Chapter 10

Prove the following statements with either induction, strong induction or proof by smallest counterexample.

Exercise (3). Prove that $1^3 + 2^3 + 3^3 + 4^3 + \cdots n^3 = \frac{n^2(n+1)^2}{4}$ for every positive integer n.

Proof: (Weak Induction)

Base Case: Observe that when $n = 1$ that $n^3 = (1)^3 = \frac{(1)^2((1)+1)^2}{4} = \frac{4}{4} = 1$ which is true.

Induction Hypothesis: Suppose there is a $k \in \mathbb{Z}$ such that $1^3 + 2^3 + 3^3 + 4^3 + \cdots k^3 = \frac{k^2(k+1)^2}{4}$.

Inductive Step: We wish to show that the statement holds for $n = k + 1$, i.e., that $1^3 + 2^3 + 3^3 + 4^3 + \cdots k^3 + (k+1)^3 = \frac{(k+1)^2((k+1)+1)^2}{4}$. Observe the following:

$$\begin{aligned}
1^3 + 2^3 + 3^3 + 4^3 + \cdots k^3 + (k+1)^3 &= [1^3 + 2^3 + 3^3 + 4^3 + \cdots k^3] + (k+1)^3 \\
&= \frac{k^2(k+1)^2}{4} + (k+1)^3 \\
&= \frac{k^2(k+1)^2}{4} + \frac{4(k+1)^3}{4} \\
&= \frac{k^2(k+1)^2 + 4(k+1)^3}{4} \\
&= \frac{(k+1)^2(k^2 + 4(k+1))}{4} \\
&= \frac{(k+1)^2(k^2 + 4k + 4))}{4} \\
&= \frac{(k+1)^2(k+2)^2}{4} \\
&= \frac{(k+1)^2((k+1)+1)^2}{4}.
\end{aligned}$$

Showing that the statement holds for $n = k + 1$.

Conclusion: Therefore, by induction on n, the statement $1^3 + 2^3 + 3^3 + 4^3 + \cdots n^3 = \frac{n^2(n+1)^2}{4}$ is true for every positive integer $n \geq 1$. □

Exercise (4). If $n \in \mathbb{N}$, then $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}$.

Proof: □

Exercise (5). If $n \in \mathbb{N}$, then $2^1 + 2^2 + 2^3 + \cdots + 2^n = 2^{n+1} - 2$.

Proof: □

Exercise (8). If $n \in \mathbb{N}$, then $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}$.

Proof: □

Exercise (10). Prove that $3 \mid (5^{2n} - 1)$ for every integer $n \geq 0$.

Proof: Write your answer here. □

Exercise (13). Prove that $6 \mid (n^3 - n)$ for every integer $n \geq 0$.

Proof: Write your answer here. □

Exercise (18). Suppose $A_1, A_2, \ldots, A_n$ are sets in some universal set U, and $n \geq 2$. Prove that $\overline{A_1 \cup A_2 \cup \cdots A_n} = \overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n}$.

Proof: Write your answer here. □

Exercise (19). Prove that $\dfrac{1}{1} + \dfrac{1}{4} + \dfrac{1}{9} + \cdots + \dfrac{1}{n^2} \leq 2 - \dfrac{1}{n}$ for every $n \in \mathbb{N}$.

Proof: Write your answer here. □

Exercise (22). If $n \in \mathbb{N}$, then

$$\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{4}\right)\left(1 - \frac{1}{8}\right)\left(1 - \frac{1}{16}\right)\cdots\left(1 - \frac{1}{2^n}\right) \geq \frac{1}{4} + \frac{1}{2^{n+1}}.$$

Proof: Write your answer here. □

Exercise (25). Concerning the Fibonacci sequence, prove that $F_1 + F_2 + F_3 + F_4 + \cdots + F_n = F_{n+2} - 1$.

Proof: Write your answer here. □

Exercise (30). Here F_n is the nth Fibonacci number. Prove that

$$F_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}.$$

Hint: There are multiple ways to do this... one is to use the fact that $a^{n-1} = \frac{a^n}{a}$, while others involve things like the fact if $\phi = \frac{1+\sqrt{5}}{2}$, then $\phi^2 - \phi - 1 = 0$.

Proof: Write your answer here. □

Exercise (33). Suppose n (infinitely long) straight lines lie on a plane in such a way that no two of the lines are parallel, and no three of the lines intersect in a single point. Show that this arrangement divides the plane into $\frac{n^2+n+2}{2}$ regions.

Proof: Write your answer here. □

Exercise (Reflection Problem).

- How long did it take you to complete each problem?

  Answer: □

- What was easy?

  Answer: □

- What was challenging? What made it challenging?

  Answer: □

- Compare your answers to the odd numbered exercises to those in the back of the textbook. What did you learn from this comparison?

  Answer: □

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