Problem 6.

$$\bigcap_{n=1}^{\infty}(0,1/n)=\emptyset.$$

Proof. Let $S=\bigcap_{n=1}^{\infty}(0,1/n)=\emptyset$. Let $x\in\mathbb{R}$. Consider the following 3 cases.
Case 1: Suppose $x\leq 0$, then $x\notin S$ as $x\notin(0,1)$.
Case 2: Suppose $x\geq 1$, then $x\notin S$ as $x\notin(0,1)$.
Case 3: Suppose $0<x<1$, Choose $n\in\mathbb{N}$ so that $n>\frac{1}{x}$. Then $x>\frac{1}{n}$. so $x\notin(0,\frac{1}{n})$, thus $x\notin S$
These cases show that an arbitrary $x\in R$ is not in S. □

Problem 7. *Given a function f and a subset A of its domain, consider the image $f(A)=\{f(x):x\in A\}$.*

(a) *An example of a function f, and two subsets A,B of the domain of f, for which $f(A\cap B)\neq f(A)\cap f(B)$ is*

$$f(x)=|x|$$

where set A is a subset of the domain defined by $A=\{-2,-1\}$ and where set B is a subset of the domain defined by $B=\{1,2\}$. Observe that $f(A\cap B)=\emptyset$ and $f(A)\cap f(B)=\{1,2\}$.

(b) *If A, B are subsets of the domain of f then $f(A\cup B)$ IS RELATED IN SOME WAY TO $f(A)\cup f(B)$.*

Proposition: If f is a function and A,B are subsets of the domain of f, then $f(A\cup B)=f(A)\cup f(B)$.

Proof. Let $y\in f(A\cup B)$. Then there exists $x\in A\cup B$ such that $f(x)=y$. Since $x\in A\cup B$, then either $x\in A$ or $x\in B$. If $x\in A$, then $y=f(x)\in f(A)\subseteq f(A)\cup f(B)$. If $x\in B$, then $y=f(x)\in f(B)\subseteq f(A)\cup f(B)$. Thus $f(A\cup B)\subseteq f(A)\cup f(B)$.

Conversely, let $y\in f(A)\cup f(B)$. Then either $y\in f(A)$ or $y\in f(B)$. If $y\in f(A)$, then there exists $x\in A$ such that $f(x)=y$. Since $A\subseteq A\cup B$, we have $x\in A\cup B$, so $y\in f(A\cup B)$. Similarly, if $y\in f(B)$, then $y\in f(A\cup B)$. Therefore $f(A)\cup f(B)\subseteq f(A\cup B)$.

Since $f(A\cup B)\subseteq f(A)\cup f(B)$ and $f(A\cup B)\supseteq f(A)\cup f(B)$ have been shown to be true, it follows that $f(A\cup B)=f(A)\cup f(B)$. □

Problem 8. *If $a\in\mathbb{R}$ is an upper bound for $A\subset\mathbb{R}$, and if a is also an element of A, then $a=\sup A$.*

Proof. Since a is an upper bound for A, we know that $x \leq a$ for all $x \in A$.

We want to show that $a = \sup A$, we need to prove that a is the least upper bound. Choose b be any upper bound for A. Since $a \in A$ and we know that a is an upper bound and b is an upper bound for A, it must be the case that $a \leq b$ and $b \leq a$ since by definition an upper bound for a subset is an element greater than or equal to every element within that subset.

Therefore, $b = a = \sup A$. $\square$

Problem 9. *(a) Let* $A = \{m/n : \ m, n \in \mathbb{N}$ *with* $m < n\}$. *Then* $\inf A = 0$ *and* $\sup A = 1$.

(b) Let $B = \{(-1)^m/n : n, m \in \mathbb{N}\}$. *Then* $\inf B = -1$ *and* $\sup B = 1$.

(c) Let $C = \{n/(3n+1) : n \in \mathbb{N}\}$. *Then* $\inf C = 1/4$ *and* $\sup C = 1/3$.

(d) Let $D = \{m/(m+n) : m, n \in \mathbb{N}\}$. *Then* $\inf D = 0$ *and* $\sup D = 1$.

Problem 10. *(a) If A and B are nonempty, bounded, and satisfy $A \subseteq B$ then* $\sup A \leq \sup B$.

Proof. Given $A \subseteq B$, every element of A is also an element of B. Since B is bounded above, $\sup B$ exists and is an upper bound for B. Therefore, $\sup B$ is also an upper bound for A.
Since $\sup A$ is the least upper bound of A, then $\sup B$ is an upper bound for A, it must be the case that $\sup A \leq \sup B$. $\square$

(b) If $\sup A < \inf B$ *for nonempty sets A and B, then there exists $c \in \mathbb{R}$ such that $a < c < b$ for all $a \in A$ and $b \in B$.*

Proof. Given that $\sup A < \inf B$, we can choose any c such that $\sup A < c < \inf B$. Choose c to be the average of $\sup A$ and $\inf B$ such that $c = \frac{\sup A + \inf B}{2}$.

Since $c > \sup A$ and $\sup A$ is an upper bound for A, we have $a \leq \sup A < c$ for all $a \in A$.
Since $c < \inf B$ and $\inf B$ is a lower bound for B, we have $c < \inf B \leq b$ for all $b \in B$.

Therefore $a < c < b$ for all $a \in A$ and $b \in B$. $\square$

(c) If there exists $c \in \mathbb{R}$ satisfying $a < c < b$ for all $a \in A$ and $b \in B$ then $\sup A < \inf B$.

This is false because we could have $\sup A = c$ *or* $\inf B = c$ *which would give us* $\sup A = \inf B$.

Problem 11. *Denote the irrational numbers by* $\mathbb{I} = \mathbb{R} \setminus \mathbb{Q}$.

(a) If $a, b \in \mathbb{Q}$ then $ab \in \mathbb{Q}$ and $a + b \in \mathbb{Q}$.

Proof. Since $a, b \in \mathbb{Q}$, we can write $a = \frac{p}{q}$ and $b = \frac{r}{s}$ where $p, r \in \mathbb{Z}$ and $q, s \in \mathbb{N}$.

For multiplication: $ab = \frac{p}{q} \cdot \frac{r}{s} = \frac{pr}{qs}$. Since $pr \in \mathbb{Z}$ (integers are closed under multiplication) and $qs \in \mathbb{N}$ (positive integers are closed under multiplication), we have $ab \in \mathbb{Q}$.

For addition: $a + b = \frac{p}{q} + \frac{r}{s} = \frac{ps+qr}{qs}$. Since $ps + qr \in \mathbb{Z}$ (integers are closed under multiplication and addition) and $qs \in \mathbb{N}$, we have $a + b \in \mathbb{Q}$. $\square$

(b) *If $a \in \mathbb{Q}$ and $t \in \mathbb{I}$ then $a + t \in \mathbb{I}$. If also $a \neq 0$ then $at \in \mathbb{I}$.*

Proof. We prove both claims by contradiction.

For addition: Suppose $a + t \in \mathbb{Q}$. Since $a \in \mathbb{Q}$ and $a + t \in \mathbb{Q}$, by part (a) we have $(a + t) + (-a) \in \mathbb{Q}$. But $(a + t) + (-a) = t$, so $t \in \mathbb{Q}$, contradicting that $t \in \mathbb{I}$. Therefore, $a + t \in \mathbb{I}$.

For multiplication (with $a \neq 0$): Suppose $at \in \mathbb{Q}$. Since $a \in \mathbb{Q}$ with $a \neq 0$, we have $\frac{1}{a} \in \mathbb{Q}$. By part (a), $(at) \cdot \frac{1}{a} \in \mathbb{Q}$. But $(at) \cdot \frac{1}{a} = t$, so $t \in \mathbb{Q}$, contradicting that $t \in \mathbb{I}$. Therefore, $at \in \mathbb{I}$. $\square$

(c) *Suppose $s, t \in \mathbb{I}$. Then PROPOSITION ABOUT WHETHER st AND $s + t$ ARE EITHER RATIONAL OR IRRATIONAL IN GENERAL.*

Proof. For $s+t$: If $s = \sqrt{2}$ and $t = -\sqrt{2}$, then $s+t = 0 \in \mathbb{Q}$. However, if $s = \sqrt{2}$ and $t = \sqrt{3}$, then $s + t = \sqrt{2} + \sqrt{3} \in \mathbb{I}$. To see this, suppose $\sqrt{2} + \sqrt{3} = r \in \mathbb{Q}$. Then $\sqrt{3} = r - \sqrt{2}$, and squaring both sides gives $3 = r^2 - 2r\sqrt{2} + 2$, which implies $\sqrt{2} = \frac{r^2-1}{2r} \in \mathbb{Q}$, a contradiction.

For st: If $s = \sqrt{2}$ and $t = \sqrt{2}$, then $st = 2 \in \mathbb{Q}$. However, if $s = \sqrt{2}$ and $t = \sqrt{3}$, then $st = \sqrt{6} \in \mathbb{I}$, since if $\sqrt{6} \in \mathbb{Q}$, then 6 would be a perfect square, which it is not.

Therefore, the irrational numbers are not closed under addition or multiplication. $\square$

Therefore, the irrational numbers are not closed under addition or multiplication.

Problem 12. *For all $n \in \mathbb{N}$, $2^n \geq n$.*

Proof. We prove by induction on n.

Base case: For $n = 1$, we have $2^1 = 2 \geq 1$, which is true.

Inductive step: Assume $2^k \geq k$ for some $k \in \mathbb{N}$. We need to show $2^{k+1} \geq k + 1$.

Starting from the inductive hypothesis:

$$\begin{aligned}
2^k &\geq k \\
2 \cdot 2^k &\geq 2k \quad \text{(multiplying both sides by 2)} \\
2^{k+1} &\geq 2k \\
2^{k+1} &\geq k + k \\
2^{k+1} &\geq k + 1 \quad \text{(since } k \geq 1 \text{ for all } k \in \mathbb{N}\text{)}
\end{aligned}$$

Therefore, by mathematical induction, $2^n \geq n$ for all $n \in \mathbb{N}$. $\square$

Problem 13. *Let $y_1 = 6$ and, for each $n \in \mathbb{N}$, let $y_{n+1} = (2y_n - 6)/3$.*

(a) For all $n \in \mathbb{N}$, $y_n \geq -6$.

Proof. We prove by induction on n.

Base case: For $n = 1$, we have $y_1 = 6 \geq -6$, which is true.

Inductive step: Assume $y_k \geq -6$ for some $k \in \mathbb{N}$. We need to show $y_{k+1} \geq -6$.

From the inductive hypothesis: $y_k \geq -6$

$$\begin{aligned} y_k &\geq -6 \\ 2y_k &\geq -12 \\ 2y_k - 6 &\geq -18 \\ \frac{2y_k - 6}{3} &\geq -6 \\ y_{k+1} &\geq -6 \end{aligned}$$

Therefore, by mathematical induction, $y_n \geq -6$ for all $n \in \mathbb{N}$. $\square$

(b) The sequence $(y_1, y_2, y_3, \ldots)$ is decreasing.

Proof. We prove by induction that $y_{n+1} < y_n$ for all $n \in \mathbb{N}$.

Base case: For $n = 1$, we have:

$$\begin{aligned} y_2 &= \frac{2y_1 - 6}{3} = \frac{2(6) - 6}{3} = \frac{6}{3} = 2 \\ y_1 &= 6 \end{aligned}$$

Since $2 < 6$, we have $y_2 < y_1$.

Inductive step: Assume $y_{k+1} < y_k$ for some $k \in \mathbb{N}$. We need to show $y_{k+2} < y_{k+1}$.

We have:

$$\begin{aligned} y_{k+2} - y_{k+1} &= \frac{2y_{k+1} - 6}{3} - y_{k+1} \\ &= \frac{2y_{k+1} - 6 - 3y_{k+1}}{3} \\ &= \frac{-y_{k+1} - 6}{3} \\ &= -\frac{y_{k+1} + 6}{3} \end{aligned}$$

From part (a), we know $y_{k+1} \geq -6$, so $y_{k+1} + 6 \geq 0$. Since $y_1 = 6 > -6$ and the recurrence relation preserves this (as shown in part (a)), we actually have $y_{k+1} > -6$, which means $y_{k+1} + 6 > 0$.

Therefore, $y_{k+2} - y_{k+1} = -\frac{y_{k+1}+6}{3} < 0$, so $y_{k+2} < y_{k+1}$.

By mathematical induction, $y_{n+1} < y_n$ for all $n \in \mathbb{N}$.

$\square$