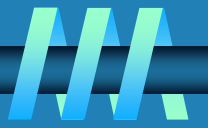
Divide-and-Conquer



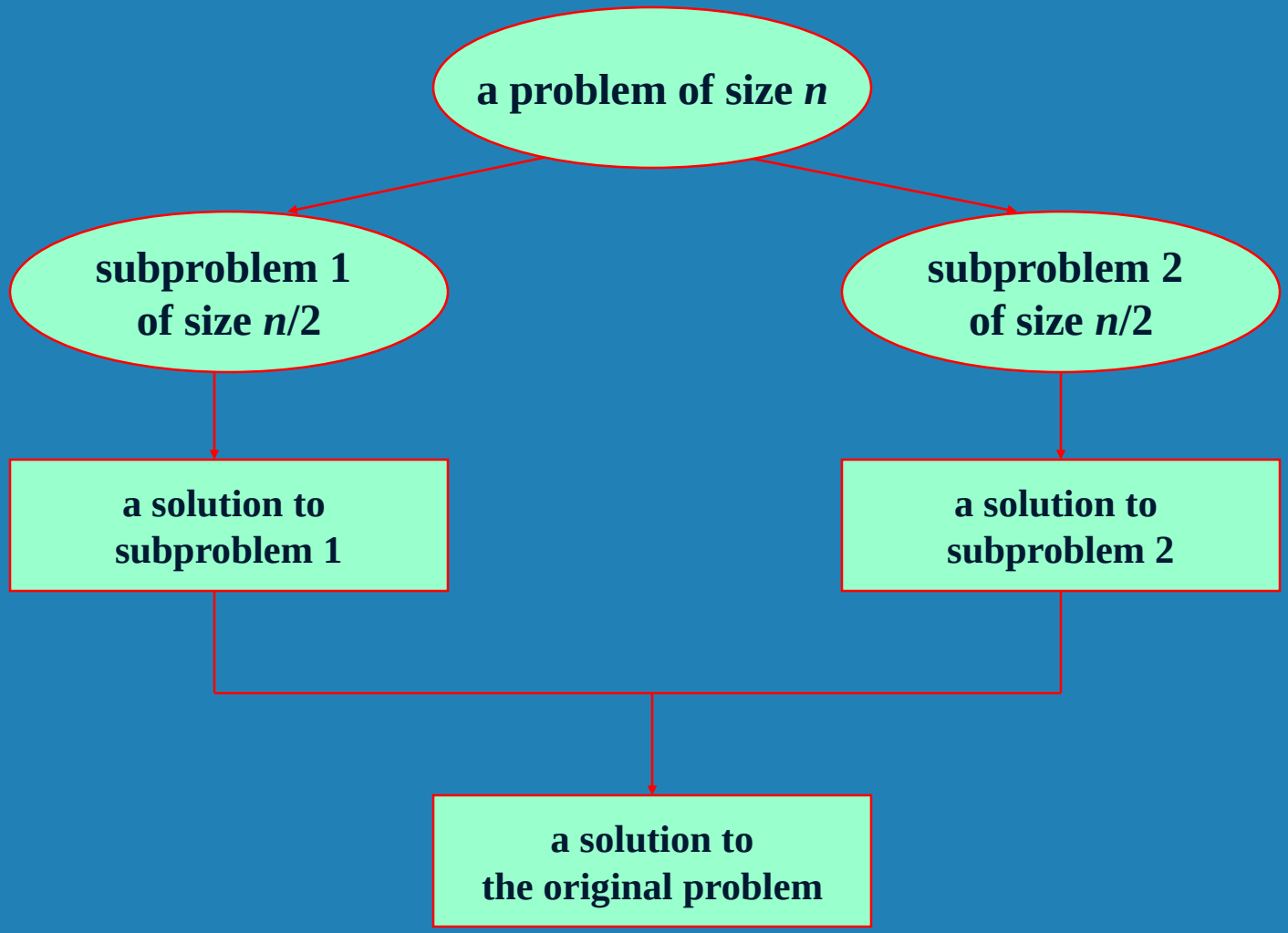
The most-well known algorithm design strategy:

1. **Divide instance of problem into two or more smaller instances**

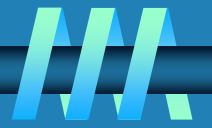
2. **Solve smaller instances recursively**

3. **Obtain solution to original (larger) instance by combining these solutions**

Divide-and-Conquer Technique (cont.)



Divide-and-Conquer Examples



- Sorting: mergesort and quicksort
- Binary tree traversals
- Multiplication of large integers
- Matrix multiplication: Strassen's algorithm
- Closest-pair and convex-hull algorithms

- Binary search: decrease-by-half (or degenerate divide&conq.)

General Divide-and-Conquer Recurrence

$T(n) = aT(n/b) + f(n)$ where $f(n) \in \Theta(n^d)$, $d \geq 0$

Master Theorem: If $a < b^d$, $T(n) \in \Theta(n^d)$

If $a = b^d$, $T(n) \in \Theta(n^d \log n)$

If $a > b^d$, $T(n) \in \Theta(n^{\log_b a})$

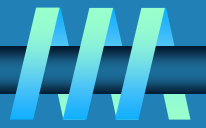
Note: The same results hold with O instead of Θ.

Examples: $T(n) = 4T(n/2) + n \Rightarrow T(n) \in ?$

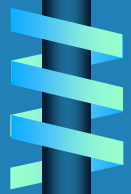
$T(n) = 4T(n/2) + n^2 \Rightarrow T(n) \in ?$

$T(n) = 4T(n/2) + n^3 \Rightarrow T(n) \in ?$

Mergesort



- Split array A[0..*n*-1] in two about equal halves and make copies of each half in arrays B and C
- Sort arrays B and C recursively
- Merge sorted arrays B and C into array A as follows:
  - Repeat the following until no elements remain in one of the arrays:
    - compare the first elements in the remaining unprocessed portions of the arrays
    - copy the smaller of the two into A, while incrementing the index indicating the unprocessed portion of that array
  - Once all elements in one of the arrays are processed, copy the remaining unprocessed elements from the other array into A.



Pseudocode of Mergesort

ALGORITHM *Mergesort*$(A[0..n-1])$

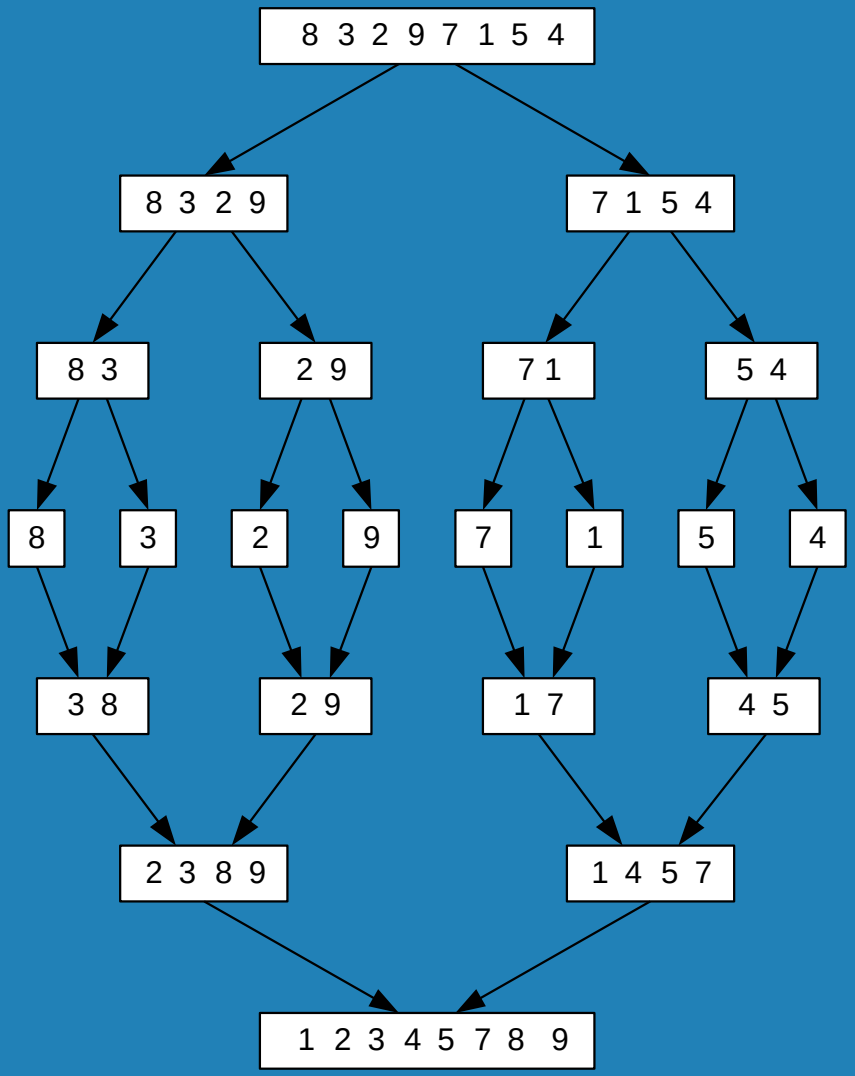
//Sorts array $A[0..n-1]$ by recursive mergesort
//Input: An array $A[0..n-1]$ of orderable elements
//Output: Array $A[0..n-1]$ sorted in nondecreasing order
if $n > 1$
    copy $A[0..\lfloor n/2 \rfloor - 1]$ to $B[0..\lfloor n/2 \rfloor - 1]$
    copy $A[\lfloor n/2 \rfloor..n-1]$ to $C[0..\lceil n/2 \rceil - 1]$
    Mergesort$(B[0..\lfloor n/2 \rfloor - 1])$
    Mergesort$(C[0..\lceil n/2 \rceil - 1])$
    Merge(B, C, A)

Pseudocode of Merge

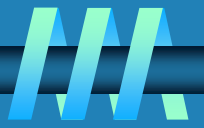
ALGORITHM $Merge(B[0..p-1], C[0..q-1], A[0..p+q-1])$

//Merges two sorted arrays into one sorted array
//Input: Arrays $B[0..p-1]$ and $C[0..q-1]$ both sorted
//Output: Sorted array $A[0..p+q-1]$ of the elements of B and C
$i \leftarrow 0;\ j \leftarrow 0;\ k \leftarrow 0$
while $i < p$ **and** $j < q$ **do**
    if $B[i] \leq C[j]$
        $A[k] \leftarrow B[i];\ i \leftarrow i + 1$
    else $A[k] \leftarrow C[j];\ j \leftarrow j + 1$
    $k \leftarrow k + 1$
if $i = p$
    copy $C[j..q-1]$ to $A[k..p+q-1]$
else copy $B[i..p-1]$ to $A[k..p+q-1]$

Mergesort Example



Analysis of Mergesort



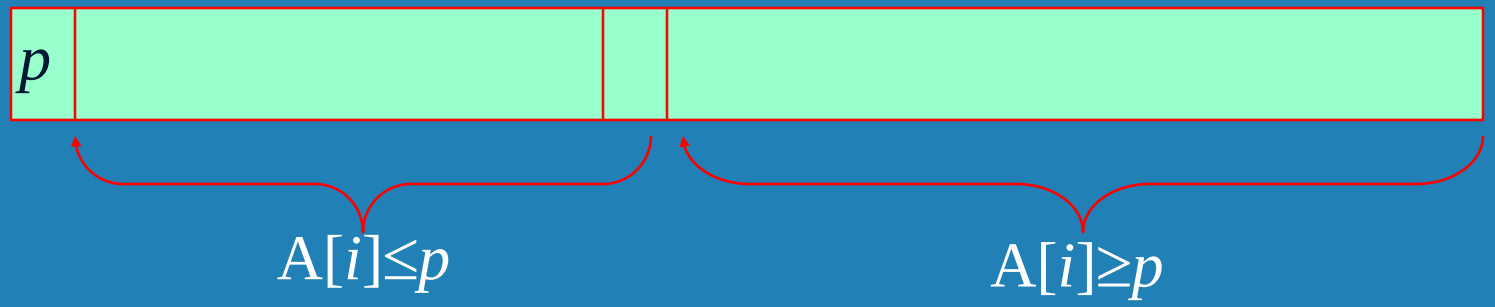
- All cases have same efficiency: $\Theta(n \log n)$
- Number of comparisons in the worst case is close to theoretical minimum for comparison-based sorting:

$$\lceil \log_2 n! \rceil \approx n \log_2 n - 1.44n$$

- Space requirement: $\Theta(n)$ (<u>not</u> in-place)
- Can be implemented without recursion (bottom-up)

Quicksort

- Select a *pivot* (partitioning element) – here, the first element
- Rearrange the list so that all the elements in the first *s* positions are smaller than or equal to the pivot and all the elements in the remaining *n-s* positions are larger than or equal to the pivot (see next slide for an algorithm)

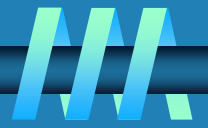


- Exchange the pivot with the last element in the first (i.e., ≤) subarray — the pivot is now in its final position
- Sort the two subarrays recursively

Hoare's Partitioning Algorithm

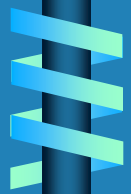
Algorithm *Partition*$(A[l..r])$
//Partitions a subarray by using its first element as a pivot
//Input: A subarray $A[l..r]$ of $A[0..n-1]$, defined by its left and right
// indices l and r $(l < r)$
//Output: A partition of $A[l..r]$, with the split position returned as
// this function's value
$p \leftarrow A[l]$
$i \leftarrow l$; $j \leftarrow r+1$
repeat
  repeat $i \leftarrow i+1$ **until** $A[i] \geq p$
  repeat $j \leftarrow j-1$ **until** $A[j] \cdot p$
  swap$(A[i], A[j])$
until $i \geq j$
swap$(A[i], A[j])$ //undo last swap when $i \geq j$
swap$(A[l], A[j])$
return j

Quicksort Algorithm



ALGORITHM *Quicksort*$(A[l..r])$

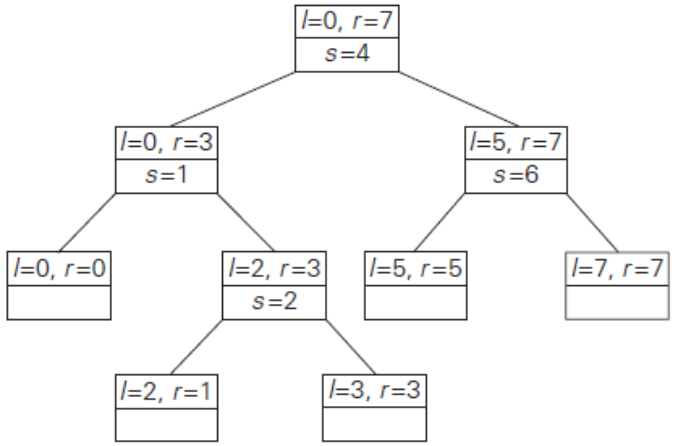
//Sorts a subarray by quicksort
//Input: Subarray of array $A[0..n-1]$, defined by its left and right
// indices l and r
//Output: Subarray $A[l..r]$ sorted in nondecreasing order
if $l < r$
  $s \leftarrow$ *Partition*$(A[l..r])$ //s is a split position
  Quicksort$(A[l..s-1])$
  Quicksort$(A[s+1..r])$



Quicksort Example

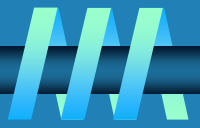
5 3 1 9 8 2 4 7

0	1	2	3	4	5	6	7
	i						*j*
5	3	1	9	8	2	4	7
			i			*j*	
5	3	1	9	8	2	4	7
			i			*j*	
5	3	1	4	8	2	9	7
				i	*j*		
5	3	1	4	8	2	9	7
				i	*j*		
5	3	1	4	2	8	9	7
				j	*i*		
5	3	1	4	2	8	9	7
2	3	1	4	**5**	8	9	7
	i		*j*				
2	3	1	4				
	i	*j*					
2	3	1	4				
	i	*j*					
2	1	3	4				
	j	*i*					
2	1	3	4				
1	**2**	3	4				
1							
			ij				
		3	4				
		j	*i*				
		3	4				
			4				
						i	*j*
					8	9	7
						i	*j*
					8	7	9
						j	*i*
					8	7	9
					7	**8**	9
					7		
							9



(b)

Analysis of Quicksort



- Best case: split in the middle — $\Theta(n \log n)$
- Worst case: sorted array! — $\Theta(n^2)$
- Average case: random arrays — $\Theta(n \log n)$

- Improvements:
  - better pivot selection: median of three partitioning
  - switch to insertion sort on small subfiles
  - elimination of recursion

  These combine to 20-25% improvement

- Considered the method of choice for internal sorting of large files ($n \geq 10000$)

Binary Tree Data Structure

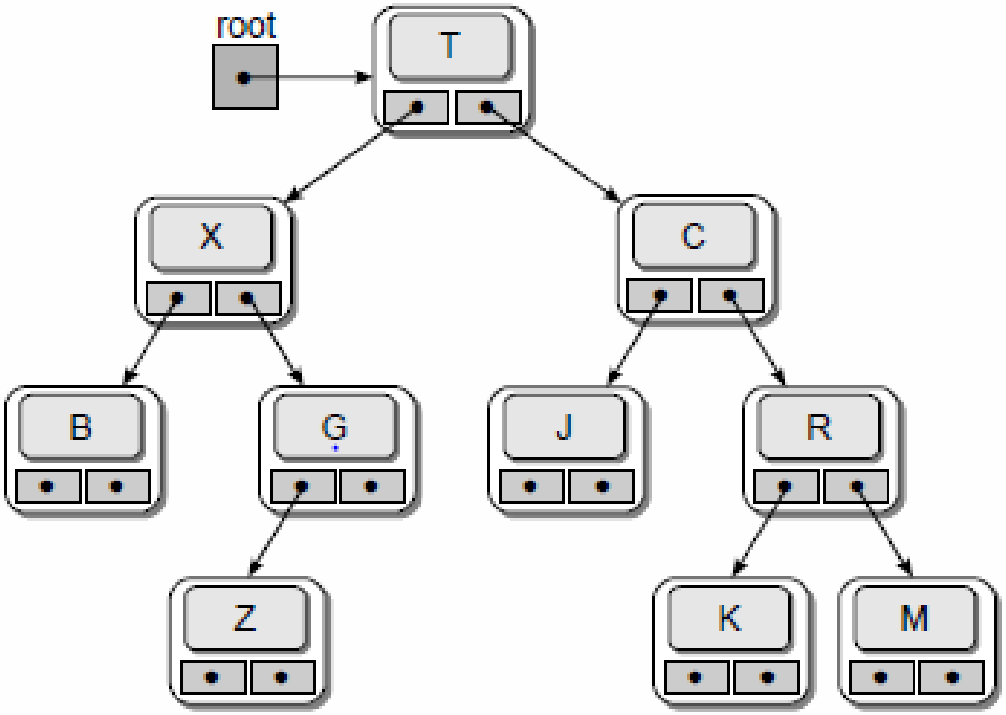
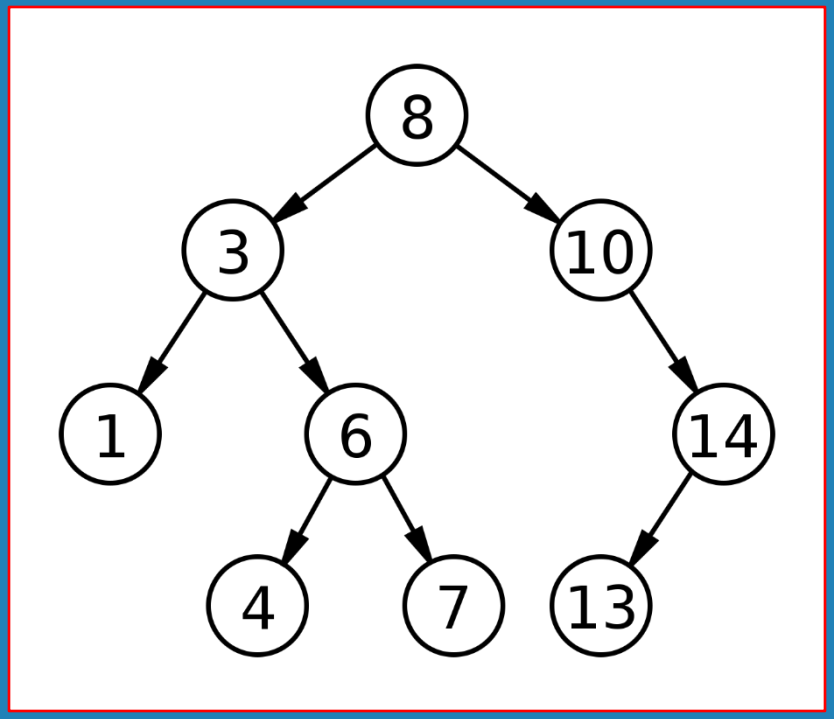


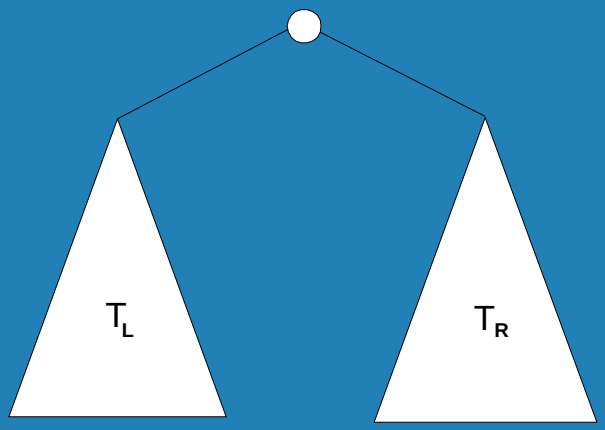
Figure 13.10: The physical implementation of a binary tree.

Binary Search Tree



Binary Tree Algorithms (cont.)

Ex. 1: Computing the height of a binary tree



ALGORITHM *Height*(T)

//Computes recursively the height of a binary tree
//Input: A binary tree T
//Output: The height of T
if $T = \varnothing$ **return** -1
else return $\max\{Height(T_{left}), Height(T_{right})\} + 1$

$h(T) = \max\{h(T_L), h(T_R)\} + 1$ if $T \neq \varnothing$ and $h(\varnothing) = -1$

Efficiency: $\Theta(n)$

How many comparisons ?

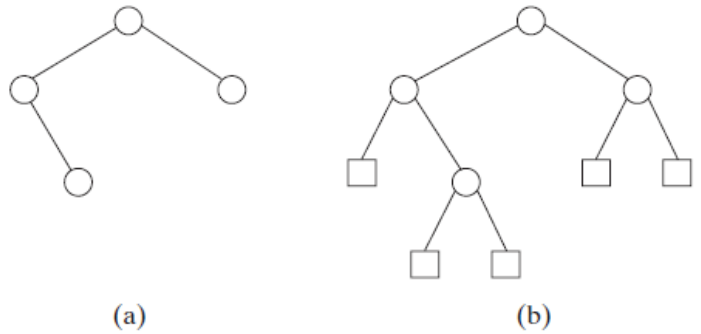


FIGURE 5.5 Binary tree (on the left) and its extension (on the right). Internal nodes are shown as circles; external nodes are shown as squares.

Binary Tree Algorithms

Binary tree is a divide-and-conquer ready structure!

Ex. 2: Classic traversals (preorder, inorder, postorder)

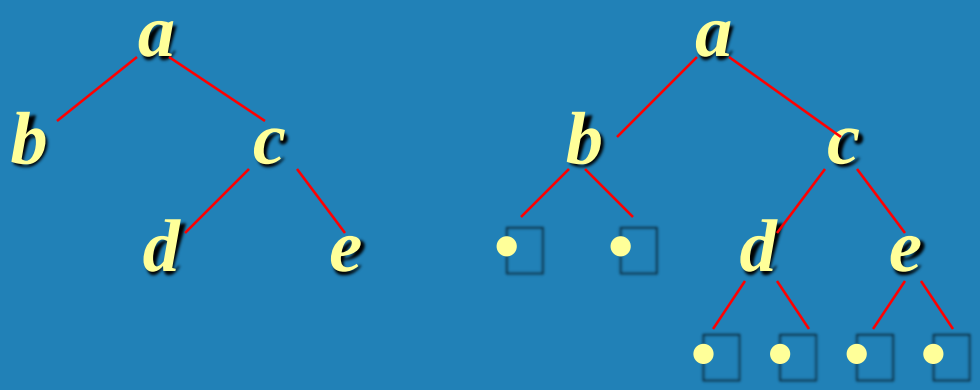
Algorithm *Inorder*(*T*)

if $T \neq \varnothing$

  Inorder(T_{left})

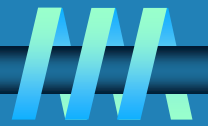
  print(root of *T*)

  Inorder(T_{right})



Efficiency: Θ(*n*)

Binary Tree Traversals and Related Properties

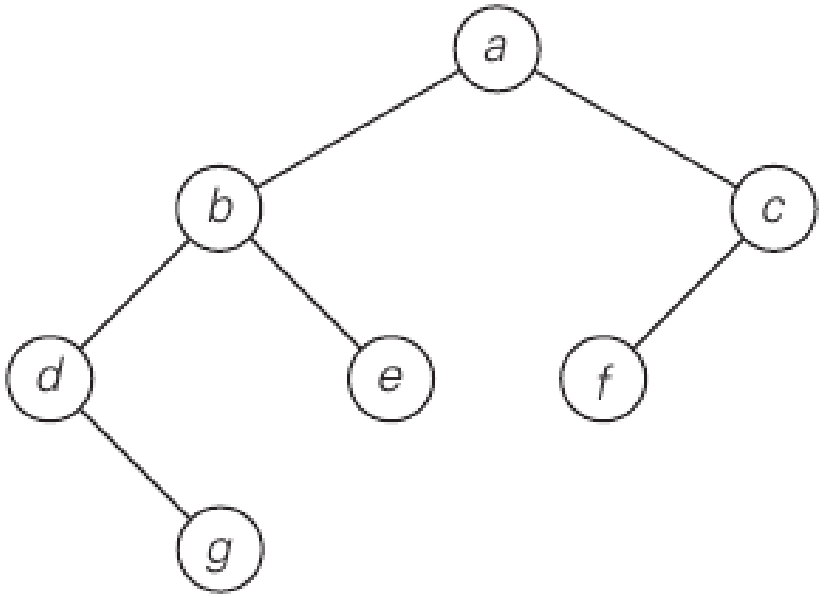


Here is a pseudocode of the preorder traversal:

Algorithm *Preorder*(T)
//Implements the preorder traversal of a binary tree
//Input: Binary tree T (with labeled vertices)
//Output: Node labels listed in preorder
if $T \neq \varnothing$
    print label of T's root
    Preorder(T_L) //T_L is the root's left subtree
    Preorder(T_R) //T_R is the root's right subtree

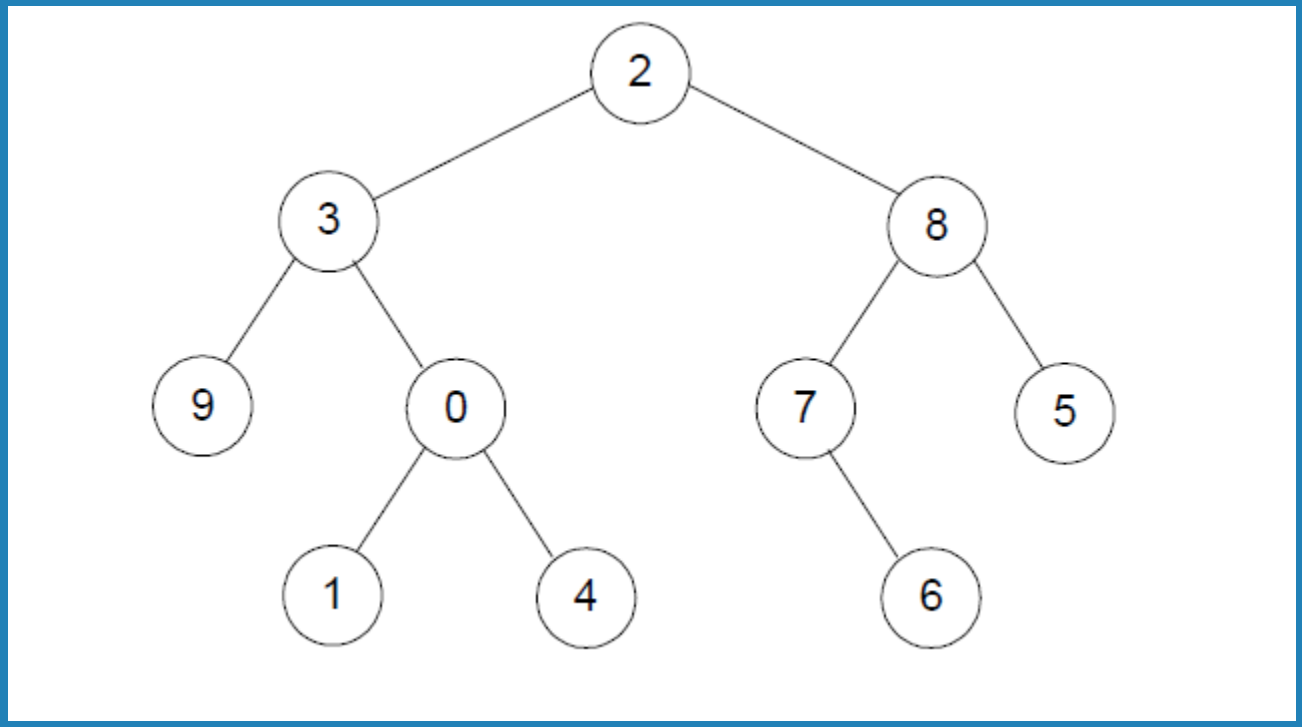
Inorder traversal of Binary Search Tree lists the numbers in order.

Binary Tree Traversals and Related Properties



preorder: *a, b, d, g, e, c, f*
inorder: *d, g, b, e, a, f, c*
postorder: *g, d, e, b, f, c, a*

Binary Tree Traversals and Related Properties

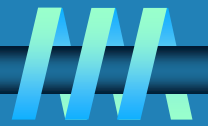


Preorder :

Inorder:

Postorder:

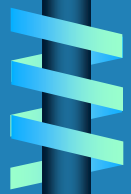
Divide-and-Conquer Question



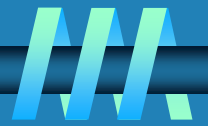
The following algorithm seeks to compute the number of leaves in a binary tree.

Algorithm *LeafCounter*(T)
//Computes recursively the number of leaves in a binary tree
//Input: A binary tree T
//Output: The number of leaves in T
if $T = \varnothing$ **return** 0
else return *LeafCounter*(T_L)+ *LeafCounter*(T_R)

Is this algorithm correct? If it is, prove it; if it is not, make an appropriate correction.



Divide-and-Conquer Solution

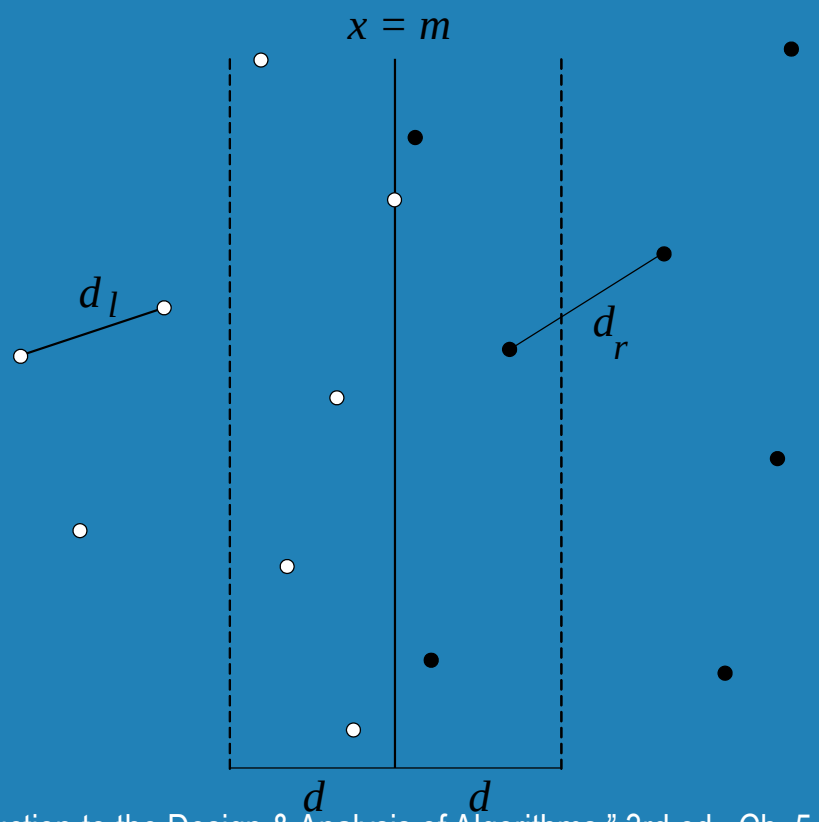


The algorithm is incorrect because it returns 0 instead of 1 for the one-node binary tree. Here is a corrected version:

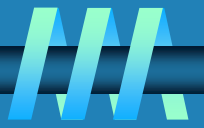
Algorithm *LeafCounter*(T)
//Computes recursively the number of leaves in a binary tree
//Input: A binary tree T
//Output: The number of leaves in T
if $T = \varnothing$ **return** 0 //empty tree
else if $T_L = \varnothing$ **and** $T_R = \varnothing$ **return** 1 //one-node tree
else return *LeafCounter*(T_L)+ *LeafCounter*(T_R) //general case

Closest-Pair Problem by Divide-and-Conquer

Step 1 Divide the points given into two subsets P_l and P_r by a vertical line $x = m$ so that half the points lie to the left or on the line and half the points lie to the right or on the line.



Closest Pair by Divide-and-Conquer



ALGORITHM *EfficientClosestPair*(P, Q)

```
//Solves the closest-pair problem by divide-and-conquer
//Input: An array P of n ≥ 2 points in the Cartesian plane sorted in
//       nondecreasing order of their x coordinates and an array Q of the
//       same points sorted in nondecreasing order of the y coordinates
//Output: Euclidean distance between the closest pair of points
if n ≤ 3
    return the minimal distance found by the brute-force algorithm
else
    copy the first ⌈n/2⌉ points of P to array P_l
    copy the same ⌈n/2⌉ points from Q to array Q_l
    copy the remaining ⌊n/2⌋ points of P to array P_r
    copy the same ⌊n/2⌋ points from Q to array Q_r
    d_l ← EfficientClosestPair(P_l, Q_l)
    d_r ← EfficientClosestPair(P_r, Q_r)
    d ← min{d_l, d_r}
    m ← P[⌈n/2⌉ − 1].x
    copy all the points of Q for which |x − m| < d into array S[0..num − 1]
    dminsq ← d^2
    for i ← 0 to num − 2 do
        k ← i + 1
        while k ≤ num − 1 and (S[k].y − S[i].y)^2 < dminsq
            dminsq ← min((S[k].x − S[i].x)^2 + (S[k].y − S[i].y)^2, dminsq)
            k ← k + 1
return sqrt(dminsq)
```

Closest Pair by Divide-and-Conquer (cont.)

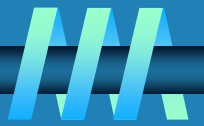
Step 2 Find recursively the closest pairs for the left and right subsets.

Step 3 Set $d = \min\{d_l, d_r\}$

We can limit our attention to the points in the symmetric vertical strip S of width $2d$ as possible closest pair. (The points are stored and processed in increasing order of their y coordinates.)

Step 4 Scan the points in the vertical strip S from the lowest up. For every point $p(x,y)$ in the strip, inspect points in in the strip that may be closer to p than d. There can be no more than 5 such points following p on the strip list!

Efficiency of the Closest-Pair Algorithm



Running time of the algorithm is described by

$$T(n) = 2T(n/2) + M(n), \text{ where } M(n) \in O(n)$$

By the Master Theorem (with $a = 2$, $b = 2$, $d = 1$)

$$T(n) \in O(n \log n)$$