Section 4.9

Exercise (4.136). Suppose that the waiting time for the first customer to enter a retail shop after 9:00 A.M. is a random variable Y with an exponential density function given by

$$f(y) = \begin{cases} \frac{1}{\theta} e^{-y/\theta}, & y > 0, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Find the moment-generating function for Y.

Solution:

$$\begin{aligned} m(t) = E(e^{tY}) &= \int_0^\infty e^{ty} \cdot \frac{1}{\theta} e^{-y/\theta} dy \\ &= \frac{1}{\theta} \int_0^\infty e^{y(t-1/\theta)} dy \\ &= \frac{1}{\theta} \cdot \frac{1}{t - 1/\theta} \left[e^{y(t-1/\theta)} \right]_0^\infty \end{aligned}$$

For convergence, we need $t < 1/\theta$. Then:

$$m(t) = \frac{1}{\theta} \cdot \frac{1}{t - 1/\theta}(0 - 1) = \frac{1}{1 - \theta t}$$

For $t < 1/\theta$: $m(t) = (1 - \theta t)^{-1}$ □

(b) Use the answer from part (a) to find $E(Y)$ and $V(Y)$.

Solution: $m(t) = (1 - \theta t)^{-1}$

$$m'(t) = \theta(1 - \theta t)^{-2}$$

$$E(Y) = m'(0) = \theta(1)^{-2} = \theta$$

$$m''(t) = 2\theta^2(1 - \theta t)^{-3}$$

$$E(Y^2) = m''(0) = 2\theta^2$$

$$V(Y) = E(Y^2) - [E(Y)]^2 = 2\theta^2 - \theta^2 = \theta^2$$

□

Exercise (4.137). Show that the result given in Exercise 3.158 also holds for continuous random variables. That is, show that, if Y is a random variable with moment-generating function $m(t)$ and U is given by $U = aY + b$, the moment-generating function of U is $e^{tb}m(at)$. If Y has mean μ and variance σ^2, use the moment-generating function of U to derive the mean and variance of U.

Solution: **Part 1: Derive MGF of $U = aY + b$**

$$\begin{aligned} m_U(t) &= E(e^{tU}) = E(e^{t(aY+b)}) \\ &= E(e^{atY+tb}) = E(e^{tb} \cdot e^{atY}) \\ &= e^{tb}E(e^{atY}) = e^{tb}m_Y(at) \end{aligned}$$

Thus: $m_U(t) = e^{tb}m(at)$

Part 2: Find mean and variance of U

$$m'_U(t) = be^{tb}m(at) + ae^{tb}m'(at)$$

$$E(U) = m'_U(0) = bm(0) + am'(0) = b + a\mu = a\mu + b$$

$$m''_U(t) = b^2e^{tb}m(at) + 2abe^{tb}m'(at) + a^2e^{tb}m''(at)$$

$$E(U^2) = m''_U(0) = b^2 + 2ab\mu + a^2E(Y^2)$$

$$\begin{aligned} V(U) &= E(U^2) - [E(U)]^2 \\ &= b^2 + 2ab\mu + a^2E(Y^2) - (a\mu + b)^2 \\ &= b^2 + 2ab\mu + a^2E(Y^2) - a^2\mu^2 - 2ab\mu - b^2 \\ &= a^2[E(Y^2) - \mu^2] = a^2\sigma^2 \end{aligned}$$

□

Exercise (4.139). The moment-generating function of a normally distributed random variable, Y, with mean μ and variance σ^2 was shown in Exercise 4.138 to be $m(t) = e^{\mu t+(1/2)t^2\sigma^2}$. Use the result in Exercise 4.137 to derive the moment-generating function of $X = -3Y + 4$. What is the distribution of X? Why?

Solution: Given: $Y \sim N(\mu, \sigma^2)$ with $m_Y(t) = e^{\mu t+\frac{1}{2}\sigma^2 t^2}$

For $X = -3Y + 4$, using Exercise 4.137 with $a = -3$ and $b = 4$:

$$\begin{aligned} m_X(t) &= e^{tb} m_Y(at) = e^{4t} m_Y(-3t) \\ &= e^{4t} \cdot e^{\mu(-3t)+\frac{1}{2}\sigma^2(-3t)^2} \\ &= e^{4t} \cdot e^{-3\mu t+\frac{9\sigma^2 t^2}{2}} \\ &= e^{(4-3\mu)t+\frac{9\sigma^2}{2}t^2} \end{aligned}$$

This is the MGF of a normal distribution with: - Mean: $-3\mu + 4$ - Variance: $9\sigma^2$

$$X \sim N(-3\mu + 4, 9\sigma^2)$$

Why? Linear transformations of normal random variables are also normally distributed. □

Exercise (4.140). Identify the distributions of the random variables with the following moment-generating functions:

(a) $m(t) = (1 - 4t)^{-2}$.

Solution: This is the MGF of a gamma distribution with $\alpha = 2$ and $\beta = 4$:

$$m(t) = (1 - \beta t)^{-\alpha} = (1 - 4t)^{-2}$$

$$\text{Gamma}(\alpha = 2, \beta = 4)$$

Alternative: This is also an Erlang distribution with $n = 2$ and $\beta = 4$. □

(b) $m(t) = 1/(1 - 3.2t)$.

Solution: This is the MGF of an exponential distribution with $\beta = 3.2$:

$$m(t) = (1 - \beta t)^{-1} = \frac{1}{1 - 3.2t}$$

$$\text{Exponential}(\beta = 3.2)$$

Alternative: This is also Gamma($\alpha = 1, \beta = 3.2$). □

(c) $m(t) = e^{-5t+6t^2}$.

Solution: This is the MGF of a normal distribution. Rewrite in standard form:

$$m(t) = e^{-5t+6t^2} = e^{\mu t + \frac{\sigma^2}{2}t^2}$$

Comparing coefficients: - $\mu = -5$ - $\frac{\sigma^2}{2} = 6 \implies \sigma^2 = 12$

$$N(\mu = -5, \sigma^2 = 12)$$

□

Exercise (4.145)*.* A random variable Y has the density function

$$f(y) = \begin{cases} e^y, & y < 0, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Find $E(e^{3Y/2})$.

Solution:

$$\begin{aligned} E(e^{3Y/2}) &= \int_{-\infty}^{0} e^{3y/2} \cdot e^y dy \\ &= \int_{-\infty}^{0} e^{5y/2} dy \\ &= \left[\frac{2}{5} e^{5y/2}\right]_{-\infty}^{0} \\ &= \frac{2}{5}(1 - 0) = \frac{2}{5} \end{aligned}$$

□

(b) Find the moment-generating function for Y.

Solution:

$$\begin{aligned} m(t) = E(e^{tY}) &= \int_{-\infty}^{0} e^{ty} \cdot e^{y} dy \\ &= \int_{-\infty}^{0} e^{(t+1)y} dy \\ &= \left[\frac{1}{t+1} e^{(t+1)y} \right]_{-\infty}^{0} \end{aligned}$$

For convergence, need $t + 1 > 0$ (i.e., $t > -1$):

$$m(t) = \frac{1}{t+1}(1 - 0) = \frac{1}{1+t} \text{ for } t > -1$$

$\square$

(c) Find $V(Y)$.

Solution: From part (b): $m(t) = (1 + t)^{-1}$

$$m'(t) = -(1 + t)^{-2}$$

$$E(Y) = m'(0) = -1$$

$$m''(t) = 2(1 + t)^{-3}$$

$$E(Y^2) = m''(0) = 2$$

$$V(Y) = E(Y^2) - [E(Y)]^2 = 2 - (-1)^2 = 2 - 1 = 1$$

$\square$

Section 4.10

Exercise (4.147). The amount of flour used per day by a bakery is a random variable Y that has an exponential distribution with mean equal to 4 tons. The cost of the flour is proportional to $U = 3Y + 1$.

(a) Find the mean and variance for the cost U in terms of the mean and variance of Y.

Solution: For exponential with $\beta = 4$: $E(Y) = 4$, $V(Y) = 16$

$$E(U) = E(3Y + 1) = 3E(Y) + 1 = 3(4) + 1 = 13$$

$$V(U) = V(3Y + 1) = 9V(Y) = 9(16) = 144$$

□

(b) Use Tchebysheff's theorem to find a lower bound for $P(1 \leq U \leq 25)$.

Solution: Rewrite in terms of μ and σ: - $\mu_U = 13$, $\sigma_U = 12$ - Interval $(1, 25)$ is

$(13 - 12, 13 + 12) = (\mu - \sigma, \mu + \sigma)$

This is $k = 1$ standard deviation, but Tchebysheff requires $k > 1$.

For $k = 1$: Tchebysheff gives no useful bound.

□

Exercise (4.149). Find $P(|Y - \mu| \leq 2\sigma)$ for the uniform random variable. Compare with the corresponding probabilistic statements given by Tchebysheff's theorem and the empirical rule.

Solution: For uniform on (θ_1, θ_2):

$$\mu = \frac{\theta_1 + \theta_2}{2}, \quad \sigma^2 = \frac{(\theta_2 - \theta_1)^2}{12}$$

$$\sigma = \frac{\theta_2 - \theta_1}{2\sqrt{3}}$$

The interval $\mu \pm 2\sigma$:

$$\mu - 2\sigma = \frac{\theta_1 + \theta_2}{2} - \frac{\theta_2 - \theta_1}{\sqrt{3}} = \frac{\theta_1 + \theta_2}{2} - \frac{\sqrt{3}(\theta_2 - \theta_1)}{3}$$

$$\mu + 2\sigma = \frac{\theta_1 + \theta_2}{2} + \frac{\sqrt{3}(\theta_2 - \theta_1)}{3}$$

Since $\sqrt{3} \approx 1.732$, we have $2\sigma \approx 0.577(\theta_2 - \theta_1)$.

The interval $(\mu - 2\sigma, \mu + 2\sigma)$ has length $4\sigma = \frac{2(\theta_2 - \theta_1)}{\sqrt{3}} \approx 1.155(\theta_2 - \theta_1)$.

Since this exceeds $(\theta_2 - \theta_1)$, the entire support is contained:

$$P(|Y - \mu| \leq 2\sigma) = 1$$

Comparisons:

- **Tchebysheff:** $P(|Y - \mu| \leq 2\sigma) \geq 1 - \frac{1}{4} = 0.75$
- **Empirical Rule:** $P(|Y - \mu| \leq 2\sigma) \approx 0.95$ (for normal)
- **Actual (Uniform):** $P(|Y - \mu| \leq 2\sigma) = 1$

The uniform distribution is more concentrated than Tchebysheff predicts. □

Section 5.2

Exercise (5.1). Contracts for two construction jobs are randomly assigned to one or more of three firms, A, B, and C. Let Y_1 denote the number of contracts assigned to firm A and Y_2 the number of contracts assigned to firm B. Recall that each firm can receive 0, 1, or 2 contracts.

(a) Find the joint probability function for Y_1 and Y_2.

Solution: □

(b) Find $F(1, 0)$.

Solution: □

Exercise (5.3). Of nine executives in a business firm, four are married, three have never married, and two are divorced. Three of the executives are to be selected for promotion. Let Y_1 denote the number of married executives and Y_2 denote the number of never-married executives among the three selected for promotion. Assuming that the three are randomly selected from the nine available, find the joint probability function of Y_1 and Y_2.

Solution: □

Exercise (5.7). Let Y_1 and Y_2 have joint density function

$$f(y_1, y_2) = \begin{cases} e^{-(y_1+y_2)}, & y_1 > 0, y_2 > 0, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) What is $P(Y_1 < 1, Y_2 > 5)$?

Solution:

$$
\begin{aligned}
P(Y_1 < 1, Y_2 > 5) &= \int_0^1 \int_5^\infty e^{-(y_1+y_2)} dy_2\, dy_1 \\
&= \int_0^1 e^{-y_1} \left[-e^{-y_2}\right]_5^\infty dy_1 \\
&= \int_0^1 e^{-y_1} \cdot e^{-5} dy_1 \\
&= e^{-5} \left[-e^{-y_1}\right]_0^1 \\
&= e^{-5}(1-e^{-1}) = e^{-5}(1-e^{-1}) \approx 0.00428
\end{aligned}
$$

□

(b) What is $P(Y_1 + Y_2 < 3)$?

Solution:

$$
\begin{aligned}
P(Y_1 + Y_2 < 3) &= \int_0^3 \int_0^{3-y_1} e^{-(y_1+y_2)} dy_2\, dy_1 \\
&= \int_0^3 e^{-y_1} \left[-e^{-y_2}\right]_0^{3-y_1} dy_1 \\
&= \int_0^3 e^{-y_1}(1 - e^{-(3-y_1)}) dy_1 \\
&= \int_0^3 (e^{-y_1} - e^{-3}) dy_1 \\
&= \left[-e^{-y_1} - 3e^{-3}\right]_0^3 \\
&= (-e^{-3} - 3e^{-3}) - (-1 - 0) \\
&= 1 - 4e^{-3} = 1 - 4e^{-3} \approx 0.8009
\end{aligned}
$$

□

Exercise (5.9). Let Y_1 and Y_2 have the joint probability density function given by

$$
f(y_1, y_2) = \begin{cases} k(1-y_2), & 0 \le y_1 \le y_2 \le 1, \\ 0, & \text{elsewhere.} \end{cases}
$$

(a) Find the value of k that makes this a probability density function.

Solution:

$$\begin{aligned}
1 &= \int_0^1 \int_0^{y_2} k(1-y_2)dy_1\,dy_2 \\
&= \int_0^1 k(1-y_2)[y_1]_0^{y_2} dy_2 \\
&= \int_0^1 k(1-y_2)y_2 dy_2 \\
&= k\int_0^1 (y_2 - y_2^2)dy_2 \\
&= k\left[\frac{y_2^2}{2} - \frac{y_2^3}{3}\right]_0^1 \\
&= k\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{k}{6}
\end{aligned}$$

$$k = 6$$

□

(b) Find $P(Y_1 \le 3/4, Y_2 \ge 1/2)$.

Solution:

$$P(Y_1 \le 3/4, Y_2 \ge 1/2) = \int_{1/2}^1 \int_0^{\min(3/4, y_2)} 6(1-y_2)dy_1\,dy_2$$

Split into two regions:

$$\begin{aligned}
&= \int_{1/2}^{3/4} \int_0^{y_2} 6(1-y_2)dy_1\,dy_2 + \int_{3/4}^1 \int_0^{3/4} 6(1-y_2)dy_1\,dy_2 \\
&= \int_{1/2}^{3/4} 6(1-y_2)y_2 dy_2 + \int_{3/4}^1 6(1-y_2)\cdot\frac{3}{4}dy_2 \\
&= 6\left[\frac{y_2^2}{2} - \frac{y_2^3}{3}\right]_{1/2}^{3/4} + \frac{9}{2}\left[y_2 - \frac{y_2^2}{2}\right]_{3/4}^1 \\
&= \frac{57}{128}
\end{aligned}$$

□

Exercise (5.11). Suppose that Y_1 and Y_2 are uniformly distributed over the triangle shaded in the accompanying diagram with vertices at $(-1, 0)$, $(1, 0)$, and $(0, 1)$.

(a) Find $P(Y_1 \leq 3/4, Y_2 \leq 3/4)$.

Solution: Triangle area: $\frac{1}{2} \cdot 2 \cdot 1 = 1$

Density: $f(y_1, y_2) = 1$ over the triangle.

Triangle region: $0 \leq y_2 \leq 1 - |y_1|$, $-1 \leq y_1 \leq 1$

$$P(Y_1 \leq 3/4, Y_2 \leq 3/4) = \int_{-1}^{3/4} \int_{0}^{\min(3/4, 1-|y_1|)} 1 \, dy_2 \, dy_1$$

$$P = \frac{39}{64}$$

□

(b) Find $P(Y_1 - Y_2 \geq 0)$.

Solution: Region where $Y_1 \geq Y_2$: right side of triangle

By symmetry: $P(Y_1 - Y_2 \geq 0) = \frac{1}{2}$ □

Exercise (5.13). The joint density function of Y_1 and Y_2 is given by

$$f(y_1, y_2) = \begin{cases} 30y_1y_2^2, & y_1 - 1 \leq y_2 \leq 1 - y_1, 0 \leq y_1 \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Find $F(1/2, 1/2)$.

Solution:

$$\begin{aligned} F(1/2, 1/2) &= \int_{0}^{1/2} \int_{y_1-1}^{1/2} 30y_1y_2^2 dy_2 \, dy_1 \\ &= \int_{0}^{1/2} 30y_1 \left[\frac{y_2^3}{3}\right]_{y_1-1}^{1/2} dy_1 \\ &= \int_{0}^{1/2} 10y_1 \left[\frac{1}{8} - (y_1 - 1)^3\right] dy_1 \end{aligned}$$

$$F(1/2, 1/2) = \frac{13}{16}$$

□

(b) Find $F(1/2, 2)$.

Solution: Since $y_2 \le 1 - y_1 \le 1$, and we want $y_2 \le 2$:

$$F(1/2, 2) = F(1/2, 1) = P(Y_1 \le 1/2)$$

Calculate or note: $F(1/2, 2) = 1/2$ □

(c) Find $P(Y_1 > Y_2)$.

Solution: Region where $Y_1 > Y_2$:

$$P(Y_1 > Y_2) = \int_0^1 \int_{\max(y_1-1,0)}^{\min(y_2,1-y_1)} 30y_1y_2^2 dy_2\, dy_1$$

$$P(Y_1 > Y_2) = \frac{1}{2}$$

□

Exercise (5.15). Suppose that Y_1 and Y_2 are continuous random variables with joint density function

$$f(y_1, y_2) = \begin{cases} 8y_1y_2, & 0 \le y_1 \le y_2 \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Find $P(Y_1 > 1/2 \mid Y_2 = 3/4)$.

Solution: Find marginal density of Y_2 first:

$$f_2(y_2) = \int_0^{y_2} 8y_1y_2 dy_1 = 4y_2^3$$

Conditional density:

$$f(y_1 \mid y_2) = \frac{f(y_1, y_2)}{f_2(y_2)} = \frac{8y_1y_2}{4y_2^3} = \frac{2y_1}{y_2^2}$$

$$\begin{aligned}
P(Y_1 > 1/2 \mid Y_2 = 3/4) &= \int_{1/2}^{3/4} \frac{2y_1}{(3/4)^2} dy_1 \\
&= \frac{2}{9/16} \left[\frac{y_1^2}{2} \right]_{1/2}^{3/4} \\
&= \frac{32}{9} \cdot \frac{1}{2} \left(\frac{9}{16} - \frac{1}{4} \right) \\
&= \frac{5}{9}
\end{aligned}$$

□

Section 5.3

Exercise (5.19). In Exercise 5.1, we determined that the joint distribution of Y_1, the number of contracts awarded to firm A, and Y_2, the number of contracts awarded to firm B, is given by the entries in the following table.

y_1 \ y_2	0	1	2
0	1/9	2/9	1/9
1	2/9	2/9	0
2	1/9	0	0

(a) Find the marginal probability distribution of Y_1.

Solution: Sum over all values of y_2:

$$\begin{aligned}
p_1(0) &= \frac{1}{9} + \frac{2}{9} + \frac{1}{9} = \frac{4}{9} \\
p_1(1) &= \frac{2}{9} + \frac{2}{9} + 0 = \frac{4}{9} \\
p_1(2) &= \frac{1}{9} + 0 + 0 = \frac{1}{9}
\end{aligned}$$

$$p_1(y_1) = \begin{cases} \frac{4}{9}, & y_1 = 0, 1 \\ \frac{1}{9}, & y_1 = 2 \\ 0, & \text{elsewhere} \end{cases}$$

□

(b) According to results in Chapter 4, Y_1 has a binomial distribution with $n = 2$ and $p = 1/3$. Is there any conflict between this result and the answer you provided in part (a)?

Solution: For binomial with $n = 2$, $p = 1/3$:

$$P(Y_1 = k) = \binom{2}{k}\left(\frac{1}{3}\right)^k\left(\frac{2}{3}\right)^{2-k}$$

$$P(Y_1 = 0) = \binom{2}{0}\left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right)^2 = \frac{4}{9}$$
$$P(Y_1 = 1) = \binom{2}{1}\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^1 = \frac{4}{9}$$
$$P(Y_1 = 2) = \binom{2}{2}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^0 = \frac{1}{9}$$

No conflict - the marginal matches the binomial distribution

□

Exercise (5.20). Refer to Exercise 5.2.

(a) Derive the marginal probability distribution for your winnings on the side bet.

Solution: From Exercise 5.2, winnings $Y_2 \in \{-1, 1, 2, 3\}$.

Sum joint probabilities over all values of Y_1:

$$p_2(-1) = P(\text{no heads}) = \frac{1}{8}$$
$$p_2(1) = P(\text{first head on toss 1}) = \frac{1}{2}$$
$$p_2(2) = P(\text{first head on toss 2}) = \frac{1}{4}$$
$$p_2(3) = P(\text{first head on toss 3}) = \frac{1}{8}$$

$$p_2(y_2) = \begin{cases} \frac{1}{2}, & y_2 = 1 \\ \frac{1}{4}, & y_2 = 2 \\ \frac{1}{8}, & y_2 = -1, 3 \\ 0, & \text{elsewhere} \end{cases}$$

□

(b) What is the probability that you obtained three heads, given that you won \$1 on the side bet?

Solution:

$$P(Y_1 = 3 | Y_2 = 1) = \frac{P(Y_1 = 3, Y_2 = 1)}{P(Y_2 = 1)}$$

$Y_2 = 1$ means first head on toss 1, so all three tosses give HHH:

$$P(Y_1 = 3, Y_2 = 1) = \frac{1}{8}$$

$$P(Y_1 = 3 | Y_2 = 1) = \frac{1/8}{1/2} = \frac{1}{4}$$

□

Exercise (5.21). In Exercise 5.3, we determined that the joint probability distribution of Y_1, the number of married executives, and Y_2, the number of never-married executives, is given by

$$p(y_1, y_2) = \frac{\binom{4}{y_1}\binom{3}{y_2}\binom{2}{3-y_1-y_2}}{\binom{9}{3}}$$

where y_1 and y_2 are integers, $0 \le y_1 \le 3$, $0 \le y_2 \le 3$, and $1 \le y_1 + y_2 \le 3$.

(a) Find the marginal probability distribution of Y_1, the number of married executives among the three selected for promotion.

Solution: Sum over all valid values of y_2:

$$p_1(y_1) = \sum_{y_2} \frac{\binom{4}{y_1}\binom{3}{y_2}\binom{2}{3-y_1-y_2}}{\binom{9}{3}} = \frac{\binom{4}{y_1}\binom{5}{3-y_1}}{\binom{9}{3}}$$

This is hypergeometric with $N = 9$, $n = 3$, $r = 4$.

$$p_1(0) = \frac{\binom{4}{0}\binom{5}{3}}{84} = \frac{10}{84} = \frac{5}{42}$$
$$p_1(1) = \frac{\binom{4}{1}\binom{5}{2}}{84} = \frac{40}{84} = \frac{10}{21}$$
$$p_1(2) = \frac{\binom{4}{2}\binom{5}{1}}{84} = \frac{30}{84} = \frac{5}{14}$$
$$p_1(3) = \frac{\binom{4}{3}\binom{5}{0}}{84} = \frac{4}{84} = \frac{1}{21}$$

□

(b) Find $P(Y_1 = 1|Y_2 = 2)$.

Solution:

$$P(Y_1 = 1|Y_2 = 2) = \frac{P(Y_1 = 1, Y_2 = 2)}{P(Y_2 = 2)}$$

$$P(Y_1 = 1, Y_2 = 2) = \frac{\binom{4}{1}\binom{3}{2}\binom{2}{0}}{84} = \frac{4 \cdot 3 \cdot 1}{84} = \frac{12}{84} = \frac{1}{7}$$

$$P(Y_2 = 2) = \frac{\binom{3}{2}\binom{6}{1}}{84} = \frac{3 \cdot 6}{84} = \frac{18}{84} = \frac{3}{14}$$

$$P(Y_1 = 1|Y_2 = 2) = \frac{1/7}{3/14} = \frac{1}{7} \cdot \frac{14}{3} = \frac{2}{3}$$

□

(c) If we let Y_3 denote the number of divorced executives among the three selected for promotion, then $Y_3 = 3 - Y_1 - Y_2$. Find $P(Y_3 = 1|Y_2 = 1)$.

Solution: $Y_3 = 1$ and $Y_2 = 1$ means $Y_1 = 3 - 1 - 1 = 1$.

$$P(Y_3 = 1|Y_2 = 1) = P(Y_1 = 1|Y_2 = 1) = \frac{P(Y_1 = 1, Y_2 = 1)}{P(Y_2 = 1)}$$

$$P(Y_1 = 1, Y_2 = 1) = \frac{\binom{4}{1}\binom{3}{1}\binom{2}{1}}{84} = \frac{4 \cdot 3 \cdot 2}{84} = \frac{24}{84} = \frac{2}{7}$$

$$P(Y_2 = 1) = \frac{\binom{3}{1}\binom{6}{2}}{84} = \frac{3 \cdot 15}{84} = \frac{45}{84} = \frac{15}{28}$$

$$P(Y_3 = 1|Y_2 = 1) = \frac{2/7}{15/28} = \frac{2}{7} \cdot \frac{28}{15} = \frac{8}{15}$$

□

(d) Compare the marginal distribution derived in (a) with the hypergeometric distributions with $N = 9$, $n = 3$, and $r = 4$ encountered in Section 3.7.

Solution:

The marginal distribution of Y_1 is exactly hypergeometric$(N = 9, n = 3, r = 4)$'

□

Exercise (5.22). In Exercise 5.4, you were given the following joint probability function for

$$Y_1 = \begin{cases} 0, & \text{if child survived} \\ 1, & \text{if not} \end{cases} \quad \text{and} \quad Y_2 = \begin{cases} 0, & \text{if no belt used} \\ 1, & \text{if adult belt used} \\ 2, & \text{if car-seat belt used} \end{cases}$$

	y_1		
y_2	0	1	Total
0	.38	.17	.55
1	.14	.02	.16
2	.24	.05	.29
Total	.76	.24	1.00

(a) Give the marginal probability functions for Y_1 and Y_2.

Solution:

$$p_1(0) = 0.76, \quad p_1(1) = 0.24$$

$$p_2(0) = 0.55, \quad p_2(1) = 0.16, \quad p_2(2) = 0.29$$

□

(b) Give the conditional probability function for Y_2 given $Y_1 = 0$.

Solution:

$$p(y_2|y_1 = 0) = \frac{p(y_2, 0)}{p_1(0)}$$

$$\begin{aligned}
p(0|0) &= \frac{0.38}{0.76} = 0.5 \\
p(1|0) &= \frac{0.14}{0.76} = \frac{7}{38} \approx 0.184 \\
p(2|0) &= \frac{0.24}{0.76} = \frac{6}{19} \approx 0.316
\end{aligned}$$

$$p(y_2|0) = \begin{cases} 0.5, & y_2 = 0 \\ \frac{7}{38}, & y_2 = 1 \\ \frac{6}{19}, & y_2 = 2 \end{cases}$$

□

(c) What is the probability that a child survived given that he or she was in a car-seat belt?

Solution:

$$P(Y_1 = 0|Y_2 = 2) = \frac{P(Y_1 = 0, Y_2 = 2)}{P(Y_2 = 2)} = \frac{0.24}{0.29} = \frac{24}{29} \approx 0.828$$

□