

Math Note

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This paper covers several topics in undergraduate mathematics.

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2.

Chapter 1

Set Theory

1.1 Map

Definition 1. Let X, Y are sets. Define a *function* X to Y is a relation

$$f \subset X \times Y$$

such that

1. For any $x \in X$, there exists $y \in Y$ such that $(x, y) \in f$.
2. If $(x, y) \in f$ and $(x, z) \in f$, then $y = z$.

Denote f as:

$$f : X \rightarrow Y : x \mapsto f(x)$$

Define *Image* of f by $A \subset X$:

$$f[A] \stackrel{\text{def}}{=} \{f(a) \mid a \in A\} \subset Y$$

And, *Preimage* of f by $B \subset Y$:

$$f^{-1}[B] \stackrel{\text{def}}{=} \{x \in X \mid f(x) \in B\} \subset X$$

$f : X \rightarrow Y$ is *Injective* if: $f(x_1) = f(x_2) \implies x_1 = x_2$.

$f : X \rightarrow Y$ is *Surjective* if: $\forall y \in Y, \exists x \in X$ s.t. $f(x) = y$.

If f is injective and surjective, called *bijective*.

If f is bijective, then define *inverse* of f as:

$$f^{-1} : Y \rightarrow X : y \mapsto x$$

where $x \in X$ is the unique elements of X such that $f(x) = y$.

Theorem 1. Let $f : X \rightarrow Y$ be a function. Then,

1. There exists $g : Y \rightarrow X$ such that $g \circ f : X \rightarrow X$ be an identity function if and only if f is injective.
2. There exists $h : Y \rightarrow X$ such that $f \circ h : Y \rightarrow Y$ be an identity function if and only if f is surjective.

Proof.

1. \implies)

Assume that $f(x_1) = f(x_2)$. Then, existence of left inverse, $g(f(x_1)) = g(f(x_2)) \implies x_1 = x_2$. Thus f injective.

1. \impliedby)

Since f is injection, for any $y \in f[X]$, there exists a unique element $x_y \in X$ such that $f(x) = y$. Now, define

$$g : Y \rightarrow X : y \mapsto \begin{cases} x_y & y \in f[X] \\ \text{any element in } X & y \notin f[X] \end{cases}$$

Then, for any $x \in X$, $g(f(x)) = g(y) = x$.

2. \implies)

Let $y \in Y$ be given. Since existence of right inverse, $f(h(y)) = y$ where $h(y) \in X$. Thus, f is surjective.

2. \impliedby)

For any $y \in Y$, there exists a $x_y \in X$ such that $f(x_y) = y$. Now, define

$$h : Y \rightarrow X : y \mapsto x_y$$

Then, for any $y \in Y$, $f \circ h(y) = f(x_y) = y$. Thus, $f \circ h$ is identity. □

Corollary 1. Let $f : X \rightarrow Y$ be a function, $\text{id}_X : X \rightarrow X : x \mapsto x$, and $\text{id}_Y : Y \rightarrow Y : y \mapsto y$.

There exists a $f^{-1} : Y \rightarrow X$ such that $f^{-1} \circ f = \text{id}_X$ and $f \circ f^{-1} = \text{id}_Y$ if and only if f is bijection.

Proof. If f is bijection, then there exists left inverse g and right inverse h .

Enough To Show that: $g = h$. Since $g \circ f = \text{id}_X$ and $f \circ h = \text{id}_Y$,

$g \circ f \circ h = g \circ \text{id}_Y$, thus $h = g$. □

Theorem 2. Let X, Y, Z are sets, $f : X \rightarrow Y$, $g : Y \rightarrow Z$ and $A \subset X, B \subset Y, C \subset Z$. Then followings are hold:

1. $g[f[A]] = (g \circ f)[A]$.
2. $f^{-1}[g^{-1}[C]] = (g \circ f)^{-1}[C]$.

Proof.

1. It is clear by definition of image:

$$\begin{aligned} g[f[A]] &\stackrel{\text{def}}{=} g[\{f(a) \mid a \in A\}] = \{g(b) \mid b \in \{f(a) \mid a \in A\}\} \\ &= \{g(b) \mid b = f(a) \text{ for some } a \in A\} = \{g(f(a)) \mid \text{for some } a \in A\} = \{g(f(a)) \mid a \in A\} \end{aligned}$$

2. It is not clear,

$$f^{-1}[g^{-1}[C]] \stackrel{\text{def}}{=} f^{-1}[\{b \in Y \mid g(b) \in C\}] = \{a \in X \mid f(a) \in \{b \in Y \mid g(b) \in C\}\} = \{a \in X \mid g(f(a)) \in C\} = (g \circ f)^{-1}[C]$$

□

Proposition 1. Let $f : X \rightarrow Y$ be a function, $A, B \subset X$ and $C, D \subset Y$.

1. If $A \subset B$, then $f[A] \subset f[B]$.
2. If $C \subset D$, then $f^{-1}[C] \subset f^{-1}[D]$

Proof.

$$\begin{aligned} y \in f[A] &\implies y = f(a) \text{ for some } a \in A \stackrel{A \subset B}{\implies} y = f(a) \text{ for some } a \in B \implies y \in f[B] \\ x \in f^{-1}[C] &\implies f(x) \in C \stackrel{C \subset D}{\implies} f(x) \in D \implies x \in f^{-1}[D] \end{aligned}$$

□

Lemma 1. Let two set X, Y be given, and $A \subset X$, $B \subset Y$, $f: X \rightarrow Y$. Then followings are holds:

1. $f^{-1}[f[A]] \supseteq A$, and equality holds if f one-to-one.
2. $f[f^{-1}[B]] \subseteq B$, and equality holds if f onto.
3. $f^{-1}[Y \setminus B] = X \setminus f^{-1}[B]$
4. $f[X] \setminus f[A] \subseteq f[X \setminus A]$, and equality holds if f one-to-one.

Proof. Proof of 4.

$$\begin{aligned} y \in f[X] \setminus f[A] &\iff y \in f[X] \text{ and } y \notin f[A] \\ &\iff \exists x \in X \text{ s.t. } y = f(x) \text{ and } \forall x \in A, y \neq f(x) \\ &\stackrel{(*)}{\implies} \exists x \in X \setminus A \text{ s.t. } y = f(x) \\ &\iff y \in f[X \setminus A] \end{aligned}$$

If f is injection, then Left Direction of $(*)$ be true: $\exists! x \in X \setminus A$ s.t. $y = f(x)$. □

Chapter 2

Group Theory

2.1 Isomorphism Theorems

Theorem 3. The First Isomorphism Theorem

Let $\varphi : G \rightarrow H$ be a Group-Homomorphism. Then,

$$G / \ker \varphi \cong \varphi[G]$$



Proof. Let $\pi : G \rightarrow G/\ker \varphi : x \mapsto x + \ker \varphi$. Then, the map $\phi : G/\ker \varphi \rightarrow \varphi[G] : a + \ker \varphi \mapsto \varphi(a)$ is isomorphism. Well-defined and Injective:

$$a + \ker \varphi = b + \ker \varphi \iff a - b \in \ker \varphi \iff \varphi(a - b) = \varphi(a) - \varphi(b) = 0$$

Surjective is clear. □

Theorem 4. The Second Isomorphism Theorem

Let G be a Group, and $H \leq G$, $N \trianglelefteq G$. Then,

$$HN/N \cong H/(H \cap N)$$

Proof. HN be a subgroup of G , being

$$HN = \bigcup_{h \in H} hN \stackrel{N \trianglelefteq G}{=} \bigcup_{h \in H} Nh = NH$$

And, $N \leq HN$ is clear, thus $N \trianglelefteq HN$.

Meanwhile, $H \cap N$ be a Normal Subgroup of H : for any $h \in H, n \in H \cap N$, $hnh^{-1} \in N$ because N is normal, and $hnh^{-1} \in H$ since h, n contained in H . Thus, $hnh^{-1} \in H \cap N$, this implies $H \cap N$ be a Normal of H .

Now, Define a Map:

$$\varphi : H \rightarrow HN/N : h \mapsto hN$$

Clearly, this map is Well-Defined and Homomorphism. And,

$$\ker \varphi = \varphi^{-1}[1] = \{h \in H \mid hN = N\} = \{h \in H \mid h \in N\} = H \cap N$$

Thus, since The 1st Isomorphism Theorem,

$$HN/N \cong H/(H \cap N)$$

□

Theorem 5. The Third Isomorphism Theorem

Let G be a Group, and $H, K \trianglelefteq G$ with $H \leq K$. Then, $K/H \trianglelefteq G/H$ and

$$(G/H)/(K/H) \cong (G/K)$$

Proof. First, show that $K/H \trianglelefteq G/H$. Let $kH \in K/H$ and $gH \in G/H$. Then,

$$(gH)(kH)(gH)^{-1} = (gH)(kH)(g^{-1}H) = (gkg^{-1})H \in K/H$$

since $gkg^{-1} \in K$, being $K \trianglelefteq G$. Now, Define a map:

$$\varphi : G/H \rightarrow G/K : gH \mapsto gK$$

1. Well-Defined.

$$g_1H = g_2H \iff g_1^{-1}g_2 \in H \xrightarrow{H \leq K} g_1^{-1}g_2 \in K \iff g_1K = g_2K$$

2. Homomorphism.

Clearly, for any $g_1H, g_2H \in G/H$,

$$\varphi(g_1H g_2H) = \varphi(g_1g_2H) = g_1g_2K = g_1K g_2K = \varphi(g_1H) \varphi(g_2H)$$

3. Surjection. Let $gK \in G/K$ be given. Then, clearly, $\varphi(gH) = gK$.

4. Kernel.

$$\ker \varphi = \{gH \in G/H \mid gK = 1\} = \{gH \in G/H \mid g \in K\} = K/H$$

Consequently, The 1st Isomorphism Theorem gives

$$(G/K) \cong (G/H)/\ker \varphi = (G/H)/(K/H)$$

□

Theorem 6. The Forth Isomorphism Theorem

Let G be a Group, and $N \trianglelefteq G$ be a Normal Subgroup. Then, there is a bijection between

$$D \stackrel{\text{def}}{=} \{H \leq G \mid N \leq H\}, \quad C \stackrel{\text{def}}{=} \{\bar{H} \leq G/N\}$$

Proof. Let $\pi : G \rightarrow G/N : g \mapsto gN$ be a natural projection. And, Define

$$\Phi : D \rightarrow C : H \mapsto \pi[H]$$

This function is well-defined: For any $H \in D$, let $aN, bN \in \pi[H]$. Then, $aN \cdot b^{-1}N = ab^{-1}N \in \pi[H]$, thus $\pi[H] \leq G/N$.

To show that one-to-one: Let $\Phi(A) = \Phi(B)$. Thus means, $\pi[A] = \pi[B]$. Let $a \in A$. Then, $\pi(a) \in \pi[A] = \pi[B]$, thus $\pi(a) = \pi(b)$ for some $b \in B$. That is, $aN = bN \iff a \in bN$. Meanwhile, $N \leq B$, thus $a \in bN \subset B$, $A \subset B$. Similarly, $B \subset A$, that is $A = B$.

To show that onto: Let $K \in C$. Then, $N \leq \pi^{-1}[K] \leq G$, thus clear.

□

2.2 Group Action

2.3 Generating subset of a Group

2.4 Commutator Subgroup

Chapter 3

Finite Group Theory

3.1 Lagrange's Theorem

3.2 The Class Equation

3.3 Cauchy's Theorem

Lemma 2. Cauchy's Theorem

Let G be a finite group, and p be a prime dividing $|G|$. Then, G has order p element.

Proof. Define a set:

$$S \stackrel{\text{def}}{=} \{(x_1, x_2, \dots, x_p) \mid x_i \in G, x_1 x_2 \cdots x_p = 1\}$$

Then, S has exactly $|G|^{p-1}$ elements because there are $|G|$ possible choices for each of the first $p-1$ elements in G .

Once x_1, \dots, x_{p-1} are chosen, then x_p is uniquely determined by the uniqueness of inverses.

Then, let $\sigma = (1, 2, \dots, p)$ be a permutation. Then, for any $\alpha \in S$, $\sigma^n(\alpha) \in S$ for all $n \in \mathbb{Z}$, being $ab = 1 \iff ba = 1$.

More precisely, let $n \in \mathbb{Z}$ be given, $\alpha = (x_1, \dots, x_n)$. Then,

$$\sigma^n(\alpha) = (x_{n+1}, x_{n+2}, \dots, x_p, x_1, x_2, \dots, x_n)$$

By $x_1 \cdots x_n x_{n+1} \cdots x_p = 1$, $x_{n+1} \cdots x_p x_1 \cdots x_n = 1$. Thus $\sigma^n(\alpha) \in S$. Now, define a relation on S as:

$$\alpha \sim \beta \text{ if and only if } \beta = \sigma^n(\alpha) \text{ for some } n \in \mathbb{Z}$$

Then, this relation be equivalent relation, thus construct a partition on S . Claim:

$$[\alpha] = \{\beta \in S \mid \beta \sim \alpha\} \text{ is singleton if and only if } \alpha = (x, \dots, x) \text{ for some } x \in G.$$

Left direction is clear, and for show that Right direction,

Suppose that $\alpha = (x_1, \dots, x_n)$ has different coordinate elements, let $x_i \neq x_j$, for some $i < j$. Then clearly

$$(x_1, \dots, x_i, \dots, x_p) \neq \sigma^{i-j}(x_1, \dots, x_i, \dots, x_j, \dots, x_p) = (\dots, \underbrace{x_j}_{i\text{'th element}}, \dots)$$

Meanwhile, if $[\alpha]$ has elements more than 1, $[\alpha]$ has exactly number of p elements. Because suppose that $\alpha = (x_1, \dots, x_p)$ has at least one different coordinate. Then,

$$\sigma^1(\alpha), \sigma^2(\alpha), \dots, \sigma^{p-1}(\alpha)$$

are mutually different: If there exist $1 \leq i < j < p$ such that $\sigma^i(\alpha) = \sigma^j(\alpha)$, that is, $\sigma^{j-i}(\alpha) = \alpha$.

Now, $j-i \mid p$, this is contradiction with p is prime. Therefore, every equivalent class has order 1 or p . Consequently,

$$|G|^{p-1} = k + pd$$

where k is a number of classes of size 1, and d is a number of classes of size p . And $(1, 1, \dots, 1) \in S$, k is at least 1.

Since p divides $|G|^{p-1} = k + pd$, thus k must be bigger than 1, thus there exists elements such that $x^p = 1$. \square

3.4 Sylow's Theorem

Theorem 7. Sylow's Theorem

Let G be a group of order $p^\alpha m$, where p is a prime such that $p \nmid m$.

A group of order p^r , ($r \geq 1$) is called a p -group, Subgroups of G which are p -groups are called p -subgroup. In particular, subgroups of order p^α is called Sylow p -subgroup of G . And, define a collection

$$\text{Syl}_p(G) \stackrel{\text{def}}{=} \{P \leq G \mid |P| = p^\alpha\}, \quad n_p(G) \stackrel{\text{def}}{=} \text{Card}(\text{Syl}_p(G))$$

The First Sylow Theorem

There exists a Sylow p -subgroup of G . i.e., $\text{Syl}_p(G) \neq \emptyset$.

The Second Sylow Theorem

If $P \in \text{Syl}_p(G)$ and $Q \leq G$ be a p -subgroup. Then, there exists $g \in G$ such that $Q \leq gPg^{-1}$.

The Third Sylow Theorem

$n_p \equiv 1 \pmod{p}$, $n_p = |G : N_G(P)|$ for any $P \in \text{Syl}_p(G)$, and $n_p \mid m$.

Before prove above statements, we show that:

Lemma 3. Let $P \in \text{Syl}_p(G)$. If Q is p -subgroup of G , then $Q \cap N_G(P) = Q \cap P$.

Proof. Put $H = Q \cap N_G(P)$. Since $P \leq G$, for any $p \in P$, $pPp^{-1} = P$, thus $p \in N_G(P)$. i.e., $P \leq N_G(P)$. Thus, Enough to Show that $H \leq Q \cap P$. Since $H \leq N_G(P)$,

$$PH = \bigcup_{h \in H} Ph = \bigcup_{h \in H} hP = HP$$

Thus, $PH \leq G$. And,

$$|PH| = \frac{|P||H|}{|P \cap H|}$$

By Lagrange's Theorem, $H \leq P$ and $P \cap H \leq P$ must have order of powers of p , so PH be a p -group. Clearly, $P \leq PH$ and P is the largest p -group of G , thus, $PH = P$. This means, $H \leq P$. \square

Proof. The First Theorem: The existence of Sylow p -subgroup. Proof by Induction:

If $|G| = 1$, there is nothing to prove.

Assume inductively the existence of Sylow p -subgroups for all groups of order less than $|G|$.

In case of $p \mid |Z(G)|$, then by Cauchy's Theorem, $Z(G)$ has a subgroup N which has order of p .

Clearly N is Normal, and $G/N = |G|/|N| = p^{\alpha-1}m$. By assumption, G/N has a subgroup P' of order $p^{\alpha-1}$.

By The Fourth Isomorphism Theorem, Let $P \leq G$ be a subgroup such that $P/N = P'$.

Then, $|P| = |P/N| \cdot |N| = p^\alpha$, Thus P be a Sylow p -subgroup of G .

In case of $p \nmid |Z(G)|$.

Let g_1, \dots, g_r be representatives of the distinct conjugacy classes of G , not contained in $Z(G)$. Then, The Class Equation gives

$$|G| = |Z(G)| + \sum_{i=1}^r |G : C_G(g_i)|$$

Since p divides $|G|$, if for all $i = 1, 2, \dots, r$, $p \mid |G : C_G(g_i)|$ then $p \mid |Z(G)|$, this is contradiction.

Thus, for some j , $p \nmid |G : C_G(g_j)|$. Put $H = C_G(g_j) < G$. Then, $|H|$ has a factor of p^α , by $p \nmid |G : C_G(g_j)|$. Now,

$$|H| = p^\alpha m' \quad (m' < m)$$

By assumption, H has a Sylow p -group, order of p^α .

Consequently, the existence of Sylow p -subgroup was shown.

The Second Theorem: Relation of p -subgroups.

The First Theorem gives existence of Sylow p -subgroups. Let $P \in \text{Syl}_p(G)$. Denote that:

$$S \stackrel{\text{def}}{=} \{gPg^{-1} \mid g \in G\} = \{P_1, \dots, P_r\}$$

Let $Q \leq G$ be an any p -subgroup of G . And, Q acts by conjugation on S . i.e.,

$$\alpha : Q \times S \rightarrow S : (q, P_i) \mapsto qP_iq^{-1}$$

Write S as a disjoint union of orbits under this action by Q :

$$S = \mathcal{O}_1 \cup \mathcal{O}_2 \cup \cdots \cup \mathcal{O}_s$$

where $r = |\mathcal{O}_1| + \cdots + |\mathcal{O}_s|$. Rearrange a set S as: $P_i \in \mathcal{O}_i$, $1 \leq i \leq s$. Now, using Definition, Lemma, and above Theorem,

$$|\mathcal{O}_i| \stackrel{\text{Thm}}{\equiv} |Q : N_Q(P_i)| \stackrel{\text{def}}{=} |Q : N_G(P_i) \cap Q| \stackrel{\text{lemma}}{\equiv} |Q : P_i \cap Q|$$

for each $1 \leq i \leq s$. Since Q was arbitrary, Let $Q = P_1$, so that $|\mathcal{O}_1| = |P_1 : P_1 \cap P_1| = 1$. And, for each $i \geq 2$, $P_i \cap P_1 < P_1$,

$$|\mathcal{O}_i| = |P_1 : P_i \cap P_1| > 1$$

Since $P_1 \in \text{Syl}_p(G)$, that is $|P_1| = p^\alpha$, $|P_1 : P_i \cap P_1| = |P_1|/|P_i \cap P_1| = p^k$ where $1 \leq k < \alpha$. This means for each $2 \leq i \leq s$, p divides $|\mathcal{O}_i|$. Thus,

$$r = |\mathcal{O}_1| + (|\mathcal{O}_2| + \cdots + |\mathcal{O}_s|) \equiv 1 \pmod{p}$$

Now, Proof by Contradiction: Let $Q \leq G$ be a p -subgroup. Suppose that for any $1 \leq i \leq r$, $Q \not\leq P_i$. Then, $P_i \cap Q < Q$ for all i , this means

$$|\mathcal{O}_i| = |Q : P_i \cap Q| > 1$$

Thus for any i , p divides $|\mathcal{O}_i|$, this is Contradiction. This proved Relation of p -subgroups. Finally, The Third Theorem:

Since Second Theorem, this gives that $S = \text{Syl}_p(G)$, thus $n_p(G) = r$. That is, $n_p \equiv 1 \pmod{p}$. Since all Sylow p -subgroups are Conjugate, for any $P \in \text{Syl}_p(G)$,

$$n_p = r = |\mathcal{O}_1| = |G : N_G(P)|$$

Consequently, Completing the Sylow Theorem. □

Chapter 4

Ring Theory

4.1 Ring of Fractions

Theorem 8. Let R be a Commutative Ring, $D \subset R$ be a subset such that $\begin{cases} \text{no zero, no zero divisors} \\ \text{closed under multiplication} \end{cases}$.

Then, there exists a Commutative Ring Q with identity satisfies:

1. R can embed in Q , and every element of D becomes unit in Q . More precisely, $Q = \{rd^{-1} \mid r \in R, d \in D\}$.
2. Q is the smallest Ring with identity such that every element of D becomes unit in Q .

Proof. Let $\mathcal{F} \stackrel{\text{def}}{=} \{(r, d) \mid r \in R, d \in D\}$ and the relation \sim on \mathcal{F} by $(r_1, d_1) \sim (r_2, d_2) \iff r_1 d_2 = r_2 d_1$.

Then, \sim is equivalent relation: reflexive and symmetirc are clear, and Suppose that $(r_1, d_1) \sim (r_2, d_2)$ and $(r_2, d_2) \sim (r_3, d_3)$.

$$r_2 d_3 = r_3 d_2 \implies r_2 d_1 d_3 = r_3 d_1 d_2 \implies r_1 d_2 d_3 = r_3 d_1 d_2 \implies d_2(r_1 d_3 - r_3 d_1) \implies r_1 d_3 = r_3 d_1$$

Thus transitivity shown. Define

$$\frac{r}{d} \stackrel{\text{def}}{=} [(r, d)] = \{(a, b) \mid (a, b) \sim (r, d)\}, \quad Q \stackrel{\text{def}}{=} \left\{ \frac{r}{d} \mid r \in R, d \in D \right\}$$

And define operations $+, \times$ on Q :

$$\frac{r_1}{d_1} + \frac{r_2}{d_2} \stackrel{\text{def}}{=} \frac{r_1 d_2 + r_2 d_1}{d_1 d_2}, \quad \frac{r_1}{d_1} \times \frac{r_2}{d_2} \stackrel{\text{def}}{=} \frac{r_1 r_2}{d_1 d_2}$$

Well-Definedness: If $\frac{r_1}{d_1} = \frac{r'_1}{d'_1}$ and $\frac{r_2}{d_2} = \frac{r'_2}{d'_2}$,

$$\frac{r_1 d_2 + r_2 d_1}{d_1 d_2} = \frac{r_1 d_2 d'_1 d'_2 + r_2 d_1 d'_1 d'_2}{d_1 d_2 d'_1 d'_2} = \frac{(r_1 d'_1) d_2 d'_2 + (r_2 d'_2) d_1 d'_1}{d_1 d_2 d'_1 d'_2} = \frac{(r'_1 d_1) d_2 d'_2 + (r'_2 d_2) d_1 d'_1}{d_1 d_2 d'_1 d'_2} = \frac{(r'_1 d'_2 + r'_2 d'_1) d_1 d_2}{d_1 d_2 d'_1 d'_2} = \frac{r'_1 d'_2 + r'_2 d'_1}{d'_1 d'_2}$$

$$\frac{r_1 r_2}{d_1 d_2} = \frac{r_1 r_2 d'_1 d'_2}{d_1 d_2 d'_1 d'_2} = \frac{(r_1 d'_1)(r_2 d'_2)}{d_1 d_2 d'_1 d'_2} = \frac{(r'_1 d_1)(r'_2 d_2)}{d_1 d_2 d'_1 d'_2} = \frac{r'_1 r'_2 d_1 d_2}{d_1 d_2 d'_1 d'_2} = \frac{r'_1 r'_2}{d'_1 d'_2}$$

Now, $(Q, +, \times)$ constructs Commutative Ring with identity: for any $d \in D$, put $0_Q \stackrel{\text{def}}{=} \frac{0}{d}$, $1_Q \stackrel{\text{def}}{=} \frac{d}{d}$. Then,

1. $(R, +, \times)$ closed under the operations since D is closed under the multiplication.

$$2. (R, +) \text{ has a zero: } \frac{r_1}{d_1} + 0_Q = \frac{r_1}{d_1} + \frac{0}{d} = \frac{r_1 d + 0 d_1}{d_1 d} = \frac{r_1 d}{d_1 d} = \frac{r_1}{d_1}.$$

$$3. (R, +) \text{ has an inverse: } \frac{r_1}{d_1} + \frac{-r_1}{d_1} = \frac{r_1 d_1 + (-r_1) d_1}{d_1 d_1} = \frac{[(r_1) + (-r_1)] d_1}{d_1 d_1} = \frac{0 d_1}{d_1 d_1} = \frac{0}{d_1 d_1} = 0_Q.$$

4. $(R, +, \times)$ satisfies distributive law:

4-1. The left law:

$$\begin{aligned} \frac{r_1}{d_1} \times \left(\frac{r_2}{d_2} + \frac{r_3}{d_3} \right) &= \frac{r_1}{d_1} \times \frac{r_2 d_3 + r_3 d_2}{d_2 d_3} = \frac{r_1 r_2 d_3 + r_1 r_3 d_2}{d_1 d_2 d_3} = \frac{r_1 r_2 d_1 d_3 + r_1 r_3 d_1 d_2}{d_1 d_2 d_1 d_3} = \frac{r_1 r_2}{d_1 d_2} + \frac{r_2 r_3}{d_2 d_3} \\ &= \frac{r_1}{d_1} \times \frac{r_2}{d_2} + \frac{r_2}{d_2} \times \frac{r_3}{d_3} \end{aligned}$$

4-2. The right law:

$$\begin{aligned} \left(\frac{r_1}{d_1} + \frac{r_2}{d_2} \right) \times \frac{r_3}{d_3} &= \frac{r_1 d_2 + r_2 d_1}{d_1 d_2} \times \frac{r_3}{d_3} = \frac{r_1 r_3 d_2 + r_2 r_3 d_1}{d_1 d_2 d_3} = \frac{r_1 r_3 d_2 d_3 + r_2 r_3 d_1 d_3}{d_1 d_3 d_2 d_3} = \frac{r_1 r_3}{d_1 d_3} + \frac{r_2 r_3}{d_2 d_3} \\ &= \frac{r_1}{d_1} \times \frac{r_3}{d_3} + \frac{r_2}{d_2} \times \frac{r_3}{d_3} \end{aligned}$$

$$5. (R, \times) \text{ has an identity: } \frac{r_1}{d_1} \times 1_Q = \frac{r_1}{d_1} \times \frac{d}{d} = \frac{r_1 d}{d_1 d} = \frac{r_1}{d_1}.$$

6. Elements of D become unit in Q : Define $\iota: R \rightarrow Q: r \mapsto \frac{rd}{d}$ where $d \in D$ is any fixed element in D .

Then, ι is Ring-Monomorphism because:

$$6-1. \text{ Well-Defined and Injective: } \iota(r_1) = \iota(r_2) \iff \frac{r_1 d}{d} = \frac{r_2 d}{d} \iff (r_1 - r_2) d = 0 \iff r_1 = r_2$$

□

Chapter 5

Polynomial Ring

Chapter 6

Field Theory

Chapter 7

Galois Theory

Chapter 8

Category

Chapter 9

General Topology

9.1 Complete Metric Space

Definition 2. Let (X, d) be a Metric Space, and $\{p_n\}$ be a Sequence in X . The Sequence $\{p_n\}$ is called *Cauchy Sequence* if:

For any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $m, n \geq N \implies d(p_m, p_n) < \epsilon$.

A Metric Space (X, d) is said to be *Complete* if every Cauchy Sequences Converge.

Lemma 4. Let $\{E_n\}$ be a sequence of closed bounded non-empty subsets in a Complete Metric Space X such that $E_n \supset E_{n+1}$. If $\lim_{n \rightarrow \infty} \text{diam} E_n = 0$, then $\bigcap_{n=1}^{\infty} E_n = \{p\}$ for some $p \in X$.

Proof. For each $n \in \mathbb{N}$, construct $p_n \in E_n$.

Let $\epsilon > 0$ be given. Since $\text{diam} E_n \rightarrow 0$, there is $N \in \mathbb{N}$ such that $\text{diam} E_n < \epsilon$.

For any $m, n \geq N$, E_N contains p_m, p_n . That is, $d(p_m, p_n) < \epsilon$. Thus, $\{p_n\}$ be a Cauchy sequence of X .

Since X is complete, there is a unique point $p \in X$ such that $p_n \rightarrow p$. Let $N \in \mathbb{N}$ be an integer such that $n \geq N \implies |p_n - p| < \epsilon$.

Now, for each $n \geq N$, E_n has a limit point as p . And for any $n \in \mathbb{N}$, E_n contains E_N, E_{N+1}, \dots , thus for all $n \in \mathbb{N}$, E_n has a limit point as p . Meanwhile, E_n closed, $p \in E_n, \forall n \in \mathbb{N}$.

Consequently, $p \in \bigcap_{n=1}^{\infty} E_n$. If there is $q \in X$ such that $p \neq q$, $q \in \bigcap_{n=1}^{\infty} E_n$. Then, $\text{diam} E_n \geq d(p, q) > 0, \forall n \in \mathbb{N}$. \square

9.1.1 Baire Category

Definition 3. The Topological Space X is called *Baire Space* if:

If $\{G_n \mid n \in \mathbb{N}\}$ be a Countable Collection of dense open sets of X , then $\overline{\bigcap_{n=1}^{\infty} G_n} = X$

In brief, every Countable intersection of dense open sets be dense in X .

Theorem 9. Locally Compact Hausdorff Space is Baire Space.

Theorem 10. Complete Metric Space is Baire Space.

Proof. Suppose that $\{G_n \mid n \in \mathbb{N}\}$ be a Countable Collection of dense open set of Complete Metric Space. Let an open $U \in \mathcal{T}$ be given. Since G_n is dense in the Space, $U \cap G_1$ is non-empty open set.

Thus, there exists a $p_1 \in U \cap G_1$ such that for some $r_1 > 0$, $B_{r_1}(p_1) \subset U \cap G_1$.
Then, automatically,

$$B_{\frac{r_1}{2}}(p_1) \subset \overline{B_{\frac{r_1}{2}}(p_1)} \subset B_{r_1}(p_1) \subset U \cap G_1$$

Set $E_1 = U$, $E_2 = B_{\frac{r_1}{2}}(p_1)$.

Suppose that E_1, \dots, E_{n-1} are chosen. Then, since $E_{n-1} \cap G_{n-1}$ is open, being intersection of opens. Thus there exists a point $p_{n-1} \in E_{n-1} \cap G_{n-1}$ and exists r_{n-1} such that

$$B_{r_{n-1}}(p_{n-1}) \subset E_{n-1} \cap G_{n-1} \subset E_{n-1}$$

This implies that

$$B_{\frac{r_{n-1}}{2}}(p_{n-1}) \subset \overline{B_{\frac{r_{n-1}}{2}}(p_{n-1})} \subset B_{r_{n-1}}(p_{n-1}) \subset E_{n-1} \cap G_{n-1} \subset E_{n-1}$$

Set $E_n = B_{\frac{r_{n-1}}{2}}(p_{n-1})$. Since inductively construction of $\{E_n\}$, $E_{n+1} \subset E_n$ and $\overline{E_n} \subset G_n$ for all $n \in \mathbb{N}$.
Consequently,

$$U \cap \left(\bigcap_{n=1}^{\infty} G_n \right) = \bigcap_{n=1}^{\infty} (U \cap G_n) \supset \bigcap_{n=1}^{\infty} (U \cap \overline{E_n}) = U \cap \left(\bigcap_{n=1}^{\infty} \overline{E_n} \right) = \bigcap_{n=1}^{\infty} \overline{E_n} \neq \emptyset$$

□

Definition 4. Let X be a Topological Space.

$A \subset X$ is said to be *nowhere dense subset* if $(\overline{A})^\circ = \emptyset$.

1. $B \subset X$ is called *first category* if B can be representative by union of countable nowhere dense subsets.
2. If the subset is not first category, then it is said to be *second category*.

9.1.2 Nowhere Differentiable function

9.1.3 Banach Fixed Point Theorem

Definition 5. Let $f : X \rightarrow X$ be any function. A point $x \in X$ is called a *fixed point* of f if $f(x) = x$.

Definition 6. Let X be a Metric Space. A map $f : X \rightarrow X$ is called *Contractive* with respect to the metric d if:

$$\text{There exists } \alpha \in (0, 1) \text{ such that for all } x, y \in X, d(f(x), f(y)) \leq \alpha d(x, y).$$

Theorem 11. Banach Fixed point Theorem

Let (X, d) be a Complete Metric Space, and $f : X \rightarrow X$ be a Contractive map.

Then, there exists a unique fixed point of f , $x^* \in X$.

Proof. Clearly,

$$\text{Contractive} \implies \text{Lipschitz Condition} \implies \text{Continuous}.$$

Thus, f is Continuous.

Let $x_0 \in X$ be arbitrary, and construct a sequence $\{x_n\}$ recursively as follows:

$$x_{n+1} \stackrel{\text{def}}{=} f(x_n), \quad n \geq 0$$

Then, for any $n \geq 0$,

$$\begin{aligned} d(x_{n+1}, x_n) &= d(f(x_n), f(x_{n-1})) \leq \alpha d(x_n, x_{n-1}) \\ &= d(f(x_{n-1}), f(x_{n-2})) \leq \alpha^2 d(x_{n-1}, x_{n-2}) \\ &\vdots \\ &\leq \alpha^n d(x_1, x_0) \end{aligned}$$

Let $\epsilon > 0$ be given. Put $N \in \mathbb{N}$ such that $\alpha^N \cdot d(x_1, x_0) < \epsilon(1 - \alpha)$. Then, $n \geq m \geq N$ implies that

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + \cdots + d(x_{m+1}, x_m) \\ &\leq \alpha^n d(x_1, x_0) + \alpha^{n-1} d(x_1, x_0) + \cdots + \alpha^{m+1} d(x_1, x_0) \\ &= \alpha^{m+1} d(x_1, x_0) \sum_{r=0}^{n-m-1} \alpha^r < \alpha^N d(x_1, x_0) \sum_{r=0}^{\infty} \alpha^r < \epsilon(1 - \alpha) \frac{1}{1 - \alpha} = \epsilon \end{aligned}$$

Therefore, $\{x_n\}$ is Cauchy sequence. Since X is Complete, for some $x^* \in X$, $\lim_{n \rightarrow \infty} x_n = x^*$. Consequently,

$$\lim_{n \rightarrow \infty} f(x_n) \stackrel{f \text{ conti.}}{=} f\left(\lim_{n \rightarrow \infty} x_n\right) = f(x^*) = \lim_{n \rightarrow \infty} x_{n+1} = x^*$$

□

9.2 Urysohn Metrization Theorem

9.2.1 Urysohn Lemma

Recall that:

Definition 7. X is T_4 if: For any disjoint closed set A and B , there exist disjoint open U, V such that $A \subseteq U$ and $B \subseteq V$.

Lemma 5. X is T_4 Space if and only if For any closed C and open U with $C \subseteq U$, there exists open O such that

$$\underset{\text{closed}}{C} \subseteq \underset{\text{open}}{O} \subseteq \underset{\text{closed}}{\overline{O}} \subseteq \underset{\text{open}}{U}$$

Proof. Proof of the left direction only.

Let X be a T_4 Space, and $C \subset X$ be a closed, U be a open containing C . Then, $C \subset U$ implies $U^c \subset C^c$, thus U^c is a closed set disjoint from C . By T_4 condition, There exist disjoint opens O, O' such that $C \subset O$ and $U^c \subset O' \iff O'^c \subset U$.

Since $O \cap O' = \emptyset \iff O \subset O'^c$, O contained in U , this implies that $C \subset O \subset U$.

Since closure is the smallest closed set such that contains it, consequently $C \subset O \subset \overline{O} \subset O'^c \subset U$. □

Definition 8. Let X be a Topological Space, and $A, B \subset X$ are disjoint closed subset.

A real-valued Continuous map $f : X \rightarrow [a, b]$ is called *Urysohn function* for A and B if: $f|_A = a$ and $f|_B = b$.

In another form,

$$f : X \rightarrow [a, b] : x \rightarrow \begin{cases} a & x \in A \\ b & x \in B \\ f(x) & x \notin A \cup B \end{cases}$$

Lemma 6. Urysohn Lemma

T_4 Space has an Urysohn function for any two disjoint closed subsets.

Proof. Generalization is the last thing to proven, first of all, prove in case of $[a, b] = [0, 1]$. This proof consists by three Step.

Let X be a T_4 Space, and $A, B \subset X$ be closed subsets.

Step 1. Construct a Chain of Open sets with Dyadic Rational Indices.

Consider a set of *Dyadic Rationals* $D \stackrel{\text{def}}{=} \left\{ \frac{k}{2^n} \mid n, k \in \mathbb{N}, k \leq 2^n - 1 \right\}$. We will show that the following statement holds:

For any $r, s \in D$ with $r < s$, there exist open sets U_r, U_s such that $A \subseteq \overline{U_r} \subseteq U_s \subseteq X \setminus B$ (*)

For this, Enough to Show that: For any $k \in \mathbb{N}$, there exists a Chain as:

$$A \subseteq U_{\frac{1}{2^k}} \subseteq \overline{U_{\frac{1}{2^k}}} \subseteq U_{\frac{2}{2^k}} \subseteq \overline{U_{\frac{2}{2^k}}} \subseteq \cdots \subseteq U_{\frac{2^{k-1}}{2^k}} \subseteq \overline{U_{\frac{2^{k-1}}{2^k}}} \subseteq X \setminus B$$

(Note that this opens in the Chain are not necessary distinct: For instance, if Ambient Space is Finite, then the Space is *Noetherian*. That is, X satisfies Ascending Chain Condition for open sets.)

Let $k = 1$. Then, By T_4 condition gives that: There exists an open set U_1 such that

$$A \subseteq U_1 \subseteq \overline{U_1} \subseteq X \setminus B$$

Now, naming this U_1 as $U_{\frac{1}{2}}$, proved when $k = 1$.

Suppose that for some $k > 1$, the Chain exists as:

$$\underset{\text{closed}}{A} \overset{*1}{\subseteq} \underset{\text{open}}{U_{\frac{1}{2^k}}} \overset{*2}{\subseteq} \underset{\text{closed}}{\overline{U_{\frac{1}{2^k}}}} \overset{*2}{\subseteq} \underset{\text{open}}{U_{\frac{2}{2^k}}} \subseteq \cdots \overset{*2^{k-1}}{\subseteq} \underset{\text{open}}{U_{\frac{2^{k-1}}{2^k}}} \overset{*2^{k-1}}{\subseteq} \underset{\text{closed}}{\overline{U_{\frac{2^{k-1}}{2^k}}}} \overset{*2^k}{\subseteq} \underset{\text{open}}{X \setminus B}$$

By repeatedly applying the T_4 condition 2^k times, as indicated by the indices $*1, *2, \dots, *2^k$, we can construct 2^k open sets such that:

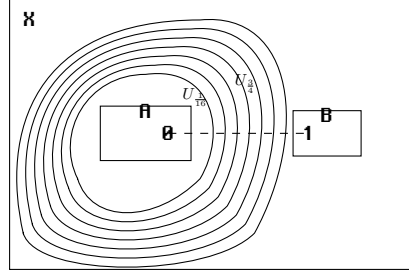
$$A \subseteq U_{\frac{1}{2^{k+1}}} \subseteq \overline{U}_{\frac{1}{2^{k+1}}} \subseteq U_{\frac{1}{2^k}} \subseteq \overline{U}_{\frac{1}{2^k}} \subseteq U_{\frac{3}{2^{k+1}}} \subseteq \overline{U}_{\frac{3}{2^{k+1}}} \subseteq U_{\frac{2}{2^k}} \subseteq \overline{U}_{\frac{2}{2^k}} \subseteq \dots \subseteq U_{\frac{2^k-1}{2^k}} \subseteq \overline{U}_{\frac{2^k-1}{2^k}} \subseteq U_{\frac{2^{k+1}-1}{2^{k+1}}} \subseteq \overline{U}_{\frac{2^{k+1}-1}{2^{k+1}}} \subseteq X \setminus B$$

Finally, Step 1 proved.

Step 2. Construct an Urysohn Function.

Define a map $f : X \rightarrow [0, 1]$ as:

$$f(x) = \begin{cases} 0 & x \in \bigcap_{t \in D} U_t \\ \sup\{t \in D \mid x \notin U_t\} & x \notin \bigcap_{t \in D} U_t \end{cases}$$



Then, this map f is well-defined by (*) and $\sup D \leq 1$. And f satisfies that:

1. $\forall r \in D, x \in A \subset U_r$. Thus, $f(x) = 0$ if $x \in A$.
2. $\forall r \in D, x \in B \subset X \setminus U_r$. Thus, $f(x) = \sup D = 1$ if $x \in B$.
3. If $x \in \overline{U}_r$, then for every $s > r, x \in \overline{U}_r \subset U_s$. Thus, $f(x) \leq r$. In Contrapositive, $f(x) > r \implies x \notin \overline{U}_r$.
(If $f(x) = \sup\{t \in D \mid x \notin U_t\} > r$, then there is $s \in D$ such that $s > r$ and $x \notin U_s$, Contradiction.)
4. If $x \notin U_r$, then, $f(x) \geq r$. In Contrapositive, $f(x) < r \implies x \in U_r$.

Now, show that this map f is Continuous map: Let $x \in X$ be fixed arbitrarily, and $\epsilon > 0$ be given.

In Case of $0 < f(x) < 1$.

Since Density of Dyadic Rationals, Choose $r, s \in D$ such that $f(x) - \epsilon < r < f(x) < s < f(x) + \epsilon$.

Now, we obtain that:

$$x \overset{(*)}{\in} U_s \setminus \overline{U}_r \overset{(**)}{\subseteq} f^{-1}[(f(x) - \epsilon, f(x) + \epsilon)]$$

(*) directly given by above properties, (**) given applying the fact that $x \in U_s \subset \overline{U}_s$ and $x \notin \overline{U}_r$.

In Case of $f(x) = 0$.

Choose $r \in D$ such that $f(x) = 0 < r < \epsilon = f(x) + \epsilon$. Then,

$$x \in U_r \subset f^{-1}[(f(x), f(x) + \epsilon)]$$

In Case of $f(x) = 1$.

Choose $r \in D$ such that $f(x) - \epsilon = 1 - \epsilon < r < 1 = f(x)$. Then,

$$x \in X \setminus U_r \subset f^{-1}[(f(x) - \epsilon, f(x))]$$

Consequently, f is Continuous map on $[0, 1]$ such that $f|_A = 0$ and $f|_B = 1$.

Step 3. Generalization.

Since $[0, 1] \cong [a, b]$ for any $a < b$, let $g : [0, 1] \rightarrow [a, b] : x \mapsto (1 - x)a + xb$ be a Homeomorphism.

Then, $h = g \circ f : X \rightarrow [a, b]$ becomes a Continuous map such that $h|_A = a$ and $h|_B = b$. □

9.2.2 Tietze Extension Theroem

Theorem 12. Tietze Extension Theroem

Let X be a T_4 Space, and $A \subseteq X$ be a closed subset.

For any Continuous map $f: A \rightarrow \mathbb{R}$, there exists a Continuous map:

$$g: X \rightarrow \mathbb{R} \quad \text{s.t.} \quad g|_A = f$$

This g is called *extension* of f .

Proof. This proof consists by three steps.

Step 1. First, we will show that:

For any Continuous map $f: A \rightarrow [-r, r]$, there is a Continuous map $h: X \rightarrow \mathbb{R}$ s.t.
$$\begin{cases} \forall x \in X, |h(x)| \leq \frac{1}{3}r \\ \forall a \in A, |f(a) - h(a)| \leq \frac{2}{3}r \end{cases} \quad (*)$$

Set

$$I_1 \stackrel{\text{def}}{=} \left[-r, -\frac{1}{3}r\right], \quad I_2 \stackrel{\text{def}}{=} \left[-\frac{1}{3}r, \frac{1}{3}r\right], \quad I_3 \stackrel{\text{def}}{=} \left[\frac{1}{3}r, r\right]$$

Then, the preimage of continuous map preserves closed and A is closed subspace of X , $f^{-1}[I_1]$ and $f^{-1}[I_3]$ are closed of X .

And, I_1 and I_3 are disjoint, thus $f^{-1}[I_1 \cap I_3] = f^{-1}[I_1] \cap f^{-1}[I_3] = \emptyset$.

Now, apply the *Urysohn Lemma*: There exists an Urysohn function $h: X \rightarrow I_2$ for $f^{-1}[I_1]$ and $f^{-1}[I_3]$.

Clearly, this map h satisfies the first condition in $(*)$. And, for show the second condition, let $a \in A$ be given.

If $a \in f^{-1}[I_1]$, then $f(a) \in I_1$ and $h(a) = -\frac{1}{3}r$, thus $|f(a) - h(a)| \leq \frac{2}{3}r$.

If $a \in f^{-1}[I_3]$, then $f(a) \in I_3$ and $h(a) = \frac{1}{3}r$, thus $|f(a) - h(a)| \leq \frac{2}{3}r$.

If $a \notin (f^{-1}[I_1] \cup f^{-1}[I_3])$, then $f(a), h(a) \in I_2$, thus $|f(a) - h(a)| \leq \frac{2}{3}r$.

Therefore, the second condition satisfied.

Step 2. We will show that: for any $f: A \rightarrow [-1, 1]$, there exists an extension of f .

Apply the result in Step 1, there exists a Continuous map:

$$h_1: X \rightarrow \mathbb{R} \quad \text{s.t.} \quad \begin{cases} \forall x \in X, |h_1(x)| \leq \frac{1}{3} \\ \forall a \in A, |f(a) - h_1(a)| \leq \frac{2}{3} \end{cases}$$

Now, the second condition of h_1 , the continuous map $f - h_1: A \rightarrow [-\frac{2}{3}, \frac{2}{3}] : x \mapsto f(x) - h_1(x)$ is well-defined.

Again, there exists a Continuous map:

$$h_2: X \rightarrow \mathbb{R} \quad \text{s.t.} \quad \begin{cases} \forall x \in X, |h_2(x)| \leq \frac{1}{3} \cdot \frac{2}{3} \\ \forall a \in A, |f(a) - h_1(a) - h_2(a)| \leq \left(\frac{2}{3}\right)^2 \end{cases}$$

Inductively, for any $n \in \mathbb{N}$, there exists a Continuous map:

$$h_n: X \rightarrow \mathbb{R} \quad \text{s.t.} \quad \begin{cases} \forall x \in X, |h_n(x)| \leq \frac{1}{3} \cdot \left(\frac{2}{3}\right)^{n-1} \\ \forall a \in A, |f(a) - h_1(a) - h_2(a) - \dots - h_n(a)| \leq \left(\frac{2}{3}\right)^n \end{cases}$$

Define a map

$$g: X \rightarrow [-1, 1] : x \mapsto \sum_{n=1}^{\infty} h_n(x)$$

For any $x \in X$,

$$|g(x)| = \left| \sum_{n=1}^{\infty} h_n(x) \right| \leq \sum_{n=1}^{\infty} |h_n(x)| \leq \sum_{n=1}^{\infty} \frac{1}{3} \cdot \left(\frac{2}{3}\right)^{n-1} = \frac{1}{3} \cdot \frac{1}{1 - \frac{2}{3}} = 1$$

Therefore, this map is well-defined. And, *Weierstrass M-test* gives that $\sum_{n=1}^{\infty} h_n(x)$ converges uniformly.

Moreover, for any $a \in A$,

$$\left| f(a) - \sum_{k=1}^n h_k(a) \right| \leq \left(\frac{2}{3}\right)^n \implies \left| f(a) - \sum_{n=1}^{\infty} h_n(a) \right| = |f(a) - g(a)| = 0$$

That is, g is Continuous on X and $g|_A = f$. Therefore, g is extension of f .

Step 3. Finally, we generalize the result in Step 2.:

Let $f: A \rightarrow [a, b]$ be a Continuous map on the closed subspace A . And, let $\varphi: [a, b] \rightarrow [-1, 1]$ be a Homeomorphism. Then, $\varphi \circ f: A \rightarrow [-1, 1]$ is Continuous map, thus there exists an extension $g: X \rightarrow [-1, 1]$ such that $g|_A = \varphi \circ f$. Now, $\varphi^{-1} \circ g: X \rightarrow [a, b]$ is Continuous, and $(\varphi^{-1} \circ g)|_A = \varphi^{-1} \circ \varphi \circ f = f$, Therefore this $\varphi^{-1} \circ g$ is the extension of f .

Let $f: A \rightarrow \mathbb{R}$ be a Continuous map on the closed subspace A .

And, let $\varphi: \mathbb{R} \rightarrow (-1, 1)$ be a Homeomorphism. Then, the map $\phi: \mathbb{R} \rightarrow [-1, 1]: x \mapsto \varphi(x)$ is still Continuous.

Now, The Continuous map $\phi \circ f: A \rightarrow [-1, 1]$ has an extension $g: X \rightarrow [-1, 1]$ such that $g|_A = \phi \circ f$.

Put $B = g^{-1}[\{-1, 1\}]$. Then B is Closed on X , and $A \cap B = \emptyset$. Now, apply the Urysohn Lemma to this, there exists an Urysohn function for A and B : Continuous map $\gamma: X \rightarrow [0, 1]$ such that $\gamma|_A = 1$ and $\gamma|_B = 0$.

Define a map $\eta: X \rightarrow (-1, 1): x \mapsto g(x)\gamma(x)$. Then, if $g(x) = 1$ or $g(x) = -1$, then $x \in B$, thus $g(x)\gamma(x) = 0$.

Therefore, η is well-defined. And, for any $a \in A$, $\eta(a) = g(a)\gamma(a) = g(a)$, thus $\eta|_A = \phi \circ f$.

Consequently, the map $\phi^{-1} \circ \eta$ is an extension of f , we wanted. □

Recall that:

Definition 9. X is T_1 if: For any distinct $x, y \in X$, there exist open sets U_x, U_y such that $\begin{cases} x \in U_x, & x \notin U_y \\ y \notin U_x, & y \in U_y \end{cases}$.

Lemma 7. X is T_1 if and only if For any $x \in X$, a singleton $\{x\}$ is closed in X .

Proof. The left direction is clear.

Let $x \in X$. Then, for any $y \in X$ with $y \neq x$, T_1 condition gives that there is an open set such that $y \in U_y$ and $x \notin U_y$.

Now, the union

$$\bigcup_{\substack{y \in X \\ y \neq x}} U_y = X \setminus \{x\}$$

is open by definition. □

9.2.3 Urysohn Metrization Theorem

Theorem 13. *Urysohn Metrization Theorem*

If X is a Second-Countable Regular Space, then X is Metrizable.

Chapter 10

Algebraic Topology

Chapter 11

Basic Analysis

11.1 Arithmetic means

Let $\{s_n\}$ be a Complex numbers Sequence. Define the *Arithmetic means* of $\{s_n\}$:

$$\sigma_n \stackrel{\text{def}}{=} \frac{s_0 + \cdots + s_n}{n+1} = \frac{1}{n+1} \left(\sum_{i=0}^n s_i \right)$$

Then, the Arithmetic means σ_n has following properties:

1). If $\lim_{n \rightarrow \infty} s_n = s$, then $\lim_{n \rightarrow \infty} \sigma_n = s$.

Proof. Let $\epsilon > 0$ be given. Then, there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $|s_n - s| < \epsilon$.
Now, for $n \geq N$,

$$\begin{aligned} |\sigma_n - s| &= \left| \frac{s_0 + \cdots + s_n}{n+1} - \frac{(n+1)s}{n+1} \right| = \left| \frac{(s_0 - s) + \cdots + (s_n - s)}{n+1} \right| \\ &\stackrel{\text{tri. ineq}}{\leq} \frac{\sum_{k=0}^{N-1} |s_k - s|}{n+1} + \frac{\sum_{k=N}^n |s_k - s|}{n+1} \\ &< \frac{\sum_{k=0}^{N-1} |s_k - s|}{n+1} + \frac{n+1-N}{n+1} \cdot \epsilon \\ &< \frac{\sum_{k=0}^{N-1} |s_k - s|}{n+1} + \epsilon \end{aligned}$$

Now, put $M \in \mathbb{N}$ satisfies $M \geq N$ and $n \geq M \implies \frac{\sum_{k=0}^{N-1} |s_k - s|}{n+1} < \epsilon$, using Archimedean property.
Then, $n \geq M$ implies $|\sigma_n - s| < \epsilon$, thus $\sigma_n \rightarrow s$. □

2). Put $a_n = s_n - s_{n-1}$, for $n \geq 1$. If $\lim_{n \rightarrow \infty} na_n = 0$ and σ_n converges, then s_n converges.

Proof. First,

$$\begin{aligned} s_n - \sigma_n &= s_n - \frac{s_0 + \cdots + s_n}{n+1} = \frac{(n+1)s_n - \sum_{k=0}^n s_k}{n+1} \\ &= \frac{1}{n+1} ((s_1 - s_0) + (2s_2 - 2s_1) + (3s_3 - 3s_2) + \cdots + (ns_n - ns_{n-1})) \\ &= \frac{1}{n+1} \sum_{k=1}^n ka_k \end{aligned}$$

Now, if $na_n \rightarrow 0$ and $\sigma_n \rightarrow \sigma$,

$$\begin{aligned} \lim_{n \rightarrow \infty} s_n &= \lim_{n \rightarrow \infty} \left(\sigma_n + \frac{1}{n+1} \sum_{k=1}^n ka_k \right) \\ &= \lim_{n \rightarrow \infty} \sigma_n + \lim_{n \rightarrow \infty} \frac{1}{n+1} \sum_{k=1}^n ka_k \stackrel{1)}{=} \sigma \end{aligned}$$

□

2) is conditional converse of 1). But, there is more weak version of the converse proposition:

3). The sequence $\{na_n\}$ bounded by $M < \infty$, and $\sigma_n \rightarrow \sigma$. Then, $s_n \rightarrow \sigma$.

Proof. First, For positive integers $m < n$,

$$\begin{aligned} s_n - \sigma_n &= s_n - \frac{\sum_{k=0}^n s_k}{n+1} = s_n - \frac{m+1}{n-m} \cdot \left(\frac{1}{m+1} - \frac{1}{n+1} \right) \sum_{k=0}^n s_k \\ &= s_n - \frac{m+1}{n-m} \cdot \left(\frac{\sum_{k=0}^m s_k + \sum_{k=m+1}^n s_k}{m+1} - \frac{\sum_{k=0}^n s_k}{n+1} \right) \\ &= s_n - \frac{m+1}{n-m} \cdot \left(\sigma_m - \sigma_n + \frac{\sum_{k=m+1}^n s_k}{m+1} \right) \\ &= \frac{m+1}{n-m} (\sigma_n - \sigma_m) + \frac{1}{n-m} \sum_{k=m+1}^n (s_n - s_k) \end{aligned}$$

Meanwhile, since for any $n \in \mathbb{N}$, $|na_n| = n|s_n - s_{n-1}| < M$, for $k = m+1, \dots, n$,

$$\begin{aligned} |s_n - s_k| &= |s_n - s_{n-1} + s_{n-1} - s_{n-2} + \dots + s_{k+1} - s_k| \\ &\leq |s_n - s_{n-1}| + |s_{n-1} - s_{n-2}| + \dots + |s_{k+1} - s_k| \\ &\leq \frac{M}{n} + \frac{M}{n-1} + \dots + \frac{M}{k+1} \leq \frac{n-k}{k+1} M \leq \frac{n-k}{m+2} M \leq \frac{n-m-1}{m+2} M \end{aligned}$$

Let $\epsilon > 0$ be given. For each $n \in \mathbb{N}$, put $m \in \mathbb{N}$ such that

$$m \leq \frac{n-\epsilon}{1+\epsilon} < m+1$$

Then,

$$m(1+\epsilon) \leq n-\epsilon \implies m+\epsilon(1+m) \leq n \implies \frac{m+1}{n-m} \leq \frac{1}{\epsilon}$$

and

$$n-\epsilon < (m+1)(1+\epsilon) \implies n+1 < (m+2)(1+\epsilon) \implies \frac{n+1}{m+2} - 1 < \epsilon \implies \frac{n-m-1}{m+2} < \epsilon$$

Now, for arbitrary $n \in \mathbb{N}$,

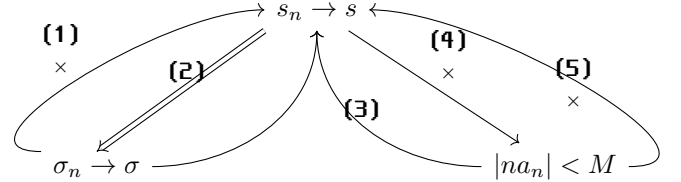
$$\begin{aligned} |s_n - \sigma| &\leq |s_n - \sigma| + |\sigma_n - \sigma| \\ \implies \limsup_{n \rightarrow \infty} |s_n - \sigma| &\leq \limsup_{n \rightarrow \infty} |s_n - \sigma_n| + \limsup_{n \rightarrow \infty} |\sigma_n - \sigma| \end{aligned}$$

And,

$$\begin{aligned} |s_n - \sigma_n| &= \frac{m+1}{n-m} |\sigma_n - \sigma_m| + \frac{1}{n-m} \sum_{k=m+1}^n |s_n - s_k| < \frac{1}{\epsilon} |\sigma_n - \sigma_m| + M\epsilon \\ \implies \limsup_{n \rightarrow \infty} |s_n - \sigma_n| &\leq \frac{1}{\epsilon} \limsup_{n \rightarrow \infty} |\sigma_n - \sigma_m| + M\epsilon = M\epsilon \end{aligned}$$

Consequently, $\limsup_{n \rightarrow \infty} |s_n - \sigma| \leq (M+1)\epsilon$, thus $s_n \rightarrow \sigma$. □

In brief, the diagram of the above conditions like this:



Examples and Counterexamples of the Diagram:

(1) Let $s_n \stackrel{\text{def}}{=} \exp(\frac{in\pi}{2})$. Then,

- s_n diverges.
- na_n diverges.
- $\sigma_n \rightarrow 0$.

(2) Let $s_n \stackrel{\text{def}}{=} \frac{1}{n}$, $s_0 = 0$.

(3) Let $s_n \stackrel{\text{def}}{=} \sum_{k=1}^n \frac{1}{k}$. Then,

- s_n diverges.
- $a_n = \frac{1}{n}$, thus $na_n \rightarrow 1$, bounded.
- If σ_n converges, then the diagram implies that s_n must converge, leading to a contradiction. Therefore, σ_n diverges.

(4) $s_n = \sum_{k=1}^n \frac{(-1)^k}{\sqrt{k}}$, $s_0 = 0$. Then,

- s_n converges, being the Alternating series Test.
- $a_n = \frac{(-1)^n}{\sqrt{n}}$, thus na_n diverges.

11.2 Taylor's Theorem

Theorem 14. Taylor's Theorem

Let $f : [a, b] \rightarrow \mathbb{R}$, and let $n \in \mathbb{N}$ be fixed. Suppose that $\begin{cases} f^{(n-1)} \text{ is Continuous.} \\ f^{(n)}(t) \text{ exists for every } t \in (a, b). \end{cases}$

Then, for any $\alpha, \beta \in [a, b]$, there exists $x \in (\alpha, \beta)$ such that

$$f(\beta) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!} (\beta - \alpha)^k + \frac{f^{(n)}(x)}{n!} (\beta - \alpha)^n$$

Proof. Put

$$M \stackrel{\text{def}}{=} \frac{1}{(\beta - \alpha)^n} \cdot \left(f(\beta) - \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!} (\beta - \alpha)^k \right)$$

That is,

$$f(\beta) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!} (\beta - \alpha)^k + M(\beta - \alpha)^n$$

and put

$$g(t) \stackrel{\text{def}}{=} f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!} (t - \alpha)^k - M(t - \alpha)^n, \quad (a \leq t \leq b)$$

If we differentiate the above equation n times,

$$g^{(n)}(t) = f^{(n)}(t) - n!M, \quad (a < t < b)$$

For each $k = 0, 1, \dots, n-1$,

$$\begin{aligned} \frac{d^r}{dt^r} \left(\sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!} (t - \alpha)^k \right) &= \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!} \cdot \frac{d^r}{dt^r} ((t - \alpha)^k) \\ &= \sum_{k=r+1}^{n-1} \frac{f^{(k)}(\alpha)}{k!} \cdot \frac{k!}{(k-r)!} (t - \alpha)^{k-r} + f^{(r)}(\alpha) \\ &= \sum_{k=r+1}^{n-1} \frac{f^{(k)}(\alpha)}{(k-r)!} (t - \alpha)^{k-r} + f^{(r)}(\alpha) \end{aligned}$$

Substituting $t = \alpha$, only the $f^{(r)}(\alpha)$ term remains. Therefore, for $r = 0, \dots, n-1$, $g(\alpha) = g'(\alpha) = \dots = g^{(n-1)}(\alpha) = 0$. Since $g(\beta) = 0$ by definition, the Mean-Value Theorem implies there exists a $x_1 \in (\alpha, \beta)$ s.t. $g'(x_1) = \frac{g(\beta) - g(\alpha)}{\beta - \alpha} = 0$. And similarly, there is $x_2 \in (x_1, \beta)$ s.t. $g''(x_2) = \frac{g'(x_1) - g'(\alpha)}{\beta - \alpha} = 0$.

Inductively, for some $x_n \in (\alpha, \beta)$, $g^{(n)}(x_n) = f^{(n)}(x_n) - n!M = 0$. That is, $M = \frac{f^{(n)}(x_n)}{n!}$.

Proof Complete by Initial Setting. □

Corollary 2. Let $f : [a, b] \rightarrow \mathbb{R}$ be an infinitely differentiable function.

Suppose that there exists a $M > 0$ such that for any $n \in \mathbb{N}$, $\sup_{t \in [a, b]} |f^{(n)}(t)| \leq M$. Then, for any $x, \alpha \in [a, b]$,

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(\alpha)}{k!} (x - \alpha)^k$$

11.3 Convexity

11.3.1 Definition

Definition 10. Let $f : (a, b) \rightarrow \mathbb{R}$ be a Real-valued function. f is said to be *convex* if: For any $x, y \in (a, b), \lambda \in (0, 1)$,

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

Convex function has following properties:

Lemma 8. Let $f : (a, b) \rightarrow \mathbb{R}$ be a Convex function, and $a < x_1 < x_2 < x_3 < b$. Then,

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(x_3) - f(x_1)}{x_3 - x_1} \leq \frac{f(x_3) - f(x_2)}{x_3 - x_2}$$

Proof. To show that first inequality, note that

$$\frac{x_2 - x_1}{x_3 - x_1} \cdot x_3 + \left(1 - \frac{x_2 - x_1}{x_3 - x_1}\right) \cdot x_1 = \frac{(x_2 - x_1)x_3 + (x_3 - x_2)x_1}{x_3 - x_1} = x_2$$

Now,

$$\begin{aligned} f(x_2) &= f\left(\frac{x_2 - x_1}{x_3 - x_1} \cdot x_3 + \left(1 - \frac{x_2 - x_1}{x_3 - x_1}\right) \cdot x_1\right) \\ &\leq \frac{x_2 - x_1}{x_3 - x_1} \cdot f(x_3) + \left(1 - \frac{x_2 - x_1}{x_3 - x_1}\right) \cdot f(x_1) = \frac{x_2 - x_1}{x_3 - x_1} \cdot f(x_3) + f(x_1) - \left(\frac{x_2 - x_1}{x_3 - x_1}\right) \cdot f(x_1) \end{aligned}$$

In brief,

$$f(x_2) - f(x_1) \leq \frac{x_2 - x_1}{x_3 - x_1} (f(x_3) - f(x_1)) \implies \frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(x_3) - f(x_1)}{x_3 - x_1}$$

And similarly,

$$\frac{x_3 - x_2}{x_3 - x_1} \cdot x_1 + \left(1 - \frac{x_3 - x_2}{x_3 - x_1}\right) x_3 = x_2$$

gives the second inequality. □

11.3.2 Properties

Proposition 2. If $f : (a, b) \rightarrow \mathbb{R}$ is Convex, then f is Continuous.

Proof. Let $\epsilon > 0$ be given, $s < t$ are fixed in (a, b) . For any $x, y \in (s, t)$ with $s < x < y < t$,

$$\frac{f(s) - f(a)}{s - a} \leq \frac{f(x) - f(s)}{x - s} \leq \frac{f(y) - f(x)}{y - x} \leq \frac{f(t) - f(y)}{t - y} \leq \frac{f(b) - f(t)}{b - t}$$

Put $M = \max \left\{ \left| \frac{f(s) - f(a)}{s - a} \right|, \left| \frac{f(b) - f(t)}{b - t} \right| \right\}$. Then, for any $x, y \in (s, t)$,

$$\left| \frac{f(y) - f(x)}{y - x} \right| \leq M$$

Now,

$$|f(y) - f(x)| \leq M|y - x| < \epsilon$$

Since $s, t \in (a, b)$ was arbitrary, f is continuous on (a, b) . □

Proposition 3. Let f is differentiable on (a, b) . Then,

f is Convex if and only if f' is monotonically increasing on (a, b) .

Proof. Prove by showing both directions: right and left.

Right Direction Let $x_1 < x_2$ in (a, b) . Then,

$$f'(x_1) = \lim_{t \rightarrow x_1} \frac{f(t) - f(x_1)}{t - x_1} \leq \frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \lim_{\tau \rightarrow x_2} \frac{f(\tau) - f(x_2)}{\tau - x_2} = f'(x_2)$$

More rigorously, put $\epsilon = \left| f'(x_1) - \frac{f(x_2) - f(x_1)}{x_2 - x_1} \right|$. (If $\epsilon = 0$, then there is nothing to prove.).

Now, there exists a $\delta > 0$ such that $|t - x_1| < \delta$ implies

$$\left| f'(x_1) - \frac{f(t) - f(x_1)}{t - x_1} \right| < \epsilon \iff -\epsilon + \frac{f(t) - f(x_1)}{t - x_1} < f'(x_1) \stackrel{(*)}{=} \epsilon + \frac{f(t) - f(x_1)}{t - x_1}$$

If $f'(x_1) - \frac{f(x_2) - f(x_1)}{x_2 - x_1} > 0$, then $(*)$ gives

$$f'(x_1) < f'(x_1) + \frac{f(t) - f(x_1)}{t - x_1} - \frac{f(x_2) - f(x_1)}{x_2 - x_1} \iff \frac{f(x_2) - f(x_1)}{x_2 - x_1} < \frac{f(t) - f(x_1)}{t - x_1} \quad \forall t \text{ s.t. } |t - x_1| < \delta$$

If $|t - x_1| < |x_2 - x_1|$, then this contradicts to Convexity.

Consequently, we obtain the first inequality, similarly can prove the second inequality.

Left Direction Let $x, y \in (a, b)$ and $\lambda \in (0, 1)$ be given. The Mean Value Theorem gives that:

$$f(\lambda x + (1 - \lambda)y) - f(x) = f'(z_1)(\lambda x + (1 - \lambda)y - x) \text{ for some } z_1 \in (x, \lambda x + (1 - \lambda)y)$$

$$f(y) - f(\lambda x + (1 - \lambda)y) = f'(z_2)(y - \lambda x + (1 - \lambda)y) \text{ for some } z_2 \in (\lambda x + (1 - \lambda)y, y)$$

Now, Monotonically increasing gives

$$\begin{aligned} \frac{f(\lambda x + (1 - \lambda)y) - f(x)}{\lambda x + (1 - \lambda)y - x} &= f'(z_1) \leq f'(z_2) = \frac{f(y) - f(\lambda x + (1 - \lambda)y)}{y - (\lambda x + (1 - \lambda)y)} \\ \implies \frac{f(\lambda x + (1 - \lambda)y) - f(x)}{(1 - x)(y - x)} &\leq \frac{f(y) - f(\lambda x + (1 - \lambda)y)}{\lambda(y - x)} \\ \implies \lambda f(\lambda x + (1 - \lambda)y) - \lambda f(x) &\leq (1 - \lambda)f(y) - (1 - \lambda)\lambda x + (1 - \lambda)y \\ \implies f(\lambda x + (1 - \lambda)y) &\leq \lambda f(x) + (1 - \lambda)f(y) \end{aligned}$$

□

Corollary 3. If $f : [a, b] \rightarrow \mathbb{R}$ is twice-differentiable, then

f is Convex if and only if $f''(x) \geq 0$ for all $x \in (a, b)$.

Theorem 15. Let $f : [a, b] \rightarrow \mathbb{R}$ be given. Then,

f is Convex if and only if f is Continuous, and Midpoint Convex.

Midpoint convex is that f satisfies $\forall x, y \in (a, b), f\left(\frac{x+y}{2}\right) \leq \frac{f(x) + f(y)}{2}$.

Proof. The right direction is clear. To show the left direction, we demonstrate that Midpoint Convexity implies Dyadic Rational Convexity. Claim: For any $n \in \mathbb{N}$,

$$f\left(\frac{\sum_{k=1}^{2^n} x_k}{2^n}\right) \leq \frac{1}{2^n} \sum_{k=1}^{2^n} f(x_k) \quad (*)$$

Using Induction: If $n = 1$, it is clear by Midpoint Convexity.

Assume that for $n \in \mathbb{N}$, $(*)$ is True. Then,

$$\begin{aligned} f\left(\frac{\sum_{k=1}^{2^{n+1}} x_k}{2^{n+1}}\right) &= f\left(\frac{1}{2} \cdot \left[\frac{\sum_{k=1}^{2^n} x_k}{2^n} + \frac{\sum_{k=2^n+1}^{2^{n+1}} x_k}{2^n}\right]\right) \\ &\stackrel{\text{m.c.}}{\leq} \frac{1}{2} \left(f\left(\frac{\sum_{k=1}^{2^n} x_k}{2^n}\right) + f\left(\frac{\sum_{k=2^n+1}^{2^{n+1}} x_k}{2^n}\right) \right) \\ &\stackrel{(*)}{\leq} \frac{1}{2} \left(\frac{1}{2^n} \sum_{k=1}^{2^n} f(x_k) + \frac{1}{2^n} \sum_{k=2^n+1}^{2^{n+1}} f(x_k) \right) \\ &= \frac{1}{2^{n+1}} \sum_{k=1}^{2^{n+1}} f(x_k) \end{aligned}$$

Consequently, we obtain the claim. Now, let $n \in \mathbb{N}$, and m be an integer such that $1 \leq m \leq 2^n$.

Put $x_1 = x_2 = \dots = x_m = x$ and $x_{m+1} = x_{m+2} = \dots = x_{2^n} = y$. Then

$$f\left(\frac{m}{2^n}x + \left(1 - \frac{m}{2^n}\right)y\right) \leq \frac{m}{2^n}f(x) + \left(1 - \frac{m}{2^n}\right)f(y)$$

For complete this discussion, Let $x, y \in (a, b), \lambda \in (0, 1)$ be given.

Since $\frac{\lfloor 2^n \lambda \rfloor}{2^n} \rightarrow \lambda$ as $n \rightarrow \infty$, for any $n \in \mathbb{N}$,

$$f\left(\frac{\lfloor 2^n \lambda \rfloor}{2^n}x + \left(1 - \frac{\lfloor 2^n \lambda \rfloor}{2^n}\right)y\right) \leq \frac{\lfloor 2^n \lambda \rfloor}{2^n}f(x) + \left(1 - \frac{\lfloor 2^n \lambda \rfloor}{2^n}\right)f(y)$$

Finally, taking limits then

$$\lim_{n \rightarrow \infty} f\left(\frac{\lfloor 2^n \lambda \rfloor}{2^n}x + \left(1 - \frac{\lfloor 2^n \lambda \rfloor}{2^n}\right)y\right) \stackrel{f \text{ cont.}}{=} f\left(\lim_{n \rightarrow \infty} \left[\frac{\lfloor 2^n \lambda \rfloor}{2^n}x + \left(1 - \frac{\lfloor 2^n \lambda \rfloor}{2^n}\right)y\right]\right) = f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

In brief, Midpoint Convexity implies Dyadic Rational Convexity, and with Continuous implies Convexity. \square

11.4 Lipschitz Condition

11.4.1 Definition

Definition 11. A real-valued function $f : (a, b) \rightarrow \mathbb{R}$ is called *Lipschitz Continuous* if:

$$\exists L \geq 0 \text{ s.t. } \forall x_1, x_2 \in (a, b), |f(x_1) - f(x_2)| \leq L \cdot |x_1 - x_2|$$

The constant L is said to be *Lipschitz Constant* of f . In particular, the constant

$$D \stackrel{\text{def}}{=} \sup_{x_1 \neq x_2} \frac{|f(x_1) - f(x_2)|}{|x_1 - x_2|}$$

is called *dilation* of f . Clearly,

$$\forall x_1, x_2 \in (a, b), |f(x_1) - f(x_2)| \leq D \cdot |x_1 - x_2|$$

and if $L > 0$ is Lipschitz Constant of f , then $D \leq L$. That is, $D = \inf\{L > 0 \mid L \text{ is Lipschitz constant of } f\}$.

11.4.2 Properties

Proposition 4. If $f : (a, b) \rightarrow \mathbb{R}$ is Lipschitz Continuous, then f is uniformly continuous.

Proof. Let $L \geq 0$ be a Lipschitz Constant of f . Then, for any $\epsilon > 0$,

$$\forall x, y \in (a, b), |x - y| < \frac{\epsilon}{L} \implies |f(x) - f(y)| \leq L|x - y| < \epsilon$$

□

Proposition 5. Let $f : (a, b) \rightarrow \mathbb{R}$ be a Differentiable function. Then,

f is Lipschitz Continuous if and only if f' is bounded in (a, b) .

Proof.

Right Direction

Let $L > 0$ be a Lipschitz constant of f , and $x \in (a, b)$ be given. Since definition of derivative,

$$f'(x) \stackrel{\text{def}}{=} \lim_{t \rightarrow x} \frac{f(x) - f(t)}{x - t}$$

Meanwhile, the assumption gives: for any distinct $x, t \in (a, b)$,

$$\frac{|f(x) - f(t)|}{|x - t|} \leq L$$

Therefore,

$$f'(x) = \lim_{t \rightarrow x} \frac{f(x) - f(t)}{x - t} \leq \lim_{t \rightarrow x} \frac{|f(x) - f(t)|}{|x - t|} \leq \lim_{t \rightarrow x} L = L$$

Left Direction

Let distinct $x, y \in (a, b)$ be given. Then, the Mean-Value Theorem gives: There exists a $z \in (x, y)$ such that

$$f(x) - f(y) = f'(z)(x - y) \implies f'(z) = \frac{f(x) - f(y)}{x - y}$$

Now,

$$\left| \frac{f(x) - f(y)}{x - y} \right| = |f'(z)| \leq L \implies |f(x) - f(y)| \leq L \cdot |x - y|$$

If $x = y$, then there is nothing to prove.

□

Note that:

$$\text{Lipschitz Continuous} \implies \text{Uniformly Continuous} \implies \text{Continuous}$$

11.4.3 Newton-Raphson Method

Theorem 16. Newton-Raphson Method

Let $f : [a, b] \rightarrow \mathbb{R}$ be a twice-differentiable, $f(a) < 0 < f(b)$. Suppose that f satisfies: for all $x \in [a, b]$,

$$f'(x) \geq \delta > 0 \text{ and } 0 \leq f''(x) \leq M$$

That is, f is strictly increasing convex function, and Lipschitz Continuous.

Further, there uniquely exists $x^* \in (a, b)$ such that $f(x^*) = 0$.

Let $x_1 \in (x^*, b)$ fixed. Define a sequence $\{x_n\}$ inductively as follows:

$$x_{n+1} \stackrel{\text{def}}{=} x_n - \frac{f(x_n)}{f'(x_n)}$$

Then, $\{x_n\}$ satisfies the following three conditions:

1. $\{x_n\}$ is decreasing sequence.
2. $x_n \rightarrow x^*$ as $n \rightarrow \infty$.
3. For any $n \in \mathbb{N}$, $0 \leq x_{n+1} - x^* \leq \left[\frac{M}{2\delta}\right]^{2^{n+1}-1} [x_1 - x^*]^{2^n}$.

Condition 3 means that for a suitable initial value x_1 , we can establish an upper bound for the error.

Proof. This proof consists by three steps.

Since f'' is non-negative, and f' is positive, f is strictly increasing convex function.

And Fundamental Theorem of Calculus gives: for any $x \in (a, b)$,

$$f'(x) \stackrel{\text{FTC}}{=} \int_a^x f''(t)dt + f'(a) \leq \int_a^x Mdt + f'(a) = M(x - a) + f'(a) \leq M(b - a) + f'(a)$$

Thus, f' is bounded on (a, b) , thus f is Lipschitz Continuous.

Step 1. f has a unique root x^* .

The existence of root given directly by Intermediate-Value theorem.

Suppose that $x^*, x' \in (a, b)$ are distinct root of f . i.e., $f(x^*) = f(x') = 0$. Then, by Mean-value theorem, there is $c \in (a, b)$ between x^* and x' such that

$$f'(c)(x^* - x') = f(x^*) - f(x') = 0$$

That is, $f'(c) = 0$. This is contradiction with f' is positive.

Step 2. $\{x_n\}$ decrease.

Proof by induction:

For $n = 1$, $f'(x_1)(x_1 - x_2) \stackrel{\text{def}}{=} f(x_1) > f(x^*) = 0$, thus $x_2 < x_1$. And,

$$\begin{aligned} f(x_2) &\stackrel{\text{MVT}}{=} f(x_1) + f'(c_1)(x_2 - x_1) \quad \text{for some } c_1 \in (x_2, x_1) \\ &> f(x_1) + f'(x_1)(x_2 - x_1) = f'(x_1)(x_1 - x_2) + f'(x_1)(x_2 - x_1) = 0 \end{aligned}$$

Now, since $f(x_2) > 0 = f(x^*)$, the Mean-Value Theorem implies that $x_2 > x^*$.

To use induction, suppose that for some $n \geq 1$, $x^* < x_{n+1} < x_n$. Then,

$$f(x_{n+1}) = f'(x_{n+1})(x_{n+1} - x_{n+2}) > 0$$

Thus $x_{n+2} < x_{n+1}$ and

$$\begin{aligned} f(x_{n+2}) &\stackrel{\text{MVT}}{=} f(x_{n+1}) + f'(c_{n+1})(x_{n+2} - x_{n+1}) \quad \text{for some } c_{n+1} \in (x_{n+2}, x_{n+1}) \\ &\geq f(x_{n+1}) + f'(x_{n+1})(x_{n+2} - x_{n+1}) \\ &= f(x_{n+1})(x_{n+1} - x_{n+2}) + f'(x_{n+1})(x_{n+2} - x_{n+1}) = 0 \end{aligned}$$

Again, the Mean-Value Theorem implies that $x_{n+2} > x^*$. Therefore, induction completes.

Now, $x_n \rightarrow x'$ as $n \rightarrow \infty$ for some $x' \in [x^*, x_1]$ since $\{x_n\}$ is Bounded below and Decreasing.

Still it remains that to show $x' = x^*$. By Continuity,

$$\begin{aligned} &f'(x_n)(x_{n+1} - x_n) + f(x_n) = 0 \\ \implies &\lim_{n \rightarrow \infty} [f'(x_n)(x_{n+1} - x_n) + f(x_n)] = f\left(\lim_{n \rightarrow \infty} x_n\right) = f(x') = 0 \end{aligned}$$

Since the root of f is unique, thus $x' = x^*$.

Step 3. Establishing the error bound.

The Taylor's Theorem implies that

$$f(x^*) = f(x_n) + f'(x_n)(x^* - x_n) + \frac{f''(t_n)}{2}(x^* - x_n)^2 \quad \text{for some } t_n \in (x^*, x_n)$$

$$\implies x_{n+1} - x^* = \frac{f''(t_n)}{2f'(x_n)}(x^* - x_n)^2$$

Consequently,

$$0 \leq x_{n+1} - x^* = \frac{f''(t_n)}{2f'(x_n)}(x_n - x^*)^2 = \frac{f''(t_n)}{2f'(x_n)} \cdot \left(\frac{f''(t_{n-1})}{2f'(x_{n-1})} \right)^2 (x_{n-1} - x^*)^4 = \dots$$

$$= \prod_{i=1}^n \left[\frac{f''(t_i)}{2f'(x_i)} \right]^{2^{(n+1-i)}} [x_1 - x^*]^{2^n} \leq \left[\frac{M}{2\delta} \right]^{2^{n+1}-1} [x_1 - x^*]^{2^n}$$

□

11.4.4 Gradient Descent

Theorem 17. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable function that satisfies the following conditions:

1. f is *Convex function*.
2. f' is *Lipschitz Continuous* with Lipschitz constant of f , $L > 0$. In this, f is called *L-Smooth*.
3. f has at least one local minimizer x^* .

Then, x^* is a Global minimizer of \mathbb{R} , and there exists a unique closed interval M containing x^* such that

$$\forall x \in M, t \notin M, f(x) = f(x^*) < f(t)$$

And, given initial point $x_0 \in \mathbb{R}$ and $0 < \gamma \leq \frac{1}{L}$, define a sequence $\{x_n\}$ inductively as follows:

$$x_{n+1} = x_n - \gamma \cdot f'(x_n)$$

Then, for any $N \in \mathbb{N}$,

$$f(x_N) - f(x^*) \leq \frac{|x_0 - x^*|^2}{2\gamma N}$$

Proof. Let $x^* \in \mathbb{R}$ be a local minimizer. That is, there exists a $\delta > 0$ such that $\forall t \in (x^* - \delta, x^* + \delta)$, $f(x^*) \leq f(t)$. Then,

$$0 \leq \lim_{t \rightarrow x^*+} \frac{f(x^*) - f(t)}{x^* - t} = f'(x^*) = \lim_{t \rightarrow x^*-} \frac{f(x^*) - f(t)}{x^* - t} \leq 0$$

thus, $f'(x^*) = 0$. And, by convexity, f' is monotonically increasing. Now, The Fundamental Theorem of Calculus gives:

$$\forall x \in \mathbb{R}, f(x) = \int_{x^*}^x f'(t)dt + f(x^*) \geq f(x^*)$$

Therefore, x^* is a Global minimizer of f .

Now, establish the closed interval M . Since f' is Lipschitz Continuous, thus f' is Continuous.

Let $D \stackrel{\text{def}}{=} \{x \in \mathbb{R} \mid f'(x) = 0\}$. (Note that: $x^* \in D$, thus D is not empty set.)

D is closed because: Let $\{x_n\}$ be a convergent sequence in D . That is, for all $n \in \mathbb{N}$, $f'(x_n) = 0$. Then, by continuity,

$$f\left(\lim_{n \rightarrow \infty} x_n\right) = \lim_{n \rightarrow \infty} f(x_n) = 0$$

The limit of $\{x_n\}$ is contained in D , thus D is closed.

And, D is interval: i.e, for any $x \in (\inf D, \sup D)$, $x \in D$ because:

Suppose that there exists $x \in (\inf D, \sup D)$ such that $x \notin D$. That is, $f'(x) \neq 0$. This is Contradiction with Monotonicity.

To set error of upper bound, we make inequalities: Let $x, y \in \mathbb{R}$ be given.

The Fundamental Theorem of Calculus and L -Smooth condition gives:

$$\begin{aligned} f(y) - f(x) &= \int_x^y f'(t)dt = \int_0^1 f'(x + (y-x)u)(y-x)du = f'(x)(y-x) + \int_0^1 (f'(x + (y-x)u) - f'(x))(y-x)du \\ &\stackrel{2.}{\leq} f'(x)(y-x) + L \cdot |y-x|^2 \int_0^1 u du = f'(x)(y-x) + \frac{L}{2}|y-x|^2 \end{aligned}$$

For any $\lambda > 0$, Put $y = x - \lambda f'(x)$. Then,

$$f(x - \lambda f'(x)) \leq f(x) - f'(x)(\lambda f'(x)) + \frac{L}{2}|\lambda f'(x)|^2 = f(x) + \lambda \left(\frac{L\lambda}{2} - 1 \right) |f'(x)|^2$$

Put $\lambda = \frac{1}{L}$, then

$$f\left(x - \frac{f'(x)}{L}\right) \leq f(x) - \frac{L}{2}|f'(x)|^2 \implies \frac{L}{2}|f'(x)|^2 \leq f(x) - f\left(x - \frac{f'(x)}{L}\right) \leq f(x) - \inf f$$

Meanwhile, the convexity gives: for any $x, y \in \mathbb{R}$,

$$f'(x)(y-x) \leq f(y) - f(x) \leq f'(y)(y-x)$$

since derivative of convex function increase monotonically. Put $z = y - \frac{1}{L}(f'(y) - f'(x))$. Then,

$$\begin{aligned} f(x) - f(y) &= f(x) - f(z) + f(z) - f(y) \\ &\leq f'(x)(x-z) + f'(y)(z-y) + \frac{L}{2}|z-y|^2 \\ &= f'(x)\left(x-y + \frac{1}{L}(f'(y) - f'(x))\right) - f'(y)\left(\frac{1}{L}(f'(y) - f'(x))\right) + \frac{L}{2}\left|\frac{1}{L}(f'(y) - f'(x))\right|^2 \\ &= f'(x)(x-y) - \frac{1}{L}|f'(y) - f'(x)|^2 + \frac{1}{2L}|f'(y) - f'(x)|^2 \\ &= f'(x)(x-y) - \frac{1}{2L}|f'(y) - f'(x)|^2 \end{aligned}$$

Now,

$$\begin{cases} \frac{1}{2L}|f'(y) - f'(x)|^2 \leq f'(x)(x-y) - (f(x) - f(y)) \\ \frac{1}{2L}|f'(x) - f'(y)|^2 \leq f'(y)(y-x) - (f(y) - f(x)) \end{cases} \implies \frac{1}{L}|f'(y) - f'(x)|^2 \leq (f'(y) - f'(x))(y-x)$$

Since above inequalities, we obtain that

$$\begin{aligned} |x_{n+1} - x^*|^2 &= |x_n - \gamma \cdot f'(x_n) - x^*|^2 = |(x_n - x^*) - \gamma \cdot f'(x_n)|^2 \\ &= |x_n - x^*|^2 - 2\gamma|f'(x_n)| \cdot |x_n - x^*| + \gamma^2|f'(x_n)|^2 \\ &\leq |x_n - x^*|^2 - 2\gamma\frac{1}{L}|f'(x_n)|^2 + \gamma^2|f'(x_n)|^2 \\ &= |x_n - x^*|^2 + \left(\gamma^2 - \frac{2\gamma}{L}\right)|f'(x_n)|^2 \leq |x_n - x^*|^2 \end{aligned}$$

Thus, $|x_n - x^*|$ decrease as $n \rightarrow \infty$. That is, $|x_n - x^*| \leq |x_0 - x^*|$ for all $n \in \mathbb{N}$.

Consider x_{n+1} and x_n . First, we obtain

$$\begin{aligned} f(x_{n+1}) &\leq f(x_n) + f'(x_n)(x_{n+1} - x_n) + \frac{L}{2}|x_{n+1} - x_n|^2 \\ &= f(x_n) - \gamma|f'(x_n)|^2 + \frac{L}{2}\gamma^2|f'(x_n)|^2 \\ &= f(x_n) - \left(\gamma - \frac{L}{2}\gamma^2\right)|f'(x_n)|^2 \end{aligned}$$

Subtracting $f(x^*)$ above, then

$$f(x_{n+1}) - f(x^*) \leq f(x_n) - f(x^*) - \left(\gamma - \frac{L}{2}\gamma^2\right)|f'(x_n)|^2$$

Meanwhile, Convexity gives

$$f(x_n) - f(x^*) \leq f'(x_n)(x_n - x^*) \leq |f'(x_n)||x_n - x^*| \leq |f'(x_n)||x_0 - x^*|$$

Combining above two inequalities,

$$f(x_{n+1}) - f(x^*) \leq f(x_n) - f(x^*) - \left(\gamma - \frac{L}{2}\gamma^2\right) \cdot \frac{|f(x_n) - f(x^*)|^2}{|x_0 - x^*|^2}$$

Dividing Both Sides by $(f(x_{n+1}) - f(x^*))(f(x_n) - f(x^*))$,

$$\begin{aligned} \frac{1}{f(x_n) - f(x^*)} &\leq \frac{1}{f(x_{n+1}) - f(x^*)} - \left(\gamma - \frac{L}{2}\gamma^2\right) \cdot \frac{f(x_n) - f(x^*)}{f(x_{n+1}) - f(x^*)} \frac{1}{|x_0 - x^*|^2} \\ \implies \left(\gamma - \frac{L}{2}\gamma^2\right) \cdot \frac{f(x_n) - f(x^*)}{f(x_{n+1}) - f(x^*)} \frac{1}{|x_0 - x^*|^2} &\leq \frac{1}{f(x_{n+1}) - f(x^*)} - \frac{1}{f(x_n) - f(x^*)} \\ \implies \left(\gamma - \frac{L}{2}\gamma^2\right) \cdot \frac{1}{|x_0 - x^*|^2} &\leq \frac{1}{f(x_{n+1}) - f(x^*)} - \frac{1}{f(x_n) - f(x^*)} \\ \implies \sum_{n=0}^{N-1} \left[\left(\gamma - \frac{L}{2}\gamma^2\right) \cdot \frac{1}{|x_0 - x^*|^2} \right] &\leq \sum_{n=0}^{N-1} \left[\frac{1}{f(x_{n+1}) - f(x^*)} - \frac{1}{f(x_n) - f(x^*)} \right] = \frac{1}{f(x_N) - f(x^*)} - \frac{1}{f(x_0) - f(x^*)} \end{aligned}$$

Consequently,

$$\frac{2\gamma N}{|x_0 - x^*|^2} \leq N \cdot \left[\left(\gamma - \frac{L}{2} \gamma^2 \right) \cdot \frac{1}{|x_0 - x^*|^2} \right] \leq \frac{1}{f(x_N) - f(x^*)} - \frac{1}{f(x_0) - f(x^*)} \leq \frac{1}{f(x_N) - f(x^*)}$$

Organizing the formula, as result:

$$f(x_N) - f(x^*) \leq \frac{|x_0 - x^*|^2}{2\gamma N}$$

□

11.5 Integral

11.5.1 Inequality of Riemann–Stieltjes Integral

Let $p, q \geq 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, and functions lying on $[a, b]$.

Lemma 9. Let $f, g \in \mathcal{R}(\alpha)$ with $f, g \geq 0$, and $\int_a^b [f(x)]^p d\alpha = \int_a^b [g(x)]^q d\alpha = 1$. Then, $\int_a^b f(x)g(x) d\alpha \leq 1$.

Proof. For any $x \in [a, b]$, the Young's Inequality gives

$$0 \leq f(x)g(x) \leq \frac{[f(x)]^p}{p} + \frac{[g(x)]^q}{q}$$

Now,

$$\int_a^b f(x)g(x) d\alpha \leq \int_a^b \frac{[f(x)]^p}{p} + \frac{[g(x)]^q}{q} d\alpha = \frac{1}{p} \int_a^b [f(x)]^p d\alpha + \frac{1}{q} \int_a^b [g(x)]^q d\alpha = \frac{1}{p} + \frac{1}{q} = 1$$

□

Definition 12. Let $f \in \mathcal{R}(\alpha)$. Define a *Norm* of f :

$$\|f\|_p \stackrel{\text{def}}{=} \left(\int_a^b |f(x)|^p d\alpha \right)^{\frac{1}{p}}$$

This becomes actually norm of set of Stieltjes Integrable functions, $\mathcal{F} \stackrel{\text{def}}{=} \{f : [a, b] \rightarrow \mathbb{C} \mid f \in \mathcal{R}(\alpha)\}$.

Lemma 10. Hölder's Inequality

Let $f, g \in \mathcal{F}$. Then,

$$\left| \int_a^b f(x)g(x) d\alpha \right| \leq \left[\int_a^b |f(x)|^p d\alpha \right]^{\frac{1}{p}} \cdot \left[\int_a^b |g(x)|^q d\alpha \right]^{\frac{1}{q}}$$

Proof. Use above definition, Rewrite:

$$\|f\|_p^p = \int_a^b |f(x)|^p d\alpha, \quad \|g\|_q^q = \int_a^b |g(x)|^q d\alpha$$

Now, we can make the condition of above lemma,

$$\int_a^b \left[\frac{|f(x)|}{\|f\|_p} \right]^p d\alpha = \frac{1}{\|f\|_p^p} \cdot \int_a^b |f(x)|^p d\alpha = 1, \quad \int_a^b \left[\frac{|g(x)|}{\|g\|_q} \right]^q d\alpha = \frac{1}{\|g\|_q^q} \cdot \int_a^b |g(x)|^q d\alpha = 1$$

And apply this,

$$\int_a^b \frac{|f(x)| \cdot |g(x)|}{\|f\|_p \|g\|_q} d\alpha \leq 1 \implies \int_a^b |f(x)| |g(x)| d\alpha \leq \|f\|_p \|g\|_q = \left[\int_a^b |f(x)|^p d\alpha \right]^{\frac{1}{p}} \cdot \left[\int_a^b |g(x)|^q d\alpha \right]^{\frac{1}{q}}$$

Finally, the general property of integral of product gives

$$\left| \int_a^b f(x)g(x) d\alpha \right| \leq \int_a^b |f(x)| |g(x)| d\alpha \leq \|f\|_p \|g\|_q = \left[\int_a^b |f(x)|^p d\alpha \right]^{\frac{1}{p}} \cdot \left[\int_a^b |g(x)|^q d\alpha \right]^{\frac{1}{q}}$$

□

Theorem 18. Minkowski inequality

Let $f, g \in \mathcal{F}$. Then, for any $p \geq 1$, $\|f + g\|_p \leq \|f\|_p + \|g\|_p$.

Proof.

$$\begin{aligned}
\|f + g\|_p^p &= \int_a^b |f + g|^p d\alpha = \int_a^b |f + g| |f + g|^{p-1} d\alpha \\
&\leq \int_a^b [|f| + |g|] |f + g|^{p-1} d\alpha \\
&= \int_a^b |f| |f + g|^{p-1} d\alpha + \int_a^b |g| |f + g|^{p-1} d\alpha \\
&\stackrel{\text{Hölder}}{\leq} \left[\int_a^b |f|^p d\alpha \right]^{\frac{1}{p}} \left[\int_a^b |f + g|^{(p-1)\frac{p}{p-1}} d\alpha \right]^{\frac{p-1}{p}} + \left[\int_a^b |g|^p d\alpha \right]^{\frac{1}{p}} \left[\int_a^b |f + g|^{(p-1)\frac{p}{p-1}} d\alpha \right]^{\frac{p-1}{p}} \\
&= \left[\int_a^b |f + g|^p d\alpha \right]^{\frac{p-1}{p}} \left(\left[\int_a^b |f|^p d\alpha \right]^{\frac{1}{p}} + \left[\int_a^b |g|^p d\alpha \right]^{\frac{1}{p}} \right) = \|f + g\|_p^{p-1} \cdot (\|f\|_p + \|g\|_p)
\end{aligned}$$

Now,

$$\|f + g\|_p^p \cdot \|f + g\|_p^{1-p} = \|f + g\|_p \leq \|f\|_p + \|g\|_p$$

□

Chapter 12

Measure

Chapter 13

Complex Analysis

Chapter 14

Differential Geometry

Chapter 15

Differential Equation

Chapter 16

Spaces

16.1 \mathbb{R}^n

16.1.1 Inner Product in \mathbb{R}

16.1.2 p -norm in \mathbb{R}^n

Definition 13. Let \mathbb{R}^n be given. Define p -norm on \mathbb{R}^n as:

$$d_p : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R} : (\mathbf{x}, \mathbf{y}) \mapsto \left(\sum_{i=1}^n |x_i - y_i|^p \right)^{\frac{1}{p}}, \quad (\mathbf{x} = (x_1, \dots, x_n), \mathbf{y} = (y_1, \dots, y_n))$$

where $p \in [1, \infty]$. In particular, p -norm is a *Metric*, being *Minkowski inequality*.

Lemma 11. Young's inequality

Let $u, v > 0$, and $p, q \in [1, \infty]$ such that $\frac{1}{p} + \frac{1}{q} = 1$.

Then,

$$uv \leq \frac{1}{p}u^p + \frac{1}{q}v^q$$

Proof. Since $f(x) = \log x$ is concave, we obtain

$$\forall \lambda \in [0, 1], \quad \lambda f(x) + (1 - \lambda)f(y) \leq f(\lambda x + (1 - \lambda)y)$$

thus,

$$\log \left(\frac{1}{p}u^p + \frac{1}{q}v^q \right) \geq \frac{1}{p} \log(u^p) + \frac{1}{q} \log(v^q) = \log(uv)$$

Since $\exp(x)$ increasing, we get

$$\exp \left(\log \left(\frac{1}{p}u^p + \frac{1}{q}v^q \right) \right) \geq \exp(\log(uv))$$

i.e.,

$$uv \leq \frac{1}{p}u^p + \frac{1}{q}v^q$$

□

Lemma 12. Holder's inequality

Let $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ be given, and $p, q \in [1, \infty]$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then,

$$\sum_{i=1}^n |x_i y_i| \leq \left(\sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}} \cdot \left(\sum_{i=1}^n |y_i|^q \right)^{\frac{1}{q}}$$

Proof. Denote that

$$\|x\|_p \stackrel{\text{def}}{=} \left(\sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}}$$

Then, since young's inequality, for each $i \in \{1, 2, \dots, n\}$,

$$\frac{|x_i|}{\|x\|_p} \cdot \frac{|y_i|}{\|y\|_q} \leq \frac{1}{p} \cdot \frac{|x_i|^p}{\|x\|_p^p} + \frac{1}{q} \cdot \frac{|y_i|^q}{\|y\|_q^q}$$

Summing for all $i = 1, 2, \dots, n$:

$$\frac{1}{\|x\|_p \|y\|_q} \cdot \sum_{i=1}^n |x_i y_i| \leq \frac{1}{p} + \frac{1}{q} = 1$$

Therefore,

$$\sum_{i=1}^n |x_i y_i| \leq \left(\sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}} \cdot \left(\sum_{i=1}^n |y_i|^q \right)^{\frac{1}{q}}$$

□

Theorem 19. Minkowski inequality

Given complex-valued sequences $\{x_i\}_{i=1}^n$ and $\{y_i\}_{i=1}^n$,

$$\left[\sum_{i=1}^n |x_i + y_i|^p \right]^{\frac{1}{p}} \leq \left[\sum_{i=1}^n |x_i|^p \right]^{\frac{1}{p}} + \left[\sum_{i=1}^n |y_i|^p \right]^{\frac{1}{p}}$$

Proof. Denote

$$|x_i + y_i|^p = |x_i + y_i| \cdot |x_i + y_i|^{p-1}$$

Then,

$$\begin{aligned} \sum_{i=1}^n |x_i + y_i|^p &= \sum_{i=1}^n |x_i + y_i| \cdot |x_i + y_i|^{p-1} \\ &= \sum_{i=1}^n (|x_i| + |y_i|) \cdot |x_i + y_i|^{p-1} \\ &= \sum_{i=1}^n |x_i| \cdot |x_i + y_i|^{p-1} + \sum_{i=1}^n |y_i| \cdot |x_i + y_i|^{p-1} \\ &\stackrel{\text{Hölder}}{\leq} \left[\sum_{i=1}^n |x_i|^p \right]^{\frac{1}{p}} \cdot \left[\sum_{i=1}^n |x_i + y_i|^p \right]^{\frac{p-1}{p}} + \left[\sum_{i=1}^n |y_i|^p \right]^{\frac{1}{p}} \cdot \left[\sum_{i=1}^n |x_i + y_i|^p \right]^{\frac{p-1}{p}} \\ &= \left[\left(\sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}} + \left(\sum_{i=1}^n |y_i|^p \right)^{\frac{1}{p}} \right] \cdot \left[\sum_{i=1}^n |x_i + y_i|^p \right]^{\frac{p-1}{p}} \end{aligned}$$

Now, Divide each side as $\left[\sum_{i=1}^n |x_i + y_i|^p\right]^{\frac{p-1}{p}}$, then we obtain

$$\left[\sum_{i=1}^n |x_i + y_i|^p\right]^{1-\frac{p-1}{p}} = \left[\sum_{i=1}^n |x_i + y_i|^p\right]^{\frac{1}{p}} \leq \left[\left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^n |y_i|^p\right)^{\frac{1}{p}}\right]$$

□

Theorem 20. Let d_{p_1}, d_{p_2} are p -norm on \mathbb{R}^n with $1 \leq p_1 < p_2 \leq \infty$. Then,

$$\exists C > 0 \text{ s.t. } \forall x, y \in \mathbb{R}^n, d_{p_2}(x, y) \leq d_{p_1}(x, y) \leq C d_{p_2}(x, y)$$

In particular, $C = n^{\frac{1}{p_1} - \frac{1}{p_2}}$.

Proof. Let $p_1 < p_2$.

For show that first-inequality,

$$1 = \sum_{i=1}^n \left[\frac{|x_i - y_i|}{\left[\sum_{i=1}^n |x_i - y_i|^{p_2}\right]^{\frac{1}{p_2}}} \right]^{p_2} \leq \sum_{i=1}^n \left[\frac{|x_i - y_i|}{\left[\sum_{i=1}^n |x_i - y_i|^{p_2}\right]^{\frac{1}{p_2}}} \right]^{p_1} = \frac{\sum_{i=1}^n |x_i - y_i|^{p_1}}{\left[\sum_{i=1}^n |x_i - y_i|^{p_2}\right]^{\frac{p_1}{p_2}}} = \left[\frac{\left[\sum_{i=1}^n |x_i - y_i|^{p_1}\right]^{\frac{1}{p_1}}}{\left[\sum_{i=1}^n |x_i - y_i|^{p_2}\right]^{\frac{1}{p_2}}} \right]^{p_1}$$

Thus, we obtain that:

$$1 \leq \left[\frac{\left[\sum_{i=1}^n |x_i - y_i|^{p_1}\right]^{\frac{1}{p_1}}}{\left[\sum_{i=1}^n |x_i - y_i|^{p_2}\right]^{\frac{1}{p_2}}} \right]^{p_1} \iff 1 \leq \frac{\left[\sum_{i=1}^n |x_i - y_i|^{p_1}\right]^{\frac{1}{p_1}}}{\left[\sum_{i=1}^n |x_i - y_i|^{p_2}\right]^{\frac{1}{p_2}}} \iff \left[\sum_{i=1}^n |x_i - y_i|^{p_2}\right]^{\frac{1}{p_2}} \leq \left[\sum_{i=1}^n |x_i - y_i|^{p_1}\right]^{\frac{1}{p_1}}$$

For show that second-inequality, using Hölder's inequality.

$$\begin{aligned} (d_{p_1}(x, y))^{p_1} &= \sum_{i=1}^n |x_i - y_i|^{p_1} = \sum_{i=1}^n |x_i - y_i|^{p_1} \cdot 1 \\ &\stackrel{\text{Hölder}}{\leq} \left[\sum_{i=1}^n \left(|x_i - y_i|^{p_1 \cdot \frac{p_2}{p_1}} \right) \right]^{\frac{p_1}{p_2}} \cdot \left[\sum_{i=1}^n 1^{\frac{p_2}{p_2 - p_1}} \right]^{1 - \frac{p_1}{p_2}} = \left[\sum_{i=1}^n (|x_i - y_i|^{p_2}) \right]^{\frac{p_1}{p_2}} \cdot n^{1 - \frac{p_1}{p_2}} \end{aligned}$$

Taking the $\frac{1}{p_1}$ -th power of both sides, then

$$d_{p_1}(x, y) \leq \left[\sum_{i=1}^n (|x_i - y_i|^{p_2}) \right]^{\frac{1}{p_2}} \cdot n^{\frac{1}{p_1} - \frac{1}{p_2}} = n^{\frac{1}{p_1} - \frac{1}{p_2}} \cdot d_{p_2}(x, y)$$

□

Corollary 4. Let \mathbb{R}^n be given as a set, and $d_{p_1}, d_{p_2} : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ are p -norm on \mathbb{R}^n . Then,

$$\mathcal{T}_{d_{p_1}} = \mathcal{T}_{d_{p_2}}$$

For every $p \geq 1$, the metric space (\mathbb{R}^n, d_p) induces the same topology as the product topology on \mathbb{R}^n . In particular, \mathbb{R}^n with the product topology coincides with \mathbb{R}^n endowed with any p -norm.

16.1.3 Open and Closed set in \mathbb{R}^n

Definition 14. For $p \in [1, \infty]$, define p -Ball in \mathbb{R}^n as:

$$B_p(x, r) \stackrel{\text{def}}{=} \{y \in \mathbb{R}^n : \|x - y\|_p < r\}$$

Since all p -norms are equivalent, for any $p \in [1, \infty]$, the collection

$$\beta_p \stackrel{\text{def}}{=} \{B_p(x, r) \mid x \in \mathbb{Q}^n, r \in \mathbb{Q}^+\} \cup \{\emptyset, \mathbb{R}^n\}$$

is Countable basis of \mathbb{R}^n . Immediately, we obtain:

Lemma 13. Every open set in \mathbb{R}^n is a countable union of p -Balls.

We call 2-Ball the *Ball*, and ∞ -Ball the *Cube*.

Theorem 21. Let $U \subseteq \mathbb{R}^n$ be an open set. Then, U is a countable union of almost disjoint closed cubes.

16.2 Topological Vector Space

16.3 Hilbert Space

Definition 15. Complete Inner product Vector Space is called *Hilbert Space*.

16.3.1 Hilbert Space in \mathbb{R}^ω

Definition 16. Define $\mathbb{R}^\omega \stackrel{\text{def}}{=} \prod_{i=1}^{\infty} \mathbb{R}$ as the countable product of Euclidean space \mathbb{R} with product topology.

And define $\mathbb{H} \stackrel{\text{def}}{=} \left\{ \{x_n\}_{n=1}^{\infty} \mid \sum_{n=1}^{\infty} x_n^2 < \infty \right\} \subset \mathbb{R}^\omega$, **Metric** on \mathbb{H} as $\mu : \mathbb{H} \times \mathbb{H} \rightarrow \mathbb{R} : (\{x_n\}, \{y_n\}) \mapsto \sqrt{\sum_{i=1}^{\infty} (x_i - y_i)^2}$.

The Metric Space (\mathbb{H}, μ) is called *Hilbert Space* or l_2 Space.

Define the operations elementwise; then $(\mathbb{H}, +, \times)$ is a Vector Space over \mathbb{R} .

Moreover, \mathbb{H} is Complete Metric Space and Inner product Vector Space.

Lemma 14. $\mu : \mathbb{H} \times \mathbb{H} \rightarrow \mathbb{R} : (\{x_n\}, \{y_n\}) \mapsto \sqrt{\sum_{i=1}^{\infty} (x_i - y_i)^2}$ is Metric function induced by the inner product.

Proof. We know that \mathbb{R}^ω is Vector Space. Moreover, $\mathbb{H} \subset \mathbb{R}^\omega$ is Subspace. Using subspace criteria:

$S \subset V$ is Subspace of Vector Space V if and only if $0 \in S$ and For any $x, y \in S$ and $a \in F$, $ax + y \in S$.

Clearly, $\{0\} \in \mathbb{H}$. Let $a \in \mathbb{R}$ and $\{x_n\}, \{y_n\} \in \mathbb{H}$ be given. Then, $a\{x_n\} + \{y_n\} = \{ax_n + y_n\} \in \mathbb{H}$ because:

$$\sum_{i=1}^{\infty} (ax_i + y_i)^2 = \sum_{i=1}^{\infty} [a^2 x_i^2 + 2ax_i y_i + y_i^2] \stackrel{(*)}{=} a^2 \sum_{i=1}^{\infty} x_i^2 + 2a \sum_{i=1}^{\infty} x_i y_i + \sum_{i=1}^{\infty} y_i^2 < \infty$$

The $(*)$ given by:

$$\sum_{i=1}^{\infty} |x_i y_i| = \sum_{i=1}^{\infty} |x_i| |y_i| \leq \sum_{i=1}^{\infty} (\max(|x_i|, |y_i|))^2 \leq \sum_{i=1}^{\infty} (x_n^2 + y_n^2) = \sum_{i=1}^{\infty} x_n^2 + \sum_{i=1}^{\infty} y_n^2 < \infty \quad (*)$$

Thus \mathbb{H} is Vector Space over \mathbb{R} . Now, define *inner product* on \mathbb{H} as:

$$\langle \cdot, \cdot \rangle : \mathbb{H} \times \mathbb{H} \rightarrow \mathbb{R} : (\{x_n\}, \{y_n\}) \mapsto \sum_{i=1}^{\infty} x_i y_i$$

This definition is well-defined since $(*)$. And, Linearity in first:

$$\langle a\{x_n\} + \{y_n\}, \{z_n\} \rangle = \langle \{ax_n + y_n\}, \{z_n\} \rangle = \sum_{i=1}^{\infty} (ax_i + y_i) z_i = a \sum_{i=1}^{\infty} x_i z_i + \sum_{i=1}^{\infty} y_i z_i = a \langle \{x_n\}, \{z_n\} \rangle + \langle \{y_n\}, \{z_n\} \rangle$$

The other conditions are clear. Thus, $(\mathbb{H}, \langle \cdot, \cdot \rangle)$ is *inner product space*.

Using *inner product*, define the *Norm* on \mathbb{H} as:

$$\|\cdot\| : \mathbb{H} \rightarrow \mathbb{R} : \{x_n\} \mapsto \sqrt{\langle \{x_n\}, \{x_n\} \rangle}$$

Finally, define *Metric* on \mathbb{H} as:

$$\mu : \mathbb{H} \times \mathbb{H} \rightarrow \mathbb{R} : (\{x_n\}, \{y_n\}) \mapsto \|\{x_n\} - \{y_n\}\| = \sqrt{\sum_{i=1}^{\infty} (x_i - y_i)^2}$$

□

Theorem 22. *Hilbert Space is Separable.*

Proof. For each $n \in \mathbb{N}$, define $D_n \stackrel{\text{def}}{=} \{\{p_n\} \mid p_i \in \mathbb{Q}, p_{n+1} = p_{n+1} = \dots = 0\}$ and $D \stackrel{\text{def}}{=} \bigcup_{n \in \mathbb{N}} D_n$.

Then, D is countable set. We will show that $\overline{D} = \mathbb{H}$.

Let $\epsilon > 0$ and $\{x_n\} \in \mathbb{H}$ be given. Since convergence, there exists $N \in \mathbb{N}$ such that

$$\sum_{i=N+1}^{\infty} x_i^2 = \sum_{i=1}^{\infty} x_i^2 - \sum_{i=1}^N x_i^2 < \frac{\epsilon^2}{2}$$

Since density of Rationals, put each $i = 1, 2, \dots, N$, $p_i \in \mathbb{Q} \mid |x_i - p_i| < \frac{\epsilon}{\sqrt{2N}}$ and $p_i = 0$ for $i \geq N + 1$.

Then, $\{p_n\} \in D_n \subset D$ and

$$\mu(\{x_n\}, \{p_n\}) = \sqrt{\sum_{i=1}^N (x_i - p_i)^2 + \sum_{i=N+1}^{\infty} (x_i - p_i)^2} = \sqrt{\sum_{i=1}^N (x_i - p_i)^2 + \sum_{i=N+1}^{\infty} x_i^2} < \sqrt{N \cdot \frac{\epsilon^2}{2N} + \frac{\epsilon^2}{2}} = \epsilon$$

□

Corollary 5. *Hilbert Space is Second-Countable.*

Theorem 23. *Hilbert Space is Complete.*

Proof. Let $\{\{x_{n,i}\}_{i=1}^{\infty}\}_{n=1}^{\infty}$ be a Cauchy sequence in \mathbb{H} . For any fixed $n, m \in \mathbb{N}$ and for each $j \in \mathbb{N}$,

$$|x_{n,j} - x_{m,j}| < \mu(\{x_{n,i}\}, \{x_{m,i}\}) = \sqrt{\sum_{i=1}^{\infty} (x_{n,i} - x_{m,i})^2}$$

That is, for each $j \in \mathbb{N}$, $\{x_{n,j}\}$ is Cauchy sequence in \mathbb{R} . Since \mathbb{R} is Complete, put $y_j \stackrel{\text{def}}{=} \lim_{n \rightarrow \infty} x_{n,j}$, each $j \in \mathbb{N}$.

Let $\epsilon > 0$ be given. Then, there exists $N \in \mathbb{N}$ such that $n, m \geq N \implies \mu(\{x_{n,i}\}, \{x_{m,i}\}) < \frac{\epsilon}{2}$.

Meanwhile, for each $k \in \mathbb{N}$,

$$\sum_{i=1}^k (x_{n,i} - x_{m,i})^2 \leq \sum_{i=1}^{\infty} (x_{n,i} - x_{m,i})^2 = [\mu(\{x_{n,i}\}, \{x_{m,i}\})]^2$$

Thus, $n, m \geq N \implies \sum_{i=1}^k (x_{n,i} - x_{m,i})^2 < \left(\frac{\epsilon}{2}\right)^2$, for each $k \in \mathbb{N}$.

Taking limit to m , then $n \geq N \implies \lim_{m \rightarrow \infty} \left(\sum_{i=1}^k (x_{n,i} - x_{m,i})^2\right) = \sum_{i=1}^k \left(x_{n,i} - \lim_{m \rightarrow \infty} x_{m,i}\right)^2 = \sum_{i=1}^k (x_{n,i} - y_i)^2 < \left(\frac{\epsilon}{2}\right)^2$.

And, for all $k \in \mathbb{N}$,

$$\sum_{i=1}^k y_i^2 = \sum_{i=1}^k (2(x_{n,i}^2 + (x_{n,i} - y_i)^2)) \leq 2\|\{x_{n,i}\}_{i=1}^{\infty}\|^2 + \left(\frac{\epsilon}{2}\right)^2$$

Thus $\{y_i\} \in \mathbb{H}$. As a result,

$$n \geq N \implies \mu(\{x_n\}, \{y_n\}) = \sqrt{\sum_{i=1}^{\infty} (x_{n,i} - y_i)^2} = \sqrt{\lim_{k \rightarrow \infty} \sum_{i=1}^k (x_{n,i} - y_i)^2} < \frac{\epsilon}{2}$$

□

Theorem 24. $\mathbb{H} \subset \mathbb{R}^\omega$ with subspace topology is Metrizable.

Proof. We will use two Lemmas:

Lemma 15. Countable Product of Metric Space is Metrizable.

Proof. Let (X_i, d_i) be a metric Space, for each $i \in \mathbb{N}$.

If $d : X \times X \rightarrow \mathbb{R}$ is a Metric, then $\frac{d}{1+d}$ is also Metric, because

$$\frac{d(x, z)}{1 + d(x, z)} \underset{\substack{\frac{x}{1+x} \\ \text{increasing}}}{\leq} \frac{d(x, y) + d(y, z)}{1 + d(x, y) + d(y, z)} \underset{d \geq 0}{\leq} \frac{d(x, y)}{1 + d(x, y)} + \frac{d(y, z)}{1 + d(y, z)} \quad (*)$$

Using this fact, define

$$d_\Pi : \prod X_i \times \prod X_i \rightarrow \mathbb{R} : (\{x_n\}_{n=1}^\infty, \{y_n\}_{n=1}^\infty) \mapsto \sum_{i=1}^\infty \left[\frac{1}{2^i} \cdot \frac{d_i(x_i, y_i)}{1 + d_i(x_i, y_i)} \right]$$

Then d_Π is a Metric because:

$$\begin{aligned} d_\Pi(\{x_n\}_{n=1}^\infty, \{z_n\}_{n=1}^\infty) &= \sum_{i=1}^\infty \left[\frac{1}{2^i} \cdot \frac{d_i(x_i, z_i)}{1 + d_i(x_i, z_i)} \right] \\ &\stackrel{(*)}{\leq} \sum_{i=1}^\infty \left[\frac{1}{2^i} \cdot \left(\frac{d_i(x_i, y_i)}{1 + d_i(x_i, y_i)} + \frac{d_i(y_i, z_i)}{1 + d_i(y_i, z_i)} \right) \right] \\ &= \sum_{i=1}^\infty \left[\frac{1}{2^i} \cdot \frac{d_i(x_i, y_i)}{1 + d_i(x_i, y_i)} \right] + \sum_{i=1}^\infty \left[\frac{1}{2^i} \cdot \frac{d_i(y_i, z_i)}{1 + d_i(y_i, z_i)} \right] \\ &= d_\Pi(\{x_n\}_{n=1}^\infty, \{y_n\}_{n=1}^\infty) + d_\Pi(\{y_n\}_{n=1}^\infty, \{z_n\}_{n=1}^\infty) \end{aligned}$$

Reflexity and symmetry are clear. □

Lemma 16. Metrizable is Hereditary.

Proof omitted.

Consequently, since $\mathbb{H} \subset \mathbb{R}^\omega$ is a subspace of a metric space, it is metrizable. □

16.4 Banach Space

16.5 L_p Space

16.6 l_p Space

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