Section 1.1

Problem 4. *Give the value of each of the following.*

(a) $\lceil 0.763 \rceil$

(b) $2\lceil 0.6 \rceil - \lceil 1.2 \rceil$

(c) $\lceil 1.1 \rceil + \lceil 3.3 \rceil$

(d) $\lfloor \sqrt{3} \rfloor - \lfloor \sqrt{3} \rfloor$

(e) $\lceil -73 \rceil - \lfloor -73 \rfloor$

Solution.

(a) $\lceil 0.763 \rceil = (1) = 1$

(b) $2\lceil 0.6 \rceil - \lceil 1.2 \rceil = 2(1) - (2) = 0$

(c) $\lceil 1.1 \rceil + \lceil 3.3 \rceil = (2) + (4) = 6$

(d) $\lfloor \sqrt{3} \rfloor - \lfloor \sqrt{3} \rfloor = (2) - (1) = 0$

(e) $\lceil -73 \rceil - \lfloor -73 \rfloor = (-73) - (-73) = 0$

$\square$

Problem 8. *How many multiples of 10 are there between the following pairs of numbers?*

(a) 1 and 80

(b) 0 and 100

(c) 9 and 2967

(d) -6 *and 34*

(e) 10^4 *and* 10^5

(f) -600 *and 3400*

Solution. (Assuming inclusive)

(a) 8

(b) $\lfloor \frac{n}{k} \rfloor - \lfloor \frac{m-1}{k} \rfloor = \lfloor \frac{100}{10} \rfloor - \lfloor \frac{-1}{10} \rfloor = 11$

(c) $\lfloor \frac{2967}{10} \rfloor - \lfloor \frac{8}{10} \rfloor = 296$

(d) $\lfloor \frac{34}{10} \rfloor - \lfloor \frac{-7}{10} \rfloor = 4$

(e) $\lfloor \frac{10{,}000}{10} \rfloor - \lfloor \frac{99{,}999}{10} \rfloor = 9001$

(f) $\lfloor \frac{3400}{10} \rfloor - \lfloor \frac{-600}{10} \rfloor = 401$

$\square$

tw+

Problem 18. *(a) What does Fact 4 say for $k = 1$? Is this statement obvious?*

(b) What does Fact 4 say for $k > n$? Is this statement obvious?

Solution.

(a) That $\lfloor n/1 \rfloor = n$ for all $n \in \mathbb{Z}$. Obvious, every positive integer is a multiple of 1 and everyone naturally knows there are n numbers between 1 and n.

(b) For $k > n$, $\lfloor n/k \rfloor = 0$, Obvious, follows from the fact that if $k > n$ then $0 \leq n/k < 1$, the floor of this range is 0 always.

$\square$

Problem 19. *(a) Give a specific example of numbers x and y for which $\lfloor x \rfloor + \lfloor y \rfloor < \lfloor x + y \rfloor$.*

(b) Give a specific example of numbers x and y for which $\lfloor x \rfloor + \lfloor y \rfloor = \lfloor x + y \rfloor$.

(c) Give a convincing argument that $\lfloor x \rfloor + \lfloor y \rfloor \leq \lfloor x + y \rfloor$ for every pair of numbers x and y.

Suggestion: Use the fact that $\lfloor x+y \rfloor$ is the largest integer less than or equal to $x+y$.

Solution.

(a) Let $x = y = 1.5$, then $\lfloor x \rfloor + \lfloor y \rfloor < \lfloor x+y \rfloor = \lfloor 1.5 \rfloor + \lfloor 1.5 \rfloor < \lfloor 1.5+1.5 \rfloor = 2 < 3$.

(b) Let $x = y = 1$, then $\lfloor x \rfloor + \lfloor y \rfloor = \lfloor x + y \rfloor = \lfloor 1 \rfloor + \lfloor 1 \rfloor = \lfloor 1 + 1 \rfloor = 2$.

(c) Proof: Let $x, y \in \mathbb{R}$. From the definition of the floor function we have $\lfloor x \rfloor \leq x$ and $\lfloor y \rfloor \leq y$. Adding these two inequalities together gives

$$\lfloor x \rfloor + \lfloor y \rfloor \leq x + y.$$

Note that $\lfloor x \rfloor + \lfloor y \rfloor$ is an integer (since it's the sum of two integers). Given that $\lfloor x + y \rfloor$ is the largest integer less than or equal to $x + y$, we have

$$\lfloor x + y \rfloor \leq x + y.$$

Since $\lfloor x \rfloor + \lfloor y \rfloor$ is an integer satisfying $\lfloor x \rfloor + \lfloor y \rfloor \leq x + y$, and $\lfloor x + y \rfloor$ is the *largest* such integer, we must have

$$\lfloor x \rfloor + \lfloor y \rfloor \leq \lfloor x + y \rfloor.$$

$\square$

Section 1.2

Problem 2. *True or False. Explain briefly.*

(a) $n \mid 1$ *for all positive integers* n.

(b) $n \mid n$ *for all positive integers* n.

(c) $n \mid n^2$ *for all positive integers* n.

Solution.

(a) **False.** For the statement $n \mid 1$ (equivalently $1 = nk$) to hold, there would need to be an integer k such that $1 = nk$. By the definition of divisibility, k must be an integer. For $n = 2$, we would need $k = 1/2$, which is not an integer. In fact, only $n = 1$ satisfies this (with $k = 1$). Therefore, the statement is false for all positive integers $n > 1$.

(b) **True.** $n \mid n$ is true for all positive integers n. This is equivalent to $n = nk$ for some integer k. Taking $k = 1$ gives $n = n \cdot 1$, which is always true.

(c) **True.** $n \mid n^2$ for all positive integers n is true. Observe that if we let $k = n$ in the equation $n^2 = nk$, we have $n^2 = n \cdot n$, which is always true.

□

Problem 14. *Suppose that* m *and* n *are integers that are multiples of* d*, say* $m = ad$ *and* $n = bd$.

(a) Explain why $d \mid lm$ *for every integer* l.

(b) Show that $m + n$ *and* $m - n$ *are multiples of* d.

(c) Must d *divide* $17m - 72n$*? Explain.*

Solution.

(a) Since $m = ad$, that means $lm = l(ad) = (la)d$. By definition of divisiblity this means $d \mid lm$.

(b) Since $m = ad$ and $n = bd$, it follows that $m + n = ad + bd = (a + b)d$. By definition of divisibility this means $d \mid m + n$. Similarly $m - n = ad - bd = (a - b)d$ so $d \mid m - n$.

(c) Yes, because $17m - 72n = 17(ad) - 72(bd) = 17ad - 72bd = (17a - 72b)d$. This means $d \mid 17m - 72n$ by definition.

□

Problem 16. *(b) List the positive integers less than 36 that are relatively prime to 36.*

Solution. A positive integer n is relatively prime to 36 if and only if $\gcd(n, 36) = 1$ holds which means n shares no prime factors with 36. Here is our list of positive integers less than 36 that have this property:

$$1, 5, 7, 11, 13, 17, 19, 23, 25, 29, 31, 35$$

□

Section 1.3

Problem 2. *List the elements in the following sets.*

(a) $\{1/n : n = 1, 2, 3, 4\}$

(b) $\{n^2 - n : n = 0, 1, 2, 3, 4\}$

(c) $\{1/n^2 : n \in \mathbb{P},\ n \textit{ is even and } n < 11\}$

(d) $\{2 + (-1)^n : n \in \mathbb{N}\}$

Solution.

(a) $\{1/n : n = 1, 2, 3, 4\} = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}\}$

(b) $\{n^2 - n : n = 0, 1, 2, 3, 4\} = \{0, 1, 2, 6, 12\}$

(c) $\{1/n^2 : n \in \mathbb{P},\ n \text{ is even and } n < 11\} = \{\frac{1}{4}\}$ ($n = 2$ is only prime even)

(d) $\{2 + (-1)^n : n \in \mathbb{N}\} = \{1, 3\}$

□

Problem 6. *Repeat Exercise 4 for the following sets.*

(a) $\{n \in \mathbb{N} : n \mid 12\}$

(b) $\{n \in \mathbb{N} : n^2 + 1 = 0\}$

(c) $\{n \in \mathbb{Z} : \lfloor \frac{n}{3} \rfloor = 8\}$

(d) $\{n \in \mathbb{N} : \lceil \frac{n}{2} \rceil = 8\}$

Solution.

(a) $\{n \in \mathbb{N} : n \mid 12\} = \{1, 2, 3, 4, 6, 12\}$

(b) $\{n \in \mathbb{N} : n^2 + 1 = 0\} = \emptyset$

(c) $\{n \in \mathbb{Z} : \lfloor \frac{n}{3} \rfloor = 8\} = \{24, 25, 26\}$

(d) $\{n \in \mathbb{N} : \lceil \frac{n}{2} \rceil = 8\} = \{15, 16\}$

□

Problem 8. *How many elements are there in the following sets? Write ∞ if the set is infinite.*

(a) $\{n \in \mathbb{N} : n^2 = 2\}$

(b) $\{n \in \mathbb{Z} : 0 \leq n \leq 73\}$

(c) $\{n \in \mathbb{Z} : 5 \leq |n| \leq 73\}$

(d) $\{n \in \mathbb{Z} : 5 < n < 73\}$

(e) $\{n \in \mathbb{Z} : n \text{ is even and } |n| \leq 73\}$

(f) $\{x \in \mathbb{Q} : 0 \leq x \leq 73\}$

(g) $\{x \in \mathbb{Q} : x^2 = 2\}$

(h) $\{x \in \mathbb{R} : x^2 = 2\}$

Solution.

(a) Cardinality of 0

(b) Cardinality of 74

(c) Cardinality of 2(73 - 5) = 136

(d) Cardinality of 73-5 = 68

(e) Cardinality of $\lfloor \frac{73}{2} \rfloor - \lfloor \frac{-73}{2} \rfloor$ = 73

(f) Cardinality of ∞, specifically $\aleph_0$ (countably infinite).

(g) Cardinality of 0

(h) Cardinality of 2

□