Math Note

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Contents

This paper covers several topics in undergraduate mathematics.

Patch Note:

$\sim$ 2025/9/28 - Drafted the initial framework of the paper.

2025/9/29 - 1. Completed proof of Ring of Fractions.
2. Transcribed Integral, Ratio, and Root Test.
3. Transcribed Tube Lemma, Lindelöf and Countably Compact product Compact.
4. Transcribed Coproduct with Continuous, open, closed map.

2025/9/30 - 1. Proved Every open set in $\mathbb{R}^n$ is countable union of closed cubes, disjointness of interiors remains.
2. Transcribed Group action.

Chapter 1

Set Theory

1.1 Map

Definition 1.1.0.1. Let X, Y are sets. Define a ***function*** X to Y is a relation

$$f \subset X \times Y$$

such that

1. For any $x \in X$, there exists $y \in Y$ such that $(x, y) \in f$.
2. If $(x, y) \in f$ and $(x, z) \in f$, then $y = z$.

Denote f as:

$$f : X \to Y : x \mapsto f(x)$$

Define ***Image*** of f by $A \subset X$:

$$f[A] \overset{\text{def}}{=} \{f(a) \mid a \in A\} \subset Y$$

And, ***Preimage*** of f by $B \subset Y$:

$$f^{-1}[B] \overset{\text{def}}{=} \{x \in X \mid f(x) \in B\} \subset X$$

$f : X \to Y$ is ***Injective*** if: $f(x_1) = f(x_2) \implies x_1 = x_2$.
$f : X \to Y$ is ***Surjective*** if: $\forall y \in Y,\ \exists x \in X$ s.t. $f(x) = y$.
If f is injective and surjective, called ***bijective***.
If f is bijective, then define ***inverse*** of f as:

$$f^{-1} : Y \to X : y \to x$$

where $x \in X$ is the unique elements of X such that $f(x) = y$.

Theorem 1.1.0.1. Let $f : X \to Y$ be a function. Then,

1. There exists $g : Y \to X$ such that $g \circ f : X \to X$ be an identity function ***if and only if*** f is injective.
2. There exists $h : Y \to X$ such that $f \circ h : Y \to Y$ be an identity function ***if and only if*** f is surjective.

Proof.
1. $\implies$)
Assume that $f(x_1) = f(x_2)$. Then, existence of left inverse, $g(f(x_1)) = g(f(x_2)) \implies x_1 = x_2$. Thus f injective.
1. $\impliedby$)

Since f is injection, for any $y \in f[X]$, there exists a unique element $x_y \in X$ such that $f(x) = y$. Now, define

$$g : Y \to X : y \mapsto \begin{cases} x_y & y \in f[X] \\ \text{any element in } X & y \notin f[X] \end{cases}$$

Then, for any $x \in X$, $g(f(x)) = g(y) = x$.
2. $\Longrightarrow$)
Let $y \in Y$ be given. Since existence of right inverse, $f(h(y)) = y$ where $h(y) \in X$. Thus, f is surjective.
2. $\Longleftarrow$)
For any $y \in Y$, there exists a $x_y \in X$ such that $f(x_y) = y$. Now, define

$$h : Y \to X : y \mapsto x_y$$

Then, for any $y \in Y$, $f \circ h(y) = f(x_y) = y$. Thus, $f \circ h$ is identity. ∎

Corollary 1.1.0.1. Let $f : X \to Y$ be a function, $\mathrm{id}_X : X \to X : x \mapsto x$, and $\mathrm{id}_Y : Y \to Y : y \mapsto y$.

There exists a $f^{-1} : Y \to X$ such that $f^{-1} \circ f = \mathrm{id}_X$ and $f \circ f^{-1} = \mathrm{id}_Y$ *if and only if* f is bijection.

Proof. If f is bijection, then there exists left inverse g and right inverse h.
Enough To Show that: $g = h$. Since $g \circ f = \mathrm{id}_X$ and $f \circ h = \mathrm{id}_Y$,
$g \circ f \circ h = g \circ \mathrm{id}_Y$, thus $h = g$. ∎

Theorem 1.1.0.2. Let X, Y, Z are sets, $f : X \to Y$, $g : Y \to Z$ and $A \subset X, B \subset Y, C \subset Z$. Then followings are hold:

1. $g[f[A]] = (g \circ f)[A]$.
2. $f^{-1}[g^{-1}[C]] = (g \circ f)^{-1}[C]$.

Proof.
1. It is clear by definition of image:

$$\begin{aligned} g[f[A]] &\overset{\text{def}}{=} g[\{f(a) \mid a \in A\}] = \{g(b) \mid b \in \{f(a) \mid a \in A\}\} \\ &= \{g(b) \mid b = f(a) \text{ for some } a \in A\} = \{g(f(a)) \mid \text{for some } a \in A\} = \{g(f(a)) \mid a \in A\} \end{aligned}$$

2. It is not clear,

$$f^{-1}[g^{-1}[C]] \overset{\text{def}}{=} f^{-1}[\{b \in Y \mid g(b) \in C\}] = \{a \in X \mid f(a) \in \{b \in Y \mid g(b) \in C\}\} = \{a \in X \mid g(f(a)) \in C\} = (g \circ f)^{-1}[C]$$

∎

Proposition 1.1.0.1. Let $f : X \to Y$ be a function, $A, B \subset X$ and $C, D \subset Y$.

1. If $A \subset B$, then $f[A] \subset f[B]$.
2. If $C \subset D$, then $f^{-1}[C] \subset f^{-1}[D]$

Proof.

$$y \in f[A] \implies y = f(a) \text{ for some } a \in A \overset{A \subset B}{\implies} y = f(a) \text{ for some } a \in B \implies y \in f[B]$$
$$x \in f^{-1}[C] \implies f(x) \in C \overset{C \subset D}{\implies} f(x) \in D \implies x \in f^{-1}[D]$$

∎

Lemma 1.1.0.1. Let two set X, Y be given, and $A \subset X$, $B \subset Y$, $f : X \to Y$. Then followings are holds:

1. $f^{-1}[f[A]] \supseteq A$, and equality holds if f one-to-one.
2. $f[f^{-1}[B]] \subseteq B$, and equality holds if f onto.
3. $f^{-1}[Y \setminus B] = X \setminus f^{-1}[B]$
4. $f[X] \setminus f[A] \subseteq f[X \setminus A]$, and equality holds if f one-to-one.

Proof. Proof of 4.

$$\begin{aligned} y \in f[X] \setminus f[A] &\iff y \in f[X] \text{ and } y \notin f[A] \\ &\iff \exists x \in X \text{ s.t. } y = f(x) \text{ and } \forall x \in A,\ y \neq f(x) \\ &\overset{(*)}{\Longrightarrow} \exists x \in X \setminus A \text{ s.t. } y = f(x) \\ &\iff y \in f[X \setminus A] \end{aligned}$$

If f is injection, then Left Direction of $(*)$ be true: $\exists! x \in X \setminus A$ s.t. $y = f(x)$. $\square$

Chapter 2

Group Theory

2.1 Isomorphism Theorems

Theorem 2.1.0.1. *The First Isomorphism Theorem*
Let $\varphi : G \to H$ be a Group-Homomorphism. Then,

$$G/\ker\varphi \cong \varphi[G]$$

$$\begin{tikzcd}
G \arrow[r, "\varphi", "\text{homomorphism}"'] \arrow[dr, "\pi"', "\text{homomorphism surjection}"'] & \varphi[G] \\
& G/\ker\varphi \arrow[u, "\phi"', "\text{isomorphism}"']
\end{tikzcd}$$

Proof. Let $\pi : G \to G/\ker\varphi : x \mapsto x + \ker\varphi$. Then, the map $\phi : G/\ker\varphi \to \varphi[G] : a + \ker\varphi \mapsto \varphi(a)$ is isomorphism.
Well-defined and Injective:

$$a + \ker\varphi = b + \ker\varphi \iff a - b \in \ker\varphi \iff \varphi(a-b) = \varphi(a) - \varphi(b) = 0$$

Surjective is clear. □

Theorem 2.1.0.2. *The Second Isomorphism Theorem*
Let G be a Group, and $H \leq G$, $N \trianglelefteq G$. Then,

$$HN/N \cong H/(H \cap N)$$

Proof. HK be a subgroup of G, being

$$HN = \bigcup_{h \in H} hN \overset{N \trianglelefteq G}{=} \bigcup_{h \in H} Nh = NH$$

And, $N \leq HN$ is clear, thus $N \trianglelefteq HN$.
Meanwhile, $H \cap N$ be a Normal Subgroup of H: for any $h \in H, n \in H \cap N$, $hnh^{-1} \in N$ because N is normal, and $hnh^{-1} \in H$ since h, n contained in H. Thus, $hnh^{-1} \in H \cap N$, this implies $H \cap N$ be a Normal of H.
Now, Define a Map:

$$\varphi : H \to HN/N : h \mapsto hN$$

Clearly, this map is Well-Defined and Homomorphism. And,

$$\ker\varphi = \varphi^{-1}[1] = \{h \in H \mid hN = N\} = \{h \in H \mid h \in N\} = H \cap N$$

Thus, since The 1st Isomorphism Theorem,

$$HN/N \cong H/(H \cap N)$$

□

Theorem 2.1.0.3. *The Third Isomorphism Theorem*
Let G be a Group, and $H, K \trianglelefteq G$ with $H \leq K$. Then, $K/H \trianglelefteq G/H$ and

$$(G/H)/(K/H) \cong (G/K)$$

Proof. First, show that $K/H \trianglelefteq G/H$. Let $kH \in K/H$ and $gH \in G/H$. Then,

$$(gH)(kH)(gH)^{-1} = (gH)(kH)(g^{-1}H) = (gkg^{-1})H \in K/H$$

since $gkg^{-1} \in K$, being $K \trianglelefteq G$. Now, Define a map:

$$\varphi : G/H \to G/K : gH \mapsto gK$$

1. Well-Defined.

$$g_1H = g_2H \iff g_1^{-1}g_2 \in H \overset{H \leq K}{\Longrightarrow} g_1^{-1}g_2 \in K \iff g_1K = g_2K$$

2. Homomorphism.
Clearly, for any $g_1H, g_2 \in G/H$,

$$\varphi(g_1Hg_2H) = \phi(g_1g_2H) = g_1g_2K = g_1Kg_2K = \varphi(g_1H)\varphi(g_2H)$$

3. Surjection. Let $gK \in G/K$ be given. Then, clearly, $\varphi(gH) = gK$.
4. Kernel.

$$\ker\varphi = \{gH \in G/H \mid gK = 1\} = \{gH \in G/H \mid g \in K\} = K/H$$

Consequently, The 1st Isomorphism Theorem gives

$$(G/K) \cong (G/H)/\ker\varphi = (G/H)/(K/H)$$

□

Theorem 2.1.0.4. *The Forth Isomorphism Theorem*
Let G be a Group, and $N \trianglelefteq G$ be a Normal Subgroup. Then, there is a bijection between

$$D \overset{\text{def}}{=} \{H \leq G \mid N \leq H\},\ C \overset{\text{def}}{=} \{\overline{H} \leq G/N\}$$

Proof. Let $\pi : G \to G/N : g \mapsto gN$ be a natural projection. And, Define

$$\Phi : D \to C : H \mapsto \pi[H]$$

This function is well-defined: For any $H \in D$, let $aN, bN \in \pi[H]$. Then, $aN \cdot b^{-1}N = ab^{-1}N \in \pi[H]$, thus $\pi[H] \leq G/N$.
To show that one-to-one: Let $\Phi(A) = \Phi(B)$. Thus means, $\pi[A] = \pi[B]$. Let $a \in A$. Then, $\pi(a) \in \pi[A] = \pi[B]$, thus $\pi(a) = \pi(b)$ for some $b \in B$. That is, $aN = bN \iff a \in bN$. Meanwhile, $N \leq B$, thus $a \in bN \subset B$, $A \subset B$. Similarly, $B \subset A$, that is $A = B$.
To show that onto: Let $K \in C$. Then, $N \leq \pi^{-1}[K] \leq G$, thus clear. □

2.2 Group Action

In this section, we follow that the notation of [Dummit and Foote, 2004, Abstract Algebra].

Definition 2.2.0.1. Let $(G, *)$ be a Group, and A be a non-empty set.
Define *Group Action* of a group G on a set A:

$$\alpha : G \times A \to A : (g, a) \mapsto g \cdot a$$

satisfies

1. For all $a \in A$, $1_G \cdot a = a$.
2. For all $g_1, g_2 \in G$, $a \in A$, $(g_1 * g_2) \cdot a = g_1 \cdot (g_2 \cdot a)$

In this, we said to be 'G acts on a set A'. Meanwhile, For each $g \in G$, Define a map

$$\sigma_g : A \to A : a \mapsto g \cdot a$$

Then, the *permutation representation*

$$\varphi : G \to S_A : g \mapsto \sigma_g$$

be a Homomorphism. Clearly, for each $g \in G$, $a \in A$,

$$\alpha(g, a) = g \cdot a = \sigma_g(a) = \varphi(g)(a)$$

Thus, there is one-to-one correspondence between group action and permutation representation.
For each $a \in A$, the *stabilizer* of a in G:

$$G_a \overset{\text{def}}{=} \{g \in G \mid g \cdot a = a\}$$

The *kernel of action*:

$$\ker \alpha \overset{\text{def}}{=} \{g \in G \mid g \cdot a = a,\ \forall a \in A\} = \bigcap_{a \in A} G_a$$

$G_a \leq G$ and $\ker \alpha \trianglelefteq G$.
If the kernel of action be trivial, the action is called *faithful*.

Definition 2.2.0.2. Let $\alpha : G \times A \to A$ be a Group Action. Define a relation on A:

$$a \sim b \iff a = g \cdot b \text{ for some } g \in G$$

Then, this relation be equivalence relation. Denote the equivalence relation, called *orbit*:

$$\mathcal{C}_a \overset{\text{def}}{=} \{b \mid b = g \cdot a \text{ for some } g \in G\} = \{g \cdot a \mid g \in G\}$$

And, the action is called *transitive* if there is only one orbit.

Lemma 2.2.0.1. For each $a \in A$,

$$|\mathcal{C}_a| = |G : G_a|$$

Proof. Since the map

$$\varphi_a : \mathcal{C}_a \to \{gG_a \mid g \in G\} : g \cdot a \mapsto gG_a$$

is well-defined, bijection.

□

Theorem 2.2.0.1. Let G be a Group, let $H \leq G$ and $A = \{gH \mid g \in G\}$, G acts by left multiplication on the set A.

$$\pi_H : G \to S_A : g \mapsto \sigma_g$$

be a permutation representation afforded by this action. Then

1. G acts transitively on A.
2. $G_{1H} = \{g \in G \mid gH = H\} = H$.
3. The kernel of the action $\ker \pi_H = \bigcap_{x \in G} xHx^{-1}$, this is the largest normal subgroup of G contained in H.

Proof. Let $aH, bH \in A$ be given. Then, for $g = ba^{-1}$, $g \cdot aH = (ga)H = bH$. Thus, $A = \mathcal{C}_a$ for any $a \in G$.
2 is clear, being $gH = H \iff g \in H$.
Now,

$$\begin{aligned}
\ker \pi_H &= \{g \in G \mid gxH = xH, \ \ \forall x \in G\} \\
&= \{g \in G \mid (x^{-1}gx)H = H, \ \ \forall x \in G\} \\
&= \{g \in G \mid x^{-1}gx \in H, \ \ \forall x \in G\} \\
&= \{g \in G \mid g \in xHx^{-1}, \ \ \forall x \in G\} = \bigcap_{x \in G} xHx^{-1}
\end{aligned}$$

And the second assertion given by:
Let N is a normal subgroup of G contained in H, then for any $x \in G$, $N = xNx^{-1} = xHx^{-1}$. Thus,

$$N \leq \bigcap_{x \in G} xHx^{-1}$$

□

Corollary 2.2.0.1. If G is a finite group of order n, p is the smallest prime dividing $|G|$.
Then, any subgroup of index p is normal.

Proof. Let $|G| = p^r p_1^{r_1} \cdots p_n^{r_n}$ be a prime decomposition, $H \leq G$ with $|G : H| = p$.
Let $K = \ker \pi_H \leq H$, $k = |H : K|$. Then, $|G : K| = |G : H||H : K| = pk$. By the First-Isomorphism Theorem,

$$G / \ker \pi_H \cong \pi_H[G] \leq S_A$$

and Since H has p left cosets, $A \cong \mathbb{Z}_p$, thus G/K is isomorphic to some subgroup of S_p.
Now, Lagrange's Theorem gives that $|G/K| = pk$ divides $|S_p| = p!$. This implies $k \mid (p-1)!$.
$|G : K| = pk$ implies $|G| = pk \cdot |K|$. Since p is the minimal prime that divides $|G|$, thus every prime divisor of k is greater than or equal to p. This implies must be $k = 1$. Thus $H = K \trianglelefteq G$. □

Definition 2.2.0.3. Let a Group action as:

$$\alpha : G \times G \to G : (g, a) \mapsto gag^{-1}$$

Now, the orbit drived from this action $[a] = \{b \in G \mid \exists g \in G$ s.t. $b = gag^{-1}\}$ is called be *Conjugacy Class*.
More generally,

$$\alpha : G \times \mathcal{P}(G) \to \mathcal{P}(G) : (g, S) \mapsto gSg^{-1}$$

Lemma 2.2.0.2. Let $\alpha : G \times \mathcal{P}(G) \to \mathcal{P}(G) : (g, S) \mapsto gSg^{-1}$ be a Group action acting as Conjugate.
Then, $G_S = N_G(S)$ and $|\mathcal{C}_S| = |G : N_G(S)|$, for any $S \subseteq G$.
In particular, if S is singleton, $S = \{g_i\}$, then $|\mathcal{C}_{\{g_i\}}| = |G : N_G(g_i)| = |G : C_G(g_i)|$.

Proof.

$$G_S = \{g \in G \mid gSg^{-1} = S\} = N_G(S)$$

Thus, for any $S \in \mathcal{P}(G)$,

$$|\mathcal{C}_S| = |G : N_G(S)|$$

□

2.2.1 Lagrange's Theorem

2.3 Generating subset of a Group

2.4 Commutator Subgroup

Chapter 3

Finite Group Theory

3.1 The Class Equation

Theorem 3.1.0.1. *The Class Equation*
Let G be a finite group, and
$g_1, \ldots, g_r$ be representatives of the distinct conjugacy classes of G not contained in the center $Z(G)$ of G.
Then,

$$|G| = |Z(G)| + \sum_{i=1}^{r} |G : C_G(g_i)|$$

3.2 Cauchy's Theorem

Lemma 3.2.0.1. *Cauchy's Theorem*
Let G be a finite group, and p be a prime dividing $|G|$. Then, G has order p element.

Proof. Define a set:

$$S \overset{\text{def}}{=} \{(x_1, x_2, \ldots, x_p) \mid x_i \in G,\ \ x_1 x_2 \cdots x_p = 1\}$$

Then, S has exactly $|G|^{p-1}$ elements because there are $|G|$ possible choices for each of the first $p-1$ elements in G.
Once $x_1, \cdots, x_{p-1}$ are chosen, then x_p is uniquely determined by the uniqueness of inverses.
Then, let $\sigma = (1, 2, \ldots, p)$ be a permutation. Then, for any $\alpha \in S$, $\sigma^n(\alpha) \in S$ for all $n \in \mathbb{Z}$, being $ab = 1 \iff ba = 1$.
More precisely, let $n \in \mathbb{Z}$ be given, $\alpha = (x_1, \cdots, x_n)$. Then,

$$\sigma^n(\alpha) = (x_{n+1}, x_{n+2}, \ldots, x_p, x_1, x_2, \ldots x_n)$$

By $x_1 \cdots x_n x_{n+1} \cdots x_p = 1$, $x_{n+1} \cdots x_p x_1 \cdots x_n = 1$. Thus $\sigma^n(\alpha) \in S$. Now, define a relation on S as:

$$\alpha \sim \beta \text{ if and only if } \beta = \sigma^n(\alpha) \text{ for some } n \in \mathbb{Z}$$

Then, this relation be equivalent relation, thus construct a partition on S. Claim:

$$[\alpha] = \{\beta \in S \mid \beta \sim \alpha\} \text{ is silngeton if and only if } \alpha = (x, \ldots, x) \text{ for some } x \in G.$$

Left direction is clear, and for show that Right direction,
Suppose that $\alpha = (x_1, \ldots, x_n)$ has different cordinate elements, let $x_i \neq x_j$, for some $i < j$. Then clearly

$$(x_1, \ldots, x_i, \ldots, x_p) \neq \sigma^{i-j}(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_p) = (\ldots, \underbrace{x_j}_{\text{i'th element}}, \ldots)$$

Meanwhile, if $[\alpha]$ has elements more than 1, $[\alpha]$ has exactly number of p elements. Because suppose that $\alpha = (x_1, \ldots, x_p)$ has at least one different coordinate. Then,

$$\sigma^1(\alpha), \sigma^2(\alpha), \cdots, \sigma^{p-1}(\alpha)$$

are mutually different: If there exist $1 \leq i < j < p$ such that $\sigma^i(\alpha) = \sigma^j(\alpha)$, that is, $\sigma^{j-i}(\alpha) = \alpha$.
Now, $j - i \mid p$, this is contradiction with p is prime. Therefore, every equivalent class has order 1 or p. Consequently,

$$|G|^{p-1} = k + pd$$

where k is a number of classes of size 1, and d is a number of classes of size p. And $(1, 1, \ldots, 1) \in S$, k is at least 1.
Since p divides $|G|^{p-1} = k + pd$, thus k must be bigger than 1, thus there exists elements such that $x^p = 1$. $\square$

3.3 Sylow's Theorem

Theorem 3.3.0.1. *Sylow's Theorem*
Let G be a group of order $p^\alpha m$, where p is a prime such that $p \nmid m$.
A group of order $p^r,\ (r \geq 1)$ is called a p-group, Subgroups of G which are p-groups are called p-subgroup.
In particular, subgroups of order p^α is called *Sylow p-subgroup* of G. And, define a collection

$$\mathtt{Syl}_p(G) \stackrel{\text{def}}{=} \{P \leq G \mid |P| = p^\alpha\},\ \ n_p(G) \stackrel{\text{def}}{=} \mathtt{Card}(\mathtt{Syl}_p(G))$$

The First Sylow Theorem
There exists a Sylow p-subgroup of G. i.e., $\mathtt{Syl}_p(G) \neq \emptyset$.

The Second Sylow Theorem
If $P \in \mathtt{Syl}_p(G)$ and $Q \leq G$ be a p-subgroup. Then, there exists $g \in G$ such that $Q \leq gPg^{-1}$.

The Third Sylow Theorem
$n_p \equiv 1(\mathrm{mod} p)$, $n_p = |G : N_G(P)|$ for any $P \in \mathtt{Syl}_p(G)$, and $n_p \mid m$.

Before prove above statments, we show that:

Lemma 3.3.0.1. Let $P \in \mathtt{Syl}_p(G)$. If Q is p-subgroup of G, then $Q \cap N_G(P) = Q \cap P$.

Proof. Put $H = Q \cap N_G(P)$. Since $P \leq G$, for any $p \in P$, $pPp^{-1} = P$, thus $p \in N_G(P)$. i.e., $P \leq N_G(P)$.
Thus, Enough to Show that $H \leq Q \cap P$. Since $H \leq N_G(P)$,

$$PH = \bigcup_{h \in H} Ph = \bigcup_{h \in H} hP = HP$$

Thus, $PH \leq G$. And,

$$|PH| = \frac{|P||H|}{|P \cap H|}$$

By Lagrange's Theorem, $H \leq P$ and $P \cap H \leq P$ must have order of powers of p, so PH be a p-group. Clearly, $P \leq PH$ and P is the largest p-group of G, thus, $PH = P$. This means, $H \leq P$. □

Proof. The First Theorem: The existence of Sylow p-subgroup. Proof by Induction:
If $|G| = 1$, there is nothing to prove.
Assume inductively the existence of Sylow p-subgroups for all groups of order less than $|G|$.
In case of $p||Z(G)|$, then by Cauchy's Theorem, $Z(G)$ has a subgroup N which has order of p.
Clearly N is Normal, and $G/N = |G|/|N| = p^{a-1}m$. By assumption, G/N has a subgroup P' of order $p^{\alpha-1}$.
By The Forth Isomorphism Theorem, Let $P \leq G$ be a subgroup such that $P/N = P'$.
Then, $|P| = |P/N| \cdot |N| = p^\alpha$, Thus P be a Sylow p-subgroup of G.
In case of $p \nmid |Z(G)|$.
Let $g_1, \ldots, g_r$ be represectatives of the distinct conjugacy classes of G, not contained in $Z(G)$. Then, The Class Equation gives

$$|G| = |Z(G)| + \sum_{i=1}^{r} |G : C_G(g_i)|$$

Since p divides $|G|$, if for all $i = 1, 2, \ldots, r$, $p \mid |G : C_G(g_i)|$ then $p \mid |Z(G)|$, this is contradiction.
Thus, for some j, $p \nmid |G : C_G(g_j)|$. Put $H = C_G(g_j) < G$. Then, $|H|$ has a factor of p^α, by $p \nmid |G : C_G(g_j)|$. Now,

$$|H| = p^\alpha m' \quad (m' < m)$$

By assumption, H has a Sylow p-group, order of p^α.
Consequently, the existence of Sylow p-subgroup was shown.
The Second Theorem: Relation of p-subgroups.
The First Theorem gives existence of Sylow p-subgroups. Let $P \in \mathtt{Syl}_p(G)$. Denote that:

$$S \stackrel{\text{def}}{=} \{gPg^{-1} \mid g \in G\} = \{P_1, \ldots, P_r\}$$

Let $Q \leq G$ be an any p-subgroup of G. And, Q acts by conjucation on S. i.e.,

$$\alpha : Q \times S \to S : (q, P_i) \mapsto qP_iq^{-1}$$

Write S as a disjoint union of orbits under this action by Q:

$$S = \mathcal{O}_1 \cup \mathcal{O}_2 \cup \cdots \cup \mathcal{O}_s$$

where $r = |\mathcal{O}_1| + \cdots + |\mathcal{O}_s|$. Rearrange a set S as: $P_i \in \mathcal{O}_i,\ 1 \leq i \leq s$. Now, using Definition, Lemma, and above Theorem,

$$|\mathcal{O}_i| \overset{\text{Thm}}{=} |Q : N_Q(P_i)| \overset{\text{def}}{=} |Q : N_G(P_i) \cap Q| \overset{\text{lemma}}{=} |Q : P_i \cap Q|$$

for each $1 \leq i \leq s$. Since Q was arbitrary, Let $Q = P_1$, so that $|\mathcal{O}_1| = |P_1 : P_1 \cap P_1| = 1$. And, for each $i \geq 2$, $P_i \cap P_1 < P_1$,

$$|\mathcal{O}_i| = |P_1 : P_i \cap P_1| > 1$$

Since $P_1 \in \mathrm{Syl}_p(G)$, that is $|P_1| = p^\alpha$, $|P_1 : P_i \cap P_1| = |P_1|/|P_i \cap P_1| = p^k$ where $1 \leq k < \alpha$.
This means for each $2 \leq i \leq s$, p divides $|\mathcal{O}_i|$. Thus,

$$r = |\mathcal{O}_1| + (|\mathcal{O}_2| + \cdots + |\mathcal{O}_s|) \equiv 1 (\mathrm{mod} p)$$

Now, Proof by Contradiction: Let $Q \leq G$ be a p-subgroup. Suppose that for any $1 \leq i \leq r$, $Q \nleq P_i$.
Then, $P_i \cap Q < Q$ for all i, this means

$$|\mathcal{O}_i| = |Q : P_i \cap Q| > 1$$

Thus for any i, p divides $|\mathcal{O}_i|$, this is Contradiction. This proved Relation of p-subgroups.
Finally, The Third Theorem:
Since Second Theorem, this gives that $S = \mathrm{Syl}_p(G)$, thus $n_p(G) = r$. That is, $n_p \equiv 1 (\mathrm{mod} p)$.
Since all Sylow p-subgroups are Conjugate, for any $P \in \mathrm{Syl}_p(G)$,

$$n_p = r = |\mathcal{O}_1| = |G : N_G(P)|$$

Consequently, Completing the Sylow Theorem. $\square$

3.4 More Theorems

> **Theorem 3.4.0.1.** n *Factorial Theroem*
> If G is simple and there is a subgroup H with $|G : H| = n$, then $|G| \mid n!$.

Proof. Let G act on $A = \{gH \mid g \in G\}$ by left multiplication. ($|A| = n$.).
Let $\varphi : G \to S_n$ be a homomorphism afforded above action. Then, $G \overset{G \text{ simp.}}{\cong} G/\ker\varphi \cong \varphi[G] \leq S_n$ □

Chapter 4

Ring Theory

4.1 Ideal

Definition 4.1.0.1. Let R be a Ring. A subset $I \subseteq R$ is called ***ideal*** of R if:

1. $I \subseteq R$ is a subgroup of R.
2. I is closed under the multiplication.
3. For any $r \in R$, $rI \subseteq I$ and $Ir \subseteq I$. (In other word, for any $r \in R, a \in I$, $ra \in I$ and $ar \in I$.)

Theorem 4.1.0.1. Let R be a Ring. Then, TFAE:

1. $I \subset R$ is an Ideal of R.
2. The additive Quotient Group $R/I \overset{\text{def}}{=} \{r + I \mid r \in R\}$ be a Ring under the operation:

$$(r + I) \times (s + I) = (rs) + I$$

Proof. Observation:

$$r_1 + I = r_2 + I \iff r_1 - r_2 \in I \iff \exists a \in I \text{ s.t. } r_1 = r_2 + a$$

Now, for well-definednes, want to show that the equality

$$\begin{aligned}&(r + I) \times (s + I) = (rs) + I\\ \overset{(*)}{=}&[(r + \alpha) + I] \times [(s + \beta) + I] = (r + \alpha)(s + \beta) + I = (rs + r\beta + \alpha s + \alpha\beta) + I\end{aligned}$$

$(*)$ holds for any $r, s \in R$, $\alpha, \beta \in I$.
If I is Ideal, then $r\beta, \alpha s, \alpha\beta \in I$. Thus closed under the addition gives $(*)$.
Conversely, if this operation is well-defined, then for any $r, s \in R$, $\alpha, \beta \in I$, $(*)$ holds.
Substitutiing zero to each r, s, α, β gives I is ideal. $\square$

4.1.1 Properties of Ideal in Ring with identity

Definition 4.1.1.1. Let R be a Ring with identity, and $A \subseteq R$. Define ***Ideal generated by*** A as:

$$(A) \overset{\text{def}}{=} \bigcap_{\substack{I \text{ ideal} \\ A \subseteq I}} I$$

And,

$$\begin{aligned}RA &\overset{\text{def}}{=} \{r_1 a_1 + \cdots + r_n a_n \mid n \in \mathbb{N}, r_i \in R, a_i \in A\}\\ AR &\overset{\text{def}}{=} \{a_1 r_1 + \cdots + a_n r_n \mid n \in \mathbb{N}, r_i \in R, a_i \in A\}\\ RAR &\overset{\text{def}}{=} \{r_1 a_1 r_1' + \cdots + r_n a_n r_n' \mid n \in \mathbb{N}, r_i, r_i' \in R, a_i \in A\}\end{aligned}$$

Lemma 4.1.1.1. Let R be a Ring with identity, and $A \subseteq R$. Then, $(A) = RAR$.

Proof. Since RAR is ideal which contains A, $(A) \subseteq RAR$.
And, conversely, if $\displaystyle\sum_{i=1}^{n} r_i a_i r_i' \in RAR$, then $\displaystyle\sum_{i=1}^{n} r_i a_i r_i' \in (A)$ because each $r_i a_i r_i'$ are contained in (A), being (A) is ideal containing A and ideal is closed under the addition. $\square$

Theorem 4.1.1.1. Let I be an ideal of Ring R with identity.

$$I = R \textbf{ if and only if } I \text{ contains a unit.}$$

Proof. Right direction is clear by $1 \in R = I$.
Denote $u \in I$ be a unit with $vu = 1$, and Let $r \in R$ be given. Then,

$$r = r1 = rvu \in I$$

□

Definition 4.1.1.2. An Ideal M of R is **Maximal ideal** if: There is no Ideal I such that $M \subsetneq I \subsetneq R$.

Theorem 4.1.1.2. Let R be a Ring with identity. Then, every proper ideal $I \subsetneq R$ is contained in a maximal ideal.

Proof. □

Lemma 4.1.1.2. Let R be a commutative Ring with identity, M, P are ideals of R.

1. M is Maximal Ideal *if and only if* R/M is a field.
2. P is Prime Ideal *if and only if* R/M is a integral domain.

4.2 Ring of Fractions

Theorem 4.2.0.1. Let R be a Commutative Ring, $D \subset R$ be a subset such that $\begin{cases} \text{no zero, no zero divisors} \\ \text{closed under multiplication} \end{cases}$.
Then, there exists a Commutative Ring Q with identity satisfies:

1. R can embed in Q, and every element of D becomes unit in Q. More precisely, $Q = \{rd^{-1} \mid r \in R,\ d \in D\}$.
2. Q is the smallest Ring contaning R with identity such that every element of D becomes unit in Q.

Proof. Let $\mathcal{F} \overset{\text{def}}{=} \{(r, d) \mid r \in R,\ d \in D\}$ and the relation $\sim$ on $\mathcal{F}$ by $(r_1, d_1) \sim (r_2, d_2) \iff r_1 d_2 = r_2 d_1$.
Then, $\sim$ is equivalent relation: reflexive and symmetirc are clear, and Suppose that $(r_1, d_1) \sim (r_2, d_2)$ and $(r_2, d_2) \sim (r_3, d_3)$.

$$r_2 d_3 = r_3 d_2 \implies r_2 d_1 d_3 = r_3 d_1 d_2 \implies r_1 d_2 d_3 = r_3 d_1 d_2 \implies d_2(r_1 d_3 - r_3 d_1) \implies r_1 d_3 = r_3 d_1$$

Thus transitivity shown. Define

$$\frac{r}{d} \overset{\text{def}}{=} [(r, d)] = \{(a, b) \mid (a, b) \sim (r, d)\}, \ \ Q \overset{\text{def}}{=} \left\{ \frac{r}{d} \ \middle|\ r \in R,\ d \in D \right\}$$

And define operations $+, \times$ on Q:

$$\frac{r_1}{d_1} + \frac{r_2}{d_2} \overset{\text{def}}{=} \frac{r_1 d_2 + r_2 d_1}{d_1 d_2}, \quad \frac{r_1}{d_1} \times \frac{r_2}{d_2} \overset{\text{def}}{=} \frac{r_1 r_2}{d_1 d_2}$$

Well-Definedness: If $\dfrac{r_1}{d_1} = \dfrac{r_1'}{d_1'}$ and $\dfrac{r_2}{d_2} = \dfrac{r_2'}{d_2'}$,

$$\frac{r_1 d_2 + r_2 d_1}{d_1 d_2} = \frac{r_1 d_2 d_1' d_2' + r_2 d_1 d_1' d_2'}{d_1 d_2 d_1' d_2'} = \frac{(r_1 d_1') d_2 d_2' + (r_2 d_2') d_1 d_1'}{d_1 d_2 d_1' d_2'} = \frac{(r_1' d_1) d_2 d_2' + (r_2' d_2) d_1 d_1'}{d_1 d_2 d_1' d_2'} = \frac{(r_1' d_2' + r_2' d_1') d_1 d_2}{d_1 d_2 d_1' d_2'} = \frac{r_1' d_2' + r_2' d_1'}{d_1' d_2'}$$

$$\frac{r_1 r_2}{d_1 d_2} = \frac{r_1 r_2 d_1' d_2'}{d_1 d_2 d_1' d_2'} = \frac{(r_1 d_1')(r_2 d_2')}{d_1 d_2 d_1' d_2'} = \frac{(r_1' d_1)(r_2' d_2)}{d_1 d_2 d_1' d_2'} = \frac{r_1' r_2' d_1 d_2}{d_1 d_2 d_1' d_2'} = \frac{r_1' r_2'}{d_1' d_2'}$$

Now, $(Q, +, \times)$ constructs Commutative Ring with identity: for any $d \in D$, put $0_Q \overset{\text{def}}{=} \dfrac{0}{d}$, $1_Q \overset{\text{def}}{=} \dfrac{d}{d}$. Then,

1. $(R, +, \times)$ closed under the operations since D is closed under the multiplication.
2. $(R, +)$ has a zero: $\dfrac{r_1}{d_1} + 0_Q = \dfrac{r_1}{d_1} + \dfrac{0}{d} = \dfrac{r_1 d + 0 d_1}{d_1 d} = \dfrac{r_1 d}{d_1 d} = \dfrac{r_1}{d_1}$.
3. $(R, +)$ has an inverse: $\dfrac{r_1}{d_1} + \dfrac{-r_1}{d_1} = \dfrac{r_1 d_1 + (-r_1) d_1}{d_1 d_1} = \dfrac{[(r_1) + (-r_1)] d_1}{d_1 d_1} = \dfrac{0 d_1}{d_1 d_1} = \dfrac{0}{d_1 d_1} = 0_Q$.
4. $(R, +, \times)$ satisfies distributive law:

   4-1. The left law:

$$\begin{aligned} \frac{r_1}{d_1} \times \left(\frac{r_2}{d_2} + \frac{r_3}{d_3} \right) &= \frac{r_1}{d_1} \times \frac{r_2 d_3 + r_3 d_2}{d_2 d_3} = \frac{r_1 r_2 d_3 + r_1 r_3 d_2}{d_1 d_2 d_3} = \frac{r_1 r_2 d_1 d_3 + r_1 r_3 d_1 d_2}{d_1 d_2 d_1 d_3} = \frac{r_1 r_2}{d_1 d_2} + \frac{r_2 r_3}{d_2 d_3} \\ &= \frac{r_1}{d_1} \times \frac{r_2}{d_2} + \frac{r_2}{d_2} \times \frac{r_3}{d_3} \end{aligned}$$

   4-2. The right law:

$$\begin{aligned} \left(\frac{r_1}{d_1} + \frac{r_2}{d_2} \right) \times \frac{r_3}{d_3} &= \frac{r_1 d_2 + r_2 d_1}{d_1 d_2} \times \frac{r_3}{d_3} = \frac{r_1 r_3 d_2 + r_2 r_3 d_1}{d_1 d_2 d_3} = \frac{r_1 r_3 d_2 d_3 + r_2 r_3 d_1 d_3}{d_1 d_3 d_2 d_3} = \frac{r_1 r_3}{d_1 d_3} + \frac{r_2 r_3}{d_2 d_3} \\ &= \frac{r_1}{d_1} \times \frac{r_3}{d_3} + \frac{r_2}{d_2} \times \frac{r_3}{d_3} \end{aligned}$$

5. $(R, \times)$ has an identity: $\dfrac{r_1}{d_1} \times 1_Q = \dfrac{r_1}{d_1} \times \dfrac{d}{d} = \dfrac{r_1 d}{d_1 d} = \dfrac{r_1}{d_1}$.

6. Elements of D become unit in Q: Define $\iota : R \to Q : r \mapsto \dfrac{rp}{p}$ where $p \in D$ is any fixed element in D.
Then, ι is Ring-Monomorphsim because:

6-1. Well-Defined and Injective: $\iota(r_1) = \iota(r_2) \iff \dfrac{r_1 p}{p} = \dfrac{r_2 f}{f} \iff (r_1 - r_2)pp = 0 \iff r_1 = r_2$

6-2. For any $d \in D$, $\iota(d)$ is a unit of Q: Put $(\iota(d))^{-1} \stackrel{\text{def}}{=} \dfrac{p}{dp}$, then

$$\iota(d) \times (\iota(d))^{-1} = \frac{dp}{p} \times \frac{p}{dp} = \frac{dpp}{dpp} = 1_Q$$

That is, ι is embedding from R into Q such that $\iota[D]$ becomes units of Q except zero.
Moreover, if $D = R \setminus \{0\}$, then Q is field.

7. Q is the *smallest* ring containing R with identity such that every element of D becomes units in Q.
Let S be an any commutative ring with identity,
and assume that $\varphi : R \to S$ is a Ring-Monomorphism such that for any $d \in D$, $\varphi(d)$ is unit in S.
Define $\phi : Q \to S : \dfrac{r}{d} \mapsto \varphi(r)\varphi(d)^{-1}$. Then, this ϕ is well-defined and injective:

$$\phi\left(\frac{r_1}{d_1}\right) = \phi\left(\frac{r_2}{d_2}\right) \iff \varphi(r_1)\varphi(d_1)^{-1} = \varphi(r_2)\varphi(d_2)^{-1} \iff \varphi(r_1)\varphi(d_2) = \varphi(r_2)\varphi(d_1)$$
$$\stackrel{\text{homomor.}}{\iff} \varphi(r_1 d_2) = \varphi(r_2 d_1) \stackrel{\text{one-to-one}}{\iff} r_1 d_2 = r_2 d_1 \iff \frac{r_1}{d_1} = \frac{r_2}{d_2}$$

That is, if a commutative ring S with identity contains a copy of R such that the denominator set D of R becomes unit in S, then S contains ring of fractions Q of R. Thus $S = Q$ is the smallest ring that satisfies these conditions.

$\square$

4.3 Commutative Ring with identity

4.3.1 Euclidean Domain

4.3.2 Principal Ideal Domain

4.3.3 Noetherian Domain

4.3.4 Factorization Domain

4.3.5 Unique Factorization Domain

4.3.6 Summary

Chapter 5

Polynomial Ring

Chapter 6

Field Theory

Chapter 7

Galois Theory

Chapter 8

Linear Algebra

Chapter 9

Category

Chapter 10

General Topology

10.1 Coproduct Space

Definition 10.1.0.1. Let $(X_\alpha, \mathcal{T}_\alpha)$ $(\alpha \in \Lambda)$ are mutually disjoint Topological Space. Define a *Coproduct Topology* $(X_\Pi, \mathcal{T}_\Pi)$:

$$X_\Pi \overset{\text{def}}{=} \bigsqcup_{\alpha\in\Lambda} X_\alpha, \ \ \mathcal{T}_\Pi \overset{\text{def}}{=} \left\{ \bigsqcup_{\alpha\in\Lambda} \mathcal{U}_\alpha \,\middle|\, \mathcal{U}_\alpha \in \mathcal{T}_\alpha \right\}$$

This actually be a Topology:
1. $\emptyset, X_\Pi \in \mathcal{T}_\Pi$ is clear,
2. Closed under union is clear.
3. Closed under finite intersection, not infinite.

Proof. Proof of *3.*
Let a finite collection

$$\left\{ \bigsqcup_{\alpha\in\Lambda} \mathcal{U}_\alpha^1, \bigsqcup_{\alpha\in\Lambda} \mathcal{U}_\alpha^2, \cdots, \bigsqcup_{\alpha\in\Lambda} \mathcal{U}_\alpha^k \right\}$$

be given. Then, their intersection be:

$$\bigcap_{j=1}^{k} \bigsqcup_{\alpha\in\Lambda} \mathcal{U}_\alpha^j = \bigsqcup_{\alpha\in\Lambda} \bigcap_{j=1}^{k} \mathcal{U}_\alpha^j \in \mathcal{T}_\Pi$$

□

Theorem 10.1.0.1. Let X_1, X_2, X_3 and Y_1, Y_2, Y_3 are mutually disjoint Topological Space, and for each $i = 1, 2, 3$,

$$f_i : X_i \to Y_i : x \mapsto f_i(x)$$

Define a function

$$f = f_1 \amalg f_2 \amalg f_3 : \bigsqcup_{i=1}^{3} X_i \to \bigsqcup_{i=1}^{3} Y_i : x \mapsto \begin{cases} f_1(x) & x \in X_1 \\ f_2(x) & x \in X_2 \\ f_3(x) & x \in X_3 \end{cases}$$

where both Domain and Codomain are Coproduct Space. (Clearly this function is well-defined.)
Suppose that:

1. f_1 is Open map, Closed map
2. f_2 is Continuous map, Open map
3. f_3 is Continuous map, Closed map

Then, The Follwings hold:

1. f_1 is Continuous map *if and only if* f is Continuous map.
2. f_2 is Open map *if and only if* f is Open map.
3. f_3 is Closed map *if and only if* f is Closed map.

Proof.
1. It follows that: For any open on Codomain $U \in \mathcal{T}_{Y_\Pi}$,

$$\begin{aligned} f^{-1}[U] &= \{x \in X \mid f(x) \in U\} = \{x \in X_1 \mid f_1(x) \in U\} \cup \{x \in X_2 \mid f_2(x) \in U\} \cup \{x \in X_3 \mid f_3(x) \in U\} \\ &= f_1^{-1}[U] \cup f_2^{-1}[U] \cup f_3^{-1}[U] \end{aligned}$$

Thus, If f_1 is Continuous, then f is Continuous map since $f^{-1}[U]$ is the union of open sets.
And, If f is Continuous, then $f^{-1}[U] \cap X_1$ be Open set and it is equal that $(f_1^{-1}[U] \cup f_2^{-1}[U] \cup f_3^{-1}[U]) \cap X_1 = f_1^{-1}[U]$.

2. It follows that: For any open on Domain $U \in \mathcal{T}_{X_\Pi}$,

$$f[U] = f_1[U] \cup f_2[U] \cup f_3[U]$$

This, if f_2 is Open map, then f is Open map since $f[U]$ is the union of open sets.
And, If f is Open, then $f[U] \cap Y_2$ be Open set and it is equal that $(f_1[U] \cup f_2[U] \cup f_3[U]) \cap Y_2 = f_2[U]$.
3. Similar to the above. $\square$

For a specific example, Define for each $i = 1, 2, 3$,

$$X_i \stackrel{\text{def}}{=} \{a_i, b_i\}, \quad \begin{cases} \mathcal{T}_{i,D} \stackrel{\text{def}}{=} \{\emptyset, X_i, \{a_i\}, \{b_i\}\} \\ \mathcal{T}_{i,I} \stackrel{\text{def}}{=} \{\emptyset, X_i\} \\ \mathcal{T}_{i,a} \stackrel{\text{def}}{=} \{\emptyset, X_i, \{a_i\}\} \\ \mathcal{T}_{i,b} \stackrel{\text{def}}{=} \{\emptyset, X_i, \{b_i\}\} \end{cases}$$

And define functions

1. $f_1 : (X_1, \mathcal{T}_{1,I}) \to (X_1, \mathcal{T}_{1,D}) : x \mapsto x$ is Not Continuous, Open, Closed.
2. $f_2 : (X_2, \mathcal{T}_{2,a}) \to (X_2, \mathcal{T}_{2,a}) : x \mapsto a_2$ is Continuous, Open, Not Closed.
3. $f_3 : (X_1, \mathcal{T}_{3,a}) \to (X_1, \mathcal{T}_{3,b}) : x \mapsto a_3$ is Continuous, Not Open, Closed.
4. $g_i : (X_i, \mathcal{T}_{i,D}) \to (X_i, \mathcal{T}_{i,D}) : x \mapsto x$ is Continuous, Open, Closed for each $i = 1, 2, 3$.

Now, from the above discussion,

1. $g_1 \amalg g_2 \amalg g_3$ is Continuous, Open, Closed.
2. $f_1 \amalg g_2 \amalg g_3$ is Not Continuous, Open, Closed.
3. $g_1 \amalg f_2 \amalg g_3$ is Continuous, Not Open, Closed.
4. $g_1 \amalg g_2 \amalg f_3$ is Continuous, Open, Not Closed.
5. $f_1 \amalg f_2 \amalg f_3$ is Not Continuous, Not Open, Not Closed.
6. $g_1 \amalg f_2 \amalg f_3$ is Continuous, Not Open, Not Closed.
7. $f_1 \amalg f_2 \amalg g_3$ is Not Continuous, Not Open, Closed.
8. $f_1 \amalg g_2 \amalg f_3$ is Not Continuous, Open, Not Closed.

No.	Map	Continuous	Open	Closed
1	$g_1 \amalg g_2 \amalg g_3$	Yes	Yes	Yes
2	$f_1 \amalg g_2 \amalg g_3$	No	No	No
3	$g_1 \amalg f_2 \amalg g_3$	Yes	No	Yes
4	$g_1 \amalg g_2 \amalg f_3$	Yes	Yes	No
5	$f_1 \amalg f_2 \amalg f_3$	No	No	No
6	$g_1 \amalg f_2 \amalg f_3$	Yes	No	No
7	$f_1 \amalg f_2 \amalg g_3$	No	No	Yes
8	$f_1 \amalg g_2 \amalg f_3$	No	Yes	No

10.2 Compact Space

Definition 10.2.0.1. A Topological Space X is *compact* if: every open cover contains a finite subcover. i.e.,

$$\text{If } X = \bigcup_{\alpha \in \Lambda} \mathcal{U}_\alpha,\ (\mathcal{U}_\alpha \in \mathcal{T}),\ \text{then there is finite subcover such that } X = \bigcup_{i=1}^{N} \mathcal{U}_{\alpha_i}$$

This is equivalent with:

$$\text{If } \emptyset = \bigcap_{\alpha \in \Lambda} \mathcal{C}_\alpha,\ (\mathcal{C}_\alpha \text{ closed}),\ \text{then there is finite subset such that } \emptyset = \bigcap_{i=1}^{N} \mathcal{C}_{\alpha_i}$$

Definition 10.2.0.2. Let X be a set. $A \subset \mathcal{P}(X)$ satisfies *finite intersection property* if:

$$\text{For all finite subset of } A,\ \{A_i \mid i = 1, 2, \ldots, n\} \subset A \text{ satisfies } \bigcap_{i=1}^{n} A_i \neq \emptyset.$$

Example.

1. $X = \mathbb{R}$, and let $A = \{(n, \infty) \mid n \in \mathbb{N}\}$. Then,

$$\bigcap_{S \in A} S = \emptyset, \quad \bigcap_{\substack{S \in F \subset A \\ |F| < \infty}} S \neq \emptyset$$

2. $X = \mathbb{R}$, and let $A = \{\mathbb{R} \setminus F \mid |F| < \aleph_0\}$.

Theorem 10.2.0.1. Let X be a Topological Space, Then, TFAE:

a) X is Compact Space.

b) If A is a collection of closed subsets of X that satisfies *FIP*, then $\bigcap_{\mathcal{C} \in A} \mathcal{C}_i \neq \emptyset$.

c) If A is a collection of subsets of X that satisfies *FIP*, then $\bigcap_{S \in A} \overline{S} \neq \emptyset$.

Proof. $a) \implies b)$. Proof by Contradiction:
Suppose that $A \subset \mathcal{P}(X)$ be a collection of closed subsets such that *FIP*.
Assume that $\bigcap_{\mathcal{C} \in A} \mathcal{C} = \emptyset$. Since X is Compact,

$$\emptyset = \bigcap_{\mathcal{C} \in A} \mathcal{C} \text{ if and only if } X = \bigcup_{\mathcal{C} \in A} (X \setminus \mathcal{C}),\ \text{where } X \setminus \mathcal{C} \text{ is open.}$$

This implies that there is a finite subcover:

$$X = \bigcup_{i=1}^{N} (X \setminus \mathcal{C}_i) \text{ if and only if } \emptyset = \bigcap_{i=1}^{N} \mathcal{C}$$

This is Contradiction with A satisfies *FIP*.
$b) \implies a)$. Proof by Contraposition:
Suppose that X is not Compact. Then, there exists an Open Cover $\mathcal{O}$ with no finite subcover: i.e.,

$$X = \bigcup_{\mathcal{U} \in \mathcal{O}} \mathcal{U} \text{ if and only if } \emptyset = \bigcap_{\mathcal{U} \in \mathcal{O}} (X \setminus \mathcal{U})$$

And,

$$\text{For any finite subset of } \mathcal{O},\ F = \{\mathcal{U}_i \mid i = 1, \ldots, N\} \text{ satisfies } X \supsetneq \bigcup_{i=1}^{N} \mathcal{U}_i \text{ if and only if } \emptyset \neq \bigcap_{i=1}^{N} (X \setminus \mathcal{U}_i)$$

Thus, $\mathcal{K} = \{X \setminus \mathcal{U} \mid \mathcal{U} \in \mathcal{O}\}$ satisfies ***FID***, but $\emptyset = \bigcap_{\mathcal{U} \in \mathcal{O}} (X \setminus \mathcal{U}) = \bigcap_{\mathcal{C} \in \mathcal{K}} \mathcal{C}$. Thus, not $a)$ implies not $b)$. $\square$

> **Theorem 10.2.0.2.** Let X is Compact Space, Y is Topological Space, and $f : X \to Y$ is Continuous Map. Then $f[X]$ is Compact.

Proof. Let $\mathcal{O}$ be an open cover of $f[X]$. i.e, $f[X] \subset \bigcup_{\mathcal{U} \in \mathcal{O}} \mathcal{U}$. Now,

$$X \subset f^{-1}[f[X]] \subset f^{-1}\left[\bigcup_{\mathcal{U} \in \mathcal{O}} \mathcal{U}\right] = \bigcup_{\mathcal{U} \in \mathcal{O}} \underbrace{f^{-1}[\mathcal{U}]}_{\text{open, } f \text{ conti.}}$$

Since X is compact, there is a finite subcover such that

$$X \subset \bigcup_{i=1}^{N} f^{-1}[\mathcal{U}_i]$$

Consequently,

$$f[X] \subset f\left[\bigcup_{i=1}^{N} f^{-1}[\mathcal{U}_i]\right] = \bigcup_{i=1}^{N} f\left[f^{-1}[\mathcal{U}_i]\right] \subset \bigcup_{i=1}^{N} \mathcal{U}_i$$

$\square$

> **Theorem 10.2.0.3.** Closed set of compact space is compact.

Proof. Let X be a compact, and $E \subset X$ be a closed subset. Let $\mathcal{O}$ be an open over of E.
Then,

$$X = E \cup (X \setminus E) \subset \left(\bigcup_{\mathcal{U} \in \mathcal{O}} \mathcal{U}\right) \cup (X \setminus E)$$

be an open cover of X. Thus, there is a finite subcover such that

$$X = \left(\bigcup_{i=1}^{N} \mathcal{U}_i\right) \cup (X \setminus E) \iff E \subset \bigcup_{i=1}^{N} \mathcal{U}_i$$

$\square$

> **Theorem 10.2.0.4.** Let X be a Topological Space, and β be a basis of X. Then, TFAE:
>
> ***a)*** X is Compact Space.
>
> ***b)*** Every open cover consisting of basis elements has a finite subcover.

Proof. $a) \implies b)$. Clear by definition of Compact.
$b) \implies a)$. Let $\{\mathcal{U}_\alpha \mid \alpha \in \Lambda\}$ be an Open cover of X. That is,

$$X = \bigcup_{\alpha \in \Lambda} \mathcal{U}_\alpha = \bigcup_{\alpha \in \Lambda} \bigcup_{\gamma \in \Gamma_\alpha} B_\alpha^\gamma$$

where $\{B_\alpha^\gamma \mid \gamma \in \Gamma_\alpha\}$ is subset of basis such that $\bigcup_{\gamma \in \Gamma_\alpha} B_\alpha^\gamma = \mathcal{U}_\alpha$. Now, by $2)$, there is finite subcover such that

$$X = \bigcup_{i=1}^{n} \bigcup_{j=1}^{m} B_{\alpha_i}^{\gamma_j} \subset \bigcup_{i=1}^{n} \mathcal{U}_{\alpha_i}$$

Thus, $\{\mathcal{U}_{\alpha_i} \mid i = 1, 2, \ldots, n\}$ be a finite subcover. $\square$

> **Theorem 10.2.0.5.** Let X, Y are Topological Space. Then, TFAE:
>
> ***a)*** $X \times Y$ is Compact.
>
> ***b)*** X and Y both are Compact.

Proof. $a) \implies b)$ is clear since projection preserves Compactness.
$b) \implies a)$ Let $\mathcal{O} \overset{\text{def}}{=} \{U \times V \mid U \in \mathcal{T}_X, V \in \mathcal{T}_Y\}$ be an Open cover of $X \times Y$.
Let $x \in X$ fix. Then, $\{x\} \times Y$ be a Compact, being $\{x\} \times Y \cong Y$ by Homeomorphism given by Projection.
Then, there is a finite subcover of $\mathcal{O}$ such that

$$\{x\} \times Y \subset \bigcup_{i=1}^{n_x} (U_i^x \times V_i^x)$$

Now, for each $x \in X$, define $U^x \overset{\text{def}}{=} \bigcap_{i=1}^{n_x} U_i^x$. Then, U^x is an open set containing x, and for any $i = 1, 2, \ldots, n_x$, $U^x \subset U_i^x$.
Since $\{U^x \mid x \in X\}$ be an open cover of X, there is a finite subcover such that

$$X = \bigcup_{i=1}^{m} U^{x_i}$$

being X is Compact. Now,

$$X \times Y = \left(\bigcup_{i=1}^{m} U^{x_i} \right) \times Y = \bigcup_{i=1}^{m} (U^{x_i} \times Y) \subset \bigcup_{i=1}^{m} \bigcup_{j=1}^{n_{x_i}} (U_j^{x_i} \times V_j^{x_i})$$

Thus, $\{U_j^{x_i} \times V_j^{x_i} \mid i = 1, 2, \ldots, m,\ \ j = 1, 2, \ldots, n_{x_i}\}$ be a finite subcover. $\square$

Tube Lemma

> Let X be a Topological Space, and Y is Compact Space.
> Then, for product space $X \times Y$, and fixed $x_0 \in X$, following statement holds:
>
> For any open $N \subset X \times Y$ with $\{x_0\} \times Y \subset N$, there is an open $W \in \mathcal{T}_X$ such that $\{x_0\} \times Y \subset W \times Y \subset N$.

Proof. Clearly, $\{x_0\} \times Y$ compact, being $\{x_0\} \times Y \simeq Y$.
For any $y \in Y$, $(x_0, y) \in \{x_0\} \times Y \subset N$, thus there exist opens $U \in \mathcal{T}_X$ and $V \in \mathcal{T}_Y$ such that $(x_0, y) \in U_y \times V_y \subset N$.
Now, Clearly $\{U_y \times V_y \subset X \times Y \mid y \in Y\}$ be an open cover of $\{x_0\} \times Y$, thus there is a finite subcover such that

$$\{x_0\} \times Y \subset \bigcup_{i=1}^{N} (U_{y_i} \times V_{y_i}) \subset N$$

Set $W = \bigcap_{i=1}^{N} U_{y_i}$. Then, clearly $x_0 \in W$, and

Let $(x, y) \in W \times Y$. Then, since $Y = \bigcup_{i=1}^{n} V_{y_i}$, there is $1 \le k \le n$ such that $y \in V_{y_k}$.
Thus, $(x, y) \in U_{y_k} \times V_{y_k} \subset N$, this implies $W \times Y \subset N$. $\square$

> **Theorem 10.2.0.6.** Let Y be a Compact Space. Then, the following statements are true, but their converses are false:
>
> 1. If X be a Lindelöf Space, then the product Topology $X \times Y$ be a Lindel$\ddot{o}$f Space.
>
> 2. If X be a Countable Compact Space, then the product Topology $X \times Y$ be a Countable Compact Space.

Proof. 1. Let $\mathcal{O}$ be an open cover of $X \times Y$.
For any $x \in X$, $\{x\} \times Y$ is compact set, being $\{x\} \times Y \simeq Y$. Thus, there is a finite subcover of $\mathcal{O}$ such that

$$\{x\} \times Y \subset \bigcup_{j=1}^{N_x} U_j^x \quad (U_j^x \in \mathcal{O})$$

Since Tube Lemma, there is an open $W_x \in \mathcal{T}_X$ such that

$$\{x\} \times Y \subset W_x \times Y \subset \bigcup_{j=1}^{N_x} U_j^x$$

Meanwhile, since X is Lindel$\ddot{o}$f, therefore for an open cover $\{W_x \mid x \in X\}$
there exists a Countable subcover such that

$$X \subset \bigcup_{i=1}^{\infty} W_{x_i}$$

Consequently,

$$X \times Y \subset \left(\bigcup_{i=1}^{\infty} W_{x_i}\right) \times Y \subset \bigcup_{i=1}^{\infty} (W_{x_i} \times Y) \subset \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{N_{x_i}} U_j^{x_i}$$

Now, $\left\{U_j^{x_i} \mid i \in \mathbb{N}, 1 \le j \le N_{x_i}\right\} \subset \mathcal{O}$ be a Countable Open Cover of $X \times Y$. □

Proof. 2. Let $\{U_n \subset X \times Y \mid n \in \mathbb{N}\}$ be a Countable open cover of $X \times Y$. For each finite subset $F \subset \mathbb{N}$, define

$$V_F \overset{\text{def}}{=} \left\{x \in X \;\middle|\; \{x\} \times Y \subset \bigcup_{n \in F} U_n\right\}$$

Then V_F satisfies:
1) V_F is open: Let a finite subset $F \subset \mathbb{N}$ fix. For each $x \in V_F$, $\{x\} \times Y \subset \bigcup_{n \in F} U_n$ by definition.

Then, there is an open $W_x \in \mathcal{T}_X$ such that $\{x\} \times Y \subset W_x \times Y \subset \bigcup_{n \in F} U_n$ by Tube Lemma.

Meanwhile, $W_x \subset V_F$ because for all $s \in W_x$, $\{s\} \times Y \subset W_x \times Y \subset \bigcup_{n \in F} U_n$, thus $s \in V_F$.
In summary, for any $x \in V_F$, there is an open $W_x \in \mathcal{T}_X$ such that $x \in W_x \subset V_F$. Consequently, V_F is open of X.
2) $\{V_F \mid F \subset \mathbb{N}, |F| < \infty\}$ is a Countable Open Cover of X:
Countability given by above set is collection of subsets of Countable set. Meanwhile,
For any $x \in X$, there is a finite subcover of $\{U_n \mid n \in \mathbb{N}\}$ such that $\{x\} \times Y \subset \bigcup_{n \in F} U_n$ where F finite.

That is, $x \in V_F$. Now, the open cover of X,

$$\{V_{F_x} \mid x \in X\} \subset \{V_F \mid F \subset \mathbb{N}\}$$

at most Countable. Since X is Countably Compact Space, there is a finite subcover such that

$$X \subset \bigcup_{i=1}^{N} V_{F_i}$$

Consequently,

$$X \times Y \subset \left(\bigcup_{i=1}^{N} V_{F_i}\right) \times Y = \bigcup_{i=1}^{N} (V_{F_i} \times Y) \subset \bigcup_{i=1}^{N} \bigcup_{n \in F_i} U_n$$

That is, $\{U_i \mid i = 1, 2, \ldots, N,\ n \in F_i\}$ be a finite subcover. □

10.2.1 Locally Compact

10.2.2 One-point Compactification

10.3 Baire Category

Definition 10.3.0.1. The Topological Space X is called ***Baire Space*** if:

If $\{G_n \mid n \in \mathbb{N}\}$ be a Countable Collection of dense open sets of X, then $\overline{\bigcap_{n=1}^{\infty} G_n} = X$

In brief, every Countable intersection of dense open sets be dense in X.

Definition 10.3.0.2. Let X be a Topological Space.
$A \subset X$ is said to be ***nowhere dense subset*** if $(\overline{A})^{\circ} = \emptyset$.

1. $B \subset X$ is called ***first category*** if B can be representive by union of countable nowhere dense subsets.
2. If the subset is not first category, then it is said to be ***second category***.

10.4 Locally Compact Hausdorff Space

Theorem 10.4.0.1. Locally Compact Hausdorff Space is Baire Space.

10.5 Complete Metric Space

Definition 10.5.0.1. Let (X, d) be a Metric Space, and $\{p_n\}$ be a Sequence in X.
The Sequence $\{p_n\}$ is called ***Cauchy Sequence*** if:

For any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $m, n \geq N \implies d(p_m, p_n) < \epsilon$.

A Metric Space (X, d) is said to be ***Complete*** if every Cauchy Sequnces Converge.

Lemma 10.5.0.1. Let $\{E_n\}$ be a sequence of closed bounded non-empty subsets in a Complete Metric Space X such that
$E_n \supset E_{n+1}$. If $\lim_{n\to\infty} \text{diam} E_n = 0$, then $\bigcap_{n=1}^{\infty} E_n = \{p\}$ for some $p \in X$.

Proof. For each $n \in \mathbb{N}$, construct $p_n \in E_n$.
Let $\epsilon > 0$ be given. Since diam$E_n \to 0$, there is $N \in \mathbb{N}$ such that diam$E_n < \epsilon$.
For any $m, n \geq M$, E_N contains p_m, p_n. That is, $d(p_m, p_n) < \epsilon$. Thus, $\{p_n\}$ be a Cauchy sequence of X.
Since X is complete, there is a unique point $p \in X$ such taht $p_n \to p$. Let $N \in \mathbb{N}$ be an integer such that $n \geq N \implies |p_n - p| < \epsilon$.
Now, for each $n \geq N$, E_n has a limit point as p. And for any $n \in \mathbb{N}$, E_n contains $E_N, E_{N+1}, \ldots$, thus for all $n \in \mathbb{N}$, E_n has a limit point as p. Meanwhile, E_n closed, $p \in E_n,\ \forall n \in \mathbb{N}$.
Consequently, $p \in \bigcap_{n=1}^{\infty} E_n$. If there is $q \in X$ such that $p \neq q$, $q \in \bigcap_{n=1}^{\infty} E_n$. Then, diam$E_n \geq d(p, q) > 0,\ \forall n \in \mathbb{N}$. ☐

Theorem 10.5.0.1. Complete Metric Space is Baire Space.

Proof. Suppose that $\{G_n \mid n \in \mathbb{N}\}$ be a Countable Collection of dense open set of Complete Metric Space.
Let an open $U \in \mathcal{T}$ be given. Since G_n is dense in the Space, $U \cap G_1$ is non-empty open set.
Thus, there exists a $p_1 \in U \cap G_1$ such that for some $r_1 > 0$, $B_{r_1}(p_1) \subset U \cap G_1$.
Then, automatically,

$$B_{\frac{r_1}{2}}(p_1) \subset \overline{B_{\frac{r_1}{2}}(p_1)} \subset B_{r_1}(p_1) \subset U \cap G_1$$

Set $E_1 = U,\ E_2 = B_{\frac{r_1}{2}}(p_1)$.
Suppose that $E_1, \ldots, E_{n-1}$ are chosen. Then, since $E_{n-1} \cap G_{n-1}$ is open, being intersection of opens. Thus there exists a point $p_{n-1} \in E_{n-1} \cap G_{n-1}$ and exists r_{n-1} such that

$$B_{r_{n-1}}(p_{n-1}) \subset E_{n-1} \cap G_{n-1} \subset E_{n-1}$$

This implies that

$$B_{\frac{r_{n-1}}{2}}(p_{n-1}) \subset \overline{B_{\frac{r_{n-1}}{2}}(p_{n-1})} \subset B_{r_{n-1}}(p_{n-1}) \subset E_{n-1} \cap G_{n-1} \subset E_{n-1}$$

Set $E_n = B_{\frac{r_{n-1}}{2}}(p_{n-1})$. Since inductively construction of $\{E_n\}$, $E_{n+1} \subset E_n$ and $\overline{E_n} \subset G_n$ for all $n \in \mathbb{N}$. Consequently,

$$U \cap \left(\bigcap_{n=1}^{\infty} G_n \right) = \bigcap_{n=1}^{\infty} (U \cap G_n) \supset \bigcap_{n=1}^{\infty} (U \cap \overline{E_n}) = U \cap \left(\bigcap_{n=1}^{\infty} \overline{E_n} \right) = \bigcap_{n=1}^{\infty} \overline{E_n} \neq \emptyset$$

□

10.5.1 Nowhere Differentiable function

10.5.2 Banach Fixed Point Theorem

Definition 10.5.2.1. Let $f : X \to X$ be any function. A point $x \in X$ is called a ***fixed point*** of f if $f(x) = x$.

Definition 10.5.2.2. Let X be a Metric Space. A map $f : X \to X$ is called *Contractive* with respect to the metric d if:

There exsits $\alpha \in (0, 1)$ such that for all $x, y \in X$, $d(f(x), f(y)) \leq \alpha d(x, y)$.

Theorem 10.5.2.1. *Banach Fixed point Theorem*
Let (X, d) be a Complete Metric Space, and $f : X \to X$ be a Contractive map.
Then, there exists a unique fixed point of f, $x^* \in X$.

Proof. Clearly,

$$\text{Contractive} \implies \text{Lipschitz Condition} \implies \text{Continuous}.$$

Thus, f is Continuous.
Let $x_0 \in X$ be arbitrary, and construct a sequence $\{x_n\}$ recursively as follows:

$$x_{n+1} \overset{\text{def}}{=} f(x_n), \ \ n \geq 0$$

Then, for any $n \geq 0$,

$$\begin{aligned} d(x_{n+1}, x_n) &= d(f(x_n), f(x_{n-1})) \leq \alpha d(x_n, x_{n-1}) \\ &= d(f(x_{n-1}), f(x_{n-2})) \leq \alpha^2 d(x_{n-1}, x_{n-2}) \\ &\quad\vdots \\ &\leq \alpha^n d(x_1, x_0) \end{aligned}$$

Let $\epsilon > 0$ be given. Put $N \in \mathbb{N}$ such that $\alpha^N \cdot d(x_1, x_0) < \epsilon(1 - \alpha)$. Then, $n \geq m \geq N$ implies that

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + \cdots + d(x_{m+1}, x_m) \\ &\leq \alpha^n d(x_1, x_0) + \alpha^{n-1} d(x_1, x_0) + \cdots + \alpha^{m+1} d(x_1, x_0) \\ &= \alpha^{m+1} d(x_1, x_0) \sum_{r=0}^{n-m-1} \alpha^r < \alpha^N d(x_1, x_0) \sum_{r=0}^{\infty} \alpha^r < \epsilon(1 - \alpha) \frac{1}{1 - \alpha} = \epsilon \end{aligned}$$

Therefore, $\{x_n\}$ is Cauchy sequence. Since X is Complete, for some $x^* \in X$, $\lim_{n\to\infty} x_n = x^*$. Consequently,

$$\lim_{n\to\infty} f(x_n) \overset{f \text{ conti.}}{=} f\left(\lim_{n\to\infty} x_n\right) = f(x^*) = \lim_{n\to\infty} x_{n+1} = x^*$$

$\square$

10.6 Urysohn Metrization Theorem

10.6.1 Urysohn Lemma

Recall that:

Definition 10.6.1.1. X is T_4 if: For any disjoint closed set A and B, there exist disjoint open U, V such that $A \subseteq U$ and $B \subseteq V$.

Lemma 10.6.1.1. X is T_4 Space if and only if For any closed C and open U with $C \subseteq U$, there exists open O such that

$$\underset{\text{closed}}{C} \subseteq \underset{\text{open}}{O} \subseteq \underset{\text{closed}}{\overline{O}} \subseteq \underset{\text{open}}{U}$$

Proof. Proof of the left direction only.
Let X be a T_4 Space, and $C \subset X$ be a closed, U be a open containing C. Then, $C \subset U$ implies $U^c \subset C^c$, thus U^c is a closed set disjoint from C. By T_4 condition, There exist disjoint opens O, O' such that $C \subset O$ and $U^c \subset O' \iff O'^c \subset U$.
Since $O \cap O' = \emptyset \iff O \subset O'^c$, O contained in U, this implies that $C \subset O \subset U$.
Since closure is the smallest closed set such that contains it, consequently $C \subset O \subset \overline{O} \subset O'^c \subset U$. □

Definition 10.6.1.2. Let X be a Toplogical Space, and $A, B \subset X$ are disjoint closed subset.
A real-valued Continuous map $f : X \to [a, b]$ is called *Urysohn function* for A and B if: $f|_A = a$ and $f|_B = b$.
In another form,

$$f : X \to [a, b] : x \to \begin{cases} a & x \in A \\ b & x \in B \\ f(x) & x \notin A \cup B \end{cases}$$

Lemma 10.6.1.2. *Urysohn Lemma*
T_4 Space has an Urysohn function for any two disjoint closed subsets.

Proof. Generalization is the last thing to proven, first of all, prove in case of $[a, b] = [0, 1]$. This proof consists by three Step.
Let X be a T_4 Space, and $A, B \subset X$ be closed subsets.
Step 1. Construct a Chain of Open sets with Dyadic Rational Indices.
Consider a set of *Dyadic Rationals* $D \overset{\text{def}}{=} \left\{ \frac{k}{2^n} \;\middle|\; n, k \in \mathbb{N},\ k \leq 2^n - 1 \right\}$. We will show that the following statement holds:

$$\text{For any } r, s \in D \text{ with } r < s\text{, there exist open sets } U_r, U_s \text{ such that } A \subseteq \overline{U}_r \subseteq U_s \subseteq X \setminus B \qquad (*)$$

For this, Enough to Show that: For any $k \in \mathbb{N}$, there exists a Chain as:

$$A \subseteq U_{\frac{1}{2^k}} \subseteq \overline{U}_{\frac{1}{2^k}} \subseteq U_{\frac{2}{2^k}} \subseteq \overline{U}_{\frac{2}{2^k}} \subseteq \cdots \subseteq U_{\frac{2^k-1}{2^k}} \subseteq \overline{U}_{\frac{2^k-1}{2^k}} \subseteq X \setminus B$$

(Note that this opens in the Chain are not necessary distinct: For instance, if Ambient Space is Finite, then the Space is *Noetherian*. That is, X satisfies Ascending Chain Condition for open sets.)
Let $k = 1$. Then, By T_4 condition gives that: There exists an open set U_1 such that

$$A \subseteq U_1 \subseteq \overline{U_1} \subseteq X \setminus B$$

Now, naming this U_1 as $U_{\frac{1}{2}}$, proved when $k = 1$.
Suppose that for some $k > 1$, the Chain exists as:

$$\underset{\text{closed}}{A} \overset{*1}{\subseteq} \underset{\text{open}}{U_{\frac{1}{2^k}}} \subseteq \underset{\text{closed}}{\overline{U}_{\frac{1}{2^k}}} \overset{*2}{\subseteq} \underset{\text{open}}{U_{\frac{2}{2^k}}} \subseteq \cdots \overset{*2^k-1}{\subseteq} \underset{\text{open}}{U_{\frac{2^k-1}{2^k}}} \subseteq \underset{\text{closed}}{\overline{U}_{\frac{2^k-1}{2^k}}} \overset{*2^k}{\subseteq} \underset{\text{open}}{X \setminus B}$$

By repeatedly applying the T_4 condition 2^k times, as indicated by the indices $*1, *2, \ldots, *2^k$, we can construct 2^k open sets such that:

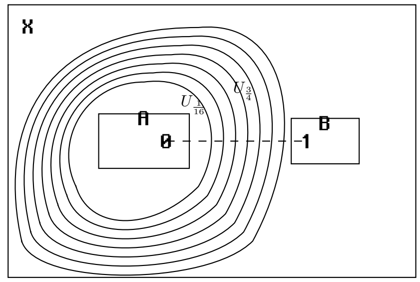
$$A \subseteq U_{\frac{1}{2^{k+1}}} \subseteq \overline{U}_{\frac{1}{2^{k+1}}} \subseteq U_{\frac{1}{2^k}} \subseteq \overline{U}_{\frac{1}{2^k}} \subseteq U_{\frac{3}{2^{k+1}}} \subseteq \overline{U}_{\frac{3}{2^{k+1}}} \subseteq U_{\frac{2}{2^k}} \subseteq \overline{U}_{\frac{2}{2^k}} \quad \subseteq \cdots \subseteq U_{\frac{2^k-1}{2^k}} \subseteq \overline{U}_{\frac{2^k-1}{2^k}} \subseteq U_{\frac{2^{k+1}-1}{2^{k+1}}} \subseteq \overline{U}_{\frac{2^{k+1}-1}{2^{k+1}}} \subseteq X \setminus B$$

Finally, Step 1 proved.

Step 2. Construct an Urysohn Function.

Define a map $f : X \to [0, 1]$ as:

$$f(x) = \begin{cases} 0 & x \in \bigcap_{t \in D} U_t \\ \sup\{t \in D \mid x \notin U_t\} & x \notin \bigcap_{t \in D} U_t \end{cases}$$



Then, this map f is well-defined by $(*)$ and $\sup D \leq 1$. And f satisfies that:

1. $\forall r \in D,\ x \in A \subset U_r$. Thus, $f(x) = 0$ if $x \in A$.
2. $\forall r \in D,\ x \in B \subset X \setminus U_r$. Thus, $f(x) = \sup D = 1$ if $x \in B$.
3. If $x \in \overline{U}_r$, then for every $s > r$, $x \in \overline{U}_r \subset U_s$. Thus, $f(x) \leq r$. In Contrapositive, $f(x) > r \implies x \notin \overline{U}_r$. (If $f(x) = \sup\{t \in D \mid x \notin U_t\} > r$, then there is $s \in D$ such that $s > r$ and $x \notin U_s$, Contradiction.)
4. If $x \notin U_r$, then, $f(x) \geq r$. In Contrapositive, $f(x) < r \implies x \in U_r$.

Now, show that this map f is Continuous map: Let $x \in X$ be fixed arbitrarlily, and $\epsilon > 0$ be given.

In Case of $0 < f(x) < 1$.

Since Density of Dyadic Rationals, Choose $r, s \in D$ such that $f(x) - \epsilon < r < f(x) < s < f(x) + \epsilon$.

Now, we obtain that:

$$x \overset{(*)}{\in} U_s \setminus \overline{U}_r \overset{(**)}{\subseteq} f^{-1}[(f(x) - \epsilon, f(x) + \epsilon)]$$

$(*)$ directly given by above properties, $(**)$ given applying the fact that $x \in U_s \subset \overline{U}_s$ and $x \notin \overline{U}_r$.

In Case of $f(x) = 0$.

Choose $r \in D$ such that $f(x) = 0 < r < \epsilon = f(x) + \epsilon$. Then,

$$x \in U_r \subset f^{-1}[(f(x), f(x) + \epsilon)]$$

In Case of $f(x) = 1$.

Choose $r \in D$ such that $f(x) - \epsilon = 1 - \epsilon < r < 1 = f(x)$. Then,

$$x \in X \setminus U_r \subset f^{-1}[(f(x) - \epsilon, f(x))]$$

Consequently, f is Continuous map on $[0, 1]$ such that $f|_A = 0$ and $f|_B = 1$.

Step 3. Generalization.

Since $[0, 1] \cong [a, b]$ for any $a < b$, let $g : [0, 1] \to [a, b] : x \mapsto (1 - x)a + xb$ be a Homeomorphism.

Then, $h = g \circ f : X \to [a, b]$ becomes a Continuous map such that $h|_A = a$ and $h|_B = b$. $\square$

10.6.2 Tietze Extension Theorem

> **Theorem 10.6.2.1.** *Tietze Extension Theroem*
> Let X be a T_4 Space, and $A \subseteq X$ be a closed subset.
> For any Continuous map $f : A \to \mathbb{R}$, there exists a Continuous map:
> $$g : X \to \mathbb{R} \text{ s.t. } \quad g|_A = f$$
> This g is called *extension* of f.

Proof. This proof consists by three steps.
Step 1. First, we will show that:

$$\text{For any Continuous map } f : A \to [-r, r] \text{, there is a Continuous map } h : X \to \mathbb{R} \text{ s.t. } \begin{cases} \forall x \in X, & |h(x)| \leq \frac{1}{3}r \\ \forall a \in A, & |f(a) - h(a)| \leq \frac{2}{3}r \end{cases} \tag{$*$}$$

Set

$$I_1 \overset{\text{def}}{=} \left[-r, -\frac{1}{3}r\right], \quad I_2 \overset{\text{def}}{=} \left[-\frac{1}{3}r, \frac{1}{3}r\right], \quad I_3 \overset{\text{def}}{=} \left[\frac{1}{3}r, r\right]$$

Then, the preimage of continuous map preserves closed and A is closed subspace of X, $f^{-1}[I_1]$ and $f^{-1}[I_3]$ are closed of X.
And, I_1 and I_3 are disjoint, thus $f^{-1}[I_1 \cap I_3] = f^{-1}[I_1] \cap f^{-1}[I_3] = \emptyset$.
Now, apply the *Urysohn Lemma*: There exists an Urysohn function $h : X \to I_2$ for $f^{-1}[I_1]$ and $f^{-1}[I_3]$.
Clearly, this map h satisfies the first condition in $(*)$. And, for show the second condition, let $a \in A$ be given.
If $a \in f^{-1}[I_1]$, then $f(a) \in I_1$ and $h(a) = -\frac{1}{3}r$, thus $|f(a) - h(a)| \leq \frac{2}{3}r$.
If $a \in f^{-1}[I_3]$, then $f(a) \in I_3$ and $h(a) = \frac{1}{3}r$, thus $|f(a) - h(a)| \leq \frac{2}{3}r$.
If $a \notin (f^{-1}[I_1] \cup f^{-1}[I_3])$, then $f(a), h(a) \in I_2$, thus $|f(a) - h(a)| \leq \frac{2}{3}r$.
Therefore, the second condition satisfied.
Step 2. We will show that: for any $f : A \to [-1, 1]$, there exists an extension of f.
Apply the result in Step 1, there exists a Continuous map:

$$h_1 : X \to \mathbb{R} \text{ s.t. } \begin{cases} \forall x \in X, & |h_1(x)| \leq \frac{1}{3} \\ \forall a \in A, & |f(a) - h_1(a)| \leq \frac{2}{3} \end{cases}$$

Now, the second condition of h_1, the continuous map $f - h_1 : A \to \left[-\frac{2}{3}, \frac{2}{3}\right] : x \to f(x) - h_1(x)$ is well-defined.
Again, there exists a Continuous map:

$$h_2 : X \to \mathbb{R} \text{ s.t. } \begin{cases} \forall x \in X, & |h_2(x)| \leq \frac{1}{3} \cdot \frac{2}{3} \\ \forall a \in A, & |f(a) - h_1(a) - h_2(a)| \leq \left(\frac{2}{3}\right)^2 \end{cases}$$

Inductively, for any $n \in \mathbb{N}$, there exists a Continuous map:

$$h_n : X \to \mathbb{R} \text{ s.t. } \begin{cases} \forall x \in X, & |h_n(x)| \leq \frac{1}{3} \cdot \left(\frac{2}{3}\right)^{n-1} \\ \forall a \in A, & |f(a) - h_1(a) - h_2(a) - \cdots - h_n(a)| \leq \left(\frac{2}{3}\right)^n \end{cases}$$

Define a map

$$g : X \to [-1, 1] : x \mapsto \sum_{n=1}^{\infty} h_n(x)$$

For any $x \in X$,

$$|g(x)| = \left|\sum_{n=1}^{\infty} h_n(x)\right| \leq \sum_{n=1}^{\infty} |h_n(x)| \leq \sum_{n=1}^{\infty} \frac{1}{3} \cdot \left(\frac{2}{3}\right)^{n-1} = \frac{1}{3} \cdot \frac{1}{1 - \frac{2}{3}} = 1$$

Therefore, this map is well-defined. And, ***Weierstrass M-test*** gives that $\sum_{n=1}^{\infty} h_n(x)$ converges uniformly.
Moreover, for any $a \in A$,

$$\left|f(a) - \sum_{k=1}^{n} h_k(a)\right| \leq \left(\frac{2}{3}\right)^n \implies \left|f(a) - \sum_{n=1}^{\infty} h_n(a)\right| = |f(a) - g(a)| = 0$$

That is, g is Continuous on X and $g|_A = f$. Therefore, g is extension of f.
Step 3. Finally, we generalize the result in Step 2.:
Let $f : A \to [a,b]$ be a Continuous map on the closed subspace A. And, let $\varphi : [a,b] \to [-1,1]$ be a Homeomorphism. Then, $\varphi \circ f : A \to [-1,1]$ is Continuous map, thus there exists an extension $g : X \to [-1,1]$ such that $g|_A = \varphi \circ f$. Now, $\varphi^{-1} \circ g : X \to [a,b]$ is Continuous, and $(\varphi^{-1} \circ g)|_A = \varphi^{-1} \circ \varphi \circ f = f$, Therefore this $\varphi^{-1} \circ g$ is the extension of f.
Let $f : A \to \mathbb{R}$ be a Continuous map on the closed subspace A.
And, let $\varphi : \mathbb{R} \to (-1,1)$ be a Homeomorphism. Then, the map $\phi : \mathbb{R} \to [-1,1] : x \mapsto \varphi(x)$ is still Continuous.
Now, The Continuous map $\phi \circ f : A \to [-1,1]$ has an extension $g : X \to [-1,1]$ such that $g|_A = \phi \circ f$.
Put $B = g^{-1}[\{-1,1\}]$. Then B is Closed on X, and $A \cap B = \emptyset$. Now, apply the Urysohn Lemma to this, there exists an Urysohn function for A and B: Continuous map $\gamma : X \to [0,1]$ such that $\gamma|_A = 1$ and $\gamma|_B = 0$.
Define a map $\eta : X \to (-1,1) : x \mapsto g(x)\gamma(x)$. Then, if $g(x) = 1$ or $g(x) = -1$, then $x \in B$, thus $g(x)\gamma(x) = 0$.
Therfore, η is well-defined. And, for any $a \in A$, $\eta(a) = g(a)\gamma(a) = g(a)$, thus $\eta|_A = \phi \circ f$.
Consequently, the map $\phi^{-1} \circ \eta$ is an extension of f, we wanted. □

Recall that:

Definition 10.6.2.1. X is T_1 if: For any distinct $x, y \in X$, there exist open sets U_x, U_y such that $\begin{cases} x \in U_x, & x \notin U_y \\ y \notin U_x, & y \in U_y \end{cases}$.

Lemma 10.6.2.1. X is T_1 if and only if For any $x \in X$, a singleton $\{x\}$ is closed in X.

Proof. The left direction is clear.
Let $x \in X$. Then, for any $y \in X$ with $y \neq x$, T_1 condition gives that there is an open set such that $y \in U_y$ and $x \notin U_y$.
Now, the union

$$\bigcup_{\substack{y \in X \\ y \neq x}} U_y = X \setminus \{x\}$$

is open by definition. □

10.6.3 Urysohn Metrization Theorem

> **Theorem 10.6.3.1.** *Urysohn Metrization Theroem*
> If X is a Second-Countable Regular Space, then X is Metrizable.

Chapter 11

Algebraic Topology

Chapter 12

Basic Analysis

12.1 Tests for Series

12.1.1 Integral Test

Theorem 12.1.1.1. Let $f : [1, \infty) \to \mathbb{R}$ be a decreasing function which satisfies $\begin{cases} \lim_{x\to\infty} f(x) = 0 \\ f > 0 \end{cases}$. Then,

$$\int_1^\infty f(x)dx \text{ converges } \textit{if and only if } \sum_{k=1}^\infty f(k) \text{ converges.}$$

Futhermore, put $d_n \overset{\text{def}}{=} \sum_{k=1}^n f(k) - \int_1^n f(x)dx$, then for any $n \in \mathbb{N}$, $0 < f(n+1) \leq d_{n+1} \leq d_n \leq f(1)$, and for any $k \in \mathbb{N}$, $0 \leq d_k - \lim_{n\to\infty} d_n \leq f(k)$. (Clearly, $\lim_{n\to\infty} d_n$ exists.)

Proof. Since

$$\int_1^{n+1} f(x)dx = \sum_{k=1}^n \int_k^{k+1} f(x) \overset{\substack{\text{decreasing} \\ \downarrow}}{\leq} \sum_{k=1}^n \int_k^{k+1} f(k) = \sum_{k=1}^n f(k)$$

$$\implies f(n+1) = \sum_{k=1}^{n+1} f(k) - \sum_{k=1}^n f(k) \leq \sum_{k=1}^{n+1} f(k) - \int_1^{n+1} f(x)dx = d_{n+1}$$

And,

$$d_n - d_{n+1} = \int_n^{n+1} f(x)dx - f(n+1) \geq \int_n^{n+1} f(n+1)dx - f(n+1) = 0$$

Immediate d_n converges, being bounded and decreasing. That is,

$$\lim_{n\to\infty} d_n = \lim_{n\to\infty} \left(\sum_{k=1}^n f(k) - \int_1^n f(x)dx \right)$$

converges. Meanwhile, since

$$0 \leq d_n - d_{n+1} = \int_n^{n+1} f(x)dx - f(n+1) \leq \int_n^{n+1} f(n)dx - f(n+1) = f(n) - f(n+1)$$

Now, telescope:

$$0 \leq d_k - \lim_{n\to\infty} d_n \leq f(k) - \lim_{n\to\infty} f(n+1) = f(k)$$

□

12.1.2 Ratio Test

Theorem 12.1.2.1. Let $\sum a_n$ be given.
$\sum_{n=1}^\infty a_n$ converges if: $\limsup_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$.
$\sum_{n=1}^\infty a_n$ diverges if: $n_0 \in \mathbb{N}$ such that $\forall n \geq n_0$, $\left| \frac{a_{n+1}}{a_n} \right| \geq 1$.

Proof. Choose $\beta < 1$ such that for some $N \in \mathbb{N}$, $n \geq N \implies \left| \frac{a_{n+1}}{a_n} \right| < \beta < 1$.
Then,

$$\begin{aligned} |a_{N+1}| &< \beta |a_N| \\ |a_{N+2}| &< \beta |a_{N+1}| < \beta^2 |a_N| \\ &\vdots \\ |a_{N+p}| &< \beta^p |a_N| \quad (p \in \mathbb{N}) \end{aligned}$$

As a result, for all $n \geq N$, $|a_n| < \beta^{n-N} |a_N|$. And, $\sum_{n=1}^\infty |a_n| \leq \sum_{n=1}^\infty \beta^{n-N} |a_N| < \infty$. □

12.1.3 Root Test

Theorem 12.1.3.1. Let $\sum a_n$ be given.

$\sum_{n=1}^{\infty} a_n$ converges if: $\limsup_{n\to\infty} \sqrt[n]{|a_n|} < 1$.

$\sum_{n=1}^{\infty} a_n$ diverges if: $\limsup_{n\to\infty} \sqrt[n]{|a_n|} < 1$.

Proof. Put $\beta \in \mathbb{R}$ such that $\limsup_{n\to\infty} \sqrt[n]{|a_n|} < \beta < 1$. Then, there is $N \in \mathbb{N}$ such that $n \geq N \implies \sqrt[n]{|a_n|} < \beta$. Now, $\sum |a_n| < \sum \beta^n < \infty$. But if $\limsup_{n\to\infty} \sqrt[n]{|a_n|} > 1$, then $a_n \nrightarrow 0$. □

12.2 Arithmetic means

Let $\{s_n\}$ be a Complex numbers Sequence. Define the ***Arithmetic means*** of $\{s_n\}$:

$$\sigma_n \overset{\text{def}}{=} \frac{s_0 + \cdots + s_n}{n+1} = \frac{1}{n+1}\left(\sum_{i=0}^{n} s_i\right)$$

Then, the Arithmetic means σ_n has the following properties:
1). If $\lim_{n\to\infty} s_n = s$, then $\lim_{n\to\infty} \sigma_n = s$.

Proof. Let $\epsilon > 0$ be given. Then, there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $|s_n - s| < \epsilon$.
Now, for $n \geq N$,

$$\begin{aligned}
|\sigma_n - s| &= \left|\frac{s_0 + \cdots + s_n}{n+1} - \frac{(n+1)s}{n+1}\right| = \left|\frac{(s_0 - s) + \cdots + (s_n - s)}{n+1}\right| \\
&\overset{\text{tri.ieq}}{\leq} \frac{\sum_{k=0}^{N-1} |s_k - s|}{n+1} + \frac{\sum_{k=N}^{n} |s_k - s|}{n+1} \\
&< \frac{\sum_{k=0}^{N-1} |s_k - s|}{n+1} + \frac{n+1-N}{n+1} \cdot \epsilon \\
&< \frac{\sum_{k=0}^{N-1} |s_k - s|}{n+1} + \epsilon
\end{aligned}$$

Now, put $M \in \mathbb{N}$ satisfies $M \geq N$ and $n \geq M \implies \frac{\sum_{k=0}^{N-1} |s_k - s|}{n+1} < \epsilon$, using Archimedean property.
Then, $n \geq M$ implies $|\sigma_n - s| < \epsilon$, thus $\sigma_n \to s$. □

2). Put $a_n = s_n - s_{n-1}$, for $n \geq 1$. If $\lim_{n\to\infty} na_n = 0$ and σ_n converges, then s_n converges.

Proof. First,

$$\begin{aligned}
s_n - \sigma_n &= s_n - \frac{s_0 + \cdots + s_n}{n+1} = \frac{(n+1)s_n - \sum_{k=0}^{n} s_k}{n+1} \\
&= \frac{1}{n+1}((s_1 - s_0) + (2s_2 - 2s_1) + (3s_3 - 3s_2) + \cdots + (ns_n - ns_{n-1})) \\
&= \frac{1}{n+1}\sum_{k=1}^{n} ka_k
\end{aligned}$$

Now, if $na_n \to 0$ and $\sigma_n \to \sigma$,

$$\begin{aligned}
\lim_{n\to\infty} s_n &= \lim_{n\to\infty}\left(\sigma_n + \frac{1}{n+1}\sum_{k=1}^{n} ka_k\right) \\
&= \lim_{n\to\infty} \sigma_n + \lim_{n\to\infty} \frac{1}{n+1}\sum_{k=1}^{n} ka_k \overset{1)}{=} \sigma
\end{aligned}$$

□

2) is conditional converse of 1). But, there is more weak version of the converse proposition:
3). The sequence $\{na_n\}$ bounded by $M < \infty$, and $\sigma_n \to \sigma$. Then, $s_n \to \sigma$.

Proof. First, For positive integers $m < n$,

$$\begin{aligned}
s_n - \sigma_n &= s_n - \frac{\sum_{k=0}^{n} s_k}{n+1} = s_n - \frac{m+1}{n-m} \cdot \left(\frac{1}{m+1} - \frac{1}{n+1}\right)\sum_{k=0}^{n} s_k \\
&= s_n - \frac{m+1}{n-m} \cdot \left(\frac{\sum_{k=0}^{m} s_k + \sum_{k=m+1}^{n} s_k}{m+1} - \frac{\sum_{k=0}^{n} s_k}{n+1}\right) \\
&= s_n - \frac{m+1}{n-m} \cdot \left(\sigma_m - \sigma_n + \frac{\sum_{k=m+1}^{n} s_k}{m+1}\right) \\
&= \frac{m+1}{n-m}(\sigma_n - \sigma_m) + \frac{1}{n-m}\sum_{k=m+1}^{n} (s_n - s_k)
\end{aligned}$$

Meanwhile, since for any $n \in \mathbb{N}$, $|na_n| = n|s_n - s_{n-1}| < M$, for $k = m+1, \ldots, n$,

$$\begin{aligned}|s_n - s_k| &= |s_n - s_{n-1} + s_{n-1} - s_{n-2} + \cdots + s_{k+1} - s_k| \\ &\le |s_n - s_{n-1}| + |s_{n-1} - s_{n-2}| + \cdots + |s_{k+1} - s_k| \\ &\le \frac{M}{n} + \frac{M}{n-1} + \cdots + \frac{M}{k+1} \le \frac{n-k}{k+1}M \le \frac{n-k}{m+2}M \le \frac{n-m-1}{m+2}M\end{aligned}$$

Let $\epsilon > 0$ be given. For each $n \in \mathbb{N}$, put $m \in \mathbb{N}$ such that

$$m \le \frac{n-\epsilon}{1+\epsilon} < m+1$$

Then,

$$m(1+\epsilon) \le n - \epsilon \implies m + \epsilon(1+m) \le n \implies \frac{m+1}{n-m} \le \frac{1}{\epsilon}$$

and

$$n - \epsilon < (m+1)(1+\epsilon) \implies n+1 < (m+2)(1+\epsilon) \implies \frac{n+1}{m+2} - 1 < \epsilon \implies \frac{n-m-1}{m+2} < \epsilon$$

Now, for arbitrary $n \in \mathbb{N}$,

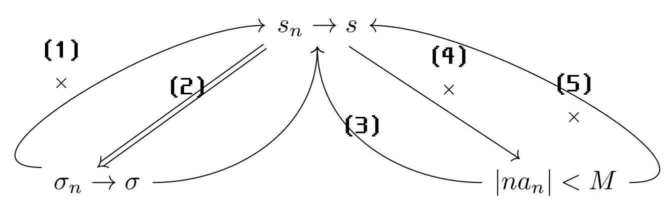
$$\begin{aligned}&|s_n - \sigma| \le |s_n - \sigma| + |\sigma_n - \sigma| \\ \implies &\limsup_{n\to\infty} |s_n - \sigma| \le \limsup_{n\to\infty} |s_n - \sigma_n| + \limsup_{n\to\infty} |\sigma_n - \sigma|\end{aligned}$$

And,

$$\begin{aligned}|s_n - \sigma_n| &= \frac{m+1}{n-m}|\sigma_n - \sigma_m| + \frac{1}{n-m}\sum_{k=m+1}^{n} |s_n - s_k| < \frac{1}{\epsilon}|\sigma_n - \sigma_m| + M\epsilon \\ \implies \limsup_{n\to\infty} |s_n - \sigma_n| &\le \frac{1}{\epsilon}\limsup_{n\to\infty} |\sigma_n - \sigma_m| + M\epsilon = M\epsilon\end{aligned}$$

Consequently, $\limsup_{n\to\infty} |s_n - \sigma| \le (M+1)\epsilon$, thus $s_n \to \sigma$. $\square$

In brief, the diagram of the above conditions like this:



Examples and Counterexamples of the Diagram:

(1) Let $s_n \overset{\text{def}}{=} \exp(\frac{in\pi}{2})$. Then,

- s_n diverges.
- na_n diverges.
- $\sigma_n \to 0$.

(2) Let $s_n \overset{\text{def}}{=} \frac{1}{n}$, $s_0 = 0$.

(3) Let $s_n \overset{\text{def}}{=} \sum_{k=1}^{n} \frac{1}{k}$. Then,

- s_n diverges.
- $a_n = \frac{1}{n}$, thus $na_n \to 1$, bounded.
- If σ_n converges, then the diagram implies that s_n must converge, leading to a contradiction. Therefore, σ_n diverges.

(4) $s_n = \sum_{k=1}^{n} \frac{(-1)^k}{\sqrt{k}}$, $s_0 = 0$. Then,

- s_n converges, being the Alternating series Test.
- $a_n = \frac{(-1)^n}{\sqrt{n}}$, thus na_n diverges.

12.3 Taylor's Theorem

Theorem 12.3.0.1. *Taylor's Theorem*

Let $f : [a, b] \to \mathbb{R}$, and let $n \in \mathbb{N}$ be fixed. Suppose that $\begin{cases} f^{(n-1)} \text{ is Continuous.} \\ f^{(n)}(t) \text{ exists for every } t \in (a, b). \end{cases}$

Then, for any $\alpha, \beta \in [a, b]$, there exists $x \in (\alpha, \beta)$ such that

$$f(\beta) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(\beta - \alpha)^k + \frac{f^{(n)}(x)}{n!}(\beta - \alpha)^n$$

Proof. Put

$$M \overset{\text{def}}{=} \frac{1}{(\beta - \alpha)^n} \cdot \left(f(\beta) - \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(\beta - \alpha)^k \right)$$

That is,

$$f(\beta) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(\beta - \alpha)^k + M(\beta - \alpha)^n$$

and put

$$g(t) \overset{\text{def}}{=} f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(t - \alpha)^k - M(t - \alpha)^n, \quad (a \le t \le b)$$

If we differentiate the above equation n times,

$$g^{(n)}(t) = f^{(n)}(t) - n!M, (a < t < b)$$

For each $k = 0, 1, \ldots, n - 1$,

$$\begin{aligned} \frac{d^r}{dt^r} \left(\sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(t - \alpha)^k \right) &= \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!} \cdot \frac{d^r}{dt^r} \left((t - \alpha)^k \right) \\ &= \sum_{k=r+1}^{n-1} \frac{f^{(k)}(\alpha)}{k!} \cdot \frac{k!}{(k - r)!}(t - \alpha)^{k-r} + f^{(r)}(\alpha) \\ &= \sum_{k=r+1}^{n-1} \frac{f^{(k)}(\alpha)}{(k - r)!}(t - \alpha)^{k-r} + f^{(r)}(\alpha) \end{aligned}$$

Substituting $t = \alpha$, only the $f^{(r)}(\alpha)$ term remains. Therefore, for $r = 0, \ldots, n-1$, $g(\alpha) = g'(\alpha) = \cdots = g^{(n-1)}(\alpha) = 0$.
Since $g(\beta) = 0$ by definition, the Mean-Value Theorem implies there exists a $x_1 \in (\alpha, \beta)$ s.t. $g'(x_1) = \frac{g(\beta) - g(\alpha)}{\beta - \alpha} = 0$.
And similarly, there is $x_2 \in (x_1, \beta)$ s.t. $g''(x_2) = \frac{g'(x_1) - g'(\alpha)}{\beta - \alpha} = 0$.

Inductively, for some $x_n \in (\alpha, \beta)$, $g^{(n)}(x_n) = f^{(n)}(x_n) - n!M = 0$. That is, $M = \dfrac{f^{(n)}(x_n)}{n!}$.
Proof Complete by Initial Setting. $\square$

Corollary 12.3.0.1. Let $f : [a, b] \to \mathbb{R}$ be an infinitely differentiable function.
Suppose that there exists a $M > 0$ such that for any $n \in \mathbb{N}$, $\sup_{t \in [a,b]} |f^{(n)}(t)| \le M$. Then, for any $x, \alpha \in [a, b]$,

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x)}{k!}(x - \alpha)^k$$

12.4 Convexity

12.4.1 Definition

Definition 12.4.1.1. Let $f:(a,b)\to\mathbb{R}$ be a Real-valued function. f is said to be *convex* if: For any $x,y\in(a,b),\lambda\in(0,1)$,

$$f(\lambda x+(1-\lambda)y)\leq\lambda f(x)+(1-\lambda)f(y)$$

Convex function has follwing properties:

Lemma 12.4.1.1. Let $f:(a,b)\to\mathbb{R}$ be a Convex function, and $a<x_1<x_2<x_3<b$. Then,

$$\frac{f(x_2)-f(x_1)}{x_2-x_1}\leq\frac{f(x_3)-f(x_1)}{x_3-x_1}\leq\frac{f(x_3)-f(x_2)}{x_3-x_2}$$

Proof. To show that first inequalty, note that

$$\frac{x_2-x_1}{x_3-x_1}\cdot x_3+\left(1-\frac{x_2-x_1}{x_3-x_1}\right)\cdot x_1=\frac{(x_2-x_1)x_3+(x_3-x_2)x_1}{x_3-x_1}=x_2$$

Now,

$$\begin{aligned}f(x_2)&=f\left(\frac{x_2-x_1}{x_3-x_1}\cdot x_3+\left(1-\frac{x_2-x_1}{x_3-x_1}\right)\cdot x_1\right)\\&\leq\frac{x_2-x_1}{x_3-x_1}\cdot f\left(x_3\right)+\left(1-\frac{x_2-x_1}{x_3-x_1}\right)\cdot f\left(x_1\right)=\frac{x_2-x_1}{x_3-x_1}\cdot f\left(x_3\right)+f(x_1)-\left(\frac{x_2-x_1}{x_3-x_1}\right)\cdot f\left(x_1\right)\end{aligned}$$

In brief,

$$f(x_2)-f(x_1)\leq\frac{x_2-x_1}{x_3-x_1}(f(x_3)-f(x_1))\implies\frac{f(x_2)-f(x_1)}{x_2-x_1}\leq\frac{f(x_3)-f(x_1)}{x_3-x_1}$$

And similarly,

$$\frac{x_3-x_2}{x_3-x_1}\cdot x_1+\left(1-\frac{x_3-x_2}{x_3-x_1}\right)x_3=x_2$$

gives the second inequality. $\square$

12.4.2 Properties

Proposition 12.4.2.1. If $f : (a,b) \to \mathbb{R}$ is Convex, then f is Continuous.

Proof. Let $\epsilon > 0$ be given, $s < t$ are fixed in (a,b). For any $x, y \in (s,t)$ with $s < x < y < t$,

$$\frac{f(s)-f(a)}{s-a} \le \frac{f(x)-f(s)}{x-s} \le \frac{f(y)-f(x)}{y-x} \le \frac{f(t)-f(y)}{t-y} \le \frac{f(b)-f(t)}{b-t}$$

Put $M = \max\left\{\left|\frac{f(s)-f(a)}{s-a}\right|, \left|\frac{f(b)-f(t)}{b-t}\right|\right\}$. Then, for any $x, y \in (s,t)$,

$$\left|\frac{f(y)-f(x)}{y-x}\right| \le M$$

Now,

$$|f(y)-f(x)| \le M|y-x| < \epsilon$$

Since $s, t \in (a,b)$ was arbitrary, f is continuous on (a,b). □

Proposition 12.4.2.2. Let f is differentiable on (a,b). Then,

f is Convex *if and only if* f' is monotonically increasing on (a,b).

Proof. Prove by showing both directions: right and left.
Right Direction Let $x_1 < x_2$ in (a,b). Then,

$$f'(x_1) = \lim_{t \to x_1} \frac{f(t)-f(x_1)}{t-x_1} \le \frac{f(x_2)-f(x_1)}{x_2-x_1} \le \lim_{\tau \to x_2} \frac{f(\tau)-f(x_2)}{\tau-x_2} = f'(x_2)$$

More rigorously, put $\epsilon = \left|f'(x_1) - \frac{f(x_2)-f(x_1)}{x_2-x_1}\right|$. (If $\epsilon = 0$, then there is nothing to prove.).
Now, there exists a $\delta > 0$ such that $|t - x_1| < \delta$ implies

$$\left|f'(x_1) - \frac{f(t)-f(x_1)}{t-x_1}\right| < \epsilon \iff -\epsilon + \frac{f(t)-f(x_1)}{t-x_1} < f'(x_1) \overset{(*)}{\le} \epsilon + \frac{f(t)-f(x_1)}{t-x_1}$$

If $f'(x_1) - \frac{f(x_2)-f(x_1)}{x_2-x_1} > 0$, then $(*)$ gives

$$f'(x_1) < f'(x_1) + \frac{f(t)-f(x_1)}{t-x_1} - \frac{f(x_2)-f(x_1)}{x_2-x_1} \iff \frac{f(x_2)-f(x_1)}{x_2-x_1} < \frac{f(t)-f(x_1)}{t-x_1} \quad \forall t \text{ s.t. } |t-x_1| < \delta$$

If $|t - x_1| < |x_2 - x_1|$, then this contradicts to Convexity.
Consequently, we obtain the first inequality, similarly can prove the second inequality.
Left Direction Let $x, y \in (a,b)$ and $\lambda \in (0,1)$ be given. The Mean Value Theorem gives that:

$$f(\lambda x + (1-\lambda)y) - f(x) = f'(z_1)(\lambda x + (1-\lambda)y - x) \text{ for some } z_1 \in (x, \lambda x + (1-\lambda)y)$$
$$f(y) - f(\lambda x + (1-\lambda)y) = f'(z_2)(y - \lambda x + (1-\lambda)y) \text{ for some } z_2 \in (\lambda x + (1-\lambda)y, y)$$

Now, Monotonically increasing gives

$$\begin{aligned}
&\frac{f(\lambda x + (1-\lambda)y) - f(x)}{\lambda x + (1-\lambda)y - x} = f'(z_1) \le f'(z_2) = \frac{f(y) - f(\lambda x + (1-\lambda)y)}{y - (\lambda x + (1-\lambda)y)} \\
&\implies \frac{f(\lambda x + (1-\lambda)y) - f(x)}{(1-x)(y-x)} \le \frac{f(y) - f(\lambda x + (1-\lambda)y)}{\lambda(y-x)} \\
&\implies \lambda f(\lambda x + (1-\lambda)y) - \lambda f(x) \le (1-\lambda)f(y) - (1-\lambda)\lambda x + (1-\lambda)y \\
&\implies f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda)f(y)
\end{aligned}$$

□

Corollary 12.4.2.1. If $f : [a, b] \to \mathbb{R}$ is twice-differentiable, then

$$f \text{ is Convex } \textit{if and only if } f''(x) \geq 0 \text{ for all } x \in (a, b).$$

Theorem 12.4.2.1. Let $f : [a.b] \to \mathbb{R}$ be given. Then,

$$f \text{ is Convex if and only if } f \text{ is Continuous, and Midpoint Convex.}$$

Midpoint convex is that f satisfies $\forall x, y \in (a, b),\ \ f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2}$.

Proof. The right direction is clear. To show the left direction, we demonstrate that *Midpoint Convexity implies Dyadic Rational Convexity*. Claim: For any $n \in \mathbb{N}$,

$$f\left(\frac{\sum_{k=1}^{2^n} x_k}{2^n}\right) \leq \frac{1}{2^n}\sum_{k=1}^{2^n} f(x_k) \tag{$*$}$$

Using Induction: If $n = 1$, it is clear by Midpoint Convexity.
Assume that for $n \in \mathbb{N}$, $(*)$ is True. Then,

$$\begin{aligned} f\left(\frac{\sum_{k=1}^{2^{n+1}} x_k}{2^{n+1}}\right) &= f\left(\frac{1}{2}\cdot\left[\frac{\sum_{k=1}^{2^n} x_k}{2^n} + \frac{\sum_{k=2^n+1}^{2^{n+1}} x_k}{2^n}\right]\right) \\ &\overset{\text{m.c}}{\leq} \frac{1}{2}\left(f\left(\frac{\sum_{k=1}^{2^n} x_k}{2^n}\right) + f\left(\frac{\sum_{k=2^n+1}^{2^{n+1}} x_k}{2^n}\right)\right) \\ &\overset{(*)}{\leq} \frac{1}{2}\left(\frac{1}{2^n}\sum_{k=1}^{2^n} f(x_k) + \frac{1}{2^n}\sum_{k=2^n+1}^{2^{n+1}} f(x_k)\right) \\ &= \frac{1}{2^{n+1}}\sum_{k=1}^{2^{n+1}} f(x_k) \end{aligned}$$

Consequently, we obtain the claim. Now, let $n \in \mathbb{N}$, and m be an integer such that $1 \leq m \leq 2^n$.
Put $x_1 = x_2 = \cdots = x_m = x$ and $x_{m+1} = x_{m+2} = \cdots = x_{2^n} = y$. Then

$$f\left(\frac{m}{2^n}x + \left(1 - \frac{m}{2^n}\right)y\right) \leq \frac{m}{2^n}f(x) + \left(1 - \frac{m}{2^n}\right)f(y)$$

For complete this discussion, Let $x, y \in (a, b), \lambda \in (0, 1)$ be given.
Since $\dfrac{\lfloor 2^n\lambda\rfloor}{2^n} \to \lambda$ as $n \to \infty$, for any $n \in \mathbb{N}$,

$$f\left(\frac{\lfloor 2^n\lambda\rfloor}{2^n}x + \left(1 - \frac{\lfloor 2^n\lambda\rfloor}{2^n}\right)y\right) \leq \frac{\lfloor 2^n\lambda\rfloor}{2^n}f(x) + \left(1 - \frac{\lfloor 2^n\lambda\rfloor}{2^n}\right)f(y)$$

Finally, taking limits then

$$\lim_{n\to\infty} f\left(\frac{\lfloor 2^n\lambda\rfloor}{2^n}x + \left(1 - \frac{\lfloor 2^n\lambda\rfloor}{2^n}\right)y\right) \overset{f\ \text{conti.}}{=} f\left(\lim_{n\to\infty}\left[\frac{\lfloor 2^n\lambda\rfloor}{2^n}x + \left(1 - \frac{\lfloor 2^n\lambda\rfloor}{2^n}\right)y\right]\right) = f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$$

In brief, Midpoint Convexity implies Dyadic Rational Convexity, and with Continuous implies Convexity. $\square$

12.5 Lipschitz Condition

12.5.1 Definition

Definition 12.5.1.1. A real-vauled function $f:(a,b)\to\mathbb{R}$ is called *Lipschitz Continuous* if:

$$\exists L\geq 0 \text{ s.t. } \forall x_1,x_2\in(a,b),\ \ |f(x_1)-f(x_2)|\leq L\cdot|x_1-x_2|$$

The constant L is said to be *Lipschitz Constant* of f. In particular, the constant

$$D\overset{\text{def}}{=}\sup_{x_1\neq x_2}\frac{|f(x_1)-f(x_2)|}{|x_1-x_2|}$$

is called *dilation* of f. Clearly,

$$\forall x_1,x_2\in(a,b),\ \ |f(x_1)-f(x_2)|\leq D\cdot|x_1-x_2|$$

and if $L>0$ is Lipschitz Constant of f, then $D\leq L$. That is, $D=\inf\{L>0\mid L\text{ is Lipschitz constant of }f\}$.

12.5.2 Properties

Proposition 12.5.2.1. If $f:(a,b)\to\mathbb{R}$ is Lipschitz Continuous, then f is uniformly continuous.

Proof. Let $L\geq 0$ be a Lipschitz Constant of f. Then, for any $\epsilon>0$,

$$\forall x,y\in(a,b),\ \ |x-y|<\frac{\epsilon}{L}\implies |f(x)-f(y)|\leq L|x-y|<\epsilon$$

□

Proposition 12.5.2.2. Let $f:(a,b)\to\mathbb{R}$ be a Differentiable function. Then,

f is Lipschitz Continuous *if and only if* f' is bounded in (a,b).

Proof.
Right Direction
Let $L>0$ be a Lipschitz constant of f, and $x\in(a,b)$ be given. Since definition of derivative,

$$f'(x)\overset{\text{def}}{=}\lim_{t\to x}\frac{f(x)-f(t)}{x-t}$$

Meanwhile, the assumption gives: for any distinct $x,t\in(a,b)$,

$$\frac{|f(x)-f(t)|}{|x-t|}\leq L$$

Therefore,

$$f'(x)=\lim_{t\to x}\frac{f(x)-f(t)}{x-t}\leq\lim_{t\to x}\frac{|f(x)-f(t)|}{|x-t|}\leq\lim_{t\to x}L=L$$

Left Direction
Let distinct $x,y\in(a,b)$ be given. Then, the Mean-Value Theorem gives: There exists a $z\in(x,y)$ such that

$$f(x)-f(y)=f'(z)(x-y)\implies f'(z)=\frac{f(x)-f(y)}{x-y}$$

Now,

$$\left|\frac{f(x)-f(y)}{x-y}\right|=|f'(z)|\leq L\implies |f(x)-f(y)|\leq L\cdot|x-y|$$

If $x=y$, then there is nothing to prove. □

Note that:

$$\text{Lipschitz Continuous}\implies\text{Uniformly Continuous}\implies\text{Continuous}$$

12.5.3 Newton-Raphson Method

Theorem 12.5.3.1. *Newton-Raphson Method*
Let $f : [a, b] \to \mathbb{R}$ be a twice-differentiable, $f(a) < 0 < f(b)$. Suppose that f satisfies: for all $x \in [a, b]$,

$$f'(x) \geq \delta > 0 \text{ and } 0 \leq f''(x) \leq M$$

That is, f is strictly increasing convex function, and Lipschitz Continuous.
Further, there uniquely exists $x^* \in (a, b)$ such that $f(x^*) = 0$.
Let $x_1 \in (x^*, b)$ fixed. Define a sequence $\{x_n\}$ inductively as follows:

$$x_{n+1} \overset{\text{def}}{=} x_n - \frac{f(x_n)}{f'(x_n)}$$

Then, $\{x_n\}$ satisfies the following three conditions:

1. $\{x_n\}$ is decreasing sequence.
2. $x_n \to x^*$ as $n \to \infty$.
3. For any $n \in \mathbb{N}$, $0 \leq x_{n+1} - x^* \leq \left[\frac{M}{2\delta}\right]^{2^{n+1}-1} [x_1 - x^*]^{2^n}$.

Condition 3 means that for a suitable initial value x_1, we can establish an upper bound for the error.

Proof. This proof consists by three steps.
Since f'' is non-negative, and f' is positive, f is strictly increasing convex function.
And Fundamental Theorem of Calculus gives: for any $x \in (a, b)$,

$$f'(x) \overset{\text{FTA}}{=} \int_a^x f''(t)dt + f'(a) \leq \int_a^x M dt + f'(a) = M(x - a) + f'(a) \leq M(b - a) + f'(a)$$

Thus, f' is bounded on (a, b), thus f is Lipschitz Continuous.
Step 1. *f has a unique root x^*.*
The existence of root given directly by Intermidate-Value theroem.
Suppose that $x^*, x' \in (a, b)$ are distinct root of f. i.e., $f(x^*) = f(x') = 0$. Then, by Mean-value theroem, there is $c \in (a, b)$ between x^* and x' such that

$$f'(c)(x^* - x') = f(x^*) - f(x') = 0$$

That is, $f'(c) = 0$. This is contradiction with f' is positive.
Step 2. *$\{x_n\}$ decrease.*
Proof by induction:
For $n = 1$, $f'(x_1)(x_1 - x_2) \overset{\text{def}}{=} f(x_1) > f(x^*) = 0$, thus $x_2 < x_1$. And,

$$\begin{aligned} f(x_2) &\overset{\text{MVT}}{=} f(x_1) + f'(c_1)(x_2 - x_1) \quad \text{for some } c_1 \in (x_2, x_1) \\ &> f(x_1) + f'(x_1)(x_2 - x_1) = f'(x_1)(x_1 - x_2) + f'(x_1)(x_2 - x_1) = 0 \end{aligned}$$

Now, since $f(x_2) > 0 = f(x^*)$, the Mean-Value Theorem implies that $x_2 > x^*$.
To use induction, suppose that for some $n \geq 1$, $x^* < x_{n+1} < x_n$. Then,

$$f(x_{n+1}) = f'(x_{n+1})(x_{n+1} - x_{n+2}) > 0$$

Thus $x_{n+2} < x_{n+1}$ and

$$\begin{aligned} f(x_{n+2}) &\overset{\text{MVT}}{=} f(x_{n+1}) + f'(c_{n+1})(x_{n+2} - x_{n+1}) \quad \text{for some } c_{n+1} \in (x_{n+2}, x_{n+1}) \\ &\geq f(x_{n+1}) + f'(x_{n+1})(x_{n+2} - x_{n+1}) \\ &= f(x_{n+1})(x_{n+1} - x_{n+2}) + f'(x_{n+1})(x_{n+2} - x_{n+1}) = 0 \end{aligned}$$

Again, the Mean-Value Theorem implies that $x_{n+2} > x^*$. Therefore, induction completes.
Now, $x_n \to x'$ as $n \to \infty$ for some $x' \in [x^*, x_1]$ since $\{x_n\}$ is Bounded below and Decreasing.
Still it remains that to show $x' = x^*$. By Continuity,

$$\begin{aligned} &f'(x_n)(x_{n+1} - x_n) + f(x_n) = 0 \\ \implies &\lim_{n\to\infty} [f'(x_n)(x_{n+1} - x_n) + f(x_n)] = f\left(\lim_{n\to\infty} x_n\right) = f(x') = 0 \end{aligned}$$

Since the root of f is unique, thus $x' = x^*$.
Step 3. Establishing the error bound.
The Taylor's Theorem implies that

$$f(x^*) = f(x_n) + f'(x_n)(x^* - x_n) + \frac{f''(t_n)}{2}(x^* - x_n)^2 \quad \text{for some } t_n \in (x^*, x_n)$$

$$\Longrightarrow x_{n+1} - x^* = \frac{f''(t_n)}{2f'(x_n)}(x^* - x_n)^2$$

Consequently,

$$\begin{aligned} 0 \le x_{n+1} - x^* &= \frac{f''(t_n)}{2f'(x_n)}(x_n - x^*)^2 = \frac{f''(t_n)}{2f'(x_n)} \cdot \left(\frac{f''(t_{n-1})}{2f'(x_{n-1})}\right)^2 (x_{n-1} - x^*)^4 = \cdots \\ &= \prod_{i=1}^{n} \left[\frac{f''(t_i)}{2f'(x_i)}\right]^{2^{(n+1-i)}} [x_1 - x^*]^{2^n} \le \left[\frac{M}{2\delta}\right]^{2^{n+1}-1} [x_1 - x^*]^{2^n} \end{aligned}$$

$\square$

12.5.4 Gradient Descent

Theorem 12.5.4.1. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function that satisfies the following conditions:

1. f is *Convex function*.
2. f' is *Lipschitz Continuous* with Lipschitz constant of f, $L > 0$. In this, f is called L*-Smooth*.
3. f has at least one local minimizer x^*.

Then, x^* is a Global minimizer of $\mathbb{R}$, and there exists a unique closed interval M containing x^* such that

$$\forall x \in M, t \notin M, \;\; f(x) = f(x^*) < f(t)$$

And, given initial point $x_0 \in \mathbb{R}$ and $0 < \gamma \leq \frac{1}{L}$, define a sequence $\{x_n\}$ inductively as follows:

$$x_{n+1} = x_t - \gamma \cdot f'(x_n)$$

Then, for any $N \in \mathbb{N}$,

$$f(x_N) - f(x^*) \leq \frac{|x_0 - x^*|^2}{2\gamma N}$$

Proof. Let $x^* \in \mathbb{R}$ be a local minimizer. That is, there exists a $\delta > 0$ such that $\forall t \in (x^* - \delta, x^* + \delta)$, $f(x^*) \leq f(t)$. Then,

$$0 \leq \lim_{t \to x^*+} \frac{f(x^*) - f(t)}{x^* - t} = f'(x^*) = \lim_{t \to x^*-} \frac{f(x^*) - f(t)}{x^* - t} \leq 0$$

thus, $f'(x^*) = 0$. And, by convextiy, f' is monotonically inceasing. Now, The Fundametal Theorem of Calculus gives:

$$\forall x \in \mathbb{R}, \;\; f(x) = \int_{x^*}^{x} f'(t)dt + f(x^*) \geq f(x^*)$$

Therefore, x^* is a Global minimizer of f.
Now, establish the closed interval M. Since f' is Lipschitz Continuous, thus f' is Continuous.
Let $D \overset{\text{def}}{=} \{x \in \mathbb{R} \mid f'(x) = 0\}$. (Note that: $x^* \in D$, thus D is not emtpyset.)
D is closed because: Let $\{x_n\}$ be a convergent sequence in D. That is, for all $n \in \mathbb{N}$, $f(x_n) = 0$. Then, by continuity,

$$f\left(\lim_{n \to \infty} x_n\right) = \lim_{n \to \infty} f(x_n) = 0$$

The limit of $\{x_n\}$ is contained in D, thus D is closed.
And, D is interval: i.e, for any $x \in (\inf D, \sup D)$, $x \in D$ because:
Suppose that there exists $x \in (\inf D, \sup D)$ such that $x \notin D$. That is, $f'(x) \neq 0$. This is Contradiction with Monotonicity.

To set error of upper bound, we make inequalities: Let $x, y \in \mathbb{R}$ be given.
The Fundamental Theorem of Calculus and $L-$Smooth condition gives:

$$\begin{aligned} f(y) - f(x) &= \int_x^y f'(t)dt = \int_0^1 f'(x + (y-x)u)(y-x)du = f'(x)(y-x) + \int_0^1 (f'(x + (y-x)u) - f'(x))(y-x)du \\ &\overset{2.}{\leq} f'(x)(y-x) + L \cdot |y-x|^2 \int_0^1 u \;\, du = f'(x)(y-x) + \frac{L}{2}|y-x|^2 \end{aligned}$$

For any $\lambda > 0$, Put $y = x - \lambda f'(x)$. Then,

$$f(x - \lambda f'(x)) \leq f(x) - f'(x)(\lambda f'(x)) + \frac{L}{2}|\lambda f'(x)|^2 = f(x) + \lambda\left(\frac{L\lambda}{2} - 1\right)|f'(x)|^2$$

Put $\lambda = \frac{1}{L}$, then

$$f\left(x - \frac{f'(x)}{L}\right) \leq f(x) - \frac{L}{2}|f'(x)|^2 \implies \frac{L}{2}|f'(x)|^2 \leq f(x) - f\left(x - \frac{f'(x)}{L}\right) \leq f(x) - \inf f$$

Meanwhile, the convexity gives: for any $x, y \in \mathbb{R}$,

$$f'(x)(y-x) \le f(y) - f(x) \le f'(y)(y-x)$$

since derivative of convex function increase monotonically. Put $z = y - \frac{1}{L}(f'(y) - f'(x))$. Then,

$$\begin{aligned} f(x) - f(y) &= f(x) - f(z) + f(z) - f(y) \\ &\le f'(x)(x-z) + f'(y)(z-y) + \frac{L}{2}|z-y|^2 \\ &= f'(x)\left(x - y + \frac{1}{L}(f'(y) - f'(x))\right) - f'(y)\left(\frac{1}{L}(f'(y) - f'(x))\right) + \frac{L}{2}\left|\frac{1}{L}(f'(y) - f'(x))\right|^2 \\ &= f'(x)(x-y) - \frac{1}{L}|f'(y) - f'(x)|^2 + \frac{1}{2L}|f'(y) - f'(x)|^2 \\ &= f'(x)(x-y) - \frac{1}{2L}|f'(y) - f'(x)|^2 \end{aligned}$$

Now,

$$\begin{cases} \frac{1}{2L}|f'(y) - f'(x)|^2 \le f'(x)(x-y) - (f(x) - f(y)) \\ \frac{1}{2L}|f'(x) - f'(y)|^2 \le f'(y)(y-x) - (f(y) - f(z)) \end{cases} \implies \quad \frac{1}{L}|f'(y) - f'(x)|^2 \le (f'(y) - f'(x))(y-x)$$

Since above inequalities, we obtain that

$$\begin{aligned} |x_{n+1} - x^*|^2 = |x_n - \gamma \cdot f'(x_n) - x^*|^2 &= |(x_n - x^*) - \gamma \cdot f'(x_n)|^2 \\ &= |x_n - x^*|^2 - 2\gamma |f'(x_n)| \cdot |x_n - x^*| + \gamma^2 |f'(x_n)|^2 \\ &\le |x_n - x^*|^2 - 2\gamma \frac{1}{L}|f'(x_n)|^2 + \gamma^2 |f'(x_n)|^2 \\ &= |x_n - x^*|^2 + \left(\gamma^2 - \frac{2\gamma}{L}\right)|f'(x_n)|^2 \le |x_n - x^*|^2 \end{aligned}$$

Thus, $|x_n - x^*|$ decrease as $n \to \infty$. That is, $|x_n - x^*| \le |x_0 - x^*|$ for all $n \in \mathbb{N}$.
Consider x_{n+1} and x_n. First, we obtain

$$\begin{aligned} f(x_{n+1}) &\le f(x_n) + f'(x_n)(x_{n+1} - x_n) + \frac{L}{2}|x_{n+1} - x_n|^2 \\ &= f(x_n) - \gamma |f'(x_n)|^2 + \frac{L}{2}\gamma^2 |f'(x_n)|^2 \\ &= f(x_n) - \left(\gamma - \frac{L}{2}\gamma^2\right)|f'(x_n)|^2 \end{aligned}$$

Subtracting $f(x^*)$ above, then

$$f(x_{n+1}) - f(x^*) \le f(x_n) - f(x^*) - \left(\gamma - \frac{L}{2}\gamma^2\right)|f'(x_n)|^2$$

Meanwhile, Convexity gives

$$f(x_n) - f(x^*) \le f'(x_n)(x_n - x^*) \le |f'(x_n)||x_n - x^*| \le |f'(x_n)||x_0 - x^*|$$

Combining abvoe two inequalities,

$$f(x_{n+1}) - f(x^*) \le f(x_n) - f(x^*) - \left(\gamma - \frac{L}{2}\gamma^2\right) \cdot \frac{|f(x_n) - f(x^*)|^2}{|x_0 - x^*|^2}$$

Dividing Both Sides by $(f(x_{n+1}) - f(x^*))(f(x_n) - f(x^*))$,

$$\begin{aligned} &\frac{1}{f(x_n) - f(x^*)} \le \frac{1}{f(x_{n+1}) - f(x^*)} - \left(\gamma - \frac{L}{2}\gamma^2\right) \cdot \frac{f(x_n) - f(x^*)}{f(x_{n+1}) - f(x^*)} \frac{1}{|x_0 - x^*|^2} \\ \implies &\left(\gamma - \frac{L}{2}\gamma^2\right) \cdot \frac{f(x_n) - f(x^*)}{f(x_{n+1}) - f(x^*)} \frac{1}{|x_0 - x^*|^2} \le \frac{1}{f(x_{n+1}) - f(x^*)} - \frac{1}{f(x_n) - f(x^*)} \\ \implies &\left(\gamma - \frac{L}{2}\gamma^2\right) \cdot \frac{1}{|x_0 - x^*|^2} \le \frac{1}{f(x_{n+1}) - f(x^*)} - \frac{1}{f(x_n) - f(x^*)} \\ \implies &\sum_{n=0}^{N-1}\left[\left(\gamma - \frac{L}{2}\gamma^2\right) \cdot \frac{1}{|x_0 - x^*|^2}\right] \le \sum_{n=0}^{N-1}\left[\frac{1}{f(x_{n+1}) - f(x^*)} - \frac{1}{f(x_n) - f(x^*)}\right] = \frac{1}{f(x_N) - f(x^*)} - \frac{1}{f(x_0) - f(x^*)} \end{aligned}$$

Consequently,

$$\frac{2\gamma N}{|x_0 - x^*|^2} \le N \cdot \left[\left(\gamma - \frac{L}{2}\gamma^2 \right) \cdot \frac{1}{|x_0 - x^*|^2} \right] \le \frac{1}{f(x_N) - f(x^*)} - \frac{1}{f(x_0) - f(x^*)} \le \frac{1}{f(x_N) - f(x^*)}$$

Organizing the formula, as result:

$$f(x_N) - f(x^*) \le \frac{|x_0 - x^*|^2}{2\gamma N}$$

$\square$

12.6 Integral

12.6.1 Inequality of Riemann-Stieltjes Integral

Let $p, q \geq 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, and functions lying on $[a, b]$.

Lemma 12.6.1.1. Let $f, g \in \mathcal{R}(\alpha)$ with $f, g \geq 0$, and $\int_a^b [f(x)]^p d\alpha = \int_a^b [g(x)]^q d\alpha = 1$. Then, $\int_a^b f(x)g(x)d\alpha \leq 1$.

Proof. For any $x \in [a, b]$, the Young's Inequality gives

$$0 \leq f(x)g(x) \leq \frac{[f(x)]^p}{p} + \frac{[g(x)]^q}{q}$$

Now,

$$\int_a^b f(x)g(x)d\alpha \leq \int_a^b \frac{[f(x)]^p}{p} + \frac{[g(x)]^q}{q} d\alpha = \frac{1}{p}\int_a^b [f(x)]^p d\alpha + \frac{1}{q}\int_a^b [g(x)]^q d\alpha = \frac{1}{p} + \frac{1}{q} = 1$$

□

Definition 12.6.1.1. Let $f \in \mathcal{R}(\alpha)$. Define a ***Norm*** of f:

$$\|f\|_p \stackrel{\text{def}}{=} \left(\int_a^b |f(x)|^p d\alpha\right)^{\frac{1}{p}}$$

This becomes actually norm of set of Stieltjes Integrable functions, $\mathcal{F} \stackrel{\text{def}}{=} \{f : [a, b] \to \mathbb{C} \mid f \in \mathcal{R}(\alpha)\}$.

Lemma 12.6.1.2. *Hölder's Inequality*
Let $f, g \in \mathcal{F}$. Then,

$$\left|\int_a^b f(x)g(x)d\alpha\right| \leq \left[\int_a^b |f(x)|^p d\alpha\right]^{\frac{1}{p}} \cdot \left[\int_a^b |g(x)|^q d\alpha\right]^{\frac{1}{q}}$$

Proof. Use above definition, Rewrite:

$$\|f\|_p^p = \int_a^b |f(x)|^p d\alpha, \quad \|g\|_q^q = \int_a^b |g(x)|^q d\alpha$$

Now, we can make the condition of above lemma,

$$\int_a^b \left[\frac{|f(x)|}{\|f\|_p}\right]^p d\alpha = \frac{1}{\|f\|_p^p} \cdot \int_a^b |f(x)|^p d\alpha = 1, \quad \int_a^b \left[\frac{|g(x)|}{\|g\|_q}\right]^q d\alpha = \frac{1}{\|g\|_q^q} \cdot \int_a^b |g(x)|^q d\alpha = 1$$

And apply this,

$$\int_a^b \frac{|f(x)| \cdot |g(x)|}{\|f\|_p \|g\|_q} d\alpha \leq 1 \implies \int_a^b |f(x)||g(x)|d\alpha \leq \|f\|_p \|g\|_q = \left[\int_a^b |f(x)|^p d\alpha\right]^{\frac{1}{p}} \cdot \left[\int_a^b |g(x)|^q d\alpha\right]^{\frac{1}{q}}$$

Finally, the general property of integral of product gives

$$\left|\int_a^b f(x)g(x)d\alpha\right| \leq \int_a^b |f(x)||g(x)|d\alpha \leq \|f\|_p \|g\|_q = \left[\int_a^b |f(x)|^p d\alpha\right]^{\frac{1}{p}} \cdot \left[\int_a^b |g(x)|^q d\alpha\right]^{\frac{1}{q}}$$

□

Theorem 12.6.1.1. *Minkowski inequality*
Let $f, g \in \mathcal{F}$. Then, for any $p \geq 1$, $\|f+g\|_p \leq \|f\|_p + \|g\|_p$.

Proof.

$$\begin{aligned}
\|f+g\|_p^p &= \int_a^b |f+g|^p d\alpha = \int_a^b |f+g||f+g|^{p-1} d\alpha \\
&\leq \int_a^b [|f|+|g|]|f+g|^{p-1} d\alpha \\
&= \int_a^b |f||f+g|^{p-1} d\alpha + \int_a^b |g||f+g|^{p-1} d\alpha \\
&\overset{\text{Hölder}}{\leq} \left[\int_a^b |f|^p d\alpha\right]^{\frac{1}{p}} \left[\int_a^b |f+g|^{(p-1)\frac{p}{p-1}} d\alpha\right]^{\frac{p-1}{p}} + \left[\int_a^b |g|^p d\alpha\right]^{\frac{1}{p}} \left[\int_a^b |f+g|^{(p-1)\frac{p}{p-1}} d\alpha\right]^{\frac{p-1}{p}} \\
&= \left[\int_a^b |f+g|^p d\alpha\right]^{\frac{p-1}{p}} \left(\left[\int_a^b |f|^p d\alpha\right]^{\frac{1}{p}} + \left[\int_a^b |g|^p d\alpha\right]^{\frac{1}{p}}\right) = \|f+g\|_p^{p-1} \cdot (\|f\|_p + \|g\|_p)
\end{aligned}$$

Now,

$$\|f+g\|_p^p \cdot \|f+g\|_p^{1-p} = \|f+g\|_p \leq \|f\|_p + \|g\|_p$$

$\square$

Chapter 13

Measure

Chapter 14

Complex Analysis

Chapter 15

Differential Geometry

Chapter 16

Differential Equation

Chapter 17

Spaces

17.1 $\mathbb{R}^n$

17.1.1 Inner Product in $\mathbb{R}$

17.1.2 p-norm in $\mathbb{R}^n$

Definition 17.1.2.1. Let $\mathbb{R}^n$ be given. Define ***p-norm*** on $\mathbb{R}^n$ as:

$$d_p : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} : (\mathbf{x}, \mathbf{y}) \mapsto \left(\sum_{i=1}^{n} |x_i - y_i|^p \right)^{\frac{1}{p}}, \quad (\mathbf{x} = (x_1, \ldots, x_n),\ \ \mathbf{y} = (y_1, \ldots, y_n))$$

where $p \in [1, \infty]$. In particular, p-norm is a ***Metric***, being ***Minkowski inequality***.

Lemma 17.1.2.1. *Young's inequality*
Let $u, v > 0$, and $p, q \in [1, \infty]$ such that $\frac{1}{p} + \frac{1}{q} = 1$.
Then,

$$uv \leq \frac{1}{p} u^p + \frac{1}{q} v^q$$

Proof. Since $f(x) = \log x$ is concave, we obtain

$$\forall \lambda \in [0, 1],\ \ \lambda f(x)(1 - \lambda) f(y) \leq f(\lambda x + (1 - \lambda) y)$$

thus,

$$\log \left(\frac{1}{p} u^p + \frac{1}{q} v^q \right) \geq \frac{1}{p} \log(u^p) + \frac{1}{q} \log(v^q) = \log(uv)$$

Since $\exp(x)$ increasing, we get

$$\exp \left(\log \left(\frac{1}{p} u^p + \frac{1}{q} v^q \right) \right) \geq \exp(\log(uv))$$

i.e.,

$$uv \leq \frac{1}{p} u^p + \frac{1}{q} v^q$$

$\square$

Lemma 17.1.2.2. *Holder's inequality*
Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ be give, and $p, q \in [1, \infty]$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then,

$$\sum_{i=1}^{n} |x_i y_i| \leq \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} \cdot \left(\sum_{i=1}^{n} |y_i|^q \right)^{\frac{1}{q}}$$

Proof. Denote that

$$\|x\|_p \overset{\text{def}}{=} \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}}$$

Then, since young's inequality, for each $i \in \{1, 2, \ldots, n\}$,

$$\frac{|x_i|}{\|x\|_p} \cdot \frac{|y_i|}{\|y\|_p} \leq \frac{1}{p} \cdot \frac{|x_i|^p}{\|x\|_p^p} + \frac{1}{q} \cdot \frac{|y_i|^q}{\|y\|_q^q}$$

Summing for all $i = 1, 2, \ldots, n$:

$$\frac{1}{\|x\|_p \|y\|_q} \cdot \sum_{i=1}^{n} |x_i y_i| \leq \frac{1}{p} + \frac{1}{q} = 1$$

Therefore,

$$\sum_{i=1}^{n} |x_i y_i| \leq \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} \cdot \left(\sum_{i=1}^{n} |y_i|^q \right)^{\frac{1}{q}}$$

$\square$

Theorem 17.1.2.1. *Minkowski inequality*
Given complex-valued sequences $\{x_i\}_{i=1}^n$ and $\{y_i\}_{i=1}^n$,

$$\left[\sum_{i=1}^{n} |x_i + y_i|^p \right]^{\frac{1}{p}} \leq \left[\sum_{i=1}^{n} |x_i|^p \right]^{\frac{1}{p}} + \left[\sum_{i=1}^{n} |y_i|^p \right]^{\frac{1}{p}}$$

Proof. Denote

$$|x_i + y_i|^p = |x_i + y_i| \cdot |x_i + y_i|^{p-1}$$

Then,

$$\begin{aligned}
\sum_{i=1}^{n} |x_i + y_i|^p &= \sum_{i=1}^{n} |x_i + y_i| \cdot |x_i + y_i|^{p-1} \\
&= \sum_{i=1}^{n} (|x_i| + |y_i|) \cdot |x_i + y_i|^{p-1} \\
&= \sum_{i=1}^{n} |x_i| \cdot |x_i + y_i|^{p-1} + \sum_{i=1}^{n} |y_i| \cdot |x_i + y_i|^{p-1} \\
&\overset{\text{Hölder}}{\leq} \left[\sum_{i=1}^{n} |x_i|^p \right]^{\frac{1}{p}} \cdot \left[\sum_{i=1}^{n} |x_i + y_i|^p \right]^{\frac{p-1}{p}} + \left[\sum_{i=1}^{n} |y_i|^p \right]^{\frac{1}{p}} \cdot \left[\sum_{i=1}^{n} |x_i + y_i|^p \right]^{\frac{p-1}{p}} \\
&= \left[\left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} |y_i|^p \right)^{\frac{1}{p}} \right] \cdot \left[\sum_{i=1}^{n} |x_i + y_i|^p \right]^{\frac{p-1}{p}}
\end{aligned}$$

Now, Divide each side as $[\sum_{i=1}^n |x_i + y_i|^p]^{\frac{p-1}{p}}$, then we obtain

$$\left[\sum_{i=1}^n |x_i + y_i|^p\right]^{1-\frac{p-1}{p}} = \left[\sum_{i=1}^n |x_i + y_i|^p\right]^{\frac{1}{p}} \le \left[\left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^n |y_i|^p\right)^{\frac{1}{p}}\right]$$

$\square$

> **Theorem 17.1.2.2.** Let d_{p_1}, d_{p_2} are p-norm on $\mathbb{R}^n$ with $1 \le p_1 < p_2 \le \infty$. Then,
>
> $$\exists C > 0 \text{ s.t. } \forall x, y \in \mathbb{R}^n,\ d_{p_2}(x,y) \le d_{p_1}(x,y) \le C d_{p_2}(x,y)$$
>
> In particular, $C = n^{\frac{1}{p_1} - \frac{1}{p_2}}$.

Proof. Let $p_1 < p_2$.
For show that first-inequality,

$$1 = \sum_{i=1}^n \left[\frac{|x_i - y_i|}{[\sum_{i=1}^n |x_i - y_i|^{p_2}]^{\frac{1}{p_2}}}\right]^{p_2} \le \sum_{i=1}^n \left[\frac{|x_i - y_i|}{[\sum_{i=1}^n |x_i - y_i|^{p_2}]^{\frac{1}{p_2}}}\right]^{p_1} = \frac{\sum_{i=1}^n |x_i - y_i|^{p_1}}{[\sum_{i=1}^n |x_i - y_i|^{p_2}]^{\frac{p_1}{p_2}}} = \left[\frac{[\sum_{i=1}^n |x_i - y_i|^{p_1}]^{\frac{1}{p_1}}}{[\sum_{i=1}^n |x_i - y_i|^{p_2}]^{\frac{1}{p_2}}}\right]^{p_1}$$

Thus, we obtain that:

$$1 \le \left[\frac{[\sum_{i=1}^n |x_i - y_i|^{p_1}]^{\frac{1}{p_1}}}{[\sum_{i=1}^n |x_i - y_i|^{p_2}]^{\frac{1}{p_2}}}\right]^{p_1} \iff 1 \le \frac{[\sum_{i=1}^n |x_i - y_i|^{p_1}]^{\frac{1}{p_1}}}{[\sum_{i=1}^n |x_i - y_i|^{p_2}]^{\frac{1}{p_2}}} \iff \left[\sum_{i=1}^n |x_i - y_i|^{p_2}\right]^{\frac{1}{p_2}} \le \left[\sum_{i=1}^n |x_i - y_i|^{p_1}\right]^{\frac{1}{p_1}}$$

For show that second-inequality, using Hölder's inequality.

$$\begin{aligned}(d_{p_1}(x,y))^{p_1} &= \sum_{i=1}^n |x_i - y_i|^{p_1} = \sum_{i=1}^n |x_i - y_i|^{p_1} \cdot 1 \\ &\overset{\text{Hölder}}{\le} \left[\sum_{i=1}^n \left(|x_i - y_i|^{p_1 \cdot \frac{p_2}{p_1}}\right)\right]^{\frac{p_1}{p_2}} \cdot \left[\sum_{i=1}^n 1^{\frac{p_2}{p_2 - p_1}}\right]^{1 - \frac{p_1}{p_2}} = \left[\sum_{i=1}^n (|x_i - y_i|^{p_2})\right]^{\frac{p_1}{p_2}} \cdot n^{1 - \frac{p_1}{p_2}}\end{aligned}$$

Taking the $\frac{1}{p_1}$-th power of both sides, then

$$d_{p_1}(x,y) \le \left[\sum_{i=1}^n (|x_i - y_i|^{p_2})\right]^{\frac{1}{p_2}} \cdot n^{\frac{1}{p_1} - \frac{1}{p_2}} = n^{\frac{1}{p_1} - \frac{1}{p_2}} \cdot d_{p_2}(x,y)$$

$\square$

> **Corollary 17.1.2.1.** Let $\mathbb{R}^n$ be given as a set, and $d_{p_1}, d_{p_2} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ are p-norm on $\mathbb{R}^n$. Then,
>
> $$\mathcal{T}_{d_{p_1}} = \mathcal{T}_{d_{p_2}}$$

For every $p \geq 1$, the metric space $(\mathbb{R}^n, d_p)$ induces the same topology as the product topology on $\mathbb{R}^n$. In particular, $\mathbb{R}^n$ with the product topology coincides with $\mathbb{R}^n$ endowed with any p-norm.

17.1.3 Open and Closed set in $\mathbb{R}^n$

Definition 17.1.3.1. For $p \in [1, \infty]$, define ***p-Ball*** in $\mathbb{R}^n$ as:

$$B_p(x, r) \overset{\text{def}}{=} \{y \in \mathbb{R}^n : \|x - y\|_p < r\}$$

Since all p-norms are equivalent, for any $p \in [1, \infty]$, the collection

$$\beta_p \overset{\text{def}}{=} \left\{B_p(x, r) \,\middle|\, x \in \mathbb{Q}^n, \;\; r \in \mathbb{Q}^+\right\}$$

is Countable basis of $\mathbb{R}^n$. Immediately, we obtain:

Lemma 17.1.3.1. Every open set in $\mathbb{R}^n$ is a countable union of p-Balls.

We call 2-Ball the ***Ball***, and ∞-Ball the ***Cube***.

Theorem 17.1.3.1. Let $U \subseteq \mathbb{R}^n$ be an open set. Then, U is a countable union of closed cubes with disjoint interiors.

Proof. Let $U \subseteq \mathbb{R}^n$ be an open set, and define the collection of ***Dyadic Cubes*** on $\mathbb{R}^n$ as: for each $k \in \mathbb{N}$,

$$Q_k \overset{\text{def}}{=} \left\{\prod_{i=1}^{n} \left[\frac{q_i}{2^k}, \frac{q_i + 1}{2^k}\right] \subset \mathbb{R}^n \,\middle|\, q_i \in \mathbb{Z}\right\}$$

Each element of Q_k is product of closed intervals, and its interiors are disjoint. For each $k \in \mathbb{N}$, construct:

$$Q_k^* \overset{\text{def}}{=} \{Q \in Q_k \mid Q \subseteq U\}$$

Then, the union $Q^* = \bigcup_{k \in \mathbb{N}} Q_k^*$ is a countable union of closed cubes, and $Q^* = U$: $Q^* \subseteq U$ is clear, and let $x \in U$. Since property of metric space, there exists $\delta > 0$ such that $x \in B_2(x, \delta) \subseteq U$. Put $k \in \mathbb{N}$ such that $\dfrac{1}{2^k} < \dfrac{\delta}{\sqrt{n}}$. Then, $x \in C \subset B_2(x, \delta) \subseteq U$ for some $C \in Q_k$, because $\operatorname{diam} C = \sqrt{n} 2^{-k}$. Since $C \subset U$, $C \in Q_k^* \subset Q^*$. i.e., $U \subseteq Q^*$. For disjointness of interiors, we will use the fact:

For any $Q_1, Q_2 \in Q^*$, either their interiors are disjoint, or one is contained in the other.

(Conti.) □

17.2 Topological Vector Space

17.3 Hilbert Space

Definition 17.3.0.1. Complete Inner product Vector Space is called *Hilbert Space*.

17.3.1 Hilbert Space in $\mathbb{R}^\omega$

Definition 17.3.1.1. Define $\mathbb{R}^\omega \stackrel{\text{def}}{=} \prod_{i=1}^{\infty} \mathbb{R}$ as the countable product of Euclidean space $\mathbb{R}$ with product topology.

And define $\mathbb{H} \stackrel{\text{def}}{=} \left\{ \{x_n\}_{n=1}^{\infty} \,\middle|\, \sum_{n=1}^{\infty} x_n^2 < \infty \right\} \subset \mathbb{R}^\omega$, ***Metric*** on $\mathbb{H}$ as $\mu : \mathbb{H} \times \mathbb{H} \to \mathbb{R} : (\{x_n\}, \{y_n\}) \mapsto \sqrt{\sum_{i=1}^{\infty} (x_i - y_i)^2}$.
The Metric Space $(\mathbb{H}, \mu)$ is called *Hilbert Space* or l_2 *Space*.

Define the operations elementwise; then $(\mathbb{H}, +, \times)$ is a Vector Space over $\mathbb{R}$.
Moreover, $\mathbb{H}$ is Complete Metric Space and Inner product Vector Space.

Lemma 17.3.1.1. $\mu : \mathbb{H} \times \mathbb{H} \to \mathbb{R} : (\{x_n\}, \{y_n\}) \mapsto \sqrt{\sum_{i=1}^{\infty} (x_i - y_i)^2}$ is Metric function induced by the inner product.

Proof. We know that $\mathbb{R}^\omega$ is Vector Space. Moreover, $\mathbb{H} \subset \mathbb{R}^\omega$ is Subspace. Using subspace criteria:

$S \subset V$ is Subspace of Vector Space V if and only if $0 \in S$ and For any $x, y \in S$ and $a \in F$, $ax + y \in S$.

Clearly, $\{0\} \in \mathbb{H}$. Let $a \in \mathbb{R}$ and $\{x_n\}, \{y_n\} \in \mathbb{H}$ be given. Then, $a\{x_n\} + \{y_n\} = \{ax_n + y_n\} \in \mathbb{H}$ because:

$$\sum_{i=1}^{\infty} (ax_i + y_i)^2 = \sum_{i=1}^{\infty} \left[a^2 x_i^2 + 2ax_iy_i + y_i^2\right] \stackrel{(*)}{=} a^2 \sum_{i=1}^{\infty} x_i^2 + 2a \sum_{i=1}^{\infty} x_iy_i + \sum_{i=1}^{\infty} y_i^2 < \infty$$

The $(*)$ given by:

$$\sum_{i=1}^{\infty} |x_iy_i| = \sum_{i=1}^{\infty} |x_i||y_i| \le \sum_{i=1}^{\infty} (\max(|x_i|, |y_i|))^2 \le \sum_{i=1}^{\infty} (x_n^2 + y_n^2) = \sum_{i=1}^{\infty} x_n^2 + \sum_{i=1}^{\infty} y_n^2 < \infty \tag{$*$}$$

Thus $\mathbb{H}$ is Vector Space over $\mathbb{R}$. Now, define ***inner product*** on $\mathbb{H}$ as:

$$\langle \ \cdot\ ,\ \cdot\ \rangle : \mathbb{H} \times \mathbb{H} \to \mathbb{R} : (\{x_n\}, \{y_n\}) \mapsto \sum_{i=1}^{\infty} x_iy_i$$

This definition is well-defined since $(*)$. And, Linearity in first:

$$\langle a\{x_n\} + \{y_n\}, \{z_n\} \rangle = \langle \{ax_n + y_n\}, \{z_n\} \rangle = \sum_{i=1}^{\infty} (ax_i + y_i)z_i = a \sum_{i=1}^{\infty} x_iz_i + \sum_{i=1}^{\infty} y_iz_i = a\langle \{x_n\}, \{z_n\} \rangle + \langle \{y_n\}, \{z_n\} \rangle$$

The other conditions are clear. Thus, $(\mathbb{H}, \langle \cdot, \cdot \rangle)$ is ***inner product space***.
Using ***ineer product***, define the ***Norm*** on $\mathbb{H}$ as:

$$\| \cdot \| : \mathbb{H} \to \mathbb{R} : \{x_n\} \mapsto \sqrt{\langle \{x_n\}, \{x_n\} \rangle}$$

Finally, define ***Metric*** on $\mathbb{H}$ as:

$$\mu : \mathbb{H} \times \mathbb{H} \to \mathbb{R} : (\{x_n\}, \{y_n\}) \mapsto \|\{x_n\} - \{y_n\}\| = \sqrt{\sum_{i=1}^{\infty} (x_i - y_i)^2}$$

$\square$

Theorem 17.3.1.1. *Hilbert Space* is Separable.

Proof. For each $n \in \mathbb{N}$, define $D_n \overset{\text{def}}{=} \{\{p_n\} \mid p_i \in \mathbb{Q},\ p_{n+1} = p_{n+1} = \cdots = 0\}$ and $D \overset{\text{def}}{=} \bigcup_{n\in\mathbb{N}} D_n$.

Then, D is countable set. We will show that $\overline{D} = \mathbb{H}$.
Let $\epsilon > 0$ and $\{x_n\} \in \mathbb{H}$ be given. Since convergence, there exists $N \in \mathbb{N}$ such that

$$\sum_{i=N+1}^{\infty} x_i^2 = \sum_{i=1}^{\infty} x_i^2 - \sum_{i=1}^{N} x_i^2 < \frac{\epsilon^2}{2}$$

Since density of Rationals, put each $i = 1, 2, \ldots, N$, $p_i \in \mathbb{Q}$ $|x_i - p_i| < \dfrac{\epsilon}{\sqrt{2N}}$ and $p_i = 0$ for $i \geq N+1$.
Then, $\{p_n\} \in D_n \subset D$ and

$$\mu(\{x_n\}, \{p_n\}) = \sqrt{\sum_{i=1}^{N}(x_i - p_i)^2 + \sum_{i=N+1}^{\infty}(x_i - p_i)^2} = \sqrt{\sum_{i=1}^{N}(x_i - p_i)^2 + \sum_{i=N+1}^{\infty} x_i^2} < \sqrt{N \cdot \frac{\epsilon^2}{2N} + \frac{\epsilon^2}{2}} = \epsilon$$

□

Corollary 17.3.1.1. *Hilbert Space* is Second-Countable.

Theorem 17.3.1.2. *Hilbert Space* is Complete.

Proof. Let $\{\{x_{n,i}\}_{i=1}^{\infty}\}_{n=1}^{\infty}$ be a Cauchy sequence in $\mathbb{H}$. For any fixed $n, m \in \mathbb{N}$ and for each $j \in \mathbb{N}$,

$$|x_{n,j} - x_{m,j}| < \mu(\{x_{n,i}\}, \{x_{m,i}\}) = \sqrt{\sum_{i=1}^{\infty}(x_{n,i} - x_{m,i})^2}$$

That is, for each $j \in \mathbb{N}$, $\{x_{n,j}\}$ is Cauchy sequence in $\mathbb{R}$. Since $\mathbb{R}$ is Complete, put $y_j \overset{\text{def}}{=} \lim_{n\to\infty} x_{n,j}$, each $j \in \mathbb{N}$.
Let $\epsilon > 0$ be given. Then, there exists $N \in \mathbb{N}$ such that $n, m \geq N \implies \mu(\{x_{n,i}\}, \{x_{m,i}\}) < \dfrac{\epsilon}{2}$.
Meanwhile, for each $k \in \mathbb{N}$,

$$\sum_{i=1}^{k}(x_{n,i} - x_{m,i})^2 \leq \sum_{i=1}^{\infty}(x_{n,i} - x_{m,i})^2 = [\mu(\{x_{n,i}\}, \{x_{m,i}\})]^2$$

Thus, $n, m \geq N \implies \displaystyle\sum_{i=1}^{k}(x_{n,i} - x_{m,i})^2 < \left(\frac{\epsilon}{2}\right)^2$, for each $k \in \mathbb{N}$.
Taking limit to m, then $n \geq N \implies \displaystyle\lim_{m\to\infty}\left(\sum_{i=1}^{k}(x_{n,i} - x_{m,i})^2\right) = \sum_{i=1}^{k}\left(x_{n,i} - \lim_{m\to\infty} x_{m,i}\right)^2 = \sum_{i=1}^{k}(x_{n,i} - y_i)^2 < \left(\frac{\epsilon}{2}\right)^2$.
And, for all $k \in \mathbb{N}$,

$$\sum_{i=1}^{k} y_i^2 = \sum_{i=1}^{k}(2(x_{n,i}^2 + (x_{n,i} - y_i)^2)) \leq 2\|\{x_{n,i}\}_{i=1}^{\infty}\|^2 + \left(\frac{\epsilon}{2}\right)^2$$

Thus $\{y_i\} \in \mathbb{H}$. As a result,

$$n \geq N \implies \mu(\{x_n\}, \{y_n\}) = \sqrt{\sum_{i=1}^{\infty}(x_{n,i} - y_i)^2} = \sqrt{\lim_{k\to\infty}\sum_{i=1}^{k}(x_{n,i} - y_i)^2} < \frac{\epsilon}{2}$$

□

Theorem 17.3.1.3. $\mathbb{H} \subset \mathbb{R}^\omega$ with subbspace topology is Metrizable.

Proof. We will use two Lemmas:

Lemma 17.3.1.2. Countable Product of Metric Space is Metrizable.

Proof. Let (X_i, d_i) be a metric Space, for each $i \in \mathbb{N}$.
If $d : X \times X \to \mathbb{R}$ is a Metric, then $\dfrac{d}{1+d}$ is also Metric, because

$$\underset{\frac{x}{1+x}}{\frac{d(x,z)}{1+d(x,z)}} \underset{\text{increasing}}{\underset{\uparrow}{\leq}} \frac{d(x,y)+d(y,z)}{1+d(x,y)+d(y,z)} \underset{d\geq 0}{\underset{\uparrow}{\leq}} \frac{d(x,y)}{1+d(x,y)} + \frac{d(y,z)}{1+d(y,z)} \tag{$*$}$$

Using this fact, define

$$d_\Pi : \prod X_i \times \prod X_i \to \mathbb{R} : (\{x_n\}_{n=1}^\infty, \{y_n\}_{n=1}^\infty) \mapsto \sum_{i=1}^\infty \left[\frac{1}{2^i} \cdot \frac{d_i(x_i, y_i)}{1+d_i(x_i,y_i)}\right]$$

Then d_Π is a Metric because:

$$\begin{aligned} d_\Pi\left(\{x_n\}_{n=1}^\infty, \{z_n\}_{n=1}^\infty\right) &= \sum_{i=1}^\infty \left[\frac{1}{2^i} \cdot \frac{d_i(x_i,z_i)}{1+d_i(x_i,z_i)}\right] \\ &\overset{(*)}{\leq} \sum_{i=1}^\infty \left[\frac{1}{2^i} \cdot \left(\frac{d_i(x_i,y_i)}{1+d_i(x_i,y_i)} + \frac{d_i(y_i,z_i)}{1+d_i(y_i,z_i)}\right)\right] \\ &= \sum_{i=1}^\infty \left[\frac{1}{2^i} \cdot \frac{d_i(x_i,y_i)}{1+d_i(x_i,y_i)}\right] + \sum_{i=1}^\infty \left[\frac{1}{2^i} \cdot \frac{d_i(y_i,z_i)}{1+d_i(y_i,z_i)}\right] \\ &= d_\Pi\left(\{x_n\}_{n=1}^\infty, \{y_n\}_{n=1}^\infty\right) + d_\Pi\left(\{y_n\}_{n=1}^\infty, \{z_n\}_{n=1}^\infty\right) \end{aligned}$$

Reflexity and symmetry are clear. ☐

Lemma 17.3.1.3. Metrizable is Hereditary.

Proof omitted.
Consequently, since $\mathbb{H} \subset \mathbb{R}^\omega$ is a subspace of a metric space, it is metrizable. ☐

17.4 Banach Space

17.5 L_p Space

17.6 l_p Space

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