4.4.2

a) We are provided the second order differential equation $y'' - 9y' = 3x^2 - 5\sin(3x)$ and we are tasked with finding the form of the particular solution, by first inspection/observation we find the form to be:

$$\boxed{y_p = (Ax^2 + Bx + C) + D\cos(3x) + E\sin(3x)} \tag{1}$$

b) We are also provided with the second order differential equation $y'' + 2y' + y = 2e^{-x} - e^x$, by inspection/observation we conclude that the form of the particular solution is:

$$\boxed{y_p = Ae^{-x} + Be^x}$$

4.4.3

We are tasked with solving the following differential equations via method of undetermined coefficients:

a) $y'' + 2y' = 2x + 5 - e^{-2x}$

$$\begin{aligned}
&y'' + 2y' = 2x + 5 - e^{-2x} \\
&m^2 + 2m + 0 = 0 \\
&m(m+2) = 0 \\
&m = 0, -2 \\
&y_c = c_1 + c_2 e^{-2x}
\end{aligned}$$

$$\begin{aligned}
y_p &= Ax^2 + Bx + Cxe^{-2x} \\
y_p' &= 2Ax + B - 2Cxe^{-2x} + Ce^{-2x} \\
y_p'' &= 2A + 4Cxe^{-2x} - 4Ce^{-2x}
\end{aligned}$$

$$\begin{aligned}
2A + 4Cxe^{-2x} - 4Ce^{-2x} + 4Ax + 2B - 4Cxe^{-2x} + 2Ce^{-2x} &= 2x + 5 - e^{-2x} \\
4Ax + 2(A+B) + 2Ce^{-2x} &= 2x + 5 - e^{-2x} \\
A = \frac{1}{2}, B = 2, C &= \frac{1}{2}
\end{aligned}$$

Our solution is:

$$y_c + y_p = \boxed{c_1 + c_2 e^{-2x} + \frac{1}{2}x^2 + 2x + \frac{1}{2}xe^{-2x}}$$

b) $y'' - 9y' = 2e^{3x}$

$$\begin{aligned}
&y'' - 9y' = 2e^{3x} \\
&m^2 - 9m = 0 \\
&m(m-9) = 0 \\
&m = 9, 0 \\
&y_c = c_1 + c_2 e^{9x}
\end{aligned}$$

$$\begin{aligned}
&y_p = Ae^{3x} \\
&y_p' = 3Ae^{3x} \\
&y_p'' = 9Ae^{3x} \\
&9Ae^{3x} - 27Ae^{3x} = 2e^{3x} \\
&-18Ae^{3x} = 2e^{3x} \\
&A = -\frac{1}{9}
\end{aligned}$$

Our solution is then:

$$y_c + y_p = \boxed{c_1 + c_2 e^{9x} - \frac{1}{9} e^{3x}}$$

c) $y'' + 4y' + 4y = (3 + x)e^{-2x}$

$$y'' + 4y' + 4y = (3 + x)e^{-2x} \qquad\qquad y_p = Ax^2 e^{-2x} + Bx^3 e^{-2x}$$

$$m^2 + 4m + 4 = 0$$

$$(m + 2)(m + 2) = 0$$

$$m = -2$$

$$y_c = c_1 e^{-2x} + c_2 x e^{-2x}$$