3.2 Exact equations

3.2.1 8 a)

Suppose $a = b = 1$ for the Gompertz differential equation: $\frac{\delta P}{\delta t} = P(a - b \ln P)$, the following are the phase portraits for cases $P_0 > e$ and $0 < P_0 < e$:

8 b)

For $a = 1, b = -1$, cases $P_0 > e^{-1}$ and $0 < P_0 < e^{-1}$:

8 c)

Explicit solution for $P(O) = P_0$.

$$\begin{aligned}
\frac{dP}{dt} &= P(a - b\ln P) \\
\int \frac{dP}{P(a - b\ln P)} &= \int dt \\
\int \frac{d(\ln P)}{(a - b\ln P)} &= \int dt \\
-\frac{1}{b}\ln|a - b\ln P| + c &= t
\end{aligned}$$

When we plug in $P(O) = P_0$ we get $c = \frac{1}{b}|a - b\ln P_0|$ so we get:

$$\begin{aligned}
t =& \quad -\frac{1}{b}\ln|a - b\ln P| + \frac{1}{b}\ln\frac{a - b\ln P_0}{a - b\ln P} \\
\ln P(t) =& \quad \frac{a}{b}(1 - e^{-bt}) + e^{-bt}\ln P_0 \\
P(t) =& \quad \boxed{e^{\frac{a}{b}(1-e^{-bt})} P_0^{e^{-bt}}}
\end{aligned}$$

3.2.2

Suppose we have the same 16 pound cannonball shot vertically upward with an initial velocity $v_0 = 300ft/s$, the differential equation for the cannonball would be:

$$m\frac{dv}{dt} = -mg - kv^2$$

Now we solve using separations of variables:

$$-\,dt = \frac{mdv}{mg + kv^2} \tag{1}$$

$$-\,dt = \frac{1}{g}\frac{dv}{1 + (\sqrt{\frac{k}{mg}}v)^2} \tag{2}$$

$$-\int dt = \int \frac{1}{g}\sqrt{\frac{mg}{k}}\frac{\sqrt{\frac{k}{mg}}}{1 + (\sqrt{\frac{k}{mg}}v)^2}dv \tag{3}$$

$$-\,t + c = \sqrt{\frac{m}{gk}}\tan^{-1}(\sqrt{\frac{k}{mg}}v) \tag{4}$$

$$-\sqrt{\frac{kg}{m}}t + c = \tan^{-1}(\sqrt{\frac{k}{mg}}v) \tag{5}$$

$$\tan(-\sqrt{\frac{kg}{m}}t + c) = (\sqrt{\frac{k}{mg}}v) \tag{6}$$

Final

$$v(t) = \sqrt{\frac{mg}{k}}\tan(-\sqrt{\frac{kg}{m}}t + c)$$

Plugging in $v(0) = 300$ we find c:

$$v(0) = \sqrt{\frac{mg}{k}}\tan(-\sqrt{\frac{kg}{m}}(0) + c) \tag{1}$$

$$300 = \sqrt{\frac{mg}{k}}\tan(c) \tag{2}$$

$$c = \tan^{-1}(300)\sqrt{\frac{k}{mg}} \tag{3}$$

Plugging this in and the mass of 16 our final solution is:

$$\boxed{v(t) = \sqrt{\frac{16g}{k}}\tan(-\sqrt{\frac{kg}{16}}t + \tan^{-1}(300)\sqrt{\frac{k}{16g}})}$$

3.4 problem 2

We are given the equation $y(x+y+1)dx + (x+2y)dy = 0$, we first determine if it is exact or not.

3.4.2 a)

$$M(x,y) = y(x+y+1) = yx + y^2 + y \qquad \frac{\delta M}{\delta y} = x + 2y + 1$$

$$N(x,y) = x + 2y \qquad \frac{\delta N}{\delta y} = 1$$

$$\frac{\delta M}{\delta y} \neq \frac{\delta N}{\delta y}$$

$$\boxed{\text{Not Exact}}$$

For this part we add $\mu(x) = e^x$ as our integrating factor to get the equation $(yxe^x + y^2e^x + ye^x)dx + (xe^x + eye^x)dy = 0$:

3.4.2 b)

$$M(x,y) = yxe^x + y^2e^x + ye^x \qquad \frac{\delta M}{\delta y} = xe^x + 2ye^x + e^x$$

$$N(x,y) = xe^x + 2ye^x \qquad \frac{\delta N}{\delta y} = e^x + xe^x + 2ye^x$$

$$\frac{\delta M}{\delta y} = \frac{\delta N}{\delta y}$$

$$\boxed{\text{Exact Equation}}$$

Now we find the solution:

3.4.2 c)

$$f(x,y) = \int (yxe^x + y^2e^x + ye^x) \qquad = e^xy^2 + e^xy + g(y) \tag{4}$$

$$\frac{\delta f}{\delta y} = 2e^xy + xe^x = xe^x + 2ye^x \qquad g(y) = 0 \tag{5}$$

$$\boxed{e^xy^2 + e^xxy = C} \tag{6}$$