

# Contents

<b>1</b>	<b>Voronoi Diagrams</b>	<b>1</b>
1.1	Introduction . . . . .	1
1.2	Euclidean Voronoi Diagrams . . . . .	5
1.3	Fortune's algorithm . . . . .	11
<b>A</b>	<b>Notation</b>	<b>13</b>

# Chapter 1

## Voronoi Diagrams

### 1.1 Introduction

Let  $\|\cdot\| : \mathbb{R}^2 \rightarrow \mathbb{R}$  be a norm. Then we define the distance function as

$$\text{dist}(p, q) = \|p - q\|. \quad (1.1)$$

For  $1 \leq p < \infty$  we define the  $L^p$  norm by

$$\|(x, y)\|_p = (|x|^p + |y|^p)^{1/p}, \quad (1.2)$$

and we note that  $\|\cdot\|_2$  is the well-known Euclidean distance. For  $p = 1$ , the above reduces to

$$\|(x, y)\|_1 = |x| + |y|. \quad (1.3)$$

Letting  $p \rightarrow \infty$ , we also obtain the norm

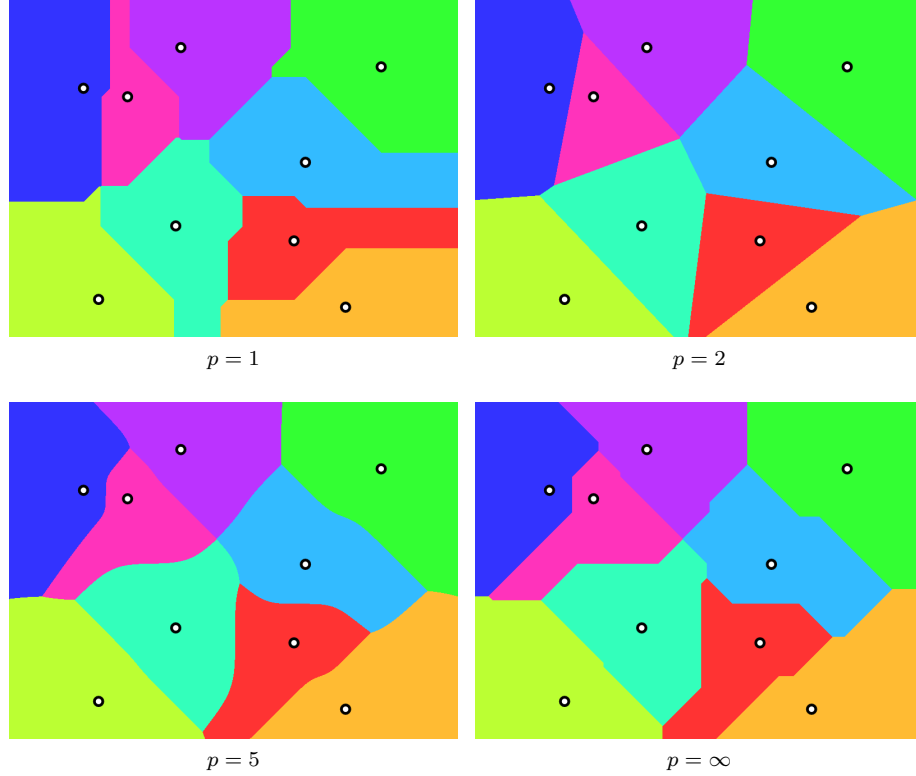
$$\|(x, y)\|_\infty = \max(|x|, |y|). \quad (1.4)$$

**Definition 1.1** (Voronoi diagram). Let  $P = \{p_1, p_2, \dots, p_n\} \subset \mathbb{R}^2$ . The cells corresponding to each point are denoted by

$$\mathcal{V}(p_i) = \{q \in \mathbb{R}^2 \mid \text{dist}(q, p_i) < \text{dist}(q, p_j) \text{ for all } i \neq j\}.$$

The Voronoi diagram of  $P$ , denoted  $\text{Vor}(P)$ , is the subdivision of  $\mathbb{R}^2$  consisting of the union of the cells  $\mathcal{V}(p_1), \mathcal{V}(p_2), \dots, \mathcal{V}(p_n)$ .

The following figure shows how the Voronoi diagram for 9 random points looks like with regards to some different  $L^p$  norms:

Figure 1.1:  $\text{Vor}(P)$  of 9 random points using different  $\|\cdot\|_p$ 

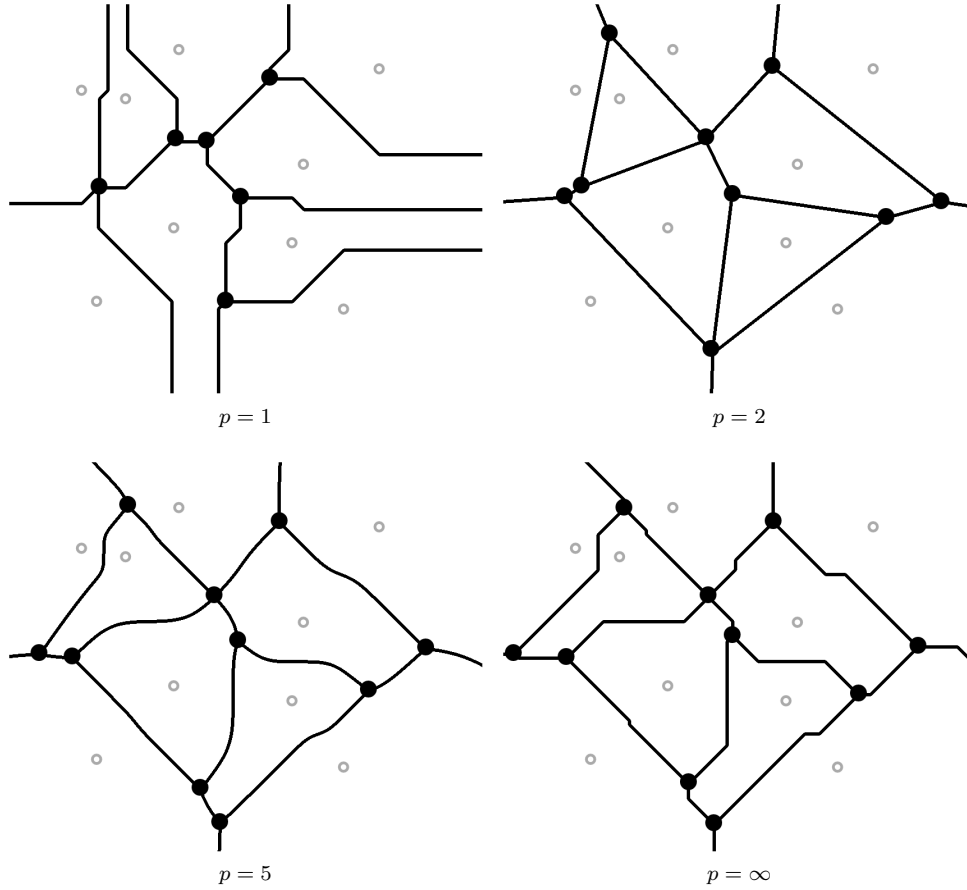
The above figures were generated using a very naive algorithm, which for each pixel determines which of the 9 points is the closest with regards to the chosen norm. A demo is available in [demos/pixel-voronoi-naive](#).

Note that some of the cells may be unbounded, for example the bottom left green cell in the above figure. For  $p = 1$  and  $p = \infty$  the boundaries of the cells  $\mathcal{V}(p_i)$  are characterised by lines, rays and segments that can only point in the 8 compass directions. For  $p = 2$  the boundaries consist of lines, rays and segments which can point in any direction. Interestingly, for  $2 < p < \infty$  it seems that the boundary consists of smooth curves that are not necessarily part of a line.

We now want to look at the graph structure of the Voronoi diagram. For  $P = \{p_1, p_2, \dots, p_n\} \subset \mathbb{R}^2$  the set

$$\text{Vor}_G(P) = \mathbb{R}^2 - \text{Vor}(P) = \{q \in \mathbb{R}^2 \mid \text{dist}(q, p_i) = \text{dist}(q, p_j) \text{ for some } i \neq j\}$$

turns out to be an embedding of a graph, where some of the edges are infinite, here's a visualization:

Figure 1.2:  $\text{Vor}_G(P)$  of the 9 random points using different  $\|\cdot\|_p$ .

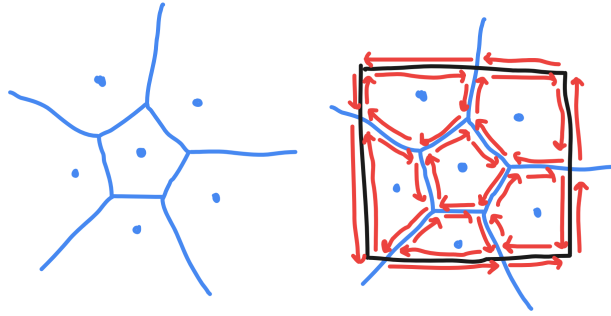
The above figures were generated by first generating the images from Figure 1.1 and then performing the following algorithm: For each pixel, we look at the surrounding pixels within a small disk about that point, and if it contains exactly 2 different colors, we know that we're looking at an edge, so we color the pixel black, and if we see 3 colors or more, we know that we're at a vertex. If we only see 1 color, then we just color the pixel white.

Note that it's the black vertices and edges which make up the graph, the gray points from  $P$  are just there for visualization. Rather than computing  $\text{Vor}(P)$ , our algorithms will actually compute  $\text{Vor}_G(P)$ , and from there be able to compute  $\text{Vor}(P)$ .

Now, a natural question arises: how do we store Voronoi diagrams? We'll need the following geometric data structure:

**Definition 1.2 (DCEL).** (TODO: Define the DCEL.)

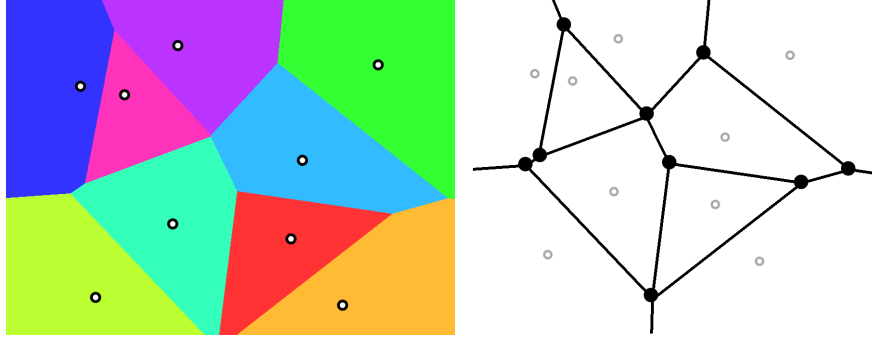
Note that the DCEL does not support infinite edges, so what we do is put a bounding box  $B$  with some padding around the vertices of  $\text{Vor}(P)$ , and then intersect the infinite edges and faces with the boundary of  $B$  and only keep the part inside the bounding box.



The aim of our algorithms will then be to calculate the DCEL in the right figure.

## 1.2 Euclidean Voronoi Diagrams

In this section we focus on proving some properties of the Voronoi diagram when the norm is the Euclidean norm, that is  $\|\cdot\|_2$ . Here is the example from earlier:



From linear algebra we know that  $\|v\|_2 = \sqrt{\langle v, v \rangle}$ , where  $\langle \cdot, \cdot \rangle$  is the usual dot product on  $\mathbb{R}^2$ . Given two points  $p, q \in \mathbb{R}^2$  then the **bisector** of  $p$  and  $q$  is denoted by  $\text{bi}(p, q) \subset \mathbb{R}^2$  and denotes the set of points on a line  $\ell$  which passes through the midpoint of  $p$  and  $q$  and is orthogonal (w.r.t.  $\langle \cdot, \cdot \rangle$ ) to the vector  $p - q$ .

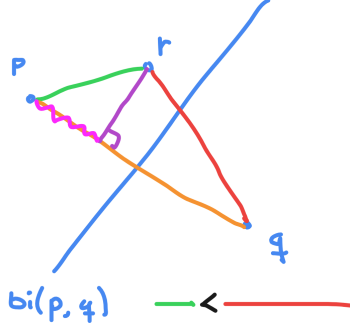


A bisector  $\text{bi}(p, q)$  splits the plane into two **half-planes**  $H_p$  and  $H_q$  such that  $p \in H_p$  and  $q \in H_q$ . We define  $h(p, q)$  to be the open half-plane which contains  $p$ , that is the interior of  $H_p$ . So we have that

$$\mathbb{R}^2 = h(p, q) \cup \text{bi}(p, q) \cup h(q, p).$$

**Proposition 1.3.**  $r \in h(p, q)$  if and only if  $\text{dist}(r, p) < \text{dist}(r, q)$ .

*Proof.*



(TODO: Formalize) Proof sketch: We want to project  $r$  onto the orange line. As long as  $r \in H_p$  then the squiggly pink segment is shorter than the orange segment, which will make the green segment shorter than the red segment (which is what we want to show).  $\square$

**Corollary 1.4.** For every Voronoi cell we have

$$\mathcal{V}(p_i) = \bigcap_{\substack{1 \leq j \leq n \\ j \neq i}} h(p_i, p_j).$$

*Proof.* “ $\subset$ ”: Let  $r \in \mathcal{V}(p_i)$ . Then  $\text{dist}(r, p_i) < \text{dist}(r, p_j)$  for all  $i \neq j$ . Prop 1.3 then gives us that this is equivalent to  $r \in h(p_i, p_j)$  for all  $i \neq j$ .

“ $\supset$ ”: This argument is symmetrical to the above argument.  $\square$

A Voronoi cell is thus the intersection of convex sets and is therefore convex. We conclude that the Voronoi cells are open and convex (possibly unbounded) polygons with at most  $n - 1$  vertices and  $n - 1$  edges.

We now look at the shape of the entire Voronoi diagram. From Corollary 1.4 it follows that the edges of  $\text{Vor}_G(P)$  are made up of parts of straight lines, namely the bisectors between different points of  $P$ . We now classify these based on the structure of the points in  $P$ :

**Theorem 1.5.** If the points in  $P$  are collinear then  $\text{Vor}_G(P)$  consists of  $n - 1$  parallel lines. Otherwise,  $\text{Vor}_G(P)$  is connected and its edges are either segments or half-lines.

*Proof.* Assume that the points in  $P$  are collinear. By applying an isometry to  $P$ , we may assume without loss of generality that the points of  $P$  lie on the  $x$ -axis:

$$P = \{(x_1, 0), (x_2, 0), \dots, (x_n, 0)\},$$

where we assume that  $x_1 < x_2 < \dots < x_n$  by rearranging the points if necessary. See the proof of Theorem 1.9 for a visualization of  $\text{Vor}(P)$ . By definition, we

have that  $p \in \text{Vor}_G(P)$  if and only if  $p \notin \mathcal{V}(x_i, 0)$  for all  $i$ . Let  $(x, y) \in \mathbb{R}^2$  such that  $x_i < x < x_{i+1}$ . Then  $(x, y) \in \text{Vor}_G(P)$  if

$$\text{dist}((x, y), (x_i, 0)) = \text{dist}((x, y), (x_{i+1}, 0)).$$

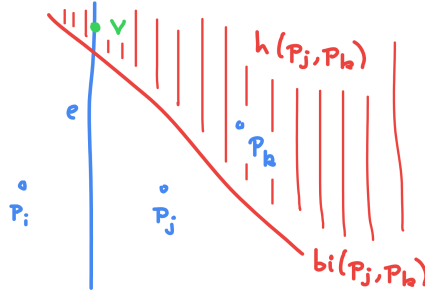
If furthermore  $(x, y) \in \text{Vor}_G(P)$  then we get

$$\begin{aligned} \|(x, y) - (x_i, 0)\| &= \|(x, y) - (x_{i+1}, 0)\| \\ \iff \sqrt{(x - x_i)^2 + y^2} &= \sqrt{(x - x_{i+1})^2 + y^2} \\ \iff |x - x_i| &= |x - x_{i+1}|. \end{aligned}$$

Thus if  $(x, 0) \in \text{Vor}_G(P)$  then  $(x, y) \in \text{Vor}_G(P)$  for all  $y \in \mathbb{R}$ . This shows that  $\text{bi}((x_i, 0), (x_{i+1}, 0)) \subset \text{Vor}_G(P)$  for all  $i < n$ . Every point of  $\text{Vor}_G(P)$  is on one of these bisectors, and the bisectors are all parallel, which proves the claim.

(TODO: Clean up above argument and consider if anything is missing.)

Assume that the points in  $P$  are not collinear. First, we show that the edges of  $\text{Vor}_G(P)$  are either segments or half-lines. Suppose for a contradiction that there is an edge  $e$  of  $\text{Vor}_G(P)$  that is a full line and assume that  $e \in \partial\mathcal{V}(p_i) \cap \partial\mathcal{V}(p_j)$ . Let  $p_k \in P$  be a point which is not collinear with  $p_i$  and  $p_j$ . Then the line  $\text{bi}(p_j, p_k)$  is not parallel to the line  $e$ , hence they have an intersection point. Then there exists a point  $v \in e \cap \circ h(p_k, p_j)$ . The situation is visualized here:



We have that  $v \in \partial\mathcal{V}(p_j)$  by definition of  $e$ . Now note that

$$\partial\mathcal{V}(p_j) = \partial \left( \bigcap_{a \neq j} h(p_j, p_a) \right) \subset^1 \bigcup_{a \neq j} \partial h(p_j, p_a) = \bigcup_{a \neq j} \text{bi}(p_j, p_a).$$

As  $v \in h(p_k, p_j)$  we have that  $\text{dist}(v, p_k) < \text{dist}(v, p_j)$ , hence  $v \notin \text{bi}(p_j, p_k)$ , so  $v \notin \partial\mathcal{V}(p_j)$  by the above characterization of  $\partial\mathcal{V}(p_j)$ . This is a contradiction, so  $e$  can't be a full line. Now we show that  $\text{Vor}_G(P)$  is connected. Assume for the sake of a contradiction that  $\text{Vor}_G(P)$  is not connected. Then there exists

<sup>1</sup>Here we used that  $\partial(A \cap B) \subset \partial A \cup \partial B$ , a proof is here: [https://proofwiki.org/wiki/Boundary\\_of\\_Intersection\\_is\\_Subset\\_of\\_Union\\_of\\_Boundaries](https://proofwiki.org/wiki/Boundary_of_Intersection_is_Subset_of_Union_of_Boundaries) (TODO: Remove this footnote and add the result to some topology appendix)



a  $\partial\mathcal{V}(p_i)$  which is not path connected. This can only happen if  $\partial\mathcal{V}(p_i)$  consists of two parallel lines (TODO: Why?). This contradicts the fact that  $\text{Vor}_G(P)$  contains no lines. Thus  $\text{Vor}_G(P)$  is connected.  $\square$

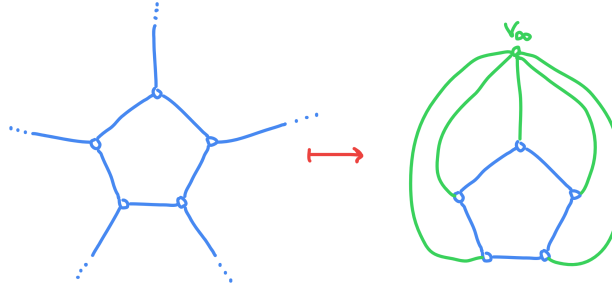
Finally, we show that the complexity of the vertices and edges is  $\mathcal{O}(n)$ :

**Theorem 1.6.** For  $n \geq 3$ , the number of vertices in  $\text{Vor}_G(P)$  is at most  $2n - 5$  and the number of edges is at most  $3n - 6$ .

*Proof.* If the points in  $P$  are collinear, then Theorem 1.5 implies the claim. Now assume that the points in  $P$  are not collinear. As a first preprocessing step, we start by transforming  $\text{Vor}_G(P)$  into an actual plane graph, as some of the edges in  $\text{Vor}_G(P)$  may be half-lines. Let  $v_1, \dots, v_k$  denote the vertices of  $\text{Vor}_G(P)$ . Let  $p = \frac{1}{k}(v_1 + v_2 + \dots + v_k) \in \mathbb{R}^2$  and let

$$r = 1 + \max\{\text{dist}(p, v_1), \text{dist}(p, v_2), \dots, \text{dist}(p, v_k)\}.$$

Then let  $B_r(p) \subset \mathbb{R}^2$  denote the open ball with center  $p$  and radius  $r$ . We have that  $B_r(p)$  contains every vertex  $v_i$  and that every half-line edge  $e$  of  $\text{Vor}_G(P)$  intersects  $\partial B_r(p)$  exactly once. Now define  $v_\infty \in \mathbb{R}^2$  as any point in  $\mathbb{R}^2 - B_r(p)$  and transform every half-line edge  $e$  into a path with finite length by connecting the half-lines to the point  $v_\infty$ . This is possible since  $\mathbb{R}^2 - B_r(p)$  only contains these half-lines, and every half-line is pointing in a unique direction so we may then transform the half-lines in order by starting with those which are closest to  $v_\infty$ . An example of this construction is given here:



In this way we can turn  $\text{Vor}_G(P)$  into a planar graph. For a planar graph  $G$ , Euler's formula<sup>2</sup> states that

$$V - E + F = 2, \tag{1.5}$$

where  $V$  is the number of vertices,  $E$  is the number of edges and  $F$  is the number of faces of  $G$ . Let  $n_v$  denote the number of vertices of the original  $\text{Vor}_G(P)$ , and let  $n_e$  denote the number of edges. In our modification, we only added a single vertex, so by plugging into (1.5) we obtain the following relationship:

$$(n_v + 1) - n_e + n = 2. \tag{1.6}$$

<sup>2</sup>(TODO: Add a reference and/or proof of Euler's formula in some topology appendix)

Note that  $n$  is the number of faces, since we have a Voronoi cell for each point in  $P$ . Every vertex  $v$  in  $G$  has  $\deg(v) \geq 3$ , otherwise there would be a  $\mathcal{V}(p_i)$  which is not convex. This means that

$$\sum_{v \in V(G)} \deg(v) \geq 3 |V(G)| = 3(n_v + 1).$$

Now we want to compute the left side of the above inequality. Given a vertex  $v$  we have that  $\deg(v)$  counts the number of edges which touch  $v$ , and in  $G$  every edge touches exactly 2 vertices, which gives us that  $\sum_{v \in V(G)} \deg(v) = 2n_e$ . Combining these facts, we obtain the inequality:

$$2n_e \geq 3(n_v + 1). \quad (1.7)$$

Multiplying (1.6) by 2, isolating  $2n_e$  and then applying (1.7) we get:

$$\begin{aligned} 2(n_v + 1) - 2n_e + 2n = 4 &\iff 2n_e = (2n_v + 1) + 2n - 4 \\ &\implies 3(n_v + 1) \leq 2(n_v + 1) + 2n - 4 \\ &\implies n_v \leq 2n - 5. \end{aligned}$$

Multiplying (1.6) by 3, isolating  $3(n_v + 1)$  and then applying (1.7) we get:

$$\begin{aligned} 3(n_v + 1) - 3n_e + 3n = 6 &\iff 3(n_v + 1) = 3n_e - 3n + 6 \\ &\implies 2n_e \geq 3n_e - 3n + 6 \\ &\implies n_e \leq 3n - 6. \end{aligned}$$

This proves the theorem.  $\square$

We thus have a linear number of vertices and edges  $\text{Vor}_G(P)$ , but we have a quadratic number of bisectors  $\text{bi}(p_i, p_j)$  of which every edge of  $\text{Vor}_G(P)$  is a subset of, and every vertex in  $\text{Vor}_G(P)$  is an intersection point of two such bisectors. Thus it would be interesting to characterize when a particular bisector is a part of  $\text{Vor}_G(P)$ . First, we need a definition:

**Definition 1.7** (Largest empty circle). For a  $q \in \mathbb{R}^2$  we define  $C_P(q)$  to be the largest empty circle of  $q$  with respect to  $P$ , which is the largest empty circle with  $q$  as its center that does not contain any point of  $P$  in its interior. Formally,

$$C_P(q) = B_r(q), \quad \text{where } r = \sup\{\lambda \in \mathbb{R}^+ \mid B_\lambda(q) \cap P = \emptyset\}.$$

**Theorem 1.8.** The bisectors and their intersections are characterized by:

- (i)  $q \in \mathbb{R}^2$  is a vertex of  $\text{Vor}_G(P)$  if and only if

$$|\partial C_P(q) \cap P| \geq 3.$$

- (ii)  $\text{bi}(p_i, p_j)$  defines an edge of  $\text{Vor}_G(P)$  if and only if

$$\exists q \in \text{bi}(p_i, p_j) : \partial C_P(q) \cap P = \{p_i, p_j\}.$$

*Proof.* We prove each statement individually:

- (i): “ $\Leftarrow$ ”: Let  $q \in \mathbb{R}^2$  and assume that  $|\partial C_P(q) \cap P| \geq 3$ . Let  $p_i, p_j, p_k$  be three distinct points from  $\partial C_P(q) \cap P$ . Since  $C_P(q) \cap P = \emptyset$  by definition, this means that  $q$  is equally close to  $p_i, p_j, p_k$  but not closer to any other points in  $P$ , so  $q \in \partial \mathcal{V}(p_i) \cap \partial \mathcal{V}(p_j) \cap \partial \mathcal{V}(p_k) \subset \text{Vor}_G(P)$ , and it is a vertex since it is at an intersection of 3 or more bisectors.

“ $\Rightarrow$ ”: Let  $q \in \mathbb{R}^2$  be a vertex of  $\text{Vor}_G(P)$ . A vertex of  $\text{Vor}_G(P)$  touches at least 3 different edges, and thus touches at least 3 distinct Voronoi cells  $\mathcal{V}(p_i), \mathcal{V}(p_j)$  and  $\mathcal{V}(p_k)$ . So  $q \in \partial \mathcal{V}(p_i) \cap \partial \mathcal{V}(p_j) \cap \partial \mathcal{V}(p_k)$ . This gives us that

$$\text{dist}(q, p_i) = \text{dist}(q, p_j) = \text{dist}(q, p_k).$$

Denote the above distance by  $D$ . Now assume for the sake of a contradiction that there exists  $p_\alpha \in P$  such that  $\text{dist}(q, p_\alpha) < D$ . Then there are parts of the bisectors  $\text{bi}(p_\alpha, p_i), \text{bi}(p_\alpha, p_j), \text{bi}(p_\alpha, p_k)$  contained inside  $B_D(q)$ , which means that  $\mathcal{V}(p_i), \mathcal{V}(p_j), \mathcal{V}(p_k)$  do not all meet at  $q$ , a contradiction. This means that  $C_P(q) \cap P = \emptyset$  and  $p_i, p_j, p_k \in \partial C_P(q)$ .

- (ii): “ $\Leftarrow$ ”: Let  $q \in \text{bi}(p_i, p_j)$  such that  $\partial C_P(q) \cap P = \{p_i, p_j\}$ . So  $C_P(q) \cap P = \emptyset$ , which by definition of  $C_P(q)$  means that

$$\text{dist}(q, p_i) = \text{dist}(q, p_j) \leq \text{dist}(q, p_k)$$

for all  $k$ . So  $q \in \text{Vor}_G(P)$  and is either a vertex or an edge. Since  $|\partial C_P(q) \cap P| < 3$  part (i) gives us that  $q$  is not a vertex, hence it must be an edge, which is a subset of  $\text{bi}(p_i, p_j)$ .

“ $\Rightarrow$ ”: Let  $e \subset \text{bi}(p_i, p_j)$  be an edge of  $\text{Vor}_G(P)$ . For  $q \in e$  we have that  $\text{dist}(q, p_i) = \text{dist}(q, p_j)$ , and that  $q$  touches  $\mathcal{V}(p_i)$  and  $\mathcal{V}(p_j)$ . By applying the same contradiction proof as in (i) “ $\Rightarrow$ ” we have that there is no point in  $P$  which is closer to  $q$  than  $p_i$  and  $p_j$ , thus  $\partial C_P(q) \cap P = \{p_i, p_j\}$ .

□

### 1.3 Fortune's algorithm

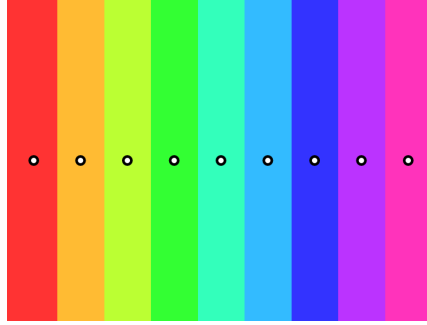
In this section we will present an algorithm which computes  $\text{Vor}(P)$  in  $\mathcal{O}(n \log n)$  time. This is actually optimal, as we can use a Voronoi diagram for sorting:

**Theorem 1.9.** We can't do better than  $\mathcal{O}(n \log n)$ .

*Proof.* Let  $A = \{a_1, a_2, \dots, a_n\} \subset \mathbb{R}$ . Now assume we have used an algorithm to compute a Voronoi diagram of the points

$$P = \{(a_1, 0), (a_2, 0), \dots, (a_n, 0)\}.$$

We obtain a diagram which looks similar to this:



We assume without loss of generality that the algorithm outputs a DCEL  $\Delta$  of  $\text{Vor}(P)$ . Assume that the **edge** pointer of every face of  $\Delta$  points to the edge to the right of the face, and that the **face** pointer of every edge of  $\Delta$  points to the face to the right. Let  $F_i$  be the face in  $\Delta$  which contains the point  $(0, a_i)$ . Let  $\ell \in \mathbb{N}$  such that  $a_\ell < a_i$  for all  $i \neq \ell$ . Let  $b_1 = a_\ell$  and if  $b_i = a_j$  and  $i < n$  then define  $b_{i+1} = a_k$  where  $k$  comes from  $F_j.\text{edge.face} = F_k$ . Then  $(b_1, b_2, \dots, b_n)$  is the elements of  $A$  in sorted order. This means that we can use the Voronoi diagram to sort, which proves the theorem.  $\square$

(TODO: The statement of the above theorem is temporary. I originally phrased it like so: “The optimal worst-case running time for computing  $\text{Vor}(P)$  is  $\Omega(n \log n)$ .” What is the proper terminology here?)

Consider a point  $p = (p_x, p_y) \in \mathbb{R}^2$  and a sweep line  $y = \ell_y$  with  $p_y > \ell_y$ . We define the distance between  $p$  and  $\ell$  as

$$\text{dist}(p, \ell) = p_y - \ell_y.$$

We can split  $\mathbb{R} \times (\ell_y, \infty)$  into two components  $R_p$  and  $R_\ell$  with a common boundary, where

$$\begin{aligned} q \in R_p &\implies \text{dist}(q, p) < \text{dist}(q, \ell) \\ q \in R_\ell &\implies \text{dist}(q, p) > \text{dist}(q, \ell) \end{aligned}$$

and where their boundary  $\partial R_p = \partial R_\ell$  is given as follows: We're interested in computing the  $q \in \mathbb{R} \times [\ell_y, \infty)$  which satisfy  $\text{dist}(q, p) = \text{dist}(q, \ell)$ . Let  $q = (x, y)$  with  $y \geq \ell_y$  be such a point. Since distances are non-negative, it is equivalent to looking at satisfying  $\text{dist}(q, p)^2 = \text{dist}(q, \ell)^2$ . We have:

$$\text{dist}(q, p)^2 = \text{dist}(q, \ell)^2 \iff (p_x - x)^2 + (p_y - y)^2 = (y - \ell_y)^2.$$

This can be transformed into the equation

$$2(p_y - \ell_y)y = x^2 - 2p_x x + p_x^2 + p_y^2 - \ell_y^2. \quad (1.8)$$

Since  $p_y \neq \ell_y$  by assumption, we obtain the parabola:

$$y = \frac{1}{2(p_y - \ell_y)}(x^2 - 2p_x x + p_x^2 + p_y^2 - \ell_y^2). \quad (1.9)$$

Given points  $P = \{p_1, p_2, \dots, p_n\}$  let  $\beta_i(x)$  denote the parabola (1.9) for the point  $p = p_i$ .

**Definition 1.10** (Beach line). The *beach line* for the points  $P$  with regards to the sweep line  $\ell$  is given by the graph of the function

$$\text{BEACH}(x) = \min\{\beta_1(x), \beta_2(x), \dots, \beta_n(x)\}.$$

**Definition 1.11** (Breakpoint). The intersection of two different  $\beta_i$  and  $\beta_j$  that lie on the beach line is called a *breakpoint*.

**Theorem 1.12.** The breakpoints exactly trace out  $\text{Vor}_G(P)$  as the sweep line  $\ell$  moves from top to bottom.

*Proof.* (TODO: Say some nice things) □

GeoGebra demo of two points and their beach line

<https://www.geogebra.org/calculator/zreutrt6>

# Appendix A

## Notation

$X - Y$	Set difference
$ X $	The number of elements in a finite set $X$ .
$\iff$	If and only if
$\implies$	Implication
$\mathbb{R}$	The real numbers.
$\mathbb{R}^n$	The vector space of $n$ -tuples of real numbers.
$\ \cdot\ $	Norm.
$\ \cdot\ _p$	The $L^p$ norm.
$ x $	Absolute value if $x$ is a number.
$\text{dist}(p, q)$	The distance between $p$ and $q$ , given by $\ p - q\ $ .
$\langle \cdot, \cdot \rangle$	An inner product.
$\subset$	Subset (not strict, e.g. $A = B \implies A \subset B$ ).
$P$	A set of points $\{p_1, p_2, \dots, p_n\}$ that we want to apply an algorithm to.
$p_i$	A point in $P$ (see above).
$n$	If not otherwise specified, $n$ is the number of points in $P$ (see above).
$\text{Vor}(P)$	The Voronoi diagram of $P$ .
$\mathcal{V}(p_i)$	The $i$ th Voronoi cell.
$\text{Vor}_G(P)$	Refers to $\mathbb{R}^2 - \text{Vor}(P)$ .
$\mathcal{O}(f(n))$	Big $O$ -notation.
$\text{bi}(p, q)$	Bisector of $p$ and $q$ .
$h(p, q)$	Open half-plane containing $p$ with $\text{bi}(p, q)$ as boundary.
$\overline{X}$	The closure of a set $X \subset \mathbb{R}^n$ , given by the union of $X$ with its limit points.
$\circ X$	The interior of a set $X \subset \mathbb{R}^n$ , given by the union of all interior points of $X$ .
$\partial X$	The boundary of a set $X \subset \mathbb{R}^n$ , given by $\overline{X} - \circ X$ .
$\overline{B}_r(p)$	$= \{x \in \mathbb{R}^n \mid \text{dist}(x, p) \leq r\}$ , the closed ball with center $p$ and radius $r$ .
$B_r(p)$	$= \{x \in \mathbb{R}^n \mid \text{dist}(x, p) < r\}$ , the open ball with center $p$ and radius $r$ .
$\partial B_r(p)$	$= \{x \in \mathbb{R}^n \mid \text{dist}(x, p) = r\}$ , the circle with center $p$ and radius $r$ .
$V(G)$	The set of vertices for the graph $G$ .
$E(G)$	The set of edges for the graph $G$ .
$\deg(v)$	The degree of a vertex $v$ in a graph, e.g. the number of edges that touch $v$ .