

Section 1.4: Sets

Problem 2. Let $A = \{1, 2, 3\}$, $B = \{n \in \mathbb{P} : n \text{ is even}\}$, and $C = \{n \in \mathbb{P} : n \text{ is odd}\}$.

- (a) Determine $A \cap B$, $B \cap C$, $B \cup C$, and $B \oplus C$.
- (b) List all subsets of A .
- (c) Which of the following sets are infinite? $A \oplus B$, $A \oplus C$, $A \setminus C$, $C \setminus A$.

Solution.

- (a) $B = \{2, 4, 6, 8, \dots\}$ (even positive integers) and $C = \{1, 3, 5, 7, \dots\}$ (odd positive integers).

$$A \cap B = \{2\}$$

$$B \cap C = \emptyset$$

$$B \cup C = \mathbb{P}$$

$$B \oplus C = (B \cup C) \setminus (B \cap C) = \mathbb{P} \setminus \emptyset = \mathbb{P}$$

- (b) The subsets of A are:

$$\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}$$

(c) $A \oplus B = (A \cup B) \setminus (A \cap B) = \{1, 3, 4, 6, 8, \dots\}$ - infinite

$$A \oplus C = (A \cup C) \setminus (A \cap C) = \{5, 7, 9, 11, \dots\}$$
 - infinite

$$A \setminus C = \{2\}$$
 - finite

$$C \setminus A = \{5, 7, 9, 11, \dots\}$$
 - infinite

□

Problem 6. The following statements about sets are false. For each statement, give an example, i.e., a choice of sets, for which the statement is false. Such examples are called counterexamples. They are examples that are counter to, i.e., contrary to, the assertion.

- (a) $A \cup B \subseteq A \cap B$ for all A, B .
- (b) $A \cap \emptyset = A$ for all A .
- (c) $A \cap (B \cup C) = (A \cap B) \cup C$ for all A, B, C .

Solution.

(a) **Counterexample:** Let $A = \{1\}$ and $B = \{2\}$.

Then $A \cup B = \{1, 2\}$ and $A \cap B = \emptyset$.

But $\{1, 2\} \not\subseteq \emptyset$, so the statement is false.

(b) **Counterexample:** Let $A = \{1, 2, 3\}$.

Then $A \cap \emptyset = \emptyset \neq \{1, 2, 3\} = A$, so the statement is false.

(In fact, $A \cap \emptyset = \emptyset$ for all sets A .)

(c) **Counterexample:** Let $A = \{1\}$, $B = \{2\}$, $C = \{3\}$.

Then $A \cap (B \cup C) = \{1\} \cap \{2, 3\} = \emptyset$.

But $(A \cap B) \cup C = \emptyset \cup \{3\} = \{3\}$.

Since $\emptyset \neq \{3\}$, the statement is false.

□

Problem 8. For the sets $A = \{1, 3, 5, 7, 9, 11\}$ and $B = \{2, 3, 5, 7, 11\}$, determine the following numbers.

(a) $|A|$

(b) $|B|$

(c) $|A \cup B|$

(d) $|A| + |B| - |A \cap B|$

(e) Do you see a general reason why the answers to (c) and (d) have to be the same?

Solution.

(a) $|A| = 6$ (counting: 1, 3, 5, 7, 9, 11)

(b) $|B| = 5$ (counting: 2, 3, 5, 7, 11)

(c) $A \cap B = \{3, 5, 7, 11\}$, so $|A \cap B| = 4$

$A \cup B = \{1, 2, 3, 5, 7, 9, 11\}$, so $|A \cup B| = 7$

(d) $|A| + |B| - |A \cap B| = 6 + 5 - 4 = 7$

(e) When we count $|A| + |B|$, we count every element in $A \cap B$ twice (once for being in A , once for being in B). Subtracting by the intersection of A and B removes the copy elements.

□

Problem 10. (a) Show that relative complementation is not commutative; that is, the equality $A \setminus B = B \setminus A$ can fail.

(b) Show that relative complementation is not associative: $(A \setminus B) \setminus C = A \setminus (B \setminus C)$ can fail.

Solution.

(a) **Counterexample:** Let $A = \{1, 2\}$ and $B = \{2, 3\}$.

Then $A \setminus B = \{1\}$ (elements in A but not in B).

And $B \setminus A = \{3\}$ (elements in B but not in A).

Since $\{1\} \neq \{3\}$, we have $A \setminus B \neq B \setminus A$.

Therefore, relative complementation is not commutative.

(b) **Counterexample:** Let $A = \{1, 2, 3\}$, $B = \{2\}$, $C = \{3\}$.

$$(A \setminus B) \setminus C = (\{1, 2, 3\} \setminus \{2\}) \setminus \{3\} = \{1, 3\} \setminus \{3\} = \{1\}$$

$$A \setminus (B \setminus C) = \{1, 2, 3\} \setminus (\{2\} \setminus \{3\}) = \{1, 2, 3\} \setminus \{2\} = \{1, 3\}$$

Since $\{1\} \neq \{1, 3\}$, we have $(A \setminus B) \setminus C \neq A \setminus (B \setminus C)$.

Therefore, relative complementation is not associative.

□

Problem 12. Let $S = \{0, 1, 2, 3, 4\}$ and $T = \{0, 2, 4\}$.

(a) How many ordered pairs are in $S \times T$? $T \times S$?

(c) List or draw the elements in the set $\{(m, n) \in T \times S : m < n\}$.

(e) List or draw the elements in the set $\{(m, n) \in T \times S : mn > 4\}$.

Solution.

(a) $|S \times T| = |S| \cdot |T| = 5 \cdot 3 = 15$ ordered pairs

$|T \times S| = |T| \cdot |S| = 3 \cdot 5 = 15$ ordered pairs

(c) $\{(m, n) \in T \times S : m < n\}$ where $T = \{0, 2, 4\}$ and $S = \{0, 1, 2, 3, 4\}$:

For $m = 0$: $(0, 1), (0, 2), (0, 3), (0, 4)$

For $m = 2$: $(2, 3), (2, 4)$

For $m = 4$: none (no element in S is greater than 4)

Answer: $\{(0, 1), (0, 2), (0, 3), (0, 4), (2, 3), (2, 4)\}$

(e) $\{(m, n) \in T \times S : mn > 4\}$:

For $m = 0$: $0 \cdot n = 0 \not> 4$ for all n

For $m = 2$: $2 \cdot 3 = 6 > 4$ and $2 \cdot 4 = 8 > 4$, so $(2, 3), (2, 4)$

For $m = 4$: $4 \cdot 2 = 8 > 4$, $4 \cdot 3 = 12 > 4$, $4 \cdot 4 = 16 > 4$, so $(4, 2), (4, 3), (4, 4)$

Answer: $\{(2, 3), (2, 4), (4, 2), (4, 3), (4, 4)\}$

□

Section 1.5: Functions

Problem 2. Consider the function $h : \mathbb{P} \rightarrow \mathbb{P}$ defined by $h(n) = |\{k \in \mathbb{N} : k \mid n\}|$ for $n \in \mathbb{P}$. In words, $h(n)$ is the number of divisors of n . Calculate $h(n)$ for $1 \leq n \leq 10$ and for $n = 73$.

(Note: In this textbook, $\mathbb{P} = \mathbb{Z}^+ = \{1, 2, 3, 4, \dots\}$)

Solution. We count the divisors of each number:

$h(1) = 1$ (divisors: 1)
 $h(2) = 2$ (divisors: 1, 2)
 $h(3) = 2$ (divisors: 1, 3)
 $h(4) = 3$ (divisors: 1, 2, 4)
 $h(5) = 2$ (divisors: 1, 5)
 $h(6) = 4$ (divisors: 1, 2, 3, 6)
 $h(7) = 2$ (divisors: 1, 7)
 $h(8) = 4$ (divisors: 1, 2, 4, 8)
 $h(9) = 3$ (divisors: 1, 3, 9)
 $h(10) = 4$ (divisors: 1, 2, 5, 10)
 $h(73) = 2$ (divisors: 1, 73, since 73 is prime)

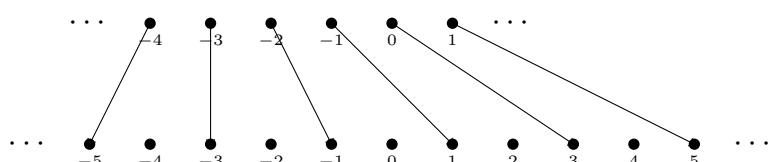
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Problem 3 (Section 1.5 Extra #1). Suppose that $f : \mathbb{Z} \rightarrow \mathbb{Z}$ by $n \mapsto \lfloor \frac{n}{2} \rfloor$ and $g : \mathbb{Z} \rightarrow \mathbb{Z}$ by $n \mapsto 3n$.

Draw portions of arrow diagrams representing each of the four functions f , g , $f \circ g$, and $g \circ f$.

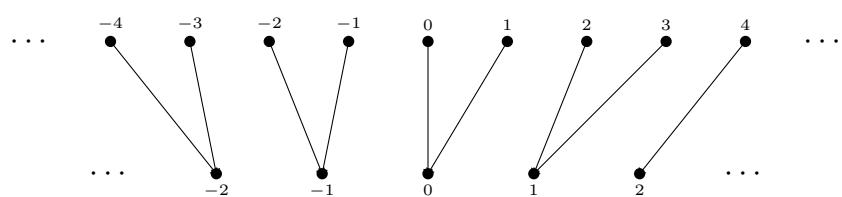
You need to make sure you include enough integers so it's clear what's going on.

Example: If $h : n \mapsto 2n + 3$, I can represent that with an arrow diagram as follows:

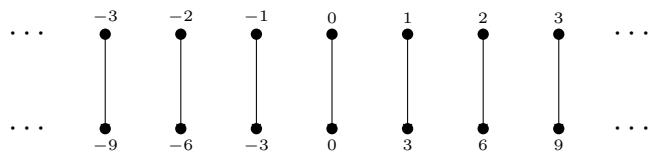


Solution.

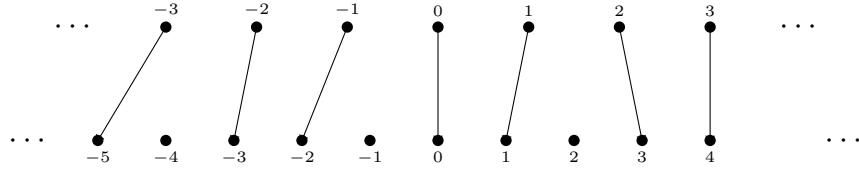
Function $f : \mathbb{Z} \rightarrow \mathbb{Z}$ by $n \mapsto |n/2|$:



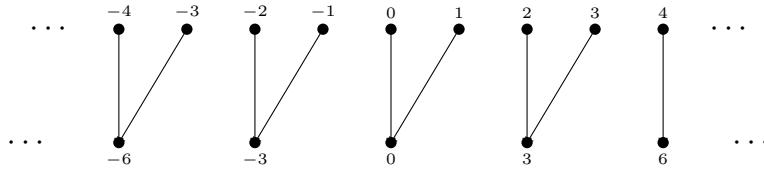
Function $g : \mathbb{Z} \rightarrow \mathbb{Z}$ by $n \mapsto 3n$:



Function $f \circ g : \mathbb{Z} \rightarrow \mathbb{Z}$ by $n \mapsto f(g(n)) = f(3n) = \lfloor 3n/2 \rfloor$:



Function $g \circ f : \mathbb{Z} \rightarrow \mathbb{Z}$ by $n \mapsto g(f(n)) = g(\lfloor n/2 \rfloor) = 3\lfloor n/2 \rfloor$:



□

Section 1.6: Sequences

Problem 2. Simplify:

$$(a) \frac{n!}{(n-1)!}$$

$$(b) \frac{(n!)^2}{(n+1)!(n-1)!}$$

Solution.

$$(a) \frac{n!}{(n-1)!} = \frac{n \cdot (n-1)!}{(n-1)!} = n$$

$$(b) \frac{(n!)^2}{(n+1)!(n-1)!} = \frac{(n!)^2}{(n+1) \cdot n! \cdot (n-1)!} = \frac{n! \cdot n!}{(n+1) \cdot n! \cdot (n-1)!} = \frac{n!}{(n+1)(n-1)!} \\ = \frac{n \cdot (n-1)!}{(n+1)(n-1)!} = \frac{n}{n+1}$$

□

Problem 4. Calculate:

$$(d) \prod_{n=1}^5 (2n + 1)$$

$$(e) \prod_{j=4}^8 (j - 1)$$

Solution.

$$(d) \prod_{n=1}^5 (2n + 1) = (2 \cdot 1 + 1)(2 \cdot 2 + 1)(2 \cdot 3 + 1)(2 \cdot 4 + 1)(2 \cdot 5 + 1) \\ = (3)(5)(7)(9)(11) = 10,395$$

$$(e) \prod_{j=4}^8 (j-1) = (4-1)(5-1)(6-1)(7-1)(8-1) \\ = (3)(4)(5)(6)(7) = 2,520$$

□

Problem 6. (a) Calculate $\sum_{k=0}^n 2^k$ for $n = 1, 2, 3, 4$, and 5.

(b) Use your answers to part (a) to guess a general formula for this sum.

Solution.

(a)

$$\begin{aligned} n = 1 : \quad & \sum_{k=0}^1 2^k = 2^0 + 2^1 = 1 + 2 = 3 \\ n = 2 : \quad & \sum_{k=0}^2 2^k = 2^0 + 2^1 + 2^2 = 1 + 2 + 4 = 7 \\ n = 3 : \quad & \sum_{k=0}^3 2^k = 1 + 2 + 4 + 8 = 15 \\ n = 4 : \quad & \sum_{k=0}^4 2^k = 1 + 2 + 4 + 8 + 16 = 31 \\ n = 5 : \quad & \sum_{k=0}^5 2^k = 1 + 2 + 4 + 8 + 16 + 32 = 63 \end{aligned}$$

(b) Looking at the pattern:

$$\begin{aligned} n = 1 : \quad & 3 = 4 - 1 = 2^2 - 1 = 2^{1+1} - 1 \\ n = 2 : \quad & 7 = 8 - 1 = 2^3 - 1 = 2^{2+1} - 1 \\ n = 3 : \quad & 15 = 16 - 1 = 2^4 - 1 = 2^{3+1} - 1 \\ n = 4 : \quad & 31 = 32 - 1 = 2^5 - 1 = 2^{4+1} - 1 \\ n = 5 : \quad & 63 = 64 - 1 = 2^6 - 1 = 2^{5+1} - 1 \end{aligned}$$

General formula: $\sum_{k=0}^n 2^k = 2^{n+1} - 1$

□

Problem 10. For $n = 1, 2, 3, \dots$, let $SSQ(n) = \sum_{i=1}^n i^2$.

(a) Calculate $SSQ(n)$ for $n = 1, 2, 3$, and 5.

(b) Observe that $SSQ(n+1) = SSQ(n) + (n+1)^2$ for $n \geq 1$.

(c) It turns out that $SSQ(73) = 132,349$. Use this to calculate $SSQ(74)$ and $SSQ(72)$.

Solution.

(a)

$$\text{SSQ}(1) = 1^2 = 1$$

$$\text{SSQ}(2) = 1^2 + 2^2 = 1 + 4 = 5$$

$$\text{SSQ}(3) = 1^2 + 2^2 + 3^2 = 1 + 4 + 9 = 14$$

$$\text{SSQ}(5) = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 1 + 4 + 9 + 16 + 25 = 55$$

(b) This follows from the definition:

$$\text{SSQ}(n+1) = \sum_{i=1}^{n+1} i^2 = \sum_{i=1}^n i^2 + (n+1)^2 = \text{SSQ}(n) + (n+1)^2$$

(c) Given $\text{SSQ}(73) = 132,349$:

$$\text{SSQ}(74) = \text{SSQ}(73) + 74^2 = 132,349 + 5,476 = 137,825$$

For $\text{SSQ}(72)$, we use: $\text{SSQ}(73) = \text{SSQ}(72) + 73^2$

$$\text{So } \text{SSQ}(72) = \text{SSQ}(73) - 73^2 = 132,349 - 5,329 = 127,020$$

□

Problem 11 (Section 1.6 Extra #1). Go to the Online Encyclopedia of Integer Sequences, <http://oeis.org>, and find an interesting sequence. Write down the sequence, explain a little about the definition of the sequence, and say why you think it is interesting.

Solution.

Sequence: The Kolakoski Sequence (OEIS A000002)

1, 2, 2, 1, 1, 2, 1, 2, 2, 1, 2, 2, 1, 1, 2, 1, 1, 2, 2, 1, 2, 1, 1, 2, 1, 2, 2, 1, ...

Definition:

The Kolakoski sequence is the unique sequence of 1's and 2's that is its own run-length encoding. That is, if you read the sequence and count the lengths of consecutive runs of the same symbol, you get the sequence itself back.

More precisely:

- The sequence starts with 1, 2, 2
- The first element (1) tells us the first run has length 1, so we have one 1
- The second element (2) tells us the second run has length 2, so we have two 2's: 1, 2, 2
- The third element (2) tells us the third run has length 2, so we have two 1's: 1, 2, 2, 1, 1

- The fourth element (1) tells us the fourth run has length 1, so we have one 2:
1, 2, 2, 1, 1, 2
- And so on...

On top of the sequence itself being interesting, the discoverer William Kolakoski himself is also an interesting figure, an artist mathematician plagued by schizophrenia. Although the sequence was named after him, it was actually discovered in 1939 by Rufus Oldenburger though it wasn't noticed at the time. It is also the second sequence in the OEIS database. \square