Chapter 10

Prove the following statements with either induction, strong induction or proof by smallest counterexample.

Exercise (3). Prove that $1^3 + 2^3 + 3^3 + 4^3 + \cdots n^3 = \frac{n^2(n+1)^2}{4}$ for every positive integer n.

Proof: (Weak Induction)

Base Case: Observe that when $n = 1$ that $n^3 = (1)^3 = \frac{(1)^2((1)+1)^2}{4} = \frac{4}{4} = 1$ which is true.

Induction Hypothesis: Suppose there is a $k \in \mathbb{Z}$ such that $1^3 + 2^3 + 3^3 + 4^3 + \cdots k^3 = \frac{k^2(k+1)^2}{4}$.

Inductive Step: We wish to show that the statement holds for $n = k+1$, i.e., that $1^3 + 2^3 + 3^3 + 4^3 + \cdots k^3 + (k+1)^3 = \frac{(k+1)^2((k+1)+1)^2}{4}$. Observe the following:

$$\begin{aligned}
1^3 + 2^3 + 3^3 + 4^3 + \cdots k^3 + (k+1)^3 &= [1^3 + 2^3 + 3^3 + 4^3 + \cdots k^3] + (k+1)^3 \\
&= \frac{k^2(k+1)^2}{4} + (k+1)^3 \\
&= \frac{k^2(k+1)^2}{4} + \frac{4(k+1)^3}{4} \\
&= \frac{k^2(k+1)^2 + 4(k+1)^3}{4} \\
&= \frac{(k+1)^2(k^2 + 4(k+1))}{4} \\
&= \frac{(k+1)^2(k^2 + 4k + 4))}{4} \\
&= \frac{(k+1)^2(k+2)^2}{4} \\
&= \frac{(k+1)^2((k+1)+1)^2}{4}.
\end{aligned}$$

Showing that the statement holds for $n = k+1$.

Conclusion: Therefore, by induction on n, the statement $1^3 + 2^3 + 3^3 + 4^3 + \cdots n^3 = \frac{n^2(n+1)^2}{4}$ is true for every positive integer $n \geq 1$. $\square$

Exercise (4). If $n \in \mathbb{N}$, then $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}$.

Proof: Base Case: Observe that when $n = 1$ that $\left[n(n+1) = \frac{n(n+1)(n+2)}{3}\right] = \left[(1)((1)+1) = \frac{(1)((1)+1)((1)+2)}{3}\right]$ $\left[(1)(2) = \frac{6}{3}\right] = 2$ is true.

Induction Hypothesis: Suppose for all all k with $1 \leq k < n$ that

$$1(2) + 2(3) + 3(4) + \cdots + k(k+1) = \frac{k(k+1)(k+2)}{3}.$$

In particular, suppose that $k = n - 1$ such that

$$1(2) + 2(3) + 3(4) + \cdots + n(n-1) = \frac{(n-1)(n)(n+1)}{3}$$

Induction Step: We need to show that $1(2)+2(3)+3(4)+...+n(n+1) = \frac{n(n+1)(n+2)}{3}$.

Observe that

$$\begin{aligned}
1(2) + 2(3) + 3(4) + \cdots + n(n+1) &= 1(2) + 2(3) + 3(4) + \cdots + (n-1)(n) + n(n+1) \\
&= \Big(1(2) + 2(3) + 3(4) + \cdots + (n-1)n\Big) + n(n+1) \\
&= \frac{(n-1)(n)(n+1)}{3} + n(n+1) \\
&= \frac{(n-1)(n)(n+1)}{3} + \frac{3n(n+1)}{3} \\
&= \frac{(n-1)(n)(n+1) + 3n(n+1)}{3} \\
&= \frac{(n(n+1))((n-1)+3)}{3} \\
&= \frac{n(n+1)(n+2)}{3}.
\end{aligned}$$

Conclusion: Therefore, by principle of mathematical induction, $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}$ is true for all $n \in \mathbb{N}$.

(Note, this one uses the induction extras problem as a skeleton.) $\square$

Exercise (5). If $n \in \mathbb{N}$, then $2^1 + 2^2 + 2^3 + \cdots + 2^n = 2^{n+1} - 2$.

Proof: Let $P(n)$ be the statement $2^1 + 2^2 + 2^3 + \cdots + 2^n = 2^{n+1} - 2$. We will demonstrate that the left hand side is equal to the right hand side.

Base Case: When $n = 1$, $P(n) = 2^{(1)} = 2^{(1)+1} - 2 = 4 - 2 = 2$. So $P(1)$ holds.

Induction Hypothesis: Suppose for all $k \in \mathbb{N}$ and $n = k \geq 1$ that $P(k)$ is true. That means that $2^1 + 2^2 + 2^3 + \cdots + 2^k = 2^{k+1} - 2$. We want to show that $P(k+1)$ holds, that is that $2^1 + 2^2 + 2^3 + \cdots + 2^{(k+1)} = 2^{(k+1)+1} - 2$.

Induction Step: Observe that when $n = k + 1$ that

$$\begin{aligned} P(n) &= 2^1 + 2^2 + 2^3 + \cdots + 2^{(k+1)} \\ &= 2^1 + 2^2 + 2^3 + \cdots + 2^k + 2^{k+1} \\ &= \left(2^1 + 2^2 + 2^3 + \cdots + 2^k\right) + 2^{k+1} \\ &= 2^{k+1} - 2 + 2^{k+1} \\ &= 2(2^{k+1}) - 2 \\ &= 2^{k+2} - 2 \\ &= 2^{(k+1)+1} - 2. \end{aligned}$$

Conclusion: Thus we have $2^1 + 2^2 + 2^3 + \cdots + 2^k + 2^{k+1} = 2^{(k+1)+1} - 2$. Hence the statement is true for $n = k + 1$, by mathematical induction $P(n)$ is true for all $n \in \mathbb{N}$. $\square$

Exercise (8). If $n \in \mathbb{N}$, then $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}$.

Proof: Let $P(n)$ be the statement $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}$

Base Case: Observe that when $n = 1$, that $P(n) = \frac{1}{((1)+1)!} = \frac{1}{2!} = \frac{1}{2} = 1 - \frac{1}{((1)+1!)}$. So $P(1)$ is true.

Induction Hypothesis: Suppose that for some $n = k \geq 1$, where $k \in \mathbb{N}$ that $P(k)$ is correct. That is to say $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{k}{(k+1)!} = 1 - \frac{1}{(k+1)!}$. We want to show that $P(k+1)$ holds.

Inductive step: Observe that when $n = k+1$ that

$$\begin{aligned}
P(n) &= \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{(k+1)}{((k+1)+1)!} \\
&= \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{k}{(k+1)!} + \frac{(k+1)}{((k+1)+1)!} \\
&= \left(\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{k}{(k+1)!}\right) + \frac{(k+1)}{((k+1)+1)!} \\
&= 1 - \frac{1}{(k+1)!} + \frac{(k+1)}{((k+1)+1)!} \\
&= 1 - \frac{((k+1)+1)}{((k+1)+1)!} + \frac{(k+1)}{((k+1)+1)!} \\
&= 1 - \frac{1}{((k+1)+1)!}.
\end{aligned}$$

Conclusion: Thus by induction we have shown $P(n) = 1 - \frac{1}{(n+1)!}$ is true for all $n \in \mathbb{N}$. $\square$

Exercise (10). Prove that $3 \mid (5^{2n} - 1)$ for every integer $n \geq 0$.

Proof: We will prove via induction on n.

Base Case: Consider the case where $n = 0$. Observe that $(5^{2n} - 1) = (5^{2(0)} - 1) = (5^0 - 1) = (1 - 1) = 0$. So we have $3 \mid 0$ which is true.

Induction Hypothesis: Now suppose the statement is true for some $n = k \geq 0$, that is to say $3 \mid (5^{2k} - 1)$. This means $5^{2k} - 1 = 3a$ for some $a \in \mathbb{Z}$. From this we get $52k = 3a + 1$

Inductive Step: Observe that

$$\begin{aligned}
5^{2(k+1)} - 1 &= 5^{2k+2} - 1 \\
&= 5^{2k}5^2 - 1 \\
&= 5^2(24a + 1) - 1 \\
&= 25(24a + 1) - 1 \\
&= 25(24a) + 25 - 1 \\
&= 25(24a) + 24 \\
&= 3(25(8a) + 8).
\end{aligned}$$

Conclusion: This shows that $5^{2(k+1)} - 1 = 3(25(8a)+8)$, which means $3 \mid (5^{2(k+1)} - 1)$. Thus by induction, $3 \mid (5^{2n} - 1)$ for all $n \in \mathbb{Z}$. $\square$

Exercise (13). Prove that $6 \mid (n^3 - n)$ for every integer $n \geq 0$.

Proof: Base Case: Consider the case where $n = 0$. Observe that $6 \mid (n^3 - n) = 6 \mid ((0)^3 - (0)) = 6 \mid 0$ which is true.

Induction Hypothesis: Assume the statement is true for $n = k \geq 0$. That is to say that $6 \mid (k^3 - k)$. This means that $k^3 - k = 6a$ for some $a \in \mathbb{Z}$. We want to show the statement is true for $n = k + 1$, that is to say $6 \mid ((k + 1)^3 - (k + 1))$.

Induction Step: Observe that

$$\begin{aligned}
(k + 1)^3 - (k + 1) &= k^3 + 3k^2 + 3k + 1 - k - 1 \\
&= (k^3 - k) + 3k^2 + 3k \\
&= 6a + 3k^2 + 3k \\
&= 6a + 3k(k + 1).
\end{aligned}$$

Conclusion: Since one of k or $(k+1)$ must be even, it follows that $k(k+1)$ is even. Thus $k(k+1) = 2b$ for some $b \in \mathbb{Z}$. So $(k+1)^3-(k+1) = 6a+3k(k+1) = 6a+3*2b = 6(a+b)$.

Therefore $6 \mid ((k+1)^3 - (k+1))$. Thus showing that $6 \mid (n^3 - n)$ for all integers $n \geq 0$. $\square$

Exercise (18). Suppose $A_1, A_2, \ldots, A_n$ are sets in some universal set U, and $n \geq 2$. Prove that $\overline{A_1 \cup A_2 \cup \cdots A_n} = \overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n}$.

Proof: Base Case: Consider the case where $n = 2$. Observe that

$$\begin{aligned}
\overline{A_1 \cup A_2} &= \{x : (x \in U) \wedge (x \notin A_1 \cup A_2)\} \\
&= \{x : (x \in U) \wedge \neg(x \in A_1 \cup A_2)\} \\
&= \{x : (x \in U) \wedge \neg((x \in A_1) \vee (x \in A_2))\} \\
&= \{x : (x \in U) \wedge (\neg(x \in A_1) \wedge \neg(x \in A_2))\} \\
&= \{x : (x \in U) \wedge ((x \notin A_1) \wedge (x \notin A_2))\} \\
&= \{x : (x \in U) \wedge (x \notin A_1) \wedge (x \in U) \wedge (x \notin A_2)\} \\
&= \{x : ((x \in U) \wedge (x \notin A_1))\} \cap \{x : ((x \in U) \wedge (x \notin A_2))\} \\
&= \overline{A_1} \cap \overline{A_2}.
\end{aligned}$$

So $\overline{A_1 \cup A_2} = \overline{A_1} \cap \overline{A_2}$.

Induction Hypothesis: Suppose the statement is true for $2 \leq k < n$ so that

$\overline{A_1 \cup A_2 \cup \cdots A_{k-1} \cup A_k} = \overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_{k-1}} \cap \overline{A_k}$.

Induction Step: Then

$$\begin{aligned}
\overline{A_1 \cup A_2 \cup \cdots A_{k-1} \cup A_k \cup A_{k+1}} &= \overline{(A_1 \cup A_2 \cup \cdots A_{k-1}) \cup A_k \cup A_{k+1}} \\
&= \overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_{k-1}} \cap \overline{A_k \cup A_{k+1}} \\
&= \overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_{k-1}} \cap \overline{A_k} \cap \overline{A_{k+1}}.
\end{aligned}$$

Conclusion: Since the statement is true for $k+1$ sets, we have shown by induction that the statement is true for all $n \geq 2$. $\square$

Exercise (19). Prove that $\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} \leq 2 - \frac{1}{n}$ for every $n \in \mathbb{N}$.

Proof: Let $P(n)$ be the statement $\dfrac{1}{1}+\dfrac{1}{4}+\dfrac{1}{9}+\cdots+\dfrac{1}{n^2}\leq 2-\dfrac{1}{n}$ for every $n\in\mathbb{N}$.

Base Case: For our $P(n)$, observe that for $n=1$ that $\frac{1}{(1)}\leq 2-\frac{1}{(1)}$ is true.

Induction Hypthesis: Suppose our statement is true for some $n\geq 1$. We will assume that $\dfrac{1}{1}+\dfrac{1}{4}+\dfrac{1}{9}+\cdots+\dfrac{1}{n^2}\leq 2-\dfrac{1}{n}$ for every $n\in\mathbb{N}$. We want to show that our statement is true for $P(n+1)$.

Induction Step: Observe that

$$\begin{aligned}
\frac{1}{1}+\frac{1}{4}+\frac{1}{9}+\cdots+\frac{1}{(n+1)^2} &\leq 2-\frac{1}{(n+1)}\\
\frac{1}{1}+\frac{1}{4}+\frac{1}{9}+\cdots+\frac{1}{n^2}+\frac{1}{(n+1)^2} &\leq 2-\frac{(n+1)}{(n+1)^2}\\
(\frac{1}{1}+\frac{1}{4}+\frac{1}{9}+\cdots+\frac{1}{n^2})+\frac{1}{(n+1)^2} &\leq 2-\frac{n^2+n}{n(n+1)^2}\\
2-\frac{1}{n}+\frac{1}{(n+1)^2} &\leq\\
2-\frac{(n+2)^2-n}{n(n+1)^2} &\leq\\
2-\frac{n^2+n+1}{n(n+1)^2} &\leq
\end{aligned}$$

So the statement holds for $P(n+1)$ as $2-\frac{n^2+n+1}{n(n+1)^2}\leq 2-\frac{n^2+n}{n(n+1)^2}=2-\frac{1}{(n+1)}$.

Conclusion: Thus $P(n)$ holds for every $n\in\mathbb{N}$, concluding our proof by induction. □

Exercise (22)*.* If $n\in\mathbb{N}$, then

$$\left(1-\frac{1}{2}\right)\left(1-\frac{1}{4}\right)\left(1-\frac{1}{8}\right)\left(1-\frac{1}{16}\right)\cdots\left(1-\frac{1}{2^n}\right)\geq\frac{1}{4}+\frac{1}{2^{n+1}}.$$

Proof: Let $P(n)$ be the statement above as presented. Base Case: Consider that for $n=1$, we get $(1-\frac{1}{2})=\frac{1}{2}\geq\frac{1}{4}+\frac{1}{2^{((1)+1)}}=\frac{1}{2}$ which is true.

Induction Hypothesis: Let $n\geq 1$ and assume that

$$\left(1-\frac{1}{2}\right)\left(1-\frac{1}{4}\right)\left(1-\frac{1}{8}\right)\left(1-\frac{1}{16}\right)\cdots\left(1-\frac{1}{2^n}\right)\geq\frac{1}{4}+\frac{1}{2^{n+1}}.$$

. We want to show that our statement is true, that

$$\left(1-\frac{1}{2}\right)\left(1-\frac{1}{4}\right)\left(1-\frac{1}{8}\right)\left(1-\frac{1}{16}\right)\cdots\left(1-\frac{1}{2^{(n+1)}}\right)\geq\frac{1}{4}+\frac{1}{2^{(n+1)+1}}.$$

Induction Step: Observe that $P(n+1)$ can be written as

$$\left[\left(1-\frac{1}{2}\right)\left(1-\frac{1}{4}\right)\left(1-\frac{1}{8}\right)\left(1-\frac{1}{16}\right)\cdots\left(1-\frac{1}{2^{n}}\right)\right]\left(1-\frac{1}{2^{n+1}}\right)\geq\left[\frac{1}{4}+\frac{1}{2^{n+1}}\right]\left(1-\frac{1}{2^{n+1}}\right)$$

Rearranging the right hand side gives

$$\begin{aligned}\left[\frac{1}{4}+\frac{1}{2^{n+1}}\right]\left(1-\frac{1}{2^{n+1}}\right)&=\frac{1}{4}+(\frac{1}{4})(\frac{1}{2^{n+1}})-\frac{1}{2^{n+1}}-(\frac{1}{2^{n+1}})(\frac{1}{2^{n+1}})\\&=\frac{1}{4}+\frac{1}{2^{n+1}}(1-\frac{1}{4}-\frac{1}{2^{n+1}})\\&=\frac{1}{4}+\frac{1}{2^{n+1}}(\frac{3}{4}-\frac{1}{2^{n+1}})\end{aligned}$$

Because $n\leq 1$, it follows that $2^{1+1}\leq 2^{n+1}$. Furthermore the inverse of this inequality gives $\frac{1}{2^{n+1}}\leq\frac{1}{2^{1+1}}=\frac{1}{4}$. So

$$\begin{aligned}\frac{1}{4}+\frac{1}{2^{n+1}}(\frac{3}{4}-\frac{1}{2^{n+1}})&\geq\frac{1}{4}+\frac{1}{2^{n+1}}(\frac{3}{4}-\frac{1}{4})\\&=\frac{1}{4}+\frac{1}{2^{n+1}}(\frac{1}{2})\\&=\frac{1}{4}+\frac{1}{2^{(n+1)+1}}\end{aligned}$$

Therefore $\left[\left(1-\frac{1}{2}\right)\left(1-\frac{1}{4}\right)\left(1-\frac{1}{8}\right)\left(1-\frac{1}{16}\right)\cdots\left(1-\frac{1}{2^{n}}\right)\right]\left(1-\frac{1}{2^{n+1}}\right)$
$\geq\left[\frac{1}{4}+\frac{1}{2^{n+1}}\right]\left(1-\frac{1}{2^{n+1}}\right)=\left[\frac{1}{4}+\frac{1}{2^{n+1}}(\frac{3}{4}-\frac{1}{2^{n+1}})\right]\geq\left[\frac{1}{4}+\frac{1}{2^{(n+1)+1}}\right].$

Conclusion: Hence by induction

$$\left(1-\frac{1}{2}\right)\left(1-\frac{1}{4}\right)\left(1-\frac{1}{8}\right)\left(1-\frac{1}{16}\right)\cdots\left(1-\frac{1}{2^{n}}\right)\geq\frac{1}{4}+\frac{1}{2^{n+1}}.$$

is true for all $n\in\mathbb{N}$. $\square$

Exercise (25). Concerning the Fibonacci sequence, prove that $F_1 + F_2 + F_3 + F_4 + \cdots + F_n = F_{n+2} - 1$ which is true.

Proof: Base Case: Consider the case when $n = 1$, then $F_1 = F_{(1)+2} - 1 = F_3 - 1 = 2 - 1 = 1$, which is true. Now consider the case where $n = 2$, then $F_1 + F_2 = F_{2+2} - 1 = F_4 - 1 = 3 - 1 = 2$, which is also true.

Induction Hypothesis: Suppose the statement is true for some $n > k \geq 1$, that $F_1 + F_2 + F_3 + F_4 + ... + F_k = F_{k+2} - 1$. We want to show that $F_1 + F_2 + F_3 + F_4 + ... + F_k + F_{k+1} = F_{k+3} - 1$.

Induction Step: Now observe the following

$$\begin{aligned} F_1 + F_2 + F_3 + F_4 + ... + F_k + F_{k+1} &= \\ (F_1 + F_2 + F_3 + F_4 + ... + F_k) + F_{k+1} &= \\ F_{k+2} - 1 + F_{k+1} &= (F_{k+1} + F_{k+2}) - 1 \\ &= F_{k+3} - 1. \end{aligned}$$

Conclusion: Thus we have shown that $F_1 + F_2 + F_3 + F_4 + ... + F_k + F_{k+1} = F_{k+3} - 1$, by induction $F_1 + F_2 + F_3 + F_4 + \cdots + F_n = F_{n+2} - 1$ is true for all $n \in \mathbb{Z}$ where $n \geq 1$. □

Exercise (30). Here F_n is the nth Fibonacci number. Prove that

$$F_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}.$$

Hint: There are multiple ways to do this... one is to use the fact that $a^{n-1} = \frac{a^n}{a}$, while others involve things like the fact if $\phi = \frac{1+\sqrt{5}}{2}$, then $\phi^2 - \phi - 1 = 0$.

Proof: Write your answer here. □

Exercise (33). Suppose n (infinitely long) straight lines lie on a plane in such a way that no two of the lines are parallel, and no three of the lines intersect in a single point. Show that this arrangement divides the plane into $\frac{n^2+n+2}{2}$ regions.

Proof: Write your answer here. □

Exercise (Reflection Problem).

- How long did it take you to complete each problem?

  Answer: □

- What was easy?

  Answer: □

- What was challenging? What made it challenging?

  Answer: □

- Compare your answers to the odd numbered exercises to those in the back of the textbook. What did you learn from this comparison?

  Answer: □

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