4.2.1

We are given the differential equation $y''+2y'+y=0$ with the solution $y_1 = xe^{-x}$ we will construct the second solution using the formula provided: $y_2 = y_1 \int \frac{e^{-\int p(x)dx}}{(y_1)^2}dx$.

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{-\int p(x)dx}}{(y_1)^2}dx && (1)\\
&= xe^{-x} \int \frac{e^{-\int 2dx}}{(xe^{-x})^2}dx \qquad p(x)=2 && (2)\\
&= xe^{-x} \int \frac{e^{-2x}}{(xe^{-x})^2}dx && (3)\\
&= xe^{-x} \int \frac{e^{-2x}}{(x^2e^{-2x})}dx && (4)\\
&= xe^{-x} \int \frac{1}{x^2}dx && (5)\\
&= xe^{-x} \int (\frac{\sec(\ln(x))}{x}) && (6)\\
&= -e^{-x} && (7)
\end{aligned}$$

Our second solution is

$$\boxed{y_2 = C_2e^{-x}}$$

4.2.2

We are given the differential equation $x^2y'' - 3xy' + 5y = 0$ with the solution $y_1 = x^2\cos(\ln(x))$ we will construct the second solution using the formula provided: $y_2 = y_1 \int \frac{e^{-\int p(x)dx}}{(y_1)^2}dx$.

First we divide our equation $x^2y'' - 3xy' + 5y = 0$ by its leading term x^2:

$$\frac{x^2y'' - 3xy' + 5y}{x^2} = y'' - \frac{3y'}{x} + \frac{5y}{x^2}$$

Our $p(x)$ term is $-\frac{3}{x}$.

$$\begin{align}
y_2 &= y_1 \int \frac{e^{-\int p(x)dx}}{(y_1)^2} dx \\
&= x^2\cos(\ln(x)) \int \frac{e^{-\int -\frac{3}{x}dx}}{(x^2\cos(\ln(x)))^2} dx & p(x) = -\frac{3}{x} \\
&= x^2\cos(\ln(x)) \int \frac{e^{3\ln(x)}}{(x^2\cos(\ln(x)))^2} dx \\
&= x^2\cos(\ln(x)) \int \frac{x^3}{(x^4\cos^2(\ln(x)))} dx \\
&= x^2\cos(\ln(x)) \int \frac{1}{xcos^2(\ln(x))} dx \\
&= x^2\cos(\ln(x)) \int \frac{\sec^2(\ln(x))}{x} \\
&= x^2\cos(\ln(x)) \int \sec^2(u)du & u = \ln(x) \qquad du = \frac{1}{x} \\
&= x^2\cos(\ln(x)) * \tan(\ln(x)) \\
&= x^2\cos(\ln(x)) * \frac{\sin(\ln(x))}{\cos(\ln(x))} \\
&= x^2\sin(\ln(x))
\end{align}$$

Meaning our second solution is

$$\boxed{y_2 = C_2x^2\sin(\ln(x))}$$

4.3