Damped and Driven Oscillations

VP160 Physics Lab Exercise 5 Report

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I. INTRODUCTION

If a periodically varying external force is applied to a damped harmonic oscillator, the resulting motion is called forced (or driven) oscillations, and the external force is called the driving force. In this experiment, forced oscillation of a balance wheel will be studied. Damping in this system is provided by air drag and electromagnets inducing eddy currents in the wheel. It is a rotating system, hence the corresponding quantities (such as the force and the position) will be replaced by their angular counterparts

When the balance wheel is acted upon a periodic driving torque $\tau_{dr} = \tau_0 \cos\omega t$ and a damping torque $\tau_f = -b\frac{d\theta}{dt}$, in addition to the restoring torque $\tau = -k\theta$, its equation of motion is of the form:

$$I\frac{d^2\theta}{dt^2} = -k\theta - b\frac{d\theta}{dt} + \tau_0 cos\omega t \quad (1)$$

where I is the moment of inertia of the balance wheel, τ_0 is the amplitude of the driving torque, and ω is angular frequency of the driving torque. Introducing the symbols: $\omega_0^2 = \frac{k}{I}$, $2\beta = \frac{b}{I}$, $\mu = \frac{\tau_0}{I}$, then Equation (1) can be rewritten as:

$$\frac{d^2\theta}{dt^2} + 2\beta\frac{d\theta}{dt} + \omega_0^2\theta = \mu cos\omega t \quad (2)$$

The solution to Equation (2), in the general case of a damped and driven system, is of the form:

$$\theta(t) = \theta_{tr}(t) + \theta_{st}\cos(\omega t + \phi) \quad (3)$$

where the former term θ_{tr} denotes the transient solution, that depends on the the initial conditions and vanishes exponentially as $t \to\infty$. The latter term describes steady-state oscillations, with the amplitude

$$\theta_{st} = \frac{\mu}{\sqrt{(\omega_0^2-\omega^2)^2+4\beta^2\omega^2}} \quad (4)$$

The phase shift ϕ can be found as:

$$\text{tg}\phi = -\frac{2\beta\omega}{\omega_0^2 - \omega^2}$$

where $-\pi \le \phi < 0$. Note again that, the amplitude and the phase shift are determined by μ, ω, ω_0, and β, and but not the initial conditions. By finding the maximum of θ_{st}, as a function of ω, we can find the *resonance angular frequency* $\omega_{res} = \sqrt{\omega_0^2 - 2\beta^2}$, and the corresponding amplitude $\theta_{res} = \theta_{st}(\omega_{res}) = \frac{\mu}{2\beta\sqrt{\omega_0^2-\beta^2}}$.

For small values of the damping coefficient β, the resonance angular frequency is close to the the natural angular frequency, and the amplitude of steady-state oscillations becomes large. The dependence of both the amplitude and the phase shift on the driving angular frequency are shown in the left and right Figure 1, respectively, for different values of the damping coefficient.

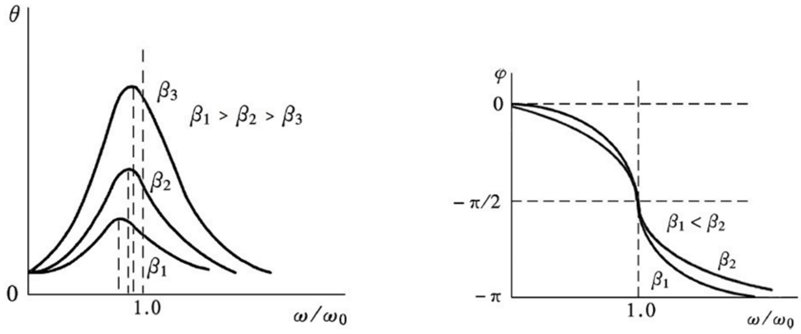
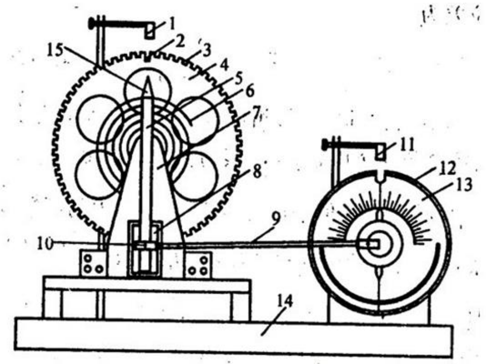


Figure 1: The dependence of the amplitude (left) and phase shift (right) of steady-state driven oscillations.

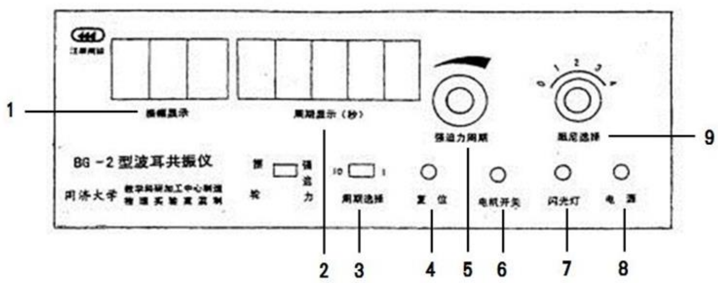
II. MEASUREMENT SETUP

The BG-2 Pohl resonator consists of two main parts: a vibrometer (see Figure 2) and a control box (see Figure 3).



1. Photoelectric detector (光电门)	2. Deep notch (深凹槽)	3. Shallow notch (浅凹槽)
4. Copper balance wheel (铜质摇轮)	5. Swinging rod (摇杆)	6. Scroll spring (蜗卷弹簧)
7. Supporting frame (支撑架)	8. Damping coil (阻尼线圈)	9. Connecting rod (连杆)
10. Adjustment screw (调节螺钉)	11. Photoelectric detector	12. Angle scale (角度盘)
13. Glass turntable (玻璃转盘)	14. Pedestal (底座)	15. Locking screw (夹持螺钉)

Figure 2: The vibrometer.



1. Amplitude display (振幅显示)	2. Period display (周期显示)	3. Period selection (周期选择)
4. Reset (复位)	5. Period of driving force (策动力周期)	6. Motor switch (电机开关)
7. Flash (闪光灯)	8. Power (电源)	9. Damping selection (阻尼选择)

Figure 3: The front panel of the control box.

There is a “Period Selection” switch and a “Period of Driving Force” knob on the electric control box, which allow to control the speed of the motor precisely. Another photoelectric detector is set above the turntable and connected to the control box to measure the period of driving force. The function “Amplitude Display” shows the oscillation amplitude of the balance wheel and “Period Display” shows the oscillation period in two modes. When the “Period Selection” switch is at position “1”, a single oscillation period will be displayed; when the “Period Selection” switch is at “10”, the time of 10 oscillation periods will be displayed. The reset button works only when the “Period Selection” button is at “10”. The “Damping Selection” knob changes the damping force by adjusting the electric current through the coils at the bottom of the wheel. There are six options, ranging from “0” (no current) to “5” (current of about 0.6 A). You will use “2”, “3” or “4” in this exercise. The strobe generates a flash that allows you to read the phase difference from the angle scale directly. To protect the strobe, you should turn on the “Strobe” switch only when measuring the phase difference. The “Motor Switch” is used to control the motor. You should turn the motor off when measuring the damping coefficient and the natural angular frequency

The chart below shows the uncertainties of the devices (see Table 1).

Table 1: The uncertainty of the devices.

Measurements	*Uncertainty*
Period Display (10 Periods)	0.001 [s]
Amplitude Display	1 [°]
Phase Lag	1 [°]

III. EXPERIMENTAL PROCUDURES

A) Natural Angular Frequency

1. We turned the Damping Selection knob to “0”.
2. We rotated the balance wheel to the initial angular position $\theta_0 \approx 150^{\circ}$ and then released it. Then recorded the time of 10 periods.
3. Later, we repeated the steps above for four times and calculated the natural angular frequency ω_0.

B) Damping Coefficient

1. We turned the Damping Selection knob to “2”, but the selection was not changed during this part.

2. We carefully rotated the balance wheel to the initial amplitude of approximately 150° and release it. We recorded the amplitude of each period (start from the second amplitude after you release the wheel) and the time of 10 periods.
3. The solution to the homogeneous equation of motion (2), with the corresponding initial conditions, is $\theta(t) = \theta_0 e^{-\beta t}\cos(\omega_f t + \alpha)$. Hence $\theta_1 = \theta_0 e^{-\beta T}$, $\theta_2 = \theta_0 e^{-\beta(2T)}, \ldots, \theta_n = \theta_0 e^{-\beta(nT)}$. The damping coefficient β was calculated as

$$\ln\frac{\theta_i}{\theta_j} = \ln\frac{\theta_0 e^{-i\beta T}}{\theta_0 e^{-j\beta T}} = (j-i)\beta T$$

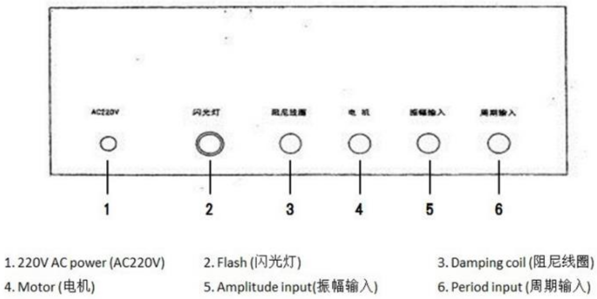


Figure 4: The rear panel of the control box.

4. The value of T should be the average period, so $\ln\frac{\theta_i}{\theta_{j+5}}$ was obtained by the successive difference method as

$$\beta = \frac{1}{5T}\ln\frac{\theta_i}{\theta_{j+5}} \quad (5)$$

C) θ_{st} **vs.** ω **and** ϕ **vs.** ω **Characteristics of Forced Oscillations**

1. We kept the Damping Selection at "2", and set the speed of the motor (record the position of the motor knob in case you need to repeat the measurement). We recorded the amplitude θ_{st}, the period T, and the phase shift ϕ when the oscillation reaches a steady state.
2. We repeated the steps above by changing the speed of the motor. It will result in a change of the phase shift ϕ (referred to as $\Delta\phi$). To make plots more accurate, we collected more data when ϕ and θ_{st} change rapidly (*e.g.* near to the resonance point).
3. We chose Damping Selection "1" and repeated the above steps.
4. We plotted the $\theta_{st}(\omega)$ characteristics, with ω/ω_0 on the horizontal axis and θ_{st} on the vertical axis.

IV. RESULTS & DATA PRESENTATION

All the data collected are shown as below (see Tables 2 – 5). All the data in the columns "ω/ω_0" are calculated from the respective data in the column "10T."

Table 2: Measurement of natural frequency

	10T [s]±0.001[s]
1	15.804
2	15.799
3	15.796
4	15.795

Table 3: Measurements for the damping coefficient.

Amplitude [°]±1[°]		Amplitude [°]±1[°]		$\ln(\theta_i / \theta_{i+5})$
θ_0	134	θ_5	85	0.4552
θ_1	122	θ_6	77	0.4602
θ_2	112	θ_7	70	0.4700
θ_3	102	θ_8	63	0.4818
θ_4	93	θ_9	58	0.4722
The average value				0.4679
Uncertainty				0.02
Relative uncertainty				5%
Final value		0.47±0.02		

Table 4: Data for Damping Selection 2.

	10T	ω/ω_0	Phase Lag [°]±1[°]	Amplitude [°]±1[°]
1	14.917	1.0591	-166	34
2	15.072	1.0482	-163	40
3	15.151	1.0428	-162	41
4	15.227	1.0376	-159	46
5	15.306	1.0322	-157	53
6	15.381	1.0272	-154	61
7	15.435	1.0236	-150	68
8	15.601	1.0127	-135	97
9	15.658	1.0090	-123	112
10	15.720	1.0050	-109	132
11	15.781	1.0011	-93	139
12	15.803	0.9997	-89	140
13	15.808	0.9994	-87	141
14	15.822	0.9985	-84	139
15	15.861	0.9961	-76	136
16	15.904	0.9934	-68	132
17	15.979	0.9887	-57	118
18	16.050	0.9844	-48	103
19	16.129	0.9795	-42	87
20	16.174	0.9768	-38	81
21	16.245	0.9725	-33	77

Table 5: Data for Damping Selection 1.

	10T	ω/ω_0	Phase Lag [°]±1[°]	Amplitude [°]±1[°]
1	14.882	1.0616	-167	32
2	15.005	1.0529	-165	38
3	15.134	1.0439	-162	44

4	15.260	1.0353	-160	54
5	15.363	1.0284	-156	65
6	15.404	1.0256	-147	81
7	15.582	1.0139	-139	105
8	15.659	1.0089	-124	132
9	15.721	1.0050	-106	151
10	15.741	1.0037	-100	156
11	15.768	1.0020	-93	160
12	15.780	1.0012	-89	160
13	15.799	1.0000	-86	160
14	15.821	0.9986	-79	158
15	15.856	0.9964	-73	153
16	15.971	0.9892	-55	128
17	16.067	0.9833	-44	108
18	16.198	0.9754	-34	84
19	16.335	0.9672	-25	69

The natural frequency is:

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{1.5799} = 3.9770 \pm 0.0018[rad \cdot s^{-1}]$$

The damping coefficient can be calculated according to Equation (5) as followings:

$$\beta = \frac{1}{5T}\ln\frac{\theta_i}{\theta_{j+5}} = \frac{2}{15.830} \cdot 0.468 = 0.059 \pm 0.003[s^{-1}]$$

By plotting on Microsoft Excel 2016®, the following figures are got (see Figures 5 & 6).

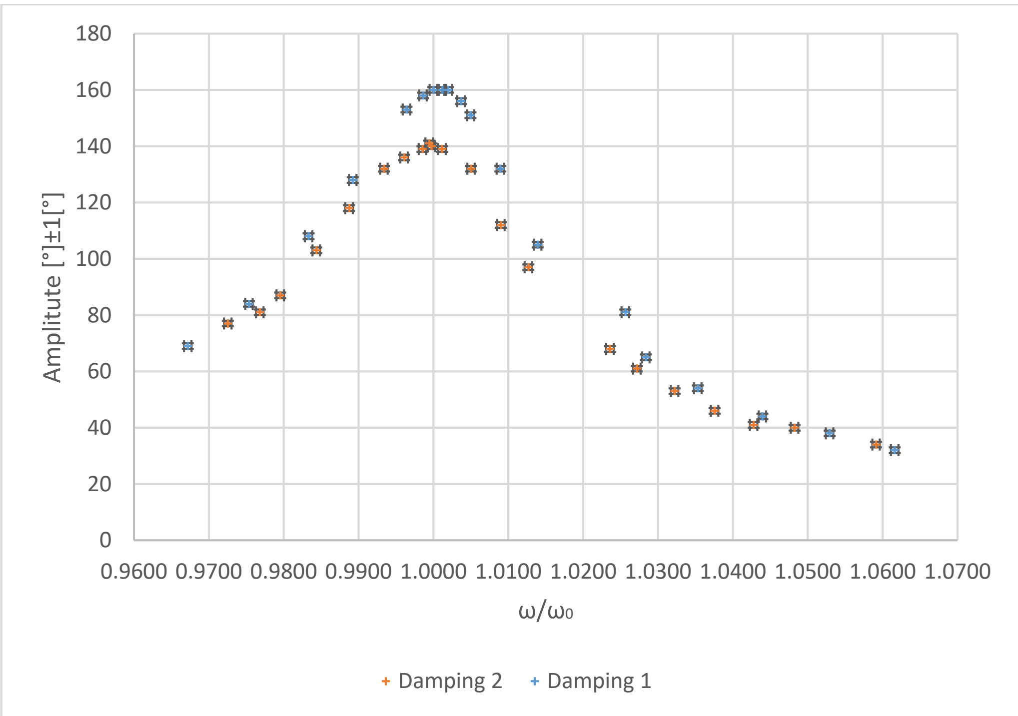


Figure 5: Amplitude ϑ vs. ω/ω_0.

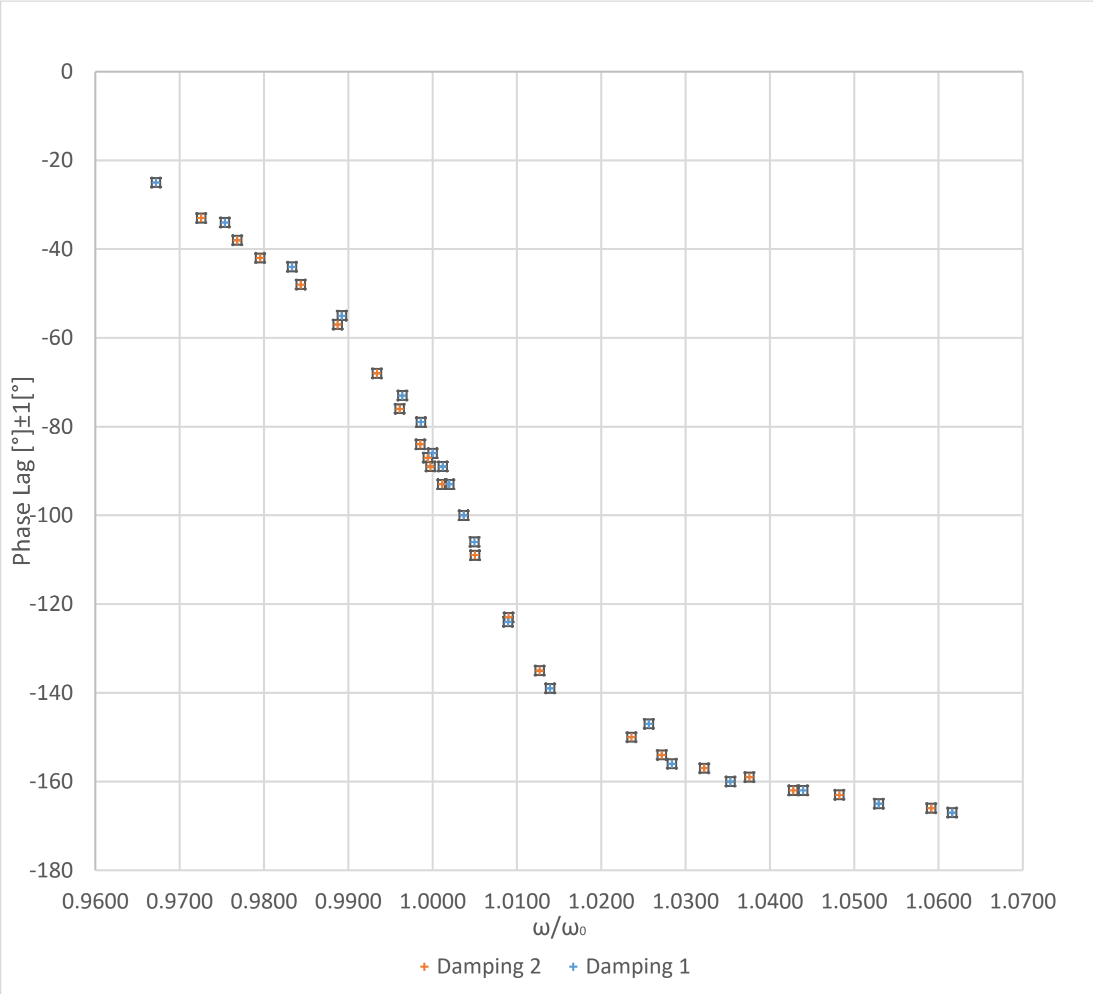


Figure 6: Phase lag ϕ vs. ω/ω_0.

It can be seen that: as the damping coefficient increases, the amplitude θ decreases, and value of ω/ω_0 at which the amplitude reaches its maximum decreases, while the plot of phase lag becomes less steep.

The uncertainties of the quotient ω/ω_0 is given in the following chart (Table 6).

Table 6: The uncertainties of quotient ω/ω_0.

	Damping 2	Uncertainty	Relative uncertainty		Damping 1	Uncertainty	Relative uncertainty
1	1.0591	0.0005	0.04%		1.0616	0.0005	0.04%
2	1.0482	0.0005	0.04%		1.0529	0.0005	0.04%
3	1.0428	0.0005	0.04%		1.0439	0.0005	0.04%
4	1.0376	0.0005	0.04%		1.0353	0.0005	0.04%
5	1.0322	0.0005	0.04%		1.0284	0.0005	0.04%
6	1.0272	0.0005	0.04%		1.0256	0.0005	0.04%
7	1.0236	0.0005	0.04%		1.0139	0.0005	0.04%
8	1.0127	0.0005	0.04%		1.0089	0.0005	0.04%
9	1.0090	0.0005	0.04%		1.0050	0.0004	0.04%
10	1.0050	0.0004	0.04%		1.0037	0.0004	0.04%

11	1.0011	0.0004	0.04%		1.0020	0.0004	0.04%
12	0.9997	0.0004	0.04%		1.0012	0.0004	0.04%
13	0.9994	0.0004	0.04%		1.0000	0.0004	0.04%
14	0.9985	0.0004	0.04%		0.9986	0.0004	0.04%
15	0.9961	0.0004	0.04%		0.9964	0.0004	0.04%
16	0.9934	0.0004	0.04%		0.9892	0.0004	0.04%
17	0.9887	0.0004	0.04%		0.9833	0.0004	0.04%
18	0.9844	0.0004	0.04%		0.9754	0.0004	0.04%
19	0.9795	0.0004	0.04%		0.9672	0.0004	0.04%
20	0.9768	0.0004	0.04%		N/A		
21	0.9725	0.0004	0.04%				

V. DISCUSSIONS

In this experiment, the inaccuracies may come from: i) not enough time waiting for the system to reach a period and amplitude; ii) environmental vibrations[1]; iii) insensitivity of the device[2]; iv) naked-eye observations on the phase lag data.

By comparison with theory, in this experiment:

- The result of natural frequency is satisfying.
- The result of natural frequency is not very satisfying, as the relative uncertainty is too large (5%). But this may mainly be the result of the device, rather than the measurements.
- The trend got in the part "amplitude θ vs. ω/ω_0" is satisfying, but the amplitude reached its maximum when ω/ω_0 is greater than 1. One probable reason may be the low precision and sensitivity of the device. Another probable reason is the insufficient time waiting for the system to reach equilibrium.
- The trend got in the part "phase lag ϕ vs. ω/ω_0" is satisfying, but the difference between two series of plots is not so impressive. One probable reason is that we chose damping selection 1 & 2, so that the two damping coefficients are close (not distinguished enough). Another probable reason is that the phase lag is observed by naked-eyes, so the results are not accurate enough.

[1] During the experiment, we discovered that whenever we put our elbows, or even a cellphone onto the table, the numbers displayed on the panel changes a lot.
[2] Sometimes, when we have waited for enough time such that the numbers become steady, and we begin our recordings, and we put our elbows onto the table, - then the numbers would jump to another "equilibrium" status. This indicates that the device is not so sensitive.

To improve this experiment, I have the following suggestions:

- When changing the speed of the motor knot, always rotate the knob with the same direction, such that the waiting time can be minimized;
- To better compare the two series in the "phase lag ϕ vs. ω/ω_0" part, there should be a large difference between the two damping coefficients.
- The device should have a strong (and uniform) magnetic field. Otherwise, the real damping coefficient may vary, due to the asymmetry of magnetic field.
- Use an electronic device to record the data, especially the data of phase lag.
- Use a device with higher precision.