

**Problem 1.** *There is no rational number whose square is 2.*

*Proof.* Assume, for contradiction, that there exist integers  $p$  and  $q$  satisfying

$$\frac{p}{q} = \sqrt{2},$$

where  $p/q$  is a rational number in lowest terms. By squaring, this is the same as  $\frac{p^2}{q^2} = 2$ , and by clearing denominators it is the same as

$$p^2 = 2q^2.$$

Thus  $p^2$  is divisible by 2, an even number. This implies that  $p$  is also divisible by 2 and can be expressed in the form  $p = 2k$  for some  $k \in \mathbb{Z}$ . If we substitute the  $p$  in  $p^2 = 2q^2$  for  $2k$ , we get

$$(2k)^2 = 4(k^2) = 2q^2$$

Further reducing this gives us

$$2(k^2) = q^2$$

Therefore  $p$  and  $q$  are both even, contradicting the assumption that  $\frac{p}{q}$  is in lowest terms. □

**Problem 2.** (a) *The negation of “For all real numbers satisfying  $a < b$ , there exists  $n \in \mathbb{N}$  such that  $a + (1/n) < b$ ” is “There exists a real number  $a, b$  satisfying  $a < b$  such that for all  $n \in \mathbb{N}$ ,  $a + (1/n) \geq b$ .”*

(b) *The negation of “There exists a real number  $x > 0$  such that  $x < 1/n$  for all  $n \in \mathbb{N}$ ” is “For all real numbers  $x > 0$ , there exists an  $n \in \mathbb{N}$  such that  $x \geq 1/n$ .”*

(b) *The negation of “Between every two distinct real numbers there is a rational number” is “There exists  $x, y \in \mathbb{R}$ , where  $x \neq y$ , such that there is no  $n \in \mathbb{Q}$  that satisfies  $x < n < y$ .”*

**Problem 3.** *Suppose  $a$  and  $b$  are real numbers. Then*

$$(a) \quad |a - b| \leq |a| + |b|$$

*Proof. Case 1:* Suppose  $a > b$ , then  $|a - b| = a - b$  (since  $a - b > 0$ ). If  $a > 0$  is true, then  $|a - b| = a - b = |a| - b$ . Since  $-b \leq |b|$  (because  $-b = |b|$  if  $b$  is negative and  $-b \leq b = |b|$  if  $b$  is non-negative), we have

$$|a - b| = |a| - b \leq |a| + |b|$$

. On the other hand, if  $a < 0$ , then

$$|a - b| = a - b \leq |a| - b \leq |a| + |b|$$

(since  $a \leq |a|$ , as before).

**Case 2:** Suppose  $b > a$ , then  $|a - b| = b - a$  (since  $b - a > 0$ ). If  $b > 0$  is also true, then  $|a - b| = b - a \leq b + |a|$  (since  $-a \leq |a|$ ).

If  $b \geq 0$ , then  $|b| = b$ , so

$$|a - b| = b - a \leq b + |a| = |a| + |b|$$

(since  $-a \leq |a|$ ).

If  $b < 0$ , then since  $b > a$ , we have  $a < b < 0$ , so  $|a| = -a$  and  $|b| = -b$ . Thus

$$|a - b| = b - a = b + (-a) \leq -b + (-a) = |b| + |a| = |a| + |b|$$

or more briefly

$$|a - b| \leq |a| + |b|$$

**Case 3:** Suppose  $a = b$ , then  $|a - b| = 0 \leq |a| + |b|$ . Since absolute values are non-negative.  $\square$

$$(b) \quad ||a| - |b|| \leq |a - b|$$

*Proof.* Observe that  $a = (a - b) + b$ . It follows that

$$|a| = |(a - b) + b| \leq |a - b| + |b|$$

by the triangle inequality. Subtracting  $|b|$  from the right and left sides of the inequality gives us  $|a| - |b| \leq |a - b|$ . Likewise observe that  $b = (b - a) - a$ . It follows that

$$|b| = |(b - a) + a| \leq |b - a| + |a|$$

by the triangle inequality. Since  $|b - a| = |a - b|$ , subtracting  $|a|$  from either sides of the inequality above gives us  $|b| - |a| \leq |a - b|$ . Since  $||a| - |b||$  is either  $|a| - |b|$  or  $|b| - |a|$ , and both are at most  $|a - b|$ , it follows that  $||a| - |b|| \leq |a - b|$ .  $\square$

**Problem 4.** Give an example of each, or state that it is impossible.

(a)  $f : \mathbb{N} \rightarrow \mathbb{N}$  that is one-to-one but not onto.

**My Answer:** The function  $f(n) = 2n$  is a mapping from  $\mathbb{N} \rightarrow \mathbb{N}$ . That is one-to-one since distinct  $n$  produce distinct even numbers, but not onto because it misses odd numbers.

(b)  $f : \mathbb{N} \rightarrow \mathbb{N}$  that is onto but not one-to-one.

**My Answer:** The function  $f(n) = \lfloor \frac{n+1}{2} \rfloor$  (For clarity's sake this is a floor function) is onto since every  $m \in \mathbb{N}$  is hit, but not one-to-one because  $f(1) = f(2) = 1$ .

(d)  $f : \mathbb{N} \rightarrow \mathbb{Z}$  that is one-to-one and onto.

**My Answer:** The piecewise function

$$f(n) = \begin{cases} \frac{n}{2} & \text{If } n \text{ is even} \\ -\frac{n-1}{2} & \text{If } n \text{ is odd} \end{cases}$$

Is a mapping from  $\mathbb{N} \rightarrow \mathbb{Z}$  that is both one-to-one and onto since it uniquely maps even  $n$  to non-negative integers and odd  $n$  to negative integers, this covers all of  $\mathbb{Z}$ .

**Problem 5.** There exists an infinite collection of sets  $A_1, A_2, A_3, \dots$  with the properties that every  $A_i$  has an infinite number of elements, and  $A_i \cap A_j = \emptyset$  for all  $i \neq j$ , and  $\bigcup_{i=1}^{\infty} A_i = \mathbb{N}$ .

*Proof.* A natural number is  $k$ -almost-prime if it has  $k$  prime factors where  $k \in \mathbb{N}$ . Let  $A_1 \subseteq \mathbb{N}$  be the union of the sets  $\{1\}$  and the set containing all  $k$ -almost-prime numbers  $\{2, 3, 5, 7, 11, 17, \dots\}$ , where  $k = 1$  so that  $A_1 = \{1, 2, 3, 5, 7, 11, 17, \dots\}$ . Let  $A_2 \subseteq \mathbb{N}$  be the set containing all  $k$ -almost-prime numbers  $\{4, 6, 9, 10, 14, 15, 21, 22, \dots\}$ , where  $k = 2$ . Let  $A_3 \subseteq \mathbb{N}$  be set containing all  $k$ -almost-prime numbers  $\{8, 12, 18, 20, 27, 28, 30, \dots\}$ , where  $k = 3$ . Similarly let  $A_k \subseteq \mathbb{N}$  be the set containing all  $k$ -almost-prime numbers for some  $k \in \mathbb{N}$ . Suppose  $m \in \mathbb{N}$  and  $m \in A_i \cap A_j$  for some  $i, j \in \mathbb{N}$ . Note that by the uniqueness of prime decompositions,  $m$  has a unique prime decomposition of a fixed length. Thus  $i = j$ . Since  $A_n \subseteq \mathbb{N}$  for all  $n \in \mathbb{N}$ , it follows that  $\bigcup_{n=1}^{\infty} A_n \subseteq \mathbb{N}$ . Similarly by fundamental theorem of arithmetic,  $n$  has a unique prime decomposition of a fixed length  $m$ . Therefore  $n \in A_m$  and  $\mathbb{N} \subseteq \bigcup_{n=1}^{\infty} A_n$ . Thus  $\bigcup_{n=1}^{\infty} A_n = \mathbb{N}$ .  $\square$