Problem 2.2:
Theorem: Let Z be an n x r null-space matrix for the matrix A. If Y is any invertible r x r matrix, $\hat{Z} = ZY$ is also a null-space matrix for A.

Proof: Given that Z is an n x r null-space matrix for matrix A, we know that $AZ = 0$, we also know that Y is an invertible matrix. We need to show that $A\hat{Z} = 0$ or $AZY = 0$ for $\hat{Z} = ZY$ to be a null-space matrix for A. Consider the following:

$$\begin{aligned} A\hat{Z} &= A(ZY) \\ A(ZY) &= (AZ)Y \\ A(ZY) &= 0Y \\ AZY &= 0 \text{ or } A\hat{Z} = 0 \end{aligned}$$

Thus proving $\hat{Z} = ZY$ is also a null-space matrix for A.

Problem 3.1: We will compute a basis for the null space for the following matrices(Denoted with A) using variable reduction:

{i}

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & -2 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \qquad R_1 - R_2 \to R_2$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & -2 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \qquad \frac{R_2}{2} + R_3 \to R_3$$

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 0 & -2x_2 - 2x_3 &= 0 & -x_3 + x_4 &= 0 & x_4 &= x_4 \\ x_4 &= t & x_3 &= t & x_2 &= -t & x_1 &= -t \end{aligned}$$

Thus null(A) $= t \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$ for $t \in \mathbb{R}$

{ii}

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$$

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 0 & x_2 &= x_2 & x_3 &= x_3 & x_4 &= x_4 \\ x_2 &= s & x_3 &= t & x_4 &= u & x_1 &= -s - t - u \end{aligned}$$

Thus null(A) $= s \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + u \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ for $s, t, u \in \mathbb{R}$

{iii}

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 0 \end{bmatrix} \qquad R_1 - R_2 \to R_2$$

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 0 & 2x_2 + 2x_3 &= 0 & x_3 &= s & x_4 &= t \\ x_2 = -x_3 &= -s & x_1 - s + s + t &= 0 & x_1 &= s - s - t & x_1 &= -t \end{aligned}$$

Thus null(A) $= s \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ for $s, t \in \mathbb{R}$

{iv}

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 0 & 0 & 2 \\ 1 & -1 & -1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 0 & 0 & 2 \\ 0 & -1 & -1 & 0 \end{bmatrix} \qquad \frac{R_2}{2} - R_3 \to R_3$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 0 \\ 0 & -1 & -1 & 0 \end{bmatrix} \qquad 2R_1 - R_2 \to R_2$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \frac{R_2}{2} + R_3 \to R_3$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \frac{R_2}{2} \to R_2$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad R_2 - R_1 \to R_1$$

$$\begin{aligned} x_1 + x_4 &= 0 & x_2 + x_3 &= 0 & x_3 &= s & x_4 &= t \\ x_2 &= -s & x_1 &= -t \end{aligned}$$

Thus null(A) $= s \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ for $s, t \in \mathbb{R}$

Problem 3.3 For the following problem we use p_2 and p_3 as our basic variables for A and we attempt the same with p_1 and p_4

$$A = \begin{bmatrix} 1 & 2 & 0 & 2 \\ 2 & 2 & 1 & 4 \end{bmatrix}$$

$$p_1 + 2p_2 + 2p_4 = 0 \qquad 2p_1 + p_2 + 2p_3 + 4p_4 = 0 \qquad p_1 = -2p_2 - 2p_4$$
$$-4p_2 - 4p_4 + p_2 + 2p_3 + 4p_4 = 0 \qquad -3p_2 + 2p_3 = 0 \qquad p_2 = \frac{2}{3}p_3$$
$$p_1 = -\frac{4}{3}p_3 - 2p_4$$

Since $p_2 = \frac{2}{3}p_3$ We end up using p_4 as our free variable

$$\text{null}(A) = p_3 \begin{bmatrix} -\frac{4}{3} \\ \frac{2}{3} \\ 1 \\ 0 \end{bmatrix} + p_4 \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

For the second part of the problem we consider p_1 and p_4 as our free variables, we similarly reduce and substitute to get our vectors:

$$p_1 + 2p_2 + 2p_4 = 0 \qquad 2p_1 + p_2 + 2p_3 + 4p_4 = 0 \qquad p_2 = -\frac{1}{2}(p_1 + 2p_4)$$
$$2p_1 - \frac{1}{2}p_1 - p_4 + 2p_3 + 4p_4 = 0 \qquad \frac{3}{2}p_1 + 2p_3 + 3p_4 = 0 \qquad 2p_3 = -\frac{3}{2} - 3p_4$$
$$p_3 = -\frac{3}{4}p_1 - \frac{3}{2}p_4$$

We can construct our solution with our $p_2 = -\frac{1}{2}(p_1 + 2p_4)$ and $p_3 = -\frac{3}{4}p_1 - \frac{3}{2}p_4$

$$\text{null}(A) = p_1 \begin{bmatrix} 1 \\ -\frac{1}{2} \\ -\frac{3}{4} \\ 0 \end{bmatrix} + p_4 \begin{bmatrix} 0 \\ -1 \\ -\frac{3}{2} \\ 1 \end{bmatrix}$$

So we can actually do it with p_1 and p_4 as basic variables.

Problem 3.4 Theorem: Let A be an m x n matrix of full row rank. The matrix AA^T is positive definite and so its inverse exists.

Proof: We want to show that $x^T AA^T x > 0$ (using the definition of positive definite) for some non-zero vector $x \in \mathbb{R}^m$, first we note that AA^T is symmetric:

$$(AA^T)^T = (A^T)^T A^T = AA^T$$

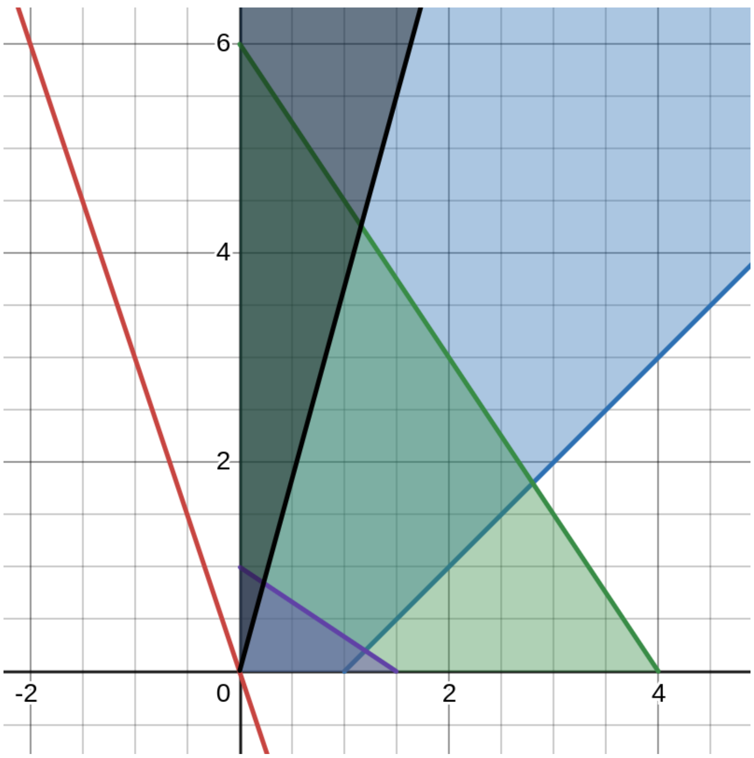
Now we consider the the case of $x^T AA^T x$

$$x^T AA^T x = (A^T x)^T (A^T x) = ||A^T x||^2$$

This is because the product of a vector and its transpose is the squared norm which is always positive. In our case as long as x is a non-zero vector, $x^T AA^T x$ is positive definite and thus invertible.

problem 1.1 We solve the following linear programs graphically:
{i}

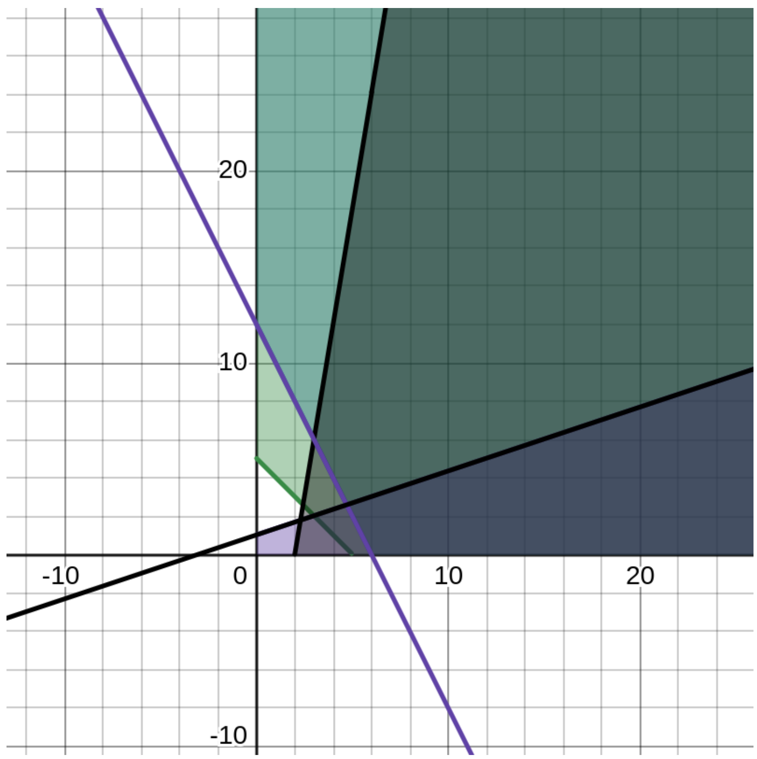
$$\begin{aligned} \text{minimize} \quad & z = 3x_1 + x_2 \\ \text{subject to} \quad & x_1 - x_2 \leq 1 \\ & 3x_1 + 2x_2 \leq 12 \\ & 2x_1 + 3x_2 \leq 3 \\ & -2x_1 + 3x_2 \geq 9 \\ & x_1, x_2 \geq 0 \end{aligned}$$



Our red line is our objective function for $z = 0$ within the boundary of our feasible set, the dark triangle in our graph, our minimum is 0 with our minimizer being $(0, 0)^T$.

{ii}

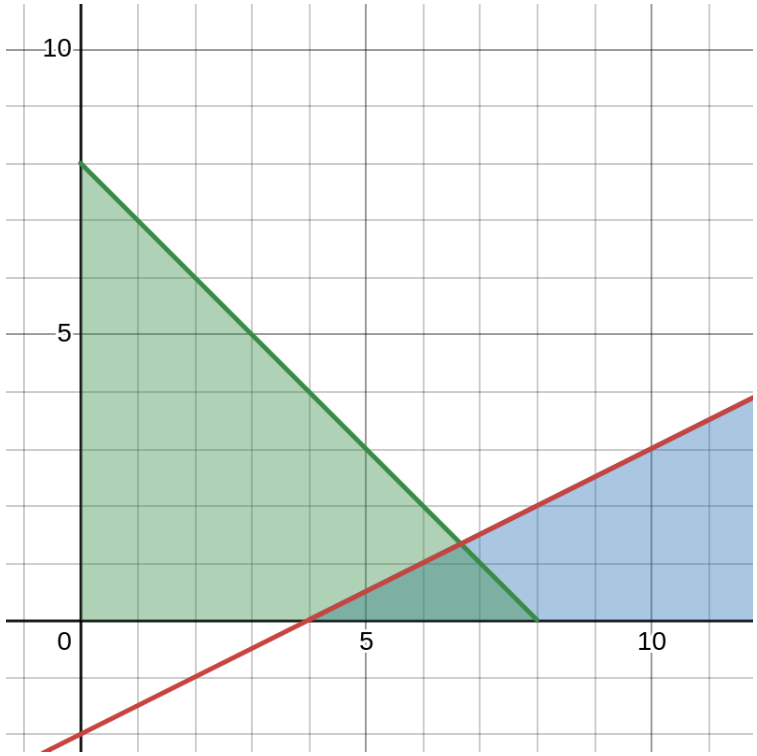
$$\begin{aligned} \text{maximize} \quad & z = x_1 + 2x_2 \\ \text{subject to} \quad & 2x_1 + x_2 \geq 12 \\ & x_1 + x_2 \geq 5 \\ & -x_1 + 3x_2 \leq 3 \\ & 6x_1 - x_2 \geq 12 \\ & x_1, x_2 \geq 0 \end{aligned}$$



Our solution to this problem is bound by the constraint $-x_1 + 3x_2 \leq 3$, Giving us our maximum value $z = 3$ along the line $-x_1 + 3x_2 = 3$. Our solution set ends up being $p_2 = \frac{p_1}{3} + 1$ for $p_1 \geq \frac{33}{7}$ and $p_2 \geq \frac{18}{7}$.

{iii}

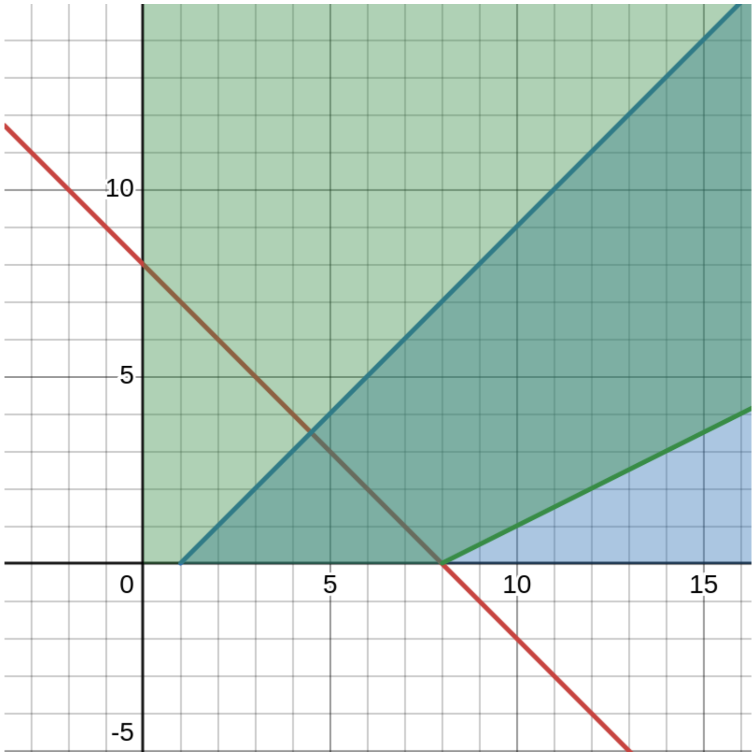
$$\begin{aligned} \text{minimize} \quad & z = x_1 - 2x_2 \\ \text{subject to} \quad & x_1 - 2x_2 \geq 4 \\ & x_1 + x_2 \leq 8 \\ & x_1, x_2 \geq 0 \end{aligned}$$



Our red line is our objective function for $z = 4$ and our feasible set is the darker green shaded triangle in our figure. our minimum value ends up being 4 with our minimizer being the set of points along the red line on the triangle. the point $(4, 0)^t$ is on the vertex is in our solution set for example. our solution set ends up being $p_2 = \frac{p_1}{2} - 2$ for $p_1 \in [4, \frac{20}{3}]$ and $p_2 \in [0, \frac{4}{3}]$.

{iv}

$$\begin{aligned} \text{minimize} \quad & z = -x_1 - x_2 \\ \text{subject to} \quad & x_1 - x_2 \geq 1 \\ & x_1 - 2x_2 \leq 8 \\ & x_1, x_2 \geq 0 \end{aligned}$$

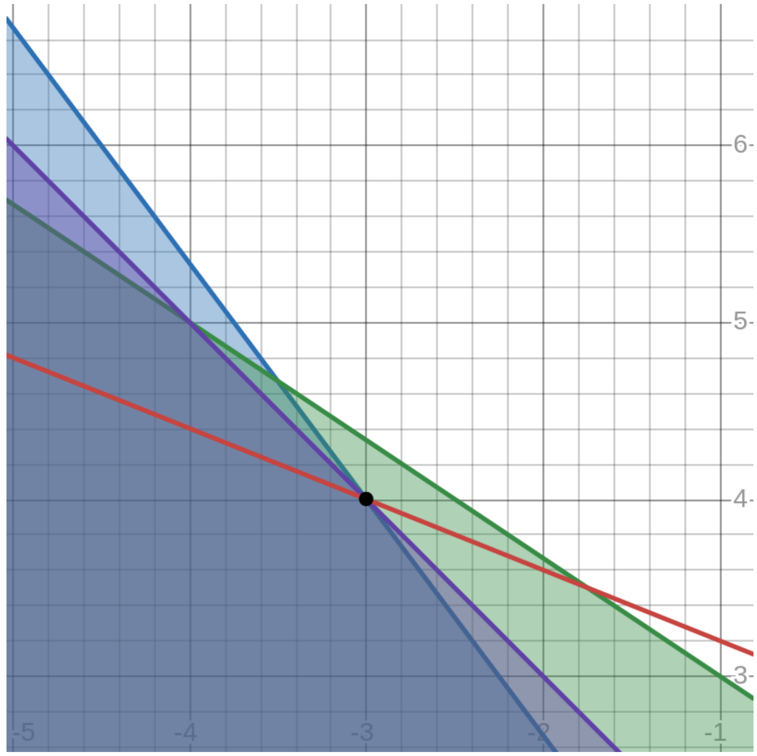


Our red line is our objective function for $z = -8$, our minimum ends up being -8 with our minimizer being $(8, 0)^T$

Problem 1.2 We are tasked with finding all values a such that $(-3, 4)^T$ us the optimal solution of the following problem:

$$\begin{array}{lr} \text{maximize} & z = ax_1 + (2-a)x_2 \\ \text{subject to} & 4x_1 + 3x_2 \leq 0 \\ & 2x_1 + 3x_2 \leq 7 \\ & x_1 + x_2 \leq 1 \end{array}$$

We graph the function below



Our red line is objective function with $a = \frac{4}{7}$ and our, we found this a by plotting our point $(-3, 4)^T$ and walking along a values. We find that our z grows larger and larger below that value of a. The range of a that maximizes our objective function for the given point we find is $a \in (-\infty, \frac{4}{7}]$.