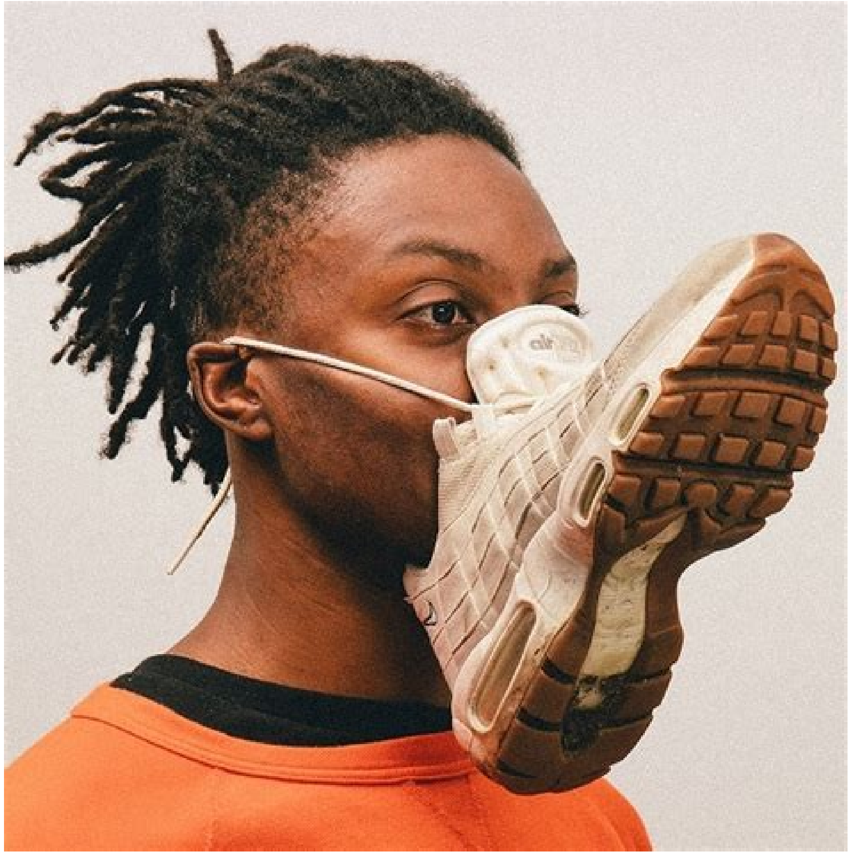
Week 11 – Big-Step Proofs



Tuples

$$\text{(TU)}\quad \frac{e_1 \Rightarrow v_1 \qquad \ldots \qquad e_k \Rightarrow v_k}{(e_1, \ldots, e_k) \Rightarrow (v_1, \ldots, v_k)}$$

Lists

$$\text{(LI)}\quad \frac{e_1 \Rightarrow v_1 \qquad e_2 \Rightarrow v_2}{e_1 :: e_2 \Rightarrow v_1 :: v_2}$$

let x = 5+3

Global definitions

$$\text{(GD)}\quad \frac{f = e \qquad e \Rightarrow v}{f \Rightarrow v}$$

$$\text{GD}\quad \frac{x = 5+3 \qquad 5+3 \Rightarrow 8}{x \Rightarrow 8}$$

Local definitions

$$(\text{LD}) \quad \frac{e_1 \Rightarrow v_1 \qquad e_0[v_1/x] \Rightarrow v_0}{\texttt{let}\ x = e_1\ \texttt{in}\ e_0 \Rightarrow v_0}$$

Function calls

$$(\text{APP}) \quad \frac{e \Rightarrow \texttt{fun}\ x\ \texttt{->}\ e_0 \qquad e_1 \Rightarrow v_1 \qquad e_0[v_1/x] \Rightarrow v_0}{e\ e_1 \Rightarrow v_0}$$

$$(\text{APP'}) \quad \frac{e_0 \Rightarrow \texttt{fun}\ x_1 \ldots x_k\ \texttt{->}\ e \qquad e_1 \Rightarrow v_1 \ \ldots\ e_k \Rightarrow v_k \qquad e[v_1/x_1, \ldots, v_k/x_k] \Rightarrow v}{e_0\ e_1\ \ldots\ e_k \Rightarrow v}$$

Pattern Matching

$$(\text{PM}) \quad \frac{e_0 \;\Rightarrow\; v' \equiv p_i[v_1/x_1, \ldots, v_k/x_k] \qquad e_i[v_1/x_1, \ldots, v_k/x_k] \;\Rightarrow\; v}{\texttt{match}\; e_0 \;\texttt{with}\; p_1 \;\texttt{->}\; e_1 \;\texttt{|}\; \ldots \;\texttt{|}\; p_m \;\texttt{->}\; e_m \;\Rightarrow\; v}$$

Built-in operators

$$(\text{OP}) \quad \frac{e_1 \;\Rightarrow\; v_1 \qquad e_2 \;\Rightarrow\; v_2 \qquad v_1 \;\mathsf{op}\; v_2 \Rightarrow v}{e_1 \;\mathsf{op}\; e_2 \;\Rightarrow\; v}$$

op

$\overline{3 + \text{foo} []}$

Unary operators are treated analogously.

Week 11 Tutorial 01 — Big Steps

fun a -> [(a+1, a-1)]

(1) ⇒ fun a -> [(a+1, a-1)]

(fun a -> [(a+1, a-1)]) 7 ⇒ [(8,6)]

```
let f = fun a -> [(a+1,a-1)] in f 7 ⇒ [(8,6)]
```

Week 11 Tutorial 01 — Big Steps

$$\Pi_0 = \overline{[(7+1,7-1)] \Rightarrow [(8,6)]}$$

$$\text{LD}\ \dfrac{\texttt{fun a -> [(a+1,a-1)]} \Rightarrow \texttt{fun a -> [(a+1,a-1)]} \qquad \text{APP}\ \dfrac{\texttt{fun a -> [(a+1,a-1)]} \Rightarrow \texttt{fun a -> [(a+1,a-1)]} \qquad 7 \Rightarrow 7 \qquad \Pi_0}{\texttt{(fun a -> [(a+1,a-1)]) 7} \Rightarrow \texttt{[(8,6)]}}}{\texttt{let f = fun a -> [(a+1,a-1)] in f 7} \Rightarrow \texttt{[(8,6)]}}$$

Week 11 Tutorial 01 — Big Steps

$$\pi_0 = \frac{\text{TA}\ \dfrac{\text{OP}\ \dfrac{7 \Rightarrow 7 \quad 1 \Rightarrow 1 \quad 7+1 \Rightarrow 8}{7+1 \Rightarrow 8} \qquad \text{OP}\ \dfrac{7 \Rightarrow 7 \quad 1 \Rightarrow 1 \quad 7-1 \Rightarrow}{7-1 \Rightarrow 6}}{(7+1,\ 7-1) \Rightarrow (8,6)} \qquad \dfrac{[\,] \Rightarrow [\,]}{}}{\texttt{[(7+1,7-1)]} \Rightarrow \texttt{[(8,6)]}}$$

$$\text{LD}\ \frac{\texttt{fun a -> [(a+1,a-1)]} \Rightarrow \texttt{fun a -> [(a+1,a-1)]} \quad \text{APP'}\ \dfrac{\texttt{fun a -> [(a+1,a-1)]} \Rightarrow \texttt{fun a -> [(a+1,a-1)]} \quad \texttt{7} \Rightarrow \texttt{7} \quad \pi_0}{\texttt{(fun a -> [(a+1,a-1)]) 7} \Rightarrow \texttt{[(8,6)]}}}{\texttt{let f = fun a -> [(a+1,a-1)] in f 7} \Rightarrow \texttt{[(8,6)]}}$$

Week 11 Tutorial 01 — Big Steps

(4)

$$\frac{7+1 \Rightarrow 8 \qquad [7-1] \Rightarrow [6]}{[7+1;\ 7-1] \Rightarrow [8;6]}$$

$$\pi_0 = \text{LI}\ \dfrac{\text{TU}\ \dfrac{\text{OP}\ \dfrac{7 \Rightarrow 7 \quad 1 \Rightarrow 1 \quad 7 + 1 \Rightarrow 8}{\texttt{7+1} \Rightarrow 8} \quad \text{OP}\ \dfrac{7 \Rightarrow 7 \quad 1 \Rightarrow 1 \quad 7 - 1 \Rightarrow 6}{\texttt{7-1} \Rightarrow 6}}{\texttt{(7+1,7-1)} \Rightarrow \texttt{(8,6)}} \quad \texttt{[]} \Rightarrow \texttt{[]}}{\texttt{[(7+1,7-1)]} \Rightarrow \texttt{[(8,6)]}}$$

$$\text{LD}\ \dfrac{\texttt{fun a -> [(a+1,a-1)]} \Rightarrow \texttt{fun a -> [(a+1,a-1)]} \quad \text{APP'}\ \dfrac{\texttt{fun a -> [(a+1,a-1)]} \Rightarrow \texttt{fun a -> [(a+1,a-1)]} \quad \texttt{7} \Rightarrow \texttt{7} \quad \pi_0}{\texttt{(fun a -> [(a+1,a-1)]) 7} \Rightarrow \texttt{[(8,6)]}}}{\texttt{let f = fun a -> [(a+1,a-1)] in f 7} \Rightarrow \texttt{[(8,6)]}}$$

Week 11 Tutorial 01 — Big Steps

```
let rec f = fun l ->
  match l with [] -> 1 | x::xs -> x + g xs
and g = fun l ->
  match l with [] -> 0 | x::xs -> x * f xs
```

$$\frac{f \Rightarrow f \quad [3;6] \Rightarrow [3;6] \qquad \text{match } [3;6] \text{ with } [\,] \Rightarrow 1 \mid x :: xs \Rightarrow x + g\ xs}{f\ [3;6] \Rightarrow 9}\ \text{APP}$$

Week 11 Tutorial 01 — Big Steps

$$\pi_f = \text{GD}\ \frac{\texttt{f} = \texttt{fun l -> match l with [] -> 1 | x::xs -> x+g xs}\quad \texttt{fun l -> match l with [] -> 1 | x::xs -> x+g xs} \Rightarrow \texttt{fun l -> match l with [] -> 1 | x::xs -> x+g xs}}{\texttt{f} \Rightarrow \texttt{fun l -> match l with [] -> 1 | x::xs -> x+g xs}}$$

$$\pi_g = \text{GD}\ \frac{\texttt{g} = \texttt{fun l -> match l with [] -> 0 | x::xs -> x*f xs}\quad \texttt{fun l -> match l with [] -> 0 | x::xs -> x*f xs} \Rightarrow \texttt{fun l -> match l with [] -> 0 | x::xs -> x*f xs}}{\texttt{g} \Rightarrow \texttt{fun l -> match l with [] -> 0 | x::xs -> x*f xs}}$$

T_f = fun l -> match l with [] -> 1 | x::xs -> x + g xs

Global Definitions

Week 11 Tutorial 01 — Big Steps

$$\pi_f = \mathrm{GD}\ \frac{\texttt{f} = \texttt{fun l -> match l with [] -> 1 | x::xs -> x+g xs} \quad \texttt{fun l -> match l with [] -> 1 | x::xs -> x+g xs} \Rightarrow \texttt{fun l -> match l with [] -> 1 | x::xs -> x+g xs}}{\texttt{f} \Rightarrow \texttt{fun l -> match l with [] -> 1 | x::xs -> x+g xs}}$$

$$\pi_g = \mathrm{GD}\ \frac{\texttt{g} = \texttt{fun l -> match l with [] -> 0 | x::xs -> x*f xs} \quad \texttt{fun l -> match l with [] -> 0 | x::xs -> x*f xs} \Rightarrow \texttt{fun l -> match l with [] -> 0 | x::xs -> x*f xs}}{\texttt{g} \Rightarrow \texttt{fun l -> match l with [] -> 0 | x::xs -> x*f xs}}$$

T_f = fun l -> match l with [] -> 1 | x::xs -> x + g xs

T_g = fun l -> match l with [] -> 0 | x::xs -> x * f xs

$$\pi_f = \mathrm{GD}\ \frac{f = T_f \qquad T_f \Rightarrow T_f}{f \Rightarrow T_f}$$

$$\pi_g = \mathrm{GD}\ \frac{g = T_g \qquad T_g \Rightarrow T_g}{g \Rightarrow T_g}$$

Global Definitions

Week 11 Tutorial 01 — Big Steps

```
let rec f = fun l ->
  match l with [] -> 1 | x::xs -> x + g xs
and g = fun l ->
  match l with [] -> 0 | x::xs -> x * f xs
```

$$
\text{APP'}\;\frac{\pi_f \quad \texttt{[3;6]} \Rightarrow \texttt{[3;6]} \quad \text{PM}\;\dfrac{\texttt{[3;6]} \Rightarrow \texttt{[3;6]} \qquad \text{OP}\;\dfrac{}{3 + g\,\texttt{[6]} \Rightarrow 9}}{\texttt{match [3;6] with [] -> 1 | x::xs -> x+g xs} \Rightarrow 9}}{\texttt{f [3;6]} \Rightarrow 9}
$$

Week 11 Tutorial 01 — Big Steps

```
let rec f = fun l ->
  match l with [] -> 1 | x::xs -> x + g xs
and g = fun l ->
  match l with [] -> 0 | x::xs -> x * f xs
```

$$\mathrm{APP'}\ \dfrac{\pi_f \quad \texttt{[3;6]} \Rightarrow \texttt{[3;6]} \quad \mathrm{PM}\ \dfrac{\texttt{[3;6]} \Rightarrow \texttt{[3;6]} \qquad \dfrac{}{\texttt{3+g [6]} \Rightarrow 9}}{\texttt{match [3;6] with [] -> 1 | x::xs -> x+g xs} \Rightarrow 9}}{\texttt{f [3;6]} \Rightarrow 9}$$

Week 11 Tutorial 01 — Big Steps

```ocaml
let rec f = fun l ->
  match l with [] -> 1 | x::xs -> x + g xs
and g = fun l ->
  match l with [] -> 0 | x::xs -> x * f xs
```

$$
\text{APP'}\ \dfrac{\pi_f \quad \texttt{[3;6]} \Rightarrow \texttt{[3;6]} \quad \text{PM}\ \dfrac{\texttt{[3;6]} \Rightarrow \texttt{[3;6]} \quad \text{OP}\ \dfrac{3 \Rightarrow 3 \quad \text{APP'}\ \dfrac{\pi_g \quad \texttt{[6]} \Rightarrow \texttt{[6]} \quad \pi_0}{\texttt{g [6]} \Rightarrow 6}}{\texttt{3+g [6]} \Rightarrow 9}}{\texttt{match [3;6] with [] -> 1 | x::xs -> x+g xs} \Rightarrow 9}}{\texttt{f [3;6]} \Rightarrow 9}
$$

Week 11 Tutorial 01 — Big Steps

```
let rec f = fun l ->
  match l with [] -> 1 | x::xs -> x + g xs
and g = fun l ->
  match l with [] -> 0 | x::xs -> x * f xs
```

$$\pi_0 = \frac{\quad}{\texttt{match [6] with [] -> 0 | x::xs -> x*f xs} \Rightarrow 6}$$

Week 11 Tutorial 01 — Big Steps

```
let rec f = fun l ->
  match l with [] -> 1 | x::xs -> x + g xs
and g = fun l ->
  match l with [] -> 0 | x::xs -> x * f xs
```

$$\pi_0 = \text{PM}\,\dfrac{\texttt{[6]} \Rightarrow \texttt{[6]} \qquad \dfrac{}{\texttt{6*f []} \Rightarrow 6}}{\texttt{match [6] with [] -> 0 | x::xs -> x*f xs} \Rightarrow 6}$$

Week 11 Tutorial 01 — Big Steps

```ocaml
let rec f = fun l ->
  match l with [] -> 1 | x::xs -> x + g xs
and g = fun l ->
  match l with [] -> 0 | x::xs -> x * f xs
```

$$\pi_0 = \mathrm{PM}\ \dfrac{[6] \Rightarrow [6]\ \ \mathrm{OP}\ \dfrac{6 \Rightarrow 6 \qquad \dfrac{}{\texttt{f []} \Rightarrow 1}}{\texttt{6*f []} \Rightarrow 6}}{\texttt{match [6] with [] -> 0 | x::xs -> x*f xs} \Rightarrow 6}$$

Week 11 Tutorial 01 — Big Steps

```
let rec f = fun l ->
  match l with [] -> 1 | x::xs -> x + g xs
and g = fun l ->
  match l with [] -> 0 | x::xs -> x * f xs
```

$$\pi_0 = \text{PM}\ \dfrac{[6] \Rightarrow [6] \quad \text{OP}\ \dfrac{6 \Rightarrow 6 \quad \text{APP'}\ \dfrac{\pi_f \quad \texttt{[]} \Rightarrow \texttt{[]} \quad \dfrac{}{\texttt{match [] with [] -> 1 | x::xs -> x+g xs} \Rightarrow 1}}{\texttt{f []} \Rightarrow 1}}{\texttt{6*f []} \Rightarrow 6}}{\texttt{match [6] with [] -> 0 | x::xs -> x*f xs} \Rightarrow 6}$$

Week 11 Tutorial 01 — Big Steps

```ocaml
let rec f = fun l ->
  match l with [] -> 1 | x::xs -> x + g xs
and g = fun l ->
  match l with [] -> 0 | x::xs -> x * f xs
```

$$\pi_0 = \text{PM}\ \dfrac{\texttt{[6]} \Rightarrow \texttt{[6]}\quad \text{OP}\ \dfrac{6 \Rightarrow 6\quad \text{APP'}\ \dfrac{\pi_f\quad \texttt{[]} \Rightarrow \texttt{[]}\quad \text{PM}\ \dfrac{\texttt{[]} \Rightarrow \texttt{[]}\quad \texttt{1} \Rightarrow \texttt{1}}{\texttt{match [] with [] -> 1 | x::xs -> x+g xs} \Rightarrow \texttt{1}}}{\texttt{f []} \Rightarrow \texttt{1}}\quad 6 * 1 \Rightarrow 6}{\texttt{6*f []} \Rightarrow \texttt{6}}}{\texttt{match [6] with [] -> 0 | x::xs -> x*f xs} \Rightarrow \texttt{6}}$$

Week 11 Tutorial 01 — Big Steps

```
let rec f = fun l ->
  match l with [] -> 1 | x::xs -> x + g xs
and g = fun l ->
  match l with [] -> 0 | x::xs -> x * f xs
```

T_f = fun l -> match l with [] -> 1 | x::xs -> x + g xs

T_g = fun l -> match l with [] -> 0 | x::xs -> x * f xs

$$\pi_f = \dfrac{f = T_f \qquad T_f \Rightarrow T_f}{f \Rightarrow T_f}$$

$$\pi_g = \dfrac{g = T_g \qquad T_g \Rightarrow T_g}{g \Rightarrow T_g}$$

$$\pi_0 = \mathrm{PM}\ \dfrac{\texttt{[6]} \Rightarrow \texttt{[6]} \quad \mathrm{OP}\ \dfrac{6 \Rightarrow 6 \quad \mathrm{APP'}\ \dfrac{\pi_f \quad \texttt{[]} \Rightarrow \texttt{[]} \quad \mathrm{PM}\ \dfrac{\texttt{[]} \Rightarrow \texttt{[]} \quad \texttt{1} \Rightarrow \texttt{1}}{\texttt{match [] with [] -> 1 | x::xs -> x+g xs} \Rightarrow \texttt{1}}}{\texttt{f []} \Rightarrow \texttt{1}} \quad 6 * 1 \Rightarrow 6}{\texttt{6*f []} \Rightarrow \texttt{6}}}{\texttt{match [6] with [] -> 0 | x::xs -> x*f xs} \Rightarrow \texttt{6}}$$

$$\mathrm{APP'}\ \dfrac{\pi_f \quad \texttt{[3;6]} \Rightarrow \texttt{[3;6]} \quad \mathrm{PM}\ \dfrac{\texttt{[3;6]} \Rightarrow \texttt{[3;6]} \quad \mathrm{OP}\ \dfrac{3 \Rightarrow 3 \quad \mathrm{APP'}\ \dfrac{\pi_g \quad \texttt{[6]} \Rightarrow \texttt{[6]} \quad \pi_0}{\texttt{g [6]} \Rightarrow \texttt{6}} \quad 3 + 6 \Rightarrow 9}{\texttt{3+g [6]} \Rightarrow \texttt{9}}}{\texttt{match [3;6] with [] -> 1 | x::xs -> x+g xs} \Rightarrow \texttt{9}}}{\texttt{f [3;6]} \Rightarrow \texttt{9}}$$

Week 11 Tutorial 01 — Big Steps

$$\text{(fun x -> x 3) (fun y z -> z y) (fun w -> w+w)} \Rightarrow 6$$

Week 11 Tutorial 01 — Big Steps

$$\pi_0 = \frac{}{\texttt{(fun x -> x 3) (fun y z -> z y)} \Rightarrow}$$

Week 11 Tutorial 01 — Big Steps

$$\pi_0 = \text{APP'}\ \dfrac{\texttt{fun x -> x 3} \Rightarrow \texttt{fun x -> x 3} \quad \texttt{fun y z -> z y} \Rightarrow \texttt{fun y z -> z y} \qquad \dfrac{}{\texttt{(fun y z -> z y) 3} \Rightarrow}}{\texttt{(fun x -> x 3) (fun y z -> z y)} \Rightarrow}$$

Week 11 Tutorial 01 — Big Steps

$$\pi_0 = \text{APP'}\ \frac{\texttt{fun x -> x 3} \Rightarrow \texttt{fun x -> x 3} \quad \texttt{fun y z -> z y} \Rightarrow \texttt{fun y z -> z y} \quad \text{APP'}\ \dfrac{\texttt{fun y z -> z y} \Rightarrow \texttt{fun y z -> z y} \quad \texttt{3} \Rightarrow \texttt{3} \quad \texttt{fun z -> z 3} \Rightarrow \texttt{fun z -> z 3}}{\texttt{(fun y z -> z y) 3} \Rightarrow \texttt{fun z -> z 3}}}{\texttt{(fun x -> x 3) (fun y z -> z y)} \Rightarrow \texttt{fun z -> z 3}}$$

Week 11 Tutorial 01 — Big Steps

$$\text{APP'}\ \frac{\pi_0}{\texttt{(fun x -> x 3) (fun y z -> z y) (fun w -> w+w)} \Rightarrow 6}$$

Week 11 Tutorial 01 — Big Steps

$$\text{APP'}\ \dfrac{\pi_0\ \ \texttt{fun w -> w+w} \Rightarrow \texttt{fun w -> w+w} \qquad \dfrac{}{\texttt{(fun w -> w+w) 3} \Rightarrow 6}}{\texttt{(fun x -> x 3) (fun y z -> z y) (fun w -> w+w)} \Rightarrow 6}$$

Week 11 Tutorial 01 — Big Steps

$$
\mathrm{APP'}\;\dfrac{\pi_0 \;\; \texttt{fun w -> w+w} \Rightarrow \texttt{fun w -> w+w} \qquad \mathrm{APP'}\;\dfrac{\texttt{fun w -> w+w} \Rightarrow \texttt{fun w -> w+w} \quad \texttt{3} \Rightarrow \texttt{3} \qquad \dfrac{}{\texttt{3+3} \Rightarrow \texttt{6}}}{\texttt{(fun w -> w+w) 3} \Rightarrow \texttt{6}}}{\texttt{(fun x -> x 3) (fun y z -> z y) (fun w -> w+w)} \Rightarrow \texttt{6}}
$$

Week 11 Tutorial 01 — Big Steps

$$
\text{APP'}\ \dfrac{\pi_0\quad \texttt{fun w -> w+w} \Rightarrow \texttt{fun w -> w+w}\quad \text{APP'}\ \dfrac{\texttt{fun w -> w+w} \Rightarrow \texttt{fun w -> w+w}\quad \texttt{3} \Rightarrow \texttt{3}\quad \text{OP}\ \dfrac{\texttt{3} \Rightarrow \texttt{3}\quad \texttt{3} \Rightarrow \texttt{3}\quad 3 + 3 \Rightarrow \texttt{6}}{\texttt{3+3} \Rightarrow \texttt{6}}}{\texttt{(fun w -> w+w) 3} \Rightarrow \texttt{6}}}{\texttt{(fun x -> x 3) (fun y z -> z y) (fun w -> w+w)} \Rightarrow \texttt{6}}
$$

Week 11 Tutorial 01 — Big Steps

$$\pi_0 = \text{APP'}\ \frac{\texttt{fun x -> x 3} \Rightarrow \texttt{fun x -> x 3} \quad \texttt{fun y z -> z y} \Rightarrow \texttt{fun y z -> z y} \quad \text{APP'}\ \dfrac{\texttt{fun y z -> z y} \Rightarrow \texttt{fun y z -> z y} \quad \texttt{3} \Rightarrow \texttt{3} \quad \texttt{fun z -> z 3} \Rightarrow \texttt{fun z -> z 3}}{\texttt{(fun y z -> z y) 3} \Rightarrow \texttt{fun z -> z 3}}}{\texttt{(fun x -> x 3) (fun y z -> z y)} \Rightarrow \texttt{fun z -> z 3}}$$

$$\text{APP'}\ \frac{\pi_0 \quad \texttt{fun w -> w+w} \Rightarrow \texttt{fun w -> w+w} \quad \text{APP'}\ \dfrac{\texttt{fun w -> w+w} \Rightarrow \texttt{fun w -> w+w} \quad \texttt{3} \Rightarrow \texttt{3} \quad \text{OP}\ \dfrac{3 \Rightarrow 3 \quad 3 \Rightarrow 3 \quad 3 + 3 \Rightarrow 6}{\texttt{3+3} \Rightarrow 6}}{\texttt{(fun w -> w+w) 3} \Rightarrow 6}}{\texttt{(fun x -> x 3) (fun y z -> z y) (fun w -> w+w)} \Rightarrow 6}$$

Week 11 Tutorial 02 — Multiplication

Prove that the function

```
let rec mul a b =
  match a with 0 -> 0 | _ -> b + mul (a-1) b
```

terminates for all inputs $a, b \geq 0$.

Week 11 Tutorial 02 — Multiplication

```
let rec mul a b =
  match a with 0 -> 0 | _ -> b + mul (a-1) b
```

Proof by Induction on n

Base: n = 0

To prove: if $a \Rightarrow 0$ and $b \Rightarrow m$, then $\texttt{mul}\ \ a\ \ b \Rightarrow 0 \cdot m$

$$\text{APP!}\ \frac{\Pi_{mul} \quad a \Rightarrow 0 \quad b \Rightarrow m \quad \overline{\text{match } 0 \text{ with } 0 \to 0 \mid \_ \to m + mul\ (0-1)\ m}}{\texttt{mul}\ \ a\ \ b \Rightarrow 0}$$

Week 11 Tutorial 02 — Multiplication

```
let rec mul a b =
  match a with 0 -> 0 | _ -> b + mul (a-1) b
```

Proof by Induction on n

Base: n = 0

To prove: if $a \Rightarrow 0$ and $b \Rightarrow m$, then $\texttt{mul}\ \ a\ \ b \Rightarrow 0 \cdot m$

$$\text{APP'}\ \frac{\pi_{mul} \quad a \Rightarrow 0 \quad b \Rightarrow m \quad \text{PM}\ \dfrac{0 \Rightarrow 0 \quad 0 \Rightarrow 0}{\texttt{match}\ 0\ \texttt{with}\ 0\ \texttt{->}\ 0\ \texttt{|}\ \texttt{\_}\ \texttt{->}\ b\ \texttt{+}\ \texttt{mul}\ (0\ \texttt{-}\ 1)\ b \Rightarrow 0}}{\texttt{mul}\ \ a\ \ b \Rightarrow 0}$$

Week 11 Tutorial 02 — Multiplication

```
let rec mul a b =
  match a with 0 -> 0 | _ -> b + mul (a-1) b
```

Proof by Induction on n

Hypothisis:

if $a \Rightarrow n$ and $b \Rightarrow m$, then $\texttt{mul}\ \ a\ \ b \Rightarrow n \cdot m$

Step:

To prove: if $a \Rightarrow n + 1$ and $b \Rightarrow m$, then $\texttt{mul}\ \ a\ \ b \Rightarrow (n + 1) \cdot m$.

$$\texttt{mul}\ \ a\ \ b \Rightarrow (n + 1) \cdot m$$

Week 11 Tutorial 02 — Multiplication

```
let rec mul a b =
  match a with 0 -> 0 | _ -> b + mul (a-1) b
```

Proof by Induction on n

Hypothisis:

if $a \Rightarrow n$ and $b \Rightarrow m$, then $\texttt{mul}\ \ a\ \ b \Rightarrow n \cdot m$

Step:

To prove: if $a \Rightarrow n+1$ and $b \Rightarrow m$, then $\texttt{mul}\ \ a\ \ b \Rightarrow (n+1) \cdot m$.

by I.H. $\overline{((n+1)-1) \Rightarrow n \qquad m \Rightarrow m}$

$$\text{APP'}\ \dfrac{\pi_{mul} \quad a \Rightarrow n+1 \quad b \Rightarrow m \quad \text{PM}\ \dfrac{n+1 \Rightarrow n+1 \quad \text{OP}\ \dfrac{m \Rightarrow m \qquad \texttt{mul}\ \ ((n+1)\ \texttt{-}\ \texttt{1})\ \ m \Rightarrow m \cdot n}{m\ \texttt{+}\ \texttt{mul}\ \ ((n+1)\ \texttt{-}\ \texttt{1})\ \ m \Rightarrow (n+1) \cdot m}}{\texttt{match}\ n+1\ \texttt{with 0 -> 0 | \_ ->}\ m\ \texttt{+ mul}\ \ ((n+1)\ \texttt{-}\ \texttt{1})\ \ m \Rightarrow (n+1) \cdot m}}{\texttt{mul}\ \ a\ \ b \Rightarrow (n+1) \cdot m}$$

$m + (m \cdot n) \Rightarrow (n+1)\, m$

Week 11 Tutorial 02 — Multiplication

```
let rec mul a b =
  match a with 0 -> 0 | _ -> b + mul (a-1) b
```

Proof by Induction on n

Hypothisis:

if $a \Rightarrow n$ and $b \Rightarrow m$, then $\texttt{mul}\ \ a\ \ b \Rightarrow n \cdot m$

Step:

To prove: if $a \Rightarrow n+1$ and $b \Rightarrow m$, then $\texttt{mul}\ \ a\ \ b \Rightarrow (n+1) \cdot m$.

$$\frac{}{\texttt{mul}\ \ ((n+1)\ \texttt{-}\ \texttt{1})\ \ m \Rightarrow n \cdot m}$$

Week 11 Tutorial 02 — Multiplication

```
let rec mul a b =
  match a with 0 -> 0 | _ -> b + mul (a-1) b
```

Proof by Induction on n

Hypothisis:

if $a \Rightarrow n$ and $b \Rightarrow m$, then $\texttt{mul}\ \ a\ \ b \Rightarrow n \cdot m$

Step:

To prove: if $a \Rightarrow n+1$ and $b \Rightarrow m$, then $\texttt{mul}\ \ a\ \ b \Rightarrow (n+1) \cdot m$.

$$\text{APP'}\ \dfrac{\pi_{mul} \quad a \Rightarrow n+1 \quad b \Rightarrow m \quad \text{PM}\ \dfrac{n+1 \Rightarrow n+1 \quad \text{OP}\ \dfrac{m \Rightarrow m \quad by\ I.H.\ \dfrac{\text{OP}\ \dfrac{n+1 \Rightarrow n+1 \quad \texttt{1} \Rightarrow 1 \quad (n+1) - 1 \Rightarrow n}{(n+1)\ \texttt{-}\ \texttt{1} \Rightarrow n} \quad m \Rightarrow m}{\texttt{mul}\ \ ((n+1)\ \texttt{-}\ \texttt{1})\ \ m \Rightarrow n \cdot m} \quad m + (n \cdot m) \Rightarrow (n+1) \cdot m}{m\ \texttt{+}\ \texttt{mul}\ \ ((n+1)\ \texttt{-}\ \texttt{1})\ \ m \Rightarrow (n+1) \cdot m}}{\texttt{match}\ n+1\ \texttt{with 0 -> 0 | \_ ->}\ m\ \texttt{+ mul}\ ((n+1)\ \texttt{-}\ \texttt{1})\ \ m \Rightarrow (n+1) \cdot m}}{\texttt{mul}\ \ a\ \ b \Rightarrow (n+1) \cdot m}$$

Week 11 Tutorial 03 — Threesum

Use big-step operational semantics to show that the function

```
let rec threesum = fun l ->
  match l with [] -> 0 | x::xs -> 3*x + threesum xs
```

terminates for all inputs and computes three times the sum of the input list's elements.

Week 11 Tutorial 03 — Threesum

```
let rec threesum = fun l ->
  match l with [] -> 0 | x::xs -> 3*x + threesum xs
```

Proof by Induction on length of List

Base: l = []

$$\text{APP}\ \frac{}{\texttt{threesum []} \Rightarrow \texttt{0}}$$

Week 11 Tutorial 03 — Threesum

```
let rec threesum = fun l ->
  match l with [] -> 0 | x::xs -> 3*x + threesum xs
```

Proof by Induction on length of List

Base: l = []

$$\text{APP}\;\frac{\pi_{ts}\quad \texttt{[]} \Rightarrow \texttt{[]}\quad \text{PM}\;\dfrac{\texttt{[]} \Rightarrow \texttt{[]}\quad \texttt{0} \Rightarrow \texttt{0}}{\texttt{match [] with [] -> 0 | x::xs -> 3*x + threesum xs} \Rightarrow \texttt{0}}}{\texttt{threesum []} \Rightarrow \texttt{0}}$$

Week 11 Tutorial 03 — Threesum

```
let rec threesum = fun l ->
  match l with [] -> 0 | x::xs -> 3*x + threesum xs
```

Proof by Induction on length of List

Hypothisis: $\text{thsm } l \Rightarrow \sum_{i=1}^{n} x_i$

Step: l = x :: xs

$$\text{APP} \quad \frac{}{\texttt{threesum } (x_{n+1}\texttt{::xs}) \Rightarrow 3\sum_{i=1}^{n+1} x_i}$$

Week 11 Tutorial 03 — Threesum

```
let rec threesum = fun l ->
  match l with [] -> 0 | x::xs -> 3*x + threesum xs
```

Proof by Induction on length of List

Hypothisis: thsm $l \Rightarrow \sum_{i=1}^{n} x_i$

Step: l = x :: xs

$$\text{APP}\ \dfrac{\pi_{ts} \quad x_{n+1}\texttt{::xs} \Rightarrow x_{n+1}\texttt{::xs}\,\text{PM} \quad \dfrac{x_{n+1}\texttt{::xs} \Rightarrow x_{n+1}\texttt{::xs}\ \text{OP} \quad \dfrac{}{\texttt{3*}x_{n+1}\ \texttt{+ threesum xs} \Rightarrow 3\sum_{i=1}^{n+1} x_i}}{\texttt{match}\ \ x_{n+1}\texttt{::xs with [] -> 0 | x::xs -> 3*x + threesum xs} \Rightarrow 3\sum_{i=1}^{n+1} x_i}}{\texttt{threesum}\ (x_{n+1}\texttt{::xs}) \Rightarrow 3\sum_{i=1}^{n+1} x_i}$$

Week 11 Tutorial 03 — Threesum

```
let rec threesum = fun l ->
  match l with [] -> 0 | x::xs -> 3*x + threesum xs
```

Proof by Induction on length of List

Hypothisis: $\text{thsm } l \Rightarrow \sum_{i=1}^{n} x_i$

Step: l = x :: xs

$$\text{APP}\ \dfrac{\pi_{ts}\quad x_{n+1}\texttt{::xs} \Rightarrow x_{n+1}\texttt{::xs}\ \text{PM}\ \dfrac{x_{n+1}\texttt{::xs} \Rightarrow x_{n+1}\texttt{::xs}\quad \text{OP}\ \dfrac{\text{OP}\ \dfrac{3 \Rightarrow 3\quad x_{n+1} \Rightarrow x_{n+1}\quad 3 * x_{n+1} \Rightarrow 3x_{n+1}}{3 * x_{n+1} \Rightarrow 3x_{n+1}}\quad \text{APP}\ \dfrac{\texttt{by I.H.}}{\texttt{threesum xs} \Rightarrow 3\sum_{i=1}^{n} x_i}\quad 3x_{n+1} + 3\sum_{i=1}^{n} x_i \Rightarrow 3\sum_{i=1}^{n+1} x_i}{\texttt{3*}x_{n+1}\ \texttt{+ threesum xs} \Rightarrow 3\sum_{i=1}^{n+1} x_i}}{\texttt{match}\ x_{n+1}\texttt{::xs with [] -> 0 | x::xs -> 3*x + threesum xs} \Rightarrow 3\sum_{i=1}^{n+1} x_i}}{\texttt{threesum}\ (x_{n+1}\texttt{::xs}) \Rightarrow 3\sum_{i=1}^{n+1} x_i}$$