Math Note

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Contents

This paper covers several topics in undergraduate mathematics.

Chapter 1

Set Theory

1.1 Map

Definition 1. Let X, Y are sets. Define a **function** X to Y is a relation

$$f \subset X \times Y$$

such that

1. For any $x \in X$, there exists $y \in Y$ such that $(x, y) \in f$.
2. If $(x, y) \in f$ and $(x, z) \in f$, then $y = z$.

Denote f as:

$$f : X \to Y : x \mapsto f(x)$$

Define **Image** of f by $A \subset X$:

$$f[A] \overset{\text{def}}{=} \{f(a) \mid a \in A\} \subset Y$$

And, **Preimage** of f by $B \subset Y$:

$$f^{-1}[B] \overset{\text{def}}{=} \{x \in X \mid f(x) \in B\} \subset X$$

$f : X \to Y$ is **Injective** if: $f(x_1) = f(x_2) \implies x_1 = x_2$.
$f : X \to Y$ is **Surjective** if: $\forall y \in Y,\ \exists x \in X$ s.t. $f(x) = y$.
If f is injective and surjective, called **bijective**.
If f is bijective, then define **inverse** of f as:

$$f^{-1} : Y \to X : y \to x$$

where $x \in X$ is the unique elements of X such that $f(x) = y$.

Theorem 1. Let $f : X \to Y$ be a function. Then,

1. There exists $g : Y \to X$ such that $g \circ f : X \to X$ be an identity function **if and only if** f is injective.
2. There exists $h : Y \to X$ such that $f \circ h : Y \to Y$ be an identity function **if and only if** f is surjective.

Proof.
1. $\implies$)
Assume that $f(x_1) = f(x_2)$. Then, existence of left inverse, $g(f(x_1)) = g(f(x_2)) \implies x_1 = x_2$. Thus f injective.
1. $\impliedby$)

Since f is injection, for any $y \in f[X]$, there exists a unique element $x_y \in X$ such that $f(x) = y$. Now, define

$$g : Y \to X : y \mapsto \begin{cases} x_y & y \in f[X] \\ \text{any element in } X & y \notin f[X] \end{cases}$$

Then, for any $x \in X$, $g(f(x)) = g(y) = x$.
2. $\implies$)
Let $y \in Y$ be given. Since existence of right inverse, $f(h(y)) = y$ where $h(y) \in X$. Thus, f is surjective.
2. $\impliedby$)
For any $y \in Y$, there exists a $x_y \in X$ such that $f(x_y) = y$. Now, define

$$h : Y \to X : y \mapsto x_y$$

Then, for any $y \in Y$, $f \circ h(y) = f(x_y) = y$. Thus, $f \circ h$ is identity. □

Corollary 1. Let $f : X \to Y$ be a function, $\mathrm{id}_X : X \to X : x \mapsto x$, and $\mathrm{id}_Y : Y \to Y : y \mapsto y$.

There exists a $f^{-1} : Y \to X$ such that $f^{-1} \circ f = \mathrm{id}_X$ and $f \circ f^{-1} = \mathrm{id}_Y$ if and only if f is bijection.

Proof. If f is bijection, then there exists left inverse g and right inverse h.
Enough To Show that: $g = h$. Since $g \circ f = \mathrm{id}_X$ and $f \circ h = \mathrm{id}_Y$,
$g \circ f \circ h = g \circ \mathrm{id}_Y$, thus $h = g$. □

Theorem 2. Let X, Y, Z are sets, $f : X \to Y$, $g : Y \to Z$ and $A \subset X, B \subset Y, C \subset Z$. Then followings are hold:

1. $g[f[A]] = (g \circ f)[A]$.
2. $f^{-1}[g^{-1}[C]] = (g \circ f)^{-1}[C]$.

Proof.
1. It is clear by definition of image:

$$\begin{aligned} g[f[A]] &\overset{\text{def}}{=} g[\{f(a) \mid a \in A\}] = \{g(b) \mid b \in \{f(a) \mid a \in A\}\} \\ &= \{g(b) \mid b = f(a) \text{ for some } a \in A\} = \{g(f(a)) \mid \text{for some } a \in A\} = \{g(f(a)) \mid a \in A\} \end{aligned}$$

2. It is not clear,

$$f^{-1}[g^{-1}[C]] \overset{\text{def}}{=} f^{-1}[\{b \in Y \mid g(b) \in C\}] = \{a \in X \mid f(a) \in \{b \in Y \mid g(b) \in C\}\} = \{a \in X \mid g(f(a)) \in C\} = (g \circ f)^{-1}[C]$$

□

Proposition 1. Let $f : X \to Y$ be a function, $A, B \subset X$ and $C, D \subset Y$.

1. If $A \subset B$, then $f[A] \subset f[B]$.
2. If $C \subset D$, then $f^{-1}[C] \subset f^{-1}[D]$

Proof.

$$\begin{aligned} &y \in f[A] \implies y = f(a) \text{ for some } a \in A \overset{A \subset B}{\implies} y = f(a) \text{ for some } a \in B \implies y \in f[B] \\ &x \in f^{-1}[C] \implies f(x) \in C \overset{C \subset D}{\implies} f(x) \in D \implies x \in f^{-1}[D] \end{aligned}$$

□

Lemma 1. Let two set X, Y be given, and $A \subset X$, $B \subset Y$, $f : X \to Y$. Then followings are holds:

1. $f^{-1}[f[A]] \supseteq A$, and equality holds if f one-to-one.
2. $f[f^{-1}[B]] \subseteq B$, and equality holds if f onto.
3. $f^{-1}[Y \setminus B] = X \setminus f^{-1}[B]$
4. $f[X] \setminus f[A] \subseteq f[X \setminus A]$, and equality holds if f one-to-one.

Proof. Proof of 4.

$$\begin{aligned} y \in f[X] \setminus f[A] &\iff y \in f[X] \text{ and } y \notin f[A] \\ &\iff \exists x \in X \text{ s.t. } y = f(x) \text{ and } \forall x \in A,\ y \neq f(x) \\ &\overset{(*)}{\Longrightarrow} \exists x \in X \setminus A \text{ s.t. } y = f(x) \\ &\iff y \in f[X \setminus A] \end{aligned}$$

If f is injection, then Left Direction of $(*)$ be true: $\exists! x \in X \setminus A$ s.t. $y = f(x)$. $\square$

Chapter 2

Group Theory

2.1 Isomorphism Theorems

Theorem 3. The First Isomorphism Theorem
Let $\varphi : G \to H$ be a Group-Homomorphism. Then,

$$G/\ker\varphi \cong \varphi[G]$$

$$\begin{tikzcd}
G \arrow[r, "\overset{\text{homomorphism}}{\varphi}"] \arrow[dr, "\substack{\pi \\ \text{homomorphism} \\ \text{surjection}}"'] & \varphi[G] \\
& G/\ker\varphi \arrow[u, "\substack{\phi \\ \text{isomorphism}}"']
\end{tikzcd}$$

Proof. Let $\pi : G \to G/\ker\varphi : x \mapsto x + \ker\varphi$. Then, the map $\phi : G/\ker\varphi \to \varphi[G] : a + \ker\varphi \mapsto \varphi(a)$ is isomorphism.
Well-defined and Injective:

$$a + \ker\varphi = b + \ker\varphi \iff a - b \in \ker\varphi \iff \varphi(a-b) = \varphi(a) - \varphi(b) = 0$$

Surjective is clear. □

Theorem 4. The Second Isomorphism Theorem
Let G be a Group, and $H \leq G$, $N \trianglelefteq G$. Then,

$$HN/N \cong H/(H \cap N)$$

Proof. HK be a subgroup of G, being

$$HN = \bigcup_{h\in H} hN \overset{N \trianglelefteq G}{=} \bigcup_{h \in H} Nh = NH$$

And, $N \leq HN$ is clear, thus $N \trianglelefteq HN$.
Meanwhile, $H \cap N$ be a Normal Subgroup of H: for any $h \in H, n \in H \cap N$, $hnh^{-1} \in N$ because N is normal, and $hnh^{-1} \in H$ since h, n contained in H. Thus, $hnh^{-1} \in H \cap N$, this implies $H \cap N$ be a Normal of H.
Now, Define a Map:

$$\varphi : H \to HN/N : h \mapsto hN$$

Clearly, this map is Well-Defined and Homomorphism. And,

$$\ker\varphi = \varphi^{-1}[1] = \{h \in H \mid hN = N\} = \{h \in H \mid h \in N\} = H \cap N$$

Thus, since The 1st Isomorphism Theorem,

$$HN/N \cong H/(H \cap N)$$

□

Theorem 5. The Third Isomorphism Theorem
Let G be a Group, and $H, K \trianglelefteq G$ with $H \leq K$. Then, $K/H \trianglelefteq G/H$ and

$$(G/H)/(K/H) \cong (G/K)$$

Proof. First, show that $K/H \trianglelefteq G/H$. Let $kH \in K/H$ and $gH \in G/H$. Then,

$$(gH)(kH)(gH)^{-1} = (gH)(kH)(g^{-1}H) = (gkg^{-1})H \in K/H$$

since $gkg^{-1} \in K$, being $K \trianglelefteq G$. Now, Define a map:

$$\varphi : G/H \to G/K : gH \mapsto gK$$

1. Well-Defined.

$$g_1H = g_2H \iff g_1^{-1}g_2 \in H \overset{H \leq K}{\Longrightarrow} g_1^{-1}g_2 \in K \iff g_1K = g_2K$$

2. Homomorphism.
Clearly, for any $g_1H, g_2 \in G/H$,

$$\varphi(g_1Hg_2H) = \phi(g_1g_2H) = g_1g_2K = g_1Kg_2K = \varphi(g_1H)\varphi(g_2H)$$

3. Surjection. Let $gK \in G/K$ be given. Then, clearly, $\varphi(gH) = gK$.
4. Kernel.

$$\ker\varphi = \{gH \in G/H \mid gK = 1\} = \{gH \in G/H \mid g \in K\} = K/H$$

Consequently, The 1st Isomorphism Theorem gives

$$(G/K) \cong (G/H)/\ker\varphi = (G/H)/(K/H)$$

$\square$

Theorem 6. The Forth Isomorphism Theorem
Let G be a Group, and $N \trianglelefteq G$ be a Normal Subgroup. Then, there is a bijection between

$$D \overset{\text{def}}{=} \{H \leq G \mid N \leq H\},\ C \overset{\text{def}}{=} \{\overline{H} \leq G/N\}$$

Proof. Let $\pi : G \to G/N : g \mapsto gN$ be a natural projection. And, Define

$$\Phi : D \to C : H \mapsto \pi[H]$$

This function is well-defined: For any $H \in D$, let $aN, bN \in \pi[H]$. Then, $aN \cdot b^{-1}N = ab^{-1}N \in \pi[H]$, thus $\pi[H] \leq G/N$.
To show that one-to-one: Let $\Phi(A) = \Phi(B)$. Thus means, $\pi[A] = \pi[B]$. Let $a \in A$. Then, $\pi(a) \in \pi[A] = \pi[B]$, thus $\pi(a) = \pi(b)$ for some $b \in B$. That is, $aN = bN \iff a \in bN$. Meanwhile, $N \leq B$, thus $a \in bN \subset B$, $A \subset B$. Similarly, $B \subset A$, that is $A = B$.
To show that onto: Let $K \in C$. Then, $N \leq \pi^{-1}[K] \leq G$, thus clear. $\square$

2.2 Group Action

2.3 Generating subset of a Group

2.4 Commutator Subgroup

Chapter 3

Finite Group Theory

3.1 Lagrange's Theorem

3.2 The Class Equation

3.3 Cauchy's Theorem

Lemma 2. Cauchy's Theorem
Let G be a finite group, and p be a prime dividing $|G|$. Then, G has order p element.

Proof. Define a set:

$$S \overset{\text{def}}{=} \{(x_1, x_2, \ldots, x_p) \mid x_i \in G,\ \ x_1 x_2 \cdots x_p = 1\}$$

Then, S has exactly $|G|^{p-1}$ elements because there are $|G|$ possible choices for each of the first $p-1$ elements in G.
Once $x_1, \cdots, x_{p-1}$ are chosen, then x_p is uniquely determined by the uniqueness of inverses.
Then, let $\sigma = (1, 2, \ldots, p)$ be a permutation. Then, for any $\alpha \in S$, $\sigma^n(\alpha) \in S$ for all $n \in \mathbb{Z}$, being $ab = 1 \iff ba = 1$.
More precisely, let $n \in \mathbb{Z}$ be given, $\alpha = (x_1, \cdots, x_n)$. Then,

$$\sigma^n(\alpha) = (x_{n+1}, x_{n+2}, \ldots, x_p, x_1, x_2, \ldots x_n)$$

By $x_1 \cdots x_n x_{n+1} \cdots x_p = 1$, $x_{n+1} \cdots x_p x_1 \cdots x_n = 1$. Thus $\sigma^n(\alpha) \in S$. Now, define a relation on S as:

$$\alpha \sim \beta \text{ if and only if } \beta = \sigma^n(\alpha) \text{ for some } n \in \mathbb{Z}$$

Then, this relation be equivalent relation, thus construct a partition on S. Claim:

$$[\alpha] = \{\beta \in S \mid \beta \sim \alpha\} \text{ is silngeton if and only if } \alpha = (x, \ldots, x) \text{ for some } x \in G.$$

Left direction is clear, and for show that Right direction,
Suppose that $\alpha = (x_1, \ldots, x_n)$ has different cordinate elements, let $x_i \neq x_j$, for some $i < j$. Then clearly

$$(x_1, \ldots, x_i, \ldots, x_p) \neq \sigma^{i-j}(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_p) = (\ldots, \underbrace{x_j}_{\text{i'th element}}, \ldots)$$

Meanwhile, if $[\alpha]$ has elements more than 1, $[\alpha]$ has exactly number of p elements. Because suppose that $\alpha = (x_1, \ldots, x_p)$ has at least one different coordinate. Then,

$$\sigma^1(\alpha), \sigma^2(\alpha), \cdots, \sigma^{p-1}(\alpha)$$

are mutually different: If there exist $1 \leq i < j < p$ such that $\sigma^i(\alpha) = \sigma^j(\alpha)$, that is, $\sigma^{j-i}(\alpha) = \alpha$.
Now, $j - i \mid p$, this is contradiction with p is prime. Therefore, every equivalent class has order 1 or p.
Consequently,

$$|G|^{p-1} = k + pd$$

where k is a number of classes of size 1, and d is a number of classes of size p. And $(1, 1, \ldots, 1) \in S$, k is at least 1.
Since p divides $|G|^{p-1} = k + pd$, thus k must be bigger than 1, thus there exists elements such that $x^p = 1$. $\square$

3.4 Sylow's Theorem

Theorem 7. Sylow's Theorem
Let G be a group of order $p^\alpha m$, where p is a prime such that $p \nmid m$.
A group of order p^r, $(r \geq 1)$ is called a p-group, Subgroups of G which are p-groups are called p-subgroup.
In particular, subgroups of order p^α is called **Sylow p-subgroup** of G. And, define a collection

$$\text{Syl}_p(G) \overset{\text{def}}{=} \{P \leq G \mid |P| = p^\alpha\}, \ n_p(G) \overset{\text{def}}{=} \text{Card}(\text{Syl}_p(G))$$

The First Sylow Theorem
There exists a Sylow p-subgroup of G. i.e., $\text{Syl}_p(G) \neq \emptyset$.

The Second Sylow Theorem
If $P \in \text{Syl}_p(G)$ and $Q \leq G$ be a p-subgroup. Then, there exists $g \in G$ such that $Q \leq gPg^{-1}$.

The Third Sylow Theorem
$n_p \equiv 1(\text{mod} p)$, $n_p = |G : N_G(P)|$ for any $P \in \text{Syl}_p(G)$, and $n_p \mid m$.

Before prove above statments, we show that:

Lemma 3. Let $P \in \text{Syl}_p(G)$. If Q is p-subgroup of G, then $Q \cap N_G(P) = Q \cap P$.

Proof. Put $H = Q \cap N_G(P)$. Since $P \leq G$, for any $p \in P$, $pPp^{-1} = P$, thus $p \in N_G(P)$. i.e., $P \leq N_G(P)$.
Thus, Enough to Show that $H \leq Q \cap P$. Since $H \leq N_G(P)$,

$$PH = \bigcup_{h \in H} Ph = \bigcup_{h \in H} hP = HP$$

Thus, $PH \leq G$. And,

$$|PH| = \frac{|P||H|}{|P \cap H|}$$

By Lagrange's Theorem, $H \leq P$ and $P \cap H \leq P$ must have order of powers of p, so PH be a p-group. Clearly, $P \leq PH$ and P is the largest p-group of G, thus, $PH = P$. This means, $H \leq P$. □

Proof. The First Theorem: The existence of Sylow p-subgroup. Proof by Induction:
If $|G| = 1$, there is nothing to prove.
Assume inductively the existence of Sylow p-subgroups for all groups of order less than $|G|$.
In case of $p||Z(G)|$, then by Cauchy's Theorem, $Z(G)$ has a subgroup N which has order of p.
Clearly N is Normal, and $G/N = |G|/|N| = p^{a-1}m$. By assumption, G/N has a subgroup P' of order $p^{\alpha-1}$.
By The Forth Isomorphism Theorem, Let $P \leq G$ be a subgroup such that $P/N = P'$.
Then, $|P| = |P/N| \cdot |N| = p^\alpha$, Thus P be a Sylow p-subgroup of G.
In case of $p \nmid |Z(G)|$.
Let $g_1, \ldots, g_r$ be represectatives of the distinct conjugacy classes of G, not contained in $Z(G)$. Then, The Class Equation gives

$$|G| = |Z(G)| + \sum_{i=1}^{r} |G : C_G(g_i)|$$

Since p divides $|G|$, if for all $i = 1, 2, \ldots, r$, $p \mid |G : C_G(g_i)|$ then $p \mid |Z(G)|$, this is contradiction.
Thus, for some j, $p \nmid |G : C_G(g_j)|$. Put $H = C_G(g_j) < G$. Then, $|H|$ has a factor of p^α, by $p \nmid |G : C_G(g_j)|$. Now,

$$|H| = p^\alpha m' \quad (m' < m)$$

By assumption, H has a Sylow p-group, order of p^α.
Consequently, the existence of Sylow p-subgroup was shown.
The Second Theorem: Relation of p-subgroups.
The First Theorem gives existence of Sylow p-subgroups. Let $P \in \text{Syl}_p(G)$. Denote that:

$$S \overset{\text{def}}{=} \{gPg^{-1} \mid g \in G\} = \{P_1, \ldots, P_r\}$$

Let $Q \leq G$ be an any p-subgroup of G. And, Q acts by conjucation on S. i.e.,

$$\alpha : Q \times S \to S : (q, P_i) \mapsto qP_iq^{-1}$$

Write S as a disjoint union of orbits under this action by Q:

$$S = \mathcal{O}_1 \cup \mathcal{O}_2 \cup \cdots \cup \mathcal{O}_s$$

where $r = |\mathcal{O}_1| + \cdots + |\mathcal{O}_s|$. Rearrange a set S as: $P_i \in \mathcal{O}_i,\ 1 \leq i \leq s$. Now, using Definition, Lemma, and above Theorem,

$$|\mathcal{O}_i| \overset{\text{Thm}}{=} |Q : N_Q(P_i)| \overset{\text{def}}{=} |Q : N_G(P_i) \cap Q| \overset{\text{lemma}}{=} |Q : P_i \cap Q|$$

for each $1 \leq i \leq s$. Since Q was arbitrary, Let $Q = P_1$, so that $|\mathcal{O}_1| = |P_1 : P_1 \cap P_1| = 1$. And, for each $i \geq 2$, $P_i \cap P_1 < P_1$,

$$|\mathcal{O}_i| = |P_1 : P_i \cap P_1| > 1$$

Since $P_1 \in \text{Syl}_p(G)$, that is $|P_1| = p^\alpha$, $|P_1 : P_i \cap P_1| = |P_1|/|P_i \cap P_1| = p^k$ where $1 \leq k < \alpha$.
This means for each $2 \leq i \leq s$, p divides $|\mathcal{O}_i|$. Thus,

$$r = |\mathcal{O}_1| + (|\mathcal{O}_2| + \cdots + |\mathcal{O}_s|) \equiv 1 (\text{mod} p)$$

Now, Proof by Contradiction: Let $Q \leq G$ be a p-subgroup. Suppose that for any $1 \leq i \leq r$, $Q \nleq P_i$.
Then, $P_i \cap Q < Q$ for all i, this means

$$|\mathcal{O}_i| = |Q : P_i \cap Q| > 1$$

Thus for any i, p divides $|\mathcal{O}_i|$, this is Contradiction. This proved Relation of p-subgroups.
Finally, The Third Theorem:
Since Second Theorem, this gives that $S = \text{Syl}_p(G)$, thus $n_p(G) = r$. That is, $n_p \equiv 1 (\text{mod} p)$.
Since all Sylow p-subgroups are Conjugate, for any $P \in \text{Syl}_p(G)$,

$$n_p = r = |\mathcal{O}_1| = |G : N_G(P)|$$

Consequently, Completing the Sylow Theorem. $\square$

Chapter 4

Ring Theory

4.1 Ring of Fractions

Theorem 8. Let R be a Commutative Ring, $D \subset R$ be a subset such that $\begin{cases}\text{no zero, no zero divisors}\\ \text{closed under multiplication}\end{cases}$.
Then, there exists a Commutative Ring Q with identity satisfies:

1. R can embed in Q, and every element of D becomes unit in Q. More precisely, $Q = \{rd^{-1} \mid r \in R,\ d \in D\}$.
2. Q is the smallest Ring with identity such that every element of D becomes unit in Q.

Proof. Let $\mathcal{F} \overset{\text{def}}{=} \{(r,d) \mid r \in R,\ d \in D\}$ and the relation $\sim$ on $\mathcal{F}$ by $(r_1,d_1) \sim (r_2,d_2) \iff r_1d_2 = r_2d_1$.
Then, $\sim$ is equivalent relation: reflexive and symmetirc are clear, and Suppose that $(r_1,d_1) \sim (r_2,d_2)$ and $(r_2,d_2) \sim (r_3,d_3)$.

$$r_2d_3 = r_3d_2 \implies r_2d_1d_3 = r_3d_1d_2 \implies r_1d_2d_3 = r_3d_1d_2 \implies d_2(r_1d_3 - r_3d_1) \implies r_1d_3 = r_3d_1$$

Thus transitivity shown. Define

$$\frac{r}{d} \overset{\text{def}}{=} [(r,d)] = \{(a,b) \mid (a,b) \sim (r,d)\},\ \ Q \overset{\text{def}}{=} \left\{\frac{r}{d}\ \middle|\ r \in R,\ d \in D\right\}$$

And define operations $+, \times$ on Q:

$$\frac{r_1}{d_1} + \frac{r_2}{d_2} \overset{\text{def}}{=} \frac{r_1d_2 + r_2d_1}{d_1d_2}, \quad \frac{r_1}{d_1} \times \frac{r_2}{d_2} \overset{\text{def}}{=} \frac{r_1r_2}{d_1d_2}$$

Well-Definedness: If $\dfrac{r_1}{d_1} = \dfrac{r_1'}{d_1'}$ and $\dfrac{r_2}{d_2} = \dfrac{r_2'}{d_2'}$,

$$\frac{r_1d_2 + r_2d_1}{d_1d_2} = \frac{r_1d_2d_1'd_2' + r_2d_1d_1'd_2'}{d_1d_2d_1'd_2'} = \frac{(r_1d_1')d_2d_2' + (r_2d_2')d_1d_1'}{d_1d_2d_1'd_2'} = \frac{(r_1'd_1)d_2d_2' + (r_2'd_2)d_1d_1'}{d_1d_2d_1'd_2'} = \frac{(r_1'd_2' + r_2'd_1')d_1d_2}{d_1d_2d_1'd_2'} = \frac{r_1'd_2' + r_2'd_1'}{d_1'd_2'}$$

$$\frac{r_1r_2}{d_1d_2} = \frac{r_1r_2d_1'd_2'}{d_1d_2d_1'd_2'} = \frac{(r_1d_1')(r_2d_2')}{d_1d_2d_1'd_2'} = \frac{(r_1'd_1)(r_2'd_2)}{d_1d_2d_1'd_2'} = \frac{r_1'r_2'd_1d_2}{d_1d_2d_1'd_2'} = \frac{r_1'r_2'}{d_1'd_2'}$$

Now, $(Q,+,\times)$ constructs Commutative Ring with identity: for any $d \in D$, put $0_Q \overset{\text{def}}{=} \dfrac{0}{d}$, $1_Q \overset{\text{def}}{=} \dfrac{d}{d}$. Then,

1. $(R,+,\times)$ closed under the operations since D is closed under the multiplication.
2. $(R,+)$ has a zero: $\dfrac{r_1}{d_1} + 0_Q = \dfrac{r_1}{d_1} + \dfrac{0}{d} = \dfrac{r_1d + 0d_1}{d_1d} = \dfrac{r_1d}{d_1d} = \dfrac{r_1}{d_1}$.
3. $(R,+)$ has an inverse: $\dfrac{r_1}{d_1} + \dfrac{-r_1}{d_1} = \dfrac{r_1d_1 + (-r_1)d_1}{d_1d_1} = \dfrac{[(r_1) + (-r_1)]d_1}{d_1d_1} = \dfrac{0d_1}{d_1d_1} = \dfrac{0}{d_1d_1} = 0_Q$.
4. $(R,+,\times)$ satisfies distributive law:

   4-1. The left law:

$$\begin{aligned}\frac{r_1}{d_1} \times \left(\frac{r_2}{d_2} + \frac{r_3}{d_3}\right) &= \frac{r_1}{d_1} \times \frac{r_2d_3 + r_3d_2}{d_2d_3} = \frac{r_1r_2d_3 + r_1r_3d_2}{d_1d_2d_3} = \frac{r_1r_2d_1d_3 + r_1r_3d_1d_2}{d_1d_2d_1d_3} = \frac{r_1r_2}{d_1d_2} + \frac{r_2r_3}{d_2d_3}\\ &= \frac{r_1}{d_1} \times \frac{r_2}{d_2} + \frac{r_2}{d_2} \times \frac{r_3}{d_3}\end{aligned}$$

   4-2. The right law:

$$\begin{aligned}\left(\frac{r_1}{d_1} + \frac{r_2}{d_2}\right) \times \frac{r_3}{d_3} &= \frac{r_1d_2 + r_2d_1}{d_1d_2} \times \frac{r_3}{d_3} = \frac{r_1r_3d_2 + r_2r_3d_1}{d_1d_2d_3} = \frac{r_1r_3d_2d_3 + r_2r_3d_1d_3}{d_1d_3d_2d_3} = \frac{r_1r_3}{d_1d_3} + \frac{r_2r_3}{d_2d_3}\\ &= \frac{r_1}{d_1} \times \frac{r_3}{d_3} + \frac{r_2}{d_2} \times \frac{r_3}{d_3}\end{aligned}$$

5. $(R,\times)$ has an identity: $\dfrac{r_1}{d_1} \times 1_Q = \dfrac{r_1}{d_1} \times \dfrac{d}{d} = \dfrac{r_1d}{d_1d} = \dfrac{r_1}{d_1}$.
6. Elements of D become unit in Q: Define $\iota : R \to Q : r \mapsto \dfrac{rd}{d}$ where $d \in D$ is any fixed element in D.
   Then, ι is Ring-Monomorphsim because:

   6-1. Well-Defined and Injective: $\iota(r_1) = \iota(r_2) \iff \dfrac{r_1d}{d} = \dfrac{r_2d}{d} \iff (r_1 - r_2)dd = 0 \iff r_1 = r_2$

□

Chapter 5

Field Theory

Chapter 6

Category

Chapter 7

General Topology

7.1 Complete Metric Space

Definition 2. Let (X, d) be a Metric Space, and $\{p_n\}$ be a Sequence in X.
The Sequence $\{p_n\}$ is called **Cauchy Sequence** if:

For any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $m, n \geq N \implies d(p_m, p_n) < \epsilon$.

A Metric Space (X, d) is said to be **Complete** if every Cauchy Sequnces Converge.

Lemma 4. Let $\{E_n\}$ be a sequence of closed bounded non-empty subsets in a Complete Metric Space X such that $E_n \supset E_{n+1}$.
If $\lim_{n\to\infty} \text{diam} E_n = 0$, then $\bigcap_{n=1}^{\infty} E_n = \{p\}$ for some $p \in X$.

Proof. For each $n \in \mathbb{N}$, construct $p_n \in E_n$.
Let $\epsilon > 0$ be given. Since $\text{diam} E_n \to 0$, there is $N \in \mathbb{N}$ such that $\text{diam} E_n < \epsilon$.
For any $m, n \geq M$, E_N contains p_m, p_n. That is, $d(p_m, p_n) < \epsilon$. Thus, $\{p_n\}$ be a Cauchy sequence of X.
Since X is complete, there is a unique point $p \in X$ such taht $p_n \to p$. Let $N \in \mathbb{N}$ be a integer such that $n \geq N \implies |p_n - p| < \epsilon$.
Now, for each $n \geq N$, E_n has a limit point as p. And for any $n \in \mathbb{N}$, E_n contains $E_N, E_{N+1}, \ldots$, thus for all $n \in \mathbb{N}$, E_n has a limit point as p. Meanwhile, E_n closed, $p \in E_n, \ \forall n \in \mathbb{N}$.
Consequently, $p \in \bigcap_{n=1}^{\infty} E_n$. If there is $q \in X$ such that $p \neq q$, $q \in \bigcap_{n=1}^{\infty} E_n$. Then, $\text{diam} E_n \geq d(p, q) > 0, \ \forall n \in \mathbb{N}$. □

7.1.1 Baire Category

Definition 3. The Topological Space X is called **Baire Space** if:

If $\{G_n \mid n \in \mathbb{N}\}$ be a Countable Collection of dense open sets of X, then $\overline{\bigcap_{n=1}^{\infty} G_n} = X$

In brief, every Countable intersection of dense open sets be dense in X.

Theorem 9. Locally Compact Hausdorff Space is Baire Space.

Theorem 10. Complete Metric Space is Baire Space.

Proof. Suppose that $\{G_n \mid n \in \mathbb{N}\}$ be a Countable Collection of dense open set of Complete Metric Space.
Let an open $U \in \mathcal{T}$ be given. Since G_n is dense in the Space, $U \cap G_1$ is non-empty open set.

Thus, there exists a $p_1 \in U \cap G_1$ such that for some $r_1 > 0$, $B_{r_1}(p_1) \subset U \cap G_1$.
Then, automatically,

$$B_{\frac{r_1}{2}}(p_1) \subset \overline{B_{\frac{r_1}{2}}(p_1)} \subset B_{r_1}(p_1) \subset U \cap G_1$$

Set $E_1 = U,\ E_2 = B_{\frac{r_1}{2}}(p_1)$.
Suppose that $E_1, \ldots, E_{n-1}$ are chosen. Then, since $E_{n-1} \cap G_{n-1}$ is open, being intersection of opens.
Thus there exists a point $p_{n-1} \in E_{n-1} \cap G_{n-1}$ and exists r_{n-1} such that

$$B_{r_{n-1}}(p_{n-1}) \subset E_{n-1} \cap G_{n-1} \subset E_{n-1}$$

This implies that

$$B_{\frac{r_{n-1}}{2}}(p_{n-1}) \subset \overline{B_{\frac{r_{n-1}}{2}}(p_{n-1})} \subset B_{r_{n-1}}(p_{n-1}) \subset E_{n-1} \cap G_{n-1} \subset E_{n-1}$$

Set $E_n = B_{\frac{r_{n-1}}{2}}(p_{n-1})$. Since inductively construction of $\{E_n\}$, $E_{n+1} \subset E_n$ and $\overline{E_n} \subset G_n$ for all $n \in \mathbb{N}$.
Consequently,

$$U \cap \left(\bigcap_{n=1}^{\infty} G_n \right) = \bigcap_{n=1}^{\infty} (U \cap G_n) \supset \bigcap_{n=1}^{\infty} (U \cap \overline{E_n}) = U \cap \left(\bigcap_{n=1}^{\infty} \overline{E_n} \right) = \bigcap_{n=1}^{\infty} \overline{E_n} \neq \emptyset$$

□

Definition 4. Let X be a Topological Space.
$A \subset X$ is said to be **nowhere dense subset** if $(\overline{A})^\circ = \emptyset$.

1. $B \subset X$ is called **first category** if B can be representive by union of countable nowhere dense subsets.

2. If the subset is not first category, then it is said to be **second category**.

7.1.2 Nowhere Differentiable function

7.1.3 Banach Fixed Point Theorem

Definition 5. Let $f : X \to X$ be any function. A point $x \in X$ is called a **fixed point** of f if $f(x) = x$.

Definition 6. Let X be a Metric Space. A map $f : X \to X$ is called **Contractive** with respect to the metric d if:

There exsits $\alpha \in (0,1)$ such that for all $x, y \in X$, $d(f(x), f(y)) \leq \alpha d(x, y)$.

Theorem 11. Banach Fixed point Theorem
Let (X, d) be a Complete Metric Space, and $f : X \to X$ be a Contractive map.
Then, there exists a unique fixed point of f, $x^* \in X$.

Proof. Clearly,

$$\text{Contractive} \implies \text{Lipschitz Condition} \implies \text{Continuous}.$$

Thus, f is Continuous.
Let $x_0 \in X$ be arbitrary, and construct a sequence $\{x_n\}$ recursively as follows:

$$x_{n+1} \stackrel{\text{def}}{=} f(x_n), \;\; n \geq 0$$

Then, for any $n \geq 0$,

$$\begin{aligned} d(x_{n+1}, x_n) &= d(f(x_n), f(x_{n-1})) \leq \alpha d(x_n, x_{n-1}) \\ &= d(f(x_{n-1}), f(x_{n-2})) \leq \alpha^2 d(x_{n-1}, x_{n-2}) \\ &\;\;\vdots \\ &\leq \alpha^n d(x_1, x_0) \end{aligned}$$

Let $\epsilon > 0$ be given. Put $N \in \mathbb{N}$ such that $\alpha^N \cdot d(x_1, x_0) < \epsilon(1-\alpha)$. Then, $n \geq m \geq N$ implies that

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + \cdots + d(x_{m+1}, x_m) \\ &\leq \alpha^n d(x_1, x_0) + \alpha^{n-1} d(x_1, x_0) + \cdots + \alpha^{m+1} d(x_1, x_0) \\ &= \alpha^{m+1} d(x_1, x_0) \sum_{r=0}^{n-m-1} \alpha^r < \alpha^N d(x_1, x_0) \sum_{r=0}^{\infty} \alpha^r < \epsilon(1-\alpha) \frac{1}{1-\alpha} = \epsilon \end{aligned}$$

Therefore, $\{x_n\}$ is Cauchy sequence. Since X is Complete, for some $x^* \in X$, $\lim_{n\to\infty} x_n = x^*$. Consequently,

$$\lim_{n\to\infty} f(x_n) \stackrel{f \text{ conti.}}{=} f\left(\lim_{n\to\infty} x_n\right) = f(x^*) = \lim_{n\to\infty} x_{n+1} = x^*$$

$\square$

7.2 Urysohn Metrization Theorem

7.2.1 Urysohn Lemma

Recall that:

Definition 7. X is T_4 if: For any disjoint closed set A and B, there exist disjoint open U, V such that $A \subseteq U$ and $B \subseteq V$.

Lemma 5. X is T_4 Space if and only if For any closed C and open U with $C \subseteq U$, there exists open O such that

$$\underset{\text{closed}}{C} \subseteq \underset{\text{open}}{O} \subseteq \underset{\text{closed}}{\overline{O}} \subseteq \underset{\text{open}}{U}$$

Proof. Proof of the left direction only.
Let X be a T_4 Space, and $C \subset X$ be a closed, U be a open containing C. Then, $C \subset U$ implies $U^c \subset C^c$, thus U^c is a closed set disjoint from C. By T_4 condition, There exist disjoint opens O, O' such that $C \subset O$ and $U^c \subset O' \iff O'^c \subset U$.
Since $O \cap O' = \emptyset \iff O \subset O'^c$, O contained in U, this implies that $C \subset O \subset U$.
Since closure is the smallest closed set such that contains it, consequently $C \subset O \subset \overline{O} \subset O'^c \subset U$. □

Definition 8. Let X be a Toplogical Space, and $A, B \subset X$ are disjoint closed subset.
A real-valued Continuous map $f : X \to [a, b]$ is called **Urysohn function** for A and B if: $f|_A = a$ and $f|_B = b$.
In another form,

$$f : X \to [a, b] : x \to \begin{cases} a & x \in A \\ b & x \in B \\ f(x) & x \notin A \cup B \end{cases}$$

Lemma 6. Urysohn Lemma
T_4 Space has an Urysohn function for any two disjoint closed subsets.

Proof. Generalization is the last thing to proven, first of all, prove in case of $[a, b] = [0, 1]$. This proof consists by three Step.
Let X be a T_4 Space, and $A, B \subset X$ be closed subsets.
Step 1. Construct a Chain of Open sets with Dyadic Rational Indices.
Consider a set of **Dyadic Rationals** $D \overset{\text{def}}{=} \left\{ \frac{k}{2^n} \,\middle|\, n, k \in \mathbb{N},\ k \leq 2^n - 1 \right\}$. We will show that the following statement holds:

$$\text{For any } r, s \in D \text{ with } r < s, \text{ there exist open sets } U_r, U_s \text{ such that } A \subseteq \overline{U}_r \subseteq U_s \subseteq X \setminus B \qquad (*)$$

For this, Enough to Show that: For any $k \in \mathbb{N}$, there exists a Chain as:

$$A \subseteq U_{\frac{1}{2^k}} \subseteq \overline{U}_{\frac{1}{2^k}} \subseteq U_{\frac{2}{2^k}} \subseteq \overline{U}_{\frac{2}{2^k}} \subseteq \cdots \subseteq U_{\frac{2^k-1}{2^k}} \subseteq \overline{U}_{\frac{2^k-1}{2^k}} \subseteq X \setminus B$$

(Note that this opens in the Chain are not necessary distinct: For instance, if Ambient Space is Finite, then the Space is **Noetherian**. That is, X satisfies Ascending Chain Condition for open sets.)
Let $k = 1$. Then, By T_4 condition gives that: There exists an open set U_1 such that

$$A \subseteq U_1 \subseteq \overline{U_1} \subseteq X \setminus B$$

Now, naming this U_1 as $U_{\frac{1}{2}}$, proved when $k = 1$.
Suppose that for some $k > 1$, the Chain exists as:

$$\underset{\text{closed}}{A} \overset{*1}{\subseteq} \underset{\text{open}}{U_{\frac{1}{2^k}}} \subseteq \underset{\text{closed}}{\overline{U}_{\frac{1}{2^k}}} \overset{*2}{\subseteq} \underset{\text{open}}{U_{\frac{2}{2^k}}} \subseteq \cdots \overset{*2^k-1}{\subseteq} \underset{\text{open}}{U_{\frac{2^k-1}{2^k}}} \subseteq \underset{\text{closed}}{\overline{U}_{\frac{2^k-1}{2^k}}} \overset{*2^k}{\subseteq} \underset{\text{open}}{X \setminus B}$$

By repeatedly applying the T_4 condition 2^k times, as indicated by the indices $*1, *2, \ldots, *2^k$, we can construct 2^k open sets such that:

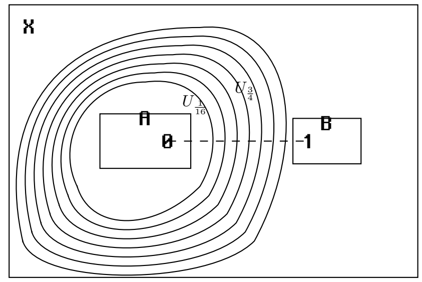
$$A \subseteq U_{\frac{1}{2^{k+1}}} \subseteq \overline{U}_{\frac{1}{2^{k+1}}} \subseteq U_{\frac{1}{2^k}} \subseteq \overline{U}_{\frac{1}{2^k}} \subseteq U_{\frac{3}{2^{k+1}}} \subseteq \overline{U}_{\frac{3}{2^{k+1}}} \subseteq U_{\frac{2}{2^k}} \subseteq \overline{U}_{\frac{2}{2^k}} \quad \subseteq \cdots \subseteq U_{\frac{2^k-1}{2^k}} \subseteq \overline{U}_{\frac{2^k-1}{2^k}} \subseteq U_{\frac{2^{k+1}-1}{2^{k+1}}} \subseteq \overline{U}_{\frac{2^{k+1}-1}{2^{k+1}}} \subseteq X \setminus B$$

Finally, Step 1 proved.

Step 2. Construct an Urysohn Function.

Define a map $f : X \to [0, 1]$ as:

$$f(x) = \begin{cases} 0 & x \in \bigcap_{t \in D} U_t \\ \sup\{t \in D \mid x \notin U_t\} & x \notin \bigcap_{t \in D} U_t \end{cases}$$



Then, this map f is well-defined by $(*)$ and $\sup D \leq 1$. And f satisfies that:

1. $\forall r \in D, \ x \in A \subset U_r$. Thus, $f(x) = 0$ if $x \in A$.
2. $\forall r \in D, \ x \in B \subset X \setminus U_r$. Thus, $f(x) = \sup D = 1$ if $x \in B$.
3. If $x \in \overline{U}_r$, then for every $s > r$, $x \in \overline{U}_r \subset U_s$. Thus, $f(x) \leq r$. In Contrapositive, $f(x) > r \implies x \notin \overline{U}_r$.
   (If $f(x) = \sup\{t \in D \mid x \notin U_t\} > r$, then there is $s \in D$ such that $s > r$ and $x \notin U_s$, Contradiction.)
4. If $x \notin U_r$, then, $f(x) \geq r$. In Contrapositive, $f(x) < r \implies x \in U_r$.

Now, show that this map f is Continuous map: Let $x \in X$ be fixed arbitrarlily, and $\epsilon > 0$ be given.
In Case of $0 < f(x) < 1$.
Since Density of Dyadic Rationals, Choose $r, s \in D$ such that $f(x) - \epsilon < r < f(x) < s < f(x) + \epsilon$.
Now, we obtain that:

$$x \overset{(*)}{\in} U_s \setminus \overline{U}_r \overset{(**)}{\subseteq} f^{-1}[(f(x) - \epsilon, f(x) + \epsilon)]$$

$(*)$ directly given by above properties, $(**)$ given applying the fact that $x \in U_s \subset \overline{U}_s$ and $x \notin \overline{U}_r$.
In Case of $f(x) = 0$.
Choose $r \in D$ such that $f(x) = 0 < r < \epsilon = f(x) + \epsilon$. Then,

$$x \in U_r \subset f^{-1}[(f(x), f(x) + \epsilon)]$$

In Case of $f(x) = 1$.
Choose $r \in D$ such that $f(x) - \epsilon = 1 - \epsilon < r < 1 = f(x)$. Then,

$$x \in X \setminus U_r \subset f^{-1}[(f(x) - \epsilon, f(x))]$$

Consequently, f is Continuous map on $[0, 1]$ such that $f|_A = 0$ and $f|_B = 1$.

Step 3. Generalization.

Since $[0, 1] \cong [a, b]$ for any $a < b$, let $g : [0, 1] \to [a, b] : x \mapsto (1 - x)a + xb$ be a Homeomorphism.
Then, $h = g \circ f : X \to [a, b]$ becomes a Continuous map such that $h|_A = a$ and $h|_B = b$. $\square$

7.2.2 Tietze Extension Theroem

> **Theorem 12. Tietze Extension Theroem**
> Let X be a T_4 Space, and $A \subseteq X$ be a closed subset.
> For any Continuous map $f : A \to \mathbb{R}$, there exists a Continuous map:
>
> $$g : X \to \mathbb{R} \ \ \textbf{s.t.} \quad g|_A = f$$
>
> This g is called **extension** of f.

Proof. This proof consists by three steps.
Step 1. First, we will show that:

$$\text{For any Continuous map } f : A \to [-r, r], \text{ there is a Continuous map } h : X \to \mathbb{R} \textbf{ s.t. } \begin{cases} \forall x \in X, \ \ |h(x)| \leq \frac{1}{3}r \\ \forall a \in A, \ \ |f(a) - h(a)| \leq \frac{2}{3}r \end{cases} \tag{$*$}$$

Set

$$I_1 \overset{\text{def}}{=} \left[-r, -\frac{1}{3}r\right], \ \ I_2 \overset{\text{def}}{=} \left[-\frac{1}{3}r, \frac{1}{3}r\right], \ \ I_3 \overset{\text{def}}{=} \left[\frac{1}{3}r, r\right]$$

Then, the preimage of continuous map preserves closed and A is closed subspace of X, $f^{-1}[I_1]$ and $f^{-1}[I_3]$ are closed of X.
And, I_1 and I_3 are disjoint, thus $f^{-1}[I_1 \cap I_3] = f^{-1}[I_1] \cap f^{-1}[I_3] = \emptyset$.
Now, apply the **Urysohn Lemma**: There exists an Urysohn function $h : X \to I_2$ for $f^{-1}[I_1]$ and $f^{-1}[I_3]$.
Clearly, this map h satisfies the first condition in $(*)$. And, for show the second condition, let $a \in A$ be given.
If $a \in f^{-1}[I_1]$, then $f(a) \in I_1$ and $h(a) = -\frac{1}{3}r$, thus $|f(a) - h(a)| \leq \frac{2}{3}r$.
If $a \in f^{-1}[I_3]$, then $f(a) \in I_3$ and $h(a) = \frac{1}{3}r$, thus $|f(a) - h(a)| \leq \frac{2}{3}r$.
If $a \notin (f^{-1}[I_1] \cup f^{-1}[I_3])$, then $f(a), h(a) \in I_2$, thus $|f(a) - h(a)| \leq \frac{2}{3}r$.
Therefore, the second condition satisfied.
Step 2. We will show that: for any $f : A \to [-1, 1]$, there exists an extension of f.
Apply the result in Step 1, there exists a Continuous map:

$$h_1 : X \to \mathbb{R} \textbf{ s.t. } \begin{cases} \forall x \in X, \ \ |h_1(x)| \leq \frac{1}{3} \\ \forall a \in A, \ \ |f(a) - h_1(a)| \leq \frac{2}{3} \end{cases}$$

Now, the second condition of h_1, the continuous map $f - h_1 : A \to \left[-\frac{2}{3}, \frac{2}{3}\right] : x \to f(x) - h_1(x)$ is well-defined.
Again, there exists a Continuous map:

$$h_2 : X \to \mathbb{R} \textbf{ s.t. } \begin{cases} \forall x \in X, \ \ |h_2(x)| \leq \frac{1}{3} \cdot \frac{2}{3} \\ \forall a \in A, \ \ |f(a) - h_1(a) - h_2(a)| \leq \left(\frac{2}{3}\right)^2 \end{cases}$$

Inductively, for any $n \in \mathbb{N}$, there exists a Continuous map:

$$h_n : X \to \mathbb{R} \textbf{ s.t. } \begin{cases} \forall x \in X, \ \ |h_n(x)| \leq \frac{1}{3} \cdot \left(\frac{2}{3}\right)^{n-1} \\ \forall a \in A, \ \ |f(a) - h_1(a) - h_2(a) - \cdots - h_n(a)| \leq \left(\frac{2}{3}\right)^n \end{cases}$$

Define a map

$$g : X \to [-1, 1] : x \mapsto \sum_{n=1}^{\infty} h_n(x)$$

For any $x \in X$,

$$|g(x)| = \left|\sum_{n=1}^{\infty} h_n(x)\right| \leq \sum_{n=1}^{\infty} |h_n(x)| \leq \sum_{n=1}^{\infty} \frac{1}{3} \cdot \left(\frac{2}{3}\right)^{n-1} = \frac{1}{3} \cdot \frac{1}{1 - \frac{2}{3}} = 1$$

Therefore, this map is well-defined. And, **Weierstrass M-test** gives that $\sum_{n=1}^{\infty} h_n(x)$ converges uniformly.
Moreover, for any $a \in A$,

$$\left|f(a) - \sum_{k=1}^{n} h_k(a)\right| \leq \left(\frac{2}{3}\right)^n \implies \left|f(a) - \sum_{n=1}^{\infty} h_n(a)\right| = |f(a) - g(a)| = 0$$

That is, g is Continuous on X and $g|_A = f$. Therefore, g is extension of f.
Step 3. Finally, we generalize the result in Step 2.:
Let $f : A \to [a, b]$ be a Continuous map on the closed subspace A. And, let $\varphi : [a, b] \to [-1, 1]$ be a Homeomorphism.
Then, $\varphi \circ f : A \to [-1, 1]$ is Continuous map, thus there exists an extension $g : X \to [-1, 1]$ such that $g|_A = \varphi \circ f$.
Now, $\varphi^{-1} \circ g : X \to [a, b]$ is Continuous, and $(\varphi^{-1} \circ g)|_A = \varphi^{-1} \circ \varphi \circ f = f$, Therefore this $\varphi^{-1} \circ g$ is the extension of f.
Let $f : A \to \mathbb{R}$ be a Continuous map on the closed subspace A.
And, let $\varphi : \mathbb{R} \to (-1, 1)$ be a Homeomorphism. Then, the map $\phi : \mathbb{R} \to [-1, 1] : x \mapsto \varphi(x)$ is still Continuous.
Now, The Continuous map $\phi \circ f : A \to [-1, 1]$ has an extension $g : X \to [-1, 1]$ such that $g|_A = \phi \circ f$.
Put $B = g^{-1}[\{-1, 1\}]$. Then B is Closed on X, and $A \cap B = \emptyset$. Now, apply the Urysohn Lemma to this, there exists an Urysohn function for A and B: Continuous map $\gamma : X \to [0, 1]$ such that $\gamma|_A = 1$ and $\gamma|_B = 0$.
Define a map $\eta : X \to (-1, 1) : x \mapsto g(x)\gamma(x)$. Then, if $g(x) = 1$ or $g(x) = -1$, then $x \in B$, thus $g(x)\gamma(x) = 0$.
Therfore, η is well-defined. And, for any $a \in A$, $\eta(a) = g(a)\gamma(a) = g(a)$, thus $\eta|_A = \phi \circ f$.
Consequently, the map $\phi^{-1} \circ \eta$ is an extension of f, we wanted. □

Recall that:

Definition 9. X is T_1 if: For any distinct $x, y \in X$, there exist open sets U_x, U_y such that $\begin{cases} x \in U_x, & x \notin U_y \\ y \notin U_x, & y \in U_y \end{cases}$.

Lemma 7. X is T_1 if and only if For any $x \in X$, a singleton $\{x\}$ is closed in X.

Proof. The left direction is clear.
Let $x \in X$. Then, for any $y \in X$ with $y \neq x$, T_1 condition gives that there is an open set such that $y \in U_y$ and $x \notin U_y$.
Now, the union

$$\bigcup_{\substack{y \in X \\ y \neq x}} U_y = X \setminus \{x\}$$

is open by definition. □

7.2.3 Urysohn Metrization Theorem

> **Theorem 13. Urysohn Metrization Theroem**
> If X is a Second-Countable Regular Space, then X is Metrizable.

Chapter 8

Algebraic Topology

Chapter 9

Basic Analysis

9.1 Arithmetic means

Let $\{s_n\}$ be a Complex numbers Sequence. Define the **Arithmetic means** of $\{s_n\}$:

$$\sigma_n \overset{\text{def}}{=} \frac{s_0 + \cdots + s_n}{n+1} = \frac{1}{n+1}\left(\sum_{i=0}^{n} s_i\right)$$

Then, the Arithmetic means σ_n has following properties:
1). If $\lim_{n\to\infty} s_n = s$, then $\lim_{n\to\infty} \sigma_n = s$.

Proof. Let $\epsilon > 0$ be given. Then, there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $|s_n - s| < \epsilon$.
Now, for $n \geq N$,

$$\begin{aligned}
|\sigma_n - s| &= \left|\frac{s_0 + \cdots + s_n}{n+1} - \frac{(n+1)s}{n+1}\right| = \left|\frac{(s_0 - s) + \cdots + (s_n - s)}{n+1}\right| \\
&\overset{\text{tri.ieq}}{\leq} \frac{\sum_{k=0}^{N-1} |s_k - s|}{n+1} + \frac{\sum_{k=N}^{n} |s_k - s|}{n+1} \\
&< \frac{\sum_{k=0}^{N-1} |s_k - s|}{n+1} + \frac{n+1-N}{n+1} \cdot \epsilon \\
&< \frac{\sum_{k=0}^{N-1} |s_k - s|}{n+1} + \epsilon
\end{aligned}$$

Now, put $M \in \mathbb{N}$ satisfies $M \geq N$ and $n \geq M \implies \frac{\sum_{k=0}^{N-1} |s_k - s|}{n+1} < \epsilon$, using Archimedean property.
Then, $n \geq M$ implies $|\sigma_n - s| < \epsilon$, thus $\sigma_n \to s$. □

2). Put $a_n = s_n - s_{n-1}$, for $n \geq 1$. If $\lim_{n\to\infty} n a_n = 0$ and σ_n converges, then s_n converges.

Proof. First,

$$\begin{aligned}
s_n - \sigma_n &= s_n - \frac{s_0 + \cdots + s_n}{n+1} = \frac{(n+1)s_n - \sum_{k=0}^{n} s_k}{n+1} \\
&= \frac{1}{n+1}((s_1 - s_0) + (2s_2 - 2s_1) + (3s_3 - 3s_2) + \cdots + (ns_n - ns_{n-1})) \\
&= \frac{1}{n+1}\sum_{k=1}^{n} k a_k
\end{aligned}$$

Now, if $na_n \to 0$ and $\sigma_n \to \sigma$,

$$\begin{aligned}
\lim_{n\to\infty} s_n &= \lim_{n\to\infty}\left(\sigma_n + \frac{1}{n+1}\sum_{k=1}^{n} k a_k\right) \\
&= \lim_{n\to\infty} \sigma_n + \lim_{n\to\infty} \frac{1}{n+1}\sum_{k=1}^{n} k a_k \overset{1)}{=} \sigma
\end{aligned}$$

□

2) is conditional converse of 1). But, there is more weak version of the converse proposition:
3). The sequence $\{na_n\}$ bounded by $M < \infty$, and $\sigma_n \to \sigma$. Then, $s_n \to \sigma$.

Proof. First, For positive integers $m < n$,

$$\begin{aligned}
s_n - \sigma_n &= s_n - \frac{\sum_{k=0}^{n} s_k}{n+1} = s_n - \frac{m+1}{n-m} \cdot \left(\frac{1}{m+1} - \frac{1}{n+1}\right)\sum_{k=0}^{n} s_k \\
&= s_n - \frac{m+1}{n-m} \cdot \left(\frac{\sum_{k=0}^{m} s_k + \sum_{k=m+1}^{n} s_k}{m+1} - \frac{\sum_{k=0}^{n} s_k}{n+1}\right) \\
&= s_n - \frac{m+1}{n-m} \cdot \left(\sigma_m - \sigma_n + \frac{\sum_{k=m+1}^{n} s_k}{m+1}\right) \\
&= \frac{m+1}{n-m}(\sigma_n - \sigma_m) + \frac{1}{n-m}\sum_{k=m+1}^{n} (s_n - s_k)
\end{aligned}$$

Meanwhile, since for any $n \in \mathbb{N}$, $|na_n| = n|s_n - s_{n-1}| < M$, for $k = m+1, \dots, n$,

$$\begin{aligned}|s_n - s_k| &= |s_n - s_{n-1} + s_{n-1} - s_{n-2} + \cdots + s_{k+1} - s_k| \\ &\le |s_n - s_{n-1}| + |s_{n-1} - s_{n-2}| + \cdots + |s_{k+1} - s_k| \\ &\le \frac{M}{n} + \frac{M}{n-1} + \cdots + \frac{M}{k+1} \le \frac{n-k}{k+1}M \le \frac{n-k}{m+2}M \le \frac{n-m-1}{m+2}M\end{aligned}$$

Let $\epsilon > 0$ be given. For each $n \in \mathbb{N}$, put $m \in \mathbb{N}$ such that

$$m \le \frac{n-\epsilon}{1+\epsilon} < m+1$$

Then,

$$m(1+\epsilon) \le n - \epsilon \implies m + \epsilon(1+m) \le n \implies \frac{m+1}{n-m} \le \frac{1}{\epsilon}$$

and

$$n - \epsilon < (m+1)(1+\epsilon) \implies n+1 < (m+2)(1+\epsilon) \implies \frac{n+1}{m+2} - 1 < \epsilon \implies \frac{n-m-1}{m+2} < \epsilon$$

Now, for arbitrary $n \in \mathbb{N}$,

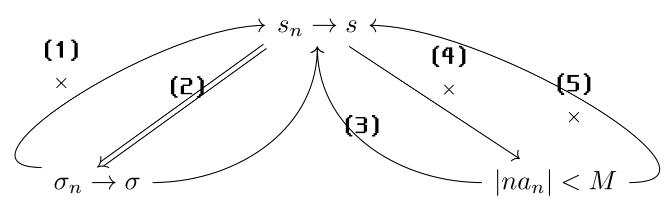
$$\begin{aligned}&|s_n - \sigma| \le |s_n - \sigma| + |\sigma_n - \sigma| \\ \implies &\limsup_{n\to\infty} |s_n - \sigma| \le \limsup_{n\to\infty} |s_n - \sigma_n| + \limsup_{n\to\infty} |\sigma_n - \sigma|\end{aligned}$$

And,

$$\begin{aligned}|s_n - \sigma_n| &= \frac{m+1}{n-m}|\sigma_n - \sigma_m| + \frac{1}{n-m}\sum_{k=m+1}^{n} |s_n - s_k| < \frac{1}{\epsilon}|\sigma_n - \sigma_m| + M\epsilon \\ \implies \limsup_{n\to\infty} |s_n - \sigma_n| &\le \frac{1}{\epsilon}\limsup_{n\to\infty} |\sigma_n - \sigma_m| + M\epsilon = M\epsilon\end{aligned}$$

Consequently, $\limsup\limits_{n\to\infty} |s_n - \sigma| \le (M+1)\epsilon$, thus $s_n \to \sigma$. $\square$

In brief, the diagram of the above conditions like this:



Examples and Counterexamples of the Diagram:

(1) Let $s_n \overset{\text{def}}{=} \exp(\frac{in\pi}{2})$. Then,

- s_n diverges.
- na_n diverges.
- $\sigma_n \to 0$.

(2) Let $s_n \overset{\text{def}}{=} \frac{1}{n}$, $s_0 = 0$.

(3) Let $s_n \overset{\text{def}}{=} \sum_{k=1}^{n} \frac{1}{k}$. Then,

- s_n diverges.
- $a_n = \frac{1}{n}$, thus $na_n \to 1$, bounded.
- If σ_n converges, then the diagram implies that s_n must converge, leading to a contradiction. Therefore, σ_n diverges.

(4) $s_n = \sum_{k=1}^{n} \frac{(-1)^k}{\sqrt{k}}$, $s_0 = 0$. Then,

- s_n converges, being the Alternating series Test.
- $a_n = \frac{(-1)^n}{\sqrt{n}}$, thus na_n diverges.

9.2 Taylor's Theorem

Theorem 14. Taylor's Theorem

Let $f : [a, b] \to \mathbb{R}$, and let $n \in \mathbb{N}$ be fixed. Suppose that $\begin{cases} f^{(n-1)} \text{ is Continuous.} \\ f^{(n)}(t) \text{ exists for every } t \in (a, b). \end{cases}$

Then, for any $\alpha, \beta \in [a, b]$, there exists $x \in (\alpha, \beta)$ such that

$$f(\beta) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(\beta - \alpha)^k + \frac{f^{(n)}(x)}{n!}(\beta - \alpha)^n$$

Proof. Put

$$M \stackrel{\text{def}}{=} \frac{1}{(\beta - \alpha)^n} \cdot \left(f(\beta) - \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(\beta - \alpha)^k \right)$$

That is,

$$f(\beta) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(\beta - \alpha)^k + M(\beta - \alpha)^n$$

and put

$$g(t) \stackrel{\text{def}}{=} f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(t - \alpha)^k - M(t - \alpha)^n, \quad (a \le t \le b)$$

If we differentiate the above equation n times,

$$g^{(n)}(t) = f^{(n)}(t) - n!M, (a < t < b)$$

For each $k = 0, 1, \ldots, n - 1$,

$$\begin{aligned} \frac{d^r}{dt^r} \left(\sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(t - \alpha)^k \right) &= \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!} \cdot \frac{d^r}{dt^r} \left((t - \alpha)^k\right) \\ &= \sum_{k=r+1}^{n-1} \frac{f^{(k)}(\alpha)}{k!} \cdot \frac{k!}{(k - r)!}(t - \alpha)^{k-r} + f^{(r)}(\alpha) \\ &= \sum_{k=r+1}^{n-1} \frac{f^{(k)}(\alpha)}{(k - r)!}(t - \alpha)^{k-r} + f^{(r)}(\alpha) \end{aligned}$$

Substituting $t = \alpha$, only the $f^{(r)}(\alpha)$ term remains. Therefore, for $r = 0, \ldots, n-1$, $g(\alpha) = g'(\alpha) = \cdots = g^{(n-1)}(\alpha) = 0$.
Since $g(\beta) = 0$ by definition, the Mean-Value Theorem implies there exists a $x_1 \in (\alpha, \beta)$ s.t. $g'(x_1) = \frac{g(\beta)-g(\alpha)}{\beta-\alpha} = 0$.
And similarly, there is $x_2 \in (x_1, \beta)$ s.t. $g''(x_2) = \frac{g'(x_1)-g'(\alpha)}{\beta-\alpha} = 0$.
Inductively, for some $x_n \in (\alpha, \beta)$, $g^{(n)}(x_n) = f^{(n)}(x_n) - n!M = 0$. That is, $M = \dfrac{f^{(n)}(x_n)}{n!}$.
Proof Complete by Initial Setting. □

Corollary 2. Let $f : [a, b] \to \mathbb{R}$ be an infinitely differentiable function.
Suppose that there exists a $M > 0$ such that for any $n \in \mathbb{N}$, $\sup_{t \in [a,b]} |f^{(n)}(t)| \le M$. Then, for any $x, \alpha \in [a, b]$,

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x)}{k!}(x - \alpha)^k$$

9.3 Convexity

9.3.1 Definition

Definition 10. Let $f:(a,b)\to\mathbb{R}$ be a Real-valued function. f is said to be **convex** if: For any $x,y\in(a,b),\lambda\in(0,1)$,

$$f(\lambda x+(1-\lambda)y)\le\lambda f(x)+(1-\lambda)f(y)$$

Convex function has follwing properties:

Lemma 8. Let $f:(a,b)\to\mathbb{R}$ be a Convex function, and $a<x_1<x_2<x_3<b$. Then,

$$\frac{f(x_2)-f(x_1)}{x_2-x_1}\le\frac{f(x_3)-f(x_1)}{x_3-x_1}\le\frac{f(x_3)-f(x_2)}{x_3-x_2}$$

Proof. To show that first inequalty, note that

$$\frac{x_2-x_1}{x_3-x_1}\cdot x_3+\left(1-\frac{x_2-x_1}{x_3-x_1}\right)\cdot x_1=\frac{(x_2-x_1)x_3+(x_3-x_2)x_1}{x_3-x_1}=x_2$$

Now,

$$\begin{aligned}f(x_2)&=f\left(\frac{x_2-x_1}{x_3-x_1}\cdot x_3+\left(1-\frac{x_2-x_1}{x_3-x_1}\right)\cdot x_1\right)\\&\le\frac{x_2-x_1}{x_3-x_1}\cdot f\left(x_3\right)+\left(1-\frac{x_2-x_1}{x_3-x_1}\right)\cdot f\left(x_1\right)=\frac{x_2-x_1}{x_3-x_1}\cdot f\left(x_3\right)+f(x_1)-\left(\frac{x_2-x_1}{x_3-x_1}\right)\cdot f\left(x_1\right)\end{aligned}$$

In brief,

$$f(x_2)-f(x_1)\le\frac{x_2-x_1}{x_3-x_1}(f(x_3)-f(x_1))\implies\frac{f(x_2)-f(x_1)}{x_2-x_1}\le\frac{f(x_3)-f(x_1)}{x_3-x_1}$$

And similarly,

$$\frac{x_3-x_2}{x_3-x_1}\cdot x_1+\left(1-\frac{x_3-x_2}{x_3-x_1}\right)x_3=x_2$$

gives the second inequality. □

9.3.2 Properties

Proposition 2. If $f:(a,b)\to\mathbb{R}$ is Convex, then f is Continuous.

Proof. Let $\epsilon>0$ be given, $s<t$ are fixed in (a,b). For any $x,y\in(s,t)$ with $s<x<y<t$,

$$\frac{f(s)-f(a)}{s-a}\le\frac{f(x)-f(s)}{x-s}\le\frac{f(y)-f(x)}{y-x}\le\frac{f(t)-f(y)}{t-y}\le\frac{f(b)-f(t)}{b-t}$$

Put $M=\max\left\{\left|\frac{f(s)-f(a)}{s-a}\right|,\left|\frac{f(b)-f(t)}{b-t}\right|\right\}$. Then, for any $x,y\in(s,t)$,

$$\left|\frac{f(y)-f(x)}{y-x}\right|\le M$$

Now,

$$|f(y)-f(x)|\le M|y-x|<\epsilon$$

Since $s,t\in(a,b)$ was arbitrary, f is continuous on (a,b). □

Proposition 3. Let f is differentiable on (a,b). Then,

f is Convex if and only if f' is monotonically increasing on (a,b).

Proof. Prove by showing both directions: right and left.
Right Direction Let $x_1<x_2$ in (a,b). Then,

$$f'(x_1)=\lim_{t\to x_1}\frac{f(t)-f(x_1)}{t-x_1}\le\frac{f(x_2)-f(x_1)}{x_2-x_1}\le\lim_{\tau\to x_2}\frac{f(\tau)-f(x_2)}{\tau-x_2}=f'(x_2)$$

More rigorously, put $\epsilon=\left|f'(x_1)-\frac{f(x_2)-f(x_1)}{x_2-x_1}\right|$. (If $\epsilon=0$, then there is nothing to prove.).
Now, there exists a $\delta>0$ such that $|t-x_1|<\delta$ implies

$$\left|f'(x_1)-\frac{f(t)-f(x_1)}{t-x_1}\right|<\epsilon\iff-\epsilon+\frac{f(t)-f(x_1)}{t-x_1}<f'(x_1)\overset{(*)}{\le}\epsilon+\frac{f(t)-f(x_1)}{t-x_1}$$

If $f'(x_1)-\frac{f(x_2)-f(x_1)}{x_2-x_1}>0$, then $(*)$ gives

$$f'(x_1)<f'(x_1)+\frac{f(t)-f(x_1)}{t-x_1}-\frac{f(x_2)-f(x_1)}{x_2-x_1}\iff\frac{f(x_2)-f(x_1)}{x_2-x_1}<\frac{f(t)-f(x_1)}{t-x_1}\quad\forall t\text{ s.t. }|t-x_1|<\delta$$

If $|t-x_1|<|x_2-x_1|$, then this contradicts to Convexity.
Consequently, we obtain the first inequality, similarly can prove the second inequality.
Left Direction Let $x,y\in(a,b)$ and $\lambda\in(0,1)$ be given. The Mean Value Theorem gives that:

$$f(\lambda x+(1-\lambda)y)-f(x)=f'(z_1)(\lambda x+(1-\lambda)y-x)\text{ for some }z_1\in(x,\lambda x+(1-\lambda)y)$$
$$f(y)-f(\lambda x+(1-\lambda)y)=f'(z_2)(y-\lambda x+(1-\lambda)y)\text{ for some }z_2\in(\lambda x+(1-\lambda)y,y)$$

Now, Monotonically increasing gives

$$\begin{aligned}&\frac{f(\lambda x+(1-\lambda)y)-f(x)}{\lambda x+(1-\lambda)y-x}=f'(z_1)\le f'(z_2)=\frac{f(y)-f(\lambda x+(1-\lambda)y)}{y-(\lambda x+(1-\lambda)y)}\\ \implies&\frac{f(\lambda x+(1-\lambda)y)-f(x)}{(1-x)(y-x)}\le\frac{f(y)-f(\lambda x+(1-\lambda)y)}{\lambda(y-x)}\\ \implies&\lambda f(\lambda x+(1-\lambda)y)-\lambda f(x)\le(1-\lambda)f(y)-(1-\lambda)\lambda x+(1-\lambda)y\\ \implies&f(\lambda x+(1-\lambda)y)\le\lambda f(x)+(1-\lambda)f(y)\end{aligned}$$

□

Corollary 3. If $f : [a, b] \to \mathbb{R}$ is twice-differentiable, then

$$f \text{ is Convex if and only if } f''(x) \geq 0 \text{ for all } x \in (a, b).$$

Theorem 15. Let $f : [a.b] \to \mathbb{R}$ be given. Then,

$$f \text{ is Convex if and only if } f \text{ is Continuous, and Midpoint Convex.}$$

Midpoint convex is that f satisfies $\forall x, y \in (a, b),\ f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2}$.

Proof. The right direction is clear. To show the left direction, we demonstrate that **Midpoint Convexity implies Dyadic Rational Convexity**. Claim: For any $n \in \mathbb{N}$,

$$f\left(\frac{\sum_{k=1}^{2^n} x_k}{2^n}\right) \leq \frac{1}{2^n}\sum_{k=1}^{2^n} f(x_k) \tag{$*$}$$

Using Induction: If $n = 1$, it is clear by Midpoint Convexity.
Assume that for $n \in \mathbb{N}$, $(*)$ is True. Then,

$$\begin{aligned}
f\left(\frac{\sum_{k=1}^{2^{n+1}} x_k}{2^{n+1}}\right) &= f\left(\frac{1}{2}\cdot\left[\frac{\sum_{k=1}^{2^n} x_k}{2^n} + \frac{\sum_{k=2^n+1}^{2^{n+1}} x_k}{2^n}\right]\right) \\
&\overset{\text{m.c}}{\leq} \frac{1}{2}\left(f\left(\frac{\sum_{k=1}^{2^n} x_k}{2^n}\right) + f\left(\frac{\sum_{k=2^n+1}^{2^{n+1}} x_k}{2^n}\right)\right) \\
&\overset{(*)}{\leq} \frac{1}{2}\left(\frac{1}{2^n}\sum_{k=1}^{2^n} f(x_k) + \frac{1}{2^n}\sum_{k=2^n+1}^{2^{n+1}} f(x_k)\right) \\
&= \frac{1}{2^{n+1}}\sum_{k=1}^{2^{n+1}} f(x_k)
\end{aligned}$$

Consequently, we obtain the claim. Now, let $n \in \mathbb{N}$, and m be an integer such that $1 \leq m \leq 2^n$.
Put $x_1 = x_2 = \cdots = x_m = x$ and $x_{m+1} = x_{m+2} = \cdots = x_{2^n} = y$. Then

$$f\left(\frac{m}{2^n}x + \left(1 - \frac{m}{2^n}\right)y\right) \leq \frac{m}{2^n}f(x) + \left(1 - \frac{m}{2^n}\right)f(y)$$

For complete this discussion, Let $x, y \in (a, b), \lambda \in (0, 1)$ be given.
Since $\frac{\lfloor 2^n\lambda\rfloor}{2^n} \to \lambda$ as $n \to \infty$, for any $n \in \mathbb{N}$,

$$f\left(\frac{\lfloor 2^n\lambda\rfloor}{2^n}x + \left(1 - \frac{\lfloor 2^n\lambda\rfloor}{2^n}\right)y\right) \leq \frac{\lfloor 2^n\lambda\rfloor}{2^n}f(x) + \left(1 - \frac{\lfloor 2^n\lambda\rfloor}{2^n}\right)f(y)$$

Finally, taking limits then

$$\lim_{n\to\infty} f\left(\frac{\lfloor 2^n\lambda\rfloor}{2^n}x + \left(1 - \frac{\lfloor 2^n\lambda\rfloor}{2^n}\right)y\right) \overset{f\ \text{conti.}}{=} f\left(\lim_{n\to\infty}\left[\frac{\lfloor 2^n\lambda\rfloor}{2^n}x + \left(1 - \frac{\lfloor 2^n\lambda\rfloor}{2^n}\right)y\right]\right) = f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$$

In brief, Midpoint Convexity implies Dyadic Rational Convexity, and with Continuous implies Convexity. □

9.4 Lipschitz Condition

9.4.1 Definition

Definition 11. A real-vauled function $f : (a, b) \to \mathbb{R}$ is called **Lipschitz Continuous** if:

$$\exists L \geq 0 \text{ s.t. } \forall x_1, x_2 \in (a, b),\ |f(x_1) - f(x_2)| \leq L \cdot |x_1 - x_2|$$

The constant L is said to be **Lipschitz Constant** of f. In particular, the constant

$$D \overset{\text{def}}{=} \sup_{x_1 \neq x_2} \frac{|f(x_1) - f(x_2)|}{|x_1 - x_2|}$$

is called **dilation** of f. Clearly,

$$\forall x_1, x_2 \in (a, b),\ |f(x_1) - f(x_2)| \leq D \cdot |x_1 - x_2|$$

and if $L > 0$ is Lipschitz Constant of f, then $D \leq L$. That is, $D = \inf\{L > 0 \mid L \text{ is Lipschitz constant of } f\}$.

9.4.2 Properties

Proposition 4. If $f : (a, b) \to \mathbb{R}$ is Lipschitz Continuous, then f is uniformly continuous.

Proof. Let $L \geq 0$ be a Lipschitz Constant of f. Then, for any $\epsilon > 0$,

$$\forall x, y \in (a, b),\ |x - y| < \frac{\epsilon}{L} \implies |f(x) - f(y)| \leq L|x - y| < \epsilon$$

□

Proposition 5. Let $f : (a, b) \to \mathbb{R}$ be a Differentiable function. Then,

f is Lipschitz Continuous **if and only if** f' is bounded in (a, b).

Proof.
Right Direction
Let $L > 0$ be a Lipschitz constant of f, and $x \in (a, b)$ be given. Since definition of derivative,

$$f'(x) \overset{\text{def}}{=} \lim_{t \to x} \frac{f(x) - f(t)}{x - t}$$

Meanwhile, the assumption gives: for any distinct $x, t \in (a, b)$,

$$\frac{|f(x) - f(t)|}{|x - t|} \leq L$$

Therefore,

$$f'(x) = \lim_{t \to x} \frac{f(x) - f(t)}{x - t} \leq \lim_{t \to x} \frac{|f(x) - f(t)|}{|x - t|} \leq \lim_{t \to x} L = L$$

Left Direction
Let distinct $x, y \in (a, b)$ be given. Then, the Mean-Value Theorem gives: There exists a $z \in (x, y)$ such that

$$f(x) - f(y) = f'(z)(x - y) \implies f'(z) = \frac{f(x) - f(y)}{x - y}$$

Now,

$$\left| \frac{f(x) - f(y)}{x - y} \right| = |f'(z)| \leq L \implies |f(x) - f(y)| \leq L \cdot |x - y|$$

If $x = y$, then there is nothing to prove. □

Note that:

Lipschitz Continuous $\implies$ Uniformly Continuous $\implies$ Continuous

9.4.3 Newton-Raphson Method

> **Theorem 16. Newton-Raphson Method**
> Let $f : [a, b] \to \mathbb{R}$ be a twice-differentiable, $f(a) < 0 < f(b)$. Suppose that f satisfies: for all $x \in [a, b]$,
>
> $$f'(x) \geq \delta > 0 \text{ and } 0 \leq f''(x) \leq M$$
>
> That is, f is strictly increasing convex function, and Lipschitz Continuous.
> Further, there uniquely exists $x^* \in (a, b)$ such that $f(x^*) = 0$.
> Let $x_1 \in (x^*, b)$ fixed. Define a sequence $\{x_n\}$ inductively as follows:
>
> $$x_{n+1} \overset{\text{def}}{=} x_n - \frac{f(x_n)}{f'(x_n)}$$
>
> Then, $\{x_n\}$ satisfies the following three conditions:
>
> 1. $\{x_n\}$ is decreasing sequence.
> 2. $x_n \to x^*$ as $n \to \infty$.
> 3. For any $n \in \mathbb{N}$, $0 \leq x_{n+1} - x^* \leq \left[\frac{M}{2\delta}\right]^{2^{n+1}-1} [x_1 - x^*]^{2^n}$.
>
> Condition 3 means that for a suitable initial value x_1, we can establish an upper bound for the error.

Proof. This proof consists by three steps.
Since f'' is non-negative, and f' is positive, f is strictly increasing convex function.
And Fundamental Theorem of Calculus gives: for any $x \in (a, b)$,

$$f'(x) \overset{\text{FTA}}{=} \int_a^x f''(t)dt + f'(a) \leq \int_a^x M dt + f'(a) = M(x - a) + f'(a) \leq M(b - a) + f'(a)$$

Thus, f' is bounded on (a, b), thus f is Lipschitz Continuous.
Step 1. f has a unique root x^*.
The existence of root given directly by Intermidate-Value theroem.
Suppose that $x^*, x' \in (a, b)$ are distinct root of f. i.e., $f(x^*) = f(x') = 0$. Then, by Mean-value theroem, there is $c \in (a, b)$ between x^* and x' such that

$$f'(c)(x^* - x') = f(x^*) - f(x') = 0$$

That is, $f'(c) = 0$. This is contradiction with f' is positive.
Step 2. $\{x_n\}$ decrease.
Proof by induction:
For $n = 1$, $f'(x_1)(x_1 - x_2) \overset{\text{def}}{=} f(x_1) > f(x^*) = 0$, thus $x_2 < x_1$. And,

$$\begin{aligned} f(x_2) &\overset{\text{MVT}}{=} f(x_1) + f'(c_1)(x_2 - x_1) \quad \text{for some } c_1 \in (x_2, x_1) \\ &> f(x_1) + f'(x_1)(x_2 - x_1) = f'(x_1)(x_1 - x_2) + f'(x_1)(x_2 - x_1) = 0 \end{aligned}$$

Now, since $f(x_2) > 0 = f(x^*)$, the Mean-Value Theorem implies that $x_2 > x^*$.
To use induction, suppose that for some $n \geq 1$, $x^* < x_{n+1} < x_n$. Then,

$$f(x_{n+1}) = f'(x_{n+1})(x_{n+1} - x_{n+2}) > 0$$

Thus $x_{n+2} < x_{n+1}$ and

$$\begin{aligned} f(x_{n+2}) &\overset{\text{MVT}}{=} f(x_{n+1}) + f'(c_{n+1})(x_{n+2} - x_{n+1}) \quad \text{for some } c_{n+1} \in (x_{n+2}, x_{n+1}) \\ &\geq f(x_{n+1}) + f'(x_{n+1})(x_{n+2} - x_{n+1}) \\ &= f(x_{n+1})(x_{n+1} - x_{n+2}) + f'(x_{n+1})(x_{n+2} - x_{n+1}) = 0 \end{aligned}$$

Again, the Mean-Value Theorem implies that $x_{n+2} > x^*$. Therefore, induction completes.
Now, $x_n \to x'$ as $n \to \infty$ for some $x' \in [x^*, x_1]$ since $\{x_n\}$ is Bounded below and Decreasing.
Still it remains that to show $x' = x^*$. By Continuity,

$$\begin{aligned} &f'(x_n)(x_{n+1} - x_n) + f(x_n) = 0 \\ \implies &\lim_{n\to\infty} [f'(x_n)(x_{n+1} - x_n) + f(x_n)] = f\left(\lim_{n\to\infty} x_n\right) = f(x') = 0 \end{aligned}$$

Since the root of f is unique, thus $x' = x^*$.
Step 3. Establishing the error bound.
The Taylor's Theorem implies that

$$f(x^*) = f(x_n) + f'(x_n)(x^* - x_n) + \frac{f''(t_n)}{2}(x^* - x_n)^2 \quad \textbf{for some } t_n \in (x^*, x_n)$$
$$\Longrightarrow x_{n+1} - x^* = \frac{f''(t_n)}{2f'(x_n)}(x^* - x_n)^2$$

Consequently,

$$0 \le x_{n+1} - x^* = \frac{f''(t_n)}{2f'(x_n)}(x_n - x^*)^2 = \frac{f''(t_n)}{2f'(x_n)} \cdot \left(\frac{f''(t_{n-1})}{2f'(x_{n-1})}\right)^2 (x_{n-1} - x^*)^4 = \cdots$$
$$= \prod_{i=1}^{n} \left[\frac{f''(t_i)}{2f'(x_i)}\right]^{2^{(n+1-i)}} [x_1 - x^*]^{2^n} \le \left[\frac{M}{2\delta}\right]^{2^{n+1}-1} [x_1 - x^*]^{2^n}$$

$\square$

9.4.4 Gradient Descent

> **Theorem 17.** Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function that satisfies the following conditions:
>
> 1. f is **Convex function**.
> 2. f' is **Lipschitz Continuous** with Lipschitz constant of f, $L > 0$. In this, f is called L-**Smooth**.
> 3. f has at least one local minimizer x^*.
>
> Then, x^* is a Global minimizer of $\mathbb{R}$, and there exists a unique closed interval M containing x^* such that
>
> $$\forall x \in M, t \notin M, \;\; f(x) = f(x^*) < f(t)$$
>
> And, given initial point $x_0 \in \mathbb{R}$ and $0 < \gamma \leq \frac{1}{L}$, define a sequence $\{x_n\}$ inductively as follows:
>
> $$x_{n+1} = x_t - \gamma \cdot f'(x_n)$$
>
> Then, for any $N \in \mathbb{N}$,
>
> $$f(x_N) - f(x^*) \leq \frac{|x_0 - x^*|^2}{2\gamma N}$$

Proof. Let $x^* \in \mathbb{R}$ be a local minimizer. That is, there exists a $\delta > 0$ such that $\forall t \in (x^* - \delta, x^* + \delta)$, $f(x^*) \leq f(t)$. Then,

$$0 \leq \lim_{t \to x^*+} \frac{f(x^*) - f(t)}{x^* - t} = f'(x^*) = \lim_{t \to x^*-} \frac{f(x^*) - f(t)}{x^* - t} \leq 0$$

thus, $f'(x^*) = 0$. And, by convextiy, f' is monotonically inceasing. Now, The Fundametal Theorem of Calculus gives:

$$\forall x \in \mathbb{R}, \;\; f(x) = \int_{x^*}^{x} f'(t)dt + f(x^*) \geq f(x^*)$$

Therefore, x^* is a Global minimizer of f.
Now, establish the closed interval M. Since f' is Lipschitz Continuous, thus f' is Continuous.
Let $D \overset{\text{def}}{=} \{x \in \mathbb{R} \mid f'(x) = 0\}$. (Note that: $x^* \in D$, thus D is not emtpyset.)
D is closed because: Let $\{x_n\}$ be a convergent sequence in D. That is, for all $n \in \mathbb{N}$, $f(x_n) = 0$. Then, by continuity,

$$f\left(\lim_{n \to \infty} x_n\right) = \lim_{n \to \infty} f(x_n) = 0$$

The limit of $\{x_n\}$ is contained in D, thus D is closed.
And, D is interval: i.e, for any $x \in (\inf D, \sup D)$, $x \in D$ because:
Suppose that there exists $x \in (\inf D, \sup D)$ such that $x \notin D$. That is, $f'(x) \neq 0$. This is Contradiction with Monotonicity.

To set error of upper bound, we make inequalities: Let $x, y \in \mathbb{R}$ be given.
The Fundamental Theorem of Calculus and $L-$Smooth condition gives:

$$\begin{aligned} f(y) - f(x) &= \int_x^y f'(t)dt = \int_0^1 f'(x + (y-x)u)(y-x)du = f'(x)(y-x) + \int_0^1 (f'(x + (y-x)u) - f'(x))(y-x)du \\ &\overset{2.}{\leq} f'(x)(y-x) + L \cdot |y-x|^2 \int_0^1 u \;\, du = f'(x)(y-x) + \frac{L}{2}|y-x|^2 \end{aligned}$$

For any $\lambda > 0$, Put $y = x - \lambda f'(x)$. Then,

$$f(x - \lambda f'(x)) \leq f(x) - f'(x)(\lambda f'(x)) + \frac{L}{2}|\lambda f'(x)|^2 = f(x) + \lambda\left(\frac{L\lambda}{2} - 1\right)|f'(x)|^2$$

Put $\lambda = \frac{1}{L}$, then

$$f\left(x - \frac{f'(x)}{L}\right) \leq f(x) - \frac{L}{2}|f'(x)|^2 \implies \frac{L}{2}|f'(x)|^2 \leq f(x) - f\left(x - \frac{f'(x)}{L}\right) \leq f(x) - \inf f$$

Meanwhile, the convexity gives: for any $x, y \in \mathbb{R}$,

$$f'(x)(y-x) \le f(y) - f(x) \le f'(y)(y-x)$$

since derivative of convex function increase monotonically. Put $z = y - \frac{1}{L}(f'(y) - f'(x))$. Then,

$$\begin{aligned} f(x) - f(y) &= f(x) - f(z) + f(z) - f(y) \\ &\le f'(x)(x-z) + f'(y)(z-y) + \frac{L}{2}|z-y|^2 \\ &= f'(x)\left(x - y + \frac{1}{L}(f'(y) - f'(x))\right) - f'(y)\left(\frac{1}{L}(f'(y) - f'(x))\right) + \frac{L}{2}\left|\frac{1}{L}(f'(y) - f'(x))\right|^2 \\ &= f'(x)(x-y) - \frac{1}{L}|f'(y) - f'(x)|^2 + \frac{1}{2L}|f'(y) - f'(x)|^2 \\ &= f'(x)(x-y) - \frac{1}{2L}|f'(y) - f'(x)|^2 \end{aligned}$$

Now,

$$\begin{cases} \frac{1}{2L}|f'(y) - f'(x)|^2 \le f'(x)(x-y) - (f(x) - f(y)) \\ \frac{1}{2L}|f'(x) - f'(y)|^2 \le f'(y)(y-x) - (f(y) - f(z)) \end{cases} \implies \frac{1}{L}|f'(y) - f'(x)|^2 \le (f'(y) - f'(x))(y-x)$$

Since above inequalities, we obtain that

$$\begin{aligned} |x_{n+1} - x^*|^2 = |x_n - \gamma \cdot f'(x_n) - x^*|^2 &= |(x_n - x^*) - \gamma \cdot f'(x_n)|^2 \\ &= |x_n - x^*|^2 - 2\gamma|f'(x_n)| \cdot |x_n - x^*| + \gamma^2|f'(x_n)|^2 \\ &\le |x_n - x^*|^2 - 2\gamma\frac{1}{L}|f'(x_n)|^2 + \gamma^2|f'(x_n)|^2 \\ &= |x_n - x^*|^2 + \left(\gamma^2 - \frac{2\gamma}{L}\right)|f'(x_n)|^2 \le |x_n - x^*|^2 \end{aligned}$$

Thus, $|x_n - x^*|$ decrease as $n \to \infty$. That is, $|x_n - x^*| \le |x_0 - x^*|$ for all $n \in \mathbb{N}$.
Consider x_{n+1} and x_n. First, we obtain

$$\begin{aligned} f(x_{n+1}) &\le f(x_n) + f'(x_n)(x_{n+1} - x_n) + \frac{L}{2}|x_{n+1} - x_n|^2 \\ &= f(x_n) - \gamma|f'(x_n)|^2 + \frac{L}{2}\gamma^2|f'(x_n)|^2 \\ &= f(x_n) - \left(\gamma - \frac{L}{2}\gamma^2\right)|f'(x_n)|^2 \end{aligned}$$

Subtracting $f(x^*)$ above, then

$$f(x_{n+1}) - f(x^*) \le f(x_n) - f(x^*) - \left(\gamma - \frac{L}{2}\gamma^2\right)|f'(x_n)|^2$$

Meanwhile, Convexity gives

$$f(x_n) - f(x^*) \le f'(x_n)(x_n - x^*) \le |f'(x_n)||x_n - x^*| \le |f'(x_n)||x_0 - x^*|$$

Combining abvoe two inequalities,

$$f(x_{n+1}) - f(x^*) \le f(x_n) - f(x^*) - \left(\gamma - \frac{L}{2}\gamma^2\right) \cdot \frac{|f(x_n) - f(x^*)|^2}{|x_0 - x^*|^2}$$

Dividing Both Sides by $(f(x_{n+1}) - f(x^*))(f(x_n) - f(x^*))$,

$$\begin{aligned} &\frac{1}{f(x_n) - f(x^*)} \le \frac{1}{f(x_{n+1}) - f(x^*)} - \left(\gamma - \frac{L}{2}\gamma^2\right) \cdot \frac{f(x_n) - f(x^*)}{f(x_{n+1}) - f(x^*)} \frac{1}{|x_0 - x^*|^2} \\ &\implies \left(\gamma - \frac{L}{2}\gamma^2\right) \cdot \frac{f(x_n) - f(x^*)}{f(x_{n+1}) - f(x^*)} \frac{1}{|x_0 - x^*|^2} \le \frac{1}{f(x_{n+1}) - f(x^*)} - \frac{1}{f(x_n) - f(x^*)} \\ &\implies \left(\gamma - \frac{L}{2}\gamma^2\right) \cdot \frac{1}{|x_0 - x^*|^2} \le \frac{1}{f(x_{n+1}) - f(x^*)} - \frac{1}{f(x_n) - f(x^*)} \\ &\implies \sum_{n=0}^{N-1}\left[\left(\gamma - \frac{L}{2}\gamma^2\right) \cdot \frac{1}{|x_0 - x^*|^2}\right] \le \sum_{n=0}^{N-1}\left[\frac{1}{f(x_{n+1}) - f(x^*)} - \frac{1}{f(x_n) - f(x^*)}\right] = \frac{1}{f(x_N) - f(x^*)} - \frac{1}{f(x_0) - f(x^*)} \end{aligned}$$

Consequently,

$$\frac{2\gamma N}{|x_0 - x^*|^2} \leq N \cdot \left[\left(\gamma - \frac{L}{2}\gamma^2 \right) \cdot \frac{1}{|x_0 - x^*|^2} \right] \leq \frac{1}{f(x_N) - f(x^*)} - \frac{1}{f(x_0) - f(x^*)} \leq \frac{1}{f(x_N) - f(x^*)}$$

Organizing the formula, as result:

$$f(x_N) - f(x^*) \leq \frac{|x_0 - x^*|^2}{2\gamma N}$$

$\square$

9.5 Integral

9.5.1 Inequality of Riemann-Stieltjes Integral

Let $p, q \geq 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, and functions lying on $[a, b]$.

Lemma 9. Let $f, g \in \mathcal{R}(\alpha)$ with $f, g \geq 0$, and $\int_a^b [f(x)]^p d\alpha = \int_a^b [g(x)]^q d\alpha = 1$. Then, $\int_a^b f(x)g(x)d\alpha \leq 1$.

Proof. For any $x \in [a, b]$, the Young's Inequality gives

$$0 \leq f(x)g(x) \leq \frac{[f(x)]^p}{p} + \frac{[g(x)]^q}{q}$$

Now,

$$\int_a^b f(x)g(x)d\alpha \leq \int_a^b \frac{[f(x)]^p}{p} + \frac{[g(x)]^q}{q} d\alpha = \frac{1}{p}\int_a^b [f(x)]^p d\alpha + \frac{1}{q}\int_a^b [g(x)]^q d\alpha = \frac{1}{p} + \frac{1}{q} = 1$$

□

Definition 12. Let $f \in \mathcal{R}(\alpha)$. Define a **Norm** of f:

$$\|f\|_p \overset{\text{def}}{=} \left(\int_a^b |f(x)|^p d\alpha\right)^{\frac{1}{p}}$$

This becomes actually norm of set of Stieltjes Integrable functions, $\mathcal{F} \overset{\text{def}}{=} \{f : [a, b] \to \mathbb{C} \mid f \in \mathcal{R}(\alpha)\}$.

Lemma 10. Hölder's Inequality
Let $f, g \in \mathcal{F}$. Then,

$$\left|\int_a^b f(x)g(x)d\alpha\right| \leq \left[\int_a^b |f(x)|^p d\alpha\right]^{\frac{1}{p}} \cdot \left[\int_a^b |g(x)|^q d\alpha\right]^{\frac{1}{q}}$$

Proof. Use above definition, Rewrite:

$$\|f\|_p^p = \int_a^b |f(x)|^p d\alpha, \quad \|g\|_q^q = \int_a^b |g(x)|^q d\alpha$$

Now, we can make the condition of above lemma,

$$\int_a^b \left[\frac{|f(x)|}{\|f\|_p}\right]^p d\alpha = \frac{1}{\|f\|_p^p} \cdot \int_a^b |f(x)|^p d\alpha = 1, \quad \int_a^b \left[\frac{|g(x)|}{\|g\|_q}\right]^q d\alpha = \frac{1}{\|g\|_q^q} \cdot \int_a^b |g(x)|^q d\alpha = 1$$

And apply this,

$$\int_a^b \frac{|f(x)| \cdot |g(x)|}{\|f\|_p \|g\|_q} d\alpha \leq 1 \implies \int_a^b |f(x)||g(x)|d\alpha \leq \|f\|_p \|g\|_q = \left[\int_a^b |f(x)|^p d\alpha\right]^{\frac{1}{p}} \cdot \left[\int_a^b |g(x)|^q d\alpha\right]^{\frac{1}{q}}$$

Finally, the general property of integral of product gives

$$\left|\int_a^b f(x)g(x)d\alpha\right| \leq \int_a^b |f(x)||g(x)|d\alpha \leq \|f\|_p \|g\|_q = \left[\int_a^b |f(x)|^p d\alpha\right]^{\frac{1}{p}} \cdot \left[\int_a^b |g(x)|^q d\alpha\right]^{\frac{1}{q}}$$

□

> **Theorem 18. Minkowski inequality**
> **Let $f, g \in \mathcal{F}$. Then, for any $p \geq 1$, $\|f+g\|_p \leq \|f\|_p + \|g\|_p$.**

Proof.

$$\begin{aligned}
\|f+g\|_p^p = \int_a^b |f+g|^p d\alpha &= \int_a^b |f+g||f+g|^{p-1} d\alpha \\
&\leq \int_a^b [|f|+|g|]|f+g|^{p-1} d\alpha \\
&= \int_a^b |f||f+g|^{p-1} d\alpha + \int_a^b |g||f+g|^{p-1} d\alpha \\
&\overset{\text{Hölder}}{\leq} \left[\int_a^b |f|^p d\alpha\right]^{\frac{1}{p}} \left[\int_a^b |f+g|^{(p-1)\frac{p}{p-1}} d\alpha\right]^{\frac{p-1}{p}} + \left[\int_a^b |g|^p d\alpha\right]^{\frac{1}{p}} \left[\int_a^b |f+g|^{(p-1)\frac{p}{p-1}} d\alpha\right]^{\frac{p-1}{p}} \\
&= \left[\int_a^b |f+g|^p d\alpha\right]^{\frac{p-1}{p}} \left(\left[\int_a^b |f|^p d\alpha\right]^{\frac{1}{p}} + \left[\int_a^b |g|^p d\alpha\right]^{\frac{1}{p}}\right) = \|f+g\|_p^{p-1} \cdot (\|f\|_p + \|g\|_p)
\end{aligned}$$

Now,

$$\|f+g\|_p^p \cdot \|f+g\|_p^{1-p} = \|f+g\|_p \leq \|f\|_p + \|g\|_p$$

□

Chapter 10

Measure

Chapter 11

Complex Analysis

Chapter 12

Differential Geometry

Chapter 13

Differential Equation

Chapter 14

Spaces

14.1 $\mathbb{R}^n$

14.1.1 Inner Product in $\mathbb{R}$

14.1.2 p-norm in $\mathbb{R}^n$

Definition 13. Let $\mathbb{R}^n$ be given. Define p**-norm** of $\mathbb{R}^n$ is metric on $\mathbb{R}$:

$$d_p : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} : (\mathbf{x}, \mathbf{y}) \mapsto \left(\sum_{i=1}^{n} |x_i - y_i|^p \right)^{\frac{1}{p}}, \quad (\mathbf{x} = (x_1, \ldots, x_n),\ \ \mathbf{y} = (y_1, \ldots, y_n))$$

where $p \in [1, \infty]$, p-norm be a **Metric from Minkowski inequality**.

Lemma 11. Holder's inequality
Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ be give, and $p, q \geq 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then,

$$\sum_{i=1}^{n} |x_i y_i| \leq \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} \cdot \left(\sum_{i=1}^{n} |y_i|^q \right)^{\frac{1}{q}}$$

Proof. Denote that

$$\|x\|_p \overset{\text{def}}{=} \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}}$$

Then, since young's inequality, for each $i \in \{1, 2, \ldots, n\}$,

$$\frac{|x_i|}{\|x\|_p} \cdot \frac{|y_i|}{\|y\|_p} \leq \frac{1}{p} \cdot \frac{|x_i|^p}{\|x\|_p^p} + \frac{1}{q} \cdot \frac{|y_i|^q}{\|y\|_q^q}$$

Summing for all $i = 1, 2, \ldots, n$:

$$\frac{1}{\|x\|_p \|y\|_q} \cdot \sum_{i=1}^{n} |x_i y_i| \leq \frac{1}{p} + \frac{1}{q} = 1$$

Therefore,

$$\sum_{i=1}^{n} |x_i y_i| \leq \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} \cdot \left(\sum_{i=1}^{n} |y_i|^q \right)^{\frac{1}{q}}$$

$\square$

Theorem 19. Minkowski inequality
Given complex-valued sequences $\{x_i\}_{i=1}^n$ and $\{y_i\}_{i=1}^n$,

$$\left[\sum_{i=1}^{n}|x_i+y_i|^p\right]^{\frac{1}{p}} \le \left[\sum_{i=1}^{n}|x_i|^p\right]^{\frac{1}{p}} + \left[\sum_{i=1}^{n}|y_i|^p\right]^{\frac{1}{p}}$$

Proof. Denote

$$|x_i+y_i|^p = |x_i+y_i|\cdot|x_i+y_i|^{p-1}$$

Then,

$$\begin{aligned}
\sum_{i=1}^{n}|x_i+y_i|^p &= \sum_{i=1}^{n}|x_i+y_i|\cdot|x_i+y_i|^{p-1}\\
&= \sum_{i=1}^{n}(|x_i|+|y_i|)\cdot|x_i+y_i|^{p-1}\\
&= \sum_{i=1}^{n}|x_i|\cdot|x_i+y_i|^{p-1} + \sum_{i=1}^{n}|y_i|\cdot|x_i+y_i|^{p-1}\\
&\overset{\text{Hölder}}{\le} \left[\sum_{i=1}^{n}|x_i|^p\right]^{\frac{1}{p}}\cdot\left[\sum_{i=1}^{n}|x_i+y_i|^p\right]^{\frac{p-1}{p}} + \left[\sum_{i=1}^{n}|y_i|^p\right]^{\frac{1}{p}}\cdot\left[\sum_{i=1}^{n}|x_i+y_i|^p\right]^{\frac{p-1}{p}}\\
&= \left[\left(\sum_{i=1}^{n}|x_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n}|y_i|^p\right)^{\frac{1}{p}}\right]\cdot\left[\sum_{i=1}^{n}|x_i+y_i|^p\right]^{\frac{p-1}{p}}
\end{aligned}$$

Now, Divide each side as $[\sum_{i=1}^{n}|x_i+y_i|^p]^{\frac{p-1}{p}}$, then we obtain

$$\left[\sum_{i=1}^{n}|x_i+y_i|^p\right]^{1-\frac{p-1}{p}} = \left[\sum_{i=1}^{n}|x_i+y_i|^p\right]^{\frac{1}{p}} \le \left[\left(\sum_{i=1}^{n}|x_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n}|y_i|^p\right)^{\frac{1}{p}}\right]$$

$\square$

Theorem 20. Let d_{p_1}, d_{p_2} are p-norm on $\mathbb{R}^n$ with $1 \le p_1 < p_2$. Then,

$$\exists C > 0 \textbf{ s.t. } \forall x, y \in \mathbb{R}^n,\ d_{p_2}(x,y) \le d_{p_1}(x,y) \le C d_{p_2}(x,y)$$

In particular, $C = n^{\frac{1}{p_1}-\frac{1}{p_2}}$.

Proof. Let $p_1 < p_2$.
For show that first-inequality,

$$1 = \sum_{i=1}^{n}\left[\frac{|x_i-y_i|}{[\sum_{i=1}^{n}|x_i-y_i|^{p_2}]^{\frac{1}{p_2}}}\right]^{p_2} \le \sum_{i=1}^{n}\left[\frac{|x_i-y_i|}{[\sum_{i=1}^{n}|x_i-y_i|^{p_2}]^{\frac{1}{p_2}}}\right]^{p_1} = \frac{\sum_{i=1}^{n}|x_i-y_i|^{p_1}}{[\sum_{i=1}^{n}|x_i-y_i|^{p_2}]^{\frac{p_1}{p_2}}} = \left[\frac{[\sum_{i=1}^{n}|x_i-y_i|^{p_1}]^{\frac{1}{p_1}}}{[\sum_{i=1}^{n}|x_i-y_i|^{p_2}]^{\frac{1}{p_2}}}\right]^{p_1}$$

Thus, we obtain that:

$$1 \le \left[\frac{[\sum_{i=1}^{n}|x_i-y_i|^{p_1}]^{\frac{1}{p_1}}}{[\sum_{i=1}^{n}|x_i-y_i|^{p_2}]^{\frac{1}{p_2}}}\right]^{p_1} \iff 1 \le \frac{[\sum_{i=1}^{n}|x_i-y_i|^{p_1}]^{\frac{1}{p_1}}}{[\sum_{i=1}^{n}|x_i-y_i|^{p_2}]^{\frac{1}{p_2}}} \iff \left[\sum_{i=1}^{n}|x_i-y_i|^{p_2}\right]^{\frac{1}{p_2}} \le \left[\sum_{i=1}^{n}|x_i-y_i|^{p_1}\right]^{\frac{1}{p_1}}$$

For show that second-inequality, using Hölder's inequality.

$$\begin{aligned}
(d_{p_1}(x,y))^{p_1} &= \sum_{i=1}^{n}|x_i-y_i|^{p_1} = \sum_{i=1}^{n}|x_i-y_i|^{p_1}\cdot 1\\
&\overset{\text{Hölder}}{\le} \left[\sum_{i=1}^{n}\left(|x_i-y_i|^{p_1\cdot\frac{p_2}{p_1}}\right)\right]^{\frac{p_1}{p_2}}\cdot\left[\sum_{i=1}^{n}1^{\frac{p_2}{p_2-p_1}}\right]^{1-\frac{p_1}{p_2}} = \left[\sum_{i=1}^{n}(|x_i-y_i|^{p_2})\right]^{\frac{p_1}{p_2}}\cdot n^{1-\frac{p_1}{p_2}}
\end{aligned}$$

Taking the $\frac{1}{p_1}$-th power of both sides, then

$$d_{p_1}(x,y) \leq \left[\sum_{i=1}^{n} \left(|x_i - y_i|^{p_2}\right)\right]^{\frac{1}{p_2}} \cdot n^{\frac{1}{p_1} - \frac{1}{p_2}} = n^{\frac{1}{p_1} - \frac{1}{p_2}} \cdot d_{p_2}(x,y)$$

$\square$

14.2 Topological Vector Space

14.3 Hilbert Space

Definition 14. Complete Inner product Vector Space is called *Hilbert Space*.

14.3.1 Hilbert Space in $\mathbb{R}^\omega$

Definition 15. Define $\mathbb{R}^\omega \overset{\text{def}}{=} \prod_{i=1}^{\infty} \mathbb{R}$ as the countable product of Euclidean space $\mathbb{R}$ with product topology.
And define *Hilbert Space* $\mathbb{H} \overset{\text{def}}{=} \left\{ \{x_n\}_{n=1}^{\infty} \,\middle|\, \sum_{n=1}^{\infty} x_n^2 < \infty \right\} \subset \mathbb{R}^\omega$ as the subspace topology.

Define the operations elementwise; then $(\mathbb{H}, +, \times)$ is a Vector Space over $\mathbb{R}$.
Moreover, $\mathbb{H}$ is Complete Metric Space and Inner product Vector Space.

Theorem 21. *Hilbert Space* $\mathbb{H}$ is Metrizable.

Proof. There are two ways to prove.
First Proof. We will use two Lemmas:

Lemma 12. Countable Product of Metric Space is Metrizable.

Proof. Let (X_i, d_i) be a metric Space, for each $i \in \mathbb{N}$.
If $d : X \times X \to \mathbb{R}$ is a Metric, then $\dfrac{d}{1+d}$ is also Metric, because

$$\frac{d(x,z)}{1+d(x,z)} \underset{\frac{x}{1+x}\ \text{increasing}}{\overset{}{\leq}} \frac{d(x,y)+d(y,z)}{1+d(x,y)+d(y,z)} \underset{d\geq 0}{\leq} \frac{d(x,y)}{1+d(x,y)} + \frac{d(y,z)}{1+d(y,z)} \tag{$*$}$$

Using this fact, define

$$d_\Pi : \prod X_i \times \prod X_i \to \mathbb{R} : (\{x_n\}_{n=1}^{\infty}, \{y_n\}_{n=1}^{\infty}) \mapsto \sum_{i=1}^{\infty} \left[\frac{1}{2^i} \cdot \frac{d_i(x_i, y_i)}{1+d_i(x_i, y_i)} \right]$$

Then d_Π is a Metric because:

$$\begin{aligned}
d_\Pi\left(\{x_n\}_{n=1}^{\infty}, \{z_n\}_{n=1}^{\infty}\right) &= \sum_{i=1}^{\infty} \left[\frac{1}{2^i} \cdot \frac{d_i(x_i, z_i)}{1+d_i(x_i, z_i)} \right] \\
&\overset{(*)}{\leq} \sum_{i=1}^{\infty} \left[\frac{1}{2^i} \cdot \left(\frac{d_i(x_i, y_i)}{1+d_i(x_i, y_i)} + \frac{d_i(y_i, z_i)}{1+d_i(y_i, z_i)} \right) \right] \\
&= \sum_{i=1}^{\infty} \left[\frac{1}{2^i} \cdot \frac{d_i(x_i, y_i)}{1+d_i(x_i, y_i)} \right] + \sum_{i=1}^{\infty} \left[\frac{1}{2^i} \cdot \frac{d_i(y_i, z_i)}{1+d_i(y_i, z_i)} \right] \\
&= d_\Pi\left(\{x_n\}_{n=1}^{\infty}, \{y_n\}_{n=1}^{\infty}\right) + d_\Pi\left(\{y_n\}_{n=1}^{\infty}, \{z_n\}_{n=1}^{\infty}\right)
\end{aligned}$$

Reflexity and symmetry are clear. □

Lemma 13. Metrizable is Hereditary.

Proof omitted.
Consequently, since $\mathbb{H} \subset \mathbb{R}^\omega$ is a subspace of a metric space, it is metrizable.

Second Proof. We know that $\mathbb{R}^\omega$ is Vector Space. Moreover, $\mathbb{H} \subset \mathbb{R}^\omega$ is Subspace. Using subspace criteria:

$S \subset V$ is Subspace of Vector Space V if and only if $0 \in S$ and for any $x, y \in S$ and $a \in F$, $ax + y \in S$.

Clearly, $\{0\} \in \mathbb{H}$. Let $a \in \mathbb{R}$ and $\{x_n\}, \{y_n\} \in \mathbb{H}$ be given. Then, $a\{x_n\} + \{y_n\} = \{ax_n + y_n\} \in \mathbb{H}$ because:

$$\sum_{i=1}^{\infty}(ax_i + y_i)^2 = \sum_{i=1}^{\infty}\left[a^2x_i^2 + 2ax_iy_i + y_i^2\right] \overset{(*)}{=} a^2\sum_{i=1}^{\infty}x_i^2 + 2a\sum_{i=1}^{\infty}x_iy_i + \sum_{i=1}^{\infty}y_i^2 < \infty$$

The $(*)$ given by:

$$\sum_{i=1}^{\infty}|x_iy_i| = \sum_{i=1}^{\infty}|x_i||y_i| \leq \sum_{i=1}^{\infty}(\max(|x_i|, |y_i|))^2 \leq \sum_{i=1}^{\infty}(x_n^2 + y_n^2) = \sum_{i=1}^{\infty}x_n^2 + \sum_{i=1}^{\infty}y_n^2 < \infty \tag{$*$}$$

Thus $\mathbb{H}$ is Vector Space over $\mathbb{R}$. Now, define ***inner product*** on $\mathbb{H}$ as:

$$\langle\ \cdot\ ,\ \cdot\ \rangle : \mathbb{H} \times \mathbb{H} \to \mathbb{R} : (\{x_n\}, \{y_n\}) \mapsto \sum_{i=1}^{\infty}x_iy_i$$

This definition is well-defined since $(*)$. $\square$

14.4 Banach Space

14.5 L_p Space

14.6 l_p Space