Cheatsheet: $\mathbb{C}$, space of the complex numbers

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Prerequisites: A decent understanding of real numbers (cf. Data sheet on $\mathbb{R}$), knowing how to read the language of set theory, as well as the definition of an algebraic field and of a vector will be useful (even if we will come back to these two notions with some examples in this sheet; cf. 01-Sets_Logic_Structures_Categories.pdf). The notion of polynomial is mentioned, but not essential in this sheet. Other notions (quotient of algebraic structures; geometric algebras) are discussed but are given as a "bonus": understanding the sections on these topics is not necessary to understand complex numbers.

1 Introduction

The space of complex numbers plays a fundamental role in several areas of mathematics. It is often referred to as the space of "imaginary numbers" because when they were invented/discovered, they were not thought to play a concrete physical role. "Complex" comes back to the idea that they are numbers "in several pieces" (precisely in 2 pieces, they are numbers that are comparable to 2-dimensional vectors). A good name for them, too infrequently used, would be to call them "transversal" numbers: this is related to the nature of the geometry that these numbers describe.

It all started from the need to solve equations containing squares or cubes (polynomial equations), but where the square seemed to be a negative number. The first guy who thought "who cares if I'm allowed or not, let's try to define a number that I can use as a square root of a negative number", at least in Europe, was Gerolamo Cardano (whose name we have kept for the method of solving polynomials of degree 3, called "Cardan's method") around the middle of the 16th century; his work was shortly followed by Bombelli, who demonstrated the necessity of the existence of i. It was above all Euler, Wessel, Argand and Gauss who formalized complex numbers in a form close to that still in use today, during the 18th century.

[I say "in Europe", because much mathematics was simultaneously discovered/invented in India, without contact or exchange with Europe, and quite often before Europe (at least until the 19th century roughly), and my culture of mathematical history isn't that advanced - oftentimes, historians themselves have difficulty with these details, given how rare historians competent in mathematics actually are].

It turns out that Cardan "was right"; that the choice of saying "there is a i number such that $i^2 = -1$" is quite valid if one positions themself in the right context. This choice generates a coherent and rich algebra. Not only that, but complex numbers find a lot of application in physics. They allow us to solve differential equations (especially in electronics), are fundamental for Fourier analysis and signal processing, and the whole of quantum mechanics can roughly be described as "the study of function spaces with complex values".

2 Algebra and geometry of the complex numbers

2.1 General presentation

Basically, complex numbers are points in a 2D plane. The peculiarity of complex numbers is that they are numbers that express translations, scalings/dilations (linear stretchings, also called "homotheties") and rotations in their operations. This combination of operations, which is said to be "conformal" (meaning that any object, even if it is magnified, shrunk, or moved, keeps the same angles), is called a "similarity". $\mathbb{C}$ is the space of choice for manipulating similarities in 2D.

Just as we often use the letter x to designate any real number, we often use the letter z to designate any complex number.

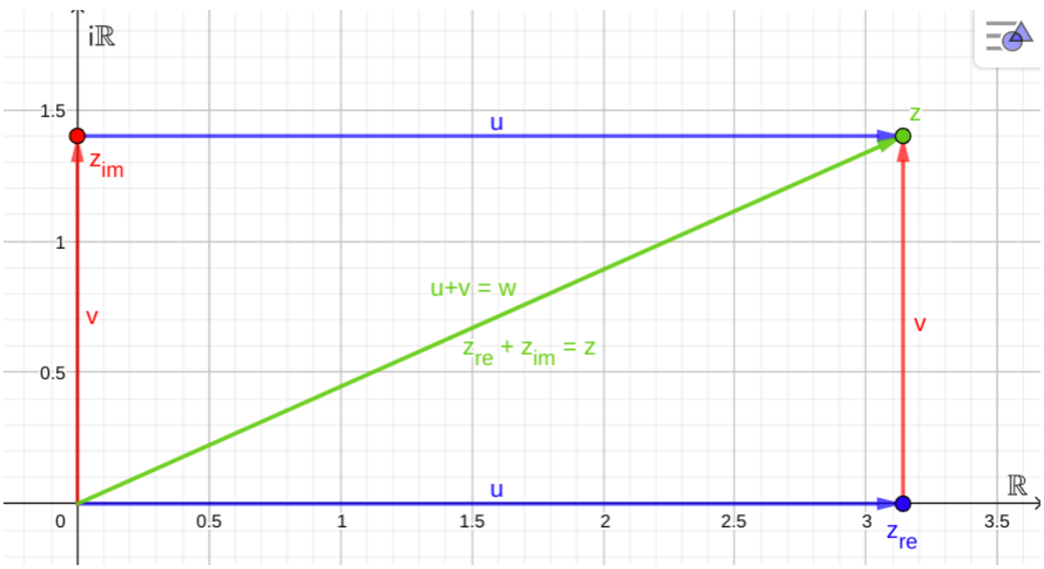

Figure 1: The complex number $z = \pi + 1.4i$. We see here how this number decomposes into a real part $\mathrm{Re}(z) = \pi$ on the $\mathbb{R}$ axis, and an imaginary part $\mathrm{Im}(z) = 1.4$ on the $i\mathbb{R}$ axis.

The horizontal axis (abscissa) is called the "real line" (it is a copy of our usual $\mathbb{R}$, even for multiplication); it is usually noted as $\mathbb{R}$. The vertical line is called the "pure imaginary line" (it is a copy of our usual $\mathbb{R}$, but it does not have the same behavior for multiplication); it is generally noted as $i\mathbb{R}$.

Indeed, the number i is called "the imaginary unit", and this number is used to construct all the roots of negative numbers, because it has the property $i^2 = -1$ for complex multiplication.

NB: We often call i "*the* (square) root of -1", but this is technically an error, because $-i$ is a second square root of -1 (and there is no other one, because the polynomial $x^2 + 1$, corresponding to the equation $x^2 = -1$, is of degree 2 so has exactly 2 roots).

NB: in physics, you'll often see the choice of the letter j rather than i, since complex numbers play a fundamental role in the theory of signals, and thus in electronics. However, in electronics, the letter i is often reserved instead for the intensity of electrical current.

2.2 Notations

For reasons that relate to algebra (what works best with the addition/translation family on the one hand; and what works best with the multiplication/scaling+rotation family on the other), there are two usual notations for complex numbers.

- The **Cartesian notation** (also called "**additive form**"): $z = x + iy$, where x and y are any real number.

- The **polar notation** (also called "**multiplicative form**"): $re^{i\theta}$, where r is a positive real number, and θ is an angle in radians (real number of the interval $[0, tau[$, where "tau", $\tau = 2\pi \approx 6.283185\ldots$).

2.2.1 Cartesian notation

The Cartesian notation (also called vector form, or additive form) of a complex number, $z = x+iy$, is the one that works well with addition. It is almost the same as the usual notation for 2D vectors, where we sum the components along the each coordinate (along a given basis) of the vector to give a total vector. If we take a vector $u = (2, 5) = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$, and basis vectors $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, then u can be written:

$$u = \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 5 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 5 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 2e_1 + 5e_2$$

Choosing to write $e_1 = 1$ and $e_2 = i$ gives us precisely the Cartesian notation of complex numbers. This analogy that we have just made between

$\mathbb{C}$ and $\mathbb{R}^2$ will be worked on in more detail below.

We call x the **real part** of z, and y its **imaginary part**. x is the displacement along the horizontal axis, y is the displacement along the vertical axis. We have the operators $\mathrm{Re}(z) = x$ and $\mathrm{Im}(z) = y$ to extract the additive components of a complex number.

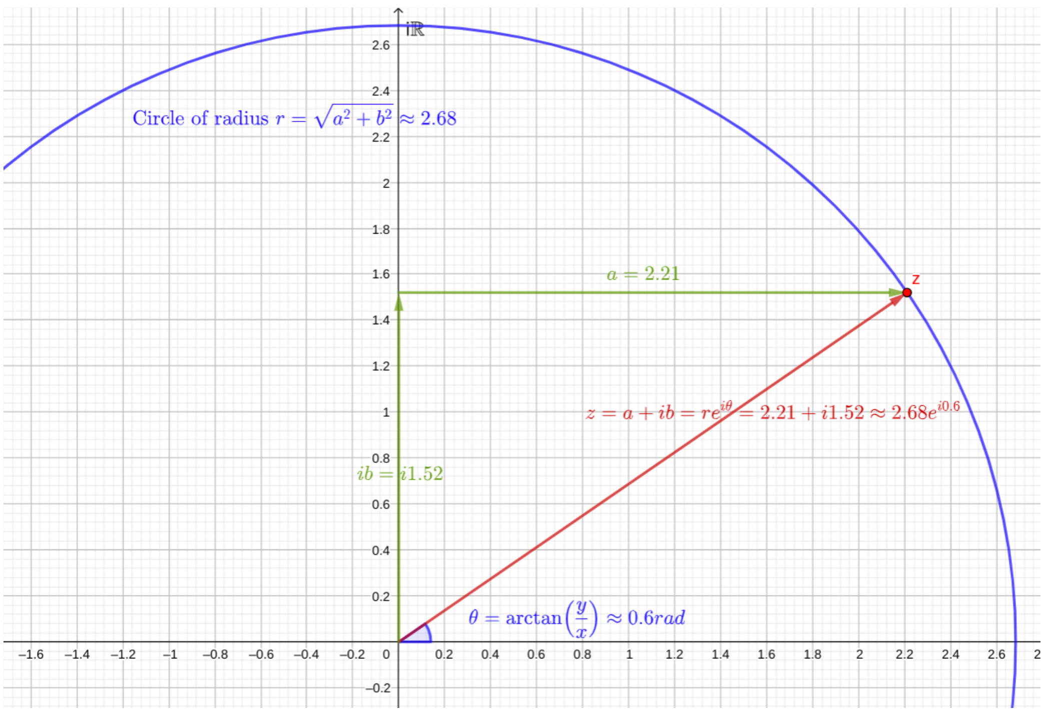

Figure 2: The number $z = 2.21 + 1.52i$ (defined in Cartesian form) and its conversion to the polar form. This grid was chosen to better illustrate the Cartesian components a and b. Note that here, the result of the conversion is not an exact, nice value; this conversion only has an approximative representation.

NB: If you see a number of the form $z = 5$, understand $z = 5 + i0$: there is a coherent correspondence between the "real part" of a complex number and the usual real numbers!

2.2.2 Polar notation

The polar notation (also called multiplicative form) of a complex number, $z = re^{i\theta}$, is the one that works well with multiplication. The term "polar" generally refers to objects moving in a coordinate system that follows rotations (cyclic, spherical, cylindrical, or hyperspherical coordinates). This is the case here.

The number r is called the **module** or the **radius** of a complex number. In physics, this may be referred to as the "magnitude" or "amplitude" of a complex number. It corresponds geometrically to the distance of a complex number from the 0 origin. The module is a positive real number ($r \in \mathbb{R}_+ = [0, \infty[$]).

The number θ is called the **argument** of a complex number. In physics, it is often referred to as the "phase" of a complex number (or a sine wave, or signal); confusingly, some authors alternatively use the term "amplitude" for this as well. The argument corresponds to the angle made by the vector of the complex number with respect to the positive real half line (positive real numbers have a null argument). The argument is a number in radians, i.e. which can technically be any real number, but which we generally prefer to note as a number between $-\pi$ and π, or between 0 and $\tau = 2\pi$. The specificity of radians is that they act a kind of "continuous clock", where two numbers are equivalent (correspond to the same argument, the same angle), if they differ precisely by a multiple of τ. For example, $z = 4th^{i(5\tau+3)} = 4th^{5\tau i}e^{3i} = 4th^{0i}e^{3i} = 4th^{3i}$. π radians is a half turn counterclockwise, and τ is a full turn, so to speak. For those who remember their (always cracked...) protractors: π rad $= 180$ and τ rad $= 360$.

The number $e \approx 2.71828\ldots$ is Euler's number, and it is the one that allows to define the generalization of the notion of power (a^n usually means multiplying a number a, n times with itself) to real numbers, then complex, by the exponential function. Indeed, it is already a bit weird to say "multiplying 7 3.724 times by itself", so let's not even talk about "multiplying 7 $4.75 + i3.724$ times by itself". For this reason, we define a function $exp(z) = e^z = \sum_{i=0}^{i=\infty} \frac{z^n}{z!} = 1 + z + \frac{z^2}{2} + \frac{z^3}{2*3} + \frac{z^4}{2*3*4} + \ldots$, which happens to correspond to the usual notion of power when z is an integer (to be precise, we have the formula $a^b = e^{b\ln(a)}$). Therefore, $e^{i\theta} = exp(i\theta)$. Don't worry, in the computer age, you should never be asked to calculate your exponentials by hand. For more details on the exponential function (including its important formulas and its relation to the logarithm), see the filesheets on $\mathbb{R} \to \mathbb{R}$

and $\mathbb{C} \to \mathbb{C}$.

Note that there exist two operators that can extract the components of the polar form. $|z| = r$, which allows to extract the modulus (it is a natural extension of the absolute value of real numbers, hence the notation), and $\arg(z) = \theta$ which allows to extract the argument (the function arg has some points in common with the logarithm, that is why it can extract a power; for this same reason, 0 has no argument and $\arg(0)$ is undefined).

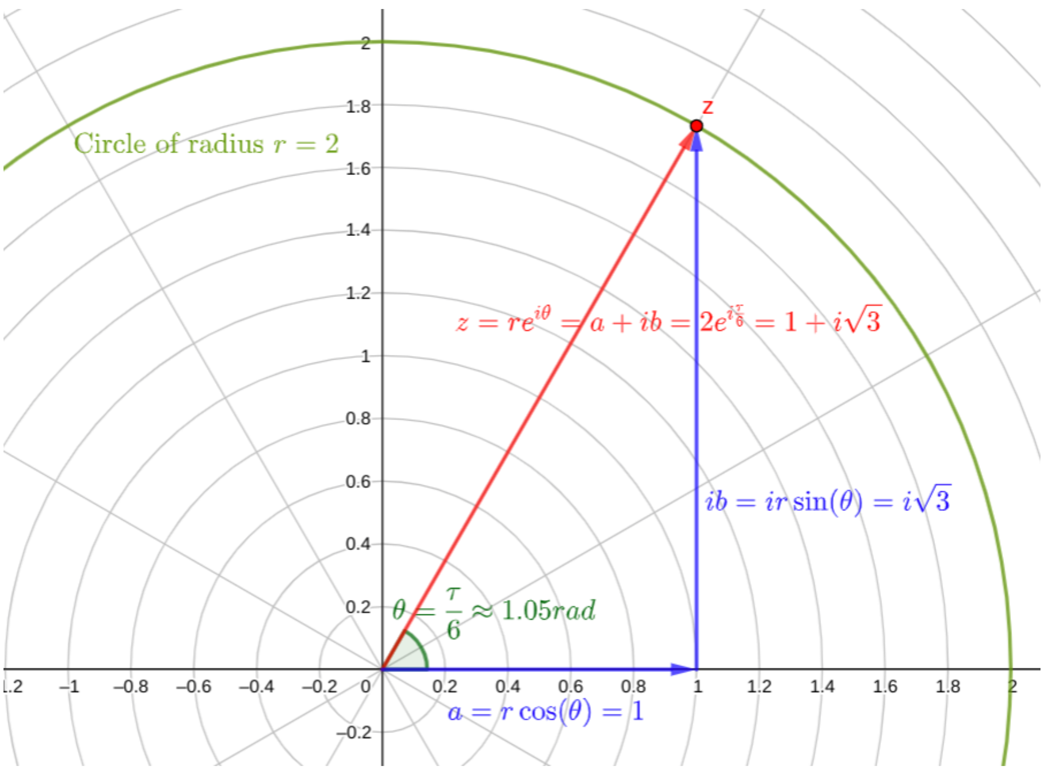

Figure 3: The number $z = 2e^{i\frac{\tau}{6}}$ (defined in polar form) and its conversion to the Cartesian form. This grid has been chosen to better illustrate the polar components r and θ. Note that here, an exact conversion exists.

NB: any number with a power of 0 is equal to 1. So if θ is zero, $re^{i\theta} = r$, a positive real number.

2.2.3 Conversion formulae

Often one will want to express a complex number with the notation with which it best expresses itself for the given context or problem. So it is a question of understanding how to compute $\mathrm{Re}(z) = x$, $\mathrm{Im}(z) = y$, $|z| = r$, and $arg(z) = \theta$ when we don't necessarily have our complex number in the form that suits us.

The trick is that the sides x, y, and r form a right-angled triangle, so we can use the tools of trigonometry to solve these problems. The three tools involved are the Pythagorean theorem (to find the hypotenuse), the arctangent (inverse function of the tangent, which generally intervenes to obtain the value in radians of an angle, when we have the two sides adjacent to a right angle of a right triangle), and Euler's formula (explained in more detail below, which connects the cosine and sine of trigonometry to the complex exponential).

We recall the fundamental mnemonic of classical trigonometry, "SOHC-AHTOA", in the context where θ is our angle, x the length of the side adjacent to the angle, y the length of the side opposite to the angle, and r the length of the hypotenuse:

$$\sin = \frac{opp}{hyp} \Leftrightarrow \sin(\theta) = \frac{y}{r}$$

$$\cos = \frac{adj}{hyp} \Leftrightarrow \cos(\theta) = \frac{x}{r}$$

$$\tan = \frac{opp}{adj} \Leftrightarrow \tan(\theta) = \frac{y}{x}$$

Our conversion formulae are then:

- $|z| = r = \sqrt{x^2 + y^2}$, with Pythagoras' theorem.

- $\arg(z) = \theta = \begin{cases} \arctan(\frac{y}{x}), & \text{if } x > 0 \\ \arctan(\frac{y}{x}) + \pi, & \text{if } x < 0 \\ \frac{\tau}{4} = \frac{\tau}{2}, & \text{if } x = 0 \text{ and } y > 0 \\ -\frac{\tau}{4} = -\frac{\pi}{2}, & \text{if } x = 0 \text{ and } y < 0 \end{cases}$

- $\mathrm{Re}(z) = x = r\cos(\theta)$, through trigonometry (CAH formula, or Euler's formula).

- $\mathrm{Im}(z) = y = r\sin(\theta)$, through trigonometry (SOH formula, or Euler's formula).

NB: there are other ways one can parameterize $\arg(z)$, for example if you want to express your angle not as an element $[0, \tau[$ but rather of $[-\pi, \pi[$. I leave this common other parameterization as exercise.

2.3 Algebraic nature of $\mathbb{C}$

There are many ways to construct or describe the space of complex numbers using other mathematical spaces. We will see four different categorization, which are generally the most useful to locate its place in the world of mathematics. We have ordered them from the simplest to the most technical. First-time readers should be able to understand the first two without any problems (if you have the aforementioned prerequisites); readers with more advanced knowledge will find that the third and fourth provide more enriching and surprising perspectives (as well as fairly memorable, and simple examples of advanced tools).

2.3.1 $\mathbb{C}$ as a field

$\mathbb{C}$ inherits most of its structure from $\mathbb{R}$ except for one detail: unlike $\mathbb{R}$, there is no single total order $\leq$ (or $\geq$) on $\mathbb{C}$ (up to isomorphism). One can define total orders on $\mathbb{C}$, but there is an infinite variety of them (that are not necessarily isomorphic). The rest of the computational rules are the standard rules of a (an algebraic) field: addition, subtraction, multiplication, division. (We also add an additional operation: conjugation, seen in a section below).

Note that the notion of field is much better understood within the framework of abstract algebra (the theory of algebraic structures), by defining this notion through the related notions of abelian group and ring. We invite you to consult the document on this subject, indicated in the prerequisites. What follows is therefore a summary of the definition of a field "by acting as if we weren't allowed to use the notion of group or ring".

$(\mathbb{C}, +, \times)$ is a field. this means that:

– $+$ is **closed** :

$$\forall (z_1, z_2) \in \mathbb{C}^2,\ z_1 + z_2 \in \mathbb{C}$$

– $+$ is **associative** :

$$\forall (z_1, z_2, z_3) \in \mathbb{C}^3,\ (z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$$

– there exists an **identity element** (written $0_{\mathbb{C}}$ or more simply 0) for $+$:

$$\exists 0 \in \mathbb{C}, \forall z \in \mathbb{C},\ z + 0 = 0 + z = z$$

– there exists an **a symmetric element** as relating to + for each z, called the "opposite" :

$$\forall z \in \mathbb{C}, \exists(-z) \in \mathbb{C},\ z + (-z) = (-z) + z = 0$$

– + is **commutative** :

$$\forall(z_1, z_2) \in \mathbb{C}^2,\ z_1 + z_2 = z_2 + z_1$$

– $\times$ is **closed**:

$$\forall(z_1, z_2) \in \mathbb{C}^2,\ z_1 \times z_2 \in \mathbb{C}$$

– $\times$ is **associative**:

$$\forall(z_1, z_2, z_3) \in \mathbb{C}^3,\ (z_1 \times z_2) \times z_3 = z_1 \times (z_2 \times z_3)$$

– there exists an **identity element** (written $1_{\mathbb{C}}$ or more simply 1) for $\times$:

$$\exists 1 \in \mathbb{C}, \forall z \in \mathbb{C},\ z \times 1 = 1 \times z = z$$

– $\mathbb{C}$ has no zero divisors

– there exists a **symmetric element** as relating to $\times$, for each nonzero z, called the "inverse":

$$\forall z \in \mathbb{C}^*, \exists z^{-1} \in \mathbb{C},\ z \times z^{-1} = z^{-1} \times z = 1$$

– $\times$ is **commutative**:

$$\forall(z_1, z_2) \in \mathbb{C}^2,\ z_1 \times z_2 = z_2 \times z_1$$

– $\times$ is **distributive on both sides** over +:

$$\forall(z_1, z_2, z_3) \in \mathbb{C}^3, z_1 \times (z_2 + z_3) = (z_1 \times z_2) + (z_1 \times z_3)$$

$$\forall(z_1, z_2, z_3) \in \mathbb{C}^3, (z_2 + z_3) \times z_1 = (z_2 \times z_1) + (z_3 \times z_1)$$

– $(\mathbb{C}, +, \times)$ is not the null ring (ie, $(\mathbb{C}, +, \times)$ has at least two elements, the identities for both its operators: $0_{\mathbb{C}}$ and $1_{\mathbb{C}}$)

Subtraction is then defined as addition by the opposite, and division as multiplication by the inverse. Often the notation of multiplication is omitted, i.e. it is noted with a dot centered "$\cdot$", or without notation, just by concatenating the letters.

Moreover, (non-constant) polynomials on $\mathbb{C}$ necessarily have at least one complex root. It is said that $\mathbb{C}$ is an algebraically closed field (which is not the case for $\mathbb{R}$: some real polynomials have complex roots, which led to the invention/discovery of complex numbers). This fact is called the "fundamental theorem of algebra" (a name that comes from a time when "algebra" rather meant "the art of solving equations" than "the theory of symbolic mathematics and mathematical spaces").

2.3.2 $\mathbb{C}$ as an $\mathbb{R}$-algebra of dimension 2

We start from $\mathbb{R}^2$, the standard vector space of dimension 2. We recall the definition of an $\mathbb{R}$-vector space:

– $\mathbb{R}$ is a field; its elements are called scalars.

– $(\mathbb{R}^2, +_{\mathbb{R}^2})$ is an abelian group (closure, associativity, identity, invertibility, commutativity); its elements are called vectors.

– We provide $\mathbb{R} \times \mathbb{R}^2$ with an operator into $\mathbb{R}^2$ called "scalar multiplication" written like usual multiplication (so with $\times$, a centered "$\cdot$", or generally without any notation, just by concatenating letters). Clearly put, $\cdot : (\mathbb{R} \times \mathbb{R}^2) \to \mathbb{R}^2$. This operator respects the following compatibility properties between $\mathbb{R}$ and $\mathbb{R}^2$:

– Pseudo-distributivity over vectors :

$$\forall \lambda \in \mathbb{R}, \forall (u, v) \in (\mathbb{R}^2)^2, \lambda(u + v) = \lambda u + \lambda v$$

– Pseudo-distributivity over scalars :

$$\forall (\lambda, \mu) \in (\mathbb{R})^2, \forall u \in \mathbb{R}^2, (\lambda + \mu)u = \lambda u + \mu u$$

– Pseudo-associativity of scalar multiplication :

$$\forall (\lambda, \mu) \in (\mathbb{R})^2, \forall u \in \mathbb{R}^2, (\lambda\mu)u = \lambda(\mu u)$$

– Compatibility of the multiplicative identity of the field with scalar multiplication:

$$\forall x \in E, 1_{\mathbb{R}} u = u$$

An element of $\mathbb{R}^2$ is written $u = (x, y) = \begin{bmatrix} x \\ y \end{bmatrix}$. We provide $\mathbb{R}^2$ with a pair of vectors, called the canonical basis, with name and value $e_1 = (1, 0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

and $e_2 = (0,1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Any vector $u \in \mathbb{R}^2$ can then be expressed as $u = xe_1 + ye_2$.

We give as alias 1 to the vector e_1 and i to the vector e_2. This makes it possible to write, for any vector:

$$u = (x,y) = (x,0) + (0,y) = x(1,0) + y(0,1) = x \times 1 + y \times i = x + iy$$

We provide our $\mathbb{R}$-vector space $\mathbb{R}^2$ with a multiplication, in order to make it into an $\mathbb{R}$-algebra, which we call $(\mathbb{C}, +, \cdot, \times)$, or more simply $\mathbb{C}$. Let be a pair of vectors, $u = (a,b)$ and $v = (c,d)$. We then define the multiplication on our algebra $\mathbb{C}$ as:

$$u \times v = (a,b) \times (c,d) = (ac - bd, ad + bc) = (ac - bd) + i(ad + bc)$$

In this system, we then have:

$$i^2 = (0.1)^2 = (0.1) \times (0.1) = (0 \times 0 - 1 \times 1, 0 \times 1 + 1 \times 0) = (-1,0) = -1$$

and i is thus one of the two roots of -1.

$\mathbb{C}$ is an $\mathbb{R}$-algebra of dimension 2; it is associative, unitary, commutative, and invertible (you can have fun proving this is true here with the parenthesis notation, but I would rather advise to do it with distribution/factorization in Cartesian notation). This means it behaves like an algebraic field.

2.3.3 $\mathbb{C}$ as a subset of $\mathbb{R}^{2\times 2}$

There is also an interpretation of the complex numbers in matrix form (called a "rotation-scaling matrix" or a "direct similarity matrix"). For any complex number $z = a + ib = re^{i\theta}$, we can represent this complex number as a matrix Z as follows:

$$Z = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

You will notice that adding, subtracting or multiplying two matrices of this form corresponds precisely to the same operations on the complex numbers. Add then matrixify, or matrixify then add, you'll get the same result:

there's an underlying commutative diagram; same for multiplication, etc. Inverting this matrix corresponds precisely to the inversion of a complex number. As for conjugation (see below), it corresponds precisely to matrix transposition.

[Side note: This explains why matrices with complex coefficients require a transconjugation rather than a normal transposition. Take a matrix with complex coefficients: first, replace each complex coordinate by the corresponding rotation-scaling matrix, then do the transposition; second, take the initial complex matrix again, do the transconjugation, then the replacement of each coordinate by its similarity matrix. In both cases you will get the same final result (commutative diagram). It turns out that conjugation and transposition are simply two perspectives on the same concept of vector spaces: dualization.]

By the way, the matrix Z can be decomposed, and thus we can have a matrix equivalent of the polar form of complex numbers:

$$Z = RS = SR$$

with two matrices:

- $R = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$, a rotation matrix;

- $S = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}$, a scaling matrix (a scaling matrix is a special type of diagonal matrix, and any diagonal matrix commutes with all other matrices, hence the validity of the relation $RS = SR$).

2.3.4 $\mathbb{C}$ as the algebraic quotient of a ring, or of an $\mathbb{R}$-algebra, i.e. $\mathbb{R}[X]/(X^2+1)$

Let $\mathbb{R}[X]$ be the ring (or $\mathbb{R}$-algebra) of polynomials with real coefficients. Let $I = (X^2+1)\mathbb{R}[X]$ be the ideal of polynomials that are multiples of (X^2+1), i.e. polynomials of the form $Q = (X^2+1)P$ where P is any polynomial. That is, $I = (X^2+1)\mathbb{R}[X] = \{(X^2+1)P \mid P \in \mathbb{R}[X]\}$.

We define $\sim_I$ as the standard equivalence relation generated by the ideal of a ring: two polynomials P and Q are equivalent by $\sim_I$ iff $P - Q \in I$. This corresponds to the fact that two polynomials are equivalent iff they exist in the same affine space parallel to I.

We then proceed to build the algebraic quotient $\mathbb{R}[X]/\sim_I$ (often noted $\mathbb{R}[X]/(X^2+1)$): let be the map (the integral domain morphism) $f : \mathbb{R}[X] \to$

$\mathbb{R}[X]/\sim_I$ which associates to each element of $\mathbb{R}[X]$ its equivalence class in $\mathbb{R}[X]/\sim_I$. This morphism is well-defined (it is indeed a morphism by the general properties of ideals); it is clearly an surjection; and it is clearly not an injection. The kernel (set of elements that have the 0 element as image) of this morphism is I as a whole: this means that in $\mathbb{R}[X]/\sim_I$, the property $X^2 + 1 = 0$ is verified by all polynomials (in the quotient space, as soon as a polynomial is a multiple of $X^2 + 1$, it is equivalent to the null polynomial). This implies the existence of an element (a polynomial) P such that $P^2 + 1 = 0 \Leftrightarrow P^2 = -1$. This polynomial is $P = X$, and it is usually called i rather than P.

There is an integral domain (and $\mathbb{R}$-algebra, and field) isomorphism between $\mathbb{R}[X]/\sim_I$ thus defined, and the space $\mathbb{C}$ of complex numbers.

[TODO: move following subsubsection to its own document; Keep here only the bare minimum (ie, Cl(0,1) and rotors)]

2.3.5 $\mathbb{C}$ and geometric algebras

NB: If you're reading this document for the first time, it may be better to skip this section, but it is very interesting once you understand complex numbers.

Some basics on geometric algebras (Clifford algebras)

We write:

- $\vee$ (often also $\cdot$) the **inner product** (generalized scalar product/dot product, NOT TO BE CONFUSED with the "interior product". Yeah, I know, sorry.)

- $\wedge$ the **outer product**, also called **exterior product** (Grassmann product)

- and multiplicatively (without symbol or $\times$; or more rarely $\cdot$ if not already chosen for the inner product) the **geometric product**.

The relationship between these different products, for vectors, is as follows: $uv = u \vee v + u \wedge v$. The inner product is the even part (symmetric, commutative) of the geometrical product, the outer product its odd part (antisymmetric, anticommutative).

The elements of a geometric algebra are called **multivectors**. These are linear combinations (generalized sums; n-sums of scalings) of k-blades, where a k**-blade** is the extension to k dimensions of what is a scalar (0-dimensional object, a point with a mass and a sign), a vector (1-dimensional object, a line with a length and an orientation). The 2-blades are pieces of oriented plane (with "mass" being the area, and "orientation" being rotational orientation, ie clockwise (counter-trigonometric) or trigonometric (counter-clockwise)). The 3-blades are pieces of oriented volume (their orientation can be described as the direction of rotation on the surface of the volume, around the vectors starting from the inside of the volume and going to the outside of the volume), etc.

All three products are associative (no need for parentheses, unless you mix them up). The inner product is commutative (symmetric), the outer product is anticommutative (antisymmetric). The geometric product is neither commutative nor anticommutative, except in the case where it can be reduced to one of its components. These cases are precisely: one, if two multivectors are parallel (in which case their exterior product is null, the geometric product is reduced to the inner product, and therefore is commutative); or two, if two multivectors are perpendicular (in which case their inner product is null, the geometric product is reduced to the outer product, and therefore is anticommutative).

The number k is called the **grade** of a blade. The inner product "consumes" (subtracts) the grades, the outer product "combines" (adds) the grades. A 2-blade is thus expressed as the outer product of two vectors ($1 + 1 = 2$, a 2-blade is then often represented as the parallelogram formed by the two vectors composing it); a 3-blade as the exterior product of a bivector and a vector etc. The inner product of two vectors (usually called the scalar product) consumes the dimensions ($1 - 1 = 0$) and returns a scalar. The inner product of a bivector and a vector returns a vector ($2 - 1 = 1$).

The inner product $u \vee v$ represents the projection of u on v (if u has a grade less than or equal to that of v; or vice-versa if the grade of v is the smallest of the two inputs) and multiplying the respective masses of the two resulting objects (mass, length, area, volume, hypervolume; oriented), keeping only the dimensions of v (if larger) which were perpendicular to u (if smaller).

[TODO: add diagram, anim and/or applet]

The outer product corresponds to moving the u object along the v object (in the different dimensions of v), and to take the object thus drawn (the mass of the final object being the product of the masses, and the orientation defined by first starting in the direction of u and completing following the direction of v), cancelling the collinear parts of u and v.

[TODO: add diagram, anim and/or applet]

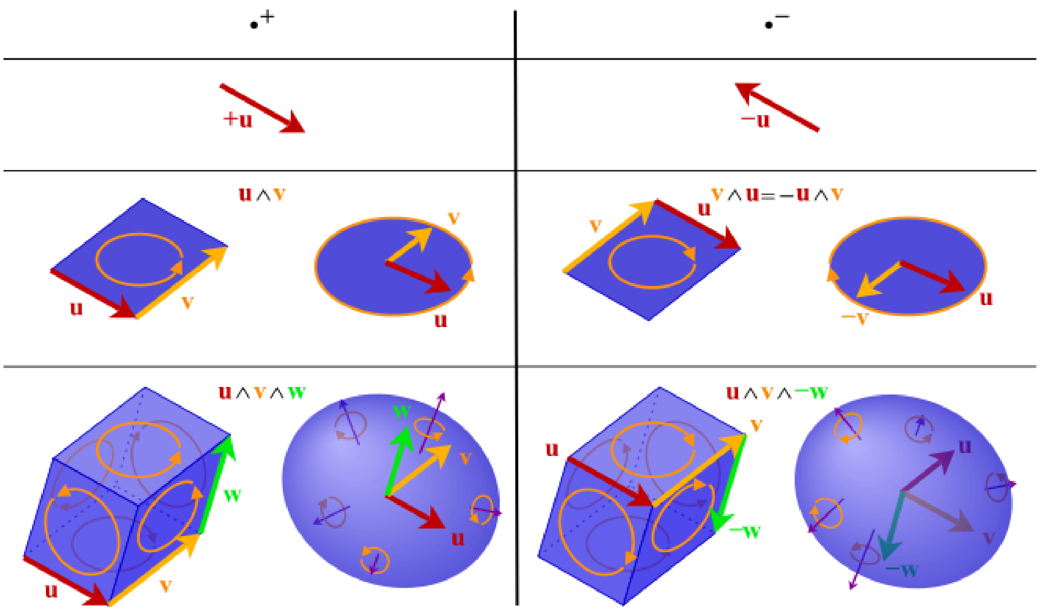

Figure 4: Examples of blades. Note that the choice of representation (ellipsoid or parallelepiped, or any other shape), or position, does not matter. Two k-blades are equal if they have the same mass, grade, orientation and direction; the rest is irrelevant.

To the notion of k-blade, we add that of k**-vector**. A k-vector is a linear combination of blades of grade k. It is a multivector of homogeneous grade. To make the distinction in one word: a k-blade can always be decomposed into a successive exterior product of k different vectors (1-blades); a k-vector cannot always be decomposed in this way. Thus, we also say that a k-blade is a "simple k-vector". Any k-vector in a space of dimension ≤ 3 is a k-blade (and vice versa, since the converse is always true); this is because a sum of bivectors (resp. trivectors) in such a space can always be expressed as a 2-blade (res. 3-blade). For this reason, you will sometimes see some language misuse confusing bivector (2-vector) and 2-blade, or trivector (3-vector) and 3-blade; this misuse is not a problem for $\mathbb{G}^n$ with $n \leq 3$. However, from a space of dimension $n \geq 4$, there are k-vectors that are not k-blades. If we take the example $\mathbb{G}^4 \overset{Vect_{\mathbb{R}}}{\cong} Cl_{0,4}(\mathbb{R})$, with $\{e_1, e_2, e_3, e_4\}$ as a basis, the 2-vector $e_1 \wedge e_2 + e_3 \wedge e_4$ is not a 2-blade.

$\mathbb{G}^n$, the geometric algebra generated by $\mathbb{R}^n$, is a vector space of dimension

2^n. For example, taking $\mathbb{R}^3$ with as basis $\{e_1, e_2, e_3\}$, we have $\mathbb{G}^3$ with the basis $\{1, e_1, e_2, e_3, e_1e_2, e_2e_3, e_3e_1, e_1e_2e_3\}$ (often noted $\{1, e_1, e_2, e_3, e_{12}, e_{23}, e_{31}, e_{123}\}$). Addition in a geometric algebra, as in any vector space, forms an abelian group; the geometric product forms a monoid (and is distributive); however, the ring thus formed also has many invertible elements. One often notes $\mathbb{G}^{p+q} = Cl_{p,q}(\mathbb{R})$, and $\mathcal{B} = \{e_1, \ldots, e_{p+q}\}$ the underlying basis for $\mathbb{R}^n$. p refers to the amount of elements of the basis that verify $e_i^2 = 1$, while q is the amount of elements of the base that verify $e_i^2 = -1$. (p, q) is called the **signature** of the geometric algebra. Note that one can have signatures of the form (p, q, r), where the elements counted by r are those that verify $e_i^2 = 0$. For example, $Cl_{(0,0,n)}(\mathbb{R})$ is the exterior algebra (Grassmann algebra) of rank n.

A blade of dimension n in $\mathbb{G}^n$ is called a pseudoscalar, an blade of dimension $n - 1$ is called a pseudovector. For example, the 3D torque, classically defined as a pseudovector, is a rotation force, therefore a bivector.

Isomorphism between $\mathbb{C}$ and $Cl_{0,1}(\mathbb{R})$

Let be $Cl_{0,1}(\mathbb{R})$, the geometric algebra over $\mathbb{R}$ generated by the basis $\{e\}$, with $e^2 = e \vee e + e \wedge e = e \vee e = -1$. Every multivector of $Cl_{0,1}(\mathbb{R})$ can be written $a + be$ with $a, b \in \mathbb{R}$.

Let be 2 multivectors $U = a + be$ and $V = c + de$. We have:

$$\begin{array}{rcll} UV & = & (a+be)(c+de) & \\ & = & ac + bce + ade + bede & \text{(distributivity)} \\ & = & ac + (bc+ad)e + bdee & \text{(commutativity of a scalar with any multivector)} \\ & = & ac + (bc+ad)e - bd & \text{(square of the basis vector)} \\ & = & (ac-bd) + (bc+ad)e & \end{array}$$

It is then clear that $Cl_{0,1}(\mathbb{R}) \overset{Alg_{\mathbb{R}}}{\cong} \mathbb{C}$.

Isomorphism between $\mathbb{C}$ and vector planes within a general geometric algebra

The geometric product uv of two vectors u et v can be described, and interpreted, as:

$$
\begin{aligned}
uv &= u \vee v + u \wedge v \\
&= \|u\|\|v\|(e_u \vee e_v + e_u \wedge e_v) \\
&= \|u\|\|v\|(\cos(u,v) + i\sin(u,v)) \\
&= r(\cos(\theta) + i\sin(\theta)) \\
&= r\exp(i\theta)
\end{aligned}
$$

where:

- e_u (resp. e_v) is the vector of norm 1 in the direction and orientation of u (resp. v),

- $i = \frac{e_u \wedge e_v}{\sin(u,v)}$ is the unit pseudoscalar in the plane containing u and v (so it is a 2-blade, possibly null if the vectors are collinear),

- $r = \|u\|\|v\|$ is the product of the norms of the two vectors, and,

- θ is the oriented angle from u to v.

This product uv can be interpreted geometrically as an arc of circle of radius r and angle θ. This arc can be rotated along the circumference of the circle without changing the interpretation, just as two vectors are interpreted as identical if they are parallel, of the same length, and are oriented the same way. Such a geometrical product of two vectors is called a **rotor**. It is then clear that any vector plane of a geometric algebra has a parameterization isomorphic to the complex plane (really, an infinity), using the basis $\{1, i = \frac{e_u \wedge e_v}{\sin(u,v)}.\}$ (and making a choice of for the positive real half-line). The rotors (by a special sandwiching called "versorial product") then allow to define simple rotations, like complex numbers; but this time in a space of arbitrary dimension.

2.3.6 Operations on complex numbers

Now we come to the "practical" part.

2.4 Addition

Complex addition is the classical vector addition in $\mathbb{R}^2$. Algebraically, it is a term-to-term addition. Geometrically, it is the sum of the paths taken by the arrows.

$$\begin{aligned} z_1 + z_2 &= (x_1 + iy_1) + (x_2 + iy_2) \\ &= x_1 + x_2 + iy_1 + iy_2 \\ &= (x_1 + x_2) + i(y_1 + y_2) \end{aligned}$$

2.4.1 Opposition and subtraction

Complex subtraction is similar just to vector subtraction. It is addition by the opposite. The opposite of a vector is the vector of the same direction and the same norm (=modulus=radius), but with opposite orientation. Geometrically the subtraction of two vectors is the vector going from the tip of the second to the head of the first.

$$\begin{aligned} -z &= -(x + iy) \\ &= -x - iy \end{aligned}$$

$$\begin{aligned} z_1 - z_2 &= (x_1 + iy_1) - (x_2 + iy_2) \\ &= x_1 - x_2 + iy_1 - iy_2 \\ &= (x_1 - x_2) + i(y_1 - y_2) \end{aligned}$$

Here's an exercise to understand the interest of the two notations (when you will have finished your first reading): try to express $z_1 + z_2$ in polar form; always staying in polar form, without ever going through x_1, x_2, y_1 or y_2, nor having them in your final result; just r_1, r_2, θ_1 and θ_2. You'll need to use the Euler formula below, of course. A royal mess, isn't it?

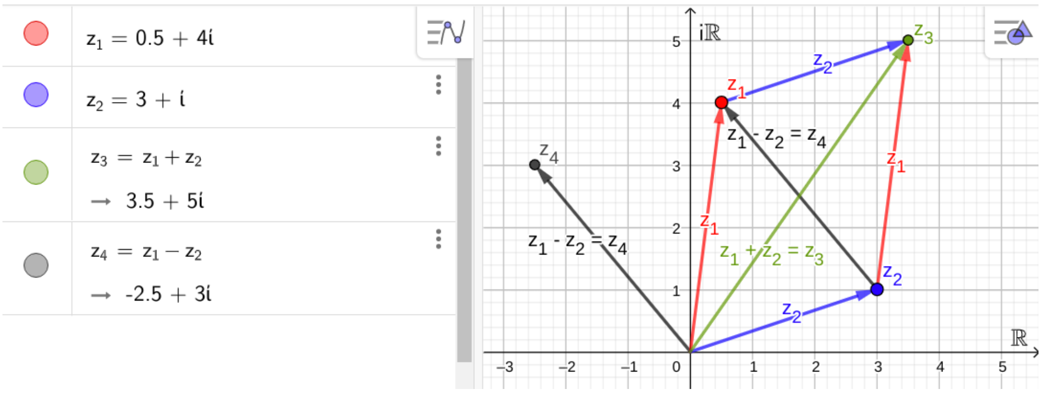

Figure 5: The classical diagram summarizing vector addition and subtraction, often called the "parallelogram diagram".

2.4.2 Multiplication

Complex multiplication is more interesting, and its behavior is specific to complex numbers. Its geometric interpretation can only be truly understood through the polar form, and this is why this polar form corresponds to multiplication (multiplication is also much easier to compute in polar form).

$$\begin{aligned} z_1 \times z_2 &= (x_1 + iy_1) \times (x_2 + iy_2) \\ &= x_1y_1 + ix_1y_2 + ix_2y_1 + i^2y_1y_2 \\ &= (x_1y_1 - y_1y_2) + i(x_1y_2 + x_2y_1) \end{aligned}$$

$$\begin{aligned} z_1 \times z_2 &= (r_1e^{i\theta_1}) \times (r_2e^{i\theta_2}) \\ &= r_1r_2e^{i\theta_1}e^{i\theta_2} \\ &= (r_1r_2)e^{i(\theta_1+\theta_2)} \end{aligned}$$

We see with the second formula that the geometrical interpretation of complex multiplication is the (usual real number) multiplication of the moduli, and the (real number, but with cyclicity of period τ) addition of the arguments. It is thus the multiplicative combination of the distances from the origin, and the additive combination of the oriented angles (of the rotations) with respect to the positive real half-line.

NB: If you have a problem with the properties of the exponential (or powers), see the sheet on the functions of $\mathbb{R} \to \mathbb{R}$ or the introduction to analysis.

NB: positive reals having a null argument, multiplying by a positive real is like scaling a complex number, without changing the argument. This is perfectly consistent with the vision of $\mathbb{C}$ as a $\mathbb{R}$-algebra. Multiplying by -1 means taking the opposite, likewise, it is perfectly consistent with the vector vision of $\mathbb{C}$.

NB: Multiplying a complex number z by $i = e^{i\frac{\tau}{4}}$ corresponds to a rotation of 90 in the trigonometric direction (positive direction, counterclockwise direction), without changing the modulus of z. Similarly, multiplication by any complex number u of modulus 1 corresponds to a simple rotation by $arg(u)$.

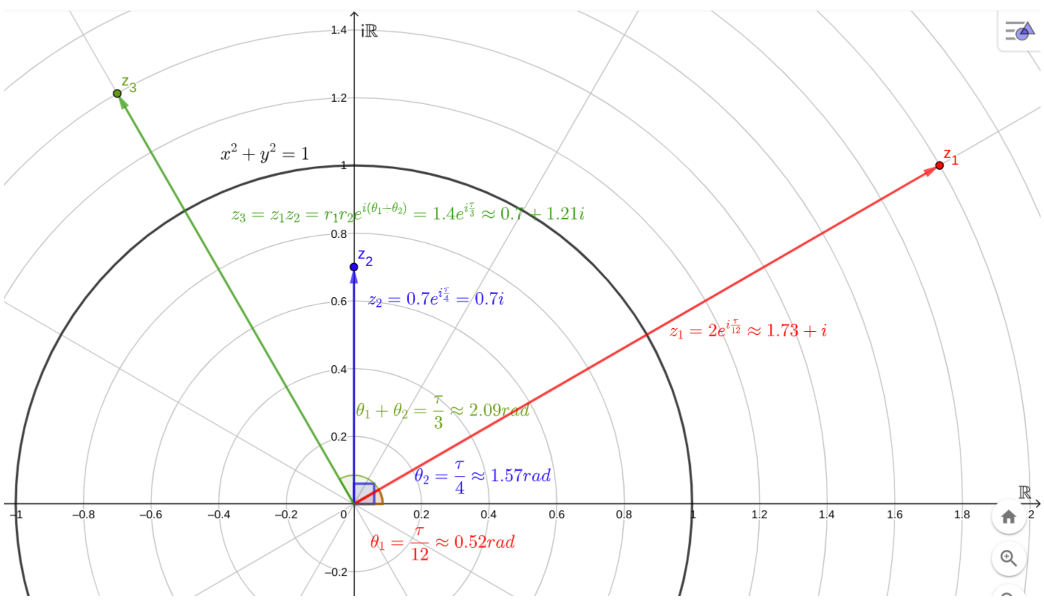

Figure 6: Image summarizing complex multiplication. Notice the multiplication of moduli (lengths), and the addition of arguments (angles).

2.4.3 Conjugation

Conjugation is an operation specific to complex numbers. It is an involution (a function that, when applied twice, returns to the starting point, like $x \to -x$ or $x \to \frac{1}{x}$). Well, to be exact, it exists in $\mathbb{C}$, so *a fortiori* exists in $\mathbb{R}$ which is a subset of $\mathbb{C}$. It turns out that real numbers are invariant by conjugation, so the conjugation on $\mathbb{R}$ is equivalent to the identity function $x \to x$. We note $\overline{z}$ the conjugate of z.

$$\begin{aligned} \overline{z} &= x - iy \\ &= re^{-i\theta} \end{aligned}$$

Geometrically, conjugation is the reflection of a complex number using the real line as the axis of symmetry.

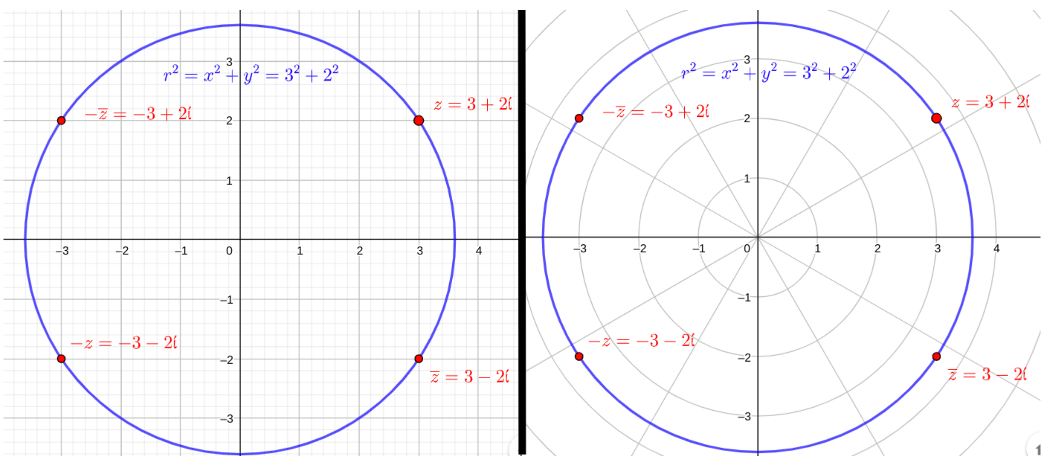

Figure 7: Image summarizing the combinations of negation and conjugation of z. The negation $(-z)$ is a central symmetry. The conjugation $(\overline{z})$ is an axial (mirror) symmetry with respect to the real line. The combination of negation and conjugation $(-\overline{z})$ is an axial (mirror) symmetry with respect to the pure imaginary line.

We have the following properties are for conjugation:

$$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$$

$$\overline{z_1 z_2} = \overline{z_1}\;\overline{z_2}$$

$$\overline{-z} = -\overline{z}$$

$$z + \overline{z} = 2\text{Re}(z)$$

$$z - \overline{z} = 2\text{Im}(z)$$

$$z\overline{z} = |z|^2 = |z^2|$$

The last three properties show that any complex operation (except division) between a number and its conjugate results in a real number. This is why polynomials of degree 2 with real coefficients, if they have complex roots, will always have conjugated roots.

2.4.4 Inversion and division

Division is defined, like usual, as multiplying by the inverse. The inverse is noted z^{-1} and is defined as:

$$\begin{aligned} z^{-1} &= \frac{1}{z} \\ &= \frac{\overline{z}}{|z|^2} \\ &= \frac{1}{r}e^{-i\theta} \end{aligned}$$

or in additive form:

$$z^{-1} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$$

Geometrically, the inverse of a complex number is therefore on the line symmetrical to its own with respect to the real line, but at a distance $\frac{1}{r}$ from the origin rather than r. If $r > 1$, then z is outside the trigonometric circle, and z^{-1} is inside. If $r < 1$, then z is inside the trigonometric circle, and z^{-1} is outside. If $r = 1$, then $\overline{z} = z^{-1}$.

There is not really any elegant way of thinking about division other than as multiplication by the inverse; but this is fine, when one has understood both complex multiplication and inversion.

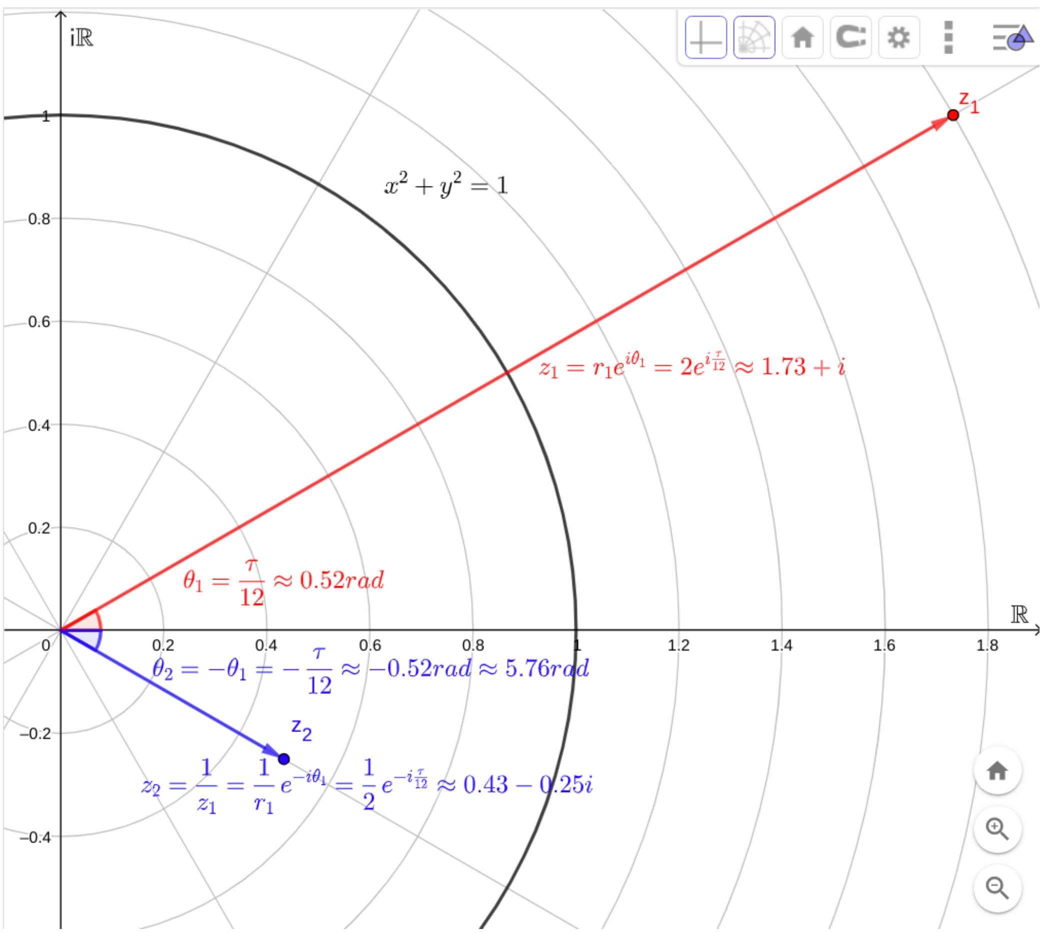

Figure 8: Image summarizing the inversion of a complex number. It is a negation of the argument and an inversion of the length

$$\begin{aligned}
\frac{z_1}{z_2} &= z_1 \frac{1}{z_2} \\
&= r_1 e^{i\theta_1} \frac{1}{r_2 e^{i\theta_2}} \\
&= r_1 e^{i\theta_1} \frac{1}{r_2} e^{-i\theta_2} \\
&= \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}
\end{aligned}$$

NB: it is often useful to know how to do a division without going through the multiplicative form. To do this, the member i must be removed from the denominator. The technique is quite simple: you have to multiply by the conjugate of the denominator at the top and at the bottom of the fraction, in order to use the formula $z\overline{z} = |z|^2$. This is the same as doing:

$$\frac{z_1}{z_2} = \frac{z_1}{z_2} \frac{\overline{z_2}}{\overline{z_2}} = \frac{z_1 \overline{z_2}}{|z_2|^2}$$

Or equivalently, with some more details in our computation, we get:

$$\begin{aligned}
\frac{a+ib}{c+id} &= \frac{a+ib}{c+id} \times 1 \\
&= \frac{a+ib}{c+id} \times \frac{c-id}{c-id} \\
&= \frac{(a+ib)(c-id)}{c^2+d^2} \\
&= \frac{(ac+bd)+i(bc-ad)}{c^2+d^2} \\
&= \frac{(ac+bd)}{c^2+d^2} + i\frac{(bc-ad)}{c^2+d^2}
\end{aligned}$$

2.4.5 Exponentiation

First, let's take a look at the basic complexe exponential function:

$$\begin{aligned} e^z &= e^{(x+iy)} \\ &= e^x e^{iy} \end{aligned}$$

The result is therefore a number of modulus $r = e^x$ (indeed a strictly positive number because x is a real number, and $exp : \mathbb{R} \to \mathbb{R}_+$) and argument $\theta = y[\tau]$ (the remainder of the division of y by τ, if we decide for the quotient of the division to be an integer).

With this, we can deal with the more general case. The calculation $z_1^{z_2}$ is best understood by taking z_1 in multiplicative form, and z_2 in additive form, in order to exploit the properties of the exponential:

$$\begin{aligned} z_1^{z_2} &= (r_1 e^{i\theta_1})^{(x_2+iy_2)} \\ &= r_1^{(x_2+iy_2)} e^{i\theta_1(x_2+iy_2)} \\ &= r_1^{x_2}\ r_1^{iy_2}\ e^{i\theta_1 x_2}\ e^{i^2\theta_1 y_2} \\ &= e^{x_2 \ln(r_1)}\ e^{iy_2 \ln(r_1)}\ e^{i\theta_1 x_2}\ e^{-\theta_1 y_2} \\ &= e^{x_2 \ln(r_1)}\ e^{-\theta_1 y_2}\ e^{iy_2 \ln(r_1)}\ e^{i\theta_1 x_2} \\ &= e^{x_2 \ln(r_1) - \theta_1 y_2}\ e^{i(y_2 \ln(r_1) - \theta_1 x_2)} \end{aligned}$$

Thus, the result of $z_1^{z_2}$ has a modulus of $r = e^{x_2 \ln(r_1) - \theta_1 y_2}$, and an argument of $\theta = y_2 \ln(r_1) - \theta_1 x_2$. Needless to say that I don't know an intuitive way to visualize the geometry of this operation. However, using the formulas $z_1^{z_2} = e^{z_2 \ln z_1}$ and $\ln z = \ln|z| + i(n\tau + arg(z))$, there is a slightly less ugly formulation of $z_1^{z_2}$. I'll let you calculate it as an exercise.

You can also check the following page for some tidbits and visuals:
https://mathworld.wolfram.com/ComplexExponentiation.html

3 Elements of general knowledge

Finally, here are some elements of general knowledge concerning complex numbers. There is way too much to say in this section, so I have limited myself in this document to a few subjects specific to "pure" complex numbers, thus reserving peripheral subjects to the filesheets for which these subjects would be best suited.

3.1 Important Theorems and Formulas

3.1.1 Euler's identity

Perhaps the best known identity in mathematics (apart from Pythagoras' theorem), described by some as "the most beautiful", is one that deals with complex numbers. It is called Euler's identity. Here it is:

$$e^{i\pi} + 1 = 0$$

If it is considered so beautiful, it is because it illustrates a deep link between different branches of mathematics, through their emblematic constants. Behind e, there is all analysis, physics, finance, etc., the theory of differential equations: the study of changes and systems with multiple interacting parts. Behind π is all geometry. Behind 1 and 0, the two usual identity elements, all of arithmetic and algebra (in the sense of the theory of algebraic structure). And behind i, all algebra (in the historical sense of the study of the resolution of equations) and polynomials. Each fundamental operator (addition, multiplication, exponentiation) appears once and only once. This is indeed quite mystifying.

However, I personally prefer the version:

$$e^{i\tau} = 1$$

which expresses in a purer and simpler way the beauty of complex numbers themselves: the idea that τ radians is 1 full turn of the complex plane.

3.2 Euler's formula

Euler's formula is perhaps the most important and useful of the complex numbers. I still prefer it to the two previous ones. On its own, it summarizes more than 80% of the usual trigonometry formulas. If you understand this formula, which encodes the links between complex addition and multiplication, the exponential, sines and cosines, you can find all the formulas you had forgotten about trigonometry by pure geometric intuition (not just the SOHCAHTOA, also the formulas for sums and products of cosine and sine). This formula also summarizes the passage from the additive form to the multiplicative form of complex numbers. Moreover, the two previous identities are consequences of this one: the first one when $\theta = \pi$, the second one when $\theta = \tau$.

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

Geometrically, this formula seems obvious. The cosine is the horizontal component; the sine is the vertical component; the exponential is our complex number of resulting modulus = 1 (the hypotenuse). This equation just describes a simple right-angled triangle. However, it hides an impressive depth. Remember the definition of the exponential as a power series (large polynomial sum) above? Well, there is a similar power series definition for cosine and sine, and these two compensate each other in their sum to give precisely the power series of the exponential.

You have a theorem from the field of analysis which says that any function can be expressed as the sum of an even function (a function such as $f(x) = f(-x)$) and an odd function (a function such as $f(x) = -f(x)$). This theorem, with this formula, induces deep links between the usual cosines and sines (cos and sin, which arise in elliptic geometry, differential equations, and the space of complex numbers) but also hyperbolic cosines and sines (cosh and sinh, which arise in hyperbolic geometry, differential equations, and the space of split-complex numbers, a close cousin where we define as base vectors 1 and j, where $j \neq -1$ and $j^2 = 1$).

So here is a series of consequences of this formula and these relations (noting also that sin is an odd function, and cos an even function):

- $e^{ix} = \cos(x) + i\sin(x)$
- $e^{-ix} = \cos(x) - i\sin(x)$
- $\cos(x) = \mathrm{Re}(e^{ix}) = \dfrac{e^{ix} + e^{-ix}}{2}$
- $\sin(x) = \mathrm{Im}(e^{ix}) = \dfrac{e^{ix} - e^{-ix}}{2i}$
- $e^{x} = \cosh(x) + \sinh(x)$

- $\cos(ix) = \cosh(x) = \dfrac{e^x + e^{-x}}{2}$
- $\sin(ix) = i\sinh(x) = i\dfrac{e^x - e^{-x}}{2}$

3.2.1 De Moivre's Formula

This is very useful to find the algebraic characteristics of the multiples of an angle. It is also often used to linearize (remove powers from) a cosine or sine (often useful for integration). It can be deduced quite easily from the previous formula:

$$(\cos(\theta) + i\sin(\theta))^n = e^{i\theta n} = \cos(n\theta) + i\sin(n\theta)$$

Do notice that with $n = -1$, we have the definition of the conjugate (equal to the inverse) of a unit-modulus complex number.

3.3 τ vs π

As you may have noticed, there has been a tendency in this document to favor the constant "tau", $\tau \approx 6.28\ldots$, rather than "pi", $\pi \approx 3.14\ldots$, which is however more well-known. The list of reasons for this is (very) long (https://tauday.com/tau-manifesto), but can be summed up fairly well by the idea that in mathematics, the main characteristic of a circle, sphere, etc (all the higher dimension versions of these objects), is not its diameter, but its *radius*. π is the ratio that corresponds to perimeter/diameter, τ the ratio perimeter/radius. Almost all mathematical formulas involving π become more expressive or simple when we replace π by $\frac{\tau}{2}$. Most of these formulas already involve 2π directly anyways...

But the case where the superiority of τ is really, I mean really, indisputable, is the handling of complex number arguments. A right angle corresponds to ± 90, or to $\pm\frac{\pi}{2}$ radians, at first glance it seems strange and arbitrary, but well, we accept, we memorize, and we copy without thinking... Now, I'm going to ask you a question. Do you know what 90 is on a clock, in minutes? 15 minutes, or a quarter of an hour. After all, $90 = \frac{360}{4}$, a quarter turn makes sense. But it is also and above all $\frac{\tau}{4}$ radians. Here is the superiority of τ over π here: angles, expressed as ratios of τ, are just fractions of the circle. Half an hour is half a turn of the circle, so half a τ:

that is $\frac{\tau}{2}$. This becomes extremely useful, and much, much more intuitive to use, especially when you consider the next section.

3.4 Roots of unity, trigonometric circle, and cyclic group $\mathbb{T}$

The set of complex numbers of unit modulus, noted $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\} = \{e^{i\theta} \mid \theta \in [0, \tau[\text{ mod } \tau\}$, which corresponds to the "trigonometric circle", is a very important set in mathematics. It is called the circular group (also called S^1, $SO(2)$, $U(1)$, or $\mathbb{R}/\mathbb{Z}$, quotient of the additive group of $\mathbb{R}$ by that of $\mathbb{Z}$). There are several ways of constructing it, and these constructions are topologically isomorphic, up to multiplication by a single scalar. We will focus on the aspects of this space that play a role for the space of complex numbers.

First of all, $(\mathbb{T}, \star)$ is an abelian group (stable, associative, unitary, invertible, commutative). You will sometimes see $+$, sometimes $\times$ instead of the star, because one can consider either the sums of θ, or the products of $e^{i\theta}$, which is exactly the same. We prefer $\times$ for what follows.

We call a torsion subgroup the set of elements of a group which, raised to a strictly positive integer power, can reach the identity element (we say that they are the elements of "finite order" where "order" refers to the power at which this number must be raised to obtain the identity). The torsion subgroup of $(\mathbb{T}, \star)$, also called the group of the roots of unity, is isomorphic to $\mathbb{Q}/\mathbb{Z}$, and is (topologically) dense in $\mathbb{T}$. This means that $\mathbb{T}$ is (topologically) separable.

An n-th root of unity is a complex number that verifies the property $z^n = 1$.

The square roots of unity are:
- $-1 = e^{i\frac{\tau}{2}}$ et
- $(-1)^2 = 1 = e^{i\frac{2\tau}{2}} = e^{i\tau} = e^{i0}$.

The cubic roots of unity are:
- $j = e^{i\frac{\tau}{3}}$,
- $j^2 = \overline{j} = e^{i\frac{2\tau}{3}}$, et
- $j^3 = 1$.

The quartic roots of unity are:
- $i = e^{i\frac{\tau}{4}}$,
- $i^2 = -1 = e^{i\frac{2\tau}{4}}$,

- $i^3 = -i = \bar{i} = e^{i\frac{3\tau}{4}}$ et

- $i^4 = 1$

NB: do not confuse the number j in the context of the roots of unity, with the j (such that $j^2 = 1$) of the split-complex numbers.

The general formula for the n-th roots of unity is $e^{\frac{ik\tau}{n}}$ with $k \in [[0, n[[$.

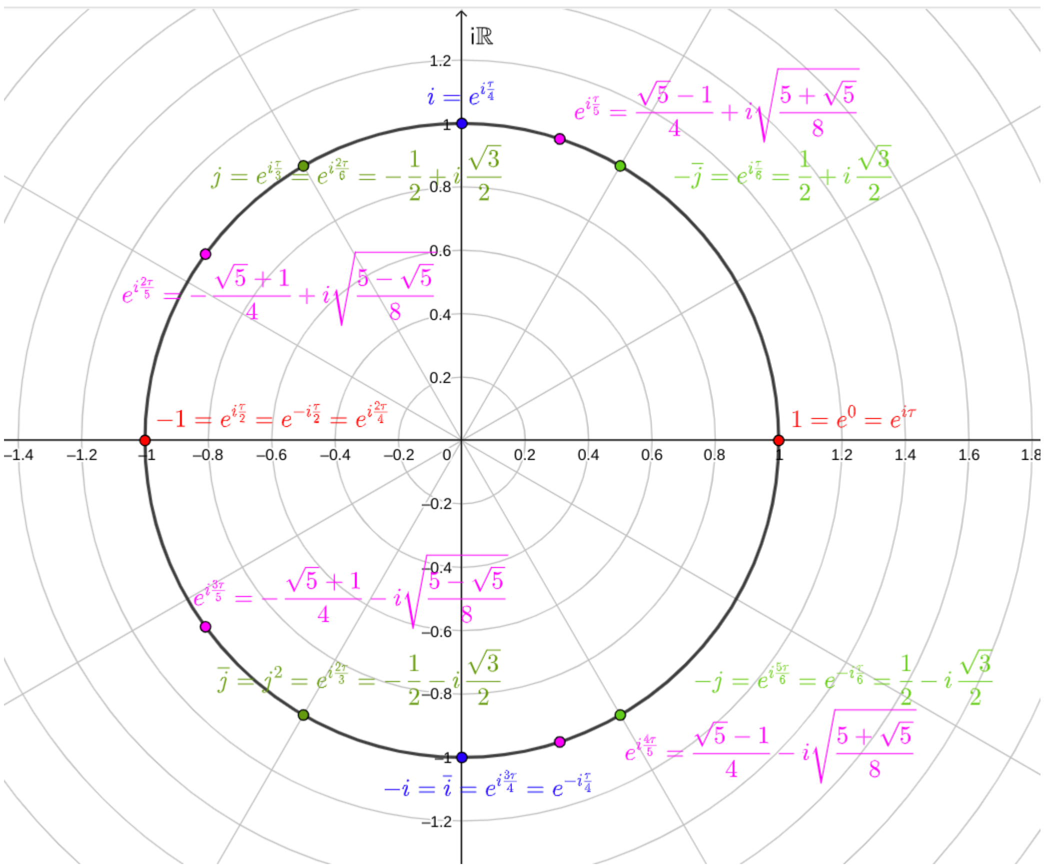

Figure 9: Image showing the roots of unity up to the 6-th order roots. Note that some of these numbers are roots for various n. For example, -1 is a n-th root of 1 for any even number n.

NB: $\mathbb{T}^n$, the product (set theoretic Cartesian product, or rather topological) of $\mathbb{T}$, n times with itself, is an n-dimensional torus.

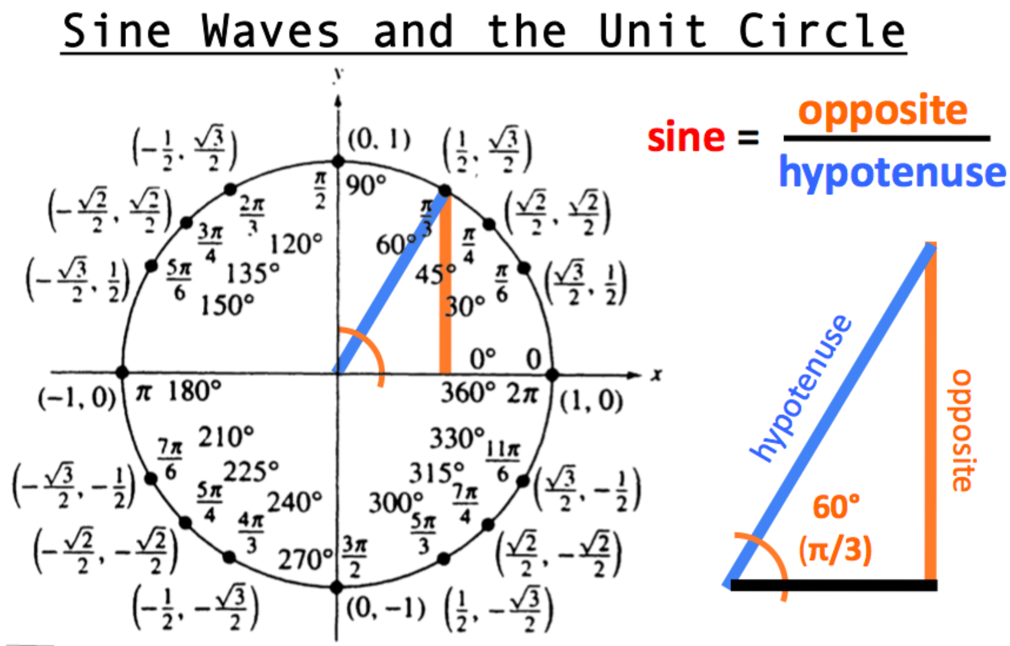

Figure 10: Here we have the best known roots of unity, which have nice, memorable values in trigonometry.

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3.5 Riemann sphere / complex projective line

The Riemann sphere, also called "extended complex plane", or "complex projective line" is the set denoted $\overline{\mathbb{C}} = P^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$. The following operations are added for any $z \in \overline{\mathbb{C}}$:

- $z + \infty = \infty$
- $z * \infty = \infty$ (for nonzero z)

The Riemann sphere is not a field because ∞ has no inverse for addition, nor for multiplication. However, we tend to define usually $\frac{z}{\infty} = 0$ (works with $z = 0$) and $\frac{z}{0} = \infty$ (works with $z = \infty$). On the other hand, $0 \times \infty$, $\infty - \infty$, $\frac{0}{0}$ and $\frac{\infty}{\infty}$ are left undefined.

The rational (polynomial) functions are all continuous over the Riemann sphere.

The transformation of the Riemann sphere into the complex plane is a stereographic projection originating from the point ∞.

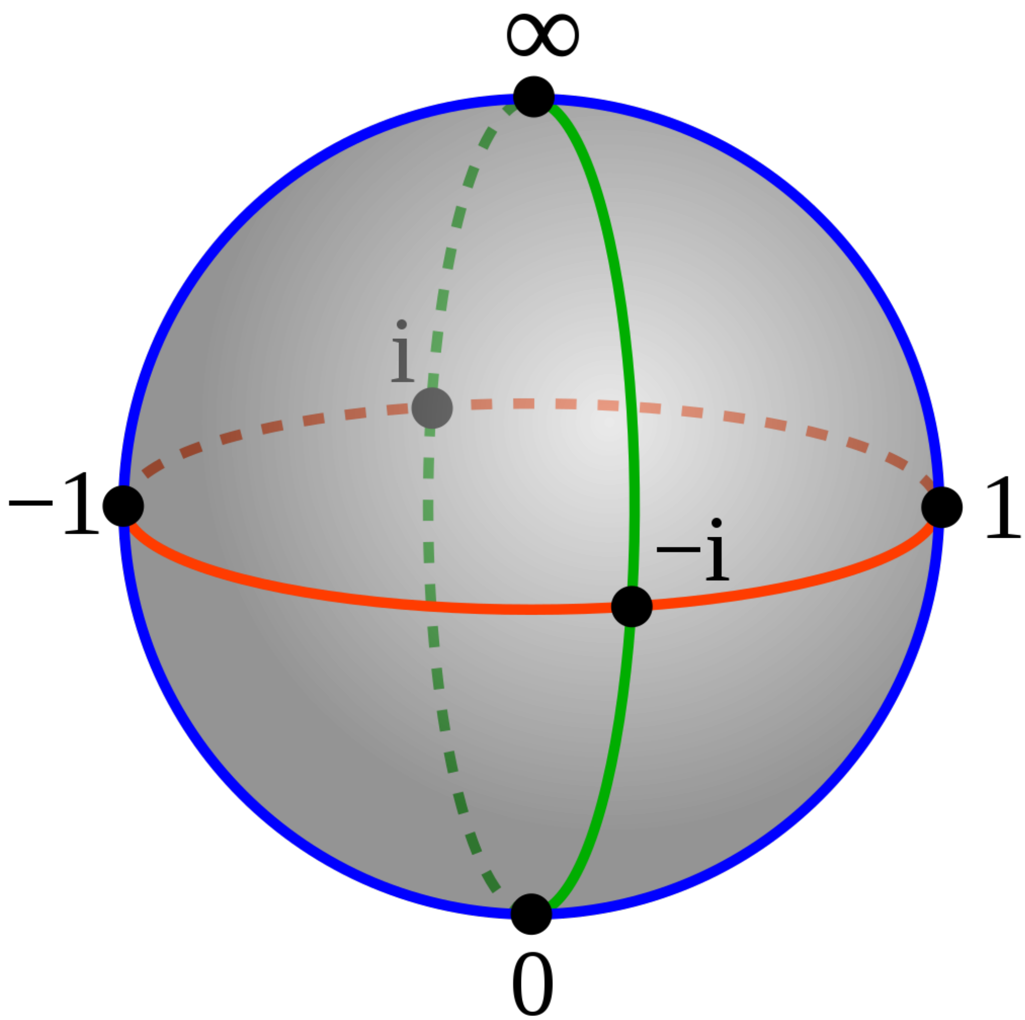

Figure 11: the Riemann Sphere

3.6 Gaussian spaces

We can construct, using i, rings and fields that are stable subspaces of $\mathbb{C}$, without needing to use $\mathbb{R}$ in our construction. These spaces are called Gaussian spaces. I leave it to you to figure out their geometry, and the geometry of their operations in your head.

- $\mathbb{Z}[i] = \{n + im \mid (n, m) \in \mathbb{Z}^2\}$, space of the Gaussian integers, sub-ring of $\mathbb{C}$.

- $\mathbb{Q}[i] = \{p + iq \mid (p, q) \in \mathbb{Q}^2\}$, space of the Gaussian rationals, sub-field of $\mathbb{C}$, and (topologically) dense in $\mathbb{C}$.

- $\mathbb{Z}/p\mathbb{Z}[i] = \{a + ib \mid (a, b) \in \mathbb{Z}/p\mathbb{Z}\}$, for $p \geq 3$, with p prime and $p \not\equiv 1[mod4]$, are finite non-prime fields (of the form $GF(p^2)$), topologically forming a product of two cyclic fields, thus a set of points covering a torus, with an analogue of complex multiplication. So $\mathbb{Z}/3\mathbb{Z}[i]$ and $\mathbb{Z}/7\mathbb{Z}[i]$ are fields, but not $\mathbb{Z}/5\mathbb{Z}[i]$. A similar construct can be done with the split-complex j, but is never a field.

3.7 To go further

There are really, really too many topics to cover that could not fit into a single paragraph in this section, so I encourage you to take a look at the following filesheets, where these topics are given due care.

Related data sheets: - $\mathbb{N} \to \mathbb{C}$, space complex-valued sequences (and important subspaces).

- $\mathbb{C} \to \mathbb{C}$, space of functions with complex values (and important subspaces).

- $\mathbb{C}^n$, standard n-dimensional vector space of complex numbers.

- $\mathbb{C}[X]$, ring of complex polynomials of finite degree (including an explanation of Escape-Time Fractals such as Julia or Mandelbrot sets).

- $\mathbb{D}$, space of split-complex numbers ($j^2 = 1$).

- $\mathbb{E}$, space of dual numbers ($\epsilon^2 = 0$).