Problem 68. *The function $f(x) = 1/x^2$ is uniformly continuous on $(1, 2)$, but it is not uniformly continuous on $(0, 1)$.*

Proof. **Part 1: Uniform Continuity on (1,2):** To show that the function $f(x) = 1/x^2$ is uniformly continuous on the set $S_1 = (1, 2)$ we must show that for every $\epsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in S_1$:

$$|x - y| < \delta \implies |f(x) - f(y)| < \epsilon$$

Let $\epsilon > 0$, Observe that

$$|f(x) - f(y)| = |\frac{1}{x^2} - \frac{1}{y^2}| = |\frac{y^2 - x^2}{x^2y^2}| = \frac{|x + y||x - y|}{x^2y^2}$$

Since $x, y \in (1, 2)$, it follows that $x^2 > 1$ and $y^2 > 1$, so $x^2y^2 > 1$. Furthermore $|x + y| < 2 + 2 = 4$. Therefore:

$$|f(x) - f(y)| = \frac{|x - y||x + y|}{x^2y^2} < \frac{|x - y| * 4}{1} = 4|x - y|$$

Choose $\delta = \frac{\epsilon}{4}$. Then whenever $|x - y| < \delta$, we have:

$$|f(x) - f(y)| < 4|x - y| < 4 * \frac{\epsilon}{4} = \epsilon$$

Since δ only depends on ϵ, f is uniformly continuous on $(1, 2)$ $\square$

Proof. **Part 2: Not Uniformly continuous on (0,1):** Suppose for the sake of contradiction that f is uniformly continuous on the set $S_2 = (0, 1)$. Choose $\epsilon = 1$. Then there exists a $\delta > 0$ such that for all $x, y \in S_2$ we have

$$|x - y| < \delta \implies |\frac{1}{x^2} - \frac{1}{y^2}| < 1$$

Choose $x \in S_2$ with $x < \delta$ and choose $y = \frac{x}{2}$. Note that $y \in S_2$ because $x \in S_2$ so $y = \frac{x}{2} < \frac{1}{2} < 1$. Then

$$|x - y| = |x - \frac{x}{2}| = |\frac{x}{2}| = \frac{x}{2} < \frac{\delta}{2} < \delta$$

So $|x - y| < \delta$ holds, it follows that

$$|\frac{1}{x^2} - \frac{1}{y^2}| = |\frac{1}{x^2} - \frac{1}{(x/2)^2}| = |\frac{1}{x^2} - \frac{1}{(x^2/4)}| = |\frac{1}{x^2} - \frac{4}{x^2}| = |\frac{-3}{x^2}| = \frac{3}{x^2} > 1$$

So $|\frac{1}{x^2} - \frac{1}{y^2}| > 1$. A contradiction, thus f is not uniformly continuous on $S_2 = (0, 1)$. $\square$

Problem 69. *We say that a function $f : A \to \mathbb{R}$ is* Lipschitz *if there exists $M > 0$ so that*

$$\frac{|f(x) - f(y)|}{|x - y|} \leq M$$

for all $x, y \in A$. If f is Lipschitz then f is uniformly continuous.

Proof. Since f is Lipschitz, there exists $M > 0$ such that $|f(x) - f(y)| \leq M|x - y|$ for all $x, y \in A$. Let $\delta = \frac{\epsilon}{M}$ and let $x, y \in A$ be any two points such that $|x - y| < \delta$. Then it follows that

$$|f(x) - f(y)| \leq M|x - y| \implies |f(x) - f(y)| < M * \delta$$

Since $|x - y| < \delta$. Observe that

$$|f(x) - f(y)| < M * \delta = M * \frac{\epsilon}{M} = \epsilon$$

Thus, whenever $|x-y| < \delta$, we have $|f(x)-f(y)| < \epsilon$. Note that M is a constant and δ only depends on ϵ. Therefore proving that if f is Lipschitz, then f is uniformly continuous. $\square$

Problem 70. *Let f and g be functions defined on an interval A. Assume both are differentiable at some point $c \in A$, and suppose $k \in \mathbb{R}$. Then*

(i) $(f + g)'(c) = f'(c) + g'(c)$

(ii) $(kf)'(c) = kf'(c)$

Proof. **(i)** Since f and g are differentiable at some point $c \in A$, then $f'(c) = \lim_{h\to 0} \frac{f(c+h)-f(c)}{h}$ and $g'(c) = \lim_{h\to 0} \frac{g(c+h)-g(c)}{h}$ by definition. We want to show that $(f + g)'(c) = f'(c) + g'(c)$. Observe that

$$\begin{aligned}
(f + g)'(c) &= \lim_{h\to 0} \frac{(f + g)(c + h) - (f + g)(c)}{h} \\
&= \lim_{h\to 0} \frac{[f(c + h) + g(c + h)] - [f(c) + g(c)]}{h} \\
&= \lim_{h\to 0} \frac{[f(c + h) - f(c)] + [g(c + h) - g(c)]}{h} \\
&= \lim_{h\to 0} \frac{f(c + h) - f(c)}{h} + \lim_{h\to 0} \frac{g(c + h) - g(c)}{h} \\
&= f'(c) + g'(c)
\end{aligned}$$

Thus $(f + g)'(c) = f'(c) + g'(c)$. $\square$

Proof. **(ii)** Note that $k \in \mathbb{R}$ is constant, then

$$\begin{aligned}
(kf)'(c) &= \lim_{h\to 0} \frac{(kf)(c+h) - (kf)(c)}{h} \\
&= \lim_{h\to 0} \frac{kf(c+h) - kf(c)}{h} \\
&= \lim_{h\to 0} \frac{k[f(c+h) - f(c)]}{h} \\
&= \lim_{h\to 0} k * \frac{f(c+h) - f(c)}{h} \\
&= k * \lim_{h\to 0} \frac{f(c+h) - f(c)}{h} \\
&= kf'(c)
\end{aligned}$$

Thus $(kf)'(c) = kf'(c)$. □

Problem 71. *Let $h(x) = 1/x$ and $\ell(x) = 1/x^2$. For $c \neq 0$, we have*

$$h'(c) = -\frac{1}{c^2}, \qquad \ell'(c) = -\frac{2}{c^3}$$

Proof. **For h(x) implies h'(c):** Suppose $h(x) = 1/x$, then by definition of derivative:

$$\begin{aligned}
h'(c) &= \lim_{h\to 0} \frac{h(c+h) - h(c)}{h} \\
&= \lim_{h\to 0} \frac{\frac{1}{c+h} - \frac{1}{c}}{h} \\
&= \lim_{h\to 0} \frac{\frac{c-c-h}{c(c+h)}}{h} \\
&= \lim_{h\to 0} \frac{\frac{-h}{c(c+h)}}{h} \\
&= \lim_{h\to 0} \frac{-h}{h * c(c+h)} \\
&= \frac{-1}{1 * c(c+0)} \\
&= \frac{-1}{c^2}
\end{aligned}$$

□

Proof. **For ℓ(x) implies ℓ'(c):** Suppose $\ell(x) = 1/x^2$, then by definition of derivative:

$$\begin{aligned}
\ell'(c) &= \lim_{h\to 0} \frac{\ell(c+h) - \ell(c)}{h} \\
&= \lim_{h\to 0} \frac{\frac{1}{(c+h)^2} - \frac{1}{c^2}}{h} \\
&= \lim_{h\to 0} \frac{\frac{c^2-(c+h)^2}{c^2(c+h)^2}}{h} \\
&= \lim_{h\to 0} \frac{\frac{c^2-(c^2+2ch+h^2)}{c^2(c+h)^2}}{h} \\
&= \lim_{h\to 0} \frac{\frac{-2ch-h^2}{c^2(c+h)^2}}{h} \\
&= \lim_{h\to 0} \frac{-2ch - h^2}{h \cdot c^2(c+h)^2} \\
&= \lim_{h\to 0} \frac{h(-2c - h)}{h \cdot c^2(c+h)^2} \\
&= \lim_{h\to 0} \frac{-2c - h}{c^2(c+h)^2} \\
&= \frac{-2c - 0}{c^2(c+0)^2} \\
&= \frac{-2c}{c^4} \\
&= \frac{-2}{c^3}
\end{aligned}$$

□

Problem 72. *Let f and g be functions defined on an interval A. Assume both are differentiable at some point $c \in A$, and suppose $g(c) \neq 0$. Then*

$$\left(\frac{f}{g}\right)'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{g(c)^2}.$$

Proof. Note that $\frac{f}{g} = f \cdot \frac{1}{g}$. Let $h(x) = \frac{1}{x}$, so $\frac{f}{g} = f \cdot (h \circ g)$. By the Product Rule (Theorem 5.2.4(iii)):

$$\begin{aligned}
\left(\frac{f}{g}\right)'(c) &= (f \cdot (h \circ g))'(c) \\
&= f'(c) \cdot (h \circ g)(c) + f(c) \cdot (h \circ g)'(c)
\end{aligned}$$

By the Chain Rule, $(h \circ g)'(c) = h'(g(c)) \cdot g'(c)$. From Problem 71, $h'(x) = -\frac{1}{x^2}$, so

$h'(g(c)) = -\frac{1}{g(c)^2}$. Thus:

$$\begin{aligned}\left(\frac{f}{g}\right)'(c) &= f'(c) \cdot \frac{1}{g(c)} + f(c) \cdot \left(-\frac{1}{g(c)^2}\right) \cdot g'(c) \\ &= \frac{f'(c)}{g(c)} - \frac{f(c)g'(c)}{g(c)^2} \\ &= \frac{f'(c)g(c) - f(c)g'(c)}{g(c)^2}\end{aligned}$$

□

Problem 73. *For $a \in \mathbb{R}$, let*

$$f_a(x) = \begin{cases} x^a, & \text{if } x > 0 \\ 0, & \text{if } x \leq 0 \end{cases}$$

(a) For which values of a is $f_a(x)$ continuous at $x = 0$?

For continuity at $x = 0$, we need $\lim_{x\to 0} f_a(x) = f_a(0) = 0$. For $x < 0$, $f_a(x) = 0$, so $\lim_{x\to 0^-} f_a(x) = 0$. For $x > 0$, $f_a(x) = x^a$, so we need $\lim_{x\to 0^+} x^a = 0$.

If $a > 0$: $\lim_{x\to 0^+} x^a = 0$.

If $a = 0$: $\lim_{x\to 0^+} x^0 = 1 \neq 0$.

If $a < 0$: $\lim_{x\to 0^+} x^a = \lim_{x\to 0^+} \frac{1}{x^{|a|}} = +\infty$.

Therefore, $f_a(x)$ is continuous at $x = 0$ if and only if $a > 0$.

(b) What is the derivative $f_a'(x)$, and what is its domain? For which values of a is $f_a(x)$ differentiable at $x = 0$? When is the derivative function $f_a'(x)$ continuous?

For $x \neq 0$: If $x > 0$, then $f_a(x) = x^a$ so $f_a'(x) = ax^{a-1}$. If $x < 0$, then $f_a(x) = 0$ so $f_a'(x) = 0$.

For $x = 0$: By definition, $f_a'(0) = \lim_{h\to 0} \frac{f_a(h)-f_a(0)}{h} = \lim_{h\to 0} \frac{f_a(h)}{h}$.

From the left: $\lim_{h\to 0^-} \frac{0}{h} = 0$.

From the right: $\lim_{h\to 0^+} \frac{h^a}{h} = \lim_{h\to 0^+} h^{a-1}$.

If $a > 1$: $\lim_{h\to 0^+} h^{a-1} = 0$, so $f_a'(0) = 0$ exists.

If $a = 1$: $\lim_{h\to 0^+} h^0 = 1 \neq 0$, so the derivative does not exist.

If $0 < a < 1$: $\lim_{h\to 0^+} h^{a-1} = +\infty$, so the derivative does not exist.

If $a \leq 0$: f_a is not continuous at 0 (from part (a)), so not differentiable.

Therefore: f_a is differentiable at $x = 0$ if and only if $a > 1$.

The derivative function is:

$$f_a'(x) = \begin{cases} ax^{a-1}, & \text{if } x > 0 \\ 0, & \text{if } x < 0 \\ 0, & \text{if } x = 0 \text{ and } a > 1 \end{cases}$$

Domain of f_a': If $a > 1$, the domain is $\mathbb{R}$. If $a \leq 1$, the domain is $(-\infty, 0) \cup (0, \infty)$.

For continuity of f_a': On $(-\infty, 0)$ and on $(0, \infty)$, the derivative f_a' is continuous for all a. At $x = 0$ (when $a > 1$):

$\lim_{x \to 0^-} f_a'(x) = 0$ *and* $\lim_{x \to 0^+} f_a'(x) = \lim_{x \to 0^+} ax^{a-1}$.

If $a > 1$: $a - 1 > 0$, so $\lim_{x \to 0^+} ax^{a-1} = 0 = f_a'(0)$. Thus f_a' is continuous at 0.

Therefore, f_a' is continuous on $\mathbb{R}$ if and only if $a > 1$.