# Math Note

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# Contents

This paper covers several topics in undergraduate mathematics.

Patch Note:

$\sim$ 2025/9/28 - Drafted the initial framework of the paper.

2025/9/29 - 1. Completed proof of Ring of Fractions.

2. Transcribed Integral Test.

# Chapter 1

# Set Theory

## 1.1 Map

**Definition 1.** Let $X, Y$ are sets. Define a ***function*** $X$ to $Y$ is a relation

$$f \subset X \times Y$$

such that

1. For any $x \in X$, there exists $y \in Y$ such that $(x, y) \in f$.
2. If $(x, y) \in f$ and $(x, z) \in f$, then $y = z$.

Denote $f$ as:

$$f : X \to Y : x \mapsto f(x)$$

Define ***Image*** of $f$ by $A \subset X$:

$$f[A] \stackrel{\text{def}}{=} \{f(a) \mid a \in A\} \subset Y$$

And, ***Preimage*** of $f$ by $B \subset Y$:

$$f^{-1}[B] \stackrel{\text{def}}{=} \{x \in X \mid f(x) \in B\} \subset X$$

$f : X \to Y$ is ***Injective*** if: $f(x_1) = f(x_2) \implies x_1 = x_2$.
$f : X \to Y$ is ***Surjective*** if: $\forall y \in Y,\ \exists x \in X$ s.t. $f(x) = y$.
If $f$ is injective and surjective, called ***bijective***.
If $f$ is bijective, then define ***inverse*** of $f$ as:

$$f^{-1} : Y \to X : y \to x$$

where $x \in X$ is the unique elements of $X$ such that $f(x) = y$.

**Theorem 1.** Let $f : X \to Y$ be a function. Then,

1. There exists $g : Y \to X$ such that $g \circ f : X \to X$ be an identity function ***if and only if*** $f$ is injective.
2. There exists $h : Y \to X$ such that $f \circ h : Y \to Y$ be an identity function ***if and only if*** $f$ is surjective.

*Proof.*
*1.* $\implies$)
Assume that $f(x_1) = f(x_2)$. Then, existence of left inverse, $g(f(x_1)) = g(f(x_2)) \implies x_1 = x_2$. Thus $f$ injective.
*1.* $\impliedby$)

Since $f$ is injection, for any $y \in f[X]$, there exists a unique element $x_y \in X$ such that $f(x) = y$. Now, define

$$g : Y \to X : y \mapsto \begin{cases} x_y & y \in f[X] \\ \text{any element in } X & y \notin f[X] \end{cases}$$

Then, for any $x \in X$, $g(f(x)) = g(y) = x$.
2. $\implies$)
Let $y \in Y$ be given. Since existence of right inverse, $f(h(y)) = y$ where $h(y) \in X$. Thus, $f$ is surjective.
2. $\impliedby$)
For any $y \in Y$, there exists a $x_y \in X$ such that $f(x_y) = y$. Now, define

$$h : Y \to X : y \mapsto x_y$$

Then, for any $y \in Y$, $f \circ h(y) = f(x_y) = y$. Thus, $f \circ h$ is identity. □

**Corollary 1.** Let $f : X \to Y$ be a function, $\mathrm{id}_X : X \to X : x \mapsto x$, and $\mathrm{id}_Y : Y \to Y : y \mapsto y$.

There exists a $f^{-1} : Y \to X$ such that $f^{-1} \circ f = \mathrm{id}_X$ and $f \circ f^{-1} = \mathrm{id}_Y$ *if and only if* $f$ is bijection.

***Proof.*** If $f$ is bijection, then there exists left inverse $g$ and right inverse $h$.
Enough To Show that: $g = h$. Since $g \circ f = \mathrm{id}_X$ and $f \circ h = \mathrm{id}_Y$,
$g \circ f \circ h = g \circ \mathrm{id}_Y$, thus $h = g$. □

**Theorem 2.** Let $X, Y, Z$ are sets, $f : X \to Y$, $g : Y \to Z$ and $A \subset X, B \subset Y, C \subset Z$. Then followings are hold:

1. $g[f[A]] = (g \circ f)[A]$.
2. $f^{-1}[g^{-1}[C]] = (g \circ f)^{-1}[C]$.

***Proof.***
1. It is clear by definition of image:

$$\begin{aligned} g[f[A]] &\overset{\text{def}}{=} g[\{f(a) \mid a \in A\}] = \{g(b) \mid b \in \{f(a) \mid a \in A\}\} \\ &= \{g(b) \mid b = f(a) \text{ for some } a \in A\} = \{g(f(a)) \mid \text{for some } a \in A\} = \{g(f(a)) \mid a \in A\} \end{aligned}$$

2. It is not clear,

$$f^{-1}[g^{-1}[C]] \overset{\text{def}}{=} f^{-1}[\{b \in Y \mid g(b) \in C\}] = \{a \in X \mid f(a) \in \{b \in Y \mid g(b) \in C\}\} = \{a \in X \mid g(f(a)) \in C\} = (g \circ f)^{-1}[C]$$

□

**Proposition 1.** Let $f : X \to Y$ be a function, $A, B \subset X$ and $C, D \subset Y$.

1. If $A \subset B$, then $f[A] \subset f[B]$.
2. If $C \subset D$, then $f^{-1}[C] \subset f^{-1}[D]$

***Proof.***

$$\begin{aligned} &y \in f[A] \implies y = f(a) \text{ for some } a \in A \overset{A \subset B}{\implies} y = f(a) \text{ for some } a \in B \implies y \in f[B] \\ &x \in f^{-1}[C] \implies f(x) \in C \overset{C \subset D}{\implies} f(x) \in D \implies x \in f^{-1}[D] \end{aligned}$$

□

**Lemma 1.** Let two set $X, Y$ be given, and $A \subset X$, $B \subset Y$, $f : X \to Y$. Then followings are holds:

1. $f^{-1}[f[A]] \supseteq A$, and equality holds if $f$ one-to-one.
2. $f[f^{-1}[B]] \subseteq B$, and equality holds if $f$ onto.
3. $f^{-1}[Y \setminus B] = X \setminus f^{-1}[B]$
4. $f[X] \setminus f[A] \subseteq f[X \setminus A]$, and equality holds if $f$ one-to-one.

*Proof.* Proof of 4.

$$\begin{aligned} y \in f[X] \setminus f[A] &\iff y \in f[X] \text{ and } y \notin f[A] \\ &\iff \exists x \in X \text{ s.t. } y = f(x) \text{ and } \forall x \in A,\ y \neq f(x) \\ &\overset{(*)}{\Longrightarrow} \exists x \in X \setminus A \text{ s.t. } y = f(x) \\ &\iff y \in f[X \setminus A] \end{aligned}$$

If $f$ is injection, then Left Direction of $(*)$ be true: $\exists! x \in X \setminus A$ s.t. $y = f(x)$. $\square$

# Chapter 2

# Group Theory

## 2.1 Isomorphism Theorems

**Theorem 3.** *The First Isomorphism Theorem*
Let $\varphi : G \to H$ be a Group-Homomorphism. Then,

$$G/\ker\varphi \cong \varphi[G]$$

$$\begin{tikzcd}
G \arrow[r, "\overset{\text{homomorphism}}{\varphi}"] \arrow[dr, "\pi \atop {\text{homomorphism} \atop \text{surjection}}"'] & \varphi[G] \\
 & G/\ker\varphi \arrow[u, "\phi \atop \text{isomorphism}"']
\end{tikzcd}$$

*Proof.* Let $\pi : G \to G/\ker\varphi : x \mapsto x + \ker\varphi$. Then, the map $\phi : G/\ker\varphi \to \varphi[G] : a + \ker\varphi \mapsto \varphi(a)$ is isomorphism.
Well-defined and Injective:

$$a + \ker\varphi = b + \ker\varphi \iff a - b \in \ker\varphi \iff \varphi(a-b) = \varphi(a) - \varphi(b) = 0$$

Surjective is clear. □

**Theorem 4.** *The Second Isomorphism Theorem*
Let $G$ be a Group, and $H \leq G$, $N \trianglelefteq G$. Then,

$$HN/N \cong H/(H \cap N)$$

*Proof.* $HK$ be a subgroup of $G$, being

$$HN = \bigcup_{h \in H} hN \overset{N \trianglelefteq G}{=} \bigcup_{h \in H} Nh = NH$$

And, $N \leq HN$ is clear, thus $N \trianglelefteq HN$.
Meanwhile, $H \cap N$ be a Normal Subgroup of $H$: for any $h \in H, n \in H \cap N$, $hnh^{-1} \in N$ because $N$ is normal, and $hnh^{-1} \in H$ since $h, n$ contained in $H$. Thus, $hnh^{-1} \in H \cap N$, this implies $H \cap N$ be a Normal of $H$.
Now, Define a Map:

$$\varphi : H \to HN/N : h \mapsto hN$$

Clearly, this map is Well-Defined and Homomorphism. And,

$$\ker\varphi = \varphi^{-1}[1] = \{h \in H \mid hN = N\} = \{h \in H \mid h \in N\} = H \cap N$$

Thus, since The 1st Isomorphism Theorem,

$$HN/N \cong H/(H \cap N)$$

□

**Theorem 5.** *The Third Isomorphism Theorem*
Let $G$ be a Group, and $H, K \trianglelefteq G$ with $H \leq K$. Then, $K/H \trianglelefteq G/H$ and

$$(G/H)/(K/H) \cong (G/K)$$

*Proof.* First, show that $K/H \trianglelefteq G/H$. Let $kH \in K/H$ and $gH \in G/H$. Then,

$$(gH)(kH)(gH)^{-1} = (gH)(kH)(g^{-1}H) = (gkg^{-1})H \in K/H$$

since $gkg^{-1} \in K$, being $K \trianglelefteq G$. Now, Define a map:

$$\varphi : G/H \to G/K : gH \mapsto gK$$

1. Well-Defined.

$$g_1H = g_2H \iff g_1^{-1}g_2 \in H \overset{H \leq K}{\implies} g_1^{-1}g_2 \in K \iff g_1K = g_2K$$

2. Homomorphism.
Clearly, for any $g_1H, g_2 \in G/H$,

$$\varphi(g_1Hg_2H) = \phi(g_1g_2H) = g_1g_2K = g_1Kg_2K = \varphi(g_1H)\varphi(g_2H)$$

3. Surjection. Let $gK \in G/K$ be given. Then, clearly, $\varphi(gH) = gK$.
4. Kernel.

$$\ker\varphi = \{gH \in G/H \mid gK = 1\} = \{gH \in G/H \mid g \in K\} = K/H$$

Consequently, The 1st Isomorphism Theorem gives

$$(G/K) \cong (G/H)/\ker\varphi = (G/H)/(K/H)$$

□

**Theorem 6.** *The Forth Isomorphism Theorem*
Let $G$ be a Group, and $N \trianglelefteq G$ be a Normal Subgroup. Then, there is a bijection between

$$D \overset{\text{def}}{=} \{H \leq G \mid N \leq H\},\ C \overset{\text{def}}{=} \{\overline{H} \leq G/N\}$$

*Proof.* Let $\pi : G \to G/N : g \mapsto gN$ be a natural projection. And, Define

$$\Phi : D \to C : H \mapsto \pi[H]$$

This function is well-defined: For any $H \in D$, let $aN, bN \in \pi[H]$. Then, $aN \cdot b^{-1}N = ab^{-1}N \in \pi[H]$, thus $\pi[H] \leq G/N$.
To show that one-to-one: Let $\Phi(A) = \Phi(B)$. Thus means, $\pi[A] = \pi[B]$. Let $a \in A$. Then, $\pi(a) \in \pi[A] = \pi[B]$, thus $\pi(a) = \pi(b)$ for some $b \in B$. That is, $aN = bN \iff a \in bN$. Meanwhile, $N \leq B$, thus $a \in bN \subset B$, $A \subset B$. Similarly, $B \subset A$, that is $A = B$.
To show that onto: Let $K \in C$. Then, $N \leq \pi^{-1}[K] \leq G$, thus clear. □

## 2.2 Group Action

### 2.2.1 Lagrange's Theorem

## 2.3 Generating subset of a Group

## 2.4 Commutator Subgroup

# Chapter 3

# Finite Group Theory

## 3.1 The Class Equation

## 3.2 Cauchy's Theorem

**Lemma 2.** *Cauchy's Theorem*
Let $G$ be a finite group, and $p$ be a prime dividing $|G|$. Then, $G$ has order $p$ element.

*Proof.* Define a set:

$$S \stackrel{\text{def}}{=} \{(x_1, x_2, \ldots, x_p) \mid x_i \in G,\ \ x_1 x_2 \cdots x_p = 1\}$$

Then, $S$ has exactly $|G|^{p-1}$ elements because there are $|G|$ possible choices for each of the first $p-1$ elements in $G$.
Once $x_1, \cdots, x_{p-1}$ are chosen, then $x_p$ is uniquely determined by the uniqueness of inverses.
Then, let $\sigma = (1, 2, \ldots, p)$ be a permutation. Then, for any $\alpha \in S$, $\sigma^n(\alpha) \in S$ for all $n \in \mathbb{Z}$, being $ab = 1 \iff ba = 1$.
More precisely, let $n \in \mathbb{Z}$ be given, $\alpha = (x_1, \cdots, x_n)$. Then,

$$\sigma^n(\alpha) = (x_{n+1}, x_{n+2}, \ldots, x_p, x_1, x_2, \ldots x_n)$$

By $x_1 \cdots x_n x_{n+1} \cdots x_p = 1$, $x_{n+1} \cdots x_p x_1 \cdots x_n = 1$. Thus $\sigma^n(\alpha) \in S$. Now, define a relation on $S$ as:

$$\alpha \sim \beta \text{ if and only if } \beta = \sigma^n(\alpha) \text{ for some } n \in \mathbb{Z}$$

Then, this relation be equivalent relation, thus construct a partition on $S$. Claim:

$$[\alpha] = \{\beta \in S \mid \beta \sim \alpha\} \text{ is sinlgeton if and only if } \alpha = (x, \ldots, x) \text{ for some } x \in G.$$

Left direction is clear, and for show that Right direction,
Suppose that $\alpha = (x_1, \ldots, x_n)$ has different cordinate elements, let $x_i \neq x_j$, for some $i < j$. Then clearly

$$(x_1, \ldots, x_i, \ldots, x_p) \neq \sigma^{i-j}(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_p) = (\ldots, \underbrace{x_j}_{\text{i'th element}}, \ldots)$$

Meanwhile, if $[\alpha]$ has elements more than 1, $[\alpha]$ has exactly number of $p$ elements. Because suppose that $\alpha = (x_1, \ldots, x_p)$ has at least one different coordinate. Then,

$$\sigma^1(\alpha), \sigma^2(\alpha), \cdots, \sigma^{p-1}(\alpha)$$

are mutually different: If there exist $1 \leq i < j < p$ such that $\sigma^i(\alpha) = \sigma^j(\alpha)$, that is, $\sigma^{j-i}(\alpha) = \alpha$.
Now, $j - i \mid p$, this is contradiction with $p$ is prime. Therefore, every equivalent class has order 1 or $p$.
Consequently,

$$|G|^{p-1} = k + pd$$

where $k$ is a number of classes of size 1, and $d$ is a number of classes of size $p$. And $(1, 1, \ldots, 1) \in S$, $k$ is at least $1$.
Since $p$ divides $|G|^{p-1} = k + pd$, thus $k$ must be bigger than $1$, thus there exists elements such that $x^p = 1$. $\square$

## 3.3 Sylow's Theorem

> **Theorem 7.** *Sylow's Theorem*
> Let $G$ be a group of order $p^\alpha m$, where $p$ is a prime such that $p \nmid m$.
> A group of order $p^r$, $(r \geq 1)$ is called a $p$-group, Subgroups of $G$ which are $p$-groups are called $p$-subgroup.
> In particular, subgroups of order $p^\alpha$ is called *Sylow $p$-subgroup* of $G$. And, define a collection
>
> $$\mathsf{Syl}_p(G) \stackrel{\text{def}}{=} \{P \leq G \mid |P| = p^\alpha\},\ n_p(G) \stackrel{\text{def}}{=} \mathsf{Card}(\mathsf{Syl}_p(G))$$
>
> *The First Sylow Theorem*
> There exists a Sylow $p$-subgroup of $G$. i.e., $\mathsf{Syl}_p(G) \neq \emptyset$.
>
> *The Second Sylow Theorem*
> If $P \in \mathsf{Syl}_p(G)$ and $Q \leq G$ be a $p$-subgroup. Then, there exists $g \in G$ such that $Q \leq gPg^{-1}$.
>
> *The Third Sylow Theorem*
> $n_p \equiv 1(\mathrm{mod} p)$, $n_p = |G : N_G(P)|$ for any $P \in \mathsf{Syl}_p(G)$, and $n_p \mid m$.

Before prove above statments, we show that:

**Lemma 3.** Let $P \in \mathsf{Syl}_p(G)$. If $Q$ is $p$-subgroup of $G$, then $Q \cap N_G(P) = Q \cap P$.

***Proof.*** Put $H = Q \cap N_G(P)$. Since $P \leq G$, for any $p \in P$, $pPp^{-1} = P$, thus $p \in N_G(P)$. i.e., $P \leq N_G(P)$.
Thus, Enough to Show that $H \leq Q \cap P$. Since $H \leq N_G(P)$,

$$PH = \bigcup_{h \in H} Ph = \bigcup_{h \in H} hP = HP$$

Thus, $PH \leq G$. And,

$$|PH| = \frac{|P||H|}{|P \cap H|}$$

By Lagrange's Theorem, $H \leq P$ and $P \cap H \leq P$ must have order of powers of $p$, so $PH$ be a $p$-group. Clearly, $P \leq PH$ and $P$ is the largest $p$-group of $G$, thus, $PH = P$. This means, $H \leq P$. □

***Proof.*** The First Theorem: The existence of Sylow $p$-subgroup. Proof by Induction:
If $|G| = 1$, there is nothing to prove.
Assume inductively the existence of Sylow $p$-subgroups for all groups of order less than $|G|$.
*In case of* $p||Z(G)|$, then by Cauchy's Theorem, $Z(G)$ has a subgroup $N$ which has order of $p$.
Clearly $N$ is Normal, and $G/N = |G|/|N| = p^{a-1}m$. By assumption, $G/N$ has a subgroup $P'$ of order $p^{\alpha-1}$.
By The Forth Isomorphism Theorem, Let $P \leq G$ be a subgroup such that $P/N = P'$.
Then, $|P| = |P/N| \cdot |N| = p^\alpha$, Thus $P$ be a Sylow $p$-subgroup of $G$.
*In case of* $p \nmid |Z(G)|$.
Let $g_1, \ldots, g_r$ be represectatives of the distinct conjugacy classes of $G$, not contained in $Z(G)$. Then, The Class Equation gives

$$|G| = |Z(G)| + \sum_{i=1}^{r} |G : C_G(g_i)|$$

Since $p$ divides $|G|$, if for all $i = 1, 2, \ldots, r$, $p \mid |G : C_G(g_i)|$ then $p \mid |Z(G)|$, this is contradiction.
Thus, for some $j$, $p \nmid |G : C_G(g_j)|$. Put $H = C_G(g_j) < G$. Then, $|H|$ has a factor of $p^\alpha$, by $p \nmid |G : C_G(g_j)|$. Now,

$$|H| = p^\alpha m' \quad (m' < m)$$

By assumption, $H$ has a Sylow $p$-group, order of $p^\alpha$.
Consequently, the existence of Sylow $p$-subgroup was shown.
The Second Theorem: Relation of $p$-subgroups.
The First Theorem gives existence of Sylow $p$-subgroups. Let $P \in \mathsf{Syl}_p(G)$. Denote that:

$$S \stackrel{\text{def}}{=} \{gPg^{-1} \mid g \in G\} = \{P_1, \ldots, P_r\}$$

Let $Q \leq G$ be an any $p$-subgroup of $G$. And, $Q$ acts by conjucation on $S$. i.e.,

$$\alpha : Q \times S \to S : (q, P_i) \mapsto qP_iq^{-1}$$

Write $S$ as a disjoint union of orbits under this action by $Q$:

$$S = \mathcal{O}_1 \cup \mathcal{O}_2 \cup \cdots \cup \mathcal{O}_s$$

where $r = |\mathcal{O}_1| + \cdots + |\mathcal{O}_s|$. Rearrange a set $S$ as: $P_i \in \mathcal{O}_i,\ 1 \leq i \leq s$. Now, using Definition, Lemma, and above Theorem,

$$|\mathcal{O}_i| \overset{\text{Thm}}{=} |Q : N_Q(P_i)| \overset{\text{def}}{=} |Q : N_G(P_i) \cap Q| \overset{\text{lemma}}{=} |Q : P_i \cap Q|$$

for each $1 \leq i \leq s$. Since $Q$ was arbitrary, Let $Q = P_1$, so that $|\mathcal{O}_1| = |P_1 : P_1 \cap P_1| = 1$. And, for each $i \geq 2$, $P_i \cap P_1 < P_1$,

$$|\mathcal{O}_i| = |P_1 : P_i \cap P_1| > 1$$

Since $P_1 \in \mathrm{Syl}_p(G)$, that is $|P_1| = p^\alpha$, $|P_1 : P_i \cap P_1| = |P_1|/|P_i \cap P_1| = p^k$ where $1 \leq k < \alpha$.
This means for each $2 \leq i \leq s$, $p$ divides $|\mathcal{O}_i|$. Thus,

$$r = |\mathcal{O}_1| + (|\mathcal{O}_2| + \cdots + |\mathcal{O}_s|) \equiv 1 (\mathrm{mod} p)$$

Now, Proof by Contradiction: Let $Q \leq G$ be a $p$-subgroup. Suppose that for any $1 \leq i \leq r$, $Q \nleq P_i$.
Then, $P_i \cap Q < Q$ for all $i$, this means

$$|\mathcal{O}_i| = |Q : P_i \cap Q| > 1$$

Thus for any $i$, $p$ divides $|\mathcal{O}_i|$, this is Contradiction. This proved Relation of $p$-subgroups.
Finally, The Third Theorem:
Since Second Theorem, this gives that $S = \mathrm{Syl}_p(G)$, thus $n_p(G) = r$. That is, $n_p \equiv 1 (\mathrm{mod} p)$.
Since all Sylow $p$-subgroups are Conjugate, for any $P \in \mathrm{Syl}_p(G)$,

$$n_p = r = |\mathcal{O}_1| = |G : N_G(P)|$$

Consequently, Completing the Sylow Theorem. $\square$

Chapter 4

# Ring Theory

## 4.1 Ring of Fractions

**Theorem 8.** Let $R$ be a Commutative Ring, $D \subset R$ be a subset such that $\begin{cases}\text{no zero, no zero divisors}\\ \text{closed under multiplication}\end{cases}$.
Then, there exists a Commutative Ring $Q$ with identity satisfies:

1. $R$ can embed in $Q$, and every element of $D$ becomes unit in $Q$. More precisely, $Q = \{rd^{-1} \mid r \in R,\ d \in D\}$.
2. $Q$ is the smallest Ring contaning $R$ with identity such that every element of $D$ becomes unit in $Q$.

*Proof.* Let $\mathcal{F} \overset{\text{def}}{=} \{(r,d) \mid r \in R,\ d \in D\}$ and the relation $\sim$ on $\mathcal{F}$ by $(r_1, d_1) \sim (r_2, d_2) \iff r_1 d_2 = r_2 d_1$.
Then, $\sim$ is equivalent relation: reflexive and symmetirc are clear, and Suppose that $(r_1, d_1) \sim (r_2, d_2)$ and $(r_2, d_2) \sim (r_3, d_3)$.

$$r_2 d_3 = r_3 d_2 \implies r_2 d_1 d_3 = r_3 d_1 d_2 \implies r_1 d_2 d_3 = r_3 d_1 d_2 \implies d_2(r_1 d_3 - r_3 d_1) \implies r_1 d_3 = r_3 d_1$$

Thus transitivity shown. Define

$$\frac{r}{d} \overset{\text{def}}{=} [(r,d)] = \{(a,b) \mid (a,b) \sim (r,d)\},\ \ Q \overset{\text{def}}{=} \left\{ \frac{r}{d} \,\middle|\, r \in R,\ d \in D \right\}$$

And define operations $+, \times$ on $Q$:

$$\frac{r_1}{d_1} + \frac{r_2}{d_2} \overset{\text{def}}{=} \frac{r_1 d_2 + r_2 d_1}{d_1 d_2}, \quad \frac{r_1}{d_1} \times \frac{r_2}{d_2} \overset{\text{def}}{=} \frac{r_1 r_2}{d_1 d_2}$$

Well-Definedness: If $\dfrac{r_1}{d_1} = \dfrac{r_1'}{d_1'}$ and $\dfrac{r_2}{d_2} = \dfrac{r_2'}{d_2'}$,

$$\frac{r_1 d_2 + r_2 d_1}{d_1 d_2} = \frac{r_1 d_2 d_1' d_2' + r_2 d_1 d_1' d_2'}{d_1 d_2 d_1' d_2'} = \frac{(r_1 d_1') d_2 d_2' + (r_2 d_2') d_1 d_1'}{d_1 d_2 d_1' d_2'} = \frac{(r_1' d_1) d_2 d_2' + (r_2' d_2) d_1 d_1'}{d_1 d_2 d_1' d_2'} = \frac{(r_1' d_2' + r_2' d_1') d_1 d_2}{d_1 d_2 d_1' d_2'} = \frac{r_1' d_2' + r_2' d_1'}{d_1' d_2'}$$

$$\frac{r_1 r_2}{d_1 d_2} = \frac{r_1 r_2 d_1' d_2'}{d_1 d_2 d_1' d_2'} = \frac{(r_1 d_1')(r_2 d_2')}{d_1 d_2 d_1' d_2'} = \frac{(r_1' d_1)(r_2' d_2)}{d_1 d_2 d_1' d_2'} = \frac{r_1' r_2' d_1 d_2}{d_1 d_2 d_1' d_2'} = \frac{r_1' r_2'}{d_1' d_2'}$$

Now, $(Q, +, \times)$ constructs Commutative Ring with identity: for any $d \in D$, put $0_Q \overset{\text{def}}{=} \dfrac{0}{d}$, $1_Q \overset{\text{def}}{=} \dfrac{d}{d}$. Then,

1. $(R, +, \times)$ closed under the operations since $D$ is closed under the multiplication.
2. $(R, +)$ has a zero: $\dfrac{r_1}{d_1} + 0_Q = \dfrac{r_1}{d_1} + \dfrac{0}{d} = \dfrac{r_1 d + 0 d_1}{d_1 d} = \dfrac{r_1 d}{d_1 d} = \dfrac{r_1}{d_1}$.
3. $(R, +)$ has an inverse: $\dfrac{r_1}{d_1} + \dfrac{-r_1}{d_1} = \dfrac{r_1 d_1 + (-r_1) d_1}{d_1 d_1} = \dfrac{[(r_1) + (-r_1)] d_1}{d_1 d_1} = \dfrac{0 d_1}{d_1 d_1} = \dfrac{0}{d_1 d_1} = 0_Q$.
4. $(R, +, \times)$ satisfies distributive law:

   4-1. The left law:

$$\begin{aligned}\frac{r_1}{d_1} \times \left(\frac{r_2}{d_2} + \frac{r_3}{d_3}\right) &= \frac{r_1}{d_1} \times \frac{r_2 d_3 + r_3 d_2}{d_2 d_3} = \frac{r_1 r_2 d_3 + r_1 r_3 d_2}{d_1 d_2 d_3} = \frac{r_1 r_2 d_1 d_3 + r_1 r_3 d_1 d_2}{d_1 d_2 d_1 d_3} = \frac{r_1 r_2}{d_1 d_2} + \frac{r_2 r_3}{d_2 d_3}\\ &= \frac{r_1}{d_1} \times \frac{r_2}{d_2} + \frac{r_2}{d_2} \times \frac{r_3}{d_3}\end{aligned}$$

   4-2. The right law:

$$\begin{aligned}\left(\frac{r_1}{d_1} + \frac{r_2}{d_2}\right) \times \frac{r_3}{d_3} &= \frac{r_1 d_2 + r_2 d_1}{d_1 d_2} \times \frac{r_3}{d_3} = \frac{r_1 r_3 d_2 + r_2 r_3 d_1}{d_1 d_2 d_3} = \frac{r_1 r_3 d_2 d_3 + r_2 r_3 d_1 d_3}{d_1 d_3 d_2 d_3} = \frac{r_1 r_3}{d_1 d_3} + \frac{r_2 r_3}{d_2 d_3}\\ &= \frac{r_1}{d_1} \times \frac{r_3}{d_3} + \frac{r_2}{d_2} \times \frac{r_3}{d_3}\end{aligned}$$

5. $(R, \times)$ has an identity: $\dfrac{r_1}{d_1} \times 1_Q = \dfrac{r_1}{d_1} \times \dfrac{d}{d} = \dfrac{r_1 d}{d_1 d} = \dfrac{r_1}{d_1}$.

6. Elements of $D$ become unit in $Q$: Define $\iota : R \to Q : r \mapsto \dfrac{rp}{p}$ where $p \in D$ is any fixed element in $D$.
Then, $\iota$ is Ring-Monomorphsim because:

6-1. Well-Defined and Injective: $\iota(r_1) = \iota(r_2) \iff \dfrac{r_1 p}{p} = \dfrac{r_2 f}{f} \iff (r_1 - r_2)pp = 0 \iff r_1 = r_2$

6-2. For any $d \in D$, $\iota(d)$ is a unit of $Q$: Put $(\iota(d))^{-1} \overset{\text{def}}{=} \dfrac{p}{dp}$, then

$$\iota(d) \times (\iota(d))^{-1} = \frac{dp}{p} \times \frac{p}{dp} = \frac{dpp}{dpp} = 1_Q$$

That is, $\iota$ is embedding from $R$ into $Q$ such that $\iota[D]$ becomes units of $Q$ except zero.
Moreover, if $D = R \setminus \{0\}$, then $Q$ is field.

7. $Q$ is the *smallest* ring containing $R$ with identity such that every element of $D$ becomes units in $Q$.
Let $S$ be an any commutative ring with identity,
and assume that $\varphi : R \to S$ is a Ring-Monomorphism such that for any $d \in D$, $\varphi(d)$ is unit in $S$.
Define $\phi : Q \to S : \dfrac{r}{d} \mapsto \varphi(r)\varphi(d)^{-1}$. Then, this $\phi$ is well-defined and injective:

$$\phi\left(\frac{r_1}{d_1}\right) = \phi\left(\frac{r_2}{d_2}\right) \iff \varphi(r_1)\varphi(d_1)^{-1} = \varphi(r_2)\varphi(d_2)^{-1} \iff \varphi(r_1)\varphi(d_2) = \varphi(r_2)\varphi(d_1)$$
$$\overset{\text{homomor.}}{\iff} \varphi(r_1 d_2) = \varphi(r_2 d_1) \overset{\text{one-to-one}}{\iff} r_1 d_2 = r_2 d_1 \iff \frac{r_1}{d_1} = \frac{r_2}{d_2}$$

That is, if a commutative ring $S$ with identity contains a copy of $R$ such that the denominator set $D$ of $R$ becomes unit in $S$, then $S$ contains ring of fractions $Q$ of $R$. Thus $S = Q$ is the smallest ring that satisfies these conditions.

$\square$

Chapter 5

# Polynomial Ring

Chapter 6

# Field Theory

Chapter 7

# Galois Theory

## Chapter 8

# Category

# Chapter 9

# General Topology

## 9.1 Complete Metric Space

**Definition 2.** Let $(X, d)$ be a Metric Space, and $\{p_n\}$ be a Sequence in $X$.
The Sequence $\{p_n\}$ is called *Cauchy Sequence* if:

$$\text{For any } \epsilon > 0 \text{, there exists } N \in \mathbb{N} \text{ such that } m, n \geq N \implies d(p_m, p_n) < \epsilon.$$

A Metric Space $(X, d)$ is said to be *Complete* if every Cauchy Sequnces Converge.

**Lemma 4.** Let $\{E_n\}$ be a sequence of closed bounded non-empty subsets in a Complete Metric Space $X$ such that $E_n \supset E_{n+1}$. If $\lim_{n\to\infty} \text{diam} E_n = 0$, then $\bigcap_{n=1}^{\infty} E_n = \{p\}$ for some $p \in X$.

*Proof.* For each $n \in \mathbb{N}$, construct $p_n \in E_n$.
Let $\epsilon > 0$ be given. Since diam$E_n \to 0$, there is $N \in \mathbb{N}$ such that diam$E_n < \epsilon$.
For any $m, n \geq M$, $E_N$ contains $p_m, p_n$. That is, $d(p_m, p_n) < \epsilon$. Thus, $\{p_n\}$ be a Cauchy sequence of $X$.
Since $X$ is complete, there is a unique point $p \in X$ such taht $p_n \to p$. Let $N \in \mathbb{N}$ be an integer such that $n \geq N \implies |p_n - p| < \epsilon$.
Now, for each $n \geq N$, $E_n$ has a limit point as $p$. And for any $n \in \mathbb{N}$, $E_n$ contains $E_N, E_{N+1}, \ldots$, thus for all $n \in \mathbb{N}$, $E_n$ has a limit point as $p$. Meanwhile, $E_n$ closed, $p \in E_n, \ \forall n \in \mathbb{N}$.
Consequently, $p \in \bigcap_{n=1}^{\infty} E_n$. If there is $q \in X$ such that $p \neq q$, $q \in \bigcap_{n=1}^{\infty} E_n$. Then, diam$E_n \geq d(p, q) > 0, \ \forall n \in \mathbb{N}$. ☐

### 9.1.1 Baire Category

**Definition 3.** The Topological Space $X$ is called *Baire Space* if:

$$\text{If } \{G_n \mid n \in \mathbb{N}\} \text{ be a Countable Collection of dense open sets of } X \text{, then } \overline{\bigcap_{n=1}^{\infty} G_n} = X$$

In brief, every Countable intersection of dense open sets be dense in $X$.

**Theorem 9.** Locally Compact Hausdorff Space is Baire Space.

**Theorem 10.** Complete Metric Space is Baire Space.

*Proof.* Suppose that $\{G_n \mid n \in \mathbb{N}\}$ be a Countable Collection of dense open set of Complete Metric Space.
Let an open $U \in \mathcal{T}$ be given. Since $G_n$ is dense in the Space, $U \cap G_1$ is non-empty open set.

Thus, there exists a $p_1 \in U \cap G_1$ such that for some $r_1 > 0$, $B_{r_1}(p_1) \subset U \cap G_1$.
Then, automatically,

$$B_{\frac{r_1}{2}}(p_1) \subset \overline{B_{\frac{r_1}{2}}(p_1)} \subset B_{r_1}(p_1) \subset U \cap G_1$$

Set $E_1 = U,\ E_2 = B_{\frac{r_1}{2}}(p_1)$.
Suppose that $E_1, \ldots, E_{n-1}$ are chosen. Then, since $E_{n-1} \cap G_{n-1}$ is open, being intersection of opens.
Thus there exists a point $p_{n-1} \in E_{n-1} \cap G_{n-1}$ and exists $r_{n-1}$ such that

$$B_{r_{n-1}}(p_{n-1}) \subset E_{n-1} \cap G_{n-1} \subset E_{n-1}$$

This implies that

$$B_{\frac{r_{n-1}}{2}}(p_{n-1}) \subset \overline{B_{\frac{r_{n-1}}{2}}(p_{n-1})} \subset B_{r_{n-1}}(p_{n-1}) \subset E_{n-1} \cap G_{n-1} \subset E_{n-1}$$

Set $E_n = B_{\frac{r_{n-1}}{2}}(p_{n-1})$. Since inductively construction of $\{E_n\}$, $E_{n+1} \subset E_n$ and $\overline{E_n} \subset G_n$ for all $n \in \mathbb{N}$.
Consequently,

$$U \cap \left(\bigcap_{n=1}^{\infty} G_n\right) = \bigcap_{n=1}^{\infty} (U \cap G_n) \supset \bigcap_{n=1}^{\infty} (U \cap \overline{E_n}) = U \cap \left(\bigcap_{n=1}^{\infty} \overline{E_n}\right) = \bigcap_{n=1}^{\infty} \overline{E_n} \neq \emptyset$$

□

**Definition 4.** Let $X$ be a Topological Space.
$A \subset X$ is said to be *nowhere dense subset* if $(\overline{A})^\circ = \emptyset$.

1. $B \subset X$ is called *first category* if $B$ can be representive by union of countable nowhere dense subsets.

2. If the subset is not first category, then it is said to be *second category*.

### 9.1.2 Nowhere Differentiable function

### 9.1.3 Banach Fixed Point Theorem

**Definition 5.** Let $f : X \to X$ be any function. A point $x \in X$ is called a ***fixed point*** of $f$ if $f(x) = x$.

**Definition 6.** Let $X$ be a Metric Space. A map $f : X \to X$ is called ***Contractive*** with respect to the metric $d$ if:

There exsits $\alpha \in (0,1)$ such that for all $x, y \in X$, $d(f(x), f(y)) \leq \alpha d(x, y)$.

**Theorem 11.** *Banach Fixed point Theorem*
Let $(X, d)$ be a Complete Metric Space, and $f : X \to X$ be a Contractive map.
Then, there exists a unique fixed point of $f$, $x^* \in X$.

*Proof.* Clearly,

$$\text{Contractive} \implies \text{Lipschitz Condition} \implies \text{Continuous.}$$

Thus, $f$ is Continuous.
Let $x_0 \in X$ be arbitrary, and construct a sequence $\{x_n\}$ recursively as follows:

$$x_{n+1} \overset{\text{def}}{=} f(x_n), \;\; n \geq 0$$

Then, for any $n \geq 0$,

$$\begin{aligned} d(x_{n+1}, x_n) &= d(f(x_n), f(x_{n-1})) \leq \alpha d(x_n, x_{n-1}) \\ &= d(f(x_{n-1}), f(x_{n-2})) \leq \alpha^2 d(x_{n-1}, x_{n-2}) \\ &\;\;\vdots \\ &\leq \alpha^n d(x_1, x_0) \end{aligned}$$

Let $\epsilon > 0$ be given. Put $N \in \mathbb{N}$ such that $\alpha^N \cdot d(x_1, x_0) < \epsilon(1 - \alpha)$. Then, $n \geq m \geq N$ implies that

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + \cdots + d(x_{m+1}, x_m) \\ &\leq \alpha^n d(x_1, x_0) + \alpha^{n-1} d(x_1, x_0) + \cdots + \alpha^{m+1} d(x_1, x_0) \\ &= \alpha^{m+1} d(x_1, x_0) \sum_{r=0}^{n-m-1} \alpha^r < \alpha^N d(x_1, x_0) \sum_{r=0}^{\infty} \alpha^r < \epsilon(1-\alpha) \frac{1}{1-\alpha} = \epsilon \end{aligned}$$

Therefore, $\{x_n\}$ is Cauchy sequence. Since $X$ is Complete, for some $x^* \in X$, $\lim_{n\to\infty} x_n = x^*$. Consequently,

$$\lim_{n\to\infty} f(x_n) \overset{f \text{ conti.}}{=} f\left(\lim_{n\to\infty} x_n\right) = f(x^*) = \lim_{n\to\infty} x_{n+1} = x^*$$

□

## 9.2 Urysohn Metrization Theorem

### 9.2.1 Urysohn Lemma

Recall that:

**Definition 7.** $X$ is $T_4$ if: For any disjoint closed set $A$ and $B$, there exist disjoint open $U, V$ such that $A \subseteq U$ and $B \subseteq V$.

**Lemma 5.** $X$ is $T_4$ Space if and only if For any closed $C$ and open $U$ with $C \subseteq U$, there exists open $O$ such that

$$\underset{\text{closed}}{C} \subseteq \underset{\text{open}}{O} \subseteq \underset{\text{closed}}{\overline{O}} \subseteq \underset{\text{open}}{U}$$

*Proof.* Proof of the left direction only.
Let $X$ be a $T_4$ Space, and $C \subset X$ be a closed, $U$ be a open containing $C$. Then, $C \subset U$ implies $U^c \subset C^c$, thus $U^c$ is a closed set disjoint from $C$. By $T_4$ condition, There exist disjoint opens $O, O'$ such that $C \subset O$ and $U^c \subset O' \iff O'^c \subset U$.
Since $O \cap O' = \emptyset \iff O \subset O'^c$, $O$ contained in $U$, this implies that $C \subset O \subset U$.
Since closure is the smallest closed set such that contains it, consequently $C \subset O \subset \overline{O} \subset O'^c \subset U$. □

**Definition 8.** Let $X$ be a Toplogical Space, and $A, B \subset X$ are disjoint closed subset.
A real-valued Continuous map $f : X \to [a, b]$ is called *Urysohn function* for $A$ and $B$ if: $f|_A = a$ and $f|_B = b$.
In another form,

$$f : X \to [a, b] : x \to \begin{cases} a & x \in A \\ b & x \in B \\ f(x) & x \notin A \cup B \end{cases}$$

**Lemma 6.** *Urysohn Lemma*
$T_4$ Space has an Urysohn function for any two disjoint closed subsets.

*Proof.* Generalization is the last thing to proven, first of all, prove in case of $[a, b] = [0, 1]$. This proof consists by three Step.
Let $X$ be a $T_4$ Space, and $A, B \subset X$ be closed subsets.
***Step 1. Construct a Chain of Open sets with Dyadic Rational Indices.***
Consider a set of ***Dyadic Rationals*** $D \overset{\text{def}}{=} \left\{ \frac{k}{2^n} \,\middle|\, n, k \in \mathbb{N},\ k \le 2^n - 1 \right\}$. We will show that the following statement holds:

$$\text{For any } r, s \in D \text{ with } r < s \text{, there exist open sets } U_r, U_s \text{ such that } A \subseteq \overline{U}_r \subseteq U_s \subseteq X \setminus B \qquad (*)$$

For this, Enough to Show that: For any $k \in \mathbb{N}$, there exists a Chain as:

$$A \subseteq U_{\frac{1}{2^k}} \subseteq \overline{U}_{\frac{1}{2^k}} \subseteq U_{\frac{2}{2^k}} \subseteq \overline{U}_{\frac{2}{2^k}} \subseteq \cdots \subseteq U_{\frac{2^k-1}{2^k}} \subseteq \overline{U}_{\frac{2^k-1}{2^k}} \subseteq X \setminus B$$

(Note that this opens in the Chain are not necessary distinct: For instance, if Ambient Space is Finite, then the Space is *Noetherian*. That is, $X$ satisfies Ascending Chain Condition for open sets.)
Let $k = 1$. Then, By $T_4$ condition gives that: There exists an open set $U_1$ such that

$$A \subseteq U_1 \subseteq \overline{U_1} \subseteq X \setminus B$$

Now, naming this $U_1$ as $U_{\frac{1}{2}}$, proved when $k = 1$.
Suppose that for some $k > 1$, the Chain exists as:

$$\underset{\text{closed}}{A} \overset{*1}{\subseteq} \underset{\text{open}}{U_{\frac{1}{2^k}}} \subseteq \underset{\text{closed}}{\overline{U}_{\frac{1}{2^k}}} \overset{*2}{\subseteq} \underset{\text{open}}{U_{\frac{2}{2^k}}} \subseteq \cdots \overset{*2^k-1}{\subseteq} \underset{\text{open}}{U_{\frac{2^k-1}{2^k}}} \subseteq \underset{\text{closed}}{\overline{U}_{\frac{2^k-1}{2^k}}} \overset{*2^k}{\subseteq} \underset{\text{open}}{X \setminus B}$$

By repeatedly applying the $T_4$ condition $2^k$ times, as indicated by the indices $*1, *2, \dots, *2^k$, we can construct $2^k$ open sets such that:

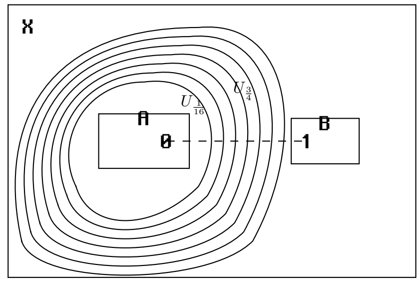
$$A \subseteq U_{\frac{1}{2^{k+1}}} \subseteq \overline{U}_{\frac{1}{2^{k+1}}} \subseteq U_{\frac{1}{2^k}} \subseteq \overline{U}_{\frac{1}{2^k}} \subseteq U_{\frac{3}{2^{k+1}}} \subseteq \overline{U}_{\frac{3}{2^{k+1}}} \subseteq U_{\frac{2}{2^k}} \subseteq \overline{U}_{\frac{2}{2^k}} \quad \subseteq \cdots \subseteq U_{\frac{2^k-1}{2^k}} \subseteq \overline{U}_{\frac{2^k-1}{2^k}} \subseteq U_{\frac{2^{k+1}-1}{2^{k+1}}} \subseteq \overline{U}_{\frac{2^{k+1}-1}{2^{k+1}}} \subseteq X \setminus B$$

Finally, Step 1 proved.

***Step 2. Construct an Urysohn Function.***

Define a map $f : X \to [0,1]$ as:

$$f(x) = \begin{cases} 0 & x \in \bigcap_{t \in D} U_t \\ \sup\{t \in D \mid x \notin U_t\} & x \notin \bigcap_{t \in D} U_t \end{cases}$$




Then, this map $f$ is well-defined by $(*)$ and $\sup D \leq 1$. And $f$ satisfies that:

1. $\forall r \in D,\ x \in A \subset U_r$. Thus, $f(x) = 0$ if $x \in A$.
2. $\forall r \in D,\ x \in B \subset X \setminus U_r$. Thus, $f(x) = \sup D = 1$ if $x \in B$.
3. If $x \in \overline{U}_r$, then for every $s > r$, $x \in \overline{U}_r \subset U_s$. Thus, $f(x) \leq r$. In Contrapositive, $f(x) > r \implies x \notin \overline{U}_r$.
   (If $f(x) = \sup\{t \in D \mid x \notin U_t\} > r$, then there is $s \in D$ such that $s > r$ and $x \notin U_s$, Contradiction.)
4. If $x \notin U_r$, then, $f(x) \geq r$. In Contrapositive, $f(x) < r \implies x \in U_r$.

Now, show that this map $f$ is Continuous map: Let $x \in X$ be fixed arbitrarlily, and $\epsilon > 0$ be given.

***In Case of*** $0 < f(x) < 1$.

Since Density of Dyadic Rationals, Choose $r, s \in D$ such that $f(x) - \epsilon < r < f(x) < s < f(x) + \epsilon$.

Now, we obtain that:

$$x \overset{(*)}{\in} U_s \setminus \overline{U}_r \overset{(**)}{\subseteq} f^{-1}[(f(x) - \epsilon, f(x) + \epsilon)]$$

$(*)$ directly given by above properties, $(**)$ given applying the fact that $x \in U_s \subset \overline{U}_s$ and $x \notin \overline{U}_r$.

***In Case of*** $f(x) = 0$.

Choose $r \in D$ such that $f(x) = 0 < r < \epsilon = f(x) + \epsilon$. Then,

$$x \in U_r \subset f^{-1}[(f(x), f(x) + \epsilon)]$$

***In Case of*** $f(x) = 1$.

Choose $r \in D$ such that $f(x) - \epsilon = 1 - \epsilon < r < 1 = f(x)$. Then,

$$x \in X \setminus U_r \subset f^{-1}[(f(x) - \epsilon, f(x))]$$

Consequently, $f$ is Continuous map on $[0,1]$ such that $f|_A = 0$ and $f|_B = 1$.

***Step 3. Generalization.***

Since $[0,1] \cong [a,b]$ for any $a < b$, let $g : [0,1] \to [a,b] : x \mapsto (1-x)a + xb$ be a Homeomorphism.

Then, $h = g \circ f : X \to [a,b]$ becomes a Continuous map such that $h|_A = a$ and $h|_B = b$. $\square$

### 9.2.2 Tietze Extension Theroem

> **Theorem 12.** *Tietze Extension Theroem*
> Let $X$ be a $T_4$ Space, and $A \subseteq X$ be a closed subset.
> For any Continuous map $f : A \to \mathbb{R}$, there exists a Continuous map:
> $$g : X \to \mathbb{R} \quad \textbf{s.t.} \quad g|_A = f$$
> This $g$ is called ***extension*** of $f$.

***Proof.*** This proof consists by three steps.
***Step 1.*** First, we will show that:

$$\text{For any Continuous map } f : A \to [-r, r]\text{, there is a Continuous map } h : X \to \mathbb{R} \textbf{ s.t. } \begin{cases} \forall x \in X, & |h(x)| \leq \frac{1}{3}r \\ \forall a \in A, & |f(a) - h(a)| \leq \frac{2}{3}r \end{cases} \tag{$*$}$$

Set

$$I_1 \overset{\text{def}}{=} \left[-r, -\frac{1}{3}r\right], \quad I_2 \overset{\text{def}}{=} \left[-\frac{1}{3}r, \frac{1}{3}r\right], \quad I_3 \overset{\text{def}}{=} \left[\frac{1}{3}r, r\right]$$

Then, the preimage of continuous map preserves closed and $A$ is closed subspace of $X$, $f^{-1}[I_1]$ and $f^{-1}[I_3]$ are closed of $X$.
And, $I_1$ and $I_3$ are disjoint, thus $f^{-1}[I_1 \cap I_3] = f^{-1}[I_1] \cap f^{-1}[I_3] = \emptyset$.
Now, apply the ***Urysohn Lemma***: There exists an Urysohn function $h : X \to I_2$ for $f^{-1}[I_1]$ and $f^{-1}[I_3]$.
Clearly, this map $h$ satisfies the first condition in $(*)$. And, for show the second condition, let $a \in A$ be given.
If $a \in f^{-1}[I_1]$, then $f(a) \in I_1$ and $h(a) = -\frac{1}{3}r$, thus $|f(a) - h(a)| \leq \frac{2}{3}r$.
If $a \in f^{-1}[I_3]$, then $f(a) \in I_3$ and $h(a) = \frac{1}{3}r$, thus $|f(a) - h(a)| \leq \frac{2}{3}r$.
If $a \notin (f^{-1}[I_1] \cup f^{-1}[I_3])$, then $f(a), h(a) \in I_2$, thus $|f(a) - h(a)| \leq \frac{2}{3}r$.
Therefore, the second condition satisfied.
***Step 2.*** We will show that: for any $f : A \to [-1, 1]$, there exists an extension of $f$.
Apply the result in Step 1, there exists a Continuous map:

$$h_1 : X \to \mathbb{R} \textbf{ s.t. } \begin{cases} \forall x \in X, & |h_1(x)| \leq \frac{1}{3} \\ \forall a \in A, & |f(a) - h_1(a)| \leq \frac{2}{3} \end{cases}$$

Now, the second condition of $h_1$, the continuous map $f - h_1 : A \to \left[-\frac{2}{3}, \frac{2}{3}\right] : x \to f(x) - h_1(x)$ is well-defined.
Again, there exists a Continuous map:

$$h_2 : X \to \mathbb{R} \textbf{ s.t. } \begin{cases} \forall x \in X, & |h_2(x)| \leq \frac{1}{3} \cdot \frac{2}{3} \\ \forall a \in A, & |f(a) - h_1(a) - h_2(a)| \leq \left(\frac{2}{3}\right)^2 \end{cases}$$

Inductively, for any $n \in \mathbb{N}$, there exists a Continuous map:

$$h_n : X \to \mathbb{R} \textbf{ s.t. } \begin{cases} \forall x \in X, & |h_n(x)| \leq \frac{1}{3} \cdot \left(\frac{2}{3}\right)^{n-1} \\ \forall a \in A, & |f(a) - h_1(a) - h_2(a) - \cdots - h_n(a)| \leq \left(\frac{2}{3}\right)^n \end{cases}$$

Define a map

$$g : X \to [-1, 1] : x \mapsto \sum_{n=1}^{\infty} h_n(x)$$

For any $x \in X$,

$$|g(x)| = \left|\sum_{n=1}^{\infty} h_n(x)\right| \leq \sum_{n=1}^{\infty} |h_n(x)| \leq \sum_{n=1}^{\infty} \frac{1}{3} \cdot \left(\frac{2}{3}\right)^{n-1} = \frac{1}{3} \cdot \frac{1}{1 - \frac{2}{3}} = 1$$

Therefore, this map is well-defined. And, ***Weierstrass M-test*** gives that $\sum_{n=1}^{\infty} h_n(x)$ converges uniformly.
Moreover, for any $a \in A$,

$$\left|f(a) - \sum_{k=1}^{n} h_k(a)\right| \leq \left(\frac{2}{3}\right)^n \implies \left|f(a) - \sum_{n=1}^{\infty} h_n(a)\right| = |f(a) - g(a)| = 0$$

That is, $g$ is Continuous on $X$ and $g|_A = f$. Therefore, $g$ is extension of $f$.
*Step 3.* Finally, we generalize the result in Step 2.:
Let $f : A \to [a, b]$ be a Continuous map on the closed subspace $A$. And, let $\varphi : [a, b] \to [-1, 1]$ be a Homeomorphism.
Then, $\varphi \circ f : A \to [-1, 1]$ is Continuous map, thus there exists an extension $g : X \to [-1, 1]$ such that $g|_A = \varphi \circ f$.
Now, $\varphi^{-1} \circ g : X \to [a, b]$ is Continuous, and $(\varphi^{-1} \circ g)|_A = \varphi^{-1} \circ \varphi \circ f = f$, Therefore this $\varphi^{-1} \circ g$ is the extension of $f$.
Let $f : A \to \mathbb{R}$ be a Continuous map on the closed subspace $A$.
And, let $\varphi : \mathbb{R} \to (-1, 1)$ be a Homeomorphism. Then, the map $\phi : \mathbb{R} \to [-1, 1] : x \mapsto \varphi(x)$ is still Continuous.
Now, The Continuous map $\phi \circ f : A \to [-1, 1]$ has an extension $g : X \to [-1, 1]$ such that $g|_A = \phi \circ f$.
Put $B = g^{-1}[\{-1, 1\}]$. Then $B$ is Closed on $X$, and $A \cap B = \emptyset$. Now, apply the Urysohn Lemma to this, there exists an Urysohn function for $A$ and $B$: Continuous map $\gamma : X \to [0, 1]$ such that $\gamma|_A = 1$ and $\gamma|_B = 0$.
Define a map $\eta : X \to (-1, 1) : x \mapsto g(x)\gamma(x)$. Then, if $g(x) = 1$ or $g(x) = -1$, then $x \in B$, thus $g(x)\gamma(x) = 0$.
Therfore, $\eta$ is well-defined. And, for any $a \in A$, $\eta(a) = g(a)\gamma(a) = g(a)$, thus $\eta|_A = \phi \circ f$.
Consequently, the map $\phi^{-1} \circ \eta$ is an extension of $f$, we wanted. □

Recall that:

**Definition 9.** $X$ is $T_1$ if: For any distinct $x, y \in X$, there exist open sets $U_x, U_y$ such that $\begin{cases} x \in U_x, & x \notin U_y \\ y \notin U_x, & y \in U_y \end{cases}$.

**Lemma 7.** $X$ is $T_1$ if and only if For any $x \in X$, a singleton $\{x\}$ is closed in $X$.

*Proof.* The left direction is clear.
Let $x \in X$. Then, for any $y \in X$ with $y \neq x$, $T_1$ condition gives that there is an open set such that $y \in U_y$ and $x \notin U_y$.
Now, the union

$$\bigcup_{\substack{y \in X \\ y \neq x}} U_y = X \setminus \{x\}$$

is open by definition. □

### 9.2.3 Urysohn Metrization Theorem

> **Theorem 13.** *Urysohn Metrization Theroem*
> If $X$ is a Second-Countable Regular Space, then $X$ is Metrizable.

Chapter 10

# Algebraic Topology

Chapter 11

# Basic Analysis

## 11.1 Tests for Series

### 11.1.1 Integral Test

**Theorem 14.** Let $f : [1, \infty) \to \mathbb{R}$ be a decreasing function which satisfies $\begin{cases} \lim\limits_{x\to\infty} f(x) = 0 \\ f > 0 \end{cases}$. Then,

$$\int_1^\infty f(x)dx \text{ converges if and only if } \sum_{k=1}^\infty f(k) \text{ converges.}$$

Futhermore, put $d_n \stackrel{\text{def}}{=} \sum\limits_{k=1}^n f(k) - \int_1^n f(x)dx$, then for any $n \in \mathbb{N}$, $0 < f(n+1) \leq d_{n+1} \leq d_n \leq f(1)$, and for any $k \in \mathbb{N}$, $0 \leq d_k - \lim\limits_{n\to\infty} d_n \leq f(k)$. (Clearly, $\lim\limits_{n\to\infty} d_n$ exists.)

*Proof.* Since

$$\int_1^{n+1} f(x)dx = \sum_{k=1}^n \int_k^{k+1} f(x) \overset{\substack{\text{decreasing} \\ \downarrow}}{\leq} \sum_{k=1}^n \int_k^{k+1} f(k) = \sum_{k=1}^n f(k)$$
$$\implies f(n+1) = \sum_{k=1}^{n+1} f(k) - \sum_{k=1}^n f(k) \leq \sum_{k=1}^{n+1} f(k) - \int_1^{n+1} f(x)dx = d_{n+1}$$

And,

$$d_n - d_{n+1} = \int_n^{n+1} f(x)dx - f(n+1) \geq \int_n^{n+1} f(n+1)dx - f(n+1) = 0$$

Immediate $d_n$ converges, being bounded and decreasing. That is,

$$\lim_{n\to\infty} d_n = \lim_{n\to\infty} \left( \sum_{k=1}^n f(k) - \int_1^n f(x)dx \right)$$

converges. Meanwhile, since

$$0 \leq d_n - d_{n+1} = \int_n^{n+1} f(x)dx - f(n+1) \leq \int_n^{n+1} f(n)dx - f(n+1) = f(n) - f(n+1)$$

Now, telescope:

$$0 \leq d_k - \lim_{n\to\infty} d_n \leq f(k) - \lim_{n\to\infty} f(n+1) = f(k)$$

□

### 11.1.2 Ratio Test

### 11.1.3 Root Test

## 11.2 Arithmetic means

Let $\{s_n\}$ be a Complex numbers Sequence. Define the ***Arithmetic means*** of $\{s_n\}$:

$$\sigma_n \overset{\text{def}}{=} \frac{s_0 + \cdots + s_n}{n+1} = \frac{1}{n+1}\left(\sum_{i=0}^{n} s_i\right)$$

Then, the Arithmetic means $\sigma_n$ has the following properties:
1). If $\lim_{n\to\infty} s_n = s$, then $\lim_{n\to\infty} \sigma_n = s$.

*Proof.* Let $\epsilon > 0$ be given. Then, there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $|s_n - s| < \epsilon$.
Now, for $n \geq N$,

$$\begin{aligned}
|\sigma_n - s| &= \left|\frac{s_0 + \cdots + s_n}{n+1} - \frac{(n+1)s}{n+1}\right| = \left|\frac{(s_0 - s) + \cdots + (s_n - s)}{n+1}\right| \\
&\overset{\text{tri.ieq}}{\leq} \frac{\sum_{k=0}^{N-1}|s_k - s|}{n+1} + \frac{\sum_{k=N}^{n}|s_k - s|}{n+1} \\
&< \frac{\sum_{k=0}^{N-1}|s_k - s|}{n+1} + \frac{n+1-N}{n+1}\cdot\epsilon \\
&< \frac{\sum_{k=0}^{N-1}|s_k - s|}{n+1} + \epsilon
\end{aligned}$$

Now, put $M \in \mathbb{N}$ satisfies $M \geq N$ and $n \geq M \implies \frac{\sum_{k=0}^{N-1}|s_k - s|}{n+1} < \epsilon$, using Archimedean property.
Then, $n \geq M$ implies $|\sigma_n - s| < \epsilon$, thus $\sigma_n \to s$. □

2). Put $a_n = s_n - s_{n-1}$, for $n \geq 1$. If $\lim_{n\to\infty} na_n = 0$ and $\sigma_n$ converges, then $s_n$ converges.

*Proof.* First,

$$\begin{aligned}
s_n - \sigma_n &= s_n - \frac{s_0 + \cdots + s_n}{n+1} = \frac{(n+1)s_n - \sum_{k=0}^{n} s_k}{n+1} \\
&= \frac{1}{n+1}((s_1 - s_0) + (2s_2 - 2s_1) + (3s_3 - 3s_2) + \cdots + (ns_n - ns_{n-1})) \\
&= \frac{1}{n+1}\sum_{k=1}^{n} ka_k
\end{aligned}$$

Now, if $na_n \to 0$ and $\sigma_n \to \sigma$,

$$\begin{aligned}
\lim_{n\to\infty} s_n &= \lim_{n\to\infty}\left(\sigma_n + \frac{1}{n+1}\sum_{k=1}^{n} ka_k\right) \\
&= \lim_{n\to\infty}\sigma_n + \lim_{n\to\infty}\frac{1}{n+1}\sum_{k=1}^{n} ka_k \overset{1)}{=} \sigma
\end{aligned}$$

□

2) is conditional converse of 1). But, there is more weak version of the converse proposition:
3). The sequence $\{na_n\}$ bounded by $M < \infty$, and $\sigma_n \to \sigma$. Then, $s_n \to \sigma$.

*Proof.* First, For positive integers $m < n$,

$$\begin{aligned}
s_n - \sigma_n &= s_n - \frac{\sum_{k=0}^{n} s_k}{n+1} = s_n - \frac{m+1}{n-m}\cdot\left(\frac{1}{m+1} - \frac{1}{n+1}\right)\sum_{k=0}^{n} s_k \\
&= s_n - \frac{m+1}{n-m}\cdot\left(\frac{\sum_{k=0}^{m} s_k + \sum_{k=m+1}^{n} s_k}{m+1} - \frac{\sum_{k=0}^{n} s_k}{n+1}\right) \\
&= s_n - \frac{m+1}{n-m}\cdot\left(\sigma_m - \sigma_n + \frac{\sum_{k=m+1}^{n} s_k}{m+1}\right) \\
&= \frac{m+1}{n-m}(\sigma_n - \sigma_m) + \frac{1}{n-m}\sum_{k=m+1}^{n}(s_n - s_k)
\end{aligned}$$

Meanwhile, since for any $n \in \mathbb{N}$, $|na_n| = n|s_n - s_{n-1}| < M$, for $k = m+1, \ldots, n$,

$$\begin{aligned}
|s_n - s_k| &= |s_n - s_{n-1} + s_{n-1} - s_{n-2} + \cdots + s_{k+1} - s_k| \\
&\le |s_n - s_{n-1}| + |s_{n-1} - s_{n-2}| + \cdots + |s_{k+1} - s_k| \\
&\le \frac{M}{n} + \frac{M}{n-1} + \cdots + \frac{M}{k+1} \le \frac{n-k}{k+1}M \le \frac{n-k}{m+2}M \le \frac{n-m-1}{m+2}M
\end{aligned}$$

Let $\epsilon > 0$ be given. For each $n \in \mathbb{N}$, put $m \in \mathbb{N}$ such that

$$m \le \frac{n-\epsilon}{1+\epsilon} < m+1$$

Then,

$$m(1+\epsilon) \le n - \epsilon \implies m + \epsilon(1+m) \le n \implies \frac{m+1}{n-m} \le \frac{1}{\epsilon}$$

and

$$n - \epsilon < (m+1)(1+\epsilon) \implies n+1 < (m+2)(1+\epsilon) \implies \frac{n+1}{m+2} - 1 < \epsilon \implies \frac{n-m-1}{m+2} < \epsilon$$

Now, for arbitrary $n \in \mathbb{N}$,

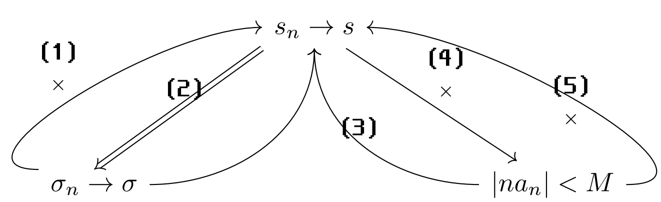
$$\begin{aligned}
&|s_n - \sigma| \le |s_n - \sigma| + |\sigma_n - \sigma| \\
\implies &\limsup_{n\to\infty} |s_n - \sigma| \le \limsup_{n\to\infty} |s_n - \sigma_n| + \limsup_{n\to\infty} |\sigma_n - \sigma|
\end{aligned}$$

And,

$$\begin{aligned}
|s_n - \sigma_n| &= \frac{m+1}{n-m}|\sigma_n - \sigma_m| + \frac{1}{n-m}\sum_{k=m+1}^{n} |s_n - s_k| < \frac{1}{\epsilon}|\sigma_n - \sigma_m| + M\epsilon \\
\implies \limsup_{n\to\infty} |s_n - \sigma_n| &\le \frac{1}{\epsilon}\limsup_{n\to\infty} |\sigma_n - \sigma_m| + M\epsilon = M\epsilon
\end{aligned}$$

Consequently, $\limsup_{n\to\infty} |s_n - \sigma| \le (M+1)\epsilon$, thus $s_n \to \sigma$. $\square$

In brief, the diagram of the above conditions like this:




Examples and Counterexamples of the Diagram:

(1) Let $s_n \overset{\text{def}}{=} \exp(\frac{in\pi}{2})$. Then,

- $s_n$ diverges.
- $na_n$ diverges.
- $\sigma_n \to 0$.

(2) Let $s_n \overset{\text{def}}{=} \frac{1}{n}$, $s_0 = 0$.

(3) Let $s_n \overset{\text{def}}{=} \sum_{k=1}^{n} \frac{1}{k}$. Then,

- $s_n$ diverges.
- $a_n = \frac{1}{n}$, thus $na_n \to 1$, bounded.
- If $\sigma_n$ converges, then the diagram implies that $s_n$ must converge, leading to a contradiction. Therefore, $\sigma_n$ diverges.

(4) $s_n = \sum_{k=1}^{n} \frac{(-1)^k}{\sqrt{k}}$, $s_0 = 0$. Then,

- $s_n$ converges, being the Alternating series Test.
- $a_n = \frac{(-1)^n}{\sqrt{n}}$, thus $na_n$ diverges.

## 11.3 Taylor's Theorem

> **Theorem 15.** *Taylor's Theorem*
>
> Let $f : [a, b] \to \mathbb{R}$, and let $n \in \mathbb{N}$ be fixed. Suppose that $\begin{cases} f^{(n-1)} \text{ is Continuous.} \\ f^{(n)}(t) \text{ exists for every } t \in (a, b). \end{cases}$
>
> Then, for any $\alpha, \beta \in [a, b]$, there exists $x \in (\alpha, \beta)$ such that
>
> $$f(\beta) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(\beta - \alpha)^k + \frac{f^{(n)}(x)}{n!}(\beta - \alpha)^n$$

*Proof.* Put

$$M \overset{\text{def}}{=} \frac{1}{(\beta - \alpha)^n} \cdot \left( f(\beta) - \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(\beta - \alpha)^k \right)$$

That is,

$$f(\beta) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(\beta - \alpha)^k + M(\beta - \alpha)^n$$

and put

$$g(t) \overset{\text{def}}{=} f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(t - \alpha)^k - M(t - \alpha)^n, \quad (a \le t \le b)$$

If we differentiate the above equation $n$ times,

$$g^{(n)}(t) = f^{(n)}(t) - n!M, (a < t < b)$$

For each $k = 0, 1, \ldots, n - 1$,

$$\begin{aligned} \frac{d^r}{dt^r} \left( \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(t - \alpha)^k \right) &= \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!} \cdot \frac{d^r}{dt^r} \left( (t - \alpha)^k \right) \\ &= \sum_{k=r+1}^{n-1} \frac{f^{(k)}(\alpha)}{k!} \cdot \frac{k!}{(k - r)!}(t - \alpha)^{k-r} + f^{(r)}(\alpha) \\ &= \sum_{k=r+1}^{n-1} \frac{f^{(k)}(\alpha)}{(k - r)!}(t - \alpha)^{k-r} + f^{(r)}(\alpha) \end{aligned}$$

Substituting $t = \alpha$, only the $f^{(r)}(\alpha)$ term remains. Therefore, for $r = 0, \ldots, n-1$, $g(\alpha) = g'(\alpha) = \cdots = g^{(n-1)}(\alpha) = 0$.
Since $g(\beta) = 0$ by definition, the Mean-Value Theorem implies there exists a $x_1 \in (\alpha, \beta)$ s.t. $g'(x_1) = \frac{g(\beta) - g(\alpha)}{\beta - \alpha} = 0$.
And similarly, there is $x_2 \in (x_1, \beta)$ s.t. $g''(x_2) = \frac{g'(x_1) - g'(\alpha)}{\beta - \alpha} = 0$.

Inductively, for some $x_n \in (\alpha, \beta)$, $g^{(n)}(x_n) = f^{(n)}(x_n) - n!M = 0$. That is, $M = \dfrac{f^{(n)}(x_n)}{n!}$.
Proof Complete by Initial Setting. □

> **Corollary 2.** Let $f : [a, b] \to \mathbb{R}$ be an infinitely differentiable function.
> Suppose that there exists a $M > 0$ such that for any $n \in \mathbb{N}$, $\sup_{t \in [a,b]} |f^{(n)}(t)| \le M$. Then, for any $x, \alpha \in [a, b]$,
>
> $$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x)}{k!}(x - \alpha)^k$$

## 11.4 Convexity

### 11.4.1 Definition

**Definition 10.** Let $f : (a,b) \to \mathbb{R}$ be a Real-valued function. $f$ is said to be ***convex*** if: For any $x, y \in (a,b), \lambda \in (0,1)$,

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$$

Convex function has follwing properties:

**Lemma 8.** Let $f : (a,b) \to \mathbb{R}$ be a Convex function, and $a < x_1 < x_2 < x_3 < b$. Then,

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(x_3) - f(x_1)}{x_3 - x_1} \leq \frac{f(x_3) - f(x_2)}{x_3 - x_2}$$

*Proof.* To show that first inequalty, note that

$$\frac{x_2 - x_1}{x_3 - x_1} \cdot x_3 + \left(1 - \frac{x_2 - x_1}{x_3 - x_1}\right) \cdot x_1 = \frac{(x_2 - x_1)x_3 + (x_3 - x_2)x_1}{x_3 - x_1} = x_2$$

Now,

$$\begin{aligned} f(x_2) &= f\left(\frac{x_2 - x_1}{x_3 - x_1} \cdot x_3 + \left(1 - \frac{x_2 - x_1}{x_3 - x_1}\right) \cdot x_1\right) \\ &\leq \frac{x_2 - x_1}{x_3 - x_1} \cdot f(x_3) + \left(1 - \frac{x_2 - x_1}{x_3 - x_1}\right) \cdot f(x_1) = \frac{x_2 - x_1}{x_3 - x_1} \cdot f(x_3) + f(x_1) - \left(\frac{x_2 - x_1}{x_3 - x_1}\right) \cdot f(x_1) \end{aligned}$$

In brief,

$$f(x_2) - f(x_1) \leq \frac{x_2 - x_1}{x_3 - x_1}(f(x_3) - f(x_1)) \implies \frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(x_3) - f(x_1)}{x_3 - x_1}$$

And similarly,

$$\frac{x_3 - x_2}{x_3 - x_1} \cdot x_1 + \left(1 - \frac{x_3 - x_2}{x_3 - x_1}\right) x_3 = x_2$$

gives the second inequality. □

### 11.4.2 Properties

**Proposition 2.** If $f:(a,b)\to\mathbb{R}$ is Convex, then $f$ is Continuous.

*Proof.* Let $\epsilon>0$ be given, $s<t$ are fixed in $(a,b)$. For any $x,y\in(s,t)$ with $s<x<y<t$,

$$\frac{f(s)-f(a)}{s-a}\le\frac{f(x)-f(s)}{x-s}\le\frac{f(y)-f(x)}{y-x}\le\frac{f(t)-f(y)}{t-y}\le\frac{f(b)-f(t)}{b-t}$$

Put $M=\max\left\{\left|\frac{f(s)-f(a)}{s-a}\right|,\left|\frac{f(b)-f(t)}{b-t}\right|\right\}$. Then, for any $x,y\in(s,t)$,

$$\left|\frac{f(y)-f(x)}{y-x}\right|\le M$$

Now,

$$|f(y)-f(x)|\le M|y-x|<\epsilon$$

Since $s,t\in(a,b)$ was arbitrary, $f$ is continuous on $(a,b)$. □

**Proposition 3.** Let $f$ is differentiable on $(a,b)$. Then,

$f$ is Convex *if and only if* $f'$ is monotonically increasing on $(a,b)$.

*Proof.* Prove by showing both directions: right and left.
***Right Direction*** Let $x_1<x_2$ in $(a,b)$. Then,

$$f'(x_1)=\lim_{t\to x_1}\frac{f(t)-f(x_1)}{t-x_1}\le\frac{f(x_2)-f(x_1)}{x_2-x_1}\le\lim_{\tau\to x_2}\frac{f(\tau)-f(x_2)}{\tau-x_2}=f'(x_2)$$

More rigorously, put $\epsilon=\left|f'(x_1)-\frac{f(x_2)-f(x_1)}{x_2-x_1}\right|$. (If $\epsilon=0$, then there is nothing to prove.).
Now, there exists a $\delta>0$ such that $|t-x_1|<\delta$ implies

$$\left|f'(x_1)-\frac{f(t)-f(x_1)}{t-x_1}\right|<\epsilon\iff-\epsilon+\frac{f(t)-f(x_1)}{t-x_1}<f'(x_1)\overset{(*)}{\le}\epsilon+\frac{f(t)-f(x_1)}{t-x_1}$$

If $f'(x_1)-\frac{f(x_2)-f(x_1)}{x_2-x_1}>0$, then $(*)$ gives

$$f'(x_1)<f'(x_1)+\frac{f(t)-f(x_1)}{t-x_1}-\frac{f(x_2)-f(x_1)}{x_2-x_1}\iff\frac{f(x_2)-f(x_1)}{x_2-x_1}<\frac{f(t)-f(x_1)}{t-x_1}\quad\forall t\ \textbf{s.t.}\ |t-x_1|<\delta$$

If $|t-x_1|<|x_2-x_1|$, then this contradicts to Convexity.
Consequently, we obtain the first inequality, similarly can prove the second inequality.
***Left Direction*** Let $x,y\in(a,b)$ and $\lambda\in(0,1)$ be given. The Mean Value Theorem gives that:

$$f(\lambda x+(1-\lambda)y)-f(x)=f'(z_1)(\lambda x+(1-\lambda)y-x)\ \text{for some}\ z_1\in(x,\lambda x+(1-\lambda)y)$$
$$f(y)-f(\lambda x+(1-\lambda)y)=f'(z_2)(y-\lambda x+(1-\lambda)y)\ \text{for some}\ z_2\in(\lambda x+(1-\lambda)y,y)$$

Now, Monotonically increasing gives

$$\begin{aligned}&\frac{f(\lambda x+(1-\lambda)y)-f(x)}{\lambda x+(1-\lambda)y-x}=f'(z_1)\le f'(z_2)=\frac{f(y)-f(\lambda x+(1-\lambda)y)}{y-(\lambda x+(1-\lambda)y)}\\&\Longrightarrow\frac{f(\lambda x+(1-\lambda)y)-f(x)}{(1-x)(y-x)}\le\frac{f(y)-f(\lambda x+(1-\lambda)y)}{\lambda(y-x)}\\&\Longrightarrow\lambda f(\lambda x+(1-\lambda)y)-\lambda f(x)\le(1-\lambda)f(y)-(1-\lambda)\lambda x+(1-\lambda)y\\&\Longrightarrow f(\lambda x+(1-\lambda)y)\le\lambda f(x)+(1-\lambda)f(y)\end{aligned}$$

□

**Corollary 3.** If $f:[a,b]\to\mathbb{R}$ is twice-differentiable, then

$$f \text{ is Convex } \textit{if and only if } f''(x)\geq 0 \text{ for all } x\in(a,b).$$

**Theorem 16.** Let $f:[a.b]\to\mathbb{R}$ be given. Then,

$$f \text{ is Convex if and only if } f \text{ is Continuous, and Midpoint Convex.}$$

Midpoint convex is that $f$ satisfies $\forall x,y\in(a,b),\ \ f\left(\dfrac{x+y}{2}\right)\leq\dfrac{f(x)+f(y)}{2}$.

*Proof.* The right direction is clear. To show the left direction, we demonstrate that *Midpoint Convexity implies Dyadic Rational Convexity*. Claim: For any $n\in\mathbb{N}$,

$$f\left(\frac{\sum_{k=1}^{2^n}x_k}{2^n}\right)\leq\frac{1}{2^n}\sum_{k=1}^{2^n}f(x_k) \tag{$*$}$$

Using Induction: If $n=1$, it is clear by Midpoint Convexity.
Assume that for $n\in\mathbb{N}$, $(*)$ is True. Then,

$$\begin{aligned}
f\left(\frac{\sum_{k=1}^{2^{n+1}}x_k}{2^{n+1}}\right) &= f\left(\frac{1}{2}\cdot\left[\frac{\sum_{k=1}^{2^n}x_k}{2^n}+\frac{\sum_{k=2^n+1}^{2^{n+1}}x_k}{2^n}\right]\right)\\
&\overset{\text{m.c}}{\leq}\frac{1}{2}\left(f\left(\frac{\sum_{k=1}^{2^n}x_k}{2^n}\right)+f\left(\frac{\sum_{k=2^n+1}^{2^{n+1}}x_k}{2^n}\right)\right)\\
&\overset{(*)}{\leq}\frac{1}{2}\left(\frac{1}{2^n}\sum_{k=1}^{2^n}f(x_k)+\frac{1}{2^n}\sum_{k=2^n+1}^{2^{n+1}}f(x_k)\right)\\
&=\frac{1}{2^{n+1}}\sum_{k=1}^{2^{n+1}}f(x_k)
\end{aligned}$$

Consequently, we obtain the claim. Now, let $n\in\mathbb{N}$, and $m$ be an integer such that $1\leq m\leq 2^n$.
Put $x_1=x_2=\cdots=x_m=x$ and $x_{m+1}=x_{m+2}=\cdots=x_{2^n}=y$. Then

$$f\left(\frac{m}{2^n}x+\left(1-\frac{m}{2^n}\right)y\right)\leq\frac{m}{2^n}f(x)+\left(1-\frac{m}{2^n}\right)f(y)$$

For complete this discussion, Let $x,y\in(a,b),\lambda\in(0,1)$ be given.
Since $\dfrac{\lfloor 2^n\lambda\rfloor}{2^n}\to\lambda$ as $n\to\infty$, for any $n\in\mathbb{N}$,

$$f\left(\frac{\lfloor 2^n\lambda\rfloor}{2^n}x+\left(1-\frac{\lfloor 2^n\lambda\rfloor}{2^n}\right)y\right)\leq\frac{\lfloor 2^n\lambda\rfloor}{2^n}f(x)+\left(1-\frac{\lfloor 2^n\lambda\rfloor}{2^n}\right)f(y)$$

Finally, taking limits then

$$\lim_{n\to\infty}f\left(\frac{\lfloor 2^n\lambda\rfloor}{2^n}x+\left(1-\frac{\lfloor 2^n\lambda\rfloor}{2^n}\right)y\right)\overset{f\ \text{conti.}}{=}f\left(\lim_{n\to\infty}\left[\frac{\lfloor 2^n\lambda\rfloor}{2^n}x+\left(1-\frac{\lfloor 2^n\lambda\rfloor}{2^n}\right)y\right]\right)=f(\lambda x+(1-\lambda)y)\leq\lambda f(x)+(1-\lambda)f(y)$$

In brief, Midpoint Convexity implies Dyadic Rational Convexity, and with Continuous implies Convexity. $\square$

## 11.5 Lipschitz Condition

### 11.5.1 Definition

**Definition 11.** A real-vauled function $f:(a,b)\to\mathbb{R}$ is called *Lipschitz Continuous* if:

$$\exists L\geq 0 \text{ s.t. } \forall x_1,x_2\in(a,b),\ \ |f(x_1)-f(x_2)|\leq L\cdot|x_1-x_2|$$

The constant $L$ is said to be *Lipschitz Constant* of $f$. In particular, the constant

$$D\overset{\text{def}}{=}\sup_{x_1\neq x_2}\frac{|f(x_1)-f(x_2)|}{|x_1-x_2|}$$

is called *dilation* of $f$. Clearly,

$$\forall x_1,x_2\in(a,b),\ \ |f(x_1)-f(x_2)|\leq D\cdot|x_1-x_2|$$

and if $L>0$ is Lipschitz Constant of $f$, then $D\leq L$. That is, $D=\inf\{L>0\mid L \text{ is Lipschitz constant of } f\}$.

### 11.5.2 Properties

**Proposition 4.** If $f:(a,b)\to\mathbb{R}$ is Lipschitz Continuous, then $f$ is uniformly continuous.

*Proof.* Let $L\geq 0$ be a Lipschitz Constant of $f$. Then, for any $\epsilon>0$,

$$\forall x,y\in(a,b),\ \ |x-y|<\frac{\epsilon}{L}\implies|f(x)-f(y)|\leq L|x-y|<\epsilon$$

□

**Proposition 5.** Let $f:(a,b)\to\mathbb{R}$ be a Differentiable function. Then,

$f$ is Lipschitz Continuous *if and only if* $f'$ is bounded in $(a,b)$.

*Proof.*
*Right Direction*
Let $L>0$ be a Lipschitz constant of $f$, and $x\in(a,b)$ be given. Since definition of derivative,

$$f'(x)\overset{\text{def}}{=}\lim_{t\to x}\frac{f(x)-f(t)}{x-t}$$

Meanwhile, the assumption gives: for any distinct $x,t\in(a,b)$,

$$\frac{|f(x)-f(t)|}{|x-t|}\leq L$$

Therefore,

$$f'(x)=\lim_{t\to x}\frac{f(x)-f(t)}{x-t}\leq\lim_{t\to x}\frac{|f(x)-f(t)|}{|x-t|}\leq\lim_{t\to x}L=L$$

*Left Direction*
Let distinct $x,y\in(a,b)$ be given. Then, the Mean-Value Theorem gives: There exists a $z\in(x,y)$ such that

$$f(x)-f(y)=f'(z)(x-y)\implies f'(z)=\frac{f(x)-f(y)}{x-y}$$

Now,

$$\left|\frac{f(x)-f(y)}{x-y}\right|=|f'(z)|\leq L\implies|f(x)-f(y)|\leq L\cdot|x-y|$$

If $x=y$, then there is nothing to prove. □

Note that:

Lipschitz Continuous $\implies$ Uniformly Continuous $\implies$ Continuous

### 11.5.3 Newton-Raphson Method

**Theorem 17.** *Newton-Raphson Method*
Let $f : [a, b] \to \mathbb{R}$ be a twice-differentiable, $f(a) < 0 < f(b)$. Suppose that $f$ satisfies: for all $x \in [a, b]$,

$$f'(x) \geq \delta > 0 \text{ and } 0 \leq f''(x) \leq M$$

That is, $f$ is strictly increasing convex function, and Lipschitz Continuous.
Further, there uniquely exists $x^* \in (a, b)$ such that $f(x^*) = 0$.
Let $x_1 \in (x^*, b)$ fixed. Define a sequence $\{x_n\}$ inductively as follows:

$$x_{n+1} \stackrel{\text{def}}{=} x_n - \frac{f(x_n)}{f'(x_n)}$$

Then, $\{x_n\}$ satisfies the following three conditions:

1. $\{x_n\}$ is decreasing sequence.
2. $x_n \to x^*$ as $n \to \infty$.
3. For any $n \in \mathbb{N}$, $0 \leq x_{n+1} - x^* \leq \left[\frac{M}{2\delta}\right]^{2^{n+1}-1} [x_1 - x^*]^{2^n}$.

Condition 3 means that for a suitable initial value $x_1$, we can establish an upper bound for the error.

***Proof.*** This proof consists by three steps.
Since $f''$ is non-negative, and $f'$ is positive, $f$ is strictly increasing convex function.
And Fundamental Theorem of Calculus gives: for any $x \in (a, b)$,

$$f'(x) \stackrel{\text{FTA}}{=} \int_a^x f''(t)dt + f'(a) \leq \int_a^x M dt + f'(a) = M(x - a) + f'(a) \leq M(b - a) + f'(a)$$

Thus, $f'$ is bounded on $(a, b)$, thus $f$ is Lipschitz Continuous.
***Step 1.*** *$f$ has a unique root $x^*$.*
The existence of root given directly by Intermidate-Value theroem.
Suppose that $x^*, x' \in (a, b)$ are distinct root of $f$. i.e., $f(x^*) = f(x') = 0$. Then, by Mean-value theroem, there is $c \in (a, b)$ between $x^*$ and $x'$ such that

$$f'(c)(x^* - x') = f(x^*) - f(x') = 0$$

That is, $f'(c) = 0$. This is contradiction with $f'$ is positive.
***Step 2.*** *$\{x_n\}$ decrease.*
Proof by induction:
For $n = 1$, $f'(x_1)(x_1 - x_2) \stackrel{\text{def}}{=} f(x_1) > f(x^*) = 0$, thus $x_2 < x_1$. And,

$$\begin{aligned} f(x_2) &\stackrel{\text{MVT}}{=} f(x_1) + f'(c_1)(x_2 - x_1) \quad \text{for some } c_1 \in (x_2, x_1) \\ &> f(x_1) + f'(x_1)(x_2 - x_1) = f'(x_1)(x_1 - x_2) + f'(x_1)(x_2 - x_1) = 0 \end{aligned}$$

Now, since $f(x_2) > 0 = f(x^*)$, the Mean-Value Theorem implies that $x_2 > x^*$.
To use induction, suppose that for some $n \geq 1$, $x^* < x_{n+1} < x_n$. Then,

$$f(x_{n+1}) = f'(x_{n+1})(x_{n+1} - x_{n+2}) > 0$$

Thus $x_{n+2} < x_{n+1}$ and

$$\begin{aligned} f(x_{n+2}) &\stackrel{\text{MVT}}{=} f(x_{n+1}) + f'(c_{n+1})(x_{n+2} - x_{n+1}) \quad \text{for some } c_{n+1} \in (x_{n+2}, x_{n+1}) \\ &\geq f(x_{n+1}) + f'(x_{n+1})(x_{n+2} - x_{n+1}) \\ &= f(x_{n+1})(x_{n+1} - x_{n+2}) + f'(x_{n+1})(x_{n+2} - x_{n+1}) = 0 \end{aligned}$$

Again, the Mean-Value Theorem implies that $x_{n+2} > x^*$. Therefore, induction completes.
Now, $x_n \to x'$ as $n \to \infty$ for some $x' \in [x^*, x_1]$ since $\{x_n\}$ is Bounded below and Decreasing.
Still it remains that to show $x' = x^*$. By Continuity,

$$\begin{aligned} &f'(x_n)(x_{n+1} - x_n) + f(x_n) = 0 \\ \Longrightarrow\ &\lim_{n\to\infty} [f'(x_n)(x_{n+1} - x_n) + f(x_n)] = f\left(\lim_{n\to\infty} x_n\right) = f(x') = 0 \end{aligned}$$

Since the root of $f$ is unique, thus $x' = x^*$.
*Step 3. Establishing the error bound.*
The Taylor's Theorem implies that

$$f(x^*) = f(x_n) + f'(x_n)(x^* - x_n) + \frac{f''(t_n)}{2}(x^* - x_n)^2 \quad \textbf{for some } t_n \in (x^*, x_n)$$
$$\Longrightarrow x_{n+1} - x^* = \frac{f''(t_n)}{2f'(x_n)}(x^* - x_n)^2$$

Consequently,

$$0 \leq x_{n+1} - x^* = \frac{f''(t_n)}{2f'(x_n)}(x_n - x^*)^2 = \frac{f''(t_n)}{2f'(x_n)} \cdot \left(\frac{f''(t_{n-1})}{2f'(x_{n-1})}\right)^2 (x_{n-1} - x^*)^4 = \cdots$$
$$= \prod_{i=1}^{n} \left[\frac{f''(t_i)}{2f'(x_i)}\right]^{2^{(n+1-i)}} [x_1 - x^*]^{2^n} \leq \left[\frac{M}{2\delta}\right]^{2^{n+1}-1} [x_1 - x^*]^{2^n}$$

$\square$

### 11.5.4 Gradient Descent

**Theorem 18.** Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function that satisfies the following conditions:

1. $f$ is *Convex function*.
2. $f'$ is *Lipschitz Continuous* with Lipschitz constant of $f$, $L > 0$. In this, $f$ is called $L$*-Smooth*.
3. $f$ has at least one local minimizer $x^*$.

Then, $x^*$ is a Global minimizer of $\mathbb{R}$, and there exists a unique closed interval $M$ containing $x^*$ such that

$$\forall x \in M, t \notin M, \ \ f(x) = f(x^*) < f(t)$$

And, given initial point $x_0 \in \mathbb{R}$ and $0 < \gamma \leq \frac{1}{L}$, define a sequence $\{x_n\}$ inductively as follows:

$$x_{n+1} = x_t - \gamma \cdot f'(x_n)$$

Then, for any $N \in \mathbb{N}$,

$$f(x_N) - f(x^*) \leq \frac{|x_0 - x^*|^2}{2\gamma N}$$

*Proof.* Let $x^* \in \mathbb{R}$ be a local minimizer. That is, there exists a $\delta > 0$ such that $\forall t \in (x^* - \delta, x^* + \delta)$, $f(x^*) \leq f(t)$. Then,

$$0 \leq \lim_{t \to x^*+} \frac{f(x^*) - f(t)}{x^* - t} = f'(x^*) = \lim_{t \to x^*-} \frac{f(x^*) - f(t)}{x^* - t} \leq 0$$

thus, $f'(x^*) = 0$. And, by convextiy, $f'$ is monotonically inceasing. Now, The Fundametal Theorem of Calculus gives:

$$\forall x \in \mathbb{R}, \ \ f(x) = \int_{x^*}^{x} f'(t)dt + f(x^*) \geq f(x^*)$$

Therefore, $x^*$ is a Global minimizer of $f$.
Now, establish the closed interval $M$. Since $f'$ is Lipschitz Continuous, thus $f'$ is Continuous.
Let $D \overset{\text{def}}{=} \{x \in \mathbb{R} \mid f'(x) = 0\}$. (Note that: $x^* \in D$, thus $D$ is not emtpyset.)
$D$ is closed because: Let $\{x_n\}$ be a convergent sequence in $D$. That is, for all $n \in \mathbb{N}$, $f(x_n) = 0$. Then, by continuity,

$$f\left(\lim_{n \to \infty} x_n\right) = \lim_{n \to \infty} f(x_n) = 0$$

The limit of $\{x_n\}$ is contained in $D$, thus $D$ is closed.
And, $D$ is interval: i.e, for any $x \in (\inf D, \sup D)$, $x \in D$ because:
Suppose that there exists $x \in (\inf D, \sup D)$ such that $x \notin D$. That is, $f'(x) \neq 0$. This is Contradiction with Monotonicity.

To set error of upper bound, we make inequalities: Let $x, y \in \mathbb{R}$ be given.
The Fundamental Theorem of Calculus and $L-$Smooth condition gives:

$$\begin{aligned} f(y) - f(x) &= \int_x^y f'(t)dt = \int_0^1 f'(x + (y-x)u)(y-x)du = f'(x)(y-x) + \int_0^1 (f'(x + (y-x)u) - f'(x))(y-x)du \\ &\overset{2.}{\leq} f'(x)(y-x) + L \cdot |y-x|^2 \int_0^1 u \ \ du = f'(x)(y-x) + \frac{L}{2}|y-x|^2 \end{aligned}$$

For any $\lambda > 0$, Put $y = x - \lambda f'(x)$. Then,

$$f(x - \lambda f'(x)) \leq f(x) - f'(x)(\lambda f'(x)) + \frac{L}{2}|\lambda f'(x)|^2 = f(x) + \lambda\left(\frac{L\lambda}{2} - 1\right)|f'(x)|^2$$

Put $\lambda = \frac{1}{L}$, then

$$f\left(x - \frac{f'(x)}{L}\right) \leq f(x) - \frac{L}{2}|f'(x)|^2 \implies \frac{L}{2}|f'(x)|^2 \leq f(x) - f\left(x - \frac{f'(x)}{L}\right) \leq f(x) - \inf f$$

Meanwhile, the convexity gives: for any $x, y \in \mathbb{R}$,

$$f'(x)(y-x) \le f(y) - f(x) \le f'(y)(y-x)$$

since derivative of convex function increase monotonically. Put $z = y - \frac{1}{L}(f'(y) - f'(x))$. Then,

$$\begin{aligned}
f(x) - f(y) &= f(x) - f(z) + f(z) - f(y) \\
&\le f'(x)(x-z) + f'(y)(z-y) + \frac{L}{2}|z-y|^2 \\
&= f'(x)\left(x - y + \frac{1}{L}(f'(y) - f'(x))\right) - f'(y)\left(\frac{1}{L}(f'(y) - f'(x))\right) + \frac{L}{2}\left|\frac{1}{L}(f'(y) - f'(x))\right|^2 \\
&= f'(x)(x-y) - \frac{1}{L}|f'(y) - f'(x)|^2 + \frac{1}{2L}|f'(y) - f'(x)|^2 \\
&= f'(x)(x-y) - \frac{1}{2L}|f'(y) - f'(x)|^2
\end{aligned}$$

Now,

$$\begin{cases} \frac{1}{2L}|f'(y) - f'(x)|^2 \le f'(x)(x-y) - (f(x) - f(y)) \\ \frac{1}{2L}|f'(x) - f'(y)|^2 \le f'(y)(y-x) - (f(y) - f(z)) \end{cases} \implies \quad \frac{1}{L}|f'(y) - f'(x)|^2 \le (f'(y) - f'(x))(y-x)$$

Since above inequalities, we obtain that

$$\begin{aligned}
|x_{n+1} - x^*|^2 = |x_n - \gamma \cdot f'(x_n) - x^*|^2 &= |(x_n - x^*) - \gamma \cdot f'(x_n)|^2 \\
&= |x_n - x^*|^2 - 2\gamma|f'(x_n)| \cdot |x_n - x^*| + \gamma^2|f'(x_n)|^2 \\
&\le |x_n - x^*|^2 - 2\gamma\frac{1}{L}|f'(x_n)|^2 + \gamma^2|f'(x_n)|^2 \\
&= |x_n - x^*|^2 + \left(\gamma^2 - \frac{2\gamma}{L}\right)|f'(x_n)|^2 \le |x_n - x^*|^2
\end{aligned}$$

Thus, $|x_n - x^*|$ decrease as $n \to \infty$. That is, $|x_n - x^*| \le |x_0 - x^*|$ for all $n \in \mathbb{N}$.
Consider $x_{n+1}$ and $x_n$. First, we obtain

$$\begin{aligned}
f(x_{n+1}) &\le f(x_n) + f'(x_n)(x_{n+1} - x_n) + \frac{L}{2}|x_{n+1} - x_n|^2 \\
&= f(x_n) - \gamma|f'(x_n)|^2 + \frac{L}{2}\gamma^2|f'(x_n)|^2 \\
&= f(x_n) - \left(\gamma - \frac{L}{2}\gamma^2\right)|f'(x_n)|^2
\end{aligned}$$

Subtracting $f(x^*)$ above, then

$$f(x_{n+1}) - f(x^*) \le f(x_n) - f(x^*) - \left(\gamma - \frac{L}{2}\gamma^2\right)|f'(x_n)|^2$$

Meanwhile, Convexity gives

$$f(x_n) - f(x^*) \le f'(x_n)(x_n - x^*) \le |f'(x_n)||x_n - x^*| \le |f'(x_n)||x_0 - x^*|$$

Combining abvoe two inequalities,

$$f(x_{n+1}) - f(x^*) \le f(x_n) - f(x^*) - \left(\gamma - \frac{L}{2}\gamma^2\right) \cdot \frac{|f(x_n) - f(x^*)|^2}{|x_0 - x^*|^2}$$

Dividing Both Sides by $(f(x_{n+1}) - f(x^*))(f(x_n) - f(x^*))$,

$$\begin{aligned}
&\frac{1}{f(x_n) - f(x^*)} \le \frac{1}{f(x_{n+1}) - f(x^*)} - \left(\gamma - \frac{L}{2}\gamma^2\right) \cdot \frac{f(x_n) - f(x^*)}{f(x_{n+1}) - f(x^*)}\frac{1}{|x_0 - x^*|^2} \\
\implies &\left(\gamma - \frac{L}{2}\gamma^2\right) \cdot \frac{f(x_n) - f(x^*)}{f(x_{n+1}) - f(x^*)}\frac{1}{|x_0 - x^*|^2} \le \frac{1}{f(x_{n+1}) - f(x^*)} - \frac{1}{f(x_n) - f(x^*)} \\
\implies &\left(\gamma - \frac{L}{2}\gamma^2\right) \cdot \frac{1}{|x_0 - x^*|^2} \le \frac{1}{f(x_{n+1}) - f(x^*)} - \frac{1}{f(x_n) - f(x^*)} \\
\implies &\sum_{n=0}^{N-1}\left[\left(\gamma - \frac{L}{2}\gamma^2\right) \cdot \frac{1}{|x_0 - x^*|^2}\right] \le \sum_{n=0}^{N-1}\left[\frac{1}{f(x_{n+1}) - f(x^*)} - \frac{1}{f(x_n) - f(x^*)}\right] = \frac{1}{f(x_N) - f(x^*)} - \frac{1}{f(x_0) - f(x^*)}
\end{aligned}$$

Consequently,

$$\frac{2\gamma N}{|x_0 - x^*|^2} \leq N \cdot \left[ \left( \gamma - \frac{L}{2}\gamma^2 \right) \cdot \frac{1}{|x_0 - x^*|^2} \right] \leq \frac{1}{f(x_N) - f(x^*)} - \frac{1}{f(x_0) - f(x^*)} \leq \frac{1}{f(x_N) - f(x^*)}$$

Organizing the formula, as result:

$$f(x_N) - f(x^*) \leq \frac{|x_0 - x^*|^2}{2\gamma N}$$

□

## 11.6 Integral

### 11.6.1 Inequality of Riemann-Stieltjes Integral

Let $p, q \geq 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, and functions lying on $[a, b]$.

**Lemma 9.** Let $f, g \in \mathcal{R}(\alpha)$ with $f, g \geq 0$, and $\displaystyle\int_a^b [f(x)]^p d\alpha = \int_a^b [g(x)]^q d\alpha = 1$. Then, $\displaystyle\int_a^b f(x)g(x)d\alpha \leq 1$.

*Proof.* For any $x \in [a, b]$, the Young's Inequality gives

$$0 \leq f(x)g(x) \leq \frac{[f(x)]^p}{p} + \frac{[g(x)]^q}{q}$$

Now,

$$\int_a^b f(x)g(x)d\alpha \leq \int_a^b \frac{[f(x)]^p}{p} + \frac{[g(x)]^q}{q} d\alpha = \frac{1}{p}\int_a^b [f(x)]^p d\alpha + \frac{1}{q}\int_a^b [g(x)]^q d\alpha = \frac{1}{p} + \frac{1}{q} = 1$$

□

**Definition 12.** Let $f \in \mathcal{R}(\alpha)$. Define a ***Norm*** of $f$:

$$\|f\|_p \overset{\text{def}}{=} \left(\int_a^b |f(x)|^p d\alpha\right)^{\frac{1}{p}}$$

This becomes actually norm of set of Stieltjes Integrable functions, $\mathcal{F} \overset{\text{def}}{=} \{f : [a, b] \to \mathbb{C} \mid f \in \mathcal{R}(\alpha)\}$.

**Lemma 10.** *Hölder's Inequality*
Let $f, g \in \mathcal{F}$. Then,

$$\left|\int_a^b f(x)g(x)d\alpha\right| \leq \left[\int_a^b |f(x)|^p d\alpha\right]^{\frac{1}{p}} \cdot \left[\int_a^b |g(x)|^q d\alpha\right]^{\frac{1}{q}}$$

*Proof.* Use above definition, Rewrite:

$$\|f\|_p^p = \int_a^b |f(x)|^p d\alpha, \quad \|g\|_q^q = \int_a^b |g(x)|^q d\alpha$$

Now, we can make the condition of above lemma,

$$\int_a^b \left[\frac{|f(x)|}{\|f\|_p}\right]^p d\alpha = \frac{1}{\|f\|_p^p} \cdot \int_a^b |f(x)|^p d\alpha = 1, \quad \int_a^b \left[\frac{|g(x)|}{\|g\|_q}\right]^q d\alpha = \frac{1}{\|g\|_q^q} \cdot \int_a^b |g(x)|^q d\alpha = 1$$

And apply this,

$$\int_a^b \frac{|f(x)| \cdot |g(x)|}{\|f\|_p \|g\|_q} d\alpha \leq 1 \implies \int_a^b |f(x)||g(x)|d\alpha \leq \|f\|_p \|g\|_q = \left[\int_a^b |f(x)|^p d\alpha\right]^{\frac{1}{p}} \cdot \left[\int_a^b |g(x)|^q d\alpha\right]^{\frac{1}{q}}$$

Finally, the general property of integral of product gives

$$\left|\int_a^b f(x)g(x)d\alpha\right| \leq \int_a^b |f(x)||g(x)|d\alpha \leq \|f\|_p \|g\|_q = \left[\int_a^b |f(x)|^p d\alpha\right]^{\frac{1}{p}} \cdot \left[\int_a^b |g(x)|^q d\alpha\right]^{\frac{1}{q}}$$

□

> **Theorem 19.** *Minkowski inequality*
> Let $f, g \in \mathcal{F}$. Then, for any $p \geq 1$, $\|f+g\|_p \leq \|f\|_p + \|g\|_p$.

*Proof.*

$$\begin{aligned}
\|f+g\|_p^p &= \int_a^b |f+g|^p d\alpha = \int_a^b |f+g||f+g|^{p-1} d\alpha \\
&\leq \int_a^b [|f|+|g|]|f+g|^{p-1} d\alpha \\
&= \int_a^b |f||f+g|^{p-1} d\alpha + \int_a^b |g||f+g|^{p-1} d\alpha \\
&\overset{\text{Hölder}}{\leq} \left[\int_a^b |f|^p d\alpha\right]^{\frac{1}{p}} \left[\int_a^b |f+g|^{(p-1)\frac{p}{p-1}} d\alpha\right]^{\frac{p-1}{p}} + \left[\int_a^b |g|^p d\alpha\right]^{\frac{1}{p}} \left[\int_a^b |f+g|^{(p-1)\frac{p}{p-1}} d\alpha\right]^{\frac{p-1}{p}} \\
&= \left[\int_a^b |f+g|^p d\alpha\right]^{\frac{p-1}{p}} \left(\left[\int_a^b |f|^p d\alpha\right]^{\frac{1}{p}} + \left[\int_a^b |g|^p d\alpha\right]^{\frac{1}{p}}\right) = \|f+g\|_p^{p-1} \cdot (\|f\|_p + \|g\|_p)
\end{aligned}$$

Now,

$$\|f+g\|_p^p \cdot \|f+g\|_p^{1-p} = \|f+g\|_p \leq \|f\|_p + \|g\|_p$$

□

## Chapter 12

# Measure

Chapter 13

# Complex Analysis

Chapter 14

# Differential Geometry

Chapter 15

# Differential Equation

# Chapter 16

# Spaces

## 16.1 $\mathbb{R}^n$

### 16.1.1 Inner Product in $\mathbb{R}$

### 16.1.2 $p$-norm in $\mathbb{R}^n$

**Definition 13.** Let $\mathbb{R}^n$ be given. Define *p-norm* on $\mathbb{R}^n$ as:

$$d_p : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} : (\mathbf{x}, \mathbf{y}) \mapsto \left( \sum_{i=1}^{n} |x_i - y_i|^p \right)^{\frac{1}{p}}, \quad (\mathbf{x} = (x_1, \dots, x_n),\ \ \mathbf{y} = (y_1, \dots, y_n))$$

where $p \in [1, \infty]$. In particular, $p$-norm is a ***Metric***, being ***Minkowski inequality***.

**Lemma 11.** *Young's inequality*
Let $u, v > 0$, and $p, q \in [1, \infty]$ such that $\frac{1}{p} + \frac{1}{q} = 1$.
Then,

$$uv \leq \frac{1}{p}u^p + \frac{1}{q}v^q$$

*Proof.* Since $f(x) = \log x$ is concave, we obtain

$$\forall \lambda \in [0, 1],\ \ \lambda f(x)(1 - \lambda)f(y) \leq f(\lambda x + (1 - \lambda)y)$$

thus,

$$\log\left(\frac{1}{p}u^p + \frac{1}{q}v^q\right) \geq \frac{1}{p}\log(u^p) + \frac{1}{q}\log(v^q) = \log(uv)$$

Since $\exp(x)$ increasing, we get

$$\exp\left(\log\left(\frac{1}{p}u^p + \frac{1}{q}v^q\right)\right) \geq \exp(\log(uv))$$

i.e.,

$$uv \leq \frac{1}{p}u^p + \frac{1}{q}v^q$$

$\square$

**Lemma 12.** *Holder's inequality*
Let $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ be give, and $p, q \in [1, \infty]$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then,

$$\sum_{i=1}^{n} |x_i y_i| \leq \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} \cdot \left(\sum_{i=1}^{n} |y_i|^q\right)^{\frac{1}{q}}$$

***Proof.*** Denote that

$$\|x\|_p \overset{\textbf{def}}{=} \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}}$$

Then, since young's inequality, for each $i \in \{1, 2, \dots, n\}$,

$$\frac{|x_i|}{\|x\|_p} \cdot \frac{|y_i|}{\|y\|_p} \leq \frac{1}{p} \cdot \frac{|x_i|^p}{\|x\|_p^p} + \frac{1}{q} \cdot \frac{|y_i|^q}{\|y\|_q^q}$$

Summing for all $i = 1, 2, \dots, n$:

$$\frac{1}{\|x\|_p \|y\|_q} \cdot \sum_{i=1}^{n} |x_i y_i| \leq \frac{1}{p} + \frac{1}{q} = 1$$

Therefore,

$$\sum_{i=1}^{n} |x_i y_i| \leq \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} \cdot \left(\sum_{i=1}^{n} |y_i|^q\right)^{\frac{1}{q}}$$

□

**Theorem 20.** *Minkowski inequality*
Given complex-valued sequences $\{x_i\}_{i=1}^n$ and $\{y_i\}_{i=1}^n$,

$$\left[\sum_{i=1}^{n} |x_i + y_i|^p\right]^{\frac{1}{p}} \leq \left[\sum_{i=1}^{n} |x_i|^p\right]^{\frac{1}{p}} + \left[\sum_{i=1}^{n} |y_i|^p\right]^{\frac{1}{p}}$$

***Proof.*** Denote

$$|x_i + y_i|^p = |x_i + y_i| \cdot |x_i + y_i|^{p-1}$$

Then,

$$\begin{aligned}
\sum_{i=1}^{n} |x_i + y_i|^p &= \sum_{i=1}^{n} |x_i + y_i| \cdot |x_i + y_i|^{p-1} \\
&= \sum_{i=1}^{n} (|x_i| + |y_i|) \cdot |x_i + y_i|^{p-1} \\
&= \sum_{i=1}^{n} |x_i| \cdot |x_i + y_i|^{p-1} + \sum_{i=1}^{n} |y_i| \cdot |x_i + y_i|^{p-1} \\
&\overset{\text{Hölder}}{\leq} \left[\sum_{i=1}^{n} |x_i|^p\right]^{\frac{1}{p}} \cdot \left[\sum_{i=1}^{n} |x_i + y_i|^p\right]^{\frac{p-1}{p}} + \left[\sum_{i=1}^{n} |y_i|^p\right]^{\frac{1}{p}} \cdot \left[\sum_{i=1}^{n} |x_i + y_i|^p\right]^{\frac{p-1}{p}} \\
&= \left[\left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} |y_i|^p\right)^{\frac{1}{p}}\right] \cdot \left[\sum_{i=1}^{n} |x_i + y_i|^p\right]^{\frac{p-1}{p}}
\end{aligned}$$

Now, Divide each side as $[\sum_{i=1}^n |x_i + y_i|^p]^{\frac{p-1}{p}}$, then we obtain

$$\left[\sum_{i=1}^n |x_i + y_i|^p\right]^{1-\frac{p-1}{p}} = \left[\sum_{i=1}^n |x_i + y_i|^p\right]^{\frac{1}{p}} \leq \left[\left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^n |y_i|^p\right)^{\frac{1}{p}}\right]$$

□

**Theorem 21.** Let $d_{p_1}, d_{p_2}$ are $p$-norm on $\mathbb{R}^n$ with $1 \leq p_1 < p_2 \leq \infty$. Then,

$$\exists C > 0 \text{ s.t. } \forall x, y \in \mathbb{R}^n,\ d_{p_2}(x,y) \leq d_{p_1}(x,y) \leq C d_{p_2}(x,y)$$

In particular, $C = n^{\frac{1}{p_1} - \frac{1}{p_2}}$.

***Proof.*** Let $p_1 < p_2$.
For show that first-inequality,

$$1 = \sum_{i=1}^n \left[\frac{|x_i - y_i|}{[\sum_{i=1}^n |x_i - y_i|^{p_2}]^{\frac{1}{p_2}}}\right]^{p_2} \leq \sum_{i=1}^n \left[\frac{|x_i - y_i|}{[\sum_{i=1}^n |x_i - y_i|^{p_2}]^{\frac{1}{p_2}}}\right]^{p_1} = \frac{\sum_{i=1}^n |x_i - y_i|^{p_1}}{[\sum_{i=1}^n |x_i - y_i|^{p_2}]^{\frac{p_1}{p_2}}} = \left[\frac{[\sum_{i=1}^n |x_i - y_i|^{p_1}]^{\frac{1}{p_1}}}{[\sum_{i=1}^n |x_i - y_i|^{p_2}]^{\frac{1}{p_2}}}\right]^{p_1}$$

Thus, we obtain that:

$$1 \leq \left[\frac{[\sum_{i=1}^n |x_i - y_i|^{p_1}]^{\frac{1}{p_1}}}{[\sum_{i=1}^n |x_i - y_i|^{p_2}]^{\frac{1}{p_2}}}\right]^{p_1} \iff 1 \leq \frac{[\sum_{i=1}^n |x_i - y_i|^{p_1}]^{\frac{1}{p_1}}}{[\sum_{i=1}^n |x_i - y_i|^{p_2}]^{\frac{1}{p_2}}} \iff \left[\sum_{i=1}^n |x_i - y_i|^{p_2}\right]^{\frac{1}{p_2}} \leq \left[\sum_{i=1}^n |x_i - y_i|^{p_1}\right]^{\frac{1}{p_1}}$$

For show that second-inequality, using Hölder's inequality.

$$\begin{aligned}(d_{p_1}(x,y))^{p_1} &= \sum_{i=1}^n |x_i - y_i|^{p_1} = \sum_{i=1}^n |x_i - y_i|^{p_1} \cdot 1 \\ &\overset{\text{Hölder}}{\leq} \left[\sum_{i=1}^n \left(|x_i - y_i|^{p_1 \cdot \frac{p_2}{p_1}}\right)\right]^{\frac{p_1}{p_2}} \cdot \left[\sum_{i=1}^n 1^{\frac{p_2}{p_2 - p_1}}\right]^{1-\frac{p_1}{p_2}} = \left[\sum_{i=1}^n (|x_i - y_i|^{p_2})\right]^{\frac{p_1}{p_2}} \cdot n^{1-\frac{p_1}{p_2}}\end{aligned}$$

Taking the $\frac{1}{p_1}$-th power of both sides, then

$$d_{p_1}(x,y) \leq \left[\sum_{i=1}^n (|x_i - y_i|^{p_2})\right]^{\frac{1}{p_2}} \cdot n^{\frac{1}{p_1} - \frac{1}{p_2}} = n^{\frac{1}{p_1} - \frac{1}{p_2}} \cdot d_{p_2}(x,y)$$

□

**Corollary 4.** Let $\mathbb{R}^n$ be given as a set, and $d_{p_1}, d_{p_2} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ are $p$-norm on $\mathbb{R}^n$. Then,

$$\mathcal{T}_{d_{p_1}} = \mathcal{T}_{d_{p_2}}$$

For every $p \geq 1$, the metric space $(\mathbb{R}^n, d_p)$ induces the same topology as the product topology on $\mathbb{R}^n$. In particular, $\mathbb{R}^n$ with the product topology coincides with $\mathbb{R}^n$ endowed with any $p$-norm.

### 16.1.3 Open and Closed set in $\mathbb{R}^n$

**Definition 14.** For $p \in [1, \infty]$, define $p$-***Ball*** in $\mathbb{R}^n$ as:

$$B_p(x, r) \overset{\text{def}}{=} \{y \in \mathbb{R}^n : \|x - y\|_p < r\}$$

Since all $p$-norms are equivalent, for any $p \in [1, \infty]$, the collection

$$\beta_p \overset{\text{def}}{=} \left\{B_p(x, r) \mid x \in \mathbb{Q}^n,\ \ r \in \mathbb{Q}^+\right\} \cup \{\emptyset, \mathbb{R}^n\}$$

is Countable basis of $\mathbb{R}^n$. Immediately, we obtain:

**Lemma 13.** Every open set in $\mathbb{R}^n$ is a countable union of $p$-Balls.

We call $2$-Ball the ***Ball***, and $\infty$-Ball the ***Cube***.

**Theorem 22.** Let $U \subseteq \mathbb{R}^n$ be an open set. Then, $U$ is a countable union of almost disjoint closed cubes.

## 16.2 Topological Vector Space

## 16.3 Hilbert Space

**Definition 15.** Complete Inner product Vector Space is called *Hilbert Space*.

### 16.3.1 Hilbert Space in $\mathbb{R}^\omega$

**Definition 16.** Define $\mathbb{R}^\omega \overset{\text{def}}{=} \prod_{i=1}^{\infty} \mathbb{R}$ as the countable product of Euclidean space $\mathbb{R}$ with product topology.
And define $\mathbb{H} \overset{\text{def}}{=} \left\{ \{x_n\}_{n=1}^{\infty} \;\middle|\; \sum_{n=1}^{\infty} x_n^2 < \infty \right\} \subset \mathbb{R}^\omega$, ***Metric*** on $\mathbb{H}$ as $\mu : \mathbb{H} \times \mathbb{H} \to \mathbb{R} : (\{x_n\}, \{y_n\}) \mapsto \sqrt{\sum_{i=1}^{\infty} (x_i - y_i)^2}$.
The Metric Space $(\mathbb{H}, \mu)$ is called *Hilbert Space* or $l_2$ *Space*.

Define the operations elementwise; then $(\mathbb{H}, +, \times)$ is a Vector Space over $\mathbb{R}$.
Moreover, $\mathbb{H}$ is Complete Metric Space and Inner product Vector Space.

**Lemma 14.** $\mu : \mathbb{H} \times \mathbb{H} \to \mathbb{R} : (\{x_n\}, \{y_n\}) \mapsto \sqrt{\sum_{i=1}^{\infty} (x_i - y_i)^2}$ is Metric function induced by the inner product.

*Proof.* We know that $\mathbb{R}^\omega$ is Vector Space. Moreover, $\mathbb{H} \subset \mathbb{R}^\omega$ is Subspace. Using subspace criteria:

$S \subset V$ is Subspace of Vector Space $V$ if and only if $0 \in S$ and for any $x, y \in S$ and $a \in F$, $ax + y \in S$.

Clearly, $\{0\} \in \mathbb{H}$. Let $a \in \mathbb{R}$ and $\{x_n\}, \{y_n\} \in \mathbb{H}$ be given. Then, $a\{x_n\} + \{y_n\} = \{ax_n + y_n\} \in \mathbb{H}$ because:

$$\sum_{i=1}^{\infty} (ax_i + y_i)^2 = \sum_{i=1}^{\infty} \left[ a^2 x_i^2 + 2a x_i y_i + y_i^2 \right] \overset{(*)}{=} a^2 \sum_{i=1}^{\infty} x_i^2 + 2a \sum_{i=1}^{\infty} x_i y_i + \sum_{i=1}^{\infty} y_i^2 < \infty$$

The $(*)$ given by:

$$\sum_{i=1}^{\infty} |x_i y_i| = \sum_{i=1}^{\infty} |x_i||y_i| \le \sum_{i=1}^{\infty} (\max(|x_i|, |y_i|))^2 \le \sum_{i=1}^{\infty} (x_n^2 + y_n^2) = \sum_{i=1}^{\infty} x_n^2 + \sum_{i=1}^{\infty} y_n^2 < \infty \tag{$*$}$$

Thus $\mathbb{H}$ is Vector Space over $\mathbb{R}$. Now, define ***inner product*** on $\mathbb{H}$ as:

$$\langle \;\cdot\; , \;\cdot\; \rangle : \mathbb{H} \times \mathbb{H} \to \mathbb{R} : (\{x_n\}, \{y_n\}) \mapsto \sum_{i=1}^{\infty} x_i y_i$$

This definition is well-defined since $(*)$. And, Linearity in first:

$$\langle a\{x_n\} + \{y_n\}, \{z_n\} \rangle = \langle \{ax_n + y_n\}, \{z_n\} \rangle = \sum_{i=1}^{\infty} (ax_i + y_i) z_i = a \sum_{i=1}^{\infty} x_i z_i + \sum_{i=1}^{\infty} y_i z_i = a \langle \{x_n\}, \{z_n\} \rangle + \langle \{y_n\}, \{z_n\} \rangle$$

The other conditions are clear. Thus, $(\mathbb{H}, \langle \cdot, \cdot \rangle)$ is ***inner product space***.
Using ***ineer product***, define the ***Norm*** on $\mathbb{H}$ as:

$$\| \cdot \| : \mathbb{H} \to \mathbb{R} : \{x_n\} \mapsto \sqrt{\langle \{x_n\}, \{x_n\} \rangle}$$

Finally, define ***Metric*** on $\mathbb{H}$ as:

$$\mu : \mathbb{H} \times \mathbb{H} \to \mathbb{R} : (\{x_n\}, \{y_n\}) \mapsto \|\{x_n\} - \{y_n\}\| = \sqrt{\sum_{i=1}^{\infty} (x_i - y_i)^2}$$

□

**Theorem 23.** *Hilbert Space* is Separable.

*Proof.* For each $n \in \mathbb{N}$, define $D_n \overset{\text{def}}{=} \{\{p_n\} \mid p_i \in \mathbb{Q},\ p_{n+1} = p_{n+1} = \cdots = 0\}$ and $D \overset{\text{def}}{=} \bigcup_{n\in\mathbb{N}} D_n$.

Then, $D$ is countable set. We will show that $\overline{D} = \mathbb{H}$.
Let $\epsilon > 0$ and $\{x_n\} \in \mathbb{H}$ be given. Since convergence, there exists $N \in \mathbb{N}$ such that

$$\sum_{i=N+1}^{\infty} x_i^2 = \sum_{i=1}^{\infty} x_i^2 - \sum_{i=1}^{N} x_i^2 < \frac{\epsilon^2}{2}$$

Since density of Rationals, put each $i = 1, 2, \ldots, N$, $p_i \in \mathbb{Q}$ $|x_i - p_i| < \dfrac{\epsilon}{\sqrt{2N}}$ and $p_i = 0$ for $i \geq N+1$.
Then, $\{p_n\} \in D_n \subset D$ and

$$\mu\left(\{x_n\}, \{p_n\}\right) = \sqrt{\sum_{i=1}^{N}(x_i - p_i)^2 + \sum_{i=N+1}^{\infty}(x_i - p_i)^2} = \sqrt{\sum_{i=1}^{N}(x_i - p_i)^2 + \sum_{i=N+1}^{\infty} x_i^2} < \sqrt{N \cdot \frac{\epsilon^2}{2N} + \frac{\epsilon^2}{2}} = \epsilon$$

□

**Corollary 5.** *Hilbert Space* is Second-Countable.

**Theorem 24.** *Hilbert Space* is Complete.

*Proof.* Let $\{\{x_{n,i}\}_{i=1}^{\infty}\}_{n=1}^{\infty}$ be a Cauchy sequence in $\mathbb{H}$. For any fixed $n, m \in \mathbb{N}$ and for each $j \in \mathbb{N}$,

$$|x_{n,j} - x_{m,j}| < \mu(\{x_{n,i}\}, \{x_{m,i}\}) = \sqrt{\sum_{i=1}^{\infty}(x_{n,i} - x_{m,i})^2}$$

That is, for each $j \in \mathbb{N}$, $\{x_{n,j}\}$ is Cauchy sequence in $\mathbb{R}$. Since $\mathbb{R}$ is Complete, put $y_j \overset{\text{def}}{=} \lim_{n\to\infty} x_{n,j}$, each $j \in \mathbb{N}$.
Let $\epsilon > 0$ be given. Then, there exists $N \in \mathbb{N}$ such that $n, m \geq N \implies \mu(\{x_{n,i}\}, \{x_{m,i}\}) < \dfrac{\epsilon}{2}$.
Meanwhile, for each $k \in \mathbb{N}$,

$$\sum_{i=1}^{k}(x_{n,i} - x_{m,i})^2 \leq \sum_{i=1}^{\infty}(x_{n,i} - x_{m,i})^2 = [\mu(\{x_{n,i}\}, \{x_{m,i}\})]^2$$

Thus, $n, m \geq N \implies \displaystyle\sum_{i=1}^{k}(x_{n,i} - x_{m,i})^2 < \left(\frac{\epsilon}{2}\right)^2$, for each $k \in \mathbb{N}$.
Taking limit to $m$, then $n \geq N \implies \displaystyle\lim_{m\to\infty}\left(\sum_{i=1}^{k}(x_{n,i} - x_{m,i})^2\right) = \sum_{i=1}^{k}\left(x_{n,i} - \lim_{m\to\infty} x_{m,i}\right)^2 = \sum_{i=1}^{k}(x_{n,i} - y_i)^2 < \left(\frac{\epsilon}{2}\right)^2$.
And, for all $k \in \mathbb{N}$,

$$\sum_{i=1}^{k} y_i^2 = \sum_{i=1}^{k}(2(x_{n,i}^2 + (x_{n,i} - y_i)^2)) \leq 2\|\{x_{n,i}\}_{i=1}^{\infty}\|^2 + \left(\frac{\epsilon}{2}\right)^2$$

Thus $\{y_i\} \in \mathbb{H}$. As a result,

$$n \geq N \implies \mu(\{x_n\}, \{y_n\}) = \sqrt{\sum_{i=1}^{\infty}(x_{n,i} - y_i)^2} = \sqrt{\lim_{k\to\infty}\sum_{i=1}^{k}(x_{n,i} - y_i)^2} < \frac{\epsilon}{2}$$

□

**Theorem 25.** $\mathbb{H} \subset \mathbb{R}^\omega$ with subbspace topology is Metrizable.

*Proof.* We will use two Lemmas:

**Lemma 15.** Countable Product of Metric Space is Metrizable.

*Proof.* Let $(X_i, d_i)$ be a metric Space, for each $i \in \mathbb{N}$.
If $d : X \times X \to \mathbb{R}$ is a Metric, then $\dfrac{d}{1+d}$ is also Metric, because

$$\underbrace{\frac{d(x,z)}{1+d(x,z)}}_{\frac{x}{1+x}} \underset{\uparrow \text{ increasing}}{\leq} \frac{d(x,y)+d(y,z)}{1+d(x,y)+d(y,z)} \underset{\substack{\uparrow \\ d \geq 0}}{\leq} \frac{d(x,y)}{1+d(x,y)} + \frac{d(y,z)}{1+d(y,z)} \tag{$*$}$$

Using this fact, define

$$d_\Pi : \prod X_i \times \prod X_i \to \mathbb{R} : (\{x_n\}_{n=1}^\infty, \{y_n\}_{n=1}^\infty) \mapsto \sum_{i=1}^\infty \left[ \frac{1}{2^i} \cdot \frac{d_i(x_i, y_i)}{1 + d_i(x_i, y_i)} \right]$$

Then $d_\Pi$ is a Metric because:

$$\begin{aligned} d_\Pi \left( \{x_n\}_{n=1}^\infty, \{z_n\}_{n=1}^\infty \right) &= \sum_{i=1}^\infty \left[ \frac{1}{2^i} \cdot \frac{d_i(x_i, z_i)}{1 + d_i(x_i, z_i)} \right] \\ &\overset{(*)}{\leq} \sum_{i=1}^\infty \left[ \frac{1}{2^i} \cdot \left( \frac{d_i(x_i, y_i)}{1 + d_i(x_i, y_i)} + \frac{d_i(y_i, z_i)}{1 + d_i(y_i, z_i)} \right) \right] \\ &= \sum_{i=1}^\infty \left[ \frac{1}{2^i} \cdot \frac{d_i(x_i, y_i)}{1 + d_i(x_i, y_i)} \right] + \sum_{i=1}^\infty \left[ \frac{1}{2^i} \cdot \frac{d_i(y_i, z_i)}{1 + d_i(y_i, z_i)} \right] \\ &= d_\Pi \left( \{x_n\}_{n=1}^\infty, \{y_n\}_{n=1}^\infty \right) + d_\Pi \left( \{y_n\}_{n=1}^\infty, \{z_n\}_{n=1}^\infty \right) \end{aligned}$$

Reflexity and symmetry are clear. □

**Lemma 16.** Metrizable is Hereditary.

Proof omitted.
Consequently, since $\mathbb{H} \subset \mathbb{R}^\omega$ is a subspace of a metric space, it is metrizable. □

## 16.4 Banach Space

## 16.5 $L_p$ Space

## 16.6 $l_p$ Space

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