Chapter 4

Exercise (2). If x is an odd integer, then x^3 is odd.

Proof. Suppose x is an odd integer. Then by definition of an odd integer, $x = 2k+1$ for some $k \in \mathbb{Z}$. Therefore $x^3 = (2k+1)(2k+1)(2k+1) = (4k^2+4k+1)(2k+1) = 8k^3+12k^2+6k+1 = 2(4k^3 + 6k^2 + 3k) + 1 = 2n + 1$, where n $= (4k^3 + 6k^2 + 3k)$. Note that n is an integer due to the closure properties under addition and multiplication in the integers. So $x^3 = 2n + 1$, where n is an integer. Thus x^3 is odd by definition of an odd number. □

Exercise (4). Suppose $x, y \in \mathbb{Z}$. If x and y are odd, then xy is odd.

Proof. Suppose x and y are odd integers. Then $x = 2m + 1$ for some $m \in \mathbb{Z}$ and $y = 2n + 1$ for some $n \in \mathbb{Z}$ by definition of odd. Therefore $xy = (2m+1)(2n+1) = 4mn+2m+2n+1 = 2(2mn + m + n) + 1 = 2p + 1$, where $p = (2mn + m + n)$. Note that p is an integer due to the closure properties under addition and multiplication in the integers. So $xy = 2p + 1$, where p is an integer. Thus xy is odd by definition of an odd number. □

Exercise (6). Suppose $a, b, c \in \mathbb{Z}$. If $a \mid b$ and $a \mid c$, then $a \mid (b + c)$.

Proof. Suppose $a \mid b$ and $a \mid c$ and $a, b, c \in \mathbb{Z}$. By definition of divisibility, we know that $a \mid b$ means $b = ak$ for some $k \in \mathbb{Z}$. Likewise $a \mid c$ means $c = al$ for some $l \in \mathbb{Z}$. Therefore $(b + c) = ak + al = a(k + l) = am$, where $m = k + l$. Note that m is an integer due to the closure properties under addition and multiplication in the integers. So $(b + c) = am$ where m is an integer. Thus $a \mid (b + c)$ by definition of divisibility. □

Exercise (11). Suppose $a, b, c, d \in \mathbb{Z}$. If $a \mid b$ and $c \mid d$, then $ac \mid bd$.

Proof. Suppose $a \mid b$ and $c \mid d$ and $a, b, c \in \mathbb{Z}$. By definition of divisibility, we know that $a \mid b$ means $b = ak$ for some $k \in \mathbb{Z}$. Likewise we know that $c \mid d$ means $d = cl$ for some $l \in \mathbb{Z}$. Thus $bd = akcl = ac(kl) = acn$, where $n = kl$. Note that n is an integer due to the closure properties under multiplication in the integers. So $bd = acn$ where $n \in \mathbb{Z}$. Thus $ac \mid bd$ by definition of divisibility. □

Exercise (12). If $x \in \mathbb{R}$ and $0 < x < 4$, then $\frac{4}{x(4-x)} \geq 1$.

Proof. Suppose $x \in \mathbb{R}$ and $0 < x < 4$, we know that any real number squared is greater than or equal to 0. Let us choose a real number $(x-2)$ in the interval $0 < x < 4$. Therefore $(x-2)^2 \geq 0$ is equivalent to $x^2 - 4x + 4 \geq 0$ which can be rewiretten as $4 \geq x(4-x)$. Dividing both sides by $x(4-x)$ we obtain $\frac{4}{x(4-x)} \geq 1$. Thus the statement holds. $\square$