Homework III - Smirnov

Problem I

1.1

Using $X_{[1]}=\min\{X_1,\ldots,X_n\}$. construct the exact confidence interval for heta, $X_i\in \mathrm{Uniform}[heta, heta+1]$

Solution

Obviously, $X_{[1]}>\theta$, so we can use it as the upper bound $T_1.$ We want to find C_{lpha} such that

$$P(X_{[1]} - C_lpha < heta < X_{[1]}) = 1 - lpha$$

which is equivalent to

$$egin{aligned} 1-lpha &= P(X_{[1]}-C_lpha < heta) = 1-P(X_{[1]}-C_lpha \geq heta) = 1-P(X_{[1]} \geq heta + C_lpha) \ &lpha &= P(X_{[1]} \geq heta + C_lpha) = (1-C_lpha)^n \ &rac{\sqrt[n]{lpha}}{C_lpha} = 1-C_lpha \ &C_lpha = 1-rac{\sqrt[n]{lpha}}{a} \end{aligned}$$

so the confidence interval is $(X_{[1]}-1+\sqrt[n]{\alpha},X_{[1]})$.

1.2

Uniform distribution on $[\theta,2\theta]$

Consider random variables $Y_i=\frac{X_i}{\theta}$ which have uniform distribution on [1,2]. The distribution of Y_i and $Y_{[1]}=\frac{X_{[1]}}{\theta}$ does not depend on θ , because $\frac{X_{[1]}}{\theta}=\min\left(\frac{X_i}{\theta}\right)=\min(Y_i)$ and Y_i has a uniform distribution on [1,2] with no dependence on θ . Since

$$Y_i = rac{X_i}{ heta}$$

for $\theta > 0$ is continuous and strictly monotonous, the <u>central function method</u> is applicable.

$$1 \leq Y_{[1]} < 2$$

Consider p_1,p_2 , $p_1=rac{lpha}{2}$, $p_2-p_1=1-lpha$. Consider

$$rac{X_{[1]}}{ heta} = x_{p_1}, rac{X_{[1]}}{ heta} = x_{p_2}$$

and denote as $T_1(x), T_2(x)$ the solutions of these equations w.r.t. θ :

$$T_1 = rac{X_{[1]}}{x_{p_1}}, T_2 = rac{X_{[1]}}{x_{p_2}}$$

where x_{p_1}, x_{p_2} are p_1, p_2 -quantiles of $\frac{X_{[1]}}{\theta}$. Then

$$P(T_2(ec{X}) < heta < T_1(ec{X})) = 1 - lpha$$

The quantiles are

$$x_{rac{lpha}{2}} = x : P\left(rac{X_{[1]}}{ heta} < x
ight) = rac{lpha}{2}$$

Obviously, 2 > x > 1.

$$egin{align} P\left(rac{X_{[1]}}{ heta} < x_{rac{lpha}{2}}
ight) &= P\left(Y_{[1]} < x_{rac{lpha}{2}}
ight) = 1 - (2-x)^n = rac{lpha}{2} \ 1 - rac{lpha}{2} &= (2-x)^n \ & \sqrt[n]{1-rac{lpha}{2}} &= 2-x \ & x_{lpha/2} &= 2 - \sqrt[n]{1-rac{lpha}{2}} \ P\left(rac{X_{[1]}}{ heta} < x_{1-rac{lpha}{2}}
ight) &= 1 - rac{lpha}{2} \ & rac{lpha}{2} &= (2-x)^n \ & x_{1-rac{lpha}{2}} &= 2 - \sqrt[n]{rac{lpha}{2}} \ & T_1(ec{X}) &= rac{X_{[1]}}{2-\sqrt[n]{1-rac{lpha}{2}}} \ & T_2(ec{X}) &= rac{X_{[1]}}{2-\sqrt[n]{rac{lpha}{2}}} \ \end{cases}$$

So the $1 - \alpha$ central confidence interval is given by

$$T_2(ec{X}) < heta < T_1(ec{X})$$

Problem II

Suppose X_1, \ldots, X_n are from $\mathcal{N}(\theta, \theta^2), \theta > 0$. Construct the exact confidence interval for θ for the confidence level $1 - \alpha$.

Consider

$$ar{Y}_n = rac{\sqrt{n}(ar{X} - heta)}{ heta} = \sqrt{n}\left(rac{ar{X}}{ heta} - 1
ight)$$

We claim that this is a central function, since it ought to have normal distribution with mean zero and variance one, independently from θ .

Denote as $x_{\alpha/2}, x_{1-\frac{\alpha}{2}}$ the quantiles of distribution of \bar{Y}_n , which has the normal distribution with parameters 0, 1.

$$egin{align} T_1(x):\sqrt{n}\left(rac{ar{X}}{ heta}-1
ight)&=x_{rac{lpha}{2}}\ &rac{x_{rac{lpha}{2}}}{\sqrt{n}}+1=rac{ar{X}}{ heta}\ &rac{ar{X}}{rac{x_{rac{lpha}{2}}}{\sqrt{n}}+1}&=T_1(ec{X})\ &x_{rac{lpha}{2}}&=-x_{1-rac{lpha}{2}} \ \end{array}$$

due to the symmetricity of Gaussian distribution. Now, $x_{\frac{\alpha}{2}}$ is the inverse error function of $\frac{\alpha}{2}$:

$$x_{rac{lpha}{2}}=x:rac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}dte^{-t^2/2}=rac{lpha}{2}$$

So the confidence interval is

$$(T_2(\vec{X}), T_1(\vec{X}))$$

where

$$T_2(ec{X}) = rac{ar{X}}{rac{x_1 - rac{lpha}{2}}{\sqrt{n}} + 1}$$

and $T_1(\vec{X})$, x_r are defined above.

Problem III

400 electric lamps were checked, 40 of them were defective. Find the confidence interval (1 $-\alpha$ =0.99) for the defect probability.

Suppose ξ_i are independent equally distributed random Bernulli variables with p equal the probability of a defect to be found. Then

$$\Xi(ec{\xi}) = \sum_{i=1}^n rac{\xi_i - p}{\sqrt{np(1-p)}}$$

has approximately Gaussian distribution with mean 0 and variance 1. Then Ξ is a central function, and we have at the confidence level $10^{-2}~x_{1-\frac{\alpha}{2}}=2.58$, so, inserting

empirical p=0.1 into the formula for $\sigma=\sqrt{np(1-p)}$, one obtains

$$p = 0.1 \pm pprox rac{2.58 \cdot 0.1}{20} pprox 0.1 \pm rac{0.258}{20} = 0.1 \pm 0.0129$$