The Hat Operator

The ***hat operator*** takes a 3x1 matrix to construct a skew-symmetric matrix according to:

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \qquad \hat{\mathbf{a}} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$$

Using this definition, the cross-product of two 3x1 vectors can be written:

$$\hat{\mathbf{a}}\mathbf{b} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} -a_3 b_2 + a_2 b_3 \\ a_3 b_1 - a_1 b_3 \\ -a_2 b_1 + a_1 b_2 \end{bmatrix} = \mathbf{a} \times \mathbf{b}$$

The cross-product (and ***hat operator***) has the properties:

$$\mathbf{R}(\mathbf{v} \times \mathbf{w}) = (\mathbf{R}\mathbf{v}) \times (\mathbf{R}\mathbf{w})$$
$$\mathbf{R}(\mathbf{w})^{\wedge}\mathbf{R}^T = (\mathbf{R}\mathbf{w})^{\wedge}$$

Rotation Matrices as Rigid Body Transformations

A rotation matrix $\mathbf{R} \in SO(3)$ is a rigid body transformation; that is:

$$\boxed{\text{1. } \mathbf{R} \text{ preserves distance: } \|\mathbf{Rq} - \mathbf{Rp}\| = \|\mathbf{q} - \mathbf{p}\| \text{ for all } \mathbf{q}, \mathbf{p} \in \mathbb{R}^3}$$

Proof: $$\|\mathbf{Rq} - \mathbf{Rp}\|^2 = \left(\mathbf{R}(\mathbf{q} - \mathbf{p})\right)^T \left(\mathbf{R}(\mathbf{q} - \mathbf{p})\right) = (\mathbf{q} - \mathbf{p})^T \mathbf{R}^T \mathbf{R}(\mathbf{q} - \mathbf{p})$$
$$= (\mathbf{q} - \mathbf{p})^T (\mathbf{q} - \mathbf{p}) = \|\mathbf{q} - \mathbf{p}\|^2$$

$$\boxed{\text{2. } \mathbf{R} \text{ preserves orientation: } \mathbf{R}(\mathbf{v} \times \mathbf{w}) = \mathbf{Rv} \times \mathbf{Rw} \text{ for all } \mathbf{v}, \mathbf{w} \in \mathbb{R}^3}$$