Spatial Velocity

Consider the rigid body transformation $\mathbf{g}_{ab}(t) \in \mathrm{SE}(3)$ in homogeneous form: $\mathbf{g}_{ab}(t) = \begin{bmatrix} \mathbf{R}_{ab}(t) & \mathbf{p}_{ab}(t) \\ \mathbf{0} & 1 \end{bmatrix}$

Similar to rotational velocity, $\dot{\mathbf{g}}_{ab}(t)$ is not useful, but $\dot{\mathbf{g}}_{ab}(t)\mathbf{g}_{ab}^{-1}(t)$ and $\mathbf{g}_{ab}^{-1}(t)\dot{\mathbf{g}}_{ab}(t)$ have significance:

$$\dot{\mathbf{g}}_{ab}(t)\mathbf{g}_{ab}^{-1}(t) = \begin{bmatrix} \dot{\mathbf{R}}_{ab} & \dot{\mathbf{p}}_{ab} \\ \mathbf{0} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{R}_{ab}^{T} & -\mathbf{R}_{ab}^{T}\mathbf{p}_{ab} \\ \mathbf{0} & 1 \end{bmatrix} = \begin{bmatrix} \dot{\mathbf{R}}_{ab}\mathbf{R}_{ab}^{T} & -\dot{\mathbf{R}}_{ab}\mathbf{R}_{ab}^{T}\mathbf{p}_{ab} + \dot{\mathbf{p}}_{ab} \\ \mathbf{0} & 0 \end{bmatrix}$$

The above has the form of a twist, so we define ***spatial velocity*** as:

$$\hat{\mathbf{V}}_{ab}^{s} = \dot{\mathbf{g}}_{ab}(t)\mathbf{g}_{ab}^{-1}(t) \qquad \mathbf{V}_{ab}^{s} = \begin{bmatrix} \mathbf{v}_{ab}^{s} \\ \boldsymbol{\omega}_{ab}^{s} \end{bmatrix} = \begin{bmatrix} -\dot{\mathbf{R}}_{ab}\mathbf{R}_{ab}^{T}\mathbf{p}_{ab} + \dot{\mathbf{p}}_{ab} \\ \left(\dot{\mathbf{R}}_{ab}\mathbf{R}_{ab}^{T}\right)^{\vee} \end{bmatrix}$$

The spatial velocity can be used to find the velocity of a point in spatial coordinates:

$$\frac{d}{dt}\left(\mathbf{q}_a(t) = \mathbf{g}_{ab}(t)\mathbf{q}_b\right) \Rightarrow \mathbf{v}_{\mathbf{q}_a} = \dot{\mathbf{q}}_a = \dot{\mathbf{g}}_{ab}\mathbf{q}_b = \dot{\mathbf{g}}_{ab}\overbrace{\mathbf{g}_{ab}^{-1}\mathbf{g}_{ab}}^{\mathbf{I}}\mathbf{q}_b = \dot{\mathbf{g}}_{ab}\mathbf{g}_{ab}^{-1}\mathbf{q}_a$$

$$\boxed{\mathbf{v}_{\mathbf{q}_a} = \hat{\mathbf{V}}_{ab}^{s}\mathbf{q}_a} = \boldsymbol{\omega}_{ab}^{s} \times \mathbf{q}_a + \mathbf{v}_{ab}^{s}$$

Body Velocity

For ***body velocity*** we still have $\mathbf{g}_{ab}(t) \in \mathrm{SE}(3)$ in homogeneous form: $\mathbf{g}_{ab}(t) = \begin{bmatrix} \mathbf{R}_{ab}(t) & \mathbf{p}_{ab}(t) \\ \mathbf{0} & 1 \end{bmatrix}$

The ***body velocity*** is defined:

$$\hat{\mathbf{V}}_{ab}^{b} = \mathbf{g}_{ab}^{-1}(t)\dot{\mathbf{g}}_{ab}(t) \qquad \mathbf{V}_{ab}^{s} = \begin{bmatrix} \mathbf{v}_{ab}^{b} \\ \boldsymbol{\omega}_{ab}^{b} \end{bmatrix} = \begin{bmatrix} \mathbf{R}_{ab}^{T}\dot{\mathbf{p}}_{ab} \\ \left(\mathbf{R}_{ab}^{T}\dot{\mathbf{R}}_{ab}\right)^{\vee} \end{bmatrix}$$

The body velocity can be used to find the velocity of a point in the body frame:

$$\mathbf{v}_{\mathbf{q}_b} = \mathbf{g}_{ab}^{-1}\mathbf{v}_{\mathbf{q}_a} = \mathbf{g}_{ab}^{-1}\dot{\mathbf{g}}_{ab}\mathbf{q}_b$$
$$\boxed{\mathbf{v}_{\mathbf{q}_b} = \hat{\mathbf{V}}_{ab}^{b}\mathbf{q}_b} = \boldsymbol{\omega}_{ab}^{b} \times \mathbf{q}_b + \mathbf{v}_{ab}^{b}$$

The ***body velocity*** takes body coordinates of point $\mathbf{q}$ and returns the velocity of the point in body coordinates.

The ***spatial velocity*** takes spatial coordinates of point $\mathbf{q}$ and returns the velocity of the point in spatial coordinates.

Interpretations of Spatial and Body Velocity

The interpretation of the ***body velocity*** is relatively straight-forward. The ***spatial velocity***, however, is less intuitive.

Body Velocity : $\mathbf{V}_{ab}^{b} = \begin{bmatrix} \mathbf{v}_{ab}^{b} \\ \boldsymbol{\omega}_{ab}^{b} \end{bmatrix}$

$\mathbf{v}_{ab}^{b}$ is the velocity of the origin of the body coordinate frame B as relative to spatial frame A, viewed from the current body frame B.

$\boldsymbol{\omega}_{ab}^{b}$ is the angular velocity of the origin of the body frame B, relative to spatial frame A, as viewed from the current body frame B.

The body velocity is not the velocity of the body relative to the body frame; this quantity is always zero.

Spatial Velocity : $\mathbf{V}_{ab}^{s} = \begin{bmatrix} \mathbf{v}_{ab}^{s} \\ \boldsymbol{\omega}_{ab}^{s} \end{bmatrix}$

$\mathbf{v}_{ab}^{s}$ is the velocity of a (possibly imaginary) point on the rigid body which is traveling through the origin of the spatial frame. That is, if one stands at the origin of the spatial frame and measures the instantaneous velocity of a point attached to the rigid body and traveling through the origin at that instant, this is $\mathbf{v}_{ab}^{s}$. *This linear component is not the velocity of the origin of the body frame!*

$\boldsymbol{\omega}_{ab}^{s}$ is the instantaneous angular velocity of the body as viewed in the spatial frame.