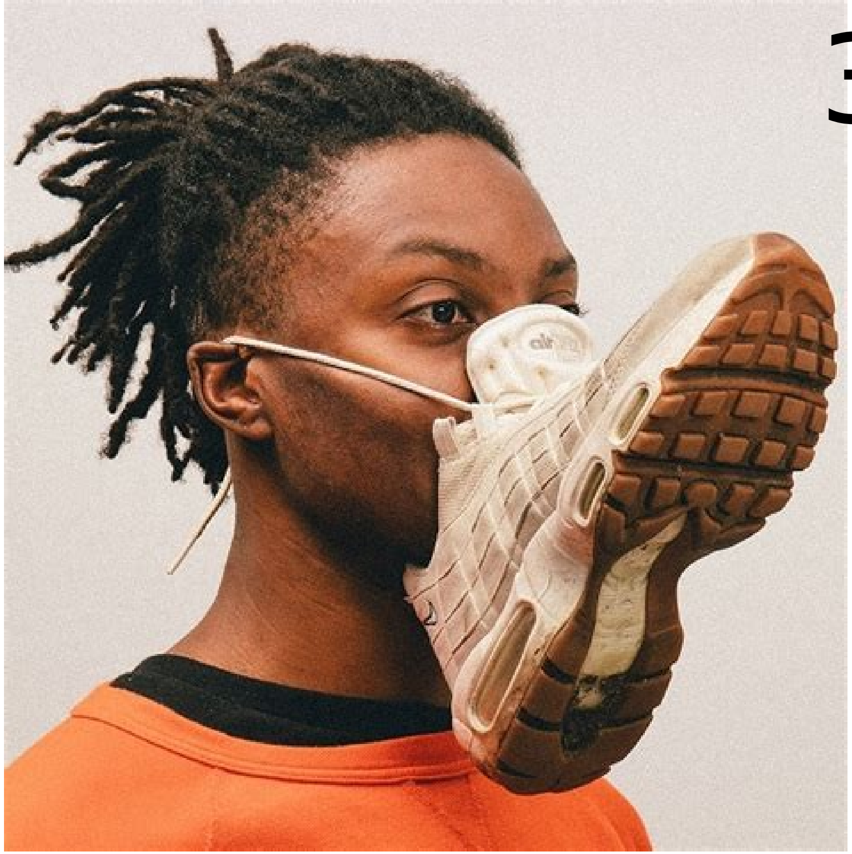
Week 11 – Big-Step Proofs



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Quiz

Tuples

$$\text{(TU)}\quad \frac{e_1 \Rightarrow v_1 \qquad \ldots \qquad e_k \Rightarrow v_k}{(e_1, \ldots, e_k) \Rightarrow (v_1, \ldots, v_k)}$$

Lists

$$\text{(LI)}\quad \frac{e_1 \Rightarrow v_1 \qquad e_2 \Rightarrow v_2}{e_1 :: e_2 \Rightarrow v_1 :: v_2}$$

Global definitions

$$\text{(GD)}\quad \frac{f = e \qquad e \Rightarrow v}{f \Rightarrow v}$$

Local definitions

$$(\text{LD}) \quad \frac{e_1 \Rightarrow v_1 \qquad e_0[v_1/x] \Rightarrow v_0}{\texttt{let}\ x = e_1\ \texttt{in}\ e_0 \quad \Rightarrow \quad v_0}$$

Function calls

$$(\text{APP}) \quad \frac{e \Rightarrow \texttt{fun}\ x\ \texttt{->}\ e_0 \quad e_1 \Rightarrow v_1 \quad e_0[v_1/x] \Rightarrow v_0}{e\ e_1 \quad \Rightarrow \quad v_0}$$

$$(\text{APP'}) \quad \frac{e_0 \Rightarrow \texttt{fun}\ x_1 \ldots x_k\ \texttt{->}\ e \quad e_1 \Rightarrow v_1 \ \ldots\ e_k \Rightarrow v_k \quad e[v_1/x_1, \ldots, v_k/x_k] \Rightarrow v}{e_0\ e_1\ \ldots\ e_k \quad \Rightarrow \quad v}$$

Pattern Matching

$$(\mathsf{PM})\quad \frac{e_0 \;\Rightarrow\; v' \equiv p_i[v_1/x_1,\ldots,v_k/x_k] \qquad e_i[v_1/x_1,\ldots,v_k/x_k] \;\Rightarrow\; v}{\texttt{match}\; e_0\; \texttt{with}\; p_1 \;\texttt{->}\; e_1 \;\texttt{|}\; \ldots \;\texttt{|}\; p_m \;\texttt{->}\; e_m \;\Rightarrow\; v}$$

Built-in operators

$$(\mathsf{OP})\quad \frac{e_1 \;\Rightarrow\; v_1 \qquad e_2 \;\Rightarrow\; v_2 \qquad v_1 \,\mathsf{op}\, v_2 \Rightarrow v}{e_1 \,\mathsf{op}\, e_2 \;\Rightarrow\; v}$$

Unary operators are treated analogously.

Week 11 Tutorial 01 — Big Steps

fun a -> [(a+1, a-1)]

LD $\Rightarrow$ fun a -> [(a+1, a-1)] (fun a -> [(a+1, a-1)]) 7 $\Rightarrow$ [(6, 8)]

```
let f = fun a -> [(a+1,a-1)] in f 7 ⇒ [(8,6)]
```

Week 11 Tutorial 01 — Big Steps

$$\Pi_0 = \frac{}{[(7+1),(7-1)] \Rightarrow [(8,6)]}$$

$$\text{LD}\ \dfrac{\texttt{fun a -> [(a+1,a-1)]} \Rightarrow \texttt{fun a -> [(a+1,a-1)]} \qquad \text{APP}\ \dfrac{\dfrac{\text{fun a -> [(a+1, a-1)]}}{\Rightarrow \text{fun a [(a+1,a-1)]}} \quad 7 \Rightarrow 7 \quad \Pi_0}{\texttt{(fun a -> [(a+1,a-1)]) 7} \Rightarrow \texttt{[(8,6)]}}}{\texttt{let f = fun a -> [(a+1,a-1)] in f 7} \Rightarrow \texttt{[(8,6)]}}$$

Week 11 Tutorial 01 — Big Steps

$$7 + 1 \Rightarrow 8 \qquad 7 - 1 \Rightarrow 6$$

$$\pi_0 = \text{Li}\ \dfrac{\text{TU}\ \dfrac{\text{OP}\ \dfrac{7 \Rightarrow 7 \quad 1 \Rightarrow 1}{7+1 \Rightarrow 8} \qquad \text{OP}\ \dfrac{7 \Rightarrow 7 \quad 1 \Rightarrow 1}{7-1 \Rightarrow 6}}{(7+1, 7-1) \Rightarrow (8,6)} \qquad \dfrac{}{[\,] \Rightarrow [\,]}}{\texttt{[(7+1,7-1)]} \Rightarrow \texttt{[(8,6)]}}$$

$$\text{LD}\ \dfrac{\texttt{fun a -> [(a+1,a-1)]} \Rightarrow \texttt{fun a -> [(a+1,a-1)]} \quad \text{APP'}\ \dfrac{\texttt{fun a -> [(a+1,a-1)]} \Rightarrow \texttt{fun a -> [(a+1,a-1)]} \quad \texttt{7} \Rightarrow \texttt{7} \quad \pi_0}{\texttt{(fun a -> [(a+1,a-1)]) 7} \Rightarrow \texttt{[(8,6)]}}}{\texttt{let f = fun a -> [(a+1,a-1)] in f 7} \Rightarrow \texttt{[(8,6)]}}$$

Week 11 Tutorial 01 — Big Steps

$$\pi_0 = \text{LI}\ \dfrac{\text{TU}\ \dfrac{\text{OP}\ \dfrac{7 \Rightarrow 7 \quad 1 \Rightarrow 1 \quad 7 + 1 \Rightarrow 8}{\texttt{7+1} \Rightarrow 8} \quad \text{OP}\ \dfrac{7 \Rightarrow 7 \quad 1 \Rightarrow 1 \quad 7 - 1 \Rightarrow 6}{\texttt{7-1} \Rightarrow 6}}{\texttt{(7+1,7-1)} \Rightarrow \texttt{(8,6)}} \quad \texttt{[]} \Rightarrow \texttt{[]}}{\texttt{[(7+1,7-1)]} \Rightarrow \texttt{[(8,6)]}}$$

$$\text{LD}\ \dfrac{\texttt{fun a -> [(a+1,a-1)]} \Rightarrow \texttt{fun a -> [(a+1,a-1)]} \quad \text{APP'}\ \dfrac{\texttt{fun a -> [(a+1,a-1)]} \Rightarrow \texttt{fun a -> [(a+1,a-1)]} \quad \texttt{7} \Rightarrow \texttt{7} \quad \pi_0}{\texttt{(fun a -> [(a+1,a-1)]) 7} \Rightarrow \texttt{[(8,6)]}}}{\texttt{let f = fun a -> [(a+1,a-1)] in f 7} \Rightarrow \texttt{[(8,6)]}}$$

Week 11 Tutorial 01 — Big Steps

```ocaml
let rec f = fun l ->
  match l with [] -> 1 | x::xs -> x + g xs
and g = fun l ->
  match l with [] -> 0 | x::xs -> x * f xs
```

$$\text{APP}\ \frac{\text{Tlp} \quad [3;6] \Rightarrow [3;6] \qquad \text{match } [3;6] \text{ with } [\,] \Rightarrow 1 \mid x::xs \Rightarrow x + g\ xs \Rightarrow 9}{\texttt{f [3;6]} \Rightarrow 9}$$

Week 11 Tutorial 01 — Big Steps

$$\pi_f = \mathrm{GD}\ \frac{\texttt{f} = \texttt{fun l -> match l with [] -> 1 | x::xs -> x+g xs} \quad \texttt{fun l -> match l with [] -> 1 | x::xs -> x+g xs} \Rightarrow \texttt{fun l -> match l with [] -> 1 | x::xs -> x+g xs}}{\texttt{f} \Rightarrow \texttt{fun l -> match l with [] -> 1 | x::xs -> x+g xs}}$$

$$\pi_g = \mathrm{GD}\ \frac{\texttt{g} = \texttt{fun l -> match l with [] -> 0 | x::xs -> x*f xs} \quad \texttt{fun l -> match l with [] -> 0 | x::xs -> x*f xs} \Rightarrow \texttt{fun l -> match l with [] -> 0 | x::xs -> x*f xs}}{\texttt{g} \Rightarrow \texttt{fun l -> match l with [] -> 0 | x::xs -> x*f xs}}$$

τ_f = fun l -> match l with | [] -> 1

| x::xs -> x+g xs

Global Definitions

$$\pi_f \ \frac{f = \tau_f \qquad \tau_f = \tau_f}{f \Rightarrow \tau_f}$$

Week 11 Tutorial 01 — Big Steps

$$\pi_f = \mathrm{GD}\ \frac{\texttt{f} = \texttt{fun l -> match l with [] -> 1 | x::xs -> x+g xs} \quad \texttt{fun l -> match l with [] -> 1 | x::xs -> x+g xs} \Rightarrow \texttt{fun l -> match l with [] -> 1 | x::xs -> x+g xs}}{\texttt{f} \Rightarrow \texttt{fun l -> match l with [] -> 1 | x::xs -> x+g xs}}$$

$$\pi_g = \mathrm{GD}\ \frac{\texttt{g} = \texttt{fun l -> match l with [] -> 0 | x::xs -> x*f xs} \quad \texttt{fun l -> match l with [] -> 0 | x::xs -> x*f xs} \Rightarrow \texttt{fun l -> match l with [] -> 0 | x::xs -> x*f xs}}{\texttt{g} \Rightarrow \texttt{fun l -> match l with [] -> 0 | x::xs -> x*f xs}}$$

T_f = fun l -> match l with [] -> 1 | x::xs -> x + g xs

T_g = fun l -> match l with [] -> 0 | x::xs -> x * f xs

$$\Pi_f = \frac{f = T_f \qquad T_f \Rightarrow T_f}{f \Rightarrow T_f}$$

$$\Pi_g = \frac{g = T_g \qquad T_g \Rightarrow T_g}{g \Rightarrow T_g}$$

Global Definitions

Week 11 Tutorial 01 — Big Steps

```
let rec f = fun l ->
  match l with [] -> 1 | x::xs -> x + g xs
and g = fun l ->
  match l with [] -> 0 | x::xs -> x * f xs
```

$$\text{APP'}\ \dfrac{\pi_f \quad [3;6] \Rightarrow [3;6] \quad \text{PM}\ \dfrac{[3;6] \Rightarrow [3;6] \qquad 3 + [6] \Rightarrow 9}{\texttt{match [3;6] with [] -> 1 | x::xs -> x+g xs} \Rightarrow 9}}{\texttt{f [3;6]} \Rightarrow 9}$$

Week 11 Tutorial 01 — Big Steps

```
let rec f = fun l ->
  match l with [] -> 1 | x::xs -> x + g xs
and g = fun l ->
  match l with [] -> 0 | x::xs -> x * f xs
```

$$
\text{APP'}\ \dfrac{\pi_f \quad \texttt{[3;6]} \Rightarrow \texttt{[3;6]} \quad \text{PM}\ \dfrac{\texttt{[3;6]} \Rightarrow \texttt{[3;6]} \qquad \dfrac{}{\texttt{3+g [6]} \Rightarrow 9}}{\texttt{match [3;6] with [] -> 1 | x::xs -> x+g xs} \Rightarrow 9}}{\texttt{f [3;6]} \Rightarrow 9}
$$

Week 11 Tutorial 01 — Big Steps

```
let rec f = fun l ->
  match l with [] -> 1 | x::xs -> x + g xs
and g = fun l ->
  match l with [] -> 0 | x::xs -> x * f xs
```

$$\text{APP'}\ \dfrac{\pi_f \quad [3;6] \Rightarrow [3;6] \quad \text{PM}\ \dfrac{[3;6] \Rightarrow [3;6] \quad \text{OP}\ \dfrac{3 \Rightarrow 3 \quad \text{APP}\ \dfrac{\pi_g \quad [6] \Rightarrow [6] \quad \pi_0}{g\ [6] \Rightarrow}}{3+g\ [6] \Rightarrow 9}}{\texttt{match [3;6] with [] -> 1 | x::xs -> x+g xs} \Rightarrow 9}}{\texttt{f [3;6]} \Rightarrow 9}$$

Week 11 Tutorial 01 — Big Steps

```
let rec f = fun l ->
  match l with [] -> 1 | x::xs -> x + g xs
and g = fun l ->
  match l with [] -> 0 | x::xs -> x * f xs
```

$$\pi_0 = \text{DM}\ \dfrac{\qquad}{\texttt{match [6] with [] -> 0 | x::xs -> x*f xs} \Rightarrow 6}$$

Week 11 Tutorial 01 — Big Steps

```
let rec f = fun l ->
  match l with [] -> 1 | x::xs -> x + g xs
and g = fun l ->
  match l with [] -> 0 | x::xs -> x * f xs
```

$$\pi_0 = \mathrm{PM}\ \frac{[6] \Rightarrow [6]\ \text{PO} \qquad \dfrac{}{\texttt{6*f [\,]} \Rightarrow 6}}{\texttt{match [6] with [\,] -> 0 | x::xs -> x*f xs} \Rightarrow 6}$$

Week 11 Tutorial 01 — Big Steps

```
let rec f = fun l ->
  match l with [] -> 1 | x::xs -> x + g xs
and g = fun l ->
  match l with [] -> 0 | x::xs -> x * f xs
```

$$\pi_0 = \text{PM}\;\dfrac{\texttt{[6]} \Rightarrow \texttt{[6]} \qquad \text{OP}\;\dfrac{6 \Rightarrow 6 \qquad \text{APP}\;\dfrac{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}{\texttt{f []} \Rightarrow \cancel{1}}\;*}{\texttt{6*f []} \Rightarrow 6}}{\texttt{match [6] with [] -> 0 | x::xs -> x*f xs} \Rightarrow 6}$$

Week 11 Tutorial 01 — Big Steps

```
let rec f = fun l ->
  match l with [] -> 1 | x::xs -> x + g xs
and g = fun l ->
  match l with [] -> 0 | x::xs -> x * f xs
```

$$\pi_0 = \text{PM}\ \dfrac{[6] \Rightarrow [6] \quad \text{OP}\ \dfrac{6 \Rightarrow 6 \quad \text{APP'}\ \dfrac{\pi_f \quad [\,] \Rightarrow [\,] \quad \text{PM}\ \dfrac{[\,] \Rightarrow [\,] \qquad 1 \Rightarrow 1}{\texttt{match [] with [] -> 1 | x::xs -> x+g xs} \Rightarrow 1}}{\texttt{f []} \Rightarrow 1}}{\texttt{6*f []} \Rightarrow 6}}{\texttt{match [6] with [] -> 0 | x::xs -> x*f xs} \Rightarrow 6}$$

$6 * 1 = 6$

Week 11 Tutorial 01 — Big Steps

```
let rec f = fun l ->
  match l with [] -> 1 | x::xs -> x + g xs
and g = fun l ->
  match l with [] -> 0 | x::xs -> x * f xs
```

$$\pi_0 = \text{PM}\ \dfrac{\texttt{[6]} \Rightarrow \texttt{[6]}\quad \text{OP}\ \dfrac{6 \Rightarrow 6 \quad \text{APP'}\ \dfrac{\pi_f \quad \texttt{[]} \Rightarrow \texttt{[]} \quad \text{PM}\ \dfrac{\texttt{[]} \Rightarrow \texttt{[]} \quad 1 \Rightarrow 1}{\texttt{match [] with [] -> 1 | x::xs -> x+g xs} \Rightarrow 1}}{\texttt{f []} \Rightarrow 1} \quad 6 * 1 \Rightarrow 6}{\texttt{6*f []} \Rightarrow 6}}{\texttt{match [6] with [] -> 0 | x::xs -> x*f xs} \Rightarrow 6}$$

Week 11 Tutorial 01 — Big Steps

```
let rec f = fun l ->
  match l with [] -> 1 | x::xs -> x + g xs
and g = fun l ->
  match l with [] -> 0 | x::xs -> x * f xs
```

T_f = fun l -> match l with [] -> 1 | x::xs -> x + g xs

T_g = fun l -> match l with [] -> 0 | x::xs -> x * f xs

$$\pi_f = \frac{f = T_f \qquad T_f \Rightarrow T_f}{f \Rightarrow T_f}$$

$$\pi_g = \frac{g = T_g \qquad T_g \Rightarrow T_g}{g \Rightarrow T_g}$$

$$\pi_0 = \text{PM}\ \dfrac{[6] \Rightarrow [6] \quad \text{OP}\ \dfrac{6 \Rightarrow 6 \quad \text{APP'}\ \dfrac{\pi_f \quad [\,] \Rightarrow [\,] \quad \text{PM}\ \dfrac{[\,] \Rightarrow [\,] \quad 1 \Rightarrow 1}{\texttt{match [] with [] -> 1 | x::xs -> x+g xs} \Rightarrow 1}}{\texttt{f []} \Rightarrow 1} \quad 6 * 1 \Rightarrow 6}{\texttt{6*f []} \Rightarrow 6}}{\texttt{match [6] with [] -> 0 | x::xs -> x*f xs} \Rightarrow 6}$$

$$\text{APP'}\ \dfrac{\pi_f \quad [3;6] \Rightarrow [3;6] \quad \text{PM}\ \dfrac{[3;6] \Rightarrow [3;6] \quad \text{OP}\ \dfrac{3 \Rightarrow 3 \quad \text{APP'}\ \dfrac{\pi_g \quad [6] \Rightarrow [6] \quad \pi_0}{\texttt{g [6]} \Rightarrow 6} \quad 3 + 6 \Rightarrow 9}{\texttt{3+g [6]} \Rightarrow 9}}{\texttt{match [3;6] with [] -> 1 | x::xs -> x+g xs} \Rightarrow 9}}{\texttt{f [3;6]} \Rightarrow 9}$$

Week 11 Tutorial 01 — Big Steps

$$\frac{}{\texttt{(fun x -> x 3) (fun y z -> z y) (fun w -> w+w)} \Rightarrow 6}$$

Week 11 Tutorial 01 — Big Steps

$$\pi_0 = \frac{}{\texttt{(fun x -> x 3) (fun y z -> z y)} \Rightarrow}$$

Week 11 Tutorial 01 — Big Steps

$$\pi_0 = \text{APP'}\ \frac{\texttt{fun x -> x 3} \Rightarrow \texttt{fun x -> x 3} \quad \texttt{fun y z -> z y} \Rightarrow \texttt{fun y z -> z y} \quad \dfrac{}{\texttt{(fun y z -> z y) 3} \Rightarrow}}{\texttt{(fun x -> x 3) (fun y z -> z y)} \Rightarrow}$$

Week 11 Tutorial 01 — Big Steps

$$\pi_0 = \text{APP'}\ \dfrac{\texttt{fun x -> x 3} \Rightarrow \texttt{fun x -> x 3} \quad \texttt{fun y z -> z y} \Rightarrow \texttt{fun y z -> z y} \quad \text{APP'}\ \dfrac{\texttt{fun y z -> z y} \Rightarrow \texttt{fun y z -> z y} \quad \texttt{3} \Rightarrow \texttt{3} \quad \texttt{fun z -> z 3} \Rightarrow \texttt{fun z -> z 3}}{\texttt{(fun y z -> z y) 3} \Rightarrow \texttt{fun z -> z 3}}}{\texttt{(fun x -> x 3) (fun y z -> z y)} \Rightarrow \texttt{fun z -> z 3}}$$

Week 11 Tutorial 01 — Big Steps

$$\text{APP'}\ \frac{\pi_0}{\texttt{(fun x -> x 3) (fun y z -> z y) (fun w -> w+w)} \Rightarrow 6}$$

Week 11 Tutorial 01 — Big Steps

$$\text{APP'}\ \dfrac{\pi_0 \quad \texttt{fun w -> w+w} \Rightarrow \texttt{fun w -> w+w} \qquad \dfrac{}{\texttt{(fun w -> w+w) 3} \Rightarrow 6}}{\texttt{(fun x -> x 3) (fun y z -> z y) (fun w -> w+w)} \Rightarrow 6}$$

Week 11 Tutorial 01 — Big Steps

$$\text{APP'}\ \dfrac{\pi_0 \quad \texttt{fun w -> w+w} \Rightarrow \texttt{fun w -> w+w} \quad \text{APP'}\ \dfrac{\texttt{fun w -> w+w} \Rightarrow \texttt{fun w -> w+w} \quad \texttt{3} \Rightarrow \texttt{3} \quad \dfrac{}{\texttt{3+3} \Rightarrow \texttt{6}}}{\texttt{(fun w -> w+w) 3} \Rightarrow \texttt{6}}}{\texttt{(fun x -> x 3) (fun y z -> z y) (fun w -> w+w)} \Rightarrow \texttt{6}}$$

Week 11 Tutorial 01 — Big Steps

$$\text{APP'}\ \dfrac{\pi_0 \quad \texttt{fun w -> w+w} \Rightarrow \texttt{fun w -> w+w} \quad \text{APP'}\ \dfrac{\texttt{fun w -> w+w} \Rightarrow \texttt{fun w -> w+w} \quad 3 \Rightarrow 3 \quad \text{OP}\ \dfrac{3 \Rightarrow 3 \quad 3 \Rightarrow 3 \quad 3 + 3 \Rightarrow 6}{\texttt{3+3} \Rightarrow 6}}{\texttt{(fun w -> w+w) 3} \Rightarrow 6}}{\texttt{(fun x -> x 3) (fun y z -> z y) (fun w -> w+w)} \Rightarrow 6}$$

Week 11 Tutorial 01 — Big Steps

$$\pi_0 = \text{APP'}\ \dfrac{\texttt{fun x -> x 3} \Rightarrow \texttt{fun x -> x 3} \quad \texttt{fun y z -> z y} \Rightarrow \texttt{fun y z -> z y} \quad \text{APP'}\ \dfrac{\texttt{fun y z -> z y} \Rightarrow \texttt{fun y z -> z y} \quad \texttt{3} \Rightarrow \texttt{3} \quad \texttt{fun z -> z 3} \Rightarrow \texttt{fun z -> z 3}}{\texttt{(fun y z -> z y) 3} \Rightarrow \texttt{fun z -> z 3}}}{\texttt{(fun x -> x 3) (fun y z -> z y)} \Rightarrow \texttt{fun z -> z 3}}$$

$$\text{APP'}\ \dfrac{\pi_0 \quad \texttt{fun w -> w+w} \Rightarrow \texttt{fun w -> w+w} \quad \text{APP'}\ \dfrac{\texttt{fun w -> w+w} \Rightarrow \texttt{fun w -> w+w} \quad \texttt{3} \Rightarrow \texttt{3} \quad \text{OP}\ \dfrac{3 \Rightarrow 3 \quad 3 \Rightarrow 3 \quad 3 + 3 \Rightarrow 6}{\texttt{3+3} \Rightarrow 6}}{\texttt{(fun w -> w+w) 3} \Rightarrow 6}}{\texttt{(fun x -> x 3) (fun y z -> z y) (fun w -> w+w)} \Rightarrow 6}$$

Week 11 Tutorial 02 — Multiplication

Prove that the function

```
let rec mul a b =
  match a with 0 -> 0 | _ -> b + mul (a-1) b
```

terminates for all inputs $a, b \geq 0$.

Week 11 Tutorial 02 — Multiplication

```
let rec mul a b =
  match a with 0 -> 0 | _ -> b + mul (a-1) b
```

Proof by Induction on a

Base: a = 0

PM

match 0 with 0 -> 0 | _ -> b + mul (0-1) b

APP T_{mul}

mul 0 b $\Rightarrow$ 0

Week 11 Tutorial 02 — Multiplication

```
let rec mul a b =
  match a with 0 -> 0 | _ -> b + mul (a-1) b
```

Proof by Induction on a

Base: a = 0

$$\dfrac{\pi_{mul} \quad \dfrac{\text{PM} \quad \overline{}}{\texttt{match 0 with 0 -> 0 | \_ -> b + mul (-1) b} \Rightarrow \texttt{0}}}{\texttt{mul 0 b} \Rightarrow \texttt{0}}\ \text{APP}$$

Week 11 Tutorial 02 — Multiplication

```
let rec mul a b =
  match a with 0 -> 0 | _ -> b + mul (a-1) b
```

Proof by Induction on a

Hypothisis:

mul a b = a*b $\wedge\ a \geq 0$

Step:

$$\text{APP}' \;\; \frac{\text{by I.H.}}{\text{mal}\,(a+1-1)\,b \Rightarrow ab}$$

$$ab + b = (a+1)\,b$$

$$\text{OP} \;\; \frac{\text{mal}\,(a+1-1)\,b \Rightarrow ab}{b + \text{mul}\,(a+1-1)\;b \Rightarrow (a+1)\,b}$$

$$\text{PM} \;\; \frac{b + \text{mul}\,(a+1-1)\;b \Rightarrow (a+1)\,b}{\text{match}\,(a+1)\ \text{with}\ 0 \to 0 \mid \_ \to b + \text{mal}\,((a+1)-1)\ b \Rightarrow (a+1)\,b}$$

$$\text{APP}' \text{Tm.al} \;\; \frac{\text{match}\,(a+1)\ \text{with}\ 0 \to 0 \mid \_ \to b + \text{mal}\,((a+1)-1)\ b \Rightarrow (a+1)\,b}{\texttt{mul (a+1) b} \Rightarrow (a+1) * b}$$

Week 11 Tutorial 02 — Multiplication

```
let rec mul a b =
  match a with 0 -> 0 | _ -> b + mul (a-1) b
```

Proof by Induction on a

Hypothisis:

    mul a b = a*b

Step:

$$
\text{APP}\ \dfrac{\pi_{mul}\quad \text{PM}\ \dfrac{\text{OP}\ \dfrac{\text{APP}\ \dfrac{\text{by I.H.}}{\texttt{mul (a+1-1) b} \Rightarrow a*b} \quad b + (a*b) \Rightarrow (a+1)*b}{\texttt{b + mul (a+1-1) b} \Rightarrow (a+1)*b}}{\texttt{match a+1 with 0 -> 0 | \_ -> b + mul (a+1-1) b} \Rightarrow (a+1)*b}}{\texttt{mul (a+1) b} \Rightarrow (a+1)*b}
$$

Week 11 Tutorial 03 — Threesum

Use big-step operational semantics to show that the function

```
let rec threesum = fun l ->
  match l with [] -> 0 | x::xs -> 3*x + threesum xs
```

terminates for all inputs and computes three times the sum of the input list's elements.

Week 11 Tutorial 03 — Threesum

```
let rec threesum = fun l ->
  match l with [] -> 0 | x::xs -> 3*x + threesum xs
```

Proof by Induction on length of List

Base: l = []

$$\text{APP}\ \frac{}{\texttt{threesum []} \Rightarrow 0}$$

Week 11 Tutorial 03 — Threesum

```
let rec threesum = fun l ->
  match l with [] -> 0 | x::xs -> 3*x + threesum xs
```

Proof by Induction on length of List

Base: l = []

$$\text{APP}\ \dfrac{\pi_{ts} \quad \texttt{[]} \Rightarrow \texttt{[]} \quad \text{PM}\ \dfrac{\texttt{[]} \Rightarrow \texttt{[]} \quad \texttt{0} \Rightarrow \texttt{0}}{\texttt{match [] with [] -> 0 | x::xs -> 3*x + threesum xs} \Rightarrow \texttt{0}}}{\texttt{threesum []} \Rightarrow \texttt{0}}$$

Week 11 Tutorial 03 — Threesum

```
let rec threesum = fun l ->
  match l with [] -> 0 | x::xs -> 3*x + threesum xs
```

Proof by Induction on length of List

Hypothisis: thsm $l \Rightarrow \sum_{i=1}^{n} x_i$

Step: l = x :: xs

APP

$$\texttt{threesum}\ (x_{n+1}\texttt{::xs}) \Rightarrow 3\sum_{i=1}^{n+1} x_i$$

Week 11 Tutorial 03 — Threesum

```
let rec threesum = fun l ->
  match l with [] -> 0 | x::xs -> 3*x + threesum xs
```

Proof by Induction on length of List

Hypothisis: thsm $l \Rightarrow \sum_{i=1}^{n} x_i$

Step: l = x :: xs

$$\dfrac{\pi_{ts} \quad x_{n+1}\texttt{::xs} \Rightarrow x_{n+1}\texttt{::xs}\,\text{PM} \qquad \dfrac{x_{n+1}\texttt{::xs} \Rightarrow x_{n+1}\texttt{::xs} \ \ \text{OP} \qquad \dfrac{}{3\texttt{*}x_{n+1} \ \texttt{+ threesum xs} \Rightarrow 3\sum_{i=1}^{n+1} x_i}}{\texttt{match} \ \ x_{n+1}\texttt{::xs with [] -> 0 | x::xs -> 3*x + threesum xs} \Rightarrow 3\sum_{i=1}^{n+1} x_i}}{\texttt{threesum} \ (x_{n+1}\texttt{::xs}) \Rightarrow 3\sum_{i=1}^{n+1} x_i} \ \text{APP}$$

Week 11 Tutorial 03 — Threesum

```
let rec threesum = fun l ->
  match l with [] -> 0 | x::xs -> 3*x + threesum xs
```

Proof by Induction on length of List

Hypothisis: $\text{thsm } l \Rightarrow \sum_{i=1}^{n} x_i$

Step: l = x :: xs

$$\text{APP}\ \dfrac{\pi_{ts}\quad x_{n+1}\texttt{::xs} \Rightarrow x_{n+1}\texttt{::xs}\ \text{PM}\ \dfrac{x_{n+1}\texttt{::xs} \Rightarrow x_{n+1}\texttt{::xs}\quad \text{OP}\ \dfrac{\text{OP}\ \dfrac{3 \Rightarrow 3\quad x_{n+1} \Rightarrow x_{n+1}\quad 3 * x_{n+1} \Rightarrow 3x_{n+1}}{3 * x_{n+1} \Rightarrow 3x_{n+1}}\quad \text{APP}\ \dfrac{\texttt{by I.H.}}{\texttt{threesum xs} \Rightarrow 3\sum_{i=1}^{n} x_i}\quad 3x_{n+1} + 3\sum_{i=1}^{n} x_i \Rightarrow 3\sum_{i=1}^{n+1} x_i}{\texttt{3*}x_{n+1}\texttt{ + threesum xs} \Rightarrow 3\sum_{i=1}^{n+1} x_i}}{\texttt{match } x_{n+1}\texttt{::xs with [] -> 0 | x::xs -> 3*x + threesum xs} \Rightarrow 3\sum_{i=1}^{n+1} x_i}}{\texttt{threesum (}x_{n+1}\texttt{::xs)} \Rightarrow 3\sum_{i=1}^{n+1} x_i}$$