Week 11 – Big-Step Proofs

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Quiz

Tuples

$$\text{(TU)}\quad \frac{e_1 \Rightarrow v_1 \qquad \ldots \qquad e_k \Rightarrow v_k}{(e_1, \ldots, e_k) \Rightarrow (v_1, \ldots, v_k)}$$

Lists

$$\text{(LI)}\quad \frac{e_1 \Rightarrow v_1 \qquad e_2 \Rightarrow v_2}{e_1 :: e_2 \Rightarrow v_1 :: v_2}$$

Global definitions

$$\text{(GD)}\quad \frac{f = e \qquad e \Rightarrow v}{f \Rightarrow v}$$

Local definitions

$$\text{(LD)}\quad \frac{e_1 \Rightarrow v_1 \qquad e_0[v_1/x] \Rightarrow v_0}{\texttt{let}\ x = e_1\ \texttt{in}\ e_0 \Rightarrow v_0}$$

Function calls

$$\text{(APP)}\quad \frac{e \Rightarrow \texttt{fun}\ x\ \texttt{->}\ e_0 \quad e_1 \Rightarrow v_1 \quad e_0[v_1/x] \Rightarrow v_0}{e\ e_1 \Rightarrow v_0}$$

$$\text{(APP')}\quad \frac{e_0 \Rightarrow \texttt{fun}\ x_1 \ldots x_k\ \texttt{->}\ e \quad e_1 \Rightarrow v_1 \ \ldots\ e_k \Rightarrow v_k \quad e[v_1/x_1, \ldots, v_k/x_k] \Rightarrow v}{e_0\ e_1\ \ldots\ e_k \Rightarrow v}$$

Pattern Matching

$$\text{(PM)}\quad \frac{e_0 \Rightarrow v' \equiv p_i[v_1/x_1, \ldots, v_k/x_k] \qquad e_i[v_1/x_1, \ldots, v_k/x_k] \Rightarrow v}{\texttt{match}\ e_0\ \texttt{with}\ p_1\ \texttt{->}\ e_1\ \texttt{|}\ \ldots\ \texttt{|}\ p_m\ \texttt{->}\ e_m \Rightarrow v}$$

Built-in operators

$$\text{(OP)}\quad \frac{e_1 \Rightarrow v_1 \qquad e_2 \Rightarrow v_2 \qquad v_1\ \mathsf{op}\ v_2 \Rightarrow v}{e_1\ \mathsf{op}\ e_2 \Rightarrow v}$$

Unary operators are treated analogously.

Week 11 Tutorial 01 — Big Steps

fun a → [(a+1, a-1)]

⇒ fun a → [(a+1, a-1)] (fun a → [(a+1, a-1)]) 7 ⇒ [(8,6)]

```
let f = fun a -> [(a+1,a-1)] in f 7 ⇒ [(8,6)]
```

Week 11 Tutorial 01 — Big Steps

$\Pi_0 \approx$ $\dfrac{}{[(7+1, 7-1)] \Rightarrow [(8,6)]}$

$\dfrac{\dfrac{}{\texttt{fun a -> [(a+1,a-1)]} \Rightarrow \texttt{fun a -> [(a+1,a-1)]}}{\texttt{(fun a -> [(a+1,a-1)]) 7} \Rightarrow \texttt{[(8,6)]}}\ \text{PP}}{\texttt{let f = fun a -> [(a+1,a-1)] in f 7} \Rightarrow \texttt{[(8,6)]}}\ \text{LD}$

$\text{fun } a \to [(a+1, a-1)]$
$\Rightarrow \text{fun } a \to [(a+1, a-1)]$ $\quad 7 \Rightarrow 7$ $\quad \Pi_0$

Week 11 Tutorial 01 — Big Steps

$$\pi_0 = \mathrm{LT}\ \frac{\dfrac{\dfrac{7 \Rightarrow 7 \quad 1 \Rightarrow 1}{7+1 \Rightarrow 8}\,\mathrm{OP} \qquad \dfrac{7 \Rightarrow 7 \quad 1 \Rightarrow 1}{7-1 \Rightarrow 6}\,\mathrm{OP}}{(7+1,\,7-1) \Rightarrow (8,6)}\,\mathrm{TU} \qquad [\,] \Rightarrow [\,]}{\texttt{[(7+1,7-1)]} \Rightarrow \texttt{[(8,6)]}}$$

$$\mathrm{LD}\ \frac{\texttt{fun a -> [(a+1,a-1)]} \Rightarrow \texttt{fun a -> [(a+1,a-1)]}\ \mathrm{APP'}\ \dfrac{\texttt{fun a -> [(a+1,a-1)]} \Rightarrow \texttt{fun a -> [(a+1,a-1)]} \quad \texttt{7} \Rightarrow \texttt{7} \quad \pi_0}{\texttt{(fun a -> [(a+1,a-1)]) 7} \Rightarrow \texttt{[(8,6)]}}}{\texttt{let f = fun a -> [(a+1,a-1)] in f 7} \Rightarrow \texttt{[(8,6)]}}$$

Week 11 Tutorial 01 — Big Steps

$$\pi_0 = \mathrm{LI}\ \frac{\mathrm{TU}\ \dfrac{\mathrm{OP}\ \dfrac{7 \Rightarrow 7\ \ 1 \Rightarrow 1\ \ 7+1 \Rightarrow 8}{\texttt{7+1} \Rightarrow \texttt{8}}\quad \mathrm{OP}\ \dfrac{7 \Rightarrow 7\ \ 1 \Rightarrow 1\ \ 7-1 \Rightarrow 6}{\texttt{7-1} \Rightarrow \texttt{6}}}{\texttt{(7+1,7-1)} \Rightarrow \texttt{(8,6)}\ \ \texttt{[]} \Rightarrow \texttt{[]}}}{\texttt{[(7+1,7-1)]} \Rightarrow \texttt{[(8,6)]}}$$

$$\mathrm{LD}\ \frac{\texttt{fun a -> [(a+1,a-1)]} \Rightarrow \texttt{fun a -> [(a+1,a-1)]}\ \ \mathrm{APP'}\ \dfrac{\texttt{fun a -> [(a+1,a-1)]} \Rightarrow \texttt{fun a -> [(a+1,a-1)]}\ \ 7 \Rightarrow 7\ \ \pi_0}{\texttt{(fun a -> [(a+1,a-1)])}\ 7 \Rightarrow \texttt{[(8,6)]}}}{\texttt{let f = fun a -> [(a+1,a-1)] in f}\ 7 \Rightarrow \texttt{[(8,6)]}}$$

Week 11 Tutorial 01 — Big Steps

```
let rec f = fun l ->
  match l with [] -> 1 | x::xs -> x + g xs
and g = fun l ->
  match l with [] -> 0 | x::xs -> x * f xs
```

$$\frac{\Pi_f\ [3;6] \Rightarrow [3;6] \qquad \text{match } [3;6] \text{ with } [] \to 7 \mid x::xs \to x + g\ xs \Rightarrow 9}{\texttt{f [3;6]} \Rightarrow 9}\ \text{APP}$$

Week 11 Tutorial 01 — Big Steps

$$\pi_f = \mathrm{GD}\ \frac{\texttt{f} = \texttt{fun l -> match l with [] -> 1 | x::xs -> x+g xs} \quad \texttt{fun l -> match l with [] -> 1 | x::xs -> x+g xs} \Rightarrow \texttt{fun l -> match l with [] -> 1 | x::xs -> x+g xs}}{\texttt{f} \Rightarrow \texttt{fun l -> match l with [] -> 1 | x::xs -> x+g xs}}$$

$$\pi_g = \mathrm{GD}\ \frac{\texttt{g} = \texttt{fun l -> match l with [] -> 0 | x::xs -> x*f xs} \quad \texttt{fun l -> match l with [] -> 0 | x::xs -> x*f xs} \Rightarrow \texttt{fun l -> match l with [] -> 0 | x::xs -> x*f xs}}{\texttt{g} \Rightarrow \texttt{fun l -> match l with [] -> 0 | x::xs -> x*f xs}}$$

T_f = fun l → match l with
| [] → 1
| x::xs → x + g xs

Global Definitions

$$\Pi_f = \frac{f = T_f \qquad T_f \Rightarrow T_f}{f \Rightarrow T_f}$$

Week 11 Tutorial 01 — Big Steps

$$\pi_f = \mathrm{GD}\ \frac{\texttt{f} = \texttt{fun l -> match l with [] -> 1 | x::xs -> x+g xs}\ \ \texttt{fun l -> match l with [] -> 1 | x::xs -> x+g xs} \Rightarrow \texttt{fun l -> match l with [] -> 1 | x::xs -> x+g xs}}{\texttt{f} \Rightarrow \texttt{fun l -> match l with [] -> 1 | x::xs -> x+g xs}}$$

$$\pi_g = \mathrm{GD}\ \frac{\texttt{g} = \texttt{fun l -> match l with [] -> 0 | x::xs -> x*f xs}\ \ \texttt{fun l -> match l with [] -> 0 | x::xs -> x*f xs} \Rightarrow \texttt{fun l -> match l with [] -> 0 | x::xs -> x*f xs}}{\texttt{g} \Rightarrow \texttt{fun l -> match l with [] -> 0 | x::xs -> x*f xs}}$$

T_f = fun l -> match l with [] -> 1 | x::xs -> x + g xs

T_g = fun l -> match l with [] -> 0 | x::xs -> x * f xs

$$\pi_f = \frac{f = T_f \qquad T_f \Rightarrow T_f}{f \Rightarrow T_f}$$

$$\pi_g = \frac{g = T_g \qquad T_g \Rightarrow T_g}{g \Rightarrow T_g}$$

Global Definitions

Week 11 Tutorial 01 — Big Steps

```
let rec f = fun l ->
  match l with [] -> 1 | x::xs -> x + g xs
and g = fun l ->
  match l with [] -> 0 | x::xs -> x * f xs
```

$$\text{APP'}\ \frac{\pi_f\ \texttt{[3;6]} \Rightarrow \texttt{[3;6]} \quad \text{PM}\ \frac{[3;6] \Rightarrow [3;6] \qquad 3 + g\ [6] \Rightarrow 9}{\texttt{match [3;6] with [] -> 1 | x::xs -> x+g xs} \Rightarrow 9}}{\texttt{f [3;6]} \Rightarrow 9}$$

Week 11 Tutorial 01 — Big Steps

```
let rec f = fun l ->
  match l with [] -> 1 | x::xs -> x + g xs
and g = fun l ->
  match l with [] -> 0 | x::xs -> x * f xs
```

$$\text{APP'}\ \frac{\pi_f\ \texttt{[3;6]} \Rightarrow \texttt{[3;6]}\ \ \text{PM}\ \dfrac{\texttt{[3;6]} \Rightarrow \texttt{[3;6]}\ \ OP \qquad \texttt{3+g [6]} \Rightarrow 9}{\texttt{match [3;6] with [] -> 1 | x::xs -> x+g xs} \Rightarrow 9}}{\texttt{f [3;6]} \Rightarrow 9}$$

Week 11 Tutorial 01 — Big Steps

```
let rec f = fun l ->
  match l with [] -> 1 | x::xs -> x + g xs
and g = fun l ->
  match l with [] -> 0 | x::xs -> x * f xs
```

$$\text{APP'}\ \frac{\pi_f\ \texttt{[3;6]} \Rightarrow \texttt{[3;6]}\ \text{PM}\ \dfrac{\texttt{[3;6]} \Rightarrow \texttt{[3;6]}\ \text{OP}\ \dfrac{3 \Rightarrow 3\ \text{APP'}\ \dfrac{\pi_g\ \texttt{[6]} \Rightarrow \texttt{[6]}\ \ \pi_0}{\texttt{g [6]} \Rightarrow 6}}{\texttt{3+g [6]} \Rightarrow 9}}{\texttt{match [3;6] with [] -> 1 | x::xs -> x+g xs} \Rightarrow 9}}{\texttt{f [3;6]} \Rightarrow 9}$$

3+6=>9

Week 11 Tutorial 01 — Big Steps

```
let rec f = fun l ->
  match l with [] -> 1 | x::xs -> x + g xs
and g = fun l ->
  match l with [] -> 0 | x::xs -> x * f xs
```

$\pi_0 =$ PM

```
match [6] with [] -> 0 | x::xs -> x*f xs ⇒ 6
```

Week 11 Tutorial 01 — Big Steps

```
let rec f = fun l ->
  match l with [] -> 1 | x::xs -> x + g xs
and g = fun l ->
  match l with [] -> 0 | x::xs -> x * f xs
```

$$\pi_0 = \mathrm{PM}\ \frac{\texttt{[6]} \Rightarrow \texttt{[6]}\ \mathrm{OP} \qquad \texttt{6*f []} \Rightarrow 6}{\texttt{match [6] with [] -> 0 | x::xs -> x*f xs} \Rightarrow 6}$$

Week 11 Tutorial 01 — Big Steps

```
let rec f = fun l ->
  match l with [] -> 1 | x::xs -> x + g xs
and g = fun l ->
  match l with [] -> 0 | x::xs -> x * f xs
```

$$\pi_0 = \mathrm{PM}\ \dfrac{[6] \Rightarrow [6]\ \ \mathrm{OP}\ \dfrac{6 \Rightarrow 6 \quad \mathrm{AP} \qquad \texttt{f [] } \Rightarrow \texttt{1}}{\texttt{6*f [] } \Rightarrow \texttt{6}}}{\texttt{match [6] with [] -> 0 | x::xs -> x*f xs} \Rightarrow \texttt{6}}$$

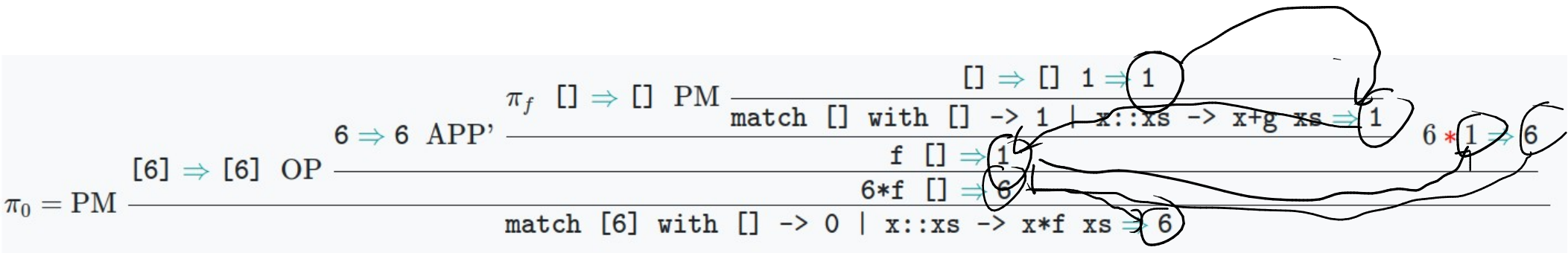
Week 11 Tutorial 01 — Big Steps

```
let rec f = fun l ->
  match l with [] -> 1 | x::xs -> x + g xs
and g = fun l ->
  match l with [] -> 0 | x::xs -> x * f xs
```

$$\pi_0 = \mathrm{PM}\ \dfrac{[6] \Rightarrow [6]\ \ \mathrm{OP} \quad \dfrac{6 \Rightarrow 6\ \ \mathrm{APP'} \quad \dfrac{\pi_f\ \ [] \Rightarrow []\ \ \not{PM} \qquad \texttt{match [] with [] -> 1 | x::xs -> x+g xs} \Rightarrow 1}{\texttt{f []} \Rightarrow 1}}{\texttt{6*f []} \Rightarrow 6}}{\texttt{match [6] with [] -> 0 | x::xs -> x*f xs} \Rightarrow 6}$$

Week 11 Tutorial 01 — Big Steps

```
let rec f = fun l ->
  match l with [] -> 1 | x::xs -> x + g xs
and g = fun l ->
  match l with [] -> 0 | x::xs -> x * f xs
```



Week 11 Tutorial 01 — Big Steps

```
let rec f = fun l ->
  match l with [] -> 1 | x::xs -> x + g xs
and g = fun l ->
  match l with [] -> 0 | x::xs -> x * f xs
```

τ_f = fun l -> match l with [] -> 1 | x::xs -> x + g xs

τ_g = fun l -> match l with [] -> 0 | x::xs -> x * f xs

$$\pi_f = \frac{f = \tau_f \quad \tau_f \Rightarrow \tau_f}{f \Rightarrow \tau_f}$$

$$\pi_g = \frac{g = \tau_g \quad \tau_g \Rightarrow \tau_g}{g \Rightarrow \tau_g}$$

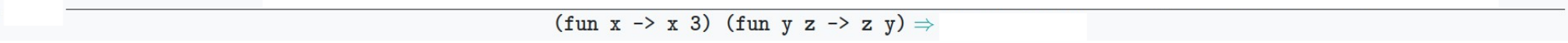
$$\pi_0 = \mathrm{PM}\ \frac{[6] \Rightarrow [6]\ \ \mathrm{OP}\ \dfrac{6 \Rightarrow 6\ \ \mathrm{APP'}\ \dfrac{\pi_f\ \ [] \Rightarrow []\ \ \mathrm{PM}\ \dfrac{[] \Rightarrow []\ \ 1 \Rightarrow 1}{\texttt{match [] with [] -> 1 | x::xs -> x+g xs} \Rightarrow 1}}{\texttt{f []} \Rightarrow 1} \quad 6 * 1 \Rightarrow 6}{\texttt{6*f []} \Rightarrow 6}}{\texttt{match [6] with [] -> 0 | x::xs -> x*f xs} \Rightarrow 6}$$

$$\mathrm{APP'}\ \frac{\pi_f\ \ [3;6] \Rightarrow [3;6]\ \ \mathrm{PM}\ \dfrac{[3;6] \Rightarrow [3;6]\ \ \mathrm{OP}\ \dfrac{3 \Rightarrow 3\ \ \mathrm{APP'}\ \dfrac{\pi_g\ \ [6] \Rightarrow [6]\ \ \pi_0}{\texttt{g [6]} \Rightarrow 6} \quad 3 + 6 \Rightarrow 9}{\texttt{3+g [6]} \Rightarrow 9}}{\texttt{match [3;6] with [] -> 1 | x::xs -> x+g xs} \Rightarrow 9}}{\texttt{f [3;6]} \Rightarrow 9}$$

Week 11 Tutorial 01 — Big Steps

```
(fun x -> x 3) (fun y z -> z y) (fun w -> w+w) ⇒ 6
```

Week 11 Tutorial 01 — Big Steps

$\pi_0 =$ 

Week 11 Tutorial 01 — Big Steps

$$\pi_0 = \text{APP'}\ \frac{\texttt{fun x -> x 3} \Rightarrow \texttt{fun x -> x 3} \quad \texttt{fun y z -> z y} \Rightarrow \texttt{fun y z -> z y} \qquad \texttt{(fun y z -> z y) 3} \Rightarrow}{\texttt{(fun x -> x 3) (fun y z -> z y)} \Rightarrow}$$

Week 11 Tutorial 01 — Big Steps

$$\pi_0 = \text{APP'}\ \frac{\texttt{fun x -> x 3} \Rightarrow \texttt{fun x -> x 3} \quad \texttt{fun y z -> z y} \Rightarrow \texttt{fun y z -> z y}\ \text{APP'}\ \dfrac{\texttt{fun y z -> z y} \Rightarrow \texttt{fun y z -> z y 3} \quad \texttt{3 fun z -> z 3} \Rightarrow \texttt{fun z -> z 3}}{\texttt{(fun y z -> z y) 3} \Rightarrow \texttt{fun z -> z 3}}}{\texttt{(fun x -> x 3) (fun y z -> z y)} \Rightarrow \texttt{fun z -> z 3}}$$

Week 11 Tutorial 01 — Big Steps

$$\text{APP'}\ \frac{\pi_0}{\texttt{(fun x -> x 3) (fun y z -> z y) (fun w -> w+w)} \Rightarrow 6}$$

Week 11 Tutorial 01 — Big Steps

$$\text{APP'}\ \frac{\pi_0\ \texttt{fun w -> w+w} \Rightarrow \texttt{fun w -> w+w} \qquad \texttt{(fun w -> w+w) 3} \Rightarrow \texttt{6}}{\texttt{(fun x -> x 3) (fun y z -> z y) (fun w -> w+w)} \Rightarrow \texttt{6}}$$

Week 11 Tutorial 01 — Big Steps

$$\text{APP'}\ \frac{\pi_0\ \texttt{fun w -> w+w} \Rightarrow \texttt{fun w -> w+w}\quad \text{APP'}\ \dfrac{\texttt{fun w -> w+w} \Rightarrow \texttt{fun w -> w+w}\quad \texttt{3} \Rightarrow \texttt{3} \qquad \texttt{3+3} \Rightarrow \texttt{6}}{\texttt{(fun w -> w+w) 3} \Rightarrow \texttt{6}}}{\texttt{(fun x -> x 3) (fun y z -> z y) (fun w -> w+w)} \Rightarrow \texttt{6}}$$

Week 11 Tutorial 01 — Big Steps

$$\text{APP'}\ \frac{\pi_0\ \texttt{fun w -> w+w} \Rightarrow \texttt{fun w -> w+w}\ \text{APP'}\ \dfrac{\texttt{fun w -> w+w} \Rightarrow \texttt{fun w -> w+w}\quad 3 \Rightarrow 3\quad \text{OP}\ \dfrac{3 \Rightarrow 3\quad 3 \Rightarrow 3\quad 3+3 \Rightarrow 6}{3+3 \Rightarrow 6}}{\texttt{(fun w -> w+w) 3} \Rightarrow 6}}{\texttt{(fun x -> x 3) (fun y z -> z y) (fun w -> w+w)} \Rightarrow 6}$$

Week 11 Tutorial 01 — Big Steps

$$\pi_0 = \text{APP'}\ \frac{\texttt{fun x -> x 3} \Rightarrow \texttt{fun x -> x 3} \quad \texttt{fun y z -> z y} \Rightarrow \texttt{fun y z -> z y} \quad \text{APP'}\ \dfrac{\texttt{fun y z -> z y} \Rightarrow \texttt{fun y z -> z y} \quad \texttt{3} \Rightarrow \texttt{3} \quad \texttt{fun z -> z 3} \Rightarrow \texttt{fun z -> z 3}}{\texttt{(fun y z -> z y) 3} \Rightarrow \texttt{fun z -> z 3}}}{\texttt{(fun x -> x 3) (fun y z -> z y)} \Rightarrow \texttt{fun z -> z 3}}$$

$$\text{APP'}\ \frac{\pi_0 \quad \texttt{fun w -> w+w} \Rightarrow \texttt{fun w -> w+w} \quad \text{APP'}\ \dfrac{\texttt{fun w -> w+w} \Rightarrow \texttt{fun w -> w+w} \quad \texttt{3} \Rightarrow \texttt{3} \quad \text{OP}\ \dfrac{3 \Rightarrow 3 \quad 3 \Rightarrow 3 \quad 3+3 \Rightarrow 6}{\texttt{3+3} \Rightarrow \texttt{6}}}{\texttt{(fun w -> w+w) 3} \Rightarrow \texttt{6}}}{\texttt{(fun x -> x 3) (fun y z -> z y) (fun w -> w+w)} \Rightarrow \texttt{6}}$$

Week 11 Tutorial 02 — Multiplication

Prove that the function

```
let rec mul a b =
  match a with 0 -> 0 | _ -> b + mul (a-1) b
```

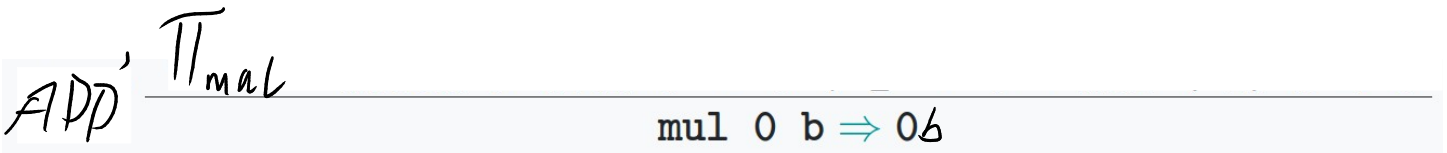
terminates for all inputs $a \geq 0$.

Week 11 Tutorial 02 — Multiplication

```
let rec mul a b =
  match a with 0 -> 0 | _ -> b + mul (a-1) b
```

Proof by Induction on a

Base: a = 0



Week 11 Tutorial 02 — Multiplication

```
let rec mul a b =
  match a with 0 -> 0 | _ -> b + mul (a-1) b
```

Proof by Induction on a

Base: a = 0

$$\text{APP}\ \frac{\pi_{mul}\quad \text{PM}\ \dfrac{}{\texttt{match 0 with 0 -> 0 | \_ -> b + mul (-1) b} \Rightarrow \texttt{0}}}{\texttt{mul 0 b} \Rightarrow \texttt{0}}$$

Week 11 Tutorial 02 — Multiplication

```
let rec mul a b =
  match a with 0 -> 0 | _ -> b + mul (a-1) b
```

Proof by Induction on a

Hypothisis:
$\forall a \geq 0$. mul a b = a*b
Step

$$\text{APP}\ \frac{\text{PM}\ \dfrac{\text{OP}\ \dfrac{\text{APP}\ \dfrac{\text{by I.H.}}{\texttt{mul}\ (a+1-1)\ b \Rightarrow ab} \qquad ab + b \Rightarrow (a+1)b}{b + \texttt{mul}\ (a+1-1)\ b \Rightarrow (a+1)b}}{\texttt{match}\ (a+1)\ \texttt{with}\ 0 \to 0 \mid \_ \to b + \texttt{mul}\ (a+1-1)\ b \Rightarrow (a+1)b}\ \Pi_{\texttt{mul}}}{\texttt{mul}\ (a+1)\ b \Rightarrow (a+1) * b}$$

Week 11 Tutorial 02 — Multiplication

```
let rec mul a b =
  match a with 0 -> 0 | _ -> b + mul (a-1) b
```

Proof by Induction on a

Hypothisis:

mul a b = a*b

Step:

$$\text{APP}\ \frac{\pi_{mul}\ \text{PM}\ \dfrac{\text{OP}\ \dfrac{\text{APP}\ \dfrac{\text{by I.H.}}{\texttt{mul (a+1-1) b} \Rightarrow a*b}\ b\ +\ (a*b) \Rightarrow (a+1)*b}{\texttt{b + mul (a+1-1) b} \Rightarrow (a+1)*b}}{\texttt{match a+1 with 0 -> 0 | \_ -> b + mul (a+1-1) b} \Rightarrow (a+1)*b}}{\texttt{mul (a+1) b} \Rightarrow (a+1)*b}$$

Week 11 Tutorial 03 — Threesum

Use big-step operational semantics to show that the function

```
let rec threesum = fun l ->
  match l with [] -> 0 | x::xs -> 3*x + threesum xs
```

terminates for all inputs and computes three times the sum of the input list's elements.

Week 11 Tutorial 03 — Threesum

```
let rec threesum = fun l ->
  match l with [] -> 0 | x::xs -> 3*x + threesum xs
```

Proof by Induction on length of List

Base: l = []

$$\text{APP}\ \frac{}{\texttt{threesum []} \Rightarrow 0}$$

Week 11 Tutorial 03 — Threesum

```
let rec threesum = fun l ->
  match l with [] -> 0 | x::xs -> 3*x + threesum xs
```

Proof by Induction on length of List

Base: l = []

$$\text{APP}\ \frac{\pi_{ts}\ \ [] \Rightarrow []\ \ \text{PM}\ \dfrac{[] \Rightarrow []\ \ 0 \Rightarrow 0}{\texttt{match [] with [] -> 0 | x::xs -> 3*x + threesum xs} \Rightarrow 0}}{\texttt{threesum []} \Rightarrow 0}$$

Week 11 Tutorial 03 — Threesum

```
let rec threesum = fun l ->
  match l with [] -> 0 | x::xs -> 3*x + threesum xs
```

Proof by Induction on length of List

Hypothisis: $threesum\ l \Rightarrow \sum_{i=1}^{n} x_i$

Step: l = x :: xs

APP

$$\texttt{threesum}\ (x_{n+1}\texttt{::xs}) \Rightarrow 3\sum_{i=1}^{n+1} x_i$$

Week 11 Tutorial 03 — Threesum

```
let rec threesum = fun l ->
  match l with [] -> 0 | x::xs -> 3*x + threesum xs
```

Proof by Induction on length of List

Hypothisis: $threesum\ l \Rightarrow \sum_{i=1}^{n} x_i$

Step: l = x :: xs

$$\text{APP}\ \frac{\pi_{ts}\ \ x_{n+1}\texttt{::xs} \Rightarrow x_{n+1}\texttt{::xs}\text{PM} \qquad \frac{x_{n+1}\texttt{::xs} \Rightarrow x_{n+1}\texttt{::xs}\ \ \text{OP}}{\texttt{3*}x_{n+1}\ \texttt{+ threesum xs} \Rightarrow 3\sum_{i=1}^{n+1} x_i}}{\frac{\texttt{match}\ \ x_{n+1}\texttt{::xs with [] -> 0 | x::xs -> 3*x + threesum xs} \Rightarrow 3\sum_{i=1}^{n+1} x_i}{\texttt{threesum}\ (x_{n+1}\texttt{::xs}) \Rightarrow 3\sum_{i=1}^{n+1} x_i}}$$

Week 11 Tutorial 03 — Threesum

```
let rec threesum = fun l ->
  match l with [] -> 0 | x::xs -> 3*x + threesum xs
```

Proof by Induction on length of List

Hypothisis: $\text{thsm } l \Rightarrow \sum_{i=1}^{n} x_i$

Step: l = x :: xs

$$\text{APP}\ \frac{\pi_{ts}\ \ x_{n+1}\texttt{::xs} \Rightarrow x_{n+1}\texttt{::xs}\text{PM} \quad x_{n+1}\texttt{::xs} \Rightarrow x_{n+1}\texttt{::xs}\ \ \text{OP}\ \dfrac{\dfrac{3 \Rightarrow 3\ \ x_{n+1} \Rightarrow x_{n+1}\ \ 3 * x_{n+1} \Rightarrow 3x_{n+1}}{3 * x_{n+1} \Rightarrow 3x_{n+1}}\ \text{APP}\ \dfrac{\texttt{by I.H.}}{\texttt{threesum xs} \Rightarrow 3\sum_{i=1}^{n} x_i}\ \ 3x_{n+1} + 3\sum_{i=1}^{n} x_i \Rightarrow 3\sum_{i=1}^{n+1} x_i}{\texttt{3*}x_{n+1}\ \texttt{+ threesum xs} \Rightarrow 3\sum_{i=1}^{n+1} x_i}}{\texttt{match}\ \ x_{n+1}\texttt{::xs with [] -> 0 | x::xs -> 3*x + threesum xs} \Rightarrow 3\sum_{i=1}^{n+1} x_i}}{\texttt{threesum (}x_{n+1}\texttt{::xs)} \Rightarrow 3\sum_{i=1}^{n+1} x_i}$$