HYDRO- OG AERODYNAMISK
LABORATORIUM

Lyngby — Denmark

Hydrodynamics
Section

Report No. Hy-5 . May 1964

Lectures on Ship Hydrodynamics - Steering and Manoeuvrability

BY

M. A. ABKOWITZ

IN COMMISSION:

VESTER FARIMAGSGADE 3 . COPENHAGEN
DENMARK

Hy-5 May 1964

Lectures on Ship Hydrodynamics

Errata

Page 3 : Line 4 f.b.: F should be $\vec{F}$

Page 8 : Line 3 f.b.: R_G should be $\vec{R}_G$

Page 10 : Line 9: $\vec{\Omega} \times \dot{\vec{R}}_G$ should be $\dot{\vec{\Omega}} \times \vec{R}_G$

Page 14 : Line 9: $p_v E$ should be p_v, E

Page 26 : Line 12 f.b.: $\mathcal{D}et.$ should be $|\mathcal{D}et.|$

Page 30 : Figure: change $r \rightarrow 0$ to $r \rightarrow$

Page 57 : Line 10 f.b.: B should be S

Page 66, top, add:

Since $(Y_v)_f$ is negative, lifting surfaces at the bow tend to decrease Y_r (make more negative or less positive) with an opposite effect for a fin located at the stern. A fin at the bow and at the stern will .....

Page 68:

read: $r = \frac{\left|\quad\right|}{\left|\quad\right|} = -\frac{\qquad}{\qquad} = -\frac{\qquad}{\qquad}$

Page 69: Line 11, read: $= -(b_3 \mathcal{D}\delta + b_4 \delta)$

Page 70 : Read: $r = -[\qquad]\delta_o$

Page 95 : Lines 18, 19, 20 should be in the foot-note after "such as"

Page 99 : Line 2 f.b.: r_1 should be f_1

Page 101: Line 5 f.b.: add $+P_1 rv\delta$ after $Pv^2\delta$

Page 108: In figure: $\underset{+N}{\overset{+\psi}{\curvearrowleft}}$ should be $\underset{+N}{\overset{+r}{\curvearrowleft}}$

Page 110: Line 6: Read $D^2 y_o - u_o D\psi$

INTRODUCTION

The Hydro- and Aerodynamics Laboratory is grateful to the author of this report for the following reasons:

Firstly because he accepted to act as a consultant to the Laboratory during his stay in Denmark.

Secondly because he agreed to having a very important part of his lectures at the Technical University of Denmark published as this HyA Report.

Thirdly for his indefatigable help and inspiration to the other research workers, and in particular to the research group developing a large Planar Motion Mechanism to be used in the investigation of steering and manoeuvring characteristics of surface-ship models, thus making evaluations for full-scale ships possible.

Finally, but certainly not least, for his extraordinary personal qualities resulting in the establishment of lasting friendships with all those who had the pleasure to co-operate with him at HyA.

Lyngby, Denmark
May, 1964

C.W. Prohaska
Director
Hydro- og Aerodynamisk
Laboratorium

PREFACE

During the academic year 1962-1963, the author, on sabbatical leave from the Massachusetts Institute of Technology, had the honor and pleasure of serving as a Fulbright Lecturer and Visiting Professor at the Technical University of Denmark. The gracious invitation of Professor Prohaska to me to lecture at the University and to participate in the research being pursued at the Hydro- and Aerodynamics Laboratory, of which he is the Director, was followed throughout the year by a sincere and friendly hospitality shown to me by the entire staff of the laboratory.

The lectures on ship hydrodynamics were concentrated in the two areas - boundary layer effects in naval architecture and ship motions. Since the lectures were presented in the English language (American version) and were concentrated in a technical area wherein there was not a readily available text, the author agreed to the suggestion of the students and laboratory staff to write, duplicate, and distribute the full text of the lectures. The text of each lecture was written, typed, proofread, figures drawn, duplicated, and readied for distribution each week. The author wishes to express his appreciation to the Hydro- and Aerodynamics Laboratory staff for a very effective job in getting the notes ready for distribution, since the job took additional time and created additional deadlines for a staff already fully occupied with busy laboratory commitments.

The realization that a modern technical treatment of the fundamentals of motion stability, maneuvering, and control (in the horizontal plane) was difficult to locate in existing texts on naval architecture, led to the suggestion by the laboratory staff and agreement by the author to publish that part of the lecture material dealing with motion, stability, and control in the horizontal plane. Hence, what is presented in this publication is essentially taken from context of material with a much larger coverage, namely Lectures 12 through 21 of a total of 27 lectures. Since the material is taken out of context, the editing and preparation for publication done by the Hydro- and Aerodynamics Laboratory staff included the proper handling of references made to those lectures not being included in the publication. The author asks the indulgence of the reader for errors of omission or commission and of overemphasis or underemphasis in the treatment of the subject. He offers only the excuse that the text was prepared under week by week deadline and he did not realize at the time of writing that the text would be published for a much more general distribution.

Ship operations are international and the profession of naval architecture has taken on this international aspect. At the Massachusetts Institute of Technology, students from many countries come to study in our Department of Naval Architecture and Marine Engineering. Through the International Towing Tank Conference, the friendship between Professor Prohaska and the author developed and the invitation resulted. The Fulbright program which sent the author to lecture in Denmark is an inter-nation program. Therefore, this publication is dedicated to the sincere spirit of cooperation and friendship that exists on an international scale in the profession of naval architecture.

Martin A. Abkowitz

TABLE OF CONTENTS

Page

CHAPTER I

Equations of Motion for a Body Moving With Six Degrees of Freedom

In the study of ship motions, on which subject we are about to embark, parameters associated with the body motion become important - such as components of the linear velocity in addition to the forward velocity, components of angular velocity, and various accelerations both linear and angular. The general field of ship motions is usually divided into the areas of a) steering and maneuverability and b) seakeeping, both areas being concerned with the concepts of motion stability and control.

a) Steering and maneuverability usually deal with the motion of a ship in the absence of excitation from the sea (calm water). The motion results from the excitation forces applied through the deflecting of control surfaces.

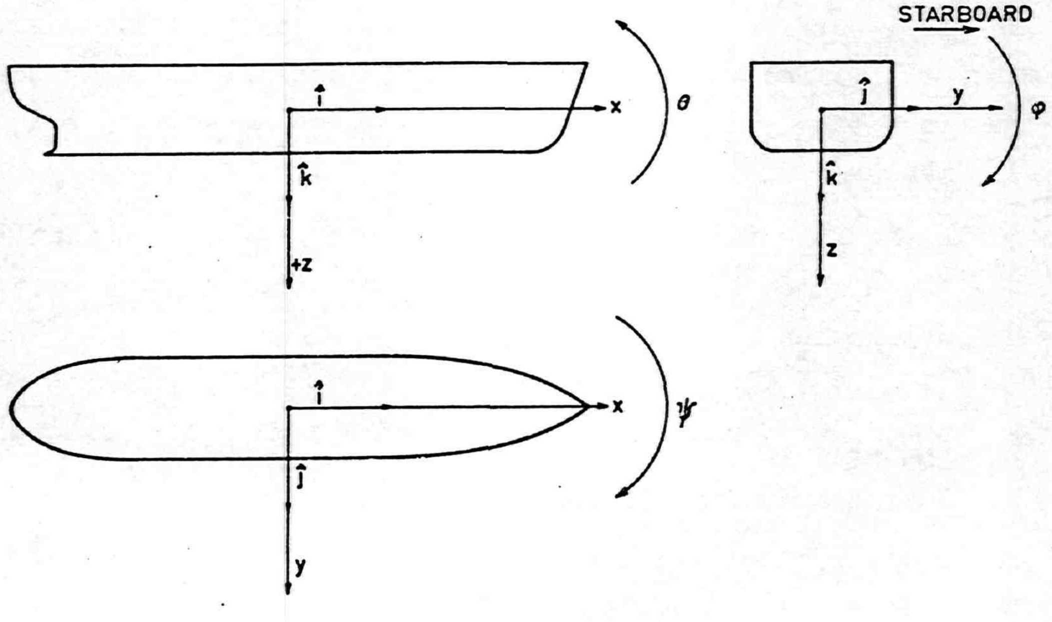
b) Seakeeping deals with the motion of a ship resulting from the excitation forces of the sea (such as waves). When control surfaces are used either to counter the sea excitation or to effect a maneuver in the presence of the sea excitation, then there is a combination of the two areas referred to as maneuvering in a seaway.

c) Motion stability deals with the aspects of ship motion in the absence of any excitation either from control surface deflection or from the seaway.

d) Motion control deals with the effects of the forces excited on the ship through manual or automatic application of control surfaces or other devices.

A ship at sea, or a body moving in a fluid, is allowed to move, and many times does move, in all the six degrees of freedom of motion -

i.e. translation along three orthogonal axes and rotating about each of the three axes. It is therefore necessary to choose an axis system to describe these motion freedoms and the choice should be one which is most convenient for the development of the motion analysis. Practically all vehicles and fluid dynamic bodies have a plane of symmetry - i.e. the centerline plane - since the port and starboard have the same geometry and represent reflections of each other in the centerline plane. This symmetry in body shape can be observed in ships, submarines, rockets, boats, torpedoes, hydrofoil boats, airplanes, dirigibles, fish, birds, etc. (Some asymmetry may be caused by a preferred direction of propeller rotation on a single screw ship but this slight diversion can be readily handled). An axis system which takes advantage of this plane symmetry is chosen. Hence, two of the three axes are in the plane of symmetry (and define the plane) and the third is perpendicular to the plane. Some bodies, such as rockets, and torpedoes, have a second plane of symmetry, where the upper and lower halves (keel and deck) are symmetrical, and this plane of symmetry is perpendicular to the other plane of symmetry. Axes, at least two of them, in the plane of symmetry are chosen, because the expressions for the hydrodynamic forces are simplified through symmetry and the equations of motion are simplified through the fact that axes oriented by symmetry are usually parallel to principal axes of inertia. The sketch below defines the axis system chosen.



x-axis = longitudinal axis in the plane of symmetry positive forward. Usually parallel to the keel or calm water line. If upper and lower half are symmetrical then the axis is the intersection of the two planes of symmetry. A unit vector along the x-axis is designated by $\hat{i}$.

y-axis = transverse axis, perpendicular to the plane of symmetry, positive to starboard. A unit vector along the y-axis is designated by $\hat{j}$.

z-axis = 'vertical axis', (perpendicular to water line planes), in the plane of symmetry, positive downward towards the keel. A unit vector along the z-axis is designated by $\hat{k}$.

These axes form a consistent right-handed coordinate system. A clockwise rotation, looking in the direction of the positive axis, would advance a right hand thread along the positive axis. Positive rotation about the x-axis tends to rotate the y-axis in the direction of the z-axis, positive rotation about the y-axis tends to rotate the z-axis towards the x-axis, and positive rotation about the z-axis tends to rotate the x-axis towards the y-axis. If φ is the roll angle, θ is the pitch angle, and ψ the yaw angle, then positive rotations are indicated in the sketch above. A consistent set of axes furnishes the convenience of being able to derive the remaining two components of a vector quantity from a general expression of the component along one of the axes, as will be demonstrated later.

In dealing with ship motion one needs to exploit the fundamentals of rigid body dynamics in order to develop the analysis. Hence, one begins with Newton's laws of motion, expressed as follows:

$$\vec{F} = \frac{d}{dt}\,(\overrightarrow{\text{Momentum}})$$

$$\vec{\mathcal{M}} = \frac{d}{dt}\,(\overrightarrow{\text{Angular momentum}})$$

F is the vector force acting on a body. The components of this force along the x, y, and z axes are X, Y, and Z respectively.

$$\vec{F} = \hat{i}X + \hat{j}Y + \hat{k}Z$$

$\vec{\mathcal{M}}$ is the vector moment acting on the body. The components along the x, y, and z axes are K, M, and N respectively.

$\vec{\mathcal{M}} = \hat{i}K + \hat{j}M + \hat{k}N.$

The origin for the axis system is taken at the center of gravity, G; this is necessary in order to write Newton's law in the form of separate force and moment equations. In addition, the axes are assumed to be the principal axes of inertia through the origin at G, thereby simplifying the momentum expressions. The force expression is written as

$$\vec{F} = \frac{d}{dt}(m\vec{U}) = m\frac{d\vec{U}}{dt} + \vec{U}\frac{dm}{dt} = m\dot{\vec{U}} + \vec{U}\frac{dm}{dt}$$

where $\vec{U}$ is the linear velocity vector, having components of u, v, w along the x, y, z axes respectively.

$\vec{U} = \hat{i}u + \hat{j}v + \hat{k}w$

m is the mass of the body.

$\frac{d}{dt}$ is the derivative with respect to time. The usual convention of denoting this derivative by a dot over the quantity is used, i.e. $\dot{\vec{U}} = \frac{d}{dt}(\vec{U})$

Since in most marine vehicles the time rate of change in mass due to fuel consumption is negligible, the mass of the body will be considered constant in time, hence $\frac{dm}{dt} = 0$. (This is not so for rockets).

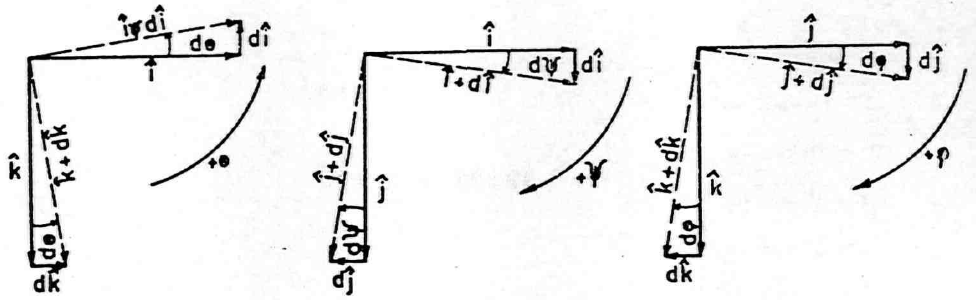
The axis system chosen is fixed in the ship in order to use the symmetry of the ship to more easily calculate the hydrodynamic and hydrostatic forces in the vector quantity, $\vec{F}$. Since the ship moves in space, the axes are moving axes which somewhat complicate the expressions for momentum change on the right hand side of the equation. This complication is minor compared to the gain effected in the ability to express $\vec{F}$ through use of symmetry considerations. Before considering the nature of $\vec{F}$ and $\vec{\mathcal{M}}$, let us develop the momentum change (right hand side of equations) for the moving axis system chosen.

On substituting the component expression for $\vec{U}$, the force equation becomes (under the constant mass assumption)

$$\vec{F} = m\frac{d}{dt}(\vec{U}) = m\frac{d}{dt}(\hat{i}u + \hat{j}v + \hat{k}w)$$

$$= m\left[\hat{i}\frac{du}{dt} + u\frac{d\hat{i}}{dt} + \hat{j}\frac{dv}{dt} + v\frac{d\hat{j}}{dt} + \hat{k}\frac{dw}{dt} + w\frac{d\hat{k}}{dt}\right]$$

A change in a vector quantity can occur only by a change in length and/or a change in direction. Since $\hat{i}$, $\hat{j}$, and $\hat{k}$ are unit vectors they do not change their length. However, their directions are along axes fixed in a moving ship and their direction change as the ship moves in space. Hence, the quantities $\frac{d\hat{i}}{dt}$, $\frac{d\hat{j}}{dt}$, and $\frac{d\hat{k}}{dt}$ are not zero for the moving axes system. The change in a unit vector is a change in direction brought about by the rotation of the body and does not depend on the translation of the body. The change in direction of the unit vectors for rotation about each of the body axes are demonstrated by the sketch below. The length of the vectors $d\hat{i}$, $d\hat{j}$, and $d\hat{k}$ are given by the unit radius multiplied by the radian measure of the small (differential) angles of rotation, The directions of $d\hat{i}$, $d\hat{j}$, $d\hat{k}$ are perpendicular to $\hat{i}$, $\hat{j}$, and $\hat{k}$ respectively.



Rotation in θ, (pitch) (about y-axis)	Rotation in ψ, (yaw) (about z-axis)	Rotation in φ, (roll) (about x-axis)
$d\hat{i} = -\hat{k}d\theta$	$d\hat{i} = \hat{j}d\psi$	$d\hat{i} = 0$
$d\hat{j} = 0$	$d\hat{j} = -\hat{i}d\psi$	$d\hat{j} = \hat{k}d\varphi$
$d\hat{k} = \hat{i}d\theta$	$d\hat{k} = 0$	$d\hat{k} = -\hat{j}d\varphi$

For a general small rotation about the three axes, the three contributions are added to give

$$d\hat{i} = \hat{i}0 + \hat{j}d\psi - \hat{k}d\theta \quad \text{or} \quad \frac{d\hat{i}}{dt} = \hat{i}0 + \hat{j}\frac{d\psi}{dt} - \hat{k}\frac{d\theta}{dt}$$

$$d\hat{j} = -\hat{i}d\psi + \hat{j}0 + \hat{k}d\varphi \quad \text{or} \quad \frac{d\hat{j}}{dt} = -\hat{i}\frac{d\psi}{dt} + \hat{j}0 + \hat{k}\frac{d\varphi}{dt}$$

$$d\hat{k} = \hat{i}d\theta - \hat{j}d\varphi + \hat{k}0 \quad \text{or} \quad \frac{d\hat{k}}{dt} = \hat{i}\frac{d\theta}{dt} - \hat{j}\frac{d\varphi}{dt} + \hat{k}0$$

The vector angular velocity is designated by $\vec{\Omega}$, and has the components of p, q, and r about the x, y, and z axes respectively, i.e.

$\vec{\Omega} = \hat{i}p + \hat{j}q + \hat{k}r$. Since $\frac{d\varphi}{dt} = p$, $\frac{d\theta}{dt} = q$, and $\frac{d\psi}{dt} = r$, one obtains

$$\frac{d\hat{i}}{dt} = \hat{i}0 + \hat{j}r - \hat{k}q \quad \text{or} \quad \frac{d\hat{i}}{dt} = \vec{\Omega} \times \hat{i} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ p & q & r \\ 1 & 0 & 0 \end{vmatrix}$$

$$\frac{d\hat{j}}{dt} = -\hat{i}r + \hat{j}0 + \hat{k}p \quad \text{or} \quad \frac{d\hat{j}}{dt} = \vec{\Omega} \times \hat{j} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ p & q & r \\ 0 & 1 & 0 \end{vmatrix}$$

$$\frac{d\hat{k}}{dt} = \hat{i}q - \hat{j}p + \hat{k}0 \quad \text{or} \quad \frac{d\hat{k}}{dt} = \vec{\Omega} \times \hat{k} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ p & q & r \\ 0 & 0 & 1 \end{vmatrix}$$

The expressions for $\frac{d\hat{j}}{dt}$ and $\frac{d\hat{k}}{dt}$ can be derived from the expression for $\frac{d\hat{i}}{dt}$ by a process of permutation of the components, a property resulting from the use of a consistent set of axes. The permutation procedure is as follows:

$$x \to y \to z \to x$$
$$\hat{i} \to \hat{j} \to \hat{k} \to \hat{i}$$
$$p \to q \to r \to p$$
$$u \to v \to w \to u$$
$$X \to Y \to Z \to X$$
$$K \to M \to N \to K$$

If one takes a general expression involving a component or a vector, the expressions for the other components, or similar vectors can be obtained by moving every item one down the index scale. For deriving $\frac{d\hat{j}}{dt}$ and $\frac{d\hat{k}}{dt}$ from $\frac{d\hat{i}}{dt}$, permute as follows:

$$\begin{array}{ccccccc} \frac{d\hat{i}}{dt} & = & \hat{i}0 & + & \hat{j}r & - & \hat{k}q \\ \downarrow & & \downarrow\downarrow & & \downarrow\downarrow & & \downarrow\downarrow \\ \frac{d\hat{j}}{dt} & = & \hat{j}0 & + & \hat{k}p & - & \hat{i}r \\ \downarrow & & \downarrow\downarrow & & \downarrow\downarrow & & \downarrow\downarrow \\ \frac{d\hat{k}}{dt} & = & \hat{k}0 & + & \hat{i}q & - & \hat{j}p \end{array}$$

Let us return to the force equation and substitute into the equation the expressions for $\frac{d\hat{i}}{dt}$, $\frac{d\hat{j}}{dt}$, and $\frac{d\hat{k}}{dt}$, using the dot over a quantity to indicate the derivative of the quantity with respect to time.

$$\vec{F} = m \left[\hat{i}\dot{u} + u(\hat{j}r - \hat{k}q) + \hat{j}\dot{v} + v(\hat{k}p - \hat{i}r) + \hat{k}\dot{w} + w(\hat{i}q - \hat{k}p)\right]$$

The quantities are grouped under the respective directional components, together with the defined components of $\vec{F}$, to give

$$\hat{i}X + \hat{j}Y + \hat{k}Z = m\left[\hat{i}(\dot{u}+qw-rv) + \hat{j}(\dot{v}+ru-pw) + \hat{k}(\dot{w}+pv-qu)\right]$$

or

$$X = m(\dot{u}+qw-rv)$$
$$Y = m(\dot{v}+ru-pw)$$
$$Z = m(\dot{w}+pv-qu)$$

The expressions for Y and Z could have been obtained from the expression for X by the process of permutation,

$$\begin{array}{l} X = m(\dot{u}+qw-rv) \\ \downarrow \qquad\quad \downarrow\ \downarrow\downarrow\ \downarrow\downarrow \\ Y = m(\dot{v}+ru-pw) \end{array}$$

Since the terms $\dot{u}$, $\dot{v}$, and $\dot{w}$ represent apparent acceleration components within the moving axis system, the terms $(qw-rv)$, $(ru-pw)$, and $(pv-qu)$ must represent the components of centripetal accelerations on the body arising from the moving coordinate system.

With axes parallel to the principal axes of inertia, the angular momentum in the moment equation can be expressed as

$$\overrightarrow{\text{Angular momentum}} = \hat{i}I_x p + \hat{j}I_y q + \hat{k}I_z r$$

where I_x, I_y, and I_z are the moments of inertia of the body about the x, y, and z axes respectively. The moment equation becomes

$$\vec{\mathcal{M}} = \frac{d}{dt}(\overrightarrow{\text{Ang.mom.}}) = \frac{d}{dt}(\hat{i}I_x p + \hat{j}I_y q + \hat{k}I_z r)$$

$$= \hat{i}\frac{d}{dt}(I_x p) + I_x p\frac{d\hat{i}}{dt} + \hat{j}\frac{d}{dt}(I_y q) + I_y q\frac{d\hat{j}}{dt} + \hat{k}\frac{d}{dt}(I_z r) + I_z r\frac{d\hat{k}}{dt}$$

Since the mass of the ship is assumed constant in time, then also the inertia of the ship (mass distribution) is assumed constant in time. Hence, $\frac{d}{dt}(I_x p) = I_x\dot{p}$, with analogous results for the similar terms. On incorporation of the expressions for $\frac{d\hat{i}}{dt}$, $\frac{d\hat{j}}{dt}$, and $\frac{d\hat{k}}{dt}$ into the moment equation, there results

$$\vec{\mathcal{M}} = \hat{i}I_x\dot{p} + I_x p(\hat{j}r-\hat{k}q) + \hat{j}I_y\dot{q} + I_y q(\hat{k}p-\hat{i}r) + \hat{k}I_z\dot{r} + I_z r(\hat{i}q-\hat{j}p)$$

With the vector components of moment defined as

$$\vec{\mathcal{M}} = \hat{i}K + \hat{j}M + \hat{k}N ,$$

the grouping of the vector quantities into components in the $\hat{i}$, $\hat{j}$, $\hat{k}$ directions gives

$$K = I_x\dot{p} + (I_z - I_y)qr$$

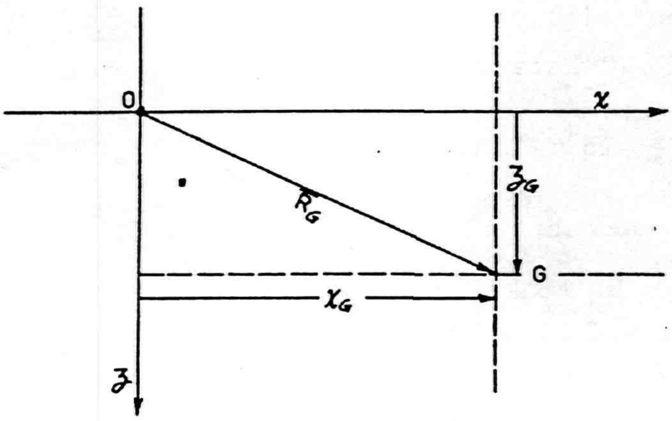
$$M = I_y\dot{q} + (I_x - I_z)rp$$

$$N = I_z\dot{r} + (I_y - I_x)pq$$

The expressions for M and N could have been deduced from the expression for K by the permutation process. Since $\dot{p}$, $\dot{q}$, $\dot{r}$ are "apparent" angular accelerations in the moving system, the terms $(I_z-I_y)qr$, $(I_x-I_z)rp$, and $(I_y-I_x)pq$ represent gyroscopic moments arising from the moving axis system.

The equations have been developed for the case of the origin located at the center of gravity of the body, but the center of gravity is not necessarily located at the center of geometry or buoyancy of the body. Since hydrostatic and hydrodynamic forces depend greatly on the geometry of the body, it would be very convenient to develop the equations for an arbitrary origin so as to provide the flexibility to choose an origin which takes advantage of body geometrical symmetries to more easily express the hydrostatic and hydrodynamic forces acting on the body. The equations of motion will be developed for an axis system parallel to the principal axes of inertia through the center of gravity, G, but for a location of the origin, O, not necessarily at the center of gravity.

In order to use the separate force and moment equations, the forces and moment acting at the center of gravity, G, will be used, but they will be expressed in terms of components measured relative to an origin O in the body. The vector distance that the center of gravity is from the origin is designated by $\bar{R}_G = \hat{i}x_G + \hat{j}y_G + \hat{k}z_G$; x_G, y_G, and z_G are the distances of the center of gravity, G, from O, along the x, y, and z axes respectively, as can be observed from the following sketch.



The equation $\vec{F} = \frac{d}{dt}(m\vec{U_G})$ is the proper Newtonian expression, since $\vec{U_G}$ refers to the velocity at the center of gravity, but it is desired to develop this expression for a velocity $\vec{U}$ as measured at the origin O. The velocity $\vec{U_G}$ at G must equal the velocity $\vec{U}$ at O plus the relative of G relative to O, or

$$\vec{U_G} = \vec{U} + \frac{d}{dt}(\vec{R_G}) = \vec{U} + \dot{\vec{R_G}}$$

Since $\vec{R_G}$ is a vector fixed in the body, it cannot change its length but only its direction as the body moves about. Hence, the velocity of G relative to O can result only from rotation - hence from the product of angular velocity and the radius. $\dot{\vec{R_G}}$ can then be expressed as

$$\dot{\vec{R_G}} = \vec{\Omega} \times \vec{R_G} .$$

This expression can be obtained by carrying through the time derivative of the vector expression of $\vec{R_G}$.

$$\dot{\vec{R_G}} = \frac{d}{dt}(\hat{i}x_G+\hat{j}y_G+\hat{k}z_G) = \hat{i}\dot{x}_G+x_G\frac{d\hat{i}}{dt}+\hat{j}\dot{y}_G+y_G\frac{d\hat{j}}{dt}+\hat{k}\dot{z}_G+z_G\frac{d\hat{k}}{dt}$$

$\dot{x}_G = \dot{y}_G = \dot{z}_G = 0$, since $\vec{R_G}$ is fixed in the body. Recall that

$$\frac{d\hat{i}}{dt} = \hat{j}r-\hat{k}q \qquad \frac{d\hat{j}}{dt} = \hat{k}p-\hat{i}r \qquad \frac{d\hat{k}}{dt} = \hat{i}q-\hat{j}p$$

and when these are substituted into the expression for $\dot{\vec{R_G}}$ there results

$$\dot{\vec{R_G}} = x_G(\hat{j}r-\hat{k}q) + y_G(\hat{k}p-\hat{i}r) + z_G(\hat{i}q-\hat{j}p)$$

This reduces to

$$\dot{\vec{R_G}} = \hat{i}(qz_G-ry_G)+\hat{j}(rx_G-pz_G)+\hat{k}(py_G-qx_G) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ p & q & r \\ x_G & y_G & z_G \end{vmatrix} = \vec{\Omega} \times \vec{R_G}$$

The force equation is therefore written as

$$\vec{F} = m\,\frac{d}{dt}(\vec{U} + \vec{\Omega}\times\vec{R_G})$$

$$= m\dot{\vec{U}} + \dot{\vec{\Omega}} \times \vec{R_G} + \vec{\Omega}\times\dot{\vec{R_G}}$$

The term $\dot{\vec{U}} = \frac{d\vec{U}}{dt}$ has been developed above and can be expressed as

$$\dot{\vec{U}} = \frac{d}{dt}(\hat{i}u+\hat{j}v+\hat{k}w) = \hat{i}(\dot{u}+qw-rv)+\hat{j}(\dot{v}+ru-pw)+\hat{k}(\dot{w}+pv-qu)$$

The components of $\dot{\vec{\Omega}}$ are developed as follows

$$\dot{\vec{\Omega}} = \frac{d}{dt}(\hat{i}p+\hat{j}q+\hat{k}r) = \hat{i}\dot{p} + p\frac{d\hat{i}}{dt} + \hat{j}\dot{q} + q\frac{d\hat{j}}{dt} + \hat{k}\dot{r} + r\frac{d\hat{k}}{dt}$$

$$p\frac{d\hat{i}}{dt} + q\frac{d\hat{j}}{dt} + r\frac{d\hat{k}}{dt} = p(\hat{j}r-\hat{k}q)+q(\hat{k}p-\hat{i}r)+r(\hat{i}q-\hat{j}p) = \hat{i}0+\hat{j}0+\hat{k}0$$

Therefore, $\dot{\vec{\Omega}} = \hat{i}\dot{p}+\hat{j}\dot{q}+\hat{k}\dot{r}$, and the expression for $\dot{\vec{\Omega}} \times \overline{R_G}$ becomes

$$\dot{\vec{\Omega}} \times \vec{R}_G = \begin{vmatrix} i & j & k \\ \dot{p} & \dot{q} & \dot{r} \\ x_G & y_G & z_G \end{vmatrix} = \hat{i}(z_G\dot{q}-y_G\dot{r})+\hat{j}(x_G\dot{r}-z_G\dot{p})+\hat{k}(y_G\dot{p}-x_G\dot{q})$$

and the expression for $\vec{\Omega} \times \dot{\vec{R}}_G$ becomes (on substituting the components of $\dot{\vec{R}}_G$)

$$\vec{\Omega} \times \dot{\vec{R}}_G = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ p & q & r \\ qz_G-ry_G & rx_G-pz_G & py_G-qx_G \end{vmatrix} = \hat{i}(qpy_G-q^2x_G-r^2x_G+prz_G)+\hat{j}(\ldots\ldots.) \\ +\hat{k}(\ldots\ldots.).$$

To obtain the expression for the X component of force, the $\hat{i}$ component of $m\dot{\vec{U}} + \dot{\vec{\Omega}}\times\vec{R}_G+\vec{\Omega}\times\dot{\vec{R}}_G$ is formulated as follows:

$$\qquad \dot{\vec{U}} \qquad\qquad \dot{\vec{\Omega}}\times\vec{R}_G \qquad\qquad\qquad \vec{\Omega}\times\dot{\vec{R}}_G$$

$$X = m\left[\dot{u}+qw-rv + z_G\dot{q}-y_G\dot{r} + qpy_G-x_G(q^2+r^2)+prz_G\right]$$

$$X = m\left[\dot{u}+qw-rv-x_G(q^2+r^2)+y_G(pq-\dot{r})+z_G(pr+\dot{q})\right]$$

The Y and Z components can be formulated by permuting the terms.

$$Y = m\left[\dot{v}+ru-pw-y_G(r^2+p^2)+z_G(qr-\dot{p})+x_G(qp+\dot{r})\right]$$

$$Z = m\left[\dot{w}+pv-qu-z_G(p^2+q^2)+x_G(rp-\dot{q})+y_G(rq+\dot{p})\right]$$

The equations for the X, Y, and Z components for an origin not at the center of gravity, differ from the equations for the origin at the center of gravity in the additional terms involving x_G, y_G, and z_G. On studying the X equation, for example, the physical significance of these additional terms become apparent. The terms resulting from $\vec{\Omega} \times \dot{\vec{R}}_G = \vec{\Omega} \times (\vec{\Omega} \times \vec{R}_G)$, represent centrifugal forces acting at the origin because of the center of gravity not being at the origin, and the terms resulting from $\overset{c}{\vec{\Omega}} \times \vec{R}_G$ represent the inertial reaction forces felt at the origin by the acceleration of the C.G. relative to the origin.

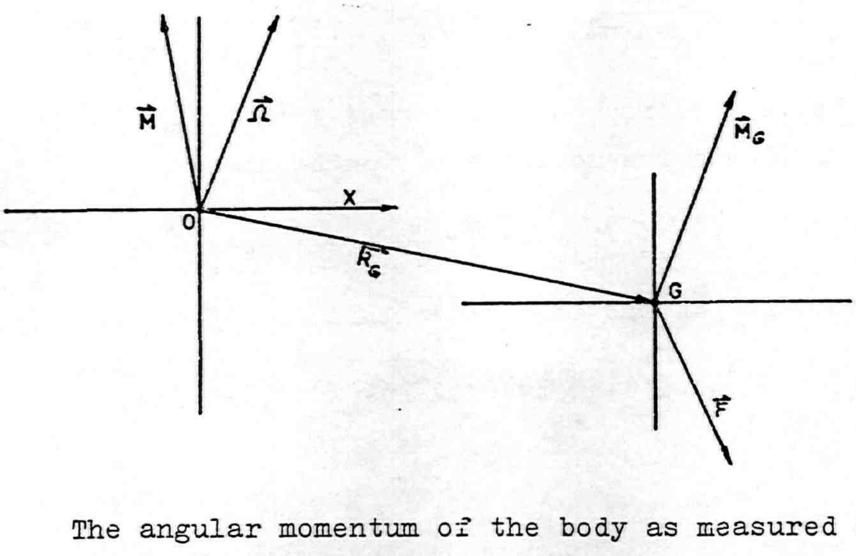
The expressions for the components of $\vec{\mathcal{M}}$ for an origin off the C.G. will now be developed, using the equation

$$\vec{\mathcal{M}}_G = \frac{d}{dt} (\overrightarrow{\text{Ang.Mom.}})_{C.G.}$$

which refers to the moment at the center of gravity. The sketch shown below serves as an aid in visualizing the relationships between the two origins.

The moment $\vec{\mathcal{M}}$ experienced at O equals the moment experienced at G plus the moment caused by the force $\vec{F}$ acting over the radius $\vec{R}_G$, i.e.

$$\vec{\mathcal{M}} = \vec{\mathcal{M}}_G + \vec{R}_G \times \vec{F}$$



The angular momentum of the body as measured at O equals the angular momentum of the body as measured at G plus the additional moment of momentum caused by the motion of G relative to O. The momentum of G relative to O is $m\dot{\vec{R}}_G = m \vec{\Omega} \times \vec{R}_G$ and the moment of momentum is given by $\vec{R}_G \times m \vec{\Omega} \times \vec{R}_G$. Another way to look at the situation is to recall that the moment of inertia of a body is minimum for an axis passing through the C.G. and an additional mR^2 must be added of the C.G. is a distance R from the axis of rotation. Hence, the R_G appears twice, in conjunction with $\vec{\Omega}$, in the product $\vec{R}_G \times m\vec{\Omega} \times \vec{R}_G$ for the additional angular momentum.

The angular momentum about 0 is given therefore by

$$\overrightarrow{(\text{Angular momentum})}_O = \overrightarrow{(\text{Angular Momentum})}_G + m\vec{R}_G \times (\vec{\Omega} \times \vec{R}_G)$$

or

$$\overrightarrow{(\text{Angular momentum})}_G = \overrightarrow{(\text{Angular Momentum})}_O - m\vec{R}_G \times (\vec{\Omega} \times \vec{R}_G)$$

The moment equation of motion now becomes

$$\vec{\mathcal{M}}_G = \vec{\mathcal{M}} - \vec{R}_G \times \vec{F} = \frac{d}{dt}\left[\overrightarrow{(\text{Ang.Momentum})}_O - m\vec{R}_G \times (\vec{\Omega} \times \vec{R}_G)\right]$$

Since $\vec{F} = m\dfrac{d}{dt}(\vec{U}_G) = m\dot{\vec{U}}_G$ and since $\vec{U}_G = \vec{U} + \vec{\Omega} \times \vec{R}_G$,

the above formulation becomes

$$\vec{\mathcal{M}} = \frac{d}{dt}\left[\overrightarrow{(\text{Ang.momentum})}_O - m\vec{R}_G \times (\vec{\Omega} \times \vec{R}_G)\right] + \vec{R}_G \times m\frac{d}{dt}(\vec{U}_G)$$

$$= \frac{d}{dt}\overrightarrow{(\text{Ang.Momentum})}_O - m\dot{\vec{R}}_G \times (\vec{\Omega} \times \vec{R}_G) - m\vec{R}_G \times \frac{d}{dt}(\vec{\Omega} \times \vec{R}_G)$$

$$+ \vec{R}_G \times m\frac{d}{dt}\vec{U} + \vec{R}_G \times m\frac{d}{dt}(\vec{\Omega} \times \vec{R}_G)$$

Since $\dot{\vec{R}}_G = \vec{\Omega} \times \vec{R}_G$

$$\vec{\mathcal{M}} = \frac{d}{dt}\overrightarrow{(\text{Ang.Momentum})}_O - m(\vec{\Omega} \times \vec{R}_G) \times (\vec{\Omega} \times \vec{R}_G) + m\vec{R}_G \times \frac{d\vec{U}}{dt}$$

Since the vector product of a vector with itself is zero, $(\vec{\Omega} \times \vec{R}_G) \times (\vec{\Omega} \times \vec{R}_G) = 0$ and the moment equation becomes

$$\vec{\mathcal{M}} = \frac{d}{dt}(\hat{i}I_x p + \hat{j}I_y q + \hat{k}I_z r) + m\vec{R}_G \times \frac{d}{dt}(\hat{i}u + \hat{j}v + \hat{k}w)$$

In previous derivations, it has been shown that

$$\dot{\vec{U}} = \frac{d}{dt}(\hat{i}u + \hat{j}v + \hat{k}w) = \hat{i}(\dot{u}+qw-rv) + \hat{j}(\dot{v}+ru-pw) + \hat{k}(\dot{w}+pv-qu)$$

and

$$\vec{R}_G \times \dot{\vec{U}} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_G & y_G & z_G \\ (\dot{u}+qw-rv) & (\dot{v}+ru-pw) & (\dot{w}+pv-qu) \end{vmatrix} = \hat{i}\left[y_G(\dot{w}+pv-qu) - z_G(\dot{v}+ru-pw)\right]$$

$$+ \hat{j}\left[\cdots\cdots\cdots\right] + \hat{k}\left[\cdots\cdots\cdots\right].$$

From previous development of the derivative of angular momentum, it has been shown that

$$\frac{d}{dt}(\hat{i}I_x p + \hat{j}I_y q + \hat{k}I_z r) = \hat{i}\left[I_x \dot{p} + (I_z - I_y)qr\right] + \hat{j}\left[\cdots\cdots\right] + \hat{k}\left[\cdots\cdots\right]$$

The various terms associated with unit vector $\hat{i}$ in the expression

$$\vec{\mathcal{M}} = \frac{d}{dt}\left(\overrightarrow{\text{Ang.momentum}}\right)_O + m\vec{R}_G \times \dot{\vec{U}}$$

are grouped together to form an expression for K, the $\hat{i}$ component of $\vec{\mathcal{M}}$.
Hence,

$$K = I_x\dot{p}+(I_z-I_y)qr+m\Big[y_G(\dot{w}+pv-qu)-z_G(\dot{v}+ru-pw)\Big]$$

What is ... not I_x

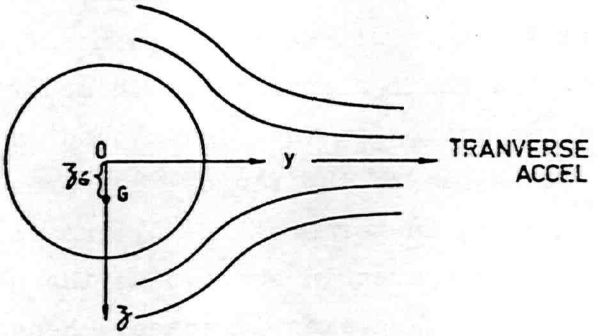
and by permutation

$$M = I_y\dot{q}+(I_x-I_z)rp+m\Big[z_G(\dot{u}+qw-rv)-x_G(\dot{w}+pv-qu)\Big]$$

$$N = I_z\dot{r}+(I_y-I_x)pq+m\Big[x_G(\dot{v}+ru-pw)-y_G(\dot{u}+qw-rv)\Big].$$

The apparent physical significance of the additional terms involving x_G, y_G, and z_G is the introduction into the moment equation of those moments resulting from inertial reaction forces caused by acceleration of the center of gravity.

The equations of motion for a body have now been expressed in a flexible form, allowing the choice of origin for the coordinate system. A simple example of the advantage of choosing an origin off the center of gravity is given by the transverse acceleration of a body like a torpedo. If the origin is chosen at O, a position of symmetry, as shown below,



then a transverse acceleration produces no hydrodynamic roll moment, K, because of the symmetry of flow relative to O. The formula gives the roll moment about O caused by G not being at the origin, - i.e. $mz_G \times$ (transverse acceleration). If the origin were at G, it would have been necessary to calculate a hydrodynamic moment.

CHAPTER II

Forces and Moments Acting on a Body

The forces and moments acting on a body, which in turn cause the ship to move, need now be studied in order to analyze the motion of a body.

Through the dependence of various phenomena on the properties of the body, properties of the motion, and properties of the fluid, the relationship for the forces and moments (in unrestricted water) become

$$\begin{matrix}\text{Forces}\\ \text{Moments}\end{matrix} = f\Big\{\underbrace{L, \text{geom}, m, \vec{R}_G, I;}_{\text{Properties of body}}\ \underbrace{\vec{R}_0, \varphi, \theta, \psi; \vec{U}, \dot{\vec{U}}, \vec{\Omega}, \dot{\vec{\Omega}}, n, \dot{n}, \delta, \dot{\delta}, \ddot{\delta};}_{\text{Properties of motion}}\ \underbrace{\rho, \mu, g, \tau, p, p_v E, \ldots}_{\text{Properties of fluid}}\Big\}$$

On reduction to non-dimensional form, the properties of the fluid were analyzed which resulted in the terms of Reynolds' number, Froude number, etc. and their significance in modelling was demonstrated. Since the fluid forces acting on the body depend on the orientation and motion of the body relative to the fluid, the parameters in the above function can be expressed in terms of the orientation and the motion of the body relative to fixed axes in space plus the orientation and motion of the fluid relative to fixed axes in space. Hence, if one prefers to call the motion properties listed in the function as referring to space axes, then additional parameters involving the orientation and motion of the fluid must be included in the parameters of the function. Such items as wave-shape, size, and particle orbital velocity would then appear in the function. One can characterize these fluid motion properties as an excitation parameter.

In order to concentrate on the effects of the dynamic parameters in the function, the dimensional form will be used and a given fluid and a given ship size will be considered. The results from the analysis of the function in dimensional form can be readily reduced to non-dimensional form for considerations of model work in maneuvering and seakeeping.

For a given ship in a given fluid, in the absence of excitation forces, one can express the general function as

$$\left.\begin{matrix}\vec{F}\\ \vec{\mathcal{M}}\end{matrix}\right\} = f(x_0, y_0, z_0, \varphi, \theta, \psi, u, v, w, p, q, r, \dot{u}, \dot{v}, \dot{w}, \dot{p}, \dot{q}, \dot{r}, \delta, \dot{\delta}, \ddot{\delta})$$

since if a function depends on $\vec{U}, \vec{\Omega}$, $\dot{\vec{U}}, \dot{\vec{\Omega}}$, then the function is also dependent on the components of these vectors. If the function were dependent on only one variable it may be possible to calculate the force and moment as the variable changed its value. For instance, in calm water with the ship not moving, and at even keel, the force exerted on the ship if it is moved vertically to the water surface is expressed as a simple function of z_0

$$\vec{F} = f_1(z_0)$$

The force is readily calculable as a vertical force equal to the change in displacement caused by increasing the draft of the ship. However, if this vertical motion is varied at the same time other variables are not zero, additional forces, not readily calculable are introduced. On the other hand, if the only variable were the forward speed, u, then even the simple function

$$\vec{F} = f_2(u)$$

is not calculable and resistance tests on models need to be run to estimate the force. It is necessary to develop the function of these many variables into a useful form for analysis purposes.

The function describing the forces and moments acting on a given ship in a given fluid involves the many motion and orientation parameters. The function can be reduced to useful mathematical form by the use of the Taylor expansion of a function of several variables. To use the expansion, the function and its derivatives need to be continuous and not go to infinity (blow up) in the region of the values of the variables under consideration. This assumption holds very well with respect to hydrodynamic bodies in the region of their operating conditions, especially ships.

Let us observe how the Taylor expansion works with one variable, say x as an example. If the value of the function f(x) is desired for a certain value of x, it can be described in terms of the value of the function and its derivatives at some other value of x, say at $x = x_0$, as follows:

$$f(x) = f(x_0)+(x-x_0)\frac{df(x_0)}{dx} + \frac{(x-x_0)^2}{2!}\frac{d^2f(x_0)}{dx^2} + \frac{(x-x_0)^3}{3!}\frac{d^3f(x_0)}{dx^3} + ---$$

where $f(x_0)$ indicates the value of the function at $x = x_0$

$\frac{d^nf(x_0)}{dx^n}$ indicates the $n^{\underline{th}}$ derivative of the function evaluated at $x = x_0$.

On introducing the differential operator $\mathcal{D}_x = \frac{d}{dx}$, $(\mathcal{D}_x^{\,n} = \frac{d^n}{dx^n})$, and $(x - x_0) = \Delta x$, then the form of the expansion becomes

$$f(x) = f(x_0) + \Delta x\,\mathcal{D}_x f(x_0) + \frac{(\Delta x\,\mathcal{D}_x)^2}{2!} f(x_0) + \frac{(\Delta x\,\mathcal{D}_x)^3}{3!} f(x_0) + -----$$

or

$$f(x) = \left[1 + (\Delta x\,\mathcal{D}_x) + \frac{(\Delta x\,\mathcal{D}_x)^2}{2!} + \frac{(\Delta x\,\mathcal{D}_x)^3}{3!} + -----\right] f(x_0).$$

This form is exactly the form for a series expansion of the exponential

$$e^a = 1 + a + \frac{a^2}{2!} + \frac{a^3}{3!} + -----$$

so that the Taylor expansion can be expressed as

$$f(x) = e^{\Delta x\,\mathcal{D}_x} \cdot f(x_0).$$

Similarly, for more than one variable, say the two variables x and y, the Taylor expansion takes the form

$$f(x,y) = e^{\Delta x\,\mathcal{D}_x + \Delta y\,\mathcal{D}_y} \cdot f(x_0,y_0)$$

where $\mathcal{D}_x = \frac{\partial}{\partial x}$ and $\mathcal{D}_y = \frac{\partial}{\partial y}$ since partial derivatives are required for more than one variable. On expanding one obtains

$$f(x,y) = f(x_0,y_0)+(\Delta x\mathcal{D}_x+\Delta y\mathcal{D}_y)f(x_0,y_0)+\frac{(\Delta x\mathcal{D}_x+\Delta y\mathcal{D}_y)^2}{2!}f(x_0,y_0)+ ---$$

$$= f(x_0,y_0)+\frac{\Delta x\,\partial f(x_0,y_0)}{\partial x}+\Delta y\frac{\partial f(x_0,y_0)}{\partial y}+\frac{1}{2!}\left[\frac{(\Delta x)^2\partial^2 f(x_0,y_0)}{\partial x^2}\right.$$

$$\left.+\frac{(\Delta y)^2\partial^2 f(x_0,y_0)}{\partial y^2}+2\,\Delta x\,\Delta y\,\frac{\partial^2 f(x_0,y_0)}{\partial x\,\partial y}\right]+ ----.$$

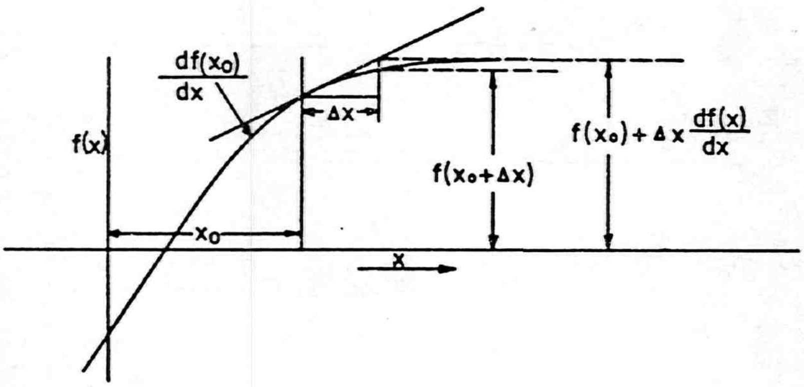
(Remember that x and y above are sample variables and bear no relationship to the variables x_0, y_0 in our function for $\vec{F}$ and $\mathcal{M}$).

The Taylor expansion for the forces and moments acting on the ship would then be expressed as

$$\text{or } \left.\begin{matrix}\vec{F}\\ \mathcal{M}\end{matrix}\right\} = \left[e^{\Delta x_0\mathcal{D}_{x_0}+\Delta y_0\mathcal{D}_{y_0}+\Delta z_0\mathcal{D}_{z_0}+\quad + --- +\Delta v\mathcal{D}_v + +\Delta\dot{r}\mathcal{D}_{\dot{r}}}\right] f\left[(x_0)_0,(y_0)_0, --- v_0, ---, \dot{r}_0\right].$$

and the expansion of the power series into the actual functional form would indeed give an extremely long, cumbersome, and almost impossible to handle expression for $\vec{F}$ and $\mathcal{M}$.

Hence, for simplicity and for the sake of reducing the equation to solvable form, the function is "linearized" about an initial equilibrium condition of motion. On linearization, only the linear terms in the change of the value of the variable from the equilibrium (initial condition) are maintained, i.e. terms of the order Δx and Δy in the example. Terms of higher order (i.e. $(\Delta x)^2$, $(\Delta y)^2$, $(\Delta x)(\Delta y)$, $(\Delta x)^3$, etc. in the example) are considered small compared to the first order terms and are neglected. This limits the validity of the analysis to relatively small changes in the variables (i.e. small Δx, Δy, etc.). Linearization, in effect, estimates the value of the function by multiplying the slope of the function by the change in the variable as shown below.



A familiar example of linearization of a function is the use of the product of metacentric height and the angle of heel to estimate the function of righting arm vs. angle of heel, which is the curve of statical stability.

In the domain of ship motions, we are mostly interested in those ship motions which depart from the condition of straight ahead motion such as various maneuvers from straight ahead motion or the pitching and heaving of the surface ship about a mean straight path. Hence, the initial condition of motion equilibrium is chosen as straight ahead motion at constant speed. This is indeed a condition of equilibrium since no forces and moments are acting on the body because there are no accelerations either angular or linear in this condition. The propeller forces are cancelling the resistance forces (through thrust deduction) with no net force acting on the body. The equilibrium condition of the function (straight ahead motion and designated by the subscript 0 on the variables) becomes

$$\left.\begin{matrix}\vec{F}_0 \\ \vec{m}_0\end{matrix}\right\} = f\left\{(x_0)_0, (y_0)_0, (z_0)_0, \varphi_0, \theta_0, \psi_0, u_0, v_0, w_0, p_0, q_0, r_0, \dot{u}_0, \ldots\ldots \dot{r}_0, \delta_0, \ldots\ldots\right\}$$

For straight ahead motion at constant speed (using a chosed orientation for reference) all the initial values (equilibrium values) of the variables are zero except for u_0 which is the value of the forward speed. Hence,

$$(x_0)_0 = (y_0)_0 = \text{-----} = \psi_0 = v_0 = w_0 = \text{-----} = \delta_0 = 0$$

$$u_0 \neq 0$$

The changes in the value of the variables from the value at the equilibrium condition already has been designated by a preceeding Δ , i.e. $\Delta u = u-u_0$, $\Delta v = v-v_0$, etc. Since all the variables have equilibrium values of 0, except for u, a change in value for all the variables, excluding u, can be written if the form

$$\Delta \text{variable} = \text{variable} - (\text{variable})_0$$

$$(\text{variable})_0 = 0$$

$$\Delta \text{variable} = \text{variable}$$

For example $\Delta v = v$, $\Delta \dot{u} = \dot{u}$, $\Delta \dot{q} = q$, etc., but $u = u_0 + \Delta u$. If the force and moment are functions of a set of variables so also are the components of the force and moment. Hence, X, Y, Z, K, M, and N can be expressed as functions of these many variables. Let us take for example

$$X = X(x_0, y_0, ----. u, v, ---, \dot{u}, \dot{v}, ---\dot{r}---)$$

indicating the X component is some function X of the variables. The linear terms of the Taylor expansion of this function would appear as follows:

$$X_0 = X_0 + \left(\frac{\partial X}{\partial x_0}\right)_0 \Delta x_0 + \left(\frac{\partial X}{\partial y_0}\right)_0 \Delta y_0 + \left(\frac{\partial X}{\partial \varphi}\right)_0 \Delta \varphi + --- + \left(\frac{\partial X}{\partial u}\right)_0 \Delta u$$

$$+ \left(\frac{\partial X}{\partial v}\right)_0 \Delta v + --- + \left(\frac{\partial X}{\partial \dot{r}}\right)_0 \Delta \dot{r} + ---.$$

A convenient notation for writing the derivative of a function taken at the equilibrium value of the function (or variable) uses a subscript to denote the variable involved in the differential, as demonstrated in the following examples.

$$\left(\frac{\partial X}{\partial u}\right)_0 = \left(\frac{\partial X}{\partial u}\right)_{\substack{u=u_0 \\ v=v_0=0 \\ w=w_0=0 \\ \vdots}} = X_u$$

$$\left(\frac{\partial X}{\partial \dot{r}}\right)_0 = \left(\frac{\partial X}{\partial \dot{r}}\right)_{\substack{\dot{r}=\dot{r}_0=0 \\ u=u_0 \\ v=v_0=0 \\ \vdots}} = X_{\dot{r}}$$

The linear expansion for the X function using this notation, together with the substitution of $\Delta v = v$, $\Delta\dot{u} = \dot{u}$, etc., as previously developed, gives

$$X = X_0+X_{x_0}x_0+X_{y_0}y_0+X_{z_0}z_0+X \quad + \text{------}+X_u\Delta u+X_v v+\text{------}+X_r r+\text{---}$$

with similar expressions for the Y, Z, K, M, and N components. In the equilibrium condition of straight ahead motion at constant speed there are no forces acting on the body, hence $X_0=Y_0=Z_0=K_0=M_0=N_0=0$. In order to keep the development of the solution of the equations of motion somewhat less complicated for the purposes of understanding the phenomenon, let us devote our efforts at the present time to the analysis of motion in the horizontal plane (maneuvering) without rolling. This involves the three degrees of motion freedom of translation along the x and y axes and rotation about the z axis, (forward, transverse and yaw motions). Under this limitation, only the following variables will appear in the function (allowing no deflection of the rudder for the present).

$$x_0,\ y_0,\ \psi,\ u,\ v,\ r,\ \dot{u},\ \dot{v},\ \dot{r}$$

and the force and moment components of interest are X, Y, and N. A comparable restriction to motion in the vertical plane (seakeeping or submarine maneuvering) would involve only X, Z, and M and the variables x_0, z_0, θ, u, w, q, $\dot{u}$, $\dot{w}$, and $\dot{q}$. The equation for roll involving K, φ, p, and $\dot{p}$ is usually taken together with the equations for motion in the horizontal plane, since this motion excites roll due to asymmetry of the hull[x)], or is treated separately as a one degree of freedom system.

The linearized force and moment functions have now been developed and it is now necessary to equate these forces and moments to the dynamic response terms - i.e. the right hand side of the equations of motion. However, since the force expression has been linearized, only the linear terms of the right side of the equation need be retained. Let us assume that the center of gravity lies in the centerline plane (since any good naval architect would design it so) and therefore $y_G = 0$. For motion in the horizontal plane (no rolling) the right side of the equations reduce to

----

x) Motion in the vertical plane (at least within the linear theory), does not excite roll because of the symmetry of port and starboard.

$$X = m(\dot{u} - rv - x_G r^2)$$

$$Y = m(\dot{v} + ru + x_G \dot{r})$$

$$N = I_z \dot{r} + m x_G (\dot{v} + ru)$$

A linearization of the right hand side of the Y equation proceeds as follows:

$$\dot{v}+ru+x_G\dot{r} = (\dot{v}_0 + \Delta\dot{v}) + (r_0 + \Delta r)(u_0 + \Delta u) + x_G(\dot{r}_0 + \Delta\dot{r})$$

$$= \Delta\dot{v} + \Delta r(u_0 + \Delta u) + x_G \Delta\dot{r} = \Delta\dot{v} + \Delta r u_0 + \Delta r\, \Delta u + x_G \Delta\dot{r}$$

since

$$\dot{v}_0 = r_0 = \dot{r}_0 = 0$$

The term $\Delta r \Delta u$ is second order and must be dropped since similar second order terms have been neglected in the force and moment function on the left side of the equation. Since $\Delta v = v - v_0 = v$, etc., the linearized right side of the Y equation becomes

$$m(\dot{v} + r u_0 + x_G \dot{r}).$$

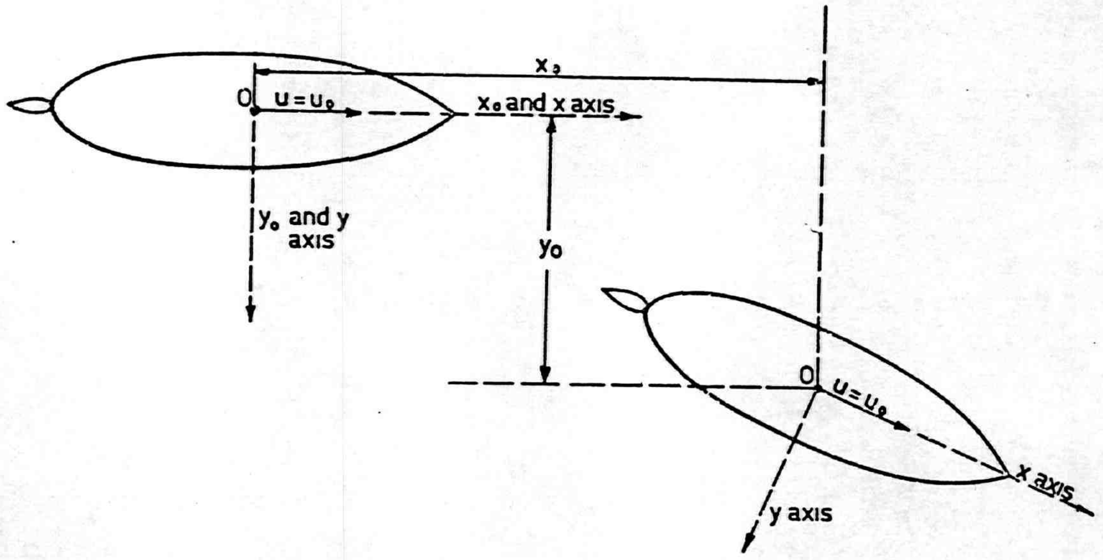
Similar linearization of the right side of the X equation gives $m\dot{u}$ and the N equation gives $I_z\dot{r} + m x_G(\dot{v} + r u_0)$. The linearized equations for motion in the horizontal plane can now be written as

$$X_{x_0} x_0 + X_{y_0} y_0 + X_\psi \psi + X_{\dot{u}} \dot{u} + X_u \Delta u + X_{\dot{v}} \dot{v} + X_v v + X_{\dot{r}} \dot{r} + X_r r = m\dot{u}$$

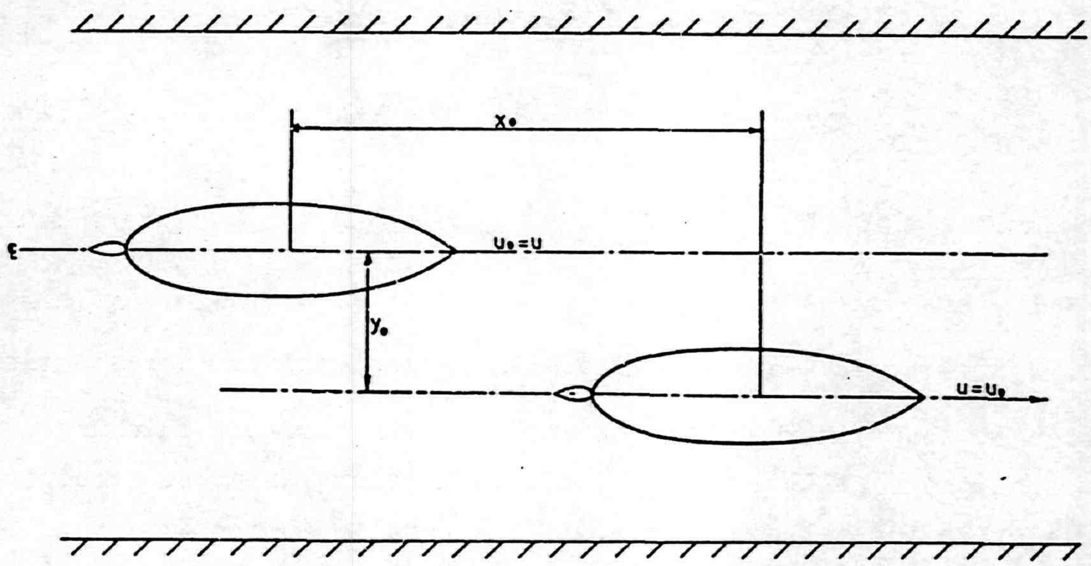
$$Y_{x_0} x_0 + Y_{y_0} y_0 + Y_\psi \psi + Y_{\dot{u}} \dot{u} + Y_u \Delta u + Y_{\dot{v}} \dot{v} + Y_v v + Y_{\dot{r}} \dot{r} + Y_r r = m\,(\dot{v} + r u_0 + x_G \dot{r})$$

$$N_{x_0} x_0 + N_{y_0} y_0 + N_\psi \psi + N_{\dot{u}} \dot{u} + N_u \Delta u + N_{\dot{v}} \dot{v} + N_v v + N_{\dot{r}} \dot{r} + N_r r = I_z \dot{r} + m x_G(\dot{v} + r u_0)$$

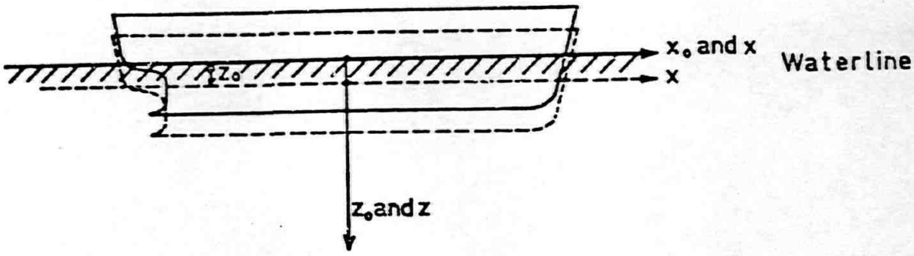
It will now be shown that the derivatives X_{x_0}, X_{y_0}, X_ψ, Y_{x_0}, Y_{y_0}, Y_ψ, N_{x_0}, N_{y_0}, N_ψ are all zero. These derivatives indicate the change brought about in the function when a given variable is changed slightly from the equilibrium value, with all other variables remaining at their equilibrium values. Hence, if the equilibrium condition is straight ahead motion at constant speed, the fact that the ship is oriented differently on the surface of the fluid, but still going straight ahead at constant speed, does not cause any forces to be exerted on the ship. For example, in the sketch below,



the orientation parameters of the ship are changed while all the remaining variables are at the equilibrium condition and it is clear that no forces or moments are exerted on the ship due to this change in orientation. This condition does not apply when a ship is sailing in a narrow canal, since, if the ship becomes oriented closer to the wall of the canal, hydrodynamic forces are created tending to draw the ship toward the near wall.



In unrestricted water, the forces on a ship due to orientation are essentially hydrostatic and hence in the vertical direction, therefore no hydrostatic forces are expected for orientation changes in the horizontal plane. For motion in the vertical plane, significant forces due to a change in vertical orientation are produced as can be seen from the example shown below for a change in vertical orientation, z_0.



A change in vertical orientation, caused by depressing the ship an amount of z_0 into the water, causes a hydrostatic force upward equal to the change in displacement resulting from the amount of orientation change, z_0. With the hydrodynamic derivatives involving the orientation parameters set equal to zero, and with the terms on the right side of the equation brought over to the left side of the equation and combined with similar terms on the left side, the equations of motion become

$$(X_{\dot{u}}-m)\dot{u}+X_u\,\Delta u+X_{\dot{v}}\dot{v}+X_v v+X_{\dot{r}}\dot{r}+X_r r = 0$$

$$Y_{\dot{u}}\dot{u}+Y_u\,\Delta u+(Y_{\dot{v}}-m)\dot{v}+Y_v v+(Y_{\dot{r}}-mx_G)\dot{r}+(Y_r-mu_0)r = 0$$

$$N_{\dot{u}}\dot{u}+N_u\,\Delta u+(N_{\dot{v}}-mx_G)\dot{v}+N_v v+(N_{\dot{r}}-I_z)\dot{r}+(N_r-mx_G u_0)r = 0$$

It is interesting to note that the coefficients of the "acceleration" terms $\dot{u}$ and $\dot{v}$ essentially have the mass increased by $X_{\dot{u}}$ and $Y_{\dot{v}}$ respectively and the coefficient of angular acceleration $\dot{r}$ has the inertia increased by $N_{\dot{r}}$. These acceleration derivatives are a result of hydrodynamic forces and represent the linear term of the Taylor expansion of the force and moment due to acceleration. $X_{\dot{u}}$, $Y_{\dot{v}}$, and $N_{\dot{r}}$ are all negative in value (as will be shown later) and therefore add in absolute magnitude to the mass or inertia in the coefficients of the accelerations. Hence, the labels of "added mass" are sometimes given to $X_{\dot{u}}$ and $Y_{\dot{v}}$ and "added inertia" to $N_{\dot{r}}$, and the combination of the mass and inertia respectively with these terms are sometimes called "virtual mass" or "virtual inertia" since the ship behaves in water with respect to acceleration as if the mass and inertia had these increased values. Some like to consider these added quantities as the amount of water the ship drags along with it as it accelerates, but this concept is physically wrong.

CHAPTER III

Solution of the Linearized Equation of Motion

We now wish to solve the three equations of motion in order to determine what the motion of the ship will be when disturbed from its original equilibrium condition of straight ahead motion. From this solution we shall develop an analysis of the motion to determine and test under what condition the motion will be stable, i.e. whether the ship can indeed maintain straight line motion with its rudder undeflected.

The solution will give as to how each of the variables, Δu, $\dot{u}$, v, $\dot{v}$, r, and $\dot{r}$ vary with time after the disturbance from the equilibrium condition. On first appearance it looks like there are six unknowns and only three equations. However, if solutions are obtained for Δu, v, and r as functions of time, then $\dot{u}$, $\dot{v}$, and $\dot{r}$ as functions of time can be obtained by differentation with respect to time of the functions Δu, v, and r. Hence, $\dot{u}$, $\dot{v}$, and $\dot{r}$ are dependent variables and there are only three independent parameters with the three equations. We are in a position now to solve the equations for the unknown Δu, v, and r as functions of time.

The equations will be solved using the operational calculus technique since only the elementary aspects of this technique need be explained to carry through the solution. Regular integral calculus or Laplace transforms could be used as well. When the differential operator, $\mathcal{D} = \frac{d}{dt}$, is introduced and used in the equations replacing the time derivatives in the manner indicated below:

$$\dot{u} = \frac{du}{dt} = \frac{d}{dt}(u_0 + \Delta u) = \frac{d}{dt}(\Delta u) = \mathcal{D}(\Delta u)$$

$$\dot{v} = \mathcal{D}v \,, \quad \dot{r} = \mathcal{D}r$$

the equations take the form

$$\overset{a_{11}}{\left[(X_{\dot{u}}-m)\mathcal{D}+X_u\right]}\Delta u+\overset{a_{12}}{\left[X_{\dot{v}}\mathcal{D}+X_v\right]}v+\overset{a_{13}}{\left[X_{\dot{r}}\mathcal{D}+X_r\right]}r=0$$

$$\overset{a_{21}}{\left[Y_{\dot{u}}\mathcal{D}+Y_u\right]}\Delta u+\overset{a_{22}}{\left[(Y_{\dot{v}}-m)\mathcal{D}+Y_v\right]}v+\overset{a_{23}}{\left[(Y_{\dot{r}}-mx_G)\mathcal{D}+(Y_r-mu_0)\right]}r=0$$

$$\overset{a_{31}}{\left[N_{\dot{u}}\mathcal{D}+N_u\right]}\Delta u+\overset{a_{32}}{\left[(N_{\dot{v}}-mx_G)\mathcal{D}+N_v\right]}v+\overset{a_{33}}{\left[(N_{\dot{r}}-I_z)\mathcal{D}+(N_r-mx_G\,u_0)\right]}r=0.$$

The letter "a" with the various subscripts are used to denote the nine coefficients of the variables u, v, and r involved in the three equations. If one could use a straightforward algebraic technique to solve these equations - i.e. if a_{11}, a_{12}, etc. were regular numerical coefficients, the solution for the variable r, for example, would be:

$$r=\frac{\begin{vmatrix} a_{11} & a_{12} & 0\\ a_{21} & a_{22} & 0\\ a_{31} & a_{32} & 0\end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{vmatrix}}=\frac{0}{\left|\;\mathit{Det.}\;\right|}$$

where *Det.* is used to designate the determinant in the denominator, with similar expressions for the solutions for Δu, and v. In operational calculus, it is shown that the operator $\mathcal{D}$ can be treated as an algebraic quantity provided the other terms in the coefficients are constants with respect to time. Since the hydrodynamic derivatives, defined as the slope of the force or moment versus a dynamic variable <u>taken at the equilibrium condition</u>, the terms in the coefficients other than $\mathcal{D}$ are constants in time. Hence, $\mathcal{D}$ can be treated as an algebraic quantity but certain interpretations with respect to the algebraic solution must be made in order to give the same result as would be obtained through formal integral calculus. To develop these interpretations, two simple examples are given below, where in each example the solution is demonstrated as calculated by the algebraic process and by formal calculus. Let us consider the differential equation

$$\frac{dz}{dt} = f(t) \quad \text{or} \quad \mathcal{D}z = f(t)$$

where z is some function of time. (z is some arbitrary variable not to be associated with the z used in the ship axis system). The algebraic solution for z is

$$z = \frac{1}{\mathcal{D}} f(t)$$

and the formal solution is

$$z = \int f(t)dt$$

If the algebraic solution is to be made equal to the formal solution, then one must interpret the operation $\frac{1}{\mathcal{D}}$ as follows,

$$\frac{1}{\mathcal{D}} = \int (\quad)dt$$

or that the inverse of differentiation is integration, as one well knows. The other example is the differential equation

$$\frac{dz}{dt} - az = f(t) \quad \text{or} \quad (\mathcal{D}-a)z = f(t)$$

The algebraic solution is given by

$$z = \frac{1}{(\mathcal{D}-a)} f(t)$$

and the formal calculus solution (as you may recall from previous mathematics) is given by

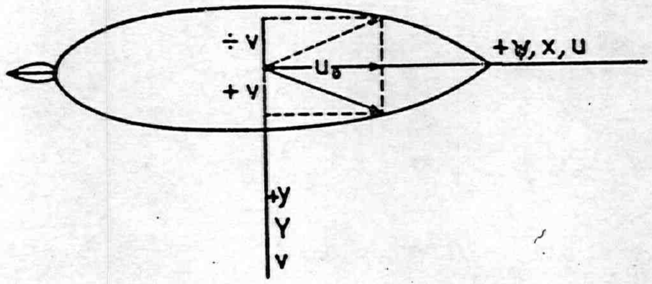
$$z = e^{at}\int e^{-at}f(t)dt$$

If the algebraic solution is to be made equal to the calculus solution, then the operation $\frac{1}{\mathcal{D}-a}$ must be interpreted as

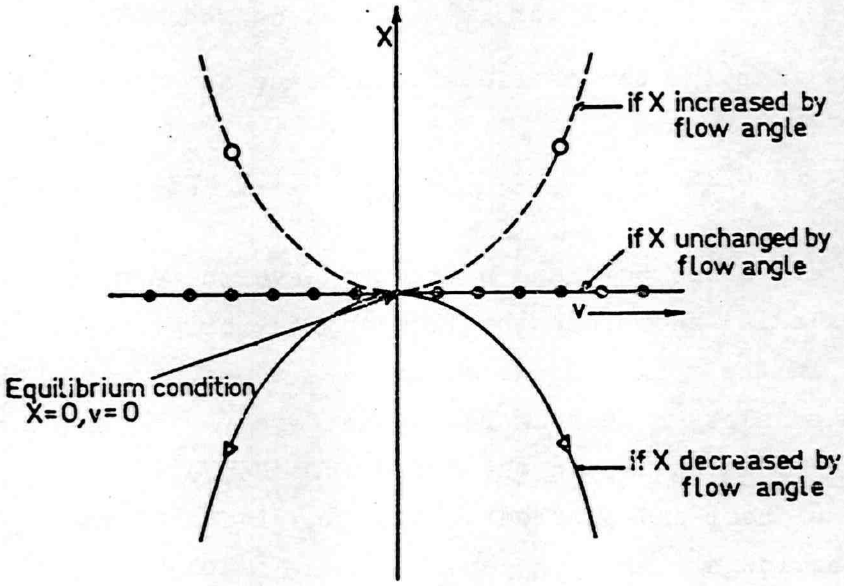
$$\left(\frac{1}{\mathcal{D}-a}\right) = e^{at}\int e^{-at}(\quad)dt$$

where whatever is operated on is inserted into the parenthesis.

Before returning to the solution of the equations of motion, we shall show that the derivatives X_v, $X_{\dot{v}}$, X_r, and $X_{\dot{r}}$ are all zero for any ship or body with symmetrical shape port and starboard. This is one of the advantages, previously mentioned, of choosing axis systems in the plane of symmetry of the body. The derivative X_v represents the slope of the X force vs. v curve taken at the equilibrium condition of $u = u_0$, $v = 0$, $\dot{u} = \dot{v} = r = \dot{r} = 0$. The sketch below indicates a ship slightly disturbed from the equilibrium condition by a small disturbance + v, and then by a disturbance - v.

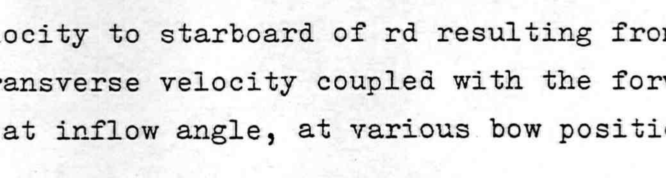


In considering how the X force varies with v, one notices that for a positive v, the approach angle of the flow to the ship is $\tan^{-1}\frac{v}{u_0}$ from starboard. Similarly, for a negative value of v, the approach angle is $\tan^{-1}\frac{v}{u_0}$ from port. Since the port and starboard side are symmetrical in shape, if an angle of flow from starboard (+v) decreases X (i.e. increases drag), then the same angle of flow from the port side (-v) must also decrease X. Similarly, if a flow from starboard increased X, then a similar flow from port will also increase X, and if flow from one side did not change X, then equal flow from the other side would not alter X either. All these deductions result from the symmetry of port and starboard. Hence for any shape, provided the port and starboard are symmetrical, the curve of X vs. v can take only one of the following shapes:

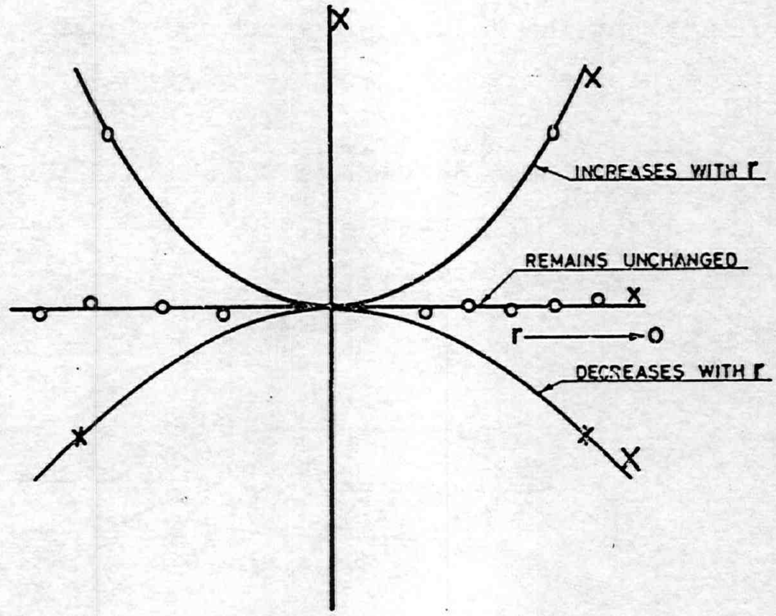


The sketch above indicates that the curve of X versus v must be symmetrical about the X axis for symmetry of port and starboard, hence the slope of the X versus v curve taken at v = 0, i.e. X_v, must be zero.

A similar situation results when considering a small disturbance in angular velocity r from the equilibrium condition, as can be seen from the following sketch.



A point B located a distance d, forward of the origin would have a transverse velocity to starboard of rd resulting from an angular velocity +r. This transverse velocity coupled with the forward velocity u_0, creates at inflow angle, at various bow positions, of $\tan^{-1}\frac{rd_1}{u_0}$, from starboard. Similarly, it can be seen that at different stern locations, the inflow angle for a +r is $\tan^{-1}\frac{rd_2}{u_0}$, from port. This type of flow, depending on the geometry of the body may increase X, decrease X, or leave it unchanged. However, for a -r, one observes that the bow sections experience an inflow angle of $\tan^{-1}\frac{rd_1}{u_0}$ from port and the stern sections an inflow angle of $\tan^{-1}\frac{rd_2}{u_0}$ from starboard. Since port and starboard have the same geometry (reflected in the x axis) and since the flow angles (or geometry of flow) are reflected in the x axis in going from +r to -r, if +r increased the X force then also -r must increase the force, with similar results for a decrease or no change in the X forces. Hence, from the symmetry properties of port and starboard, the function of X versus r must take one of the following three shapes.



Again since the function X versus r is symmetrical about the X axis, (even function), the derivative of X versus r taken at r = 0, must be zero. Hence, $X_r = 0$.

With respect to the derivatives $X_{\dot{v}}$ and $X_{\dot{r}}$, similar arguments can be presented to show that the functions X vs. $\dot{v}$ and X vs. $\dot{r}$ must be even functions because of symmetry of port and starboard. Therefore, $X_v = X_r = X_{\dot{v}} = X_{\dot{r}} = 0$ and the coefficients a_{12} and a_{13} in the equations of motion are thereby also zero.

The solution for r can now be written as

$$r = \frac{0}{\begin{vmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}} = \frac{0}{a_{11}(a_{22}a_{33}-a_{23}a_{32})}$$

If the denominator in this expression is other than zero, the solution for r (and subsequently v, and Δu) would be identically zero for all time for a small disturbance from the equilibrium condition. This is physically impossible, hence a solution exists only if the denominator is equal to zero. Setting the denominator equal to zero gives

$$a_{11}(a_{22}a_{33}-a_{23}a_{32}) = 0$$

and this condition is satisfied only if

$$a_{11} = 0 \text{ or } a_{22}a_{33}-a_{23}a_{32} = 0$$

$$a_{11} = (X_{\dot{u}}-m)\mathscr{D} + X_u = 0 \text{ or } (X_{\dot{u}}-m)\left(\mathscr{D} + \frac{X_u}{X_{\dot{u}}-m}\right) = (X_{\dot{u}}-m)(\mathscr{D}-\sigma_3) = 0$$

where we define

$$\sigma_3 = -\frac{X_u}{X_{\dot{u}}-m} .$$

If one expands the product $a_{22}a_{33}-a_{23}a_{32}$, the product contains terms in $\mathscr{D}^2$, $\mathscr{D}$, and independent of $\mathscr{D}$.

The product as expanded and set equal to zero becomes

$$a_{22}a_{33}-a_{23}a_{32} = A\mathscr{D}^2 + B\mathscr{D} + C = 0$$

where

$$A = (Y_{\dot{v}}-m)(N_{\dot{r}}-I_z)-(Y_{\dot{r}}-mx_G)(N_{\dot{v}}-mx_G)$$

$$B = (Y_{\dot{v}}-m)(N_r-mx_Gu_0)+(N_{\dot{r}}-I_z)Y_v-(Y_{\dot{r}}-mx_G)N_v-(N_{\dot{v}}-mx_G)(Y_r-mu_0)$$

$$C = Y_v(N_r-mx_Gu_0)-N_v(Y_r-mu_0).$$

The quadratic equation in $\mathscr{D}$ can be written in the form

$$A\mathcal{D}^2+B\mathcal{D}+C = A(\mathcal{D}-\sigma_1)(\mathcal{D}-\sigma_2) = A\mathcal{D}^2-A(\sigma_1+\sigma_2)\mathcal{D}+A\sigma_1\sigma_2$$

where σ_1 and σ_2 are the roots of the equation as given by the well known quadratic solution

$$\left.\begin{array}{c}\sigma_1\\ \sigma_2\end{array}\right\} = \frac{-\frac{B}{A} \pm \sqrt{\left(\frac{B}{A}\right)^2 - \frac{4C}{A}}}{2}$$

where it is clear that $A\sigma_1\sigma_2 = C$, and $-A(\sigma_1+\sigma_2) = B$. The solution for r now becomes

$$r = \frac{0}{A(X_u-m)(\mathcal{D}-\sigma_3)(\mathcal{D}-\sigma_2)(\mathcal{D}-\sigma_1)} = \left(\frac{1}{\mathcal{D}-\sigma_3}\right)\left(\frac{1}{\mathcal{D}-\sigma_2}\right)\left(\frac{1}{\mathcal{D}-\sigma_1}\right)(\ 0\ )$$

The solution will result from a sequence of operations of the form $\left(\frac{1}{\mathcal{D}-\sigma}\right)$ on the value 0. The first operation gives (using the definition of the operator previously developed)

$$\left(\frac{1}{\mathcal{D}-\sigma_1}\right) 0 = e^{\sigma_1 t}\int e^{-\sigma_1 t} 0\, dt = e^{\sigma_1 t}\int 0\, dt = e^{\sigma_1 t}\left[C_1\right] = C_1 e^{\sigma_1 t}$$

where C_1 is a constant of integration.

Continuing the operation, one obtains

$$\left(\frac{1}{\mathcal{D}-\sigma_2}\right)(C_1 e^{\sigma_1 t}) = e^{\sigma_2 t}\int e^{-\sigma_2 t} C_1 e^{\sigma_1 t} dt = e^{\sigma_2 t}\int C_1 e^{(\sigma_1-\sigma_2)t} dt$$

$$= e^{\sigma_2 t}\left[\left(\frac{C_1}{\sigma_1-\sigma_2}\right) e^{(\sigma_1-\sigma_2)t} + C_2\right] = C_1^1 e^{\sigma_1 t} + C_2 e^{\sigma_2 t}$$

where C_1^1 is an arbitrary constant of integration since C_1, being arbitrary, divided by a fixed quantity $\sigma_1 - \sigma_2$, is also arbitrary. (In the exceptional case where $\sigma_1 = \sigma_2$, the integration

$$\int e^{(\sigma_1-\sigma_2)t} dt = \int e^0 dt = \int dt = t+C_2$$

and this results in the form $C_1 te^{\ 2^t} + C_2 e^{\ 2^t}$).
The final operation gives the solution for r.

$$r = \left(\frac{1}{\mathcal{D}-\sigma_3}\right)\left(C_1{}^1 e^{\sigma_1 t} + C_2 e^{\sigma_2 t}\right) = e^{\sigma_3 t}\int e^{-\sigma_3 t}\left[C_1{}^1 e^{\sigma_1 t} + C_2 e^{\sigma_2 t}\right] dt$$

$$= e^{\sigma_3 t}\int\left[C_1{}^1 e^{(\sigma_1-\sigma_3)t} + C_2 e^{(\sigma_2-\sigma_3)t}\right] dt$$

$$= \left(\frac{C_1{}^1}{\sigma_1-\sigma_3}\right) e^{\sigma_1 t} + \left(\frac{C_2}{\sigma_2-\sigma_3}\right) e^{\sigma_2 t} + C_3 e^{\sigma_3 t}$$

$$r = r_1 e^{\sigma_1 t} + r_2 e^{\sigma_2 t} + r_3 e^{\sigma_3 t}$$

where r_1, r_2, and r_3 are constants of integration. Since the algebraic solutions for Δu and v are the same form as the solution for r, namely

$$\left.\begin{matrix}\Delta u\\ v\\ r\end{matrix}\right\} = \frac{0}{(\mathcal{D}-\sigma_3)(\mathcal{D}-\sigma_2)(\mathcal{D}-\sigma_1)}$$

then the actual solution for Δu and r is

$$\Delta u = u_1 e^{\sigma_1 t} + u_2 e^{\sigma_2 t} + u_3 e^{\sigma_3 t}$$

$$v = v_1 e^{\sigma_1 t} + v_2 e^{\sigma_2 t} + v_3 e^{\sigma_3 t}$$

where u_1, u_2, u_3, v_1, v_2, and v_3 are constants of integration.

The solutions obtained describe how the motion of the ship will vary with time after an initial disturbance from straight line motion. Analysis of this solution leads us to determine under what conditions the ship will be stable in straight line motion and will furnish us with a criteria for this stability.

CHAPTER IV

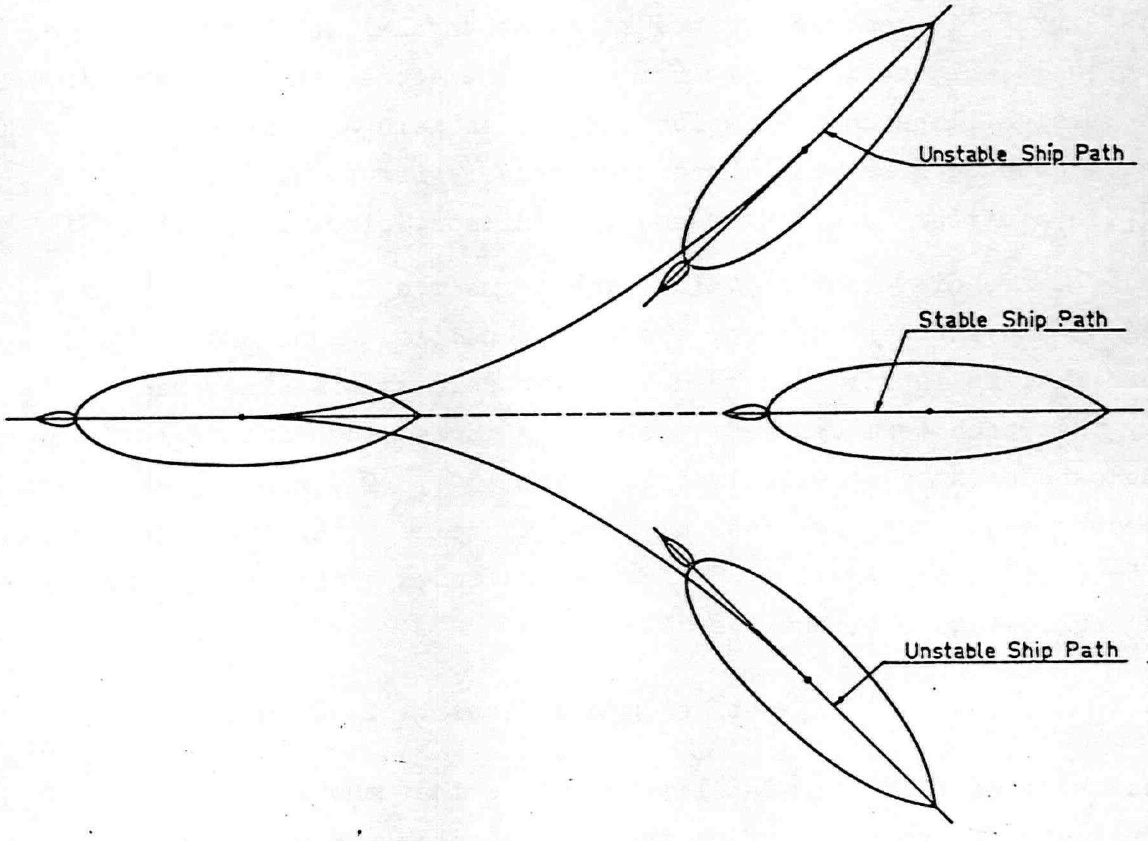
Stability of a Ship in Straight Ahead Motion

The test for any type of stability is to establish an equilibrium situation, and determine whether the system returns to the original condition of equilibrium after a disturbance of the smallest amount (infinitesimal disturbance). If it returns, or tends to return, to the original equilibrium condition when the disturbance is removed, it is stable. If it departs, or has the tendency to depart, from the original equilibrium condition, the original equilibrium condition is unstable. The usual case for a body which is unstable in a given condition of equilibrium, is to depart from that condition until it reaches another equilibrium condition (not the original one) which is a stable one. This is the way one goes about testing a ship for stability in roll. The ship is disturbed slightly from its upright equilibrium position and when the disturbance is removed the tendency to return to the original upright position is observed. If it returns, it is stable, if it departs, it is unstable. An unstable ship in heel, cannot remain in the upright equilibrium condition, but, in the absence of disturbance, heels (flops) either to starboard or port until a new angle of heel is reached which is a stable one (new position of equilibrium).

In this manner, one tests the equilibrium condition of straight ahead motion at constant speed for stability[*)]. Just as in the case of stability in roll where an unstable ship cannot remain upright when there is no heeling moment, a ship which is dynamically unstable in straight

[*)] Straight ahead motion at constant speed is a condition of equilibrium, since there are no linear or angular accelerations and therefore no net forces or moments acting on the body.

line motion cannot maintain straight line motion when there is no rudder deflection. The unstable ship will go into a starboard or port turn without any rudder deflection as indicated by the sketch below.



The ship which is dynamically unstable in straight line motion, can maintain a straight course (on the average) only by continuous use of the rudder.

As mentioned earlier, the linearized equations of motion were solved, furnishing certain parameters of the ship motion as functions of time. These solutions will now be used to analyse whether a ship is capable of maintaining straight ahead motion (without rudder application) and thereby determine whether it is dynamically stable in this motion. The solution for the angular velocity, r, as a function of time was

$$r = r_1 e^{\sigma_1 t} + r_2 e^{\sigma_2 t} + r_3 e^{\sigma_3 t}$$

where r_1, r_2, and r_3 were arbitrary constants of integration depending on initial conditions, and the roots σ_1, σ_2, and σ_3 were expressed in terms of certain combinations of the various hydrodynamic derivatives.

(In the case of equal roots, the solution was $r = r_1 e^{\sigma_1 t} + r_2 t e^{\sigma_1 t} + r_3 e^{\sigma_3 t}$). The equations represented the ship motions in the absence of any disturbance and therefore represent the behavior of the ship when a (slight) disturbance is removed. Since r is the angular velocity, straight ahead motion is only satisfied when r = 0. Therefore, the test for stability in straight line motion is for r to go to zero as time increases (time being counted from when the disturbance is removed). Since r_1, r_2, and r_3 are arbitrary constants, and in addition, since in general σ_1, σ_2, and σ_3 are different in value, the terms $r_1 e^{\sigma_1 t}$, $r_2 e^{\sigma_2 t}$, and $r_3 e^{\sigma_3 t}$ cannot negate one another. Hence, the only condition under which r will go to zero in time is for each term to go to zero as time increases. The only way that each term can go to zero with increasing time is for each of the exponents to be negative, i.e. that σ_1, σ_2, and σ_3 all be negative quantities if they are real numbers, since as t increases to infinity, $e^{kt} \rightarrow 0$, if k is negative. If σ is a complex number in the form $\sigma = a+ib$, the following relationships hold

$$e^{\sigma t} = e^{(a+ib)t} = e^{at} e^{ibt} = e^{at}(\cos bt + i \sin bt)$$

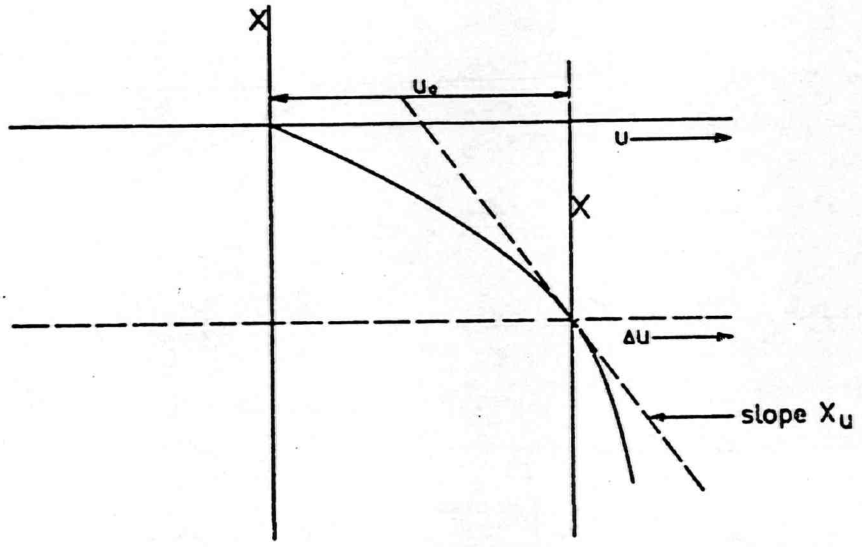
and the condition for stability requires that the real parts of σ_1, σ_2, and σ_3 be negative as they are complex numbers. (The imaginary part of the number indicates the angular frequency of oscillation, in which the motion dies down or is amplified). (In the case of equal roots, $\sigma_1 = \sigma_2$, the term $r_2 t e^{\sigma_2 t}$ goes to zero as $t \rightarrow \infty$ for σ_2 negative, since $t e^{-|\sigma_2| t} \rightarrow 0$ as $t \rightarrow \infty$).

We shall now analyse under what conditions, all three roots σ_1, σ_2, and σ_3 will be negative, if real, or have real parts which are negative, if complex. This will furnish us with a criteria for determining whether a given ship is stable or not in straight line motion.

The value of σ_3 was previously determined as

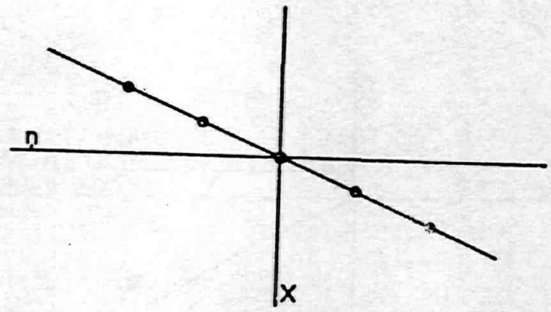
$$\sigma_3 = -\frac{X_u}{X_{\dot{u}} - m}$$

and it will be shown that σ_3 is always negative for the regular displacement type ship because of the basic nature of hydrodynamic drag. Since the direction of the positive X force is in the opposite direction to the direction of the drag (or resistance) force, the plot of X vs. speed, in the absence of any propeller thrust, would appear as follows,



with the drag increasing (X force decreasing) with speed u in some power function of u (approximately u^2 if the resistance coefficient does not change significantly). At the equilibrium speed of u_o, the propeller thrust, through the thrust deduction factor, will overcome the resistance so that at a speed u_o, the X force is zero when propeller effect is included. Hence, the plot of X vs. Δu appears to the right and below the axis for the minus drag vs. u curve. The derivative X_u is the slope of this curve taken at $\Delta u = 0$, and X_u will be negative as long as the net drag increases with speed (propeller effect included in net drag - say at constant r.p.m.). Since the net drag for displacement ships increases markedly with speed, X_u will be a relatively large negative number. For the case of a planing boat just about to plane, the value of X_u may be positive if a decrease in drag results from an increase in speed.

The derivative $X_{\dot{u}}$ is a negative quantity because of the following hydrodynamic reasons. The term $X_{\dot{u}}\dot{u}$ represents the force that a body experiences in the x direction as the result of an acceleration in the x direction. The body must accelerate the water and there is an inertial reaction force of the water (because of its density) on the body. This reaction force is in the opposite direction to the acceleration. Hence, the plot of X vs. $\dot{u}$ will look as follows



and the value of $X_{\dot{u}}$ will be negative - i.e. $\left(\frac{\partial X}{\partial \dot{u}}\right)_{\dot{u}=o} = \ominus$. For elongated bodies, such as normal ship types $X_{\dot{u}}$ is the order of about 5-10% of the mass of the ship. The value of σ_3 is then

$$\sigma_3 = \frac{-X_u}{X_{\dot{u}}-m} = - \frac{-|X_u|}{-|X_{\dot{u}}|-m} = - \frac{|X_u|}{|X_{\dot{u}}|+m} = \ominus$$

$\ominus$ indicates a minus quantity. Since X_u and $X_{\dot{u}}$ have been shown to be negative, these quantities can be designated by a minus sign times their absolute values. (Absolute value denoted by $|\ |$). Hence, σ_3 is real and negative and therefore stable. It indicates that a disturbance in speed (Δu) will tend to zero after the disturbing forces are removed - i.e. ship will return to the original equilibrium condition.

With σ_3 always a stable root for normal ship types, the stability then depends on the value of σ_1 and σ_2. Therefore, one must consider under what conditions of the coefficients A, B, and C (defined earlier) will the roots σ_1 and σ_2 have real parts which are negative. The solution for these roots were

$$\left.\begin{matrix}\sigma_1\\ \sigma_2\end{matrix}\right\} = \frac{\frac{-B}{A} \pm \sqrt{\left(\frac{B}{A}\right)^2 - \frac{4C}{A}}}{2}$$

a) For any value of $\frac{B}{A}$, whether a positive or a negative value, if $\frac{C}{A}$ is negative (i.e. $\frac{C}{A} < 0$), then

$$\left.\begin{matrix}\sigma_1\\ \sigma_2\end{matrix}\right\} = \frac{1}{2}\left(- \frac{B}{A} \pm Q\right) \text{ when } Q > \left|\frac{B}{A}\right| \quad \text{since, if } \frac{C}{A} < 0,\ - \frac{4C}{A} \text{ is a}$$

positive quantity and this is added to a positive quantity $\left(\frac{B}{A}\right)^2$. The

quantity under the radical sign is positive and equals Q^2.
($Q^2 = (\frac{B}{A})^2 + \frac{4C}{A}$). The roots σ_1 and σ_2 are real quantities. However, since $Q > \frac{B}{A}$, whether $\frac{B}{A}$ is positive or negative, one of the roots, σ_1 or σ_2, must be a positive quantity and therefore unstable. Therefore, one of the conditions for stability is that $\frac{C}{A}$ not be negative - i.e. $\frac{C}{A} > 0$ for stability.

b) If $\frac{C}{A} > 0$, and $\frac{4C}{A} < (\frac{B}{A})^2$, then

$$\left.\begin{matrix}\sigma_1\\ \sigma_2\end{matrix}\right\} = \frac{1}{2}(- \frac{B}{A} \mp Q) \text{ where } Q < \left|\frac{B}{A}\right|$$

If $\frac{B}{A}$ is negative, then both roots will be real and positive. If $\frac{B}{A}$ is positive then both roots will be real and negative. Hence, an additional condition for stability (over and above $\frac{C}{A} > 0$) is that $\frac{B}{A} > 0$.

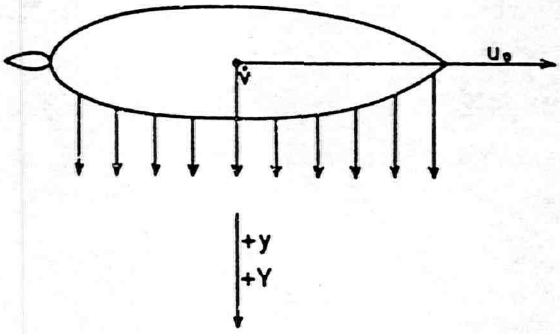
c) If $\frac{C}{A} > 0$ and $\frac{4C}{A} > (\frac{B}{A})^2$, then the roots are complex of the form

$$\left.\begin{matrix}\sigma_1\\ \sigma_2\end{matrix}\right\} = \frac{1}{2}(- \frac{B}{A} \pm iQ) \text{ where } Q = \sqrt{\frac{4C}{A} - (\frac{B}{A})^2}\ .$$

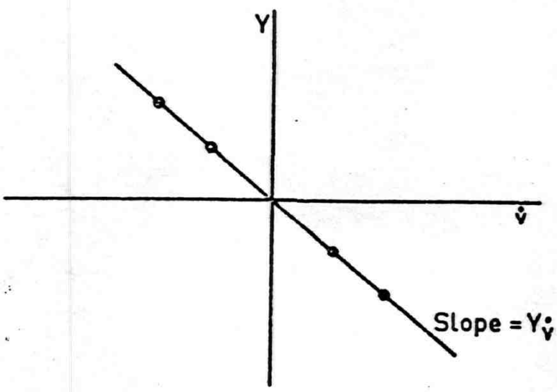
The real part of the roots is $-\frac{B}{A}$. Hence, $\frac{B}{A}$ has to be positive for stability (i.e. $-\frac{B}{A}$ has to be negative).

The conditions for stability have now been reduced to the requirements that $\frac{B}{A}$ and $\frac{C}{A}$ must both be positive quantities. The hydrodynamic derivatives appearing in the definitions of A, B, and C will now be analysed to see under what conditions $\frac{B}{A}$ and $\frac{C}{A}$ are positive and thereby develop a criterion for stability. It is necessary to establish the order of magnitude and the sign (whether positive or negative) of the various derivatives. The analysis is intended to show that for ships, A and B are always positive quantities and that the condition of stability rests on $\frac{C}{A}$ (or C) being positive.

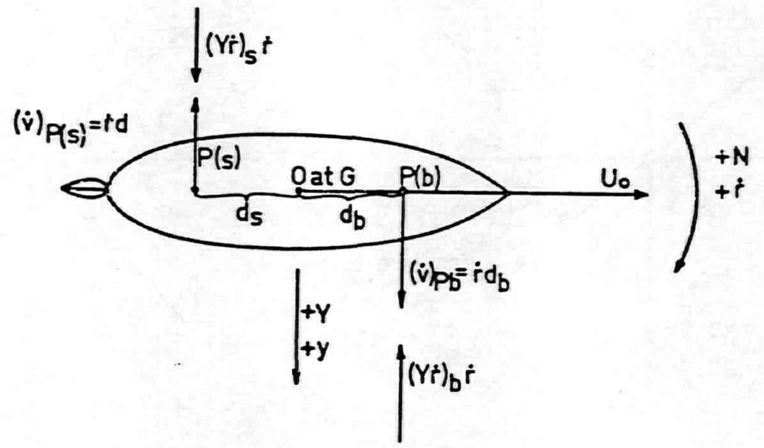
The term $Y_{\dot{v}}\dot{v}$ represents the linear approximation of the Y force resulting from an acceleration in the y direction. The sketch shown below represents the ship with an acceleration $+\dot{v}$.



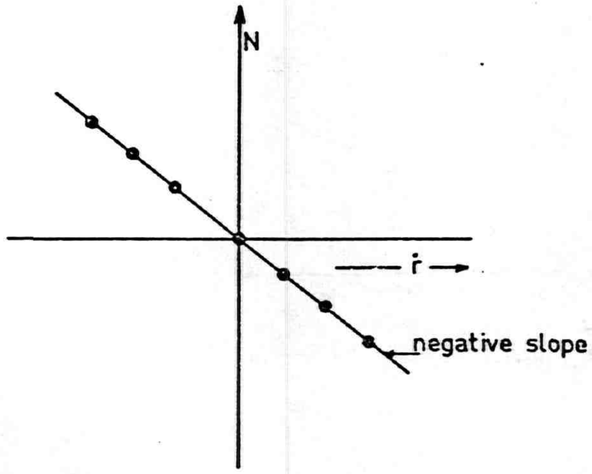
Both bow and stern experience a $\dot{v}$ acceleration in the y direction. Inertial reaction pressures of the water being accelerated by the hull produce forces in the negative y direction on both the bow and stern. Hence bow and stern effects add to give a relatively large negative Y force resulting from a positive $\dot{v}$. If a disturbance of a negative $\dot{v}$ is placed on the ship, the inertial pressures on bow and stern add together to give a relatively large Y force in the positive y direction. Hence the plot of Y versus $\dot{v}$ would appear as follows.

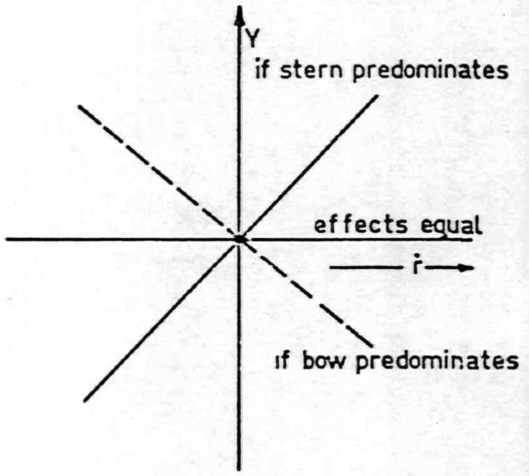


The slope taken at $\dot{v} = 0$, would be a negative value of relatively large magnitude. For elongated bodies, like ships with usual length to beam ratios, the magnitude of $Y_{\dot{v}}$ is approximately that of the ship's mass, m. (Ship in neutral buoyancy). For example, theoretically calculated (potential theory) values of $Y_{\dot{v}}$ for ellipsoids give values of -0.5 m for $\frac{L}{B} = 1$, -0.9 m for $\frac{L}{B} = 5$, -0.95 m for $\frac{L}{B} = 8.5$, and -1.0 m for $\frac{L}{B} = \infty$. Since $Y_{\dot{v}}$ for most ships are of the order of -m, then the term $(Y_{\dot{v}}-m)$ is about -2 m and represents a relatively large negative number. Similarly the derivative $N_{\dot{r}}$ is negative and relatively large as demonstrated in the following sketch.

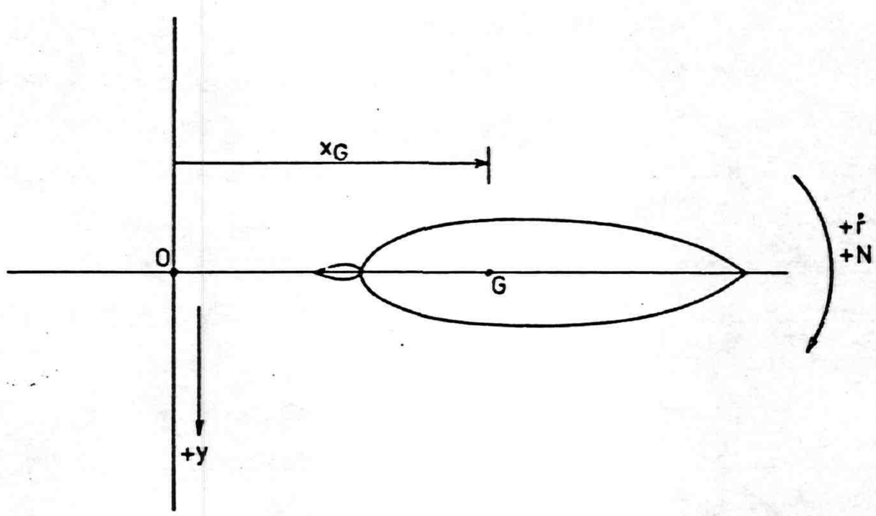


A positive $\dot{r}$ produces a local $\dot{v}$ of $+\dot{r}d_b$ at a point P(b) at the bow and a $-\dot{r}d_S$ at the stern. The hydrodynamic force is an inertial reaction force of the water on the hull, in the opposite direction to the local acceleration. Hence, for a positive $\dot{r}$, the bow experiences a force in the negative y direction, producing a negative N moment and the stern experiences a positive Y force, but also producing a negative N moment. Hence, the moments produced at the bow and stern add to give a significant negative value for a positive $\dot{r}$. However, bow and stern produce Y forces in the opposite direction to one another - i.e. bow fights stern. A similar situation arises for a negative $\dot{r}$ - a large positive moment (bow adds to stern) and a small net Y force. Hence, $N_{\dot{r}}$ is a relatively good-size negative quantity. If bow and stern have equal effect, then $Y_{\dot{r}}$ is 0, if the bow predominates (greater pressure distribution) over the stern $Y_{\dot{r}}$ is negative. If the stern predominates, then $Y_{\dot{r}}$ is positive. Since, bow fights stern, whether positive or negative, $Y_{\dot{r}}$ will be a relatively small quantity. A sample plot of N vs. $\dot{r}$ and Y vs. $\dot{r}$ are given below.





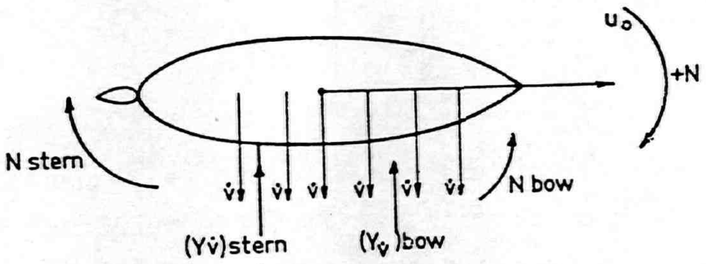
In the example discussed, the origin of the axis system was taken at the center of gravity, G, so that the value of $Y_{\dot{r}}$ represented the magnitude of the term $(Y_{\dot{r}}-mx_G)$ since $x_G = 0$. One can choose the origin arbitrarily, the resulting expressions, and the relative magnitudes of the terms in A, B, C, should not change. For example, the term $Y_{\dot{r}}-mx_G$ can be made a very large quantity by choosing an origin so that x_G is large, as indicated in the following sketch.



Under the above situation $Y_{\dot{r}}$ is made a very large negative quantity, something of the order of $(Y_{\dot{v}})x_G \approx -mx_G$, since a positive $\dot{r}$ produces a local $\dot{v}$ at G of $x_G\dot{r}$, and the hydrodynamic reaction is a large force in the negative Y direction. Hence, with a large x_G, the term $Y_{\dot{r}}$ can be made very large merely by choice of origin. However, at the same time $N_{\dot{r}}$

is increased in negative value by about $Y_{\dot{v}}x_G^2$ and I_z is increased by mx_G^2 so that the term $(N_{\dot{r}}-I_z)$ is increased by a much larger amount than $(Y_{\dot{r}}-mx_G)$. Hence, the same relationship is maintained - $(N_{\dot{r}}-I_z)$ is a relatively large quantity (negative) and $(Y_{\dot{r}}-mx_G)$ is a relatively small quantity. A similar situation obtains with the other derivatives and terms when the choice of origin is changed. Therefore, an analysis of the magnitude of the terms, with the c.g. at the origin or x_G small (the case of practically all displacement type ships) will give a proper indication of the relative magnitude of the various derivatives.

As was indicated in the analysis of the derivative $Y_{\dot{v}}$, both bow and stern add to contribute to a negative $Y_{\dot{v}}$. In the case of $N_{\dot{v}}$, the bow fights the stern as can be seen from the sketch below.



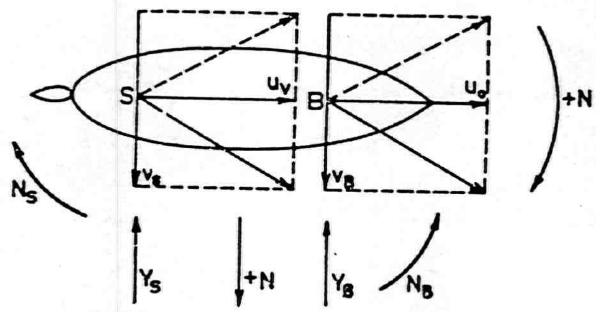
For a positive $\dot{v}$ the bow contributes to a negative N value whereas the stern contributes to a positive, and for a negative $\dot{v}$ the bow contributes to a positive value of N and the stern to a negative value. Hence, bow and stern fight each other and $N_{\dot{v}}$ is expected to be a relatively small quantity, positive if the stern predominates and negative if the bow predominates.

Since

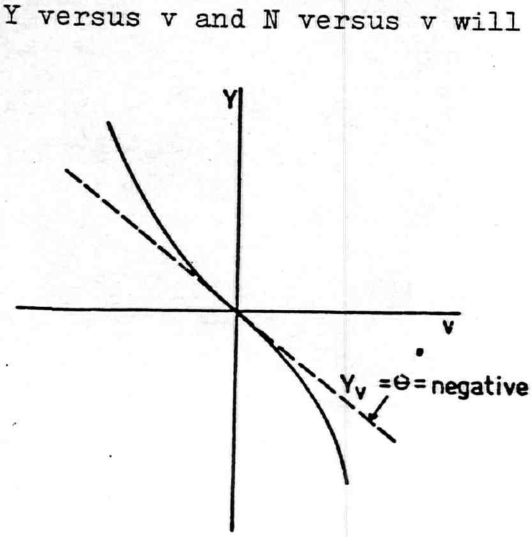
$$A = \underbrace{(Y_{\dot{v}}-m)}_{\substack{\text{large negative}\\ \text{about } -2\text{ m}}}\ \underbrace{(N_{\dot{r}}-I_z)}_{\substack{\text{large negative}\\ \text{about } -1.8\, I_z}} - \underbrace{(Y_{\dot{r}}-mx_G)}_{\substack{\text{small positive}\\ \text{or negative}}}\ \underbrace{(N_{\dot{v}}-mx_G)}_{\substack{\text{small positive}\\ \text{or negative}}}$$

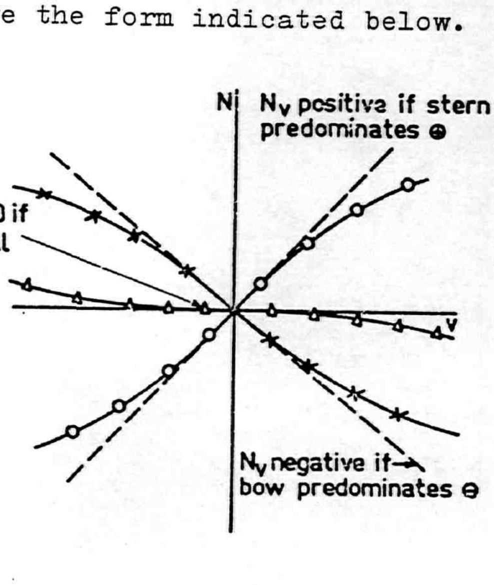
it is clear from the nature of the derivatives that the product of the first two terms is a much larger quantity than the product of the second two terms. Since the product of the first two terms is very large and positive, A must be a substantial positive quantity. Therefore, the conditions for stability of $\frac{B}{A} > 0$ and $\frac{C}{A} > 0$ become $B > 0$ and $C > 0$.

To evaluate the relative magnitudes of B and C it is necessary to look at the nature of the derivatives Y_v, N_v, Y_r, and N_r. In the following sketch the nature of the forces acting on a body with a velocity v added to a forward velocity u_o, is shown.

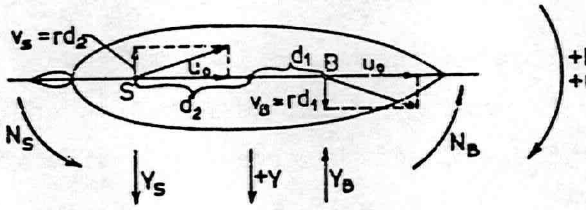


At a point B on the bow, a positive v together with a u_o give an angle of attack on the bow section, the resulting lift of which produces a force in the negative Y direction (Y_B) and a moment in the negative N direction (N_B). At point S on the stern, a positive v produces an angle of attack resulting in negative Y forces at the stern (Y_S) and a positive N moment at the stern (N_S). A negative v will reverse the direction of all the above forces and moments (angle of attack is changed to negative). Hence, in the case of Y_v, bow and stern add to give a substantial negative value for Y_v whereas bow and stern fight each other in the case of N_v. Therefore N_v is relatively small (for a moment) and is positive if the stern predominates, is negative if the bow predominates, and is zero if the contributions of bow and stern are equal. The plots of Y versus v and N versus v will have the form indicated below.

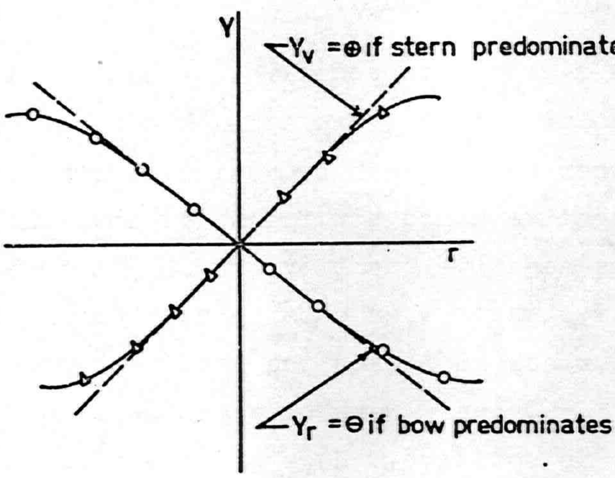


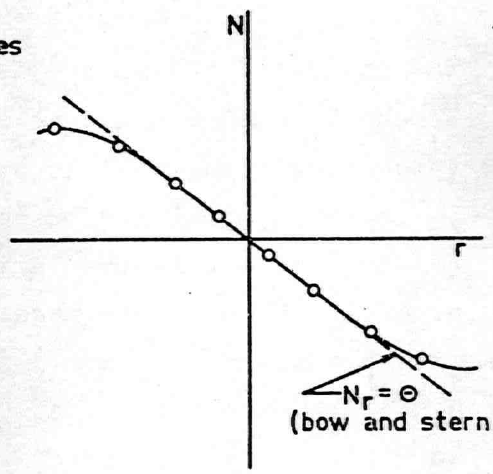


In analyzing the effect of an angular velocity r on Y and N, again a location B on the bow and S on the stern are followed.



When the ship is moving ahead with a velocity u_o and an angular velocity +r is added, point B at the bow has an angle of attack from starboard ($\approx \frac{rd_1}{u_o}$ for small r) producing a negative Y force and a negative N moment on the bow. For a +r, point S at the stern experiences an angle of attack from the port side producing a positive Y force at the stern and a negative N moment. Hence, bow and stern add to give a large negative N for a positive r, whereas bow and stern fight each other to give either a positive or negative Y force for a positive r, negative if the bow predominates. For a negative r, angles of attack change to opposite sides and hence the force and moment contributions change sign. Sample curves are indicated in the plots below.





Hence, N_r is always a substantial negative quantity (for ships) since bow and stern effects add whereas Y_r is a relatively small quantity since bow fights stern and is positive if the stern predominates and negative if the bow predominates. Also, the quantity Y_r is always in combination with mu_o and is very small compared to mu_o.

The term, B, has been defined as

$$B = (Y_{\dot{v}}-m)(N_r-mx_Gu_o)+(N_{\dot{r}}-I_z)Y_v-(Y_{\dot{r}}-mx_G)N_v-(N_{\dot{v}}-mx_G)(Y_r-mu_o)$$

It has been indicated that $Y_{\dot{v}}-m$ is a large negative term and this term is multiplied by $(N_r-mx_Gu_o)$ which is negative and relatively large since in N_r both bow and stern add to give a negative value.*) Hence the product $(Y_{\dot{v}}-m)(N_r-mx_Gu_o)$ is a large positive number. To this is added another large positive number, that is the product $(N_{\dot{r}}-I_z)Y_v$, since it was already shown that $(N_{\dot{r}}-I_z)$ is a large negative term of the order of $-1.8\ I_z$ and that Y_v is substantially negative since bow and stern effects add to give a negative value. On the other hand the products $(Y_{\dot{r}}-mx_G)N_v$ and $(N_{\dot{v}}-mx_G)(Y_r-mu_o)$ are small and either positive or negative by virtue of the fact that the terms $(Y_{\dot{r}}-mx_G)$ and $(N_{\dot{v}}-mx_G)$ are relatively small, since bow fights stern effects in these derivatives. Hence, the sum of the first two large positive products in B greatly outweigh the possible magnitude of the last two products; therefore, B is always a positive quantity for ships. For ships, therefore, one of the conditions for stability, i.e. $B > 0$, is satisfied.

With both $A > 0$ and $B > 0$ for ships, the condition for dynamic stability in straight line motion essentially reduces to the condition that $C > 0$. The term C was defined as

$$C = Y_v(N_r-mx_Gu_o)-N_v(Y_r-mu_o).$$

The product $Y_v(N_r-mx_Gu_o)$ is a good size positive quantity in that both Y_v and N_r are negative in value with bow and stern adding to give negative values for both. On the other hand N_v can be positive or negative since Y_r is small (positive or negative since bow fights stern), the term (Y_r-mu_o) is a relative large negative quantity. If N_v is positive, the product $N_v(Y_r-mu_o)$ is negative which when subtracted from the large positive product $Y_v(N_r-mx_Gu_o)$ further increases the positive value of C. Hence, the condition that $C > 0$ can always be satisfied (and hence insure stability) provided N_v is positive. However, it is not necessary

*) x_G is usually small. Also in a previous discussion it was shown that arbitrary choices of axes well removed from the center of gravity or midship section do not change relative values.

for N_v to be positive for C to be positive and in the usual case for ships N_v is not positive. With A and B positive for ships, the criteria for dynamic stability in straight line motion becomes

$$Y_v(N_{\dot{r}}-mx_G u_o)-N_v(Y_r-mu_o) > 0.$$

Whereas in the single degree of freedom for roll motion the criterion rested on the value of one term i.e. metacentric height being positive or negative, in the case of the two degrees of motion freedom for motion in the horizontal plane (Y and N), the four terms Y_v, N_r, N_v, and Y_r are involved in the criterion for stability. The hydrodynamic analysis of these four derivatives, carried out previously indicated the following.

Y_v bow and stern effects add to give negative values.

N_r bow and stern effects add to give negative values.

N_v bow and stern effects fight each other. If bow predominates N_v is negative, if the stern predominates, N_v is positive.

Y_r bow and stern effects fight each other. If bow predominates Y_r is negative, if stern predominates Y_r is positive. Y_r is not a sensitive parameter since mu_o is the predominant term in (Y_r-mu_o).

Since making C more positive improves the stability, design changes which in effect add positive amounts to C will improve the stability. If the bow lifting forces are increased relative to those at the stern, by a lines change or by the addition of lifting surfaces or fins at the bow, then N_r and Y_v are made somewhat more negative and their product more positive. However, additional forces at the bow make N_v less positive or more negative. Since in N_v, bow fights stern, small magnitude changes in N_v are large percentage changes so that the product $N_v(Y_r-mu_o)$ can be changed markedly in relative magnitude as well as changing sign. Y_v and N_r changes by adding lifting surfaces are smaller percentages than occur with N_v.

Hence, lifting surfaces at the bow tend to make C more negative (or less positive) and hence tend to reduce stability. If additional lifting forces are produced at the stern, Y_v and N_r are made more negative, as in the case of forces at the bow, but N_v is made more positive (or less negative) thereby improving the stability. Therefore, adding lifting surfaces at the bow tends to destabilize, adding lifting surfaces at the stern tends to stabilize. Hence, a fine stern with neat flow lines (no separation), and deadwood or stabilizing fins aft will improve stability. Since small length/beam ratios prevent fine sterns, a tendency to instability may exist on ships with small length to beam ratios.

One may be tempted to design a ship so that N_v is positive and hence guaranteeing a very stable ship. It must be remembered that stability indicates the tendency to go in straight line motion when subjected to small transient disturbances. Since steering of a ship is effected by producing disturbing force and moment by a rudder deflection, a too stable ship will not turn as tight as a somewhat less stable ship. Hence, a too stable ship may compromise the maneuverability of the ship. On the other hand, an unstable ship will not be able to go straight but will require constant use of the rudder. A ship should be designed for a moderate amount of stability so as to be able to go straight but not so much stability as to compromise maneuverability. This situation is similar to having a positive metacentric height to ensure stability, but not too large a metacentric height so as to have too rapid a rolling motion and rolling accelerations.

It can be seen that the further forward the center of gravity is from the center of geometry (midship section) the more stable the situation by observing the effect of increasing the positive value of x_G in the term $(N_r - m x_G u_o)$. In fact, qualitatively speaking, if the center of dynamic pressure is aft of the center of gravity, the ship will be stable in straight line motion. This is analogous to the condition in roll stability that the center of hydrostatic pressure (metacenter) has to be above the center of gravity for the ship to be stable in roll (i.e. positive metacentric height).

Before indicating how the actual values of the various derivatives are obtained for a given ship design, a rather simple stability analysis of the very familiar case of a ship in roll will be carried out by means of the more formal approach taken in the analysis of the more complicated case of ship motion in the horizontal plane. This is done in order to show, in a formal way, the criterion of positive metacentric height for stability. For the single degree of motion freedom in roll, the linearized equation of motion becomes

$$K = K_{\dot{p}}\dot{p} + K_p p + K_\varphi \varphi = I_x \dot{p}$$

where it may be recalled that

K is the roll moment

φ is the angle of roll (or heel)

p is the angular velocity of roll, i.e. $p = \dot{\varphi} = \mathcal{D}\varphi$ *)

$\dot{p}$ is the angular acceleration, i.e. $\dot{p} = \ddot{\varphi} = \mathcal{D}^2\varphi$

I_x is the moment of inertia about the x axis

The roll moment experienced by the ship is a function of the orientation and motion variables φ, p, and $\dot{p}$ (single degree of freedom system) and the linearization of the function of these variables gives the derivatives $K_{\dot{p}}$, K_p, and K_φ.

The roll equation can be written as

$$\Big[\underbrace{(K_{\dot{p}}-I_x)}_{A}\mathcal{D}^2+\underbrace{K_p}_{B}\mathcal{D}+\underbrace{K_\varphi}_{C}\Big]\varphi = 0$$

or

$$\left[A\,\mathcal{D}^2+B\mathcal{D}+C\right]\varphi = 0.$$

Following the system of solving linear differential equations with constant coefficients by use of the operator $\mathcal{D}$ as was done previously, the solution for φ as a function of time is given by

$$\varphi = \frac{0}{A\mathcal{D}^2+B\mathcal{D}+C} = \frac{0}{A(\mathcal{D}-\sigma_1)(\mathcal{D}-\sigma_2)} = \varphi_1 e^{\sigma_1 t} + \varphi_2 e^{\sigma_2 t}$$

where σ_1 and σ_2 are the roots of the quadratic in the denominator and φ_1 and φ_2 are constants of integration.

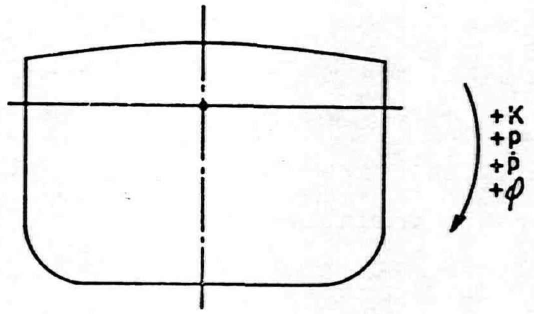
$$\left.\begin{matrix}\sigma_1\\ \sigma_2\end{matrix}\right\} = \frac{1}{2}\left[-\frac{B}{A} \pm \sqrt{\left(\frac{B}{A}\right)^2 - \frac{4C}{A}}\right]$$

Again, as in the previous case, the roots σ_1 and σ_2 will be stable roots if $\frac{B}{A} > 0$ and $\frac{C}{A} > 0$. These conditions are

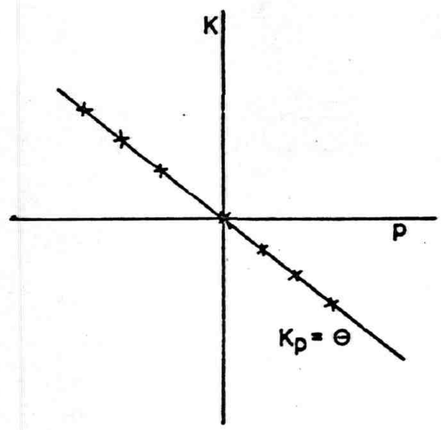
$$\frac{K_p}{K_{\dot{p}}-I_x} > 0 \quad \text{and} \quad \frac{K_\varphi}{K_{\dot{p}}-I_x} > 0.$$

*) $\mathcal{D}$ is the differential operator, $\mathcal{D} = \frac{d}{dt}$.

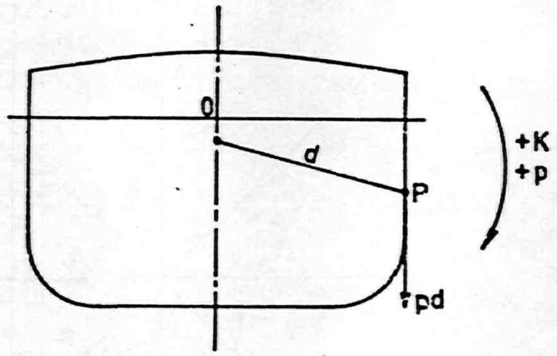
To evaluate the nature of K_p, one observes the hydrodynamic effect of an acceleration in roll using the sketch below.



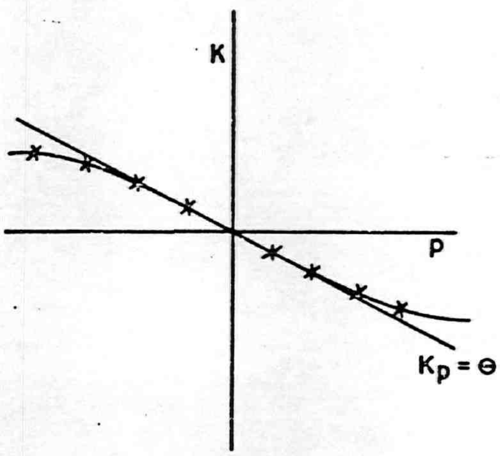
For a positive p, the hydrodynamic pressure on the hull results from the inertial reaction of the hull accelerating fluid along with it. Hence, the reaction moment is opposed to the direction of acceleration. The plot of K versus p would appear as follows (positive p producing a negative K and a negative p producing a positive K).



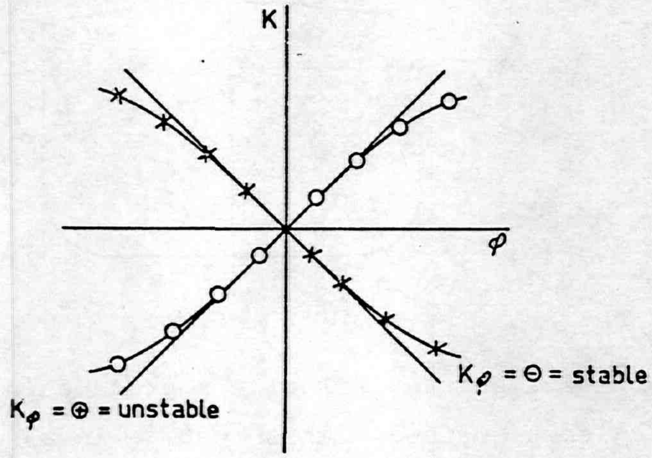
The derivative $K_{\dot{p}}$ is therefore negative and since I_x is a positive quantity, $(K_{\dot{p}}-I_x)$ is always negative in value. The condition for stability then becomes that K_p and K_φ are both negative in value. The derivative K_p can be shown to be negative for the following reasons.



A positive angular velocity p produces a linear velocity pd at point **P** on the ship surface. This results in a skin friction force opposite to pd which produces a rolling moment opposite in direction to p. Also eddy resistance around the bilge will produce a moment opposite to p. In addition, these pressures can cause surface waves to be generated and radiated from the ship (this is energy dissipation and indicates damping). Hence, the curve of K versus p will have a negative slope (i.e. K_p is negative in value) and will appear as



With K_p always negative, the criterion for stability now becomes the condition that K_φ be negative. This means that in the plot of K versus φ, the slope K_φ must be negative for stability.



Since for a positive angle of roll φ , the righting moment is opposite to the direction of roll, and since a positive metacentric height produces a positive righting moment, the condition that K_{φ} be negative is identical to the condition that the metacentric height be positive. Hence, the criteria for roll stability of positive metacentric height (or negative K_{φ}) has been demonstrated in the more formal manner.

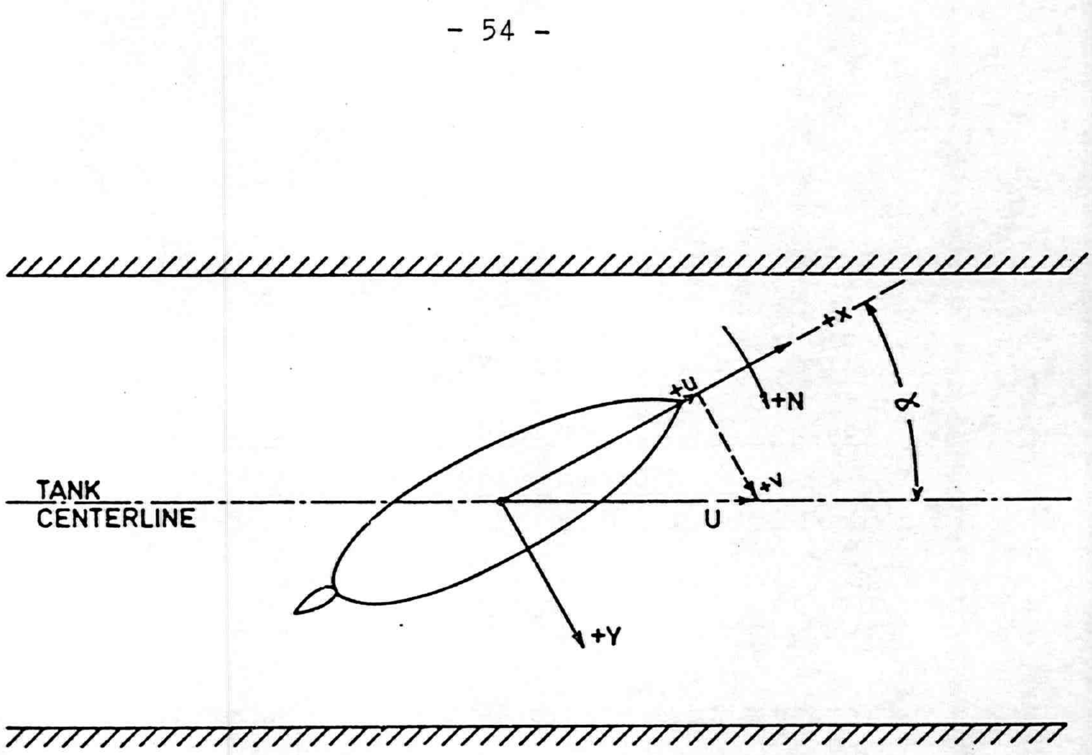
CHAPTER V

Testing Techniques Used for the Measurement of Hydrodynamic Derivatives

The various hydrodynamic derivatives, which appear in the equations of motion and in the criteria for motion stability have numerical values which depend on the geometry (i.e. design) of the ship. In the case of stability criteria in roll, the metacentric height can be calculated from the ship lines by rather simple hydrostatic theory. In the case of dynamical stability in straight line motion, the various derivatives involve calculating forces and moments acting on a given ship design not only while it is moving ahead with a velocity u_o but also while it is experiencing sidewise velocity and angular velocities as well. In the case of the relatively simple motion case of straight ahead motion at constant speed, no adequate theory or calculation exists to predict the resistance of the ship, and resort is made to testing ship models to obtain the necessary information. The more complicated motion involving velocity components in addition to straight ahead motion has not been rendered solvable by present theories or calculations, hence resort to ship model tests of a special nature must be made in order to determine certain hydrodynamic derivatives for a given design.

Some of the model testing techniques used for measuring the derivatives, especially the four derivatives which appear in the stability criterion, will be discussed.

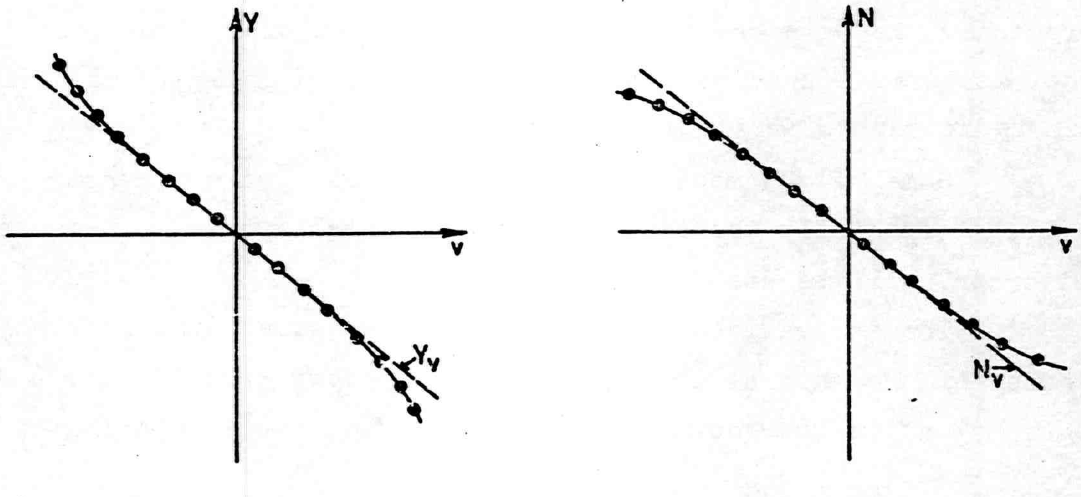
The derivatives Y_v and N_v can be measured on a model by towing a model of the ship at the proper speed (proper Froude number) at various angles of attack to the model path. The sketch below indicates the nature of the model test.



From the sketch, it can readily be observed when a model is towed down the centerline of the tank with a velocity U and at an angle of attack α from starboard, that a velocity component v along the positive y axis is produced such that

$$v = U \sin\alpha$$

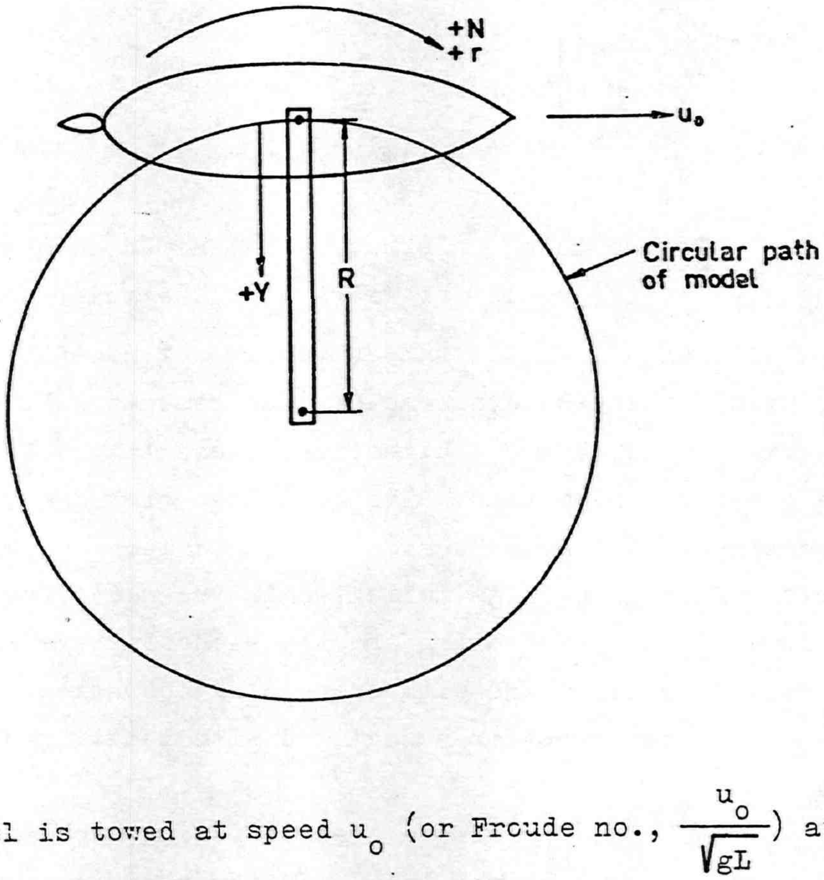
A dynamometer measures the force Y and moment N experienced by the model at each of a series of angles of attack. These measurements are then plotted versus v, producing the typical plots shown below.



The slopes of these plots, taken at v = 0, give numerical values for the derivatives Y_v and N_v for the model. These derivatives can be reduced to non-dimensional form or converted to ship dimensions using the dimensions of length L, speed U, and density ρ. as will be indicated later.

Since a rotating propeller acts as a lifting surface, the various model tests should be conducted with propellers operating, preferably at the ship propulsion points. Since the undeflected rudder acts as a lifting surface, model tests should include the rudder in the undeflected position.

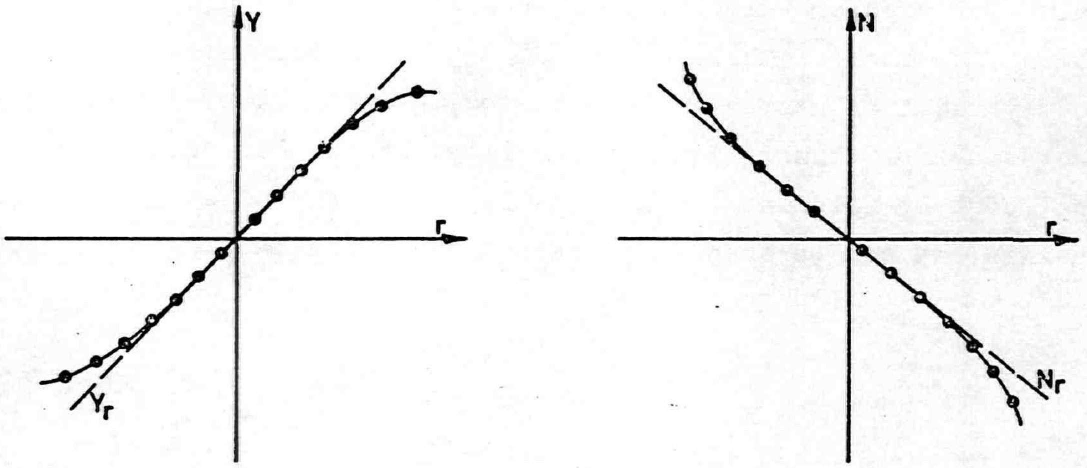
In order to measure the derivatives Y_r and N_r on a model, it is necessary to tow the model at regular forward speed and impose various values of angular velocity r on the model and measure the Y force and N moment for each of these different angular velocities. To do this directly requires a special type of towing tank and apparatus called the "rotating arm" tank. In this facility, an angular velocity is imposed on the model by rotating it in a circle at the end of an arm rotating about an axis, as can be seen from the following sketch.



The model is towed at speed u_o (or Froude no., $\frac{u_o}{\sqrt{gL}}$) at various radii R, and a dynamometer measures the force Y and moment N during each of these tests. Since, for a given model speed u_o (or Froude no. $\frac{u_o}{\sqrt{gL}}$), the angular velocity r is given by

$$r = \frac{u_o}{R}$$

the only way to vary r (at constant $\frac{u_o}{\sqrt{gL}}$) is to vary R. Hence, tests are carried out at various R values. Typical plots of the resulting measurements (after model inertial effects are deducted) are shown below, and the derivatives Y_r and N_r are obtained by evaluating the slope at $r = 0$.



If the model is properly ballasted so that its weight equals its buoyancy and the center of gravity of the model is at the same geometrical position as that of the ship (i.e. $\frac{x_G}{L}$ of model is the same as that of the ship), then the dynamometer measurements will provide values for $(N_r - mx_G u_o)$ and $(Y_r - mu_o)$ directly for use in the stability criteria and the equations of motion. The reduction of these derivative values to non-dimensional form or expansion to ship dimensions will be indicated later. For the same reasons as indicated in the model tests for N_v and Y_v, the model used in the rotating arm tests should have the rudder in the undeflected position and with propellers operating.

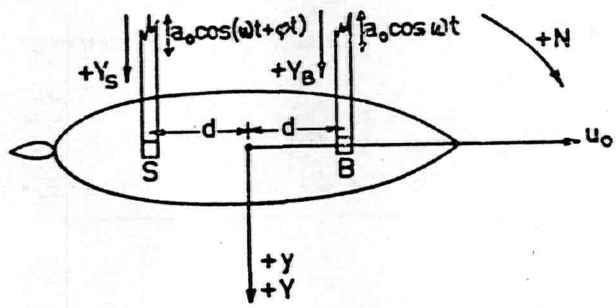
A few problems associated with rotating arm tests and techniques are

1. They require a special towing tank and cannot use the usual long and narrow tank used for resistance testing.

2. The model must be accelerated and data obtained before one revolution. Otherwise, the model will be running in its own wake and the actual u_o will not be known, or the speed will need to be corrected for this wake.

3. In order to obtain the derivatives (i.e. the slope at $r = 0$) sufficient data at small values of r are necessary. This means the radius of turn R, or more correctly, the ratio of R to model length L must be large. For large models, a large facility is required. Smaller models require a smaller tank, but too small models will lead to scale effects in the ship predictions.

In order to avoid the large expense of an additional facility such as the rotating arm tank, a device known as a planar motions mechanism was devised for use in the regular long and narrow towing tank to measure the derivatives Y_r and N_r and some of the other derivatives, such as the acceleration derivatives $N_{\dot{v}}$, $Y_{\dot{r}}$, $Y_{\dot{v}}$, and $N_{\dot{r}}$. The apparatus consists of two oscillators, one produces a transverse oscillation at the bow and the other a transverse oscillation at the stern of the model while the model moves down the towing tank at the speed u_o.



The bow at a point B located a distance d forward of the origin (usually ⊗) is oscillated transversely with a small amplitude a_o and with a circular frequency ω. Point B on the stern at a distance d aft of the origin is oscillated transversely with the same amplitude a_o and frequency ω but the phasing of the oscillation of the stern relative to the bow can be adjusted and is indicated by the phase angle φ. If $\varphi = 0$, then bow and stern have the same transverse displacement and the model experiences a pure transverse oscillation of the form

$$y = a_o \cos \omega t$$

$$\frac{dy}{dt} = v = -a_o \omega \sin \omega t$$

$$\dot{v} = -a_o \omega^2 \cos \omega t$$

Dynamometers at the bow and stern measure the oscillatory Y forces experienced by the model at the bow and stern, i.e. Y_B and Y_S. Since the velocity v (sine function) is out of phase with the displacement y, (cosine function), then the out of phase measurements of Y_B and Y_S are forces arising from the effects of v. Since the acceleration $\dot{v}$ (cosine function) is in phase with y (cosine function), the in phase measurements of Y_B and Y_S denote forces originating form $\dot{v}$. The derivatives Y_v and N_v are obtained by the following relationship

$$Y_v = \frac{\partial Y}{\partial v} = \pm \frac{\text{Out of phase amplitude of } (Y_B+Y_S)}{-a_o \omega}$$

$$N_v = \frac{\partial N}{\partial v} = \pm \frac{\text{Out of phase amplitude of } (Y_B-Y_S)d}{-a_o \omega}$$

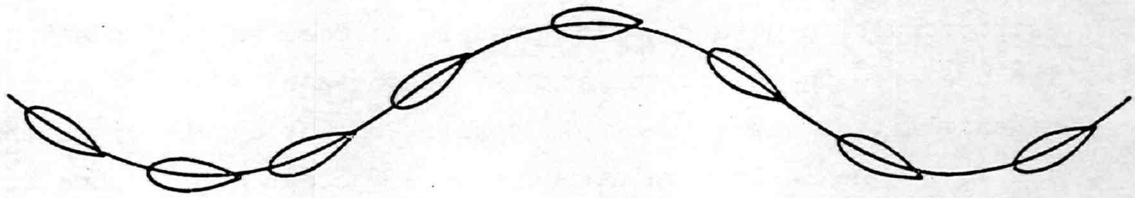
By testing at various frequencies, ω, the frequency dependence of these derivatives can be determined. The derivatives N_v and Y_v can be obtained without oscillating by towing at different angles of attack, i.e. zero frequency as indicated previously. The derivatives $Y_{\dot{v}}$ and $N_{\dot{v}}$ can be obtained by measuring the in phase components of Y_B and Y_S (model inertial forces must be removed).

$$Y_{\dot{v}} = \pm \frac{\text{In phase amplitude of } (Y_B+Y_S)}{-\omega^2 a_o}$$

$$N_{\dot{v}} = \pm \frac{\text{In phase amplitude of } (Y_B-Y_S)d}{-\omega^2 a_o}$$

If the model is properly ballasted ($\frac{x_G}{L}$ same as ship and weight equals buoyancy) then the in phase components will furnish the correct value for the terms $(Y_{\dot{v}}-m)$ and $(N_{\dot{v}}-mx_G)$. Frequency dependence can be determined by testing at different ω values. The time the out of phase Y_B and Y_S are measured, are at the peak values of v, i.e. when $\dot{v} = 0$, and when the in phase components are measured $\dot{v}$ is at its maximum and $v = 0$. Since there is no angular velocity or acceleration, the measurements at the proper phasing are made when $u = u_o$ and only the one variable involved has a value other than zero.

In order to obtain the derivatives Y_r and N_r, the measurements must be made at the time or phasing when $\dot{r} = 0$ and $v = \dot{v} = 0$. Similarly for $Y_{\dot{r}}$ and $N_{\dot{r}}$, the measurements must be taken when $r = 0$, and $v = \dot{v} = 0$. In order to impose an angular velocity and angular acceleration on the body with v and $\dot{v}$ equal to zero, the model must travel down the tank with the centerline of the model always tangent to the path - (this means there is no v component since u = U). The path is oscillatory as shown below.



This type of path will be followed by the model if the phase angle φ , between the bow and stern oscillators, satisfies the condition

$$\frac{\varphi}{2} = \tan^{-1} \frac{\omega d}{U} .$$

With the phase angle set at this value and thereby $v = \dot{v} = 0$ assured, the out of phase components of Y_B and Y_S will provide the forces and moment due to r and the in phase components will provide the forces and moment resulting from $\dot{r}$. If ψ is the orientation angle, then

$$\psi = \psi_o \cos \omega t$$

$$r = - \psi_o \omega \sin \omega t$$

$$r = - \psi_o \omega^2 \cos \omega t$$

Hence, r is out of phase with ψ and $\dot{r}$ is in phase with ψ. ψ_o is determined from the amplitude a_o, the distance d, and the phase φ . The derivative values are then, including the inertial effects of the model,

$$(Y_r - mu_o) = \frac{\pm \text{ Out of phase amplitude of } (Y_B + Y_S)}{- \psi_o \omega}$$

$$(N_r - mx_G u_o) = \frac{\pm \text{ Out of phase amplitude of } (Y_B - Y_S)d}{- \psi_o \omega}$$

$$(Y_{\dot{r}} - mx_G) = \frac{\pm \text{ In phase amplitude of } (Y_B + Y_S)}{- \psi_o \omega^2}$$

$$(N_{\dot{r}} - I_z) = \frac{\pm \text{ In phase amplitude of } (Y_B - Y_S)d}{- \psi_o \omega^2}$$

If the model is properly ballasted so that $\frac{x_G}{L}$ of the model and ship are the same and the weight equals the buoyancy, then the above measured values can be directly non-dimensionalized or scaled up to the ship. As in the other tests, the model should be propelled preferably at the ship propulsion point and the rudder included in the undeflected position. The use of $\pm$ in the above terms are associated with the term "amplitude" of oscillation which is always positive. The direction of the forces Y_B and Y_S at the maximum values determine whether + or - should be used.

Some precaution is necessary in applying planar motions tests. Since the ship model is at the water surface, oscillatory motions can create waves whose properties depend on the frequency of generation, hence the derivatives may be frequency dependent. The actual maneuver of a ship going into a turn is at 0 frequency, hence low frequencies are of interest. Since $r = -\omega \psi_0 \sin \omega t$, then small ω gives small r which is desirable, since the tests should be carried out at small values of r. The rotating arm test will give data free of frequency effects.

In the case of a deeply submerged submarine model, where surface frequency effects disappear, other frequency effects called "unsteady effects" caused by circulation and lift considerations come into play. The parameter $\frac{\omega L}{U}$ called "reduced frequency" is important for these effects. However, unsteady effects are felt only at high values $\frac{\omega L}{U}$ which are well out of the range of those frequencies used for ship models.

The use of model test data immediately brings to mind the possibility of scale effects. The Froude number is to be satisfied, hence the Reynolds' number will not be satisfied. Since in determining the Y force and N moments, the lift and circulation effects are involved, low aspect ratio airfoil theory indicates very little scale effect on the slope of lift coefficient vs. angle of attack. However, separation or breakdown of lift occurs at a lower angle of attack at the lower Reynolds' number. Fortunately, the various derivatives are determined at the small values of v and r and hence at the small angles of attack before any separation effects come into play. Consideration should be given to possible scale effects if measurements are made at larger values of v and r when obtaining information to be used in any non-linear equations.

It has been indicated that the velocity derivatives N_v, Y_v, N_r, and Y_r for a ship could not be readily calculated and that resort was made to measuring these derivatives by means of special model tests using special dynamometers. Also, the velocity derivatives play the dominant role in the criteria for dynamic stability. On the other hand, the acceleration derivatives appearing in the equations of motion, i.e. $Y_{\dot{v}}$, $N_{\dot{r}}$, $Y_{\dot{r}}$, and $N_{\dot{v}}$, are not involved directly in the criteria and are either small in magnitude as the case of $N_{\dot{v}}$ and $Y_{\dot{r}}$, or are combined with terms of about equal magnitude as $Y_{\dot{v}}$ and $N_{\dot{r}}$ are. The acceleration derivatives, or more specifically the forces arising from acceleration in the fluid, are the result of the inertial properties of the fluid with little, if any, dependence on the viscous properties. Hence, potential theory may be readily employed to estimate these acceleration derivatives - provided such a theory is valid for ship - like bodies and that the free surface is taken into account for bodies operating at the water surface. Calculation for the acceleration derivatives for various submerged bodies of revolution (submarine hulls) have been made by machine calculation, but the theory for arbitrary surface ships remains limited to thin ship theory. Of course, as was the result of the other derivatives for surface ships, they may be frequency dependent.

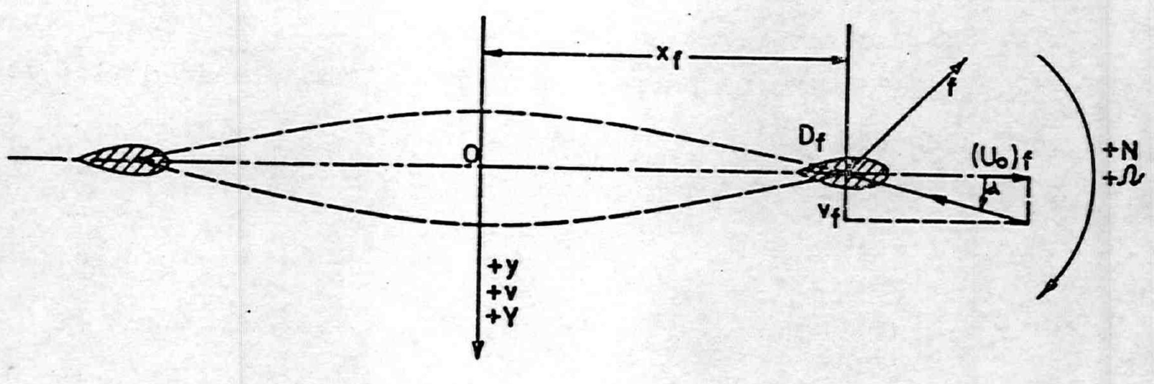
However, a) since we are looking for values of the acceleration derivatives at essentially zero frequency, b) since the significant derivatives $N_{\dot{r}}$ and $Y_{\dot{v}}$ occur in the combinations $(N_{\dot{r}}-I_z)$ and $(Y_{\dot{v}}-m)$ so that a given error in the derivative results in only half that error in the term, and c) since the derivatives are not directly a part of the criteria for stability but are more involved in the magnitude of the roots and the resulting trajectories, the theoretical values for these derivatives may be used. Typical values for use are those of ellipsoids of different length-beam (or length-draft) ratios as calculated and published in Lamb's Hydrodynamics.

The acceleration derivatives can also be readily measured from the record of the in phase components of the planar motion tests on ship models as was mentioned earlier.

CHAPTER VI

Isolated Lifting Surfaces

A special type of body for which the varicus derivatives can be calculated is the isolated lifting surface, or hydrofoil, located some distance from the axis. The approach is essentially to calculate the angle of attack developed at the surface as the result of a transverse velocity v and an angular velocity r. From the lift characteristics at the resulting angle of attack, the force and moment produced on the body is attached as an appendage to a larger hull, (say a rudder or stabilizing fin), then the effect of the interference of the hull on the water approaching the appendage must be taken into account. Items such as wake and propeller race are examples of such interference. The example is for a lifting surface or fin located either well forward or aft of the origin, a distance at least several foil chord lengths away. The sketch below indicates such a fin arrangement.



The development is for a general location and therefore the example deals with a fin forward a distance $+x_f$ from the origin. Any resulting relationship or formulation from the analysis would give the effect of a fin aft of the origin provided a negative value of x_f is used. The subscript f is used to indicate local condition at the fin. If the body is given a transverse velocity disturbance +v, then the fin also experiences a transverse velocity of v, i.e.

$$v_f = v$$

If at the time of this transverse disturbance the forward velocity component on the fin is $(U_o)_f$, then the change in angle of attack at the fin caused by the transverse velocity v is

$$\alpha = \tan^{-1} \frac{v_f}{(U_o)_f}$$

and the Y force and N moment produced at the fin by this angle of attack is given by

$$Y_f = -(L_f \cos\alpha + D_f \sin\alpha)$$

$$N_f = Y_f x_f$$

where L_f and D_f are the lift and drag forces on the fin. The lift and drag of a foil are usually expressed in terms of the lift coefficient C_L and drag coefficient C_D as defined below

$$L_f = (C_L)_f \, 1/2 \rho A_f U_f^2 = (C_L)_f \, 1/2 \, A_f \left[(U_o)_f^2 + v_f^2\right]$$

$$D_f = (C_D)_f \, 1/2 \rho A_f \left[(U_o)_f^2 + v_f^2\right]$$

where ρ is the fluid density and A_f is the fin (projected) area. Hence

$$Y_f = -1/2 \rho A_f (U_o)_f^2 \left[(C_L)_f \sec\alpha + (C_D)_f \tan\alpha \sec\alpha\right]$$

since

$$\left[(U_o)_f^2 + v_f^2\right] \cos\alpha = ((U_o)_f^2 \; \frac{\left[(U_o)_f^2 + v_f^2\right]}{(U_o)_f^2} \; \frac{(U_o)_f}{\left[(U_o)_f^2 + v_f^2\right]^{1/2}}$$

$$= (U_o)_f^2 \frac{\left[(U_o)_f^2 + v_f^2\right]^{1/2}}{(U_o)_f} = (U_o)_f^2 \sec\alpha$$

and by similar substitution

$$\left[(U_o)_f^{\,2}+v_f^{\,2}\right]\sin\alpha \doteq (U_o)_f^{\,2}\operatorname{cosec}^2\alpha\sin\alpha = (U_o)_f^{\,2}\tan\alpha\ \sec\alpha$$

From the relationship $v_f = (U_o)_f \tan\alpha$

$$\frac{dv_f}{d\alpha} = (U_o)_f \sec^2\alpha \qquad \text{or} \qquad \frac{d\alpha}{dv_f} = \frac{\cos^2\alpha}{(U_o)_f}$$

and

$$(Y_v)_f = \frac{\partial Y_f}{\partial v_f} = \left(\frac{\partial Y_f}{\partial\alpha}\right)\frac{d\alpha}{dv_f} = -\frac{\rho}{2}A_f(U_o)_f\cos^2\alpha\,\frac{d}{d\alpha}\left[(C_L)_f\sec\alpha+(C_D)_f\tan\alpha\sec\alpha\right]$$

evaluated at $\alpha = 0$. The derivative of the expression contained within the brackets becomes

$$\frac{d(C_L)_f}{d\alpha}\sec\alpha+(C_L)_f\tan\alpha\sec\alpha+\frac{d(C_D)_f}{d}\tan\alpha\sec\alpha+(C_D)_f(\sec^3\alpha+\tan^2\alpha\sec\alpha)$$

and at $\alpha = 0$, one obtains

$$(Y_v)_f = -\frac{\rho}{2}A_f(U_o)_f\left[\frac{d(C_L)_f}{d\alpha}+(C_D)_f\right]$$

and obviously

$$(N_v)_f = x_f(Y_v)_f\,.$$

Since $v = v_f$, the above represent contributions of the fin to the overall derivative.

From airfoil and hydrofoil theory the slope of the lift coefficient curve versus angle of attack is given by

$$\frac{\partial C_L}{\partial\alpha} = \frac{2\pi}{1+\frac{2}{AR}}$$

where AR is the aspect ratio, which is the ratio of foil span to chord*). The theoretical value of lift can be used for obtaining numerical values or else the slope as obtained from wind tunnel or towing test on foils (readily obtainable in published literature) can be used. The drag coefficient, D_f, can be estimated as essentially the skin friction drag of the foil at the local Reynolds' number on the foil corresponding to a velocity $(U_o)_f$ or else can be obtained from published data.

If there are no wake effects, or propeller race effects, or no other flow interference effects with the hull, then $(U_o)_f = U_o$ (i.e. inflow to fin same as forward velocity of ship).

If for example, a fin is in the wake of the hull and it is estimated that there is a wake factor of 20% on the fin, then

$$(U_o)_f = (1-0.20)U_o = 0.80\ U_o.$$

From the expression for the contribution of the fin to the Y_v derivative (i.e. $(Y_v)_f$) it is clear that $(Y_v)_f$ is always negative, whether at bow or stern, but that a fin forward decreases the value of N_v and a fin aft increases (makes more positive or less negative) the value of N_v. This confirms an earlier presentation of this point.

The contribution of a fin to the derivatives Y_r and N_r are readily calculable from the expressions already developed. It is clear from the sketch used above, that a small positive angular velocity r produces a linear transverse velocity at the fin, given by

$$v_f = x_f r$$

The force and moment produced at the fin as the result of r is

$$Y_f \text{ due to } r = (Y_v)_f v_f = (Y_v)_f x_f r$$

$$N_f \text{ due to } r = (N_v)_f v_f = x_f(Y_v)_f x_f r = Y_v x_f^2 r$$

On taking the derivative with respect to r, one obtains

$$(Y_r)_f = x_f(Y_v)_f$$

$$(N_r)_f = x_f^2(Y_v)_f$$

*) In determining the proper aspect ratio, consideration must be given to the actual tip losses that occur.

tend to make N_r more negative. This confirms the results indicated previously.

The method developed gives a way of estimating quantitatively, the improvement to be expected from lifting surface addition to a hull. Since a rotating propeller acts as a lifting surface, if the lift characteristics are known, the contribution of propeller to the overall ship derivatives can be calculated.

A similar method of analysis can be used to develop the contribution of the fin to the acceleration derivatives of the body. The analysis results in the following formulations.

$$(N_{\dot{v}})_f = x_f (Y_{\dot{v}})_f$$

$$(Y_{\dot{r}})_f = x_f (Y_{\dot{v}})_f$$

$$(N_{\dot{r}})_f = x_f^2 (Y_{\dot{v}})_f$$

$$(Y_{\dot{v}})_f = - \frac{\pi \rho s^2 c^2}{\sqrt{s^2 + c^2}}$$

where s is the span and c is the chord of the fin (i.e. dimensions of the fin)*). The formulation for $(Y_{\dot{v}})_f$ is taken from the calculation of the "added mass" of a flat plate of dimensions s and c for acceleration perpendicular to the plate. It should be noted and stressed that the contribution of a fin like appendage to the acceleration derivatives of a body are small and of minor significance, whereas the contribution of such fins to the velocity derivatives are major, very significant, and often times decisive in their contribution to dynamical stability.

*) For a fin in which one edge does not allow water to flow over it, one must consider using twice this length and taking one half the result.

CHAPTER VII

Solution of Motion Equations for Control Surface Deflections

Let us now turn to the area of control of ship motion. Control forces and moments can be produced on the ship by a deflection of the control surface such as a rudder. If δ designates the deflection of the control surface, then the forces X, Y, and moment N produced on the ship by this deflection, as indicated by the linear terms in the Taylor expansion, are

$$X_\delta \delta$$

$$Y_\delta \delta$$

$$N_\delta \delta$$

It is assumed that the forces and moments produced on the ship as the result of $\dot{\delta}$ and $\ddot{\delta}$ are negligible, although these variables are not necessarily negligible in determining the torque on the rudder stock during a maneuver. It can readily be shown that $X_\delta = 0$ since the rudder is symmetric port and starboard. The equations of motion, including the rudder effect now become

$$a_{11}\,\Delta u + 0v + 0r = -X_\delta \delta = 0$$

$$a_{21}\,\Delta u + a_{22}v + a_{23}r = -Y_\delta \delta$$

$$a_{31}\,\Delta u + a_{32}v + a_{33}r = -N_\delta \delta$$

(Since a_{12} and a_{13} are zero by symmetry, it can be shown that a_{21} and a_{31} must also be zero).

The linear solution for the angular velocity r is readily expressed as

$$r = \frac{\begin{vmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & -Y_\delta \delta \\ a_{31} & a_{32} & -N_\delta \delta \end{vmatrix}}{\begin{vmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}} = -\frac{a_{11}(a_{22}N_\delta - a_{32}Y_\delta)\delta}{a_{11}(a_{22}a_{33} - a_{23}a_{32})} = -\frac{\left[a_{22}N_\delta - a_{32}Y_\delta\right]\delta}{A(\mathcal{D} - \sigma_1)(\mathcal{D} - \sigma_2)}$$

where A, the coefficient of $\mathcal{D}^2$ has been expressed in terms of derivatives previously

$$v = \frac{\begin{vmatrix} a_{11} & 0 & 0 \\ a_{21} & -Y_\delta \delta & a_{23} \\ a_{31} & -N_\delta \delta & a_{33} \end{vmatrix}}{\text{Det.}} = \frac{\left[a_{23}N_\delta - a_{33}Y_\delta\right]\delta}{A(\mathcal{D} - \sigma_1)(\mathcal{D} - \sigma_2)}$$

$$u = \frac{\begin{vmatrix} 0 & 0 & 0 \\ -Y_\delta \delta & a_{22} & a_{23} \\ -N_\delta \delta & a_{32} & a_{33} \end{vmatrix}}{\text{Det.}} = \frac{0}{(X_{\dot{u}} - m)A(\mathcal{D} - \sigma_1)(\mathcal{D} - \sigma_2)(\mathcal{D} - \sigma_3)}.$$

The solution for Δu is the homogeneous solution obtained previously, i.e.

$$\Delta u = u_1 e^{\sigma_1 t} + u_2 e^{\sigma_2 t} + u_3 e^{\sigma_3 t}.$$

However, from the nature of the solution for v and r above (and from the fact that $a_{12} = a_{13} = 0$), the X equation can be decoupled from the Y and N equations, so that it can be shown that the constants of integration u_1 and u_2 are zero. This gives

$$\Delta u = u_3 e^{\sigma_3 t}$$

as the solution of the linear equations with or without rudder deflection. This decoupling, a result of the linear theory, indicates no speed loss in a maneuver and is a limitation of the linear theory. Non-linear equations can make the situation more realistic as will be indicated later.

In the solution for v, the numerator is developed by substituting for a_{23} and a_{33} the appropriate expressions.

$$\left[a_{23}N_\delta - a_{33}Y_\delta\right]\delta = \left[(Y_{\dot r}-mx_G)\mathcal{D}+(Y_r-mu_o)\right]N_\delta\delta - \left[(N_{\dot r}-I_z)\mathcal{D}+(N_r-mx_Gu_o)\right]Y_\delta\delta$$

$$= \overbrace{\left[N_\delta\,(Y_{\dot r}-mx_G)-Y_\delta\,(N_{\dot r}-I_z)\right]}^{b_1}\mathcal{D}\delta + \overbrace{\left[N_\delta\,(Y_r-mu_o)-Y_\delta\,(N_r-mx_Gu_o)\right]}^{b_2}\delta$$

$$= b_1\mathcal{D}\delta + b_2\delta$$

where b_1 and b_2 are defined as the terms within the brackets.

Similarly, the numerator in the solution for r, becomes

$$\left[a_{22}N_\delta - a_{32}Y_\delta\right]\delta = \left[(Y_{\dot v}-m)\mathcal{D}+Y_v\right]N_\delta\delta - \left[(N_{\dot v}-mx_G)\mathcal{D}+N_v\right]Y_\delta\delta$$

$$= \overbrace{\left[(Y_{\dot v}-m)N_\delta - (N_{\dot v}-mx_G)Y_\delta\right]}^{b_3}\mathcal{D}\delta + \overbrace{\left[Y_vN_\delta - N_vY_\delta\right]}^{b_4}\delta$$

$$= -(b_3\mathcal{D}\delta + b_4\delta)$$

For the case where the rudder deflection does not vary with time, as in the case of the rudder being deflected to a position $\delta = \delta_o$ at time t = 0, and held at δ_o, then $\mathcal{D}\delta = \frac{d\delta}{dt} = 0$ and the solution for r develops as follows

$$r = \frac{b_4\,\delta_o}{A(\mathcal{D}-\sigma_1)(\mathcal{D}-\sigma_2)} = \frac{1}{A}\left(\frac{1}{\mathcal{D}-\sigma_1}\right)e^{\sigma_2 t}\int e^{-\sigma_2 t}(b_4\,\delta_o)dt$$

$$= \frac{b_4\,\delta_o}{A}\left(\frac{1}{\mathcal{D}-\sigma_1}\right)e^{\sigma_2 t}\left[\int e^{-\sigma_2 t}dt\right] = \frac{b_4\,\delta_o}{A}\left(\frac{1}{\mathcal{D}-\sigma_1}\right)\left[\frac{e^{-\sigma_2 t}}{-\sigma_2}+C_2\right]e^{\sigma_2 t}$$

$$= \frac{b_4\,\delta_o}{A}\,e^{\sigma_1 t}\int e^{-\sigma_1 t}\left(\frac{1}{-\sigma_2}+C_2e^{\sigma_2 t}\right)dt = \frac{b_4\,\delta_o}{A}\left[\frac{1}{\sigma_1\sigma_2}+\frac{C_2e^{\sigma_2 t}}{\sigma_2-\sigma_1}+C_1e^{\sigma_1 t}\right]$$

$$r = \frac{b_4\,\delta_o}{A\sigma_1\sigma_2} + r_1e^{\sigma_1 t} + r_2e^{\sigma_2 t}$$

where r_1 and r_2 are constants of integration.

The result gives the homogeneous solution (stability equation solution) plus a particular solution $\frac{b_4\,\delta_o}{A\sigma_1\sigma_2}$.

If the ship is dynamically unstable, the angular velocity will increase in time reaching no steady state angular velocity - as would be the case without rudder deflection. However, if the ship is stable (σ_1 and σ_2 are negative), then the angular velocity changes in time according to the equation, reaching the steady state angular velocity of $\frac{b_4 \delta_o}{A\sigma_1\sigma_2}$ as the transient terms $C_2^{1} e^{\sigma_2 t} + C_1^{1} e^{\sigma_1 t}$ go to zero as time goes on, for the stable ship.

The solution for v is readily written down (since a similar process of integration is involved) as

$$v = \frac{b_2 \delta_o}{A\sigma_1\sigma_2} + v_1 e^{\sigma_1 t} + v_2 e^{\sigma_2 t}.$$

Recalling that $A\, \sigma_1\, \sigma_2 = C$, the coefficient independent of $\mathcal{D}$ in the quadratic equation $A\mathcal{D}^2+B\mathcal{D}+C$, and that $C = Y_v(N_r-mx_G u_o)-N_v(Y_r-mu_o)$, the steady state angular velocity and transverse velocity, for a rudder deflection δ_o is expressed by

$$r = \left[\frac{Y_v N_\delta - N_v Y_\delta}{Y_v(N_r-mx_G u_o)-N_v(Y_r-mu_o)}\right]\delta_o$$

$$v = \left[\frac{N_\delta\,(Y_r-mu_o)-Y_\delta\,(N_r-mx_G u_o)}{Y_v(N_r-mx_G u_o)-N_v(Y_r-mu_o)}\right]\delta_o$$

Within the linear theory, the angular velocity and transverse velocity in the steady turn are proportional to the rudder deflection. Since, within the linear theory there is no speed loss, the radius of the turn is given by

$$\text{Radius} = \frac{u_o}{r}$$

It must be recalled here that the deflection of a rudder is positive in the same sense of rotation as r and N, i.e. if one looks from above a clockwise rotation of r,N, and δ is positive. The above expressions for r and v have little meaning if the ship is unstable - no more than knowing the magnitude of a negative metacentric height can give the angle of heel for a given heeling moment. If the values are negative, i.e. stable, the above expressions give rather good estimates for small rudder deflections. However, for large rudder deflections and tight turns it becomes necessary to solve the non-linear equations - either by computational technique or by free model turning tests in a maneuvering basin.

In the case where the rudder deflection varies with time such as an exponential rudder deflection

$$\delta = \delta_o(1-e^{-at})$$

or sinusoidal oscillation

$$\delta = \delta_o \cos \ wt = \text{Real part of } \delta_o e^{iwt} .$$

one must carry through the differentation in the numerator of the algebraic solution before operating with $(\frac{1}{\mathcal{D}-\sigma})$. For example, in the solution for r for an exponential deflection, one has

$$r = \frac{b_3 \mathcal{D}\delta + b_4 \delta}{A(\mathcal{D}-\sigma_1)(\mathcal{D}-\sigma_2)} = \frac{b_3 \mathcal{D}\left[\delta_o(1-e^{-at})\right] + b_4 \ \delta_o(1-e^{-at})}{A(\mathcal{D}-\sigma_1)(\mathcal{D}-\sigma_2)}$$

$$= \frac{b_3 \ \delta_o ae^{-at} + b_4 \ \delta_o(1-e^{-at})}{A(\mathcal{D}-\sigma_1)(\mathcal{D}-\sigma_2)} = \frac{\delta_o\left[b_4 + (b_3 a - b_4)e^{-at}\right]}{A(\mathcal{D}-\sigma_1)(\mathcal{D}-\sigma_2)} .$$

The integrations resulting from operating with $(\frac{1}{\mathcal{D}-\sigma_1})$ and $(\frac{1}{\mathcal{D}-\sigma_2})$ can be readily accomplished since the integrand is an exponential function of time. In a similar manner, for the case of the sinusoidally oscillating rudder, the complex exponential e^{iwt} can be readily integrated during the process of operating with $\frac{1}{\mathcal{D}-\sigma_2}$ and $\frac{1}{\mathcal{D}-\sigma_1}$.

CHAPTER VIII

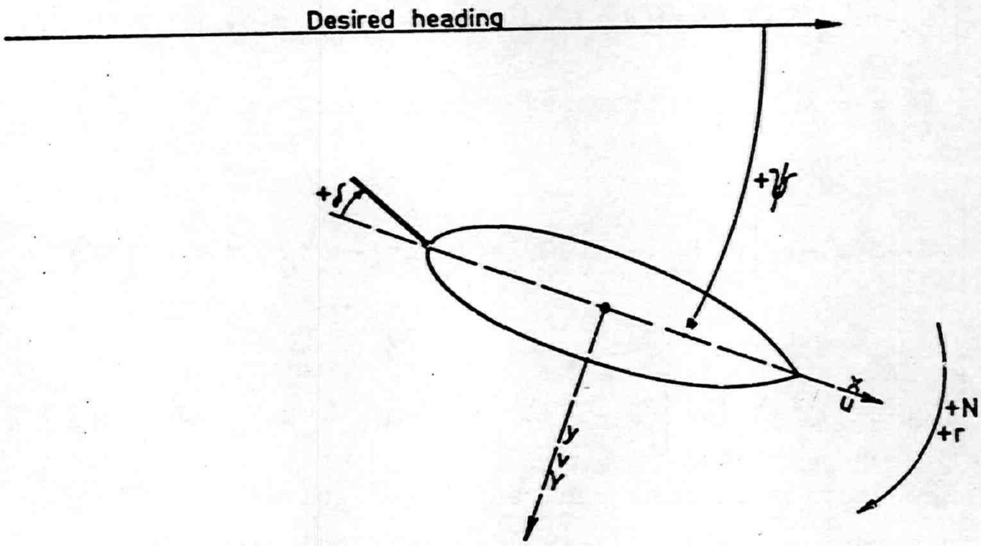
Automatic Steering Control -

Rudder Deflection as a Function of Ship Motion

The linearized equations of motion with rudder deflection were solved previously for the case where the rudder was deflected as some prescribed function of time, such as a constant, exponential, or sinusoidal deflection. The solution of the resulting linearized equations were indicated by using the operational technique previously developed. The LaPlace Transform technique could also have readily been used to obtain the solutions for these equations.

The deflection of the rudder in this sort of prescribed manner is essentially associated with the maneuvering aspect of ships. However, if a rudder or a control surface is deflected as a function of some parameter associated with the ship motion or trajectory, such as ship heading, velocity, etc., the rudder deflection varies with time only as that parameter varies with time. If the control surface is thus automatically deflected according to the value of a parameter or several parameters, then one is dealing with the domain of automatic control, and for motion in the horizontal plane with the rudder being the control surface, one is involved in the area of automatic steering control.

The usual desire is to keep the ship on a desired heading - i.e. a given angular orientation (such as north). Let us designate the angle ψ as the error or deviation from the desired heading. It is obvious that $\dot{\psi} = \frac{d\psi}{dt} = r$, the angular velocity. The rudder deflection, denoted by δ , is positive in the same sense of rotation as ψ , r, and N, as indicated in the following sketch.



In order to deflect the rudder as a function of ψ , one must be able to measure ψ continuously and use the signal from this measurement to activate the rudder mechanism. Since the heading angle is readily measurable by a gyro compass, a directional control signal is readily available. A good helmsman, in attempting to maintain course, will not only deflect the rudder in accordance with his sensing of the deviation from course, but will also ease off on the rudder and perhaps apply a little opposite rudder to meet the "swing of the ship" in order to prevent the angular velocity of the ship from overshooting (swinging the ship beyond) the desired heading. Hence, in addition to a sensitivity to heading, a good control system should have a sensitivity to angular velocity. Let us, therefore, deflect the rudder proportional to the heading error and the angular velocity. (The rudder can be deflected according to any measurable parameter - within the limits of the rudder system - which results in a signal capable of activating the rudder). The equation for the rudder deflection under the conditions mentioned above becomes

$$\delta = k_1 \psi + k_2 r$$

where k_1 and k_2 are the constants of proportionality of the control system. On substitution of this expression for δ into the linearized equations, (after recalling that $X_\delta = 0$, and that the X equation and Δu can be decoupled from the Y and N equations), one obtains

$$\left[(Y_{\dot{v}}-m)\mathcal{D} + Y_v\right] v + \left[(Y_{\dot{r}}-mx_G)\mathcal{D} + (Y_r - mu_o)\right] r + Y_\delta\ (k_1\psi + k_2 r) = 0$$

$$\left[(N_{\dot{v}}-mx_G)\mathscr{D}+N_v\right]v+\left[(N_{\dot{r}}-I_z)\mathscr{D}+(N_r-mx_Gu_o)\right]r+N_\delta\ (k_1\psi+k_2r)=0$$

Since $r = \dot{\psi} = \mathscr{D}\psi$, the above equations take the form

$$\overset{a_{22}}{\left[(Y_{\dot{v}}-m)\mathscr{D}-Y_v\right]}v+\overset{a_{23}}{\left[(Y_{\dot{r}}-mx_G)\mathscr{D}^2+(Y_r-mu_o+k_2Y_\delta)\mathscr{D}+k_1Y_\delta\right]}\psi=0$$

$$\overset{a_{32}}{\left[(N_{\dot{v}}-mx_G)\mathscr{D}+N_v\right]}v+\overset{a_{33}}{\left[(N_{\dot{r}}-I_z)\mathscr{D}^2+(N_r-mx_Gu_o+k_2N_\delta)\mathscr{D}+k_1N_\delta\right]}\psi=0.$$

The above equations can readily be solved for ψ as a function of time in a manner similar to the previous solution for r.

$$\psi = \frac{0}{\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}} = \frac{0}{A_1\mathscr{D}^3+B_1\mathscr{D}^2+C_1\mathscr{D}+D}$$

$$= \frac{0}{A_1(\mathscr{D}-\sigma_1^1)(\mathscr{D}-\sigma_2^1)(\mathscr{D}-\sigma_3^1)}$$

where σ_1^{1}, σ_2^{1}, and σ_3^{1} are the roots of the cubic equation in $\mathscr{D}$ obtained by multiplying out the determinant. The fundamental differences between the solution for ψ above and the solution for r in the dynamical stability equations, solved previously, are that

a) the stability roots resulting from the Y and N equations (X equation decoupled) are three in number (σ_1^{1}, σ_2^{1}, and σ_3^{1}) as compared to the two roots (σ_1 and σ_2).

b) the automatic controls have introduced a sensitivity to the heading angle ψ which is not inherent in the hull dynamics. Hence ψ is a basic variable where formerly r was the basic variable.

c) obtaining the three roots of the cubic equation is more complicated and less explicit than the simple quadratic solution for getting the two roots of a quadratic equation.

From our experience with the operational technique, the solution for the linearized equations can readily be written in the form

$$\psi = \psi_1 e^{\sigma_1{}^1 t} + \psi_2 e^{\sigma_2{}^1 t} + \psi_3 e^{\sigma_3{}^1 t}$$

$$v = v_1 e^{\sigma_1{}^1 t} + v_2 e^{\sigma_2{}^1 t} + v_3 e^{\sigma_3{}^1 t}$$

$$\Delta u = u_1 e^{\sigma_4{}^1 t}$$

where, because the x equation is decoupled, $e^{\sigma_4{}^1 t}$ does not appear in the ψ and v equations and where $\sigma_4{}^1$ equals the σ_3 value previously obtained from the x equation during the analysis of stability of straight ahead motion. ψ_1, ψ_2, ψ_3, v_1, v_2, v_3, and u_1 are constants of integration depending on initial conditions. The condition for directional stability - i.e. that the ship returns to the original direction of motion as well as straight line motion, after an arbitrary disturbance - is that all three roots $\sigma_1{}^1$, $\sigma_2{}^1$, and $\sigma_3{}^1$ are negative if they are real or have negative real parts if they are complex. ($\sigma_4{}^1$ has already been shown to be real and negative in previous discussions involving σ_3).

By using the quadratic equation solution, it was relatively easy to show in the case of stability of straight line motion, that the terms $\frac{B}{A}$ and $\frac{C}{A}$ had to be positive in order for the σ values to be negative, stable roots. In the case of the cubic equation involving the coefficients A_1, B_1, C_1, and D_1, the condition for all three roots ($\sigma_1{}^1$, $\sigma_2{}^1$, and $\sigma_3{}^1$), to be negative (or real parts negative) is that

$$\frac{B_1}{A_1} > 0, \quad \frac{C_1}{A_1} > 0, \quad \frac{D_1}{A_1} > 0 \quad \text{and} \quad \frac{B_1 C_1 - A_1 D_1}{A_1{}^2} > 0.$$

The last condition is often referred to as Routh's discriminant for the cubic equation. (these conditions result from an analysis of stability in dynamical systems as presented by Routh). The discriminant can be written in the form

$$\frac{B_1}{A_1}\frac{C_1}{A_1} - \frac{D_1}{A_1} > 0$$

so that if $\frac{B_1}{A_1}$ is positive and $\frac{D_1}{A_1}$ is positive, the discriminant has the possibility of becoming positive only if $\frac{C_1}{A_1}$ is positive. Hence, the condition $\frac{C_1}{A_1} > 0$ is a redundant condition.

By designing the proper values of the control constants k_1 and k_2 in the control mechanisms (k_1 and k_2 can be negative as well as positive quantities), the ship will be suitably automatically controlled to maintain a given heading when subjected to disturbances (or reasonable magnitude) from the desired heading. The solution of the equations of motion can be carried out (by digital or analogue computer) for various values of k_1 and k_2 and for various expected disturbances and the resulting trajectories can be analyzed to provide the proper choice of these control constants. Of course, the design and construction of the control hardware should allow for some range of adjustment of the constants to be properly adjusted after installation.

By the use of automatic controls, the equations of motion, as compared to the equations without control, have changed in two major respects. There is a sensitivity to the orientation of the ship, ψ, which is not inherent in the hull hydromechanics and secondly certain terms which readily appear in the criteria for straight line motion have been altered in value by the controls. For example, the former term $(Y_r - mu_o)$ now appears as $(Y_r - mu_o + k_2 Y_\delta)$ and what was formerly $(N_r - mx_G u_o)$ now appears as $(N_r - mx_G u_o + k_2 N_\delta)$. If the transverse velocity v could be readily measured and the control made sensitive to v then, also the terms N_v and Y_v would have additional terms added to them. Hence, the second effect of automatic controls is to make the ship behave as if the ship possessed different values of the hydrodynamic derivatives - i.e. as if the hull had different inherent properties. It can be surmised that a dynamically unstable ship can be made dynamically stable (straight line motion) and directionally stable by the use of automatic control (within certain limits). However, the controls need be less sensitive and less worked and in addition the ship can be handled readily without controls, if the ship is dynamically stable in straight line motion to begin with.

The rudder and steering mechanism represent a reasonable amount of inertia and a certain amount of time is necessary to deflect the rudder, once the signal is given. Therefore, although ψ is measured at time t, it requires some time, say Δt, for the rudder to actually reach the deflection $k_1 \psi(t_1)$. Hence, the deflection of the rudder at time t, i.e. $\delta(t)$ is proportional to ψ at time t_1 (where $t_1 = t - \Delta t$). In functional form, our control system appears as

$$\delta(t) = k_1 \psi(t - \Delta t) + k_2 r(t - \Delta t)$$

where Δt is referred to as the time lag of the control system.

Since, in the linearized equations of motion the variables ψ and r appear as functions of time t, i.e. $\psi(t)$ and $r(t)$, it is necessary to express $\delta(t)$ in terms of $\psi(t)$ and $r(t)$ before introducing the control forces into the equations of motion. A convenient way to handle this is to use the Taylor expansion of the function in order to express $\psi(t-\Delta t)$ and $r(t-\Delta t)$ in terms of $\psi(t)$ and $r(t)$ respectively.

When the Taylor expansion was used previously to develop the nature of the hydrodynamic function, it was shown that a convenient way to write the expansion was in exponential form using a differential operator. Hence, $\psi(t-\Delta t)$ can be written as follows

$$\psi(t-\Delta t) = e^{-\Delta t \mathcal{D}} \cdot \psi(t) \qquad \text{where } \mathcal{D} = \frac{d}{dt}$$

and on expanding the exponential in a series, one obtains

$$\psi(t-\Delta t) = 1+(-\Delta t)\mathcal{D} + \frac{(-\Delta t)^2}{2\,!}\mathcal{D}^2 + \frac{(-\Delta t)^3}{3\,!}\mathcal{D}^3 + \ldots\ldots \quad (t)$$

$$= \psi(t) - \Delta t\, \mathcal{D}\psi(t) + \frac{(\Delta t)^2}{2\,!}\mathcal{D}^2\, \psi(t) + \ldots\ldots$$

If it is assumed that the time lag Δt is small, the terms greater than the linear term contribute little and the expression up to and including the linear term is sufficient to describe the lag effect. Since $\mathcal{D}\psi(t) = r(t)$, and $\mathcal{D}r(t) = \dot{r}(t) = \mathcal{D}^2\,\psi(t)$, the equation for the rudder deflection, including the linear term in the time lag expression, becomes

$$\delta(t) = k_1\psi(t-\Delta t)+k_2 r(t-\Delta t) = k_1\left[\psi(t)-\Delta t r(t)\right] + k_2\left[r(t)-\Delta t\dot{r}(t\right.$$

When this formulation for δ is substituted back into the two equations of motion, the following form results

$$\left[(Y_{\dot{v}}-m)\mathcal{D}+Y_v\right]v + \left[(Y_{\dot{r}}-mx_G-\Delta t k_2 Y_\delta)\mathcal{D}^2 + (Y_r - mu_o + k_2 Y_\delta - \Delta t k_1 Y_\delta)\mathcal{D} + k_1 Y_\delta\right]\psi =$$

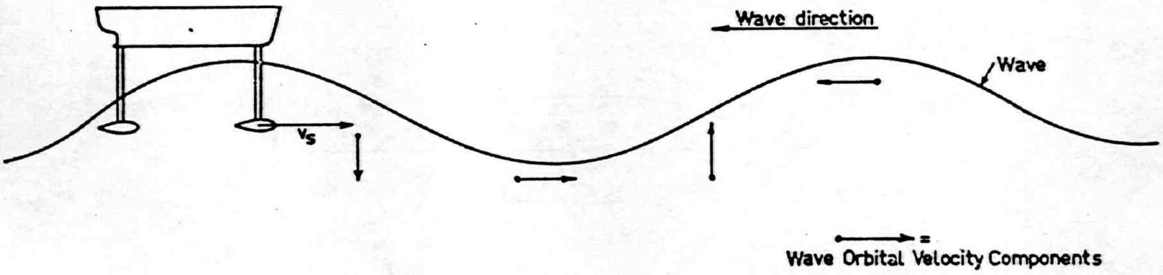
$$\left[(N_{\dot{v}}-mx_G)\mathcal{D}+N_v\right]v + \left[(N_{\dot{r}}-I_z-\Delta t k_2 N_\delta)\mathcal{D}^2 + (N_r - mx_G u_o + k_2 N_\delta - \Delta t k_1 N_\delta)\mathcal{D} + k_1 N_\delta\right]\psi =$$

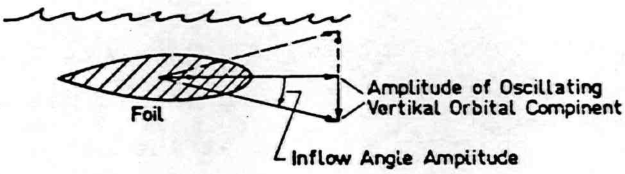
The above equations can be solved in the usual manner and the three roots of the cubic in $\mathcal{D}$ can be obtained or the criteria for stability can be tested on the values of A_1, B_1, C_1, and D_1. Since the rudder is usually located at the stern, a positive rudder deflection produces a negative N moment (and vice versa), hence N_δ is negative and for proper automatic control k_1 is a positive quantity. It can be seen that the term $-\Delta t k_1 N_\delta$ is positive in quantity and detracts from the negative value of N_r, the damping coefficient. Recalling that a large negative value of N_r encourages stability, it can be concluded that a system with relatively large time lag, at the time it introduces a sensitivity to direction, can also tend to degrade the dynamical stability. In the case above, designing the controls sensitive to r can introduce the term $k_2 N_\delta$ to compensate for the lag effect. The additional term $-\Delta t k_2 N_\delta$ is rather insignificant compared to the large and relatively insensitive quantity $N_{\dot{r}} - I_z$.

Sloppy control systems with unnecessarily large time lags are undesirable. The time lag Δt in any system is rather difficult to ascertain, but certainly depends on the rudder rate. It may depend on the variable ψ (or its derivatives) and in that case non-linear equations result. The purpose of discussing time lags is to indicate qualitatively, rather than quantitatively, the effect of such lags on the performance of the automatic control and the motion stability of the ship.

A more accurate and realistic, but much more complicated, analysis of the lags in control mechanisms can be accomplished by writing the equations which describe the actual operation of the mechanism. For example, the electrical equation of the build up of voltage (or amperage) as the result of the gyro measurement of the deviation, the equations describing the actual method of amplification of the signal to produce the power to activate the rudder motor, the equations to describe the electro-mechanical response of the electric motor activating the rudder system, and the equations of motion of the rudder system. These equations are then coupled with the ship motion equations and the overall response analyzed. The results will give a complete test of the stability of the overall system, ship and controls. The controls themselves, if not proper, can introduce instability into the system. The complete control analysis is rather complicated with several more unknowns and equations, resulting in a greater number of roots.

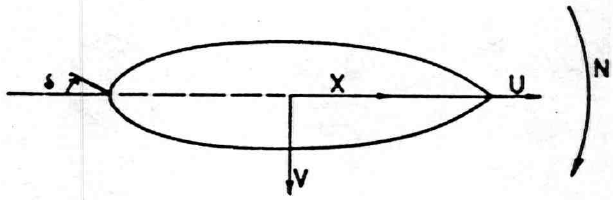
Time lags in control systems become rather important in high speed craft subject to relatively large disturbing forces at sea. Such a vehicle is the hydrofoil boat running into good size waves. The excitation of a hydrofoil boat, with completely submerged foils, is brought on by the varying angle of attack on the foils resulting from the combination of the forward motion of the foil and the vertical component of orbital motion of the water below the wave surface, as demonstrated in the sketch below.



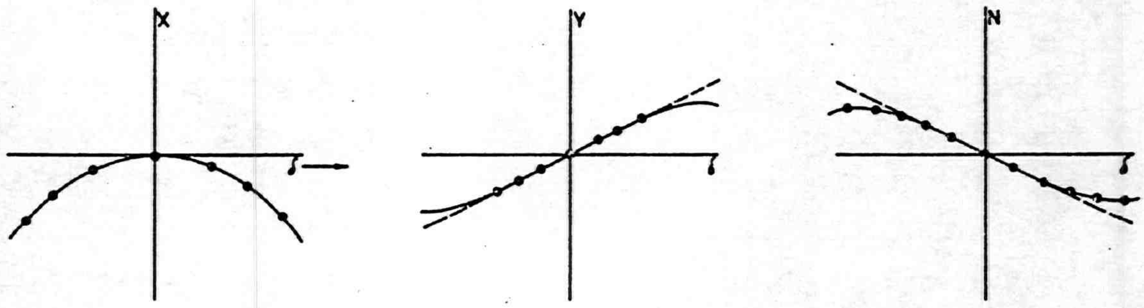


The inflow angle varies approximately sinusoidally in the period of encounter. The period of encounter is rather short in time since, in head seas, the relative speed between the wave and ship is the sum of the ship speed plus the wave speed. Since the hydrofoil, for efficient lift-drag considerations is designed for lifting the weight of the hull with a small angle of attack, small changes in angle of attack brought about by the orbital velocity produce significant exciting forces tending to oscillate the boat in pitch and heave. With such a high speed of encounter, excitations occur so fast that even small time lags still are sufficient to compromise the automatic control based on sensing ship motion parameters. By using a device (acoustic type - or radar principle) that measures the oncoming wave before it excites the ship and coupling this measurement with a control system that deflects the foil by the time the wave excitation reaches the foil, the control system can be made effective. In this case, the measuring or sensing system has enough lead time designed into it to cancel the effect of lag time.

In discussing the effect of controls, the control derivatives X_δ , Y_δ , and N_δ were used to represent the linearization of the force and moment functions resulting from the control surface deflection. The values for these derivatives, for any given design, can be estimated from ship model tests, in a similar fashion as was used to evaluate the derivatives Y_v and N_v. The model (usually with propellers operating) is towed in the towing tank, in straight ahead motion, and a dynamometer capable of measuring resistance, sideforce, and moment, measures these components for various settings of the rudder angle δ. The following is a sketch of such a test.



The measured forces and moments are then plotted versus δ and the slope, taken at $\delta = 0$, will furnish the values of the derivatives. If the rudder (as is usual) is located at the stern, the plots will look something like these shown below.



From symmetry considerations, X is zero. The nature of the forces indicate a positive slope, Y_δ, and negative slope, N_δ, for a rudder located aft. Since rudder forces depend on the inflow velocity, and therefore on any race effects of the propellers, it is more realistic to conduct such tests with model propellers rotating at the ship's corresponding point. The values obtained for the control derivatives may suffer somewhat from scale effect since the rudder operates in the wake region behind the hull. Certain corrections can be estimated to account for the expected difference in wake between the model and the full scale ship. On the other hand, it may be possible to estimate the value of the derivatives based on a lifting surface sufficiently removed from the origin by the use of hydrofoil theory as was demonstrated earlier. An estimate of wake and propeller race effects is necessary with this approach.

Although the low Reynolds' number of the model may not significantly affect the value of the control derivative, the range of linearity of the Y and N versus δ curves (i.e. value of δ when curve departs from the tangent) may be somewhat lower at model Reynolds' number. This results from the fact that separation of flow on a foil will occur at a lower angle of attack at the lower Reynolds' number. In linear theory, there is little concern about this difference. However, in handling the non-linear equations, some consideration should be given to the possible existence of this form of scale effect. Also in dealing with non-linear effects, consideration should be given to the fact that in a maneuver the angle of attack on the rudder is a combination (addition) of the rudder deflection, the drift angle $\frac{v}{U}$, and the inflow angle caused by the angular velocity, $\frac{rd}{U}$, (d is distance the rudder is from the origin).

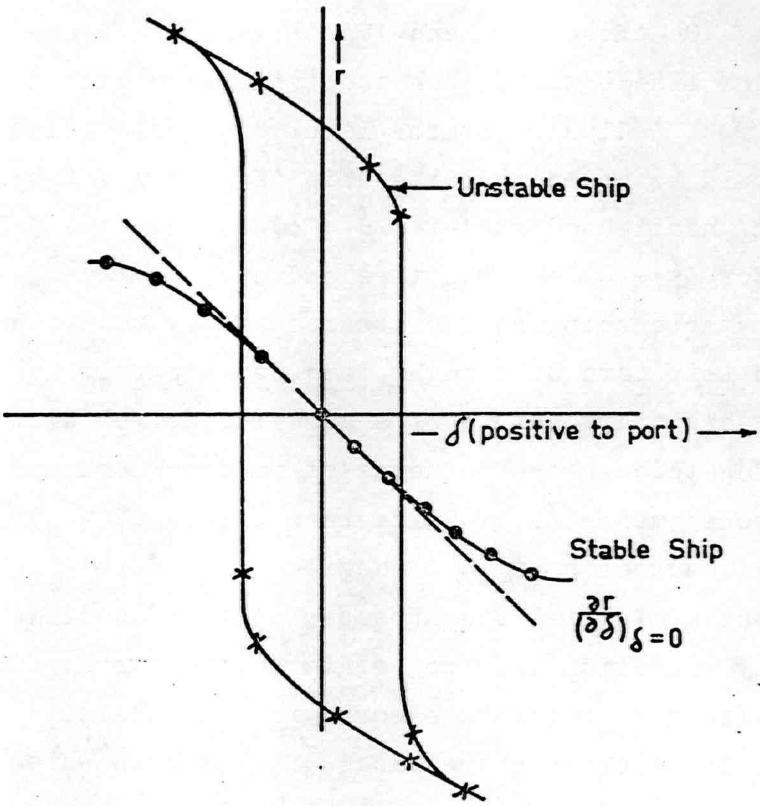
CHAPTER IX

Full Scale Trials for Evaluation of Steering and Maneuvring Characteristics

The theory of dynamic stability and control has been discussed and methods of evaluating these items for a given ship design have been described, using model tests and/or theoretical procedures. The determination of the stability and maneuvering qualities of the ship, as built, requires certain full scale trials and the nature of these trials will be briefly discussed.

Just as in the domain of resistance and powering of ships, where towing the full size ship to find its resistance is out of the question, it is virtually impossible to conduct tests on the ship which duplicate the model tests in order to determine the various hydrodynamic derivatives for the full size ship. Such restrained body tests as planar motions, rotating arm, etc., on the full size ship is inconceivable. The nature of the tests for the ship must necessarily involve an unrestrained, free running ship. Just as the first ship trial is an inclining experiment to determine the value of the metacentric height and the determination of the ship's stability in roll, an analogous type of trial referred to as the spiral maneuver is designed to determine the dynamical stability of the ship.

The spiral maneuver consists of the following operation of the ship. The ship executes a large rudder deflection to one side, say 25 degrees rudder to starboard. The rudder is held in this position until a constant angular velocity is obtained and this angular velocity is recorded. The rudder deflection is then reduced to say 20 degrees starboard and held until a steady angular velocity is reached and recorded. This procedure is continued down through 0 deflection and continues up through say 25 degrees port rudder. Although 5 degree intervals are desired at the large rudder deflections, deflection increments of 1 or 2 degrees are desired in the range between 5 degrees starboard and 5 degrees port rudder. The process is then repeated starting with say 25 degrees port rudder and ending up at 25 degrees starboard rudder. A plot of the measured angular velocity versus the rudder deflection is made as shown below.



If the ship is stable, then for each rudder deflection there is only one steady state angular velocity (i.e. one turning radius) and the curve appears as that indicated by the "stable ship" label in the above figure. Since, for small rudder deflections linear theory holds, the solution of the linearized equations for a rudder deflection should give us a prediction of the slope of the r versus δ curve of the trial (i.e. $(\frac{\partial r}{\partial \delta})_{\delta=o}$. Since the solution for r was

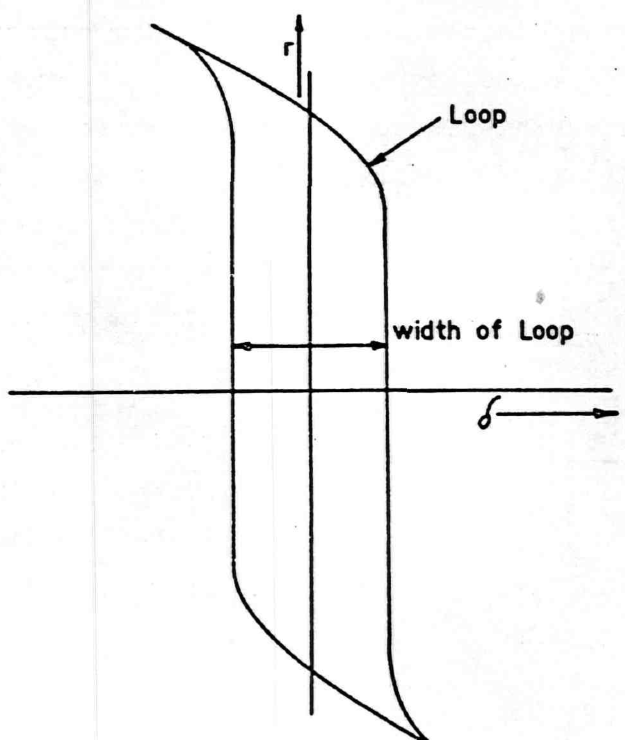
$$r = \left[\frac{N_v Y_\delta - Y_v N_\delta}{Y_v(N_r - mx_G u_o) - N_v(Y_r - mu_o)}\right] \delta_o$$

then

$$(\frac{\partial r}{\partial \delta})_{\delta=o} = \frac{N_v Y_\delta - Y_v N_\delta}{Y_v(N_r - mx_G u_o) - N_v(Y_r - mu_o)} .$$

The solution of the linear equations gives r as a linear function of δ. Hence, the range of validity of the linear theory is determined by observing just where the curve of r vs. δ departs from the tangent as drawn to this curve at $\delta = 0$. The use of non-linear equations and their solution is necessary to predict the curve of r versus δ at the larger rudder deflections.

On the other hand, if the ship is dynamically unstable in straight line motion, then the typical result of the spiral tests will be as that indicated by the label "unstable ship" in the figure above. It should be noted that the ship is unable to go straight ahead (r = o) for an undeflected rudder (δ = o) but has two values of angular velocity for δ = o, one positive and one negative. This indicates that the unstable ship, in the absence of any rudder deflection, may go into a left turn or a right turn, on a purely arbitrary basis depending on the arbitrary nature of any infinitesimal disturbance. The process of going through the sequence of rudder increments twice, including a reverse sequence (i.e. 25 degrees starboard to 25 degrees port and then from 25 degrees port to 25 degrees starboard) is needed in order to set up opposite initial disturbances for the condition of δ = o in order to clearly establish the two points on the curve. This reverse procedure establishes a loop in the curve for the unstable ship. The width of this loop determines the range of δ within which the ship may turn against its rudder. Hence the width of the loop, as shown below, is an indication of the magnitude of the instability of the ship.



When the linearized equations of motion, with rudder deflection, were solved, it was indicated that the solution for the case of the unstable ship was rather useless in that no steady state angular velocity was obtained since the transient terms involving $e^{\sigma_1 t}$ and $e^{\sigma_2 t}$ did not go to zero in time but increased. Hence, for the unstable ship, linear theory is unable to predict the nature of the loop but it can indicate whether a loop will exist (because of instability) and give some information as to the relative width of the loop. Non-linear theory and equations and their solutions are required in order to predict the size and shape of the loop for the unstable ship.

Some understanding of the significance of the spiral tests and the loop can be obtained by making an analogy between the spiral test plot and the curve of statical stability for a ship. The curves below are analogous - a righting or heeling moment (or arm) is the ordinate in the curve of statical stability and a minus rudder deflection*) (or positive rudder turning moment since with rudder at stern N_δ is negative) is the ordinate in the curve resulting from the spiral tests (have rotated the axis from the previous figure).

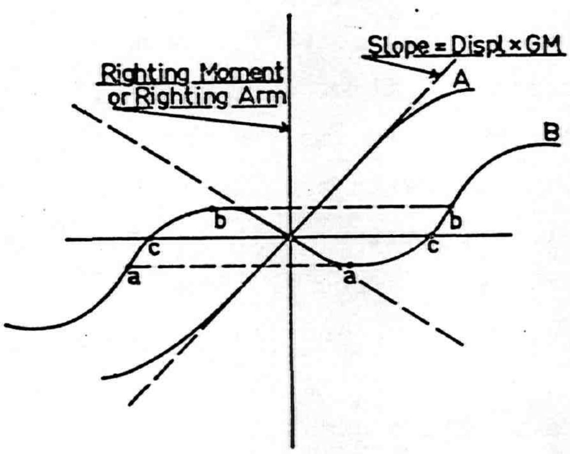


FIG.1

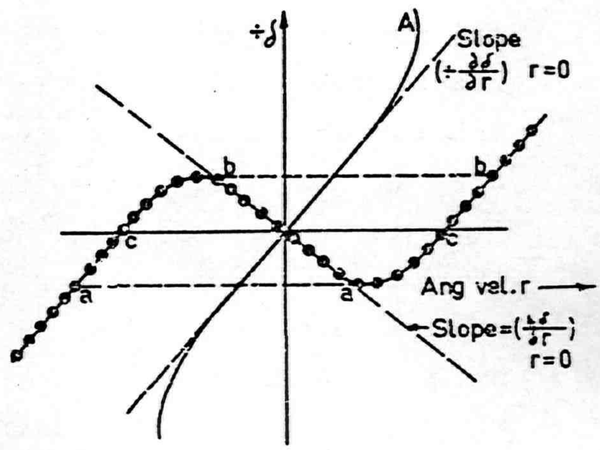


FIG. 2

In the case of the curve of statical stability the abscissa is the angle of heel φ , produced by the heeling moment, whereas in the spi-

*) Since for a rudder at the stern, a deflection to starboard is negative δ and yet an angular velocity of the ship to starboard is a positive r, then labelling the positive ordinate "rudder deflection to starboard" and the positive abscissa "angular velocity to starboard" will produce the proper graph.

ral tests the abscissa is the angular velocity resulting from the turning force and moment produced by the rudder deflection.

In the case of the ship stable in heel, curve A in Figure 1, the slope of the righting moment versus φ (at $\varphi = 0$) curve is positive, indicating stability, and, since this slope equals the ship's displacement multiplied by the metacentric height, it represents a positive metacentric height which is the criteria for stability in heel. Similarly, for the ship which is dynamically stable in straight line motion, curve A in Fig. 2, the slope of the $-\delta$ versus r (at r = 0) is positive - i.e. $\left(\frac{-\partial\delta}{\partial r}\right)_{r=0} > 0$. From the solution of r versus δ from the linearized equations of motion, we have for the equilibrium, or steady turning condition, for a given rudder deflection,

$$-\delta = \left[\frac{Y_v(N_r - mx_G u_o) - N_v(Y_r - mu_o)}{Y_v N_\delta \quad - N_v Y_\delta}\right] r$$

$$\left(\frac{-\partial\delta}{\partial r}\right)_{r=0} = \left[\frac{Y_v(N_r - mx_G u_o) - N_v(Y_r - mu_o)}{Y_v N_\delta \quad - N_v Y_\delta}\right].$$

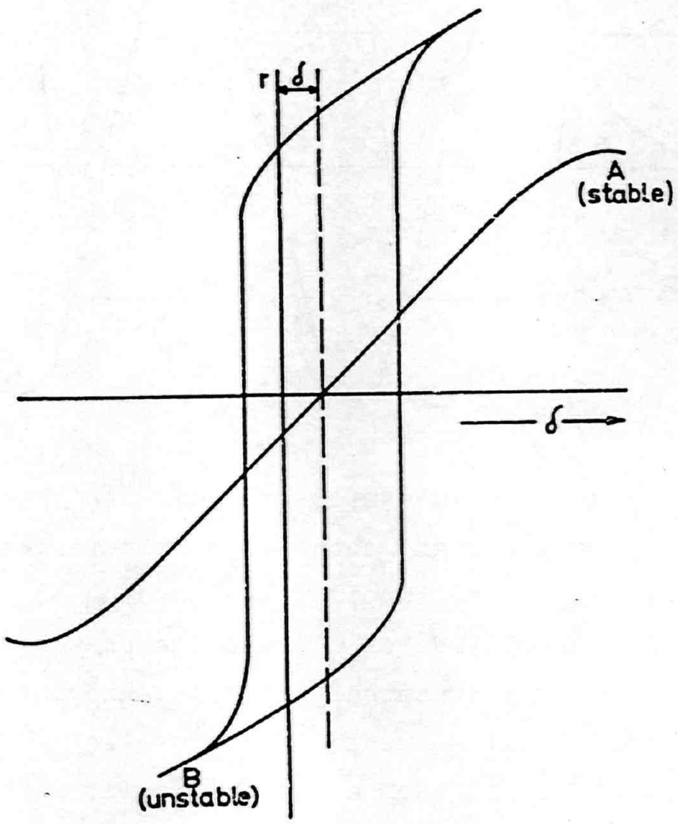
Since Y_v is negative and since N_δ is negative for a rudder at the stern (and this analysis has been developed for a rudder at the stern), the product $Y_v N_\delta$ is positive. Also, since for practically all ships N_v is slightly negative and since Y_δ is positive, the product $N_v Y_\delta$ is negative. Hence, the two terms in the denominator add to assure that the denominator, (in the above equation) is a positive quantity. (Very seldom, for ships, if ever, is N_v a positive quantity. If in some extreme case it is, it is only slightly so and the first term in the denominator will be predominant, thereby causing the denominator to be positive). With the denominator a positive quantity, whether the slope $\left(\frac{-\partial\delta}{\partial r}\right)_{r=0}$ is positive or negative depends on whether the numerator is positive or negative. But the numerator is exactly the criteria for dynamical stability in straight line motion as was derived from the solution of the homogeneous linear equations of motion. The criteria for stability, as previously developed, is that $Y_v(N_r - mx_G u_o) - N_v(Y_r - mu_o)$ be positive. Hence, if this term is positive, then the slope $\left(\frac{-\partial\delta}{\partial r}\right)_{r=0}$ is positive, and a positive slope of the curve of $-\delta$ versus r (for rudder aft) indicates stability of straight line motion. It has been demonstrated that a rather direct analogy exists between the curve of statical stability and the curve of $-\delta$ versus r (results of spiral trials) in that the criteria for stability, that GM be positive, determines the nature of the initial slope of the curve of statical stability and the criteria for dynamical stability determines the nature of the slope of the curve of $-\delta$ versus r. Similarly, if the ship is unstable, curve B in Figures 1 and 2, the slope at the origin of each of the B curves is negative, resulting in the expression in the criteria becoming negative, a negative GM in the case of the curve of statical stability and a negative value for $Y_v(N_r-mx_Gmu_o)-N_v(Y_r-mu_o)$ in the curve of $-\delta$ versus r. The larger the negative values of these slopes, the more unstable the ship is.

In the case of the stable ship (A in Figs. 1 and 2) there is only one resulting angle of heel for any given heeling disturbance and there is only one angular velocity (or turning radius) for any given rudder deflection. In the case of the unstable ship (B in Figures 1 and 2), there are regions, (between the lines aa_1 and bb_1) where there is more than one equilibrium angle of heel for a given heeling moment in the case of Fig. 1 and more than one turning angular velocity for a given rudder deflection in the case of Fig. 2. For the unstable ship in roll, there is a region in which a ship can heel against the direction of the heeling moment. Similarly, for the dynamically unstable ship, there is a region in which the ship can turn against its rudder.

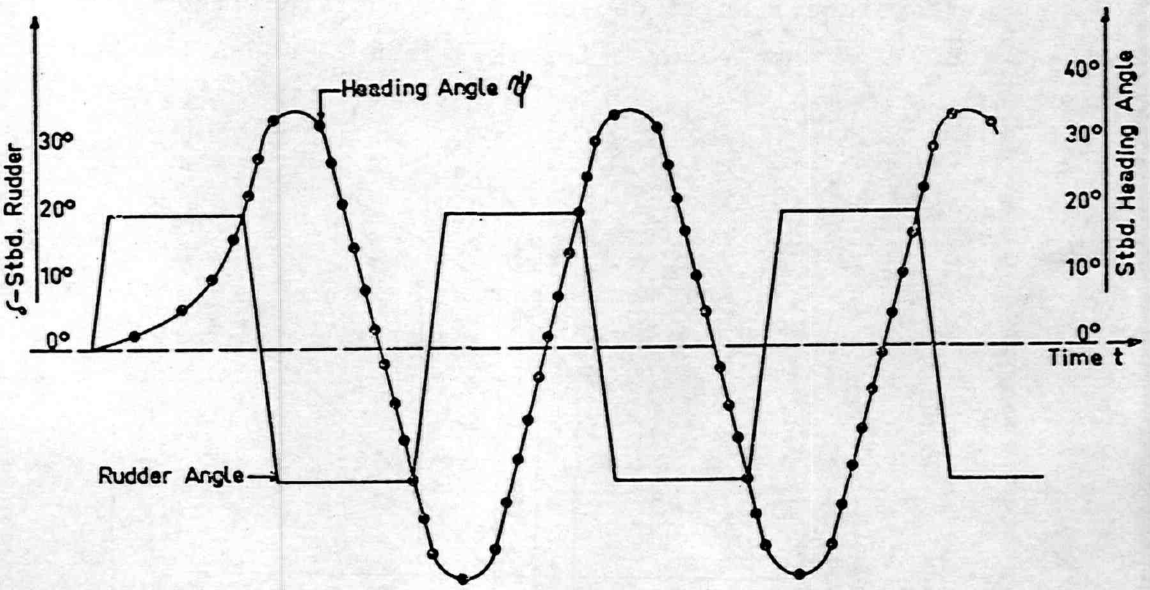
During the spiral tests, no data could be obtained for the unstable ship in the region between a and b on curve B in Figure 2 (marked by points, O) even though these are points of equilibrium for the given rudder deflections. The reason for this is that these are points where the equilibrium is unstable and the ship cannot remain at the unstable equilibrium position but moves to a position of stable equilibrium for that rudder deflection. Hence in the absence of any turning disturbance at $\delta = 0$, the ship cannot go straight ahead ($r = 0$) since this equilibrium condition is unstable, and therefore it goes into a turn to either port or starboard ending up at a turning angular velocity (turning radius) denoted by points c or c_1 (on curve B Figure 2), which are <u>stable</u> equilibrium conditions for zero rudder deflection. (Note that the slope of the $-\delta$ versus r curve at these points are positive i.e. stable equilibrium). For a dynamically unstable ship with a large rudder deflection $-\delta$, the ship has an angular velocity denoted by point d on curve B of Fig. 2. As the rudder angle is reduced, the angular velocity is reduced following curve B, until at zero rudder deflection an angular velocity indicated by point c is obtained. On further reducing the rudder angle (i.e. deflecting the rudder to the other side), we find that the ship still continues to turn in the original direction against the direction of the rudder deflection until a rudder deflection and angular velocity indicated by 'a' on curve B. Any further change of the rudder angle will cause the ship to shoot over to a large angular velocity in the opposite direction as indicated by point a_1 (and perhaps temporarily overshoot a_1). Hence the unstable ship can turn against its rudder up to a point and then suddenly it will swing to the other direction to a stable condition for the rudder deflection. Similarly, as a spiral test progresses from point c_1 to b further increase in rudder angle causes the ship to quickly swing as fast as its inertia will let it to an angular velocity indicated by b.

This type of behavior of the dynamically unstable ship is analogous to the ship which is unstable in roll. The unstable ship cannot remain upright even in the absence of a heeling disturbance. It will heel either to port or starboard reaching an angle of heel (for no disturbing moment) indicated by either c or c_1 on curve B Figure 1, which are positions of equilibrium (slope of the curve is positive at these points but negative at $\varphi = 0$). For a ship, heeled at the angle to starboard denoted by point c, if a port heeling moment is applied the angle of heel is reduced, but the ship is now heeled against the direction of the heeling moment. When this port heeling moment is increased the heel angle moves along curve B until point a is reached. Any further increase in the heel moment to port will cause the ship to swing quickly from point a to point a_1, which is a stable position of large heel to port. (Of course, the heel angle will overshoot a_1 but will finally settle down at a_1). Hence, the ship behaves in its inclining experiment in a fashion quite analogous to that of the ship in its spiral tests. No points in the unstable region between a and b, on curve B Figure 1, can be obtained during the inclining experiment.

For the case of a single screw ship wherein the direction of rotation of a single propeller in any asymmetry which produces a turning moment (and slight side force) on the ship, a certain amount of rudder deflection is required to overcome this moment and produce an equilibrium condition. The results of any spiral tests on this type of ship would cause a shift of the curve of r versus δ in the δ direction by the amount of rudder angle necessary to counteract the single screw effect (δ_e). A sample curve for this case is shown below.



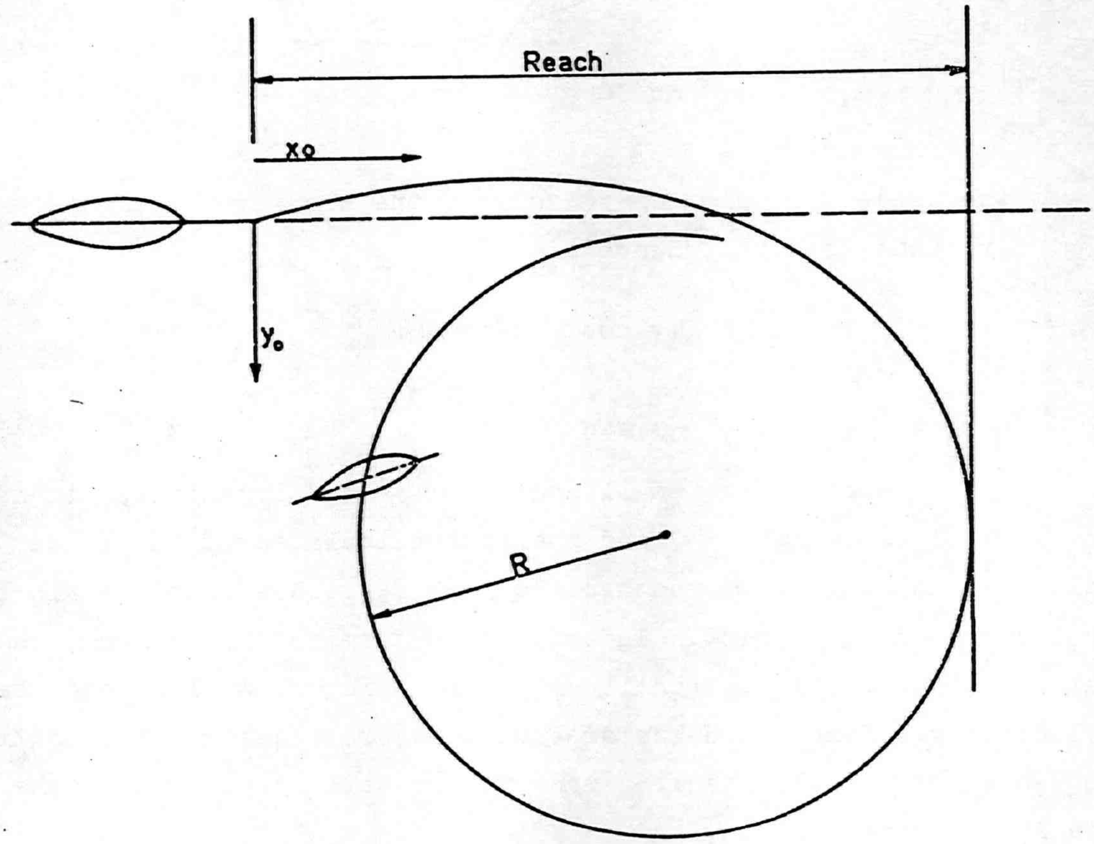
Another type of ship maneuvering trial is the Z maneuver consisting of the following steering sequence. With the ship proceeding at speed with zero rudder, the rudder is deflected to say 20 degrees to starboard (numbers are nominal and can be chosen to best suit the individual tests) and is held in this position until the ship has changed its heading to say 20° starboard. At this point the rudder is then suddenly deflected 20 degrees to port and held in that position until the ship reaches a heading of 20° to port at which time the rudder is suddenly deflected to 20° starboard. This sequence is repeated through several cycles, and the rudder angle and heading angle are recorded as a function of time during these maneuvers. The following sketch indicates typical measured response for the trials described above.



The z maneuver measures or indicates the ability of a ship to rectify the motion, brought about by a maneuver, by a countermaneuver. Hence, the degree of overshoot of the heading angle curve (ratio of amplitude of ψ to amplitude of δ) and the phasing between the two curves are indicative of the dynamical stability and maneuverability of the ship. One notices the almost sinusoidal response in ψ for this maneuver. The response of the ship to such a rudder program may be predicted from the sinusoidal rudder oscillation equations and solution which was indicated previously. Or a more accurate calculation can be made by solving the equations for a square wave input. The linearized equations should give a fairly good estimate of the ship response provided moderate rudder angles and changes of heading are used in the trials.

A ship which is slightly unstable can turn rather quickly to starboard for application of starboard rudder (although it has an ability to turn against the rudder for certain small rudder deflections). Since it turns in the absence of any rudder, a first maneuver can be accomplished somewhat more quickly by the less stable ship. However, in a second countering maneuver it takes more time for the less stable ship to pull out of its original turn and go into the turn called for by the second maneuver. Hence the less stable ship tends to overshoot more during a Z maneuver. The ability to react properly during countering maneuvers is important when sailing in restricted waters (harbor or canal) in that in taking action to avoid one situation, the ship may find itself in a second situation from which it cannot pull out in sufficient time. Hence, besides desiring dynamical stability for the purposes of wanting to go straight when no rudder angle is applied, one also desires to have such stability to improve the ship's ability in countermaneuvers.

Of course the normal steering trial of ships is well-known, wherein the ship proceeds at speed in straight line and then suddenly deflects the rudder and the resulting turning trajectory is measured. A typical ship's path for such a maneuver is shown below.



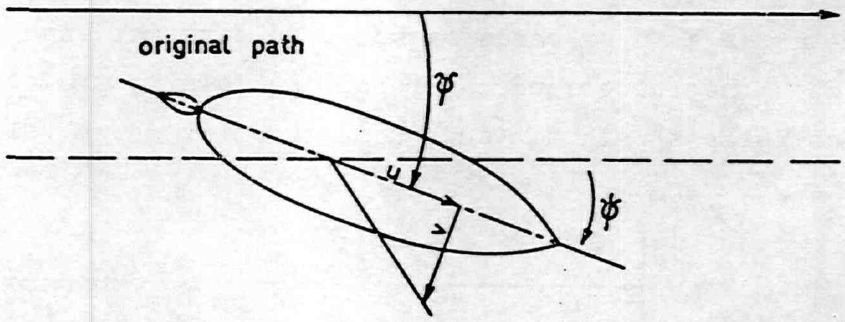
The trajectory of the ship y_o versus x_o is plotted and from this the radius of turn, R, and the reach is determined. For small rudder angles (not too tight a turn) the linear theory may give a pretty good prediction of the turning circle radius. Linear theory does not give the speed reduction in the turn nor can it readily give the trajectory.

The rate of motion sidewise to the original path is given by $\frac{dy_o}{dt} = \dot{y}_o$ and along the direction of the original path by $\frac{dx_o}{dt} = \dot{x}_o$. It can be seen from the sketch below that

$$\dot{y}_o = u \sin\psi + v \cos\psi$$

$$\dot{x}_o = u \cos\psi - v \sin\psi$$

where ψ is the heading angle of the ship relative to the original path.



To get the trajectory plot of y_o vs. x_o, one needs to integrate $\dot{y}_o$ and $\dot{x}_o$ with respect to time. Hence

$$y_o = \int (u \sin\psi + v \cos\psi)\,dt$$

$$x_o = \int (u \cos\psi - v \sin\psi)\,dt\,.$$

If one considers small angles of ψ and linearizes, then $\sin\psi \approx \psi$, $\cos\psi \approx 1$, and the path calculated by the linear theory would only hold until ψ reached about 15 degrees and then would be quite invalid beyond this point. In order to determine the "reach", the integrals must be carried out until $\psi = 90^o$ which is well beyond the range of linearity. Hence, non-linear equations are necessary for estimating the trajectory of the steering maneuver as well as predicting the speed loss in the maneuver.

CHAPTER X

Non-Linear Equation of Motion

The usefulness and the limitations of the linear equations for steering, stability, and maneuvering and their solutions are mentioned previously. It was indicated that certain important information in the prediction of certain ship maneuvers would require the use and solution of non-linear equations of motion. The nature or form of these non-linear equations will be developed and discussed from the point of view of certain hydrodynamic aspects. First let us consider the equation for the X force. The linear equation was

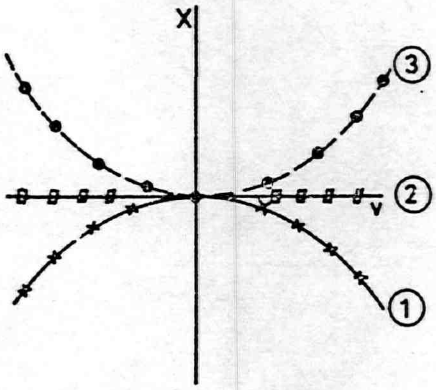
$$X_{\dot{u}}\dot{u}+X_u\,\Delta u+X_{\dot{v}}\dot{v}+X_v v+X_{\dot{r}}\dot{r}+X_r r+X_\delta \delta = m\dot{u}.$$

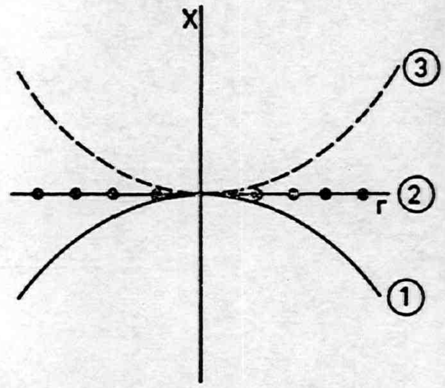
On the right hand side of the equation, the non-linear terms, as previously developed in the general equations of motion, become

$$m\left[\dot{u}-rv-x_G r^2\right]$$

for the condition of motion in the horizontal plane, without roll, with $y_G = 0$ (G on centerline plane).

It was shown, previously, that the derivatives X_v, X_r, X_δ, $X_{\dot{v}}$, and $X_{\dot{r}}$, were all zero because of the symmetry of port and starboard (present the same shape to the flow). Because of these symmetry properties, it was shown that the functions, from which each of the above derivatives were obtained, had to be even functions of the variables, that is, the graph was symmetrical relative to the ordinate (X) axis, as indicated by the sketches below, (general types of curves 1, 2, or 3.





Similar symmetrical graphs result for the X force as functions of δ, $\dot{r}$, and $\dot{v}$. Let us take as a typical case, the curve of X versus v. If X is to be expressed by an expansion in powers of v beyond the first power - i.e. in non-linear polynomial form - then, because X is an even function of v, only the even powers of v can appear in the expansion and the coefficients of the odd powers must be zero. Hence X as a function of v takes the form

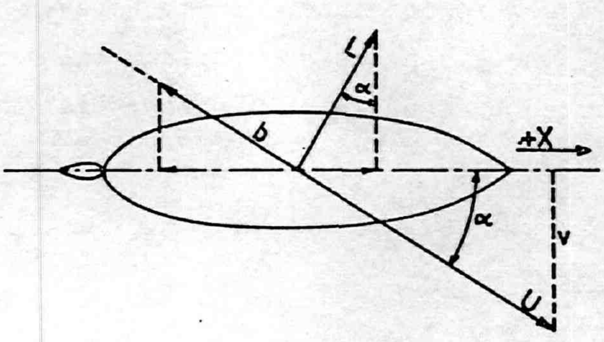
$$X(v) = a_2v^2+a_4v^4+a_6v^6+........$$

where the a's are coefficients depending on body shape.

For a given u_o, as v is changed, the angle of attack of the flow to the hull changes with v. From the hydrodynamics of low aspect ratio foils, when a body is placed at an angle of attack to the flow, in addition to a lift force, there is also created an increase in drag resulting from an increase in profile drag and the production of induced drag due to flow over the foil edge (in this case - the keel). The X force in terms of lift (L) and drag (D) can be expressed as

$$X = L \sin\alpha - D \cos\alpha$$

and $\alpha = \sin^{-1} \frac{v}{U}$.



Since the lift is linear in α for small α and since the drag varies approximately as the square of α, the X versus v sould initially take the form of a quadratic (and perhaps higher even powers), probably of the shape indicated by 1, since it is expected that the drag force term is the larger of the two terms. Since an angular velocity, r, produces a local flow approach angle $\frac{rd}{u_o}$ at a distance d from the origin, there will be "localized L and D" at each point with a net force on the body from the integral effect. Hence, the nature of the X versus r curve will be similar to those of the X versus v curve because similar hydrodynamic effects are present. The curve of X versus r is an even function*) because of symmetry, and, because of the reasons just mentioned, the curve will probably have the form indicated by 1 in the sketch above. Similar reasons hold for the X versus δ curve. Hence we have

$$X(r) = b_2r^2+b_4r^4+b_6r^6+\ldots\ldots$$

$$X(\delta) = c_2\delta^2+c_4\delta^4+c_6\delta^6+\ldots\ldots$$

where the b and c are various coefficients depending on the body shape.

Actually, in the general situation, δ, v, and r vary at the same time. The localized flow approach angle along the hull is $\frac{v}{U}+\frac{rd}{U}$, and this angle at the rudder is $\delta+\frac{v}{U}+\frac{rd}{U}$. Considering the symmetry of port and starboard (calling for even power expansion) and the hydrodynamic aspects discussed above), the form of X as a function of δ, v, and r will be the even power expansion of the effective sum of these parameters as shown below.

$$X\ (r,v,\delta) = (k_1r+k_2v+k_3\delta)^2+(k_1r+k_2v+k_3\delta)^4+\ \ldots\ldots$$

*) Even functions can be produced using absolute values of odd powers, such as

$$a_1|v|+a_2v^2+a_3|v^3|+a_4v^4+\ldots\ldots$$

but the term $|v|$ makes the function discontinuous at v = 0 and the use of absolute values seems unnatural and further complicates computation.

The expansion gives terms in

$$r^2, v^2, \delta^2, rv, r\delta, v\delta, r^4, v^4, \delta^4, r^3v, + \ldots\ldots$$

For the purpose of calculating or predicting ship maneuvers, it is sufficient to carry the non-linearity through the 3rd order, or cubic term. Hence, the 4th order terms are to be dropped and the non-linear terms of interest are

$$r^2, v^2, \delta^2, rv, r\delta, v\delta .$$

Since the hydrodynamic forces contributing to X are, in part, attributable to angle of attack situations, and since the angle of attack and hydrodynamic forces depend on the forward velocity u, the forces arising from r, v, and δ may vary as u varies. If for a constant velocity u_o, X varies, say, as kr^2 (where k is a constant), then for a different velocity $u_o + \Delta u$ (and therefore a different Froude number, $\frac{u}{\sqrt{gL}}$, the force at a given r will change as u changes. If this change in force brought about by a change in forward velocity component, Δu, is expressed as a power series expansion in Δu, one obtains

$$X = kr^2 \left[1^{*)} + k_1 \Delta u + k_2 (\Delta u)^2 + \ldots\ldots \right]$$

Since only terms through the cubic are to be retained, the above expression reduces to

$$kr^2(1+k_1 \Delta u) = kr^2 + kk_1 r^2 \Delta u = kr^2 + k_2 r^2 \Delta u$$

where the k's are constants depending on ship geometry and size. The term r^2 was taken as an example. Similar terms will be derived from considering v^2, δ^2, rv, $r\delta$, and $v\delta$. (For example, one obtains terms $k_3 \delta v + k_4 \delta v \Delta u$). The non-linear equation in X (through the cubic term) will include terms in

$$v^2, r^2, \delta^2, rv, r\delta, \delta v, v^2 \Delta u, r^2 \Delta u, \delta^2 \Delta u, rv\,\Delta u, r\,\delta\,\Delta u, \text{ and } \delta v\,\Delta u.$$

In the linear theory, $X_u \Delta u$ gives the variation of X with Δu. The variation of the X force with Δu can be expressed as a power expres-

*) This 1 takes care of the value when $u = u_o$, i.e. when $\Delta u = 0$.

sion in Δu; hence, one expects terms in Δu, $(\Delta u)^2$, $(\Delta u)^3$, etc. Such a form can be shown to exist for the actual ship. The X force on a ship moving straight ahead is the difference between the propeller thrust (working with the thrust deduction) and the resistance (drag) of the ship. In equation form this becomes

$$X = K_t \rho n^2 d^4 (1-t) - 1/2\ C_R \rho S u^2$$

where

K_t is the propeller thrust coefficient

d is the propeller diameter

ρ is the water density

n is the revolutions per second of the propeller

t is the thrust deduction factor

C_R is the resistance coefficient

S is the wetted surface of the ship

u is the forward speed component of the ship

The coefficient K_t is a function of the speed coefficient, $\mathcal{J} = \frac{(1-w)u}{nd}$, where w is the wake fraction. The resistance coefficient, C_R, is a function of u (or the Froude number, $\frac{u}{\sqrt{gL}}$). The K_t vs. $\mathcal{J}$ curve is quite linear in the region of the design speed. Hence, the change in K_t with u, in the region of a speed change caused by a maneuver, can be expressed as $\frac{\partial K_t}{\partial \Delta u} \Delta u = \frac{\partial K_t}{\partial \mathcal{J}} \frac{\partial \mathcal{J}}{\partial \Delta u} \Delta u$.

Similarly, let us assume, and it is sufficiently valid for the Δu occurring in a maneuver, that the variation in C_R with u (i.e. with Froude number) can be expressed by $\frac{\partial C_R}{\partial \Delta u} \Delta u$, (this means the slope of the curve multiplied by the change in u). Assuming that the revolutions of the propeller do not change during a maneuver, the expression for the X force for a disturbance, Δu, becomes (assuming t does not vary with Δu)

$$u_o^2 + 2u_o \Delta u + (\Delta u)^2$$

$$X(\Delta u) = \left[(K_t)_o + \left(\frac{\partial K_t}{\partial \Delta u}\right)_{u=o} \Delta u \right] \rho n^2 d^4 (1-t) - \left[(C_R)_o + \left(\frac{\partial C_R}{\partial \Delta u}\right)_{u=o} \Delta u \right] 1/2\, \rho S (u_o + \Delta u)^2$$

where $(K_t)_o$ is the thrust coefficient at $u = u_o$ and $(C_R)_o$ is the resistance coefficient at $u = u_o$. Since at the speed u_o, the ship is in an equilibrium condition, i.e. constant speed, then

$$(K_t)_o \rho n^2 d^4 (1-t) - (C_R)_o 1/2 \rho S u_o^{\ 2} = 0.$$

Also, $\frac{\partial J}{\partial \Delta u} = \frac{\partial}{\partial \Delta u} \frac{(u_o + \Delta u)}{nd} (1-w) = \frac{(1-w)}{nd}$ where w is the wake fraction.

The expression for $X(\Delta u)$ becomes

$$X(\Delta u) = \left[\left[\left(\frac{\partial K_t}{\partial J}\right)_{u=u_o} \left(\frac{1-w}{nd}\right)(1-t)\rho n^2 d^4\right] - 1/2 \rho S \left[2u_o (C_R)_o + \left(\frac{\partial C_R}{\partial \Delta u}\right)_{u=u_o} u_o^{\ 2}\right]\right] \Delta u$$

$$-1/2 \rho S \left[(C_R)_o + 2u_o \left(\frac{\partial C_R}{\partial \Delta u}\right)_{u=u_o}\right] (\Delta u)^2 - 1/2 \rho S \left[\left(\frac{\partial C_R}{\partial \Delta u}\right)_{u=u_o}\right] (\Delta u)^3.$$

Hence, this analysis indicates that a power function in Δu is required and the series is carried through the cubic term. In any practical example, the K_t vs. J curve for the ship's propeller along with the C_R versus u for the ship, can be used to estimate the coefficients of Δu, $(\Delta u)^2$, and $(\Delta u)^3$. The coefficients can also be estimated by running self propelled model tests, but consideration should be given to scale effects and the equivalent propulsion point of the ship. The X equation will therefore include terms in Δu, $(\Delta u)^2$, and $(\Delta u)^3$.

The X force also depends on the acceleration parameters $\dot{u}$, $\dot{v}$, and $\dot{r}$. In the linear equations, one obtained for the linear form

$$X(\dot{u}) \approx X_{\dot{u}} \dot{u}$$

$$X(\dot{v}) \approx X_{\dot{v}} \dot{v} = 0$$

$$X(\dot{r}) \approx X_{\dot{r}} \dot{r} = 0$$

since, as was shown previously, X_v and X_r were zero because of the symmetry of port and starboard. In considering the non-linear form for these parameters one would form a power series in these accelerations, i.e. $\dot{u}^2$, $\dot{u}^3$, $\dot{v}^2$, $\dot{v}^3$, etc. It is expected that the coefficients of the higher power acceleration terms will be zero (or negligibly small) from the following considerations.

The acceleration forces are essentially the result of the inertia property, i.e. density, of the fluid. Since there is no significant interaction between the inertial and viscous forces, it was indicated previously that calculation of the hydrodynamic acceleration forces by potential theory gave adequate values. Non-linear equations of potential theory when applied to submerged bodies give forces resulting from accelerations which are linear in the acceleration. Hence, one expects no second or higher order terms in the acceleration parameters. Also, on the right hand side of the general equations which represent the inertial properties of the body, acceleration terms appear only in the linear term.

In a general Taylor expansion of the variables to the higher order terms, terms of the form $\dot{r}v^2$, $\dot{u}rv$, etc. would appear as cubic terms. These terms represent interaction between acceleration and velocity parameters. Since these terms a) do not appear in the non-linear potential theory solution, b) are essentially interference between viscous and inertial aspects and c) do not appear on the right side of the equation (body inertial forces), then the coefficients of these type of terms are considered zero or at least negligibly small.

An additional argument is that, because of the magnitude of the inertia of the ship, during even tight maneuvers the accelerations remain small.

The expected form of the non-linear equation for X becomes

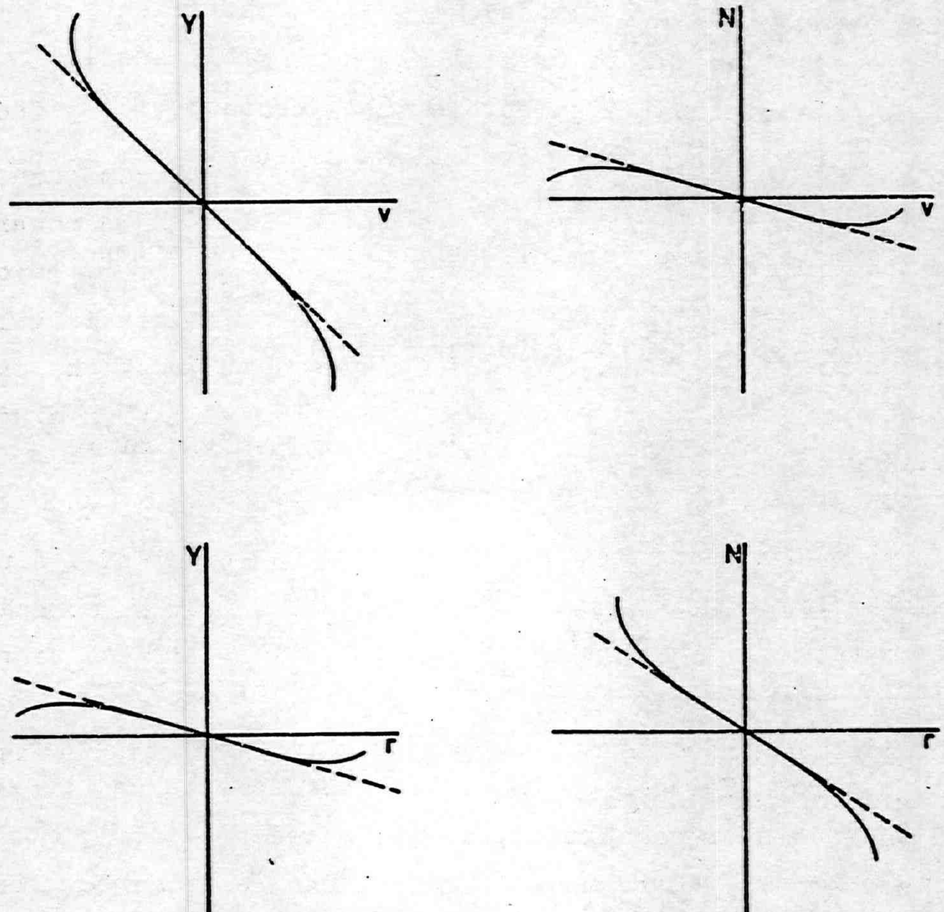
$$a\dot{u}+b\Delta u+c(\Delta u)^2+d(\Delta u)^3+ev^2+fr^2+g\delta^2+hrv+jr\delta+kv\delta+e_1v^2\Delta u$$

$$+f_1r^2\Delta u+g_1\delta^2\Delta u+h_1rv\Delta u+j_1r\delta\Delta u+k_1v\delta\Delta u = 0.$$

The coefficients a, f, and h include those terms brought over from the right hand side of the X equation.

Some of the coefficients in the above equation are small and difficult to measure in a model experiment or to calculate and little is lost from the accuracy of prediction if these coefficients are set equal to zero. Certain of the other coefficients can be readily measured on models or calculated, as indicated previously for the case of a, b, c, and d. The terms e, f, and g can be obtained from model tests by measuring the X force at various values of the parameter (v, r, and δ), and determining the value of the coefficient from the resulting curve (essentially parabolic). Similarly, by running these model tests at different speeds, the coefficients e_1, f_1 and g_1 can be determined. Considerations of inflow angle at the rudder may lead to an estimate of the coefficients j, k, and possibly j_1 and k_1 (although these terms may be negligibly small). (The counterparts of these terms in the Y and N equation can similarly be estimated).

The arguments for the form of the non-linear equations for Y and N are similar to those used in the X equation. With regard to the acceleration parameters only the linear terms will appear in the non-linear equations for the same reasons as were discussed for the acceleration terms in the X equation. The terms involving r, v, and δ , in the Y and N equations will be odd functions of these parameters, that is the graph of the function (say Y versus v) will be reflected about the origin. This results again from the symmetry of port and starboard of the ship. The curves of Y vs. v, N vs. v, Y vs. r, and N vs. r will appear something like the sketches shown below.



The curve of Y versus v will be taken as an example for discussion, realizing that similar conclusions will be obtained from consideration of the other curves.

Any expansion of the $Y(v)$ in a power series in v will give 0 for the coefficients of the even powers of v, since the function is an odd function of v. Hence, for $Y(v)$ one obtains

$$Y(v) = s_1 v + s_2 v^3 + s_3 v^5 + \ldots\ldots$$

It is obvious that the coefficient s_1 represents the slope of the curve at $v = 0$ and therefore $s_1 = Y_v$. In the discussion of the X equation, it was indicated that the hydrodynamic forces due to (v, r, and δ) would be power expansions of the combination of these terms. Therefore, one expects in the Y and N functions an odd power series in the sum of these variables of the following form

$$(k_5 v + k_6 r + k_7 \delta) + (k_8 v + k_9 r + k_{10} \delta)^3 + (k_{11} v + k_{12} r + k_{13} \delta)^5 + \ldots\ldots$$

If the series is stopped after the cubic term, then the following terms appear in the expansion of Y and N in r, v, and δ.

$$r, v, \delta, r^3, v^3, \delta^3, r^2\delta, \delta^2 r, r^2 v, v^2 r, \delta^2 v, v^2\delta, \text{ and } \delta r v.$$

As indicated in the discussion of the X equation, the change in velocity, Δu, will affect the forces involved and a power series in Δu was indicated. Hence, in the Y and N equations, a power expansion in (Δu) is used to describe the effect of a change in velocity Δu from the original velocity u_o. If terms up through the cubic are retained, then terms like $r\Delta u$, $r(\Delta u)^2$, etc. will be included in the function.

If for the case of zero rudder deflection, the ship has a turning moment N_o and a force Y_o due to the asymmetry of a single screw propeller (rotating in one direction), this initial moment and force must be taken into account. Since, the forces due to single screw action may be altered by a change in forward speed, other terms in Y_o and N_o in combination with (Δu) may be present. The non-linear equation for Y takes the following form on the basis of the previous discussion.

$$A\dot{v}+B\dot{r}+Cv+Dr+E\delta+Fv^3+Gr^3+H\delta^3+Ir^2\delta+J_1\delta^2 r+Kv^2 r+Mr^2 v+N_1\delta^2 v+Pv^2\delta$$

$$+Qv\Delta u+Rv(\Delta u)^2+Sr\Delta u+Tr(\Delta u)^2+V\delta\Delta u+W\delta(\Delta u)^2+Y_o\left[1+A_1\Delta u+B_1(\Delta u)^2\right]=0$$

The coefficients A, B, D, and S incorporate terms brought over from the right hand side of the equation of motion (i.e. body inertial forces).

The form of the N equation is identical, with different coefficients of course, and the coefficients corresponding to A, B, D, and S will also contain terms brought over from the right hand side.

The coefficients C and F can be determined from the plot of model test results on the planar motions mechanism (or three component dynamometer). The coefficients D and G can be obtained by the plot of model test results from the rotating arm tests or planar motions test. The coefficients E and H can be determined by the model tests described previously where the measurement of Y_δ and N_δ was discussed. By testing at different speeds, the coefficients Q, R, S, T, V, and W, A_1 and B_1 can be estimated. A and B are derivable from planar motions model tests. The order of magnitude of other coefficients may be estimated from hydrodynamic considerations and thereby it can be determined if they are small enough to neglect. The more important <u>non-linear</u> terms in the Y and N equations are the terms in v^3, r^3, and δ^3 followed in importance by perhaps some terms in Δu. In the X equation the important <u>non-linear</u> terms are the terms in $(\Delta u)^2$, v^2, r^2, and perhaps rv followed by some of the terms involving Δu.

The solution of the non-linear equations for ship maneuvering has been programmed for the computer. Using the various coefficients determined by model tests or calculated, the path, velocity, angular velocity, etc. are calculated as a function of time for a ship in the several types of maneuver (or trials) such as the spiral, turning circle, and oscillating rudder maneuvers.

The equations of motion have been developed and solved in dimensional form. For example, Y has the dimensions of a force, v has the dimensions of a velocity, $\dot{r}$ has the dimensions of angular acceleration. Since the values of many of the coefficients for a given ship are derived from model tests, it may be advantageous in certain cases to express these coefficients, the equations of motion, and their solution in non-dimensional form. As indicated earlier, the physical quantities of length (L), mass density (ρ), and velocity (U) are the most convenient quantities to use to produce dimensionless coefficients in hydrodynamics. ρ has the dimension of mass per unit volume $(\frac{M}{L^3})$, L has the dimension of length, and U has the dimension of lenght per unit time $(\frac{L}{T})$. Hence, time, T, has the dimension of $\frac{L}{U}$, $(\frac{L}{\frac{L}{T}} = T)$.

The non-dimensional form of the various derivatives can be obtained by writing both the numerator and denominator of the derivative in non-dimensional form.

For example, if the derivative Y_v in non-dimensional form is expressed by $(Y_v)^1$, then

$$(Y_v)^1 = \frac{\partial Y^1}{\partial v^1} = \frac{\partial\left[\frac{Y}{\frac{1}{2}\rho L^2U^2}\right]}{\partial\left[\frac{v}{U}\right]} = \frac{\partial Y}{\partial v}\left[\frac{1}{\frac{1}{2}\rho L^2U}\right] = \frac{Y_v}{\frac{1}{2}\rho L^2U}$$

since $\frac{1}{2}\rho L^2U^2$ has the dimension of a force and v has the dimension of a velocity. Hence, dividing Y_v by $\frac{1}{2}\rho L^2U$ will reduce it to non-dimensional form. The coefficient $(Y_{\dot{v}}-m)$ is reduced to non-dimensional form by dividing $\dot{v}$ by $\frac{U^2}{L}$ which has the dimensions of an acceleration, since $\frac{U^2}{L} = \frac{L^2}{T^2} \cdot \frac{1}{L} = \frac{L}{T^2}$ = dimensions of acceleration. Therefore

$$(Y_{\dot{v}})^1 = \frac{\partial Y^1}{\partial \dot{v}^1} = \frac{\partial\left[\frac{Y}{\frac{1}{2}\rho L^2U^2}\right]}{\partial\left(\frac{\dot{v}L}{U^2}\right)} = \frac{\partial Y}{\partial \dot{v}}\ \frac{1}{\frac{1}{2}\rho L^3} = \frac{Y_{\dot{v}}}{\frac{1}{2}\rho L^3}$$

and the mass m is non-dimensionalized by dividing by the dimension of mass - i.e. $m^1 = \frac{m}{\frac{1}{2}\rho L^3}$. It is clear that $Y_{\dot{v}}$ has the dimensions of mass, as it should, and the total term $Y_{\dot{v}}-m$ reduces to the non-dimensional form as follows.

$$(Y_{\dot{v}})^1 - m^1 = (Y_{\dot{v}}-m)^1 = \frac{(Y_{\dot{v}}-m)}{\frac{1}{2}\rho L^3}.$$

When a similar procedure is used to reduce $N_r - mx_Gu_o$ to non-dimensional form, one obtains (since r has the dimension of $\frac{1}{T}$ and N has the dimensions of a moment = force · length)

$$(N_r)^1 = \frac{\partial\left(\frac{N}{\frac{1}{2}\rho L^3U^2}\right)}{\partial\left(\frac{rL}{U}\right)} = \frac{N_r}{\frac{1}{2}\rho L^4U}$$

$$(mx_Gu_o)^1 = \frac{m}{\frac{1}{2}\rho L^3} \cdot \frac{x_G}{L} \cdot \frac{u_o}{U} = \frac{mx_Gu_o}{\frac{1}{2}\rho L^4U}$$

$$(N_r - mx_G u_o)^1 = \frac{N_r - mx_G u_o}{\frac{1}{2}\rho L^4 U}$$

The non-dimensional forms of the other derivatives can be derived in the same manner. In using the non-dimensional form of the equations, the Froude number, $\frac{U}{\sqrt{gL}}$, (or $\frac{u_o}{\sqrt{gL}}$) is to be used as the parameter for speed. For the maneuvering of a submerged submarine in the horizontal plane, where there are no free surface effects, and therefore no dependence on Froude number, the non-dimensional equations become independent of speed.

CHAPTER XI

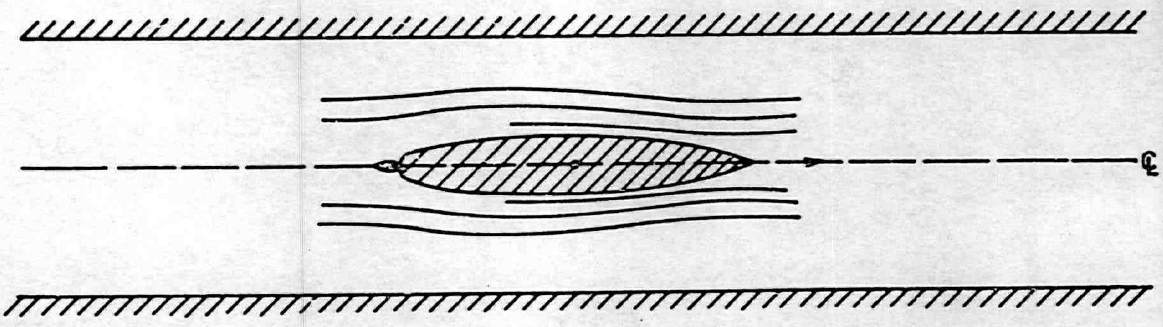
Maneuvring in Restricted Waters and Stability of Towed Bodies

Up to the present, the discussions have dealt with the ship in unrestricted water both in water surface size and water depth. Also, we have considered the ship as a free body with no extraneous forces acting on it, only hydrodynamic forces arising from its motion and control surface deflection.

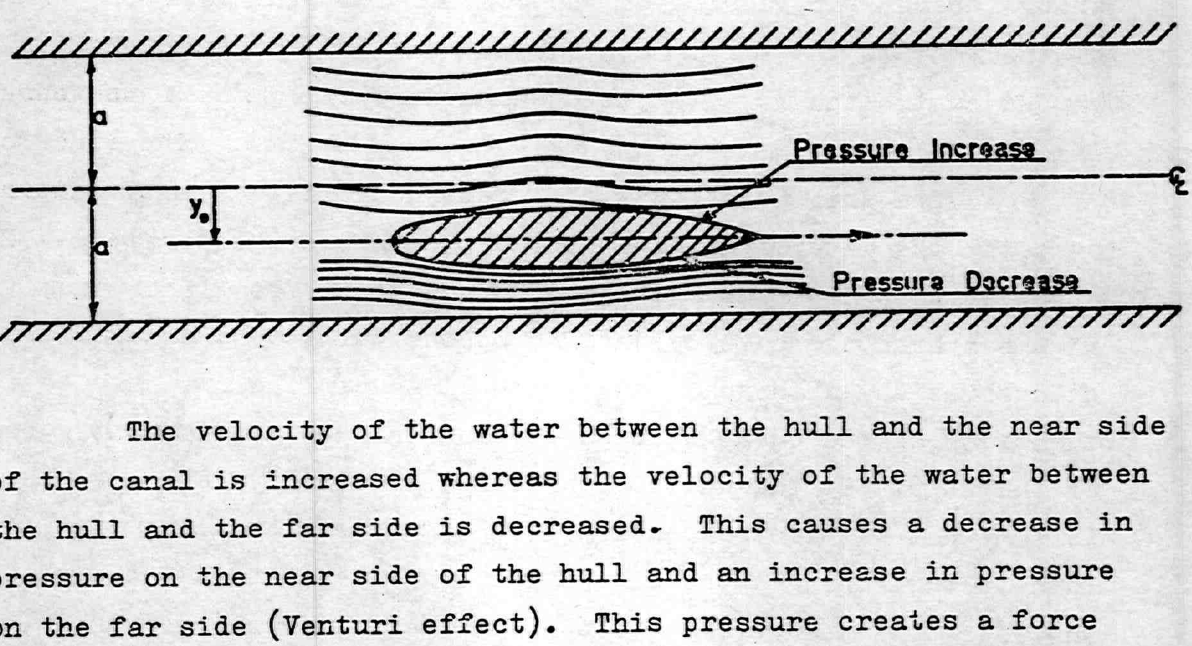
If a ship (barge) is being towed, forces are applied to the ship, through the tension in the tow line, at the point of attachment of the line to the ship. Hence, one must add the force and moment caused by the tow rope to the usual hydrodynamic force and moment acting on the ship.

If a ship moves in water, restricted either in depth or width or both, the water flow lines about the ship are altered from the situation of deep depth and width. The change in flow lines cause a change in the hydrodynamic forces acting on the hull. For example, when a ship moves ahead through water, the lines of flow not only go around the sides of the ship but also go down the bow and along the bottom of the ship. If the water is shallow, the water flow under the hull is somewhat restricted causing more water to flow along the sides. This in turn changes the side force and moment acting on a ship and therefore can change the hydrodynamic derivatives (such as Y_v, N_v, etc.). Hence, shallow water may change the value of the derivatives for a given hull and, if maneuvering in very shallow water is being considered, model tests to measure these derivatives, should be carried out in shallow water.

If a ship moves in water where close boundaries restrict the motion, such as in a canal, then the flow around the sides of the ship are altered from those existing in unrestricted water. This in turn causes certain hydrodynamic forces on the hull. Consider a ship moving in a canal as indicated in the sketch below.



If the ship is on the centerline of the canal, there is symmetrical flow on port and starboard side, hence, no moment or side force. If the ship is moving along the canal off of the centerline and closer to one wall, the symmetry of flow is disturbed, as indicated in the following sketch.

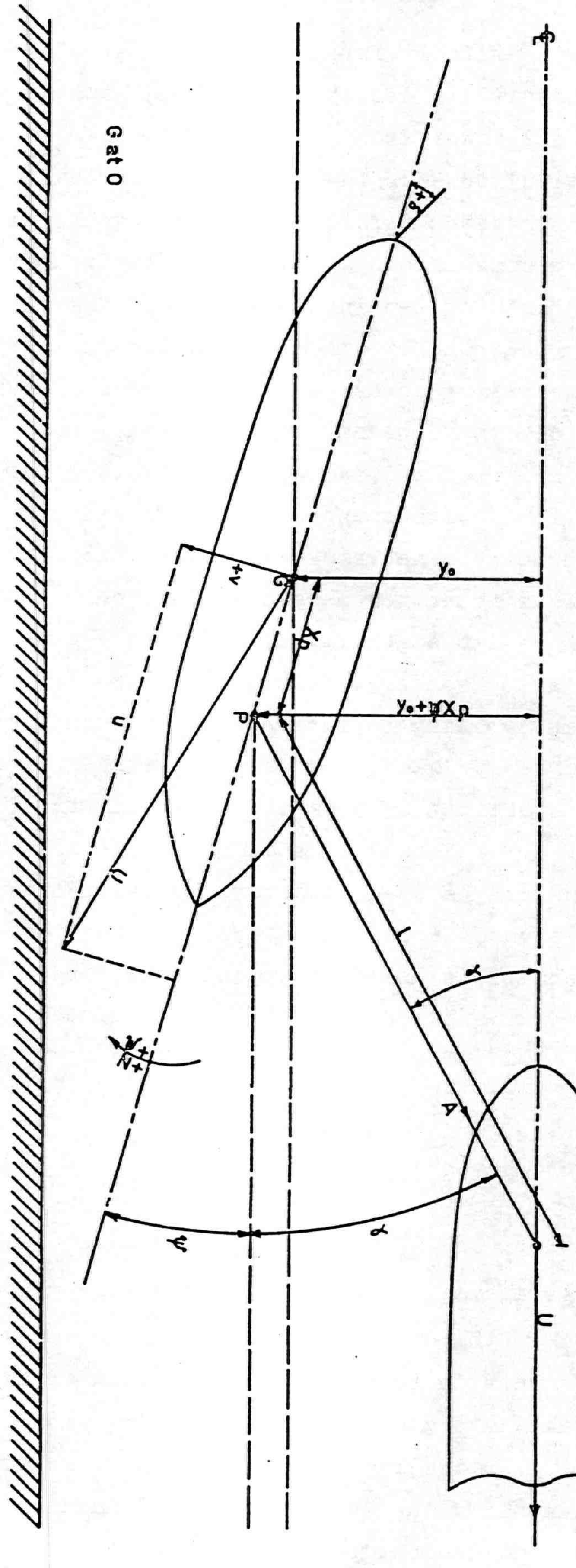


The velocity of the water between the hull and the near side of the canal is increased whereas the velocity of the water between the hull and the far side is decreased. This causes a decrease in pressure on the near side of the hull and an increase in pressure on the far side (Venturi effect). This pressure creates a force drawing the ship towards the near wall of the canal and a moment tending to swing the bow away from the near canal wall. If the distance from the centerline of the canal to the origin on the centerline of the ship is designated by y_o, then in a narrow canal there is a force and moment produced on the ship as a function of y_o (i.e. orientation in space) and the linear approximations (linear term in the Taylor expansion of the function) are $Y_{y_o} y_o$ and $N_{y_o} y_o$. The derivatives Y_{y_o} and N_{y_o} are zero for the case of a ship in open ocean but become very significant for motion of a ship in a canal since they represent essentially destabilizing (and hazardous) effects - tending to draw the ship away from the center of the canal (original equilibrium condition) towards the near bank and tending to swing the ship away from its heading straight down the canal and head it toward the far bank. This hazardous situation calls for very experienced piloting since a maneuver tending to avoid hitting the near side (or even without any rudder application) may cause the ship to head for the other side requiring an appropriate counter-maneuver to avoid hitting the canal side. Here we have a case where not only the stability of the ship has been reduced because of its location in a canal but the ship is faced with counter-maneuvers and a greater tendency to overshoot on these maneuvers because of this reduced stability. A ship with insufficient or marginal dynamical stability in the open ocean, may have its stability degraded in a canal, where stability and control are essential, to a point where the pilot has extreme difficulty in avoiding collision with the canal wall.

In order to handle the case of stability of towed bodies and the case of stability in a canal, a general treatment of the case of stability of a towed body in a canal is carried out. The equations and solutions can be used to cover any specialized case by setting equal to zero those terms which are zero for the special case. Hence, the linear equations and development hold for

1. Towed or partially towed bodies in a canal.
2. Towed bodies in open sea.
3. Ship moving in a canal (under own power).
4. Ship moving in open sea.
5. Any of the above, with or without automatic control.

In order to make the treatment of the problem more realistic, the rudder is deflected under automatic controls sensitive to the heading angle ψ, the distance off the centerline y_o, and the angular velocity r. This is to simulate the action of an experienced pilot who would take into account these items in ordering the various rudder applications. A time lag t^1 is used for the control system. The following sketch shows the ship being towed in a canal with the ship oriented in some general position in the canal. It is assumed that the towing vessel moves at all times down the center of the tank,


G at O

wherein the towed vessel is free to move over the water surface. (A more complicated analysis can be made for the case where the towing vessel has the freedom to move off the centerline).

The following is a definition of certain symbols indicated in the sketch.

ℓ is the length of the towline.

x_p is the location of the tow point (point of line attachment) relative to the origin, G. x_p is positive if forward of G and negative if aft of G.

G is the location of the center of gravity which has been chosen as the origin in this development.

P is the point of attachment of the tow line to the hull.

ψ is the heading angle of the ship relative to canal centerline.

γ is the angle between the canal direction and the towline direction.

T is the tension in the towline.

With the origin located at the center of gravity, the linearized equations for Y and N become

$$Y = m(\dot{v}+ru_o) = Y_{\dot{v}}\dot{v}+Y_v v+Y_{\dot{r}}\dot{r}+Y_r r+Y_\psi \psi+Y_{y_o} y_o+Y_\delta \delta - T\sin(\gamma+\psi) \tag{1}$$

$$N = I_z \dot{r} = N_{\dot{v}}\dot{v}+N_v v+N_r r+N_{\dot{r}}\dot{r}+N_\psi \psi+N_{y_o} y_o+N_\delta \delta - T\sin(\gamma+\psi)x_p. \tag{2}$$

The X equation has been decoupled since, as shown in a previous development, certain derivatives in the X equation are zero because of port and starboard symmetry.

For small ψ and γ:

$$\sin(\gamma+\psi) = \psi + \left(\frac{y_o+x_p\psi}{\ell}\right) = \left(1+\frac{x_p}{\ell}\right) + \frac{y_o}{\ell}.$$

If we have controls proportional to ψ, r, and y_o with an equivalent time lag t^1, then the control function becomes:

$$\delta = k_1(1-t^1D)\psi + k_2(1-t^1D)r + k_3(1-t^1D)y_o \tag{3}$$

where D is the operator $\frac{d}{dt}$.

$$\dot{y}_o = \frac{dy_o}{dt} = v\cos\psi + u\sin\psi.$$

For small ψ, within the linear theory $\cos\psi \approx 1$, $\sin\psi \approx \psi$, and $u \approx u_o$.

Then

$$\dot{y}_o \approx v + u_o\psi$$

$$v = \dot{y}_o - u_o\psi = Dy_o - u_o\psi \tag{4}$$

$$\dot{v} = \ddot{y}_o - u_o\dot{\psi} = D^2y_o - u_oD \tag{5}$$

$$r = \dot{\psi} = D\psi, \quad \dot{r} = D^2\psi.$$

Substituting equations (3), (4), and (5) into equations (1) and (2) results in:

$$\begin{aligned}&(Y_{\dot{v}}-m)(D^2y_o-u_oD\psi)+Y_v(D_{y_o}-u_o\psi)+\left[Y_{\dot{r}}(D^2)+(Y_r-mu_o)D+Y_\psi\right]\psi+Y_{y_o}y_o\\&+Y_\delta\left[k_1(1-t^1D)\psi+k_2(1-t^1D)D\psi+k_3(1-t^1D)y_o\right]-T\left(1+\frac{x_p}{\ell}\right)\psi-\frac{Ty_o}{\ell}=0\end{aligned} \tag{6}$$

$$\begin{aligned}&N_{\dot{v}}(D^2y_o-u_oD\psi)+N_v(Dy_o-u_o\psi)+\left[(N_{\dot{r}}-I_z)D^2+N_rD+N_\psi\right]\psi+N_{y_o}y_o\\&+N_\delta\left[k_1(1-t^1D)\psi+k_2(1-t^1D)D\psi+k_3(1-t^1D)y_o\right]-Tx_p\left(1+\frac{x_p}{\ell}\right)\psi-\frac{Tx_py_o}{\ell}=0\end{aligned} \tag{7}$$

When terms are regrouped, there results

$$\begin{aligned}&\left[D^2(Y_{\dot{v}}-m)+D(Y_v-k_3Y_\delta t^1)+\left(Y_{y_o}-\frac{T}{\ell}+k_3Y_\delta\right)\right]y_o+\left[D^2(Y_{\dot{r}}-k_2Y_\delta t^1)\right.\\&\left.+D\left\{-u_o(Y_{\dot{v}}-m)+(Y_r-mu_o)-k_1Y_\delta t^1+k_2Y_\delta\right\}+\left\{-Y_vu_o+Y_\psi+k_1Y_\delta-T\left(1+\frac{x_p}{\ell}\right)\right\}\right]\psi=0\end{aligned} \tag{6a}$$

$$\begin{aligned}&\left[D^2(N_{\dot{v}})+D(N_v-N_\delta k_3t^1)+\left(N_{y_o}+k_3N_\delta-\frac{Tx_p}{\ell}\right)\right]y_o\\&+\left[D^2(N_{\dot{r}}-I_z-N_\delta k_2t^1)+D(-N_{\dot{v}}u_o+N_r-k_1N_\delta t^1+k_2N_\delta)\right.\\&\left.+\left\{-N_vu_o+N_\psi+N_\delta k_1-Tx_p\left(1+\frac{x_p}{\ell}\right)\right\}\right]\psi=0.\end{aligned} \tag{7a}$$

The determinant of the coefficients of equations (6a) and (7a) is:

$$aD^4 + bD^3 + cD^2 + dD + e = 0$$

where:

$$a = (Y_{\dot{v}}-m)(N_{\dot{r}}-I_z-N_{\delta}k_2t^1)-(N_{\dot{v}})(Y_{\dot{r}}-k_2Y_{\delta}t^1)$$

$$b = (Y_{\dot{v}}-m)(-N_{\dot{v}}u_o+N_r-k_1N_{\delta}\,t^1+k_2N_{\delta}\,)+(Y_v-k_3Y_{\delta}\,t^1)(N_{\dot{r}}-I_z-N_{\delta}\,k_2t^1)$$

$$-(-u_o\{Y_{\dot{v}}-m\}+\{Y_r-mu_o\}-k_1Y_{\delta}t^1+k_2Y_{\delta})(N_{\dot{v}})-(Y_{\dot{r}}-k_2Y_{\delta}t^1)(N_v-N_{\delta}k_3t^1)$$

$$c = (Y_v-k_3Y_{\delta}t^1)(-N_{\dot{v}}u_o+N_r-k_1N_{\delta}t^1+k_2N_{\delta})+(N_{\dot{r}}-I_z-N_{\delta}k_2t^1)(Y_{y_o}-T/\ell+k_3Y_{\delta})$$

$$+(Y_{\dot{v}}-m)(-N_vu_o+N_{\psi}+N_{\delta}k_1-Tx_p\{1+\frac{x_p}{\ell}\}\,)-(-u_o\{Y_{\dot{v}}-m\}+(Y_r-mu_o)$$

$$-k_1Y_{\delta}t^1+k_2Y_{\delta})(N_v-N_{\delta}k_3t^1)-(Y_{\dot{r}}-k_2Y_{\delta}t^1)(N_{y_o}+k_3N_{\delta}-\frac{Tx_p}{\ell})$$

$$-(N_{\dot{v}})(-Y_vu_o+Y_{\psi}+k_1Y_{\delta}-T(1+\frac{x_p}{\ell})$$

$$d = (Y_v-k_3Y_{\delta}t^1)(-N_vu_o+N_{\psi}+N_{\delta}k_1-Tx_p(1+\frac{x_p}{\ell}))+(Y_{y_o}-\frac{T}{\ell}+k_3Y_{\delta}\,)$$

$$(-N_{\dot{v}}u_o+N_r-k_1N_{\delta}t^1+k_2N_{\delta})-(-u_o(Y_{\dot{v}}-m)+(Y_r-mu_o)-k_1Y_{\delta}t^1+k_2Y_{\delta})$$

$$(N_{y_o}+k_3N_{\delta}-\frac{Tx_p}{\ell})-(-Y_vu_o+Y_{\psi}+k_1Y_{\delta}-T(1+\frac{x_p}{\ell}))(N_v-N_{\delta}k_3t^1)$$

$$e = (Y_{y_o}-\frac{T}{\ell}+k_3Y_{\delta}\,)(-N_vu_o+N_{\psi}+N_{\delta}k_1-Tx_p(1+\frac{x_p}{\ell}))$$

$$-(-Y_vu_o+Y_{\psi}+k_1Y_{\delta}-T(1+\frac{x_p}{\ell}))(N_{y_o}+k_3N_{\delta}-T\frac{x_p}{\ell}).$$

The fourth order equation in the differential operator D, when the determinant of the coefficients in the Y and N equations are set equal to zero, will give four roots.

$$a(D-\sigma_1)(D-\sigma_2)(D-\sigma_3)(D-\sigma_4) = 0.$$

The condition for stability, as developed previously, is that the all four roots (σ_1, σ_2, σ_3, and σ_4) must be negative if real or the real part must be negative if the root is complex. The condition of equilibrium in this general case is straight ahead motion at constant speed, in the direction of the canal centerline and on the canal centerline. A stable condition would result in the ship returning to the centerline of the canal, and heading in the canal direction, after a disturbance from this condition. For a fourth degree equation in D, the following necessary and sufficient conditions for stability (negative roots) have been established (by Routh).

$$\frac{b}{a} > 0, \ \frac{d}{a} > 0, \ \frac{e}{a} > 0 \text{ and } \frac{bcd-ad^2-b^2e}{a^3} > 0$$

($\frac{c}{a} > 0$ is implied in the last condition).

The equations of motion can be solved, to give the trajectory resulting from any given disturbance. This more general case can be readily reduced to give the solution for any of the more specialized cases.

1. To obtain the case of a ship freely moving in a canal under its own power (i.e. not being towed), the term T is set equal to zero.

2. To obtain the case of a ship (or barge) being towed in the open sea (unrestricted waters), the derivatives, Y_{y_o}, N_{y_o}, Y_{ψ}, and N_{ψ} are also set equal to zero.

3. To obtain the case of a ship moving on its own in unrestricted water, the terms Y_{y_o}, N_{y_o}, Y_{ψ}, N_{ψ}, and T are set equal to zero.

4. In each of the cases above, to obtain the case of a ship without automatic controls, the values of k_1, k_2, and k_3 are set equal to zero, or if the controls are not sensitive to certain of the parameters indicated, then the appropriate k is set equal to zero.

In cases 1 and 2 above, there will be four roots as in the more general case developed. In case 3, if the controls are sensitive to heading ψ, and not y_o, there will be three roots. The fourth root will be D = 0, since the ship is not sensitive to y_o (location in space). If there are no controls in case 3, then one obtains the two roots and the criterion developed previously for this special case of stability in straight line motion. In this latter case, a factor $D^2 = 0$ can be taken from the determinant equation in D. (This indicates insensitivity to both ψ and y_o).

One should remember that the hydrodynamic derivatives used must be evaluated under the conditions of the ship, i.e. shallow water, restricted water, etc.

An analysis of the equations and solution, for a body being towed, is that the stability, as it depends on the tow line length ℓ and the towpoint P, is improved as ℓ decreases and as x_p increased. The graph below is indicative of the trend in stability brought about by the parameters of towline length for the practical case of a ship being towed with the tow point at the extreme bow.