Chapter 7

Prove the following statements.

Exercise (12). There exists a positive real number x for which $x^2 < \sqrt{x}$.

Proof: Suppose that $x = \frac{1}{4}$. Observe that substituting for x in our inequality $x^2 < \sqrt{x}$ gives $(\frac{1}{4})^2 = \frac{1}{16} < \frac{1}{2} = \sqrt{\frac{1}{4}}$. Thus $x = \frac{1}{4}$ is such a positive real number. $\square$

Exercise (18). There is a set X for which $\mathbb{N} \in X$ and $\mathbb{N} \subseteq X$.

Proof: Suppose that $X = \mathbb{N} \cup \{\mathbb{N}\}$. Observe that $\mathbb{N} \in X$ and that $\mathbb{N} \subseteq X$. Thus $X = \mathbb{N} \cup \{\mathbb{N}\}$ is such a set. $\square$

Exercise (21). Every real solution of $x^3 + x + 3 = 0$ is irrational.

Proof: (By Contradiction) Suppose for the sake of contradiction that there exists a rational solution to $x^3 + x + 3 = 0$, that is to say that there is an $x = \frac{a}{b}$ where $a, b \in \mathbb{Z}$ in its most reduced form such that $(\frac{a}{b})^3 + \frac{a}{b} + 3 = 0$. Observe that multiplying our equation by b^3 gives $a^3 + ab^2 + 3b^3 = 0$. Consider these 3 cases:

<u>Case 1:</u> Suppose a is odd and b is odd. Then the left-hand side is a sum of 3 odd numbers, which is odd, meaning 0 is odd. This is a contradiction.

<u>Case 2:</u> Suppose a is odd and b is even. Then the left-hand side is a sum of 2 even numbers and an odd number, meaning 0 is odd. This is also contradiction.

<u>Case 3:</u> Suppose a is even and b is odd, likewise the left-hand side is a sum of 2 even numbers and an odd number, meaning 0 is odd. This is yet again another contradiction.

Thus it follows that every real solution of $x^3 + x + 3 = 0$ must be irrational. $\square$

Exercise (31). If $n \in \mathbb{Z}$, then $\gcd(n, n+1) = 1$.

Proof: Suppose d is an integer and that $d \mid n$ and $d \mid (n+1)$. Then it follows that $d \mid (n+1) - n$ which implies $d \mid 1$. Thus the greatest common divisor of n and $n+1$ is in fact 1. $\square$

Exercise (35). Suppose $a, b \in \mathbb{N}$. Then $a = \gcd(a, b)$ if and only if $a \mid b$.

Proof: Suppose $a = \gcd(a, b)$. Then by definition $a \mid a$ and more importantly $a \mid b$.

Conversely suppose $a \mid b$. Then it must be the case that $a \leq \gcd(a, b)$ since a divides itself and $a \mid b$. Since $\gcd(a,b) \mid a$ then $a = \gcd(a,b) * x$ where $x \in \mathbb{Z}$. As all integers are positive, it follows that $a \geq \gcd(a,b)$.

Since $a \leq \gcd(a,b)$ and $a \geq \gcd(a,b)$, then $a = \gcd(a,b)$. □

Chapter 8

Use the methods introduced in this chapter to prove the following statements.

Exercise (4). If $m, n \in \mathbb{Z}$, then $\{x \in \mathbb{Z} : mn \mid x\} \subseteq (\{x \in \mathbb{Z} : m \mid x\} \cap \{x \in \mathbb{Z} : n \mid x\}$.

Proof: Suppose $a \in \{x \in \mathbb{Z} : mn \mid x\}$. This means $a \in \mathbb{Z}$ and $mn \mid a$. By definition of divisibility, there is an integer k such that $a = mn * k$. Therefore $a = m(n * k)$ and $a = n(m * k)$. From $a = m(n * k)$, it follows that $m \mid a$ so that $a \in \{x \in \mathbb{Z} : m \mid x\}$. Similarly from $a = n(m * k)$, it follows that $n \mid a$ so that $a \in \{x \in \mathbb{Z} : n \mid x\}$. Thus by the definition of the intersection of two sets, we have $a \in \{x \in \mathbb{Z} : m \mid x\} \cap \{x \in \mathbb{Z} : n \mid x\}$. Thus $\{x \in \mathbb{Z} : mn \mid x\} \subseteq (\{x \in \mathbb{Z} : m \mid x\} \cap \{x \in \mathbb{Z} : n \mid x\}$. □

Exercise (6). Suppose A, B and C are sets. Prove that if $A \subseteq B$, then $A - C \subseteq B - C$.

Proof: Suppose $A \subseteq B$. Let $x \in (A - C)$, by definition this means $x \in A \wedge x \notin C$. Since $x \in A$ and $A \subseteq B$, this means $x \in B$. Since $x \in B$ and $x \notin C$ it follows that $x \in B - C$. Thus $A - C \subseteq B - C$. □

Exercise (7). Suppose A, B and C are sets. If $B \subseteq C$, then $A \times B \subseteq A \times C$.

Proof: Suppose $B \subseteq C$ and let $(x, y) \in A \times B$. Then by definition of the Cartesian product $x \in A$ and $y \in B$. Since $B \subseteq C$ it follows that $y \in C$. Thus $x \in A$ and $y \in C$ implies $(x, y) \in A \times C$. Therefore $(x, y) \in A \times B$ implies $(x, y) \in A \times C$. Hence $A \times B \subseteq A \times C$. □

Exercise (9). If A, B and C are sets, then $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Proof: Observe the following:

$$\begin{aligned}
A \cap (B \cup C) &= \{x : x \in A \wedge x \in (B \cup C)\} && \text{definition of interesection} \\
&= \{x : x \in A \wedge (x \in B \vee x \in C\} && \text{definition of union} \\
&= \{x : (x \in A \wedge x \in B) \vee (x \in A \wedge x \in C)\} && \text{distributive law} \\
&= \{x : (x \in A \cap B) \vee (x \in A \cap C)\} && \text{definition of intersection} \\
&= (A \cap B) \cup (A \cap C) && \text{definition of union}
\end{aligned}$$

Thus completing the proof. □

Exercise (10). If A and B are sets in a universal set U, then $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

Proof: Observe the following:

$$\begin{aligned}
\overline{A \cap B} &= U - (A \cap B) && \text{definition of compliment} \\
&= \{x : (x \in U) \wedge (x \notin A \cap B) && \text{definition of negation} \\
&= \{x : (x \in U) \wedge \neg((x \in A) \wedge (x \in B)) && \text{definition of interesection} \\
&= \{x : (x \in U) \wedge (\neg(x \in A) \vee \neg(x \in B)) && \text{demorgans law} \\
&= \{x : (x \in U) \wedge (x \in U) \wedge ((x \notin A) \vee (x \notin B)) && (x \in U) = (x \in U) \wedge (x \in U) \\
&= \{x : (x \in U) \wedge ((x \notin A) \vee (x \in U) \wedge (x \notin B)) && \text{regroup} \\
&= \{x : (x \in U) \wedge ((x \notin A)\} \cup \{x : (x \in U) \wedge (x \notin B)\} && \text{definition of union} \\
&= (U - A) \cup (U - B) && \text{definition of negation} \\
&= \overline{A} \cup \overline{B} && \text{definition of compliment}
\end{aligned}$$

Thus completing the proof. □

Exercise (14). If A, B and C are sets, then $(A \cup B) - C = (A - C) \cup (B - C)$.

Proof: Observe the following:

$$
\begin{aligned}
(A \cup B) - C &= \{x : (x \in a \vee x \in B) \wedge x \notin C\} && \text{def of union and negation} \\
&= \{x : (x \in a) \wedge (x \notin C) \vee (x \in B) \wedge (x \notin C)\} && \text{regroup} \\
&= \{x : ((x \in a) \wedge (x \notin C)) \vee ((x \in B) \wedge (x \notin C))\} && \text{regroup} \\
&= \{x : ((x \in a) \wedge (x \notin C))\} \cup \{x : ((x \in B) \wedge (x \notin C))\} && \text{definition of union} \\
&= (A - C) \cup (B - C) && \text{definition of negation}
\end{aligned}
$$

Thus completing the proof. □

Exercise (Reflection Problem).

- How long did it take you to complete each problem? What part of the assignment took the most time? Why?

  Response: This one went rather smoothly, a couple of minutes at most for each. The ones that took the most time were the last problems in chapter 8 due to formatting and writing it all out. □

- What was easy for you? Why do you think that was so?

  Response: Anything involving existence proofs felt really easy, felt like a little dopamine boost. Probably the most relaxing in a newspaper puzzle sort of way but perhaps thats the ultimate aim. To have it all feel that relaxing. □

- What was challenging for you? What made it challenging?

  Response: That proof involving greatest common divisors(problem 31) weirdly enough, I got hung up on it and my solution differs from the one in the back of the book. Perhaps I'm missing something but I feel like my logic was good albeit simple. I'd like to know why my solution wouldn't work if it doesn't so I stuck to my initial solution. □

- Compare your answers to the odd numbered exercises to those in the back of the textbook. What did you learn from this comparison?

  Response: I'm not sure this time around to be honest, I'm starting to grow skeptical about whether I should even be consulting them. The most helpful solutions this time around I felt were for the problems that weren't on the assignment as it demonstrated how one should present. For example, 10 and 14 are very similar problems to 11 and 15. $\square$