Multiversity *Algebra Chapter 0* Reading Group

# Summaries

Chapter I)

Section 1) Explains fundamentals of set theory and basic set operations

Section 2) Explains set relations, set functions and some more advanced set operations

Section 3) Presents categories, and multiple examples of categories. Some are simple, some are advanced.

# Group Weekly Reports

Week 1 : Today we mostly talked about the first chapter first section’s reading; going over the vocabulary term by term (see the lexicon on the github repo), and going more in depth over certain concepts (particularly relating to set relations). We also saw a bit of a "teaser" of how these notions are used. We did not go over the exercises since not everyone had done them.

Week 2 : Today we continued on discussing the first chapter, it was mostly freeform. We mostly talked about foundations of set theory (mostly stemming from the discussion of exercise 1 on russell’s paradox), why we use function notation the way we do, and about some of the operators over sets themselves (including through some examples from linear algebra and things like the subobject classifier which is seen at the end of section 3).

Week 3 : Today we finished discussing the first chapter. We went over all exercises. We mostly spoke about equivalence relations and partitions. We also spoke about the geometry/topology of quotients of sets by equivalence relations. This was naturally related to exercises 1.2 to 1.7.

Week 4 : We went over monomorphisms and epimorphisms in more depth. We corrected exercises 2.1 to 2.3 (included)

Week 5 : We went in depth over the distinction between isomorphisms and bijections (foreshadowing a bunch of category theory while we were at it) and corrected exercises 2.4 and 2.5.

Week 6 : We went in depth over the notion of section. We corrected exercises 2.6 and 2.7. For the latter exercise, we understood Tristan’s solution by ourselves ! (written by Amric)

# Notes

## Chapter 1, Section 1

Go check out the extra exercises on set notation.

## Chapter 1, Section 2

### On injections and surjections

#### Injections

Injections (which aren’t also surjections) have multiple left-inverses (post-inverses). Eg:

It is precisely the free element with no antecedent in (here, ) which leaves room for multiple choices, but doesn’t affect the overall inversion process.

#### Surjections

Surjections (which aren’t also injections) have multiple right-inverses (pre-inverses), called sections.

It is precisely the fact that there are multiple elements that map to the same element (here, and to ) which leaves room for multiple choices, but doesn’t affect the overall inversion process.

#### Cancellations

Function Cancellation Lemma: If a composition of functions cancels out, then the first of the pair is an injection, and the second of the pair is a surjection. Algebraically:

Corollary 1: any post-inverse of an injection is a surjection.

Corollary 2: any pre-inverse of a surjection is an injection.

Proof: Let be

a) Suppose is not an injection. This means:

However, with such an , we have:

This means that is an injection. Contradiction.

Conclusion: in this context, must be an injection.

b) Suppose is not a surjection. This means:

Since , that means that , this means that . Contradiction.

Conclusion: in this context, must be a surjection.

### On sections and fibers

Section example with a tangent bundle.

Consider the cylinder , and the function , the projection onto the circle. The cylinder is itself the space in which one can easily represent maps of . Each such map corresponds to a section.

Let be

We have

(TODO add diagrams for S1xR, g1 and g2)

A fiber is the preimage of a singleton. In the case of above, for every , is the copy of the real line on the cylinder that passes by .

(TODO add diagram)

### Alternative characterization of a bijection

" is bijective" ("every element of has a non-empty fiber" (surjection) and "every fiber is a singleton" (injection))

### On monomorphisms and epimorphisms

#### Failing the mono/epi condition

##### Example of failing the monomorphism definition for a non-injection

Monomorphism definition:

$$\text{$f : A \to B$ is a monomorphism}
\\ \Leftrightarrow \\
\forall Z \in \text{Obj}(\mathcal{C}), \;
\forall g\_1, g\_2 \in \text{Hom}(Z, A), \;
(f \circ g\_1 = f \circ g\_2 \Rightarrow g\_1 = g\_2)$$

The multiple choice of elements (here, and ) in which map to in is precisely what allows the overall composition to be equal, even when . This provides insight into a proof of " is a monomorphism implies that is an injection". If you suppose that is a monomorphism and not an injection, you can easily reach a contradiction, since you can use elements like and that both map to the same to construct a counter-example to the implication that defines a monomorphism.

##### Example of failing the epimorphism definition for a non-surjection

Epimorphism definition:

$$\text{$f : A \to B$ is an epimorphism}
\\ \Leftrightarrow \\
\forall Z \in \text{Obj}(\mathcal{C}), \;
\forall g\_1, g\_2 \in \text{Hom}(B, Z), \;
(g\_1 \circ f = g\_2 \circ f \Rightarrow g\_1 = g\_2)$$

The element in not being reached by is precisely that which provides the opportunity to build such that they compose to the same result with , since the output of for them doesn’t affect the overall composition. This provides insight into a proof of " is an epimorphism implies that is a surjection". If you suppose that is an epimorphism and not a surjection, you can easily reach a contradiction, since you can use elements like that are not reached by to construct a counter-example to the implication that defines an epimorphism.

#### Proofs of mono/inj and epi/surj equivalence

Let .

The parts which are "Injection => Monomorphism" and "Surjection => Epimorphism" both use the respective sided inverses to prove the implication.

The other parts use the following tautology to prove the implication by contradiction. "Suppose that and , show that it leads to a contradiction".

##### Injection => Monomorphism

Suppose that is an injection. It thus has post-inverses.

From there:

We conclude that is also a monomorphism.

##### Surjection => Epimorphism

Suppose that is a surjection. It thus has pre-inverses.

From there:

We conclude that is also an epimorphism.

##### Monomorphism => Injection

Suppose that is a monomorphism.

Suppose now that is not an injection. Algebraically, this means that:

We can construct and such that but , using such a pair . Thereby, we prove that is not an monomorphism and arrive at a contradiction.

(If is the empty set, being initial in , there is only 1 map into (the empty map) and always hold. Therefore, any counterexample to the epimorphism definition needs to have at least 1 element.)

Let .

Clearly, . However, we also have:

This means that is not a monomorphism: contradiction.

Conclusion: is an injection.

##### Epimorphism => Surjection

Suppose that is an epimorphism.

Suppose now that isn’t a surjection. Algebraically, this means that:

We can construct and such that but , using such an . Thereby, we prove that is not an epimorphism and arrive at a contradiction.

(If is the singleton set, being terminal in , there is only 1 map into and always hold. Therefore, any counterexample to the epimorphism definition needs to have at least 2 elements. We will however use a 3-element set, since it makes things more intuitive and pedagogical.)

Let .

Clearly, . However, since is the domain of , of , and of , we have:

This means that is not an epimorphism: contradiction.

Conclusion: is a surjection.

## Chapter 1, Section 3

### On terminal and initial objects in **Set**

If $\empty$ is initial and is terminal, it is because a function in (in categorical terms) must always have an output for every input. Ie, in category theory, all functions are maps ("applications").

Said algebraically:

$$\forall A, B \in \text{Obj}(\bold{Set}), \;
\forall a \in A, \;
\forall f \in \text{Hom}(A, B), \;
\exists f(a) \in B$$

In the case of $\empty$ as the input set, and there is only one function $f: \empty \to Z$ for any : is the empty mapping. But any $Z \to \empty$ (expept $\empty \to \empty$) contains no mapping (since we’d necessarily be ignoring at least one element of ).

Similarly, in the case of the (unique up-to-isomorphism) singleton set as the output, you’d have multiple functions (precisely ) into it, if you could ignore some elements of the input set. However, if all elements of the input set are required, that leaves you with only one function possible from : the function which maps all elements to .

### Restrictions and extensions of functions, and its consequences on a function’s nature

8 possibilities to study, based on the following binary dichotomies:

* injection or surjection
* enlarging or restricting
* domain or codomain

Note that enlarging the domain sometimes implies enlarging the codomain, and restricting the codomain sometimes implies restricting the domain.

Legend: Yes, No, Depends

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  | enlarge dom | restrict dom | enlarge cod | restrict cod |
| injection | D | Y | Y | Y |
| surjection | Y | D | N | Y |

Theorems:

A) if , then , the restriction of the function to the corresponding smaller codomain, is also an injection (resp. surjection).

B) if , then (with the the canonical injection of into its superset ), is also an injection (resp. is never a surjection).

C) if , then , we have that is also an injection. However, one can construct such that stops being a surjection.

D) if , then , we have that is also a surjection. However, one can construct such that stops being a injection.

Proof: TODO

# Lexicon

## Chapter 1

### Section 1

* Set (not a multiset)
* ∅: the empty set, containing no elements;
* N: the set of natural numbers (that is, nonnegative integers);
* Z: the set of integers;
* Q: the set of rational numbers;
* R: the set of real numbers;
* C: the set of complex numbers.
* Singleton:
* ∃: existential quantifier, "there exists"
* ∀: universal quantifier, "for all"
* inclusion:
* subset:
* cardinal:
* powerset:
* ∪: the union:
* ∩: the intersection:
* $\\$: the difference:
* : the disjoint union:
* ×: the (Cartesian) product:
* complement of a subset
* relation
* order relation
* equivalence relation
* reflexivity
* symmetry
* antisymmetry
* transitivity
* partition
* quotient by an equivalence relation

### Section 2

* function
* graph
* (categorical, function) diagram
* identity function
* kernel (of a function)
* image (of a function)
* restriction (of a function to a subset)
* multiset
* composition
* commutative (diagram)
* injection
* surjection
* bijection
* isomorphism
* inverse
* pre-inverse, right-inverse
* post-inverse, left-inverse
* monomorphism
* epimorphism
* natural projection
* natural injection
* canonical decomposition (of a function)

### Section 3

* category
* object
* morphism
* endomorphism
* operation
* discrete category
* small category
* locally small category
* slice category
* coslice category
* comma category (mentioned, undefined)
* pointed set
* category ??