Question.2-10

$\alpha, \vec{\beta}, \vec{\gamma}, \vec{\delta}$가 다음과 같이 주어졌다.

$$\vec{\beta} = \begin{pmatrix} \beta_1(\vec{\gamma}) \\ \beta_2(\vec{\gamma}) \\ \beta_3(\vec{\gamma}) \end{pmatrix} \qquad \vec{\gamma} = \begin{pmatrix} r_1(\vec{\delta}) \\ r_2(\vec{\delta}) \end{pmatrix} \qquad \vec{\delta} = \begin{pmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{pmatrix}$$

$$\alpha = \frac{1}{3}\sum_{i=1}^{3}\beta_i \qquad \vec{\beta}(\vec{\gamma}) = \begin{pmatrix} \beta_1(\vec{\gamma}) \\ \beta_2(\vec{\gamma}) \\ \beta_3(\vec{\gamma}) \end{pmatrix} = \begin{pmatrix} (r_1)^2 + 2r_2 \\ 2(r_1)^2 - 4(r_2)^2 \\ r_1 + 3(r_2)^2 \end{pmatrix} \qquad \vec{\gamma}(\vec{\delta}) = \begin{pmatrix} \gamma_1(\vec{\delta}) \\ \gamma_2(\vec{\delta}) \end{pmatrix} = \begin{pmatrix} sin(\delta_1) + cos(\delta_2) + tan(\delta_3) \\ e^{\delta_1} - e^{2\delta_2} + ln(\delta_3) \end{pmatrix}$$

이때 $\dfrac{\partial \alpha}{\partial \vec{\delta}}$를 구하시오.

주어진 상황에서 $\dfrac{\partial \alpha}{\partial \vec{\delta}}$를 구하기 위해선 $\dfrac{\partial \alpha}{\partial \vec{\beta}}, \dfrac{\partial \vec{\beta}}{\partial \vec{\gamma}}, \dfrac{\partial \vec{\gamma}}{\partial \vec{\delta}}$를 이용해서

$$\frac{\partial \alpha}{\partial \vec{\delta}} = \frac{\partial \alpha}{\partial \vec{\beta}} \cdot \frac{\partial \vec{\beta}}{\partial \vec{\gamma}} \cdot \frac{\partial \vec{\gamma}}{\partial \vec{\delta}}$$

위와 같이 구할 수 있다. 이 Jacobian들을 차례로 구하면 다음과 같다.

$$\frac{\partial \alpha}{\partial \vec{\beta}} = \begin{pmatrix} \frac{\partial}{\partial \beta_1}\left[\frac{1}{3}\sum_{i=1}^{3}\beta_i\right] & \frac{\partial}{\partial \beta_2}\left[\frac{1}{3}\sum_{i=1}^{3}\beta_i\right] & \frac{\partial}{\partial \beta_3}\left[\frac{1}{3}\sum_{i=1}^{3}\beta_i\right] \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

$$\frac{\partial \vec{\beta}}{\partial \vec{\gamma}} = \begin{pmatrix} \frac{\partial}{\partial \gamma_1}[(\gamma_1)^2 + 2\gamma_2] & \frac{\partial}{\partial \gamma_2}[(\gamma_1)^2 + 2\gamma_2] \\ \frac{\partial}{\partial \gamma_1}[2(\gamma_1)^2 - 4(\gamma_2)^2] & \frac{\partial}{\partial \gamma_2}[2(\gamma_1)^2 - 4(\gamma_2)^2] \\ \frac{\partial}{\partial \gamma_1}[\gamma_1 + 3(\gamma_2)^2] & \frac{\partial}{\partial \gamma_2}[\gamma_1 + 3(\gamma_2)^2] \end{pmatrix} = \begin{pmatrix} 2\gamma_1 & 2 \\ 4\gamma_1 & -8\gamma_2 \\ 1 & 6\gamma_2 \end{pmatrix}$$

$$\frac{\partial \vec{\gamma}}{\partial \vec{\delta}} = \begin{pmatrix} \frac{\partial}{\partial \delta_1}[sin(\delta_1) + cos(\delta_2) + tan(\delta_3)] & \frac{\partial}{\partial \delta_2}[sin(\delta_1) + cos(\delta_2) + tan(\delta_3)] & \frac{\partial}{\partial \delta_3}[sin(\delta_1) + cos(\delta_2) + tan(\delta_3)] \\ \frac{\partial}{\partial \delta_1}[e^{\delta_1} - e^{2\delta_2} + ln(\delta_3)] & \frac{\partial}{\partial \delta_1}[e^{\delta_1} - e^{2\delta_2} + ln(\delta_3)] & \frac{\partial}{\partial \delta_1}[e^{\delta_1} - e^{2\delta_2} + ln(\delta_3)] \end{pmatrix}$$

$$= \begin{pmatrix} cos(\delta_1) & -sin(\delta_2) & sec^2(\delta_3) \\ e^{\delta_1} & -2e^{2\delta_2} & \frac{1}{\delta_3} \end{pmatrix}$$

따라서 $\dfrac{\partial \alpha}{\partial \vec{\delta}}$를 구하면 다음과 같다.

$$\frac{\partial \alpha}{\partial \vec{\delta}} = \frac{\partial \alpha}{\partial \vec{\beta}} \cdot \frac{\partial \vec{\beta}}{\partial \vec{\gamma}} \cdot \frac{\partial \vec{\gamma}}{\partial \vec{\delta}} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 2\gamma_1 & 2 \\ 4\gamma_1 & -8\gamma_2 \\ 1 & 6\gamma_2 \end{pmatrix} \begin{pmatrix} cos(\delta_1) & -sin(\delta_2) & sec^2(\delta_3) \\ e^{\delta_1} & -2e^{2\delta_2} & \frac{1}{\delta_3} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{3}(6\gamma_1 + 1) & \frac{1}{3}(-2\gamma_2 + 2) \end{pmatrix} \begin{pmatrix} cos(\delta_1) & -sin(\delta_2) & sec^2(\delta_3) \\ e^{\delta_1} & -2e^{2\delta_2} & \frac{1}{\delta_3} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{3}\left[(6\gamma_1+1)cos(\delta_1) - (2\gamma_2-2)e^{\delta_1}\right] & -\frac{1}{3}\left[(6\gamma_1+1)sin(\delta_2) - 2(2\gamma_2-2)e^{2\delta_2}\right] & \frac{1}{3}\left[(6\gamma_1+1)sec^2(\delta_3) - (2\gamma_2-2)\frac{1}{\delta_3}\right] \end{pmatrix}$$