P. PARIS [1]
Associate Professor of Mechanics and Assistant Director of the Institute of Research, Lehigh University, Bethlehem, Pa.

F. ERDOGAN
Professor of Mechanical Engineering, Lehigh University, Bethlehem, Pa. Mem. ASME

A Critical Analysis of Crack Propagation Laws

The practice of attempting validation of crack-propagation laws (i.e., the laws of Head, Frost and Dugdale, McEvily and Illg, Liu, and Paris) with a small amount of data, such as a few single specimen test results, is questioned. It is shown that all the laws, though they are mutually contradictory, can be in agreement with the same small sample of data. It is suggested that agreement with a wide selection of data from many specimens and over many orders of magnitudes of crack-extension rates may be necessary to validate crack-propagation laws. For such a wide comparison of data a new simple empirical law is given which fits the broad trend of the data.

Introduction

Several crack-propagation laws have been presented in the past few years, which claim to be verified by the experimental data analyzed in their respective papers. They are specifically the work of Head [1],[2] Frost and Dugdale [2], McEvily and Illg [3], Liu [4, 5], and Paris, Gomez, and Anderson [6, 7, 8].

This paper will attempt to show that basing validation of a crack-propagation theory on a limited amount of data is a poor test of any theory. Moreover, a wide range of test data is now available in a convenient form [9] which may be employed to analyze critically these several crack propagation laws.

Therefore the paper will consist of three parts which will include:

1 A review and comparison of existing crack-propagation laws.

2 The erroneous results obtained by comparing each law with a limited range of test data.

3 The results of comparing crack-propagation laws with a wide range of data.

A Review of Existing Crack-Propagation Laws

Crack-propagation laws given in the literature take many forms. In general they treat cracks in infinite sheets subjected to a uniform stress perpendicular to the crack (or can be applied to that configuration) and they relate the crack length, $2a$, to the number of cycles of load applied, N, with the stress range σ, and material constants C_i.[3] The single form in which all crack-propagation laws may be written is

$$\frac{da}{dN} = f(\sigma, a, C_i) \qquad (1)$$

Therefore this format will be employed in the discussion to follow.

Chronologically the first crack-propagation law which drew wide attention was that of Head [1] in 1953. He employed a mechanical model which considered rigid-plastic work-hardening elements ahead of a crack tip and elastic elements over the remainder of the infinite sheet. The model required extensive calculations and deductions to obtain a law which may be written as

$$\frac{da}{dN} = \frac{C_1\sigma^3 a^{3/2}}{(C_2 - \sigma)\omega_0^{1/2}} \quad \text{(Head's law)} \qquad (2)$$

where C_1 depends upon the strain-hardening modulus, the modulus of elasticity, the yield stress and the fracture stress of the material, and C_2 is the yield strength of the material. Head defined ω_0 as the size of the plastic zone near the crack tip and presumed it was constant during crack propagation. However, Frost [2] noticed that the plastic-zone size increased in direct proportion to the crack length in his tests. Moreover, Irwin [10] has recently pointed out from analytical considerations that

$$\omega_0 \approx \sigma^2 a \qquad (3)$$

for the configuration treated here which is in agreement with Frost's conclusions. Therefore, though Head adopted equation (2) with ω_0 considered constant as his crack-propagation law, we are forced here to introduce equation (3) into equation (2) to obtain a modified or corrected form of Head's crack-propagation law; i.e.,

$$\frac{da}{dN} = \frac{C_3\sigma^2 a}{(C_2 - \sigma)} \quad \text{(Head's corrected law)} \qquad (4)$$

Frost and Dugdale [2] in 1958 presented a new approach to crack-propagation laws. They observed as was introduced in equation (4), that Head's law should be corrected for the variation of plastic-zone size with crack length. They deduced that the corrected result, equation (4), depends linearly on the crack length a. However, they also argued by dimensional analysis

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[1] Also Consultant to the Boeing Company, Transport Division, Renton, Wash.

[2] Numbers in brackets designate References at end of paper.

[3] In this discussion σ will imply the stress-fluctuation range and C_i will vary slightly with mean stress in a general interpretation.

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Nomenclature

a = half crack length
a_0 = initial half crack length
B = a function of stress range (and mean stress)
C = a numerical constant
C_i = constants (which vary slightly with mean stress)
D_i = constants which depend on stress level
$f(\ )$ = a function of
$F\{\ \}$ = a function of
$G\{\ \}$ = a function of
k = stress-intensity factor (range)
K_N = stress-concentration factor
m = a numerical exponent
M = a material constant
n = a numerical exponent
N = cycle number
N_0, N_F = initial and final cycle number, respectively
$\frac{da}{dN}$ = crack extension per cycle of load
Δk = stress-intensity-factor range
ρ, ρ_1 = crack-tip radius
σ = applied stress (range)
σ_0 = stress at a crack tip (with finite radius ρ)
σ_{net} = net section stress
ω_0 = plastic-zone size

that the incremental increase in crack length da, for an incremental number of cycles dN, should be directly proportional to the crack length a. Hence they concluded that (independent of Head's model)

$$\frac{da}{dN} = Ba \tag{5}$$

where B is a function of the applied stresses. Then they observed that in order to fit their experimental data:

$$B = \frac{\sigma^3}{C_i} \tag{6}$$

Combining equations (5) and (6) they obtained the law:

$$\frac{da}{dN} = \frac{\sigma^3 a}{C_4} \quad \text{(Frost and Dugdale's law)} \tag{7}$$

About the same time McEvily and Illg [3] modified a method of analysis of static strength of plates with cracks used at NASA to obtain a theory of crack propagation. Their arguments were as follows: Presuming that a crack tip in a material has a characteristic (fictitious) radius ρ_1, which allows computation of the stress σ_0, in the element at the crack tip using elastic stress-concentration factor concepts, the stress σ_0 is

$$\sigma_0 = K_N \sigma_{\text{net}} \tag{8}$$

where K_N is the stress-concentration factor and σ_{net} is the net area stress at the cracked section. For the configuration used here, i.e., an infinite plate with uniform stress σ

$$K_N = 1 + 2(a/\rho_1)^{1/2} \tag{9}$$

which is based on the elastic solution for an elliptical hole of semimajor axis a and end radius ρ_1.

$$\sigma_{\text{net}} = \sigma \tag{10}$$

and substituting equations (9) and (10) into (8) gives

$$\sigma_0 = \sigma[1 + 2(a/\rho_1)^{1/2}] \tag{11}$$

Based on considerations that under cyclic loading work-hardening at the crack tip will raise the local stress to a fracture stress, they concluded that the crack-extension rate will be a function of σ_0 or

$$\frac{da}{dN} = F\{\sigma_0\} = F\{K_N \sigma_{\text{net}}\} \tag{12}$$

(McEvily and Illg's law)

Therefore for the special configuration of interest here, i.e., introducing equations (8), (9), (10), and (11) into (12), we have

$$\frac{da}{dN} = F\left\{\sigma\left[1 + 2\left(\frac{a}{\rho_1}\right)^{1/2}\right]\right\} \tag{13}$$

which is the desired form in this discussion upon considering ρ_1 to be a material constant, in likeness to the C_i.

McEvily and Illg go on in an empirical manner to obtain the form of the function $F\{\ \}$, and suggest

$$\log_{10}\left(\frac{da}{dN}\right) = 0.00509\, K_N \sigma_{\text{net}} - 5.472 - \frac{34}{K_N \sigma_{\text{net}} - 34} \tag{14}$$

(McEvily and Illg's law empirically extended)

It should be noted by the reader that McEvily and Illg's laws, equations (12) and (14), are not restricted to the special configuration here, as was the case for the laws of Head, and Frost and Dugdale. The applicability to other configurations and an additional similarity to Paris' work [6, 7, 8] will warrant later comments.

Independent of McEvily and Illg, Paris [11] proposed a crack-propagation theory at about the same time. It is based on the following arguments: Irwin's [12] stress-intensity-factor k reflects the effect of external load and configuration on the intensity of the whole stress field around a crack tip. Moreover, for various configurations the crack-tip stress field always has the same form (i.e., distribution). Therefore it was reasoned that the intensity of the crack-tip stress field as represented by k should control the rate of crack extension.[4] That is to say:

$$\frac{da}{dN} = G\{k\} \tag{15}$$

In treating the special configuration of interest in this discussion, it should first be observed that (11)

$$k = \sigma a^{1/2}, \tag{16}$$

whereupon equation (15) may be specialized to read:

$$\frac{da}{dN} = G\{\sigma a^{1/2}\}. \tag{17}$$

Somewhat later, Liu [4] restated Frost and Dugdale's [2] dimensional analysis in a much more elegant form and argued that the crack-growth rate should depend linearly on the crack length; i.e.,

$$\frac{da}{dN} = Ba \quad \text{(Liu's law)} \tag{18}$$

which is the same result as equation (5). Liu then presumed that B was in general a function of stress range σ (and mean stress); i.e.,

$$B = B(\sigma) \tag{19}$$

In a subsequent work, Liu [5] notes that mean stress is of secondary influence and, using a model of crack extension employing an idealized elastic-plastic stress-strain diagram and a concept of total hysteresis energy absorption to failure, reasons that

$$B(\sigma) = C_5 \sigma^2 \tag{20}$$

which combined with equation (18) gives

$$\frac{da}{dN} = C_5 \sigma^2 a \quad \text{(Liu's modified law)} \tag{21}$$

The Equivalence of $K_N \sigma_{\text{net}}$ to k

Hardrath [14] observed that $K_N \sigma_{\text{net}}$ for the special configuration employed here, from equations (8) and (10),

$$K_N \sigma_{\text{net}} = \sigma[1 + 2(a/\rho_1)^{1/2}] \tag{22}$$

is similar to the stress-intensity factor, equation (16),

$$k = \sigma a^{1/2} \tag{23}$$

if ρ_1 is small compared to a. For aluminum alloys (2024T3 and 7075T6) his colleagues McEvily and Illg [3] had already observed that ρ_1 is less than 0.005 in. so that the condition $\rho_1 \ll a$ is in fact present for cracks of a readily observable length in crack-propagation tests. Thus $K_N \sigma_{\text{net}}$ and k are known to concur for this special configuration and in addition a proof that they are equivalent in general will be offered here.

Irwin [12] observed that the equations for the stress field surrounding the tip of a sharp crack contains the factor $k/(2r)^{1/2}$ which implies a singularity of stress at the crack tip. Now, if a hole of radius ρ is drilled at the crack tip, the maximum stress on the periphery of the hole will be proportional to $k/(2\rho)^{1/2}$ or

$$\sigma_0 = Ck/(2\rho)^{1/2} \tag{24}$$

which applies for any ρ which is small compared to other planar

[4] At about this same time Martin and Sinclair [13] attempted unsuccessfully to correlate crack-extension rates using a similar parameter but did not observe a correlation which does in fact occur.

dimensions of any configuration considered, and C is a constant independent of the configuration. Finally, σ_0 in equation (24) may be interpreted to be the same as $K_N\sigma_{net}$, from equation (8) and its accompanying discussion, if ρ is taken equal to ρ_1. Then

$$K_N\sigma_{net} = \sigma_0 \frac{Ck}{(2\rho_1)^{1/2}} \tag{25}$$

In order to evaluate C, the results for the special configuration considered herein may be employed, i.e., substituting equations (22) and (23) into (25) and noting equivalence for small ρ_1 gives

$$C = \lim_{\rho_1\to 0} \frac{K_N\,\sigma_{net}\,(2\rho_1)^{1/2}}{k}$$

$$= \lim_{\rho_1\to 0} \frac{\sigma\left[1 + 2\left(\frac{a}{\rho_1}\right)^{1/2}\right](2\rho_1)^{1/2}}{a} = 2\sqrt{2} \tag{26}$$

Putting this result into equation (25) and rearranging,

$$k = \lim_{\rho_1\to 0} \frac{K_N\,\sigma_{net}\,\rho_1^{\,1/2}}{2} \tag{27}$$

which implies the general equivalence of $K_N\sigma_{net}$ and k.

Similarities Between Crack-Propagation Laws

The result of the foregoing discussion, equation (27), implies the direct similarity of McEvily and Illg's law, equation (12), and Paris' result, equation (15). A choice between the two is strictly dependent on a matter of convenience and clarity of accompanying concepts in employing one or the other.[5]

The laws of Head, equation (4), Frost and Dugdale, equation (7), and Liu, equations (18) and (21), can all be approximated by the form

$$\frac{da}{dN} = \frac{\sigma^n a^m}{C_0} \tag{28}$$

for the special configuration which is treated. Now, it is evident that Paris' result for this configuration, equation (17), implies:

$$m = \frac{n}{2} \tag{29}$$

which can also be derived from McEvily and Illg's result, equation (13), for ρ_1 small compared to a.

The laws of Head, and Frost and Dugdale do not quite agree with applying equation (29) to equation (28), however, Liu's law does concur with the specified form.

It is pertinent to now show that determining m and n from a limited quantity of data is a doubtful practice. That is to say that plotting data from single test specimens on a logarithmic or semilogarithmic graph on which laws such as Head's, Frost's, and Liu's predict straight line relationships is not a reasonable test of the validity of a crack propagation law.

Erroneous Evaluation of Data From Single Tests

A typical crack-propagation test consists of a wide plate with a central crack of length $2a$, subjected to uniform tension σ, repeatedly applied. During a single test then σ is a constant and data consisting of crack lengths and corresponding cycle numbers are obtained. Let the problem of examining each of the previous crack-propagation laws for a particular test be formulated.

In a constant maximum repeated stress-level test Head's law, equation (2), is reduced to

$$\frac{da}{dN} = D_1\, a^{3/2} \tag{30}$$

[5] For very small cracks there is a difference between the two theories which is left unresolved.

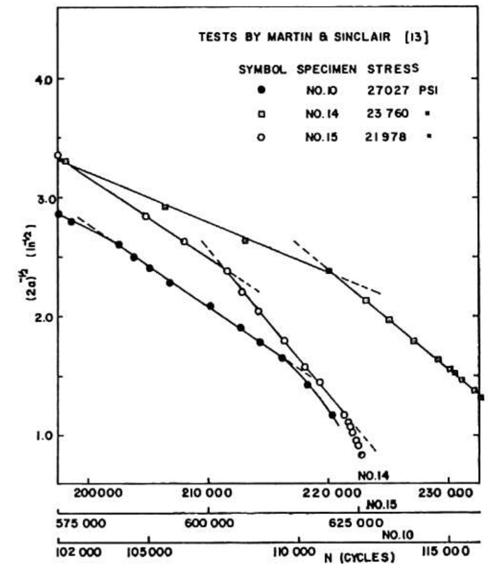

Fig. 1 Typical data from Martin and Sinclair on Head's suggested plot

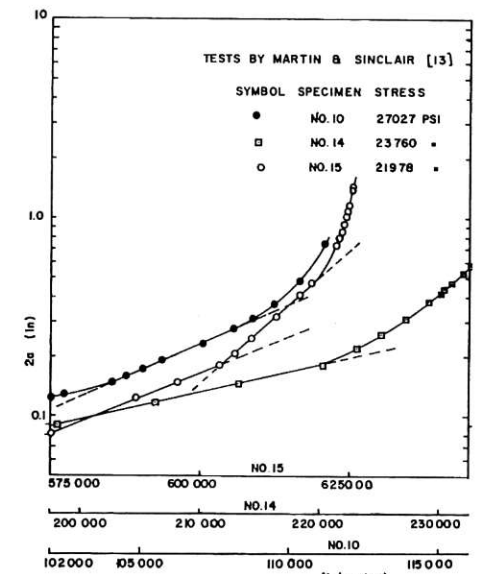

Fig. 2 The same data from Martin and Sinclair on Frost's or Liu's suggested plot

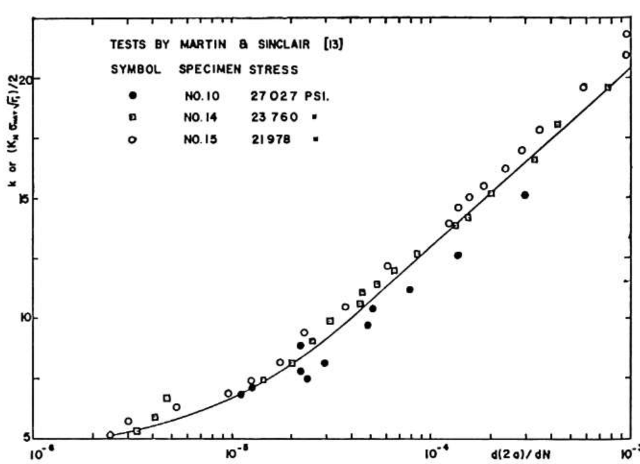

Fig. 3 Again the same data from Martin and Sinclair on McEvily's or Paris' suggested plot

where D_1 is a constant for a given stress range σ. Integrating this expression gives

$$-\frac{2}{a^{1/2}} = D_1N + \text{const} \tag{31}$$

Therefore Head suggested plotting $1/a^{1/2}$ versus N and, observing equation (31), implied that obtaining a straight line for each specimen whose data are plotted this way indicates verification of his law.

The data of Martin and Sinclair [13] are employed here as an unbiased source. From their data specimens nos. 10, 14, and 15 were chosen since those specimens were run at medium stress levels and the greatest number of data points per specimen was recorded for them. Fig. 1 shows the type of plot suggested by Head and it is noticed that portions of the data do form straight lines. Does this mean that Head's theory is verified? The reader is warned that this might be a hasty conclusion.

First, consider the corrected law of Head, Frost and Dugdale's law, and Liu's law, equations (4), (7), (18), and (19), respectively. For a constant stress-range test all these laws reduce to the form

$$\frac{da}{dN} = D_2a \tag{32}$$

Integrating this result gives:

$$\log a = D_2N + \text{const} \tag{33}$$

Therefore plotting $\log a$ versus N and obtaining straight lines has been accepted as verifying these theories. Fig. 2 is this type of plot employing the very same test data point by point of Martin and Sinclair shown in Fig. 1. Again, the test data have straight-line portions. Therefore it now seems to verify both Head's law equation (30) and the other laws with the form of equation (32). But the theories represented by equations (30) and (32) are not the same!

Now let the laws of McEvily and Illg, and Paris, equations (12) and (15), be examined in the light of these same data. Recalling equation (27) these two laws are equivalently expressed by

$$\frac{da}{dN} = G\{k\} \tag{34}$$

for a given material. To test these theories, data from several specimens should be plotted on a k versus da/dN (or $\log da/dN$) graph and these laws predict that the data of several specimens will form a single curve. Again the same data of Martin and Sinclair are plotted in Fig. 3. Notice that the points which do not fall on the straight-line portions of Figs. 1 and 2 seem to be perfectly acceptable here. Since the data were differentiated to obtain da/dN, an additional amount of scatter is introduced. Therefore the data also imply verification of these two laws as well.

Hence, all of the laws agree with the data and, since these laws are not identical, the method of verification of crack-propagation theories from a limited amount of test data is evidently in error. An alternative approach employing a wider range of test data must be employed.

Comparison of Crack-Propagation Laws With a Wide Range of Test Data

In previous work [6, 8] it was observed that data from several sources [3, 9, 13] may be plotted in the form of Fig. 3 (semi-logarithmic) to obtain a single curve on a range of stress-intensity factor, Δk (corresponding to stress range) versus $\log (da/dN)$ diagram, where da/dN covers as many as 6 log cycles. However, replotting these data on a $\log \Delta k$ versus $\log (da/dN)$ graph reveals some pertinent results. The data are replotted in Fig. 4 and the three specimens from Martin and Sinclair's work used in the foregoing for the purpose of illustration are shown to indicate their concurrence with the general trend.

The authors are hesitant but cannot resist the temptation to draw the straight line of slope $^1/_4$ through the data in Fig. 4. The equation of this line is observed to be

$$\frac{da}{dN} = \frac{(\Delta k)^4}{M} \tag{35}$$

Equation (35) fits the data almost as well as McEvily and Illg's extended law, equation (14), and is considerably simpler in form.

The advantage of this form is clarified by considering its application to the configuration employed earlier, in which case it becomes

$$\frac{da}{dN} = \frac{\sigma^4a^2}{M} \tag{36}$$

The laws of Head, Frost and Dugdale, and Liu depend on a in a manner other than a^2. Clearly, their laws are at variance with the data trend in Fig. 4; i.e., Head predicts the slope of $^1/_3$ and Frost and Dugdale as well as Liu would predict $^1/_2$ as indicated in the figure.

Moreover, if one might be so bold as to attempt to integrate equation (36) it becomes

$$\frac{1}{a_0} - \frac{1}{a} = \frac{\sigma^4}{M}(N - N_0) \tag{37}$$

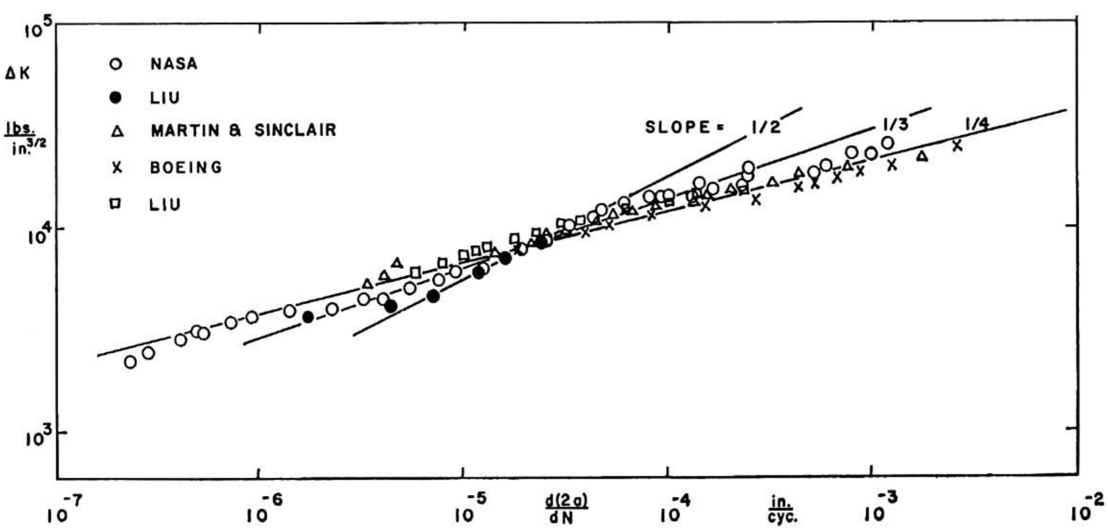

Fig. 4 Broad trend of crack-growth data on 2024-T3 aluminum alloy

where a_0 and N_0 are the initial crack size and cycle number, respectively. For final failure of a specimen, the number of cycles may be equated to N_F, as a approaches infinity whereupon equation (37) reduces to

$$N_F = \frac{M}{\sigma^4 a_0} \tag{38}$$

which does in fact resemble an S-N diagram. The authors do not wish to imply that equation (38) is directly useful, but derive it here to show that such considerations do not immediately lead to the conclusion that equations (35) or (36) are in error.

Data From Wedge-Force Tests

The results of wedge-force tests of the configuration shown in Fig. 5 are useful in critically examining the dimensional analyses of Frost and Dugdale, and Liu in deriving their crack-propagation laws. If the sheet is infinite, the only characteristic dimension of the problem is a. As the crack grows, strictly dimensional arguments imply that for different crack lengths a_1 and a_2 the incremental rates of crack extension are

$$\frac{da_1}{dN} \cdot \frac{1}{a_1} = \frac{da_2}{dN} \cdot \frac{1}{a_2} = B \tag{39}$$

which might be thought valid for all crack lengths or

$$\frac{da}{dN} = Ba \tag{40}$$

This result is the basis of equations (5) and (18).

In wedge-force tests Donaldson and Anderson [9] observe that the crack grows slower as it gets larger. This is contrary to equation (40) and hence weakens any arguments based on dimensional analyses which do not include further considerations. Therefore the crack propagation laws of Head, Frost and Dugdale, and Liu must be considered as unclarified, if not totally in error.

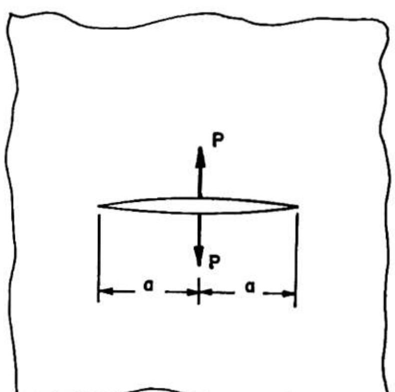

Fig. 5 Wedge-force test configuration

Moreover, Fig. 6 is a graph, similar in form to Fig. 4, but for a different material; i.e., 7075T6 aluminum alloy. The data shown are from two independent sources [3 and 9] and in addition wedge-force test data from tests employing configuration of Fig. 5 are also plotted. No further comment seems necessary.

Conclusion

The results here indicate that the practice of using data from single test specimens is not a sensitive evaluation of a crack-propagation law's validity. Randomly chosen data from single specimens analyzed here leads to an apparent agreement of several contradictory laws to the same test data.

For that reason the authors suggest that laws which correlate a wide range of test data from many specimens are perhaps the "correct" laws. The results at least indicate that hasty conclusions have been drawn in many earlier works which should be re-examined before any given crack-propagation law is accepted as valid.

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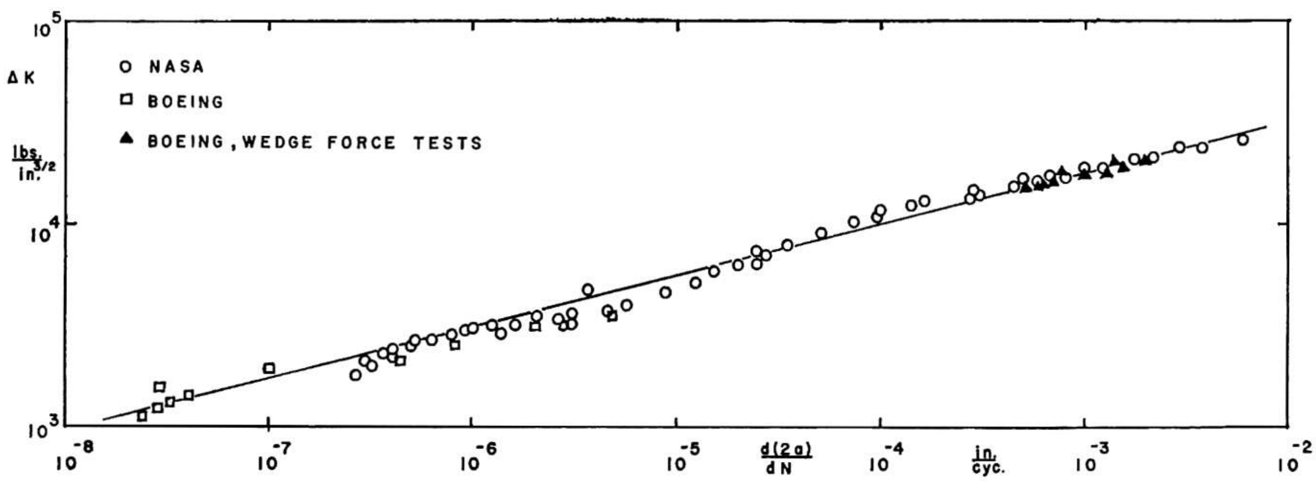

Fig. 6 Broad trend of crack-growth data on 7075-T6 aluminum alloy including wedge-force tests

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