Université catholique de Louvain

LINMA2471
Optimization models and methods II

Course notes

Students:
Nicolas Boutet, Benjamin Chiem, Antoine de Comité, Alexandre de Touzalin, Céline Gérard, Andine Havelange, Florimond Houssiau, Sébastien Lagae, Adissa Laurent, Laura Motte, Caroline Sautelet & Vincent Schellekens

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A. Linear and Convex modeling

In this chapter we will see some important classes of optimization problems, and some techniques to re-write an optimization problem into another form.

1. Definitions and motivation

First of all, let's recall what the model of an optimization problem looks like.

Definition A.1. *A **general model** has the following form :*

$$\min_{x \in X \subseteq \mathbb{R}^n} f(x) \tag{A.1}$$

*where x are the **variables**, X is the **feasible set** (also called domain or feasible region) and f is the **objective function**.*

Note that the feasible set X is a subset of a *finite* dimensional space. Optimization within infinite-dimensional spaces are not covered in this course.

Let us next introduce two very important classes of models : the *linear* and *convex* models.

Definition A.2. *A model is called a **linear model** if :*

1. *The objective function is linear/affine[1], that is, of the form $c^T x / c^T x + d$.*
2. *The feasible set X is a **polyhedron**. A polyhedron is an intersection of a finite[2] number of closed **half-spaces**. As a reminder, a half-space is the set of points that lie on one side of a hyperplane; in an algebraïc form : $\{x \in \mathbb{R}^n | a^T x \geq b\}$ or $\{x \in \mathbb{R}^n | a^T x \leq b\}$.*

Definition A.3. *A model is called a **convex model** if :*

1. *The objective function f is convex (see below).*
2. *The feasible set X is convex (see below).*

We still need to define what are convex functions and sets :

Definition A.4. *A set X is a **convex set** if it contains the segments between every pair of its points.*

[1] Note that the independent term of an affine function (d) can be easily dropped out, because it doesn't affect the optimal solution in any way. Therefore, every affine function can be replaced with a purely linear one.

[2] For example, a sphere is therefore *not* a polyhedron, because it is an intersection of an *infinite* number of closed half-planes.

Definition A.5. *A function f is a **convex function** if its **epigraph** is convex. The epigraph*[3] *of f is the set of points above (and including) the graph of f. Formally, we write this as : epi* $f := \{(x,t)|t \geq f(x)\}$. *This notion is illustrated on Figure A.1.*

For the definition of convex functions, we didn't used the concept of derivative, because we want our definition to be as general as possible. In other words, a non-differentiable function can be convex[4]. Note also that *every linear model is a particular case of a convex model.*

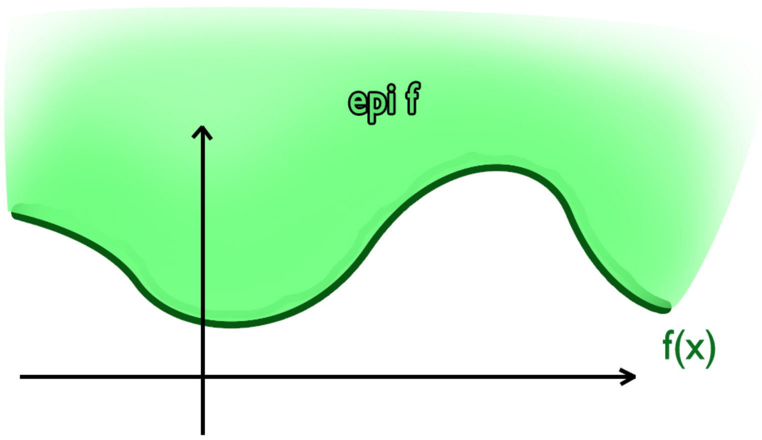


Figure A.1.: Illustration of the epigraph of some one-dimensional (and non-convex) function. The epigraph of f is the region above the graph of f.

Now that we know the definitions of linear and convex models, what is the **motivation** to study such models? First, these models are useful to develop *efficient algorithms* with *guarantees* about the exactitude of the optimal solution and the speed of the algorithm.

Also, one could argue that studying linear/convex models is a restriction to the number of problems we will be able to solve, but it turns out that convex problems are *not so rare* in practice. Many problems are, or can be formulated as convex problems. Some problems can even be solved by using an equivalent convex problem : for example, the *branch and bound* algorithm transforms a discrete (and therefore non-convex) problem into a sequence of linear problems. One last -informal- reason to restrict ourselves to convex problem is that, for non-convex problems, there is nothing interesting we can really do or say.

2. Standard forms

As the general formulation of convex and linear problems can be very hard to use in order to develop a theory about them, due (mostly) to the variety of constraint types, it is important to define **standard forms**. Standard forms define a unique, specific, formulation of these problems, that is much simpler than the general form.

[3]Note that if $f : \mathbb{R}^n \to \mathbb{R}$ then *epi* $f \subseteq \mathbb{R}^{n+1}$

[4]For example, the norm function defined by $f : \mathbb{R} \to \mathbb{R} : x \to |x|$ is convex

a. Linear case: re-writing objective function, variables and constraints

Before we define the standard form, let us observe a number of transformations that can be applied to a linear problem without changing its solution (that is, the two problems will be equivalent).

Maximization and minimization In general, a maximization problem can easily be formulated as a minimization problem. Indeed, maximizing a function f is equivalent to minimizing its opposite $-f$. If the solution x is the same, the value of the objective function $-f(x)$ is simply the opposite of that of the original problem.

$$\boxed{\textbf{Max}} \xleftrightarrow{\text{max } f = \text{- min (-}f\text{)}} \boxed{\textbf{Min}}$$

Non-negative variables The variables used in the linear standard form are non-negative, that is, they are free with the implicit constraint $x \geq 0$. It is possible to transform free variables in non-negative variables by a process known as **splitting**.

$$\boxed{\textbf{X non-negative}} \xleftarrow{\text{splitting}} \boxed{\textbf{X free}}$$

Splitting is done by, for every free variable x_i, adding two non-negative variables x_i^+ and x_i^-, defined by the relationship $x_i = x_i^+ - x_i^-$. Then, x_i is substitued by $x_i^+ - x_i^-$ everywhere in the problem.

However, this can be severely inefficient, since it adds a variable for every free variable. In a problem with n free variables, n additionnal variables are created, thus doubling the size of the problem. As this can be a major performance issue, it is important to do the splitting without creating too many variables.

A simple observation about the current method that can be made is that, if we assume x_i has a unique value in the solution, the values of x_i^+ and x_i^- are not uniquely defined (in some sense, there is one too many degree of freedom). By exploiting this notion, another formulation can be proposed: if a problem has n free variables x_i, we subtitute these by $x_i^+ - x^-$, where x_i^+ and x^- are non-negative variable, and x^- is **the same for all the variables**. This method is better since it only creates **one additional** variable!

Why does this work? If and whenever a solution x^* is found, the value of $x^{-,*}$ will be at most equal to the smaller (i.e. most negative) x_i^*. Then, by definition, $x_i^{+,*} = x_i^* + x^{-,*}$ (with $x^{-,*} \geq 0$), and since $x_i^* \geq -x^{-,*}$, we get that $x_i^{+,*} \geq 0$, and so $x_i^{+,*}$ is indeed non-negative.

As an example, suppose a linear problem with 3 variables, all of them free, has as unique solution $(x_1^*, x_2^*, x_3^*) = (-3, 7, -10)$. A solution, in term of non-negative variables, is thus $(x_1^{+,*}, x_2^{+,*}, x_3^{+,*}, x^{-,*}) = (7, 17, 0, -10)$.

The reverse is also possible, although not really interesting.

$$\boxed{\textbf{X non-negative}} \rightarrow \begin{cases} \textbf{Constraint: } X \geq 0 \\ \textbf{X free} \end{cases}$$

Equalities and inequalities After treating the objective and the variables, it is important to treat the constraints (which offer the widest range of varieties). Firstly, turning equalities into inequalities is very straightforward:

$$\boxed{\textbf{Equality: } f = g} \longrightarrow \boxed{\textbf{Two inequalities: } f \geq g \textbf{ and } f \leq g}$$

The opposite is both more useful and more subtle:

$$\boxed{\begin{array}{l}\textbf{Equality: } f + s = g\\ \textbf{and slack variable } s\end{array}} \xleftarrow{\text{Slack Variable}} \boxed{\textbf{Inequality } f \geq g}$$

To do this, we must introduce the concept of **slack variable** (*variable d'écart* in French). A slack variable is a non-negative variable added on the greater side of the inequality to make it an equality. Basically, its value is $f - g$, the slack between f and g, representing the *margin* before the constraint becomes an equality.

One last case to be treated is the **lower bound constraint**. This can of course be treated as a constraint, but is redundant with the nonnegativity of the variables in the standard form. A fairly simple solution is to subtitute the variable X with $X - l$, where l is the lower bound.

$$\boxed{\textbf{Lower Bound } X \geq l} \xrightarrow{\text{Substitution}} \boxed{\begin{array}{l}\textbf{non-negative}\\ X' \textbf{ : } X = l + X'\end{array}}$$

b. Standard form for linear models

The standard form of a **linear optimization problem** is:

$$\boxed{\begin{array}{ll}\min_X & c^T X\\ & AX = b\\ & X \geq 0 \quad (\Longleftrightarrow X\text{non-negative})\end{array}}$$

If the problem has n variables and m constraints, then $c \in \mathbb{R}^{n\times 1}$, $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^{m\times 1}$. These constitute the only data needed to uniquely define the problem.

The transformations exposed in the previous section illustrate the fact that it is possible to turn any linear optimization problem in the standard form.

c. Standard form for convex models

There is **no known standard form** for convex optimization problems. This can be understood in the sense that the objective function can be very general, as well as the set X. Since, by definition, the set X is uncountable, it is very hard to represent it in a way that can be, for example, treated by a computer. However, it is possible, under certain assumptions, to represent the set X in a purely functional way, as a set of inequalities involving functions.

Most often, the set X is defined as a set of constraints. Let's suppose that there exists a set of functions g_i, $i \in \{1 \geq \cdots \geq m\}$, h_j, $j \in \{1 \geq \cdots \geq l\}$, such that:

$$X = \{x \in \mathbb{R}^n | g_1(x) \leq 0, \dots, g_m(x) \leq 0 \text{ and } h_1(x) = 0, \dots, h_l(x) = 0\}$$

This form is general, since the $g_i(x) \geq 0$ constraint is equivalent to $-g_i(x) \leq 0$. The one exception lies in the fact that **strict inequalities** cannot be treated. But, as is explained later, this is not a real issue.

In general, this does not suffice to guarantee that X is convex. The following conditions are **sufficient** to ensure the convexity of the set X:

- the g_i functions are **convex** (this results from the choice that g_i should be smaller than zero: if the opposite were chosen, then the functions should be concave)
- the h_j functions are **linear**: it's tempting to say that convexity is enough, but it is not the case. For example, the function $h : \mathbb{R}^2 \to \mathbb{R} : (x, y) \to x^2 + y^2 - 1$ defined as ensemble X the circle (and not the discus!) of radius 1, which is obviously not convex[5].

In some cases, the constraints $h_j(x) = 0$ can be relaxed to inequalities $h_j(x) = 0$ (Shouldn't it be an inequality?), thus relaxing the linearity constraint on h_j. One of these cases, which will be developed in the next section, is when the objective function is linear.

A note on $\neq$ The standard forms developed in this section do not allow for strict inequalities to be considered. This is because strict inequalities tend to make solutions *disappear*: in linear optimization, solutions are always located on the boundary of a closed polyhedron. By making this polyhedron open (with strict inequalities), the solutions disappear, as there is not admissible point with a minimal value (it is always possible to get *closer* to the boundary, thus reducing the objective function).

A solution to this is to treat strict inequalities as non-strict ones by introducing a *tolerance* $\epsilon > 0$, that describes how close to the open boundary the solutions can lie. The constraint $f > g$ then becomes $f \geq g + \epsilon$. The choice of the tolerance depends on the context of the problem (and is to be discussed with the client, for example).

d. Transforming *any* problem into a convex problem

We can turn any optimization problem into a convex problem by following two "easy" steps. First, *the objective function can be made linear* by adding a new variable. Then, *the constraints can be made convex* by an operation called taking the convex hull. Let us see these operations in detail.

First step : making the objective function linear Let us assume the following (general) model :

$$\min_{x \in X \subseteq \mathbb{R}^n} f(x)$$

[5] In fact, each segment binding two points of the circle doesn't include any other point of the circle that its extremity...

The trick is to re-write this problem introducing a new variable t that is greater or equal[6] to $f(x)$. We now have :

$$\min_{x\in\mathbb{R}^n,\ t\in\mathbb{R}} t \text{ with } x \in X,\ (x,t) \in \textit{epi } f$$

The new objective function is just t and is clearly linear, but the domain is now more complicated. In other words, we have traded simplicity in the objective function (wich is a good thing) by adding complexity in the domain (wich is usually already complex anyway, so it isn't that bad). Note that if the original problem was convex, the new problem is still convex (because it means *epi* f is convex).

Second step : making the constraints convex The feasible set can be made convex by taking the convex hull of the set.

Definition A.6. *The **convex hull** of a set X (denoted by conv X) is the **smallest** convex set containing X. The smallest set means : the intersection of all possible convex sets containing X.*

> **Example A.1.** *If we have a simple (non-convex) set containing two points in space, the convex hull of this set is the segment between those two points. This example is illustrated figure A.2. This gives us an (infeasible in practice, see "the catch" to make every problem convex, page 8) algorithm to take the convex hull of any set : just take every pair of points in the set and add the segment between those two points!*

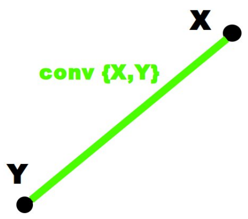


Figure A.2.: Illustration of the convex hull of a set X. Note that the convex hull also includes the original set, wich is not very well represented on the figure.

Taking the convex hull of the feasible set X obviously gives us a new optimization problem. What are the optimal solutions of this new problem?

Theorem A.1. *Any optimal solution x^* to the original problem :*

$$\min_{x\in X} c^T x$$

is also optimal for the new problem :

$$\min_{x\in \textit{conv } X} c^T x$$

[6]It is intuitively more appealing to impose that $t = f(x)$, so we still have "exactly the same" objective function. But since equality constraints are harder to handle, we prefer the inequality $t \geq f(x)$. So instead of minimizing $f(x)$, we minimize *a higher bound* to $f(x)$, wich is equivalent.

This also means that the *optimal values* of the two problems are the same. But because $X \subseteq conv\ X$, some optimal solutions of the new problem won't be in the original set X. So, in order to find the original optimal solutions, once we have solved the convex problem, we should always reject the optimal solutions of the new problems that aren't in X. Mathematically speaking :

$$\{x^*_{original\,problem}\} = \{x^*_{new\,convex\,problem}\} \cap X$$

So, to summarize, we can make *every problem in the world* convex, and we can (not yet, but after finishing this course) solve convex problems with good algorithms! This implies that basically any optimization problem can be solved easily! It seems too good to be true, and it is, since there is a **catch**. Although the definition of complex hull is rather simple, taking the convex hull of a general (that is, a little sophisticated) set is a very difficult operation.

So, in general, this approach is useless. There are specific cases, however, where this can be very useful!

The first, is a special case of linear optimization with **or** constraints. In such a problem, the admissible set is the union of (possibly disjoint) polyhedra. Computing the convex hull of the union of polyhedra can be done very efficiently, if the vertices are known, since the only task is to compute the vertices that will stay extreme in the union. The figure A.3 illustrates this example.

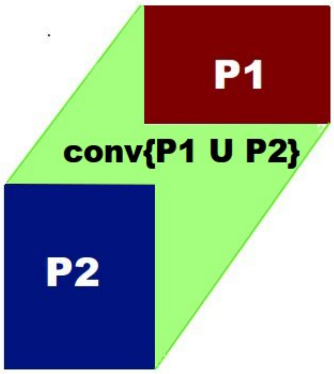


Figure A.3.: Two (simple) disjoint polyhedron and the convex hull (in green) of the union.

An interesting point to be raised here, is that such a problem could also be solved by computing the solution on every polyhedron, then choosing the best one. For small problems, this is of course valid, but for high number of dimensions, the cost of solving the problem on each polyhedron becomes prohibitive, while computing the convex hull remains a relatively cheap operation.

The second case where taking the convex hull is useful is for (some) discrete models. Indeed:

$$\min_{x_i \in \{-1,1\}} c^T x \iff \begin{array}{l} min_x c^T x \\ -1 \leq x_i \leq 1 \end{array}$$

The convex hull of $\{-1, 1\}^n$ is $[-1, 1]^n$, and so both problems have the same solutions. The flow problem with integers (as seen in LINMA1702), which retains integer only solutions with relaxation, is an example of taking the convex hull of discrete problems without changing the nature of solutions.

e. Approximate *any* convex problem by a linear problem

It is possible to approximate convex problems by linear ones. Since the objective can always be converted to a linear function by adding a variable, the only work to be done concerns the constraints. Basically, the idea is to approximate the (closed[7]) set X by a finite intersection of half-spaces (thus, linear constraints).

The way to do this is to use **projections** of points on the convex space. The projection of a point u on a set X is defined as the point $u_p \in X$ that minimizes the distance between itself and u.

Theorem A.2. ***Uniqueness and existence of projection.*** *Let X be a* ***closed, non-empty, convex*** *set in $\mathbb{R}^n$, the projection of any exterior point on X exists and is unique.*

It is easy to see that the closeness and non-emptiness guarantee the existence of such projection. The unicity, however, is ensured by the convexity of the set. As an example, it is easy to see that a non-convex set such as the unit circle as an infinite number of projections of the origin.

With this concept of projection, we can introduce the **separation property**: for every exterior point u of a convex closed set X, there exists a plane that *separates* u from X, that is, such that every element of X is on *one side* of the plane, while u is on the *other side*. This results directly from the uniqueness and existence of the projection of u on X, although this was not demonstrated in class. Intuitively, this separation plane can be built perpendicular to the segment joining u and its projection, without intersecting the convex plane.

So, to approximate a convex set by a linear one, the following method should be applied: for every point that should not be in X, create a separation plane. Such plane defines a half-space containing all of X. The intersection of all the half-spaces obtained this way forms a polyhedron containing X.

Moreover, it is interesting to note that an infinite number of points will create the convex set itself! This yields a new definition for a convex set: a convex set can always be written as the infinite intersection of half-spaces. This definition also proves immediately that every polyhedra is convex.

[7] Why not open? Because an open set corresponds to strict inequalities, which cannot and will not be treated with the common optimization tools (see "A note on $\neq$", page 6)

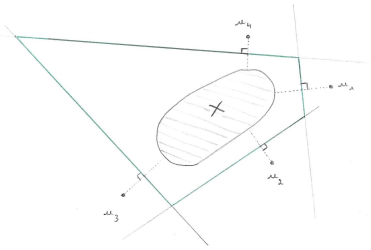


Figure A.4.: Approximation of a non-linear problem by a linear one.

3. Modelling Tricks

After studying the standard form of some optimization problems, we will see some modelling tricks which permit to transform a non-linear or non-convex problem into a linear or convex optimization problem.

a. Monotonicity

Definition A.7. *A monotonic function over an interval is a function that is either increasing or decreasing over this interval.*

The transformation that is described in the following lines uses monotonic functions to turn an optimization problem into a simpler one. These operations do not change the problem but can change the solution and the optimal value of the objective of the problem. Let's see some examples.

Example A.2. $\min\ \|x\|_2$ *with* $x \in X$ *is equivalent to* $\min\ \|x\|_2^2$ *with* $x \in X$*. In fact,* $z \to z^2$ *is a monotonic function (increasing in this case) over* $\mathbb{R}$*. By solving this problem, we will find the same optimal solution than the first model but the value of the objective function will be different.*

Example A.3. *These functions are monotonic and can be used like in the previous example to simplify the problem:*

- $z \to e^z$
- $z \to \log(z),\ (z > 0)$
- $z \to -\frac{1}{z},\ (z > 0)$ *for example, we can change* $\min\ \frac{1}{\|x\|_2}$ *to* $\min\ -\|x\|_2$

We not only use this trick to modify the objective function, but also some constraints. For example:

$$f(x) \leq b \Leftrightarrow e^{f(x)} \leq e^b$$

Example A.4. ***Advertisement for bank account:*** *This example allows us to look at the effects of monotonicity in real life optimization problems. We put money into an account where we can't take our money back sooner than 5 years after. The bank guarantees that we have a high percentage when we take back our money after 5 years and assures the three rates over 3 years given on Figure A.5.*
What is the worst global rate compatible with this? Therefore, we are looking for the cumulative effect of the 5 rates (given over a year).

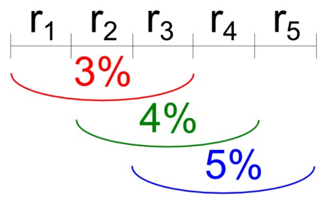


Figure A.5.: Rates

To solve this problem, we introduce the variables r_i corresponding to the rate per year where $i = 1, ..., 5$. We don't want a rate which is negative or null so we have the following optimization problem:

$$\min_{r_i > 0} r_1 r_2 r_3 r_4 r_5$$

under the constraints:

$$r_1 r_2 r_3 \geq 1.03$$

$$r_2 r_3 r_4 \geq 1.04$$

$$r_3 r_4 r_5 \geq 1.05$$

This problem is not linear. We can transform it into a linear problem if we use the logarithm function. Indeed, let $y_i = \log(r_i)\ \forall i = 1, ..., 5$. We can do this because the logarithm is a monotonic increasing function. The problem can be written as:

$$\min\ y_1 + y_2 + y_3 + y_4 + y_5$$

under the constraints:

$$y_1 + y_2 + y_3 \geq \log(1.03)$$

$$y_2 + y_3 + y_4 \geq \log(1.04)$$

$$y_3 + y_4 + y_5 \geq \log(1.05)$$

This problem is now linear (and thus convex).

***Remark**: If we don't make the change of variable $y_i = \log(r_i)\ \forall i = 1, ..., 5$, the variables r_i appear in* log *so the problem isn't linear.*

b. Change of variables

We use change of variables in order to transform the problem into a linear or convex problem. For example, if every variable appears in a logarithm, then we can use the change of variable to remove the logarithm.
Remark: every appearance of the variables needs to match the change of variables!

For example, signomials[8] can be converted into convex functions thanks of this trick. Let's consider the signomial $\frac{x_1 x_2^2}{x_3^{\frac{1}{2}}}$. Let $x_i = e^{y_i}$, we obtain the following expression by change

[8] A signomial is an algebraic function of one or more variables of the form:
$f(x_1, x_2, ...x_n) = \sum_i \left(c_i \prod_j x_j^{a_{ij}}\right)$

of variable $e^{y_1+2y_2-\frac{1}{2}y_3}$. Moreover, the exponential of a linear function is convex and the change of variables conserves the convexity (see later Should add correct reference).

c. Misleading/Deceptive appearances

In this part, we are looking for a polynomial $p(x),\ x \in \mathbb{R}$ of degree D which fulfils some characteristic (see Figure A.6):

- For $x \in [0; f_1]$, $p(x) \geq 3$
- For $x \in [f_2; f_3]$, $p(x) \leq 0.5$

Let $p(x) = \sum_{i=0}^{D} a_i x^i$. On the Figure A.6, we observe that the polynomial will not be linear or convex. We then have the impression that the optimization problem is not linear. But it is not the case, given that the variables are not the x_i but the a_i. Furthermore, the constraints (2 examples of constraints are given below) are linear.

$$\sum_{i=0}^{D} a_i 50^i \geq 3 \Leftrightarrow p(50) \geq 3$$

$$\sum_{i=0}^{D} a_i 100^i = 1 \Leftrightarrow p(100) = 1$$

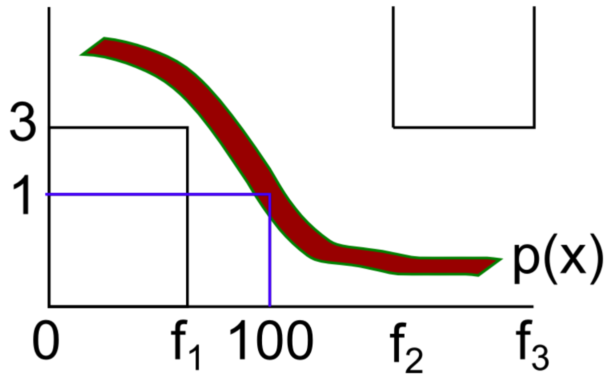


Figure A.6.: Polynomial $p(x)$

d. Flexibility

The flexibility of an optimization problem is its capacity to be solved in different ways depending on which criterion we really want to optimize. Let us consider an approximation problem. We have a set of points (x_i, y_i) in $\mathbb{R}^2$, and we want to find the function (which is often a polynomial) which approximates *at best* these points. In other words, we want to find a function f that minimizes the errors ϵ_i at the approximation points:

$$\epsilon_i \quad = \quad |y_i - f(x_i)|$$

The question now is to choose the way to minimize errors ϵ_i. Indeed, we could choose to minimize overall errors with a least square criterion:

$$\min ||\epsilon||_2 \quad = \quad \sqrt{\sum_i \epsilon_i^2}$$

Another choice would be to minimize the sum of the errors:

$$\min ||\epsilon||_1 \quad = \quad \sum_i |\epsilon_i|$$

Finally, a last (often used) way to minimize errors is to minimize the maximum error:

$$\min \ \max_i \epsilon_i$$

All these considerations prove that, depending on the criterion we choose to optimize (i.e. depending on the context), the result could be different. For example, the problem

$$\begin{aligned} \text{max} \quad & \text{safety} \\ \text{cost} \quad & \leq \quad m \end{aligned}$$

will not lead to the same solution than

$$\begin{aligned} \text{min} \quad & cost \\ safety \quad & \geq \quad b \end{aligned}$$

e. Charnes and Cooper

Consider the non-linear problem which is a division of two linear expression:

$$\min \frac{c^T x + d}{f^T x + g}$$

under the constraints:

$$Ax \leq b$$

In this case, taking the logarithm of the objective function will not change anything: we will have the same problem.

We make the hypothesis that $f^T x + g > 0 \ \forall x$ such that $Ax \leq b$. If not, we have the solution $-\infty$ which is not an interesting solution.

To solve this problem and make it linear, we are going to homogenize the objective function. Let $x = \frac{y}{t}$ with $y \in \mathbb{R}^n$ and $t > 0 \in \mathbb{R}$. If we take $t = 1$ then we get back to the original problem. So, one solution in x correspond to several solutions in (y, t) (for example: $(x, 1)$, $(2x, 2)$, ... $(\lambda x, \lambda)$).
We can write the problem as:

$$\min \frac{\frac{c^T y}{t} + d}{\frac{f^T y}{t} + g}$$

with:

$$A\frac{y}{t} \leq b$$

By simplification, we obtain the following problem:

$$\min \frac{c^T y + dt}{f^T y + gt}$$

with:

$$Ay \leq bt$$

Now, we notice that the objective function's numerator and denominator are linear as the constraint. This problem has a property called homogeneity. If you take any solution, you can multiply any component by the same constant and nothing changes. Mathematically: (y, t) solution $\Rightarrow (\lambda y, \lambda t)$ solution $\forall \lambda \neq 0$

We are going to choose solutions satisfying: $f^T y + gt = 1$. We can do this using the property of homogeneity. This step results in selecting one solution among the collection of solutions multiple of each other. The objective function gets simpler and linear and we add one constraint. So, we have the following linear optimization problem:

$$\min c^T y + dt$$

with:

$$Ay - bt \leq 0$$
$$f^T y + gt = 1$$
$$t \geq 0$$

We compute y^* and t^* from this problem and take $x^* = \frac{y^*}{t^*}$ the solution of the original problem.
Remark: If we have $t = 0$ at the optimum then the problem is unbounded and $x^* \to \infty$ (see Example A.5).

> **Example A.5.**
>
> $$\min \frac{1}{x}$$
>
> *with:*
>
> $$x \geq 1$$
>
> *This problem have an optimal value of 0 so the solution $x^* \to +\infty$.*

4. Convex Optimization: Theorems and properties

a. Convex sets

i. Definition and examples

Definition A.8. *A set X is convex if and only if*

$$x,\ y \in X \Rightarrow \lambda x + (1-\lambda)y \in X \quad \forall\, 0 \leq \lambda \leq 1$$

Thus, a set is convex if and only if it contains all the segments joining any pair of its points.

Example A.6. *Here are several examples:*

- $\mathbb{R}^n$, $\mathbb{R}^n_+$, $\emptyset$
- *Hyperplans* ($\{x|\ b^T x = \beta\}$)
- *Open or closed half-spaces* ($\{x|\ b^T x < \beta\}$ *and* $\{x|\ b^T x \leq \beta\}$)
- *Open and closed balls* ($\{x|\ \|x - a\| < r\}$ *and* $\{x|\ \|x - a\| \leq r\}$)

ii. Properties

Property A.1. *Given a collection of convex sets* $\{C_i\}_{i \in I} \subseteq \mathbb{R}^n$ *(I can be arbitrary), then* $\bigcap\limits_{i \in I} C_i$ *is convex too.*

It follows that polyhedrons are convex because they are intersection of half-spaces.

Property A.2. *Given a collection of convex sets* $C_1,\ C_2,\ C_3,\ ....\ C_n$*, their cartesian product* $C_1 \times C_2 \times C_3 \times ... \times C_n$ *is convex too.*

Property A.3. *If* $X \subseteq \mathbb{R}^n$ *and* $X \subseteq \mathbb{R}^n$ *are convex then the Minkowski sum of* X *and* Y*,* $X + Y = \{x + y \mid x \in X \text{ and } y \in Y\}$ *is convex too.*

Remark: The union of convex sets is not always convex!

b. Convex functions

i. Definition and examples

Definition A.9. *A function f with domain D is a convex function of and only if*

$$D \text{ is convex and}$$
$$x, y\ \in D \Rightarrow f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y) \quad \forall\, 0 \leq \lambda \leq 1$$

Example A.7. *Here are several examples:*

- *Linear and affine functions are convex* ($x \to \alpha x$, $x \to b^T x$ *and* $x \to b^T x + \alpha$)
- *The norm function and the square of the norm function are convex functions* ($x \to \|x\|$ *and* $x \to \|x\|^2$)
- *Quadratic forms* ($x \to x^T Q x$) *are convex functions when the matrix* $Q \in \mathbb{R}^{n \times n}$ *is semi positive definite*
- *The functions* $x \to e^x$, $x \to log(x)$ *and* $x \to |x|^p$ $(1 \leq p)$ *are convex*

Definition A.10. *A function* f *is concave* $\Leftrightarrow -f$ *is convex.*

Remark: Linear and affine functions are convex and concave.

ii. Properties

To know whether a function is convex or not, we have to transform it into its epigraph and check if it is convex or not. But there are some useful properties of convex functions that we can use to spare time.

Property A.4. *If f is a convex function and $c \in \mathbb{R}_0^+$, then cf is convex.*

Property A.5. *If f and g are convex functions, then $f+g$ is convex.*

Property A.6. *Given a collection of convex functions $\{f_i\}_{i\in I} : \mathbb{R}^n \to \mathbb{R}$, $\sup\limits_{i\in I} f_i$ is convex too.*
With $\left[\sup\limits_{i\in I} f_i\right](x) = \sup\limits_{i\in I} f_i(x)$

Property A.7. *Given $f(x,s)$ (with $x \in \mathbb{R}^n$ and $s \in \mathbb{R}$, a parameter) such that $x \to f(x,s)$ is convex for any s,*

$$\int_{s\in S} f(x,s)ds \ \ \textit{is} \ \ \textit{convex}$$

c. Properties of convex functions

i. Convexity and differential calculus

Property A.8. *Let f be a differentiable function of which the domain D is open. f is convex if and only if D is convex and*

$$\forall x, y \ \in D, \ f(y) \geq f(x) + \nabla f(x)^T (y-x)$$

This property means that, if f is convex, it will be above all its Taylor approximations of first order. This signifies that at any point, the tangent of the function is under the function. This is useful to make a piecewise approximation of f by linear functions (an example of such an approximation is shown on Figure A.7). In order to obtain such an approximation, we have to choose n points at which we calculate the tangent of the function f and then take the maximum of the n tangents (in the example, $n = 3$).
Besides, the value of the approximation is lower than the real value on any point. It allows us to obtain lower bounds.

Property A.9. *Let f be a twice differentiable function of which the domain D is open. f is convex if and only if D is convex and*

$$\forall x \in D, \ \nabla^2 f(x) \geq 0$$

ii. Convexity and linear transformations

Linear transformations preserve convexity. Indeed,

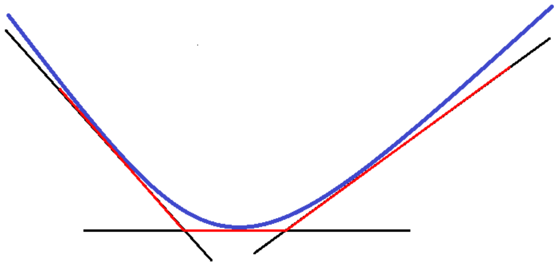


Figure A.7.: Illustration of the piecewise linear approximation of a convex function

Property A.10. *If $S \subseteq \mathbb{R}^n$ is convex and $\Phi : \mathbb{R}^n \to \mathbb{R}^m : x \to Ax + b$ is a linear transformation, then the image of S by Φ,*

$$\Phi(S) = \{\Phi(x) | \; x \in S\}$$

is convex too.

Property A.11. *If $\Phi : \mathbb{R}^n \to \mathbb{R}^m : x \to Ax + b$ is a linear transformation and $f : x \to f(x)$ is a convex function, then the composition*

$$f \circ \Phi = f(\Phi(x)) = f(Ax + b)$$

is convex too.

Property A.12. *If $S \subseteq \mathbb{R}^n$ is convex and $\Theta : \mathbb{R}^m \to \mathbb{R}^n : x \to ax + b$ is a linear transformation, then the image of S by the inverse of Θ,*

$$\Theta^{-1}(S) = \{x| \; \Theta(x) \in S\}$$

is convex too.

Property A.13. *Given a convex and affine transformation $x \mapsto Ax + b$, the composition $x \mapsto f(Ax + b)$ is also convex.*

> **Example A.8.** e^{2x-y+z} *is convex because the exponential is convex and $2x - y + z$ is a linear transformation of x, y and z.*

> **Example A.9. *Convex functions***
> *Any norm $x \mapsto ||x||$ is convex, thus the distance $||x - y||$ between two points x and y is convex because $x - y$ is a linear transformation.*
> *The maximum distance between a set S and a point x is a convex function. Indeed, taking the maximum between a point and a set requires to take the maximum of all the distances between the point and any point in the set (distance between two points is a convex function): $f_{S,max} = \max_{s \in S}\{||x - s||\}$*

iii. Partial minimization

Property A.14. *(Partial minimization) If the function $f : (x, y) \mapsto f(x, y)$ is convex, then $f_x(y) = \inf_x f(x, y)$ is convex.*

Example A.10. *If a set S is convex, then the minimum distance function between a point x and the set S is convex. Indeed, one can write the function as follow:*

$$f(x, s) = ||x - s||$$

Since this is a norm, f is convex. Since the restriction of a convex function stays convex as long as the feasible region stays convex and S is a convex set, property A.14 gives that:

$$f_S(x) = \inf_S f(x, s)$$

is a convex function.

Remark: Property A.14 is a one side property. A counter-example for the reverse side is given by:

$$f_x(y) + \sqrt{||x||}$$

iv. Extended real valued functions

Most of theorems to prove the convexity of a function require the convexity of the domain. However, it is possible to extend a function to tackle this problem.

Example A.11. *Let's take the function $f : \mathbb{R}_+ \mapsto \mathbb{R} : x \mapsto \frac{1}{x}$ and extend it such that its domain becomes the whole real line. One consider:*

$$f_e : \mathbb{R} \mapsto \mathbb{R} \cup \{+\infty\} : x \mapsto \begin{cases} \frac{1}{x} \text{ if } x > 0 \\ +\infty \text{ elsewhere} \end{cases}$$

One can see that the extended function is convex over the whole real line. The epigraph definition still holds since there isn't any point above $+\infty$.

v. Composition and product

Property A.15. *If g is a convex function and f is a convex, increasing and one-dimensional function then the composition function $h \circ g : x \mapsto h(g(x))$ is also convex.*

Proof Let's prove this proposition for a simple case. We assume that f and g are both one-dimensional functions and that $f, g \in \mathcal{C}^2$. The general case requires a more difficult proof.
Since f and g are 2 times differentiable, one has:

$$[h(g(x))]'' = [h'(g(x))g'(x)]' = \underbrace{h''(x)(g'(x))^2}_{\text{A}} + \underbrace{h'(g(x))g''(x)}_{\text{B}}$$

Since h is convex, its second derivative is positive and given that a square is positive, one has that A is positive. Furthermore, since g is also convex and h is increasing, one also has that B is positive. One conclude that the second derivative of $h \circ g$ is positive and thus, the function $h \circ g$ is convex.

QED

Remark: Sometimes we need to square a value but also to keep convexity (for example, we don't care about negative deviations on a budget) Should be better expressed. However the traditional square function is not convex on the real line. Let's introduce a restricted square function as follows:

$$f : x \mapsto (x_+)^2 = (\frac{x + |x|}{2})^2$$

We easily see (Figure A.8) that this restricted square function is convex.

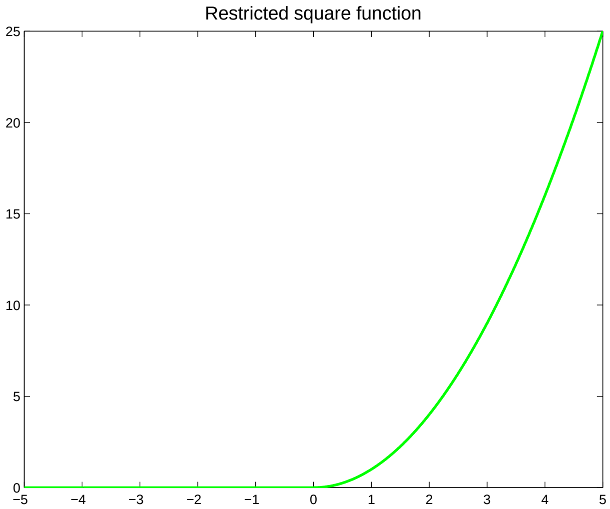


Figure A.8.: Restricted square function

> **Example A.12.** *The function* $\left[\log(x+y)_+\right]^2$ *is convex. Indeed, the restricted square and* $-\log$ *are convex functions. Therefore, their composition is convex. Since* $x + y$ *is a linear transformation, it preserves convexity and the whole function is convex.*

Property A.16. *If f and g are both convex, positive and increasing then their product is convex.*

Proof Again, one proves it in the simple differentiable, one-dimensional case. One has:

$$(fg)'' = [f'g + fg']' = f''g + 2f'g' + fg''$$

The result follows immediately since by assumptions one has $f, g, f', g', f'', g'' \geq 0$.

QED

Remark: The previous proof tends to indicate variants of Property A.16. One can see that if f and g are both concave, decreasing and negative then the proposition still holds.

d. Advantage of convex problems

Property A.17. *Let's recall that* $\min_{x \in X} f(x)$ *is convex if* f *is convex,* X *is convex and we are looking for a minima. We study the properties of a convex problem:*

MODEL	*METHODS*
- *Local minima are also global* - *The set of optimal solutions is convex* - *Using duality we can get guarantees*	- *Methods which only work on convex problems: first order, second order...*

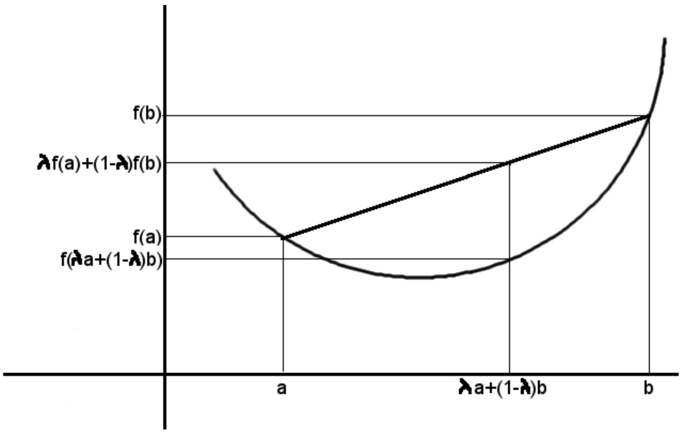
e. Variants of convex functions

It's a difficult thing to know if a problem has a unique solution. There is one class of problems with only one solution: minimization of strictly convex functions.

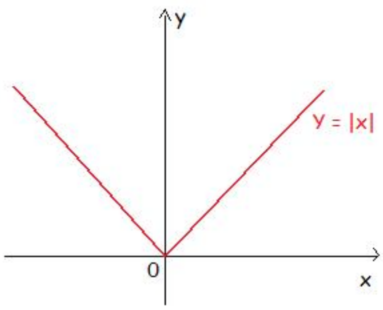
Definition A.11. *Strict convexity: f is strictly convex if and only if*

- *The domain is convex*
- $f(\lambda x + (1-\lambda)y) < \lambda f(x) + (1-\lambda)f(y)$ $\forall x, y \in \mathrm{Dom}$, $\forall \lambda \in]0, 1[$

Example A.13. *On the following graph, we can observe that the function is strictly convex.*



Example A.14. *The absolute function is not strictly convex. In fact, when we have a flat part in the graph, it can not be strictly convex.*



Example A.15. *The function $x \mapsto ||x||_2 = \sqrt{\sum_i x_i^2}$ is convex but not strictly convex.*

Property A.18. *If we have* $\min f(x)_{x\in X}$ *with* f *strictly convex and* X *a convex set then the problem admit at most one solution.*

Property A.19. *If* $f \in C^2$ *and* $\nabla^2 f(x) > 0$ *then* f *is strictly convex. (*$\lambda_i > 0\ \forall i$*)*

Property A.20. *If* f *is convex, then* $f + ||x||_2^2$ *is strictly convex.*

Proof Assume $f \in C_2$
then

$$\nabla^2(f + ||x||^2) = \nabla^2 f + \nabla^2 ||x||_2^2 = \nabla^2 f + 2I$$

where $||x||_2^2 = \sum_i x_i^2$ and $\lambda_i \geq 0$.

QED

Property A.21.

$$(1)\ \lambda \textit{ is an eigenvalue of } M$$

$$\Leftrightarrow$$

$$(2)\ \lambda + \Delta \textit{ is an eigenvalue of } M + \Delta I$$

for any $\Delta \in \mathbb{R}$ *and* $M \in \mathbb{R}^{n\times n}$ *symmetric.*

Proof

$$(1)\ \exists v\ \ Mv = \lambda v$$
$$(2)\ \exists v\ \ (M + \Delta I)v = (\lambda + \Delta)v$$

QED

Remark: We can have the same propositions and the same proof while adding $\mu > 0$ anywhere: $f + \mu ||x||^2$. It is a regularization to make it strictly convex.
Remark: There are functions that have no derivative and are strictly convex.

B. First-order methods

C. Conic modeling and duality

D. Interior-point methods

Part II.

Labs

Not yet controlled

1. Introduction : AMPL

AMPL (which stands for "A mathematical programming language") is an algebraic modelling language which enables the solving of high complexity problems for large (or small) scale models and will be used throughout this course. Knowing or identifying what type of problem we're dealing with is important and allows us to decide which will be the most adequate method to solve it. Certain methods work well with certain types of models, some are better than others, etc.

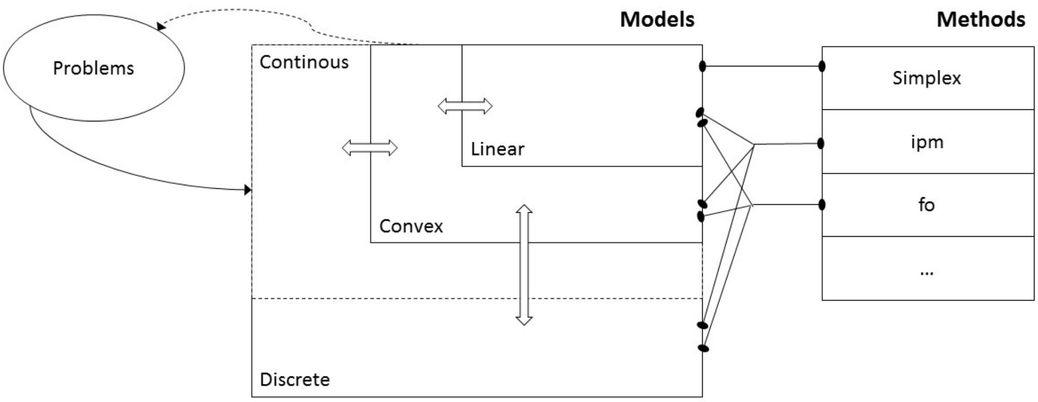


Figure D.1.: Visualization of the relation between models and methods

AMPL works by first reading a text file which contains all the useful information of the model, this file usually ends in ".mod", it then parses it and tries to solve the problem. Parameters, variables, objective function and constraints are all defined in this file. The solving part is done by communicating with a solver, AMPL gives it all the information, the solver then sends back the solution. Consider the following optimization problem :

$$\min c^T x$$
$$Ay \leq b$$
$$l \leq y \leq u$$

For this problem AMPL will give the solver the usefull values (c,A,b,l,u,...) it neads to solve the problem. There exists quite a few of these solvers, some mode adapted to certain model types (linear, convex,...). We have for example :

- minos (basic solver for linear and nonlinear problems)
- cplex (can be used for linear, convex, mixted integer models)
- gurobi (very similar to cplex)
- knitro (good for nonlinear models)
- $\cdots$

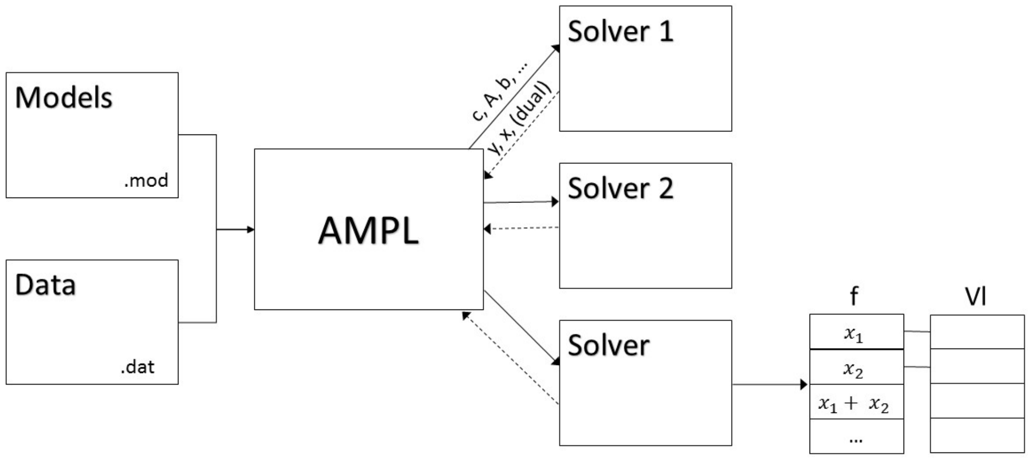


Figure D.2.

AMPL can also work with an additionnal data file (".dat") which is used when parameters are left in the model file. This allows us to avoid changing the entire file when looking at different values of parameters, and only having to change them once in the data file.

A few examples and the basic syntax for AMPL can be found in the Tutorials Dropbox, given on Moodle.

Everything in AMPL has a name, whether it be variables, constants, or even constraints (which represent a dual variable) and each command and declaration ends with a semi-colon (";"). Certain commands are worth being reminded here, for example :

- Changing solvers : **option solver ... ;**
- Displaying dual variable : **display Protein;**
- Displaying variable : **display Protein.body;**
- Reset the whole model : **reset;**

- Chosing a model file : **data data1.dat;**
- Chosing a data file : **model data1.mod;**
- Solving the chosen model : **solve;**

Part III.

Exercices