Problem 1. *There is no rational number whose square is* 2.

Proof. Assume, for contradiction, that there exist integers p and q satisfying

$$\frac{p}{q} = \sqrt{2},$$

where p/q is a rational number in lowest terms. By squaring, this is the same as $\frac{p^2}{q^2} = 2$, and by clearing denominators it is the same as

$$p^2 = 2q^2.$$

Thus p^2 is divisible by 2, an even number. This implies that p is also divisible by 2 and can be expressed in the form $p = 2k$ for some $k \in \mathbb{Z}$. If we sbustitute the p in $p^2 = 2q^2$ for $2k$, we get

$$(2k)^2 = 4(k^2) = 2q^2$$

Further reducing this gives us

$$2(k^2) = q^2$$

This is a contradiction as the result implies that q^2 is also even and thus q is even. Therefore p and q are both even and are irreducible.

□

Problem 2. *(a) The negation of "For all real numbers satisfying $a < b$, there exists $n \in \mathbb{N}$ such that $a + (1/n) < b$" is "There exists a real number satisfying $a < b$ such that for all $n \in \mathbb{N}$, $a + (1/n) \geq b$.*

(b) The negation of "There exists a real number $x > 0$ such that $x < 1/n$ for all $n \in \mathbb{N}$" is "For all real numbers $x > 0$, there exists an $n \in \mathbb{N}$ such that $x \geq 1/n$.

(b) The negation of "Between every two distinct real numbers there is a rational number" is "There exists an $x, y \in \mathbb{R}$, where $x \neq y$, such that there is no $n \in \mathbb{Q}$ that satisfies $x < n < y$.

Problem 3. *Suppose a and b are real numbers. Then*

(a) $|a - b| \leq |a| + |b|$

Proof. **Case 1:** Suppose $a > b$, then $|a - b| = a - b$ (since $a - b > 0$). If $a > 0$ is true, then $|a - b| = a - b = |a| - b$. But $-b \leq |b|$ since $-b = b$ if b is negative and $-b \leq 0 \leq b$ if b is non-negative, So $|a - b| = |a| - b \leq |a| + |b|$. On the other hand, if $a < 0$, then $|a - b| = a - b \leq |a| - b \leq |a| - |b|$ (since $a \leq |a|$, as before).

Case 2: Suppose $b > a$, then $|a - b| = b - a$ (since $b - a > 0$). If $b > 0$ is

also true, then $|a-b| = b-a \leq b+|a|$ (since $-a \leq |a|$). If $b \geq 0$, then $|b| = b$, so $|a-b| = b-a \leq b+|a| = |a|+|b|$ (since $-a \leq |a|$). If $b < 0$, then since $b > a$, we have $a < b < 0$, so $|a| = -a$ and $|b| = -b$. Thus, $|a-b| = b-a = b+(-a) \leq -b+(-a) = |b|+|a|$.
Case 3: Suppose $a = b$, then $|a-b| = 0 \leq |a|+|b|$ Since absolute values are non-negative. □

(b) $\big||a|-|b|\big| \leq |a-b|$

Proof. □

Problem 4. *Give an example of each, or state that it is impossible.*

(a) $f : \mathbb{N} \to \mathbb{N}$ *that is one-to-one but not onto.*

My Answer: *The function* $f(n) = 2n$ *is a mapping from* $\mathbb{N} \to \mathbb{N}$ *That is one-to-one but not onto.*

(b) $f : \mathbb{N} \to \mathbb{N}$ *that is onto but not one-to-one.*

My Answer: *The function* $f(n) = [(n+1)/2]$ *is onto but not one-to one.*

(d) $f : \mathbb{N} \to \mathbb{Z}$ *that is one-to-one and onto.*

My Answer: *The piecewise function*

$$f(n) = \begin{cases} \frac{n}{2} & \textit{If n is even} \\ -\frac{n+1}{2} & \textit{If n is odd} \end{cases}$$

Is a mapping from $\mathbb{N} \to \mathbb{Z}$ *that is both one-to-one and onto.*

Problem 5. *There exists an infinite collection of sets* $A_1, A_2, A_3, \ldots$ *with the properties that every* A_i *has an infinite number of elements, and* $A_i \cap A_j = \emptyset$ *for all* $i \neq j$, *and* $\bigcup_{i=1}^{\infty} A_i = \mathbb{N}$.

Proof. □