Math 4553 Final Project: Optimization of a Reparable Multi-State System

1. INTRODUCTION

The mathematical model of a reparable multi-state device with arbitrarily distributed repair time has been introduced in Chung (1981). Here consider the simplified model, which can be described by the ordinary differential equations as follows.

$$\begin{cases} \dfrac{dp_0(t)}{dt} = -\lambda_0 p_0(t) + \mu_1 p_1(t) \\ \dfrac{dp_1(t)}{dt} = -\mu_1 p_1(t) + \lambda_0 p_0(t), \quad t \in [0, T_0], \end{cases} \tag{1}$$

and initial conditions

$$p_0(0) = 1, \quad p_1(0) = 0. \tag{2}$$

Here

(1) $p_0(t)$ Probability that the device is in good mode 0 at time t;
(2) $p_1(t)$ Probability that the failed device is in failure mode 1 at time t;
(3) $\lambda_0 = 0.3$ Constant failure rate of the device for failure mode 1;
(4) $\mu_1 = 0.65$ Constant repair rate when the device is in state 1;

The following assumptions are associated with the device:

(1) The failure rates are constant;
(2) All failures are statistically independent;
(3) There is only one mode of failure and 0 implies the good state;
(4) Repair is to like-new and it does not cause damage to any other part of the system.
(5) Transitions are permitted only between states 0 and 1;
(6) The repair process begins soon after the device is in failure state;
(7) The repaired device is as good as new;
(8) No further failure can occur when the device has been down.

2. OPTIMAL CONTROL DESIGN FOR THE REPARABLE SYSTEM

In order to maintain the system with desired probability distributions, consider the following optimization problem

$$\begin{cases} \dfrac{dp_0(t)}{dt} = -0.3 p_0(t) + 0.65 p_1(t) + u(t), \\ \dfrac{dp_1(t)}{dt} = -0.65 p_1(t) + 0.3 p_0(t) - u(t), \end{cases} \tag{3}$$

where function $u(t)$ is the control input, which represents the system maintenance, through which the system can be steered to a desired probability distribution over the time interval $[0, T_0]$. To this purpose, consider the following minimization problem over $[0, T_0]$. Minimize

$$\begin{aligned} J(p_0(t); p_1(t); u(t)) = & \|p_0(t) - p_0^*\|_2^2 + \|p_1(t) - p_1^*\|_2^2 \\ & + \|u(t)\|_2^2 \end{aligned} \tag{4}$$

subject to system (3) with initial conditions (2). The goal of this project is to minimize the objective function with optimal control input $u(t)$. This provides insight into designing the optimal maintenance policy for the system. Here set the time interval $[0, T_0] = [0, 5]$. Assume that the desired probability of the system in good state is $p_0^* = 90\%$ and the desired probability of the system in failure state is $p_1^* = 10\%$. We expect to obtain an optimal maintenance $u(t)$ to minimize the objective function (4).

Note that system (3) can be rewritten in matrix-vector form

$$\frac{d}{dt}\begin{bmatrix} p_0(t) \\ p_1(t) \end{bmatrix} = A \begin{bmatrix} p_0(t) \\ p_1(t) \end{bmatrix} + Bu(t), \tag{5}$$

where

$$A = \begin{bmatrix} -0.3 & 0.65 \\ 0.3 & -0.65 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

By using the **backward Euler's method** for discretizing the linear system of ordinary differential equations, we can reduce the controlled system (5) to the following linear system of algebraic equations

$$\begin{bmatrix} \dfrac{p_0(t_{i+1}) - p_0(t_i)}{\Delta t} \\ \dfrac{p_1(t_{i+1}) - p_1(t_i)}{\Delta t} \end{bmatrix} = A \begin{bmatrix} p_0(t_{i+1}) \\ p_1(t_{i+1}) \end{bmatrix} + Bu(t_{i+1}), \tag{6}$$

for $i = 0, 1, \dots, 99$. Here divide the interval $[0, T_0] = [0, 5]$ into 100 equal subintervals $[t_{i-1}, t_i]$, $i = 1, 2, \dots, 100$, and set $\Delta t = \frac{1}{20}$, then $t_i = i\Delta t$ and

$$0 = t_0 < t_1 < \cdots < t_i < \cdots < t_{100} = 5.$$

Denoted by $p_{0,i} = p_0(t_i)$, $p_{1,i} = p_1(t_i)$ and $u_i = u(t_i)$, $i = 0, 1, \dots, 100$, system (6) can be rewritten as

$$\begin{bmatrix} \dfrac{p_{0,i+1} - p_{0,i}}{\Delta t} \\ \dfrac{p_{1,i+1} - p_{1,i}}{\Delta t} \end{bmatrix} = A \begin{bmatrix} p_{0,i+1} \\ p_{1,i+1} \end{bmatrix} + Bu_{i+1}, \tag{7}$$

for $i = 0, 1, \dots, 99$. Let

$$P_0 = [p_{0,1}, p_{0,2}, \dots, p_{0,100}]^T, \quad P_1 = [p_{1,1}, p_{1,2}, \dots, p_{1,100}]^T$$

and

$$\boldsymbol{u} = [u_1, u_2, \dots, u_{100}]^T,$$

then the objective function (4) becomes

$$J(P_0; P_1; \boldsymbol{u}) = \left\| \begin{bmatrix} P_0 \\ P_1 \end{bmatrix} - \begin{bmatrix} P_0^* \\ P_1^* \end{bmatrix} \right\|_2^2 + \|\boldsymbol{u}\|_2^2, \tag{8}$$

where

$$P_0^* = [90\%, 90\%, \dots, 90\%]^T_{100\times 1}$$

and

$$P_1^* = [10\%, 10\%, \dots, 10\%]^T_{100\times 1}.$$

3. GOALS

(1) Derive the analytical optimal solution by applying Lagrange's Theorem.
(2) Construct a numerical algorithm to solve the problem and write a MATLAB code to implement the algorithm.
(3) Compare the numerical solution with the analytical solution and show the difference of the associated objective functions.

You are more than welcome to solve this problem by using any other methods.

4. WHAT TO HAND IN?

Please follow instructions in the template and show **ALL** your work to support your results. The figures of the discretized probabilities P_0, P_1 and control input $\boldsymbol{u}$ are required to provide in your report. The grading policy is as follows:

(1) Problem description: 10%;
(2) Methodology: 20%;
(3) Results: 45%;
(4) Observation and Conclusions: 25%.

REFERENCES

W.K. Chung. A Reparable Multi-state Device with Arbitrarily Distributed Repair Times *Micro. Reliab.*, volume 21, number 2, pages. 255–256, 1981.

E.K.P. Chong and S.H. Zak, An Introduction to Optimization, 3rd Ed., Wiley, 2008.

M.C. Ferris, O.L. Mangasarian and S.J. Wright, Linear Programming with Matlab, MPS-SIAM Series on Optimization, 2007.