We will begin by building theorems from the ground up from basic rules

Definition 1. *Convergence: For $A_n \to L$ means: For all $\epsilon > 0$, there exists N such that for all $n > N$, implies $\mid a_n - L \mid < \epsilon$.*

Definition 2. *Bounded: (a_n) is bounded if there exists $M > 0$ such that $|a_n| \leq M$ for all n.*

Definition 3. *Triangle inequality*

$$\begin{aligned} \textit{Triangle inequality :} \quad & \mid a+b \mid \leq \mid a \mid + \mid b \mid \\ \textit{Reverse triangle :} \quad & ||a| - |b|| \leq |a-b| \\ \textit{Product bound :} \quad & |ab| = |a||b| \end{aligned}$$

Theorem 0.1. *If (a_n) converges to L, then (a_n) is bounded.*

Proof. Since (a_n) converges to L, this means that for an $\epsilon > 0$, there exists N such that for all $n > N$, implies $\mid a_n - L \mid < \epsilon$. From this we get the following inequality:

$$|a_n| = |a_n - L + L| \leq |a_n - L| + |L| \qquad \text{By Triangle Inequality}$$

Now let $\epsilon = 1$, then there exists an $n > N$ such that $|a_n - L| < 1$, it follows from this that

$$|a_n| = |a_n - L + L| \leq |a_n - L| + |L| < 1 + |L|$$

for $n \leq N$, let $M_1 = \max\{|a_1|, |a_2|, \cdots, |a_N|\}$. Now let $M = \{M_1, 1 + |L|\}$. Then $|a_n| \leq M$ for all n. □

Theorem 0.2. *(Uniqueness of Limits) If $a_n \to L$ and $a_n \to M$ then $L = M$.*

Proof. Let $\epsilon > 0$ be arbitrary. Since $a_n \to L$ there exists an N_1 such that for all $n \geq N : |a_n - L| < \frac{\epsilon}{2}$.

Likewise since $a_n \to M$, there exists N_2 such that for all $n \geq N_2 : |a_n - M| < \frac{\epsilon}{2}$.

Let $N = \max\{N_1, N_2\}$. For $n \geq N$:

$$|L - M| = |L - a_n + a_n - M| \leq |a_n - L| + |a_n + M| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Since this holds for arbitrary $\epsilon > 0$, we must have $|L - M| = 0$, so L = M.

□

Definition 4.