Hammack Exercises - Chapter 10

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1 Preface

i dont really have anything to say

2 Induction

Proposition 10.1.

$$1 + 2 + 3 + 4 + \cdots + n = \frac{n^2 + n}{2}$$

for some positive integer n.

Proof. Denote $S_n : 1 + 2 + 3 + \cdots + n = \frac{n^2+n}{2}$. We will prove this using induction.

Basis step. If $n = 1$, then the equation is true.

Inductive hypothesis. For $k \geq 1$, suppose S_k is true; that is, $1 + 2 + 3 + \cdots + k = \frac{k^2+k}{2}$. We wish to prove S_{k+1} is true.

Induction step. Observe that:

$$\begin{aligned} 1 + 2 + 3 + \cdots + k + 1 &= 1 + 2 + 3 + \cdots + k + k + 1 \\ &= \frac{k^2 + k}{2} + k + 1 \\ &= \frac{k^2 + 3k + 2}{2} \\ &= \frac{(k+1)^2 + (k+1)}{2}. \end{aligned}$$

Thus $1 + 2 + 3 + \cdots + k + 1 = \frac{(k+1)^2+(k+1)}{2}$, which implies $S_k \implies S_{k+1}$. This completes our proof.

Proposition 10.2.

$$1^2 + 2^2 + 3^2 + 4^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

for every positive integer n.

Proof. Denote $S_n : 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$. We will prove this using induction.

Basis step. If $n = 1$, then S_1 is true.

Inductive hypothesis. For $k \geq 1$, suppose S_k is true; that is, $1^2 + 2^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}$. We wish to prove S_{k+1} is true.

Induction step. Observe that:

$$\begin{aligned} 1^2 + 2^2 + \cdots + (k+1)^2 = 1^2 + 2^2 + \cdots + k^2 + (k+1)^2 &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\ &= \frac{(k+1)(k(2k+1)+6k+6)}{6} = \frac{(k+1)(2k^2+7k+6)}{6} \\ &= \frac{(k+1)(k+2)(2k+3)}{6} = \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}. \end{aligned}$$

Thus $1^2 + 2^2 + \cdots + (k+1)^2 = \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$, which implies $S_k \implies S_{k+1}$. This completes our proof.

Proposition 10.3.

$$1^3 + 2^3 + 3^3 + 4^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$$

for every positive integer n.

Proof. Denote $S_n : 1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$. We shall prove this by induction.

Basis step. If $n = 1$, then the equation is true.

Inductive hypothesis. For $k \geq 1$, suppose S_k is true; that is, $1^3 + 2^3 + \cdots + k^3 = \frac{k^2(k+1)^2}{4}$. We wish to prove S_{k+1} is true.

Induction step. Observe that:

$$\begin{aligned} 1^3 + 2^3 + \cdots + (k+1)^3 = 1^3 + 2^3 + \cdots + k^3 + (k+1)^3 &= \frac{k^2(k+1)^2}{4} + (k+1)^3 \\ &= \frac{(k+1)^2(k^2+4k+4)}{4} = \frac{(k+1)^2(k+2)^2}{4} \\ &= \frac{(k+1)^2((k+1)+1)^2}{4}. \end{aligned}$$

Thus $1^3 + 2^3 + \cdots + (k+1)^3 = \frac{(k+1)^2((k+1)+1)^2}{4}$, which implies $S_k \implies S_{k+1}$. This completes our proof.

Proposition 10.4. *If $n \in \mathbb{N}$, then*

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + 4 \cdot 5 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}.$$

Proof. Denote $S_n : 1 \cdot 2 + 2 \cdot 3 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}$. We will prove this using induction.

Basis step. If $n = 1$, then the equation is true.

Inductive hypothesis. For $k \geq 1$, suppose S_k is true; that is, $1 \cdot 2 + 2 \cdot 3 + \cdots + k(k+1) = \frac{k(k+1)(k+2)}{3}$. We wish to prove S_{k+1} is true.

Induction step. Observe that:

$$\begin{aligned}1 \cdot 2 + 2 \cdot 3 + \cdots + (k+1)(k+2) &= 1 \cdot 2 + 2 \cdot 3 + \cdots + k(k+1) + (k+1)(k+2) \\ &= \frac{k(k+1)(k+2)}{3} + (k+1)(k+2) \\ &= \frac{(k+1)(k+2)(k+3)}{3} \\ &= \frac{(k+1)((k+1)+1)((k+1)+2)}{3}.\end{aligned}$$

Thus $1 \cdot 2 + 2 \cdot 3 + \cdots + (k+1)(k+2) = \frac{(k+1)((k+1)+1)((k+1)+2)}{3}$, which implies $S_k \implies S_{k+1}$. This completes our proof.

Proposition 10.5. *If $n \in \mathbb{N}$, then*

$$2^1 + 2^2 + 2^3 + \cdots + 2^n = 2^{n+1} - 2.$$

Proof. Denote $S_n : 2^1 + 2^2 + \cdots + 2^n = 2^{n+1} - 2$. We shall prove this using induction.

Basis step. If $n = 1$, then the equation is true.

Inductive hypothesis. For $k \geq 1$, suppose S_k is true; that is, $2^1 + 2^2 + \cdots + 2^k = 2^{k+1} - 2$. We wish to show that S_{k+1} is true.

Induction step. Observe that:

$$\begin{aligned}2^1 + 2^2 + \cdots + 2^{k+1} &= 2^1 + 2^2 + \cdots + 2^k + 2^{k+1} \\ &= 2^{k+1} - 2 + 2^{k+1} \\ &= 2^{k+2} - 2 \\ &= 2^{(k+1)+1} - 2.\end{aligned}$$

Thus $2^1 + 2^2 + \cdots + 2^{k+1} = 2^{(k+1)+1} - 2$, which implies $S_k \implies S_{k+1}$. This completes our proof.

Proposition 10.6.

$$\sum_{i=1}^{n} (8i - 5) = 4n^2 - n$$

for every positive integer n.

Proof. Denote the following:

$$S_n : \sum_{i=1}^{n} (8i - 5) = 4n^2 - n.$$

We shall prove this by using induction.

Basis case. If $n = 1$, then the equation is true.

Inductive hypothesis. For $k \geq 1$, suppose S_k is true; that is,

$$\sum_{i=1}^{k} (8i - 5) = 4k^2 - k.$$

We wish to prove that S_{k+1} is true.

Induction step. Observe the following:

$$\begin{aligned}\sum_{i=1}^{k+1}(8i-5) &= \sum_{i=1}^{k}(8i-5) + (8(k+1)-5)\\ &= 4k^2 - k + 8k + 8 - 5\\ &= 4k^2 + 8k + 4 - k - 1\\ &= 4(k+1)^2 - (k+1).\end{aligned}$$

Thus S_k implies S_{k+1}. This completes our proof.

Proposition 10.7. *If $n \in \mathbb{N}$, then*

$$1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + 4 \cdot 6 + \cdots + n(n+2) = \frac{n(n+1)(2n+7)}{6}.$$

Proof. Denote $S_n : 1 \cdot 3 + 2 \cdot 4 + \cdots + n(n+2) = \frac{n(n+1)(2n+7)}{6}$. We shall prove this using induction.

Basis step. If $n = 1$, then the equation is true.

Inductive hypothesis. For $k \geq n$, suppose S_k is true; that is, $1 \cdot 3 + 2 \cdot 4 + \cdots + k(k+2) = \frac{k(k+1)(2k+7)}{6}$. We wish to prove S_{k+1} is true.

Induction step. Observe the following:

$$\begin{aligned}1 \cdot 3 + 2 \cdot 4 + \cdots + (k+1)(k+3) &= 1 \cdot 3 + 2 \cdot 4 + \cdots + k(k+2) + (k+1)(k+3)\\ &= \frac{k(k+1)(2k+7)}{6} + (k+1)(k+3)\\ &= \frac{(k+1)(2k^2+13k+18)}{6}\\ &= \frac{(k+1)(k+2)(2k+9)}{6}\\ &= \frac{(k+1)((k+1)+1)(2(k+1)+7)}{6}.\end{aligned}$$

Thus S_k implies S_{k+1}. This completes our proof.

Proposition 10.8. *If $n \in \mathbb{N}$, then*

$$\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}.$$

Proof. We shall prove this using induction. Denote S_n to be the following:

$$S_n : \frac{1}{2!} + \frac{2}{3!} + \cdots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}.$$

Basis step. The equation holds if $n = 1$.

Inductive hypothesis. For $k \geq 1$, suppose S_k is true; that is,

$$\frac{1}{2!} + \frac{2}{3!} + \cdots + \frac{k}{(k+1)!} = 1 - \frac{1}{(k+1)!}.$$

We wish to prove S_{k+1} is true.

Induction step. Observe the following:

$$\begin{aligned}
\frac{1}{2!} + \frac{2}{3!} + \cdots + \frac{k+1}{(k+2)!} &= \frac{1}{2!} + \frac{2}{3!} + \cdots + \frac{k}{(k+1)!} + \frac{k+1}{(k+2)!} \\
&= 1 - \frac{1}{(k+1)!} + \frac{k+1}{(k+2)!} \\
&= 1 - \frac{k+2-k-1}{(k+2)!} \\
&= 1 - \frac{1}{((k+1)+1)!}.
\end{aligned}$$

Thus S_k implies S_{k+1}. This completes our proof.

Proposition 10.9. $24 \mid (5^{2n} - 1)$ *for every integer* $n \geq 0$.

Proof. Denote $S_n : 24 \mid (5^{2n} - 1)$. We shall prove this using induction.

Basis step. If $n = 0$, then we have $24 \mid 0$, which is true.

Inductive hypothesis. For $k \geq 0$, suppose S_k is true; that is, $24 \mid (5^{2k} - 1)$. We wish to show that S_{k+1} is true.

Induction step. Observe that

$$\begin{aligned}
5^{2(k+1)} - 1 - (5^{2k} - 1) &= 25^{k+1} - 1 - 25^k + 1 \\
&= 25 \cdot 25^k - 25^k \\
&= 24 \cdot 25^k.
\end{aligned}$$

We can see that $5^{2(k+1)} - 1$ and $5^{2k} - 1$ differ by a multiple of 24. Since 24 divides the latter, 24 must also divides the former. Thus S_{k+1} is true, as desired.

Proposition 10.10. $3 \mid (5^{2n} - 1)$ *for every integer* $n \geq 0$.

Proof. Denote $S_n : 24 \mid (5^{2n} - 1)$. We shall prove this using induction.

Basis step. If $n = 0$, then we have $3 \mid 0$, which is true.

Inductive hypothesis. For some $k \geq 0$, suppose S_k is true; that is, $3 \mid (5^{2k} - 1)$. We wish to show that S_{k+1} is true.

Induction step. Observe that

$$\begin{aligned}
5^{2(k+1)} - 1 - (5^{2k} - 1) &= 25^{k+1} - 1 - 25^k + 1 \\
&= 25 \cdot 25^k - 25^k \\
&= 3 \cdot 8 \cdot 25^k.
\end{aligned}$$

We can see that $5^{2(k+1)} - 1$ and $5^{2k} - 1$ differ by a multiple of 3. Since 3 divides the latter, 3 must also divide the former. Thus S_{k+1} is true, as desired.

Proposition 10.11. $3 \mid (n^3 + 5n + 6)$ *for every integer* $n \geq 0$.

Proof. Denote $S_n : 3 \mid (n^3 + 5n + 6)$. We will prove this using induction.

Basis step. If $n = 0$, then we have $3 \mid 6$, which is true.

Inductive hypothesis. For some $k \geq 0$, suppose S_k is true; that is, $3 \mid (k^3 + 5k + 6)$. We wish to prove S_{k+1} is true.

Induction step. Observe that

$$\begin{aligned} ((k+1)^3 + 5(k+1) + 6) - (k^3 + 5k + 6) &= k^3 + 3k^2 + 3k + 1 + 5k + 5 + 6 - k^3 - 5k - 6 \\ &= 3k^2 + 3k + 6 \\ &= 3(k^2 + k + 2). \end{aligned}$$

Note that $(k+1)^3 + 5(k+1) + 6$ and $k^3 + 5k + 6$ differ by a multiple of 3. Since 3 divides the latter as supposed, 3 must also divide the former. Thus S_{k+1} is true, as desired.

Proposition 10.12. $9 \mid (4^{3n} + 8)$ *for every integer* $n \geq 0$.

Proof. Denote $S_n : 9 \mid (4^{3n} + 8)$. We will prove this using induction.

Basis step. If $n = 0$, then we have $3 \mid 9$, which is true.

Inductive hypothesis. For some $k \geq 0$, suppose S_k is true; that is, $9 \mid (4^{3k} + 8)$. We wish to prove S_{k+1} is true.

Induction step. Observe that

$$\begin{aligned} (4^{3(k+1)} + 8) - (4^{3k} + 8) &= 64 \cdot 4^{3k} + 8 - 4^{3k} - 8 \\ &= 9 \cdot 7 \cdot 4^{3k}. \end{aligned}$$

Note that $4^{3(k+1)} + 8$ and $4^{3k} + 8$ differ by a multiple of 9. Since 9 divides the latter as supposed, 9 must also divide the former. Thus S_{k+1} is true, as desired.

Proposition 10.13. $6 \mid (n^3 - n)$ *for every integer* $n \geq 0$.

Proof. Denote $S_n : 6 \mid (n^3 - n)$. We will prove this using induction.

Basis step. Note that the statement is true for the first four positive integers.
If $n = 0$, then $6 \mid 0$, which is true. If $n = 2$, then $6 \mid 6$, which is true.
If $n = 1$, then $6 \mid 0$, which is true. If $n = 3$, then $6 \mid 24$, which is true.

Inductive hypothesis. For $0 \leq m \leq k$, suppose S_m is true. We wish to show that S_{k+1} is true. Note that S_{k-3} is true implies $6 \mid ((k-3)^3 - (k-3))$. Let $x = k - 3$; thus $6 \mid (x^3 - x)$ implies $x^3 - x = 6a$ for some integer a, and $x + 4 = k + 1$. We are now ready for the induction step.

Induction step. Observe that

$$\begin{aligned} (k+1)^3 - (k+1) &= (x+4)^3 - (x+4) \\ &= x^3 + 12x^2 + 48x + 64 - x - 4 \\ &= 6a + 12x^2 + 48x + 60 \\ &= 6(a + 2x^2 + 8x + 10). \end{aligned}$$

Thus $6 \mid ((k+1)^3 - (k+1))$ means S_{k+1} is true, as desired.

Proposition 10.14. *Suppose* $a \in \mathbb{Z}$. *Then* $5 \mid 2^n a$ *implies* $5 \mid a$ *for any* $n \in \mathbb{N}$.

Proof. Denote $S_n : 5 \mid 2^n a \implies 5 \mid a$. We will prove this using induction.

Basis step. If $n = 1$, then S_1 is true by Proposition 4.8.

Inductive hypothesis. For $k \geq 1$, suppose S_k is true; that is, $5 \mid 2^k a$ implies $5 \mid a$. We wish to show that S_{k+1} is true.

Induction step. Suppose $5 \mid 2^{k+1}a$. Then it follows that $5 \mid 2 \cdot 2^k a$ and then $5 \mid 2^k a$ by Proposition 4.8. From our initial assumption, we have $5 \mid 2^k a$ implies $5 \mid a$. Thus S_{k+1} is true, as desired.

Proposition 10.15. *If $n \in \mathbb{N}$, then*

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} + \cdots + \frac{1}{n(n+1)} = 1 - \frac{1}{n+1}.$$

Proof. Denote $S_n : \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)} = 1 - \frac{1}{n+1}$.

Basis step. If $n = 1$, then the equation is true.

Inductive hypothesis. For some $k \leq 1$, suppose S_k is true. We wish to prove S_{k+1} is true.

Induction step. Observe that

$$\begin{aligned}
\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{(k+1)(k+2)} &= \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)} \\
&= 1 - \frac{1}{k+1} + \frac{1}{(k+1)(k+2)} \\
&= 1 - \frac{k+1}{(k+1)(k+2)}.
\end{aligned}$$

Thus S_{k+1} is true, as desired.

Proposition 10.16. $2^n + 1 \leq 3^n$ *for every positive integer n.*

Proof. Denote $S_n : 2^n + 1 \leq 3^n$. We will prove this by using proof by smallest counterexample.

Basis step. If $n = 1$, then we have $3 \leq 3$, which is true.

Inductive hypothesis. Suppose for the sake of contradiction that there exists $k > n$ such that S_k is not true; that is, $2^k + 1 > 3^k$. Thus S_{k-1} is true, which means $2^{k-1} + 1 \leq 3^{k-1}$.

Induction step. Observe that

$$\begin{aligned}
2^{k-1} + 1 \leq 3^{k-1} &\implies \frac{2^k}{2} + 1 \leq \frac{3^k}{3} \\
&\implies \frac{2^k + 2}{2} \leq \frac{3^k}{3} \\
&\implies 2^k + 2 \leq \frac{2}{3} \cdot 3^k \\
&\implies 2^k + 1 \leq 3^k.
\end{aligned}$$

Thus it is a contradiction that $2^k + 1 > 3^k$ and $2^k + 1 \leq 3^k$.

Proposition 10.17. *Suppose $A_1, A_2, \ldots, A_n$ are sets in some universal set U, and $n \geq 2$. Prove that*

$$\overline{A_1 \cap A_2 \cap \cdots \cap A_n} = \overline{A_1} \cup \overline{A_2} \cup \cdots \cup \overline{A_n}.$$

Proof. We denote the following:

$$S_n : \overline{A_1 \cap A_2 \cap \cdots \cap A_n} = \overline{A_1} \cup \overline{A_2} \cup \cdots \cup \overline{A_n}.$$

We will prove this using induction.

Basis step. If $n = 2$, then $\overline{A_1 \cap A_2} = \overline{A_1} \cup \overline{A_2}$. This is true by Proposition 8.10.

Inductive hypothesis. For some $k \geq 2$, suppose S_k is true; that is,

$$\overline{A_1 \cap A_2 \cap \cdots \cap A_k} = \overline{A_1} \cup \overline{A_2} \cup \cdots \cup \overline{A_k}.$$

We wish to prove S_{k+1} is true.

Induction step. Observe that

$$\begin{aligned}
\overline{A_1 \cap A_2 \cap \cdots \cap A_{k+1}} &= \overline{(A_1 \cap A_2 \cap \cdots \cap A_k) \cap A_{k+1}} \\
&= \overline{A_1 \cap A_2 \cap \cdots \cap A_k} \cup \overline{A_{k+1}} \\
&= \overline{A_1} \cup \overline{A_2} \cup \cdots \cup \overline{A_k} \cup \overline{A_{k+1}}.
\end{aligned}$$

Thus S_{k+1} is true, and we are done.

Proposition 10.18. *Suppose $A_1, A_2, \ldots, A_n$ are sets in some universal set U, and $n \geq 2$. Prove that*

$$\overline{A_1 \cup A_2 \cup \cdots \cup A_n} = \overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n}.$$

Proof. We denote the following:

$$S_n : \overline{A_1 \cup A_2 \cup \cdots \cup A_n} = \overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n}.$$

We will prove this using induction.

Basis step. If $n = 2$, then $\overline{A_1 \cup A_2} = \overline{A_1} \cap \overline{A_2}$. This is true by Proposition 8.11.

Inductive hypothesis. For some $k \geq 2$, suppose S_k is true; that is,

$$\overline{A_1 \cup A_2 \cup \cdots \cup A_k} = \overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_k}.$$

We wish to prove S_{k+1} is true.

Induction step. Observe that

$$\begin{aligned}
\overline{A_1 \cup A_2 \cup \cdots \cup A_{k+1}} &= \overline{(A_1 \cup A_2 \cup \cdots \cup A_k) \cup A_{k+1}} \\
&= \overline{A_1 \cup A_2 \cup \cdots \cup A_k} \cap \overline{A_{k+1}} \\
&= \overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_k} \cap \overline{A_{k+1}}.
\end{aligned}$$

Thus S_{k+1} is true, and we are done.

Proposition 10.19.

$$\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} \leq 2 - \frac{1}{n}$$

for every $n \in \mathbb{N}$.

Proof. Denote S_n to be

$$S_n : \frac{1}{1} + \frac{1}{4} + \cdots + \frac{1}{n^2} \leq 2 - \frac{1}{n}.$$

We will prove this using proof by smallest counterexample.

Basis step. If $n = 1$, we have $1 \leq 1$, which is true.

Inductive hypothesis. Suppose for the sake of contradiction that there exists $k > n$ for which S_k is false:

$$\frac{1}{1} + \frac{1}{4} + \cdots + \frac{1}{k^2} > 2 - \frac{1}{k}.$$

Let k be the smallest integer such that this happens. Then S_{k-1} is true.

Induction step. Observe that

$$\begin{aligned}
\frac{1}{1} + \frac{1}{4} + \cdots + \frac{1}{(k-1)^2} \leq 2 - \frac{1}{k-1} &\implies \frac{1}{1} + \frac{1}{4} + \cdots + \frac{1}{(k-1)^2} + \frac{1}{k^2} \leq 2 - \frac{1}{k-1} + \frac{1}{k^2} \\
&\implies \frac{1}{1} + \frac{1}{4} + \cdots + \frac{1}{k^2} \leq 2 - \frac{k^2}{k^2(k-1)} + \frac{k-1}{k^2(k-1)} + \frac{1}{k^2(k-1)} \\
&\implies \frac{1}{1} + \frac{1}{4} + \cdots + \frac{1}{k^2} \leq 2 - \frac{k^2-k}{k^2(k-1)} \\
&\implies \frac{1}{1} + \frac{1}{4} + \cdots + \frac{1}{k^2} \leq 2 - \frac{1}{k}.
\end{aligned}$$

Thus S_k is true, a contradiction to our initial assumption that S_k is false.

> **Proposition 10.20.**
> $$(1 + 2 + 3 + \cdots + n)^2 = 1^3 + 2^3 + \cdots + n^3$$
> *for every* $n \in \mathbb{N}$.

Proof. Denote $S_n : (1 + 2 + \cdots + n)^2 = 1^3 + 2^3 + \cdots + n^3$. We will prove this using induction.

Basis step. If $n = 1$, then we have $1^2 = 1^3$, which is true.

Inductive hypothesis. For some $k \geq 1$, suppose S_k is true; that is, $(1+2+\cdots+k)^2 = 1^3+2^3+\cdots+k^3$. We wish to show S_k is true.

Induction step. Observe that:

$$\begin{aligned}
(1 + 2 + \cdots + (k+1))^2 &= ((1 + 2 + \cdots + k) + (k+1))^2 \\
&= (1 + 2 + \cdots + k)^2 + 2(k+1)(1 + 2 + \cdots + k) + (k+1)^2 \\
&= 1^3 + 2^3 + \cdots + k^3 + k(k+1)^2 + (k+1)^2 && \text{(Prop. 10.1)} \\
&= 1^3 + 2^3 + \cdots + k^3 + (k+1)^3.
\end{aligned}$$

Thus S_{k+1} is true, and we are done.

> **Proposition 10.21.** *If* $n \in \mathbb{N}$*, then*
> $$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{2^n - 1} + \frac{1}{2^n} \geq 1 + \frac{n}{2}.$$

Proof. Denote S_n to be the following:

$$S_n : \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{2^n} \geq 1 + \frac{n}{2}.$$

We will prove this using induction.

Basis step. If $n = 1$, then we have $\frac{3}{2} \geq \frac{3}{2}$, which is true.

Inductive hypothesis. For some $k \geq 1$, suppose S_k is true; that is,

$$\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{2^k} \geq 1 + \frac{k}{2}.$$

We wish to prove S_{k+1} is true.

Induction step. Observe that

$$\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{2^{k+1}} = \sum_{i=1}^{2^{k+1}} \frac{1}{i} = \sum_{i=1}^{2^k} \frac{1}{i} + \sum_{i=2^k+1}^{2^{k+1}} \frac{1}{i}.$$

Note that on the right hand side, the first sum has a lower bound of $1 + \frac{k}{2}$, as supposed by the induction hypothesis. The second sum consists of 2^k terms, each of which has a lower bound of $\frac{1}{2^{k+1}}$. And so their total will be bounded below by $\frac{2^k}{2^{k+1}} = \frac{1}{2}$. As such, the total of two sums will have a lower bound of $1 + \frac{k}{2} + \frac{1}{2} = 1 + \frac{k+1}{2}$. Thus S_{k+1} is true, and we are done.

Proposition 10.22. *If $n \in \mathbb{N}$, then*

$$\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{4}\right)\left(1 - \frac{1}{8}\right)\left(1 - \frac{1}{16}\right)\cdots\left(1 - \frac{1}{2^n}\right) \geq \frac{1}{4} + \frac{1}{2^{n+1}}.$$

Proof. Denote S_n to be the following:

$$S_n : \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{4}\right)\cdots\left(1 - \frac{1}{2^n}\right) \geq \frac{1}{4} + \frac{1}{2^{n+1}}.$$

We will prove this using induction.

Basis step. If $n = 1$, then we have $\frac{1}{2} \geq \frac{1}{2}$, which is true.

Inductive hypothesis. For some $k \geq 1$, suppose S_k is true; that is,

$$\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{4}\right)\cdots\left(1 - \frac{1}{2^k}\right) \geq \frac{1}{4} + \frac{1}{2^{k+1}}.$$

We wish to prove S_{k+1} is true.

Induction step. Observe that

$$\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{4}\right)\cdots\left(1 - \frac{1}{2^{k+1}}\right) = \prod_{i=1}^{k}\left(1 - \frac{1}{2^i}\right) \cdot \left(1 - \frac{1}{2^{k+1}}\right).$$

The first term of the right hand side product has a lower bound of $\frac{1}{4} + \frac{1}{2^{k+1}}$, as supposed by the inductive hypothesis. Therefore

$$\begin{aligned}\prod_{i=1}^{k}\left(1 - \frac{1}{2^i}\right) \geq \frac{1}{4} + \frac{1}{2^{k+1}} \implies \prod_{i=1}^{k}\left(1 - \frac{1}{2^i}\right) \cdot \left(1 - \frac{1}{2^{k+1}}\right) &\geq \left(\frac{1}{4} + \frac{1}{2^{k+1}}\right)\left(1 - \frac{1}{2^{k+1}}\right) \\ &\geq \frac{1}{4} + \frac{3}{4} \cdot \frac{1}{2^{k+1}} - \frac{1}{2^{2k+2}} \\ &\geq \frac{1}{4} + \frac{3}{2^{k+3}} - \frac{1}{2^{2k+2}}\end{aligned}$$

$$
\begin{aligned}
&\geq \frac{1}{4} + \frac{1}{2^{k+2}} + \frac{1}{2^{k+3}} - \frac{1}{2^{2k+2}} \\
&\geq \frac{1}{4} + \frac{1}{2^{k+2}}.
\end{aligned}
$$

Thus S_1 is true, and we are done.

Proposition 10.23 (Binomial Theorem). *If n is a non-negative integer, then*

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \binom{n}{3}x^{n-3}y^3 + \cdots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n.$$

Proof. Denote S_n to be the following:

$$S_n : (x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i.$$

We will prove this using induction.

Basis step. If $n = 0$, then the equation is true.

Inductive hypothesis. For some $k \geq 0$, suppose that S_k is true; that is,

$$(x+y)^k = \sum_{i=0}^{k} \binom{k}{i} x^{k-i} y^i.$$

We wish to prove S_{k+1} is true.

Induction step. Observe the following:

$$
\begin{aligned}
(x+y)^{k+1} &= (x+y)(x+y)^k \\
&= (x+y)\sum_{i=0}^{k} \binom{k}{i} x^{k-i} y^i \\
&= x\sum_{i=0}^{k} \binom{k}{i} x^{k-i} y^i + y\sum_{i=0}^{k} \binom{k}{i} x^{k-i} y^i \\
&= x\left(\sum_{i=1}^{k} \binom{k}{i} x^{k-i} y^i + x^k\right) + y\left(\sum_{i=0}^{k-1} \binom{k}{i} x^{k-i} y^i + y^k\right) \\
&= x\sum_{i=1}^{k} \binom{k}{i} x^{k-i} y^i + y\sum_{i=0}^{k-1} \binom{k}{i} x^{k-i} y^i + x^{k+1} + y^{k+1} \\
&= \sum_{i=1}^{k} \binom{k}{i} x^{k+1-i} y^i + \sum_{i=0}^{k-1} \binom{k}{i} x^{k-i} y^{i+1} + x^{k+1} + y^{k+1}. \qquad (\;)
\end{aligned}
$$

Consider the second term of ( ). Let $i' = i + 1$; the second term becomes

$$\sum_{i=0}^{k-1} \binom{k}{i} x^{k-i} y^{i+1} = \sum_{i'=1}^{k} \binom{k}{i'-1} x^{k+1-i'} y^{i'}.$$

Thus we have the following:

$$(\;) = \sum_{i=1}^{k} \binom{k}{i} x^{k+1-i} y^i + \sum_{i=1}^{k} \binom{k}{i-1} x^{k+1-i} y^i + x^{k+1} + y^{k+1}$$

$$
\begin{aligned}
&= \sum_{i=1}^{k} \left(\binom{k}{i} + \binom{k}{i-1} \right) x^{k+1-i} y^i + x^{k+1} + y^{k+1} \\
&= \sum_{i=1}^{k} \binom{k+1}{i} x^{k+1-i} y^i + x^{k+1} + y^{k+1} \\
&= \sum_{i=0}^{k+1} \binom{k+1}{i} x^{k+1-i} y^i.
\end{aligned}
$$

Thus S_{k+1} is true, as desired.

Proposition 10.24.

$$\sum_{k=1}^{n} k \binom{n}{k} = n2^{n-1}$$

for each natural number n.

Proof. Denote S_n to be the following:

$$S_n : \sum_{k=1}^{n} k \binom{n}{k} = n2^{n-1}.$$

We will prove this using induction.

Basis step. If $n = 1$, then we have $1 = 1$, which is true.

Inductive hypothesis. For some $m \geq 1$, suppose S_m is true. We wish to show that S_{m+1} is true.

Induction step. Observe that:

$$
\begin{aligned}
\sum_{k=1}^{m+1} k \binom{m+1}{k} &= \sum_{k=1}^{m} \left(\binom{m}{k-1} k + \binom{m}{k} k \right) + m + 1 \\
&= \sum_{k=1}^{m+1} \binom{m}{k-1} k + \sum_{k=1}^{m} \binom{m}{k} k \\
&= \sum_{k=0}^{m} \binom{m}{k} (k+1) + m2^{m-1} \\
&= \sum_{k=0}^{m} \binom{m}{k} k + \sum_{k=0}^{m} \binom{m}{k} + m2^{m-1} \\
&= 2m2^{m-1} + 2^m \\
&= (m+1)2^m.
\end{aligned}
$$

Thus S_{m+1} is true, as desired.

Proposition 10.25.

$$F_1 + F_2 + F_3 + F_4 + \cdots + F_n = F_{n+2} - 1.$$

Proof. Denote S_n to be $S_n : F_1 + F_2 + \cdots + F_n = F_{n+2} - 1$. We will prove this using induction.

Basis step. If $n = 1$, then the equation is true.

Inductive hypothesis. For some $k \geq 1$, suppose S_k is true, that is, $F_1 + F_2 + \cdots + F_k = F_{k+2} - 1$. We wish to show that S_{k+1} is true.

Induction step. Observe that

$$\begin{aligned} F_1 + F_2 + \cdots + F_{k+1} &= F_1 + F_2 + \cdots + F_k + F_{k+1} \\ &= F_{k+2} - 1 + F_{k+1} \\ &= F_{k+3} - 1 \\ &= F(k+1) + 2 - 1. \end{aligned}$$

Thus S_{k+1} is true, and we are done.

Proposition 10.26.

$$\sum_{k=1}^{n} F_k^2 = F_n F_{n+1}.$$

Proof. Denote S_n to be the following:

$$S_n : \sum_{k=1}^{n} F_k^2 = F_n F_{n+1}.$$

We will prove this using induction.

Basis step. If $n = 1$, then the equation holds.

Inductive hypothesis. For some $m \geq n$, suppose S_m is true; that is,

$$\sum_{k=1}^{m} F_k^2 = F_m F_{m+1}.$$

We wish to show that S_{k+1} is true.

Induction step. Observe that

$$\begin{aligned} \sum_{k=1}^{m+1} F_k^2 &= \sum_{k=1}^{m} F_k^2 + F_{m+1}^2 \\ &= F_m F_{m+1} + F_{m+1}^2 \\ &= F_{m+1}(F_m + F_{m+1}) \\ &= F_{m+1} F_{m+2}. \end{aligned}$$

Thus S_{m+1} is true, and we are done.

Proposition 10.27.

$$F_1 + F_3 + F_5 + F_7 + \cdots + F_{2n-1} = F_{2n}.$$

Proof. Denote S_n to be $S_n : F_1 + F_3 + \cdots + F_{2n-1} = F_{2n}$. We will prove this using induction.

Basis step. If $n = 1$, then the equation holds.

Inductive hypothesis. For some $k \geq n$, suppose F_k is true; that is, $F_1 + F_3 + \cdots + F_{2k-1} = F_{2k}$. We wish to show that F_{k+1} is true.

Induction step. Observe that

$$\begin{aligned}
F_1 + F_3 + \cdots + F_{2(k+1)-1} &= F_1 + F_3 + \cdots + F_{2k+1} \\
&= F_1 + F_3 + \cdots + F_{2k-1} + F_{2k+1} \\
&= F_2 k + F_{2k+1} \\
&= F_{2k+2}.
\end{aligned}$$

Thus S_{k+1} is true, as desired.

Proposition 10.28.

$$F_2 + F_4 + F_6 + F_8 + \cdots + F_{2n} = F_{2n+1} - 1.$$

Proof. Denote S_n to be $S_n : F_2 + F_4 + \cdots + F_{2n} = F_{2n+1} - 1$. We will prove this using induction.

Basis step. If $n = 1$, then we have $F_2 = F_3 - 1$, which is true.

Inductive hypothesis. For some $k \geq n$, suppose S_k is true; that is, $F_2 + F_4 + \cdots + F_{2k} = F_{2k+1} - 1$. We wish to show that F_{k+1} is true.

Induction step. Observe that

$$\begin{aligned}
F_2 + F_4 + \cdots + F_{2(k+1)} &= F_2 + F_4 + \cdots + F_{2k} + F_{2k+2} \\
&= F_{2k+1} - 1 + F_{2k+2} \\
&= F_{2k+3} - 1 \\
&= F_{2(k+1)+1} - 1.
\end{aligned}$$

Thus S_{k+1} is true, and we are done.

Proposition 10.29. *The diagonals of Pascal's triangle sum to Fibonacci numbers.*

Proof. The expression that represents the sum of the Pascal's triangle's diagonals is

$$\sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-k}{k}.$$

Denote S_n to be the following:

$$S_n : F_n = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-k}{k}.$$

We will prove this using induction.

Basis step. If $n = 1$, then we have $F_1 = \binom{0}{0} = 1$, which is true. If $n = 2$, then we have

Inductive hypothesis. For some $m \geq 2$, suppose S_m and S_{m-1} is true; that is,

$$S_m : F_m = \sum_{k=0}^{\lfloor \frac{m-1}{2} \rfloor} \binom{m-1-k}{k}.$$

$$S_{m-1} : F_{m-1} = \sum_{k=0}^{\lfloor \frac{m-2}{2} \rfloor} \binom{m-2-k}{k}.$$

We wish to prove S_{m+1} is true.

Induction step. Observe the following:

$$\begin{aligned}
F_{m+1} = F_{m-1} + F_m &= \sum_{k=0}^{\lfloor\frac{m-2}{2}\rfloor} \binom{m-k-2}{k} + \sum_{k=0}^{\lfloor\frac{m-1}{2}\rfloor} \binom{m-k-1}{k} \\
&= \sum_{k=1}^{\lfloor\frac{m-1}{2}\rfloor} \binom{m-k-1}{k-1} + \sum_{k=0}^{\lfloor\frac{m-1}{2}\rfloor} \binom{m-k-1}{k}. \qquad (\ )
\end{aligned}$$

Without loss of generality, assume m is odd. Then from ( ) it follows that:

$$\begin{aligned}
F_{m+1} &= \sum_{k=1}^{\frac{m-1}{2}} \binom{m-k-1}{k-1} + \sum_{k=0}^{\frac{m-1}{2}} \binom{m-k-1}{k} \\
&= \sum_{k=1}^{\frac{m-1}{2}} \binom{m-k-1}{k-1} + \sum_{k=1}^{\frac{m-1}{2}} \binom{m-k-1}{k} + 1 \\
&= \sum_{k=1}^{\frac{m-1}{2}} \binom{m-k}{k} + \binom{m-0}{0} \\
&= \sum_{k=1}^{\frac{m-1}{2}} \binom{m-k}{k}.
\end{aligned}$$

The proof is complete.

Proposition 10.30.

$$F_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}$$

Proof. Denote S_n to be

$$S_n : F_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}$$

. We will prove this using induction.

Basis step. If $n = 1$, then we have $F_1 = \frac{\sqrt{5}}{\sqrt{5}} = 1$, which is true. If $n = 2$, then we have $F_2 = \frac{2\sqrt{5}}{\sqrt{5}}$, which is true.

Inductive hypothesis. For some $k \geq 2$, suppose S_k and S_{k-1} is true. We wish to show that S_{k+1} is true.

Induction step. Observe that

$$\begin{aligned}
F_{k+1} = F_k + F_{k-1} &= \frac{\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k}{\sqrt{5}} + \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{k-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k-1}}{\sqrt{5}} \\
&= \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{k-1} \frac{3+\sqrt{5}}{2} - \left(\frac{1-\sqrt{5}}{2}\right)^{k-1} \frac{3-\sqrt{5}}{2}}{\sqrt{5}} \\
&= \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{k-1} \left(\frac{(1+\sqrt{5})^2}{4}\right) - \left(\frac{1-\sqrt{5}}{2}\right)^{k-1} \left(\frac{(1-\sqrt{5})^2}{4}\right)}{\sqrt{5}}
\end{aligned}$$

$$= \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+1}}{\sqrt{5}}.$$

Thus S_{k+1} is true, as desired.

Proposition 10.31.
$$\sum_{k=0}^{n} \binom{k}{r} = \binom{n+1}{r+1},$$
where $1 \leq r \leq n$.

Proof. Denote S_n to be
$$S_n : \sum_{k=0}^{n} \binom{k}{r} = \binom{n+1}{r+1}.$$

We will prove this using induction.

Basis step. If $n = 1$, then we have $\binom{0}{r} + \binom{1}{r} = \binom{2}{r+1} = 2$, which is true.

Inductive hypothesis. For some $m \geq 1$, suppose S_m is true. We wish to show that S_{m+1} is true.

Induction step. Observe that:
$$\begin{aligned}\sum_{k=0}^{m+1} \binom{k}{r} &= \sum_{k=0}^{m} \binom{k}{r} + \binom{m+1}{r} \\ &= \binom{m+1}{r+1} + \binom{m+1}{r} \\ &= \binom{m+2}{r+1}.\end{aligned}$$

Thus S_{m+1} is true.

Proposition 10.32. *The number of n-digit binary numbers that have no consecutive 1's is the Fibonacci number F_{n+2}.*

Proof. Let A_n be the set of all numbers of n-digit binary numbers that have no consecutive 1's. Denote S_n to be
$$S_n : |A_n| = F_{n+2}.$$

We will prove this using induction.

Basis step. If $n = 0$, there is only one 0-digit binary number with no consecutive ones. Thus $|A_0| = F_2 = 1$ is true. If $n = 1$, there are only two 1-digit binary numbers with no consecutive ones (i.e. 01 and 10). Thus $|A_1| = F_3 = 2$ is true.

Inductive hypothesis. For some $k \geq 1$, suppose S_k and S_{k-1} is true; that is, $A_k = F_{k+2}$ and $A_{k-1} = F_{k+1}$. We wish to prove S_{k+1} is true.

Induction step. Every element of A_{k+1} can either end with 0 or 1. We can establish a one-to-one correspondence between the former and A_k (the elements of A_{k+1} with the ending 0 removed are the corresponding elements in A_k), as well as a one-to-one correspondence between the latter and A_{k-1} (the elements of A_{k+1} with the ending 1 removed are the elements ending with 0 in A_k, which correspond to elements of A_{k-2}). Thus we have the following relation:

$$|A_{k-1}| + |A_k| = |A_{k+1}|.$$

Observe that

$$F_{k+3} = F_{k+1} + F_{k+2} = |A_{k-1}| + |A_k| = |A_{k+1}|.$$

Thus S_{k+1} is true, and we are done.

Proposition 10.33. *Suppose n (infinitely long) straight lines lie on a plane in such a way that no two of the lines are parallel, and no three of the lines intersect at a single point. Then this arrangement divides the plane into $\frac{n^2+n+2}{2}$ regions.*

Proof. Let r_n be the number of regions created by n non-parallel lines, no three of which intersect at a single point. Denote by S_n the statement $S_n : r_n = \frac{n^2+n+2}{2}$. We shall proceed using induction.

Basis step. If $n = 0$, then there is only $\frac{0^2+0+2}{2} = 1$ region, i.e. the plane itself.

Inductive hypothesis. For some $k \geq 0$, suppose S_k is true; that is, $r_k = \frac{k^2+k+2}{2}$. We wish to prove S_{k+1} is true.

Induction step. Note that as a line passes through two regions, it creates an intersection with the regions' common boundary line. Since all lines are non-parallel and there are no three-way intersections, the $(k+1)$-th line must create k intersections (one with each line), and thus pass through $k+1$ regions. As such, the $(k+1)$-th line creates $k+1$ more regions from those that it passes through, and we get the following relation:

$$r_{k+1} = r_k + k + 1.$$

It follows that

$$\begin{aligned} r_{k+1} = r_k + k + 1 &= \frac{k^2+k+2}{2} + k + 1 \\ &= \frac{k^2+3k+4}{2} \\ &= \frac{(k+1)^2+k+1+2}{2}. \end{aligned}$$

Thus S_{k+1} is true, and we are done.

Proposition 10.34.

$$3^1 + 3^2 + 3^3 + 3^4 + \cdots + 3^n = \frac{3^{n+1}-3}{2}$$

for every $n \in \mathbb{N}$.

Proof. Denote by S_n the statement

$$S_n : \sum_{k=1}^{n} 3^k = \frac{3^{n+1}-3}{2}.$$

We shall proceed using induction.

Basis step. If $n = 1$, then the equation holds.

Inductive hypothesis. For some $m \geq 0$, suppose S_m is true. We wish to show that S_{m+1} is true.

Induction step. Observe that

$$\begin{aligned}\sum_{k=1}^{m+1} 3^k &= \sum_{k=1}^{m} 3^k + 3^{m+1} \\ &= \frac{3^{m+1} - 3}{2} + 3^{m+1} \\ &= \frac{3^{m+2} - 3}{2}.\end{aligned}$$

Thus S_{k+1} is true.

Proposition 10.35. *If $n, k \in \mathbb{N}$, and n is even and k is odd, then $\binom{n}{k}$ is even.*

Proof. Denote by S_n the statement S_n : n is even $\implies$ $\binom{n}{k}$ is even for some odd $1 \leq k \leq n$. We shall proceed using induction.

Basis step. If $n = 2$, then k only has one possible value, i.e. 1; so $\binom{2}{1} = 2$, which is even.

Inductive hypothesis. For some even $m \geq 4$, suppose S_m is even. We wish to show S_{m+2} is even.

Induction step. Observe that

$$\begin{aligned}\binom{m+2}{k} &= \binom{m+1}{k-1} + \binom{m+1}{k} \\ &= \binom{m}{k-2} + \binom{m}{k-1} + \binom{m}{k-1} + \binom{m}{k} \\ &= \binom{m}{k-2} + 2\binom{m}{k-1} + \binom{m}{k}.\end{aligned}$$

Note that $k-2$ is odd, and so the first and third term are even, as supposed by the induction hypothesis. The second term is obviously even, so we have the sum of three even numbers is an even number. Thus S_{m+2} is even, as desired.

Proposition 10.36. *If $n = 2^k - 1$ for $k \in \mathbb{N}$, then every entry in Row n of Pascal's Triangle is odd.*

Proof. Note that the r-th entry of the Pascal's Triangle of row $n+1$ is the sum of two entries above to the left and right of the n-th row:

$$\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}.$$

We wish to show that the row $n = 2^k - 1$ consists of only odd entries. To this end, we shall prove the row 2^k has even numbers in every its entry but the first and last. Denote by S_k the statement

$$S_k : \binom{2^k}{r} \text{ is even,}$$

for some $1 \leq r \leq 2^k - 1$. By symmetry, this range can be halved to $1 \leq r \leq 2^{k-1}$. We shall proceed using induction.

Basis step. If $k = 1$, then $\binom{2}{1} = \binom{2}{2} = 2$, which is even.

Inductive hypothesis. For some $m \geq 1$, suppose S_m is true. We wish to show that S_{m+1} is true.

Induction step. By Proposition 3.10.7, we have the following:

$$\binom{2^{m+1}}{r} = \binom{2^m + 2^m}{r} = \sum_{i=0}^{r} \binom{2^m}{r}\binom{2^m}{r-i}.$$

Consider the case when $r = 2^m$. Then the summation term at $i = 0$ and $i = 2^m$ evaluate to 1, which then add up to become 2. Since either i or $r - i$ must fall below 2^m, and $\binom{2^m}{r}$ must be even for $1 \leq r \leq 2^m$ as assumed in the induction, the summation is in fact a summation of even numbers, which is itself even. Similarly, when $r \neq 2^m$, at least one of the coefficients in the summation is even, and so the terms themselves are also even. This shows that S_{m+1} is true, as desired.

Proposition 10.37. *If $m, n \in \mathbb{N}$, then*

$$\sum_{k=0}^{n} k\binom{m+k}{m} = n\binom{m+n+1}{m+1} - \binom{m+n+1}{m+2}.$$

Proof. Denote by S_n the statement

$$S_n : \sum_{k=0}^{n} k\binom{m+k}{m} = n\binom{m+n+1}{m+1} - \binom{m+n+1}{m+2}.$$

We shall proceed using induction.

Basis step. If $n = 1$ then we have the following:

$$\begin{aligned} \sum_{k=0}^{1} k\binom{m+k}{m} &= \binom{m+2}{m+1} - \binom{m+2}{m+2} \\ \binom{m+1}{m} + \binom{m+1}{m+1} &= \binom{m+2}{m+1} \\ \binom{m+2}{m+1} &= \binom{m+2}{m+1}. \end{aligned}$$

Thus S_1 is true.

Inductive hypothesis. For some $p \geq 1$, suppose S_p is true; that is,

$$\sum_{k=0}^{p} k\binom{m+k}{m} = p\binom{m+p+1}{m+1} - \binom{m+p+1}{m+2}.$$

We wish to show that S_{p+1} is true.

Induction step. Observe that

$$\begin{aligned}
\sum_{k=0}^{p+1} k\binom{m+k}{m} &= \sum_{k=0}^{p} k\binom{m+k}{m} + (p+1)\binom{m+p+1}{m} \\
&= p\binom{m+p+1}{m+1} - \binom{m+p+1}{m+2} + (p+1)\binom{m+p+1}{m} \\
&= p\binom{m+p+1}{m+1} + \binom{m+p+1}{m+1} + (p+1)\binom{m+p+1}{m} - \left(\binom{m+p+1}{m+2} + \binom{m+p+1}{m+1}\right) \\
&= (p+1)\binom{m+p+2}{m+1} - \binom{m+p+2}{m+2}.
\end{aligned}$$

Thus S_{p+1} is true, as desired.

Proposition 10.38.

$$\sum_{k=0}^{p} \binom{m}{k}\binom{n}{p-k} = \binom{m+n}{p}.$$

for non-negative integers m, n and p.

Proof. Denote by S_m the statement

$$S_m : \sum_{k=0}^{p} \binom{m}{k}\binom{n}{p-k} = \binom{m+n}{p}.$$

We shall proceed using induction.

Basis step. Consider the case $m = 0$. Note that $\binom{0}{k} = 0$ for all integer k, save for the case $k = 0$ where it equals to 1. Thus

$$\sum_{k=0}^{p} \binom{m}{k}\binom{n}{p-k} = \binom{n}{p} = \binom{0+n}{p}.$$

Consider the case $m = 1$. Note that $\binom{1}{k} = 0$ for all integer k, save for the case $k = 0$ and $k = 1$ where it equals to 1. Thus

$$\sum_{k=0}^{p} \binom{m}{k}\binom{n}{p-k} = \binom{n}{p} + \binom{n}{p-1} = \binom{1+n}{p}.$$

Inductive hypothesis. For some $q \geq 1$, suppose S_q and S_{q-1} is true. We wish to show that S_{q+1} is true.

Induction step. Observe the following:

$$\begin{aligned}
\sum_{k=0}^{p} \binom{q+1}{k}\binom{n}{p-k} &= \sum_{k=0}^{p} \left(\binom{q}{k-1} + \binom{q}{k}\right)\binom{n}{p-k} \\
&= \sum_{k=0}^{p} \binom{q}{k}\binom{n}{p-k} + \sum_{k=0}^{p} \binom{q}{k-1}\binom{n}{p-k}
\end{aligned}$$

$$\begin{aligned}
&= \binom{q+n}{p} + \sum_{k=1}^{p} \binom{q}{k-1}\binom{n}{p-k} \\
&= \binom{q+n}{p} + \sum_{k=0}^{p-1} \binom{q}{k}\binom{n}{p-1-k} \\
&= \binom{q+n}{p} + \binom{q+n}{p-1} \\
&= \binom{q+n+1}{p}.
\end{aligned}$$

Thus S_{q+1} is true, and we are done.

Proposition 10.39.

$$\sum_{k=0}^{m} \binom{m}{k}\binom{n}{p+k} = \binom{m+n}{m+p}.$$

for non-negative integers m, n and p.

Proof. Denote by S_n the statement

$$\sum_{k=0}^{m} \binom{m}{k}\binom{n}{p+k} = \binom{m+n}{m+p}.$$

We will proceed using induction.

Basis step. Consider the case $n = 0$. If $p > 0$, then

$$\sum_{k=0}^{m} \binom{m}{k}\binom{0}{p+k} = 0 = \binom{m}{m+p}.$$

If $p = 0$, then

$$\sum_{k=0}^{m} \binom{m}{k}\binom{0}{k} = 1 = \binom{m}{m}.$$

Inductive hypothesis. For some $q \geq 0$, suppose S_q is true. We wish to show that S_{q+1} is true.

Induction step. Observe that

$$\begin{aligned}
\sum_{k=0}^{m} \binom{m}{k}\binom{q+1}{p+k} &= \sum_{k=0}^{m} \left(\binom{q}{p+k} + \binom{q}{p+k-1}\right)\binom{m}{k} \\
&= \sum_{k=0}^{m} \binom{q}{p+k}\binom{m}{k} + \sum_{k=0}^{m} \binom{q}{p+k-1}\binom{m}{k} \\
&= \binom{m+q}{m+p} + \binom{m+q}{m+p-1} \\
&= \binom{m+q+1}{m+p}.
\end{aligned}$$

Thus S_{q+1} is true, and we are done.

Proposition 10.40. *If $n \in \mathbb{N}$, then*

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}.$$

Proof. Denote by S_n the statement

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.$$

We shall proceed using induction.

Basis step. If $n = 1$, then the equation holds.

Inductive hypothesis. For some $m \geq 1$, suppose S_m is true. We wish to prove S_{m+1} is true.

Induction step. Observe the following:

$$\begin{aligned} \binom{2(n+1)}{n+1} = \binom{n+1+n+1}{n+1} &= \sum_{k=0}^{n+1} \binom{n+1}{k}\binom{n+1}{n+1-k} && \text{(Prop. 10.38)} \\ &= \sum_{k=0}^{n+1} \binom{n+1}{k}^2. \end{aligned}$$

Thus S_{k+1} is true, as desired.

Proposition 10.41. *If n and k are non-negative integers, then*

$$\binom{n+0}{0} + \binom{n+1}{1} + \binom{n+2}{2} + \cdots + \binom{n+k}{k} = \binom{n+k+1}{k}.$$

Proof. Denote by S_k the statement

$$\sum_{i=0}^{k} \binom{n+i}{i} = \binom{n+k+1}{k}.$$

We shall proceed using induction.

Basis step. If $k = 0$, then we have $1 = 1$, which is true.

Inductive hypothesis. For some $m \geq 0$, suppose S_m is true. We wish to show that S_{m+1} is true.

Induction step. Observe that

$$\begin{aligned} \sum_{i=0}^{m+1} \binom{n+i}{i} &= \sum_{i=0}^{m} \binom{n+i}{i} + \binom{n+m+1}{m+1} \\ &= \binom{n+m+1}{m} + \binom{n+m+1}{m+1} \\ &= \binom{n+m+2}{m+1}. \end{aligned}$$

Thus S_{m+1} is true, and we are done.

Proposition 10.42. *The n-th Fibonacci number F_n is even if and only if $3 \mid n$.*

Proof. ($\Longleftarrow$) Since $n = 3a$ for some integer a, we denote by S_n the statement S_a: F_{3a} is even. We shall proceed using induction.

Basis step. If $a = 1$, then $F_3 = 2$ is even.

Inductive hypothesis. For some $b \geq 1$, suppose S_b is true. We wish to show S_{b+1} is true.

Induction step. Observe that:

$$F_{3b+3} = F_{3b+1} + F_{3b+2} = 2F_{3b+1} + F_{3b}.$$

The first term is clearly even, whereas the second is even by the inductive hypothesis. Thus their sum is even itself.

($\Longrightarrow$) Denote by S_n the statement $S_n : F_n$ is even if $3 \mid n$. We shall proceed using induction.

Basis step. If $n = 3$, $F_3 = 2$ is even.

Inductive hypothesis. Suppose for the sake of contradiction that there exists smallest $k > n$ such that F_k is even and $3 \nmid k$. We wish to show that there exists no such k.

Induction step. Since $3 \nmid k$, there exist $a \in \mathbb{Z}$ such that $k = 3a + 1$ or $k = 3a + 2$, and F_k is even. We divide into two cases as follow:

Case 1. Consider the case $k = 3a + 1$. Then $F_{3a+1} = F_{3a-1} + F_{3a}$. The second term is even, and since an even number is the sum of two even numbers, it follows that F_{3a-1} is even. But $3 \nmid (3a - 1)$, and this contradicts the fact that $k = 3a + 1$ is the smallest k that satisfies F_k is even.

Case 2. Consider the case $k = 3a + 2$. Then $F_{3a+2} = F_{3a} + F_{3a+1}$. The fisrt term is even, and since an even number is the sum of two even numbers, it follows that F_{3a+a} is even. But $3 \nmid (3a + 1)$, and this contradicts the fact that $k = 3a + 2$ is the smallest k that satisfies F_k is even.

Thus there exists no such k, and the original statement is true.