

**Problem 22.** If  $f : A \rightarrow B$  has an inverse function then  $f$  is onto and  $f$  is one-to-one.

*Proof.* Suppose  $f : A \rightarrow B$  has an inverse function  $g : B \rightarrow A$ . By definition of an inverse function this means that

$$\begin{aligned} g(f(a)) &= a, \text{ for all } a \in A \\ f(g(b)) &= b, \text{ for all } b \in B \end{aligned}$$

**one-to-one :** Let  $a_1, a_2 \in A$  such that  $f(a_1) = f(a_2)$ . To prove  $f$  is one-to-one (injective) we must show that  $a_1 = a_2$ . Observe that composing the inverse  $g$  with  $f$  gives

$$g(f(a_1)) = g(f(a_2))$$

As a result of applying the definition of inverse  $g(f(a)) = a$ , we get

$$a_1 = a_2$$

Therefore  $f$  is injective.

**onto :** To prove that  $f$  is onto, we must show that there exists an  $a \in A$  such that  $f(a) = b$ . Since  $g : B \rightarrow A$ , we know that  $g(b) \in A$ . Let  $a = g(b)$ . Substituting  $a$  in  $f(a)$  gives

$$f(a) = f(g(b)) = b$$

As a result of applying the definition of inverse  $f(g(b)) = b$ . Therefore, for every  $b \in B$ , there exists  $a = g(b) \in A$  such that  $f(a) = b$ . Hence  $f$  is onto.

Since  $f$  is both one-to-one and onto, it follows that  $f$  is bijective. □

**Problem 23.** A real number  $x \in \mathbb{R}$  is called algebraic if there exists  $a_0, a_1, \dots, a_{n-1}, a_n \in \mathbb{Z}$ , not all zero, so that

$$a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 = 0$$

That is, a real number is algebraic if it is a root of a polynomial equation with integer coefficients.

(a) The numbers  $\sqrt{2}$ ,  $\sqrt[3]{2}$ , and  $\sqrt{3} + \sqrt{2}$  are algebraic.

*Proof.* □

(b) For fixed  $n \in \mathbb{N}$ , let  $A_n$  be the set of algebraic numbers which are roots of polynomials, with integer coefficients, of degree  $n$ . Then  $A_n$  is countable.

*Proof.* □

(c) The set of all algebraic numbers is countable.

*Proof.* □

**Problem 24.** *There is an onto function  $f : (0, 1) \rightarrow S$  where  $S = \{(x, y) : 0 < x, y < 1\}$  is the unit square in the plane  $\mathbb{R}^2$ .*

*Proof.* □

**Problem 25.** (a)  $\lim_{n \rightarrow \infty} \frac{2n+1}{5n+3} = \frac{2}{5}$

*Proof.* Let  $\epsilon > 0$ . □

(b)  $\lim_{n \rightarrow \infty} \frac{2n^2}{n^3+1} = 0$

*Proof.* Let  $\epsilon > 0$ . □

(c)  $\lim_{n \rightarrow \infty} \frac{\sin(n)}{\sqrt{n}} = 0$

*Proof.* Let  $\epsilon > 0$ . □

**Problem 26.** (a) *A sequence with an infinite number of ones that does not converge to one.*

(b) *A sequence with an infinite number of ones that converges to a limit not equal to one.*

**Problem 27.** *Let  $(x_n)$  be a sequence that converges to  $x$ . Suppose  $p(x)$  is a polynomial. Then*

$$\lim_{n \rightarrow \infty} p(x_n) = p(x).$$

*Proof.* □

**Problem 28.** *Consider three sequences  $(x_n)$ ,  $(y_n)$ , and  $(z_n)$  for which  $x_n \leq y_n \leq z_n$  for each  $n$ . If  $x_n \rightarrow \ell$  and  $z_n \rightarrow \ell$  then  $y_n \rightarrow \ell$ .*

*Proof.* □