

SECTION 5.3: MARGINAL AND CONDITIONAL PROBABILITY DISTRIBUTIONS

Exercise (5.23). In Example 5.4 and Exercise 5.5, we considered the joint density of Y_1 , the proportion of the capacity of the tank that is stocked at the beginning of the week, and Y_2 , the proportion of the capacity sold during the week, given by

$$f(y_1, y_2) = \begin{cases} 3y_1, & 0 \leq y_2 \leq y_1 \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

- (a) Find the marginal density function for Y_2 .

Solution:

$$\begin{aligned} f_2(y_2) &= \int_{y_2}^1 3y_1 dy_1 = \left[\frac{3y_1^2}{2} \right]_{y_2}^1 = \frac{3}{2} - \frac{3y_2^2}{2} \\ &= \frac{3}{2}(1 - y_2^2) \end{aligned}$$

□

- (b) For what values of y_2 is the conditional density $f(y_1|y_2)$ defined?

Solution:

$$\begin{aligned} f(y_1|y_2) &= \frac{f(y_1, y_2)}{f_2(y_2)} = \frac{3y_1}{\frac{3}{2}(1 - y_2^2)} = \frac{2y_1}{1 - y_2^2} \\ &= \begin{cases} \frac{2y_1}{1 - y_2^2}, & y_2 \leq y_1 \leq 1 \\ 0, & \text{elsewhere} \end{cases} \end{aligned}$$

So its defined for $0 \leq y_2 < 1$ to avoid division by 0.

□

- (c) What is the probability that more than half a tank is sold given that three-fourths of a tank is stocked?

Solution: We have $P(Y_2 > 1/2 | Y_1 = 3/4)$

$$\begin{aligned} f_1(y_1) &= \int_0^{y_1} 3y_1 dy_2 = 3y_1 y_2 \Big|_0^{y_1} = 3y_1^2 \\ f(y_2|y_1) &= \frac{3y_1}{3y_1^2} = \frac{1}{y_1} \\ P(Y_2 > 1/2 | Y_1 = 3/4) &= \int_{1/2}^{3/4} \frac{1}{3/4} dy_2 = \int_{1/2}^{3/4} \frac{4}{3} dy_2 = \frac{4}{3} y_2 \Big|_{1/2}^{3/4} \\ &= \left(\frac{4}{3} * \frac{3}{4}\right) - \left(\frac{4}{3} * \frac{1}{2}\right) = 1 - \frac{4}{6} = \frac{1}{3} \end{aligned}$$

□

Exercise (5.24). In Exercise 5.6, we assumed that if a radioactive particle is randomly located in a square with sides of unit length, a reasonable model for the joint density function for Y_1 and Y_2 is

$$f(y_1, y_2) = \begin{cases} 1, & 0 \leq y_1 \leq 1, 0 \leq y_2 \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

- (a) Find the marginal density functions for Y_1 and Y_2 .

Solution:

$$f_1(y_1) = 1 \text{ for } 0 \leq y_1 \leq 1$$

$$f_2(y_2) = 1 \text{ for } 0 \leq y_2 \leq 1$$

□

- (b) What is $P(.3 < Y_1 < .5)$? $P(.3 < Y_2 < .5)$?

Solution: Since its a rectangular region for each

$$P(.3 < Y_1 < .5) = P(.3 < Y_2 < .5) = .2 * 1 = .2$$

□

- (c) For what values of y_2 is the conditional density $f(y_1|y_2)$ defined?

Solution:

$$f(y_1|y_2) = \frac{f(y_1, y_2)}{f_2(y_2)} = \frac{1}{1} = 1$$

Defined for $0 \leq y_2 \leq 1$. □

- (d) For any y_2 , $0 \leq y_2 \leq 1$ what is the conditional density function of Y_1 given that $Y_2 = y_2$?

Solution: The conditional probability we calculated in the last one so

$$f(y_1|y_2) = \begin{cases} 1, & 0 \leq y_1 \leq 1 \\ 0, & \text{elsewhere} \end{cases}$$

For any y_2 where $0 \leq y_2 \leq 1$. □

- (e) Find $P(.3 < Y_1 < .5|Y_2 = .3)$.

Solution: $P(.3 < Y_1 < .5|Y_2 = .3) = .2/1 = .2$ □

- (f) Find $P(.3 < Y_1 < .5|Y_2 = .5)$.

Solution: $P(.3 < Y_1 < .5|Y_2 = .5) = .2/1 = .2$ □

- (g) Compare the answers that you obtained in parts (a), (d), and (e). For any y_2 , $0 \leq y_2 \leq 1$ how does $P(.3 < Y_1 < .5)$ compare to $P(.3 < Y_1 < .5|Y_2 = y_2)$?

Solution: There are all the same answers. □

Exercise (5.25). Let Y_1 and Y_2 have joint density function first encountered in Exercise 5.7:

$$f(y_1, y_2) = \begin{cases} e^{-(y_1+y_2)}, & y_1 > 0, y_2 > 0, \\ 0, & \text{elsewhere.} \end{cases}$$

- (a) Find the marginal density functions for Y_1 and Y_2 . Identify these densities as one of those studied in Chapter 4.

Solution: □

- (b) What is $P(1 < Y_1 < 2.5)? P(1 < Y_2 < 2.5)?$

Solution: □

- (c) For what values of y_2 is the conditional density $f(y_1|y_2)$ defined?

Solution: □

(d) For any $y_2 > 0$, what is the conditional density function of Y_1 given that $Y_2 = y_2$?

Solution:

□

(e) For any $y_1 > 0$, what is the conditional density function of Y_2 given that $Y_1 = y_1$?

Solution:

□

(f) For any $y_2 > 0$, how does the conditional density function $f(y_1|y_2)$ that you obtained in part (d) compare to the marginal density function $f_1(y_1)$ found in part (a)?

Solution:

□

(g) What does your answer to part (f) imply about marginal and conditional probabilities that Y_1 falls in any interval?

Solution:

□

Exercise (5.27). In Exercise 5.9, we determined that

$$f(y_1, y_2) = \begin{cases} 6(1 - y_2), & 0 \leq y_1 \leq y_2 \leq 1, \\ 0, & \text{elsewhere} \end{cases}$$

is a valid joint probability density function. Find

(a) the marginal density functions for Y_1 and Y_2 .

Solution:

□

(b) $P(Y_2 \leq 1/2 | Y_1 \leq 3/4)$.

Solution:

□

(c) the conditional density function of Y_1 given $Y_2 = y_2$.

Solution:

□

(d) the conditional density function of Y_2 given $Y_1 = y_1$.

Solution:

□

(e) $P(Y_2 \geq 3/4 | Y_1 = 1/2)$.

Solution:

□

Exercise (5.29). Refer to Exercise 5.11. Find

(a) the marginal density functions for Y_1 and Y_2 .

Solution:

□

(b) $P(Y_2 > 1/2 | Y_1 = 1/4)$.

Solution:

□

Exercise (5.32). Suppose that the random variables Y_1 and Y_2 have joint probability density function, $f(y_1, y_2)$, given by (see Exercise 5.14)

$$f(y_1, y_2) = \begin{cases} 6y_1^2 y_2, & 0 \leq y_1 \leq y_2, y_1 + y_2 \leq 2, \\ 0, & \text{elsewhere.} \end{cases}$$

- (a) Show that the marginal density of Y_1 is a beta density with $\alpha = 3$ and $\beta = 2$.

Solution:

□

- (b) Derive the marginal density of Y_2 .

Solution:

□

- (c) Derive the conditional density of Y_2 given $Y_1 = y_1$.

Solution:

□

- (d) Find $P(Y_2 < 1.1 | Y_1 = .60)$.

Solution:

□

Exercise (5.33). Suppose that Y_1 is the total time between a customer's arrival in the store and departure from the service window, Y_2 is the time spent in line before reaching the window, and the joint density of these variables (as was given in Exercise 5.15) is

$$f(y_1, y_2) = \begin{cases} e^{-y_1}, & 0 \leq y_2 \leq y_1 \leq \infty, \\ 0, & \text{elsewhere.} \end{cases}$$

- (a) Find the marginal density functions for Y_1 and Y_2 .

Solution:

□

- (b) What is the conditional density function of Y_1 given that $Y_2 = y_2$? Be sure to specify the values of y_2 for which this conditional density is defined.

Solution:

□

- (c) What is the conditional density function of Y_2 given that $Y_1 = y_1$? Be sure to specify the values of y_1 for which this conditional density is defined.

Solution:

□

- (d) Is the conditional density function $f(y_1|y_2)$ that you obtained in part (b) the same as the marginal density function $f_1(y_1)$ found in part (a)?

Solution:

□

- (e) What does your answer to part (d) imply about marginal and conditional probabilities that Y_1 falls in any interval?

Solution:

□

Exercise (5.35). Refer to Exercise 5.33. If two minutes elapse between a customer's arrival at the store and his departure from the service window, find the probability that he waited in line less than one minute to reach the window.

Solution:

□

Exercise (5.37). In Exercise 5.18, Y_1 and Y_2 denoted the lengths of life, in hundreds of hours, for components of types I and II, respectively, in an electronic system. The joint density of Y_1 and Y_2 is given by

$$f(y_1, y_2) = \begin{cases} (1/8)y_1 e^{-(y_1+y_2)/2}, & y_1 > 0, y_2 > 0 \\ 0, & \text{elsewhere.} \end{cases}$$

Find the probability that a component of type II will have a life length in excess of 200 hours.

Solution:

□

SECTION 5.4: INDEPENDENT RANDOM VARIABLES

Exercise (5.43). Let Y_1 and Y_2 have joint density function $f(y_1, y_2)$ and marginal densities $f_1(y_1)$ and $f_2(y_2)$, respectively. Show that Y_1 and Y_2 are independent if and only if $f(y_1|y_2) = f_1(y_1)$ for all values of y_1 and for all y_2 such that $f_2(y_2) > 0$. A completely analogous argument establishes that Y_1 and Y_2 are independent if and only if $f(y_2|y_1) = f_2(y_2)$ for all values of y_2 and for all y_1 such that $f_1(y_1) > 0$.

Solution:

□

Exercise (5.45). In Exercise 5.1, we determined that the joint distribution of Y_1 , the number of contracts awarded to firm A, and Y_2 , the number of contracts awarded to firm B, is given by the entries in the following table.

y_1	y_2		
	0	1	2
0	1/9	2/9	1/9
1	2/9	2/9	0
2	1/9	0	0

The marginal probability function of Y_1 was derived in Exercise 5.19 to be binomial with $n = 2$ and $p = 1/3$. Are Y_1 and Y_2 independent? Why?

Solution:

□

Exercise (5.47). In Exercise 5.3, we determined that the joint probability distribution of Y_1 , the number of married executives, and Y_2 , the number of never-married executives, is given

by

$$p(y_1, y_2) = \frac{\binom{4}{y_1} \binom{3}{y_2} \binom{2}{3-y_1-y_2}}{\binom{9}{3}},$$

where y_1 and y_2 are integers, $0 \leq y_1 \leq 3$, $0 \leq y_2 \leq 3$, and $1 \leq y_1 + y_2 \leq 3$. Are Y_1 and Y_2 independent? (Recall your answer to Exercise 5.21.)

Solution:

□

Exercise (5.49). In Example 5.4 and Exercise 5.5, we considered the joint density of Y_1 , the proportion of the capacity of the tank that is stocked at the beginning of the week and Y_2 , the proportion of the capacity sold during the week, given by

$$f(y_1, y_2) = \begin{cases} 3y_1, & 0 \leq y_2 \leq y_1 \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Show that Y_1 and Y_2 are dependent.

Solution:

□

Exercise (5.51). In Exercise 5.7, we considered Y_1 and Y_2 with joint density function

$$f(y_1, y_2) = \begin{cases} e^{-(y_1+y_2)}, & y_1 > 0, y_2 > 0, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Are Y_1 and Y_2 independent?

Solution:

□

(b) Does the result from part (a) explain the results you obtained in Exercise 5.25 (d)–(f)? Why?

Solution:

□

Exercise (5.53). In Exercise 5.9, we determined that

$$f(y_1, y_2) = \begin{cases} 6(1 - y_2), & 0 \leq y_1 \leq y_2 \leq 1, \\ 0, & \text{elsewhere} \end{cases}$$

is a valid joint probability density function. Are Y_1 and Y_2 independent?

Solution:

□

Exercise (5.55). Suppose that, as in Exercise 5.11, Y_1 and Y_2 are uniformly distributed over the triangle shaded in the accompanying diagram. Are Y_1 and Y_2 independent?

Solution:

□

Exercise (5.57). In Exercises 5.13 and 5.31, the joint density function of Y_1 and Y_2 was given by

$$f(y_1, y_2) = \begin{cases} 30y_1y_2^2, & y_1 - 1 \leq y_2 \leq 1 - y_1, 0 \leq y_1 \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Are the random variables Y_1 and Y_2 independent?

Solution:

□

Exercise (5.59). If Y_1 is the total time between a customer's arrival in the store and leaving the service window and if Y_2 is the time spent in line before reaching the window, the joint density of these variables, according to Exercise 5.15, is

$$f(y_1, y_2) = \begin{cases} e^{-y_1}, & 0 \leq y_2 \leq y_1 \leq \infty \\ 0, & \text{elsewhere.} \end{cases}$$

Are Y_1 and Y_2 independent?

Solution:

□

Exercise (5.61). In Exercise 5.18, Y_1 and Y_2 denoted the lengths of life, in hundreds of hours, for components of types I and II, respectively, in an electronic system. The joint density of Y_1 and Y_2 is

$$f(y_1, y_2) = \begin{cases} (1/8)y_1e^{-(y_1+y_2)/2}, & y_1 > 0, y_2 > 0, \\ 0, & \text{elsewhere.} \end{cases}$$

Are Y_1 and Y_2 independent?

Solution:

□

Exercise (5.63). Let Y_1 and Y_2 be independent exponentially distributed random variables, each with mean 1. Find $P(Y_1 > Y_2 | Y_1 < 2Y_2)$.

Solution:

□

SECTION 5.6: EXPECTED VALUE OF A FUNCTION OF RANDOM VARIABLES

Exercise (5.73). In Exercise 5.3, we determined that the joint probability distribution of Y_1 , the number of married executives, and Y_2 , the number of never-married executives, is given by

$$p(y_1, y_2) = \frac{\binom{4}{y_1} \binom{3}{y_2} \binom{2}{3-y_1-y_2}}{\binom{9}{3}},$$

where y_1 and y_2 are integers, $0 \leq y_1 \leq 3$, $0 \leq y_2 \leq 3$, and $1 \leq y_1 + y_2 \leq 3$. Find the expected number of married executives among the three selected for promotion. (See Exercise 5.21.)

Solution:

□

Exercise (5.75). Refer to Exercises 5.7, 5.25, and 5.51. Let Y_1 and Y_2 have joint density function

$$f(y_1, y_2) = \begin{cases} e^{-(y_1+y_2)}, & y_1 > 0, y_2 > 0 \\ 0, & \text{elsewhere.} \end{cases}$$

- (a) What are $E(Y_1 + Y_2)$ and $V(Y_1 + Y_2)$?

Solution:

□

- (b) What is $P(Y_1 - Y_2 > 3)$?

Solution:

□

- (c) What is $P(Y_1 - Y_2 < -3)$?

Solution:

□

- (d) What are $E(Y_1 - Y_2)$ and $V(Y_1 - Y_2)$?

Solution:

□

- (e) What do you notice about $V(Y_1 + Y_2)$ and $V(Y_1 - Y_2)$?

Solution:

□

Exercise (5.77). In Exercise 5.9, we determined that

$$f(y_1, y_2) = \begin{cases} 6(1 - y_2), & 0 \leq y_1 \leq y_2 \leq 1, \\ 0, & \text{elsewhere} \end{cases}$$

is a valid joint probability density function. Find

- (a) $E(Y_1)$ and $E(Y_2)$.

Solution:

□

- (b) $V(Y_1)$ and $V(Y_2)$.

Solution:

□

- (c) $E(Y_1 - 3Y_2)$.

Solution:

□

Exercise (5.79). Suppose that, as in Exercise 5.11, Y_1 and Y_2 are uniformly distributed over the triangle shaded in the accompanying diagram with vertices at $(-1, 0)$, $(1, 0)$, and $(0, 1)$. Find $E(Y_1 Y_2)$.

Solution:

□

Exercise (5.81). In Exercise 5.18, Y_1 and Y_2 denoted the lengths of life, in hundreds of hours, for components of types I and II, respectively, in an electronic system. The joint density of Y_1 and Y_2 is

$$f(y_1, y_2) = \begin{cases} (1/8)y_1 e^{-(y_1+y_2)/2}, & y_1 > 0, y_2 > 0, \\ 0, & \text{elsewhere.} \end{cases}$$

One way to measure the relative efficiency of the two components is to compute the ratio Y_2/Y_1 . Find $E(Y_2/Y_1)$. [Hint: In Exercise 5.61, we proved that Y_1 and Y_2 are independent.]

Solution:

□

Exercise (5.87). Suppose that Y_1 and Y_2 are independent χ^2 random variables with ν_1 and ν_2 degrees of freedom, respectively. Find

- (a) $E(Y_1 + Y_2)$.

Solution:

□

- (b) $V(Y_1 + Y_2)$. [Hint: Use Theorem 5.9 and the result of Exercise 4.112(a).]

Solution:

□

SECTION 5.7: THE COVARIANCE OF TWO RANDOM VARIABLES

Exercise (5.89). In Exercise 5.1, we determined that the joint distribution of Y_1 , the number of contracts awarded to firm A, and Y_2 , the number of contracts awarded to firm B, is given by the entries in the following table.

		y_2		
		0	1	2
y_1	0	1/9	2/9	1/9
	1	2/9	2/9	0
	2	1/9	0	0

Find $\text{Cov}(Y_1, Y_2)$. Does it surprise you that $\text{Cov}(Y_1, Y_2)$ is negative? Why?

Solution:

□

Exercise (5.91). In Exercise 5.8, we derived the fact that

$$f(y_1, y_2) = \begin{cases} 4y_1 y_2, & 0 \leq y_1 \leq 1, 0 \leq y_2 \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Show that $\text{Cov}(Y_1, Y_2) = 0$. Does it surprise you that $\text{Cov}(Y_1, Y_2)$ is zero? Why?

Solution:

□

Exercise (5.92). In Exercise 5.9, we determined that

$$f(y_1, y_2) = \begin{cases} 6(1 - y_2), & 0 \leq y_1 \leq y_2 \leq 1, \\ 0, & \text{elsewhere} \end{cases}$$

is a valid joint probability density function. Find $\text{Cov}(Y_1, Y_2)$. Are Y_1 and Y_2 independent?

Solution: □

Exercise (5.93). Let the discrete random variables Y_1 and Y_2 have the joint probability function

$$p(y_1, y_2) = 1/3, \text{ for } (y_1, y_2) = (-1, 0), (0, 1), (1, 0).$$

Find $\text{Cov}(Y_1, Y_2)$. Notice that Y_1 and Y_2 are dependent. (Why?) This is another example of uncorrelated random variables that are not independent.

Solution: □

Exercise (5.95). Suppose that, as in Exercises 5.11 and 5.79, Y_1 and Y_2 are uniformly distributed over the triangle shaded in the accompanying diagram with vertices at $(-1, 0)$, $(1, 0)$, and $(0, 1)$.

- (a) Find $\text{Cov}(Y_1, Y_2)$.

Solution: □

- (b) Are Y_1 and Y_2 independent? (See Exercise 5.55.)

Solution: □

- (c) Find the coefficient of correlation for Y_1 and Y_2 .

Solution: □

- (d) Does your answer to part (b) lead you to doubt your answer to part (a)? Why or why not?

Solution: □

Exercise (5.99). If c is any constant and Y is a random variable such that $E(Y)$ exists, show that $\text{Cov}(c, Y) = 0$.

Solution: □