Problem 6.

$$\bigcap_{n=1}^{\infty}(0,1/n)=\emptyset.$$

Proof. Let $S=\bigcap_{i=1}^{\infty}(0,1/n)=\emptyset$. Let $x\in\mathbb{R}$. Consider the following 3 cases.
Case 1: Suppose $x\leq 0$, then $x\notin S$ as $x\notin(0,1)$.
Case 2: Suppose $x\geq 0$, then $x\notin S$ as $x\notin(0,1)$.
Case 3: Suppose $0<x<1$, Choose $n\in\mathbb{N}$ so that $n>\frac{1}{x}$. Then $x>\frac{1}{n}$. so $x\in(0,\frac{1}{n})$, thus $x\notin S$
These cases show that an arbitrary $x\in R$ is not in S. □

Problem 7. *Given a function f and a subset A of its domain, consider the image $f(A)=\{f(x):x\in A\}$.*

(a) *An example of a function f, and two subsets A,B of the domain of f, for which $f(A\cap B)\neq f(A)\cap f(B)$ is*

$$f(x)=|x|$$

where set A is a subset of the domain defined by $A=\{x\in\mathbb{R}\mid 0<x\}$ and where set B is a subset of the domain defined by $B=\{x\in\mathbb{R}\mid x\geq 0\}$.

(b) *If A, B are subsets of the domain of f then $f(A\cup B)$ IS RELATED IN SOME WAY TO $f(A)\cup f(B)$.*

Proof. □

Problem 8. *If $a\in\mathbb{R}$ is an upper bound for $A\subset\mathbb{R}$, and if a is also an element of A, then $a=\sup A$.*

Proof. □

Problem 9. *(a) Let $A=\{m/n:\ m,n\in\mathbb{N}$ with $m<n\}$. Then $\inf A=$ and $\sup A=$.*

(b) Let $B=\{(-1)^m/n:n,m\in\mathbb{N}\}$. Then $\inf B=$ and $\sup B=$.

(c) Let $C=\{n/(3n+1):n\in\mathbb{N}\}$. Then $\inf C=$ and $\sup C=$.

(d) Let $D=\{m/(m+n):m,n\in\mathbb{N}\}$. Then $\inf D=$ and $\sup D=$.

Problem 10. *(a) If A and B are nonempty, bounded, and satisfy $A\subseteq B$ then $\sup A\leq\sup B$.*

(b) If $\sup A<\inf B$ for nonempty sets A and B, then there exists $c\in\mathbb{R}$ such that $a<c<b$ for all $a\in A$ and $b\in B$.

(c) *If there exists $c \in \mathbb{R}$ satisfying $a < c < b$ for all $a \in A$ and $b \in B$ then $\sup A < \inf B$.*

Problem 11. *Denote the irrational numbers by $\mathbb{I} = \mathbb{R} \setminus \mathbb{Q}$.*

(a) *If $a, b \in \mathbb{Q}$ then $ab \in \mathbb{Q}$ and $a + b \in \mathbb{Q}$.*

Proof. □

(b) *If $a \in \mathbb{Q}$ and $t \in \mathbb{I}$ then $a + t \in \mathbb{I}$. If also $a \neq 0$ then $at \in \mathbb{I}$.*

Proof. □

(c) *Suppose $s, t \in \mathbb{I}$. Then PROPOSITION ABOUT WHETHER st AND $s + t$ ARE EITHER RATIONAL OR IRRATIONAL IN GENERAL.*

Problem 12. *For all $n \in \mathbb{N}$, $2^n \geq n$.*

Proof. □

Problem 13. *Let $y_1 = 6$ and, for each $n \in \mathbb{N}$, let $y_{n+1} = (2y_n - 6)/3$.*

(a) *For all $n \in \mathbb{N}$, $y_n \geq -6$.*

Proof. □

(b) *The sequence $(y_1, y_2, y_3, \ldots)$ is decreasing.*

Proof. □