Math Note

Jong Won

University of Seoul, Mathematics

Contents

This paper covers several topics in undergraduate mathematics.

Chapter 1

Set Theory

1.1 Map

Definition 1. Let X, Y are sets. Define a **function** X to Y is a relation

$$f \subset X \times Y$$

such that

1. For any $x \in X$, there exists $y \in Y$ such that $(x, y) \in f$.

2. If $(x, y) \in f$ and $(x, z) \in f$, then $y = z$.

Denote f as:

$$f : X \to Y : x \mapsto f(x)$$

Define **Image** of f by $A \subset X$:

$$f[A] \overset{\text{def}}{=} \{f(a) \mid a \in A\} \subset Y$$

And, **Preimage** of f by $B \subset Y$:

$$f^{-1}[B] \overset{\text{def}}{=} \{x \in X \mid f(x) \in B\} \subset X$$

$f : X \to Y$ is **Injective** if: $f(x_1) = f(x_2) \implies x_1 = x_2$.
$f : X \to Y$ is **Surjective** if: $\forall y \in Y, \ \exists x \in X$ s.t. $f(x) = y$.
If f is injective and surjective, called **bijective**.
If f is bijective, then define **inverse** of f as:

$$f^{-1} : Y \to X : y \to x$$

where $x \in X$ is the unique elements of X such that $f(x) = y$.

Theorem 1. Let $f : X \to Y$ be a function. Then,

1. There exists $g : Y \to X$ such that $g \circ f : X \to X$ be an identity function **if and only if** f is injective.

2. There exists $h : Y \to X$ such that $f \circ h : Y \to Y$ be an identity function **if and only if** f is surjective.

Proof.
1. $\implies$)
Assume that $f(x_1) = f(x_2)$. Then, existence of left inverse, $g(f(x_1)) = g(f(x_2)) \implies x_1 = x_2$. Thus f injective.
1. $\impliedby$)

Since f is injection, for any $y \in f[X]$, there exists a unique element $x_y \in X$ such that $f(x) = y$. Now, define

$$g : Y \to X : y \mapsto \begin{cases} x_y & y \in f[X] \\ \text{any element in } X & y \notin f[X] \end{cases}$$

Then, for any $x \in X$, $g(f(x)) = g(y) = x$.

2. $\Longrightarrow$)
Let $y \in Y$ be given. Since existence of right inverse, $f(h(y)) = y$ where $h(y) \in X$. Thus, f is surjective.

2. $\Longleftarrow$)
For any $y \in Y$, there exists a $x_y \in X$ such that $f(x_y) = y$. Now, define

$$h : Y \to X : y \mapsto x_y$$

Then, for any $y \in Y$, $f \circ h(y) = f(x_y) = y$. Thus, $f \circ h$ is identity. $\square$

Corollary 1. Let $f : X \to Y$ be a function, $\text{id}_X : X \to X : x \mapsto x$, and $\text{id}_Y : Y \to Y : y \mapsto y$.

There exists a $f^{-1} : Y \to X$ such that $f^{-1} \circ f = \text{id}_X$ and $f \circ f^{-1} = \text{id}_Y$ **if and only if** f is bijection.

Proof. If f is bijection, then there exists left inverse g and right inverse h.
Enough To Show that: $g = h$. Since $g \circ f = \text{id}_X$ and $f \circ h = \text{id}_Y$,
$g \circ f \circ h = g \circ \text{id}_Y$, thus $h = g$. $\square$

Theorem 2. Let X, Y, Z are sets, $f : X \to Y$, $g : Y \to Z$ and $A \subset X, B \subset Y, C \subset Z$. Then followings are hold:

1. $g[f[A]] = (g \circ f)[A]$.

2. $f^{-1}[g^{-1}[C]] = (g \circ f)^{-1}[C]$.

Proof.
1. It is clear by definition of image:

$$g[f[A]] \stackrel{\text{def}}{=} g[\{f(a) \mid a \in A\}] = \{g(b) \mid b \in \{f(a) \mid a \in A\}\}$$
$$= \{g(b) \mid b = f(a) \text{ for some } a \in A\} = \{g(f(a)) \mid \text{for some } a \in A\} = \{g(f(a)) \mid a \in A\}$$

2. It is not clear,

$$f^{-1}[g^{-1}[C]] \stackrel{\text{def}}{=} f^{-1}[\{b \in Y \mid g(b) \in C\}] = \{a \in X \mid f(a) \in \{b \in Y \mid g(b) \in C\}\} = \{a \in X \mid g(f(a)) \in C\} = (g \circ f)^{-1}[C]$$

$\square$

Proposition 1. Let $f : X \to Y$ be a function, $A, B \subset X$ and $C, D \subset Y$.

1. If $A \subset B$, then $f[A] \subset f[B]$.

2. If $C \subset D$, then $f^{-1}[C] \subset f^{-1}[D]$

Proof.

$$y \in f[A] \implies y = f(a) \text{ for some } a \in A \stackrel{A \subset B}{\implies} y = f(a) \text{ for some } a \in B \implies y \in f[B]$$
$$x \in f^{-1}[C] \implies f(x) \in C \stackrel{C \subset D}{\implies} f(x) \in D \implies x \in f^{-1}[D]$$

$\square$

Lemma 1. Let two set X, Y be given, and $A \subset X$, $B \subset Y$, $f : X \to Y$. Then followings are holds:

1. $f^{-1}[f[A]] \supseteq A$, and equality holds if f one-to-one.

2. $f[f^{-1}[B]] \subseteq B$, and equality holds if f onto.

3. $f^{-1}[Y \setminus B] = X \setminus f^{-1}[B]$

4. $f[X] \setminus f[A] \subseteq f[X \setminus A]$, and equality holds if f one-to-one.

Proof. Proof of 4.

$$
\begin{aligned}
y \in f[X] \setminus f[A] &\iff y \in f[X] \text{ and } y \notin f[A] \\
&\iff \exists x \in X \text{ s.t. } y = f(x) \text{ and } \forall x \in A, \ y \neq f(x) \\
&\overset{(*)}{\implies} \exists x \in X \setminus A \text{ s.t. } y = f(x) \\
&\iff y \in f[X \setminus A]
\end{aligned}
$$

If f is injection, then Left Direction of $(*)$ be true: $\exists! x \in X \setminus A$ s.t. $y = f(x)$. $\qquad \square$

Chapter 2

Group Theory

2.1 Isomorphism Theorems

Theorem 3. The First Isomorphism Theorem
Let $\varphi : G \to H$ be a Group-Homomorphism. Then,

$$G/\ker\varphi \cong \varphi[G]$$

$$G \xrightarrow{\overset{homomorphism}{\varphi}} \varphi[G]$$

π
homomorphism
surjection $\quad G/\ker\varphi$

ϕ
isomorphism

Proof. Let $\pi : G \to G/\ker\varphi : x \mapsto x + \ker\varphi$. Then, the map $\phi : G/\ker\varphi \to \varphi[G] : a + \ker\varphi \mapsto \varphi(a)$ is isomorphism. Well-defined and Injective:

$$a + \ker\varphi = b + \ker\varphi \iff a - b \in \ker\varphi \iff \varphi(a - b) = \varphi(a) - \varphi(b) = 0$$

Surjective is clear. $\qquad\square$

Theorem 4. The Second Isomorphism Theorem
Let G be a Group, and $H \leq G$, $N \trianglelefteq G$. Then,

$$HN/N \cong H/(H \cap N)$$

Proof. HK be a subgroup of G, being

$$HN = \bigcup_{h \in H} hN \overset{N \trianglelefteq G}{=\!=} \bigcup_{h \in H} Nh = NH$$

And, $N \leq HN$ is clear, thus $N \trianglelefteq HN$.
Meanwhile, $H \cap N$ be a Normal Subgroup of H: for any $h \in H, n \in H \cap N$, $hnh^{-1} \in N$ because N is normal, and $hnh^{-1} \in H$ since h,n contained in H. Thus, $hnh^{-1} \in H \cap N$, this implies $H \cap N$ be a Normal of H.
Now, Define a Map:

$$\varphi : H \to HN/N : h \mapsto hN$$

Clearly, this map is Well-Defined and Homomorphism. And,

$$\ker\varphi = \varphi^{-1}[1] = \{h \in H \mid hN = N\} = \{h \in H \mid h \in N\} = H \cap N$$

Thus, since The 1st Isomorphism Theorem,

$$HN/N \cong H/(H \cap N)$$

$\qquad\square$

Theorem 5. The Third Isomorphism Theorem
Let G be a Group, and $H, K \trianglelefteq G$ with $H \leq K$. Then, $K/H \trianglelefteq G/H$ and

$$(G/H)/(K/H) \cong (G/K)$$

Proof. First, show that $K/H \trianglelefteq G/H$. Let $kH \in K/H$ and $gH \in G/H$. Then,

$$(gH)(kH)(gH)^{-1} = (gH)(kH)(g^{-1}H) = (gkg^{-1})H \in K/H$$

since $gkg^{-1} \in K$, being $K \trianglelefteq G$. Now, Define a map:

$$\varphi : G/H \to G/K : gH \mapsto gK$$

1. Well-Defined.

$$g_1H = g_2H \iff g_1^{-1}g_2 \in H \overset{H \leq K}{\implies} g_1^{-1}g_2 \in K \iff g_1K = g_2K$$

2. Homomorphism.
Clearly, for any $g_1H, g_2 \in G/H$,

$$\varphi(g_1Hg_2H) = \phi(g_1g_2H) = g_1g_2K = g_1Kg_2K = \varphi(g_1H)\varphi(g_2H)$$

3. Surjection. Let $gK \in G/K$ be given. Then, clearly, $\varphi(gH) = gK$.
4. Kernel.

$$\ker \varphi = \{gH \in G/H \mid gK = 1\} = \{gH \in G/H \mid g \in K\} = K/H$$

Consequently, The 1st Isomorphism Theorem gives

$$(G/K) \cong (G/H)/\ker \varphi = (G/H)/(K/H)$$

$\square$

Theorem 6. The Forth Isomorphism Theorem
Let G be a Group, and $N \trianglelefteq G$ be a Normal Subgroup. Then, there is a bijection between

$$D \overset{\text{def}}{=} \{H \leq G \mid N \leq H\}, \quad C \overset{\text{def}}{=} \{\overline{H} \leq G/N\}$$

Proof. Let $\pi : G \to G/N : g \mapsto gN$ be a natural projection. And, Define

$$\Phi : D \to C : H \mapsto \pi[H]$$

This function is well-defined: For any $H \in D$, let $aN, bN \in \pi[H]$. Then, $aN \cdot b^{-1}N = ab^{-1}N \in \pi[H]$, thus $\pi[H] \leq G/N$.
To show that one-to-one: Let $\Phi(A) = \Phi(B)$. Thus means, $\pi[A] = \pi[B]$. Let $a \in A$. Then, $\pi(a) \in \pi[A] = \pi[B]$, thus $\pi(a) = \pi(b)$ for some $b \in B$. That is, $aN = bN \iff a \in bN$. Meanwhile, $N \leq B$, thus $a \in bN \subset B$, $A \subset B$. Similarly, $B \subset A$, that is $A = B$.
To show that onto: Let $K \in C$. Then, $N \leq \pi^{-1}[K] \leq G$, thus clear. $\square$

Chapter 3

Ring Theory

3.1 Ring of Fractions

Theorem 7. Let R be a Commutative Ring, $D \subset R$ be a subset such that $\begin{cases} \text{no zero, no zero divisors} \\ \text{closed under multiplication} \end{cases}$.

Then, there exists a Commutative Ring Q with identity satisfies:

1. R can embed in Q, and every element of D becomes unit in Q. More precisely, $Q = \{rd^{-1} \mid r \in R, \ d \in D\}$.

2. Q is the smallest Ring with identity such that every element of D becomes unit in Q.

Proof. Let $\mathcal{F} \overset{\text{def}}{=} \{(r, d) \mid r \in R, \ d \in D\}$ and the relation $\sim$ on $\mathcal{F}$ by $(r_1, d_1) \sim (r_2, d_2) \iff r_1 d_2 = r_2 d_1$.

Then, $\sim$ is equivalent relation: reflexive and symmetirc are clear, and Suppose that $(r_1, d_1) \sim (r_2, d_2)$ and $(r_2, d_2) \sim (r_3, d_3)$.

$$r_2 d_3 = r_3 d_2 \implies r_2 d_1 d_3 = r_3 d_1 d_2 \implies r_1 d_2 d_3 = r_3 d_1 d_2 \implies d_2(r_1 d_3 - r_3 d_1) \implies r_1 d_3 = r_3 d_1$$

Thus transitivity shown. Define

$$\frac{r}{d} \overset{\text{def}}{=} [(r, d)] = \{(a, b) \mid (a, b) \sim (r, d)\}, \quad Q \overset{\text{def}}{=} \left\{ \frac{r}{d} \ \middle| \ r \in R, \ d \in D \right\}$$

And define operations $+, \times$ on Q:

$$\frac{r_1}{d_1} + \frac{r_2}{d_2} \overset{\text{def}}{=} \frac{r_1 d_2 + r_2 d_1}{d_1 d_2}, \quad \frac{r_1}{d_1} \times \frac{r_2}{d_2} \overset{\text{def}}{=} \frac{r_1 r_2}{d_1 d_2}$$

Well-Definedness: If $\dfrac{r_1}{d_1} = \dfrac{r_1'}{d_1'}$ and $\dfrac{r_2}{d_2} = \dfrac{r_2'}{d_2'}$,

$$\frac{r_1 d_2 + r_2 d_1}{d_1 d_2} = \frac{r_1 d_2 d_1' d_2' + r_2 d_1 d_1' d_2'}{d_1 d_2 d_1' d_2'} = \frac{(r_1 d_1') d_2 d_2' + (r_2 d_2') d_1 d_1'}{d_1 d_2 d_1' d_2'} = \frac{(r_1' d_1) d_2 d_2' + (r_2' d_2) d_1 d_1'}{d_1 d_2 d_1' d_2'} = \frac{(r_1' d_2' + r_2' d_1') d_1 d_2}{d_1 d_2 d_1' d_2'} = \frac{r_1' d_2' + r_2' d_1'}{d_1' d_2'}$$

$$\frac{r_1 r_2}{d_1 d_2} = \frac{r_1 r_2 d_1' d_2'}{d_1 d_2 d_1' d_2'} = \frac{(r_1 d_1')(r_2 d_2')}{d_1 d_2 d_1' d_2'} = \frac{(r_1' d_1)(r_2' d_2)}{d_1 d_2 d_1' d_2'} = \frac{r_1' r_2' d_1 d_2}{d_1 d_2 d_1' d_2'} = \frac{r_1' r_2'}{d_1' d_2'}$$

Now, $(Q, +, \times)$ constructs Commutative Ring with identity: for any $d \in D$, put $0_Q \overset{\text{def}}{=} \dfrac{0}{d}$, $1_Q \overset{\text{def}}{=} \dfrac{d}{d}$. Then,

1. $(R, +, \times)$ closed under the operations since D is closed under the multiplication.

2. $(R, +)$ has a zero: $\dfrac{r_1}{d_1} + 0_Q = \dfrac{r_1}{d_1} + \dfrac{0}{d} = \dfrac{r_1 d + 0 d_1}{d_1 d} = \dfrac{r_1 d}{d_1 d} = \dfrac{r_1}{d_1}$.

3. $(R, +)$ has an inverse: $\dfrac{r_1}{d_1} + \dfrac{-r_1}{d_1} = \dfrac{r_1 d_1 + (-r_1) d_1}{d_1 d_1} = \dfrac{[(r_1) + (-r_1)] d_1}{d_1 d_1} = \dfrac{0 d_1}{d_1 d_1} = \dfrac{0}{d_1 d_1} = 0_Q$.

4. $(R, +, \times)$ satisfies distributive law:

   4-1. The left law:
   $$\frac{r_1}{d_1} \times \left(\frac{r_2}{d_2} + \frac{r_3}{d_3} \right) = \frac{r_1}{d_1} \times \frac{r_2 d_3 + r_3 d_2}{d_2 d_3} = \frac{r_1 r_2 d_3 + r_1 r_3 d_2}{d_1 d_2 d_3} = \frac{r_1 r_2 d_1 d_3 + r_1 r_3 d_1 d_2}{d_1 d_2 d_1 d_3} = \frac{r_1 r_2}{d_1 d_2} + \frac{r_2 r_3}{d_2 d_3}$$
   $$= \frac{r_1}{d_1} \times \frac{r_2}{d_2} + \frac{r_2}{d_2} \times \frac{r_3}{d_3}$$

   4-2. The right law:
   $$\left(\frac{r_1}{d_1} + \frac{r_2}{d_2} \right) \times \frac{r_3}{d_3} = \frac{r_1 d_2 + r_2 d_1}{d_1 d_2} \times \frac{r_3}{d_3} = \frac{r_1 r_3 d_2 + r_2 r_3 d_1}{d_1 d_2 d_3} = \frac{r_1 r_3 d_2 d_3 + r_2 r_3 d_1 d_3}{d_1 d_3 d_2 d_3} = \frac{r_1 r_3}{d_1 d_3} + \frac{r_2 r_3}{d_2 d_3}$$
   $$= \frac{r_1}{d_1} \times \frac{r_3}{d_3} + \frac{r_2}{d_2} \times \frac{r_3}{d_3}$$

5. $(R, \times)$ has an identity: $\dfrac{r_1}{d_1} \times 1_Q = \dfrac{r_1}{d_1} \times \dfrac{d}{d} = \dfrac{r_1 d}{d_1 d} = \dfrac{r_1}{d_1}$.

6. Elements of D become unit in Q: Define $\iota : R \to Q : r \mapsto \dfrac{rd}{d}$ where $d \in D$ is any fixed element in D.
   Then, ι is Ring-Monomorphsim because:

   6-1. Well-Defined and Injective: $\iota(r_1) = \iota(r_2) \iff \dfrac{r_1 d}{d} = \dfrac{r_2 d}{d} \iff (r_1 - r_2) dd = 0 \iff r_1 = r_2$

$\square$

Chapter 4

Field Theory

Chapter 5

Category

Chapter 6

General Topology

6.1 Complete Metric Space

> **Definition 2.** Let (X, d) be a Metric Space, and $\{p_n\}$ be a Sequence in X.
> The Sequence $\{p_n\}$ is called **Cauchy Sequence** if:
>
> $$\text{For any } \epsilon > 0, \text{ there exists } N \in \mathbb{N} \text{ such that } m, n \geq N \implies d(p_m, p_n) < \epsilon.$$
>
> A Metric Space (X, d) is said to be **Complete** if every Cauchy Sequnces Converge.

> **Lemma 2.** Let $\{E_n\}$ be a sequence of closed bounded non-empty subsets in a Complete Metric Space X such that $E_n \supset E_{n+1}$.
>
> If $\lim\limits_{n \to \infty} \operatorname{diam} E_n = 0$, then $\bigcap\limits_{n=1}^{\infty} E_n = \{p\}$ for some $p \in X$.

Proof. For each $n \in \mathbb{N}$, construct $p_n \in E_n$.
Let $\epsilon > 0$ be given. Since $\operatorname{diam} E_n \to 0$, there is $N \in \mathbb{N}$ such that $\operatorname{diam} E_n < \epsilon$.
For any $m, n \geq M$, E_N contains p_m, p_n. That is, $d(p_m, p_n) < \epsilon$. Thus, $\{p_n\}$ be a Cauchy sequence of X.
Since X is complete, there is a unique point $p \in X$ such taht $p_n \to p$. Let $N \in \mathbb{N}$ be a integer such that $n \geq N \implies |p_n - p| < \epsilon$.
Now, for each $n \geq N$, E_n has a limit point as p. And for any $n \in \mathbb{N}$, E_n contains $E_N, E_{N+1}, \ldots$, thus for all $n \in \mathbb{N}$, E_n has a limit point as p. Meanwhile, E_n closed, $p \in E_n$, $\forall n \in \mathbb{N}$.
Consequently, $p \in \bigcap\limits_{n=1}^{\infty} E_n$. If there is $q \in X$ such that $p \neq q$, $q \in \bigcap\limits_{n=1}^{\infty} E_n$. Then, $\operatorname{diam} E_n \geq d(p, q) > 0$, $\forall n \in \mathbb{N}$. $\square$

6.1.1 Baire Category

> **Definition 3.** The Topological Space X is called **Baire Space** if:
>
> $$\text{If } \{G_n \mid n \in \mathbb{N}\} \text{ be a Countable Collection of dense open sets of } X, \text{ then } \overline{\bigcap\limits_{n=1}^{\infty} G_n} = X$$
>
> In brief, every Countable intersection of dense open sets be dense in X.

> **Theorem 8.** Locally Compact Hausdorff Space is Baire Space.

> **Theorem 9.** Complete Metric Space is Baire Space.

Proof. Suppose that $\{G_n \mid n \in \mathbb{N}\}$ be a Countable Collection of dense open set of Complete Metric Space.
Let an open $U \in \mathcal{T}$ be given. Since G_n is dense in the Space, $U \cap G_1$ is non-empty open set.

Thus, there exists a $p_1 \in U \cap G_1$ such that for some $r_1 > 0$, $B_{r_1}(p_1) \subset U \cap G_1$.
Then, automatically,

$$B_{\frac{r_1}{2}}(p_1) \subset \overline{B_{\frac{r_1}{2}}(p_1)} \subset B_{r_1}(p_1) \subset U \cap G_1$$

Set $E_1 = U$, $E_2 = B_{\frac{r_1}{2}}(p_1)$.
Suppose that $E_1, \ldots, E_{n-1}$ are chosen. Then, since $E_{n-1} \cap G_{n-1}$ is open, being intersection of opens.
Thus there exists a point $p_{n-1} \in E_{n-1} \cap G_{n-1}$ and exists r_{n-1} such that

$$B_{r_{n-1}}(p_{n-1}) \subset E_{n-1} \cap G_{n-1} \subset E_{n-1}$$

This implies that

$$B_{\frac{r_{n-1}}{2}}(p_{n-1}) \subset \overline{B_{\frac{r_{n-1}}{2}}(p_{n-1})} \subset B_{r_{n-1}}(p_{n-1}) \subset E_{n-1} \cap G_{n-1} \subset E_{n-1}$$

Set $E_n = B_{\frac{r_{n-1}}{2}}(p_{n-1})$. Since inductively construction of $\{E_n\}$, $E_{n+1} \subset E_n$ and $\overline{E_n} \subset G_n$ for all $n \in \mathbb{N}$.
Consequently,

$$U \cap \left(\bigcap_{n=1}^{\infty} G_n \right) = \bigcap_{n=1}^{\infty} (U \cap G_n) \supset \bigcap_{n=1}^{\infty} (U \cap \overline{E_n}) = U \cap \left(\bigcap_{n=1}^{\infty} \overline{E_n} \right) = \bigcap_{n=1}^{\infty} \overline{E_n} \neq \emptyset$$

$\square$

Definition 4. Let X be a Topological Space.
$A \subset X$ is said to be **nowhere dense subset** if $(\overline{A})^\circ = \emptyset$.

1. $B \subset X$ is called **first category** if B can be representive by union of countable nowhere dense subsets.

2. If the subset is not first category, then it is said to be **second category**.

6.1.2 Nowhere Differentiable function

6.2 Urysohn Metrization Theorem

6.2.1 Urysohn Metrization Theroem

Recall that:

Definition 5. X is T_4 if: For any disjoint closed set A and B, there exist disjoint open U, V such that $A \subseteq U$ and $B \subseteq V$.

Lemma 3. X is T_4 Space if and only if For any closed C and open U with $C \subseteq U$, there exists open O such that

$$\underset{\text{closed}}{C} \subseteq \underset{\text{open}}{O} \subseteq \underset{\text{closed}}{\overline{O}} \subseteq \underset{\text{open}}{U}$$

Proof. Proof of the left direction only.
Let X be a T_4 Space, and $C \subset X$ be a closed, U be a open containing C. Then, $C \subset U$ implies $U^c \subset C^c$, thus U^c is a closed set disjoint from C. By T_4 condition, There exist disjoint opens O, O' such that $C \subset O$ and $U^c \subset O' \iff O'^c \subset U$.
Since $O \cap O' = \emptyset \iff O \subset O'^c$, O contained in U, this implies that $C \subset O \subset U$.
Since closure is the smallest closed set such that contains it, consequently $C \subset O \subset \overline{O} \subset O'^c \subset U$. $\square$

Definition 6. Let X be a Toplogical Space, and $A, B \subset X$ are disjoint closed subset.
A real-valued Continuous map $f : X \to [a, b]$ is called **Urysohn function** for A and B if: $f|_A = a$ and $f|_B = b$.
In another form,
$$f : X \to [a, b] : x \to \begin{cases} a & x \in A \\ b & x \in B \\ f(x) & x \notin A \cup B \end{cases}$$

Lemma 4. Urysohn Lemma
T_4 Space has an Urysohn function for any two disjoint closed subsets.

Proof. Generalization is the last thing to proven, first of all, prove in case of $[a, b] = [0, 1]$. This proof consists by three Step.
Let X be a T_4 Space, and $A, B \subset X$ be closed subsets.
Step 1. Construct a Chain of Open sets with Dyadic Rational Indices.
Consider a set of **Dyadic Rationals** $D \overset{\text{def}}{=} \left\{ \frac{k}{2^n} \mid n, k \in \mathbb{N}, \ k \leq 2^n - 1 \right\}$. We will show that the following statement holds:

$$\text{For any } r, s \in D \text{ with } r < s, \text{ there exist open sets } U_r, U_s \text{ such that } A \subseteq \overline{U}_r \subseteq U_s \subseteq X \setminus B \tag{$*$}$$

For this, Enough to Show that: For any $k \in \mathbb{N}$, there exists a Chain as:

$$A \subseteq U_{\frac{1}{2^k}} \subseteq \overline{U}_{\frac{1}{2^k}} \subseteq U_{\frac{2}{2^k}} \subseteq \overline{U}_{\frac{2}{2^k}} \subseteq \cdots \subseteq U_{\frac{2^k-1}{2^k}} \subseteq \overline{U}_{\frac{2^k-1}{2^k}} \subseteq X \setminus B$$

(Note that this opens in the Chain are not necessary distinct: For instance, if Ambient Space is Finite, then the Space is **Noetherian**. That is, X satisfies Ascending Chain Condition for open sets.)
Let $k = 1$. Then, By T_4 condition gives that: There exists an open set U_1 such that

$$A \subseteq U_1 \subseteq \overline{U_1} \subseteq X \setminus B$$

Now, naming this U_1 as $U_{\frac{1}{2}}$, proved when $k = 1$.
Suppose that for some $k > 1$, the Chain exists as:

$$\underset{\text{closed}}{A} \overset{*1}{\subseteq} U_{\frac{1}{2^k}} \subseteq \overline{U}_{\frac{1}{2^k}} \overset{*2}{\subseteq} U_{\frac{2}{2^k}} \subseteq \cdots \overset{*2^k-1}{\subseteq} U_{\frac{2^k-1}{2^k}} \subseteq \overline{U}_{\frac{2^k-1}{2^k}} \overset{*2^k}{\subseteq} \underset{\text{open}}{X \setminus B}$$

By repeatedly applying the T_4 condition 2^k times, as indicated by the indices $*1, *2, \ldots, *2^k$, we can construct 2^k open sets such that:
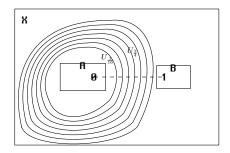
$$A \subseteq U_{\frac{1}{2^{k+1}}} \subseteq \overline{U}_{\frac{1}{2^{k+1}}} \subseteq U_{\frac{1}{2^k}} \subseteq \overline{U}_{\frac{1}{2^k}} \subseteq U_{\frac{3}{2^{k+1}}} \subseteq \overline{U}_{\frac{3}{2^{k+1}}} \subseteq U_{\frac{2}{2^k}} \subseteq \overline{U}_{\frac{2}{2^k}} \quad \subseteq \cdots \subseteq U_{\frac{2^k-1}{2^k}} \subseteq \overline{U}_{\frac{2^k-1}{2^k}} \subseteq U_{\frac{2^{k+1}-1}{2^{k+1}}} \subseteq \overline{U}_{\frac{2^{k+1}-1}{2^{k+1}}} \subseteq X \setminus B$$

Finally, Step 1 proved.

Step 2. Construct an Urysohn Function.

Define a map $f : X \to [0,1]$ as:

$$f(x) = \begin{cases} 0 & x \in \bigcap_{t \in D} U_t \\ \sup\{t \in D \mid x \notin U_t\} & x \notin \bigcap_{t \in D} U_t \end{cases}$$

Then, this map f is well-defined by $(*)$ and $\sup D \leq 1$. And f satisfies that:

1. $\forall r \in D, \ x \in A \subset U_r$. Thus, $f(x) = 0$ if $x \in A$.

2. $\forall r \in D, \ x \in B \subset X \setminus U_r$. Thus, $f(x) = \sup D = 1$ if $x \in B$.

3. If $x \in \overline{U}_r$, then for every $s > r$, $x \in \overline{U}_r \subset U_s$. Thus, $f(x) \leq r$. In Contrapositive, $f(x) > r \implies x \notin \overline{U}_r$.
   (If $f(x) = \sup\{t \in D \mid x \notin U_t\} > r$, then there is $s \in D$ such that $s > r$ and $x \notin U_s$, Contradiction.)

4. If $x \notin U_r$, then, $f(x) \geq r$. In Contrapositive, $f(x) < r \implies x \in U_r$.

Now, show that this map f is Continuous map: Let $x \in X$ be fixed arbitrarlily, and $\epsilon > 0$ be given.
In Case of $0 < f(x) < 1$.
Since Density of Dyadic Rationals, Choose $r, s \in D$ such that $f(x) - \epsilon < r < f(x) < s < f(x) + \epsilon$.
Now, we obtain that:

$$x \overset{(*)}{\in} U_s \setminus \overline{U}_r \overset{(**)}{\subseteq} f^{-1}[(f(x) - \epsilon, f(x) + \epsilon)]$$

$(*)$ directly given by above properties, $(**)$ given applying the fact that $x \in U_s \subset \overline{U}_s$ and $x \notin \overline{U}_r$.
In Case of $f(x) = 0$.
Choose $r \in D$ such that $f(x) = 0 < r < \epsilon = f(x) + \epsilon$. Then,

$$x \in U_r \subset f^{-1}[(f(x), f(x) + \epsilon)]$$

In Case of $f(x) = 1$.
Choose $r \in D$ such that $f(x) - \epsilon = 1 - \epsilon < r < 1 = f(x)$. Then,

$$x \in X \setminus U_r \subset f^{-1}[(f(x) - \epsilon, f(x))]$$

Consequently, f is Continuous map on $[0,1]$ such that $f|_A = 0$ and $f|_B = 1$.
Step 3. Generalization.
Since $[0,1] \cong [a,b]$ for any $a < b$, let $g : [0,1] \to [a,b] : x \mapsto (1-x)a + xb$ be a Homeomorphism.
Then, $h = g \circ f : X \to [a,b]$ becomes a Continuous map such that $h|_A = a$ and $h|_B = b$. □

Chapter 7

Algebraic Topology

Chapter 8

Basic Analysis

8.1 Taylor's Theorem

> **Theorem 10. Taylor's Theorem**
>
> Let $f : [a,b] \to \mathbb{R}$, and let $n \in \mathbb{N}$ be fixed. Suppose that $\begin{cases} f^{(n-1)} \text{ is Continuous.} \\ f^{(n)}(t) \text{ exists for every } t \in (a,b). \end{cases}$
>
> Then, for any $\alpha, \beta \in [a,b]$, there exists $x \in (\alpha, \beta)$ such that
>
> $$f(\beta) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(\beta - \alpha)^k + \frac{f^{(n)}(x)}{n!}(\beta - \alpha)^n$$

Proof. Put

$$M \stackrel{\text{def}}{=} \frac{1}{(\beta - \alpha)^n} \cdot \left(f(\beta) - \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(\beta - \alpha)^k \right)$$

That is,

$$f(\beta) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(\beta - \alpha)^k + M(\beta - \alpha)^n$$

and put

$$g(t) \stackrel{\text{def}}{=} f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(t - \alpha)^k - M(t - \alpha)^n, \quad (a \leq t \leq b)$$

If we differentiate the above equation n times,

$$g^{(n)}(t) = f^{(n)}(t) - n!M, (a < t < b)$$

For each $k = 0, 1, \ldots, n-1$,

$$\frac{d^r}{dt^r} \left(\sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(t - \alpha)^k \right) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!} \cdot \frac{d^r}{dt^r}\left((t - \alpha)^k \right)$$

$$= \sum_{k=r+1}^{n-1} \frac{f^{(k)}(\alpha)}{k!} \cdot \frac{k!}{(k-r)!}(t - \alpha)^{k-r} + f^{(r)}(\alpha)$$

$$= \sum_{k=r+1}^{n-1} \frac{f^{(k)}(\alpha)}{(k-r)!}(t - \alpha)^{k-r} + f^{(r)}(\alpha)$$

Substituting $t = \alpha$, only the $f^{(r)}(\alpha)$ term remains. Therefore, for $r = 0, \ldots, n-1$, $g(\alpha) = g'(\alpha) = \cdots = g^{(n-1)}(\alpha) = 0$.
Since $g(\beta) = 0$ by definition, the Mean-Value Theorem implies there exists a $x_1 \in (\alpha, \beta)$ s.t. $g'(x_1) = \frac{g(\beta) - g(\alpha)}{\beta - \alpha} = 0$.
And similarly, there is $x_2 \in (x_1, \beta)$ s.t. $g''(x_2) = \frac{g'(x_1) - g'(\alpha)}{\beta - \alpha} = 0$.

Inductively, for some $x_n \in (\alpha, \beta)$, $g^{(n)}(x_n) = f^{(n)}(x_n) - n!M = 0$. That is, $M = \frac{f^{(n)}(x_n)}{n!}$.
Proof Complete by Initial Setting. □

> **Corollary 2.** Let $f : [a,b] \to \mathbb{R}$ be an infinitely differentiable function.
> Suppose that there exists a $M > 0$ such that for any $n \in \mathbb{N}$, $\sup_{t \in [a,b]} |f^{(n)}(t)| \leq M$. Then, for any $x, \alpha \in [a,b]$,
>
> $$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x)}{k!}(x - \alpha)^k$$

8.2 Convexity

8.2.1 Definition

Definition 7. Let $f : (a,b) \to \mathbb{R}$ be a Real-valued function. f is said to be **convex** if: For any $x, y \in (a,b), \lambda \in (0,1)$,

$$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda)f(y)$$

Convex function has follwing properties:

Lemma 5. Let $f : (a,b) \to \mathbb{R}$ be a Convex function, and $a < x_1 < x_2 < x_3 < b$. Then,

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \le \frac{f(x_3) - f(x_1)}{x_3 - x_1} \le \frac{f(x_3) - f(x_2)}{x_3 - x_2}$$

Proof. To show that first inequalty, note that

$$\frac{x_2 - x_1}{x_3 - x_1} \cdot x_3 + \left(1 - \frac{x_2 - x_1}{x_3 - x_1}\right) \cdot x_1 = \frac{(x_2 - x_1)x_3 + (x_3 - x_2)x_1}{x_3 - x_1} = x_2$$

Now,

$$f(x_2) = f\left(\frac{x_2 - x_1}{x_3 - x_1} \cdot x_3 + \left(1 - \frac{x_2 - x_1}{x_3 - x_1}\right) \cdot x_1\right)$$
$$\le \frac{x_2 - x_1}{x_3 - x_1} \cdot f(x_3) + \left(1 - \frac{x_2 - x_1}{x_3 - x_1}\right) \cdot f(x_1) = \frac{x_2 - x_1}{x_3 - x_1} \cdot f(x_3) + f(x_1) - \left(\frac{x_2 - x_1}{x_3 - x_1}\right) \cdot f(x_1)$$

In brief,

$$f(x_2) - f(x_1) \le \frac{x_2 - x_1}{x_3 - x_1}(f(x_3) - f(x_1)) \implies \frac{f(x_2) - f(x_1)}{x_2 - x_1} \le \frac{f(x_3) - f(x_1)}{x_3 - x_1}$$

And similarly,

$$\frac{x_3 - x_2}{x_3 - x_1} \cdot x_1 + \left(1 - \frac{x_3 - x_2}{x_3 - x_1}\right) x_3 = x_2$$

gives the second inequality. $\square$

8.2.2 Properties

> **Proposition 2.** If $f : (a, b) \to \mathbb{R}$ is Convex, then f is Continuous.

Proof. Let $\epsilon > 0$ be given, $s < t$ are fixed in (a, b). For any $x, y \in (s, t)$ with $s < x < y < t$,

$$\frac{f(s) - f(a)}{s - a} \leq \frac{f(x) - f(s)}{x - s} \leq \frac{f(y) - f(x)}{y - x} \leq \frac{f(t) - f(y)}{t - y} \leq \frac{f(b) - f(t)}{b - t}$$

Put $M = \max \left\{ \left| \frac{f(s) - f(a)}{s - a} \right|, \left| \frac{f(b) - f(t)}{b - t} \right| \right\}$. Then, for any $x, y \in (s, t)$,

$$\left| \frac{f(y) - f(x)}{y - x} \right| \leq M$$

Now,

$$|f(y) - f(x)| \leq M|y - x| < \epsilon$$

Since $s, t \in (a, b)$ was arbitrary, f is continuous on (a, b). $\qquad \square$

> **Proposition 3.** Let f is differentiable on (a, b). Then,
>
> f is Convex **if and only if** f' is monotonically increasing on (a, b).

Proof. Prove by showing both directions: right and left.
Right Direction Let $x_1 < x_2$ in (a, b). Then,

$$f'(x_1) = \lim_{t \to x_1} \frac{f(t) - f(x_1)}{t - x_1} \leq \frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \lim_{\tau \to x_2} \frac{f(\tau) - f(x_2)}{\tau - x_2} = f'(x_2)$$

More rigorously, put $\epsilon = \left| f'(x_1) - \frac{f(x_2) - f(x_1)}{x_2 - x_1} \right|$. (If $\epsilon = 0$, then there is nothing to prove.).
Now, there exists a $\delta > 0$ such that $|t - x_1| < \delta$ implies

$$\left| f'(x_1) - \frac{f(t) - f(x_1)}{t - x_1} \right| < \epsilon \iff -\epsilon + \frac{f(t) - f(x_1)}{t - x_1} < f'(x_1) \stackrel{(*)}{=} \epsilon + \frac{f(t) - f(x_1)}{t - x_1}$$

If $f'(x_1) - \dfrac{f(x_2) - f(x_1)}{x_2 - x_1} > 0$, then $(*)$ gives

$$f'(x_1) < f'(x_1) + \frac{f(t) - f(x_1)}{t - x_1} - \frac{f(x_2) - f(x_1)}{x_2 - x_1} \iff \frac{f(x_2) - f(x_1)}{x_2 - x_1} < \frac{f(t) - f(x_1)}{t - x_1} \quad \forall t \text{ s.t. } |t - x_1| < \delta$$

If $|t - x_1| < |x_2 - x_1|$, then this contradicts to Convexity.
Consequently, we obtain the first inequality, similarly can prove the second inequality.
Left Direction Let $x, y \in (a, b)$ and $\lambda \in (0, 1)$ be given. The Mean Value Theorem gives that:

$$f(\lambda x + (1 - \lambda)y) - f(x) = f'(z_1)(\lambda x + (1 - \lambda)y - x) \text{ for some } z_1 \in (x, \lambda x + (1 - \lambda)y)$$
$$f(y) - f(\lambda x + (1 - \lambda)y) = f'(z_2)(y - \lambda x + (1 - \lambda)y) \text{ for some } z_2 \in (\lambda x + (1 - \lambda)y, y)$$

Now, Monotonically increasing gives

$$\frac{f(\lambda x + (1 - \lambda)y) - f(x)}{\lambda x + (1 - \lambda)y - x} = f'(z_1) \leq f'(z_2) = \frac{f(y) - f(\lambda x + (1 - \lambda)y)}{y - (\lambda x + (1 - \lambda)y)}$$
$$\implies \frac{f(\lambda x + (1 - \lambda)y) - f(x)}{(1 - x)(y - x)} \leq \frac{f(y) - f(\lambda x + (1 - \lambda)y)}{\lambda(y - x)}$$
$$\implies \lambda f(\lambda x + (1 - \lambda)y) - \lambda f(x) \leq (1 - \lambda)f(y) - (1 - \lambda)\lambda x + (1 - \lambda)y$$
$$\implies f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

$\qquad \square$

Corollary 3. If $f : [a,b] \to \mathbb{R}$ is twice-differentiable, then

$$f \text{ is Convex if and only if } f''(x) \geq 0 \text{ for all } x \in (a,b).$$

Theorem 11. Let $f : [a.b] \to \mathbb{R}$ be given. Then,

$$f \text{ is Convex if and only if } f \text{ is Continuous, and Midpoint Convex.}$$

Midpoint convex is that f satisfies $\forall x, y \in (a,b)$, $f\left(\dfrac{x+y}{2}\right) \leq \dfrac{f(x)+f(y)}{2}$.

Proof. The right direction is clear. To show the left direction, we demonstrate that
Midpoint Convexity implies Dyadic Rational Convexity. Claim: For any $n \in \mathbb{N}$,

$$f\left(\frac{\sum_{k=1}^{2^n} x_k}{2^n}\right) \leq \frac{1}{2^n} \sum_{k=1}^{2^n} f(x_k) \tag{$*$}$$

Using Induction: If $n = 1$, it is clear by Midpoint Convexity.
Assume that for $n \in \mathbb{N}$, $(*)$ is True. Then,

$$
\begin{aligned}
f\left(\frac{\sum_{k=1}^{2^{n+1}} x_k}{2^{n+1}}\right) &= f\left(\frac{1}{2} \cdot \left[\frac{\sum_{k=1}^{2^n} x_k}{2^n} + \frac{\sum_{k=2^n+1}^{2^{n+1}} x_k}{2^n}\right]\right) \\
&\overset{\text{m.c}}{\leq} \frac{1}{2}\left(f\left(\frac{\sum_{k=1}^{2^n} x_k}{2^n}\right) + f\left(\frac{\sum_{k=2^n+1}^{2^{n+1}} x_k}{2^n}\right)\right) \\
&\overset{(*)}{\leq} \frac{1}{2}\left(\frac{1}{2^n} \sum_{k=1}^{2^n} f(x_k) + \frac{1}{2^n} \sum_{k=2^n+1}^{2^{n+1}} f(x_k)\right) \\
&= \frac{1}{2^{n+1}} \sum_{k=1}^{2^{n+1}} f(x_k)
\end{aligned}
$$

Consequently, we obtain the claim. Now, let $n \in \mathbb{N}$, and m be an integer such that $1 \leq m \leq 2^n$.
Put $x_1 = x_2 = \cdots = x_m = x$ and $x_{m+1} = x_{m+2} = \cdots = x_{2^n} = y$. Then

$$f\left(\frac{m}{2^n}x + \left(1 - \frac{m}{2^n}\right)y\right) \leq \frac{m}{2^n}f(x) + \left(1 - \frac{m}{2^n}\right)f(y)$$

For complete this discussion, Let $x, y \in (a,b), \lambda \in (0,1)$ be given.
Since $\dfrac{\lfloor 2^n \lambda \rfloor}{2^n} \to \lambda$ as $n \to \infty$, for any $n \in \mathbb{N}$,

$$f\left(\frac{\lfloor 2^n \lambda \rfloor}{2^n}x + \left(1 - \frac{\lfloor 2^n \lambda \rfloor}{2^n}\right)y\right) \leq \frac{\lfloor 2^n \lambda \rfloor}{2^n}f(x) + \left(1 - \frac{\lfloor 2^n \lambda \rfloor}{2^n}\right)f(y)$$

Finally, taking limits then

$$\lim_{n \to \infty} f\left(\frac{\lfloor 2^n \lambda \rfloor}{2^n}x + \left(1 - \frac{\lfloor 2^n \lambda \rfloor}{2^n}\right)y\right) \overset{f \text{ conti.}}{=} f\left(\lim_{n \to \infty}\left[\frac{\lfloor 2^n \lambda \rfloor}{2^n}x + \left(1 - \frac{\lfloor 2^n \lambda \rfloor}{2^n}\right)y\right]\right) = f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$$

In brief, Midpoint Convexity implies Dyadic Rational Convexity, and with Continuous implies Convexity. $\square$

8.3 Lipschitz Condition

8.3.1 Definition

Definition 8. A real-vaulued function $f : (a, b) \to \mathbb{R}$ is called **Lipschitz Continuous** if:

$$\exists L \geq 0 \text{ s.t. } \forall x_1, x_2 \in (a, b), \ |f(x_1) - f(x_2)| \leq L \cdot |x_1 - x_2|$$

The constant L is said to be **Lipschitz Constant** of f. In particular, the constant

$$D \overset{\text{def}}{=} \sup_{x_1 \neq x_2} \frac{|f(x_1) - f(x_2)|}{|x_1 - x_2|}$$

is called **dilation** of f. Clearly,

$$\forall x_1, x_2 \in (a, b), \ |f(x_1) - f(x_2)| \leq D \cdot |x_1 - x_2|$$

and if $L > 0$ is Lipschitz Constant of f, then $D \leq L$. That is, $D = \inf\{L > 0 \mid L \text{ is Lipschitz constant of } f\}$.

8.3.2 Properties

Proposition 4. If $f : (a, b) \to \mathbb{R}$ is Lipschitz Continuous, then f is uniformly continuous.

Proof. Let $L \geq 0$ be a Lipschitz Constant of f. Then, for any $\epsilon > 0$,

$$\forall x, y \in (a, b), \ |x - y| < \frac{\epsilon}{L} \implies |f(x) - f(y)| \leq L|x - y| < \epsilon$$

$\square$

Proposition 5. Let $f : (a, b) \to \mathbb{R}$ be a Differentiable function. Then,

$$f \text{ is Lipschitz Continuous } \textbf{if and only if } f' \text{ is bounded in } (a, b).$$

Proof.
Right Direction
Let $L > 0$ be a Lipschitz constant of f, and $x \in (a, b)$ be given. Since definition of derivative,

$$f'(x) \overset{\text{def}}{=} \lim_{t \to x} \frac{f(x) - f(t)}{x - t}$$

Meanwhile, the assumption gives: for any distinct $x, t \in (a, b)$,

$$\frac{|f(x) - f(t)|}{|x - t|} \leq L$$

Therefore,

$$f'(x) = \lim_{t \to x} \frac{f(x) - f(t)}{x - t} \leq \lim_{t \to x} \frac{|f(x) - f(t)|}{|x - t|} \leq \lim_{t \to x} L = L$$

Left Direction
Let distinct $x, y \in (a, b)$ be given. Then, the Mean-Value Theorem gives: There exists a $z \in (x, y)$ such that

$$f(x) - f(y) = f'(z)(x - y) \implies f'(z) = \frac{f(x) - f(y)}{x - y}$$

Now,

$$\left| \frac{f(x) - f(y)}{x - y} \right| = |f'(z)| \leq L \implies |f(x) - f(y)| \leq L \cdot |x - y|$$

If $x = y$, then there is nothing to prove. $\square$

Note that:

$$\text{Lipschitz Continuous} \implies \text{Uniformly Continuous} \implies \text{Continuous}$$

8.3.3 Newton-Raphson Method

> **Theorem 12. Newton-Raphson Method**
> Let $f : [a, b] \to \mathbb{R}$ be a twice-differentiable, $f(a) < 0 < f(b)$. Suppose that f satisfies: for all $x \in [a, b]$,
>
> $$f'(x) \geq \delta > 0 \text{ and } 0 \leq f''(x) \leq M$$
>
> That is, f is strictly increasing convex function, and Lipschitz Continuous.
> Further, there uniquely exists $x^* \in (a, b)$ such that $f(x^*) = 0$.
> Let $x_1 \in (x^*, b)$ fixed. Define a sequence $\{x_n\}$ inductively as follows:
>
> $$x_{n+1} \stackrel{\text{def}}{=} x_n - \frac{f(x_n)}{f'(x_n)}$$
>
> Then, $\{x_n\}$ satisfies the following three conditions:
>
> 1. $\{x_n\}$ is decreasing sequence.
>
> 2. $x_n \to x^*$ as $n \to \infty$.
>
> 3. For any $n \in \mathbb{N}$, $0 \leq x_{n+1} - x^* \leq \left[\frac{M}{2\delta}\right]^{2^{n+1}-1} [x_1 - x^*]^{2^n}$.
>
> Condition 3 means that for a suitable initial value x_1, we can establish an upper bound for the error.

Proof. This proof consists by three steps.
Since f'' is non-negative, and f' is positive, f is strictly increasing convex function.
And Fundamental Theorem of Calculus gives: for any $x \in (a, b)$,

$$f'(x) \stackrel{\text{FTA}}{=} \int_a^x f''(t)dt + f'(a) \leq \int_a^x Mdt + f'(a) = M(x-a) + f'(a) \leq M(b-a) + f'(a)$$

Thus, f' is bounded on (a, b), thus f is Lipschitz Continuous.
Step 1. f **has a unique root** x^*.
The existence of root given directly by Intermidate-Value theroem.
Suppose that $x^*, x' \in (a, b)$ are distinct root of f. i.e., $f(x^*) = f(x') = 0$. Then, by Mean-value theroem, there is $c \in (a, b)$ between x^* and x' such that

$$f'(c)(x^* - x') = f(x^*) - f(x') = 0$$

That is, $f'(c) = 0$. This is contradiction with f' is positive.
Step 2. $\{x_n\}$ **decrease**.
Proof by induction:
For $n = 1$, $f'(x_1)(x_1 - x_2) \stackrel{\text{def}}{=} f(x_1) > f(x^*) = 0$, thus $x_2 < x_1$. And,

$$\begin{aligned}
f(x_2) &\stackrel{\text{MVT}}{=} f(x_1) + f'(c_1)(x_2 - x_1) \quad \text{for some } c_1 \in (x_2, x_1) \\
&> f(x_1) + f'(x_1)(x_2 - x_1) = f'(x_1)(x_1 - x_2) + f'(x_1)(x_2 - x_1) = 0
\end{aligned}$$

Now, since $f(x_2) > 0 = f(x^*)$, the Mean-Value Theorem implies that $x_2 > x^*$.
To use induction, suppose that for some $n \geq 1$, $x^* < x_{n+1} < x_n$. Then,

$$f(x_{n+1}) = f'(x_{n+1})(x_{n+1} - x_{n+2}) > 0$$

Thus $x_{n+2} < x_{n+1}$ and

$$\begin{aligned}
f(x_{n+2}) &\stackrel{\text{MVT}}{=} f(x_{n+1}) + f'(c_{n+1})(x_{n+2} - x_{n+1}) \quad \text{for some } c_{n+1} \in (x_{n+2}, x_{n+1}) \\
&\geq f(x_{n+1}) + f'(x_{n+1})(x_{n+2} - x_{n+1}) \\
&= f(x_{n+1})(x_{n+1} - x_{n+2}) + f'(x_{n+1})(x_{n+2} - x_{n+1}) = 0
\end{aligned}$$

Again, the Mean-Value Theorem implies that $x_{n+2} > x^*$. Therefore, induction completes.
Now, $x_n \to x'$ as $n \to \infty$ for some $x' \in [x^*, x_1]$ since $\{x_n\}$ is Bounded below and Decreasing.
Still it remains that to show $x' = x^*$. By Continuity,

$$f'(x_n)(x_{n+1} - x_n) + f(x_n) = 0$$
$$\implies \lim_{n \to \infty} [f'(x_n)(x_{n+1} - x_n) + f(x_n)] = f\left(\lim_{n \to \infty} x_n\right) = f(x') = 0$$

Since the root of f is unique, thus $x' = x^*$.

Step 3. Establishing the error bound.

The Taylor's Theorem implies that

$$f(x^*) = f(x_n) + f'(x_n)(x^* - x_n) + \frac{f''(t_n)}{2}(x^* - x_n)^2 \quad \text{for some } t_n \in (x^*, x_n)$$

$$\implies x_{n+1} - x^* = \frac{f''(t_n)}{2f'(x_n)}(x^* - x_n)^2$$

Consequently,

$$0 \leq x_{n+1} - x^* = \frac{f''(t_n)}{2f'(x_n)}(x_n - x^*)^2 = \frac{f''(t_n)}{2f'(x_n)} \cdot \left(\frac{f''(t_{n-1})}{2f'(x_{n-1})}\right)^2 (x_{n-1} - x^*)^4 = \cdots$$

$$= \prod_{i=1}^{n} \left[\frac{f''(t_i)}{2f'(x_i)}\right]^{2^{(n+1-i)}} [x_1 - x^*]^{2^n} \leq \left[\frac{M}{2\delta}\right]^{2^{n+1}-1} [x_1 - x^*]^{2^n}$$

$\square$

8.3.4 Gradient Descent

> **Theorem 13.** Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function that satisfies the following conditions:
>
> 1. f is **Convex function**.
>
> 2. f' is **Lipschitz Continuous** with Lipschitz constant of f, $L > 0$. In this, f is called L-**Smooth**.
>
> 3. f has at least one local minimizer x^*.
>
> Then, x^* is a Global minimizer of $\mathbb{R}$, and there exists a unique closed interval M containing x^* such that
>
> $$\forall x \in M, t \notin M, \ f(x) = f(x^*) < f(t)$$
>
> And, given initial point $x_0 \in \mathbb{R}$ and $0 < \gamma \le \frac{1}{L}$, define a sequence $\{x_n\}$ inductively as follows:
>
> $$x_{n+1} = x_t - \gamma \cdot f'(x_n)$$
>
> Then, for any $N \in \mathbb{N}$,
>
> $$f(x_N) - f(x^*) \le \frac{|x_0 - x^*|^2}{2\gamma N}$$

Proof. Let $x^* \in \mathbb{R}$ be a local minimizer. That is, there exists a $\delta > 0$ such that $\forall t \in (x^* - \delta, x^* + \delta)$, $f(x^*) \le f(t)$. Then,

$$0 \le \lim_{t \to x^*+} \frac{f(x^*) - f(t)}{x^* - t} = f'(x^*) = \lim_{t \to x^*-} \frac{f(x^*) - f(t)}{x^* - t} \le 0$$

thus, $f'(x^*) = 0$. And, by convextiy, f' is monotonically inceasing. Now, The Fundametal Theorem of Calculus gives:

$$\forall x \in \mathbb{R}, \ f(x) = \int_{x^*}^{x} f'(t)dt + f(x^*) \ge f(x^*)$$

Therefore, x^* is a Global minimizer of f.
Now, establish the closed interval M. Since f' is Lipschitz Continuous, thus f' is Continuous.
Let $D \overset{\text{def}}{=} \{x \in \mathbb{R} \mid f'(x) = 0\}$. (Note that: $x^* \in D$, thus D is not emtpyset.)
D is closed because: Let $\{x_n\}$ be a convergent sequence in D. That is, for all $n \in \mathbb{N}$, $f(x_n) = 0$. Then, by continuity,

$$f\left(\lim_{n \to \infty} x_n\right) = \lim_{n \to \infty} f(x_n) = 0$$

The limit of $\{x_n\}$ is contained in D, thus D is closed.
And, D is interval: i.e, for any $x \in (\inf D, \sup D)$, $x \in D$ because:
Suppose that there exists $x \in (\inf D, \sup D)$ such that $x \notin D$. That is, $f'(x) \ne 0$. This is Contradiction with Monotonicity.

To set error of upper bound, we make inequalities: Let $x, y \in \mathbb{R}$ be given.
The Fundamental Theorem of Calculus and L-Smooth condition gives:

$$f(y) - f(x) = \int_x^y f'(t)dt = \int_0^1 f'(x + (y-x)u)(y-x)du = f'(x)(y-x) + \int_0^1 (f'(x + (y-x)u) - f'(x))(y-x)du$$

$$\overset{2.}{\le} f'(x)(y-x) + L \cdot |y-x|^2 \int_0^1 u \ du = f'(x)(y-x) + \frac{L}{2}|y-x|^2$$

For any $\lambda > 0$, Put $y = x - \lambda f'(x)$. Then,

$$f(x - \lambda f'(x)) \le f(x) - f'(x)(\lambda f'(x)) + \frac{L}{2}|\lambda f'(x)|^2 = f(x) + \lambda\left(\frac{L\lambda}{2} - 1\right)|f'(x)|^2$$

Put $\lambda = \frac{1}{L}$, then

$$f\left(x - \frac{f'(x)}{L}\right) \le f(x) - \frac{L}{2}|f'(x)|^2 \implies \frac{L}{2}|f'(x)|^2 \le f(x) - f\left(x - \frac{f'(x)}{L}\right) \le f(x) - \inf f$$

Meanwhile, the convexity gives: for any $x, y \in \mathbb{R}$,

$$f'(x)(y - x) \leq f(y) - f(x) \leq f'(y)(y - x)$$

since derivative of convex function increase monotonically. Put $z = y - \frac{1}{L}(f'(y) - f'(x))$. Then,

$$\begin{aligned}
f(x) - f(y) &= f(x) - f(z) + f(z) - f(y) \\
&\leq f'(x)(x - z) + f'(y)(z - y) + \frac{L}{2}|z - y|^2 \\
&= f'(x)\left(x - y + \frac{1}{L}(f'(y) - f'(x))\right) - f'(y)\left(\frac{1}{L}(f'(y) - f'(x))\right) + \frac{L}{2}\left|\frac{1}{L}(f'(y) - f'(x))\right|^2 \\
&= f'(x)(x - y) - \frac{1}{L}|f'(y) - f'(x)|^2 + \frac{1}{2L}|f'(y) - f'(x)|^2 \\
&= f'(x)(x - y) - \frac{1}{2L}|f'(y) - f'(x)|^2
\end{aligned}$$

Now,

$$\begin{cases} \frac{1}{2L}|f'(y) - f'(x)|^2 \leq f'(x)(x - y) - (f(x) - f(y)) \\ \frac{1}{2L}|f'(x) - f'(y)|^2 \leq f'(y)(y - x) - (f(y) - f(z)) \end{cases} \implies \frac{1}{L}|f'(y) - f'(x)|^2 \leq (f'(y) - f'(x))(y - x)$$

Since above inequalities, we obtain that

$$\begin{aligned}
|x_{n+1} - x^*|^2 = |x_n - \gamma \cdot f'(x_n) - x^*|^2 &= |(x_n - x^*) - \gamma \cdot f'(x_n)|^2 \\
&= |x_n - x^*|^2 - 2\gamma|f'(x_n)| \cdot |x_n - x^*| + \gamma^2 |f'(x_n)|^2 \\
&\leq |x_n - x^*|^2 - 2\gamma \frac{1}{L}|f'(x_n)|^2 + \gamma^2 |f'(x_n)|^2 \\
&= |x_n - x^*|^2 + \left(\gamma^2 - \frac{2\gamma}{L}\right)|f'(x_n)|^2 \leq |x_n - x^*|^2
\end{aligned}$$

Thus, $|x_n - x^*|$ decrease as $n \to \infty$. That is, $|x_n - x^*| \leq |x_0 - x^*|$ for all $n \in \mathbb{N}$.
Consider x_{n+1} and x_n. First, we obtain

$$\begin{aligned}
f(x_{n+1}) &\leq f(x_n) + f'(x_n)(x_{n+1} - x_n) + \frac{L}{2}|x_{n+1} - x_n|^2 \\
&= f(x_n) - \gamma|f'(x_n)|^2 + \frac{L}{2}\gamma^2|f'(x_n)|^2 \\
&= f(x_n) - \left(\gamma - \frac{L}{2}\gamma^2\right)|f'(x_n)|^2
\end{aligned}$$

Subtracting $f(x^*)$ above, then

$$f(x_{n+1}) - f(x^*) \leq f(x_n) - f(x^*) - \left(\gamma - \frac{L}{2}\gamma^2\right)|f'(x_n)|^2$$

Meanwhile, Convexity gives

$$f(x_n) - f(x^*) \leq f'(x_n)(x_n - x^*) \leq |f'(x_n)||x_n - x^*| \leq |f'(x_n)||x_0 - x^*|$$

Combining abvoe two inequalities,

$$f(x_{n+1}) - f(x^*) \leq f(x_n) - f(x^*) - \left(\gamma - \frac{L}{2}\gamma^2\right) \cdot \frac{|f(x_n) - f(x^*)|^2}{|x_0 - x^*|^2}$$

Dividing Both Sides by $(f(x_{n+1}) - f(x^*))(f(x_n) - f(x^*))$,

$$\begin{aligned}
&\frac{1}{f(x_n) - f(x^*)} \leq \frac{1}{f(x_{n+1}) - f(x^*)} - \left(\gamma - \frac{L}{2}\gamma^2\right) \cdot \frac{f(x_n) - f(x^*)}{f(x_{n+1}) - f(x^*)}\frac{1}{|x_0 - x^*|^2} \\
\implies &\left(\gamma - \frac{L}{2}\gamma^2\right) \cdot \frac{f(x_n) - f(x^*)}{f(x_{n+1}) - f(x^*)}\frac{1}{|x_0 - x^*|^2} \leq \frac{1}{f(x_{n+1}) - f(x^*)} - \frac{1}{f(x_n) - f(x^*)} \\
\implies &\left(\gamma - \frac{L}{2}\gamma^2\right) \cdot \frac{1}{|x_0 - x^*|^2} \leq \frac{1}{f(x_{n+1}) - f(x^*)} - \frac{1}{f(x_n) - f(x^*)} \\
\implies &\sum_{n=0}^{N-1}\left[\left(\gamma - \frac{L}{2}\gamma^2\right) \cdot \frac{1}{|x_0 - x^*|^2}\right] \leq \sum_{n=0}^{N-1}\left[\frac{1}{f(x_{n+1}) - f(x^*)} - \frac{1}{f(x_n) - f(x^*)}\right] = \frac{1}{f(x_N) - f(x^*)} - \frac{1}{f(x_0) - f(x^*)}
\end{aligned}$$

Consequently,

$$\frac{2\gamma N}{|x_0 - x^*|^2} \leq N \cdot \left[\left(\gamma - \frac{L}{2}\gamma^2\right) \cdot \frac{1}{|x_0 - x^*|^2}\right] \leq \frac{1}{f(x_N) - f(x^*)} - \frac{1}{f(x_0) - f(x^*)} \leq \frac{1}{f(x_N) - f(x^*)}$$

Organizing the formula, as result:

$$f(x_N) - f(x^*) \leq \frac{|x_0 - x^*|^2}{2\gamma N}$$

□

8.4 Integral

8.4.1 Inequality of Riemann-Stieltjes Integral

Let $p, q \geq 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, and functions lying on $[a, b]$.

Lemma 6. Let $f, g \in \mathcal{R}(\alpha)$ with $f, g \geq 0$, and $\int_a^b [f(x)]^p d\alpha = \int_a^b [g(x)]^q d\alpha = 1$. Then, $\int_a^b f(x)g(x)d\alpha \leq 1$.

Proof. For any $x \in [a, b]$, the Young's Inequality gives

$$0 \leq f(x)g(x) \leq \frac{[f(x)]^p}{p} + \frac{[g(x)]^q}{q}$$

Now,

$$\int_a^b f(x)g(x)d\alpha \leq \int_a^b \frac{[f(x)]^p}{p} + \frac{[g(x)]^q}{q} d\alpha = \frac{1}{p}\int_a^b [f(x)]^p d\alpha + \frac{1}{q}\int_a^b [g(x)]^q d\alpha = \frac{1}{p} + \frac{1}{q} = 1$$

$\square$

Definition 9. Let $f \in \mathcal{R}(\alpha)$. Define a **Norm** of f:

$$\|f\|_p \overset{\text{def}}{=} \left(\int_a^b |f(x)|^p d\alpha \right)^{\frac{1}{p}}$$

This becomes actually norm of set of Stieltjes Integrable functions, $\mathcal{F} \overset{\text{def}}{=} \{f : [a, b] \to \mathbb{C} \mid f \in \mathcal{R}(\alpha)\}$.

Lemma 7. Hölder's Inequality
Let $f, g \in \mathcal{F}$. Then,

$$\left| \int_a^b f(x)g(x)d\alpha \right| \leq \left[\int_a^b |f(x)|^p d\alpha \right]^{\frac{1}{p}} \cdot \left[\int_a^b |g(x)|^q d\alpha \right]^{\frac{1}{q}}$$

Proof. Use above definition, Rewrite:

$$\|f\|_p^p = \int_a^b |f(x)|^p d\alpha, \quad \|g\|_q^q = \int_a^b |g(x)|^q d\alpha$$

Now, we can make the condition of above lemma,

$$\int_a^b \left[\frac{|f(x)|}{\|f\|_p} \right]^p d\alpha = \frac{1}{\|f\|_p^p} \cdot \int_a^b |f(x)|^p d\alpha = 1, \quad \int_a^b \left[\frac{|g(x)|}{\|g\|_q} \right]^q d\alpha = \frac{1}{\|g\|_q^q} \cdot \int_a^b |g(x)|^q d\alpha = 1$$

And apply this,

$$\int_a^b \frac{|f(x)| \cdot |g(x)|}{\|f\|_p \|g\|_q} d\alpha \leq 1 \implies \int_a^b |f(x)||g(x)|d\alpha \leq \|f\|_p \|g\|_q = \left[\int_a^b |f(x)|^p d\alpha \right]^{\frac{1}{p}} \cdot \left[\int_a^b |g(x)|^q d\alpha \right]^{\frac{1}{q}}$$

Finally, the general property of integral of product gives

$$\left| \int_a^b f(x)g(x)d\alpha \right| \leq \int_a^b |f(x)||g(x)|d\alpha \leq \|f\|_p \|g\|_q = \left[\int_a^b |f(x)|^p d\alpha \right]^{\frac{1}{p}} \cdot \left[\int_a^b |g(x)|^q d\alpha \right]^{\frac{1}{q}}$$

$\square$

> **Theorem 14. Minkowski inequality**
> Let $f, g \in \mathcal{F}$. Then, for any $p \geq 1$, $\|f + g\|_p \leq \|f\|_p + \|g\|_p$.

Proof.

$$\|f + g\|_p^p = \int_a^b |f + g|^p d\alpha = \int_a^b |f + g||f + g|^{p-1} d\alpha$$

$$\leq \int_a^b [|f| + |g|]|f + g|^{p-1} d\alpha$$

$$= \int_a^b |f||f + g|^{p-1} d\alpha + \int_a^b |g||f + g|^{p-1} d\alpha$$

$$\overset{\text{Hölder}}{\leq} \left[\int_a^b |f|^p d\alpha \right]^{\frac{1}{p}} \left[\int_a^b |f + g|^{(p-1)\frac{p}{p-1}} d\alpha \right]^{\frac{p-1}{p}} + \left[\int_a^b |g|^p d\alpha \right]^{\frac{1}{p}} \left[\int_a^b |f + g|^{(p-1)\frac{p}{p-1}} d\alpha \right]^{\frac{p-1}{p}}$$

$$= \left[\int_a^b |f + g|^p d\alpha \right]^{\frac{p-1}{p}} \left(\left[\int_a^b |f|^p d\alpha \right]^{\frac{1}{p}} + \left[\int_a^b |g|^p d\alpha \right]^{\frac{1}{p}} \right) = \|f + g\|_p^{p-1} \cdot (\|f\|_p + \|g\|_p)$$

Now,

$$\|f + g\|_p^p \cdot \|f + g\|_p^{1-p} = \|f + g\|_p \leq \|f\|_p + \|g\|_p$$

$\square$

Chapter 9

Measure

Chapter 10

Complex Analysis

Chapter 11

Differential Geometry

Chapter 12

Differential Equation

Chapter 13

Spaces

13.1 $\mathbb{R}^n$

13.1.1 Inner Product in $\mathbb{R}$

13.1.2 p-norm in $\mathbb{R}^n$

Definition 10. Let $\mathbb{R}^n$ be given. Define p-norm of $\mathbb{R}^n$ is metric on $\mathbb{R}$:

$$d_p : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} : (\mathbf{x}, \mathbf{y}) \mapsto \left(\sum_{i=1}^{n} |x_i - y_i|^p \right)^{\frac{1}{p}}, \quad (\mathbf{x} = (x_1, \ldots, x_n), \ \mathbf{y} = (y_1, \ldots, y_n))$$

where $p \in [1, \infty]$, p-norm be a **Metric** from **Minkowski inequality**.

Lemma 8. Holder's inequality
Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ be give, and $p, q \geq 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then,

$$\sum_{i=1}^{n} |x_i y_i| \leq \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} \cdot \left(\sum_{i=1}^{n} |y_i|^q \right)^{\frac{1}{q}}$$

Proof. Denote that

$$\|x\|_p \stackrel{\mathsf{def}}{=} \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}}$$

Then, since young's inequality, for each $i \in \{1, 2, \ldots, n\}$,

$$\frac{|x_i|}{\|x\|_p} \cdot \frac{|y_i|}{\|y\|_p} \leq \frac{1}{p} \cdot \frac{|x_i|^p}{\|x\|_p^p} + \frac{1}{q} \cdot \frac{|y_i|^q}{\|y\|_q^q}$$

Summing for all $i = 1, 2, \ldots, n$:

$$\frac{1}{\|x\|_p \|y\|_q} \cdot \sum_{i=1}^{n} |x_i y_i| \leq \frac{1}{p} + \frac{1}{q} = 1$$

Therefore,

$$\sum_{i=1}^{n} |x_i y_i| \leq \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} \cdot \left(\sum_{i=1}^{n} |y_i|^q \right)^{\frac{1}{q}}$$

$\square$

Theorem 15. Minkowski inequality
Given complex-valued sequences $\{x_i\}_{i=1}^n$ and $\{y_i\}_{i=1}^n$,

$$\left[\sum_{i=1}^n |x_i+y_i|^p\right]^{\frac{1}{p}} \le \left[\sum_{i=1}^n |x_i|^p\right]^{\frac{1}{p}} + \left[\sum_{i=1}^n |y_i|^p\right]^{\frac{1}{p}}$$

Proof. Denote

$$|x_i+y_i|^p = |x_i+y_i| \cdot |x_i+y_i|^{p-1}$$

Then,

$$\sum_{i=1}^n |x_i+y_i|^p = \sum_{i=1}^n |x_i+y_i| \cdot |x_i+y_i|^{p-1}$$

$$= \sum_{i=1}^n (|x_i|+|y_i|) \cdot |x_i+y_i|^{p-1}$$

$$= \sum_{i=1}^n |x_i| \cdot |x_i+y_i|^{p-1} + \sum_{i=1}^n |y_i| \cdot |x_i+y_i|^{p-1}$$

$$\overset{\text{Hölder}}{\le} \left[\sum_{i=1}^n |x_i|^p\right]^{\frac{1}{p}} \cdot \left[\sum_{i=1}^n |x_i+y_i|^p\right]^{\frac{p-1}{p}} + \left[\sum_{i=1}^n |y_i|^p\right]^{\frac{1}{p}} \cdot \left[\sum_{i=1}^n |x_i+y_i|^p\right]^{\frac{p-1}{p}}$$

$$= \left[\left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^n |y_i|^p\right)^{\frac{1}{p}}\right] \cdot \left[\sum_{i=1}^n |x_i+y_i|^p\right]^{\frac{p-1}{p}}$$

Now, Divide each side as $\left[\sum_{i=1}^n |x_i+y_i|^p\right]^{\frac{p-1}{p}}$**, then we obtain**

$$\left[\sum_{i=1}^n |x_i+y_i|^p\right]^{1-\frac{p-1}{p}} = \left[\sum_{i=1}^n |x_i+y_i|^p\right]^{\frac{1}{p}} \le \left[\left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^n |y_i|^p\right)^{\frac{1}{p}}\right]$$

$\square$

Theorem 16. Let d_{p_1}, d_{p_2} are p-norm on $\mathbb{R}^n$ with $1 \le p_1 < p_2$. Then,

$$\exists C > 0 \text{ s.t. } \forall x, y \in \mathbb{R}^n, \ d_{p_2}(x,y) \le d_{p_1}(x,y) \le C d_{p_2}(x,y)$$

In particular, $C = n^{\frac{1}{p_1}-\frac{1}{p_2}}$.

Proof. Let $p_1 < p_2$.
For show that first-inequality,

$$1 = \sum_{i=1}^n \left[\frac{|x_i-y_i|}{\left[\sum_{i=1}^n |x_i-y_i|^{p_2}\right]^{\frac{1}{p_2}}}\right]^{p_2} \le \sum_{i=1}^n \left[\frac{|x_i-y_i|}{\left[\sum_{i=1}^n |x_i-y_i|^{p_2}\right]^{\frac{1}{p_2}}}\right]^{p_1} = \frac{\sum_{i=1}^n |x_i-y_i|^{p_1}}{\left[\sum_{i=1}^n |x_i-y_i|^{p_2}\right]^{\frac{p_1}{p_2}}} = \left[\frac{\left[\sum_{i=1}^n |x_i-y_i|^{p_1}\right]^{\frac{1}{p_1}}}{\left[\sum_{i=1}^n |x_i-y_i|^{p_2}\right]^{\frac{1}{p_2}}}\right]^{p_1}$$

Thus, we obtain that:

$$1 \le \left[\frac{\left[\sum_{i=1}^n |x_i-y_i|^{p_1}\right]^{\frac{1}{p_1}}}{\left[\sum_{i=1}^n |x_i-y_i|^{p_2}\right]^{\frac{1}{p_2}}}\right]^{p_1} \iff 1 \le \frac{\left[\sum_{i=1}^n |x_i-y_i|^{p_1}\right]^{\frac{1}{p_1}}}{\left[\sum_{i=1}^n |x_i-y_i|^{p_2}\right]^{\frac{1}{p_2}}} \iff \left[\sum_{i=1}^n |x_i-y_i|^{p_2}\right]^{\frac{1}{p_2}} \le \left[\sum_{i=1}^n |x_i-y_i|^{p_1}\right]^{\frac{1}{p_1}}$$

For show that second-inequality, using Hölder's inequality.

$$(d_{p_1}(x,y))^{p_1} = \sum_{i=1}^n |x_i-y_i|^{p_1} = \sum_{i=1}^n |x_i-y_i|^{p_1} \cdot 1$$

$$\overset{\text{Hölder}}{\le} \left[\sum_{i=1}^n \left(|x_i-y_i|^{p_1 \cdot \frac{p_2}{p_1}}\right)\right]^{\frac{p_1}{p_2}} \cdot \left[\sum_{i=1}^n 1^{\frac{p_2}{p_2-p_1}}\right]^{1-\frac{p_1}{p_2}} = \left[\sum_{i=1}^n \left(|x_i-y_i|^{p_2}\right)\right]^{\frac{p_1}{p_2}} \cdot n^{1-\frac{p_1}{p_2}}$$

Taking the $\frac{1}{p_1}$-th power of both sides, then

$$d_{p_1}(x, y) \leq \left[\sum_{i=1}^{n} \left(|x_i - y_i|^{p_2} \right) \right]^{\frac{1}{p_2}} \cdot n^{\frac{1}{p_1} - \frac{1}{p_2}} = n^{\frac{1}{p_1} - \frac{1}{p_2}} \cdot d_{p_2}(x, y)$$

□

13.2 Topological Vector Space

13.3 Hilbert Space

13.4 Banach Space

13.5 L_p Space

13.6 l_p Space