Section 3.1

Problem 2. *Let $A = \{0, 1, 2\}$. Each of the statements below defines a relation R on A by $(m, n) \in R$ if the statement is true for m and n. Write each of the relations as a set of ordered pairs.*

(a) $m \leq n$

(b) $m < n$

(f) $m + n \in A$

(h) $m^2 + n^2 = 3$

Solution.

(a) $m \leq n$

Answer: $R = \{(0, 0), (0, 1), (0, 2), (1, 1), (1, 2), (2, 2)\}$

(b) $m < n$

Answer: $R = \{(0, 1), (0, 2), (1, 2)\}$

(f) $m + n \in A$

Answer: $R = \{(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (2, 0)\}$

(h) $m^2 + n^2 = 3$

Answer: $R = \emptyset$ (the empty set)

□

Problem 6. *Consider the relation R on $\mathbb{Z}$ defined by $(m, n) \in R$ if and only if $m^3 - n^3 \equiv 0 \pmod 5$. Which of the properties (R), (AR), (S), (AS), and (T) are satisfied by R?*

Note: (R) = Reflexive, (AR) = Antireflexive, (S) = Symmetric, (AS) = Antisymmetric, (T) = Transitive

Solution. R is reflexive, symmetric and transitive making R an equivalence relation. In fact module n partitions the integers into equivalence classes. □

Problem 7. *Define the "divides" relation R on $\mathbb{N}$ by $(m,n) \in R$ if $m \mid n$.*

(a) Which of the properties (R), (AR), (S), (AS), and (T) does R satisfy?

Solution. R satisfies reflexivity and transitivity but does not satisfy symmetry making it anti-symmetric, thus it is a partial order. □

Problem 10. *Give an example of a relation that is:*

(a) antisymmetric and transitive but not reflexive,

(b) symmetric but not reflexive or transitive.

Solution.

(a) **Antisymmetric and transitive but not reflexive:**

The greater than relation $>$ on $\mathbb{R}$ (or on any set of numbers).

Define R on $\mathbb{R}$ by: $(a,b) \in R$ if and only if $a > b$.

Not Reflexive: For any $a \in \mathbb{R}$, we have $a \not> a$, so $(a,a) \notin R$.

Antisymmetric: If $(a,b) \in R$ and $(b,a) \in R$, then $a > b$ and $b > a$, which is impossible. So vacuously, the implication holds (antisymmetric).

Transitive: If $a > b$ and $b > c$, then $a > c$ by the transitivity of $>$. So $(a,b) \in R$ and $(b,c) \in R$ implies $(a,c) \in R$.

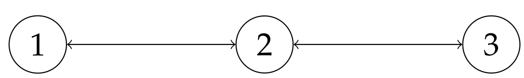
(b) **Symmetric but not reflexive or transitive:**

Let $S = \{1,2,3\}$ and $R = \{(1,2),(2,1),(2,3),(3,2)\}$.

Not Reflexive: $(1,1) \notin R$, $(2,2) \notin R$, $(3,3) \notin R$.

Symmetric: $(1,2) \in R$ and $(2,1) \in R$ ✓. $(2,3) \in R$ and $(3,2) \in R$ ✓.

Not Transitive: $(1,2) \in R$ and $(2,3) \in R$, but $(1,3) \notin R$.



□

Problem 18. *Draw pictures of each of the relations in Exercise 2 for parts (a), (b), (f), and (h). Use double-headed arrows or two arrows when the relation is symmetric.*

From Exercise 2 on $A = \{0, 1, 2\}$*:*

(a) $m \leq n$*:* $R = \{(0,0), (0,1), (0,2), (1,1), (1,2), (2,2)\}$

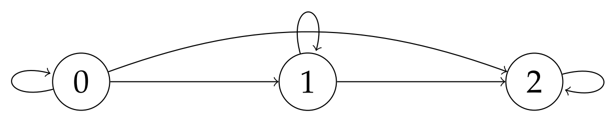
(b) $m < n$*:* $R = \{(0,1), (0,2), (1,2)\}$

(f) $m + n \in A$*:* $R = \{(0,0), (0,1), (0,2), (1,0), (1,1), (2,0)\}$

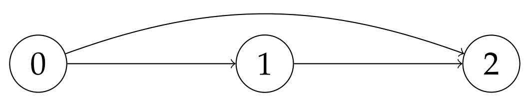
(h) $m^2 + n^2 = 3$*:* $R = \emptyset$

Solution.

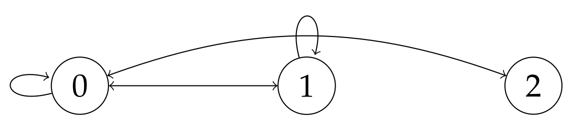
(a) $m \leq n$**:** NOT symmetric (e.g., $(0,1) \in R$ but $(1,0) \notin R$).



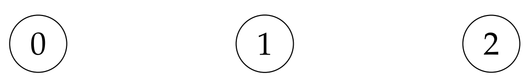
(b) $m < n$**:** NOT symmetric.



(f) $m + n \in A$**:** IS symmetric: $(0,1)$ and $(1,0)$ both in R; $(0,2)$ and $(2,0)$ both in R.



(h) $m^2 + n^2 = 3$**:** Empty relation, no edges.



□

Section 3.2

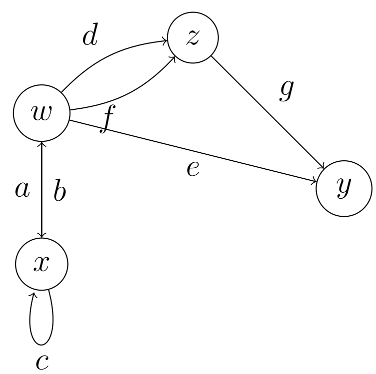
Problem 2. *Draw a picture of the digraph G with vertex set $V(G) = \{w, x, y, z\}$, edge set $E(G) = \{a, b, c, d, e, f, g\}$, and γ given by:*

e	a	b	c	d	e	f	g
$\gamma(e)$	(x, w)	(w, x)	(x, x)	(w, z)	(w, y)	(w, z)	(z, y)

Solution.

The digraph has edges:

- Edge a: $x \to w$
- Edge b: $w \to x$
- Edge c: $x \to x$ (loop)
- Edge d: $w \to z$
- Edge e: $w \to y$
- Edge f: $w \to z$ (parallel to d)
- Edge g: $z \to y$



□

Problem 3. *Which of the following vertex sequences describe paths in the digraph pictured in Figure 7(a)?*

(a) $z\ y\ v\ w\ t$

(b) $x\ z\ w\ t$

(c) $v\ s\ t\ x$

(d) $z\ y\ s\ u$

(e) $x\ z\ y\ v\ s$

(f) $s\ u\ x\ t$

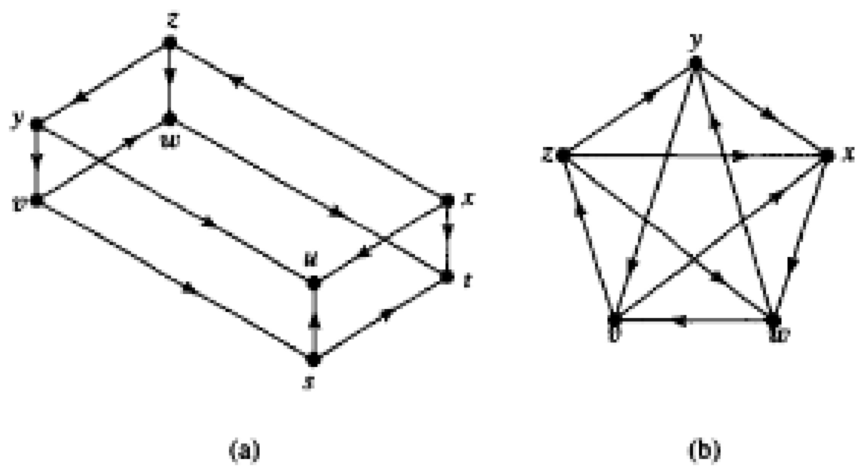


Figure 7 ▲

Solution. Below we checkmark the valid paths and x mark the invalid ones:

(a) $z\ y\ v\ w\ t$✓

(b) $x\ z\ w\ t$✓

(c) $v\ s\ t\ x\times$

(d) $z\ y\ s\ u\times$

(e) $x\ z\ y\ v\ s$✓

(f) $s\ u\ x\ t\times$

□

Problem 5. *Consider the digraph pictured in Figure 7(b). Describe an acyclic path*

(a) from x to y

(e) from z to v

Solution.

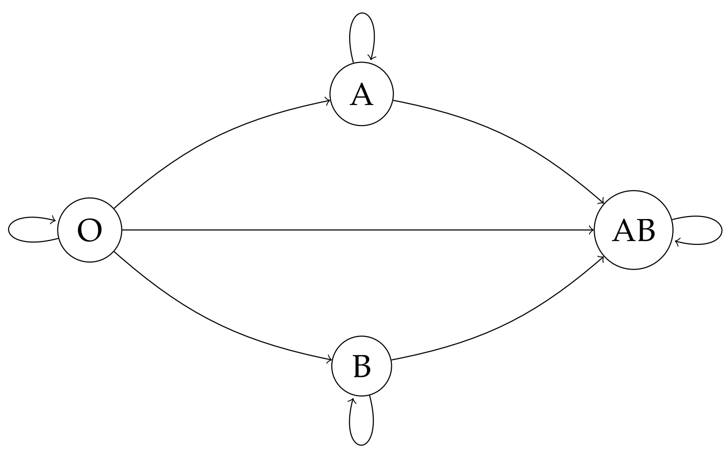
from x to y : x w y

from z to v : z y v □

Problem 6. *There are four basic blood types: A, B, AB, and O. Type O can donate to any of the four types, A and B can donate to AB as well as to their own types, but type AB can only donate to AB. Draw a digraph that presents this information. Is the digraph acyclic?*

Solution.

Vertices: $\{O, A, B, AB\}$



Graph is not acyclic, Each vertex has a self-loop (cycle of length 1).

□

Problem 8. *Determine the reachability relation for the digraphs in Figures 6(a), (c), and (d).*

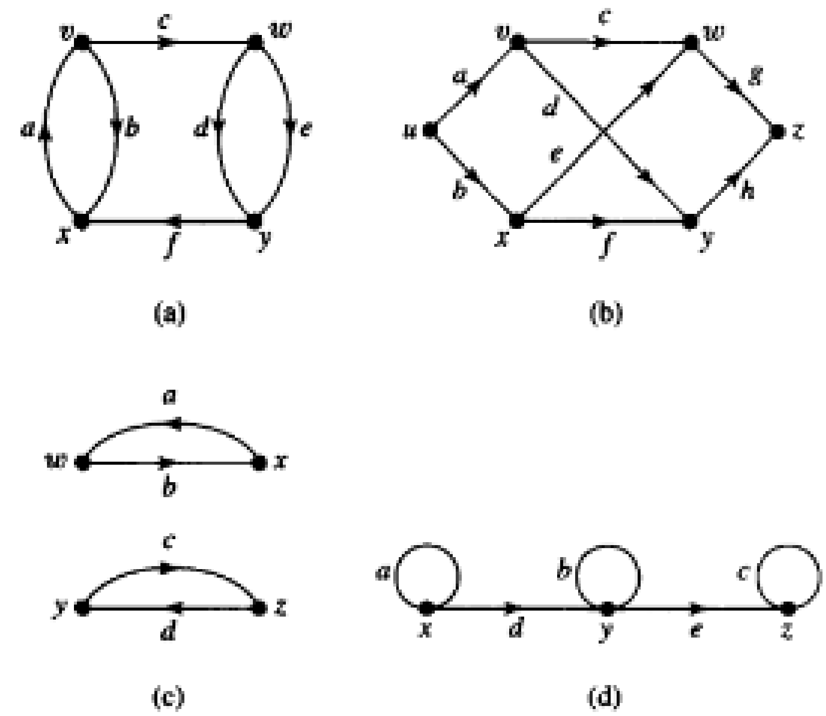


Figure 6 ▲

Solution.

$$R(a) = \{(v,w),(v,y),(v,x),(v,v),(w,y),(w,x),(w,v),$$
$$(w,w)(y,x),(y,v),(y,w),(y,y),(x,v),(x,w),(x,y),(x,x)\}$$

$$R(c) = \{(w,x),(x,w),(y,z),(z,y),(w,w),(x,x),(y,y),(z,z)\}$$

$$R(d) = \{(x,x),(y,y),(z,z),(x,y),(x,z),(y,z)\}$$

□

Problem 15. *Give the adjacency relation A and the reachable relation R for each of the graphs of Figure 8.*

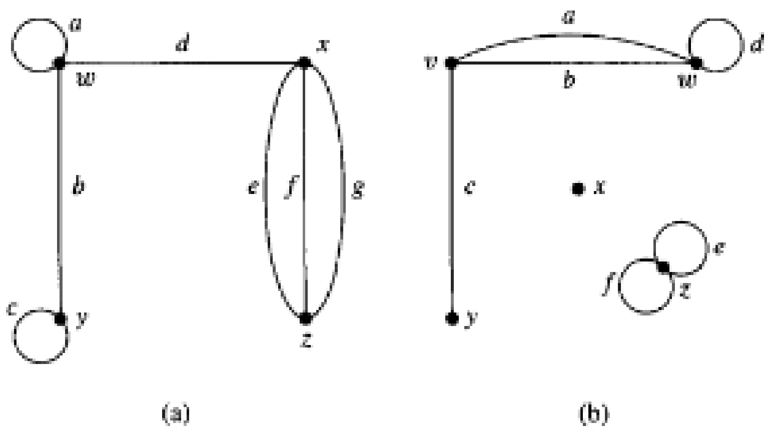


Figure 8 ▲

Solution.

$$A(a) = \{(y,y),(w,w),(y,w),(w,y),(w,x),(x,w),(x,z),(z,x)\}$$
$$R(a) = \{(y,y),(w,w),(x,x),(z,z),(y,w),(y,x),(y,z),(w,y),(w,x),(w,z)$$
$$(x,w),(x,y),(x,z),(z,x),(z,w),(z,y)\}$$

$$A(b) = \{(y,v)(v,w),(w,w),(w,v),(v,y),(z,z)\}$$
$$R(b) = \{(y,y),(v,v),(w,w),(y,v),(y,w),(v,y),(v,w),(w,v),(w,y),(z,z)\}$$

□

Problem 16. *For the graph in Figure 8(a), give an example of each of the following. Be sure to specify the edge sequence and the vertex sequence.*

(a) a path of length 2 from w to z.

(b) a path of length 4 from z to itself.

(c) a path of length 5 from z to itself.

(d) a path of length 3 from w to x.

Solution.

(a)	$V(a) = (w,z),(x,z)$	$E(a) = d, f$
(b)	$V(b) = (z,x),(x,z),(z,x),(x,z)$	$E(b) = g, e, f, g$
(c)	$V(c) = (z,x),(x,w),(w,w),(w,x),(x,z)$	$E(c) = f, d, a, d, f$
(d)	$V(d) = (w,w),(w,w),(w,x)$	$E(d) = a, a, d$

□

Section 3.3

Problem 15. *Give the matrices for the digraphs in Figure 3.*

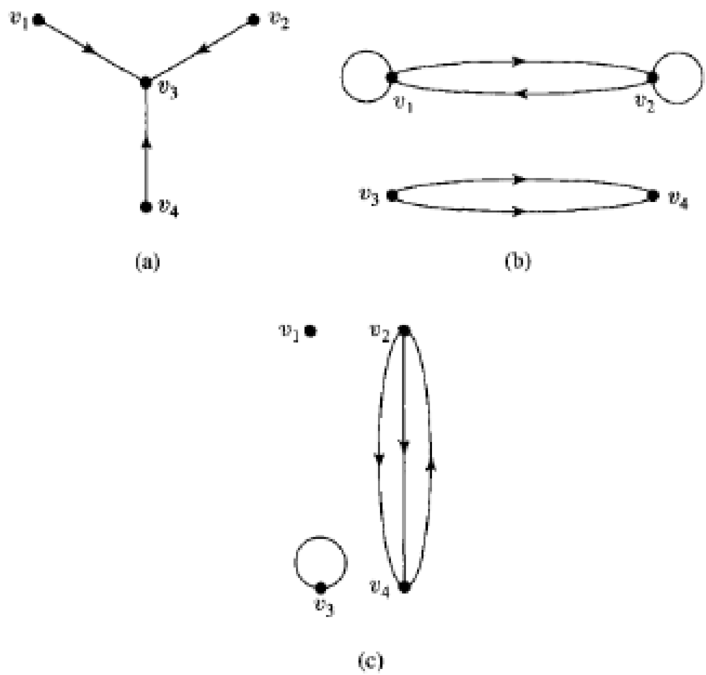


Figure 3 ▲

Solution.

$$(a) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} (b) = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} (c) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

□

Problem 16. *Write matrices for the graphs in Figure 4.*

(c) *(Figure 4(c))*

(d) *(Figure 4(d))*

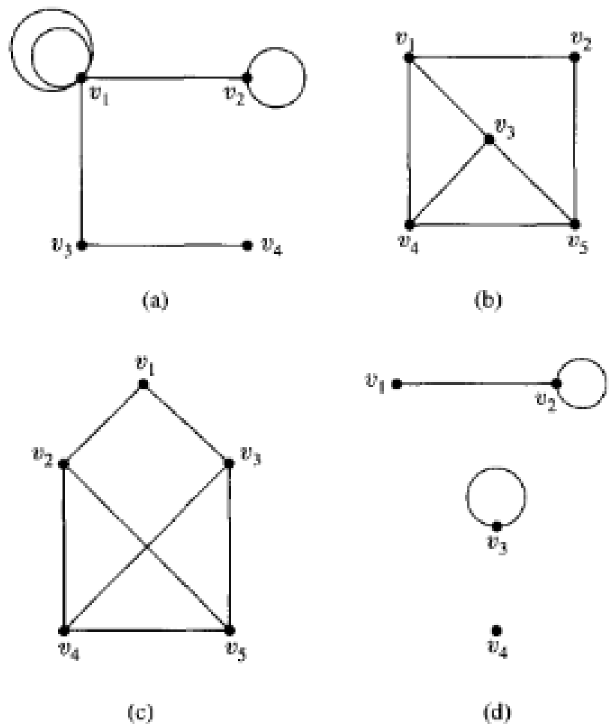


Figure 4 ▲

Solution.

$$(c) = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix} \quad (d) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

□

Problem 17. *For each matrix in Figure 5, draw a digraph having the matrix.*

(b) *(Matrix from Figure 5(b))*

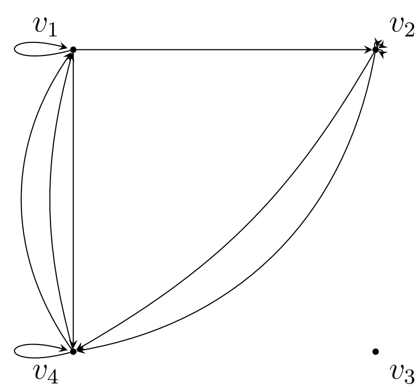
(c) *(Matrix from Figure 5(c))*

$$\begin{bmatrix} 0 & 0 & 2 & 1 \\ 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 3 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

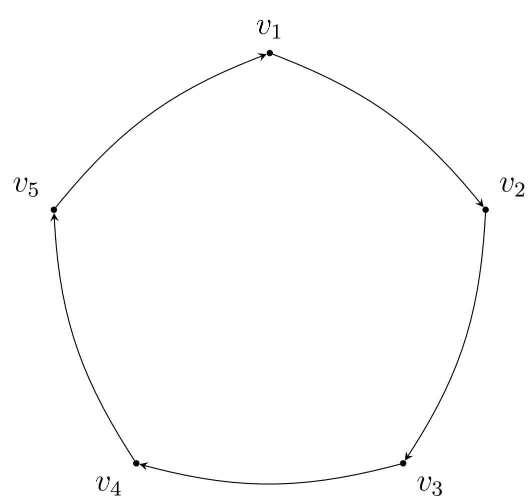
(a) (b) (c)

Figure 5 ▲

Solution. (b)



(c)



□

Problem 18. *For each matrix in Figure 6, draw a graph having the matrix.*

(c) (Matrix from Figure 6(c)

(d) (Matrix from Figure 6(d))

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 \end{bmatrix}$$

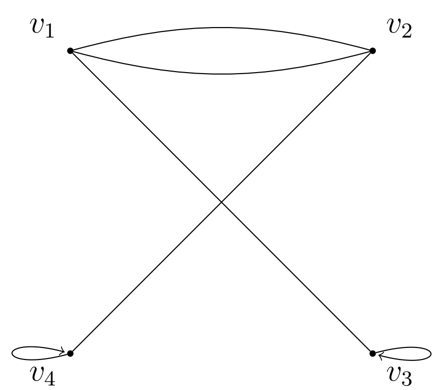
(a) (b)

$$\begin{bmatrix} 0 & 2 & 1 & 0 \\ 2 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

(c) (d)

Figure 6 ▲

Solution. (c)



(d) (Trust me, there are two loops for v1 and v2, just tiny)

□