Problem 60. *Let f, g, h satisfy $f(x) \leq g(x) \leq h(x)$ for all x in some common domain A. Assume c is a limit point of A. If $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} h(x) = L$ then $\lim_{x \to c} g(x) = L$.*

Proof. ☐

Problem 61. *If $h : \mathbb{R} \to \mathbb{R}$ is a continuous function then the set $K = \{x \in \mathbb{R} : h(x) = 0\}$ is closed.*

Proof. ☐

Problem 62. *If c is an isolated point of $A \subset \mathbb{R}$, and if $f : A \to \mathbb{R}$ is a function, then f is continuous at c.*

Proof. ☐

Problem 63. *The function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = \sqrt[3]{x}$ is continuous.*

Proof. ☐

Problem 64. *Dirichlet's function from Section 4.1, namely*

$$g(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

is not continuous at any $c \in \mathbb{R}$.

Proof. ☐

Problem 65. *The function*

$$h(x) = \begin{cases} 0 & \text{if } x = 0, \\ \sqrt{|x|} \cos(1/x) & \text{otherwise}, \end{cases}$$

shown in the figure below, is continuous at zero.

Proof. ☐

Problem 66. *Thomae's function from Section 4.1, namely*

$$t(x) = \begin{cases} 1 & \text{if } x = 0, \\ 1/n & \text{if } x \in \mathbb{Q} \setminus \{0\} \text{ and } x = \pm m/n \text{ in lowest terms, with } n > 0, \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

is not continuous at any rational point $c \in \mathbb{Q}$.

Proof. ☐

Problem 67. *Suppose $f : A \to \mathbb{R}$ is continuous at $c \in A$. Suppose that $g : B \to \mathbb{R}$ has a domain satisfying $f(A) \subset B$, and that g is continuous at $f(c)$. Let*

$$h(x) = (g \circ f)(x) = g(f(x))$$

be the composition of functions. Then h is continuous at c.

Proof. □

Problem 68. *The function $f(x) = 1/x^2$ is uniformly continuous on $(1, 2)$, but it is not uniformly continuous on $(0, 1)$.*

Proof. □

Problem 69. *We say that a function $f : A \to \mathbb{R}$ is Lipschitz if there exists $M > 0$ so that*

$$\frac{|f(x) - f(y)|}{|x - y|} \leq M$$

for all $x, y \in A$. If f is Lipschitz then f is uniformly continuous.

Proof. □