Wrenches

A **wrench** is a generalized force acting on a rigid body consisting of a linear force and angular moment:

$$\mathbf{F} = \begin{bmatrix} \mathbf{f} \\ \boldsymbol{\tau} \end{bmatrix}$$

$\mathbf{F} \in \mathbb{R}^6$ is a wrench; a combination of forces and torques

$\mathbf{f} \in \mathbb{R}^3$ is a linear force vector

$\boldsymbol{\tau} \in \mathbb{R}^3$ is a torque vector

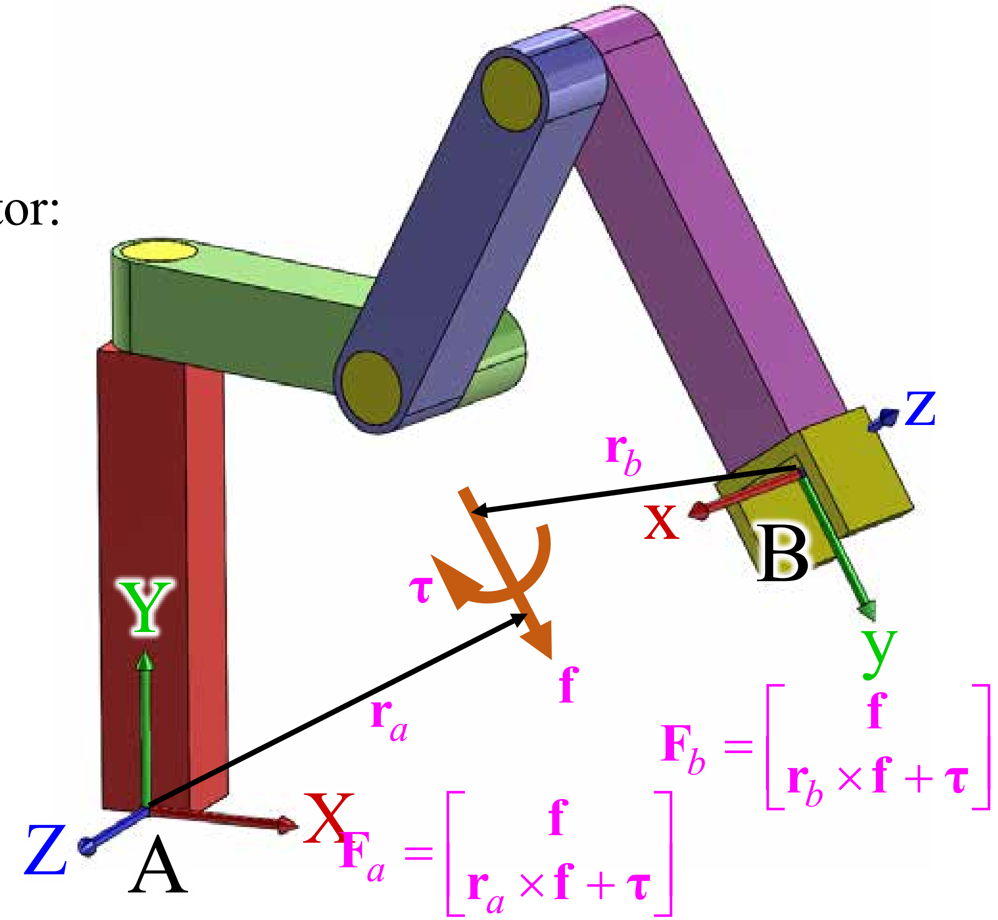
Wrenches can be defined in both the body frame and the spatial frame.
Both describe forces/moments applied to the rigid body at the end-effector:

$$\mathbf{F}_b = \begin{bmatrix} \mathbf{f}_b \\ \boldsymbol{\tau}_b \end{bmatrix} = \text{body wrench}$$

The body representation of an applied wrench represents the equivalent force and moment applied at the origin of the B frame and written in B's coordinates.

$$\mathbf{F}_a = \begin{bmatrix} \mathbf{f}_a \\ \boldsymbol{\tau}_a \end{bmatrix} = \text{spatial wrench}$$

The spatial representation of an applied wrench represents the equivalent force and moment applied at the origin of the A frame and written in A's coordinates.



Instantaneous Work

$$\mathbf{F}_b = \begin{bmatrix} \mathbf{f}_b \\ \boldsymbol{\tau}_b \end{bmatrix} = \text{body wrench}$$

$$\mathbf{F}_a = \begin{bmatrix} \mathbf{f}_a \\ \boldsymbol{\tau}_a \end{bmatrix} = \text{spatial wrench}$$

Wrenches combine with twists to define instantaneous work. Consider:

$\mathbf{F}_b$ is the body wrench

$\mathbf{V}_{ab}^b \in \mathbb{R}^6$ is the body velocity

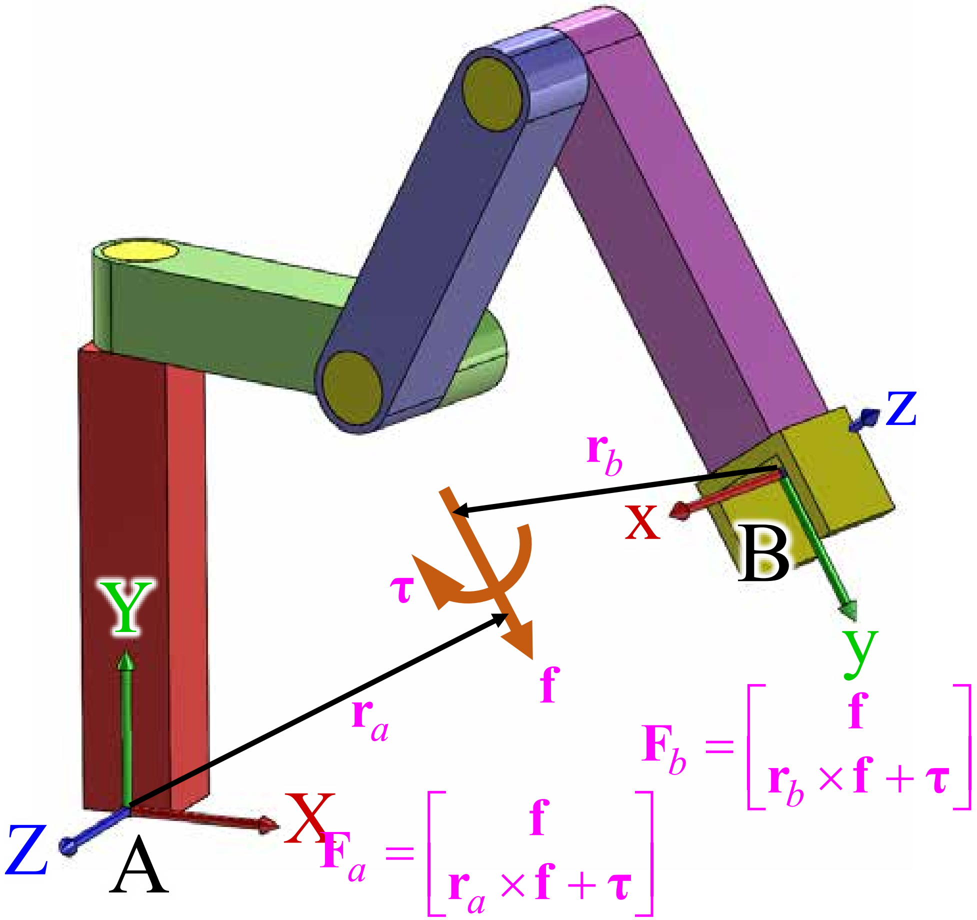
Infinitesimal work can be defined:

$$\delta w = \mathbf{V}_{ab}^b \cdot \mathbf{F}_b = (\mathbf{v} \cdot \mathbf{f} + \boldsymbol{\omega} \cdot \boldsymbol{\tau})$$

Net work is then:

$$W = \int_{t_1}^{t_2} \left(\mathbf{V}_{ab}^b \cdot \mathbf{F}_b \right) dt$$

Two wrenches are said to be equivalent if they generate the same work for every possible rigid motion.



Equivalent Wrenches

$$\mathbf{F}_b = \begin{bmatrix} \mathbf{f}_b \\ \boldsymbol{\tau}_b \end{bmatrix} = \text{body wrench}$$

$$\mathbf{F}_a = \begin{bmatrix} \mathbf{f}_a \\ \boldsymbol{\tau}_a \end{bmatrix} = \text{spatial wrench}$$

By equating the instantaneous work done by the spatial and body wrenches:

$$\begin{aligned}
\mathbf{V}^b \cdot \mathbf{F}_b &= \mathbf{V}^a \cdot \mathbf{F}_a \\
&= \mathrm{Ad}_{\mathbf{g}_{ab}} \mathbf{V}^b \cdot \mathbf{F}_a \\
&= \left(\mathrm{Ad}_{\mathbf{g}_{ab}} \mathbf{V}^b\right)^T \mathbf{F}_a \\
&= \left(\mathbf{V}^b\right)^T \mathbf{Ad}^T_{\mathbf{g}_{ab}} \mathbf{F}_a \\
\mathbf{V}^b \cdot \mathbf{F}_b &= \mathbf{V}^b \cdot \mathbf{Ad}^T_{\mathbf{g}_{ab}} \mathbf{F}_a
\end{aligned}$$

Now because $\mathbf{V}^b$ is free: $\mathbf{F}_b = \mathbf{Ad}^T_{\mathbf{g}_{ab}} \mathbf{F}_a$

Or alternatively:

$$\begin{aligned}
\mathbf{F}_a &= \mathbf{Ad}^T_{\mathbf{g}^{-1}_{ab}} \mathbf{F}_b \\
&= \mathbf{Ad}^T_{\mathbf{g}_{ba}} \mathbf{F}_b \quad \because \quad \mathbf{g}_{ba} = \mathbf{g}^{-1}_{ab}
\end{aligned}$$