Problem 36. *Give a justified example of each, or argue (prove) that it is impossible.*

(a) *A sequence that has a subsequence that is bounded, but which contains no subsequence which converges.*

This is impossible by Bolzano Weierstrass. Every bounded sequence has at least one convergent subsequence.

(b) *A sequence that does not contain* 0 *or* 1 *as a term, but which contains subsequences which converge to each of these values.*

$$a_n = \frac{1 + (-1)^n}{2} + \frac{1}{n}$$

is such a sequence, we can set n *to even or odd numbers to converge to* 0 *or* 1.

(c) *A sequence that contains subsequences converging to every point in the infinite set* $\{1, 1/2, 1/3, 1/4, \dots\}$.

Consider that we can construct a subsequence that convergest to a chosen arbitrary value with $k - \frac{1}{n}$ *where* k *is any number we want to converge to and* $\frac{1}{n}$ *just going to zero. Let our sequence be defined by* $a_n = \frac{1}{k} - \frac{1}{n}$. *For* $k, n \in \mathbb{N}$ *this converges to every point in the infinite set.*

incomplete

Problem 37. *Let* (a_n) *be a bounded sequence. Define the set*

$$S = \{x \in \mathbb{R} : x < a_n \text{ for infinitely many terms } a_n\}.$$

Then S *is bounded above, and there exists a subsequence* (a_{n_k}) *which converges to* $\sup S$.

Proof. Since (a_n) is a bounded sequence, there exists an $M \in \mathbb{R}$ such that $a_n \leq M$ for all $n \in \mathbb{N}$. From this we have

$$x < a_n < M$$

by transitivity $x < M$ for all $x \in S$, so S is bounded above by M. Since S is a non-empty real set and bounded above, By Axiom of completeness, $s = \sup S$ exists.

Choose an arbitrary $k \in \mathbb{N}$ so that we create an interval around the supremum s:

$$s - \frac{1}{k} < s < s + \frac{1}{k}$$

Since any number smaller than s is not an upper bound of S, there exists an $s' \in S$ so that $s - \frac{1}{k} < s'$(s' is in the interval below s). Since $s' \in S$, it follows by transitivity that $s - \frac{1}{k} < s' < a_n$, thus $s - \frac{1}{k} < a_n$ for infinitely many terms a_n. So we have

$$s - \frac{1}{k} < a_n < s + \frac{1}{k}$$

Satisfied by every $k \in \mathbb{N}$. We construct the subsequence a_{n_k} recursively. For $k = 1$, choose any $n_1 \in \mathbb{N}$ such that $s - 1 < a_{n_1} \leq s + 1$. Having chosen $n_1 < n_2 < \cdots < n_k$, we choose $n_{k+1} > n_k$ such that

$$s - \frac{1}{k+1} < a_{n_{k+1}} < s + \frac{1}{k+1}.$$

Now we show convergence, Let $\epsilon > 0$, choose $K \in \mathbb{N}$ such that $\frac{1}{K} < \epsilon$. Then for all $k \geq K$, we have

$$\frac{1}{k} \leq \frac{1}{K} < \epsilon$$

By construction

$$s = \frac{1}{k} < a_{n_k} \leq s + \frac{1}{k}$$

Since $\frac{1}{k} < \epsilon$, we have

$$s - \epsilon < s - \frac{1}{k} < a_{n_k} \leq s + \frac{1}{k} < s + \epsilon$$

thus $|a_{n_k} - s| < \epsilon$ meaning by definition there is a subsequence a_{n_k} that converges to $\sup S$. □

Problem 38. *Every convergent sequence is a Cauchy sequence.*

Proof. A sequence is Cauchy iff for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for every $m, n \in \mathbb{N}$ when $m, n > N$ we have $|a_n - a_m| < \epsilon$.

Let (a_n) be a convergent sequence and let $(a_n) \to a$. By definition this means that for $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that when $n > N$ we have $|a_n - a| < \frac{\epsilon}{2}$. We now show that this is a Cauchy sequence. Let $\epsilon > 0$ and let $m, n > N$. Observe that

$$\begin{aligned} \mid a_n - a_m \mid &= \mid (a_n - a) + (a - a_m) \mid \\ &= \mid a_n - a \mid + \mid a - a_m \mid \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

Thus (a_n) is a Cauchy sequence. □

Problem 39. *Give a justified example of each, or argue (prove) that it is impossible.*

(a) A Cauchy sequence that is not monotone.

Since all convergent sequences are Cauchy sequences, we just need to find any sequence that converges that is not monotone. Let $a_n = \frac{(-1)^n}{n}$.

(b) A Cauchy sequence containing an unbounded subsequence.

Boundedness is a criteria for convergence so this is impossible

(c) *An unbounded sequence containing a Cauchy subsequence.*

Impossible for the same reason as above

Problem 40. *Give a justified example of each, or explain (prove) why the request is impossible, by referencing the proper theorem(s).*

(a) *Two series $\sum x_n$ and $\sum y_n$ which both diverge, but where $\sum x_n y_n$ converges.*

Let $x_n = y_n = \frac{1}{n}$, then $\sum x_n$ and $\sum y_n$ diverge. Consider the product $\sum x_n y_n = \sum \frac{1}{n} * \frac{1}{n} = \frac{1}{n^2}$. This is a p series where $p > 1$ and thus converges.

(b) *A convergent series $\sum x_n$ and a bounded sequence (y_n), such that $\sum x_n y_n$ diverges.*

Let $x_n = \frac{(-1)^n}{n}$ and let $y_n = (-1)^n$. The sum $\sum x_n = \sum \frac{(-1)^n}{n}$ is an alternating harmonic series so it converges.. The sum $\sum y_n = \sum(-1)^n$ just flips between -1 if odd and 1 if even, this is also the greatest lower bound and least upper bound respectively.The product $\sum x_n y_n = \sum[\frac{(-1)^n}{n} * (-1)^n] = \sum[\frac{(-1)^{2n}}{n}] = \sum \frac{1}{n}$ diverges.

(c) *Two sequences (x_n) and (y_n) where $\sum x_n$ and $\sum(x_n + y_n)$ both converge, but $\sum y_n$ diverges.*

Impossible

Proof. Suppose (x_n) converges and $\sum(x_n + y_n)$ converge with (y_n) diverging. Observe that $\sum y_n = \sum(x_n + y_n) - \sum x_n$, by the algebraic rule for series. A consequence is that (y_n) converges. But this is a contradiction since we assumed (y_n) diverges. Thus this is impossible. $\square$

(d) *A sequence (x_n) satisfying $0 \leq x_n \leq 1/n$ where $\sum(-1)^n x_n$ diverges.*

Let

$$x_n = \begin{cases} \frac{1}{n} & \text{if n is odd} \\ 0 & \text{if n even} \end{cases}$$

This is just $\sum(-\frac{1}{n})$ which diverges without the even numbers.

Problem 41. *If $\sum a_n$ converges absolutely then $\sum a_n^2$ converges absolutely.*

Proof. If $\sum |a_n|$ converges then $\lim |a_n| = 0$. It follows that there exists an $N \in \mathbb{N}$ such that for all $n \geq N$ that $|a_n| < 1$. Since $|a_n|$ is positive(absolute value) we have

$$0 < |a_n| < 1$$

Being that $|a_n|$ is between 0 and 1, we have

$$|a_n^2| = |a_n|^2 \leq |a_n|$$

Since $\sum |a_n|$ converges and $|a_n^2| < |a_n|$, then by the Comparison Test, for all sufficiently large n, the sum $\sum |a_n^2|$ converges. Thus $\sum a_n^2$ converges absolutely. $\square$

Problem 42. *Ratio test: For a series $\sum a_n$, if the sequence of terms (a_n) satisfies $a_n \neq 0$ for all n, and if*

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = r < 1,$$

then the series converges absolutely.

Proof. Suppose $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = r < 1$

Choose r' such that $r < r' < 1$ and let $\epsilon =' r - r > 0$. Since

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = r < 1$$

by the definition of limit, there exists $N \in \mathbb{N}$ such that for all $n \geq N$.

$$\mid \frac{|a_{n+1}|}{|a_n|} - r \mid < \epsilon = r' - r$$

This implies

$$\frac{|a_{n+1}|}{|a_n|} < r + (r' - r) = r'$$

Thus for all $n \geq N$

$$|a_{n+1}| \leq r'|a_n|$$

We want to show that for all $n \geq N$ that

$$|a_n| \leq |a_N| * (r')^{n-N}$$

We show this by induction:
Base case Suppose $n = N$, then

$$|a_N| \leq |a_N| * (r')^{n-N} = |a_N| * (r')^0 = |a_N|$$

This gives is $|a_N| \leq |a_N|$ so the inequality holds. **Inductive hypothesis** Suppose the statement holds for some particular $n \geq N$, that

$$|a_n| \leq |a_N| * (r')^{n-N}$$

Inductive step We want to show that the statemen holds for $n+1$. Observe that

$$a_{n+1} \leq r'|a_n| \leq r' * [|a_N| * (r')^{n-N}]$$

rearranging gives

$$\begin{aligned} |a_{n+1}| &\leq |a_N| * r' * (r')^{n-N} \\ &= |a_N| * (r')^{1+n-N} \\ &= |a_N| * (r')^{(n+1)-N} \end{aligned}$$

Thus by induction $|a_n| \leq |a_N| * (r')^{n-N}$ holds for all $n \geq N$. Taking the sums of the inequality we get

$$\sum_{n=N}^{\infty} |a_n| \leq \sum_{n=N}^{\infty} |a_N| \cdot (r')^{n-N} = |a_N| \sum_{n=N}^{\infty} (r')^{n-N}$$

Let $k = n - N$ so we get

$$|a_n| \sum_{k=0}^{\infty} (r')^k$$

Since $0 < r' < 1$, this is a convergent geometric series with sum $\frac{1}{1-r}$. Thus

$$\sum_{n=N}^{\infty} |a_n| \leq |a_N| * \frac{1}{1-r'} < \infty$$

Since the sume of $\sum_{n=N}^{\infty} |a_n|$ converges and the first $N-1$ are a finite sum we have

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{N-1} |a_n| + \sum_{n=N}^{\infty} |a_n| < \infty$$

Therefore $\sum a_n$ converges absolutely.

$\square$

Problem 43. *Do the following series converge or diverge? A careful proof is not needed, but a logical and correct justification or explanation is required, possibly using Theorems from Sections 2.1–2.7, or Problems above.*

(a) $\sum_{n=1}^{\infty} \frac{1}{2^n+n}$

(b) $\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$

(c) $1 - \frac{3}{4} + \frac{4}{6} - \frac{5}{8} + \frac{6}{10} - \frac{7}{12} + \frac{8}{14} + \dots$

(d) $1 - \frac{1}{2^2} + \frac{1}{3} - \frac{1}{4^2} + \frac{1}{5} - \frac{1}{6^2} + \frac{1}{7} - \frac{1}{8^2} + \frac{1}{9} - \dots$

(e) $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$