**Problem 1.** *There is no rational number whose square is* 2.

*Proof.* Assume, for contradiction, that there exist integers p and q satisfying

$$\frac{p}{q} = \sqrt{2},$$

where p/q is a rational number in lowest terms. By squaring, this is the same as  $\frac{p^2}{a^2} = 2$ , and by clearing denominators it is the same as

$$p^2 = 2q^2.$$

Thus  $p^2$  is divisible by 2, an even number. This implies that p is also divisible by 2 and can be expressed in the form p = 2k for some  $k \in \mathbb{Z}$ . If we sbustitute the p in  $p^2 = 2q^2$  for 2k, we get

$$(2k)^2 = 4(k^2) = 2q^2$$

Further reducing this gives us

$$2(k^2) = q^2$$

This is a contradiction as the result implies that  $q^2$  is also even and thus q is even. Therefore p and q are both even and are irreducible.

- (a) The negation of "For all real numbers satisfying a < b, there exists  $n \in \mathbb{N}$  such that a + (1/n) < b'' is "There exists a real number satisfying a < b such that for all  $n \in \mathbb{N}$ ,  $a + (1/n) \ge b$ .
  - (b) The negation of "There exists a real number x > 0 such that x < 1/n for all  $n \in \mathbb{N}$ " is "For all real numbers x > 0, there exists an  $n \in \mathbb{N}$  such that  $x \ge 1/n$ .
  - (b) The negation of "Between every two distinct real numbers there is a rational number" is "There exists an  $x,y \in \mathbb{R}$ , where  $x \neq y$ , such that there is no  $n \in \mathbb{Q}$  that satisfies x < n < y.

**Problem 3.** Suppose a and b are real numbers. Then

(a) 
$$|a - b| \le |a| + |b|$$

*Proof.* Case 1: Suppose a > b, then |a - b| = a - b (since a - b > 0). If a > 0is true, then |a-b|=a-b=|a|-b. Since  $-b \leq |b|$  (because -b=|b| if b is negative and  $-b \le b = |b|$  if b is non-negative), we have

$$|a - b| = |a| - b \le |a| + |b|$$

. On the other hand, if a < 0, then

$$|a - b| = a - b \le |a| - b \le |a| + |b|$$

(since  $a \leq |a|$ , as before).

**Case 2:** Suppose  $b \ge a$ , then |a - b| = b - a (since b - a > 0). If b > 0 is also true, then  $|a - b| = b - a \le b + |a|$  (since  $-a \le |a|$ ). If  $b \ge 0$ , then |b| = b, so

$$|a - b| = b - a < b + |a| = |a| + |b|$$

(since  $-a \leq |a|$ ).

If b < 0, then since b > a, we have a < b < 0, so |a| = -a and |b| = -b. Thus

$$|a-b| = b-a = b + (-a) \le -b + (-a) = |b| + |a| = |a| + |b|$$

or more briefly

$$|a - b| \le |a| + |b|$$

**Case 3:** Suppose a = b, then  $|a - b| = 0 \le |a| + |b|$  Since absolute values are non-negative.

(b)  $||a| - |b|| \le |a - b|$ 

*Proof.* Observe that a = (a - b) + b. It follows that

$$|a| = |(a - b) + b| \le |a - b| + |b|$$

by the triangle inequality. Subtracting |b| from the right and left side of the inequality gives us  $|a|-|b| \leq |a-b|$ . Likewise observe that b=(b-a)-a. It follows that

$$|b| = |(b-a) + a| \le |b-a| + |a|$$

by the triangle inequality. Since |b-a|=|a-b|, subtracting |a| from either side of the inequality above gives us  $|b|-|a| \le |a-b|$ . Since ||a||-|b|| is either |a|-|b| or |b|-|a|, and both are at most |a-b|, it follows that  $||a|-|b|| \le |a-b|$ .  $\square$ 

**Problem 4.** Give an example of each, or state that it is impossible.

(a)  $f: \mathbb{N} \to \mathbb{N}$  that is one-to-one but not onto.

**My Answer:** The function f(n) = 2n is a mapping from  $\mathbb{N} \to \mathbb{N}$  That is one-to-one since distinct n produce distinct even numbers, but not onto because it misses odd numbers.

(b)  $f: \mathbb{N} \to \mathbb{N}$  that is onto but not one-to-one.

**My Answer:** The function  $f(n) = \lfloor \frac{n+1}{2} \rfloor$  (For clarity sake this is a floor function) is onto since every  $m \in \mathbb{N}$  is hit, but not one-to one because f(1) = f(2) = 1.

(d)  $f: \mathbb{N} \to \mathbb{Z}$  that is one-to-one and onto.

My Answer: The piecewise function

$$f(n) = \begin{cases} \frac{n}{2} & \text{If n is even} \\ -\frac{n-1}{2} & \text{If n is odd} \end{cases}$$

Is a mapping from  $\mathbb{N} \to \mathbb{Z}$  that is both one-to-one and onto since it uniquely maps even n to non-negative integers and ood n to negative integers, this covers all of  $\mathbb{Z}$ .

**Problem 5.** There exists an infinite collection of sets  $A_1, A_2, A_3, \ldots$  with the properties that every  $A_i$  has an infinite number of elements, and  $A_i \cap A_j = \emptyset$  for all  $i \neq j$ , and  $\bigcup_{i=1}^{\infty} A_i = \mathbb{N}$ .

*Proof.* A natural number is k-almost-prime if it has k prime factors where  $k \in \mathbb{N}$ . Let  $A_1 \subseteq \mathbb{N}$  be the union of the sets  $\{1\}$  and the set containing all k-almost-prime numbers  $\{2,3,5,7,11,17,...\}$ , where k=1 so that  $A_1=\{1,2,3,5,6,11,17,...\}$ . Let  $A_2 \subset \mathbb{N}$  be the set containing all k-almost-prime numbers  $\{4,6,9,10,14,15,21,22,...\}$ , where k=2. Let  $A_3 \subseteq \mathbb{N}$  be set containing all k-almost-prime numbers  $\{8,12,18,20,27,28,30,...\}$ , where k=3. Similarly let  $A_k \subseteq \mathbb{N}$  be the set containing all k-almost-prime numbers for some  $k \in \mathbb{N}$ . Suppose  $m \in \mathbb{N}$  and  $m \in A_i \cap A_j$  for some  $i,j \in \mathbb{N}$ . Note that by the uniqueness of prime decompositions, m has a unique prime decomposition of a fixed length. Thus i=j. Since  $A_n \subseteq \mathbb{N}$  for all  $n \in \mathbb{N}$ , it follows that  $\bigcup_{n=1}^{\infty} A_n \subseteq \mathbb{N}$ . Similarly by fundamental theorem of arithmetic, n has a unique prime decomposition of a fixed length m. Therefore  $n \in A_m$  and  $\mathbb{N} \subseteq \bigcup_{n=1}^{\infty} A_n$ . Thus  $\bigcup_{n=1}^{\infty} A_n = \mathbb{N}$ .