

## CHAPTER 10

Prove the following statements with either induction, strong induction or proof by smallest counterexample.

*Exercise (3).* Prove that  $1^3 + 2^3 + 3^3 + 4^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$  for every positive integer  $n$ .

*Proof:* (Weak Induction)

Base Case: Observe that when  $n = 1$  that  $n^3 = (1)^3 = \frac{(1)^2((1)+1)^2}{4} = \frac{4}{4} = 1$  which is true.

Induction Hypothesis: Suppose there is a  $k \in \mathbb{Z}$  such that  $1^3 + 2^3 + 3^3 + 4^3 + \cdots + k^3 = \frac{k^2(k+1)^2}{4}$ .

Inductive Step: We wish to show that the statement holds for  $n = k + 1$ , i.e., that  $1^3 + 2^3 + 3^3 + 4^3 + \cdots + k^3 + (k + 1)^3 = \frac{(k+1)^2((k+1)+1)^2}{4}$ . Observe the following:

$$\begin{aligned}
 1^3 + 2^3 + 3^3 + 4^3 + \cdots + k^3 + (k + 1)^3 &= [1^3 + 2^3 + 3^3 + 4^3 + \cdots + k^3] + (k + 1)^3 \\
 &= \frac{k^2(k + 1)^2}{4} + (k + 1)^3 \\
 &= \frac{k^2(k + 1)^2}{4} + \frac{4(k + 1)^3}{4} \\
 &= \frac{k^2(k + 1)^2 + 4(k + 1)^3}{4} \\
 &= \frac{(k + 1)^2(k^2 + 4(k + 1))}{4} \\
 &= \frac{(k + 1)^2(k^2 + 4k + 4)}{4} \\
 &= \frac{(k + 1)^2(k + 2)^2}{4} \\
 &= \frac{(k + 1)^2((k + 1) + 1)^2}{4}.
 \end{aligned}$$

Showing that the statement holds for  $n = k + 1$ .

Conclusion: Therefore, by induction on  $n$ , the statement  $1^3 + 2^3 + 3^3 + 4^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$  is true for every positive integer  $n \geq 1$ .  $\square$

*Exercise (4).* If  $n \in \mathbb{N}$ , then  $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n + 1) = \frac{n(n+1)(n+2)}{3}$ .

*Proof:* Base Case: Observe that when  $n = 1$  that  $\left[ n(n+1) = \frac{n(n+1)(n+2)}{3} \right] = \left[ (1)((1)+1) = \frac{(1)((1)+1)((1)+2)}{3} \right]$   
 $\left[ (1)(2) = \frac{6}{3} \right] = 2$  is true.

Induction Hypothesis: Suppose for all  $k$  with  $1 \leq k < n$  that

$$1(2) + 2(3) + 3(4) + \cdots + k(k+1) = \frac{k(k+1)(k+2)}{3}.$$

In particular, suppose that  $k = n - 1$  such that

$$1(2) + 2(3) + 3(4) + \cdots + n(n-1) = \frac{(n-1)(n)(n+1)}{3}$$

Induction Step: We need to show that  $1(2) + 2(3) + 3(4) + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}$ .

Observe that

$$\begin{aligned} 1(2) + 2(3) + 3(4) + \cdots + n(n+1) &= 1(2) + 2(3) + 3(4) + \cdots + (n-1)(n) + n(n+1) \\ &= \left( 1(2) + 2(3) + 3(4) + \cdots + (n-1)(n) \right) + n(n+1) \\ &= \frac{(n-1)(n)(n+1)}{3} + n(n+1) \\ &= \frac{(n-1)(n)(n+1)}{3} + \frac{3n(n+1)}{3} \\ &= \frac{(n-1)(n)(n+1) + 3n(n+1)}{3} \\ &= \frac{(n(n+1))((n-1)+3)}{3} \\ &= \frac{n(n+1)(n+2)}{3} \end{aligned}$$

Therefore, by principle of mathematical induction,  $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}$  is true for all  $n \in \mathbb{N}$ .

(Note, this one uses the induction extras problem as a skeleton.) □

*Exercise (5).* If  $n \in \mathbb{N}$ , then  $2^1 + 2^2 + 2^3 + \cdots + 2^n = 2^{n+1} - 2$ .

*Proof:* Let  $P(n)$  be the statement  $2^1 + 2^2 + 2^3 + \cdots + 2^n = 2^{n+1} - 2$ . We will demonstrate that the left hand side is equal to the right hand side.

Base Case: When  $n = 1$ ,  $P(n) = 2^{(1)} = 2^{(1)+1} - 2 = 4 - 2 = 2$ . So  $P(1)$  holds.

Induction Hypothesis: Suppose for all  $k \in \mathbb{N}$  and  $n = k \geq 1$  that  $P(k)$  is true. That means that  $2^1 + 2^2 + 2^3 + \cdots + 2^k = 2^{k+1} - 2$ . We want to show that  $P(k+1)$  holds, that is that  $2^1 + 2^2 + 2^3 + \cdots + 2^{(k+1)} = 2^{(k+1)+1} - 2$ .

Induction Step: Observe that when  $n = k + 1$  that

$$\begin{aligned}
 P(n) &= 2^1 + 2^2 + 2^3 + \cdots + 2^{(k+1)} \\
 &= 2^1 + 2^2 + 2^3 + \cdots + 2^k + 2^{k+1} \\
 &= \left( 2^1 + 2^2 + 2^3 + \cdots + 2^k \right) + 2^{k+1} \\
 &= 2^{k+1} - 2 + 2^{k+1} \\
 &= 2(2^{k+1}) - 2 \\
 &= 2^{k+2} - 2 \\
 &= 2^{(k+1)+1} - 2
 \end{aligned}$$

Thus we have  $2^1 + 2^2 + 2^3 + \cdots + 2^k + 2^{k+1} = 2^{(k+1)+1} - 2$ . Hence the statement is true for  $n = k + 1$ , by mathematical induction  $P(n)$  is true for all  $n \in \mathbb{N}$ .  $\square$

*Exercise (8).* If  $n \in \mathbb{N}$ , then  $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}$ .

*Proof:* Let  $P(n)$  be the statement  $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}$

Base Case: Observe that when  $n = 1$ , that  $P(n) = \frac{1}{((1)+1)!} = \frac{1}{2!} = \frac{1}{2} = 1 - \frac{1}{((1)+1)!}$ . So  $P(1)$  is true.

Induction Hypothesis: Suppose that for some  $n = k \geq 1$ , where  $k \in \mathbb{N}$  that  $P(k)$  is correct. That is to say  $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{k}{(k+1)!} = 1 - \frac{1}{(k+1)!}$ . We want to show that  $P(k+1)$  holds.

Inductive step: Observe that when  $n = k + 1$  that

$$\begin{aligned}
P(n) &= \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{(k+1)}{((k+1)+1)!} \\
&= \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{k}{(k+1)!} + \frac{(k+1)}{((k+1)+1)!} \\
&= \left( \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{k}{(k+1)!} \right) + \frac{(k+1)}{((k+1)+1)!} \\
&= 1 - \frac{1}{(k+1)!} + \frac{(k+1)}{((k+1)+1)!} \\
&= 1 - \frac{((k+1)+1)}{((k+1)+1)!} + \frac{(k+1)}{((k+1)+1)!} \\
&= 1 - \frac{1}{((k+1)+1)!}
\end{aligned}$$

Thus by induction we have shown  $P(n) = 1 - \frac{1}{(n+1)!}$  is true for all  $n \in \mathbb{N}$ .  $\square$

*Exercise (10).* Prove that  $3 \mid (5^{2n} - 1)$  for every integer  $n \geq 0$ .

*Proof:* We will prove via induction on  $n$ .

Base Case: Consider the case where  $n = 0$ . Observe that  $(5^{2n} - 1) = (5^{2(0)} - 1) = (5^0 - 1) = (1 - 1) = 0$ . So we have  $3 \mid 0$  which is true.

Induction Hypothesis: Now suppose the statement is true for some  $n = k \geq 0$ , that is to say  $3 \mid (5^{2k} - 1)$ . This means  $5^{2k} - 1 = 3a$  for some  $a \in \mathbb{Z}$ . From this we get  $52k = 3a + 1$

Inductive Step: Observe that

$$\begin{aligned}
 5^{2(k+1)} - 1 &= 5^{2k+2} - 1 \\
 &= 5^{2k} 5^2 - 1 \\
 &= 5^2(24a + 1) - 1 \\
 &= 25(24a + 1) - 1 \\
 &= 25(24a) + 25 - 1 \\
 &= 25(24a) + 24 \\
 &= 3(25(8a) + 8)
 \end{aligned}$$

This shows that  $5^{2(k+1)} - 1 = 3(25(8a) + 8)$ , which means  $3 \mid (5^{2(k+1)} - 1)$ . Thus  $3 \mid (5^{2n} - 1)$  for all  $n \in \mathbb{Z}$ . □

*Exercise (13).* Prove that  $6 \mid (n^3 - n)$  for every integer  $n \geq 0$ .

*Proof:* Write your answer here. □

*Exercise (18).* Suppose  $A_1, A_2, \dots, A_n$  are sets in some universal set  $U$ , and  $n \geq 2$ . Prove that  $A_1 \cup A_2 \cup \dots \cup A_n = A_1 \cap A_2 \cap \dots \cap A_n$ .

*Proof:* Write your answer here. □

*Exercise (19).* Prove that  $\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \dots + \frac{1}{n^2} \leq 2 - \frac{1}{n}$  for every  $n \in \mathbb{N}$ .

*Proof:* Write your answer here. □

*Exercise (22).* If  $n \in \mathbb{N}$ , then

$$\left(1 - \frac{1}{2}\right) \left(1 - \frac{1}{4}\right) \left(1 - \frac{1}{8}\right) \left(1 - \frac{1}{16}\right) \cdots \left(1 - \frac{1}{2^n}\right) \geq \frac{1}{4} + \frac{1}{2^{n+1}}.$$

*Proof:* Write your answer here. □

*Exercise (25).* Concerning the Fibonacci sequence, prove that  $F_1 + F_2 + F_3 + F_4 + \dots + F_n = F_{n+2} - 1$ .

*Proof:* Write your answer here. □

*Exercise (30).* Here  $F_n$  is the  $n$ th Fibonacci number. Prove that

$$F_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}.$$

*Hint:* There are multiple ways to do this... one is to use the fact that  $a^{n-1} = \frac{a^n}{a}$ , while others involve things like the fact if  $\phi = \frac{1+\sqrt{5}}{2}$ , then  $\phi^2 - \phi - 1 = 0$ .

*Proof:* Write your answer here. □

*Exercise (33).* Suppose  $n$  (infinitely long) straight lines lie on a plane in such a way that no two of the lines are parallel, and no three of the lines intersect in a single point. Show that this arrangement divides the plane into  $\frac{n^2+n+2}{2}$  regions.

*Proof:* Write your answer here. □

*Exercise (Reflection Problem).*

- How long did it take you to complete each problem?

*Answer:* □

- What was easy?

*Answer:* □

- What was challenging? What made it challenging?

*Answer:* □

- Compare your answers to the odd numbered exercises to those in the back of the textbook. What did you learn from this comparison?

*Answer:* □