

Wholemeal (functional) programming

Peter Borovanský
I-18

<http://dai.fmph.uniba.sk/courses/FPRO/>



Wholemeal in functional

- dnes na príklade Sudoku Solvera (podľa: Richard Bird)

The wholemeal approach often offers new insights or provides new perspectives on a given problem. It is nicely complemented by the idea of projective programming:

first

- solve a more general problem,

then

- extract the interesting bits and pieces by transforming the general program into more specialised ones."

<http://www.cs.tufts.edu/~nr/comp150fp/archive/richard-bird/sudoku.pdf>

rôzne sudoku solvery (v Haskell)

<http://www.haskell.org/haskellwiki/Sudoku>



Sudoku

```

type Matrix a      = [Row a]
type Row a         = [a]
type Value         = Char

type Grid          = Matrix Value
-- de facto [[Char]]

easy              :: Grid
easy              =
[    "2....1.38",
      ".....5",
      ".7...6...",
      ".....13",
      ".981..257",
      "31....8..",
      "9..8...2.",
      ".5..69784",
      "4..25...."]

```

```

solve              :: Grid -> [Grid]  -- nájdi všetky riešenia

```

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 8 | | | 4 | | 6 | | | 7 |
| | | | | | | 4 | | |
| | 1 | | | | | 6 | 5 | |
| 5 | | 9 | | 3 | | 7 | 8 | |
| | | | | 7 | | | | |
| | 4 | 8 | | 2 | | 1 | | 3 |
| | 5 | 2 | | | | | 9 | |
| | | 1 | | | | | | |
| 3 | | | 9 | | 2 | | | 5 |

```

-- [String] = [[Char]]

```

```

-- String = [Char]

```

```
rows . rows = id
cols . cols = id
```



Základné definície

```
boxsize      :: Int      -- 9 štvorcov 3x3
boxsize      = 3

values       :: [Value]  -- prípustné hodnoty
values       = ['1'..'9']

empty       :: Value -> Bool      -- nevyplnené ?
empty       = (== '.')

blank       :: Grid      -- vytvor prázdny štvorec
blank       = replicate n (replicate n '.')
              where n = boxsize ^ 2
replicate n x = [ x | i<-[1..n] ]

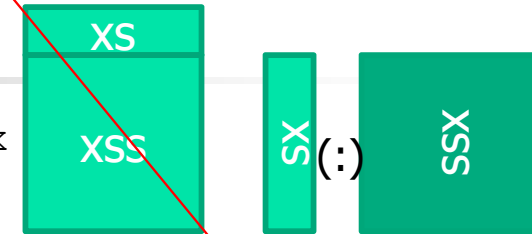
rows        :: Matrix a -> [Row a] -- zoznam riadkov
rows        = id

cols        :: Matrix a -> [Row a] -- zoznam stĺpcov
cols        = transpose
```

Trasponovanie matice

(stĺpce sa stanú riadkami)

```
transpose'      :: Matrix a -> Matrix a
transpose' [xs] = [[x] | x <- xs]
transpose' (xs:xss) = zipWith (:) xs (transpose' xss)
```



```
zipWith          :: (a -> b -> c) -> [a] -> [b] -> [c]
zipWith f (x:xs) (y:ys) = f x y : (zipWith f xs ys)
zipWith _ _ _          = []
```

- pokúsme sa transpose' prepísať pomocou foldr:

```
transpose'' xss = foldr (\xs -> \rek -> zipWith (:) xs rek)
                        -- (replicate (length xss) [])
                        [ [] | _ <- [1..(length xss)]]
                        xss
```

- a funguje to ?
- vieme napísať transpose pomocou foldl

all/any :: (a->Bool) -> [a] -> Bool

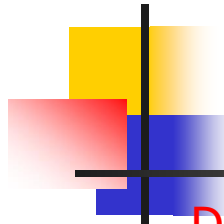


Korektné riešenie

```
valid      :: Grid -> Bool    -- bezosporné riešenie
valid g    = all nodups (rows g) &&
              all nodups (cols g) &&
              all nodups (boxs g)

nodups     :: Eq a => [a] -> Bool -- bez duplikátov
nodups []  = True
nodups (x:xs) = not (elem x xs) && nodups xs

boxs       :: Matrix a -> [Row a] -- zoznam 3x3 štvorcov
boxs       = unpack . map cols . pack
            where
              unpack = map concat . concat
              pack   = group3 . map group3
              group3  = group boxsize
              group   :: Int -> [a] -> [[a]]
              group n [] = []
              group n xs = take n xs : group n (drop n xs)
```



Turbo - SudokuStvorce

Definujte vlastnú verziu boxes, ktorá implementuje:

Nech toto je `e::Grid = [[9*i+j+1 | j <- [0..8]] | i <- [0..8]]`

```
[[1, 2, 3, 4, 5, 6, 7, 8, 9],  
[10, 11, 12, 13, 14, 15, 16, 17, 18],  
[19, 20, 21, 22, 23, 24, 25, 26, 27],  
[28, 29, 30, 31, 32, 33, 34, 35, 36],  
[37, 38, 39, 40, 41, 42, 43, 44, 45],  
[46, 47, 48, 49, 50, 51, 52, 53, 54],  
[55, 56, 57, 58, 59, 60, 61, 62, 63],  
[64, 65, 66, 67, 68, 69, 70, 71, 72],  
[73, 74, 75, 76, 77, 78, 79, 80, 81]]
```

```
Main> boxes e  
[[1, 2, 3, 10, 11, 12, 19, 20, 21],  
[4, 5, 6, 13, 14, 15, 22, 23, 24],  
[7, 8, 9, 16, 17, 18, 25, 26, 27],  
[28, 29, 30, 37, 38, 39, 46, 47, 48],  
[31, 32, 33, 40, 41, 42, 49, 50, 51],  
[34, 35, 36, 43, 44, 45, 52, 53, 54],  
[55, 56, 57, 64, 65, 66, 73, 74, 75],  
[58, 59, 60, 67, 68, 69, 76, 77, 78],  
[61, 62, 63, 70, 71, 72, 79, 80, 81]]
```



Riešenie s indexovaním

```
sudokuStvorce :: [[Int]] -> [[Int]]
sudokuStvorce m =
  concat [
    [
      [ m!!(3*i+k)!!(3*j+1) | k <- [0..2], l <- [0..2] ]
      | j <- [0..2]
    ] | i <- [0..2]
  ]
```

- riešenie ViktorS, DavidK-veľmi podobné:

```
blocks :: [[Int]]
blocks = [[0..2],[3..5],[6..8]]
sudokuStvorce :: [[Int]] -> [[Int]]
sudokuStvorce xs = [
  [xs!!r!!c | r <- b1, c <- b2] | b1<-blocks, b2<-blocks ]
```




Boxs

(krok 1 - pack)

```
boxs = unpack . map cols . pack
where
  unpack = map concat . concat
  pack   = group3 . map group3
  group3 = group boxsize
  group  :: Int -> [a] -> [[a]]
  group n [] = []
  group n xs = take n xs : group n (drop n xs)
```

```
Main > group 3 [1..9]
[ [1,2,3], [4,5,6], [7,8,9] ]
```

-- toto robí group3

```
Main > (group3 . map group3) e
[[[ 1, 2, 3 ], [ 4, 5, 6 ], [ 7, 8, 9 ]],
 [ [10,11,12 ], [13,14,15 ], [16,17,18 ]],
 [ [19,20,21 ], [22,23,24 ], [25,26,27 ]]],
 [[ [28,29,30 ], [31,32,33 ], [34,35,36 ]],
 [ [37,38,39 ], [40,41,42 ], [43,44,45 ]],
 [ [46,47,48 ], [49,50,51 ], [52,53,54 ]] ],
 [[ [55,56,57 ], [58,59,60 ], [61,62,63 ]],
 [ [64,65,66 ], [67,68,69 ], [70,71,72 ]],
 [ [73,74,75 ], [76,77,78 ], [79,80,81 ]]]]
```

-- iný zápis pre group3 (map group3 e)



Boxs

(krok 2 – map cols)

```
boxs = unpack . map cols . pack
where
  unpack = map concat . concat
  pack   = group3 . map group3
  group3 = group boxsize
  group  :: Int -> [a] -> [[a]]
  group n [] = []
  group n xs = take n xs : group n (drop n xs)
```

```
Main > ((map cols) . (group3. map group3)) e
```

```
[ [ [ 1, 2, 3 ], [ 10,11,12 ], [ 19,20,21 ] ],
  [ [ 4, 5, 6 ], [ 13,14,15 ], [ 22,23,24 ] ],
  [ [ 7, 8, 9 ], [ 16,17,18 ], [ 25,26,27 ] ] ],
[ [ [ 28,29,30 ], [ 37,38,39 ], [ 46,47,48 ] ],
  [ [ 31,32,33 ], [ 40,41,42 ], [ 49,50,51 ] ],
  [ [ 34,35,36 ], [ 43,44,45 ], [ 52,53,54 ] ] ],
[ [ [ 55,56,57 ], [ 64,65,66 ], [ 73,74,75 ] ],
  [ [ 58,59,60 ], [ 67,68,69 ], [ 76,77,78 ] ],
  [ [ 61,62,63 ], [ 70,71,72 ], [ 79,80,81 ] ] ] ]
```



Boxs

(krok 3 - unpack)

```
boxs = unpack . map cols . pack
      where
        unpack = map concat . concat
        pack   = group3 . map group3
        group3  = group boxsize
        group   :: Int -> [a] -> [[a]]
        group n [] = []
        group n xs = take n xs : group n (drop n xs)
```

```
concat :: [[a]] -> [a]
```

```
concat [[1,2,3],[4,5],[6]] = [1,2,3,4,5,6]
```

```
Main > ((map concat . concat) . (map cols) . (group3 . map group3))
```

e

```
[[ 1, 2, 3 , 10,11,12 , 19,20,21 ],
 [ 4, 5, 6 , 13,14,15 , 22,23,24 ],
 [ 7, 8, 9 , 16,17,18 , 25,26,27 ],
 [ 28,29,30 , 37,38,39 , 46,47,48 ],
 [ 31,32,33 , 40,41,42 , 49,50,51 ],
 [ 34,35,36 , 43,44,45 , 52,53,54 ],
 [ 55,56,57 , 64,65,66 , 73,74,75 ],
 [ 58,59,60 , 67,68,69 , 76,77,78 ],
 [ 61,62,63 , 70,71,72 , 79,80,81 ] ]
```

```
[ [ [[ 1,, 2,, 3 ],, [ 10,,11,,12 ],, [ 19,,20,,21 ] ],,
    [ [ 4, 5, 6 ], [ 13,14,15 ], [ 22,23,24 ] ],,
    [ [ 7, 8, 9 ], [ 16,17,18 ], [ 25,26,27 ] ] ],
 [ [ [ 28,29,30 ], [ 37,38,39 ], [ 46,47,48 ] ],,
    [ [ 31,32,33 ], [ 40,41,42 ], [ 49,50,51 ] ],,
    [ [ 34,35,36 ], [ 43,44,45 ], [ 52,53,54 ] ] ],
 [ [ [ 55,56,57 ], [ 64,65,66 ], [ 73,74,75 ] ],,
    [ [ 58,59,60 ], [ 67,68,69 ], [ 76,77,78 ] ],,
    [ [ 61,62,63 ], [ 70,71,72 ], [ 79,80,81 ] ] ] ]
```

`concat . group3 = id`
`group3 . concat = id`



Vlastnosti

Platí, že:

```
rows . rows = id
cols . cols = id
boxs . boxs = id,      kde boxs = unpack . map cols . pack
```

```
(unpack . map cols . pack) . (unpack . map cols . pack) =
```

dosadíme:

```
(map concat . concat) . map cols . (group3 . map group3) . -- pokračuje nižšie
(map concat . concat) . map cols . (group3 . map group3) =
```

asociativnosť

```
map concat . concat . map cols . group3 . map group3 .
map concat . concat . map cols . group3 . map group3 =
```

```
map concat . concat . map cols . group3 .
concat . map cols . group3 . map group3 =
```

```
map concat . concat . map cols . map cols . group3 . map group3 =
map concat . concat . group3 . map group3 =
map concat . map group3 =
id ☺
```

Dokážte, či vyvráťte, že `group3 . concat = id`



Na príklade

Riešenie Turbo - pre kontrolu

```
[1,2,3,10,11,12,19,20,21],  
[4,5,6,13,14,15,22,23,24],  
[7,8,9,16,17,18,25,26,27],  
[28,29,30,37,38,39,46,47,48],  
[31,32,33,40,41,42,49,50,51],  
[34,35,36,43,44,45,52,53,54],  
[55,56,57,64,65,66,73,74,75],  
[58,59,60,67,68,69,76,77,78],  
[61,62,63,70,71,72,79,80,81]]
```

```
Main> e -- kde e::Grid = [[9*i+j+1 | j <- [0..8]] | i <- [0..8]]
```

```
■ [[1,2,3,4,5,6,7,8,9],[10,11,12,13,14,15,16,17,18],[19,20,21,22,23,24,25,26,27],[28,29,  
30,31,32,33,34,35,36],[37,38,39,40,41,42,43,44,45],[46,47,48,49,50,51,52,53,54],[55,56,  
57,58,59,60,61,62,63],[64,65,66,67,68,69,70,71,72],[73,74,75,76,77,78,79,80,81]]
```

```
Main> map group3 e
```

```
■ [[[1,2,3],[4,5,6],[7,8,9]],[[10,11,12],[13,14,15],[16,17,18]],[[19,20,21],[22,23,24],[  
25,26,27]],[[28,29,30],[31,32,33],[34,35,36]],[[37,38,39],[40,41,42],[43,44,45]],[[46,  
47,48],[49,50,51],[52,53,54]],[[55,56,57],[58,59,60],[61,62,63]],[[64,65,66],[67,68,69  
],[70,71,72]],[[73,74,75],[76,77,78],[79,80,81]]]
```

```
Main> (group3.map group3) e
```

```
■ [[[[[1,2,3],[4,5,6],[7,8,9]],[[10,11,12],[13,14,15],[16,17,18]],[[19,20,21],[22,23,24],[  
25,26,27]]],[[[28,29,30],[31,32,33],[34,35,36]],[[37,38,39],[40,41,42],[43,44,45]],[[  
46,47,48],[49,50,51],[52,53,54]]],[[[55,56,57],[58,59,60],[61,62,63]],[[64,65,66],[67,  
68,69],[70,71,72]],[[73,74,75],[76,77,78],[79,80,81]]]]]
```

```
Main> ((map cols).(group3.map group3)) e
```

```
■ [[[[[1,2,3],[10,11,12],[19,20,21]],[[4,5,6],[13,14,15],[22,23,24]],[[7,8,9],[16,17,18],[  
25,26,27]]],[[[28,29,30],[37,38,39],[46,47,48]],[[31,32,33],[40,41,42],[49,50,51]],[[  
34,35,36],[43,44,45],[52,53,54]]],[[[55,56,57],[64,65,66],[73,74,75]],[[58,59,60],[67,  
68,69],[76,77,78]],[[61,62,63],[70,71,72],[79,80,81]]]]]
```

```
Main> (concat.(map cols).(group3.map group3)) e
```

```
■ [[1,2,3],[10,11,12],[19,20,21],[4,5,6],[13,14,15],[22,23,24],[7,8,9],[16,17,18],[  
25,26,27],[28,29,30],[37,38,39],[46,47,48],[31,32,33],[40,41,42],[49,50,51],[34,  
35,36],[43,44,45],[52,53,54],[55,56,57],[64,65,66],[73,74,75],[58,59,60],[67,68,69  
],[76,77,78],[61,62,63],[70,71,72],[79,80,81]]]
```

```
Main> ((map concat.concat).(map cols).(group3.map group3)) e
```

```
■ [1,2,3,10,11,12,19,20,21],[4,5,6,13,14,15,22,23,24],[7,8,9,16,17,18,25,26,27],[28,29,  
30,37,38,39,46,47,48],[31,32,33,40,41,42,49,50,51],[34,35,36,43,44,45,52,53,54],[55,56  
57,64,65,66,73,74,75],[58,59,60,67,68,69,76,77,78],[61,62,63,70,71,72,79,80,81]]]
```



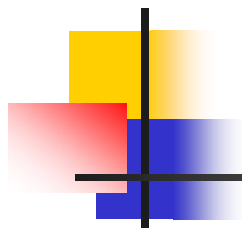
Nájdenie všetkých riešení

```
type Choices      = [Value]    -- zoznam možností jedného políčka
```

-- do každého políčka, kde je '.', vpíšeme úplne všetky možnosti

```
choices           :: Grid -> Matrix Choices
choices           = map (map choice)
                  where
                    choice v = if empty v then values else [v]
```

```
Main> easy
["2....1.38",".....5",".7...6...",".....13",".981..257","31....8..","9..8...2.",
 ".5..69784","4..25...."]
Main> choices easy
[["2","123456789","123456789","123456789","123456789","1","123456789","3","8"],["1234
56789","123456789","123456789","123456789","123456789","123456789","123456789","12
3456789","5"],["123456789","7","123456789","123456789","123456789","6","123456789"
,"123456789","123456789"],["123456789","123456789","123456789","123456789","123456
789","123456789","123456789","1","3"],["123456789","9","8","1","123456789","123456
789","2","5","7"],["3","1","123456789","123456789","123456789","123456789","8","12
3456789","123456789"],["9","123456789","123456789","8","123456789","123456789","12
3456789","2","123456789"],["123456789","5","123456789","123456789","6","9","7","8"
,"4"],["4","123456789","123456789","2","5","123456789","123456789","123456789","12
3456789"]]
```



Choices

"123456789"

27 singles

$81 - 27 = 54$ empty

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 8 | | | 4 | | 6 | | | 7 |
| | | | | | | | 4 | |
| | 1 | | | | | 6 | 5 | |
| 5 | | 9 | | 3 | | 7 | 8 | |
| | | | | 7 | | | | |
| | 4 | 8 | | 2 | | 1 | | 3 |
| | 5 | 2 | | | | | 9 | |
| | | 1 | | | | | | |
| 3 | | | 9 | | 2 | | | 5 |

$9^{54} = 3_381_391_913_522_726_342_930_221_472_392_241_170_198_527_451_848_561$ možností



Nájdienie všetkých riešení

-- kartézsky súčin všetkých možností v jednom riadku

```
cp                :: [[a]] -> [[a]]      -- Row[a] -> Row[a]
cp []             =  [[]]
cp (xs:xss)       =  [y:ys | y<-xs, ys<-cp xss]
```

```
Main > cp [ [1,2,3], [4,5], [6] ]
[[1,4,6],[1,5,6],[2,4,6],[2,5,6],[3,4,6],[3,5,6]]
```

A potrebujeme cp aj na matici...

```
collapse          :: Matrix [a] -> [Matrix a]
collapse          =  cp . map cp
```

collapse vytvorí z matice možností, zoznam všetkých potenciálnych riešení



Naivné riešenie

```
Main > collapse (choices easy)
??? Koľko ich je ???
```

```
Main> easy
```

```
["2....1.38", "...5", ".7...6...", "...13", ".981..257", "
 31....8..", "9..8...2.", ".5..69784", "4..25...."]
```

```
Main> map (map (\x->if empty x then 9 else 1)) easy
```

```
[[1,9,9,9,9,1,9,1,1],[9,9,9,9,9,9,9,9,1],[9,1,9,9,9,1,9,9,9],[
 9,9,9,9,9,9,9,1,1],[9,1,1,1,9,9,1,1,1],[1,1,9,9,9,9,1,9,9],
 [1,9,9,1,9,9,9,1,9],[9,1,9,9,1,1,1,1,1],[1,9,9,1,1,9,9,9,9]
]
```

```
Main> (product . map product)
```

```
      (map (map (\x->if empty x then 9 else 1)) easy)
```

```
4638397686588101979328150167890591454318967698009 ☹
```

```
solve :: Grid -> [Grid]
```

```
solve = filter valid . collapse . choices
```

rows . rows = id
cols . cols = id
boxs . boxs = id

Orezávanie možností

Zredukujme tie možnosti, ktoré sa vylučujú so single-možnosťami

```
prune      :: Matrix Choices -> Matrix Choices
prune      = pruneBy boxs . pruneBy cols . pruneBy rows
              where pruneBy f = f . map reduce . f
```

```
reduce      :: Row Choices -> Row Choices
reduce xss   = [xs `minus` singles | xs <- xss]
              where singles = concat (filter single xss)
                      -- singles zoznam použitých single-možností v riadku
```

```
Main> reduce [ "123","2","567","7" ]
["13","2","56","7"]
```

```
minus      :: Choices -> Choices -> Choices
xs `minus` ys = if single xs then xs else xs \\ ys
```

```
solve2      :: Grid -> [Grid]
solve2      = filter valid . collapse . prune . choices
```

Koľko možností má (prune . choices) grid (napr.easy)?
Definujte funkciu v Haskell, ktorá to spočíta...



prune.choices

"239" "359" "347" "246" "59"

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 8 | | | 4 | | 6 | | | 7 |
| | | | | | | 4 | | |
| | 1 | | | | | 6 | 5 | |
| 5 | | 9 | | 3 | | 7 | 8 | |
| | | | | 7 | | | | |
| | 4 | 8 | | 2 | | 1 | | 3 |
| | 5 | 2 | | | | | 9 | |
| | | 1 | | | | | | |
| 3 | | | 9 | | 2 | | | 5 |

rows . rows = id
cols . cols = id
boxs . boxs = id



Opakované orezávanie

prune . prune ... prune, až kým je čo orezať ...

```
filter valid . collapse . prune . prune ... prune . prune . choices
```

```
solve3      :: Grid -> [Grid]
```

```
solve3      = filter valid . collapse . fix prune . choices
```

```
fix          :: Eq a => (a -> a) -> a -> a
```

```
fix f x      = if x == x' then x else fix f x'  
              where x' = f x
```

Koľko možností má (fix prune. choices) pre easy, resp. gentle, ...



Vlastnosti matíc

```
complete :: Matrix Choices -> Bool
complete = all (all single)
```

-- matica možností predstavuje
-- jediné riešenie

```
Main> (all (all single)) (choices easy)
False
```

```
void :: Matrix Choices -> Bool
void = any (any null)
```

-- neexistuje riešenie, lebo
-- niektorá z možností je null

```
safe :: Matrix Choices -> Bool
safe m = all consistent (rows m) &&
         all consistent (cols m) &&
         all consistent (boxs m)
```

-- konzistencia na singletony
-- na riadkoch
-- na stĺpcoch
-- v štvorcoch

```
consistent :: Row Choices -> Bool
consistent = nodups . concat . filter single
```

```
Main> consistent [ "12", "2", "34", "3", "2" ]
False
```

```
blocked :: Matrix Choices -> Bool
blocked m = void m || not (safe m)
```

-- zlá možnosť



Constraint propagation

```
solve4 :: Grid -> [Grid]
solve4 = search . prune . choices

search :: Matrix Choices -> [Grid]
search m
  | blocked m      = []
  | complete m     = collapse m
  | otherwise      = [g | m' <- expand m
                        , g  <- search (prune m')]

-- zober niektorú/prvú možnosť, ktorá nie je singleton, a rozpiš ju
expand :: Matrix Choices -> [Matrix Choices]
expand m =
  [rows1 ++ [row1 ++ [c] : row2] ++ rows2 | c <- cs]
  where
    (rows1,row:rows2) = break (any (not . single)) m
    (row1,cs:row2)    = break (not . single) row
```

zistite, čo robí break a definujte vlastnú implementáciu



"239"

| | | | | | |
|---|---|---|---|---|---|
| 8 | | 4 | 6 | | 7 |
| | | | | 4 | |
| | 1 | | | 6 | 5 |
| 5 | 9 | | 3 | 7 | 8 |
| | | | 7 | | |
| | 4 | 8 | 2 | 1 | 3 |
| | 5 | 2 | | | 9 |
| | | 1 | | | |
| 3 | | 9 | 2 | | 5 |

| | | | | | | |
|---|---|---|---|---|---|---|
| 8 | 2 | | 4 | 6 | | 7 |
| | | | | | 4 | |
| | 1 | | | | 6 | 5 |
| 5 | | 9 | | 3 | 7 | 8 |
| | | | | 7 | | |
| | 4 | 8 | | 2 | 1 | 3 |
| | 5 | 2 | | | | 9 |
| | | 1 | | | | |
| 3 | | | 9 | 2 | | 5 |

| | | | | | | |
|---|---|---|---|---|---|---|
| 8 | 3 | | 4 | 6 | | 7 |
| | | | | | 4 | |
| | 1 | | | | 6 | 5 |
| 5 | | 9 | | 3 | 7 | 8 |
| | | | | 7 | | |
| | 4 | 8 | | 2 | 1 | 3 |
| | 5 | 2 | | | | 9 |
| | | 1 | | | | |
| 3 | | | 9 | 2 | | 5 |

| | | | | | | |
|---|---|---|---|---|---|---|
| 8 | 9 | | 4 | 6 | | 7 |
| | | | | | 4 | |
| | 1 | | | | 6 | 5 |
| 5 | | 9 | | 3 | 7 | 8 |
| | | | | 7 | | |
| | 4 | 8 | | 2 | 1 | 3 |
| | 5 | 2 | | | | 9 |
| | | 1 | | | | |
| 3 | | | 9 | 2 | | 5 |



Minimum možností

Domáca úloha: upravte expand na

```
expandMin :: Matrix Choices -> [Matrix Choices]
```

ktorá expanduje maticu podľa políčka s minimálnym počtom možností