Section 1.7

Problem 4. *Consider the following functions from* $\mathbb{N}$ *into* $\mathbb{N}$*:*
$I_{\mathbb{N}}(n) = n$, $f(n) = 3n$, $g(n) = n + (-1)^n$, $h(n) = \min\{n, 100\}$, $k(n) = \max\{0, n - 5\}$.

(a) Which of these functions are one-to-one?

(b) Which of these functions map $\mathbb{N}$ *onto* $\mathbb{N}$*?*

Solution.

(a) $I_{\mathbb{N}}$: One-to-one since if $I_{\mathbb{N}}(a) = I_{\mathbb{N}}(b)$, then $a = b$.

f: One-to-one since if $3a = 3b$, then $a = b$.

g: One-to-one. For odd n, $g(n) = n-1$. For even n, $g(n) = n+1$. If $g(a) = g(b)$, either both are odd or both are even (otherwise the outputs have different parities), and in either case $a = b$.

h: Not one-to-one. $h(100) = h(101) = 100$ but $100 \neq 101$.

k: Not one-to-one. $k(0) = k(5) = 0$ but $0 \neq 5$.

(b) $I_{\mathbb{N}}$: Onto since for any $m \in \mathbb{N}$, $I_{\mathbb{N}}(m) = m$.

f: Not onto. There is no n such that $f(n) = 1$.

g: Onto. The range is $\{1, 0, 3, 2, 5, 4, \ldots\}$ which covers all of $\mathbb{N}$.

h: Not onto. There is no n such that $h(n) = 101$.

k: Onto. For any $m \in \mathbb{N}$, $k(m + 5) = m$.

□

Problem 5. *Here are two "shift functions" mapping* $\mathbb{N}$ *into* $\mathbb{N}$*:*
$f(n) = n + 1$ *and* $g(n) = \max\{0, n - 1\}$ *for* $n \in \mathbb{N}$.

(c) Show that f *is one-to-one but does not map* $\mathbb{N}$ *onto* $\mathbb{N}$.

(d) Show that g *maps* $\mathbb{N}$ *onto* $\mathbb{N}$ *but is not one-to-one.*

(e) Show that $g \circ f(n) = n$ *for all* n*, but that* $f \circ g(n) = n$ *does not hold for all* n.

Solution.

(c) f is one-to-one: If $f(a) = f(b)$, then $a + 1 = b + 1$, so $a = b$.

f is not onto: The range of f is $\{1, 2, 3, \ldots\}$, so 0 is not in the range.

(d) g is onto: For any $m \in \mathbb{N}$, take $n = m + 1$. Then $g(m + 1) = \max\{0, m\} = m$.
g is not one-to-one: $g(0) = g(1) = 0$ but $0 \neq 1$.

(e) For any $n \in \mathbb{N}$, $g(f(n)) = g(n + 1) = \max\{0, n\} = n$.
But $f(g(0)) = f(0) = 1 \neq 0$, so $f \circ g$ is not the identity.

□

Problem 6. *Let $\Sigma = \{a, b, c\}$ and let Σ^* be the set of all words w using letters from Σ. Define $L(w) = length(w)$ for all $w \in \Sigma^*$.*

(b) Is L a one-to-one function? Explain.

(c) The function L maps Σ^* into $\mathbb{N}$. Does L map Σ^* onto $\mathbb{N}$? Explain.

Solution.

(b) No, L is not one-to-one. Many words have the same length, e.g., $L(ab) = L(ca) = 2$ but $ab \neq ca$.

(c) Yes, L is onto. For $n = 0$, take the empty word λ. For $n > 0$, take the word $aa \ldots a$ (n times). Then $L(w) = n$ for any $n \in \mathbb{N}$.

□

Problem 13. *Let $f : S \to T$ and $g : T \to U$ be one-to-one functions. Show that the function $g \circ f : S \to U$ is one-to-one.*

Proof. Suppose $(g \circ f)(a) = (g \circ f)(b)$. Then $g(f(a)) = g(f(b))$. Since g is one-to-one, $f(a) = f(b)$. Since f is one-to-one, $a = b$. Therefore $g \circ f$ is one-to-one. □

Section 2.1

Problem 2. *Let p, q, and r be the following propositions:*
$p =$ *"it is raining,"*
$q =$ *"the sun is shining,"*
$r =$ *"there are clouds in the sky."*
Translate the following into English sentences.

(a) $(p \wedge q) \to r$

(b) $\neg p \leftrightarrow (q \vee r)$

Solution.

(a) $(p \wedge q) \to r$

"If it is raining and the sun is shining, then there are clouds in the sky."

(b) $\neg p \leftrightarrow (q \vee r)$

”It is not raining if and only if the sun is shining or there are clouds in the sky.”

□

Problem 4. *Which of the following are propositions? Give the truth values of the propositions.*

(a) $x^2 = x$ *for all* $x \in \mathbb{R}$.

(b) $x^2 = x$ *for some* $x \in \mathbb{R}$.

Solution.

(a) This is a proposition with truth value **False**. When $x = 2$, we have $x^2 = 4 \neq 2$.

(b) This is a proposition with truth value **True**. For $x = 0$, we have $0^2 = 0$.

□

Problem 6. *Give the converses of the following propositions.*

(b) If I am smart, then I am rich.

(c) If $x^2 = x$, *then* $x = 0$ *or* $x = 1$.

Solution.
The converse of ”$p \to q$” is ”$q \to p$”.

(b) **Converse:** If I am rich, then I am smart.

(c) **Converse:** If $x = 0$ or $x = 1$, then $x^2 = x$.

□

Problem 7. *Give the contrapositives of the propositions in Exercise 6.*

(b) If I am smart, then I am rich.

(c) If $x^2 = x$, *then* $x = 0$ *or* $x = 1$.

Solution.
The contrapositive of ”$p \to q$” is ”$\neg q \to \neg p$”.

(b) **Contrapositive:** If I am not rich, then I am not smart.

(c) **Contrapositive:** If $x \neq 0$ and $x \neq 1$, then $x^2 \neq x$.
(Or equivalently: If it is not the case that $x = 0$ or $x = 1$, then $x^2 \neq x$.)

$\square$

Problem 12. *Find counterexamples to the following assertions.*

(a) $2^n - 1$ *is prime for every* $n > 2$.

(b) $2^n + 3^n$ *is prime for all* $n \in \mathbb{N}$.

(c) $2^n + n$ *is prime for every positive odd integer* n.

Solution.

(a) $n = 4$: $2^4 - 1 = 15 = 3 \times 5$ is composite.

(b) $n = 3$: $2^3 + 3^3 = 8 + 27 = 35 = 5 \times 7$ is composite.

(c) $n = 7$: $2^7 + 7 = 128 + 7 = 135 = 5 \times 27$ is composite.

$\square$

Problem 14. *Let* S *be a nonempty set. Determine which of the following assertions are true. For the true ones, give a reason. For the false ones, provide a counterexample.*

(a) $A \cup B = B \cup A$ *for all* $A, B \in \mathcal{P}(S)$.

(b) $(A \setminus B) \cup B = A$ *for all* $A, B \in \mathcal{P}(S)$.

(c) $(A \cup B) \setminus A = B$ *for all* $A, B \in \mathcal{P}(S)$.

(d) $(A \cap B) \cap C = A \cap (B \cap C)$ *for all* $A, B, C \in \mathcal{P}(S)$.

Solution.

(a) TRUE. Union is commutative.

(b) FALSE. Let $A = \{1, 2\}$ and $B = \{2, 3\}$. Then $(A \setminus B) \cup B = \{1\} \cup \{2, 3\} = \{1, 2, 3\} \neq A$.

(c) FALSE. Let $A = \{1, 2\}$ and $B = \{2, 3\}$. Then $(A \cup B) \setminus A = \{1, 2, 3\} \setminus \{1, 2\} = \{3\} \neq B$.

(d) TRUE. Intersection is associative.

$\square$

Section 2.2

Problem 2. *Let p, q, and r be as in Exercise 1. Translate the following into English sentences.*

(a) $(p \wedge q) \rightarrow r$

(b) $(p \rightarrow r) \rightarrow q$

(c) $\neg p \leftrightarrow (q \vee r)$

(d) $\neg(p \rightarrow (q \vee r))$

(e) $\neg(p \vee q) \wedge r$

Solution.

Recall: $p =$ "it is raining," $q =$ "the sun is shining," $r =$ "there are clouds in the sky."

(a) "If it is raining and the sun is shining, then there are clouds in the sky."

(b) "If (it raining implies there are clouds in the sky), then the sun is shining."

(c) "It is not raining if and only if (the sun is shining or there are clouds in the sky)."

(d) "It is not the case that (if it is raining then the sun is shining or there are clouds in the sky)."

(e) "It is not raining and the sun is not shining, and there are clouds in the sky."

□

Problem 3. *Consider the following propositions:*
$p \rightarrow q$, $p \wedge \neg q$, $\neg p \rightarrow q$, $\neg p \vee q$, $q \rightarrow p$, $\neg q \vee p$, $\neg q \rightarrow \neg p$, $p \wedge \neg q$.

(a) Which proposition is the converse of $p \rightarrow q$?

(b) Which proposition is the contrapositive of $p \rightarrow q$?

(c) Which propositions are logically equivalent to $p \rightarrow q$?

Solution.

(a) $q \rightarrow p$

(b) $\neg q \rightarrow \neg p$

(c) $\neg p \vee q$ and $\neg q \rightarrow \neg p$

□

Problem 11. *Construct truth tables for*

(a) $\neg(p \vee q) \to r$

(b) $\neg((p \vee q) \to r)$

This exercise shows that one must be careful with parentheses.

Solution.

(a) Truth table for $\neg(p \vee q) \to r$:

p	q	r	$p \vee q$	$\neg(p \vee q)$	$\neg(p \vee q) \to r$
T	T	T	T	F	T
T	T	F	T	F	T
T	F	T	T	F	T
T	F	F	T	F	T
F	T	T	T	F	T
F	T	F	T	F	T
F	F	T	F	T	T
F	F	F	F	T	F

(b) Truth table for $\neg((p \vee q) \to r)$:

p	q	r	$p \vee q$	$(p \vee q) \to r$	$\neg((p \vee q) \to r)$
T	T	T	T	T	F
T	T	F	T	F	T
T	F	T	T	T	F
T	F	F	T	F	T
F	T	T	T	T	F
F	T	F	T	F	T
F	F	T	F	T	F
F	F	F	F	T	F

□

Problem 12. *In which of the following statements is the "or" an "inclusive or"?*

(a) *Choice of soup or salad.*

(b) *To enter the university, a student must have taken a year of chemistry or physics in high school.*

(c) *Publish or perish.*

(d) *Experience with C++ or Java is desirable.*

(e) The task will be completed on Thursday or Friday.

(f) Discounts are available to persons under 20 or over 60.

(g) No fishing or hunting allowed.

(h) The school will not be open in July or August.

Solution.

(a) Exclusive. You don't get both.

(b) Inclusive. Taking both satisfies the requirement.

(c) Inclusive. You can both publish and not perish.

(d) Inclusive. Having both is better.

(e) Inclusive. The task could span both days.

(f) Inclusive (though impossible to satisfy both).

(g) Inclusive. Both are prohibited.

(h) Inclusive. Closed both months.

□

Problem 23. *Prove or disprove the following. Don't forget that only one line of the truth table is needed to show that a proposition is not a tautology.*

(d) $(q \to p) \Rightarrow (p \wedge q)$

(e) $(p \wedge \neg q) \Rightarrow (p \to q)$

(f) $(p \wedge q) \Rightarrow (p \vee q)$

Note: $A \Rightarrow B$ *means "*A *implies* B*" or equivalently "*$A \to B$ *is a tautology."*

Solution.

(d) FALSE. Let $p = T$ and $q = F$. Then $q \to p$ is true but $p \wedge q$ is false.

(e) FALSE. Let $p = T$ and $q = F$. Then $p \wedge \neg q$ is true but $p \to q$ is false.

(f) TRUE. If $p \wedge q$ is true, then both p and q are true, so $p \vee q$ is true.

□

Problem 23. *A logician told her son "If you don't finish your dinner, you will not get to stay up and watch TV." He finished his dinner and then was sent straight to bed. Discuss.*

Solution. Let $p =$ "finish dinner" and $q =$ "watch TV." The mother said $\neg p \to \neg q$, which is equivalent to $q \to p$. The son incorrectly interpreted this as $p \to q$. The mother's statement only guarantees what happens if he doesn't finish dinner - it says nothing about what happens if he does. The converse is not implied, so the son confused the statement with its converse. □

Problem 24. *Consider the statement "Concrete does not grow if you do not water it."*

(a) Give the contrapositive.

(b) Give the converse.

(c) Give the converse of the contrapositive.

(d) Which among the original statement and the ones in parts (a), (b), and (c) are true?

Solution.
Let $p =$ "water concrete" and $q =$ "concrete grows."
Original: $\neg p \to \neg q$

(a) Contrapositive: $q \to p$ ("If concrete grows, then you watered it")

(b) Converse: $\neg q \to \neg p$ ("If concrete doesn't grow, then you didn't water it")

(c) Converse of contrapositive: $p \to q$ ("If you water concrete, then it grows")

(d) Original and contrapositive are TRUE (concrete never grows regardless, so the conclusion is always true). Converse and converse of contrapositive are FALSE (you can water concrete and it still doesn't grow).

□

Other Exercises

For the following statements: (i) translate into symbols, (ii) write the negation in words, (iii) write the contrapositive in words.

Problem 1. *For all functions $f : S \to T$, if $f : S \to T$ is onto, then for all $s \in S$ there exists a $t \in T$ such that $f(s) = t$.*

Solution.
(i) $\forall f : [\text{onto}(f) \to \forall s \in S \, \exists t \in T : f(s) = t]$
(ii) "There exists a function $f : S \to T$ such that f is onto and there exists $s \in S$ with $f(s) \neq t$ for all $t \in T$."
(iii) "For all functions $f : S \to T$, if there exists $s \in S$ that is not mapped to any $t \in T$, then f is not onto." □

Problem 2. *For all graphs G, if G is finite and connected, then G has a spanning tree.*

Solution.

(i) $\forall G : [(\text{finite}(G) \wedge \text{connected}(G)) \rightarrow \text{hasSpanningTree}(G)]$

(ii) "There exists a graph G such that G is finite and connected, but G does not have a spanning tree."

(iii) "For all graphs G, if G does not have a spanning tree, then G is infinite or disconnected." □

Problem 3. *If G is a finite connected graph and every vertex has even degree, then G is Eulerian.*

Solution.

(i) $\forall G : [(\text{finite}(G) \wedge \text{connected}(G) \wedge \forall v \in V(G) : \text{deg}(v) \text{ is even}) \rightarrow \text{Eulerian}(G)]$

(ii) "There exists a graph G such that G is finite and connected and every vertex has even degree, but G is not Eulerian."

(iii) "If a graph is not Eulerian, then it is infinite, disconnected, or has a vertex of odd degree." □

Problem 4. *For all graphs G with no loops or parallel edges, if $|V(G)| = n \geq 3$ and $\deg(v) \geq n/2$ for each vertex of G, then G is Hamiltonian.*

(Note the domain here is {graphs with no loops or parallel edges}.)

Solution.

(i) $\forall G : [(|V(G)| = n \geq 3 \wedge \forall v \in V(G) : \deg(v) \geq n/2) \rightarrow \text{Hamiltonian}(G)]$

(ii) "There exists a graph G with no loops or parallel edges such that $|V(G)| \geq 3$ and every vertex has degree at least $n/2$, but G is not Hamiltonian."

(iii) "For all simple graphs G, if G is not Hamiltonian, then either $|V(G)| < 3$ or some vertex has degree less than $n/2$." □