INFDEV036A - Algorithms Lesson Unit 2

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Today

- ~~Why is my code slow?~~
  - ~~**Empirical and complexity analysis**~~
- How do I order my data?
  - **Sorting algorithms**
- How do I structure my data?
  - **Linear, tabular, recursive data structures**
- How do I represent relationship networks?
  - **Graphs**

Sorting algorithms

Insertion sort, Merge sort

http://www.youtube.com/watch?v=INHF_5RIxTE

Sorting algorithms

- Algorithms that put elements of a sequence in a certain order (numerical/lexicographical)
  - fundamental problem in computer science
    - as a standalone algorithm (i.e., producing human readable output)
    - as part of more complex algorithms which require sorted data (i.e., binary search!)
  - usually, data is considered to be stored in an array

Sorting algorithms

- Popular sorting algorithms
  - Simple sorts
    - Insertion sort, selection sort
  - Efficient sorts
    - Merge sort, Quick sort, Heap sort
  - Bubble sort and variants
    - Bubble sort, Shell sort, Comb sort
  - Distribution sort
    - Counting sort, Bucket sort, Radix sort
- In practice, a few algorithms predominate

Sorting algorithms

- When studying algorithms, we are interested in many aspects
  - **Correctness**, first and foremost
  - **Performance**, especially in certain domains (i.e., games)
    - **Time** & Space

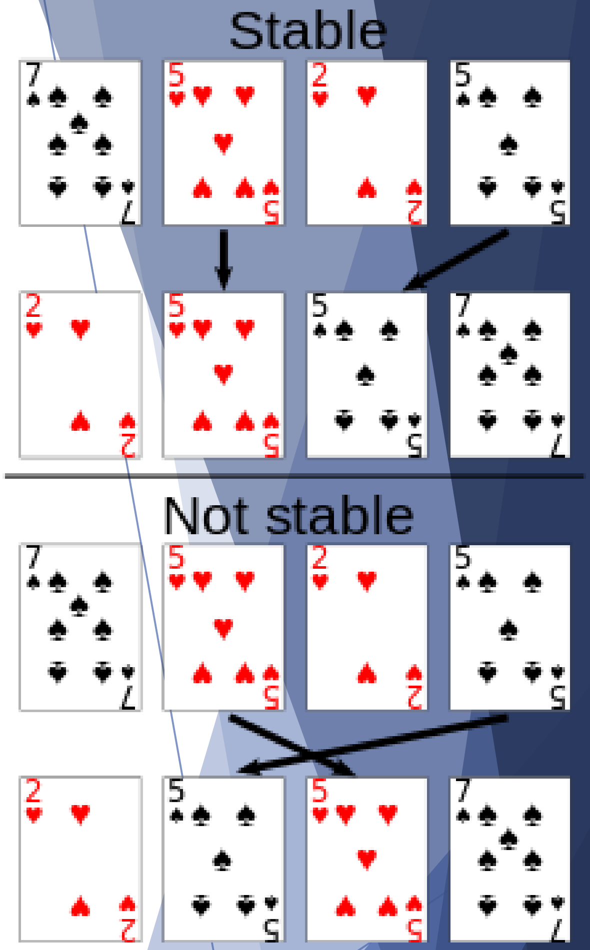
- Time complexity… remember from last lesson?
  - Asymptotic notation to express the relationship between computing time and input size
  - Example: suppose an algorithm runs in $0.1n^2 + 10.0n$ milliseconds for an input of size n → its complexity is $O(n^2)$
  - Valid asymptotically (i.e., not for small inputs)

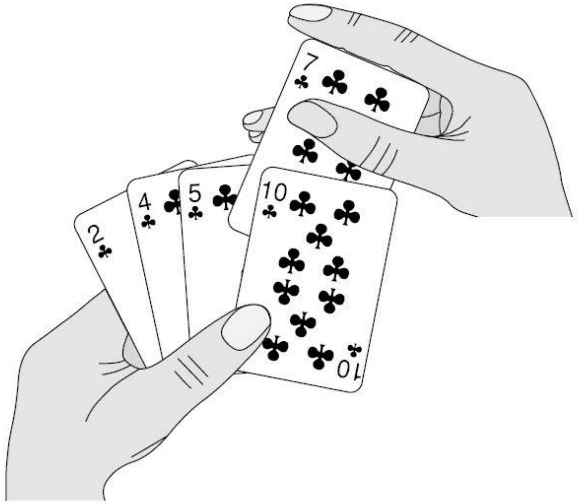
Sorting algorithms

- **Input**
  - a sequence of n numbers $a_1, \ldots, a_n$
- **Output**
  - a permutation (reordering) $a_{1'}, \ldots, a_{n'}$ of the input sequence …
  - … in non-decreasing order (i.e., such that $a_{1'} \leq \cdots \leq a_{n'}$)
- The numbers that we wish to sort are also known as the **keys**

Sorting algorithms
Properties

- Interesting properties for a sorting algorithm
  - STABILITY → maintain the relative order of elements with equal keys
  - COMPUTATIONAL COMPLEXITY → how many elements comparisons in terms of the size of the sequence
    - Good behavior is $O(n \log n)$
  - MEMORY USAGE → *in-place* algorithms need only $O(1)$ memory beyond the items being sorted
  - RECURSION → recursive or not
  - ADAPTABILITY → the presortedness of the input affects the running time

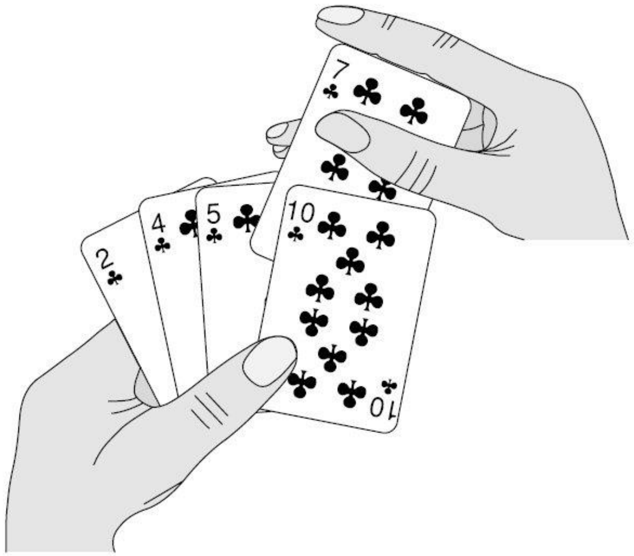




Insertion sort

Insertion sort

- Basic idea
  - When people manually sort something (for example, a deck of playing cards), most use a method that is similar to insertion sort
  - Put one element at a time in its right position in the sorted sub-array
  - The final sorted array (or list) is built one item at a time



Insertion sort

- Advantages
  - very intuitive algorithm ; simple implementation
  - efficient for (quite) small data sets
    - very efficient for data sets that are already substantially sorted and more efficient in practice than most other simple quadratic (i.e., $O(n^2)$) algorithms; the best case (nearly sorted input) is $O(n)$
  - stable
  - in-place
  - online (can sort a sequence *as* it receives it, one element at a time)

Insertion sort

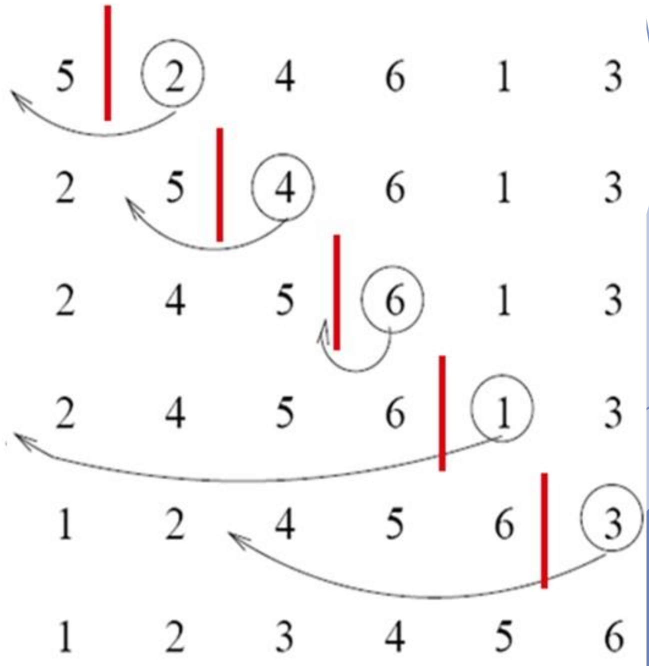
- Graphical example

6 5 3 1 8 7 2 4

Insertion sort

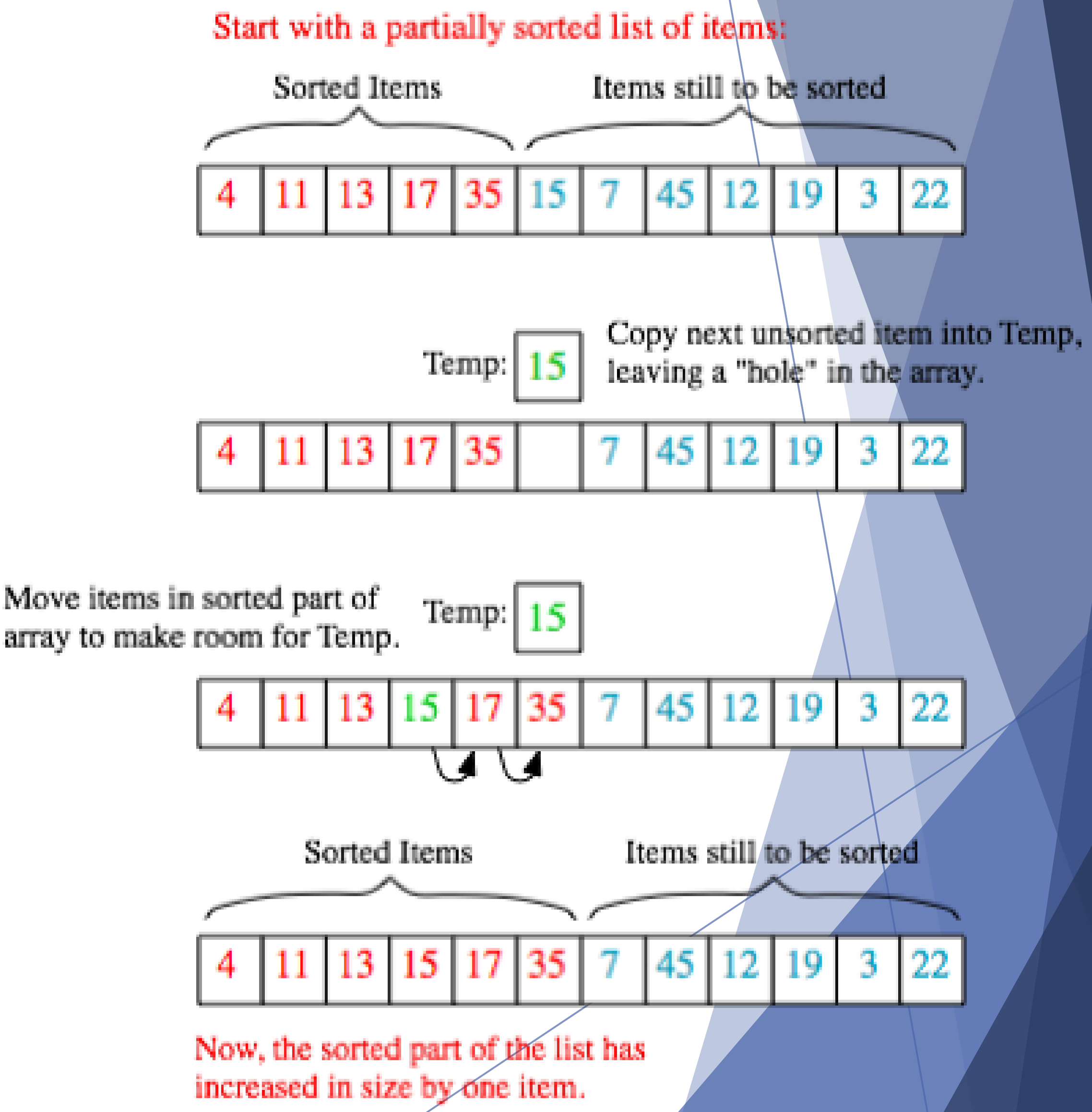
- Iterative algorithm
  - At each iteration one input element is consumed, growing a sorted output sequence

- Iteration
  i. remove one element from the input data
  ii. find the location it belongs within the sorted sequence
  iii. insert it there
- Repeat until no input elements remain

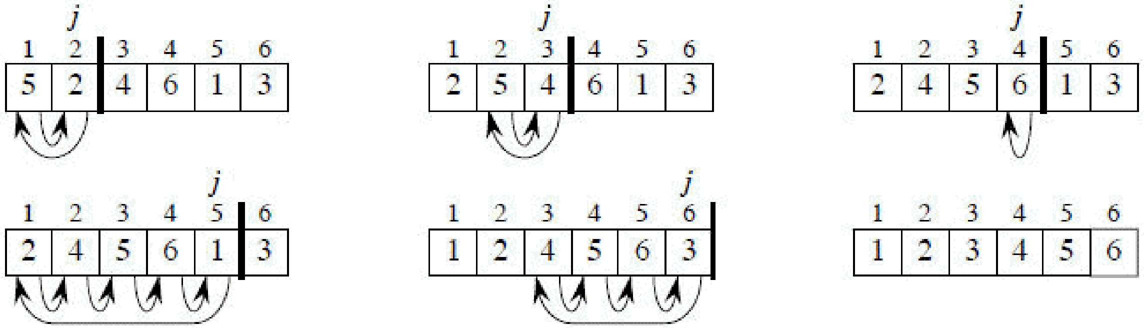


Insertion sort

- Sorting is typically done in-place
- For each unsorted item
  - shift all the larger values up to make a space
  - then insert it into the correct position



Insertion sort



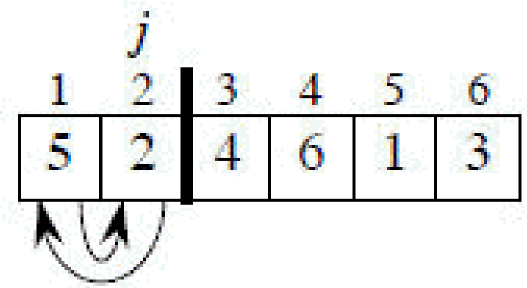
- Pseudo-code of the algorithm (supposing the origin of the array is 1)

```
FOR j = 2 to length(A)
  key = A[j]
  % put A[j] into the sorted sequence A[1..j-1]
  i = j - 1
  WHILE i > 0 and A[i] > key
    A[i+1] = A[i]
    i = i - 1
  A[i+1] = key
```

Insertion sort

```
FOR j = 2 to length(A)
    key = A[j]
    % put A[j] into the sorted sequence A[1..j-1]
    i = j - 1
    WHILE i > 0 and A[i] > key
        A[i+1] = A[i]
        i = i – 1
    A[i+1] = key
```

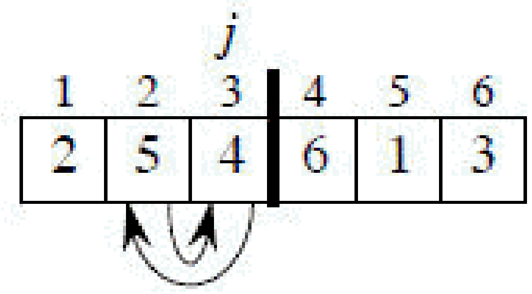
- First iteration trace
  - j = 2
  - key = A[2] = 2
  - i = 1
    - i > 0 && A[i] > 2 ? YES
      - A[2] = A[1] → A[2] = 5
      - i = i - 1 = 0
    - i > 0 && A[i] > 2 ? NO because i = 0
  - A[i+1] = 2 → A[1] = 2



Insertion sort

```
FOR j = 2 to length(A)
    key = A[j]
    % put A[j] into the sorted sequence A[1..j-1]
    i = j - 1
    WHILE i > 0 and A[i] > key
         A[i+1] = A[i]
         i = i – 1
    A[i+1] = key
```

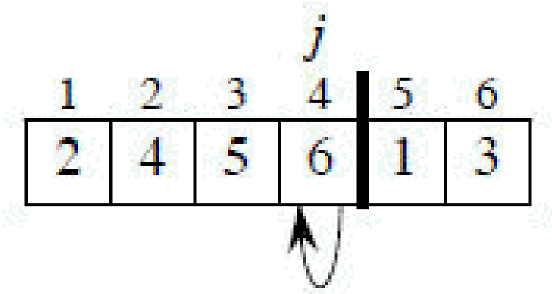
- Second iteration trace
  - j = 3
  - key = A[3] = 4
  - i = 2
    - i > 0 && A[i] > 4 ? YES
      - A[3] = A[2] → A[3] = 5
      - i = i - 1 = 1
    - i > 0 && A[i] > 4 ? NO because 2 > 4 is false
  - A[i+1] = 4 → A[2] = 4



Insertion sort

```
FOR j = 2 to length(A)
    key = A[j]
    % put A[j] into the sorted sequence A[1..j-1]
    i = j - 1
    WHILE i > 0 and A[i] > key
         A[i+1] = A[i]
         i = i – 1
    A[i+1] = key
```

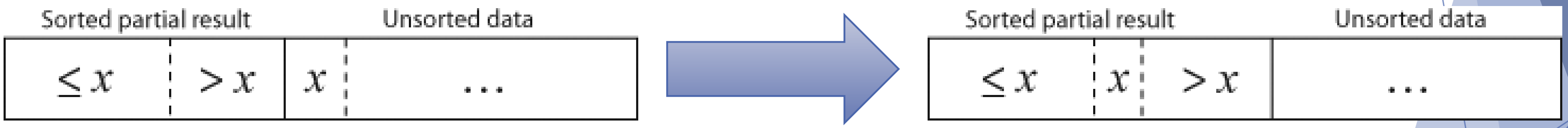
- Third iteration trace
  - j = 4
  - key = A[4] = 6
  - i = 3
    - i > 0 && A[i] > 6 ? NO because 5 > 6 is false
  - A[i+1] = 6 → A[4] = 6



- … and so on …

Insertion sort

- In each iteration the first remaining entry of the input is removed, and inserted into the result at the correct position, thus extending the result



Insertion sort

- **Correctness**
  - After k iterations, the following property holds:

    <u>*The first* $k+1$ *entries are sorted*</u>

    ("+1" because the first entry is skipped)
  - this property (called **INVARIANT**) holds true for every k, i.e. for the whole run of the algorithm
    - can be proved formally by induction (for us, intuition only)

- How many iterations does the algorithm?
  - $length - 1$
  - After the last iteration, then: "*The first* $(length - 1) + 1$ *entries are sorted*" → "*The first* $length$ *entries are sorted*" → All entries are sorted!!!

Insertion sort

- **Performance**

- Even with the same input *size*, runtime may differ
  - Depends on the *shape* of the data!
    - what varies is how many times we execute the loop test (t_j)
    - we can distinguish best, **worst**, average case
    - for each one, we can use the Big O notation (upper bound)

Insertion sort

```
FOR j = 2 to length(A)
    key = A[j]
    % put A[j] into the sorted sequence A[1..j-1]
    i = j - 1
    WHILE i > 0 and A[i] > key
         A[i+1] = A[i]
         i = i – 1
    A[i+1] = key
```

- Best case (when the array is already sorted)

2	4	5	7

  - The condition of the **while** loop then is false (the body is not executed)
  - We just verify that every element is in the correct position (through the "for" loop)
  - Complexity $O(n)$

Insertion sort

```
FOR j = 2 to length(A)
    key = A[j]
    % put A[j] into the sorted sequence A[1..j-1]
    i = j - 1
    WHILE i > 0 and A[i] > key
         A[i+1] = A[i]
         i = i – 1
    A[i+1] = key
```

- Worst case (when the array is in reverse sorted order)

7	5	4	2

  - The **while** loop is executed the maximum possible # of times
  - For each element we have to insert it into the right position by shifting all the elements to its left
  - Complexity $O(n^2)$

Insertion sort

- **Complexity** analysis: summary
  - *Best case* → sequence already sorted

3	5	6	9	11	15

$$O(n)$$

  - *Worst case* → sequence in reverse sorted order

15	11	9	6	5	3

$$O(n^2)$$

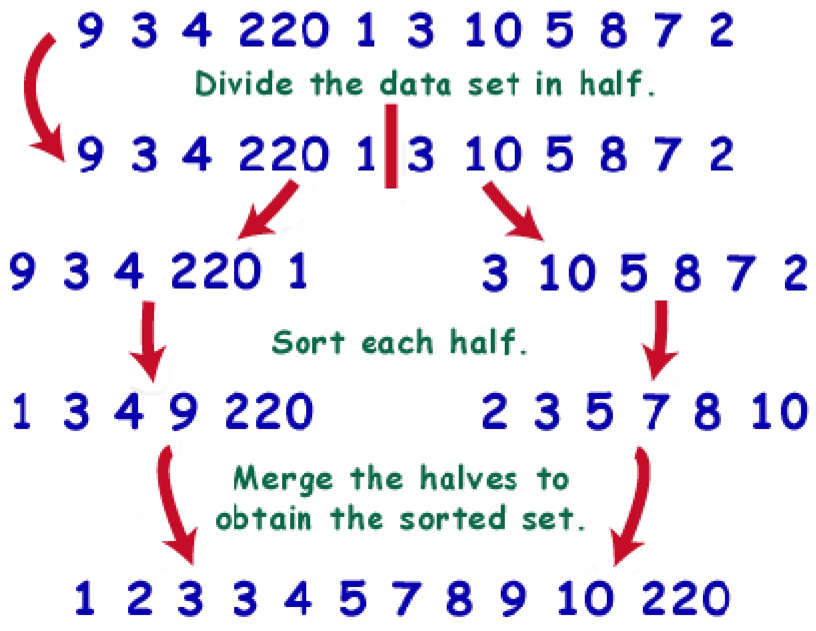
Bubble sort

- A “similar” simple sorting algorithm is called Bubble Sort
- It works by repeatedly swapping adjacent elements that are out of order

- Pseudocode:

```
FOR i = 1 to length(A)
    FOR j = length(A) downto i+1
        if A[j] < A[j-1]
            exchange A[j] with A[j-1]
```

- What is the complexity of this algorithm?


9 3 4 220 1 3 10 5 8 7 2
Divide the data set in half.
9 3 4 220 1 3 10 5 8 7 2
9 3 4 220 1
3 10 5 8 7 2
Sort each half.
1 3 4 9 220
2 3 5 7 8 10
Merge the halves to obtain the sorted set.
1 2 3 3 4 5 7 8 9 10 220

Merge sort

Merge sort

- “Divide-and-conquer” approach
  1. [DIVIDE] Break problem into smaller sub-problems
  2. [CONQUER] Solve the sub-problems recursively
     - If the sub-problems are small enough just solve them straightforwardly
  3. [COMBINE] Combine the solutions of the sub-problems

Merge sort

- “Divide-and-conquer” approach
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- Merge sort idea
  1. [DIVIDE] Split the sequence to sort (n elements) into two sub-sequences ($\frac{n}{2}$ elements each)
  2. [CONQUER] Sort the two sub-sequences recursively (using merge sort)
     - If the sub-sequence has length 1, it is already sorted (recursion stops here)
  3. [COMBINE] Merge the two sorted sub-sequences to produce the complete sorted sequence

Merge sort

- Merge sort idea
  1. [DIVIDE] Split the sequence to sort into two sub-sequences ($\frac{n}{2}$ elements each)
  2. [CONQUER] Sort the two sub-sequences recursively (using merge sort)
  3. [COMBINE] Merge the two sorted sub-sequences to produce the complete sorted sequence

- Pseudocode

```
MERGE-SORT(A,p,r)
   if p < r
      q = (p + r) / 2
      MERGE-SORT(A,p,q)
      MERGE-SORT(A,q+1,r)
      MERGE(A,p,q,r)
```

Merge sort

- Merge sort idea
  1. [DIVIDE] Split the sequence to sort into two sub-sequences ($\frac{n}{2}$ elements each)
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Merge sort

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```

Merge sort

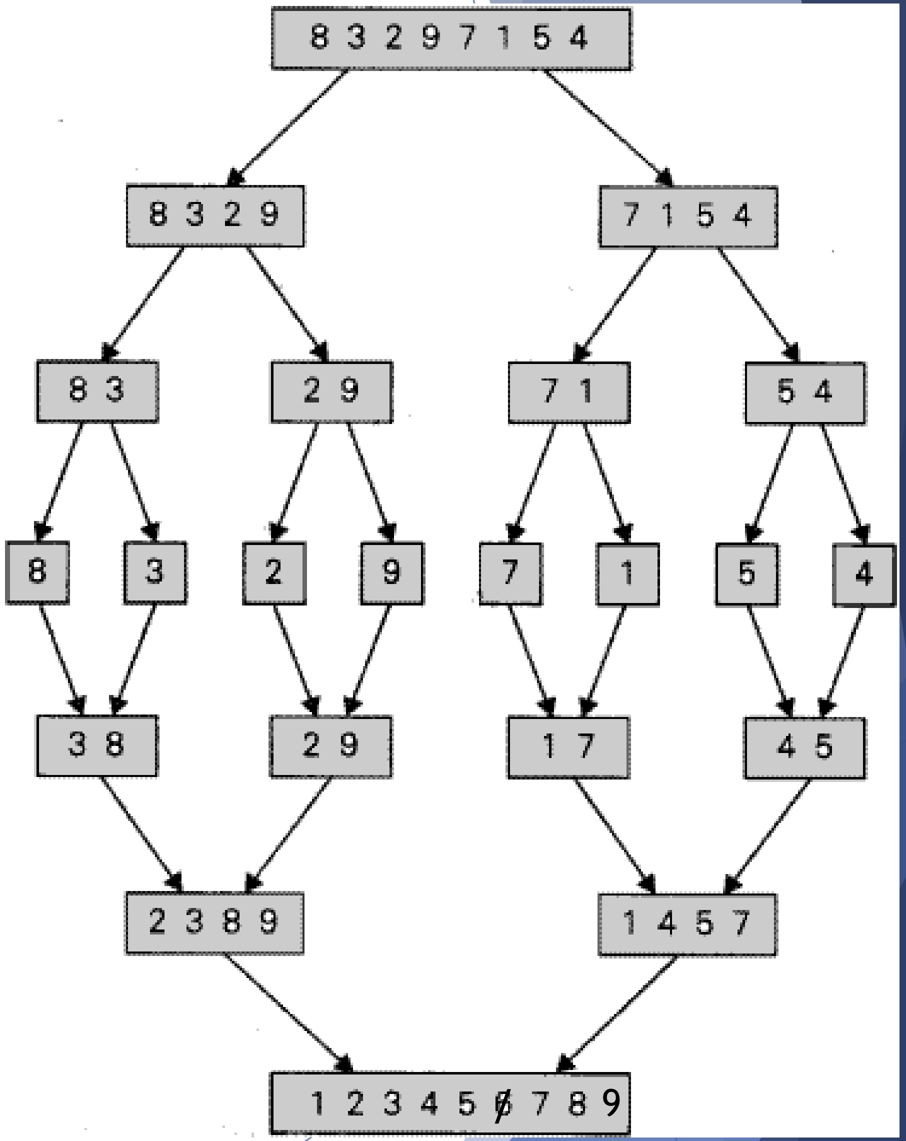
- How to combine two sorted sub-sequences into one?
- Example
  - two sorted piles of cards (face-up; smallest card on top)
  - we want to merge them into a single sorted pile

Merge sort

- How to combine two sorted sub-sequences into one?
- Example
  - two sorted piles of cards (face-up; smallest card on top)
  - we want to merge them into a single sorted pile
- Procedure
  - Choose the smallest of the two cards on top and remove it from its pile
    - Place this card into the output pile (face down)
  - Repeat the previous step until one of the two piles is empty
  - Take the remaining input pile and move it into the output one

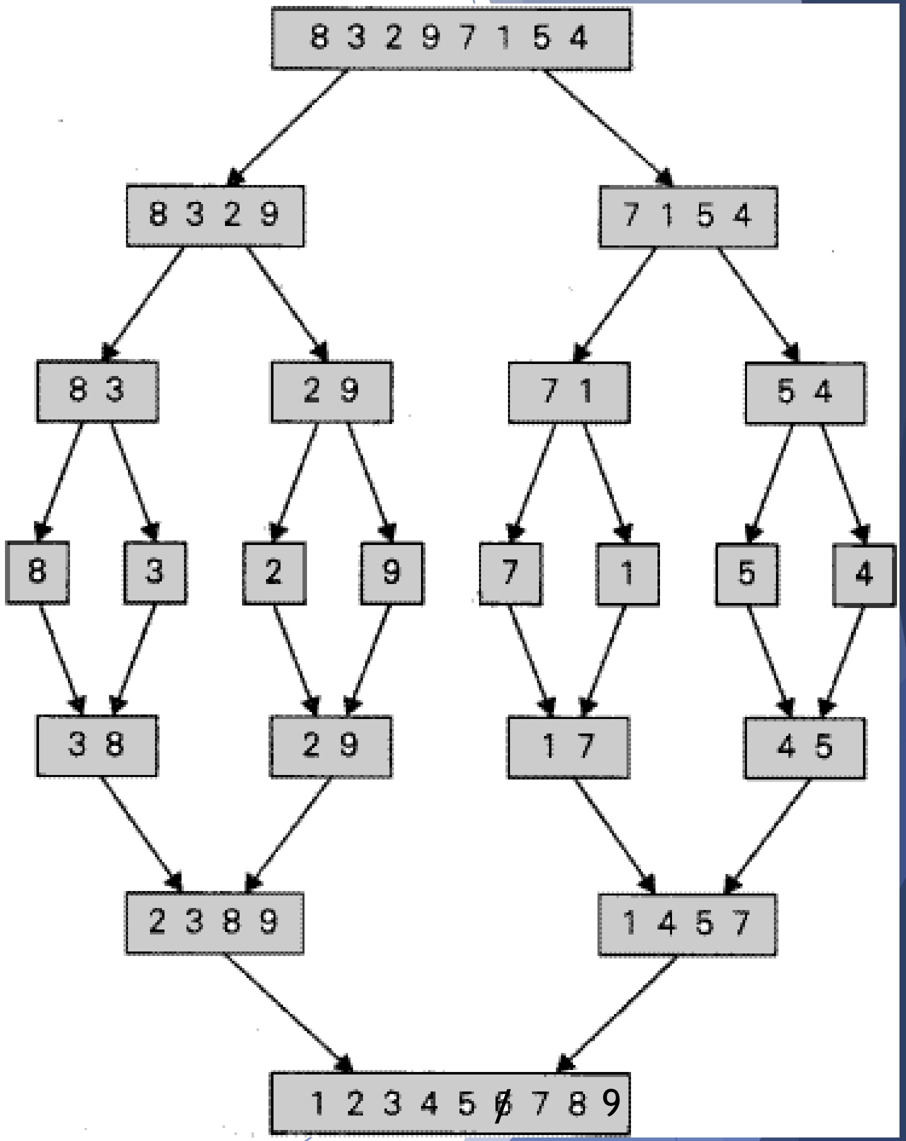
Merge sort

- Example: merging the two sequences 2 3 8 9 and 1 4 5 7
  - Compare 2 and 1 → Put 1 into the output sequence
    - S1: 2 3 8 9; S2: 4 5 7; OUTPUT: **1**
  - Compare 2 and 4 → Put 2 into the output sequence
    - S1: 3 8 9; S2: 4 5 7; OUTPUT: **1 2**
  - Compare 3 and 4 → Put 3 into the output sequence
    - S1: 8 9; S2: 4 5 7; OUTPUT: **1 2 3**
  - Compare 8 and 4 → Put 4 into the output sequence
    - S1: 8 9; S2: 5 7; OUTPUT: **1 2 3 4**
  - …



Merge sort

- Example: merging the two sequences 2 3 8 9 and 1 4 5 7
  - …
    - S1: 8 9; S2: 5 7; OUTPUT: **1 2 3 4**
  - Compare 8 and 5 → Put 5 into the output sequence
    - S1: 8 9; S2: 7; OUTPUT: **1 2 3 4 5**
  - Compare 8 and 7 → Put 7 into the output sequence
    - S1: 8 9; S2: ∅; OUTPUT: **1 2 3 4 5 7**
  - One pile is empty
    - Put all the elements of the other pile (8 9) in the output sequence
    - S1: ∅; S2: ∅; OUTPUT: **1 2 3 4 5 7 8 9**

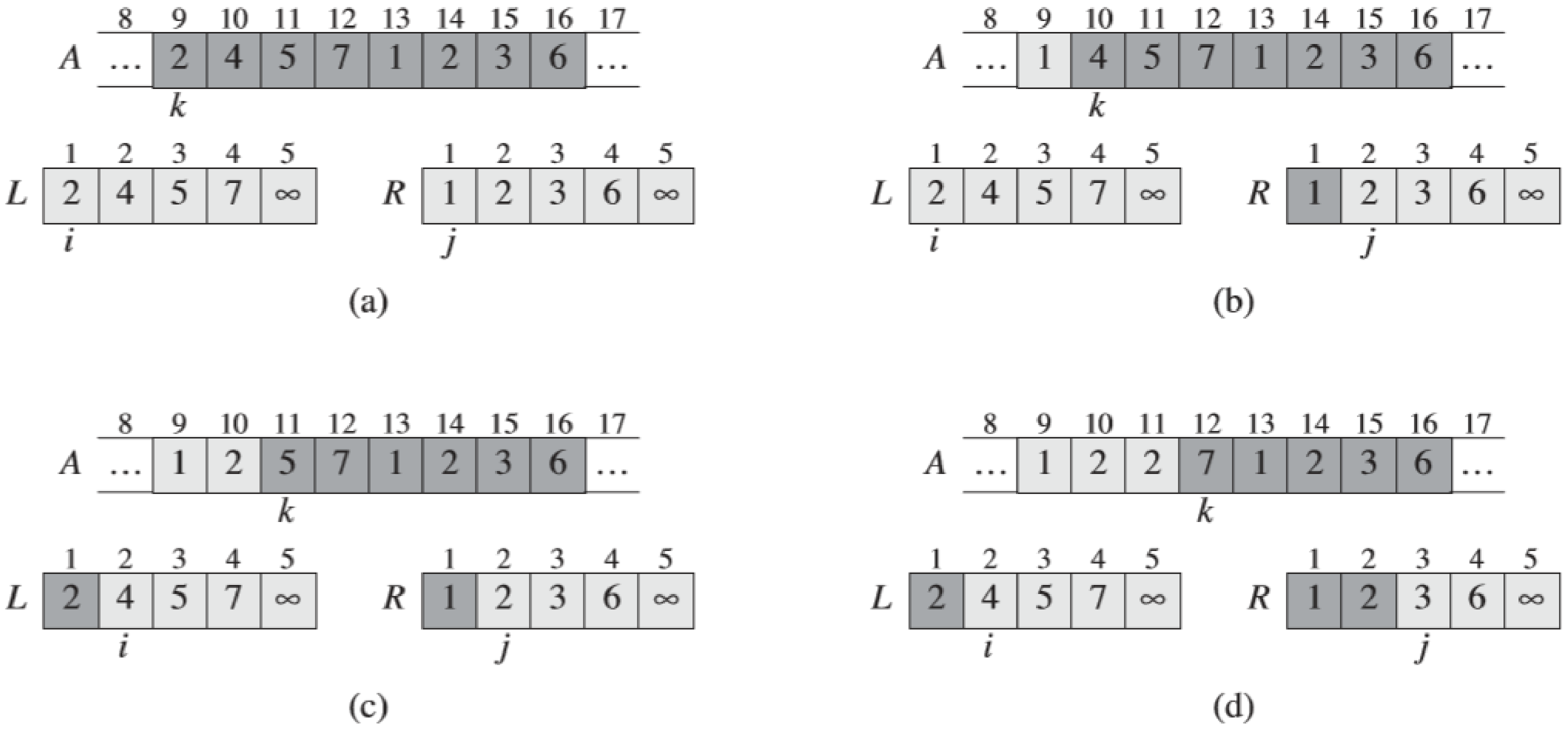


Merge sort

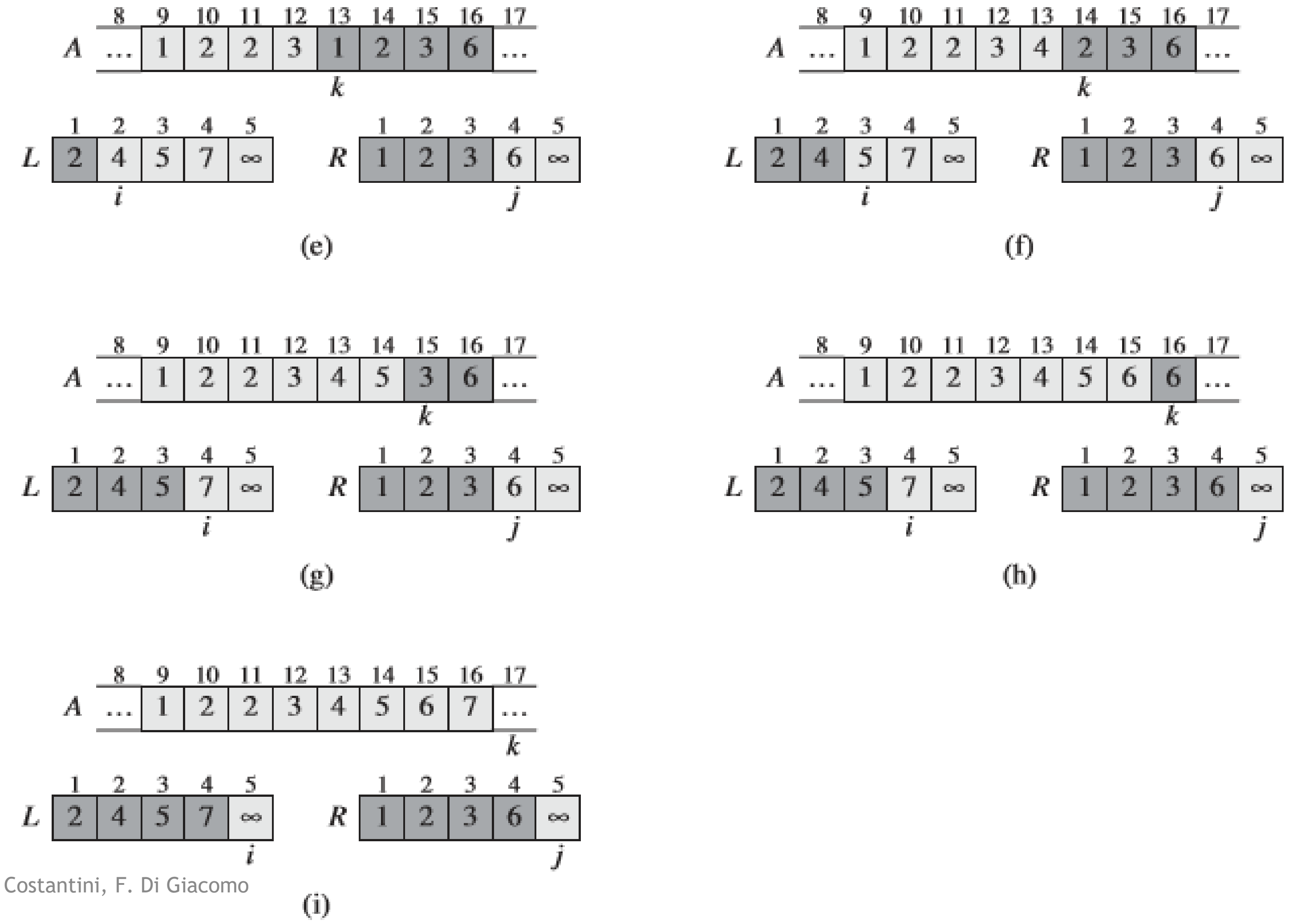
- A possible way to put it into code
  - Start by creating two arrays L,R
    - L contains the left part of A (one pile)
    - R contains the right part of A (other pile)
    - Both parts are sorted!
    - Last element of both L/R is ∞
  - Then, choose & copy the smallest element from the two arrays
    - No need to check if one part is empty, thanks to the use of ∞

```
MERGE(A, p, q, r)
n₁ = q − p + 1
n₂ = r − q
let L[1..n₁ + 1 ] and L[1..n₂ + 1] be new arrays
FOR i = 1 TO n₁
  L[i] = A[p + i - 1]
FOR j = 1 TO n₂
  R[j] = A[q + j]
L[n₁ + 1] = ∞
R[n₂ + 1] = ∞
i = 1
j = 1
FOR k = p TO r
  IF L[i] ≤ R[j]
    A[k] = L[i]
    i = i + 1
  ELSE
    A[k] = R[j]
    j = j + 1
```

Merge sort

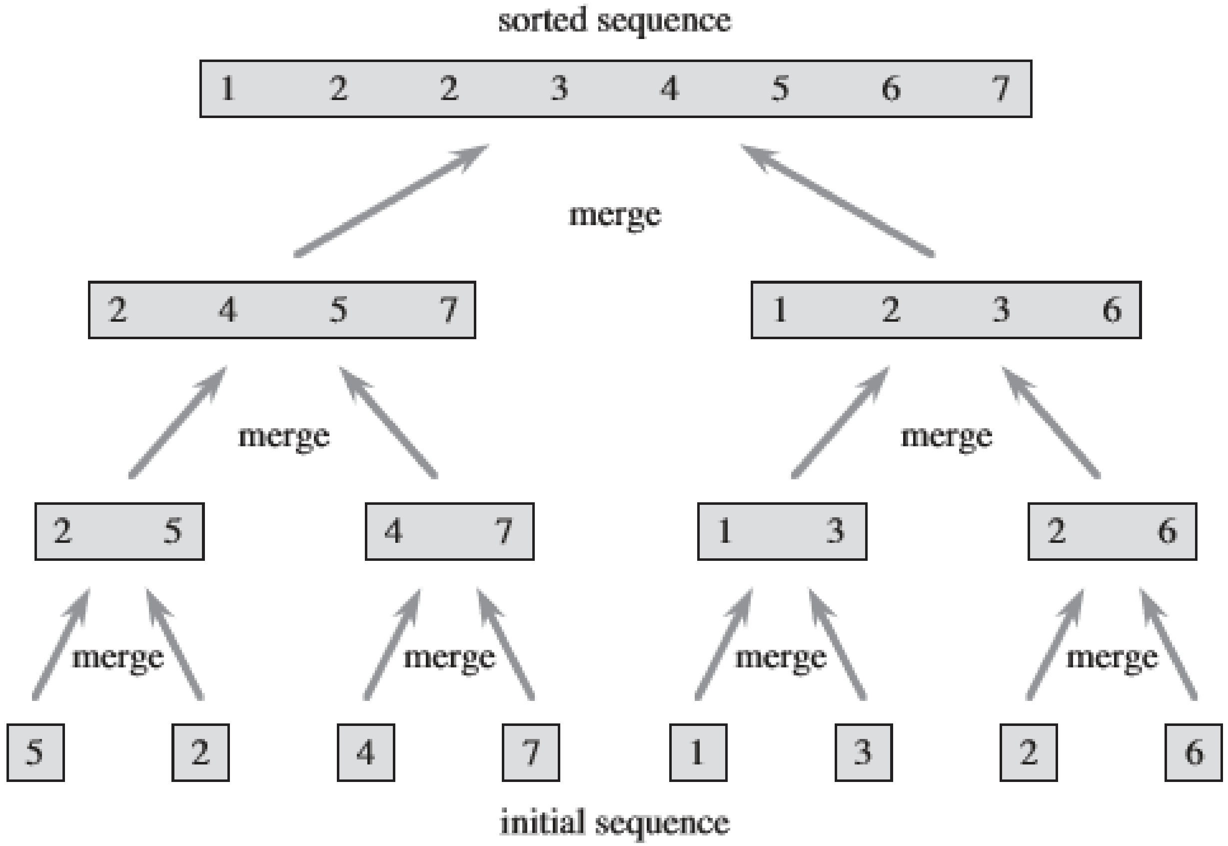

8 9 10 11 12 13 14 15 16 17
A … 2 4 5 7 1 2 3 6 …
k
1 2 3 4 5
L 2 4 5 7 ∞
i
1 2 3 4 5
R 1 2 3 6 ∞
j
(a)
8 9 10 11 12 13 14 15 16 17
A … 1 4 5 7 1 2 3 6 …
k
1 2 3 4 5
L 2 4 5 7 ∞
i
1 2 3 4 5
R 1 2 3 6 ∞
j
(b)
8 9 10 11 12 13 14 15 16 17
A … 1 2 5 7 1 2 3 6 …
k
1 2 3 4 5
L 2 4 5 7 ∞
i
1 2 3 4 5
R 1 2 3 6 ∞
j
(c)
8 9 10 11 12 13 14 15 16 17
A … 1 2 2 7 1 2 3 6 …
k
1 2 3 4 5
L 2 4 5 7 ∞
i
1 2 3 4 5
R 1 2 3 6 ∞
j
(d)

Merge sort



8 9 10 11 12 13 14 15 16 17
A … 1 2 2 3 1 2 3 6 …
k
1 2 3 4 5
L 2 4 5 7 ∞
i
1 2 3 4 5
R 1 2 3 6 ∞
j
(e)
8 9 10 11 12 13 14 15 16 17
A … 1 2 2 3 4 2 3 6 …
k
1 2 3 4 5
L 2 4 5 7 ∞
i
1 2 3 4 5
R 1 2 3 6 ∞
j
(f)
8 9 10 11 12 13 14 15 16 17
A … 1 2 2 3 4 5 3 6 …
k
1 2 3 4 5
L 2 4 5 7 ∞
i
1 2 3 4 5
R 1 2 3 6 ∞
j
(g)
8 9 10 11 12 13 14 15 16 17
A … 1 2 2 3 4 5 6 6 …
k
1 2 3 4 5
L 2 4 5 7 ∞
i
1 2 3 4 5
R 1 2 3 6 ∞
j
(h)
8 9 10 11 12 13 14 15 16 17
A … 1 2 2 3 4 5 6 7 …
k
1 2 3 4 5
L 2 4 5 7 ∞
i
1 2 3 4 5
R 1 2 3 6 ∞
j
(i)

Merge sort


sorted sequence
1 2 2 3 4 5 6 7
merge
2 4 5 7
1 2 3 6
merge
merge
2 5
4 7
1 3
2 6
merge
merge
merge
merge
5
2
4
7
1
3
2
6
initial sequence

Merge sort

- **Performance**
  - $T(n)$ → running time of Merge Sort on an input of size n
  - The total running time is the sum of…
    - [DIVIDE] compute the middle of the subarray → $O(1)$
    - [CONQUER] recursively solve the two sub-problems, each of size $\frac{n}{2}$ → $2 \times T\left(\frac{n}{2}\right)$
    - [COMBINE] n iterations of the loop, each of which takes constant time → $O(n)$

$$\boldsymbol{T(n) = 2 \times T\left(\frac{n}{2}\right) + n}$$

  - To solve this recurrence, we would need the "master theorem"…
  - … here only intuition! (pfiuuuuu ☺)

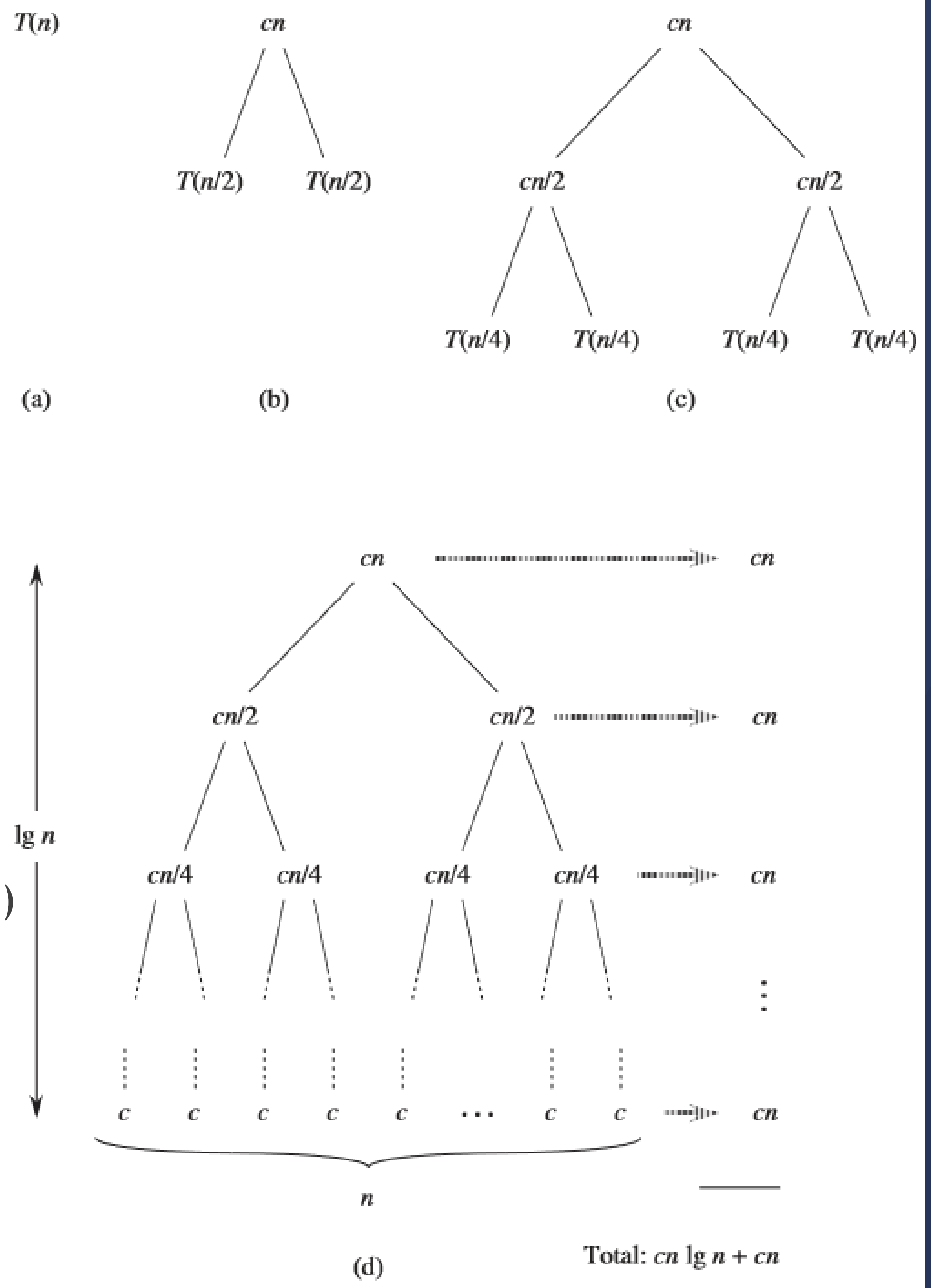
Merge sort

- Performance

$$T(n) = 2 \times T\left(\frac{n}{2}\right) + n$$

  - for convenience, assume that n is a power of 2

- Tree representing the recurrence
  - Root = top level of recursion
  - Each node = cost of merging plus cost of sub-problems (subtrees)
  - Leaves = problems of size 1 (recursion stops)



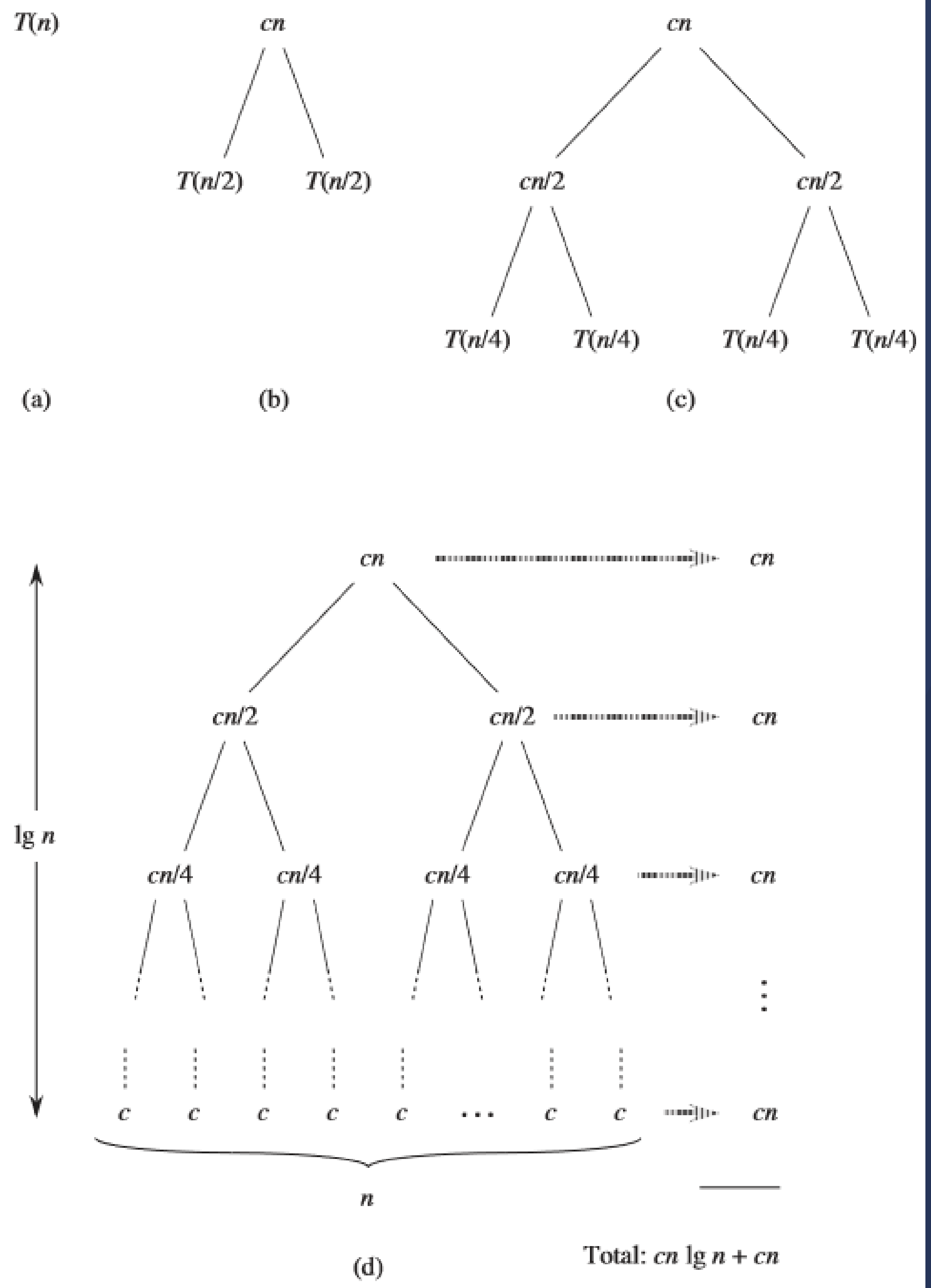
Merge sort

- Performance

$$T(n) = 2 \times T\left(\frac{n}{2}\right) + n$$

  - for convenience, assume that n is a power of 2

- Total cost of the tree?
  - Cost of each level multiplied by number of levels
  - What is the cost of each level?
    - $O(n)$
  - What is the height of the tree?
    - $\log_2 n$ because, by definition, $x = \log_2 n \Rightarrow 2^x = n \Rightarrow \frac{n}{2^x} = 1$



Merge sort

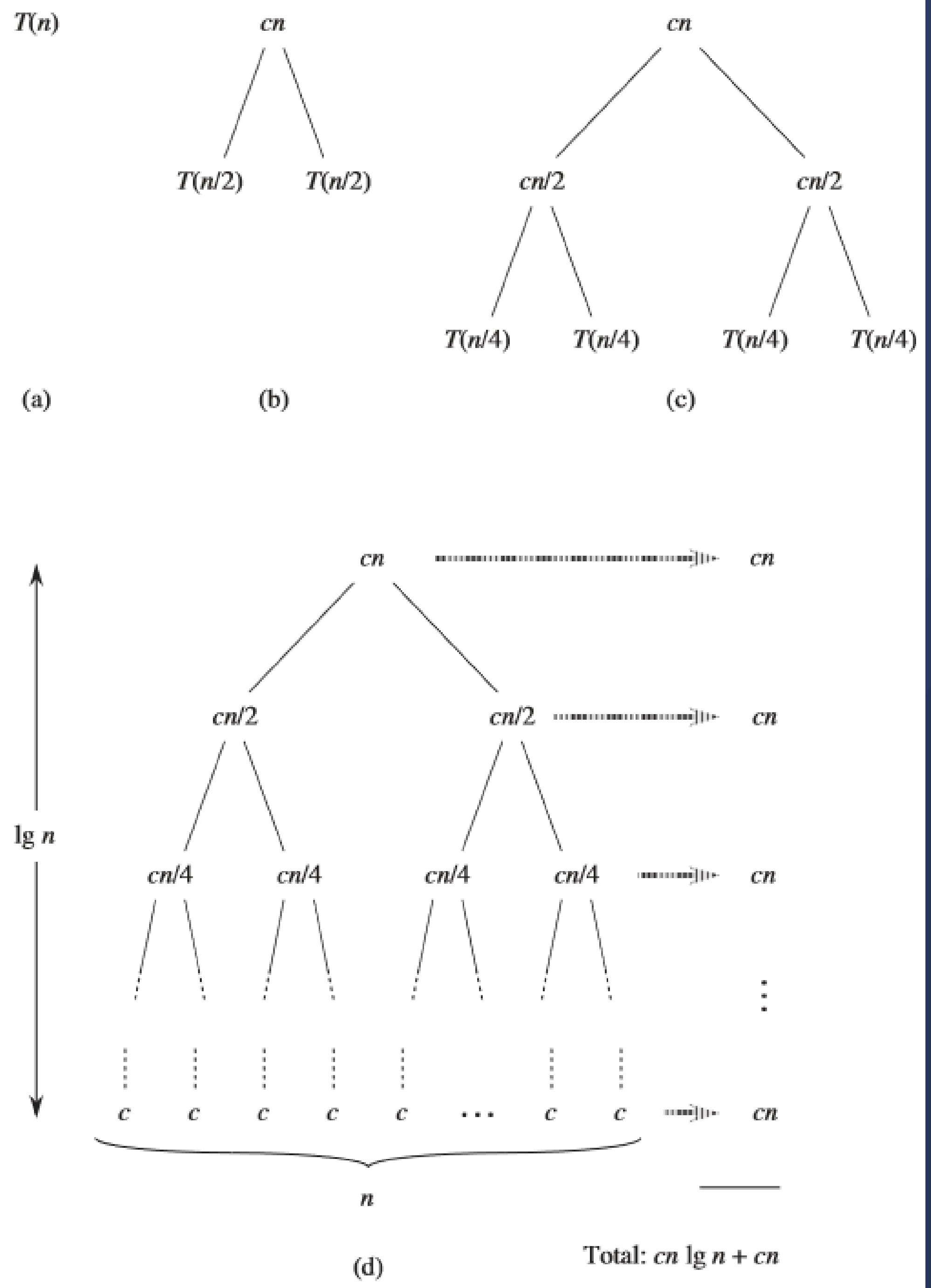
- Performance

$$T(n) = 2 \times T\left(\frac{n}{2}\right) + n$$

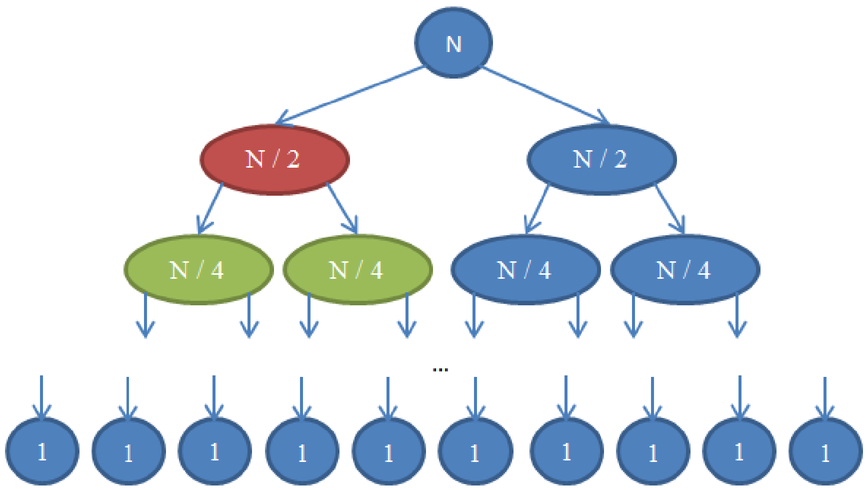
  - for convenience, assume that n is a power of 2

- Total cost of the tree?
  - Cost of each level → $O(n)$
  - How many levels? → $\log_2 n + 1$
  - Ignoring the constant 1 we get

$$\boldsymbol{T(n) = O(n \log n)}$$

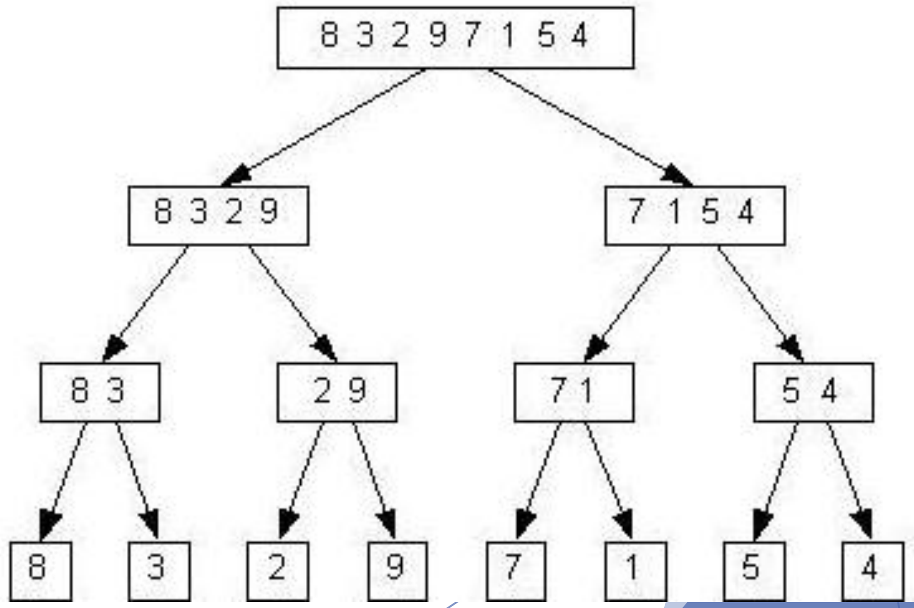


Merge sort

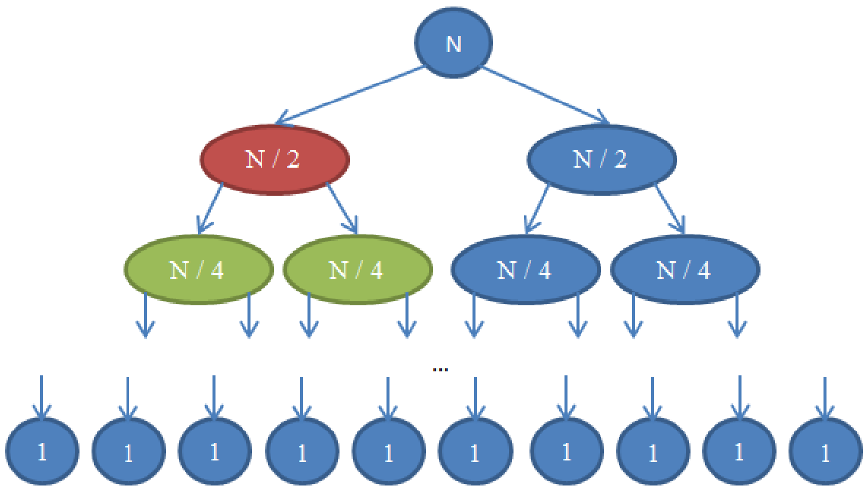


- Recursion tree, example

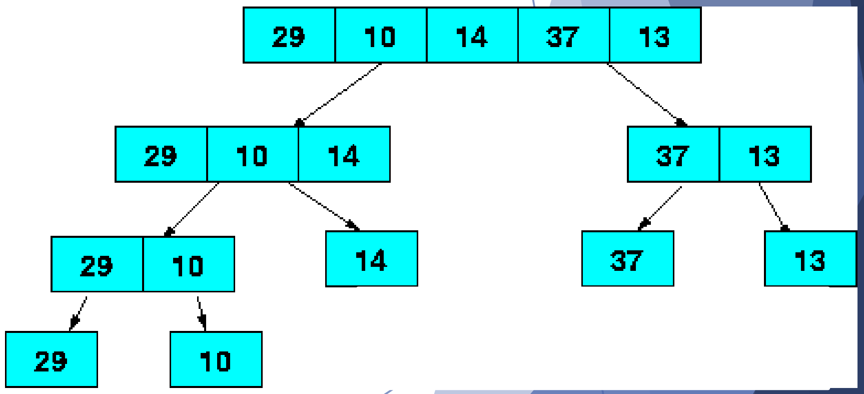
  - Let $n = 8$
  - Levels of the recursion tree?
    - First level: 1 node with 8 elements
    - Second level: 2 nodes with 4 elements each
    - Third level: 4 nodes with 2 elements each
    - Fourth (and last) level: 8 nodes with 1 element each
  - $\log_2 8 = 3$ because $2^3 = 2 \times 2 \times 2 = 8$
  - $(\log_2 8) + 1 = 4 \rightarrow 4$ levels



Merge sort



- Recursion tree, example
  - … what if n is not a power of 2?
  - Let $n = 5$
  - Levels of the recursion tree?
    - First level: 1 node with 5 elements
    - Second level: 2 nodes with max 3 elements each
    - Third level: 4 nodes with max 2 elements each
    - Fourth (and last) level: some nodes with max 1 element each
  - $\log_2 5 = 2.321 \ldots$ → we round it to the next integer (3)!
  - $\lceil \log_2 5 \rceil + 1 = 4$ → 4 levels



Homework

- Study the slides
- Multiple choice questions on GrandeOmega
- Implement the two sorting algorithms (Insertion sort and Merge sort)
  - *Facultative*: try to make it generic with respect to the type of the elements being sorted (using a comparator)
  - Implement also Bubble sort
- … See you next week ☺