Particle Physics Note

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Chapter 1

Decay Rate & Cross Sections

§ 1.1 Fermi Golden Rule

Consider time-dependent Schrödinger equation,

$$i\frac{d\psi}{dt} = \hat{H}(t)\psi. \tag{1.1}$$

We can separate Hamiltonian operator into two parts, one is time-independent, and else is time-dependent,

$$\hat{H}(t) = \hat{H}_0 + \hat{H}'(\mathrm{x}, t) \tag{1.2}$$

For time-independent part we can use common method to solve it,

$$\hat{H}_0\psi_k = E_k\psi_k, \quad \text{and} \quad \langle\psi_j|\psi_k\rangle = \delta_{jk} \tag{1.3}$$

Then the wavefunction can be expressed in terms of complete set of states of the unperturbed Hamiltonian as

$$\psi(\mathrm{x}, t) = \sum_k c_k(t)\psi(\mathrm{x})e^{-iE_kt}. \tag{1.4}$$

Then we can substitute Eq. 1.4 into Eq. 1.1, we can get

$$\begin{aligned} i\sum_k\left[\frac{\mathrm{d}c_k}{\mathrm{d}t}\psi_k e^{-iE_kt} - iE_kc_k\psi_k e^{-iE_kt}\right] &= \sum_k c_k\hat{H}_0\psi_k e^{-iE_kt} + \sum_k \hat{H}'c_k\psi_k e^{-iE_kt} \\ \Rightarrow \quad i\sum_k\frac{\mathrm{d}c_k}{\mathrm{d}t}\psi_k e^{-iE_kt} &= \sum_k \hat{H}'c_k(t)\psi_k e^{-iE_kt} \end{aligned} \tag{1.5}$$

Suppose the initial state is i for $\psi_i \equiv |i\rangle$, that is

$$\psi(\mathrm{x}, 0) = \sum_k c_k(0)\psi_k = c_i(0)\psi_i. \tag{1.6}$$

We can easily imply that $c_k = \delta_{ik}$. Assume that time-dependent Hamiltonian is perturbed, which implies $c_{i\neq k} \ll 1, c_i \sim 1$. And Eq. 1.5 can reduce to

$$i \sum_k \frac{\mathrm{d}c_k}{\mathrm{d}t} \left|k\right\rangle e^{-iE_k t} \approx \hat{H}' \left|i\right\rangle e^{-iE_i t} \tag{1.7}$$

Then we need to get to final state, so we can use relation of inner product, let $\langle f|$ act on Eq. 1.7, we can get the coefficient c_f

$$\frac{dc_f}{dt} = -i \left\langle f|\hat{H}'|i\right\rangle e^{-i(E_i-E_f)t}, \tag{1.8}$$

where $\langle f|\hat{H}'|i\rangle \equiv T_{fi}$ is **transition matrix element**, can be calculated by

$$\left\langle f|\hat{H}'|i\right\rangle = \int_V \psi_f^*(\mathrm{x})\hat{H}'\psi_i(\mathrm{x})\,\mathrm{d}^3\mathrm{x}\,. \tag{1.9}$$

Then we can calculate the coefficient c_f, at time $t = T$

$$c_f(T) = -i\int_0^T T_{fi} e^{-i(E_i-E_f)t}\,\mathrm{d}t \tag{1.10}$$

The probability for a transition to the state $|f\rangle$ is given by

$$P_{if} = c_f^*(t)c_f(t) = \left|T_{fi}\right|^2 \int_0^T\int_0^T e^{-i(E_i-E_f)t}e^{-i(E_i-E_f)t'}\,\mathrm{d}t\,\mathrm{d}t' \tag{1.11}$$

And we have **transition rate** $\mathrm{d}\Gamma_{fi}$ from the initial state $|i\rangle$ to the single final state $|f\rangle$

$$\mathrm{d}\Gamma_{fi} = \frac{P_{fi}}{T} = \frac{1}{T}\left|T_{fi}\right|^2 \int_{-\frac{T}{2}}^{\frac{T}{2}}\int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-i(E_i-E_f)t}e^{-i(E_i-E_f)t'}\,\mathrm{d}t\,\mathrm{d}t' \tag{1.12}$$

If we consider the time is long enough comparing to transition process, we can take a limit,

$$\mathrm{d}\Gamma_{fi} = \left|T_{fi}\right|^2 \lim_{T\to\infty}\left\{\frac{1}{T}\int_{-\frac{T}{2}}^{+\frac{T}{2}}\int_{-\frac{T}{2}}^{+\frac{T}{2}} e^{i\left(E_f-E_i\right)t}e^{-i\left(E_f-E_i\right)t'}\mathrm{d}t\,\mathrm{d}t'\right\}. \tag{1.13}$$

Associated with the definition of Dirac delta-function, the integral over $\mathrm{d}t'$ can be replaced by $2\pi\delta(E_f - E_i)$ and thus

$$\Gamma_{fi} = 2\pi\left|T_{fi}\right|^2 \left|\frac{\mathrm{d}n}{\mathrm{d}E_f}\right|_{E_i}, \tag{1.14}$$

where $\left|\frac{\mathrm{d}n}{\mathrm{d}E_f}\right|_{E_i}$ is called **density of states**, it can be also written as

$$\rho(E_i) = \left|\frac{\mathrm{d}n}{\mathrm{d}E_f}\right|_{E_i} \tag{1.15}$$

Fermi's golden rule for the total transition rate is therefore

$$\Gamma_{fi} = 2\pi \left|T_{fi}\right|^2 \rho(E_i) \tag{1.16}$$

However, we take assumption $c_{i\neq f}(t) \approx 0$, if we need to get more precise information, we should expand more terms of transition matrix element, we start from Eq. 1.8

$$\frac{\mathrm{d}c_f}{\mathrm{d}t} \approx -i\langle f|\hat{H}|i\rangle e^{i\left(E_f-E_i\right)t} + (-i)^2 \sum_{k\neq i} \langle f\left|\hat{H}'\right|k\rangle e^{i\left(E_f-E_k\right)t} \int_0^t \langle k\left|\hat{H}'\right|i\rangle e^{i(E_k-E_i)t'} \mathrm{d}t' \tag{1.17}$$

Therefore, the improved approximation for the evolution of the coefficients $c_f(t)$ is given by

$$\frac{\mathrm{d}c_f}{\mathrm{d}t} = -i\left(\langle f|\hat{H}|i\rangle + \sum_{k\neq i} \frac{\langle f\left|\hat{H}'\right|k\rangle \langle k\left|\hat{H}'\right|i\rangle}{E_i - E_k}\right) e^{i\left(E_f-E_i\right)t}. \tag{1.18}$$

For second order of transition matrix element, we have

$$T_{fi} = \langle f|\hat{H}|i\rangle + \sum_{k\neq i} \frac{\langle f\left|\hat{H}'\right|k\rangle \langle k\left|\hat{H}'\right|i\rangle}{E_i - E_k} \tag{1.19}$$

§ 1.2 Decay Rate

Fermi's golden rule can be written as an alternative forma

$$\Gamma_{fi} = 2\pi \int \left|T_{fi}\right|^2 \delta(E_i - E)\,\mathrm{d}n \tag{1.20}$$

Firstly consider the decay rate for the process $a \to 1+2$ in non-relativistic situation, related Fermi's golden rule, we write transition matrix element

$$T_{fi} = \langle \psi_1\psi_2|\hat{H}'|\psi_a\rangle \tag{1.21}$$

$$= \int_V \psi_1^*\psi_2^*\hat{H}'\psi_a\,\mathrm{d}^3x \tag{1.22}$$

In the Born approximation, the perturbation is taken to be small and the initial- and final-state particles are represented by plane waves of the form

$$\psi(\mathrm{x}, t) = Ae^{i(\mathbf{p}\cdot\mathrm{x}-Et)}, \tag{1.23}$$

where $A^2 = \frac{1}{V}$ determines wavefunction normalization. We use such condition

$$\psi(x+a, y, z) = \psi(x, y, z), \quad \text{etc.}, \tag{1.24}$$

The periodic boundary conditions on the wavefunction imply that the components of momentum are quantised to

$$(p_x, p_y, p_z) = (n_x, n_y, n_z)\frac{2\pi}{a} \tag{1.25}$$

Therefore, we can get the density of states,

$$dn = dV(p)\,\frac{V}{(2\pi)^3} = 4\pi p^2\,dp\,\frac{V}{(2\pi)^3} \tag{1.26}$$

The density of states in Fermis golden rule then can be obtained from

$$\rho(E) = \frac{dn}{dE} = \frac{dn}{dp}\left|\frac{dp}{dE}\right| \tag{1.27}$$

1.2.1 Lorentz-invariant Form

To keep wavefunction normalized, a unit volume should decrease with particle energy $E = \gamma m$ increasing. For convenience, we usually take $2E$ as normalization volume

$$\int_V \psi'^* \psi'\,d^3x = 2E, \tag{1.28}$$

and therefore

$$\psi' = \sqrt{2E}\psi. \tag{1.29}$$

Therefore, we can get Lorentz-invariant form of transition matrix element

$$\mathcal{M}_{fi} = \langle \psi'_1 \psi'_2 \cdots | \hat{H}' | \psi'_a \psi'_b \cdots \rangle = \sqrt{2E_1 \cdot 2E_2 \cdot \cdots \cdot 2E_a \cdot 2E_b \cdots} T_{fi} \tag{1.30}$$

We go back to Fermi's golden rule. Combining with Eq. 1.20, we can get

$$\Gamma_{fi} = \frac{(2\pi)^4}{2E_a} \int |\mathcal{M}_{fi}|^2 \,\delta\left(E_a - E_1 - E_2\right) \delta^3\left(\mathbf{p}_a - \mathbf{p}_1 - \mathbf{p}_2\right) \frac{d^3\mathbf{p}_1}{(2\pi)^3 2E_1} \frac{d^3\mathbf{p}_2}{(2\pi)^3 2E_2}. \tag{1.31}$$

It should be noticed that $\frac{d^3\mathbf{p_i}}{E_i}$ is Lorentz-invariant.

1.2.2 N-body Decay

For N-body decay, we should generalize phase space, and the element of phase space can be expressed by

$$dV_{LIPS} = \prod_{i=1}^{N} \frac{d^3\mathbf{p}_i}{(2\pi)^3 2E_i}, \tag{1.32}$$

where LIPS is known as Lorentz-invariant phase space.

With the definition of Dirac-delta function, we can imply that

$$\int \delta(E_i^2 - \mathbf{p}_i^2 - m^2)dE_i = \frac{1}{E_i}. \tag{1.33}$$

So, we have

$$\int \cdots dV_{LIPS} = \int \cdots \prod_{i=1}^{N} (2\pi)^{-3} \delta(E_i^2 - \mathbf{p}_i^2 - m_i^2)\,d^3\mathbf{p}_i\,dE_i \tag{1.34}$$

Therefore, we can write element for $a \to 1 + 2$ decay

$$\Gamma_{fi} = \frac{(2\pi)^4}{2E_a} \int (2\pi)^{-6} |\mathcal{M}_{fi}|^2 \delta^4(p_a - p_a - p_2) \delta(p_1^2 - m_1^2) \delta(p_2^2 - m_2^2)\,d^4p_1\,d^4p_2 \tag{1.35}$$

§ 1.3

Cross-Sections

Chapter 2

Klein-Gordon Equation