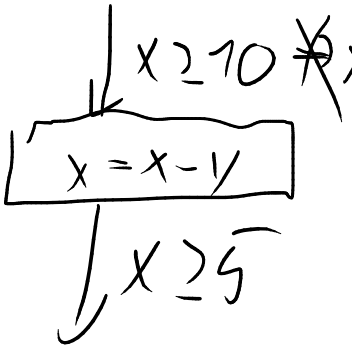
Week 3: Loop Invariants



x - y ≥ 5

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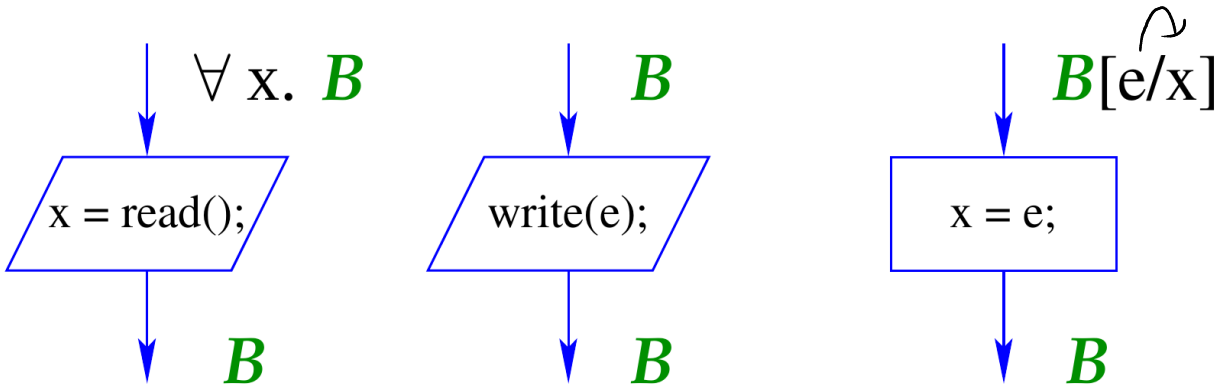
Quiz

Week 03 Tutorial 01 — MiniJava 2.0

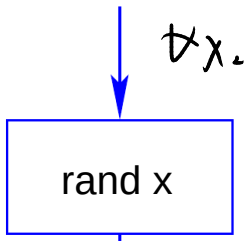


Week 03 Tutorial 01 — MiniJava 2.0

1. `rand x`:
   Assigns a random value to variable x,
2. `x = either` $e_0, \ldots, e_k$:
   Assigns one of the values of the expressions $e_0, \ldots, e_k$ to variable `x` non-deterministically,
3. `x = e in a, b`:
   Assigns the value 1 to variable `x`, if the value of expression `e` is in the range $[a, b]$ and 0 if `e` is not in the range or the range is empty $(a > b)$,
4. `stop`:
   Immediately stops the program.

Define the weakest precondition operator $\mathbf{WP}[\![\ldots]\!](B)$ for each of these statements.

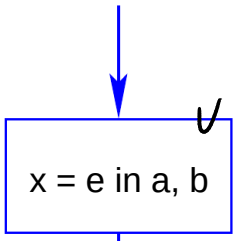
Week 03 Tutorial 01 — MiniJava 2.0



$\forall x.\ B \equiv \forall x.\ y > 2$

$\equiv y > 2$

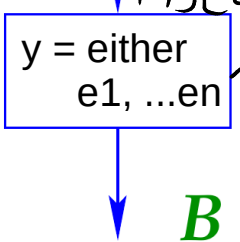
$B \equiv y > 2$



$(B[1/x] \wedge a \le e \le b)$

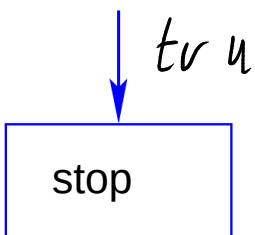
$\vee\ (B[0/x] \wedge (e < a \vee b < e))$

$B \equiv y + x = z$



$B[e_1/y] \wedge B[e_2/x] \equiv e_1 > 2 \wedge B[e_2/x] \equiv e_2 > 2 \wedge \ldots$

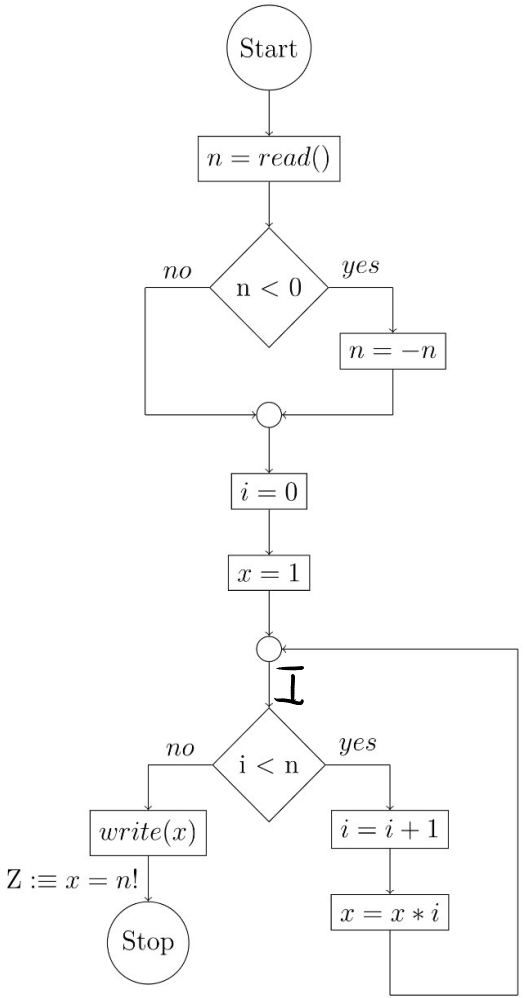
$B \equiv y > 2$



$true$

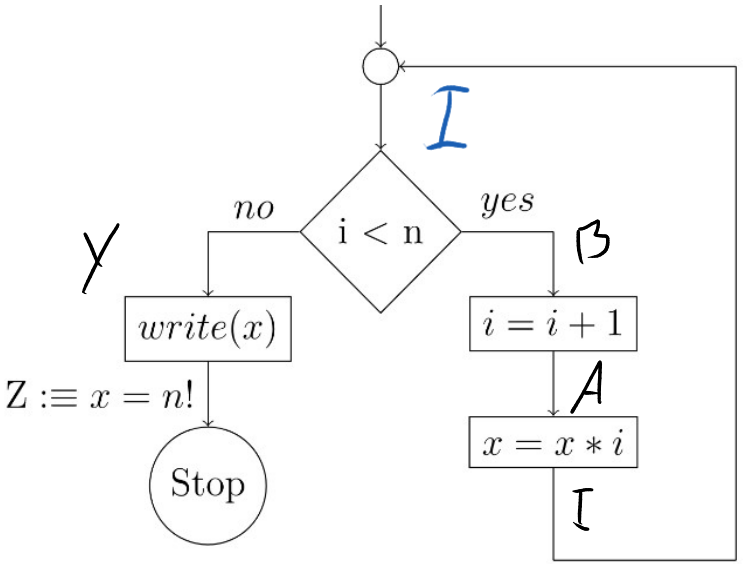
$B \equiv y > 2$

Week 03 Tutorial 02 — Loop Invariants



1. Discuss the problem that arises when computing weakest preconditions to prove Z.
2. How can you use weakest preconditions to prove Z anyway?

Week 03 Tutorial 02 — Loop Invariants



$y \equiv z$

$I :\equiv x \geq 0$

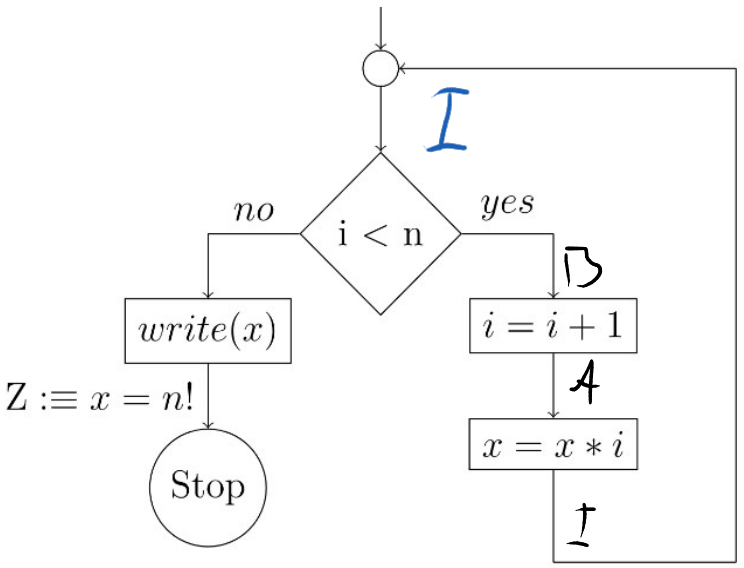
$A :\equiv wp[x := x \cdot i](x \geq 0) \equiv x \cdot i \geq 0$

$B :\equiv x(i+1) \geq 0$

$I \wedge i < n \overset{!}{\Rightarrow} B$

$x \geq 0 \wedge i < n \not\Rightarrow x(i+1) \geq 0$

Week 03 Tutorial 02 — Loop Invariants



$I :\equiv i=0 \wedge x=1 \wedge n=0$

$A :\equiv i=0 \wedge xi=1 \wedge n=0$

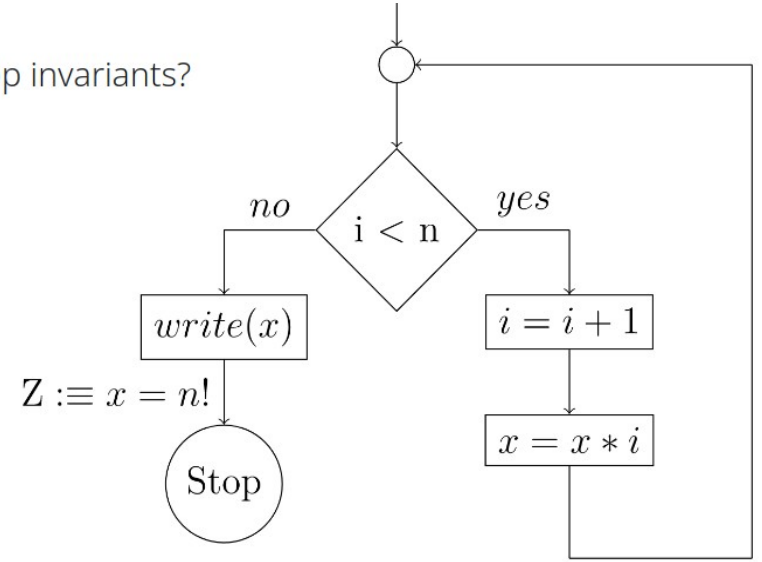
$B :\equiv i+1=0 \wedge x(i+1)=1 \wedge n=0$

$I \wedge i<n \overset{!}{\Rightarrow} B$

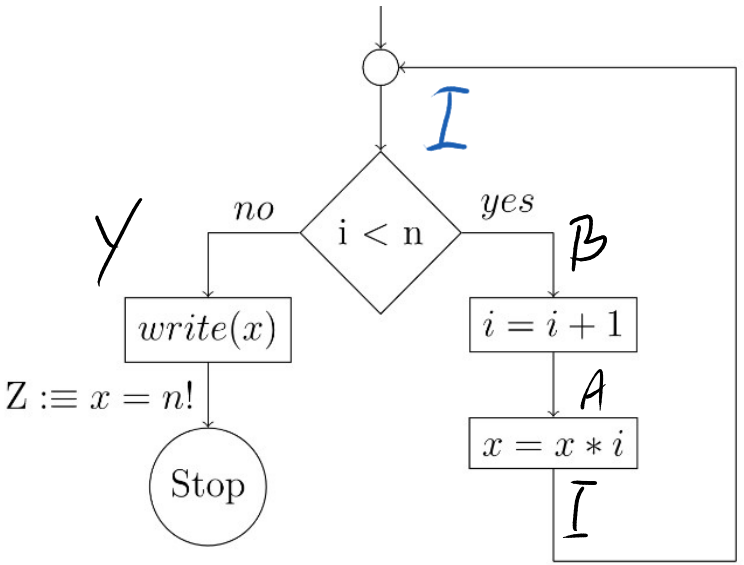
$i=0 \wedge x=1 \wedge n=0 \wedge i<n \overset{!!}{\not\Rightarrow} i=-1 \wedge x(i+1)=1 \wedge n=0$

Week 03 Tutorial 02 — Loop Invariants

- a) How has a useful loop invariant be related to Z?
- b) What happens if the loop invariant is chosen too strong?
- c) What happens if the loop invariant is chosen too weak?
- d) Can you give a meaningful lower and upper bound for useful loop invariants?



Week 03 Tutorial 02 — Loop Invariants



$Y \equiv Z$

$I :\equiv x = i!$ $\quad \wedge \; i \le n$

$A :\equiv x\,i = i!$ $\quad \wedge \; i \le n$

$B :\equiv x(i+1) = (i+1)!$ $\quad \wedge \; i+1 \le n$

$\equiv x(i+1) = (i+1) \cdot i!$ ...

$\equiv x = i!$ $\quad \wedge \; i+1 \le n$

Proof LC:

yes: $I \wedge i < n \Rightarrow B$

$x = i! \wedge i \le n \wedge i < n \Rightarrow x = i! \wedge i+1 \le n$

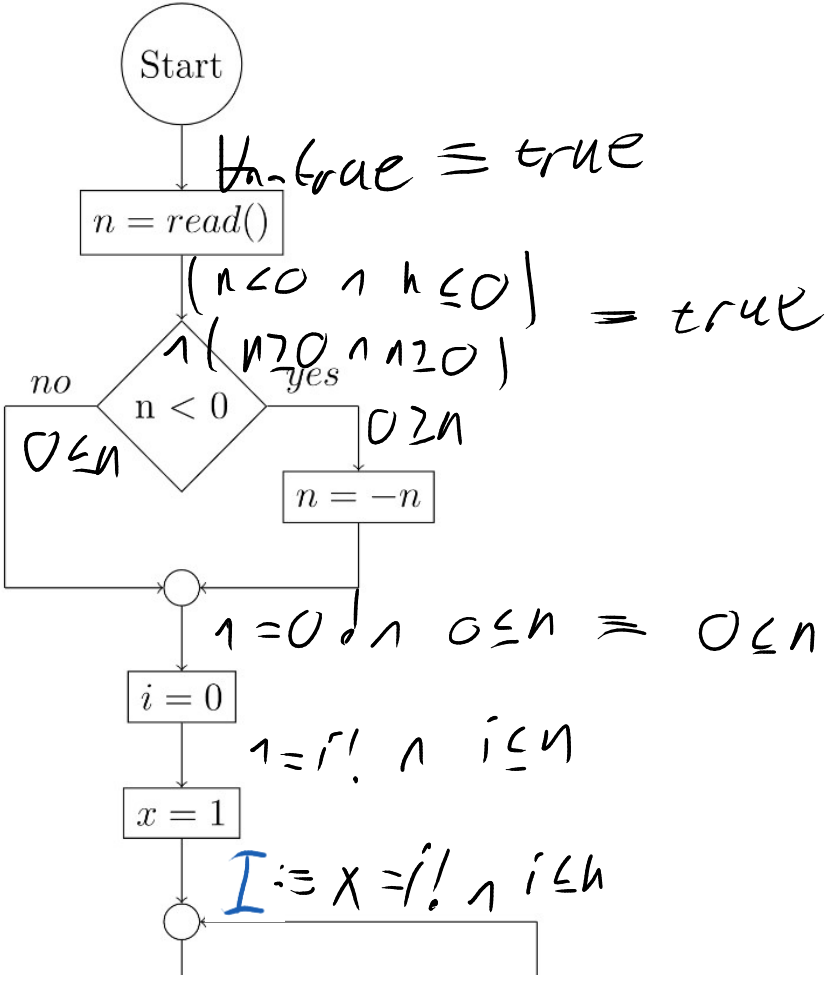
no: $I \wedge i \ge n \overset{!}{\Rightarrow} Y$

$x = i! \wedge i \ge n \Rightarrow x = n!$

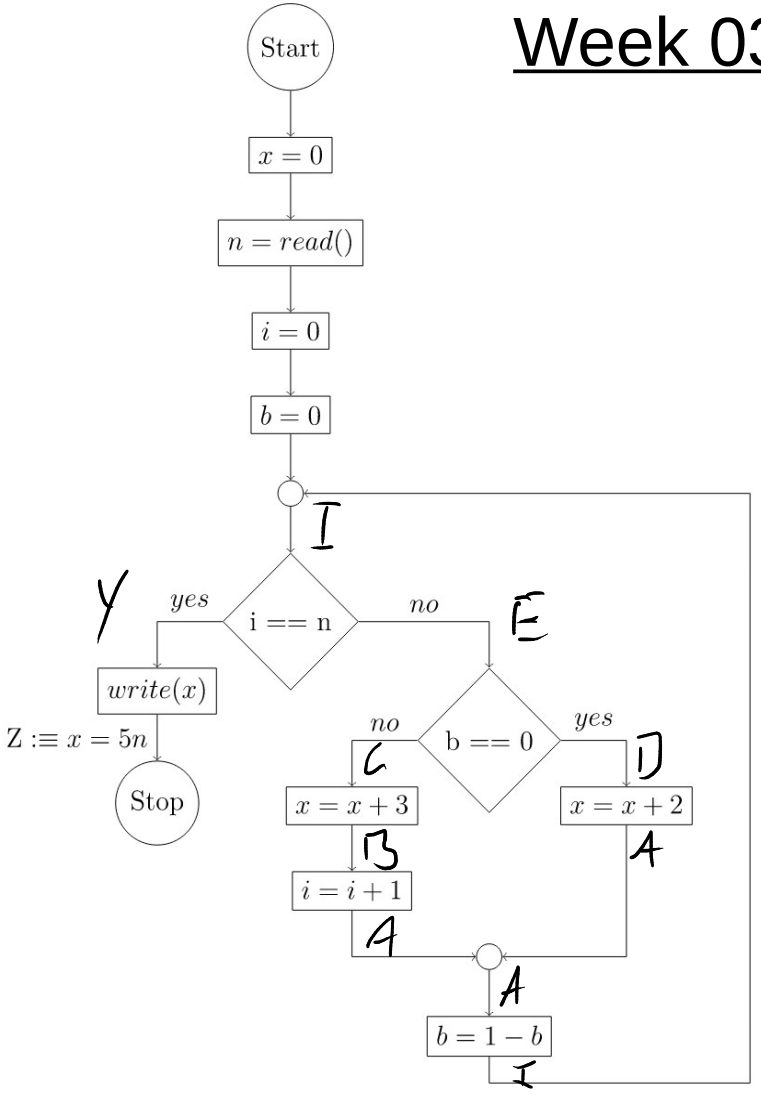
$\wedge \; i \le n$

$\equiv x = i! \wedge i = n \Rightarrow x = n!$

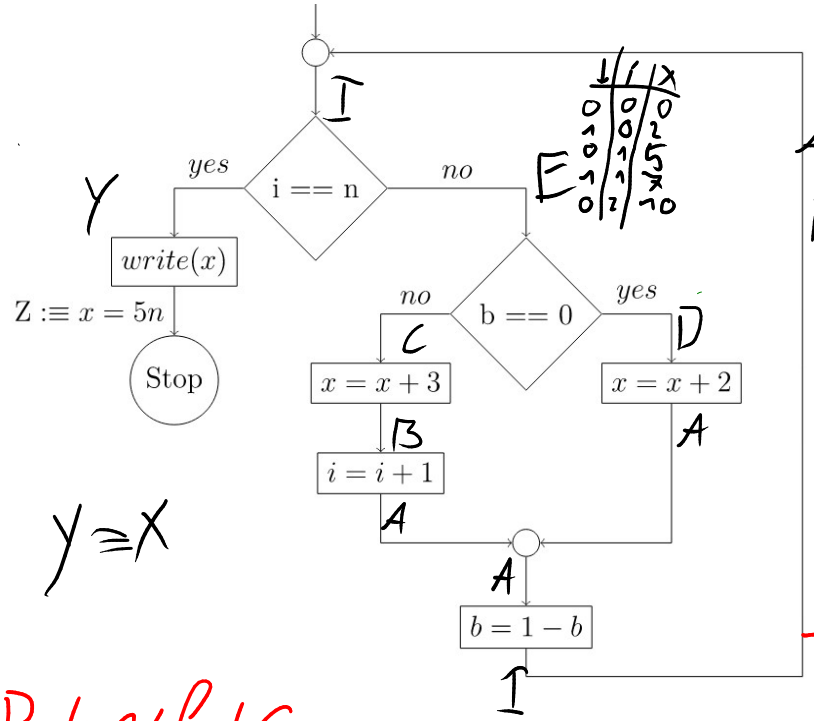
Week 03 Tutorial 02 — Loop Invariants



Week 03 Tutorial 03 — Two b, or Not Two b



Week 03 Tutorial 03 — Two b, or Not Two b



$Y :\equiv X$

$I :\equiv x = 5i + 26$

$A :\equiv x = 5i + 2 - 26$

$B :\equiv x = 5i + 7 - 26$

$C :\equiv x = 5i + 4 - 26$

$D :\equiv x = 5i - 26$

$E :\equiv (b = 0 \wedge x = 5i) \vee (b \neq 0 \wedge x = 5i + 4 - 26)$

$\rightarrow \equiv (b = 0 \wedge x = 5i) \vee (b = 1 \wedge x = 5i + 2)$

$\wedge (b = 0 \vee b = 1) \wedge (i = n \Rightarrow b = 0)$

$\wedge (b = 1 \vee b = 0) \wedge (i = n \Rightarrow b = 1)$

$\wedge (b = 1 \vee b = 0) \wedge (i + 1 = n \Rightarrow b = 1)$

$\wedge (b = 1 \vee b = 0) \wedge (i + 1 \neq n \Rightarrow b = 1)$

$\wedge (b = 1 \vee b = 0) \wedge (i = n \Rightarrow b = 1)$

$\Rightarrow (b = 0 \wedge x = 5i \wedge i \neq n) \vee (b = 1 \wedge x = 5i + 2)$

Proof of LC

yes
- $I \wedge i \neq n \overset{!}{\Rightarrow} E$
- $I \wedge i = n \overset{!}{\Rightarrow} E$ ✓

no
- $I \wedge i = n \overset{!}{\Rightarrow} Y$
- $I \wedge i = n \overset{!}{\Rightarrow} Y$ ✓
  - $x = 5i + 26 \wedge (b = 1 \vee b = 0) \wedge i = n \Rightarrow x = 5n$
  - $\wedge\ i = n \Rightarrow b = 0$

$x = 5i + 26 \wedge (b = 1 \vee b = 0) \wedge i = n \Rightarrow x = 5n$