Section 14.1

Show that the two given sets have equal cardinality by describing a bijection from one to the other. Describe your bijection with a formula (not as a table).

Exercise (1). $\mathbb{R}$ and $(0, \infty)$.

Solution: We need to show bijectivity from $\mathbb{R}$ and $(0, \infty)$. Consider the function $f(a) = e^a$. We observe that $f(a) = f(b)$ implies that $e^a = e^b$ and that taking the natural log of both sides gives $a = b$. This demonstrates the function injective. Because $b \in (0, \infty)$ and $f(\ln(b) = b$, the function is is surjective. Thus $\mathbb{R}$ and $(0, \infty)$ have equal cardinality. □

Exercise (3). $\mathbb{R}$ and $(0, 1)$.

Solution: Let $f(a) = \frac{1}{1+e^a}$. Observe that $f(a) = f(b)$ implies $\frac{1}{1+e^a} = \frac{1}{1+2^b}$. Cross-multiplying both sides gives $1 + e^a = 1 + e^b$, so $e^a = e^b$. Taking the natural log of both sides gives $a = b$. Thus f is injective. Let $a \in (0, \infty)$ and $b = \frac{1}{1+e^a}$. Solving for e^a gives $e^a = \frac{1-b}{b}$. Taking the natural log of both sides gives $a = \ln\left(\frac{1-b}{b}\right)$ which will always be greater than 0 as $a \in (0, 1)$ and $a \in \mathbb{R}$. This demonstrates that f is surjective. Thus f is bijective and $\mathbb{R}$ and $(0, 1)$ share the same cardinality. □

Exercise (4). The set of even integers and the set of odd integers.

Proof: Let $A = \{2k : k \in \mathbb{Z}\}$ and $B = \{2n + 1 : n \in \mathbb{Z}\}$. Lets define our function $f : A \to B$ by $f(x) = x + 1$. Suppose $f(x) = f(y)$. Then $x + 1 = y + 1$, so $x = y$. Thus f is injective. Let $x = 2k \in \mathbb{Z}$, then $f(x) = f(2k) = 2k + 1 = y$. Thus f is surjective. Therefore f is bijective and the set of even and odd integers share the same cardinality. □

Exercise (12). $\mathbb{N}$ and $\mathbb{Z}$ (SuggestionL: use Exercise 18 from §12.2.)

Solution: Lets define our function $f : \mathbb{N} \to \mathbb{Z}$ by

$$f(x) = \begin{cases} \frac{x}{2} & \text{if x is even,} \\ -\frac{n-1}{2} & \text{if n is odd} \end{cases}$$

Consider the cases where both are even, both are odd or one is even and one is odd. Suppose x and y are even, then $\frac{x}{2} = \frac{y}{2}$. $x = y$. Suppose x and y are odd, then $-\frac{x-1}{2} = -\frac{y-1}{2}$, so $x - 1 = y - 1$ meaning $x = y$. Suppose x is even and y is odd, then $\frac{x}{2} = -\frac{y-1}{2}$. This leads to a contradiction as the left is non-negative and the right is negative unless we choose $x = 0$ and $y = 1$ in order to make both sides equal 0 which is unique. Thus f is injective. Since our even values $n = 2k$ maps to $k \geq 0$ and odd values $n = 2k + 1$ maps to $-k \leq 0$, the integers are all mapped. Thus f is surjective meaning f is bijective, $\mathbb{N}$ and $\mathbb{Z}$ share the same cardinality. □

Exercise (13). $\mathcal{P}(\mathbb{N})$ and $\mathcal{P}(\mathbb{Z})$. (Suggestion: use Exercise 12, above.)

Proof: Define function $f : P(\mathbb{N}) \to P(\mathbb{Z})$ by

$$f(A) = \{g(n) : n \in \mathring{A}\}$$

where our $g(n)$ is the function from our previous exercise, proven to be bijective. If $f(A_1) = f(A_2)$, then $\{g(n) : n \in A_1\} = \{g(n) : n \in A_2\}$. Since we've shown g was injective, $A_1 = A_2$ and f is injective. For any $B \subseteq \mathbb{Z}$, let $A = \{n \in \mathbb{N} : g(n) \in B\}$. Then $f(A) = B$, since g was proven to be surjective, f has to be too in our case. Thus f is bijective and $P(\mathbb{N})$ and $P(\mathbb{Z})$ share the same cardinality. □

Section 14.2

Exercise (1). Prove that the set $A = \{\ln(n) : n \in \mathbb{N}\} \subseteq \mathbb{R}$ is countably infinite.

Proof: The elements of the set can be written as an infinite list of

$$\ln(1) \to \ln(2) \to \ln(3) \cdots$$

. Thus A is countably infinite. □

Exercise (2). Prove that the set $A = \{(m, n) \in \mathbb{N} \times \mathbb{N} : m \leq n\}$ is countably infinite.

Proof: Write your answer here. □

Exercise (7). Prove or disprove: The set $\mathbb{Q}^{100}$ is countably infinite.

Proof: Write your answer here. □

Section 14.3

Exercise (1). Suppose B is an uncountable set and A is a set. Given that there is a surjective function $f : A \to B$, what can be said about the cardinality of A?

Solution: Write your answer here. □

Exercise (3). Prove or disprove: If A is uncountable, then $|A| = |\mathbb{R}|$.

Proof: Write your answer here. □

Exercise (7). Prove or disprove: If $A \subseteq B$ and A is countably infinite and B is uncountable, then $B - A$ is uncountable.

Proof: Write your answer here. □

Section 14.4

Exercise (1). Show that if $A \subseteq B$ and there is an injection $g : B \to A$, then $|A| = |B|$.

Proof: Write your answer here. □

Exercise (2). Show that $|\mathbb{R}^2| = |\mathbb{R}|$. Suggestion: Begin by showing $|(0, 1) \times (0, 1)| = |(0, 1)|$.

Proof: Write your answer here. □

Reflection

Exercise (Reflection Problem).

Answers:

How long did it take you to complete each problem?:

What was easy?:

What was challenging? What made it challenging?:

What did you learn from comparing your answers to those in the book?:

□

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