Problem 1. *Definition. A set $A \subseteq \mathbb{R}$ is open if*
For any $x \in A$, there is an $\epsilon > 0$ such that $V_\epsilon(x) = |x - \epsilon, x + \epsilon| \subseteq eqA$.

Problem 2. *Prove that a finite intersection of open sets is open*

Proof. Suppose $A_1, \cdots A_n$ are a collection open sets. Let $A = \bigcap^n A_n$ and let x be an arbitrary element of A_n. Since A_n is open then by definition there exists $V_\epsilon(x) \subseteq A_n$. Since x is in all A_n it is the intersection of all A_n. Thus $V_\epsilon(x) \subseteq A$.(rough) □

Problem 3. *State the Monotone Convergence Theorem*
If a sequence is bounded and if it is monotone, then it converges.

Problem 4. *Prove the Monotone Convergence Theorem, but restrict your proof to the case that the sequence is increasing.*

Proof. Suppose (a_n) is bounded and increasing. Let

$$A = \{a_n : n \in \mathbb{N}\}$$

Since (a_n) is bounded, then A is bounded. Let $s = \sup A$. Let $\epsilon > 0$. From the property of sup, there is an $a_N \in A$ such that $a_N > s - \epsilon$. Also $A_N \leq s$. If $n \geq N$, then because (a_n) is increasing we have

$$s - \epsilon < a_N \leq a_n$$

Note that $s - a_n < \epsilon$. But also

$$-\epsilon < 0 \leq s - a_n$$

because s os an upper bound on A. Thus

$$-\epsilon < 0 \leq s - a_n$$

or

$$|s - a_n| < \epsilon$$

. Thus (a_n) converges to s. □

Problem 5. *State the Bolzano-Weierstrauss Theorem*
A bounded sequence has convergent subsequence

Problem 6. *Prove that if (a_n) is a convergent sequence then it is Cauchy.*

Proof. Let $\epsilon > 0$. Since (a_n) converges to a a point we denote as $a \in \mathbb{R}$, there exists an $N \in \mathbb{N}$ such that for $n \geq N$ we have

$$|a_n - a| < \epsilon/2$$

Choose $m, n \geq N$. Then by the triangle inequality

$$\begin{aligned} |a_n - a_m| &= |a_n - a + a - a_m| \\ &\leq |a_n - a| + |a - a_m| \\ &< \epsilon/2 + \epsilon/2 = \epsilon \end{aligned}$$

Thus (a_n) is Cauchy. □

Problem 7. *(a) Compute and simplify the partial sums of the geometric series $\sum_{n=0}^{\infty} ar^n$, assuming $r \neq 1$.*

Proof.

$$\begin{aligned} s_m = \sum_{n=0}^{m} ar^n &= a(1 + r + r^2 + \cdots + r^m) \\ rs_m &= a(r + r^2 + \cdots + r^{m+1}) \\ (1-r)s_m &= a(1 - r^{m+1}) \\ s_m &= \frac{a(1 - r^{m+1}}{1-r} \end{aligned}$$

□

(b) Under what assumptions does the geometric series in part (a) converge? State a theorem and prove it.

Theorem: The geometric series $\sum_{n=0}^{\infty} ar^n$ converges if $|r| < 1$, and it converges to $a/(1-r)$.

Proof. By The Algebraic Limit Theorem

$$\begin{aligned} \lim_{m\to\infty} s_m &= \lim_{m\to\infty} \frac{a(1 - r^{m+1})}{1-r} \\ &= \frac{a(1-0)}{1-r} = \frac{a}{1-r} \end{aligned}$$

since $\lim_{m\to\infty} r^{m+1} = 0$. □

Problem 8. *Consider an infinite series $\sum_{n=1}^{\infty} a_n$. Define what it means for it to converge.*

Definition: Let $s_m = \sum_{n=1}^{m} a_n$. The series converges if $\lim_{m\to\infty} s_m$ exists, in other words if the sequence of partiaul sums converges.

Problem 9. *Give a justified example, or argue (prove) that it is impossible*
(a) A uninon of closed sets which is not closed

Proof. Consider the closed set $F_n = [\frac{1}{n}, 1 - \frac{1}{n}]$ for $n = 2, 3, 4, \cdots$. Observe that $F = \cup_{n=2}^{\infty} F_n = (0, 1)$ is not closed and that 0 is a limit point of F but $0 \notin F$. $\square$

(b) A sequence (y_n) satisfying $0 \leq y_n \leq \frac{1}{n}$ where $\sum_{n=1}^{\infty}(-1)^{n+1} y_n$ diverges.
Consider the sequence

$$y_n = \begin{cases} \frac{1}{n}, & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$$

then

$$\sum_{n=1}^{\infty}(-1)^{n+1} y_n = \sum_{k=1}^{\infty}(-1)^{2k+1}\frac{1}{2k}$$
$$= -\frac{1}{2}(\sum_{k=1}^{\infty}\frac{1}{k})$$

But the harmonic series diverges.

Problem 10. *State the alternating Series Test*
Theorem: If (a_n) is a nonnegative sqeuence, and if $\lim_{n\to\infty} a_n = 0$ and (a_n) is decreasing then $\sum_{n=1}^{\infty}(-1)^n a_n$ converges.

Problem 11. *Prove that if x is a limit point of set $A \subset \mathbb{R}$ then there is a sequence (a_n) such that $a_n \in A$ and $a_n \neq x$, for all n such that $a_n \to x$.*

Proof. Suppose $x \in A$ is a limit point. Let $\epsilon_1 = 1$. Then there is $a_1 \in A$ such that $a_1 \neq x$ and $a_1 \in V_\epsilon(x) = V_1(x)$, so $|a_1 - x| < 1$.

Let $\epsilon_2 = \frac{1}{2}$. Then there is an $a_2 \in A$ such that $a_2 \neq x$ and $a_2 \in V_\epsilon(x) = V_2(x)$, so $|a_2 - x| < \frac{1}{2}$. Continuing in this way we construct a sequence (a_n) so that $a_n \neq x$ for all x, and $|a_n = x| < \frac{1}{n}$. Let $\epsilon > 0$, and choose N such that $\frac{1}{N} < \epsilon$. For $n \geq N$ we have $|a_n - x| < \frac{1}{n} < \frac{1}{N} < \epsilon$. Thus (a_n) converges to x. $\square$