

Problem 68. *The function $f(x) = 1/x^2$ is uniformly continuous on $(1, 2)$, but it is not uniformly continuous on $(0, 1)$.*

Proof. Part 1: Uniform Continuity on $(1, 2)$: To show that the function $f(x) = 1/x^2$ is uniformly continuous on the set $S_1 = (1, 2)$ we must show that for every $\epsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in S_1$:

$$|x - y| < \delta \implies |f(x) - f(y)| < \epsilon$$

Let $\epsilon > 0$, Observe that

$$|f(x) - f(y)| = \left| \frac{1}{x^2} - \frac{1}{y^2} \right| = \left| \frac{y^2 - x^2}{x^2 y^2} \right| = \frac{|x + y||x - y|}{x^2 y^2}$$

Since $x, y \in (1, 2)$, it follows that $x^2 > 1$ and $y^2 > 1$, so $x^2 y^2 > 1$. Furthermore $|x + y| < 2 + 2 = 4$. Therefore:

$$|f(x) - f(y)| = \frac{|x - y||x + y|}{x^2 y^2} < \frac{|x - y| * 4}{1} = 4|x - y|$$

Choose $\delta = \frac{\epsilon}{4}$. Then whenever $|x - y| < \delta$, we have:

$$|f(x) - f(y)| < 4|x - y| < 4 * \frac{\epsilon}{4} = \epsilon$$

Since δ only depends on ϵ , f is uniformly continuous on $(1, 2)$ \square

Proof. Part 2: Not Uniformly continuous on $(0, 1)$: Suppose for the sake of contradiction that f is uniformly continuous on the set $S_2 = (0, 1)$. Choose $\epsilon = 1$. Then there exists a $\delta > 0$ such that for all $x, y \in S_2$ we have

$$|x - y| < \delta \implies \left| \frac{1}{x^2} - \frac{1}{y^2} \right| < 1$$

Choose $x \in S_2$ with $x < \delta$ and choose $y = \frac{x}{2}$. Note that $y \in S_2$ because $x \in S_2$ so $y = \frac{x}{2} < \frac{1}{2} < 1$. Then

$$|x - y| = |x - \frac{x}{2}| = |\frac{x}{2}| = \frac{x}{2} < \frac{\delta}{2} < \delta$$

So $|x - y| < \delta$ holds, it follows that

$$\left| \frac{1}{x^2} - \frac{1}{y^2} \right| = \left| \frac{1}{x^2} - \frac{1}{(x/2)^2} \right| = \left| \frac{1}{x^2} - \frac{1}{(x^2/4)} \right| = \left| \frac{1}{x^2} - \frac{4}{x^2} \right| = \left| \frac{-3}{x^2} \right| = \frac{3}{x^2} > 1$$

So $\left| \frac{1}{x^2} - \frac{1}{y^2} \right| > 1$. A contradiction, thus f is not uniformly continuous on $S_2 = (0, 1)$. \square

Problem 69. We say that a function $f : A \rightarrow \mathbb{R}$ is Lipschitz if there exists $M > 0$ so that

$$\frac{|f(x) - f(y)|}{|x - y|} \leq M$$

for all $x, y \in A$. If f is Lipschitz then f is uniformly continuous.

Proof. Since f is Lipschitz, there exists $M > 0$ such that $|f(x) - f(y)| \leq M|x - y|$ for all $x, y \in A$. Let $\delta = \frac{\epsilon}{M}$ and let $x, y \in A$ be any two points such that $|x - y| < \delta$. Then it follows that

$$|f(x) - f(y)| \leq M|x - y| \implies |f(x) - f(y)| < M * \delta$$

Since $|x - y| < \delta$. Observe that

$$|f(x) - f(y)| < M * \delta = M * \frac{\epsilon}{M} = \epsilon$$

Thus, whenever $|x - y| < \delta$, we have $|f(x) - f(y)| < \epsilon$. Note that M is a constant and δ only depends on ϵ . Therefore proving that if f is Lipschitz, then f is uniformly continuous. \square

Problem 70. Let f and g be functions defined on an interval A . Assume both are differentiable at some point $c \in A$, and suppose $k \in \mathbb{R}$. Then

$$(i) (f + g)'(c) = f'(c) + g'(c)$$

$$(ii) (kf)'(c) = kf'(c)$$

Proof. (i) Since f and g are differentiable at some point $c \in A$, then $f'(c) = \lim_{h \rightarrow 0} \frac{f(c+h) - f(c)}{h}$ and $g'(c) = \lim_{h \rightarrow 0} \frac{g(c+h) - g(c)}{h}$ \square

Problem 71. Let $h(x) = 1/x$ and $\ell(x) = 1/x^2$. For $c \neq 0$, we have

$$h'(c) = -\frac{1}{c^2}, \quad \ell'(c) = -\frac{2}{c^3}$$

Proof. \square

Problem 72. Let f and g be functions defined on an interval A . Assume both are differentiable at some point $c \in A$, and suppose $g(c) \neq 0$. Then

$$\left(\frac{f}{g}\right)'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{g(c)^2}.$$

Proof. \square

Problem 73. For $a \in \mathbb{R}$, let

$$f_a(x) = \begin{cases} x^a, & \text{if } x > 0 \\ 0, & \text{if } x \leq 0 \end{cases}$$

- (a) For which values of a is $f_a(x)$ continuous at $x = 0$?
- (b) What is the derivative $f'_a(x)$, and what is its domain? For which values of a is $f_a(x)$ differentiable at $x = 0$? When is the derivative function $f'_a(x)$ continuous?