

4.4.2

a) We are provided the second order differential equation $y'' - 9y' = 3x^2 - 5\sin(3x)$ and we are tasked with finding the form of the particular solution, by first inspection/observation we find the form to be:

$$\boxed{y_p = (Ax^2 + Bx + C) + D\cos(3x) + E\sin(3x)} \quad (1)$$

b) We are also provided with the second order differential equation $y'' + 2y' + y = 2e^{-x} - e^x$, by inspection/observation we conclude that the form of the particular solution is:

$$\boxed{y_p = Ae^{-x} + Be^x}$$

4.4.3

We are tasked with solving the following differential equations via method of undetermined coefficients:

a) $y'' + 2y' = 2x + 5 - e^{-2x}$

$$y'' + 2y' = 2x + 5 - e^{-2x}$$

$$m^2 + 2m + 0 = 0$$

$$m(m + 2) = 0$$

$$m = 0, -2$$

$$y_c = c_1 + c_2e^{-2x}$$

$$y_p = Ax^2 + Bx + Cxe^{-2x}$$

$$y'_p = 2Ax + B - 2Cxe^{-2x} + Ce^{-2x}$$

$$y''_p = 2A + 4Cxe^{-2x} - 4Ce^{-2x}$$

$$2A + 4Cxe^{-2x} - 4Ce^{-2x} + 4Ax + 2B - 4Cxe^{-2x} + 2Ce^{-2x} = 2x + 5 - e^{-2x}$$

$$4Ax + 2(A + B) + 2Ce^{-2x} = 2x + 5 - e^{-2x}$$

$$A = \frac{1}{2}, B = 2, C = \frac{1}{2}$$

Our solution is:

$$y_c + y_p = \boxed{c_1 + c_2e^{-2x} + \frac{1}{2}x^2 + 2x + \frac{1}{2}xe^{-2x}}$$

b) $y'' - 9y' = 2e^{3x}$

$$y'' - 9y' = 2e^{3x}$$

$$m^2 - 9m = 0$$

$$m(m - 9) = 0$$

$$m = 9, 0$$

$$y_c = c_1 + c_2e^{9x}$$

$$y_p = Ae^{3x}$$

$$y'_p = 3Ae^{3x}$$

$$y''_p = 9Ae^{3x}$$

$$9Ae^{3x} - 27Ae^{3x} = 2e^{3x}$$

$$-18Ae^{3x} = 2e^{3x}$$

$$A = -\frac{1}{9}$$

Our solution is then:

$$y_c + y_p = \boxed{c_1 + c_2 e^{9x} - \frac{1}{9} e^{3x}}$$

c) $y'' + 4y' + 4y = (3 + x)e^{-2x}$

$$\begin{aligned} y'' + 4y' + 4y &= (3 + x)e^{-2x} & y_p &= Ax^2e^{-2x} + Bx^3e^{-2x} \\ m^2 + 4m + 4 &= 0 & y'_p &= 2Axe^{-2x} - 2Ax^2e^{-2x} + 3Bx^2e^{-2x} - 2Bx^3e^{-2x} \\ (m + 2)(m + 2) &= 0 & y''_p &= 2Ae^{-2x} + 4Ae^{-2x} - 8Ae^{-2x} \\ m &= -2 & &+ 4Bx^3e^{-2x} - 12Bx^2e^{-2x} + 6Bxe^{-2x} \\ y_c &= c_1e^{-2x} + c_2xe^{-2x} \end{aligned}$$

Here is the nasty after plugging in:

$$\begin{aligned} &2Ae^{-2x} + 4Ae^{-2x} - 8Ae^{-2x} + 4Bx^3e^{-2x} - 12Bx^2e^{-2x} + 6Bxe^{-2x} \\ &+ 8Axe^{-2x} - 8Ax^2e^{-2x} + 12Bx^2e^{-2x} - 8Bx^3e^{-2x} \\ &+ Ax^2e^{-2x} + Bx^3e^{-2x} \\ &= 3e^{-2x} + xe^{-2x} \end{aligned}$$

Simplifies beautifully into:

$$\begin{aligned} 2Ae^{-2x} + 6Bxe^{-2x} &= 3e^{-2x} + xe^{-2x} \\ A = \frac{3}{2}, B &= \frac{1}{6} \end{aligned}$$

Our solution is then :

$$y_c + y_p = \boxed{c_1 e^{-2x} + c_2 x e^{-2x} + \frac{3}{2} x^2 e^{-2x} + \frac{1}{6} x^3 e^{-2x}}$$

4.6.1

We proceed to solve the problem $3y'' + y' = 9x$ using variation of parameters where our $y_p = u_1 y_1 + u_2 y_2$.

$$\begin{aligned}
 y'' + \frac{y'}{3} &= 3x & W &= \begin{vmatrix} 1 & e^{-\frac{x}{3}} \\ 0 & -\frac{1}{3}e^{-\frac{x}{3}} \end{vmatrix} = -\frac{1}{3}e^{-\frac{x}{3}} \\
 m^2 + \frac{m}{3} &= 0 & W_1 &= \begin{vmatrix} 0 & e^{-\frac{x}{3}} \\ 3x & -\frac{1}{3}e^{-\frac{x}{3}} \end{vmatrix} = -3xe^{-\frac{x}{3}} \\
 m(m + \frac{1}{3}) &= 0 & W_2 &= \begin{vmatrix} 1 & 0 \\ 0 & 3x \end{vmatrix} = 3x \\
 m = 0, m = -\frac{1}{3} & & u_1 &= \int \frac{-3xe^{-\frac{x}{3}}}{-\frac{1}{3}e^{-\frac{x}{3}}} = \frac{9x^2}{2} \\
 y_c = c_1 + c_2e^{-\frac{x}{3}} & & u_2 &= \int \frac{3x}{-\frac{1}{3}e^{-\frac{x}{3}}} = -27e^{\frac{x}{3}}(x - 3)
 \end{aligned}$$

Our y_p should be $u_1 * 1 + u_2 * e^{-\frac{x}{3}}$. We proceed to plug in for u_1 and u_2 , I'm assuming the 81 below gets absorbed into c_1 since its a constant too, our answer should be:

$$y_c + y_p = c_1 + c_2e^{-\frac{x}{3}} + \frac{9x^2}{3} - 27x + 81$$

$$c_1 + c_2e^{-\frac{x}{3}} + \frac{9x^2}{3} - 27x$$

4.6.2

We solve for the initial value problem $y'' - 4y' + 4y = (6x^2 - 12x)e^{2x}$, $y(0) = 1$, $y'(0) = 0$. Gonna skip the bits with finding our complimentary solution since the roots to $m^2 - 4m + 4$ is 2, $y_c = c_1e^{2x} + c_2xe^{2x}$

$$y_c = c_1e^{2x} + c_2xe^{2x}$$

$$y_p = u_1e^{2x} + u_2xe^{2x}$$

Now lets find u_1 and u_2 :

$$W_1 = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & 2xe^{2x} + e^{2x} \end{vmatrix} = (2xe^{4x} + e^{4x}) - 2xe^{4x} = e^{4x} \quad (1)$$

$$W_2 = \begin{vmatrix} 0 & xe^{2x} \\ (6x^2 - 12x)e^{2x} & 2xe^{2x} + e^{2x} \end{vmatrix} = -xe^{4x}(6x^2 - 12x) \quad (2)$$

$$W_3 = \begin{vmatrix} e^{2x} & 0 \\ 2e^{2x} & (6x^2 - 12x)e^{2x} \end{vmatrix} = e^{4x}(6x^2 - 12x) \quad (3)$$

$$u_1 = \int \frac{-xe^{4x}(6x^2 - 12x)}{e^{4x}} dx = \int (-6x^3 + 12x^2) dx = -\frac{3}{2}x^4 + 4x^3 \quad (4)$$

$$u_2 = \int \frac{e^{4x}(6x^2 - 12x)}{e^{4x}} dx = \int (6x^2 - 12x) dx = 2x^3 - 6x^2 \quad (5)$$

$$y_c + y_p = c_1e^{2x} + c_2xe^{2x} + (-\frac{3}{2}x^4 + 4x^3)e^{2x} + (2x^3 - 6x^2)xe^{2x} \quad (6)$$

$$= c_1e^{2x} + c_2xe^{2x} + \frac{1}{2}x^4e^{2x} + 2x^3e^{2x} \quad (7)$$

Now that we have our general solution, its time to find its derivative, plug in our initial conditions and solve the subsequent system of equations for c_1 and c_2

$$\begin{aligned}y &= c_1 e^{2x} + c_2 x e^{2x} + \frac{1}{2} x^4 e^{2x} + 2x^3 e^{2x} \\y' &= 2c_1 e^{2x} + c_2 (2x e^{2x} + e^{2x}) + \frac{1}{4} (4x^3 e^{2x} + 2x^4 e^{2x}) + 2(3x^2 e^{2x} + 2x^3 e^{2x}) \\y(0) &= c_1 = 1 \\y'(0) &= 2 + c_2 = 0, c_2 = -2\end{aligned}$$

Turned out not to have been much of a system since plugging in gave us the solution thankfully(or I might be a complete idiot). Our solution is:

$$y = e^{2x} - 2x e^{2x} + \frac{1}{2} x^4 e^{2x} + 2x^3 e^{2x}$$

4.7.1

We are tasked with solving the following differential equations:

a) $4x^2 y'' + y = 0$

$$\begin{aligned}y &= x^m \\y' &= m x^{m-1} \\y'' &= m(m-1) x^{m-2}\end{aligned}$$