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Contents

This paper covers several topics in undergraduate mathematics.

Chapter 1

Set Theory

1.1 Map

Definition 1. Let X, Y are sets. Define a **function** X to Y is a relation

$$f \subset X \times Y$$

such that

1. For any $x \in X$, there exists $y \in Y$ such that $(x, y) \in f$.
2. If $(x, y) \in f$ and $(x, z) \in f$, then $y = z$.

Denote f as:

$$f : X \to Y : x \mapsto f(x)$$

Define **Image** of f by $A \subset X$:

$$f[A] \overset{\text{def}}{=} \{f(a) \mid a \in A\} \subset Y$$

And, **Preimage** of f by $B \subset Y$:

$$f^{-1}[B] \overset{\text{def}}{=} \{x \in X \mid f(x) \in B\} \subset X$$

$f : X \to Y$ is **Injective** if: $f(x_1) = f(x_2) \implies x_1 = x_2$.
$f : X \to Y$ is **Surjective** if: $\forall y \in Y,\ \exists x \in X$ s.t. $f(x) = y$.
If f is injective and surjective, called **bijective**.
If f is bijective, then define **inverse** of f as:

$$f^{-1} : Y \to X : y \to x$$

where $x \in X$ is the unique elements of X such that $f(x) = y$.

Theorem 1. Let $f : X \to Y$ be a function. Then,

1. There exists $g : Y \to X$ such that $g \circ f : X \to X$ be an identity function **if and only if** f is injective.
2. There exists $h : Y \to X$ such that $f \circ h : Y \to Y$ be an identity function **if and only if** f is surjective.

Proof.
1. $\implies$)
Assume that $f(x_1) = f(x_2)$. Then, existence of left inverse, $g(f(x_1)) = g(f(x_2)) \implies x_1 = x_2$. Thus f injective.
1. $\impliedby$)
Since f is injection, for any $y \in f[X]$, there exists a unique element $x_y \in X$ such that $f(x) = y$. Now, define

$$g : Y \to X : y \mapsto \begin{cases} x_y & y \in f[X] \\ \text{any element in } X & y \notin f[X] \end{cases}$$

Then, for any $x \in X$, $g(f(x)) = g(y) = x$.
2. $\implies$)

Let $y \in Y$ be given. Since existence of right inverse, $f(h(y)) = y$ where $h(y) \in X$. Thus, f is surjective.
2. $\Longleftarrow$)
For any $y \in Y$, there exists a $x_y \in X$ such that $f(x_y) = y$. Now, define

$$h : Y \to X : y \mapsto x_y$$

Then, for any $y \in Y$, $f \circ h(y) = f(x_y) = y$. Thus, $f \circ h$ is identity. □

Corollary 2. Let $f : X \to Y$ be a function, $\mathrm{id}_X : X \to X : x \mapsto x$, and $\mathrm{id}_Y : Y \to Y : y \mapsto y$.

There exists a $f^{-1} : Y \to X$ such that $f^{-1} \circ f = \mathrm{id}_X$ and $f \circ f^{-1} = \mathrm{id}_Y$ if and only if f is bijection.

Proof. If f is bijection, then there exists left inverse g and right inverse h.
Enough To Show that: $g = h$. Since $g \circ f = \mathrm{id}_X$ and $f \circ h = \mathrm{id}_Y$,
$g \circ f \circ h = g \circ \mathrm{id}_Y$, thus $h = g$. □

Theorem 3. Let X, Y, Z are sets, $f : X \to Y$, $g : Y \to Z$ and $A \subset X, B \subset Y, C \subset Z$. Then followings are hold:

1. $g[f[A]] = (g \circ f)[A]$.
2. $f^{-1}[g^{-1}[C]] = (g \circ f)^{-1}[C]$.

Proof.
1. It is clear by definition of image:

$$\begin{aligned} g[f[A]] &\overset{\text{def}}{=} g[\{f(a) \mid a \in A\}] = \{g(b) \mid b \in \{f(a) \mid a \in A\}\} \\ &= \{g(b) \mid b = f(a) \text{ for some } a \in A\} = \{g(f(a)) \mid \text{for some } a \in A\} = \{g(f(a)) \mid a \in A\} \end{aligned}$$

2. It is not clear,

$$f^{-1}[g^{-1}[C]] \overset{\text{def}}{=} f^{-1}[\{b \in Y \mid g(b) \in C\}] = \{a \in X \mid f(a) \in \{b \in Y \mid g(b) \in C\}\} = \{a \in X \mid g(f(a)) \in C\} = (g \circ f)^{-1}[C]$$

□

Proposition 1. Let $f : X \to Y$ be a function, $A, B \subset X$ and $C, D \subset Y$.

1. If $A \subset B$, then $f[A] \subset f[B]$.
2. If $C \subset D$, then $f^{-1}[C] \subset f^{-1}[D]$

Proof.

$$\begin{aligned} &y \in f[A] \implies y = f(a) \text{ for some } a \in A \overset{A \subset B}{\implies} y = f(a) \text{ for some } a \in B \implies y \in f[B] \\ &x \in f^{-1}[C] \implies f(x) \in C \overset{C \subset D}{\implies} f(x) \in D \implies x \in f^{-1}[D] \end{aligned}$$

□

Lemma 4. Let two set X, Y be given, and $A \subset X$, $B \subset Y$, $f : X \to Y$. Then followings are holds:

1. $f^{-1}[f[A]] \supseteq A$, and equality holds if f one-to-one.
2. $f[f^{-1}[B]] \subseteq B$, and equality holds if f onto.
3. $f^{-1}[Y \setminus B] = X \setminus f^{-1}[B]$
4. $f[X] \setminus f[A] \subseteq f[X \setminus A]$, and equality holds if f one-to-one.

Proof. Proof of 4.

$$\begin{aligned} y \in f[X] \setminus f[A] &\iff y \in f[X] \text{ and } y \notin f[A] \\ &\iff \exists x \in X \text{ s.t. } y = f(x) \text{ and } \forall x \in A,\ y \neq f(x) \\ &\overset{(*)}{\implies} \exists x \in X \setminus A \text{ s.t. } y = f(x) \\ &\iff y \in f[X \setminus A] \end{aligned}$$

If f is injection, then Left Direction of $(*)$ be true: $\exists! x \in X \setminus A$ s.t. $y = f(x)$. □

Chapter 2

Group Theory

Example. Dihedral Group

Chapter 3

Ring Theory

3.1 Ring of Fractions

Theorem 5. Let R be a Commutative Ring, $D \subset R$ be a subset such that $\begin{cases} \text{no zero, no zero divisors} \\ \text{closed under multiplication} \end{cases}$.
Then, there exists a Commutative Ring Q with identity satisfies:

1. R can embed in Q, and every element of D becomes unit in Q. More precisely, $Q = \{rd^{-1} \mid r \in R,\ \ d \in D\}$.
2. Q is the smallest Ring with identity such that every element of D becomes unit in Q.

Proof. Let $\mathcal{F} \overset{\text{def}}{=} \{(r,d) \mid r \in R,\ \ d \in D\}$ and the relation $\sim$ on $\mathcal{F}$ by $(r_1,d_1) \sim (r_2,d_2) \iff r_1d_2 = r_2d_1$.
Then, $\sim$ is equivalent relation: reflexive and symmetirc are clear, and Suppose that $(r_1,d_1) \sim (r_2,d_2)$ and $(r_2,d_2) \sim (r_3,d_3)$.

$$r_2d_3 = r_3d_2 \implies r_2d_1d_3 = r_3d_1d_2 \implies r_1d_2d_3 = r_3d_1d_2 \implies d_2(r_1d_3 - r_3d_1) \implies r_1d_3 = r_3d_1$$

Thus transitivity shown. Define

$$\frac{r}{d} \overset{\text{def}}{=} [(r,d)] = \{(a,b) \mid (a,b) \sim (r,d)\},\ \ Q \overset{\text{def}}{=} \left\{\frac{r}{d} \,\middle|\, r \in R,\ \ d \in D\right\}$$

And define operations $+, \times$ on Q:

$$\frac{r_1}{d_1} + \frac{r_2}{d_2} \overset{\text{def}}{=} \frac{r_1d_2 + r_2d_1}{d_1d_2}, \quad \frac{r_1}{d_1} \times \frac{r_2}{d_2} \overset{\text{def}}{=} \frac{r_1r_2}{d_1d_2}$$

Well-Definedness: If $\dfrac{r_1}{d_1} = \dfrac{r_1'}{d_1'}$ and $\dfrac{r_2}{d_2} = \dfrac{r_2'}{d_2'}$,

$$\frac{r_1d_2 + r_2d_1}{d_1d_2} = \frac{r_1d_2d_1'd_2' + r_2d_1d_1'd_2'}{d_1d_2d_1'd_2'} = \frac{(r_1d_1')d_2d_2' + (r_2d_2')d_1d_1'}{d_1d_2d_1'd_2'} = \frac{(r_1'd_1)d_2d_2' + (r_2'd_2)d_1d_1'}{d_1d_2d_1'd_2'} = \frac{(r_1'd_2' + r_2'd_1')d_1d_2}{d_1d_2d_1'd_2'} = \frac{r_1'd_2' + r_2'd_1'}{d_1'd_2'}$$

$$\frac{r_1r_2}{d_1d_2} = \frac{r_1r_2d_1'd_2'}{d_1d_2d_1'd_2'} = \frac{(r_1d_1')(r_2d_2')}{d_1d_2d_1'd_2'} = \frac{(r_1'd_1)(r_2'd_2)}{d_1d_2d_1'd_2'} = \frac{r_1'r_2'd_1d_2}{d_1d_2d_1'd_2'} = \frac{r_1'r_2'}{d_1'd_2'}$$

Now, $(Q,+,\times)$ constructs Commutative Ring with identity: for any $d \in D$, put $0_Q \overset{\text{def}}{=} \dfrac{0}{d}$, $1_Q \overset{\text{def}}{=} \dfrac{d}{d}$. Then,

1. $(R,+,\times)$ closed under the operations since D is closed under the multiplication.
2. $(R,+)$ has a zero: $\dfrac{r_1}{d_1} + 0_Q = \dfrac{r_1}{d_1} + \dfrac{0}{d} = \dfrac{r_1d + 0d_1}{d_1d} = \dfrac{r_1d}{d_1d} = \dfrac{r_1}{d_1}$.
3. $(R,+)$ has an inverse: $\dfrac{r_1}{d_1} + \dfrac{-r_1}{d_1} = \dfrac{r_1d_1 + (-r_1)d_1}{d_1d_1} = \dfrac{[(r_1) + (-r_1)]d_1}{d_1d_1} = \dfrac{0d_1}{d_1d_1} = \dfrac{0}{d_1d_1} = 0_Q$.
4. $(R,+,\times)$ satisfies distributive law:

   4-1. The left law:

$$\begin{aligned}\frac{r_1}{d_1} \times \left(\frac{r_2}{d_2} + \frac{r_3}{d_3}\right) &= \frac{r_1}{d_1} \times \frac{r_2d_3 + r_3d_2}{d_2d_3} = \frac{r_1r_2d_3 + r_1r_3d_2}{d_1d_2d_3} = \frac{r_1r_2d_1d_3 + r_1r_3d_1d_2}{d_1d_2d_1d_3} = \frac{r_1r_2}{d_1d_2} + \frac{r_2r_3}{d_2d_3} \\ &= \frac{r_1}{d_1} \times \frac{r_2}{d_2} + \frac{r_2}{d_2} \times \frac{r_3}{d_3}\end{aligned}$$

   4-2. The right law:

$$\begin{aligned}\left(\frac{r_1}{d_1} + \frac{r_2}{d_2}\right) \times \frac{r_3}{d_3} &= \frac{r_1d_2 + r_2d_1}{d_1d_2} \times \frac{r_3}{d_3} = \frac{r_1r_3d_2 + r_2r_3d_1}{d_1d_2d_3} = \frac{r_1r_3d_2d_3 + r_2r_3d_1d_3}{d_1d_3d_2d_3} = \frac{r_1r_3}{d_1d_3} + \frac{r_2r_3}{d_2d_3} \\ &= \frac{r_1}{d_1} \times \frac{r_3}{d_3} + \frac{r_2}{d_2} \times \frac{r_3}{d_3}\end{aligned}$$

5. $(R,\times)$ has an identity: $\dfrac{r_1}{d_1} \times 1_Q = \dfrac{r_1}{d_1} \times \dfrac{d}{d} = \dfrac{r_1d}{d_1d} = \dfrac{r_1}{d_1}$.
6. Elements of D become unit in Q: Define $\iota : R \to Q : r \mapsto \dfrac{rd}{d}$ where $d \in D$ is any fixed element in D.
   Then, ι is Ring-Monomorphsim because:

   6-1. Well-Defined and Injective: $\iota(r_1) = \iota(r_2) \iff \dfrac{r_1d}{d} = \dfrac{r_2d}{d} \iff (r_1 - r_2)dd = 0 \iff r_1 = r_2$

$\square$

Chapter 4

Field Theory

Chapter 5

Category

Chapter 6

General Topology

6.1 Complete Metric Space

Definition 2. Let (X, d) be a Metric Space, and $\{p_n\}$ be a Sequence in X.
The Sequence $\{p_n\}$ is called **Cauchy Sequence** if:

$$\text{For any } \epsilon > 0, \text{ there exists } N \in \mathbb{N} \text{ such that } m, n \geq N \implies d(p_m, p_n) < \epsilon.$$

A Metric Space (X, d) is said to be **Complete** if every Cauchy Sequnces Converge.

Lemma 6. Let $\{E_n\}$ be a sequence of closed bounded non-empty subsets in a Complete Metric Space X such that $E_n \supset E_{n+1}$.
If $\lim_{n\to\infty} \text{diam} E_n = 0$, then $\bigcap_{n=1}^{\infty} E_n = \{p\}$ for some $p \in X$.

Proof. For each $n \in \mathbb{N}$, construct $p_n \in E_n$.
Let $\epsilon > 0$ be given. Since diam$E_n \to 0$, there is $N \in \mathbb{N}$ such that diam$E_n < \epsilon$.
For any $m, n \geq M$, E_N contains p_m, p_n. That is, $d(p_m, p_n) < \epsilon$. Thus, $\{p_n\}$ be a Cauchy sequence of X.
Since X is complete, there is a unique point $p \in X$ such taht $p_n \to p$. Let $N \in \mathbb{N}$ be a integer such that $n \geq N \implies |p_n - p| < \epsilon$.
Now, for each $n \geq N$, E_n has a limit point as p. And for any $n \in \mathbb{N}$, E_n contains $E_N, E_{N+1}, \ldots$, thus for all $n \in \mathbb{N}$, E_n has a limit point as p. Meanwhile, E_n closed, $p \in E_n,\ \forall n \in \mathbb{N}$.
Consequently, $p \in \bigcap_{n=1}^{\infty} E_n$. If there is $q \in X$ such that $p \neq q$, $q \in \bigcap_{n=1}^{\infty} E_n$. Then, diam$E_n \geq d(p, q) > 0,\ \forall n \in \mathbb{N}$. ☐

6.1.1 Baire Category

Definition 3. The Topological Space X is called **Baire Space** if:

$$\text{If } \{G_n \mid n \in \mathbb{N}\} \text{ be a Countable Collection of dense open sets of } X, \text{ then } \overline{\bigcap_{n=1}^{\infty} G_n} = X$$

In brief, every Countable intersection of dense open sets be dense in X.

Theorem 7. Locally Compact Hausdorff Space is Baire Space.

Theorem 8. Complete Metric Space is Baire Space.

Proof. Suppose that $\{G_n \mid n \in \mathbb{N}\}$ be a Countable Collection of dense open set of Complete Metric Space.
Let an open $U \in \mathcal{T}$ be given. Since G_n is dense in the Space, $U \cap G_1$ is non-empty open set.
Thus, there exists a $p_1 \in U \cap G_1$ such that for some $r_1 > 0$, $B_{r_1}(p_1) \subset U \cap G_1$.
Then, automatically,

$$B_{\frac{r_1}{2}}(p_1) \subset \overline{B_{\frac{r_1}{2}}(p_1)} \subset B_{r_1}(p_1) \subset U \cap G_1$$

Set $E_1 = U,\ E_2 = B_{\frac{r_1}{2}}(p_1)$.
Suppose that $E_1, \ldots, E_{n-1}$ are chosen. Then, since $E_{n-1} \cap G_{n-1}$ is open, being intersection of opens.
Thus there exists a point $p_{n-1} \in E_{n-1} \cap G_{n-1}$ and exists r_{n-1} such that

$$B_{r_{n-1}}(p_{n-1}) \subset E_{n-1} \cap G_{n-1} \subset E_{n-1}$$

This implies that

$$B_{\frac{r_{n-1}}{2}}(p_{n-1}) \subset \overline{B_{\frac{r_{n-1}}{2}}(p_{n-1})} \subset B_{r_{n-1}}(p_{n-1}) \subset E_{n-1} \cap G_{n-1} \subset E_{n-1}$$

Set $E_n = B_{\frac{r_{n-1}}{2}}(p_{n-1})$. Since inductively construction of $\{E_n\}$, $E_{n+1} \subset E_n$ and $\overline{E_n} \subset G_n$ for all $n \in \mathbb{N}$. Consequently,

$$U \cap \left(\bigcap_{n=1}^{\infty} G_n \right) = \bigcap_{n=1}^{\infty} (U \cap G_n) \supset \bigcap_{n=1}^{\infty} (U \cap \overline{E_n}) = U \cap \left(\bigcap_{n=1}^{\infty} \overline{E_n} \right) = \bigcap_{n=1}^{\infty} \overline{E_n} \neq \emptyset$$

□

Definition 4. Let X be a Topological Space.
$A \subset X$ is said to be **nowhere dense subset** if $(\overline{A})^\circ = \emptyset$.

1. $B \subset X$ is called **first category** if B can be representive by union of countable nowhere dense subsets.

2. If the subset is not first category, then it is said to be **second category**.

6.1.2 Nowhere Differentiable function

6.2 Urysohn Metrization Theorem

6.2.1 Urysohn Metrization Theroem

Recall that:

Definition 5. X is T_4 if: For any disjoint closed set A and B, there exist disjoint open U, V such that $A \subseteq U$ and $B \subseteq V$.

Lemma 9. X is T_4 Space if and only if For any closed C and open U with $C \subseteq U$, there exists open O such that

$$\underset{\text{closed}}{C} \subseteq \underset{\text{open}}{O} \subseteq \underset{\text{closed}}{\overline{O}} \subseteq \underset{\text{open}}{U}$$

Proof. Proof of the left direction only.
Let X be a T_4 Space, and $C \subset X$ be a closed, U be a open containing C. Then, $C \subset U$ implies $U^c \subset C^c$, thus U^c is a closed set disjoint from C. By T_4 condition, There exist disjoint opens O, O' such that $C \subset O$ and $U^c \subset O' \iff O'^c \subset U$.
Since $O \cap O' = \emptyset \iff O \subset O'^c$, O contained in U, this implies that $C \subset O \subset U$.
Since closure is the smallest closed set such that contains it, consequently $C \subset O \subset \overline{O} \subset O'^c \subset U$. □

Definition 6. Let X be a Toplogical Space, and $A, B \subset X$ are disjoint closed subset.
A real-valued Continuous map $f : X \to [a, b]$ is called **Urysohn function** for A and B if: $f|_A = a$ and $f|_B = b$.
In another form,

$$f : X \to [a, b] : x \to \begin{cases} a & x \in A \\ b & x \in B \\ f(x) & x \notin A \cup B \end{cases}$$

Lemma 10. Urysohn Lemma
T_4 Space has an Urysohn function for any two disjoint closed subsets.

Proof. Generalization is the last thing to proven, first of all, prove in case of $[a, b] = [0, 1]$. This proof consists by three Step.
Let X be a T_4 Space, and $A, B \subset X$ be closed subsets.
Step 1. Construct a Chain of Open sets with Dyadic Rational Indices.
Consider a set of **Dyadic Rationals** $D \overset{\text{def}}{=} \left\{ \frac{k}{2^n} \;\middle|\; n, k \in \mathbb{N},\ k \leq 2^n - 1 \right\}$. We will show that the following statement holds:

$$\text{For any } r, s \in D \text{ with } r < s\text{, there exist open sets } U_r, U_s \text{ such that } A \subseteq \overline{U}_r \subseteq U_s \subseteq X \setminus B \qquad (*)$$

For this, Enough to Show that: For any $k \in \mathbb{N}$, there exists a Chain as:

$$A \subseteq U_{\frac{1}{2^k}} \subseteq \overline{U}_{\frac{1}{2^k}} \subseteq U_{\frac{2}{2^k}} \subseteq \overline{U}_{\frac{2}{2^k}} \subseteq \cdots \subseteq U_{\frac{2^k-1}{2^k}} \subseteq \overline{U}_{\frac{2^k-1}{2^k}} \subseteq X \setminus B$$

(Note that this opens in the Chain are not necessary distinct: For instance, if Ambient Space is Finite, then the Space is **Noetherian**. That is, X satisfies Ascending Chain Condition for open sets.)
Let $k = 1$. Then, By T_4 condition gives that: There exists an open set U_1 such that

$$A \subseteq U_1 \subseteq \overline{U_1} \subseteq X \setminus B$$

Now, naming this U_1 as $U_{\frac{1}{2}}$, proved when $k = 1$.
Suppose that for some $k > 1$, the Chain exists as:

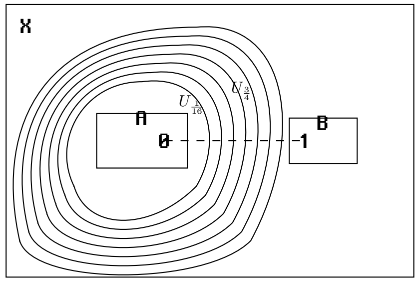
$$\underset{\text{closed}}{A} \overset{*1}{\subseteq} \underset{\text{open}}{U_{\frac{1}{2^k}}} \subseteq \underset{\text{closed}}{\overline{U}_{\frac{1}{2^k}}} \overset{*2}{\subseteq} \underset{\text{open}}{U_{\frac{2}{2^k}}} \subseteq \cdots \overset{*2^k-1}{\subseteq} \underset{\text{open}}{U_{\frac{2^k-1}{2^k}}} \subseteq \underset{\text{closed}}{\overline{U}_{\frac{2^k-1}{2^k}}} \overset{*2^k}{\subseteq} \underset{\text{open}}{X \setminus B}$$

By repeatedly applying the T_4 condition 2^k times, as indicated by the indices $*1, *2, \ldots, *2^k$, we can construct 2^k open sets such that:

$$A \subseteq U_{\frac{1}{2^{k+1}}} \subseteq \overline{U}_{\frac{1}{2^{k+1}}} \subseteq U_{\frac{1}{2^k}} \subseteq \overline{U}_{\frac{1}{2^k}} \subseteq U_{\frac{3}{2^{k+1}}} \subseteq \overline{U}_{\frac{3}{2^{k+1}}} \subseteq U_{\frac{2}{2^k}} \subseteq \overline{U}_{\frac{2}{2^k}} \quad \subseteq \cdots \subseteq U_{\frac{2^k-1}{2^k}} \subseteq \overline{U}_{\frac{2^k-1}{2^k}} \subseteq U_{\frac{2^{k+1}-1}{2^{k+1}}} \subseteq \overline{U}_{\frac{2^{k+1}-1}{2^{k+1}}} \subseteq X \setminus B$$

Finally, Step 1 proved.

Step 2. Construct an Urysohn Function.



Define a map $f : X \to [0,1]$ as:

$$f(x) = \begin{cases} 0 & x \in \bigcap\limits_{t \in D} U_t \\ \sup\{t \in D \mid x \notin U_t\} & x \notin \bigcap\limits_{t \in D} U_t \end{cases}$$

Then, this map f is well-defined by $(*)$ and $\sup D \leq 1$. And f satisfies that:

1. $\forall r \in D,\ x \in A \subset U_r$. Thus, $f(x) = 0$ if $x \in A$.
2. $\forall r \in D,\ x \in B \subset X \setminus U_r$. Thus, $f(x) = \sup D = 1$ if $x \in B$.
3. If $x \in \overline{U}_r$, then for every $s > r$, $x \in \overline{U}_r \subset U_s$. Thus, $f(x) \leq r$. In Contrapositive, $f(x) > r \implies x \notin \overline{U}_r$. (If $f(x) = \sup\{t \in D \mid x \notin U_t\} > r$, then there is $s \in D$ such that $s > r$ and $x \notin U_s$, Contradiction.)
4. If $x \notin U_r$, then, $f(x) \geq r$. In Contrapositive, $f(x) < r \implies x \in U_r$.

Now, show that this map f is Continuous map: Let $x \in X$ be fixed arbitrarlily, and $\epsilon > 0$ be given.
In Case of $0 < f(x) < 1$.
Since Density of Dyadic Rationals, Choose $r, s \in D$ such that $f(x) - \epsilon < r < f(x) < s < f(x) + \epsilon$.
Now, we obtain that:

$$x \overset{(*)}{\in} U_s \setminus \overline{U}_r \overset{(**)}{\subseteq} f^{-1}[(f(x) - \epsilon, f(x) + \epsilon)]$$

$(*)$ directly given by above properties, $(**)$ given applying the fact that $x \in U_s \subset \overline{U}_s$ and $x \notin \overline{U}_r$.
In Case of $f(x) = 0$.
Choose $r \in D$ such that $f(x) = 0 < r < \epsilon = f(x) + \epsilon$. Then,

$$x \in U_r \subset f^{-1}[(f(x), f(x) + \epsilon)]$$

In Case of $f(x) = 1$.
Choose $r \in D$ such that $f(x) - \epsilon = 1 - \epsilon < r < 1 = f(x)$. Then,

$$x \in X \setminus U_r \subset f^{-1}[(f(x) - \epsilon, f(x))]$$

Consequently, f is Continuous map on $[0,1]$ such that $f|_A = 0$ and $f|_B = 1$.
Step 3. Generalization.
Since $[0,1] \cong [a,b]$ for any $a < b$, let $g : [0,1] \to [a,b] : x \mapsto (1-x)a + xb$ be a Homeomorphism.
Then, $h = g \circ f : X \to [a,b]$ becomes a Continuous map such that $h|_A = a$ and $h|_B = b$. $\square$

Chapter 7

Algebraic Topology

Chapter 8

Real Analysis

Chapter 9

Measure

Chapter 10

Complex Analysis

Chapter 11

Differential Geometry

Chapter 12

Differential Equation

Chapter 13

Spaces

13.1 $\mathbb{R}^n$

13.1.1 Inner Product in $\mathbb{R}$

13.1.2 p-norm in $\mathbb{R}^n$

Definition 7. Let $\mathbb{R}^n$ be given. Define **p-norm** of $\mathbb{R}^n$ is metric on $\mathbb{R}$:

$$d_p : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} : (\mathbf{x}, \mathbf{y}) \mapsto \left(\sum_{i=1}^{n} |x_i - y_i|^p \right)^{\frac{1}{p}}, \quad (\mathbf{x} = (x_1, \ldots, x_n),\ \ \mathbf{y} = (y_1, \ldots, y_n))$$

where $p \in [1, \infty]$, p-norm be a **Metric from Minkowski inequality**.

Lemma 11. Holder's inequality
Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ be give, and $p, q \geq 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then,

$$\sum_{i=1}^{n} |x_i y_i| \leq \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} \cdot \left(\sum_{i=1}^{n} |y_i|^q \right)^{\frac{1}{q}}$$

Proof. Denote that

$$\|x\|_p \overset{\text{def}}{=} \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}}$$

Then, since young's inequality, for each $i \in \{1, 2, \ldots, n\}$,

$$\frac{|x_i|}{\|x\|_p} \cdot \frac{|y_i|}{\|y\|_p} \leq \frac{1}{p} \cdot \frac{|x_i|^p}{\|x\|_p^p} + \frac{1}{q} \cdot \frac{|y_i|^q}{\|y\|_q^q}$$

Summing for all $i = 1, 2, \ldots, n$:

$$\frac{1}{\|x\|_p \|y\|_q} \cdot \sum_{i=1}^{n} |x_i y_i| \leq \frac{1}{p} + \frac{1}{q} = 1$$

Therefore,

$$\sum_{i=1}^{n} |x_i y_i| \leq \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} \cdot \left(\sum_{i=1}^{n} |y_i|^q \right)^{\frac{1}{q}}$$

□

Theorem 12. Minkowski inequality
Given complex-valued sequences $\{x_i\}_{i=1}^n$ and $\{y_i\}_{i=1}^n$,

$$\left[\sum_{i=1}^{n} |x_i + y_i|^p \right]^{\frac{1}{p}} \leq \left[\sum_{i=1}^{n} |x_i|^p \right]^{\frac{1}{p}} + \left[\sum_{i=1}^{n} |y_i|^p \right]^{\frac{1}{p}}$$

Proof. Denote

$$|x_i+y_i|^p = |x_i+y_i|\cdot|x_i+y_i|^{p-1}$$

Then,

$$\begin{aligned}
\sum_{i=1}^{n}|x_i+y_i|^p &= \sum_{i=1}^{n}|x_i+y_i|\cdot|x_i+y_i|^{p-1}\\
&= \sum_{i=1}^{n}(|x_i|+|y_i|)\cdot|x_i+y_i|^{p-1}\\
&= \sum_{i=1}^{n}|x_i|\cdot|x_i+y_i|^{p-1}+\sum_{i=1}^{n}|y_i|\cdot|x_i+y_i|^{p-1}\\
&\overset{\text{Hölder}}{\leq} \left[\sum_{i=1}^{n}|x_i|^p\right]^{\frac{1}{p}}\cdot\left[\sum_{i=1}^{n}|x_i+y_i|^p\right]^{\frac{p-1}{p}}+\left[\sum_{i=1}^{n}|y_i|^p\right]^{\frac{1}{p}}\cdot\left[\sum_{i=1}^{n}|x_i+y_i|^p\right]^{\frac{p-1}{p}}\\
&= \left[\left(\sum_{i=1}^{n}|x_i|^p\right)^{\frac{1}{p}}+\left(\sum_{i=1}^{n}|y_i|^p\right)^{\frac{1}{p}}\right]\cdot\left[\sum_{i=1}^{n}|x_i+y_i|^p\right]^{\frac{p-1}{p}}
\end{aligned}$$

Now, Divide each side as $[\sum_{i=1}^{n}|x_i+y_i|^p]^{\frac{p-1}{p}}$, then we obtain

$$\left[\sum_{i=1}^{n}|x_i+y_i|^p\right]^{1-\frac{p-1}{p}} = \left[\sum_{i=1}^{n}|x_i+y_i|^p\right]^{\frac{1}{p}} \leq \left[\left(\sum_{i=1}^{n}|x_i|^p\right)^{\frac{1}{p}}+\left(\sum_{i=1}^{n}|y_i|^p\right)^{\frac{1}{p}}\right]$$

$\square$

Theorem 13. Let d_{p_1}, d_{p_2} are p-norm on $\mathbb{R}^n$ with $1\leq p_1<p_2$. Then,

$$\exists C>0 \text{ s.t. } \forall x,y\in\mathbb{R}^n,\ d_{p_2}(x,y)\leq d_{p_1}(x,y)\leq Cd_{p_2}(x,y)$$

In particular, $C=n^{\frac{1}{p_1}-\frac{1}{p_2}}$.

Proof. Let $p_1<p_2$.
For show that first-inequality,

$$1=\sum_{i=1}^{n}\left[\frac{|x_i-y_i|}{[\sum_{i=1}^{n}|x_i-y_i|^{p_2}]^{\frac{1}{p_2}}}\right]^{p_2}\leq\sum_{i=1}^{n}\left[\frac{|x_i-y_i|}{[\sum_{i=1}^{n}|x_i-y_i|^{p_2}]^{\frac{1}{p_2}}}\right]^{p_1}=\frac{\sum_{i=1}^{n}|x_i-y_i|^{p_1}}{[\sum_{i=1}^{n}|x_i-y_i|^{p_2}]^{\frac{p_1}{p_2}}}=\left[\frac{[\sum_{i=1}^{n}|x_i-y_i|^{p_1}]^{\frac{1}{p_1}}}{[\sum_{i=1}^{n}|x_i-y_i|^{p_2}]^{\frac{1}{p_2}}}\right]^{p_1}$$

Thus, we obtain that:

$$1\leq\left[\frac{[\sum_{i=1}^{n}|x_i-y_i|^{p_1}]^{\frac{1}{p_1}}}{[\sum_{i=1}^{n}|x_i-y_i|^{p_2}]^{\frac{1}{p_2}}}\right]^{p_1}\iff 1\leq\frac{[\sum_{i=1}^{n}|x_i-y_i|^{p_1}]^{\frac{1}{p_1}}}{[\sum_{i=1}^{n}|x_i-y_i|^{p_2}]^{\frac{1}{p_2}}}\iff\left[\sum_{i=1}^{n}|x_i-y_i|^{p_2}\right]^{\frac{1}{p_2}}\leq\left[\sum_{i=1}^{n}|x_i-y_i|^{p_1}\right]^{\frac{1}{p_1}}$$

For show that second-inequality, using Hölder's inequality.

$$\begin{aligned}
(d_{p_1}(x,y))^{p_1} &= \sum_{i=1}^{n}|x_i-y_i|^{p_1}=\sum_{i=1}^{n}|x_i-y_i|^{p_1}\cdot 1\\
&\overset{\text{Hölder}}{\leq}\left[\sum_{i=1}^{n}\left(|x_i-y_i|^{p_1\cdot\frac{p_2}{p_1}}\right)\right]^{\frac{p_1}{p_2}}\cdot\left[\sum_{i=1}^{n}1^{\frac{p_2}{p_2-p_1}}\right]^{1-\frac{p_1}{p_2}}=\left[\sum_{i=1}^{n}(|x_i-y_i|^{p_2})\right]^{\frac{p_1}{p_2}}\cdot n^{1-\frac{p_1}{p_2}}
\end{aligned}$$

Taking the $\frac{1}{p_1}$-th power of both sides, then

$$d_{p_1}(x,y)\leq\left[\sum_{i=1}^{n}(|x_i-y_i|^{p_2})\right]^{\frac{1}{p_2}}\cdot n^{\frac{1}{p_1}-\frac{1}{p_2}}=n^{\frac{1}{p_1}-\frac{1}{p_2}}\cdot d_{p_2}(x,y)$$

$\square$

13.2 Topological Vector Space

13.3 Hilbert Space

13.4 Banach Space

13.5 L_p Space

13.6 l_p Space