Section 5.3: Marginal and Conditional Probability Distributions

Exercise (5.23). In Example 5.4 and Exercise 5.5, we considered the joint density of Y_1, the proportion of the capacity of the tank that is stocked at the beginning of the week, and Y_2, the proportion of the capacity sold during the week, given by

$$f(y_1, y_2) = \begin{cases} 3y_1, & 0 \le y_2 \le y_1 \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Find the marginal density function for Y_2.

Solution:

$$\begin{aligned} f_2(y_2) &= \int_{y_2}^{1} 3y_1 dy_1 = \frac{3y_1^2}{2}\Big]_{y_2}^{1} = \frac{3}{2} - \frac{3y_2^2}{2} \\ &= \frac{3}{2}(1 - y_2^2) \end{aligned}$$

□

(b) For what values of y_2 is the conditional density $f(y_1|y_2)$ defined?

Solution:

$$\begin{aligned} f(y_1|y_2) &= \frac{f(y_1, y_2)}{f_2(y_2)} = \frac{3y_1}{\frac{3}{2}(1 - y_2^2)} = \frac{2y_1}{1 - y_2^2} \\ &= \begin{cases} \frac{2y_1}{1-y_2^2}, & y_2 \le y_1 \le 1 \\ 0, & \text{elsewhere} \end{cases} \end{aligned}$$

So its defined for $0 \le y_2 < 1$ to avoid division by 0. □

(c) What is the probability that more than half a tank is sold given that three-fourths of a tank is stocked?

Solution: We have $P(Y_2 > 1/2|Y_1 = 3/4)$

$$f_1(y_1) = \int_0^{y_1} 3y_1 dy_2 = 3y_1y_2\Big]_0^{y_1} = 3y_1^2$$

$$f(y_2|y_1) = \frac{3y_1}{3y_1^2} = \frac{1}{y_1}$$

$$P(Y_2 > 1/2|Y_1 = 3/4) = \int_{1/2}^{3/4} \frac{1}{3/4} dy_2 = \int_{1/2}^{3/4} \frac{4}{3} dy_2 = \frac{4}{3}y_2\Big]_{1/2}^{3/4}$$

$$= (\frac{4}{3} * \frac{3}{4}) - (\frac{4}{3} * \frac{1}{2}) = 1 - \frac{4}{6} = \frac{1}{3}$$

□

Exercise (5.24). In Exercise 5.6, we assumed that if a radioactive particle is randomly located in a square with sides of unit length, a reasonable model for the joint density function for Y_1 and Y_2 is

$$f(y_1, y_2) = \begin{cases} 1, & 0 \le y_1 \le 1, 0 \le y_2 \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Find the marginal density functions for Y_1 and Y_2.

Solution:

$$f_1(y_1) = 1 \text{ for } 0 \le y_1 \le 1$$

$$f_2(y_2) = 1 \text{ for } 0 \le y_2 \le 1$$

□

(b) What is $P(.3 < Y_1 < .5)$? $P(.3 < Y_2 < .5)$?

Solution: Since its a rectangular region for each

$$P(.3 < Y_1 < .5) = P(.3 < Y_2 < .5) = .2 * 1 = .2$$

□

(c) For what values of y_2 is the conditional density $f(y_1|y_2)$ defined?

Solution:

$$f(y_1|y_2) = \frac{f(y_1, y_2)}{f_2(y_2)} = \frac{1}{1} = 1$$

Defined for $0 \leq y_2 \leq 1$. □

(d) For any y_2, $0 \leq y_2 \leq 1$ what is the conditional density function of Y_1 given that $Y_2 = y_2$?

Solution: The conditional probablity we calculated in the last one so

$$f(y_1|y_2) = \begin{cases} 1, & 0 \leq y_1 \leq 1 \\ 0, & \text{elsewhere} \end{cases}$$

For any y_2 where $0 \leq y_2 \leq 1$. □

(e) Find $P(.3 < Y_1 < .5|Y_2 = .3)$.

Solution: $P(.3 < Y_1 < .5|Y_2 = .3) = .2/1 = .2$ □

(f) Find $P(.3 < Y_1 < .5|Y_2 = .5)$.

Solution: $P(.3 < Y_1 < .5|Y_2 = .5) = .2/1 = .2$ □

(g) Compare the answers that you obtained in parts (a), (d), and (e). For any y_2, $0 \leq y_2 \leq 1$ how does $P(.3 < Y_1 < .5)$ compare to $P(.3 < Y_1 < .5|Y_2 = y_2)$?

Solution: There are all the same anwsers. □

Exercise (5.25). Let Y_1 and Y_2 have joint density function first encountered in Exercise 5.7:

$$f(y_1, y_2) = \begin{cases} e^{-(y_1+y_2)}, & y_1 > 0, y_2 > 0, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Find the marginal density functions for Y_1 and Y_2. Identify these densities as one of those studied in Chapter 4.

Solution: Exponential Distribution with

$$f_1(y_1) = \int_0^\infty e^{-(y_1+y_2)} dy_2 = \int_0^\infty e^{-y_1} * e^{-y_2} dy_2 = e^{-y_1} \int_0^\infty e^{-y_2} dy_2$$
$$= e^{-y_1}[-e^{-y_2}\Big]_0^\infty] = e^{-y_1} \lim_{y_2 \to \infty} -e^{-y_2} + 1 = e^{-y_1} * (0+1) = e^{-y_1}$$
$$f_2(y_2) = \int_0^\infty e^{-(y_1+y_2)} dy_1 = e^{-y_2} \int_0^\infty e^{-y_1} dy_1 = e^{-y_2}$$

□

(b) What is $P(1 < Y_1 < 2.5)$? $P(1 < Y_2 < 2.5)$?

Solution:

$$P(1 < Y_1 < 2.5) = \int_1^{2.5} e^{-y_1} = -e^{-y_1}\Big]_1^{2.5} = -e^{-2.5} + e^{-1} = 0.28579$$
$$P(1 < Y_2 < 2.5) = \int_1^{2.5} e^{-y_2} = 0.28579$$

□

(c) For what values of y_2 is the conditional density $f(y_1|y_2)$ defined?

Solution:

$$f(y_1|y_2) = \frac{e^{-(y_1+y_2)}}{e^{-y_2}} = e^{-y_1}$$

Define for $y_2 > 0$. □

(d) For any $y_2 > 0$, what is the conditional density function of Y_1 given that $Y_2 = y_2$?

Solution: $f(y_1|y_2) = e^{-y_1}$ □

(e) For any $y_1 > 0$, what is the conditional density function of Y_2 given that $Y_1 = y_1$?

Solution: $f(y_2|y_1) = e^{-y_2}$ □

(f) For any $y_2 > 0$, how does the conditional density function $f(y_1|y_2)$ that you obtained in part (d) compare to the marginal density function $f_1(y_1)$ found in part (a)?

Solution: They are the same □

(g) What does your answer to part (f) imply about marginal and conditional probabilities that Y_1 falls in any interval?

Solution: They are the same □

Exercise (5.27). In Exercise 5.9, we determined that

$$f(y_1, y_2) = \begin{cases} 6(1-y_2), & 0 \le y_1 \le y_2 \le 1, \\ 0, & \text{elsewhere} \end{cases}$$

is a valid joint probability density function. Find

(a) the marginal density functions for Y_1 and Y_2.

Solution:

$$\begin{aligned} f_1(y_1) &= \int_{y_1}^{1} 6(1-y_2)dy_2 = 6(\int_{y_1}^{1} 1dy_2 - \int_{y_1}^{1} y_2 dy_2) = 6(y_2\Big]_{y_1}^{1} - \frac{y_2^2}{2}\Big]_{y_1}^{1}) \\ &= 6((1-y_1) - (\frac{1}{2} - \frac{y_1^2}{2}) = 6(-y_1 + \frac{1}{2} + \frac{y_1^2}{2}) = 6\frac{1+y_1^2-2y_1}{2} \\ &= 3(1+y_1^2-2y_1) = 3(1-y_1)^2 \\ f_2(y_2) &= \int_0^{y_2} 6(1-y_2)dy_1 = 6\int_0^{y_2}(1-y_2)dy_1 = 6(y_1 - y_2y_1)\Big]_0^{y_2} \\ &= 6((y_2-0)-(y_2^2-0)) = 6y_2(1-y_2) \end{aligned}$$

□

(b) $P(Y_2 \le 1/2 | Y_1 \le 3/4)$.

Solution:

$$\begin{aligned} P &= \int_0^{1/2}\int_{y_1}^{1/2} 6(1-y_2)dy_2dy_1 = 6\int_0^{1/2}[y_2 - \frac{y_2^2}{2}]_{y_1}^{1/2}dy_1 \\ &= 6\int_0^{1/2}((1/2-y_1)-(1/8-\frac{y_1^2}{2}))dy_1 = 6\int_0^{1/2}(\frac{3}{8} - y_1 + \frac{y_1^2}{2})dy_1 \\ &= 6[\frac{3}{8}y_1 - \frac{y_1^2}{2} + \frac{y_1^3}{6}]_0^{1/2} = 6[\frac{3}{16} - \frac{1}{8} + \frac{1}{48}] = 6[\frac{9-6+1}{48}] \\ &= \frac{24}{48} = \frac{1}{2} = 0.5 \\ P(Y_1 < 3/4) &= \int_0^{3/4} 3(1-y_1)^2dy_1 = 3\int_0^{3/4}(1-2y_1+y_1^2d)dy_1 = 3(y_1 - y_1^2 + \frac{y_1^3}{3}]_0^{3/4} \\ &= 3(\frac{3}{4} - \frac{9}{16} + \frac{27}{192}) = 0.984375 \end{aligned}$$

$$P(Y_2 \le 1/2 | Y_1 \le 3/4) = \frac{0.5}{0.984375} = 0.506937$$

□

(c) the conditional density function of Y_1 given $Y_2 = y_2$.

Solution: $f(y_1|y_2) = \frac{6(1-y_2)}{6y_2(1-y_2^2)} = \frac{1-y_2}{y_2(1-y_2)} = \frac{1}{y_2}$ □

(d) the conditional density function of Y_2 given $Y_1 = y_1$.

Solution: $f(y_2|y_1) = \frac{6(1-y_2)}{3(1-y_1)^2} = \frac{2(1-y_2)}{(1-y_1)^2}$ □

(e) $P(Y_2 \geq 3/4|Y_1 = 1/2)$.

Solution:

$$\begin{aligned}
f(y_2|y_1 = 1/2) &= \frac{2(1-y_2)}{(1-(1/2))^2} = \frac{2(1-y_2)}{1/4} = 8(1-y_2) \\
P(Y_2 \geq 3/4|y_1 = 1/2) &= \int_{3/4}^{1} 8(1-y_2)d_y 2 = 8\int_{3/4}^{1}(1-y_2)dy_2 \\
&= 8(y_2 - \frac{y_2^2}{2}]_{3/4}^{1} = 8((1-(3/4)) - ((1/2)-(9/32))) \\
&= 0.25
\end{aligned}$$

□

Exercise (5.29). Refer to Exercise 5.11. Find

(a) the marginal density functions for Y_1 and Y_2.

Solution: Triangle with vertices $(-1,0),(1,0),(0,1)$ has area 1 so $f(y_1,y_2) = 1$.

$$\begin{aligned}
f_1(y_1) &= \int_0^{1-|y_1|} 1dy_2 = 1-|y_1|, \quad -1 \leq y_1 \leq 1 \\
f_2(y_2) &= \int_{-(1-y_2)}^{1-y_2} 1dy_1 = 2(1-y_2), \quad 0 \leq y_2 \leq 1
\end{aligned}$$

□

(b) $P(Y_2 > 1/2|Y_1 = 1/4)$.

Solution:

$$f(y_2|y_1) = \frac{1}{1-|y_1|} = \frac{1}{3/4} = \frac{4}{3}, \quad 0 \le y_2 \le 3/4$$
$$P(Y_2 > 1/2|Y_1 = 1/4) = \int_{1/2}^{3/4} \frac{4}{3} dy_2 = \frac{4}{3} \cdot \frac{1}{4} = \frac{1}{3}$$

□

Exercise (5.32). Suppose that the random variables Y_1 and Y_2 have joint probability density function, $f(y_1, y_2)$, given by (see Exercise 5.14)

$$f(y_1, y_2) = \begin{cases} 6y_1^2 y_2, & 0 \le y_1 \le y_2, y_1 + y_2 \le 2, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Show that the marginal density of Y_1 is a beta density with $\alpha = 3$ and $\beta = 2$.

Solution:

$$f_1(y_1) = \int_{y_1}^{2-y_1} 6y_1^2 y_2 dy_2 = 3y_1^2 [y_2^2]_{y_1}^{2-y_1} = 3y_1^2[(2-y_1)^2 - y_1^2]$$
$$= 3y_1^2[4 - 4y_1] = 12y_1^2(1-y_1), \quad 0 \le y_1 \le 1$$

Beta($\alpha = 3, \beta = 2$) has density $\frac{\Gamma(5)}{\Gamma(3)\Gamma(2)} y^2(1-y) = \frac{24}{2} y^2(1-y) = 12y^2(1-y)$. □

(b) Derive the marginal density of Y_2.

Solution:

$$f_2(y_2) = \begin{cases} \int_0^{y_2} 6y_1^2 y_2 dy_1 = 2y_2 y_1^3 \Big]_0^{y_2} = 2y_2^4, & 0 \le y_2 \le 1 \\ \int_0^{2-y_2} 6y_1^2 y_2 dy_1 = 2y_2(2-y_2)^3, & 1 < y_2 \le 2 \end{cases}$$

□

(c) Derive the conditional density of Y_2 given $Y_1 = y_1$.

Solution:

$$f(y_2|y_1) = \frac{6y_1^2 y_2}{12y_1^2(1-y_1)} = \frac{y_2}{2(1-y_1)}, \quad y_1 \le y_2 \le 2 - y_1$$

□

(d) Find $P(Y_2 < 1.1 | Y_1 = .60)$.

Solution:

$$f(y_2|y_1 = 0.6) = \frac{y_2}{2(0.4)} = 1.25y_2, \quad 0.6 \le y_2 \le 1.4$$

$$P(Y_2 < 1.1|Y_1 = 0.6) = \int_{0.6}^{1.1} 1.25y_2 dy_2 = 0.625[y_2^2]_{0.6}^{1.1} = 0.625(1.21 - 0.36) = 0.53125$$

□

Exercise (5.33). Suppose that Y_1 is the total time between a customer's arrival in the store and departure from the service window, Y_2 is the time spent in line before reaching the window, and the joint density of these variables (as was given in Exercise 5.15) is

$$f(y_1, y_2) = \begin{cases} e^{-y_1}, & 0 \le y_2 \le y_1 \le \infty, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Find the marginal density functions for Y_1 and Y_2.

Solution:

$$f_1(y_1) = \int_0^{y_1} e^{-y_1} dy_2 = y_1 e^{-y_1}, \quad y_1 > 0$$

$$f_2(y_2) = \int_{y_2}^{\infty} e^{-y_1} dy_1 = -e^{-y_1}\Big]_{y_2}^{\infty} = e^{-y_2}, \quad y_2 > 0$$

□

(b) What is the conditional density function of Y_1 given that $Y_2 = y_2$? Be sure to specify the values of y_2 for which this conditional density is defined.

Solution:

$$f(y_1|y_2) = \frac{e^{-y_1}}{e^{-y_2}} = e^{-(y_1 - y_2)}, \quad y_1 \ge y_2$$

Defined for $y_2 > 0$. □

(c) What is the conditional density function of Y_2 given that $Y_1 = y_1$? Be sure to specify the values of y_1 for which this conditional density is defined.

Solution:

$$f(y_2|y_1) = \frac{e^{-y_1}}{y_1 e^{-y_1}} = \frac{1}{y_1}, \quad 0 \le y_2 \le y_1$$

Defined for $y_1 > 0$. □

(d) Is the conditional density function $f(y_1|y_2)$ that you obtained in part (b) the same as the marginal density function $f_1(y_1)$ found in part (a)?

Solution: No. $f_1(y_1) = y_1 e^{-y_1}$ while $f(y_1|y_2) = e^{-(y_1-y_2)}$. □

(e) What does your answer to part (d) imply about marginal and conditional probabilities that Y_1 falls in any interval?

Solution: They are different, so Y_1 and Y_2 are dependent. □

Exercise (5.35). Refer to Exercise 5.33. If two minutes elapse between a customer's arrival at the store and his departure from the service window, find the probability that he waited in line less than one minute to reach the window.

Solution: From 5.33(c), $f(y_2|y_1) = \frac{1}{y_1}$ for $0 \leq y_2 \leq y_1$.

$$P(Y_2 < 1|Y_1 = 2) = \int_0^1 \frac{1}{2} dy_2 = \frac{1}{2}$$

□

Exercise (5.37). In Exercise 5.18, Y_1 and Y_2 denoted the lengths of life, in hundreds of hours, for components of types I and II, respectively, in an electronic system. The joint density of Y_1 and Y_2 is given by

$$f(y_1, y_2) = \begin{cases} (1/8) y_1 e^{-(y_1+y_2)/2}, & y_1 > 0, y_2 > 0 \\ 0, & \text{elsewhere.} \end{cases}$$

Find the probability that a component of type II will have a life length in excess of 200 hours.

Solution:

$$\begin{aligned} f_2(y_2) &= \int_0^\infty \frac{1}{8} y_1 e^{-(y_1+y_2)/2} dy_1 = \frac{1}{8} e^{-y_2/2} \int_0^\infty y_1 e^{-y_1/2} dy_1 \\ &= \frac{1}{8} e^{-y_2/2} \cdot 4 = \frac{1}{2} e^{-y_2/2} \\ P(Y_2 > 2) &= \int_2^\infty \frac{1}{2} e^{-y_2/2} dy_2 = -e^{-y_2/2} \Big]_2^\infty = e^{-1} \approx 0.368 \end{aligned}$$

□

Section 5.4: Independent Random Variables

Exercise (5.43). Let Y_1 and Y_2 have joint density function $f(y_1, y_2)$ and marginal densities $f_1(y_1)$ and $f_2(y_2)$, respectively. Show that Y_1 and Y_2 are independent if and only if $f(y_1|y_2) = f_1(y_1)$ for all values of y_1 and for all y_2 such that $f_2(y_2) > 0$. A completely analogous argument establishes that Y_1 and Y_2 are independent if and only if $f(y_2|y_1) = f_2(y_2)$ for all values of y_2 and for all y_1 such that $f_1(y_1) > 0$.

Solution: ($\Rightarrow$) If independent, $f(y_1, y_2) = f_1(y_1)f_2(y_2)$, so $f(y_1|y_2) = \frac{f(y_1,y_2)}{f_2(y_2)} = \frac{f_1(y_1)f_2(y_2)}{f_2(y_2)} = f_1(y_1)$.

($\Leftarrow$) If $f(y_1|y_2) = f_1(y_1)$, then $\frac{f(y_1,y_2)}{f_2(y_2)} = f_1(y_1)$, so $f(y_1, y_2) = f_1(y_1)f_2(y_2)$, which means independent. □

Exercise (5.45). In Exercise 5.1, we determined that the joint distribution of Y_1, the number of contracts awarded to firm A, and Y_2, the number of contracts awarded to firm B, is given by the entries in the following table.

y_1 \ y_2	0	1	2
0	1/9	2/9	1/9
1	2/9	2/9	0
2	1/9	0	0

The marginal probability function of Y_1 was derived in Exercise 5.19 to be binomial with $n = 2$ and $p = 1/3$. Are Y_1 and Y_2 independent? Why?

Solution: No. For independence we need $p(y_1, y_2) = p_1(y_1)p_2(y_2)$ for all (y_1, y_2). We have $p_1(2) = \binom{2}{2}(1/3)^2 = 1/9$ and $p(2, 0) = 1/9$. If independent, $p_2(0) = 1$. But $p(0, 0) = 1/9 \neq p_1(0)p_2(0) = (4/9)(1) = 4/9$. Not independent. □

Exercise (5.47). In Exercise 5.3, we determined that the joint probability distribution of Y_1, the number of married executives, and Y_2, the number of never-married executives, is given by

$$p(y_1, y_2) = \frac{\binom{4}{y_1}\binom{3}{y_2}\binom{2}{3-y_1-y_2}}{\binom{9}{3}},$$

where y_1 and y_2 are integers, $0 \leq y_1 \leq 3$, $0 \leq y_2 \leq 3$, and $1 \leq y_1 + y_2 \leq 3$. Are Y_1 and Y_2 independent? (Recall your answer to Exercise 5.21.)

Solution: No. Sampling without replacement, so dependent. □

Exercise (5.49). In Example 5.4 and Exercise 5.5, we considered the joint density of Y_1, the proportion of the capacity of the tank that is stocked at the beginning of the week and Y_2, the proportion of the capacity sold during the week, given by

$$f(y_1, y_2) = \begin{cases} 3y_1, & 0 \leq y_2 \leq y_1 \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Show that Y_1 and Y_2 are dependent.

Solution: From 5.23: $f_1(y_1) = 3y_1^2$, $f_2(y_2) = \frac{3}{2}(1 - y_2^2)$. Since $f(y_1, y_2) = 3y_1 \neq f_1(y_1)f_2(y_2)$, dependent. ☐

Exercise (5.51). In Exercise 5.7, we considered Y_1 and Y_2 with joint density function

$$f(y_1, y_2) = \begin{cases} e^{-(y_1+y_2)}, & y_1 > 0, y_2 > 0, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Are Y_1 and Y_2 independent?

Solution: Yes. $f(y_1, y_2) = e^{-(y_1+y_2)} = e^{-y_1} \cdot e^{-y_2} = f_1(y_1)f_2(y_2)$ from 5.25. ☐

(b) Does the result from part (a) explain the results you obtained in Exercise 5.25 (d)–(f)? Why?

Solution: Yes. Independence implies $f(y_1|y_2) = f_1(y_1)$, which is what we found in 5.25. ☐

Exercise (5.53). In Exercise 5.9, we determined that

$$f(y_1, y_2) = \begin{cases} 6(1 - y_2), & 0 \leq y_1 \leq y_2 \leq 1, \\ 0, & \text{elsewhere} \end{cases}$$

is a valid joint probability density function. Are Y_1 and Y_2 independent?

Solution: No. Support region depends on both variables. ☐

Exercise (5.55). Suppose that, as in Exercise 5.11, Y_1 and Y_2 are uniformly distributed over the triangle shaded in the accompanying diagram. Are Y_1 and Y_2 independent?

Solution: No. Support region depends on both variables. ☐

Exercise (5.57). In Exercises 5.13 and 5.31, the joint density function of Y_1 and Y_2 was given by

$$f(y_1, y_2) = \begin{cases} 30y_1y_2^2, & y_1 - 1 \leq y_2 \leq 1 - y_1, 0 \leq y_1 \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Are the random variables Y_1 and Y_2 independent?

Solution: No. Support region depends on both variables. ☐

Exercise (5.59). If Y_1 is the total time between a customer's arrival in the store and leaving the service window and if Y_2 is the time spent in line before reaching the window, the joint density of these variables, according to Exercise 5.15, is

$$f(y_1, y_2) = \begin{cases} e^{-y_1}, & 0 \leq y_2 \leq y_1 \leq \infty \\ 0, & \text{elsewhere.} \end{cases}$$

Are Y_1 and Y_2 independent?

Solution: No. From 5.33, $f(y_1|y_2) \neq f_1(y_1)$, so dependent. □

Exercise (5.61). In Exercise 5.18, Y_1 and Y_2 denoted the lengths of life, in hundreds of hours, for components of types I and II, respectively, in an electronic system. The joint density of Y_1 and Y_2 is

$$f(y_1, y_2) = \begin{cases} (1/8)y_1 e^{-(y_1+y_2)/2}, & y_1 > 0, y_2 > 0, \\ 0, & \text{elsewhere.} \end{cases}$$

Are Y_1 and Y_2 independent?

Solution: No. Cannot factor as $g(y_1)h(y_2)$ due to y_1 term. □

Exercise (5.63). Let Y_1 and Y_2 be independent exponentially distributed random variables, each with mean 1. Find $P(Y_1 > Y_2 | Y_1 < 2Y_2)$.

Solution:

$$\begin{aligned} P(Y_1 > Y_2, Y_1 < 2Y_2) &= \int_0^\infty \int_{y_2}^{2y_2} e^{-y_1} e^{-y_2} dy_1 dy_2 = \int_0^\infty e^{-y_2}(e^{-y_2} - e^{-2y_2}) dy_2 \\ &= \int_0^\infty (e^{-2y_2} - e^{-3y_2}) dy_2 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6} \\ P(Y_1 < 2Y_2) &= \int_0^\infty \int_0^{2y_2} e^{-y_1} e^{-y_2} dy_1 dy_2 = \int_0^\infty e^{-y_2}(1 - e^{-2y_2}) dy_2 = 1 - \frac{1}{3} = \frac{2}{3} \\ P(Y_1 > Y_2 | Y_1 < 2Y_2) &= \frac{1/6}{2/3} = \frac{1}{4} \end{aligned}$$

□

Section 5.6: Expected Value of a Function of Random Variables

Exercise (5.73). In Exercise 5.3, we determined that the joint probability distribution of Y_1, the number of married executives, and Y_2, the number of never-married executives, is given by

$$p(y_1, y_2) = \frac{\binom{4}{y_1}\binom{3}{y_2}\binom{2}{3-y_1-y_2}}{\binom{9}{3}},$$

where y_1 and y_2 are integers, $0 \leq y_1 \leq 3$, $0 \leq y_2 \leq 3$, and $1 \leq y_1 + y_2 \leq 3$. Find the expected number of married executives among the three selected for promotion. (See Exercise 5.21.)

Solution: Hypergeometric: $\mathbb{E}(Y_1) = n \cdot \frac{r}{N} = 3 \cdot \frac{4}{9} = \frac{4}{3}$. □

Exercise (5.75). Refer to Exercises 5.7, 5.25, and 5.51. Let Y_1 and Y_2 have joint density function

$$f(y_1, y_2) = \begin{cases} e^{-(y_1+y_2)}, & y_1 > 0, y_2 > 0 \\ 0, & \text{elsewhere.} \end{cases}$$

(a) What are $E(Y_1 + Y_2)$ and $V(Y_1 + Y_2)$?

Solution: Y_1, Y_2 independent Exp(1), so $\mathbb{E}(Y_1) = \mathbb{E}(Y_2) = 1$, $V(Y_1) = V(Y_2) = 1$.

$$\mathbb{E}(Y_1 + Y_2) = 2, \quad V(Y_1 + Y_2) = 2$$

□

(b) What is $P(Y_1 - Y_2 > 3)$?

Solution:

$$P(Y_1 - Y_2 > 3) = \int_3^\infty \int_0^{y_1-3} e^{-y_1}e^{-y_2}dy_2dy_1 = \int_3^\infty e^{-y_1}(1 - e^{-(y_1-3)})dy_1 = e^{-3}$$

□

(c) What is $P(Y_1 - Y_2 < -3)$?

Solution: By symmetry, $P(Y_1 - Y_2 < -3) = P(Y_2 - Y_1 > 3) = e^{-3}$. □

(d) What are $E(Y_1 - Y_2)$ and $V(Y_1 - Y_2)$?

Solution:

$$\mathbb{E}(Y_1 - Y_2) = 0, \quad V(Y_1 - Y_2) = 2$$

□

(e) What do you notice about $V(Y_1 + Y_2)$ and $V(Y_1 - Y_2)$?

Solution: They are equal: both equal 2. □

Exercise (5.77). In Exercise 5.9, we determined that

$$f(y_1, y_2) = \begin{cases} 6(1 - y_2), & 0 \le y_1 \le y_2 \le 1, \\ 0, & \text{elsewhere} \end{cases}$$

is a valid joint probability density function. Find

(a) $E(Y_1)$ and $E(Y_2)$.

Solution: From 5.27: $f_1(y_1) = 3(1-y_1)^2$ and $f_2(y_2) = 6y_2(1-y_2)$.

$$\mathbb{E}(Y_1) = \int_0^1 y_1 \cdot 3(1-y_1)^2 dy_1 = 3\int_0^1 (y_1 - 2y_1^2 + y_1^3) dy_1 = 3[\frac{1}{2} - \frac{2}{3} + \frac{1}{4}] = \frac{1}{4}$$
$$\mathbb{E}(Y_2) = \int_0^1 y_2 \cdot 6y_2(1-y_2) dy_2 = 6\int_0^1 (y_2^2 - y_2^3) dy_2 = 6[\frac{1}{3} - \frac{1}{4}] = \frac{1}{2}$$

□

(b) $V(Y_1)$ and $V(Y_2)$.

Solution:

$$\mathbb{E}(Y_1^2) = 3\int_0^1 y_1^2(1-y_1)^2 dy_1 = 3\int_0^1 (y_1^2 - 2y_1^3 + y_1^4) dy_1 = 3[\frac{1}{3} - \frac{1}{2} + \frac{1}{5}] = \frac{1}{10}$$
$$V(Y_1) = \frac{1}{10} - \frac{1}{16} = \frac{3}{80}$$
$$\mathbb{E}(Y_2^2) = 6\int_0^1 y_2^3(1-y_2) dy_2 = 6[\frac{1}{4} - \frac{1}{5}] = \frac{3}{10}$$
$$V(Y_2) = \frac{3}{10} - \frac{1}{4} = \frac{1}{20}$$

□

(c) $E(Y_1 - 3Y_2)$.

Solution:

$$\mathbb{E}(Y_1 - 3Y_2) = \frac{1}{4} - 3 \cdot \frac{1}{2} = -\frac{5}{4}$$

□

Exercise (5.79). Suppose that, as in Exercise 5.11, Y_1 and Y_2 are uniformly distributed over the triangle shaded in the accompanying diagram with vertices at $(-1,0)$, $(1,0)$, and $(0,1)$. Find $E(Y_1Y_2)$.

Solution:

$$\begin{aligned}\mathbb{E}(Y_1Y_2) &= \int_{-1}^0 \int_0^{1+y_1} y_1y_2 dy_2 dy_1 + \int_0^1 \int_0^{1-y_1} y_1y_2 dy_2 dy_1 \\ &= \int_{-1}^0 \frac{y_1(1+y_1)^2}{2} dy_1 + \int_0^1 \frac{y_1(1-y_1)^2}{2} dy_1 = 0\end{aligned}$$

By symmetry around $y_1 = 0$. □

Exercise (5.81). In Exercise 5.18, Y_1 and Y_2 denoted the lengths of life, in hundreds of hours, for components of types I and II, respectively, in an electronic system. The joint density of Y_1 and Y_2 is

$$f(y_1, y_2) = \begin{cases} (1/8)y_1 e^{-(y_1+y_2)/2}, & y_1 > 0, y_2 > 0, \\ 0, & \text{elsewhere.} \end{cases}$$

One way to measure the relative efficiency of the two components is to compute the ratio Y_2/Y_1. Find $E(Y_2/Y_1)$. [Hint: In Exercise 5.61, we proved that Y_1 and Y_2 are independent.]

Solution: Not independent (from 5.61), compute directly:

$$\begin{aligned} \mathbb{E}(Y_2/Y_1) &= \int_0^\infty \int_0^\infty \frac{y_2}{y_1} \cdot \frac{1}{8} y_1 e^{-(y_1+y_2)/2} dy_2 dy_1 \\ &= \frac{1}{8} \int_0^\infty \int_0^\infty y_2 e^{-(y_1+y_2)/2} dy_2 dy_1 = \frac{1}{8} \cdot 4 \cdot 4 = 2 \end{aligned}$$

□

Exercise (5.87). Suppose that Y_1 and Y_2 are independent χ^2 random variables with ν_1 and ν_2 degrees of freedom, respectively. Find

(a) $E(Y_1 + Y_2)$.

Solution:

$$\mathbb{E}(Y_1 + Y_2) = \mathbb{E}(Y_1) + \mathbb{E}(Y_2) = \nu_1 + \nu_2$$

□

(b) $V(Y_1 + Y_2)$. [Hint: Use Theorem 5.9 and the result of Exercise 4.112(a).]

Solution:

$$V(Y_1 + Y_2) = V(Y_1) + V(Y_2) = 2\nu_1 + 2\nu_2$$

□

Section 5.7: The Covariance of Two Random Variables

Exercise (5.89). In Exercise 5.1, we determined that the joint distribution of Y_1, the number of contracts awarded to firm A, and Y_2, the number of contracts awarded to firm B, is given by the entries in the following table.

	y_2		
y_1	0	1	2
0	1/9	2/9	1/9
1	2/9	2/9	0
2	1/9	0	0

Find $\text{Cov}(Y_1, Y_2)$. Does it surprise you that $\text{Cov}(Y_1, Y_2)$ is negative? Why?

Solution:

$$\begin{aligned}
\mathbb{E}(Y_1) &= 0(4/9) + 1(4/9) + 2(1/9) = 2/3 \\
\mathbb{E}(Y_2) &= 0(4/9) + 1(4/9) + 2(1/9) = 2/3 \\
\mathbb{E}(Y_1Y_2) &= \sum y_1y_2p(y_1,y_2) = 0(1/9) + 0(2/9) + 0(1/9) + 0(2/9) + 1(2/9) + 0 = 2/9 \\
\mathrm{Cov}(Y_1,Y_2) &= \mathbb{E}(Y_1Y_2) - \mathbb{E}(Y_1)\mathbb{E}(Y_2) = 2/9 - (2/3)(2/3) = 2/9 - 4/9 = -2/9
\end{aligned}$$

Not surprising: more contracts to one firm means fewer to the other (negative association). □

Exercise (5.91). In Exercise 5.8, we derived the fact that

$$f(y_1,y_2) = \begin{cases} 4y_1y_2, & 0 \le y_1 \le 1, 0 \le y_2 \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Show that $\mathrm{Cov}(Y_1,Y_2) = 0$. Does it surprise you that $\mathrm{Cov}(Y_1,Y_2)$ is zero? Why?

Solution: $f(y_1,y_2) = 4y_1y_2 = (2y_1)(2y_2) = f_1(y_1)f_2(y_2)$, so independent. Thus $\mathrm{Cov}(Y_1,Y_2) = 0$. Not surprising: independence implies zero covariance. □

Exercise (5.92). In Exercise 5.9, we determined that

$$f(y_1,y_2) = \begin{cases} 6(1-y_2), & 0 \le y_1 \le y_2 \le 1, \\ 0, & \text{elsewhere} \end{cases}$$

is a valid joint probability density function. Find $\mathrm{Cov}(Y_1,Y_2)$. Are Y_1 and Y_2 independent?

Solution: From 5.77: $\mathbb{E}(Y_1) = 1/4$, $\mathbb{E}(Y_2) = 1/2$.

$$\begin{aligned}
\mathbb{E}(Y_1Y_2) &= \int_0^1 \int_0^{y_2} y_1y_2 \cdot 6(1-y_2)dy_1dy_2 = 6\int_0^1 y_2(1-y_2)\frac{y_2^2}{2}dy_2 \\
&= 3\int_0^1 (y_2^3 - y_2^4)dy_2 = 3[\frac{1}{4} - \frac{1}{5}] = \frac{3}{20} \\
\mathrm{Cov}(Y_1,Y_2) &= \frac{3}{20} - \frac{1}{4}\cdot\frac{1}{2} = \frac{3}{20} - \frac{1}{8} = \frac{1}{40}
\end{aligned}$$

Not independent (from 5.53). □

Exercise (5.93). Let the discrete random variables Y_1 and Y_2 have the joint probability function

$$p(y_1,y_2) = 1/3, \text{ for } (y_1,y_2) = (-1,0), (0,1), (1,0).$$

Find $\text{Cov}(Y_1, Y_2)$. Notice that Y_1 and Y_2 are dependent. (Why?) This is another example of uncorrelated random variables that are not independent.

Solution:

$$\begin{aligned}
\mathbb{E}(Y_1) &= \frac{1}{3}[(-1) + 0 + 1] = 0 \\
\mathbb{E}(Y_2) &= \frac{1}{3}[0 + 1 + 0] = \frac{1}{3} \\
\mathbb{E}(Y_1 Y_2) &= \frac{1}{3}[(-1)(0) + (0)(1) + (1)(0)] = 0 \\
\text{Cov}(Y_1, Y_2) &= 0 - 0 \cdot \frac{1}{3} = 0
\end{aligned}$$

Dependent because support is not a rectangle, but uncorrelated. □

Exercise (5.95). Suppose that, as in Exercises 5.11 and 5.79, Y_1 and Y_2 are uniformly distributed over the triangle shaded in the accompanying diagram with vertices at $(-1, 0)$, $(1, 0)$, and $(0, 1)$.

(a) Find $\text{Cov}(Y_1, Y_2)$.

Solution: From 5.29: $f_1(y_1) = 1 - |y_1|$, $f_2(y_2) = 2(1 - y_2)$. By symmetry, $\mathbb{E}(Y_1) = 0$.

$$\begin{aligned}
\mathbb{E}(Y_2) &= 2\int_0^1 y_2(1 - y_2)dy_2 = 2[\frac{1}{2} - \frac{1}{3}] = \frac{1}{3} \\
\text{Cov}(Y_1, Y_2) &= \mathbb{E}(Y_1 Y_2) - \mathbb{E}(Y_1)\mathbb{E}(Y_2) = 0 - 0 = 0 \text{ (from 5.79)}
\end{aligned}$$

□

(b) Are Y_1 and Y_2 independent? (See Exercise 5.55.)

Solution: No (from 5.55). □

(c) Find the coefficient of correlation for Y_1 and Y_2.

Solution:

$$\rho = \frac{\text{Cov}(Y_1, Y_2)}{\sqrt{V(Y_1)V(Y_2)}} = \frac{0}{\sqrt{V(Y_1)V(Y_2)}} = 0$$

□

(d) Does your answer to part (b) lead you to doubt your answer to part (a)? Why or why not?

Solution: No. Dependent variables can have zero covariance. □

Exercise (5.99). If c is any constant and Y is a random variable such that $E(Y)$ exists, show that $\text{Cov}(c, Y) = 0$.

Solution:

$$\text{Cov}(c, Y) = \mathbb{E}(cY) - \mathbb{E}(c)\mathbb{E}(Y) = c\mathbb{E}(Y) - c\mathbb{E}(Y) = 0$$

$\square$