Problem 60. *Let f, g, h satisfy $f(x) \leq g(x) \leq h(x)$ for all x in some common domain A. Assume c is a limit point of A. If $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} h(x) = L$ then $\lim_{x\to c} g(x) = L$.*

Proof. Let $\epsilon > 0$. Since $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} h(x) = L$ then by definition of limit in the context of functions, for every $\epsilon > 0$, there exists $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$0 < |x - c| < \delta_1 \Rightarrow L - \epsilon < f(x) < L + \epsilon$$
$$0 < |x - c| < \delta_2 \Rightarrow L - \epsilon < h(x) < L + \epsilon$$

for all $x \in A$. Given $f(x) \leq g(x) \leq h(x)$, if we let $\delta = \min\{\delta_1, \delta_2\}$, then for all $x \in A$ with $0 < |x - c| < \delta$ we have

$$L - \epsilon < f(x) \leq g(x) \leq h(x) < L + \epsilon$$

Therefore $L - \epsilon < g(x) < L + \epsilon$ or equivalently $|g(x) - L| < \epsilon$. Since ϵ was arbitrary we conclude that

$$\lim_{x\to c} g(x) = L.$$

□

Problem 61. *If $h : \mathbb{R} \to \mathbb{R}$ is a continuous function then the set $K = \{x \in \mathbb{R} : h(x) = 0\}$ is closed.*

Proof. Recall that a set is closed if it contains all of its limit points.

Let $c \in K$ be a limit point, since $K = \{x \in \mathbb{R} : h(x) = 0\}$, we want to show that $h(c) = 0$. Since c is a limit point of K, then there exists a sequence (x_n) in K with $x_n \to c$. Since each $x_n \in K$, it follows that $h(x_n) = 0$ for all n.

Since $h : \mathbb{R} \to \mathbb{R}$ is a continous function at c and $x_n \to c$, we have

$$\lim_{n\to\infty} h(x_n) = h(c)$$

Since $h(x_n) = 0$ for all n, we have

$$\lim_{n\to\infty} h(x_n) = 0$$

Therefore $h(c) = 0$, which means $c \in K$. Thus every limit point of K belongs to K meaning the set K is closed. □

Problem 62. *If c is an isolated point of $A \subset \mathbb{R}$, and if $f : A \to \mathbb{R}$ is a function, then f is continuous at c.*

Proof. Suppose c is an isolated point for $A \subset \mathbb{R}$. Then by definition there exists $\delta_0 > 0$ such that $(c-\delta_0, c+\delta_0) \cap A = \{c\}$ We want to show that f is continous at c. For this we let $\epsilon > 0$, we must find $\delta > 0$ such that for all $x \in A$ with $|x-c| < \delta$, we have $|f(x) - f(c)| < \epsilon$.

Choose $\delta = \delta_0$ and suppose $x \in A$ such that $|x-c| < \delta = \delta_0$. Then $x \in (c-\delta_0, c+\delta_0) \cap A = \{c\}$, which means $x = c$. Therefore when $x \in A$ and $|x-c| < \delta$, we have $x = c$, so

$$|f(x) - f(c)| = |f(c) - f(c)| = 0 < \epsilon$$

Thus f is continuous at c. □

Problem 63. *The function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = \sqrt[3]{x}$ is continuous.*

Proof. We need to show that $g(x) = \sqrt[3]{x}$ is continuous at every point $c \in \mathbb{R}$.
Case 1: $c = 0$**:**

Let $\epsilon > 0$. We need to find $\delta > 0$ such that $|x - 0| < \delta$ implies $|\sqrt[3]{x} - 0| < \epsilon$. Choose $\delta = \epsilon^3$. Then if $|x| < \delta = \epsilon^3$, we have

$$|\sqrt[3]{x}| = \sqrt[3]{|x|} < \sqrt[3]{\epsilon^3} = \epsilon$$

Thus g is continuous at $c = 0$.
Case 2: $c > 0$**:**

Let $\epsilon > 0$. Using the identity $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$ with $a = \sqrt[3]{x}$ and $b = \sqrt[3]{c}$, we have

$$x - c = (\sqrt[3]{x} - \sqrt[3]{c})((\sqrt[3]{x})^2 + \sqrt[3]{x}\sqrt[3]{c} + (\sqrt[3]{c})^2)$$

Therefore

$$\sqrt[3]{x} - \sqrt[3]{c} = \frac{x - c}{(\sqrt[3]{x})^2 + \sqrt[3]{x}\sqrt[3]{c} + (\sqrt[3]{c})^2}$$

Choose $\delta = \frac{c}{2}$. For $|x - c| < \delta = \frac{c}{2}$, we have $x > \frac{c}{2} > 0$, so $\sqrt[3]{x} > 0$ and $\sqrt[3]{c} > 0$. Thus

$$(\sqrt[3]{x})^2 + \sqrt[3]{x}\sqrt[3]{c} + (\sqrt[3]{c})^2 > (\sqrt[3]{c})^2$$

So for $|x - c| < \min\left\{\frac{c}{2}, \epsilon(\sqrt[3]{c})^2\right\}$, we have

$$|\sqrt[3]{x} - \sqrt[3]{c}| = \frac{|x - c|}{(\sqrt[3]{x})^2 + \sqrt[3]{x}\sqrt[3]{c} + (\sqrt[3]{c})^2} < \frac{\epsilon(\sqrt[3]{c})^2}{(\sqrt[3]{c})^2} = \epsilon$$

Thus g is continuous at every $c > 0$.
Case 3: $c < 0$**:**

Let $\epsilon > 0$. Using the same identity as in Case 2, for $|x - c| < \frac{|c|}{2}$, we have $x < \frac{c}{2} < 0$, so $\sqrt[3]{x} < 0$ and $\sqrt[3]{c} < 0$. Thus

$$(\sqrt[3]{x})^2 + \sqrt[3]{x}\sqrt[3]{c} + (\sqrt[3]{c})^2 > (\sqrt[3]{c})^2$$

So for $|x - c| < \min\left\{\frac{|c|}{2}, \epsilon(\sqrt[3]{c})^2\right\}$, we have

$$|\sqrt[3]{x} - \sqrt[3]{c}| < \frac{\epsilon(\sqrt[3]{c})^2}{(\sqrt[3]{c})^2} = \epsilon$$

Thus g is continuous at every $c < 0$.

Since g is continuous at every point $c \in \mathbb{R}$, we conclude that g is continuous on $\mathbb{R}$. $\square$

Problem 64. *Dirichlet's function from Section 4.1, namely*

$$g(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

is not continuous at any $c \in \mathbb{R}$.

Proof. Recall the Criterion for Discontinuity: Let $f : A \to \mathbb{R}$, and let $c \in A$ be a limit point of A. If there exists a sequence $(x_n) \subseteq A$ where $(x_n) \to c$ but such that $f(x_n)$ does not converge to $f(c)$, we may conclude that f is not continuous at c.

Let $c \in \mathbb{R}$ be arbitrary. Note that c is a limit point of $\mathbb{R}$ since every neighborhood of c contains infinitely many points of $\mathbb{R}$. By the density of rationals in $\mathbb{R}$, there exists a sequence (r_n) of rational numbers with $r_n \to c$. Since $r_n \in \mathbb{Q}$ for all n, we have $g(r_n) = 1$ for all n. Consider the following cases

Case 1: $c \in \mathbb{Q}$:

Then $g(c) = 1$. But by the density of irrationals in $\mathbb{R}$, there exists a sequence (s_n) of irrational numbers with $s_n \to c$. Since $s_n \notin \mathbb{Q}$ for all n, we have $g(s_n) = 0$ for all n. Thus $g(s_n) \to 0 \neq 1 = g(c)$.

By the Criterion for Discontinuity, g is not continuous at c.

Case 2: $c \notin \mathbb{Q}$:

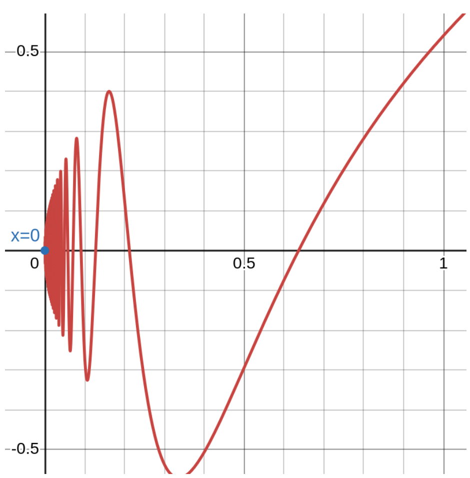
Then $g(c) = 0$. Since $r_n \to c$ and $g(r_n) = 1$ for all n, we have $g(r_n) \to 1 \neq 0 = g(c)$.

By the Criterion for Discontinuity, g is not continuous at c. Since c was arbitrary, g is not continuous at any point in $\mathbb{R}$. $\square$

Problem 65. *The function*

$$h(x) = \begin{cases} 0 & \text{if } x = 0, \\ \sqrt{|x|}\cos(1/x) & \text{otherwise,} \end{cases}$$

shown in the figure below, is continuous at zero.



Proof. Let $\epsilon > 0$.

We must find $\delta > 0$ such that $|x - 0| < \delta$ implies $|h(x) - h(0)| < \epsilon$.

Since $h(0) = 0$, we need to show that $|h(x)| < \epsilon$ when $|x| < \delta$.

For $x \neq 0$, we have $h(x) = \sqrt{|x|}\cos(1/x)$. Since $|\cos(1/x)| \leq 1$ for all $x \neq 0$, we have

$$|h(x)| = |\sqrt{|x|}\cos(1/x)| = \sqrt{|x|}|\cos(1/x)| \leq \sqrt{|x|} \cdot 1 = \sqrt{|x|}$$

Choose $\delta = \epsilon^2$. Then if $|x - 0| = |x| < \delta = \epsilon^2$, we have

$$|h(x) - h(0)| = |h(x)| \leq \sqrt{|x|} < \sqrt{\epsilon^2} = \epsilon$$

Since ϵ was arbitrary, we conclude that h is continuous at 0. □

Problem 66. *Thomae's function from Section 4.1, namely*

$$t(x) = \begin{cases} 1 & \text{if } x = 0, \\ 1/n & \text{if } x \in \mathbb{Q} \setminus \{0\} \text{ and } x = \pm m/n \text{ in lowest terms, with } n > 0, \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

is not continuous at any rational point $c \in \mathbb{Q}$.

Proof. We will use the Criterion for Discontinuity again in this problem. Let $f : A \to \mathbb{R}$, and let $c \in A$ be a limit point of A. If there exists a sequence $(x_n) \subseteq A$ where $(x_n) \to c$ but such that $f(x_n)$ does not converge to $f(c)$, we may conclude that f is not continuous at c.

Let $c \in \mathbb{Q}$ be arbitrary. We will show that t is not continuous at c using the Criterion for Discontinuity.

First, we show that every rational is the limit of an irrational sequence. For each $n \in \mathbb{N}$, define $s_n = c + \frac{\sqrt{2}}{n}$. Since $\sqrt{2}$ is irrational and c is rational, each s_n is irrational. Clearly $s_n \to c$ as $n \to \infty$ since

$$|s_n - c| = \left|\frac{\sqrt{2}}{n}\right| = \frac{\sqrt{2}}{n} \to 0$$

Since each $s_n \notin \mathbb{Q}$, we have $t(s_n) = 0$ for all n. Thus $t(s_n) \to 0$ as $n \to \infty$.

However, since $c \in \mathbb{Q}$, we have either $c = 0$ (in which case $t(c) = 1$) or $c = \pm m/n$ in lowest terms with $n > 0$ (in which case $t(c) = 1/n > 0$). In either case, $t(c) > 0$.

Since $t(s_n) \to 0$ but $t(c) > 0$, we have $t(s_n)$ does not converge to $t(c)$.

By the Criterion for Discontinuity, t is not continuous at c.

Since c was an arbitrary rational point, t is not continuous at any rational point. □

Problem 67. *Suppose $f : A \to \mathbb{R}$ is continuous at $c \in A$. Suppose that $g : B \to \mathbb{R}$ has a domain satisfying $f(A) \subset B$, and that g is continuous at $f(c)$. Let*

$$h(x) = (g \circ f)(x) = g(f(x))$$

be the composition of functions. Then h is continuous at c.

Proof. We need to show that h is continuous at c. Let $\epsilon > 0$.

Since g is continuous at $f(c)$ and $f(c) \in B$, there exists $\delta_1 > 0$ such that for all $y \in B$,

$$|y - f(c)| < \delta_1 \implies |g(y) - g(f(c))| < \epsilon$$

Since f is continuous at $c \in A$, using $\delta_1 > 0$ from above, there exists $\delta > 0$ such that for all $x \in A$,

$$|x - c| < \delta \implies |f(x) - f(c)| < \delta_1$$

Now suppose $x \in A$ and $|x - c| < \delta$. Then by continuity of f at c, we have

$$|f(x) - f(c)| < \delta_1$$

Since $f(A) \subset B$, we have $f(x) \in B$. Thus we can apply the continuity of g at $f(c)$ with $y = f(x)$ to obtain

$$|g(f(x)) - g(f(c))| < \epsilon$$

That is, $|h(x) - h(c)| < \epsilon$.

Since ϵ was arbitrary, we conclude that h is continuous at c. $\square$