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# Chapter 1

# Voronoi Diagrams

## 1.1 Introduction

Let $\|\cdot\| : \mathbb{R}^2 \to \mathbb{R}$ be a norm. Then we define the distance function as

$$\operatorname{dist}(p, q) = \|p - q\| . \tag{1.1}$$

For $1 \leq p < \infty$ we define the $L^p$ norm by

$$\|(x, y)\|_p = \left( |x|^p + |y|^p \right)^{1/p}, \tag{1.2}$$

and we note that $\|\cdot\|_2$ is the well-known Euclidean distance. For $p = 1$, the above reduces to

$$\|(x, y)\|_1 = |x| + |y| . \tag{1.3}$$

Letting $p \to \infty$, we also obtain the norm

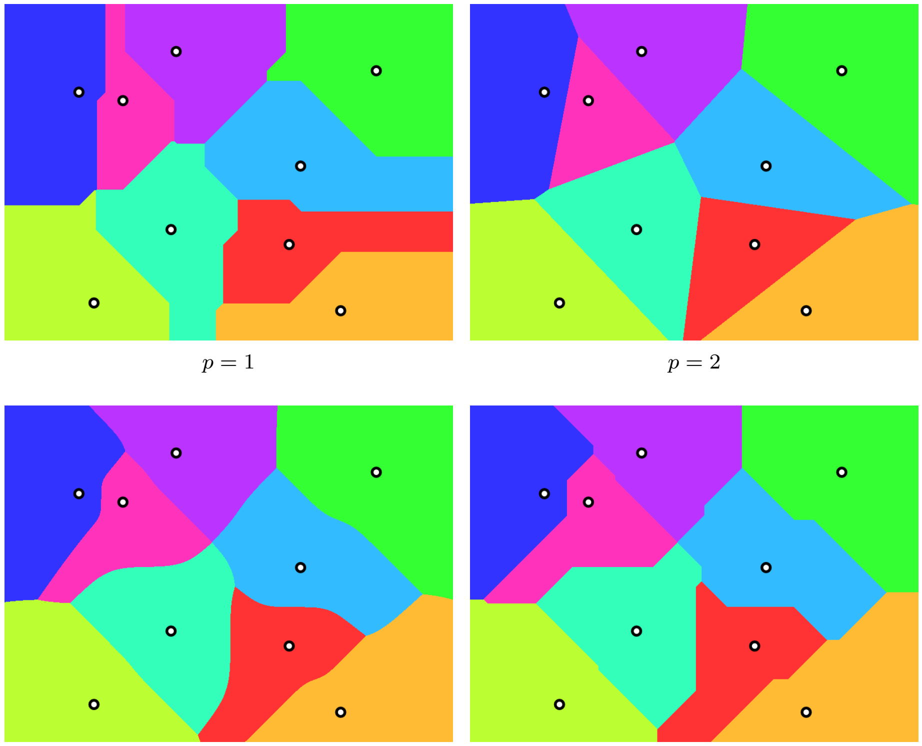
$$\|(x, y)\|_\infty = \max\left( |x| , |y| \right). \tag{1.4}$$

**Definition 1.1** (Voronoi diagram)**.** Let $P = \{p_1, p_2, \ldots, p_n\} \subset \mathbb{R}^2$. The cells corresponding to each point are denoted by

$$\mathcal{V}(p_i) = \{q \in \mathbb{R}^2 \mid \operatorname{dist}(q, p_i) < \operatorname{dist}(q, p_j) \text{ for all } i \neq j\}.$$

The Voronoi diagram of $P$, denoted $\operatorname{Vor}(P)$, is the subdivision of $\mathbb{R}^2$ consisting of the union of the cells $\mathcal{V}(p_1), \mathcal{V}(p_2), \ldots, \mathcal{V}(p_n)$.

The following figure shows how the Voronoi diagram for 9 random points looks like with regards to some different $L^p$ norms:



$p = 1$ $p = 2$

$p = 5$ $p = \infty$

Figure 1.1: $\mathrm{Vor}(P)$ of 9 random points using different $\|\cdot\|_p$

The above figures were generated using a very naive algorithm, which for each each pixel determinates which of the 9 points is the closest with regards to the chosen norm. A demo is available in demos/pixel-voronoi-naive.

Note that some of the cells may be unbounded, for example the bottom left green cell in the above figure. For $p = 1$ and $p = \infty$ the boundaries of the cells $\mathcal{V}(p_i)$ are characterised by lines, rays and segments that can only point in the 8 compass directions. For $p = 2$ the boundaries consist of lines, rays and segments which can point in any direction. Interestingly, for $2 < p < \infty$ it seems that the boundary consists of smooth curves that are not necessarily part of a line.

We now want to look at the graph structure of the Voronoi diagram. For $P = \{p_1, p_2, \ldots, p_n\} \subset \mathbb{R}^2$ the set

$$\mathrm{Vor}_{\mathrm{G}}(P) = \mathbb{R}^2 - \mathrm{Vor}(P) = \{q \in \mathbb{R}^2 \mid \mathrm{dist}(q, p_i) = \mathrm{dist}(q, p_j) \text{ for some } i \neq j\}$$

turns out to be an embedding of a graph, where some of the edges are infinite, here's a visualization:

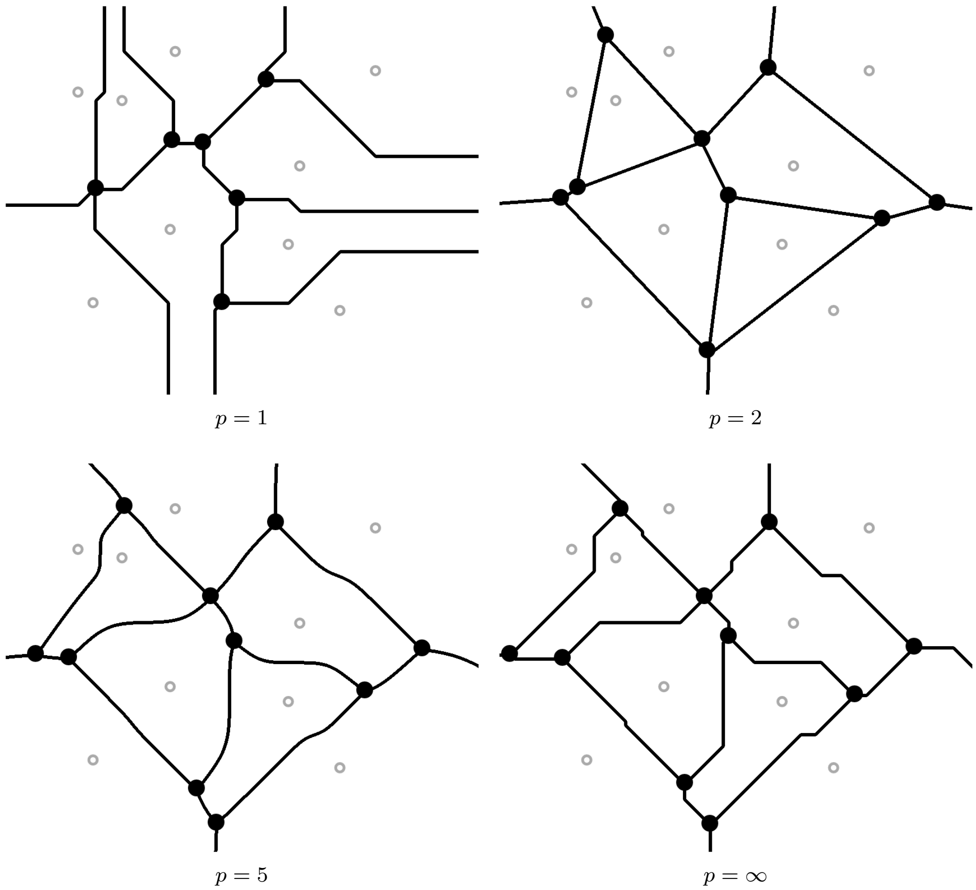



Figure 1.2: $\mathrm{Vor}_{\mathrm{G}}(P)$ of the 9 random points using different $\|\cdot\|_p$.

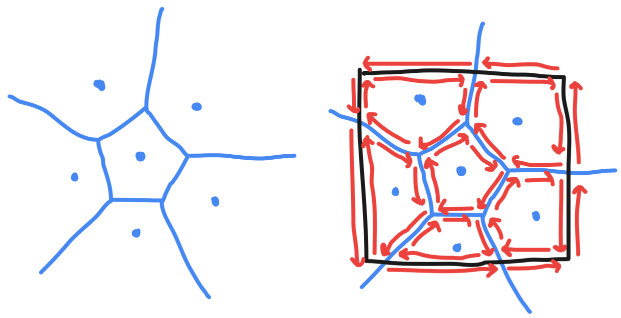
The above figures were generated by first generating the images from Figure 1.1 and then performing the following algorithm: For each pixel, we look at the surrounding pixels within a small disk about that point, and if it contains exactly 2 different colors, we know that we're looking at an edge, so we color the pixel black, and if we see 3 colors or more, we know that we're at a vertex. If we only see 1 color, then we just color the pixel white.

Note that it's the black vertices and edges which make up the graph, the gray points from $P$ are just there for visualization. Rather than computing $\mathrm{Vor}(P)$, our algorithms will actually compute $\mathrm{Vor}_{\mathrm{G}}(P)$, and from there be able to compute $\mathrm{Vor}(P)$.

Now, a natural question arises: how do we store Voronoi diagrams? We'll need the following geometric data structure:

**Definition 1.2** (DCEL). (TODO: Define the DCEL.)

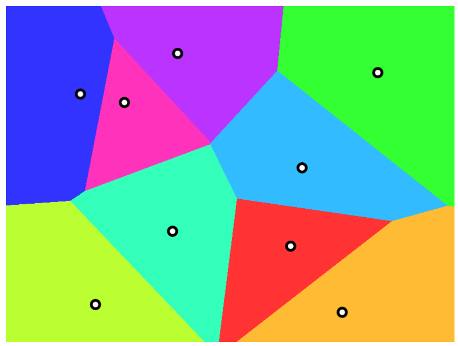
Note that the DCEL does not support infinite edges, so what we do is put a bounding box $B$ with some padding around the vertices of $\mathrm{Vor}(P)$, and then intersect the infinite edges and faces with the boundary of $B$ and only keep the part inside the bounding box.

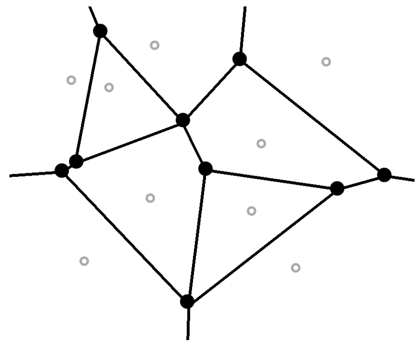


The aim of our algorithms will then be to calculate the DCEL in the right figure.

## 1.2 Euclidean Voronoi Diagrams

In this section we focus on proving some properties of the Voronoi diagram when the norm is the Euclidean norm, that is $\|\cdot\|_2$. Here is the example from earlier:





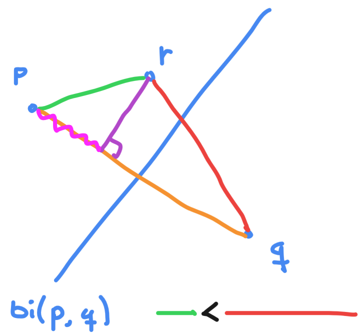
From linear algebra we know that $\|v\|_2 = \sqrt{\langle v, v\rangle}$, where $\langle\,\cdot\,,\,\cdot\,\rangle$ is the usual dot product on $\mathbb{R}^2$. Given two points $p, q \in \mathbb{R}^2$ then the **bisector** of $p$ and $q$ is denoted by $\mathrm{bi}(p,q) \subset \mathbb{R}^2$ and denotes the set of points on a line $\ell$ which passes through the midpoint of $p$ and $q$ and is orthogonal (w.r.t. $\langle\,\cdot\,,\,\cdot\,\rangle$) to the vector $p - q$.


A bisector $\mathrm{bi}(p,q)$ splits the plane into two **half-planes** $H_p$ and $H_q$ such that $p \in H_p$ and $q \in H_q$. We define $h(p,q)$ to be the open half-plane which contains $p$, that is the interior of $H_p$. So we have that

$$\mathbb{R}^2 = h(p,q) \cup \mathrm{bi}(p,q) \cup h(q,p).$$

**Proposition 1.3.** $r \in h(p,q)$ if and only if $\mathrm{dist}(r,p) < \mathrm{dist}(r,q)$.

*Proof.*




(TODO: Formalize) Proof sketch: We want to project $r$ onto the orange line. As long as $r \in H_p$ then the squiggly pink segment is shorter than the orange segment, which will make the green segment shorter than the red segment (which is what we want to show). □

**Corollary 1.4.** For every Voronoi cell we have

$$\mathcal{V}(p_i) = \bigcap_{\substack{1 \leq j \leq n \\ j \neq i}} h(p_i, p_j).$$

*Proof.* "⊂": Let $r \in \mathcal{V}(p_i)$. Then $\mathrm{dist}(r, p_i) < \mathrm{dist}(r, p_j)$ for all $i \neq j$. Prop 1.3 then gives us that this is equivalent to $r \in h(p_i, p_j)$ for all $i \neq j$.

"⊃": This argument is symmetrical to the above argument. □

A Voronoi cell is thus the intersection of convex sets and is therefore convex. We conclude that the Voronoi cells are open and convex (possibly unbounded) polygons with at most $n-1$ vertices and $n-1$ edges.

We now look at the shape of the entire Voronoi diagram. From Corollary 1.4 it follows that the edges of $\mathrm{Vor_G}(P)$ are made up of parts of straight lines, namely the bisectors between different points of $P$. We now classify these based on the structure of the points in $P$:

**Theorem 1.5.** If the points in $P$ are collinear then $\mathrm{Vor_G}(P)$ consists of $n-1$ parallel lines. Otherwise, $\mathrm{Vor_G}(P)$ is connected and its edges are either segments or half-lines.

*Proof.* Assume that the points in $P$ are collinear. By applying an isometry to $P$, we may assume without loss of generality that the points of $P$ lie on the $x$-axis:

$$P = \{(x_1, 0), (x_2, 0), \ldots, (x_n, 0)\},$$

where we assume that $x_1 < x_2 < \cdots < x_n$ by rearranging the points if necessary. See the proof of Theorem 1.9 for a visualization of $\mathrm{Vor}(P)$. By definition, we have that $p \in \mathrm{Vor}_\mathrm{G}(P)$ if and only if $p \notin \mathcal{V}(x_i, 0)$ for all $i$. Let $(x, y) \in \mathbb{R}^2$ such that $x_i < x < x_{i+1}$. Then $(x, y) \in \mathrm{Vor}_\mathrm{G}(P)$ if

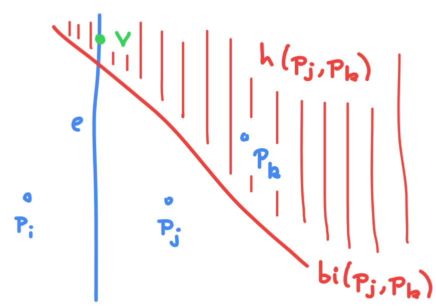
$$\mathrm{dist}((x, y), (x_i, 0)) = \mathrm{dist}((x, y), (x_{i+1}, 0)).$$

If furthermore $(x, y) \in \mathrm{Vor}_\mathrm{G}(P)$ then we get

$$\begin{aligned} & \|(x, y) - (x_i, 0)\| = \|(x, y) - (x_{i+1}, 0)\| \\ \iff & \sqrt{(x - x_i)^2 + y^2} = \sqrt{(x - x_{i+1})^2 + y^2} \\ \iff & |x - x_i| = |x - x_{i+1}| . \end{aligned}$$

Thus if $(x, 0) \in \mathrm{Vor}_\mathrm{G}(P)$ then $(x, y) \in \mathrm{Vor}_\mathrm{G}(P)$ for all $y \in \mathbb{R}$. This shows that $\mathrm{bi}((x_i, 0), (x_{i+1}, 0)) \subset \mathrm{Vor}_\mathrm{G}(P)$ for all $i < n$. Every point of $\mathrm{Vor}_\mathrm{G}(P)$ is on one of these bisectors, and the bisectors are all parallel, which proves the claim. (TODO: Clean up above argument and consider if anything is missing.)

Assume that the points in $P$ are not collinear. First, we show that the edges of $\mathrm{Vor}_\mathrm{G}(P)$ are either segments or half-lines. Suppose for a contradiction that there is an edge $e$ of $\mathrm{Vor}_\mathrm{G}(P)$ that is a full line and assume that $e \in \partial\mathcal{V}(p_i) \cap \partial\mathcal{V}(p_j)$. Let $p_k \in P$ be a point which is not collinear with $p_i$ and $p_j$. Then the line $\mathrm{bi}(p_j, p_k)$ is not parallel to the line $e$, hence they have an intersection point. Then there exists a point $v \in e \cap {}^\circ h(p_k, p_j)$. The situation is visualized here:




We have that $v \in \partial\mathcal{V}(p_j)$ by definition of $e$. Now note that

$$\partial\mathcal{V}(p_j) = \partial\left(\bigcap_{a \neq j} h(p_j, p_a)\right) \subset^1 \bigcup_{a \neq j} \partial h(p_j, p_a) = \bigcup_{a \neq j} \mathrm{bi}(p_j, p_a).$$

As $v \in h(p_k, p_j)$ we have that $\mathrm{dist}(v, p_k) < \mathrm{dist}(v, p_j)$, hence $v \notin \mathrm{bi}(p_j, p_k)$, so $v \notin \partial V(v_j)$ by the above characterization of $\partial\mathcal{V}(p_j)$. This is a contradiction, so $e$ can't be a full line. Now we show that $\mathrm{Vor}_\mathrm{G}(P)$ is connected. Assume for the sake of a contradiction that $\mathrm{Vor}_\mathrm{G}(P)$ is not connected. Then there exists

[1]Here we used that $\partial(A \cap B) \subset \partial A \cup \partial B$, a proof is here: `https://proofwiki.org/wiki/Boundary_of_Intersection_is_Subset_of_Union_of_Boundaries` (TODO: Remove this footnote and add the result to some topology appendix)

a $\partial\mathcal{V}(p_i)$ which is not path connected. This can only happen if $\partial\mathcal{V}(p_i)$ consists of two parallel lines (TODO: Why?). This contradicts the fact that $\mathrm{Vor}_{\mathrm{G}}(P)$ contains no lines. Thus $\mathrm{Vor}_{\mathrm{G}}(P)$ is connected. □

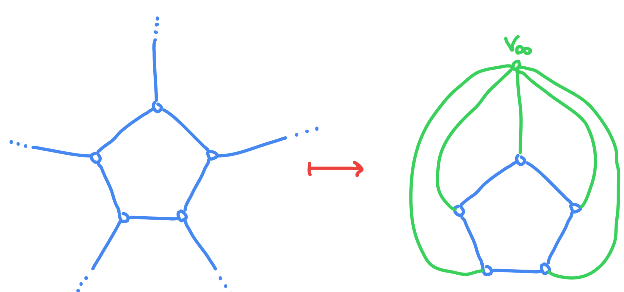
Finally, we show that that the complexity of the vertices and edges is $\mathcal{O}(n)$:

**Theorem 1.6.** For $n \geq 3$, the number of vertices in $\mathrm{Vor}_{\mathrm{G}}(P)$ is at most $2n-5$ and the number of edges is at most $3n-6$.

*Proof.* If the points in $P$ are collinear, then Theorem 1.5 implies the claim. Now assume that the points in $P$ are not collinear. As a first preprocessing step, we start by transforming $\mathrm{Vor}_{\mathrm{G}}(P)$ into an actual plane graph, as some of the edges in $\mathrm{Vor}_{\mathrm{G}}(P)$ may be half-lines. Let $v_1, \dots, v_k$ denote the vertices of $\mathrm{Vor}_{\mathrm{G}}(P)$. Let $p = \frac{1}{k}(v_1 + v_2 + \cdots + v_k) \in \mathbb{R}^2$ and let

$$r = 1 + \max\{\mathrm{dist}(p, v_1), \mathrm{dist}(p, v_2), \dots, \mathrm{dist}(p, v_k)\}.$$

Then let $B_r(p) \subset \mathbb{R}^2$ denote the open ball with center $p$ and radius $r$. We have that $B_r(p)$ contains every vertex $v_i$ and that every half-line edge $e$ of $\mathrm{Vor}_{\mathrm{G}}(P)$ intersects $\partial B_r(p)$ exactly once. Now define $v_\infty \in \mathbb{R}^2$ as any point in $\mathbb{R}^2 - B_r(p)$ and transform every half-line edge $e$ into a path with finite length by connecting the half-lines to the point $v_\infty$. This is possible since $\mathbb{R}^2 - \overline{B_p(r)}$ only contains these half-lines, and every half-line is pointing in a unique direction so we may then transform the half-lines in order by starting with those which are closest to $v_\infty$. An example of this construction is given here:




In this way we can turn $\mathrm{Vor}_{\mathrm{G}}(P)$ into a planar graph. For a planar graph $G$, Euler's formula[2] states that

$$V - E + F = 2, \tag{1.5}$$

where $V$ is the number of vertices, $E$ is the number of edges and $F$ is the number of faces of $G$. Let $n_v$ denote the number of vertices of the original $\mathrm{Vor}_{\mathrm{G}}(P)$, and let $n_e$ denote the number of edges. In our modification, we only added a single vertex, so by plugging into (1.5) we obtain the following relationship:

$$(n_v + 1) - n_e + n = 2. \tag{1.6}$$

[2](TODO: Add a reference and/or proof of Euler's formula in some topology appendix)

Note that $n$ is the number of faces, since we have a Voronoi cell for each point in $P$. Every vertex $v$ in $G$ has $\deg(v) \geq 3$, otherwise there would be a $\mathcal{V}(p_i)$ which is not convex. This means that

$$\sum_{v \in V(G)} \deg(v) \geq 3\,|V(G)| = 3(n_v + 1).$$

Now we want to compute the left side of the above inequality. Given a vertex $v$ we have that $\deg(v)$ counts the number of edges which touch $v$, and in $G$ every edge touches exactly 2 vertices, which gives us that $\sum_{v \in V(G)} \deg(v) = 2n_e$. Combining these facts, we obtain the inequality:

$$2n_e \geq 3(n_v + 1). \tag{1.7}$$

Multiplying (1.6) by 2, isolating $2n_e$ and then applying (1.7) we get:

$$\begin{aligned} 2(n_v + 1) - 2n_e + 2n = 4 &\iff 2n_e = (2n_v + 1) + 2n - 4 \\ &\implies 3(n_v + 1) \leq 2(n_v + 1) + 2n - 4 \\ &\implies n_v \leq 2n - 5. \end{aligned}$$

Multiplying (1.6) by 3, isolating $3(n_v + 1)$ and then applying (1.7) we get:

$$\begin{aligned} 3(n_v + 1) - 3n_e + 3n = 6 &\iff 3(n_v + 1) = 3n_e - 3n + 6 \\ &\implies 2n_e \geq 3n_e - 3n + 6 \\ &\implies n_e \leq 3n - 6. \end{aligned}$$

This proves the theorem. □

We thus have a linear number of vertices and edges $\mathrm{Vor}_{\mathrm{G}}(P)$, but we have a quadratic number of bisectors $\mathrm{bi}(p_i, p_j)$ of which every edge of $\mathrm{Vor}_{\mathrm{G}}(P)$ is a subset of, and every vertex in $\mathrm{Vor}_{\mathrm{G}}(P)$ is an intersection point of two such bisectors. Thus it would be interesting to characterize when a particular bisector is a part of $\mathrm{Vor}_{\mathrm{G}}(P)$. First, we need a definition:

**Definition 1.7** (Largest empty circle)**.** For a $q \in \mathbb{R}^2$ we define $C_P(q)$ to be *the largest empty circle of* $q$ *with respect to* $P$, which is the largest empty circle with $q$ as its center that does not contain any point of $P$ in its interior. Formally,

$$C_P(q) = B_r(q), \quad \text{where} \quad r = \sup\{\lambda \in \mathbb{R}^+ \mid B_\lambda(q) \cap P = \varnothing\}.$$

**Theorem 1.8.** The bisectors and their intersections are characterized by:

(i) $q \in \mathbb{R}^2$ is a vertex of $\mathrm{Vor_G}(P)$ if and only if

$$|\partial C_P(q) \cap P| \geq 3.$$

(ii) $\mathrm{bi}(p_i, p_j)$ defines an edge of $\mathrm{Vor_G}(P)$ if and only if

$$\exists q \in \mathrm{bi}(p_i, p_j) \colon \partial C_P(q) \cap P = \{p_i, p_j\}.$$

*Proof.* We prove each statement individually:

(i): "$\Leftarrow$": Let $q \in \mathbb{R}^2$ and assume that $|\partial C_P(q) \cap P| \geq 3$. Let $p_i, p_j, p_k$ be three distinct points from $\partial C_P(q) \cap P$. Since $C_P(q) \cap P = \varnothing$ by definition, this means that $q$ is equally close to $p_i, p_j, p_k$ but not closer to any other points in $P$, so $q \in \partial\mathcal{V}(p_i) \cap \partial\mathcal{V}(p_j) \cap \partial\mathcal{V}(p_k) \subset \mathrm{Vor_G}(P)$, and it is a vertex since it is at an intersection of 3 or more bisectors.

"$\Rightarrow$": Let $q \in \mathbb{R}^2$ be a vertex of $\mathrm{Vor_G}(P)$. A vertex of $\mathrm{Vor_G}(P)$ touches at least 3 different edges, and thus touches at least 3 distinct Voronoi cells $\mathcal{V}(p_i), \mathcal{V}(p_j)$ and $\mathcal{V}(p_k)$. So $q \in \partial\mathcal{V}(p_i) \cap \partial\mathcal{V}(p_j) \cap \partial\mathcal{V}(p_k)$. This gives us that

$$\mathrm{dist}(q, p_i) = \mathrm{dist}(q, p_j) = \mathrm{dist}(q, p_k).$$

Denote the above distance by $D$. Now assume for the sake of a contradiction that there exists $p_\alpha \in P$ such that $\mathrm{dist}(q, p_\alpha) < D$. Then there are parts of the bisectors $\mathrm{bi}(p_\alpha, p_i), \mathrm{bi}(p_\alpha, p_j), \mathrm{bi}(p_\alpha, p_k)$ contained inside $B_D(q)$, which means that $\mathcal{V}(p_i), \mathcal{V}(p_j), \mathcal{V}(p_k)$ do not all meet at $q$, a contradiction. This means that $C_P(q) \cap P = \varnothing$ and $p_i, p_j, p_k \in \partial C_P(q)$.

(ii): "$\Leftarrow$": Let $q \in \mathrm{bi}(p_i, p_j)$ such that $\partial C_P(q) \cap P = \{p_i, p_j\}$. So $C_P(q) \cap P = \varnothing$, which by definition of $C_P(q)$ means that

$$\mathrm{dist}(q, p_i) = \mathrm{dist}(q, p_j) \leq \mathrm{dist}(q, p_k)$$

for all $k$. So $q \in \mathrm{Vor_G}(P)$ and is either a vertex or an edge. Since $|\partial C_P(q) \cap P| < 3$ part (i) gives us that $q$ is not a vertex, hence it must be an edge, which is a subset of $\mathrm{bi}(p_i, p_j)$.

"$\Rightarrow$": Let $e \subset \mathrm{bi}(p_i, p_j)$ be an edge of $\mathrm{Vor_G}(P)$. For $q \in e$ we have that $\mathrm{dist}(q, p_i) = \mathrm{dist}(q, p_j)$, and that $q$ touches $\mathcal{V}(p_i)$ and $\mathcal{V}(p_j)$. By applying the same contradiction proof as in (i) "$\Rightarrow$" we have that there is no point in $P$ which is closer to $q$ than $p_i$ and $p_j$, thus $\partial C_P(q) \cap P = \{p_i, p_j\}$.

$\square$

## 1.3 Fortune's algorithm

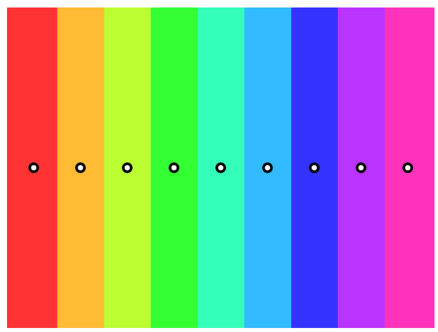
In this section we will present an algorithm which computes $\mathrm{Vor}(P)$ in $\mathcal{O}(n \log n)$ time. This is actually optimal, as we can use a Voronoi diagram for sorting:

**Theorem 1.9.** We can't do better than $\mathcal{O}(n \log n)$.

*Proof.* Let $A = \{a_1, a_2, \ldots, a_n\} \subset \mathbb{R}$. Now assume we have used an algorithm to compute a Voronoi diagram of the points

$$P = \{(a_1, 0), (a_2, 0), \ldots, (a_n, 0)\}.$$

We obtain a diagram which looks similar to this:



We assume without loss of generality that the algorithm outputs a DCEL $\Delta$ of $\mathrm{Vor}(P)$. Assume that the `edge` pointer of every face of $\Delta$ points to the edge to the right of the face, and that the `face` pointer of every edge of $\Delta$ points to the face to the right. Let $F_i$ be the face in $\Delta$ which contains the point $(0, a_i)$. Let $\ell \in \mathbb{N}$ such that $a_\ell < a_i$ for all $i \neq \ell$. Let $b_1 = a_\ell$ and if $b_i = a_j$ and $i < n$ then define $b_{i+1} = a_k$ where $k$ comes from $F_j.\mathsf{edge}.\mathsf{face} = F_k$. Then $(b_1, b_2, \ldots, b_n)$ is the elements of $A$ in sorted order. This means that we can use the Voronoi diagram to sort, which proves the theorem. $\square$

(TODO: The statement of the above theorem is temporary. I originally phrased it like so: "The optimal worst-case running time for computing $\mathrm{Vor}(P)$ is $\Omega(n \log n)$." What is the proper terminology here?)

We now present Fortune's algorithm. It is a sweep line algorithm which maintains a horizontal sweep line $\ell \colon y = \ell_y$, and $\ell$ sweeps the plane from top to bottom in order to uncover the structure of the Voronoi diagram.

For a point $p = (p_x, p_y) \in \mathbb{R}^2$ and a sweep line $\ell \colon y = \ell_y$ the distance between $p$ and $\ell$ is

$$\mathrm{dist}(p, \ell) = |p_y - \ell_y| \,.$$

Define

$$B_i = \{q \in \mathbb{R}^2 \mid \mathrm{dist}(q, p_i) = \mathrm{dist}(q, \ell)\}$$

for all $i$. If $(p_i)_y > \ell_y$, it turns out we may parametrize $B_i$ by a parabola: Let $p = (p_x, p_y)$ denote $p_i$ and let $q = (x, y) \in B_i$. Since distances are non-negative, it is equivalent to looking at satisfying $\operatorname{dist}(q,p)^2 = \operatorname{dist}(q,\ell)^2$. We have:

$$\operatorname{dist}(q,p)^2 = \operatorname{dist}(q,\ell)^2 \iff (p_x - x)^2 + (p_y - y)^2 = (y - \ell_y)^2.$$

This can be transformed into the equation

$$2(p_y - \ell_y)y = x^2 - 2p_x x + p_x^2 + p_y^2 - \ell_y^2. \tag{1.8}$$

Since $p_y \neq \ell_y$ by assumption, we obtain the parabola:

$$y = \frac{1}{2(p_y - \ell_y)}(x^2 - 2p_x x + p_x^2 + p_y^2 - \ell_y^2), \tag{1.9}$$

which parametrizes $B_i$ if $(p_i)_y > \ell_y$. Now we look at the situation where $(p_i)_y = \ell_y$. Then

$$\operatorname{dist}(q,p)^2 = \operatorname{dist}(q,\ell)^2 \iff (p_x - x)^2 + (p_y - y)^2 = (p_y - y)^2.$$

Then it must be the case that $p_x = x$, so $B_i$ is a subset of a vertical line, and is a line segment if there is some $B_k$ above $B_i$ and a half-line which starts at $p_i$ otherwise. Finally, if $(p_i)_y < \ell_y$, we let $B_i = \varnothing$. We now for all $i$ define the maps

$$\beta_i(x) = \begin{cases} \dfrac{x^2 - 2(p_i)_x x + (p_i)_x^2 + (p_i)_y^2 - \ell_y^2}{2((p_i)_y - \ell_y)} & \text{if } (p_i)_y > \ell_y, \\ \infty & \text{otherwise.} \end{cases}$$

Let $\mathsf{LB}(x)$ denote the map which takes the minimum of each $\beta_i$, i.e.

$$\mathsf{LB}(x) = \min\{\beta_1(x), \beta_2(x), \ldots, \beta_n(x)\}.$$

**Definition 1.10** (Beach line)**.** The *beach line for the points $P$ with regards to the sweep line $\ell$* is given by the following subset of $\mathbb{R}^2$:

$$G \cup V,$$

where $G$ is the graph of $\mathsf{LB}$ when it is finite

$$G = \{(x, \mathsf{LB}(x)) \in \mathbb{R}^2 \mid \mathsf{LB}(x) < \infty\},$$

and $V$ is all the vertical parts not hidden behind other parabolas

$$V = \{B_i - \{(p_i)_x\} \times (\mathsf{LB}((p_i)_x), \infty) \mid i = 1, \ldots, n \text{ where } (p_i)_y = \ell_y\}.$$

**Remark 1.11.** From the definition we see that the beach line consists of parts of parabolas, and vertical line segments or half-lines. For this reason, it is easy to see that the intersection between any vertical line and the beach line has at most one component.

**Remark 1.12.** For a sweep line $\ell$ which does not intersect any of the points in $P$, it follows from the definition of beach line that the map $\mathsf{LB}(x)$ parametrizes the beach line. This was used to make a simple demo visualing the beach line, which can be found in demos/beachline.

**Definition 1.13** (Breakpoint)**.** Every point $q$ on the beach line such that $q \in B_i \cap B_j$ for two different $i, j$ is called a *breakpoint*.

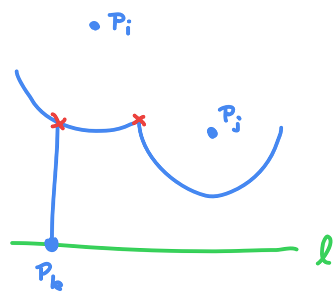



Figure 1.3: The red crosses indicate breakpoints and the blue lines represent the beach line.

Now we show that the breakpoints exactly trace out $\mathrm{Vor}_{\mathrm{G}}(P)$ as the sweep line $\ell$ moves from top to bottom.

**Proposition 1.14.** We have the following:

(i) For every sweep line $\ell$: $y = \ell_y$ each breakpoint lies on $\mathrm{Vor}_{\mathrm{G}}(P)$.

(ii) For every point $q$ in $\mathrm{Vor}_{\mathrm{G}}(P)$ there is a position of the sweep line $\ell$ such that $q$ is a breakpoint.

*Proof.* We prove each statement individually:

(i): Let $\ell$ be the sweep line, and assume that it has one or more breakpoints. Let $q \in \mathbb{R}^2$ be such a breakpoint. Then $q \in B_i \cap B_j$ for some $i \neq j$, which means that

$$\mathrm{dist}(q, \ell) = \mathrm{dist}(q, p_i) = \mathrm{dist}(q, p_j).$$

The last equality gives us that $q \notin \mathcal{V}(p_k)$ for all $k$, hence $q \in \mathrm{Vor}_{\mathrm{G}}(P)$.

(ii): Let $q = (q_x, q_y) \in \mathrm{Vor}_{\mathrm{G}}(P)$. Since $q$ is either an edge or a vertex, Theorem 1.8 gives us that $\partial C_P(q) \cap P$ has at least two elements, so let $p_i, p_j \in \partial C_P(q) \cap P$ be two different elements. We have $\mathrm{dist}(q, p_i) = \mathrm{dist}(q, p_j)$ by definition of $C_P(q)$, and then we may set

$$\ell_y := q_y - \mathrm{dist}(q, p_i),$$

and obtain

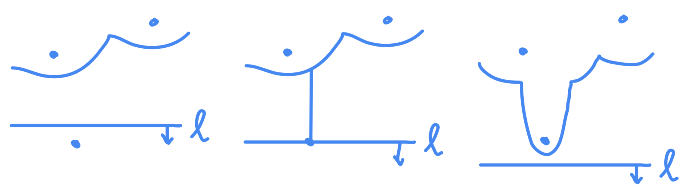
$$\operatorname{dist}(q, \ell) = \operatorname{dist}(q, p_i) = \operatorname{dist}(q, p_j).$$

Then $B_i$ and $B_j$ intersect at $q$, and $q$ is on the beach line since there is no $B_k$ with a point $p_k$ closer to $q$ than $p_i$ and $p_j$, by definition of $C_P(q)$.

□

As the sweep line $\ell$ sweeps the plane from top to bottom, the combinatorial structure of the beach line changes. We'll categorize these changes into *events*.

First we will consider when new arcs appear on the beach line. As $\ell$ sweeps down and hits a point, a vertical segment is added to the beach line, and then as $\ell$ continues to move, the vertical line spreads out into a new parabolic arc, as seen in this figure:




**Definition 1.15** (Site event)**.** When $\ell$ encounters a point $p_i \in P$, that is when $\ell_y = (p_i)_y$, we say that we encounter a *site event*.

**Lemma 1.16.** If $(p_i)_y = (p_j)_y$ then the parabolas $\beta_i$ and $\beta_j$ intersect in exactly one point. If $(p_i)_y \neq (q_i)_y$ then $\beta_i$ and $\beta_j$ intersect in 2 different points.

*Proof.* Let $p = (p_x, p_y)$ denote $p_i$ and $q = (q_x, q_y)$ denote $p_j$. If $p_y = q_y$ then

$$\beta_i(x) - \beta_j(x) = \left(\frac{p_x - q_x}{\ell_y - p_y}\right) x + \frac{q_x^2 + q_y^2 - p_x^2 - p_y^2}{2(\ell_y - p_y)} \tag{1.10}$$

is a line with non-zero slope, since $p_y \neq \ell_y$ and $p_x \neq q_x$ since otherwise it would be the case that $p_i = p_j$. Such a line intersects the $x$-axis exactly once.

If $p_y \neq q_y$ then $\beta_i(x) - \beta_j(x)$ is a second degree polynomial with discriminant

$$D = \frac{(p_x - q_x)^2 + (p_y - q_y)^2}{(p_y - \ell_y)(q_y - \ell_y)}. \tag{1.11}$$

Since $p_y, q_y > \ell_y$ the denominator is strictly positive, and since $p_i \neq p_j$ the numerator is also strictly positive, so $D > 0$. This means that $\beta_i$ and $\beta_j$ intersect in two different points. □

**Lemma 1.17.** The only way in which a new arc can appear on the beach line is through a site event.

*Proof.* (TODO: working on this on paper) □

# Appendix A

# Notation

| | |
|---|---|
| $X - Y$ | Set difference |
| $\|X\|$ | The number of elements in a finite set $X$. |
| $\iff$ | If and only if |
| $\implies$ | Implication |
| $\mathbb{R}$ | The real numbers. |
| $\mathbb{R}^n$ | The vector space of $n$-tuples of real numbers. |
| $\\|\cdot\\|$ | Norm. |
| $\\|\cdot\\|_p$ | The $L^p$ norm. |
| $\|x\|$ | Absolute value if $x$ is a number. |
| $\mathrm{dist}(p,q)$ | The distance between $p$ and $q$, given by $\\|p-q\\|$. |
| $\langle \cdot, \cdot \rangle$ | An inner product. |
| $\subset$ | Subset (not strict, e.g. $A = B \implies A \subset B$). |
| $P$ | A set of points $\{p_1, p_2, \ldots, p_n\}$ that we want to apply an algorithm to. |
| $p_i$ | A point in $P$ (see above). |
| $n$ | If not otherwise specified, $n$ is the number of points in $P$ (see above). |
| $\mathrm{Vor}(P)$ | The Voronoi diagram of $P$. |
| $\mathcal{V}(p_i)$ | The $i$th Voronoi cell. |
| $\mathrm{Vor}_{\mathrm{G}}(P)$ | Refers to $\mathbb{R}^2 - \mathrm{Vor}(P)$. |
| $\mathcal{O}(f(n))$ | Big $O$-notation. |
| $\mathrm{bi}(p,q)$ | Bisector of $p$ and $q$. |
| $h(p,q)$ | Open half-plane containing $p$ with $\mathrm{bi}(p,q)$ as boundary. |
| $\overline{X}$ | The closure of a set $X \subset \mathbb{R}^n$, given by the union of $X$ with its limit points. |
| $^\circ X$ | The interior of a set $X \subset \mathbb{R}^n$, given by the union of all interior points of $X$. |
| $\partial X$ | The boundary of a set $X \subset \mathbb{R}^n$, given by $\overline{X} - {}^\circ X$. |
| $\overline{B_r(p)}$ | $= \{x \in \mathbb{R}^n \mid \mathrm{dist}(x,p) \leq r\}$, the closed ball with center $p$ and radius $r$. |
| $B_r(p)$ | $= \{x \in \mathbb{R}^n \mid \mathrm{dist}(x,p) < r\}$, the open ball with center $p$ and radius $r$. |
| $\partial B_r(p)$ | $= \{x \in \mathbb{R}^n \mid \mathrm{dist}(x,p) = r\}$, the circle with center $p$ and radius $r$. |
| $V(G)$ | The set of vertices for the graph $G$. |
| $E(G)$ | The set of edges for the graph $G$. |
| $\deg(v)$ | The degree of a vertex $v$ in a graph, e.g. the number of edges that touch $v$. |