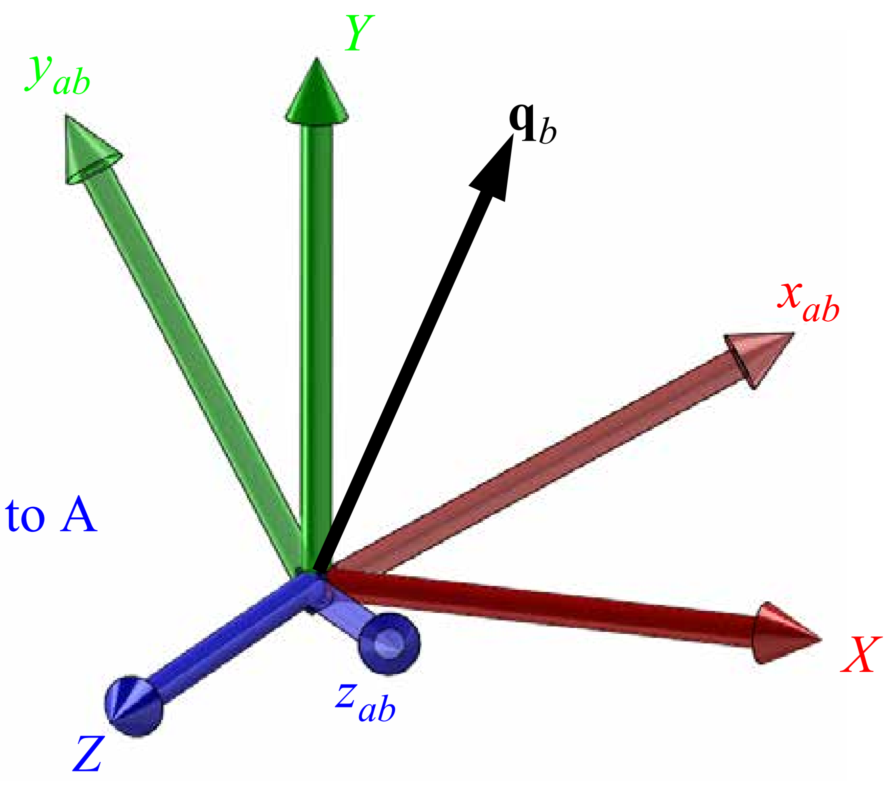
The Rotation Matrix



A **Rotation Matrix** can: ***define the orientation of a rigid body***, or ***transform points from one frame to another***

Definitions. From the shown figure:

$\mathbf{x}_{ab}, \mathbf{y}_{ab}, \mathbf{z}_{ab} \in \mathbb{R}^3$ are the coordinates of the principle axes of B relative to A

$\mathbf{q}_b = \begin{bmatrix} x_b \\ y_b \\ z_b \end{bmatrix}$ are the coordinates of $\mathbf{q}$ relative to frame B

$x_b, y_b, z_b \in \mathbb{R}$ are the projections of $\mathbf{q}$ onto coordinate axes of frame B which have coordinates $\mathbf{x}_{ab}, \mathbf{y}_{ab}, \mathbf{z}_{ab} \in \mathbb{R}^3$ w.r.t. frame A

A is the Inertial Frame

B is the Body Frame (here rotated about origin)

We can find point $\mathbf{q}$ in inertial frame A using:

$$\mathbf{q}_a = \mathbf{x}_{ab} x_b + \mathbf{y}_{ab} y_b + \mathbf{z}_{ab} z_b$$

$$\mathbf{q}_a = \underbrace{\begin{bmatrix} \mathbf{x}_{ab} & \mathbf{y}_{ab} & \mathbf{z}_{ab} \end{bmatrix}}_{\mathbf{R}_{ab}} \underbrace{\begin{bmatrix} x_b \\ y_b \\ z_b \end{bmatrix}}_{\mathbf{q}_b}$$

$\mathbf{R}_{ab}$ is a 3×3 rotation matrix, mapping coordinates of a point specified in frame B to frame A

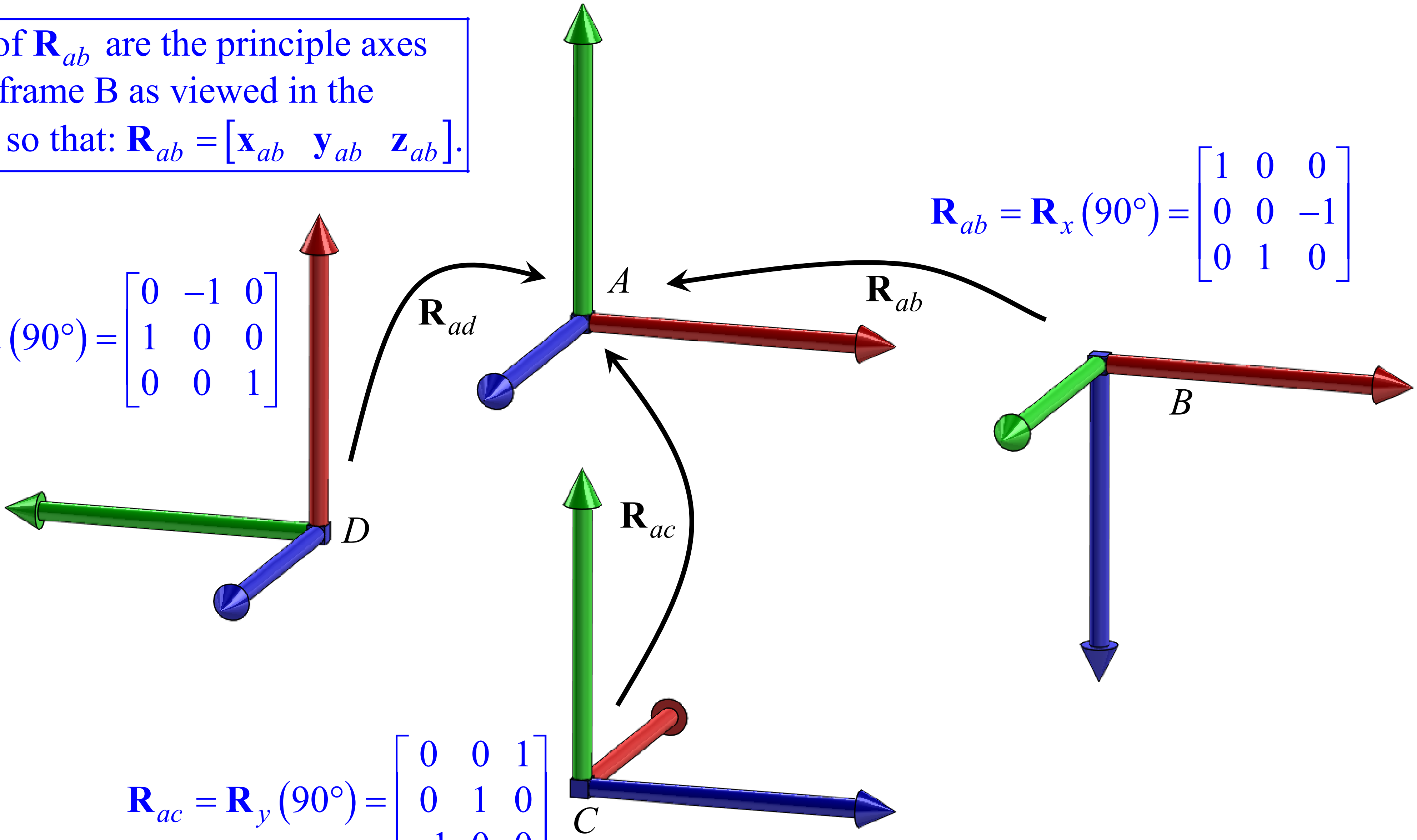
Simple 90° Rotation Matrices

The columns of $\mathbf{R}_{ab}$ are the principle axes of the rotated frame B as viewed in the fixed frame A so that: $\mathbf{R}_{ab} = \left[\mathbf{x}_{ab} \;\; \mathbf{y}_{ab} \;\; \mathbf{z}_{ab}\right]$.

$$\mathbf{R}_{ab} = \mathbf{R}_x(90°) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

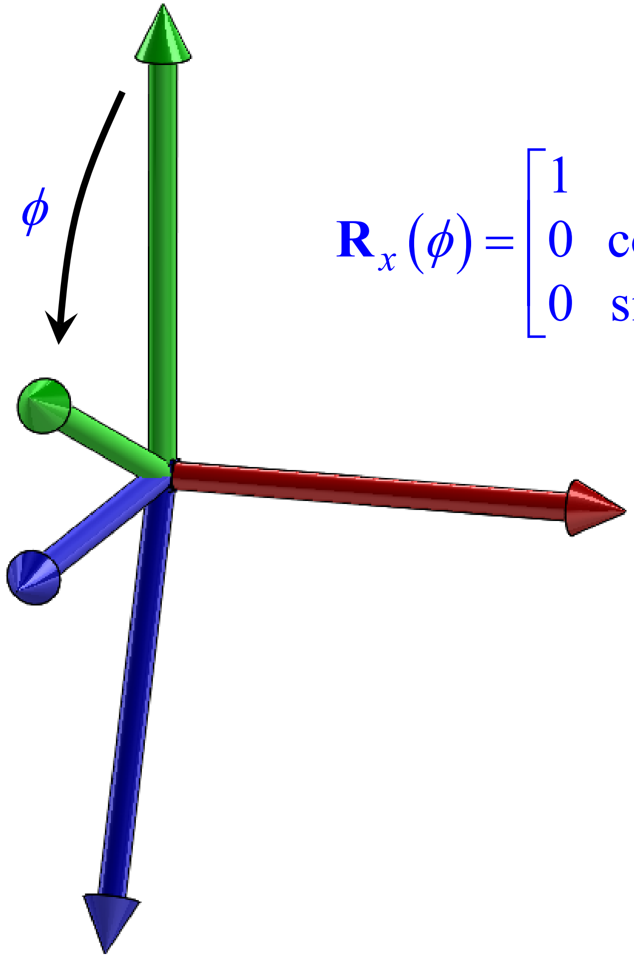
$$\mathbf{R}_{ad} = \mathbf{R}_z(90°) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{R}_{ac} = \mathbf{R}_y(90°) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$



Simple Rotation Matrices

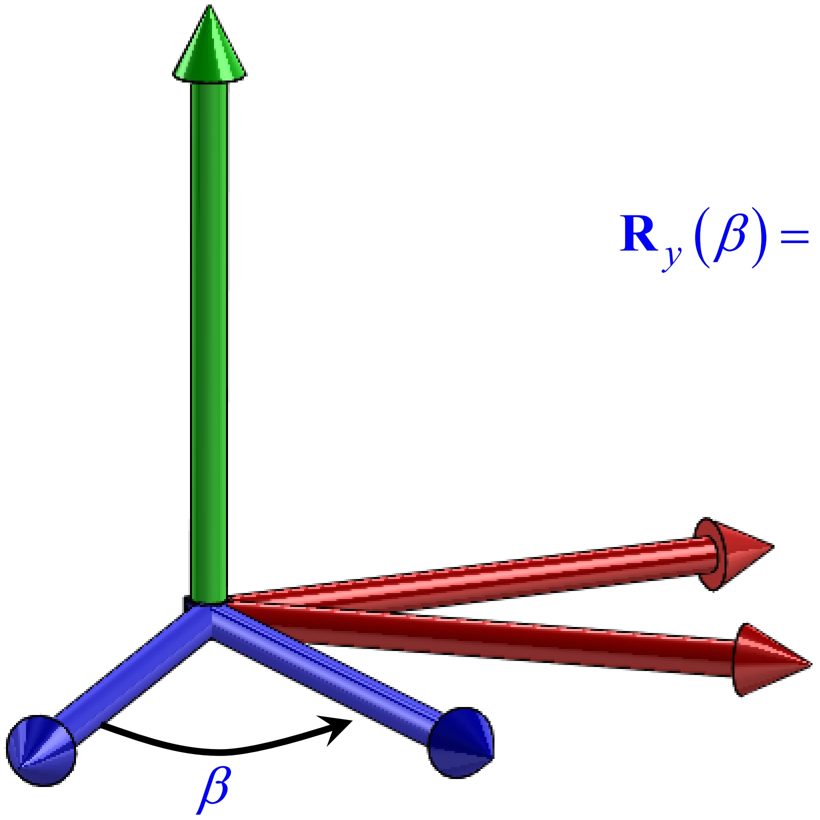
Rotation about the x axis:



$$\mathbf{R}_x(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\phi & -s\phi \\ 0 & s\phi & c\phi \end{bmatrix}$$

Simple Rotation Matrices

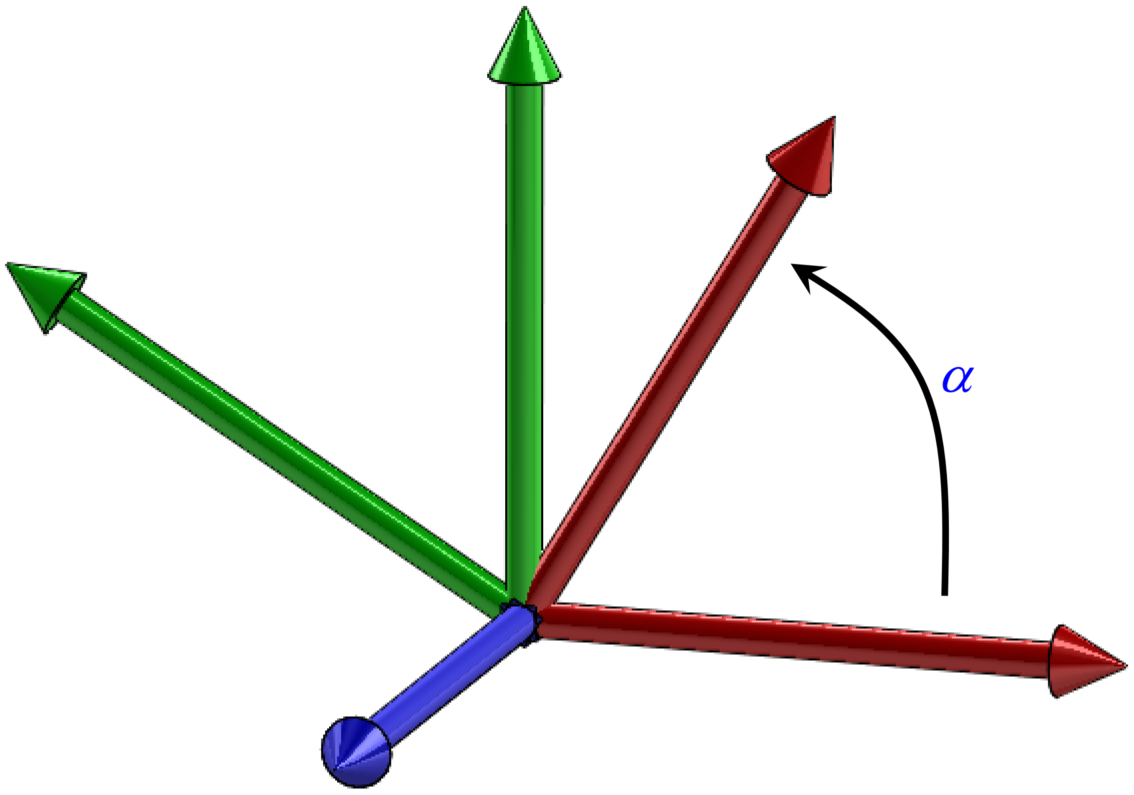
Rotation about the Y axis:



$$\mathbf{R}_y(\beta) = \begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix}$$

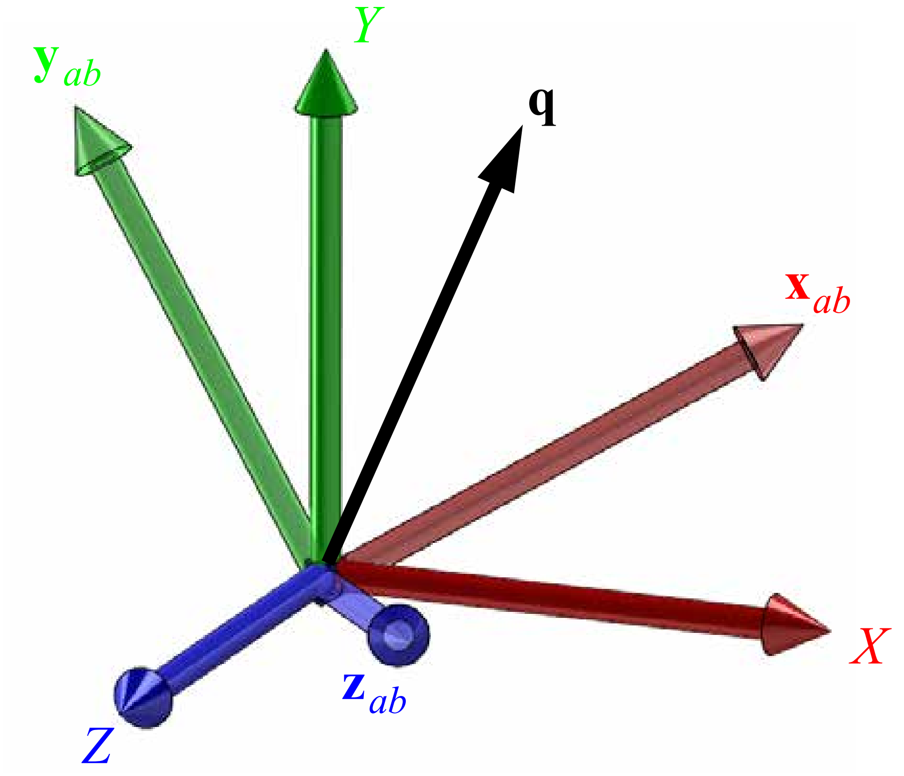
Simple Rotation Matrices

Rotation about the Z axis:



$$\mathbf{R}_z(\alpha) = \begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

SUMMARY



A **Rotation Matrix** can: ***transform points from one frame to another***

$$\mathbf{q}_a = \mathbf{x}_{ab} x_b + \mathbf{y}_{ab} y_b + \mathbf{z}_{ab} z_b$$

$$\mathbf{q}_a = \underbrace{\begin{bmatrix} \mathbf{x}_{ab} & \mathbf{y}_{ab} & \mathbf{z}_{ab} \end{bmatrix}}_{\mathbf{R}_{ab}} \begin{bmatrix} x_b \\ y_b \\ z_b \end{bmatrix}$$

$$\boxed{\mathbf{q}_a = \mathbf{R}_{ab} \mathbf{q}_b}$$

$\mathbf{R}_{ab} \in \mathbb{R}^3$ maps point from rotated frame B to fixed frame A.

$\mathbf{x}_{ab}$, $\mathbf{y}_{ab}$, $\mathbf{z}_{ab}$ are the coordinates of the principle axes of B relative to A.

A is the Inertial Frame

B is the Body Frame

(here rotated about origin)

A **Rotation Matrix** can also: ***define the orientation of a rigid body***