Rotational Velocity

Consider a pure rotation: $\mathbf{R}_{ab}(t) \in SO(3)$

We have two reference frames: Spatial Coordinate Frame A (fixed to an inertial reference, i.e. earth)
Body Coordinate Frame B (fixed to the moving body)

For any point **q**: $\mathbf{q}_a(t) = \mathbf{R}_{ab}(t)\mathbf{q}_b$ (point **q** fixed in body frame B)

The velocity of point **q** in spatial coordinates is:

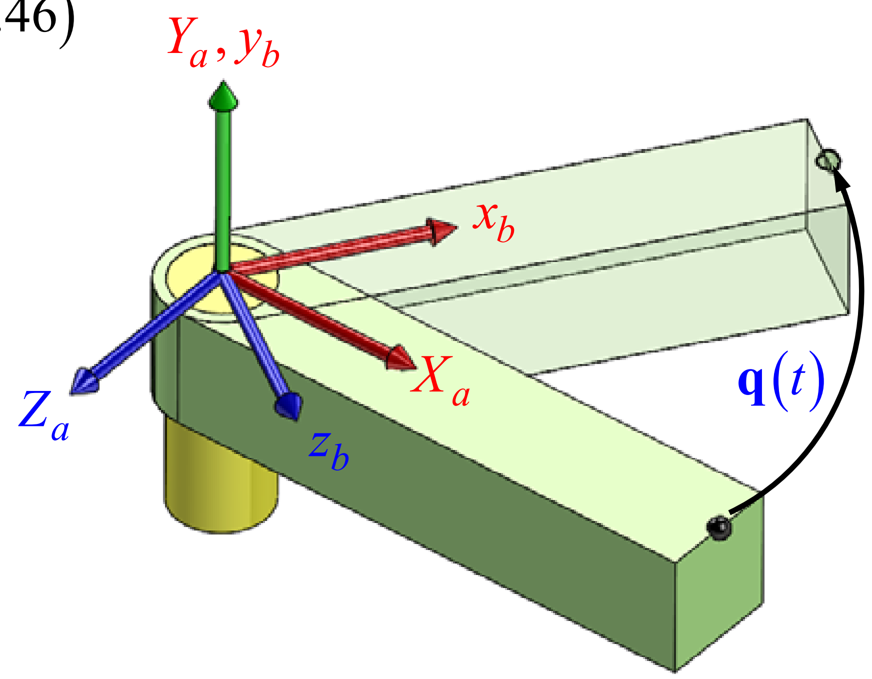
$$\mathbf{v}_{\mathbf{q}_a}(t) = \frac{d}{dt}\left(\mathbf{q}_a(t)\right) = \frac{d}{dt}\left(\mathbf{R}_{ab}(t)\mathbf{q}_b\right) = \dot{\mathbf{R}}_{ab}(t)\mathbf{q}_b \quad (2.46)$$

$\dot{\mathbf{R}}_{ab}(t)$ maps body coordinates of a point to the spatial velocity of the point.

This is inefficient. Requires 9 numbers!

By expanding (2.46), we have:

$$\mathbf{v}_{\mathbf{q}_a}(t) = \dot{\mathbf{R}}_{ab}(t)\overbrace{\mathbf{R}_{ab}^{-1}(t)\mathbf{R}_{ab}(t)}^{\mathbf{I}}\mathbf{q}_b$$
$$= \underbrace{\dot{\mathbf{R}}_{ab}(t)\mathbf{R}_{ab}^{-1}(t)}_{skew-symetric}\mathbf{R}_{ab}(t)\mathbf{q}_b$$



Rotational Velocity

In order to show that $\dot{\mathbf{R}}_{ab}(t)\mathbf{R}_{ab}^{-1}(t)$ is skew-symmetric, we consider the following identity:

$$\mathbf{R}\mathbf{R}^T = \mathbf{I} \qquad \mathbf{R}{=}\mathbf{R}(t) \quad \text{(time dependency dropped for simplicity)}$$

Differentiating the above identity gives:

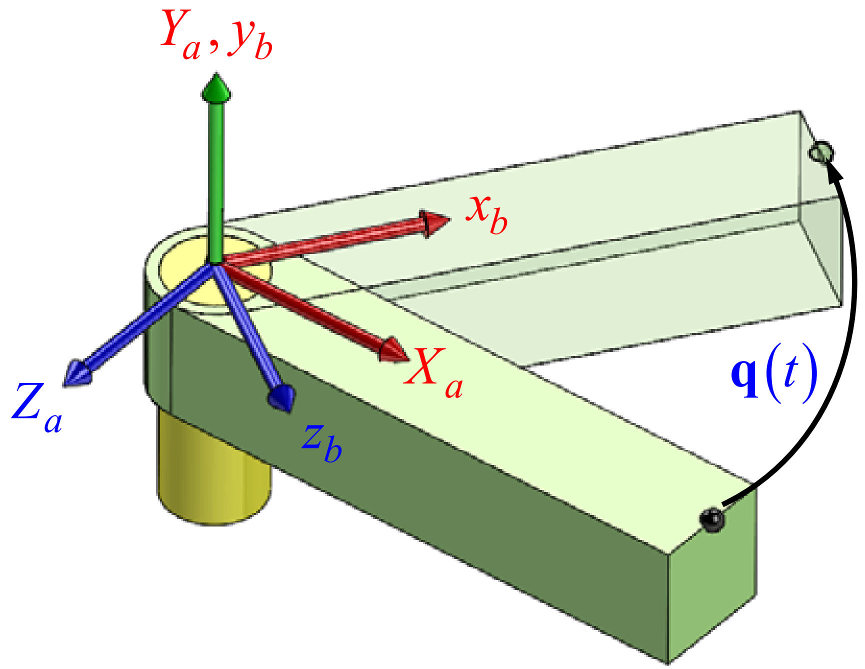
$$\dot{\mathbf{R}}\mathbf{R}^T + \mathbf{R}\dot{\mathbf{R}}^T = 0$$

Rearranging gives:

$$\dot{\mathbf{R}}\mathbf{R}^T = -\mathbf{R}\dot{\mathbf{R}}^T$$
$$\dot{\mathbf{R}}\mathbf{R}^T = -(\dot{\mathbf{R}}\mathbf{R}^T)^T$$

This result is the definition of skew-symmetric, with:

$$\mathbf{A} = -\mathbf{A}^T$$



We can now represent the velocity of a rotating body using a vector in $\mathbb{R}^3$!

Rotational Velocity

We can now represent the velocity of a rotating body using a vector in $\mathbb{R}^3$. We define the following:

Instantaneous Spatial Angular Velocity

$$\hat{\boldsymbol{\omega}}^s_{ab} := \dot{\mathbf{R}}_{ab}\mathbf{R}_{ab}^{-1} \qquad (2.48)$$

$\boldsymbol{\omega}^s_{ab}$ is the instantaneous angular velocity as seen from the spatial (A) frame

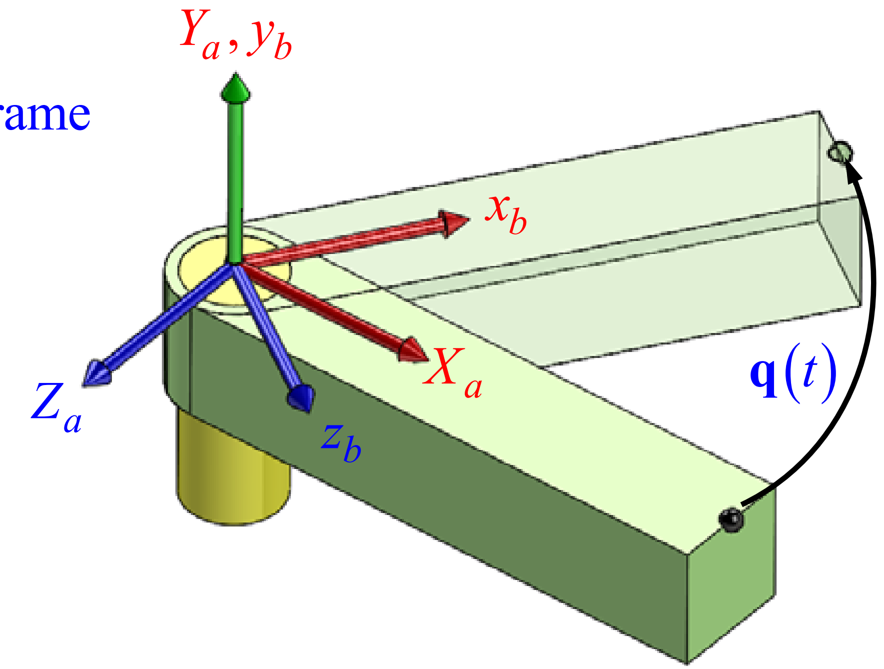
Instantaneous Body Angular Velocity

$$\hat{\boldsymbol{\omega}}^b_{ab} := \mathbf{R}_{ab}^{-1}\dot{\mathbf{R}}_{ab} \qquad (2.49)$$

$\boldsymbol{\omega}^b_{ab}$ is the instantaneous angular velocity as seen from the instantaneous body (B) frame

It can be easily shown that these are related according to:

$$\hat{\boldsymbol{\omega}}^b_{ab} = \mathbf{R}_{ab}^{-1}\hat{\boldsymbol{\omega}}^s_{ab}\mathbf{R}_{ab} \quad \text{or} \quad \boldsymbol{\omega}^b_{ab} = \mathbf{R}_{ab}^{-1}\boldsymbol{\omega}^s_{ab} \qquad (2.50)$$



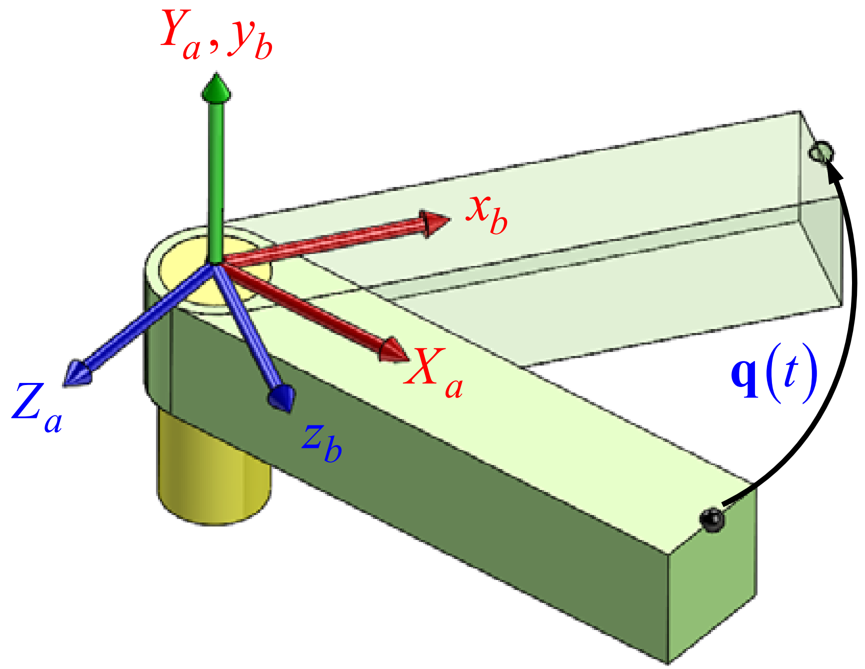
Rotational Velocity

The **instantaneous spatial angular velocity** can be used to find the velocity of a point in the spatial frame:

$$\begin{aligned}\mathbf{v}_{\mathbf{q}_a}(t) &= \dot{\mathbf{R}}_{ab}(t)\mathbf{R}_{ab}^{-1}(t)\mathbf{R}_{ab}(t)\mathbf{q}_b \\ &= \hat{\boldsymbol{\omega}}_{ab}^{s}\mathbf{R}_{ab}(t)\mathbf{q}_b = \hat{\boldsymbol{\omega}}_{ab}^{s}\mathbf{q}_a(t) = \boldsymbol{\omega}_{ab}^{s}\times\mathbf{q}_a(t)\end{aligned}$$

Alternatively, we can use the **instantaneous body angular velocity** to find the velocity of a point in the body frame:

$$\begin{aligned}\mathbf{v}_{\mathbf{q}_b}(t) &= \mathbf{R}_{ab}^{T}(t)\mathbf{v}_{\mathbf{q}_a}(t) \\ &= \underbrace{\mathbf{R}_{ab}^{T}(t)\dot{\mathbf{R}}_{ab}(t)}_{\hat{\boldsymbol{\omega}}_{ab}^{b}}\underbrace{\mathbf{R}_{ab}^{-1}(t)\mathbf{R}_{ab}(t)}_{\mathbf{I}}\mathbf{q}_b \\ &= \hat{\boldsymbol{\omega}}_{ab}^{b}\mathbf{q}_b = \boldsymbol{\omega}_{ab}^{b}\times\mathbf{q}_b\end{aligned}$$



Remember: $\mathbf{R}^T = \mathbf{R}^{-1}$