

Problem 6.

$$\bigcap_{n=1}^{\infty} (0, 1/n) = \emptyset.$$

Proof. Let $S = \bigcap_{n=1}^{\infty} (0, 1/n) = \emptyset$. Let $x \in \mathbb{R}$. Consider the following 3 cases.

Case 1: Suppose $x \leq 0$, then $x \notin S$ as $x \notin (0, 1)$.

Case 2: Suppose $x \geq 1$, then $x \notin S$ as $x \notin (0, 1)$.

Case 3: Suppose $0 < x < 1$, Choose $n \in \mathbb{N}$ so that $n > \frac{1}{x}$. Then $x > \frac{1}{n}$. so $x \notin (0, \frac{1}{n})$, thus $x \notin S$

These cases show that an arbitrary $x \in \mathbb{R}$ is not in S . \square

Problem 7. Given a function f and a subset A of its domain, consider the image $f(A) = \{f(x) : x \in A\}$.

- (a) An example of a function f , and two subsets A, B of the domain of f , for which $f(A \cap B) \neq f(A) \cap f(B)$ is

$$f(x) = |x|$$

where set A is a subset of the domain defined by $A = \{-2, -1\}$ and where set B is a subset of the domain defined by $B = \{1, 2\}$. Observe that $f(A \cap B) = \emptyset$ and $f(A) \cap f(B) = \{1, 2\}$.

- (b) If A, B are subsets of the domain of f then $f(A \cup B)$ IS RELATED IN SOME WAY TO $f(A) \cup f(B)$.

Proposition: If f is a function and A, B are subsets of the domain of f , then $f(A \cup B) = f(A) \cup f(B)$.

Proof. Let $y \in f(A \cup B)$. Then there exists $x \in A \cup B$ such that $f(x) = y$. Since $x \in A \cup B$, then either $x \in A$ or $x \in B$. If $x \in A$, then $y = f(x) \in f(A) \subseteq f(A) \cup f(B)$. If $x \in B$, then $y = f(x) \in f(B) \subseteq f(A) \cup f(B)$. Thus $f(A \cup B) \subseteq f(A) \cup f(B)$.

Conversely, let $y \in f(A) \cup f(B)$. Then either $y \in f(A)$ or $y \in f(B)$. If $y \in f(A)$, then there exists $x \in A$ such that $f(x) = y$. Since $A \subseteq A \cup B$, we have $x \in A \cup B$, so $y \in f(A \cup B)$. Similarly, if $y \in f(B)$, then $y \in f(A \cup B)$. Therefore $f(A) \cup f(B) \subseteq f(A \cup B)$.

Since $f(A \cup B) \subseteq f(A) \cup f(B)$ and $f(A \cup B) \supseteq f(A) \cup f(B)$ have been shown to be true, it follows that $f(A \cup B) = f(A) \cup f(B)$. \square

Problem 8. If $a \in \mathbb{R}$ is an upper bound for $A \subset \mathbb{R}$, and if a is also an element of A , then $a = \sup A$.

Proof. Since a is an upper bound for A , we know that $x \leq a$ for all $x \in A$.

We want to show that $a = \sup A$, we need to prove that a is the least upper bound. Choose b be any upper bound for A . Since $a \in A$ and we know that a is an upper bound and b is an upper bound for A , it must be the case that $a \leq b$ and $b \leq a$ since by definition an upper bound for a subset is an element greater than or equal to every element within that subset.

Therefore, $b = a = \sup A$. □

Problem 9. (a) Let $A = \{m/n : m, n \in \mathbb{N} \text{ with } m < n\}$. Then $\inf A = 0$ and $\sup A = 1$.

(b) Let $B = \{(-1)^m/n : n, m \in \mathbb{N}\}$. Then $\inf B = -1$ and $\sup B = 1$.

(c) Let $C = \{n/(3n+1) : n \in \mathbb{N}\}$. Then $\inf C = 1/4$ and $\sup C = 1/3$.

(d) Let $D = \{m/(m+n) : m, n \in \mathbb{N}\}$. Then $\inf D = 0$ and $\sup D = 1$.

Problem 10. (a) If A and B are nonempty, bounded, and satisfy $A \subseteq B$ then $\sup A \leq \sup B$.

Proof. Given $A \subseteq B$, every element of A is also an element of B . Since B is bounded above, $\sup B$ exists and is an upper bound for B . Therefore, $\sup B$ is also an upper bound for A .

Since $\sup A$ is the least upper bound of A , then $\sup B$ is an upper bound for A , it must be the case that $\sup A \leq \sup B$. □

(b) If $\sup A < \inf B$ for nonempty sets A and B , then there exists $c \in \mathbb{R}$ such that $a < c < b$ for all $a \in A$ and $b \in B$.

Proof. Given that $\sup A < \inf B$, we can choose any c such that $\sup A < c < \inf B$. Choose c to be the average of $\sup A$ and $\inf B$ such that $c = \frac{\sup A + \inf B}{2}$.

Since $c > \sup A$ and $\sup A$ is an upper bound for A , we have $a \leq \sup A < c$ for all $a \in A$.

Since $c < \inf B$ and $\inf B$ is a lower bound for B , we have $c < \inf B \leq b$ for all $b \in B$.

Therefore $a < c < b$ for all $a \in A$ and $b \in B$. □

(c) If there exists $c \in \mathbb{R}$ satisfying $a < c < b$ for all $a \in A$ and $b \in B$ then $\sup A < \inf B$.

This is false because we could have $\sup A = c$ or $\inf B = c$ which would give us $\sup A = \inf B$.

Problem 11. Denote the irrational numbers by $\mathbb{I} = \mathbb{R} \setminus \mathbb{Q}$.

(a) If $a, b \in \mathbb{Q}$ then $ab \in \mathbb{Q}$ and $a + b \in \mathbb{Q}$.

Proof. Since $a, b \in \mathbb{Q}$, we can write $a = \frac{p}{q}$ and $b = \frac{r}{s}$ where $p, r \in \mathbb{Z}$ and $q, s \in \mathbb{N}$.

For multiplication: $ab = \frac{p}{q} \cdot \frac{r}{s} = \frac{pr}{qs}$. Since $pr \in \mathbb{Z}$ (integers are closed under multiplication) and $qs \in \mathbb{N}$ (positive integers are closed under multiplication), we have $ab \in \mathbb{Q}$.

For addition: $a + b = \frac{p}{q} + \frac{r}{s} = \frac{ps+qr}{qs}$. Since $ps+qr \in \mathbb{Z}$ (integers are closed under multiplication and addition) and $qs \in \mathbb{N}$, we have $a + b \in \mathbb{Q}$. \square

(b) If $a \in \mathbb{Q}$ and $t \in \mathbb{I}$ then $a + t \in \mathbb{I}$. If also $a \neq 0$ then $at \in \mathbb{I}$.

Proof. We prove both claims by contradiction.

For addition: Suppose $a + t \in \mathbb{Q}$. Since $a \in \mathbb{Q}$ and $a + t \in \mathbb{Q}$, by part (a) we have $(a + t) + (-a) \in \mathbb{Q}$. But $(a + t) + (-a) = t$, so $t \in \mathbb{Q}$, contradicting that $t \in \mathbb{I}$. Therefore, $a + t \in \mathbb{I}$.

For multiplication (with $a \neq 0$): Suppose $at \in \mathbb{Q}$. Since $a \in \mathbb{Q}$ with $a \neq 0$, we have $\frac{1}{a} \in \mathbb{Q}$. By part (a), $(at) \cdot \frac{1}{a} \in \mathbb{Q}$. But $(at) \cdot \frac{1}{a} = t$, so $t \in \mathbb{Q}$, contradicting that $t \in \mathbb{I}$. Therefore, $at \in \mathbb{I}$. \square

(c) Suppose $s, t \in \mathbb{I}$. Then PROPOSITION ABOUT WHETHER st AND $s + t$ ARE EITHER RATIONAL OR IRRATIONAL IN GENERAL.

Proof. For $s + t$: If $s = \sqrt{2}$ and $t = -\sqrt{2}$, then $s + t = 0 \in \mathbb{Q}$. However, if $s = \sqrt{2}$ and $t = \sqrt{3}$, then $s + t = \sqrt{2} + \sqrt{3} \in \mathbb{I}$. To see this, suppose $\sqrt{2} + \sqrt{3} = r \in \mathbb{Q}$. Then $\sqrt{3} = r - \sqrt{2}$, and squaring both sides gives $3 = r^2 - 2r\sqrt{2} + 2$, which implies $\sqrt{2} = \frac{r^2-1}{2r} \in \mathbb{Q}$, a contradiction.

For st : If $s = \sqrt{2}$ and $t = \sqrt{2}$, then $st = 2 \in \mathbb{Q}$. However, if $s = \sqrt{2}$ and $t = \sqrt{3}$, then $st = \sqrt{6} \in \mathbb{I}$, since if $\sqrt{6} \in \mathbb{Q}$, then 6 would be a perfect square, which it is not.

Therefore, the irrational numbers are not closed under addition or multiplication. \square

Therefore, the irrational numbers are not closed under addition or multiplication.

Problem 12. For all $n \in \mathbb{N}$, $2^n \geq n$.

Proof. We prove by induction on n .

Base case: For $n = 1$, we have $2^1 = 2 \geq 1$, which is true.

Inductive step: Assume $2^k \geq k$ for some $k \in \mathbb{N}$. We need to show $2^{k+1} \geq k + 1$.

Starting from the inductive hypothesis:

$$\begin{aligned} 2^k &\geq k \\ 2 \cdot 2^k &\geq 2k \quad (\text{multiplying both sides by 2}) \\ 2^{k+1} &\geq 2k \\ 2^{k+1} &\geq k + k \\ 2^{k+1} &\geq k + 1 \quad (\text{since } k \geq 1 \text{ for all } k \in \mathbb{N}) \end{aligned}$$

Therefore, by mathematical induction, $2^n \geq n$ for all $n \in \mathbb{N}$. □

Problem 13. Let $y_1 = 6$ and, for each $n \in \mathbb{N}$, let $y_{n+1} = (2y_n - 6)/3$.

(a) For all $n \in \mathbb{N}$, $y_n \geq -6$.

Proof. We prove by induction on n .

Base case: For $n = 1$, we have $y_1 = 6 \geq -6$, which is true.

Inductive step: Assume $y_k \geq -6$ for some $k \in \mathbb{N}$. We need to show $y_{k+1} \geq -6$.

From the inductive hypothesis: $y_k \geq -6$

$$\begin{aligned} y_k &\geq -6 \\ 2y_k &\geq -12 \\ 2y_k - 6 &\geq -18 \\ \frac{2y_k - 6}{3} &\geq -6 \\ y_{k+1} &\geq -6 \end{aligned}$$

Therefore, by mathematical induction, $y_n \geq -6$ for all $n \in \mathbb{N}$. □

(b) The sequence (y_1, y_2, y_3, \dots) is decreasing.

Proof. We prove by induction that $y_{n+1} < y_n$ for all $n \in \mathbb{N}$.

Base case: For $n = 1$, we have:

$$\begin{aligned} y_2 &= \frac{2y_1 - 6}{3} = \frac{2(6) - 6}{3} = \frac{6}{3} = 2 \\ y_1 &= 6 \end{aligned}$$

Since $2 < 6$, we have $y_2 < y_1$.

Inductive step: Assume $y_{k+1} < y_k$ for some $k \in \mathbb{N}$. We need to show $y_{k+2} < y_{k+1}$.

We have:

$$\begin{aligned} y_{k+2} - y_{k+1} &= \frac{2y_{k+1} - 6}{3} - y_{k+1} \\ &= \frac{2y_{k+1} - 6 - 3y_{k+1}}{3} \\ &= \frac{-y_{k+1} - 6}{3} \\ &= -\frac{y_{k+1} + 6}{3} \end{aligned}$$

From part (a), we know $y_{k+1} \geq -6$, so $y_{k+1} + 6 \geq 0$. Since $y_1 = 6 > -6$ and the recurrence relation preserves this (as shown in part (a)), we actually have $y_{k+1} > -6$, which means $y_{k+1} + 6 > 0$.

Therefore, $y_{k+2} - y_{k+1} = -\frac{y_{k+1}+6}{3} < 0$, so $y_{k+2} < y_{k+1}$.

By mathematical induction, $y_{n+1} < y_n$ for all $n \in \mathbb{N}$.

□