## Chapter 10

Prove the following statements with either induction, strong induction or proof by smallest counterexample.

*Exercise* (3). Prove that $1^3 + 2^3 + 3^3 + 4^3 + \cdots n^3 = \frac{n^2(n+1)^2}{4}$ for every positive integer $n$.

*Proof:* (Weak Induction)

Base Case: Observe that when $n = 1$ that $n^3 = (1)^3 = \frac{(1)^2((1)+1)^2}{4} = \frac{4}{4} = 1$ which is true.

Induction Hypothesis: Suppose there is a $k \in \mathbb{Z}$ such that $1^3 + 2^3 + 3^3 + 4^3 + \cdots k^3 = \frac{k^2(k+1)^2}{4}$.

Inductive Step: We wish to show that the statement holds for $n = k+1$, i.e., that $1^3 + 2^3 + 3^3 + 4^3 + \cdots k^3 + (k+1)^3 = \frac{(k+1)^2((k+1)+1)^2}{4}$. Observe the following:

$$\begin{aligned}
1^3 + 2^3 + 3^3 + 4^3 + \cdots k^3 + (k+1)^3 &= [1^3 + 2^3 + 3^3 + 4^3 + \cdots k^3] + (k+1)^3 \\
&= \frac{k^2(k+1)^2}{4} + (k+1)^3 \\
&= \frac{k^2(k+1)^2}{4} + \frac{4(k+1)^3}{4} \\
&= \frac{k^2(k+1)^2 + 4(k+1)^3}{4} \\
&= \frac{(k+1)^2(k^2 + 4(k+1))}{4} \\
&= \frac{(k+1)^2(k^2 + 4k + 4))}{4} \\
&= \frac{(k+1)^2(k+2)^2}{4} \\
&= \frac{(k+1)^2((k+1)+1)^2}{4}.
\end{aligned}$$

Showing that the statement holds for $n = k+1$.

Conclusion: Therefore, by induction on $n$, the statement $1^3 + 2^3 + 3^3 + 4^3 + \cdots n^3 = \frac{n^2(n+1)^2}{4}$ is true for every positive integer $n \geq 1$. □

*Exercise* (4). If $n \in \mathbb{N}$, then $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}$.

*Proof:* Base Case: Observe that when $n = 1$ that $\left[n(n+1) = \frac{n(n+1)(n+2)}{3}\right] = \left[(1)((1)+1) = \frac{(1)((1)+1)((1)+2)}{3}\right]$ $\left[(1)(2) = \frac{6}{3}\right] = 2$ is true.

Induction Hypothesis: Suppose for all all $k$ with $1 \le k < n$ that

$$1(2) + 2(3) + 3(4) + \cdots + k(k+1) = \frac{k(k+1)(k+2)}{3}.$$

In particular, suppose that $k = n - 1$ such that

$$1(2) + 2(3) + 3(4) + \cdots + n(n-1) = \frac{(n-1)(n)(n+1)}{3}$$

Induction Step: We need to show that $1(2)+2(3)+3(4)+...+n(n+1) = \frac{n(n+1)(n+2)}{3}$.

Observe that

$$\begin{aligned}
1(2) + 2(3) + 3(4) + \cdots + n(n+1) &= 1(2) + 2(3) + 3(4) + \cdots + (n-1)(n) + n(n+1) \\
&= \Big(1(2) + 2(3) + 3(4) + \cdots + (n-1)n\Big) + n(n+1) \\
&= \frac{(n-1)(n)(n+1)}{3} + n(n+1) \\
&= \frac{(n-1)(n)(n+1)}{3} + \frac{3n(n+1)}{3} \\
&= \frac{(n-1)(n)(n+1) + 3n(n+1)}{3} \\
&= \frac{(n(n+1))((n-1)+3)}{3} \\
&= \frac{n(n+1)(n+2)}{3}
\end{aligned}$$

Therefore, by principle of mathematical induction, $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}$ is true for all $n \in \mathbb{N}$.

(Note, this one uses the induction extras problem as a skeleton.) □

*Exercise* (5). If $n \in \mathbb{N}$, then $2^1 + 2^2 + 2^3 + \cdots + 2^n = 2^{n+1} - 2$.

*Proof:* Let $P(n)$ be the statement $2^1 + 2^2 + 2^3 + \cdots + 2^n = 2^{n+1} - 2$. We will demonstrate that the left hand side is equal to the right hand side.

Base Case: When $n = 1$, $P(n) = 2^{(1)} = 2^{(1)+1} - 2 = 4 - 2 = 2$. So $P(1)$ holds.

Induction Hypothesis: Suppose for all $k \in \mathbb{N}$ and $n = k \geq 1$ that $P(k)$ is true. That means that $2^1 + 2^2 + 2^3 + \cdots + 2^k = 2^{k+1} - 2$. We want to show that $P(k+1)$ holds, that is that $2^1 + 2^2 + 2^3 + \cdots + 2^{(k+1)} = 2^{(k+1)+1} - 2$.

Induction Step: Observe that when $n = k + 1$ that

$$\begin{aligned}
P(n) &= 2^1 + 2^2 + 2^3 + \cdots + 2^{(k+1)} \\
&= 2^1 + 2^2 + 2^3 + \cdots + 2^k + 2^{k+1} \\
&= \left(2^1 + 2^2 + 2^3 + \cdots + 2^k\right) + 2^{k+1} \\
&= 2^{k+1} - 2 + 2^{k+1} \\
&= 2(2^{k+1}) - 2 \\
&= 2^{k+2} - 2 \\
&= 2^{(k+1)+1} - 2
\end{aligned}$$

Thus we have $2^1 + 2^2 + 2^3 + \cdots + 2^k + 2^{k+1} = 2^{(k+1)+1} - 2$. Hence the statement is true for $n = k + 1$, by mathematical induction $P(n)$ is true for all $n \in \mathbb{N}$. $\square$

*Exercise* (8). If $n \in \mathbb{N}$, then $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}$.

*Proof:* Let $P(n)$ be the statement $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}$

Base Case: Observe that when $n = 1$, that $P(n) = \frac{1}{((1)+1)!} = \frac{1}{2!} = \frac{1}{2} = 1 - \frac{1}{((1)+1!)}$. So $P(1)$ is true.

Induction Hypothesis: Suppose that for some $n = k \geq 1$, where $k \in \mathbb{N}$ that $P(k)$ is correct. That is to say $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{k}{(k+1)!} = 1 - \frac{1}{(k+1)!}$. We want to show that $P(k+1)$ holds.

Inductive step: Observe that when $n = k + 1$ that

$$\begin{aligned}
P(n) &= \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{(k+1)}{((k+1)+1)!} \\
&= \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{k}{(k+1)!} + \frac{(k+1)}{((k+1)+1)!} \\
&= \left(\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{k}{(k+1)!}\right) + \frac{(k+1)}{((k+1)+1)!} \\
&= 1 - \frac{1}{(k+1)!} + \frac{(k+1)}{((k+1)+1)!} \\
&= 1 - \frac{((k+1)+1)}{((k+1)+1)!} + \frac{(k+1)}{((k+1)+1)!} \\
&= 1 - \frac{1}{((k+1)+1)!}
\end{aligned}$$

Thus by induction we have shown $P(n) = 1 - \frac{1}{(n+1)!}$ is true for all $n \in \mathbb{N}$. □

*Exercise* (10). Prove that $3 \mid (5^{2n} - 1)$ for every integer $n \geq 0$.

*Proof:* Base Case:

Induction Hypothesis:

Inductive Step:

□

*Exercise* (13). Prove that $6 \mid (n^3 - n)$ for every integer $n \geq 0$.

*Proof:* Write your answer here. □

*Exercise* (18). Suppose $A_1, A_2, \ldots, A_n$ are sets in some universal set $U$, and $n \geq 2$. Prove that $\overline{A_1 \cup A_2 \cup \cdots A_n} = \overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n}$.

*Proof:* Write your answer here. □

*Exercise* (19). Prove that $\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} \leq 2 - \frac{1}{n}$ for every $n \in \mathbb{N}$.

*Proof:* Write your answer here. □

*Exercise* (22). If $n \in \mathbb{N}$, then

$$\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{4}\right)\left(1 - \frac{1}{8}\right)\left(1 - \frac{1}{16}\right) \cdots \left(1 - \frac{1}{2^n}\right) \geq \frac{1}{4} + \frac{1}{2^{n+1}}.$$

*Proof:* Write your answer here. □

*Exercise* (25). Concerning the Fibonacci sequence, prove that $F_1 + F_2 + F_3 + F_4 + \cdots + F_n = F_{n+2} - 1$.

*Proof:* Write your answer here. □

*Exercise* (30). Here $F_n$ is the $n$th Fibonacci number. Prove that

$$F_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}.$$

*Hint:* There are multiple ways to do this... one is to use the fact that $a^{n-1} = \frac{a^n}{a}$, while others involve things like the fact if $\phi = \frac{1+\sqrt{5}}{2}$, then $\phi^2 - \phi - 1 = 0$.

*Proof:* Write your answer here. □

*Exercise* (33). Suppose $n$ (infinitely long) straight lines lie on a plane in such a way that no two of the lines are parallel, and no three of the lines intersect in a single point. Show that this arrangement divides the plane into $\frac{n^2+n+2}{2}$ regions.

*Proof:* Write your answer here. □

*Exercise* (Reflection Problem).

- How long did it take you to complete each problem?

  *Answer:* □

- What was easy?

  *Answer:* □

- What was challenging? What made it challenging?

  *Answer:* □

- Compare your answers to the odd numbered exercises to those in the back of the textbook. What did you learn from this comparison?

  *Answer:* □

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