Velocity of Screw Motion

To find the velocity of screw motion without differentiation, consider the homogeneous transformation:

$$\mathbf{g}_{ab}(\theta) = e^{\hat{\xi}\theta}\mathbf{g}_{ab}(0)$$

Taking the derivative of the above, we find:

$$\dot{\mathbf{g}}_{ab}(\theta) = \frac{d}{dt}\left(e^{\hat{\xi}\theta}\right)\mathbf{g}_{ab}(0) = \hat{\xi}\dot{\theta}e^{\hat{\xi}\theta}\mathbf{g}_{ab}(0)$$

We can use this expression in the equation for spatial velocity:

$$\hat{\mathbf{V}}_{ab}^{s} = \dot{\mathbf{g}}_{ab}(\theta)\mathbf{g}_{ab}^{-1}(\theta)$$

$$\hat{\mathbf{V}}_{ab}^{s} = \hat{\xi}\dot{\theta}e^{\hat{\xi}\theta}\mathbf{g}_{ab}(0)\mathbf{g}_{ab}^{-1}(\theta) \qquad \because \quad \dot{\mathbf{g}}_{ab}(\theta) = \hat{\xi}\dot{\theta}e^{\hat{\xi}\theta}\mathbf{g}_{ab}(0)$$

$$\hat{\mathbf{V}}_{ab}^{s} = \hat{\xi}\dot{\theta}e^{\hat{\xi}\theta}\underbrace{\mathbf{g}_{ab}(0)\mathbf{g}_{ab}^{-1}(0)}_{\mathbf{I}}e^{-\hat{\xi}\theta} \qquad \because \quad \mathbf{g}_{ab}^{-1}(\theta) = \left(e^{\hat{\xi}\theta}\mathbf{g}_{ab}(0)\right)^{-1} = \mathbf{g}_{ab}^{-1}(0)e^{-\hat{\xi}\theta}$$

$$\hat{\mathbf{V}}_{ab}^{s} = \hat{\xi}\dot{\theta}\underbrace{e^{\hat{\xi}\theta}e^{-\hat{\xi}\theta}}_{\mathbf{I}}$$

$$\boxed{\hat{\mathbf{V}}_{ab}^{s} = \hat{\xi}\dot{\theta}}$$

Velocity of Screw Motion

Body velocity can be found in a similar fashion:

$$\hat{\mathbf{V}}_{ab}^{b} = \mathbf{g}_{ab}^{-1}(\theta)\dot{\mathbf{g}}_{ab}(\theta)$$

$$\hat{\mathbf{V}}_{ab}^{b} = \mathbf{g}_{ab}^{-1}(0)e^{-\hat{\xi}\theta}\dot{\mathbf{g}}_{ab}(\theta) \qquad \because \quad \mathbf{g}_{ab}^{-1}(\theta) = \mathbf{g}_{ab}^{-1}(0)e^{-\hat{\xi}\theta}$$

$$\hat{\mathbf{V}}_{ab}^{b} = \mathbf{g}_{ab}^{-1}(0)e^{-\hat{\xi}\theta}\hat{\xi}\dot{\theta}e^{\hat{\xi}\theta}\mathbf{g}_{ab}(0) \qquad \because \quad \dot{\mathbf{g}}_{ab}(\theta) = \hat{\xi}\dot{\theta}e^{\hat{\xi}\theta}\mathbf{g}_{ab}(0)$$

$$\hat{\mathbf{V}}_{ab}^{b} = \mathbf{g}_{ab}^{-1}(0)\left(e^{-\hat{\xi}\theta}\hat{\xi}e^{\hat{\xi}\theta}\right)\mathbf{g}_{ab}(0)\dot{\theta}$$

$$\hat{\mathbf{V}}_{ab}^{b} = \mathbf{g}_{ab}^{-1}(0)\hat{\xi}\mathbf{g}_{ab}(0)\dot{\theta} \qquad \because \quad e^{-\hat{\xi}\theta}\hat{\xi}e^{\hat{\xi}\theta} = \hat{\xi}$$

$$\boxed{\hat{\mathbf{V}}_{ab}^{b} = \left(\mathrm{Ad}_{\mathbf{g}_{ab}^{-1}(0)}\xi\right)^{\wedge}\dot{\theta}}$$