2.4 Exact equations

2.4.1 problem a)

Consider the differential equation: $(\sin(y) - y\sin(x))dx + (\cos(x) + x\cos(y) - 3y)dy = 0$, we solve via the method of exact equations:

$$M(x,y) = (\sin(y) - y\sin(x)) \qquad \frac{\delta M}{\delta y} = \cos(y) - \sin(x)$$

$$N(x,y) = (\cos(x) + x\cos(y) - 3y) \qquad \frac{\delta N}{\delta y} = \cos(y) - \sin(x)$$

Now we solve:

$$f(x,y) = \int (\sin(y) - y\sin(x))dx \qquad = x\sin(y) + y\cos(x) + g(y) \tag{1}$$

$$\frac{\delta f}{\delta y} = x\cos(y) + \cos(x) + g'(y) = \cos(x) + x\cos(y) - 3y \qquad \to g'(y) = -3y \tag{2}$$

$$\int g'(y)dy = \int -3ydy \tag{3}$$

$$g(y) = -\frac{3}{2}y^2 \tag{4}$$

Thus our solution is:

$$\boxed{x\sin(y) + y\cos(x) - \frac{3}{2}y^2 = C} \tag{5}$$

2.4.1 problem b)

We are given the differential equation $(y\ln(y) + e^{xy})dx + (\frac{1}{y} + x\ln(y))dy = 0$ first we determine if it is exact:

$$M(x,y) = y\ln(y) + e^{xy} \qquad \frac{\delta M}{\delta y} = \ln(y) + 1 + xe^{xy}$$

$$N(x,y) = \frac{1}{y} + x\ln(y) \qquad \frac{\delta N}{\delta y} = \ln(y)$$

Thus we can conclude:

$$\boxed{\text{The Equation is not exact}}$$

2.4 problem 2

We are given the equation $y(x+y+1)dx+(x+2y)dy=0$, we first determine if it is exact or not.

2.4.2 a)

$$M(x,y) = y(x+y+1) = yx + y^2 + y \qquad \frac{\delta M}{\delta y} = x + 2y + 1$$

$$N(x,y) = x + 2y \qquad \frac{\delta N}{\delta y} = 1$$

$$\frac{\delta M}{\delta y} \neq \frac{\delta N}{\delta y}$$
$$\boxed{\text{Not Exact}}$$

For this part we add $\mu(x) = e^x$ as our integrating factor to get the equation $(yxe^x+y^2e^x+ye^x)dx+(xe^x+eye^x)dy=0$:

2.4.2 b)

$$M(x,y) = yxe^x + y^2e^x + ye^x \qquad \frac{\delta M}{\delta y} = xe^x + 2ye^x + e^x$$

$$N(x,y) = xe^x + 2ye^x \qquad \frac{\delta N}{\delta y} = e^x + xe^x + 2ye^x$$

$$\frac{\delta M}{\delta y} = \frac{\delta N}{\delta y}$$
$$\boxed{\text{Exact Equation}}$$

Now we find the solution:

2.4.2 c)

$$f(x,y) = \int (yxe^x + y^2e^x + ye^x) \qquad = e^xy^2 + e^xy + g(y) \tag{1}$$

$$\frac{\delta f}{\delta y} = 2e^xy + xe^x = xe^x + 2ye^x \qquad g(y) = 0 \tag{2}$$

$$\boxed{e^xy^2 + e^xxy = C} \tag{3}$$