

## SECTION 5.3: MARGINAL AND CONDITIONAL PROBABILITY DISTRIBUTIONS

*Exercise (5.23).* In Example 5.4 and Exercise 5.5, we considered the joint density of  $Y_1$ , the proportion of the capacity of the tank that is stocked at the beginning of the week, and  $Y_2$ , the proportion of the capacity sold during the week, given by

$$f(y_1, y_2) = \begin{cases} 3y_1, & 0 \leq y_2 \leq y_1 \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

- (a) Find the marginal density function for  $Y_2$ .

*Solution:*

$$\begin{aligned} f_2(y_2) &= \int_{y_2}^1 3y_1 dy_1 = \left. \frac{3y_1^2}{2} \right]_{y_2}^1 = \frac{3}{2} - \frac{3y_2^2}{2} \\ &= \frac{3}{2}(1 - y_2^2) \end{aligned}$$

□

- (b) For what values of  $y_2$  is the conditional density  $f(y_1|y_2)$  defined?

*Solution:*

$$\begin{aligned} f(y_1|y_2) &= \frac{f(y_1, y_2)}{f_2(y_2)} = \frac{3y_1}{\frac{3}{2}(1 - y_2^2)} = \frac{2y_1}{1 - y_2^2} \\ &= \begin{cases} \frac{2y_1}{1 - y_2^2}, & y_2 \leq y_1 \leq 1 \\ 0, & \text{elsewhere} \end{cases} \end{aligned}$$

So its defined for  $0 \leq y_2 < 1$  to avoid division by 0.

□

- (c) What is the probability that more than half a tank is sold given that three-fourths of a tank is stocked?

*Solution:* We have  $P(Y_2 > 1/2 | Y_1 = 3/4)$

$$\begin{aligned}
 f_1(y_1) &= \int_0^{y_1} 3y_1 dy_2 = 3y_1 y_2 \Big|_0^{y_1} = 3y_1^2 \\
 f(y_2 | y_1) &= \frac{3y_1}{3y_1^2} = \frac{1}{y_1} \\
 P(Y_2 > 1/2 | Y_1 = 3/4) &= \int_{1/2}^{3/4} \frac{1}{3/4} dy_2 = \int_{1/2}^{3/4} \frac{4}{3} dy_2 = \frac{4}{3} y_2 \Big|_{1/2}^{3/4} \\
 &= \left(\frac{4}{3} * \frac{3}{4}\right) - \left(\frac{4}{3} * \frac{1}{2}\right) = 1 - \frac{4}{6} = \frac{1}{3}
 \end{aligned}$$

□

*Exercise (5.24).* In Exercise 5.6, we assumed that if a radioactive particle is randomly located in a square with sides of unit length, a reasonable model for the joint density function for  $Y_1$  and  $Y_2$  is

$$f(y_1, y_2) = \begin{cases} 1, & 0 \leq y_1 \leq 1, 0 \leq y_2 \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

- (a) Find the marginal density functions for  $Y_1$  and  $Y_2$ .

*Solution:*

$$f_1(y_1) = 1 \text{ for } 0 \leq y_1 \leq 1$$

$$f_2(y_2) = 1 \text{ for } 0 \leq y_2 \leq 1$$

□

- (b) What is  $P(.3 < Y_1 < .5)$ ?  $P(.3 < Y_2 < .5)$ ?

*Solution:* Since its a rectangular region for each

$$P(.3 < Y_1 < .5) = P(.3 < Y_2 < .5) = .2 * 1 = .2$$

□

- (c) For what values of  $y_2$  is the conditional density  $f(y_1 | y_2)$  defined?

*Solution:*

$$f(y_1|y_2) = \frac{f(y_1, y_2)}{f_2(y_2)} = \frac{1}{1} = 1$$

Defined for  $0 \leq y_2 \leq 1$ . □

- (d) For any  $y_2$ ,  $0 \leq y_2 \leq 1$  what is the conditional density function of  $Y_1$  given that  $Y_2 = y_2$ ?

*Solution:* The conditional probability we calculated in the last one so

$$f(y_1|y_2) = \begin{cases} 1, & 0 \leq y_1 \leq 1 \\ 0, & \text{elsewhere} \end{cases}$$

For any  $y_2$  where  $0 \leq y_2 \leq 1$ . □

- (e) Find  $P(.3 < Y_1 < .5|Y_2 = .3)$ .

*Solution:*  $P(.3 < Y_1 < .5|Y_2 = .3) = .2/1 = .2$  □

- (f) Find  $P(.3 < Y_1 < .5|Y_2 = .5)$ .

*Solution:*  $P(.3 < Y_1 < .5|Y_2 = .5) = .2/1 = .2$  □

- (g) Compare the answers that you obtained in parts (a), (d), and (e). For any  $y_2$ ,  $0 \leq y_2 \leq 1$  how does  $P(.3 < Y_1 < .5)$  compare to  $P(.3 < Y_1 < .5|Y_2 = y_2)$ ?

*Solution:* There are all the same answers. □

*Exercise (5.25).* Let  $Y_1$  and  $Y_2$  have joint density function first encountered in Exercise 5.7:

$$f(y_1, y_2) = \begin{cases} e^{-(y_1+y_2)}, & y_1 > 0, y_2 > 0, \\ 0, & \text{elsewhere.} \end{cases}$$

- (a) Find the marginal density functions for  $Y_1$  and  $Y_2$ . Identify these densities as one of those studied in Chapter 4.

*Solution:* □

- (b) What is  $P(1 < Y_1 < 2.5)$ ?  $P(1 < Y_2 < 2.5)$ ?

*Solution:* □

- (c) For what values of  $y_2$  is the conditional density  $f(y_1|y_2)$  defined?

*Solution:* □

- (d) For any  $y_2 > 0$ , what is the conditional density function of  $Y_1$  given that  $Y_2 = y_2$ ?

*Solution:* □

- (e) For any  $y_1 > 0$ , what is the conditional density function of  $Y_2$  given that  $Y_1 = y_1$ ?

*Solution:* □

- (f) For any  $y_2 > 0$ , how does the conditional density function  $f(y_1|y_2)$  that you obtained in part (d) compare to the marginal density function  $f_1(y_1)$  found in part (a)?

*Solution:* □

- (g) What does your answer to part (f) imply about marginal and conditional probabilities that  $Y_1$  falls in any interval?

*Solution:* □

*Exercise (5.27).* In Exercise 5.9, we determined that

$$f(y_1, y_2) = \begin{cases} 6(1 - y_2), & 0 \leq y_1 \leq y_2 \leq 1, \\ 0, & \text{elsewhere} \end{cases}$$

is a valid joint probability density function. Find

- (a) the marginal density functions for  $Y_1$  and  $Y_2$ .

*Solution:* □

- (b)  $P(Y_2 \leq 1/2 | Y_1 \leq 3/4)$ .

*Solution:* □

- (c) the conditional density function of  $Y_1$  given  $Y_2 = y_2$ .

*Solution:* □

- (d) the conditional density function of  $Y_2$  given  $Y_1 = y_1$ .

*Solution:* □

- (e)  $P(Y_2 \geq 3/4 | Y_1 = 1/2)$ .

*Solution:* □

*Exercise (5.29).* Refer to Exercise 5.11. Find

- (a) the marginal density functions for  $Y_1$  and  $Y_2$ .

*Solution:* □

- (b)  $P(Y_2 > 1/2 | Y_1 = 1/4)$ .

*Solution:*

□

*Exercise (5.32).* Suppose that the random variables  $Y_1$  and  $Y_2$  have joint probability density function,  $f(y_1, y_2)$ , given by (see Exercise 5.14)

$$f(y_1, y_2) = \begin{cases} 6y_1^2 y_2, & 0 \leq y_1 \leq y_2, y_1 + y_2 \leq 2, \\ 0, & \text{elsewhere.} \end{cases}$$

- (a) Show that the marginal density of  $Y_1$  is a beta density with  $\alpha = 3$  and  $\beta = 2$ .

*Solution:*

□

- (b) Derive the marginal density of  $Y_2$ .

*Solution:*

□

- (c) Derive the conditional density of  $Y_2$  given  $Y_1 = y_1$ .

*Solution:*

□

- (d) Find  $P(Y_2 < 1.1 | Y_1 = .60)$ .

*Solution:*

□

*Exercise (5.33).* Suppose that  $Y_1$  is the total time between a customer's arrival in the store and departure from the service window,  $Y_2$  is the time spent in line before reaching the window, and the joint density of these variables (as was given in Exercise 5.15) is

$$f(y_1, y_2) = \begin{cases} e^{-y_1}, & 0 \leq y_2 \leq y_1 \leq \infty, \\ 0, & \text{elsewhere.} \end{cases}$$

- (a) Find the marginal density functions for  $Y_1$  and  $Y_2$ .

*Solution:*

□

- (b) What is the conditional density function of  $Y_1$  given that  $Y_2 = y_2$ ? Be sure to specify the values of  $y_2$  for which this conditional density is defined.

*Solution:*

□

- (c) What is the conditional density function of  $Y_2$  given that  $Y_1 = y_1$ ? Be sure to specify the values of  $y_1$  for which this conditional density is defined.

*Solution:*

□

- (d) Is the conditional density function  $f(y_1 | y_2)$  that you obtained in part (b) the same as the marginal density function  $f_1(y_1)$  found in part (a)?

*Solution:*

□

- (e) What does your answer to part (d) imply about marginal and conditional probabilities that  $Y_1$  falls in any interval?

*Solution:*

□

*Exercise (5.35).* Refer to Exercise 5.33. If two minutes elapse between a customer's arrival at the store and his departure from the service window, find the probability that he waited in line less than one minute to reach the window.

*Solution:*

□

*Exercise (5.37).* In Exercise 5.18,  $Y_1$  and  $Y_2$  denoted the lengths of life, in hundreds of hours, for components of types I and II, respectively, in an electronic system. The joint density of  $Y_1$  and  $Y_2$  is given by

$$f(y_1, y_2) = \begin{cases} (1/8)y_1 e^{-(y_1+y_2)/2}, & y_1 > 0, y_2 > 0 \\ 0, & \text{elsewhere.} \end{cases}$$

Find the probability that a component of type II will have a life length in excess of 200 hours.

*Solution:*

□

#### SECTION 5.4: INDEPENDENT RANDOM VARIABLES

*Exercise (5.43).* Let  $Y_1$  and  $Y_2$  have joint density function  $f(y_1, y_2)$  and marginal densities  $f_1(y_1)$  and  $f_2(y_2)$ , respectively. Show that  $Y_1$  and  $Y_2$  are independent if and only if  $f(y_1|y_2) = f_1(y_1)$  for all values of  $y_1$  and for all  $y_2$  such that  $f_2(y_2) > 0$ . A completely analogous argument establishes that  $Y_1$  and  $Y_2$  are independent if and only if  $f(y_2|y_1) = f_2(y_2)$  for all values of  $y_2$  and for all  $y_1$  such that  $f_1(y_1) > 0$ .

*Solution:*

□

*Exercise (5.45).* In Exercise 5.1, we determined that the joint distribution of  $Y_1$ , the number of contracts awarded to firm A, and  $Y_2$ , the number of contracts awarded to firm B, is given by the entries in the following table.

	$y_2$		
$y_1$	0	1	2
0	1/9	2/9	1/9
1	2/9	2/9	0
2	1/9	0	0

The marginal probability function of  $Y_1$  was derived in Exercise 5.19 to be binomial with  $n = 2$  and  $p = 1/3$ . Are  $Y_1$  and  $Y_2$  independent? Why?

*Solution:*

□

*Exercise (5.47).* In Exercise 5.3, we determined that the joint probability distribution of  $Y_1$ , the number of married executives, and  $Y_2$ , the number of never-married executives, is given

by

$$p(y_1, y_2) = \frac{\binom{4}{y_1} \binom{3}{y_2} \binom{2}{3-y_1-y_2}}{\binom{9}{3}},$$

where  $y_1$  and  $y_2$  are integers,  $0 \leq y_1 \leq 3$ ,  $0 \leq y_2 \leq 3$ , and  $1 \leq y_1 + y_2 \leq 3$ . Are  $Y_1$  and  $Y_2$  independent? (Recall your answer to Exercise 5.21.)

*Solution:*

□

*Exercise (5.49).* In Example 5.4 and Exercise 5.5, we considered the joint density of  $Y_1$ , the proportion of the capacity of the tank that is stocked at the beginning of the week and  $Y_2$ , the proportion of the capacity sold during the week, given by

$$f(y_1, y_2) = \begin{cases} 3y_1, & 0 \leq y_2 \leq y_1 \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Show that  $Y_1$  and  $Y_2$  are dependent.

*Solution:*

□

*Exercise (5.51).* In Exercise 5.7, we considered  $Y_1$  and  $Y_2$  with joint density function

$$f(y_1, y_2) = \begin{cases} e^{-(y_1+y_2)}, & y_1 > 0, y_2 > 0, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Are  $Y_1$  and  $Y_2$  independent?

*Solution:*

□

(b) Does the result from part (a) explain the results you obtained in Exercise 5.25 (d)–(f)? Why?

*Solution:*

□

*Exercise (5.53).* In Exercise 5.9, we determined that

$$f(y_1, y_2) = \begin{cases} 6(1 - y_2), & 0 \leq y_1 \leq y_2 \leq 1, \\ 0, & \text{elsewhere} \end{cases}$$

is a valid joint probability density function. Are  $Y_1$  and  $Y_2$  independent?

*Solution:*

□

*Exercise (5.55).* Suppose that, as in Exercise 5.11,  $Y_1$  and  $Y_2$  are uniformly distributed over the triangle shaded in the accompanying diagram. Are  $Y_1$  and  $Y_2$  independent?

*Solution:*

□

*Exercise (5.57).* In Exercises 5.13 and 5.31, the joint density function of  $Y_1$  and  $Y_2$  was given by

$$f(y_1, y_2) = \begin{cases} 30y_1y_2^2, & y_1 - 1 \leq y_2 \leq 1 - y_1, 0 \leq y_1 \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Are the random variables  $Y_1$  and  $Y_2$  independent?

*Solution:*

□

*Exercise (5.59).* If  $Y_1$  is the total time between a customer's arrival in the store and leaving the service window and if  $Y_2$  is the time spent in line before reaching the window, the joint density of these variables, according to Exercise 5.15, is

$$f(y_1, y_2) = \begin{cases} e^{-y_1}, & 0 \leq y_2 \leq y_1 \leq \infty \\ 0, & \text{elsewhere.} \end{cases}$$

Are  $Y_1$  and  $Y_2$  independent?

*Solution:*

□

*Exercise (5.61).* In Exercise 5.18,  $Y_1$  and  $Y_2$  denoted the lengths of life, in hundreds of hours, for components of types I and II, respectively, in an electronic system. The joint density of  $Y_1$  and  $Y_2$  is

$$f(y_1, y_2) = \begin{cases} (1/8)y_1e^{-(y_1+y_2)/2}, & y_1 > 0, y_2 > 0, \\ 0, & \text{elsewhere.} \end{cases}$$

Are  $Y_1$  and  $Y_2$  independent?

*Solution:*

□

*Exercise (5.63).* Let  $Y_1$  and  $Y_2$  be independent exponentially distributed random variables, each with mean 1. Find  $P(Y_1 > Y_2 | Y_1 < 2Y_2)$ .

*Solution:*

□

## SECTION 5.6: EXPECTED VALUE OF A FUNCTION OF RANDOM VARIABLES

*Exercise (5.73).* In Exercise 5.3, we determined that the joint probability distribution of  $Y_1$ , the number of married executives, and  $Y_2$ , the number of never-married executives, is given by

$$p(y_1, y_2) = \frac{\binom{4}{y_1} \binom{3}{y_2} \binom{2}{3-y_1-y_2}}{\binom{9}{3}},$$

where  $y_1$  and  $y_2$  are integers,  $0 \leq y_1 \leq 3$ ,  $0 \leq y_2 \leq 3$ , and  $1 \leq y_1 + y_2 \leq 3$ . Find the expected number of married executives among the three selected for promotion. (See Exercise 5.21.)

*Solution:*

□



*Exercise (5.75).* Refer to Exercises 5.7, 5.25, and 5.51. Let  $Y_1$  and  $Y_2$  have joint density function

$$f(y_1, y_2) = \begin{cases} e^{-(y_1+y_2)}, & y_1 > 0, y_2 > 0 \\ 0, & \text{elsewhere.} \end{cases}$$

- (a) What are  $E(Y_1 + Y_2)$  and  $V(Y_1 + Y_2)$ ?

*Solution:*

□

- (b) What is  $P(Y_1 - Y_2 > 3)$ ?

*Solution:*

□

- (c) What is  $P(Y_1 - Y_2 < -3)$ ?

*Solution:*

□

- (d) What are  $E(Y_1 - Y_2)$  and  $V(Y_1 - Y_2)$ ?

*Solution:*

□

- (e) What do you notice about  $V(Y_1 + Y_2)$  and  $V(Y_1 - Y_2)$ ?

*Solution:*

□

*Exercise (5.77).* In Exercise 5.9, we determined that

$$f(y_1, y_2) = \begin{cases} 6(1 - y_2), & 0 \leq y_1 \leq y_2 \leq 1, \\ 0, & \text{elsewhere} \end{cases}$$

is a valid joint probability density function. Find

- (a)  $E(Y_1)$  and  $E(Y_2)$ .

*Solution:*

□

- (b)  $V(Y_1)$  and  $V(Y_2)$ .

*Solution:*

□

- (c)  $E(Y_1 - 3Y_2)$ .

*Solution:*

□

*Exercise (5.79).* Suppose that, as in Exercise 5.11,  $Y_1$  and  $Y_2$  are uniformly distributed over the triangle shaded in the accompanying diagram with vertices at  $(-1, 0)$ ,  $(1, 0)$ , and  $(0, 1)$ . Find  $E(Y_1 Y_2)$ .

*Solution:*

□

*Exercise (5.81).* In Exercise 5.18,  $Y_1$  and  $Y_2$  denoted the lengths of life, in hundreds of hours, for components of types I and II, respectively, in an electronic system. The joint density of  $Y_1$  and  $Y_2$  is

$$f(y_1, y_2) = \begin{cases} (1/8)y_1 e^{-(y_1+y_2)/2}, & y_1 > 0, y_2 > 0, \\ 0, & \text{elsewhere.} \end{cases}$$

One way to measure the relative efficiency of the two components is to compute the ratio  $Y_2/Y_1$ . Find  $E(Y_2/Y_1)$ . [Hint: In Exercise 5.61, we proved that  $Y_1$  and  $Y_2$  are independent.]

*Solution:* □

*Exercise (5.87).* Suppose that  $Y_1$  and  $Y_2$  are independent  $\chi^2$  random variables with  $\nu_1$  and  $\nu_2$  degrees of freedom, respectively. Find

(a)  $E(Y_1 + Y_2)$ .

*Solution:* □

(b)  $V(Y_1 + Y_2)$ . [Hint: Use Theorem 5.9 and the result of Exercise 4.112(a).]

*Solution:* □

#### SECTION 5.7: THE COVARIANCE OF TWO RANDOM VARIABLES

*Exercise (5.89).* In Exercise 5.1, we determined that the joint distribution of  $Y_1$ , the number of contracts awarded to firm A, and  $Y_2$ , the number of contracts awarded to firm B, is given by the entries in the following table.

$y_1$	$y_2$		
	0	1	2
0	1/9	2/9	1/9
1	2/9	2/9	0
2	1/9	0	0

Find  $\text{Cov}(Y_1, Y_2)$ . Does it surprise you that  $\text{Cov}(Y_1, Y_2)$  is negative? Why?

*Solution:* □

*Exercise (5.91).* In Exercise 5.8, we derived the fact that

$$f(y_1, y_2) = \begin{cases} 4y_1 y_2, & 0 \leq y_1 \leq 1, 0 \leq y_2 \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Show that  $\text{Cov}(Y_1, Y_2) = 0$ . Does it surprise you that  $\text{Cov}(Y_1, Y_2)$  is zero? Why?

*Solution:* □

*Exercise (5.92).* In Exercise 5.9, we determined that

$$f(y_1, y_2) = \begin{cases} 6(1 - y_2), & 0 \leq y_1 \leq y_2 \leq 1, \\ 0, & \text{elsewhere} \end{cases}$$

is a valid joint probability density function. Find  $\text{Cov}(Y_1, Y_2)$ . Are  $Y_1$  and  $Y_2$  independent?

*Solution:*

□

*Exercise (5.93).* Let the discrete random variables  $Y_1$  and  $Y_2$  have the joint probability function

$$p(y_1, y_2) = 1/3, \text{ for } (y_1, y_2) = (-1, 0), (0, 1), (1, 0).$$

Find  $\text{Cov}(Y_1, Y_2)$ . Notice that  $Y_1$  and  $Y_2$  are dependent. (Why?) This is another example of uncorrelated random variables that are not independent.

*Solution:*

□

*Exercise (5.95).* Suppose that, as in Exercises 5.11 and 5.79,  $Y_1$  and  $Y_2$  are uniformly distributed over the triangle shaded in the accompanying diagram with vertices at  $(-1, 0)$ ,  $(1, 0)$ , and  $(0, 1)$ .

(a) Find  $\text{Cov}(Y_1, Y_2)$ .

*Solution:*

□

(b) Are  $Y_1$  and  $Y_2$  independent? (See Exercise 5.55.)

*Solution:*

□

(c) Find the coefficient of correlation for  $Y_1$  and  $Y_2$ .

*Solution:*

□

(d) Does your answer to part (b) lead you to doubt your answer to part (a)? Why or why not?

*Solution:*

□

*Exercise (5.99).* If  $c$  is any constant and  $Y$  is a random variable such that  $E(Y)$  exists, show that  $\text{Cov}(c, Y) = 0$ .

*Solution:*

□