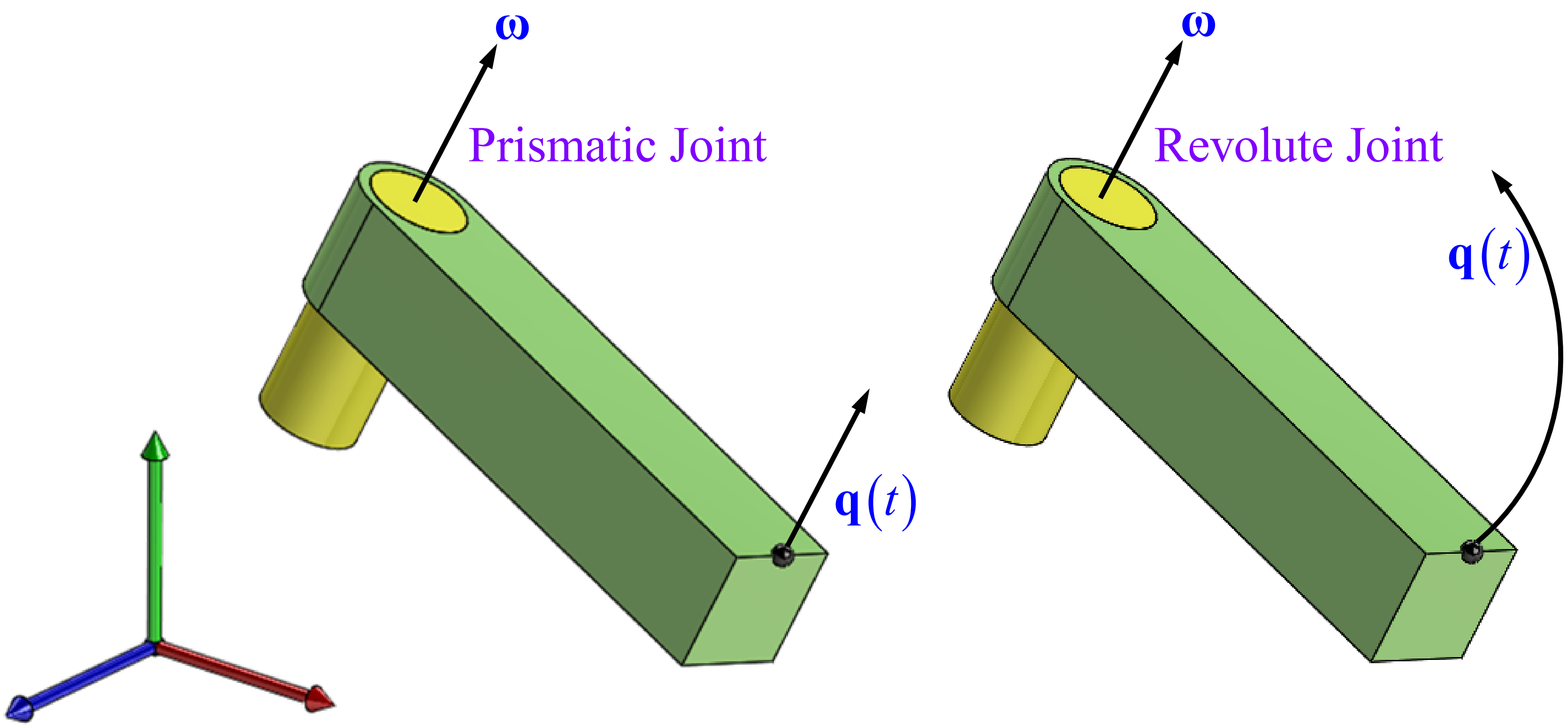
Exponential Coordinates for Rigid Motion

Now consider rotation about an axis that does not pass through the origin of the A frame.

We are primarily interested in revolute joints and prismatic joints.

$\boldsymbol{\omega} \in \mathbb{R}^3$

$\|\boldsymbol{\omega}\| = 1$



Exponential Coordinates for Rigid Motion

For revolute joints, consider a point $\mathbf{p}$, rotation about axis $\boldsymbol{\omega}$, which passes through point $\mathbf{q}$:

$$\boldsymbol{\omega} \in \mathbb{R}^3 = \text{ axis of rotation}$$
$$\mathbf{q} \in \mathbb{R}^3 = \text{ point (any!) on the axis of rotation relative to the A frame}$$
$$\mathbf{p} \in \mathbb{R}^3 = \text{ a point on the end of the rotating body}$$

Assuming the link rotates with unit velocity, then the velocity of the tip point is:

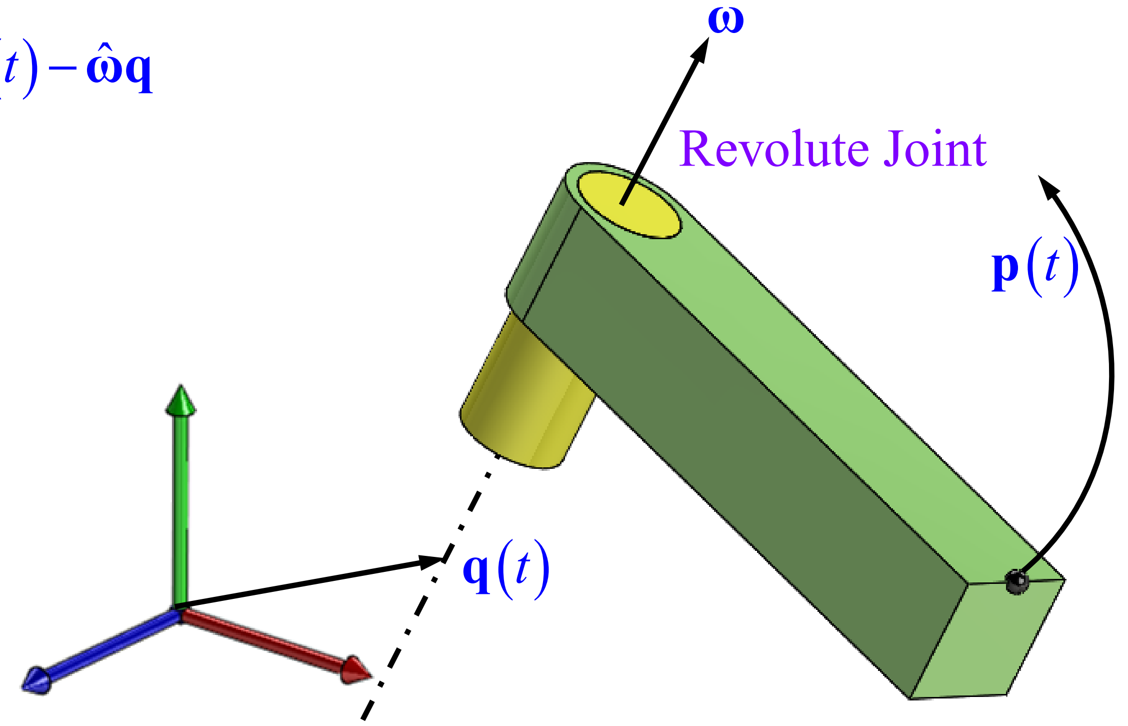
$$\dot{\mathbf{p}}(t) = \boldsymbol{\omega} \times (\mathbf{p}(t) - \mathbf{q}) = \boldsymbol{\omega} \times \mathbf{p}(t) - \boldsymbol{\omega} \times \mathbf{q} = \hat{\boldsymbol{\omega}}\mathbf{p}(t) - \hat{\boldsymbol{\omega}}\mathbf{q}$$

This equation can be put into homogeneous form by defining:

$$\hat{\boldsymbol{\xi}} = \begin{bmatrix} \hat{\boldsymbol{\omega}} & \mathbf{v} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \hat{\boldsymbol{\omega}} & -\boldsymbol{\omega} \times \mathbf{q} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \hat{\boldsymbol{\omega}} & -\hat{\boldsymbol{\omega}}\mathbf{q} \\ 0 & 0 \end{bmatrix}$$

Using the above definition, we have:

$$\begin{bmatrix} \dot{\mathbf{p}} \\ 0 \end{bmatrix} = \begin{bmatrix} \hat{\boldsymbol{\omega}} & -\hat{\boldsymbol{\omega}}\mathbf{q} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{p} \\ 1 \end{bmatrix} \Rightarrow \dot{\bar{\mathbf{p}}} = \hat{\boldsymbol{\xi}}\bar{\mathbf{p}}$$
$$= \hat{\boldsymbol{\omega}}\mathbf{p}(t) - \hat{\boldsymbol{\omega}}\mathbf{q}$$



Exponential Coordinates for Rigid Motion

To find point $\mathbf{p}$ after a rotation, we solve the following first order differential equation:

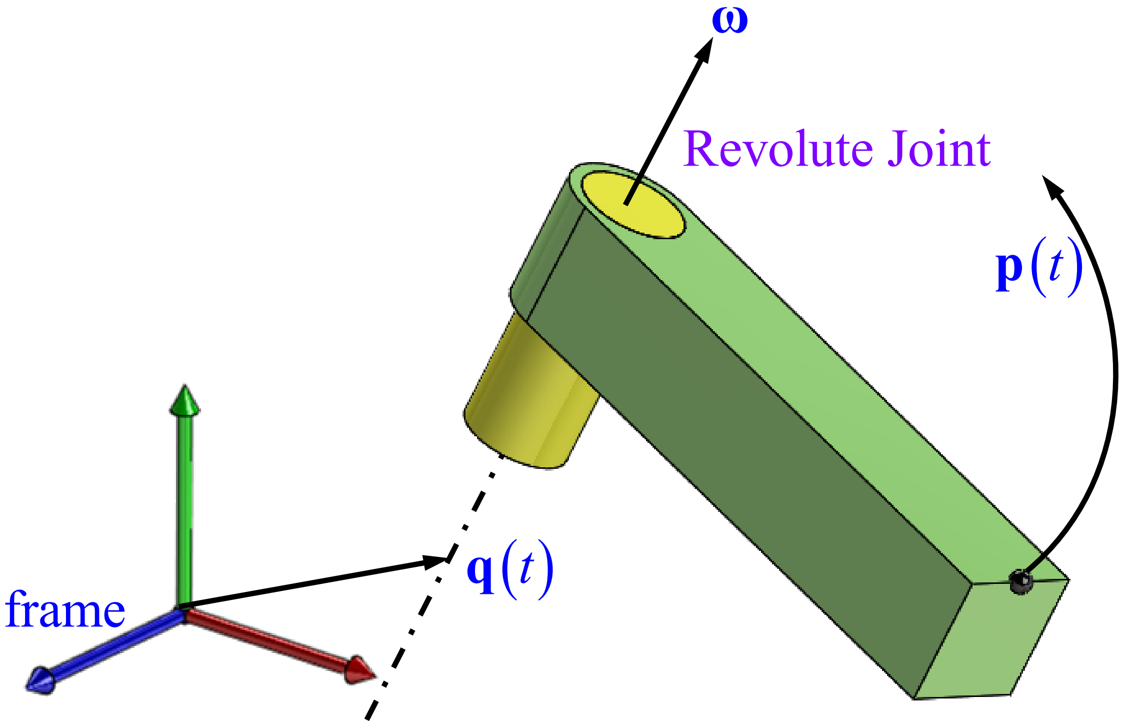
$$\dot{\bar{\mathbf{p}}} = \hat{\boldsymbol{\xi}}\bar{\mathbf{p}} \qquad \hat{\boldsymbol{\xi}} = \begin{bmatrix} \hat{\boldsymbol{\omega}} & -\hat{\boldsymbol{\omega}}\mathbf{q} \\ 0 & 0 \end{bmatrix}$$ for pure rotation!

The solution to the above equation is:

$$\mathbf{p}(t) = e^{\hat{\xi}t}\mathbf{p}(0)$$

And the matrix exponential is:

$$e^{\hat{\xi}t} = \mathbf{I} + \hat{\xi}t + \frac{\left(\hat{\xi}t\right)^2}{2!} + \frac{\left(\hat{\xi}t\right)^3}{3!} + \cdots$$



$\boldsymbol{\omega} \in \mathbb{R}^3 =$ axis of rotation

$\mathbf{q} \in \mathbb{R}^3 =$ point (any!) on the axis of rotation relative to the A frame

$\mathbf{p} \in \mathbb{R}^3 =$ a point on the end of the rotating body

Exponential Coordinates for Rigid Motion

For prismatic joints, point **p** translates directly along axis **v**. In this case, we have:

$$\hat{\xi} = \begin{bmatrix} \hat{\boldsymbol{\omega}} & \mathbf{v} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \mathbf{v} \\ 0 & 0 \end{bmatrix} \text{ for pure translation}$$

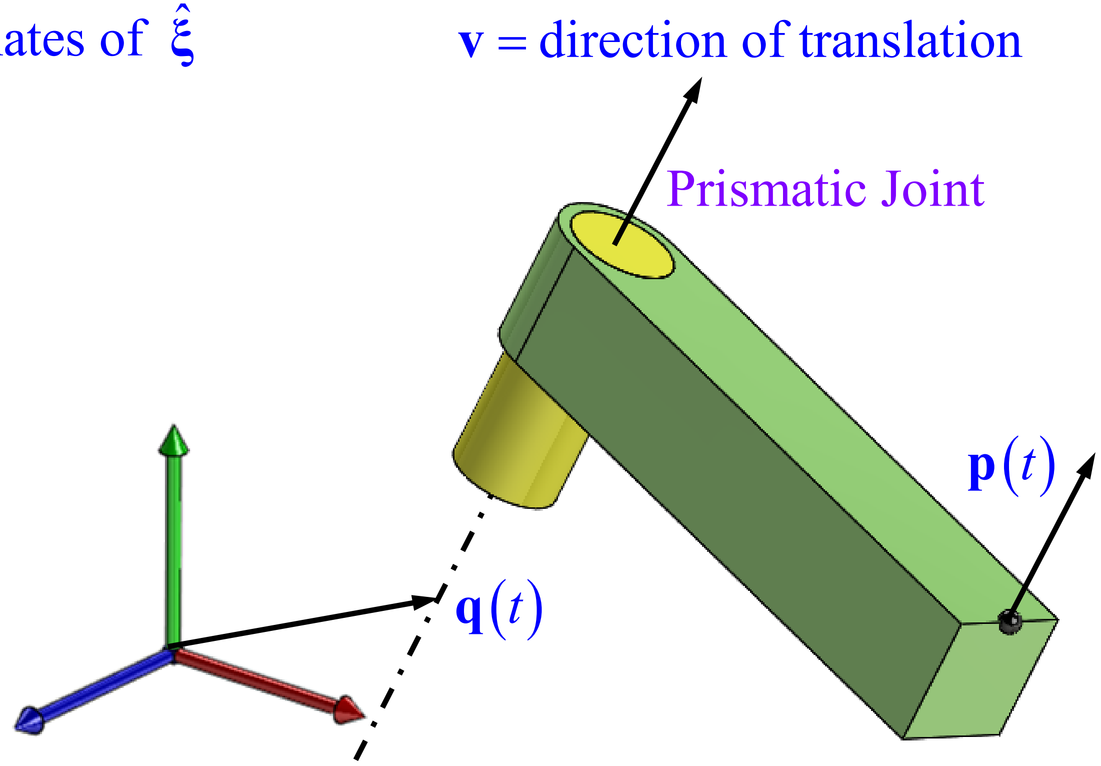
For rotation joints we have:

$$\hat{\xi} = \begin{bmatrix} \hat{\boldsymbol{\omega}} & \mathbf{v} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \hat{\boldsymbol{\omega}} & -\hat{\boldsymbol{\omega}}\mathbf{q} \\ 0 & 0 \end{bmatrix} \text{ for pure rotation!}$$

The pair ($\mathbf{v}$, $\boldsymbol{\omega}$) is called a twist. $\xi = \begin{bmatrix} \mathbf{v} \\ \boldsymbol{\omega} \end{bmatrix}$ are the twist coordinates of $\hat{\xi}$

The "wedge" and "vee" operators transform twists to and from homogeneous representation:

$$\underbrace{\begin{bmatrix} \mathbf{v} \\ \boldsymbol{\omega} \end{bmatrix}}_{6\times 1}^{\wedge} = \underbrace{\begin{bmatrix} \hat{\boldsymbol{\omega}} & \mathbf{v} \\ 0 & 0 \end{bmatrix}}_{4\times 4} \text{ and } \begin{bmatrix} \hat{\boldsymbol{\omega}} & \mathbf{v} \\ 0 & 0 \end{bmatrix}^{\vee} = \begin{bmatrix} \mathbf{v} \\ \boldsymbol{\omega} \end{bmatrix}$$



Exponential Coordinates for Rigid Motion

We again consider a rotation about $\boldsymbol{\omega}$ at unit velocity for θ units of time:
This is a relative mapping (same frame).

$$\mathbf{p}_a(t) = e^{\hat{\xi}\theta}\mathbf{p}_a(0) \qquad \hat{\xi} = \begin{bmatrix} \hat{\boldsymbol{\omega}} & -\hat{\boldsymbol{\omega}}\mathbf{q} \\ 0 & 0 \end{bmatrix}$$

The above equation maps point from their original coordinates to their coordinates after the rigid motion is applied.

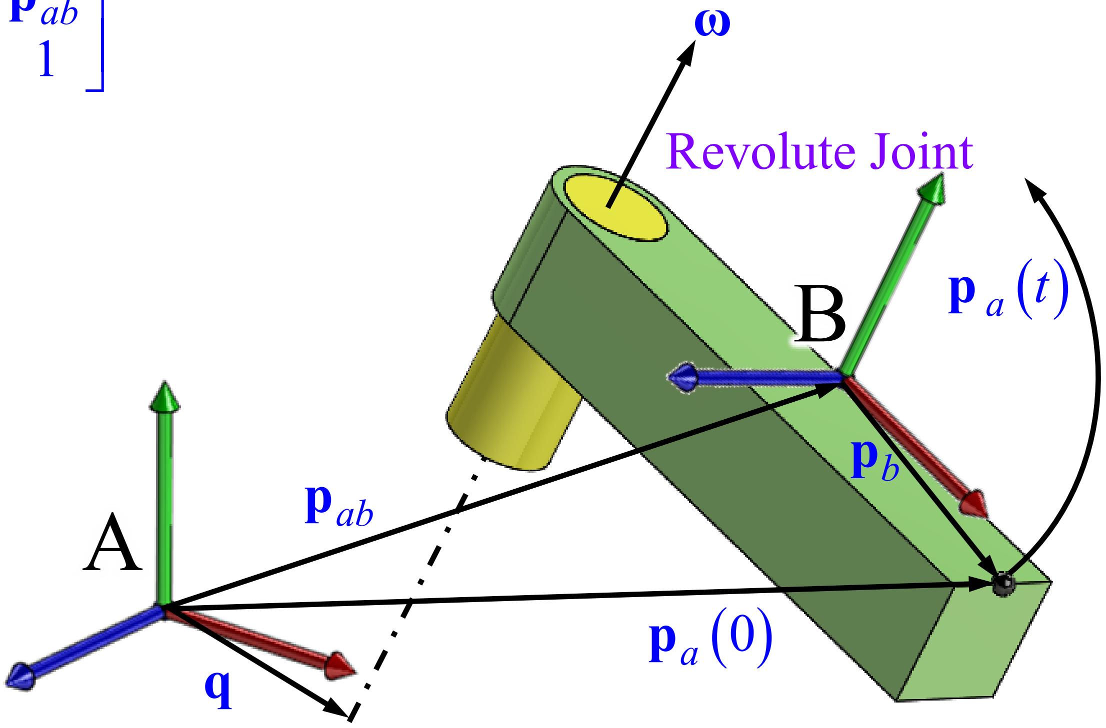
Alternatively, we can define a frame that rotates with (i.e. is fixed to) the rotating body. In this case, we define the original configuration of the rigid body as:

$$\mathbf{g}_{ab}(0) = \begin{bmatrix} \mathbf{R}_{ab} & \mathbf{p}_{ab} \\ 0 & 1 \end{bmatrix}$$

With this definition, we can define a transformation from frame B to frame A:

$$\mathbf{p}_a(t) = e^{\hat{\xi}\theta}\mathbf{g}_{ab}(0)\mathbf{p}_b$$

$$\boxed{\mathbf{p}_a(t) = \mathbf{g}_{ab}(\theta)\mathbf{p}_b} \quad \text{where} \quad \boxed{\mathbf{g}_{ab}(\theta) = e^{\hat{\xi}\theta}\mathbf{g}_{ab}(0)}$$



$\boldsymbol{\omega} \in \mathbb{R}^3 =$ axis of rotation

$\mathbf{q} \in \mathbb{R}^3 =$ any point on $\boldsymbol{\omega}$ defined relative to frame A

$\mathbf{p} \in \mathbb{R}^3 =$ a point on the end of the rotating body