Problem 29. *Suppose $(x_n)_{n=1}^{\infty}$ converges. Let $k \in \mathbb{N}$. The new sequence $(x_{n+k})_{n=1}^{\infty}$ also converges, and to the same limit.*

Proof. Let $\epsilon > 0$. Since the sequence $(x_n)_{n=1}^{\infty}$ converges to L, there exists $N \in \mathbb{N}$ such that for all $n > N$, $|x_n - L| < \epsilon$. Now choose $M = N$ for our shifted sequence. Then for all $n > M$, we have $n + k > N$(since $k \geq 1$), so $|x_{n+k} - L| < \epsilon$. Therefore (x_{n+k}) converges to L. $\square$

Problem 30. *Give an example of each of the following, or state that such a request is impossible. In the latter case, identify specific theorem(s) that justify your statement.*

(a) *sequences (x_n) and (y_n), which both diverge, where the sum $(x_n + y_n)$ converges*

We take the alternating harmonic series $\sum_{k=1}^{\infty} \frac{(-1)^{n+1}}{n}$ which famously converges to $\ln(2)$ and define x_n as the sequence of positive terms and y_n as the sequence of negative terms.

$$x_n = \begin{cases} \frac{1}{n} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases} \qquad y_n = \begin{cases} \frac{-1}{n} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

These two sequence of partial sums converge when combined and each diverge when split this way.

(b) *a convergent sequence (x_n), and a divergent sequence (y_n), where (x_n+y_n) converges*

This is impossible, a consequence of the Algebraic Limit Theorem. If we suppose (x_n) converges and $(x_n + y_n)$ converges, then $y_n = (x_n + y_n) - x_n$ must also converge. This leads to a consequence since we assumed (y_n) does not converege.

(c) *a convergent sequence (b_n), with $b_n \neq 0$ for all n, such that $(1/b_n)$ diverges*

This one is also impossible as a consequence of the Algebraic Limit Theorem. If we suppose (b_n) converges to b_n and $b_n \neq 0$ and choose (a_n) to converge to 1, then according to the Algebraic Limit Theorem $\frac{(a_n)}{(b_n)} = \frac{1}{(b_n)}$ must also converge.

(d) *sequences (x_n) and (y_n), where $(x_n y_n)$ and (x_n) converge but (y_n) does not*

If we let $(x_n) = \frac{1}{n^3}$ which converges and $(y_n) = n$ which diverges, we get $(x_n y_n) = \frac{1}{n^2}$ which converges.

Problem 31. *If $a \geq 0$ and $b \geq 0$ then $\sqrt{ab} \leq \dfrac{1}{2}(a+b)$.*

Proof. Suppose $a \geq 0$ and $b \geq 0$, then it follows that $(a-b)^2 \geq 0$. Expanding this gives

$$a^2 - 2ab + b^2 \geq 0$$

add $2ab$ to both sides

$$a^2 + b^2 \geq 2ab$$

add another $2ab$ to both sides

$$a^2 + 2ab + b^2 \geq 4ab$$

Since $a + b$ and $\sqrt{ab}$ are non-negative, we can take square roots and get

$$a + b \geq 2\sqrt{ab}$$

Dividing by 2:

$$\frac{1}{2}(a + b) \geq \sqrt{ab}$$

Therefore $\sqrt{ab} \leq \dfrac{1}{2}(a + b)$. □

Problem 32. *Consider the real sequence generated by setting $x_1 = 2$ and then*

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right).$$

(a) The sequence (x_n) is bounded below by $\sqrt{2}$.

Proof. We will prove by induction that $x_n \geq \sqrt{2}$ for all $n \geq 1$.

Base case: For $n = 1$, we have $x_1 = 2 > \sqrt{2}$ since $2 > 1.414...$

Inductive step: Assume $x_n \geq \sqrt{2}$ for some $n \geq 1$. We must show $x_{n+1} \geq \sqrt{2}$.

By the recurrence relation, $x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right)$. Since $x_n \geq \sqrt{2} > 0$ by the inductive hypothesis, both x_n and $\frac{2}{x_n}$ are positive. By the Arithmetic-Geometric Mean Inequality (Problem 31) with $a = x_n$ and $b = \frac{2}{x_n}$:

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right) \geq \sqrt{x_n \cdot \frac{2}{x_n}} = \sqrt{2}$$

Therefore $x_{n+1} \geq \sqrt{2}$. By induction, $x_n \geq \sqrt{2}$ for all $n \geq 1$. □

(b) $\lim_{n\to\infty} x_n = \sqrt{2}$.

Proof. We will show that (x_n) is monotone decreasing and bounded below, which by the Monotone Convergence Theorem implies the limit exists.

We first show (x_n) is decreasing for $n \geq 1$. We need $x_{n+1} \leq x_n$, which is equivalent to:

$$\frac{1}{2}\left(x_n + \frac{2}{x_n}\right) \leq x_n$$

Multiplying both sides by $2x_n > 0$:

$$x_n^2 + 2 \leq 2x_n^2$$

$$2 \leq x_n^2$$

This holds since $x_n \geq \sqrt{2}$ by part (a), so $x_n^2 \geq 2$. Thus (x_n) is monotone decreasing.

We know from part (a), (x_n) is bounded below by $\sqrt{2}$.

By the Monotone Convergence Theorem, (x_n) converges. Let $L = \lim_{n\to\infty} x_n$.

We now take the limit of both sides of the recurrence relation:

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \frac{1}{2}\left(x_n + \frac{2}{x_n}\right)$$

by problem 29 we know that, $\lim_{n\to\infty} x_{n+1} = L$. By the Algebraic Limit Theorem:

$$L = \frac{1}{2}\left(L + \frac{2}{L}\right)$$

Multiplying by $2L$ (note $L \geq \sqrt{2} > 0$):

$$2L^2 = L^2 + 2$$

$$L^2 = 2$$

$$L = \pm\sqrt{2}$$

Since $x_n \geq \sqrt{2} > 0$ for all n, we have $L > 0$, so $L = \sqrt{2}$. $\square$

Problem 33. *The sequence* $\sqrt{2}, \sqrt{2+\sqrt{2}}, \sqrt{2+\sqrt{2+\sqrt{2}}}, \ldots$ *converges to* X.

Proof. Let $x_1 = \sqrt{2}$ and $x_{n+1} = \sqrt{2 + x_n}$ for $n \geq 1$. This gives the recurrence relation for our sequence.

We will first show that (x_n) is bounded above. We claim $x_n < 2$ for all n.

For $n = 1$: $x_1 = \sqrt{2} < 2$.

Assume $x_n < 2$. Then $x_{n+1} = \sqrt{2 + x_n} < \sqrt{2+2} = \sqrt{4} = 2$. By induction, $x_n < 2$ for all n. We next show that (x_n) is increasing. We need $x_{n+1} > x_n$, i.e., $\sqrt{2 + x_n} > x_n$.

Squaring both sides (valid since both are positive):

$$2 + x_n > x_n^2$$

$$x_n^2 - x_n - 2 < 0$$

$$(x_n - 2)(x_n + 1) < 0$$

Since $x_n > 0$, we have $x_n + 1 > 0$, so we need $x_n - 2 < 0$, i.e., $x_n < 2$. This holds by Step 1, so (x_n) is increasing.

By the Monotone Convergence Theorem, (x_n) converges. Let $X = \lim_{n\to\infty} x_n$. Taking the limit of $x_{n+1} = \sqrt{2 + x_n}$:

$$X = \sqrt{2 + X}$$

Squaring both sides:

$$X^2 = 2 + X$$
$$X^2 - X - 2 = 0$$
$$(X - 2)(X + 1) = 0$$

So $X = 2$ or $X = -1$. Since $x_n > 0$ for all n, we have $X > 0$, thus $X = 2$. □

Problem 34. *For each series, find an explicit formula for the partial sums, and determine if the series converges.*

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}$

This is a geometric series with $r = \frac{1}{2}$. The partial sums are:

$$S_N = \sum_{n=1}^{N} \frac{1}{2^n} = \frac{1/2(1 - (1/2)^N)}{1 - 1/2} = 1 - \frac{1}{2^N}$$

As $N \to \infty$, $\frac{1}{2^N} \to 0$, so $\lim_{N\to\infty} S_N = 1$.

The series converges to 1.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

Using partial fractions: $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$.

The partial sums are:

$$\begin{aligned} S_N &= \sum_{n=1}^{N} \frac{1}{n(n+1)} = \sum_{n=1}^{N} \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{N} - \frac{1}{N+1}\right) \\ &= 1 - \frac{1}{N+1} \end{aligned}$$

This is a telescoping series. As $N \to \infty$, $\frac{1}{N+1} \to 0$, so $\lim_{N\to\infty} S_N = 1$.

The series converges to 1.

(c) $\displaystyle\sum_{n=1}^{\infty} \log\left(\frac{n+1}{n}\right)$

Using logarithm properties: $\log\left(\frac{n+1}{n}\right) = \log(n+1) - \log(n)$.*(email says its halal)*

The partial sums are:

$$\begin{aligned}
S_N = \sum_{n=1}^{N} \log\left(\frac{n+1}{n}\right) &= \sum_{n=1}^{N}(\log(n+1) - \log(n)) \\
&= (\log(2) - \log(1)) + (\log(3) - \log(2)) + \cdots + (\log(N+1) - \log(N)) \\
&= \log(N+1) - \log(1) = \log(N+1)
\end{aligned}$$

This is a telescoping series. As $N \to \infty$, $\log(N+1) \to \infty$.

The series diverges.

Problem 35.

(a) *Suppose* $0 \le a_n \le b_n$*. If* $\displaystyle\sum_{n=1}^{\infty} a_n$ *diverges then* $\displaystyle\sum_{n=1}^{\infty} b_n$ *diverges.*

Proof. Let $A_N = \sum_{n=1}^{N} a_n$ and $B_N = \sum_{n=1}^{N} b_n$ be the sequences of partial sums.

Since $0 \le a_n \le b_n$ for all n, summing from $n = 1$ to N gives:

$$A_N = \sum_{n=1}^{N} a_n \le \sum_{n=1}^{N} b_n = B_N$$

Both (A_N) and (B_N) are monotone increasing sequences since the terms are non-negative.

Suppose $\sum_{n=1}^{\infty} a_n$ diverges. Then $A_N \to \infty$ as $N \to \infty$, meaning (A_N) is unbounded.

Since $A_N \le B_N$ for all N and (A_N) is unbounded, (B_N) must also be unbounded. Therefore $B_N \to \infty$ as $N \to \infty$, so $\sum_{n=1}^{\infty} b_n$ diverges $\square$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ *diverges.*

Proof. We know the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

For all $n \ge 1$, we have $\sqrt{n} \le n$, so $\frac{1}{n} \le \frac{1}{\sqrt{n}}$.

By part (a), since $0 \le \frac{1}{n} \le \frac{1}{\sqrt{n}}$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, we conclude that $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges. $\square$