SECTION 5.3: MARGINAL AND CONDITIONAL PROBABILITY DISTRIBUTIONS

Exercise (5.23). In Example 5.4 and Exercise 5.5, we considered the joint density of Y_1, the proportion of the capacity of the tank that is stocked at the beginning of the week, and Y_2, the proportion of the capacity sold during the week, given by

$$f(y_1, y_2) = \begin{cases} 3y_1, & 0 \leq y_2 \leq y_1 \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Find the marginal density function for Y_2.

Solution:

$$f_2(y_2) = \int_{y_2}^{1} 3y_1 dy_1 = \frac{3y_1^2}{2}\Bigg]_{y_2}^{1} = \frac{3}{2} - \frac{3y_2^2}{2}$$

$$= \frac{3}{2}(1 - y_2^2)$$

$\square$

(b) For what values of y_2 is the conditional density $f(y_1|y_2)$ defined?

Solution:

$$f(y_1|y_2) = \frac{f(y_1, y_2)}{f_2(y_2)} = \frac{3y_1}{\frac{3}{2}(1 - y_2^2)} = \frac{2y_1}{1 - y_2^2}$$

$$= \begin{cases} \frac{2y_1}{1-y_2^2}, & y_2 \leq y_1 \leq 1 \\ 0, & \text{elsewhere} \end{cases}$$

So its defined for $0 \leq y_2 < 1$ to avoid division by 0. $\square$

(c) What is the probability that more than half a tank is sold given that three-fourths of a tank is stocked?

Solution: We have $P(Y_2 > 1/2 | Y_1 = 3/4)$

$$f_1(y_1) = \int_0^{y_1} 3y_1 dy_2 = 3y_1 y_2 \Big]_0^{y_1} = 3y_1^2$$

$$f(y_2|y_1) = \frac{3y_1}{3y_1^2} = \frac{1}{y_1}$$

$$P(Y_2 > 1/2 | Y_1 = 3/4) = \int_{1/2}^{3/4} \frac{1}{3/4} dy_2 = \int_{1/2}^{3/4} \frac{4}{3} dy_2 = \frac{4}{3} y_2 \Big]_{1/2}^{3/4}$$

$$= (\frac{4}{3} * \frac{3}{4}) - (\frac{4}{3} * \frac{1}{2}) = 1 - \frac{4}{6} = \frac{1}{3}$$

$\square$

Exercise (5.24). In Exercise 5.6, we assumed that if a radioactive particle is randomly located in a square with sides of unit length, a reasonable model for the joint density function for Y_1 and Y_2 is

$$f(y_1, y_2) = \begin{cases} 1, & 0 \le y_1 \le 1, 0 \le y_2 \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Find the marginal density functions for Y_1 and Y_2.

    Solution:

$$f_1(y_1) = 1 \text{ for } 0 \le y_1 \le 1$$

$$f_2(y_2) = 1 \text{ for } 0 \le y_2 \le 1$$

$\square$

(b) What is $P(.3 < Y_1 < .5)$? $P(.3 < Y_2 < .5)$?

    Solution: Since its a rectangular region for each

$$P(.3 < Y_1 < .5) = P(.3 < Y_2 < .5) = .2 * 1 = .2$$

$\square$

(c) For what values of y_2 is the conditional density $f(y_1|y_2)$ defined?

Solution:

$$f(y_1|y_2) = \frac{f(y_1, y_2)}{f_2(y_2)} = \frac{1}{1} = 1$$

Defined for $0 \leq y_2 \leq 1$.   □

(d) For any y_2, $0 \leq y_2 \leq 1$ what is the conditional density function of Y_1 given that $Y_2 = y_2$?

Solution: The conditional probablity we calculated in the last one so

$$f(y_1|y_2) = \begin{cases} 1, & 0 \leq y_1 \leq 1 \\ 0, & \text{elsewhere} \end{cases}$$

For any y_2 where $0 \leq y_2 \leq 1$.   □

(e) Find $P(.3 < Y_1 < .5|Y_2 = .3)$.

Solution: $P(.3 < Y_1 < .5|Y_2 = .3) = .2/1 = .2$ □

(f) Find $P(.3 < Y_1 < .5|Y_2 = .5)$.

Solution: $P(.3 < Y_1 < .5|Y_2 = .5) = .2/1 = .2$ □

(g) Compare the answers that you obtained in parts (a), (d), and (e). For any y_2, $0 \leq y_2 \leq 1$ how does $P(.3 < Y_1 < .5)$ compare to $P(.3 < Y_1 < .5|Y_2 = y_2)$?

Solution: There are all the same anwsers.   □

Exercise (5.25). Let Y_1 and Y_2 have joint density function first encountered in Exercise 5.7:

$$f(y_1, y_2) = \begin{cases} e^{-(y_1+y_2)}, & y_1 > 0, y_2 > 0, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Find the marginal density functions for Y_1 and Y_2. Identify these densities as one of those studied in Chapter 4.

Solution: Exponential Distribution with

□

(b) What is $P(1 < Y_1 < 2.5)$? $P(1 < Y_2 < 2.5)$?

Solution: □

(c) For what values of y_2 is the conditional density $f(y_1|y_2)$ defined?

Solution: □

(d) For any $y_2 > 0$, what is the conditional density function of Y_1 given that $Y_2 = y_2$?

Solution: □

(e) For any $y_1 > 0$, what is the conditional density function of Y_2 given that $Y_1 = y_1$?

Solution: □

(f) For any $y_2 > 0$, how does the conditional density function $f(y_1|y_2)$ that you obtained in part (d) compare to the marginal density function $f_1(y_1)$ found in part (a)?

Solution: □

(g) What does your answer to part (f) imply about marginal and conditional probabilities that Y_1 falls in any interval?

Solution: □

Exercise (5.27). In Exercise 5.9, we determined that

$$f(y_1, y_2) = \begin{cases} 6(1 - y_2), & 0 \le y_1 \le y_2 \le 1, \\ 0, & \text{elsewhere} \end{cases}$$

is a valid joint probability density function. Find

(a) the marginal density functions for Y_1 and Y_2.

Solution: □

(b) $P(Y_2 \le 1/2|Y_1 \le 3/4)$.

Solution: □

(c) the conditional density function of Y_1 given $Y_2 = y_2$.

Solution: □

(d) the conditional density function of Y_2 given $Y_1 = y_1$.

Solution: □

(e) $P(Y_2 \ge 3/4|Y_1 = 1/2)$.

Solution: □

Exercise (5.29). Refer to Exercise 5.11. Find

    (a) the marginal density functions for Y_1 and Y_2.

        Solution: □

    (b) $P(Y_2 > 1/2 | Y_1 = 1/4)$.

        Solution: □

Exercise (5.32). Suppose that the random variables Y_1 and Y_2 have joint probability density function, $f(y_1, y_2)$, given by (see Exercise 5.14)

$$f(y_1, y_2) = \begin{cases} 6y_1^2 y_2, & 0 \le y_1 \le y_2, y_1 + y_2 \le 2, \\ 0, & \text{elsewhere.} \end{cases}$$

    (a) Show that the marginal density of Y_1 is a beta density with $\alpha = 3$ and $\beta = 2$.

        Solution: □

    (b) Derive the marginal density of Y_2.

        Solution: □

    (c) Derive the conditional density of Y_2 given $Y_1 = y_1$.

        Solution: □

    (d) Find $P(Y_2 < 1.1 | Y_1 = .60)$.

        Solution: □

Exercise (5.33). Suppose that Y_1 is the total time between a customer's arrival in the store and departure from the service window, Y_2 is the time spent in line before reaching the window, and the joint density of these variables (as was given in Exercise 5.15) is

$$f(y_1, y_2) = \begin{cases} e^{-y_1}, & 0 \le y_2 \le y_1 \le \infty, \\ 0, & \text{elsewhere.} \end{cases}$$

    (a) Find the marginal density functions for Y_1 and Y_2.

        Solution: □

    (b) What is the conditional density function of Y_1 given that $Y_2 = y_2$? Be sure to specify the values of y_2 for which this conditional density is defined.

        Solution: □

    (c) What is the conditional density function of Y_2 given that $Y_1 = y_1$? Be sure to specify the values of y_1 for which this conditional density is defined.

        Solution: □

(d) Is the conditional density function $f(y_1|y_2)$ that you obtained in part (b) the same as the marginal density function $f_1(y_1)$ found in part (a)?

Solution: □

(e) What does your answer to part (d) imply about marginal and conditional probabilities that Y_1 falls in any interval?

Solution: □

Exercise (5.35). Refer to Exercise 5.33. If two minutes elapse between a customer's arrival at the store and his departure from the service window, find the probability that he waited in line less than one minute to reach the window.

Solution: □

Exercise (5.37). In Exercise 5.18, Y_1 and Y_2 denoted the lengths of life, in hundreds of hours, for components of types I and II, respectively, in an electronic system. The joint density of Y_1 and Y_2 is given by

$$f(y_1, y_2) = \begin{cases} (1/8)y_1 e^{-(y_1+y_2)/2}, & y_1 > 0, y_2 > 0 \\ 0, & \text{elsewhere.} \end{cases}$$

Find the probability that a component of type II will have a life length in excess of 200 hours.

Solution: □

SECTION 5.4: INDEPENDENT RANDOM VARIABLES

Exercise (5.43). Let Y_1 and Y_2 have joint density function $f(y_1, y_2)$ and marginal densities $f_1(y_1)$ and $f_2(y_2)$, respectively. Show that Y_1 and Y_2 are independent if and only if $f(y_1|y_2) = f_1(y_1)$ for all values of y_1 and for all y_2 such that $f_2(y_2) > 0$. A completely analogous argument establishes that Y_1 and Y_2 are independent if and only if $f(y_2|y_1) = f_2(y_2)$ for all values of y_2 and for all y_1 such that $f_1(y_1) > 0$.

Solution: □

Exercise (5.45). In Exercise 5.1, we determined that the joint distribution of Y_1, the number of contracts awarded to firm A, and Y_2, the number of contracts awarded to firm B, is given by the entries in the following table.

		y_2	
y_1	0	1	2
0	1/9	2/9	1/9
1	2/9	2/9	0
2	1/9	0	0

The marginal probability function of Y_1 was derived in Exercise 5.19 to be binomial with $n = 2$ and $p = 1/3$. Are Y_1 and Y_2 independent? Why?

Solution: ☐

Exercise (5.47). In Exercise 5.3, we determined that the joint probability distribution of Y_1, the number of married executives, and Y_2, the number of never-married executives, is given by

$$p(y_1, y_2) = \frac{\binom{4}{y_1}\binom{3}{y_2}\binom{2}{3-y_1-y_2}}{\binom{9}{3}},$$

where y_1 and y_2 are integers, $0 \leq y_1 \leq 3$, $0 \leq y_2 \leq 3$, and $1 \leq y_1 + y_2 \leq 3$. Are Y_1 and Y_2 independent? (Recall your answer to Exercise 5.21.)

Solution: ☐

Exercise (5.49). In Example 5.4 and Exercise 5.5, we considered the joint density of Y_1, the proportion of the capacity of the tank that is stocked at the beginning of the week and Y_2, the proportion of the capacity sold during the week, given by

$$f(y_1, y_2) = \begin{cases} 3y_1, & 0 \leq y_2 \leq y_1 \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Show that Y_1 and Y_2 are dependent.

Solution: ☐

Exercise (5.51). In Exercise 5.7, we considered Y_1 and Y_2 with joint density function

$$f(y_1, y_2) = \begin{cases} e^{-(y_1+y_2)}, & y_1 > 0, y_2 > 0, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Are Y_1 and Y_2 independent?

    Solution: ☐

(b) Does the result from part (a) explain the results you obtained in Exercise 5.25 (d)–(f)? Why?

    Solution: ☐

Exercise (5.53). In Exercise 5.9, we determined that

$$f(y_1, y_2) = \begin{cases} 6(1 - y_2), & 0 \leq y_1 \leq y_2 \leq 1, \\ 0, & \text{elsewhere} \end{cases}$$

is a valid joint probability density function. Are Y_1 and Y_2 independent?

Solution: ☐

Exercise (5.55). Suppose that, as in Exercise 5.11, Y_1 and Y_2 are uniformly distributed over the triangle shaded in the accompanying diagram. Are Y_1 and Y_2 independent?

Solution: □

Exercise (5.57). In Exercises 5.13 and 5.31, the joint density function of Y_1 and Y_2 was given by

$$f(y_1, y_2) = \begin{cases} 30y_1 y_2^2, & y_1 - 1 \le y_2 \le 1 - y_1, 0 \le y_1 \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Are the random variables Y_1 and Y_2 independent?

Solution: □

Exercise (5.59). If Y_1 is the total time between a customer's arrival in the store and leaving the service window and if Y_2 is the time spent in line before reaching the window, the joint density of these variables, according to Exercise 5.15, is

$$f(y_1, y_2) = \begin{cases} e^{-y_1}, & 0 \le y_2 \le y_1 \le \infty \\ 0, & \text{elsewhere.} \end{cases}$$

Are Y_1 and Y_2 independent?

Solution: □

Exercise (5.61). In Exercise 5.18, Y_1 and Y_2 denoted the lengths of life, in hundreds of hours, for components of types I and II, respectively, in an electronic system. The joint density of Y_1 and Y_2 is

$$f(y_1, y_2) = \begin{cases} (1/8)y_1 e^{-(y_1+y_2)/2}, & y_1 > 0, y_2 > 0, \\ 0, & \text{elsewhere.} \end{cases}$$

Are Y_1 and Y_2 independent?

Solution: □

Exercise (5.63). Let Y_1 and Y_2 be independent exponentially distributed random variables, each with mean 1. Find $P(Y_1 > Y_2 | Y_1 < 2Y_2)$.

Solution: □

SECTION 5.6: EXPECTED VALUE OF A FUNCTION OF RANDOM VARIABLES

Exercise (5.73). In Exercise 5.3, we determined that the joint probability distribution of Y_1, the number of married executives, and Y_2, the number of never-married executives, is given by

$$p(y_1, y_2) = \frac{\binom{4}{y_1}\binom{3}{y_2}\binom{2}{3-y_1-y_2}}{\binom{9}{3}},$$

where y_1 and y_2 are integers, $0 \le y_1 \le 3$, $0 \le y_2 \le 3$, and $1 \le y_1 + y_2 \le 3$. Find the expected number of married executives among the three selected for promotion. (See Exercise 5.21.)

Solution: □

Exercise (5.75). Refer to Exercises 5.7, 5.25, and 5.51. Let Y_1 and Y_2 have joint density function

$$f(y_1, y_2) = \begin{cases} e^{-(y_1+y_2)}, & y_1 > 0, y_2 > 0 \\ 0, & \text{elsewhere.} \end{cases}$$

(a) What are $E(Y_1 + Y_2)$ and $V(Y_1 + Y_2)$?

Solution: □

(b) What is $P(Y_1 - Y_2 > 3)$?

Solution: □

(c) What is $P(Y_1 - Y_2 < -3)$?

Solution: □

(d) What are $E(Y_1 - Y_2)$ and $V(Y_1 - Y_2)$?

Solution: □

(e) What do you notice about $V(Y_1 + Y_2)$ and $V(Y_1 - Y_2)$?

Solution: □

Exercise (5.77). In Exercise 5.9, we determined that

$$f(y_1, y_2) = \begin{cases} 6(1 - y_2), & 0 \le y_1 \le y_2 \le 1, \\ 0, & \text{elsewhere} \end{cases}$$

is a valid joint probability density function. Find

(a) $E(Y_1)$ and $E(Y_2)$.

Solution: □

(b) $V(Y_1)$ and $V(Y_2)$.

Solution: □

(c) $E(Y_1 - 3Y_2)$.

Solution: □

Exercise (5.79). Suppose that, as in Exercise 5.11, Y_1 and Y_2 are uniformly distributed over the triangle shaded in the accompanying diagram with vertices at $(-1, 0)$, $(1, 0)$, and $(0, 1)$. Find $E(Y_1 Y_2)$.

Solution: □

Exercise (5.81). In Exercise 5.18, Y_1 and Y_2 denoted the lengths of life, in hundreds of hours, for components of types I and II, respectively, in an electronic system. The joint density of Y_1 and Y_2 is

$$f(y_1, y_2) = \begin{cases} (1/8)y_1 e^{-(y_1+y_2)/2}, & y_1 > 0, y_2 > 0, \\ 0, & \text{elsewhere.} \end{cases}$$

One way to measure the relative efficiency of the two components is to compute the ratio Y_2/Y_1. Find $E(Y_2/Y_1)$. [Hint: In Exercise 5.61, we proved that Y_1 and Y_2 are independent.]

Solution: □

Exercise (5.87). Suppose that Y_1 and Y_2 are independent χ^2 random variables with ν_1 and ν_2 degrees of freedom, respectively. Find

(a) $E(Y_1 + Y_2)$.

     Solution: □

(b) $V(Y_1 + Y_2)$. [Hint: Use Theorem 5.9 and the result of Exercise 4.112(a).]

     Solution: □

SECTION 5.7: THE COVARIANCE OF TWO RANDOM VARIABLES

Exercise (5.89). In Exercise 5.1, we determined that the joint distribution of Y_1, the number of contracts awarded to firm A, and Y_2, the number of contracts awarded to firm B, is given by the entries in the following table.

y_1	y_2 = 0	1	2
0	1/9	2/9	1/9
1	2/9	2/9	0
2	1/9	0	0

Find $\text{Cov}(Y_1, Y_2)$. Does it surprise you that $\text{Cov}(Y_1, Y_2)$ is negative? Why?

Solution: □

Exercise (5.91). In Exercise 5.8, we derived the fact that

$$f(y_1, y_2) = \begin{cases} 4y_1 y_2, & 0 \le y_1 \le 1, 0 \le y_2 \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Show that $\text{Cov}(Y_1, Y_2) = 0$. Does it surprise you that $\text{Cov}(Y_1, Y_2)$ is zero? Why?

Solution: □

Exercise (5.92). In Exercise 5.9, we determined that

$$f(y_1, y_2) = \begin{cases} 6(1 - y_2), & 0 \le y_1 \le y_2 \le 1, \\ 0, & \text{elsewhere} \end{cases}$$

is a valid joint probability density function. Find $\text{Cov}(Y_1, Y_2)$. Are Y_1 and Y_2 independent?

Solution: ☐

Exercise (5.93). Let the discrete random variables Y_1 and Y_2 have the joint probability function
$$p(y_1, y_2) = 1/3, \text{ for } (y_1, y_2) = (-1, 0), (0, 1), (1, 0).$$
Find $\text{Cov}(Y_1, Y_2)$. Notice that Y_1 and Y_2 are dependent. (Why?) This is another example of uncorrelated random variables that are not independent.

Solution: ☐

Exercise (5.95). Suppose that, as in Exercises 5.11 and 5.79, Y_1 and Y_2 are uniformly distributed over the triangle shaded in the accompanying diagram with vertices at $(-1, 0)$, $(1, 0)$, and $(0, 1)$.

   (a) Find $\text{Cov}(Y_1, Y_2)$.

   Solution: ☐

   (b) Are Y_1 and Y_2 independent? (See Exercise 5.55.)

   Solution: ☐

   (c) Find the coefficient of correlation for Y_1 and Y_2.

   Solution: ☐

   (d) Does your answer to part (b) lead you to doubt your answer to part (a)? Why or why not?

   Solution: ☐

Exercise (5.99). If c is any constant and Y is a random variable such that $E(Y)$ exists, show that $\text{Cov}(c, Y) = 0$.

Solution: ☐