Rodrigues' Formula

We desired a closed form solution for the matrix exponential. We can start with:

$$e^{\hat{\boldsymbol{\omega}}\theta} = \mathbf{I} + \hat{\boldsymbol{\omega}}\theta + \frac{(\hat{\boldsymbol{\omega}}\theta)^2}{2!} + \frac{(\hat{\boldsymbol{\omega}}\theta)^3}{3!} + \cdots$$

Applying the previous skew-symmetric matrix properties, we have:

$$\hat{\mathbf{a}}^2 = \mathbf{a}\mathbf{a}^T - \|\mathbf{a}\|^2\mathbf{I} \qquad \hat{\mathbf{a}}^3 = -\|\mathbf{a}\|^2\hat{\mathbf{a}} \qquad \boxed{\|\boldsymbol{\omega}\| = 1} \qquad \hat{\boldsymbol{\omega}}^T = -\hat{\boldsymbol{\omega}}$$

$$\hat{\boldsymbol{\omega}}^2 = \boldsymbol{\omega}\boldsymbol{\omega}^T - \mathbf{I} \qquad \hat{\boldsymbol{\omega}}^3 = -\hat{\boldsymbol{\omega}}$$

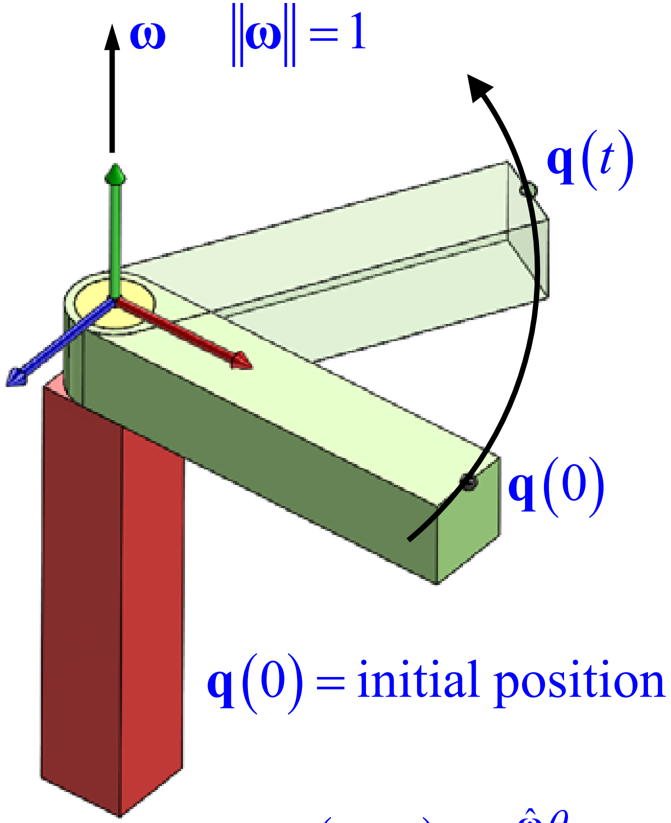
$$\hat{\boldsymbol{\omega}}^4 = -\hat{\boldsymbol{\omega}}^2 \qquad \hat{\boldsymbol{\omega}}^5 = \hat{\boldsymbol{\omega}}$$

So now the matrix exponential for our rotation matrix is:

$$e^{\hat{\boldsymbol{\omega}}\theta} = \mathbf{I} + \hat{\boldsymbol{\omega}}\theta + \frac{\theta^2}{2!}\hat{\boldsymbol{\omega}}^2 + \frac{\theta^3}{3!}\hat{\boldsymbol{\omega}}^3 + \frac{\theta^4}{4!}\hat{\boldsymbol{\omega}}^4 + \frac{\theta^5}{5!}\hat{\boldsymbol{\omega}}^5 + \cdots\cdots$$

$$e^{\hat{\boldsymbol{\omega}}\theta} = \mathbf{I} + \hat{\boldsymbol{\omega}}\theta + \frac{\theta^2}{2!}\hat{\boldsymbol{\omega}}^2 + \frac{\theta^3}{3!}(-\hat{\boldsymbol{\omega}}) + \frac{\theta^4}{4!}\left(-\hat{\boldsymbol{\omega}}^2\right) + \frac{\theta^5}{5!}\hat{\boldsymbol{\omega}} + \cdots$$

$$e^{\hat{\boldsymbol{\omega}}\theta} = \mathbf{I} + \left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)\hat{\boldsymbol{\omega}} + \left(\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \frac{\theta^6}{6!}\cdots\right)\hat{\boldsymbol{\omega}}^2$$



$\mathbf{q}(0) =$ initial position

$$\mathbf{R}(\boldsymbol{\omega},\theta) = e^{\hat{\boldsymbol{\omega}}\theta}$$

$$\hat{\boldsymbol{\omega}} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

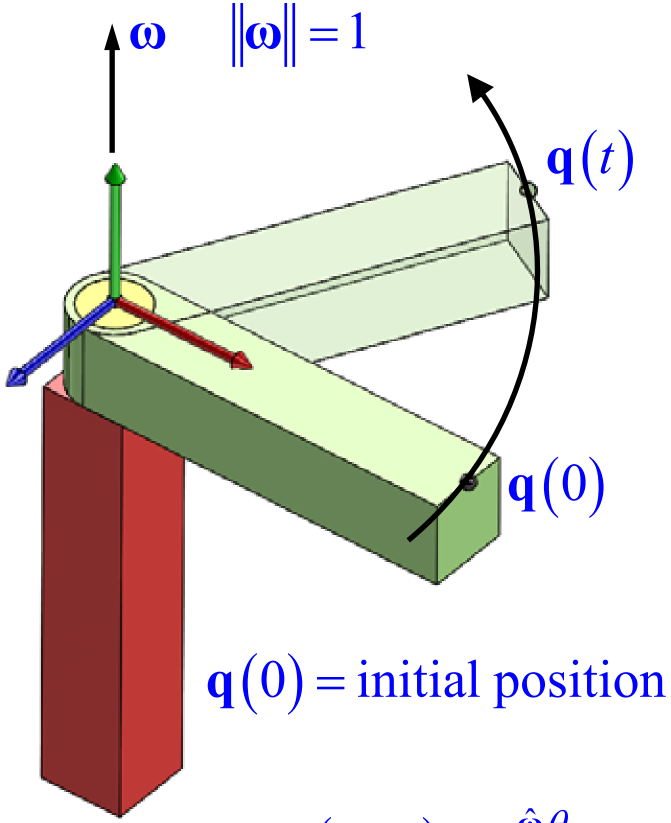
Rodrigues' Formula

From the previous slide:

$$e^{\hat{\boldsymbol{\omega}}\theta} = \mathbf{I} + \left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)\hat{\boldsymbol{\omega}} + \left(\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \frac{\theta^6}{6!}\cdots\right)\hat{\boldsymbol{\omega}}^2$$

The quantities in the parenthesis above can be reduced using trigonometric identities. The result is known as Rodrigues' Formula:

$$\boxed{e^{\hat{\boldsymbol{\omega}}\theta} = \mathbf{I} + \hat{\boldsymbol{\omega}}\sin(\theta) + \hat{\boldsymbol{\omega}}^2\left(1 - \cos(\theta)\right)}$$



$\mathbf{q}(0) =$ initial position

$$\mathbf{R}(\boldsymbol{\omega}, \theta) = e^{\hat{\boldsymbol{\omega}}\theta}$$

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