Homework III - Smirnov

Problem I

I.1

> Using $X_{[1]} = \min\{X_1, \dots, X_n\}$. construct the exact confidence interval for θ, $X_i \in \mathrm{Uniform}[\theta, \theta+1]$

Solution

Obviously, $X_{[1]} > \theta$, so we can use it as the upper bound T_1. We want to find C_α such that

$$P(X_{[1]} - C_\alpha < \theta < X_{[1]}) = 1 - \alpha$$

which is equivalent to

$$1 - \alpha = P(X_{[1]} - C_\alpha < \theta) = 1 - P(X_{[1]} - C_\alpha \geq \theta) = 1 - P(X_{[1]} \geq \theta + C_\alpha)$$

$$\alpha = P(X_{[1]} \geq \theta + C_\alpha) = (1 - C_\alpha)^n$$

$$\sqrt[n]{\alpha} = 1 - C_\alpha$$

$$C_\alpha = 1 - \sqrt[n]{\alpha}$$

so the confidence interval is $(X_{[1]} - 1 + \sqrt[n]{\alpha}, X_{[1]})$.

I.2

> Uniform distribution on $[\theta, 2\theta]$

Consider random variables $Y_i = \frac{X_i}{\theta}$ which have uniform distribution on $[1, 2]$. The distribution of Y_i and $Y_{[1]} = \frac{X_{[1]}}{\theta}$ does not depend on θ, because $\frac{X_{[1]}}{\theta} = \min\left(\frac{X_i}{\theta}\right) = \min(Y_i)$ and Y_i has a uniform distribution on $[1, 2]$ with no dependence on θ. Since

$$Y_i = \frac{X_i}{\theta}$$

for $\theta > 0$ is continuous and strictly monotonous, the central function method is applicable.

$$1 \leq Y_{[1]} < 2$$

Consider p_1, p_2, $p_1 = \frac{\alpha}{2}$, $p_2 - p_1 = 1 - \alpha$. Consider

$$\frac{X_{[1]}}{\theta} = x_{p_1}, \frac{X_{[1]}}{\theta} = x_{p_2}$$

and denote as $T_1(x), T_2(x)$ the solutions of these equations w.r.t. θ:

$$T_1 = \frac{X_{[1]}}{x_{p_1}}, T_2 = \frac{X_{[1]}}{x_{p_2}}$$

where x_{p_1}, x_{p_2} are p_1, p_2-quantiles of $\frac{X_{[1]}}{\theta}$. Then

$$P(T_2(\vec{X}) < \theta < T_1(\vec{X})) = 1 - \alpha$$

The quantiles are

$$x_{\frac{\alpha}{2}} = x : P\left(\frac{X_{[1]}}{\theta} < x\right) = \frac{\alpha}{2}$$

Obviously, $2 > x > 1$.

$$P\left(\frac{X_{[1]}}{\theta} < x_{\frac{\alpha}{2}}\right) = P\left(Y_{[1]} < x_{\frac{\alpha}{2}}\right) = 1 - (2-x)^n = \frac{\alpha}{2}$$

$$1 - \frac{\alpha}{2} = (2-x)^n$$

$$\sqrt[n]{1 - \frac{\alpha}{2}} = 2 - x$$

$$x_{\alpha/2} = 2 - \sqrt[n]{1 - \frac{\alpha}{2}}$$

$$P\left(\frac{X_{[1]}}{\theta} < x_{1-\frac{\alpha}{2}}\right) = 1 - \frac{\alpha}{2}$$

$$\frac{\alpha}{2} = (2-x)^n$$

$$x_{1-\frac{\alpha}{2}} = 2 - \sqrt[n]{\frac{\alpha}{2}}$$

$$T_1(\vec{X}) = \frac{X_{[1]}}{2 - \sqrt[n]{1 - \frac{\alpha}{2}}}$$

$$T_2(\vec{X}) = \frac{X_{[1]}}{2 - \sqrt[n]{\frac{\alpha}{2}}}$$

So the $1 - \alpha$ central confidence interval is given by

$$T_2(\vec{X}) < \theta < T_1(\vec{X})$$

Problem II

> Suppose $X_1, \ldots, X_n$ are from $\mathcal{N}(\theta, \theta^2), \theta > 0$. Construct the exact confidence interval for θ for the confidence level $1 - \alpha$.

Consider

$$\bar{Y}_n = \frac{\sqrt{n}(\bar{X}-\theta)}{\theta} = \sqrt{n}\left(\frac{\bar{X}}{\theta} - 1\right)$$

We claim that this is a central function, since it ought to have normal distribution with mean zero and variance one, independently from θ.
Denote as $x_{\alpha/2}, x_{1-\frac{\alpha}{2}}$ the quantiles of distribution of $\bar{Y}_n$, which has the normal distribution with parameters $0, 1$.

$$T_1(x) : \sqrt{n}\left(\frac{\bar{X}}{\theta} - 1\right) = x_{\frac{\alpha}{2}}$$

$$\frac{x_{\frac{\alpha}{2}}}{\sqrt{n}} + 1 = \frac{\bar{X}}{\theta}$$

$$\frac{\bar{X}}{\frac{x_{\frac{\alpha}{2}}}{\sqrt{n}} + 1} = T_1(\vec{X})$$

$$x_{\frac{\alpha}{2}} = -x_{1-\frac{\alpha}{2}}$$

due to the symmetricity of Gaussian distribution. Now, $x_{\frac{\alpha}{2}}$ is the inverse error function of $\frac{\alpha}{2}$:

$$x_{\frac{\alpha}{2}} = x : \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} dte^{-t^2/2} = \frac{\alpha}{2}$$

So the confidence interval is

$$(T_2(\vec{X}), T_1(\vec{X}))$$

where

$$T_2(\vec{X}) = \frac{\bar{X}}{\frac{x_{1-\frac{\alpha}{2}}}{\sqrt{n}} + 1}$$

and $T_1(\vec{X})$, x_r are defined above.

Problem III

> 400 electric lamps were checked, 40 of them were defective. Find the confidence interval ($1-\alpha$=0.99) for the defect probability.

Suppose ξ_i are independent equally distributed random Bernulli variables with p equal the probability of a defect to be found. Then

$$\Xi(\vec{\xi}) = \sum_{i=1}^{n} \frac{\xi_i - p}{\sqrt{np(1-p)}}$$

has approximately Gaussian distribution with mean 0 and variance 1. Then Ξ is a central function, and we have at the confidence level 10^{-2} $x_{1-\frac{\alpha}{2}} = 2.58$, so, inserting empirical $p = 0.1$ into the formula for $\sigma = \sqrt{np(1-p)}$, one obtains

$$p = 0.1 \pm \approx \frac{2.58 \cdot 0.1}{20} \approx 0.1 \pm \frac{0.258}{20} = 0.1 \pm 0.0129$$