

Problem 2.2:

Theorem: Let Z be an $n \times r$ null-space matrix for the matrix A . If Y is any invertible $r \times r$ matrix, $\hat{Z} = ZY$ is also a null-space matrix for A .

Proof: Given that Z is an $n \times r$ null-space matrix for matrix A , we know that $AZ = 0$, we also know that Y is an invertible matrix. We need to show that $A\hat{Z} = 0$ or $AZY = 0$ for $\hat{Z} = ZY$ to be a null-space matrix for A . Consider the following:

$$\begin{aligned} A\hat{Z} &= A(ZY) \\ A(ZY) &= (AZ)Y \\ A(ZY) &= 0Y \\ AZY &= 0 \text{ or } A\hat{Z} = 0 \end{aligned}$$

Thus proving $\hat{Z} = ZY$ is also a null-space matrix for A .

Problem 3.1: We will compute a basis for the null space for the following matrices(Denoted with A) using variable reduction:

{i}

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & -2 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \quad R_1 - R_2 \rightarrow R_2$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & -2 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \quad \frac{R_2}{2} + R_3 \rightarrow R_3$$

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 0 & -2x_2 - 2x_3 &= 0 & -x_3 + x_4 &= 0 & x_4 &= x_4 \\ x_4 &= t & x_3 &= t & x_2 &= -t & x_1 &= -t \end{aligned}$$

$$\text{Thus null}(A) = t \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix} \text{ for } t \in \mathbb{R}$$

{ii}

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$$

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 0 & x_2 &= x_2 & x_3 &= x_3 & x_4 &= x_4 \\ x_2 &= s & x_3 &= t & x_4 &= u & x_1 &= -s - t - u \end{aligned}$$

$$\text{Thus null}(A) = s \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + u \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \text{ for } s, t, u \in \mathbb{R}$$

{iii}

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 0 \end{bmatrix}$$

$$R_1 - R_2 \rightarrow R_2$$

$$x_1 + x_2 + x_3 + x_4 = 0$$

$$2x_2 + 2x_3 = 0$$

$$x_3 = s$$

$$x_4 = t$$

$$x_2 = -x_3 = -s$$

$$x_1 - s + s + t = 0$$

$$x_1 = s - s - t$$

$$x_1 = -t$$

$$\text{Thus null}(A) = s \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \text{ for } s, t \in \mathbb{R}$$

{iv}

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 0 & 0 & 2 \\ 1 & -1 & -1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 0 & 0 & 2 \\ 0 & -1 & -1 & 0 \end{bmatrix}$$

$$\frac{R_2}{2} - R_3 \rightarrow R_3$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 0 \\ 0 & -1 & -1 & 0 \end{bmatrix}$$

$$2R_1 - R_2 \rightarrow R_2$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\frac{R_2}{2} + R_3 \rightarrow R_3$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\frac{R_2}{2} \rightarrow R_2$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R_2 - R_1 \rightarrow R_1$$

$$x_1 + x_4 = 0$$

$$x_2 + x_3 = 0$$

$$x_3 = s$$

$$x_4 = t$$

$$x_2 = -s$$

$$x_1 = -t$$

$$\text{Thus null}(A) = s \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \text{ for } s, t \in \mathbb{R}$$

Problem 3.3

$$A = \begin{bmatrix} 1 & 2 & 0 & 2 \\ 2 & 2 & 1 & 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 1 & -0.5 & 0 \end{bmatrix}$$