Section 3.8

Exercise (3.121). Let Y denote a random variable that has a Poisson distribution with mean $\lambda = 2$. Find:

(a) $P(Y = 4)$

Solution: Poisson distribution function is

$$P(Y = y) = \frac{e^{-\lambda}\lambda^y}{y!}$$

$$P(Y = 4) = \frac{e^{-2}(-2)^4}{4!} = 0.09022352$$

□

(b) $P(Y \geq 4)$

Solution:

$$\begin{aligned} P(Y \geq 4) &= 1 - [P(Y = 3) + P(Y = 2) + P(Y = 1) + P(Y = 0)] \\ &= 1 - [\frac{e^{-2}(-2)^3}{3!} + \frac{e^{-2}(-2)^2}{2!} + \frac{e^{-2}(-2)^1}{1!} + \frac{e^{-2}(-2)^0}{0!}] \\ &= 1 - [0.180447 + 0.2706706 + 0.2706706 + 0.1353353] \\ &= 0.1428765 \end{aligned}$$

(table gives me 1-0.857 = 0.143 but wanted to do it full at least once) □

(c) $P(Y < 4)$

Solution:

$$P(Y < 4) = P(Y \leq 3) = 0.857$$

(table) □

(d) $P(Y \geq 4 \mid Y \geq 2)$

Solution:

$$P(Y \geq 4 \mid Y \geq 2) = \frac{P(Y \geq 4) \cap P(Y \geq 2)}{P(Y \geq 2)} = \frac{P(Y \geq 4)}{P(Y \geq 2)} = \frac{0.143}{1 - 0.406} = 0.241$$

(table and prev prob) □

Exercise (3.122). Customers arrive at a checkout counter in a department store according to a Poisson distribution at an average of seven per hour. During a given hour, what are the probabilities that:

(a) no more than three customers arrive?

Solution:

$$P(Y \leq 3) = 0.082$$

(table) □

(b) at least two customers arrive?

Solution:

$$P(Y \geq 2) = 1 - P(Y \leq 1) = 1 - 0.007 = 0.993$$

□

(c) exactly five customers arrive?

Solution:

$$P(Y = 5) = \frac{e^{-7}7^5}{5!} = 0.1277167$$

□

Exercise (3.125). Refer to Exercise 3.122. If it takes approximately ten minutes to serve each customer, find the mean and variance of the total service time for customers arriving during a 1-hour period. (Assume that a sufficient number of servers are available so that no customer must wait for service.) Is it likely that the total service time will exceed 2.5 hours?

Solution: Note that 10 minutes = 1/6 of an hour, $E(Y) = \lambda$ and $V(Y) = \lambda$ for Poisson.

$$E(\frac{1}{6}Y) = \frac{1}{6}E(Y) = \frac{1}{6} * 7 = 1.166667$$
$$V(\frac{1}{6}Y) = (\frac{1}{6})^2 V(Y) = \frac{1}{36} * 7 = 0.1944444$$

Not likely to exceed 2.5 hours of total service time since the variance is already so small, the standard deviation would be smaller □

Exercise (3.127). The number of typing errors made by a typist has a Poisson distribution with an average of four errors per page. If more than four errors appear on a given page, the typist must retype the whole page. What is the probability that a randomly selected page does not need to be retyped?

Solution: $\lambda = 4$ and $Y \leq 4$ for this one.

$$P(Y \leq 4) = 0.629$$

(table) □

Exercise (3.131). The number of knots in a particular type of wood has a Poisson distribution with an average of 1.5 knots in 10 cubic feet of the wood. Find the probability that a 10-cubic-foot block of the wood has at most 1 knot.

Solution: $\lambda = 1.5$ and $Y \leq 1$.

$$P(Y \leq 1) = 0.558$$

(table really has everything doesn't it) □

Exercise (3.134). Consider a binomial experiment for $n = 20$, $p = .05$. Use Table 1, Appendix 3, to calculate the binomial probabilities for $Y = 0, 1, 2, 3,$ and 4. Calculate the same probabilities by using the Poisson approximation with $\lambda = np$. Compare.

Solution: Binomial Distribution for $P(Y \leq 4)$ for $n = 20$ and $p = 0.5$ is 0.997 according to back table. For our Poisson Distribution $\lambda = 20 * 0.05 = 1$, we refer to the table in the back and calculate it.

$$\begin{aligned}
P(Y \leq 4) &= 0.996 \text{ (table)} \\
P(Y \leq 4) &= P(Y = 0) + P(Y = 1) + P(Y = 2) + P(Y = 3) + P(Y = 4) \\
&= e^{-1}[\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!}] \\
&= 0.9963402 \text{ (full calculation)}
\end{aligned}$$

Depending on who you are, they are pretty close. □

Exercise (3.135). A salesperson has found that the probability of a sale on a single contact is approximately .03. If the salesperson contacts 100 prospects, what is the approximate probability of making at least one sale?

Solution: We use Poisson Approximation

$$P(Y \geq 1) = 1 - P(Y = 0) = 1 - \frac{e^{-3}3^0}{0!} = e^{-3} = 0.9502129$$

□

Exercise (3.139). In the daily production of a certain kind of rope, the number of defects per foot Y is assumed to have a Poisson distribution with mean $\lambda = 2$. The profit per foot when the rope is sold is given by X, where $X = 50 - 2Y - Y^2$. Find the expected profit per foot.

Solution:

$$\begin{aligned} E(X) &= E[50 - 2Y - Y^2] \\ &= E(50) - 2E(Y) - E(Y^2) \\ &= 50 - 2(2) - [V(Y) + [E(Y)]^2] \\ &= 50 - 4 - (2 + 4) \\ &= 40 \end{aligned}$$

□

Exercise (3.141). A food manufacturer uses an extruder (a machine that produces bite-size cookies and snack food) that yields revenue for the firm at a rate of \$200 per hour when in operation. However, the extruder breaks down an average of two times every day it operates. If Y denotes the number of breakdowns per day, the daily revenue generated by the machine is $R = 1600 - 50Y^2$. Find the expected daily revenue for the extruder.

Solution:

$$\begin{aligned} E(R) &= E[1600 - 50Y^2] \\ &= E[1600] - 50E[Y^2] \\ &= 1600 - 50[V(Y) + E(Y)^2] \\ &= 1600 - 50[2 + 4] \\ &= 1300 \end{aligned}$$

□

Section 3.9

Exercise (3.145). If Y has a binomial distribution with n trials and probability of success p, show that the moment-generating function for Y is

$$m(t) = (pe^t + q)^n, \text{ where } q = 1 - p.$$

Solution: The moment-generating function is $m(t) = E(e^{tY})$.

For binomial distribution:

$$\begin{aligned} m(t) = E(e^{tY}) &= \sum_{y=0}^{n} e^{ty} \binom{n}{y} p^y q^{n-y} \\ &= \sum_{y=0}^{n} \binom{n}{y} (pe^t)^y q^{n-y} \\ &= (pe^t + q)^n \text{ by the binomial theorem} \end{aligned}$$

□

Exercise (3.146). Differentiate the moment-generating function in Exercise 3.145 to find $E(Y)$ and $E(Y^2)$. Then find $V(Y)$.

Solution: $m(t) = (pe^t + q)^n$.

First derivative:

$$m'(t) = n(pe^t + q)^{n-1} \cdot pe^t$$

$$m'(0) = n(p + q)^{n-1} \cdot p = np = E(Y)$$

Second derivative:

$$m''(t) = n(n-1)(pe^t + q)^{n-2}(pe^t)^2 + n(pe^t + q)^{n-1}pe^t$$

$$m''(0) = n(n-1)p^2 + np = E(Y^2)$$

Therefore:

$$\begin{aligned} V(Y) &= E(Y^2) - [E(Y)]^2 \\ &= n(n-1)p^2 + np - n^2p^2 \\ &= n^2p^2 - np^2 + np - n^2p^2 \\ &= np - np^2 = np(1-p) = npq \end{aligned}$$

□

Exercise (3.147). If Y has a geometric distribution with probability of success p, show that the moment-generating function for Y is

$$m(t) = \frac{pe^t}{1 - qe^t}, \text{ where } q = 1 - p.$$

Solution: For geometric distribution, $P(Y = y) = pq^{y-1}$ for $y = 1, 2, 3, \ldots$

$$\begin{aligned} m(t) = E(e^{tY}) &= \sum_{y=1}^{\infty} e^{ty} pq^{y-1} \\ &= pe^t \sum_{y=1}^{\infty} (qe^t)^{y-1} \\ &= pe^t \sum_{k=0}^{\infty} (qe^t)^k \text{ (let } k = y - 1) \\ &= pe^t \cdot \frac{1}{1 - qe^t} \text{ (geometric series, } |qe^t| < 1) \\ &= \frac{pe^t}{1 - qe^t} \end{aligned}$$

□

Exercise (3.149). Refer to Exercise 3.145. Use the uniqueness of moment-generating functions to give the distribution of a random variable with moment-generating function $m(t) = (0.6e^t + 0.4)^3$.

Solution: Comparing $m(t) = (0.6e^t + 0.4)^3$ with the binomial MGF $m(t) = (pe^t + q)^n$:

We have $p = 0.6$, $q = 0.4$, and $n = 3$.

Therefore, Y has a **binomial distribution with** $n = 3$ **and** $p = 0.6$. □

Exercise (3.151). Refer to Exercise 3.145. If Y has moment-generating function $m(t) = (0.7e^t + 0.3)^{10}$, what is $P(Y \leq 5)$?

Solution: From the MGF, Y has a binomial distribution with $n = 10$ and $p = 0.7$.

Using Table 1 (Binomial table) with $n = 10$ and $p = 0.7$:

$$P(Y \leq 5) = 0.150$$

□

Exercise (3.153). Find the distributions of the random variables that have each of the following moment-generating functions:

(a) $m(t) = [(1/3)e^t + (2/3)]^5$

Solution: This is a binomial MGF with $p = 1/3$, $q = 2/3$, and $n = 5$.

Therefore, $Y \sim \text{Binomial}(n = 5, p = 1/3)$. □

(b) $m(t) = \frac{e^t}{2-e^t}$

Solution: Rewrite: $m(t) = \frac{e^t}{2-e^t} = \frac{(1/2)e^t}{1-(1/2)e^t}$

This matches the geometric MGF with $p = 1/2$ and $q = 1/2$.

Therefore, $Y \sim \text{Geometric}(p = 1/2)$. □

(c) $m(t) = e^{2(e^t-1)}$

Solution: The Poisson MGF is $m(t) = e^{\lambda(e^t-1)}$.

Comparing with $m(t) = e^{2(e^t-1)}$, we have $\lambda = 2$.

Therefore, $Y \sim \text{Poisson}(\lambda = 2)$. □

Exercise (3.155). Let $m(t) = \frac{1}{6}e^t + \frac{2}{6}e^{2t} + \frac{3}{6}e^{3t}$. Find the following:

(a) $E(Y)$

Solution: $m'(t) = \frac{1}{6}e^t + \frac{4}{6}e^{2t} + \frac{9}{6}e^{3t}$

$E(Y) = m'(0) = \frac{1}{6} + \frac{4}{6} + \frac{9}{6} = \frac{14}{6} = \frac{7}{3}$ □

(b) $V(Y)$

Solution: $m''(t) = \frac{1}{6}e^t + \frac{8}{6}e^{2t} + \frac{27}{6}e^{3t}$

$E(Y^2) = m''(0) = \frac{1}{6} + \frac{8}{6} + \frac{27}{6} = \frac{36}{6} = 6$

$$V(Y) = E(Y^2) - [E(Y)]^2 = 6 - \left(\tfrac{7}{3}\right)^2 = 6 - \tfrac{49}{9} = \tfrac{54-49}{9} = \tfrac{5}{9}$$ □

(c) The distribution of Y

Solution: The MGF has the form $m(t) = \sum p(y)e^{yt}$.

Comparing: $p(1) = 1/6$, $p(2) = 2/6 = 1/3$, $p(3) = 3/6 = 1/2$

Therefore, Y is a discrete random variable with probability function:

$$p(y) = \begin{cases} 1/6, & y = 1 \\ 1/3, & y = 2 \\ 1/2, & y = 3 \\ 0, & \text{otherwise} \end{cases}$$

□

Section 3.11

Exercise (3.167). Let Y be a random variable with mean 11 and variance 9. Using Tchebysheff's theorem, find:

(a) a lower bound for $P(6 < Y < 16)$

Solution: $\mu = 11$, $\sigma^2 = 9$, so $\sigma = 3$.

The interval $(6, 16)$ is $\mu \pm 5 = 11 \pm 5$.

So $k = 5/3$ standard deviations.

By Tchebysheff's theorem:

$$P(|Y - \mu| < k\sigma) \geq 1 - \frac{1}{k^2} = 1 - \frac{1}{(5/3)^2} = 1 - \frac{9}{25} = \frac{16}{25} = 0.64$$

□

(b) the value of C such that $P(|Y - 11| \geq C) \leq 0.09$

Solution: By Tchebysheff's theorem: $P(|Y - \mu| \geq k\sigma) \leq \frac{1}{k^2}$

We want $\frac{1}{k^2} = 0.09$, so $k^2 = \frac{1}{0.09} = \frac{100}{9}$, thus $k = \frac{10}{3}$.

Therefore:

$$C = k\sigma = \frac{10}{3} \cdot 3 = 10$$

□

Exercise (3.168). Would you rather take a multiple-choice test or a full-recall test? If you have absolutely no knowledge of the test material, you will score zero on a full-recall test. However, if you are given 5 choices for each multiple-choice question, you have at least one chance in five of guessing each correct answer! Suppose that a multiple-choice exam contains 100 questions, each with 5 possible answers, and you guess the answer to each of the questions.

(a) What is the expected value of the number Y of questions that will be correctly answered?

Solution: $Y \sim \text{Binomial}(n = 100, p = 1/5 = 0.2)$

$$E(Y) = np = 100(0.2) = 20$$

□

(b) Find the standard deviation of Y.

Solution: $V(Y) = npq = 100(0.2)(0.8) = 16$

$$\sigma = \sqrt{16} = 4$$

□

(c) Calculate the intervals $\mu \pm 2\sigma$ and $\mu \pm 3\sigma$.

Solution: $\mu \pm 2\sigma = 20 \pm 2(4) = 20 \pm 8 = (12, 28)$

$$\mu \pm 3\sigma = 20 \pm 3(4) = 20 \pm 12 = (8, 32)$$

□

(d) If the results of the exam are curved so that 50 correct answers is a passing score, are you likely to receive a passing score? Explain.

Solution: 50 is $\frac{50-20}{4} = 7.5$ standard deviations above the mean.

By Tchebysheff's theorem, $P(|Y - 20| \geq 30) \leq \frac{1}{(7.5)^2} = \frac{1}{56.25} \approx 0.018$.

Therefore, it is **very unlikely** to receive a passing score (probability less than 2%).

□

Exercise (3.171). For a certain type of soil the number of wireworms per cubic foot has a mean of 100. Assuming a Poisson distribution of wireworms, give an interval that will include at least 5/9 of the sample values of wireworm counts obtained from a large number of 1-cubic-foot samples.

Solution: For Poisson: $\mu = \lambda = 100$ and $\sigma^2 = \lambda = 100$, so $\sigma = 10$.

By Tchebysheff's theorem: $P(|Y - \mu| < k\sigma) \geq 1 - \frac{1}{k^2}$

We want $1 - \frac{1}{k^2} = \frac{5}{9}$, so $\frac{1}{k^2} = \frac{4}{9}$, thus $k^2 = \frac{9}{4}$ and $k = \frac{3}{2}$.

The interval is:

$$\mu \pm k\sigma = 100 \pm \frac{3}{2}(10) = 100 \pm 15 = (85, 115)$$

$\square$

4.2

Exercise (4.1). Let Y be a random variable with $p(y)$ given in the table below.

y	1	2	3	4
$p(y)$	.4	.3	.2	.1

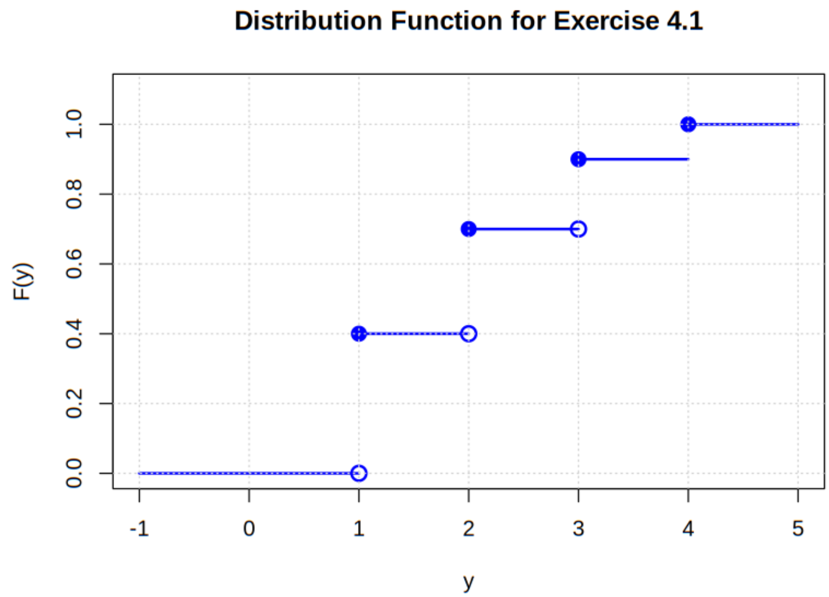
(a) Give the distribution function, $F(y)$. Be sure to specify the value of $F(y)$ for all y, $-\infty < y < \infty$.

Solution:

$$F(y) = P(Y \leq y) = \begin{cases} 0, & y < 1 \\ 0.4, & 1 \leq y < 2 \\ 0.7, & 2 \leq y < 3 \\ 0.9, & 3 \leq y < 4 \\ 1, & y \geq 4 \end{cases}$$

$\square$

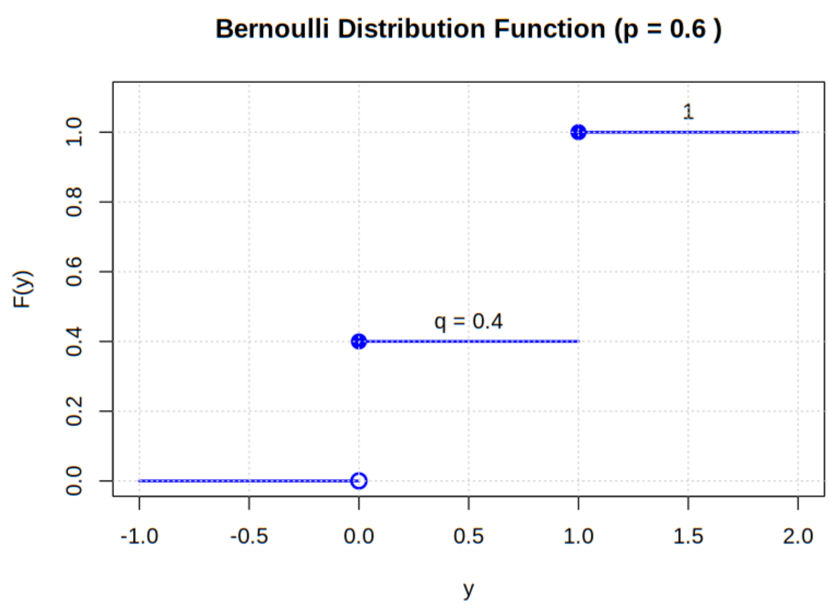
(b) Sketch the distribution function given in part (a).



Solution: ☐

Exercise (4.3). A Bernoulli random variable is one that assumes only two values, 0 and 1 with $p(1) = p$ and $p(0) = 1 - p \equiv q$.

(a) Sketch the corresponding distribution function.



Solution: ☐

(b) Show that this distribution function has the properties given in Theorem 4.1.

Solution: Theorem 4.1 properties:

(a) $F(-\infty) = 0$: Yes, $\lim_{y\to-\infty} F(y) = 0$

(b) $F(\infty) = 1$: Yes, $\lim_{y \to \infty} F(y) = 1$

(c) F is non-decreasing: $0 \leq q \leq 1$ since $q = 1 - p$ and $0 \leq p \leq 1$

(d) F is right-continuous: At each point, F equals its right-hand limit

□

Exercise (4.5). Suppose that Y is a random variable that takes on only integer values $1, 2, \ldots$ and has distribution function $F(y)$. Show that the probability function $p(y) = P(Y = y)$ is given by

$$p(y) = \begin{cases} F(1), & y = 1, \\ F(y) - F(y-1), & y = 2, 3, \ldots \end{cases}$$

Solution: For $y = 1$: $P(Y = 1) = P(Y \leq 1) = F(1)$ since Y takes integer values ≥ 1.

For $y \geq 2$:

$$P(Y = y) = P(Y \leq y) - P(Y \leq y - 1)$$
$$= F(y) - F(y - 1)$$

□

Exercise (4.7). Let Y be a binomial random variable with $n = 10$ and $p = 0.2$.

(a) Use Table 1, Appendix 3, to obtain $P(2 < Y < 5)$ and $P(2 \leq Y < 5)$. Are the probabilities that Y falls in the intervals $(2, 5)$ and $[2, 5)$ equal? Why or why not?

Solution: $P(2 < Y < 5) = P(Y = 3) + P(Y = 4) = P(Y \leq 4) - P(Y \leq 2) = 0.967 - 0.678 = 0.289$

$P(2 \leq Y < 5) = P(Y = 2) + P(Y = 3) + P(Y = 4) = P(Y \leq 4) - P(Y \leq 1) = 0.967 - 0.376 = 0.591$

not equal because Y is discrete. For discrete variables, $(2, 5)$ excludes 2 while $[2, 5)$ includes 2. □

(b) Use Table 1, Appendix 3, to obtain $P(2 < Y \leq 5)$ and $P(2 \leq Y \leq 5)$. Are these two probabilities equal? Why or why not?

Solution: $P(2 < Y \leq 5) = P(Y = 3) + P(Y = 4) + P(Y = 5) = P(Y \leq 5) - P(Y \leq 2) = 0.994 - 0.678 = 0.316$

$P(2 \leq Y \leq 5) = P(Y \leq 5) - P(Y \leq 1) = 0.994 - 0.376 = 0.618$

not equal because including/excluding $Y = 2$ makes a difference for discrete variables. $\square$

(c) Earlier in this section, we argued that if Y is continuous and $a < b$, then $P(a < Y < b) = P(a \leq Y < b)$. Does the result in part (a) contradict this claim? Why?

Solution: no this does not contradict the claim. The claim is specifically for continuous random variables, where $P(Y = a) = 0$ for any specific value a. Here, Y is **discrete**, so $P(Y = 2) \neq 0$, and therefore the inclusion/exclusion of endpoints matters. $\square$

Exercise (4.9). A random variable Y has the following distribution function:

$$F(y) = P(Y \leq y) = \begin{cases} 0, & \text{for } y < 2, \\ 1/8, & \text{for } 2 \leq y < 2.5, \\ 3/16, & \text{for } 2.5 \leq y < 4, \\ 1/2, & \text{for } 4 \leq y < 5.5, \\ 5/8, & \text{for } 5.5 \leq y < 6, \\ 11/16, & \text{for } 6 \leq y < 7, \\ 1, & \text{for } y \geq 7. \end{cases}$$

(a) Is Y a continuous or discrete random variable? Why?

Solution: $\square$

(b) What values of Y are assigned positive probabilities?

Solution: $\square$

(c) Find the probability function for Y.

Solution: $\square$

(d) What is the median, $\phi_{.5}$, of Y?

Solution: $\square$

Exercise (4.11). Suppose that Y possesses the density function

$$f(y) = \begin{cases} cy, & 0 \leq y \leq 2, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Find the value of c that makes $f(y)$ a probability density function.

Solution: For $f(y)$ to be a pdf: $\int_{-\infty}^{\infty} f(y)dy = 1$

$$\int_0^2 cy\,dy = c\left[\frac{y^2}{2}\right]_0^2 = c \cdot 2 = 1$$

Therefore, $c = 1/2$. □

(b) Find $F(y)$.

Solution: For $y < 0$: $F(y) = 0$

For $0 \le y \le 2$:

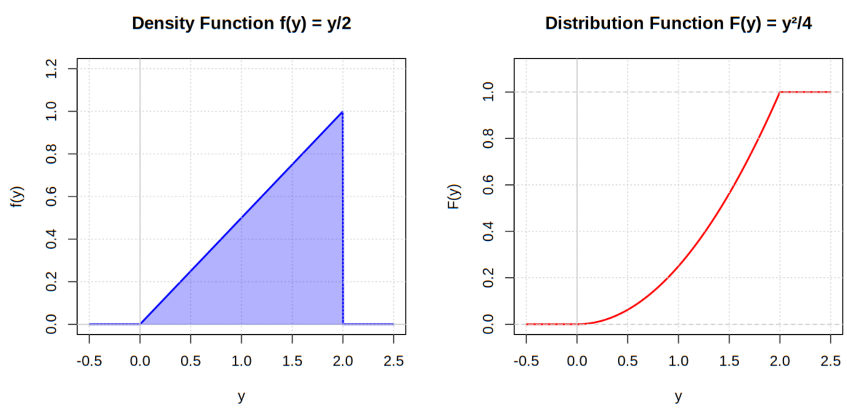
$$F(y) = \int_0^y \frac{1}{2}t\,dt = \frac{1}{2} \cdot \frac{y^2}{2} = \frac{y^2}{4}$$

For $y > 2$: $F(y) = 1$

$$F(y) = \begin{cases} 0, & y < 0 \\ y^2/4, & 0 \le y \le 2 \\ 1, & y > 2 \end{cases}$$

□

(c) Graph $f(y)$ and $F(y)$.



Solution: □

(d) Use $F(y)$ to find $P(1 \le Y \le 2)$.

Solution:

$$P(1 \le Y \le 2) = F(2) - F(1) = 1 - \frac{1}{4} = \frac{3}{4}$$

□

(e) Use $f(y)$ and geometry to find $P(1 \le Y \le 2)$.

Solution:

$$P(1 \le Y \le 2) = \int_1^2 \frac{y}{2} dy = \frac{1}{2}\left[\frac{y^2}{2}\right]_1^2 = \frac{1}{4}(4-1) = \frac{3}{4}$$

Geometrically, this is the area of a trapezoid with parallel sides $f(1) = 1/2$ and $f(2) = 1$, and height 1:

$$\text{Area} = \frac{1}{2}(1/2 + 1)(1) = \frac{3}{4}$$

□

Exercise (4.13). A supplier of kerosene has a 150-gallon tank that is filled at the beginning of each week. His weekly demand shows a relative frequency behavior that increases steadily up to 100 gallons and then levels off between 100 and 150 gallons. If Y denotes weekly demand in hundreds of gallons, the relative frequency of demand can be modeled by

$$f(y) = \begin{cases} y, & 0 \le y \le 1, \\ 1, & 1 < y \le 1.5, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Find $F(y)$.

Solution: For $y < 0$: $F(y) = 0$

For $0 \le y \le 1$: $F(y) = \int_0^y t\,dt = \frac{y^2}{2}$

For $1 < y \le 1.5$: $F(y) = \int_0^1 t\,dt + \int_1^y 1\,dt = \frac{1}{2} + (y-1) = y - \frac{1}{2}$ For $y > 1.5$: $F(y) = 1$

$$F(y) = \begin{cases} 0, & y < 0 \\ y^2/2, & 0 \le y \le 1 \\ y - 1/2, & 1 < y \le 1.5 \\ 1, & y > 1.5 \end{cases}$$

□

(b) Find $P(0 \le Y \le 0.5)$.

Solution:

$$P(0 \leq Y \leq 0.5) = F(0.5) - F(0) = \frac{(0.5)^2}{2} - 0 = \frac{0.25}{2} = 0.125$$

□

(c) Find $P(0.5 \leq Y \leq 1.2)$.

Solution:

$$P(0.5 \leq Y \leq 1.2) = F(1.2) - F(0.5) = \left(1.2 - \frac{1}{2}\right) - \frac{(0.5)^2}{2} = 0.7 - 0.125 = 0.575$$

□

Exercise (4.17). The length of time required by students to complete a one-hour exam is a random variable with a density function given by

$$f(y) = \begin{cases} cy^2 + y, & 0 \leq y \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Find c.

Solution: For $f(y)$ to be a pdf: $\int_{-\infty}^{\infty} f(y)dy = 1$

$$\int_0^1 (cy^2 + y)dy = \left[\frac{cy^3}{3} + \frac{y^2}{2}\right]_0^1 = \frac{c}{3} + \frac{1}{2} = 1$$

$$\frac{c}{3} = \frac{1}{2} \implies c = \frac{3}{2}$$

□

(b) Find $F(y)$.

Solution: For $y < 0$: $F(y) = 0$

For $0 \leq y \leq 1$:

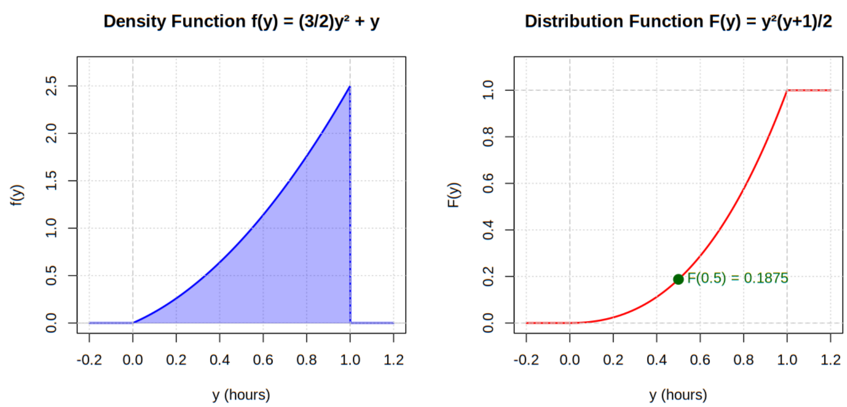
$$F(y) = \int_0^y \left(\frac{3}{2}t^2 + t\right) dt = \left[\frac{t^3}{2} + \frac{t^2}{2}\right]_0^y = \frac{y^3}{2} + \frac{y^2}{2} = \frac{y^2(y+1)}{2}$$

For $y > 1$: $F(y) = 1$

$$F(y) = \begin{cases} 0, & y < 0 \\ \frac{y^2(y+1)}{2}, & 0 \le y \le 1 \\ 1, & y > 1 \end{cases}$$

□

(c) Graph $f(y)$ and $F(y)$.



Solution: □

(d) Use $F(y)$ in part (b) to find $F(-1)$, $F(0)$, and $F(1)$.

Solution:

$$F(-1) = 0$$

$$F(0) = 0$$

$$F(1) = \frac{1^2(1+1)}{2} = 1$$

□

(e) Find the probability that a randomly selected student will finish in less than half an hour.

Solution:

$$P(Y < 0.5) = F(0.5) = \frac{(0.5)^2(0.5+1)}{2} = \frac{0.25 \cdot 1.5}{2} = \frac{0.375}{2} = 0.1875$$

□

(f) Given that a particular student needs at least 15 minutes to complete the exam, find the probability that she will require at least 30 minutes to finish.

Solution: 15 minutes = 0.25 hours, 30 minutes = 0.5 hours

$$P(Y \geq 0.5|Y \geq 0.25) = \frac{P(Y \geq 0.5)}{P(Y \geq 0.25)} = \frac{1 - F(0.5)}{1 - F(0.25)}$$

$$F(0.25) = \frac{(0.25)^2(1.25)}{2} = \frac{0.078125}{2} = 0.0390625$$

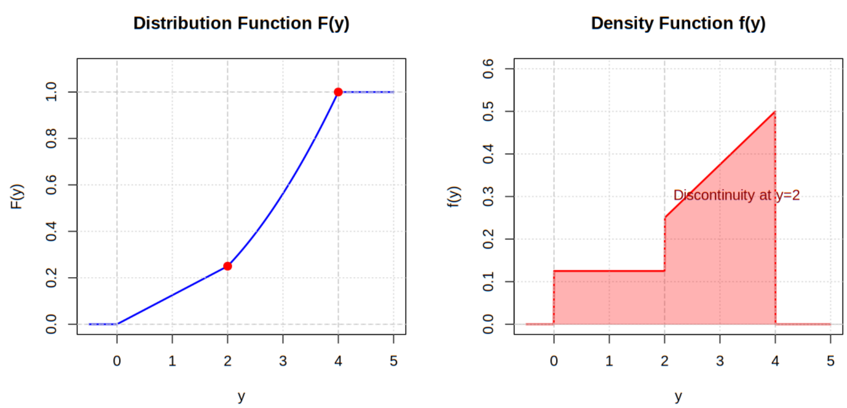
$$P(Y \geq 0.5|Y \geq 0.25) = \frac{1 - 0.1875}{1 - 0.0390625} = \frac{0.8125}{0.9609375} \approx 0.846$$

$\square$

Exercise (4.19). Let the distribution function of a random variable Y be

$$F(y) = \begin{cases} 0, & y \leq 0, \\ \frac{y}{8}, & 0 < y < 2, \\ \frac{y^2}{16}, & 2 \leq y < 4, \\ 1, & y \geq 4. \end{cases}$$

(a) Find the density function of Y.



Solution: Note: At $y = 2$, F is not differentiable (there's a corner), so the density is undefined there.

$\square$

(b) Find $P(1 \leq Y \leq 3)$.

Solution:

$$P(1 \leq Y \leq 3) = F(3) - F(1) = \frac{9}{16} - \frac{1}{8} = \frac{9}{16} - \frac{2}{16} = \frac{7}{16}$$

$\square$

(c) Find $P(Y \geq 1.5)$.

Solution:

$$P(Y \geq 1.5) = 1 - F(1.5) = 1 - \frac{1.5}{8} = 1 - \frac{3}{16} = \frac{13}{16}$$

$\square$

(d) Find $P(Y \geq 1 \mid Y \leq 3)$.

Solution:

$$P(Y \geq 1 | Y \leq 3) = \frac{P(1 \leq Y \leq 3)}{P(Y \leq 3)} = \frac{F(3) - F(1)}{F(3)} = \frac{9/16 - 1/8}{9/16} = \frac{7/16}{9/16} = \frac{7}{9}$$

$\square$