# Hammack Exercises - Chapter 9

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## 1 Preface

i dont really have anything to say

## 2 Conjectures

**Conjecture 9.1.** *If* $x, y \in \mathbb{R}$, *then* $|x+y| = |x| + |y|$.

*Disproof.* This conjecture is false due to the following counterexample. Let $x = -69$ and $y = 420$. Then $|x+y| = 351$, whereas $|x| + |y| = 489$. Thus $|x+y| \neq |x| + |y|$.

**Conjecture 9.2.** *For every natural number* $n$, *the integer* $2n^2 - 4n + 31$ *is prime.*

*Disproof.* This conjecture is false due to the following counterexample. Let $n = 31$. Then $2n^2 - 4n + 31 = 1829 = 31 \cdot 59$. Thus $2n^2 - 4n + 31$ is not prime.

**Conjecture 9.3.** *If* $n \in \mathbb{Z}$ *and* $n^5 - n$ *is even, then* $n$ *is even.*

*Disproof.* This conjecture is false due to the following counterexample. Let $n = 1$, which is odd. Then $n^5 - n = 0$, which is even. Thus $n^5 - n$ is even, but $n$ itself is odd.

**Conjecture 9.4.** *For every natural number* $n$, *the integer* $n^2 + 17n + 17$.

*Disproof.* This conjecture is false due to the following counterexample. Let $n = 17$. Then $n^2 + 17n + 17 = 595 = 35 \cdot 17$. Thus $n^2 + 17n + 17$ is not prime.

**Conjecture 9.5.** *If* $A, B, C$ *and* $D$ *are sets, then* $(A \times B) \cup (C \times D) = (A \cup C) \times (B \cup D)$.

*Disproof.* This conjecture is false due to the following counterexample. Let $A = \{1\}, B = \{1,2\}, C = \{3\}, D = \{2,3\}$. Then $(A \times B) \cup (C \times D) = \{(1,1),(1,2),(3,2),(3,3)\}$, whereas $(A \cup C) \times (B \cup D) = \{(1,1),(1,2),(1,3),(3,1),(3,2),(3,3)\}$. Thus $(A \times B) \cup (C \times D) \neq (A \cup C) \times (B \cup D)$.

**Conjecture 9.6.** *If* $A, B, C$ *and* $D$ *are sets, then* $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$.

This conjecture was proven earlier in the form of Lemma 8.1.

**Conjecture 9.7.** *If* $A, B$ *and* $C$ *are sets, and* $A \times C = B \times C$, *then* $A = B$.

*Disproof.* This conjecture is false due to the following counter example. Let $A = \{1\}$, $B = \{2\}$ and $C = \varnothing$. Then $A \neq B$, but $A \times C = B \times C = \varnothing$. Thus $A \times C = B \times C$ does not imply $A = B$.

**Conjecture 9.8.** *If $A, B$ and $C$ are sets, then $A - (B \cup C) = (A - B) \cup (A - C)$.*

*Disproof.* This conjecture is false due to the following counterexample. Let $A = \{1\}$, $B = \{1, 2\}$, $C = \varnothing$. Note that $A - (B \cup C) = \varnothing$, whereas $(A - B) \cup (A - C) = \{1\}$. Thus $A - (B \cup C) \neq (A - B) \cup (A - C)$.

**Conjecture 9.9.** *If $A$ and $B$ are sets, then $\mathscr{P}(A) - \mathscr{P}(B) \subseteq \mathscr{P}(A - B)$.*

*Disproof.* This conjecture is false due to the following counterexample. Let $A = \{1, 2\}$ and $B = \{2\}$. Note that $\mathscr{P}(A) - \mathscr{P}(B) = \{\{1\}, \{1, 2\}\}$, whereas $\mathscr{P}(A - B) = \{\varnothing, \{1\}\}$. Thus $\mathscr{P}(A) - \mathscr{P}(B) \nsubseteq \mathscr{P}(A - B)$.

**Conjecture 9.10.** *If $A$ and $B$ are sets and $A \cap B = \varnothing$, then $\mathscr{P}(A) - \mathscr{P}(B) \subseteq \mathscr{P}(A - B)$.*

This conjecture is true. To prove it, we shall suppose $A \cap B = \varnothing$ for arbitrary sets $A$ and $B$, and show that $\mathscr{P}(A) - \mathscr{P}(B) \subseteq \mathscr{P}(A - B)$. A proof follows.

*Proof.* Suppose $A \cap B = \varnothing$; then there does not exist $x$ such that $x \in A \wedge x \in B$. We have the following:

$$A - B = \{x : x \in A \wedge x \notin B\} = A.$$

Thus $\mathscr{P}(A) - \mathscr{P}(B) \subseteq \mathscr{P}(A) = \mathscr{P}(A - B)$.

**Conjecture 9.11.** *If $a, b \in \mathbb{N}$, then $a + b < ab$.*

*Disproof.* This conjecture is false due to the following counterexample. Let $a = b = 1$. Note that $a + b = 2$, whereas $ab = 1$. Thus $a + b > ab$.

**Conjecture 9.12.** *If $a, b, c \in \mathbb{N}$ and $ab$, $bc$ and $ac$ all have the same parity, then $a, b$ and $c$ all have the same parity.*

*Disproof.* This conjecture is false due to the following counterexample. Let $a = 2$, $b = 3$ and $c = 4$. Note that $ab = 6$, $bc = 12$ and $ac = 8$; and so $ab$, $bc$ and $ac$ have the same parity while $a$, $b$ and $c$ themselves do not.

**Conjecture 9.13.** *There exists a set $X$ for which $\mathbb{R} \subseteq X$ and $\varnothing \in X$.*

This conjecture is true. We shall provide an example set $X$ for which $\mathbb{R} \subseteq X$ and $\varnothing \in X$. A proof follows.

*Proof.* Consider $X = \mathbb{R} \cup \{\varnothing\}$. Note that $R \subseteq X$ and $\varnothing \in X$. We have found a set $X$ that satisfies such properties.

**Conjecture 9.14.** *If $A$ and $B$ are sets, then $\mathscr{P}(A) \cap \mathscr{P}(B) = \mathscr{P}(A \cap B)$.*

This conjecture is true. To prove this, we shall show that $\mathscr{P}(A) \cap \mathscr{P}(B) \subseteq \mathscr{P}(A \cap B)$ and vice versa. A proof follows.

*Proof.* ($\subseteq$) Suppose $X \in \mathscr{P}(A) \cap \mathscr{P}(B)$. Then $X \subseteq A$ and $X \subseteq B$. Let $x \in X$; since $x \in A$ and $x \in B$, we have $x \in A \cap B$. This implies $X \subseteq A \cap B$, and so $X \in \mathscr{P}(A \cap B)$.

($\supseteq$) Suppose $X \in \mathscr{P}(A \cap B)$. Then $X \subseteq A \cap B$. Let $x \in X$; since $x \in A \cap B$, we have $x \in A$ and $x \in B$. This implies $X \subseteq A$ and $X \subseteq B$, and so $X \in \mathscr{P}(A) \cap \mathscr{P}(B)$. This completes our proof.

**Conjecture 9.15.** *Every odd integer is the sum of three odd integers.*

This conjecture is true. A proof follows.

*Proof.* Let $a, b, c \in \mathbb{Z}$ be odd. Then there exist $x, y, z \in \mathbb{Z}$ such that $a = 2x+1$, $b = 2y+1$, $c = 2z+1$. Thus we have the following:

$$a + b + c = 2x + 2y + 2z + 3 = 2(x + y + z + 1) + 1.$$

This completes our proof.

**Conjecture 9.16.** *If $A$ and $B$ are finite sets, then $|A \cup B| = |A| + |B|$.*

*Disproof.* This conjecture is false due to the following counterexample. Let $A = \{1\}$ and $B = \{1, 2\}$. Note that $|A \cup B| = 2$, whereas $|A| + |B| = 3$. Thus $|A \cup B| \neq |A| + |B|$.

**Conjecture 9.17.** *For all sets $A$ and $B$, if $A - B = \varnothing$, then $B \neq \varnothing$.*

*Disproof.* This conjecture is false due to the following counterexample. Let $A = B = \varnothing$. Note that $A - B = \varnothing$. Thus $A - B = \varnothing$ does not imply $B \neq \varnothing$.

**Conjecture 9.18.** *If $a, b, c \in \mathbb{N}$, then at least one of $a - b$, $a + c$ and $b - c$ is even*

This conjecture is true. To prove it, we suppose the conjecture is false, and find a contradiction that results from the assumption. A proof follows.

*Proof.* Suppose to the contrary that $a - b$, $a + c$ and $b - c$ are all odd for natural $a, b, c$. Note that the sum of three odd integers are odd, but $a - b + a + c + b - c = 2a$, which is even. Thus we have a contradiction.

**Conjecture 9.19.** *For every $r, s \in \mathbb{Q}$ with $r < s$, there is an irrational number $u$ for which $r < u < s$.*

This conjecture is true. To show why, we shall provide an irrational number $u$ such that $r < u < s$. A proof follows.

*Proof.* Let $r, s \in \mathbb{Q}$ such that $r < s$. Observe the following:

$$0 < \frac{\sqrt{2}}{2} < 1 \iff 0 < \frac{(s-r)\sqrt{2}}{2} < s - r \iff r < \frac{(s-r)\sqrt{2}}{2} + r < s.$$

Note that $\frac{\sqrt{2}}{2}$ is irrational (otherwise, its product with the rational number 2 would be rational, but $\sqrt{2}$ is irrational). Also note that $\frac{(s-r)\sqrt{2}}{2} + r$ is irrational (suppose to the contrary that it is rational; then it can be rewritten as $\frac{p}{q}$ for some integer $p, q$. Then we have $\frac{p-qr}{q(s-r)} = \frac{\sqrt{2}}{2}$. The left hand side is rational, while the right hand side is irrational, thus a contradiction). Therefore, there exists irrational $u = \frac{(s-r)\sqrt{2}}{2} + r$ such that $r < u < s$, and we are done.

**Conjecture 9.20.** *There exist prime numbers $p$ and $q$ for which $p - q = 1000$.*

*Proof.* This conjecture is true due to the following example. Consider $p = 1013$ and $q = 13$. Note that both $p$ and $q$ are prime, and $p - q = 1000$.

**Conjecture 9.21.** *There exist prime numbers $p$ and $q$ for which $p - q = 97$.*

*Disproof.* This conjecture is false. To see why, suppose to the contrary that there exist prime numbers $p$ and $q$ such that $p - q = 97$. Note that a sum of two integers is odd if and only if the two integers themselves have opposite parity. As such, either $p$ or $q$ is even. Thus $q = 2$ implies $p = 99$, which contradicts the fact that $p$ is prime. If $q \neq 2$, then it is also a contradiction as all even numbers above 2 are composite.

**Conjecture 9.22.** *If $p$ and $q$ are prime numbers for which $p < q$, then $2p + q^2$ is odd.*

*Proof.* This conjecture is true. To prove this, note that there is no even prime below 2, which is the only even prime. We divide into the following two cases:

**Case 1.** If $p$ and $q$ have opposite parity, then $p = 2$ and $q$ is odd. Observe that $2p$ is even, whereas $q^2$ is odd. As such, their sum is odd.

**Case 2.** If $p$ and $q$ have opposite parity, then $p$ and $q$ are both odd. Observe that $2p$ is even, whereas $q^2$ is odd. As such, their sum is odd.

This completes our proof.

**Conjecture 9.23.** *If $x, y \in \mathbb{R}$ and $x^3 < y^3$, then $x < y$.*

*Proof.* This conjecture is true. To see why, suppose that $x^3 < y^3$; thus $x^3 - y^3 < 0$. Observe the following:

$$\begin{aligned} x^3 - y^3 = (x-y)(x^2 + xy + y^2) &= (x-y)\left(\frac{3}{4}x^2 + \left(\frac{1}{4}x^2 + 2 \cdot \frac{1}{2}xy + y^2\right)\right) \\ &= (x-y)\left(\frac{3}{4}x^2 + \left(\frac{1}{2}x + y\right)^2\right). \end{aligned}$$

For $x^3 - y^3 < 0$, it must be that $x - y < 0$, since the second term in the product is always positive. Thus $x < y$, and we are done.

**Conjecture 9.24.** *The inequality $2^x \geq x + 1$ is true for all positive real number $x$.*

*Disproof.* This conjecture is false due to the following counterexample. Let $x = \frac{1}{2}$. Note that $2^{\frac{1}{2}} < 1 + \frac{1}{2}$, thus the inequality is false.

**Conjecture 9.25.** *For all $a, b, c \in \mathbb{Z}$, if $a \mid bc$, then $a \mid b$ or $a \mid c$.*

*Disproof.* This conjecture is false due to the following counterexample. Let $a = 4$, $b = 2$ and $c = 6$. Note that $a \mid bc$, but $a \nmid b$ and $a \nmid c$. Thus $a \mid bc$ does not imply $a \mid b$ or $a \mid c$ for all $a, b, c \in \mathbb{Z}$.

**Conjecture 9.26.** *Suppose $A$, $B$ and $C$ are sets. If $A = B - C$, then $B = A \cup C$.*

*Disproof.* This conjecture is false due to the following counterexample. Let $A = \varnothing$, $B = \{3, 5\}$ and $C = \{3, 5, 2\}$. Note that $A = B - C$, but $B \neq A \cup C$. Thus $A = B - C$ does not imply $B = A \cup C$.

**Conjecture 9.27.** *The equation $x^2 = 2^x$ has three real solutions.*

*Proof.* This conjecture is true. The function whose three roots we wish to find is $f(x) = x^2 - 2^x$. To this end, observe that the function has two trivial roots, i.e. 2 and 4. Consider $f(-1) = \frac{1}{2}$ and $f(1) = -1$. By the intermediate value theorem, there exists $-1 \leq m \leq 1$ such that $f(m) = 0$. Thus $m$ is the third root we need to find.

**Conjecture 9.28.** *Suppose $a, b \in \mathbb{Z}$. If $a \mid b$ and $b \mid a$, then $a = b$.*

*Disproof.* This conjecture is false due to the following counterexample. Let $a = 3$ and $b = -3$. Note that $a \mid b$ and $b \mid a$, but $a \neq b$. Thus $a \mid b$ and $b \mid a$ does not imply $a = b$.

**Conjecture 9.29.** *If $x, y \in \mathbb{R}$ and $|x + y| = |x - y|$, then $y = 0$.*

*Disproof.* This conjecture is false due to the following counterexample. Let $x = 0$ and $y = 1$. Note that $|x + y| = |x - y| = 1$. Thus $|x + y| = |x - y|$ for real $x, y$ does not imply $y = 0$.

**Conjecture 9.30.** *There exist integers $a$ and $b$ for which $42a + 7b = 1$.*

*Disproof.* This conjecture is false. To show why, suppose to the contrary that this conjecture is true. Then there exist integers $a$ and $b$ such that $42a + 7b = 1$. Observe that $42a + 7b = 1$ implies $7(6a + b) = 1$. Thus it is a contradiction that $7 \mid 1$.

**Conjecture 9.31.** *No number (other than 1) appears in Pascal's triangle more than four times.*

*Disproof.* This conjecture is false due to the following counterexample. Consider the number 120. Observe that $120 = \binom{10}{3} = \binom{10}{7} = \binom{16}{2} = \binom{16}{14} = \binom{120}{1} = \binom{120}{119}$. Thus the number 120 appears in the Pascal's triangle six times.

**Conjecture 9.32.** *If $n, k \in \mathbb{N}$ and $\binom{n}{k}$ is a prime number, then $k = 1$ or $k = n - 1$.*

*Proof.* This conjecture is true. To see why, suppose to the contrary that there exists $2 \leq k \leq n - 1$ such that $\binom{n}{k}$ is prime. Let $p = \binom{n}{k}$, we have the following:

$$p = \frac{n!}{k!(n-k)!} \iff p \cdot k! = \frac{n!}{(n-k)!} = n(n-1)(n-2)\cdots(n-k+1).$$

Thus $p$ must divide one of the terms in the product since $p$ is prime. Therefore $p \leq n$. On the other hand, observe that for $0 \leq r < \frac{n}{2}$, we have:

$$\binom{n}{r+1} = \frac{n-r}{r+1}\binom{n}{r} \geq \binom{n}{r}$$

At $r = 0$, we have $\binom{n}{r+1} = n$. And so for arbitrary $k$, we have $p > n$ (the inequality is strict at $r = 0$). Therefore it is a contradiction that $p \leq n$ and $p > n$.

**Conjecture 9.33.** *Suppose $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ is a polynomial of degree 1 or greater, and for which each coefficient $a_i$ is in $\mathbb{N}$. Then there is a $k \in \mathbb{N}$ for which the integer $f(k)$ is not prime.*

*Proof.* This conjecture is true. To show why, let $p = f(1) = a_0 + a_1 + \cdots + a_n$. If $p$ is composite, we are done. If $p$ is prime, let there be $m \in \mathbb{N}$. Consider the following:

$$\begin{aligned}
f(mp+1) &= a_0 + a_1(mp+1) + a_2(mp+1)^2 + \cdots + a_n(mp+1)^n \\
&= a_0 + a_1 \sum_{j=0}^{1} \binom{1}{j}(m^j p^j) + a_2 \sum_{j=0}^{2} \binom{2}{j}(m^j p^j) + \cdots + a_n \sum_{j=0}^{n} \binom{n}{j}(m^j p^j) \\
&= a_0 + a_1 + a_2 + \cdots + a_n + a_1 \sum_{j=1}^{1} \binom{1}{j}(m^j p^j) + a_2 \sum_{j=1}^{2} \binom{2}{j}(m^j p^j) + \cdots + a_n \sum_{j=1}^{n} \binom{n}{j}(m^j p^j) \\
&= p + a_1 mp + a_2 \sum_{j=1}^{2} \binom{2}{j}(m^j p^j) + \cdots + a_n \sum_{j=1}^{n} \binom{n}{j}(m^j p^j) \\
&= p + mp(a_1 + a_2 \sum_{j=2}^{2} \binom{2}{j}(m^j p^j) + \cdots + a^n \sum_{j=2}^{n} \binom{n}{j}(m^j p^j)) \\
&= p(1 + m(a_1 + a_2 \sum_{j=2}^{2} \binom{2}{j}(m^j p^j) + \cdots + a^n \sum_{j=2}^{n} \binom{n}{j}(m^j p^j))).
\end{aligned}$$

Note that neither $p$ nor the term after it is equal to 1. Thus for $k = mp + 1$, $f(k)$ is composite. This completes our proof.

**Conjecture 9.34.** *If $X \subseteq A \cup B$, then $X \subseteq A$ or $X \subseteq B$.*

*Disproof.* This conjecture is false due to the following counterexample. Let $A = \{1, 2, 5\}$, $B = \{2, 4, 5\}$ and $X = \{1, 2, 4\}$. Note that $X \subseteq A \cup B$, but $X \nsubseteq A$ and $X \nsubseteq B$. Thus $X \subseteq A \cup B$ does not imply $X \subseteq A$ or $X \subseteq B$.

**Conjecture 9.35.** *If $n$ is prime, then $2^n - 1$ is prime.*

*Disproof.* This conjecture is false due to the following counterexample. Let $n = 11$, then $2^n - 1 = 2047 = 23 \cdot 87$. Thus $2^n - 1$ is not prime.