Problem 4.2:
In this problem we explore descent directions for the linear function $f(x) = x_1 - 2x_2 + 3x_3$ and determine whether our solutions depend on the value of x. Let us consider the gradient of our linear function:

$$\nabla f(x) = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$$

The descent direction is any direction that satisfies $\nabla f(x) * d < 0$ where d is a direction. Plugging in our values for our gradient we get $1d_1 - 2d_2 + 3d_3 < 0$. Note the absence of any x variables, meaning our answer does not depend on our value of x, it must only satisfy the inequality, in our case: $1d_1 - 2d_2 + 3d_3 < 0$. This should be true for all linear functions or constant gradient.

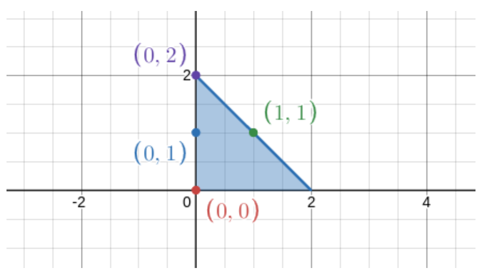
Problem 4.3: Consider the following problem:

$$\begin{aligned} \text{minimize} \quad & f(x) = -x_1 - x_2 \\ \text{subject to} \quad & x_1 + x_2 \leq 2 \\ & x_1, x_2 \geq 0 \end{aligned}$$

We will do the following:

{i} Determine the feasible directions at $x = (0,0)^T, (0,1)^T, (1,1)^T$ and $(0,2)^T$

{ii} Determine whether there exist feasible descent direction at these points, and hence determine which (if any) of the point can be local minimizers.



{i} Solution, below we let p be a set of vectors.

$$\begin{aligned} \bar{x} = (0,0)^T: \quad & \{p \in \mathbb{R}^2 | p_1 \geq 0, p_2 \geq 0\} \\ \bar{x} = (0,1)^T: \quad & \{p \in \mathbb{R}^2 | p_1 \geq 0, p_1 \leq 1\} \\ \bar{x} = (1,1)^T: \quad & \{p \in \mathbb{R}^2 | p_1 \leq 1, p_2 \leq 1\} \\ \bar{x} = (0,2)^T: \quad & \{p \in \mathbb{R}^2 | p_1 \geq 0, p_2 \leq 2\} \end{aligned}$$

{ii}
For the point $(0,0)^T$, every direction $p_1 \geq 0$ and $p_2 \geq 0$ is a feasible descent direction. Its not a local minimizer.

For the point $(0,1)^T$, there are feasible directions if $0 \leq p_1$ such that $p_1 + p_2 \leq 2$, not a local minimizer.
For point $(1,1)$, there are no feasible descent directions, this is local minimizer
For point $(0,2)$ there are no feasible descent directions, also a local minimizer

Problem 6.4: We find the first three terms of the taylor series at point $x_0 = (1,-1)^T$ for:

$$f(x_1,x_2) = 3x_1^4 - 2x_1^3x_2 - 4x_1^2x_2^2 + 5x_1x_2^3 + 2x_2^4$$

We first start with finding the gradient, our partial derivatives with respect to x_1 then x_2.

$$\nabla f(x) = \begin{pmatrix} 12x_1^3 - 6x_2x_1^2 - 8x_2^2x_1 + 5x_2^3 \\ -2x_1^3 - 8x_2x_1^2 + 15x_2^2x_1 + 8x_2^3 \end{pmatrix}$$

Followed by our Hessian:

$$\nabla^2 f(x) = \begin{pmatrix} 36x_1^2 - 12x_2x_1 - 8x_2^2 & -6x_1^2 - 16x_2x_1 + 15x_2^2 \\ -6x_1^2 - 16x_2x_1 + 15x_2^2 & -8x_1^2 + 30x_2x_1 + 24x_2^2 \end{pmatrix}$$

At the point $x_0 = (1,-1)$ our original function, gradient and Hessian have the following values:

$$f(1,-1) = -2 \qquad \nabla f(1,-1) = \begin{pmatrix} 5 \\ 13 \end{pmatrix} \qquad \nabla^2 f(1,-1) = \begin{pmatrix} 40 & 25 \\ 25 & -14 \end{pmatrix}$$

We then plug into our general form formula for second-order Taylor expansion
$f(x_1,x_2) \approx f(a) + \nabla f(a)^T(x-a) + \frac{1}{2}(x-a)^T\nabla^2(a)(x-a)$ to get our approximation:

$$f(1,-1) \approx -2 + \begin{pmatrix} 5 \\ 13 \end{pmatrix}\begin{pmatrix} x_1 - 1 \\ x_2 + 1 \end{pmatrix} + \frac{1}{2}(x_1 - 1\; x_2 + 1)\begin{pmatrix} 40 & 25 \\ 25 & -14 \end{pmatrix}\begin{pmatrix} x_1 - 1 \\ x_2 + 1 \end{pmatrix}$$

Evaluating the series for $f(x_0 + p)$ gives us the value -1.81621. Now we compare this to our series form:

$$\begin{aligned} f(1.01,-0.99) &\approx -2 + \begin{pmatrix} 5 \\ 13 \end{pmatrix}\begin{pmatrix} 0.1 \\ 0.01 \end{pmatrix} + \frac{1}{2}(0.1\; 0.01)\begin{pmatrix} 40 & 25 \\ 25 & -14 \end{pmatrix}\begin{pmatrix} 0.1 \\ 0.01 \end{pmatrix} \\ &= -2 + 0.63 - \frac{1}{2}0.4486 \\ &= -1.1457 \end{aligned}$$

Which is not close enough to -1.81621 for my liking so I screwed up somewhere.

Problem 6.5: For the function $f(x_1,x_2) = \sqrt{x_1^2 + x_2^2}$, we find the first three terms of the Taylor series at point $x_0 = (3.4)^T$, below are our computed Gradient and Hessian:

$$\nabla f(x) = \begin{pmatrix} \frac{x_1}{\sqrt{x_1^2+x_2^2}} \\ \frac{x_2}{\sqrt{x_1^2+x_2^2}} \end{pmatrix} \qquad \nabla^2 f(x) = \begin{pmatrix} \frac{1}{\sqrt{x_1^2+x_2^2}} - \frac{x_1^2}{(x_1^2+x_2^2)^{3/2}} & \frac{-x_1x_2}{(x_1^2+x_2^2)^{3/2}} \\ \frac{-x_1x_2}{(x_1^2+x_2^2)^{3/2}} & \frac{1}{\sqrt{x_1^2+x_2^2}} - \frac{x_2^2}{(x_1^2+x_2^2)^{3/2}} \end{pmatrix}$$

Now we can compute our terms and get our taylor expansion:

$$f(3,4) = 5 \qquad \nabla f(3,4) = (\frac{3}{5}, \frac{4}{5})^T \qquad \nabla^2 f(3,4) = \begin{pmatrix} \frac{16}{125} & -\frac{12}{125} \\ -\frac{12}{125} & \frac{16}{125} \end{pmatrix}$$

Which gets us:

$$f(x_1, x_2) \approx 5 + \begin{pmatrix} \frac{3}{5} \\ \frac{4}{5} \end{pmatrix} \begin{pmatrix} x_1 - 3 \\ x_2 - 4 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} x_1 - 3 \\ x_2 - 4 \end{pmatrix} \begin{pmatrix} \frac{16}{125} & -\frac{12}{125} \\ -\frac{12}{125} & \frac{16}{125} \end{pmatrix} \begin{pmatrix} x_1 - 3 \\ x_2 - 4 \end{pmatrix}$$

This is our taylor expansion
Problem 6.6 Theorem: If $p^T \nabla f(x_k) < 0$ then $f(x_k + \epsilon p) < 0$ for $\epsilon > 0$

Proof. First we expand $f(x_k + \epsilon p)$ in a Taylor series about the point x_k

$$f(x_k + \epsilon p) \approx f(x_k) + (\epsilon p)^t \nabla f(x_0) + \frac{1}{2} (\epsilon p)^T \nabla^2 f(x_k)(\epsilon p)$$

because ϵ is super tiny, the quadratic terms becomes negligible compared to the linear terms because ϵ is multiplyed by itself in the quadratic term, we can remove that term and do the following:

$$f(x_k + \epsilon p) \approx f(x_k) + (\epsilon p)^t \nabla f(x_0)$$
$$f(x_k + \epsilon p) - f(x_k) \approx (\epsilon p)^t \nabla f(x_0)$$

since $p^T \nabla f(x_k) < 0$ or negative and $\epsilon > 0$ or positive, $(\epsilon p)^t \nabla f(x_0) < 0$, Thus:

$$f(x_k + \epsilon p) - f(x_k) \approx (\epsilon p)^t \nabla f(x_0) < 0$$
$$f(x_k + \epsilon p) - f(x_k) < 0$$

□

Problem 1.2 We consider the set defined by the constraints $x_1 + x_2 = 1, x_1 \geq 0, x_2 \geq 0$ at the points (a)$(0,1)^T$; (b)$(1,0)^T$ and (c)$(0.5)(0.5)^T$ and determine the feasible directions, let p be a set of vectors in $\mathbb{R}^2$:

(a) For the point $(0,1)^T$, our only feasible direction is on the line at the end in one direction increasing our x_1 and decreasing our x_2. in order to stay within our constraints:
$\{p \in \mathbb{R}^2 | p_1 = -p_2, p_1 \geq 0, p_2 \leq 1\}$

(b) For the point $(1,0)^T$ we run into a similar case where our only feasible direction is on the line, decreasing our x_1 while increasing our x_2, in order to stay within our constraints:
$\{p \in \mathbb{R}^2 | -p_1 = p_2, p_1 \leq 1, p_2 \geq 0\}$

(c) For the point $(0.5, 0.5)^T$ we are also bounded to the line, we're right in the middle of it. We can move in 2 directions , up and down the line where $p_1 = -p_2$. We'll be more explicit about our constraint: $\{p \in \mathbb{R}^2 | p_1 + p_2 = 0\}$

Problem 1.3 We consider the system of inequality constraints $Ax \geq b$:

$$A = \begin{pmatrix} 9 & 4 & 1 & 9 & -7 \\ 6 & -7 & 8 & -4 & -6 \\ 1 & 6 & 3 & -7 & 6 \end{pmatrix} \qquad \text{and } b = \begin{pmatrix} -15 \\ -30 \\ -20 \end{pmatrix}$$

We will perform a ratio test $\bar{\alpha} = \min(A^T\bar{x} - b_i)/(-\alpha_i^T p)$ to determine the max step length $\bar{\alpha}$ such that $x + \bar{\alpha}p$ remains feasible for the following values of x and p:

(i) $x = (8, 4, -3, 4, 1)^T$ and $p = (1, 1, 1, 1, 1)^T$

$$Ax = \begin{pmatrix} 9 & 4 & 1 & 9 & -7 \\ 6 & -7 & 8 & -4 & -6 \\ 1 & 6 & 3 & -7 & 6 \end{pmatrix} \begin{pmatrix} 8 \\ 4 \\ -3 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 114 \\ -26 \\ 1 \end{pmatrix}$$

$$Ap = \begin{pmatrix} 9 & 4 & 1 & 9 & -7 \\ 6 & -7 & 8 & -4 & -6 \\ 1 & 6 & 3 & -7 & 6 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 16 \\ -3 \\ 9 \end{pmatrix}$$

$$b - Ax = \begin{pmatrix} -15 \\ -30 \\ 20 \end{pmatrix} - \begin{pmatrix} 114 \\ -26 \\ 1 \end{pmatrix} = \begin{pmatrix} -129 \\ -4 \\ 19 \end{pmatrix}$$

Now we can begin to solve the inequality for $\bar{a} \geq \frac{b-Ax}{Ap}$ note that $\bar{\alpha}$ needs to be non-negative for there to be a feasible step.

$$\bar{a} \geq \frac{-129}{16} \approx -8.0625 \tag{1}$$

$$\bar{a} \geq \frac{-4}{-3} \approx 1.333\bar{3} \tag{2}$$

$$\bar{a} \geq \frac{19}{9} \approx 2.111\bar{1} \tag{3}$$

Thus we find the maximum step length to be $\bar{a} \approx 2.111\bar{1}$

(ii) $x = (7, -4, -3, -3, 3)^T$ and $p = (3, 2, 0, 1, -2)^T$

$$Ax = \begin{pmatrix} 9 & 4 & 1 & 9 & -7 \\ 6 & -7 & 8 & -4 & -6 \\ 1 & 6 & 3 & -7 & 6 \end{pmatrix} \begin{pmatrix} 7 \\ -4 \\ -3 \\ -3 \\ 3 \end{pmatrix} = \begin{pmatrix} -4 \\ 40 \\ 13 \end{pmatrix}$$

$$Ap = \begin{pmatrix} 9 & 4 & 1 & 9 & -7 \\ 6 & -7 & 8 & -4 & -6 \\ 1 & 6 & 3 & -7 & 6 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \\ 0 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 58 \\ 12 \\ -4 \end{pmatrix}$$

$$b - Ax = \begin{pmatrix} -15 \\ -30 \\ 20 \end{pmatrix} - \begin{pmatrix} -4 \\ -40 \\ 13 \end{pmatrix} = \begin{pmatrix} -11 \\ -10 \\ 7 \end{pmatrix}$$

$$\bar{a} \geq \frac{-11}{58} \approx -0.1896 \tag{1}$$
$$\bar{a} \geq \frac{-10}{12} \approx -0.833\bar{3} \tag{2}$$
$$\bar{a} \geq \frac{7}{-4} \approx -1.7500 \tag{3}$$

There are no feasible step, all these values are negative.

(iii) $x = (5, 0, -6, -8, 3)^T$ and $p = (5, 0, 5, 1, 3)^T$

$$Ax = \begin{pmatrix} 9 & 4 & 1 & 9 & -7 \\ 6 & -7 & 8 & -4 & -6 \\ 1 & 6 & 3 & -7 & 6 \end{pmatrix} \begin{pmatrix} 5 \\ 0 \\ -6 \\ -8 \\ -3 \end{pmatrix} = \begin{pmatrix} -12 \\ 32 \\ 25 \end{pmatrix}$$

$$Ap = \begin{pmatrix} 9 & 4 & 1 & 9 & -7 \\ 6 & -7 & 8 & -4 & -6 \\ 1 & 6 & 3 & -7 & 6 \end{pmatrix} \begin{pmatrix} 5 \\ 0 \\ 5 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 38 \\ 48 \\ 31 \end{pmatrix}$$

$$b - Ax = \begin{pmatrix} -15 \\ -30 \\ 20 \end{pmatrix} - \begin{pmatrix} -12 \\ -32 \\ 25 \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \\ -5 \end{pmatrix}$$

$$\bar{a} \geq \frac{-3}{38} \approx -0.0790 \tag{1}$$
$$\bar{a} \geq \frac{2}{48} \approx 0.041\bar{6} \tag{2}$$
$$\bar{a} \geq \frac{-5}{31} \approx -0.1613 \tag{3}$$

We find our maximum step length to be $\bar{a} \approx 0.041\bar{6}$

(iv) $x = (9, 1, -1, 6, 3)^T$ and $p = (-4, -2, 4, -2, 2)^T$

$$Ax = \begin{pmatrix} 9 & 4 & 1 & 9 & -7 \\ 6 & -7 & 8 & -4 & -6 \\ 1 & 6 & 3 & -7 & 6 \end{pmatrix} \begin{pmatrix} 9 \\ 1 \\ -1 \\ 6 \\ 3 \end{pmatrix} = \begin{pmatrix} 117 \\ -3 \\ -12 \end{pmatrix}$$

$$Ap = \begin{pmatrix} 9 & 4 & 1 & 9 & -7 \\ 6 & -7 & 8 & -4 & -6 \\ 1 & 6 & 3 & -7 & 6 \end{pmatrix} \begin{pmatrix} -4 \\ -2 \\ 4 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} -72 \\ 18 \\ 22 \end{pmatrix}$$

$$b - Ax = \begin{pmatrix} -15 \\ -30 \\ 20 \end{pmatrix} - \begin{pmatrix} 117 \\ -3 \\ -12 \end{pmatrix} = \begin{pmatrix} -132 \\ -27 \\ 32 \end{pmatrix}$$

$$\bar{a} \geq \frac{-132}{-72} \approx 1.833\bar{3} \tag{1}$$

$$\bar{a} \geq \frac{-26}{18} \approx -1.444\bar{4} \tag{2}$$

$$\bar{a} \geq \frac{32}{-12} \approx -2.6666 \tag{3}$$

We find our maximum step length to be $\bar{a} \approx 1.833\bar{3}$