4.4.2

a) We are provided the second order differential equation $y'' - 9y' = 3x^2 - 5\sin(3x)$ and we are tasked with finding the form of the particular solution, by first inspection/observation we find the form to be:

$$\boxed{y_p = (Ax^2 + Bx + C) + D\cos(3x) + E\sin(3x)} \tag{1}$$

b) We are also provided with the second order differential equation $y'' + 2y' + y = 2e^{-x} - e^x$, by inspection/observation we conclude that the form of the particular solution is:

$$\boxed{y_p = Ae^{-x} + Be^x}$$

4.4.3

We are tasked with solving the following differential equations via method of undetermined coefficients:

a) $y'' + 2y' = 2x + 5 - e^{-2x}$

$$\begin{aligned} y'' + 2y' &= 2x + 5 - e^{-2x} \\ m^2 + 2m + 0 &= 0 \\ m(m+2) &= 0 \\ m &= 0, -2 \\ y_c &= c_1 + c_2e^{-2x} \end{aligned}$$

$$\begin{aligned} y_p &= Ax^2 + Bx + Cxe^{-2x} \\ y_p' &= 2Ax + B - 2Cxe^{-2x} + Ce^{-2x} \\ y_p'' &= 2A + 4Cxe^{-2x} - 4Ce^{-2x} \end{aligned}$$

$$\begin{aligned} 2A + 4Cxe^{-2x} - 4Ce^{-2x} + 4Ax + 2B - 4Cxe^{-2x} + 2Ce^{-2x} &= 2x + 5 - e^{-2x} \\ 4Ax + 2(A+B) + 2Ce^{-2x} &= 2x + 5 - e^{-2x} \\ A = \frac{1}{2}, B = 2, C &= \frac{1}{2} \end{aligned}$$

Our solution is:

$$y_c + y_p = \boxed{c_1 + c_2e^{-2x} + \frac{1}{2}x^2 + 2x + \frac{1}{2}xe^{-2x}}$$

b) $y'' - 9y' = 2e^{3x}$

$$\begin{aligned} y'' - 9y' &= 2e^{3x} \\ m^2 - 9m &= 0 \\ m(m-9) &= 0 \\ m &= 9, 0 \\ y_c &= c_1 + c_2e^{9x} \end{aligned}$$

$$\begin{aligned} y_p &= Ae^{3x} \\ y_p' &= 3Ae^{3x} \\ y_p'' &= 9Ae^{3x} \end{aligned}$$

$$\begin{aligned} 9Ae^{3x} - 27Ae^{3x} &= 2e^{3x} \\ -18Ae^{3x} &= 2e^{3x} \\ A &= -\frac{1}{9} \end{aligned}$$

Our solution is then:

$$y_c + y_p = \boxed{c_1 + c_2 e^{9x} - \frac{1}{9}e^{3x}}$$

c) $y'' + 4y' + 4y = (3+x)e^{-2x}$

$$\begin{aligned}
&y'' + 4y' + 4y = (3+x)e^{-2x} \\
&m^2 + 4m + 4 = 0 \\
&(m+2)(m+2) = 0 \\
&m = -2 \\
&y_c = c_1 e^{-2x} + c_2 x e^{-2x}
\end{aligned}$$

$$\begin{aligned}
&y_p = Ax^2 e^{-2x} + Bx^3 e^{-2x} \\
&y_p' = 2Axe^{-2x} - 2Ax^2 e^{-2x} + 3Bx^2 e^{-2x} - 2Bx^3 e^{-2x} \\
&y_p'' = 2Ae^{-2x} + 4Ae^{-2x} - 8Ae^{-2x} \\
&\quad + 4Bx^3 e^{-2x} - 12Bx^2 e^{-2x} + 6Bxe^{-2x}
\end{aligned}$$

Here is the nasty after plugging in:

$$\begin{aligned}
&2Ae^{-2x} + 4Ae^{-2x} - 8Ae^{-2x} + 4Bx^3 e^{-2x} - 12Bx^2 e^{-2x} + 6Bxe^{-2x} \\
&+ 8Axe^{-2x} - 8Ax^2 e^{-2x} + 12Bx^2 e^{-2x} - 8Bx^3 e^{-2x} \\
&+ Ax^2 e^{-2x} + Bx^3 e^{-2x} \\
&= 3e^{-2x} + xe^{-2x}
\end{aligned}$$

Simplifies beautifully into:

$$\begin{aligned}
&2Ae^{-2x} + 6Bxe^{-2x} = 3e^{-2x} + xe^{-2x} \\
&A = \frac{3}{2}, B = \frac{1}{6}
\end{aligned}$$

Our solution is then :

$$y_c + y_p = \boxed{c_1 e^{-2x} + c_2 x e^{-2x} + \frac{3}{2}x^2 e^{-2x} + \frac{1}{6}x^3 e^{-2x}}$$

4.6.1

We proceed to solve the problem $3y'' + y' = 9x$ using variation of parameters where our $y_p = u_1 y_1 + u_2 y_2$.

$$y'' + \frac{y'}{3} = 3x \qquad W = \begin{vmatrix} 1 & e^{\frac{-x}{3}} \\ 0 & -\frac{1}{3}e^{\frac{-x}{3}} \end{vmatrix} = -\frac{1}{3}e^{\frac{-x}{3}}$$

$$m^2 + \frac{m}{3} = 0 \qquad W_1 = \begin{vmatrix} 0 & e^{\frac{-x}{3}} \\ 3x & -\frac{1}{3}e^{\frac{-x}{3}} \end{vmatrix} = -3xe^{\frac{-x}{3}}$$

$$m(m + \frac{1}{3}) = 0 \qquad W_2 = \begin{vmatrix} 1 & 0 \\ 0 & 3x \end{vmatrix} = 3x$$

$$m = 0, m = -\frac{1}{3} \qquad u_1 = \int \frac{-3xe^{\frac{-x}{3}}}{-\frac{1}{3}e^{\frac{-x}{3}}} = \frac{9x^2}{2}$$

$$y_c = c_1 + c_2e^{\frac{-x}{3}} \qquad u_2 = \int \frac{3x}{-\frac{1}{3}e^{\frac{-x}{3}}} = -27e^{\frac{x}{3}}(x-3)$$

Our y_p should be $u_1 * 1 + u_2 * e^{\frac{-x}{3}}$. We proceed to plug in for u_1 and u_2, I'm assuming the 81 below gets absorbed into c_1 since its a constant too, our answer should be:

$$y_c + y_p = c_1 + c_2e^{-\frac{x}{3}} + \frac{9x^2}{3} - 27x + 81$$

$$\boxed{c_1 + c_2e^{-\frac{x}{3}} + \frac{9x^2}{3} - 27x}$$

4.6.2

We solve for the initial value problem $y'' - 4y' + 4y = (6x^2 - 12x)e^{2x}$, $y(0) = 1$, $y'(0) = 0$. Gonna skip the bits with finding our complimentary solution since the roots to $m^2 - 4m + 4$ is 2, $y_c = c_1e^{2x} + c_2xe^{2x}$

$$y_c = c_1e^{2x} + c_2xe^{2x} \qquad y_p = u_1e^{2x} + u_2xe^{2x}$$

Now lets find u_1 and u_2:

$$W_1 = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & 2xe^{2x} + e^{2x} \end{vmatrix} = (2xe^{4x} + e^{4x}) - 2xe^{4x} = e^{4x} \tag{1}$$

$$W_2 = \begin{vmatrix} 0 & xe^{2x} \\ (6x^2 - 12x)e^{2x} & 2xe^{2x} + e^{2x} \end{vmatrix} = -xe^{4x}(6x^2 - 12x) \tag{2}$$

$$W_3 = \begin{vmatrix} e^{2x} & 0 \\ 2e^{2x} & (6x^2 - 12x)e^{2x} \end{vmatrix} = e^{4x}(6x^2 - 12x) \tag{3}$$

$$u_1 = \int \frac{-xe^{4x}(6x^2 - 12x)}{e^{4x}}dx = \int(-6x^3 + 12x^2)dx = -\frac{3}{2}x^4 + 4x^3 \tag{4}$$

$$u_2 = \int \frac{e^{4x}(6x^2 - 12x)}{e^{4x}}dx = \int(6x^2 - 12x)dx = 2x^3 - 6x^2 \tag{5}$$

$$y_c + y_p = c_1e^{2x} + c_2xe^{2x} + (-\frac{3}{2}x^4 + 4x^3)e^{2x} + (2x^3 - 6x^2)xe^{2x} \tag{6}$$

$$= c_1e^{2x} + c_2xe^{2x} + \frac{1}{2}x^4e^{2x} + 2x^3e^{2x} \tag{7}$$

Now that we have our general solution, its time to find its derivative, plug in our initial conditions and solve the subsequent system of equations for c_1 and c_2

$$\begin{aligned}
y &= c_1e^{2x} + c_2xe^{2x} + \frac{1}{2}x^4e^{2x} + 2x^3e^{2x} \\
y' &= 2c_1e^{2x} + c_2(2xe^{2x} + e^{2x}) + \frac{1}{4}(4x^3e^{2x} + 2x^4e^{2x}) + 2(3x^2e^{2x} + 2x^3e^{2x}) \\
y(0) &= c_1 = 1 \\
y'(0) &= 2 + c_2 = 0, c_2 = -2
\end{aligned}$$

Turned out not to have been much of a system since plugging in gave us the solution thankfully(or I might be a complete idiot). Our solution is:

$$\boxed{y = e^{2x} - 2xe^{2x} + \frac{1}{2}x^4e^{2x} + 2x^3e^{2x}}$$

4.7.1

We are tasked with solving the following differential equations:
a) $4x^2y'' + y = 0$

$$\begin{aligned}
y &= x^m & 4m^2x^m - 4mx^m + x^m &= 0 \\
y' &= mx^{m-1} & (4m^2 - 4m + 1)x^m &= 0 \\
y'' &= m(m-1)x^{m-2} & (2m-1)^2 &= 0 \\
& & m &= \frac{1}{2}
\end{aligned}$$

Simple enough, we end up with the solution:

$$y = \boxed{c_1\sqrt{x} + c_2\sqrt{x}\ln x}$$

b) $x^2y'' - 7xy' + 41y = 0$

$$\begin{aligned}
y &= x^m & m^2x^m - 8mx^m + 41x^m &= 0 \\
y' &= mx^{m-1} & (m^2 - 8m + 41)x^m &= 0 \\
y'' &= m(m-1)x^{m-2} & m^2 - 8m + 41 &= 0
\end{aligned}$$

We're gonna throw that quadratic into a calculator because I'm out of time, we get our roots and then our solution:

$$4 + 5i \qquad\qquad 4 - 5i$$

Our answer is then:

$$y = \boxed{c_1x^4\cos(5\ln(x)) + c_2x^4\sin(5\ln(x))}$$