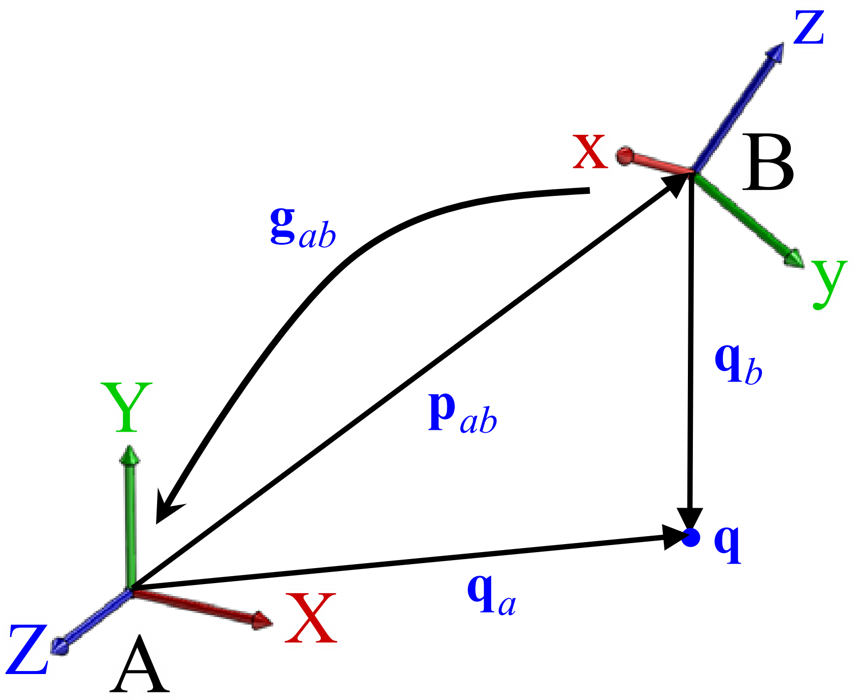
Homogeneous Representation of RBTs

Rigid body transformations can be represented with matrices in $\mathbb{R}^{4x4}$ and vectors in $\mathbb{R}^4$.

First, we define points and vectors as:

$$\bar{\mathbf{q}} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ 1 \end{bmatrix} = \text{homogenous representation of point } \mathbf{q} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

$$\bar{\mathbf{v}} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ 0 \end{bmatrix} = \text{homogenous representation of vector } \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$



Points and vectors:

1) Sums and differences of vectors are vectors
2) The sum of a vector and a point is a point
3) The difference between two points is a vector
4) The sum of two points is meaningless

Homogeneous Representation of RBTs

The transformation $\mathbf{g}_{ab}=(\mathbf{p}_{ab},\mathbf{R}_{ab})$ is an affine transformation and may be represented in linear form as:

$$\bar{\mathbf{g}}_{ab} = \begin{bmatrix} \mathbf{R}_{ab} & \mathbf{p}_{ab} \\ \mathbf{0} & 1 \end{bmatrix}^{4\times 4} = \text{the homogeneous representation of } \mathbf{g}_{ab}$$

This homogeneous representation can be used to find $\mathbf{q}$ in frame A from its coordinates in frame B:

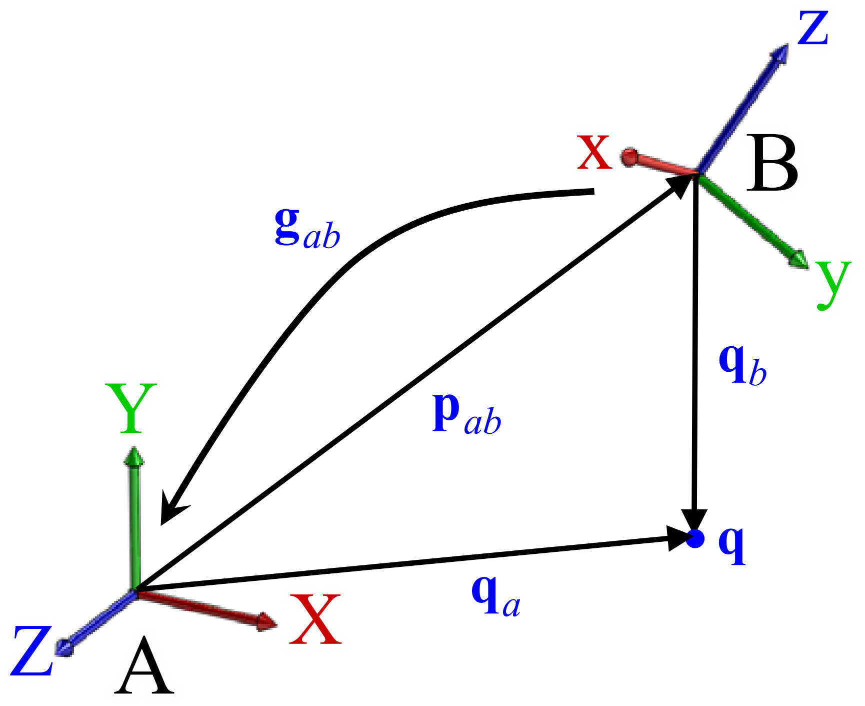
$$\bar{\mathbf{q}}_a = \begin{bmatrix} \mathbf{q}_a \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{R}_{ab}\mathbf{q}_b + \mathbf{p}_{ab} \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{R}_{ab} & \mathbf{p}_{ab} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{q}_b \\ 1 \end{bmatrix}$$

$$\bar{\mathbf{q}}_a = \bar{\mathbf{g}}_{ab}\bar{\mathbf{q}}_b$$

For vectors, a 0 replaces the 1 in the 4th row:

$$\mathbf{v} = \mathbf{s} - \mathbf{r} \qquad \bar{\mathbf{g}}_*\bar{\mathbf{v}} = \bar{\mathbf{g}}\bar{\mathbf{s}} - \bar{\mathbf{g}}\bar{\mathbf{r}} = \begin{bmatrix} \mathbf{R} & \mathbf{p} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ 0 \end{bmatrix}$$

More generally: If $\mathbf{g} = (\mathbf{p}, \mathbf{R}) \in SE(3)$, then: $\bar{\mathbf{g}} = \begin{bmatrix} \mathbf{R} & \mathbf{p} \\ 0 & 1 \end{bmatrix}$



Composition Rule for RBTs

Rigid body transformation can be combined to form new rigid body transformations:

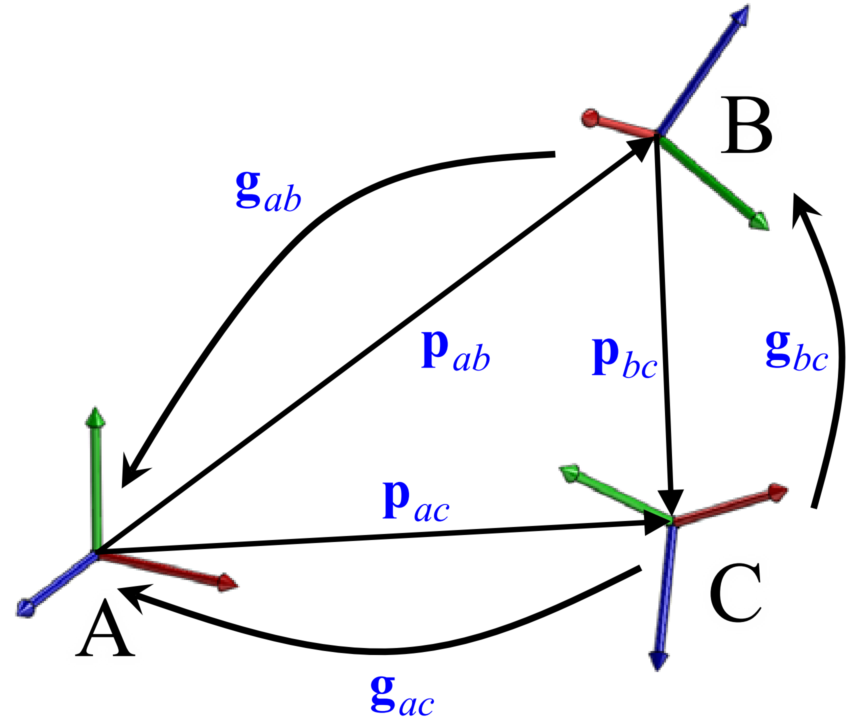
$\mathbf{g}_{bc} \in SE(3)$ is the configuration of C relative to B
$\mathbf{g}_{ab} \in SE(3)$ is the configuration of B relative to A

The product of two RBTs represents the total transformation:

$$\bar{\mathbf{g}}_{ac} = \bar{\mathbf{g}}_{ab}\bar{\mathbf{g}}_{bc} = \begin{bmatrix} \mathbf{R}_{ab} & \mathbf{p}_{ab} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \mathbf{R}_{bc} & \mathbf{p}_{bc} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{R}_{ac}\mathbf{R}_{bc} & \mathbf{R}_{ab}\mathbf{p}_{bc} + \mathbf{p}_{ab} \\ 0 & 1 \end{bmatrix}$$

HGTs are a group:

1. CLOSURE: if $\mathbf{g}_1, \mathbf{g}_2 \in SE(3)$ then $\mathbf{g}_1\mathbf{g}_2 \in SE(3)$
2. IDENTITY: 4×4 identity element $\mathbf{I}$ is in $SE(3)$
3. INVERSE: if $\mathbf{g} \in SE(3)$, then $\mathbf{g}^{-1} = \begin{bmatrix} \mathbf{R}^T & -\mathbf{R}^T\mathbf{p} \\ 0 & 1 \end{bmatrix}$, so $\mathbf{g}^{-1} = \left(-\mathbf{R}^T\mathbf{p}, \mathbf{R}^T\right)$
4. ASSOCIATIVE: $(\mathbf{g}_1\mathbf{g}_2)\mathbf{g}_3 = \mathbf{g}_1(\mathbf{g}_2\mathbf{g}_3)$



Inverse of an HGT

The inverse of a homogeneous transformation defined: if $\mathbf{g} \in SE(3)$, then $\mathbf{g}^{-1} = \begin{bmatrix} \mathbf{R}^T & -\mathbf{R}^T\mathbf{p} \\ 0 & 1 \end{bmatrix}$

This may be shown by:

$$\mathbf{I} = \begin{bmatrix} \mathbf{R} & \mathbf{p} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{R}^T & -\mathbf{R}^T\mathbf{p} \\ 0 & 1 \end{bmatrix} = \mathbf{g}\mathbf{g}^{-1}$$

$$\mathbf{I} = \begin{bmatrix} \mathbf{R} & \mathbf{p} \\ [0 \quad 0 \quad 0] & 1 \end{bmatrix} \begin{bmatrix} \mathbf{R}^T & -\mathbf{R}^T\mathbf{p} \\ [0 \quad 0 \quad 0] & 1 \end{bmatrix} = \mathbf{g}\mathbf{g}^{-1}$$

$$\mathbf{I} = \begin{bmatrix} \mathbf{R}\mathbf{R}^T + \mathbf{p}[0 \quad 0 \quad 0] & -\mathbf{R}\mathbf{R}^T\mathbf{p} + \mathbf{p} \\ [0 \quad 0 \quad 0]\mathbf{R}^T + 1[0 \quad 0 \quad 0] & -[0 \quad 0 \quad 0]\mathbf{R}^T\mathbf{p} + 1 \end{bmatrix} = \mathbf{g}\mathbf{g}^{-1}$$

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{R}\mathbf{R}^T & \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ [0 \quad 0 \quad 0] & 1 \end{bmatrix} = \mathbf{g}\mathbf{g}^{-1}$$