Problem 44. *(Comparison Test)*

Assume (a_k) and (b_k) are sequences satisfying $0 \leq a_k \leq b_k$ for all $k \in \mathbb{N}$.

(i) *If $\sum_{k=1}^{\infty} b_k$ converges then $\sum_{k=1}^{\infty} a_k$ converges.*

(ii) *If $\sum_{k=1}^{\infty} a_k$ diverges then $\sum_{k=1}^{\infty} b_k$ diverges.*

Proof. □

Problem 45. *(Alternating Series Test)*

Suppose (a_n) is a nonnegative *sequence which satisfies*

(i) *(a_n) is decreasing, and*

(ii) $\lim_{n\to\infty} a_n = 0$.

Then the alternating series $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ *converges.*

Proof. □

Problem 46. *For each of the subsets of $\mathbb{R}$ below, decide whether it is open, closed, or neither. If a set is not open, find a point in the set for which there is no ϵ-neighborhood contained in the set. If a set is not closed, find a limit point that is not contained in the set.*

(a) $\mathbb{Q}$

(b) $\mathbb{N}$

(c) $\{x \in \mathbb{R} : x \neq 0\}$

(d) $\{1 + 1/4 + 1/9 + \cdots + 1/n^2 : n \in \mathbb{N}\}$

(e) $\{1 + 1/2 + 1/3 + \cdots + 1/n : n \in \mathbb{N}\}$

Problem 47. *Let $A \subset \mathbb{R}$ be nonempty and bounded above, and let $s = \sup A$. Then*

(i) *$s \in \overline{A}$, but*

(ii) *if A is open then $s \notin A$.*

Proof. □

Problem 48. *Decide whether the following statements are true or false. Provide proofs for those that are true, and counterexamples for those that are false.*

(a) *Every nonempty open set contains a rational number.*

(b) *The Cantor set is closed.*

(c) *If $A \subseteq \mathbb{R}$ is an open set which contains every rational ($\mathbb{Q} \subset A$) then $A = \mathbb{R}$.*

Problem 49. *(De Morgan's Laws for arbitrary unions and intersections)*
Let X be a set, which we call the universe set. For any $A \subset X$ we write

$$A^c = \{x \in X \,:\, x \notin A\}$$

for the complement set. Also let Λ be any set, which will be used as a set of indices. Consider

$$\mathcal{E} = \{E_\lambda \subset X \,:\, \lambda \in \Lambda\},$$

a collection of sets. The following equalities hold:

(i) $\displaystyle \left(\bigcup_{\lambda\in\Lambda} E_\lambda\right)^c = \bigcap_{\lambda\in\Lambda} E_\lambda^{\,c}$

(ii) $\displaystyle \left(\bigcap_{\lambda\in\Lambda} E_\lambda\right)^c = \bigcup_{\lambda\in\Lambda} E_\lambda^{\,c}$

Proof. □

Problem 50. *If $A \subset \mathbb{R}$ is both open and closed then either $A = \emptyset$ or $A = \mathbb{R}$.*

Proof. □