5.1.3

A mass weighing 4 pounds is attached to a spring whose constant is 2 lb/ft. The medium offers a damping force that is numerically equal to the instantaneous velocity. The mass is initially released from a point 1 foot above the equilibrium position with a downward velocity of 8 ft/s. Determine the time at which the mass passes through the equilibrium position. Find the time at which the mass attains its extreme displacement from the equilibrium position. What is the position of the mass at this instant?

We have the initial conditions $x(0) = -1$ and $\dot{x}(0) = 8$, we first establish the equation to solve:

$$m\ddot{x} = -kx - \beta\dot{x} \qquad\qquad m = \frac{4}{32} = \frac{1}{8}; k = 2; \beta = 1 \tag{1}$$

$$\frac{1}{8}\ddot{x} = -2x - \dot{x} \tag{2}$$

$$\ddot{x} = -16x - 8\dot{x} \tag{3}$$

$$\ddot{x} + 8\dot{x} + 16x = 0 \tag{4}$$

We can begin solving the auxillary equation:

$$m^2 + 8m + 16 = 0 \tag{5}$$

$$(m + 4)(m + 4) = 0 \tag{6}$$

$$m = -4 \tag{7}$$

meaning our general solution is $c_1 e^{-4t} + c_2 t e^{-4t}$, we proceed to plug in our initial conditions and solve for c_1 and c_2

$$x = c_1 e^{-4t} + c_2 t e^{-4t} \qquad\qquad \dot{x} = -4c_1 e^{-4t} + c_2(e^{-4t} - 4te^{-4t}) \tag{8}$$

$$x(0) = c_1 e^{-4(0)} + c_2 t e^{-4(0)} \qquad\qquad \dot{x}(0) = -4c_1 e^{-4(0)} + c_2(e^{-4(0)} - 4te^{-4(0)}) \tag{9}$$

$$-1 = c_1 \qquad\qquad 8 = -4c_1 + c_2 \tag{10}$$

$$c_1 = -1 \qquad\qquad c_2 = 4 \tag{11}$$

Our solution is $x = -e^{-4t} + 4te^{-4t}$, Now we solve for t when $x = 0$ and $\dot{x} = 0$ to find time where the mass passes through the equilibrium position and its extreme displacement from equilibrium position, note the derivative of our solution is $\dot{x} = 8e^{-4t} - 16te^{-4t}$

$$0 = -e^{-4t} + 4te^{-4t} \qquad\qquad 0 = 8e^{-4t} - 16te^{-4t} \tag{12}$$

$$0 = e^{-4t}(4t - 1) \qquad\qquad 0 = 8e^{-4t}(1 - 2t) \tag{13}$$

$$t = \frac{1}{4}s \qquad\qquad t = \frac{1}{2}s \tag{14}$$

To find the mass at the instance of extreme displacement, we plug in our time for it into our solution:

$$x(\frac{1}{2}) = -e^{-2} + 2e^{-2} \tag{15}$$

$$= e^{-2} \tag{16}$$

Our solutions are

$$\boxed{t = \frac{1}{4}s}$$ \quad Time mass passes through equilibrium position

$$\boxed{t = \frac{1}{2}s}$$ \quad Time mass attainsextreme displacement from equilibrium

$$\boxed{e^{-2}}$$ \quad position of mass at the instant of extreme displacement

5.1.4

A mass weighing 10 pounds stretches a spring 2 feet. The mass is attached to a dashpot device that offers a damping force numerically equal to $\beta(\beta > 0)$ times the instantaneous velocity. Determine the values of the damping constant β so that the subsequent motion is overdamped, critically damped and underdamped.

We have our equations for overdamped, critically damped and underdamped below and constants:

$$\beta^2 - 4mk > 0 \qquad\qquad m = \frac{10}{32} = \frac{5}{16}$$

$$\beta^2 - 4mk = 0 \qquad\qquad k = \frac{10}{2} = 5$$

$$\beta^2 - 4mk < 0$$

Now we just plug in and solve for each:

$$\beta^2 - 4 * \frac{5}{16} * 5 > 0 \qquad \beta^2 - 4 * \frac{5}{16} * 5 = 0 \qquad \beta^2 - 4 * \frac{5}{16} * 5 < 0$$

$$\beta^2 > \frac{100}{16} \qquad\qquad \beta^2 = \frac{100}{16} \qquad \beta^2 < \frac{100}{16}$$

$$\beta > \frac{5}{2} \qquad\qquad\qquad \beta = \frac{5}{2} \qquad\qquad \beta < \frac{5}{2}$$

Our values are

$$\boxed{\beta > \frac{5}{2}}$$ \quad For overdamped

$$\boxed{\beta = \frac{5}{2}}$$ \quad For critically damped

$$\boxed{0 < \beta < \frac{5}{2}}$$ \quad For underdamped