4.4.2

a) We are provided the second order differential equation $y'' - 9y' = 3x^2 - 5\sin(3x)$ and we are tasked with finding the form of the particular solution, by first inspection/observation we find the form to be:

$$\boxed{y_p = (Ax^2 + Bx + C) + D\cos(3x) + E\sin(3x)} \tag{1}$$

b) We are also provided with the second order differential equation $y'' + 2y' + y = 2e^{-x} - e^x$, by inspection/observation we conclude that the form of the particular solution is:

$$\boxed{y_p = Ae^{-x} + Be^x}$$

4.4.3

We are tasked with solving the following differential equations via method of undetermined coefficients:

a) $y'' + 2y' = 2x + 5 - e^{-2x}$

$$y'' + 2y' = 2x + 5 - e^{-2x}$$
$$m^2 + 2m + 0 = 0$$
$$m(m+2) = 0$$
$$m = 0, -2$$
$$y_c = c_1 + c_2e^{-2x}$$

$$y_p = Ax^2 + Bx + Cxe^{-2x}$$
$$y_p' = 2Ax + B - 2Cxe^{-2x} + Ce^{-2x}$$
$$y_p'' = 2A + 4Cxe^{-2x} - 4Ce^{-2x}$$

$$2A + 4Cxe^{-2x} - 4Ce^{-2x} + 4Ax + 2B - 4Cxe^{-2x} + 2Ce^{-2x} = 2x + 5 - e^{-2x}$$
$$4Ax + 2(A+B) + 2Ce^{-2x} = 2x + 5 - e^{-2x}$$
$$A = \frac{1}{2}, B = 2, C = \frac{1}{2}$$

Our solution is:

$$y_c + y_p = \boxed{c_1 + c_2e^{-2x} + \frac{1}{2}x^2 + 2x + \frac{1}{2}xe^{-2x}}$$

b) $y'' - 9y' = 2e^{3x}$

$$y'' - 9y' = 2e^{3x}$$
$$m^2 - 9m = 0$$
$$m(m-9) = 0$$
$$m = 9, 0$$
$$y_c = c_1 + c_2e^{9x}$$

$$y_p = Ae^{3x}$$
$$y_p' = 3Ae^{3x}$$
$$y_p'' = 9Ae^{3x}$$

$$9Ae^{3x} - 27Ae^{3x} = 2e^{3x}$$
$$-18Ae^{3x} = 2e^{3x}$$
$$A = -\frac{1}{9}$$

Our solution is then:

$$y_c + y_p = \boxed{c_1 + c_2 e^{9x} - \frac{1}{9} e^{3x}}$$

c) $y'' + 4y' + 4y = (3 + x)e^{-2x}$

$$
\begin{aligned}
&y'' + 4y' + 4y = (3 + x)e^{-2x} && y_p = Ax^2e^{-2x} + Bx^3e^{-2x} \\
&m^2 + 4m + 4 = 0 && y_p' = 2Axe^{-2x} - 2Ax^2e^{-2x} + 3Bx^2e^{-2x} - 2Bx^3e^{-2x} \\
&(m + 2)(m + 2) = 0 && y_p'' = 2Ae^{-2x} + 4Ae^{-2x} - 8Ae^{-2x} \\
&m = -2 && + 4Bx^3e^{-2x} - 12Bx^2e^{-2x} + 6Bxe^{-2x} \\
&y_c = c_1e^{-2x} + c_2xe^{-2x} &&
\end{aligned}
$$

Why did you choose to torture me like this, here is the nasty after plugging in:

$$
\begin{aligned}
&2Ae^{-2x} + 4Ae^{-2x} - 8Ae^{-2x} + 4Bx^3e^{-2x} - 12Bx^2e^{-2x} + 6Bxe^{-2x} \\
&+ 8Axe^{-2x} - 8Ax^2e^{-2x} + 12Bx^2e^{-2x} - 8Bx^3e^{-2x} \\
&+ Ax^2e^{-2x} + Bx^3e^{-2x} \\
&= 3e^{-2x} + xe^{-2x}
\end{aligned}
$$

Simplifies beautifully into:

$$
\begin{aligned}
&2Ae^{-2x} + 6Bxe^{-2x} = 3e^{-2x} + xe^{-2x} \\
&A = \frac{3}{2}, B = \frac{1}{6}
\end{aligned}
$$

Our solution is then :

$$y_c + y_p = \boxed{c_1e^{-2x} + c_2xe^{-2x} + \frac{3}{2}x^2e^{-2x} + \frac{1}{6}x^3e^{-2x}}$$

4.6.1

We proceed to solve the problem $3y'' + y' = 9x$ using variation of parameters where our $y_p = u_1y_1 + u_2y_2$.

$$
\begin{aligned}
& y'' + \frac{y'}{3} = 3x && W = \begin{vmatrix} 1 & e^{\frac{-x}{3}} \\ 0 & -\frac{1}{3}e^{\frac{-x}{3}} \end{vmatrix} = -\frac{1}{3}e^{\frac{-x}{3}} \\
& m^2 + \frac{m}{3} = 0 && W_1 = \begin{vmatrix} 0 & e^{\frac{-x}{3}} \\ 3x & -\frac{1}{3}e^{\frac{-x}{3}} \end{vmatrix} = -3xe^{\frac{-x}{3}} \\
& m(m + \frac{1}{3}) = 0 && W_2 = \begin{vmatrix} 1 & 0 \\ 0 & 3x \end{vmatrix} = 3x \\
& m = 0, m = -\frac{1}{3} && u_1 = \int \frac{-3xe^{\frac{-x}{3}}}{-\frac{1}{3}e^{\frac{-x}{3}}} = \frac{9x^2}{2} \\
& y_c = c_1 + c_2 e^{\frac{-x}{3}} && u_2 = \int \frac{3x}{-\frac{1}{3}e^{\frac{-x}{3}}} = -27e^{\frac{x}{3}}(x - 3)
\end{aligned}
$$

Our y_p should be $u_1 * 1 + u_2 * e^{\frac{-x}{3}}$. We proceed to plug in for u_1 and u_2, I'm assuming the 81 below gets absorbed into c_1 since its a constant too, our answer should be:

$$
y_c + y_p = c_1 + c_2 e^{-\frac{x}{3}} + \frac{9x^2}{3} - 27x + 81
$$

$$
\boxed{c_1 + c_2 e^{-\frac{x}{3}} + \frac{9x^2}{3} - 27x}
$$

4.6.2

We solve for the initial value problem $y'' - 4y' + 4y = (6x^2 - 12x)e^{2x}$, $y(0) = 1$, $y'(0) = 0$.