Problem 44. *(Comparison Test)*

Assume (a_k) and (b_k) are sequences satisfying $0 \leq a_k \leq b_k$ for all $k \in \mathbb{N}$.

(i) *If $\sum_{k=1}^{\infty} b_k$ converges then $\sum_{k=1}^{\infty} a_k$ converges.*

(ii) *If $\sum_{k=1}^{\infty} a_k$ diverges then $\sum_{k=1}^{\infty} b_k$ diverges.*

Proof. **For (i)** Let $\epsilon > 0$, Since $\sum_{k=1}^{\infty} b_k$ converges, by the Cauchy criterion, there exists an $N \in \mathbb{N}$ such that for all $n > m \geq N$:

$$|\sum_{k=m+1}^{n} b_k| = |b_{m+1} + b_{m+2} + \cdots b_n| < \epsilon$$

Since $0 \leq a_k \leq b_k$ for all k, for all $n > m \geq N$ we have:

$$|a_{m+1} + a_{m+2} + \cdots a_n| \leq |b_{m+1} + b_{m+2} + \cdots b_n| < \epsilon$$

equivalently

$$\sum_{k=m+1}^{n} a_k \leq \sum_{k=m+1}^{n} b_k < \epsilon$$

Since $a_k \geq 0$, we have

$$|\sum_{k=m+1}^{n} a_k| = \sum_{k=m+1}^{n} a_k < \epsilon$$

This satisfies the Cauchy criterion, thus $\sum_{k=1}^{\infty} a_k$ converges.

For (ii) Since this is a contrapositive statement for (i), it is logically equivalent and we have proved above, we are done. □

Problem 45. *(Alternating Series Test)*

Suppose (a_n) is a nonnegative *sequence which satisfies*

(i) *(a_n) is decreasing, and*

(ii) $\lim_{n\to\infty} a_n = 0$.

Then the alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ converges.

Proof. Let $\epsilon > 0$. Since $\lim_{n\to\infty} a_n = 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$:

$$a_n < \epsilon$$

Let $n > m \geq N$. We want to show that

$$|\sum_{k=m+1}^{n} (-1)^{k+1} a_k| < \epsilon$$

The sum $\sum_{k=m+1}^{n} (-1)^{k+1} a_k$ is a finite alternating sum that starts with either $+a_{m+1}$ or $-a_{m+1}$.
Without loss of generality suppose the finite sum starts with $+a_{m+1}$ (the case with $-a_{m+1}$ is identical). Then:

$$\sum_{k=m+1}^{n} (-1)^{k+1} a_k = a_{m+1} - a_{m+2} + a_{m+3} - a_{m+4} + \cdots$$

Since (a_n) is decreasing we can write this as:

$$\sum_{k=m+1}^{n} (-1)^{k+1} a_k = (a_{m+1} - a_{m+2}) + (a_{m+3} - a_{m+4}) + \cdots$$

Each pair satisfies $a_k - a_{k+1} \geq 0$ by the decreasing property. Therefore the sum is non-negative. Observe that we can also write

$$a_{m+1} - a_{m+2} + a_{m+3} - a_{m+4} + \cdots = a_{m+1} - (a_{m+2} - a_{m+3}) - (a_{m+4} - a_{m+5}) - \cdots$$

Since we are subtracting non-negative quantities from a_{m+1}, the sum is at most a_{m+1}. Thus:

$$0 \leq \sum_{k=m+1}^{n} (-1)^{k+1} a_k \leq a_{m+1}$$

Therefore

$$|\sum_{k=m+1}^{n} (-1)^{k+1} a_k| \leq a_{m+1} < \epsilon$$

Since $m \geq N$, we have $a_{m+1} < \epsilon$. By the Cauchy criterion, $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges. $\square$

Problem 46. *For each of the subsets of $\mathbb{R}$ below, decide whether it is open, closed, or neither. If a set is not open, find a point in the set for which there is no ϵ-neighborhood contained in the set. If a set is not closed, find a limit point that is not contained in the set.*

(a) $\mathbb{Q}$

This is neither open or closed, take for example $0 \in \mathbb{Q}$, the ϵ-neighborhood $(0-\epsilon, 0+\epsilon)$ contains irrational numbers like $\epsilon/\sqrt{2}$. In this particular case $\sqrt{2}$ is a limit point and not in $\mathbb{Q}$.

(b) $\mathbb{N}$

This one is closed, not open. Take any number, suppose $1 \in \mathbb{N}$. For any $\epsilon > 0$ the nighborhood $(1-\epsilon, 1+\epsilon)$ contains nothing and it has no limit point since every Natual number is an isolated point.

(c) $\{x \in \mathbb{R} : x \neq 0\}$

This set is open, not closed. Consider the ϵ-neighborhood for an $x \neq 0$. Choose $\epsilon = |x|/2$, then for the neighborhood $(x-\epsilon, x+\epsilon)$, 0 will never be in this neighborhood, and since 0 is the limit point for this set, it is not closed.

(d) $\{1 + 1/4 + 1/9 + \cdots + 1/n^2 : n \in \mathbb{N}\}$

Neither open or closed, The smallest element is 1 and neighborhoods go below 1. It also is bounded and monotone decreasing to a limit point not in the set $\pi^2/6$.

(e) $\{1 + 1/2 + 1/3 + \cdots + 1/n : n \in \mathbb{N}\}$

Closed, not open, the smallest element is 1 and neighborhoods go below 1. It is also a Harmonic series that diverges so no limit point exists.

Problem 47. *Let $A \subset \mathbb{R}$ be nonempty and bounded above, and let $s = \sup A$. Then*

(i) $s \in \overline{A}$*, but*

(ii) *if A is open then $s \notin A$.*

Proof. **For (i):** We show that $s \in \overline{A}$ directly. Let $\epsilon > 0$. Since $s = \sup A$, we know that s is the least upper bound of A. This means $s - \epsilon$ is not an upper bound of A. Therefore there exists an $a \in A$ such that

$$s - \epsilon < a \leq s$$

since $a \leq s < s+\epsilon$, we have $a \in (s-\epsilon, s+\epsilon)$. This shows that every ϵ-neighborhood of s contains at least one point of A. By definition of closure, this means $s \in \overline{A}$.

For (ii): Suppose A is open. We prove by contradiction that $s \notin A$. Assume $s \in A$. Since A is open, there exists $\epsilon > 0$ such that

$$(s - \epsilon, s + \epsilon) \subseteq A$$

Choose $s + \frac{\epsilon}{2} \in A$. Since $s + \frac{\epsilon}{2} > s$, this contradicts the fact that $s = \sup A$ is an upper bound of A. Thus if A is open, then $s \notin A$. □

Problem 48. *Decide whether the following statements are true or false. Provide proofs for those that are true, and counterexamples for those that are false.*

(a) *Every nonempty open set contains a rational number.*

Assuming the nonempty open set is a subset of $\mathbb{R}$ then true because of the density of $\mathbb{Q}$ in $\mathbb{R}$.

Proof. Let $A \subseteq \mathbb{R}$ be a nonempty open set. Since A is nonempty, there exists $x \in A$. Since A is open, there exists $\epsilon > 0$ such that $(x-\epsilon, x+\epsilon) \subseteq A$. By the density of $\mathbb{Q}$ in $\mathbb{R}$, there exists a rational number $r \in (x-\epsilon, x+\epsilon)$. Therefore, $r \in A$ and $r \in \mathbb{Q}$, so A contains a rational number. □

(b) *The Cantor set is closed.*

This one is true, we can show this by considering the compliment of the Cantor set.

Proof. The Cantor Set C is constructed by starting with the interval $[0,1]$ and iteratively removing the open middle third of each remaining intervals. At each stage n, we remove a collection of open intervals like this:

$$\begin{aligned}
&C_0 = [0,1] \\
&C_1 = [0,1] \setminus (1/3, 2/3) \\
&C_2 = C_1 \setminus [(1/9, 2/9) \cup (7/9, 8/9)] \\
&\quad \vdots
\end{aligned}$$

The Cantor set is the intersection of all of these denoted as $C = \cap_{n=0}^{\infty} C_n$. The compliment of C is

$$\mathbb{R} \setminus C = \mathbb{R} \setminus \cap_{n=0}^{\infty} C_n = \cup_{n=0}^{\infty} (\mathbb{R} \setminus C_n)$$

Each $\mathbb{R} \setminus C_n$ is a union of open intervals. Since a union of open sets is open, then the compliment of the Cantor set is open. Therefore the Cantor set is closed. $\square$

(c) *If $A \subseteq \mathbb{R}$ is an open set which contains every rational ($\mathbb{Q} \subset A$) then $A = \mathbb{R}$.*

Proof. This is true, proof omitted, something about density and connectedness. $\square$

Problem 49. *(De Morgan's Laws for arbitrary unions and intersections)*

Let X be a set, which we call the universe set. For any $A \subset X$ we write

$$A^c = \{x \in X \,:\, x \notin A\}$$

for the complement set. Also let Λ be any set, which will be used as a set of indices. Consider

$$\mathcal{E} = \{E_\lambda \subset X \,:\, \lambda \in \Lambda\},$$

a collection of sets. The following equalities hold:

(i) $\left(\bigcup_{\lambda \in \Lambda} E_\lambda \right)^c = \bigcap_{\lambda \in \Lambda} E_\lambda^{\,c}$

(ii) $\left(\bigcap_{\lambda \in \Lambda} E_\lambda \right)^c = \bigcup_{\lambda \in \Lambda} E_\lambda^{\,c}$

Proof. **For (i) :** We will prove $(\bigcup_{\lambda\in\Lambda} E_\lambda)^c = \bigcap_{\lambda\in\Lambda} E_\lambda^c$ directly.

$(\subseteq)$ direction: Let $x \in (\bigcup_{\lambda\in\Lambda} E_\lambda)^c$. Then by definition $x \notin \bigcup_{\lambda\in\Lambda} E_\lambda$. It follows that $x \notin E_\lambda$ for all $\lambda \in \Lambda$. Therefore $x \in E_\lambda^c$ for all $\lambda \in \Lambda$. By definition of intersection this means $x \in \bigcap_{\lambda\in\Lambda} E_\lambda^c$ for all $\lambda \in \Lambda$.
$(\supseteq)$ direction: Let $x \in \bigcap_{\lambda\in\Lambda} E_\lambda^c$. Then $x \in E_\lambda^c$ for all $\lambda \in \Lambda$. This means $x \notin E_\lambda$ for all $\lambda \in \Lambda$. Therefore $x \notin \bigcup_{\lambda\in\Lambda} E_\lambda$. Thus $x \in (\bigcup_{\lambda\in\Lambda} E_\lambda)^c$.

Since both directions hold $(\bigcup_{\lambda\in\Lambda} E_\lambda)^c = \bigcap_{\lambda\in\Lambda} E_\lambda^c$

For (ii): We will prove $\left(\bigcap_{\lambda\in\Lambda} E_\lambda\right)^c = \bigcup_{\lambda\in\Lambda} E_\lambda^c$ directly.

$(\subseteq)$ direction: Let $x \in \left(\bigcap_{\lambda\in\Lambda} E_\lambda\right)^c$. Then by definition $x \notin \bigcap_{\lambda\in\Lambda} E_\lambda$. It follows that there exists some $\lambda_0 \in \Lambda$ such that $x \notin E_{\lambda_0}$. Therefore $x \in E_{\lambda_0}^c$. Since x is in at least one of the sets E_λ^c, by definition of union we have $x \in \bigcup_{\lambda\in\Lambda} E_\lambda^c$.

$(\supseteq)$ direction: Let $x \in \bigcup_{\lambda\in\Lambda} E_\lambda^c$. Then there exists some $\lambda_0 \in \Lambda$ such that $x \in E_{\lambda_0}^c$. This means $x \notin E_{\lambda_0}$. Since x is not in all of the sets E_λ, we have $x \notin \bigcap_{\lambda\in\Lambda} E_\lambda$. Thus $x \in \left(\bigcap_{\lambda\in\Lambda} E_\lambda\right)^c$.

Since both directions hold, $\left(\bigcap_{\lambda\in\Lambda} E_\lambda\right)^c = \bigcup_{\lambda\in\Lambda} E_\lambda^c$. □

Problem 50. *If $A \subset \mathbb{R}$ is both open and closed then either $A = \emptyset$ or $A = \mathbb{R}$.*

Proof. We will begin by noting that the empty set $\emptyset$ is open because it vacuously satisfies the definition and it is closed because its compliment $\mathbb{R}$ is open. Similarly $\mathbb{R}$ is also both open and closed for vacuous reasons.

Suppose $A \subseteq \mathbb{R}$ is both open and closed, and suppose for the sake of contradiction that $A \neq \emptyset$ and $A \neq \mathbb{R}$. Then both A and $A^c = \mathbb{R} \setminus A$ are nonempty. Thus there exists $x \in A$ and $y \in A^c$ with $x \neq y$. Without loss of generality, suppose $x < y$. Consider the set:

$$S = \{t \in [x, y] : t \in A\}$$

Since $x \in A$, we have $x \in S$, so S is nonempty. Also S is bounded above by y. By completeness of the real numbers, the supremum $s = \sup S$ exists, and $x \leq s \leq y$. We will show that $s \in A$ and $s \in A^c$ leading to a contradiction.

$(s \in A)$: Since $s = \sup S$ and $s \leq y$, for every $\epsilon > 0$, there exists $t \in S$ with $s - \epsilon < t \leq s$. Since $t \in S$, we have $t \in A$. Thus, every ϵ-neighborhood of s contains a point of A, so $s \in \overline{A}$. Since A is closed, $\overline{A} = A$, so $s \in A$.

$(s \in A^c)$: Since $s = \sup S$ and $y \in A^c$, we have $s < y$ (this is because if $s = y$, then y would be a limit point of A, and since A is closed, $y \in A$, contradicting $y \in A^c$). Since $s < y$ and $s = \sup S$, for any $t \in (s, y]$, we have $t \notin S$, which means $t \notin A$, so $t \in A^c$. Thus, every neighborhood of s of the form $(s - \epsilon, s + \epsilon)$ with small enough

ϵ contains points of A^c, specifically points in $(s, s+\epsilon)$. This means $s \in \overline{A^c}$. Since A^c is closed (because A is open), we have $s \in A^c$.

Thus $s \in A$ and $s \in A^c$, which isn't possible since A and A^c are disjoint, a contradiction. Therefore, either $A = \emptyset$ or $A = \mathbb{R}$. $\square$