GR multipole extraction

Kernels

In[•]:=
$$\mathbb{E}[i\_, j\_, l\_] := \frac{k_i^2\, k_j^2}{k_l^4}\left(3 + 2 * \text{Cos}[\text{Theta}[i, j, l]]\left(\frac{k_i}{k_j} + \frac{k_j}{k_i}\right) + (\text{Cos}[\text{Theta}[i, j, l]])^2\right);$$

$$\text{F}[i\_, j\_, l\_] := \frac{10}{7} + \text{Cos}[\text{Theta}[i, j, l]]\left(\frac{k_i}{k_j} + \frac{k_j}{k_i}\right) + \left(1 - \frac{3}{7}\right)(\text{Cos}[\text{Theta}[i, j, l]])^2;$$

$$\text{G}[i\_, j\_, l\_] := \frac{6}{7} + \text{Cos}[\text{Theta}[i, j, l]]\left(\frac{k_i}{k_j} + \frac{k_j}{k_i}\right) + \left(2 - \frac{6}{7}\right)(\text{Cos}[\text{Theta}[i, j, l]])^2$$

In[•]:=
$$\text{KN1}[i\_] := \text{b1} + \text{f}\, \mu_i^2$$

In[•]:=
$$\text{KGR1}[i\_] := i\, \mu_i \frac{\Upsilon_1}{k_i} + \frac{\Upsilon_2}{k_i^2}$$

$$\text{KN2}[i\_, j\_, l\_] := \text{b1}\, \text{F}[i, j, l] + \text{b2} + \text{f}\, \text{G}[i, j, l]\, \mu_l^2 - \frac{4}{7}(\text{b1} - 1) * \left((\text{Cos}[\text{Theta}[i, j, l]])^2 - \frac{1}{3}\right) + \text{f}^2 \frac{\mu_i\, \mu_j}{k_i\, k_j}(\mu_i\, k_i + \mu_j\, k_j)^2 + \text{b1}\frac{\text{f}}{k_i\, k_j}\left((\mu_i^2 + \mu_j^2)\, k_i\, k_j + \mu_i\, \mu_j\, (k_i^2 + k_j^2)\right)$$

In[•]:=
$$\begin{aligned}\text{KGR2}[i\_, j\_, l\_] := \frac{1}{k_i^2\, k_j^2}\Bigg(&\beta_1 + \mathbb{E}[i, j, l]\, \beta_2 + i\left((\mu_i\, k_i + \mu_j\, k_j)\, \beta_3 + \mu_l\, k_l\, (\beta_4 + \mathbb{E}[i, j, l]\, \beta_5)\right) + \\ &\frac{k_i^2\, k_j^2}{k_l^2}(\text{F}[i, j, l]\, \beta_6 + \text{G}[i, j, l]\, \beta_7) + (\mu_i\, k_i\, \mu_j\, k_j)\, \beta_8 + \\ &\mu_l^2\, k_l^2\, (\beta_9 + \mathbb{E}[i, j, l]\, \beta_{10}) + (k_i * k_j * \text{Cos}[\text{Theta}[i, j, l]])\, \beta_{11} + \\ &(k_i^2 + k_j^2)\, \beta_{12} + (\mu_i^2\, k_i^2 + \mu_j^2\, k_j^2)\, \beta_{13} + i\Big((\mu_i\, k_i^3 + \mu_j\, k_j^3)\, \beta_{14} + \\ &\quad (\mu_i\, k_i + \mu_j\, k_j)\, (k_i * k_j * \text{Cos}[\text{Theta}[i, j, l]])\, \beta_{15} + k_i\, k_j\, (\mu_i\, k_j + \mu_j\, k_i)\, \beta_{16} + \\ &\quad (\mu_i^3\, k_i^3 + \mu_j^3\, k_j^3)\, \beta_{17} + \mu_i\, \mu_j\, k_i\, k_j\, (\mu_i\, k_i + \mu_j\, k_j)\, \beta_{18} + \mu_l \frac{k_i^2\, k_j^2}{k_l}\, \text{G}[i, j, l]\, \beta_{19}\Big)\Bigg)\end{aligned}$$

In[•]:=
$$\text{BSperm}[i\_, j\_, l\_, gr\_] := (\text{KN1}[i] + gr * \text{KGR1}[i])\, (\text{KN1}[j] + gr * \text{KGR1}[j])\, (\text{KN2}[i, j, l] + gr * \text{KGR2}[i, j, l])\, \text{Pm}[k_i]\, \text{Pm}[k_j]$$

In[•]:=
$$\text{test123} = \text{KN2}[1, 2, 3] * \text{KGR2}[1, 2, 3] \text{ //. } \left\{\mu_3 \to \frac{-k_1}{k_3}\mu_1 - \frac{k_2}{k_3}\mu_2\right\};$$

In[∘]:=
```
perm123 = ExpandAll[BSperm[1, 2, 3, 1]] //. {μ3 → -k1/k3 μ1 - k2/k3 μ2}
```

$$\text{perm123} = \text{ExpandAll}[\text{BSperm}[1,2,3,1]] \;//. \left\{\mu_3 \to \frac{-k_1}{k_3}\mu_1 - \frac{k_2}{k_3}\mu_2\right\}$$

Out[∘]=

$$-\frac{2}{21}\,b1^2\,\text{Pm}[k_1]\,\text{Pm}[k_2] + \frac{32}{21}\,b1^3\,\text{Pm}[k_1]\,\text{Pm}[k_2] + \cdots 1151 \cdots + \frac{2\,f^2\,\text{Cos}[\text{Theta}[1,2,3]]\,\text{Pm}[k_1]\,\text{Pm}[k_2]\,k_2\,\beta_{10}\,\mu_1^2\,\mu_2^2\left(-\frac{k_1\mu_1}{k_3}-\frac{k_2\mu_2}{k_3}\right)^2}{k_1\,k_3^2}$$

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In[∘]:=
```
coeffrules = CoefficientRules[test123, {μ1, μ2}];
keys = Keys[coeffrules];
vals = Values[coeffrules];
```

In[∘]:=
```
Module[{list1, values = {}, length = 1, a},
 list1 = vals;
 While[length ≤ Length[list1],
  a = Simplify[vals[[length]]];
  AppendTo[values, a];
  length++
 ];
 result = values;
 values
]
```

In[∘]:=
```
Length[result]
```

Out[∘]= 32

In[∘]:=
```
Length[keys]
```

Out[∘]= 32

In[∘]:= result[[30]] //. {Theta[1, 2, 3] → θ} // Simplify

Out[∘]=

$$
\begin{aligned}
&\frac{1}{294\, k_1^3\, k_2^2\, k_3^6}\Big(294\,\text{Cos}[\theta]^2\, k_1^5\, k_2^2\, \left(2\, f\, \beta_2 + k_3^2\, \left(f\, \beta_6 + f\, \beta_7 + 2\, b1\, \beta_{10}\right)\right) + \\
&\quad 7\,\text{Cos}[\theta]\, k_1^4\, k_2\, \Big(k_2^2\, \Big(3\, f\, \left(89 + 23\,\text{Cos}[2\,\theta]\right)\, \beta_2 + k_3^2\, \Big(12\, f\, \left(11 + 3\,\text{Cos}[2\,\theta]\right)\, \beta_6 + \\
&\qquad\qquad 24\, f\, \left(5 + 2\,\text{Cos}[2\,\theta]\right)\, \beta_7 + \left(4 + 287\, b1 + 84\, b2 + 3\, \left(4 + 11\, b1\right)\, \text{Cos}[2\,\theta]\right)\, \beta_{10}\Big)\Big) + \\
&\qquad 42\, f\, k_3^2\, \left(2\, b1\, \beta_2 + k_3^2\, \left(b1\, \beta_6 + b1\, \beta_7 + \beta_{12}\right)\right)\Big) + \\
&\quad k_1^3\, \Big(2\, k_2^4\, \Big(21\, f\, \left(50 + 33\,\text{Cos}[2\,\theta] + \text{Cos}[4\,\theta]\right)\, \beta_2 + k_3^2\, \Big(3\, f\, \left(173 + 117\,\text{Cos}[2\,\theta] + 4\,\text{Cos}[4\,\theta]\right)\, \beta_6 + \\
&\qquad\qquad 3\, f\, \left(157 + 129\,\text{Cos}[2\,\theta] + 8\,\text{Cos}[4\,\theta]\right)\, \beta_7 + \frac{7}{4}\, \Big(17 + 661\, b1 + 294 \\
&\qquad\qquad\quad b2 + \left(44 + 280\, b1 + 42\, b2\right)\, \text{Cos}[2\,\theta] + 3\, \left(1 + b1\right)\, \text{Cos}[4\,\theta]\Big)\, \beta_{10}\Big)\Big) + \\
&\qquad 294\, b1\, f\, k_3^6\, \beta_{12} + 21\, f\, k_2^2\, k_3^2\, \Big(7\, b1\, \left(7 + \text{Cos}[2\,\theta]\right)\, \beta_2 + k_3^2\, \Big(4\, b1\, \left(6 + \text{Cos}[2\,\theta]\right)\, \beta_6 + \\
&\qquad\qquad 4\, b1\, \left(5 + 2\,\text{Cos}[2\,\theta]\right)\, \beta_7 + 7\, \beta_{11} + 7\,\text{Cos}[2\,\theta]\, \beta_{11} + 20\, \beta_{12} + 8\,\text{Cos}[2\,\theta]\, \beta_{12}\Big)\Big)\Big) + \\
&\quad 294\,\text{Cos}[\theta]\, k_2^3\, k_3^4\, \left(f\, \beta_1 + f\, k_2^2\, \beta_{12} + b1\, k_3^2\, \left(\beta_9 + \beta_{13}\right)\right) + \\
&\quad 7 \\
&\quad\ \text{Cos}[\theta] \\
&\quad\ k_1^2 \\
&\quad\ k_2 \\
&\quad\ \Big(3\, f\, \left(89 + 23\,\text{Cos}[2\,\theta]\right)\, k_2^4\, \beta_2 + \\
&\qquad k_2^2\, k_3^2\, \Big(84\, b1\, f\, \beta_2 + k_2^2\, \Big(12\, f\, \left(11 + 3\,\text{Cos}[2\,\theta]\right)\, \beta_6 + 24\, f\, \left(5 + 2\,\text{Cos}[2\,\theta]\right)\, \beta_7 + \\
&\qquad\qquad \left(4 + 287\, b1 + 84\, b2 + 3\, \left(4 + 11\, b1\right)\, \text{Cos}[2\,\theta]\right)\, \beta_{10}\Big)\Big) + \\
&\qquad 6\, f\, k_3^4\, \left(7\, \beta_1 + k_2^2\, \left(7\, b1\, \beta_6 + 7\, b1\, \beta_7 + 10\, \beta_{11} + 4\,\text{Cos}[2\,\theta]\, \beta_{11} + 14\, \beta_{12}\right)\right) + \\
&\qquad 42\, b1\, k_3^6\, \left(\beta_9 + f\, \beta_{11} + \beta_{13}\right)\Big) + \\
&\quad 14\, k_1\, \Big(21\, b1\, f\, k_3^6\, \beta_1 + 21\,\text{Cos}[\theta]^2\, k_2^6\, \left(2\, f\, \beta_2 + k_3^2\, \left(f\, \beta_6 + f\, \beta_7 + 2\, b1\, \beta_{10}\right)\right) + \\
&\qquad 3\, f\, k_2^4\, k_3^4\, \left(7\,\text{Cos}[\theta]^2\, \beta_{11} + 2\, \left(5 + 2\,\text{Cos}[2\,\theta]\right)\, \beta_{12}\right) + \\
&\qquad k_2^2\, k_3^4\, \Big(6\, f\, \left(5 + 2\,\text{Cos}[2\,\theta]\right)\, \beta_1 + k_3^2\, \Big(\left(1 + 35\, b1 + 21\, b2 + 3\, \left(1 + b1\right)\, \text{Cos}[2\,\theta]\right)\, \beta_9 + \\
&\qquad\qquad 21\, b1\, f\, \beta_{12} + \left(1 + 35\, b1 + 21\, b2 + 3\, \left(1 + b1\right)\, \text{Cos}[2\,\theta]\right)\, \beta_{13}\Big)\Big)\Big)\Big)
\end{aligned}
$$

In[∘]:= BS[gr_] := BSperm[1, 2, 3, gr] + BSperm[1, 3, 2, gr] + BSperm[2, 3, 1, gr]

In[∘]:= BSgr = BS[1](*//.{μ_1→mu1, μ_2→mu2, μ_3→mu3}*) //. {μ_1 → mu1, μ_2 → mu2, μ_3 → mu3};

In[∘]:= BB[a_, b_, c_] := $B_{a,b,c}$

In[∘]:= sum =
Sum[Sum[Sum[BB[a, b, c] * $(\mathbb{i}\,mu1)^a\, (\mathbb{i}\,mu2)^b\, (\mathbb{i}\,mu3)^c$, {a, 0, 8}], {b, 0, 8}], {c, 0, 8}];

In[∘]:= crap = MonomialList[Expand[BSgr - sum], {mu1, mu2, mu3}];

In[∘]:= morecrap = CoefficientRules[crap, {mu1, mu2, mu3}];

In[∘]:= tosolve = Reverse[Values[morecrap]];

(*sum2=Flatten[Table[BB[a,b,c],{a,0,8},{b,0,8}]];*)

In[∘]:= sum2 = List @@ Sum[Sum[Sum[BB[a, b, c], {a, 0, 8}], {b, 0, 8}], {c, 0, 8}];

In[∘]:=
```
Module[{list, list2, values = {}, length = 1, a},
 (*Random list with 100 numbers*)
 list = tosolve;
 list2 = sum2;

 (*A little function were you can create another list of the same size as the
  previous one with functions applied to the values of the previous one*)
 While[length ≤ Length[list],
  a = Reduce[list[[length]] == 0, list2[[length]]];

  AppendTo[values, Last[a]];
  length++
 ];
 result = values;
 (*Just printing the resulting values*)
 values]
```

Out[∘]=

$$\Big\{B_{0,0,0} == \frac{1}{21\,k_1^4\,k_2^4\,k_3^4}\Big(21\,b1^3\,\mathrm{Cos}[\mathrm{Theta}[1,3,2]]\,\mathrm{Pm}[k_1]\,\mathrm{Pm}[k_3]\,k_1^5\,k_2^4\,k_3^3 + \cdots 274 \cdots + 21\,\mathrm{Pm}[k_1]\,\mathrm{Pm}[k_2]\,k_2^2\,k_3^4\,\beta_{12}\,\gamma_2^2\Big),$$

$$B_{0,1,0} == \frac{1}{21\,k_1^4\,k_2^3\,k_3^4}\Big(-21\,b1^2\,\mathrm{Pm}[k_1]\,\mathrm{Pm}[k_3]\,k_1^2\,k_2^4\,k_3^2\,\beta_3 + \cdots 363 \cdots + 21\,\mathrm{Cos}[\mathrm{Theta}[1,3,2]]\,\cdots 5 \cdots\,\beta_{19}\,\gamma_2^2\Big),$$

$$B_{0,2,0} == \frac{\cdots 1 \cdots}{21\,k_1^4\,k_2^2\,k_3^4},\ B_{0,3,0} == \frac{\cdots 1 \cdots}{7\,\cdots 2 \cdots\,k_3^4},\ B_{\cdots 1 \cdots} == \cdots 1 \cdots,$$

$$\cdots 71 \cdots, 0, 0, 0, 0, 0\Big\}$$

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In[∘]:=
```
nonzeroabcoeff = DeleteCases[result, 0];
```

In[∘]:=
```
(*if we want to simplify the Bab coefficients,*)
(*Module[{startlist,values={},length=1,a},
 startlist=nonzeroabcoeff;
 While[length≤Length[startlist],
  a=Simplify[startlist[[length]]];
  AppendTo[values,a];
  length++
 ];
 simpleabcoeff=values;
 values
]*)
```

In[∘]:=
```
(*Export[
 "/home/elinemaaike/Documents/PhD/nonzero_ab_coefficients.txt",simpleabcoeff]*)
```

In[∘]:=
```
(*Module[{list,list2,values={},length=1,a},
 (*Random list with 100 numbers*)
 list=tosolve;
 list2=sum2;

 (*A little function were you can create another list of the same size as the
  previous one with functions applied to the values of the previous one*)
 While[length≤Length[list],
  a=Reduce[list⟦length⟧==0,list2⟦length⟧];

  AppendTo[values,Last[a]];
  length++
 ];
 (*Just printing the resulting values*)
 values;]
(*Export["/home/elinemaaike/Documents/list.txt",values]*)*)
```

Sums for multipole expansion;

In[∘]:=

$$\text{SimplIntPosm}[a\_, b\_, c\_, l\_, m\_] := \sqrt{\frac{(2\,l+1)}{4\,\pi}}$$

$$\text{Sum}\Big[\text{Sum}\Big[\text{Sum}\Big[(-1)^{\left(\frac{11}{2}m+\frac{3}{2}(h+n)\right)}\ \text{Binomial}[b, h] * \text{Binomial}[c, j] * \text{Binomial}[j, n] * \left(\frac{-k_1}{k_3}\right) \wedge (c-j) * \left(\frac{-k_2}{k_3}\right) \wedge j * \text{Cos}[\theta] \wedge (b-h+j-n) * \text{Sin}[\theta] \wedge (h+n) * \text{Boole}[\text{EvenQ}[h+n+m]] * \left(1 / ((2\,i) \wedge (h+n))\right) * \text{Binomial}\big[(h+n), ((h+n)+m)/2\big] * \pi * \sqrt{\frac{(l-m)\,!}{(l+m)\,!}} * 2 \wedge l * \text{Sum}\big[\text{Binomial}[l, g] * \text{Binomial}\big[(l+g-1)/2, l\big] * \left(\text{Factorial}[g] / \text{Factorial}[g-m]\right) * \left(1 + (-1) \wedge ((a+b-h+c-n)+g-m)\right) * \left(\left(\text{Gamma}\big[((a+b-h+c-n)+g-m+1)/2\big]\ \text{Gamma}\big[((h+n)+m+2)/2\big]\right) / \text{Gamma}\big[((a+b+c)+g)/2 + 3/2\big]\right), \{g, m, l\}\big], \{n, 0, j\}\Big], \{j, 0, c\}\Big], \{h, 0, b\}\Big]$$

In[•]:= SimplIntNegm[a_, b_, c_, l_, m_] :=

$$\sqrt{\frac{(2\,l+1)}{4\,\pi}}\,\sqrt{\frac{(l+m)\,!}{(l-m)\,!}}\ \text{Sum}\Big[\text{Sum}\Big[\text{Sum}\Big[(-1)^{m+\frac{3}{2}\text{Abs}[m]+\frac{1}{2}(h+n)}\ \text{Binomial}[b, h] *$$

$$\text{Binomial}[c, j] * \text{Binomial}[j, n] * \left(\frac{-k_1}{k_3}\right)\text{^}(c-j) * \left(\frac{-k_2}{k_3}\right)\text{^}j *$$

```
Cos[θ] ^ (b - h + j - n) * Sin[θ] ^ (h + n) * Boole[EvenQ[h + n + Abs[m]]] *
(1 / ((2 i) ^ (h + n))) * Binomial[(h + n), ((h + n) + Abs[m]) / 2] * π * 2^l *
Sum[Binomial[l, g] * Binomial[(l + g - 1) / 2, l] * Boole[Abs[m] ≤ g] * (Factorial[
      g] / Factorial[g - Abs[m]]) * (1 + (-1) ^ ((a + b - h + c - n) + g - Abs[m])) *
    ((Gamma[((a + b - h + c - n) + g - Abs[m] + 1) / 2] Gamma[((h + n) + Abs[m] + 2) / 2]) /
      Gamma[((a + b + c) + g) / 2 + 3 / 2]),
  {g, Abs[m], l}] , {n, 0, j}], {j, 0, c}], {h, 0, b}]
```

In[•]:= GuessInt[a_, b_, c_, l_, m_] :=

$$\text{If}\big[m<0,\ i^{a+b+c}\ \text{SimplIntNegm}[a, b, c, l, m],\ i^{a+b+c}\ \text{SimplIntPosm}[a, b, c, l, m]\big]$$

```
(*so below, I am calling 'GuessInt' with a,b,0,l,m. I put in 1,1 for l,
m manually, but is there a way to make this module a function of l,
m, or somehow save all lists 'values' for l,
m up to say l=8? it would make plotting different multipoles
  a big faster (but its really not a big deal if not possible)*)
```

In[•]:=
```
bspec[l_, m_] := Module[{list, values = {}, length = 1, a, b, coeff},
  list = nonzeroabcoeff;
  While[length ≤ Length[list],
   a = list[[length, 1]][[2]];
   b = list[[length, 1]][[3]];
   coeff = list[[length, 2]];
   AppendTo[values, GuessInt[a, b, 0, l, m] * coeff];
   length++
  ];
  res = values
 ]
```

In[•]:=
```
bspec[l_, m_] := Module[{list, values = {}, length = 1, a, b, coeff},
  list = nonzeroabcoeff;
  While[length ≤ Length[list],
   a = list[[length, 1]][[2]];
   b = list[[length, 1]][[3]];
   coeff = list[[length, 2]];
   AppendTo[values, testformula[a, b, l, m] * coeff];
   length++
  ];
  res = values
 ]
```

In[ ]:=
```
l0m0 = Plus @@ bspec[0, 0]
```

In[ ]:=
```
l1m1 = Plus @@ bspec[1, 1];
```

```
Module[{list, values = {}, length = 1, a},
 list = nonzeroabcoeff;
 While[length ≤ Length[list],
  a = list⟦length, 1⟧⟦2⟧;
  AppendTo[values, a];
  length++
 ];
 avalues = values;
 (*Export[
  "/home/elinemaaike/Documents/PhD/Plots/Notebooks/avalues.txt", avalues]*)
]
```

Out[ ]=
```
/home/elinemaaike/Documents/PhD/Plots/Notebooks/avalues.txt
```

```
Module[{list, values = {}, length = 1, b},
 list = nonzeroabcoeff;
 While[length ≤ Length[list],
  b = list⟦length, 1⟧⟦3⟧;
  AppendTo[values, b];
  length++
 ];
 bvalues = values;
 (*Export[
  "/home/elinemaaike/Documents/PhD/Plots/Notebooks/bvalues.txt", bvalues]*)
]
```

In[ ]:=
```
Module[{list, values = {}, length = 1, coeff},
 list = nonzeroabcoeff;
 While[length ≤ Length[list],
  coeff = list⟦length, 2⟧;
  AppendTo[values, coeff];
  length++
 ];
 coeffvalues = values;
 (*Export["/home/elinemaaike/Documents/PhD/Plots/Notebooks/coeffvalues.txt",
  coeffvalues]*)
]
```

In[∘]:=
```
save = Riffle[coeffvalues, "<div>"]
```

Out[∘]=

$\left\{\frac{21\ b1^3\ \mathrm{Cos}[\mathrm{Theta}[1,3,2]]\ \mathrm{Pm}[k_1]\ \mathrm{Pm}[k_3]\ k_1^5\ k_2^4\ k_3^3 + \cdots 299 \cdots}{21\ k_1^4\ k_2^4\ k_3^4}, \text{<div>}, \frac{-21\ b1^2\ \mathrm{Pm}[k_1]\ \mathrm{Pm}[k_3]\ k_1^2\ k_2^4\ k_3^2\ \beta_3 + \cdots 373 \cdots}{21\ k_1^4\ k_2^3\ k_3^4}, \text{<div>}, \frac{\cdots 1 \cdots}{21\ k_1^4\ k_2^2\ k_3^4}, \cdots 67 \cdots, \frac{\cdots 1 \cdots \cdots 1 \cdots \cdots 1 \cdots}{k_3^4}, \text{<div>}, \frac{3\ f^2\ \mathrm{Pm}[k_1]\ \mathrm{Pm}[k_3]\ k_1^2\ k_2\ \beta_{17} - f^2\ \mathrm{Pm}[k_1] \cdots 2 \cdots k_2\ \beta_{18}}{k_3^4}, \text{<div>}, -\frac{f^4\ \mathrm{Pm}[k_1]\ \mathrm{Pm}[k_3]\ k_1^2\ k_2^2}{k_3^4}\right\}$

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In[∘]:=
```
Export["/home/elinemaaike/Documents/PhD/Plots/Notebooks/coeffvalues.txt", save]
```

Out[∘]= /home/elinemaaike/Documents/PhD/Plots/Notebooks/coeffvalues.txt