The Adjoint Transformation

The spatial and body velocity of a rigid motion are related by a similarity transformation. Start by noting:

$$\hat{\mathbf{V}}_{ab}^{s} = \dot{\mathbf{g}}_{ab}\mathbf{g}_{ab}^{-1} = \underbrace{\left(\mathbf{g}_{ab}\mathbf{g}_{ab}^{-1}\right)}_{\mathbf{I}}\dot{\mathbf{g}}_{ab}\mathbf{g}_{ab}^{-1} = \mathbf{g}_{ab}\underbrace{\left(\mathbf{g}_{ab}^{-1}\dot{\mathbf{g}}_{ab}\right)}_{\hat{\mathbf{V}}_{ab}^{b}}\mathbf{g}_{ab}^{-1}$$

$$\hat{\mathbf{V}}_{ab}^{s} = \mathbf{g}_{ab}\hat{\mathbf{V}}_{ab}^{b}\mathbf{g}_{ab}^{-1}$$

The above the result can be rewritten:

$$\hat{\mathbf{V}}_{ab}^{s} = \mathbf{g}_{ab}\hat{\mathbf{V}}_{ab}^{b}\mathbf{g}_{ab}^{-1} = \begin{bmatrix} \mathbf{R}_{ab} & \mathbf{p}_{ab} \\ \mathbf{0} & 1 \end{bmatrix}\begin{bmatrix} \hat{\boldsymbol{\omega}}_{ab}^{b} & \mathbf{v}_{ab}^{b} \\ \mathbf{0} & 0 \end{bmatrix}\begin{bmatrix} \mathbf{R}_{ab}^{T} & -\mathbf{R}_{ab}^{T}\mathbf{p}_{ab} \\ \mathbf{0} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{R}_{ab}\hat{\boldsymbol{\omega}}_{ab}^{b}\mathbf{R}_{ab}^{T} & -\mathbf{R}_{ab}\hat{\boldsymbol{\omega}}_{ab}^{b}\mathbf{R}_{ab}^{T}\mathbf{p}_{ab} + \mathbf{R}_{ab}\mathbf{v}_{ab}^{b} \\ \mathbf{0} & 1 \end{bmatrix}$$

If we "unwedge" the above result, you see that:

$$\boldsymbol{\omega}_{ab}^{s} = \mathbf{R}_{ab}\boldsymbol{\omega}_{ab}^{b} \qquad \because \left(\hat{\boldsymbol{\omega}}_{ab}^{s} = \mathbf{R}_{ab}\hat{\boldsymbol{\omega}}_{ab}^{b}\mathbf{R}_{ab}^{T}\right) \equiv \left(\boldsymbol{\omega}_{ab}^{s} = \mathbf{R}_{ab}\boldsymbol{\omega}_{ab}^{b}\right)$$

$$\mathbf{v}_{ab}^{s} = \hat{\mathbf{p}}_{ab}\mathbf{R}_{ab}\boldsymbol{\omega}_{ab}^{b} + \mathbf{R}_{ab}\mathbf{v}_{ab}^{b}$$

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From the previous page:

$$\boldsymbol{\omega}_{ab}^{s} = \mathbf{R}_{ab}\boldsymbol{\omega}_{ab}^{b} \qquad \because \left(\hat{\boldsymbol{\omega}}_{ab}^{s} = \mathbf{R}_{ab}\hat{\boldsymbol{\omega}}_{ab}^{b}\mathbf{R}_{ab}^{T}\right) \equiv \left(\boldsymbol{\omega}_{ab}^{s} = \mathbf{R}_{ab}\boldsymbol{\omega}_{ab}^{b}\right)$$

$$\mathbf{v}_{ab}^{s} = \hat{\mathbf{p}}_{ab}\mathbf{R}_{ab}\boldsymbol{\omega}_{ab}^{b} + \mathbf{R}_{ab}\mathbf{v}_{ab}^{b}$$

We now see that:

$$\mathbf{V}_{ab}^{s} = \begin{bmatrix} \mathbf{v}_{ab}^{s} \\ \boldsymbol{\omega}_{ab}^{s} \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{p}}_{ab}\mathbf{R}_{ab}\boldsymbol{\omega}_{ab}^{b} + \mathbf{R}_{ab}\mathbf{v}_{ab}^{b} \\ \mathbf{R}_{ab}\boldsymbol{\omega}_{ab}^{b} \end{bmatrix} \qquad \mathbf{V}_{ab}^{s} = \begin{bmatrix} \mathbf{R}_{ab} & \hat{\mathbf{p}}_{ab}\mathbf{R}_{ab} \\ \mathbf{0} & \mathbf{R}_{ab} \end{bmatrix} \begin{bmatrix} \mathbf{v}_{ab}^{b} \\ \boldsymbol{\omega}_{ab}^{b} \end{bmatrix} = \mathbf{Ad}_{\mathbf{g}_{ab}} \mathbf{V}_{ab}^{b}$$

The 6x6 matrix which transforms twists from one coordinate frame to another is referred to as the Adjoint transformation associated with g, written Ad_g:

$$\mathrm{Ad}_g = \begin{bmatrix} \mathbf{R} & \hat{\mathbf{p}}\mathbf{R} \\ \mathbf{0} & \mathbf{R} \end{bmatrix}$$

It is important to note that the “inverse of the Adjoint of g” is equal to the “Adjoint of the inverse of g”:

$$\left(\mathrm{Ad}_g\right)^{-1} = \begin{bmatrix} \mathbf{R}^T & -(\mathbf{R}\mathbf{p})^{\wedge}\mathbf{R}^T \\ \mathbf{0} & \mathbf{R}^T \end{bmatrix} = \begin{bmatrix} \mathbf{R}^T & -\mathbf{R}^T\hat{\mathbf{p}} \\ \mathbf{0} & \mathbf{R}^T \end{bmatrix} = \mathrm{Ad}_{g^{-1}} \qquad \mathbf{g}^{-1} = \begin{bmatrix} \mathbf{R}^T & -\mathbf{R}^T\mathbf{p} \\ 0\ 0\ 0 & 1 \end{bmatrix}$$