We will begin by building theorems from the ground up from basic rules

**Definition 1.** *Convergence: For $A_n \to L$ means: For all $\epsilon > 0$, there exists $N$ such that for all $n > N$, implies $\mid a_n - L \mid < \epsilon$.*

**Definition 2.** *Bounded: $(a_n)$ is bounded if there exists $M > 0$ such that $|a_n| \leq M$ for all $n$.*

**Definition 3.** *Triangle inequality*

$$\begin{aligned} \textit{Triangle inequality :} \quad & \mid a + b \mid \leq \mid a \mid + \mid b \mid \\ \textit{Reverse triangle :} \quad & ||a| - |b|| \leq |a - b| \\ \textit{Product bound :} \quad & |ab| = |a||b| \end{aligned}$$

**Theorem 0.1.** *If $(a_n)$ converges to $L$, then $(a_n)$ is bounded.*

*Proof.* Since $(a_n)$ converges to $L$, this means that for an $\epsilon > 0$, there exists $N$ such that for all $n > N$, implies $\mid a_n - L \mid < \epsilon$. From this we get the following inequality:

$$|a_n| = |a_n - L + L| \leq |a_n - L| + |L| \qquad \text{By Triangle Inequality}$$

Now let $\epsilon = 1$, then there exists an $n > N$ such that $|a_n - L| < 1$, it follows from this that

$$|a_n| = |a_n - L + L| \leq |a_n - L| + |L| < 1 + |L|$$

for $n \leq N$, let $M_1 = \max\{|a_1|, |a_2|, \cdots, |a_N|\}$. Now let $M = \{M_1, 1 + |L|\}$. Then $|a_n| \leq M$ for all $n$. □

**Theorem 0.2.** *(Uniqueness of Limits) If $a_n \to L$ and $a_n \to M$ then $L = M$.*

*Proof.* Let $\epsilon > 0$ be arbitrary. Since $a_n \to L$ there exists an $N_1$ such that for all $n \geq N : |a_n - L| < \frac{\epsilon}{2}$.

Likewise since $a_n \to M$, there exists $N_2$ such that for all $n \geq N_2 : |a_n - M| < \frac{\epsilon}{2}$.

Let $N = \max\{N_1, N_2\}$. For $n \geq N$:

$$|L - M| = |L - a_n + a_n - M| \leq |a_n - L| + |a_n + M| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Since this holds for arbitrary $\epsilon > 0$, we must have $|L - M| = 0$, so L = M.

□

**Theorem 0.3.** *(Algebraic Limit Theorem) If $x_n \to a$ and $y_n \to b$, then the algebraic limit theorem states*

$$\textbf{\textit{Sum:}} \quad \lim(x_n + y_n) = a + b \tag{1}$$

$$\textbf{\textit{Scalar:}} \quad \lim(cx_n) = ca \tag{2}$$

$$\textbf{\textit{Product:}} \quad \lim(x_n * y_n) = a * b \tag{3}$$

$$\textbf{\textit{Quotient:}} \quad \lim(\frac{x_n}{y_n}) = \frac{a}{b} \qquad \textit{for } b \neq 0 \tag{4}$$

*Proof.* Sum: Recall that a sequence $(s_n)$ converges to $L$ if for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that if $n \geq N$ then $|s_n - L| < \epsilon$. Given $x_n \to a$ and $y_n \to b$, it follows that there exists $N_1, N_2 \in \mathbb{N}$ such that if $n \geq N$ we have $n \geq N_1$ and $n \geq N_2$ such that $|x_n - a| < \epsilon/2$ and $|y_n - b| < \epsilon/2$. Let $N = \max\{N_1, N_2\}$. In order to show that $\lim(x_n + y_n) = a + b$, we need to show that $|(x_n + y_n) - (a + b)| < \epsilon$ (epsilon definition of equality). Observe that

$$|(x_n + y_n) - (a + b)| = |(x_n - a) + (y_n - b)| \leq |x_n - a| + |y_n - b| < \epsilon/2 + \epsilon/2 = \epsilon$$

By the triangle inequality. Thus $\lim(x_n + y_n) = a + b$. $\square$

*Proof.* Scalar: $\square$