and hence that the $H_n(x)$ satisfy the Hermite equation

$$y^n - 2xy' + 2ny = 0,$$

where n is an integer $\ngeq 0$.

Use Φ to prove that

(a) $H_n'(x) = 2nH_{n-1}(x)$,

(b) $H_{n+1}(x) - 2xH_n(x) + 2nH_{n-1}(x) = 0$.

18.6 A charge $+2q$ is situated at the origin and charged of $-q$ are situated at distances $\pm a$ from it among the polar axis. By relating it to the generating function for the Legendre polynomials, show that the electrostatic potential Φ at a point (r, θ, ϕ) with $r > a$ is given by

$$\Phi(r, \theta, \phi) = \frac{2q}{4\pi\epsilon_0 r} \sum_{s=1}^{\infty} (\frac{a}{r})^2 P_{2s}(cos\theta).$$

18.7 For the associated Laguerre polynomials, carry through the following exercises.

(a) Prove the Rodrigues' formula

$$L_n^m(x) = \frac{e^x x^{-m}}{n!} \frac{d^n}{dx^n}(x^{n+m}e^{-x}),$$

taking the polynomials to be defined by

$$L_n^m(x) = \sum_{k=0}^{n} (-1)^k \frac{(n+m)!}{k!(n-k)!(k+m)!} x^k.$$

(b) Prove the recurrence relations

$$(n+1)L_{n+1}^m(x) = (2n+m+1-x)L_n^m(x) - (n+m)L_{n-1}^m(x),$$
$$x(L_n^m)'(x) = nL_n^m(x) - (n+m)L_{n-1}^m(x),$$

but this time taking the polynomial as defined by

$$L_n^m(x) = (-1)^m \frac{d^m}{dx^m} L_{n+m}(x)$$

or the generating fun

18.8 The quantum mechanical wavefunction for a one-dimensional simple harmonic oscillator in its nth energy level is of the form

$$\psi(x) = exp(-x^2/2)H_n(x),$$

where $H_n(x)$ is nth Hermite polynomial. The generating function for the polynomials is

$$G(x, h) = e^{2hx-h^2} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} h^n.$$

(a) Find $H_i(x)$ for $i = 1, 2, 3, 4$.

(b) Evaluate by direct calculation

$$\int_{-\infty}^{\infty} e^{-x^2} H_p(x)H_q(x)dx,$$

(i) for $p = 2$, $q = 3$; (ii) for $p = 2, q = 4$; (iii) for $p = q = 3$. Check your answers against the expected values $2^p p! \sqrt{\pi}\delta_{pq}$.