

Funkcie a funkcionály

referečná transparentnosť

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I-18

<http://dai.fmph.uniba.sk/courses/FPRO/>



Zoznamová rekurzia

```
-- vyber prvých n prvkov zo zoznamu
take      :: Int -> [a] -> [a]
take 0 _  = []
take _ [] = []
take n (x:xs) = x : take (n-1) xs
```

```
-- dĺžka zoznamu
```

```
length      :: [a] -> Int
length []   = 0
length (x:xs) = 1 + length xs
```

Hypotéza (pre ľubovoľné n a xs):

- $\text{length (take } n \text{ xs)} = n$
- $\text{length } \$ \text{ take } n \text{ xs} = n$
- $(\text{length} . \text{take } n) \text{ xs} = n$

```
"?: " take 5 [1,3..100]
```

```
[1,3,5,7,9]
```

```
"?: " length (take 5 [1,3..100])
```

```
5
```

```
"?: " length $ take 5 [1,3..100]
```

```
5
```



Dôkaz - $\text{length} (\text{take } n \text{ xs}) = n$

Indukcia (vzhľadom na dĺžku xs):

- $xs = []$

$\text{length} (\text{take } n []) = 0$

$0 = 0$

č.b.t.d.

- $xs = (y:ys)$

$\text{length} (\text{take } n (y:ys)) = n$

$\text{length} (y:\text{take } (n-1) ys) = n$

$1 + \text{length} (\text{take } (n-1) ys) = n$

indukčný predpoklad, $|ys| < |xs|$

$1 + (n-1) = n$

č.b.t.d.



QuickCheck

Elegantný nástroj na testovanie (!!! nie dôkaz !!!) hypotéz

```
"?: " import Test.QuickCheck
```

```
"?: " quickCheck ((\xs,n) -> (length (take n xs)) == n))
```

```
*** Failed! Falsifiable (after 2 tests and 1 shrink):
```

```
"?: " verboseCheck ((\xs,n) -> (length (take n xs)) == n))
```

```
Passed:
```

```
([],0)
```

```
Passed:
```

```
([()],1)
```

```
Failed:
```

```
([],-1)
```

```
*** Failed! Failed:
```

Neplatí to pre n záporne, lebo napr. `take (-3) [1..100] = []`,
resp. naša definícia nepokrýva prípad $n < 0$





QuickCheck

Podmienka: miesto písania

if **n** >= **0** then (length (take n s)) == n else True

"?: " verboseCheck ((\xs,n) -> **n>=0 ==>** (length (take n xs)) == n))

Passed:

([],0)

Failed:

([()],2)

Neplatí to pre ak length xs < n ☹️

"?: " quickCheck ((\xs,n) -> n>=0 && length xs == n ==>

(length (take n xs)) == n))

*** Gave up! Passed only 35 tests.



Tvrdenie sme **overili** na niekoľkých prípadoch, ale to **nie je dôkaz**.

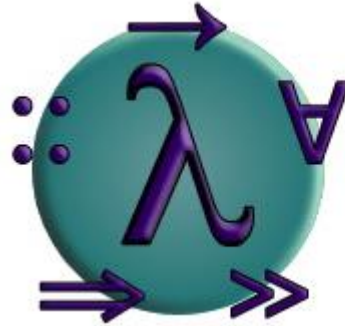
V dôkaze môžeme urobiť chybu (ako na slajde 2), QuickCheck slúži ako nástroj na hľadanie/odhaľovanie kontrapríkladov, kedy naše tvrdenie neplatí.

Kvíz - platí/neplatí ?

(neseriózný prístup ale intuíciu treba tiež trénovať)

- `length [m..n] == n-m+1` 😞
"?: " `quickCheck ((\ (n,m) -> length [m..n] == n-m+1))`
*** Failed! Falsifiable (after 3 tests and 1 shrink):
"?: " `quickCheck ((\ (n,m) -> m <= n ==> length [m..n] == n-m+1))` 😊
+++ OK, passed 100 tests.
- `length (xs ++ ys) == length xs + length ys` 😊
"?: " `quickCheck((\xs->\ys->(length (xs++ys)==length xs + length ys)))`
+++ OK, passed 100 tests.
- `length (reverse xs) == length xs` 😊
`quickCheck((\xs -> (length (reverse xs) == length xs)))`
+++ OK, passed 100 tests.
- `(xs, ys) == unzip (zip xs ys)` 😞
`quickCheck((\xs -> \ys -> ((xs, ys) == unzip (zip xs ys))))`
*** Failed! Falsifiable (after 3 tests and 1 shrink):
`quickCheck((\xs -> \ys -> (length xs == length ys ==>`
`(xs, ys) == unzip (zip xs ys))))` 😊

Funkcia (predikát) argumentom



Haskell

A Purely Functional Language
featuring static typing, higher-order functions,
polymorphism, type classes and monadic effects

- zober zo zoznamu tie prvky, ktoré spĺňajú podmienku (test)
Booleovská podmienka príde ako argument funkcie a má typ (a -> Bool):

filter :: (a -> Bool) -> [a] -> [a]

filter p xs = [x | x <- xs, p x]

**> filter even [1..10]
[2,4,6,8,10]**

alternatívna definícia:

filter p [] = []

filter p (x:xs) = if p x then x:(filter p xs) else filter p xs

vlastnosti (zväčša úplne zrejmé):

- filter True xs = xs ... [x | x <- xs, True] = [x | x <- xs] = xs
- filter False xs = [] ... [x | x <- xs, False] = []
- filter p1 (filter p2 xs) = filter (p1 && p2) xs
- (filter p1 xs) ++ (filter p2 xs) = filter (p1 || p2) xs

$$\begin{aligned}\text{filter } p \ [] &= [] \\ \text{filter } p \ (x:xs) &= \text{if } p \ x \text{ then } x:(\text{filter } p \ xs) \text{ else } \text{filter } p \ xs\end{aligned}$$

Dôkaz

$\text{filter } p1 \ (\text{filter } p2 \ xs) = \text{filter } (p1 \ \&\& \ p2) \ xs$

Indukcia vzhľadom na parameter xs

- []
L.S. = $\text{filter } p1 \ (\text{filter } p2 \ []) = \text{filter } p1 \ [] = [] = \text{filter } (p1 \ \&\& \ p2) \ [] = \text{P.S.}$
- (x:xs)
L.S. = $\text{filter } p1 \ (\text{filter } p2 \ (x:xs)) = \dots \text{definícia}$
 $\text{filter } p1 \ (\text{if } p2 \ x \text{ then } x:(\text{filter } p2 \ xs) \text{ else } \text{filter } p2 \ xs) = \dots \text{if-then-else}$
 $\text{if } p2 \ x \text{ then } \text{filter } p1 \ (x:(\text{filter } p2 \ xs)) \text{ else } \text{filter } p1 \ (\text{filter } p2 \ xs) = \dots \text{indukcia}$
 $\text{if } p2 \ x \text{ then } \text{filter } p1 \ (x:(\text{filter } p2 \ xs)) \text{ else } \text{filter } (p1 \ \&\& \ p2) \ xs = \dots \text{definícia}$
 $\text{if } p2 \ x \text{ then}$
 - $\text{if } p1 \ x \text{ then } x:(\text{filter } p1 \ (\text{filter } p2 \ xs)) \text{ else } \text{filter } p1 \ (\text{filter } p2 \ xs)$ $\text{else } \text{filter } (p1 \ \&\& \ p2) \ xs = \dots \text{2 x indukcia}$
 $\text{if } p2 \ x \text{ then}$
 - $\text{if } p1 \ x \text{ then } x:(\text{filter } (p1 \ \&\& \ p2) \ xs) \text{ else } \text{filter } (p1 \ \&\& \ p2) \ xs$ $\text{else } \text{filter } (p1 \ \&\& \ p2) \ xs =$

$\text{filter } p [] = []$
 $\text{filter } p (x:xs) = \text{if } p \ x \text{ then } x:(\text{filter } p \ xs) \text{ else } \text{filter } p \ xs$

Dôkaz

$\text{filter } p1 (\text{filter } p2 \ xs) = \text{filter } (p1 \ \&\& \ p2) \ xs$

if $p2 \ x$ then

if $p1 \ x$ then $x:(\text{filter } (p1 \ \&\& \ p2) \ xs)$ else $\text{filter } (p1 \ \&\& \ p2) \ xs$

else $\text{filter } (p1 \ \&\& \ p2) \ xs = \dots$ **požívame vlastnosť if-then-else**

if A then

if $A \ \&\& \ B$ then C

if B then C

else D

else D

else D

if $(p1 \ \&\& \ p2) \ x$ then $x:(\text{filter } (p1 \ \&\& \ p2) \ xs)$ else $\text{filter } (p1 \ \&\& \ p2) \ xs = \dots$ **def.**

$\text{filter } (p1 \ \&\& \ p2) \ (x:xs) = \text{P.S.}$



QuickCheck a funkcie

Funkcie sú hodnoty ako každé iné
Ako vie QuickCheck pracovať s funkciami ?

- je skladanie funkcií komutatívne ?

```
"?: " import Text.Show.Functions
```



```
"?: " quickCheck(
```

```
  (\x -> \f -> \g -> (f.g) x == (g.f) x)::Int->(Int->Int)->(Int->Int)->Bool)
```

```
*** Failed! Falsifiable (after 2 tests):
```

- je skladanie funkcií asociatívne ?

```
"?: " quickCheck(
```

```
  (\x -> \f -> \g -> \h -> (f.(g.h)) x == ((f.g).h) x)
```

```
  ::Int->(Int->Int)->(Int->Int)->(Int->Int)->Bool)
```



```
+++ OK, passed 100 tests.
```

Opäť to NIE je DÔKAZ, len 100 pokusov.

Funkcia argumentom

map

- funktor, ktorý aplikuje funkciu (1.argument) na všetky prvky zoznamu

`map` $:: (a \rightarrow b) \rightarrow [a] \rightarrow [b]$

`map f []` $= []$

`map f (x:xs)` $= f\ x : \text{map } f\ xs$

`map f xs` $= [f\ x \mid x \leftarrow xs]$

- Príklady:

`map (+1) [1,2,3,4,5]` $= [2,3,4,5,6]$

`map odd [1,2,3,4,5]` $= [\text{True}, \text{False}, \text{True}, \text{False}, \text{True}]$

`and (map odd [1,2,3,4,5])` $= \text{False}$

`map head [[1,0,0], [2,1,0], [3,0,1]]` $= [1, 2, 3]$

`map tail [[1,0,0], [2,1,0], [3,0,1]]` $= [[0,0], [1,0], [0,1]]$

`map (0:) [[1],[2],[3]]` $= [[0,1],[0,2],[0,3]]$



Vlastnosti map

- $\text{map id } xs = xs$ ☒ $\text{map id} = \text{id}$
- $\text{map } (f.g) \text{ } xs = \text{map } f (\text{map } g \text{ } xs)$ ☒ $\text{map } f . \text{map } g = \text{map } (f.g)$
- ~~$\text{head } (\text{map } f \text{ } xs) = f (\text{head } xs)$~~ ☒ ~~$\text{head} . \text{map } f = f . \text{head}$~~
- ~~$\text{tail } (\text{map } f \text{ } xs) = \text{map } f (\text{tail } xs)$~~ ☒ ~~$\text{tail} . \text{map } f = \text{map } f . \text{tail}$~~
- $\text{map } f (xs ++ ys) = \text{map } f \text{ } xs ++ \text{map } f \text{ } ys$ ☒
- $\text{length } (\text{map } f \text{ } xs) = \text{length } xs$ ☒ $\text{length} . \text{map } f = \text{length}$
- $\text{map } f (\text{reverse } xs) = \text{reverse } (\text{map } f \text{ } xs)$ ☒ $\text{map } f . \text{reverse} = \text{reverse} . \text{map } f$
- ~~$\text{sort } (\text{map } f \text{ } xs) = \text{map } f (\text{sort } xs)$~~ ☒ ~~$\text{sort} . \text{map } f = \text{map } f . \text{sort}$~~
- $\text{map } f (\text{concat } xss) = \text{concat } (\text{map } (\text{map } f) \text{ } xss)$ ☒

$\text{map } f . \text{concat} = \text{concat} . \text{map } (\text{map } f)$

$\text{concat} :: [[a]] \rightarrow [a]$

$\text{concat } [] = []$



$\text{concat } (xs:xss) = xs ++ \text{concat } xss$

$\text{concat } [[1], [2,3], [4,5,6], []] = [1,2,3,4,5,6]$



Vlastnosti map, filter

Na zamyslenie:

- `filter p (map f xs)` = `???` `(filter (p.f) xs)` 
- `filter p (map f xs)` = `map f (filter (p.f) xs)` 
- `filter p . map f` = `map f . filter (p.f)`

`filter p (map f xs)`

= `filter p [f x | x<-xs]`

= `[y | y <- [f x | x<-xs], p y]`

= `[f x | x<-xs, p (f x)]`

= `map f [x | x<-xs, p (f x)]`

= `map f (filter (p.f))`



Quíz - prémia

nájdite pravdivé a zdôvodnite

- $\text{map } f . \text{take } n = \text{take } n . \text{map } f$
- $\text{map } f . \text{reverse} = \text{reverse} . \text{map } f$
- $\text{map } f . \text{filter } p = \text{map } \text{fst} . \text{filter } \text{snd} . \text{map } (\text{fork } (f,p))$
where $\text{fork} :: (a \rightarrow b, a \rightarrow c) \rightarrow a \rightarrow (b,c)$
 $\text{fork } (f,g) x = (f x, g x)$
- $\text{filter } (p . g) = \text{map } (\text{inverzna_g}) . \text{filter } p . \text{map } g$ ak $\text{inverzna_g} . g = \text{id}$.
- $\text{reverse} . \text{concat} = \text{concat} . \text{reverse} . \text{map } \text{reverse}$
- $\text{filter } p . \text{concat} = \text{concat} . \text{map } (\text{filter } p)$

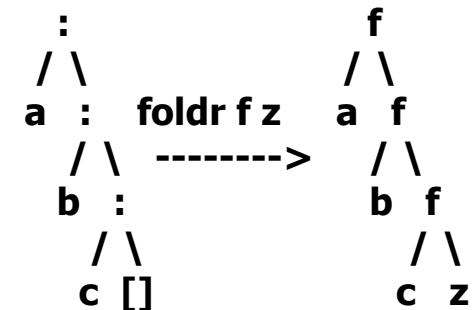
Haskell – foldr

`foldr :: (a -> b -> b) -> b -> [a] -> b`

`foldr f z [] = z`

`foldr f z (x:xs) = f x (foldr f z xs)`

`a : b : c : [] -> f a (f b (f c z))`



```
Main> foldr (+) 0 [1..100]
```

```
5050
```

```
Main> foldr (\x y->10*y+x) 0 [1,2,3,4]
```

```
4321
```

-- g je vnorená lokálna funkcia

`foldr :: (a -> b -> b) -> b -> [a] -> b`

`foldr f z = g`

where `g [] = z`

`g (x:xs) = f x (g xs)`



Haskell – foldl

`foldl` :: (a -> b -> a) -> a -> [b] -> a

`foldl f z []` = `z`

`foldl f z (x:xs)` = `foldl f (f z x) xs`

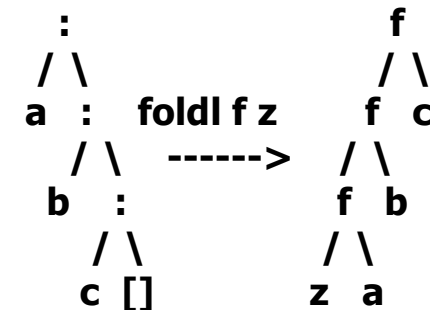
`a : b : c : [] -> f (f (f z a) b) c`

`Main> foldl (+) 0 [1..100]`

`5050`

`Main> foldl (\x y->10*x+y) 0 [1,2,3,4]`

`1234`





Vypočítajte

- `foldr max (-999) [1,2,3,4]`
`foldl max (-999) [1,2,3,4]`
- `foldr (_ -> \y ->(y+1)) 0 [3,2,1,2,4]`
`foldl (\x -> _ ->(x+1)) 0 [3,2,1,2,4]`

- `foldr (-) 0 [1..100] =`

$$(1-(2-(3-(4-\dots-(100-0)))))) = 1-2 + 3-4 + 5-6 + \dots + (99-100) = -50$$

- `foldl (-) 0 [1..100] =`

$$(\dots(((0-1)-2)-3) \dots - 100) = -5050$$



Funkcia je hodnotou

- $[a \rightarrow a]$ je zoznam funkcií typu $a \rightarrow a$
napríklad: $[(+1), (+2), (*3)]$ je $[\backslash x \rightarrow x+1, \backslash x \rightarrow x+2, \backslash x \rightarrow x*3]$

- čo je foldr (.) id $[(+1), (+2), (*3)]$??

akého je typu

foldr (.) id $[(+1), (+2), (*3)]$ 100

foldl (.) id $[(+1), (+2), (*3)]$ 100

$[a \rightarrow a]$

303

???

lebo skladanie fcií je asociatívne:

- $((f \cdot g) \cdot h) x = (f \cdot g) (h x) = f (g (h x)) = f ((g \cdot h) x) = (f \cdot (g \cdot h)) x$
- funkcie nevieme porovnávať, napr. $\text{head } [(+1), (+2), (*3)] = \text{id}$
- funkcie vieme permutovať, length \$ permutations $[(+1), (+2), (*3), (^2)]$



Maximálna permutácia funkcií

- zoznam funkcií aplikujeme na zoznam argumentov

```
apply      :: [a -> b] -> [a] -> [b]  
apply fs args = [ f a | f <- fs, a <- args]
```

```
apply [(+1),(+2),(*3)] [100, 200]  
[101,201,102,202,300,600]
```

- čo počíta tento výraz

```
maximum $  
  apply  
    (map (foldr (.) id) (permutations [(+1),(^2),(*3),(+2),(/3)]))  
    [100]
```

31827

- $(+1).(+2).(*3).(^2).(/3) 100$

3336.333333333334

- $((/3).(^2).(*3).(+2).(+1)) 100$

31827.0

Kvíz

`foldr (:) [] xs = xs`

`foldr (:) ys xs = xs++ys`

`foldr ? ? xs = reverse xs`

<http://foldl.com/>



Pre tých, čo zvládli kvíz, odmena !

kliknite si podľa vašej politickej orientácie

<http://foldr.com/>





Vlastnosti

Fussion Law:

Ak platí

$$f \ z1 = z2 \ \&\& \ f \ (g1 \ a \ b) = g2 \ a \ (f \ b)$$

potom platí

$$f \ . \ foldr \ g1 \ z1 = foldr \ g2 \ z2$$

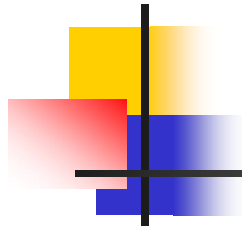
Príklad použitia Fussion Law:

$$(n^*). \ foldr \ (+) \ 0 = foldr \ (+). (n^*) \ 0$$

Dôkaz (pomocou Fussion Law): overíme predpoklady

$$f = (n^*), \ z1 = z2 = 0, \ g1 = (+), \ g1 = g2 = (+). \ (n^*)$$

- $(n^*) \ 0 = 0$
- $(n^*) \ (a+b) = (n^*a + n^*b) = (+). (n^*) \ a \ ((n^*) \ b)$



Vlastnosti

Acid Rain (fold/build/deforestation theorem)

$$\text{foldr } f \ z \ . \ g \ (\ :) \ [] = g \ f \ z$$

Intuícia: Keď máme vytvoriť zoznam pomocou funkcie g zo zoznamových konštruktorov $(\ :) \ []$, na ktorý následne pustíme foldr , ktorý nahradí $(\ :)$ za f a $[]$ za z , namiesto toho môžeme konštruovať priamo výsledný zoznam pomocou $g \ f \ z$.

Otypujme si to:

Ak $z :: u$, potom $f :: x \rightarrow u \rightarrow u$, $\text{foldr } f \ z :: [x] \rightarrow u$

Ľavá strana: $([x] \rightarrow u) \cdot (t \rightarrow [x])$ výsledkom je typ $t \rightarrow u$

Pravá strana: $g :: (x \rightarrow u \rightarrow u) \rightarrow u \rightarrow (t \rightarrow u)$



`length . map _ = length`

`foldr f z . g (:) [] = g f z`

`map :: (a -> b) -> [a] -> [b]`

`map h = foldr ((:) . h) []`

`= (\f z -> foldr (f . h) z) (:) []`

`-- (:) . h a as = (:) (h a as) = h a : as`

`length :: [a] -> Int`

`length = foldr (_ -> \n -> n+1) 0`

`length . map h = length`

`(foldr (_ -> \n -> n+1) 0) . (foldr ((:) . h) []) =`

`= podľa Acid Rain theorem (f = (_ -> \n -> n+1), z = 0, ale čo je g ? ...`

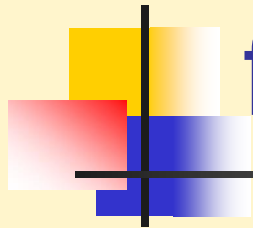
`g x y = (foldr (x . h) y)`

`g f z = (foldr (f . h) z) = foldr ((_ -> \n -> n+1) . h) 0 =`

`foldr ((_ -> \n -> n+1)) 0 = length`

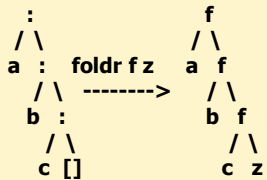
`lebo`

`((_ -> \n -> n+1) . h) x y = (_ -> \n -> n+1) (h x) y = (\n -> n+1) y = y+1`

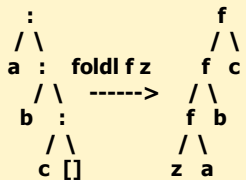


foldr a foldl pre pokročilejších

definujte foldl pomocou foldr, alebo naopak:



$\text{myfoldl } f \ z \ xs = \text{foldr } (\backslash x \rightarrow \backslash y \rightarrow (f \ y \ x)) \ z \ (\text{myReverse } xs)$
 $\text{myfoldr } f \ z \ xs = \text{foldl } (\backslash x \rightarrow \backslash y \rightarrow (f \ y \ x)) \ z \ (\text{myReverse } xs)$



- odstránime myReverse

$\text{myReverse } xs = \text{foldr } (\backslash x \rightarrow \backslash y \rightarrow (y ++ [x])) \ [] \ xs$

$\text{myfoldl}' f \ z \ xs = \text{foldr } (\backslash x \rightarrow \backslash y \rightarrow (f \ y \ x)) \ z$
 $(\text{foldr } (\backslash x \rightarrow \backslash y \rightarrow (y ++ [x])) \ [] \ xs)$

- odstránime ++

$xs ++ ys = \text{foldr } (:) \ ys \ xs$

$\text{myfoldl}'' f \ z \ xs = \text{foldr } (\backslash x \rightarrow \backslash y \rightarrow (f \ y \ x)) \ z$
 $(\text{foldr } (\backslash x \rightarrow \backslash y \rightarrow (\text{foldr } (:) \ [x] \ y)) \ [] \ xs)$

hmmm..., teoreticky (možno) zaujímavé, prakticky nepoužiteľné ...

foldr a foldl posledný krát

Zamyslime sa, ako z foldr urobíme foldl:

induktívne predpokladajme, že rekurzívne volanie foldr nám vráti výsledok, t.j. hodnotu y , ktorá zodpovedá foldl:

- $y = \text{myfoldl } f \ [b,c] = \lambda z \rightarrow f (f z b) c$

nech x je ďalší prvok zoznamu, t.j.

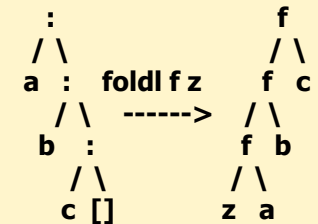
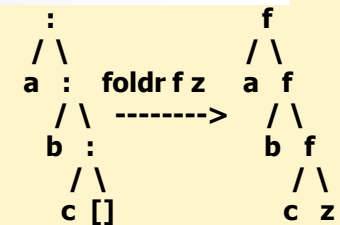
- $x = a$

ako musí vyzerat' funkcia $?$, ktorou fold-r-ujeme, aby sme dostali $\text{myfoldl } f \ [a,b,c] = \lambda z' \rightarrow f (f (f z' a) b) c = ? \ x \ y$

- $? = (\lambda x \ y \ z' \rightarrow y (f z' x))$

dosad'me:

- $(\lambda z' \rightarrow (\lambda z \rightarrow f (f z b) c) (f z' a)) =$
- $(\lambda z' \rightarrow f (f (f z' a) b) c) =$
- $\lambda z' \rightarrow f (f (f z' a) b) c$



Pre tých, čo neveria, fakt posledný krát

$$? = (\backslash x y z' \rightarrow y (f z' x))$$

$$\blacksquare \text{ myfoldl'''' } f \text{ xs } z = \text{ foldr } (\backslash x y z \rightarrow y (f z x)) \text{ id } \text{ xs } z$$

- $\text{myfoldl'''' } f [] = \text{id}$
- $\text{myfoldl'''' } f [c] = (\backslash x y z \rightarrow y (f z x)) c \text{ id} = \backslash z \rightarrow f z c$
- $\text{myfoldl'''' } f [b,c] = (\backslash x y z \rightarrow y (f z x)) b (\backslash w \rightarrow f w c) =$
 $\backslash z \rightarrow (\backslash w \rightarrow f w c) (f z b) =$
 $\backslash z \rightarrow f (f z b) c$
- $\text{myfoldl'''' } f [a,b,c] = (\backslash x y z \rightarrow y (f z x)) a (\backslash w \rightarrow f (f w b) c) =$
 $\backslash z \rightarrow (\backslash w \rightarrow f (f w b) c) (f z a) =$
 $\backslash z \rightarrow f (f (f z a) b) c$
- $\text{myfoldl'''' } f z \text{ xs} = \text{ foldr } (\backslash x y z \rightarrow y (f x z)) \text{ id } \text{ xs } z$

... doma skúste foldr pomocou foldl ...