LINMA2471: Optimization models and methods: course 4 (07/10/2015)

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1 Properties of convex functions

1.1 Linear transformation on the variables

Proposition 1. *Given a convex and affine transformation $x \mapsto Ax + b$, the composition $x \mapsto f(Ax + b)$ is also convex.*

Example 1.1. *e^{2x-y+z} is convex because the exponential is convex and $2x - y + z$ is a linear transformation of x, y and z.*

Example 1.2 (Convex functions)**.** *Any norm $x \mapsto ||x||$ is convex, thus the distance $||x - y||$ between two points x and y is convex because $x - y$ is a linear transformation.*
The maximum distance between a set S and a point x is a convex function. Indeed, taking the maximum between a point and a set requires to take the maximum of all the distances between the point and any point in the set (distance between two points is a convex function) : $f_{S,max} = \max_{s \in S}\{||x - s||\}$

1.2 Partial minimization

Proposition 2. *(Partial minimization) If the function $f : (x, y) \mapsto f(x, y)$ is convex, then $f_x(y) = \inf_x f(x, y)$ is convex.*

Example 1.3. *If a set S is convex, then the minimum distance function between a point x and the set S is convex. Indeed, one can write the function as follows :*

$$f(x, s) = ||x - s||$$

Since this is a norm, f is convex. Since the restriction of a convex function stays convex as long as the feasible region stays convex and S is a convex set, property 2 gives that:

$$f_S(x) = \inf_S f(x, s)$$

is a convex function.

Remark 1. *Property 2 is a one side property. A counter-example for the reverse side is given by :*

$$f_x(y) + \sqrt{||x||}$$

1.3 Extended real valued functions

Most of theorems to prove the convexity of a function require the convexity of the domain. However, it is possible to extend a function to tackle this problem.

Example 1.4. *Let's take the function $f : \mathbb{R}_+ \mapsto \mathbb{R} : x \mapsto \frac{1}{x}$ and extend it such that its domain becomes the whole real line. One consider :*

$$f_e : \mathbb{R} \mapsto \mathbb{R} \cup \{+\infty\} : x \mapsto \begin{cases} \frac{1}{x} \text{ if } x > 0 \\ +\infty \text{ elsewhere} \end{cases}$$

One can see that the extended function is convex over the whole real line. The epigraph definition still holds since there isn't any point above $+\infty$.

1.4 Composition and product

Proposition 3. *If g is a convex function and f is a convex, increasing and one-dimensional function then the composition function $h \circ g : x \mapsto h(g(x))$ is also convex.*

Proof. Let's prove this proposition for a simple case. We assume that f and g are both one-dimensional functions and that $f, g \in \mathcal{C}^2$. The general case requires a more difficult proof. Since f and g are 2 times differentiable, one has:

$$\big[h(g(x))\big]'' = \big[h'(g(x))g'(x)\big]' = \underbrace{h''(x)(g'(x))^2}_{\text{A}} + \underbrace{h'(g(x))g''(x)}_{\text{B}}$$

Since h is convex, its second derivative is positive and given that a square is positive, one has that A is positive. Furthermore, since g is also convex and h is increasing, one also has that B is positive. One conclude that the second derivative of $h \circ g$ is positive and thus, the function $h \circ g$ is convex. □

Remark 2. *Sometimes we need to square a value but also to keep convexity (for example, we don't care about negative deviations on a budget). However the traditional square function is not convex on the real line. Let's introduce a restricted square function as follows:*

$$f : x \mapsto (x_+)^2 = (\frac{x + |x|}{2})^2$$

We easily see (figure 1) that this restricted square function is convex.

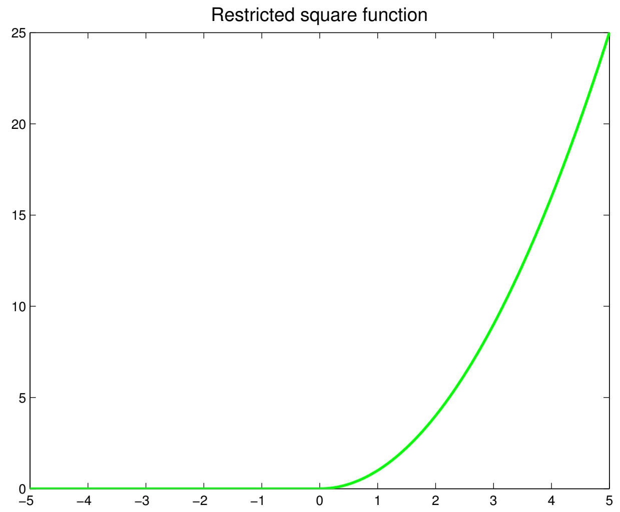


Figure 1: Restricted square function

Example 1.5. *The function* $\left[\log(x+y)_+\right]^2$ *is convex. Indeed, the restricted square and* $-\log$ *are convex functions. Therefore, their composition is convex. Since* $x+y$ *is a linear transformation, it preserves convexity and the whole function is convex.*

Proposition 4. *If* f *and* g *are both convex, positive and increasing then their product is convex.*

Proof. Again, one proves it in the simple differentiable, one-dimensional case. One has:

$$(fg)'' = \left[f'g + fg'\right]' = f''g + 2f'g' + fg''$$

The result follows immediately since by assumptions one has $f, g, f', g', f'', g'' \geq 0$. □

Remark 3. *The previous proof tends to indicate variants of proposition 4. One can see that if* f *and* g *are both concave, decreasing and negative then the proposition still holds.*

1.5 Advantage of convex problems

Properties 1. *Let's recall that* $\min_{x\in X} f(x)$ *is convex if* f *is convex,* X *is convex and we are looking for a minima. We study the properties of a convex problem :*

MODEL	*METHODS*
- Local minima are also global	*- Methods which only work on convex problems : first order, second order...*
- The set of optimal solutions is convex	
- Using duality we can get guarantees	

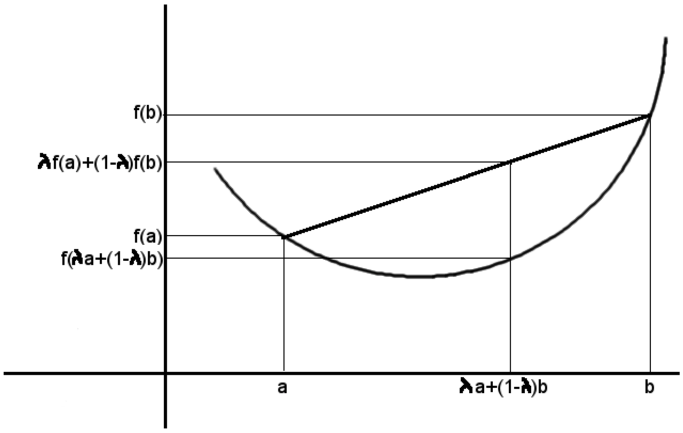
1.6 Variants of convex functions

It's a difficult thing to know if a problem have a unique solution. There is one class of problems with only one solution : minimization of strictly convex functions.

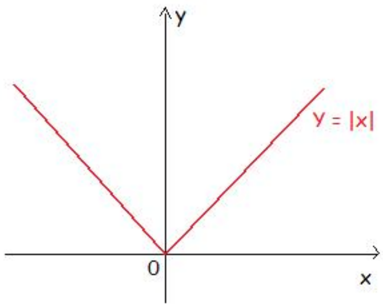
Definition 1. *Strict convexity : f is strictly convex if and only if*

- *the domain is convex*
- $f(\lambda x + (1-\lambda)y) < \lambda f(x) + (1-\lambda)f(y)\ \forall x, y \in dom\ ,\ \forall \lambda \in]0,1[$

Example 1.6. *On the following graph, we can observe that the function is strictly convex.*



Example 1.7. *The absolute function is not strictly convex, in fact, when we have a flat part in the graph, it can not be strictly convex.*



Example 1.8. *The function $x \mapsto ||x||_2 = \sqrt{\sum_i x_i^2}$ is convex but not strictly convex.*

Proposition 5. *If we have $\min f(x)_{x\in X}$ with f strictly convex and X a convex set then the problem admit at most one solution.*

Proposition 6. *If $f \in C^2$ and $\nabla^2 f(x) > 0$ then f is strictly convex. ($\lambda_i > 0\ \forall i$)*

Proposition 7. *If f is convex, then $f + ||x||_2^2$ is strictly convex.*

Proof. Assume $f \in C_2$
then

$$\nabla^2(f + ||x||^2) = \nabla^2 f + \nabla^2 ||x||_2^2 = \nabla^2 f + 2I$$

where $||x||_2^2 = \sum_i = x_i^2$ and $\lambda_i \geq 0$. □

Proposition 8.

$$(1)\ \lambda \textit{ is an eigenvalue of } M$$

$$\Leftrightarrow$$

$$(2)\ \lambda + \Delta \textit{ is an eigenvalue of } M + \Delta I$$

for any $\Delta \in \mathbb{R}$ *and* $M \in \mathbb{R}^{n \times n}$ *symmetric.*

Proof.

$$(1)\ \exists v\ :\ Mv = \lambda v$$

$$(2)\ \exists v\ :\ (M + \Delta I)v = (\lambda + \Delta)v$$

□

Remark 4. *We can have the same propositions and the same proof while adding* $\mu > 0$ *anywhere :* $f + \mu||x||^2$*. It is a regularization to make it strictly convex.*

Remark 5. *There are functions that have no derivative and are strictly convex.*