Transformation of Spatial Velocities

Consider the motion of three coordinates frames, A, B, and C. The relationship between their spatial velocities can be found by starting with the configuration of frame C relative to frame A:

$$\mathbf{g}_{ac} = \mathbf{g}_{ab}\mathbf{g}_{bc}$$

By definition and the chain rule:

$$\begin{aligned}
\mathbf{g}_{ac} &= \mathbf{g}_{ab}\mathbf{g}_{bc} \\
\hat{\mathbf{V}}_{ac}^{s} &= \dot{\mathbf{g}}_{ac}\mathbf{g}_{ac}^{-1} \\
&= \left(\dot{\mathbf{g}}_{ab}\mathbf{g}_{bc} + \mathbf{g}_{ab}\dot{\mathbf{g}}_{bc}\right)\left(\mathbf{g}_{bc}^{-1}\mathbf{g}_{ab}^{-1}\right) \\
&= \dot{\mathbf{g}}_{ab}\mathbf{g}_{ab}^{-1} + \mathbf{g}_{ab}\dot{\mathbf{g}}_{bc}\mathbf{g}_{bc}^{-1}\mathbf{g}_{ab}^{-1} \\
&= \hat{\mathbf{V}}_{ab}^{s} + \mathbf{g}_{ab}\hat{\mathbf{V}}_{bc}^{s}\mathbf{g}_{ab}^{-1}
\end{aligned}$$

Converting to twist coordinates gives:

$$\boxed{\mathbf{V}_{ac}^{s} = \mathbf{V}_{ab}^{s} + \mathrm{Ad}_{g_{ab}}\mathbf{V}_{ab}^{s}}$$

Transformation of Spatial Velocities

If $\boldsymbol{\xi}$ is a twist which represents motion of a screw and we move that screw by applying a rigid motion, $\mathbf{g} \in SE(3)$, then:

$$\boldsymbol{\xi}' = \mathrm{Ad}_{\mathbf{g}}\boldsymbol{\xi} \quad \text{or} \quad \hat{\boldsymbol{\xi}}' = \mathbf{g}\hat{\boldsymbol{\xi}}\mathbf{g}^{-1}$$