. For I to be initial, this is true if and only if $m^n = 1$ for all m, and so if and only if $n = 0$. We recall that the empty set is the only set with $|\emptyset| = 0$, therefore $I = \emptyset$.

Now this is already enough to prove unicity, but let us spell it out for pedagogy's sake.

Unicity: We recall that two objects of **Set** are isomorphic if, and only if, there exists a bijection between them. This is equivalent to saying that two sets have the same cardinal. We once again recall that the empty set is the only set with $|\emptyset| = 0$; there are no bijections relating to the empty set, other than its identity, the unique morphism in $Hom_{\mathbf{Set}}(\emptyset, \emptyset)$. Using proposition 5.4 (that terminal objects are unique up-to-isomorphism), we finally deduce that $\emptyset$ is the unique initial object in **Set**.

NB: the unique function in $Z^{\emptyset}$ is always the empty function.

5.3)

Prove that final objects are unique up to isomorphism.

Let us suppose we have a category $\mathcal{C}$ with two final objects, F_1 and F_2.

For every object A of $\mathcal{C}$ there is at least one element in $Hom_{\mathcal{C}}(A, A)$, namely the identity 1_A. If F is final, then there is a unique morphism $F \to F$, which therefore must be the identity 1_F.

Now assume F_1 and F_2 are both final in $\mathcal{C}$. Since F_1 is final, there is a unique morphism $f : F_2 \to F_1$ in $\mathcal{C}$. Since F_2 is final, there is a unique morphism $g : F_1 \to F_2$ in $\mathcal{C}$. Consider $gf : F_1 \to F_1$; as observed, necessarily $gf = 1_{F_2}$ since F_1 is final. By the same token $fg = 1F_1$, proving f is an isomorphism. Thus $F_1 \simeq F_2$.

5.4)

What are initial and final objects in the category of "pointed sets" (Example 3.8)? Are they unique?

We recall that a pointed set is just a regular set with a special, identified point, and that the category of pointed sets **Set*** is built upon the same objects as **Set**, but where each object A in **Set** is multiplied into $|A|$ copies of itself in **Set*** (one for each choice of special point; this implies that the empty set is not a part of **Set***, since it has no point). Morphisms in **Set*** are set functions, but with the restriction of mapping the special point in the domain to the special point in the codomain.

Given this information, we will prove that the initial and final objects in **Set*** are the singleton sets.

Let (O, o) be a singleton set in **Set***. Let o be the single element of O; it is necessarily also the special point, as there is no other choice. For any codomain (Z, z_0) in **Set***, the condition that "special points map to special points" restricts our choice of function to the unique function (o, z_0), thus, O is initial. If O had more than one element, there would exist some Z (non-singletons) for which the other element would allow another degree of freedom (and thus O would not be initial).

Similarly, now studying Z as a domain and O as a codomain, we see that that only function from Z to O is (like in **Set**) the function which maps everything (including Z's special point) to o. Thus, O is final. If O had more than one element, there would similarly be many choices for any Z of cardinal ≥ 2, so long as the special point maps to the special point.

Every singleton pointed set is both initial and final in **Set*** and is thus a zero object. These are also the only such pointed sets.

5.5)

What are the final objects in the category considered in §5.3?

The category considered in paragraph 5.3 is the coslice category over some set A. However, what is presented in this paragraph is some extra structure that arises when considering the statement "The quotient $A/\sim$ is universal with respect to the property of mapping A to a set in such a way that equivalent elements have the same image". We thus give some equivalence relation $\sim$ on A and study the quotient set $A/\sim$ in the general coslice category; to do this, we consider the subcategory $\mathcal{Q}$ of $\mathcal{C}_A$ where only φ such that "equivalence is preserved" (i.e., such that $\forall a', a'' \in A, a' \sim a'' \Rightarrow \varphi(a') = \varphi(a'')$).

With:

- s the canonical surjection of A onto its quotient $A/\sim$,
- φ_1 (resp. φ_2) being some arbitrary function from A to some arbitrary Z_1 (resp. Z_2),

- σ any function (if it exists) such that $\sigma\varphi_1 = \varphi_2$
- f_1 (resp. f_2) is the (unique!) function such that $sf_1 = \varphi_1$ (resp. $sf_2 = \varphi_2$)

The following diagram commutes, and summarizes the situation.

$$\begin{tikzcd}
A \arrow[r, "\varphi_2"] \arrow[d, "s"'] \arrow[dr, "\varphi_1" near end] & Z_2 \\
A/\sim \arrow[r, "f_1"'] \arrow[ur, "f_2"' near end] & Z_1 \arrow[u, "\sigma"']
\end{tikzcd}$$

Objects in this category are denoted as (φ, Z) and are obtained from what used to be *morphisms* (regular functions) in **Set**. Morphisms are mappings $\sigma_{\mathcal{Q}} : (\varphi_1, Z_1) \to (\varphi_2, Z_2)$ such that one exists if and only if $\exists \sigma \in (Z_1 \to Z_2), \sigma\varphi_1 = \varphi_2$, and $\forall a', a'' \in A, a' \sim a'' \Rightarrow \varphi(a') = \varphi(a'')$.

Since the textbook also asks whether such a category has initial objects, we will first also answer this and consider all terminal objects.

The initial object of a general coslice category is id_A. This is easily verified by doing $\varphi_1 = id_A$, necessarily $\sigma\varphi_1 = \sigma id_A = \sigma = \varphi_2$, and so σ always exists and is unique. We also see that this object satisfies the "equivalence preservation" condition, hence it exists in $\mathcal{Q}$, and is also the initial object in $\mathcal{Q}$.

A general coslice category has a final object (t, F) (or many final objects (t_i, F_i)) iff $\mathcal{C}$ has a final object F (or many final objects F_i). In that case, any final object (t, F) in $\mathcal{C}_A$ corresponds to the unique morphism from A to F (for any final F) in $\mathcal{C}$. Let us verify this.

Let F be final in $\mathcal{C}$, and t be the unique morphism $t \in Hom_{\mathcal{C}}(A, F)$. Let (φ, Z) be an arbitrary object of $\mathcal{C}_A$. Let be σ such that $\sigma\varphi = t$. We consider the following diagram:

$$\begin{tikzcd}
A \arrow[r, "t"] \arrow[d, "\varphi"'] & F \\
Z \arrow[ur, "\sigma"'] &
\end{tikzcd}$$

Since F is final in $\mathcal{C}$, σ is unique and always exists. Also, since σ is unique and always exist, the choice of φ is irrelevant: this same σ works for all choices of φ for a given arbitrary Z. This proves that $\sigma_{\mathcal{C}_A}$ exists and is unique for all (φ, Z). Finally, since σ works for all choices of φ, it works for those that satisfy the "equivalence preservation" condition, and so does t: this means that (t, F) is indeed a final object in $\mathcal{Q}$.

5.6)

Consider the category corresponding to endowing (as in Example 3.3) the set $\mathbb{Z}^+$ of positive integers with the divisibility relation. Thus there is exactly one morphism $d \to m$ in this category if and only if d divides m without remainder; there is no morphism between d and m otherwise. Show that this category has products and coproducts. What are their 'conventional' names? [§VII.5.1]

Like example 3.3, this is a case of "category made from an order relation over a set", since divisibility is an order relation (reflexive, antisymmetric, transitive).

Let us remind the definition of categorical products and coproducts. We consider some general category $\mathcal{C}$.

An object $A \prod B$ is the product of two objects A and B iff there is a unique morphism π_A (resp. π_B) from $A \prod B$ to A (resp. B), and for every Z in $\mathcal{C}$, and for every pair of morphisms $f_A : Z \to A$ and $f_B : Z \to B$, there exists a single morphism $\sigma = f_A \prod f_B$ such that $\pi_A \sigma = f_A$ and $\pi_B \sigma = f_B$. This is summarized in the following commutative diagram.

$$\begin{tikzcd}
 & Z \arrow[dl, "f_A"'] \arrow[d, dashed, "\sigma = f_A \prod f_B"] \arrow[dr, "f_B"] & \\
A & A \prod B \arrow[l, "\pi_A"] \arrow[r, "\pi_B"'] & B
\end{tikzcd}$$

An object $A \coprod B$ is the coproduct of two objects A and B iff there is a unique morphism i_A (resp. i_B) from A (resp. B) into $A \coprod B$, and for every Z in $\mathcal{C}$, and for every pair of morphisms $f_A : A \to Z$ and $f_B : B \to Z$, there exists a single $\sigma = f_A \coprod f_B$) such that $\sigma i_A = f_A$ and $\sigma i_B = f_B$. This is summarized in the following commutative diagram.

$$\begin{tikzcd}
A \arrow[r, "i_A"] \arrow[dr, "f_A"'] & A \coprod B \arrow[d, dashed, "\sigma = f_A \coprod f_B"] & B \arrow[l, "i_B"'] \arrow[dl, "f_B"] \\
 & Z &
\end{tikzcd}$$

We now return to our "divisibility order category". We write its objects as simple integers, and the (if it exists, unique) morphism representing "divisibility of m by n" as $(n|m)$. The conventional name of the product for this category is "greatest common divisor" (or "meet"), and of the coproduct, "least common multiple" (or "join").

The following commutative diagrams represent this fact. Take two arbitrary naturals m and n. Any number k which divides both m and n also divides their GCD. Likewise, if k is a multiple of both n and m, then it is a multiple of their LCM.

$$\begin{tikzcd}
 & k \arrow[dl] \arrow[d] \arrow[dr] & \\
n \arrow[r] & GCD(n,m) & m \arrow[l]
\end{tikzcd}$$

$$\begin{tikzcd}
n \arrow[r] \arrow[dr] & LCM(n,m) \arrow[d] & m \arrow[l] \arrow[dl] \\
 & k &
\end{tikzcd}$$

5.7)

Redo Exercise 2.9 ("Show that if $A \simeq A'$ and $B \simeq B'$, and further $A \cap B = \emptyset$ and $A' \cap B' = \emptyset$, then $A \cup B \simeq A' \cup B'$. Conclude that the operation $A \coprod B$ (as described in §1.4) is well-defined up to isomorphism") this time using Proposition 5.4. (the unicity up-to-isomorphism of terminal objects).

We define $\mathbf{Set}^{A,B}$ as the "bicoslice category of A and B over $\mathbf{Set}$". Objects in this category are pairs of morphisms (f, g) from A and B, respectively, into sets Z. They can be diagrammed as follows.

$$\begin{tikzcd}
A \arrow[dr, "f"] & \\
 & Z \\
B \arrow[ur, "g"] &
\end{tikzcd}$$

Morphisms are defined between objects as

$$\begin{tikzcd}
A \arrow[dr, "f_1"] & & & & A \arrow[dr, "f_2"] & \\
 & Z_1 & \arrow[r, "\sigma^{A,B}"] & {} & & Z_2 \\
B \arrow[ur, "g_1"'] & & & & B \arrow[ur, "g_2"'] &
\end{tikzcd}$$

such that the following diagram commutes

$$\begin{tikzcd}
A \arrow[d, "f_1"'] \arrow[dr, "f_2"] & \\
Z_1 \arrow[r, "\sigma"] & Z_2 \\
B \arrow[u, "g_1"] \arrow[ur, "g_2"'] &
\end{tikzcd}$$

Let us call I the following object of $\mathbf{Set}^{A,B}$, where $A \coprod B$ is any choice of valid disjoint union of A and B:

$$\begin{tikzcd}
A \arrow[dr, "i_A"] & \\
& A \coprod B \\
B \arrow[ur, "i_B"'] &
\end{tikzcd}$$

By definition of a coproduct, we know that in such a configuration, a morphism $\sigma^{A,B}$ from this object into any other object of $\mathbf{Set}^{A,B}$ exists and is unique, and so is the σ on which it is based. This means that I is initial in $\mathbf{Set}^{A,B}$. Consequently, using prop 5.4., the fact that if an initial object exists, it is unique up-to-isomorphism, we conclude that $A \coprod B$ is unique up-to-isomorphism.

5.8)

Part V

Extra exercises by/for the group

Chapter I) 1) Set notation)

Write the following in set notation (as a list of numbers, and algebraically):

- the set of all odd integers
- the set of all integers that are not multiples of 3
- the set of integers from 10 (included) to 20 (included)
- the set of integers from 10 (included) to 20 (excluded)
- the set of pairs of integers with both elements of the same value
- the set of triplets of real numbers that together sum to 1
- the set of pairs of positive real numbers that together sum to 1
- the set of n-tuplets (for any n) of real number that together sum to 1
- the set of all natural numbers such that there exists at least one triplet of positive even numbers which are all different and which sum to that number.

Now take the sets in their algebraic notation, and represent them both as a list of numbers (as a logical sequence or just a couple of examples), and as a "description" of what they are:

- $\{3n + 2 \mid n \in \mathbb{N}\}$
- $\{3k + 2 \mid k \in \mathbb{Z}\}$
- $\{2^i \mid i \in [[0, 10]]\}$
- $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$
- $\{x \in \mathbb{R} \mid -2 \leq x \leq 2\}$
- $\{(m, n, p) \in \mathbb{N}^3 \mid m + n + p = 10\}$

Part VI

Notes

Chapter 1, Section 1

Go check out the extra exercises on set notation.

Chapter 1, Section 2

On injections and surjections

Injections

Injections (which aren't also surjections) have multiple left-inverses (post-inverses). Eg:

$$A = \{a, b\}$$

$$B = \{1, 2, 3\}$$

$$f : A \to B = \{(a, 2), (b, 3)\}, \text{ injective}$$

$$g_1 : B \to A = \{(1, a), (2, a), (3, b)\}$$

$$g_2 : B \to A = \{(1, b), (2, a), (3, b)\}$$

$$g_1 \circ f = g_2 \circ f = id_A$$

It is precisely the free element with no antecedent in B (here, 1) which leaves room for multiple choices, but doesn't affect the overall inversion process.

Surjections

Surjections (which aren't also injections) have multiple right-inverses (pre-inverses), called sections.

$$B = \{1, 2, 3\}$$

$$A = \{a, b\}$$
$$f : B \to A = \{(1, a), (2, a), (3, b)\}, \text{ surjective}$$

$$g_1 : A \to B = \{(a, 1), (b, 3)\}$$
$$g_2 : A \to B = \{(a, 2), (b, 3)\}$$

$$f \circ g_1 = f \circ g_2 = id_A$$

It is precisely the fact that there are multiple elements that map to the same element (here, 1 and 2 to a) which leaves room for multiple choices, but doesn't affect the overall inversion process.

Cancellations

Function Cancellation Lemma: If a composition of functions cancels out, then the first of the pair is an injection, and the second of the pair is a surjection. Algebraically:

$$\forall A, B \in Obj(\mathbf{Set}), f \in (A \to B), g \in (B \to A),\ g \circ f = id_A \Rightarrow \begin{cases} f \text{ is injective} \\ g \text{ is surjective} \end{cases}$$

Corollary 1: any post-inverse of an injection is a surjection.

Corollary 2: any pre-inverse of a surjection is an injection.

Proof: Let be

$$A, B \in Obj(\mathbf{Set}), f \in (A \to B), g \in (B \to A),\ g \circ f = id_A$$

a) Suppose f is not an injection. This means:

$$\exists x, y \in B,\ x \neq y \text{ and } g(x) = g(y)$$

However, with such an f, we have:

$$g(x) = g(y) \Rightarrow f(g(x)) = f(g(y)) = id_A(x) = id_A(y) = x = y$$

This means that f is an injection. Contradiction.

Conclusion: in this context, f must be an injection.

b) Suppose g is not a surjection. This means:

$$\exists a \in A,\ a \notin g(B)$$

Since $g \circ f = id_A$, that means that $g(f(a)) = id_A(a) = a$, this means that $a \in g(B)$. Contradiction.

Conclusion: in this context, g must be a surjection.

On sections and fibers

Section example with a tangent bundle.

Consider the cylinder $S^1 \times \mathbb{R}$, and the function $f : S^1 \times \mathbb{R} \to S^1$, the projection onto the circle. The cylinder is itself the space in which one can easily represent maps of $(S^1 \to \mathbb{R})$. Each such map corresponds to a section.

Let be

$$\begin{aligned} g_1 : S^1 &\longrightarrow S^1 \times \mathbb{R} \\ \theta &\longmapsto (\theta, 1) \end{aligned}$$

$$\begin{aligned} g_2 : S^1 &\longrightarrow S^1 \times \mathbb{R} \\ \theta &\longmapsto (\theta, cos(\theta)) \end{aligned}$$

We have

$$f \circ g_1 = f \circ g_2 = id_{S^1}$$

(TODO add diagrams for S1xR, g1 and g2)

A fiber is the preimage of a singleton. In the case of f above, for every $q \in S^1$, $f^{-1}(q)$ is the copy of the real line on the cylinder that passes by q.

(TODO add diagram)

Alternative characterization of a bijection

"f is bijective" $\Leftrightarrow$ ("every element of B has a non-empty fiber" (surjection) and "every fiber is a singleton" (injection))

On monomorphisms and epimorphisms

Failing the mono/epi condition

Example of failing the monomorphism definition for a non-injection

Monomorphism definition:

$$f : A \to B \text{ is a monomorphism} \Leftrightarrow \forall Z \in \mathrm{Obj}(\mathcal{C}),\ \forall g_1, g_2 \in \mathrm{Hom}(Z, A),\ (f \circ g_1 = f \circ g_2 \Rightarrow g_1 = g$$

$$A = \{a, b, c\}$$

$$B = \{1, 2\}$$
$$Z = \{x, y\}$$
$$f : A \to B = \{(a, 1), (b, 1), (c, 2)\}, \text{ not injective}$$

$$g_1 : Z \to A = \{(x, a), (y, c)\}$$
$$g_2 : Z \to A = \{(x, b), (y, c)\}$$

$$f \circ g_1 = f \circ g_2 = \{(x, 1), (y, 2)\} \in (Z \to B)$$

The multiple choice of elements (here, a and b) in A which map to 1 in B is precisely what allows the overall composition to be equal, even when $g_1 \neq g_2$. This provides insight into a proof of "f is a monomorphism implies that f is an injection". If you suppose that f is a monomorphism and not an injection, you can easily reach a contradiction, since you can use elements like 1 and 2 that both map to the same a to construct a counter-example to the implication that defines a monomorphism.

Example of failing the epimorphism definition for a non-surjection

Epimorphism definition:

$$f : A \to B \text{ is an epimorphism} \Leftrightarrow \forall Z \in \mathrm{Obj}(\mathcal{C}),\ \forall g_1, g_2 \in \mathrm{Hom}(B, Z),\ (g_1 \circ f = g_2 \circ f \Rightarrow g_1 = g_2)$$

$$g_1 : Z \to A = \{(x, a), (y, c)\}$$
$$g_2 : Z \to A = \{(x, b), (y, c)\}$$

$$f \circ g_1 = f \circ g_2 = \{(x, 1), (y, 2)\} \in (Z \to B)$$

$$A = \{a, b\}$$
$$B = \{1, 2, 3\}$$
$$Z = \{x, y\}$$
$$f : A \to B = \{(a, 1), (b, 2)\}, \text{ not surjective}$$

$$g_1 : B \to Z = \{(1, x), (2, y), (3, x)\}$$
$$g_2 : B \to Z = \{(1, x), (2, y), (3, y)\}$$

$$g_1 \circ f = g_2 \circ f = \{(a, x), (b, y)\} \in (A \to Z)$$

The element 3 in B not being reached by f is precisely that which provides the opportunity to build $g_1 \neq g_2$ such that they compose to the same result with f, since the output of 3 for them doesn't affect the overall composition. This provides insight into a proof of "f is an epimorphism implies that f is a surjection". If you suppose that f is an epimorphism and not a surjection, you can easily reach a contradiction, since you can use elements like 3 that are not reached by f to construct a counter-example to the implication that defines an epimorphism.

Proofs of mono/inj and epi/surj equivalence

Let $f : A \to B$.

The parts which are "Injection =¿ Monomorphism" and "Surjection =¿ Epimorphism" both use the respective sided inverses to prove the implication.

The other parts use the following tautology to prove the implication by contradiction. "Suppose that p and $\neg q$, show that it leads to a contradiction".

$$(p \Rightarrow q) \Leftrightarrow ((\neg p) \cup q) \Leftrightarrow (\neg(p \cap \neg q))$$

Injection =¿ Monomorphism

Suppose that f is an injection. It thus has post-inverses.

$$\exists g \in (B \to A), g \circ f = id_A$$

From there:

$$\forall Z \in \mathrm{Obj}(\mathcal{C}),\ \forall a, b \in \mathrm{Hom}(Z, A),$$

$$\begin{array}{rcccc} f \circ a = f \circ b & \Rightarrow & g \circ (f \circ a) & = & g \circ (f \circ b) \\ & = & (g \circ f) \circ a & = & (g \circ f) \circ b \\ & = & id_A \circ a & = & id_A \circ b \\ & = & a & = & b \end{array}$$

We conclude that f is also a monomorphism.

Surjection =¿ Epimorphism

Suppose that f is a surjection. It thus has pre-inverses.

$$\exists g \in (B \to A), f \circ g = id_B$$

From there:

$$\forall Z \in \mathrm{Obj}(\mathcal{C}),\ \forall a, b \in \mathrm{Hom}(B, Z),$$

$$\begin{array}{rccccc} a \circ f = b \circ f & \Rightarrow & (a \circ f) \circ g & = & (b \circ f) \circ g \\ & = & a \circ (f \circ g) & = & b \circ (f \circ g) \\ & = & a \circ id_B & = & b \circ id_B \\ & = & a & = & b \end{array}$$

We conclude that f is also an epimorphism.

Monomorphism =¿ Injection

Suppose that f is a monomorphism.

$$\forall Z \in \mathrm{Obj}(\mathcal{C}),\ \forall g_1, g_2 \in \mathrm{Hom}(Z, A),\ f \circ g_1 = f \circ g_2 \Rightarrow g_1 = g_2$$

Suppose now that f is not an injection. Algebraically, this means that:

$$\exists (x, y) \in A^2, \text{ such that } x \neq y \text{ and } f(x) = f(y)$$

We can construct g_1 and g_2 such that $f \circ g_1 = f \circ g_2$ but $g_1 \neq g_2$, using such a pair (x, y). Thereby, we prove that f is not an monomorphism and arrive at a contradiction.

(If Z is the empty set, being initial in **Set**, there is only 1 map into A (the empty map) and $a = b$ always hold. Therefore, any counterexample to the epimorphism definition needs to have at least 1 element.)

Let $Z = \{a\}$.

$$g_1(a) = x$$

$$g_2(a) = y$$

Clearly, $g_1 \neq g_2$. However, we also have:

$$f(g_1(a)) = f(x) = f(y) = f(g_2(a)) \Rightarrow f \circ g_1 = f \circ g_2$$

This means that f is not a monomorphism: contradiction.
Conclusion: f is an injection.

Epimorphism =¿ Surjection

Suppose that f is an epimorphism.

$$\forall Z \in \mathrm{Obj}(\mathcal{C}),\ \forall g_1, g_2 \in \mathrm{Hom}(B, Z),\ g_1 \circ f = g_2 \circ f \Rightarrow g_1 = g_2$$

Suppose now that f isn't a surjection. Algebraically, this means that:

$$\exists x \in B, x \notin f(A)$$

We can construct g_1 and g_2 such that $g_1 \circ f = g_2 \circ f$ but $g_1 \neq g_2$, using such an $x \notin f(A)$. Thereby, we prove that f is not an epimorphism and arrive at a contradiction.

(If Z is the singleton set, being terminal in **Set**, there is only 1 map into Z and $a = b$ always hold. Therefore, any counterexample to the epimorphism definition needs to have at least 2 elements. We will however use a 3-element set, since it makes things more intuitive and pedagogical.)

Let $Z = \{a, b, c\}$.

$$g_1 = \begin{cases} \forall x \in f(A), g_1(x) = a \\ \forall x \notin f(A), g_1(x) = b \end{cases}$$

$$g_2 = \begin{cases} \forall x \in f(A), g_2(x) = a \\ \forall x \notin f(A), g_2(x) = c \end{cases}$$

Clearly, $g_1 \neq g_2$. However, since A is the domain of f, of $g_1 \circ f$, and of $g_2 \circ f$, we have:

$$g_1 \circ f = g_2 \circ f = (x \mapsto a) \in (A \to Z)$$

This means that f is not an epimorphism: contradiction.
Conclusion: f is a surjection.

Chapter 1, Section 3

Example summary

- (3.2): Set, category of sets as objects and set functions as morphisms.
- (3.3): preorder (or order, or equivalence relation) over a (single) set, transformed into a category; elements of the set as objects, and elements of the preorder (which is a relation, hence a subset of the cartesian product) as morphisms.
- (3.4): the powerset with the inclusion operator, transformed into a category; elements of the powerset (i.e., subsets of the set) as objects, and inclusion relations as morphisms (this is just an example of a preorder / order / equivalence category seen in 3.3).
- (3.5): slice categories $\mathcal{C}_A$, categories which isolate a specific object A of a given category $\mathcal{C}$, and studies the morphisms into that object; an object of $\mathcal{C}_A$ is any morphism from any arbitrary objet Z into A (not the homset $Hom(Z, A)$ itself !) and a morphism in $\mathcal{C}_A$ (from $z_1 \in Z_1 \to A$ to $z_2 \in Z_2 \to A$) is a "raising" σ_A into $\mathcal{C}_A$ of a morphism $\sigma \in Z_1 \to Z_2$ in $\mathcal{C}$ that preserves composition on morphisms in $\mathcal{C}$ (i.e., $z_1 = z_2\sigma \Rightarrow \sigma_A z_1 = z_2$).
- (3.6): combining examples 3.3 and 3.5, first start with an order category on the set $\mathbb{Z}$ (there is a morphism $m \to n$ iff $m \leq n$), then select a specific object (here, $A = 3$) then study all morphisms of the category into A (so the relation $n \leq 3$ for any $Z = n$); the morphisms $\sigma_3 = (m, 3) \to (n, 3)$ are then simply given by the transitivity of $\leq$, i.e., $m \leq n \leq 3$ ($(m, 3) \to (n, 3)$ corresponds to $m \leq 3 \Rightarrow n \leq 3$, meaning our $z_1 = z_2\sigma$ transforming into $\sigma_A z_1 = z_2$, here, corresponds to ($m \leq$

$3) = (n \leq 3) \cap (m \leq n)$ is transformed into $(m \leq 3 \Rightarrow n \leq 3) \cap (m \leq 3) \Leftrightarrow (n \leq 3))$.

- (3.7): coslice categories (morphisms out of a chosen object).

- (3.8): the category $\mathbf{Set}^\star$ of pointed sets, a coslice category over **Set** and any singleton set $\{\star\}$. Objects in $\mathbf{Set}^\star$ are regular sets, but with a unique distinguished element; morphisms are any set functions that map the domain's distinguised element to the codomain's distinguished element.

- (3.9): "bislice" and "bicoslice" categories, basically a similar construct as slice and coslice, but taking two objects of the starting category, and studying pairs of morphisms (from a common domain, resp codomain) into (resp from) this pair.

- (3.10): "fibered bislice" and "fibered bicoslice" categories, once again a similair construct, but this time taking two *morphisms* into a common set C (resp. from a common set C).

On terminal and initial objects in Set

If is initial and $\{\star\}$ is terminal, it is because a function in Set (in categorical terms) must always have an output for every input. Ie, in category theory, all functions are maps ("applications").

Said algebraically:

$$\forall A, B \in \mathrm{Obj}(\mathbf{Set}),\ \forall a \in A,\ \forall f \in \mathrm{Hom}(A, B),\ \exists f(a) \in B$$

In the case of as the input set, and there is only one function $f : \to Z$ for any Z: f is the empty mapping. But any $Z \to$ (expept $\to$) contains no mapping (since we'd necessarily be ignoring at least one element of Z).

Similarly, in the case of the (unique up-to-isomorphism) singleton set $\{\star\}$ as the output, you'd have multiple functions (precisely $2^{|Z|}$) into it, if you could ignore some elements of the input set. However, if all elements of the input set are required, that leaves you with only one function possible from $Z \to \{\star\}$: the function which maps all elements to $\star$.

Restrictions and extensions of functions, and its consequences on a function's nature

8 possibilities to study, based on the following binary dichotomies:

- injection or surjection
- enlarging or restricting
- domain or codomain

Note that enlarging the domain sometimes implies enlarging the codomain, and restricting the codomain sometimes implies restricting the domain.

Legend: Yes, No, Depends

	enlarge dom	restrict dom	enlarge cod	restrict cod
injection	D	Y	Y	Y
surjection	Y	D	N	Y

Theorems:

A) if $f \in (A \to B), f$ injective (resp. surjective), then $\forall Z \subseteq B, \hat{f} \in ((f^{-1}(Z) \subseteq A) \to Z), \hat{f} = f|_{f^{-1}(Z)}$, the restriction of the function to the corresponding smaller codomain, is also an injection (resp. surjection).

B) if $f \in (A \to B), f$ injective (resp. surjective), then $\forall Z \supseteq B, \hat{f} \in (A \to Z), \hat{f} = \iota \circ f$ (with the ι the canonical injection of $b \in B$ into its superset Z), is also an injection (resp. is never a surjection).

C) if $f \in (A \to B), f$ injective, then $\forall Z \subseteq A, \hat{f} \in (Z \to B), f = \iota_{(Z \to A)} \circ \hat{f}$, we have that $\hat{f}$ is also an injection. However, one can construct $Z \subseteq A$ such that f stops being a surjection.

D) if $f \in (A \to B), f$ surjective, then $\forall Z \supseteq A, \hat{f} \in (Z \to (B \cup f(Z))), f = \iota_{(Z \to A)} \circ \hat{f}$, we have that $\hat{f}$ is also a surjection. However, one can construct $Z \subseteq A$ such that f stops being a injection.

Proof: TODO

On the morphisms of slice and coslice categories

Given a base category $\mathcal{C}$, and some set A we wish to study the homsets of the slice (resp. coslice) category $\mathcal{C}_A$ (resp. $\mathcal{C}^A$. **These homsets might be empty, or have more than one element.**

We remind that slice categories consider *morphisms to* A as their *objects* (written as (Z, φ) for any $\varphi : Z \to A$), while coslice categories consider *morphisms from* A as their *objects* (written as (φ, Z) for any $\varphi : A \to Z$. *Morphisms*, of the form $\sigma_A : (Z_1, \varphi_1) \to (Z_2, \varphi_2)$ (resp. $\sigma^A : (\varphi_1, Z_1) \to (\varphi_2, Z_2)$) in a slice category $\mathcal{C}_A$ (resp. coslice category $\mathcal{C}^A$) map such objects to one another if and only if there exists a morphism in $\mathcal{C}$ (the base category!) such that $\varphi_1 = \varphi_2\sigma$ (resp. $\sigma\varphi_1 = \varphi_2$).

Example of no σ for a slice category

For example, note that there exist pairs of morphisms $f_1 \in (Z_1 \to A)$ and $f_2 \in (Z_2 \to A)$ between which there is no morphism that exists in the slice category. One such example we can make is in $(Vect_{\mathbb{R}})_{\mathbb{R}^2}$. If we take the maps:

$$f_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in \mathcal{L}(\mathbb{R}^2)$$

$$f_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in \mathcal{L}(\mathbb{R}^2)$$

There exists no map σ such that the following diagram commutes (since the output of f_1 will always be null in its second coordinate, and the output of f_2 will always be null in the first):

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "\sigma"] \arrow[d, "f_1"'] & \mathbb{R}^2 \arrow[dl, "f_2"] \\ \mathbb{R}^2 & \end{tikzcd}$$

Example of no σ for a coslice category

TODO add potato diagram

Take $A = Z_2 = \{a, b, c\}$, $\varphi_2 = id_A$, and $Z_1 = \{1, 2\}$. Since $|Z_1| = |dom(\sigma)| < |cod(\sigma)| = |Z_2|$, there is no possible case in which $\mathrm{Im}(\sigma) = \mathrm{Im}(id_A)$ (even if φ_1 is epi).

Example of multiple σ

TODO add potato diagram

Another example, this time in a coslice category, with $A = \{a, b, c\}$, we take $\mathbf{Set}_A$. We take $Z_1 = \{1, 2, 3\}$, $Z_2 = \{T, F\}$. For $\varphi_1 : A \to Z_1 = \{(a, 1), (b, 1), (c, 3)\}$ and $\varphi_2 : A \to Z_2 = \{(a, T), (b, T), (c, F)\}$ there exists two elements in $Hom_{\mathcal{S}\rceil\sqcup^{A}}(Z_1, Z_2)$. These originate from two functions in **Set**: $\sigma_\alpha = \{(1, T), (2, T), (3, F)\}$ and $\sigma_\beta = \{(1, T), (2, F), (3, F)\}$. The free element in $2 \in Z_1$ which is not in $\mathrm{Im}(\varphi_1)$ offers a degree of freedom.

Explanations on the conditions for σ

In the first example, what causes the issue is the fact that the images of f_1 and f_2 in A are distinct.

In the second example, what causes the issue is the fact that we've reduced our common domain A to an insufficient intermediary object Z_1.

The "element which isn't mapped to offers a degree of freedom and thus breaks unicity" in the last example should remind you of the notes on surjections. Indeed, when comparing σ_α and σ_β, we're in a context which is reminiscent of the definition of an epimorphism.

$$\varphi_2 = \sigma_\alpha \varphi_1 = \sigma_\alpha \varphi_1$$

We can be assured of being able to cancel $\varphi 1$ iff φ_1 is an epimorphism, and thus σ, if it exists, is unique. The proof for slice categories is similar, but given the reverse orders, uses monomorphisms.

More generally:

- a) in a slice category, there will be no σ if $\mathrm{Im}(\varphi_1) \neq \mathrm{Im}(\varphi_2)$ (proof ? and in more general categories than concrete categories ? TODO: iff ?)
- b) in a coslice category, there will be no σ if $\mathrm{Pr}(\mathrm{Im}(\varphi_1) \neq \mathrm{Pr}(\mathrm{Im}(\varphi_2))$ (proof ? and in more general categories than concrete categories ? TODO: iff ?)
- c) in a slice category, there is at most a single σ iff φ_2 is mono
- d) in a coslice category, there is at most a single σ iff φ_1 is epi

(see perhaps https://ncatlab.org/nlab/show/over+category and https://ncatlab.org/nlab/show/und
)

Chapter 1, Section 4

Notes on counterintuitive rules

- in some categories (such as $\mathbb{Z}$ with $\leq$; or **Ring**), "mono and epi" does not imply "iso"
- in every *abelian* category, we have that "iso $\Leftrightarrow$ epi and mono" (and though **Set** is not abelian, the property still holds)
- while in **Set**, a function is an epimorphism (surjective) iff it has a pre-inverse, in **Grp**, some epimorphisms do not have right inverses.

Chapter 1, Section 5

Initial and terminal objects

- there are categories without either initial or terminal objects, such as the preorder category of $\mathbb{Z}$ with $\leq$.
- there are categories with multiple initial or terminal objects (for example, in **Set**, every singleton set is a terminal object); however, these are respectively unique up to isomorphism
- any object which is both initial and terminal is called a zero object.

Universal properties

"Normal" universal properties

Verbatim: "The most natural context in which to introduce universal properties requires a good familiarity with the language of functors, which we will only introduce at a later stage. [...] We say that a construction satisfies a universal property (or: 'is the solution to a universal problem') when it may be viewed as a terminal object of a category."

Then: "The declaration/explanation of a universal property generally follows the pattern 'object X is universal with respect to the following property: for any Y such that..., there exists a unique morphism Y $\rightarrow$ X such that...'; this explanation hides the definition of an accessory category, and the statement that X is terminal."

This is a complicated way to say: there is some construct to decompose a morphism which is "universal" (always exists) and reduces the rest of the information of the morphism into something "unique" (hence terminal object of some subcategory).

Dual universal properties