STOCHASTIC DIFFERENTIAL EQUATIONS: ITO CALCULUS

SDE, Langevin equation

$$\frac{dx}{dt} = a(x,t) + b(x,t)\xi(t)\ , \quad x \in \mathbb{R}^n$$

(but we start with $x \in \mathbb{R}^1$)

Examples

- Brownian motion
- Stock-market
- Options: Black-Scholes
- Astro: stochastic supernova explosions, gamma rays
- Turbulence: transport of particles
- QFT: stochastic quantisation

.

.

.

Is this important ? YES !!!

What is our focus ?

long time behaviour or *fast fluctuations:*

$$\langle \xi(t)\xi(t') \rangle = \delta(t - t') \ , \ \ \langle \xi(t) \rangle = 0$$

Is the SDE defined ? We need:

$$W(t) = \int_0^t dt' \xi(t')$$

must exist (and must be continuous)

This is called *Wiener* noise.

We use: $dW = W(t+dt) - W(t) = \xi(t)dt$

and integrate the SDE

$$x(t) - x(0) = \int_0^t a(x(s), s)ds + \int_0^t b(x(s), s)\xi(s)ds$$

$$= \int_0^t a(x(s), s)ds + \int_0^t b(x(s), s)dW(s)$$

in short: $\boxed{dx = adt + bdW}$

LITERATURE

- Gardiner:
  Handbook of Stochastic Methods (Springer)
- van Kampen:
  Stochastic Processes in Physics and Chemistry (North-Holland)
- Evans:
  An Introduction to Stochastic Differential Equations (AMS)

We follow the treatment of Gardiner.

STOCHASTIC INTEGRAL

What is $\int_{t_0}^{t} G(t')dW(t')$? ($G(t)$ some function)

Riemann-Stieltjes integral: devide $[t_0, t]$ into n subintervals

$$t_0 < t_1 < t_2 < \ldots < t_{n-1} < t$$

and define intermediate points τ_i : $t_{i-1} \leqslant \tau_i \leqslant t_i$

Limit of partial sums

$$S_n = \sum_{i=1}^{n} G(\tau_i)[W(t_i) - W(t_{i-1})]$$

Now, S_n depends on the particular choice of the intermediate points τ_i

Example: take $G(\tau_i) = W(\tau_i)$

$$\langle S_n \rangle = \left\langle \sum_{i=1}^{n} W\left(\tau_i\right)\left[W\left(t_i\right) - W\left(t_{i-1}\right)\right] \right\rangle$$

$$= \sum_{i=1}^{n} \left[\min(\tau_i, t_i) - \min(\tau_i, t_{i-1})\right] = \sum_{i=1}^{n} (\tau_i - t_{i-1})$$

Now make the choice $\tau_i = \alpha t_i + (1-\alpha) t_{i-1}$

$$\implies S_n = \sum_{i=1}^{n} (t_i - t_{i-1})\,\alpha = (t - t_0)\alpha$$

$\implies$ anything between 0 and $t - t_0$

Remark:

$$\langle W(t)W(t')\rangle = \left\langle \int_0^t ds \int_0^{t'} ds' \xi(s)\xi(s') \right\rangle = \int_0^t ds \int_0^{t'} ds' \delta(s-s')$$

i) $t < t' : \tilde{s} := s' - s \Rightarrow t' - s > 0$

$$\langle W(t)W(t')\rangle = \int_0^t ds \int_{-s}^{t'-s} d\tilde{s}\, \delta(\tilde{s}) = \int_0^t ds = t$$

ii) $t > t' : \tilde{s} = s - s'$

$$\langle W(t)W(t')\rangle = \int_0^{t'} ds \int_{-s}^{t-s} d\tilde{s}\, \delta(\tilde{s}) = \int_0^{t'} ds = t'$$

$$\Longrightarrow \langle W(t)W(t')\rangle = \min(t, t') \quad \checkmark$$

Definition: Ito stochastic integral $(\alpha = 0)$

$$\tau_i = t_{i-1}\ , \quad \int_{t_0}^{t} G(t')\, dW(t') = \underset{n\to\infty}{\mathrm{ms}\ \lim} \sum_{i=1}^{n} G(t_{i-1})\left[W(t_i) - W(t_{i-1})\right]$$

Example: $\displaystyle\int_{t_0}^{t} W(t')\, dW(t')$ $\qquad W_i = W(t_i)$

$$S_n = \sum_{i=1}^{n} W_i (W_i - W_{i-1}) = \sum_{i=1}^{n} W_i \Delta W_i \qquad \Delta W_i = W_i - W_{i-1}$$

$$= \frac{1}{2}\sum_{i=1}^{n}\left[(W_{i-1} + \Delta W_i)^2 - W_{i-1}^2 - (\Delta W_i)^2\right]$$

$$= \frac{1}{2}\left[W(t)^2 - W(t_0)^2\right] - \frac{1}{2}\sum_{i=1}^{n}(\Delta W_i)^2$$

Definition: Ito stochastic integral $(\alpha = 0)$

mean square limit of last term

$$\left\langle \sum_{i=1}^{n} (\Delta W_i)^2 \right\rangle = \sum_{i=1}^{n} (t_i - 2t_{i-1} + t_{i-1}) = t - t_0$$

Now look at

$$\left\langle \left[\sum_i (W_i - W_{i-1})^2 - (t - t_0) \right]^2 \right\rangle$$
$$= \left\langle \sum_i (W_i - W_{i-1})^4 + 2 \sum_{i>j} (W_i - W_{i-1})^2 (W_j - W_{j-1})^2 - 2(t - t_0) \sum_i (W_i - W_{i-1})^2 + (t - t_0)^2 \right\rangle$$

$W_i - W_{i-1}$ is independent of $W_j - W_{j-1}\ ,\ \ i \neq j$

$$\Longrightarrow \left\langle (W_i - W_{i-1})^2 \, (W_j - W_{j-1})^2 \right\rangle = (t_i - t_{i-1})(t_j - t_{j-1})$$

ΔW_i is a Gaussian variable

$$\Longrightarrow \left\langle (W_i - W_{i-1})^4 \right\rangle = 3 \left\langle (W_i - W_{i-1})^2 \right\rangle^2 = 3(t_i - t_{i-1})^2$$

Definition: Ito stochastic integral $(\alpha = 0)$

$$\Longrightarrow \left\langle \left[\sum_i (W_i - W_{i-1})^2 - (t - t_0) \right]^2 \right\rangle$$

$$= 3 \sum_i (t_i - t_{i-1})^2 + 2 \sum_{i>j} (t_i - t_{i-1})(t_j - t_{j-1}) - 2(t - t_0) \sum (t_i - t_{i-1}) + (t - t_0)^2$$

$$= 2 \sum_i (t_i - t_{i-1})^2 + \sum_{i,j} (t_i - t_{i-1})(t_j - t_{j-1}) - 2(t - t_0) \sum_i (t_i - t_{i-1}) + (t - t_0)^2$$

$$= 2 \sum_i (t_i - t_{i-1})^2 + \sum_{i,j} [(t_i - t_{i-1}) - (t - t_0)] [(t_j - t_{j-1}) - (t - t_0)]$$

$$= 2 \sum_{i=1}^{h} (t_i - t_{i-1})^2 = 2 \sum_{i=1}^{n} \frac{1}{n^2} (t - t_0)^2 \to 0 \text{ for } n \to \infty$$

This is the definition of the mean square limit and thus $\operatorname*{ms\,lim}_{n\to\infty} \sum_i (W_i - W_{i-1})^2 = t - t_0$

$$\Longrightarrow \quad \int_{t_0}^{t} W(t')\, dW(t') = \frac{1}{2} \left[W(t)^2 - W(t_0)^2 \right] - (t - t_0)$$

new

Definition: Ito stochastic integral $(\alpha = 0)$

Comment:

$$\left\langle \int_{t_0}^{t} W(t)\, dW(t) \right\rangle = \frac{1}{2}\left[\left\langle W(t)^2 \right\rangle - \left\langle W(0)^2 \right\rangle - (t - t_0)\right] = 0$$

since $\left\langle W_{i-1}\, \Delta W_i \right\rangle = 0$ because W_{i-1} and ΔW_i are statistically independent

Remark: Gaussian $p(x) = \dfrac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2}$

$$\langle 1 \rangle = \int_{-\infty}^{\infty} 1\; p(x) = 1$$

$$\langle x \rangle = \int_{-\infty}^{\infty} x\; p(x) = 0$$

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} x^2\; p(x) = \sigma^2$$

$$\langle x^4 \rangle = \int_{-\infty}^{\infty} x^4\; p(x) = 3\sigma^4$$

The Stratonovich Integral

Evaluate the integral as an average

Ito: $W(t_{i-1})$

Stratonovich: $\frac{1}{2}\left(W(t_i) + W(t_{i-1})\right)$

$$\overset{\text{easy}}{\Longrightarrow} S\int_{t_0}^{t} W(t')\,dW(t') = \frac{1}{2}\left[W(t)^2 - W(t_0)^2\right]$$

(no extra term)

denotes Stratonovich integral

Non-anticipating Functions

We consider SDEs driven by a Wiener process $W(t)$

$$x(t) - x(t_0) = \int_{t_0}^{t} a\left(x(t'), t'\right) dt' + \int_{t_0}^{t} b\left(x(t'), t'\right) dW(x')$$

$G(t)$ is called non-anticipating iff

$\forall s, t \in \mathbb{R}\ ,\ \ t < s : G(t)$ statistically independent of $W(s) - W(t)$

This means *causality*.

We will only consider non-anticipating functions.

Ito: $dW(t)^2 = dt$ and $dW(t)^{2+N} = 0$

That means:

$$\int_{t_0}^{t} \left[dW(t')\right]^{2+N} G(t') = \mathrm{ms}\lim_{n\to\infty} \sum_i G_{i-1}\Delta W_i^{2+N} = \begin{cases} \int_{t_0}^{t} dt' G(t') & N = 0 \\ 0 & N > 0 \end{cases}$$

for non-anticipating $G(t)$

Proof for $N = 0$:

$$I := \lim_{n\to\infty} \left\langle \left[\sum_i G_{i-1}\left(\Delta W_i^2 - \Delta t_i\right)\right]^2 \right\rangle$$

$$= \lim_{n\to\infty} \left\{ \left\langle \sum_i G_{i-1}^2 \underbrace{\left(\Delta W_i^2 - \Delta t_i\right)^2}_{\text{independent}} + \sum_{i>j} 2G_{i-1}\underbrace{G_{j-1}\left(\Delta W_j^2 - \Delta t_j\right)\left(\Delta W_i^2 - \Delta t_i\right)}_{\text{independent}} \right\rangle \right\}$$

Ito: $dW(t)^2 = dt$ and $dW(t)^{2+N} = 0$

Using

i. $\left\langle \Delta W_i^2 \right\rangle = \Delta t_i$

ii.
$$\begin{aligned}\left\langle \left(\Delta W_i^2 - \Delta t_i\right)^2 \right\rangle &= \left\langle \Delta W_i^4 \right\rangle - 2\Delta t_i \left\langle \Delta W_i^2 \right\rangle + \Delta t_i^2 \\ &= 3\Delta t_i^2 \quad - 2\Delta t_i^2 \qquad + \Delta t_i^2 \\ &= 2\Delta t_i^2\end{aligned}$$

we get $I = 2 \lim_{n\to\infty} \left[\sum_i \Delta t_i^2 G_{i-1}^2 \right] \to 0$ (if G is bounded)

$$\Longrightarrow \operatorname{ms}\lim_{n\to\infty} \sum_i G_{i-1} \Delta W_i^2 = \operatorname{ms}\lim \sum_i G_{i-1} \Delta t_i$$

$$\Longrightarrow \int_{t_0}^{t} \left[dW(t')\right]^2 G(t') = \int_{t_0}^{t} dt'\, G(t')$$

Comments

i. Proof for $N > 0$ similar

ii. result **not** valid for Stratonovich: independence not guaranteed

Properties of the Ito Stochastic Integral

i. Existence if $G(t)$ is continuous and non-anticipating

ii. Integration of polynomials

$$d[W(t)]^n = [W(t) + dW(t)]^n - W(t)^n = \sum_{r=1}^{n} \binom{n}{r} W(t)^{n-r} dW(t)^r$$

$$= nW(t)^{n-1} dW(t) + \frac{n(n-1)}{2} W(t)^{n-2} dt$$

$$\Longrightarrow \int_{t_0}^{t} W(t')^n \, dW(t') = \frac{1}{n+1} \left[W(t)^{n+1} - W\left(t_0\right)^{n+1} \right] - \frac{n}{2} \int_{t_0}^{t} W(t)^{n-1} dt$$

iii. exponential function

$$d\{\exp[W(t)]\} = \exp[W(t) + dW(t)] - \exp[W(t)]$$

$$= \exp[W(t)] \left[dW(t) + \frac{1}{2} dW(t)^2 \right] = \exp[W(t)] \left[dW(t) + \frac{1}{2} dt \right]$$

Properties of the Ito Stochastic Integral

or more generally

$$df[W(t),t] = \frac{\partial f}{\partial t}dt + \frac{1}{2}\frac{\partial^2 f}{\partial t^2}(dt)^2 + \frac{\partial f}{\partial W}dW(t) + \frac{1}{2}\frac{\partial^2 f}{\partial W^2}[dW(t)]^2 + \frac{\partial^2 f}{\partial W \partial t}dtdW(t) + \ldots$$

using $(dt)^2 \to 0$, $dtdW(t) \to 0$, $[dW(t)]^2 = dt$

$$\Longrightarrow df[W(t),t] = \left(\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial W^2}\right)dt + \frac{\partial f}{\partial W}dW(t)$$

iv. Mean value formula

$$\left\langle \int_{t_0}^{t} G(t')\, dW(t') \right\rangle = 0$$ for causal $G(t)$

since $\left\langle \sum_i G_{i-1}\Delta W_i \right\rangle = \sum_i \langle G_{i-1}\rangle\langle \Delta W_i\rangle = 0$ Not true for Stratonovich !!!

Properties of the Ito Stochastic Integral

v. Correlation formula

$$\left\langle \int_{t_0}^{t} G(t')\, dW(t') \int_{t_0}^{t} H(t'')\, dW(t'') \right\rangle = \int_{t_0}^{t} dt' \left\langle G(t')\, H(t') \right\rangle$$

Proof: $\left\langle \sum_i G_{i-1} \Delta W_i \sum_j H_{j-1} \Delta W_j \right\rangle = \left\langle \sum_i G_{i-1} H_{i-1} \left(\Delta W_i\right)^2 \right\rangle$

$$+ \left\langle \sum_{i>j} \left(G_{i-1} H_{j-1} + G_{j-1} H_{i-1}\right) \Delta W_j \Delta W_i \right\rangle$$

ΔW_i is independent of all other terms since $j < i$

factorise out the term $\langle \Delta W_i \rangle = 0$ and $\langle \Delta W_i^2 \rangle = dt$

Properties of the Ito Stochastic Integral

Examples with some integrals

$$I_1 = \int_{t_1}^{t_2} dt f(t) \delta\left(t - t_1\right)$$

$$I_2 = \int_{t_1}^{t_2} dt f(t) \delta\left(t - t_2\right)$$

Ito: $I_1 = f(t_1)$

$$I_2 = 0$$

Stratonovich: $I_1 = \dfrac{1}{2} f(t_1)$

$$I_2 = \frac{1}{2} f(t_2)$$

Ito Stochastic Differential Equation

$$dx(t) = a[x(t), t]dt + b[x(t), t]dW(t)$$

$$x(t) = x\left(t_0\right) + \int_{t_0}^{t} dt' a\left[x\left(t'\right), t'\right] + \int_{t_0}^{t} dW\left(t'\right) b\left[x\left(t'\right), t'\right]$$

mesh points

$$t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = t$$

$$x_{i+1} = x_i + a\left(x_i, t_i\right)\Delta t_i + b\left(x_i, t_i\right)\Delta W_i$$

with

$$x_i = x\left(t_i\right)\ ,\ \ \Delta t_i = t_{i+1} - t_i\ ,\ \ \Delta W_i = W\left(t_{i+1}\right) - W\left(t_i\right)$$

Change of Variables: Ito's Formula

$$
\begin{aligned}
df[x(t)] = f[x(t) + dx(t)] - f[x(t)] \\
&= f'[x(t)]dx(t) + \frac{1}{2}f''[x(t)]dx(t)^2 + \ldots \\
&= f'[x(t)]\{a[x(t),t]dt + b[x(t),t]dW(t)\} + \frac{1}{2}f''[x(t)]b[x(t),t]^2[dW(t)]^2 + \ldots
\end{aligned}
$$

$$
\Longrightarrow df[x(t)] = \left\{ a[x(t),t]f'[x(t)] + \frac{1}{2}b[x(t),t]^2 f''[x(t)] \right\} dt + b[x(t),t]f'[x(t)]dW(t)
$$

Many variables

$$
d\boldsymbol{x} = \boldsymbol{A}(\boldsymbol{x},t)dt + \boldsymbol{B}(\boldsymbol{x},t)d\boldsymbol{W}(t)
$$

with

$$
df(\boldsymbol{x}) = \left\{ \sum_i A_i(\boldsymbol{x},t)\partial_i f(\boldsymbol{x}) + \frac{1}{2}\sum_{i,j}\left[\boldsymbol{B}(\boldsymbol{x},t)\boldsymbol{B}^{\mathrm{T}}(\boldsymbol{x},t)\right]_{ij}\partial_i\partial_j f(\boldsymbol{x}) \right\} dt + \sum_{i,j} B_{ij}(\boldsymbol{x},t)\partial_i f(\boldsymbol{x})dW_j(t)
$$

Fokker-Planck equation

arbitrary function $f(x)$

$$\langle df[x(t)]\rangle/dt = \left\langle \frac{df[x(t)]}{dt} \right\rangle = \frac{d}{dt}\langle f[x(t)]\rangle = \left\langle a[x(t),t]\partial_x f + \frac{1}{2}b[x(t),t]^2\partial_x^2 f \right\rangle$$

conditional probability density $p(x,t\,|\,x_0,t_0)$

$$\frac{d}{dt}\langle f[x(t)]\rangle = \int dx f(x)\partial_t p\left(x,t\,|\,x_0,t_0\right) = \int dx \left[a(x,t)\partial_x f + \frac{1}{2}b(x,t)^2\partial_x^2 f\right] p\left(x,t\,|\,x_0,t_0\right)$$

Integration by parts

$$\int dx f(x)\partial_t p = \int dx f(x)\left\{ -\partial_x[a(x,t)p] + \frac{1}{2}\partial_x^2\left[b(x,t)^2 p\right] \right\}$$

$f(x)$ arbitrary

$$\partial_t p\left(x,t\,|\,x_0,t_0\right) = -\,\partial_x\left[a(x,t)p\left(x,t\,|\,x_0,t_0\right)\right] + \frac{1}{2}\partial_x^2\left[b(x,t)^2 p\left(x,t\,|\,x_0,t_0\right)\right]$$

Stratonovich SDE

$$S\int_{t_0}^{t} G\left[x(t'),t'\right] dW(t') = \text{ms}\lim_{n\to\infty}\sum_{i=1}^{n} G\left\{\frac{x(t_i)+x(t_{i-1})}{2}, t_{i-1}\right\}\left[W(t_i)-W(t_{i-1})\right]$$

SDE with Stratonovich integral

$$x(t) = x(t_0) + \int_{t_0}^{t} dt'\alpha\left[x(t'),t'\right] + S\int_{t_0}^{t} dW(t')\beta\left[x(t'),t'\right]$$

Let us assume that $x(t)$ is a solution of

$$dx(t) = a[x(t),t]dt + b[x(t),t]dW(t)$$

connection between $S\int_{t_0}^{t} dW(t')\,\beta\left[x(t'),t'\right]$ and $\int_{t_0}^{t} dW(t')\,b\left[x(t'),t'\right]$???

Stratonovich SDE

$$S\int_{t_0}^{t} dW(t')\,\beta\left[x(t'),t'\right] \simeq \sum_i \beta\left[\frac{x(t_i)+x(t_{i-1})}{2},t_{i-1}\right]\left[W(t_i)-W(t_{i-1})\right]$$

We write $x(t_i) = x(t_{i-1}) + dx(t_{i-1})$

and use Ito SDE to get

$$dx(t_i) = a\left[x(t_{i-1}),t_{i-1}\right](t_i - t_{i-1}) + b\left[x(t_{i-1}),t_{i-1}\right]\left[W(t_i)-W(t_{i-1})\right]$$

$$\beta\left[\frac{x(t_i)+x(t_{i-1})}{2},t_{i-1}\right] = \beta\left[x(t_{i-1}) + \frac{1}{2}dx(t_{i-1}),t_{i-1}\right]$$

$$= \beta(t_{i-1}) + \left[a(t_{i-1})\,\partial_x\beta(t_{i-1}) + \frac{1}{4}b^2(t_{i-1})\right]\left[\frac{1}{2}(t_i - t_{i-1})\right]$$

$$+\frac{1}{2}b(t_{i-1})\,\partial_x\beta(t_{i-1})\left[W(t_i)-W(t_{i-1})\right]$$

$$\Longrightarrow S\int_{t_0}^{t}\beta\left[x(t'),t'\right]dW(t') = \int_{t_0}^{t}\beta\left[x(t'),t'\right]dW(t') + \frac{1}{2}\int_{t_0}^{t} b\left[x(t'),t'\right]\partial_x\beta\left[x(t'),t'\right]dt'$$

Stratonovich SDE

We see that the Ito SDE $$dx = a dt + b dW(t)$$

is the same as the Stratonovich SDE $$dx = \left[a - \frac{1}{2} b \partial_x b\right] dt + b dW(t)$$

and the Stratonovich SDE $$dx = \alpha dt + \beta dW(t)$$

is the same as the Ito SDE $$dx = \left[\alpha + \tfrac{1}{2} \beta \partial_x \beta\right] dt + \beta dW(t)$$

Stratonovich: change of variables as in normal calculus

Fokker-Planck equation

$$\partial_t p = - \textstyle\sum_i \partial_i \left\{A_i^s p\right\} + \tfrac{1}{2} \textstyle\sum_{i,k} \partial_i \left\{ B_{ik}^s \partial_j \left[B_{jk}^s p \right] \right\}$$

Stratonovich SDE

Desmond J. Higham: *An Algorithmic Introduction toNumerical Simulation of Stochastic Differential Equations* SIAM Review 43 (2001) 525

Matlab: http://www.maths.strath.ac.uk/~aas96106/algfiles.html
Python: see in Moodle class

Brownian Motion or standard Wiener process, over $[0,T]$ is a random variable $W(t)$ that depends continuously on $t \in [0,T]$ and satisfies the following three conditions.

1. $W(0) = 0$ (with probability 1).

2. For $0 \leq s < t \leq T$ the random variable given by the increment $W(t) - W(s)$ is normally distributed with mean zero and variance $t - s$; equivalently, $W(t) - W(s) \sim \sqrt{t-s}N(0,1)$, where $N(0,1)$ denotes a normally distributed random variable with zero mean and unit variance.

3. For $0 \leq s < t < u < v \leq T$ the increments $W(t) - W(s)$ and $W(v) - W(u)$ are independent.

Stratonovich SDE

discretized Brownian motion: $W(t)$ is specified at discrete t values.

$\delta t = T/N$ for some positive integer N

$W_j = W(t_j)$ with $t_j = j\delta t$

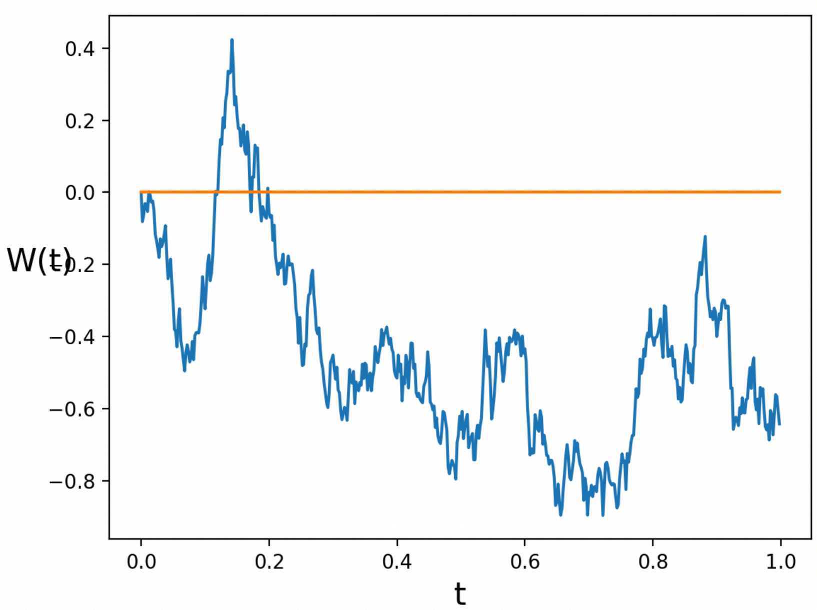
$W_0 = 0$ with probability 1

$W_j = W_{j-1} + \mathrm{d}W_j, \quad j = 1,2,\ldots,N$

where each $\mathrm{d}W_j$ is an independent random variable of the form $\sqrt{\delta t}N(0,1)$.

python code:

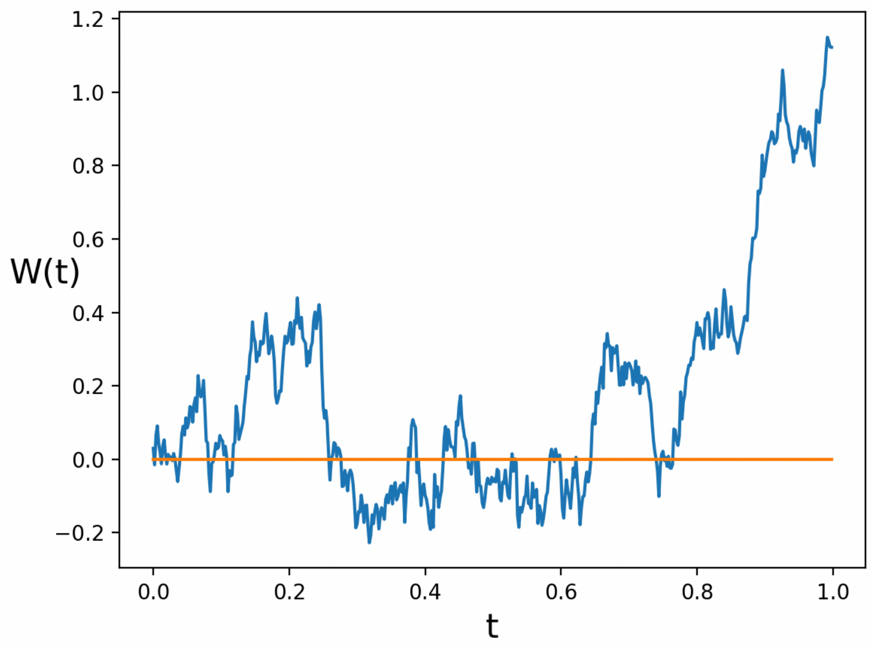
```
import numpy as np
import matplotlib.pyplot as plt

T = 1.; N = 500; dt = T/N
dW = np.zeros(N)
W = np.zeros(N)

dW[0] = np.sqrt(dt) * np.random.randn()
W[0] = dW[0]
for j in np.arange(1,N):
    dW[j] = np.sqrt(dt) * np.random.randn()
    W[j] = W[j-1] + dW[j]

time = np.arange(0,T,dt)
null = np.zeros(N)
plt.plot(time,W)
plt.plot(time,null)
plt.xlabel('t', fontsize=16)
plt.ylabel('W(t)', fontsize=16,rotation=0)

plt.show()
```



Stratonovich SDE

vectorized Brownian path

```
import numpy as np
import matplotlib.pyplot as plt

T = 1.; N = 500; dt = T/N

dW = np.sqrt(dt) * np.random.randn(N)
W = np.cumsum(dW)

time = np.arange(0,T,dt)
null = np.zeros(N)
plt.plot(time,W)
plt.plot(time,null)
plt.xlabel('t', fontsize=16)
plt.ylabel('W(t)', fontsize=16,rotation=0)

plt.show()
```



Function of Wiener noise

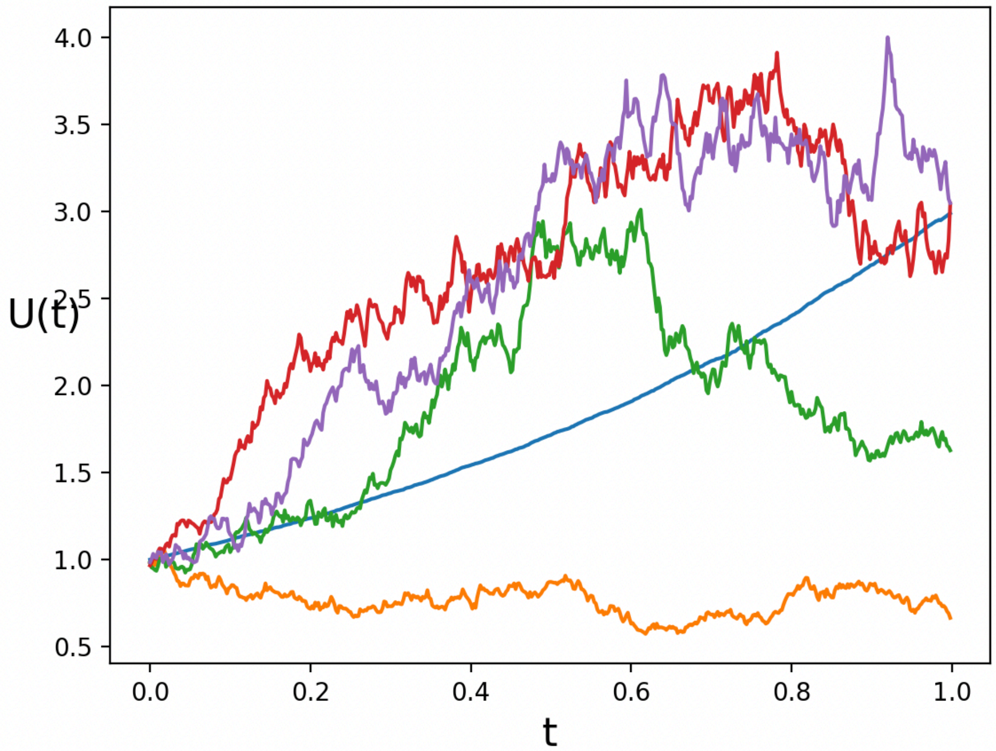
consider $u(W(t)) = \exp\left(t + \frac{1}{2}W(t)\right)$

mean: $\langle u(W(t))\rangle = \langle \exp\left(t + \frac{1}{2}W(t)\right)\rangle = \exp(9t/8)$

```
Integrate[Exp[x/2] Exp[-1/2 x^2/t]/(Sqrt[2 Pi t]), {x, -Infinity, Infinity}]
```

Stratonovich SDE

averaged Brownian path




```
import numpy as np
import numpy.matlib
import matplotlib.pyplot as plt

T = 1.; N = 500; dt = T/N;
time = np.arange(0,T,dt)

M = 1000;
dW = np.sqrt(dt) * np.random.randn(M,N)
W = np.cumsum(dW,1)
U = np.exp(np.matlib.repmat(time, M, 1) + 0.5*W)
Umean = np.mean(U,0)

null = np.zeros(N)
plt.plot(time,Umean)
plt.plot(np.transpose(np.matlib.repmat(time,4,1)),np.transpose(U[0:4,:]))
plt.xlabel('t', fontsize=16)
plt.ylabel('U(t)', fontsize=16,rotation=0)

plt.show()

print(np.max(np.abs(Umean-np.exp(9*time/8.))))
```

Stratonovich SDE

Stochastic Integrals

Ito: $$\sum_{j=0}^{N-1} h(t_j)(W(t_{j+1}) - W(t_j))$$

Stratonovich: $$\sum_{j=0}^{N-1} h\left(\frac{t_j + t_{j+1}}{2}\right)(W(t_{j+1}) - W(t_j))$$

```
Ito integral:            -0.05610518876586967
Stratonovich integral:    0.4026119859010535
Ito error:                0.04020028760298173
Stratonovich error:       0.00108253773009509
```

Caution: results are very different !!!

Ito: $$\int_0^T W(t)dW(t) = \frac{1}{2}W(T)^2 - \frac{1}{2}T$$

Stratonovich: $$S\int_0^T W(t)dW(t) = \frac{1}{2}W(T)^2$$

```
import numpy as np


T = 1.; N = 500; dt = T/N;

dW = np.sqrt(dt) * np.random.randn(N);
W = np.cumsum(dW)

Wshift=np.pad(W,(1,0),mode='constant')[:-1]
ito = np.sum(Wshift*dW)
print('Ito integral:          ', ito)
strat = sum((0.5*(Wshift+W) + 0.5*np.sqrt(dt)*np.random.randn(N))*dW)
print('Stratonovich integral:', strat)
itoerr = np.abs(ito - 0.5*(W[-1]**2-T))
print('Ito error:          ',itoerr)
straterr = abs(strat - 0.5*W[-1]**2)
print('Stratonovich error:', straterr)
```

The Euler-Maruyama Method

Ito:
$$dX(t) = f(X(t))dt + g(X(t))dW(t), \quad X(0) = X_0, \quad 0 \leq t \leq T$$

$$X_j = X_{j-1} + f(X_{j-1})\Delta t + g(X_{j-1})(W(\tau_j) - W(\tau_{j-1})), \quad j = 1,2,\ldots,L$$

consider linear SDE

$$dX(t) = \lambda X(t)dt + \mu X(t)dW(t), \quad X(0) = X_0$$

we know the exact solution

$$X(t) = X(0)\exp\left(\left(\lambda - \frac{1}{2}\mu^2\right)t + \mu W(t)\right)$$

remember:
$$df[W(t), t] = \left(\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial W^2}\right)dt + \frac{\partial f}{\partial W}dW(t)$$

Python-file em.py: $\lambda = 2$, $\mu = 1$, and $X_0 = 1$ (constant).
discretized Brownian path over $[0,1]$ with $\delta t = 2^{-8}$. Use analytic solution and name it “Xtrue” (magenta).

apply EM using a stepsize $\Delta t = R\delta t,\ R = 4$. On a general step the EM requires the increment $W(\tau_j) - W(\tau_{j-1})$, which is given by

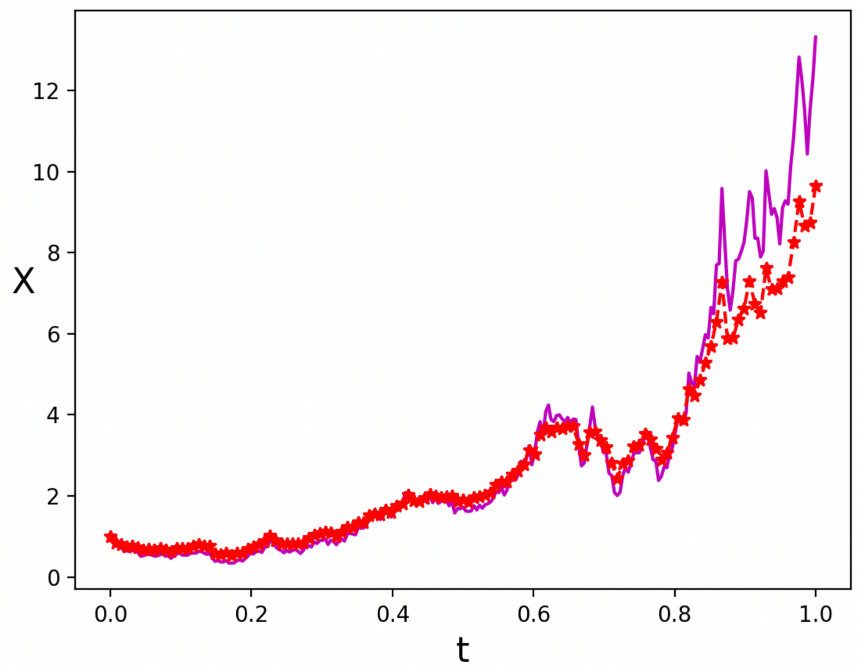
$$W(\tau_j) - W(\tau_{j-1}) = W(jR\delta t) - W((j-1)R\delta t) = \sum_{k=jR-R+1}^{jR} \mathrm{d}W_k$$

Winc in Python

Discrepancy between the exact solution and the EM solution at the endpoint $t = T$, computed as emerr, was found to be 0.6907 . Taking $\Delta t = R\delta t$ with smaller R values of 2 and 1 produced endpoint errors of 0.1595 and 0.0821.

Stratonovich SDE

```
import numpy as np
import numpy.matlib
import matplotlib.pyplot as plt

#np.random.seed(100)
lamda = 2; mu = 1; Xzero = 1
T = 1.; N = 2**8; dt = T/N
dW = np.sqrt(dt) * np.random.randn(N)
W = np.cumsum(dW)

Xtrue = Xzero*np.exp((lamda-0.5*mu**2)*(np.arange(dt,T+dt,dt)
+mu*W))
Xtrue=np.concatenate(([Xzero],Xtrue))
plt.plot(np.arange(0,T+dt,dt),Xtrue,'m-')

R = 2; Dt = R*dt; L = N//R
Xem = np.zeros(L)
Xtemp = Xzero;
for j in range(L):
   Winc = np.sum(dW[R*j:R*(j+1)])
   Xtemp = Xtemp + Dt*lamda*Xtemp + mu*Xtemp*Winc
   Xem[j] = Xtemp

plt.plot(np.arange(0,T+Dt,Dt),np.concatenate(([Xzero],Xem)),'r--
*')
plt.xlabel('t', fontsize=16)
plt.ylabel('X', fontsize=16, rotation=0)
plt.show()

emerr = np.abs(Xem[-1]-Xtrue[-1])
print('Euler-Maruyama error:', emerr)
```



Strong and Weak Convergence of the EM Method

strong order of convergence equal to γ

$$\mathbb{E}\left|X_n - X(\tau)\right| \leq C\Delta t^\gamma$$

for any fixed $\tau = n\Delta t \in [0,T]$ and Δt sufficiently small.

EM has strong order $\gamma = \frac{1}{2}$.

weak order of convergence equal to γ

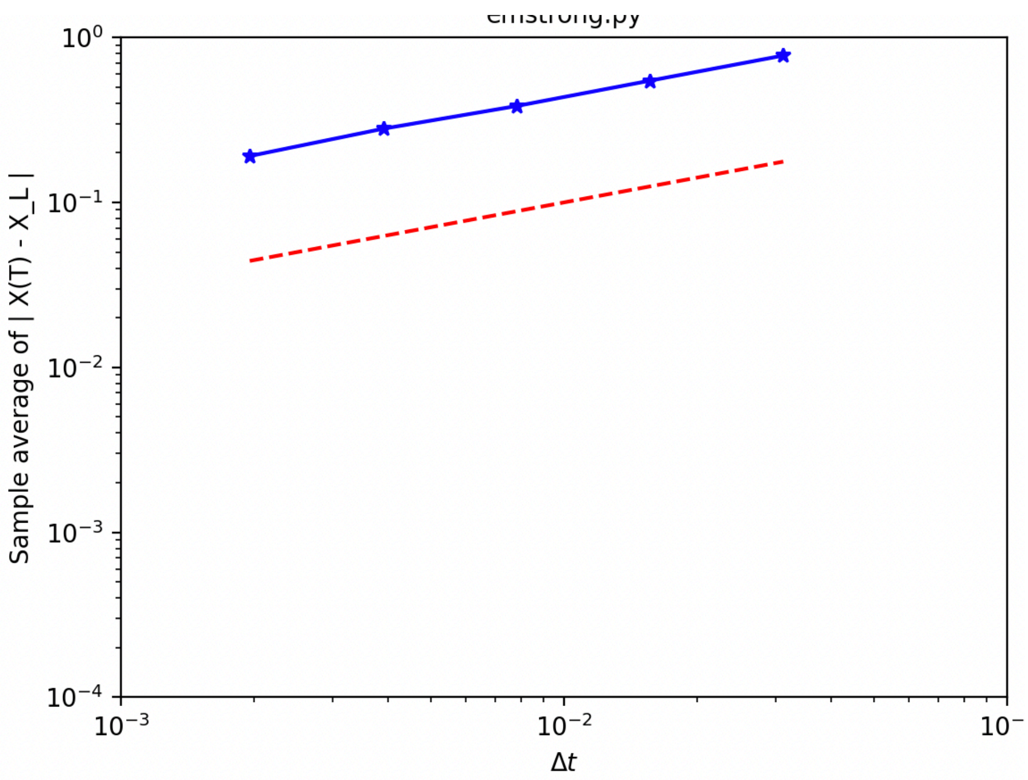
$$\left|\mathbb{E}p\left(X_n\right) - \mathbb{E}p(X(\tau))\right| \leq C\Delta t^\gamma$$, p is some smooth function with polynomial growth

Here $p = id$. EM has weak order $\gamma = 1$.

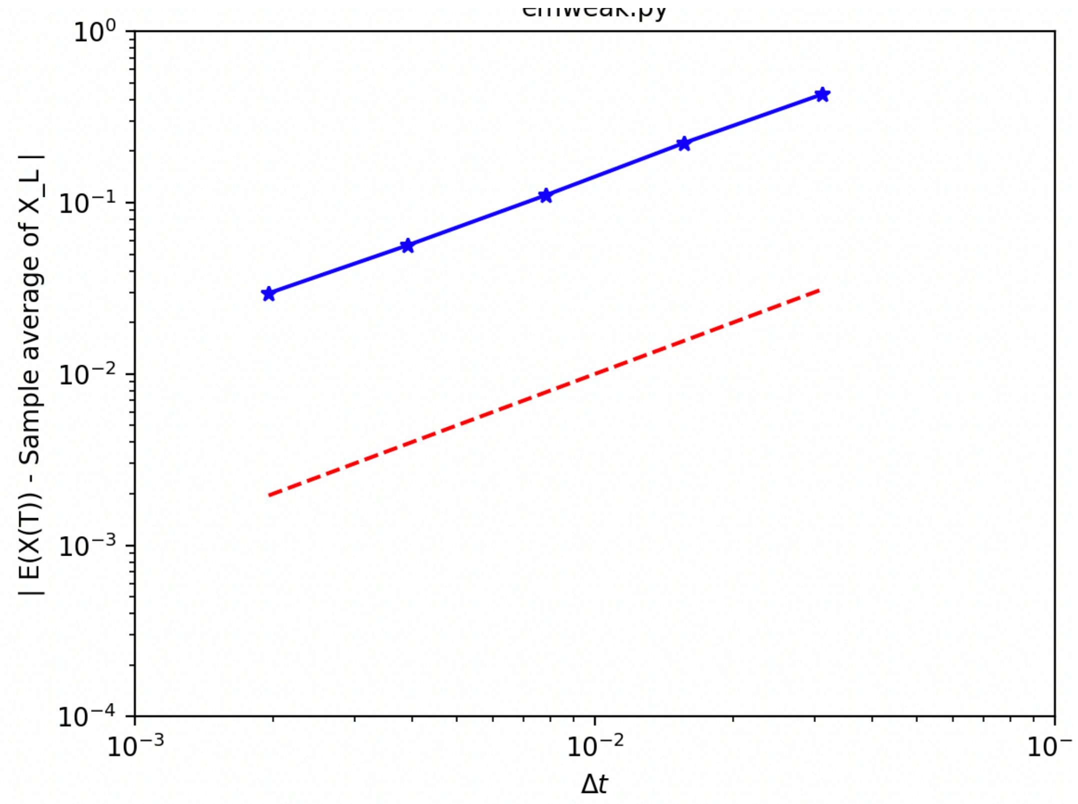
Stratonovich SDE

Strong Convergence

```
import numpy as np
import numpy.matlib
import matplotlib.pyplot as plt

#np.random.seed(100)
lamda = 2; mu = 1; Xzero = 1
T = 1.; N = 2**9; dt = T/N
M = 1000

Xerr = np.zeros([M,5])
for s in range(M):
    dW = np.sqrt(dt)*np.random.randn(N)
    W = np.cumsum(dW)
    Xtrue = Xzero*np.exp((lamda-0.5*mu**2)*T+mu*W[-1])
    for p in range(5):
        R = 2**p; Dt = R*dt; L = N//R
        Xtemp = Xzero
        for j in range(L):
            Winc = np.sum(dW[R*j:R*(j+1)])
            Xtemp = Xtemp + Dt*lamda*Xtemp + mu*Xtemp*Winc
        print('Xtrue:',Xtrue,'Xtemp:',Xtemp)
        Xerr[s,p] = np.abs(Xtemp - Xtrue)

mean=np.mean(Xerr,axis=0)
print('mean:',mean)
Dtvals = dt*2.**np.arange(5)
plt.loglog(Dtvals,mean,'b*-')
plt.loglog(Dtvals,(Dtvals**(.5)),'r--')
plt.axis([1e-3, 1e-1, 1e-4, 1])
plt.xlabel('$\Delta t$'); plt.ylabel('Sample average of | X(T) - X_L |')
plt.title('emstrong.py', fontsize=10)
plt.show()
```



Weak Convergence




```
import numpy as np
import numpy.matlib
import matplotlib.pyplot as plt

#np.random.seed(100)
lamda = 2; mu = 0.1; Xzero = 1; T = 1.
M = 50000

Xem = np.zeros(5)
for p in range(5):
    Dt = 2.**(p-9); L = T/Dt
    Xtemp = Xzero*np.ones(M)
    for j in range(int(round(L))):
        Winc = np.sqrt(Dt)*np.random.randn(M)
        #Winc = np.sqrt(Dt)*np.sign(np.random.randn(M)); ## use for weak E-M
        Xtemp = Xtemp + Dt*lamda*Xtemp + mu*Xtemp*Winc
    Xem[p] = np.mean(Xtemp)

Xerr = np.abs(Xem - np.exp(lamda))

Dtvals = 2.**(np.arange(5)-9)
plt.loglog(Dtvals,Xerr,'b*-')
plt.loglog(Dtvals,Dtvals,'r--')
plt.axis([1e-3, 1e-1, 1e-4, 1])
plt.xlabel('$\Delta t$'); plt.ylabel('| E(X(T)) - Sample average of X_L |')
plt.title('emweak.py', fontsize=10)
plt.show()
```

Milstein's Higher Order Method: use Ito-Taylor (Ito calculus)

$$X_j = X_{j-1} + \Delta t f\left(X_{j-1}\right) + g\left(X_{j-1}\right)\left(W\left(\tau_j\right) - W\left(\tau_{j-1}\right)\right)$$
$$+\frac{1}{2} g\left(X_{j-1}\right) g'\left(X_{j-1}\right)\left(\left(W\left(\tau_j\right) - W\left(\tau_{j-1}\right)\right)^2 - \Delta t\right), \quad j = 1,2,\ldots,L$$

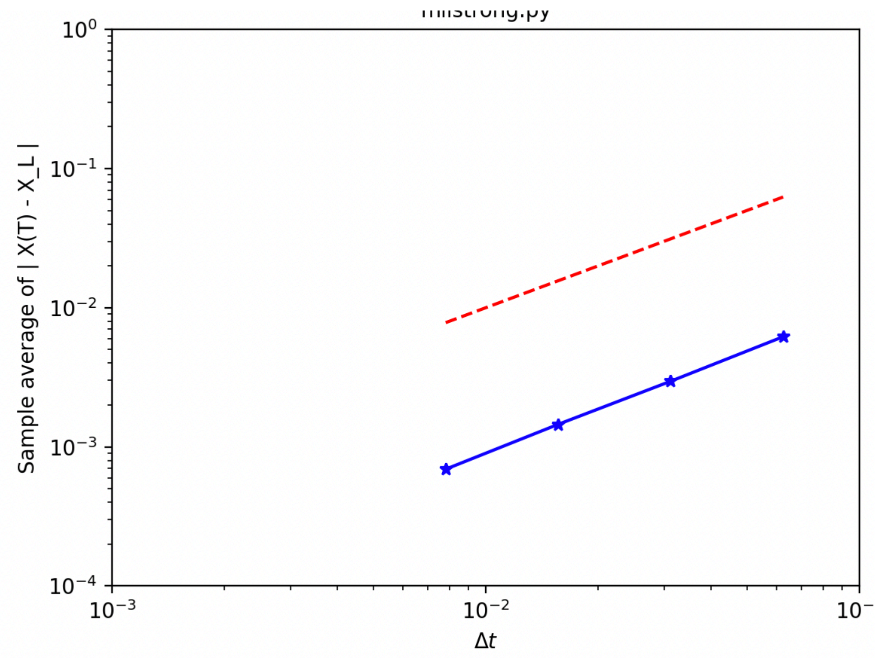
consider linear SDE

$$dX(t) = rX(t)(K - X(t))dt + \beta X(t)dW(t), \quad X(0) = X_0$$

Milstein is of strong order $\gamma = 1$

Stratonovich SDE

```
import numpy as np
import numpy.matlib
import matplotlib.pyplot as plt

#np.random.seed(100)
r = 2; K = 1; beta = 0.25; Xzero = 0.5
T = 1.; N = 2**11; dt = T/N
M = 500
R = np.array([1, 16, 32, 64, 128])

dW = np.sqrt(dt)*np.random.randn(M,N);
Xmil = np.zeros([M,5])
for p in range(5):
    Dt = R[p]*dt; L = N//R[p]
    Xtemp = Xzero*np.ones(M)
    for j in range(L):
        Winc = np.sum(dW[:,R[p]*j:R[p]*(j+1)],axis=1)
        Xtemp = Xtemp + Dt*r*Xtemp*(K-Xtemp) + beta*Xtemp*Winc \
        + 0.5*beta**2*Xtemp*(Winc**2 - Dt)

    Xmil[:,p] = Xtemp

Xref = Xmil[:,0]
Xerr = abs(Xmil[:,1:] - np.transpose(np.matlib.repmat(Xref,4,1)))

mean=np.mean(Xerr,axis=0)
print('mean:',mean)
Dtvals = dt*R[1:]
plt.loglog(Dtvals,mean,'b*-')
plt.loglog(Dtvals,Dtvals,'r--')
plt.axis([1e-3, 1e-1, 1e-4, 1])
plt.xlabel('$\Delta t$')
plt.ylabel('Sample average of | X(T) - X_L |')
plt.title('milstrong.py', fontsize=10)
plt.show()
```



Stratonovich SDE

Stochastic Chain Rule

$$dV(X(t)) = \frac{dV(X(t))}{dX}dX + \frac{1}{2}g(X(t))^2\frac{d^2V(X(t))}{dX^2}dt$$

$$\implies dV(X(t)) = \left(f(X(t))\frac{dV(X(t))}{dX} + \frac{1}{2}g(X(t))^2\frac{d^2V(X(t))}{dX^2}\right)dt + g(X(t))\frac{dV(X(t))}{dX}dW(t)$$

we consider SDE

$$dX(t) = (\alpha - X(t))dt + \beta\sqrt{X(t)}dW(t), \quad X(0) = X_0$$

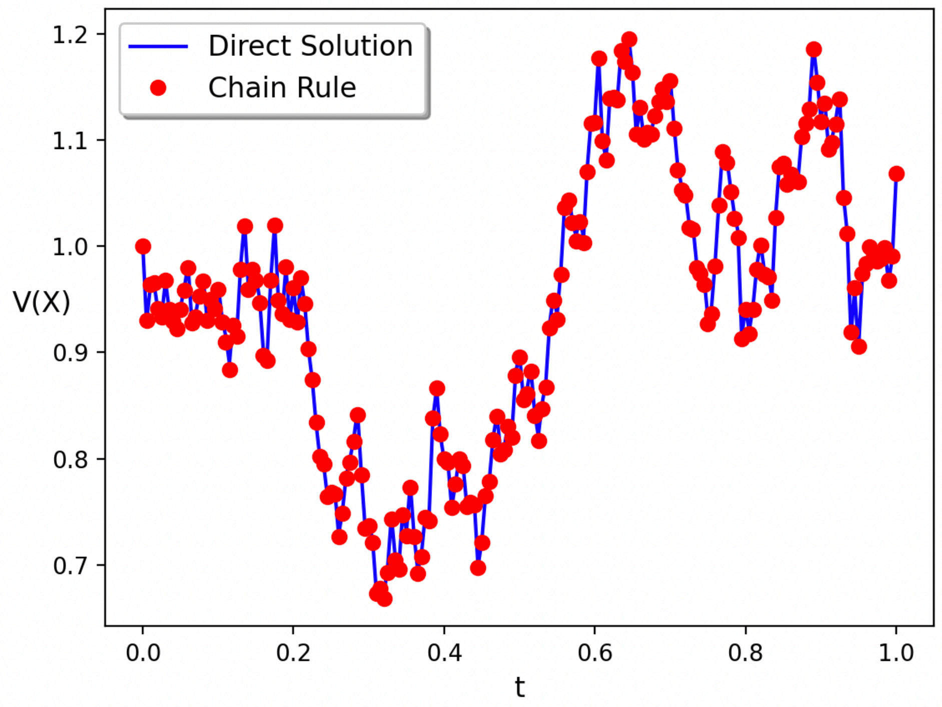
we choose: $V(X) = \sqrt{X}$

$$dV(t) = \left(\frac{4\alpha - \beta^2}{8V(t)} - \frac{1}{2}V(t)\right)dt + \frac{1}{2}\beta dW(t)$$

We compare square root of the original solution $X(t)$ with solution for $V(t)$.

Stratonovich SDE

```
import numpy as np
import numpy.matlib
import matplotlib.pyplot as plt

#np.random.seed(100)
alpha = 2.; beta = 1.; T = 1.; N = 200; dt = T/N
Xzero = 1.; Xzero2 = np.sqrt(Xzero)

Dt = dt
Xem1 = np.zeros(N); Xem2 = np.zeros(N)
Xtemp1 = Xzero;  Xtemp2 = Xzero2
for j in range(N):
    Winc = np.sqrt(dt)*np.random.randn()
    f1 = (alpha-Xtemp1)
    g1 = beta*np.sqrt(np.abs(Xtemp1))
    Xtemp1 = Xtemp1 + Dt*f1 + Winc*g1
    Xem1[j] = Xtemp1
    f2 = (4*alpha-beta**2)/(8*Xtemp2) - Xtemp2/2.
    g2 = beta/2
    Xtemp2 = Xtemp2 + Dt*f2 + Winc*g2
    Xem2[j] = Xtemp2

time = np.arange(0,T+Dt,Dt)
plt.plot(time,np.sqrt(np.concatenate(([Xzero],np.abs(Xem1)))),'b-',label='Direct Solution')
plt.plot(time,np.concatenate(([Xzero2],Xem2)),'ro',label='Chain Rule')
plt.legend(loc='upper left', shadow=True, fontsize='large')
plt.xlabel('t', fontsize=12)
plt.ylabel('V(X)', fontsize=12, rotation=0)
plt.show()

Xdiff = np.max(np.abs(np.sqrt(Xem1) - Xem2))
print('Difference:', Xdiff)
~
```



RUHR-UNIVERSITÄT BOCHUM

RUB

Tensor Networks and Machine Learning

Chaos, Turbulence and Stochastic Systems SS 2025

Rainer Grauer
TP I RUB

RUB

Outline

- DL/ML as function fitting
  - linear regression
  - feature maps, support vector machines: going to higher dimensions
  - neural nets, backpropagation and stochastic gradient
  - linear versus non-linear: examples
- What are Tensor Networks ?
  - Tensor networks: the physics view
    - MPS, PEPS, MERA, area law
    - DMRG
  - Tensor networks: the math view
    - CP, Tucker, TT, HTucker
    - MALS
- Applications at TP1 (optional)
  - Vlasov simulations
  - Instanton fluctuations
- Machine Learning with tensor networks
- Summary and Literature

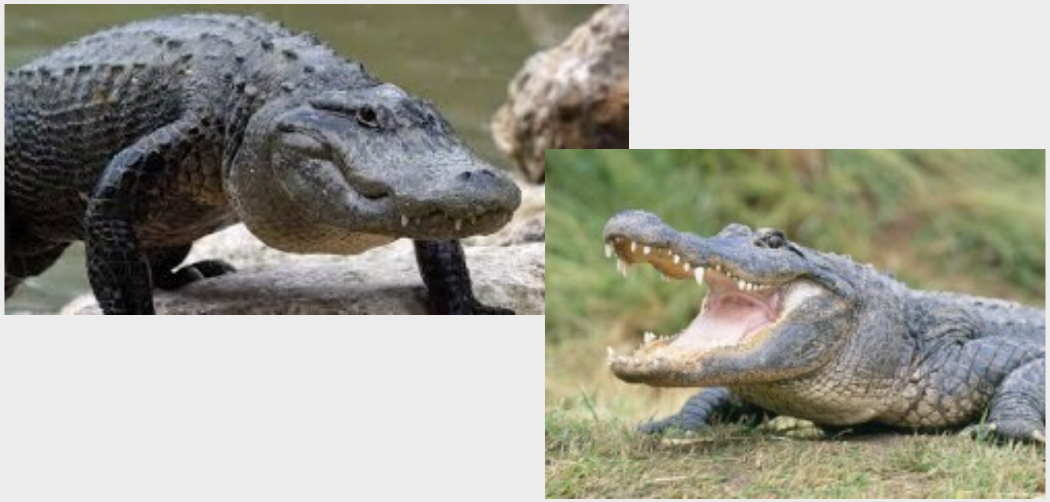
- DL/ML as function fitting
  - today's focus: Supervised Learning

    Given labeled training data (labels A and B)

    Find *decision function* $f(\mathbf{x})$

$$f(\mathbf{x}) > 0 \qquad \mathbf{x} \in A \qquad\qquad f(\mathbf{x}) < 0 \qquad \mathbf{x} \in B$$

Example: identify photos of alligators and bears



What are Tensor Networks ?

Physics

MPS, PEPS, MERA
DMRG

St. White 1992
J. I. Cirac, F. Verstraete, 2004
U. Schollwöck 2005
.
.
.
DMRG

Math

TT, Tucker
MALS

I. V. Oseledets 2008
W. Hackbusch 2009
L. Grasedyck 2013
.
.
.
ALS, MALS

Roman Orus *A Practical Introduction to Tensor Networks: Matrix Product States and Projected Entangled Pair States* 2014

L. Grasedyck, D. Kressner, C. Tobler *A literature survey of low-rank tensor approximation techniques* 2013

RUB

Physics

Many-body wavefunction in compact form

$$|\psi\rangle = \sum_s \psi^{s_1 s_2 s_3 \cdots s_d} |s_1 s_2 s_3 \cdots s_d\rangle$$

amplitudes

basis

$\psi^{s_1 s_2 s_3 \cdots s_d}$ amplitude tensor

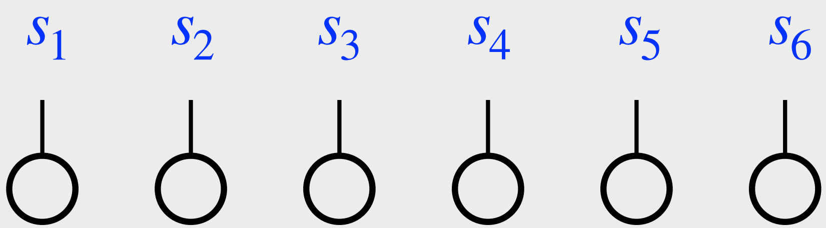
high dimensional monster

Neglect any correlations

$$\psi^{s_1 s_2 s_3 s_4 s_5 s_6} \simeq \psi^{s_1}\psi^{s_2}\psi^{s_3}\psi^{s_4}\psi^{s_5}\psi^{s_6}$$

single-body wave function

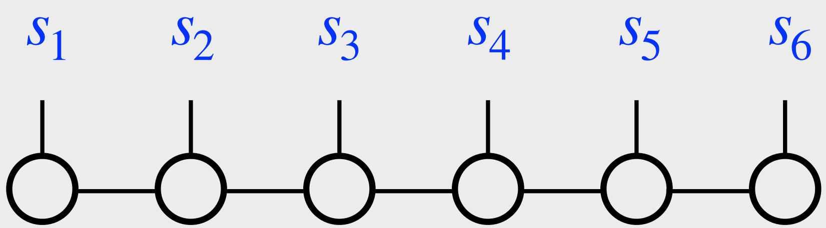
pictorial way:



Restore correlations (locally)

$$\psi^{s_1 s_2 s_3 s_4 s_5 s_6} \simeq \psi^{s_1}_{i_1} \psi^{s_2}_{i_1 i_2} \psi^{s_3}_{i_2 i_3} \psi^{s_4}_{i_3 i_4} \psi^{s_5}_{i_4 i_5} \psi^{s_6}_{i_5}$$

pictorial way:



matrix product state (MPS)

✓ exponentially decaying correlations

Matrix Product State compresses a tensor

$$\psi^{s_1 s_2 s_3 s_4 s_5} = M_1^{s_1} M_2^{s_2} M_3^{s_3} M_4^{s_4} M_5^{s_5}$$

Complexity: # parameters for matrix size $m \times m$ (m: bond dimension)

exponential

linear

$$n^d \longrightarrow 2nm + (d-2)nm^2$$

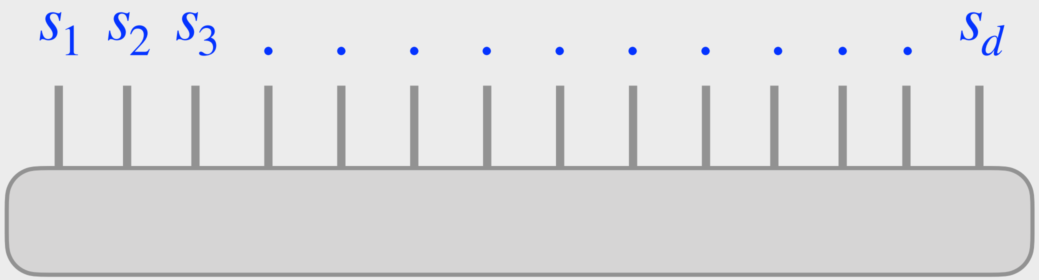
discrete basis: spins $n = 2$

spatial discretisation $n =$ # mesh points in one direction

Tensor diagrams

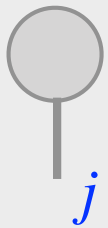
draw d-index tensor as “blob” with d lines

$$\psi^{s_1 s_2 s_3 \cdots s_d} =$$

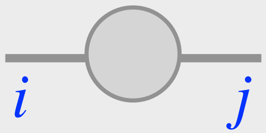


simple tensors:

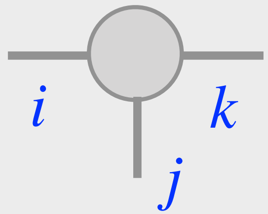
vector v_j



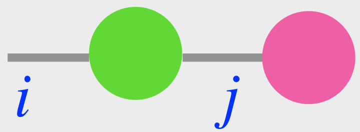
matrix M_{ij}



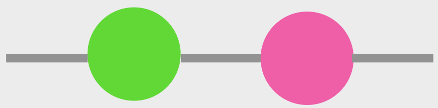
3-index tensor T_{ijk}



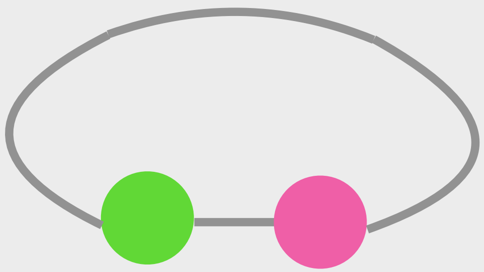
joining lines means contraction



$$\sum_j M_{ij}\, v_j$$



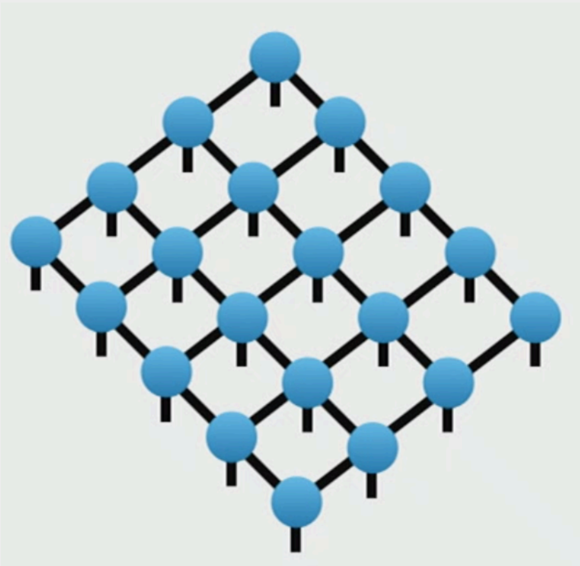
$$A_{ij}\, B_{jk} = AB$$



$$A_{ij}\, B_{ji} = \mathrm{Tr}\,[AB]$$

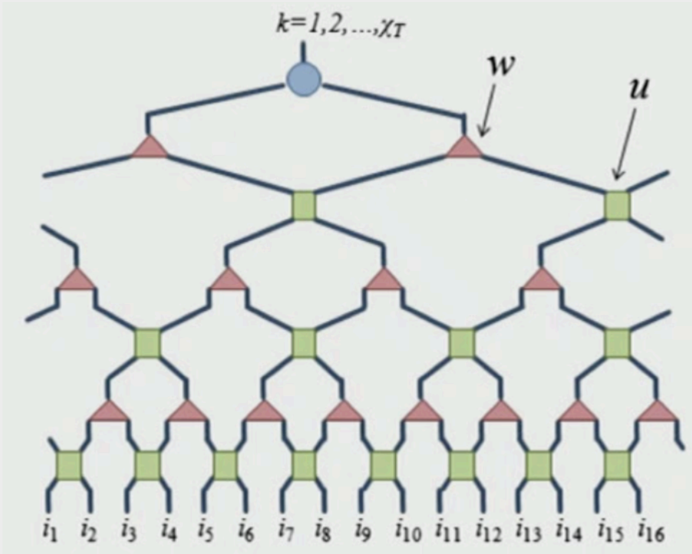
Extendable to other types tensor networks

2D systems



PEPS: projected entangled pair states

power law correlations



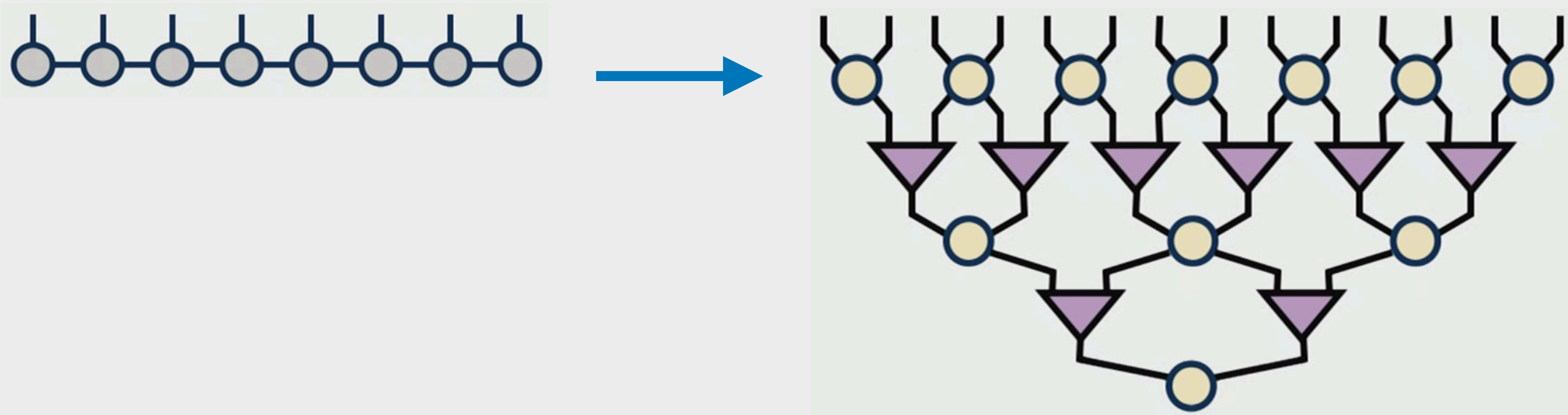
MERA: multi-scale entanglement renormalization ansatz

Evenbly, Vidal, PRB 79 (2009)144108

Verstraete, Cirac, cond-mat/0407066 (2004)

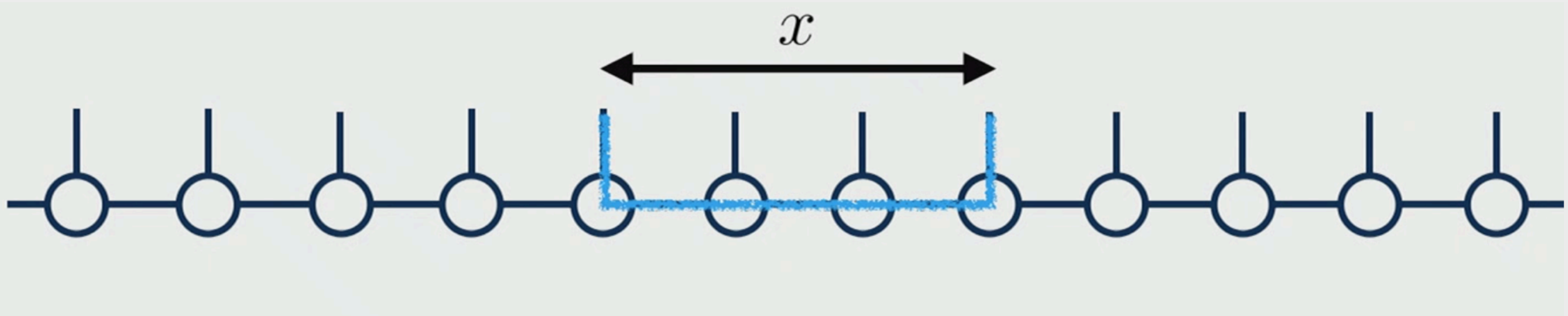
MERA Tensor Network

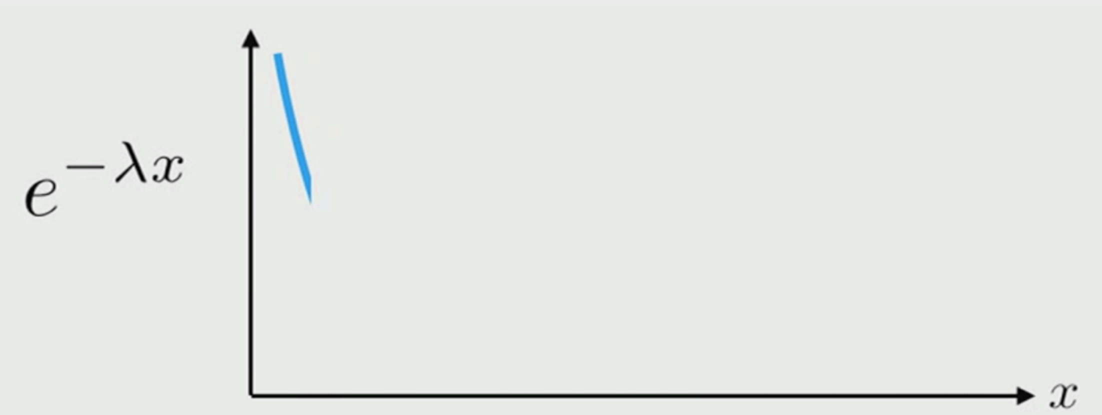
… generalizes matrix product state to a layered structure



MERA Tensor Network

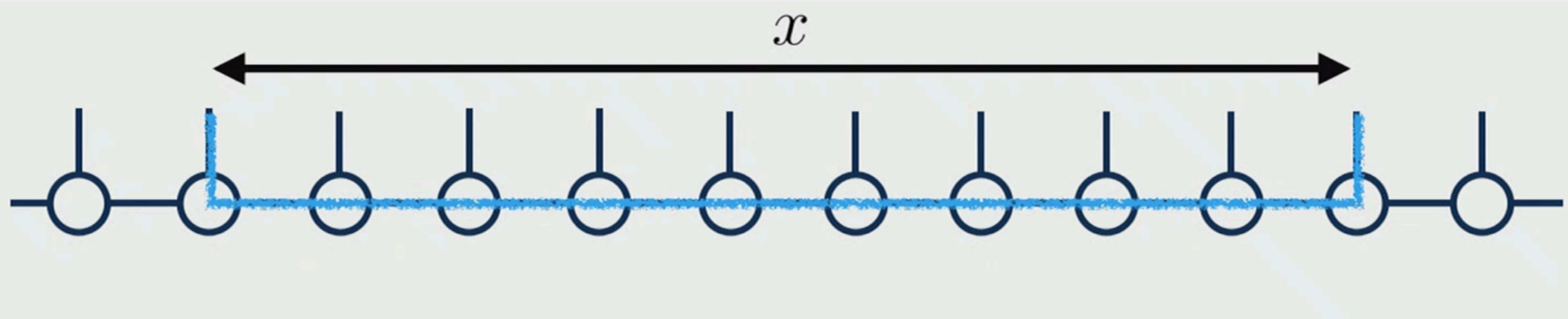
matrix product state captures only exponentially decaying correlations

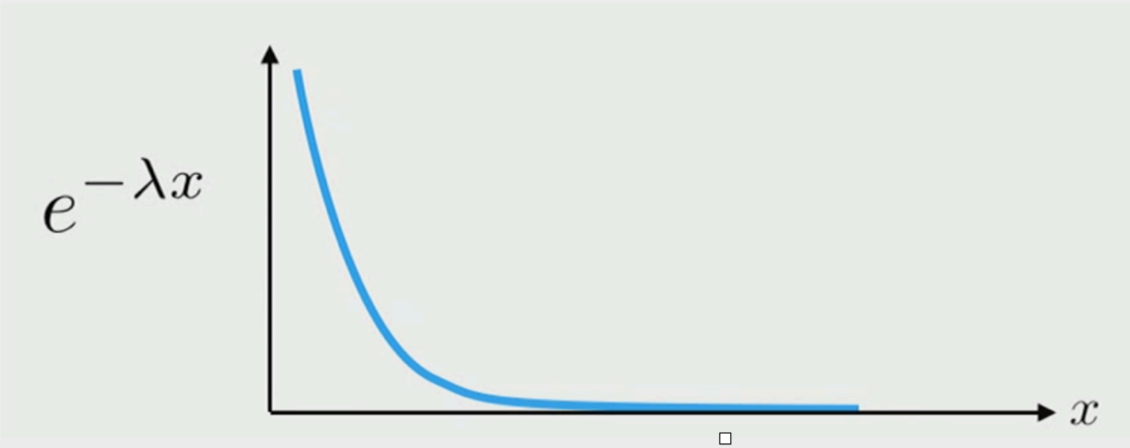




MERA Tensor Network

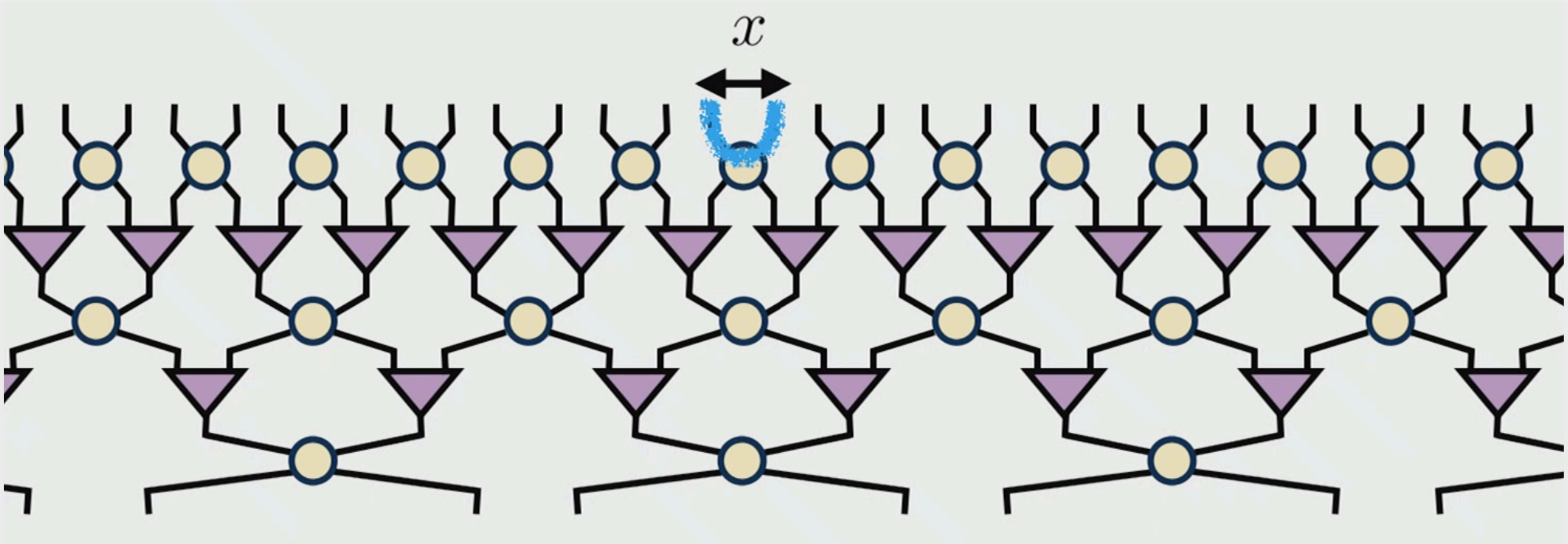
matrix product state captures only exponentially decaying correlations





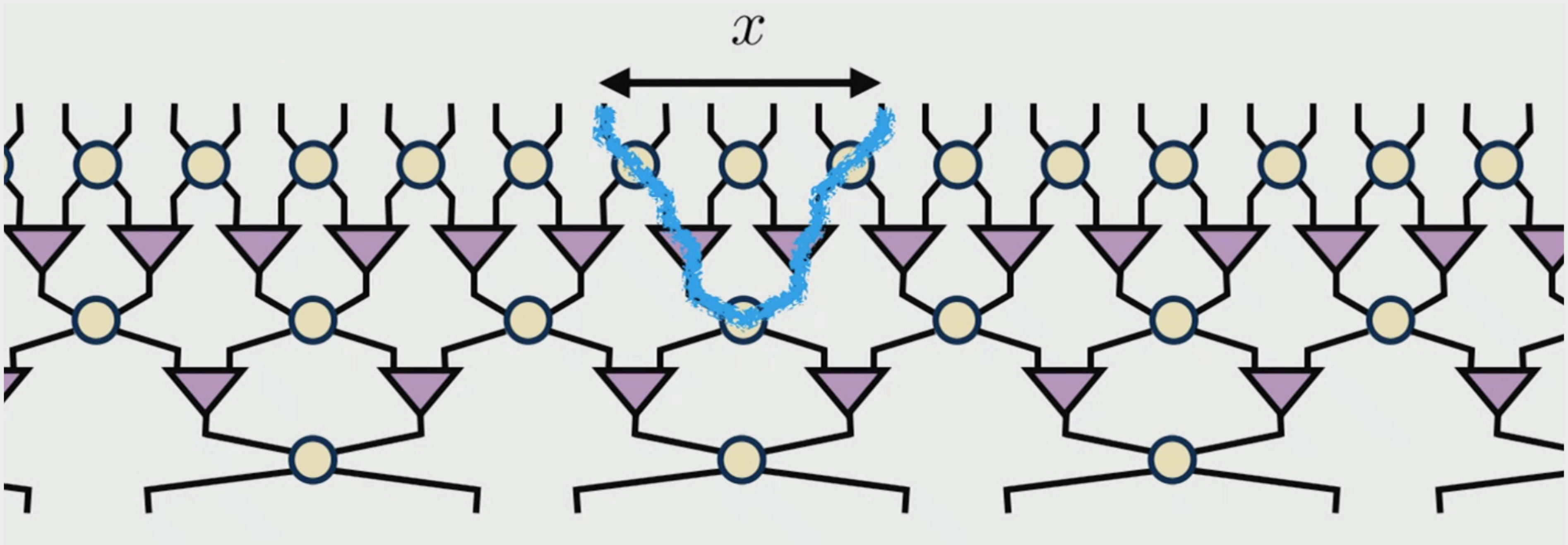
MERA Tensor Network

MERA layers architecture captures power-law correlations



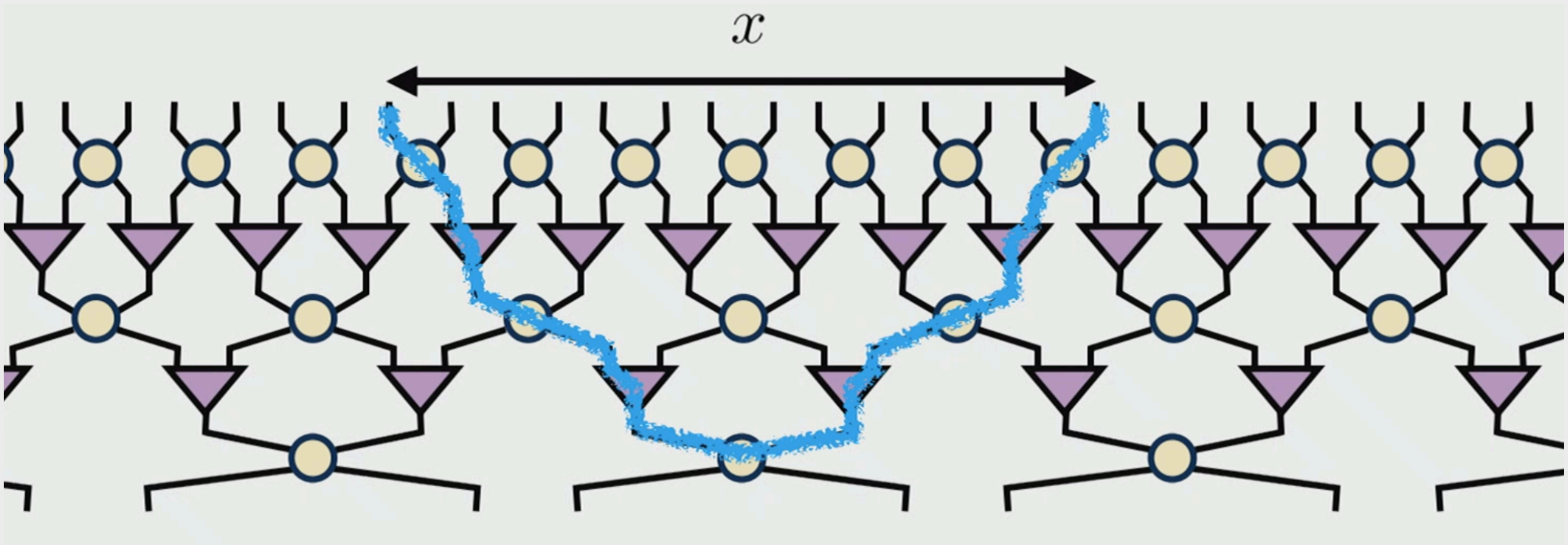
MERA Tensor Network

MERA layers architecture captures power-law correlations



MERA Tensor Network

MERA layers architecture captures power-law correlations



Examples of Matrix Product States

RUB

Example: single-particle state

$$|\Psi\rangle = \phi_1 |100000\rangle + \phi_2 |010000\rangle + \phi_3 |001000\rangle + \phi_4 |000100\rangle + \ldots \quad = \sum_j \phi_j c_j^\dagger |0\rangle$$

$$= \begin{bmatrix} |\phi_1|1\rangle & |0\rangle \end{bmatrix} \begin{bmatrix} |0\rangle & 0 \\ \phi_2|1\rangle & |0\rangle \end{bmatrix} \begin{bmatrix} |0\rangle & 0 \\ \phi_3|1\rangle & |0\rangle \end{bmatrix} \begin{bmatrix} |0\rangle & 0 \\ \phi_4|1\rangle & |0\rangle \end{bmatrix} \cdots \begin{bmatrix} |0\rangle & 0 \\ \phi_{N-1}|1\rangle & |0\rangle \end{bmatrix} \begin{bmatrix} |0\rangle \\ \phi_N|1\rangle \end{bmatrix}$$

$$= [\phi_1|10\rangle + \phi_2|01\rangle \qquad |00\rangle] \begin{bmatrix} |0\rangle & 0 \\ \phi_3|1\rangle & |0\rangle \end{bmatrix} \begin{bmatrix} |0\rangle & 0 \\ \phi_4|1\rangle & |0\rangle \end{bmatrix} \cdots \begin{bmatrix} |0\rangle & 0 \\ \phi_{N-1}|1\rangle & |0\rangle \end{bmatrix} \begin{bmatrix} |0\rangle \\ \phi_N|1\rangle \end{bmatrix}$$

$$= \begin{bmatrix} \phi_1|100\rangle + \phi_2|010\rangle + \phi_3|001\rangle & |000\rangle \end{bmatrix} \begin{bmatrix} |0\rangle & 0 \\ \phi_4|1\rangle & |0\rangle \end{bmatrix} \cdots \begin{bmatrix} |0\rangle & 0 \\ \phi_{N-1}|1\rangle & |0\rangle \end{bmatrix} \begin{bmatrix} |0\rangle \\ \phi_N|1\rangle \end{bmatrix}$$

$$= [\phi_1|1000\rangle + \phi_2|0100\rangle + \phi_3|0010\rangle + \phi_4|0001\rangle \quad |0000\rangle] \cdots \begin{bmatrix} |0\rangle & 0 \\ \phi_{N-1}|1\rangle & |0\rangle \end{bmatrix} \begin{bmatrix} |0\rangle \\ \phi_N|1\rangle \end{bmatrix}$$

other exact MPS states: Kitaev chain ground states

RUB

Math

pictures/ideas stolen from D. Kressner

A d-th order **tensor** $\mathcal{X}$ of size $n_1 \times n_2 \times \ldots \times n_d$ is a d-dimensional array with entries

$$\mathcal{X}_{i_1, i_2, \ldots, i_d}, \quad i_\mu \in \{1, \ldots, n_\mu\}, \quad \mu = 1, \ldots, d$$

The **vectorization** of $\mathcal{X}$ is denoted by $\text{vec}(\mathcal{X})$, where

$$\text{vec} : \mathbb{R}^{n_1 \times n_2 \times \ldots \times n_d} \longrightarrow \mathbb{R}^{n_1 \cdot n_s \cdots n_d}$$

stacks the entries of a tensor in reverse lexicographical order into a long column vector.

Example:

$$A = \left[\begin{array}{c|c} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{array}\right] \quad \Rightarrow \quad \text{vec}(A) = \left[\begin{array}{c} a_{11} \\ a_{21} \\ a_{31} \\ \hline a_{12} \\ a_{22} \\ a_{32} \end{array}\right]$$

Matricization

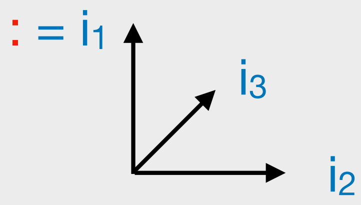
- A matrix has two modes (column mode and row mode).
- A dth-order tensor χ has d modes ($\mu = 1, \mu = 2, \ldots, \mu = d$).

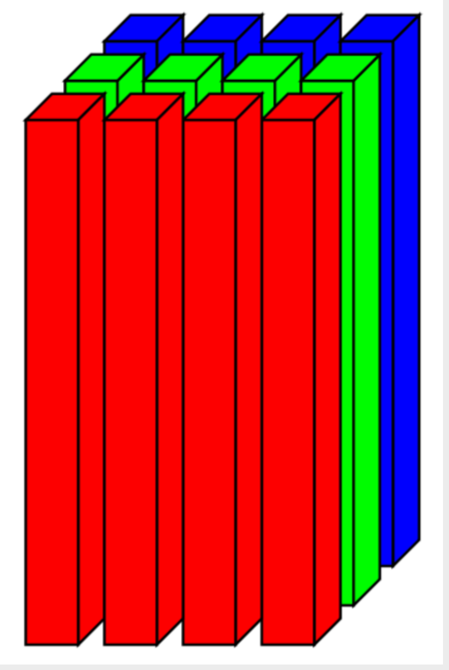
Let us fix all but one mode, e.g. $\mu = 1$: Then

$\chi(:, i_2, i_3, \ldots, i_d)$ (Matlab notation)

is a **vector** of length n_1 for each choice of $i_2, \ldots, i_d$.

$\Longrightarrow$ view tensor χ as a bunch of column vectors:

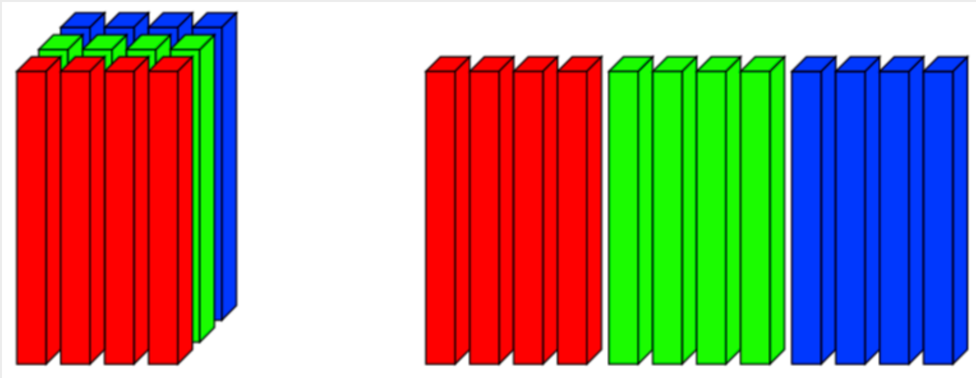




Matricization

stack vectors into an $n_1 \times (n_2 \cdots n_d)$ matrix:

$\mathscr{X} \in \mathbb{R}^{n_1 \times n_2 \times \ldots \times n_d}$



$X^{(1)} \in \mathbb{R}^{n_1 \times (n_2 n_3 \cdots n_d)}$

For $\mu = 1, \ldots, d$, the μ-mode matriculation of $\mathscr{X}$ is a matrix

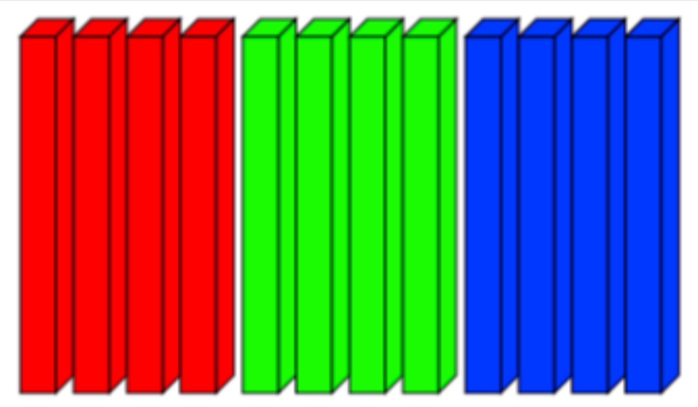
$$X^{(\mu)} \in \mathbb{R}^{n_\mu \times (n_1 \cdots n_{\mu-1} n_{\mu+1} \cdots n_d)}$$

with entries

$$\left(X^{(\mu)}\right)_{i_{\mu 1},(i_1,\ldots,i_{\mu-1},i_{\mu+1}\ldots i_\alpha)} = \chi_i \quad \forall i \in \mathfrak{I}$$

For a matrix $A \in \mathbb{R}^{n_1 \times n_2}$: $\quad A^{(1)} = A \quad A^{(2)} = A^T$

μ-mode matrix products



Consider 1-mode matricization

$X^{(1)} \in \mathbb{R}^{n_1 \times (n_2 n_3 \cdots n_d)}$:

Seems to make sense to multiply an $m \times n_1$ matrix A from the left:

$$Y^{(1)} := AX^{(1)} \in \mathbb{R}^{m \times (n_2 \cdots n_d)}$$

Can rearrange $Y^{(1)}$ back into an $m \times n_2 \times \cdots \times n_d$ tensor $\mathcal{Y}$.
This is called 1-mode matrix multiplication

$$\mathcal{Y} = A \circ_1 \mathcal{X} \quad \Longleftrightarrow \quad Y^{(1)} = AX^{(1)}$$

More formally

$$\mathcal{Y}_{i_1, i_2, \ldots, i_d} = \sum_{k=1}^{n_1} a_{i_1, k} \mathcal{X}_{k, i_2, \ldots, i_d}$$

μ-mode matrix products

General definition of a μ-mode matrix product with $A \in \mathbb{R}^{m \times n_\mu}$:

$$\mathcal{Y} = A \circ_\mu \mathcal{X} \quad \Longleftrightarrow \quad Y^{(\mu)} = AX^{(\mu)}.$$

More formally:

$$\mathcal{Y}_{i_1, i_2, \ldots, i_d} = \sum_{k=1}^{n_\mu} a_{i_\mu, k} \mathcal{X}_{i_1, \ldots, i_{\mu-1}, k, i_{\mu+1}, \ldots, i_d}$$

For matrices:

- 1-mode multiplication = multiplication from the left:

$$Y = A \circ_1 X = AX.$$

- 2-mode multiplication = transposed multiplication from the right:

$$Y = A \circ_2 X = XA^T.$$

RUB

Kronecker product

For $m \times n$ matrix A and $k \times l$ matrix B, the Kronecker product is defined as

$$B \otimes A := \begin{bmatrix} b_{11}A & \cdots & b_{1l}A \\ \vdots & \vdots & \\ b_{k1}A & \cdots & b_{kl}A \end{bmatrix} \in \mathbb{R}^{km \times ln}$$

Most important properties (for our purposes):

1. vec$(AX) = (I \otimes A)$vec(X)
2. vec$(XA^T) = (A \otimes I)$vec(X)
3. $(B \otimes A)(D \otimes C) = (BD \otimes AC)$
4. $I_m \otimes I_n = I_{mn}$

μ-mode matrix products and vectorisation

By definition, $\text{vec}(\mathscr{X}) = \text{vec}(X^{(1)})$.

Consequently, also $\text{vec}(A \circ_1 \mathscr{X}) = \text{vec}(AX^{(1)})$.

$\Longrightarrow$ Vectorized version of 1-mode matrix product:

$$\text{vec}(A \circ_1 \mathscr{X}) = (I_{n_2 \cdots n_d} \otimes A)\text{vec}(\mathscr{X}) = (I_{n_d} \otimes \cdots \otimes I_{n_2} \otimes A)\text{vec}(\mathscr{X})$$

Relation between μ-mode matrix product and matrix-vector product:

$$\boxed{\text{vec}(A \circ_\mu \mathscr{X}) = (I_{n_d} \otimes \cdots \otimes I_{n_{\mu+1}} \otimes A \otimes I_{n_{\mu-1}} \otimes \cdots \otimes I_{n_1})\text{vec}(\mathscr{X})}$$

Low-rank approximation

matrix: singular value decomposition

SVD: Let matrix $X \in \mathbb{R}^{n \times m}$ and $k = \min\{m, n\}$. Then $\exists$ orthonormal matrices

$$U = [u_1, u_2, \ldots, u_k] \in \mathbb{R}^{n \times k}, \; V = [v_1, v_2, \ldots, v_k] \in \mathbf{R}^{m \times k},$$

such that

$$X = U \Sigma V^T \;, \;\; \Sigma = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_k).$$

Choose $r \leq k$ and partition

$$X = [U_1, U_2] \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix} [V_1, V_2]^T = \underbrace{U_1 \Sigma_1}_{=:R} \; \underbrace{V_1^T}_{=:S^T} + U_2 \Sigma_2 V_2.$$

Then $\|X - RS^T\|_2 = |\,\Sigma_2\,|_2 = \sigma_{r+1}$.

Good low rank approximation if singular values decay sufficiently fast.

Low-rank **tensor** approximations

- CP (canonical polyadic decomposition)
- Tucker
- HOSVD (higher-order SVD)
- Tensor Networks (TT, HTucker)

CP: Motivation with matrix $X \in \mathbb{R}^{n_1 \times n_2}$

$$X = \left[a_1, a_2, \ldots, a_R\right] \left[b_1, b_2, \ldots, b_R\right]^T = a_1 b_1^T + a_2 b_2^T + \cdots + a_R b_R^T \quad a_j \in \mathbb{R}^{n_1}, \; b_j \in \mathbb{R}^{n_2}$$

$\Longrightarrow$ vectorisation $\quad \mathrm{vec}(X) = b_1 \otimes a_1 + b_2 \otimes a_2 + \cdots + b_R \otimes a_R$

Canonical **P**olyadic decomposition of a tensor $X \in \mathbb{R}^{n_1 \times n_2 \times n_3}$

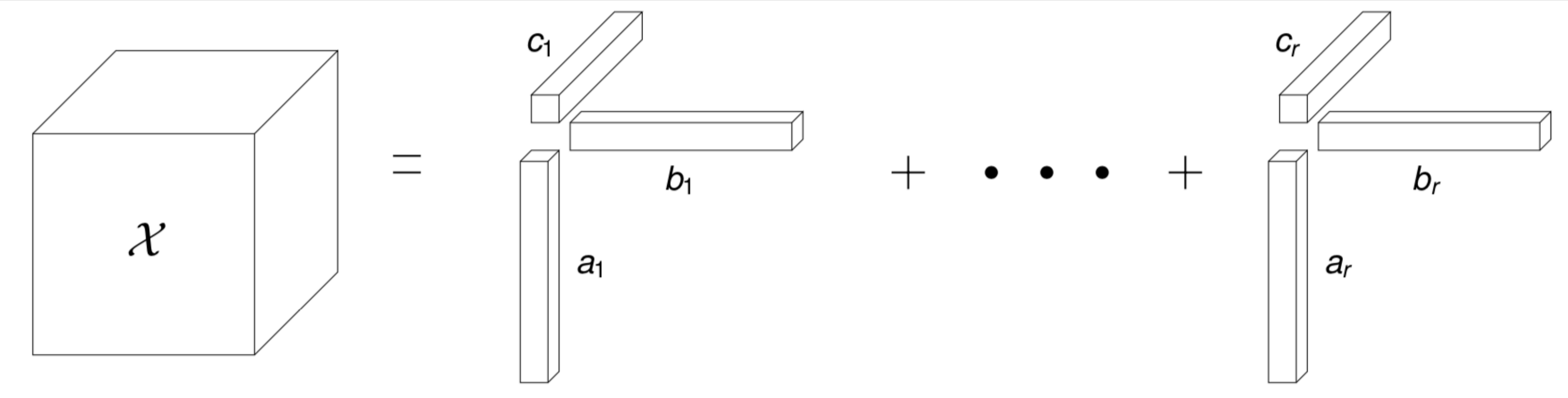
$$\mathrm{vec}(X) = c_1 \otimes b_1 \otimes a_1 + c_2 \otimes b_2 \otimes a_2 + \cdots + c_R \otimes b_R \otimes a_R$$

for vectors $a_j \in \mathbb{R}^{n_1}, \; b_j \in \mathbb{R}^{n_2}, \; c_j \in \mathbb{R}^{n_3}$.

semi-separable approximation

CP decomposition

$$\text{vec}(X) = c_1 \otimes b_1 \otimes a_1 + c_2 \otimes b_2 \otimes a_2 + \cdots + c_R \otimes b_R \otimes a_R$$



nice: linear scaling in d
Problem: lack of closedness (no minimum, only infimum, divergent norm

Tucker decomposition

Aim: Generalize concept of low rank to matrices motivated by

$$A = U \cdot \Sigma \cdot V^T\,,\;\; U \in \mathbb{R}^{n_1 \times r}\,,\;\; V \in \mathbb{R}^{n_2 \times r}\,,\;\; \Sigma \in \mathbb{R}^{r \times r}$$

$\Longrightarrow$ vectorisation $\quad \text{vec}(X) = (V \otimes U) \cdot \text{vec}(\Sigma)$

ignore diagonal structure of Σ and call it $\mathscr{C}$

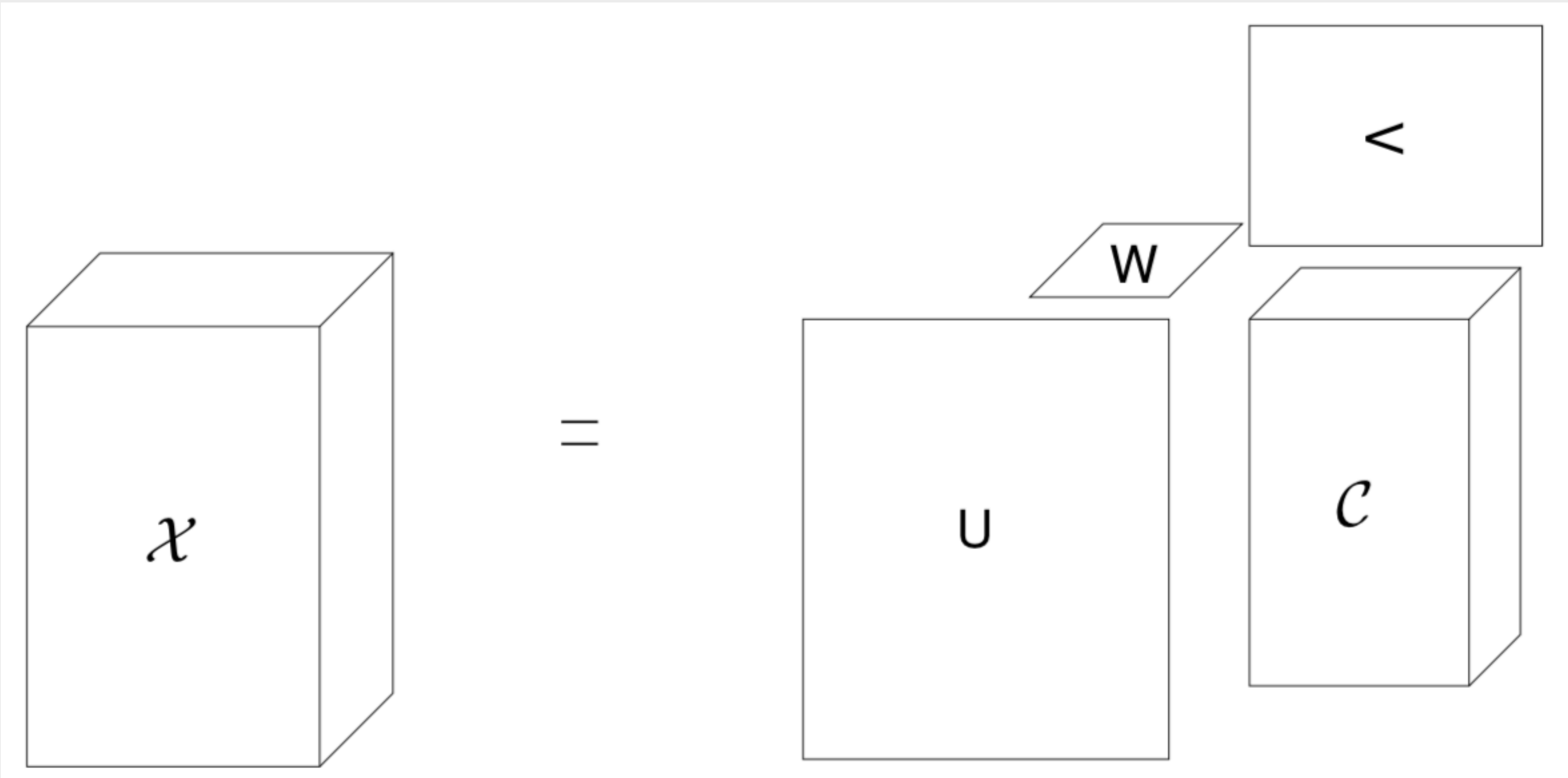
Tucker decomposition of tensor $\mathscr{X} \in \mathbb{R}^{n_1 \times n_2 \times n_3}$ defined via

$$\text{vec}(\mathscr{X}) = (W \otimes V \otimes U) \cdot \text{vec}(\mathscr{C})$$

with $U \in \mathbb{R}^{n_1 \times r_1}\,,\;\; V \in \mathbb{R}^{n_2 \times r_2}\,,\;\; W \in \mathbb{R}^{n_3 \times r_3}$ and core tensor $\mathscr{C} \in \mathbb{R}^{r_1 \times r_2 \times r_3}$.

μ-mode matrix product: $\quad \mathscr{X} = U \circ_1 V \circ_2 W \circ_3 \mathscr{C} =: (U, V, W) \circ \mathscr{C}$

Tucker decomposition: illustration



RUB

Higher-order SVD (HOSVD)

Goal: approximate given tensor $\mathscr{X}$ by Tucker decomposition (example in 3D)

- Calculate SVD of all matricizations: $X^{(\mu)} = \tilde{U}_\mu \tilde{\Sigma}_\mu \tilde{V}_\mu^T$ for $\mu = 1,2,3$
- Truncate basis matrices: $U_\mu := \tilde{U}_\mu \left(: ,1 : r_\mu \right)$ for $\mu = 1,2,3$
- Form core tensor: $\text{vec}(\mathscr{C}) := \left(U_3^T \otimes U_2^T \otimes U_1^T \right) \cdot \text{vec}(\mathscr{X})$
- Truncated tensor produced by HOSVD: $\text{vec}(\tilde{\mathscr{X}}) := \left(U_3 \otimes U_2 \otimes U_1 \right) \cdot \text{vec}(\mathscr{C})$

Drawback: storage of core tensor $\mathscr{C} \sim r^d$ (curse of dimensionality)

Hierarchical Tucker Decomposition:

do we have:

- CP offers low data complexity but difficult truncation
- Tucker offers simple truncation but high data complexity

Alternatives:

- MPS $\equiv$ Tensor Train (TT)
- Hierarchical Tucker (HT)

L. Grasedyck. Hierarchical singular value decomposition of tensors. SIAM J. Matrix Anal. Appl., 31(4):2029–2054, 2010.

W. Hackbusch and S. Kühn. A new scheme for the tensor representation. J. Fourier Anal. Appl., 15(5):706–722, 2009.

D. Kressner and C. Tobler. htucker – A MATLAB toolbox for the hierarchical Tucker decomposition.

More general matricizations:

Recall: μ-mode matricization for tensor $\mathcal{X}$:

$$X^{(\mu)} \in \mathbb{R}^{n_\mu \times \left(n_1 \cdots n_{\mu-1} n_{\mu+1} \cdots n_d\right)}, \quad \mu = 1, \ldots, d$$

General matricization for mode decomposition $\{1, \ldots, d\} = t \cup s$:

$$X^{(t)} \in \mathbb{R}^{(n_{t_1} \cdots n_{t_k}) \times (n_{s_1} \cdots n_{s_{d-k}})} \qquad \text{with} \qquad \left(X^{(t)}\right)_{\left(i_{t_1}, \ldots, i_{t_k}\right), \left(i_{s_1}, \ldots, i_{s_{d-k}}\right)} := \mathcal{X}_{i_1, \ldots, i_d}$$

hierarchical construction: tree structure and SVD of $X^{(t)}$

construction in a second

RUB

tensors: example 4D

$$M^{d_1 \times d_2 \times d_3 \times d_4} \longrightarrow M^{d_1 \cdot d_2 \times d_3 \cdot d_4} = U_{34}^{d_3 d_4 \times r_{34}} \, U_{12}^{d_1 d_2 \times r_{12}} \, B_{1234}^{r_{34} \times r_{12}}$$

$$M = (U_{34} \otimes U_{12}) B_{1234}$$

$$U_{12} = (U_2 \otimes U_1) B_{12} \qquad U_{34} = (U_4 \otimes U_3) B_{34}$$

$$\boxed{M = (U_4 \otimes U_3 \times U_2 \otimes U_1)(B_{34} \times B_{12}) B_{1234}}$$

RUB

Now more precise: Lars Grasedyck SIAM Journal on Matrix Analysis and Applications 31 (2010) 2029-2054

$$M^{n_1\times n_2\times n_3\times n_4} \longrightarrow \begin{cases} M^{n_1\cdot n_2\times n_3\cdot n_4} = U_{12}^{n_1n_2\times n_1n_2}\,\Sigma_{12}^{n_1n_2\times n_3n_4}\,V_{12}^{n_3n_4\times n_3n_4} \\ M^{n_3\cdot n_4\times n_1\cdot n_2} = U_{34}^{n_3n_4\times n_3n_4}\,\Sigma_{34}^{n_3n_4\times n_1n_2}\,V_{34}^{n_1n_2\times n_1n_2} \end{cases}$$

RUB

Now more precise: Lars Grasedyck SIAM Journal on Matrix Analysis and Applications 31 (2010) 2029-2054

$$M^{n_1\times n_2\times n_3\times n_4} \longrightarrow \begin{cases} M^{n_1\cdot n_2\times n_3\cdot n_4} = U_{12}^{n_1n_2\times r_{12}}\ \Sigma_{12}^{r_{12}\times n_3n_4}\ V_{12}^{n_3n_4\times n_3n_4} \\ M^{n_3\cdot n_4\times n_1\cdot n_2} = U_{34}^{n_3n_4\times r_{34}}\ \Sigma_{34}^{r_{34}\times n_1n_2}\ V_{34}^{n_1n_2\times n_1n_2} \end{cases}$$

calculate

$$B_{1234}^{r_{12}r_{34}} = U_{34}^{n_3n_4\times r_{34}}U_{12}^{n_1n_2\times r_{12}}M^{n_1n_2n_3n_4} \Longrightarrow M_{1234}^{n_1n_2n_3n_4} \approx U_{34}^{n_3n_4\times r_{34}}U_{12}^{n_1n_2\times r_{12}}B_{1234}^{r_{12}r_{34}}$$

next

$$U_{12}^{n_1n_2\times r_{12}} \longrightarrow \begin{cases} U_{12}^{n_1\times n_2\cdot r_{12}} = U_1^{n_1\times r_1}\ \Sigma_1^{r_1\times n_2r_{12}}\ V_1^{n_2r_{12}\times n_2r_{12}} \\ U_{12}^{n_2\times n_1\cdot r_{12}} = U_2^{n_2\times r_2}\ \Sigma_2^{r_2\times n_1r_{12}}\ V_2^{n_1r_{12}\times n_1r_{12}} \end{cases}$$

calculate (and also for 3,4)

$$B_{12}^{r_1r_2\times r_{12}} = U_2^{n_2\times r_2}U_1^{n_1\times r_1}\ U_{12}^{n_1n_2\times r_{12}} \Longrightarrow U_{12}^{n_1n_2\times r_{12}} = U_2^{n_2\times r_2}U_1^{n_1\times r_1}\ B_{12}^{r_1r_2\times r_{12}}$$

$$B_{34}^{r_3r_4\times r_{34}} = U_4^{d_4\times r_4}U_3^{d_3\times r_3}\ U_{34}^{d_3d_4\times r_{34}} \Longrightarrow U_{34}^{n_3n_4\times r_{34}} = U_4^{n_4\times r_4}U_3^{n_3\times r_3}\ B_{34}^{r_3r_4\times r_{34}}$$

reshape

$$\mathsf{B}_{12} \in \mathbb{R}^{r_1 r_2 \times r_{12}} \Rightarrow \mathscr{B}_{12} \in \mathbb{R}^{r_1 \times r_2 \times r_{12}}$$

$$\mathsf{B}_{34} \in \mathbb{R}^{r_3 r_4 \times r_{34}} \Rightarrow \mathscr{B}_{34} \in \mathbb{R}^{r_3 \times r_4 \times r_{34}}$$

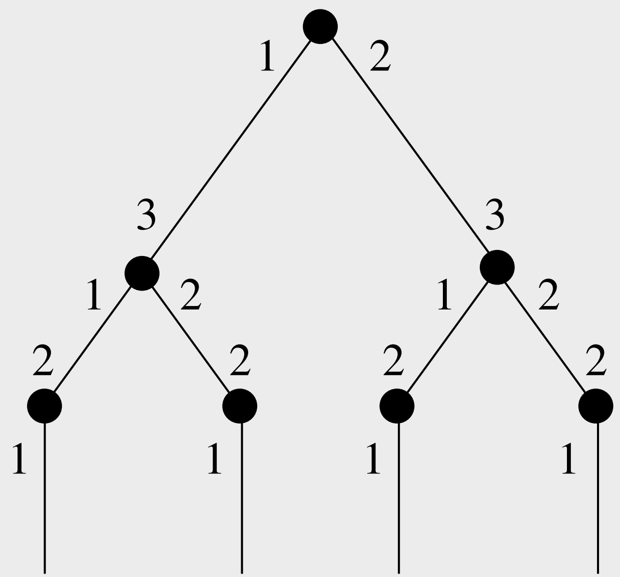
$$\mathsf{B}_{1234} \in \mathbb{R}^{r_{12} r_{34} \times 1} \Rightarrow \mathscr{B}_{1234} \in \mathbb{R}^{r_{12} \times r_{34}}$$

store only $\mathscr{B}_{1234}, \mathscr{B}_{12}, \mathscr{B}_{34}, \mathsf{U}_1, \mathsf{U}_2, \mathsf{U}_3, \mathsf{U}_4$

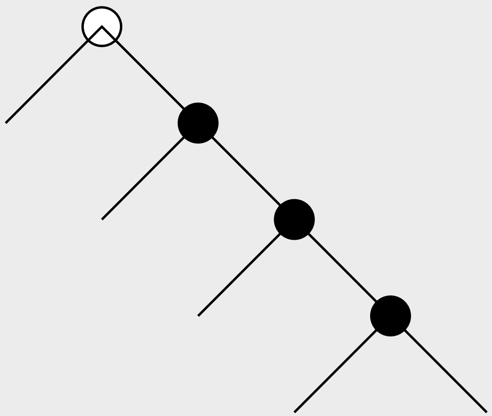
storage requirements

$$\mathcal{O}(dnr) + \mathcal{O}(dr^3) \quad r = \max\{r_t\}\ ,\ n = \max\{dr^3\}$$

hierarchical Tucker HT



tensor train TT



Clever techniques:

- linear algebra with tensor networks
  additions, tensor multiplications, contractions
  applications:
  - direct integration of path integrals
  - eigenvalue problems
  - chemical master equation
  - ….
- tensor completion and cross approximations
  implemented in library t3f upon tensorflow
  closely related to “learning”

Wavelets and Wavelet packets

Why wavelets ?

Example: Navier-Stokes

$$\partial_t \mathbf{u} + \underbrace{\mathbf{u} \cdot \nabla \mathbf{u}}_{\text{local in real space}} + \underbrace{\nabla p}_{\text{local in Fourier space}} = \mathbf{F} + \nu \Delta \mathbf{u}, \nabla \cdot \mathbf{u} = 0$$

somewhere in between

Applications

- turbulence
- coherent structures
- numerics
- alternative to Fourier bounds
- acoustics, Filter
- signal theory (EEG, EKG)

Highlights

- discrete orthonormal basis

- fast wavelet-transform $\mathcal{O}(\mathrm{N})$ (FFT $\mathcal{O}(N \ln N)$)

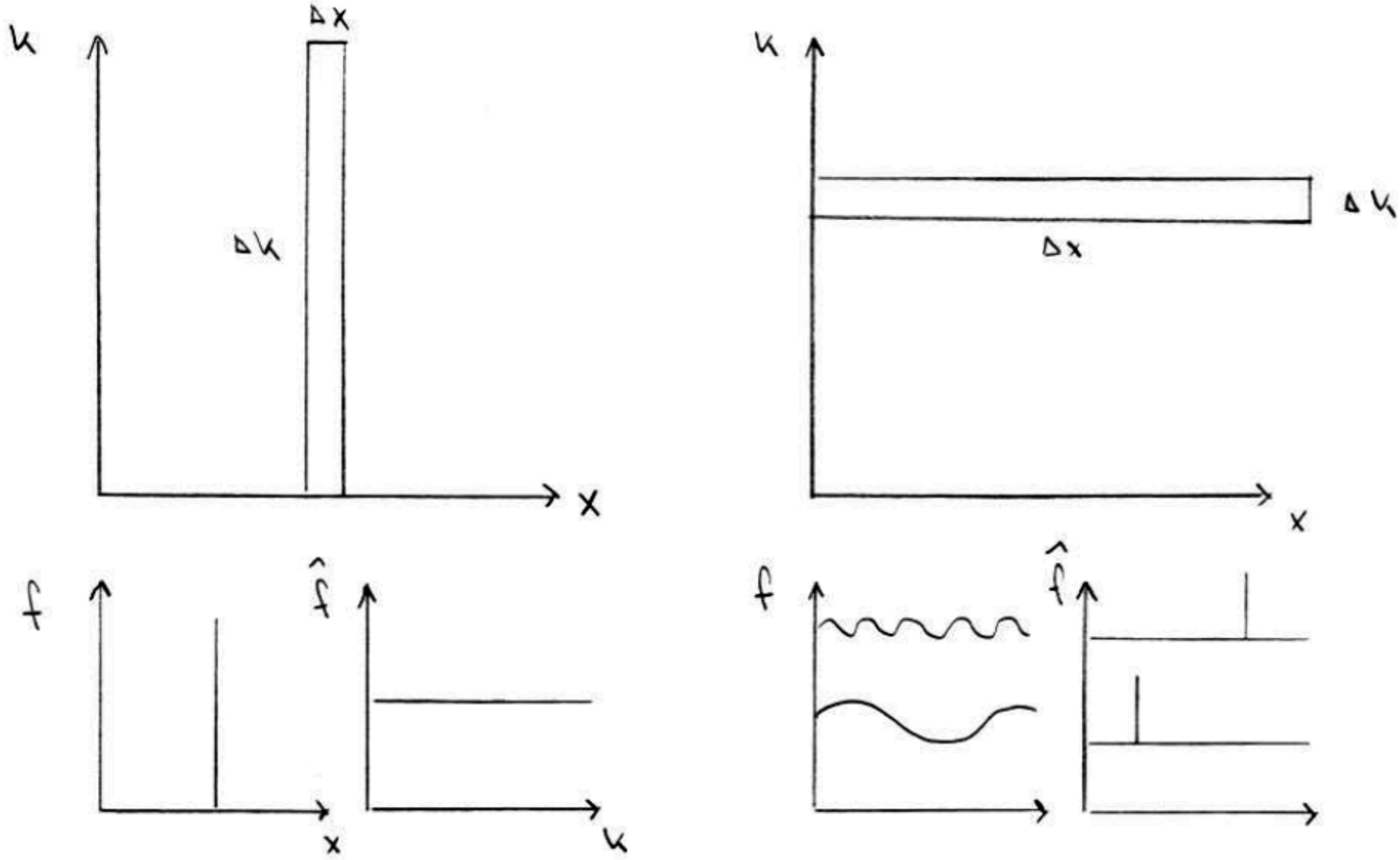
- less error-prone than Fourier
  (small phase errors $\longrightarrow$ large error in localization

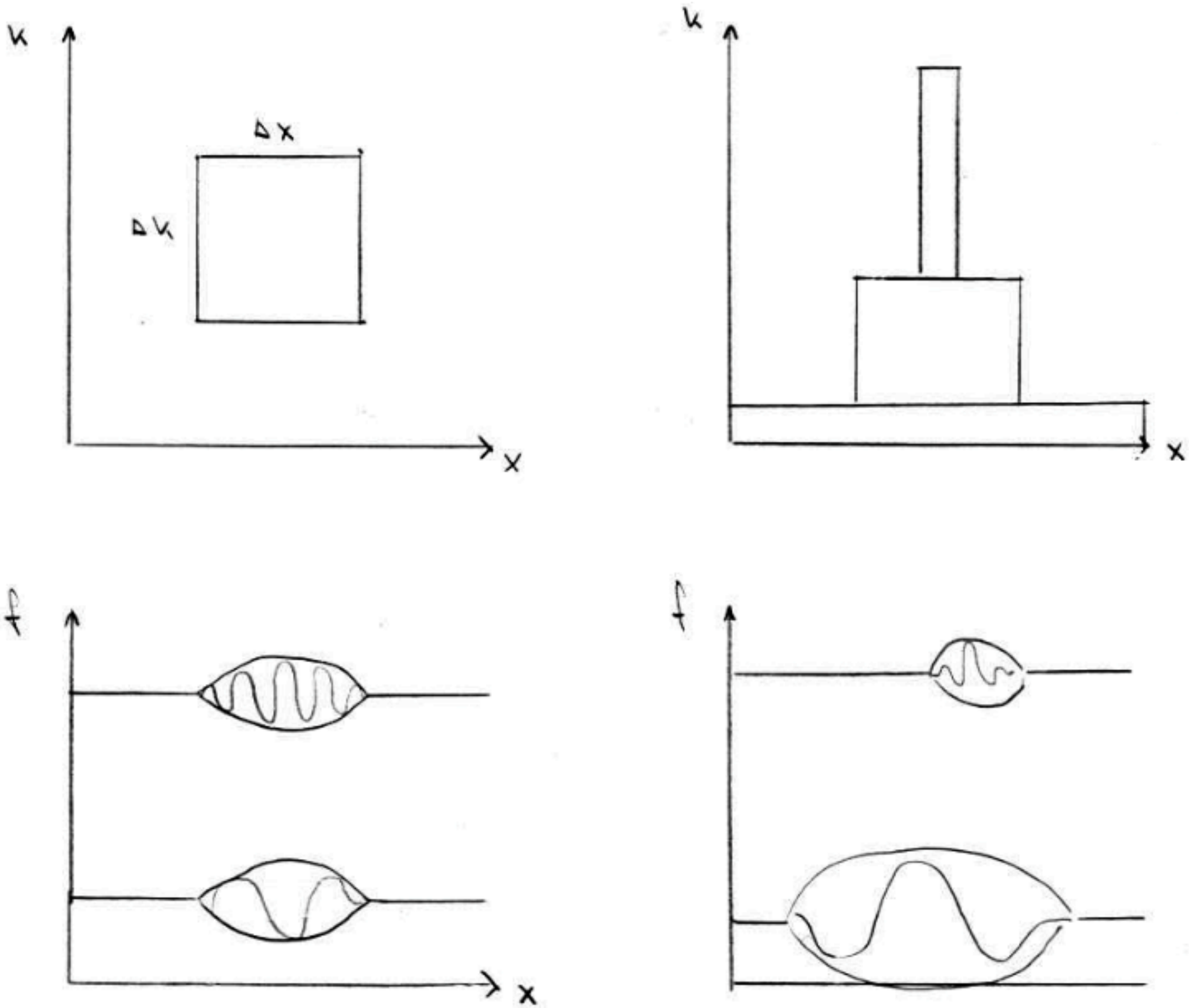
- minimal uncertainty

Literature:

Adaptive Wavelet-Analysis, M.V. Wickershauser, Vieweg

Why not Fouriertransform with “window” ?





wavelets

small scales: spatial resolution is bounded
large scales: resolution in momentum space is bounded

History

1984: Goupillard, Grossmann, Morlet
continuous wavelet trafo (Ondelette)
$\Longleftrightarrow$ Calderóns reproducing formula (1964)

1986: Daubechies, Grossmann, Meyer: wavelet frames
discrete quasi-orthogonal basis von $\mathrm{L}^2\left(\mathbb{R}^n\right)$

1986: Meyer: regular discrete orthogonal wavelets

1988: Meyer, Mallat: multi-scale-analysis

1988: Daubechies: regular discrete wavelets with compact support

1989: Mallat: fast wavelet trafo

What is a wavelet ?

A function $\psi \in \mathrm{L}^2(\mathbb{R})$, that fulfills the admissibility condition

$$0 < c_\psi := 2\pi \int_{\mathbb{R}} \frac{|\hat{\psi}(\omega)|^2}{|\omega|} d\omega < \infty$$

is called a wavelet. It follows immediately $0 = \int_{\mathbb{R}} \psi(t) dt$.

The wavelet trafo of $f \in L^2(\mathbb{R})$ with the wavelet ψ reads

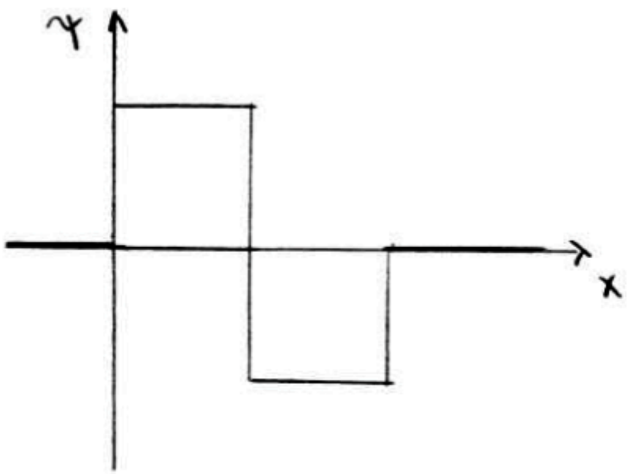
$$\mathrm{L}_\psi f(a, b) = \frac{1}{\sqrt{c_\psi}} |a|^{-1/2} \int_{\mathbb{R}} f(t) \psi\left(\frac{t-b}{a}\right) dt$$

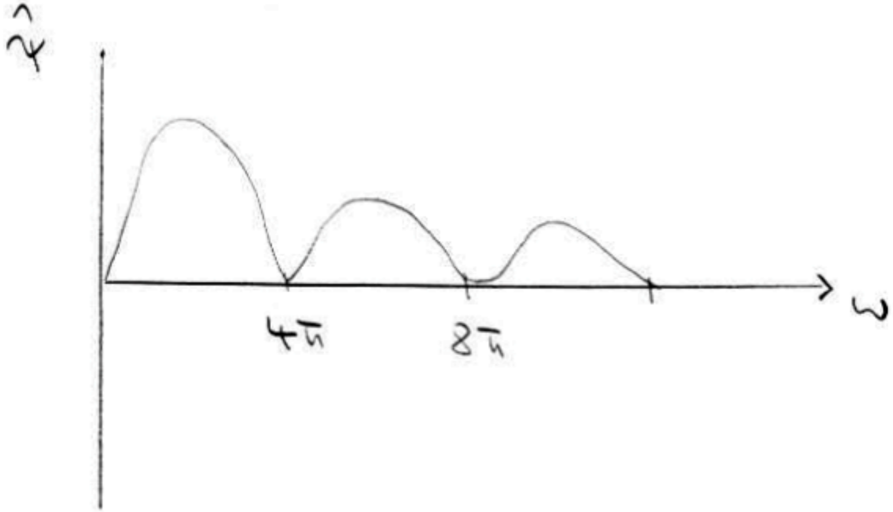
with $a \in \mathbb{R} \backslash \{0\}, b \in \mathbb{R}$.

Examples

i) Let $\psi(t) = \begin{cases} 1: & 0 \le t < \frac{1}{2} \\ -1: & \frac{1}{2} \le t \le 1 \\ 0: & \text{else} \end{cases}$ Haar wavelet (1909)

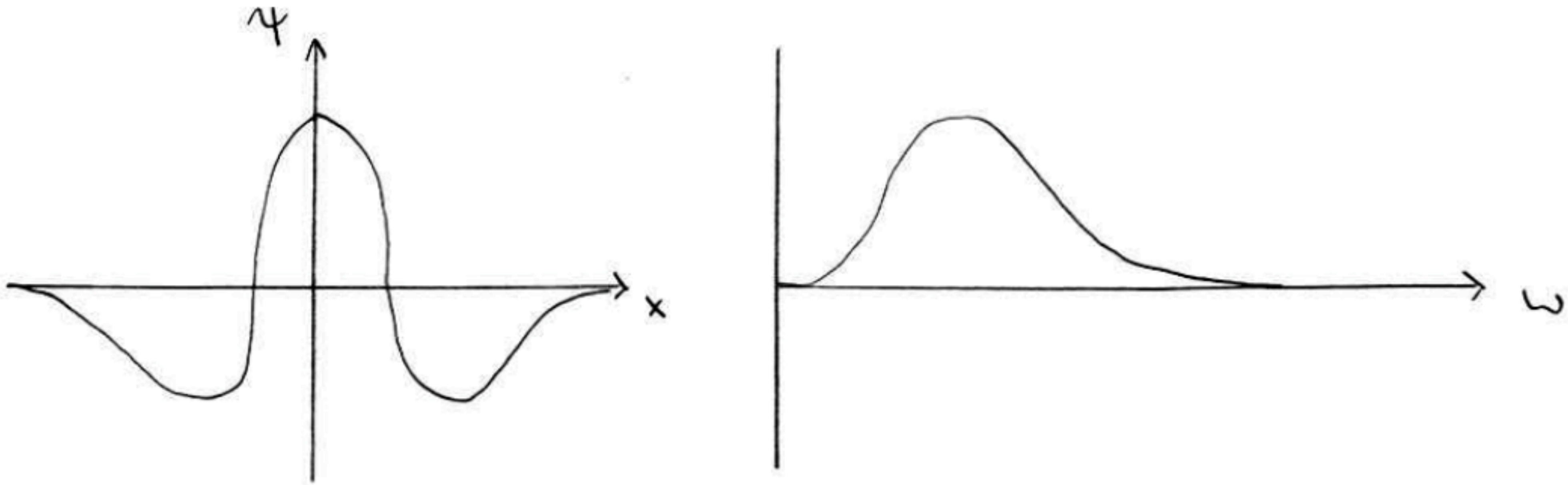
$$\hat{\psi}(\omega) = ie^{-i\omega/2} \sin^2\left(\frac{\omega}{4}\right) \frac{1}{x\sqrt{2\pi}} \qquad c_\psi = 2\ln 2$$





ii) Let $\psi(x) = -\,\partial_x^2 e^{-x^2/2} = \left(1 - x^2\right) e^{-x^2/2} \in C^\infty$ mexican hat

$$\hat{\psi}(x) = \omega^2 e^{-\omega^2/2}/\sqrt{2} \qquad c_\psi = 1$$



Theorem: The wavelet transform

$$\mathrm{L}_{\psi} : \mathrm{L}^2(\mathbb{R}) \to \mathrm{L}^2(\mathbb{R}^2, \underbrace{\frac{da\,db}{a^2}}_{weight})$$

is an isometry.

Proof: $$\| L_\psi f \|^2_{L^2(\mathbb{R}, \frac{da\,db}{a^2})} = \int_{\mathbb{R}}\int_{\mathbb{R}} | L_\psi f(a,b)|^2 \, \frac{da\,db}{a^2}$$

$$= \int_{\mathbb{R}}\int_{\mathbb{R}} | \widehat{L_\psi f}(a,\omega)|^2 \, \frac{da\,d\omega}{a^2}$$

We have $$\widehat{L_\psi f}(a,\omega) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} db \, \frac{|a|^{-1/2}}{\sqrt{c_\psi}} \int_{\mathbb{R}} f(t)\, \bar{\psi}\left(\frac{t-b}{a}\right) dt \; e^{-ib\omega}$$

$$\left(s = \frac{b-t}{a} \quad \Rightarrow \right)$$

$$= (2\pi)^{1/2} \, |a|^{1/2} \, c_\psi^{-1/2} \, \hat{\bar{\psi}}(-a\omega)\, \hat{f}(\omega)$$

$$\hat{\bar{\psi}}(-a\omega) = \bar{\hat{\psi}}(a\omega)$$

It follows:

$$\| L_\psi f \|^2 = \frac{2\pi}{c_\psi} \int_{\mathbb{R}} \int_{\mathbb{R}} |a| \, |\hat{\psi}(a\omega)|^2 \, |\hat{f}(\omega)|^2 \frac{da\, d\omega}{a^2}$$

$$= \frac{2\pi}{c_\psi} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|\hat{\psi}(\tau)|^2}{|\tau|} \, |\hat{f}(\omega)|^2 \, d\tau\, d\omega$$

$$= \| f \|^2_{L^2} \quad \checkmark$$

This gives us the inversion formula, because the following applies:

Let H_1 and H_2 be Hilbert spaces and $L : H_1 \to H_2$ an isometry, then it holds:

$$L^*L = \mathrm{Id}_{H_1}\ , \ \ LL^* = P_{\mathrm{image}(L)}\ .$$

Proof:

$$\langle f+g, f+g\rangle = \langle f,f\rangle + \langle g,f\rangle + \langle f,g\rangle + \langle g,g\rangle$$
$$= \langle f,f\rangle + 2\,\mathrm{Re}(\langle f,g\rangle) + \langle g,g\rangle$$

$$\Rightarrow 2\,\mathrm{Re}(\langle f,g\rangle) = \langle f+g, f+g\rangle - \langle f,f\rangle - \langle g,g\rangle$$

$$= \langle L(f+g), L(f+g)\rangle - \langle Lf, Lf\rangle - \langle Lg, Lg\rangle$$

$$= 2\,\mathrm{Re}(\langle Lf, Lg\rangle) = 2\,\mathrm{Re}(\langle L^*Lf, g\rangle) \checkmark$$

This results in the inversion formula $\langle g, L_\psi f \rangle_{L^2(\mathbb{R}^2, \frac{da\,db}{a^2})} = \langle L_\psi^* g, f \rangle,$

thus

$$\langle g, L_\psi f \rangle_{L^2(\mathbb{R}^2, \frac{da\,db}{a^2})} = \int_{\mathbb{R}}\int_{\mathbb{R}} L_\psi f(a,b)\, g(a,b)\, \frac{da\,db}{a^2}$$

$$= \int_{\mathbb{R}}\int_{\mathbb{R}} c_\psi^{-1/2} \int_{\mathbb{R}} f(t)\, |a|^{-1/2}\, \psi\left(\frac{t-b}{a}\right) g(a,b)\, dt\, \frac{da\,db}{a^2}$$

$$= \int_{\mathbb{R}} f(t)\, L_\psi^* g(t)\, dt$$

$$L_\psi^* : L^2\left(\mathbb{R}^2, \frac{da\,db}{a^2}\right) \to L^2(\mathbb{R})$$

$$\boxed{g \mapsto c_\psi^{-1/2} \int_{\mathbb{R}}\int_{\mathbb{R}} |a|^{-1/2}\, \psi\left(\frac{t-b}{a}\right) g(a,b)\, \frac{da\,db}{a^2}}$$

Definition: A wavelet ψ is called of order $N \in \mathbb{N}$, if

i) $$\int_{\mathbb{R}} x^k \psi(x)\, dx = 0 \quad , \; 0 \leq k \leq N-1$$

ii) $$\int_{\mathbb{R}} x^N \psi(x)\, dx \neq 0 .$$

We would like to have discrete orthonormal wavelets, thus:

Quadrature filters

Aperiodic filter

Suppose that $f = \{f(n) : n \in \mathbb{Z}\}$ is an absolutely summable sequence.

Define convolution-decimation operator F and its adjoint F^*

$$Fu(i) = \sum_{j=-\infty}^{\infty} f(2i-j)u(j) = \sum_{j=-\infty}^{\infty} f(j)u(2i-j), \quad i \in \mathbb{Z}$$

$$F^*u(j) = \sum_{i=-\infty}^{\infty} \bar{f}(2i-j)u(i) = \begin{cases} \sum_{i=-\infty}^{\infty} \bar{f}(2i)u\left(i+\frac{j}{2}\right), & j \in \mathbb{Z} \text{ even} \\ \sum_{i=-\infty}^{\infty} \bar{f}(2i+1)u\left(i+\frac{j+1}{2}\right), & j \in \mathbb{Z} \text{ odd} \end{cases}$$

Periodic filter

Let f_{2q} be a $2q$-periodic sequence

F_{2q}: $2q$-periodic sequence $\longrightarrow$ q-periodic sequence

F^*_{2q}: q-periodic sequence $\longrightarrow$ $2q$-periodic sequence

$$F_{2q}u(i) = \sum_{j=0}^{2q-1} f_{2q}(2i-j)u(j) = \sum_{j=0}^{2q-1} f_{2q}(j)u(2i-j), \quad 0 \le i < q$$

$$F^*_{2q}u(j) = \sum_{i=0}^{q-1} \bar{f}_{2q}(2i-j)u(i)$$

$$= \begin{cases} \sum_{i=0}^{q-1} \bar{f}_{2q}(2i)u\left(i+\frac{j}{2}\right), & \text{if } j \in [0,2q-2] \text{ is even} \\ \sum_{i=0}^{q-1} \bar{f}_{2q}(2i+1)u\left(i+\frac{j+1}{2}\right), & \text{if } j \in [1,2q-1] \text{ is odd.} \end{cases}$$

Orthogonal QFs

Two filter H, G are called pair of orthogonal QFs, iff

Self-duality: $HH^* = GG^* = I$

Independence: $GH^* = HG^* = 0$

Exact reconstruction: $H^*H + G^*G = I$

Normalization: $H1 = \sqrt{2}1$, where $1 = \{\ldots,1,1,1,\ldots\}$

Multiresolution analysis MRA

A MRA of $L^2(\mathbf{R})$ *is a chain of subspaces* $\left\{ V_j : j \in \mathbf{Z} \right\}$ *satisfying the following conditions:*

Containment: $V_j \subset V_{j-1} \subset L^2$ for all $j \in \mathbf{Z}$

Decrease: $\lim\limits_{j \to \infty} V_j = 0$, i.e., $\bigcap\limits_{j>N} V_j = \{0\}$ for all N

Increase: $\lim\limits_{j \to -\infty} V_j = L^2$, i.e., $\bigcup\limits_{j<N} V_j = L^2$ for all N

Dilation: $v(2t) \in V_{j-1} \Longleftrightarrow v(t) \in V_j$

Generator: There is a function $\phi \in V_0$ whose translates $\{\phi(t-k) : k \in \mathbf{Z}\}$ form a Riesz basis for V_0.

$\implies \left\{\phi\left(2^{-L}t-k\right) : k \in \mathbf{Z}\right\}$ is a Riesz basis for V_L

Definition of Riesz basis:

A function system $\left\{\phi_n \in H : h \in \mathbb{Z}\right\}$ is called frame, iff

$$A\|f\|^2 \leq \sum_{n=-\infty}^{\infty} \left|\langle f, \phi_n \rangle\right|^2 \leq B\|f\|^2 \quad \forall f \in H$$

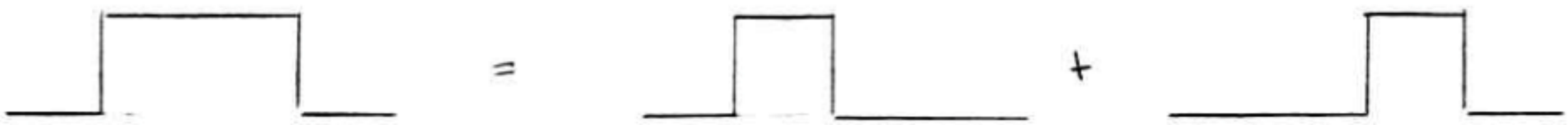
with $0 < A \leqslant B < \infty$. Let $\left\{\phi_n : n \in \mathbb{Z}\right\}$ a frame and be linear independent, then this is called a Riesz basis.

Scale relation

$$\phi(t) = \sqrt{2}\sum_{k\in 2} h(k)\phi(2t-k) = H\phi(t)$$

$\in V_j$ filter $\in V_{j-1} \supset V_j$

Example: Haar scale function



complementary subspaces $W_j = V_{j-1} - V_j\,,\;\; V_{j-1} = V_j + W_j$

"increase" property: $L^2 = \sum_{j\in\mathbb{Z}} W_j$ (wavelet decomposition)

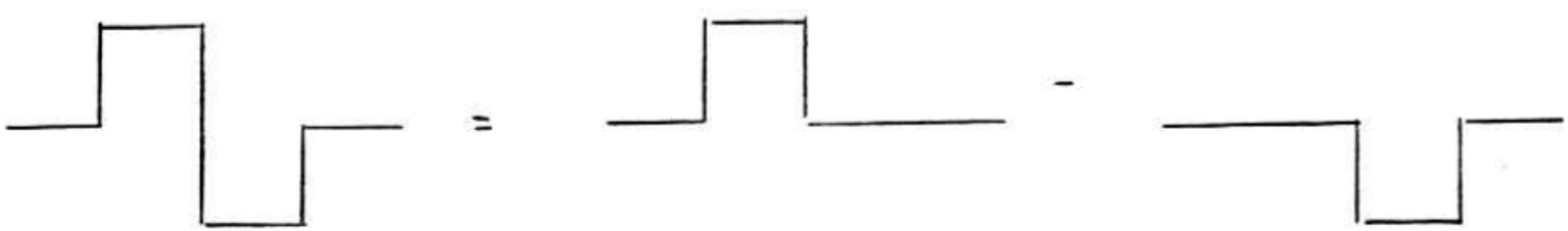
Wavelet equation:

$$\psi(t) = \sqrt{2} \sum_{k \in \mathbb{Z}} g(k)\phi(2t-k) = G\phi(t)\ , \quad g(k) = (-1)^k \bar{h}(1-k)$$

$\in W_j$ filter $\in V_{j-1} \supset W_j$

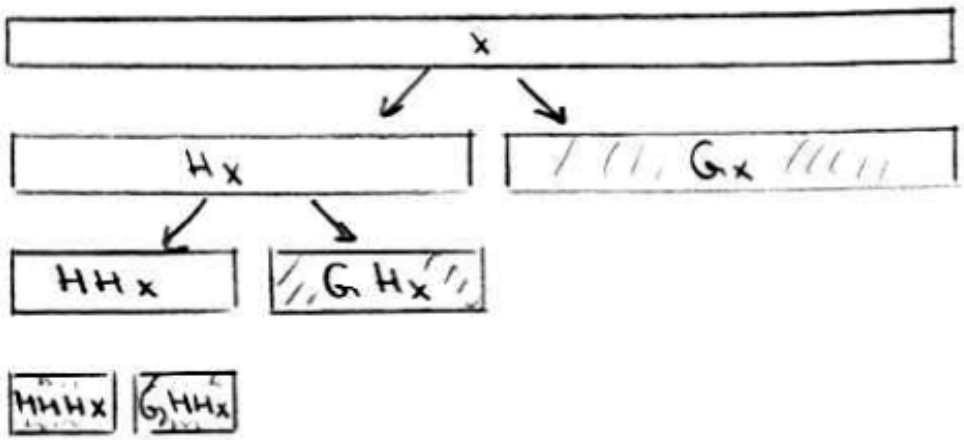
ψ is called “mother” wavelet

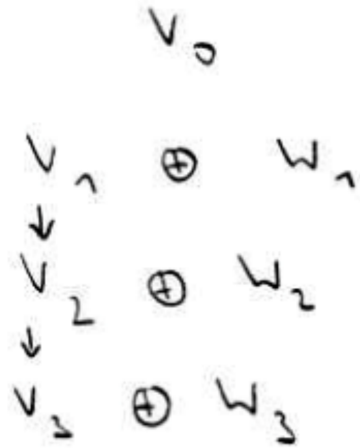
Example: Haar mother wavelet



orthogonal MRA $\Longrightarrow$ $\left\{2^{-j/2}\psi\left(2^{-j}t-k\right) : j,k \in \mathbf{Z}\right\}$ is onb: $L^2 = \bigoplus_{j\in\mathbb{Z}} W_j$

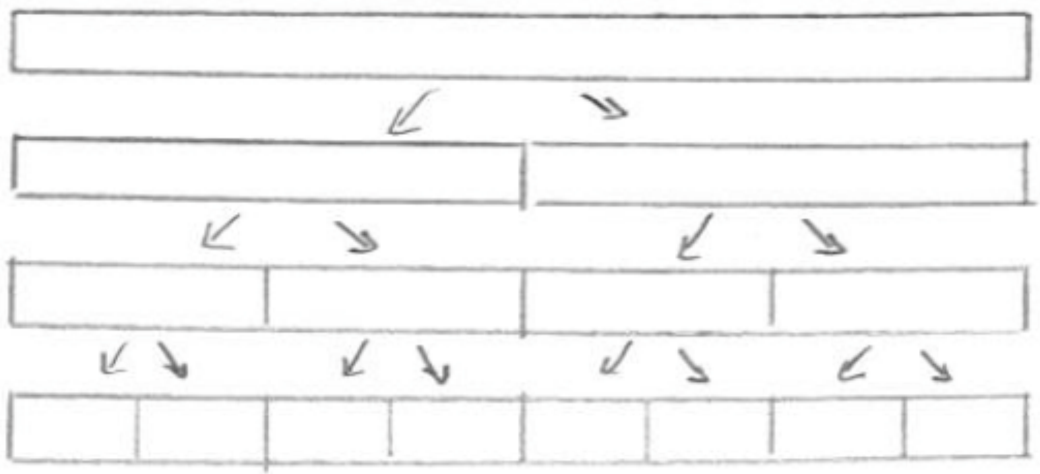
finite resolution: $L^2 = \bigoplus_{j=0}^{l} W_j + V_l$

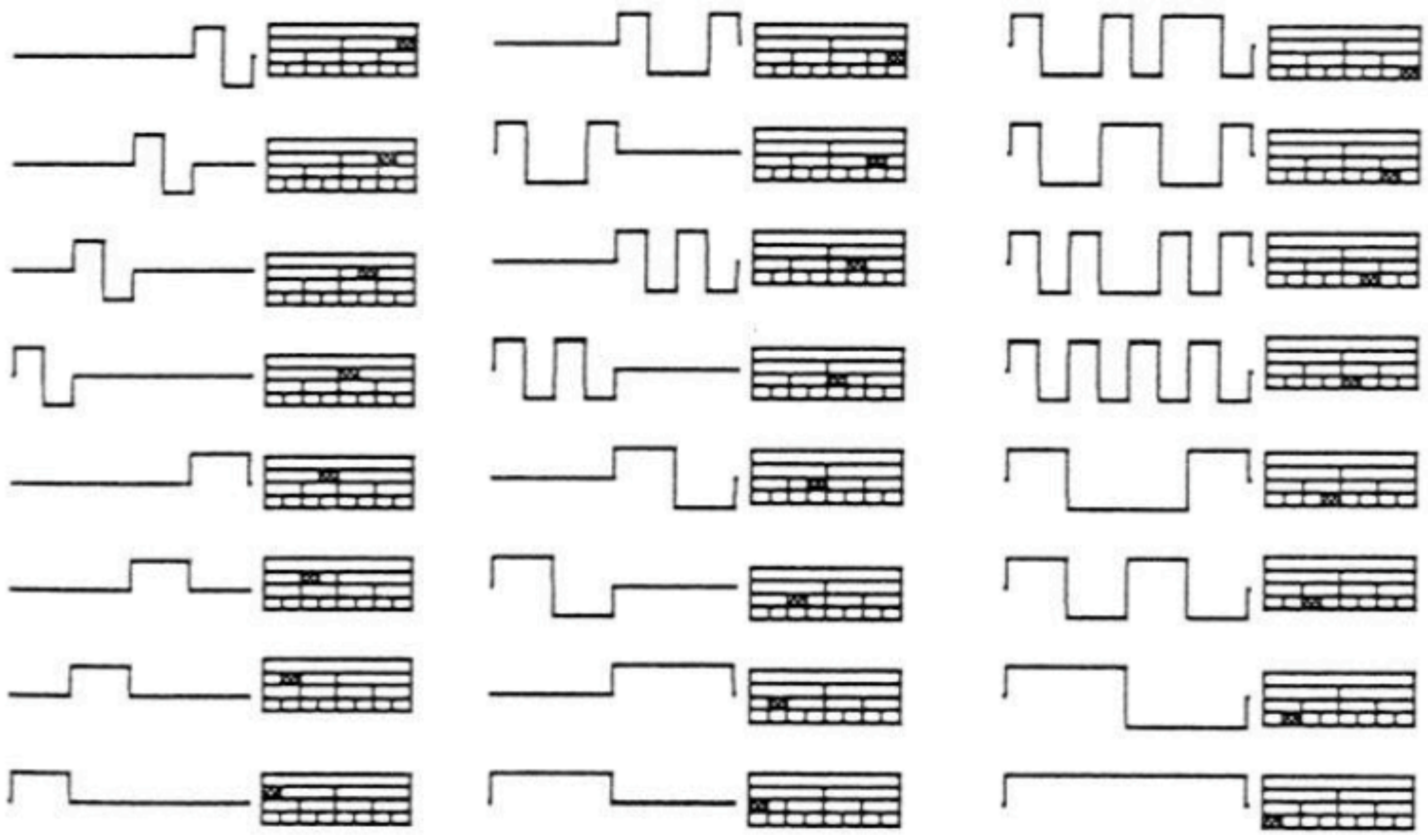




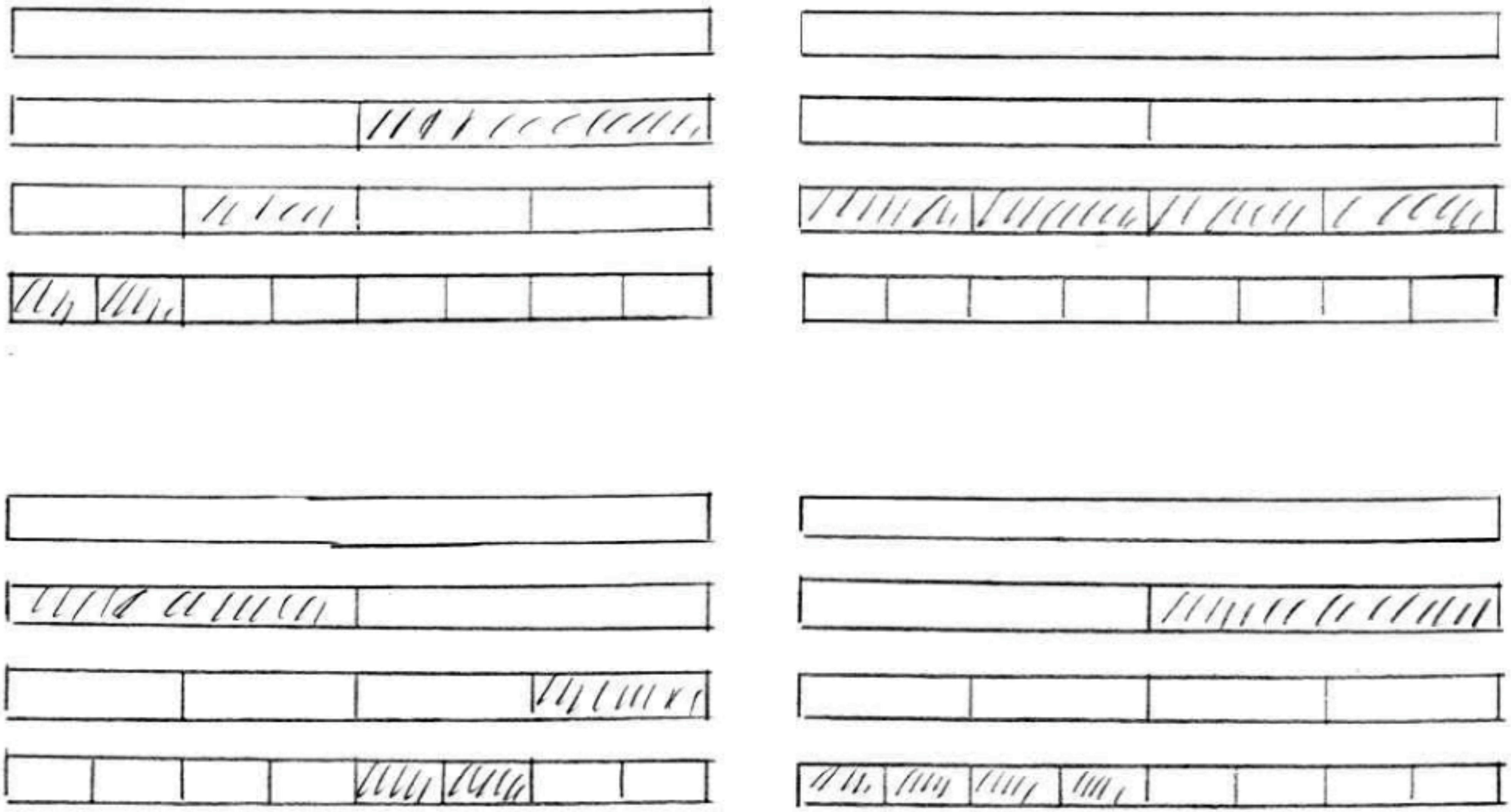
thus $V_0 = W_1 + W_2 + W_3 + V_3$

area of the original signal = hatched area





Orthonormal basis



area of original signal = hatched area

Best basis algorithm

additive cost functional

$$M(u) = \sum_{k \in Z} \mu(\,|\,u(k)\,|\,)\,, \qquad \mu(0) = 0$$

Example:

i) Number of elements above a threshold

$$\mu(w) = \begin{cases} |\,w\,|, & \text{if } |\,w\,| \geq \epsilon \\ 0, & \text{if } |\,w\,| < \epsilon \end{cases}$$

ii) Concentration in l^p

$$\mu(\omega) = |\,\omega\,|^p \quad \Longrightarrow \quad M(u) = \|u\|_p^p \quad 0 < p < 2$$

iii) Entropy

$$\mathscr{H}(u) = \sum_k p(k) \log \frac{1}{p(k)} \qquad p(k) = |u(k)|^2 / \|u\|^2$$

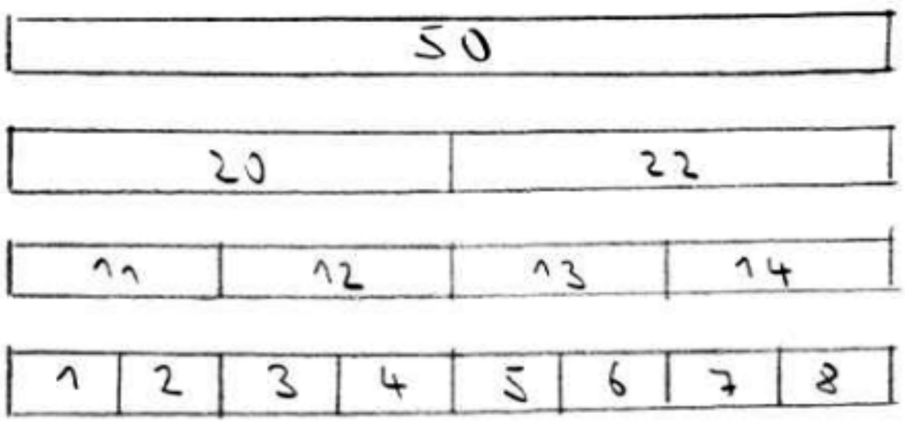
for $p = 0$ set $p \log \dfrac{1}{p} = 0$

no cost function but

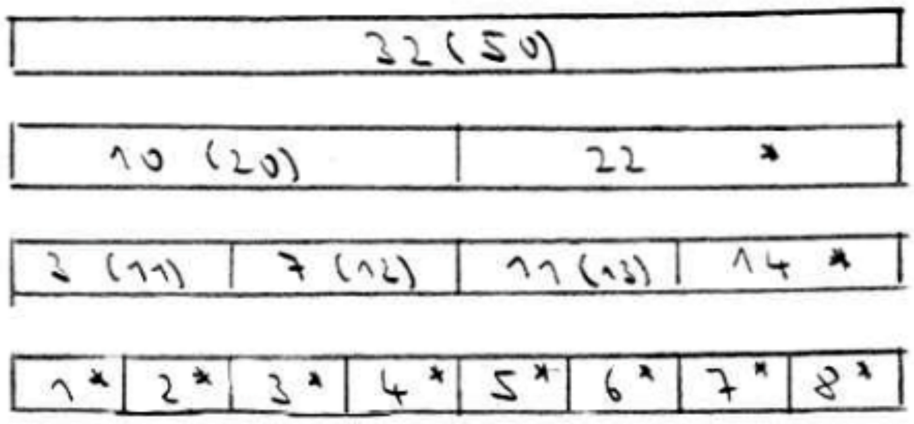
$$l(u) = \sum_k |u(k)|^2 \log \left(1/|u(k)|^2 \right)$$

is one.

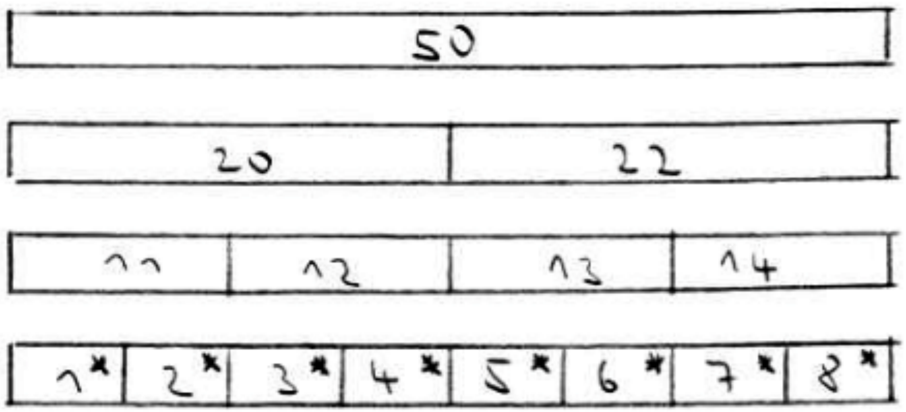
iv) Logarithm of energy $\quad M(u) = \sum_{k=1}^{N} \log |u(k)|^2$



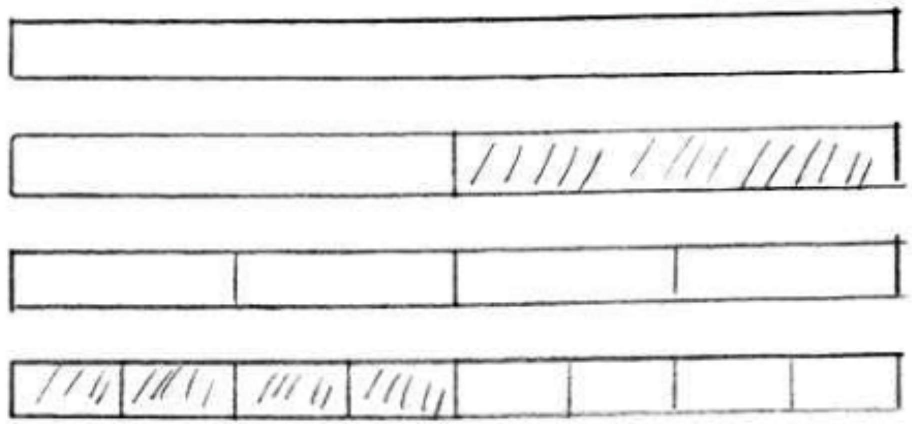
cost of a tree



2. step

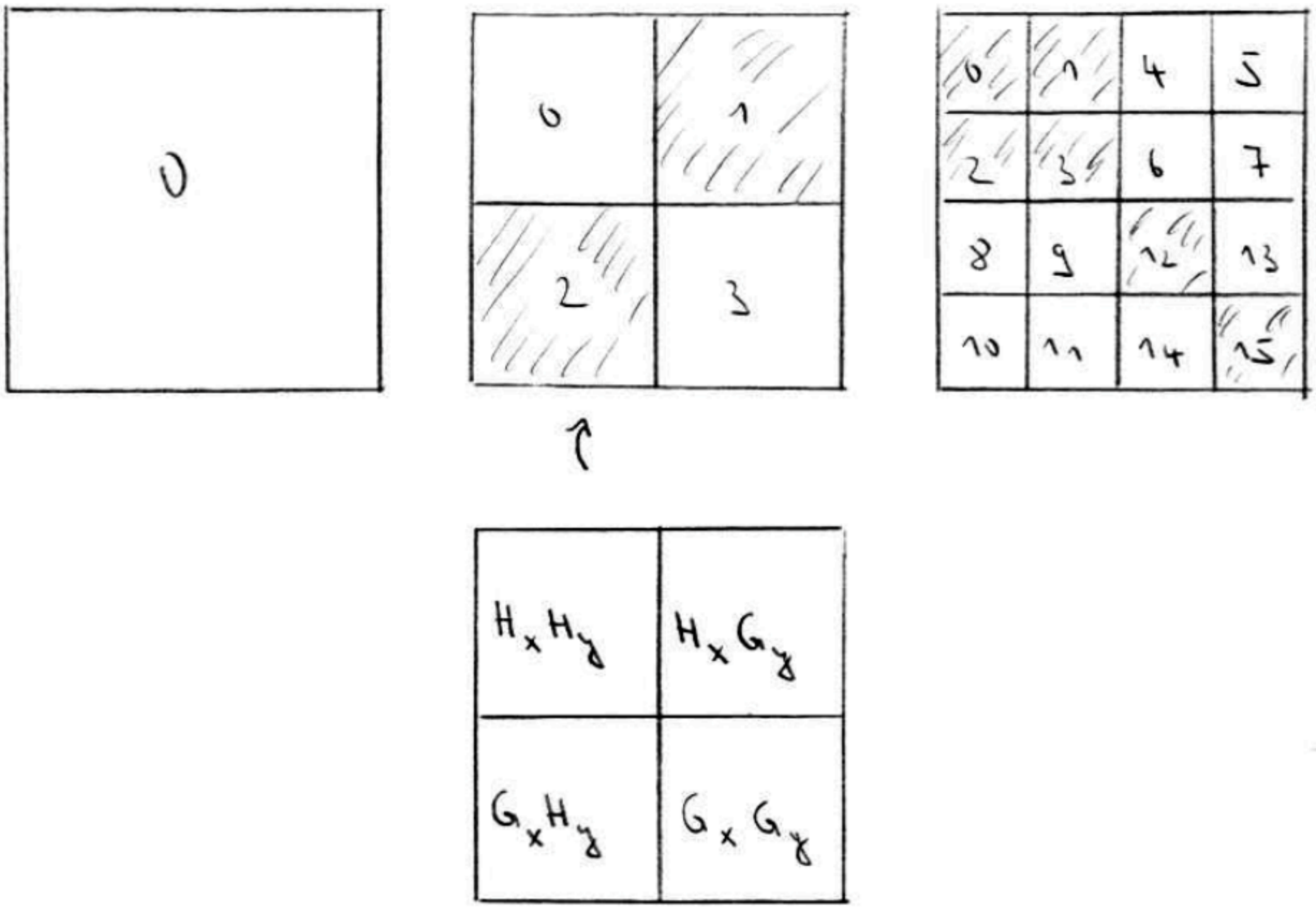


1. step



3. step

Multidimensional libraries



Extraction of $\longrightarrow$ structures

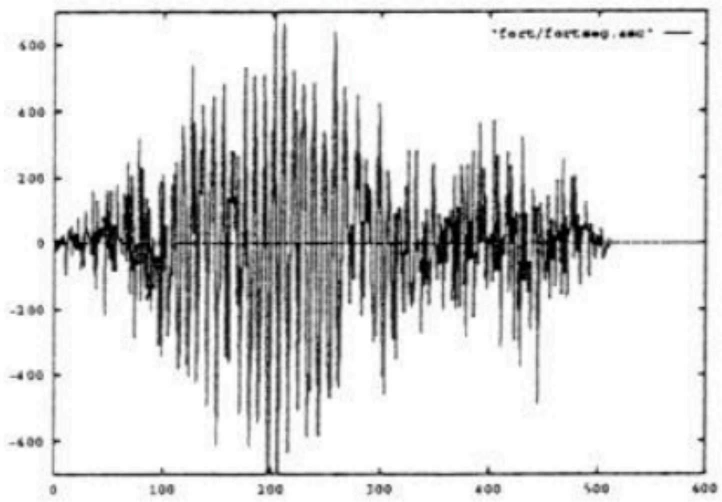
signal $\longrightarrow$ best basis $\longrightarrow$ largest coefficients

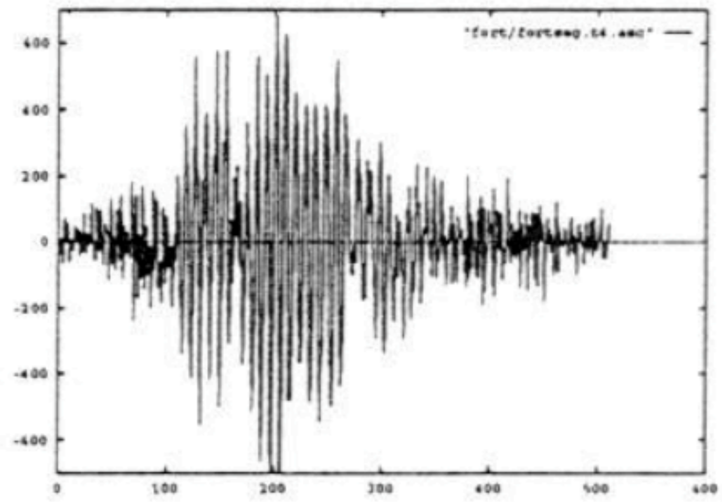
$\Longrightarrow$ coherent +

incoherent part

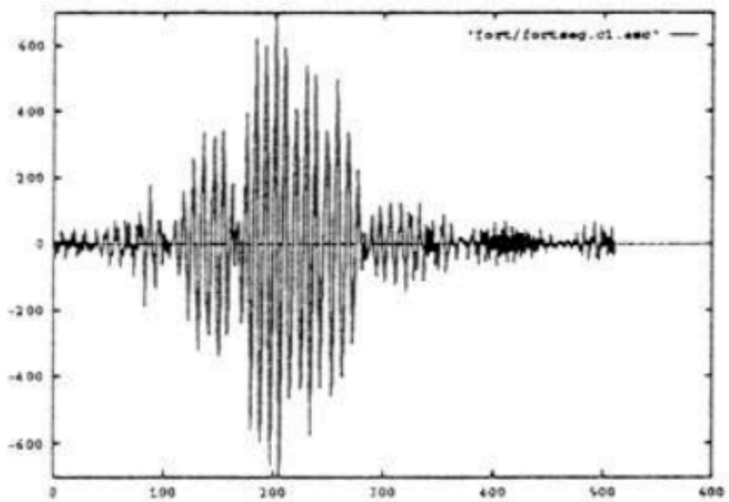
stopping criteria: entropy(incoherent part) $> \; H_{\text{ref}}$

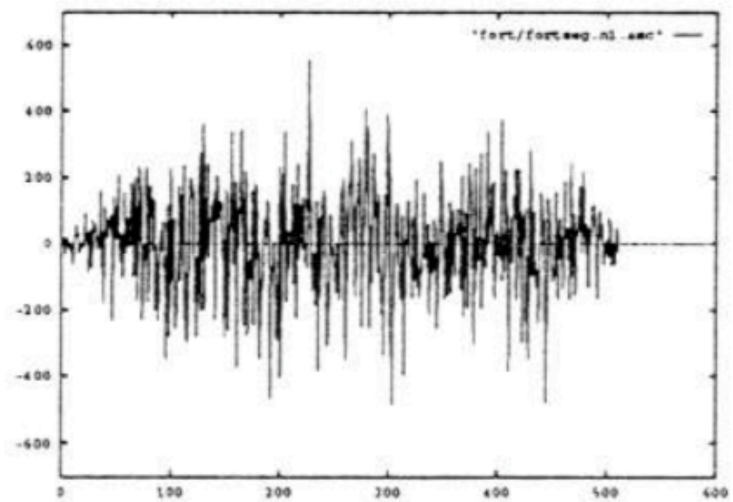
$$H_{\text{ref}} = \log N - \varepsilon$$



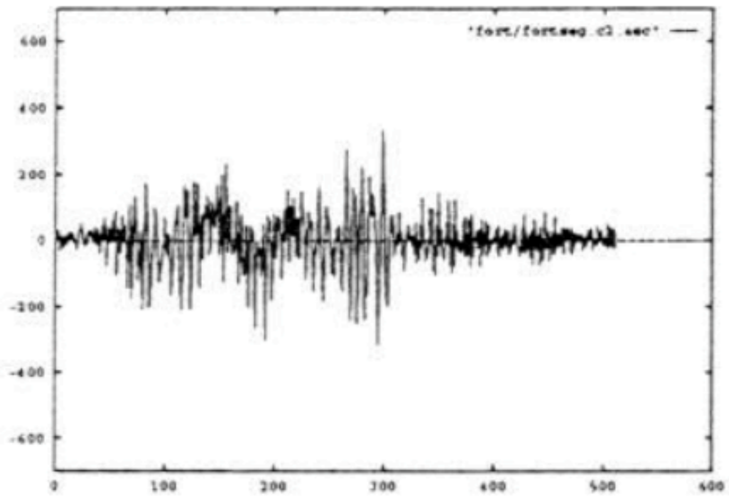


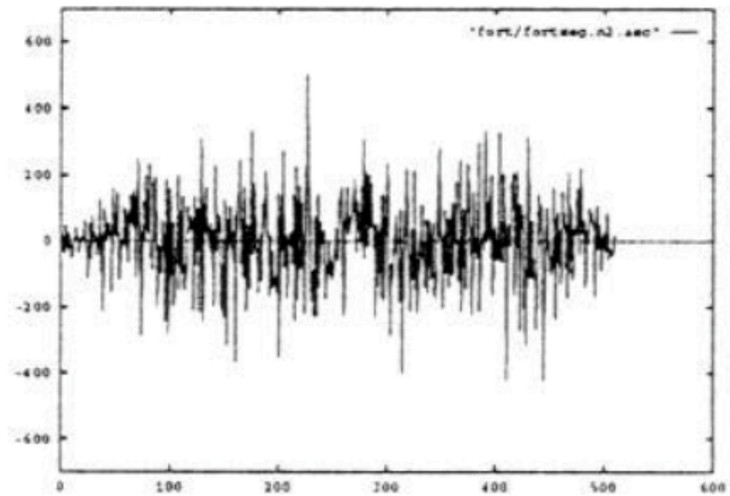
noisy and original signal



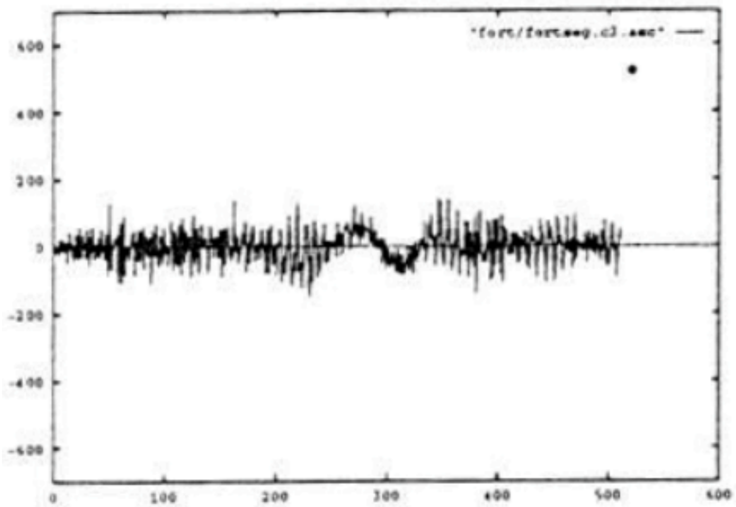


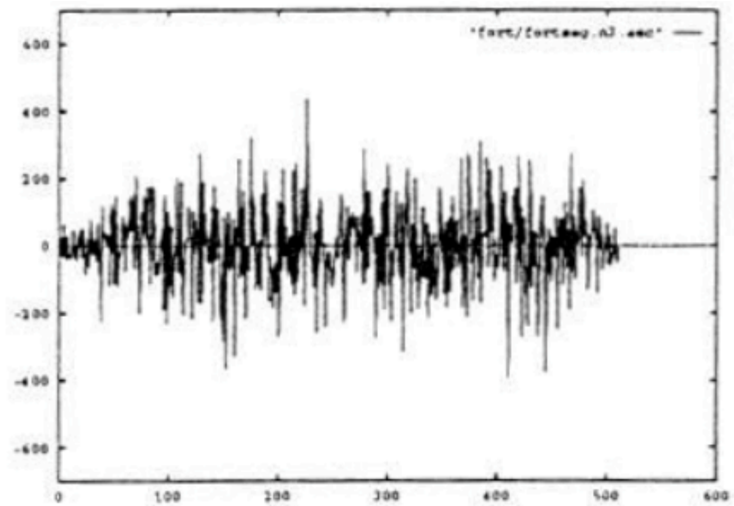
first coherent and first residuum



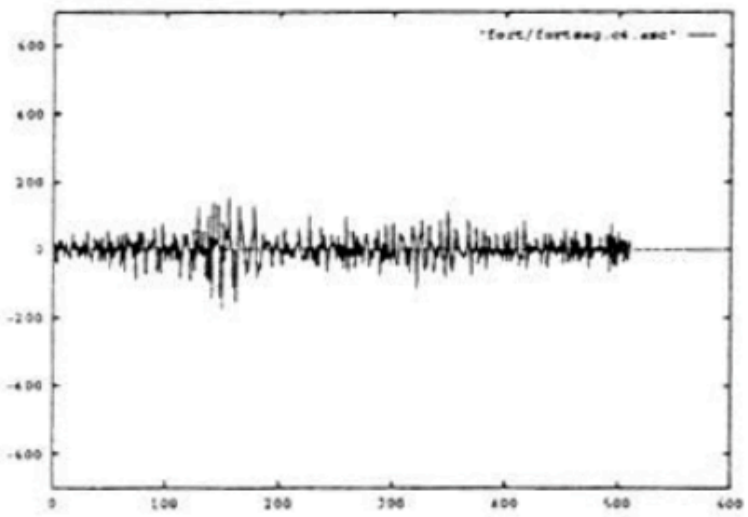


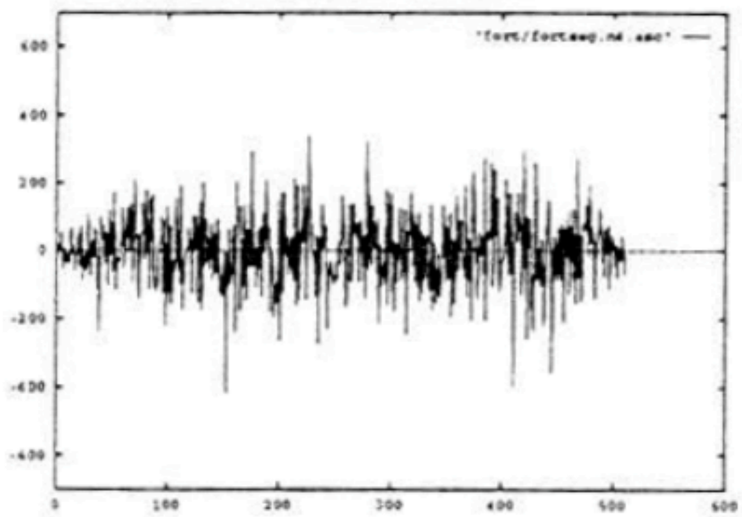
second coherent and second residuum



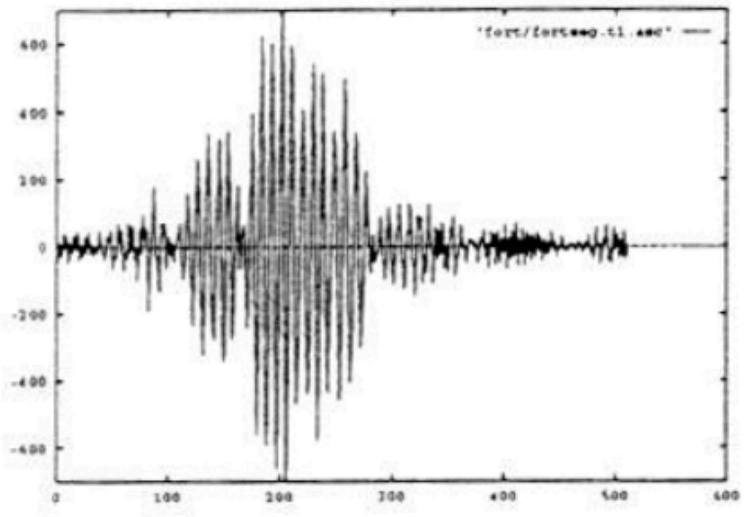
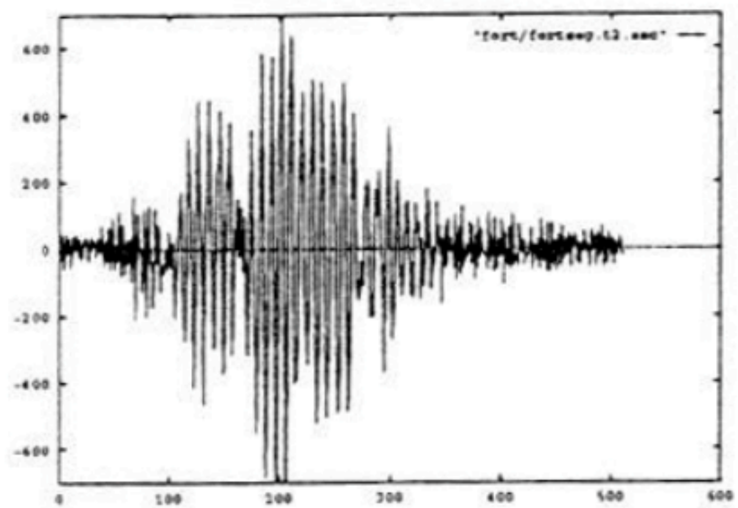


third coherent and third residuum

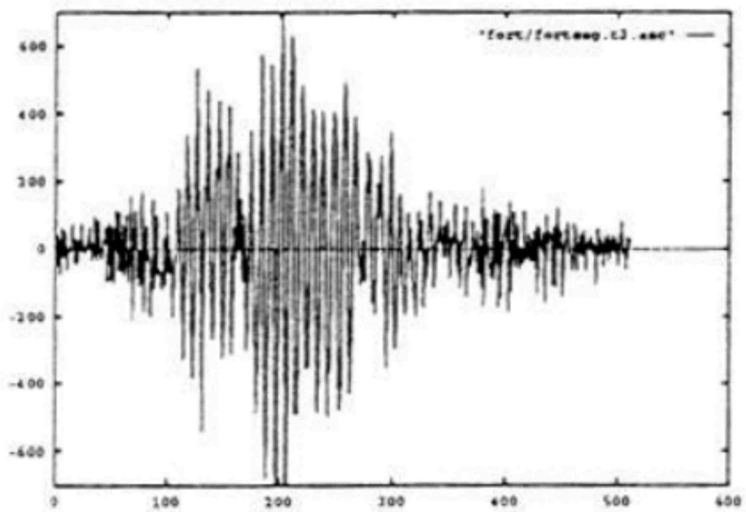
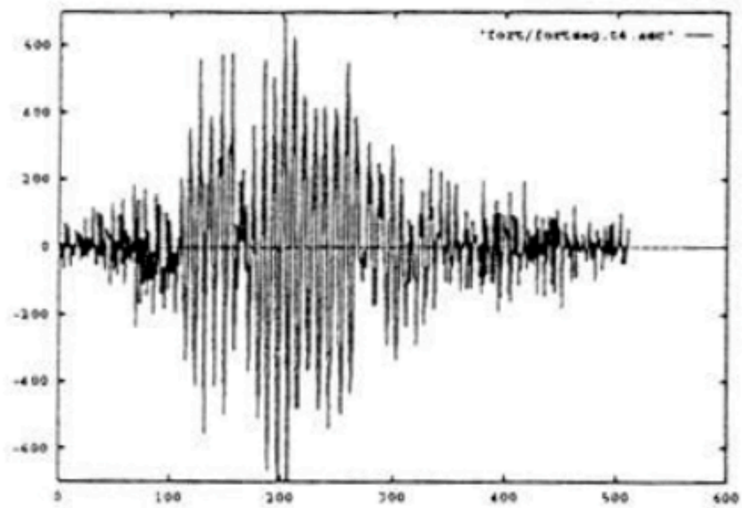




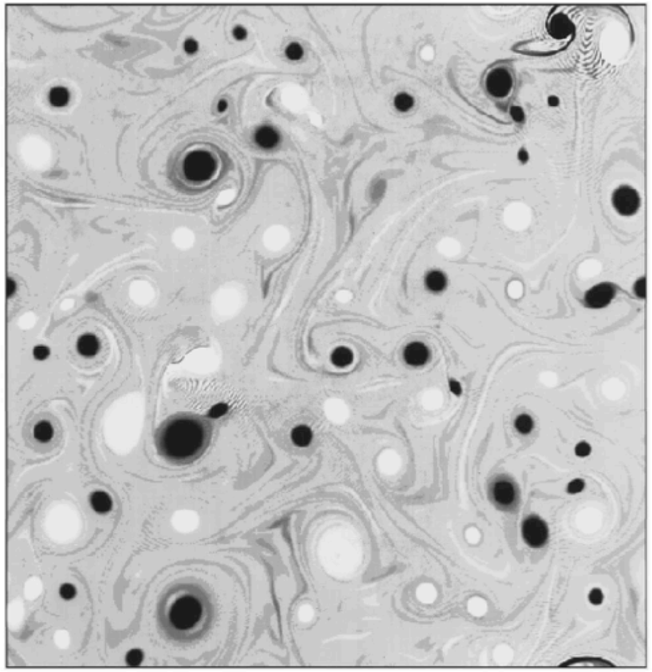
forth coherent and forth residuum

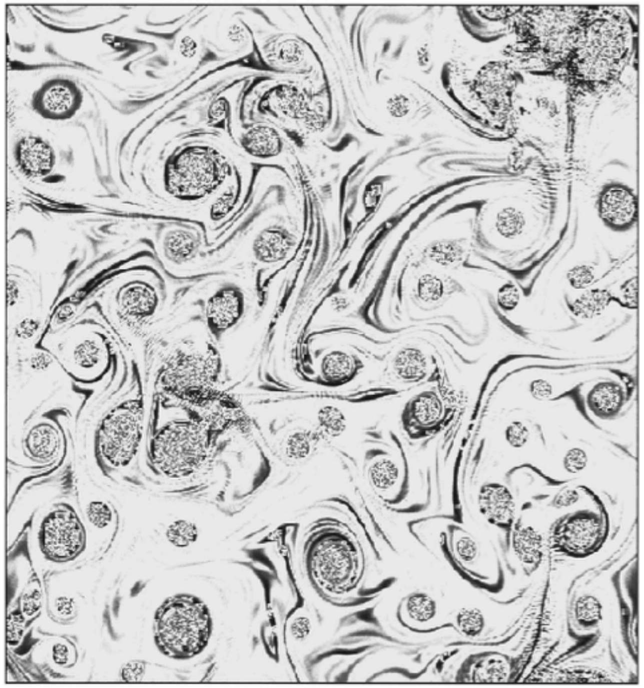



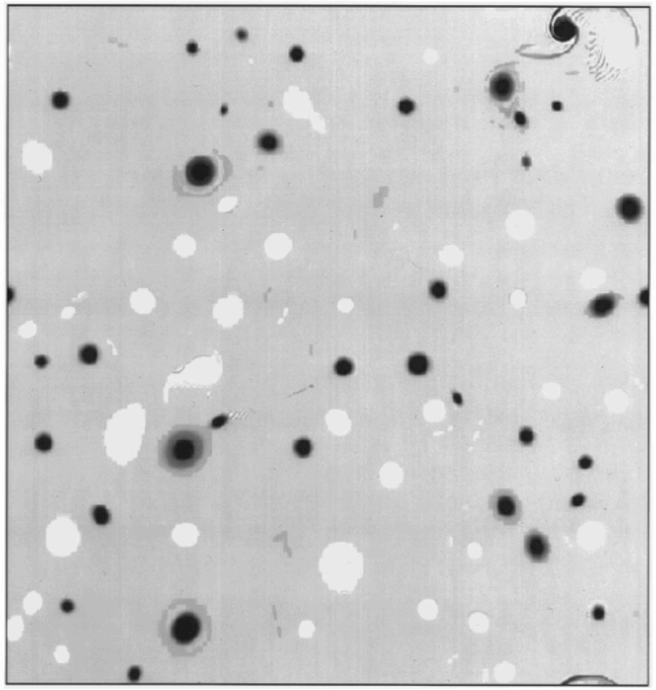
first and second reconstruction

forth reconstruction and original







A wavelet-packet census algorithm for calculating vortex statistics

A. Siegel, J. Weiss Phys. Fluids 9 (1997) 1988.

Original (262179 Bytes)

JPEG:

(4302 Bytes)

(2245 Bytes)

(1714 Bytes)

Wavelet-Compression:

(4272 Bytes)

(2256 Bytes)

(1708 Bytes)

Some basics on flow maps and circulation for Euler $\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = 0$

Define flow map $\phi : \mathbf{x} \to \boldsymbol{\phi}(\mathbf{x}, t)$ defining the trajectory of a fluid particle
Jacobian

$$J(\mathbf{x}, t) = \det\left(\nabla \boldsymbol{\phi}(\mathbf{x}, t)\right)$$

Incompressibility $\Longrightarrow$

$$0 = \frac{d}{dt}\int_{W(t)} dV = \frac{d}{dt}\int_{W(0)} J(\mathbf{x}, t) dV = \int_{W(0)} \frac{\partial}{\partial t} J(\mathbf{x}, t) dV$$

$$\Longrightarrow \boxed{J(\mathbf{x}, t) = 1} \quad \text{since} \quad \boldsymbol{\phi}(\mathbf{x}, 0) = \mathbf{x} \quad \Longrightarrow \quad J(\mathbf{x}, 0) = 1$$

Conserved quantities:

<u>Energy</u> $= \dfrac{1}{2}\displaystyle\int_\Omega |\, \mathbf{u} \,|^2 \, dV$

$$\frac{d}{dt}E = \int_\Omega \mathbf{u} \cdot \mathbf{u}_t \, dV = \int_\Omega \{-\mathbf{u} \cdot [\mathbf{u} \cdot \nabla \mathbf{u}] - \mathbf{u} \cdot \nabla p\} \, dV = 0$$

<u>Helicity</u> $= \displaystyle\int_\Omega \mathbf{u} \cdot \boldsymbol{\omega} \, dV$

Caution: holds only for smooth velocity fields

Vorticity $= \boldsymbol{\omega} = \nabla \times \mathbf{u}$

Taylor-expansion:

$$\mathbf{u}(\mathbf{x}+\mathbf{h}) = \mathbf{u}(\mathbf{x}) + (\nabla \mathbf{u})\mathbf{h} + \mathcal{O}(\mid h \mid^2)$$

split $\nabla \mathbf{u}$ in antisymmetric and symmetric parts:

$$\boldsymbol{\Omega} = \frac{1}{2}\left(\nabla \mathbf{u} - {}^T\nabla \mathbf{u}\right) \quad , \quad \mathbf{D} = \frac{1}{2}\left(\nabla \mathbf{u} + {}^T\nabla \mathbf{u}\right)$$

with identity:

$$\boldsymbol{\Omega}\mathbf{h} = \frac{1}{2}\boldsymbol{\omega} \times \mathbf{h}$$

$\Longrightarrow$

$$\mathbf{u}(\mathbf{x}+\mathbf{h}) = \mathbf{u}(\mathbf{x}) + \frac{1}{2}\boldsymbol{\omega} \times \mathbf{h} + D\mathbf{h} + \mathcal{O}(\mid h \mid^2)$$

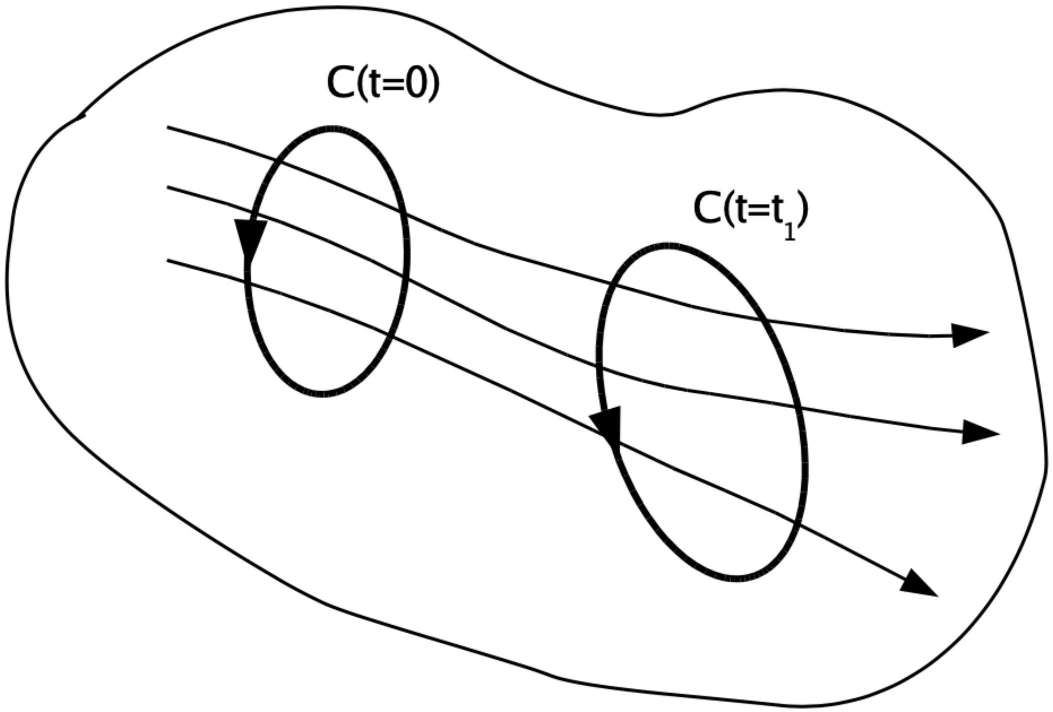
D: rate of strain, deformation
D is symmetric, so can be transformed to diagonal form with eigenvalues $\lambda_1, \lambda_2, \lambda_3$. It holds

$$\lambda_1 + \lambda_2 + \lambda_3 = \text{trace}\, D = \nabla \cdot \mathbf{u} = 0$$

Circulation $\Gamma_{C(t)}$

Let $C(t)$ be a closed curve: $C(t) = \boldsymbol{\phi}(C(t=0), t)$, $\boldsymbol{\phi}$ being the flow map.
Circulation is defined as

$$\Gamma_{C(t)} := \oint_{C(t)} \mathbf{u} \cdot d\mathbf{s}$$



Kelvin's Theorem: Circulation is conserved !

To proof this we the following Lemma:

$$\frac{d}{dt}\oint_{C(t)} \mathbf{u}\cdot d\mathbf{s} = \oint_{C(t)} \frac{D\mathbf{u}}{Dt}\cdot d\mathbf{s}$$

with $\frac{D}{Dt} = \frac{\partial}{\partial_t} + \mathbf{u}\cdot\nabla$

Proof: Parametrization $\mathbf{x}(s)$ of $C(0)$ $\;,0 \leq s \leq 1$

$$\begin{aligned}
\frac{d}{dt}\oint_{C(t)} \mathbf{u}\cdot d\mathbf{s} &= \frac{d}{dt}\int_0^1 \mathbf{u}(\boldsymbol{\phi}(\mathbf{x}(s),t),t)\cdot\frac{\partial}{\partial s}\boldsymbol{\phi}(\mathbf{x}(s),t)\,ds \\
&= \int_0^1 \frac{d}{dt}\mathbf{u}(\boldsymbol{\phi}(\mathbf{x}(s),t),t)\cdot\frac{\partial}{\partial s}\boldsymbol{\phi}(\mathbf{x}(s),t)\,ds \\
&+ \int_0^1 \mathbf{u}(\boldsymbol{\phi}(\mathbf{x}(s),t),t)\cdot\frac{\partial}{\partial t}\frac{\partial}{\partial s}\boldsymbol{\phi}(\mathbf{x}(s),t)\,ds \\
&= \int_0^1 \left[\frac{D\mathbf{u}}{Dt}\right](\boldsymbol{\phi}(\mathbf{x}(s),t),t)\cdot\frac{\partial}{\partial s}\boldsymbol{\phi}(\mathbf{x}(s),t)\,ds \\
&+ \int_0^1 \left(\mathbf{u}\cdot\frac{\partial}{\partial s}\mathbf{u}\right)(\boldsymbol{\phi}(\mathbf{x}(s),t),t)\,ds \\
&= \int_{C(t)} \frac{D\mathbf{u}}{Dt}\cdot d\mathbf{s}
\end{aligned}$$

Some basics on flow maps and circulation

... and it follows Kelvin's theorem:

$$\frac{d}{dt}\Gamma_{C(t)} = \frac{d}{dt}\int_{C(t)} \mathbf{u}\cdot d\mathbf{s} = \int_{C(t)} \frac{D\mathbf{u}}{Dt}\cdot d\mathbf{s} = -\int_{C(t)} \nabla p\cdot d\mathbf{s} = 0$$

Using Stokes we obtain

$$\Gamma_{C(t)} = \int_{C(t)} \mathbf{u}\cdot d\mathbf{s} = \int_{\Sigma(t)} \nabla\times\mathbf{u}\cdot d\mathbf{A} = \int_{\Sigma(t)} \boldsymbol{\omega}\cdot d\mathbf{A} \quad ,$$

where $\Sigma(t)$ is a surface with boundary $C(t)$.

That means:
Vorticity flux is constant through a surface moving with the flow

Vortexsurface, Vortexline:

A vortexsurface (vortexline) is a surface (line) tangential to $\boldsymbol{\omega}$

Theorem: If a surface S (or curve C) is moving with an incompressible flow and if this surface (or curve) is at time $t = 0$ a vortexsurface (vortexline) then it is for all times.

Proof: Let $\mathbf{n}$ be normal to S. At time $t = 0$ we have $\boldsymbol{\omega} \cdot \mathbf{n} = 0$. Kelvin $\Longrightarrow$ for every small piece of surface $\tilde{S}(t)$ we have

$$\int_{\tilde{S}(t)} \boldsymbol{\omega} \cdot \mathbf{n} \, dA = 0 \quad \Longrightarrow \quad \boldsymbol{\omega} \cdot \mathbf{n} = 0$$

Theorem: Evolution of vorticity

$$\frac{D\boldsymbol{\omega}}{Dt} = \boldsymbol{\omega} \cdot \nabla \mathbf{u}$$

and

$$\boldsymbol{\omega}(\boldsymbol{\phi}(\mathbf{x}, t), t) = \nabla \boldsymbol{\phi}(\mathbf{x}, t) \cdot \boldsymbol{\omega}(\mathbf{x}, 0)$$

Proof: Use identity $\frac{1}{2}\nabla(\mathbf{u} \cdot \mathbf{u}) = \mathbf{u} \times (\nabla \times \mathbf{u}) + \mathbf{u} \cdot \nabla \mathbf{u}$

$$\Longrightarrow \quad \frac{\partial}{\partial t}\mathbf{u} + \frac{1}{2}\nabla(\mathbf{u} \cdot \mathbf{u}) - \mathbf{u} \times \boldsymbol{\omega} = -\nabla p$$

$$\Longrightarrow \quad \frac{\partial}{\partial t}\boldsymbol{\omega} - \nabla \times (\mathbf{u} \times \boldsymbol{\omega}) = 0$$

$$\Longrightarrow \quad \frac{\partial}{\partial t}\boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \nabla \mathbf{u} - \mathbf{u} \cdot \nabla \boldsymbol{\omega}) = 0 \quad \Longrightarrow \quad \frac{D\boldsymbol{\omega}}{Dt} = \boldsymbol{\omega} \cdot \nabla \mathbf{u}$$

Eigenvalues and direction of eigenvectors of deformation matrix are important for vortex amplification !

Some basics on flow maps and circulation

For the proof of the second statement we define

$$\mathbf{F}(\mathbf{x},t) = \boldsymbol{\omega}(\boldsymbol{\phi}(\mathbf{x},t),t) \quad , \quad \mathbf{G}(\mathbf{x},t) = \nabla\boldsymbol{\phi}(\mathbf{x},t)\cdot\boldsymbol{\omega}(\mathbf{x},0)$$

$\mathbf{F}$ and $\mathbf{G}$ satisfy

$$\frac{\partial \mathbf{F}}{\partial t} = (\mathbf{F}\cdot\nabla)\mathbf{u} \ \text{ due to } \frac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega}\cdot\nabla)\mathbf{u}$$

$$\frac{\partial \mathbf{G}}{\partial t} = \nabla\left[\frac{\partial \boldsymbol{\phi}}{\partial t}(\mathbf{x},t)\right]\cdot\boldsymbol{\omega}(\mathbf{x},0) = \nabla(\mathbf{u}(\varphi(\mathbf{x},t),t))\cdot\boldsymbol{\omega}(\mathbf{x},0)$$

$$= (\nabla\mathbf{u})\cdot(\nabla\boldsymbol{\phi}(\mathbf{x},t)\cdot\boldsymbol{\omega}(\mathbf{x},0)) = (\mathbf{G}\cdot\nabla)\mathbf{u}$$

Both quantities satisfy the same equation and

$$\mathbf{F}(\mathbf{x},0) = \mathbf{G}(\mathbf{x},0) \quad\Longrightarrow\quad \mathbf{F}(\mathbf{x},t) = \mathbf{G}(\mathbf{x},t)$$

In two dimensions we have $\mathbf{u} = (u,v,0)$, $\boldsymbol{\omega} = (0,0,\omega)$ and therefore

$$\frac{D\boldsymbol{\omega}}{Dt} = 0 \quad\Longrightarrow\quad \boldsymbol{\omega}(\boldsymbol{\phi}(\mathbf{x},t),t) = \boldsymbol{\omega}(\mathbf{x},0)$$

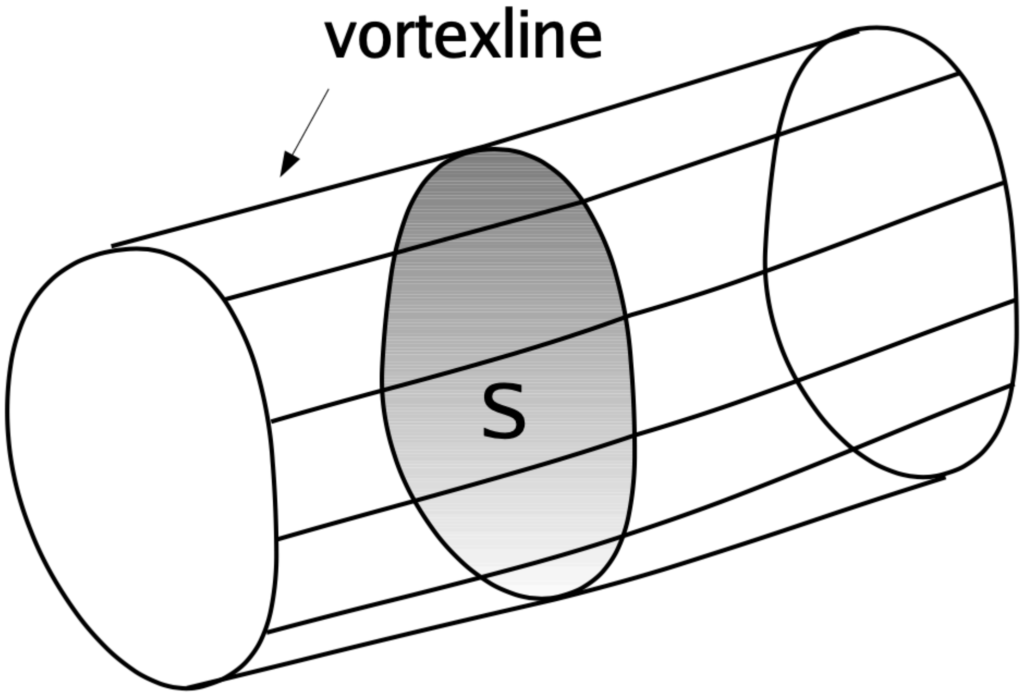
Vorticity is only convected !

2D: existence and uniqueness is clear

3D: existence and uniqueness is still an open problem

Some basics on flow maps and circulation

Let S be a 2D surface which is nowhere tangential to $\boldsymbol{\omega}$. Now consider vortexlines through the boundary ∂S. In this way you obtain a vortextube.



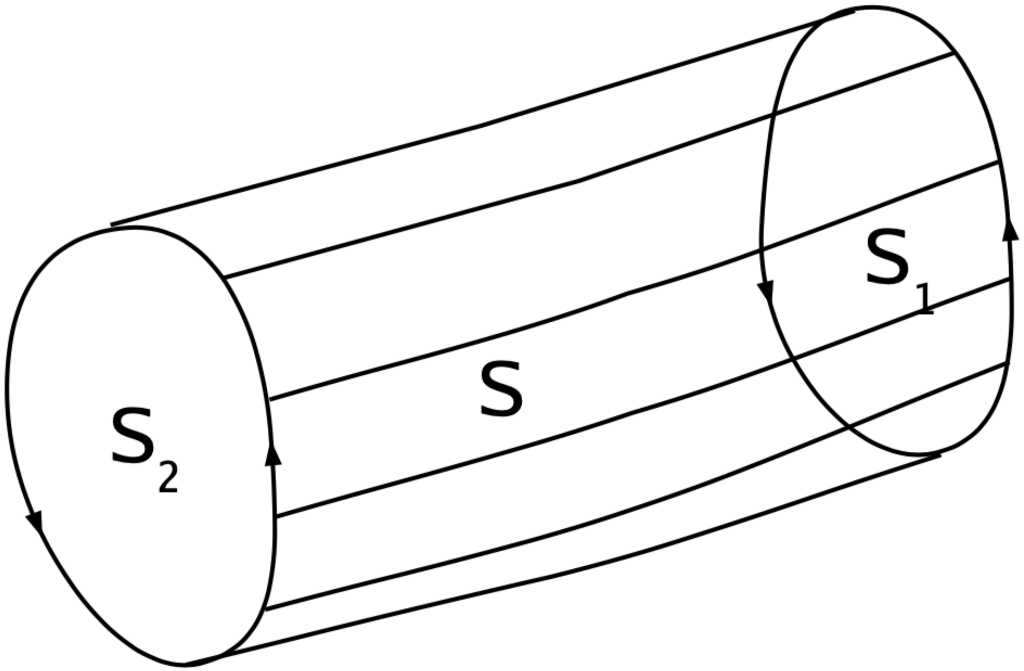
Helmholtz Theorem:

1. Let C_1 and C_2 two curves surrounding the vortextube. Then it holds

$$\int_{C_1} \mathbf{u} \cdot d\mathbf{s} = \int_{C_2} \mathbf{u} \cdot d\mathbf{s} = \Gamma$$

2. Γ is constant in time

Proof:



$$0 = \int_V \nabla \cdot \boldsymbol{\omega}\, dV \;=\; \int_{S_1} \boldsymbol{\omega} \cdot d\mathbf{A} + \int_{S_2} \boldsymbol{\omega} \cdot d\mathbf{A} + \int_S \boldsymbol{\omega} \cdot d\mathbf{A}$$
$$= \int_{C_1} \mathbf{u} \cdot d\mathbf{s} - \int_{C_2} \mathbf{u} \cdot d\mathbf{s} \;+\; 0$$

The second statement follows from Kelvin's theorem.

Consequence: stretching of vortextube $\Longrightarrow$ amplification of vorticity

Proposition: Let $D(t)$ be a symmetric 3×3 matrix with $\mathrm{trace} D(t) = 0$.
Choose the vorticity to satisfy the linear ODE

$$\frac{d\boldsymbol{\omega}(t)}{dt} = D(t)\boldsymbol{\omega}(t) \quad , \quad \boldsymbol{\omega}(0) = \boldsymbol{\omega}_0$$

Form the velocity vector $\mathbf{u}$, by setting

$$\mathbf{u}(\mathbf{x}, t) = \frac{1}{2}\boldsymbol{\omega} \times \mathbf{x} + D(t)\mathbf{x}$$

Then, there is an explicit quadratic pressure $p(\mathbf{x}, t)$ such that $\mathbf{u}$ and p define a solution of the incompressible Euler equations.

Proof: Define $V = \nabla \mathbf{u}$. V satisfies

$$\frac{DV}{Dt} + V^2 = -P \quad , \quad \mathrm{trace} V = 0 \quad , \quad P = \frac{\partial^2 p}{\partial x_i \partial x_j}$$

Decompose $V = D + \Omega$. Now $D(t)$ satisfies

$$\frac{\partial D(t)}{\partial t} + D(t)^2 + \Omega(t)^2 = -P(t) \quad .$$

Left hand side is a known symmetric matrix (by assumption) defining $P(t)$.
The correspond pressure is $p(\mathbf{x}, t) = \frac{1}{2}\mathbf{x} \cdot \mathbf{x}$.

Example: A swirling drain

$$D = \begin{pmatrix} -\frac{1}{2}\gamma & 0 & 0 \\ 0 & -\frac{1}{2}\gamma & 0 \\ 0 & 0 & \gamma \end{pmatrix} \implies \boldsymbol{\omega}(t) = (0, 0, e^{\gamma t}\boldsymbol{\omega}_0)$$

The vorticity increases when its the direction aligns with an eigenvector associated with a positive eigenvalue of D.

Navier-Stokes turbulence: Kolmogorov theory

incompressible Navier-Stokes equations

$$\frac{D\mathbf{u}}{\Delta t} = -\nabla p' + \nu \Delta \mathbf{u}\ , \quad \nabla \cdot \mathbf{u} = 0\ ; \quad p' = \frac{p}{\rho_0}, \nu = \frac{\mu}{\rho_0}$$

Second order in $x \Longrightarrow$ boundary conditions: $\mathbf{u} = 0$ or periodic at the boundary

Definition of the Reynolds number: L: typical length , U: typical velocity

define: $t' = \frac{U}{L} t\ , \quad \mathbf{x}' = \frac{\mathbf{x}}{L}\ , \quad \mathbf{u}' = \frac{\mathbf{u}}{U} \Longrightarrow$

$$\frac{\partial \mathbf{u}'}{\partial t'} + \mathbf{u}' \cdot \nabla' \mathbf{u}' = -\nabla' \underline{p}' + \frac{\underline{\nu}}{LU} \Delta' \mathbf{u}'\ , \quad p' = \frac{\rho}{\rho_0 u^2}$$

dimensionless parameter: $R = \frac{LU}{\nu}$ Reynolds number

scaling important for experimental models

Navier-Stokes turbulence: Kolmogorov theory

We show now that the pressure is defined by $\nabla \cdot \mathbf{u} = 0$. We need the following Theorem:

Any vectorfield $\mathbf{w}$ on some domain D can uniquely be decomposed:

$$\mathbf{w} = \mathbf{u} + \nabla p\,, \;\; \nabla \cdot \mathbf{u} = 0\,, \;\; \mathbf{u} \cdot \mathbf{n} = 0 \text{ on } \partial D$$

Proof: We have $\displaystyle\int_D \mathbf{u} \cdot \nabla p \, d\Omega = 0$ since

$$\nabla \cdot (p\mathbf{u}) = (\nabla \cdot \mathbf{u})p + \mathbf{u} \cdot \nabla p\,, \;\; \nabla \cdot \underline{u} = 0 \Longrightarrow$$

$$\int_D \mathbf{u} \cdot \nabla p d\Omega = \int_D \nabla \cdot (p\mathbf{u}) d\Omega - \int_{\partial D} p\mathbf{u} \cdot \mathbf{n} dA = 0\,, \text{ since } \mathbf{u} \cdot \mathbf{n} = 0 \text{ on } \partial D.$$

To show uniqueness, we use orthogonality. Let $\mathbf{w} = \mathbf{u}_1 + \nabla p_1 = \mathbf{u}_2 + \nabla p_2$

$$\Longrightarrow \quad 0 = \mathbf{u}_1 - \mathbf{u}_2 + \nabla \left(p_1 - p_2\right)$$

$$\Longrightarrow 0 = \int_D \Big(|\,\mathbf{u}_1 - \mathbf{u}_2\,|^2 + \left(\mathbf{u}_1 - \mathbf{u}_2\right) \nabla \left(p_1 - p_2\right) \Big)\, d\Omega = \int_D |\,\mathbf{u}_1 - \mathbf{u}_2\,|^2 \, d\Omega$$

$$\Longrightarrow \quad \mathbf{u}_1 = \mathbf{u}_2 \Longrightarrow \nabla p_1 = \nabla p_2$$

Navier-Stokes turbulence: Kolmogorov theory

Now we show existence.

$$\mathbf{w} = \mathbf{u} + \nabla p \implies \nabla \cdot \mathbf{w} = \Delta p\,,\ \mathbf{w} \cdot \mathbf{n} = \mathbf{n} \cdot \nabla p$$

Given $\mathbf{w} \implies p$ solution of $\Delta p = \nabla \cdot \mathbf{w}$ in D, $\dfrac{\partial p}{\partial n} = \mathbf{w} \cdot \mathbf{n}$ on ∂D

The solution to this Neumann problem is unique except for a constant.

Now define $\mathbf{u} = \mathbf{w} - \nabla p \implies \nabla \cdot \mathbf{u} = 0$ in D, $\mathbf{u} \cdot \mathbf{n} = 0$ on ∂D ✓

We denote the operator $\mathbb{P}$ which projects the vectorfield $\mathbf{w}$ on its divergence free part.

$$\mathbb{P}\mathbf{w} = \mathbf{u}\,,\ \nabla \cdot \mathbf{u} = 0 \text{ in } D\,,\ \mathbf{u} \cdot \mathbf{n} = 0 \text{ on } \partial D \quad \implies \mathbf{w} = \mathbb{P}\mathbf{w} + \nabla p$$

In addition, we have: $\mathbb{P}\mathbf{u} = \mathbf{u}$ if $\nabla \cdot \mathbf{u} = 0$ in D, $\mathbf{u} \cdot \mathbf{n} = 0$ on ∂D, $\mathbb{P}(\nabla p) = 0$

Now, the Navier-Stokes equations take the form $\dfrac{\partial}{\partial t}\mathbf{u} = \mathbb{P}\left(-\mathbf{u} \cdot \nabla \mathbf{u} + \dfrac{1}{R}\Delta \mathbf{u}\right)$

(Caution: $\mathbb{P}\left(\dfrac{1}{R}\Delta \mathbf{u}\right) \neq \dfrac{1}{R}\Delta \mathbf{u}$ in general, although $\nabla \cdot \Delta \mathbf{u} = 0$; but $\Delta \mathbf{u} \cdot \mathbf{n} \neq 0$ in general.)

Interpretation of pressure: p is gauge field to to keep the velocity field divergence free $\nabla \cdot \mathbf{u} = 0$

Finally, we come to the energy theorem for the incompressible Navier-Stokes equations.

$$\frac{d}{dt}E = \rho_0 \frac{1}{2}\frac{d}{dt}\int_D |\mathbf{u}|^2 d\Omega = \rho_0 \int \mathbf{u} \cdot \mathbf{u}_t d\Omega$$

$$= \rho_0 \int \left[-\mathbf{u} \cdot (\mathbf{u} \cdot \nabla \mathbf{u}) - \mathbf{u} \cdot \nabla p + \frac{1}{R} \mathbf{u} \cdot \Delta \mathbf{u} \right] d\Omega$$

$$= \rho_0 \int \left[-\frac{1}{2} \mathbf{u} \cdot \nabla \left(\mathbf{u}^2 \right) - \mathbf{u} \cdot \nabla p + \frac{1}{R} \mathbf{u} \cdot \Delta \mathbf{u} \right] d\Omega$$

$$= \frac{\rho_0}{R} \int \mathbf{u} \Delta \mathbf{u} d\Omega$$

$$= -\frac{\rho_0}{R} \int |\nabla \mathbf{u}|^2 d\Omega \leqslant 0$$

Navier-Stokes turbulence: Kolmogorov theory

Navier-Stokes

$$\partial_t u_i + u_m \frac{\partial u_i}{\partial x_m} - \nu \Delta u_i = -\frac{\partial p}{\partial x_i}, \frac{\partial u_i}{\partial x_i} = 0 \implies \partial_t u_i + \frac{\partial}{\partial x_m}\left(u_i u_m\right) - \nu \Delta u_i = -\frac{\partial p}{\partial x_i}$$

take divergence

$$\Delta p = -\frac{\partial^2}{\partial x_i \partial x_m}\left(u_i u_m\right), p = -\Delta^{-1}\frac{\partial^2}{\partial x_i \partial x_m}\left(u_i u_m\right)$$

It then follows that

$$\partial_t u_i - \nu \Delta u_i = -\frac{1}{2} P_{ijm}(\nabla)\left(u_j u_m\right) \qquad \text{with } P_{ijm}(\nabla) = \frac{\partial}{\partial x_m} P_{ij}(\nabla) + \frac{\partial}{\partial x_j} P_{im}(\nabla)$$

$$\text{and } P_{ij}(\nabla) = \delta_{ij} - \Delta^{-1}\frac{\partial^2}{\partial x_i \partial x_j}$$

Proof: $$-\frac{1}{2} P_{ijm}(\nabla)\left(u_j u_m\right) = -\frac{1}{2}\frac{\partial}{\partial x_m}\left(\delta_{ij} - \Delta^{-1}\frac{\partial^2}{\partial x_i \partial x_j}\right) u_j u_m$$

$$-\frac{1}{2}\frac{\partial}{\partial x_j}\left(\delta_{im} - \Delta^{-1}\frac{\partial^2}{\partial x_i \partial x_m}\right) u_j u_m = -\frac{\partial}{\partial x_m} u_i u_m - \frac{\partial}{\partial x_i} p \quad \checkmark$$

Navier-Stokes turbulence: Kolmogorov theory

Homogenous isotropic turbulence

averages $\langle \ldots \rangle$ are ensemble averages (taken over many realization of the system)

homogenous means: $E_0 = \frac{1}{2}\left\langle u_i(\mathbf{x}) u_i(\mathbf{x}) \right\rangle$ is independent of $\mathbf{x}$

$q_{ij}(\mathbf{r}) = \left\langle u_i(\mathbf{x}) u_j(\mathbf{x} + \mathbf{r}) \right\rangle$ is independent of $\mathbf{x}$

isotropic means: no direction is preferred

Fourier transform of the Navier-Stokes equations

$$u_i(\mathbf{x}) = \int u_i(\mathbf{k}) \exp(i\mathbf{k} \cdot \mathbf{x}) d^3k$$ (infinite volume)

$$u_i(\mathbf{k}) = (2\pi)^{-3} \int u_i(\mathbf{x}) \exp(-i\mathbf{k} \cdot \mathbf{x}) d^3x \,, \;\; \delta(\mathbf{x}) = (2\pi)^{-3} \int \exp(i\mathbf{k} \cdot \mathbf{x}) d^3k$$

Navier-Stokes turbulence: Kolmogorov theory

The only "difficult" term is the nonlinear term $u_j u_m$

$$(2\pi)^{-3}\int \exp(-i\mathbf{k}\cdot\mathbf{x})u_j(\mathbf{x})u_m(\mathbf{x})d^3x$$

$$= (2\pi)^{-3}\iiint d^3x d^3p d^3q \exp(-i(\mathbf{k}-\mathbf{p}-\mathbf{q})\cdot\mathbf{x})u_j(\mathbf{p})u_m(\mathbf{q})$$

$$= \iint d^3p d^3r u_j(\mathbf{p})u_m(\mathbf{q})\delta(\mathbf{k}-\mathbf{p}-\mathbf{q}) \;=: \sum^{\Delta} u_j(\underline{p})u_m(\underline{q})$$

(Δ denotes triangular interaction)

Thus we get the Fourier transformed Navier-Stokes equation

$$\left(\partial_t + \nu k^2\right) u_i(\mathbf{k}) = M_{ijm}(\mathbf{k}) \sum_{\mathbf{k}=\mathbf{p}+\mathbf{q}}^{\Delta} u_j(\mathbf{p})u_m(\mathbf{q}) \qquad (*)$$

with the definitions

$$M_{ijm}(\underline{k}) = -\frac{i}{2}P_{ijm}(\underline{k})\,,\;\; P_{ijm}(\mathbf{k}) = k_m P_{ij}(\mathbf{k}) + k_j P_{im}(\mathbf{k})\,,\;\; P_{ij}(\underline{k}) = \delta_{ij} - \frac{k_i k_j}{k^2}$$

Navier-Stokes turbulence: Kolmogorov theory

stationary turbulence: mean values are independent of time. This is not possible in equation (*). Therefore addition of a hypothetical homogeneous, isotropic force f_i. Thus the equation to study is

$$\left(\partial_t + \nu k^2\right) u_i(\mathbf{k}) = M_{ijm} \sum u_j(\mathbf{p}) u_m(\mathbf{r}) + f_i(\mathbf{k})$$

Correlations

turbulent fields characterized by hierarchy of correlations

$$\left\langle u_i(\mathbf{x}) \right\rangle, \left\langle u_i(\mathbf{x}) u_j(\mathbf{x}') \right\rangle, \left\langle u_i(\mathbf{x}) u_j(\mathbf{x}') u_x(\mathbf{x}'') \right\rangle, \ldots$$

homogenous turbulence $\Longrightarrow \left\langle u_i(\mathbf{x}) \right\rangle = U_i$

now we calculate the Fourier transform of the two-point correlation

$$\left\langle u_i(\mathbf{k}) u_j(\mathbf{k}') \right\rangle = \frac{1}{(2\pi)^6} \iint q_{ij}(\mathbf{x} - \mathbf{x}') \exp\left[-\mathbf{i}\left(\mathbf{k} \cdot \mathbf{x} + \mathbf{k}' \cdot \mathbf{x}\right)\right] d^3x d^3x'$$

with $q_{ij}(\mathbf{x} - \mathbf{x}') := \left\langle u_i(\mathbf{x}) u_j(\mathbf{x}') \right\rangle$ (homogeneity $\Longrightarrow q_{ij} = q_{ij}(\mathbf{x} - \mathbf{x}')$)

Navier-Stokes turbulence: Kolmogorov theory

define $\quad \chi := \frac{1}{2}(\mathbf{x}+\mathbf{x}')\ , \ \ \xi = \mathbf{x}-\mathbf{x}'$

$$\Longrightarrow \left\langle u_i(\mathbf{k}) u_j(\mathbf{k}') \right\rangle = \frac{1}{(2\pi)^3}\left[\int d^3\chi \exp\left[-i\left(\mathbf{k}+\mathbf{k}'\right)\cdot\chi\right]\right]$$

$$* \frac{1}{(2\pi)^3}\left[\int d^3\xi q_{ij}(\boldsymbol{\xi})\exp\left[-\frac{1}{2}i\left(\mathbf{k}-\mathbf{k}'\right)\cdot\boldsymbol{\xi}\right]\right]$$

$$= \delta\left(\mathbf{k}+\mathbf{k}'\right) q_{ij}(\mathbf{k})$$

Until now, we have used homogeneity. Now we consider isotropy. Isotropy means independence from spatial direction.

Example: isotropic vector $\mathbf{U}(\mathbf{r})$. Let $\mathbf{a}$ be an arbitrary vector

$\Longrightarrow \mathbf{U}(\mathbf{r})\cdot\mathbf{a} =: \tilde{U}(\mathbf{r},\mathbf{a})$ is scalar linear in $\mathbf{a}$

isotropic means $\quad \mathbf{U}(S\mathbf{r})\cdot S\mathbf{a} = \mathbf{U}(\mathbf{r})\cdot\mathbf{a}\ $ for $S\in O(3)$

Navier-Stokes turbulence: Kolmogorov theory

The only scalars that can be formed from $\mathbf{r}$ and $\mathbf{a}$ *and* are invariant under rotations and reflections *and* that are linear in $\mathbf{a}$ are given by:

$$|\mathbf{r}|\,,\ \ \mathbf{r}\cdot\mathbf{a} \qquad \Longrightarrow \qquad \tilde{U}(\mathbf{r},\mathbf{a}) = \tilde{\tilde{U}}(|\mathbf{r}|)\mathbf{r}\cdot\mathbf{a}$$

$$\Longrightarrow \qquad \mathbf{U}(\mathbf{r}) = \tilde{\tilde{U}}(|\mathbf{r}|)\mathbf{r}$$

homogeneity and isotropy $\Longrightarrow$ $\mathbf{U}(\mathbf{r}) = 0$

Second example: isotropic tensor $q_{ij}(\mathbf{r})$. Let $\mathbf{a},\mathbf{b}$ be vectors

$\Longrightarrow \tilde{q}(\mathbf{r},\mathbf{a},\mathbf{b}) = q_{ij}(\mathbf{r})a_ib_j$ is scalar linear in $\mathbf{a},\mathbf{b}$

isotropy means

$\tilde{q}(\mathbf{r},\mathbf{a},\mathbf{b}) = \tilde{q}(S\mathbf{r},S\mathbf{a},S\mathbf{b})$ with $S \in O(3)$

The only isotropic scalars formed from $\mathbf{r},\mathbf{a},\mathbf{b}$ *and* linear in $\mathbf{a},\mathbf{b}$ are given by:

$$|\mathbf{r}|,\mathbf{r}\cdot\mathbf{a},\mathbf{r}\cdot\mathbf{b},\mathbf{a}\cdot\mathbf{b} \qquad \Longrightarrow \qquad \tilde{q} = q^{(1)}(|\mathbf{r}|)(\mathbf{r}\cdot\mathbf{a})(\mathbf{r}\cdot\mathbf{b}) + q^{(2)}(|\mathbf{r}|)\mathbf{a}\cdot\mathbf{b}$$

$$\Longrightarrow \qquad \boxed{q_{ij}(r) = q^{(1)}(r)r_ir_j + q^{(2)}(r)\delta_{ij}}$$

Navier-Stokes turbulence: Kolmogorov theory

The Fourier transform of an isotropic tensor gives

$$q_{ij}(\mathbf{k}) = q^{(1)}(k)k_i k_j + q^{(2)}(k)\delta_{ij}\,. \qquad \text{(Richardson)}$$

The correlation tensor $\langle u_i(\mathbf{k})u_j(\mathbf{k}')\rangle$ must fulfill $\nabla \cdot \mathbf{u} = 0 \Longrightarrow$

$$\langle u_i(\mathbf{k})u_j(\mathbf{k}')\rangle = \delta(\mathbf{k}+\mathbf{k}')q_{ij}(\mathbf{k}) = \delta(\mathbf{k}+\mathbf{k}')P_{ij}(\mathbf{k})\left[q^{(1)}(k)k_l k_j + q^{(2)}(k)\delta_{lj}\right]$$

$$\Longrightarrow \quad \boxed{q_{ik}(\mathbf{k}) = P_{ij}(\mathbf{k})q(k)}$$

Back to real space:

$$\langle u_i(\mathbf{x})u_j(\mathbf{x})\rangle = q_{ij}(0) = \int d^3k' \int d^3k \langle u_i(\mathbf{k})u_j(\mathbf{k}')\rangle \exp(i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{x})$$

$$= \int d^3k\, q_{ij}(\mathbf{k}) = \int d^3k \left(\delta_{ij} - \frac{k_i k_j}{k^2}\right) q(k)$$

$$= \left(\frac{8\pi}{3}\right)\delta_{ij}\int_0^\infty k^2 q(k)dk$$

Navier-Stokes turbulence: Kolmogorov theory

Energy density:

$$\frac{1}{2}\langle u_i(\mathbf{x})u_i(\mathbf{x})\rangle = \int_0^\infty E(k)dk \quad \text{with} \quad E(k) = 4\pi k^2 q(k)$$

$E(k)dk$ is the energy of the modes in the wavenumber shell $[k, k+dk]$.
$q^{(k)}$ is the energy of a mode with wavenumber $|\mathbf{k}| = k$.

Correlation of velocity measured at two points:

$$\langle u_i(\mathbf{x})u_j(\mathbf{x}+\boldsymbol{\rho})\rangle = \int \exp(-i\mathbf{k}\cdot\boldsymbol{\rho})P_{ij}(\mathbf{k})q(k)d^3k$$

and (will be important later)

$$\frac{1}{2}\langle u_i(\mathbf{x})u_i(\mathbf{x}+\boldsymbol{\rho})\rangle = \int_0^\infty \frac{\sin k\rho}{k\rho}E(k)dk$$

Thus, knowledge of $E(k)$ is fundamental.

Fourier transform of the energy equation

stationarity (in time)

$$\langle u_i(\mathbf{k}, t) u_j(\mathbf{k}', t)\rangle = \delta(\mathbf{k} + \mathbf{k}') q_{ij}(\mathbf{k})$$

Now we want a time dependent energy tensor

$$\langle u_i(\mathbf{k}, t) u_j(\mathbf{k}', t')\rangle = \delta(\mathbf{k} + \mathbf{k}') Q_{ij}(\mathbf{k}, t, t') = \delta(\mathbf{k} + \mathbf{k}') Q_{ij}(\mathbf{k}, t - t') \qquad \text{if stationary}$$

Fourier transformed Navier-Stokes equation

$$\left(\partial_t + \nu k^2\right) u_i(\mathbf{k}, t) = M_{ijm}(\mathbf{k}) \iint d^3p\, d^3q\, u_j(\mathbf{p}, t) u_m(\mathbf{q}, t) \delta(\mathbf{k} - \mathbf{p} - \mathbf{q}) + f_i(\mathbf{k})$$

Navier-Stokes turbulence: Kolmogorov theory

multiply with $u_n(-\mathbf{k},t)$ and add the same but interchanged $i \leftrightarrow n$

$$(\partial_t + 2\nu k^2)\langle u_i(\mathbf{k},t)u_n(-\mathbf{k},t)\rangle$$

$$= M_{ijm}(\mathbf{k})\iint d^3p d^3q \langle u_j(\mathbf{p},t)u_m(\mathbf{q},t)u_n(-\mathbf{k},t)\rangle \delta(\mathbf{k}-\mathbf{p}-\mathbf{q})$$

$$-M_{njm}(-\mathbf{k})\iint d^3p d^3q \left\langle u_j(\mathbf{p},t)u_m(\mathbf{q},t)u_i(\mathbf{k},t)\right\rangle \delta(\mathbf{k}+\mathbf{p}+\mathbf{q})$$

$$+\langle f_i(\mathbf{k})u_n(-\mathbf{k},t)\rangle + \langle f_n(-\mathbf{k})u_n(\mathbf{k},t)\rangle$$

$$= M_{ijm}(\mathbf{k})\int d^3p \langle u_j(\mathbf{p},t)u_m(\mathbf{k}-\mathbf{p})u_n(-\mathbf{k},t)\rangle$$

$$-M_{njm}(\mathbf{k})\int d^3p \langle u_j(\mathbf{p},t)u_m(-\mathbf{k}-\mathbf{p})u_i(\mathbf{k},t)\rangle$$

+ force terms

Navier-Stokes turbulence: Kolmogorov theory

We demonstrate the connection between finite box $\longleftrightarrow$ infinite system

$$\langle u_i(\mathbf{k},t)u_j(\mathbf{k}',t')\rangle = \left(\frac{L}{2\pi}\right)^3 \delta_{\mathbf{k}+\mathbf{k}',0} Q_{ij}(\mathbf{k},t,t') \overset{L\to\infty}{=} \delta(\underline{k}+\underline{k}')Q_{ij}(\underline{k},t,t')$$

$$\lim_{L\to\infty}\left(\frac{2\pi}{L}\right)^3 \sum_{\underline{k}} = \int d^3k \qquad \Longrightarrow \qquad Q_{ijk}(\mathbf{p},\mathbf{q},\mathbf{k}) = \left(\frac{L}{2\pi}\right)^3 \langle u_i(\mathbf{p})u_j(\mathbf{q})u_k(\mathbf{k})\rangle$$

With this we get

$$\left(\frac{\partial}{\partial t} + 2\nu k^2\right) P_{in}(\mathbf{k})Q(k,t)$$

$$= M_{ijm}(\mathbf{k})\int d^3p Q_{jmn}(\mathbf{p},\mathbf{u}-\mathbf{p},-\mathbf{k};t) - M_{njm}(\mathbf{k})\int d^3p Q_{jmi}(\mathbf{p},-\mathbf{u}-\mathbf{p},\mathbf{k};t)$$

+ force terms

Navier-Stokes turbulence: Kolmogorov theory

contract $\Longrightarrow$

$$\left(\frac{\partial}{\partial t} + 2\nu k^2\right) E(k,t) = T(k,t) + \text{ force terms}$$

$$T(k,t) = 2\pi k^2 M_{ijm}(\mathbf{k}) \int d^3p \left\{ Q_{jmn}\left(\mathbf{p}, \mathbf{k}-\mathbf{p}, \mathbf{k}; t\right) - Q_{jmi}\left(\mathbf{p}, -\mathbf{k}-\mathbf{p}, \mathbf{k}; t\right) \right\}$$

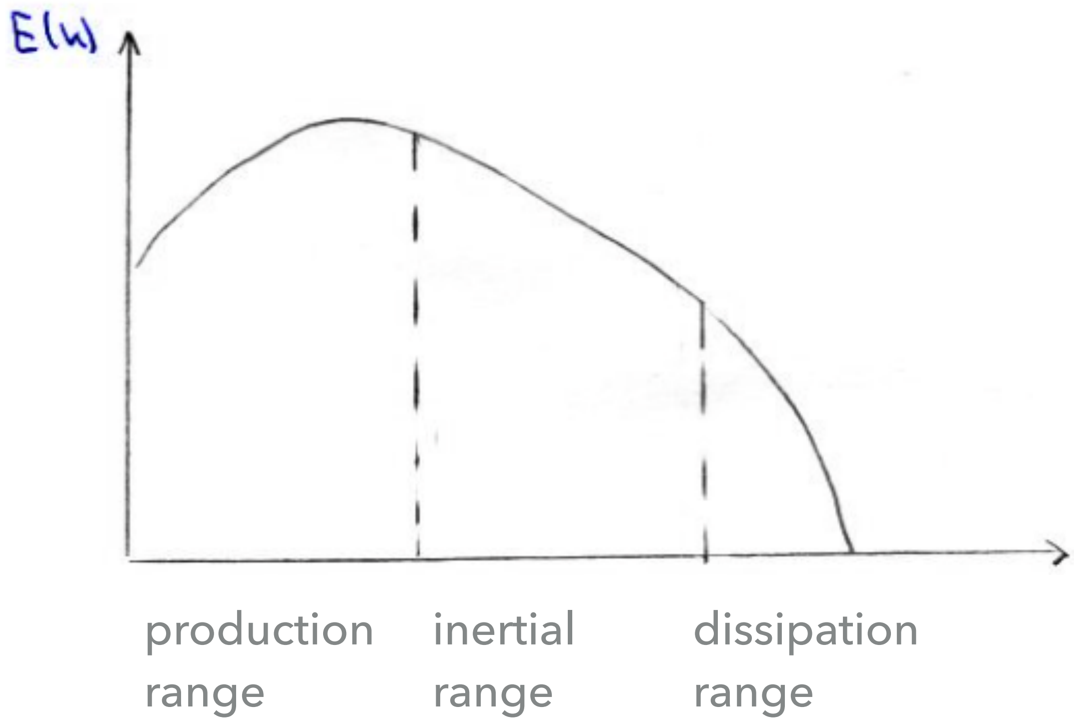
$T(k,t)$ contains the convection terms and the pressure $\Longrightarrow$ $T(k,t)$ redistributes energy

$$\int d^3k T(k,t) = 0$$

total dissipation rate: $\varepsilon = \int_\omega d^3k 2\nu k^2 Q(k) = \int_0^\infty 2\nu k^2 \cdot 4\pi k^2 Q(k) dk = \int_0^\infty 2\nu k^2 E(k)$

Navier-Stokes turbulence: Kolmogorov theory

energy spectrum divided into three parts:



1st Kolmogorov hypothesis: The ranges are independent. Inertial and dissipation ranges are independent of how energy is put into the production range. There exists a stationary state with an energy cascade from large to small length scales $\Longrightarrow$ production rate = dissipation rate ε.

$\Pi(k)$ is the flow of turbulent energy through the wavenumber k from wavenumbers smaller than k to wavenumbers larger than k.

Navier-Stokes turbulence: Kolmogorov theory

2nd Kolmogorov hypothesis: The energy cascade is local with respect to the wavenumber

$$\Longrightarrow \quad \Pi(k) = \Pi = \varepsilon \quad \text{(due to energy conservation)}$$

In addition, we neglect fluctuations (see discussion on intermittency below)

$$\Longrightarrow \quad E(k) = E(k, \nu, \pi = \varepsilon)$$

dimensional analysis:

$$[E(k)] = \frac{m^3}{s^2}\,,\; [\varepsilon] = \frac{m^2}{s^3}\,,\; [\nu] = \frac{m^2}{s} \qquad \Longrightarrow \qquad E(k) = g(\nu^{3/4}\varepsilon^{-\frac{1}{4}}k)\varepsilon^{2/3}k^{-5/3}$$

For high Reynolds numbers there exists a range, called inertial range such that the energy spectrum is not depending on viscosity:

$$E(k) = k_0 \varepsilon^{2/3} k^{-5/3}$$

k_0: Kolmogorov constant (can't be deduced from scaling arguments)

Navier-Stokes turbulence: Kolmogorov theory

Length scales

i) dissipation scale k_D^{-1}: $k_D = k_D(\varepsilon, \nu)$

dimensional analysis: $\left[k_D\right] = \frac{1}{m}, [\nu] = \frac{m^2}{s}, [\varepsilon] = \frac{m^2}{s^3}$

$$\Longrightarrow \quad k_D \sim \left(\frac{\varepsilon}{\nu^3}\right)^{\frac{1}{4}}$$

$\Longrightarrow$ Definition: Kolmogorov microscale $\quad k_D \sim \left(\frac{\varepsilon}{\nu^3}\right)^{\frac{1}{4}}$

ii) L = length of the physical system

iii) integral scale k_I^{-1}:

Let $E = \frac{1}{2}\left\langle \mathbf{u}^2 \right\rangle$. Then define $k_I = \frac{\varepsilon}{E^{3/2}}$

inertial range: $k_I \ll k \ll k_D$

iv) Taylor microscale: k_T^{-1}: $\quad k_T^{-2} = \frac{\left\langle \mathbf{u}^2 \right\rangle}{\left\langle (\nabla \times \mathbf{u})^2 \right\rangle} = 2\frac{\nu}{\varepsilon}E$

Navier-Stokes turbulence: Kolmogorov theory

Reynolds number Re: (based on integral scale)

$$R_e = \frac{u_{rms} l}{\nu}, \; u_{rms} = (2E)^{1/2}, \; l = k_I^{-1} = \sqrt{2}\left(\frac{k_D}{K_I}\right)^{4/3}$$

Reynolds number Re_T: (based on Taylor microscale)

$$Re_T = \frac{u_{rms}\lambda_T}{\nu}, \; u_{rms} = (2E)^{1/2}, \; \lambda_T = k_T^{-1} \qquad \Longrightarrow \qquad Re_T \sim Re^{\frac{1}{2}}$$

Number of degrees of freedom

$$N \sim \left(\frac{l}{l_D}\right)^3 \sim Re^{9/4} \qquad \text{(3D)}$$

In 2D we have an enstrophy cascade $\Longrightarrow \quad k_D = k_D(\eta, v) \quad \Longrightarrow \quad k_D = \left(\frac{\eta}{\nu^3}\right)^{1/6}$

The integral takes the form $\quad k_I = \frac{\eta^{1/3}}{E} \quad \Longrightarrow \quad Re = 2\left(\frac{k_D}{k_I}\right)^2 \quad \Longrightarrow \quad N \sim Re$

Navier-Stokes turbulence: Kolmogorov theory

different derivation of the K41 scaling law

scale invariance of the Navier-Stokes equation

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \nu \Delta \mathbf{u}\,, \quad \nabla \cdot \mathbf{u} = 0$$

invariant under the scaling transformation $\mathscr{D}_h$

$$\mathbf{r} \to \lambda \mathbf{r}\,,\ \mathbf{u} \to \lambda^h \mathbf{u}\,,\ t \to \lambda^{1-h} t\,,\ p \to \lambda^{2h} p\,,\ ,\nu \to \lambda^{1+h} \nu\,,\ \lambda > 0$$

Kolmogorov's assumption: ε is a local quantity, does not depend on scale

thus ε is invariant under $\mathscr{D}_h$

This uniquely defines the scaling exponent:

$$\varepsilon = 2\nu \int |\nabla u|^2 d\Omega = \lambda^{1+h+2h-2} \varepsilon \quad \Longrightarrow \boxed{h = \frac{1}{3}}$$

Navier-Stokes turbulence: Kolmogorov theory

This has consequences for the structure functions $\langle(\delta u(l))^p\rangle$

$$\langle(\delta \mathrm{u}(l))^p\rangle := \langle(u(\mathbf{r}+\mathbf{l}) - u(\mathbf{r}))^p\rangle$$

where u denotes the component $\mathbf{u}||\mathbf{l}$.

We assume that the structure function show a power law for small l.

$$\langle(\delta u(l))^p\rangle \sim l^{\xi_p}$$

Thus from the invariance under $\mathscr{D}_h$ follows

$$\langle(u(\mathbf{r}+\mathbf{l}) - u(\mathbf{r}))^p\rangle = \lambda^{-ph}\langle(u'(\lambda\mathbf{r}+\lambda\mathbf{l}) - u'(\lambda\mathbf{r})^p\rangle$$

$$\Longrightarrow \quad l^{\xi_p} \quad \sim \lambda^{-ph}(\lambda l)^{\xi_p} \quad \Longrightarrow \quad \boxed{\xi_p = ph = \frac{p}{3}}$$

Navier-Stokes turbulence: Kolmogorov theory

We now want to establish the connection with the energy spectrum. To do this, we consider the case $p = 2$.

$$\langle(\delta u(l))^2\rangle = 2(\langle u^2\rangle - \langle u(\mathbf{r}+\mathbf{l})u(\mathbf{r})\rangle)$$

It holds

$$\frac{1}{2}\langle u_i(\mathbf{r}+\mathbf{l})u_i(\mathbf{l})\rangle = \int d^3k P_{ii}(u)q(k)\exp\left(-i\mathbf{k}\cdot\mathbf{l}\right)$$

$$= \int_0^\infty k^2 q(u)du \oint \exp(-i\mathbf{k}\cdot\mathbf{l})\sin\theta d\theta d\phi \quad (\mathbf{u}\cdot\mathbf{l} = \nu l\cos\theta)$$

$$= \int_0^\infty k^2 q(k)dk \int_0^{2\pi} \exp(-ikl\cos\theta)\sin\theta d\theta \int_0^{2\pi} d\phi$$

$$= \int_0^\infty \frac{\sin kl}{ul} E(k)dk$$

Navier-Stokes turbulence: Kolmogorov theory

This means for the structure functions

$$\langle (\delta u(l))^2 \rangle \sim \int_0^\infty \left(1 - \frac{\sin kl}{kl}\right) E(k) dk$$

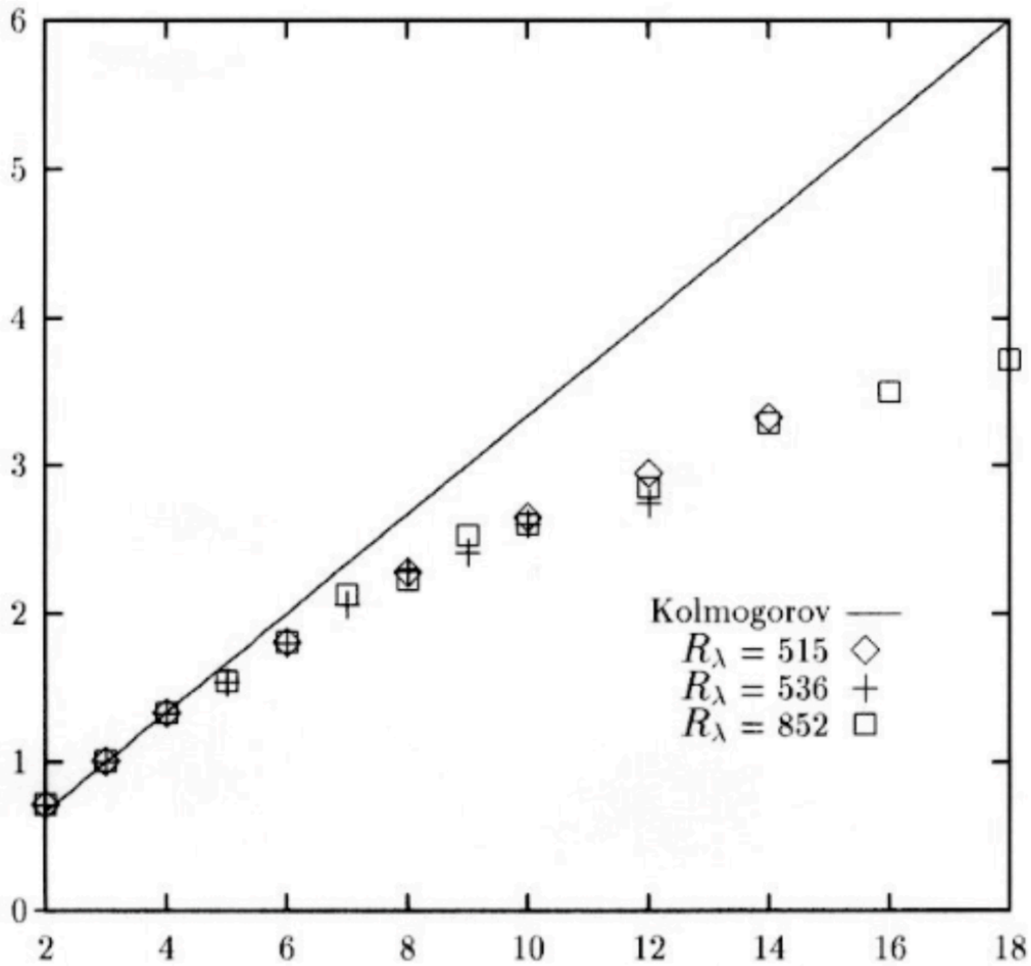
($E(k) \sim k^{-\gamma}$, substitution $m = kl \Longrightarrow$)

$$\sim \int_0^\infty \left(1 - \frac{\sin m}{m}\right) m^{-\gamma} dm l^{\gamma-1} \sim l^{\gamma-1}$$

$$\langle (\delta u(l))^2 \rangle \sim l^{\gamma-1} \sim l^{2/3} \Longrightarrow \gamma = \frac{5}{3} \Longrightarrow E(k) \sim k^{-5/3}$$

Navier-Stokes turbulence: Kolmogorov theory

The exponents can be easily determined experimentally. The results of Anselmet, Gagne, Hopfinger and Antonia (2006) are shown shown in the following graph.



Local scale-invariance:

There exist range of scaling exponents $I = (h_{min}, h_{max})$. For each $h \in I$ there is a set $\mathcal{S}_h \subset \mathbf{R}^3$ of fractal dimension $D(h)$ such that for $l \to 0$

$$\frac{\delta_l u(\mathbf{r})}{u_0} \sim \left(\frac{l}{l_0}\right)^h \quad , \quad \mathbf{r} \in \mathcal{S}_h$$

Structure functions:

$$\frac{S_p(l)}{u_0^p} \equiv \frac{\langle(\delta_l u)^p\rangle}{u_0^p} \sim \int_I d\mu(h) \left(\frac{l}{l_0}\right)^{ph+3-D(h)} \sim \left(\frac{l}{l_0}\right)^{\zeta_p}$$

$d\mu(h)$	gives weight of the different exponents.
$(l/l_0)^{ph}$	comes from the multifractal ansatz
$(l/l_0)^{3-D(h)}$	probality of being within a distance $\sim l$ of the set $\mathcal{S}_h$ of dimension $D(h)$

In the limit $l \to 0$ power law with smallest exponent survives

$$\zeta_p = \mathrm{inf}_{h\in I}[ph + 3 - D(h)]$$

Legendre trafo:

$$D(h) = \mathrm{inf}_p[ph + 3 - \zeta_p]$$

Navier-Stokes turbulence: She-Leveque model (1994)

Energy dissipation averaged over a ball of radius l

$$\langle \epsilon_l^p \rangle \sim l^{\tau_p}$$

K41:

$$\langle (\delta_l u)^p \rangle \sim (\epsilon l)^{p/3}$$

Kolmogorovs refined similarity hypothesis (1962):

$$\langle (\delta_l u)^p \rangle \sim (\epsilon_l l)^{p/3} \Longrightarrow \zeta_p = p/3 + \tau_{p/3}$$

K41: $\zeta_p = p/3$ and $\tau_p = 0$ because

$$\langle \epsilon_l^p \rangle \sim \bar{\epsilon}^p \ \text{ independent of } l$$

Define

$$\epsilon_l^{(p)} = \frac{\langle \epsilon_l^{p+1} \rangle}{\langle \epsilon_l^p \rangle}$$

$\epsilon_l^{(\infty)}$: most singular structures

Navier-Stokes turbulence: She-Leveque model (1994)

$$\epsilon_l^{(\infty)} = \delta E^\infty / t_l \quad , \quad t_l = \bar{\epsilon}^{-1/3} l^{2/3} \quad \text{K41 assumption}$$

$$\Longrightarrow \epsilon^{(\infty)} \sim l^{-2/3} \Longrightarrow \tau_{p+1} - \tau_p \to -\frac{2}{3} \Longrightarrow \tau_p \to -\frac{2}{3}p + C_0$$

Codimension: $C_0 = 3 - D(h^*)$

Ansatz:

$$\epsilon_l^{(p+1)} = A_p {\epsilon_l^{(p)}}^\beta {\epsilon_l^{(\infty)}}^{1-\beta} \quad (0 < \beta < 1)$$

$$\tau_{p+2} - (1+\beta)\tau_{p+1} + \beta\tau_p + \frac{2}{3}(1-\beta) = 0$$

$\tau_0 = \tau_1 = 0$, $C_0 = 2$ (fluxtubes), $\beta = 2/3$

$$\Longrightarrow \tau_p = -\frac{2}{3}p + 2\left[1 - \left(\frac{2}{3}\right)^p\right]$$

$$\boxed{\zeta_p = \frac{p}{9} + 2\left[1 - \left(\frac{2}{3}\right)^{p/3}\right]}$$

Incompressible MHD equations

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = \mathbf{B} \cdot \nabla \mathbf{B} - \nabla p - \nu_h (-\Delta)^h \mathbf{u} + \mathbf{f}, \quad \nabla \cdot \mathbf{u} = 0$$

$$\frac{\partial \mathbf{B}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{B} - \mathbf{B} \cdot \nabla \mathbf{u} = -\eta_h (-\Delta)^h \mathbf{B}, \quad \nabla \cdot \mathbf{B} = 0$$

MHD turbulence

Three quadratic invariants in 3D MHD:

1. Energy E

$$\frac{dE}{dt} = \frac{d}{dt}\int \frac{1}{2}\left(v^2 + B^2\right) dV = -\eta \int j^2 dB - \nu \int \omega^2 dV$$

2. cross-helicity H^c

$$\frac{dH^C}{dt} = \frac{d}{dt}\int \mathbf{v}\cdot\mathbf{B}dV = -(\nu+\eta)\int \mathbf{j}\cdot\omega dV$$

3. magnetic helicity H^M

$$\frac{dH^M}{dt} = \frac{d}{dt}\int \mathbf{A}\cdot\mathbf{B}dV = -\eta\int \mathbf{j}\cdot\mathbf{B}dV$$

Decay of Energy E much faster than decay of magnetic helicity H^M, and cross-helicity H^C

If H^M is most important constraint, then

$$\delta\left(\int \frac{1}{2}\left(v^2 + B^2\right) dV - \lambda \int \mathbf{A} \cdot \mathbf{B} dV\right) = 0$$

Variation with respect to $\mathbf{A}$ and $\mathbf{v}$ gives

$$\nabla \times \mathbf{B} - \lambda \mathbf{B} = 0, \quad \mathbf{v} = 0$$

current $\mathbf{j}$ parallel to magnetic field $\mathbf{B}$ (important for coronal loops)

If H^c is most important constraint, then

$$\delta\left(\int \frac{1}{2}\left(v^2 + B^2\right) dV - \lambda' \int \mathbf{v} \cdot \mathbf{B}\, dV\right) = 0$$

Variation with respect to $\mathbf{v}$ and $\mathbf{B}$ gives

$$\mathbf{v} - \lambda'\mathbf{B} = 0\ , \quad \mathbf{B} - \lambda'\mathbf{v} = 0$$

$\implies \quad \lambda'^2 = 1$ such that

$\mathbf{v} = \pm\, \mathbf{B}$ aligned state: Alfvén wave

Which relaxation processes dominates depends on initial condition:

- strongly helical system $\implies$ force free field
- sufficient large initial alignment $\implies$ Alfvénic state

Dynamic alignment

MHD in Elsässer variables: $\mathbf{z}^{\pm} = \mathbf{v} \pm \mathbf{B}$

$$\partial_t \mathbf{z}^{\pm} + \mathbf{z}^{\mp} \cdot \nabla \mathbf{z}^{\pm} + \nabla p = 0$$

Two time scales:

i. Alfvén time: $\tau_A = \frac{l}{v_A}, \quad v_A = B_0$

ii. nonlinear time scale: $\tau_l^{\pm} = \frac{l}{\delta z_l^{\mp}}$

with $\tau_A \ll \tau_l^{\pm}$

Distortion of $\delta z_l^{\pm}$ during one collision of Alfvénic wave packets:

$$\frac{\partial z_l^{\pm}}{\partial t} \sim \frac{\Delta \delta z_l^{\pm}}{l / v_A} = \frac{\delta z_l^{-} \delta z_l^{+}}{l} \sim z_l^{\mp} \nabla z_l^{\pm}$$

random scattering of wave packets (diffusion like) } $\Longrightarrow$ need

$$N \sim \left(\frac{\delta z_I^\pm}{\Delta \delta z_I^\pm}\right)^2 \sim \left(\frac{v_A}{\delta z_I^\mp}\right)^2$$

collisions to have substantial change of wave packet.

This leads to an modified energy-transfer time $\quad T_I^\pm \sim N\tau_A \sim \frac{\left(\tau_I^\pm\right)^2}{\tau_A}$

Define $E^\pm = \frac{1}{4}\int \left(z^\pm\right)^2 dV \quad \Longrightarrow \qquad E = E^+ + E^-\,, \quad H^C = \frac{1}{2}\left(E^+ - E^-\right)$

Thus E^+ and E^- are ideal invariants which cascade (like energy in Navier-Stokes) with the fluxes

$$\epsilon_I^\pm \sim \frac{\left(\delta z_I^\pm\right)^2}{T_I^\pm} \sim \left(\delta z_I^+\right)^2 \left(\delta z_I^-\right)^2 \tau_A / l^2$$

no intermittency: $\epsilon_I^\pm = \epsilon^\pm$, symmetric in $\pm \quad \Longrightarrow \quad \epsilon^+ = \epsilon^-$

no cross-helicity is dissipated

$$dH^C/dt = \frac{1}{4}\left(\epsilon^+ - \epsilon^-\right) = 0$$

This is important: *assume* that initially $\delta z_I^+ > \delta z_I^-$.

Still it holds by symmetry that $\epsilon^+ = \epsilon^-$.

But transfer times are different:

$$\frac{T_I^+}{T_I^-} = \left(\frac{\delta z_I^+}{\delta z_I^-}\right)^2 > 1$$

$\Longrightarrow$ transfer (and therefore damping) of δz_I^- (minority field) is more rapid and increasing the ratio E^+/E^-:

Dissipation to an Alfvénic state: $E^- = 0$.

Numerically verified by Pouquet *et al.* (1988) and Biskamp and Welter (1989).

Energy spectra I: Iroshnikov-Kraichnan

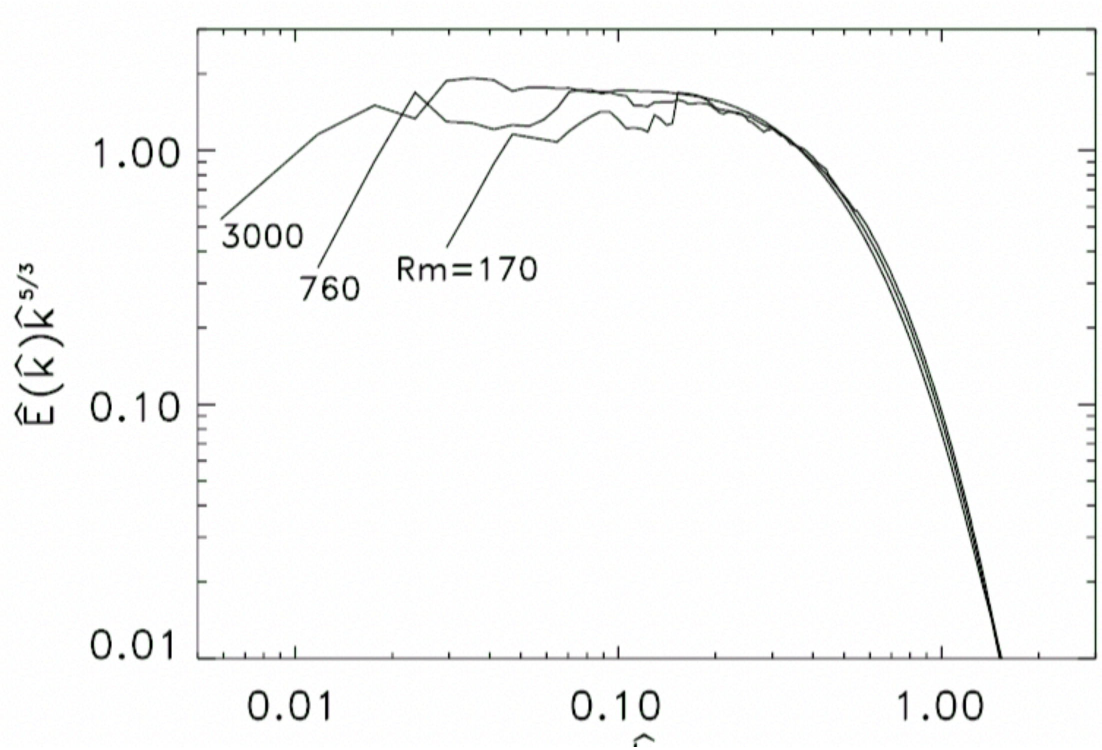
$$\epsilon_l^{\pm} \sim \frac{\left(\delta z_l^{\pm}\right)^2}{T_l^{\pm}} \sim \left(\delta z_l^{+}\right)^2 \left(\delta z_l^{-}\right)^2 \tau_A / l^2$$

now restrict discussion to weak velocity-magnetic-field correlation:

$$\delta z_l^{+} \approx \delta z_l^{-} := \delta z_l \Longrightarrow \epsilon_l^{\pm} \sim \left(\delta z_l\right)^4 \tau_A / l^2 \Longrightarrow$$

$$\delta z_l \sim \left(\epsilon v_A\right)^{1/4} l^{1/4} \Longrightarrow E_k = C_{IK} \left(\epsilon V_A\right)^{1/2} k^{-3/2}$$ Iroshnikov (1963), Kraichnan (1965)

But this is **not** seen in numerics: Müller and Biskamp (2000)



Energy spectra II: Goldreich-Sridhar

wave-packets (eddys) are strongly anisotropic:

field lines are difficult to bend

$\Longrightarrow$ shrinking (energy transfer) mostly $\perp B$

$\Longrightarrow$ elongated eddys along the large-scale magnetic field lines

Goldreich and Sridhar (1995): critical balance
$$\frac{\delta z_{l_\perp}}{l_\perp} \sim \frac{v_A}{l_\parallel}$$

energy transfer:
$$\epsilon \sim \frac{\left(\delta z_{l_\perp}\right)^3}{l_\perp}$$

parallel length scale
$$l_\parallel \sim \frac{v_A}{\epsilon^{1/3}} l_\perp^{2/3} \sim L^{1/3} l_\perp^{2/3}$$

with integral scale
$$L = v_A^3/\epsilon$$

Anisotropy increases with k:
$$\frac{k_\perp}{k_\parallel} \sim \left(L k_\perp\right)^{1/3}$$

Kolmogorov spectrum perpendicular to local field:

$$E\left(k_\perp\right) \sim \epsilon^{2/3} k_\perp^{-5/3}$$

parallel spectrum

$$E\left(k_\parallel\right) \sim \epsilon^{3/2} v_A^{-5/2} k_\parallel^{-5/2}$$

No magnetic mean field $\Longrightarrow$ parallel contribution negligible $\Longrightarrow$ K41 ✓

Fits with numerics !

Large magnetic mean field $\Longrightarrow$ perpendicular spectrum should be K41

Does not fit with numerics with guide field.

Numerics shows KI !

(Bale et al 2005, Müller, Grapin 2005, Mason et al 2008)

Problem not solved by GS!

Energy spectra III: Boldyrev

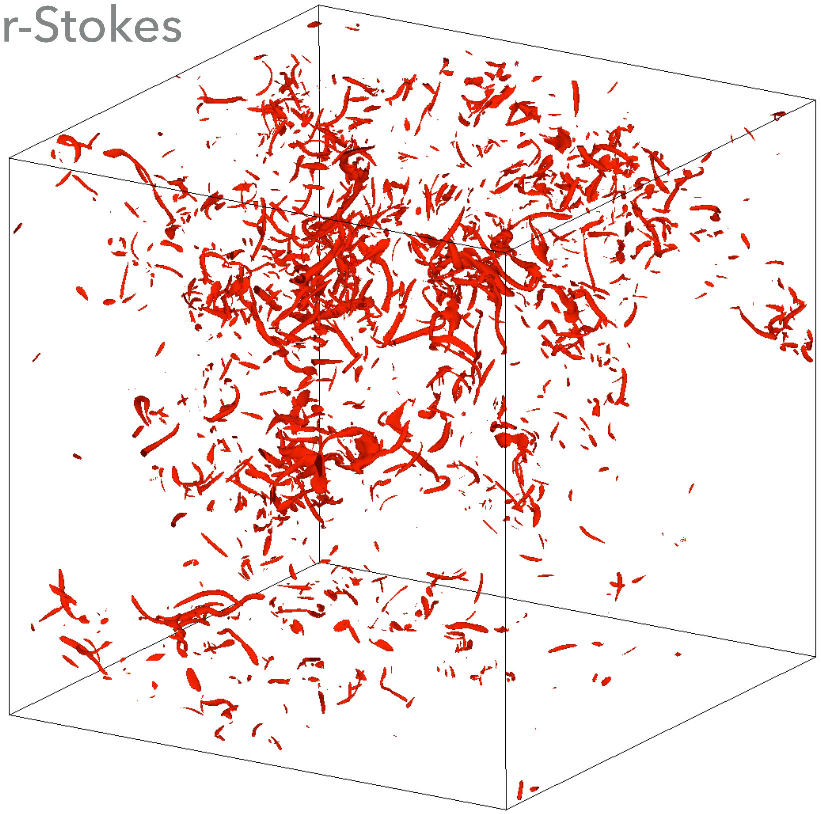
From Goldreich-Sridhar follows:

as scale decreases structures turn into filaments

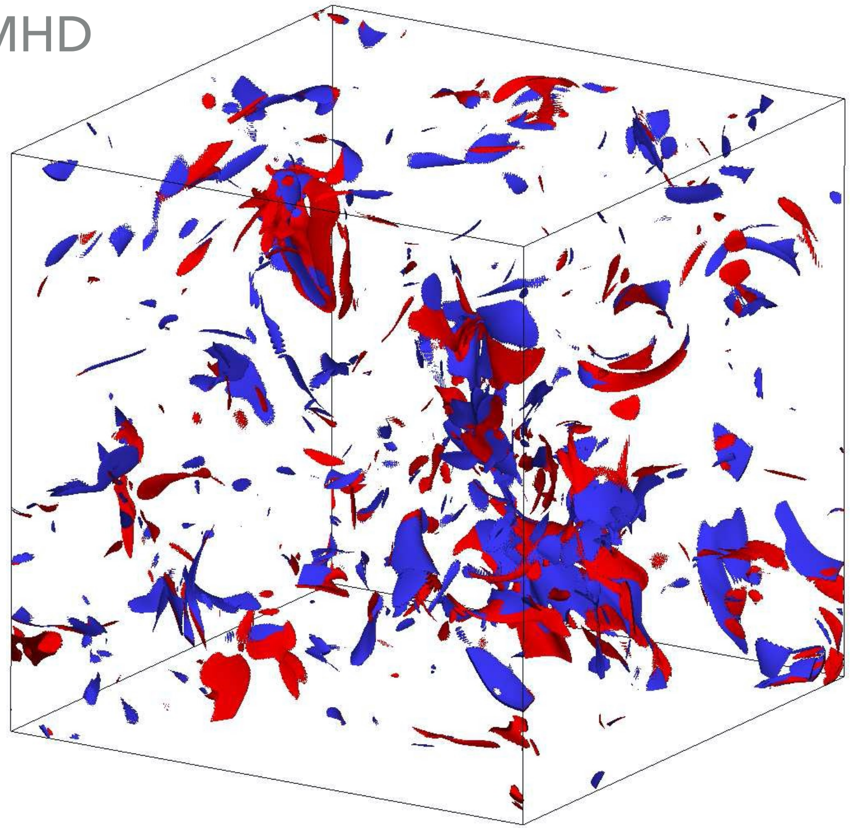
but

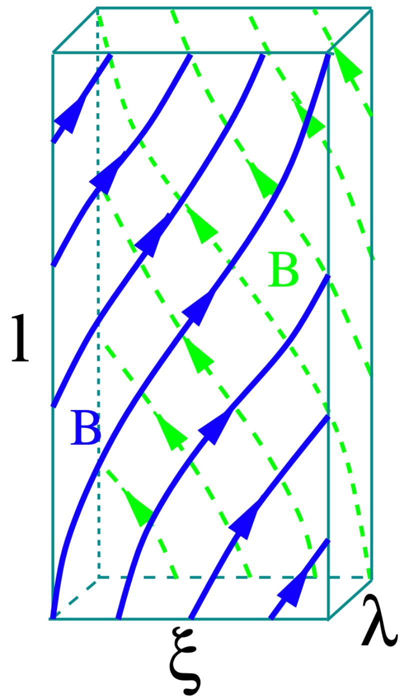
structures are current sheets !!!

Navier-Stokes



MHD





3 lengthscales:

l : size of an eddy parallel to the guide field

ξ : larger size of an eddy perpendicular to the guide field

λ : smaller size of an eddy perpendicular to the guide field

current sheet: $l \gg \xi \gg \lambda$

$\mathbf{B}(\mathbf{x}, t) = \mathbf{B}_0 + \mathbf{b}(\mathbf{x}, t)$

$\mathbf{B}_0$: averaged large scale magnetic field, guide field

$\mathbf{b}(\mathbf{x}, t)$: fluctuations

$\mathbf{z} = \mathbf{v} - \mathbf{b}\,, \quad \mathbf{w} = \mathbf{v} + \mathbf{b}$

$$\partial_t \mathbf{z} + \mathbf{V_A} \cdot \nabla \mathbf{z} + \mathbf{w} \cdot \nabla \mathbf{z} + \nabla p = 0$$
$$\partial_t \mathbf{w} - \mathbf{V_A} \cdot \nabla \mathbf{w} + \mathbf{z} \cdot \nabla \mathbf{w} + \nabla p = 0$$

cross-helicity: $\mathbf{v}(\mathbf{x}) = \pm\, \mathbf{b}(\mathbf{x})$

The nonlinear interaction is zero in the aligned state I!

Turbulence: alignment is not perfect and scale-dependent

$\delta\mathbf{v}_\lambda$ and $\pm\delta\mathbf{b}_\lambda$ are aligned within some small angle θ_λ

Depletion of nonlinearity: $\mathbf{w}\cdot\nabla\mathbf{z} \sim \mathbf{z}\cdot\nabla\mathbf{w} \sim \dfrac{\theta_\lambda \delta v_\lambda^2}{\lambda}$

As in Goldreich-Sridhar, balance linear and nonlinear terms and you get

$$\tau_N = \frac{l}{V_A} \sim \frac{\lambda}{\delta v_\lambda \theta_\lambda}$$

Kolmogorov idea: energy flux scale independent

$$\frac{\delta v_\lambda^2}{\tau_N} = \text{const} \implies \delta v_\lambda \propto \left(\frac{\lambda}{\theta}\right)^{1/3}$$

Displacement of magnetic field lines

mean magnetic field $\mathbf{B}_0 = B_0\mathbf{e}_z$, transverse fluctuations $\delta\mathbf{b}_\perp$: $\mathbf{B} = B_0\mathbf{e}_z + \delta\mathbf{b}_\perp$

direction of magnetic field line: $\dfrac{d\mathbf{r}}{ds} = \dfrac{\mathbf{B}}{|\mathbf{B}|} \approx \mathbf{e}_z + \dfrac{\delta\mathbf{b}_\perp}{B_0}$

So as a particle or field line moves a distance l along $\mathbf{e}_z$, the transverse displacement ξ is approximately:

$$\xi \sim \frac{\delta b_\perp}{B_0} l$$

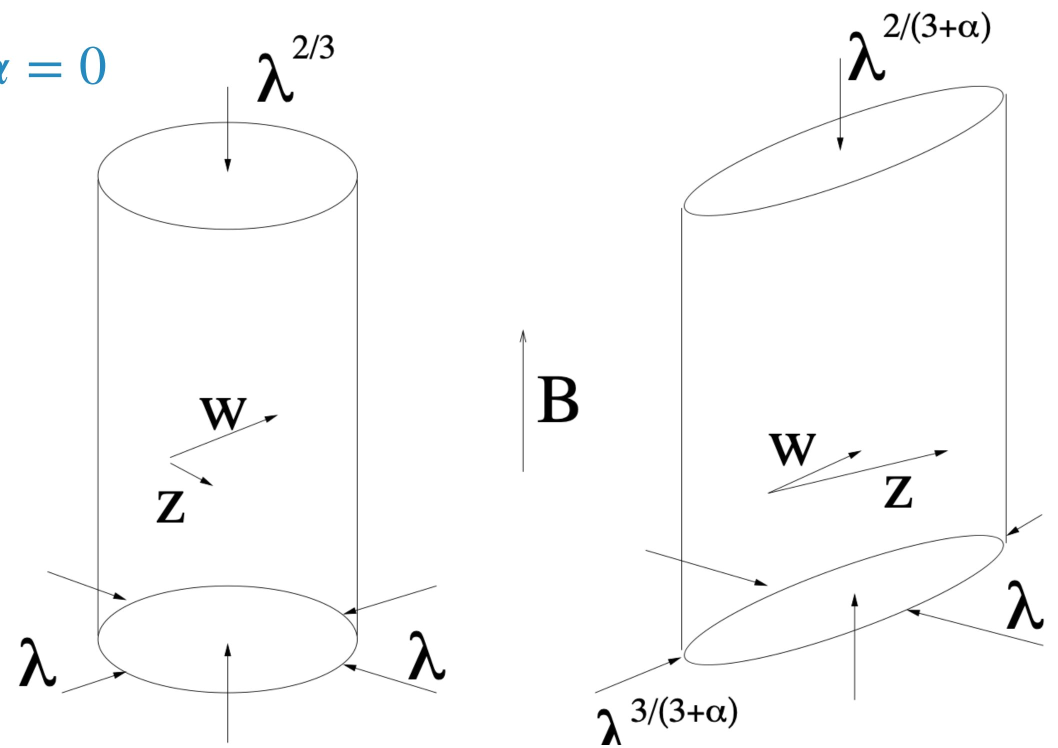
strong turbulence $\delta b_\perp \sim \delta v_\lambda \Longrightarrow \dfrac{\delta b_\perp}{B_0} = \dfrac{\delta v_\lambda}{V_A} \Longrightarrow \xi \sim \dfrac{\delta v_\lambda}{V_A} l \Longrightarrow \dfrac{\lambda}{\xi} \sim \theta_\lambda$

Displacement of magnetic field line: $\xi \sim \frac{\delta v_\lambda l}{V_A} \Longrightarrow \frac{\lambda}{\xi} \sim \theta_\lambda$

Assume power law scaling: $\theta_\lambda \propto \lambda^{\alpha/(3+\alpha)}, \quad \alpha \geq 0$

$$\Longrightarrow \delta v_\lambda \propto \lambda^{1/(3+\alpha)}, \quad \xi \propto \lambda^{3/(3+\alpha)}, \quad l \propto \lambda^{2/(3+\alpha)}$$

Goldreich-Sridhar: $\alpha = 0$
Lazarian …



Boldyrev: $\alpha \neq 0$

Howto to determine the value of α ?

Use cross-helicity and minimize the total mismatch angle !!!

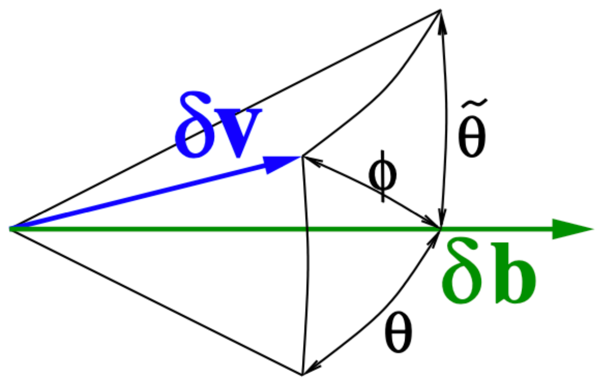
Mismatch angle in the field-perpendicular plane: $\theta_\lambda \propto \lambda^{\alpha/(3+\alpha)}$

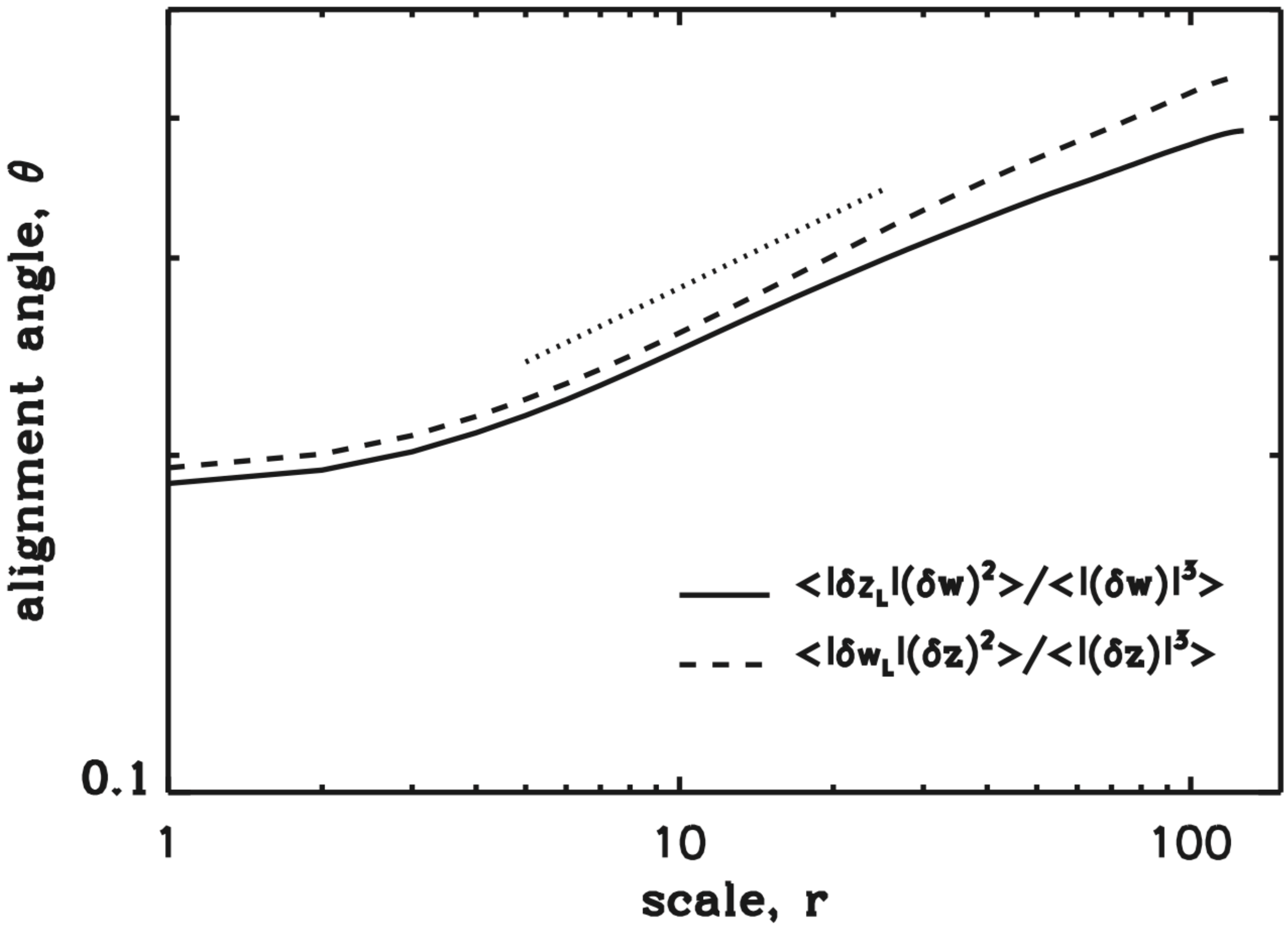
But, also mismatch in the vertical direction
with vertical mismatch angle: $\tilde{\theta}_\lambda \sim \frac{\xi}{l} \propto \lambda^{1/(3+\alpha)}$

total mismatch angle: $\phi_\lambda = \sqrt{\theta_\lambda^2 + \tilde{\theta}_\lambda^{\,2}}$

cross-helicity (alignment): this must be minimized !!! $\Longrightarrow \theta_\lambda = \tilde{\theta}_\lambda \Longrightarrow \alpha = 1$

$\Longrightarrow \delta v_\lambda \propto \lambda^{1/4} \Longrightarrow E\left(k_\perp\right) \propto k_\perp^{-3/2}$ Boldyrev (2005,2006)





Mason et al (2006)

Navier-Stokes: $$\left\langle \delta v_L^3(\mathbf{r}) \right\rangle = -\frac{4}{5}\epsilon r$$ (exact from NS)

MHD: $$S_{3L}^{w}(r) \equiv \left\langle \delta z_L (\delta \mathbf{w})^2 \right\rangle = -\frac{4}{3}\epsilon^{w} r$$

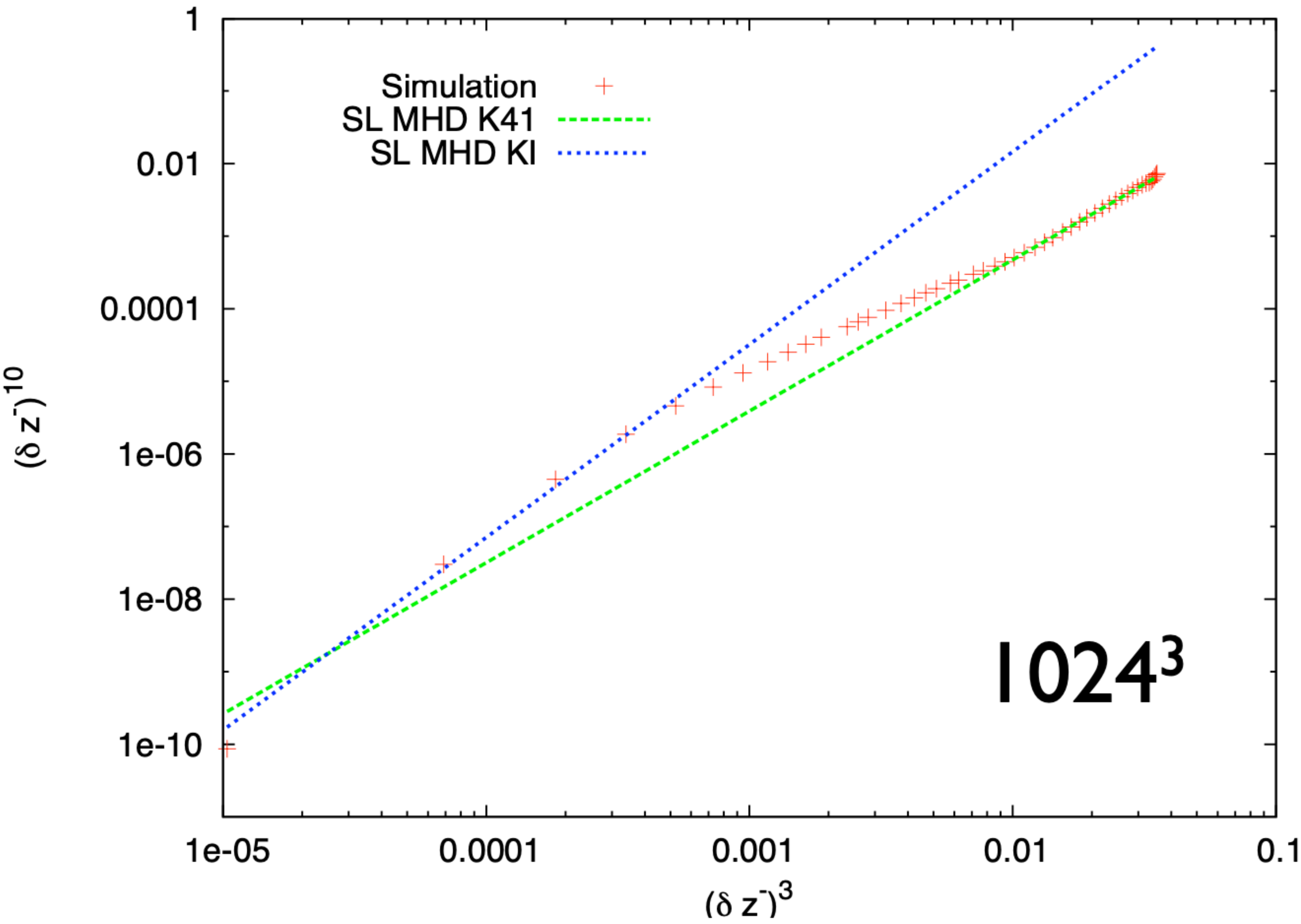
Politano, Pouquet 1998 $$S_{3L}^{z}(r) \equiv \left\langle \delta w_L (\delta \mathbf{z})^2 \right\rangle = -\frac{4}{3}\epsilon^{z} r$$

How does this fit ?

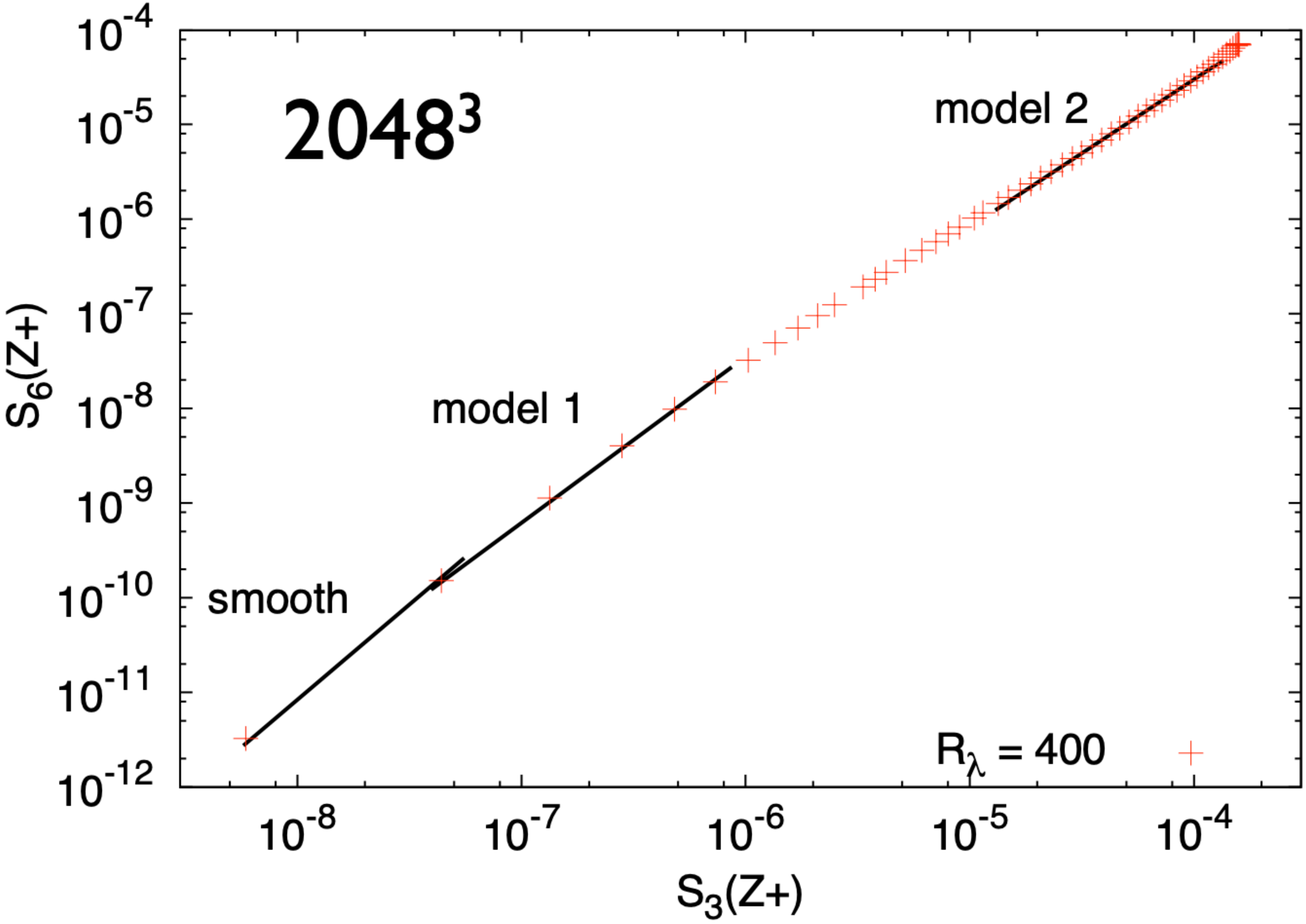
$$\left\langle \delta z_L (\delta w)^2 \right\rangle \sim \theta_\lambda \delta v_\lambda^3 \propto \lambda, \quad \theta_\lambda \propto \lambda^{1/4} \Longrightarrow \delta v_\lambda \propto \lambda^{1/4} \quad \checkmark$$

Some issues:

- unbalanced turbulence
- solar wind shows K41, WHY ?
- where is the crossover scale from K41 → IK-Boldyrev



Simulation
SL MHD K41
SL MHD KI
1024³
1
0.01
0.0001
1e-06
1e-08
1e-10
1e-05
0.0001
0.001
0.01
0.1
(δ z⁻)¹⁰
(δ z⁻)³



Homann, Grauer 2010 work in progress

MHD Turbulence: A Biased Review

Alexander A. Schekochihin

Journal of Plasma Physics. 88 (2022) 155880501

1D Burgers equation

$$\partial_t u = -\frac{1}{2}\partial_x u^2 + \nu\partial_{xx} u + f$$

We want to solve this in Fourier space:

$$\partial_t \hat{u}_k = -\frac{ik}{2}\widehat{(u^2)}_k - \nu k^2 \hat{u}_k + \hat{f}_k$$

We want to solve the dissipation explicitely $\Longrightarrow$ Ansatz:

$$\hat{u}_k(t) = \hat{\tilde{u}}_k(t) e^{-\nu k^2 t}$$

$\Longrightarrow$

$$\partial_t \hat{\tilde{u}}_k = \left(-\frac{ik}{2}\widehat{(u^2)}_k + \hat{f}_k\right) e^{\nu k^2 t}$$

Now we have no dissipation anymore at the expense of a time-dependent right hand side.

Euler Step

Euler step in general:

$$u^{(1)} = u^{(0)} + \Delta t L[u^{(0)}, t = t^{(0)}]$$

Now for our problem:

$$\hat{\tilde{u}}_k^{(1)} = \hat{\tilde{u}}_k^{(0)} + \Delta t \left(-\frac{ik}{2} \widehat{\left({u^{(0)}}^2 \right)}_k + \hat{f}_k^{(0)} \right) e^{\nu k^2 0}$$

or in our original $\hat{u}_k(t)$:

$$\hat{u}_k^{(1)} = e^{-\nu k^2 \Delta t} \left[\hat{u}_k^{(0)} + \Delta t \left(-\frac{ik}{2} \widehat{\left({u^{(0)}}^2 \right)}_k + \hat{f}_k^{(0)} \right) \right]$$

Pseudospectral steps

- Initialisation
  - initial condition in real space $u_0(x)$
  - FFT $\Longrightarrow \hat{u}_k^{(0)}$
  - same for forcing $\Longrightarrow \hat{f}_k^{(0)}$
- Euler steps
  - aliasing $\hat{u}_k^{(0)}$
  - transform $\hat{u}_k^{(0)}$ in real space $\Longrightarrow u^{(0)}(x)$
  - square it and store it in temporary field: $\mathrm{tmp}(x) = \frac{1}{2}{u^{(0)}}^2(x)$
  - FFT of $\mathrm{tmp}(x) \Longrightarrow \widehat{\mathrm{tmp}}_k$
  - $\hat{u}_k^{(1)} = e^{-\nu k^2 \Delta t}\left[\hat{u}_k^{(0)} + \Delta t\left(-\frac{ik}{2}\widehat{\mathrm{tmp}}_k + \hat{f}_k^{(0)}\right)\right]$

stable RK2: Heuns method

Heuns method in general:

$$
\begin{aligned}
u^{(1)} &= u^{(0)} + \Delta t L[u^{(0)}, t = t^{(0)}] \\
u^{(2)} &= \frac{1}{2} u^{(0)} + \frac{1}{2} u^{(1)} + \frac{1}{2} \Delta t L[u^{(1)}, t = t^{(1)}]
\end{aligned}
$$

Now for our problem:

$$
\begin{aligned}
\hat{u}_k^{(1)} &= e^{-\nu k^2 \Delta t} \left[\hat{u}_k^{(0)} + \Delta t \left(-\frac{ik}{2} \widehat{\left({u^{(0)}}^2 \right)}_k + \hat{f}_k^{(0)} \right) \right] \\
\hat{\tilde{u}}_k^{(2)} &= \frac{1}{2} \hat{u}_k^{(0)} + \frac{1}{2} \hat{\tilde{u}}_k^{(1)} + \frac{1}{2} e^{\nu k^2 \Delta t} \Delta t \left[-\frac{ik}{2} \widehat{\left({u^{(1)}}^2 \right)}_k + \hat{f}_k^{(1)} \right]
\end{aligned}
$$

or

$$
\hat{u}_k^{(2)} = \frac{1}{2} e^{-\nu k^2 \Delta t} \hat{u}_k^{(0)} + \frac{1}{2} \left[\hat{u}_k^{(1)} + \Delta t \left(-\frac{ik}{2} \widehat{\left({u^{(1)}}^2 \right)}_k + \hat{f}_k^{(1)} \right) \right]
$$

Pseudospectral steps

- aliasing $\hat{u}_k^{(0)}$
- transform $\hat{u}_k^{(0)}$ in real space $\Longrightarrow u^{(0)}(x)$
- square it and store it in temporary field: $\mathrm{tmp}_1(x) = \frac{1}{2} {u^{(0)}}^2(x)$
- FFT of $\mathrm{tmp}(x) \Longrightarrow \widehat{\mathrm{tmp}}_k$
- $\hat{u}_k^{(1)} = e^{-\nu k^2 \Delta t} \left[\hat{u}_k^{(0)} + \Delta t \left(-\frac{ik}{2} \widehat{\mathrm{tmp}_1}_k + \hat{f}_k^{(0)} \right) \right]$
- aliasing $\hat{u}_k^{(1)}$
- transform $\hat{u}_k^{(1)}$ in real space $\Longrightarrow u^{(1)}(x)$
- square it and store it in temporary field: $\mathrm{tmp}_2(x) = \frac{1}{2} {\tilde{u}^{(1)}}^2(x)$
- FFT of $\mathrm{tmp}_2(x) \Longrightarrow \widehat{\mathrm{tmp}_2}_k$
- $\hat{u}_k^{(2)} = \frac{1}{2} e^{-\nu k^2 \Delta t} \hat{u}_k^{(0)} + \frac{1}{2} \left[\hat{u}_k^{(1)} + \Delta t \left(-\frac{ik}{2} \widehat{\mathrm{tmp}_2}_k + \hat{f}_k^{(1)} \right) \right]$

stable RK3: Shu and Osher

Shu and Osher in general:

$$
\begin{aligned}
u^{(1)} &= u^{(0)} + \Delta t L[u^{(0)}, t^{(0)}] \\
u^{(2)} &= \frac{3}{4} u^{(0)} + \frac{1}{4} u^{(1)} + \frac{1}{4} \Delta t L[u^{(1)}, t^{(1)}] \\
u^{(3)} &= \frac{1}{3} u^{(0)} + \frac{2}{3} u^{(2)} + \frac{2}{3} \Delta t L[u^{(2)}, t^{(2)}]
\end{aligned}
$$

Now it is important to know at what times the $u^{(i)}$ live:

$$
\begin{aligned}
u^{(0)} &: \quad t = 0 \\
u^{(1)} &: \quad t = \Delta t \\
u^{(2)} &: \quad t = \frac{1}{2} \Delta t \\
u^{(3)} &: \quad t = \Delta t
\end{aligned}
$$

Now for our problem:

$$
\begin{aligned}
\hat{u}_k^{(1)} &= e^{-\nu k^2 \Delta t} \left[\hat{u}_k^{(0)} + \Delta t \left(-\frac{ik}{2} \widehat{\left(u^{(0)^2} \right)}_k + \hat{f}_k^{(0)} \right) \right] \\
\tilde{\hat{u}}_k^{(2)} &= \frac{3}{4} \hat{u}_k^{(0)} + \frac{1}{4} \tilde{\hat{u}}_k^{(1)} + \frac{1}{4} e^{\nu k^2 \Delta t} \Delta t \left(-\frac{ik}{2} \widehat{\left(u^{(1)^2} \right)}_k + \hat{f}_k^{(1)} \right)
\end{aligned}
$$

or

$$
\begin{aligned}
\hat{u}_k^{(2)} &= \frac{3}{4} e^{-\nu k^2 \frac{\Delta t}{2}} \hat{u}_k^{(0)} + \frac{1}{4} e^{\nu k^2 \frac{\Delta t}{2}} \left[\hat{u}_k^{(1)} + \Delta t \left(-\frac{ik}{2} \widehat{\left(u^{(1)^2} \right)}_k + \hat{f}_k^{(1)} \right) \right] \\
\tilde{\hat{u}}_k^{(3)} &= \frac{1}{3} \hat{u}_k^{(0)} + \frac{2}{3} \tilde{\hat{u}}_k^{(2)} + \frac{2}{3} e^{\nu k^2 \frac{\Delta t}{2}} \Delta t \left(-\frac{ik}{2} \widehat{\left(u^{(2)^2} \right)}_k + \hat{f}_k^{(2)} \right)
\end{aligned}
$$

or

$$
\hat{u}_k^{(3)} = \frac{1}{3} e^{-\nu k^2 \Delta t} \hat{u}_k^{(0)} + \frac{2}{3} e^{-\nu k^2 \frac{\Delta t}{2}} \left[\hat{u}_k^{(2)} + \Delta t \left(-\frac{ik}{2} \widehat{\left(u^{(2)^2} \right)}_k + \hat{f}_k^{(2)} \right) \right]
$$

2D Navier-Stokes

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \nu \Delta \mathbf{u} + \mathbf{f} \quad , \quad \nabla \cdot \mathbf{u} = 0$$

u depends only on x, y

$$\Longrightarrow \nabla \times \mathbf{u} = (\partial_x u_y - \partial_y u_x)\mathbf{e}_z = \omega \mathbf{e}_z$$

time evolution for ω:

$$\partial_t \omega + \mathbf{u} \cdot \nabla \omega = \nu \Delta \omega + (\nabla \times \mathbf{f})_z$$

For simplicity forget forcing. Now in Fourierspace:

$$\partial_t \hat{\omega}_{\mathbf{k}} + (\widehat{\mathbf{u} \cdot \nabla \omega})_{\mathbf{k}} = -k^2 \hat{\omega}_{\mathbf{k}}$$

In Fourierspace we have:

$$\hat{\omega}(\mathbf{k}) = ik_x \hat{u}_y(\mathbf{k}) - ik_y \hat{u}_x(\mathbf{k}) \quad , \quad k_x \hat{u}_x(\mathbf{k}) + k_y \hat{u}_y(\mathbf{k}) = 0$$

And for the velocity:

$$\hat{u}_x(\mathbf{k}) = ik_y \frac{\hat{\omega}(\mathbf{k})}{k^2} \quad , \quad \hat{u}_y(\mathbf{k}) = -ik_x \frac{\hat{\omega}(\mathbf{k})}{k^2}$$

We can write (because of $\nabla \cdot \mathbf{u} = 0$):

$$\mathbf{u} \cdot \nabla\omega = \nabla \cdot (\mathbf{u}\omega)$$

For the pseudospectral method we have to inverse Fouriertransform:

$$\hat{u}_x(\mathbf{k}) \ , \ \ \hat{u}_y(\mathbf{k}) \ , \ \ \hat{\omega}(\mathbf{k})$$

The rest works as in Burgers !!!

1 MHD equations

We use $\rho_0 = 1$, $\mu_0 = 1$:

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v} - \mathbf{B}\cdot\nabla\mathbf{B} = -\nabla p + \nu\Delta\mathbf{v} \ , \qquad \nabla\cdot\mathbf{v} = 0 \tag{1}$$

$$\frac{\partial \mathbf{B}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{B} - \mathbf{B}\cdot\nabla\mathbf{v} = \eta\Delta\mathbf{B} \ , \qquad \nabla\cdot\mathbf{B} = 0 \tag{2}$$

Now 2D:

$$\mathbf{B} = \mathbf{e}_z \times \nabla\psi \ , \ \mathbf{j} = \nabla\times\mathbf{B} = j(x,y;t)\mathbf{e}_z = \Delta\psi\mathbf{e}_z \tag{3}$$

$$\boldsymbol{\omega} = \omega(x,y;t)\mathbf{e}_z \ , \ \frac{\partial v_x}{\partial x} = -\frac{\partial v_y}{\partial y} \ , \ \Delta v_x = -\frac{\partial \omega}{\partial y} \ , \ \Delta v_y = \frac{\partial \omega}{\partial x} \tag{4}$$

2D MHD:

$$\frac{\partial \omega}{\partial t} + (\mathbf{v}\cdot\nabla)\omega = (\mathbf{B}\cdot\nabla)j + \nu\Delta\omega \tag{5}$$

$$\frac{\partial \psi}{\partial t} + (\mathbf{v}\cdot\nabla)\psi = \eta\Delta\psi \tag{6}$$

Since $\nabla\cdot\mathbf{v} = 0$, $\nabla\cdot\mathbf{B} = 0$ we can write (conservative formulation):

$$\frac{\partial \omega}{\partial t} + \nabla\cdot(\mathbf{v}\omega) = \nabla\cdot(\mathbf{B}j) + \nu\Delta\omega \tag{7}$$

$$\frac{\partial \psi}{\partial t} + \nabla\cdot(\mathbf{v}\psi) = \eta\Delta\psi \tag{8}$$

2 Spectral method

2.1 Vorticity-stream function formulation

1. start with $\hat{\omega}^n$, $\hat{\psi}^n$ ($\hat{\ldots}$ denotes Fourier space, n denotes time)
2. calculate $v_x = -\Delta^{-1}\frac{\partial\omega}{\partial y}$, $v_y = \Delta^{-1}\frac{\partial\omega}{\partial x}$ in Fourierspace
3. inverse Fourier transform: now we have $\mathbf{v}$, $\mathbf{B}$ ω, ψ and j in real space (at time n)
4. calculate nonlinearities: $\mathbf{v}\omega$, $\mathbf{v}\psi$, $\mathbf{B}j$
5. Fourier transform of nonlinearities, calculate $\nabla\cdot$ in Fourier space, calculate right hand side and Euler substep from Runge-Kutta SSP3.

Costs: 7 inverse FFTs, 6 direct FFTs

2.2 Velocity-magnetic field formulation

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla\cdot\mathbf{v}\mathbf{v} - \nabla\cdot\mathbf{B}\mathbf{B} = -\nabla p + \nu\Delta\mathbf{v} \ , \qquad \nabla\cdot\mathbf{v} = 0 \tag{9}$$

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla\cdot\mathbf{v}\mathbf{B} - \nabla\cdot\mathbf{B}\mathbf{v} = \eta\Delta\mathbf{B} \ , \qquad \nabla\cdot\mathbf{B} = 0 \tag{10}$$

Procedure similar as above, needs a projection step after each Euler step

Costs: 4 inverse FFTs, 10 direct FFTs

2.3 Velocity-stream function formulation

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla \cdot \mathbf{v}\mathbf{v} - \nabla \cdot \mathbf{B}\mathbf{B} = -\nabla p + \nu \Delta \mathbf{v} \ , \qquad \nabla \cdot \mathbf{v} = 0 \tag{11}$$

$$\frac{\partial \psi}{\partial t} + \nabla \cdot (\mathbf{v}\psi) = \eta \Delta \psi \tag{12}$$

Costs: 5 inverse FFTs, 8 direct FFTs

2.4 Elsässer Variables

$$\partial_t \mathbf{z}^{\pm} + \mathbf{z}^{\mp} \cdot \nabla \mathbf{z}^{\pm} = -\boldsymbol{\nabla} p + \nu_+ \nabla^2 \mathbf{z}^{\pm} + \nu_- \nabla^2 \mathbf{z}^{\mp} \ , \qquad \nabla \cdot \mathbf{z}^{\pm} = 0 \tag{13}$$

with $\nu_{\pm} = \frac{1}{2}(\nu \pm \eta)$

or better

$$\partial_t \mathbf{z}^{\pm} + \nabla \cdot \mathbf{z}^{\mp} \mathbf{z}^{\pm} = -\boldsymbol{\nabla} p + \nu_+ \nabla^2 \mathbf{z}^{\pm} + \nu_- \nabla^2 \mathbf{z}^{\mp} \ , \qquad \nabla \cdot \mathbf{z}^{\pm} = 0 \tag{14}$$

Costs: 4 inverse FFTs, 4 direct FFTs

and the winner is ?

Suggestion for 2D MHD:

Ensure: $\hat{\omega}^{\pm}$ at time n

$\hat{\mathbf{z}}^{\pm} \leftarrow \hat{\omega}^{\pm}$: $\hat{z}_x^{\pm} = \frac{ik_y}{k^2}\hat{\omega}^{\pm}$, $\hat{z}_y^{\pm} = \frac{-ik_x}{k^2}\hat{\omega}^{\pm}$

iFFT: $\mathbf{z}^{\pm} \leftarrow \hat{\mathbf{z}}^{\pm}$

calculate nonlinearities: $z_x^+ z_x^-, z_x^+ z_y^-, z_y^+ z_x^-, z_y^+ z_y^-$

FFT: $\widehat{z_x^+ z_x^-}, \widehat{z_x^+ z_y^-}, \widehat{z_y^+ z_x^-}, \widehat{z_y^+ z_y^-} \leftarrow z_x^+ z_x^-, z_x^+ z_y^-, z_y^+ z_x^-, z_y^+ z_y^-$

calculate $\nabla \times \nabla \cdot \mathbf{z}^- \mathbf{z}^+$ and $\nabla \times \nabla \cdot \mathbf{z}^+ \mathbf{z}^-$ in Fourier space:

$$i\mathbf{k} \times i\mathbf{k} \cdot \widehat{\mathbf{z}^- \mathbf{z}^+} = +k_x k_y \widehat{z_x^- z_x^+} + k_y k_y \widehat{z_y^- z_x^+} - k_x k_x \widehat{z_x^- z_y^+} - k_x k_y \widehat{z_y^- z_y^+}$$

$$i\mathbf{k} \times i\mathbf{k} \cdot \widehat{\mathbf{z}^+ \mathbf{z}^-} = +k_x k_y \widehat{z_x^- z_x^+} + k_y k_y \widehat{z_x^- z_y^+} - k_x k_x \widehat{z_y^- z_x^+} - k_x k_y \widehat{z_y^- z_y^+}$$

calculate right hand side and then Euler substep from Runge-Kutta SSP3

Closure Theories

Symbolic notation:

$$\partial_t u(k) = uu - \nu k^2 u(k)$$

averaging and since $\langle u \rangle = 0 \Longrightarrow 0 = 0$

We want to calculate spectral tensor $\langle u(k)u(k')\rangle$:

$$\left(\partial_t + \nu\left(k^2 + k^2\right)\right)\left\langle u(k)u\left(k'\right)\right\rangle = \langle uuu \rangle$$

not closed: need evolution of $\langle uuu \rangle$

$$\left(\partial_t + \nu\left(k^2 + p^2 + q^2\right)\right)\langle u(k)u(p)u(q)\rangle = \langle uuuu \rangle$$

also not closed: what's next ?

Try to manipulate $\langle uuuu \rangle$!

Closure Theories

proposed by Chou (1940), Millionshtchikov (1941
worked out by Proudman and Reid (1954), Tatsumi (1957)

Consider Gaussian random functions:

Let X be a four-dimensional variable (3 space +1 time), and let $g(X)$ be a random function of X (g might also be a vector) of zero mean. g is a Gaussian random function if, given N arbitrary numbers α_i and N values X_i of X, the linear combination $\sum \alpha_i X_i$ is a Gaussian random variable.

If $g(X)$ is a Gaussian random variable, then

- the odd moments of g are zero. (Now you understand why we need the fourth-order moment.)
- the even moments can be expressed in terms of the second-order moments,e.g.

$$\begin{aligned}\left\langle g\left(X_1\right) g\left(X_2\right) g\left(X_3\right) g\left(X_4\right)\right\rangle = &\left\langle g\left(X_1\right) g\left(X_2\right)\right\rangle\left\langle g\left(X_3\right) g\left(X_4\right)\right\rangle \\ &+\left\langle g\left(X_1\right) g\left(X_3\right)\right\rangle\left\langle g\left(X_2\right) g\left(X_4\right)\right\rangle \\ &+\left\langle g\left(X_1\right) g\left(X_4\right)\right\rangle\left\langle g\left(X_2\right) g\left(X_3\right)\right\rangle\end{aligned}$$

Derivations from the Gaussian behavior are captured in the cummulants.

Closure Theories

Idea of the Quasi-Normal approximation (Q.N.):

replace $\langle uuuu\rangle$ by the Gaussian values $\sum \langle uu\rangle\langle uu\rangle$

$$\left(\partial_t + \nu\left(k^2 + p^2 + q^2\right)\right)\langle u(k)u(p)u(q)\rangle = \sum \langle uu\rangle\langle uu\rangle$$

integrate in time and plug in equation for the spectral tensor:

$$\left(\partial_t + 2\nu k^2\right) U_{ij}(k,t) = \int_0^t d\tau \int_{p+q=k} dp \exp\left[-\nu\left(k^2 + p^2 + q^2\right)(t-\tau)\right]$$

isotropic turbulence without helicity:

$$\left(\partial_t + 2\nu k^2\right) E(k,t) = \int_0^t d\tau \int_{p+q=k} dp \exp\left[-\nu\left(k^2 + p^2 + q^2\right)(t-\tau)\right]$$

where $b(k,p,q)$ is a known function.

Note: this is an easy equation !!!

Closure Theories

Numerical simulation of Ogura (1963):
negative energy spectra of large eddies $\Longrightarrow$ USELESS

Reason for failure identified by Orszag (1970, 1977):

Q.N. builds up to high third-order moments

fourth-order cummulants (discarded in Q.N.) provide damping to saturate third-order moments.

This is the motivation of the

Eddy-Damped Quasi-Normal type theories

Orszag (1970, 1977): approximate fourth-oder cumulants (neglected in Q.N.) by a linear damping term μ_{kpq}

$$\left(\partial_t + \nu\left(k^2 + p^2 + q^2\right) + \mu_{kpq}\right)\langle u(k)u(p)u(q)\rangle = \sum \langle uu\rangle\langle uu\rangle$$

where μ_{kpq} is a damping rate associated to the triad (k, p, q) having the dimension of an inverse time.

Closure Theories

Isotropic turbulence: $\mu_{kpq} = \mu_k + \mu_p + \mu_q$

Orszag (1970): $\mu_k \sim \left[k^3 E(k)\right]^{1/2}$

μ_k is kind of frequency, therefore should be higher for higher k

$\Longrightarrow$ Does not work for decaying turbulence where spectrum can be rapidly decreasing for large k and thus producing low frequency.

Much better ansatz: Frisch (1974) and Pouquet et al (1975)

$$\mu_k \sim \left[\int_0^k p^2 E(p,t) dp\right]^{1/2}$$

takes into account large eddies and produces always growing μ_k

Eddy-Damped Quasi-Normal approximation (E.D.Q.N)

$$\left(\partial_t + 2\nu k^2\right) U_{ij}(k,t) = \int_0^t d\tau \int_{p+q=k} dp \exp\left[-\mu_{kpq} - \nu\left(k^2 + p^2 + q^2\right)(t-\tau)\right] \sum \langle uu\rangle\langle uu\rangle(\tau)$$

E.D.Q.N. physically more acceptable, but does not guarantee realizability (positivity of energy spectrum)

Closure Theories

Orszag (1977): minor modification, called Markovianization

exponential term in integrand varies with much smaller time scale than nonlinear term $\sum \langle uu \rangle \langle uu \rangle$ of the order of large eddy turnover time

$\Longrightarrow$ change

$$\sum \langle uu \rangle \langle uu \rangle (\tau) \longrightarrow \sum \langle uu \rangle \langle uu \rangle (t)$$

Energy spectrum of Eddy-Damped Quasi-Normal Markovian approximation (E.D.Q.N.M)

$$\left(\partial_t + 2\nu k^2\right) E(k,t) = \int_{p+q=k} dp \theta_{kpq} \frac{k}{pq} b(k,p,q) \left[k^2 E(p,t) - p^2 E(k,t)\right] E(q,t)$$

with

$$\theta_{kpq} = \int_0^t \exp\left[-\left(\mu_{kpq} + \nu \left(k^2 + p^2 + q^2\right)\right)\right](t-\tau) d\tau$$

Orszag (1977): E.D.Q.N.M has positive energy spectrum

Renormalized Perturbation Theory

Perturbation expansion of the Navier-Stokes equation

Kraichnan (1959):	direct-interaction approximation
Wyld (1961):	partial summation for a simplified scalar model
Lee (1965):	partial summation for Navier-Stokes and MHD

The zero-order isotropic propagators

$$\left(\frac{\partial}{\partial t} + \nu k^2\right) u_\alpha(\mathbf{k}, t) = \lambda M_{\alpha\beta\gamma}(\mathbf{k}) \sum_j u_\beta(\mathbf{j}, t) u_\gamma(\mathbf{k} - \mathbf{j}, t) + D_{\alpha\beta}(\mathbf{k}) f_\beta(\mathbf{k}, t)$$

λ: bookkeeping parameter, $D_{\alpha\beta}$: projector on divergence free part

zero-order Green tensor: $$\left(\frac{\partial}{\partial t} + \nu k^2\right) G^{(0)}_{\alpha\beta}(\mathbf{k}; t, t') = D_{\alpha\beta}(\mathbf{k}) \delta(t - t')$$

with $$G^{(0)}_{\alpha\beta}(\mathbf{k}; t, t') = D_{\alpha\beta}(\mathbf{k}) G_0\left(k; t, t'\right)$$

so that $$\left(\frac{\partial}{\partial t} + \nu k^2\right) G_0\left(k; t, t'\right) = \delta(t - t')$$

Renormalized Perturbation Theory

zero-order velocity:

$$u_\alpha^{(0)}(\mathbf{k},t) = \int \mathrm{d}t' G_{\alpha\beta}^{(0)}(\mathbf{k};t,t') f_\beta(\mathbf{k},t') = D_{\alpha\beta}(\mathbf{k}) \int \mathrm{d}t' G_0\left(k;t,t'\right) f_\beta(\mathbf{k},t')$$

Fourier transform in time:

$$u_\alpha(\mathbf{x},t) = \sum_{\mathbf{k}} \sum_{\omega} u_\alpha(\mathbf{k},\omega) \exp\{i\mathbf{k}\cdot\mathbf{x} + i\omega t\}$$

$\Longrightarrow$ Navier-Stokes

$$\left(\mathrm{i}\omega + \nu k^2\right) u_\alpha(\mathbf{k},\omega) = \lambda M_{\alpha\beta\gamma}(\mathbf{k}) \sum_{\mathbf{j}} \sum_{\omega'} u_\beta\left(\mathbf{j},\omega'\right) u_\gamma\left(\mathbf{k}-\mathbf{j},\omega-\omega'\right) + D_{\alpha\beta}(\mathbf{k}) f_\beta(\mathbf{k},\omega)$$

zero-order propagator is given by

$$G_0(\mathbf{k},\omega) = \frac{1}{\mathrm{i}\omega + \nu k^2}$$

The primitive perturbation expansion

Stirring force $$\left(\frac{L}{2\pi}\right)^3 \left\langle f_\alpha(\mathbf{k},t) f_\beta(-\mathbf{k},t')\right\rangle = D_{\alpha\beta}(\mathbf{k}) w\left(k;t,t'\right)$$

after Fourier-trafo $$\left(\frac{L}{2\pi}\right)^3 \left(\frac{T}{2\pi}\right) \left\langle f_\alpha(\mathbf{k},\omega) f_\beta(-\mathbf{k},\omega')\right\rangle = D_{\alpha\beta}(\mathbf{k}) w\left(k;\omega,\omega'\right)$$

zero-order velocity field after in Fourier-space $$u_\alpha^{(0)}(\mathbf{k},\omega) = D_{\alpha\beta}(\mathbf{k}) G_0(k,\omega) f_\beta(\mathbf{k},\omega)$$

which is the zero-order term in expansion

$$u_\alpha(\mathbf{k},\omega) = u_\alpha^{(0)}(\mathbf{k},\omega) + \lambda u_\alpha^{(1)}(\mathbf{k},\omega) + \lambda^2 u_\alpha^{(2)}(\mathbf{k},\omega) + \cdots + -\lambda^n u_\alpha^{(n)}(\mathbf{k},\omega) + \cdots$$

correlation $Q_{\alpha\beta}(\mathbf{k};\omega,\omega')$

$$\left\langle u_\alpha^{(0)}(\mathbf{k},\omega) u_\beta^{(0)}(-\mathbf{k},\omega')\right\rangle = D_{\alpha\gamma}(\mathbf{k}) D_{\beta\sigma}(-\mathbf{k}) G_0(k,\omega) G_0\left(-k,\omega'\right) \left\langle f_\gamma(\mathbf{k},\omega) f_\sigma(-\mathbf{k},\omega')\right\rangle$$

$$= D_{\alpha\gamma}(\mathbf{k}) D_{\beta\sigma}(\mathbf{k}) D_{\gamma\sigma}(\mathbf{k}) G_0(k,\omega) G_0\left(k,\omega'\right) \left(\frac{2\pi}{L}\right)^3 w\left(k;\omega,\omega'\right)$$

$$= \left(\frac{2\pi}{L}\right)^3 D_{\alpha\beta}(\mathbf{k}) G_0(\mathbf{k},\omega) G_0\left(k,\omega'\right) w\left(k;\omega,\omega'\right)$$

Renormalized Perturbation Theory

homogeneous isotropic turbulence

$$\left(\frac{L}{2\pi}\right)^3 \left\langle u_\alpha^{(0)}(\mathbf{k},\omega) u_\beta^{(0)}(-\mathbf{k},\omega')\right\rangle = D_{\alpha\beta}(\mathbf{k}) Q_0\left(k;\omega,\omega'\right)$$

and

$$Q_0\left(k;\omega,\omega'\right) = G_0(k,\omega) G_0\left(k,\omega'\right) w\left(k;\omega,\omega'\right)$$

Perturbation expansion for Q

$$D_{\alpha\beta}(\mathbf{k}) Q\left(k;\omega,\omega'\right) = \left(\frac{2\pi}{L}\right)^3 \left\langle u_\alpha(\mathbf{k},\omega) u_\beta(-\mathbf{k},\omega')\right\rangle$$

$$= \left(\frac{2\pi}{L}\right)^3 \left\{ \left\langle u_\alpha^{(0)}(\mathbf{k},\omega) u_\beta^{(0)}(-\mathbf{k},\omega')\right\rangle + \right.$$

$$\left. + \lambda^2 \left[\left\langle u_\alpha^{(0)} u_\beta^{(2)} \right\rangle + \left\langle u_\alpha^{(1)} u_\beta^{(1)} \right\rangle + \left\langle u_\alpha^{(2)} u_\beta^{(0)} \right\rangle \right] + O\left(\lambda^4\right) \right\}$$

$$= D_{\alpha\beta}(\mathbf{k}) Q_0\left(k;\omega,\omega'\right) + \left(\frac{2\pi}{L}\right)^3 \left\{ \lambda^2 \left[\left\langle u_\alpha^{(0)} u_\beta^{(2)} \right\rangle + \right.\right.$$

$$\left.\left. + \left\langle u_\alpha^{(1)} u_\beta^{(1)} \right\rangle + \left\langle u_\alpha^{(2)} u_\beta^{(0)} \right\rangle \right] + O\left(\lambda^4\right) \right\}$$

Renormalized Perturbation Theory

Take Fourier transformed NS-equation, invert linear part and substitute

forcing by $u^{(0)}$

$$u_\alpha(\mathbf{k},\omega) = u_\alpha^{(0)}(\mathbf{k},\omega) + \lambda G_0(k,\omega) M_{\alpha\beta\gamma}(\mathbf{k}) \sum_{\mathbf{j}} \sum_{\omega'} u_\beta\left(\mathbf{j},\omega'\right) u_\gamma\left(\mathbf{k}-\mathbf{j},\omega-\omega'\right)$$

Now substitute perturbation expansion for u

$$\begin{aligned} & u_\alpha^{(0)}(\mathbf{k},\omega) + \lambda u_\alpha^{(1)}(\mathbf{k},\omega) + \lambda^2 u_\alpha^{(2)}(\mathbf{k},\omega) + \cdots \\ & \quad = u_\alpha^{(0)}(\mathbf{k},\omega) + \lambda G_0(k,\omega) M_{\alpha\beta\gamma}(\mathbf{k}) \times \sum_{\mathbf{j}} \sum_{\omega'} \left\{ u_\beta^{(0)}\left(\mathbf{j},\omega'\right) u_\gamma^{(0)}\left(\mathbf{k}-\mathbf{j},\omega-\omega'\right) \right. \\ & \quad \left. +\lambda \left[u_\beta^{(0)}\left(\mathbf{j},\omega'\right) u_\gamma^{(1)}\left(\mathbf{k}-\mathbf{j},\omega-\omega'\right) + u_\beta^{(1)}\left(\mathbf{j},\omega'\right) u_\gamma^{(0)}\left(\mathbf{k}-\mathbf{j},\omega-\omega'\right) \right] + O\left(\lambda^2\right) \right\} \end{aligned}$$

equating coefficients:

$$u_\alpha^{(1)}(\mathbf{k},\omega) = G_0(k,\omega) M_{\alpha\beta\gamma}(\mathbf{k}) \sum_{\mathbf{j}} \sum_{\omega'} u_\beta^{(0)}\left(\mathbf{j},\omega'\right) u_\gamma^{(0)}\left(\mathbf{k}-\mathbf{j},\omega-\omega'\right)$$

$$u_\alpha^{(2)}(\mathbf{k},\omega) = 2G_0(k,\omega) M_{\alpha\beta\gamma}(\mathbf{k}) \sum_{\mathbf{j}} \sum_{\omega'} u_\beta^{(1)}\left(\mathbf{j},\omega'\right) u_\gamma^{(0)}\left(\mathbf{k}-\mathbf{j},\omega-\omega'\right)$$

Renormalized Perturbation Theory

$u^{(2)}$ can be expressed by $u^{(0)}$

$$u_\alpha^{(2)}(\mathbf{k},\omega) = 2G_0(k,\omega)M_{\alpha\beta\gamma}(\mathbf{k})\sum_{\mathbf{j}}\sum_{\omega'} G_0\left(\mathbf{j},\omega'\right)M_{\beta\rho\sigma}(\mathbf{j}) \times$$

$$\times \sum_{\mathbf{l}}\sum_{\omega''} u_\rho^{(0)}\left(\mathbf{l},\omega''\right) u_\sigma^{(0)}\left(\mathbf{j}-\mathbf{l},\omega'-\omega''\right) u_\gamma^{(0)}\left(\mathbf{k}-\mathbf{j},\omega-\omega'\right)$$

stirring Gaussian $\implies u^{(0)}$ Gaussian

$\implies \left\langle u^{(0)}u^{(0)}u^{(0)}u^{(0)}\right\rangle$ can be factored as in Quasi-Normal approximation

Renormalized Perturbation Theory

second-order correlation tensor:

$$
\begin{aligned}
D_{\alpha\beta}(\mathbf{k})Q\left(k;\omega,\omega'\right) &= D_{\alpha\beta}(\mathbf{k})Q_0\left(k;\omega,\omega'\right)+\\
&+\lambda^2\Big[4G_0\left(k,\omega'\right)M_{\beta\delta\gamma}(-\mathbf{k})\int \mathrm{d}^3j\int \mathrm{d}\omega''\int \mathrm{d}\omega'''G_0\left(j,\omega''\right)\times\\
&\times M_{\delta\rho\sigma}(\mathbf{j})D_{\alpha\rho}(\mathbf{k})D_{\gamma\sigma}(\mathbf{k}+\mathbf{j})\times Q_0\left(k;\omega,\omega'''\right)Q_0\left(|\,\mathbf{k}+\mathbf{j}\,|;\omega'-\omega'',\omega''-\omega'''\right)+\\
&\quad +2G_0(k,\omega)M_{\alpha\delta\gamma}(\mathbf{k})\int \mathrm{d}^3j\int \mathrm{d}\omega''\int \mathrm{d}\omega'''G_0\left(k,\omega'\right)\times\\
&\times M_{\beta\rho\sigma}(-\mathbf{k})D_{\delta\sigma}(\mathbf{j})D_{\gamma\rho}(\mathbf{k}-\mathbf{j})\times Q_0\left(|\,\mathbf{k}-\mathbf{j}\,|;\omega-\omega'',\omega'''\right)Q_0\left(j;\omega'',\omega'-\omega'''\right)+\\
&\quad +4G_0(k,\omega)M_{\alpha\delta\gamma}(\mathbf{k})\int \mathrm{d}^3j\int \mathrm{d}\omega''\int \mathrm{d}\omega'''G_0\left(j,\omega''\right)\times\\
&\times M_{\delta\rho\sigma}(\mathbf{j})D_{\beta\rho}(\mathbf{k})D_{\sigma\gamma}(\mathbf{k}-\mathbf{j})\times Q_0\left(k;\omega',\omega'''\right)Q_0\left(|\,\mathbf{k}-\mathbf{j}\,|;\omega-\omega'',\omega''-\omega'''\right)\Big]+O\left(\lambda^4\right)
\end{aligned}
$$

Renormalized Perturbation Theory

Graphical representation of the perturbation series

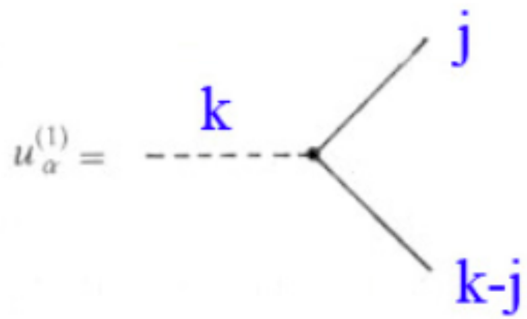
all orders can be expressed by zero-order terms, but possible divergent series

three main constituents: $u^{(0)}$, G_0 and M

full line	$\leftrightarrow u^{(0)}$
broken line	$\leftrightarrow G_0$
point (vertex)	$\leftrightarrow M$

zero-order: $u_\alpha^{(0)}(\mathbf{k}, t) =$ 

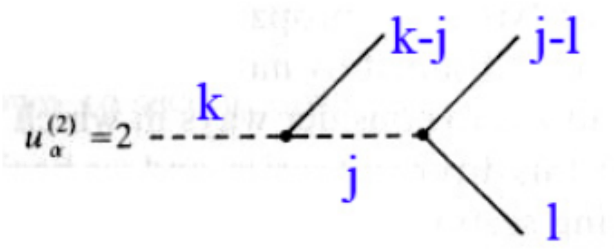
first-order: wavenumber conservation



$$u_\alpha^{(1)}(\mathbf{k}, \omega) = G_0(k, \omega) M_{\alpha\beta\gamma}(\mathbf{k}) \sum_{\mathbf{j}} \sum_{\omega'} u_\beta^{(0)}\left(\mathbf{j}, \omega'\right) u_\gamma^{(0)}\left(\mathbf{k} - \mathbf{j}, \omega - \omega'\right)$$

Renormalized Perturbation Theory

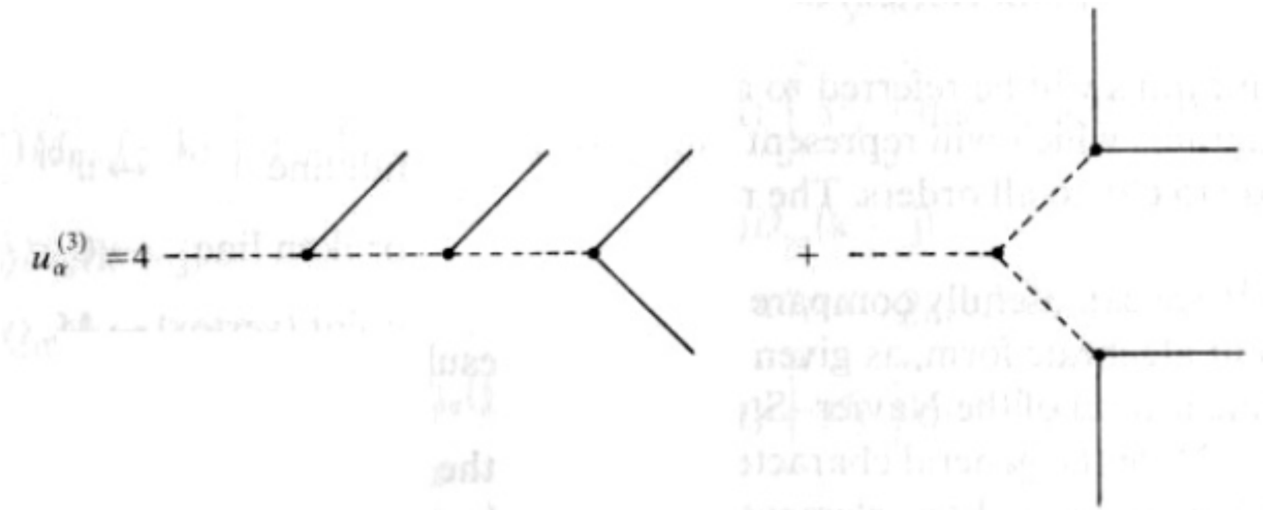
second-order: two M factors:



$$u_\alpha^{(2)}(\mathbf{k},\omega) = 2G_0(k,\omega)M_{\alpha\beta\gamma}(\mathbf{k})\sum_{\mathbf{j}}\sum_{\omega'} G_0\left(\mathbf{j},\omega'\right) M_{\beta\rho\sigma}(\mathbf{j}) \times$$

$$\times \sum_{\mathbf{l}}\sum_{\omega''} u_\rho^{(0)}\left(\mathbf{l},\omega''\right) u_\sigma^{(0)}\left(\mathbf{j}-\mathbf{l},\omega'-\omega''\right) u_\gamma^{(0)}\left(\mathbf{k}-\mathbf{j},\omega-\omega'\right)$$

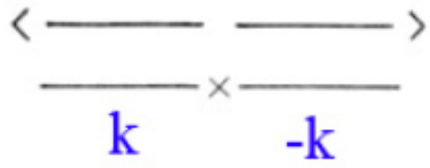
third-order: three M factors:



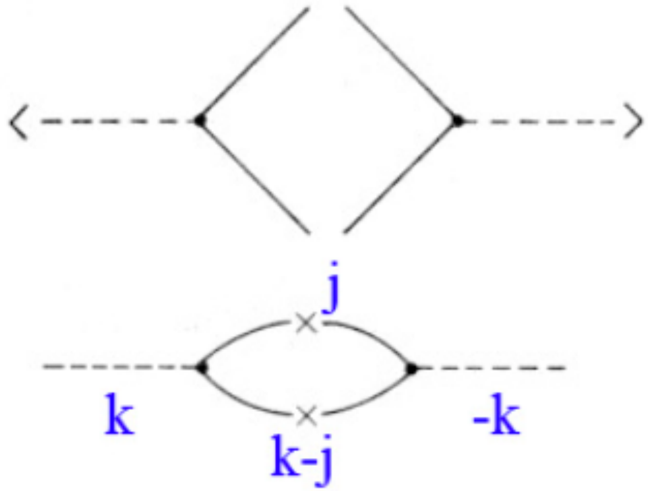
Renormalized Perturbation Theory

graphical expansion for correlation tensor

zero-order: $\langle u_\alpha^{(0)}(\mathbf{k})\, {}_\beta^{(0)}(-\mathbf{k})\rangle =$ $=$



second order: $\langle u_\alpha^{(1)}(\mathbf{k})\, u_\beta^{(1)}(-\mathbf{k})\rangle =$ $= 2$

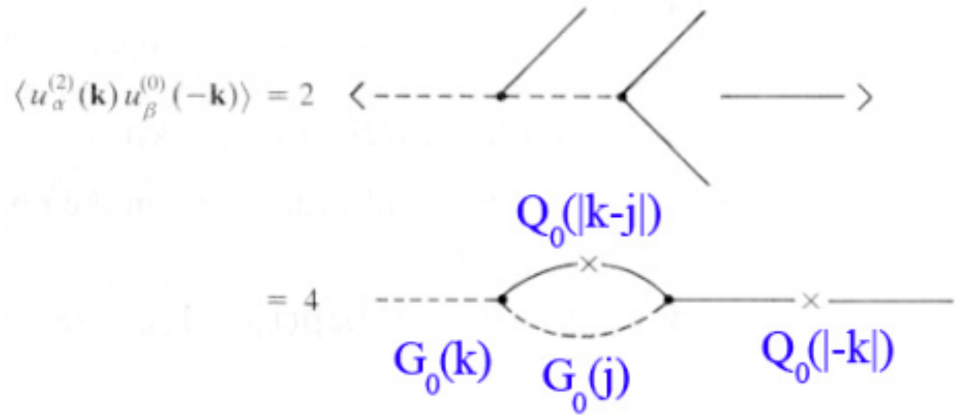


this is middle second-order term:

$$+2G_0(k,\omega)M_{\alpha\delta\gamma}(\mathbf{k})\int \mathrm{d}^3 j \int \mathrm{d}\omega'' \int \mathrm{d}\omega''' G_0\left(k,\omega'\right) \times$$

$$\times M_{\beta\rho\sigma}(-\mathbf{k})D_{\delta\sigma}(\mathbf{j})D_{\gamma\rho}(\mathbf{k}-\mathbf{j}) \times Q_0\left(|\mathbf{k}-\mathbf{j}|;\omega-\omega'',\omega'''\right) Q_0\left(j;\omega'',\omega'-\omega'''\right)+$$

Renormalized Perturbation Theory



this is the last second-order term:

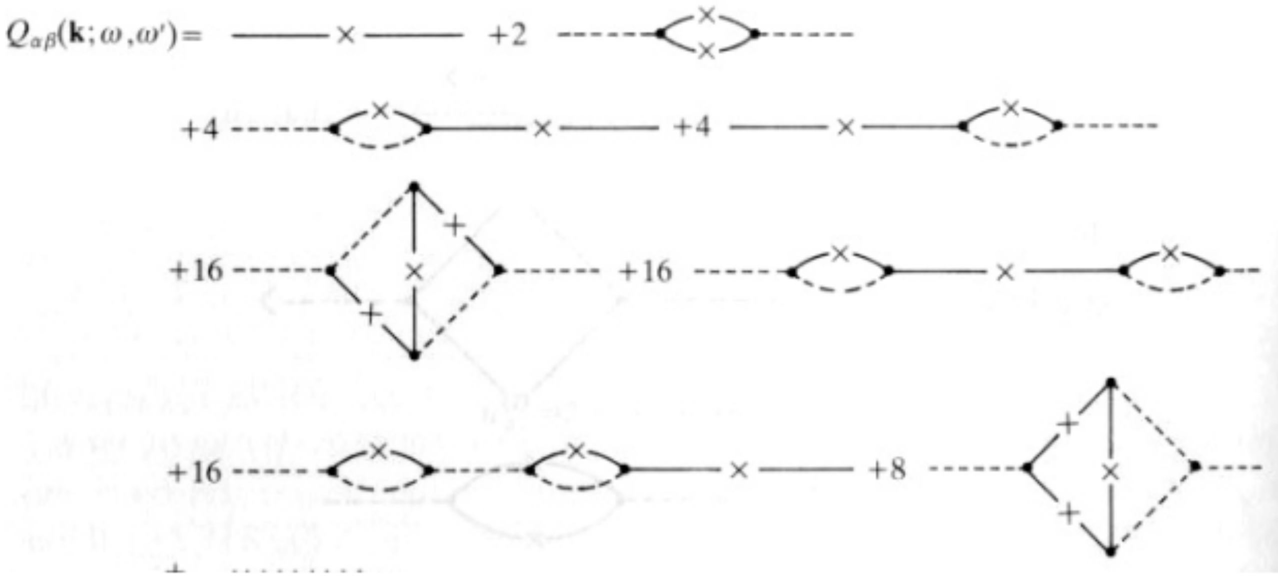
$$+4G_0(k,\omega)M_{\alpha\delta\gamma}(\mathbf{k})\int \mathrm{d}^3 j \int \mathrm{d}\omega'' \int \mathrm{d}\omega''' G_0\left(j,\omega''\right) \times$$

$$\times\, M_{\delta\rho\sigma}(\mathbf{j}) D_{\beta\rho}(\mathbf{k}) D_{\sigma\gamma}(\mathbf{k}-\mathbf{j}) \times Q_0\left(k;\omega',\omega'''\right) Q_0\left(\,|\,\mathbf{k}-\mathbf{j}\,|\,;\omega-\omega'',\omega''-\omega'''\right)\Big]$$

The third is a mirror image of this one.

Renormalized Perturbation Theory

fourth-order showing four of the 29 fourth-order diagrams:



Now **resummation** (renormalisation): new diagram elements

thick full line	$\leftrightarrow$ u (exact velocity field)
thick broken line	$\leftrightarrow$ G (renormalized propagator)
open circle	$\leftrightarrow$ (renormalized vertex)

Write correlation tensor as:

$$Q_{\alpha\beta}(\mathbf{k};\omega,\omega') = \underbrace{Q_{\alpha\beta}(\mathbf{k};\omega,\omega')_{\mathrm{A}}}_{\text{Class A diagrams}} + \underbrace{Q_{\alpha\beta}(\mathbf{k};\omega,\omega')_{B}}_{\text{Class B diagrams}}$$

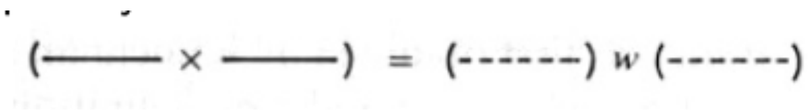
Renormalized Perturbation Theory

Class A diagram: the renormalized propagator

Wyld (1961): Class A diagrams are those diagrams which can be split into two pieces by cutting a single Q_0 line.

zero-order: Q_0 can be expressed in terms of two zero-order propagators acting on the spectrum of the stirring forces $w\left(k;\omega,\omega'\right)$

This looks graphically like



Now second order:

Renormalized Perturbation Theory

Let's summarize: at zero order, we have w with a G_0 on each side. At second order, w has a G_0 on one side and a diagram which connects like a G_0 on the other. This holds for all orders. Thus we have a generalization

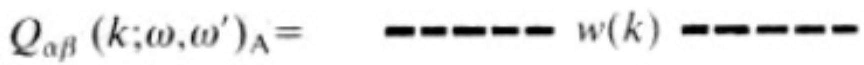
$$Q_0\left(k;\omega,\omega'\right) = G_0(k,\omega)G_0\left(k,\omega'\right)w\left(k;\omega,\omega'\right)$$

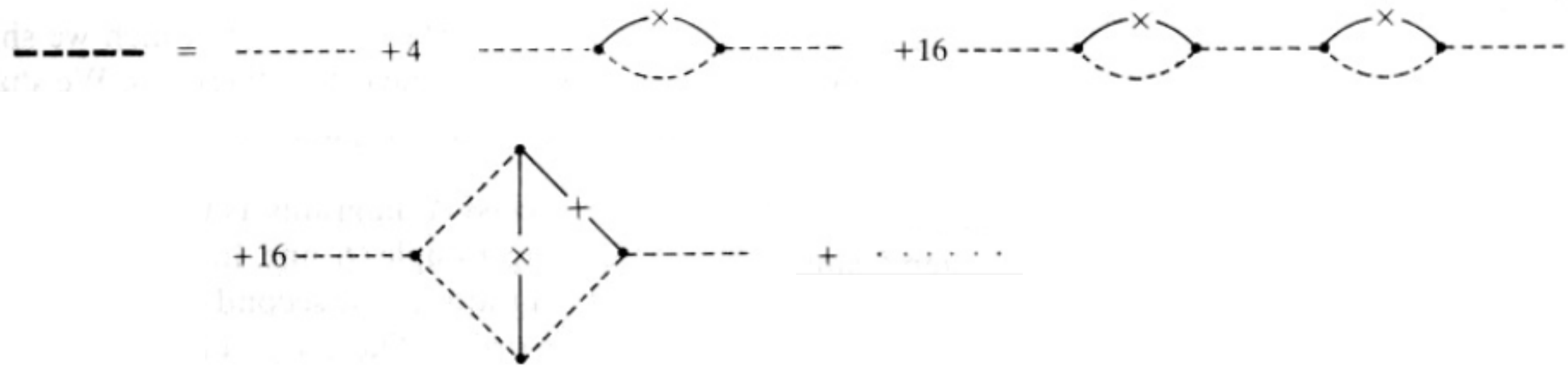
which reads

$$Q\left(k;\omega,\omega'\right)_{\mathrm{A}} = G(k,\omega)G\left(k,\omega'\right)w\left(k;\omega,\omega'\right)$$

where $G(k,\omega)$ is the renormalized propagator.

Graphically, this corresponds to





Renormalized Perturbation Theory

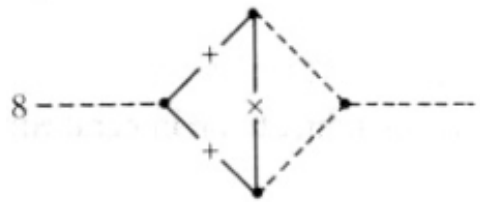
Class B diagrams: renormalized perturbation series

Class B diagrams can't be split into two by cutting a single Q_0 line.

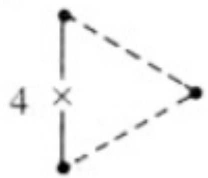
In the class A diagrams, certain diagram parts were propagator like, that is, they connected like G_0 : renormalize G_0 by adding up all diagrams which connect like G_0.

Renormalize vertex: add up all diagrams which connect like a vertex

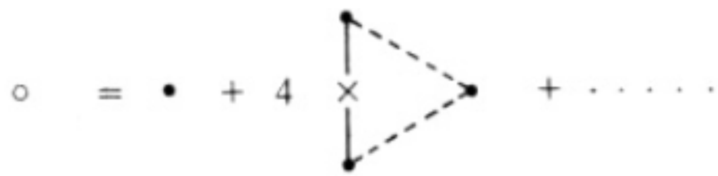
Example: consider fourth-order diagram



The part

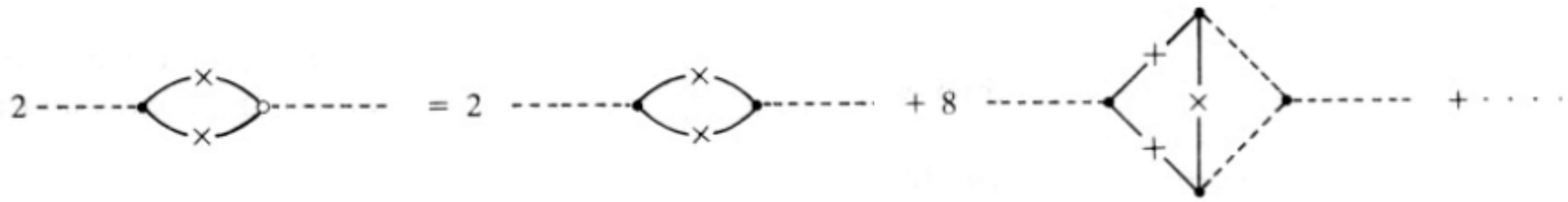


connects like a point vertex $\Longrightarrow$ renormalized vertex



Renormalized Perturbation Theory

replace vertex by renormalized vertex:



Therefore the key to the class B diagrams is as follows:

1. Find those diagrams which cannot be reduced to a lower order by replacing diagram parts.

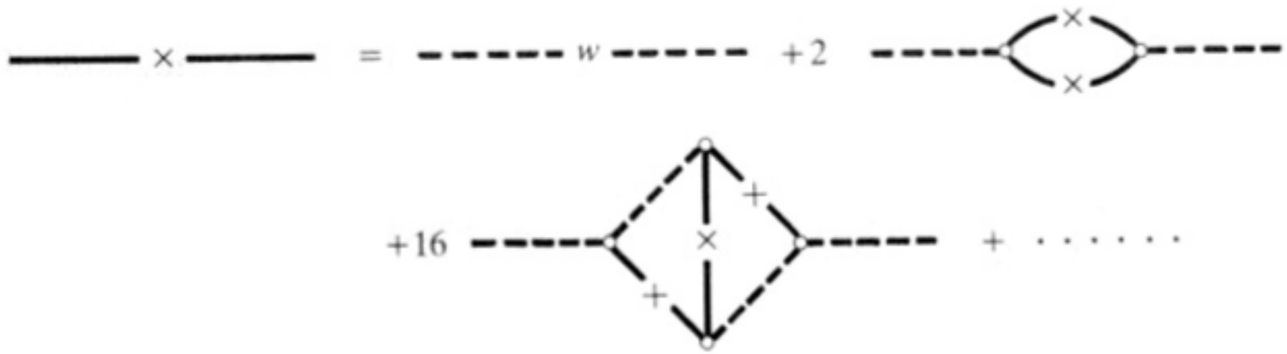
2. Call these the irreducible diagrams.

3. Replace all elements in the irreducible diagrams by their renormalized forms.

4. Write down all these modified diagrams in order, thus generating a *renormalized* perturbation expansion.

Renormalized Perturbation Theory

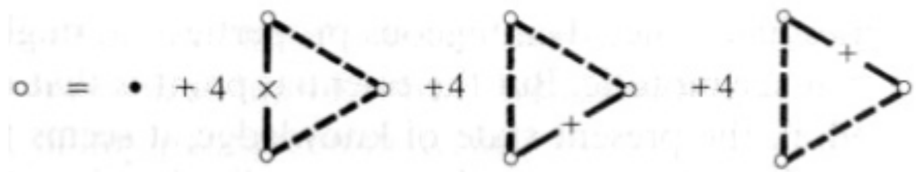
Result for $Q\left(k;\omega,\omega'\right)$



This is an integral equation for $Q(k;\omega,\omega)$

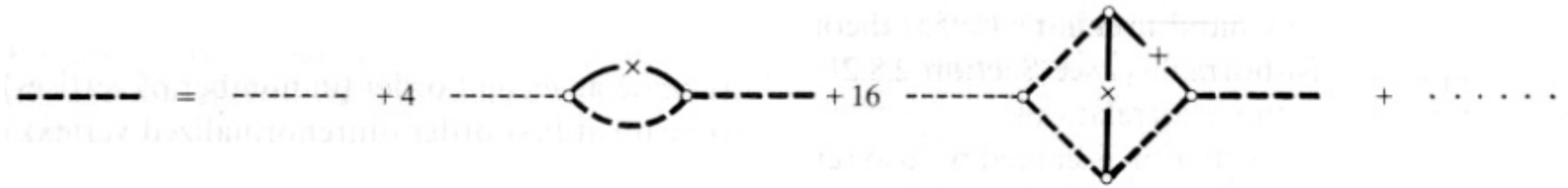
<u>Combine vertex and propagator expansions:</u>

Integral equation for the renormalized vertex



Renormalized Perturbation Theory

Integral equation for the renormalized propagator $G(k,\omega)$



Pecularity of this diagram:

unrenormalized propagator emerging from the left !!!

Reason for this: symbolic form of Navier-Stokes

$$L_0 u(k) = \lambda M(k) u(j) u(k-j), \quad L_0 = \partial_t + \nu k^2$$

and renormalize r.h.s., then invert L_0 which results in G_0

Second-order closures

We replaced a wildly divergent series with one of unknown properties !

We have hope that it might be assymptotic, but we simple don't know !

Well known examples recovered from this Wyld (1961) formulation:

Renormalized Perturbation Theory

Example 1:

correlation tensor:	truncate at second order	(in number vertices)
vertex:	truncate at first order	(unrenormalized vertex)
propagator:	truncate at zero order	(unrenormalized propagator)

This is Chandrasekhar's theory (1955) which is the two-time analog of quasi-normality.

Example 2:

correlation tensor:	truncate at second order
vertex:	truncate at first order
propagator:	truncate at second order

This is the pioneering direct-interaction approximation (DIA) by Kraichnan (1959): second-order closure with line and with no vertex renormalization.