Problem 22. *If $f : A \to B$ has an inverse function then f is onto and f is one-to-one.*

Proof. Suppose $f : A \to B$ has an inverse function $g : B \to A$. By definition of an inverse function this means that

$$g(f(a)) = a, \text{ for all } a \in A$$
$$f(g(b)) = b, \text{ for all } b \in B$$

one-to-one : Let $a_1, a_2 \in A$ such that $f(a_1) = f(a_2)$. To prove f is one-to-one(injective) we must show that $a_1 = a_2$. Observe that composing the inverse g with f gives

$$g(f(a_1) = g(f(a_2))$$

As a result of applying the definition of inverse $g(f(a)) = a$, we get

$$a_1 = a_2$$

Therefore f is injective.
onto : To prove that f is onto, we must show that there exists an $a \in A$ such that $f(a) = b$. Since $g : B \to A$, we know that $g(b) \in A$. Let $a = g(b)$. Substituting a in $f(a)$ gives

$$f(a) = f(g(b)) = b$$

As a result of applying the definition of inverse $f(g(b)) = b$. Therefore, for every $b \in B$, there exists $a = g(b) \in A$ such that $f(a) = b$. Hence f is onto.

Since f is both one-to-one and onto, it follows that f is bijective. □

Problem 23. *A real number $x \in \mathbb{R}$ is called* algebraic *if there exists $a_0, a_1, \ldots, a_{n-1}, a_n \in \mathbb{Z}$, not all zero, so that*

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$$

That is, a real number is algebraic if it is a root of a polynomial equation with integer coefficients.

(a) *The numbers $\sqrt{2}$, $\sqrt[3]{2}$, and $\sqrt{3} + \sqrt{2}$ are algebraic.*
For $\sqrt{2}$:

Proof. Let $p(x) = x^2 - 2$. We verify that $\sqrt{2}$ is a root of this polynomial:

$$p(\sqrt{2}) = (\sqrt{2})^2 - 2 = 2 - 2 = 0$$

Thus $\sqrt{2}$ is algebraic. □

For $\sqrt[3]{2}$:

Proof. Let $p(x) = x^3 - 2$. We verify that $\sqrt[3]{2}$ is a root of this polynomial:

$$p(\sqrt[3]{2}) = (\sqrt[3]{2})^3 - 2 = 2 - 2 = 0$$

Therefore $\sqrt[3]{2}$ is algebraic. $\square$

For $\sqrt{3} + \sqrt{2}$***:***

Proof. Let $p(x) = x^4 - 10x^2 + 1$. We verify that $\sqrt{3} + \sqrt{2}$ is a root of this polynomial:

$$\begin{aligned} p(\sqrt{3} + \sqrt{2}) &= (\sqrt{3} + \sqrt{2})^4 - 10(\sqrt{3} + \sqrt{2})^2 + 1 \\ &= (3 + 2\sqrt{6} + 2)^2 - 10(3 + 2\sqrt{6} + 2) + 1 \\ &= (5 + 2\sqrt{6})^2 - 30 - 20\sqrt{6} - 20 + 1 \\ &= 25 + 20\sqrt{6} + 24 - 30 - 20\sqrt{6} - 20 + 1 \\ &= 50 + 20\sqrt{6} - 50 - 20\sqrt{6} \\ &= 0 \end{aligned}$$

Thus $\sqrt{3} + \sqrt{2}$ is algebraic. $\square$

(b) *For fixed $n \in \mathbb{N}$, let A_n be the set of algebraic numbers which are roots of polynomials, with integer coefficients, of degree n. Then A_n is countable.*

Proof. A polynomial $P \in P_n$ of degree n with integer coefficients is defined as

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$$

where $a_i \in \mathbb{Z}$ and $a_n \neq 0$. Let P_n denote the set of all such degree-n polynomials. Each polynomial is uniquely determined by the $(n+1)$-tuple $(a_n, a_{n-1}, \ldots, a_1, a_0)$ where $a_n \neq 0$. This is a bijection between P_n and a subset of Z^{n+1}. Since Z^{n+1} is countable, any subset of Z^{n+1} is at most countable. Therefore P_n is countable. We can thus enumerate $P_n = \{P_1, P_2, P_3, \ldots\}$.

By the Fundamental Theorem of Algebra, each polynomial of degree n has at most n roots. For each $i \in \mathbb{N}$, let R_i denote the set of all roots of P_i. Then $|R_i| \leq n < \infty$.

Every algebraic number in A_n is, by definition a root of some polynomial in P_n. Therefore $A_n = \cup_{i=1}^{\infty} R_i$. Since A_n is a countable union of finite sets, A_n is countable. $\square$

(c) *The set of all algebraic numbers is countable.*

Proof. For each $n \in \mathbb{N}$, let A_n be the set of roots of all degree-n polynomials with integer coefficients. From the previous problem we know that each A_n is countable. The set of all algebraic numbers is

$$A = \cup_{n=1}^{\infty} A_n$$

Since A is a countable union of countable sets, A is countable. Therefore the set of all algebraic numbers is countable. $\square$

Problem 24. *There is an onto function $f : (0,1) \to S$ where $S = \{(x,y) : 0 < x, y < 1\}$ is the unit square in the plane $\mathbb{R}^2$.*

Proof. Let $(u,v) \in S$ be arbitrary. Since $u, v \in (0,1)$, each has a decimal expansion:

$$u = 0.a_1a_2a_3a_4\dots$$
$$v = 0.b_1b_2b_3b_4\dots$$

where $a_i, b_i \in \{0,1,2,\dots,9\}$. If u or v has two representations, we choose the non-terminating one.
We construct $x \in (0,1)$ by interleaving the decimal digits:

$$x = 0.a_1b_1a_2b_2a_3b_3a_4b_4\dots$$

Now define $f : (0,1) \to S$ as follows: For any $x = 0.d_1d_2d_3d_4\dots \in (0,1)$, let

$$f(x) = (0.d_1d_3d_5d_7\dots, 0.d_2d_4d_6d_8\dots)$$

Then f(x) = (u,v) by construction, since the odd-positioned digits of x are precisely $a_1, a_2, a_3, \dots$ and the even-positioned digits are $b_1, b_2, b_3, \dots$. Since (u,v) was arbitrary, f is onto. $\square$

Problem 25. *(a)* $\lim_{n\to\infty} \frac{2n+1}{5n+3} = \frac{2}{5}$

Proof. Let $\epsilon > 0$. We compute:

$$\begin{aligned} \left|\frac{2n+1}{5n+3} - \frac{2}{5}\right| &= \left|\frac{5(2n+1) - 2(5n+3)}{5(5n+3)}\right| \\ &= \frac{1}{5(5n+3)} \\ &= \frac{1}{25n+15} \end{aligned}$$

For $n \geq 1$, we have $25n + 15 \geq 25n$, so:

$$\frac{1}{25n+15} \leq \frac{1}{25n} < \epsilon$$

The last inequality holds when $n > \frac{1}{25\epsilon}$.
Choose $N \in \mathbb{N}$ such that $N > \frac{1}{25\epsilon}$.
Then for all $n \geq N$, we have $\left|\frac{2n+1}{5n+3} - \frac{2}{5}\right| < \epsilon$. $\square$

(b) $\lim_{n\to\infty} \frac{2n^2}{n^3+1} = 0$

Proof. Let $\epsilon > 0$. We compute:

$$\left|\frac{2n^2}{n^3+1} - 0\right| = \frac{2n^2}{n^3+1}$$

For $n \geq 1$, we have $n^3 + 1 \geq n^3$, so:

$$\frac{2n^2}{n^3+1} \leq \frac{2n^2}{n^3} = \frac{2}{n} < \epsilon$$

The last inequality holds when $n > \frac{2}{\epsilon}$.

Choose $N \in \mathbb{N}$ such that $N > \frac{2}{\epsilon}$.

Then for all $n \geq N$, we have $\left|\frac{2n^2}{n^3+1}\right| < \epsilon$.

Therefore $\lim_{n\to\infty} \frac{2n^2}{n^3+1} = 0$. □

(c) $\lim_{n\to\infty} \frac{\sin(n)}{\sqrt{n}} = 0$

Proof. Let $\epsilon > 0$. We compute:

$$\left|\frac{\sin(n)}{\sqrt{n}} - 0\right| = \frac{|\sin(n)|}{\sqrt{n}}$$

Since $|\sin(n)| \leq 1$ for all n, we have:

$$\frac{|\sin(n)|}{\sqrt{n}} \leq \frac{1}{\sqrt{n}} < \epsilon$$

The last inequality holds when $\sqrt{n} > \frac{1}{\epsilon}$, or equivalently, $n > \frac{1}{\epsilon^2}$.

Choose $N \in \mathbb{N}$ such that $N > \frac{1}{\epsilon^2}$.

Then for all $n \geq N$, we have $\left|\frac{\sin(n)}{\sqrt{n}}\right| < \epsilon$.

Therefore $\lim_{n\to\infty} \frac{\sin(n)}{\sqrt{n}} = 0$. □

Problem 26. *(a) A sequence with an infinite number of ones that does not converge to one.*

Define $a_n = \sin(\frac{n\pi}{2})$ *where* $n \in \mathbb{N}$*. This sequence looks like* $1, 0, -1, 0, 1, 0, -1, 0, \ldots$ *and contains infinitely many ones when* $n = 1, 5, 9, 13, \ldots$.

Proof. We show this sequence does not converge to 1. Let $\epsilon = \frac{1}{2}$. For any $N \in \mathbb{N}$, we can find $n > N$ with $n \equiv 3 \pmod 4$. For such n, $a_n = \sin\left(\frac{n\pi}{2}\right) = -1$, so $|a_n - 1| = |-1-1| = 2 > \epsilon$. Therefore the sequence does not converge to 1. □

(b) *A sequence with an infinite number of ones that converges to a limit not equal to one.*

Proof. Suppose, for the sake of contradiction, that there exists a sequence (a_n) that:

(a) Contains infinitely many ones, and

(b) Converges to some limit L where $L \neq 1$

Since $L \neq 1$, the distance between L and 1 is positive. Let $d = |L-1| > 0$.

Choose $\epsilon = \frac{d}{2} = \frac{|L-1|}{2}$. This is half the distance from L to 1.

Since (a_n) converges to L, by definition of convergence, there exists some $N \in \mathbb{N}$ such that:

$$\text{for all } n \geq N, \quad |a_n - L| < \epsilon$$

This means that all terms after position N must lie within distance ϵ of L.

However, the sequence contains infinitely many ones. Therefore, there must exist some $n \geq N$ where $a_n = 1$.

For this particular n, we compute:

$$|a_n - L| = |1 - L| = |L - 1| = d = 2\epsilon$$

But this contradicts the requirement that $|a_n - L| < \epsilon$.

Therefore, our assumption must be false. No such sequence can exist. □

Problem 27. *Let (x_n) be a sequence that converges to x. Suppose $p(x)$ is a polynomial. Then*

$$\lim_{n\to\infty} p(x_n) = p(x).$$

Proof. We prove by induction on the degree k of the polynomial $p(x)$.
Base case ($k=0$): If $p(x) = a_0$ is a constant polynomial, then $p(x_n) = a_0$ for all n, so:

$$\lim_{n\to\infty} p(x_n) = \lim_{n\to\infty} a_0 = a_0 = p(x)$$

Inductive step: Assume the result holds for all polynomials of degree at most $k-1$. Let $p(x)$ be a polynomial of degree k:

$$p(x) = a_k x^k + q(x)$$

where $q(x) = a_{k-1}x^{k-1} + \cdots + a_1 x + a_0$ is a polynomial of degree at most $k-1$.
By the inductive hypothesis, $\lim_{n\to\infty} q(x_n) = q(x)$.
For the term $a_k x^k$, we can write:

$$x_n^k = x_n \cdot x_n^{k-1}$$

By the Algebraic Limit Theorem (product rule), since $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} x_n^{k-1} = x^{k-1}$ (by applying the product rule inductively), we have:

$$\lim_{n\to\infty} x_n^k = x \cdot x^{k-1} = x^k$$

By the Algebraic Limit Theorem (constant multiple rule):

$$\lim_{n\to\infty} a_k x_n^k = a_k x^k$$

Finally, by the Algebraic Limit Theorem (sum rule):

$$\lim_{n\to\infty} p(x_n) = \lim_{n\to\infty} (a_k x_n^k + q(x_n)) = a_k x^k + q(x) = p(x)$$

By induction, the result holds for all polynomials. □

Problem 28. *Consider three sequences (x_n), (y_n), and (z_n) for which $x_n \leq y_n \leq z_n$ for each n. If $x_n \to \ell$ and $z_n \to \ell$ then $y_n \to \ell$.*

Proof. Let $\epsilon > 0$. We must show there exists N such that for all $n \geq N$, $|y_n - \ell| < \epsilon$.

Since $x_n \to \ell$, there exists $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$:

$$|x_n - \ell| < \epsilon$$

which implies:

$$-\epsilon < x_n - \ell < \epsilon$$

or equivalently:

$$\ell - \epsilon < x_n < \ell + \epsilon$$

Since $z_n \to \ell$, there exists $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$:

$$|z_n - \ell| < \epsilon$$

which implies:

$$-\epsilon < z_n - \ell < \epsilon$$

or equivalently:

$$\ell - \epsilon < z_n < \ell + \epsilon$$

Choose $N = \max\{N_1, N_2\}$. For any $n \geq N$, we have both $n \geq N_1$ and $n \geq N_2$. Therefore, using the given condition $x_n \leq y_n \leq z_n$:

$$\ell - \epsilon < x_n \leq y_n \leq z_n < \ell + \epsilon$$

This gives us:

$$\ell - \epsilon < y_n < \ell + \epsilon$$

which is equivalent to:

$$|y_n - \ell| < \epsilon$$

Since ϵ was arbitrary, we conclude that $y_n \to \ell$. $\square$