Chapter 6

Exercise (2). Suppose $n \in \mathbb{Z}$. If n^2 is odd, then n is odd.

Proof: Suppose for the sake of contradiction that n^2 is odd and n is not odd, Then n is even, so $n = 2k$ for some $k \in \mathbb{Z}$. Therefore $n^2 = (2k)^2 = 4k^2 = 2(2k^2) = 2b$, where $b \in \mathbb{Z}$ by closure properties of the integers. So n^2 is even, this is a contradiction. So it must be the case that if n^2 is odd then n is odd. □

Exercise (3). Prove that $\sqrt[3]{2}$ is irrational.

Proof: Suppose for the sake of contradiction that $\sqrt[3]{2}$ is not irrational. Then $\sqrt[3]{2}$ is a rational number which can be expressed in the form $\sqrt[3]{2} = \frac{a}{b}$ where $a, b \in \mathbb{Z}$ and a, b do not sure factors so that $\frac{a}{b}$ is in its lowest form. Observe that when cubing both sides, $2 = (\frac{a}{b})^3 = \frac{a^3}{b^3}$ so that $2b^3 = a^3$. The integer a^3 being an even number implies that a is an even number and divisible by 2. So $a = 2k$ for some $k \in \mathbb{Z}$. Substituting for a gives $2b^3 = (2k)^3 = 8k^3$. Thus $b^3 = 4k^3 = 2(2k^3)$. Because b^3 is even, this implies that b is also an even number which is contraditory to $\frac{a}{b}$ existing in its lowest forms. Thus $\sqrt[3]{2}$ must be irrational. □

Exercise (4). Prove that $\sqrt{6}$ is irrational.

Proof: Suppose for the sake of contradiction $\sqrt{6}$ is a rational number. Then there exists an $a, b \in \mathbb{Z}$ such that $\sqrt{6} = \frac{a}{b}$ and $\frac{a}{b}$ is in its lowest form with no common factors. Squaring both sides gives $6 = (\frac{a}{b})^2 = \frac{a^2}{b^2}$. Thus $6b^2 = a^2$ which implies that a^2 is divisible by 6 and thus a is divisible by 6 or $a = 6k$ for some $k \in \mathbb{Z}$. Substituting in our original equation gives $6b^2 = (6k)^2 = 36k^2$. Simplifying this equation by division of 6 shows that $b^2 = 6k^2$, this implies that b^2 is divisible by 6. Since it follows that b is also divisible by 6 and that a is divisible by 6 we have a contradiction as $\frac{a}{b}$ cannot be in its lowest form. Thus $\sqrt{6}$ is an irrational number. □

Exercise (8). Suppose $a, b, c \in \mathbb{Z}$. If $a^2 + b^2 = c^2$, then a or b is even.

Proof: For the sake of contradiction, suppose that $a, b, c \in \mathbb{Z}$ such that $a^2 = b^2 = c^2$ and a and b are odd. Then there exists an $k, j \in \mathbb{Z}$ such that $a = 2k + 1$ and $b = 2j + 1$.

We know that an odd number added to another odd number is an even number, so that c^2 is even and thus c is also even, so $c = 2x$ for some $x \in \mathbb{Z}$. Observe that when we substitute a, b, c on both sides of the equations, $a^2 + b^2 = (2k+1)^2 + (2j+1)^2 = 4k^2 + 4k + 1 + 4j^2 + 4j + 1 = 4k^2 + 4j^2 + 4k + 4j + 2 = 2(2k^2 + 2j^2 + 2k + 2j + 1)$ and $c^2 = (2x)^2 = 4x^2$, so that $2(k^2 + 2j^2 + 2k + 2j + 1) = 4x^2$. Dividing both sides by 2 gives $2k^2 + 2j^2 + 2k + 2j + 1 = 2x^2$. Note that the left hand side is odd and the right hand side is even. This is an impossibility since an odd number cannot equal an even number. Thus if $a^2 + b^2 = c^2$, then a or b must be even. $\square$

Exercise (9). Suppose $a, b \in \mathbb{R}$. If a is rational and ab is irrational, then b is irrational.

Proof: Suppose for the sake of contradiction that a is rational, ab is irrational, and b is not irrational. Then b is a rational number and there exists $c, d \in \mathbb{Z}$ such that $b = \frac{c}{d}$. Similarly there exists an $x, y \in \mathbb{Z}$ such that $a = \frac{x}{y}$ since a is rational. Observe that $ab = \frac{cx}{dy}$ and that cx and dy are integers by closure properties of the integers. Then ab is a rational number by definition which is a contradiction to our original premise that ab is irrational. Thus if a is rational and ab is irrational, then b must be irrational. $\square$

Exercise (11). There exist no integers a and b for which $18a + 6b = 1$.

Proof: Suppose for the sake of contradiction that there does exist integers a and b for which $18a + 6b = 1$. Then $2(9a + 3b) = 1$ which means 1 is even, a contradiction. Thus there exists no $a, b \in \mathbb{Z}$ that satisfies $18a + 6b = 1$. $\square$

Exercise (12). For every positive $x \in \mathbb{Q}$, there is a positive $y \in \mathbb{Q}$ for which $y < x$.

Proof: Suppose for the sake of contradiction that there exists a positive $x \in \mathbb{Q}$, such that for all positive $y \in \mathbb{Q}$ that $y \geq x$. Lets consider the possibility that $y = \frac{x}{2}$, then $y = \frac{x}{2} \geq x$. This is a contradiction because obviously $x > \frac{x}{2}$. $\square$

Exercise (16). If a and b are positive real numbers, then $a + b \geq 2\sqrt{ab}$.

Proof: Suppose for the sake of contradiction that a and b are positive real numbers and that $a + b \geq 2\sqrt{ab}$ is false. That is to say $a + b < 2\sqrt{ab}$. Squaring both sides gives

$(a+b)^2 = a^2+2ab+b^2 < (2\sqrt{ab})^2 = 4ab$. Subtracting both sides of the inequality by $4ab$ gives $a^2 - ab^2 + b^2 = (a-b)^2 < 0$. This is a contradiction since any real number squared must be greater than or equal to 0, $(a-b)^2$ cannot be less than 0. □

Exercise (19). The product of any five consecutive integers is divisible by 120. (For example, the product of 3, 4, 5, 6 and 7 is 2520, and $2520 = 120 \cdot 21$.)

Proof: Suppose we have a product of 5 consecutive integers, this product of consecutive integers may be expressed as $n(n-1)(n-2)(n-3)(n-4)$ for some $n \in \mathbb{Z}$. Observe that $\binom{n}{5}$ is an integer and that $\binom{n}{5} = \frac{n!}{5!(n-5)!} = \frac{n!}{120(n-5)!} = \frac{n(n-1)(n-2)(n-3)(n-4)}{120}$. Thus 120 divides our product of 5 consecutive integers. □

Chapter 7

Exercise (1). Suppose $x \in \mathbb{Z}$. Then x is even if and only if $3x+5$ is odd.

Proof: Suppose that x is even, then $x = 2k$ for some $k \in \mathbb{Z}$. Substituting for x gives, $3x+5 = 3(2k)+5 = 6k+5 = 2(3k+2)+1$, an odd number. Showing that if x is even then $3x+5$ is odd. Conversely using contraposition, suppose that x is not even. Then x is odd and there exists a $k \in \mathbb{Z}$ such that $x = 2k+1$. Substituting for x gives $3x+5 = 3(2k+1)+5 = 6k+3+5 = 6k+8 = 2(3k+4)$, an even number. Showing that if x is odd then $3x+5$ is even. By contraposition it must be the case that if $3x+5$ is odd then x is even. □

Exercise (4). Let a be an integer. Then a^2+4a+5 is odd if and only if a is even.

Proof: Lets suppose that a^2+4a+5 is odd, then $a^2+4a+5 = 2k+1$ for some $k \in \mathbb{Z}$. When we isolate a^2 we get $a^2 = 2k-4a-4 = 2(k-2a-2)$, so a^2 is even. This implies that a is even. Thus if a^2+4a+5 is odd then a is even. Conversely if we suppose by contraposition that a is odd, then there exists a $k \in \mathbb{Z}$ such that $a = 2k+1$. Thus $a^2+4a+5 = (2k+1)^2+4(2k+1)+5 = 4k^2+4k+1+8k+4+5 = 4k^2+12k+9 = 2(2k^2+6k+5)$, an even number. Since when a is odd, a^2+4a+5 is even, it must follow by contraposition that if a^2+4a+5 is odd then a is even. □

Exercise (7). Suppose $x, y \in \mathbb{R}$. Then $(x+y)^2 = x^2+y^2$ if and only if $x = 0$ or $y = 0$.

Proof: Suppose that $(x+y)^2 = x^2 + y^2$. Expanding out gives $x^2 + 2xy + y^2 = x^2 + y^2$, so $2xy = 0$. It follows that $xy = 0$. Thus either $x = 0$ or $y = 0$. Conversely lets suppose $x = 0$ or $y = 0$, then we have 2 cases to show.

Case 1: Suppose $x = 0$, then $(x+y)^2 = 0 + y^2 = y^2$ and $x^2 + y^2 = 0 + y^2 = y^2$. So $(x+y)^2 = x^2 + y^2$ holds.

Case 2: Suppose $y = 0$, then $(x+y)^2 = x^2 + 0 = x^2$ and $x^2 + y^2 = x^2 + 0 = x^2$. So $(x+y)^2 = x^2 + y^2$ holds for this case as well.

In all cases where either $x = 0$ or $y = 0$, the equation $(x+y)^2 = x^2 + y^2$ holds true. □

Exercise (Reflection Problem). *Proof:* Length of time: This too me significantly less time than the previous homework, 1 to 2 minutes on some being able to quickly deduce strategy and what moves to make. Its often the case that more time was spent figuring out how I was going to write it down rather than finding an argument itself.

Difficulty: Again significantlly easier time than the previous homeworks, I would say the most challenging problem was actually problem 19 in chapter 6. I was not able to provide an argument without some assistance from the back of the book. My initial strategy was to multiply out $n(n+1)(n+2)(n+3)(n+4)$ but it did not give me anything oobvious I could work with. I even attempted to express consecutive integers as $n(n-1)(n+1)(n-2)(n+2)$ but ran into a similar wall.

Challenges:Besides problem 19, I had challenges in balancing the time, but I am proud to say I've caught up after falling behind.

Comparison: Again, aped Problem 19, my proofs involving irrationality I questioned after looking at the back. But I've come around to sticking with what I wrote. □

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