Math Note

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This paper covers several topics in undergraduate mathematics.

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Ring Theory

3.1 Ring of Fractions

Theorem 1. Let R be a Commutative Ring, $D \subset R$ be a subset such that $\begin{cases} \text{no zero, no zero divisors} \\ \text{closed under multiplication} \end{cases}$.

Then, there exists a Commutative Ring Q with identity satisfies:

1. R can embed in Q, and every element of D becomes unit in Q. More precisely, $Q = \{rd^{-1} \mid r \in R,\ d \in D\}$.
2. Q is the smallest Ring with identity such that every element of D becomes unit in Q.

Proof. Let $\mathcal{F} \overset{\text{def}}{=} \{(r,d) \mid r \in R,\ d \in D\}$ and the relation $\sim$ on $\mathcal{F}$ by $(r_1,d_1) \sim (r_2,d_2) \iff r_1d_2 = r_2d_1$.
Then, $\sim$ is equivalent relation: reflexive and symmetirc are clear, and Suppose that $(r_1,d_1) \sim (r_2,d_2)$ and $(r_2,d_2) \sim (r_3,d_3)$.

$$r_2d_3 = r_3d_2 \implies r_2d_1d_3 = r_3d_1d_2 \implies r_1d_2d_3 = r_3d_1d_2 \implies d_2(r_1d_3 - r_3d_1) \implies r_1d_3 = r_3d_1$$

Thus transitivity shown. Define

$$\frac{r}{d} \overset{\text{def}}{=} [(r,d)] = \{(a,b) \mid (a,b) \sim (r,d)\},\ \ Q \overset{\text{def}}{=} \left\{ \frac{r}{d} \;\middle|\; r \in R,\ d \in D \right\}$$

And define operations $+, \times$ on Q:

$$\frac{r_1}{d_1} + \frac{r_2}{d_2} \overset{\text{def}}{=} \frac{r_1d_2 + r_2d_1}{d_1d_2}, \quad \frac{r_1}{d_1} \times \frac{r_2}{d_2} \overset{\text{def}}{=} \frac{r_1r_2}{d_1d_2}$$

Well-Definedness: If $\dfrac{r_1}{d_1} = \dfrac{r_1'}{d_1'}$ and $\dfrac{r_2}{d_2} = \dfrac{r_2'}{d_2'}$,

$$\frac{r_1d_2 + r_2d_1}{d_1d_2} = \frac{r_1d_2d_1'd_2' + r_2d_1d_1'd_2'}{d_1d_2d_1'd_2'} = \frac{(r_1d_1')d_2d_2' + (r_2d_2')d_1d_1'}{d_1d_2d_1'd_2'} = \frac{(r_1'd_1)d_2d_2' + (r_2'd_2)d_1d_1'}{d_1d_2d_1'd_2'} = \frac{(r_1'd_2' + r_2'd_1')d_1d_2}{d_1d_2d_1'd_2'} = \frac{r_1'd_2' + r_2'd_1'}{d_1'd_2'}$$

$$\frac{r_1r_2}{d_1d_2} = \frac{r_1r_2d_1'd_2'}{d_1d_2d_1'd_2'} = \frac{(r_1d_1')(r_2d_2')}{d_1d_2d_1'd_2'} = \frac{(r_1'd_1)(r_2'd_2)}{d_1d_2d_1'd_2'} = \frac{r_1'r_2'd_1d_2}{d_1d_2d_1'd_2'} = \frac{r_1'r_2'}{d_1'd_2'}$$

Now, $(Q,+,\times)$ constructs Commutative Ring with identity: for any $d \in D$, put $0_Q \overset{\text{def}}{=} \dfrac{0}{d}$, $1_Q \overset{\text{def}}{=} \dfrac{d}{d}$. Then,

1. $(R,+,\times)$ closed under the operations since D is closed under the multiplication.
2. $(R,+)$ has a zero: $\dfrac{r_1}{d_1} + 0_Q = \dfrac{r_1}{d_1} + \dfrac{0}{d} = \dfrac{r_1d + 0d_1}{d_1d} = \dfrac{r_1d}{d_1d} = \dfrac{r_1}{d_1}$.
3. $(R,+)$ has an inverse: $\dfrac{r_1}{d_1} + \dfrac{-r_1}{d_1} = \dfrac{r_1d_1 + (-r_1)d_1}{d_1d_1} = \dfrac{[(r_1) + (-r_1)]d_1}{d_1d_1} = \dfrac{0d_1}{d_1d_1} = \dfrac{0}{d_1d_1} = 0_Q$.
4. $(R,+,\times)$ satisfies distributive law:

   4-1. The left law:

$$\begin{aligned} \frac{r_1}{d_1} \times \left(\frac{r_2}{d_2} + \frac{r_3}{d_3}\right) &= \frac{r_1}{d_1} \times \frac{r_2d_3 + r_3d_2}{d_2d_3} = \frac{r_1r_2d_3 + r_1r_3d_2}{d_1d_2d_3} = \frac{r_1r_2d_1d_3 + r_1r_3d_1d_2}{d_1d_2d_1d_3} = \frac{r_1r_2}{d_1d_2} + \frac{r_2r_3}{d_2d_3} \\ &= \frac{r_1}{d_1} \times \frac{r_2}{d_2} + \frac{r_2}{d_2} \times \frac{r_3}{d_3} \end{aligned}$$

   4-2. The right law:

$$\begin{aligned} \left(\frac{r_1}{d_1} + \frac{r_2}{d_2}\right) \times \frac{r_3}{d_3} &= \frac{r_1d_2 + r_2d_1}{d_1d_2} \times \frac{r_3}{d_3} = \frac{r_1r_3d_2 + r_2r_3d_1}{d_1d_2d_3} = \frac{r_1r_3d_2d_3 + r_2r_3d_1d_3}{d_1d_3d_2d_3} = \frac{r_1r_3}{d_1d_3} + \frac{r_2r_3}{d_2d_3} \\ &= \frac{r_1}{d_1} \times \frac{r_3}{d_3} + \frac{r_2}{d_2} \times \frac{r_3}{d_3} \end{aligned}$$

5. $(R,\times)$ has an identity: $\dfrac{r_1}{d_1} \times 1_Q = \dfrac{r_1}{d_1} \times \dfrac{d}{d} = \dfrac{r_1d}{d_1d} = \dfrac{r_1}{d_1}$.
6. Elements of D become unit in Q: Define $\iota : R \to Q : r \mapsto \dfrac{rd}{d}$ where $d \in D$ is any fixed element in D.
   Then, ι is Ring-Monomorphsim because:

   6-1. Well-Defined and Injective: $\iota(r_1) = \iota(r_2) \iff \dfrac{r_1d}{d} = \dfrac{r_2d}{d} \iff (r_1 - r_2)dd = 0 \iff r_1 = r_2$

$\square$

Chapter 4

Field Theory

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Chapter 13

Spaces

13.1 $\mathbb{R}^n$

13.1.1 p-norm in $\mathbb{R}^n$

Definition 1. Let $\mathbb{R}^n$ be given. Define p**-norm** of $\mathbb{R}^n$ is metric on $\mathbb{R}$:

$$d_p : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} : (\mathbf{x}, \mathbf{y}) \mapsto \left(\sum_{i=1}^{n} |x_i - y_i|^p \right)^{\frac{1}{p}}, \quad (\mathbf{x} = (x_1, \dots, x_n),\ \ \mathbf{y} = (y_1, \dots, y_n))$$

where $p \in [1, \infty]$, p-norm be a **Metric from Minkowski inequality**.

Lemma 2. Holder's inequality
Let $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ be give, and $p, q \geq 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then,

$$\sum_{i=1}^{n} |x_i y_i| \leq \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} \cdot \left(\sum_{i=1}^{n} |y_i|^q \right)^{\frac{1}{q}}$$

Proof. Denote that

$$\|x\|_p \overset{\text{def}}{=} \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}}$$

Then, since young's inequality, for each $i \in \{1, 2, \dots, n\}$,

$$\frac{|x_i|}{\|x\|_p} \cdot \frac{|y_i|}{\|y\|_p} \leq \frac{1}{p} \cdot \frac{|x_i|^p}{\|x\|_p^p} + \frac{1}{q} \cdot \frac{|y_i|^q}{\|y\|_q^q}$$

Summing for all $i = 1, 2, \dots, n$:

$$\frac{1}{\|x\|_p \|y\|_q} \cdot \sum_{i=1}^{n} |x_i y_i| \leq \frac{1}{p} + \frac{1}{q} = 1$$

Therefore,

$$\sum_{i=1}^{n} |x_i y_i| \leq \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} \cdot \left(\sum_{i=1}^{n} |y_i|^q \right)^{\frac{1}{q}}$$

□

Theorem 3. Minkowski inequality
Given complex-valued sequences $\{x_i\}_{i=1}^{n}$ and $\{y_i\}_{i=1}^{n}$,

$$\left[\sum_{i=1}^{n} |x_i + y_i|^p \right]^{\frac{1}{p}} \leq \left[\sum_{i=1}^{n} |x_i|^p \right]^{\frac{1}{p}} + \left[\sum_{i=1}^{n} |y_i|^p \right]^{\frac{1}{p}}$$

Proof. Denote

$$|x_i + y_i|^p = |x_i + y_i| \cdot |x_i + y_i|^{p-1}$$

Then,

$$\begin{aligned}
\sum_{i=1}^n |x_i + y_i|^p &= \sum_{i=1}^n |x_i + y_i| \cdot |x_i + y_i|^{p-1} \\
&= \sum_{i=1}^n (|x_i| + |y_i|) \cdot |x_i + y_i|^{p-1} \\
&= \sum_{i=1}^n |x_i| \cdot |x_i + y_i|^{p-1} + \sum_{i=1}^n |y_i| \cdot |x_i + y_i|^{p-1} \\
&\overset{\text{Hölder}}{\leq} \left[\sum_{i=1}^n |x_i|^p\right]^{\frac{1}{p}} \cdot \left[\sum_{i=1}^n |x_i + y_i|^p\right]^{\frac{p-1}{p}} + \left[\sum_{i=1}^n |y_i|^p\right]^{\frac{1}{p}} \cdot \left[\sum_{i=1}^n |x_i + y_i|^p\right]^{\frac{p-1}{p}} \\
&= \left[\left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^n |y_i|^p\right)^{\frac{1}{p}}\right] \cdot \left[\sum_{i=1}^n |x_i + y_i|^p\right]^{\frac{p-1}{p}}
\end{aligned}$$

Now, Divide each side as $[\sum_{i=1}^n |x_i + y_i|^p]^{\frac{p-1}{p}}$, then we obtain

$$\left[\sum_{i=1}^n |x_i + y_i|^p\right]^{1-\frac{p-1}{p}} = \left[\sum_{i=1}^n |x_i + y_i|^p\right]^{\frac{1}{p}} \leq \left[\left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^n |y_i|^p\right)^{\frac{1}{p}}\right]$$

□

Theorem 4. Let d_{p_1}, d_{p_2} are p-norm on $\mathbb{R}^n$ with $1 \leq p_1 < p_2$. Then,

$$\exists C > 0 \text{ s.t. } \forall x, y \in \mathbb{R}^n, \ d_{p_2}(x,y) \leq d_{p_1}(x,y) \leq C d_{p_2}(x,y)$$

In particular, $C = n^{\frac{1}{p_1} - \frac{1}{p_2}}$.

Proof. Let $p_1 < p_2$.
For show that first-inequality,

$$1 = \sum_{i=1}^n \left[\frac{|x_i - y_i|}{[\sum_{i=1}^n |x_i - y_i|^{p_2}]^{\frac{1}{p_2}}}\right]^{p_2} \leq \sum_{i=1}^n \left[\frac{|x_i - y_i|}{[\sum_{i=1}^n |x_i - y_i|^{p_2}]^{\frac{1}{p_2}}}\right]^{p_1} = \frac{\sum_{i=1}^n |x_i - y_i|^{p_1}}{[\sum_{i=1}^n |x_i - y_i|^{p_2}]^{\frac{p_1}{p_2}}} = \left[\frac{[\sum_{i=1}^n |x_i - y_i|^{p_1}]^{\frac{1}{p_1}}}{[\sum_{i=1}^n |x_i - y_i|^{p_2}]^{\frac{1}{p_2}}}\right]^{p_1}$$

Thus, we obtain that:

$$1 \leq \left[\frac{[\sum_{i=1}^n |x_i - y_i|^{p_1}]^{\frac{1}{p_1}}}{[\sum_{i=1}^n |x_i - y_i|^{p_2}]^{\frac{1}{p_2}}}\right]^{p_1} \iff 1 \leq \frac{[\sum_{i=1}^n |x_i - y_i|^{p_1}]^{\frac{1}{p_1}}}{[\sum_{i=1}^n |x_i - y_i|^{p_2}]^{\frac{1}{p_2}}} \iff \left[\sum_{i=1}^n |x_i - y_i|^{p_2}\right]^{\frac{1}{p_2}} \leq \left[\sum_{i=1}^n |x_i - y_i|^{p_1}\right]^{\frac{1}{p_1}}$$

For show that second-inequality, using Hölder's inequality.

$$\begin{aligned}
(d_{p_1}(x,y))^{p_1} &= \sum_{i=1}^n |x_i - y_i|^{p_1} = \sum_{i=1}^n |x_i - y_i|^{p_1} \cdot 1 \\
&\overset{\text{Hölder}}{\leq} \left[\sum_{i=1}^n \left(|x_i - y_i|^{p_1 \cdot \frac{p_2}{p_1}}\right)\right]^{\frac{p_1}{p_2}} \cdot \left[\sum_{i=1}^n 1^{\frac{p_2}{p_2 - p_1}}\right]^{1-\frac{p_1}{p_2}} = \left[\sum_{i=1}^n (|x_i - y_i|^{p_2})\right]^{\frac{p_1}{p_2}} \cdot n^{1-\frac{p_1}{p_2}}
\end{aligned}$$

Taking the $\frac{1}{p_1}$-th power of both sides, then

$$d_{p_1}(x,y) \leq \left[\sum_{i=1}^n (|x_i - y_i|^{p_2})\right]^{\frac{1}{p_2}} \cdot n^{\frac{1}{p_1} - \frac{1}{p_2}} = n^{\frac{1}{p_1} - \frac{1}{p_2}} \cdot d_{p_2}(x,y)$$

□

13.2 Topological Vector Space

13.3 Hilbert Space

13.4 Banach Space

13.5 L_p Space

13.6 l_p Space