Chapter 8

Prove the following statements.

Exercise (16). If A, B and C are sets, then $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

Proof: Observe the following sequence of equalities:

$$\begin{aligned}
A \times (B \cup C) &= \{(x, y) : (x \in A) \wedge (y \in B \cup C)\} && \text{(def. of } \times\text{)} \\
&= \{(x, y) : (x \in A) \wedge (y \in B) \vee (y \in C)\} && \text{(def. of } \cup\text{)} \\
&= \{(x, y) : (x \in A) \wedge (x \in A) \wedge (y \in B) \vee (y \in C)\} && (A = A \wedge A) \\
&= \{(x, y) : (x \in A) \wedge (y \in B) \vee (x \in A) \wedge (y \in C)\} && \text{(distrib, law for sets)} \\
&= \{(x, y) : (x \in A) \wedge (y \in B)\} \cup \{(x, y) : (x \in A) \wedge (y \in C)\} && \text{(def. of } \cup\text{)} \\
&= (A \times B) \cup (A \times C) && \text{(def. of } \times\text{)}
\end{aligned}$$

Thus completes the proof. □

Exercise (22). Let A and B be sets. Prove that $A \subseteq B$ if and only if $A \cap B = A$.

Proof: Suppose $A \subseteq B$. Then by definition, for an arbitrary $x \in A$, then $x \in B$. Since $x \in A$ and $x \in B$ then by definition of the intersection of sets, $x \in A \cap B$. Given that $x \in A \cap B$ and $x \in A$, it follows that $A \cap B \subseteq A$. Furthermore $A \subseteq A \cap B$ since all elements A are in $A \cap B$ as B is a superset of A. Thus if $A \subseteq B$ then $A \cap B = A$.

Conversely if we suppose $A \cap B = A$, then there exists $x \in A$ and $x \in B$ such that all elements of A are in B. Thus $A \subseteq B$. □

Exercise (26). Prove that $\{4k + 5 : k \in \mathbb{Z}\} = \{4k + 1 : k \in \mathbb{Z}\}$.

Proof: Suppose $x \in \{4k + 5 : k \in \mathbb{Z}\}$. Then $x = 4a + 5$ for some $a \in \mathbb{Z}$. From this we get $x = 4(a + 1) + 1$. So $x = 4k + 1$ where $k = (a + 1)$ and $k \in \mathbb{Z}$ by closure properties of the integers. Hence $x \in \{4k + 1 : k \in \mathbb{Z}\}$. Subseqentially this means $\{4k + 5 : k \in \mathbb{Z}\} \subseteq \{4k + 1 : k \in \mathbb{Z}\}$

Conversely, suppose $x \in \{4k + 1 : k \in \mathbb{Z}\}$. Then $x = 4a + 1$ for some $a \in \mathbb{Z}$. If we let $a = b+1$, where $b \in \mathbb{Z}$, then we get $x = 4(b+1)+1 = 4b+5$. So $x \in \{4k+5 : k \in \mathbb{Z}\}$.

Thus $\{4k+1 : k \in \mathbb{Z}\} \subseteq \{4k+5 : k \in \mathbb{Z}\}$.

Since we established that $\{4k+5 : k \in \mathbb{Z}\} \subseteq \{4k+1 : k \in \mathbb{Z}\}$ and $\{4k+1 : k \in \mathbb{Z}\} \subseteq \{4k+5 : k \in \mathbb{Z}\}$. By definition of equality $\{4k+5 : k \in \mathbb{Z}\} = \{4k+1 : k \in \mathbb{Z}\}$. □

Chapter 9

Each of the following statements is either true or false. If a statement is true, prove it. If a statement is false, disprove it.

Exercise (3). If $n \in \mathbb{Z}$ and $n^5 - n$ is even, then n is even.

Proof: □

Exercise (5). If A, B, C and D are sets, then $(A \times B) \cup (C \times D) = (A \cup C) \times (B \cup D)$.

Proof: □

Exercise (8). If A, B and C are sets, and $A - (B \cup C) = (A - B) \cup (A - C)$.

Proof: □

Exercise (9). If A and B are sets, then $\mathcal{P}(A) - \mathcal{P}(B) \subseteq \mathcal{P}(A \setminus B)$.

Proof: □

Exercise (12). If $a, b, c \in \mathbb{N}$ and ab, bc and ac all have the same parity, then a, b and c all have the same parity.

Proof: □

Exercise (30). There exist integers a and b for which $42a + 7b = 1$.

Proof: □

Exercise (34). If $X \subseteq A \cup B$, then $X \subseteq A$ or $X \subseteq B$.

Proof: □

Exercise (Reflection Problem). Answer the following questions:

Proof:

- How long did it take you to complete each problem?

  Write your answer here.

- What was easy?

  Write your answer here.

- What was challenging? What made it challenging?

  Write your answer here.

- Compare your answers to the odd numbered exercises to those in the back of the textbook. What did you learn from this comparison?

  Write your answer here.

□