Hammack Exercises - Part IV

FungusDesu

September 30th 2024

1 Preface

i dont really have anything to say

2 Proofs

Problem 12.2.4. A function $f : \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ is defined as $f(n) = (2n, n+3)$. Verify whether this function is injective or surjective.

The function f is injective. To prove this, we shall show that $f(a) = f(b)$ implies $a = b$ for any integer a, b. Then we have the following system of equations

$$\begin{cases} 2a = 2b \\ a + 3 = b + 3 \end{cases}$$

$$\begin{cases} a = b \\ a = b \end{cases}$$

The function f is not surjective, as there exists $(x, y) = (69, 420) \in \mathbb{Z} \times \mathbb{Z}$ for which $2n \neq 69$ for every $n \in \mathbb{Z}$, and so $(69, 420)$ is not in the image of f.

Problem 12.2.5. A function $f : \mathbb{Z} \to \mathbb{Z}$ is defined as $f(n) = 2n + 1$. Verify whether this function is injective or surjective.

The function f is injective. To see why, we show $f(a) = f(b)$ implies $a = b$ for any integer a, b. Then we have $2a + 1 = 2b + 1$ implies $a = b$.

The function f is not surjective, since there does not exist $n \in \mathbb{Z}$ such that $2n + 1 = 420$.

Problem 12.2.6. A function $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ is defined as $f(m, n) = 3n - 4m$. Verify whether this function is injective or surjective.

The function f is not injective, as there exist unequal elements $(0, 2)$ and $(3, 6)$ in $\mathbb{Z} \times \mathbb{Z}$ such that $f(0, 2) = f(3, 6) = 6$.

The function f is surjective. To see why, consider an arbitrary integer a. We need to show that there exists $(m, n) \in \mathbb{Z} \times \mathbb{Z}$ such that $f(m, n) = 3n - 4m = a$. By Proposition 7.1, there exists $m', n' \in \mathbb{Z}$ such that $3n' - 4m' = \gcd(-4, 3) = 1$. Thus $3 \cdot (an') - 4 \cdot (am') = a$. Therefore for $m = am'$ and $n = an'$, we have $f(m, n) = a$.

Problem 12.2.7. A function $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ is defined as $f(m, n) = 2n - 4m$. Verify whether this function is injective or surjective.

The function f is not injective, as there exist unequal elements $(0,0)$ and $(1,2)$ in $\mathbb{Z}\times\mathbb{Z}$ such that $f(0,0)=f(1,2)=0$.

The function f is not surjective, as there does not exist $m,n\in\mathbb{Z}$ such that $f(m,n)=2n-4m=1$ (the sum of two even numbers cannot be an odd number).

> **Problem 12.2.8.** A function $f:\mathbb{Z}\times\mathbb{Z}\to\mathbb{Z}\times\mathbb{Z}$ is defined as $f(m,n)=(m+n,2m+n)$. Verify whether this function is injective or surjective.

The function f is injective. To show why, we shall prove that $f(m,n)=f(m',n')$ implies $(m,n)=(m',n')$ for any $(m,n),(m',n')\in\mathbb{Z}^2$. Thus we have the following system of equations:

$$\begin{cases} m+n=m'+n' & (1)\\ 2m+n=2m'+n' & (2)\end{cases}$$

Subtracting equation (1) from equation (2) gives $m=m'$, and subsequently $n=n'$. Thus $(m,n)=(m',n')$.

The function f is surjective. To show why, consider an arbitrary $(a,b)\in\mathbb{Z}\times\mathbb{Z}$. We shall show that there exists $(x,y)\in\mathbb{Z}\times\mathbb{Z}$ for which $f(x,y)=(a,b)$. Thus we have the following system of equations:

$$\begin{cases} x+y=a\\ 2x+y=b\end{cases} \iff \begin{cases} x=b-a\\ x+y=a\end{cases} \iff \begin{cases} x=b-a\\ y=2a-b\end{cases}$$

Thus for any (a,b), there exists $(x,y)=(b-a,2a-b)$ such that $f(x,y)=(a,b)$.

> **Problem 12.2.9.** Prove that the function $f:\mathbb{R}\setminus\{2\}\to\mathbb{R}\setminus\{5\}$ defined by $f(x)=\frac{5x-1}{x-2}$ is bijective.

Proof. We first show that f is injective. To this end, we show that $f(a)=f(b)$ implies $a=b$ for any real number $a,b\neq 2$. Then

$$\begin{aligned}\frac{5a-1}{a-2}&=\frac{5b-1}{b-2}\\ (5a-1)(b-2)&=(5b-1)(a-2)\\ 5ab-10a-b+2&=5ab-10b-a+2\\ -10a+a&=-10b+b\\ a&=b.\end{aligned}$$

We now show that f is surjective. Consider an arbitrary element y such that $y\in\mathbb{R}\setminus\{5\}$. We wish to prove there exists $x\in\mathbb{R}\setminus\{2\}$ for which $f(x)=y$. Then

$$\frac{5x-1}{x-2}=y \implies 5x-1=xy-2y \implies x(5-y)=1-2y \implies x=\frac{1-2y}{5-y}.$$

And so $f(\frac{1-2y}{5-y})=y$ for arbitrary $y\in\mathbb{R}\setminus\{5\}$. Since f is both injective and surjective, it is also bijective.

> **Problem 12.2.10.** Prove the function $f:\mathbb{R}\setminus\{1\}\to\mathbb{R}\setminus\{1\}$ defined by $f(x)=\left(\frac{x+1}{x-1}\right)^3$ is bijective.

Proof. We first show that f is injective. To this end, we show that $f(a)=f(b)$ implies $a=b$ for any real number $a,b\neq 1$. Note that the function $g:\mathbb{R}\to\mathbb{R}$ defined by $g(x)=x^3$ is injective. To show this, we prove that for $m,n\in\mathbb{R}$, we have $g(m)=g(n)$ implies $m=n$. Thus we have the following:

$$\begin{aligned}&m^3=n^3\\ \implies &(m-n)(m^2+mn+n^2)=0\end{aligned}$$

$$\implies (m-n)\left(\left(\frac{m}{2}+n\right)^2+\frac{3m^2}{4}\right)=0.$$

If $m=n=0$, then we are done. If $a \neq 0$ and $b \neq 0$, then we have $m-n=0$ implies $m=n$. Using this fact and $f(a)=f(b)$, we deduce that

$$\begin{aligned}
&\left(\frac{a+1}{a-1}\right)^3=\left(\frac{b+1}{b-1}\right)^3\\
\implies &\frac{a+1}{a-1}=\frac{b+1}{b-1}\\
\implies &(a+1)(b-1)=(a-1)(b+1)\\
\implies &ab-a+b-1=ab+a-b-1\\
\implies &a=b.
\end{aligned}$$

We now show that f is surjective. Consider an arbitrary element $y \in \mathbb{R} \setminus \{1\}$; we wish to show that there exists some $x \in \mathbb{R} \setminus \{1\}$ such that $f(x)=y$. Thus $x^3=y$ implies $x=\sqrt[3]{y}$, which is the x value we wish to find. Since f is injective and surjective, it is also bijective.

Problem 12.2.11. Consider the function $\theta : \{0,1\} \times \mathbb{N} \to \mathbb{Z}$ defined as $\theta(a,b)=(-1)^a b$. Is θ injective? Surjective? Bijective? Explain.

The function θ is injective. To see why, we show that $\theta(x,y)=\theta(x',y')$ implies $(x,y)=(x',y')$. Then we have

$$\begin{aligned}
(-1)^x y &= (-1)^{x'} y'\\
(-1)^{x-x'} &= \frac{y}{y'}.
\end{aligned}$$

Without loss of generality, assume $x=0$ and $x'=1$. Then $\frac{y}{y'}$. But since $y, y' \in \mathbb{N}$, this is not possible. Thus $x=x'$, which implies $y=y'$.

The function θ is not surjective, since there does not exist $a, b \in \{0,1\} \times \mathbb{N}$ for which $(-1)^a b = 0$. Since θ is injective, but not surjective, it is not bijective.

Problem 12.2.12. Consider the function $\theta : \{0,1\} \times \mathbb{N} \to \mathbb{Z}$ defined as $\theta(a,b)=a-2ab+b$. Is θ injective? Surjective? Bijective? Explain.

The function θ is injective. To see why, we show that $\theta(x,y)=\theta(x',y')$ implies $(x,y)=(x',y')$. Then we have

$$\begin{aligned}
&x-2xy+y=x'-2x'y'+y'\\
\implies &x(1-2y)+y=x'(1-2y')+y'.
\end{aligned}$$

If $x=x'$, then it implies that $y=y'$. Without loss of generality, assume $x=0$ and $x'=1$, then it implies that $y=1-y'$. The left hand side has a lower bound of 1, whereas the right hand side as an upper bound of 0; thus the equality does not hold for any $y, y' \in \mathbb{N}$. Thus $(x,y)=(x',y')$.

The function θ is surjective. To see why, we show that there exists some $(a,b) \in \{0,1\} \times \mathbb{N}$ for which $\theta(a,b) \in \mathbb{Z}$. We have the following:

$$a-2ab+b=a(1-2b)+b. \tag{3}$$

If $a=0$, then $(3)=b \geq 1$. If $a=1$, then $(3)=1-b \leq 0$. So there always exists some (a,b) for which $\theta(a,b) \in \mathbb{Z}$. Since θ is injective and surjective, it is also bijective.

Problem 12.2.13. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by the formula $f(x, y) = (xy, x^3)$. Is f injective? Surjective? Bijective? Explain.

The function f is not injective, as there exist unequal elements $(0, 0)$ and $(0, 1)$ for which $f(0, 0) = f(0, 1) = (0, 0)$.

The function f is not surjective, as there does not exists $(x, y) \in \mathbb{R}^2$ for which $f(x, y) = (-1, 0)$ ($(xy, x^3) = (0, 0)$ implies $x = 0$, and so $0y = 0$ for all real y). Since f is not injective and not surjective, it is not bijective.

Problem 12.2.14. Consider the function $\theta : \mathscr{P}(\mathbb{Z}) \to \mathscr{P}(\mathbb{Z})$ defined as $\theta(X) = \overline{X}$. Is θ injective? Surjective? Bijective? Explain.

The function θ is injective. To see why, we shall show that $\theta(A) = \theta(B)$ implies $A = B$ for any $A, B \in \mathscr{P}(Z)$. Then

$$\overline{A} = \overline{B} \implies \overline{\overline{A}} = \overline{\overline{B}} \implies A = B.$$

The function θ is surjective. To see why, suppose an arbitrary set $Y \subseteq \mathbb{Z}$; we wish to show there exists some $X \subseteq \mathbb{Z}$ for which $\theta(X) = Y$. Then $\overline{X} = Y$ implies $X = \overline{Y}$, and so this is the set X for which $\theta(X) = Y$.

Problem 12.2.18. Prove that the function $f : \mathbb{N} \to \mathbb{Z}$ defined as $f(n) = \frac{(-1)^n(2n-1)+1}{4}$ is bijective.

Proof. We first show that f is injective. To this end, we show that $f(a) = f(b)$ implies $a = b$ for any natural a, b. Suppose $f(a) = f(b)$; then

$$\begin{aligned}
\frac{(-1)^a(2a-1)+1}{4} = \frac{(-1)^b(2b-1)+1}{4} &\implies (-1)^a(2a-1) = (-1)^b(2b-1) \\
&\implies |(-1)^a(2a-1)| = |(-1)^b(2b-1)| \\
&\implies |2a-1| = |2b-1|
\end{aligned}$$

Since $2a - 1$ and $2b - 1$ are always positive for natural a, b, it follows that

$$2a - 1 = 2b - 1 \implies a = b.$$

We then show that f is surjective. Consider an arbitrary element $y \in \mathbb{Z}$. We seek an $x \in \mathbb{N}$ for which $f(x) = y$, that is, for which

$$\frac{(-1)^x(2x-1)+1}{4} = y.$$

If $y = 0$, then solving the equality gets $x = 1$. If $y > 0$, then $x = 2y \in \mathbb{N}$ is the solution to the equality. Observe that

$$f(2y) = \frac{(-1)^{2y}(4y-1)+1}{4} = y.$$

If $y < 0$, then $x = -2y + 1 \in \mathbb{N}$ is the solution to the equality. Observe that

$$f(-2y+1) = \frac{(-1)^{-2y+1}(2(-2y+1)-1)+1}{4} = \frac{4y-2+1+1}{4} = y.$$

Thus for any $y \in \mathbb{Z}$, there exists $x \in \mathbb{N}$ such that $f(x) = y$, so f is surjective. Since f is both injective and surjective, it is also bijective.

Problem 12.3.1. Prove that if six integers are chosen at random, then at least two of them will have the same remainder when divided by 5.

Proof. Let $X \subseteq \mathbb{Z}$ be the set of any six integers. Let $Y = \{0, 1, 2, 3, 4\}$ be the set of possible remainders an arbitrary integer can have when divided by 5. Consider the function

$$f : X \to Y,$$

where $f(x)$ is the remainder of x when divided by 5. As $|X| = 6 > 5 = |Y|$, it follows from the pigeonhole principle that f is not injective. Thus there exists $a, b \in X$ for which $a \neq b$ such that $f(a) = f(b)$.

Proposition 12.6.5. Consider a function $f : A \to B$ and a subset $X \subseteq A$. Then $X \subseteq f^{-1}(f(X))$.

Proof. Suppose $x \in X$; then $f(x) \in f(X)$. Since $x \in X \subseteq A$, we have $x \in A$. By definition, the preimage of $f(X)$ is $f^{-1}(f(X)) = \{y \in A : f(y) \in f(X)\}$. Thus $x \in f^{-1}(f(X))$, and so $X \subseteq f^{-1}(f(X))$.

Conjecture 12.6.6. Given a function $f : A \to B$ and a subset $Y \subseteq B$. Then $f(f^{-1}(Y)) = Y$.

Disproof. This conjecture is false due to the following counterexample. Let $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$. Let $Y = \{-1, 1\} \in \mathbb{R}$. Note that $f(f^{-1}(Y)) = \{1\} \neq Y$, and we are done.

Problem 13.2.1. Prove that

$$\lim_{x \to 5} (8x - 3) = 37.$$

Proof. Choose any $\varepsilon > 0$. Observe that

$$|(8x - 3) - 37| = |8(x - 5)| = 8|x - 5|.$$

Choose $\delta = \frac{\varepsilon}{8} > 0$, then $0 < |x - 5| < \delta$ yields $0 < 8|x - 5| = |(8x - 3) - 37| < \frac{8\varepsilon}{8} = \varepsilon$. Thus

$$\lim_{x \to 5} (8x - 3) = 37.$$

Problem 13.2.2. Prove that

$$\lim_{x \to -1} (4x + 6) = 2.$$

Proof. Choose any $\varepsilon > 0$. Observe that

$$|(4x + 6) - 2| = 4|x - (-1)|.$$

Choose $\delta = \frac{\varepsilon}{4} > 0$, then $0 < |x - (-1)| < \delta$ yields $0 < 4|x - (-1)| = |(4x + 6) - 2| < \frac{4\varepsilon}{4} = \varepsilon$. Thus

$$\lim_{x \to -1} (4x + 6) = 2.$$

Problem 13.2.3. Prove that

$$\lim_{x \to 0} (x + 2) = 2.$$

Proof. Choose any $\varepsilon > 0$. Observe that

$$|(x+2)-2| = |x-0|.$$

Choose $\delta = \varepsilon > 0$. Then $0 < |x-0| < \delta$ yields $0 < |x-0| < |x-0| = |(x+2)-2| < \varepsilon$. Thus

$$\lim_{x \to 0}(x+2) = 2.$$

Problem 13.2.4. Prove that

$$\lim_{x \to 8}(2x-7) = 9.$$

Proof. Choose any $\varepsilon > 0$. Observe that $|(2x-7)-9| = 2|x-8|$. Choose $\delta = \frac{\varepsilon}{2} > 0$. Suppose $0 < |x-8| < \delta$, then $0 < 2|x-8| = |(2x-7)-9| < \frac{2\varepsilon}{2} = \varepsilon$. Thus

$$\lim_{x \to 8}(2x-7) = 9.$$