4.2.1

We are given the differential equation $y''+2y'+y=0$ with the solution $y_1 = xe^{-x}$ we will construct the second solution using the formula provided: $y_2 = y_1 \int \frac{e^{-\int p(x)dx}}{(y_1)^2} dx$.

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{-\int p(x)dx}}{(y_1)^2} dx && && (1)\\
&= xe^{-x} \int \frac{e^{-\int 2dx}}{(xe^{-x})^2} dx && p(x) = 2 && (2)\\
&= xe^{-x} \int \frac{e^{-2x}}{(xe^{-x})^2} dx && && (3)\\
&= xe^{-x} \int \frac{e^{-2x}}{(x^2e^{-2x})} dx && && (4)\\
&= xe^{-x} \int \frac{1}{x^2} dx && && (5)\\
&= xe^{-x} \int (\frac{\sec(\ln(x))}{x}) && && (6)\\
&= -e^{-x} && && (7)
\end{aligned}$$

Our second solution is

$$\boxed{y_2 = C_2e^{-x}}$$

4.2.2

We are given the differential equation $x^2y'' - 3xy' + 5y = 0$ with the solution $y_1 = x^2\cos(\ln(x))$ we will construct the second solution using the formula provided: $y_2 = y_1 \int \frac{e^{-\int p(x)dx}}{(y_1)^2} dx$.

First we divide our equation $x^2y'' - 3xy' + 5y = 0$ by its leading term x^2:

$$\frac{x^2y'' - 3xy' + 5y}{x^2} = y'' - \frac{3y'}{x} + \frac{5y}{x^2}$$

Our $p(x)$ term is now $-\frac{3}{x}$, now we just plug in and solve:

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{-\int p(x)dx}}{(y_1)^2} dx && (1)\\
&= x^2\cos(\ln(x)) \int \frac{e^{-\int -\frac{3}{x}dx}}{(x^2\cos(\ln(x)))^2} dx \qquad p(x) = -\frac{3}{x} && (2)\\
&= x^2\cos(\ln(x)) \int \frac{e^{3\ln(x)}}{(x^2\cos(\ln(x)))^2} dx && (3)\\
&= x^2\cos(\ln(x)) \int \frac{x^3}{(x^4\cos^2(\ln(x)))} dx && (4)\\
&= x^2\cos(\ln(x)) \int \frac{1}{xcos^2(\ln(x))} dx && (5)\\
&= x^2\cos(\ln(x)) \int \frac{\sec^2(\ln(x))}{x} && (6)\\
&= x^2\cos(\ln(x)) \int \sec^2(u)du \qquad u = \ln(x) \qquad du = \frac{1}{x} && (7)\\
&= x^2\cos(\ln(x)) * \tan(\ln(x)) && (8)\\
&= x^2\cos(\ln(x)) * \frac{\sin(\ln(x))}{\cos(\ln(x))} && (9)\\
&= x^2\sin(\ln(x)) && (10)
\end{aligned}$$

Thus our second solution is:

$$\boxed{y_2 = C_2x^2\sin(\ln(x))}$$

4.2.3

We are given the differential equation $y'' + 16y = 0$ with the solution of $y_1 = \sin(4x)$. We will find the second solution using the reduction of order technique assuming $y_2 = uy_1$, first some derivatives

$$\begin{aligned}
y_2 &= uy_1 & y_1 &= \sin(4x)\\
y_2' &= u'y_1 + uy_1'' & y_1' &= 4\cos(4x)\\
y_2'' &= u''y_1 + 2u'y_1' + uy_1'' & y_1'' &= -16\sin(4x)
\end{aligned}$$

Next we plug in our derivatives into the original equation:

$$u''y_1 + 2u'y_1' + uy_1'' + 16(uy_1) = 0 \qquad (1)$$

and our y's, simplify and solve:

$$\begin{aligned}
u'' \sin(4x) + 8u' \cos(4x) - 16u \sin(4x) + 16u \sin(4x) &= 0 && (2)\\
u'' \sin(4x) + 8u' \cos(4x) + 16u(\sin(4x) - \sin(4x)) &= 0 && (3)\\
u'' \sin(4x) + 8u' \cos(4x) &= 0 \qquad w' = u'', w = u' && (4)\\
w' \sin(4x) + 8w \cos(4x) &= 0 && (5)\\
w' + 8w \cot(4x) &= 0 && (6)\\
& && (7)
\end{aligned}$$

Got stuck

4.3

4.3.1

The following is mostly factoring and algebra:
a) $20y'' - y' - y = 0$

$$20m^2 - m - 1 = 0$$

$$(4m-1)(5m+1) = 0 \qquad m = \frac{1}{4}, -\frac{1}{5}$$

$$\boxed{y = c_1 e^{\frac{-x}{5}} + c_2 e^{\frac{x}{4}}}$$

b) $y'' - 10y' + 25y = 0$

$$m^2 - 10m + 25 = 0$$
$$(m-5)(m-5) = 0 \qquad m = 5$$
$$\boxed{y = c_1 e^{5x} + c_2 x e^{5x}}$$

c) $5y'' + y = 0$

$$5m^2 + 1 = 0$$
$$m^2 = -\frac{1}{5} \qquad m = \frac{i}{\sqrt{5}}, -\frac{i}{\sqrt{5}}$$
$$\boxed{y = c_1 \cos(\frac{x}{\sqrt{5}}) + c_2 \sin(\frac{x}{\sqrt{5}})}$$

d) $y^{(4)} - 2y'' + y = 0$

$$m^4 - 2m^2 + 1 = 0$$
$$(m+1)^2(m-1)^2 \qquad m = 1, -1$$
$$\boxed{y = c_1 e^{-x} + c_2 x e^{-x} + c_3 e^{x} + c_4 x e^{x}}$$

e) $y^{(5)} + y^{(4)} - y''' = 0$

$$m^5 + m^4 - m^3 = 0$$

$$m^3(m^2 + m - 1) \qquad \text{Use quadratic formula for right side}$$

$$m = 0, \frac{\sqrt{5}-1}{2}, \frac{-\sqrt{5}-1}{2}$$

$$\boxed{y = c_1 + c_2 e^{\frac{x\sqrt{5}-1}{2}} + c_3 e^{\frac{-x\sqrt{5}-1}{2}}}$$

I hope that's the answer

4.3.2

We can find a differential equation whos solution is $c_1 e^{9x} + c_2 x e^{9x}$ by finding a quadratic equation whos root is 9 and 9 alone, so I'm going to go ahead and steal problem be from 4.3.1.

$$(m-9)(m-9)$$

$$m^2 - 18m + 81 = 0$$

$$\boxed{y'' - 18y' + 81y = 0}$$

4.3.3

Here we solve the initial value problem $y'' + 2y' + y = 0$ with $y(0) = 5$ and $y'(0) = 10$

$$m^2 + 2m + 1 = 0$$

$$(m+1)(m+1) = 0 \qquad m = -1$$

$$y = c_1 e^{-x} + c_2 x e^{-x}$$

now for our actual solution:

$$y = c_1 e^{-x} + c_2 x e^{-x}$$

$$y' = -c_1 e^{-x} + c_2(e^{-x} - xe^{-x}) \qquad 10 = c_2$$

$$\boxed{y = 5e^{-x} + 10xe^{-x}}$$

Ran out of time for the rest