Linear optimization: Mandatory exercise

Johannes Jensen, 201505594@post.au.dk

24. januar 2022

Opg 1

(a)

Maximize $5p_1 + 5p_2 + 8p_3$
s.t.

$$\begin{aligned} p_1 + 2p_2 + p_3 &\leq -2, \\ p_1 + p_2 + 2p_3 &\leq -3, \\ p_1 + 2p_2 + 2p_3 &\leq -4, \\ -p_2 &\leq 0, \\ p_3 &\leq 0, \end{aligned}$$

(b)

First we set up the auxillary linear program

Minimize $x_6 + x_7 + x_8$
s.t.

$$\begin{aligned} x_1 + x_2 + x_3 + x_6 &= 5, \\ 2x_1 + x_2 + 2x_3 - x_4 + x_7 &= 5, \\ x_1 + 2x_2 + 2x_3 + x_5 + x_8 &= 8, \\ x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8 &\geq 0. \end{aligned}$$

		x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
	-18	-4	-4	-5	1	-1	0	0	0
$x_6 =$	5	1	1	1	0	0	1	0	0
$x_7 =$	5	2	1	2	-1	0	0	1	0
$x_8 =$	8	1	2	2	0	1	0	0	1

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
-8	0	-2	-1	-1	-1	0	2	0
$x_6 =$ $\frac{5}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	1	$-\frac{1}{2}$	0
$x_1 =$ $\frac{5}{2}$	1	$\frac{1}{2}$	1	$-\frac{1}{2}$	0	0	$\frac{1}{2}$	0
$x_8 =$ $\frac{11}{2}$	0	$\frac{3}{2}$	1	$\frac{1}{2}$	1	0	$-\frac{1}{2}$	1

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
$-\frac{2}{3}$	0	0	$\frac{1}{3}$	$-\frac{1}{3}$	$\frac{1}{3}$	0	$\frac{4}{3}$	$\frac{4}{3}$
$x_6 =$ $\frac{2}{3}$	0	0	$-\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{3}$	1	$-\frac{1}{3}$	$-\frac{1}{3}$
$x_1 =$ $\frac{2}{3}$	1	0	$\frac{2}{3}$	$-\frac{2}{3}$	$-\frac{1}{3}$	0	$\frac{2}{3}$	$-\frac{1}{3}$
$x_2 =$ $\frac{11}{3}$	0	1	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	0	$-\frac{1}{3}$	$\frac{2}{3}$

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
0	0	0	0	0	0	1	1	1
$x_4 =$ 2	0	0	-1	1	-1	3	-1	-1
$x_1 =$ 2	1	0	0	0	-1	2	0	-1
$x_2 =$ 3	0	1	1	0	1	-1	0	1

Optimal solution is

$$(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = (2, 3, 0, 2, 0, 0, 0, 0)$$

Optimal solution is Q(0), so the problem is feasible, and we now find the initial basis None of the new variables are in the basis, so we have found an initial basic feasible solution!

(c)

	x_1	x_2	x_3	x_4	x_5
13	0	0	-1	0	1
$x_4 =$ 2	0	0	-1	1	-1
$x_1 =$ 2	1	0	0	0	-1
$x_2 =$ 3	0	1	1	0	1

	x_1	x_2	x_3	x_4	x_5
	16	0	1	0	0	2
$x_4 =$	5	0	1	0	1	0
$x_1 =$	2	1	0	0	0	-1
$x_3 =$	3	0	1	1	0	1

Optimal solution is

$$(x_1, x_2, x_3, x_4, x_5) = (2, 0, 3, 5, 0)$$

(g)

Standard form

Minimize $-2x_1 - 3x_2 - 4x_3$
s.t.

$$\begin{aligned} x_1 + x_2 + x_3 &= 5, \\ 2x_1 + x_2 + 2x_3 - x_4 &= 5, \\ x_1 + 2x_2 + 2x_3 + x_5 &= 8, \\ x_1, x_2, x_3, x_4, x_5 &\geq 0. \end{aligned}$$

New standard form

Minimize $-2x_1 - x_2 - x_3 - 3x_6$
s.t.

$$\begin{aligned} x_1 + x_2 + x_3 + x_6 &= 5, \\ 2x_1 + x_2 + 2x_3 - x_4 + x_6 &= 5, \\ x_1 + 2x_2 + 2x_3 + x_5 + x_6 &= 8, \\ x_1, x_2, x_3, x_4, x_5, x_6 &\geq 0. \end{aligned}$$

		x_1	x_2	x_3	x_4	x_5	x_6
	7	0	0	0	0	-1	-1
$x_4 =$	5	0	1	0	1	0	1
$x_1 =$	2	1	0	0	0	-1	1
$x_3 =$	3	0	1	1	0	1	0

		x_1	x_2	x_3	x_4	x_5	x_6
	10	0	1	1	0	0	-1
$x_4 =$	5	0	1	0	1	0	1
$x_1 =$	5	1	1	1	0	0	1
$x_5 =$	3	0	1	1	0	1	0

		x_1	x_2	x_3	x_4	x_5	x_6
	15	1	2	2	0	0	0
$x_4 =$	0	-1	0	-1	1	0	0
$x_6 =$	5	1	1	1	0	0	1
$x_5 =$	3	0	1	1	0	1	0

Optimal solution is

$$(x_1, x_2, x_3, x_4, x_5, x_6) = (0, 0, 0, 0, 3, 5)$$

Opg 2

(a)

		x_1	x_2	x_3	x_4	x_5	x_6
	$\frac{3}{2}$	0	0	0	0	$\frac{1}{2}$	0
$x_2 =$	$\frac{1}{2}$	0	1	0	0	$\frac{1}{2}$	0
$x_3 =$	$\frac{1}{2}$	0	0	1	0	$-\frac{1}{2}$	0
$x_4 =$	$\frac{1}{2}$	0	0	0	1	$\frac{1}{2}$	0
$x_6 =$	$\frac{1}{2}$	1	0	0	0	$\frac{1}{2}$	1

Optimal solution is

$$(x_1, x_2, x_3, x_4, x_5, x_6) = (0, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, 0, \tfrac{1}{2})$$

(d)

Minimize $-x_2 - x_3 - x_4$
s.t.

$$\begin{aligned} x_1 + x_2 + x_5 + x_6 &= 1, \\ x_2 + x_3 &= 1, \\ x_3 + x_4 &= 1, \\ x_2 + x_4 + x_5 &= 1, \\ x_2 &= 0, \\ x_3 - x_5 &= 0, \\ x_4 &= 0, \\ x_1 + x_6 &= 0, \\ x_1, x_2, x_3, x_4, x_5, x_6 &\geq 0. \end{aligned}$$

Fjerner de sidste 3 ligninger da de er redundante, og løser: First we set up the auxillary linear program

Minimize $x_7 + x_8 + x_9 + x_10 + x_11$
s.t.

$$\begin{aligned} x_1 + x_2 + x_5 + x_6 + x_7 &= 1, \\ x_2 + x_3 + x_8 &= 1, \\ x_3 + x_4 + x_9 &= 1, \\ x_2 + x_4 + x_5 + x_10 &= 1, \\ x_2 + x_11 &= 0, \\ x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_10, x_11 &\geq 0. \end{aligned}$$

		x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_10	x_11
	-4	-1	-4	-2	-2	-2	-1	0	0	0	0	0
$x_7 =$	1	1	1	0	0	1	1	1	0	0	0	0
$x_8 =$	1	0	1	1	0	0	0	0	1	0	0	0
$x_9 =$	1	0	0	1	1	0	0	0	0	1	0	0
$x_10 =$	1	0	1	0	1	1	0	0	0	0	1	0
$x_11 =$	0	0	1	0	0	0	0	0	0	0	0	1

		x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_10	x_11
	-3	0	-3	-2	-2	-1	0	1	0	0	0	0
$x_1 =$	1	1	1	0	0	1	1	1	0	0	0	0
$x_8 =$	1	0	1	1	0	0	0	0	1	0	0	0
$x_9 =$	1	0	0	1	1	0	0	0	0	1	0	0
$x_10 =$	1	0	1	0	1	1	0	0	0	0	1	0
$x_11 =$	0	0	1	0	0	0	0	0	0	0	0	1

		x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_10	x_11
	-3	0	0	-2	-2	-1	0	1	0	0	0	3
$x_1 =$	1	1	0	0	0	1	1	1	0	0	0	-1
$x_8 =$	1	0	0	1	0	0	0	0	1	0	0	-1
$x_9 =$	1	0	0	1	1	0	0	0	0	1	0	0
$x_10 =$	1	0	0	0	1	1	0	0	0	0	1	-1
$x_2 =$	0	0	1	0	0	0	0	0	0	0	0	1

		x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_10	x_11
	-1	0	0	0	0	-1	0	1	0	2	0	3
$x_1 =$	1	1	0	0	0	1	1	1	0	0	0	-1
$x_8 =$	0	0	0	0	-1	0	0	0	1	-1	0	-1
$x_3 =$	1	0	0	1	1	0	0	0	0	1	0	0
$x_10 =$	1	0	0	0	1	1	0	0	0	0	1	-1
$x_2 =$	0	0	1	0	0	0	0	0	0	0	0	1

		x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_10	x_11
	0	0	0	0	1	0	0	1	0	2	1	2
$x_1 =$	0	1	0	0	-1	0	1	1	0	0	-1	0
$x_8 =$	0	0	0	0	-1	0	0	0	1	-1	0	-1
$x_3 =$	1	0	0	1	1	0	0	0	0	1	0	0
$x_5 =$	1	0	0	0	1	1	0	0	0	0	1	-1
$x_2 =$	0	0	1	0	0	0	0	0	0	0	0	1

Optimal solution is

$$(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_10, x_11) = (0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0)$$

Optimal solution is Q(0), so the problem is feasible, and we now find the initial basis

		x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_10	x_11
	0	0	0	0	0	0	0	1	1	1	1	1
$x_1 =$	0	1	0	0	0	0	1	1	-1	1	-1	1
$x_4 =$	0	0	0	0	1	0	0	0	-1	1	0	1
$x_3 =$	1	0	0	1	0	0	0	0	1	0	0	-1
$x_5 =$	1	0	0	0	0	1	0	0	1	-1	1	-2
$x_2 =$	0	0	1	0	0	0	0	0	0	0	0	1

None of the new variables are in the basis, so we have found an initial basic feasible solution!

		x_1	x_2	x_3	x_4	x_5	x_6
	1	0	0	0	0	0	0
$x_1 =$	0	1	0	0	0	0	1
$x_4 =$	0	0	0	0	1	0	0
$x_3 =$	1	0	0	1	0	0	0
$x_5 =$	1	0	0	0	0	1	0
$x_2 =$	0	0	1	0	0	0	0

Optimal solution is

$$(x_1, x_2, x_3, x_4, x_5, x_6) = (0, 0, 1, 0, 1, 0)$$

Opg 3

(a)

We now have the tree T is $(s1, d1)$, $(s1, d4)$, $(s2, d1)$, $(s2, d3)$, $(s3, d2)$, $(s3, d3)$
This gives us the basic feasible solution
$f_{s1,d1} = 20$, $f_{s1,d2} = 0$, $f_{s1,d3} = 0$, $f_{s1,d4} = 16$, $f_{s2,d1} = 25$, $f_{s2,d2} = 0$, $f_{s2,d3} = 17$, $f_{s2,d4} = 0$, $f_{s3,d1} = 0$, $f_{s3,d2} = 23$, $f_{s3,d3} = 19$, $f_{s3,d4} = 0$
Now we compute the dual vector. The equation system is
$p_{s1} - p_{d1} = 6$,
$p_{s1} - p_{d4} = 4$,
$p_{s2} - p_{d1} = 2$,
$p_{s2} - p_{d3} = 9$,
$p_{s3} - p_{d2} = 3$,
$p_{s3} - p_{d3} = 8$,
$p_{d4} = 0$.
This has the solution $p = (4, 0, -1, -2, -4, -9, 0)$.
The reduced costs $\overline{c}_{ij}$ are:
$\overline{c}_{s1,d1} = 0$, $\overline{c}_{s1,d2} = -3$, $\overline{c}_{s1,d3} = -2$, $\overline{c}_{s1,d4} = 0$
$\overline{c}_{s2,d1} = 0$, $\overline{c}_{s2,d2} = 4$, $\overline{c}_{s2,d3} = 0$, $\overline{c}_{s2,d4} = 6$
$\overline{c}_{s3,d1} = 9$, $\overline{c}_{s3,d2} = 0$, $\overline{c}_{s3,d3} = 0$, $\overline{c}_{s3,d4} = 3$
Some of the reduced costs outside our tree are negative, so the f_{ij} are not optimal. Associated to one such negative cost is the edge (s1, d2) which we add to T.

(c)

We now have the tree T is $(s1, d1)$, $(s1, d4)$, $(s2, d1)$, $(s2, d3)$, $(s3, d2)$, $(s3, d3)$
This gives us the basic feasible solution
$f_{s1,d1} = 20$, $f_{s1,d2} = 0$, $f_{s1,d3} = 0$, $f_{s1,d4} = 16$, $f_{s2,d1} = 25$, $f_{s2,d2} = 0$, $f_{s2,d3} = 17$, $f_{s2,d4} = 0$, $f_{s3,d1} = 0$, $f_{s3,d2} = 23$, $f_{s3,d3} = 19$, $f_{s3,d4} = 0$
Now we compute the dual vector. The equation system is
$p_{s1} - p_{d1} = 6$,
$p_{s1} - p_{d4} = 4$,
$p_{s2} - p_{d1} = 2$,
$p_{s2} - p_{d3} = 9$,
$p_{s3} - p_{d2} = 3$,
$p_{s3} - p_{d3} = 8$,
$p_{d4} = 0$.
This has the solution $p = (4, 0, -1, -2, -4, -9, 0)$.
The reduced costs $\overline{c}_{ij}$ are:
$\overline{c}_{s1,d1} = 0$, $\overline{c}_{s1,d2} = -3$, $\overline{c}_{s1,d3} = -2$, $\overline{c}_{s1,d4} = 0$
$\overline{c}_{s2,d1} = 0$, $\overline{c}_{s2,d2} = 4$, $\overline{c}_{s2,d3} = 0$, $\overline{c}_{s2,d4} = 6$
$\overline{c}_{s3,d1} = 9$, $\overline{c}_{s3,d2} = 0$, $\overline{c}_{s3,d3} = 0$, $\overline{c}_{s3,d4} = 3$
Some of the reduced costs outside our tree are negative, so the f_{ij} are not optimal. Associated to one such negative cost is the edge (s1, d2) which we add

to T. We have the following cycle: $(s1, d2)$, $(s3, d2)$, $(s3, d3)$, $(s2, d3)$, $(s2, d1)$, $(s1, d1)$, where
F is $(s1, d2)$, $(s3, d3)$, $(s2, d1)$
B is $(s3, d2)$, $(s2, d3)$, $(s1, d1)$
We get $\theta^* = 17$ and adjust the flow accordingly. We remove (s2, d3) from T, and we now have a tree once again. We now have the tree T is $(s1, d1)$, $(s1, d4)$, $(s2, d1)$, $(s3, d2)$, $(s3, d3)$, $(s1, d2)$
This gives us the basic feasible solution
$f_{s1,d1} = 3$, $f_{s1,d2} = 17$, $f_{s1,d3} = 0$, $f_{s1,d4} = 16$, $f_{s2,d1} = 42$, $f_{s2,d2} = 0$, $f_{s2,d3} = 0$, $f_{s2,d4} = 0$, $f_{s3,d1} = 0$, $f_{s3,d2} = 6$, $f_{s3,d3} = 36$, $f_{s3,d4} = 0$
Now we compute the dual vector. The equation system is
$p_{s1} - p_{d1} = 6$,
$p_{s1} - p_{d4} = 4$,
$p_{s2} - p_{d1} = 2$,
$p_{s3} - p_{d2} = 3$,
$p_{s3} - p_{d3} = 8$,
$p_{s1} - p_{d2} = 5$,
$p_{d4} = 0$.
This has the solution $p = (4, 0, 2, -2, -1, -6, 0)$.
The reduced costs $\overline{c}_{ij}$ are:
$\overline{c}_{s1,d1} = 0$, $\overline{c}_{s1,d2} = 0$, $\overline{c}_{s1,d3} = 1$, $\overline{c}_{s1,d4} = 0$
$\overline{c}_{s2,d1} = 0$, $\overline{c}_{s2,d2} = 7$, $\overline{c}_{s2,d3} = 3$, $\overline{c}_{s2,d4} = 6$
$\overline{c}_{s3,d1} = 6$, $\overline{c}_{s3,d2} = 0$, $\overline{c}_{s3,d3} = 0$, $\overline{c}_{s3,d4} = 0$
All reduced costs outside our tree are nonnegative, which means that the basic feasible solution f_{ij} is optimal. $[3, 17, 0, 16, 42, 0, 0, 0, 0, 6, 36, 0]$