Problem 29. *Suppose $(x_n)_{n=1}^{\infty}$ converges. Let $k \in \mathbb{N}$. The new sequence $(x_{n+k})_{n=1}^{\infty}$ also converges, and to the same limit.*

Proof. Let $\epsilon > 0$. Since the sequence $(x_n)_{n=1}^{\infty}$ converges to L, there exists $N \in \mathbb{N}$ such that for all $n > N$, $|x_n - L| < \epsilon$. Now choose $M = N$ for our shifted sequence. Then for all $n > M$, we have $n + k > N$(since $k \geq 1$), so $|x_{n+k} - L| < \epsilon$. Therefore (x_{n+k}) converges to L. $\square$

Problem 30. *Give an example of each of the following, or state that such a request is impossible. In the latter case, identify specific theorem(s) that justify your statement.*

(a) *sequences (x_n) and (y_n), which both diverge, where the sum $(x_n + y_n)$ converges*

We take the alternating harmonic series $\sum_{k=1}^{\infty} \frac{(-1)^{n+1}}{n}$ which famously converges to $\ln(2)$ and define x_n as the sequence of positive terms and y_n as the sequence of negative terms.

$$x_n = \begin{cases} \frac{1}{n} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases} \qquad y_n = \begin{cases} \frac{-1}{n} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

These two sequence of partial sums converge when combined and each diverge when split this way.

(b) *a convergent sequence (x_n), and a divergent sequence (y_n), where $(x_n + y_n)$ converges*

(c) *a convergent sequence (b_n), with $b_n \neq 0$ for all n, such that $(1/b_n)$ diverges*

(d) *sequences (x_n) and (y_n), where $(x_n y_n)$ and (x_n) converge but (y_n) does not*

Problem 31. *If $a \geq 0$ and $b \geq 0$ then $\sqrt{ab} \leq \frac{1}{2}(a + b)$.*

Proof. $\square$

Problem 32. *Consider the real sequence generated by setting $x_1 = 2$ and then*

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right).$$

(a) *The sequence (x_n) is bounded below by $\sqrt{2}$.*

Proof. $\square$

(b) $\lim_{n\to\infty} x_n = \sqrt{2}$.

Proof. $\square$

Problem 33. *The sequence* $\sqrt{2}, \sqrt{2+\sqrt{2}}, \sqrt{2+\sqrt{2+\sqrt{2}}}, \ldots$ *converges to* X.

Proof. $\square$

Problem 34. *For each series, find an explicit formula for the partial sums, and determine if the series converges.*

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

(c) $\displaystyle\sum_{n=1}^{\infty} \log\left(\frac{n+1}{n}\right)$

Problem 35.

(a) *Suppose* $0 \leq a_n \leq b_n$. *If* $\displaystyle\sum_{n=1}^{\infty} a_n$ *diverges then* $\displaystyle\sum_{n=1}^{\infty} b_n$ *diverges.*

Proof. $\square$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ *diverges.*

Proof. $\square$