Hammack Exercises - Chapter 6

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September 6st 2024

1 Preface

i dont really have anything to say

2 Section A - Proof by contradiction only

Proposition 6.1. *Suppose $n \in \mathbb{Z}$. If n is odd, then n^2 is odd.*

Proof. Suppose for the sake of contradiction that n is odd and n^2 is even. Then there exist $x, y \in \mathbb{Z}$ such that $n = 2x + 1$ and $n^2 = 2y$. Thus we have $4x^2 + 4x + 1 = 2y$ implies $1 = 2(-2x^2 - 2x + y)$. Since $-2x^2 - 2x + y \in \mathbb{Z}$, we have 1 is even, a contradiction.

Proposition 6.2. *Suppose $n \in \mathbb{Z}$. If n^2 is odd, then n is odd.*

Proof. Suppose for the sake of contradiction that n^2 is odd and n is even. Then there exist $x, y \in \mathbb{Z}$ such that $n^2 = 2x + 1$ and $n = 2y$. Thus we have $4y^2 = 2x + 1$ implies $1 = 2(2y^2 - x)$. Since $2y^2 - x \in \mathbb{Z}$, we have 1 is even, a contradiction.

Proposition 6.3. *$\sqrt[3]{2}$ is irrational.*

Proof. Suppose for the sake of contradiction that $\sqrt[3]{2}$ is rational. Then there exist $a, b \in \mathbb{Z}$ such that $\sqrt[3]{2} = \frac{a}{b}$. Let $\frac{a}{b}$ be irreducible; it follows that

$$2b^3 = a^3, \tag{1}$$

which makes a^3 an even number. Thus a must be even; because a and b cannot be both even, we have b is odd. Since a is even, there exist $c \in \mathbb{Z}$ such that $a = 2c$. Substituting that into Equation (1), we get $b^3 = 2(2c^3)$. Therefore b^3 is an even number, which implies b is even. Thus we have a contradiction that b is both odd and even.

Proposition 6.4. *$\sqrt{6}$ is irrational.*

Proof. Suppose for the sake of contradiction that $\sqrt{6}$ is rational. Then there exist $a, b \in \mathbb{Z}$ such that $\sqrt{6} = \frac{a}{b}$. Let $\frac{a}{b}$ be irreducible; it follows that

$$a^2 = 2(3b^2), \tag{2}$$

which makes a^2 an even number. Thus a must be even; because a and b cannot be both even, we have b is odd. Since a is even, there exist $c \in \mathbb{Z}$ such that $a = 2c$. Substituting that into Equation (2), we get

$2c^2 = 3b^2$. Therefore $3b^2$ is even, which implies b is even. Thus we have a contradiction that b is both odd and even.

Proposition 6.5. *$\sqrt{3}$ is irrational.*

Proof. Suppose for the sake of contradiction that $\sqrt{3}$ is rational. Then there exist $a, b \in \mathbb{Z}$ such that $\sqrt{3} = \frac{a}{b}$. Let $\frac{a}{b}$ be irreducible; it follows that

$$a^2 = 3b^2, \tag{3}$$

implying $3 \mid a^2$. Because 3 is prime and divides a^2, it follows that a must contain 3 in its prime factorization, thus $3 \mid a$. Therefore $a = 3x$ for some $x \in \mathbb{Z}$. Substituting it into Equation (3), we yield $b^2 = 3x^2$. Similarly, we can also conclude that $3 \mid b$. Thus it is a contradiction that a and b share a common divisor of 3 when $\frac{a}{b}$ is irreducible.

Proposition 6.6. *If $a, b \in \mathbb{Z}$, then $a^2 - 4b - 2 \neq 0$.*

Proof. Suppose for the sake of contradiction that $a^2 - 4b - 2 = 0$ for some $a, b \in \mathbb{Z}$. Then we have $a^2 = 2(2b - 1)$, which implies a^2 is even. Thus a is even and can be expressed as $a = 2k$ for some $k \in \mathbb{Z}$. Substituting that into the original equation, we get $2k^2 - 2b - 1 = 0$ implies $1 = 2(k^2 - b)$. Therefore it is a contradiction that 1 is an even number.

Proposition 6.7. *If $a, b \in \mathbb{Z}$, then $a^2 - 4b - 3 \neq 0$.*

Proof. Suppose for the sake of contradiction that $a^2 - 4b - 3 = 0$ for some $a, b \in \mathbb{Z}$. Then we have $a^2 = 2(2b - 2) + 1$, which implies a^2 is odd. Thus a is odd and can be expressed as $a = 2k + 1$ for some $k \in \mathbb{Z}$. Substituting that into the original equation, we yield $2k^2 + 2k - 2b - 1 = 0$ implies $1 = 2(k^2 + k - b)$. Therefore it is a contradiction that 1 is an even number.

Proposition 6.8. *Suppose $a, b, c \in \mathbb{Z}$. If $a^2 + b^2 = c^2$, then a or b is even.*

Proof. Suppose for the sake of contradiction that there exist odd a and b such that $a^2 + b^2 = c^2$ for some $c \in \mathbb{Z}$. Since a and b is odd, we have $a = 2m + 1$ and $b = 2n + 1$ for some $m, n \in \mathbb{Z}$. Substituting those into the original equation, we yield

$$4m^2 + 4m + 4n^2 + 4n + 2 = c^2 \tag{4}$$

$$c^2 = 2(2m^2 + 2m + 2n^2 + 2n + 1), \tag{5}$$

implying c^2, and consequently c, is even. Thus $c = 2k$ for some $k \in \mathbb{Z}$; substituting that into Equation (5) yields

$$\begin{aligned} 2k^2 &= 2m^2 + 2m + 2n^2 + 2n + 1 \\ 1 &= 2(k^2 - m^2 - m - n^2 - n). \end{aligned}$$

Thus it is a contradiction that 1 is even.

Proposition 6.9. *Suppose $a, b, \in \mathbb{R}$. If a is rational and ab is irrational, then b is irrational.*

Proof. Suppose for the sake of contradiction that a, b is rational and their product is irrational. Then $a = \frac{m_a}{n_a}$ and $b = \frac{m_b}{n_b}$ for some $m_a, n_a, m_b, n_b \in \mathbb{Z}$. Thus $ab = \frac{m_a m_b}{n_a n_b}$, contradicting the fact that ab is irrational.

Proposition 6.10. *There exist no integers a and b for which $21a + 30b = 1$.*

Proof. Suppose for the sake of contradiction that there exist $a, b \in \mathbb{Z}$ such that $21a + 30b = 1$. Then $7a + 10b = \frac{1}{3}$. Thus it is a contradiction that the sum of two integers is a non-integer.

Proposition 6.11. *There exist no integers a and b for which $18a + 6b = 1$.*

Proof. Suppose for the sake of contradiction that there exist $a, b \in \mathbb{Z}$ such that $18a+6b = 1$. Then $3a+b = \frac{1}{6}$. Thus it is a contradiction that the sum of two integers is a non-integer.

Proposition 6.12. *For every positive $x \in \mathbb{Q}$, there is a positive $y \in \mathbb{Q}$ for which $y < x$.*

Proof. Suppose for the sake of contradiction that there exists a positive $x \in \mathbb{Q}$ such that for all positive $y \in \mathbb{Q}$, we have $y \geq x$. At $y = \frac{x}{2}$, we have $0 < y < x$. Thus it is a contradiction that $y \geq x$ and $y < x$.

Proposition 6.13. *For every $x \in \left[\frac{\pi}{2}, \pi\right]$, $\sin x - \cos x \geq 1$.*

Proof. Suppose for the sake of contradiction that there exists $x \in \left[\frac{\pi}{2}, \pi\right]$ for which $\sin x - \cos x < 1$. Since $x \in \left[\frac{\pi}{2}, \pi\right]$, we have $\sin x \geq 0$ and $\cos x \leq 0$; consequently, their product is never positive. Thus we have:

$$0 \leq \sin x - \cos x < 1.$$

Squaring each side of the inequality, we obtain:

$$\begin{aligned} 0 \leq \sin^2 x - 2\sin x \cos x + \cos^2 x &< 1 \\ 0 < \sin x \cos x &\leq \frac{1}{2}. \end{aligned}$$

Thus we have a contradiction.

Proposition 6.14. *If A and B are sets, then $A \cap (B - A) = \varnothing$.*

Proof. Suppose for the sake of contradiction that $A \cap (B - A) \neq \varnothing$ for some sets A, B. Then there exists $x \in (A \cap (B - A))$ and so $x \in A \wedge (x \in B \wedge x \notin A)$. Thus it is a contradiction that there exists an element that both belongs to A and not belongs to A.

Proposition 6.15. *If $b \in \mathbb{Z}$ and $b \nmid k$ for every $k \in \mathbb{N}$, then $b = 0$.*

Proof. Suppose for the sake of contradiction that there exists an integer $b \neq 0$ such that $b \nmid k$ for all $k \in \mathbb{N}$. Then $b \mid |b|$; thus contradicting the fact that it does not divide any natural number.

Proposition 6.16. *If a and b are positive real numbers, then $a + b \geq 2\sqrt{ab}$.*

Proof. Suppose for the sake of contradiction that $a + b < 2\sqrt{ab}$ for positive real a and b. Then we have:

$$\begin{aligned}(a+b)^2 &< 4ab\\ a^2 + 2ab + b^2 &< 4ab\\ a^2 - 2ab + b^2 &< 0\\ (a-b)^2 &< 0.\end{aligned}$$

Because $a - b \in \mathbb{R}$, it is a contradiction that the square of a real number is negative.

Proposition 6.17. *For every $n \in \mathbb{Z}$, $4 \nmid (n^2 + 2)$.*

Proof. Suppose for the sake of contradiction that $4 \mid n^2 + 2$ for some $n \in \mathbb{Z}$. Then there exists $x \in \mathbb{Z}$ such that $n^2 = 4x - 2$. As proven in **Proposition 5.28**, because only $n^2 \equiv 0 \pmod 4$ or $n^2 \equiv 1 \pmod 4$ is true and $(4x - 2) \equiv 2 \pmod 4$, it is contradictory that $n^2 \equiv 2 \pmod 4$.

Proposition 6.18. *Suppose $a, b \in \mathbb{Z}$. If $4 \mid (a^2 + b^2)$, then a and b are not both odd.*

Proof. Suppose for the sake of contradiction that if $4 \mid (a^2 + b^2)$, then a and b are both odd. Then $a = 2m+1$ and $b = 2n + 1$ for some $m, n \in \mathbb{Z}$. Thus we have

$$\begin{aligned}4 &\mid ((2m+1)^2 + (2n+1)^2)\\ 4 &\mid (4(m^2 + m + n^2 + n) + 2).\end{aligned}$$

Therefore we have a contradiction.

3 Section B - Direct, contrapositive and contradiction

Proposition 6.19. *The product of any five consecutive integers is divisible by* 120.

Proof. Consider any given five consecutive numbers. Notice that among them,

- there is always one number divisible by 5,
- there are at least two numbers divisible by 2, one of which is also divisible by 4,
- there is at least one number divisible by 3.

The multiples of 2 contribute 2 and 4 as the factors, multiples of 3 contribute 3 and multiple of 5 contributes 5. And because $2 \cdot 3 \cdot 4 \cdot 5 = 120$, it is evident their product is always a multiple of 120, as desired.

Proposition 6.20. *The curve $x^2 + y^2 - 3 = 0$ has no rational points.*

Proof. Suppose for the sake of contradiction that there exists a rational point in the curve $x^2 + y^2 - 3 = 0$. Let $P = (x_0, y_0)$ be such rational point; then $x_0 = \frac{m}{q}$ and $y_0 = \frac{n}{q}$ for some $m, n, q \in \mathbb{Z}$. Substituting P into the curve, we obtain:

$$\begin{aligned}\frac{m^2}{q^2} + \frac{n^2}{q^2} &= 3\\ m^2 + n^2 &= 3q^2.\end{aligned}$$

We define the **p-adic valuation** of a non-zero integer n to be the highest exponent of the prime number p in the factorization of n, denoted as $v_p(n)$. Thus

$$v_p(n) = \max(\{k \in \mathbb{N} : p^k \mid n\}).$$

Let a, b, c be the 3-adic valuation of m^2, n^2 and q^2 respectively. Let z^2 and w^2 be arbitrary square numbers. To complete the proof, we shall show that $v_3(z^2 + w^2) = \min(v_3(z^2), v_3(w^2))$. We divide into two cases as follow:

Case 1. If $v_3(z^2) \neq v_3(w^2)$, then

$$\begin{aligned}
&v_3(z^2 + w^2) \\
&= v_3(2^{v_2(z^2)} \cdot 3^{v_3(z^2)} \cdot 5^{v_5(z^2)} \cdots + 2^{v_2(w^2)} + 3^{v_3(w^2)} + 5^{v_5(w^2)} \cdots) \\
&= v_3(3^{\min(v_3(z^2),v_3(w^2))} \cdot (2^{v_2(z^2)} \cdot 3^{v_3(z^2)-\min(v_3(z^2),v_3(w^2))} \cdot 5^{v_5(z^2)} \cdots \\
&+ 2^{v_2(w^2)} + 3^{v_3(w^2-\min(v_3(z^2),v_3(w^2)))} + 5^{v_5(w^2)} \cdots)).
\end{aligned}$$

Because factoring $3^{\min(v_3(z^2),v_3(w^2))}$ leaves the sum with exactly 1 term divisible by 3, the sum is no longer a multiple of 3. Thus by definition, $v_3(z^2 + w^2) = \min(v_3(z^2), v_3(w^2))$.

Case 2. If $v_3(z^2) = v_3(w^2)$, we use the same process of reduction in **Case 1** to reduce the sum into one where both terms are not divisible by 3. Here we consider the divisibility of 3 on the sum itself. Because only either $a^2 \equiv 0 \pmod 3$ or $a^2 \equiv 1 \pmod 3$ is true for arbitrary integer a, if a^2 is not divisible by 3, then only $a^2 \equiv 1 \pmod 3$ is true. Thus $(\frac{z^2}{3^{v_3(z^2)}} + \frac{w^2}{3^{v_3(w^2)}}) \equiv 2 \pmod 3$. By the same principle in **Case 1**, it is also true in this case that $v_3(z^2 + w^2) = \min(v_3(z^2), v_3(w^2))$.

Thus $m^2 + n^2 = 3q^2$ is a sufficient condition for $v_3(m^2 + n^2) = v_3(3q^2)$, which implies $\min(a, b) = c + 1$. On the other hand, observe that a, b, c must all be even since all the terms' powers in their prime factorizations are even. Thus $\min(a, b)$ and $c + 1$ have opposite parity, implying they cannot be equal. Therefore we have a contradiction.

> **Proposition 6.21.** $\sqrt{3}$ *is irrational because the curve* $x^2 + y^2 - 3 = 0$ *has no rational points.*

Proof. Consider the case where $(x, y) = (q, 0)$ is a point on the curve $x^2 + y^2 - 3 = 0$. Thus we have $q^2 = 3$ implies $q = \sqrt{3}$. As proven in **Proposition 6.20** that the curve $x^2 + y^2 - 3 = 0$ has no rational points, q and 0 must not be both rational. 0 is rational, thus $q = \sqrt{3}$ is irrational, and we are done.

> **Proposition 6.22.** $x^2 + y^2 - 3 = 0$ *not having any rational solutions implies* $x^2 + y^2 - 3^k = 0$ *has no rational solutions for* k *an odd, positive integer.*

Proof. Becuase k is odd, we have $x^2 + y^2 = 3^{2n+1}$ for some integer n. Dividing both sides by 3^{2n} yields $\frac{x^2}{3^{2n}} + \frac{y^2}{3^{2n}} = 3$. Let $u = \frac{x}{3^n}$ and $v = \frac{y}{3^n}$, we see that u and v are rational if and only if x and y are rational. As such, we can form a bijection between the rational solutions of $\left(\frac{x}{3^n}\right)^2 + \left(\frac{y}{3^n}\right)^2 = 3$ and $u^2 + v^2 = 3$. However, **Proposition 6.20** says the latter has no rational solutions. Therefore neither does the former.

> **Proposition 6.23.** $\sqrt{3^k}$ *is irrational for all odd, positive* k.

Proof. Consider the case where $(x, y) = (q, 0)$ is a point on the curve $x^2 + y^2 - 3^k = 0$. Thus we have $q^2 = 3^k$ implies $q = \sqrt{3^k}$. As proven in **Proposition 6.22** that the curve $x^2 + y^2 - 3^k = 0$ has no rational points for positive odd k, we have q and 0 must not be both rational. 0 is rational, thus $q = \sqrt{3^k}$ is irrational, and we are done.

Proposition 6.24. *The number* $\log_2 3$ *is irrational.*

Proof. Suppose for the sake of contradiction that $\log_2 3$ is rational. For the sake of simplicity, there exist $m, n \in \mathbb{Z}$ such that $\log_2 3 = \frac{m}{n}$, and $m, n > 0$ since the fraction is positive ($\log_2 3$ is positive because $\log_2 3 > \log_2 2 = 1$). Let this fraction be irreducible; thus $3 = 2^{\frac{m}{n}}$, implying $3^n = 2^m$. For all $m, n \in \mathbb{Z}_{>0}$, we can see that 3^n is always odd, while 2^m is always even. Thus it is a contradiction that two numbers of opposite parity are equal.