Part I

Exercise solutions

Chapter I)

Section 1)

1.1)

In a nutshell, Russell's paradox proves, by contradiction, that certain mathematical collections cannot be sets. It posits the existence of a "set of all sets that don't contain themselves". Such a set can neither contain itself (since in that case, it would be a "set that does contain itself", and should be excluded); nor can it exclude it itself (since in that case, it would be a "set that doesn't contain itself", and should be included).

1.2)

Prove that any equivalence relation over a set S defines a partition of $\mathcal{P}_S$.

a) $\mathcal{P}_S$ has no empty elements: any element in S is part of at least one equivalence class, the class containing at least that element itself. Since there is no equivalence class constructed independently from elements, there are no empty equivalence classes.

b) Elements of $\mathcal{P}_S$ are disjoint: suppose there is an element x that is part of A and B, two distinct equivalence classes. $\forall a \in A, x \sim a$ and $\forall b \in B, x \sim b$. By transivity through x, $\forall a \in A, \forall b \in B, a \sim b$. Therefore, A and B are the same equivalence class: $A = B$. Contradiction. Therefore all elements of $\mathcal{P}_S$ are disjoint subsets of S.

c) The union of all elements of $\mathcal{P}_S$ makes up S: suppose $\exists x \in S$ such that $x \notin \bigcup_{S_i \in \mathcal{P}_S} S_i$. From the argument made in (a), x exists in at least one equivalence class, the class which contains only x itself. This is one of ou S_i sets. Contradiction. Therefore, $\bigcup_{S_i \in \mathcal{P}_S} S_i = S$

1.3)

Given a partition $\mathcal{P}$ on a set S, show how to define a relation $\sim$ on S such that P is the corresponding partition.

The insight here is to build an equivalence relation such that two elements are equivalent if and only if they are part of the same subset of S, which is understood as their common equivalence class.

We define $\sim$ such that $\forall S_i, S_j \in \mathcal{P}, \forall x \in S_i, \forall y \in S_j, x \sim y \Leftrightarrow S_i = S_j$.

Let us prove that $\sim$ is an equivalence relation.

a) Reflexivity:

$$\forall A \in \mathcal{P}, \forall x \in A, A = A \Rightarrow x \sim x$$

b) Symmetry:

$$\forall S_i, S_j \in \mathcal{P}, \forall x \in S_i, \forall y \in S_j, x \sim y \Leftrightarrow S_i = S_j \Leftrightarrow S_j = S_i \Leftrightarrow y \sim x$$

c) Transitivity:

$$\forall S_i, S_j, S_k \in \mathcal{P}, \forall x \in S_i, \forall y \in S_j, \forall z \in S_k, x \sim y \cap y \sim z \Leftrightarrow S_i = S_j \cap S_j = S_k \Rightarrow S_i = S_k \Leftrightarrow x \sim z$$

Therefore, $\sim$ is indeed an equivalence relation, and is generated uniquely by the partition.

1.4)

How many different equivalence relations may be defined on the set $\{1, 2, 3\}$?

If we start with the 1 element set, we have only one possible partition, one possible equivalence class.

With the 2 element set, there are 2 partitions, $\{\{1, 2\}\}$ and $\{\{1\}, \{2\}\}$.

With the 3 element set, there is:

- 1 partition of type 1-1-1: $\{\{1\}, \{2\}, \{3\}\}$.

- 3 partitions of type 2-1: $\{\{1\}, \{2, 3\}\}$, $\{\{2\}, \{1, 3\}\}$, and $\{\{3\}, \{1, 2\}\}$.

- 1 partition of type 3: $\{\{1, 2, 3\}\}$.

Hence, there are five equivalence classes on the 3 element set.
See the Bell numbers: https://oeis.org/A000110

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1.5)

Give an example of a relation that is reflexive and symmetric, but not transitive. What happens if you attempt to use this relation to define a partition on the set?

In terms of graph theory, if we express a set with an internal relation as a graph, we can represent elements are nodes and relationships are edges. Reflexivity means that every node has a loop (unary, self-edge). Symmetry means that the graph is not directed (since every relationship goes both ways). Transitivity means that every connected subset of nodes is a maximal clique (synonymously, every connected component is a complete subgraph).

In a relation which is reflexive and symmetric, but not transitive, you would have connected components of this graph which are not cliques. For these, there is ambiguity as to how you would group their nodes. Two obvious choices would be either:

- to remove the minimal number of edges to obtain n distinct cliques (thereby gaining the transitive restriction of the relation) from a given non-clique;

- to complete the connected subgraph into a clique (thereby gaining the transitive closure of the relation).

1.6)

Define a relation $\sim$ on the set $\mathbb{R}$ of real numbers, by setting $a \sim b \Leftrightarrow b - a \in \mathbb{Z}$. Prove that this is an equivalence relation, and find a 'compelling' description for $\mathbb{R}/\sim$. Do the same for the relation $\approx$ on the plane $\mathbb{R} \times \mathbb{R}$ defined by declaring $(a_1, a_2) \approx (b_1, b_2) \Leftrightarrow b_1 - a_1 \in \mathbb{Z}$ and $b_2 - a_2 \in \mathbb{Z}$.

$b - a \in \mathbb{Z}$ means that 2 real numbers differ by an integral amount. This means that the equivalence relation algebraically describes the idea that "with this relation, 2 real numbers are the same iff they have the same fractional component x (or $1 - x$ for negative numbers)". Eg, $4.76 \sim 1024.76 \sim -5.34$, since $-5.34 + 10 = 4.76$, etc.

To make an algebraic quotient of a set by an equivalence relation, we take the function which maps each element to its corresponding equivalence class, in the set (partition) containing these equivalence class. Intuitively, this is similar to keeping only one representative element per equivalence class. For the example class above, we can keep the representative 0.76. There is such

an equivalence class for every fractional part possible, that is, one for every number in $[0, 1[$. The corresponding map is the "real remainder of division modulo 1". This map is well-defined because each real number has only one output for this map, and all real numbers that are equivalent through $\sim$ are mapped to the same value in the output set.

We should also notice that since $0 \sim 1$, this space loops around on itself. Intuitively, if you increase linearly in the input space $\mathbb{R}$, it goes back to 0 after $0.9999999...$ in the output space. This output space is thus a circle of perimeter 1.

Similarly, $b_1 - a_1 \in \mathbb{Z}$ and $b_2 - a_2 \in \mathbb{Z}$ means that 2 points in the 2D plane are the same iff they differ in each coordinate by an integral amount. This boils down to combining two such loops from the first part of the exercise: one in the x direction and one in the y direction: what this gives is the small square $[0, 1[\times [0, 1[$, which loops to $x = 0$ (resp. $y = 0$) when $x = 1$ (resp. $y = 1$) is reached. This space functions like a small torus, of area 1.

Section 2)

2.1)

How many different bijections are there between a set S with n elements and itself?

Any bijection is a choice of a pairs from 2 sets of the same size, where each element is used only once, and each pair has one element from each set. At first there are n choices in each set. If we pick a pair, we pick from n^2 possibilities

There are then $(n - 1)^2$ choices of pairs left, etc.

Ccl°: $\prod_{i=1}^{i=n} i^2 = (n!)^2$

2.2)

Prove that a function has a right-inverse iff it is surjective (can use AC)

Let $f \in (A \to B)$.

2.2.a) $\Rightarrow$: Suppose that f has a right-inverse (pre-inverse).

We have $\exists g \in (B \to A), f \circ g = id_B$

Suppose that f is not a a surjection. This means $\exists b \in B, \nexists a \in A, b = f(a)$

$f(g(b)) = id_B(b) = b$ Necessarily, $g(b)$ is such an a, so $\exists a \in A, b = f(a)$. Contradiction.

Ccl°:: f is a surjection.

2.2.b) $\Leftarrow$: Suppose that f is a surjection.

$\forall b \in B, \exists a \in A, b = f(a)$

We will construct a pre-inverse for f.

The insight here is to realize that a surjection divides its input set into a partition, where each 2-by-2 disjoint subset corresponds to $f^{-1}(\{q\})$, for every q in the output set. More formally, each "fiber" (preimage of a singleton) is a disjoint subset of the input set, and the union of fibers is the input set itself. You can see this in the following diagram:

(add diagram) 1234 to ab 1a 2a (fiber from a) 3b 4b (fiber from b) https://tex.stackexchange.com/questions/157450/producing-a-diagram-showing-relations-between-sets https://tex.stackexchange.com/questions/79009/drawing-the-mapping-of-elements-for-sets-in-latex

Using AC, we select a single element from each such fiber. For each $q \in B$, we name $p_q \in f^{-1}(\{q\})$ the chosen element. We define g as $g \in (B \to A), g = (q \mapsto p_q)$. With this, $\forall b \in B, f \circ g(b) = b$, and so $f \circ g = id_A$. Thus, f has a preinverse.

A summary of this idea: all surjection preinverses are simply a choice of a representative for each fiber of the surjection as the output to the respective singleton.

2.3)

Prove that the inverse of a bijection is a bijection, and that the composition of two bijections is a bijection.

2.3.a) Using the fact that a function is a bijection iff it has a two-sided

inverse (

Corollary 2.2)

we can see from this defining fact, $f \in (A \to B)$bijective $\Leftrightarrow, \exists f^{-1} \in (B \to A), f^{-1} \circ f = id_A$ and $f \circ f^{-1} = id_B$ that f is naturally f^{-1}'s (unique) two-

sided inverse, and so f^{-1} is also a bijection.

2.3.b) Let be $f \in (A \to B), g \in (B \to C)$, both bijective (hence with inverses in the respective function spaces). Let $h \in (A \to C), h = g \circ f$ and $h^{-1} \in (C \to A), h^{-1} = f^{-1} \circ g^{-1}$. We have:

$$
\begin{aligned}
h^{-1} \circ h &= (f^{-1} \circ g^{-1}) \circ (g \circ f) \\
&= f^{-1} \circ g^{-1} \circ g \circ f \\
&= f^{-1} \circ id_B \circ f \\
&= f^{-1} \circ f \\
&= id_A
\end{aligned}
$$

$$
\begin{aligned}
h \circ h^{-1} &= (g \circ f) \circ (f^{-1} \circ g^{-1}) \\
&= g \circ f \circ f^{-1} \circ g^{-1} \\
&= g \circ id_B \circ g^{-1} \\
&= g \circ g^{-1} \\
&= id_C
\end{aligned}
$$

Therefore h and h^{-1} are two-sided inverses of each other, and thus bijections. From this we conclude that the composition of any two bijections is also a bijection.

2.4)

Prove that 'isomorphism' is an equivalence relation (on any set of sets).

2.4.a) Problem statement

Let $\mathcal{A}$ be a set of sets. We define the relation $\simeq$ between the elements of $\mathcal{A}$ as the following:

$$\forall X, Y \in \mathcal{A}, \ X \simeq Y \Leftrightarrow \text{there exists a bijection between } X \text{ and } Y$$

Let us show that $\simeq$ is an equivalence relation.

2.4.b) Reflexivity

For any set $A \in \mathcal{A}$, the identity mapping on A is a bijection. This means that $\forall A \in \mathcal{A}, A \simeq A$, ie, $\simeq$ is reflexive.

2.4.c) Symmetry

$$\forall X, Y \in \mathcal{A}, \ X \simeq Y \Rightarrow \exists f \in (X \to Y) \text{ bijective}$$
$$\Rightarrow \exists f^{-1} \in (Y \to X) \text{ bijective}$$
$$\Rightarrow Y \simeq X$$

Therefore, $\simeq$ is symmetric.

2.4.d) Transitivity

Let be $X, Y, Z \in \mathcal{A}$. Suppose that $X \simeq Y$ and $Y \simeq Z$. This means $\exists f \in (X \to Y), g \in (Y \to Z)$, both bijections. Let be $h \in (X \to Z), h = g \circ f$. h is also a bijection since the composition of

two bijections is also a bijection (exercise 2.3)

The existence of h implies $X \simeq Z$.

Therefore $\simeq$ is transitive.

2.4.e) Conclusion

Isomorphism, $\simeq$, is a relation on an arbitrary set (of sets) which is always reflexive, symmetric and transitive. It is thus an equivalence relation.

2.5)

Formulate a notion of epimorphism and prove that epimorphisms and surjections are equivalent.

See "notes" file: section "Proofs of mono/inj and epi/surj equivalence".

2.6)

With notation as in Example 2.4, explain how any function $f \in (A \to B)$ determines a section of π_A.

A section is the preinverse of a surjection. Here, the surjection in question is π_A the projection of $A \times B$ onto A.

Let $f \in (A \to B)$.

We now consider the function which maps an input $a \in A$ of f to its "geometric representation" (its coordinates in the enclosing space $A \times B$, corresponding to a point of the graph Γ_f).

$$\hat{f} \in (A \to (A \times B)), \hat{f} = (\ a \mapsto (a, f(a))\ )$$

We notice that $\hat{f}(A) = \Gamma_f$.

Naturally, $\pi_A \circ \hat{f} = (a \mapsto a) = id_A$, therefore, $\hat{f}$ is a pre-inverse (section) of π_A.

This set of relationships can be expressed in the following commutative diagram:

PS: see "On sections and fibers" in the "notes" file for a worked example.

2.7)

Let $f = (A \to B)$ be any function. Prove that the graph Γ_f of f is isomorphic to A.

Using the elements from the previous exercise, we know that $\hat{f}$ is injective from A into $A \times B$. This property is inherited to any restriction of the codomain $Z \subseteq B$, and corresponding implied restriction of the domain to $Y = f^{-1}(Z) \subseteq A$. In particular, here, $Y = A$ and $Z = \Gamma_f = \hat{f}(A)$. We now consider $\overline{f} \in (A \to \Gamma_f), \overline{f} = (a \mapsto \hat{f}(a))$. We can see that $\overline{f}$ is injective from being a restriction of an injective function to a smaller codomain. We also know that $\overline{f}$ is surjective, since its domain is its image. Therefore, $\overline{f}$ is a bijection. This means that $A \simeq \Gamma_f$.

2.8)

Describe as explicitly as you can all terms in the canonical decomposition of the function $f \in (\mathbb{R} \to \mathbb{C})$ defined by $f = (r \mapsto e^{2\pi i r})$. (This exercise matches one assigned previously, which one?)

Firstly, elements of $\mathbb{R}$ are equivalent by this map (they have the same output) if they vary by 1 from each other. This is a reference to the equivalence relation $\sim$ in exercise 1.6. Therefore, we will use $\mathbb{R}/\sim \simeq S^1$ in our decomposition. Obviously, the map from $(\mathbb{R} \to \mathbb{R}/\sim)$, which maps each element of $\mathbb{R}$ to respective their equivalence class is a surjection (since there's no empty equivalence class).

Secondly, as mentioned, we have a bijection $\tilde{f}$ between $\mathbb{R}/\sim$ and S^1, the circle group of unit complex numbers, namely $\tilde{f} = (x \mapsto e^{2\pi i x}$, where each element x of $\mathbb{R}/\sim$ can be understood to correspond to a (class representative) value in the interval $[0, 1[$.

Finally, we do the canonical injection of S^1 into its superset $\mathbb{C}$.

2.9)

Show that if $A \simeq A'$ and $B \simeq B'$, and further $A \cap B = \emptyset$ and $A' \cap B' = \emptyset$, then $A \cup B \simeq A' \cup B'$. Conclude that the operation $A \coprod B$ (as described in §1.4) is well-defined up to isomorphism.

We suppose the aforementioned.

Let f_A be a bijection from $A \to A'$, and f_B be a bijection from $B \to B'$. We define the following:

$$f \in (A \cup B \to A' \cup B'), \text{ such that } \begin{cases} \forall a \in A, \ f(a) = f_A(a) \\ \forall b \in B, \ f(b) = f_B(b) \end{cases}$$

This function is a well-defined function, since $A \cap B = \emptyset$: every element of the domain has one, and only one, possible image.

Similarly, we define:

$$g \in (A' \cup B' \to A \cup B), \text{ such that } \begin{cases} \forall a \in A', \ g(a) = f_A^{-1}(a) \\ \forall b \in B', \ g(b) = f_B^{-1}(b) \end{cases}$$

Similarly, because $A' \cap B' = \emptyset$, g is well-defined.

Let us study $g \circ f$. We have:

$$\begin{cases} \forall a \in A, \ g(f(a)) = f_A^{-1}(f_A(a)) = a \\ \forall b \in B, \ g(f(b)) = f_B^{-1}(f_B(b)) = b \end{cases}$$

Hence, $g \circ f = id_{A \cup B}$. Similarly, $f \circ g = id_{A' \cup B'}$. Therefore, $g = f^{-1}$, f is a bijection, and $A \cup B \simeq A' \cup B'$.

We'll now do a shift in notation. Let be some arbitrary sets A and B. Let be A_1, A_2, B_1, B_2 such that $A_1 = \{1\} \times A$, $A_2 = \{2\} \times A$, $B_1 = \{1\} \times B$, and $B_2 = \{2\} \times B$. This means $A \simeq A_1$, $A \simeq A_2$, $B \simeq B_1$, and $B \simeq B_2$. It also means $A_1 \cap A_2 = \emptyset$ and $B_1 \cap B_2 = \emptyset$. From the above, this implies $A_1 \cup B_1 \simeq A_2 \cup B_2$.

This means that the disjoint union of A and B is indeed well-defined, up to isomorphism: so long as 2 respective copies of A and B are made in a way that their intersection is empty, the 2 respective unions of 1 copy each will be isomorphic.

2.10)

Show that if A and B are finite sets, then $|B^A| = |B|^{|A|}$.

The number of $|B^A|$ functions in $B^A = (A \to B)$ can be counted in the following way.

For each element a of A, of which there are $|A|$, we can pick any element of $|B|$ as the image. This means $|B| \times ... \times |B|$, a total of $|A|$ times. Hence, $|B^A| = |B|^{|A|}$.

2.11)

In view of Exercise 2.10, it is not unreasonable to use 2^A to denote the set of functions from an arbitrary set A to a set with 2 elements (say $\mathbb{B} = \{0, 1\}$). Prove that there is a bijection between 2^A and the power set $\mathcal{P}(A)$ of A.

Simply put, every subset A_i of A is built through a series of $|A|$ choices: for each element a in A, do we keep the element a in our subset A_i (output 1) or do we remove it (output 0) ? It is then easy to see that such a series of choices can easily be encoded as a unique function in $A \to \mathbb{B}$. The totality of such series of choices thus corresponds both to the space $A \to \mathbb{B}$, and to the powerset $\mathcal{A}$, and there is a bijection between the two.

Section 3)

3.1)

Let $\mathcal{C}$ be a category. Consider a structure $\mathcal{C}^{op}$ with: - $Obj(\mathcal{C}^{op}) := Obj(\mathcal{C})$; - for A, B objects of $\mathcal{C}^{op}$ (hence, objects of $\mathcal{C}$), $Hom_{\mathcal{C}^{op}}(A, B) := Hom_{\mathcal{C}}(B, A)$ Show how to make this into a category.

3.1.a) Composition

First, to make things clearer and more rigorous, let us distinguish composition in $\mathcal{C}$ as $\circ$ and composition in $\mathcal{C}^{op}$ as $\star$. We define $\star$ as:

$$
\begin{aligned}
&\forall f \in Hom_{\mathcal{C}^{op}}(B, A) = Hom_{\mathcal{C}}(A, B), \\
&\forall g \in Hom_{\mathcal{C}^{op}}(C, B) = Hom_{\mathcal{C}}(B, C), \\
&\exists h \in Hom_{\mathcal{C}^{op}}(C, A) = Hom_{\mathcal{C}}(A, C), \\
&f \star g := g \circ f = h
\end{aligned}
$$

We will now show that $\mathcal{C}^{op}$ with $\star$ verifies the other axioms of a category (namely identity and assossiativity of composition).

3.1.b) Identity

Since $\mathcal{C}$ is a category, since $\mathcal{C}^{op}$ has the same objects, and since, by definition, for all object A, we have $Hom_{\mathcal{C}^{op}}(A, A) = Hom_{\mathcal{C}}(A, A)$, we can take every $id_A \in Hom_{\mathcal{C}}(A, A)$ as the same identity in $\mathcal{C}^{op}$. We can verify that this is compatible with $\star$:

$$
\begin{aligned}
\forall A, B \in Obj(\mathcal{C}) \quad &= Obj(\mathcal{C}^{op}), \\
\exists id_A \in Hom_{\mathcal{C}}(A, A) \quad &= Hom_{\mathcal{C}^{op}}(A, A), \\
\exists id_B \in Hom_{\mathcal{C}}(B, B) \quad &= Hom_{\mathcal{C}^{op}}(B, B), \\
\forall f \in Hom_{\mathcal{C}}(A, B) \quad &= Hom_{\mathcal{C}^{op}}(B, A), \\
f = f \circ id_A \quad &= id_A \star f, \\
f = id_B \circ f \quad &= f \star id_B
\end{aligned}
$$

3.1.c) Associativity

Using associativity in $\mathcal{C}$, we have:

$$\begin{aligned}
\forall A, B, C, D \in Obj(\mathcal{C}) \quad &= Obj(\mathcal{C}^{op}), \\
\forall f \in Hom_{\mathcal{C}}(A, B) \quad &= Hom_{\mathcal{C}^{op}}(B, A), \\
\forall g \in Hom_{\mathcal{C}}(B, C) \quad &= Hom_{\mathcal{C}^{op}}(C, B), \\
\forall h \in Hom_{\mathcal{C}}(C, D) \quad &= Hom_{\mathcal{C}^{op}}(D, C), \\
h \star (g \star f) = \quad &h \star (f \circ g) \\
= \quad &(f \circ g) \circ h \\
= \quad &f \circ (g \circ h) \\
= \quad &(g \circ h) \star f \\
= \quad &(h \star g) \star f
\end{aligned}$$

Therefore, $\star$ is associative.

We conclude that $\mathcal{C}^{op}$ is a category.

3.2)

If A is a finite set, how large is $End_{\text{Set}}(A)$?

We know that, in Set, $End_{\text{Set}}(A) = (A \to A) = A^A$. From a previous exercise, we know that $|B^A| = |B|^{|A|}$, therefore $|End_{\text{Set}}(A)| = |A|^{|A|}$.

3.3)

Formulate precisely what it means to say that "1_a is an identity with respect to composition" in Example 3.3, and prove this assertion.

Example 3.3 is that of a category over a set S with a (reflexive, transitive) relation $\sim$, where the objects of the category are the elements of S, and the homset between two elements a and b is the singleton (a, b) if $a \sim b$, and $\emptyset$ otherwise. Composition $\circ$ is given by transitivity of $\sim$, where $(b, c) \circ (a, b) = (a, c)$. Reflexivity gives the identities ($id_a = (a, a)$ for any element a).

In this context, to say that "1_a is an identity with respect to composition" means that we can cancel out an element of the form (a, a) from a composition.

Formally, we have:

$$\forall a, b \in S, (b, b) \circ (a, b) = (a, b) = (a, b) \circ (a, a)$$

proving that (b, b) is indeed a post-identity, and (a, a) a pre-identity, in this context.

3.4)

Can we define a category in the style of Example 3.3, using the relation $<$ on the set $\mathbb{Z}$?

(Description of example 3.3 in the exercise 3.3 just above.)

Naively, saying like in example 3.3 "there is a singleton homset $\text{Hom}(a, b)$ each time we have $a < b$", we cannot define such a category, since $<$ is not reflexive, and we would thus lack identity morphisms.

However, in a roundabout way, we can define a category over the *negation* of $<$: "there is a singleton homset $\text{Hom}(a, b)$ each time we DO NOT have $a < b$". Namely this corresponds to the relation $\geq$, which is, itself, reflexive, transitive (and antisymmetric), and is a valid instance of the kind of category presented in example 3.3.

In fact, the pair $(\mathbb{Z}, \geq)$ is an instance of what is called a "totally ordered set", which is a more restrictive kind of "partially ordered set" (also called "poset" for short). Consequently, this kind of category is called a "poset category".

3.5)

Explain in what sense Example 3.4 is an instance of the categories considered in Example 3.3.

(Description of example 3.3 in the exercise 3.3 just above.)

Example 3.4 describes a category $\hat{S}$ where the objects are the subsets of a set S (equivalently: elements of the powerset $\mathcal{P}(S)$ of S), and morphisms between two subsets A and B of S are singleton (or empty) homsets based on whether the inclusion is true (or false).

Inclusion of sets, $\subset$, is also an order relation, this time between the elements of a set of sets (here, $\mathcal{P}(S)$). This means inclusion is reflexive, transitive, and antisymmetric. This makes $\hat{S}$ a poset category, and thus another instance of example 3.3.

3.6)

Define a category V by taking $Obj(V) = \mathbb{N}$, and $Hom_V(n, m) = Mat_{\mathbb{R}}(m, n)$, the set of $m \times n$ matrices with real entries, for all $n, m \in \mathbb{N}$. (I will leave the reader the task to make sense of a matrix with 0 rows or columns.) Use

product of matrices to define composition. Does this category 'feel' familiar
?

The formulation of the exercise is strange. It says to use the product
of matrices to define composition, and to have homsets be sets of matrices,
but objects of the category are supposed to be integers. I don't know of any
matrix with real entries that maps an integer to an integer in this way.

We thus infer that the meaning of the exercise can be one of two things.

Either we suppose the set of objects could rather be understood as "some-
thing isomorphic to $\mathbb{N}$", ie, the collection of real vector spaces with finite bases
(ie, $\forall n \in \mathbb{N}, \mathbb{R}^n$). In which case, this is just the category of real vector spaces
with finite basis (and linear maps as morphisms), which is a subcategory of
the category real vector spaces (commonly called $Vect_{\mathbb{R}}$). In this context,
any morphism starting from $0 \simeq \mathbb{R}^0 = \{0\}$ is just the injection of the ori-
gin into the codomain; and any morphism ending at 0 is the mapping of all
elements to the origin.

Otherwise, we understand this as "yes, the objects of the category are
integers: this means you should ignore the actual content of the matrices,
and instead consider only their effect on the dimensionality of domains and
codomains". In this case, this category is a complete directed graph over $\mathbb{N}$
where each edge corresponds to the change in dimension (from domain to
codomain) caused by a given linear map.

3.7)

Define carefully objects and morphisms in Example 3.7, and draw the dia-
gram corresponding to composition.

Example 3.7 (on coslice categories) refers to example 3.5 (on slice cate-
gories). Let's go over slice categories (since example 3.5 asks the reader to
"check all [their various properties]").

3.7.1) Slice categories

Slice categories are categories made by singling out an object (say A) in
some parent (larger) category (say $\mathcal{C}$), and studying all morphisms into that
object. These morphisms become the objects of a new category (ie, for any
Z of $\mathcal{C}$, $f \in (Z \to A)$ is an object of the slice category, called $\mathcal{C}_A$ in this
context). In the slice category, morphisms are defined as those morphism in
$\mathcal{C}$ that preserve composition between 2 morphisms into A.

Note that there exist pairs of morphisms $f_1 \in (Z_1 \to A)$ and $f_2 \in (Z_2 \to A)$ between which there is no morphism that exists in the slice category. One such example we can make is in $(Vect_\mathbb{R})_{\mathbb{R}^2}$. If we take the maps:

$$f_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in \mathcal{L}(\mathbb{R}^2)$$

$$f_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in \mathcal{L}(\mathbb{R}^2)$$

There exists no map σ such that the following diagram commutes (since the output of f_1 will always be null in its second coordinate, and the output of f_2 will always be null in the first):

$$\begin{array}{ccc} \mathbb{R}^2 & \xrightarrow{\sigma} & \mathbb{R}^2 \\ f_1 \downarrow & \swarrow f_2 & \\ \mathbb{R}^2 & & \end{array}$$

Now, let us prove that $\mathcal{C}_A$ is indeed a category for an arbitrary object A of an arbitrary category $\mathcal{C}$.

3.7.1.a) Identity

A generic identity morphism is expressed diagrammatically in $\mathcal{C}_A$ as:

$$\begin{array}{ccc} Z & \xrightarrow{id_Z} & Z \\ f \downarrow & \swarrow f & \\ A & & \end{array}$$
$$\circlearrowright$$
$$id_A$$

We can see that since $f = f \circ id_Z$ in $\mathcal{C}$, this is compatible with the definition of a (pre-/right-)unit morphism in $\mathcal{C}_A$. Also, since the only maps post-f are maps from $A \to A$, we have id_A as the (post-/left-)unit for every morphism f (ie, $f = id_A \circ f$.

3.7.1.b) Composition

Taking 3 objects of the slice category ($f_1 \in (Z_1 \to A)$, $f_2 \in (Z_2 \to A)$ and $f_3 \in (Z_3 \to A)$), and two morphisms (σ_A mapping f_1 to f_2 via a $\mathcal{C}$-morphism $\sigma \in (Z_1 \to Z_2)$, and τ_A mapping f_2 to f_3 via a $\mathcal{C}$-morphism $\tau \in (Z_2 \to Z_3)$), we have that $f_1 = f_2 \circ \sigma$ and $f_2 = f_3 \circ \tau$. This is expressed as the following commutative diagram.

$$Z_1 \xrightarrow{\ \sigma\ } Z_2 \xrightarrow{\ \tau\ } Z_3$$

$$f_1 \searrow \quad \downarrow f_2 \quad \swarrow f_3$$

$$A$$

Composition of morphisms is then defined as $\tau_A \circ_A \sigma_A$ as a mapping from f_1 to f_3, such that $f_1 = f_3 \circ (\tau \circ \sigma)$. This can be understood through the following commutative diagram:

$$Z_1 \xrightarrow{\ \tau \circ \sigma\ } Z_3$$

$$f_1 \searrow \quad \swarrow f_3$$

$$A$$

Which commutes, because we have:

$$\begin{aligned}
f_1 &= & f_2 \circ \sigma \\
&= & (f_3 \circ \tau) \circ \sigma \\
&= & f_3 \circ (\tau \circ \sigma)
\end{aligned}$$

Thus, we have a working composition of morphisms.

3.7.1.c) Associativity

We take 4 objects of the slice category ($f_1 \in (Z_1 \to A)$, $f_2 \in (Z_2 \to A)$, $f_3 \in (Z_3 \to A)$ and $f_4 \in (Z_4 \to A)$), and three morphisms (σ_A mapping f_1 to f_2, τ_A mapping f_2 to f_3, and υ_A mapping f_3 to f_4). Using composition defined as above, we have

$$\begin{aligned}
f_1 &= & f_4 \circ (\upsilon \circ (\tau \circ \sigma)) \\
&= & f_4 \circ ((\upsilon \circ \tau) \circ \sigma) \\
\Rightarrow & & \\
& & \upsilon_A \circ (\tau_A \circ \sigma_A) \\
&= & (\upsilon_A \circ \tau_A) \circ \sigma_A
\end{aligned}$$

Through associativity in $\mathcal{C}$.

3.7.2) Coslice categories

A coslice category $\mathcal{C}^A$ is similar, except it takes the morphisms coming *from* a chosen object A, rather than those going *to* this object A. Below is a commutative diagram in the style of the one of the textbook for slice categories.

$$\begin{array}{ccc} & A & \\ {\scriptstyle f_1}\swarrow & & \searrow{\scriptstyle f_2} \\ Z_1 & \xrightarrow{\ \ \sigma\ \ } & Z_2 \end{array}$$

We can similarly show that this also defines a category.

3.7.2.a) Identity

A generic identity morphism is expressed diagrammatically in $\mathcal{C}^A$ as:

$$\begin{array}{ccc} & id_A & \\ & \curvearrowright & \\ & A & \\ {\scriptstyle f}\downarrow & \searrow{\scriptstyle f} & \\ Z & \xrightarrow{id_Z} & Z \end{array}$$

We can see that since $f = id_Z \circ f$ in $\mathcal{C}$, this is compatible with the definition of a (post-/left-)unit morphism in $\mathcal{C}^A$. Also, since the only maps pre-f are maps from $A \to A$, we have id_A as the (pre-/right-)unit for every morphism f (ie, $f = f \circ id_A$.

3.7.2.b) Composition

Taking 3 objects of the slice category ($f_1 \in (A \to Z_1)$, $f_2 \in (A \to Z_2)$ and $f_3 \in (A \to Z_3)$), and two morphisms (σ^A mapping f_1 to f_2 via a $\mathcal{C}$-morphism $\sigma \in (Z_1 \to Z_2)$, and τ^A mapping f_2 to f_3 via a $\mathcal{C}$-morphism $\tau \in (Z_2 \to Z_3)$), we have that $f_1 = \sigma \circ f_2$ and $f_2 = \tau \circ f_3$. This is expressed as the following commutative diagram.

$$\begin{array}{ccccc} & & A & & \\ & {\scriptstyle f_1}\swarrow & {\scriptstyle f_2}\downarrow & \searrow{\scriptstyle f_3} & \\ Z_1 & \xrightarrow{\ \sigma\ } & Z_2 & \xrightarrow{\ \tau\ } & Z_3 \end{array}$$

Composition of morphisms is then defined as $\tau^A \circ^A \sigma^A$ as a mapping from f_1 to f_3, such that $f_3 = (\tau \circ \sigma) \circ f_1$. This can be understood through the following commutative diagram:

$$\begin{array}{ccc} & A & \\ {\scriptstyle f_1}\swarrow & & \searrow{\scriptstyle f_3} \\ Z_1 & \xrightarrow{\ \tau\circ\sigma\ } & Z_3 \end{array}$$

Which commutes, because we have:

$$f_3 = \tau \circ f_2$$
$$= \tau \circ (\sigma \circ f_1)$$
$$= (\tau \circ \sigma) \circ f_1$$

Thus, we have a working composition of morphisms.

3.7.2.c) Associativity

We take 4 objects of the slice category ($f_1 \in (A \to Z_1)$, $f_2 \in (A \to Z_2)$, $f_3 \in (A \to Z_3)$ and $f_4 \in (A \to Z_4)$), and three morphisms (σ^A mapping f_1 to f_2, τ^A mapping f_2 to f_3, and υ^A mapping f_3 to f_4). Using composition defined as above, we have

$$f_4 = (\upsilon \circ (\tau \circ \sigma)) \circ f_1$$
$$= ((\upsilon \circ \tau) \circ \sigma) \circ f_1$$
$$\Rightarrow$$
$$\upsilon^A \circ (\tau^A \circ \sigma^A)$$
$$= (\upsilon^A \circ \tau^A) \circ \sigma^A$$

Through associativity in $\mathcal{C}$.

3.8)

A subcategory $\mathcal{C}'$ of a category $\mathcal{C}$ consists of a collection of objects of $\mathcal{C}$, with morphisms $Hom_{\mathcal{C}'}(A, B) \subseteq Hom_{\mathcal{C}}(A, B)$ for all objects A, B in $Obj(\mathcal{C}')$, such that identities and compositions in $\mathcal{C}$ make $\mathcal{C}'$ into a category. A subcategory $\mathcal{C}'$ is *full* if $Hom_{\mathcal{C}'}(A, B) = Hom_{\mathcal{C}}(A, B)$ for all A, B in $Obj(\mathcal{C}')$. Construct a category of *infinite sets* and explain how it may be viewed as a full subcategory of **Set**.

To put it less technically, a "subcategory" $\mathcal{C}'$ is just "picking only certain items of a base category $\mathcal{C}$, and making sure that things stay closed uneder morphism composition". It is "full" if *all* morphisms between the objects that remain are also conserved.

We can construct a category **InfSet** of infinite sets by taking all the objects A of **Set** such that $\nexists n \in \mathbb{N}, |A| = n$, and only homsets between these objects. This is clearly a subcategory of **Set**, since it inherits all identity morphisms, composition works the same, and so does associativity; also, restricting the choice of homsets makes it so that the category is closed (you

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can't reach a finite set via a homset that went from an infinite to a finite set).

For this category to not be full, there would need to be some homset that loses a morphism, or fully disappears, in the ordeal. However, there is no restriction as to the kind of morphism that is conserved, so any homset that is kept is identical to its original version. Finally, homsets between infinite sets are also infinite sets, so they don't disappear in this operation.

Consequently **InfSet** defined as such is a full subcategory of **Set**.

3.9)

An alternative to the notion of multiset introduced in §2.2 is obtained by considering sets endowed with equivalence relations; equivalent elements are taken to be multiple instances of elements 'of the same kind'. Define a notion of morphism between such enhanced sets, obtaining a category **MSet** containing (a 'copy' of) **Set** as a full subcategory. (There may be more than one reasonable way to do this! This is intentionally an open-ended exercise.) Which objects in **MSet** determine ordinary multisets as defined in §2.2, and how? Spell out what a morphism of multisets would be from this point of view. (There are several natural notions of morphisms of multisets. Try to define morphisms in MSet so that the notion you obtain for ordinary multisets captures your intuitive understanding of these objects.) [§2.2, §3.2, 4.5]

Let us recall how multisets were defined in §2.2. Since duplicate elements do not exist in sets, multisets were instead defined as functions from a set S to $\mathbb{N}*$, the set of (nonzero) positive integers. This allows each element in S to have a "count", thereby encoding the intuitive notion of multiset. A similar, and equivalent (isomorphic), way of defining it is *via* pairs $(s, n) \in S \times \mathbb{N}*$, which is simpler to think about. We'll call this category **CMSet**, for "count multiset" (TODO: probably has a conventional and better name, but I don't know it). As for morphisms in **CMSet**, we can consider that for any multisets $A = S_A \times \mathbb{N}*$ and $B = S_B \times \mathbb{N}*$, the homset from A to B is simply the set functions from $S_A \times \mathbb{N}*$ to $S_B \times \mathbb{N}*$ as usual.

We first notice that if we restrict **CMSet** to only the objects for which all elements have a count of 1, and where morphisms only ever output to $\{1\}$ in the second coordinate (a subcategory we can call **C1MSet**, for example), we get a "copy" of **Set**: **C1MSet** and **Set** are isomorphic. This is a full subcategory because there are no morphisms that map counts to anything

else than $\{1\}$ if we restrict our objects to this form; so all morphisms between the kept objects are also kept.

Now let us do a similar construction, but based on equivalence classes instead. We know that each equivalence class over a set corresponds uniquely to a partition of that set. By considering only these partitions (these "sets of sets") as objects, we can build a category **EMSet** (for "equivalence multiset"). The "count" corresponds simply to the cardinal of a top-level element in the partition. For example, the top-level elements of $M = \{S_1, S_2, S_3\} = \{\{a\}, \{b, c\}, \{d, e, f\}\}$ would be understood to have counts $|S_1| = 1$, $|S_2| = 2$ and $|S_3| = 3$ respectively.

As for morphisms in **EMSet**, they simply map each top-level element of the domain multiset (a distinct subset of the original set) to some other top-level elements in the codomain multiset. This has precisely the same effect as mapping pairs of "value and count" as seen in the previous **CMSet** construction.

In this example, any set itself, when "injected" (by a functor) into **EMSet** would just nest all of its elements into singletons. I.e., $S = \{a, b, c\}$ in **Set** would become $S = \{\{a\}, \{b\}, \{c\}\}$ in **EMSet**. This also shows how restricting **EMSet** to "only objects that are a set of (toplevel) singletons" makes **EMSet** have a "copy" of **Set** as a full subcategory (for similar arguments as above).

3.10)

Since the objects of a category $\mathcal{C}$ are not (necessarily) sets, it is not clear how to make sense of a notion of 'subobject' in general. In some situations it does make sense to talk about subobjects, and the subobjects of any given object A in $\mathcal{C}$ are in one-to-one correspondence with the morphisms $A \to \Omega$ for a fixed, special object Ω of $\mathcal{C}$, called a subobject classifier. Show that **Set** has a subobject classifier.

We define the set $\mathbb{B} = \{0, 1\}$, aka the binary alphabet or booleans, as the subobject classifier of **Set**. For any subset A of B, there is a unique map $f : B \to \mathbb{B}$, such that $\forall b \in B, f(b) = 1 \Leftrightarrow b \in A$ (otherwise $f(b) = 0$, of course, as the equivalence and lack of alternatives to 0 as an output imply). The map f always fully describes A from its relationship with B.

3.11)

Draw the relevant diagrams and define composition and identities for the category $\mathcal{C}^{A,B}$ mentioned in Example 3.9. Do the same for the category $\mathcal{C}^{\alpha,\beta}$ mentioned in Example 3.10. [§5.5, 5.12]

For lack of a better term, we will refer to the categories of the form $\mathcal{C}_{A,B}$ represented by Example 3.9 as "bi-slice categories". The first part of the exercise is thus asking us to define and explain what "bi-coslice categories" (of the form $\mathcal{C}^{A,B}$) are.

Similarly, we will refer to the categories of the form $\mathcal{C}_{\alpha,\beta}$ represented by Example 3.10 as "fibered bi-slice categories". The second part of the exercise is thus asking us to define and explain what "fibered bi-coslice categories" (of the form $\mathcal{C}^{\alpha,\beta}$) are.

We will, of course, attempt to make more formal and pedagogical all definitions broached in the textbook's examples as well.

3.11.1) Bi-slice categories

3.11.1.a) Objects and morphisms

To make a bi-slice category $\mathcal{C}_{A,B}$, we pick 2 objects A and B of a base category $\mathcal{C}$, and consider for all other objects Z of $\mathcal{C}$, all pairs of morphisms $(f,g) \in (Z \to A) \times (Z \to B)$. These pairs of morphisms are the objects of the bi-slice category $\mathcal{C}_{A,B}$. Morphisms $\sigma_{A,B}$ are defined from an object $p_1 = (f_1, g_1) \in (Z_1 \to A) \times (Z_1 \to B)$ to an object $p_2 = (f_2, g_2) \in (Z_2 \to A) \times (Z_2 \to B)$ so that we have both $f_1 = f_2 \circ \sigma$ and $g_1 = g_2 \circ \sigma$, for some $\sigma \in (Z_1 \to Z_2)$.
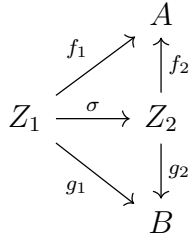
A generic object in $\mathcal{C}_{A,B}$ is of the form:
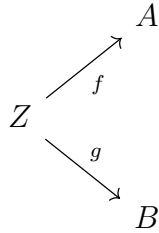
3.11.1.b) Morphisms
Morphisms are defined between objects as
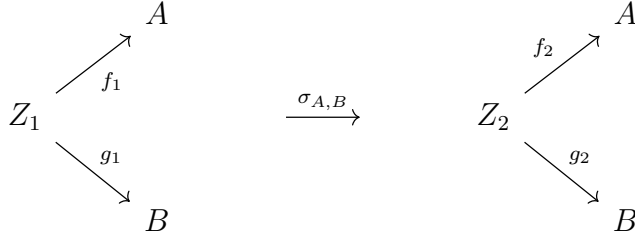
such that the following diagram commutes


3.11.1.c) Identity

It is clear that identity morphisms exist for all objects, simply by taking $Z = Z_1 = Z_2$, $f_1 = f_2$, $g_1 = g_2$ and $\sigma = id_Z$, in the diagram above.

3.11.1.d) Composition

Let be 3 objects of $\mathcal{C}_{A,B}$, which we will name p_1, p_2 and p_3 (and define with the respective (Z_n, f_n, g_n) triplet for p_n).

Composition $\tau_{A,B} \circ \sigma_{A,B} = p_1 \mapsto p_3$ of two morphisms $\sigma_{A,B} = p_1 \mapsto p_2$ and $\tau_{A,B} = p_2 \mapsto p_3$ is defined so that the following diagram commutes.
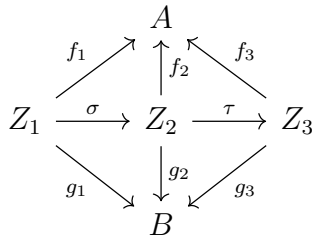
3.11.1.e) Associativity

Associativity follows from associativity of morphisms in $\mathcal{C}$, similarly to what was done for slice categories in exercise 3.7 .

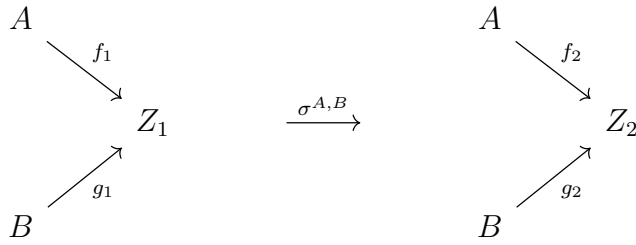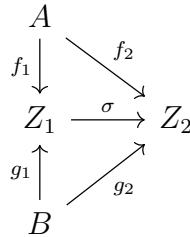**3.11.2) Bi-coslice categories**

3.11.2.a) Objects and morphisms

To make a bi-coslice category $\mathcal{C}^{A,B}$, we similarly pick 2 objects A and B of our base category $\mathcal{C}$, but instead consider, for all other objects Z of $\mathcal{C}$, all pairs of morphisms $(f, g) \in (A \to Z) \times (B \to Z)$.

A generic object in $\mathcal{C}^{A,B}$ is of the form:
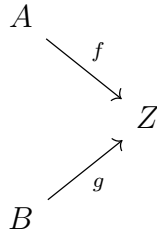
$$
\begin{array}{c}
A \\
\searrow{\scriptstyle f} \\
Z \\
\nearrow{\scriptstyle g} \\
B
\end{array}
$$

3.11.2.b) Morphisms

Morphisms are defined between objects as

$$
\begin{array}{ccc}
A & & A \\
\searrow{\scriptstyle f_1} & & \searrow{\scriptstyle f_2} \\
Z_1 \quad\xrightarrow{\ \sigma^{A,B}\ }\quad & & Z_2 \\
\nearrow{\scriptstyle g_1} & & \nearrow{\scriptstyle g_2} \\
B & & B
\end{array}
$$

such that the following diagram commutes

$$
\begin{array}{c}
A \\
{\scriptstyle f_1}\downarrow \quad \searrow{\scriptstyle f_2} \\
Z_1 \xrightarrow{\ \sigma\ } Z_2 \\
{\scriptstyle g_1}\uparrow \quad \nearrow{\scriptstyle g_2} \\
B
\end{array}
$$

3.11.2.c) Identity

It is clear that identity morphisms exist for all objects, simply by taking $Z = Z_1 = Z_2$, $f_1 = f_2$, $g_1 = g_2$ and $\sigma = id_Z$, in the diagram above.

3.11.2.d) Composition

Let be 3 objects of $\mathcal{C}^{A,B}$, which we will name p_1, p_2 and p_3 (and define with the respective (Z_n, f_n, g_n) triplet for p_n).

Composition $\tau^{A,B} \circ \sigma^{A,B} = p_1 \mapsto p_3$ of two morphisms $\sigma^{A,B} = p_1 \mapsto p_2$ and $\tau^{A,B} = p_2 \mapsto p_3$ is defined so that the following diagram commutes.

$$
\begin{array}{ccccc}
 & & A & & \\
 & \overset{f_1}{\swarrow} & \downarrow f_2 & \overset{f_3}{\searrow} & \\
Z_1 & \overset{\sigma}{\longrightarrow} & Z_2 & \overset{\tau}{\longrightarrow} & Z_3 \\
 & \underset{g_1}{\nwarrow} & \uparrow g_2 & \underset{g_3}{\nearrow} & \\
 & & B & &
\end{array}
$$

<u>3.11.2.e) Associativity</u>

Associativity follows from associativity of morphisms in $\mathcal{C}$, similarly to what was done for slice categories in exercise 3.7 .

3.11.3) Fibered bi-slice categories

<u>3.11.3.a) Objects</u>

To build a fibered bi-slice category $\mathcal{C}_{\alpha,\beta}$, one takes a base category $\mathcal{C}$, as well as a fixed pair of morphisms $\alpha : A \to C$ and $\beta : B \to C$ in $\mathcal{C}$, that point to a common object C of $\mathcal{C}$. Our basic "fixed construct" from $\mathcal{C}$ looks like so:

$$
\begin{array}{ccc}
A & & \\
 & \overset{\alpha}{\searrow} & \\
 & & C \\
 & \overset{\beta}{\nearrow} & \\
B & &
\end{array}
$$

The role of the category $\mathcal{C}_{\alpha,\beta}$ is now to study the morphisms into this construct. A generic object from this new category looks like so:

$$
\begin{array}{ccccc}
 & & A & & \\
 & \overset{f}{\nearrow} & & \overset{\alpha}{\searrow} & \\
Z & & & & C \\
 & \underset{g}{\searrow} & & \underset{\beta}{\nearrow} & \\
 & & B & &
\end{array}
$$

such that the diagram commutes. This means that valid object in $\mathcal{C}_{\alpha,\beta}$ are triplets (Z, f, g), with $f : Z \to A$ and $g : Z \to B$, such that $\alpha \circ f = \beta \circ g$. In a caricatural way, this boils down to studying "the comparison of the different paths one can use to reach C, knowing that the last steps are on one hand, α, and on the other, β".

3.11.3.b) Morphisms

Morphisms are defined between objects as:
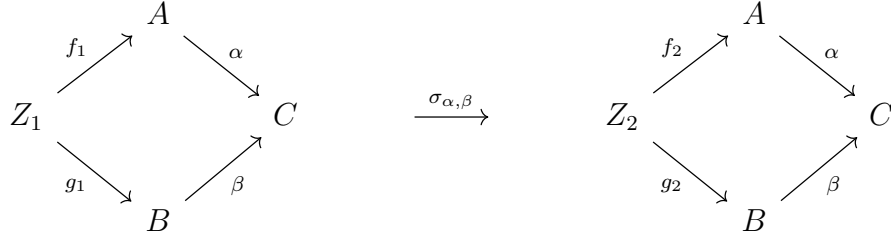
such that the following diagram commutes
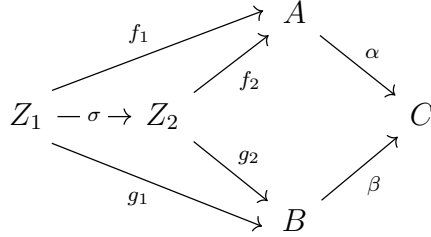
3.11.3.c) Identity

Once again, it is clear that identity morphisms exist for all objects, simply by taking $Z = Z_1 = Z_2$, $f_1 = f_2$, $g_1 = g_2$ and $\sigma = id_Z$, in the diagram above.

3.11.3.d) Composition

Let be 3 objects of $\mathcal{C}_{\alpha,\beta}$, which we will name p_1, p_2 and p_3 (and define with the respective (Z_n, f_n, g_n) triplet for p_n).

Composition $\tau_{\alpha,\beta} \circ \sigma_{\alpha,\beta} = p_1 \mapsto p_3$ of two morphisms $\sigma_{\alpha,\beta} = p_1 \mapsto p_2$ and $\tau_{\alpha,\beta} = p_2 \mapsto p_3$ is defined so that the following diagram commutes.
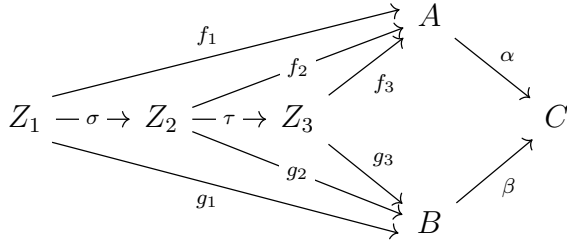
Associativity follows from associativity of morphisms in $\mathcal{C}$, similarly to what was done for slice categories in exercise 3.7 .

3.11.4) Fibered bi-coslice categories

3.11.4.a) Objects

To build a fibered bi-coslice category $\mathcal{C}^{\alpha,\beta}$, one takes a base category $\mathcal{C}$, as well as a fixed pair of morphisms $\alpha : C \to A$ and $\beta : C \to B$ in $\mathcal{C}$, that originate from a common object C of $\mathcal{C}$. Our basic "fixed construct" from $\mathcal{C}$ looks like so:
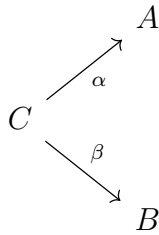
The role of the category $\mathcal{C}^{\alpha,\beta}$ is now to study the morphisms from this construct. A generic object from this new category looks like so:
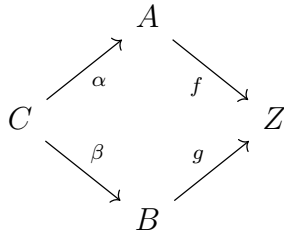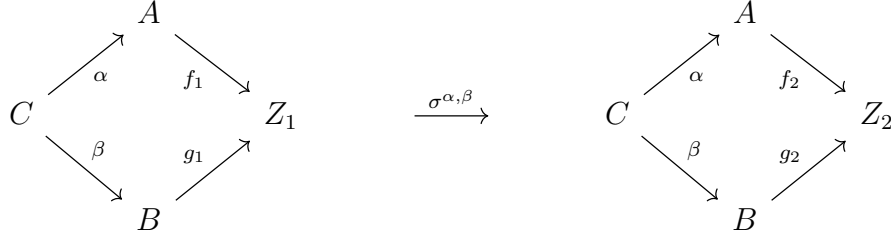

such that the diagram commutes. This means that valid object in $\mathcal{C}^{\alpha,\beta}$ are triplets (Z, f, g), with $f : A \to Z$ and $g : B \to Z$, such that $f \circ \alpha = g \circ \beta$. In a caricatural way, this boils down to studying "the comparison of the
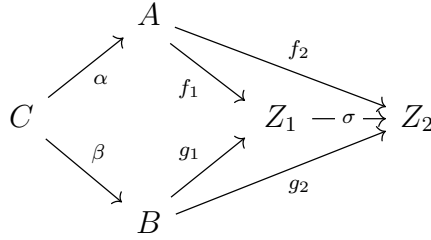
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different paths one can build by starting from C, knowing that the choice of first step is on one hand, α, and on the other, β".

3.11.4.b) Morphisms
Morphisms are defined between objects as:

such that the following diagram commutes


3.11.4.c) Identity
Once again, it is clear that identity morphisms exist for all objects, simply by taking $Z = Z_1 = Z_2$, $f_1 = f_2$, $g_1 = g_2$ and $\sigma = id_Z$, in the diagram above.

3.11.4.d) Composition
Let be 3 objects of $\mathcal{C}^{\alpha,\beta}$, which we will name p_1, p_2 and p_3 (and define with the respective (Z_n, f_n, g_n) triplet for p_n).

Composition $\tau^{\alpha,\beta} \circ \sigma^{\alpha,\beta} = p_1 \mapsto p_3$ of two morphisms $\sigma^{\alpha,\beta} = p_1 \mapsto p_2$ and $\tau^{\alpha,\beta} = p_2 \mapsto p_3$ is defined so that the following diagram commutes.
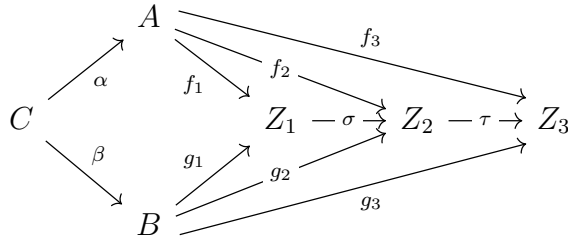
3.11.4.e) Associativity

Associativity follows from associativity of morphisms in $\mathcal{C}$, similarly to what was done for slice categories in exercise 3.7 .

Part II

Extra exercises by/for the group

Chapter I) 1) Set notation)

Write the following in set notation (as a list of numbers, and algebraically):

- the set of all odd integers

- the set of all integers that are not multiples of 3

- the set of integers from 10 (included) to 20 (included)

- the set of integers from 10 (included) to 20 (excluded)

- the set of pairs of integers with both elements of the same value

- the set of triplets of real numbers that together sum to 1

- the set of pairs of positive real numbers that together sum to 1

- the set of n-tuplets (for any n) of real number that together sum to 1

- the set of all natural numbers such that there exists at least one triplet of positive even numbers which are all different and which sum to that number.

Now take the sets in their algebraic notation, and represent them both as a list of numbers (as a logical sequence or just a couple of examples), and as a "description" of what they are:

- $\{3n + 2 \mid n \in \mathbb{N}\}$

- $\{3k + 2 \mid k \in \mathbb{Z}\}$

- $\{2^i \mid i \in [[0, 10]]\}$

- $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$

- $\{x \in \mathbb{R} \mid -2 \le x \le 2\}$

- $\{(m, n, p) \in \mathbb{N}^3 \mid m + n + p = 10\}$