Problem 2.1 We covert the following linear program to standard form:

$$\text{maximize} \quad z = 3x_1 + 5x_2 - 4x_3 \tag{1}$$
$$\text{subject to} \quad 7x_1 - 2x_2 - 3x_3 \geq 4 \tag{2}$$
$$- 2x_1 + 4x_2 + 8x_3 = -3 \tag{3}$$
$$5x_1 - 3x_2 - 2x_3 \leq 9 \tag{4}$$
$$x_1 \geq 1, x_2 \leq 7, x_3 \geq 0 \tag{5}$$

We convert the problem into a minimization problem and multiplying (1) by -1, adding slack to (2), multiplying (3) by -1 adding surplus to (4), and turning our general constraints into equalities in 5, We will be using the notation in the book. Our linear program in standard form ends up being:

$$\text{minimize} \quad \hat{z} = -3x_1 - 5x_2 + 4x_3 \tag{1}$$
$$\text{subject to} \quad 7x_1 - 2x_2 - 3x_3 - e_4 = 4 \tag{2}$$
$$2x_1 - 4x_2 - 8x_3 = 3 \tag{3}$$
$$5x_1 - 3x_2 - 2x_3 + s_5 = 9 \tag{4}$$
$$x_1 - e_6 = 1 \tag{5}$$
$$x_2 + s_7 = 7 \tag{6}$$
$$x_1, x_2, x_3, e_4, s_5, e_6, s_7 \geq 0 \tag{7}$$

Or in our matrix-vector form:

$$\text{minimize} \quad z = c^T x$$
$$\text{subject to} \quad Ax = b$$
$$x \geq 0$$

Where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^n$, $x \in \mathbb{R}^n$ such that:

$$c = \begin{bmatrix} -3 & -5 & 4 & 0 & 0 & 0 & 0 \end{bmatrix}^T \qquad A = \begin{bmatrix} 7 & -2 & -3 & -1 & 0 & 0 & 0 \\ 2 & -4 & -8 & 0 & 0 & 0 & 0 \\ 5 & -3 & -2 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 & x_2 & x_3 & e_4 & s_5 & e_6 & s_7 \end{bmatrix}^T \qquad b = \begin{bmatrix} 4 & 3 & 9 & 1 & 7 \end{bmatrix}^T$$

Problem 2.2 We convert the flowing linear program to standard form:

$$\text{minimize} \quad z = x_1 - 5x_2 - 7x_3 \tag{1}$$
$$\text{subject to} \quad 5x_1 - 2x_2 + 6x_3 \geq 5 \tag{2}$$
$$3x_1 + 4x_2 - 9x_3 = 3 \tag{3}$$
$$7x_1 + 3x_2 + 5x_3 \leq 9 \tag{4}$$
$$x_1 \geq -2, x_2, x_3 \text{ free} \tag{5}$$

For this problem we start with substituting for the free variables first so we can inject those values into our function, we substitute $x_2 = x_2' - x_2''$ and $x_3 = x_3' - x_3''$, we also add excess to (2) and surplus to (4) and substitute our general constraints at (5) which then becomes:

$$\text{minimize} \quad \hat{z} = x_1 - 5x_2' + 5x_2'' - 7x_3' + 7x_3'' \tag{1}$$
$$\text{subject to} \quad 5x_1 - 2x_2' + 2x_2'' + 6x_3' - 6x_3'' - e_4 = 5 \tag{2}$$
$$3x_1 + 4x_2' - 4x_2'' - 9x_3' + 9x_3'' = 3 \tag{3}$$
$$7x_1 + 3x_2' - 3x_2'' + 5x_3' - 5x_3'' + s_5 = 9 \tag{4}$$
$$-x_1 - s_6 = 2 \tag{5}$$
$$x_1, x_2', x_2'', x_3', x_3'', e_4, s_5, s_6, \geq 0 \tag{6}$$

Or in our matrix-vector form:

$$\text{minimize} \quad z = c^T x$$
$$\text{subject to} \quad Ax = b$$
$$x \geq 0$$

Where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^n$, $x, e, s \in \mathbb{R}^n$ such that:

$$c = \begin{bmatrix} 1 & -5 & 5 & -7 & 7 & 0 & 0 & 0 \end{bmatrix}^T \quad A = \begin{bmatrix} 5 & -2 & 2 & 6 & -6 & -1 & 0 & 0 \\ 2 & 4 & -4 & -9 & 9 & 0 & 0 & 0 \\ 7 & 3 & -3 & 5 & -5 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 & x_2' & x_2'' & x_3' & x_3'' & e_4 & s_5 & s_6 \end{bmatrix}^T \qquad b = \begin{bmatrix} 5 & 3 & 9 & 2 \end{bmatrix}$$

Problem 2.4 We consider the following linear program and convert to standard form, this time without using the $x_3 = x_3' - x_3''$ substitution, rather we show that we can replace the problem with an equivalent problem with one less variable and one less constraint through eliminating x_3.

$$\text{maximize} \quad z = -5x_1 - 3x_2 + 7x_3 \tag{1}$$
$$\text{subject to} \quad 2x_1 + 4x_2 + 6x_3 = 7 \tag{2}$$
$$3x_1 - 5x_2 + 3x_3 \leq 5 \tag{3}$$
$$-4x_1 - 9x_2 + 4x_3 \leq -4 \tag{4}$$
$$x_1 \geq -2, 0 \leq x_2 \leq 4, x_3 \text{ free} \tag{5}$$

We solve for x_3 in (2) and get $x_3 = \frac{7 - 2x_1 - 4x_2}{6}$ and replace x_3 everywhere, multiply (1) and (4) by

-1, add slack to (3) and (4), and excess to x_1 and slack to x_2 in (5).

$$\text{minimize} \quad \hat{z} = 5x_1 + 3x_2 - 7\left(\frac{7}{6} - \frac{2x_1}{6} - \frac{4x_2}{6}\right) \tag{1}$$

$$\text{subject to} \quad 2x_1 + 4x_2 + 6\left(\frac{7}{6} - \frac{2x_1}{6} - \frac{4x_2}{6}\right) = 7 \tag{2}$$

$$3x_1 - 5x_2 + 3\left(\frac{7}{6} - \frac{2x_1}{6} - \frac{4x_2}{6}\right) + s_3 = 5 \tag{3}$$

$$4x_1 + 9x_2 - 4\left(\frac{7}{6} - \frac{2x_1}{6} - \frac{4x_2}{6}\right) - e_4 = 4 \tag{4}$$

$$- x_1 + s_5 = 2, \tag{5}$$

$$x_2 + s_6 = 4 \tag{6}$$

$$x_1, x_2, s_3, e_4, s_5, s_6 \geq 0 \tag{7}$$

We consolidate and simplify, replacing our objective function with $z = \hat{z} - \frac{49}{6}$, note that (2) gets eliminated above:

$$\text{minimize} \quad z' = \frac{22x_1}{3} + \frac{23x_2}{3}$$

$$\text{subject to} \quad - 2x_1 + 7x_2 - s_3 = \frac{3}{2}$$

$$\frac{16x_1}{3} + \frac{35x_2}{3} - e_4 = \frac{26}{3}$$

$$- x_1 + s_5 = 2,$$

$$x_2 + s_6 = 4$$

$$x_1, x_2, s_3, e_4, s_5, s_6 \geq 0$$

And in our matrix-vector form

$$\text{minimize} \quad z = c^T x$$

$$\text{subject to} \quad Ax = b$$

$$x \geq 0$$

Where $A \in \mathbb{R}^{mxn}$, $b \in \mathbb{R}^n$, $x, e, s \in \mathbb{R}^n$ such that:

$$c = \begin{bmatrix} \frac{22}{3} & \frac{23}{3} & 0 & 0 & 0 & 0 \end{bmatrix}^T \qquad A = \begin{bmatrix} -2 & 7 & -1 & 0 & 0 & 0 \\ \frac{16}{3} & \frac{35}{3} & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 & x_2 & s_3 & e_4 & s_5 & s_6 \end{bmatrix}^T \qquad b = \begin{bmatrix} \frac{3}{2} & \frac{26}{3} & 2 & 4 \end{bmatrix}$$

The reason this techinque cannot be used to eliminate variables with nonnegative constraints is because we can't ensure that our substitution for x is greater than or equal to zero, we may actually violate the original problems constraints if we don't add new constraints. I have not tested if this is equivalent to the original problem, though a concern is the more calculations or steps a human has to accomplish, the more likely there will be an error.