

Math Note

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This paper covers several topics in undergraduate mathematics.

Chapter 1

Set Theory

Chapter 2

Group Theory

Example. Dihedral Group

Chapter 3

Ring Theory

3.1 Ring of Fractions

Theorem 1. Let R be a Commutative Ring, $D \subset R$ be a subset such that $\begin{cases} \text{no zero, no zero divisors} \\ \text{closed under multiplication} \end{cases}$.

Then, there exists a Commutative Ring Q with identity satisfies:

1. R can embed in Q , and every element of D becomes unit in Q . More precisely, $Q = \{rd^{-1} \mid r \in R, d \in D\}$.
2. Q is the smallest Ring with identity such that every element of D becomes unit in Q .

Proof. Let $\mathcal{F} \stackrel{\text{def}}{=} \{(r, d) \mid r \in R, d \in D\}$ and the relation \sim on \mathcal{F} by $(r_1, d_1) \sim (r_2, d_2) \iff r_1 d_2 = r_2 d_1$.

Then, \sim is equivalent relation: reflexive and symetric are clear, and Suppose that $(r_1, d_1) \sim (r_2, d_2)$ and $(r_2, d_2) \sim (r_3, d_3)$.

$$r_2 d_3 = r_3 d_2 \implies r_2 d_1 d_3 = r_3 d_1 d_2 \implies r_1 d_2 d_3 = r_3 d_1 d_2 \implies d_2(r_1 d_3 - r_3 d_1) \implies r_1 d_3 = r_3 d_1$$

Thus transitivity shown. Define

$$\frac{r}{d} \stackrel{\text{def}}{=} [(r, d)] = \{(a, b) \mid (a, b) \sim (r, d)\}, \quad Q \stackrel{\text{def}}{=} \left\{ \frac{r}{d} \mid r \in R, d \in D \right\}$$

And define operations $+$, \times on Q :

$$\frac{r_1}{d_1} + \frac{r_2}{d_2} \stackrel{\text{def}}{=} \frac{r_1 d_2 + r_2 d_1}{d_1 d_2}, \quad \frac{r_1}{d_1} \times \frac{r_2}{d_2} \stackrel{\text{def}}{=} \frac{r_1 r_2}{d_1 d_2}$$

Well-Definedness: If $\frac{r_1}{d_1} = \frac{r'_1}{d'_1}$ and $\frac{r_2}{d_2} = \frac{r'_2}{d'_2}$,

$$\frac{r_1 d_2 + r_2 d_1}{d_1 d_2} = \frac{r_1 d_2 d'_1 d'_2 + r_2 d_1 d'_1 d'_2}{d_1 d_2 d'_1 d'_2} = \frac{(r_1 d'_1) d_2 d'_2 + (r_2 d'_2) d_1 d'_1}{d_1 d_2 d'_1 d'_2} = \frac{(r'_1 d_1) d_2 d'_2 + (r'_2 d_2) d_1 d'_1}{d_1 d_2 d'_1 d'_2} = \frac{(r'_1 d'_2 + r'_2 d'_1) d_1 d_2}{d_1 d_2 d'_1 d'_2} = \frac{r'_1 d'_2 + r'_2 d'_1}{d'_1 d'_2}$$

$$\frac{r_1 r_2}{d_1 d_2} = \frac{r_1 r_2 d'_1 d'_2}{d_1 d_2 d'_1 d'_2} = \frac{(r_1 d'_1)(r_2 d'_2)}{d_1 d_2 d'_1 d'_2} = \frac{(r'_1 d_1)(r'_2 d_2)}{d_1 d_2 d'_1 d'_2} = \frac{r'_1 r'_2 d_1 d_2}{d_1 d_2 d'_1 d'_2} = \frac{r'_1 r'_2}{d'_1 d'_2}$$

Now, $(Q, +, \times)$ constructs Commutative Ring with identity: for any $d \in D$, put $0_Q \stackrel{\text{def}}{=} \frac{0}{d}$, $1_Q \stackrel{\text{def}}{=} \frac{d}{d}$. Then,

1. $(R, +, \times)$ closed under the operations since D is closed under the multiplication.

$$2. (R, +) \text{ has a zero: } \frac{r_1}{d_1} + 0_Q = \frac{r_1}{d_1} + \frac{0}{d} = \frac{r_1 d + 0 d_1}{d_1 d} = \frac{r_1 d}{d_1 d} = \frac{r_1}{d_1}.$$

$$3. (R, +) \text{ has an inverse: } \frac{r_1}{d_1} + \frac{-r_1}{d_1} = \frac{r_1 d_1 + (-r_1) d_1}{d_1 d_1} = \frac{[(r_1) + (-r_1)] d_1}{d_1 d_1} = \frac{0 d_1}{d_1 d_1} = \frac{0}{d_1 d_1} = 0_Q.$$

4. $(R, +, \times)$ satisfies distributive law:

4-1. The left law:

$$\begin{aligned} \frac{r_1}{d_1} \times \left(\frac{r_2}{d_2} + \frac{r_3}{d_3} \right) &= \frac{r_1}{d_1} \times \frac{r_2 d_3 + r_3 d_2}{d_2 d_3} = \frac{r_1 r_2 d_3 + r_1 r_3 d_2}{d_1 d_2 d_3} = \frac{r_1 r_2 d_1 d_3 + r_1 r_3 d_1 d_2}{d_1 d_2 d_1 d_3} = \frac{r_1 r_2}{d_1 d_2} + \frac{r_2 r_3}{d_2 d_3} \\ &= \frac{r_1}{d_1} \times \frac{r_2}{d_2} + \frac{r_1}{d_1} \times \frac{r_3}{d_3} \end{aligned}$$

4-2. The right law:

$$\begin{aligned} \left(\frac{r_1}{d_1} + \frac{r_2}{d_2} \right) \times \frac{r_3}{d_3} &= \frac{r_1 d_2 + r_2 d_1}{d_1 d_2} \times \frac{r_3}{d_3} = \frac{r_1 r_3 d_2 + r_2 r_3 d_1}{d_1 d_2 d_3} = \frac{r_1 r_3 d_2 d_3 + r_2 r_3 d_1 d_3}{d_1 d_3 d_2 d_3} = \frac{r_1 r_3}{d_1 d_3} + \frac{r_2 r_3}{d_2 d_3} \\ &= \frac{r_1}{d_1} \times \frac{r_3}{d_3} + \frac{r_2}{d_2} \times \frac{r_3}{d_3} \end{aligned}$$

$$5. (R, \times) \text{ has an identity: } \frac{r_1}{d_1} \times 1_Q = \frac{r_1}{d_1} \times \frac{d}{d} = \frac{r_1 d}{d_1 d} = \frac{r_1}{d_1}.$$

6. Elements of D become unit in Q : Define $\iota: R \rightarrow Q: r \mapsto \frac{rd}{d}$ where $d \in D$ is any fixed element in D . Then, ι is Ring-Monomorphism because:

$$6-1. \text{ Well-Defined and Injective: } \iota(r_1) = \iota(r_2) \iff \frac{r_1 d}{d} = \frac{r_2 d}{d} \iff (r_1 - r_2) d = 0 \iff r_1 = r_2$$

□

Chapter 4

Field Theory

Chapter 5

Category

Chapter 6

General Topology

6.1 Complete Metric Space

Definition 1. Let (X, d) be a Metric Space, and $\{p_n\}$ be a Sequence in X . The Sequence $\{p_n\}$ is called **Cauchy Sequence** if:

$$\text{For any } \epsilon > 0, \text{ there exists } N \in \mathbb{N} \text{ such that } m, n \geq N \implies d(p_m, p_n) < \epsilon.$$

A Metric Space (X, d) is said to be **Complete** if every Cauchy Sequences Converge.

Lemma 2. Let $\{E_n\}$ be a sequence of closed bounded non-empty subsets in a Complete Metric Space X such that $E_n \supset E_{n+1}$.

If $\lim_{n \rightarrow \infty} \text{diam} E_n = 0$, then $\bigcap_{n=1}^{\infty} E_n = \{p\}$ for some $p \in X$.

Proof. For each $n \in \mathbb{N}$, construct $p_n \in E_n$.

Let $\epsilon > 0$ be given. Since $\text{diam} E_n \rightarrow 0$, there is $N \in \mathbb{N}$ such that $\text{diam} E_n < \epsilon$.

For any $m, n \geq N$, E_N contains p_m, p_n . That is, $d(p_m, p_n) < \epsilon$. Thus, $\{p_n\}$ be a Cauchy sequence of X .

Since X is complete, there is a unique point $p \in X$ such that $p_n \rightarrow p$. Let $N \in \mathbb{N}$ be a integer such that $n \geq N \implies |p_n - p| < \epsilon$.

Now, for each $n \geq N$, E_n has a limit point as p . And for any $n \in \mathbb{N}$, E_n contains E_N, E_{N+1}, \dots , thus for all $n \in \mathbb{N}$, E_n has a limit point as p . Meanwhile, E_n closed, $p \in E_n, \forall n \in \mathbb{N}$.

Consequently, $p \in \bigcap_{n=1}^{\infty} E_n$. If there is $q \in X$ such that $p \neq q$, $q \in \bigcap_{n=1}^{\infty} E_n$. Then, $\text{diam} E_n \geq d(p, q) > 0, \forall n \in \mathbb{N}$. \square

6.1.1 Baire Category

Definition 2. The Topological Space X is called **Baire Space** if:

$$\text{If } \{G_n \mid n \in \mathbb{N}\} \text{ be a Countable Collection of dense open sets of } X, \text{ then } \overline{\bigcap_{n=1}^{\infty} G_n} = X$$

In brief, every Countable intersection of dense open sets be dense in X .

Theorem 3. Locally Compact Hausdorff Space is Baire Space.

Theorem 4. Complete Metric Space is Baire Space.

Proof. Suppose that $\{G_n \mid n \in \mathbb{N}\}$ be a Countable Collection of dense open set of Complete Metric Space.

Let an open $U \in \mathcal{T}$ be given. Since G_n is dense in the Space, $U \cap G_1$ is non-empty open set.

Thus, there exists a $p_1 \in U \cap G_1$ such that for some $r_1 > 0$, $B_{r_1}(p_1) \subset U \cap G_1$.

Then, automatically,

$$B_{\frac{r_1}{2}}(p_1) \subset \overline{B_{\frac{r_1}{2}}(p_1)} \subset B_{r_1}(p_1) \subset U \cap G_1$$

Set $E_1 = U$, $E_2 = B_{\frac{r_1}{2}}(p_1)$.

Suppose that E_1, \dots, E_{n-1} are chosen. Then, since $E_{n-1} \cap G_{n-1}$ is open, being intersection of opens.

Thus there exists a point $p_{n-1} \in E_{n-1} \cap G_{n-1}$ and exists r_{n-1} such that

$$B_{r_{n-1}}(p_{n-1}) \subset E_{n-1} \cap G_{n-1} \subset E_{n-1}$$

This implies that

$$B_{\frac{r_{n-1}}{2}}(p_{n-1}) \subset \overline{B_{\frac{r_{n-1}}{2}}(p_{n-1})} \subset B_{r_{n-1}}(p_{n-1}) \subset E_{n-1} \cap G_{n-1} \subset E_{n-1}$$

Set $E_n = B_{\frac{r_{n-1}}{2}}(p_{n-1})$. Since inductively construction of $\{E_n\}$, $E_{n+1} \subset E_n$ and $\overline{E_n} \subset G_n$ for all $n \in \mathbb{N}$. Consequently,

$$U \cap \left(\bigcap_{n=1}^{\infty} G_n \right) = \bigcap_{n=1}^{\infty} (U \cap G_n) \supset \bigcap_{n=1}^{\infty} (U \cap \overline{E_n}) = U \cap \left(\bigcap_{n=1}^{\infty} \overline{E_n} \right) = \bigcap_{n=1}^{\infty} \overline{E_n} \neq \emptyset$$

□

Definition 3. Let X be a Topological Space.

$A \subset X$ is said to be **nowhere dense subset** if $(\overline{A})^\circ = \emptyset$.

1. $B \subset X$ is called **first category** if B can be representative by union of countable nowhere dense subsets.
2. If the subset is not first category, then it is said to be **second category**.

6.1.2 Nowhere Differentiable function

6.2 Urysohn Metrization Theorem

6.2.1 Urysohn Metrization Theroem

Recall that:

Definition 4. X is T_4 if: For any disjoint closed set A and B , there exist disjoint open U, V such that $A \subseteq U$ and $B \subseteq V$.

Lemma 5. X is T_4 Space if and only if For any closed C and open U with $C \subseteq U$, there exists open O such that

$$\underset{\text{closed}}{C} \subseteq \underset{\text{open}}{O} \subseteq \underset{\text{closed}}{\overline{O}} \subseteq \underset{\text{open}}{U}$$

Proof. Proof of the left direction only.

Let X be a T_4 Space, and $C \subset X$ be a closed, U be a open containing C . Then, $C \subset U$ implies $U^c \subset C^c$, thus U^c is a closed set disjoint from C . By T_4 condition, There exist disjoint opens O, O' such that $C \subset O$ and $U^c \subset O' \iff O'^c \subset U$.

Since $O \cap O' = \emptyset \iff O \subset O'^c$, O contained in U , this implies that $C \subset O \subset U$.

Since closure is the smallest closed set such that contains it, consequently $C \subset O \subset \overline{O} \subset O'^c \subset U$. \square

Definition 5. Let X be a Topological Space, and $A, B \subset X$ are disjoint closed subset.

A real-valued Continuous map $f: X \rightarrow [a, b]$ is called **Urysohn function** for A and B if: $f|_A = a$ and $f|_B = b$.

In another form,

$$f: X \rightarrow [a, b] : x \rightarrow \begin{cases} a & x \in A \\ b & x \in B \\ f(x) & x \notin A \cup B \end{cases}$$

Lemma 6. Urysohn Lemma

T_4 Space has an Urysohn function for any two disjoint closed subsets.

Proof. Generalization is the last thing to proven, first of all, prove in case of $[a, b] = [0, 1]$. This proof consists by three Step.

Let X be a T_4 Space, and $A, B \subset X$ be closed subsets.

Step 1. Construct a Chain of Open sets with Dyadic Rational Indices.

Consider a set of **Dyadic Rationals** $D \stackrel{\text{def}}{=} \left\{ \frac{k}{2^n} \mid n, k \in \mathbb{N}, k \leq 2^n - 1 \right\}$. We will show that the following statement holds:

$$\text{For any } r, s \in D \text{ with } r < s, \text{ there exist open sets } U_r, U_s \text{ such that } A \subseteq \overline{U_r} \subseteq U_s \subseteq X \setminus B \quad (*)$$

For this, Enough to Show that: For any $k \in \mathbb{N}$, there exists a Chain as:

$$A \subseteq U_{\frac{1}{2^k}} \subseteq \overline{U_{\frac{1}{2^k}}} \subseteq U_{\frac{2}{2^k}} \subseteq \overline{U_{\frac{2}{2^k}}} \subseteq \cdots \subseteq U_{\frac{2^k-1}{2^k}} \subseteq \overline{U_{\frac{2^k-1}{2^k}}} \subseteq X \setminus B$$

(Note that this opens in the Chain are not necessary distinct: For instance, if Ambient Space is Finite, then the Space is **Noetherian**. That is, X satisfies Ascending Chain Condition for open sets.)

Let $k = 1$. Then, By T_4 condition gives that: There exists an open set U_1 such that

$$A \subseteq U_1 \subseteq \overline{U_1} \subseteq X \setminus B$$

Now, naming this U_1 as $U_{\frac{1}{2}}$, proved when $k = 1$.

Suppose that for some $k > 1$, the Chain exists as:

$$\underset{\text{closed}}{A} \subseteq \overset{*1}{\underset{\text{open}}{U_{\frac{1}{2^k}}}} \subseteq \underset{\text{closed}}{\overline{U_{\frac{1}{2^k}}}} \subseteq \overset{*2}{\underset{\text{open}}{U_{\frac{2}{2^k}}}} \subseteq \cdots \subseteq \overset{*2^k-1}{\underset{\text{open}}{U_{\frac{2^k-1}{2^k}}}} \subseteq \underset{\text{closed}}{\overline{U_{\frac{2^k-1}{2^k}}}} \overset{*2^k}{\subseteq} \underset{\text{open}}{X \setminus B}$$

By repeatedly applying the T_4 condition 2^k times, as indicated by the indices $*1, *2, \dots, *2^k$, we can construct 2^k open sets such that:

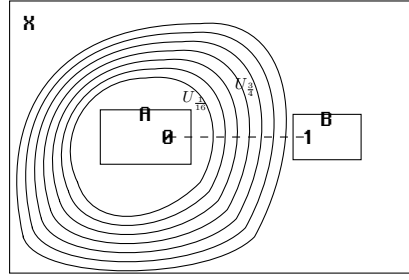
$$A \subseteq U_{\frac{1}{2^{k+1}}} \subseteq \overline{U_{\frac{1}{2^{k+1}}}} \subseteq U_{\frac{1}{2^k}} \subseteq \overline{U_{\frac{1}{2^k}}} \subseteq U_{\frac{3}{2^{k+1}}} \subseteq \overline{U_{\frac{3}{2^{k+1}}}} \subseteq U_{\frac{2}{2^k}} \subseteq \overline{U_{\frac{2}{2^k}}} \subseteq \cdots \subseteq U_{\frac{2^k-1}{2^k}} \subseteq \overline{U_{\frac{2^k-1}{2^k}}} \subseteq U_{\frac{2^{k+1}-1}{2^{k+1}}} \subseteq \overline{U_{\frac{2^{k+1}-1}{2^{k+1}}}} \subseteq X \setminus B$$

Finally, Step 1 proved.

Step 2. Construct an Urysohn Function.

Define a map $f : X \rightarrow [0, 1]$ as:

$$f(x) = \begin{cases} 0 & x \in \bigcap_{t \in D} U_t \\ \sup\{t \in D \mid x \notin U_t\} & x \notin \bigcap_{t \in D} U_t \end{cases}$$



Then, this map f is well-defined by (*) and $\sup D \leq 1$. And f satisfies that:

1. $\forall r \in D, x \in A \subset U_r$. Thus, $f(x) = 0$ if $x \in A$.
2. $\forall r \in D, x \in B \subset X \setminus U_r$. Thus, $f(x) = \sup D = 1$ if $x \in B$.
3. If $x \in \overline{U_r}$, then for every $s > r, x \in \overline{U_r} \subset U_s$. Thus, $f(x) \leq r$. In Contrapositive, $f(x) > r \implies x \notin \overline{U_r}$.
(If $f(x) = \sup\{t \in D \mid x \notin U_t\} > r$, then there is $s \in D$ such that $s > r$ and $x \notin U_s$, Contradiction.)
4. If $x \notin U_r$, then, $f(x) \geq r$. In Contrapositive, $f(x) < r \implies x \in U_r$.

Now, show that this map f is Continuous map: Let $x \in X$ be fixed arbitrarily, and $\epsilon > 0$ be given.

In Case of $0 < f(x) < 1$.

Since Density of Dyadic Rationals, Choose $r, s \in D$ such that $f(x) - \epsilon < r < f(x) < s < f(x) + \epsilon$.

Now, we obtain that:

$$x \stackrel{(*)}{\in} U_s \setminus \overline{U_r} \stackrel{(**)}{\subseteq} f^{-1}[(f(x) - \epsilon, f(x) + \epsilon)]$$

(*) directly given by above properties, (**) given applying the fact that $x \in U_s \subset \overline{U_s}$ and $x \notin \overline{U_r}$.

In Case of $f(x) = 0$.

Choose $r \in D$ such that $f(x) = 0 < r < \epsilon = f(x) + \epsilon$. Then,

$$x \in U_r \subset f^{-1}[(f(x), f(x) + \epsilon)]$$

In Case of $f(x) = 1$.

Choose $r \in D$ such that $f(x) - \epsilon = 1 - \epsilon < r < 1 = f(x)$. Then,

$$x \in X \setminus U_r \subset f^{-1}[(f(x) - \epsilon, f(x))]$$

Consequently, f is Continuous map on $[0, 1]$ such that $f|_A = 0$ and $f|_B = 1$.

Step 3. Generalization.

Since $[0, 1] \cong [a, b]$ for any $a < b$, let $g : [0, 1] \rightarrow [a, b] : x \mapsto (1 - x)a + xb$ be a Homeomorphism.

Then, $h = g \circ f : X \rightarrow [a, b]$ becomes a Continuous map such that $h|_A = a$ and $h|_B = b$. □

Chapter 7

Algebraic Topology

Chapter 8

Real Analysis

Chapter 9

Measure

Chapter 10

Complex Analysis

Chapter 11

Differential Geometry

Chapter 12

Differential Equation

Chapter 13

Spaces

13.1 \mathbb{R}^n

13.1.1 p -norm in \mathbb{R}^n

Definition 6. Let \mathbb{R}^n be given. Define p -norm of \mathbb{R}^n is metric on \mathbb{R}^n :

$$d_p : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R} : (\mathbf{x}, \mathbf{y}) \mapsto \left(\sum_{i=1}^n |x_i - y_i|^p \right)^{\frac{1}{p}}, \quad (\mathbf{x} = (x_1, \dots, x_n), \mathbf{y} = (y_1, \dots, y_n))$$

where $p \in [1, \infty]$, p -norm be a Metric from Minkowski inequality.

Lemma 7. Holder's inequality

Let $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ be give, and $p, q \geq 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then,

$$\sum_{i=1}^n |x_i y_i| \leq \left(\sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}} \cdot \left(\sum_{i=1}^n |y_i|^q \right)^{\frac{1}{q}}$$

Proof. Denote that

$$\|x\|_p \stackrel{\text{def}}{=} \left(\sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}}$$

Then, since young's inequality, for each $i \in \{1, 2, \dots, n\}$,

$$\frac{|x_i|}{\|x\|_p} \cdot \frac{|y_i|}{\|y\|_q} \leq \frac{1}{p} \cdot \frac{|x_i|^p}{\|x\|_p^p} + \frac{1}{q} \cdot \frac{|y_i|^q}{\|y\|_q^q}$$

Summing for all $i = 1, 2, \dots, n$:

$$\frac{1}{\|x\|_p \|y\|_q} \cdot \sum_{i=1}^n |x_i y_i| \leq \frac{1}{p} + \frac{1}{q} = 1$$

Therefore,

$$\sum_{i=1}^n |x_i y_i| \leq \left(\sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}} \cdot \left(\sum_{i=1}^n |y_i|^q \right)^{\frac{1}{q}}$$

□

Theorem 8. Minkowski inequality

Given complex-valued sequences $\{x_i\}_{i=1}^n$ and $\{y_i\}_{i=1}^n$,

$$\left[\sum_{i=1}^n |x_i + y_i|^p \right]^{\frac{1}{p}} \leq \left[\sum_{i=1}^n |x_i|^p \right]^{\frac{1}{p}} + \left[\sum_{i=1}^n |y_i|^p \right]^{\frac{1}{p}}$$

Proof. Denote

$$|x_i + y_i|^p = |x_i + y_i| \cdot |x_i + y_i|^{p-1}$$

Then,

$$\begin{aligned} \sum_{i=1}^n |x_i + y_i|^p &= \sum_{i=1}^n |x_i + y_i| \cdot |x_i + y_i|^{p-1} \\ &= \sum_{i=1}^n (|x_i| + |y_i|) \cdot |x_i + y_i|^{p-1} \\ &= \sum_{i=1}^n |x_i| \cdot |x_i + y_i|^{p-1} + \sum_{i=1}^n |y_i| \cdot |x_i + y_i|^{p-1} \\ &\stackrel{\text{Hölder}}{\leq} \left[\sum_{i=1}^n |x_i|^p \right]^{\frac{1}{p}} \cdot \left[\sum_{i=1}^n |x_i + y_i|^p \right]^{\frac{p-1}{p}} + \left[\sum_{i=1}^n |y_i|^p \right]^{\frac{1}{p}} \cdot \left[\sum_{i=1}^n |x_i + y_i|^p \right]^{\frac{p-1}{p}} \\ &= \left[\left(\sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}} + \left(\sum_{i=1}^n |y_i|^p \right)^{\frac{1}{p}} \right] \cdot \left[\sum_{i=1}^n |x_i + y_i|^p \right]^{\frac{p-1}{p}} \end{aligned}$$

Now, Divide each side as $\left[\sum_{i=1}^n |x_i + y_i|^p \right]^{\frac{p-1}{p}}$, then we obtain

$$\left[\sum_{i=1}^n |x_i + y_i|^p \right]^{1 - \frac{p-1}{p}} = \left[\sum_{i=1}^n |x_i + y_i|^p \right]^{\frac{1}{p}} \leq \left[\left(\sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}} + \left(\sum_{i=1}^n |y_i|^p \right)^{\frac{1}{p}} \right]$$

□

Theorem 9. Let d_{p_1}, d_{p_2} are p -norm on \mathbb{R}^n with $1 \leq p_1 < p_2$. Then,

$$\exists C > 0 \text{ s.t. } \forall x, y \in \mathbb{R}^n, d_{p_2}(x, y) \leq d_{p_1}(x, y) \leq C d_{p_2}(x, y)$$

In particular, $C = n^{\frac{1}{p_1} - \frac{1}{p_2}}$.

Proof. Let $p_1 < p_2$.

For show that first-inequality,

$$1 = \sum_{i=1}^n \left[\frac{|x_i - y_i|}{\left[\sum_{i=1}^n |x_i - y_i|^{p_2} \right]^{\frac{1}{p_2}}} \right]^{p_2} \leq \sum_{i=1}^n \left[\frac{|x_i - y_i|}{\left[\sum_{i=1}^n |x_i - y_i|^{p_2} \right]^{\frac{1}{p_2}}} \right]^{p_1} = \frac{\sum_{i=1}^n |x_i - y_i|^{p_1}}{\left[\sum_{i=1}^n |x_i - y_i|^{p_2} \right]^{\frac{p_1}{p_2}}} = \left[\frac{\left[\sum_{i=1}^n |x_i - y_i|^{p_1} \right]^{\frac{1}{p_1}}}{\left[\sum_{i=1}^n |x_i - y_i|^{p_2} \right]^{\frac{1}{p_2}}} \right]^{p_1}$$

Thus, we obtain that:

$$1 \leq \left[\frac{\left[\sum_{i=1}^n |x_i - y_i|^{p_1} \right]^{\frac{1}{p_1}}}{\left[\sum_{i=1}^n |x_i - y_i|^{p_2} \right]^{\frac{1}{p_2}}} \right]^{p_1} \iff 1 \leq \frac{\left[\sum_{i=1}^n |x_i - y_i|^{p_1} \right]^{\frac{1}{p_1}}}{\left[\sum_{i=1}^n |x_i - y_i|^{p_2} \right]^{\frac{1}{p_2}}} \iff \left[\sum_{i=1}^n |x_i - y_i|^{p_2} \right]^{\frac{1}{p_2}} \leq \left[\sum_{i=1}^n |x_i - y_i|^{p_1} \right]^{\frac{1}{p_1}}$$

For show that second-inequality, using Hölder's inequality.

$$\begin{aligned} (d_{p_1}(x, y))^{p_1} &= \sum_{i=1}^n |x_i - y_i|^{p_1} = \sum_{i=1}^n |x_i - y_i|^{p_1} \cdot 1 \\ &\stackrel{\text{Hölder}}{\leq} \left[\sum_{i=1}^n \left(|x_i - y_i|^{p_1 \cdot \frac{p_2}{p_1}} \right) \right]^{\frac{p_1}{p_2}} \cdot \left[\sum_{i=1}^n 1^{\frac{p_2}{p_2 - p_1}} \right]^{1 - \frac{p_1}{p_2}} = \left[\sum_{i=1}^n (|x_i - y_i|^{p_2}) \right]^{\frac{p_1}{p_2}} \cdot n^{1 - \frac{p_1}{p_2}} \end{aligned}$$

Taking the $\frac{1}{p_1}$ -th power of both sides, then

$$d_{p_1}(x, y) \leq \left[\sum_{i=1}^n (|x_i - y_i|^{p_2}) \right]^{\frac{1}{p_2}} \cdot n^{\frac{1}{p_1} - \frac{1}{p_2}} = n^{\frac{1}{p_1} - \frac{1}{p_2}} \cdot d_{p_2}(x, y)$$

□

13.2 Topological Vector Space

13.3 Hilbert Space

13.4 Banach Space

13.5 L_p Space

13.6 l_p Space