

# Math Note

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This paper contains independent topics in undergraduate mathematics.

## Chapter 1

## Set Theory

## Chapter 2

# Group Theory

Example. Dihedral Group

## Chapter 3

# Ring Theory

### 3.1 Ring of Fractions

**Theorem 1.** Let  $R$  be a Commutative Ring,  $D \subset R$  be a subset such that  $\begin{cases} \text{no zero, no zero divisors} \\ \text{closed under multiplication} \end{cases}$ .

Then, there exists a Commutative Ring  $Q$  with identity satisfies:

1.  $R$  can embed in  $Q$ , and every element of  $D$  becomes unit in  $Q$ . More precisely,  $Q = \{rd^{-1} \mid r \in R, d \in D\}$ .
2.  $Q$  is the smallest Ring with identity such that every element of  $D$  becomes unit in  $Q$ .

**Proof.** Let  $\mathcal{F} \stackrel{\text{def}}{=} \{(r, d) \mid r \in R, d \in D\}$  and the relation  $\sim$  on  $\mathcal{F}$  by  $(r_1, d_1) \sim (r_2, d_2) \iff r_1d_2 = r_2d_1$ .

Then,  $\sim$  is equivalent relation: reflexive and symetric are clear, and Suppose that  $(r_1, d_1) \sim (r_2, d_2)$  and  $(r_2, d_2) \sim (r_3, d_3)$ .

$$r_2d_3 = r_3d_2 \implies r_2d_1d_3 = r_3d_1d_2 \implies r_1d_2d_3 = r_3d_1d_2 \implies d_2(r_1d_3 - r_3d_1) \implies r_1d_3 = r_3d_1$$

Thus transitivity shown. Define

$$\frac{r}{d} \stackrel{\text{def}}{=} [(r, d)] = \{(a, b) \mid (a, b) \sim (r, d)\}, \quad Q \stackrel{\text{def}}{=} \left\{ \frac{r}{d} \mid r \in R, d \in D \right\}$$

And define operations  $+, \times$  on  $Q$ :

$$\frac{r_1}{d_1} + \frac{r_2}{d_2} \stackrel{\text{def}}{=} \frac{r_1d_2 + r_2d_1}{d_1d_2}, \quad \frac{r_1}{d_1} \times \frac{r_2}{d_2} \stackrel{\text{def}}{=} \frac{r_1r_2}{d_1d_2}$$

**Well-Definedness:** If  $\frac{r_1}{d_1} = \frac{r'_1}{d'_1}$  and  $\frac{r_2}{d_2} = \frac{r'_2}{d'_2}$ ,

$$\frac{r_1d_2 + r_2d_1}{d_1d_2} = \frac{r_1d_2d'_1d'_2 + r_2d_1d'_1d'_2}{d_1d_2d'_1d'_2} = \frac{(r_1d'_1)d_2d'_2 + (r_2d'_2)d_1d'_1}{d_1d_2d'_1d'_2} = \frac{(r'_1d_1)d_2d'_2 + (r'_2d_2)d_1d'_1}{d_1d_2d'_1d'_2} = \frac{(r'_1d'_2 + r'_2d'_1)d_1d_2}{d_1d_2d'_1d'_2} = \frac{r'_1d'_2 + r'_2d'_1}{d'_1d'_2}$$

$$\frac{r_1r_2}{d_1d_2} = \frac{r_1r_2d'_1d'_2}{d_1d_2d'_1d'_2} = \frac{(r_1d'_1)(r_2d'_2)}{d_1d_2d'_1d'_2} = \frac{(r'_1d_1)(r'_2d_2)}{d_1d_2d'_1d'_2} = \frac{r'_1r'_2d_1d_2}{d_1d_2d'_1d'_2} = \frac{r'_1r'_2}{d'_1d'_2}$$

Now,  $(Q, +, \times)$  constructs Commutative Ring with identity: for any  $d \in D$ , put  $0_Q \stackrel{\text{def}}{=} \frac{0}{d}$ ,  $1_Q \stackrel{\text{def}}{=} \frac{d}{d}$ . Then,

1.  $(R, +, \times)$  closed under the operations since  $D$  is closed under the multiplication.

$$2. (R, +) \text{ has a zero: } \frac{r_1}{d_1} + 0_Q = \frac{r_1}{d_1} + \frac{0}{d} = \frac{r_1d + 0d_1}{d_1d} = \frac{r_1d}{d_1d} = \frac{r_1}{d_1}.$$

$$3. (R, +) \text{ has an inverse: } \frac{r_1}{d_1} + \frac{-r_1}{d_1} = \frac{r_1d_1 + (-r_1)d_1}{d_1d_1} = \frac{[(r_1) + (-r_1)]d_1}{d_1d_1} = \frac{0d_1}{d_1d_1} = \frac{0}{d_1d_1} = 0_Q.$$

4.  $(R, +, \times)$  satisfies distributive law:

4-1. The left law:

$$\begin{aligned} \frac{r_1}{d_1} \times \left( \frac{r_2}{d_2} + \frac{r_3}{d_3} \right) &= \frac{r_1}{d_1} \times \frac{r_2d_3 + r_3d_2}{d_2d_3} = \frac{r_1r_2d_3 + r_1r_3d_2}{d_1d_2d_3} = \frac{r_1r_2d_1d_3 + r_1r_3d_1d_2}{d_1d_2d_1d_3} = \frac{r_1r_2}{d_1d_2} + \frac{r_2r_3}{d_2d_3} \\ &= \frac{r_1}{d_1} \times \frac{r_2}{d_2} + \frac{r_2}{d_2} \times \frac{r_3}{d_3} \end{aligned}$$

4-2. The right law:

$$\begin{aligned} \left( \frac{r_1}{d_1} + \frac{r_2}{d_2} \right) \times \frac{r_3}{d_3} &= \frac{r_1d_2 + r_2d_1}{d_1d_2} \times \frac{r_3}{d_3} = \frac{r_1r_3d_2 + r_2r_3d_1}{d_1d_2d_3} = \frac{r_1r_3d_2d_3 + r_2r_3d_1d_3}{d_1d_3d_2d_3} = \frac{r_1r_3}{d_1d_3} + \frac{r_2r_3}{d_2d_3} \\ &= \frac{r_1}{d_1} \times \frac{r_3}{d_3} + \frac{r_2}{d_2} \times \frac{r_3}{d_3} \end{aligned}$$

$$5. (R, \times) \text{ has an identity: } \frac{r_1}{d_1} \times 1_Q = \frac{r_1}{d_1} \times \frac{d}{d} = \frac{r_1d}{d_1d} = \frac{r_1}{d_1}.$$

6. Elements of  $D$  become unit in  $Q$ : Define  $\iota: R \rightarrow Q: r \mapsto \frac{rd}{d}$  where  $d \in D$  is any fixed element in  $D$ . Then,  $\iota$  is Ring-Monomorphism because:

$$6-1. \text{ Well-Defined and Injective: } \iota(r_1) = \iota(r_2) \iff \frac{r_1d}{d} = \frac{r_2d}{d} \iff (r_1 - r_2)d = 0 \iff r_1 = r_2$$

□

## Chapter 4

# Field Theory

**Chapter 5**

**Category**



## Chapter 6

# General Topology

## Chapter 7

# Algebraic Topology

## Chapter 8

# Real Analysis

## Chapter 9

### Measure

## Chapter 10

# Complex Analysis

## Chapter 11

# Differential Geometry

## Chapter 12

# Differential Equation