Problem 2.2:
Theorem: Let Z be an n x r null-space matrix for the matrix A. If Y is any invertible r x r matrix, $\hat{Z} = ZY$ is also a null-space matrix for A.

Proof: Given that Z is an n x r null-space matrix for matrix A, we know that $AZ = 0$, we also know that Y is an invertible matrix. We need to show that $A\hat{Z} = 0$ or $AZY = 0$ for $\hat{Z} = ZY$ to be a null-space matrix for A. Consider the following:

$$\begin{aligned} A\hat{Z} &= A(ZY) \\ A(ZY) &= (AZ)Y \\ A(ZY) &= 0Y \\ AZY = 0 &\text{ or } A\hat{Z} = 0 \end{aligned}$$

Thus proving $\hat{Z} = ZY$ is also a null-space matrix for A.

Problem 3.1: We will compute a basis for the null space for the following matrices(Denoted with A) using variable reduction:

{i}

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & -2 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \qquad R_1 - R_2 \to R_2$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & -2 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \qquad \frac{R_2}{2} + R_3 \to R_3$$

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 0 & -2x_2 - 2x_3 &= 0 & -x_3 + x_4 &= 0 & x_4 &= x_4 \\ x_4 &= t & x_3 &= t & x_2 &= -t & x_1 &= -t \end{aligned}$$

Thus null(A) $= t \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$ for $t \in \mathbb{R}$

{ii}

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$$

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 0 & x_2 &= x_2 & x_3 &= x_3 & x_4 &= x_4 \\ x_2 &= s & x_3 &= t & x_4 &= u & x_1 &= -s - t - u \end{aligned}$$

Thus null(A) $= s \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + u \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ for $s, t, u \in \mathbb{R}$

{iii}

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 0 \end{bmatrix} \qquad R_1 - R_2 \to R_2$$

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 0 & 2x_2 + 2x_3 &= 0 & x_3 &= s & x_4 &= t \\ x_2 = -x_3 &= -s & x_1 - s + s + t &= 0 & x_1 &= s - s - t & x_1 &= -t \end{aligned}$$

Thus null(A) $= s \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ for $s, t \in \mathbb{R}$

{iv}

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 0 & 0 & 2 \\ 1 & -1 & -1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 0 & 0 & 2 \\ 0 & -1 & -1 & 0 \end{bmatrix} \qquad \frac{R_2}{2} - R_3 \to R_3$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 0 \\ 0 & -1 & -1 & 0 \end{bmatrix} \qquad 2R_1 - R_2 \to R_2$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \frac{R_2}{2} + R_3 \to R_3$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \frac{R_2}{2} \to R_2$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad R_2 - R_1 \to R_1$$

$$\begin{aligned} x_1 + x_4 &= 0 & x_2 + x_3 &= 0 & x_3 &= s & x_4 &= t \\ x_2 &= -s & x_1 &= -t \end{aligned}$$

Thus null(A) $= s \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ for $s, t \in \mathbb{R}$