Section 14.1

Show that the two given sets have equal cardinality by describing a bijection from one to the other. Describe your bijection with a formula (not as a table).

Exercise (1). $\mathbb{R}$ and $(0, \infty)$.

Solution: We need to show bijectivity from $\mathbb{R}$ and $(0, \infty)$. Consider the function $f(a) = e^a$. We observe that $f(a) = f(b)$ implies that $e^a = e^b$ and that taking the natural log of both sides gives $a = b$. This demonstrates the function injective. Because $b \in (0, \infty)$ and $f(\ln(b) = b$, the function is is surjective. Thus $\mathbb{R}$ and $(0, \infty)$ have equal cardinality. $\square$

Exercise (3). $\mathbb{R}$ and $(0, 1)$.

Solution: Let $f(a) = \frac{1}{1+e^a}$. Observe that $f(a) = f(b)$ implies $\frac{1}{1+e^a} = \frac{1}{1+2^b}$. Cross-multiplying both sides gives $1 + e^a = 1 + e^b$, so $e^a = e^b$. Taking the natural log of both sides gives $a = b$. Thus f is injective. Let $a \in (0, \infty)$ and $b = \frac{1}{1+e^a}$. Solving for e^a gives $e^a = \frac{1-b}{b}$. Taking the natural log of both sides gives $a = \ln(\frac{1-b}{b})$ which will always be greater than 0 as $a \in (0, 1)$ and $a \in \mathbb{R}$. This demonstrates that f is surjective. Thus f is bijective and $\mathbb{R}$ and $(0, 1)$ share the same cardinality. $\square$

Exercise (4). The set of even integers and the set of odd integers.

Proof: Let $A = \{2k : k \in \mathbb{Z}\}$ and $B = \{2n + 1 : n \in \mathbb{Z}\}$ $\square$

Exercise (12). $\mathbb{N}$ and $\mathbb{Z}$ (SuggestionL: use Exercise 18 from §12.2.)

Solution: Write your answer here. $\square$

Exercise (13). $\mathcal{P}(\mathbb{N})$ and $\mathcal{P}(\mathbb{Z})$. (Suggestion: use Exercise 12, above.)

Proof: Write your answer here. $\square$

Section 14.2

Exercise (1). Prove that the set $A = \{\ln(n) : n \in \mathbb{N}\} \subseteq \mathbb{R}$ is countably infinite.

Proof: Write your answer here. $\square$

Exercise (2). Prove that the set $A = \{(m, n) \in \mathbb{N} \times \mathbb{N} : m \leq n\}$ is countably infinite.

Proof: Write your answer here. □

Exercise (7). Prove or disprove: The set $\mathbb{Q}^{100}$ is countably infinite.

Proof: Write your answer here. □

Section 14.3

Exercise (1). Suppose B is an uncountable set and A is a set. Given that there is a surjective function $f : A \to B$, what can be said about the cardinality of A?

Solution: Write your answer here. □

Exercise (3). Prove or disprove: If A is uncountable, then $|A| = |\mathbb{R}|$.

Proof: Write your answer here. □

Exercise (7). Prove or disprove: If $A \subseteq B$ and A is countably infinite and B is uncountable, then $B - A$ is uncountable.

Proof: Write your answer here. □

Section 14.4

Exercise (1). Show that if $A \subseteq B$ and there is an injection $g : B \to A$, then $|A| = |B|$.

Proof: Write your answer here. □

Exercise (2). Show that $|\mathbb{R}^2| = |\mathbb{R}|$. Suggestion: Begin by showing $|(0,1) \times (0,1)| = |(0,1)|$.

Proof: Write your answer here. □

Reflection

Exercise (Reflection Problem).

Answers:

How long did it take you to complete each problem?:

What was easy?:

What was challenging? What made it challenging?:

What did you learn from comparing your answers to those in the book?:

$\square$

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