Problem 14. *Suppose A, B are disjoint sets with $A \cup B = \mathbb{R}$, and suppose that $a < b$ for all $a \in A$ and $b \in B$. Then there exists $c \in \mathbb{R}$ such that $x \leq c$ for $x \in A$ and $x \geq c$ for $x \in B$.*

Proof. Since A and B are non-empty sets and $a < b$ for all $a \in A$ and $b \in B$, any $b \in B$ is an upper bound for A. This means that by the least upper bound property of the Axiom of Completeness, A has a supremum that we will denote as $c = \sup A$. We want to show that for any $x \in A$, that $x \leq c$ and for any $x \in B$, that $x \geq c$.

For any $x \in A$, we have $x \leq c$ by definition of supremum.

For any $x \in B$, suppose for contradiction that $x < c$. Since c is the least upper bound of A, there exists some $a \in A$ with $x < a \leq c$. But this contradicts the given condition that $a < b$ for all $a \in A$ and $b \in B$. Therefore $x \geq c$ for all $x \in B$. □

Problem 15. *Here is an example which shows that the claim in Problem 14 is false if $\mathbb{R}$ is replaced, in both instances, by the set of rationals $\mathbb{Q}$:*

Let $A = \{x \in \mathbb{Q} : x < \pi\}$ and $B = \{x \in \mathbb{Q} : x > \pi\}$.

Note that $A \cup B = \mathbb{Q}$ and that $A \cap B = \emptyset$ meaning A and B are disjoint sets. Consider that for all $a \in A$ and $b \in B$ we have $a < \pi < b$, so that $a < b$ like our previous problem. Unlike our previous problem, there is no c that satisfies $x \leq c$ for $x \in A$ and $x \geq c$ for $x \in B$ because although π is a supremum for A and an infimum for B, it does not exist in $\mathbb{Q}$. Thus the claim in Problem 14 is false for $\mathbb{Q}$.

Problem 16. *Let $a < b$ be real numbers. Define the set $T = \mathbb{Q} \cap [a, b]$. Then $\sup T = b$.*

Proof. For any $t \in T = \mathbb{Q} \cap [a, b]$, we have $t \in [a, b]$, so $t \leq b$. Thus b is an upper bound of T.

To show $b = \sup T$, suppose N is an upper bound with $N < b$. By density of rationals, there exists $r \in \mathbb{Q}$ with $N < r < b$. Since we can choose r close enough to b, we have $r \in [a, b]$, so $r \in T$. But then $r > N$, contradicting that N is an upper bound.

Therefore $b = \sup T$. □

Problem 17. *By definition, a set $C \subseteq \mathbb{R}$ is* dense *if for any real numbers $a < b$ there is $c \in C$ so that $a < c < b$. Let T be the set of all rational numbers p/q, with $p \in \mathbb{Z}$, for which $q = 2^k$ for some $k \in \mathbb{N}$. Then T is dense.*

Proof. □

Problem 18.

(a) *An example of two real sets A, B with $A \cap B = \emptyset$, $\sup A = \sup B$, $\sup A \notin A$, and $\sup B \notin B$ is*

(b) *An example of a sequence of nested open intervals $J_1 \supseteq J_2 \subseteq J_3 \supseteq \ldots$, with $S = \bigcap_{n=1}^{\infty} J_n$ nonempty and of finite cardinality, is*

(c) *By definition, an unbounded closed interval is of the form $[a, \infty) = \{x \in \mathbb{R} : x \geq a\}$. An example of a sequence of nested unbounded closed intervals $L_1 \supseteq L_2 \subseteq L_3 \supseteq \ldots$, with $\bigcap_{n=1}^{\infty} L_n = \emptyset$, is*

Problem 19. *If $A \subseteq B$ and B is countable then A is either countable or finite.*

Proof. Assume B is countable. If $|A| < \infty$ then A is finite and we are done. So we will consider an infinite subset $A \subseteq B$ and show it is countable. □

Problem 20.

(a) *For any $a < b$ it follows that $(a, b) \sim \mathbb{R}$.*

Proof. □

(b) $[0, 1) \sim (0, 1)$

Proof. □

Problem 21. *If $A \sim B$ and $B \sim C$ then $A \sim C$.*

Proof. □