We will begin by building theorems from the ground up from basic rules

Definition 1. *Convergence: For $A_n \to L$ means: For all $\epsilon > 0$, there exists N such that for all $n > N$, implies $\mid a_n - L \mid < \epsilon$.*

Definition 2. *Bounded: (a_n) is bounded if there exists $M > 0$ such that $|a_n| \leq M$ for all n.*

Definition 3. *Triangle inequality*

Triangle inequality :	$\mid a + b \mid \leq \mid a \mid + \mid b \mid$
Reverse triangle :	$\|\|a\| - \|b\|\| \leq \|a - b\|$
Product bound :	$\|ab\| = \|a\|\|b\|$

Theorem 0.1. *If (a_n) converges to L, then (a_n) is bounded.*

Proof. Since (a_n) converges to L, this means that for an $\epsilon > 0$, there exists N such that for all $n > N$, implies $\mid a_n - L \mid < \epsilon$. From this we get the following inequality:

$$|a_n| = |a_n - L + L| \leq |a_n - L| + |L| \qquad \text{By Triangle Inequality}$$

Now let $\epsilon = 1$, then there exists an $n > N$ such that $|a_n - L| < 1$, it follows from this that

$$|a_n| = |a_n - L + L| \leq |a_n - L| + |L| < 1 + |L|$$

for $n \leq N$, let $M_1 = \max\{|a_1|, |a_2|, \cdots, |a_N|\}$. Now let $M = \{M_1, 1 + |L|\}$. Then $|a_n| \leq M$ for all n. □

Theorem 0.2. *(Uniqueness of Limits) If $a_n \to L$ and $a_n \to M$ then $L = M$.*

Proof. Let $\epsilon > 0$ be arbitrary. Since $a_n \to L$ there exists an N_1 such that for all $n \geq N : |a_n - L| < \frac{\epsilon}{2}$.

Likewise since $a_n \to M$, there exists N_2 such that for all $n \geq N_2 : |a_n - M| < \frac{\epsilon}{2}$.

Let $N = \max\{N_1, N_2\}$. For $n \geq N$:

$$|L - M| = |L - a_n + a_n - M| \leq |a_n - L| + |a_n + M| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Since this holds for arbitrary $\epsilon > 0$, we must have $|L - M| = 0$, so L = M.

□

Theorem 0.3. *(Algebraic Limit Theorem) If $x_n \to a$ and $y_n \to b$, then the algebraic limit theorem states*

Sum:	$\lim(x_n + y_n) = a + b$		(1)
Scalar:	$\lim(cx_n) = ca$		(2)
Product:	$\lim(x_n * y_n) = a * b$		(3)
Quotient:	$\lim(\frac{x_n}{y_n}) = \frac{a}{b}$	*for* $b \neq 0$	(4)

Proof. Sum: Recall that a sequence (s_n) converges to L if for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that if $n \geq N$ then $|s_n - L| < \epsilon$. Given $x_n \to a$ and $y_n \to b$, it follows that there exists $N_1, N_2 \in \mathbb{N}$ such that if $n \geq N$ we have $n \geq N_1$ and $n \geq N_2$ such that $|x_n - a| < \epsilon/2$ and $|y_n - b| < \epsilon/2$. Let $N = \max\{N_1, N_2\}$. In order to show that $\lim(x_n + y_n) = a + b$, we need to show that $|(x_n + y_n) - (a + b)| < \epsilon$ (epsilon definition of equality). Observe that

$$|(x_n + y_n) - (a + b)| = |(x_n - a) + (y_n - b)| \leq |x_n - a| + |y_n - b| < \epsilon/2 + \epsilon/2 = \epsilon$$

By the triangle inequality. Thus $\lim(x_n + y_n) = a + b$. □

Proof. Scalar: □

Theorem 0.4. *Monotone Convergence Theorem: If a sequence is monotone and bounded then it converges.*

Proof. Suppose (a_n) is a monotone and bounded sequence. Without loss of generality assume (a_n) is monotone increasing. Since (a_n) is bounded above, the set $S = \{a_n : n \in \mathbb{N}\}$ is a nonempty set of real numbers that is bounded above. By the Axiom of completeness, S has a supremum. Let $s = \sup S$.

We want to show that $\lim a_n = s$.

Let $\epsilon > 0$ Since s is the supremum of S, the number $s - \epsilon$ is not an upper bound for S. Therefore there exists $N \in \mathbb{N}$ such that

$$s - e < A_n \leq s$$

Since (a_n) is monotone increasing, for all $n \geq N$ we have

$$a_N \leq a_n \leq s$$

combining these inequalities, for all $n \geq N$ we obtain

$$s - \epsilon < a_N \leq a_n \leq s < s + \epsilon$$

Which implies $|a_n - s| < \epsilon$ Since $\epsilon > 0$ was arbitrary, we conclude that $\lim a_n = s$. □

Theorem 0.5. *Bolzano-Wierstrass Theorem: Every bounded sequence contains a convergent subsequence.*

Proof. Suppose (a_n) is a bounded sequence. Then by definition there exists $m, M \in \mathbb{R}$ such that $m \leq a_n \leq M$.

Let $I_1 = [m, M]$. Bisect this interval at its midpoint $\frac{M-m}{2}$ so we get the intervals $[m, \frac{M-m}{2}]$ and $[\frac{M-m}{2}, M]$. Because I_1 contained a_n for infinitely many n, it follows that at least one of the 2 intervals we constructed also contains a_n for infinitely many terms of n. Choose that interval and denote it as I_2. Continue this process inductively.

Suppose I_k is an interval containing a_n for infinitely many n. Bisect this interval by its midpoint to obtain two closed intervals, choose one that contains a_n for infinitely many n and denote this interval by I_{k+1}. This construction yields the nested sequence of closed intervals

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$$

Where each $I_k = [c_k, d_k]$ satisfying $d_k - c_k = \frac{M-m}{2^{k-1}}$ and each I_k contains an infinitely many terms of the sequence (a_n). By the Nested Interval Property there exists $x \in \bigcap_{k=1}^{\infty} I_k$. Now construct a a convergent subsequence of (a_n) as follows. Choose $n_1 \in \mathbb{N}$ such that $a_{n_1} \in I_1$. Since I_2 contains infinitely many terms of the sequence, choose $n_2 > n_1$ such that $a_{n_2} \in I_2$. Continue this process inductively. We get $n_1 < n_2 < \cdots < n_k$ with $a_{n_j} \in I_j$ for $j = 1, 2, \cdots k$. Choose $n_{k+1} > n_k$ such that $a_{n_{k+1}} > a_{n_k}$. such that $a_{n_{k+1}} \in I_{k+1}$. This produces a subsequence (a_{n_k}) with the property that $a_{n_k} \in I_k$ for all $k \in \mathbb{N}$.

We want to show that $a_{n_k} \to x$.

Let $\epsilon > 0$.

Choose $K \in \mathbb{N}$ such that $\frac{M-m}{2^{K-1}} < \epsilon$. Then for all $k \geq K$, both a_{n_k} and x like in I_k which has the length

$$\frac{M-m}{2^{k-1}} \leq \frac{M-m}{2^{K-1}} < \epsilon$$

It follows then that

$$|a_{n_k} - x| \leq \frac{M-m}{2^{k-1}} < \epsilon$$

Thus $|a_{n_k} - x| < \epsilon$ meaning $a_{n_k} \to x$. □

Theorem 0.6. *Density of $\mathbb{Q}$: For every two real numbers a and b with $a < b$, there exists a rational number $r \in \mathbb{Q}$ satisfying $a < r < b$.*

Proof. Let a and b be real numbers with $a < b$. Then $b - a > 0$. By the Archimedean property there exists $n \in \mathbb{N}$ such that

$$n(b-a) > 1$$

Rearranging gives

$$nb > na + 1$$

Now suppose we have a set $S = \{k \in \mathbb{Z} : k > na\}$. By the well-ordered properties of the integers and because S is bounded by $k > na$. The set S has a least element. Denote this element as as $m = \min S$. Note that since $m \in S$, it follows that $m > na$ and $m - 1 \notin S$. So $m - 1 \leq na$. From this we get

$$m \leq na + 1$$

noticeably

$$m \leq na + 1 < nb$$

Together with $m > na$ we get

$$na < m < nb$$

Dividing by n we get

$$a < \frac{m}{n} < b$$

Setting $r = \frac{m}{n} \in \mathbb{Q}$, we conclude that $a < r < b$. □

Theorem 0.7. *Continous preserves compactness: Let $f : A \to \mathbb{R}$ be continous on A. If $K \subseteq A$ is compact then $f(K)$ is compact.*

Proof. Let $K \subseteq A$ be compact. We will show that $f(K)$ is compact by showing that every sequence in $f(K)$ has a subsequence that converges to a point in $f(K)$. Let (y_n) be a sequence in $f(K)$.

By definition of $f(K)$, for each $n \in \mathbb{N}$ there exists $f(x_n) = y_n$.

Since K is compact and (x_n) is a sequence in K, there exists a subsequence (x_{n_k}) of (x_n) and a point $x \in K$ such that

$$\lim_{k\to\infty} x_{n_k} = x$$

Since f is continous on A and $x \in K \subseteq A$, the function f is continuous at x.

Therefore by the sequential characterization of continuity,

$$\lim_{k\to\infty} f(x_{n_k}) = f(x)$$

But $f(x_{n_k}) = y_{n_k}$ for all k, so we have

$$\lim_{k\to\infty} y_{n_k} = f(x)$$

Since $x \in K$, we have $f(x) \in f(K)$.

Thus, we have found a subsequence (y_{n_k}) of (y_n) that converges to a point $f(x) \in f(K)$. Since (y_n) was an arbitrary sequence in $f(K)$, we have concluded that every sequence in $f(K)$ has a subsequence converging to a point in $f(K)$. $\square$