Section 6.3: Method of Transformations

Exercise (6.1). Let Y be a random variable with probability density function given by

$$f(y) = \begin{cases} 2(1-y), & 0 \le y \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Find the density function of $U_1 = 2Y - 1$.

Solution:

$$U_1 = 2Y - 1 \qquad \frac{dy}{du_1} = \frac{1}{2}$$

$$2Y = U_1 + 1 \qquad f_{u_1(}(u_1) = 2(1 - \frac{u_1+1}{2})\frac{1}{2}$$

$$Y = \frac{U_1 + 1}{2} \qquad = \frac{1-u_1}{2}$$

$$f_{U_1} = \begin{cases} \frac{1+u_1}{2}, & -1 \le u_1 \le 1 \\ 0, & \text{elsewhere} \end{cases}$$

$\square$

(b) Find the density function of $U_2 = 1 - 2Y$.

Solution:

$$U_2 = 1 - 2Y \qquad \frac{dy}{du_2} = -\frac{1}{2}$$

$$2Y = 1 - U_2 \qquad f_{U_2} = 2(1 - \frac{1-u_2}{2} * \frac{1}{2}$$

$$Y = \frac{1 - U_2}{2} \qquad = \frac{1+u_2}{2}$$

$$f_{U_2}(u_2) = \begin{cases} \frac{1+u_2}{2}, & -1 \le u_2 \le 1 \\ 0, & \text{elsewhere} \end{cases}$$

$\square$

(c) Find the density function of $U_3 = Y^2$.

Solution:

$$U_3 = Y^2 \qquad \frac{dy}{du_3} = \frac{1}{2\sqrt{u_3}}$$

$$Y = \sqrt{U_3} \qquad f_{U_3}(u_3) = 2(1-\sqrt{u_3}) \cdot \frac{1}{2\sqrt{u_3}}$$

$$= \frac{1-\sqrt{u_3}}{\sqrt{u_3}}$$

$$f_{U_3}(u_3) = \begin{cases} \frac{1-\sqrt{u_3}}{\sqrt{u_3}}, & 0 \le u_3 \le 1 \\ 0, & \text{elsewhere} \end{cases}$$

$\square$

(d) Find $E(U_1)$, $E(U_2)$, and $E(U_3)$ by using the derived density functions for these random variables.

Solution:

$$E(U_1) = \int_{-1}^{1} u_1 \cdot \frac{1-u_1}{2} du_1 = \frac{1}{2}\left[\frac{u_1^2}{2} - \frac{u_1^3}{3}\right]_{-1}^{1} = \frac{1}{2}\left[-\frac{2}{3}\right] = -\frac{1}{3}$$

$$E(U_2) = \int_{-1}^{1} u_2 \cdot \frac{1+u_2}{2} du_2 = \frac{1}{2}\left[\frac{u_2^2}{2} + \frac{u_2^3}{3}\right]_{-1}^{1} = \frac{1}{2}\left[\frac{2}{3}\right] = \frac{1}{3}$$

$$E(U_3) = \int_{0}^{1} u_3 \cdot \frac{1-\sqrt{u_3}}{\sqrt{u_3}} du_3 = \int_0^1 (\sqrt{u_3} - u_3) du_3 = \left[\frac{2u_3^{3/2}}{3} - \frac{u_3^2}{2}\right]_0^1 = \frac{2}{3} - \frac{1}{2} = \frac{1}{6}$$

$\square$

(e) Find $E(U_1)$, $E(U_2)$, and $E(U_3)$ by the methods of Chapter 4.

Solution: First, $E(Y) = \int_0^1 y \cdot 2(1-y)dy = 2\left[\frac{y^2}{2} - \frac{y^3}{3}\right]_0^1 = \frac{1}{3}$

$$E(U_1) = E(2Y-1) = 2E(Y) - 1 = 2 \cdot \frac{1}{3} - 1 = -\frac{1}{3}$$

$$E(U_2) = E(1-2Y) = 1 - 2E(Y) = 1 - \frac{2}{3} = \frac{1}{3}$$

$$E(U_3) = E(Y^2) = \int_0^1 y^2 \cdot 2(1-y)dy = 2\left[\frac{y^3}{3} - \frac{y^4}{4}\right]_0^1 = \frac{2}{3} - \frac{1}{2} = \frac{1}{6}$$

$\square$

Exercise (6.6). The joint distribution of amount of pollutant emitted from a smokestack without a cleaning device (Y_1) and a similar smokestack with a cleaning device (Y_2) was given in Exercise 5.10 to be

$$f(y_1, y_2) = \begin{cases} 1, & 0 \le y_1 \le 2, 0 \le y_2 \le 1, 2y_2 \le y_1, \\ 0, & \text{elsewhere.} \end{cases}$$

The reduction in amount of pollutant due to the cleaning device is given by $U = Y_1 - Y_2$.

(a) Find the probability density function for U.

Solution: Let $U = Y_1 - Y_2$ and $V = Y_2$. Then $Y_1 = U + V$, $Y_2 = V$, and $|J| = 1$.

The region becomes: $0 \le v \le 1$, $2v \le u + v \le 2$, which gives $v \le u$ and $v \le 2 - u$.

$$f_U(u) = \int_0^{\min(u,1,2-u)} 1\, dv$$

$$f_U(u) = \begin{cases} u, & 0 \le u \le 1 \\ 2 - u, & 1 < u \le 2 \\ 0, & \text{elsewhere} \end{cases}$$

□

(b) Use the answer in part (a) to find $E(U)$. Compare your results with those of Exercise 5.78(c).

Solution:

$$\begin{aligned} E(U) &= \int_0^1 u \cdot u\, du + \int_1^2 u(2-u)\, du \\ &= \left[\frac{u^3}{3}\right]_0^1 + \left[u^2 - \frac{u^3}{3}\right]_1^2 \\ &= \frac{1}{3} + \left[\left(4 - \frac{8}{3}\right) - \left(1 - \frac{1}{3}\right)\right] \\ &= \frac{1}{3} + \frac{2}{3} = 1 \end{aligned}$$

□

Exercise (6.7). Suppose that Z has a standard normal distribution.

(a) Find the density function of $U = Z^2$.

Solution: For $Z \sim N(0,1)$, $f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$.

Since $u = z^2$ gives $z = \pm\sqrt{u}$ and $\left|\frac{dz}{du}\right| = \frac{1}{2\sqrt{u}}$:

$$\begin{aligned} f_U(u) &= f_Z(\sqrt{u}) \cdot \frac{1}{2\sqrt{u}} + f_Z(-\sqrt{u}) \cdot \frac{1}{2\sqrt{u}} \\ &= 2 \cdot \frac{1}{\sqrt{2\pi}} e^{-u/2} \cdot \frac{1}{2\sqrt{u}} = \frac{1}{\sqrt{2\pi u}} e^{-u/2}, \quad u > 0 \end{aligned}$$

□

(b) Does U have a gamma distribution? What are the values of α and β?

Solution: Yes. Rewriting $f_U(u) = \frac{1}{\sqrt{2\pi}} u^{-1/2} e^{-u/2}$ and using $\Gamma(1/2) = \sqrt{\pi}$:

$$f_U(u) = \frac{1}{\Gamma(1/2) \cdot 2^{1/2}} u^{1/2-1} e^{-u/2}$$

Thus $\alpha = 1/2$ and $\beta = 2$. □

(c) What is another name for the distribution of U?

Solution: Chi-square distribution with 1 degree of freedom, $\chi^2(1)$. □

Exercise (6.8). Assume that Y has a beta distribution with parameters α and β.

(a) Find the density function of $U = 1 - Y$.

Solution: For $Y \sim \text{Beta}(\alpha, \beta)$, $f_Y(y) = \frac{1}{B(\alpha,\beta)} y^{\alpha-1}(1-y)^{\beta-1}$.

Since $u = 1 - y$, we have $y = 1 - u$ and $\left|\frac{dy}{du}\right| = 1$:

$$f_U(u) = \frac{1}{B(\alpha, \beta)} (1-u)^{\alpha-1} u^{\beta-1}, \quad 0 \le u \le 1$$

□

(b) Identify the density of U as one of the types we studied in Chapter 4. Be sure to identify any parameter values.

Solution: U has a beta distribution with parameters β and α: $U \sim \text{Beta}(\beta, \alpha)$. □

(c) How is $E(U)$ related to $E(Y)$?

Solution: $E(U) = E(1 - Y) = 1 - E(Y)$, or equivalently $E(U) = \frac{\beta}{\alpha+\beta}$ and $E(Y) = \frac{\alpha}{\alpha+\beta}$. □

(d) How is $V(U)$ related to $V(Y)$?

Solution: $V(U) = V(1-Y) = V(Y)$, since variance is unaffected by sign changes. □

Exercise (6.9). Suppose that a unit of mineral ore contains a proportion Y_1 of metal A and a proportion Y_2 of metal B. Experience has shown that the joint probability density function of Y_1 and Y_2 is uniform over the region $0 \le y_1 \le 1$, $0 \le y_2 \le 1$, $0 \le y_1 + y_2 \le 1$. Let $U = Y_1 + Y_2$, the proportion of either metal A or B per unit. Find

(a) the probability density function for U.

Solution: The region has area 1/2, so $f(y_1, y_2) = 2$ on the region. Let $U = Y_1 + Y_2$ and $V = Y_2$.

Then $Y_1 = U - V$, $Y_2 = V$, and $|J| = 1$. For $0 \le u \le 1$, v ranges from 0 to u:

$$f_U(u) = \int_0^u 2\,dv = 2u, \quad 0 \le u \le 1$$

□

(b) $E(U)$ by using the answer to part (a).

Solution:

$$E(U) = \int_0^1 u \cdot 2u\,du = 2\left[\frac{u^3}{3}\right]_0^1 = \frac{2}{3}$$

□

(c) $E(U)$ by using only the marginal densities of Y_1 and Y_2.

Solution:

$$f_1(y_1) = \int_0^{1-y_1} 2\,dy_2 = 2(1-y_1), \quad 0 \le y_1 \le 1$$

Similarly, $f_2(y_2) = 2(1-y_2)$. Then:

$$E(Y_1) = \int_0^1 y_1 \cdot 2(1-y_1)\,dy_1 = 2\left[\frac{y_1^2}{2} - \frac{y_1^3}{3}\right]_0^1 = \frac{1}{3}$$

By symmetry, $E(Y_2) = \frac{1}{3}$. Thus $E(U) = E(Y_1) + E(Y_2) = \frac{2}{3}$. □

Exercise (6.10). The total time from arrival to completion of service at a fast-food outlet, Y_1, and the time spent waiting in line before arriving at the service window, Y_2, were given in Exercise 5.15 with joint density function

$$f(y_1, y_2) = \begin{cases} e^{-y_1}, & 0 \le y_2 \le y_1 < \infty, \\ 0, & \text{elsewhere.} \end{cases}$$

Another random variable of interest is $U = Y_1 - Y_2$, the time spent at the service window. Find

(a) the probability density function for U.

Solution: Let $U = Y_1 - Y_2$ and $V = Y_2$. Then $Y_1 = U + V$, $Y_2 = V$, and $|J| = 1$.

The region $0 \leq y_2 \leq y_1$ becomes $0 \leq v$ and $v \leq u + v$, so $u \geq 0$:

$$f_U(u) = \int_0^\infty e^{-(u+v)}\, dv = e^{-u} \int_0^\infty e^{-v}\, dv = e^{-u}, \quad u \geq 0$$

□

(b) $E(U)$ and $V(U)$. Compare your answers with the results of Exercise 5.108.

Solution: U is exponential with parameter 1, so $E(U) = 1$ and $V(U) = 1$. □

Exercise (6.11). Suppose that two electronic components in the guidance system for a missile operate independently and that each has a length of life governed by the exponential distribution with mean 1 (with measurements in hundreds of hours). Find the

(a) probability density function for the average length of life of the two components.

Solution: Let $W = Y_1 + Y_2$. Since Y_1, Y_2 are independent exponential(1), $W \sim$ Gamma$(2, 1)$ with $f_W(w) = we^{-w}$ for $w \geq 0$.

For $U = W/2$, we have $w = 2u$ and $\frac{dw}{du} = 2$:

$$f_U(u) = f_W(2u) \cdot 2 = (2u)e^{-2u} \cdot 2 = 4ue^{-2u}, \quad u \geq 0$$

□

(b) mean and variance of this average, using the answer in part (a). Check your answer by computing the mean and variance, using Theorem 5.12.

Solution:

$$E(U) = \int_0^\infty u \cdot 4ue^{-2u}\, du = 4 \cdot \frac{2!}{2^3} = 1$$
$$E(U^2) = \int_0^\infty u^2 \cdot 4ue^{-2u}\, du = 4 \cdot \frac{3!}{2^4} = \frac{3}{2}$$
$$V(U) = E(U^2) - [E(U)]^2 = \frac{3}{2} - 1 = \frac{1}{2}$$

Using Theorem 5.12: $E(U) = \frac{E(Y_1)+E(Y_2)}{2} = 1$ and $V(U) = \frac{V(Y_1)+V(Y_2)}{4} = \frac{1}{2}$. □

Exercise (6.13). If Y_1 and Y_2 are independent exponential random variables, both with mean β, find the density function for their sum. (In Exercise 5.7, we considered two independent exponential random variables, both with mean 1 and determined $P(Y_1 + Y_2 \leq 3)$.)

Solution: Since Y_1, Y_2 are independent exponential with mean β, their sum $U = Y_1 + Y_2$ has a gamma distribution with $\alpha = 2$ and parameter β:

$$f_U(u) = \frac{1}{\Gamma(2)\beta^2} u^{2-1} e^{-u/\beta} = \frac{u}{\beta^2} e^{-u/\beta}, \quad u \geq 0$$

□

Exercise (6.14). In a process of sintering (heating) two types of copper powder (see Exercise 5.152), the density function for Y_1, the volume proportion of solid copper in a sample, was given by

$$f_1(y_1) = \begin{cases} 6y_1(1-y_1), & 0 \leq y_1 \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

The density function for Y_2, the proportion of type A crystals among the solid copper, was given as

$$f_2(y_2) = \begin{cases} 3y_2^2, & 0 \leq y_2 \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

The variable $U = Y_1 Y_2$ gives the proportion of the sample volume due to type A crystals. If Y_1 and Y_2 are independent, find the probability density function for U.

Solution: Let $U = Y_1 Y_2$ and $V = Y_2$. Then $Y_1 = U/V$, $Y_2 = V$, and $|J| = \frac{1}{v}$.

Since Y_1, Y_2 are independent:

$$\begin{aligned} f_{U,V}(u,v) &= f_1(u/v) \cdot f_2(v) \cdot \frac{1}{v} \\ &= 6(u/v)(1-u/v) \cdot 3v^2 \cdot \frac{1}{v} \\ &= 18u - \frac{18u^2}{v} \end{aligned}$$

For $0 \leq u \leq 1$, v ranges from u to 1 (from $u/v \leq 1$ and $v \leq 1$):

$$\begin{aligned} f_U(u) &= \int_u^1 \left(18u - \frac{18u^2}{v}\right) dv \\ &= 18u(1-u) - 18u^2[\ln v]_u^1 \\ &= 18u(1-u) + 18u^2 \ln u, \quad 0 \leq u \leq 1 \end{aligned}$$

$\square$