Problem 51. *If $K \subset \mathbb{R}$ is compact and nonempty, then $\sup K$ and $\inf K$ both exist and are elements of K.*

Proof. Recall that a set is compact if it is closed and bounded, meaning that the set K contains all its limit points, is non-empty and is bounded both above and below. It follows by the Axiom of completeness that $\sup K$ and $\inf K$ both exist from the definitions. □

Problem 52. *What of the following sets are compact? For those that are compact, give a brief justification. For those that are not, show how the book's definition of compact (Definition 3.3.1) breaks down. That is, give an example sequence in the set that does not contain a subsequence which converges to a point in the set.*

(a) $\mathbb{N}$

This is not compact because $\mathbb{N}$ is unbounded. This means that all sequences and subsequences of the set $\mathbb{N}$ diverge.

(b) $\mathbb{Q} \cap [0, 1]$

This is not compact because $\mathbb{Q} \cap [0, 1]$ does not contain all its limit points, we can construct a sequence of rational numbers converging to $1/\pi$ since this is an irrational number, it is not in the set.

(c) The Cantor set C

The Cantor set is closed, since it consists of compliments of the union of open sets, which is closed and is bounded by 0 and 1 which contained in the set. By Heine-Borel C is compact.

(d) $\left\{1 + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \cdots + \dfrac{1}{n^2} \,:\, n \in \mathbb{N}\right\}$

We should note that the set is a p-series, with the value for $p = 2$. This sum converges to $\frac{\pi^2}{6}$. Consider the range of the sequence of partial sum, if $(s_n) = \left\{1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \,:\, n \in \mathbb{N}\right\}$, we define it as the set $S = \{s_1, s_2, s_3, \cdots s_n\}$. We observe that $\frac{\pi^2}{6}$ is not in S but that S approaches $\frac{\pi^2}{6}$ as n goes to infinity, this means that S does not contain all its limit points and thus S is not compact.

(e) $\left\{1, \dfrac{1}{2}, \dfrac{2}{3}, \dfrac{4}{5}, \dots\right\}$

Observe that this sequence is equivalent to $S = \{1\} \cup \{n/(n+1) : n \geq 1\}$. We note that $\{n/(n+1) : n \geq 1\}$ with its bounds defined by $[0, 1)$ converges to 1 as n approaches infinity , all of its subsequences also converge to $\{1\}$ similary. Normally this set would not be closed since it wouldn't contain all its limit point, but since $\{1\}$ is unioned to this set, it contains all its limit points and the bounds are defined by $[0, 1]$. Thus the set is compact.

Problem 53. *If a set $K \subset \mathbb{R}$ is closed and bounded, then it is compact.*

Proof. Since K is closed and bounded, we can use Bolzano-Weierstrass to construct a sequence in K in which its subsequence converges.

Let (x_n) be an arbitrary sequence in K. Since K is bounded, there exists $a, b \in \mathbb{R}$ such that $K \subseteq [a, b]$. Thus (x_n) is a bounded sequence. By the Bolzano-Weierstrass Theorem, there exists a subsequence (x_{n_k}) and $L \in \mathbb{R}$ such that $x_{n_k} \to L$. Since K is closed it follows that $L \in K$. Thus (x_n) has a subsequence converging to a point in K, so K is compact. $\square$

Problem 54. *Decide whether the following propositions are true or false. If the claim is valid, supply a short proof. If the claim is false, provide a counterexample.*

(a) The arbitrary union of compact sets is compact.

False: Since we are speaking of arbitrary union, suppose an infinite union of compact sets such as $K_n = [n, n+1]$ for $n \in \mathbb{N}$. Each of the K_n are closed and bounded and thus compact, if we union an an infinite amount of these K_n we get an unbounded set and thus not compact.

(b) The arbitrary intersection of compact sets is compact.

True:

Proof. $\square$

(c) Let A be arbitrary, and let K be compact. Then $A \cap K$ is compact.

(d) If $F_1 \supseteq F_2 \supseteq F_3 \supseteq \ldots$ is a nested sequence of nonempty closed sets then the intersection $F = \bigcap_{n=1}^{\infty} F_n$ is nonempty.

Problem 55. *Let A and B be nonempty subsets of $\mathbb{R}$. If there exist disjoint open sets U and V with $A \subseteq U$ and $B \subseteq V$, then A and B are separated.*

Proof. $\square$

Problem 56. *A set $E \subset \mathbb{R}$ is* totally disconnected *if, given any two distinct points $x, y \in E$, there exist separated sets A and B with $x \in A$ and $y \in B$ and $E = A \cup B$.*

The Cantor set C is totally disconnected.

Proof. $\square$

Problem 57. *For each stated limit, find the largest possible δ-neighborhood that is a proper response to the given ϵ challenge. Note that $[[x]]$ denotes the greatest integer which is less than or equal to x.*

(a) $\lim_{x \to 3} 5x - 6 = 9$, where $\epsilon = 1$

(b) $\lim_{x \to 4} \sqrt{x} = 2$, where $\epsilon = 0.5$

(c) $\lim_{x \to \pi} [[x]] = 3$, where $\epsilon = 0.5$

Problem 58. *Use the definition of functional limit in the textbook (Definition 4.2.1) to prove the following limit statements.*

(a) $\lim_{x \to 2} 3x + 4 = 10$

Proof. □

(b) $\lim_{x \to 2} x^2 + x - 1 = 5$

Proof. □

(c) $\lim_{x \to 3} \frac{1}{x} = \frac{1}{3}$

Proof. □

(d) $\lim_{x \to 3} \frac{x^2 - 9}{x - 3} = 6$

Proof. □

Problem 59. *Let $g : A \to \mathbb{R}$ and assume that f is a bounded function on A. Assume c is a limit point of A. If $\lim_{x \to c} g(x) = 0$ then $\lim_{x \to c} f(x)g(x) = 0$.*

Proof. □