Chapter 10

Prove the following statements with either induction, strong induction or proof by smallest counterexample.

Exercise (3). Prove that $1^3 + 2^3 + 3^3 + 4^3 + \cdots n^3 = \frac{n^2(n+1)^2}{4}$ for every positive integer n.

Proof: (Weak Induction)

Base Case: Observe that when $n = 1$ that $n^3 = (1)^3 = \frac{(1)^2((1)+1)^2}{4} = \frac{4}{4} = 1$ which is true.

Induction Hypothesis: Suppose there is a $k \in \mathbb{Z}$ such that $1^3 + 2^3 + 3^3 + 4^3 + \cdots k^3 = \frac{k^2(k+1)^2}{4}$.

Inductive Step: We wish to show that the statement holds for $n = k + 1$, i.e., that $1^3 + 2^3 + 3^3 + 4^3 + \cdots k^3 + (k+1)^3 = \frac{(k+1)^2((k+1)+1)^2}{4}$. Observe the following:

$$\begin{aligned}
1^3 + 2^3 + 3^3 + 4^3 + \cdots k^3 + (k+1)^3 &= [1^3 + 2^3 + 3^3 + 4^3 + \cdots k^3] + (k+1)^3 \\
&= \frac{k^2(k+1)^2}{4} + (k+1)^3 \\
&= \frac{k^2(k+1)^2}{4} + \frac{4(k+1)^3}{4} \\
&= \frac{k^2(k+1)^2 + 4(k+1)^3}{4} \\
&= \frac{(k+1)^2(k^2 + 4(k+1))}{4} \\
&= \frac{(k+1)^2(k^2 + 4k + 4))}{4} \\
&= \frac{(k+1)^2(k+2)^2}{4} \\
&= \frac{(k+1)^2((k+1)+1)^2}{4}.
\end{aligned}$$

Showing that the statement holds for $n = k + 1$.

Conclusion: Therefore, by induction on n, the statement $1^3 + 2^3 + 3^3 + 4^3 + \cdots n^3 = \frac{n^2(n+1)^2}{4}$ is true for every positive integer $n \geq 1$. □

Exercise (4). If $n \in \mathbb{N}$, then $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}$.

Proof: Base Case: Observe that when $n = 1$ that $\left[n(n+1) = \frac{n(n+1)(n+2)}{3}\right] = \left[(1)((1)+1) = \frac{(1)((1)+1)((1)+2)}{3}\right]$ $\left[(1)(2) = \frac{6}{3}\right] = 2$ is true.

Induction Hypothesis: Suppose for all all k with $1 \leq k < n$ that

$$1(2) + 2(3) + 3(4) + \cdots + k(k+1) = \frac{k(k+1)(k+2)}{3}.$$

In particular, suppose that $k = n-1$ such that

$$1(2) + 2(3) + 3(4) + \cdots + n(n-1) = \frac{(n-1)(n)(n+1)}{3}$$

Induction Step: We need to show that $1(2) + 2(3) + 3(4) + ... + n(n+1) = \frac{n(n+1)(n+2)}{3}$.

Observe that

$$\begin{aligned}
1(2) + 2(3) + 3(4) + \cdots + n(n+1) &= 1(2) + 2(3) + 3(4) + \cdots + (n-1)(n) + n(n+1) \\
&= \Big(1(2) + 2(3) + 3(4) + \cdots + (n-1)n\Big) + n(n+1) \\
&= \frac{(n-1)(n)(n+1)}{3} + n(n+1) \\
&= \frac{(n-1)(n)(n+1)}{3} + \frac{3n(n+1)}{3} \\
&= \frac{(n-1)(n)(n+1) + 3n(n+1)}{3} \\
&= \frac{(n(n+1))((n-1)+3)}{3} \\
&= \frac{n(n+1)(n+2)}{3}.
\end{aligned}$$

Conclusion: Therefore, by principle of mathematical induction, $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}$ is true for all $n \in \mathbb{N}$.

(Note, this one uses the induction extras problem as a skeleton.) $\square$

Exercise (5). If $n \in \mathbb{N}$, then $2^1 + 2^2 + 2^3 + \cdots + 2^n = 2^{n+1} - 2$.

Proof: Let $P(n)$ be the statement $2^1 + 2^2 + 2^3 + \cdots + 2^n = 2^{n+1} - 2$. We will demonstrate that the left hand side is equal to the right hand side.

Base Case: When $n = 1$, $P(n) = 2^{(1)} = 2^{(1)+1} - 2 = 4 - 2 = 2$. So $P(1)$ holds.

Induction Hypothesis: Suppose for all $k \in \mathbb{N}$ and $n = k \geq 1$ that $P(k)$ is true. That means that $2^1 + 2^2 + 2^3 + \cdots + 2^k = 2^{k+1} - 2$. We want to show that $P(k+1)$ holds, that is that $2^1 + 2^2 + 2^3 + \cdots + 2^{(k+1)} = 2^{(k+1)+1} - 2$.

Induction Step: Observe that when $n = k + 1$ that

$$\begin{aligned}
P(n) &= 2^1 + 2^2 + 2^3 + \cdots + 2^{(k+1)} \\
&= 2^1 + 2^2 + 2^3 + \cdots + 2^k + 2^{k+1} \\
&= \left(2^1 + 2^2 + 2^3 + \cdots + 2^k\right) + 2^{k+1} \\
&= 2^{k+1} - 2 + 2^{k+1} \\
&= 2(2^{k+1}) - 2 \\
&= 2^{k+2} - 2 \\
&= 2^{(k+1)+1} - 2.
\end{aligned}$$

Conclusion: Thus we have $2^1 + 2^2 + 2^3 + \cdots + 2^k + 2^{k+1} = 2^{(k+1)+1} - 2$. Hence the statement is true for $n = k + 1$, by mathematical induction $P(n)$ is true for all $n \in \mathbb{N}$. $\square$

Exercise (8). If $n \in \mathbb{N}$, then $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}$.

Proof: Let $P(n)$ be the statement $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}$

Base Case: Observe that when $n = 1$, that $P(n) = \frac{1}{((1)+1)!} = \frac{1}{2!} = \frac{1}{2} = 1 - \frac{1}{((1)+1!)}$. So $P(1)$ is true.

Induction Hypothesis: Suppose that for some $n = k \geq 1$, where $k \in \mathbb{N}$ that $P(k)$ is correct. That is to say $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{k}{(k+1)!} = 1 - \frac{1}{(k+1)!}$. We want to show that $P(k+1)$ holds.

Inductive step: Observe that when $n = k + 1$ that

$$\begin{aligned}
P(n) &= \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{(k+1)}{((k+1)+1)!} \\
&= \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{k}{(k+1)!} + \frac{(k+1)}{((k+1)+1)!} \\
&= \left(\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{k}{(k+1)!}\right) + \frac{(k+1)}{((k+1)+1)!} \\
&= 1 - \frac{1}{(k+1)!} + \frac{(k+1)}{((k+1)+1)!} \\
&= 1 - \frac{((k+1)+1)}{((k+1)+1)!} + \frac{(k+1)}{((k+1)+1)!} \\
&= 1 - \frac{1}{((k+1)+1)!}.
\end{aligned}$$

Conclusion: Thus by induction we have shown $P(n) = 1 - \frac{1}{(n+1)!}$ is true for all $n \in \mathbb{N}$. $\square$

Exercise (10). Prove that $3 \mid (5^{2n} - 1)$ for every integer $n \geq 0$.

Proof: We will prove via induction on n.

Base Case: Consider the case where $n = 0$. Observe that $(5^{2n} - 1) = (5^{2(0)} - 1) = (5^0 - 1) = (1 - 1) = 0$. So we have $3 \mid 0$ which is true.

Induction Hypothesis: Now suppose the statement is true for some $n = k \geq 0$, that is to say $3 \mid (5^{2k} - 1)$. This means $5^{2k} - 1 = 3a$ for some $a \in \mathbb{Z}$. From this we get $52k = 3a + 1$

Inductive Step: Observe that

$$\begin{aligned} 5^{2(k+1)} - 1 &= 5^{2k+2} - 1 \\ &= 5^{2k}5^2 - 1 \\ &= 5^2(24a+1) - 1 \\ &= 25(24a+1) - 1 \\ &= 25(24a) + 25 - 1 \\ &= 25(24a) + 24 \\ &= 3(25(8a) + 8). \end{aligned}$$

Conclusion: This shows that $5^{2(k+1)} - 1 = 3(25(8a)+8)$, which means $3 \mid (5^{2(k+1)} - 1)$. Thus by induction, $3 \mid (5^{2n} - 1)$ for all $n \in \mathbb{Z}$. $\square$

Exercise (13). Prove that $6 \mid (n^3 - n)$ for every integer $n \geq 0$.

Proof: Base Case: Consider the case where $n = 0$. Observe that $6 \mid (n^3 - n) = 6 \mid ((0)^3 - (0)) = 6 \mid 0$ which is true.

Induction Hypothesis: Assume the statement is true for $n = k \geq 0$. That is to say that $6 \mid (k^3 - k)$. This means that $k^3 - k = 6a$ for some $a \in \mathbb{Z}$. We want to show the statement is true for $n = k+1$, that is to say $6 \mid ((k+1)^3 - (k+1))$.

Induction Step: Observe that

$$\begin{aligned} (k+1)^3 - (k+1) &= k^3 + 3k^2 + 3k + 1 - k - 1 \\ &= (k^3 - k) + 3k^2 + 3k \\ &= 6a + 3k^2 + 3k \\ &= 6a + 3k(k+1). \end{aligned}$$

Conclusion: Since one of k or $(k+1)$ must be even, it follows that $k(k+1)$ is even. Thus $k(k+1) = 2b$ for some $b \in \mathbb{Z}$. So $(k+1)^3 - (k+1) = 6a + 3k(k+1) = 6a + 3*2b = 6(a+b)$.

Therefore $6 \mid ((k+1)^3 - (k+1))$. Thus showing that $6 \mid (n^3 - n)$ for all integers $n \geq 0$. $\square$

Exercise (18). Suppose $A_1, A_2, \ldots, A_n$ are sets in some universal set U, and $n \geq 2$. Prove that $\overline{A_1 \cup A_2 \cup \cdots A_n} = \overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n}$.

Proof: <u>Base Case:</u> Consider the case where $n = 2$. Observe that

$$\begin{aligned}
\overline{A_1 \cup A_2} &= \{x : (x \in U) \wedge (x \notin A_1 \cup A_2)\} \\
&= \{x : (x \in U) \wedge \neg(x \in A_1 \cup A_2)\} \\
&= \{x : (x \in U) \wedge \neg((x \in A_1) \vee (x \in A_2))\} \\
&= \{x : (x \in U) \wedge (\neg(x \in A_1) \wedge \neg(x \in A_2))\} \\
&= \{x : (x \in U) \wedge ((x \notin A_1) \wedge (x \notin A_2))\} \\
&= \{x : (x \in U) \wedge (x \notin A_1) \wedge (x \in U) \wedge (x \notin A_2)\} \\
&= \{x : ((x \in U) \wedge (x \notin A_1))\} \cap \{x : ((x \in U) \wedge (x \notin A_2))\} \\
&= \overline{A_1} \cap \overline{A_2}.
\end{aligned}$$

So $\overline{A_1 \cup A_2} = \overline{A_1} \cap \overline{A_2}$.

<u>Induction Hypothesis:</u> Suppose the statement is true for $2 \leq k < n$ so that

$\overline{A_1 \cup A_2 \cup \cdots A_{k-1} \cup A_k} = \overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_{k-1}} \cap \overline{A_k}$.

<u>Induction Step:</u> Then

$$\begin{aligned}
\overline{A_1 \cup A_2 \cup \cdots A_{k-1} \cup A_k \cup A_{k+1}} &= \overline{(A_1 \cup A_2 \cup \cdots A_{k-1}) \cup A_k \cup A_{k+1}} \\
&= \overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_{k-1}} \cap \overline{A_k \cup A_{k+1}} \\
&= \overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_{k-1}} \cap \overline{A_k} \cap \overline{A_{k+1}}.
\end{aligned}$$

<u>Conclusion:</u> Since the statement is true for $k+1$ sets, we have shown by induction that the statement is true for all $n \geq 2$. $\square$

Exercise (19). Prove that $\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} \leq 2 - \frac{1}{n}$ for every $n \in \mathbb{N}$.

Proof: Let $P(n)$ be the statement $\frac{1}{1}+\frac{1}{4}+\frac{1}{9}+\cdots+\frac{1}{n^2} \leq 2-\frac{1}{n}$ for every $n \in \mathbb{N}$.

Base Case: For our $P(n)$, observe that for $n=1$ that $\frac{1}{(1)} \leq 2-\frac{1}{(1)}$ is true.

Induction Hypthesis: Suppose our statement is true for some $n \geq 1$. We will assume that $\frac{1}{1}+\frac{1}{4}+\frac{1}{9}+\cdots+\frac{1}{n^2} \leq 2-\frac{1}{n}$ for every $n \in \mathbb{N}$. We want to show that our statement is true for $P(n+1)$.

Induction Step: Observe that

$$\begin{aligned}
\frac{1}{1}+\frac{1}{4}+\frac{1}{9}+\cdots+\frac{1}{(n+1)^2} &\leq 2-\frac{1}{(n+1)} \\
\frac{1}{1}+\frac{1}{4}+\frac{1}{9}+\cdots+\frac{1}{n^2}+\frac{1}{(n+1)^2} &\leq 2-\frac{(n+1)}{(n+1)^2} \\
(\frac{1}{1}+\frac{1}{4}+\frac{1}{9}+\cdots+\frac{1}{n^2})+\frac{1}{(n+1)^2} &\leq 2-\frac{n^2+n}{n(n+1)^2} \\
2-\frac{1}{n}+\frac{1}{(n+1)^2} &\leq \\
2-\frac{(n+2)^2-n}{n(n+1)^2} &\leq \\
2-\frac{n^2+n+1}{n(n+1)^2} &\leq
\end{aligned}$$

So the statement holds for $P(n+1)$ as $2-\frac{n^2+n+1}{n(n+1)^2} \leq 2-\frac{n^2+n}{n(n+1)^2} = 2-\frac{1}{(n+1)}$.

Conclusion: Thus $P(n)$ holds for every $n \in \mathbb{N}$, concluding our proof by induction. □

Exercise (22). If $n \in \mathbb{N}$, then

$$\left(1-\frac{1}{2}\right)\left(1-\frac{1}{4}\right)\left(1-\frac{1}{8}\right)\left(1-\frac{1}{16}\right)\cdots\left(1-\frac{1}{2^n}\right) \geq \frac{1}{4}+\frac{1}{2^{n+1}}.$$

Proof: Let $P(n)$ be the statement above as presented. Base Case: Consider that for $n=1$, we get $(1-\frac{1}{2}) = \frac{1}{2} \geq \frac{1}{4}+\frac{1}{2^{((1)+1)}} = \frac{1}{2}$ which is true.

Induction Hypothesis: Let $n \geq 1$ and assume that

$$\left(1-\frac{1}{2}\right)\left(1-\frac{1}{4}\right)\left(1-\frac{1}{8}\right)\left(1-\frac{1}{16}\right)\cdots\left(1-\frac{1}{2^n}\right) \geq \frac{1}{4}+\frac{1}{2^{n+1}}.$$

. We want to show that our statement is true, that

$$\left(1-\frac{1}{2}\right)\left(1-\frac{1}{4}\right)\left(1-\frac{1}{8}\right)\left(1-\frac{1}{16}\right)\cdots\left(1-\frac{1}{2^{(n+1)}}\right)\geq\frac{1}{4}+\frac{1}{2^{(n+1)+1}}.$$

Induction Step: Observe that $P(n+1)$ can be written as

$$\left[\left(1-\frac{1}{2}\right)\left(1-\frac{1}{4}\right)\left(1-\frac{1}{8}\right)\left(1-\frac{1}{16}\right)\cdots\left(1-\frac{1}{2^n}\right)\right]\left(1-\frac{1}{2^{n+1}}\right)\geq\left[\frac{1}{4}+\frac{1}{2^{n+1}}\right]\left(1-\frac{1}{2^{n+1}}\right)$$

Rearranging the right hand side gives

$$\begin{aligned}\left[\frac{1}{4}+\frac{1}{2^{n+1}}\right]\left(1-\frac{1}{2^{n+1}}\right)&=\frac{1}{4}+(\frac{1}{4})(\frac{1}{2^{n+1}})-\frac{1}{2^{n+1}}-(\frac{1}{2^{n+1}})(\frac{1}{2^{n+1}})\\&=\frac{1}{4}+\frac{1}{2^{n+1}}(1-\frac{1}{4}-\frac{1}{2^{n+1}})\\&=\frac{1}{4}+\frac{1}{2^{n+1}}(\frac{3}{4}-\frac{1}{2^{n+1}})\end{aligned}$$

Because $n\leq 1$, it follows that $2^{1+1}\leq 2^{n+1}$. Furthermore the inverse of this inequality gives $\frac{1}{2^{n+1}}\leq\frac{1}{2^{1+1}}=\frac{1}{4}$. So

$$\begin{aligned}\frac{1}{4}+\frac{1}{2^{n+1}}(\frac{3}{4}-\frac{1}{2^{n+1}})&\geq\frac{1}{4}+\frac{1}{2^{n+1}}(\frac{3}{4}-\frac{1}{4})\\&=\frac{1}{4}+\frac{1}{2^{n+1}}(\frac{1}{2})\\&=\frac{1}{4}+\frac{1}{2^{(n+1)+1}}\end{aligned}$$

Therefore $\left[\left(1-\frac{1}{2}\right)\left(1-\frac{1}{4}\right)\left(1-\frac{1}{8}\right)\left(1-\frac{1}{16}\right)\cdots\left(1-\frac{1}{2^n}\right)\right]\left(1-\frac{1}{2^{n+1}}\right)$
$\geq\left[\frac{1}{4}+\frac{1}{2^{n+1}}\right]\left(1-\frac{1}{2^{n+1}}\right)=\left[\frac{1}{4}+\frac{1}{2^{n+1}}(\frac{3}{4}-\frac{1}{2^{n+1}})\right]\geq\left[\frac{1}{4}+\frac{1}{2^{(n+1)+1}}\right]$.
Conclusion: Hence by induction

$$\left(1-\frac{1}{2}\right)\left(1-\frac{1}{4}\right)\left(1-\frac{1}{8}\right)\left(1-\frac{1}{16}\right)\cdots\left(1-\frac{1}{2^n}\right)\geq\frac{1}{4}+\frac{1}{2^{n+1}}.$$

is true for all $n\in\mathbb{N}$. $\square$

Exercise (25). Concerning the Fibonacci sequence, prove that $F_1+F_2+F_3+F_4+\cdots+F_n = F_{n+2}-1$ which is true.

Proof: Base Case: Consider the case when $n=1$, then $F_1 = F_{(1)+2}-1 = F_3-1 = 2-1 = 1$, which is true. Now consider the case where $n=2$, then $F_1+F_2 = F_{2+2}-1 = F_4-1 = 3-1 = 2$, which is also true.

Induction Hypothesis: Suppose the statement is true for some $n > k \geq 1$, that $F_1 + F_2+F_3+F_4+...+F_k = F_{k+2}-1$. We want to show that $F_1+F_2+F_3+F_4+...+F_k+F_{k+1} = F_{k+3}-1$.

Induction Step: Now observe the following

$$\begin{aligned} F_1+F_2+F_3+F_4+...+F_k+F_{k+1} &= \\ (F_1+F_2+F_3+F_4+...+F_k)+F_{k+1} &= \\ F_{k+2}-1+F_{k+1} &= (F_{k+1}+F_{k+2})-1 \\ &= F_{k+3}-1. \end{aligned}$$

Conclusion: Thus we have shown that $F_1+F_2+F_3+F_4+...+F_k+F_{k+1} = F_{k+3}-1$, by induction $F_1+F_2+F_3+F_4+\cdots+F_n = F_{n+2}-1$ is true for all $n \in \mathbb{Z}$ where $n \geq 1$. □

Exercise (30). Here F_n is the nth Fibonacci number. Prove that

$$F_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}.$$

Hint: There are multiple ways to do this... one is to use the fact that $a^{n-1} = \frac{a^n}{a}$, while others involve things like the fact if $\phi = \frac{1+\sqrt{5}}{2}$, then $\phi^2-\phi-1=0$.

Proof: Let $P(n)$ be our statement

$$F_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}.$$

Base Case: For the case where $n=1$, observe that the statement $P(n)$ holds, note that F_1 in the fibonacci sequence is 1.

$$F_1 = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^1 - \left(\frac{1-\sqrt{5}}{2}\right)^1}{\sqrt{5}} = \frac{\frac{2\sqrt{5}}{2}}{\sqrt{5}} = \frac{\sqrt{5}}{\sqrt{5}} = 1$$

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Induction hypothesis: Suppose $P(n)$ is true for $n \geq 1$, note that the Fibonacci numbers may be defined by the recurrence relation $F_n = F_{n-1} + F_{n-2}$. In particular, $F_{n+1} = F_n + F_{n-1}$ for $n > 1$. We want to show that

$$F_{n+1} = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}}{\sqrt{5}}$$

Induction Step: Observe that

$$\begin{aligned} F_{n+1} &= \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}} + \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n-1}}{\sqrt{5}} \\ &= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n\left(1+\frac{2}{1+\sqrt{5}}\right) - \left(\frac{1-\sqrt{5}}{2}\right)^n\left(1+\frac{2}{1-\sqrt{5}}\right)\right] \\ &= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n\left(1+\frac{2}{1+\sqrt{5}}\right) - \left(\frac{1-\sqrt{5}}{2}\right)^n\left(1+\frac{2}{1-\sqrt{5}}\right)\right] \end{aligned}$$

Note that $\left(1+\frac{2}{1+\sqrt{5}}\right)\left(\frac{1-\sqrt{5}}{1-\sqrt{5}}\right) = \frac{1+\sqrt{5}}{2}$ and that $\left(1+\frac{2}{1-\sqrt{5}}\right)\left(\frac{1+\sqrt{5}}{1+\sqrt{5}}\right) = \frac{1-\sqrt{5}}{2}$, when we substitute in our equation we get

$$\begin{aligned} &= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n\left(\frac{1+\sqrt{5}}{2}\right) - \left(\frac{1-\sqrt{5}}{2}\right)^n\left(\frac{1-\sqrt{5}}{2}\right)\right] \\ &= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right] \\ &= \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}}{\sqrt{5}}. \end{aligned}$$

Conclusion: So $P(n+1)$ holds, thus by induction we have proven that $P(n)$ is is true for all $n \in \mathbb{Z}$. $\square$

Exercise (33). Suppose n (infinitely long) straight lines lie on a plane in such a way that no two of the lines are parallel, and no three of the lines intersect in a single point. Show that this arrangement divides the plane into $\frac{n^2+n+2}{2}$ regions.

Proof: Base Case: For the case $n = 1$, where we have a single infinite line on a plane, the plane is divided in two. Substituting in our expression we get, $\frac{1^2+1+2}{2} = \frac{4}{2} = 2$. So the statement holds for $n = 1$.

Induction Hypothesis: Suppose our statement is true for $n \geq 1$. We would like to show that the statement is true for $n+1$ lines. That is adding $n+1$ lines gives $\frac{(n+1)^2+(n+1)^2+2}{2}$.

Induction Step: Observe that adding $n+1$ lines to our original expression gives

$$(n+1) = \frac{n^2+n+2}{2} = \frac{2n+2+n^2+n+2}{2} = \frac{(n+1)^2+(n+1)^2+2}{2}.$$

Conclusion: Thus the statement holds for $n+1$ lines. Thus by induction we conclude that the statement is true for any n lines.(Sorry no pictures, had trouble getting tikz.) $\square$

Exercise (Reflection Problem).

- How long did it take you to complete each problem?

  Answer: Waaaaaayy too long, I get held up in minor things, some of them philosophical, some of it more nuanced. But most of all it was the careful thinking and writnig out of the proofs. Lots of trial and error algebra were involved and some of wikipedia definitions for the fibonacci sequence. Starting to see what doing pure mathematics actually entails. I started out really slow, sometimes getting bogged down for days for the earlier problems, then it took hours, then it took minutes, and now I have some muscle memory for it. I fell so far behind its embarassing. $\square$

- What was easy?

Answer: It was hard at first, then it got easy as I got the hang of it, then the fibonacci stuff at the end made it hard again, then it became easy again. □

- What was challenging? What made it challenging?

  Answer: The time consuming nature of writing in TeX, the "figuring it out", induction didn't click right away for me, then it suddenly did. □

- Compare your answers to the odd numbered exercises to those in the back of the textbook. What did you learn from this comparison?

  Answer: I think I did pretty good all things considered □

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