Properties of Rotation Matrices

Consider a rotation matrix, with 3 columns defined:

$$\mathbf{R} \in \mathbb{R}^{3\times 3},\ \mathbf{R} = \begin{bmatrix} \mathbf{r}_1 & \mathbf{r}_2 & \mathbf{r}_3 \end{bmatrix} \text{ with columns } \mathbf{r}_1,\ \mathbf{r}_2,\ \mathbf{r}_3 \in \mathbb{R}^3$$

The columns of R are mutually orthogonal, thus:

$$\mathbf{r}_i^T \mathbf{r}_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

This result can be rewritten as:

$$\mathbf{R}^T \mathbf{R} = \begin{bmatrix} \mathbf{r}_1^T \\ \mathbf{r}_2^T \\ \mathbf{r}_3^T \end{bmatrix} \begin{bmatrix} \mathbf{r}_1 & \mathbf{r}_2 & \mathbf{r}_3 \end{bmatrix}$$

$$\mathbf{R}\mathbf{R}^T = \mathbf{R}^T \mathbf{R} = \mathbf{I}$$

Which implies: $\det(\mathbf{R}) = \pm 1 \quad (\det(\mathbf{R}) = 1 \text{ for right hand systems})$

Special Orthogonal 3 Group (Rotation Matrices)

Special Orthogonal 3 Group, SO(3), is the set of 3x3 rotation matrices that satisfy the properties:

$$\mathbf{R}\mathbf{R}^T = \mathbf{R}^T\mathbf{R} = \mathbf{I} \quad \& \quad \det(\mathbf{R}) = +1, \quad \mathbf{R} \in \mathbb{R}^{3\times 3}$$

For more general n^th order rotation matrices:

$$\mathbf{SO}(n) = \left\{\mathbf{R} \in \mathbb{R}^{n\times n} : \mathbf{R}\mathbf{R}^T = \mathbf{I}\ ,\ \det(\mathbf{R}) = +1\right\}$$

when $n = 2$ we have planar rotation matrices

when $n = 3$ we have spatial rotation matrices

$\mathbf{SO}(3)$ is the rotation group of $\mathbb{R}^3$

Rotation & Group Properties

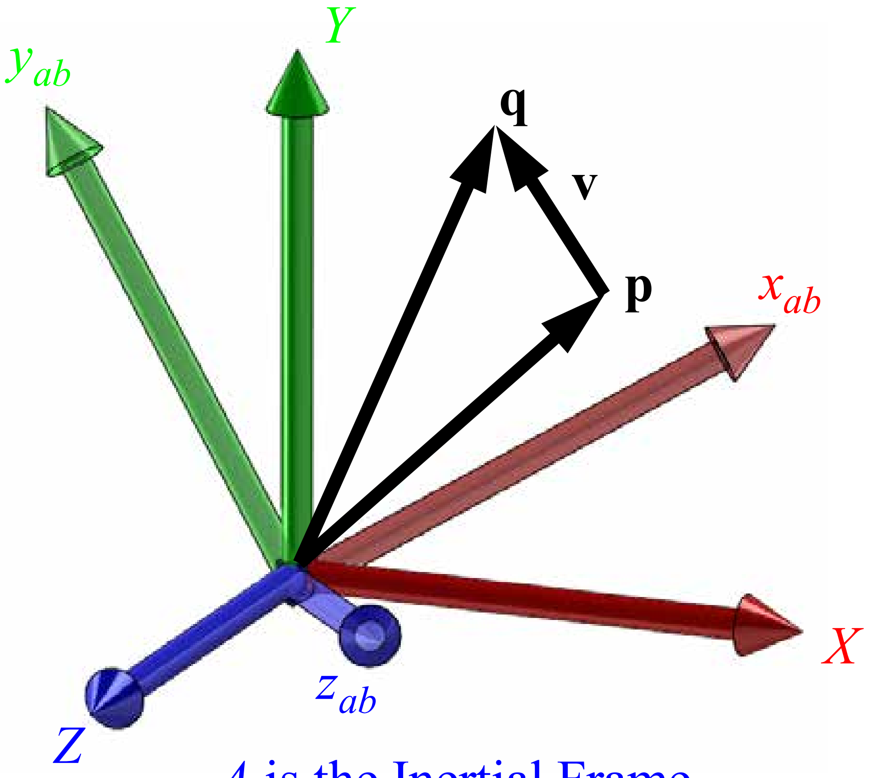
A set **G** together with a binary operation ○ defined on elements of **G** is called a **group** if it satisfies the properties tabulated below. Also shown are the equivalent definitions and properties for rotation group SO(3):

Property	GENERAL	ROTATION MATRICES
Element and Group	$g \in \mathrm{G}$	$\mathrm{R} \in \mathrm{SO}(3)$
1. CLOSURE	If $g_1,\ g_2 \in \mathrm{G}$, then $g_1 \circ g_2 \in \mathrm{G}$	If $\mathbf{R}_1,\ \mathbf{R}_2 \in \mathrm{SO}(3)$, then $\mathbf{R}_1\mathbf{R}_2 \in \mathrm{SO}(3)$ *
2. IDENTITY	There exists and identify element, e, such that $g \circ e = e \circ g = g\ \forall\ g \in \mathrm{G}$	$e = \mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
3. INVERSE	For each $g \in \mathrm{G}$, there exists a unique inverse g^{-1} such that $g \circ g^{-1} = g^{-1} \circ g = e$	$\mathbf{R}^{-1} = \mathbf{R}^T \quad \because \quad \mathbf{R}^T\mathbf{R} = \mathbf{I}$
3. ASSOCIATIVITY	If $g_1,\ g_2,\ g_3 \in \mathrm{G}$, then $(g_1 \circ g_2) \circ\ g_3 = g_1 \circ (g_2 \circ\ g_3)$	$(\mathbf{R}_1\mathbf{R}_2)\mathbf{R}_3 = \mathbf{R}_1(\mathbf{R}_2\mathbf{R}_3)$

$$* \; \mathbf{R}_1\mathbf{R}_2(\mathbf{R}_1\mathbf{R}_2)^T = \mathbf{R}_1 \underbrace{\mathbf{R}_2\mathbf{R}_2^T}_{I} \mathbf{R}_1^T = \mathbf{R}_1\mathbf{R}_1^T = \mathbf{I} \quad \& \quad \det(\mathbf{R}_1\mathbf{R}_2) = \det(\mathbf{R}_1)\det(\mathbf{R}_2) = +1$$

Rotational Motion in $\mathbb{R}^3$

A **Rotation Matrix** can: ***define the orientation of a rigid body***, or ***transform points from one frame to another***



A is the Inertial Frame
B is the Body Frame
(here rotated about origin)

Vectors can also be rotated from frame B to frame A using a rotation matrix:

$$\mathbf{v} = \mathbf{q} - \mathbf{p} \qquad \mathbf{R}_{ab}\mathbf{v}_b = \mathbf{R}_{ab}\mathbf{q}_b - \mathbf{R}_{ab}\mathbf{p}_b = \mathbf{q}_a - \mathbf{p}_a = \mathbf{v}_a$$

The ***composition rule*** for rotation matrices: Rotation matrices can be combined to form new rotation matrices using matrix multiplication:

$$\mathbf{R}_{ac} = \mathbf{R}_{ab}\mathbf{R}_{bc}$$

$\mathbf{R}_{ac}$ maps $\mathbb{R}^3$ to $\mathbb{R}^3$, rotating coordinates of a point from frame C to frame A, by first rotating from C to B and then B to A.

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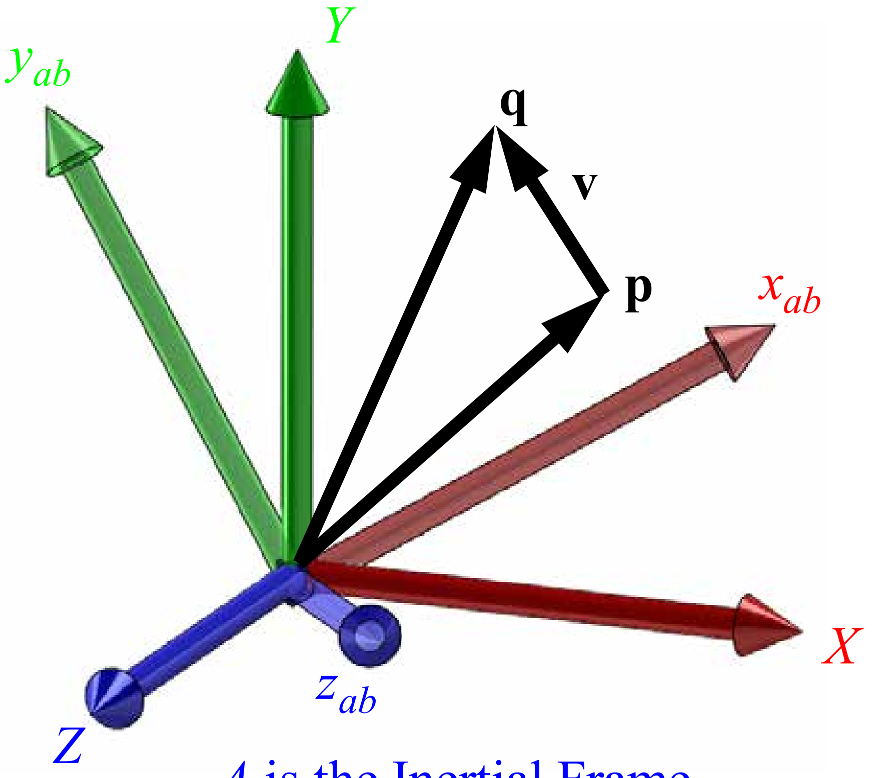

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