Problem 68. *The function $f(x) = 1/x^2$ is uniformly continuous on $(1,2)$, but it is not uniformly continuous on $(0,1)$.*

Proof. **Part 1: Uniform Continuity on (1,2):** To show that the function $f(x) = 1/x^2$ is uniformly continuous on the set $S_1 = (1,2)$ we must show that for every $\epsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in S_1$:

$$|x - y| < \delta \implies |f(x) - f(y)| < \epsilon$$

Let $\epsilon > 0$, Observe that

$$|f(x) - f(y)| = |\frac{1}{x^2} - \frac{1}{y^2}| = |\frac{y_2 - x^2}{x^2 y_2}| = \frac{|x+y||x-y|}{x^2 y^2}$$

Since $x, y \in (1,2)$, it follows that $x^2 > 1$ and $y^2 > 1$, so $x^2y^2 > 1$. Furthermore $|x + y| < 2 + 2 = 4$. Therefore:

$$|f(x) - f(y)| = \frac{|x-y||x+y|}{x^2y^2} < \frac{|x-y| * 4}{1} = 4|x-y|$$

Choose $\delta = \frac{\epsilon}{4}$. Then whenever $|x - y| < \delta$, we have:

$$|f(x) - f(y)| < 4|x - y| < 4 * \frac{\epsilon}{4} = \epsilon$$

Since δ only depends on ϵ, f is uniformly continous on $(1,2)$ $\square$

Proof. **Part 2: Not Uniformly continuous on (0,1):** Suppose for the sake of contradiction that f is uniformly continuous on the set $S_2 = (0,1)$. Choose $\epsilon = 1$. Then there exists a $\delta > 0$ such that for all $x, y \in S_2$ we have

$$|x - y| < \delta \implies |\frac{1}{x^2} - \frac{1}{y^2}| < 1$$

Choose $x \in S_2$ with $x < \delta$ and choose $y = \frac{x}{2}$. Note that $y \in S_2$ because $x \in S_2$ so $y = \frac{x}{2} < \frac{1}{2} < 1$. Then

$$|x - y| = |x - \frac{x}{2}| = |\frac{x}{2}| = \frac{x}{2} < \frac{\delta}{2} < \delta$$

So $|x - y| < \delta$ holds, it follows that

$$|\frac{1}{x^2} - \frac{1}{y^2}| = |\frac{1}{x^2} - \frac{1}{(x/2)^2}| = |\frac{1}{x^2} - \frac{1}{(x^2/4)}| = |\frac{1}{x^2} - \frac{4}{x^2}| = |\frac{-3}{x^2}| = \frac{3}{x^2} > 1$$

So $|\frac{1}{x^2} - \frac{1}{y_2}| > 1$. A contradiction, thus f is not uniformly continuous on $S_2 = (0,1)$. $\square$

Problem 69. *We say that a function $f : A \to \mathbb{R}$ is* Lipschitz *if there exists $M > 0$ so that*

$$\frac{|f(x) - f(y)|}{|x - y|} \leq M$$

for all $x, y \in A$. If f is Lipschitz then f is uniformly continuous.

Proof. Since f is Lipschitz, there exists $M > 0$ such that $|f(x) - f(y)| \leq M|x - y|$ for all $x, y \in A$. Let $\delta = \frac{\epsilon}{M}$ and let $x, y \in A$ be any two points such that $|x - y| < \delta$. Then it follows that

$$|f(x) - f(y)| \leq M|x - y| \implies |f(x) - f(y)| < M * \delta$$

Since $|x - y| < \delta$. Observe that

$$|f(x) - f(y)| < M * \delta = M * \frac{\epsilon}{M} = \epsilon$$

Thus, whenever $|x - y| < \delta$, we have $|f(x) - f(y) < \epsilon$. Note that M is a constant and δ only depends on ϵ. Therefore proving that if f is Lipschitz, then f is uniformly continuous. □

Problem 70. *Let f and g be functions defined on an interval A. Assume both are differentiable at some point $c \in A$, and suppose $k \in \mathbb{R}$. Then*

(i) $(f + g)'(c) = f'(c) + g'(c)$

(ii) $(kf)'(c) = kf'(c)$

Proof. **(i)** Since f and g are differentiable at some point $c \in A$, then $f'(c) = \lim_{h\to\infty} \frac{f(c+h)-f(c)}{h}$ and $g'(c) = \lim_{h\to\infty} \frac{g(c+h)-g(c)}{h}$ by definition. We want to show that $(f + g)'(c) = f'(c) + g'(c)$. Observe that

$$\begin{aligned}
(f + g)'(c) &= \lim_{h\to 0} \frac{(f + g)(c + h) - (f + g)(c)}{h} \\
&= \lim_{h\to 0} \frac{[f(c + h) + g(c + h)] - [f(c) + g(c)]}{h} \\
&= \lim_{h\to 0} \frac{[f(c + h) - f(c)] + [g(c + h) - g(c)]}{h} \\
&= \lim_{h\to 0} \frac{f(c + h) - f(c)}{h} + \lim_{h\to 0} \frac{g(c + h) - g(c)}{h} \\
&= f'(c) + g'(c)
\end{aligned}$$

Thus $(f + g)'(c) = f'(c) + g'(c)$. □

Proof. **(ii)** □

Problem 71. *Let $h(x) = 1/x$ and $\ell(x) = 1/x^2$. For $c \neq 0$, we have*

$$h'(c) = -\frac{1}{c^2}, \qquad \ell'(c) = -\frac{2}{c^3}$$

Proof. □

Problem 72. *Let f and g be functions defined on an interval A. Assume both are differentiable at some point $c \in A$, and suppose $g(c) \neq 0$. Then*

$$\left(\frac{f}{g}\right)'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{g(c)^2}.$$

Proof. □

Problem 73. *For $a \in \mathbb{R}$, let*

$$f_a(x) = \begin{cases} x^a, & \text{if } x > 0 \\ 0, & \text{if } x \leq 0 \end{cases}$$

(a) For which values of a is $f_a(x)$ continuous at $x = 0$?

(b) What is the derivative $f_a'(x)$, and what is its domain? For which values of a is $f_a(x)$ differentiable at $x = 0$? When is the derivative function $f_a'(x)$ continuous?