3.2

3.2.1 8 a)

Suppose $a = b = 1$ for the Gompertz differential equation: $\frac{\delta P}{\delta t} = P(a - b\ln P)$, the following are the phase portraits for cases $P_0 > e$ and $0 < P_0 < e$:

8 b)

For $a = 1, b = -1$, cases $P_0 > e^{-1}$ and $0 < P_0 < e^{-1}$:

az Explicit solution for $P(O) = P_0$.

$$
\begin{aligned}
\frac{dP}{dt} &= P(a - b\ln P) \\
\int \frac{dP}{P(a - b\ln P)} &= \int dt \\
\int \frac{d(\ln P)}{(a - b\ln P)} &= \int dt \\
-\frac{1}{b}\ln|a - b\ln P| + c &= t
\end{aligned}
$$

When we plug in $P(O) = P_0$ we get $c = \frac{1}{b}|a - b\ln P_0|$ so we get:

$$
\begin{aligned}
t = & -\frac{1}{b}\ln|a - b\ln P| + \frac{1}{b}\ln\frac{a - b\ln P_0}{a - b\ln P} \\
\ln P(t) = & \frac{a}{b}(1 - e^{-bt}) + e^{-bt}\ln P_0 \\
P(t) = & \boxed{e^{\frac{a}{b}(1-e^{-bt})}P_0^{e^{-bt}}}
\end{aligned}
$$

3.2.2

Suppose we have the same 16 pound cannonball shot vertically upward with an initial velocity $v_0 = 300 ft/s$, the differential equation for the cannonball would be:

$$m\frac{dv}{dt} = -mg - kv^2$$

Now we solve using separations of variables:

$$-dt = \frac{mdv}{mg + kv^2} \tag{1}$$

$$-dt = \frac{1}{g}\frac{dv}{1 + (\sqrt{\frac{k}{mg}}v)^2} \tag{2}$$

$$-\int dt = \int \frac{1}{g}\sqrt{\frac{mg}{k}}\frac{\sqrt{\frac{k}{mg}}}{1 + (\sqrt{\frac{k}{mg}}v)^2}dv \tag{3}$$

$$-t + c = \sqrt{\frac{m}{gk}}\tan^{-1}(\sqrt{\frac{k}{mg}}v) \tag{4}$$

$$-\sqrt{\frac{kg}{m}}t + c = \tan^{-1}(\sqrt{\frac{k}{mg}}v) \tag{5}$$

$$\tan(-\sqrt{\frac{kg}{m}}t + c) = (\sqrt{\frac{k}{mg}}v) \tag{6}$$

Final

$$v(t) = \sqrt{\frac{mg}{k}}\tan(-\sqrt{\frac{kg}{m}}t + c)$$

Plugging in $v(0) = 300$ we find c:

$$v(0) = \sqrt{\frac{mg}{k}}\tan(-\sqrt{\frac{kg}{m}}(0) + c) \tag{1}$$

$$300 = \sqrt{\frac{mg}{k}}\tan(c) \tag{2}$$

$$c = \tan^{-1}(300)\sqrt{\frac{k}{mg}} \tag{3}$$

Plugging this in and the mass of 16 our final solution is:

$$\boxed{v(t) = \sqrt{\frac{16g}{k}}\tan(-\sqrt{\frac{kg}{16}}t + \tan^{-1}(300)\sqrt{\frac{k}{16g}})}$$

4.1

4.1.1

We are given the equation $x^2y;;-xy'+y=0$ and its solution $y=c_1x+c_2x\ln(x),(0,\infty)$ we are tasked with finding the member that satisfies y(1) = 3 and y'(1) = -1, first we derive and plug:

$$y = c_1x + c_2x\ln(x) \qquad 3 = c_1$$
$$y' = c_1 + c_2(\ln(x)+1) \qquad -1 = c_1 + c_2$$
$$y'' = c_2\frac{1}{x}$$

We get $c_1 = 3$ and $c_2 = -4$, our solution is:

$$\boxed{3x - 4x\ln(x)}$$

4.1.2

We determine if $f_1 = 1+x, f_2 = 3x, f_3 = -x^2$ are linearly independant using wronskian:

$$\begin{bmatrix} 1+x & 3x & -x^2 \\ 1 & 3 & -2x \\ 0 & 0 & -2 \end{bmatrix} = \boxed{-6} \tag{4}$$

Thus the functions are linearly independant since -6 $\neq$ 0

4.1.3

We verify that $y_1 = e^{\frac{x}{3}} and y_2 = xe^{\frac{x}{3}}$ form a fundamental set of solutions for $9y''+6y'+y=0$

$$y_1 = e^{\frac{x}{3}} \qquad y_1'' = \frac{e^{\frac{x}{3}}}{3} \qquad y_1'' = \frac{e^{\frac{x}{3}}}{9}$$
$$y_2 = xe^{\frac{x}{3}} \qquad y_2' = \frac{1}{3}e^{\frac{x}{3}}(x+3) \qquad y_2'' = \frac{1}{9}e^{\frac{x}{3}}(x+6)$$

plugging both equations into $9y''+6y'+y=0$ we get:

Get to this later