Hammack Exercises - Chapter 8

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1 Preface

i dont really have anything to say

2 Proofs Involving Sets

Proposition 8.1. $\{12n : n \in \mathbb{Z}\} \subseteq \{2n : n \in \mathbb{Z}\} \cap \{3n : n \in \mathbb{Z}\}$.

Proof. Suppose $a \in \{12n : n \in \mathbb{Z}\}$. Then $a = 12n$ for some $n \in \mathbb{Z}$. Consequently, $a = 2 \cdot (6n)$ implies $a \in \{2n : n \in \mathbb{Z}\}$, and $a = 3 \cdot (4n)$ implies $a \in \{3n : n \in \mathbb{Z}\}$. Thus $a \in \{2n : n \in \mathbb{Z}\} \cap \{3n : n \in \mathbb{Z}\}$. This completes the proof.

Proposition 8.2. $\{6n : n \in \mathbb{Z}\} = \{2n : n \in \mathbb{Z}\} \cap \{3n : n \in \mathbb{Z}\}$.

Proof. ($\subseteq$) Suppose $a \in \{6n : n \in \mathbb{Z}\}$. Then $a = 6n$ for some $n \in \mathbb{Z}$. Thus $a = 2 \cdot (3n)$ implies $a \in \{2n : n \in \mathbb{Z}\}$, and $a = 3 \cdot (2n)$ implies $a \in \{3n : n \in \mathbb{Z}\}$. Thus $a \in \{2n : n \in \mathbb{Z}\} \cap \{3n : n \in \mathbb{Z}\}$.

($\supseteq$) Suppose $a \in \{2n : n \in \mathbb{Z}\} \cap \{3n : n \in \mathbb{Z}\}$. Then $a = 2x = 3y$ for some $x, y \in \mathbb{Z}$. $2x$ is even, thus $3y$ must also be even implies y is even. We have $y = 2k$ for some integer k, thus $a = 3y = 6k$. So a is a multiple of 6; as such, we get $a \in \{6n : n \in \mathbb{Z}\}$. This completes the proof.

Proposition 8.3. *If $k \in \mathbb{Z}$, then $\{n \in \mathbb{Z} : n \mid k\} \subseteq \{n \in \mathbb{Z} : n \mid k^2\}$.*

Proof. Suppose $a \in \{n \in \mathbb{Z} : n \mid k\}$. Then $a \mid k$ implies $k = ax$ for some $x \in \mathbb{Z}$. Thus $k^2 = a \cdot ax^2$; so $a \mid k^2$. Therefore $a \in \{n \in \mathbb{Z} : n \mid k^2\}$.

Proposition 8.4. *If $m, n \in \mathbb{Z}$, then $\{x \in \mathbb{Z} : mn \mid x\} \subseteq \{x \in \mathbb{Z} : m \mid x\} \cap \{x \in \mathbb{Z} : n \mid x\}$.*

Proof. Suppose $a \in \{x \in \mathbb{Z} : mn \mid x\}$. Then $mn \mid a$ implies $a = mnk$ for some integer k. Thus $a = m \cdot (nk) = n \cdot (mk)$ implies $m \mid a$ and $n \mid a$. Therefore $a \in \{x \in \mathbb{Z} : m \mid x\}$ and $a \in \{x \in \mathbb{Z} : n \mid x\}$; and so $a \in \{x \in \mathbb{Z} : m \mid x\} \cap \{x \in \mathbb{Z} : n \mid x\}$.

Proposition 8.5. *If p and q are positive integers, then $\{pn : n \in \mathbb{N}\} \cap \{qn : n \in \mathbb{N}\} \neq \varnothing$.*

Proof. Note that pq is both a element of $\{pn : n \in \mathbb{N}\}$ and $\{qn : n \in \mathbb{N}\}$. Thus $\{pn : n \in \mathbb{N}\} \cap \{qn : n \in \mathbb{N}\}$ always has at least one element, namely pq.

Proposition 8.6. *Suppose A, B and C are sets. If $A \subseteq B$, then $A - C \subseteq B - C$.*

Proof. Suppose $a \in A - C$. Then $a \in A$ and $a \notin C$. But because $A \subseteq B$, we have $a \in B$ and $a \notin C$. Thus $a \in B - C$. And so $A - C \subseteq B - C$, as desired.

Proposition 8.7. *Suppose A, B and C are sets. If $B \subseteq C$, then $A \times B \subseteq A \times C$.*

Proof. Let $a \in A$ and $b \in B$. Then we have $(a, b) \in A \times B$. Since $B \subseteq C$, we have $b \in B$ implies $b \in C$. Thus it is also true that $(a, b) \in A \times C$. Therefore $(a, b) \in A \times B$ implies $(a, b) \in A \times C$, and so $A \times B \subseteq A \times C$.

Proposition 8.8. *If A, B and C are sets, then $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.*

Proof. Observe that:

$$\begin{aligned} A \cup (B \cap C) &= \{x : x \in A \vee x \in B \cap C\} \\ &= \{x : x \in A \vee (x \in B \wedge x \in C)\} \\ &= \{x : (x \in A \vee x \in B) \wedge (x \in A \vee x \in C)\}. \end{aligned}$$

We can confirm the previous line is indeed true by the following truth table:

$x \in A$	$x \in B$	$x \in C$	$x \in A \vee (x \in B \wedge x \in C)$	$(x \in A \vee x \in B) \wedge (x \in A \vee x \in C)$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	T	T
F	T	T	T	T
F	T	F	F	F
F	F	T	F	F
F	F	F	F	F

Thus, it follows that:

$$\begin{aligned} A \cup (B \cap C) &= \{x : (x \in A \vee x \in B) \wedge (x \in A \vee x \in C)\} \\ &= \{x : (x \in A \vee x \in B)\} \cap \{x : (x \in A \vee x \in c)\} \\ &= (A \cup B) \cap (A \cup C). \end{aligned}$$

This completes our proof.

Proposition 8.9. *If A, B and C are sets, then $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.*

Proof. Observe that:

$$\begin{aligned} A \cap (B \cup C) &= \{x : x \in A \wedge (x \in B \vee x \in C)\} \\ &= \{x : (x \in A \wedge x \in B) \vee (x \in A \wedge x \in C)\}. \end{aligned}$$

The following truth table justifies this equality:

$x \in A$	$x \in B$	$x \in C$	$x \in A \wedge (x \in B \vee x \in C)$	$(x \in A \wedge x \in B) \vee (x \in A \wedge x \in C)$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	F	F
F	T	T	F	F
F	T	F	F	F
F	F	T	F	F
F	F	F	F	F

Thus, it follows that:

$$\begin{aligned} A \cap (B \cup C) &= \{x : (x \in A \wedge x \in B) \vee (x \in A \wedge x \in C)\} \\ &= (A \cap B) \cup (A \cap C). \end{aligned}$$

This completes our proof.

Proposition 8.10. *If A and B are sets in a universal set U, then $\overline{A \cap B} = \overline{A} \cup \overline{B}$.*

Proof. Simply observe the following chain of equalities:

$$\begin{aligned} \overline{A \cap B} &= \{x : \neg(x \in A \wedge x \in B)\} \\ &= \{x : \neg(x \in A) \vee \neg(x \in B)\} \\ &= \overline{A} \cup \overline{B}. \end{aligned}$$

Proposition 8.11. *If A and B are sets in a universal set U, then $\overline{A \cup B} = \overline{A} \cap \overline{B}$.*

Proof. Simply observe the following chain of equalities:

$$\begin{aligned} \overline{A \cup B} &= \{x : \neg(x \in A \vee x \in B)\} \\ &= \{x : \neg(x \in A) \wedge \neg(x \in B)\} \\ &= \overline{A} \cap \overline{B}. \end{aligned}$$

Proposition 8.12. *If A, B and C are sets, then $A - (B \cap C) = (A - B) \cup (A - C)$*

Proof. Simply observe the following chain of equalities:

$$\begin{aligned} A - (B \cap C) &= \{x : x \in A \wedge (\neg(x \in B \wedge x \in C))\} \\ &= \{x : x \in A \wedge (\neg(x \in B) \vee \neg(x \in C))\} \\ &= \{x : (x \in A \wedge \neg(x \in B)) \vee (x \in A \wedge \neg(x \in C))\} \\ &= \{x : x \in (A - B) \vee x \in (A - C)\} \\ &= (A - B) \cup (A - C). \end{aligned}$$

Proposition 8.13. *If A, B and C are sets, then $A - (B \cup C) = (A - B) \cap (A - C)$*

Proof. Simply observe the following chain of equalities:

$$\begin{aligned} A - (B \cup C) &= \{x : x \in A \wedge (\neg(x \in B \vee x \in C))\} \\ &= \{x : x \in A \wedge x \in A \wedge \neg(x \in B) \wedge \neg(x \in C)\} \\ &= \{x : (x \in A \wedge \neg(x \in B)) \wedge (x \in A \wedge \neg(x \in C))\} \\ &= \{x : x \in (A - B) \wedge x \in (A - C)\} \\ &= (A - B) \cap (A - C). \end{aligned}$$

Proposition 8.14. *If A, B and C are sets, then $(A \cup B) - C = (A - C) \cup (B - C)$.*

Proof. Simply observe the following chain of equalities:

$$\begin{aligned}(A \cup B) - C &= \{x : (x \in A \vee x \in B) \wedge \neg(x \in C)\} \\ &= \{x : (x \in A \wedge \neg(x \in C)) \vee (x \in B \wedge \neg(x \in C))\} \\ &= \{x : (x \in A - C) \vee (x \in B - C)\} \\ &= (A - C) \cup (B - C).\end{aligned}$$

Proposition 8.15. *If A, B and C are sets, then $(A \cap B) - C = (A - C) \cap (B - C)$.*

Proof. Simply observe the following chain of equalities:

$$\begin{aligned}(A \cap B) - C &= \{x : x \in A \wedge x \in B \wedge \neg(x \in C)\} \\ &= \{x : (x \in A \wedge \neg(x \in C)) \wedge (x \in B \wedge \neg(x \in C))\} \\ &= \{x : (x \in A - C) \wedge (x \in B - C)\} \\ &= (A - C) \cap (B - C).\end{aligned}$$

Proposition 8.16. *If A, B and C are sets, then $A \times (B \cup C) = (A \times B) \cup (A \times C)$.*

Proof. Simply observe the following chain of equalities:

$$\begin{aligned}A \times (B \cup C) &= \{(x, y) : x \in A \wedge (y \in B \vee y \in C)\} \\ &= \{(x, y) : (x \in A \wedge y \in B) \vee (x \in A \wedge y \in C)\} \\ &= \{(x, y) : x \in A \wedge y \in B\} \cup \{(x, y) : x \in A \wedge y \in C\} \\ &= (A \times B) \cup (A \times C).\end{aligned}$$

Proposition 8.17. *If A, B and C are sets, then $A \times (B \cap C) = (A \times B) \cap (A \times C)$.*

Proof. This is proven in Example 8.13.

Proposition 8.18. *If A, B and C are sets, then $A \times (B - C) = (A \times B) - (A \times C)$.*

Proof. Simply observe the following chain of equalities:

$$\begin{aligned}A \times (B - C) &= \{(x, y) : x \in A \wedge y \in B \wedge \neg(y \in C)\} \\ &= \{(x, y) : (x \in A \wedge y \in B) \wedge \neg(x \in A \wedge y \in C)\}.\end{aligned}$$

This can be justified by the following truth table:

$x \in A$	$y \in B$	$y \in C$	$x \in A \wedge y \in B \wedge \neg(y \in C)$	$(x \in A \wedge y \in B) \wedge \neg(x \in A \wedge y \in C)$
T	T	T	F	F
T	T	F	T	T
T	F	T	F	F
T	F	F	F	F
F	T	T	F	F
F	T	F	F	F
F	F	T	F	F
F	F	F	F	F

Thus, it follows that

$$\begin{aligned} A \times (B - C) &= \{(x, y) : (x \in A \wedge y \in B) \wedge \neg(x \in A \wedge y \in C)\} \\ &= \{(x, y) : x \in A \wedge y \in B\} \cap \{(x, y) : x \notin A \vee y \notin C\} \\ &= \{(x, y) : (x, y) \in (A \times B)\} \cap \{(x, y) : (x, y) \notin (A \times C)\} \\ &= (A \times B) - (A \times C). \end{aligned}$$

This completes our proof.

Proposition 8.19. $\{9^n : n \in \mathbb{Z}\} \subseteq \{3^n : n \in \mathbb{Z}\}$, *but* $\{9^n : n \in \mathbb{Z}\} \neq \{3^n : n \in \mathbb{Z}\}$.

Proof. Let $a \in \{9^n : n \in \mathbb{Z}\}$. Then $a = 9^n$ for some integer n. Note that $a = 9^n = 3^{2n}$. Thus $a \in \{3^n : n \in \mathbb{Z}\}$, and consequently $\{9^n : n \in \mathbb{Z}\} \subseteq \{3^n : n \in \mathbb{Z}\}$. Notice that 3 is an element of $\{3^n : n \in \mathbb{Z}\}$. Since there does not exist $x \in \mathbb{Z}$ such that $9^x = 3$, we know $3 \notin \{9^n : n \in \mathbb{Z}\}$. Thus $\{9^n : n \in \mathbb{Z}\} \neq \{3^n : n \in \mathbb{Z}\}$, and we are done.

Proposition 8.20. $\{9^n : n \in \mathbb{Q}\} = \{3^n : n \in \mathbb{Q}\}$.

Proof. ($\subseteq$) Suppose $x \in \{9^n : n \in \mathbb{Q}\}$. Then $x = 9^n$ for some rational n. Note that $x = 9^n = 3^{2n}$, and so $x \in \{3^n : n \in \mathbb{Q}\}$.

($\supseteq$) Suppose $x \in \{3^n : n \in \mathbb{Q}\}$. Then $x = 3^n$ for some rational n. Note that $x = 3^n = 9^{\frac{n}{2}}$, and so $x \in \{9^n : n \in \mathbb{Q}\}$. This completes our proof.

Proposition 8.21. *Suppose A and B are sets. $A \subseteq B$ if and only if $A - B = \varnothing$.*

Proof. ($\implies$) Suppose to the contrary that there exists $x \in A - B$. Then $x \in A$ and $x \notin B$. But we also have $A \subseteq B$, which contradicts this fact.

($\impliedby$) Suppose to the contrary that $A \nsubseteq B$. Then there exists $x \in A$ and $x \notin B$. But this contradicts the fact that $A - B = \varnothing$. This completes our proof.

Proposition 8.22. *Let A and B be sets. $A \subseteq B$ if and only if $A \cap B = A$.*

Proof. ($\implies$) Suppose to the contrary that $A \cap B \neq A$. Then there exists $x \in A$ and $x \notin B$. But this contradicts the fact that $A \subseteq B$.

($\impliedby$) Suppose to the contrary that $A \nsubseteq B$. Then there exists $x \in A$ and $x \notin B$. But this contradicts the fact that $A \cap B = A$. This completes our proof.

Proposition 8.23. *For each $a \in \mathbb{R}$, let $A_a = \{(x, a(x^2 - 1)) \in \mathbb{R}^2 : x \in \mathbb{R}\}$. Then*

$$\bigcap_{a \in \mathbb{R}} A_a = \{(-1, 0), (1, 0)\}.$$

Proof. Note that $A_a = \{(x, a(x^2-1)) \in \mathbb{R}^2 : x \in \mathbb{R}\}$ represents the graph of $y = a(x^2-1)$ for some real a; and so $\bigcap_{a\in\mathbb{R}} A_a$ represents the set of points that A_a always passes through for all real a. We shall first show that $\{(-1,0),(1,0)\}$ are two members of $\bigcap_{a\in\mathbb{R}} A_a$.

($\supseteq$) Suppose $(x,y) \in \{(-1,0),(1,0)\}$. Then for all real a, we can see that $(x,y) \in A_a$ $(0 = a((-1)^2-1) = 0, 0 = a((1)^2-1) = 0)$. Thus

$$(x,y) \in \bigcap_{a\in\mathbb{R}} A_a.$$

($\subseteq$) Suppose $(x,y) \in \bigcap_{a\in\mathbb{R}} A_a$. Consider $A_{420} \cap A_{0.69}$. This represents the set of common points of $y = 420(x^2-1)$ and $y = 0.69(x^2-1)$; in other words, the roots of $420(x^2-1) = 0.69(x^2-1)$. Thus:

$$420(x^2-1) = 0.69(x^2-1) \iff 419.31x^2 = 419.31 \iff x^2 = 1 \iff x = 1 \vee x = -1$$

At $x = 1$ or $x = -1$, we have $y = 0$. Thus $(x,y) \in \{(-1,0),(1,0)\}$. This completes our proof.

Proposition 8.24.

$$\bigcap_{x\in\mathbb{R}} [3-x^2, 5+x^2] = [3,5].$$

Proof. ($\supseteq$) Suppose $a \in [3,5]$. For all real x, we have $3 - x^2 \leq 3$ and $5 + x^2 \geq 5$. Thus $[3,5] \subseteq [3-x^2, 5+x^2]$ for all real x. Therefore

$$a \in \bigcap_{x\in\mathbb{R}} [3-x^2, 5+x^2].$$

($\subseteq$) Suppose $a \in \bigcap_{x\in\mathbb{R}}[3-x^2, 5+x^2]$. Let $_x = [3-x^2, 5+x^2]$; consider $_0$. Then $_0 = [3-0^2, 5+0^2] = [3,5] \subseteq [3,5]$. Thus $a \in [3,5]$. This completes our proof.

Proposition 8.25. *Suppose A, B, C and D are sets. Then $(A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D)$.*

Proof. Suppose $(x,y) \in (A \times B) \cup (C \times D)$. Then we have the following:

$$\begin{aligned}
(x,y) \in (A \times B) \cup (C \times D) &\implies (x,y) \in (A \times B) \vee (x,y) \in (C \times D)\\
&\implies (x \in A \wedge y \in B) \vee (x \in C \wedge y \in D)\\
&\implies (x \in A \vee x \in C) \wedge (y \in B \vee y \in D)\\
&\implies x \in A \cup C \wedge y \in B \cup D\\
&\implies (x,y) \in (A \cup C) \times (B \cup D),
\end{aligned}$$

and we are done.

Proposition 8.26. $\{4k+5 : k \in \mathbb{Z}\} = \{4k+1 : k \in \mathbb{Z}\}$.

Proof. ($\subseteq$) Suppose $\in \{4k+5 : k \in \mathbb{Z}\}$. Thus $= 4k+5$ for some integer k. Note that $= 4k+5 = 4(k+1)+1$, thus $\in \{4k+1 : k \in \mathbb{Z}\}$.

($\supseteq$) Suppose $\in \{4k+1 : k \in \mathbb{Z}\}$. Thus $= 4k+1$ for some integer k. Note that $= 4k+1 = 4(k-1)+5$, thus $\in \{4k+5 : k \in \mathbb{Z}\}$. This completes our proof.

Proposition 8.27. $\{12a+4b : a,b \in \mathbb{Z}\} = \{4c : c \in \mathbb{Z}\}$.

Proof. ($\subseteq$) Suppose $x \in \{12a+4b : a,b \in \mathbb{Z}\}$. Then $x = 12a+4b$ for some integer a, b. Note that $x = 12a+4b = 4(3a+b)$. Thus $x \in \{4c : c \in \mathbb{Z}\}$.

($\supseteq$) Suppose $x \in \{4c : c \in \mathbb{Z}\}$. Then $x = 4c$ for some integer c. Note that $x = 4c = 12 \cdot 0 + 4c$. Thus $x \in \{12a+4b : a,b \in \mathbb{Z}\}$, and we are done.

Proposition 8.28. $\{12a + 25b : a, b \in \mathbb{Z}\} = \mathbb{Z}$.

Proof. ($\subseteq$). Suppose $k \in \{12a + 25b : a, b \in \mathbb{Z}\}$. Thus $k = 12a + 25b$ for some integer a, b. Therefore $k \in \mathbb{Z}$.

($\supseteq$). Suppose $k \in \mathbb{Z}$. Note that there exist x, y such that $12x + 25y = \gcd(12, 25) = 1$ by Proposition 7.1. Thus we have $12xk + 25yk = k$. Thus $k \in \{12a + 25b : a, b \in \mathbb{Z}\}$. This completes our proof.

Proposition 8.29. *Suppose $A \neq \varnothing$. Then $A \times B \subseteq A \times C$ if and only if $B \subseteq C$.*

Proof. ($\implies$) Suppose $x \in B$. Because $A \neq \varnothing$, there exists $a \in A$. Thus $(a, x) \in (A \times B)$. But because $A \times B \subseteq A \times C$, we have $(a, x) \in (A \times C)$. Thus $x \in C$. Therefore $B \subseteq C$.

($\impliedby$) Suppose $(x, y) \in (A \times B)$; thus $x \in A$ and $y \in B$. Since $B \subseteq C$, we have $y \in C$, and so $(x, y) \in (A \times C)$. Therefore $A \times B \subseteq A \times C$, and we are done.

Lemma 8.1. *Suppose A, B, C and D are sets. Then $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$.*

Proof. Suppose $(x, y) \in (A \times B) \cap (C \times D)$. Then we have the following:

$$\begin{aligned} (x, y) \in (A \times B) \cap (C \times D) &\iff x \in A \wedge y \in B \wedge x \in C \wedge y \in D \\ &\iff (x \in A \cap C) \cap (y \in B \cap D) \\ &\iff (x, y) \in (A \cap C) \times (B \cap D). \end{aligned}$$

Proposition 8.30. $(\mathbb{Z} \times \mathbb{N}) \cap (\mathbb{N} \times \mathbb{Z}) = \mathbb{N} \times \mathbb{N}$.

Proof. By Lemma 8.1, we have the following:

$$(\mathbb{Z} \times \mathbb{N}) \cap (\mathbb{N} \times \mathbb{Z}) = (\mathbb{Z} \cap \mathbb{N}) \times (\mathbb{N} \cap \mathbb{Z}) = \mathbb{N} \times \mathbb{N}.$$

Proposition 8.31. *Suppose $B \neq \varnothing$ and $A \times B \subseteq B \times C$. Then $A \subseteq C$.*

Proof. Suppose $x \in A$. Since $B \neq \varnothing$, there exists $y \in B$. Thus $(x, y) \in (A \times B)$; because $(A \times B) \subseteq (B \times C)$, we have $(x, y) \in (A \times C)$. Therefore $y \in C$, and we are done.