

2.4 Exact equations

2.4.1 problem a)

Consider the differential equation: $(\sin(y) - y \sin(x))dx + (\cos(x) + x \cos(y) - 3y)dy = 0$, we solve via the method of exact equations:

$$\begin{aligned} M(x, y) &= (\sin(y) - y \sin(x)) & \frac{\delta M}{\delta y} &= \cos(y) - \sin(x) \\ N(x, y) &= (\cos(x) + x \cos(y) - 3y) & \frac{\delta N}{\delta y} &= \cos(y) - \sin(x) \end{aligned}$$

Now we solve:

$$f(x, y) = \int (\sin(y) - y \sin(x))dx = x \sin(y) + y \cos(x) + g(y) \quad (1)$$

$$\frac{\delta f}{\delta y} = x \cos(y) + \cos(x) + g'(y) = \cos(x) + x \cos(y) - 3y \quad \rightarrow g'(y) = -3y \quad (2)$$

$$\int g'(y)dy = \int -3ydy \quad (3)$$

$$g(y) = -\frac{3}{2}y^2 \quad (4)$$

Thus our solution is:

$$\boxed{x \sin(y) + y \cos(x) - \frac{3}{2}y^2 = C} \quad (5)$$

2.4.1 problem b)

We are given the differential equation $(y \ln(y) + e^{xy})dx + (\frac{1}{y} + x \ln(y))dy = 0$ first we determine if it is exact:

$$\begin{aligned} M(x, y) &= y \ln(y) + e^{xy} & \frac{\delta M}{\delta y} &= \ln(y) + 1 + xe^{xy} \\ N(x, y) &= \frac{1}{y} + x \ln(y) & \frac{\delta N}{\delta y} &= \ln(y) \end{aligned}$$

Thus we can conclude:

$$\boxed{\text{The Equation is not exact}}$$

2.4 problem 2

We are given the equation $y(x + y + 1)dx + (x + 2y)dy = 0$, we first determine if it is exact or not.

2.4.2 a)

$$\begin{aligned} M(x, y) &= y(x + y + 1) = yx + y^2 + y & \frac{\delta M}{\delta y} &= x + 2y + 1 \\ N(x, y) &= x + 2y & \frac{\delta N}{\delta y} &= 1 \end{aligned}$$

$$\frac{\delta M}{\delta y} \neq \frac{\delta N}{\delta y}$$

Not Exact

For this part we add $\mu(x) = e^x$ as our integrating factor to get the equation $(yxe^x + y^2e^x + ye^x)dx + (xe^x + e^x)dy = 0$:

2.4.2 b)

$$\begin{aligned} M(x, y) &= yxe^x + y^2e^x + ye^x & \frac{\delta M}{\delta y} &= xe^x + 2ye^x + e^x \\ N(x, y) &= xe^x + e^x & \frac{\delta N}{\delta y} &= e^x + xe^x + 2ye^x \end{aligned}$$

$$\frac{\delta M}{\delta y} = \frac{\delta N}{\delta y}$$

Exact Equation

Now we find the solution:

2.4.2 c)

$$f(x, y) = \int (yxe^x + y^2e^x + ye^x) = e^xy^2 + e^xy + g(y) \quad (1)$$

$$\frac{\delta f}{\delta y} = 2e^xy + xe^x = xe^x + 2ye^x \quad g(y) = 0 \quad (2)$$

$$e^xy^2 + e^xy = C \quad (3)$$