Problem 2.7 We are given the problem

$$\text{minimize} \quad f(x_1, x_2) = (x_2 - x_1^2)(x_2 - 2x_1^2)$$

And are tasked with showing the first and second-order necessary conditions for optimality are satisfied at $(0,0)^T$ (i), showing that the origin is a local minimizer f along any line passing through the origin ($x_2 = mx_1$)(ii) and to show that the origin is not local minimizer of f.

(i) We first find the critical points by computing the gradient of $f(x_1, x_2)$:

$$f(x_1, x_2) = (x_2 - x_1^2)(x_2 - 2x_1^2) = x_2^2 - 3x_1^2 x_2 - 2x_1^4$$

$$\frac{\partial f}{\partial x_1} = -6x_1x_2 + 8x_1^3 \qquad \frac{\partial f}{\partial x_2} = 2x_2 - 3x_1^2$$

$$\nabla_f = \begin{bmatrix} -6x_1x_2 + 8x_1^3 \\ 2x_2 - 3x_1^2 \end{bmatrix} \qquad \nabla_f(0,0) = \begin{bmatrix} -6(0)(0) + 8(0) \\ 2(0) - 3(0) \end{bmatrix} = 0$$

Since $\nabla_f(0,0) = 0$, we satisfy the first order condition, now for the second order condition we find the hessian:

$$\frac{\partial^2 f}{\partial x_1^2} = 24x_1^2 - 6x_2 \qquad \frac{\partial^2 f}{\partial x_1 \partial x_2} = -6x_1$$

$$\frac{\partial^2 f}{\partial x_2 \partial x_1} = -6x_1 \qquad \frac{\partial^2 f}{\partial x_2^2} = 2$$

Evaluating for the $(0,0)^T$ and finding the eigenvalues we get:

$$\nabla_f^2 = \begin{bmatrix} 24x_1^2 - 6x_2 & -6x_1 \\ -6x_1 & 2 \end{bmatrix} \qquad \nabla_f^2(0,0) = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$$

$$|\nabla_f^2(0,0) - \lambda I| = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = 0 \qquad \lambda(2 - \lambda) = 0$$

The values for our eigenvalues, λ are 2 and 0, meaning it is positive semi-definite at this point, it could be a non-strict local minimizer.

(ii)