Chapter 6

Exercise (2). Suppose $n \in \mathbb{Z}$. If n^2 is odd, then n is odd.

Proof: Suppose for the sake of contradiction that n^2 is odd and n is not odd, Then n is even, so $n = 2k$ for some $k \in \mathbb{Z}$. Therefore $n^2 = (2k)^2 = 4k^2 = 2(2k^2) = 2b$, where $b \in \mathbb{Z}$ by closure properties of the integers. So n^2 is even, this is a contradiction. So it must be the case that if n^2 is odd then n is odd. $\square$

Exercise (3). Prove that $\sqrt[3]{2}$ is irrational.

Proof: Suppose for the sake of contradiction that $\sqrt[3]{2}$ is not irrational. Then $\sqrt[3]{2}$ is a rational and is in the form $\sqrt[3]{2} = \frac{a}{b}$ where $a, b \in \mathbb{Z}$ and the ratio a, b do not sure factors so that $\frac{a}{b}$ is in its lowest form. Observe that when cubing both sides, $2 = (\frac{a}{b})^3 = \frac{a^3}{b^3}$ so that $2b^3 = a^3$. This implies that a is an even number and divisible by 2. So $a = 2k$ for some $k \in \mathbb{Z}$. Substituting for a gives $2b^3 = (2k)^3 = 8k^3$. Thus $b^3 = 4k^3 = 2(2k^3)$. Because b^3 is even, this implies that b is also an even number which is contraditory to $\frac{a}{b}$ existing in its lowest forms. Thus $\sqrt[3]{2}$ must be irrational. $\square$

Exercise (4). Prove that $\sqrt{6}$ is irrational.

Proof: Suppose for the sake of contradiction $\sqrt{6}$ is a rational number. Then there exists an $a, b \in \mathbb{Z}$ such that $\sqrt{6} = \frac{a}{b}$ and $\frac{a}{b}$ is in its lowest form with no common factors. Squaring both sides gives $6 = (\frac{a}{b})^2 = \frac{a^2}{b^2}$. Thus $6b^2 = a^2$ which implies that a^2 is divisible by 6 and thus a is divisible by 6 or $a = 6k$ for some $k \in \mathbb{Z}$. Substituting in our original equation gives $6b^2 = (6k)^2 = 36k^2$. Simplifying this equation by division of 6 shows that $b^2 = 6k^2$, this implies that b^2 is divisible by 6. Since it follows that b is also divisible by 6 and that a is divisible by 6 we have a contradiction as $\frac{a}{b}$ cannot be in its lowest form. Thus $\sqrt{6}$ is an irrational number. $\square$

Exercise (8). Suppose $a, b, c \in \mathbb{Z}$. If $a^2 + b^2 = c^2$, then a or b is even.

Proof: For the sake of contradiction, suppose that $a, b, c \in \mathbb{Z}$ such that $a^2 = b^2 = c^2$ and a and b are odd. Then there exists an $k, j \in \mathbb{Z}$ such that $a = 2k + 1$ and $b = 2j + 1$. We know that an odd number added to another odd number is an even number, so that c^2 is even and thus c is even so that, $c = 2x$ for some $x \in \mathbb{Z}$. Observe that when we substitute a, b, c on both sides of the equations, $a^2 + b^2 = (2k+1)^2 + (2j+1)^2 = 4k^2 + 4k + 1 + 4j^2 + 4j + 1 = 4k^2 + 4j^2 + 4k + 4j + 2 = 2(2k^2 + 2j^2 + 2k + 2j + 1)$ and $c^2 = (2x)^2 = 4x^2$, so that $2(k^2 + 2j^2 + 2k + 2j + 1) = 4x^2$. Dividing both sides by 2 gives $2k^2 + 2j^2 + 2k + 2j + 1 = 2x^2$. Note that the left hand side is odd and the right hand side is even. This is an impossibility since an odd number cannot equal an even number. Thus if $a^2 + b^2 = c^2$, then a or b must be even. □

Exercise (9). Suppose $a, b \in \mathbb{R}$. If a is rational and ab is irrational, then b is irrational.

Proof: Write your answer here. □

Exercise (11). There exist no integers a and b for which $18a + 6b = 1$.

Proof: Write your answer here. □

Exercise (12). For every positive $x \in \mathbb{Q}$, there is a positive $y \in \mathbb{Q}$ for which $y < x$.

Proof: Write your answer here. □

Exercise (16). If a and b are positive real numbers, then $a + b \geq 2\sqrt{ab}$.

Proof: Write your answer here. □

Exercise (19). The product of any five consecutive integers is divisible by 120. (For example, the product of 3, 4, 5, 6 and 7 is 2520, and $2520 = 120 \cdot 21$.)

Proof: Write your answer here. □

Chapter 7

Exercise (1). Suppose $x \in \mathbb{Z}$. Then x is even if and only if $3x + 5$ is odd.

Proof: Write your answer here. □

Exercise (4). Let a be an integer. Then $a^2 + 4a + 5$ is odd if an d only if a is even.

Proof: Write your answer here. □

Exercise (7). Suppose $x, y \in \mathbb{R}$. Then $(x+y)^2 = x^2 + y^2$ if and only if $x = 0$ or $y = 0$.

Proof: Write your answer here. □

Exercise (Reflection Problem). *Proof:* Write your answer here. □

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