

# Math Note

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This paper covers several topics in undergraduate mathematics.

# Chapter 1

## Set Theory

### 1.1 Map

**Definition 1.** Let  $X, Y$  are sets. Define a *function*  $X$  to  $Y$  is a relation

$$f \subset X \times Y$$

such that

1. For any  $x \in X$ , there exists  $y \in Y$  such that  $(x, y) \in f$ .
2. If  $(x, y) \in f$  and  $(x, z) \in f$ , then  $y = z$ .

Denote  $f$  as:

$$f : X \rightarrow Y : x \mapsto f(x)$$

Define *Image* of  $f$  by  $A \subset X$ :

$$f[A] \stackrel{\text{def}}{=} \{f(a) \mid a \in A\} \subset Y$$

And, *Preimage* of  $f$  by  $B \subset Y$ :

$$f^{-1}[B] \stackrel{\text{def}}{=} \{x \in X \mid f(x) \in B\} \subset X$$

$f : X \rightarrow Y$  is *Injective* if:  $f(x_1) = f(x_2) \implies x_1 = x_2$ .

$f : X \rightarrow Y$  is *Surjective* if:  $\forall y \in Y, \exists x \in X$  s.t.  $f(x) = y$ .

If  $f$  is injective and surjective, called *bijective*.

If  $f$  is bijective, then define *inverse* of  $f$  as:

$$f^{-1} : Y \rightarrow X : y \mapsto x$$

where  $x \in X$  is the unique elements of  $X$  such that  $f(x) = y$ .

**Theorem 1.** Let  $f : X \rightarrow Y$  be a function. Then,

1. There exists  $g : Y \rightarrow X$  such that  $g \circ f : X \rightarrow X$  be an identity function if and only if  $f$  is injective.
2. There exists  $h : Y \rightarrow X$  such that  $f \circ h : Y \rightarrow Y$  be an identity function if and only if  $f$  is surjective.

*Proof.*

1.  $\implies$  )

Assume that  $f(x_1) = f(x_2)$ . Then, existence of left inverse,  $g(f(x_1)) = g(f(x_2)) \implies x_1 = x_2$ . Thus  $f$  injective.

1.  $\impliedby$  )

Since  $f$  is injection, for any  $y \in f[X]$ , there exists a unique element  $x_y \in X$  such that  $f(x) = y$ . Now, define

$$g : Y \rightarrow X : y \mapsto \begin{cases} x_y & y \in f[X] \\ \text{any element in } X & y \notin f[X] \end{cases}$$

Then, for any  $x \in X$ ,  $g(f(x)) = g(y) = x$ .

2.  $\implies$  )

Let  $y \in Y$  be given. Since existence of right inverse,  $f(h(y)) = y$  where  $h(y) \in X$ . Thus,  $f$  is surjective.

2.  $\impliedby$  )

For any  $y \in Y$ , there exists a  $x_y \in X$  such that  $f(x_y) = y$ . Now, define

$$h : Y \rightarrow X : y \mapsto x_y$$

Then, for any  $y \in Y$ ,  $f \circ h(y) = f(x_y) = y$ . Thus,  $f \circ h$  is identity. □

**Corollary 1.** Let  $f : X \rightarrow Y$  be a function,  $\text{id}_X : X \rightarrow X : x \mapsto x$ , and  $\text{id}_Y : Y \rightarrow Y : y \mapsto y$ .

There exists a  $f^{-1} : Y \rightarrow X$  such that  $f^{-1} \circ f = \text{id}_X$  and  $f \circ f^{-1} = \text{id}_Y$  if and only if  $f$  is bijection.

*Proof.* If  $f$  is bijection, then there exists left inverse  $g$  and right inverse  $h$ .

Enough To Show that:  $g = h$ . Since  $g \circ f = \text{id}_X$  and  $f \circ h = \text{id}_Y$ ,

$g \circ f \circ h = g \circ \text{id}_Y$ , thus  $h = g$ . □

**Theorem 2.** Let  $X, Y, Z$  are sets,  $f : X \rightarrow Y$ ,  $g : Y \rightarrow Z$  and  $A \subset X, B \subset Y, C \subset Z$ . Then followings are hold:

1.  $g[f[A]] = (g \circ f)[A]$ .
2.  $f^{-1}[g^{-1}[C]] = (g \circ f)^{-1}[C]$ .

*Proof.*

1. It is clear by definition of image:

$$\begin{aligned} g[f[A]] &\stackrel{\text{def}}{=} g[\{f(a) \mid a \in A\}] = \{g(b) \mid b \in \{f(a) \mid a \in A\}\} \\ &= \{g(b) \mid b = f(a) \text{ for some } a \in A\} = \{g(f(a)) \mid \text{for some } a \in A\} = \{g(f(a)) \mid a \in A\} \end{aligned}$$

2. It is not clear,

$$f^{-1}[g^{-1}[C]] \stackrel{\text{def}}{=} f^{-1}[\{b \in Y \mid g(b) \in C\}] = \{a \in X \mid f(a) \in \{b \in Y \mid g(b) \in C\}\} = \{a \in X \mid g(f(a)) \in C\} = (g \circ f)^{-1}[C]$$

□

**Proposition 1.** Let  $f : X \rightarrow Y$  be a function,  $A, B \subset X$  and  $C, D \subset Y$ .

1. If  $A \subset B$ , then  $f[A] \subset f[B]$ .
2. If  $C \subset D$ , then  $f^{-1}[C] \subset f^{-1}[D]$

*Proof.*

$$\begin{aligned} y \in f[A] &\implies y = f(a) \text{ for some } a \in A \stackrel{A \subset B}{\implies} y = f(a) \text{ for some } a \in B \implies y \in f[B] \\ x \in f^{-1}[C] &\implies f(x) \in C \stackrel{C \subset D}{\implies} f(x) \in D \implies x \in f^{-1}[D] \end{aligned}$$

□

**Lemma 1.** Let two set  $X, Y$  be given, and  $A \subset X$ ,  $B \subset Y$ ,  $f: X \rightarrow Y$ . Then followings are holds:

1.  $f^{-1}[f[A]] \supseteq A$ , and equality holds if  $f$  one-to-one.
2.  $f[f^{-1}[B]] \subseteq B$ , and equality holds if  $f$  onto.
3.  $f^{-1}[Y \setminus B] = X \setminus f^{-1}[B]$
4.  $f[X] \setminus f[A] \subseteq f[X \setminus A]$ , and equality holds if  $f$  one-to-one.

**Proof.** Proof of 4.

$$\begin{aligned} y \in f[X] \setminus f[A] &\iff y \in f[X] \text{ and } y \notin f[A] \\ &\iff \exists x \in X \text{ s.t. } y = f(x) \text{ and } \forall x \in A, y \neq f(x) \\ &\stackrel{(*)}{\implies} \exists x \in X \setminus A \text{ s.t. } y = f(x) \\ &\iff y \in f[X \setminus A] \end{aligned}$$

If  $f$  is injection, then Left Direction of  $(*)$  be true:  $\exists! x \in X \setminus A$  s.t.  $y = f(x)$ . □

# Chapter 2

## Group Theory

### 2.1 Isomorphism Theorems

#### Theorem 3. The First Isomorphism Theorem

Let  $\varphi : G \rightarrow H$  be a Group-Homomorphism. Then,

$$G / \ker \varphi \cong \varphi[G]$$



**Proof.** Let  $\pi : G \rightarrow G/\ker \varphi : x \mapsto x + \ker \varphi$ . Then, the map  $\phi : G/\ker \varphi \rightarrow \varphi[G] : a + \ker \varphi \mapsto \varphi(a)$  is isomorphism. Well-defined and Injective:

$$a + \ker \varphi = b + \ker \varphi \iff a - b \in \ker \varphi \iff \varphi(a - b) = \varphi(a) - \varphi(b) = 0$$

Surjective is clear. □

#### Theorem 4. The Second Isomorphism Theorem

Let  $G$  be a Group, and  $H \leq G$ ,  $N \trianglelefteq G$ . Then,

$$HN/N \cong H/(H \cap N)$$

**Proof.**  $HN$  be a subgroup of  $G$ , being

$$HN = \bigcup_{h \in H} hN \stackrel{N \trianglelefteq G}{=} \bigcup_{h \in H} Nh = NH$$

And,  $N \leq HN$  is clear, thus  $N \trianglelefteq HN$ .

Meanwhile,  $H \cap N$  be a Normal Subgroup of  $H$ : for any  $h \in H, n \in H \cap N$ ,  $hnh^{-1} \in N$  because  $N$  is normal, and  $hnh^{-1} \in H$  since  $h, n$  contained in  $H$ . Thus,  $hnh^{-1} \in H \cap N$ , this implies  $H \cap N$  be a Normal of  $H$ .

Now, Define a Map:

$$\varphi : H \rightarrow HN/N : h \mapsto hN$$

Clearly, this map is Well-Defined and Homomorphism. And,

$$\ker \varphi = \varphi^{-1}[1] = \{h \in H \mid hN = N\} = \{h \in H \mid h \in N\} = H \cap N$$

Thus, since The 1st Isomorphism Theorem,

$$HN/N \cong H/(H \cap N)$$

□

**Theorem 5. The Third Isomorphism Theorem**

Let  $G$  be a Group, and  $H, K \trianglelefteq G$  with  $H \leq K$ . Then,  $K/H \trianglelefteq G/H$  and

$$(G/H)/(K/H) \cong (G/K)$$

**Proof.** First, show that  $K/H \trianglelefteq G/H$ . Let  $kH \in K/H$  and  $gH \in G/H$ . Then,

$$(gH)(kH)(gH)^{-1} = (gH)(kH)(g^{-1}H) = (gkg^{-1})H \in K/H$$

since  $gkg^{-1} \in K$ , being  $K \trianglelefteq G$ . Now, Define a map:

$$\varphi : G/H \rightarrow G/K : gH \mapsto gK$$

1. Well-Defined.

$$g_1H = g_2H \iff g_1^{-1}g_2 \in H \xrightarrow{H \leq K} g_1^{-1}g_2 \in K \iff g_1K = g_2K$$

2. Homomorphism.

Clearly, for any  $g_1H, g_2H \in G/H$ ,

$$\varphi(g_1H g_2H) = \varphi(g_1g_2H) = g_1g_2K = g_1K g_2K = \varphi(g_1H) \varphi(g_2H)$$

3. Surjection. Let  $gK \in G/K$  be given. Then, clearly,  $\varphi(gH) = gK$ .

4. Kernel.

$$\ker \varphi = \{gH \in G/H \mid gK = 1\} = \{gH \in G/H \mid g \in K\} = K/H$$

Consequently, The 1st Isomorphism Theorem gives

$$(G/K) \cong (G/H)/\ker \varphi = (G/H)/(K/H)$$

□

**Theorem 6. The Forth Isomorphism Theorem**

Let  $G$  be a Group, and  $N \trianglelefteq G$  be a Normal Subgroup. Then, there is a bijection between

$$D \stackrel{\text{def}}{=} \{H \leq G \mid N \leq H\}, \quad C \stackrel{\text{def}}{=} \{\bar{H} \leq G/N\}$$

**Proof.** Let  $\pi : G \rightarrow G/N : g \mapsto gN$  be a natural projection. And, Define

$$\Phi : D \rightarrow C : H \mapsto \pi[H]$$

This function is well-defined: For any  $H \in D$ , let  $aN, bN \in \pi[H]$ . Then,  $aN \cdot b^{-1}N = ab^{-1}N \in \pi[H]$ , thus  $\pi[H] \leq G/N$ .

To show that one-to-one: Let  $\Phi(A) = \Phi(B)$ . Thus means,  $\pi[A] = \pi[B]$ . Let  $a \in A$ . Then,  $\pi(a) \in \pi[A] = \pi[B]$ , thus  $\pi(a) = \pi(b)$  for some  $b \in B$ . That is,  $aN = bN \iff a \in bN$ . Meanwhile,  $N \leq B$ , thus  $a \in bN \subset B$ ,  $A \subset B$ . Similarly,  $B \subset A$ , that is  $A = B$ .

To show that onto: Let  $K \in C$ . Then,  $N \leq \pi^{-1}[K] \leq G$ , thus clear.

□



## 2.2 Group Action

## 2.3 Generating subset of a Group

## 2.4 Commutator Subgroup

## Chapter 3

# Finite Group Theory

### 3.1 Lagrange's Theorem

### 3.2 The Class Equation

### 3.3 Cauchy's Theorem

**Lemma 2. Cauchy's Theorem**

Let  $G$  be a finite group, and  $p$  be a prime dividing  $|G|$ . Then,  $G$  has order  $p$  element.

*Proof.* Define a set:

$$S \stackrel{\text{def}}{=} \{(x_1, x_2, \dots, x_p) \mid x_i \in G, x_1 x_2 \cdots x_p = 1\}$$

Then,  $S$  has exactly  $|G|^{p-1}$  elements because there are  $|G|$  possible choices for each of the first  $p-1$  elements in  $G$ .

Once  $x_1, \dots, x_{p-1}$  are chosen, then  $x_p$  is uniquely determined by the uniqueness of inverses.

Then, let  $\sigma = (1, 2, \dots, p)$  be a permutation. Then, for any  $\alpha \in S$ ,  $\sigma^n(\alpha) \in S$  for all  $n \in \mathbb{Z}$ , being  $ab = 1 \iff ba = 1$ .

More precisely, let  $n \in \mathbb{Z}$  be given,  $\alpha = (x_1, \dots, x_n)$ . Then,

$$\sigma^n(\alpha) = (x_{n+1}, x_{n+2}, \dots, x_p, x_1, x_2, \dots, x_n)$$

By  $x_1 \cdots x_n x_{n+1} \cdots x_p = 1$ ,  $x_{n+1} \cdots x_p x_1 \cdots x_n = 1$ . Thus  $\sigma^n(\alpha) \in S$ . Now, define a relation on  $S$  as:

$$\alpha \sim \beta \text{ if and only if } \beta = \sigma^n(\alpha) \text{ for some } n \in \mathbb{Z}$$

Then, this relation be equivalent relation, thus construct a partition on  $S$ . Claim:

$$[\alpha] = \{\beta \in S \mid \beta \sim \alpha\} \text{ is singleton if and only if } \alpha = (x, \dots, x) \text{ for some } x \in G.$$

Left direction is clear, and for show that Right direction,

Suppose that  $\alpha = (x_1, \dots, x_n)$  has different coordinate elements, let  $x_i \neq x_j$ , for some  $i < j$ . Then clearly

$$(x_1, \dots, x_i, \dots, x_p) \neq \sigma^{i-j}(x_1, \dots, x_i, \dots, x_j, \dots, x_p) = (\dots, \underbrace{x_j}_{i\text{'th element}}, \dots)$$

Meanwhile, if  $[\alpha]$  has elements more than 1,  $[\alpha]$  has exactly number of  $p$  elements. Because suppose that  $\alpha = (x_1, \dots, x_p)$  has at least one different coordinate. Then,

$$\sigma^1(\alpha), \sigma^2(\alpha), \dots, \sigma^{p-1}(\alpha)$$

are mutually different: If there exist  $1 \leq i < j < p$  such that  $\sigma^i(\alpha) = \sigma^j(\alpha)$ , that is,  $\sigma^{j-i}(\alpha) = \alpha$ .

Now,  $j-i \mid p$ , this is contradiction with  $p$  is prime. Therefore, every equivalent class has order 1 or  $p$ . Consequently,

$$|G|^{p-1} = k + pd$$

where  $k$  is a number of classes of size 1, and  $d$  is a number of classes of size  $p$ . And  $(1, 1, \dots, 1) \in S$ ,  $k$  is at least 1.

Since  $p$  divides  $|G|^{p-1} = k + pd$ , thus  $k$  must be bigger than 1, thus there exists elements such that  $x^p = 1$ .  $\square$

### 3.4 Sylow's Theorem

#### Theorem 7. Sylow's Theorem

Let  $G$  be a group of order  $p^\alpha m$ , where  $p$  is a prime such that  $p \nmid m$ .

A group of order  $p^r$ , ( $r \geq 1$ ) is called a  $p$ -group, Subgroups of  $G$  which are  $p$ -groups are called  $p$ -subgroup. In particular, subgroups of order  $p^\alpha$  is called Sylow  $p$ -subgroup of  $G$ . And, define a collection

$$\text{Syl}_p(G) \stackrel{\text{def}}{=} \{P \leq G \mid |P| = p^\alpha\}, \quad n_p(G) \stackrel{\text{def}}{=} \text{Card}(\text{Syl}_p(G))$$

#### The First Sylow Theorem

There exists a Sylow  $p$ -subgroup of  $G$ . i.e.,  $\text{Syl}_p(G) \neq \emptyset$ .

#### The Second Sylow Theorem

If  $P \in \text{Syl}_p(G)$  and  $Q \leq G$  be a  $p$ -subgroup. Then, there exists  $g \in G$  such that  $Q \leq gPg^{-1}$ .

#### The Third Sylow Theorem

$n_p \equiv 1 \pmod{p}$ ,  $n_p = |G : N_G(P)|$  for any  $P \in \text{Syl}_p(G)$ , and  $n_p \mid m$ .

Before prove above statements, we show that:

**Lemma 3.** Let  $P \in \text{Syl}_p(G)$ . If  $Q$  is  $p$ -subgroup of  $G$ , then  $Q \cap N_G(P) = Q \cap P$ .

*Proof.* Put  $H = Q \cap N_G(P)$ . Since  $P \leq G$ , for any  $p \in P$ ,  $pPp^{-1} = P$ , thus  $p \in N_G(P)$ . i.e.,  $P \leq N_G(P)$ . Thus, Enough to Show that  $H \leq Q \cap P$ . Since  $H \leq N_G(P)$ ,

$$PH = \bigcup_{h \in H} Ph = \bigcup_{h \in H} hP = HP$$

Thus,  $PH \leq G$ . And,

$$|PH| = \frac{|P||H|}{|P \cap H|}$$

By Lagrange's Theorem,  $H \leq P$  and  $P \cap H \leq P$  must have order of powers of  $p$ , so  $PH$  be a  $p$ -group. Clearly,  $P \leq PH$  and  $P$  is the largest  $p$ -group of  $G$ , thus,  $PH = P$ . This means,  $H \leq P$ .  $\square$

*Proof.* The First Theorem: The existence of Sylow  $p$ -subgroup. Proof by Induction:

If  $|G| = 1$ , there is nothing to prove.

Assume inductively the existence of Sylow  $p$ -subgroups for all groups of order less than  $|G|$ .

In case of  $p \mid |Z(G)|$ , then by Cauchy's Theorem,  $Z(G)$  has a subgroup  $N$  which has order of  $p$ .

Clearly  $N$  is Normal, and  $G/N = |G|/|N| = p^{\alpha-1}m$ . By assumption,  $G/N$  has a subgroup  $P'$  of order  $p^{\alpha-1}$ .

By The Fourth Isomorphism Theorem, Let  $P \leq G$  be a subgroup such that  $P/N = P'$ .

Then,  $|P| = |P/N| \cdot |N| = p^\alpha$ , Thus  $P$  be a Sylow  $p$ -subgroup of  $G$ .

In case of  $p \nmid |Z(G)|$ .

Let  $g_1, \dots, g_r$  be representatives of the distinct conjugacy classes of  $G$ , not contained in  $Z(G)$ . Then, The Class Equation gives

$$|G| = |Z(G)| + \sum_{i=1}^r |G : C_G(g_i)|$$

Since  $p$  divides  $|G|$ , if for all  $i = 1, 2, \dots, r$ ,  $p \mid |G : C_G(g_i)|$  then  $p \mid |Z(G)|$ , this is contradiction.

Thus, for some  $j$ ,  $p \nmid |G : C_G(g_j)|$ . Put  $H = C_G(g_j) < G$ . Then,  $|H|$  has a factor of  $p^\alpha$ , by  $p \nmid |G : C_G(g_j)|$ . Now,

$$|H| = p^\alpha m' \quad (m' < m)$$

By assumption,  $H$  has a Sylow  $p$ -group, order of  $p^\alpha$ .

Consequently, the existence of Sylow  $p$ -subgroup was shown.

The Second Theorem: Relation of  $p$ -subgroups.

The First Theorem gives existence of Sylow  $p$ -subgroups. Let  $P \in \text{Syl}_p(G)$ . Denote that:

$$S \stackrel{\text{def}}{=} \{gPg^{-1} \mid g \in G\} = \{P_1, \dots, P_r\}$$

Let  $Q \leq G$  be an any  $p$ -subgroup of  $G$ . And,  $Q$  acts by conjugation on  $S$ . i.e.,

$$\alpha : Q \times S \rightarrow S : (q, P_i) \mapsto qP_iq^{-1}$$

Write  $S$  as a disjoint union of orbits under this action by  $Q$ :

$$S = \mathcal{O}_1 \cup \mathcal{O}_2 \cup \cdots \cup \mathcal{O}_s$$

where  $r = |\mathcal{O}_1| + \cdots + |\mathcal{O}_s|$ . Rearrange a set  $S$  as:  $P_i \in \mathcal{O}_i$ ,  $1 \leq i \leq s$ . Now, using Definition, Lemma, and above Theorem,

$$|\mathcal{O}_i| \stackrel{\text{Thm}}{\equiv} |Q : N_Q(P_i)| \stackrel{\text{def}}{=} |Q : N_G(P_i) \cap Q| \stackrel{\text{lemma}}{\equiv} |Q : P_i \cap Q|$$

for each  $1 \leq i \leq s$ . Since  $Q$  was arbitrary, Let  $Q = P_1$ , so that  $|\mathcal{O}_1| = |P_1 : P_1 \cap P_1| = 1$ . And, for each  $i \geq 2$ ,  $P_i \cap P_1 < P_1$ ,

$$|\mathcal{O}_i| = |P_1 : P_i \cap P_1| > 1$$

Since  $P_1 \in \text{Syl}_p(G)$ , that is  $|P_1| = p^\alpha$ ,  $|P_1 : P_i \cap P_1| = |P_1|/|P_i \cap P_1| = p^k$  where  $1 \leq k < \alpha$ . This means for each  $2 \leq i \leq s$ ,  $p$  divides  $|\mathcal{O}_i|$ . Thus,

$$r = |\mathcal{O}_1| + (|\mathcal{O}_2| + \cdots + |\mathcal{O}_s|) \equiv 1 \pmod{p}$$

Now, Proof by Contradiction: Let  $Q \leq G$  be a  $p$ -subgroup. Suppose that for any  $1 \leq i \leq r$ ,  $Q \not\leq P_i$ . Then,  $P_i \cap Q < Q$  for all  $i$ , this means

$$|\mathcal{O}_i| = |Q : P_i \cap Q| > 1$$

Thus for any  $i$ ,  $p$  divides  $|\mathcal{O}_i|$ , this is Contradiction. This proved Relation of  $p$ -subgroups. Finally, The Third Theorem:

Since Second Theorem, this gives that  $S = \text{Syl}_p(G)$ , thus  $n_p(G) = r$ . That is,  $n_p \equiv 1 \pmod{p}$ . Since all Sylow  $p$ -subgroups are Conjugate, for any  $P \in \text{Syl}_p(G)$ ,

$$n_p = r = |\mathcal{O}_1| = |G : N_G(P)|$$

Consequently, Completing the Sylow Theorem. □

## Chapter 4

# Ring Theory

## 4.1 Ring of Fractions

**Theorem 8.** Let  $R$  be a Commutative Ring,  $D \subset R$  be a subset such that  $\begin{cases} \text{no zero, no zero divisors} \\ \text{closed under multiplication} \end{cases}$ .

Then, there exists a Commutative Ring  $Q$  with identity satisfies:

1.  $R$  can embed in  $Q$ , and every element of  $D$  becomes unit in  $Q$ . More precisely,  $Q = \{rd^{-1} \mid r \in R, d \in D\}$ .
2.  $Q$  is the smallest Ring with identity such that every element of  $D$  becomes unit in  $Q$ .

**Proof.** Let  $\mathcal{F} \stackrel{\text{def}}{=} \{(r, d) \mid r \in R, d \in D\}$  and the relation  $\sim$  on  $\mathcal{F}$  by  $(r_1, d_1) \sim (r_2, d_2) \iff r_1 d_2 = r_2 d_1$ .

Then,  $\sim$  is equivalent relation: reflexive and symmetirc are clear, and Suppose that  $(r_1, d_1) \sim (r_2, d_2)$  and  $(r_2, d_2) \sim (r_3, d_3)$ .

$$r_2 d_3 = r_3 d_2 \implies r_2 d_1 d_3 = r_3 d_1 d_2 \implies r_1 d_2 d_3 = r_3 d_1 d_2 \implies d_2(r_1 d_3 - r_3 d_1) \implies r_1 d_3 = r_3 d_1$$

Thus transitivity shown. Define

$$\frac{r}{d} \stackrel{\text{def}}{=} [(r, d)] = \{(a, b) \mid (a, b) \sim (r, d)\}, \quad Q \stackrel{\text{def}}{=} \left\{ \frac{r}{d} \mid r \in R, d \in D \right\}$$

And define operations  $+, \times$  on  $Q$ :

$$\frac{r_1}{d_1} + \frac{r_2}{d_2} \stackrel{\text{def}}{=} \frac{r_1 d_2 + r_2 d_1}{d_1 d_2}, \quad \frac{r_1}{d_1} \times \frac{r_2}{d_2} \stackrel{\text{def}}{=} \frac{r_1 r_2}{d_1 d_2}$$

**Well-Definedness:** If  $\frac{r_1}{d_1} = \frac{r'_1}{d'_1}$  and  $\frac{r_2}{d_2} = \frac{r'_2}{d'_2}$ ,

$$\frac{r_1 d_2 + r_2 d_1}{d_1 d_2} = \frac{r_1 d_2 d'_1 d'_2 + r_2 d_1 d'_1 d'_2}{d_1 d_2 d'_1 d'_2} = \frac{(r_1 d'_1) d_2 d'_2 + (r_2 d'_2) d_1 d'_1}{d_1 d_2 d'_1 d'_2} = \frac{(r'_1 d_1) d_2 d'_2 + (r'_2 d_2) d_1 d'_1}{d_1 d_2 d'_1 d'_2} = \frac{(r'_1 d'_2 + r'_2 d'_1) d_1 d_2}{d_1 d_2 d'_1 d'_2} = \frac{r'_1 d'_2 + r'_2 d'_1}{d'_1 d'_2}$$

$$\frac{r_1 r_2}{d_1 d_2} = \frac{r_1 r_2 d'_1 d'_2}{d_1 d_2 d'_1 d'_2} = \frac{(r_1 d'_1)(r_2 d'_2)}{d_1 d_2 d'_1 d'_2} = \frac{(r'_1 d_1)(r'_2 d_2)}{d_1 d_2 d'_1 d'_2} = \frac{r'_1 r'_2 d_1 d_2}{d_1 d_2 d'_1 d'_2} = \frac{r'_1 r'_2}{d'_1 d'_2}$$

Now,  $(Q, +, \times)$  constructs Commutative Ring with identity: for any  $d \in D$ , put  $0_Q \stackrel{\text{def}}{=} \frac{0}{d}$ ,  $1_Q \stackrel{\text{def}}{=} \frac{d}{d}$ . Then,

1.  $(R, +, \times)$  closed under the operations since  $D$  is closed under the multiplication.

$$2. (R, +) \text{ has a zero: } \frac{r_1}{d_1} + 0_Q = \frac{r_1}{d_1} + \frac{0}{d} = \frac{r_1 d + 0 d_1}{d_1 d} = \frac{r_1 d}{d_1 d} = \frac{r_1}{d_1}.$$

$$3. (R, +) \text{ has an inverse: } \frac{r_1}{d_1} + \frac{-r_1}{d_1} = \frac{r_1 d_1 + (-r_1) d_1}{d_1 d_1} = \frac{[(r_1) + (-r_1)] d_1}{d_1 d_1} = \frac{0 d_1}{d_1 d_1} = \frac{0}{d_1 d_1} = 0_Q.$$

4.  $(R, +, \times)$  satisfies distributive law:

4-1. The left law:

$$\begin{aligned} \frac{r_1}{d_1} \times \left( \frac{r_2}{d_2} + \frac{r_3}{d_3} \right) &= \frac{r_1}{d_1} \times \frac{r_2 d_3 + r_3 d_2}{d_2 d_3} = \frac{r_1 r_2 d_3 + r_1 r_3 d_2}{d_1 d_2 d_3} = \frac{r_1 r_2 d_1 d_3 + r_1 r_3 d_1 d_2}{d_1 d_2 d_1 d_3} = \frac{r_1 r_2}{d_1 d_2} + \frac{r_2 r_3}{d_2 d_3} \\ &= \frac{r_1}{d_1} \times \frac{r_2}{d_2} + \frac{r_2}{d_2} \times \frac{r_3}{d_3} \end{aligned}$$

4-2. The right law:

$$\begin{aligned} \left( \frac{r_1}{d_1} + \frac{r_2}{d_2} \right) \times \frac{r_3}{d_3} &= \frac{r_1 d_2 + r_2 d_1}{d_1 d_2} \times \frac{r_3}{d_3} = \frac{r_1 r_3 d_2 + r_2 r_3 d_1}{d_1 d_2 d_3} = \frac{r_1 r_3 d_2 d_3 + r_2 r_3 d_1 d_3}{d_1 d_3 d_2 d_3} = \frac{r_1 r_3}{d_1 d_3} + \frac{r_2 r_3}{d_2 d_3} \\ &= \frac{r_1}{d_1} \times \frac{r_3}{d_3} + \frac{r_2}{d_2} \times \frac{r_3}{d_3} \end{aligned}$$

$$5. (R, \times) \text{ has an identity: } \frac{r_1}{d_1} \times 1_Q = \frac{r_1}{d_1} \times \frac{d}{d} = \frac{r_1 d}{d_1 d} = \frac{r_1}{d_1}.$$

6. Elements of  $D$  become unit in  $Q$ : Define  $\iota: R \rightarrow Q: r \mapsto \frac{rd}{d}$  where  $d \in D$  is any fixed element in  $D$ .

Then,  $\iota$  is Ring-Monomorphism because:

$$6-1. \text{ Well-Defined and Injective: } \iota(r_1) = \iota(r_2) \iff \frac{r_1 d}{d} = \frac{r_2 d}{d} \iff (r_1 - r_2) d = 0 \iff r_1 = r_2$$

□



## Chapter 5

# Field Theory

**Chapter 6**

**Category**

# Chapter 7

## General Topology

### 7.1 Complete Metric Space

**Definition 2.** Let  $(X, d)$  be a Metric Space, and  $\{p_n\}$  be a Sequence in  $X$ . The Sequence  $\{p_n\}$  is called *Cauchy Sequence* if:

For any  $\epsilon > 0$ , there exists  $N \in \mathbb{N}$  such that  $m, n \geq N \implies d(p_m, p_n) < \epsilon$ .

A Metric Space  $(X, d)$  is said to be *Complete* if every Cauchy Sequences Converge.

**Lemma 4.** Let  $\{E_n\}$  be a sequence of closed bounded non-empty subsets in a Complete Metric Space  $X$  such that  $E_n \supset E_{n+1}$ . If  $\lim_{n \rightarrow \infty} \text{diam} E_n = 0$ , then  $\bigcap_{n=1}^{\infty} E_n = \{p\}$  for some  $p \in X$ .

*Proof.* For each  $n \in \mathbb{N}$ , construct  $p_n \in E_n$ .

Let  $\epsilon > 0$  be given. Since  $\text{diam} E_n \rightarrow 0$ , there is  $N \in \mathbb{N}$  such that  $\text{diam} E_n < \epsilon$ .

For any  $m, n \geq N$ ,  $E_N$  contains  $p_m, p_n$ . That is,  $d(p_m, p_n) < \epsilon$ . Thus,  $\{p_n\}$  be a Cauchy sequence of  $X$ .

Since  $X$  is complete, there is a unique point  $p \in X$  such that  $p_n \rightarrow p$ . Let  $N \in \mathbb{N}$  be an integer such that  $n \geq N \implies |p_n - p| < \epsilon$ .

Now, for each  $n \geq N$ ,  $E_n$  has a limit point as  $p$ . And for any  $n \in \mathbb{N}$ ,  $E_n$  contains  $E_N, E_{N+1}, \dots$ , thus for all  $n \in \mathbb{N}$ ,  $E_n$  has a limit point as  $p$ . Meanwhile,  $E_n$  closed,  $p \in E_n, \forall n \in \mathbb{N}$ .

Consequently,  $p \in \bigcap_{n=1}^{\infty} E_n$ . If there is  $q \in X$  such that  $p \neq q$ ,  $q \in \bigcap_{n=1}^{\infty} E_n$ . Then,  $\text{diam} E_n \geq d(p, q) > 0, \forall n \in \mathbb{N}$ .  $\square$

#### 7.1.1 Baire Category

**Definition 3.** The Topological Space  $X$  is called *Baire Space* if:

If  $\{G_n \mid n \in \mathbb{N}\}$  be a Countable Collection of dense open sets of  $X$ , then  $\overline{\bigcap_{n=1}^{\infty} G_n} = X$

In brief, every Countable intersection of dense open sets be dense in  $X$ .

**Theorem 9.** Locally Compact Hausdorff Space is Baire Space.

**Theorem 10.** Complete Metric Space is Baire Space.

*Proof.* Suppose that  $\{G_n \mid n \in \mathbb{N}\}$  be a Countable Collection of dense open set of Complete Metric Space. Let an open  $U \in \mathcal{T}$  be given. Since  $G_n$  is dense in the Space,  $U \cap G_1$  is non-empty open set.

Thus, there exists a  $p_1 \in U \cap G_1$  such that for some  $r_1 > 0$ ,  $B_{r_1}(p_1) \subset U \cap G_1$ .  
Then, automatically,

$$B_{\frac{r_1}{2}}(p_1) \subset \overline{B_{\frac{r_1}{2}}(p_1)} \subset B_{r_1}(p_1) \subset U \cap G_1$$

Set  $E_1 = U$ ,  $E_2 = B_{\frac{r_1}{2}}(p_1)$ .

Suppose that  $E_1, \dots, E_{n-1}$  are chosen. Then, since  $E_{n-1} \cap G_{n-1}$  is open, being intersection of opens. Thus there exists a point  $p_{n-1} \in E_{n-1} \cap G_{n-1}$  and exists  $r_{n-1}$  such that

$$B_{r_{n-1}}(p_{n-1}) \subset E_{n-1} \cap G_{n-1} \subset E_{n-1}$$

This implies that

$$B_{\frac{r_{n-1}}{2}}(p_{n-1}) \subset \overline{B_{\frac{r_{n-1}}{2}}(p_{n-1})} \subset B_{r_{n-1}}(p_{n-1}) \subset E_{n-1} \cap G_{n-1} \subset E_{n-1}$$

Set  $E_n = B_{\frac{r_{n-1}}{2}}(p_{n-1})$ . Since inductively construction of  $\{E_n\}$ ,  $E_{n+1} \subset E_n$  and  $\overline{E_n} \subset G_n$  for all  $n \in \mathbb{N}$ .  
Consequently,

$$U \cap \left( \bigcap_{n=1}^{\infty} G_n \right) = \bigcap_{n=1}^{\infty} (U \cap G_n) \supset \bigcap_{n=1}^{\infty} (U \cap \overline{E_n}) = U \cap \left( \bigcap_{n=1}^{\infty} \overline{E_n} \right) = \bigcap_{n=1}^{\infty} \overline{E_n} \neq \emptyset$$

□

**Definition 4.** Let  $X$  be a Topological Space.

$A \subset X$  is said to be *nowhere dense subset* if  $(\overline{A})^\circ = \emptyset$ .

1.  $B \subset X$  is called *first category* if  $B$  can be representative by union of countable nowhere dense subsets.
2. If the subset is not first category, then it is said to be *second category*.

### 7.1.2 Nowhere Differentiable function

### 7.1.3 Banach Fixed Point Theorem

**Definition 5.** Let  $f : X \rightarrow X$  be any function. A point  $x \in X$  is called a *fixed point* of  $f$  if  $f(x) = x$ .

**Definition 6.** Let  $X$  be a Metric Space. A map  $f : X \rightarrow X$  is called *Contractive* with respect to the metric  $d$  if:

$$\text{There exists } \alpha \in (0, 1) \text{ such that for all } x, y \in X, d(f(x), f(y)) \leq \alpha d(x, y).$$

**Theorem 11. Banach Fixed point Theorem**

Let  $(X, d)$  be a Complete Metric Space, and  $f : X \rightarrow X$  be a Contractive map.

Then, there exists a unique fixed point of  $f$ ,  $x^* \in X$ .

*Proof.* Clearly,

$$\text{Contractive} \implies \text{Lipschitz Condition} \implies \text{Continuous}.$$

Thus,  $f$  is Continuous.

Let  $x_0 \in X$  be arbitrary, and construct a sequence  $\{x_n\}$  recursively as follows:

$$x_{n+1} \stackrel{\text{def}}{=} f(x_n), \quad n \geq 0$$

Then, for any  $n \geq 0$ ,

$$\begin{aligned} d(x_{n+1}, x_n) &= d(f(x_n), f(x_{n-1})) \leq \alpha d(x_n, x_{n-1}) \\ &= d(f(x_{n-1}), f(x_{n-2})) \leq \alpha^2 d(x_{n-1}, x_{n-2}) \\ &\vdots \\ &\leq \alpha^n d(x_1, x_0) \end{aligned}$$

Let  $\epsilon > 0$  be given. Put  $N \in \mathbb{N}$  such that  $\alpha^N \cdot d(x_1, x_0) < \epsilon(1 - \alpha)$ . Then,  $n \geq m \geq N$  implies that

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + \cdots + d(x_{m+1}, x_m) \\ &\leq \alpha^n d(x_1, x_0) + \alpha^{n-1} d(x_1, x_0) + \cdots + \alpha^{m+1} d(x_1, x_0) \\ &= \alpha^{m+1} d(x_1, x_0) \sum_{r=0}^{n-m-1} \alpha^r < \alpha^N d(x_1, x_0) \sum_{r=0}^{\infty} \alpha^r < \epsilon(1 - \alpha) \frac{1}{1 - \alpha} = \epsilon \end{aligned}$$

Therefore,  $\{x_n\}$  is Cauchy sequence. Since  $X$  is Complete, for some  $x^* \in X$ ,  $\lim_{n \rightarrow \infty} x_n = x^*$ . Consequently,

$$\lim_{n \rightarrow \infty} f(x_n) \stackrel{f \text{ conti.}}{=} f\left(\lim_{n \rightarrow \infty} x_n\right) = f(x^*) = \lim_{n \rightarrow \infty} x_{n+1} = x^*$$

□

## 7.2 Urysohn Metrization Theorem

### 7.2.1 Urysohn Lemma

Recall that:

**Definition 7.**  $X$  is  $T_4$  if: For any disjoint closed set  $A$  and  $B$ , there exist disjoint open  $U, V$  such that  $A \subseteq U$  and  $B \subseteq V$ .

**Lemma 5.**  $X$  is  $T_4$  Space if and only if For any closed  $C$  and open  $U$  with  $C \subseteq U$ , there exists open  $O$  such that

$$\underset{\text{closed}}{C} \subseteq \underset{\text{open}}{O} \subseteq \underset{\text{closed}}{\overline{O}} \subseteq \underset{\text{open}}{U}$$

*Proof.* Proof of the left direction only.

Let  $X$  be a  $T_4$  Space, and  $C \subset X$  be a closed,  $U$  be a open containing  $C$ . Then,  $C \subset U$  implies  $U^c \subset C^c$ , thus  $U^c$  is a closed set disjoint from  $C$ . By  $T_4$  condition, There exist disjoint opens  $O, O'$  such that  $C \subset O$  and  $U^c \subset O' \iff O'^c \subset U$ .

Since  $O \cap O' = \emptyset \iff O \subset O'^c$ ,  $O$  contained in  $U$ , this implies that  $C \subset O \subset U$ .

Since closure is the smallest closed set such that contains it, consequently  $C \subset O \subset \overline{O} \subset O'^c \subset U$ . □

**Definition 8.** Let  $X$  be a Topological Space, and  $A, B \subset X$  are disjoint closed subset.

A real-valued Continuous map  $f : X \rightarrow [a, b]$  is called *Urysohn function* for  $A$  and  $B$  if:  $f|_A = a$  and  $f|_B = b$ .

In another form,

$$f : X \rightarrow [a, b] : x \rightarrow \begin{cases} a & x \in A \\ b & x \in B \\ f(x) & x \notin A \cup B \end{cases}$$

**Lemma 6. Urysohn Lemma**

$T_4$  Space has an Urysohn function for any two disjoint closed subsets.

*Proof.* Generalization is the last thing to proven, first of all, prove in case of  $[a, b] = [0, 1]$ . This proof consists by three Step.

Let  $X$  be a  $T_4$  Space, and  $A, B \subset X$  be closed subsets.

**Step 1. Construct a Chain of Open sets with Dyadic Rational Indices.**

Consider a set of *Dyadic Rationals*  $D \stackrel{\text{def}}{=} \left\{ \frac{k}{2^n} \mid n, k \in \mathbb{N}, k \leq 2^n - 1 \right\}$ . We will show that the following statement holds:

For any  $r, s \in D$  with  $r < s$ , there exist open sets  $U_r, U_s$  such that  $A \subseteq \overline{U_r} \subseteq U_s \subseteq X \setminus B$  (\*)

For this, Enough to Show that: For any  $k \in \mathbb{N}$ , there exists a Chain as:

$$A \subseteq U_{\frac{1}{2^k}} \subseteq \overline{U_{\frac{1}{2^k}}} \subseteq U_{\frac{2}{2^k}} \subseteq \overline{U_{\frac{2}{2^k}}} \subseteq \cdots \subseteq U_{\frac{2^{k-1}}{2^k}} \subseteq \overline{U_{\frac{2^{k-1}}{2^k}}} \subseteq X \setminus B$$

(Note that this opens in the Chain are not necessary distinct: For instance, if Ambient Space is Finite, then the Space is *Noetherian*. That is,  $X$  satisfies Ascending Chain Condition for open sets.)

Let  $k = 1$ . Then, By  $T_4$  condition gives that: There exists an open set  $U_1$  such that

$$A \subseteq U_1 \subseteq \overline{U_1} \subseteq X \setminus B$$

Now, naming this  $U_1$  as  $U_{\frac{1}{2}}$ , proved when  $k = 1$ .

Suppose that for some  $k > 1$ , the Chain exists as:

$$\underset{\text{closed}}{A} \subseteq \overset{*1}{\underset{\text{open}}{U_{\frac{1}{2^k}}}} \subseteq \underset{\text{closed}}{\overline{U_{\frac{1}{2^k}}}} \subseteq \overset{*2}{\underset{\text{open}}{U_{\frac{2}{2^k}}}} \subseteq \cdots \subseteq \overset{*2^{k-1}}{\underset{\text{open}}{U_{\frac{2^{k-1}}{2^k}}}} \subseteq \underset{\text{closed}}{\overline{U_{\frac{2^{k-1}}{2^k}}}} \subseteq \overset{*2^k}{\underset{\text{open}}{X \setminus B}}$$

By repeatedly applying the  $T_4$  condition  $2^k$  times, as indicated by the indices  $*1, *2, \dots, *2^k$ , we can construct  $2^k$  open sets such that:

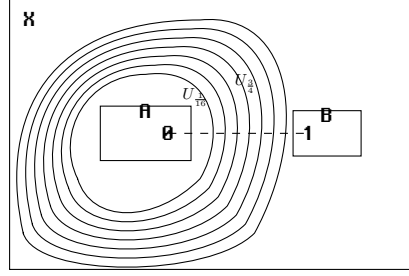
$$A \subseteq U_{\frac{1}{2^{k+1}}} \subseteq \overline{U}_{\frac{1}{2^{k+1}}} \subseteq U_{\frac{1}{2^k}} \subseteq \overline{U}_{\frac{1}{2^k}} \subseteq U_{\frac{3}{2^{k+1}}} \subseteq \overline{U}_{\frac{3}{2^{k+1}}} \subseteq U_{\frac{2}{2^k}} \subseteq \overline{U}_{\frac{2}{2^k}} \subseteq \dots \subseteq U_{\frac{2^k-1}{2^k}} \subseteq \overline{U}_{\frac{2^k-1}{2^k}} \subseteq U_{\frac{2^{k+1}-1}{2^{k+1}}} \subseteq \overline{U}_{\frac{2^{k+1}-1}{2^{k+1}}} \subseteq X \setminus B$$

Finally, Step 1 proved.

**Step 2. Construct an Urysohn Function.**

Define a map  $f : X \rightarrow [0, 1]$  as:

$$f(x) = \begin{cases} 0 & x \in \bigcap_{t \in D} U_t \\ \sup\{t \in D \mid x \notin U_t\} & x \notin \bigcap_{t \in D} U_t \end{cases}$$



Then, this map  $f$  is well-defined by (\*) and  $\sup D \leq 1$ . And  $f$  satisfies that:

1.  $\forall r \in D, x \in A \subset U_r$ . Thus,  $f(x) = 0$  if  $x \in A$ .
2.  $\forall r \in D, x \in B \subset X \setminus U_r$ . Thus,  $f(x) = \sup D = 1$  if  $x \in B$ .
3. If  $x \in \overline{U}_r$ , then for every  $s > r$ ,  $x \in \overline{U}_r \subset U_s$ . Thus,  $f(x) \leq r$ . In Contrapositive,  $f(x) > r \implies x \notin \overline{U}_r$ .  
(If  $f(x) = \sup\{t \in D \mid x \notin U_t\} > r$ , then there is  $s \in D$  such that  $s > r$  and  $x \notin U_s$ , Contradiction.)
4. If  $x \notin U_r$ , then,  $f(x) \geq r$ . In Contrapositive,  $f(x) < r \implies x \in U_r$ .

Now, show that this map  $f$  is Continuous map: Let  $x \in X$  be fixed arbitrarily, and  $\epsilon > 0$  be given.

In Case of  $0 < f(x) < 1$ .

Since Density of Dyadic Rationals, Choose  $r, s \in D$  such that  $f(x) - \epsilon < r < f(x) < s < f(x) + \epsilon$ .

Now, we obtain that:

$$x \overset{(*)}{\in} U_s \setminus \overline{U}_r \overset{(**)}{\subseteq} f^{-1}[(f(x) - \epsilon, f(x) + \epsilon)]$$

(\*) directly given by above properties, (\*\*) given applying the fact that  $x \in U_s \subset \overline{U}_s$  and  $x \notin \overline{U}_r$ .

In Case of  $f(x) = 0$ .

Choose  $r \in D$  such that  $f(x) = 0 < r < \epsilon = f(x) + \epsilon$ . Then,

$$x \in U_r \subset f^{-1}[(f(x), f(x) + \epsilon)]$$

In Case of  $f(x) = 1$ .

Choose  $r \in D$  such that  $f(x) - \epsilon = 1 - \epsilon < r < 1 = f(x)$ . Then,

$$x \in X \setminus U_r \subset f^{-1}[(f(x) - \epsilon, f(x))]$$

Consequently,  $f$  is Continuous map on  $[0, 1]$  such that  $f|_A = 0$  and  $f|_B = 1$ .

**Step 3. Generalization.**

Since  $[0, 1] \cong [a, b]$  for any  $a < b$ , let  $g : [0, 1] \rightarrow [a, b] : x \mapsto (1 - x)a + xb$  be a Homeomorphism.

Then,  $h = g \circ f : X \rightarrow [a, b]$  becomes a Continuous map such that  $h|_A = a$  and  $h|_B = b$ . □

### 7.2.2 Tietze Extension Theroem

#### Theorem 12. Tietze Extension Theroem

Let  $X$  be a  $T_4$  Space, and  $A \subseteq X$  be a closed subset.

For any Continuous map  $f : A \rightarrow \mathbb{R}$ , there exists a Continuous map:

$$g : X \rightarrow \mathbb{R} \quad \text{s.t.} \quad g|_A = f$$

This  $g$  is called *extension* of  $f$ .

*Proof.* This proof consists by three steps.

**Step 1.** First, we will show that:

For any Continuous map  $f : A \rightarrow [-r, r]$ , there is a Continuous map  $h : X \rightarrow \mathbb{R}$  s.t. 
$$\begin{cases} \forall x \in X, |h(x)| \leq \frac{1}{3}r \\ \forall a \in A, |f(a) - h(a)| \leq \frac{2}{3}r \end{cases} \quad (*)$$

Set

$$I_1 \stackrel{\text{def}}{=} \left[-r, -\frac{1}{3}r\right], \quad I_2 \stackrel{\text{def}}{=} \left[-\frac{1}{3}r, \frac{1}{3}r\right], \quad I_3 \stackrel{\text{def}}{=} \left[\frac{1}{3}r, r\right]$$

Then, the preimage of continuous map preserves closed and  $A$  is closed subspace of  $X$ ,  $f^{-1}[I_1]$  and  $f^{-1}[I_3]$  are closed of  $X$ .

And,  $I_1$  and  $I_3$  are disjoint, thus  $f^{-1}[I_1 \cap I_3] = f^{-1}[I_1] \cap f^{-1}[I_3] = \emptyset$ .

Now, apply the *Urysohn Lemma*: There exists an Urysohn function  $h : X \rightarrow I_2$  for  $f^{-1}[I_1]$  and  $f^{-1}[I_3]$ .

Clearly, this map  $h$  satisfies the first condition in  $(*)$ . And, for show the second condition, let  $a \in A$  be given.

If  $a \in f^{-1}[I_1]$ , then  $f(a) \in I_1$  and  $h(a) = -\frac{1}{3}r$ , thus  $|f(a) - h(a)| \leq \frac{2}{3}r$ .

If  $a \in f^{-1}[I_3]$ , then  $f(a) \in I_3$  and  $h(a) = \frac{1}{3}r$ , thus  $|f(a) - h(a)| \leq \frac{2}{3}r$ .

If  $a \notin (f^{-1}[I_1] \cup f^{-1}[I_3])$ , then  $f(a), h(a) \in I_2$ , thus  $|f(a) - h(a)| \leq \frac{2}{3}r$ .

Therefore, the second condition satisfied.

**Step 2.** We will show that: for any  $f : A \rightarrow [-1, 1]$ , there exists an extension of  $f$ .

Apply the result in Step 1, there exists a Continuous map:

$$h_1 : X \rightarrow \mathbb{R} \quad \text{s.t.} \quad \begin{cases} \forall x \in X, |h_1(x)| \leq \frac{1}{3} \\ \forall a \in A, |f(a) - h_1(a)| \leq \frac{2}{3} \end{cases}$$

Now, the second condition of  $h_1$ , the continuous map  $f - h_1 : A \rightarrow [-\frac{2}{3}, \frac{2}{3}] : x \mapsto f(x) - h_1(x)$  is well-defined.

Again, there exists a Continuous map:

$$h_2 : X \rightarrow \mathbb{R} \quad \text{s.t.} \quad \begin{cases} \forall x \in X, |h_2(x)| \leq \frac{1}{3} \cdot \frac{2}{3} \\ \forall a \in A, |f(a) - h_1(a) - h_2(a)| \leq \left(\frac{2}{3}\right)^2 \end{cases}$$

Inductively, for any  $n \in \mathbb{N}$ , there exists a Continuous map:

$$h_n : X \rightarrow \mathbb{R} \quad \text{s.t.} \quad \begin{cases} \forall x \in X, |h_n(x)| \leq \frac{1}{3} \cdot \left(\frac{2}{3}\right)^{n-1} \\ \forall a \in A, |f(a) - h_1(a) - h_2(a) - \dots - h_n(a)| \leq \left(\frac{2}{3}\right)^n \end{cases}$$

Define a map

$$g : X \rightarrow [-1, 1] : x \mapsto \sum_{n=1}^{\infty} h_n(x)$$

For any  $x \in X$ ,

$$|g(x)| = \left| \sum_{n=1}^{\infty} h_n(x) \right| \leq \sum_{n=1}^{\infty} |h_n(x)| \leq \sum_{n=1}^{\infty} \frac{1}{3} \cdot \left(\frac{2}{3}\right)^{n-1} = \frac{1}{3} \cdot \frac{1}{1 - \frac{2}{3}} = 1$$

Therefore, this map is well-defined. And, *Weierstrass M-test* gives that  $\sum_{n=1}^{\infty} h_n(x)$  converges uniformly.

Moreover, for any  $a \in A$ ,

$$\left| f(a) - \sum_{k=1}^n h_k(a) \right| \leq \left(\frac{2}{3}\right)^n \implies \left| f(a) - \sum_{n=1}^{\infty} h_n(a) \right| = |f(a) - g(a)| = 0$$



That is,  $g$  is Continuous on  $X$  and  $g|_A = f$ . Therefore,  $g$  is extension of  $f$ .

**Step 3.** Finally, we generalize the result in Step 2.:

Let  $f: A \rightarrow [a, b]$  be a Continuous map on the closed subspace  $A$ . And, let  $\varphi: [a, b] \rightarrow [-1, 1]$  be a Homeomorphism. Then,  $\varphi \circ f: A \rightarrow [-1, 1]$  is Continuous map, thus there exists an extension  $g: X \rightarrow [-1, 1]$  such that  $g|_A = \varphi \circ f$ . Now,  $\varphi^{-1} \circ g: X \rightarrow [a, b]$  is Continuous, and  $(\varphi^{-1} \circ g)|_A = \varphi^{-1} \circ \varphi \circ f = f$ , Therefore this  $\varphi^{-1} \circ g$  is the extension of  $f$ .

Let  $f: A \rightarrow \mathbb{R}$  be a Continuous map on the closed subspace  $A$ .

And, let  $\varphi: \mathbb{R} \rightarrow (-1, 1)$  be a Homeomorphism. Then, the map  $\phi: \mathbb{R} \rightarrow [-1, 1]: x \mapsto \varphi(x)$  is still Continuous.

Now, The Continuous map  $\phi \circ f: A \rightarrow [-1, 1]$  has an extension  $g: X \rightarrow [-1, 1]$  such that  $g|_A = \phi \circ f$ .

Put  $B = g^{-1}[\{-1, 1\}]$ . Then  $B$  is Closed on  $X$ , and  $A \cap B = \emptyset$ . Now, apply the Urysohn Lemma to this, there exists an Urysohn function for  $A$  and  $B$ : Continuous map  $\gamma: X \rightarrow [0, 1]$  such that  $\gamma|_A = 1$  and  $\gamma|_B = 0$ .

Define a map  $\eta: X \rightarrow (-1, 1): x \mapsto g(x)\gamma(x)$ . Then, if  $g(x) = 1$  or  $g(x) = -1$ , then  $x \in B$ , thus  $g(x)\gamma(x) = 0$ .

Therefore,  $\eta$  is well-defined. And, for any  $a \in A$ ,  $\eta(a) = g(a)\gamma(a) = g(a)$ , thus  $\eta|_A = \phi \circ f$ .

Consequently, the map  $\phi^{-1} \circ \eta$  is an extension of  $f$ , we wanted. □

Recall that:

**Definition 9.**  $X$  is  $T_1$  if: For any distinct  $x, y \in X$ , there exist open sets  $U_x, U_y$  such that  $\begin{cases} x \in U_x, & x \notin U_y \\ y \notin U_x, & y \in U_y \end{cases}$ .

**Lemma 7.**  $X$  is  $T_1$  if and only if For any  $x \in X$ , a singleton  $\{x\}$  is closed in  $X$ .

*Proof.* The left direction is clear.

Let  $x \in X$ . Then, for any  $y \in X$  with  $y \neq x$ ,  $T_1$  condition gives that there is an open set such that  $y \in U_y$  and  $x \notin U_y$ .

Now, the union

$$\bigcup_{\substack{y \in X \\ y \neq x}} U_y = X \setminus \{x\}$$

is open by definition. □

### 7.2.3 Urysohn Metrization Theorem

**Theorem 13.** *Urysohn Metrization Theorem*

If  $X$  is a Second-Countable Regular Space, then  $X$  is Metrizable.

## Chapter 8

# Algebraic Topology

## Chapter 9

# Basic Analysis

## 9.1 Arithmetic means

Let  $\{s_n\}$  be a Complex numbers Sequence. Define the *Arithmetic means* of  $\{s_n\}$ :

$$\sigma_n \stackrel{\text{def}}{=} \frac{s_0 + \cdots + s_n}{n+1} = \frac{1}{n+1} \left( \sum_{i=0}^n s_i \right)$$

Then, the Arithmetic means  $\sigma_n$  has following properties:

1). If  $\lim_{n \rightarrow \infty} s_n = s$ , then  $\lim_{n \rightarrow \infty} \sigma_n = s$ .

**Proof.** Let  $\epsilon > 0$  be given. Then, there exists  $N \in \mathbb{N}$  such that  $n \geq N$  implies  $|s_n - s| < \epsilon$ .  
Now, for  $n \geq N$ ,

$$\begin{aligned} |\sigma_n - s| &= \left| \frac{s_0 + \cdots + s_n}{n+1} - \frac{(n+1)s}{n+1} \right| = \left| \frac{(s_0 - s) + \cdots + (s_n - s)}{n+1} \right| \\ &\stackrel{\text{tri. ineq}}{\leq} \frac{\sum_{k=0}^{N-1} |s_k - s|}{n+1} + \frac{\sum_{k=N}^n |s_k - s|}{n+1} \\ &< \frac{\sum_{k=0}^{N-1} |s_k - s|}{n+1} + \frac{n+1-N}{n+1} \cdot \epsilon \\ &< \frac{\sum_{k=0}^{N-1} |s_k - s|}{n+1} + \epsilon \end{aligned}$$

Now, put  $M \in \mathbb{N}$  satisfies  $M \geq N$  and  $n \geq M \implies \frac{\sum_{k=0}^{N-1} |s_k - s|}{n+1} < \epsilon$ , using Archimedean property.  
Then,  $n \geq M$  implies  $|\sigma_n - s| < \epsilon$ , thus  $\sigma_n \rightarrow s$ . □

2). Put  $a_n = s_n - s_{n-1}$ , for  $n \geq 1$ . If  $\lim_{n \rightarrow \infty} na_n = 0$  and  $\sigma_n$  converges, then  $s_n$  converges.

**Proof.** First,

$$\begin{aligned} s_n - \sigma_n &= s_n - \frac{s_0 + \cdots + s_n}{n+1} = \frac{(n+1)s_n - \sum_{k=0}^n s_k}{n+1} \\ &= \frac{1}{n+1} ((s_1 - s_0) + (2s_2 - 2s_1) + (3s_3 - 3s_2) + \cdots + (ns_n - ns_{n-1})) \\ &= \frac{1}{n+1} \sum_{k=1}^n ka_k \end{aligned}$$

Now, if  $na_n \rightarrow 0$  and  $\sigma_n \rightarrow \sigma$ ,

$$\begin{aligned} \lim_{n \rightarrow \infty} s_n &= \lim_{n \rightarrow \infty} \left( \sigma_n + \frac{1}{n+1} \sum_{k=1}^n ka_k \right) \\ &= \lim_{n \rightarrow \infty} \sigma_n + \lim_{n \rightarrow \infty} \frac{1}{n+1} \sum_{k=1}^n ka_k \stackrel{1)}{=} \sigma \end{aligned}$$

□

2) is conditional converse of 1). But, there is more weak version of the converse proposition:

3). The sequence  $\{na_n\}$  bounded by  $M < \infty$ , and  $\sigma_n \rightarrow \sigma$ . Then,  $s_n \rightarrow \sigma$ .

**Proof.** First, For positive integers  $m < n$ ,

$$\begin{aligned} s_n - \sigma_n &= s_n - \frac{\sum_{k=0}^n s_k}{n+1} = s_n - \frac{m+1}{n-m} \cdot \left( \frac{1}{m+1} - \frac{1}{n+1} \right) \sum_{k=0}^n s_k \\ &= s_n - \frac{m+1}{n-m} \cdot \left( \frac{\sum_{k=0}^m s_k + \sum_{k=m+1}^n s_k}{m+1} - \frac{\sum_{k=0}^n s_k}{n+1} \right) \\ &= s_n - \frac{m+1}{n-m} \cdot \left( \sigma_m - \sigma_n + \frac{\sum_{k=m+1}^n s_k}{m+1} \right) \\ &= \frac{m+1}{n-m} (\sigma_n - \sigma_m) + \frac{1}{n-m} \sum_{k=m+1}^n (s_n - s_k) \end{aligned}$$

Meanwhile, since for any  $n \in \mathbb{N}$ ,  $|na_n| = n|s_n - s_{n-1}| < M$ , for  $k = m+1, \dots, n$ ,

$$\begin{aligned} |s_n - s_k| &= |s_n - s_{n-1} + s_{n-1} - s_{n-2} + \dots + s_{k+1} - s_k| \\ &\leq |s_n - s_{n-1}| + |s_{n-1} - s_{n-2}| + \dots + |s_{k+1} - s_k| \\ &\leq \frac{M}{n} + \frac{M}{n-1} + \dots + \frac{M}{k+1} \leq \frac{n-k}{k+1} M \leq \frac{n-k}{m+2} M \leq \frac{n-m-1}{m+2} M \end{aligned}$$

Let  $\epsilon > 0$  be given. For each  $n \in \mathbb{N}$ , put  $m \in \mathbb{N}$  such that

$$m \leq \frac{n-\epsilon}{1+\epsilon} < m+1$$

Then,

$$m(1+\epsilon) \leq n-\epsilon \implies m+\epsilon(1+m) \leq n \implies \frac{m+1}{n-m} \leq \frac{1}{\epsilon}$$

and

$$n-\epsilon < (m+1)(1+\epsilon) \implies n+1 < (m+2)(1+\epsilon) \implies \frac{n+1}{m+2} - 1 < \epsilon \implies \frac{n-m-1}{m+2} < \epsilon$$

Now, for arbitrary  $n \in \mathbb{N}$ ,

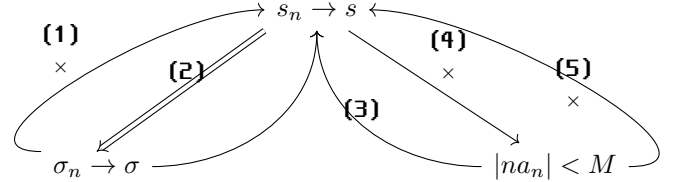
$$\begin{aligned} |s_n - \sigma| &\leq |s_n - \sigma| + |\sigma_n - \sigma| \\ \implies \limsup_{n \rightarrow \infty} |s_n - \sigma| &\leq \limsup_{n \rightarrow \infty} |s_n - \sigma_n| + \limsup_{n \rightarrow \infty} |\sigma_n - \sigma| \end{aligned}$$

And,

$$\begin{aligned} |s_n - \sigma_n| &= \frac{m+1}{n-m} |\sigma_n - \sigma_m| + \frac{1}{n-m} \sum_{k=m+1}^n |s_n - s_k| < \frac{1}{\epsilon} |\sigma_n - \sigma_m| + M\epsilon \\ \implies \limsup_{n \rightarrow \infty} |s_n - \sigma_n| &\leq \frac{1}{\epsilon} \limsup_{n \rightarrow \infty} |\sigma_n - \sigma_m| + M\epsilon = M\epsilon \end{aligned}$$

Consequently,  $\limsup_{n \rightarrow \infty} |s_n - \sigma| \leq (M+1)\epsilon$ , thus  $s_n \rightarrow \sigma$ . □

In brief, the diagram of the above conditions like this:



Examples and Counterexamples of the Diagram:

(1) Let  $s_n \stackrel{\text{def}}{=} \exp(\frac{in\pi}{2})$ . Then,

- $s_n$  diverges.
- $na_n$  diverges.
- $\sigma_n \rightarrow 0$ .

(2) Let  $s_n \stackrel{\text{def}}{=} \frac{1}{n}$ ,  $s_0 = 0$ .

(3) Let  $s_n \stackrel{\text{def}}{=} \sum_{k=1}^n \frac{1}{k}$ . Then,

- $s_n$  diverges.
- $a_n = \frac{1}{n}$ , thus  $na_n \rightarrow 1$ , bounded.
- If  $\sigma_n$  converges, then the diagram implies that  $s_n$  must converge, leading to a contradiction. Therefore,  $\sigma_n$  diverges.

(4)  $s_n = \sum_{k=1}^n \frac{(-1)^k}{\sqrt{k}}$ ,  $s_0 = 0$ . Then,

- $s_n$  converges, being the Alternating series Test.
- $a_n = \frac{(-1)^n}{\sqrt{n}}$ , thus  $na_n$  diverges.

## 9.2 Taylor's Theorem

### Theorem 14. Taylor's Theorem

Let  $f : [a, b] \rightarrow \mathbb{R}$ , and let  $n \in \mathbb{N}$  be fixed. Suppose that  $\begin{cases} f^{(n-1)} \text{ is Continuous.} \\ f^{(n)}(t) \text{ exists for every } t \in (a, b). \end{cases}$

Then, for any  $\alpha, \beta \in [a, b]$ , there exists  $x \in (\alpha, \beta)$  such that

$$f(\beta) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!} (\beta - \alpha)^k + \frac{f^{(n)}(x)}{n!} (\beta - \alpha)^n$$

**Proof.** Put

$$M \stackrel{\text{def}}{=} \frac{1}{(\beta - \alpha)^n} \cdot \left( f(\beta) - \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!} (\beta - \alpha)^k \right)$$

That is,

$$f(\beta) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!} (\beta - \alpha)^k + M(\beta - \alpha)^n$$

and put

$$g(t) \stackrel{\text{def}}{=} f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!} (t - \alpha)^k - M(t - \alpha)^n, \quad (a \leq t \leq b)$$

If we differentiate the above equation  $n$  times,

$$g^{(n)}(t) = f^{(n)}(t) - n!M, \quad (a < t < b)$$

For each  $k = 0, 1, \dots, n-1$ ,

$$\begin{aligned} \frac{d^r}{dt^r} \left( \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!} (t - \alpha)^k \right) &= \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!} \cdot \frac{d^r}{dt^r} ((t - \alpha)^k) \\ &= \sum_{k=r+1}^{n-1} \frac{f^{(k)}(\alpha)}{k!} \cdot \frac{k!}{(k-r)!} (t - \alpha)^{k-r} + f^{(r)}(\alpha) \\ &= \sum_{k=r+1}^{n-1} \frac{f^{(k)}(\alpha)}{(k-r)!} (t - \alpha)^{k-r} + f^{(r)}(\alpha) \end{aligned}$$

Substituting  $t = \alpha$ , only the  $f^{(r)}(\alpha)$  term remains. Therefore, for  $r = 0, \dots, n-1$ ,  $g(\alpha) = g'(\alpha) = \dots = g^{(n-1)}(\alpha) = 0$ . Since  $g(\beta) = 0$  by definition, the Mean-Value Theorem implies there exists a  $x_1 \in (\alpha, \beta)$  s.t.  $g'(x_1) = \frac{g(\beta) - g(\alpha)}{\beta - \alpha} = 0$ .

And similarly, there is  $x_2 \in (x_1, \beta)$  s.t.  $g''(x_2) = \frac{g'(x_1) - g'(\alpha)}{\beta - \alpha} = 0$ .

Inductively, for some  $x_n \in (\alpha, \beta)$ ,  $g^{(n)}(x_n) = f^{(n)}(x_n) - n!M = 0$ . That is,  $M = \frac{f^{(n)}(x_n)}{n!}$ .

Proof Complete by Initial Setting. □

**Corollary 2.** Let  $f : [a, b] \rightarrow \mathbb{R}$  be an infinitely differentiable function.

Suppose that there exists a  $M > 0$  such that for any  $n \in \mathbb{N}$ ,  $\sup_{t \in [a, b]} |f^{(n)}(t)| \leq M$ . Then, for any  $x, \alpha \in [a, b]$ ,

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(\alpha)}{k!} (x - \alpha)^k$$

## 9.3 Convexity

### 9.3.1 Definition

**Definition 10.** Let  $f : (a, b) \rightarrow \mathbb{R}$  be a Real-valued function.  $f$  is said to be *convex* if: For any  $x, y \in (a, b), \lambda \in (0, 1)$ ,

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

Convex function has following properties:

**Lemma 8.** Let  $f : (a, b) \rightarrow \mathbb{R}$  be a Convex function, and  $a < x_1 < x_2 < x_3 < b$ . Then,

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(x_3) - f(x_1)}{x_3 - x_1} \leq \frac{f(x_3) - f(x_2)}{x_3 - x_2}$$

*Proof.* To show that first inequality, note that

$$\frac{x_2 - x_1}{x_3 - x_1} \cdot x_3 + \left(1 - \frac{x_2 - x_1}{x_3 - x_1}\right) \cdot x_1 = \frac{(x_2 - x_1)x_3 + (x_3 - x_2)x_1}{x_3 - x_1} = x_2$$

Now,

$$\begin{aligned} f(x_2) &= f\left(\frac{x_2 - x_1}{x_3 - x_1} \cdot x_3 + \left(1 - \frac{x_2 - x_1}{x_3 - x_1}\right) \cdot x_1\right) \\ &\leq \frac{x_2 - x_1}{x_3 - x_1} \cdot f(x_3) + \left(1 - \frac{x_2 - x_1}{x_3 - x_1}\right) \cdot f(x_1) = \frac{x_2 - x_1}{x_3 - x_1} \cdot f(x_3) + f(x_1) - \left(\frac{x_2 - x_1}{x_3 - x_1}\right) \cdot f(x_1) \end{aligned}$$

In brief,

$$f(x_2) - f(x_1) \leq \frac{x_2 - x_1}{x_3 - x_1} (f(x_3) - f(x_1)) \implies \frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(x_3) - f(x_1)}{x_3 - x_1}$$

And similarly,

$$\frac{x_3 - x_2}{x_3 - x_1} \cdot x_1 + \left(1 - \frac{x_3 - x_2}{x_3 - x_1}\right) x_3 = x_2$$

gives the second inequality. □



### 9.3.2 Properties

**Proposition 2.** If  $f : (a, b) \rightarrow \mathbb{R}$  is Convex, then  $f$  is Continuous.

**Proof.** Let  $\epsilon > 0$  be given,  $s < t$  are fixed in  $(a, b)$ . For any  $x, y \in (s, t)$  with  $s < x < y < t$ ,

$$\frac{f(s) - f(a)}{s - a} \leq \frac{f(x) - f(s)}{x - s} \leq \frac{f(y) - f(x)}{y - x} \leq \frac{f(t) - f(y)}{t - y} \leq \frac{f(b) - f(t)}{b - t}$$

Put  $M = \max \left\{ \left| \frac{f(s) - f(a)}{s - a} \right|, \left| \frac{f(b) - f(t)}{b - t} \right| \right\}$ . Then, for any  $x, y \in (s, t)$ ,

$$\left| \frac{f(y) - f(x)}{y - x} \right| \leq M$$

Now,

$$|f(y) - f(x)| \leq M|y - x| < \epsilon$$

Since  $s, t \in (a, b)$  was arbitrary,  $f$  is continuous on  $(a, b)$ . □

**Proposition 3.** Let  $f$  is differentiable on  $(a, b)$ . Then,

$f$  is Convex if and only if  $f'$  is monotonically increasing on  $(a, b)$ .

**Proof.** Prove by showing both directions: right and left.

**Right Direction** Let  $x_1 < x_2$  in  $(a, b)$ . Then,

$$f'(x_1) = \lim_{t \rightarrow x_1} \frac{f(t) - f(x_1)}{t - x_1} \leq \frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \lim_{\tau \rightarrow x_2} \frac{f(\tau) - f(x_2)}{\tau - x_2} = f'(x_2)$$

More rigorously, put  $\epsilon = \left| f'(x_1) - \frac{f(x_2) - f(x_1)}{x_2 - x_1} \right|$ . (If  $\epsilon = 0$ , then there is nothing to prove.).

Now, there exists a  $\delta > 0$  such that  $|t - x_1| < \delta$  implies

$$\left| f'(x_1) - \frac{f(t) - f(x_1)}{t - x_1} \right| < \epsilon \iff -\epsilon + \frac{f(t) - f(x_1)}{t - x_1} < f'(x_1) \stackrel{(*)}{=} \epsilon + \frac{f(t) - f(x_1)}{t - x_1}$$

If  $f'(x_1) - \frac{f(x_2) - f(x_1)}{x_2 - x_1} > 0$ , then  $(*)$  gives

$$f'(x_1) < f'(x_1) + \frac{f(t) - f(x_1)}{t - x_1} - \frac{f(x_2) - f(x_1)}{x_2 - x_1} \iff \frac{f(x_2) - f(x_1)}{x_2 - x_1} < \frac{f(t) - f(x_1)}{t - x_1} \quad \forall t \text{ s.t. } |t - x_1| < \delta$$

If  $|t - x_1| < |x_2 - x_1|$ , then this contradicts to Convexity.

Consequently, we obtain the first inequality, similarly can prove the second inequality.

**Left Direction** Let  $x, y \in (a, b)$  and  $\lambda \in (0, 1)$  be given. The Mean Value Theorem gives that:

$$f(\lambda x + (1 - \lambda)y) - f(x) = f'(z_1)(\lambda x + (1 - \lambda)y - x) \text{ for some } z_1 \in (x, \lambda x + (1 - \lambda)y)$$

$$f(y) - f(\lambda x + (1 - \lambda)y) = f'(z_2)(y - \lambda x + (1 - \lambda)y) \text{ for some } z_2 \in (\lambda x + (1 - \lambda)y, y)$$

Now, Monotonically increasing gives

$$\begin{aligned} \frac{f(\lambda x + (1 - \lambda)y) - f(x)}{\lambda x + (1 - \lambda)y - x} &= f'(z_1) \leq f'(z_2) = \frac{f(y) - f(\lambda x + (1 - \lambda)y)}{y - (\lambda x + (1 - \lambda)y)} \\ \implies \frac{f(\lambda x + (1 - \lambda)y) - f(x)}{(1 - x)(y - x)} &\leq \frac{f(y) - f(\lambda x + (1 - \lambda)y)}{\lambda(y - x)} \\ \implies \lambda f(\lambda x + (1 - \lambda)y) - \lambda f(x) &\leq (1 - \lambda)f(y) - (1 - \lambda)\lambda x + (1 - \lambda)y \\ \implies f(\lambda x + (1 - \lambda)y) &\leq \lambda f(x) + (1 - \lambda)f(y) \end{aligned}$$

□

**Corollary 3.** If  $f : [a, b] \rightarrow \mathbb{R}$  is twice-differentiable, then

$f$  is Convex if and only if  $f''(x) \geq 0$  for all  $x \in (a, b)$ .

**Theorem 15.** Let  $f : [a, b] \rightarrow \mathbb{R}$  be given. Then,

$f$  is Convex if and only if  $f$  is Continuous, and Midpoint Convex.

Midpoint convex is that  $f$  satisfies  $\forall x, y \in (a, b), f\left(\frac{x+y}{2}\right) \leq \frac{f(x) + f(y)}{2}$ .

*Proof.* The right direction is clear. To show the left direction, we demonstrate that Midpoint Convexity implies Dyadic Rational Convexity. Claim: For any  $n \in \mathbb{N}$ ,

$$f\left(\frac{\sum_{k=1}^{2^n} x_k}{2^n}\right) \leq \frac{1}{2^n} \sum_{k=1}^{2^n} f(x_k) \quad (*)$$

Using Induction: If  $n = 1$ , it is clear by Midpoint Convexity.

Assume that for  $n \in \mathbb{N}$ ,  $(*)$  is True. Then,

$$\begin{aligned} f\left(\frac{\sum_{k=1}^{2^{n+1}} x_k}{2^{n+1}}\right) &= f\left(\frac{1}{2} \cdot \left[\frac{\sum_{k=1}^{2^n} x_k}{2^n} + \frac{\sum_{k=2^n+1}^{2^{n+1}} x_k}{2^n}\right]\right) \\ &\stackrel{\text{m.c.}}{\leq} \frac{1}{2} \left( f\left(\frac{\sum_{k=1}^{2^n} x_k}{2^n}\right) + f\left(\frac{\sum_{k=2^n+1}^{2^{n+1}} x_k}{2^n}\right) \right) \\ &\stackrel{(*)}{\leq} \frac{1}{2} \left( \frac{1}{2^n} \sum_{k=1}^{2^n} f(x_k) + \frac{1}{2^n} \sum_{k=2^n+1}^{2^{n+1}} f(x_k) \right) \\ &= \frac{1}{2^{n+1}} \sum_{k=1}^{2^{n+1}} f(x_k) \end{aligned}$$

Consequently, we obtain the claim. Now, let  $n \in \mathbb{N}$ , and  $m$  be an integer such that  $1 \leq m \leq 2^n$ .

Put  $x_1 = x_2 = \dots = x_m = x$  and  $x_{m+1} = x_{m+2} = \dots = x_{2^n} = y$ . Then

$$f\left(\frac{m}{2^n}x + \left(1 - \frac{m}{2^n}\right)y\right) \leq \frac{m}{2^n}f(x) + \left(1 - \frac{m}{2^n}\right)f(y)$$

For complete this discussion, Let  $x, y \in (a, b), \lambda \in (0, 1)$  be given.

Since  $\frac{\lfloor 2^n \lambda \rfloor}{2^n} \rightarrow \lambda$  as  $n \rightarrow \infty$ , for any  $n \in \mathbb{N}$ ,

$$f\left(\frac{\lfloor 2^n \lambda \rfloor}{2^n}x + \left(1 - \frac{\lfloor 2^n \lambda \rfloor}{2^n}\right)y\right) \leq \frac{\lfloor 2^n \lambda \rfloor}{2^n}f(x) + \left(1 - \frac{\lfloor 2^n \lambda \rfloor}{2^n}\right)f(y)$$

Finally, taking limits then

$$\lim_{n \rightarrow \infty} f\left(\frac{\lfloor 2^n \lambda \rfloor}{2^n}x + \left(1 - \frac{\lfloor 2^n \lambda \rfloor}{2^n}\right)y\right) \stackrel{f \text{ cont.}}{=} f\left(\lim_{n \rightarrow \infty} \left[\frac{\lfloor 2^n \lambda \rfloor}{2^n}x + \left(1 - \frac{\lfloor 2^n \lambda \rfloor}{2^n}\right)y\right]\right) = f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

In brief, Midpoint Convexity implies Dyadic Rational Convexity, and with Continuous implies Convexity.  $\square$

## 9.4 Lipschitz Condition

### 9.4.1 Definition

**Definition 11.** A real-valued function  $f : (a, b) \rightarrow \mathbb{R}$  is called *Lipschitz Continuous* if:

$$\exists L \geq 0 \text{ s.t. } \forall x_1, x_2 \in (a, b), |f(x_1) - f(x_2)| \leq L \cdot |x_1 - x_2|$$

The constant  $L$  is said to be *Lipschitz Constant* of  $f$ . In particular, the constant

$$D \stackrel{\text{def}}{=} \sup_{x_1 \neq x_2} \frac{|f(x_1) - f(x_2)|}{|x_1 - x_2|}$$

is called *dilation* of  $f$ . Clearly,

$$\forall x_1, x_2 \in (a, b), |f(x_1) - f(x_2)| \leq D \cdot |x_1 - x_2|$$

and if  $L > 0$  is Lipschitz Constant of  $f$ , then  $D \leq L$ . That is,  $D = \inf\{L > 0 \mid L \text{ is Lipschitz constant of } f\}$ .

### 9.4.2 Properties

**Proposition 4.** If  $f : (a, b) \rightarrow \mathbb{R}$  is Lipschitz Continuous, then  $f$  is uniformly continuous.

*Proof.* Let  $L \geq 0$  be a Lipschitz Constant of  $f$ . Then, for any  $\epsilon > 0$ ,

$$\forall x, y \in (a, b), |x - y| < \frac{\epsilon}{L} \implies |f(x) - f(y)| \leq L|x - y| < \epsilon$$

□

**Proposition 5.** Let  $f : (a, b) \rightarrow \mathbb{R}$  be a Differentiable function. Then,

$f$  is Lipschitz Continuous if and only if  $f'$  is bounded in  $(a, b)$ .

*Proof.*

*Right Direction*

Let  $L > 0$  be a Lipschitz constant of  $f$ , and  $x \in (a, b)$  be given. Since definition of derivative,

$$f'(x) \stackrel{\text{def}}{=} \lim_{t \rightarrow x} \frac{f(x) - f(t)}{x - t}$$

Meanwhile, the assumption gives: for any distinct  $x, t \in (a, b)$ ,

$$\frac{|f(x) - f(t)|}{|x - t|} \leq L$$

Therefore,

$$f'(x) = \lim_{t \rightarrow x} \frac{f(x) - f(t)}{x - t} \leq \lim_{t \rightarrow x} \frac{|f(x) - f(t)|}{|x - t|} \leq \lim_{t \rightarrow x} L = L$$

*Left Direction*

Let distinct  $x, y \in (a, b)$  be given. Then, the Mean-Value Theorem gives: There exists a  $z \in (x, y)$  such that

$$f(x) - f(y) = f'(z)(x - y) \implies f'(z) = \frac{f(x) - f(y)}{x - y}$$

Now,

$$\left| \frac{f(x) - f(y)}{x - y} \right| = |f'(z)| \leq L \implies |f(x) - f(y)| \leq L \cdot |x - y|$$

If  $x = y$ , then there is nothing to prove.

□

Note that:

$$\text{Lipschitz Continuous} \implies \text{Uniformly Continuous} \implies \text{Continuous}$$

### 9.4.3 Newton-Raphson Method

#### Theorem 16. Newton-Raphson Method

Let  $f : [a, b] \rightarrow \mathbb{R}$  be a twice-differentiable,  $f(a) < 0 < f(b)$ . Suppose that  $f$  satisfies: for all  $x \in [a, b]$ ,

$$f'(x) \geq \delta > 0 \text{ and } 0 \leq f''(x) \leq M$$

That is,  $f$  is strictly increasing convex function, and Lipschitz Continuous.

Further, there uniquely exists  $x^* \in (a, b)$  such that  $f(x^*) = 0$ .

Let  $x_1 \in (x^*, b)$  fixed. Define a sequence  $\{x_n\}$  inductively as follows:

$$x_{n+1} \stackrel{\text{def}}{=} x_n - \frac{f(x_n)}{f'(x_n)}$$

Then,  $\{x_n\}$  satisfies the following three conditions:

1.  $\{x_n\}$  is decreasing sequence.
2.  $x_n \rightarrow x^*$  as  $n \rightarrow \infty$ .
3. For any  $n \in \mathbb{N}$ ,  $0 \leq x_{n+1} - x^* \leq \left[\frac{M}{2\delta}\right]^{2^{n+1}-1} [x_1 - x^*]^{2^n}$ .

Condition 3 means that for a suitable initial value  $x_1$ , we can establish an upper bound for the error.

*Proof.* This proof consists by three steps.

Since  $f''$  is non-negative, and  $f'$  is positive,  $f$  is strictly increasing convex function.

And Fundamental Theorem of Calculus gives: for any  $x \in (a, b)$ ,

$$f'(x) \stackrel{\text{FTC}}{=} \int_a^x f''(t)dt + f'(a) \leq \int_a^x Mdt + f'(a) = M(x - a) + f'(a) \leq M(b - a) + f'(a)$$

Thus,  $f'$  is bounded on  $(a, b)$ , thus  $f$  is Lipschitz Continuous.

*Step 1.*  $f$  has a unique root  $x^*$ .

The existence of root given directly by Intermediate-Value theorem.

Suppose that  $x^*, x' \in (a, b)$  are distinct root of  $f$ . i.e.,  $f(x^*) = f(x') = 0$ . Then, by Mean-value theorem, there is  $c \in (a, b)$  between  $x^*$  and  $x'$  such that

$$f'(c)(x^* - x') = f(x^*) - f(x') = 0$$

That is,  $f'(c) = 0$ . This is contradiction with  $f'$  is positive.

*Step 2.*  $\{x_n\}$  decrease.

Proof by induction:

For  $n = 1$ ,  $f'(x_1)(x_1 - x_2) \stackrel{\text{def}}{=} f(x_1) > f(x^*) = 0$ , thus  $x_2 < x_1$ . And,

$$\begin{aligned} f(x_2) &\stackrel{\text{MVT}}{=} f(x_1) + f'(c_1)(x_2 - x_1) \quad \text{for some } c_1 \in (x_2, x_1) \\ &> f(x_1) + f'(x_1)(x_2 - x_1) = f'(x_1)(x_1 - x_2) + f'(x_1)(x_2 - x_1) = 0 \end{aligned}$$

Now, since  $f(x_2) > 0 = f(x^*)$ , the Mean-Value Theorem implies that  $x_2 > x^*$ .

To use induction, suppose that for some  $n \geq 1$ ,  $x^* < x_{n+1} < x_n$ . Then,

$$f(x_{n+1}) = f'(x_{n+1})(x_{n+1} - x_{n+2}) > 0$$

Thus  $x_{n+2} < x_{n+1}$  and

$$\begin{aligned} f(x_{n+2}) &\stackrel{\text{MVT}}{=} f(x_{n+1}) + f'(c_{n+1})(x_{n+2} - x_{n+1}) \quad \text{for some } c_{n+1} \in (x_{n+2}, x_{n+1}) \\ &\geq f(x_{n+1}) + f'(x_{n+1})(x_{n+2} - x_{n+1}) \\ &= f(x_{n+1})(x_{n+1} - x_{n+2}) + f'(x_{n+1})(x_{n+2} - x_{n+1}) = 0 \end{aligned}$$

Again, the Mean-Value Theorem implies that  $x_{n+2} > x^*$ . Therefore, induction completes.

Now,  $x_n \rightarrow x'$  as  $n \rightarrow \infty$  for some  $x' \in [x^*, x_1]$  since  $\{x_n\}$  is Bounded below and Decreasing.

Still it remains that to show  $x' = x^*$ . By Continuity,

$$\begin{aligned} &f'(x_n)(x_{n+1} - x_n) + f(x_n) = 0 \\ \implies &\lim_{n \rightarrow \infty} [f'(x_n)(x_{n+1} - x_n) + f(x_n)] = f\left(\lim_{n \rightarrow \infty} x_n\right) = f(x') = 0 \end{aligned}$$

Since the root of  $f$  is unique, thus  $x' = x^*$ .

*Step 3. Establishing the error bound.*

The Taylor's Theorem implies that

$$f(x^*) = f(x_n) + f'(x_n)(x^* - x_n) + \frac{f''(t_n)}{2}(x^* - x_n)^2 \quad \text{for some } t_n \in (x^*, x_n)$$

$$\implies x_{n+1} - x^* = \frac{f''(t_n)}{2f'(x_n)}(x^* - x_n)^2$$

Consequently,

$$0 \leq x_{n+1} - x^* = \frac{f''(t_n)}{2f'(x_n)}(x_n - x^*)^2 = \frac{f''(t_n)}{2f'(x_n)} \cdot \left( \frac{f''(t_{n-1})}{2f'(x_{n-1})} \right)^2 (x_{n-1} - x^*)^4 = \dots$$

$$= \prod_{i=1}^n \left[ \frac{f''(t_i)}{2f'(x_i)} \right]^{2^{(n+1-i)}} [x_1 - x^*]^{2^n} \leq \left[ \frac{M}{2\delta} \right]^{2^{n+1}-1} [x_1 - x^*]^{2^n}$$

□

### 9.4.4 Gradient Descent

**Theorem 17.** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a differentiable function that satisfies the following conditions:

1.  $f$  is *Convex function*.
2.  $f'$  is *Lipschitz Continuous* with Lipschitz constant of  $f$ ,  $L > 0$ . In this,  $f$  is called *L-Smooth*.
3.  $f$  has at least one local minimizer  $x^*$ .

Then,  $x^*$  is a Global minimizer of  $\mathbb{R}$ , and there exists a unique closed interval  $M$  containing  $x^*$  such that

$$\forall x \in M, t \notin M, f(x) = f(x^*) < f(t)$$

And, given initial point  $x_0 \in \mathbb{R}$  and  $0 < \gamma \leq \frac{1}{L}$ , define a sequence  $\{x_n\}$  inductively as follows:

$$x_{n+1} = x_n - \gamma \cdot f'(x_n)$$

Then, for any  $N \in \mathbb{N}$ ,

$$f(x_N) - f(x^*) \leq \frac{|x_0 - x^*|^2}{2\gamma N}$$

**Proof.** Let  $x^* \in \mathbb{R}$  be a local minimizer. That is, there exists a  $\delta > 0$  such that  $\forall t \in (x^* - \delta, x^* + \delta)$ ,  $f(x^*) \leq f(t)$ . Then,

$$0 \leq \lim_{t \rightarrow x^*+} \frac{f(x^*) - f(t)}{x^* - t} = f'(x^*) = \lim_{t \rightarrow x^*-} \frac{f(x^*) - f(t)}{x^* - t} \leq 0$$

thus,  $f'(x^*) = 0$ . And, by convexity,  $f'$  is monotonically increasing. Now, The Fundamental Theorem of Calculus gives:

$$\forall x \in \mathbb{R}, f(x) = \int_{x^*}^x f'(t)dt + f(x^*) \geq f(x^*)$$

Therefore,  $x^*$  is a Global minimizer of  $f$ .

Now, establish the closed interval  $M$ . Since  $f'$  is Lipschitz Continuous, thus  $f'$  is Continuous.

Let  $D \stackrel{\text{def}}{=} \{x \in \mathbb{R} \mid f'(x) = 0\}$ . (Note that:  $x^* \in D$ , thus  $D$  is not empty set.)

$D$  is closed because: Let  $\{x_n\}$  be a convergent sequence in  $D$ . That is, for all  $n \in \mathbb{N}$ ,  $f'(x_n) = 0$ . Then, by continuity,

$$f' \left( \lim_{n \rightarrow \infty} x_n \right) = \lim_{n \rightarrow \infty} f'(x_n) = 0$$

The limit of  $\{x_n\}$  is contained in  $D$ , thus  $D$  is closed.

And,  $D$  is interval: i.e, for any  $x \in (\inf D, \sup D)$ ,  $x \in D$  because:

Suppose that there exists  $x \in (\inf D, \sup D)$  such that  $x \notin D$ . That is,  $f'(x) \neq 0$ . This is Contradiction with Monotonicity.

To set error of upper bound, we make inequalities: Let  $x, y \in \mathbb{R}$  be given.

The Fundamental Theorem of Calculus and  $L$ -Smooth condition gives:

$$\begin{aligned} f(y) - f(x) &= \int_x^y f'(t)dt = \int_0^1 f'(x + (y-x)u)(y-x)du = f'(x)(y-x) + \int_0^1 (f'(x + (y-x)u) - f'(x))(y-x)du \\ &\stackrel{2.}{\leq} f'(x)(y-x) + L \cdot |y-x|^2 \int_0^1 u \, du = f'(x)(y-x) + \frac{L}{2}|y-x|^2 \end{aligned}$$

For any  $\lambda > 0$ , Put  $y = x - \lambda f'(x)$ . Then,

$$f(x - \lambda f'(x)) \leq f(x) - f'(x)(\lambda f'(x)) + \frac{L}{2}|\lambda f'(x)|^2 = f(x) + \lambda \left( \frac{L\lambda}{2} - 1 \right) |f'(x)|^2$$

Put  $\lambda = \frac{1}{L}$ , then

$$f \left( x - \frac{f'(x)}{L} \right) \leq f(x) - \frac{L}{2}|f'(x)|^2 \implies \frac{L}{2}|f'(x)|^2 \leq f(x) - f \left( x - \frac{f'(x)}{L} \right) \leq f(x) - \inf f$$

Meanwhile, the convexity gives: for any  $x, y \in \mathbb{R}$ ,

$$f'(x)(y-x) \leq f(y) - f(x) \leq f'(y)(y-x)$$

since derivative of convex function increase monotonically. Put  $z = y - \frac{1}{L}(f'(y) - f'(x))$ . Then,

$$\begin{aligned} f(x) - f(y) &= f(x) - f(z) + f(z) - f(y) \\ &\leq f'(x)(x-z) + f'(y)(z-y) + \frac{L}{2}|z-y|^2 \\ &= f'(x)\left(x-y + \frac{1}{L}(f'(y) - f'(x))\right) - f'(y)\left(\frac{1}{L}(f'(y) - f'(x))\right) + \frac{L}{2}\left|\frac{1}{L}(f'(y) - f'(x))\right|^2 \\ &= f'(x)(x-y) - \frac{1}{L}|f'(y) - f'(x)|^2 + \frac{1}{2L}|f'(y) - f'(x)|^2 \\ &= f'(x)(x-y) - \frac{1}{2L}|f'(y) - f'(x)|^2 \end{aligned}$$

Now,

$$\begin{cases} \frac{1}{2L}|f'(y) - f'(x)|^2 \leq f'(x)(x-y) - (f(x) - f(y)) \\ \frac{1}{2L}|f'(x) - f'(y)|^2 \leq f'(y)(y-x) - (f(y) - f(x)) \end{cases} \implies \frac{1}{L}|f'(y) - f'(x)|^2 \leq (f'(y) - f'(x))(y-x)$$

Since above inequalities, we obtain that

$$\begin{aligned} |x_{n+1} - x^*|^2 &= |x_n - \gamma \cdot f'(x_n) - x^*|^2 = |(x_n - x^*) - \gamma \cdot f'(x_n)|^2 \\ &= |x_n - x^*|^2 - 2\gamma|f'(x_n)| \cdot |x_n - x^*| + \gamma^2|f'(x_n)|^2 \\ &\leq |x_n - x^*|^2 - 2\gamma\frac{1}{L}|f'(x_n)|^2 + \gamma^2|f'(x_n)|^2 \\ &= |x_n - x^*|^2 + \left(\gamma^2 - \frac{2\gamma}{L}\right)|f'(x_n)|^2 \leq |x_n - x^*|^2 \end{aligned}$$

Thus,  $|x_n - x^*|$  decrease as  $n \rightarrow \infty$ . That is,  $|x_n - x^*| \leq |x_0 - x^*|$  for all  $n \in \mathbb{N}$ .

Consider  $x_{n+1}$  and  $x_n$ . First, we obtain

$$\begin{aligned} f(x_{n+1}) &\leq f(x_n) + f'(x_n)(x_{n+1} - x_n) + \frac{L}{2}|x_{n+1} - x_n|^2 \\ &= f(x_n) - \gamma|f'(x_n)|^2 + \frac{L}{2}\gamma^2|f'(x_n)|^2 \\ &= f(x_n) - \left(\gamma - \frac{L}{2}\gamma^2\right)|f'(x_n)|^2 \end{aligned}$$

Subtracting  $f(x^*)$  above, then

$$f(x_{n+1}) - f(x^*) \leq f(x_n) - f(x^*) - \left(\gamma - \frac{L}{2}\gamma^2\right)|f'(x_n)|^2$$

Meanwhile, Convexity gives

$$f(x_n) - f(x^*) \leq f'(x_n)(x_n - x^*) \leq |f'(x_n)||x_n - x^*| \leq |f'(x_n)||x_0 - x^*|$$

Combining above two inequalities,

$$f(x_{n+1}) - f(x^*) \leq f(x_n) - f(x^*) - \left(\gamma - \frac{L}{2}\gamma^2\right) \cdot \frac{|f(x_n) - f(x^*)|^2}{|x_0 - x^*|^2}$$

Dividing Both Sides by  $(f(x_{n+1}) - f(x^*))(f(x_n) - f(x^*))$ ,

$$\begin{aligned} \frac{1}{f(x_n) - f(x^*)} &\leq \frac{1}{f(x_{n+1}) - f(x^*)} - \left(\gamma - \frac{L}{2}\gamma^2\right) \cdot \frac{f(x_n) - f(x^*)}{f(x_{n+1}) - f(x^*)} \frac{1}{|x_0 - x^*|^2} \\ \implies \left(\gamma - \frac{L}{2}\gamma^2\right) \cdot \frac{f(x_n) - f(x^*)}{f(x_{n+1}) - f(x^*)} \frac{1}{|x_0 - x^*|^2} &\leq \frac{1}{f(x_{n+1}) - f(x^*)} - \frac{1}{f(x_n) - f(x^*)} \\ \implies \left(\gamma - \frac{L}{2}\gamma^2\right) \cdot \frac{1}{|x_0 - x^*|^2} &\leq \frac{1}{f(x_{n+1}) - f(x^*)} - \frac{1}{f(x_n) - f(x^*)} \\ \implies \sum_{n=0}^{N-1} \left[ \left(\gamma - \frac{L}{2}\gamma^2\right) \cdot \frac{1}{|x_0 - x^*|^2} \right] &\leq \sum_{n=0}^{N-1} \left[ \frac{1}{f(x_{n+1}) - f(x^*)} - \frac{1}{f(x_n) - f(x^*)} \right] = \frac{1}{f(x_N) - f(x^*)} - \frac{1}{f(x_0) - f(x^*)} \end{aligned}$$

Consequently,

$$\frac{2\gamma N}{|x_0 - x^*|^2} \leq N \cdot \left[ \left( \gamma - \frac{L}{2} \gamma^2 \right) \cdot \frac{1}{|x_0 - x^*|^2} \right] \leq \frac{1}{f(x_N) - f(x^*)} - \frac{1}{f(x_0) - f(x^*)} \leq \frac{1}{f(x_N) - f(x^*)}$$

Organizing the formula, as result:

$$f(x_N) - f(x^*) \leq \frac{|x_0 - x^*|^2}{2\gamma N}$$

□



## 9.5 Integral

### 9.5.1 Inequality of Riemann–Stieltjes Integral

Let  $p, q \geq 1$  such that  $\frac{1}{p} + \frac{1}{q} = 1$ , and functions lying on  $[a, b]$ .

**Lemma 9.** Let  $f, g \in \mathcal{R}(\alpha)$  with  $f, g \geq 0$ , and  $\int_a^b [f(x)]^p d\alpha = \int_a^b [g(x)]^q d\alpha = 1$ . Then,  $\int_a^b f(x)g(x) d\alpha \leq 1$ .

*Proof.* For any  $x \in [a, b]$ , the Young's Inequality gives

$$0 \leq f(x)g(x) \leq \frac{[f(x)]^p}{p} + \frac{[g(x)]^q}{q}$$

Now,

$$\int_a^b f(x)g(x) d\alpha \leq \int_a^b \frac{[f(x)]^p}{p} + \frac{[g(x)]^q}{q} d\alpha = \frac{1}{p} \int_a^b [f(x)]^p d\alpha + \frac{1}{q} \int_a^b [g(x)]^q d\alpha = \frac{1}{p} + \frac{1}{q} = 1$$

□

**Definition 12.** Let  $f \in \mathcal{R}(\alpha)$ . Define a *Norm* of  $f$ :

$$\|f\|_p \stackrel{\text{def}}{=} \left( \int_a^b |f(x)|^p d\alpha \right)^{\frac{1}{p}}$$

This becomes actually norm of set of Stieltjes Integrable functions,  $\mathcal{F} \stackrel{\text{def}}{=} \{f : [a, b] \rightarrow \mathbb{C} \mid f \in \mathcal{R}(\alpha)\}$ .

**Lemma 10. Hölder's Inequality**

Let  $f, g \in \mathcal{F}$ . Then,

$$\left| \int_a^b f(x)g(x) d\alpha \right| \leq \left[ \int_a^b |f(x)|^p d\alpha \right]^{\frac{1}{p}} \cdot \left[ \int_a^b |g(x)|^q d\alpha \right]^{\frac{1}{q}}$$

*Proof.* Use above definition, Rewrite:

$$\|f\|_p^p = \int_a^b |f(x)|^p d\alpha, \quad \|g\|_q^q = \int_a^b |g(x)|^q d\alpha$$

Now, we can make the condition of above lemma,

$$\int_a^b \left[ \frac{|f(x)|}{\|f\|_p} \right]^p d\alpha = \frac{1}{\|f\|_p^p} \cdot \int_a^b |f(x)|^p d\alpha = 1, \quad \int_a^b \left[ \frac{|g(x)|}{\|g\|_q} \right]^q d\alpha = \frac{1}{\|g\|_q^q} \cdot \int_a^b |g(x)|^q d\alpha = 1$$

And apply this,

$$\int_a^b \frac{|f(x)| \cdot |g(x)|}{\|f\|_p \|g\|_q} d\alpha \leq 1 \implies \int_a^b |f(x)| |g(x)| d\alpha \leq \|f\|_p \|g\|_q = \left[ \int_a^b |f(x)|^p d\alpha \right]^{\frac{1}{p}} \cdot \left[ \int_a^b |g(x)|^q d\alpha \right]^{\frac{1}{q}}$$

Finally, the general property of integral of product gives

$$\left| \int_a^b f(x)g(x) d\alpha \right| \leq \int_a^b |f(x)| |g(x)| d\alpha \leq \|f\|_p \|g\|_q = \left[ \int_a^b |f(x)|^p d\alpha \right]^{\frac{1}{p}} \cdot \left[ \int_a^b |g(x)|^q d\alpha \right]^{\frac{1}{q}}$$

□

**Theorem 18. Minkowski inequality**

Let  $f, g \in \mathcal{F}$ . Then, for any  $p \geq 1$ ,  $\|f + g\|_p \leq \|f\|_p + \|g\|_p$ .

*Proof.*

$$\begin{aligned}
\|f + g\|_p^p &= \int_a^b |f + g|^p d\alpha = \int_a^b |f + g| |f + g|^{p-1} d\alpha \\
&\leq \int_a^b [|f| + |g|] |f + g|^{p-1} d\alpha \\
&= \int_a^b |f| |f + g|^{p-1} d\alpha + \int_a^b |g| |f + g|^{p-1} d\alpha \\
&\stackrel{\text{Hölder}}{\leq} \left[ \int_a^b |f|^p d\alpha \right]^{\frac{1}{p}} \left[ \int_a^b |f + g|^{(p-1)\frac{p}{p-1}} d\alpha \right]^{\frac{p-1}{p}} + \left[ \int_a^b |g|^p d\alpha \right]^{\frac{1}{p}} \left[ \int_a^b |f + g|^{(p-1)\frac{p}{p-1}} d\alpha \right]^{\frac{p-1}{p}} \\
&= \left[ \int_a^b |f + g|^p d\alpha \right]^{\frac{p-1}{p}} \left( \left[ \int_a^b |f|^p d\alpha \right]^{\frac{1}{p}} + \left[ \int_a^b |g|^p d\alpha \right]^{\frac{1}{p}} \right) = \|f + g\|_p^{p-1} \cdot (\|f\|_p + \|g\|_p)
\end{aligned}$$

Now,

$$\|f + g\|_p^p \cdot \|f + g\|_p^{1-p} = \|f + g\|_p \leq \|f\|_p + \|g\|_p$$

□

## Chapter 10

### Measure

## Chapter 11

# Complex Analysis

## Chapter 12

# Differential Geometry

## Chapter 13

# Differential Equation

# Chapter 14

## Spaces

### 14.1 $\mathbb{R}^n$

#### 14.1.1 Inner Product in $\mathbb{R}$

#### 14.1.2 $p$ -norm in $\mathbb{R}^n$

**Definition 13.** Let  $\mathbb{R}^n$  be given. Define  $p$ -norm on  $\mathbb{R}^n$  as:

$$d_p : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R} : (\mathbf{x}, \mathbf{y}) \mapsto \left( \sum_{i=1}^n |x_i - y_i|^p \right)^{\frac{1}{p}}, \quad (\mathbf{x} = (x_1, \dots, x_n), \mathbf{y} = (y_1, \dots, y_n))$$

where  $p \in [1, \infty]$ . In particular,  $p$ -norm is a *Metric*, being *Minkowski inequality*.

**Lemma 11. Young's inequality**

Let  $u, v > 0$ , and  $p, q \in [1, \infty]$  such that  $\frac{1}{p} + \frac{1}{q} = 1$ .

Then,

$$uv \leq \frac{1}{p}u^p + \frac{1}{q}v^q$$

**Proof.** Since  $f(x) = \log x$  is concave, we obtain

$$\forall \lambda \in [0, 1], \quad \lambda f(x) + (1 - \lambda)f(y) \leq f(\lambda x + (1 - \lambda)y)$$

thus,

$$\log \left( \frac{1}{p}u^p + \frac{1}{q}v^q \right) \geq \frac{1}{p} \log(u^p) + \frac{1}{q} \log(v^q) = \log(uv)$$

Since  $\exp(x)$  increasing, we get

$$\exp \left( \log \left( \frac{1}{p}u^p + \frac{1}{q}v^q \right) \right) \geq \exp(\log(uv))$$

i.e.,

$$uv \leq \frac{1}{p}u^p + \frac{1}{q}v^q$$

□

**Lemma 12. Holder's inequality**

Let  $x = (x_1, \dots, x_n)$  and  $y = (y_1, \dots, y_n)$  be given, and  $p, q \in [1, \infty]$  such that  $\frac{1}{p} + \frac{1}{q} = 1$ . Then,

$$\sum_{i=1}^n |x_i y_i| \leq \left( \sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}} \cdot \left( \sum_{i=1}^n |y_i|^q \right)^{\frac{1}{q}}$$

*Proof.* Denote that

$$\|x\|_p \stackrel{\text{def}}{=} \left( \sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}}$$

Then, since young's inequality, for each  $i \in \{1, 2, \dots, n\}$ ,

$$\frac{|x_i|}{\|x\|_p} \cdot \frac{|y_i|}{\|y\|_q} \leq \frac{1}{p} \cdot \frac{|x_i|^p}{\|x\|_p^p} + \frac{1}{q} \cdot \frac{|y_i|^q}{\|y\|_q^q}$$

Summing for all  $i = 1, 2, \dots, n$ :

$$\frac{1}{\|x\|_p \|y\|_q} \cdot \sum_{i=1}^n |x_i y_i| \leq \frac{1}{p} + \frac{1}{q} = 1$$

Therefore,

$$\sum_{i=1}^n |x_i y_i| \leq \left( \sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}} \cdot \left( \sum_{i=1}^n |y_i|^q \right)^{\frac{1}{q}}$$

□

**Theorem 19. Minkowski inequality**

Given complex-valued sequences  $\{x_i\}_{i=1}^n$  and  $\{y_i\}_{i=1}^n$ ,

$$\left[ \sum_{i=1}^n |x_i + y_i|^p \right]^{\frac{1}{p}} \leq \left[ \sum_{i=1}^n |x_i|^p \right]^{\frac{1}{p}} + \left[ \sum_{i=1}^n |y_i|^p \right]^{\frac{1}{p}}$$

*Proof.* Denote

$$|x_i + y_i|^p = |x_i + y_i| \cdot |x_i + y_i|^{p-1}$$

Then,

$$\begin{aligned} \sum_{i=1}^n |x_i + y_i|^p &= \sum_{i=1}^n |x_i + y_i| \cdot |x_i + y_i|^{p-1} \\ &= \sum_{i=1}^n (|x_i| + |y_i|) \cdot |x_i + y_i|^{p-1} \\ &= \sum_{i=1}^n |x_i| \cdot |x_i + y_i|^{p-1} + \sum_{i=1}^n |y_i| \cdot |x_i + y_i|^{p-1} \\ &\stackrel{\text{Hölder}}{\leq} \left[ \sum_{i=1}^n |x_i|^p \right]^{\frac{1}{p}} \cdot \left[ \sum_{i=1}^n |x_i + y_i|^p \right]^{\frac{p-1}{p}} + \left[ \sum_{i=1}^n |y_i|^p \right]^{\frac{1}{p}} \cdot \left[ \sum_{i=1}^n |x_i + y_i|^p \right]^{\frac{p-1}{p}} \\ &= \left[ \left( \sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}} + \left( \sum_{i=1}^n |y_i|^p \right)^{\frac{1}{p}} \right] \cdot \left[ \sum_{i=1}^n |x_i + y_i|^p \right]^{\frac{p-1}{p}} \end{aligned}$$



Now, Divide each side as  $\left[\sum_{i=1}^n |x_i + y_i|^p\right]^{\frac{p-1}{p}}$ , then we obtain

$$\left[\sum_{i=1}^n |x_i + y_i|^p\right]^{1-\frac{p-1}{p}} = \left[\sum_{i=1}^n |x_i + y_i|^p\right]^{\frac{1}{p}} \leq \left[\left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^n |y_i|^p\right)^{\frac{1}{p}}\right]$$

□

**Theorem 20.** Let  $d_{p_1}, d_{p_2}$  are  $p$ -norm on  $\mathbb{R}^n$  with  $1 \leq p_1 < p_2 \leq \infty$ . Then,

$$\exists C > 0 \text{ s.t. } \forall x, y \in \mathbb{R}^n, d_{p_2}(x, y) \leq d_{p_1}(x, y) \leq C d_{p_2}(x, y)$$

In particular,  $C = n^{\frac{1}{p_1} - \frac{1}{p_2}}$ .

**Proof.** Let  $p_1 < p_2$ .

For show that first-inequality,

$$1 = \sum_{i=1}^n \left[ \frac{|x_i - y_i|}{\left[\sum_{i=1}^n |x_i - y_i|^{p_2}\right]^{\frac{1}{p_2}}} \right]^{p_2} \leq \sum_{i=1}^n \left[ \frac{|x_i - y_i|}{\left[\sum_{i=1}^n |x_i - y_i|^{p_2}\right]^{\frac{1}{p_2}}} \right]^{p_1} = \frac{\sum_{i=1}^n |x_i - y_i|^{p_1}}{\left[\sum_{i=1}^n |x_i - y_i|^{p_2}\right]^{\frac{p_1}{p_2}}} = \left[ \frac{\left[\sum_{i=1}^n |x_i - y_i|^{p_1}\right]^{\frac{1}{p_1}}}{\left[\sum_{i=1}^n |x_i - y_i|^{p_2}\right]^{\frac{1}{p_2}}} \right]^{p_1}$$

Thus, we obtain that:

$$1 \leq \left[ \frac{\left[\sum_{i=1}^n |x_i - y_i|^{p_1}\right]^{\frac{1}{p_1}}}{\left[\sum_{i=1}^n |x_i - y_i|^{p_2}\right]^{\frac{1}{p_2}}} \right]^{p_1} \iff 1 \leq \frac{\left[\sum_{i=1}^n |x_i - y_i|^{p_1}\right]^{\frac{1}{p_1}}}{\left[\sum_{i=1}^n |x_i - y_i|^{p_2}\right]^{\frac{1}{p_2}}} \iff \left[\sum_{i=1}^n |x_i - y_i|^{p_2}\right]^{\frac{1}{p_2}} \leq \left[\sum_{i=1}^n |x_i - y_i|^{p_1}\right]^{\frac{1}{p_1}}$$

For show that second-inequality, using Hölder's inequality.

$$\begin{aligned} (d_{p_1}(x, y))^{p_1} &= \sum_{i=1}^n |x_i - y_i|^{p_1} = \sum_{i=1}^n |x_i - y_i|^{p_1} \cdot 1 \\ &\stackrel{\text{Hölder}}{\leq} \left[ \sum_{i=1}^n \left( |x_i - y_i|^{p_1 \cdot \frac{p_2}{p_1}} \right) \right]^{\frac{p_1}{p_2}} \cdot \left[ \sum_{i=1}^n 1^{\frac{p_2}{p_2 - p_1}} \right]^{1 - \frac{p_1}{p_2}} = \left[ \sum_{i=1}^n (|x_i - y_i|^{p_2}) \right]^{\frac{p_1}{p_2}} \cdot n^{1 - \frac{p_1}{p_2}} \end{aligned}$$

Taking the  $\frac{1}{p_1}$ -th power of both sides, then

$$d_{p_1}(x, y) \leq \left[ \sum_{i=1}^n (|x_i - y_i|^{p_2}) \right]^{\frac{1}{p_2}} \cdot n^{\frac{1}{p_1} - \frac{1}{p_2}} = n^{\frac{1}{p_1} - \frac{1}{p_2}} \cdot d_{p_2}(x, y)$$

□

**Corollary 4.** Let  $\mathbb{R}^n$  be given as a set, and  $d_{p_1}, d_{p_2} : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$  are  $p$ -norm on  $\mathbb{R}^n$ . Then,

$$\mathcal{T}_{d_{p_1}} = \mathcal{T}_{d_{p_2}}$$

For every  $p \geq 1$ , the metric space  $(\mathbb{R}^n, d_p)$  induces the same topology as the product topology on  $\mathbb{R}^n$ . In particular,  $\mathbb{R}^n$  with the product topology coincides with  $\mathbb{R}^n$  endowed with any  $p$ -norm.

### 14.1.3 Open and Closed set in $\mathbb{R}^n$

**Definition 14.** For  $p \in [1, \infty]$ , define  $p$ -Ball in  $\mathbb{R}^n$  as:

$$B_p(x, r) \stackrel{\text{def}}{=} \{y \in \mathbb{R}^n : \|x - y\|_p < r\}$$

Since all  $p$ -norms are equivalent, for any  $p \in [1, \infty]$ , the collection

$$\beta_p \stackrel{\text{def}}{=} \{B_p(x, r) \mid x \in \mathbb{Q}^n, r \in \mathbb{Q}^+\} \cup \{\emptyset, \mathbb{R}^n\}$$

is Countable basis of  $\mathbb{R}^n$ . Immediately, we obtain:

**Lemma 13.** Every open set in  $\mathbb{R}^n$  is a countable union of  $p$ -Balls.

## 14.2 Topological Vector Space

## 14.3 Hilbert Space

**Definition 15.** Complete Inner product Vector Space is called *Hilbert Space*.

### 14.3.1 Hilbert Space in $\mathbb{R}^\omega$

**Definition 16.** Define  $\mathbb{R}^\omega \stackrel{\text{def}}{=} \prod_{i=1}^{\infty} \mathbb{R}$  as the countable product of Euclidean space  $\mathbb{R}$  with product topology.

And define  $\mathbb{H} \stackrel{\text{def}}{=} \left\{ \{x_n\}_{n=1}^{\infty} \mid \sum_{n=1}^{\infty} x_n^2 < \infty \right\} \subset \mathbb{R}^\omega$ , **Metric** on  $\mathbb{H}$  as  $\mu : \mathbb{H} \times \mathbb{H} \rightarrow \mathbb{R} : (\{x_n\}, \{y_n\}) \mapsto \sqrt{\sum_{i=1}^{\infty} (x_i - y_i)^2}$ .

The Metric Space  $(\mathbb{H}, \mu)$  is called *Hilbert Space* or  $l_2$  Space.

Define the operations elementwise; then  $(\mathbb{H}, +, \times)$  is a Vector Space over  $\mathbb{R}$ .

Moreover,  $\mathbb{H}$  is Complete Metric Space and Inner product Vector Space.

**Lemma 14.**  $\mu : \mathbb{H} \times \mathbb{H} \rightarrow \mathbb{R} : (\{x_n\}, \{y_n\}) \mapsto \sqrt{\sum_{i=1}^{\infty} (x_i - y_i)^2}$  is Metric function induced by the inner product.

**Proof.** We know that  $\mathbb{R}^\omega$  is Vector Space. Moreover,  $\mathbb{H} \subset \mathbb{R}^\omega$  is Subspace. Using subspace criteria:

$S \subset V$  is Subspace of Vector Space  $V$  if and only if  $0 \in S$  and For any  $x, y \in S$  and  $a \in F$ ,  $ax + y \in S$ .

Clearly,  $\{0\} \in \mathbb{H}$ . Let  $a \in \mathbb{R}$  and  $\{x_n\}, \{y_n\} \in \mathbb{H}$  be given. Then,  $a\{x_n\} + \{y_n\} = \{ax_n + y_n\} \in \mathbb{H}$  because:

$$\sum_{i=1}^{\infty} (ax_i + y_i)^2 = \sum_{i=1}^{\infty} [a^2 x_i^2 + 2ax_i y_i + y_i^2] \stackrel{(*)}{=} a^2 \sum_{i=1}^{\infty} x_i^2 + 2a \sum_{i=1}^{\infty} x_i y_i + \sum_{i=1}^{\infty} y_i^2 < \infty$$

The  $(*)$  given by:

$$\sum_{i=1}^{\infty} |x_i y_i| = \sum_{i=1}^{\infty} |x_i| |y_i| \leq \sum_{i=1}^{\infty} (\max(|x_i|, |y_i|))^2 \leq \sum_{i=1}^{\infty} (x_n^2 + y_n^2) = \sum_{i=1}^{\infty} x_n^2 + \sum_{i=1}^{\infty} y_n^2 < \infty \quad (*)$$

Thus  $\mathbb{H}$  is Vector Space over  $\mathbb{R}$ . Now, define *inner product* on  $\mathbb{H}$  as:

$$\langle \cdot, \cdot \rangle : \mathbb{H} \times \mathbb{H} \rightarrow \mathbb{R} : (\{x_n\}, \{y_n\}) \mapsto \sum_{i=1}^{\infty} x_i y_i$$

This definition is well-defined since  $(*)$ . And, Linearity in first:

$$\langle a\{x_n\} + \{y_n\}, \{z_n\} \rangle = \langle \{ax_n + y_n\}, \{z_n\} \rangle = \sum_{i=1}^{\infty} (ax_i + y_i) z_i = a \sum_{i=1}^{\infty} x_i z_i + \sum_{i=1}^{\infty} y_i z_i = a \langle \{x_n\}, \{z_n\} \rangle + \langle \{y_n\}, \{z_n\} \rangle$$

The other conditions are clear. Thus,  $(\mathbb{H}, \langle \cdot, \cdot \rangle)$  is *inner product space*.

Using *inner product*, define the *Norm* on  $\mathbb{H}$  as:

$$\|\cdot\| : \mathbb{H} \rightarrow \mathbb{R} : \{x_n\} \mapsto \sqrt{\langle \{x_n\}, \{x_n\} \rangle}$$

Finally, define *Metric* on  $\mathbb{H}$  as:

$$\mu : \mathbb{H} \times \mathbb{H} \rightarrow \mathbb{R} : (\{x_n\}, \{y_n\}) \mapsto \|\{x_n\} - \{y_n\}\| = \sqrt{\sum_{i=1}^{\infty} (x_i - y_i)^2}$$

□

**Theorem 21.** *Hilbert Space is Separable.*

**Proof.** For each  $n \in \mathbb{N}$ , define  $D_n \stackrel{\text{def}}{=} \{\{p_n\} \mid p_i \in \mathbb{Q}, p_{n+1} = p_{n+1} = \dots = 0\}$  and  $D \stackrel{\text{def}}{=} \bigcup_{n \in \mathbb{N}} D_n$ .

Then,  $D$  is countable set. We will show that  $\overline{D} = \mathbb{H}$ .

Let  $\epsilon > 0$  and  $\{x_n\} \in \mathbb{H}$  be given. Since convergence, there exists  $N \in \mathbb{N}$  such that

$$\sum_{i=N+1}^{\infty} x_i^2 = \sum_{i=1}^{\infty} x_i^2 - \sum_{i=1}^N x_i^2 < \frac{\epsilon^2}{2}$$

Since density of Rationals, put each  $i = 1, 2, \dots, N$ ,  $p_i \in \mathbb{Q} \mid |x_i - p_i| < \frac{\epsilon}{\sqrt{2N}}$  and  $p_i = 0$  for  $i \geq N+1$ .

Then,  $\{p_n\} \in D_n \subset D$  and

$$\mu(\{x_n\}, \{p_n\}) = \sqrt{\sum_{i=1}^N (x_i - p_i)^2 + \sum_{i=N+1}^{\infty} (x_i - p_i)^2} = \sqrt{\sum_{i=1}^N (x_i - p_i)^2 + \sum_{i=N+1}^{\infty} x_i^2} < \sqrt{N \cdot \frac{\epsilon^2}{2N} + \frac{\epsilon^2}{2}} = \epsilon$$

□

**Corollary 5.** *Hilbert Space is Second-Countable.*

**Theorem 22.** *Hilbert Space is Complete.*

**Proof.** Let  $\{\{x_{n,i}\}_{i=1}^{\infty}\}_{n=1}^{\infty}$  be a Cauchy sequence in  $\mathbb{H}$ . For any fixed  $n, m \in \mathbb{N}$  and for each  $j \in \mathbb{N}$ ,

$$|x_{n,j} - x_{m,j}| < \mu(\{x_{n,i}\}, \{x_{m,i}\}) = \sqrt{\sum_{i=1}^{\infty} (x_{n,i} - x_{m,i})^2}$$

That is, for each  $j \in \mathbb{N}$ ,  $\{x_{n,j}\}$  is Cauchy sequence in  $\mathbb{R}$ . Since  $\mathbb{R}$  is Complete, put  $y_j \stackrel{\text{def}}{=} \lim_{n \rightarrow \infty} x_{n,j}$ , each  $j \in \mathbb{N}$ .

Let  $\epsilon > 0$  be given. Then, there exists  $N \in \mathbb{N}$  such that  $n, m \geq N \implies \mu(\{x_{n,i}\}, \{x_{m,i}\}) < \frac{\epsilon}{2}$ .

Meanwhile, for each  $k \in \mathbb{N}$ ,

$$\sum_{i=1}^k (x_{n,i} - x_{m,i})^2 \leq \sum_{i=1}^{\infty} (x_{n,i} - x_{m,i})^2 = [\mu(\{x_{n,i}\}, \{x_{m,i}\})]^2$$

Thus,  $n, m \geq N \implies \sum_{i=1}^k (x_{n,i} - x_{m,i})^2 < \left(\frac{\epsilon}{2}\right)^2$ , for each  $k \in \mathbb{N}$ .

Taking limit to  $m$ , then  $n \geq N \implies \lim_{m \rightarrow \infty} \left(\sum_{i=1}^k (x_{n,i} - x_{m,i})^2\right) = \sum_{i=1}^k \left(x_{n,i} - \lim_{m \rightarrow \infty} x_{m,i}\right)^2 = \sum_{i=1}^k (x_{n,i} - y_i)^2 < \left(\frac{\epsilon}{2}\right)^2$ .

And, for all  $k \in \mathbb{N}$ ,

$$\sum_{i=1}^k y_i^2 = \sum_{i=1}^k (2(x_{n,i}^2 + (x_{n,i} - y_i)^2)) \leq 2\|\{x_{n,i}\}_{i=1}^{\infty}\|^2 + \left(\frac{\epsilon}{2}\right)^2$$

Thus  $\{y_i\} \in \mathbb{H}$ . As a result,

$$n \geq N \implies \mu(\{x_n\}, \{y_n\}) = \sqrt{\sum_{i=1}^{\infty} (x_{n,i} - y_i)^2} = \sqrt{\lim_{k \rightarrow \infty} \sum_{i=1}^k (x_{n,i} - y_i)^2} < \frac{\epsilon}{2}$$

□

**Theorem 23.**  $\mathbb{H} \subset \mathbb{R}^\omega$  with subspace topology is Metrizable.

*Proof.* We will use two Lemmas:

**Lemma 15.** Countable Product of Metric Space is Metrizable.

*Proof.* Let  $(X_i, d_i)$  be a metric Space, for each  $i \in \mathbb{N}$ .

If  $d : X \times X \rightarrow \mathbb{R}$  is a Metric, then  $\frac{d}{1+d}$  is also Metric, because

$$\frac{d(x, z)}{1 + d(x, z)} \underset{\substack{\frac{x}{1+x} \\ \text{increasing}}}{\leq} \frac{d(x, y) + d(y, z)}{1 + d(x, y) + d(y, z)} \underset{d \geq 0}{\leq} \frac{d(x, y)}{1 + d(x, y)} + \frac{d(y, z)}{1 + d(y, z)} \quad (*)$$

Using this fact, define

$$d_\Pi : \prod X_i \times \prod X_i \rightarrow \mathbb{R} : (\{x_n\}_{n=1}^\infty, \{y_n\}_{n=1}^\infty) \mapsto \sum_{i=1}^\infty \left[ \frac{1}{2^i} \cdot \frac{d_i(x_i, y_i)}{1 + d_i(x_i, y_i)} \right]$$

Then  $d_\Pi$  is a Metric because:

$$\begin{aligned} d_\Pi(\{x_n\}_{n=1}^\infty, \{z_n\}_{n=1}^\infty) &= \sum_{i=1}^\infty \left[ \frac{1}{2^i} \cdot \frac{d_i(x_i, z_i)}{1 + d_i(x_i, z_i)} \right] \\ &\stackrel{(*)}{\leq} \sum_{i=1}^\infty \left[ \frac{1}{2^i} \cdot \left( \frac{d_i(x_i, y_i)}{1 + d_i(x_i, y_i)} + \frac{d_i(y_i, z_i)}{1 + d_i(y_i, z_i)} \right) \right] \\ &= \sum_{i=1}^\infty \left[ \frac{1}{2^i} \cdot \frac{d_i(x_i, y_i)}{1 + d_i(x_i, y_i)} \right] + \sum_{i=1}^\infty \left[ \frac{1}{2^i} \cdot \frac{d_i(y_i, z_i)}{1 + d_i(y_i, z_i)} \right] \\ &= d_\Pi(\{x_n\}_{n=1}^\infty, \{y_n\}_{n=1}^\infty) + d_\Pi(\{y_n\}_{n=1}^\infty, \{z_n\}_{n=1}^\infty) \end{aligned}$$

Reflexity and symmetry are clear. □

**Lemma 16.** Metrizable is Hereditary.

*Proof omitted.*

Consequently, since  $\mathbb{H} \subset \mathbb{R}^\omega$  is a subspace of a metric space, it is metrizable. □

#### **14.4 Banach Space**

#### **14.5 $L_p$ Space**

#### **14.6 $l_p$ Space**

# Bibliography

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