

**Problem 6.**

$$\bigcap_{n=1}^{\infty} (0, 1/n) = \emptyset.$$

*Proof.* Let  $S = \bigcap_{i=1}^{\infty} (0, 1/n) = \emptyset$ . Let  $x \in \mathbb{R}$ . Consider the following 3 cases.

**Case 1:** Suppose  $x \leq 0$ , then  $x \notin S$  as  $x \notin (0, 1)$ .

**Case 2:** Suppose  $x \geq 1$ , then  $x \notin S$  as  $x \notin (0, 1)$ .

**Case 3:** Suppose  $0 < x < 1$ , Choose  $n \in \mathbb{N}$  so that  $n > \frac{1}{x}$ . Then  $x > \frac{1}{n}$ . so  $x \notin (0, \frac{1}{n})$ , thus  $x \notin S$

These cases show that an arbitrary  $x \in \mathbb{R}$  is not in  $S$ .  $\square$

**Problem 7.** Given a function  $f$  and a subset  $A$  of its domain, consider the image  $f(A) = \{f(x) : x \in A\}$ .

- (a) An example of a function  $f$ , and two subsets  $A, B$  of the domain of  $f$ , for which  $f(A \cap B) \neq f(A) \cap f(B)$  is

$$f(x) = |x|$$

where set  $A$  is a subset of the domain defined by  $A = \{-2, -1\}$  and where set  $B$  is a subset of the domain defined by  $B = \{1, 2\}$ . Observe that  $f(A \cap B) = \emptyset$  and  $f(A) \cap f(B) = \{1, 2\}$ .

- (b) If  $A, B$  are subsets of the domain of  $f$  then  $f(A \cup B)$  IS RELATED IN SOME WAY TO  $f(A) \cup f(B)$ .

*Proposition:* If  $f$  is a function and  $A, B$  are subsets of the domain of  $f$ , then  $f(A \cup B) = f(A) \cup f(B)$ .

*Proof.* Let  $y \in f(A \cup B)$ . Then there exists  $x \in A \cup B$  such that  $f(x) = y$ . Since  $x \in A \cup B$ , then either  $x \in A$  or  $x \in B$ . If  $x \in A$ , then  $y = f(x) \in f(A) \subseteq f(A) \cup f(B)$ . If  $x \in B$ , then  $y = f(x) \in f(B) \subseteq f(A) \cup f(B)$ . Thus  $f(A \cup B) \subseteq f(A) \cup f(B)$ .

Conversely, let  $y \in f(A) \cup f(B)$ . Then either  $y \in f(A)$  or  $y \in f(B)$ . If  $y \in f(A)$ , then there exists  $x \in A$  such that  $f(x) = y$ . Since  $A \subseteq A \cup B$ , we have  $x \in A \cup B$ , so  $y \in f(A \cup B)$ . Similarly, if  $y \in f(B)$ , then  $y \in f(A \cup B)$ . Therefore  $f(A) \cup f(B) \subseteq f(A \cup B)$ .

Since  $f(A \cup B) \subseteq f(A) \cup f(B)$  and  $f(A \cup B) \supseteq f(A) \cup f(B)$  have been shown to be true, it follows that  $f(A \cup B) = f(A) \cup f(B)$ .  $\square$

**Problem 8.** If  $a \in \mathbb{R}$  is an upper bound for  $A \subset \mathbb{R}$ , and if  $a$  is also an element of  $A$ , then  $a = \sup A$ .

*Proof.* Since  $a$  is an upper bound for  $A$ , we know that  $x \leq a$  for all  $x \in A$ .

We want to show that  $a = \sup A$ , we need to prove that  $a$  is the least upper bound. Choose  $b$  be any upper bound for  $A$ . Since  $a \in A$  and we know that  $a$  is an upper bound and  $b$  is an upper bound for  $A$ , it must be the case that  $a \leq b$  and  $b \leq a$  since by definition an upper bound for a subset is an element greater than or equal to every element within that subset.

Therefore,  $b = a = \sup A$ . □

**Problem 9.** (a) Let  $A = \{m/n : m, n \in \mathbb{N} \text{ with } m < n\}$ . Then  $\inf A = 0$  and  $\sup A = 1$ .

(b) Let  $B = \{(-1)^m/n : n, m \in \mathbb{N}\}$ . Then  $\inf B = -1$  and  $\sup B = 1$ .

(c) Let  $C = \{n/(3n+1) : n \in \mathbb{N}\}$ . Then  $\inf C = 1/4$  and  $\sup C = 1/3$ .

(d) Let  $D = \{m/(m+n) : m, n \in \mathbb{N}\}$ . Then  $\inf D = 0$  and  $\sup D = 1$ .

**Problem 10.** (a) If  $A$  and  $B$  are nonempty, bounded, and satisfy  $A \subseteq B$  then  $\sup A \leq \sup B$ .

*Proof.* Given  $A \subseteq B$ , every element of  $A$  is also an element of  $B$ . Since  $B$  is bounded above,  $\sup B$  exists and is an upper bound for  $B$ . Therefore,  $\sup B$  is also an upper bound for  $A$ .

Since  $\sup A$  is the least upper bound of  $A$ , then  $\sup B$  is an upper bound for  $A$ , it must be the case that  $\sup A \leq \sup B$ . □

(b) If  $\sup A < \inf B$  for nonempty sets  $A$  and  $B$ , then there exists  $c \in \mathbb{R}$  such that  $a < c < b$  for all  $a \in A$  and  $b \in B$ .

*Proof.* Given that  $\sup A < \inf B$ , we can choose any  $c$  such that  $\sup A < c < \inf B$ . Choose  $c$  to be the average of  $\sup A$  and  $\inf B$  such that  $c = \frac{\sup A + \inf B}{2}$ .

Since  $c > \sup A$  and  $\sup A$  is an upper bound for  $A$ , we have  $a \leq \sup A < c$  for all  $a \in A$ .

Since  $c < \inf B$  and  $\inf B$  is a lower bound for  $B$ , we have  $c < \inf B \leq b$  for all  $b \in B$ .

Therefore  $a < c < b$  for all  $a \in A$  and  $b \in B$ . □

(c) If there exists  $c \in \mathbb{R}$  satisfying  $a < c < b$  for all  $a \in A$  and  $b \in B$  then  $\sup A < \inf B$ .

*This is false because we could have  $\sup A = c$  or  $\inf B = c$  which would give us  $\sup A = \inf B$ .*

**Problem 11.** Denote the irrational numbers by  $\mathbb{I} = \mathbb{R} \setminus \mathbb{Q}$ .

(a) If  $a, b \in \mathbb{Q}$  then  $ab \in \mathbb{Q}$  and  $a + b \in \mathbb{Q}$ .

*Proof.* Since  $a, b \in \mathbb{Q}$ , we can write  $a = \frac{p}{q}$  and  $b = \frac{r}{s}$  where  $p, r \in \mathbb{Z}$  and  $q, s \in \mathbb{N}$ .

For multiplication:  $ab = \frac{p}{q} \cdot \frac{r}{s} = \frac{pr}{qs}$ . Since  $pr \in \mathbb{Z}$  (integers are closed under multiplication) and  $qs \in \mathbb{N}$  (positive integers are closed under multiplication), we have  $ab \in \mathbb{Q}$ .

For addition:  $a + b = \frac{p}{q} + \frac{r}{s} = \frac{ps+qr}{qs}$ . Since  $ps+qr \in \mathbb{Z}$  (integers are closed under multiplication and addition) and  $qs \in \mathbb{N}$ , we have  $a + b \in \mathbb{Q}$ .  $\square$

(b) If  $a \in \mathbb{Q}$  and  $t \in \mathbb{I}$  then  $a + t \in \mathbb{I}$ . If also  $a \neq 0$  then  $at \in \mathbb{I}$ .

*Proof.* We prove both claims by contradiction.

For addition: Suppose  $a + t \in \mathbb{Q}$ . Since  $a \in \mathbb{Q}$  and  $a + t \in \mathbb{Q}$ , by part (a) we have  $(a + t) + (-a) \in \mathbb{Q}$ . But  $(a + t) + (-a) = t$ , so  $t \in \mathbb{Q}$ , contradicting that  $t \in \mathbb{I}$ . Therefore,  $a + t \in \mathbb{I}$ .

For multiplication (with  $a \neq 0$ ): Suppose  $at \in \mathbb{Q}$ . Since  $a \in \mathbb{Q}$  with  $a \neq 0$ , we have  $\frac{1}{a} \in \mathbb{Q}$ . By part (a),  $(at) \cdot \frac{1}{a} \in \mathbb{Q}$ . But  $(at) \cdot \frac{1}{a} = t$ , so  $t \in \mathbb{Q}$ , contradicting that  $t \in \mathbb{I}$ . Therefore,  $at \in \mathbb{I}$ .  $\square$

(c) Suppose  $s, t \in \mathbb{I}$ . Then PROPOSITION ABOUT WHETHER  $st$  AND  $s + t$  ARE EITHER RATIONAL OR IRRATIONAL IN GENERAL.

*Proof.* For  $s + t$ : If  $s = \sqrt{2}$  and  $t = -\sqrt{2}$ , then  $s + t = 0 \in \mathbb{Q}$ . However, if  $s = \sqrt{2}$  and  $t = \sqrt{3}$ , then  $s + t = \sqrt{2} + \sqrt{3} \in \mathbb{I}$ . To see this, suppose  $\sqrt{2} + \sqrt{3} = r \in \mathbb{Q}$ . Then  $\sqrt{3} = r - \sqrt{2}$ , and squaring both sides gives  $3 = r^2 - 2r\sqrt{2} + 2$ , which implies  $\sqrt{2} = \frac{r^2-1}{2r} \in \mathbb{Q}$ , a contradiction.

For  $st$ : If  $s = \sqrt{2}$  and  $t = \sqrt{2}$ , then  $st = 2 \in \mathbb{Q}$ . However, if  $s = \sqrt{2}$  and  $t = \sqrt{3}$ , then  $st = \sqrt{6} \in \mathbb{I}$ , since if  $\sqrt{6} \in \mathbb{Q}$ , then 6 would be a perfect square, which it is not.

Therefore, the irrational numbers are not closed under addition or multiplication.  $\square$

Therefore, the irrational numbers are not closed under addition or multiplication.

**Problem 12.** For all  $n \in \mathbb{N}$ ,  $2^n \geq n$ .

*Proof.* We prove by induction on  $n$ .

Base case: For  $n = 1$ , we have  $2^1 = 2 \geq 1$ , which is true.

Inductive step: Assume  $2^k \geq k$  for some  $k \in \mathbb{N}$ . We need to show  $2^{k+1} \geq k + 1$ .

Starting from the inductive hypothesis:

$$\begin{aligned} 2^k &\geq k \\ 2 \cdot 2^k &\geq 2k \quad (\text{multiplying both sides by 2}) \\ 2^{k+1} &\geq 2k \\ 2^{k+1} &\geq k + k \\ 2^{k+1} &\geq k + 1 \quad (\text{since } k \geq 1 \text{ for all } k \in \mathbb{N}) \end{aligned}$$

Therefore, by mathematical induction,  $2^n \geq n$  for all  $n \in \mathbb{N}$ . □

**Problem 13.** Let  $y_1 = 6$  and, for each  $n \in \mathbb{N}$ , let  $y_{n+1} = (2y_n - 6)/3$ .

(a) For all  $n \in \mathbb{N}$ ,  $y_n \geq -6$ .

*Proof.* We prove by induction on  $n$ .

Base case: For  $n = 1$ , we have  $y_1 = 6 \geq -6$ , which is true.

Inductive step: Assume  $y_k \geq -6$  for some  $k \in \mathbb{N}$ . We need to show  $y_{k+1} \geq -6$ .

From the inductive hypothesis:  $y_k \geq -6$

$$\begin{aligned} y_k &\geq -6 \\ 2y_k &\geq -12 \\ 2y_k - 6 &\geq -18 \\ \frac{2y_k - 6}{3} &\geq -6 \\ y_{k+1} &\geq -6 \end{aligned}$$

Therefore, by mathematical induction,  $y_n \geq -6$  for all  $n \in \mathbb{N}$ . □

(b) The sequence  $(y_1, y_2, y_3, \dots)$  is decreasing.

*Proof.* We prove by induction that  $y_{n+1} < y_n$  for all  $n \in \mathbb{N}$ .

Base case: For  $n = 1$ , we have:

$$\begin{aligned} y_2 &= \frac{2y_1 - 6}{3} = \frac{2(6) - 6}{3} = \frac{6}{3} = 2 \\ y_1 &= 6 \end{aligned}$$

Since  $2 < 6$ , we have  $y_2 < y_1$ .

Inductive step: Assume  $y_{k+1} < y_k$  for some  $k \in \mathbb{N}$ . We need to show  $y_{k+2} < y_{k+1}$ .

We have:

$$\begin{aligned} y_{k+2} - y_{k+1} &= \frac{2y_{k+1} - 6}{3} - y_{k+1} \\ &= \frac{2y_{k+1} - 6 - 3y_{k+1}}{3} \\ &= \frac{-y_{k+1} - 6}{3} \\ &= -\frac{y_{k+1} + 6}{3} \end{aligned}$$

From part (a), we know  $y_{k+1} \geq -6$ , so  $y_{k+1} + 6 \geq 0$ . Since  $y_1 = 6 > -6$  and the recurrence relation preserves this (as shown in part (a)), we actually have  $y_{k+1} > -6$ , which means  $y_{k+1} + 6 > 0$ .

Therefore,  $y_{k+2} - y_{k+1} = -\frac{y_{k+1}+6}{3} < 0$ , so  $y_{k+2} < y_{k+1}$ .

By mathematical induction,  $y_{n+1} < y_n$  for all  $n \in \mathbb{N}$ .

□