

# Math Note

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This paper covers several topics in undergraduate mathematics.

#### Patch Note:

- ~ 2025/9/28 - Drafted the initial framework of the paper, and Transcribed previous works.
- 2025/09/29 - 1. Completed proof of Ring of Fractions.  
2. Transcribed Integral, Ratio, and Root Test.  
3. Transcribed Tube Lemma, Lindelöf and Countably Compact product Compact.  
4. Transcribed Coproduct with Continuous, open, closed map.
- 2025/09/30 - 1. Proved Every open set in  $\mathbb{R}^n$  is countable union of closed cubes, disjoint of interiors remains.  
2. Transcribed Group action.
- 2025/10/01 - 1. Transcribed One-point Compactification.  
2. Transcribed Definitions of subbasis, Borel set.
- 2025/10/02 - 1. Proved Euclidean Domain  
2. Proving Existence of Nowhere-differentiable function.
- 2025/10/03 - 1. Drafted definition and propositions of Quotient Space.
- 2025/10/04 - 1. Studying Quotient Map.
- 2025/10/05 - 1. Studied Basic Properties of the Quotient Map, and Drew quotient map diagram.
- 2025/10/06 - 1. Drafted basic functions in a Metric space.
- 2025/10/07 - 1. Proved basic properties of Completely regular space.
- 2025/10/08 - 1. Proved Compact Hausdorff Space is Normal.  
2. Proved Equivalent Conditions of Completely Regular Space.  
3. Proving the Urysohn Metrization Theorem.  
4. Understanding relations and characteristic of Domains.
- 2025/10/10 - 1. Transcribed basic statements of Polynomial Ring.
- 2025/10/11 - 1. Proved Gauss's Lemma in Polynomial Ring, and Transcribed.  
2. Proving  $R$  U.F.D. iff  $R[x]$  U.F.D.
- 2025/10/13 - 1. Understanding U.F.D and irreducible.  
2. Proved  $R$  U.F.D. iff  $R[x]$  U.F.D.  
3. Drew the Diagram of Domains.

# Chapter 1

## Set Theory

### 1.1 Map

**Definition 1.1.0.1.** Let  $X, Y$  are sets. Define a *function*  $X$  to  $Y$  is a relation

$$f \subset X \times Y$$

such that

1. For any  $x \in X$ , there exists  $y \in Y$  such that  $(x, y) \in f$ .
2. If  $(x, y) \in f$  and  $(x, z) \in f$ , then  $y = z$ .

Denote  $f$  as:

$$f : X \rightarrow Y : x \mapsto f(x)$$

Define *Image* of  $f$  by  $A \subset X$ :

$$f[A] \stackrel{\text{def}}{=} \{f(a) \mid a \in A\} \subset Y$$

And, *Preimage* of  $f$  by  $B \subset Y$ :

$$f^{-1}[B] \stackrel{\text{def}}{=} \{x \in X \mid f(x) \in B\} \subset X$$

$f : X \rightarrow Y$  is *Injective* if:  $f(x_1) = f(x_2) \implies x_1 = x_2$ .

$f : X \rightarrow Y$  is *Surjective* if:  $\forall y \in Y, \exists x \in X$  s.t.  $f(x) = y$ .

If  $f$  is injective and surjective, called *bijective*.

If  $f$  is bijective, then define *inverse* of  $f$  as:

$$f^{-1} : Y \rightarrow X : y \mapsto x$$

where  $x \in X$  is the unique elements of  $X$  such that  $f(x) = y$ .

**Theorem 1.1.0.1.** Let  $f : X \rightarrow Y$  be a function. Then,

1. There exists  $g : Y \rightarrow X$  such that  $g \circ f : X \rightarrow X$  be an identity function if and only if  $f$  is injective.
2. There exists  $h : Y \rightarrow X$  such that  $f \circ h : Y \rightarrow Y$  be an identity function if and only if  $f$  is surjective.

*Proof.*

1.  $\implies$  )

Assume that  $f(x_1) = f(x_2)$ . Then, existence of left inverse,  $g(f(x_1)) = g(f(x_2)) \implies x_1 = x_2$ . Thus  $f$  injective.

1.  $\impliedby$  )

Since  $f$  is injection, for any  $y \in f[X]$ , there exists a unique element  $x_y \in X$  such that  $f(x) = y$ . Now, define

$$g : Y \rightarrow X : y \mapsto \begin{cases} x_y & y \in f[X] \\ \text{any element in } X & y \notin f[X] \end{cases}$$

Then, for any  $x \in X$ ,  $g(f(x)) = g(y) = x$ .

2.  $\implies$  )

Let  $y \in Y$  be given. Since existence of right inverse,  $f(h(y)) = y$  where  $h(y) \in X$ . Thus,  $f$  is surjective.

2.  $\impliedby$  )

For any  $y \in Y$ , there exists a  $x_y \in X$  such that  $f(x_y) = y$ . Now, define

$$h : Y \rightarrow X : y \mapsto x_y$$

Then, for any  $y \in Y$ ,  $f \circ h(y) = f(x_y) = y$ . Thus,  $f \circ h$  is identity. □

**Corollary 1.1.0.1.** Let  $f : X \rightarrow Y$  be a function,  $\text{id}_X : X \rightarrow X : x \mapsto x$ , and  $\text{id}_Y : Y \rightarrow Y : y \mapsto y$ .

There exists a  $f^{-1} : Y \rightarrow X$  such that  $f^{-1} \circ f = \text{id}_X$  and  $f \circ f^{-1} = \text{id}_Y$  if and only if  $f$  is bijection.

*Proof.* If  $f$  is bijection, then there exists left inverse  $g$  and right inverse  $h$ .

Enough To Show that:  $g = h$ . Since  $g \circ f = \text{id}_X$  and  $f \circ h = \text{id}_Y$ ,

$g \circ f \circ h = g \circ \text{id}_Y$ , thus  $h = g$ . □

**Theorem 1.1.0.2.** Let  $X, Y, Z$  are sets,  $f : X \rightarrow Y$ ,  $g : Y \rightarrow Z$  and  $A \subset X, B \subset Y, C \subset Z$ . Then followings are hold:

1.  $g[f[A]] = (g \circ f)[A]$ .
2.  $f^{-1}[g^{-1}[C]] = (g \circ f)^{-1}[C]$ .

*Proof.*

1. It is clear by definition of image:

$$\begin{aligned} g[f[A]] &\stackrel{\text{def}}{=} g[\{f(a) \mid a \in A\}] = \{g(b) \mid b \in \{f(a) \mid a \in A\}\} \\ &= \{g(b) \mid b = f(a) \text{ for some } a \in A\} = \{g(f(a)) \mid \text{for some } a \in A\} = \{g(f(a)) \mid a \in A\} \end{aligned}$$

2. It is not clear,

$$f^{-1}[g^{-1}[C]] \stackrel{\text{def}}{=} f^{-1}[\{b \in Y \mid g(b) \in C\}] = \{a \in X \mid f(a) \in \{b \in Y \mid g(b) \in C\}\} = \{a \in X \mid g(f(a)) \in C\} = (g \circ f)^{-1}[C]$$

□

**Proposition 1.1.0.1.** Let  $f : X \rightarrow Y$  be a function,  $A, B \subset X$  and  $C, D \subset Y$ .

1. If  $A \subset B$ , then  $f[A] \subset f[B]$ .
2. If  $C \subset D$ , then  $f^{-1}[C] \subset f^{-1}[D]$

*Proof.*

$$\begin{aligned} y \in f[A] &\implies y = f(a) \text{ for some } a \in A \stackrel{A \subset B}{\implies} y = f(a) \text{ for some } a \in B \implies y \in f[B] \\ x \in f^{-1}[C] &\implies f(x) \in C \stackrel{C \subset D}{\implies} f(x) \in D \implies x \in f^{-1}[D] \end{aligned}$$

□

**Lemma 1.1.0.1.** Let two set  $X, Y$  be given, and  $A \subset X$ ,  $B \subset Y$ ,  $f: X \rightarrow Y$ . Then followings are holds:

1.  $f^{-1}[f[A]] \supseteq A$ , and equality holds if  $f$  one-to-one.
2.  $f[f^{-1}[B]] \subseteq B$ , and equality holds if  $f$  onto.
3.  $f^{-1}[Y \setminus B] = X \setminus f^{-1}[B]$
4.  $f[X] \setminus f[A] \subseteq f[X \setminus A]$ , and equality holds if  $f$  one-to-one.

**Proof.** Proof of 4.

$$\begin{aligned} y \in f[X] \setminus f[A] &\iff y \in f[X] \text{ and } y \notin f[A] \\ &\iff \exists x \in X \text{ s.t. } y = f(x) \text{ and } \forall x \in A, y \neq f(x) \\ &\stackrel{(*)}{\implies} \exists x \in X \setminus A \text{ s.t. } y = f(x) \\ &\iff y \in f[X \setminus A] \end{aligned}$$

If  $f$  is injection, then Left Direction of  $(*)$  be true:  $\exists! x \in X \setminus A$  s.t.  $y = f(x)$ . □

# Chapter 2

## Group Theory

### 2.1 Isomorphism Theorems

**Theorem 2.1.0.1. The First Isomorphism Theorem**

Let  $\varphi : G \rightarrow H$  be a Group-Homomorphism. Then,

$$G / \ker \varphi \cong \varphi[G]$$



**Proof.** Let  $\pi : G \rightarrow G / \ker \varphi : x \mapsto x + \ker \varphi$ . Then, the map  $\phi : G / \ker \varphi \rightarrow \varphi[G] : a + \ker \varphi \mapsto \varphi(a)$  is isomorphism. Well-defined and Injective:

$$a + \ker \varphi = b + \ker \varphi \iff a - b \in \ker \varphi \iff \varphi(a - b) = \varphi(a) - \varphi(b) = 0$$

Surjective is clear. □

**Theorem 2.1.0.2. The Second Isomorphism Theorem**

Let  $G$  be a Group, and  $H \leq G$ ,  $N \trianglelefteq G$ . Then,

$$HN/N \cong H/(H \cap N)$$

**Proof.**  $HN$  be a subgroup of  $G$ , being

$$HN = \bigcup_{h \in H} hN \stackrel{N \trianglelefteq G}{\cong} \bigcup_{h \in H} Nh = NH$$

And,  $N \leq HN$  is clear, thus  $N \trianglelefteq HN$ .

Meanwhile,  $H \cap N$  be a Normal Subgroup of  $H$ : for any  $h \in H, n \in H \cap N$ ,  $hnh^{-1} \in N$  because  $N$  is normal, and  $hnh^{-1} \in H$  since  $h, n$  contained in  $H$ . Thus,  $hnh^{-1} \in H \cap N$ , this implies  $H \cap N$  be a Normal of  $H$ .

Now, Define a Map:

$$\varphi : H \rightarrow HN/N : h \mapsto hN$$

Clearly, this map is Well-Defined and Homomorphism. And,

$$\ker \varphi = \varphi^{-1}[1] = \{h \in H \mid hN = N\} = \{h \in H \mid h \in N\} = H \cap N$$

Thus, since The 1st Isomorphism Theorem,

$$HN/N \cong H/(H \cap N)$$

□



**Theorem 2.1.0.3. The Third Isomorphism Theorem**

Let  $G$  be a Group, and  $H, K \trianglelefteq G$  with  $H \leq K$ . Then,  $K/H \trianglelefteq G/H$  and

$$(G/H)/(K/H) \cong (G/K)$$

**Proof.** First, show that  $K/H \trianglelefteq G/H$ . Let  $kH \in K/H$  and  $gH \in G/H$ . Then,

$$(gH)(kH)(gH)^{-1} = (gH)(kH)(g^{-1}H) = (gkg^{-1})H \in K/H$$

since  $gkg^{-1} \in K$ , being  $K \trianglelefteq G$ . Now, Define a map:

$$\varphi : G/H \rightarrow G/K : gH \mapsto gK$$

1. Well-Defined.

$$g_1H = g_2H \iff g_1^{-1}g_2 \in H \xrightarrow{H \leq K} g_1^{-1}g_2 \in K \iff g_1K = g_2K$$

2. Homomorphism.

Clearly, for any  $g_1H, g_2H \in G/H$ ,

$$\varphi(g_1H g_2H) = \varphi(g_1g_2H) = g_1g_2K = g_1K g_2K = \varphi(g_1H) \varphi(g_2H)$$

3. Surjection. Let  $gK \in G/K$  be given. Then, clearly,  $\varphi(gH) = gK$ .

4. Kernel.

$$\ker \varphi = \{gH \in G/H \mid gK = 1\} = \{gH \in G/H \mid g \in K\} = K/H$$

Consequently, The 1st Isomorphism Theorem gives

$$(G/K) \cong (G/H)/\ker \varphi = (G/H)/(K/H)$$

□

**Theorem 2.1.0.4. The Fourth Isomorphism Theorem**

Let  $G$  be a Group, and  $N \trianglelefteq G$  be a Normal Subgroup. Then, there is a bijection between

$$D \stackrel{\text{def}}{=} \{H \leq G \mid N \leq H\}, \quad C \stackrel{\text{def}}{=} \{\bar{H} \leq G/N\}$$

**Proof.** Let  $\pi : G \rightarrow G/N : g \mapsto gN$  be a natural projection. And, Define

$$\Phi : D \rightarrow C : H \mapsto \pi[H]$$

This function is well-defined: For any  $H \in D$ , let  $aN, bN \in \pi[H]$ . Then,  $aN \cdot b^{-1}N = ab^{-1}N \in \pi[H]$ , thus  $\pi[H] \leq G/N$ .

To show that one-to-one: Let  $\Phi(A) = \Phi(B)$ . Thus means,  $\pi[A] = \pi[B]$ . Let  $a \in A$ . Then,  $\pi(a) \in \pi[A] = \pi[B]$ , thus  $\pi(a) = \pi(b)$  for some  $b \in B$ . That is,  $aN = bN \iff a \in bN$ . Meanwhile,  $N \leq B$ , thus  $a \in bN \subset B$ ,  $A \subset B$ . Similarly,  $B \subset A$ , that is  $A = B$ .

To show that onto: Let  $K \in C$ . Then,  $N \leq \pi^{-1}[K] \leq G$ , thus clear.

□

## 2.2 Group Action

In this section, we follow that the notation of [Dummit and Foote, 2004, Abstract Algebra].

**Definition 2.2.0.1.** Let  $(G, *)$  be a Group, and  $A$  be a non-empty set. Define *Group Action* of a group  $G$  on a set  $A$ :

$$\alpha : G \times A \rightarrow A : (g, a) \mapsto g \cdot a$$

satisfies

1. For all  $a \in A$ ,  $1_G \cdot a = a$ .
2. For all  $g_1, g_2 \in G$ ,  $a \in A$ ,  $(g_1 * g_2) \cdot a = g_1 \cdot (g_2 \cdot a)$

In this, we said to be ' $G$  acts on a set  $A$ '. Meanwhile, For each  $g \in G$ , Define a map

$$\sigma_g : A \rightarrow A : a \mapsto g \cdot a$$

Then, the *permutation representation*

$$\varphi : G \rightarrow S_A : g \mapsto \sigma_g$$

be a Homomorphism. Clearly, for each  $g \in G$ ,  $a \in A$ ,

$$\alpha(g, a) = g \cdot a = \sigma_g(a) = \varphi(g)(a)$$

Thus, there is one-to-one correspondence between group action and permutation representation. For each  $a \in A$ , the *stabilizer* of  $a$  in  $G$ :

$$G_a \stackrel{\text{def}}{=} \{g \in G \mid g \cdot a = a\}$$

The *kernel of action*:

$$\ker \alpha \stackrel{\text{def}}{=} \{g \in G \mid g \cdot a = a, \forall a \in A\} = \bigcap_{a \in A} G_a$$

$G_a \leq G$  and  $\ker \alpha \leq G$ .

If the kernel of action be trivial, the action is called *faithful*.

**Definition 2.2.0.2.** Let  $\alpha : G \times A \rightarrow A$  be a Group Action. Define a relation on  $A$ :

$$a \sim b \iff a = g \cdot b \text{ for some } g \in G$$

Then, this relation be equivalence relation. Denote the equivalence relation, called *orbit*:

$$\mathcal{C}_a \stackrel{\text{def}}{=} \{b \mid b = g \cdot a \text{ for some } g \in G\} = \{g \cdot a \mid g \in G\}$$

And, the action is called *transitive* if there is only one orbit.

**Lemma 2.2.0.1.** For each  $a \in A$ ,

$$|\mathcal{C}_a| = |G : G_a|$$

*Proof.* Since the map

$$\varphi_a : \mathcal{C}_a \rightarrow \{gG_a \mid g \in G\} : g \cdot a \mapsto gG_a$$

is well-defined, bijection.

□

**Theorem 2.2.0.1.** Let  $G$  be a Group, let  $H \leq G$  and  $A = \{gH \mid g \in G\}$ ,  $G$  acts by left multiplication on the set  $A$ .

$$\pi_H : G \rightarrow S_A : g \mapsto \sigma_g$$

be a permutation representation afforded by this action. Then

1.  $G$  acts transitively on  $A$ .
2.  $G_{1H} = \{g \in G \mid gH = H\} = H$ .
3. The kernel of the action  $\ker \pi_H = \bigcap_{x \in G} xHx^{-1}$ , this is the largest normal subgroup of  $G$  contained in  $H$ .

**Proof.** Let  $aH, bH \in A$  be given. Then, for  $g = ba^{-1}$ ,  $g \cdot aH = (ga)H = bH$ . Thus,  $A = C_a$  for any  $a \in G$ .  
 It is clear, being  $gH = H \iff g \in H$ .

Now,

$$\begin{aligned} \ker \pi_H &= \{g \in G \mid gxH = xH, \forall x \in G\} \\ &= \{g \in G \mid (x^{-1}gx)H = H, \forall x \in G\} \\ &= \{g \in G \mid x^{-1}gx \in H, \forall x \in G\} \\ &= \{g \in G \mid g \in xHx^{-1}, \forall x \in G\} = \bigcap_{x \in G} xHx^{-1} \end{aligned}$$

And the second assertion given by:

Let  $N$  is a normal subgroup of  $G$  contained in  $H$ , then for any  $x \in G$ ,  $N = xNx^{-1} = xHx^{-1}$ . Thus,

$$N \leq \bigcap_{x \in G} xHx^{-1}$$

□

**Corollary 2.2.0.1.** If  $G$  is a finite group of order  $n$ ,  $p$  is the smallest prime dividing  $|G|$ . Then, any subgroup of index  $p$  is normal.

**Proof.** Let  $|G| = p_1^{r_1} \cdots p_n^{r_n}$  be a prime decomposition,  $H \leq G$  with  $|G : H| = p$ .

Let  $K = \ker \pi_H \leq H$ ,  $k = |H : K|$ . Then,  $|G : K| = |G : H||H : K| = pk$ . By the First-Isomorphism Theorem,

$$G/\ker \pi_H \cong \pi_H[G] \leq S_A$$

and Since  $H$  has  $p$  left cosets,  $A \cong \mathbb{Z}_p$ , thus  $G/K$  is isomorphic to some subgroup of  $S_p$ .

Now, Lagrange's Theorem gives that  $|G/K| = pk$  divides  $|S_p| = p!$ . This implies  $k \mid (p-1)!$ .

$|G : K| = pk$  implies  $|G| = pk \cdot |K|$ . Since  $p$  is the minimal prime that divides  $|G|$ , thus every prime divisor of  $k$  is greater than or equal to  $p$ . This implies must be  $k = 1$ . Thus  $H = K \trianglelefteq G$ . □

**Definition 2.2.0.3.** Let a Group action as:

$$\alpha : G \times G \rightarrow G : (g, a) \mapsto gag^{-1}$$

Now, the orbit derived from this action  $[a] = \{b \in G \mid \exists g \in G \text{ s.t. } b = gag^{-1}\}$  is called be *Conjugacy Class*.  
 More generally,

$$\alpha : G \times \mathcal{P}(G) \rightarrow \mathcal{P}(G) : (g, S) \mapsto gSg^{-1}$$

**Lemma 2.2.0.2.** Let  $\alpha : G \times \mathcal{P}(G) \rightarrow \mathcal{P}(G) : (g, S) \mapsto gSg^{-1}$  be a Group action acting as Conjugate. Then,  $G_S = N_G(S)$  and  $|\mathcal{C}_S| = |G : N_G(S)|$ , for any  $S \subseteq G$ . In particular, if  $S$  is singleton,  $S = \{g_i\}$ , then  $|\mathcal{C}_{\{g_i\}}| = |G : N_G(g_i)| = |G : C_G(g_i)|$ .

*Proof.*

$$G_S = \{g \in G \mid gSg^{-1} = S\} = N_G(S)$$

Thus, for any  $S \in \mathcal{P}(G)$ ,

$$|\mathcal{C}_S| = |G : N_G(S)|$$

□

### 2.2.1 Lagrange's Theorem

## 2.3 Generating subset of a Group

## 2.4 Commutator Subgroup

## Chapter 3

# Finite Group Theory

### 3.1 The Class Equation

**Theorem 3.1.0.1. The Class Equation**

Let  $G$  be a finite group, and

$g_1, \dots, g_r$  be representatives of the distinct conjugacy classes of  $G$  not contained in the center  $Z(G)$  of  $G$ .

Then,

$$|G| = |Z(G)| + \sum_{i=1}^r |G : C_G(g_i)|$$

### 3.2 Cauchy's Theorem

**Lemma 3.2.0.1. Cauchy's Theorem**

Let  $G$  be a finite group, and  $p$  be a prime dividing  $|G|$ . Then,  $G$  has order  $p$  element.

*Proof.* Define a set:

$$S \stackrel{\text{def}}{=} \{(x_1, x_2, \dots, x_p) \mid x_i \in G, x_1 x_2 \cdots x_p = 1\}$$

Then,  $S$  has exactly  $|G|^{p-1}$  elements because there are  $|G|$  possible choices for each of the first  $p-1$  elements in  $G$ .

Once  $x_1, \dots, x_{p-1}$  are chosen, then  $x_p$  is uniquely determined by the uniqueness of inverses.

Then, let  $\sigma = (1, 2, \dots, p)$  be a permutation. Then, for any  $\alpha \in S$ ,  $\sigma^n(\alpha) \in S$  for all  $n \in \mathbb{Z}$ , being  $ab = 1 \iff ba = 1$ .

More precisely, let  $n \in \mathbb{Z}$  be given,  $\alpha = (x_1, \dots, x_n)$ . Then,

$$\sigma^n(\alpha) = (x_{n+1}, x_{n+2}, \dots, x_p, x_1, x_2, \dots, x_n)$$

By  $x_1 \cdots x_n x_{n+1} \cdots x_p = 1$ ,  $x_{n+1} \cdots x_p x_1 \cdots x_n = 1$ . Thus  $\sigma^n(\alpha) \in S$ . Now, define a relation on  $S$  as:

$$\alpha \sim \beta \text{ if and only if } \beta = \sigma^n(\alpha) \text{ for some } n \in \mathbb{Z}$$

Then, this relation be equivalent relation, thus construct a partition on  $S$ . Claim:

$$[\alpha] = \{\beta \in S \mid \beta \sim \alpha\} \text{ is singleton if and only if } \alpha = (x, \dots, x) \text{ for some } x \in G.$$

Left direction is clear, and for show that Right direction,

Suppose that  $\alpha = (x_1, \dots, x_n)$  has different coordinate elements, let  $x_i \neq x_j$ , for some  $i < j$ . Then clearly

$$(x_1, \dots, x_i, \dots, x_p) \neq \sigma^{i-j}(x_1, \dots, x_i, \dots, x_j, \dots, x_p) = (\dots, \underbrace{x_j}_{i\text{'th element}}, \dots)$$

Meanwhile, if  $[\alpha]$  has elements more than 1,  $[\alpha]$  has exactly number of  $p$  elements. Because suppose that  $\alpha = (x_1, \dots, x_p)$  has at least one different coordinate. Then,

$$\sigma^1(\alpha), \sigma^2(\alpha), \dots, \sigma^{p-1}(\alpha)$$

are mutually different: If there exist  $1 \leq i < j < p$  such that  $\sigma^i(\alpha) = \sigma^j(\alpha)$ , that is,  $\sigma^{j-i}(\alpha) = \alpha$ .

Now,  $j - i \mid p$ , this is contradiction with  $p$  is prime. Therefore, every equivalent class has order 1 or  $p$ . Consequently,

$$|G|^{p-1} = k + pd$$

where  $k$  is a number of classes of size 1, and  $d$  is a number of classes of size  $p$ . And  $(1, 1, \dots, 1) \in S$ ,  $k$  is at least 1.

Since  $p$  divides  $|G|^{p-1} = k + pd$ , thus  $k$  must be bigger than 1, thus there exists elements such that  $x^p = 1$ .  $\square$



### 3.3 Sylow's Theorem

#### Theorem 3.3.0.1. Sylow's Theorem

Let  $G$  be a group of order  $p^\alpha m$ , where  $p$  is a prime such that  $p \nmid m$ .

A group of order  $p^r$ , ( $r \geq 1$ ) is called a  $p$ -group, Subgroups of  $G$  which are  $p$ -groups are called  $p$ -subgroup. In particular, subgroups of order  $p^\alpha$  is called Sylow  $p$ -subgroup of  $G$ . And, define a collection

$$\text{Syl}_p(G) \stackrel{\text{def}}{=} \{P \leq G \mid |P| = p^\alpha\}, \quad n_p(G) \stackrel{\text{def}}{=} \text{Card}(\text{Syl}_p(G))$$

#### The First Sylow Theorem

There exists a Sylow  $p$ -subgroup of  $G$ . i.e.,  $\text{Syl}_p(G) \neq \emptyset$ .

#### The Second Sylow Theorem

If  $P \in \text{Syl}_p(G)$  and  $Q \leq G$  be a  $p$ -subgroup. Then, there exists  $g \in G$  such that  $Q \leq gPg^{-1}$ .

#### The Third Sylow Theorem

$n_p \equiv 1 \pmod{p}$ ,  $n_p = |G : N_G(P)|$  for any  $P \in \text{Syl}_p(G)$ , and  $n_p \mid m$ .

Before prove above statements, we show that:

**Lemma 3.3.0.1.** Let  $P \in \text{Syl}_p(G)$ . If  $Q$  is  $p$ -subgroup of  $G$ , then  $Q \cap N_G(P) = Q \cap P$ .

*Proof.* Put  $H = Q \cap N_G(P)$ . Since  $P \leq G$ , for any  $p \in P$ ,  $pPp^{-1} = P$ , thus  $p \in N_G(P)$ . i.e.,  $P \leq N_G(P)$ . Thus, Enough to Show that  $H \leq Q \cap P$ . Since  $H \leq N_G(P)$ ,

$$PH = \bigcup_{h \in H} Ph = \bigcup_{h \in H} hP = HP$$

Thus,  $PH \leq G$ . And,

$$|PH| = \frac{|P||H|}{|P \cap H|}$$

By Lagrange's Theorem,  $H \leq P$  and  $P \cap H \leq P$  must have order of powers of  $p$ , so  $PH$  be a  $p$ -group. Clearly,  $P \leq PH$  and  $P$  is the largest  $p$ -group of  $G$ , thus,  $PH = P$ . This means,  $H \leq P$ .  $\square$

*Proof.* The First Theorem: The existence of Sylow  $p$ -subgroup. Proof by Induction:

If  $|G| = 1$ , there is nothing to prove.

Assume inductively the existence of Sylow  $p$ -subgroups for all groups of order less than  $|G|$ .

In case of  $p \mid |Z(G)|$ , then by Cauchy's Theorem,  $Z(G)$  has a subgroup  $N$  which has order of  $p$ .

Clearly  $N$  is Normal, and  $G/N = |G|/|N| = p^{\alpha-1}m$ . By assumption,  $G/N$  has a subgroup  $P'$  of order  $p^{\alpha-1}$ .

By The Forth Isomorphism Theorem, Let  $P \leq G$  be a subgroup such that  $P/N = P'$ .

Then,  $|P| = |P/N| \cdot |N| = p^\alpha$ , Thus  $P$  be a Sylow  $p$ -subgroup of  $G$ .

In case of  $p \nmid |Z(G)|$ .

Let  $g_1, \dots, g_r$  be represectatives of the distinct conjugacy classes of  $G$ , not contained in  $Z(G)$ . Then, The Class Equation gives

$$|G| = |Z(G)| + \sum_{i=1}^r |G : C_G(g_i)|$$

Since  $p$  divides  $|G|$ , if for all  $i = 1, 2, \dots, r$ ,  $p \mid |G : C_G(g_i)|$  then  $p \mid |Z(G)|$ , this is contradiction.

Thus, for some  $j$ ,  $p \nmid |G : C_G(g_j)|$ . Put  $H = C_G(g_j) < G$ . Then,  $|H|$  has a factor of  $p^\alpha$ , by  $p \nmid |G : C_G(g_j)|$ . Now,

$$|H| = p^\alpha m' \quad (m' < m)$$

By assumption,  $H$  has a Sylow  $p$ -group, order of  $p^\alpha$ .

Consequently, the existence of Sylow  $p$ -subgroup was shown.

The Second Theorem: Relation of  $p$ -subgroups.

The First Theorem gives existence of Sylow  $p$ -subgroups. Let  $P \in \text{Syl}_p(G)$ . Denote that:

$$S \stackrel{\text{def}}{=} \{gPg^{-1} \mid g \in G\} = \{P_1, \dots, P_r\}$$

Let  $Q \leq G$  be an any  $p$ -subgroup of  $G$ . And,  $Q$  acts by conjugation on  $S$ . i.e.,

$$\alpha : Q \times S \rightarrow S : (q, P_i) \mapsto qP_iq^{-1}$$

Write  $S$  as a disjoint union of orbits under this action by  $Q$ :

$$S = \mathcal{O}_1 \cup \mathcal{O}_2 \cup \cdots \cup \mathcal{O}_s$$

where  $r = |\mathcal{O}_1| + \cdots + |\mathcal{O}_s|$ . Rearrange a set  $S$  as:  $P_i \in \mathcal{O}_i$ ,  $1 \leq i \leq s$ . Now, using Definition, Lemma, and above Theorem,

$$|\mathcal{O}_i| \stackrel{\text{Thm}}{\equiv} |Q : N_Q(P_i)| \stackrel{\text{def}}{=} |Q : N_G(P_i) \cap Q| \stackrel{\text{lemma}}{\equiv} |Q : P_i \cap Q|$$

for each  $1 \leq i \leq s$ . Since  $Q$  was arbitrary, Let  $Q = P_1$ , so that  $|\mathcal{O}_1| = |P_1 : P_1 \cap P_1| = 1$ . And, for each  $i \geq 2$ ,  $P_i \cap P_1 < P_1$ ,

$$|\mathcal{O}_i| = |P_1 : P_i \cap P_1| > 1$$

Since  $P_1 \in \text{Syl}_p(G)$ , that is  $|P_1| = p^\alpha$ ,  $|P_1 : P_i \cap P_1| = |P_1|/|P_i \cap P_1| = p^k$  where  $1 \leq k < \alpha$ . This means for each  $2 \leq i \leq s$ ,  $p$  divides  $|\mathcal{O}_i|$ . Thus,

$$r = |\mathcal{O}_1| + (|\mathcal{O}_2| + \cdots + |\mathcal{O}_s|) \equiv 1 \pmod{p}$$

Now, Proof by Contradiction: Let  $Q \leq G$  be a  $p$ -subgroup. Suppose that for any  $1 \leq i \leq r$ ,  $Q \not\leq P_i$ . Then,  $P_i \cap Q < Q$  for all  $i$ , this means

$$|\mathcal{O}_i| = |Q : P_i \cap Q| > 1$$

Thus for any  $i$ ,  $p$  divides  $|\mathcal{O}_i|$ , this is Contradiction. This proved Relation of  $p$ -subgroups. Finally, The Third Theorem:

Since Second Theorem, this gives that  $S = \text{Syl}_p(G)$ , thus  $n_p(G) = r$ . That is,  $n_p \equiv 1 \pmod{p}$ . Since all Sylow  $p$ -subgroups are Conjugate, for any  $P \in \text{Syl}_p(G)$ ,

$$n_p = r = |\mathcal{O}_1| = |G : N_G(P)|$$

Consequently, Completing the Sylow Theorem. □

## 3.4 More Theorems

### Theorem 3.4.0.1. *n* Factorial Theroem

If  $G$  is simple and there is a subgroup  $H$  with  $|G:H| = n$ , then  $|G| \mid n!$ .

*Proof.* Let  $G$  act on  $A = \{gH \mid g \in G\}$  by left multiplication. ( $|A| = n$ ).

Let  $\varphi: G \rightarrow S_n$  be a homomorphism afforded above action. Then,  $G \stackrel{G \text{ simp.}}{\cong} G/\ker \varphi \cong \varphi[G] \leq S_n$  □

## 3.5 Simple groups

## 3.6 Cyclic Group

## 3.7 Symmetric Group

## 3.8 Dihedral Group

## Chapter 4

# Ring Theory

## 4.1 Ideal

**Definition 4.1.0.1.** Let  $R$  be a Ring. A subset  $I \subseteq R$  is called *ideal* of  $R$  if:

1.  $I \subseteq R$  is a subgroup of  $R$ .
2.  $I$  is closed under the multiplication.
3. For any  $r \in R$ ,  $rI \subseteq I$  and  $Ir \subseteq I$ . (In other word, for any  $r \in R, a \in I$ ,  $ra \in I$  and  $ar \in I$ .)

**Theorem 4.1.0.1.** Let  $R$  be a Ring. Then, TFAE:

1.  $I \subseteq R$  is an Ideal of  $R$ .
2. The additive Quotient Group  $R/I \stackrel{\text{def}}{=} \{r + I \mid r \in R\}$  be a Ring under the operation:

$$(r + I) \times (s + I) = (rs) + I$$

**Proof.** Observation:

$$r_1 + I = r_2 + I \iff r_1 - r_2 \in I \iff \exists a \in I \text{ s.t. } r_1 = r_2 + a$$

Now, for well-definednes, want to show that the equality

$$(r + I) \times (s + I) = (rs) + I \\ \stackrel{(*)}{=} [(r + \alpha) + I] \times [(s + \beta) + I] = (r + \alpha)(s + \beta) + I = (rs + r\beta + \alpha s + \alpha\beta) + I$$

(\*) holds for any  $r, s \in R$ ,  $\alpha, \beta \in I$ .

If  $I$  is Ideal, then  $r\beta, \alpha s, \alpha\beta \in I$ . Thus closed under the addition gives (\*).

Conversely, if this operation is well-defined, then for any  $r, s \in R$ ,  $\alpha, \beta \in I$ , (\*) holds.

Substituting zero to each  $r, s, \alpha, \beta$  gives  $I$  is ideal. □

### 4.1.1 Properties of Ideal in Ring with identity

**Definition 4.1.1.1.** Let  $R$  be a Ring with identity, and  $A \subseteq R$ . Define *Ideal generated by  $A$*  as:

$$(A) \stackrel{\text{def}}{=} \bigcap_{\substack{I \text{ ideal} \\ A \subseteq I}} I$$

And,

$$RA \stackrel{\text{def}}{=} \{r_1a_1 + \cdots + r_na_n \mid n \in \mathbb{N}, r_i \in R, a_i \in A\}$$

$$AR \stackrel{\text{def}}{=} \{a_1r_1 + \cdots + a_nr_n \mid n \in \mathbb{N}, r_i \in R, a_i \in A\}$$

$$RAR \stackrel{\text{def}}{=} \{r_1a_1r'_1 + \cdots + r_na_nr'_n \mid n \in \mathbb{N}, r_i, r'_i \in R, a_i \in A\}$$

**Lemma 4.1.1.1.** Let  $R$  be a Ring with identity, and  $A \subseteq R$ . Then,  $(A) = RAR$ .

*Proof.* Since  $RAR$  is ideal which contains  $A$ ,  $(A) \subseteq RAR$ .

And, conversely, if  $\sum_{i=1}^n r_ia_ir'_i \in RAR$ , then  $\sum_{i=1}^n r_ia_ir'_i \in (A)$  because each  $r_ia_ir'_i$  are contained in  $(A)$ , being  $(A)$  is ideal containing  $A$  and ideal is closed under the addition. □

**Theorem 4.1.1.1.** Let  $I$  be an ideal of Ring  $R$  with identity.

$$I = R \text{ if and only if } I \text{ contains a unit.}$$

*Proof.* Right direction is clear by  $1 \in R = I$ .

Denote  $u \in I$  be a unit with  $vu = 1$ , and Let  $r \in R$  be given. Then,

$$r = r1 = rvu \in I$$

□

**Definition 4.1.1.2.** An Ideal  $M$  of  $R$  is *Maximal ideal* if: There is no Ideal  $I$  such that  $M \subsetneq I \subsetneq R$ .

**Theorem 4.1.1.2.** Let  $R$  be a Ring with identity.

Then, every proper ideal  $I \subsetneq R$  is contained in a maximal ideal.

*Proof.*

□

**Lemma 4.1.1.2.** Let  $R$  be a commutative Ring with identity,  $M, P$  are proper ideals of  $R$ .

1.  $M$  is Maximal Ideal if and only if  $R/M$  is a field.
2.  $P$  is Prime Ideal if and only if  $R/M$  is an integral domain.

*Proof. Summary:*  $M$  maximal  $\iff R/M$  field  $\implies R/M$  integral domain  $\iff M$  prime.

$$\begin{aligned} M \text{ is maximal} &\iff \text{There is no ideal } I \text{ such that } M \subsetneq I \subsetneq R \\ &\iff \text{There are Ideals of } R/M \text{ only } 0 \text{ and } R/M \\ &\iff R/M \text{ is field} \end{aligned}$$

$$\begin{aligned} P \text{ is Prime Ideal} &\iff \text{If } ab \in P, \text{ then } a \in P \text{ or } b \in P \\ &\iff \text{If } ab + P = P, \text{ then } a + P = P \text{ or } b + P = P \\ &\iff \text{If } \bar{a}\bar{b} = \bar{0}, \text{ then } \bar{a} = \bar{0} \text{ or } \bar{b} = \bar{0} \\ &\iff R/P \text{ is integral domain} \end{aligned}$$

□

## 4.2 Ring of Fractions

**Theorem 4.2.0.1.** Let  $R$  be a Commutative Ring,  $D \subset R$  be a subset such that  $\begin{cases} \text{no zero, no zero divisors} \\ \text{closed under multiplication} \end{cases}$ .

Then, there exists a Commutative Ring  $Q$  with identity satisfies:

1.  $R$  can embed in  $Q$ , and every element of  $D$  becomes unit in  $Q$ . More precisely,  $Q = \{rd^{-1} \mid r \in R, d \in D\}$ .
2.  $Q$  is the smallest Ring containing  $R$  with identity such that every element of  $D$  becomes unit in  $Q$ .

**Proof.** Let  $\mathcal{F} \stackrel{\text{def}}{=} \{(r, d) \mid r \in R, d \in D\}$  and the relation  $\sim$  on  $\mathcal{F}$  by  $(r_1, d_1) \sim (r_2, d_2) \iff r_1 d_2 = r_2 d_1$ . Then,  $\sim$  is equivalent relation: reflexive and symmetirc are clear, and Suppose that  $(r_1, d_1) \sim (r_2, d_2)$  and  $(r_2, d_2) \sim (r_3, d_3)$ .

$$r_2 d_3 = r_3 d_2 \implies r_2 d_1 d_3 = r_3 d_1 d_2 \implies r_1 d_2 d_3 = r_3 d_1 d_2 \implies d_2(r_1 d_3 - r_3 d_1) \implies r_1 d_3 = r_3 d_1$$

Thus transitivity shown. Define

$$\frac{r}{d} \stackrel{\text{def}}{=} [(r, d)] = \{(a, b) \mid (a, b) \sim (r, d)\}, \quad Q \stackrel{\text{def}}{=} \left\{ \frac{r}{d} \mid r \in R, d \in D \right\}$$

And define operations  $+, \times$  on  $Q$ :

$$\frac{r_1}{d_1} + \frac{r_2}{d_2} \stackrel{\text{def}}{=} \frac{r_1 d_2 + r_2 d_1}{d_1 d_2}, \quad \frac{r_1}{d_1} \times \frac{r_2}{d_2} \stackrel{\text{def}}{=} \frac{r_1 r_2}{d_1 d_2}$$

**Well-Definedness:** If  $\frac{r_1}{d_1} = \frac{r'_1}{d'_1}$  and  $\frac{r_2}{d_2} = \frac{r'_2}{d'_2}$ ,

$$\begin{aligned} \frac{r_1 d_2 + r_2 d_1}{d_1 d_2} &= \frac{r_1 d_2 d'_1 d'_2 + r_2 d_1 d'_1 d'_2}{d_1 d_2 d'_1 d'_2} = \frac{(r_1 d'_1) d_2 d'_2 + (r_2 d'_2) d_1 d'_1}{d_1 d_2 d'_1 d'_2} = \frac{(r'_1 d_1) d_2 d'_2 + (r'_2 d_2) d_1 d'_1}{d_1 d_2 d'_1 d'_2} = \frac{(r'_1 d'_2 + r'_2 d'_1) d_1 d_2}{d_1 d_2 d'_1 d'_2} = \frac{r'_1 d'_2 + r'_2 d'_1}{d'_1 d'_2} \\ \frac{r_1 r_2}{d_1 d_2} &= \frac{r_1 r_2 d'_1 d'_2}{d_1 d_2 d'_1 d'_2} = \frac{(r_1 d'_1)(r_2 d'_2)}{d_1 d_2 d'_1 d'_2} = \frac{(r'_1 d_1)(r'_2 d_2)}{d_1 d_2 d'_1 d'_2} = \frac{r'_1 r'_2 d_1 d_2}{d_1 d_2 d'_1 d'_2} = \frac{r'_1 r'_2}{d'_1 d'_2} \end{aligned}$$

Now,  $(Q, +, \times)$  constructs Commutative Ring with identity: for any  $d \in D$ , put  $0_Q \stackrel{\text{def}}{=} \frac{0}{d}$ ,  $1_Q \stackrel{\text{def}}{=} \frac{d}{d}$ . Then,

1.  $(R, +, \times)$  closed under the operations since  $D$  is closed under the multiplication.

$$2. (R, +) \text{ has a zero: } \frac{r_1}{d_1} + 0_Q = \frac{r_1}{d_1} + \frac{0}{d} = \frac{r_1 d + 0 d_1}{d_1 d} = \frac{r_1 d}{d_1 d} = \frac{r_1}{d_1}.$$

$$3. (R, +) \text{ has an inverse: } \frac{r_1}{d_1} + \frac{-r_1}{d_1} = \frac{r_1 d_1 + (-r_1) d_1}{d_1 d_1} = \frac{[(r_1) + (-r_1)] d_1}{d_1 d_1} = \frac{0 d_1}{d_1 d_1} = \frac{0}{d_1 d_1} = 0_Q.$$

4.  $(R, +, \times)$  satisfies distributive law:

4-1. The left law:

$$\begin{aligned} \frac{r_1}{d_1} \times \left( \frac{r_2}{d_2} + \frac{r_3}{d_3} \right) &= \frac{r_1}{d_1} \times \frac{r_2 d_3 + r_3 d_2}{d_2 d_3} = \frac{r_1 r_2 d_3 + r_1 r_3 d_2}{d_1 d_2 d_3} = \frac{r_1 r_2 d_1 d_3 + r_1 r_3 d_1 d_2}{d_1 d_2 d_1 d_3} = \frac{r_1 r_2}{d_1 d_2} + \frac{r_1 r_3}{d_2 d_3} \\ &= \frac{r_1}{d_1} \times \frac{r_2}{d_2} + \frac{r_1}{d_1} \times \frac{r_3}{d_3} \end{aligned}$$

4-2. The right law:

$$\begin{aligned} \left( \frac{r_1}{d_1} + \frac{r_2}{d_2} \right) \times \frac{r_3}{d_3} &= \frac{r_1 d_2 + r_2 d_1}{d_1 d_2} \times \frac{r_3}{d_3} = \frac{r_1 r_3 d_2 + r_2 r_3 d_1}{d_1 d_2 d_3} = \frac{r_1 r_3 d_2 d_3 + r_2 r_3 d_1 d_3}{d_1 d_3 d_2 d_3} = \frac{r_1 r_3}{d_1 d_3} + \frac{r_2 r_3}{d_2 d_3} \\ &= \frac{r_1}{d_1} \times \frac{r_3}{d_3} + \frac{r_2}{d_2} \times \frac{r_3}{d_3} \end{aligned}$$

$$5. (R, \times) \text{ has an identity: } \frac{r_1}{d_1} \times 1_Q = \frac{r_1}{d_1} \times \frac{d}{d} = \frac{r_1 d}{d_1 d} = \frac{r_1}{d_1}.$$



6. Elements of  $D$  become unit in  $Q$ : Define  $\iota: R \rightarrow Q: r \mapsto \frac{rp}{p}$  where  $p \in D$  is any fixed element in  $D$ .

Then,  $\iota$  is Ring-Monomorphism because:

$$6-1. \text{ Well-Defined and Injective: } \iota(r_1) = \iota(r_2) \iff \frac{r_1 p}{p} = \frac{r_2 p}{p} \iff (r_1 - r_2)p = 0 \iff r_1 = r_2$$

6-2. For any  $d \in D$ ,  $\iota(d)$  is a unit of  $Q$ : Put  $(\iota(d))^{-1} \stackrel{\text{def}}{=} \frac{p}{dp}$ , then

$$\iota(d) \times (\iota(d))^{-1} = \frac{dp}{p} \times \frac{p}{dp} = \frac{dpp}{dpp} = 1_Q$$

That is,  $\iota$  is embedding from  $R$  into  $Q$  such that  $\iota[D]$  becomes units of  $Q$  except zero.  
Moreover, if  $D = R \setminus \{0\}$ , then  $Q$  is field.

7.  $Q$  is the *smallest* ring containing  $R$  with identity such that every element of  $D$  becomes units in  $Q$ .

Let  $S$  be an any commutative ring with identity,

and assume that  $\varphi: R \rightarrow S$  is a Ring-Monomorphism such that for any  $d \in D$ ,  $\varphi(d)$  is unit in  $S$ .

Define  $\phi: Q \rightarrow S: \frac{r}{d} \mapsto \varphi(r)\varphi(d)^{-1}$ . Then, this  $\phi$  is well-defined and injective:

$$\begin{aligned} \phi\left(\frac{r_1}{d_1}\right) = \phi\left(\frac{r_2}{d_2}\right) &\iff \varphi(r_1)\varphi(d_1)^{-1} = \varphi(r_2)\varphi(d_2)^{-1} \iff \varphi(r_1)\varphi(d_2) = \varphi(r_2)\varphi(d_1) \\ &\stackrel{\text{homom.}}{\iff} \varphi(r_1 d_2) = \varphi(r_2 d_1) \stackrel{\text{one-to-one}}{\iff} r_1 d_2 = r_2 d_1 \iff \frac{r_1}{d_1} = \frac{r_2}{d_2} \end{aligned}$$

That is, if a commutative ring  $S$  with identity contains a copy of  $R$  such that the denominator set  $D$  of  $R$  becomes unit in  $S$ , then  $S$  contains ring of fractions  $Q$  of  $R$ . Thus  $S = Q$  is the smallest ring that satisfies these conditions.

□

### 4.3 Commutative Ring with identity

**Lemma 4.3.0.1.** Let  $R$  be a Commutative Ring,  $a, b \in R$  with  $b \neq 0$ .

$$a = bx \text{ for some } x \in R \stackrel{\text{def}}{\iff} b \mid a \iff a \in (b) \iff (a) \subseteq (b)$$

**Lemma 4.3.0.2.** Let  $a, b$  be non-zero elements in a Commutative Ring  $R$ .

If  $(a, b) = (d)$ , then  $d$  is the greatest common divisor of  $a$  and  $b$ .

**Theorem 4.3.0.1.** Let  $R$  be an integral domain. If  $(d) = (d')$ , then  $d' = ud$  for some unit  $u \in R$ .

Particular,  $d$  and  $d'$  are greatest common divisor of  $a$  and  $b$ , then  $(d) = (d')$ , thus  $d' = ud$  for some unit  $u \in R$ .

*Proof.* If either  $d$  or  $d'$  is zero, then there is nothing to prove. Thus, Suppose that neither  $d$  nor  $d'$  is non-zero. Since  $(d) \subseteq (d')$  and  $(d) \supseteq (d')$ ,  $d' = dx$  for some  $x \in R$  and  $d = d'y$  for some  $y \in R$ . Combining above, then  $d' = dx = (d'y)x = d'(yx)$ , this implies  $d'(1 - yx) = 0$ . Since  $d'$  is non-zero and  $d'$  chosen in the integral domain,  $1 - yx = 0$ . Now, both  $x$  and  $y$  are unit, we obtain the result. Second assertion is clear by the First.  $\square$

#### 4.3.1 Euclidean Domain

**Definition 4.3.1.1.** An integral domain  $R$  is called *Euclidean Domain* if: there exists a norm  $N$  such that:

for any  $a, b \in R$  with  $b \neq 0$ , there exist  $q, r \in R$  with  $a = qb + r$  with  $r = 0$  or  $N(r) < N(b)$ .

This definition allows us the *Euclidean Algorithm* on an integral domain  $R$ : for any  $a, b \in R$  with  $b \neq 0$ ,

$$\begin{aligned} a &= q_0b + r_0 \\ b &= q_1r_0 + r_1 \\ r_0 &= q_2r_1 + r_2 \\ r_1 &= q_3r_2 + r_3 \\ &\vdots \\ r_k &= q_{k+2}r_{k+1} + r_{k+2} \\ &\vdots \\ r_{n-2} &= q_nr_{n-1} + r_n \\ r_{n-1} &= q_{n+1}r_n \end{aligned}$$

This process gives a chain:

$$N(r_n) < N(r_{n-1}) < \cdots < N(r_2) < N(r_1) < N(r_0)$$

and this process terminates in finite iteration, since well-ordering principle.

**Theorem 4.3.1.1.** Let  $I$  be an ideal of a Euclidean Domain  $R$ . Then,  $I$  is principal ideal.

*Proof.* If  $I$  is zero ideal, there is nothing to prove. Let  $I$  be a non-zero ideal.

Since the set  $\{N(a) \mid a \in I \setminus \{0\}\}$  has a minimum by Well-Ordering Principle,

choose  $d \in I$  such that  $N(d) \leq N(a)$ ,  $\forall a \in I \setminus \{0\}$ . Clearly,  $(d) \subseteq I$ . Let  $a \in I$ . Then, there is  $q, r \in R$  such that

$$a = qd + r \text{ with } r = 0 \text{ or } N(r) < N(d)$$

Since  $r = a - qd \in I$  by  $a, d \in I$ , thus closed under the multiplication gives  $r \in I$ .

But, by minimality of  $d$ ,  $r$  must be 0. Now,  $a = qd + r = qd \in (d)$ .  $\square$

**Theorem 4.3.1.2. Euclidean Algorithm**

Let  $R$  be a Euclidean Domain,  $a, b \in R$  be non-zero.

Denote  $d = r_n$  where  $r_n$  is the last nonzero remainder in the Euclidean Algorithm for  $a$  and  $b$ .

Then,  $d$  is the greatest common integer of  $a$  and  $b$ . And,  $(d) = (a, b)$ . That is, there exist  $x, y \in R$  such that

$$d = ax + by$$

**Proof.** Note that:  $(a, b)$  is principal in Euclidean Domain.

Moreover,  $(a, b)$  is the smallest ideal containing  $(a)$  and  $(b)$ . That is,

If  $(a) \subseteq (x)$  and  $(b) \subseteq (x)$ , then  $(a, b) \subseteq (x)$ . Now, Enough to Show:

1.  $(a), (b) \subseteq (d)$ . (It follows that  $(a, b) \subseteq (d)$ )

2.  $(d) \subseteq (a, b)$ . (That is,  $(d) = (a, b)$ )

Since  $(a), (b) \subseteq (d)$  if and only if  $d \mid a, b$ , show that  $d$  divides  $a, b$ .

In the last equation,  $r_{n-1} = q_{n+1}r_n = q_{n+1}d$ . Thus,  $d \mid r_{n-1}$ .

Clearly,  $r_n \mid r_n$ , thus  $d \mid r_{n-2}$ . Repeat this to finite times, then we obtain:  $\forall 1 \leq i \leq n, d \mid r_i$ . As result,  $d \mid a$  and  $d \mid b$ . This proved 1.

For to show that 2., we will prove  $d \in (a, b)$ .

The first equation gives directly  $r_0 \in (a, b)$ .

That is,  $(r_0) \subseteq (a, b)$ , thus  $r_1 = b - q_1r_0 \in (a, b)$ .

Inductively,  $r_n = d \in (a, b)$ , theorem completed. □

$$a = q_0b + r_0$$

$$b = q_1r_0 + r_1$$

$$r_0 = q_2r_1 + r_2$$

$$r_1 = q_3r_2 + r_3$$

$$\vdots$$

$$r_{n-2} = q_nr_{n-1} + r_n$$

$$r_{n-1} = q_{n+1}r_n + 0$$

**4.3.2 Principal Ideal Domain**

**Definition 4.3.2.1.** An integral domain  $R$  is called *Principal Ideal Domain* if: every ideal of  $R$  is principal.

**Theorem 4.3.2.1.** Let  $R$  be a Principal Ideal Domain, and  $a, b \in R$  be non-zero.

Let  $d$  be a generator for the principal ideal  $(a, b)$ . Then,

$d$  is the greatest common divisor of  $a$  and  $b$ , and unique up to multiplication of unit of  $R$ .

**Theorem 4.3.2.2.** Every non-zero Prime Ideal in a Principal Domain is Maximal Ideal.

**Proof.** Let  $(p)$  be a non-zero Prime Ideal.

Let  $I = (m)$  be an Ideal such that  $(p) \subseteq (m)$ . Since  $p \in (m)$ , there is a  $x \in R$  such that  $p = mx$ .

But,  $p = mx \in (p)$ , Prime Ideal,  $m \in (p)$  or  $x \in (p)$ .

If  $m \in (p)$ ,  $m = py$  for some  $y \in R$ . That is,  $m = py \in (p)$ ,  $(p) = (m)$ .

If  $x \in (p)$ ,  $x = pz$  for some  $z \in R$ . That is,  $p = mx = mpz = p(mz)$ ,  $m$  becomes a unit.

The Ideal  $(m)$  containing unit implies  $(m) = R$ . □

### 4.3.3 Noetherian Domain

**Definition 4.3.3.1.** The Ring  $R$  is said to be *Noetherian Ring* if:  $R$  satisfies *Ascending Chain Condition* on ideals.

The Integral Domain  $R$  with Noetherian is called *Noetherian Domain*.

**Theorem 4.3.3.1.** Principal Ideal Domain is Noetherian Domain.

*Proof.* Suppose that there is an ascending chain of ideals,

$$I_1 \subseteq I_2 \subseteq \cdots \subseteq R$$

(Considering only countable Chain: Since m.stackexchange 4265544.)

Put  $I \stackrel{\text{def}}{=} \bigcup_{i=1}^{\infty} I_i$ . Since for any  $r \in R$  and  $a \in I$ , there is  $i \in \mathbb{N}$  such that  $a \in I_i$ . Thus,  $ra \in I_i \subseteq I$ ,  $I$  is ideal.

Since  $R$  is Principal, for some  $a \in R$ ,  $(a) = I$ . That is,  $a \in I$ . This implies there exists  $n \in \mathbb{N}$  such that  $a \in I_n$ . Now,  $(a) \subseteq I_n \subseteq I = (a)$ , This  $I_n = (a) = I$ . Consequently,  $R$  is Noetherian.  $\square$

### 4.3.4 Unique Factorization Domain

**Definition 4.3.4.1.** Let  $R$  be an Integral Domain.

1. A non-zero, not unit  $r \in R$  is called *irreducible* of  $R$  if: If  $r = ab$  for some  $a, b \in R$ , then either  $a$  or  $b$  is a unit.
2. A non-zero, not unit  $p \in R$  is called *prime* of  $R$  if: If  $p \mid ab$  for some  $a, b \in R$ , then either  $p \mid a$  or  $p \mid b$ .

Clearly,  $p$  is prime if and only if  $(p)$  is Prime Ideal.

**Theorem 4.3.4.1.** Let  $R$  be an Integral Domain. Then, every prime element is irreducible.

*Proof.* Let  $R$  be an Integral Domain,  $p \in R$  be a prime. Suppose that  $p = ab$  for some  $a, b \in R$ . Then, clearly  $p \mid ab$ , thus  $p \mid a$  or  $p \mid b$ . If  $p \mid a$ , then  $a = px$  for some  $x \in R$ . Now,  $p = ab = pxb$ ,  $p(1 - xb) = 0$ . Since  $R$  is integral domain and  $p$  is non-zero,  $xb = 1$ . That is,  $b$  is a unit, thus  $p$  is irreducible.  $\square$

**Definition 4.3.4.2.** Let  $R$  be an integral domain and let  $r \in R$  be a nonzero, nonunit element.

1. We say that  $r$  is *factorizable* if there exist irreducible elements  $p_1, \dots, p_n$  ( $n \geq 1$ ) such that

$$r = p_1 p_2 \cdots p_n.$$

Any such expression is called an *irreducible factorization* of  $r$ .

2. An irreducible factorization is *unique up to associates* if for any two irreducible factorizations

$$r = p_1 \cdots p_n = q_1 \cdots q_m,$$

we have  $n = m$  and there exist a permutation  $\sigma \in S_n$  and units  $u_1, \dots, u_n \in R^\times$  such that

$$q_i = u_i p_{\sigma(i)} \quad (i = 1, \dots, n),$$

equivalently,  $q_i$  is associate to  $p_{\sigma(i)}$  for each  $i$ .

The domain  $R$  is called a *factorization domain* (also: *atomic*) if every nonzero, nonunit element of  $R$  is factorizable. If, in addition, irreducible factorizations are unique up to associates, then  $R$  is called a *unique factorization domain (UFD)*.

**Theorem 4.3.4.2.** Noetherian Domain is Factorization Domain.

*Proof.* Let  $R$  be a Noetherian Domain. And, let  $r \in R$  be a non-zero, not unit.

There exist onyl two possibility:  $r$  is irreducible or not irreducible.

If  $r$  is irreducible, then there is nothing to prove. If  $r$  is not irreducible, then there exist not unit  $r_1, r_2 \in R$  such that  $r = r_1 r_2$ .

If  $r_1$  and  $r_2$  are irreducible, prove end. If  $r_1$  is reducible, then there exist not unit  $r_{1,1}, r_{1,2} \in R$  such that  $r_1 = r_{1,1} r_{1,2}$ .

If this process never terminates, then, there is a infinite strictly ascending chain:

$$(r) \subsetneq (r_1) \subsetneq (r_{1,1}) \subsetneq \cdots \subsetneq R$$

Strictly given by  $r = r_1 r_2$  and  $r_2$  is not a unit.

More precisely, if  $(r) = (r_1)$ , then  $r_1 = rk$  for some  $k \in R$ ,  $r_1 = rk = r_1 r_2 k$ ,  $r_1$  becomes a unit. Contradiction.  $\square$

**Theorem 4.3.4.3.**

1. In Principal Ideal Domain, every irreducible element is prime.
2. In Unique Factorization Domain, every irreducible element is prime.

*Proof.* Let  $R$  be a Principal Ideal Domain, and  $r \in R$  be an irreducible.

Suppose that  $(m)$  is an ideal of  $R$  such that  $(r) \subseteq (m)$ .

Then,  $r \in (m)$  implies  $r = mx$  for some  $x \in R$ , now irreducibility gives either  $m$  or  $x$  is a unit.

If  $m$  is a unit, then  $(m) = R$ . If  $x$  is a unit,  $r = mx$  implies  $rx^{-1} = m$  implies  $m \in (r)$  implies  $(m) \subseteq (r)$  implies  $(m) = (r)$ .

Consequently,  $(r)$  is maximal ideal in the Principal Ideal Domain,

$$(r) \text{ is a maximal} \iff R/(r) \text{ is a field} \implies R/(r) \text{ is an integral domain} \iff (r) \text{ is Prime.}$$

Let  $R$  be a Unique Factorization Domain, and  $r \in R$  be an irreducible. Suppose that  $r \mid ab$  for some  $a, b \in R$ .

If either  $a$  or  $b$  is unit, then  $r \mid ab$  implies  $r$  divides  $a$  or  $b$ , there is nothing to prove.

If neither  $a$  nor  $b$  is a unit, write as factorization form:  $a = a_1 \cdots a_n$  and  $b = b_1 \cdots b_m$ , being  $a, b$  in U.F.D.

Since  $r$  divides  $ab = a_1 \cdots a_n b_1 \cdots b_m$ , there exists  $x \in R$  such that

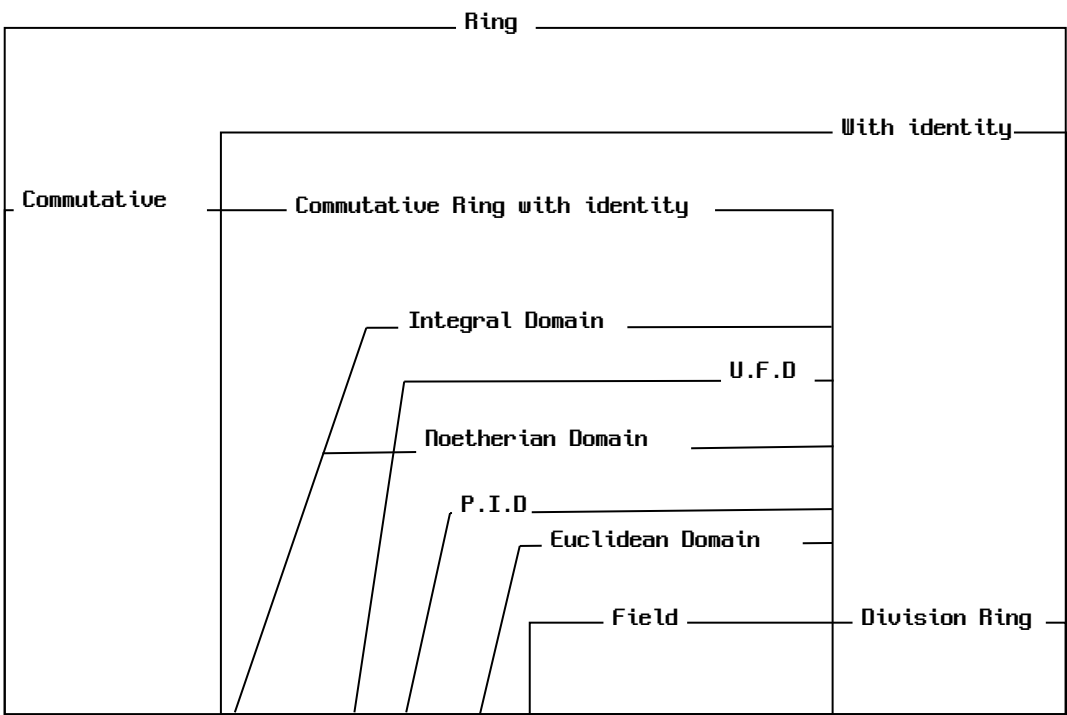
$$rx = a_1 \cdots a_n b_1 \cdots b_m$$

If  $x$  is a unit, then  $r = x^{-1}a_1 \cdots a_n b_1 \cdots b_m$ . But, the uniqueness gives contradiction. Thus  $x$  is not unit.

Now,  $x$  has irreducible factorization, the uniqueness gives  $r = a_i$  for some  $1 \leq i \leq n$  or  $r = b_j$  for some  $1 \leq j \leq m$ .

This means  $r$  divides  $a$  or  $b$ . □

4.3.5 Summary



## Chapter 5

# Polynomial Ring Theory

**Definition 5.0.0.1.** Let  $R$  be a Commutative Ring with unity. Define *Polynomial Ring*:

$$R[x] \stackrel{\text{def}}{=} \left\{ \sum_{i=0}^n a_i x^i \mid n \geq 0, a_i \in R \right\}$$

Addition defined by pointwise, and Multiplication defined by:

$$\left( \sum_{i=0}^n a_i x^i \right) \times \left( \sum_{i=0}^m b_i x^i \right) = \sum_{k=0}^{n+m} \left( \sum_{i=0}^k a_i b_{k-i} \right) x^k$$

**Proposition 5.0.0.1.** Let  $R$  be an integral domain, and  $p, q \in R[x]$  be non-zero elements.

1.  $\deg(pq) = \deg p + \deg q$ .
2.  $R[x]$  is an integral domain.
3. If  $p \in R[x]$  is unit, then  $\deg p = 0$  and  $p$  is unit in  $R$ .

*Proof.*

$$\left( \sum_{i=0}^n a_i x^i \right) \left( \sum_{i=0}^m b_i x^i \right) = a_n b_m x^{n+m} + \dots,$$

This proves the statement (1) immediately, and assume not zero, then proves 2).

And, if  $p \in R[x]$  is unit, then

$$0 = \deg 1 = \deg(pp^{-1}) = \deg p + \deg p^{-1}$$

Thus,  $\deg p = \deg p^{-1} = 0$  and this implies  $p, p^{-1} \in R$ . □



## 5.1 Basic Theorems

**Theorem 5.1.0.1.** Let  $I$  be an ideal of the Commutative Ring  $R$  with unity, and  $(I) \subseteq R[x]$ . Then,  $R[x]/(I) \cong (R/I)[x]$ . In particular, if  $I$  is prime ideal in  $R$ , then  $(I)$  is prime ideal in  $R[x]$ .

*Proof.* First, establish that  $(I) = I[x]$ . Since properties of Ideal,  $(I) = IR[x] = I[x]$  directly.

Now, define a map  $\varphi : R[x] \rightarrow (R/I)[x] : \sum_{i=0}^n a_i x^i \mapsto \sum_{i=0}^n (a_i + I)x^i$ . Then,  $\varphi$  is homomorphism with  $\ker \varphi = I[x]$ .

The first-iso. Thm gives  $R[x]/(I) = R[x]/I[x] \cong (R/I)[x]$ . Particular,

$$I \text{ prime ideal} \iff R/I \text{ integral domain} \implies (R/I)[x] = R[x]/(I) \text{ integral domain} \iff (I) \text{ prime ideal.}$$

□

**Theorem 5.1.0.2.** If  $F$  is a field, then  $F[x]$  is Euclidean domain.

Specifically, assume  $R$  is Commutative Ring with unity,  $f, g \in R[x]$  with  $\deg f, \deg g \geq 0$ .

If leading coefficient of  $g$  is unit in  $R$ , then there exists unique  $q, r \in R[x]$  such that

$$f(x) = g(x)q(x) + r(x) \quad (\deg r(x) < \deg g(x))$$

*Proof.* If  $\deg f < \deg g$ , put  $g(x) = 0$  and  $r(x) = f(x)$ . Then proved. Suppose that  $\deg f \geq \deg g$ , and using induction. If  $\deg f = 0$ , then put  $g(x) = 0$ , write leading coefficient of  $g$  as  $b$  and of  $f$  as  $a$ . Then, put  $q = b^{-1}a$ ,  $r = 0$ . If  $\deg f \geq 1$ , put  $n = \deg f$ ,  $m = \deg g$ . Then  $n \geq m$ . Write:

$$\begin{aligned} f(x) &= a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \\ g(x) &= b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x + b_0 \end{aligned}$$

Then, by induction,

$$\begin{aligned} f(x) &= a_n b_m^{-1} x^{n-m} g(x) + f_1(x) & (\deg f_1 < n-1) \\ &= a_n b_m^{-1} x^{n-m} g(x) + q_1(x)g(x) + r(x) & (\deg r < \deg g) \\ &= (a_n b_m^{-1} x^{n-m} q_1(x))g(x) + r(x) & (\deg r < \deg g) \end{aligned}$$

To show the uniqueness,

$$f = q_1 g + r_1 = q_2 g + r_2 \implies g(q_1 - q_2) = r_2 - r_1 \implies \deg g(q_1 - q_2) = \deg g + \deg(q_1 - q_2) = \deg(r_2 - r_1) < \deg g$$

□

## 5.2 Polynomial Ring over Unique Factorization Domain

### Lemma 5.2.0.1. Gauss's Lemma

Let  $R$  be a Unique Factorization Domain with field of fractions  $F$ .

If  $p(x) \in R[x]$  is reducible in  $F[x]$ , then  $p(x)$  is reducible in  $R[x]$ .

**Proof.** Let  $p(x) \in R[x]$  be reducible in  $F[x]$ . i.e.,

$$p(x) = A(x)B(x) \text{ for some } A(x), B(x) \in F[x] \text{ with } A(x), B(x) \text{ are both non-zero and non-units.}$$

Both  $\deg A$  and  $\deg B$  are at least 1: if either degree were zero, then lie in  $F$ , hence a unit - contradiction.

Write  $A(x) = \sum_{i=0}^n \frac{r_i}{a_i} x^i$  and  $B(x) = \sum_{i=0}^m \frac{s_i}{b_i} x^i$ , and put  $d_1 = a_1 \cdots a_n$ ,  $d_2 = b_1 \cdots b_m$ .

Now,  $d_1 d_2 p(x) = d_1 A(x) d_2 B(x)$  where  $d_1 A(x), d_2 B(x) \in R[x]$ .

If  $d = d_1 d_2$  is unit in  $R$ , then  $p(x) = (d^{-1} d_1 A(x))(d_2 B(x))$  where  $d^{-1} d_1 A(x), d_2 B(x) \in R[x]$ , both are non-unit.

Suppose that  $d$  is not unit. Write  $d = p_1 p_2 \cdots p_n$  is factorization of  $d$ .

$p_1$  is prime, being irreducible in U.F.D.  $(p_1) = p_1 R[x]$  is prime,  $R[x]/p_1 R[x] \cong (R/p_1 R)[x]$  is an integral domain.

Since  $dp(x) = p_1 \cdots p_n p(x) \in p_1 R[x]$ ,

$$\bar{0} = dp(x) + p_1 R[x] = d_1 A(x) d_2 B(x) + p_1 R[x] = \overline{d_1 A(x)} \times \overline{d_2 B(x)}$$

Since  $p_1 R[x]$  is an integral domain, either  $\overline{d_1 A(x)}$  or  $\overline{d_2 B(x)}$  is zero. WLOG, let  $\overline{d_1 A(x)} = d_1 A(x) + p_1 R[x] = \bar{0}$ .

This means all coefficient of  $d_1 A(x)$  lies in  $p_1 R$ . Thus, we can cancel  $p_1$  in the equation  $dp(x) = d_1 A(x) d_2 B(x)$ .

In finite process, we obtain  $p(x) = A'(x)B'(x)$  where  $A'(x), B'(x) \in R[x]$  with

$$A'(x) = rA(x), \quad B'(x) = sB(x) \text{ where } r, s \in F$$

□

**Corollary 5.2.0.1.** Let  $R$  be a Unique Factorization Domain with field of fractions  $F$ .

Suppose that the greatest common divisor of the coefficients of  $p(x) \in R[x]$  is 1. Then,

$$p(x) \text{ is irreducible in } R[x] \text{ if and only if } p(x) \text{ is irreducible in } F[x]$$

In particular, if  $p(x)$  is an irreducible monic polynomial in  $R[x]$ , then it is also irreducible in  $F[x]$ .

**Proof.** By Contraposition of Gauss's Lemma, if  $p(x)$  is irreducible in  $R[x]$ , then  $p(x)$  is irreducible in  $F[x]$ .

Conversely, suppose that  $p(x)$  is reducible in  $R[x]$ , and the greatest common divisor of coefficients of  $p(x)$  is 1.

Write  $p(x) = a(x)b(x)$  where neither  $a(x)$  nor  $b(x)$  are not unit in  $R[x]$ , being reducible.

And, both  $a(x)$  and  $b(x)$  are not constant: because g.c.d. is 1. Thus, both are not unit in  $F[x]$ .

□

**Theorem 5.2.0.1.**  *$R$  is Unique Factorization Domain if and only if  $R[x]$  is Unique Factorization Domain.*

*Proof.* Suppose that  $R$  is Unique Factorization Domain with field of fractions  $F$ .

Let  $p(x) \in R[x]$  be non-zero element, and  $d \in R$  be the greatest common divisor of coefficients of  $p(x)$ .

Then,  $p(x) = dp'(x)$  where g.c.d. of coefficient of  $p'(x)$  is 1. More precisely, write  $p(x) = \sum_{i=0}^n a_i x^i$ , ( $a_i \in R$ ).

$$p(x) = \sum_{i=0}^n a_i x^i = \sum_{i=0}^n da'_i x^i = d \left( \sum_{i=0}^n a'_i x^i \right)$$

for some  $a'_i \in R$  such that  $a_i = da'_i$ . Put g.c.d of  $a'_i$ 's to  $d' \in R$ . Then,  $a_i = da'_i = dd'a''_i$ .

This implies  $dd'$  divides every  $a_i$ ; hence  $dd'$  divides  $d$ . That is,  $d'$  is unit, thus  $d'$  must be 1.

Since  $F[x]$  is U.F.D, let  $p'(x) = p_1(x)p_2(x) \cdots p_n(x)$  be a factorization of  $p(x)$  in  $F[x]$ .

The g.c.d of  $p'(x)$  is 1, thus g.c.d. of each  $p_i(x)$  is 1.

Now, the corollary of the Gauss's Lemma gives that every  $p_i(x)$  is irreducible in  $R[x]$ .

□

## Chapter 6

# Field Theory

## Chapter 7

# Galois Theory

## Chapter 8

# Linear Algebra

**Chapter 9**

**Category**

## Chapter 10

# General Topology

In this chapter, we follow the notations of [Steen et al., 1978, COUNTEREXAMPLES IN TOPOLOGY].

### 10.1 Basis

#### 10.1.1 Subbasis

**Definition 10.1.1.1.** Let  $X$  be a set.

A collection  $\mathcal{S} \subseteq \mathcal{P}(X)$  is called *subbasis* if:  $X = \bigcup_{S \in \mathcal{S}} S$ . (That is,  $\forall x \in X, \exists S \in \mathcal{S}$  s.t.  $x \in S$ )

$\beta_{\mathcal{S}}$  is called *Basis generated by the subbasis  $\mathcal{S}$* .

Note that:  $\tau_{\beta_{\mathcal{S}}}$  is the smallest Topology such that containing  $\mathcal{S}$ .



## 10.2 Coproduct Space

**Definition 10.2.0.1.** Let  $(X_\alpha, \mathcal{T}_\alpha)$  ( $\alpha \in \Lambda$ ) are mutually disjoint Topological Space. Define a *Coproduct Topology*  $(X_\Pi, \mathcal{T}_\Pi)$ :

$$X_\Pi \stackrel{\text{def}}{=} \bigsqcup_{\alpha \in \Lambda} X_\alpha, \quad \mathcal{T}_\Pi \stackrel{\text{def}}{=} \left\{ \bigsqcup_{\alpha \in \Lambda} \mathcal{U}_\alpha \mid \mathcal{U}_\alpha \in \mathcal{T}_\alpha \right\}$$

This actually be a Topology:

1.  $\emptyset, X_\Pi \in \mathcal{T}_\Pi$  is clear,
2. Closed under union is clear.
3. Closed under finite intersection, not infinite.

*Proof.* Proof of 3.

Let a finite collection

$$\left\{ \bigsqcup_{\alpha \in \Lambda} \mathcal{U}_\alpha^1, \bigsqcup_{\alpha \in \Lambda} \mathcal{U}_\alpha^2, \dots, \bigsqcup_{\alpha \in \Lambda} \mathcal{U}_\alpha^k \right\}$$

be given. Then, their intersection be:

$$\bigcap_{j=1}^k \bigsqcup_{\alpha \in \Lambda} \mathcal{U}_\alpha^j = \bigsqcup_{\alpha \in \Lambda} \bigcap_{j=1}^k \mathcal{U}_\alpha^j \in \mathcal{T}_\Pi$$

□

**Theorem 10.2.0.1.** Let  $X_1, X_2, X_3$  and  $Y_1, Y_2, Y_3$  are mutually disjoint Topological Space, and for each  $i = 1, 2, 3$ ,

$$f_i : X_i \rightarrow Y_i : x \mapsto f_i(x)$$

Define a function

$$f = f_1 \amalg f_2 \amalg f_3 : \bigsqcup_{i=1}^3 X_i \rightarrow \bigsqcup_{i=1}^3 Y_i : x \mapsto \begin{cases} f_1(x) & x \in X_1 \\ f_2(x) & x \in X_2 \\ f_3(x) & x \in X_3 \end{cases}$$

where both Domain and Codomain are Coproduct Space. (Clearly this function is well-defined.)

Suppose that:

1.  $f_1$  is Open map, Closed map
2.  $f_2$  is Continuous map, Open map
3.  $f_3$  is Continuous map, Closed map

Then, The Followings hold:

1.  $f_1$  is Continuous map if and only if  $f$  is Continuous map.
2.  $f_2$  is Open map if and only if  $f$  is Open map.
3.  $f_3$  is Closed map if and only if  $f$  is Closed map.

*Proof.*

1. It follows that: For any open on Codomain  $U \in \mathcal{T}_{Y_\Pi}$ ,

$$\begin{aligned} f^{-1}[U] &= \{x \in X \mid f(x) \in U\} = \{x \in X_1 \mid f_1(x) \in U\} \cup \{x \in X_2 \mid f_2(x) \in U\} \cup \{x \in X_3 \mid f_3(x) \in U\} \\ &= f_1^{-1}[U] \cup f_2^{-1}[U] \cup f_3^{-1}[U] \end{aligned}$$

Thus, If  $f_1$  is Continuous, then  $f$  is Continuous map since  $f^{-1}[U]$  is the union of open sets.

And, If  $f$  is Continuous, then  $f^{-1}[U] \cap X_1$  be Open set and it is equal that  $(f_1^{-1}[U] \cup f_2^{-1}[U] \cup f_3^{-1}[U]) \cap X_1 = f_1^{-1}[U]$ .

2. It follows that: For any open on Domain  $U \in \mathcal{T}_{X_{II}}$ ,

$$f[U] = f_1[U] \cup f_2[U] \cup f_3[U]$$

This, if  $f_2$  is Open map, then  $f$  is Open map since  $f[U]$  is the union of open sets.

And, If  $f$  is Open, then  $f[U] \cap Y_2$  be Open set and it is equal that  $(f_1[U] \cup f_2[U] \cup f_3[U]) \cap Y_2 = f_2[U]$ .

3. Similar to the above. □

For a specific example, Define for each  $i = 1, 2, 3$ ,

$$X_i \stackrel{\text{def}}{=} \{a_i, b_i\}, \quad \begin{cases} \mathcal{T}_{i,D} \stackrel{\text{def}}{=} \{\emptyset, X_i, \{a_i\}, \{b_i\}\} \\ \mathcal{T}_{i,I} \stackrel{\text{def}}{=} \{\emptyset, X_i\} \\ \mathcal{T}_{i,a} \stackrel{\text{def}}{=} \{\emptyset, X_i, \{a_i\}\} \\ \mathcal{T}_{i,b} \stackrel{\text{def}}{=} \{\emptyset, X_i, \{b_i\}\} \end{cases}$$

And define functions

1.  $f_1 : (X_1, \mathcal{T}_{1,I}) \rightarrow (X_1, \mathcal{T}_{1,D}) : x \mapsto x$  is Not Continuous, Open, Closed.
2.  $f_2 : (X_2, \mathcal{T}_{2,a}) \rightarrow (X_2, \mathcal{T}_{2,a}) : x \mapsto a_2$  is Continuous, Open, Not Closed.
3.  $f_3 : (X_1, \mathcal{T}_{3,a}) \rightarrow (X_1, \mathcal{T}_{3,b}) : x \mapsto a_3$  is Continuous, Not Open, Closed.
4.  $g_i : (X_i, \mathcal{T}_{i,D}) \rightarrow (X_i, \mathcal{T}_{i,D}) : x \mapsto x$  is Continuous, Open, Closed for each  $i = 1, 2, 3$ .

Now, from the above discussion,

1.  $g_1 \amalg g_2 \amalg g_3$  is Continuous, Open, Closed.
2.  $f_1 \amalg g_2 \amalg g_3$  is Not Continuous, Open, Closed.
3.  $g_1 \amalg f_2 \amalg g_3$  is Continuous, Not Open, Closed.
4.  $g_1 \amalg g_2 \amalg f_3$  is Continuous, Open, Not Closed.
5.  $f_1 \amalg f_2 \amalg f_3$  is Not Continuous, Not Open, Not Closed.
6.  $g_1 \amalg f_2 \amalg f_3$  is Continuous, Not Open, Not Closed.
7.  $f_1 \amalg f_2 \amalg g_3$  is Not Continuous, Not Open, Closed.
8.  $f_1 \amalg g_2 \amalg f_3$  is Not Continuous, Open, Not Closed.

No.	Map	Continuous	Open	Closed
1	$g_1 \amalg g_2 \amalg g_3$	Yes	Yes	Yes
2	$f_1 \amalg g_2 \amalg g_3$	No	No	No
3	$g_1 \amalg f_2 \amalg g_3$	Yes	No	Yes
4	$g_1 \amalg g_2 \amalg f_3$	Yes	Yes	No
5	$f_1 \amalg f_2 \amalg f_3$	No	No	No
6	$g_1 \amalg f_2 \amalg f_3$	Yes	No	No
7	$f_1 \amalg f_2 \amalg g_3$	No	No	Yes
8	$f_1 \amalg g_2 \amalg f_3$	No	Yes	No

## 10.3 Compact Space

**Definition 10.3.0.1.** A Topological Space  $X$  is *compact* if: every open cover contains a finite subcover. i.e.,

$$\text{If } X = \bigcup_{\alpha \in \Lambda} \mathcal{U}_\alpha, (\mathcal{U}_\alpha \in \mathcal{T}), \text{ then there is finite subcover such that } X = \bigcup_{i=1}^N \mathcal{U}_{\alpha_i}$$

This is equivalent with:

$$\text{If } \emptyset = \bigcap_{\alpha \in \Lambda} \mathcal{C}_\alpha, (\mathcal{C}_\alpha \text{ closed}), \text{ then there is finite subset such that } \emptyset = \bigcap_{i=1}^N \mathcal{C}_{\alpha_i}$$

**Definition 10.3.0.2.** Let  $X$  be a set.  $A \subset \mathcal{P}(X)$  satisfies *finite intersection property* if:

$$\text{For all finite subset of } A, \{A_i \mid i = 1, 2, \dots, n\} \subset A \text{ satisfies } \bigcap_{i=1}^n A_i \neq \emptyset.$$

**Example.** 1.  $X = \mathbb{R}$ , and let  $A = \{(n, \infty) \mid n \in \mathbb{N}\}$ . Then,

$$\bigcap_{S \in A} S = \emptyset, \quad \bigcap_{\substack{S \in F \subset A \\ |F| < \infty}} S \neq \emptyset$$

2.  $X = \mathbb{R}$ , and let  $A = \{\mathbb{R} \setminus F \mid |F| < \aleph_0\}$ .

**Theorem 10.3.0.1.** Let  $X$  be a Topological Space, Then, TFAE:

a)  $X$  is Compact Space.

b) If  $A$  is a collection of closed subsets of  $X$  that satisfies *FID*, then  $\bigcap_{C \in A} C \neq \emptyset$ .

c) If  $A$  is a collection of subsets of  $X$  that satisfies *FID*, then  $\bigcap_{S \in A} \bar{S} \neq \emptyset$ .

**Proof.** a)  $\implies$  b). **Proof by Contradiction:**

Suppose that  $A \subset \mathcal{P}(X)$  be a collection of closed subsets such that *FID*.

Assume that  $\bigcap_{C \in A} C = \emptyset$ . Since  $X$  is Compact,

$$\emptyset = \bigcap_{C \in A} C \text{ if and only if } X = \bigcup_{C \in A} (X \setminus C), \text{ where } X \setminus C \text{ is open.}$$

This implies that there is a finite subcover:

$$X = \bigcup_{i=1}^N (X \setminus C_i) \text{ if and only if } \emptyset = \bigcap_{i=1}^N C$$

This is Contradiction with  $A$  satisfies *FID*.

b)  $\implies$  a). **Proof by Contraposition:**

Suppose that  $X$  is not Compact. Then, there exists an Open Cover  $\mathcal{O}$  with no finite subcover: i.e.,

$$X = \bigcup_{\mathcal{U} \in \mathcal{O}} \mathcal{U} \text{ if and only if } \emptyset = \bigcap_{\mathcal{U} \in \mathcal{O}} (X \setminus \mathcal{U})$$

And,

$$\text{For any finite subset of } \mathcal{O}, F = \{\mathcal{U}_i \mid i = 1, \dots, N\} \text{ satisfies } X \supsetneq \bigcup_{i=1}^N \mathcal{U}_i \text{ if and only if } \emptyset \neq \bigcap_{i=1}^N (X \setminus \mathcal{U}_i)$$

Thus,  $\mathcal{K} = \{X \setminus \mathcal{U} \mid \mathcal{U} \in \mathcal{O}\}$  satisfies *FID*, but  $\emptyset = \bigcap_{\mathcal{U} \in \mathcal{O}} (X \setminus \mathcal{U}) = \bigcap_{\mathcal{C} \in \mathcal{K}} \mathcal{C}$ . Thus, not *a*) implies not *b*). □

**Theorem 10.3.0.2.** Let  $X$  is Compact Space,  $Y$  is Topological Space.  
If  $f: X \rightarrow Y$  is Continuous Map, then  $f[X]$  is Compact.

*Proof.* Let  $\mathcal{O}$  be an open cover of  $f[X]$ . i.e,  $f[X] \subset \bigcup_{\mathcal{U} \in \mathcal{O}} \mathcal{U}$ . Now,

$$X \subset f^{-1}[f[X]] \subset f^{-1} \left[ \bigcup_{\mathcal{U} \in \mathcal{O}} \mathcal{U} \right] = \bigcup_{\mathcal{U} \in \mathcal{O} \text{ open, } f \text{ conti.}} f^{-1}[\mathcal{U}]$$

Since  $X$  is compact, there is a finite subcover such that

$$X \subset \bigcup_{i=1}^N f^{-1}[\mathcal{U}_i]$$

Consequently,

$$f[X] \subset f \left[ \bigcup_{i=1}^N f^{-1}[\mathcal{U}_i] \right] = \bigcup_{i=1}^N f[f^{-1}[\mathcal{U}_i]] \subset \bigcup_{i=1}^N \mathcal{U}_i$$

□

**Theorem 10.3.0.3.** Closed set of compact space is compact.

*Proof.* Let  $X$  be a compact, and  $E \subset X$  be a closed subset. Let  $\mathcal{O}$  be an open over of  $E$ .  
Then,

$$X = E \cup (X \setminus E) \subset \left( \bigcup_{\mathcal{U} \in \mathcal{O}} \mathcal{U} \right) \cup (X \setminus E)$$

be an open cover of  $X$ . Thus, there is a finite subcover such that

$$X = \left( \bigcup_{i=1}^N \mathcal{U}_i \right) \cup (X \setminus E) \iff E \subset \bigcup_{i=1}^N \mathcal{U}_i$$

□

**Theorem 10.3.0.4.** Let  $X$  be a Topological Space, and  $\beta$  be a basis of  $X$ . Then, TFAE:

- a)  $X$  is Compact Space.
- b) Every open cover consisting of basis elements has a finite subcover.

*Proof.* *a*)  $\implies$  *b*). Clear by definition of Compact.

*b*)  $\implies$  *a*). Let  $\{\mathcal{U}_\alpha \mid \alpha \in \Lambda\}$  be an Open cover of  $X$ . That is,

$$X = \bigcup_{\alpha \in \Lambda} \mathcal{U}_\alpha = \bigcup_{\alpha \in \Lambda} \bigcup_{\gamma \in \Gamma_\alpha} B_\alpha^\gamma$$

where  $\{B_\alpha^\gamma \mid \gamma \in \Gamma_\alpha\}$  is subset of basis such that  $\bigcup_{\gamma \in \Gamma_\alpha} B_\alpha^\gamma = \mathcal{U}_\alpha$ . Now, by 2), there is finite subcover such that

$$X = \bigcup_{i=1}^n \bigcup_{j=1}^m B_{\alpha_i}^{\gamma_j} \subset \bigcup_{i=1}^n \mathcal{U}_{\alpha_i}$$

Thus,  $\{\mathcal{U}_{\alpha_i} \mid i = 1, 2, \dots, n\}$  be a finite subcover. □

**Theorem 10.3.0.5.** Let  $X, Y$  are Topological Space. Then, TFAE:

- a)  $X \times Y$  is Compact.
- b)  $X$  and  $Y$  both are Compact.

**Proof.** a)  $\implies$  b) is clear since projection preserves Compactness.

b)  $\implies$  a) Let  $\mathcal{O} \stackrel{\text{def}}{=} \{U \times V \mid U \in \mathcal{T}_X, V \in \mathcal{T}_Y\}$  be an Open cover of  $X \times Y$ .

Let  $x \in X$  fix. Then,  $\{x\} \times Y$  be a Compact, being  $\{x\} \times Y \cong Y$  by Homeomorphism given by Projection. Then, there is a finite subcover of  $\mathcal{O}$  such that

$$\{x\} \times Y \subset \bigcup_{i=1}^{n_x} (U_i^x \times V_i^x)$$

Now, for each  $x \in X$ , define  $U^x \stackrel{\text{def}}{=} \bigcup_{i=1}^{n_x} U_i^x$ . Then,  $U^x$  is an open set containing  $x$ , and for any  $i = 1, 2, \dots, n_x$ ,  $U^x \subset U_i^x$ .

Since  $\{U^x \mid x \in X\}$  be an open cover of  $X$ , there is a finite subcover such that

$$X = \bigcup_{i=1}^m U^{x_i}$$

being  $X$  is Compact. Now,

$$X \times Y = \left( \bigcup_{i=1}^m U^{x_i} \right) \times Y = \bigcup_{i=1}^m (U^{x_i} \times Y) \subset \bigcup_{i=1}^m \bigcup_{j=1}^{n_{x_i}} (U_j^{x_i} \times V_j^{x_i})$$

Thus,  $\{U_j^{x_i} \times V_j^{x_i} \mid i = 1, 2, \dots, m, j = 1, 2, \dots, n_{x_i}\}$  be a finite subcover. □

#### Tube Lemma

Let  $X$  be a Topological Space, and  $Y$  is Compact Space.

Then, for product space  $X \times Y$ , and fixed  $x_0 \in X$ , following statement holds:

For any open  $N \subset X \times Y$  with  $\{x_0\} \times Y \subset N$ , there is an open  $W \in \mathcal{T}_X$  such that  $\{x_0\} \times Y \subset W \times Y \subset N$ .

**Proof.** Clearly,  $\{x_0\} \times Y$  compact, being  $\{x_0\} \times Y \cong Y$ .

For any  $y \in Y$ ,  $(x_0, y) \in \{x_0\} \times Y \subset N$ , thus there exist opens  $U \in \mathcal{T}_X$  and  $V \in \mathcal{T}_Y$  such that  $(x_0, y) \in U \times V \subset N$ . Now, Clearly  $\{U_y \times V_y \subset X \times Y \mid y \in Y\}$  be an open cover of  $\{x_0\} \times Y$ , thus there is a finite subcover such that

$$\{x_0\} \times Y \subset \bigcup_{i=1}^N (U_{y_i} \times V_{y_i}) \subset N$$

Set  $W = \bigcap_{i=1}^N U_{y_i}$ . Then, clearly  $x_0 \in W$ , and

Let  $(x, y) \in W \times Y$ . Then, since  $Y = \bigcup_{i=1}^n V_{y_i}$ , there is  $1 \leq k \leq n$  such that  $y \in V_{y_k}$ .

Thus,  $(x, y) \in U_{y_k} \times V_{y_k} \subset N$ , this implies  $W \times Y \subset N$ . □

**Theorem 10.3.0.6.** Let  $Y$  be a Compact Space. Then, the following statements are true, but their converses are false:

1. If  $X$  be a Lindelöf Space, then the product Topology  $X \times Y$  be a Lindelöf Space.
2. If  $X$  be a Countable Compact Space, then the product Topology  $X \times Y$  be a Countable Compact Space.

*Proof.* 1. Let  $\mathcal{O}$  be an open cover of  $X \times Y$ .

For any  $x \in X$ ,  $\{x\} \times Y$  is compact set, being  $\{x\} \times Y \simeq Y$ . Thus, there is a finite subcover of  $\mathcal{O}$  such that

$$\{x\} \times Y \subset \bigcup_{j=1}^{N_x} U_j^x \quad (U_j^x \in \mathcal{O})$$

Since Tube Lemma, there is an open  $W_x \in \mathcal{T}_X$  such that

$$\{x\} \times Y \subset W_x \times Y \subset \bigcup_{j=1}^{N_x} U_j^x$$

Meanwhile, since  $X$  is Lindelöf, therefore for an open cover  $\{W_x \mid x \in X\}$  there exists a Countable subcover such that

$$X \subset \bigcup_{i=1}^{\infty} W_{x_i}$$

Consequently,

$$X \times Y \subset \left( \bigcup_{i=1}^{\infty} W_{x_i} \right) \times Y \subset \bigcup_{i=1}^{\infty} (W_{x_i} \times Y) \subset \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{N_{x_i}} U_j^{x_i}$$

Now,  $\{U_j^{x_i} \mid i \in \mathbb{N}, 1 \leq j \leq N_{x_i}\} \subset \mathcal{O}$  be a Countable Open Cover of  $X \times Y$ . □

*Proof.* 2. Let  $\{U_n \subset X \times Y \mid n \in \mathbb{N}\}$  be a Countable open cover of  $X \times Y$ . For each finite subset  $F \subset \mathbb{N}$ , define

$$V_F \stackrel{\text{def}}{=} \left\{ x \in X \mid \{x\} \times Y \subset \bigcup_{n \in F} U_n \right\}$$

Then  $V_F$  satisfies:

1)  $V_F$  is open: Let a finite subset  $F \subset \mathbb{N}$  fix. For each  $x \in V_F$ ,  $\{x\} \times Y \subset \bigcup_{n \in F} U_n$  by definition.

Then, there is an open  $W_x \in \mathcal{T}_X$  such that  $\{x\} \times Y \subset W_x \times Y \subset \bigcup_{n \in F} U_n$  by Tube Lemma.

Meanwhile,  $W_x \subset V_F$  because for all  $s \in W_x$ ,  $\{s\} \times Y \subset W_x \times Y \subset \bigcup_{n \in F} U_n$ , thus  $s \in V_F$ .

In summary, for any  $x \in V_F$ , there is an open  $W_x \in \mathcal{T}_X$  such that  $x \in W_x \subset V_F$ . Consequently,  $V_F$  is open of  $X$ .

2)  $\{V_F \mid F \subset \mathbb{N}, |F| < \infty\}$  is a Countable Open Cover of  $X$ :

Countability given by above set is collection of subsets of Countable set. Meanwhile,

For any  $x \in X$ , there is a finite subcover of  $\{U_n \mid n \in \mathbb{N}\}$  such that  $\{x\} \times Y \subset \bigcup_{n \in F} U_n$  where  $F$  finite.

That is,  $x \in V_F$ . Now, the open cover of  $X$ ,

$$\{V_{F_x} \mid x \in X\} \subset \{V_F \mid F \subset \mathbb{N}\}$$

at most Countable. Since  $X$  is Countably Compact Space, there is a finite subcover such that

$$X \subset \bigcup_{i=1}^N V_{F_i}$$

Consequently,

$$X \times Y \subset \left( \bigcup_{i=1}^N V_{F_i} \right) \times Y = \bigcup_{i=1}^N (V_{F_i} \times Y) \subset \bigcup_{i=1}^N \bigcup_{n \in F_i} U_n$$

That is,  $\{U_i \mid i = 1, 2, \dots, N, n \in F_i\}$  be a finite subcover. □

### 10.3.1 Locally Compact

**Definition 10.3.1.1.** A Space  $X$  is called *Locally Compact* if:

For any  $x \in X$ , there exist open  $U$  and compact  $C$  such that  $x \in U \subseteq C$ .

**Lemma 10.3.1.1.** Let  $X$  be a Hausdorff Space. TFAE:

1.  $X$  is Locally-compact space.
2. For any  $x \in X$ , there exists an open  $U$  with  $x \in U$  such that the closure  $\overline{U}$  is Compact in  $X$ .

### 10.3.2 One-point Compactification

**Definition 10.3.2.1.** Let  $(X, \mathcal{T})$  be a Space.

Define  $X_\infty \stackrel{\text{def}}{=} X \sqcup \{\infty\}$  and  $\mathcal{T}_\infty \stackrel{\text{def}}{=} \mathcal{T} \sqcup \{U \subseteq X_\infty \mid \infty \in U, X_\infty \setminus U \text{ is compact in } X\}$ .

This  $(X_\infty, \mathcal{T}_\infty)$  is called **one-point compactification** of  $X$ .

**Theorem 10.3.2.1.** Let  $(X, \infty)$  be a Locally-Compact Hausdorff Space, but not Compact.

Then, one-point compactification  $(X_\infty, \mathcal{T}_\infty)$  of  $X$  is Compact Hausdorff Space.

*Proof.* This proof consisted of five steps.

1). Claim:  $\mathcal{T}_\infty$  is Topology on  $X_\infty$ . (Using  $X$  is Hausdorff)

Let  $U_\gamma \in \Gamma$ ,  $(\gamma \in \Gamma)$  be elements of  $\mathcal{T}_\infty$ .

Define  $\Gamma_1 \stackrel{\text{def}}{=} \{\alpha \in \Gamma \mid U_\alpha \in \mathcal{T}\}$ , and  $\Gamma_2 \stackrel{\text{def}}{=} \Gamma \setminus \Gamma_1 = \{\beta \in \Gamma \mid \infty \in U_\beta, X_\infty \setminus U_\beta \text{ is compact in } X\}$ .

Then,  $\bigcup_{\gamma \in \Gamma} U_\gamma = \left( \bigcup_{\alpha \in \Gamma_1} U_\alpha \right) \cup \left( \bigcup_{\beta \in \Gamma_2} U_\beta \right)$ . The left term is open in  $X$  clearly.

And, put  $C_\beta = X_\infty \setminus U_\beta$  for each  $\beta \in \Gamma_2$ . Then,  $C_\beta$  is Compact in  $X$  by definition, thus closed by  $X$  is Hausdorff.

$$\bigcup_{\beta \in \Gamma_2} U_\beta = \bigcup_{\beta \in \Gamma_2} X_\infty \setminus C_\beta = X_\infty \setminus \left( \bigcap_{\beta \in \Gamma_2} C_\beta \right)$$

This intersection of  $C_\beta$  is compact, being any intersection of closed is closed and closed subset of compact. That is, it is compact in  $X$ , therefore this union of  $U_\beta$  is contained in  $\mathcal{T}_\infty$ .

Let  $U_1, U_2 \in \mathcal{T}$ , and  $V_1, V_2 \in \mathcal{T}_\infty \setminus \mathcal{T}$ . Put  $C_i \stackrel{\text{def}}{=} X_\infty \setminus V_i$ ,  $(i = 1, 2)$ . Then,  $C_i$  is compact. Now,

$$U_1 \cap U_2 \in \mathcal{T} \subset \mathcal{T}_\infty$$

$$U_1 \cap V_1 = U_1 \cap (X_\infty \setminus C_1) = U_1 \cap X_\infty \cap C_1^c = U_1 \cap C_1^c = U_1 \setminus C_1 \in \mathcal{T} \subset \mathcal{T}_\infty$$

$$V_1 \cap V_2 = (X_\infty \setminus C_1) \cap (X_\infty \setminus C_2) = X_\infty \setminus (C_1 \cap C_2) \in \mathcal{T}_\infty$$

Thus closed under the arbitrary union and finite intersection.

2). Claim:  $(X, \mathcal{T})$  is a Subspace of  $(X_\infty, \mathcal{T}_\infty)$ . That is,  $\mathcal{T} = \{U \cap X \mid U \in \mathcal{T}_\infty\}$ . (Using  $X$  is Hausdorff)

The right inclusion is clear:  $U \in \mathcal{T} \implies U \in \mathcal{T}_\infty$ . Thus  $U = X \cap U \in \{U \cap X \mid U \in \mathcal{T}_\infty\}$ .

To show the left inclusion: Let  $U \in \mathcal{T}_\infty$ . If  $U \in \mathcal{T}$ , then  $X \cap U = U \in \mathcal{T}$ .

If  $U \notin \mathcal{T}$ , then  $X_\infty \setminus U$  is compact in  $X$ . Now,  $X \cap U = X \setminus (X_\infty \setminus U) \in \mathcal{T}$ .

compact in  $T_2 \implies$  closed

3). Claim:  $\overline{X} = X_\infty$ . That is, closure of  $X$  is  $X_\infty$ . (Using  $X$  is not compact)

Let  $U \in \mathcal{T}_\infty$  with  $\infty \in U$ . Then,  $X_\infty \setminus U$  is compact of  $X$ , thus  $X_\infty \setminus U \subsetneq X$  because  $X$  is not compact.

4). Claim:  $X_\infty$  is Compact Space.

Let  $\mathcal{O} = \{U_\alpha \mid \alpha \in \Lambda\}$  be an open cover of  $X_\infty$ . Since  $\infty \in X_\infty = \bigcup_{\alpha \in \Lambda} U_\alpha$ , there is  $\alpha_0 \in \Lambda$  such that  $\infty \in U_{\alpha_0}$ .

$C \stackrel{\text{def}}{=} X_\infty \setminus U_{\alpha_0}$  is compact in  $X$ , thus so in  $X_\infty$ . And,  $C \subseteq \bigcup_{\alpha \in \Lambda \setminus \{\alpha_0\}} U_\alpha$ , thus there is finite subcover of  $C$ .

Finally, union of finite subcover of  $C$  and  $U_{\alpha_0}$  is finite subcover of  $X_\infty$ .

5). Claim:  $X_\infty$  is Hausdorff. (Using  $X$  is Locally-Compact)

Let  $x, y \in X_\infty$ . If both  $x, y$  are contained  $X$ , then there is nothing to prove, being  $X$  is hausdorff.

If  $x \in X$  and  $y = \infty$ , then there is open  $U$  and compact  $C$  of  $X$  such that  $x \in U \subseteq C$ , by Locaaly-Compact.

Now,  $x \in U$  and  $\infty \in X_\infty \setminus C$ , both are open of  $X_\infty$  with  $U \cap (X_\infty \setminus C) = \emptyset$ . □



### 10.3.3 Stereographic projection

## 10.4 Borel Set

**Definition 10.4.0.1.** Let  $X$  be a Topological Space.

1.  $F \subseteq X$  is called  $F_\sigma$ -set if:  $F$  can be represented as countable union of closed sets.
2.  $G \subseteq X$  is called  $G_\delta$ -set if:  $G$  can be represented as countable intersection of open sets.

**Proposition 10.4.0.1.** Let  $X$  be a Topological Space.

1. If  $F \subseteq X$  is  $F_\sigma$ -set, then there exists sequence of closed sets  $\{F_n\}_{n \in \mathbb{N}}$  such that

$$F = \bigcup_{n=1}^{\infty} F_n, \quad F_1 \subseteq F_2 \subseteq \cdots \subseteq F_n \subseteq \cdots$$

2. If  $G \subseteq X$  is  $G_\delta$ -set, then there exists sequence of open sets  $\{G_n\}_{n \in \mathbb{N}}$  such that

$$G = \bigcap_{n=1}^{\infty} G_n, \quad G_1 \supseteq G_2 \supseteq \cdots \supseteq G_n \supseteq \cdots$$

3. Countable union of  $F_\sigma$ -sets is  $F_\sigma$ .
4. Finite intersection of  $F_\sigma$ -sets is  $F_\sigma$ .
5. Countable intersection of  $G_\delta$ -sets is  $G_\delta$ .
6. Finite union of  $G_\delta$ -sets is  $G_\delta$ .
7. Complement of  $F_\sigma$ -set is  $G_\delta$ .

## 10.5 Baire Category

**Definition 10.5.0.1.** The Topological Space  $X$  is called *Baire Space* if:

If  $\{G_n \mid n \in \mathbb{N}\}$  be a Countable Collection of dense open sets of  $X$ , then  $\overline{\bigcap_{n=1}^{\infty} G_n} = X$

In brief, every Countable intersection of dense open sets be dense in  $X$ .

**Definition 10.5.0.2.** Let  $X$  be a Topological Space.

$A \subset X$  is said to be *nowhere dense subset* if  $(\overline{A})^\circ = \emptyset$ .

1.  $B \subset X$  is called *first category* if  $B$  can be representative by union of countable nowhere dense subsets.
2. If the subset is not first category, then it is said to be *second category*.

## 10.6 Locally Compact Hausdorff Space

**Theorem 10.6.0.1.** Locally Compact Hausdorff Space is Baire Space.

## 10.7 Complete Metric Space

**Definition 10.7.0.1.** Let  $(X, d)$  be a Metric Space, and  $\{p_n\}$  be a Sequence in  $X$ .

The Sequence  $\{p_n\}$  is called *Cauchy Sequence* if:

For any  $\epsilon > 0$ , there exists  $N \in \mathbb{N}$  such that  $m, n \geq N \implies d(p_m, p_n) < \epsilon$ .

A Metric Space  $(X, d)$  is said to be *Complete* if every Cauchy Sequences Converge.

**Lemma 10.7.0.1.** Let  $\{E_n\}$  be a sequence of closed bounded non-empty subsets in a Complete Metric Space  $X$  such that

$E_n \supset E_{n+1}$ . If  $\lim_{n \rightarrow \infty} \text{diam} E_n = 0$ , then  $\bigcap_{n=1}^{\infty} E_n = \{p\}$  for some  $p \in X$ .

**Proof.** For each  $n \in \mathbb{N}$ , construct  $p_n \in E_n$ .

Let  $\epsilon > 0$  be given. Since  $\text{diam} E_n \rightarrow 0$ , there is  $N \in \mathbb{N}$  such that  $\text{diam} E_n < \epsilon$ .

For any  $m, n \geq N$ ,  $E_N$  contains  $p_m, p_n$ . That is,  $d(p_m, p_n) < \epsilon$ . Thus,  $\{p_n\}$  be a Cauchy sequence of  $X$ .

Since  $X$  is complete, there is a unique point  $p \in X$  such that  $p_n \rightarrow p$ . Let  $N \in \mathbb{N}$  be an integer such that  $n \geq N \implies |p_n - p| < \epsilon$ .

Now, for each  $n \geq N$ ,  $E_n$  has a limit point as  $p$ . And for any  $n \in \mathbb{N}$ ,  $E_n$  contains  $E_N, E_{N+1}, \dots$ , thus for all  $n \in \mathbb{N}$ ,  $E_n$  has a limit point as  $p$ . Meanwhile,  $E_n$  closed,  $p \in E_n, \forall n \in \mathbb{N}$ .

Consequently,  $p \in \bigcap_{n=1}^{\infty} E_n$ . If there is  $q \in X$  such that  $p \neq q$ ,  $q \in \bigcap_{n=1}^{\infty} E_n$ . Then,  $\text{diam} E_n \geq d(p, q) > 0, \forall n \in \mathbb{N}$ .  $\square$

**Theorem 10.7.0.1.** Complete Metric Space is Baire Space.

**Proof.** Suppose that  $\{G_n \mid n \in \mathbb{N}\}$  be a Countable Collection of dense open set of Complete Metric Space.

Let an open  $U \in \mathcal{T}$  be given. Since  $G_n$  is dense in the Space,  $U \cap G_1$  is non-empty open set.

Thus, there exists a  $p_1 \in U \cap G_1$  such that for some  $r_1 > 0$ ,  $B_{r_1}(p_1) \subset U \cap G_1$ .

Then, automatically,

$$B_{r_1}(p_1) \subset \overline{B_{r_1}(p_1)} \subset B_{r_1}(p_1) \subset U \cap G_1$$

Set  $E_1 = U$ ,  $E_2 = B_{\frac{r_1}{2}}(p_1)$ .

Suppose that  $E_1, \dots, E_{n-1}$  are chosen. Then, since  $E_{n-1} \cap G_{n-1}$  is open, being intersection of opens.

Thus there exists a point  $p_{n-1} \in E_{n-1} \cap G_{n-1}$  and exists  $r_{n-1}$  such that

$$B_{r_{n-1}}(p_{n-1}) \subset E_{n-1} \cap G_{n-1} \subset E_{n-1}$$

This implies that

$$B_{\frac{r_{n-1}}{2}}(p_{n-1}) \subset \overline{B_{\frac{r_{n-1}}{2}}(p_{n-1})} \subset B_{r_{n-1}}(p_{n-1}) \subset E_{n-1} \cap G_{n-1} \subset E_{n-1}$$

Set  $E_n = B_{\frac{r_{n-1}}{2}}(p_{n-1})$ . Since inductively construction of  $\{E_n\}$ ,  $E_{n+1} \subset E_n$  and  $\overline{E_n} \subset G_n$  for all  $n \in \mathbb{N}$ . Consequently,

$$U \cap \left( \bigcap_{n=1}^{\infty} G_n \right) = \bigcap_{n=1}^{\infty} (U \cap G_n) \supset \bigcap_{n=1}^{\infty} (U \cap \overline{E_n}) = U \cap \left( \bigcap_{n=1}^{\infty} \overline{E_n} \right) = \bigcap_{n=1}^{\infty} \overline{E_n} \neq \emptyset$$

□

### 10.7.1 Nowhere Differentiable function

**Theorem 10.7.1.1.** Let  $\mathcal{C}[\mathbb{R}] \stackrel{\text{def}}{=} \{f : \mathbb{R} \rightarrow \mathbb{R} \mid f \text{ is continuous.}\}$  and  $d : \mathcal{C}[\mathbb{R}] \times \mathcal{C}[\mathbb{R}] \rightarrow \mathbb{R} : (f, g) \mapsto \sup_{t \in \mathbb{R}} |f(t) - g(t)|$ . Then,  $(\mathcal{C}[\mathbb{R}], d)$  is Complete Metric Space, and set of Nowhere-Differentiable functions is dense in  $\mathcal{C}[\mathbb{R}]$ .

*Proof.* First, show that  $d$  satisfies triangle inequality: let  $f, g, h \in \mathcal{C}[\mathbb{R}]$  be given.

For any  $t \in \mathbb{R}$ ,  $|f(t) - g(t)| \leq |f(t) - h(t)| + |h(t) - g(t)|$ . Thus,

$$d(f, g) = \sup_{t \in \mathbb{R}} |f(t) - g(t)| \leq \sup_{t \in \mathbb{R}} [|f(t) - h(t)| + |h(t) - g(t)|] \leq \sup_{t \in \mathbb{R}} |f(t) - h(t)| + \sup_{t \in \mathbb{R}} |h(t) - g(t)| = d(f, h) + d(h, g)$$

□

## 10.7.2 Banach Fixed Point Theorem

**Definition 10.7.2.1.** Let  $f : X \rightarrow X$  be any function. A point  $x \in X$  is called a *fixed point* of  $f$  if  $f(x) = x$ .

**Definition 10.7.2.2.** Let  $X$  be a Metric Space. A map  $f : X \rightarrow X$  is called *Contractive* with respect to the metric  $d$  if:

There exists  $\alpha \in (0, 1)$  such that for all  $x, y \in X$ ,  $d(f(x), f(y)) \leq \alpha d(x, y)$ .

**Theorem 10.7.2.1. Banach Fixed point Theorem**

Let  $(X, d)$  be a Complete Metric Space, and  $f : X \rightarrow X$  be a Contractive map.

Then, there exists a unique fixed point of  $f$ ,  $x^* \in X$ .

*Proof.* Clearly,

Contractive  $\implies$  Lipschitz Condition  $\implies$  Continuous.

Thus,  $f$  is Continuous.

Let  $x_0 \in X$  be arbitrary, and construct a sequence  $\{x_n\}$  recursively as follows:

$$x_{n+1} \stackrel{\text{def}}{=} f(x_n), \quad n \geq 0$$

Then, for any  $n \geq 0$ ,

$$\begin{aligned} d(x_{n+1}, x_n) &= d(f(x_n), f(x_{n-1})) \leq \alpha d(x_n, x_{n-1}) \\ &= d(f(x_{n-1}), f(x_{n-2})) \leq \alpha^2 d(x_{n-1}, x_{n-2}) \\ &\vdots \\ &\leq \alpha^n d(x_1, x_0) \end{aligned}$$

Let  $\epsilon > 0$  be given. Put  $N \in \mathbb{N}$  such that  $\alpha^N \cdot d(x_1, x_0) < \epsilon(1 - \alpha)$ . Then,  $n \geq m \geq N$  implies that

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + \cdots + d(x_{m+1}, x_m) \\ &\leq \alpha^n d(x_1, x_0) + \alpha^{n-1} d(x_1, x_0) + \cdots + \alpha^{m+1} d(x_1, x_0) \\ &= \alpha^{m+1} d(x_1, x_0) \sum_{r=0}^{n-m-1} \alpha^r < \alpha^N d(x_1, x_0) \sum_{r=0}^{\infty} \alpha^r < \epsilon(1 - \alpha) \frac{1}{1 - \alpha} = \epsilon \end{aligned}$$

Therefore,  $\{x_n\}$  is Cauchy sequence. Since  $X$  is Complete, for some  $x^* \in X$ ,  $\lim_{n \rightarrow \infty} x_n = x^*$ . Consequently,

$$\lim_{n \rightarrow \infty} f(x_n) \stackrel{f \text{ conti.}}{=} f\left(\lim_{n \rightarrow \infty} x_n\right) = f(x^*) = \lim_{n \rightarrow \infty} x_{n+1} = x^*$$

□

## 10.8 Maps in Metric Space

In this section,  $(X, d_X)$  and  $(Y, d_Y)$  are metric spaces.

### 10.8.1 Metric

**Definition 10.8.1.1.** A *metric* on a set  $X$  is a map  $d : X \times X \rightarrow [0, \infty)$  such that for all  $x, y, z \in X$ :

1.  $d(x, y) = 0 \iff x = y$ .
2.  $d(x, y) = d(y, x)$ .
3.  $d(x, z) \leq d(x, y) + d(y, z)$ .

We call  $(X, d)$  a *metric space*.

**Theorem 10.8.1.1.** The map  $d : X \times X \rightarrow \mathbb{R}$  is continuous.

**Proof.** Let  $(x, y) \in X \times X$  and  $\varepsilon > 0$ . For  $U = B_{\varepsilon/2}(x) \times B_{\varepsilon/2}(y)$  and any  $(p, q) \in U$ ,

$$\begin{aligned} d(p, q) &\leq d(p, x) + d(x, y) + d(y, q) < d(x, y) + \varepsilon \\ d(x, y) &\leq d(x, p) + d(p, q) + d(q, y) < d(p, q) + \varepsilon \end{aligned}$$

so  $|d(p, q) - d(x, y)| < \varepsilon$ . □

### 10.8.2 Diameter

**Definition 10.8.2.1.** For  $E \subseteq X$ , the *diameter* is

$$\text{diam } E \stackrel{\text{def}}{=} \sup_{x, y \in E} d(x, y).$$

**Theorem 10.8.2.1.** For any  $E \subseteq X$ ,  $\text{diam } E = \text{diam } \overline{E}$ .

**Proof.** Clearly,  $\text{diam } E \leq \text{diam } \overline{E}$ . Let  $\epsilon > 0$  be given. Then, there exist  $a, b \in \overline{E}$  such that

$$\text{diam } \overline{E} - \frac{\epsilon}{2} \leq d(a, b) < \text{diam } \overline{E}$$

Meanwhile,  $a, b \in \overline{E}$  implies:  $B_{\frac{\epsilon}{2}}(a) \cap E \neq \emptyset$ ,  $B_{\frac{\epsilon}{2}}(b) \cap E \neq \emptyset$ .

Put  $p \in B_{\frac{\epsilon}{2}}(a) \cap E$  and  $q \in B_{\frac{\epsilon}{2}}(b) \cap E$ . Now, the triangle inequality gives

$$\text{diam } \overline{E} - \frac{\epsilon}{2} \leq d(a, b) \leq d(a, p) + d(p, q) + d(q, b) \leq \frac{\epsilon}{2} + \text{diam } E + \frac{\epsilon}{2} = \text{diam } E + \epsilon$$

Since  $\epsilon$  is chosen arbitrarily,  $\text{diam } \overline{E} \leq \text{diam } E$ . □

### 10.8.3 Distance

**Definition 10.8.3.1.** For nonempty  $E \subseteq X$ , define  $\rho_E : X \rightarrow [0, \infty)$  by

$$\rho_E(x) \stackrel{\text{def}}{=} \inf_{t \in E} d(x, t).$$

**Proposition 10.8.3.1.** For all  $x \in X$ ,  $\rho_E(x) = 0$  if and only if  $x \in \overline{E}$ .

*Proof.*

$$\rho_E(x) = 0 \stackrel{\text{by def.}}{\iff} \inf_{t \in E} d(x, t) = 0 \iff \forall \epsilon > 0, \exists p \in E \text{ s.t. } 0 < d(x, p) \leq \epsilon \iff \forall \epsilon > 0, B_\epsilon(x) \cap E \neq \emptyset$$

□

**Theorem 10.8.3.1.** The distance  $\rho_E$  satisfies *Lipschitz Condition*. Furthermore, *Uniformly Continuous*.

*Proof.* Let  $x, y \in X$  be given. Then, for any  $z \in E$ ,

$$\rho_E(x) = \inf_{t \in E} d(x, t) \leq d(x, z) \leq d(x, y) + d(y, z)$$

Since  $z \in E$  given arbitrarily,

$$\rho_E(x) \leq d(x, y) + \rho_E(y)$$

Thus  $\rho_E(x) - \rho_E(y) \leq d(x, y)$ . Similarly,  $\rho_E(y) - \rho_E(x) \leq d(x, y)$ . That is, For any  $x, y \in X$ ,  $|\rho_E(x) - \rho_E(y)| \leq d(x, y)$ . Now, for any  $\epsilon > 0$ , put  $\delta = \epsilon$ . Then,

$$d(x, y) < \delta \implies |\rho_E(x) - \rho_E(y)| \leq d(x, y) < \delta = \epsilon$$

□

**Theorem 10.8.3.2.** Let  $C \subseteq X$  be compact,  $F \subseteq X$  closed, and  $C \cap F = \emptyset$ . Then there exists  $\delta > 0$  such that

$$d(p, q) \geq \delta \text{ for all } p \in C, q \in F.$$

*Proof.*

□

#### 10.8.4 Isometry

**Definition 10.8.4.1.** An onto map  $f : (X, d_X) \rightarrow (Y, d_Y)$  is an *isometry* if: for all  $x, y \in X$ ,

$$d_X(x, y) = d_Y(f(x), f(y))$$



## 10.9 Separation Axioms

## 10.10 Urysohn Metrization Theorem

### 10.10.1 Urysohn Lemma

Recall that:

**Definition 10.10.1.1.**  $X$  is  $T_4$  if: For any disjoint closed set  $A$  and  $B$ , there exist disjoint open  $U, V$  such that  $A \subseteq U$  and  $B \subseteq V$ .

**Lemma 10.10.1.1.**  $X$  is  $T_4$  Space if and only if For any closed  $C$  and open  $U$  with  $C \subseteq U$ , there exists open  $O$  such that

$$\underset{\text{closed}}{C} \subseteq \underset{\text{open}}{O} \subseteq \underset{\text{closed}}{\overline{O}} \subseteq \underset{\text{open}}{U}$$

*Proof.* Proof of the left direction only.

Let  $X$  be a  $T_4$  Space, and  $C \subset X$  be a closed,  $U$  be a open containing  $C$ . Then,  $C \subset U$  implies  $U^c \subset C^c$ , thus  $U^c$  is a closed set disjoint from  $C$ . By  $T_4$  condition, There exist disjoint opens  $O, O'$  such that  $C \subset O$  and  $U^c \subset O' \iff O'^c \subset U$ .

Since  $O \cap O' = \emptyset \iff O \subset O'^c$ ,  $O$  contained in  $U$ , this implies that  $C \subset O \subset U$ .

Since closure is the smallest closed set such that contains it, consequently  $C \subset O \subset \overline{O} \subset O'^c \subset U$ . □

**Definition 10.10.1.2.** Let  $X$  be a Topological Space, and  $A, B \subset X$  are disjoint closed subset.

A real-valued Continuous map  $f : X \rightarrow [a, b]$  is called *Urysohn function* for  $A$  and  $B$  if:  $f|_A = a$  and  $f|_B = b$ .

In another form,

$$f : X \rightarrow [a, b] : x \rightarrow \begin{cases} a & x \in A \\ b & x \in B \\ f(x) & x \notin A \cup B \end{cases}$$

**Lemma 10.10.1.2. Urysohn Lemma**

$T_4$  Space has an Urysohn function for any two disjoint closed subsets.

*Proof.* Generalization is the last thing to proven, first of all, prove in case of  $[a, b] = [0, 1]$ . This proof consists by three Step.

Let  $X$  be a  $T_4$  Space, and  $A, B \subset X$  be closed subsets.

**Step 1. Construct a Chain of Open sets with Dyadic Rational Indices.**

Consider a set of *Dyadic Rationals*  $D \stackrel{\text{def}}{=} \left\{ \frac{k}{2^n} \mid n, k \in \mathbb{N}, k \leq 2^n - 1 \right\}$ . We will show that the following statement holds:

For any  $r, s \in D$  with  $r < s$ , there exist open sets  $U_r, U_s$  such that  $A \subseteq \overline{U_r} \subseteq U_s \subseteq X \setminus B$  (\*)

For this, Enough to Show that: For any  $k \in \mathbb{N}$ , there exists a Chain as:

$$A \subseteq U_{\frac{1}{2^k}} \subseteq \overline{U_{\frac{1}{2^k}}} \subseteq U_{\frac{2}{2^k}} \subseteq \overline{U_{\frac{2}{2^k}}} \subseteq \cdots \subseteq U_{\frac{2^{k-1}}{2^k}} \subseteq \overline{U_{\frac{2^{k-1}}{2^k}}} \subseteq X \setminus B$$

(Note that this opens in the Chain are not necessary distinct: For instance, if Ambient Space is Finite, then the Space is *Noetherian*. That is,  $X$  satisfies Ascending Chain Condition for open sets.)

Let  $k = 1$ . Then, By  $T_4$  condition gives that: There exists an open set  $U_1$  such that

$$A \subseteq U_1 \subseteq \overline{U_1} \subseteq X \setminus B$$

Now, naming this  $U_1$  as  $U_{\frac{1}{2}}$ , proved when  $k = 1$ .

Suppose that for some  $k > 1$ , the Chain exists as:

$$\underset{\text{closed}}{A} \subseteq \overset{*1}{\underset{\text{open}}{U_{\frac{1}{2^k}}}} \subseteq \underset{\text{closed}}{\overline{U_{\frac{1}{2^k}}}} \subseteq \overset{*2}{\underset{\text{open}}{U_{\frac{2}{2^k}}}} \subseteq \cdots \subseteq \overset{*2^k-1}{\underset{\text{open}}{U_{\frac{2^{k-1}}{2^k}}}} \subseteq \underset{\text{closed}}{\overline{U_{\frac{2^{k-1}}{2^k}}}} \subseteq \overset{*2^k}{\underset{\text{open}}{X \setminus B}}$$

By repeatedly applying the  $T_4$  condition  $2^k$  times, as indicated by the indices  $*1, *2, \dots, *2^k$ , we can construct  $2^k$  open sets such that:

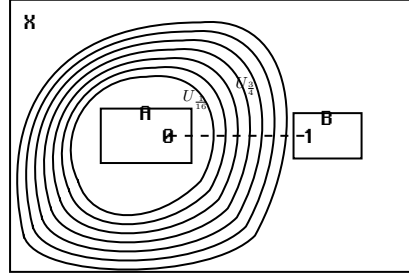
$$A \subseteq U_{\frac{1}{2^{k+1}}} \subseteq \overline{U}_{\frac{1}{2^{k+1}}} \subseteq U_{\frac{1}{2^k}} \subseteq \overline{U}_{\frac{1}{2^k}} \subseteq U_{\frac{3}{2^{k+1}}} \subseteq \overline{U}_{\frac{3}{2^{k+1}}} \subseteq U_{\frac{2}{2^k}} \subseteq \overline{U}_{\frac{2}{2^k}} \subseteq \dots \subseteq U_{\frac{2^k-1}{2^k}} \subseteq \overline{U}_{\frac{2^k-1}{2^k}} \subseteq U_{\frac{2^{k+1}-1}{2^{k+1}}} \subseteq \overline{U}_{\frac{2^{k+1}-1}{2^{k+1}}} \subseteq X \setminus B$$

Finally, Step 1 proved.

**Step 2. Construct an Urysohn Function.**

Define a map  $f : X \rightarrow [0, 1]$  as:

$$f(x) = \begin{cases} 0 & x \in \bigcap_{t \in D} U_t \\ \sup\{t \in D \mid x \notin U_t\} & x \notin \bigcap_{t \in D} U_t \end{cases}$$



Then, this map  $f$  is well-defined by (\*) and  $\sup D \leq 1$ . And  $f$  satisfies that:

1.  $\forall r \in D, x \in A \subset U_r$ . Thus,  $f(x) = 0$  if  $x \in A$ .
2.  $\forall r \in D, x \in B \subset X \setminus U_r$ . Thus,  $f(x) = \sup D = 1$  if  $x \in B$ .
3. If  $x \in \overline{U}_r$ , then for every  $s > r, x \in \overline{U}_r \subset U_s$ . Thus,  $f(x) \leq r$ . In Contrapositive,  $f(x) > r \implies x \notin \overline{U}_r$ .  
(If  $f(x) = \sup\{t \in D \mid x \notin U_t\} > r$ , then there is  $s \in D$  such that  $s > r$  and  $x \notin U_s$ , Contradiction.)
4. If  $x \notin U_r$ , then,  $f(x) \geq r$ . In Contrapositive,  $f(x) < r \implies x \in U_r$ .

Now, show that this map  $f$  is Continuous map: Let  $x \in X$  be fixed arbitrarily, and  $\epsilon > 0$  be given.

In Case of  $0 < f(x) < 1$ .

Since Density of Dyadic Rationals, Choose  $r, s \in D$  such that  $f(x) - \epsilon < r < f(x) < s < f(x) + \epsilon$ .

Now, we obtain that:

$$x \stackrel{(*)}{\in} U_s \setminus \overline{U}_r \stackrel{(**)}{\subseteq} f^{-1}[(f(x) - \epsilon, f(x) + \epsilon)]$$

(\*) directly given by above properties, (\*\*) given applying the fact that  $x \in U_s \subset \overline{U}_s$  and  $x \notin \overline{U}_r$ .

In Case of  $f(x) = 0$ .

Choose  $r \in D$  such that  $f(x) = 0 < r < \epsilon = f(x) + \epsilon$ . Then,

$$x \in U_r \subset f^{-1}[(f(x), f(x) + \epsilon)]$$

In Case of  $f(x) = 1$ .

Choose  $r \in D$  such that  $f(x) - \epsilon = 1 - \epsilon < r < 1 = f(x)$ . Then,

$$x \in X \setminus U_r \subset f^{-1}[(f(x) - \epsilon, f(x))]$$

Consequently,  $f$  is Continuous map on  $[0, 1]$  such that  $f|_A = 0$  and  $f|_B = 1$ .

**Step 3. Generalization.**

Since  $[0, 1] \cong [a, b]$  for any  $a < b$ , let  $g : [0, 1] \rightarrow [a, b] : x \mapsto (1 - x)a + xb$  be a Homeomorphism.

Then,  $h = g \circ f : X \rightarrow [a, b]$  becomes a Continuous map such that  $h|_A = a$  and  $h|_B = b$ . □

## 10.10.2 Tietze Extension Theorem

### Theorem 10.10.2.1. Tietze Extension Theorem

Let  $X$  be a  $T_4$  Space, and  $A \subseteq X$  be a closed subset.

For any Continuous map  $f : A \rightarrow \mathbb{R}$ , there exists a Continuous map:

$$g : X \rightarrow \mathbb{R} \quad \text{s.t.} \quad g|_A = f$$

This  $g$  is called *extension* of  $f$ .

*Proof.* This proof consists by three steps.

**Step 1.** First, we will show that:

For any Continuous map  $f : A \rightarrow [-r, r]$ , there is a Continuous map  $h : X \rightarrow \mathbb{R}$  s.t. 
$$\begin{cases} \forall x \in X, |h(x)| \leq \frac{1}{3}r \\ \forall a \in A, |f(a) - h(a)| \leq \frac{2}{3}r \end{cases} \quad (*)$$

Set

$$I_1 \stackrel{\text{def}}{=} \left[-r, -\frac{1}{3}r\right], \quad I_2 \stackrel{\text{def}}{=} \left[-\frac{1}{3}r, \frac{1}{3}r\right], \quad I_3 \stackrel{\text{def}}{=} \left[\frac{1}{3}r, r\right]$$

Then, the preimage of continuous map preserves closed and  $A$  is closed subspace of  $X$ ,  $f^{-1}[I_1]$  and  $f^{-1}[I_3]$  are closed of  $X$ .

And,  $I_1$  and  $I_3$  are disjoint, thus  $f^{-1}[I_1 \cap I_3] = f^{-1}[I_1] \cap f^{-1}[I_3] = \emptyset$ .

Now, apply the *Urysohn Lemma*: There exists an Urysohn function  $h : X \rightarrow I_2$  for  $f^{-1}[I_1]$  and  $f^{-1}[I_3]$ .

Clearly, this map  $h$  satisfies the first condition in  $(*)$ . And, for show the second condition, let  $a \in A$  be given.

If  $a \in f^{-1}[I_1]$ , then  $f(a) \in I_1$  and  $h(a) = -\frac{1}{3}r$ , thus  $|f(a) - h(a)| \leq \frac{2}{3}r$ .

If  $a \in f^{-1}[I_3]$ , then  $f(a) \in I_3$  and  $h(a) = \frac{1}{3}r$ , thus  $|f(a) - h(a)| \leq \frac{2}{3}r$ .

If  $a \notin (f^{-1}[I_1] \cup f^{-1}[I_3])$ , then  $f(a), h(a) \in I_2$ , thus  $|f(a) - h(a)| \leq \frac{2}{3}r$ .

Therefore, the second condition satisfied.

**Step 2.** We will show that: for any  $f : A \rightarrow [-1, 1]$ , there exists an extension of  $f$ .

Apply the result in Step 1, there exists a Continuous map:

$$h_1 : X \rightarrow \mathbb{R} \quad \text{s.t.} \quad \begin{cases} \forall x \in X, |h_1(x)| \leq \frac{1}{3} \\ \forall a \in A, |f(a) - h_1(a)| \leq \frac{2}{3} \end{cases}$$

Now, the second condition of  $h_1$ , the continuous map  $f - h_1 : A \rightarrow [-\frac{2}{3}, \frac{2}{3}] : x \mapsto f(x) - h_1(x)$  is well-defined.

Again, there exists a Continuous map:

$$h_2 : X \rightarrow \mathbb{R} \quad \text{s.t.} \quad \begin{cases} \forall x \in X, |h_2(x)| \leq \frac{1}{3} \cdot \frac{2}{3} \\ \forall a \in A, |f(a) - h_1(a) - h_2(a)| \leq \left(\frac{2}{3}\right)^2 \end{cases}$$

Inductively, for any  $n \in \mathbb{N}$ , there exists a Continuous map:

$$h_n : X \rightarrow \mathbb{R} \quad \text{s.t.} \quad \begin{cases} \forall x \in X, |h_n(x)| \leq \frac{1}{3} \cdot \left(\frac{2}{3}\right)^{n-1} \\ \forall a \in A, |f(a) - h_1(a) - h_2(a) - \dots - h_n(a)| \leq \left(\frac{2}{3}\right)^n \end{cases}$$

Define a map

$$g : X \rightarrow [-1, 1] : x \mapsto \sum_{n=1}^{\infty} h_n(x)$$

For any  $x \in X$ ,

$$|g(x)| = \left| \sum_{n=1}^{\infty} h_n(x) \right| \leq \sum_{n=1}^{\infty} |h_n(x)| \leq \sum_{n=1}^{\infty} \frac{1}{3} \cdot \left(\frac{2}{3}\right)^{n-1} = \frac{1}{3} \cdot \frac{1}{1 - \frac{2}{3}} = 1$$

Therefore, this map is well-defined. And, *Weierstrass M-test* gives that  $\sum_{n=1}^{\infty} h_n(x)$  converges uniformly.

Moreover, for any  $a \in A$ ,

$$\left| f(a) - \sum_{k=1}^n h_k(a) \right| \leq \left(\frac{2}{3}\right)^n \implies \left| f(a) - \sum_{n=1}^{\infty} h_n(a) \right| = |f(a) - g(a)| = 0$$

That is,  $g$  is Continuous on  $X$  and  $g|_A = f$ . Therefore,  $g$  is extension of  $f$ .

**Step 3.** Finally, we generalize the result in Step 2.:

Let  $f: A \rightarrow [a, b]$  be a Continuous map on the closed subspace  $A$ . And, let  $\varphi: [a, b] \rightarrow [-1, 1]$  be a Homeomorphism. Then,  $\varphi \circ f: A \rightarrow [-1, 1]$  is Continuous map, thus there exists an extension  $g: X \rightarrow [-1, 1]$  such that  $g|_A = \varphi \circ f$ . Now,  $\varphi^{-1} \circ g: X \rightarrow [a, b]$  is Continuous, and  $(\varphi^{-1} \circ g)|_A = \varphi^{-1} \circ \varphi \circ f = f$ , Therefore this  $\varphi^{-1} \circ g$  is the extension of  $f$ .

Let  $f: A \rightarrow \mathbb{R}$  be a Continuous map on the closed subspace  $A$ .

And, let  $\varphi: \mathbb{R} \rightarrow (-1, 1)$  be a Homeomorphism. Then, the map  $\phi: \mathbb{R} \rightarrow [-1, 1]: x \mapsto \varphi(x)$  is still Continuous.

Now, The Continuous map  $\phi \circ f: A \rightarrow [-1, 1]$  has an extension  $g: X \rightarrow [-1, 1]$  such that  $g|_A = \phi \circ f$ .

Put  $B = g^{-1}[\{-1, 1\}]$ . Then  $B$  is Closed on  $X$ , and  $A \cap B = \emptyset$ . Now, apply the Urysohn Lemma to this, there exists an Urysohn function for  $A$  and  $B$ : Continuous map  $\gamma: X \rightarrow [0, 1]$  such that  $\gamma|_A = 1$  and  $\gamma|_B = 0$ .

Define a map  $\eta: X \rightarrow (-1, 1): x \mapsto g(x)\gamma(x)$ . Then, if  $g(x) = 1$  or  $g(x) = -1$ , then  $x \in B$ , thus  $g(x)\gamma(x) = 0$ .

Therefore,  $\eta$  is well-defined. And, for any  $a \in A$ ,  $\eta(a) = g(a)\gamma(a) = g(a)$ , thus  $\eta|_A = \phi \circ f$ .

Consequently, the map  $\phi^{-1} \circ \eta$  is an extension of  $f$ , we wanted. □

Recall that:

**Definition 10.10.2.1.**  $X$  is  $T_1$  if:

For any distinct  $x, y \in X$ , there exist open sets  $U_x, U_y$  such that  $\begin{cases} x \in U_x, & x \notin U_y \\ y \notin U_x, & y \in U_y \end{cases}$ .

**Lemma 10.10.2.1.**  $X$  is  $T_1$  if and only if For any  $x \in X$ , a singleton  $\{x\}$  is closed in  $X$ .

**Proof.** The left direction is clear.

Let  $x \in X$ . Then, for any  $y \in X$  with  $y \neq x$ ,  $T_1$  condition gives that there is an open set such that  $y \in U_y$  and  $x \notin U_y$ .

Now, the union

$$\bigcup_{\substack{y \in X \\ y \neq x}} U_y = X \setminus \{x\}$$

is open by definition. □

### 10.10.3 Urysohn Metrization Theorem

**Definition 10.10.3.1.** A space  $X$  is called *Completely Regular* if:  $X$  is  $T_1$  and  $T_{3\frac{1}{2}}$  where

$T_{3\frac{1}{2}}$  Condition: For any closed set  $C \subset X$  and  $x \in X \setminus C$ , there exists an *Urysohn function* for  $\{x\}$  and  $C$ .

Completely regular space is sometimes called *Tychonoff Space*.

**Proposition 10.10.3.1.** Normal Space  $\implies$  Completely Regular Space  $\implies$  Regular Space.

*Proof.* If  $X$  is Normal space, then every singleton is closed by  $T_1$ . And, the *Urysohn Lemma* gives Urysohn map. If  $X$  is Completely Regular, then for closed  $C \subset X$  and  $x \in X \setminus C$ , there exists a continuous map  $f: X \rightarrow [0, 1]$  s.t

$$f[\{x\}] = 0 \text{ and } f[C] = \{1\}$$

Then,

$$\{x\} \subseteq f^{-1}\left[\left[0, \frac{1}{2}\right)\right], \quad C \subseteq f^{-1}\left[\left(\frac{1}{2}, 1\right]\right]$$

□

**Theorem 10.10.3.1.**  $T_{3\frac{1}{2}}$  is Hereditary. Furthermore, *Completely Regular* is hereditary since  $T_1$  is hereditary.

*Proof.* Let  $X$  be a  $T_{3\frac{1}{2}}$  Space, and  $Y \subseteq X$  be a subspace of  $X$ . Let  $C \subseteq Y$  is closed set of  $Y$ , and  $x \in Y \setminus C$ . Note that:

$$C = \text{Closure of } C \text{ in } Y = \bigcap_{\substack{F \text{ closed in } Y \\ C \subseteq F}} F = \bigcap_{\substack{F' \text{ closed in } X \\ \text{s.t. } F = F' \cap Y}} F' \cap Y = (\text{Closure of } C \text{ in } X) \cap Y$$

Since  $x$  is contained in  $Y$  but not  $C$ , thus  $x$  is not contained in Closure of  $C$  in  $X$ . Now, since  $X$  is  $T_{3\frac{1}{2}}$ ,

There exists a Continuous map  $f: X \rightarrow [0, 1]$  s.t.  $f(x) = 0$ ,  $f|_{\text{cl}_X(C)} = 1$

The restriction  $f_Y$  is continuous, and Urysohn function for  $x$  and  $C$ .

□

**Theorem 10.10.3.2.** Arbitrary product space of  $T_{3\frac{1}{2}}$  space is  $T_{3\frac{1}{2}}$ .

*Proof.* Let  $X_\gamma$  ( $\gamma \in \Gamma$ ) be  $T_{3\frac{1}{2}}$  Spaces. Put  $X = \prod_{\gamma \in \Gamma} X_\gamma$ . Suppose that  $C \subset X$  is closed set, and  $x \in X \setminus C$ .

Since  $X \setminus C$  is open, there exists an open  $U$  in  $X$  such that  $x \in U \subset X \setminus C$ .

Put  $F = \{\alpha \in \Gamma \mid X_\alpha \neq \pi_\alpha[U]\}$ . By definition of product space, this  $F$  is a finite index set. Note that:

$$\forall \alpha \in F, \pi_\alpha(x) \notin X_\alpha \setminus \pi_\alpha[U]$$

And, for each  $\alpha \in F$ ,  $X_\alpha \setminus \pi_\alpha[U]$  are non-empty closed set in  $X_\alpha$ , there exist continuous maps  $f_\alpha$  such that

$$f_\alpha: X_\alpha \rightarrow [0, 1], \quad f_\alpha|_{X \setminus \pi_\alpha[U]} = 0, \quad f_\alpha|_{\pi_\alpha(x)} = 1$$

And, the composition  $f_\alpha \circ \pi_\alpha$  ( $\alpha \in F$ ) is continuous, and

$$(f_\alpha \circ \pi_\alpha)[X \setminus \pi_\alpha^{-1}[\pi_\alpha[U]]] = (f_\alpha \circ \pi_\alpha)[\pi_\alpha^{-1}[X_\alpha \setminus \pi_\alpha[U]]] \subseteq f_\alpha[X_\alpha \setminus \pi_\alpha[U]] = \{0\}$$

Now, the map

$$\Psi: X \rightarrow [0, 1] : t \mapsto \prod_{\alpha \in F} (f_\alpha \circ \pi_\alpha)(t)$$

is Continuous, and  $\Psi(x) = 1$  and  $\Psi[C] \subseteq \Psi[X \setminus U] = \{0\}$ .

□

**Theorem 10.10.3.3.** If  $X$  is *Completely Regular*, then for some index set  $\Lambda$ ,  $X$  can be embedded in  $[0, 1]^\Lambda$ .

*Proof.* Denote that:

$$\{f_\alpha \mid \alpha \in \Lambda\} = \{f : X \rightarrow [0, 1] \mid f \text{ is continuous}\}$$

Claim: the following function is embedding  $X$  into  $[0, 1]^\Lambda$ .

$$F : X \rightarrow [0, 1]^\Lambda : x \mapsto (f_\alpha(x))_{\alpha \in \Lambda}$$

1.  $F$  is Continuous, since each  $f_\alpha$  is Continuous.
2.  $F$  is injective: Let  $x \neq y$  in  $X$ . Then,  $\{x\}$  and  $\{y\}$  are closed by  $T_1$ .  
By  $T_{3\frac{1}{2}}$ , there exists a continuous map  $f : X \rightarrow [0, 1]$  such that  $f(x) = 0$  and  $f(y) = 1$ .  
Since  $f = f_\beta$  for some  $\beta \in \Lambda$ ,  $F(x) \neq F(y)$ .
3.  $F$  is Open map: Let  $U \subseteq X$  be an open set, and let  $y \in F[U]$ . That is, for some  $x \in U$ ,  $F(x) = y$ .  
Since  $X \setminus U$  is closed,  $x \notin X \setminus U$ , and  $T_{3\frac{1}{2}}$ , there exists a continuous map

$$f_\alpha : X \rightarrow [0, 1] \text{ s.t. } f_\alpha(x) = 0, f_\alpha|_{X \setminus U} = 1$$

Meanwhile, put  $V \stackrel{\text{def}}{=} \pi_\alpha^{-1}([0, 1]) \subseteq [0, 1]^\Lambda$ , and  $W \stackrel{\text{def}}{=} V \cap F[X]$ . Then,  $W$  is open in the subspace  $F[X]$ .  
And,  $\pi_\alpha(y) = \pi_\alpha(F(x)) = (\pi_\alpha \circ F)(x) = f_\alpha(x) = 0$ , thus  $y \in W$ . Now, there remains to prove:  $W \subseteq F[U]$ .  
Let  $z \in W$ . Then,  $z \in V$  and  $z = F(x)$  for some  $x \in X$ , this implies  $\pi_\alpha(z) \in [0, 1]$ , i.e.,  $\pi_\alpha(F(x)) = f_\alpha(x) \neq 1$ .  
Now,  $x \in U$ , that is  $F(x) = z \in F[U]$ . Thus,  $W \subseteq F[U]$ , consequently  $F$  is embedding.

□

**Corollary 10.10.3.1.** Let  $X$  is a Topological Space. TFAE:

- a)  $X$  is *Completely Regular Space*.
- b)  $X$  can be embedded in *Compact Hausdorff Space*.
- c)  $X$  can be embedded in *Normal Space*.

*Proof.*

1. a)  $\implies$  b).  $X$  can be embedded in  $[0, 1]^\Lambda$ . And  $[0, 1]$  is *Compact Hausdorff*.
2. b)  $\implies$  c). Every *Compact Hausdorff Space* is *Normal*.
3. c)  $\implies$  a). *Normal Space* is *Completely Regular*, and *Completely Regular* is hereditary.

□

**Lemma 10.10.3.1.** Every *Compact Hausdorff Space* is *Normal*.

*Proof.* Let  $X$  be a *Compact Hausdorff Space*, and  $C, D \subset X$  be disjoint closed subsets.

Since  $X$  is *Compact*,  $C$  and  $D$  are *Compact*. Fix  $x \in C$ . Then, for any  $y \in D$ ,

There exist disjoint opens  $U_y, V_y$  such that  $x \in U_y$  and  $y \in V_y$ .

Since  $\{V_y \mid y \in D\}$  is open cover of  $D$ , there is a finite subcover  $\{V_y^i \mid 1 \leq i \leq n\}$ . That is,  $D \subseteq \bigcup_{i=1}^n V_y^i$ .

Now,  $\bigcap_{i=1}^n U_y^i$  is open set containing  $x$ , and

$$\left( \bigcup_{i=1}^n V_y^i \right) \cap \left( \bigcap_{i=1}^n U_y^i \right) = \bigcup_{i=1}^n \left( V_y^i \cap \left( \bigcap_{i=1}^n U_y^i \right) \right) = \bigcup_{i=1}^n \emptyset = \emptyset$$

In summary, for any  $x \in C$ , there exist disjoint open  $U_x, V_x$  such that  $x \in U_x$  and  $D \subset V_x$ .

Using this: Let  $\{U_x \mid x \in C\}$  be an open cover, then compactness gives the finite subcover  $\{U_x^i \mid 1 \leq i \leq n\}$ . Now,

$$C \subseteq \bigcup_{i=1}^n U_x^i, D \subseteq \bigcap_{i=1}^n V_x^i, \left( \bigcup_{i=1}^n U_x^i \right) \cap \left( \bigcap_{i=1}^n V_x^i \right) = \bigcup_{i=1}^n \left( U_x^i \cap \left( \bigcap_{i=1}^n V_x^i \right) \right) = \emptyset$$

□

**Theorem 10.10.3.4. Embedding Theorem**

Let  $X$  be a  $T_1$  Space. Denote  $\{f_\alpha \mid \alpha \in \Lambda\} = \{f : X \rightarrow \mathbb{R} \mid f \text{ is continuous}\}$ .

Suppose that for any  $x \in X$  and open neighborhood  $U$  of  $x$ , there exists  $\alpha \in \Lambda$  such that  $f_\alpha(x) > 0$ ,  $f_\alpha|_{X \setminus U} = 0$ . Then, the map  $F : X \rightarrow \mathbb{R}^\Lambda : x \mapsto (f_\alpha(x))_{\alpha \in \Lambda}$  is embedding.

**Theorem 10.10.3.5. Suppose that  $X$  is Second-Countable Regular Space.**

Then there exists a Countable collection  $\{f_n : X \rightarrow [0, 1] \mid n \in \mathbb{N}\}$  such that:

For any open  $U \subset X$  and  $x \in U$ , there exists  $n \in \mathbb{N}$  such that  $f_n(x) > 0$  and  $f_n|_{X \setminus U} = 0$ .



**Theorem 10.10.3.6. Urysohn Metrization Theroem**

If  $X$  is a Second-Countable Regular Space, then  $X$  is Metrizable.

## 10.11 Examples

**Proposition 10.11.0.1.** Lower Limit Topology  $(\mathbb{R}, \mathcal{T}_l)$  is  $T_1$  and  $T_4$  Space.

*Proof.*  $T_1$  is clear, because: let  $x, y \in \mathbb{R}$  be a distinct two points. Without Loss of Generality, assume  $x < y$ . Then,

$$\begin{cases} x \in \left[ x, \frac{x+y}{2} \right), & y \in [y, y+1) \\ y \notin \left[ x, \frac{x+y}{2} \right), & x \notin [y, y+1) \end{cases}$$

Thus,  $T_1$  satisfied. And, to show  $T_4$ , Let  $C, D \subseteq \mathbb{R}$  be disjoint closed subsets. Note that: for any  $\epsilon > 0$ ,

$$C \subseteq \bigcup_{x \in C} ([x, x + \epsilon) \setminus D), \quad D \subseteq \bigcup_{y \in D} ([y, y + \epsilon) \setminus C)$$

are open sets. To show disjointness, suppose that their intersection is not empty. Choose  $t$ :

$$t \in \left( \bigcup_{x \in C} ([x, x + \epsilon) \setminus D) \right) \cap \left( \bigcup_{y \in D} ([y, y + \epsilon) \setminus C) \right)$$

Then, for some  $x \in C$  and  $y \in D$ ,  $t \in [x, x + \epsilon) \setminus D$  and  $t \in [y, y + \epsilon) \setminus C$ .

□

## 10.12 Quotient Space

**Definition 10.12.0.1.** Let  $(X, \mathcal{T})$  be a Topological Space,  $Y$  be a set, and  $f: X \rightarrow Y$  be an onto map. Define *Quotient Toplogy on  $Y$  induced by  $f$* :  $\mathcal{T}_Q \stackrel{\text{def}}{=} \{U \subseteq Y \mid f^{-1}[U] \in \mathcal{T}\}$ . This is the largest topology on  $Y$  such that  $f$  is continuous map.

**Definition 10.12.0.2.** Let  $X$  be a Topological Space, and  $\sim$  be an equivalent relation on  $X$ . Define *Canonical map on  $X$* :  $\pi: X \rightarrow X/\sim: x \mapsto [x]$ , and define *Quotient Space  $(X/\sim, \mathcal{T}_Q)$*  where  $\mathcal{T}_Q$  is quotient topology on  $X/\sim$  induced by  $\pi$ .

$X$  Topological Space,  $\sim$  equivalent relation on  $X$ ,  $\pi: X \rightarrow X/\sim: x \mapsto [x]$  canonical map.

**Lemma 10.12.0.1.** For any topological space  $Z$  and a map  $g: X/\sim \rightarrow Z$ ,

$g$  is continuous if and only if  $g \circ \pi$  is continuous.

*Proof.* Let  $g \circ \pi$  be continuous map. Then, for any open  $U \subseteq Z$ ,

$$(g \circ \pi)^{-1}[U] = \pi^{-1}[g^{-1}[U]]$$

is open, thus  $g^{-1}[U]$  is open in  $X/\sim$ . That is,  $g$  is continuous. □

**Lemma 10.12.0.2.** Let  $Z$  be a topological space.

If given continuous map  $f: X \rightarrow Z$  satisfies  $x \sim y \implies f(x) = f(y)$ , then  $\tilde{f}: X/\sim \rightarrow Z: [x] \mapsto f(x)$  is continuous, and unique map such that  $\tilde{f} \circ \pi = f$ .

*Proof.* Well-Defined because:  $[x] = [y] \iff x \sim y \implies f(x) = f(y)$ .

$\tilde{f} \circ \pi = f$ : for any  $x \in X$ ,  $(\tilde{f} \circ \pi)(x) = \tilde{f}(\pi(x)) = \tilde{f}([x]) = f(x)$ , thus  $\tilde{f}$  is continuous since above lemma.

Uniqueness: if  $g: X/\sim \rightarrow Z$  satisfies  $g \circ \pi = f$ , then for any  $[x] \in X/\sim$ ,

$$g([x]) = g(\pi(x)) = (g \circ \pi)(x) = f(x) = \tilde{f}([x])$$

□

**Lemma 10.12.0.3.** Let  $Z$  be a topological space.

If given continuous onto map  $f: X \rightarrow Z$  satisfies  $x \sim y \iff f(x) = f(y)$ , and  $f$  is either open or closed map, then  $\tilde{f}: X/\sim \rightarrow Z: [x] \mapsto f(x)$  is Homeomorphism.

## 10.13 Quotient Map

**Definition 10.13.0.1.** Let  $X, Y$  be Topological Space.

A continuous onto map  $f: X \rightarrow Y$  is called *quotient map* if:

$$U \subseteq Y \text{ is open if and only if } f^{-1}[U] \subseteq X \text{ is open.}$$

### 10.13.1 Basic Properties

**Proposition 10.13.1.1.** Composition of quotient maps is quotient map.

**Proposition 10.13.1.2.** Continuous onto map is quotient map if either open or closed map.

**Theorem 10.13.1.1.** If  $f: X \rightarrow Y$  is quotient map, then  $X/\sim \cong Y$  where

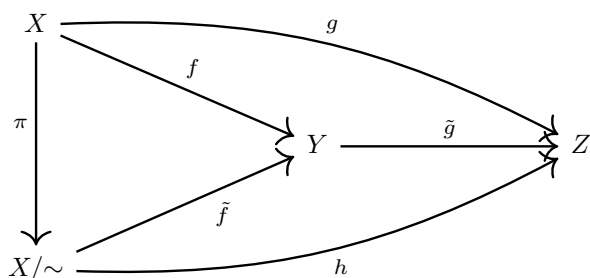
$$x \sim y \iff f(x) = f(y)$$

Moreover, if continuous map  $g: X \rightarrow Z$  satisfies

$$f(x) = f(y) \implies g(x) = g(y)$$

Then,  $\tilde{g}: Y \rightarrow Z: f(x) \mapsto g(x)$  is the unique continuous map such that  $\tilde{g} \circ f = g$ .

### 10.13.2 Quotient map Diagram



- $f: X \rightarrow Y$  is quotient map.
- $g: X \rightarrow Z$  is continuous map s.t  $f(x) = f(y) \implies g(x) = g(y)$ .
- $\pi: X \rightarrow X/\sim: x \mapsto [x]$ .
- $X/\sim$  is quotient topology induced by  $\pi$ .

In this setting,  $\tilde{f}$  is Homeomorphism between  $X/\sim$  and  $Y$ , and  $h = \tilde{g} \circ \tilde{f}$  is Continuous map between  $X/\sim$  and  $Z$ .

## Chapter 11

# Algebraic Topology

## Chapter 12

# Basic Analysis

## 12.1 Tests for Series

### 12.1.1 Integral Test

**Theorem 12.1.1.1.** Let  $f : [1, \infty) \rightarrow \mathbb{R}$  be a decreasing function which satisfies  $\begin{cases} \lim_{x \rightarrow \infty} f(x) = 0 \\ f > 0 \end{cases}$ . Then,

$$\int_1^{\infty} f(x)dx \text{ converges if and only if } \sum_{k=1}^{\infty} f(k) \text{ converges.}$$

Futhermore, put  $d_n \stackrel{\text{def}}{=} \sum_{k=1}^n f(k) - \int_1^n f(x)dx$ , then for any  $n \in \mathbb{N}$ ,  $0 < f(n+1) \leq d_{n+1} \leq d_n \leq f(1)$ , and for any  $k \in \mathbb{N}$ ,  $0 \leq d_k - \lim_{n \rightarrow \infty} d_n \leq f(k)$ . (Clearly,  $\lim_{n \rightarrow \infty} d_n$  exists.)

**Proof.** Since

$$\begin{aligned} \int_1^{n+1} f(x)dx &= \sum_{k=1}^n \int_k^{k+1} f(x)dx \leq \sum_{k=1}^n \int_k^{k+1} f(k)dx = \sum_{k=1}^n f(k) \\ \implies f(n+1) &= \sum_{k=1}^{n+1} f(k) - \sum_{k=1}^n f(k) \leq \sum_{k=1}^{n+1} f(k) - \int_1^{n+1} f(x)dx = d_{n+1} \end{aligned}$$

And,

$$d_n - d_{n+1} = \int_n^{n+1} f(x)dx - f(n+1) \geq \int_n^{n+1} f(n+1)dx - f(n+1) = 0$$

Immediate  $d_n$  converges, being bounded and decreasing. That is,

$$\lim_{n \rightarrow \infty} d_n = \lim_{n \rightarrow \infty} \left( \sum_{k=1}^n f(k) - \int_1^n f(x)dx \right)$$

converges. Meanwhile, since

$$0 \leq d_n - d_{n+1} = \int_n^{n+1} f(x)dx - f(n+1) \leq \int_n^{n+1} f(n)dx - f(n+1) = f(n) - f(n+1)$$

Now, telescope:

$$0 \leq d_k - \lim_{n \rightarrow \infty} d_n \leq f(k) - \lim_{n \rightarrow \infty} f(n+1) = f(k)$$

□

### 12.1.2 Ratio Test

**Theorem 12.1.2.1.** Let  $\sum a_n$  be given.

$$\sum_{n=1}^{\infty} a_n \text{ converges if: } \limsup_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1.$$

$$\sum_{n=1}^{\infty} a_n \text{ diverges if: } n_0 \in \mathbb{N} \text{ such that } \forall n \geq n_0, \left| \frac{a_{n+1}}{a_n} \right| \geq 1.$$

**Proof.** Choose  $\beta < 1$  such that for some  $N \in \mathbb{N}$ ,  $n \geq N \implies \left| \frac{a_{n+1}}{a_n} \right| < \beta < 1$ .

Then,

$$\begin{aligned} |a_{N+1}| &< \beta |a_N| \\ |a_{N+2}| &< \beta |a_{N+1}| < \beta^2 |a_N| \\ &\vdots \\ |a_{N+p}| &< \beta^p |a_N| \quad (p \in \mathbb{N}) \end{aligned}$$

As a result, for all  $n \geq N$ ,  $|a_n| < \beta^{n-N} |a_N|$ . And,  $\sum_{n=1}^{\infty} |a_n| \leq \sum_{n=1}^{\infty} \beta^{n-N} |a_N| < \infty$ .

□

### 12.1.3 Root Test

**Theorem 12.1.3.1.** Let  $\sum a_n$  be given.

$\sum_{n=1}^{\infty} a_n$  **converges if:**  $\limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|} < 1$ .

$\sum_{n=1}^{\infty} a_n$  **diverges if:**  $\limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|} > 1$ .

**Proof.** Put  $\beta \in \mathbb{R}$  such that  $\limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|} < \beta < 1$ . Then, there is  $N \in \mathbb{N}$  such that  $n \geq N \implies \sqrt[n]{|a_n|} < \beta$ .

Now,  $\sum |a_n| < \sum \beta^n < \infty$ . But if  $\limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|} > 1$ , then  $a_n \not\rightarrow 0$ . □



## 12.2 Arithmetic means

Let  $\{s_n\}$  be a Complex numbers Sequence. Define the *Arithmetic means* of  $\{s_n\}$ :

$$\sigma_n \stackrel{\text{def}}{=} \frac{s_0 + \cdots + s_n}{n+1} = \frac{1}{n+1} \left( \sum_{i=0}^n s_i \right)$$

Then, the Arithmetic means  $\sigma_n$  has the following properties:

1). If  $\lim_{n \rightarrow \infty} s_n = s$ , then  $\lim_{n \rightarrow \infty} \sigma_n = s$ .

*Proof.* Let  $\epsilon > 0$  be given. Then, there exists  $N \in \mathbb{N}$  such that  $n \geq N$  implies  $|s_n - s| < \epsilon$ .  
Now, for  $n \geq N$ ,

$$\begin{aligned} |\sigma_n - s| &= \left| \frac{s_0 + \cdots + s_n}{n+1} - \frac{(n+1)s}{n+1} \right| = \left| \frac{(s_0 - s) + \cdots + (s_n - s)}{n+1} \right| \\ &\stackrel{\text{tri. ineq}}{\leq} \frac{\sum_{k=0}^{N-1} |s_k - s|}{n+1} + \frac{\sum_{k=N}^n |s_k - s|}{n+1} \\ &< \frac{\sum_{k=0}^{N-1} |s_k - s|}{n+1} + \frac{n+1-N}{n+1} \cdot \epsilon \\ &< \frac{\sum_{k=0}^{N-1} |s_k - s|}{n+1} + \epsilon \end{aligned}$$

Now, put  $M \in \mathbb{N}$  satisfies  $M \geq N$  and  $n \geq M \implies \frac{\sum_{k=0}^{N-1} |s_k - s|}{n+1} < \epsilon$ , using Archimedean property.  
Then,  $n \geq M$  implies  $|\sigma_n - s| < \epsilon$ , thus  $\sigma_n \rightarrow s$ . □

2). Put  $a_n = s_n - s_{n-1}$ , for  $n \geq 1$ . If  $\lim_{n \rightarrow \infty} na_n = 0$  and  $\sigma_n$  converges, then  $s_n$  converges.

*Proof.* First,

$$\begin{aligned} s_n - \sigma_n &= s_n - \frac{s_0 + \cdots + s_n}{n+1} = \frac{(n+1)s_n - \sum_{k=0}^n s_k}{n+1} \\ &= \frac{1}{n+1} ((s_1 - s_0) + (2s_2 - 2s_1) + (3s_3 - 3s_2) + \cdots + (ns_n - ns_{n-1})) \\ &= \frac{1}{n+1} \sum_{k=1}^n ka_k \end{aligned}$$

Now, if  $na_n \rightarrow 0$  and  $\sigma_n \rightarrow \sigma$ ,

$$\begin{aligned} \lim_{n \rightarrow \infty} s_n &= \lim_{n \rightarrow \infty} \left( \sigma_n + \frac{1}{n+1} \sum_{k=1}^n ka_k \right) \\ &= \lim_{n \rightarrow \infty} \sigma_n + \lim_{n \rightarrow \infty} \frac{1}{n+1} \sum_{k=1}^n ka_k \stackrel{1)}{=} \sigma \end{aligned}$$

□

2) is conditional converse of 1). But, there is more weak version of the converse proposition:

3). The sequence  $\{na_n\}$  bounded by  $M < \infty$ , and  $\sigma_n \rightarrow \sigma$ . Then,  $s_n \rightarrow \sigma$ .

*Proof.* First, For positive integers  $m < n$ ,

$$\begin{aligned} s_n - \sigma_n &= s_n - \frac{\sum_{k=0}^n s_k}{n+1} = s_n - \frac{m+1}{n-m} \cdot \left( \frac{1}{m+1} - \frac{1}{n+1} \right) \sum_{k=0}^n s_k \\ &= s_n - \frac{m+1}{n-m} \cdot \left( \frac{\sum_{k=0}^m s_k + \sum_{k=m+1}^n s_k}{m+1} - \frac{\sum_{k=0}^n s_k}{n+1} \right) \\ &= s_n - \frac{m+1}{n-m} \cdot \left( \sigma_m - \sigma_n + \frac{\sum_{k=m+1}^n s_k}{m+1} \right) \\ &= \frac{m+1}{n-m} (\sigma_n - \sigma_m) + \frac{1}{n-m} \sum_{k=m+1}^n (s_n - s_k) \end{aligned}$$

Meanwhile, since for any  $n \in \mathbb{N}$ ,  $|na_n| = n|s_n - s_{n-1}| < M$ , for  $k = m+1, \dots, n$ ,

$$\begin{aligned} |s_n - s_k| &= |s_n - s_{n-1} + s_{n-1} - s_{n-2} + \dots + s_{k+1} - s_k| \\ &\leq |s_n - s_{n-1}| + |s_{n-1} - s_{n-2}| + \dots + |s_{k+1} - s_k| \\ &\leq \frac{M}{n} + \frac{M}{n-1} + \dots + \frac{M}{k+1} \leq \frac{n-k}{k+1} M \leq \frac{n-k}{m+2} M \leq \frac{n-m-1}{m+2} M \end{aligned}$$

Let  $\epsilon > 0$  be given. For each  $n \in \mathbb{N}$ , put  $m \in \mathbb{N}$  such that

$$m \leq \frac{n-\epsilon}{1+\epsilon} < m+1$$

Then,

$$m(1+\epsilon) \leq n-\epsilon \implies m+\epsilon(1+m) \leq n \implies \frac{m+1}{n-m} \leq \frac{1}{\epsilon}$$

and

$$n-\epsilon < (m+1)(1+\epsilon) \implies n+1 < (m+2)(1+\epsilon) \implies \frac{n+1}{m+2} - 1 < \epsilon \implies \frac{n-m-1}{m+2} < \epsilon$$

Now, for arbitrary  $n \in \mathbb{N}$ ,

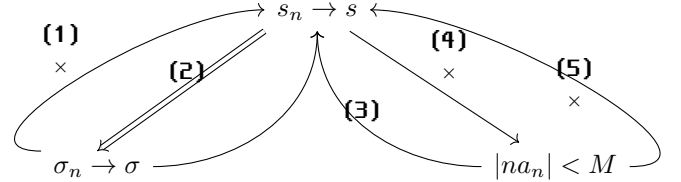
$$\begin{aligned} |s_n - \sigma| &\leq |s_n - \sigma| + |\sigma_n - \sigma| \\ \implies \limsup_{n \rightarrow \infty} |s_n - \sigma| &\leq \limsup_{n \rightarrow \infty} |s_n - \sigma_n| + \limsup_{n \rightarrow \infty} |\sigma_n - \sigma| \end{aligned}$$

And,

$$\begin{aligned} |s_n - \sigma_n| &= \frac{m+1}{n-m} |\sigma_n - \sigma_m| + \frac{1}{n-m} \sum_{k=m+1}^n |s_n - s_k| < \frac{1}{\epsilon} |\sigma_n - \sigma_m| + M\epsilon \\ \implies \limsup_{n \rightarrow \infty} |s_n - \sigma_n| &\leq \frac{1}{\epsilon} \limsup_{n \rightarrow \infty} |\sigma_n - \sigma_m| + M\epsilon = M\epsilon \end{aligned}$$

Consequently,  $\limsup_{n \rightarrow \infty} |s_n - \sigma| \leq (M+1)\epsilon$ , thus  $s_n \rightarrow \sigma$ . □

In brief, the diagram of the above conditions like this:



Examples and Counterexamples of the Diagram:

(1) Let  $s_n \stackrel{\text{def}}{=} \exp(\frac{in\pi}{2})$ . Then,

- $s_n$  diverges.
- $na_n$  diverges.
- $\sigma_n \rightarrow 0$ .

(2) Let  $s_n \stackrel{\text{def}}{=} \frac{1}{n}$ ,  $s_0 = 0$ .

(3) Let  $s_n \stackrel{\text{def}}{=} \sum_{k=1}^n \frac{1}{k}$ . Then,

- $s_n$  diverges.
- $a_n = \frac{1}{n}$ , thus  $na_n \rightarrow 1$ , bounded.
- If  $\sigma_n$  converges, then the diagram implies that  $s_n$  must converge, leading to a contradiction. Therefore,  $\sigma_n$  diverges.

(4)  $s_n = \sum_{k=1}^n \frac{(-1)^k}{\sqrt{k}}$ ,  $s_0 = 0$ . Then,

- $s_n$  converges, being the Alternating series Test.
- $a_n = \frac{(-1)^n}{\sqrt{n}}$ , thus  $na_n$  diverges.

## 12.3 Taylor's Theorem

### Theorem 12.3.0.1. Taylor's Theorem

Let  $f : [a, b] \rightarrow \mathbb{R}$ , and let  $n \in \mathbb{N}$  be fixed. Suppose that  $\begin{cases} f^{(n-1)} \text{ is Continuous.} \\ f^{(n)}(t) \text{ exists for every } t \in (a, b). \end{cases}$

Then, for any  $\alpha, \beta \in [a, b]$ , there exists  $x \in (\alpha, \beta)$  such that

$$f(\beta) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!} (\beta - \alpha)^k + \frac{f^{(n)}(x)}{n!} (\beta - \alpha)^n$$

**Proof.** Put

$$M \stackrel{\text{def}}{=} \frac{1}{(\beta - \alpha)^n} \cdot \left( f(\beta) - \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!} (\beta - \alpha)^k \right)$$

That is,

$$f(\beta) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!} (\beta - \alpha)^k + M(\beta - \alpha)^n$$

and put

$$g(t) \stackrel{\text{def}}{=} f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!} (t - \alpha)^k - M(t - \alpha)^n, \quad (a \leq t \leq b)$$

If we differentiate the above equation  $n$  times,

$$g^{(n)}(t) = f^{(n)}(t) - n!M, \quad (a < t < b)$$

For each  $k = 0, 1, \dots, n-1$ ,

$$\begin{aligned} \frac{d^r}{dt^r} \left( \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!} (t - \alpha)^k \right) &= \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!} \cdot \frac{d^r}{dt^r} ((t - \alpha)^k) \\ &= \sum_{k=r+1}^{n-1} \frac{f^{(k)}(\alpha)}{k!} \cdot \frac{k!}{(k-r)!} (t - \alpha)^{k-r} + f^{(r)}(\alpha) \\ &= \sum_{k=r+1}^{n-1} \frac{f^{(k)}(\alpha)}{(k-r)!} (t - \alpha)^{k-r} + f^{(r)}(\alpha) \end{aligned}$$

Substituting  $t = \alpha$ , only the  $f^{(r)}(\alpha)$  term remains. Therefore, for  $r = 0, \dots, n-1$ ,  $g(\alpha) = g'(\alpha) = \dots = g^{(n-1)}(\alpha) = 0$ . Since  $g(\beta) = 0$  by definition, the Mean-Value Theorem implies there exists a  $x_1 \in (\alpha, \beta)$  s.t.  $g'(x_1) = \frac{g(\beta) - g(\alpha)}{\beta - \alpha} = 0$ . And similarly, there is  $x_2 \in (x_1, \beta)$  s.t.  $g''(x_2) = \frac{g'(x_1) - g'(\alpha)}{\beta - \alpha} = 0$ .

Inductively, for some  $x_n \in (\alpha, \beta)$ ,  $g^{(n)}(x_n) = f^{(n)}(x_n) - n!M = 0$ . That is,  $M = \frac{f^{(n)}(x_n)}{n!}$ .

Proof Complete by Initial Setting. □

**Corollary 12.3.0.1.** Let  $f : [a, b] \rightarrow \mathbb{R}$  be an infinitely differentiable function.

Suppose that there exists a  $M > 0$  such that for any  $n \in \mathbb{N}$ ,  $\sup_{t \in [a, b]} |f^{(n)}(t)| \leq M$ . Then, for any  $x, \alpha \in [a, b]$ ,

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(\alpha)}{k!} (x - \alpha)^k$$

## 12.4 Convexity

### 12.4.1 Definition

**Definition 12.4.1.1.** Let  $f : (a, b) \rightarrow \mathbb{R}$  be a Real-valued function.  $f$  is said to be *convex* if: For any  $x, y \in (a, b), \lambda \in (0, 1)$ ,

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

Convex function has following properties:

**Lemma 12.4.1.1.** Let  $f : (a, b) \rightarrow \mathbb{R}$  be a Convex function, and  $a < x_1 < x_2 < x_3 < b$ . Then,

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(x_3) - f(x_1)}{x_3 - x_1} \leq \frac{f(x_3) - f(x_2)}{x_3 - x_2}$$

*Proof.* To show that first inequality, note that

$$\frac{x_2 - x_1}{x_3 - x_1} \cdot x_3 + \left(1 - \frac{x_2 - x_1}{x_3 - x_1}\right) \cdot x_1 = \frac{(x_2 - x_1)x_3 + (x_3 - x_2)x_1}{x_3 - x_1} = x_2$$

Now,

$$\begin{aligned} f(x_2) &= f\left(\frac{x_2 - x_1}{x_3 - x_1} \cdot x_3 + \left(1 - \frac{x_2 - x_1}{x_3 - x_1}\right) \cdot x_1\right) \\ &\leq \frac{x_2 - x_1}{x_3 - x_1} \cdot f(x_3) + \left(1 - \frac{x_2 - x_1}{x_3 - x_1}\right) \cdot f(x_1) = \frac{x_2 - x_1}{x_3 - x_1} \cdot f(x_3) + f(x_1) - \left(\frac{x_2 - x_1}{x_3 - x_1}\right) \cdot f(x_1) \end{aligned}$$

In brief,

$$f(x_2) - f(x_1) \leq \frac{x_2 - x_1}{x_3 - x_1} (f(x_3) - f(x_1)) \implies \frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(x_3) - f(x_1)}{x_3 - x_1}$$

And similarly,

$$\frac{x_3 - x_2}{x_3 - x_1} \cdot x_1 + \left(1 - \frac{x_3 - x_2}{x_3 - x_1}\right) x_3 = x_2$$

gives the second inequality. □

## 12.4.2 Properties

**Proposition 12.4.2.1.** If  $f : (a, b) \rightarrow \mathbb{R}$  is Convex, then  $f$  is Continuous.

*Proof.* Let  $\epsilon > 0$  be given,  $s < t$  are fixed in  $(a, b)$ . For any  $x, y \in (s, t)$  with  $s < x < y < t$ ,

$$\frac{f(s) - f(a)}{s - a} \leq \frac{f(x) - f(s)}{x - s} \leq \frac{f(y) - f(x)}{y - x} \leq \frac{f(t) - f(y)}{t - y} \leq \frac{f(b) - f(t)}{b - t}$$

Put  $M = \max \left\{ \left| \frac{f(s) - f(a)}{s - a} \right|, \left| \frac{f(b) - f(t)}{b - t} \right| \right\}$ . Then, for any  $x, y \in (s, t)$ ,

$$\left| \frac{f(y) - f(x)}{y - x} \right| \leq M$$

Now,

$$|f(y) - f(x)| \leq M|y - x| < \epsilon$$

Since  $s, t \in (a, b)$  was arbitrary,  $f$  is continuous on  $(a, b)$ . □

**Proposition 12.4.2.2.** Let  $f$  is differentiable on  $(a, b)$ . Then,

$f$  is Convex if and only if  $f'$  is monotonically increasing on  $(a, b)$ .

*Proof.* Prove by showing both directions: right and left.

*Right Direction* Let  $x_1 < x_2$  in  $(a, b)$ . Then,

$$f'(x_1) = \lim_{t \rightarrow x_1} \frac{f(t) - f(x_1)}{t - x_1} \leq \frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \lim_{\tau \rightarrow x_2} \frac{f(\tau) - f(x_2)}{\tau - x_2} = f'(x_2)$$

More rigorously, put  $\epsilon = \left| f'(x_1) - \frac{f(x_2) - f(x_1)}{x_2 - x_1} \right|$ . (If  $\epsilon = 0$ , then there is nothing to prove.).

Now, there exists a  $\delta > 0$  such that  $|t - x_1| < \delta$  implies

$$\left| f'(x_1) - \frac{f(t) - f(x_1)}{t - x_1} \right| < \epsilon \iff -\epsilon + \frac{f(t) - f(x_1)}{t - x_1} < f'(x_1) \stackrel{(*)}{=} \epsilon + \frac{f(t) - f(x_1)}{t - x_1}$$

If  $f'(x_1) - \frac{f(x_2) - f(x_1)}{x_2 - x_1} > 0$ , then  $(*)$  gives

$$f'(x_1) < f'(x_1) + \frac{f(t) - f(x_1)}{t - x_1} - \frac{f(x_2) - f(x_1)}{x_2 - x_1} \iff \frac{f(x_2) - f(x_1)}{x_2 - x_1} < \frac{f(t) - f(x_1)}{t - x_1} \quad \forall t \text{ s.t. } |t - x_1| < \delta$$

If  $|t - x_1| < |x_2 - x_1|$ , then this contradicts to Convexity.

Consequently, we obtain the first inequality, similarly can prove the second inequality.

*Left Direction* Let  $x, y \in (a, b)$  and  $\lambda \in (0, 1)$  be given. The Mean Value Theorem gives that:

$$f(\lambda x + (1 - \lambda)y) - f(x) = f'(z_1)(\lambda x + (1 - \lambda)y - x) \text{ for some } z_1 \in (x, \lambda x + (1 - \lambda)y)$$

$$f(y) - f(\lambda x + (1 - \lambda)y) = f'(z_2)(y - \lambda x + (1 - \lambda)y) \text{ for some } z_2 \in (\lambda x + (1 - \lambda)y, y)$$

Now, Monotonically increasing gives

$$\begin{aligned} \frac{f(\lambda x + (1 - \lambda)y) - f(x)}{\lambda x + (1 - \lambda)y - x} &= f'(z_1) \leq f'(z_2) = \frac{f(y) - f(\lambda x + (1 - \lambda)y)}{y - (\lambda x + (1 - \lambda)y)} \\ \implies \frac{f(\lambda x + (1 - \lambda)y) - f(x)}{(1 - x)(y - x)} &\leq \frac{f(y) - f(\lambda x + (1 - \lambda)y)}{\lambda(y - x)} \\ \implies \lambda f(\lambda x + (1 - \lambda)y) - \lambda f(x) &\leq (1 - \lambda)f(y) - (1 - \lambda)\lambda x + (1 - \lambda)y \\ \implies f(\lambda x + (1 - \lambda)y) &\leq \lambda f(x) + (1 - \lambda)f(y) \end{aligned}$$

□

**Corollary 12.4.2.1.** If  $f : [a, b] \rightarrow \mathbb{R}$  is twice-differentiable, then

$f$  is Convex if and only if  $f''(x) \geq 0$  for all  $x \in (a, b)$ .

**Theorem 12.4.2.1.** Let  $f : [a, b] \rightarrow \mathbb{R}$  be given. Then,

$f$  is Convex if and only if  $f$  is Continuous, and Midpoint Convex.

Midpoint convex is that  $f$  satisfies  $\forall x, y \in (a, b), f\left(\frac{x+y}{2}\right) \leq \frac{f(x) + f(y)}{2}$ .

*Proof.* The right direction is clear. To show the left direction, we demonstrate that Midpoint Convexity implies Dyadic Rational Convexity. Claim: For any  $n \in \mathbb{N}$ ,

$$f\left(\frac{\sum_{k=1}^{2^n} x_k}{2^n}\right) \leq \frac{1}{2^n} \sum_{k=1}^{2^n} f(x_k) \quad (*)$$

Using Induction: If  $n = 1$ , it is clear by Midpoint Convexity.

Assume that for  $n \in \mathbb{N}$ ,  $(*)$  is True. Then,

$$\begin{aligned} f\left(\frac{\sum_{k=1}^{2^{n+1}} x_k}{2^{n+1}}\right) &= f\left(\frac{1}{2} \cdot \left[\frac{\sum_{k=1}^{2^n} x_k}{2^n} + \frac{\sum_{k=2^n+1}^{2^{n+1}} x_k}{2^n}\right]\right) \\ &\stackrel{\text{m.c.}}{\leq} \frac{1}{2} \left( f\left(\frac{\sum_{k=1}^{2^n} x_k}{2^n}\right) + f\left(\frac{\sum_{k=2^n+1}^{2^{n+1}} x_k}{2^n}\right) \right) \\ &\stackrel{(*)}{\leq} \frac{1}{2} \left( \frac{1}{2^n} \sum_{k=1}^{2^n} f(x_k) + \frac{1}{2^n} \sum_{k=2^n+1}^{2^{n+1}} f(x_k) \right) \\ &= \frac{1}{2^{n+1}} \sum_{k=1}^{2^{n+1}} f(x_k) \end{aligned}$$

Consequently, we obtain the claim. Now, let  $n \in \mathbb{N}$ , and  $m$  be an integer such that  $1 \leq m \leq 2^n$ .

Put  $x_1 = x_2 = \dots = x_m = x$  and  $x_{m+1} = x_{m+2} = \dots = x_{2^n} = y$ . Then

$$f\left(\frac{m}{2^n}x + \left(1 - \frac{m}{2^n}\right)y\right) \leq \frac{m}{2^n}f(x) + \left(1 - \frac{m}{2^n}\right)f(y)$$

For complete this discussion, Let  $x, y \in (a, b), \lambda \in (0, 1)$  be given.

Since  $\frac{\lfloor 2^n \lambda \rfloor}{2^n} \rightarrow \lambda$  as  $n \rightarrow \infty$ , for any  $n \in \mathbb{N}$ ,

$$f\left(\frac{\lfloor 2^n \lambda \rfloor}{2^n}x + \left(1 - \frac{\lfloor 2^n \lambda \rfloor}{2^n}\right)y\right) \leq \frac{\lfloor 2^n \lambda \rfloor}{2^n}f(x) + \left(1 - \frac{\lfloor 2^n \lambda \rfloor}{2^n}\right)f(y)$$

Finally, taking limits then

$$\lim_{n \rightarrow \infty} f\left(\frac{\lfloor 2^n \lambda \rfloor}{2^n}x + \left(1 - \frac{\lfloor 2^n \lambda \rfloor}{2^n}\right)y\right) \stackrel{f \text{ cont.}}{=} f\left(\lim_{n \rightarrow \infty} \left[\frac{\lfloor 2^n \lambda \rfloor}{2^n}x + \left(1 - \frac{\lfloor 2^n \lambda \rfloor}{2^n}\right)y\right]\right) = f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

In brief, Midpoint Convexity implies Dyadic Rational Convexity, and with Continuous implies Convexity.  $\square$

## 12.5 Lipschitz Condition

### 12.5.1 Definition

**Definition 12.5.1.1.** A real-valued function  $f : (a, b) \rightarrow \mathbb{R}$  is called *Lipschitz Continuous* if:

$$\exists L \geq 0 \text{ s.t. } \forall x_1, x_2 \in (a, b), |f(x_1) - f(x_2)| \leq L \cdot |x_1 - x_2|$$

The constant  $L$  is said to be *Lipschitz Constant* of  $f$ . In particular, the constant

$$D \stackrel{\text{def}}{=} \sup_{x_1 \neq x_2} \frac{|f(x_1) - f(x_2)|}{|x_1 - x_2|}$$

is called *dilation* of  $f$ . Clearly,

$$\forall x_1, x_2 \in (a, b), |f(x_1) - f(x_2)| \leq D \cdot |x_1 - x_2|$$

and if  $L > 0$  is Lipschitz Constant of  $f$ , then  $D \leq L$ . That is,  $D = \inf\{L > 0 \mid L \text{ is Lipschitz constant of } f\}$ .

### 12.5.2 Properties

**Proposition 12.5.2.1.** If  $f : (a, b) \rightarrow \mathbb{R}$  is Lipschitz Continuous, then  $f$  is uniformly continuous.

*Proof.* Let  $L \geq 0$  be a Lipschitz Constant of  $f$ . Then, for any  $\epsilon > 0$ ,

$$\forall x, y \in (a, b), |x - y| < \frac{\epsilon}{L} \implies |f(x) - f(y)| \leq L|x - y| < \epsilon$$

□

**Proposition 12.5.2.2.** Let  $f : (a, b) \rightarrow \mathbb{R}$  be a Differentiable function. Then,

$f$  is Lipschitz Continuous if and only if  $f'$  is bounded in  $(a, b)$ .

*Proof.*

*Right Direction*

Let  $L > 0$  be a Lipschitz constant of  $f$ , and  $x \in (a, b)$  be given. Since definition of derivative,

$$f'(x) \stackrel{\text{def}}{=} \lim_{t \rightarrow x} \frac{f(x) - f(t)}{x - t}$$

Meanwhile, the assumption gives: for any distinct  $x, t \in (a, b)$ ,

$$\frac{|f(x) - f(t)|}{|x - t|} \leq L$$

Therefore,

$$f'(x) = \lim_{t \rightarrow x} \frac{f(x) - f(t)}{x - t} \leq \lim_{t \rightarrow x} \frac{|f(x) - f(t)|}{|x - t|} \leq \lim_{t \rightarrow x} L = L$$

*Left Direction*

Let distinct  $x, y \in (a, b)$  be given. Then, the Mean-Value Theorem gives: There exists a  $z \in (x, y)$  such that

$$f(x) - f(y) = f'(z)(x - y) \implies f'(z) = \frac{f(x) - f(y)}{x - y}$$

Now,

$$\left| \frac{f(x) - f(y)}{x - y} \right| = |f'(z)| \leq L \implies |f(x) - f(y)| \leq L \cdot |x - y|$$

If  $x = y$ , then there is nothing to prove.

□

Note that:

$$\text{Lipschitz Continuous} \implies \text{Uniformly Continuous} \implies \text{Continuous}$$

## 12.6 Optimization Methods

### 12.6.1 Newton-Raphson Method

#### Theorem 12.6.1.1. Newton-Raphson Method

Let  $f : [a, b] \rightarrow \mathbb{R}$  be a twice-differentiable,  $f(a) < 0 < f(b)$ . Suppose that  $f$  satisfies: for all  $x \in [a, b]$ ,

$$f'(x) \geq \delta > 0 \text{ and } 0 \leq f''(x) \leq M$$

That is,  $f$  is strictly increasing convex function, and Lipschitz Continuous.

Further, there uniquely exists  $x^* \in (a, b)$  such that  $f(x^*) = 0$ .

Let  $x_1 \in (x^*, b)$  fixed. Define a sequence  $\{x_n\}$  inductively as follows:

$$x_{n+1} \stackrel{\text{def}}{=} x_n - \frac{f(x_n)}{f'(x_n)}$$

Then,  $\{x_n\}$  satisfies the following three conditions:

1.  $\{x_n\}$  is decreasing sequence.
2.  $x_n \rightarrow x^*$  as  $n \rightarrow \infty$ .
3. For any  $n \in \mathbb{N}$ ,  $0 \leq x_{n+1} - x^* \leq \left[\frac{M}{2\delta}\right]^{2^{n+1}-1} [x_1 - x^*]^{2^n}$ .

Condition 3 means that for a suitable initial value  $x_1$ , we can establish an upper bound for the error.

*Proof.* This proof consists by three steps.

Since  $f''$  is non-negative, and  $f'$  is positive,  $f$  is strictly increasing convex function.

And Fundamental Theorem of Calculus gives: for any  $x \in (a, b)$ ,

$$f'(x) \stackrel{\text{FTC}}{=} \int_a^x f''(t)dt + f'(a) \leq \int_a^x Mdt + f'(a) = M(x - a) + f'(a) \leq M(b - a) + f'(a)$$

Thus,  $f'$  is bounded on  $(a, b)$ , thus  $f$  is Lipschitz Continuous.

*Step 1.*  $f$  has a unique root  $x^*$ .

The existence of root given directly by Intermediate-Value theorem.

Suppose that  $x^*, x' \in (a, b)$  are distinct root of  $f$ . i.e.,  $f(x^*) = f(x') = 0$ . Then, by Mean-value theorem, there is  $c \in (a, b)$  between  $x^*$  and  $x'$  such that

$$f'(c)(x^* - x') = f(x^*) - f(x') = 0$$

That is,  $f'(c) = 0$ . This is contradiction with  $f'$  is positive.

*Step 2.*  $\{x_n\}$  decrease.

*Proof by induction:*

For  $n = 1$ ,  $f'(x_1)(x_1 - x_2) \stackrel{\text{def}}{=} f(x_1) > f(x^*) = 0$ , thus  $x_2 < x_1$ . And,

$$\begin{aligned} f(x_2) &\stackrel{\text{MVT}}{=} f(x_1) + f'(c_1)(x_2 - x_1) \quad \text{for some } c_1 \in (x_2, x_1) \\ &> f(x_1) + f'(x_1)(x_2 - x_1) = f'(x_1)(x_1 - x_2) + f'(x_1)(x_2 - x_1) = 0 \end{aligned}$$

Now, since  $f(x_2) > 0 = f(x^*)$ , the Mean-Value Theorem implies that  $x_2 > x^*$ .

To use induction, suppose that for some  $n \geq 1$ ,  $x^* < x_{n+1} < x_n$ . Then,

$$f(x_{n+1}) = f'(x_{n+1})(x_{n+1} - x_{n+2}) > 0$$

Thus  $x_{n+2} < x_{n+1}$  and

$$\begin{aligned} f(x_{n+2}) &\stackrel{\text{MVT}}{=} f(x_{n+1}) + f'(c_{n+1})(x_{n+2} - x_{n+1}) \quad \text{for some } c_{n+1} \in (x_{n+2}, x_{n+1}) \\ &\geq f(x_{n+1}) + f'(x_{n+1})(x_{n+2} - x_{n+1}) \\ &= f(x_{n+1})(x_{n+1} - x_{n+2}) + f'(x_{n+1})(x_{n+2} - x_{n+1}) = 0 \end{aligned}$$

Again, the Mean-Value Theorem implies that  $x_{n+2} > x^*$ . Therefore, induction completes.

Now,  $x_n \rightarrow x^*$  as  $n \rightarrow \infty$  for some  $x' \in [x^*, x_1]$  since  $\{x_n\}$  is Bounded below and Decreasing.



Still it remains that to show  $x' = x^*$ . By Continuity,

$$\begin{aligned} f'(x_n)(x_{n+1} - x_n) + f(x_n) &= 0 \\ \implies \lim_{n \rightarrow \infty} [f'(x_n)(x_{n+1} - x_n) + f(x_n)] &= f\left(\lim_{n \rightarrow \infty} x_n\right) = f(x') = 0 \end{aligned}$$

Since the root of  $f$  is unique, thus  $x' = x^*$ .

**Step 3. Establishing the error bound.**

The Taylor's Theorem implies that

$$\begin{aligned} f(x^*) &= f(x_n) + f'(x_n)(x^* - x_n) + \frac{f''(t_n)}{2}(x^* - x_n)^2 \quad \text{for some } t_n \in (x^*, x_n) \\ \implies x_{n+1} - x^* &= \frac{f''(t_n)}{2f'(x_n)}(x^* - x_n)^2 \end{aligned}$$

Consequently,

$$\begin{aligned} 0 \leq x_{n+1} - x^* &= \frac{f''(t_n)}{2f'(x_n)}(x_n - x^*)^2 = \frac{f''(t_n)}{2f'(x_n)} \cdot \left(\frac{f''(t_{n-1})}{2f'(x_{n-1})}\right)^2 (x_{n-1} - x^*)^4 = \dots \\ &= \prod_{i=1}^n \left[\frac{f''(t_i)}{2f'(x_i)}\right]^{2^{(n+1-i)}} [x_1 - x^*]^{2^n} \leq \left[\frac{M}{2\delta}\right]^{2^{n+1}-1} [x_1 - x^*]^{2^n} \end{aligned}$$

□

## 12.6.2 Gradient Descent

**Theorem 12.6.2.1.** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a differentiable function that satisfies the following conditions:

1.  $f$  is *Convex function*.
2.  $f'$  is *Lipschitz Continuous* with Lipschitz constant of  $f$ ,  $L > 0$ . In this,  $f$  is called *L-Smooth*.
3.  $f$  has at least one local minimizer  $x^*$ .

Then,  $x^*$  is a Global minimizer of  $\mathbb{R}$ , and there exists a unique closed interval  $M$  containing  $x^*$  such that

$$\forall x \in M, t \notin M, f(x) = f(x^*) < f(t)$$

And, given initial point  $x_0 \in \mathbb{R}$  and  $0 < \gamma \leq \frac{1}{L}$ , define a sequence  $\{x_n\}$  inductively as follows:

$$x_{n+1} = x_n - \gamma \cdot f'(x_n)$$

Then, for any  $N \in \mathbb{N}$ ,

$$f(x_N) - f(x^*) \leq \frac{|x_0 - x^*|^2}{2\gamma N}$$

**Proof.** Let  $x^* \in \mathbb{R}$  be a local minimizer. That is, there exists a  $\delta > 0$  such that  $\forall t \in (x^* - \delta, x^* + \delta)$ ,  $f(x^*) \leq f(t)$ . Then,

$$0 \leq \lim_{t \rightarrow x^*+} \frac{f(x^*) - f(t)}{x^* - t} = f'(x^*) = \lim_{t \rightarrow x^*-} \frac{f(x^*) - f(t)}{x^* - t} \leq 0$$

thus,  $f'(x^*) = 0$ . And, by convexity,  $f'$  is monotonically increasing. Now, The Fundamental Theorem of Calculus gives:

$$\forall x \in \mathbb{R}, f(x) = \int_{x^*}^x f'(t)dt + f(x^*) \geq f(x^*)$$

Therefore,  $x^*$  is a Global minimizer of  $f$ .

Now, establish the closed interval  $M$ . Since  $f'$  is Lipschitz Continuous, thus  $f'$  is Continuous.

Let  $D \stackrel{\text{def}}{=} \{x \in \mathbb{R} \mid f'(x) = 0\}$ . (Note that:  $x^* \in D$ , thus  $D$  is not empty set.)

$D$  is closed because: Let  $\{x_n\}$  be a convergent sequence in  $D$ . That is, for all  $n \in \mathbb{N}$ ,  $f'(x_n) = 0$ . Then, by continuity,

$$f'\left(\lim_{n \rightarrow \infty} x_n\right) = \lim_{n \rightarrow \infty} f'(x_n) = 0$$

The limit of  $\{x_n\}$  is contained in  $D$ , thus  $D$  is closed.

And,  $D$  is interval: i.e, for any  $x \in (\inf D, \sup D)$ ,  $x \in D$  because:

Suppose that there exists  $x \in (\inf D, \sup D)$  such that  $x \notin D$ . That is,  $f'(x) \neq 0$ . This is Contradiction with Monotonicity.

To set error of upper bound, we make inequalities: Let  $x, y \in \mathbb{R}$  be given.

The Fundamental Theorem of Calculus and  $L$ -Smooth condition gives:

$$\begin{aligned} f(y) - f(x) &= \int_x^y f'(t)dt = \int_0^1 f'(x + (y-x)u)(y-x)du = f'(x)(y-x) + \int_0^1 (f'(x + (y-x)u) - f'(x))(y-x)du \\ &\stackrel{2.}{\leq} f'(x)(y-x) + L \cdot |y-x|^2 \int_0^1 u du = f'(x)(y-x) + \frac{L}{2}|y-x|^2 \end{aligned}$$

For any  $\lambda > 0$ , Put  $y = x - \lambda f'(x)$ . Then,

$$f(x - \lambda f'(x)) \leq f(x) - f'(x)(\lambda f'(x)) + \frac{L}{2}|\lambda f'(x)|^2 = f(x) + \lambda \left( \frac{L\lambda}{2} - 1 \right) |f'(x)|^2$$

Put  $\lambda = \frac{1}{L}$ , then

$$f\left(x - \frac{f'(x)}{L}\right) \leq f(x) - \frac{L}{2}|f'(x)|^2 \implies \frac{L}{2}|f'(x)|^2 \leq f(x) - f\left(x - \frac{f'(x)}{L}\right) \leq f(x) - \inf f$$

Meanwhile, the convexity gives: for any  $x, y \in \mathbb{R}$ ,

$$f'(x)(y - x) \leq f(y) - f(x) \leq f'(y)(y - x)$$

since derivative of convex function increase monotonically. Put  $z = y - \frac{1}{L}(f'(y) - f'(x))$ . Then,

$$\begin{aligned} f(x) - f(y) &= f(x) - f(z) + f(z) - f(y) \\ &\leq f'(x)(x - z) + f'(y)(z - y) + \frac{L}{2}|z - y|^2 \\ &= f'(x) \left( x - y + \frac{1}{L}(f'(y) - f'(x)) \right) - f'(y) \left( \frac{1}{L}(f'(y) - f'(x)) \right) + \frac{L}{2} \left| \frac{1}{L}(f'(y) - f'(x)) \right|^2 \\ &= f'(x)(x - y) - \frac{1}{L}|f'(y) - f'(x)|^2 + \frac{1}{2L}|f'(y) - f'(x)|^2 \\ &= f'(x)(x - y) - \frac{1}{2L}|f'(y) - f'(x)|^2 \end{aligned}$$

Now,

$$\begin{cases} \frac{1}{2L}|f'(y) - f'(x)|^2 \leq f'(x)(x - y) - (f(x) - f(y)) \\ \frac{1}{2L}|f'(x) - f'(y)|^2 \leq f'(y)(y - x) - (f(y) - f(x)) \end{cases} \implies \frac{1}{L}|f'(y) - f'(x)|^2 \leq (f'(y) - f'(x))(y - x)$$

Since above inequalities, we obtain that

$$\begin{aligned} |x_{n+1} - x^*|^2 &= |x_n - \gamma \cdot f'(x_n) - x^*|^2 = |(x_n - x^*) - \gamma \cdot f'(x_n)|^2 \\ &= |x_n - x^*|^2 - 2\gamma|f'(x_n)| \cdot |x_n - x^*| + \gamma^2|f'(x_n)|^2 \\ &\leq |x_n - x^*|^2 - 2\gamma\frac{1}{L}|f'(x_n)|^2 + \gamma^2|f'(x_n)|^2 \\ &= |x_n - x^*|^2 + \left( \gamma^2 - \frac{2\gamma}{L} \right) |f'(x_n)|^2 \leq |x_n - x^*|^2 \end{aligned}$$

Thus,  $|x_n - x^*|$  decrease as  $n \rightarrow \infty$ . That is,  $|x_n - x^*| \leq |x_0 - x^*|$  for all  $n \in \mathbb{N}$ .

Consider  $x_{n+1}$  and  $x_n$ . First, we obtain

$$\begin{aligned} f(x_{n+1}) &\leq f(x_n) + f'(x_n)(x_{n+1} - x_n) + \frac{L}{2}|x_{n+1} - x_n|^2 \\ &= f(x_n) - \gamma|f'(x_n)|^2 + \frac{L}{2}\gamma^2|f'(x_n)|^2 \\ &= f(x_n) - \left( \gamma - \frac{L}{2}\gamma^2 \right) |f'(x_n)|^2 \end{aligned}$$

Subtracting  $f(x^*)$  above, then

$$f(x_{n+1}) - f(x^*) \leq f(x_n) - f(x^*) - \left( \gamma - \frac{L}{2}\gamma^2 \right) |f'(x_n)|^2$$

Meanwhile, Convexity gives

$$f(x_n) - f(x^*) \leq f'(x_n)(x_n - x^*) \leq |f'(x_n)||x_n - x^*| \leq |f'(x_n)||x_0 - x^*|$$

Combining above two inequalities,

$$f(x_{n+1}) - f(x^*) \leq f(x_n) - f(x^*) - \left( \gamma - \frac{L}{2}\gamma^2 \right) \cdot \frac{|f(x_n) - f(x^*)|^2}{|x_0 - x^*|^2}$$

Dividing Both Sides by  $(f(x_{n+1}) - f(x^*))(f(x_n) - f(x^*))$ ,

$$\begin{aligned} \frac{1}{f(x_n) - f(x^*)} &\leq \frac{1}{f(x_{n+1}) - f(x^*)} - \left( \gamma - \frac{L}{2}\gamma^2 \right) \cdot \frac{f(x_n) - f(x^*)}{f(x_{n+1}) - f(x^*)} \frac{1}{|x_0 - x^*|^2} \\ \implies \left( \gamma - \frac{L}{2}\gamma^2 \right) \cdot \frac{f(x_n) - f(x^*)}{f(x_{n+1}) - f(x^*)} \frac{1}{|x_0 - x^*|^2} &\leq \frac{1}{f(x_{n+1}) - f(x^*)} - \frac{1}{f(x_n) - f(x^*)} \\ \implies \left( \gamma - \frac{L}{2}\gamma^2 \right) \cdot \frac{1}{|x_0 - x^*|^2} &\leq \frac{1}{f(x_{n+1}) - f(x^*)} - \frac{1}{f(x_n) - f(x^*)} \\ \implies \sum_{n=0}^{N-1} \left[ \left( \gamma - \frac{L}{2}\gamma^2 \right) \cdot \frac{1}{|x_0 - x^*|^2} \right] &\leq \sum_{n=0}^{N-1} \left[ \frac{1}{f(x_{n+1}) - f(x^*)} - \frac{1}{f(x_n) - f(x^*)} \right] = \frac{1}{f(x_N) - f(x^*)} - \frac{1}{f(x_0) - f(x^*)} \end{aligned}$$

Consequently,

$$\frac{2\gamma N}{|x_0 - x^*|^2} \leq N \cdot \left[ \left( \gamma - \frac{L}{2} \gamma^2 \right) \cdot \frac{1}{|x_0 - x^*|^2} \right] \leq \frac{1}{f(x_N) - f(x^*)} - \frac{1}{f(x_0) - f(x^*)} \leq \frac{1}{f(x_N) - f(x^*)}$$

Organizing the formula, as result:

$$f(x_N) - f(x^*) \leq \frac{|x_0 - x^*|^2}{2\gamma N}$$

□

## 12.7 Integral

### 12.7.1 Inequality of Riemann–Stieltjes Integral

Let  $p, q \geq 1$  such that  $\frac{1}{p} + \frac{1}{q} = 1$ , and functions lying on  $[a, b]$ .

**Lemma 12.7.1.1.** Let  $f, g \in \mathcal{R}(\alpha)$  with  $f, g \geq 0$ , and  $\int_a^b [f(x)]^p d\alpha = \int_a^b [g(x)]^q d\alpha = 1$ . Then,  $\int_a^b f(x)g(x) d\alpha \leq 1$ .

*Proof.* For any  $x \in [a, b]$ , the Young's Inequality gives

$$0 \leq f(x)g(x) \leq \frac{[f(x)]^p}{p} + \frac{[g(x)]^q}{q}$$

Now,

$$\int_a^b f(x)g(x) d\alpha \leq \int_a^b \frac{[f(x)]^p}{p} + \frac{[g(x)]^q}{q} d\alpha = \frac{1}{p} \int_a^b [f(x)]^p d\alpha + \frac{1}{q} \int_a^b [g(x)]^q d\alpha = \frac{1}{p} + \frac{1}{q} = 1$$

□

**Definition 12.7.1.1.** Let  $f \in \mathcal{R}(\alpha)$ . Define a *Norm* of  $f$ :

$$\|f\|_p \stackrel{\text{def}}{=} \left( \int_a^b |f(x)|^p d\alpha \right)^{\frac{1}{p}}$$

This becomes actually norm of set of Stieltjes Integrable functions,  $\mathcal{F} \stackrel{\text{def}}{=} \{f : [a, b] \rightarrow \mathbb{C} \mid f \in \mathcal{R}(\alpha)\}$ .

**Lemma 12.7.1.2. Hölder's Inequality**

Let  $f, g \in \mathcal{F}$ . Then,

$$\left| \int_a^b f(x)g(x) d\alpha \right| \leq \left[ \int_a^b |f(x)|^p d\alpha \right]^{\frac{1}{p}} \cdot \left[ \int_a^b |g(x)|^q d\alpha \right]^{\frac{1}{q}}$$

*Proof.* Use above definition, Rewrite:

$$\|f\|_p^p = \int_a^b |f(x)|^p d\alpha, \quad \|g\|_q^q = \int_a^b |g(x)|^q d\alpha$$

Now, we can make the condition of above lemma,

$$\int_a^b \left[ \frac{|f(x)|}{\|f\|_p} \right]^p d\alpha = \frac{1}{\|f\|_p^p} \cdot \int_a^b |f(x)|^p d\alpha = 1, \quad \int_a^b \left[ \frac{|g(x)|}{\|g\|_q} \right]^q d\alpha = \frac{1}{\|g\|_q^q} \cdot \int_a^b |g(x)|^q d\alpha = 1$$

And apply this,

$$\int_a^b \frac{|f(x)| \cdot |g(x)|}{\|f\|_p \|g\|_q} d\alpha \leq 1 \implies \int_a^b |f(x)| |g(x)| d\alpha \leq \|f\|_p \|g\|_q = \left[ \int_a^b |f(x)|^p d\alpha \right]^{\frac{1}{p}} \cdot \left[ \int_a^b |g(x)|^q d\alpha \right]^{\frac{1}{q}}$$

Finally, the general property of integral of product gives

$$\left| \int_a^b f(x)g(x) d\alpha \right| \leq \int_a^b |f(x)| |g(x)| d\alpha \leq \|f\|_p \|g\|_q = \left[ \int_a^b |f(x)|^p d\alpha \right]^{\frac{1}{p}} \cdot \left[ \int_a^b |g(x)|^q d\alpha \right]^{\frac{1}{q}}$$

□

**Theorem 12.7.1.1. Minkowski inequality**

Let  $f, g \in \mathcal{F}$ . Then, for any  $p \geq 1$ ,  $\|f + g\|_p \leq \|f\|_p + \|g\|_p$ .

*Proof.*

$$\begin{aligned}
 \|f + g\|_p^p &= \int_a^b |f + g|^p d\alpha = \int_a^b |f + g| |f + g|^{p-1} d\alpha \\
 &\leq \int_a^b [|f| + |g|] |f + g|^{p-1} d\alpha \\
 &= \int_a^b |f| |f + g|^{p-1} d\alpha + \int_a^b |g| |f + g|^{p-1} d\alpha \\
 &\stackrel{\text{Hölder}}{\leq} \left[ \int_a^b |f|^p d\alpha \right]^{\frac{1}{p}} \left[ \int_a^b |f + g|^{(p-1)\frac{p}{p-1}} d\alpha \right]^{\frac{p-1}{p}} + \left[ \int_a^b |g|^p d\alpha \right]^{\frac{1}{p}} \left[ \int_a^b |f + g|^{(p-1)\frac{p}{p-1}} d\alpha \right]^{\frac{p-1}{p}} \\
 &= \left[ \int_a^b |f + g|^p d\alpha \right]^{\frac{p-1}{p}} \left( \left[ \int_a^b |f|^p d\alpha \right]^{\frac{1}{p}} + \left[ \int_a^b |g|^p d\alpha \right]^{\frac{1}{p}} \right) = \|f + g\|_p^{p-1} \cdot (\|f\|_p + \|g\|_p)
 \end{aligned}$$

Now,

$$\|f + g\|_p^p \cdot \|f + g\|_p^{1-p} = \|f + g\|_p \leq \|f\|_p + \|g\|_p$$

□

## Chapter 13

### Measure

# Chapter 14

## Complex Analysis

### 14.1 Series

#### Theorem 14.1.0.1. Laurent's theorem

Suppose that  $f$  is analytic on annular domain  $D = \{z \in \mathbb{C} \mid R_1 < |z - z_0| < R_2\}$ , and  $C$  is simple closed contour around  $z_0$  and lying in that domain  $D$ . Then each point in  $D$ ,  $f(z)$  can express that:

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \cdot \sum_{n=0}^{\infty} \left( \int_C \frac{f(s)}{(s - z_0)^{n+1}} ds \cdot (z - z_0)^n \right) + \frac{1}{2\pi i} \cdot \sum_{n=1}^{\infty} \left( \int_C \frac{f(s)}{(s - z_0)^{-n+1}} ds \cdot \frac{1}{(z - z_0)^n} \right) \\ &= \frac{1}{2\pi i} \cdot \sum_{n=-\infty}^{\infty} \left( \int_C \frac{f(s)}{(s - z_0)^{n+1}} ds \cdot (z - z_0)^n \right), \quad (R_1 < |z - z_0| < R_2) \end{aligned}$$

In particular, If  $f(s)$  is analytic inside and on circle  $C$ ,

$\forall n \in \mathbb{N}$ ,  $f(s) \cdot (s - z_0)^{n-1}$  is analytic too. then by *Cauchy-Goursat Thm*, term (2) is zero, thus we can write that:

$$f(z) = \frac{1}{2\pi i} \cdot \sum_{n=0}^{\infty} \left( \int_C \frac{f(s)}{(s - z_0)^{n+1}} ds \cdot (z - z_0)^n \right)$$

and, since  $f$  is analytic on  $C$ , applies *Cauchy integral theorem*:

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} \cdot (z - z_0)^n$$

This is what we already know as the *Taylor Series* form. Therefore, we can say *Laurent's theorem* is generalization form of *Taylor Theorem*.

*Proof.*

In case of  $z_0 = 0$ .

First, since  $C$  is lying in annular  $R_1 < |z| < R_2$ ,

can construct annular  $A: r_1 < |z| < r_2$  such that  $A$  contains circle  $C$ .

Let write  $C_1: |z| = r_1$ ,  $C_2: |z| = r_2$ , each circles are positively oriented.

Now, construct circle  $\gamma$  such that positively oriented and lying in annular  $A: r_1 < |z| < r_2$ .

Then by *multiply connected theorem*, we get that:

$$\int_{C_2} \frac{f(s)}{s - z} ds = \int_{\gamma} \frac{f(s)}{s - z} ds + \int_{C_1} \frac{f(s)}{s - z} ds$$

Inside and on  $\gamma$ ,  $f$  is analytic, thus we can apply *Cauchy integral theorem*:

$$\begin{aligned} \int_{\gamma} \frac{f(s)}{s - z} ds &= 2\pi i \cdot f(z) = \int_{C_2} \frac{f(s)}{s - z} ds - \int_{C_1} \frac{f(s)}{s - z} ds = \int_{C_2} \frac{f(s)}{s - z} ds + \int_{C_1} \frac{f(s)}{z - s} ds \\ \Rightarrow f(z) &= \frac{1}{2\pi i} \int_{C_2} \frac{f(s)}{s - z} ds + \frac{1}{2\pi i} \int_{C_1} \frac{f(s)}{z - s} ds \end{aligned}$$



And we already know in proof of *Taylor theorem*,

$$\frac{1}{s-z} = \sum_{n=0}^{N-1} \frac{z^n}{s^{n+1}} + \frac{z^N}{(s-z)s^N}$$

and also

$$\begin{aligned} \frac{1}{z-s} &= \sum_{n=0}^{N-1} \frac{s^n}{z^{n+1}} + \frac{s^N}{(z-s)z^N} \\ &= \sum_{n=1}^N \frac{s^{n-1}}{z^n} + \frac{s^N}{(z-s)z^N} \\ &= \sum_{n=1}^N \frac{1}{s^{-n+1} \cdot z^n} + \frac{s^N}{(z-s)z^N} \end{aligned}$$

Now we can write that:

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_{C_2} \frac{f(s)}{s-z} ds + \frac{1}{2\pi i} \int_{C_1} \frac{f(s)}{z-s} ds \\ &= \frac{1}{2\pi i} \int_{C_2} \left( \sum_{n=0}^{N-1} \frac{z^n}{s^{n+1}} f(s) + \frac{z^N}{(s-z)s^N} f(s) \right) ds + \frac{1}{2\pi i} \int_{C_1} \left( \sum_{n=1}^N \frac{f(s)}{s^{-n+1} \cdot z^n} + \frac{s^N}{(z-s)z^N} f(s) \right) ds \\ &= \frac{1}{2\pi i} \sum_{n=0}^{N-1} \int_{C_2} \frac{f(s)}{s^{n+1}} ds \cdot z^n + \frac{z^N}{2\pi i} \int_{C_2} \frac{f(s)}{(s-z)s^N} ds + \frac{1}{2\pi i} \sum_{n=1}^N \int_{C_1} \frac{f(s)}{s^{-n+1}} ds \cdot \frac{1}{z^n} + \frac{1}{2\pi i \cdot z^N} \int_{C_1} \frac{s^N f(s)}{(z-s)} ds \\ &= \frac{1}{2\pi i} \sum_{n=0}^{N-1} \int_{C_2} \frac{f(s)}{s^{n+1}} ds \cdot z^n + \frac{1}{2\pi i} \sum_{n=1}^N \int_{C_1} \frac{f(s)}{s^{-n+1}} ds \cdot z^{-n} + \frac{z^N}{2\pi i} \int_{C_2} \frac{f(s)}{(s-z)s^N} ds + \frac{1}{2\pi i \cdot z^N} \int_{C_1} \frac{s^N f(s)}{(z-s)} ds \end{aligned}$$

And by construction of  $C$ ,  $C_1$ ,  $C_2$ ,  $f$  is analytic between  $C$  and  $C_1$ , also  $C$  and  $C_2$ .

Thus applies *multiply connected*:

$$\begin{aligned} &= \frac{1}{2\pi i} \sum_{n=0}^{N-1} \int_C \frac{f(s)}{s^{n+1}} ds \cdot z^n + \frac{1}{2\pi i} \sum_{n=1}^N \int_C \frac{f(s)}{s^{-n+1}} ds \cdot z^{-n} + \frac{z^N}{2\pi i} \int_{C_2} \frac{f(s)}{(s-z)s^N} ds + \frac{1}{2\pi i \cdot z^N} \int_{C_1} \frac{s^N f(s)}{(z-s)} ds \\ &= \frac{1}{2\pi i} \sum_{n=-N}^{N-1} \int_C \frac{f(s)}{s^{n+1}} ds \cdot z^n + \frac{z^N}{2\pi i} \int_{C_2} \frac{f(s)}{(s-z)s^N} ds + \frac{1}{2\pi i \cdot z^N} \int_{C_1} \frac{s^N f(s)}{(z-s)} ds \end{aligned}$$

Now, enough to show

$$\begin{aligned} \frac{z^N}{2\pi i} \int_{C_2} \frac{f(s)}{(s-z)s^N} ds &\rightarrow 0 \text{ as } N \rightarrow \infty \\ \frac{1}{2\pi i \cdot z^N} \int_{C_1} \frac{s^N f(s)}{z-s} ds &\rightarrow 0 \text{ as } N \rightarrow \infty \end{aligned}$$

Let  $|z| = r$ . Then  $r_1 < r < r_2$ . And, Let  $M = \max \left\{ \max_{z \in C_1} f(z), \max_{z \in C_2} f(z) \right\}$ . And,

for  $s$  on  $C_2$ ,  $|s-z| \geq ||s| - |z|| = r_2 - r$ , for  $s$  on  $C_1$ ,  $|z-s| \geq ||z| - |s|| = r - r_1$ .

Finally, since *ML inequality*,

$$\begin{aligned} \left| \frac{z^N}{2\pi i} \int_{C_2} \frac{f(s)}{(s-z)s^N} ds \right| &\leq \frac{|z^N|}{2\pi} \int_{C_2} \left| \frac{f(s)}{(s-z)s^N} \right| ds \leq \frac{r^N}{2\pi} \frac{M \cdot 2\pi r_2}{(r_2 - r)(r_2)^N} = \frac{Mr_2}{r_2 - r} \left( \frac{r}{r_2} \right)^N \\ \left| \frac{1}{2\pi i \cdot z^N} \int_{C_1} \frac{s^N f(s)}{z-s} ds \right| &\leq \frac{1}{2\pi \cdot r^N} \int_{C_1} \left| \frac{s^N f(s)}{z-s} \right| ds \leq \frac{1}{2\pi \cdot r^N} \frac{(r_1)^N \cdot M \cdot 2\pi r_1}{r - r_1} = \frac{Mr_1}{r - r_1} \left( \frac{r_1}{r} \right)^N \end{aligned}$$

Consequently, since  $\left( \frac{r}{r_2} \right) < 1$ ,  $\left( \frac{r_1}{r} \right) < 1$ , we get result.

In case of  $z_0 \neq 0$ .

Let  $f$  be analytic throughout annular  $R_1 < |z - z_0| < R_2$ .

Then  $g(z) = f(z + z_0)$  is analytic throughout  $R_1 < |(z + z_0) - z_0| < R_2$ .

Now let  $C : z = z(t) \quad (a \leq t \leq b)$  is closed simple contour, following by statement.

Then  $\forall t \in [a, b], \quad R_1 < |z(t) - z_0| < R_2$  and

for  $\Gamma : z = z(t) - z_0 \quad (a \leq t \leq b)$  is lying in  $R_1 < |z| < R_2$ . Now since In  $z_0 = 0$  case,

$$g(z) = \frac{1}{2\pi i} \sum_{n=-\infty}^{\infty} \int_{\Gamma} \frac{g(s)}{s^{n+1}} ds \cdot z^n \quad (R_1 < |z| < R_2)$$

This is equal that:

$$f(z + z_0) = \frac{1}{2\pi i} \sum_{n=-\infty}^{\infty} \int_{\Gamma} \frac{g(s)}{s^{n+1}} ds \cdot z^n \quad (R_1 < |z| < R_2)$$

Finally, change  $z$  to  $z - z_0$  then:

$$f(z) = \frac{1}{2\pi i} \sum_{n=-\infty}^{\infty} \int_{\Gamma} \frac{g(s)}{s^{n+1}} ds \cdot (z - z_0)^n \quad (R_1 < |z - z_0| < R_2)$$

And

$$\int_{\Gamma} \frac{g(s)}{s^{n+1}} ds = \int_a^b \frac{f(z(t) - z_0 + z_0)}{(z(t) - z_0)^{n+1}} \cdot z'(t) dt = \int_a^b \frac{f(z(t))}{(z(t) - z_0)^{n+1}} \cdot z'(t) dt = \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz$$

Consequently we get

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \sum_{n=-\infty}^{\infty} \int_{\Gamma} \frac{g(s)}{s^{n+1}} ds \cdot (z - z_0)^n \\ &= \frac{1}{2\pi i} \sum_{n=-\infty}^{\infty} \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz \cdot (z - z_0)^n \quad (R_1 < |z - z_0| < R_2) \end{aligned}$$

□

## Chapter 15

# Multivariable Analysis

## Chapter 16

# Differential Geometry

## Chapter 17

# Differential Equation

## Chapter 18

# Differential Form

# Chapter 19

## Spaces

### 19.1 $\mathbb{R}^n$

#### 19.1.1 Inner Product in $\mathbb{R}$

#### 19.1.2 $p$ -norm in $\mathbb{R}^n$

**Definition 19.1.2.1.** Let  $\mathbb{R}^n$  be given. Define  $p$ -norm on  $\mathbb{R}^n$  as:

$$d_p : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R} : (\mathbf{x}, \mathbf{y}) \mapsto \left( \sum_{i=1}^n |x_i - y_i|^p \right)^{\frac{1}{p}}, \quad (\mathbf{x} = (x_1, \dots, x_n), \mathbf{y} = (y_1, \dots, y_n))$$

where  $p \in [1, \infty]$ . In particular,  $p$ -norm is a *Metric*, being *Minkowski inequality*.

**Lemma 19.1.2.1. Young's inequality**

Let  $u, v > 0$ , and  $p, q \in [1, \infty]$  such that  $\frac{1}{p} + \frac{1}{q} = 1$ .

Then,

$$uv \leq \frac{1}{p}u^p + \frac{1}{q}v^q$$

**Proof.** Since  $f(x) = \log x$  is concave, we obtain

$$\forall \lambda \in [0, 1], \quad \lambda f(x) + (1 - \lambda)f(y) \leq f(\lambda x + (1 - \lambda)y)$$

thus,

$$\log \left( \frac{1}{p}u^p + \frac{1}{q}v^q \right) \geq \frac{1}{p} \log(u^p) + \frac{1}{q} \log(v^q) = \log(uv)$$

Since  $\exp(x)$  increasing, we get

$$\exp \left( \log \left( \frac{1}{p}u^p + \frac{1}{q}v^q \right) \right) \geq \exp(\log(uv))$$

i.e.,

$$uv \leq \frac{1}{p}u^p + \frac{1}{q}v^q$$

□

**Lemma 19.1.2.2. Holder's inequality**

Let  $x = (x_1, \dots, x_n)$  and  $y = (y_1, \dots, y_n)$  be given, and  $p, q \in [1, \infty]$  such that  $\frac{1}{p} + \frac{1}{q} = 1$ . Then,

$$\sum_{i=1}^n |x_i y_i| \leq \left( \sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}} \cdot \left( \sum_{i=1}^n |y_i|^q \right)^{\frac{1}{q}}$$

*Proof.* Denote that

$$\|x\|_p \stackrel{\text{def}}{=} \left( \sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}}$$

Then, since young's inequality, for each  $i \in \{1, 2, \dots, n\}$ ,

$$\frac{|x_i|}{\|x\|_p} \cdot \frac{|y_i|}{\|y\|_q} \leq \frac{1}{p} \cdot \frac{|x_i|^p}{\|x\|_p^p} + \frac{1}{q} \cdot \frac{|y_i|^q}{\|y\|_q^q}$$

Summing for all  $i = 1, 2, \dots, n$ :

$$\frac{1}{\|x\|_p \|y\|_q} \cdot \sum_{i=1}^n |x_i y_i| \leq \frac{1}{p} + \frac{1}{q} = 1$$

Therefore,

$$\sum_{i=1}^n |x_i y_i| \leq \left( \sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}} \cdot \left( \sum_{i=1}^n |y_i|^q \right)^{\frac{1}{q}}$$

□

**Theorem 19.1.2.1. Minkowski inequality**

Given complex-valued sequences  $\{x_i\}_{i=1}^n$  and  $\{y_i\}_{i=1}^n$ ,

$$\left[ \sum_{i=1}^n |x_i + y_i|^p \right]^{\frac{1}{p}} \leq \left[ \sum_{i=1}^n |x_i|^p \right]^{\frac{1}{p}} + \left[ \sum_{i=1}^n |y_i|^p \right]^{\frac{1}{p}}$$

*Proof.* Denote

$$|x_i + y_i|^p = |x_i + y_i| \cdot |x_i + y_i|^{p-1}$$

Then,

$$\begin{aligned} \sum_{i=1}^n |x_i + y_i|^p &= \sum_{i=1}^n |x_i + y_i| \cdot |x_i + y_i|^{p-1} \\ &= \sum_{i=1}^n (|x_i| + |y_i|) \cdot |x_i + y_i|^{p-1} \\ &= \sum_{i=1}^n |x_i| \cdot |x_i + y_i|^{p-1} + \sum_{i=1}^n |y_i| \cdot |x_i + y_i|^{p-1} \\ &\stackrel{\text{Hölder}}{\leq} \left[ \sum_{i=1}^n |x_i|^p \right]^{\frac{1}{p}} \cdot \left[ \sum_{i=1}^n |x_i + y_i|^p \right]^{\frac{p-1}{p}} + \left[ \sum_{i=1}^n |y_i|^p \right]^{\frac{1}{p}} \cdot \left[ \sum_{i=1}^n |x_i + y_i|^p \right]^{\frac{p-1}{p}} \\ &= \left[ \left( \sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}} + \left( \sum_{i=1}^n |y_i|^p \right)^{\frac{1}{p}} \right] \cdot \left[ \sum_{i=1}^n |x_i + y_i|^p \right]^{\frac{p-1}{p}} \end{aligned}$$



Now, Divide each side as  $\left[\sum_{i=1}^n |x_i + y_i|^p\right]^{\frac{p-1}{p}}$ , then we obtain

$$\left[\sum_{i=1}^n |x_i + y_i|^p\right]^{1-\frac{p-1}{p}} = \left[\sum_{i=1}^n |x_i + y_i|^p\right]^{\frac{1}{p}} \leq \left[\left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^n |y_i|^p\right)^{\frac{1}{p}}\right]$$

□

**Theorem 19.1.2.2.** Let  $d_{p_1}, d_{p_2}$  are  $p$ -norm on  $\mathbb{R}^n$  with  $1 \leq p_1 < p_2 \leq \infty$ . Then,

$$\exists C > 0 \text{ s.t. } \forall x, y \in \mathbb{R}^n, d_{p_2}(x, y) \leq d_{p_1}(x, y) \leq C d_{p_2}(x, y)$$

In particular,  $C = n^{\frac{1}{p_1} - \frac{1}{p_2}}$ .

**Proof.** Let  $p_1 < p_2$ .

For show that first-inequality,

$$1 = \sum_{i=1}^n \left[ \frac{|x_i - y_i|}{\left[\sum_{i=1}^n |x_i - y_i|^{p_2}\right]^{\frac{1}{p_2}}} \right]^{p_2} \leq \sum_{i=1}^n \left[ \frac{|x_i - y_i|}{\left[\sum_{i=1}^n |x_i - y_i|^{p_2}\right]^{\frac{1}{p_2}}} \right]^{p_1} = \frac{\sum_{i=1}^n |x_i - y_i|^{p_1}}{\left[\sum_{i=1}^n |x_i - y_i|^{p_2}\right]^{\frac{p_1}{p_2}}} = \left[ \frac{\left[\sum_{i=1}^n |x_i - y_i|^{p_1}\right]^{\frac{1}{p_1}}}{\left[\sum_{i=1}^n |x_i - y_i|^{p_2}\right]^{\frac{1}{p_2}}} \right]^{p_1}$$

Thus, we obtain that:

$$1 \leq \left[ \frac{\left[\sum_{i=1}^n |x_i - y_i|^{p_1}\right]^{\frac{1}{p_1}}}{\left[\sum_{i=1}^n |x_i - y_i|^{p_2}\right]^{\frac{1}{p_2}}} \right]^{p_1} \iff 1 \leq \frac{\left[\sum_{i=1}^n |x_i - y_i|^{p_1}\right]^{\frac{1}{p_1}}}{\left[\sum_{i=1}^n |x_i - y_i|^{p_2}\right]^{\frac{1}{p_2}}} \iff \left[\sum_{i=1}^n |x_i - y_i|^{p_2}\right]^{\frac{1}{p_2}} \leq \left[\sum_{i=1}^n |x_i - y_i|^{p_1}\right]^{\frac{1}{p_1}}$$

For show that second-inequality, using Hölder's inequality.

$$\begin{aligned} (d_{p_1}(x, y))^{p_1} &= \sum_{i=1}^n |x_i - y_i|^{p_1} = \sum_{i=1}^n |x_i - y_i|^{p_1} \cdot 1 \\ &\stackrel{\text{Hölder}}{\leq} \left[ \sum_{i=1}^n \left( |x_i - y_i|^{p_1 \cdot \frac{p_2}{p_1}} \right) \right]^{\frac{p_1}{p_2}} \cdot \left[ \sum_{i=1}^n 1^{\frac{p_2}{p_2 - p_1}} \right]^{1 - \frac{p_1}{p_2}} = \left[ \sum_{i=1}^n (|x_i - y_i|^{p_2}) \right]^{\frac{p_1}{p_2}} \cdot n^{1 - \frac{p_1}{p_2}} \end{aligned}$$

Taking the  $\frac{1}{p_1}$ -th power of both sides, then

$$d_{p_1}(x, y) \leq \left[ \sum_{i=1}^n (|x_i - y_i|^{p_2}) \right]^{\frac{1}{p_2}} \cdot n^{\frac{1}{p_1} - \frac{1}{p_2}} = n^{\frac{1}{p_1} - \frac{1}{p_2}} \cdot d_{p_2}(x, y)$$

□

**Corollary 19.1.2.1.** Let  $\mathbb{R}^n$  be given as a set, and  $d_{p_1}, d_{p_2} : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$  are  $p$ -norm on  $\mathbb{R}^n$ . Then,

$$\mathcal{T}_{d_{p_1}} = \mathcal{T}_{d_{p_2}}$$

For every  $p \geq 1$ , the metric space  $(\mathbb{R}^n, d_p)$  induces the same topology as the product topology on  $\mathbb{R}^n$ . In particular,  $\mathbb{R}^n$  with the product topology coincides with  $\mathbb{R}^n$  endowed with any  $p$ -norm.

### 19.1.3 Open and Closed set in $\mathbb{R}^n$

**Definition 19.1.3.1.** For  $p \in [1, \infty]$ , define  $p$ -Ball in  $\mathbb{R}^n$  as:

$$B_p(x, r) \stackrel{\text{def}}{=} \{y \in \mathbb{R}^n : \|x - y\|_p < r\}$$

Since all  $p$ -norms are equivalent, for any  $p \in [1, \infty]$ , the collection

$$\beta_p \stackrel{\text{def}}{=} \{B_p(x, r) \mid x \in \mathbb{Q}^n, r \in \mathbb{Q}^+\}$$

is Countable basis of  $\mathbb{R}^n$ . Immediately, we obtain:

**Lemma 19.1.3.1.** Every open set in  $\mathbb{R}^n$  is a countable union of  $p$ -Balls.

We call 2-Ball the *Ball*, and  $\infty$ -Ball the *Cube*.

**Theorem 19.1.3.1.** Let  $U \subseteq \mathbb{R}^n$  be an open set. Then,  $U$  is a countable union of closed cubes with disjoint interiors.

*Proof.* Let  $U \subseteq \mathbb{R}^n$  be an open set, and define the collection of *Dyadic Cubes* on  $\mathbb{R}^n$  as: for each  $k \in \mathbb{N}$ ,

$$Q_k \stackrel{\text{def}}{=} \left\{ \prod_{i=1}^n \left[ \frac{q_i}{2^k}, \frac{q_i + 1}{2^k} \right] \subset \mathbb{R}^n \mid q_i \in \mathbb{Z} \right\}$$

Each element of  $Q_k$  is product of closed intervals, and its interiors are disjoint. For each  $k \in \mathbb{N}$ , construct:

$$Q_k^* \stackrel{\text{def}}{=} \{Q \in Q_k \mid Q \subseteq U\}$$

Then, the union  $Q^* = \bigcup_{k \in \mathbb{N}} Q_k^*$  is a countable union of closed cubes, and  $Q^* = U$ :  $Q^* \subseteq U$  is clear, and let  $x \in U$ .

Since property of metric space, there exists  $\delta > 0$  such that  $x \in B_2(x, \delta) \subseteq U$ . Put  $k \in \mathbb{N}$  such that  $\frac{1}{2^k} < \frac{\delta}{\sqrt{n}}$ .

Then,  $x \in C \subset B_2(x, \delta) \subseteq U$  for some  $C \in Q_k$ , because  $\text{diam } C = \sqrt{n}2^{-k}$ . Since  $C \subset U$ ,  $C \in Q_k^* \subset Q^*$ . i.e.,  $U \subseteq Q^*$ . For disjointness of interiors, we will use the fact:

For any  $Q_1, Q_2 \in Q^*$ , either their interiors are disjoint, or one is contained in the other.

(Conti.)

□

**19.2 Topological Vector Space**

## 19.3 Hilbert Space

**Definition 19.3.0.1.** Complete Inner product Vector Space is called *Hilbert Space*.

### 19.3.1 Hilbert Space in $\mathbb{R}^\omega$

**Definition 19.3.1.1.** Define  $\mathbb{R}^\omega \stackrel{\text{def}}{=} \prod_{i=1}^{\infty} \mathbb{R}$  as the countable product of Euclidean space  $\mathbb{R}$  with product topology.

And define  $\mathbb{H} \stackrel{\text{def}}{=} \left\{ \{x_n\}_{n=1}^{\infty} \mid \sum_{n=1}^{\infty} x_n^2 < \infty \right\} \subset \mathbb{R}^\omega$ , **Metric** on  $\mathbb{H}$  as  $\mu : \mathbb{H} \times \mathbb{H} \rightarrow \mathbb{R} : (\{x_n\}, \{y_n\}) \mapsto \sqrt{\sum_{i=1}^{\infty} (x_i - y_i)^2}$ .

The Metric Space  $(\mathbb{H}, \mu)$  is called *Hilbert Space* or  $l_2$  Space.

Define the operations elementwise; then  $(\mathbb{H}, +, \times)$  is a Vector Space over  $\mathbb{R}$ .

Moreover,  $\mathbb{H}$  is Complete Metric Space and Inner product Vector Space.

**Lemma 19.3.1.1.**  $\mu : \mathbb{H} \times \mathbb{H} \rightarrow \mathbb{R} : (\{x_n\}, \{y_n\}) \mapsto \sqrt{\sum_{i=1}^{\infty} (x_i - y_i)^2}$  is Metric function induced by the inner product.

**Proof.** We know that  $\mathbb{R}^\omega$  is Vector Space. Moreover,  $\mathbb{H} \subset \mathbb{R}^\omega$  is Subspace. Using subspace criteria:

$S \subset V$  is Subspace of Vector Space  $V$  if and only if  $0 \in S$  and For any  $x, y \in S$  and  $a \in F$ ,  $ax + y \in S$ .

Clearly,  $\{0\} \in \mathbb{H}$ . Let  $a \in \mathbb{R}$  and  $\{x_n\}, \{y_n\} \in \mathbb{H}$  be given. Then,  $a\{x_n\} + \{y_n\} = \{ax_n + y_n\} \in \mathbb{H}$  because:

$$\sum_{i=1}^{\infty} (ax_i + y_i)^2 = \sum_{i=1}^{\infty} [a^2 x_i^2 + 2ax_i y_i + y_i^2] \stackrel{(*)}{=} a^2 \sum_{i=1}^{\infty} x_i^2 + 2a \sum_{i=1}^{\infty} x_i y_i + \sum_{i=1}^{\infty} y_i^2 < \infty$$

The  $(*)$  given by:

$$\sum_{i=1}^{\infty} |x_i y_i| = \sum_{i=1}^{\infty} |x_i| |y_i| \leq \sum_{i=1}^{\infty} (\max(|x_i|, |y_i|))^2 \leq \sum_{i=1}^{\infty} (x_n^2 + y_n^2) = \sum_{i=1}^{\infty} x_n^2 + \sum_{i=1}^{\infty} y_n^2 < \infty \quad (*)$$

Thus  $\mathbb{H}$  is Vector Space over  $\mathbb{R}$ . Now, define *inner product* on  $\mathbb{H}$  as:

$$\langle \cdot, \cdot \rangle : \mathbb{H} \times \mathbb{H} \rightarrow \mathbb{R} : (\{x_n\}, \{y_n\}) \mapsto \sum_{i=1}^{\infty} x_i y_i$$

This definition is well-defined since  $(*)$ . And, Linearity in first:

$$\langle a\{x_n\} + \{y_n\}, \{z_n\} \rangle = \langle \{ax_n + y_n\}, \{z_n\} \rangle = \sum_{i=1}^{\infty} (ax_i + y_i) z_i = a \sum_{i=1}^{\infty} x_i z_i + \sum_{i=1}^{\infty} y_i z_i = a \langle \{x_n\}, \{z_n\} \rangle + \langle \{y_n\}, \{z_n\} \rangle$$

The other conditions are clear. Thus,  $(\mathbb{H}, \langle \cdot, \cdot \rangle)$  is *inner product space*.

Using *inner product*, define the *Norm* on  $\mathbb{H}$  as:

$$\|\cdot\| : \mathbb{H} \rightarrow \mathbb{R} : \{x_n\} \mapsto \sqrt{\langle \{x_n\}, \{x_n\} \rangle}$$

Finally, define *Metric* on  $\mathbb{H}$  as:

$$\mu : \mathbb{H} \times \mathbb{H} \rightarrow \mathbb{R} : (\{x_n\}, \{y_n\}) \mapsto \|\{x_n\} - \{y_n\}\| = \sqrt{\sum_{i=1}^{\infty} (x_i - y_i)^2}$$

□

**Theorem 19.3.1.1. Hilbert Space is Separable.**

**Proof.** For each  $n \in \mathbb{N}$ , define  $D_n \stackrel{\text{def}}{=} \{\{p_n\} \mid p_i \in \mathbb{Q}, p_{n+1} = p_{n+1} = \dots = 0\}$  and  $D \stackrel{\text{def}}{=} \bigcup_{n \in \mathbb{N}} D_n$ .

Then,  $D$  is countable set. We will show that  $\overline{D} = \mathbb{H}$ .

Let  $\epsilon > 0$  and  $\{x_n\} \in \mathbb{H}$  be given. Since convergence, there exists  $N \in \mathbb{N}$  such that

$$\sum_{i=N+1}^{\infty} x_i^2 = \sum_{i=1}^{\infty} x_i^2 - \sum_{i=1}^N x_i^2 < \frac{\epsilon^2}{2}$$

Since density of Rationals, put each  $i = 1, 2, \dots, N$ ,  $p_i \in \mathbb{Q} \mid |x_i - p_i| < \frac{\epsilon}{\sqrt{2N}}$  and  $p_i = 0$  for  $i \geq N+1$ .

Then,  $\{p_n\} \in D_n \subset D$  and

$$\mu(\{x_n\}, \{p_n\}) = \sqrt{\sum_{i=1}^N (x_i - p_i)^2 + \sum_{i=N+1}^{\infty} (x_i - p_i)^2} = \sqrt{\sum_{i=1}^N (x_i - p_i)^2 + \sum_{i=N+1}^{\infty} x_i^2} < \sqrt{N \cdot \frac{\epsilon^2}{2N} + \frac{\epsilon^2}{2}} = \epsilon$$

□

**Corollary 19.3.1.1. Hilbert Space is Second-Countable.**

**Theorem 19.3.1.2. Hilbert Space is Complete.**

**Proof.** Let  $\{\{x_{n,i}\}_{i=1}^{\infty}\}_{n=1}^{\infty}$  be a Cauchy sequence in  $\mathbb{H}$ . For any fixed  $n, m \in \mathbb{N}$  and for each  $j \in \mathbb{N}$ ,

$$|x_{n,j} - x_{m,j}| < \mu(\{x_{n,i}\}, \{x_{m,i}\}) = \sqrt{\sum_{i=1}^{\infty} (x_{n,i} - x_{m,i})^2}$$

That is, for each  $j \in \mathbb{N}$ ,  $\{x_{n,j}\}$  is Cauchy sequence in  $\mathbb{R}$ . Since  $\mathbb{R}$  is Complete, put  $y_j \stackrel{\text{def}}{=} \lim_{n \rightarrow \infty} x_{n,j}$ , each  $j \in \mathbb{N}$ .

Let  $\epsilon > 0$  be given. Then, there exists  $N \in \mathbb{N}$  such that  $n, m \geq N \implies \mu(\{x_{n,i}\}, \{x_{m,i}\}) < \frac{\epsilon}{2}$ .

Meanwhile, for each  $k \in \mathbb{N}$ ,

$$\sum_{i=1}^k (x_{n,i} - x_{m,i})^2 \leq \sum_{i=1}^{\infty} (x_{n,i} - x_{m,i})^2 = [\mu(\{x_{n,i}\}, \{x_{m,i}\})]^2$$

Thus,  $n, m \geq N \implies \sum_{i=1}^k (x_{n,i} - x_{m,i})^2 < \left(\frac{\epsilon}{2}\right)^2$ , for each  $k \in \mathbb{N}$ .

Taking limit to  $m$ , then  $n \geq N \implies \lim_{m \rightarrow \infty} \left(\sum_{i=1}^k (x_{n,i} - x_{m,i})^2\right) = \sum_{i=1}^k \left(x_{n,i} - \lim_{m \rightarrow \infty} x_{m,i}\right)^2 = \sum_{i=1}^k (x_{n,i} - y_i)^2 < \left(\frac{\epsilon}{2}\right)^2$ .

And, for all  $k \in \mathbb{N}$ ,

$$\sum_{i=1}^k y_i^2 = \sum_{i=1}^k (2(x_{n,i}^2 + (x_{n,i} - y_i)^2)) \leq 2\|\{x_{n,i}\}_{i=1}^{\infty}\|^2 + \left(\frac{\epsilon}{2}\right)^2$$

Thus  $\{y_i\} \in \mathbb{H}$ . As a result,

$$n \geq N \implies \mu(\{x_n\}, \{y_n\}) = \sqrt{\sum_{i=1}^{\infty} (x_{n,i} - y_i)^2} = \sqrt{\lim_{k \rightarrow \infty} \sum_{i=1}^k (x_{n,i} - y_i)^2} < \frac{\epsilon}{2}$$

□

**Theorem 19.3.1.3.**  $\mathbb{H} \subset \mathbb{R}^\omega$  with subspace topology is Metrizable.

*Proof.* We will use two Lemmas:

**Lemma 19.3.1.2.** Countable Product of Metric Space is Metrizable.

*Proof.* Let  $(X_i, d_i)$  be a metric Space, for each  $i \in \mathbb{N}$ .

If  $d : X \times X \rightarrow \mathbb{R}$  is a Metric, then  $\frac{d}{1+d}$  is also Metric, because

$$\frac{d(x, z)}{1 + d(x, z)} \underset{\substack{\frac{x}{1+x} \\ \text{increasing}}}{\leq} \frac{d(x, y) + d(y, z)}{1 + d(x, y) + d(y, z)} \underset{d \geq 0}{\leq} \frac{d(x, y)}{1 + d(x, y)} + \frac{d(y, z)}{1 + d(y, z)} \quad (*)$$

Using this fact, define

$$d_\Pi : \prod X_i \times \prod X_i \rightarrow \mathbb{R} : (\{x_n\}_{n=1}^\infty, \{y_n\}_{n=1}^\infty) \mapsto \sum_{i=1}^\infty \left[ \frac{1}{2^i} \cdot \frac{d_i(x_i, y_i)}{1 + d_i(x_i, y_i)} \right]$$

Then  $d_\Pi$  is a Metric because:

$$\begin{aligned} d_\Pi(\{x_n\}_{n=1}^\infty, \{z_n\}_{n=1}^\infty) &= \sum_{i=1}^\infty \left[ \frac{1}{2^i} \cdot \frac{d_i(x_i, z_i)}{1 + d_i(x_i, z_i)} \right] \\ &\stackrel{(*)}{\leq} \sum_{i=1}^\infty \left[ \frac{1}{2^i} \cdot \left( \frac{d_i(x_i, y_i)}{1 + d_i(x_i, y_i)} + \frac{d_i(y_i, z_i)}{1 + d_i(y_i, z_i)} \right) \right] \\ &= \sum_{i=1}^\infty \left[ \frac{1}{2^i} \cdot \frac{d_i(x_i, y_i)}{1 + d_i(x_i, y_i)} \right] + \sum_{i=1}^\infty \left[ \frac{1}{2^i} \cdot \frac{d_i(y_i, z_i)}{1 + d_i(y_i, z_i)} \right] \\ &= d_\Pi(\{x_n\}_{n=1}^\infty, \{y_n\}_{n=1}^\infty) + d_\Pi(\{y_n\}_{n=1}^\infty, \{z_n\}_{n=1}^\infty) \end{aligned}$$

Reflexity and symmetry are clear. □

**Lemma 19.3.1.3.** Metrizable is Hereditary.

*Proof omitted.*

Consequently, since  $\mathbb{H} \subset \mathbb{R}^\omega$  is a subspace of a metric space, it is metrizable. □

## 19.4 Banach Space

## 19.5 $L_p$ Space

## 19.6 $l_p$ Space

[Athreya et al., 2019] [Croom, 2002] [Dummit and Foote, 2004]

# Bibliography

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