Problem 1. *There is no rational number whose square is* 2*.*

Proof. Assume, for contradiction, that there exist integers p and q satisfying

$$\frac{p}{q} = \sqrt{2},$$

where p/q is a rational number in lowest terms. By squaring, this is the same as $\frac{p^2}{q^2} = 2$, and by clearing denominators it is the same as

$$p^2 = 2q^2.$$

Thus p^2 is divisible by 2, an even number. This implies that p is also divisible by 2 and can be expressed in the form $p = 2k$ for some $k \in \mathbb{Z}$. If we sbustitute the p in $p^2 = 2q^2$ for $2k$, we get

$$(2k)^2 = 4(k^2) = 2q^2$$

Further reducing this gives us

$$2(k^2) = q^2$$

This is a contradiction as the result implies that q^2 is also even and thus q is even. Therefore p and q are both even and are irreducible.

□

Problem 2. *(a) The negation of "For all real numbers satisfying* $a < b$*, there exists* $n \in \mathbb{N}$ *such that* $a + (1/n) < b$*" is "There exists a real number satisfying* $a < b$ *such that for all* $n \in \mathbb{N}$*,* $a + (1/n) \geq b$*.*

(b) The negation of "There exists a real number $x > 0$ *such that* $x < 1/n$ *for all* $n \in \mathbb{N}$*" is "For all real numbers* $x > 0$*, there exists an* $n \in \mathbb{N}$ *such that* $x \geq 1/n$*.*

(b) The negation of "Between every two distinct real numbers there is a rational number" is "There exists an $x, y \in \mathbb{R}$*, where* $x \neq y$*, such that there is no* $n \in \mathbb{Q}$ *that satisfies* $x < n < y$*.*

Problem 3. *Suppose* a *and* b *are real numbers. Then*

(a) $|a - b| \leq |a| + |b|$

Proof. **Case 1:** Suppose $a > b$, then $|a - b| = a - b$ (since $a - b > 0$). If $a > 0$ is true, then $|a - b| = a - b = |a| - b$. Since $-b \leq |b|$(because $-b = |b|$ if b is negative and $-b \leq b = |b|$ if b is non-negative), we have

$$|a - b| = |a| - b \leq |a| + |b|$$

. On the other hand, if $a < 0$, then

$$|a - b| = a - b \leq |a| - b \leq |a| + |b|$$

(since $a \leq |a|$, as before).
Case 2: Suppose $b \geq a$, then $|a - b| = b - a$ (since $b - a > 0$). If $b > 0$ is also true, then $|a - b| = b - a \leq b + |a|$ (since $-a \leq |a|$).
If $b \geq 0$, then $|b| = b$, so

$$|a - b| = b - a \leq b + |a| = |a| + |b|$$

(since $-a \leq |a|$).
If $b < 0$, then since $b > a$, we have $a < b < 0$, so $|a| = -a$ and $|b| = -b$. Thus

$$|a - b| = b - a = b + (-a) \leq -b + (-a) = |b| + |a| = |a| + |b|$$

or more briefly

$$|a - b| \leq |a| + |b|$$

.
Case 3: Suppose $a = b$, then $|a - b| = 0 \leq |a| + |b|$ Since absolute values are non-negative. □

(b) $\big||a| - |b|\big| \leq |a - b|$

Proof. Observe that $a = (a - b) + b$. It follows that

$$|a| = |(a - b) + b| \leq |a - b| + |b|$$

by the triangle inequality. Subtracting $|b|$ from the right and left side of the inequality gives us $|a| - |b| \leq |a - b|$. Likewise observe that $b = (b - a) - a$. It follows that

$$|b| = |(b - a) + a| \leq |b - a| + |a|$$

by the triangle inequality. Since $|b - a| = |a - b|$, subtracting $|a|$ from either side of the inequality above gives us $|b| - |a| \leq |a - b|$. Since $||a|| - |b||$ is either $|a| - |b|$ or $|b| - |a|$, and both are at most $|a - b|$, it follows that $||a| - |b|| \leq |a - b|$. □

Problem 4. *Give an example of each, or state that it is impossible.*

(a) $f : \mathbb{N} \to \mathbb{N}$ *that is one-to-one but not onto.*

My Answer: *The function* $f(n) = 2n$ *is a mapping from* $\mathbb{N} \to \mathbb{N}$ *That is one-to-one but not onto.*

(b) $f : \mathbb{N} \to \mathbb{N}$ *that is onto but not one-to-one.*

My Answer: *The function* $f(n) = [(n + 1)/2]$ *is onto but not one-to one.*

(d) $f : \mathbb{N} \to \mathbb{Z}$ *that is one-to-one and onto.*

My Answer: *The piecewise function*

$$f(n) = \begin{cases} \frac{n}{2} & \textit{If n is even} \\ -\frac{n+1}{2} & \textit{If n is odd} \end{cases}$$

Is a mapping from $\mathbb{N} \to \mathbb{Z}$ *that is both one-to-one and onto.*

Problem 5. *There exists an infinite collection of sets* $A_1, A_2, A_3, \ldots$ *with the properties that every* A_i *has an infinite number of elements, and* $A_i \cap A_j = \emptyset$ *for all* $i \neq j$*, and* $\bigcup_{i=1}^{\infty} A_i = \mathbb{N}$.

Proof. A natural number is k-almost-prime if it has k prime factors where $k \in \mathbb{N}$. Let $A_1 \subseteq \mathbb{N}$ be the union of the sets $\{1\}$ and the set containing all k-almost-prime numbers $\{2, 3, 5, 7, 11, 17, ...\}$, where $k = 1$ so that $A_1 = \{1, 2, 3, 5, 6, 11, 17, ...\}$. Let $A_2 \subset \mathbb{N}$ be the set containing all k-almost-prime numbers $\{4, 6, 9, 10, 14, 15, 21, 22, ...\}$, where $k = 2$. Let $A_3 \subseteq \mathbb{N}$ be set containing all k-almost-prime numbers $\{8, 12, 18, 20, 27, 28, 30, \ldots\}$, where $k = 3$. Similarly let $A_k \subseteq \mathbb{N}$ be the set containing all k-almost-prime numbers for some $k \in \mathbb{N}$. Suppose $m \in \mathbb{N}$ and $m \in A_i \cap A_j$ for some $i, j \in \mathbb{N}$. Note that by the uniqueness of prime decompositions, m has a unique prime decomposition of a fixed length. Thus $i = j$. Since $A_n \subseteq \mathbb{N}$ for all $n \in \mathbb{N}$, it follows that $\cup_{n=1}^{\infty} A_n \subseteq \mathbb{N}$. Similarly by fundamental theorem of arithmetic, n has a unique prime decomposition of a fixed length m. Therefore $n \in A_m$ and $\mathbb{N} \subseteq \cup_{n=1}^{\infty} A_n$. Thus $\cup_{n=1}^{\infty} A_n = \mathbb{N}$. $\square$