Problem 2.1 We covert the following linear program to standard form:

$$\begin{aligned} \text{maximize} \quad & z = 3x_1 + 5x_2 - 4x_3 && (1)\\ \text{subject to} \quad & 7x_1 - 2x_2 - 3x_3 \geq 4 && (2)\\ & -2x_1 + 4x_2 + 8x_3 = -3 && (3)\\ & 5x_1 - 3x_2 - 2x_3 \leq 9 && (4)\\ & x_1 \geq 1, x_2 \leq 7, x_3 \geq 0 && (5) \end{aligned}$$

We convert the problem into a minimization problem and multiplying (1) by -1, adding slack to (2), multiplying (3) by -1 adding surplus to (4), and turning our general constraints into equalities in 5, We will be using the notation in the book. Our linear program in standard form ends up being:

$$\begin{aligned} \text{minimize} \quad & \hat{z} = -3x_1 - 5x_2 + 4x_3 && (1)\\ \text{subject to} \quad & 7x_1 - 2x_2 - 3x_3 - e_4 = 4 && (2)\\ & 2x_1 - 4x_2 - 8x_3 = 3 && (3)\\ & 5x_1 - 3x_2 - 2x_3 + s_5 = 9 && (4)\\ & x_1 - e_6 = 1 && (5)\\ & x_2 + s_7 = 7 && (6)\\ & x_1, x_2, x_3, e_4, s_5, e_6, s_7 \geq 0 && (7) \end{aligned}$$

Or in our matrix-vector form:

$$\begin{aligned} \text{minimize} \quad & z = c^T x\\ \text{subject to} \quad & Ax = b\\ & x \geq 0 \end{aligned}$$

Where $A \in \mathbb{R}^{mxn}$, $b \in \mathbb{R}^n$, $x \in \mathbb{R}^n$ such that:

$$c = \begin{bmatrix} -3 & -5 & 4 & 0 & 0 & 0 & 0 \end{bmatrix}^T \qquad A = \begin{bmatrix} 7 & -2 & -3 & -1 & 0 & 0 & 0\\ 2 & -4 & -8 & 0 & 0 & 0 & 0\\ 5 & -3 & -2 & 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0 & 0 & -1 & 0\\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 & x_2 & x_3 & e_4 & s_5 & e_6 & s_7 \end{bmatrix}^T \qquad b = \begin{bmatrix} 4 & 3 & 9 & 1 & 7 \end{bmatrix}^T$$

Problem 2.2 We convert the flowing linear program to standard form:

$$\begin{aligned} \text{minimize} \quad & z = x_1 - 5x_2 - 7x_3 && (1)\\ \text{subject to} \quad & 5x_1 - 2x_2 + 6x_3 \geq 5 && (2)\\ & 3x_1 + 4x_2 - 9x_3 = 3 && (3)\\ & 7x_1 + 3x_2 + 5x_3 \leq 9 && (4)\\ & x_1 \geq -2, x_2, x_3 \text{ free} && (5) \end{aligned}$$

For this problem we start with substituting for the free variables first so we can inject those values into our function, we substitute $x_2 = x_2' - x_2''$ and $x_3 = x_3' - x_3''$, we also add excess to (2) and surplus to (4) and substitute our general constraints at (5) which then becomes:

$$\begin{aligned} \text{minimize} \quad & \hat{z} = x_1 - 5x_2' + 5x_2'' - 7x_3' + 7x_3'' && (1)\\ \text{subject to} \quad & 5x_1 - 2x_2' + 2x_2'' + 6x_3' - 6x_3'' - e_4 = 5 && (2)\\ & 3x_1 + 4x_2' - 4x_2'' - 9x_3' + 9x_3'' = 3 && (3)\\ & 7x_1 + 3x_2' - 3x_2'' + 5x_3' - 5x_3'' + s_5 = 9 && (4)\\ & -x_1 - s_6 = 2 && (5)\\ & x_1, x_2', x_2'', x_3', x_3'', e_4, s_5, s_6, \geq 0 && (6) \end{aligned}$$

Or in our matrix-vector form:

$$\begin{aligned} \text{minimize} \quad & z = c^T x\\ \text{subject to} \quad & Ax = b\\ & x \geq 0 \end{aligned}$$

Where $A \in \mathbb{R}^{mxn}$, $b \in \mathbb{R}^n$, $x, e, s \in \mathbb{R}^n$ such that:

$$c = \begin{bmatrix} 1 & -5 & 5 & -7 & 7 & 0 & 0 & 0 \end{bmatrix}^T \qquad A = \begin{bmatrix} 5 & -2 & 2 & 6 & -6 & -1 & 0 & 0\\ 2 & 4 & -4 & -9 & 9 & 0 & 0 & 0\\ 7 & 3 & -3 & 5 & -5 & 0 & 1 & 0\\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 & x_2' & x_2'' & x_3' & x_3'' & e_4 & s_5 & s_6 \end{bmatrix}^T \qquad b = \begin{bmatrix} 5 & 3 & 9 & 2 \end{bmatrix}$$

Problem 2.4 We consider the following linear program and convert to standard form, this time without using the $x_3 = x_3' - x_3''$ substitution, rather we show that we can replace the problem with an equivalent problem with one less variable and one less constraint through eliminating x_3.

$$\begin{aligned} \text{maximize} \quad & z = -5x_1 - 3x_2 + 7x_3 && (1)\\ \text{subject to} \quad & 2x_1 + 4x_2 + 6x_3 = 7 && (2)\\ & 3x_1 - 5x_2 + 3x_3 \leq 5 && (3)\\ & -4x_1 - 9x_2 + 4x_3 \leq -4 && (4)\\ & x_1 \geq -2, 0 \leq x_2 \leq 4, x_3 \text{ free} && (5) \end{aligned}$$

We solve for x_3 in (2) and get $x_3 = \frac{7-2x_1-4x_2}{6}$ and replace x_3 everywhere, multiply (1) and (4) by

-1, add slack to (3) and (4), and excess to x_1 and slack to x_2 in (5).

$$\begin{aligned}
\text{minimize} \quad & \hat{z} = 5x_1 + 3x_2 - 7(\frac{7}{6} - \frac{2x_1}{6} - \frac{4x_2}{6}) && (1)\\
\text{subject to} \quad & 2x_1 + 4x_2 + 6(\frac{7}{6} - \frac{2x_1}{6} - \frac{4x_2}{6}) = 7 && (2)\\
& 3x_1 - 5x_2 + 3(\frac{7}{6} - \frac{2x_1}{6} - \frac{4x_2}{6}) + s_3 = 5 && (3)\\
& 4x_1 + 9x_2 - 4(\frac{7}{6} - \frac{2x_1}{6} - \frac{4x_2}{6}) - e_4 = 4 && (4)\\
& -x_1 + s_5 = 2, && (5)\\
& x_2 + s_6 = 4 && (6)\\
& x_1, x_2, s_3, e_4, s_5, s_6 \geq 0 && (7)
\end{aligned}$$

We consolidate and simplify, replacing our objective function with $z = \hat{z} - \frac{49}{6}$, note that (2) gets eliminated above:

$$\begin{aligned}
\text{minimize} \quad & z' = \frac{22x_1}{3} + \frac{23x_2}{3}\\
\text{subject to} \quad & -2x_1 + 7x_2 - s_3 = \frac{3}{2}\\
& \frac{16x_1}{3} + \frac{35x_2}{3} - e_4 = \frac{26}{3}\\
& -x_1 + s_5 = 2,\\
& x_2 + s_6 = 4\\
& x_1, x_2, s_3, e_4, s_5, s_6 \geq 0
\end{aligned}$$

And in our matrix-vector form

$$\begin{aligned}
\text{minimize} \quad & z = c^T x\\
\text{subject to} \quad & Ax = b\\
& x \geq 0
\end{aligned}$$

Where $A \in \mathbb{R}^{mxn}$, $b \in \mathbb{R}^n$, $x, e, s \in \mathbb{R}^n$ such that:

$$c = \begin{bmatrix} \frac{22}{3} & \frac{23}{3} & 0 & 0 & 0 & 0 \end{bmatrix}^T \qquad A = \begin{bmatrix} -2 & 7 & -1 & 0 & 0 & 0 \\ \frac{16}{3} & \frac{35}{3} & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 & x_2 & s_3 & e_4 & s_5 & s_6 \end{bmatrix}^T \qquad b = \begin{bmatrix} \frac{3}{2} & \frac{26}{3} & 2 & 4 \end{bmatrix}$$

The reason this technique cannot be used to eliminate variables with nonnegative constraints is because we can't ensure that our substitution for x is greater than or equal to zero, we may actually violate the original problems constraints if we don't add new constraints.

Problem 2.5 We consider the following linear program below

$$\begin{aligned} \text{minimize} \quad & z = c^T x \\ \text{subject to} \quad & Ax \leq b \\ & e^T x = 1 \\ & x_1, ..., x_{n-1} \geq 0, x_n \text{free} \end{aligned}$$

Where $e = (1, ..., 1)^T$, b and c are arbitrary vectors of length n, and A is the matrix with entries $a_{i,i} = a_{i,n} = 1$ for $i = 1, ...n$ and all other entries being zero. We use the constraint $e^T x = 1$ to eliminate the free variable x_n.

Lets first show what A actually looks for visual reference and convenience:

$$A = \begin{bmatrix} 1 & 0 & 0 & ... & 0 & 1 \\ 0 & 1 & 0 & ... & 0 & 1 \\ 0 & 0 & 1 & ... & 0 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & ... & 1 & 1 \end{bmatrix}$$

Now lets consider the constraint $e^T x = 1$:

$$x_1 + x_2 + ... + x_n = 1$$

and solve for x_n:

$$x_n = 1 - (x_1 + x_2 + ... + x_{n-1})$$

We can substitute this into our objective function $z = c^T x$ and into our constraints like the previous problem we did, for the substitution in our objective function:

$$\begin{aligned} \hat{z} &= c_1 x_1 + c_2 x_2 + ...c_{n-1} x_{n-1} + c_n(1 - (x_1 + x_2 + ... + x_{n-1})) \\ &= (c_1 - c_n)x_1 + (c_2 - c_n)x_2 + ... + (c_{n-1} - c_n)x_{n-1} + c_n \end{aligned}$$

Now we do it for our $Ax \leq b$

$$\begin{bmatrix} x_1 + x_n \\ x_2 + x_n \\ \vdots \\ x_{n-1} + x_n \\ x_n + x_n \end{bmatrix} \leq b \qquad \Rightarrow \qquad \begin{bmatrix} 1 - x_1 \\ 1 - x_2 \\ \vdots \\ 1 - x_{n-1} \\ 2 - (x_1 + x_2 ... + x_{n-1}) \end{bmatrix} \leq b$$

so our constraints end up becoming:

$$\begin{aligned} & 1 - x_i \leq b_i \text{ for } i = 1, ..., n-1 \\ & 2 - (x_1 + x_2 + ... + x_{n-1}) \leq b_n \text{ for } n = i \end{aligned}$$

This might be a pretty good approach when n is large, I can't imagine that this would significantly improve performance for a solver though. It might be a marginal improvement, the reason I have my doubts is because the nature of this problem is linear and it will always remain linear. Though until I test this I won't know how much of an improvement if there is one. I'm a little hestitant to say if its a good approach or not considering the time and effort taken to remove x_n compared to maybe a marginal improvement in computation speed. This is a matter of the cost of human time and the cost of computation time.

Problem 3.1 Lets consider the system of linear constraints and find some basic feasible solutions and extreme points:

$$2x_1 + x_2 \le 100 \tag{1}$$
$$x_1 + x_2 \le 80 \tag{2}$$
$$x_1 \le 40 \tag{3}$$
$$x_1, x_2 \ge 0 \tag{4}$$

We convert it to standard form by adding slack variables to (1), (2), (3).

$$\begin{aligned} 2x_1 + x_2 + s_3 &= 100 \\ x_1 + x_2 + s_4 &= 80 \\ x_1 + s_5 &= 40 \\ x_1, x_2, s_3, s_4, s_5 &\ge 0 \end{aligned}$$

Part(i): Since there are 3 constraints we can set 3 variables for basis. There should be 5 choose 3, or 10 possibilities to try. Below is what we find.

$$x_1, x_2, s_3 \quad \Rightarrow \quad x_1 = 40, x_2 = 40, s_3 = -60 \tag{1}$$
$$x_1, x_2, s_4 \quad \Rightarrow \quad x_1 = 40, x_2 = 20, s_4 = 20 \tag{2}$$
$$x_1, x_2, s_5 \quad \Rightarrow \quad x_1 = 20, x_2 = 60, s_5 = 20 \tag{3}$$
$$x_1, s_3, s_4 \quad \Rightarrow \quad x_1 = 40, s_3 = 20, s_4 = 20 \tag{4}$$
$$x_1, s_3, s_5 \quad \Rightarrow \quad x_1 = 80, s_3 = -60, s_5 = -40 \tag{5}$$
$$x_1, s_4, s_5 \quad \Rightarrow \quad x_1 = 50, s_4 = 30, s_5 = -10 \tag{6}$$
$$x_2, s_3, s_4 \quad \Rightarrow \quad \text{No Solution} \tag{7}$$
$$x_2, s_3, s_5 \quad \Rightarrow \quad x_2 = 80, s_3 = 20, s_5 = 40 \tag{8}$$
$$x_2, s_4, s_5 \quad \Rightarrow \quad x_2 = 100, s_4 = -20, s_5 = 40 \tag{9}$$
$$s_3, s_4, s_5 \quad \Rightarrow \quad s_3 = 100, s_4 = 80, s_5 = 40 \tag{10}$$

Given our constraints, the following are the basic feasible solutions, the rest can be binned.

$$\begin{aligned} (x_1, x_2, s_3, s_4, s_5)^T &= (40, 20, 0, 20, 0)^T \\ (x_1, x_2, s_3, s_4, s_5)^T &= (20, 60, 0, 0, 20)^T \\ (x_1, x_2, s_3, s_4, s_5)^T &= (40, 0, 20, 20, 0)^T \\ (x_1, x_2, s_3, s_4, s_5)^T &= (0, 80, 20, 0, 40)^T \end{aligned}$$

As for extreme points they are $(0, 0), (0, 80), (20, 60), (40, 20), (40, 0)$ These are the vertices of our feasible region.