4.4.2

a) We are provided the second order differential equation $y'' - 9y' = 3x^2 - 5\sin(3x)$ and we are tasked with finding the form of the particular solution, by first inspection/observation we find the form to be:

$$\boxed{y_p = (Ax^2 + Bx + C) + D\cos(3x) + E\sin(3x)} \tag{1}$$

b) We are also provided with the second order differential equation $y'' + 2y' + y = 2e^{-x} - e^{x}$, by inspection/observation we conclude that the form of the particular solution is:

$$\boxed{y_p = Ae^{-x} + Be^{x}}$$

4.4.3

We are tasked with solving the following differential equations via method of undetermined coefficients:

a) $y'' + 2y' = 2x + 5 - e^{-2x}$

$y'' + 2y' = 2x + 5 - e^{-2x}$

$m^2 + 2m + 0 = 0$

$m(m+2) = 0$

$m = 0, -2$

$y_c = c_1 + c_2e^{-2x}$

$y_p = Ax^2 + Bx + Cxe^{-2x}$

$y_p' = 2Ax + B - 2Cxe^{-2x} + Ce^{-2x}$

$y_p'' = 2A + 4Cxe^{-2x} - 4Ce^{-2x}$

$$2A + 4Cxe^{-2x} - 4Ce^{-2x} + 4Ax + 2B - 4Cxe^{-2x} + 2Ce^{-2x} = 2x + 5 - e^{-2x}$$

$$4Ax + 2(A + B) + 2Ce^{-2x} = 2x + 5 - e^{-2x}$$

$$A = \frac{1}{2}, B = 2, C = \frac{1}{2}$$

Our solution is:

$$y_c + y_p = \boxed{c_1 + c_2e^{-2x} + \frac{1}{2}x^2 + 2x + \frac{1}{2}xe^{-2x}}$$

b) $y'' - 9y' = 2e^{3x}$

$y'' - 9y' = 2e^{3x}$

$m^2 - 9m = 0$

$m(m-9) = 0$

$m = 9, 0$

$y_c = c_1 + c_2e^{9x}$

$y_p = Ae^{3x}$

$y_p' = 3Ae^{3x}$

$y_p'' = 9Ae^{3x}$

$9Ae^{3x} - 27Ae^{3x} = 2e^{3x}$

$-18Ae^{3x} = 2e^{3x}$

$A = -\frac{1}{9}$

Our solution is then:

$$y_c + y_p = \boxed{c_1 + c_2 e^{9x} - \frac{1}{9}e^{3x}}$$

c) $y'' + 4y' + 4y = (3+x)e^{-2x}$

$$\begin{aligned}
&y'' + 4y' + 4y = (3+x)e^{-2x} && y_p = Ax^2e^{-2x} + Bx^3e^{-2x} \\
&m^2 + 4m + 4 = 0 && y_p' = 2Axe^{-2x} - 2Ax^2e^{-2x} + 3Bx^2e^{-2x} - 2Bx^3e^{-2x} \\
&(m+2)(m+2) = 0 && y_p'' = 2Ae^{-2x} + 4Ae^{-2x} - 8Ae^{-2x} \\
&m = -2 && \quad + 4Bx^3e^{-2x} - 12Bx^2e^{-2x} + 6Bxe^{-2x} \\
&y_c = c_1e^{-2x} + c_2xe^{-2x}
\end{aligned}$$

Why did you choose to torture me like this, here is the nasty after plugging in:

$$\begin{aligned}
&2Ae^{-2x} + 4Ae^{-2x} - 8Ae^{-2x} + 4Bx^3e^{-2x} - 12Bx^2e^{-2x} + 6Bxe^{-2x} \\
&+ 8Axe^{-2x} - 8Ax^2e^{-2x} + 12Bx^2e^{-2x} - 8Bx^3e^{-2x} \\
&+ Ax^2e^{-2x} + Bx^3e^{-2x} \\
&= 3e^{-2x} + xe^{-2x}
\end{aligned}$$

Simplifies beautifully into:

$$\begin{aligned}
&2Ae^{-2x} + 6Bxe^{-2x} = 3e^{-2x} + xe^{-2x} \\
&A = \frac{3}{2}, B = \frac{1}{6}
\end{aligned}$$

Our solution is then :

$$y_c + y_p = \boxed{c_1e^{-2x} + c_2xe^{-2x} + \frac{3}{2}x^2e^{-2x} + \frac{1}{6}x^3e^{-2x}}$$