Contents

1 Voronoi Diagrams **1**
1.1 Introduction 1
1.2 Euclidean Voronoi Diagrams 4
1.3 Fortune's algorithm 6

Chapter 1

Voronoi Diagrams

1.1 Introduction

Let $\|\cdot\| : \mathbb{R}^2 \to \mathbb{R}$ be a norm. Then we define the distance function as

$$\operatorname{dist}(p, q) = \|p - q\| . \tag{1.1}$$

For $1 \leq p < \infty$ we define the L^p norm by

$$\|(x, y)\|_p = \left(|x|^p + |y|^p \right)^{1/p}, \tag{1.2}$$

and we note that $\|\cdot\|_2$ is the well-known Euclidean distance. For $p = 1$, the above reduces to

$$\|(x, y)\|_1 = |x| + |y| . \tag{1.3}$$

Letting $p \to \infty$, we also obtain the norm

$$\|(x, y)\|_\infty = \max \left(|x| , |y| \right). \tag{1.4}$$

Definition 1.1 (Voronoi diagram)**.** Let $P = \{p_1, p_2, \ldots, p_n\} \subset \mathbb{R}^2$. The cells corresponding to each point are denoted by

$$\mathcal{V}(p_i) = \{q \in \mathbb{R}^2 \mid \operatorname{dist}(q, p_i) < \operatorname{dist}(q, p_j) \text{ for all } i \neq j\}.$$

The Voronoi diagram of P, denoted $\operatorname{Vor}(P)$, is the subdivision of $\mathbb{R}^2$ consisting of the cells $\mathcal{V}(p_1), \mathcal{V}(p_2), \ldots, \mathcal{V}(p_n)$.

The following figure shows how the Voronoi diagram for 9 random points looks like with regards to some different L^p norms:

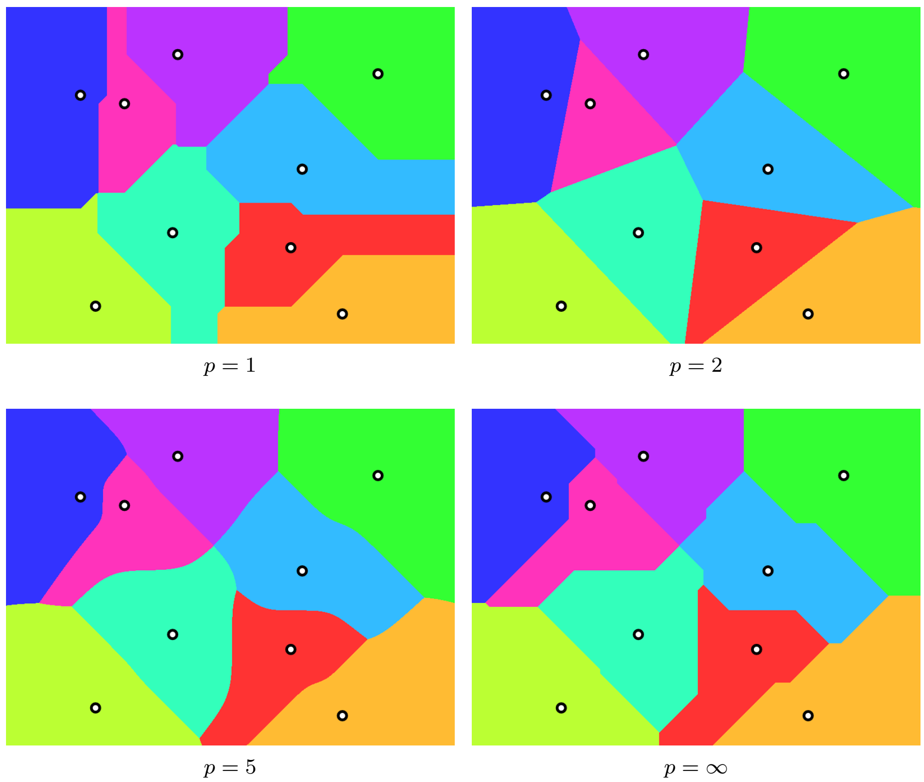


Figure 1.1: Voronoi diagrams of 9 random points using different $\|\cdot\|_p$

The above figures were generated using a very naive algorithm, which for each each pixel determinates which of the 9 points is the closest with regards to the chosen norm. A demo is available in `demos/pixel-voronoi-naive`.

Note that some of the cells may be unbounded, for example the bottom left green cell in the above figure. For $p = 1$ and $p = \infty$ the boundaries of the cells $\mathcal{V}(p_i)$ are characterised by lines, rays and segments that can only point in the 8 compass directions. For $p = 2$ the boundaries consist of lines, rays and segments which can point in any direction. Interestingly, for $2 < p < \infty$ it seems that the boundary consists of smooth curves that are not necessarily part of a line.

We now want to look at the graph structure of the Voronoi diagram. For $P = \{p_1, p_2, \ldots, p_n\} \subset \mathbb{R}^2$ the set

$$G_P = \mathbb{R}^2 - \bigcup_{i=1}^{n} \mathcal{V}(p_i) = \{q \in \mathbb{R}^2 \mid \operatorname{dist}(q, p_i) = \operatorname{dist}(q, p_j) \text{ for some } i \neq j\}$$

turns out to be an embedding of a graph, here's a visualization:

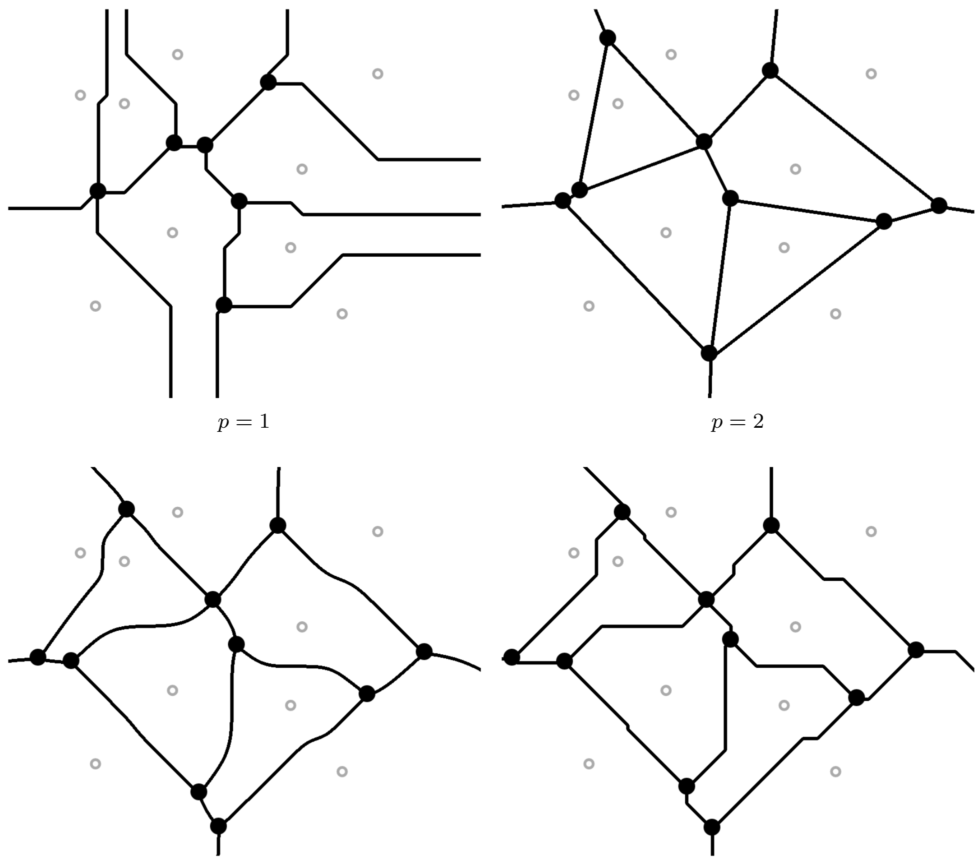
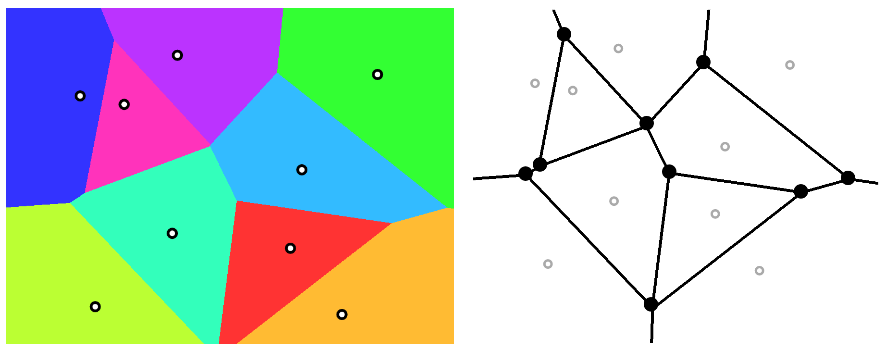


Figure 1.2: G_P of the 9 random points using different $\|\cdot\|_p$.

Note that it's the black vertices and edges which make up the graph, the gray points from P are just there for visualization. Rather than computing $\mathrm{Vor}(P)$, our algorithms will actually compute G_P, and from there be able to compute $\mathrm{Vor}(P)$. For this reason we shall actually refer to G_P as $\mathrm{Vor}(P)$ from now on.

1.2 Euclidean Voronoi Diagrams

In this section we focus on the Voronoi diagram when the norm is the Euclidean norm, that is $\|\cdot\|_2$. Here is the example from earlier:



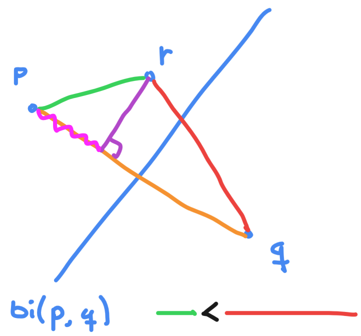
From linear algebra we know that $\|v\|_2 = \sqrt{\langle v, v\rangle}$, where $\langle\,\cdot\,,\,\cdot\,\rangle$ is the usual dot product on $\mathbb{R}^2$. Given two points $p, q \in \mathbb{R}^2$ then the **bisector** of p and q is denoted by $\mathrm{bi}(p, q) \subset \mathbb{R}^2$ and denotes the set of points on a line ℓ which passes through the midpoint of p and q and is orthogonal (w.r.t. $\langle\,\cdot\,,\,\cdot\,\rangle$) to the vector $p - q$.

A bisector $\mathrm{bi}(p, q)$ splits the plane into two **half-planes** H_p and H_q such that $p \in H_p$ and $q \in H_q$. We define $h(p, q)$ to be the open half-plane which contains p, that is the interior of H_p. So we have that

$$\mathbb{R}^2 = h(p, q) \cup \mathrm{bi}(p, q) \cup h(q, p).$$

Proposition 1.2. $r \in h(p, q)$ if and only if $\mathrm{dist}(r, p) < \mathrm{dist}(r, q)$.

Proof.



(TODO: Formalize) Proof sketch: We want to project r onto the orange line. As long as $r \in H_p$ then the squiggly pink segment is shorter than the orange segment, which will make the green segment shorter than the red segment (which is what we want to show). $\square$

Proposition 1.3. For every Voronoi cell we have

$$\mathcal{V}(p_i) = \bigcap_{\substack{1 \leq j \leq n \\ j \neq i}} h(p_i, p_j).$$

Proof. "$\subset$": Let $r \in \mathcal{V}(p_i)$. Then $\mathrm{dist}(r, p_i) < \mathrm{dist}(r, p_j)$ for all $i \neq j$. Prop 1.2 then gives us that this is equivalent to $r \in h(p_i, p_j)$ for all $i \neq j$.

"$\supset$": This argument is symmetrical to the above argument. $\square$

A Voronoi cell is thus the intersection of convex sets and is therefore convex. We conclude that the Voronoi cells are open and convex (possibly unbounded) polygons with at most $n-1$ vertices and $n-1$ edges.

We now look at the shape of the entire Voronoi diagram:

Theorem 1.4. If the points in P are collinear then $\mathrm{Vor}(P)$ consists of $n-1$ parallel lines. Otherwise, $\mathrm{Vor}(P)$ is connected and its edges are either segments or half-lines.

Proof. (TODO: .) $\square$

Finally, we show that that the complexity of the vertices and edges is $\mathcal{O}(n)$:

Theorem 1.5. For $n \geq 3$, the number of vertices in $\mathrm{Vor}(P)$ at most $2n-5$ and the number of edges is at most $3n-6$.

Proof. (TODO: .) $\square$

1.3 Fortune's algorithm

Hello world.