* Number Sets:

→ $\mathbb{N} = \{1, 2, 3, \ldots\}$ or $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$ (Natural numbers)

→ $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ (Integers)

→ $\mathbb{Q} = \{p/q \mid p, q \in \mathbb{Z},\ GCD(p, q) = 1\}$ (Rational numbers)

→ $\mathbb{R} = \mathbb{Q} \cup \{\text{Irrational numbers}\}$ (Real numbers)

* Set Theory:

→ Cardinality: count of items in finite set

Ex: Cardinality of power set of a set with n no. of element is 2^n.

$A = \{a, b, c\}$

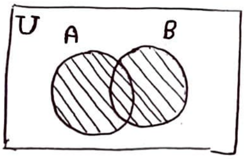
Powerset of $A = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}\}$

→ Sub set: $A \subseteq B$

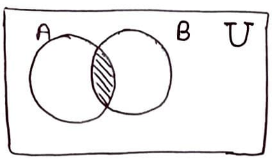
Proper sub set: $A \subset B$ but $A \neq B$ i.e. $A \subsetneq B$

→ Empty set ϕ: For every set A, $\phi \subset A$

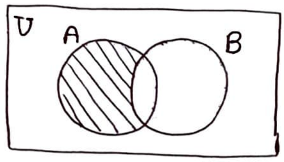
→ Union: $A \cup B$



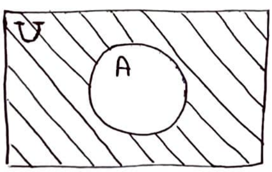
Intersection: $A \cap B$



Difference: $A - B$ or $A \setminus B$



Complement: A' or A^c or $\overline{A}$



* Functions:

$$f : X \longrightarrow Y$$

f ↑ function X ↑ Domain Y ↑ Codomain

→ One-to-One (Injective) function : If $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$

→ onto (Surjective) function : $\forall\, y \in$ codomain (f), there is $\exists x \in$ Dom.(f
Such that $f(x) = y$.

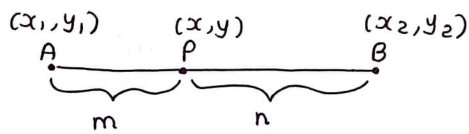
→ Bijective function : If function is one-to-one & onto also, it is called bijective function.

* Prime Number Theorem:

→ $\pi(x) =$ No. of primes $\leq x$

→ For larger x ; $\pi(x) \simeq \dfrac{x}{\ln(x)}$

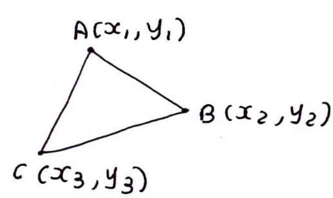
* Section Formula:



$$(x, y) = \left(\frac{mx_2 + nx_1}{m+n} , \frac{my_2 + ny_1}{m+n} \right)$$

* Area of Triangle:

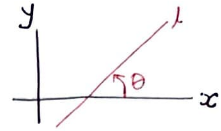
$$A(\triangle ABC) = \frac{1}{2} \left\| \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \right\|$$



* Slope's Relation :

Line l_1 : m_1 slope (non-verticle)
Line l_2 : m_2 slope (non-verticle)

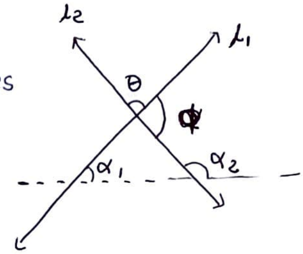
$m = \tan\theta$
↑ Slope ↑ Inclination



→ l_1 & l_2 are parallel if and only if $m_1 = m_2$.

→ l_1 & l_2 are perpendicular if and only if $m_1 \cdot m_2 = -1$.

→ l_1 & l_2 are intersecting making angles θ & α.



$\theta = \alpha_2 - \alpha_1$; $\phi = 180^\circ - \theta$

$$\tan\theta = \frac{m_2 - m_1}{1 + m_2 m_1}$$

$$\tan\phi = -\tan\theta$$

* Equations of Line :

→ General eq^n : $ax + by + c = 0$; a & $b \neq 0$ simultoneously

→ Horizontal line : $y = a$

→ Verticle line : $x = a$

→ Point slope form : $y = m(x - x_0) + y_0$

→ Two point form : $\dfrac{y - y_1}{x - x_1} = \dfrac{y_2 - y_1}{x_2 - x_1}$

→ Slope intercept form : $y = mx + c$ (y intercept)
$y = m(x - d)$ (x intercept)

→ Intercept form : $\dfrac{x}{a} + \dfrac{y}{b} = 1$

a = x - intercept
b = y - intercept

* Distance of Point from Line :

$$\text{Distance} = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2+b^2}}$$ where $ax+by+c=0$ is line. (x_1, y_1) is point.

* Distance b/w Two Parallel Line :

$$\text{Distance} = \frac{|c_1 - c_2|}{\sqrt{a^2+b^2}}$$ where $ax+by+c_1=0$ is line-1. $ax+by+c_2=0$ is line-2.

* Straight line Fit :

→ n observation in form of (x_i, y_i) where $i = 1, 2, \ldots n$

Let $y = mx + c$ is fitted line.

SSE (squared sum error) $= \sum_{i=1}^{n} (y_i - y)^2 = \sum_{i=1}^{n} (y_i - mx_i - c)^2$

* Quadratic Function :

→ General form: ax^2+bx+c ; $a \neq 0$

→ Slope of ax^2+bx+c : $2ax+b$

→ Axis of symmetry : $x = \frac{-b}{2a}$

→ Intercept form: $f(x) = a(x-p)(x-q)$ where p, q are x-intercepts.

* Solⁿ. of Quadratic function:

→ $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$ where $\sqrt{b^2-4ac} = D$ called discriminant.

$D = 0 \Rightarrow$ Equal real roots.

$D > 0 \Rightarrow$ Distinct real roots.

$D < 0 \Rightarrow$ Distinct imaginary roots.

* Parabola :

→ vertex form : $y = a(x-h)^2 + k$ where (h,k) is vertex.

* Polynomial :

→ Formal Definition : An algebric expression in which only arithetic is
+ (addition)
− (subtraction)
× (multiplication)
^ (natural exponent)
of variables.

→ ~~constant is also polynomial called monomial.~~
monomial is also polynomial.
Ex: $4x^2y$, 30, y^4 etc.

→ One variable polynomial :

$$a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 = \sum_{m=0}^{n} a_m x^m$$

exponent
variable
coefficient

→ Degree of polynomial :

Ex: $3x^3 + 4x^2y^2 + 10y + 1$ $\qquad$ Deg (poly.) = max (3, 4, 1, 0) = 4

$$\underset{3}{3x^3} + \underset{4}{4x^2y^2} + \underset{1}{10y} + \underset{0}{1}$$

→ For polynomia 0, degee is not defined.

→ If $\deg(P(x)) = n$ and $\deg(q(x)) = m$ then ;
$\deg(P(x) \pm q(x)) = \max(m, n)$.
$\deg(P(x) \cdot q(x)) = m + n$

* Divison of Polynomial :

$$\frac{p(x)}{g(x)} = q(x) + \frac{r(x)}{g(x)}$$

($g(x)$ ↑ Divisor; $q(x)$ ↑ Quotient; $r(x)$ ← Remainder)

→ multiplication of 2 polynomial : Always polynomial

Division of 2 polynomial : can or can not be polynomial

* Graphs of Polynomial :

→ smooth and continuous.

→ Let's $f(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0$ can be factorised as

$f(x) = (x-a_1)^p (x-a_2)^q \dots (x-a_m)^s$

If p is Even ⇒ Graph Bonce-off at $x = a_1$

If p is odd ⇒ Graph crosses x-axis at $x = a_1$

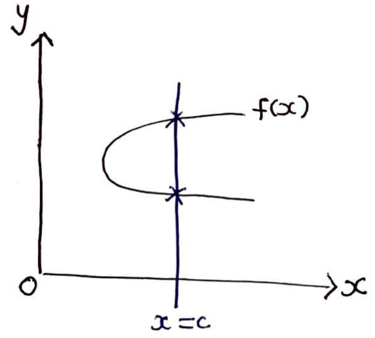
} same for other factors also.

→ For n^{th} degree polynomial, max. no of turning points $\leq n-1$

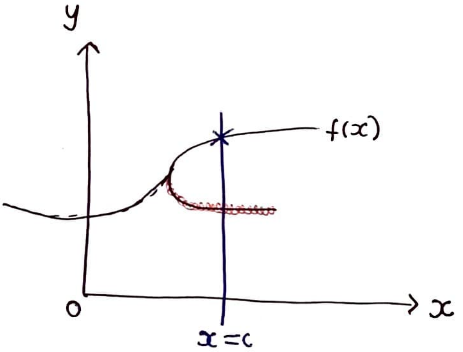
* Intermediate Value Theorem :

If $f(a)$ and $f(b)$ have opposite sign, then $\exists c$ such that $f(c) = 0$.

* Verticle Line Test : (one-to-one test)

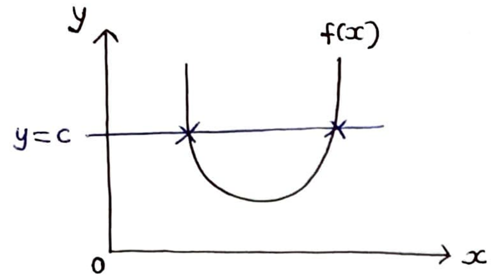


(one-to-many)

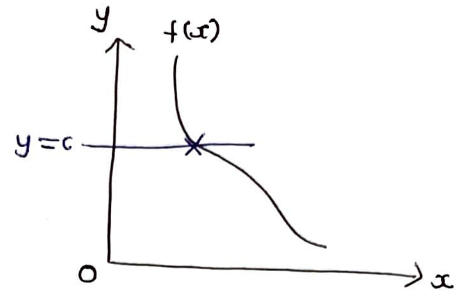


(one-to-one)

* Horizontal Line Test : (Inversible test)



(Non-Inversible)
(many-to-one)



(Inversible)
(one-to-one)

* One-to-One functions :

→ If function passes Both Horizontal & Verticle line test, it is one-to-one function.

→ If any function is increasing / decreasing, it is one-to-one function.

* Composite Function :

$$\begin{tikzcd} x \arrow[r, "g"'] \arrow[r, bend left, "g(x)"] \arrow[rr, bend right, "f(g(x))"'] & g(x) \arrow[r, "f"'] \arrow[r, bend left, "f(g)"] & fog(x) \end{tikzcd}$$

→ Substitute following values from dom(g) to find dom(fog(x)).

(i) $\{x \mid g(x) \notin dom(f)\}$

(ii) $\forall x$ for which $g(x)$ is not defined.

* Inverse Function :

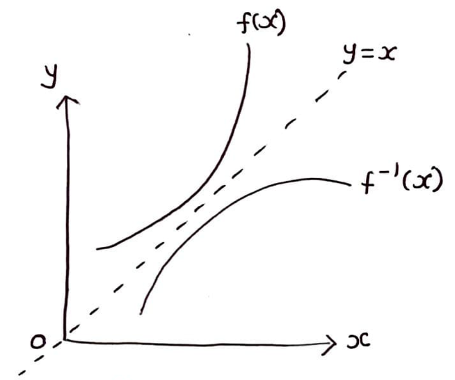
→ f^{-1} is inverse of f if and only if,

(i) $f^{-1} o f\ (x) = x$ and $f o f^{-1}(x) = x$

(ii) f is one-to-one function.

→ $Range(f) = Dom(f^{-1})$

$Dom(f) = Range(f^{-1})$



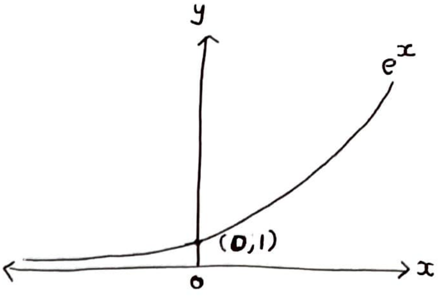
(mirror)

* Exponential Function :

→ $f(x) = a^x$, where $a>0$ and $a \neq 1$; $x \in R$

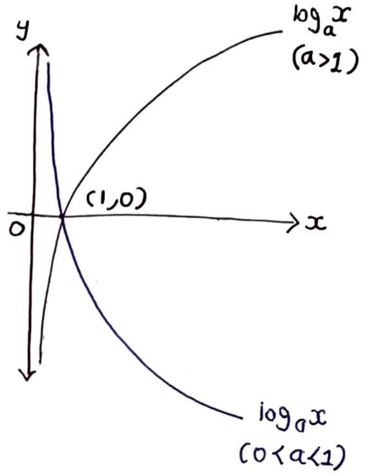
$Dom(f) = R$

$Range(f) = (0, \infty)$



→ $\lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n = e^x$ [$e \approx 2.718$]

* Logarithmic Function :



→ $y = \log_a x$ where $a>0$ and $a \neq 1$; $x>0$

$Dom(y) = (0, \infty)$

$Range(y) = R$

→ $y = \log_a x \iff a^y = x$

→ If $a \neq 1$ and $a^u = a^v \Rightarrow u = v$

→ For $a>1$; $\log_a x$ is increasing function.

For $0<a<1$; $\log_a x$ is decreasing function.

→ $\log_{10} x$ is common log. while $\log_e x$ ($\ln x$) is natural log.

$\log_a 1 = 0$

$\log_a a = 1$

$a^{\log_a x} = x$

$\log_a a^x = x$

$m^{\log_a n} = n^{\log_a m}$

$\log_a x = \dfrac{\log_b x}{\log_b a}$

$\log_a(mn) = \log_a m + \log_a n$

$\log_a(m/n) = \log_a m - \log_a n$

$\log_a(1/m) = -\log_a m$

$\log_a(m^n) = n \log_a m$

* Graph :

→ Collection of two sets V and E ; $G = (V, E)$

V : set of vertices or nodes

E : set of edges $\{E \subseteq V \times V\}$

* Walk :

→ Any sequence of vertices and edges as we travese a graph.

- Vertex and Edges can be repeated

* Path :

→ Travesing graph without repeating any vertex and edge.

* Cycle :

→ Path whose starting and ending vertex is same. Can repeat vertex but not an edge.

* Tree :

→ Minimally connected acyclic graph.

(i) tree on 'n' vertices has exactly 'n-1' edges

(ii) Adding an edge to a tree, creates cycle

(iii) Every pair of vertices connected by a unique path

* Minimum Spanning Tree (MST) :

→ MST for a weighted, connected and undirerceted graph is a spanning tree ~~without~~ weight less or equal to the weight of every other spanning tree.

(i) Prim's algoritm

(ii) Kruskal's algorithm

} (Minimum Separator Lemma)

→ Prim's incrementally grows minimum spanning tree.

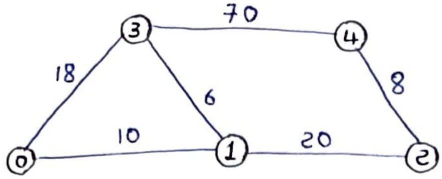
→ Kruskal's build MST by adding edges in increasing order.

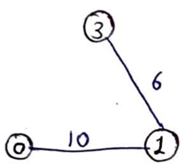
* Prim's Algorithm:

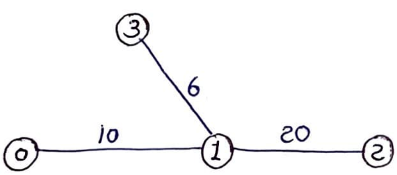
(start with any vertex)

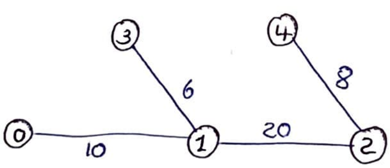
* Kruskal's Algoritm:

(start with minimum weighted edge)









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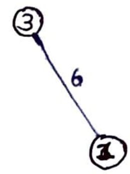
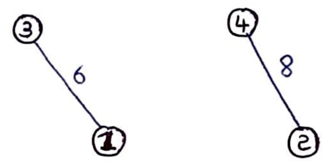
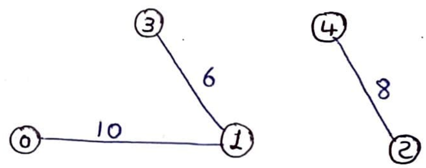
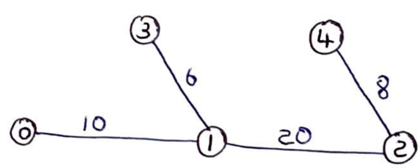








(END)

* Weighted shortest Path:

→ For weighted shorted path, length of a path is sum of the weights, not no. of edges.

→ BFS gives shorted path in term of no. of edges, which need not be weighted shortest path.

Weighted shortest Path

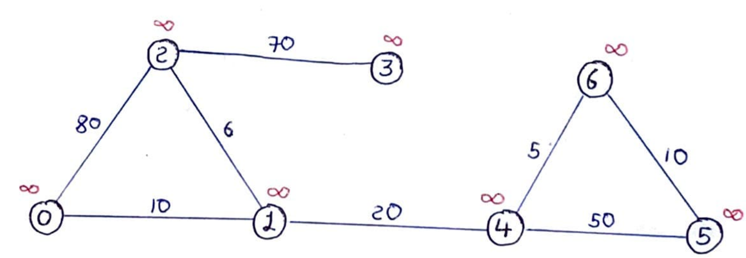
→ Single source
- Dijkstra's Algorithm (non-negative edge)
- Bellman-Ford Algorithm (negative/non-negative edge)

→ All pairs
- Floyd-Warshall's Algorithm

→ For a graph with negative cycle (not negative edge) weighted shortest path is not defined.

* Dijkstra's Algorithm:

(start)



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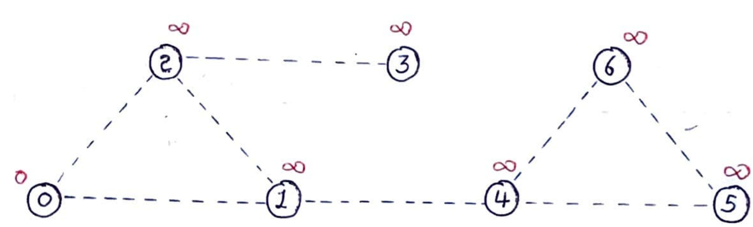

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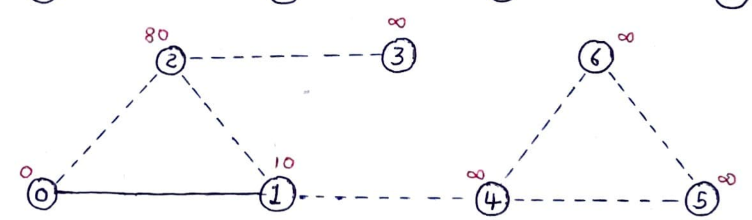

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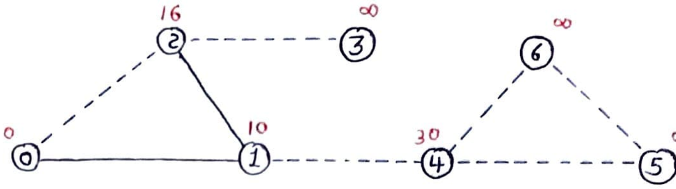

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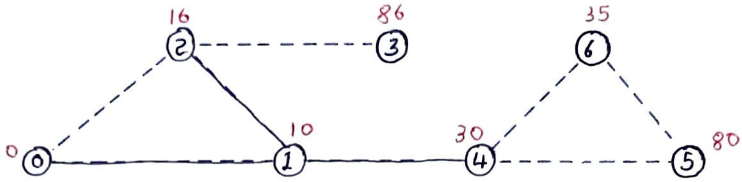

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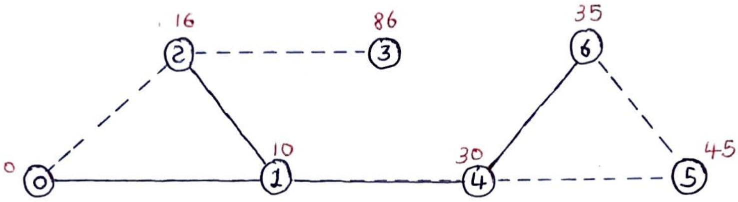

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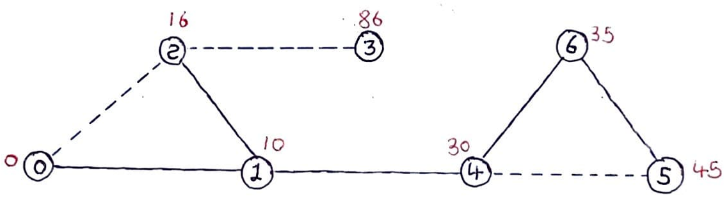

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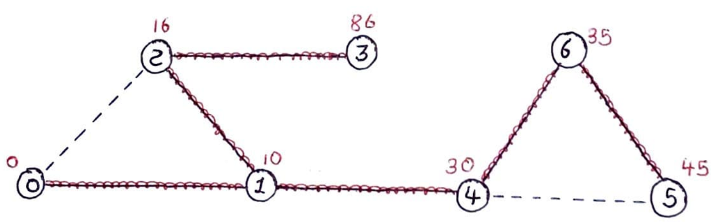
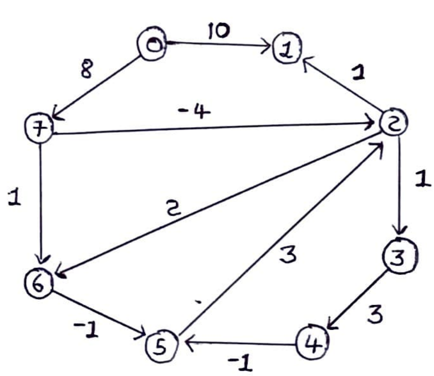


* Bell-man-ford Algorithm :

→ Stabilize after n-1 iterations.

V	D(V)						
0	0	0	0	0	0	0	0
1	∞	10	10	5	5	5	5
2	∞	∞	4	4	4	4	4
3	∞	∞	∞	5	5	5	5
4	∞	∞	∞	∞	8	8	8
5	∞	∞	∞	8	7	5	5
6	∞	∞	9	6	6	6	6
7	∞	8	8	8	8	8	8



→ If Bellman-Ford algorithm does not converge after n-1 iterations, there is a negative cycle.

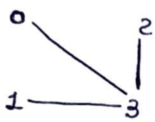
* Minimum Coloring:

→ Assign colours to nodes such that endpoints of edge have different colours.

→ Four color Theorem: Geographical derived graph at max 4 color needed.

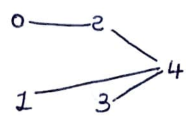
* Vertex cover:

→ Smallest subset of V such that making V covers all edges of graph.



VC = {3}

|VC(G)| = 1



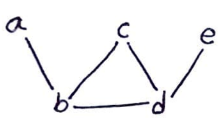
VC = {4, 2} or {4, 0}

|VC(G)| = 2

* Independent set:

→ Subset of V such that no two are connected by an edge.

→ We are generally interested in finding maximum idependent set.



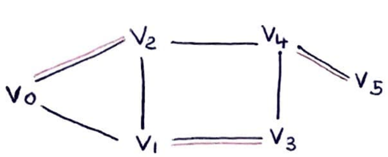
Max. IS(G) = {a, c, e}

|Max. IS(G)| = 3

* Matching:

→ A matching is a subset of E such that $\{m \subseteq E \mid m \in M\}$ are mutually disjoin edges.

→ We are generally interested in finding Maximum mactching.

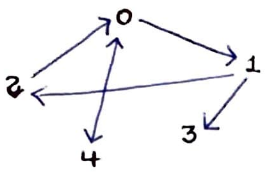


$M = \{(v_0, v_2), (v_1, v_3), (v_4, v_5)\}$

$|M(G)| = 3$

→ Perfect matching not always possible.

* Adjacency Matrix :



	0	1	2	3	4
0	0	1	0	0	1
1	0	0	1	1	0
2	1	0	0	0	0
3	0	0	0	0	0
4	1	0	0	0	0

→ Row represend outgoint edges
Column represent incoming edges

→ Degree of vertex : No. of incoming/outgoing edges.
→ $O(n^2)$ (BFS or DFS)

* Adjacency List :

0	{1,4}
1	{2,3}
2	{0}
3	{ϕ}
4	{0}

→ List of neighbour of each vertex.
→ $O(n+m)$ (where n vertices
(BFS or DFS) m edges)

* More about Graph :

→ For undirected graph no. of edges $|E| \leq \binom{n}{2}$

→ For directed graph no. of edges $|E| \leq 2 \times \binom{n}{2}$

→ Connected Graph : If every vertex is reachable from every other vertex.
→ Complete Graph : If every vertex is connected to every other vertex; (has n-1 degress)

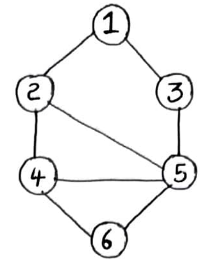
→ Sum of degrees of all vertices is always even.

→ No. of vertices of odd degree is always even.

* Breadth First Search:

→ BFS explores graph level-by-level.

→ Give shortest path in terms of edges.



V	visited?	Level	Parent
1	F	-1	
2	F	-1	
3	F	-1	
4	F	-1	
5	F	-1	
6	F	-1	

To explore queue						

V	visited?	Level	Parent
1	T	0	
2	F	-1	
3	F	-1	
4	F	-1	
5	F	-1	
6	F	-1	

To explore queue					
1					

V	visited?	Level	Parent
1	T	0	
2	T	1	1
3	T	1	1
4	F	-1	
5	F	-1	
6	F	-1	

To explore queue					
2	3				

V	visited?	Level	Parent
1	T	0	
2	T	1	1
3	T	1	1
4	T	2	2
5	T	2	2
6	F	-1	

To explore queue					
3	4	5			

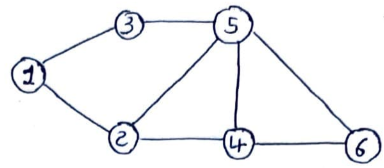
V	visited?	Level	Parent
1	T	0	
2	T	1	1
3	T	1	1
4	T	2	2
5	T	2	2
6	T	3	4

To explore queue					
5	6				

(Now empty queue)

→ In BFS search tree, any vertex apart more than by 1 level can-not be an edge in original graph.

* Depth First serch:



→ DFS path are not shortest path.

V	1	2	3	4	5	6
visited ?	F	F	F	F	F	F

Stack of suspended vertices					

V	1	2	3	4	5	6
visited ?	T	F	F	F	F	F

Stack					
1					

V	1	2	3	4	5	6
visited ?	T	T	F	F	F	F

Stack					
1	2				

V	1	2	3	4	5	6
visited ?	T	T	F	T	F	F

Stack					
1	2	4			

V	1	2	3	4	5	6
visited ?	T	T	F	T	T	F

Stack					
1	2	4	5		

V	1	2	3	4	5	6
visited?	T	T	T	T	T	F

Stack					
1	2	4	5	3	

V	1	2	3	4	5	6
visited ?	T	T	T	T	T	F

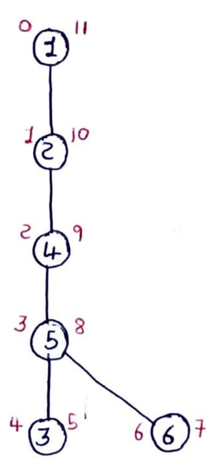
Stack					
1	2	4	5		

V	1	2	3	4	5	6
visited ?	T	T	T	T	T	T

Stack					
1	2	4	5	6	

(Now empty stack)

→ Assign each vertex an entry and exit no. to create DFS tree.



* Strongly connected components:

→ Directed graphs can be decomposed into SCCs.

→ In SCC, each pair of vertices is strongly connected (i.e. if there is path from i to j then there is alway path from j to i)

* Cut vertex:

→ In undirected connected graph, a cut vertex is an articulation point whose removal disconnects the graph.

* Bridge (cut vertex):

→ An edge in undirected connected graph, whose removal disconnects the graph.

→ An edge in undirected disconnected graph, whose removal increases no. of disconnected components.

* Directed Acyclic Graph (DAG):

→ A graph without cycle and also directed.

(i) Topological sorting

(ii) Longest path

→ It's natural way to represent dependencies

* Topological Sorting:

→ A graph with cycle can not be topologically sorted.

→ Topological sorting means sequencing vertices such that for any $(i,j) \in E$, i appears before j.

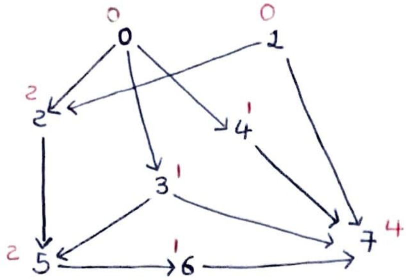
→ Compute indegree of each vertex.

Remove vertex with indegree 0, from DAG.

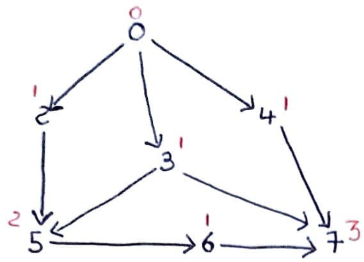
Update indegrees.

Repeat till all vertices are listed.

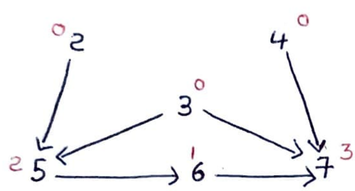
→ Topological sorting can or can not give unique sequence.



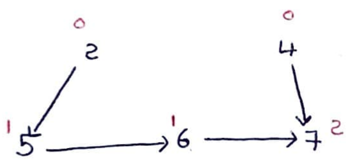
sequence : ϕ



sequence : 1

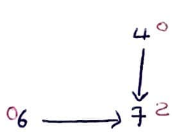


sequence : 1, 0



Sequence : 1, 0, 3

sequence : 1, 0, 3, 2



Sequence : 1, 0, 3, 2, 5

sequence : 1, 0, 3, 2, 5, 4

7

Sequence : 1, 0, 3, 2, 5, 4, 6

ϕ

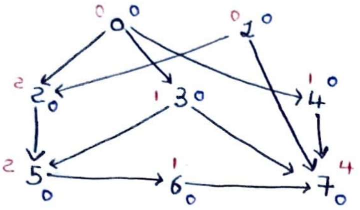
Sequence : 1, 0, 3, 2, 5, 4, 6, 7

* Longest Path in DAG:

→ Take '0' as longest path for all vertices initally.

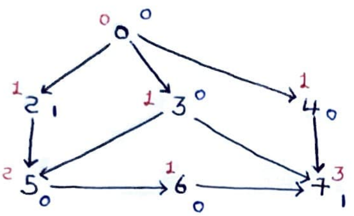
→ Remove vertex with 0 indegree and update longest path and indegrees.

→ Repeat untill all vertices listed.

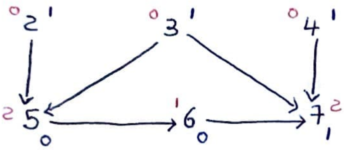


Topological order:

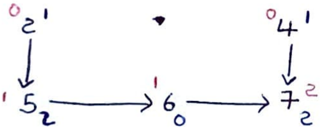
Longest path to:



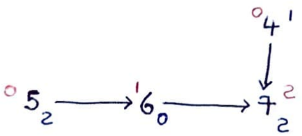
Topological order:	1
Longest path to:	0



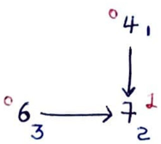
Topological order:	1	0
Longest path to:	0	0



Topological order:	1	0	3
Longest path to:	0	0	1

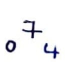


Topological order:	1	0	3	2
Longest path to:	0	0	1	1



Topological order:	1	0	3	2	5
Longest path to:	0	0	1	1	2

Topological order:	1	0	3	2	5	6
Longest path to:	0	0	1	1	2	3



Topological order:	1	0	3	2	5	6	4	7
Longest path to:	0	0	1	1	2	3	1	4

* Transitive closer :

→ $A^+[i,j]$ is 1 if and only if there is path from i to j.

$A^+ = A + A^2 + \ldots + A^{n-1}$ (where '+' is logical 'or' operator)

* matric multiplication :

→ A is $m \times n$

B is $n \times p$

C is $A \times B$ which $m \times p$

→ $C[i,j] = \sum_{k=0}^{n-1} A[i,k] \cdot B[k,j]$