Homework II

Theoretical part

Problem 1

> Let $\hat{X}_1, \dots, \hat{X}_n$ be a random sample from the biparametric distribution with PDF
>
> $$\rho(t) = \frac{1}{\alpha} e^{-\frac{t-\beta}{\alpha}},\ t > \beta,\ 0 \text{ otherwise}$$
>
> Using the method of moments, produce an estimate for the parameters α, β, $\alpha > 0$

Solution

Due to the convergence properties of the exponent, all moments of ρ exist. Let's find moments as functions of parameters. Calculations of integrals were performed with Wolfram Alpha, because I am too much sleepy to do them correctly at the moment:

$$\mathbb{E}[t] = \int_{-\infty}^{\infty} t\rho(t)dt = \frac{1}{\alpha}\int_{\beta}^{\infty} te^{-\frac{t-\beta}{\alpha}}dt = \alpha + \beta$$

$$\mathbb{E}[t^2] = \int_{-\infty}^{\infty} t^2\rho(t)dt = \frac{1}{\alpha}\int_{-\infty}^{\infty} t^2 e^{-\frac{t-\beta}{\alpha}}dt = 2\alpha^2 + 2\alpha\beta + \beta^2$$

$$\hat{\alpha} + \hat{\beta} = \sum_{i=1}^{N} \frac{\hat{X}_i}{N} = \hat{M}_1$$

$$2\hat{\alpha}^2 + 2\hat{\alpha}\hat{\beta} + \hat{\beta}^2 = \frac{\sum_{i=1}^{N} \hat{X}_i^2}{N} = \hat{M}_2$$

Solving for $\hat{\alpha}, \hat{\beta}$, with the condition $\alpha > 0$ (again, using Wolfram, since I am unable to perform effective computations at the moment), one obtains

$$a = -\sqrt{\hat{M}_2 - \hat{M}_1^2},\ b = \hat{M}_1 - \sqrt{\hat{M}_2 - \hat{M}_1}$$

where $\hat{M}_1$, $\hat{M}_2$ are empirical first and second moments defined above.

Problem 2

> Let $\hat{X}_1, \dots, \hat{X}_n$ be a random sample from the Pareto distribution with distribution function
>
> $$F(t) = 1 - t^{-\gamma},\ t \geq 1, 0 \text{ otherwise}$$
>
> Estimate the γ using maximal likelihood method.

Solution

The probability density function is

$$f(t) = \frac{d}{dt}(F(t)) = \gamma t^{-(\gamma+1)},\ t \geq 1$$

The likelihood (density) function is

$$L(\gamma, \hat{X}_i) = \prod_{i=1}^{N} f(\hat{X}_i, \gamma) = \gamma \prod_{i=1}^{N} \hat{X}_i^{-(\gamma+1)}$$

if we assume that $\hat{X}_i > 1$. To find the maximum of the likelihood function, it is sufficient to find the maximum of its logarithm, due to the monotonicity of the latter. So

$$\overset{\text{argmax}}{\gamma} \gamma \prod_{i=1}^{N} \hat{X}_i^{-(\gamma+1)} = \overset{\text{argmax}}{\gamma} \ln(\ldots) = \overset{\text{argmax}}{\gamma} \left(\ln \gamma - \sum_{i=1}^{N} (\gamma+1) \ln \hat{X}_i \right) =$$

$$\overset{\text{argmax}}{\gamma} \left(\ln \gamma - (\gamma+1) \sum_{i=1}^{N} \ln \hat{X}_i \right)$$

differentiating w.r.t. γ, and putting $\frac{d}{d\gamma}(\ldots) = 0$ one obtains

$$\frac{1}{\gamma_{\max}} - \sum_{i=1}^{N} \ln \hat{X}_i = 0$$

$$\gamma_{\max} = \frac{1}{\sum_{i=1}^{N} \ln \hat{X}_i}$$

which is the maximal likelihood estimate of γ (one also needs to prove that γ is a maximum and not a minimum or an inflection point, but this is done trivially by calculating the second derivative w.r.t. γ at the point $\gamma = \gamma_{max}$:

$$-\frac{1}{\gamma_{\max}^2} = -\left(\sum_{i=1}^{N} \ln \hat{X}_i \right)^2$$

which is clearly a negative number, so the second derivative is negative and $\gamma_{\max}$ is a local maximum).

Problem 3

> Find the maximal likelihood estimates of the parameter $\theta > 0$ for the PDFs
>
> 1. $\theta t^{\theta-1},\ t \in [0, 1]$
> 2. $\frac{2t}{\theta^2}\ t \in [0, \theta]$
> 3. $\frac{\theta e^{-\frac{\theta^2}{2t}}}{\sqrt{2\pi t^3}},\ t > 0$
> 4. $\frac{\theta(\ln t)^{\theta-1}}{t},\ t \in [1, e]$
> 5. $\frac{e^{-|t|}}{2(1-e^{-\theta})}, t \in [-\theta, \theta]$

Solution

1)

$$\theta t^{\theta-1},\ t \in [0,1]$$

Logarithm of maximal likelihood function:

$$\ln L(\theta, \hat{X}_i) = \sum_{i=1}^{N} \left(\ln(\theta) + (\theta-1)\ln \hat{X}_i\right)$$

$$\frac{d}{d\theta}\ln L(\theta, \hat{X}_i) = \sum_{i=1}^{N}\left(\frac{1}{\theta} + \ln \hat{X}_i\right) = \frac{N}{\theta} - \sum_{i=1}^{N}\ln \hat{X}_i = 0$$

$$\frac{N}{\sum_{i=1}^{N}\ln \hat{X}_i} = \theta_{max}$$

the second derivative is positive for the same reason it is positive in Problem 2.

2)

$$\frac{2t}{\theta^2}, t \in [0,\theta]$$

Since parameter also defines the domain of the function, to make the dependence explicit, we rewrite

$$\frac{2t}{\theta^2}\Theta(t(\theta-t))$$

where $\Theta(x) = \int_{-\infty}^{x}\delta(t)dt$, and $\delta(x)$ is the Dirac delta function.

$$L(\theta, \hat{X}_i) = \prod_{i=1}^{N}\frac{2\hat{X}_i}{\theta^2}\Theta(\hat{X}_i(\theta-\hat{X}_i))$$

$$\ln L(\theta, \hat{X}_i) = \sum_{i=1}^{N}\left(\ln 2 + \ln \hat{X}_i - 2\ln\theta + \ln\Theta\left(\hat{X}_i(\theta-\hat{X}_i)\right)\right)$$

$$\frac{d}{d\theta}\ln L(\theta, \hat{X}_i) = \sum_{i=1}^{N}\left(-\frac{2}{\theta} + \frac{1}{|\hat{X}_i|}\frac{\delta(\hat{X}_i-\theta)}{\Theta(\hat{X}_i(\theta-\hat{X}_i))}\right)$$

This function, as just as $\ln L(\theta, \hat{X}_i)$, is defined only for $\theta > \hat{X}_{max}$, because otherwise at least in one term we have division by zero/ logarithm of zero, so for a monotonously decreasing

$$-2\ln\theta$$

we have

$$\arg\max_{\theta}\ln L(\theta, \hat{X}_i) = \hat{X}_{max}$$

3)

$$\frac{\theta e^{-\frac{\theta^2}{2t}}}{\sqrt{2\pi t^3}},\ t > 0$$

$$\ln L(\theta, \hat{X}_i) = \sum_{i=1}^{N}\left(\ln\theta - \frac{\theta^2}{2\hat{X}_i} - \frac{3}{2}\ln 2\pi - \frac{3}{2}\ln\hat{X}_i\right)$$

$$\frac{d}{dt}L(\theta, \hat{X}_i) = \sum_{i=1}^{N}\left(\frac{1}{\theta} - \frac{\theta}{\hat{X}_i}\right) = \frac{N}{\theta} - N\theta\sum_{i=1}^{N}\frac{1}{\hat{X}_i} = 0$$

$$\sum_{i=1}^{N}\frac{1}{\hat{X}_i} = \frac{1}{\theta^2}$$

$$\hat{\theta}_{max} = \sqrt{\frac{1}{\sum_{i=1}^{N}\frac{1}{\hat{X}_i}}}$$

4)

$$\frac{\theta(\ln t)^{\theta-1}}{t},\ t \in [1, e]$$

$$\ln L(\theta, \hat{X}_i) = \sum_{i=1}^{N}(\ln\theta + (\theta-1)\ln\ln\hat{X}_i - \ln\hat{X}_i)$$

$$\frac{d}{d\theta}L(\theta, \hat{X}_i) = \sum_{i=1}^{N}\left(\frac{1}{\theta} + \ln\ln\hat{X}_i\right) = 0$$

$$\frac{N}{\theta} + \sum_{i=1}^{N}\ln\ln\hat{X}_i = 0 \iff \theta_{max} = -\frac{N}{\sum_{i=1}^{N}\ln\ln\hat{X}_i}$$

5)

$$\frac{e^{-|t|}}{2(1-e^{-\theta})}, t \in [-\theta, \theta]$$

$$\ln L(\theta, \hat{X}_i) = \sum_{i=1}^{N}(-|\hat{X}_i| - \ln 2 - \ln(1-e^{-\theta}))$$

$$\frac{d}{d\theta}\ln L(\theta, \hat{X}_i) = -\sum_{i=1}^{N}\frac{e^{-\theta}}{1-e^{-\theta}} = N(\dots)$$

We note the analogy with 2). We also note that the function clearly has a singularity at

$$\theta = 0$$

, so there is no maximum value, and if we expand

$$\frac{e^{-\theta}}{1-e^{-\theta}} = e^{-\theta}(1 + e^{-\theta} + e^{-2\theta} + \dots)$$

in the formal power series (or just perform some school algebra), we notice that

$$\frac{e^{-\theta}}{1-e^{-\theta}} \cong \frac{1}{1-e^{-\theta}} - 1$$

and this expression has no zeros at the real line.