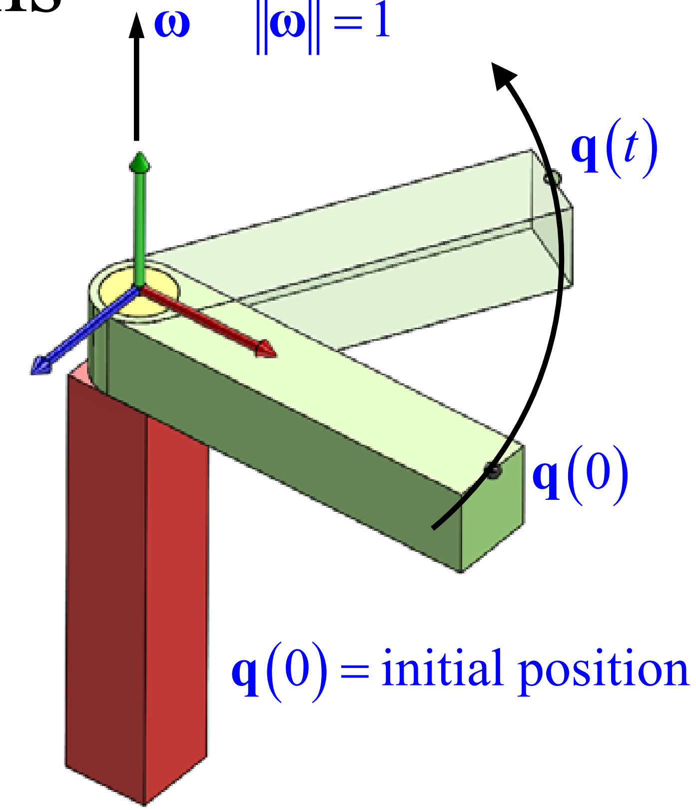
Exponential Coordinates for Rotations



Consider the velocity of a point $\mathbf{q}$ attached to a rotating body rotating about axis $\boldsymbol{\omega}$:

If we rotate the body at a constant unit velocity about the axis, the velocity of the point may be written as:

$$\dot{\mathbf{q}}(t) = \boldsymbol{\omega} \times \mathbf{q}(t) = \hat{\boldsymbol{\omega}}\mathbf{q}(t)$$

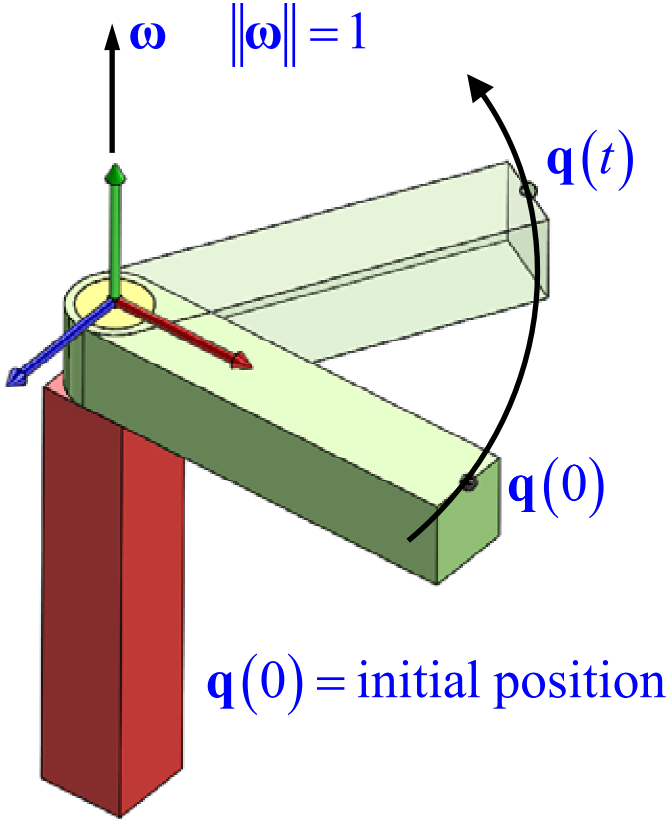
This is a time-invariant linear differential equation which may be integrated to give:

$$\mathbf{q}(t) = e^{\hat{\boldsymbol{\omega}}t}\mathbf{q}(0) \quad \text{where } e^{\hat{\boldsymbol{\omega}}t} \text{ is the matrix exponential}$$

The Matrix Exponential

The ***matrix exponential*** can be found by starting with a Tayler series expansion of the exponential function:

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^p}{p!} + \cdots$$



The ***matrix exponential*** is defined as:

$$e^{\mathbf{A}} = \sum_{k=0}^{\infty} \frac{1}{k!}\mathbf{A}^k = \mathbf{I} + \mathbf{A} + \frac{1}{2!}\mathbf{A}^2 + \frac{1}{3!}\mathbf{A}^3 + \cdots + \frac{1}{p!}\mathbf{A}^p + \cdots$$

In the case of constant unit angular velocity:

$$e^{\hat{\boldsymbol{\omega}}t} = I + \hat{\boldsymbol{\omega}}t + \frac{(\hat{\boldsymbol{\omega}}t)^2}{2!} + \frac{(\hat{\boldsymbol{\omega}}t)^3}{3!} + \cdots$$

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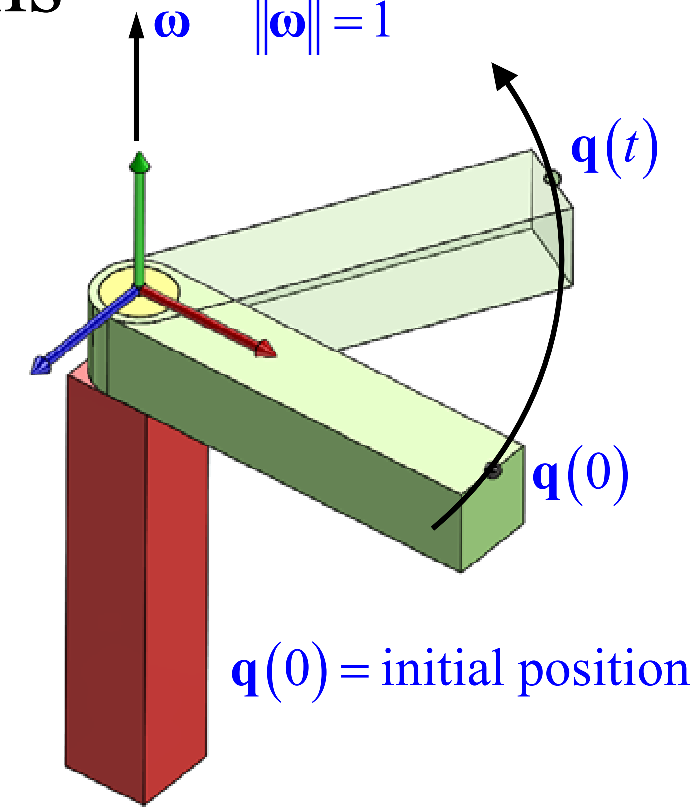

If we rotate the body at a constant unit velocity about the axis, the velocity of the point may be written as:

$$\dot{\mathbf{q}}(t) = \boldsymbol{\omega} \times \mathbf{q}(t) = \hat{\boldsymbol{\omega}}\mathbf{q}(t) \quad \text{which has the solution} \quad \mathbf{q}(t) = e^{\hat{\boldsymbol{\omega}}t}\mathbf{q}(0)$$

So if we rotate about axis $\boldsymbol{\omega}$ @ unit velocity for θ units of time, the net rotation is:

$$\boxed{\mathbf{R}(\boldsymbol{\omega}, \theta) = e^{\hat{\boldsymbol{\omega}}\theta}} \qquad \hat{\boldsymbol{\omega}} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

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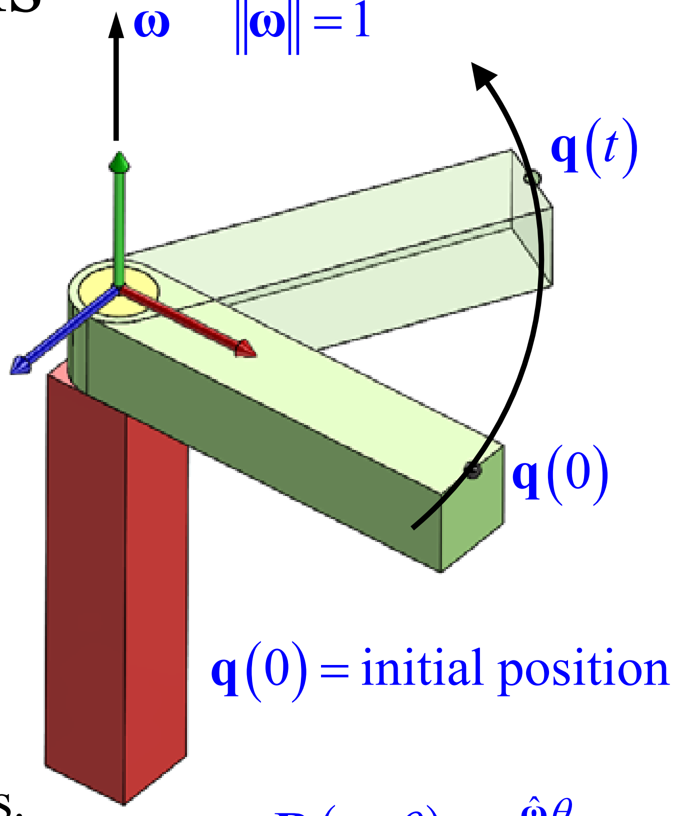

The matrix $\hat{\boldsymbol{\omega}}$ is skew-symmetric, such that:

$$\hat{\boldsymbol{\omega}}^T = -\hat{\boldsymbol{\omega}}$$

The vector space of all 3x3 skew-symmetric matrices is denoted **so(3)**.
More generally:

$$\mathrm{so}(n) = \left\{ \mathbf{s} \in \mathbb{R}^{n\times n} : \mathbf{s}^T = -\mathbf{s} \right.$$

We are typically interested in the planar rotation (n=2) and spatial rotation (n=3) cases.

$$\mathbf{R}(\boldsymbol{\omega}, \theta) = e^{\hat{\boldsymbol{\omega}}\theta}$$

Given $\hat{\mathbf{a}} \in \mathrm{so}(3)$, we have:

$$\hat{\mathbf{a}}^2 = \mathbf{a}\mathbf{a}^T - \|\mathbf{a}\|^2 \mathbf{I}$$

$$\hat{\mathbf{a}}^3 = -\|\mathbf{a}\|^2 \hat{\mathbf{a}}$$

$$\hat{\boldsymbol{\omega}} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$