Chapter 4

Exercise (14). If $n \in \mathbb{Z}$, then $5n^2 + 3n + 7$ is odd. (Try cases.)

Proof. Suppose $n \in \mathbb{Z}$. Then n must be either an even or odd integer.

Case 1: Lets suppose that n is an even integer. Then by the definition of an even integer, n can be expressed as $n = 2k$, where $k \in \mathbb{Z}$. Therefore $5n^2 + 3n + 7 = 5(2k)^2 + 3(2k) + 7 = 20k^2 + 6k + 7 = 2(10k^2 + 3k + 3) + 1 = 2m + 1$, where $m = 10k^2 + 3k + 3$. Note that m is an integer because of the closure properties of the integers. Since $5n^2 + 3n + 7 = 2m + 1$, then $5n^2 + 3n + 7$ an odd integer by the definition of odd. Thus when n is even, then $5n^2 + 3n + 7$ is odd.

Case 2: Suppose that n is an odd integer. Then by the definition of an odd integer, n can be expressed as $n = 2k + 1$, where $k \in \mathbb{Z}$. Therefore $5n^2 + 3n + 7 = 5(2k + 1)^2 + 3(2k + 1) + 7 = 5(4k^2+4k+1)+6k+3+7 = 20k^2+20k+5+6k+3+7 = 20k^2+26k+15 = 2(10k^2+13k+7)+1 = 2m + 1$, where m $= 10k^2 + 13k + 7$ and likewise $m \in \mathbb{Z}$. Since $5n^2 + 3n + 7 = 2m + 1$, then $5n^2 + 3n + 7$ is odd by definition. Thus when n is odd, then $5n^2 + 3n + 7$ is odd.

In each case $5n^2 + 3n + 7$ is odd, satisfying all possible integer values for n. □

Exercise (16). If two integers have the same parity, then their sum is even. (Try cases.)

Proof. Suppose we have $x, y \in \mathbb{Z}$ such that they share the same parity, that is to say either x and y are both even or x and y are both odd.

Case 1: Suppose x is even and y is even, then they can be express as $x = 2p$ and $y = 2q$ for some $p, q \in \mathbb{Z}$. Therefore $x + y = (2p) + (2q) = 2p + 2q = 2(p + q) = 2n$, where $n = p + q$ and $n \in \mathbb{Z}$ because of the closure properties of addition under the integers. Because $x + y = 2n$, that makes $x + y$ even by definition whenever x and y are even. Case 2: Suppose x is odd and y is odd, then $x = 2p + 1$ and $y = 2q + 1$ for some $p, q \in \mathbb{Z}$ □

Exercise (18). Suppose x and y are positive real numbers. If $x < y$, then $x^2 < y^2$.

Proof. Write your answer here. □

Exercise (20). If a is an integer and $a^2 \mid a$, then $a \in \{-1, 0, 1\}$.

Proof. Write your answer here. □

Exercise (26). Every odd integer is a difference of two squares.

Proof. Write your answer here. □

Exercise (28). Let $a, b, c \in \mathbb{Z}$. Suppose a and b are not both zero, and $c \neq 0$. Prove that $c \gcd(a, b) \leq gcd(ca, cb)$.

Proof. Write your answer here. □

Chapter 5

Exercise (4). Suppose $a, b, c \in \mathbb{Z}$. If a does not divice bc, then a does not divide b.

Proof. Write your answer here. □

Exercise (5). Suppose $x \in \mathbb{R}$. If $x^2 + 5x < -$ then $x < 0$.

Proof. Write your answer here. □

Exercise (6). Suppose $x \in \mathbb{R}$. If $x^3 - x > 0$ then $x > -1$.

Proof. Write your answer here. □

Exercise (7). Suppose $a, b \in \mathbb{Z}$. If both ab and $a+b$ are even, then both a and b are even.

Proof. Write your answer here. $\square$

Exercise (9). Suppose $n \in \mathbb{Z}$. If $3 \nmid n^2$, then $3 \nmid n$.

Proof. Write your answer here. $\square$

Exercise (10). Suppose $x, y, z \in \mathbb{Z}$ and $x \neq 0$. If $x \nmid yz$, then $x \nmid y$ and $x \nmid z$.

Proof. Write your answer here. $\square$

Exercise (16). Suppose $x, y \in \mathbb{Z}$. If $x+y$ is even, then x and y have the same parity.

Proof. Write your answer here. $\square$

Exercise (18). If $a, b \in \mathbb{Z}$, then $(a+b)^3 \equiv a^3 + b^3 \pmod{3}$.

Proof. Write your answer here. $\square$

Exercise (19). Let $a, b, c \in \mathbb{Z}$ and $n \in \mathbb{N}$. If $a \equiv b \pmod{n}$ and $a \equiv c \pmod{n}$, then $c \equiv b \pmod{n}$.

Proof. Write your answer here. $\square$

Exercise (22). Let $a \in \mathbb{Z}, n \in \mathbb{N}$. If a has remainder r when divided by n, then $a \equiv r \pmod{n}$.

Proof. Write your answer here. $\square$

Exercise (24). If $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$, then $ac \equiv bd \pmod{n}$.

Proof. Write your answer here. $\square$

Exercise (25). Let $n \in \mathbb{N}$. If $2^n - 1$ is prime, then n is prime.

Proof. Write your answer here. $\square$

Exercise (32). If $a \equiv b \pmod{n}$, then a and b have the same remainder when divided by n.

Proof. Write your answer here. $\square$