Problem 22. *If $f : A \to B$ has an inverse function then f is onto and f is one-to-one.*

Proof. Suppose $f : A \to B$ has an inverse function $g : B \to A$. By definition of an inverse function this means that

$$g(f(a)) = a, \text{ for all } a \in A$$
$$f(g(b)) = b, \text{ for all } b \in B$$

one-to-one : Let $a_1, a_2 \in A$ such that $f(a_1) = f(a_2)$. To prove f is one-to-one(injective) we must show that $a_1 = a_2$. Observe that composing the inverse g with f gives

$$g(f(a_1) = g(f(a_2))$$

As a result of applying the definition of inverse $g(f(a)) = a$, we get

$$a_1 = a_2$$

Therefore f is injective.
onto : To prove that f is onto, we must show that there exists an $a \in A$ such that $f(a) = b$. Since $g : B \to A$, we know that $g(b) \in A$. Let $a = g(b)$. Substituting a in $f(a)$ gives

$$f(a) = f(g(b)) = b$$

As a result of applying the definition of inverse $f(g(b)) = b$. Therefore, for every $b \in B$, there exists $a = g(b) \in A$ such that $f(a) = b$. Hence f is onto.

Since f is both one-to-one and onto, it follows that f is bijective. □

Problem 23. *A real number $x \in \mathbb{R}$ is called* algebraic *if there exists $a_0, a_1, \ldots, a_{n-1}, a_n \in \mathbb{Z}$, not all zero, so that*

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0$$

That is, a real number is algebraic if it is a root of a polynomial equation with integer coefficients.

(a) *The numbers $\sqrt{2}$, $\sqrt[3]{2}$, and $\sqrt{3} + \sqrt{2}$ are algebraic.*
For $\sqrt{2}$:

Proof. Let $p(x) = x^2 - 2$. We verify that $\sqrt{2}$ is a root of this polynomial:

$$p(\sqrt{2}) = (\sqrt{2})^2 - 2 = 2 - 2 = 0$$

Thus $\sqrt{2}$ is algebraic. □

For $\sqrt[3]{2}$:

Proof. Let $p(x) = x^3 - 2$. We verify that $\sqrt[3]{2}$ is a root of this polynomial:

$$p(\sqrt[3]{2}) = (\sqrt[3]{2})^3 - 2 = 2 - 2 = 0$$

Therefore $\sqrt[3]{2}$ is algebraic. □

For $\sqrt{3} + \sqrt{2}$***:***

Proof. Let $p(x) = x^4 - 10x^2 + 1$. We verify that $\sqrt{3} + \sqrt{2}$ is a root of this polynomial:

$$\begin{aligned} p(\sqrt{3} + \sqrt{2}) &= (\sqrt{3} + \sqrt{2})^4 - 10(\sqrt{3} + \sqrt{2})^2 + 1 \\ &= (3 + 2\sqrt{6} + 2)^2 - 10(3 + 2\sqrt{6} + 2) + 1 \\ &= (5 + 2\sqrt{6})^2 - 30 - 20\sqrt{6} - 20 + 1 \\ &= 25 + 20\sqrt{6} + 24 - 30 - 20\sqrt{6} - 20 + 1 \\ &= 50 + 20\sqrt{6} - 50 - 20\sqrt{6} \\ &= 0 \end{aligned}$$

Thus $\sqrt{3} + \sqrt{2}$ is algebraic. □

(b) *For fixed $n \in \mathbb{N}$, let A_n be the set of algebraic numbers which are roots of polynomials, with integer coefficients, of degree n. Then A_n is countable.*

Proof. A polynomial $P \in P_n$ of degree n with integer coefficients is defined as

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$$

where $a_i \in \mathbb{Z}$ and $a_n \neq 0$. Let P_n denote the set of all such degree-n polynomials. Each polynomial is uniquely determined by the $(n+1)$-tuple $(a_n, a_{n-1}, \ldots, a_1, a_0)$ where $a_n \neq 0$. This is a bijection between P_n and a subset of Z^{n+1}. Since Z^{n+1} is countable, any subset of Z^{n+1} is at most countable. Therefore P_n is countable. We can thus enumerate $P_n = \{P_1, P_2, P_3, \ldots\}$.

By the Fundamental Theorem of Algebra, each polynomial of degree n has at most n roots. For each $i \in \mathbb{N}$, let R_i denote the set of all roots of P_i. Then $|R_i| \leq n \leq \infty$.

Every algebraic number in A_n is, by definition a root of some polynomial in P_n. Therefore $A_n = \cup_{i=0}^{\infty} R_i$. □

(c) *The set of all algebraic numbers is countable.*

Proof. □

Problem 24. *There is an onto function $f : (0,1) \to S$ where $S = \{(x,y) : 0 < x, y < 1\}$ is the unit square in the plane $\mathbb{R}^2$.*

Proof. $\square$

Problem 25. (a) $\lim_{n\to\infty} \frac{2n+1}{5n+3} = \frac{2}{5}$

Proof. Let $\epsilon > 0$. $\square$

(b) $\lim_{n\to\infty} \frac{2n^2}{n^3+1} = 0$

Proof. Let $\epsilon > 0$. $\square$

(c) $\lim_{n\to\infty} \frac{\sin(n)}{\sqrt{n}} = o$

Proof. Let $\epsilon > 0$. $\square$

Problem 26. (a) *A sequence with an infinite number of ones that does not converge to one.*

(b) *A sequence with an infinite number of ones that converges to a limit not equal to one.*

Problem 27. *Let (x_n) be a sequence that converges to x. Suppose $p(x)$ is a polynomial. Then*

$$\lim_{n\to\infty} p(x_n) = p(x).$$

Proof. $\square$

Problem 28. *Consider three sequences (x_n), (y_n), and (z_n) for which $x_n \leq y_n \leq z_n$ for each n. If $x_n \to \ell$ and $z_n \to \ell$ then $y_n \to \ell$.*

Proof. $\square$