CSE214 – Analysis of Algorithms

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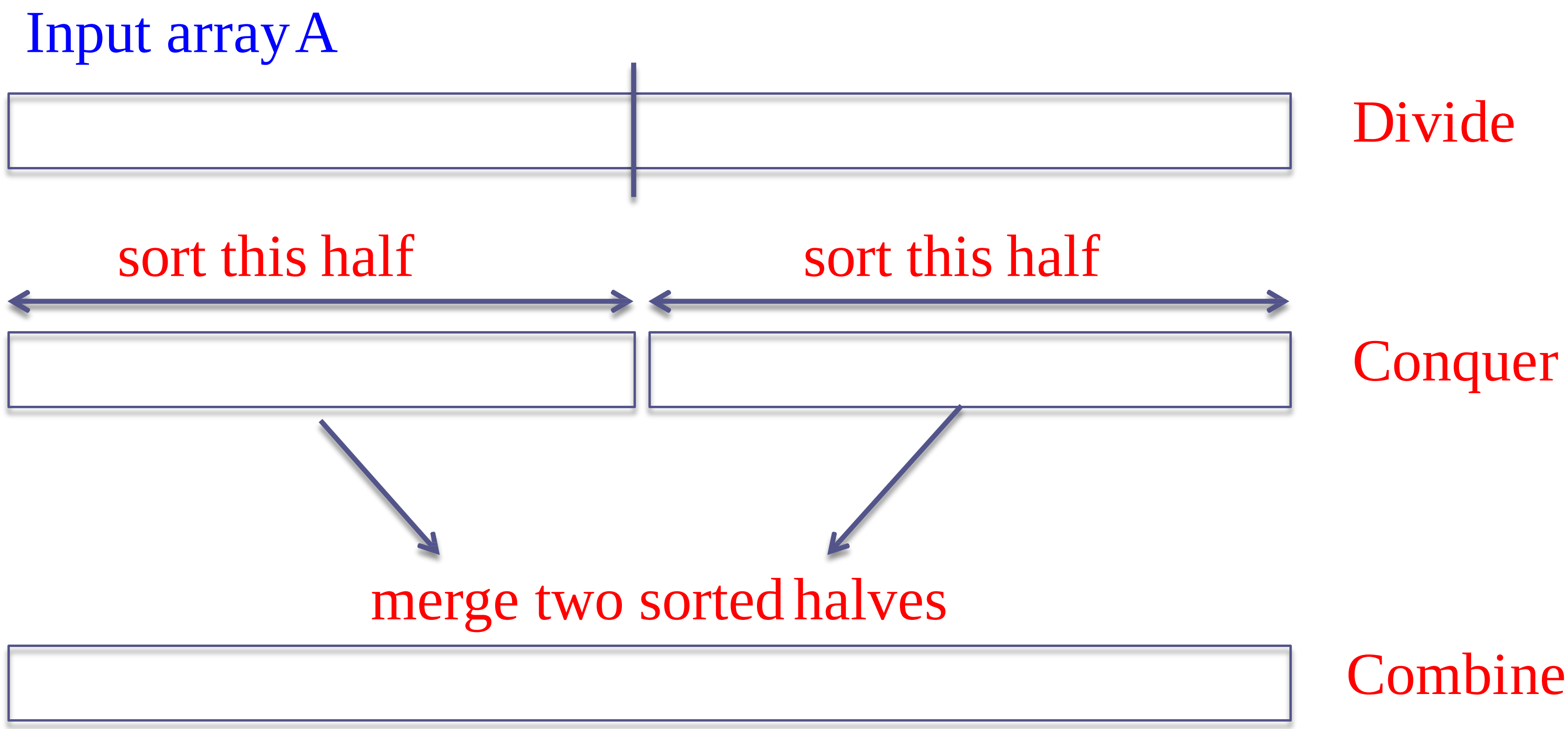
https://github.com/FurkanGozukara/Analysis-of-Algorithms-2019

Lecture 4

The Divide-and-Conquer Design Paradigm

Based on Cevdet Aykanat's and Mustafa Ozdal's Lecture Notes - Bilkent

Reminder: Merge Sort



The Divide-and-Conquer Design Paradigm

1. ***Divide*** **the problem (instance) into subproblems.**

2. ***Conquer*** **the subproblems by solving them recursively.**

3. ***Combine*** **subproblem solutions.**

Example: Merge Sort

1. ***Divide*:** Trivial.
2. ***Conquer*:** Recursively sort 2 subarrays.
3. ***Combine*:** Linear- time merge.

$$T(n) = \underbrace{2}_{\text{\# subproblems}} T(\underbrace{n/2}_{\text{subproblem size}}) + \underbrace{\Theta(n)}_{\text{work dividing and combining}}$$

Master Theorem: Reminder

T(n) = aT(n/b) + f(n)

Case 1: $\dfrac{n^{\log_b a}}{f(n)} = \Omega(n^{\varepsilon})$ → $T(n) = \Theta(n^{\log_b a})$

Case 2: $\dfrac{f(n)}{n^{\log_b a}} = \Theta(\lg^k n)$ → $T(n) = \Theta(n^{\log_b a} \lg^{k+1} n)$

Case 3: $\dfrac{f(n)}{n^{\log_b a}} = \Omega(n^{\varepsilon})$ → $T(n) = \Theta(f(n))$

and $a\, f(n/b) \le c\, f(n)$ for $c < 1$

Merge Sort: Solving the Recurrence

$$T(n) = 2\,T(n/2) + \Theta(n)$$

$a = 2, \quad b = 2, \quad f(n) = \Theta(n), \quad n^{\log_b a} = n$

Case 2: $\dfrac{f(n)}{n^{\log_b a}} = \Theta(\lg^k n) \Rightarrow T(n) = \Theta(n^{\log_b a} \lg^{k+1} n)$

holds for k = 0

$T(n) = \Theta(n\lg n)$

Binary Search

Find an element in a **sorted** array:

1. <u>Divide</u>: Check middle element.

2. <u>Conquer</u>: Recursively search 1 subarray.

3. <u>Combine</u>: Trivial.

Example: Find **9**

3	**5**	**7**	**8**	**9**	**12**	**15**

Binary Search

Find an element in a **sorted** array:

1. ***Divide*****:** Check middle element.

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3	**5**	**7**	**8**	**9**	**12**	**15**

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Binary Search

Find an element in a **sorted** array:

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Example: Find **9**

3 5 7 8 9 12 15

Binary Search

Find an element in a **sorted** array:

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Example: Find **9**

3	**5**	**7**	**8**	$\boxed{\mathbf{9}}$	**12**	**15**

Binary Search

Find an element in a **sorted** array:

1. Divide: Check middle element.

2. Conquer: Recursively search 1 subarray.

3. Combine: Trivial.

Example: Find **9**

3	**5**	**7**	**8**	$\boxed{\mathbf{9}}$	**12**	**15**

Recurrence for Binary Search

$$T(n) = 1\, T(n/2) + \Theta(1)$$

subproblems

subproblem size

work dividing and combining

Binary Search: Solving the Recurrence

$$T(n) = T(n/2) + \Theta(1)$$

$\Rightarrow$ $a = 1$, $b = 2$, $f(n) = \Theta(1)$, $n^{\log_b a} = n^0 = 1$

Case 2: $\dfrac{f(n)}{n^{\log_b a}} = \Theta(\lg^k n)$ $\Rightarrow$ $T(n) = \Theta\left(n^{\log_b a} \lg^{k+1} n\right)$

holds for $k = 0$

$\Rightarrow$ $T(n) = \Theta(\lg n)$

Powering a Number

- Problem: Compute a^n, where n is a natural number

```
Naive-Power (a, n)
    powerVal ← 1
    for i ← 1 to n
        powerVal ← powerVal . a
    return powerVal
```

- What is the complexity?

$T(n) = \Theta(n)$

Powering a Number: Divide & Conquer

Basic idea:

$$a^n = \begin{cases} a^{n/2} \cdot a^{n/2} & \text{if n is \underline{even}} \\ a^{(n-1)/2} \cdot a^{(n-1)/2} \cdot a & \text{if n is \underline{odd}} \end{cases}$$

Example: $3^7 = 3^3 \times 3^3 \times 3$

Example: $3^8 = 3^4 \times 3^4$

Powering a Number: Divide & Conquer

```
POWER (a, n)
        if n = 0 then return 1

        else if n is even then
                val ← POWER (a, n/2)
                return val * val

        else if n is odd then
                val ← POWER (a, (n-1)/2)
                return val * val * a
```

Powering a Number: Solving the Recurrence

$$T(n) = T(n/2) + \Theta(1)$$

$\Rightarrow$ $a = 1$, $b = 2$, $f(n) = \Theta(1)$, $n^{\log_b a} = n^0 = 1$

Case 2: $\dfrac{f(n)}{n^{\log_b a}} = \Theta(\lg^k n)$ $\Rightarrow$ $T(n) = \Theta\left(n^{\log_b a} \lg^{k+1} n\right)$

holds for $k = 0$

$\Rightarrow$ $T(n) = \Theta(\lg n)$

Matrix Multiplication

Input : $A = [a_{ij}]$, $B = [b_{ij}]$.
Output: $C = [c_{ij}] = A \cdot B$.

$\Big\}$ $i, j = 1, 2, \ldots, n$.

$$\begin{pmatrix} c_{11} & c_{12} & \ldots & c_{1n} \\ \boxed{c_{21}} & c_{22} & \ldots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \ldots & c_{nn} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix} . \begin{pmatrix} b_{11} & b_{12} & \ldots & b_{1n} \\ b_{21} & b_{22} & \ldots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \ldots & b_{nn} \end{pmatrix}$$

$$c_{ij} = \sum_{1 \le k \le n} a_{ik} . b_{kj}$$

If we multiply a 2×3 matrix with a 3×1 matrix, the product matrix is 2×1

$$2 \times \boxed{3} = \boxed{3} \times 1 \qquad 2 \times 1$$

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \end{bmatrix} \times \begin{bmatrix} t_{11} \\ t_{21} \\ t_{31} \end{bmatrix} = \begin{bmatrix} M_{11} \\ M_{12} \end{bmatrix}$$

Here is how we get M_{11} and M_{12} in the product.

$$M_{11} = r_{11} \times t_{11} + r_{12} \times t_{21} + r_{13} \times t_{31}$$
$$M_{12} = r_{21} \times t_{11} + r_{22} \times t_{21} + r_{23} \times t_{31}$$

$$\begin{bmatrix} 3 & 12 & 4 \\ 5 & 6 & 8 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 7 & 3 & 8 \\ 11 & 9 & 5 \\ 6 & 8 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} \underline{3*7+12*11+4*6} & 3*3+12*9+4*8 & 3*8+12*5+4*4 \\ 5*7+6*11+8*6 & \underline{5*3+6*9+8*8} & 5*8+6*5+8*4 \\ 1*7+0*11+2*6 & 1*3+0*9+2*8 & 1*8+0*5+2*4 \end{bmatrix}$$

$$= \begin{bmatrix} 177 & 149 & 100 \\ 149 & 133 & 102 \\ 19 & 19 & 16 \end{bmatrix}$$

Standard Algorithm

```
for i ← 1 to n
    do for j ← 1 to n
        do c_ij ← 0
            for k ← 1 to n
                do c_ij ← c_ij + a_ik . b_kj
```

Running time = $\Theta(n^3)$

Matrix Multiplication: Divide & Conquer

IDEA: Divide the n x n matrix into

2x2 matrix of (n/2)x(n/2) submatrices

$$
\begin{array}{ccccc}
C & & A & & B \\
\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} & = & \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} & \cdot & \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}
\end{array}
$$

$$c_{11} = a_{11}b_{11} + a_{12}b_{21}$$

Matrix Multiplication: Divide & Conquer

IDEA: Divide the n x n matrix into

2x2 matrix of (n/2)x(n/2) submatrices

$$\overset{C}{\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}} = \overset{A}{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}} \cdot \overset{B}{\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}}$$

$$c_{12} = a_{11} b_{12} + a_{12} b_{22}$$

Matrix Multiplication: Divide & Conquer

IDEA: Divide the n x n matrix into

2x2 matrix of $(n/2)x(n/2)$ submatrices

$$\begin{array}{ccccc} C & & A & & B \\ \left(\begin{array}{c|c} c_{11} & c_{12} \\ \hline c_{21} & c_{22} \end{array}\right) & = & \left(\begin{array}{c|c} a_{11} & a_{12} \\ \hline a_{21} & a_{22} \end{array}\right) & \cdot & \left(\begin{array}{c|c} b_{11} & b_{12} \\ \hline b_{21} & b_{22} \end{array}\right) \end{array}$$

$$c_{21} = a_{21}b_{11} + a_{22}b_{21}$$

Matrix Multiplication: Divide & Conquer

IDEA: Divide the n x n matrix into

2x2 matrix of (n/2)x(n/2) submatrices

$$\begin{array}{ccccc} C & & A & & B \\ \left(\begin{array}{c|c} c_{11} & c_{12} \\ \hline c_{21} & \boxed{c_{22}} \end{array}\right) & = & \left(\begin{array}{c|c} a_{11} & a_{12} \\ \hline \boxed{a_{21}} & \boxed{a_{22}} \end{array}\right) & \cdot & \left(\begin{array}{c|c} b_{11} & \boxed{b_{12}} \\ \hline b_{21} & \boxed{b_{22}} \end{array}\right) \end{array}$$

$$c_{22} = a_{21}b_{12} + a_{22}b_{22}$$

Matrix Multiplication: Divide & Conquer

$$\overset{C}{\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}} = \overset{A}{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}} \cdot \overset{B}{\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}}$$

$$\left.\begin{aligned} c_{11} &= a_{11} b_{11} + a_{12} b_{21} \\ c_{12} &= a_{11} b_{12} + a_{12} b_{22} \\ c_{21} &= a_{21} b_{11} + a_{22} b_{21} \\ c_{22} &= a_{21} b_{12} + a_{22} b_{22} \end{aligned}\right\}$$

8 mults of (n/2)x(n/2) submatrices

4 adds of (n/2)x(n/2) submatrices

Matrix Multiplication: Divide & Conquer

MATRIX-MULTIPLY (A, B)

// Assuming that both A and B are nxn matrices

if n = 1 **then return** A * B

else

partition A, B, and C as shown before

c_{11} = MATRIX-MULTIPLY(a_{11}, b_{11}) + MATRIX-MULTIPLY(a_{12}, b_{21})

c_{12} = MATRIX-MULTIPLY(a_{11}, b_{12}) + MATRIX-MULTIPLY(a_{12}, b_{22})

c_{21} = MATRIX-MULTIPLY(a_{21}, b_{11}) + MATRIX-MULTIPLY(a_{22}, b_{21})

c_{22} = MATRIX-MULTIPLY(a_{21}, b_{12}) + MATRIX-MULTIPLY(a_{22}, b_{22})

return C

Matrix Multiplication: Divide & Conquer Analysis

$$T(n) = 8\,T(n/2) + \Theta(n^2)$$

8 recursive calls

each subproblem has size n/2

submatrix addition

Matrix Multiplication: Solving the Recurrence

$$T(n) = 8\,T(n/2) + \Theta(n^2)$$

$a = 8$, $b = 2$, $f(n) = \Theta(n^2)$, $n^{\log_b a} = n^3$

Case 1: $\dfrac{n^{\log_b a}}{f(n)} = \Omega(n^{\varepsilon})$ $\Rightarrow$ $T(n) = \Theta(n^{\log_b a})$

$T(n) = \Theta(n^3)$

No better than the ordinary algorithm!

Matrix Multiplication: Strassen's Idea

$$\overset{C}{\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}} = \overset{A}{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}} \cdot \overset{B}{\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}}$$

Compute $\mathbf{c_{11}}$, $\mathbf{c_{12}}$, $\mathbf{c_{21}}$, and $\mathbf{c_{22}}$ using **7** recursive multiplications

Matrix Multiplication: Strassen's Idea

$P_1 = a_{11} \times (b_{12} - b_{22})$

$P_2 = (a_{11} + a_{12}) \times b_{22}$

$P_3 = (a_{21} + a_{22}) \times b_{11}$

$P_4 = a_{22} \times (b_{21} - b_{11})$

$P_5 = (a_{11} + a_{22}) \times (b_{11} + b_{22})$

$P_6 = (a_{12} - a_{22}) \times (b_{21} + b_{22})$

$P_7 = (a_{11} - a_{21}) \times (b_{11} + b_{12})$

Reminder: Each submatrix is of size $(n/2)x(n/2)$

Each add/sub operation takes $\Theta(n^2)$ time

Compute $P_1..P_7$ using 7 recursive calls to matrix-multiply

How to compute c_{ij} using $P_1.. P_7$?

Matrix Multiplication: Strassen's Idea

$P_1 = a_{11} \times (b_{12} - b_{22})$

$P_2 = (a_{11} + a_{12}) \times b_{22}$

$P_3 = (a_{21} + a_{22}) \times b_{11}$

$P_4 = a_{22} \times (b_{21} - b_{11})$

$P_5 = (a_{11} + a_{22}) \times (b_{11} + b_{22})$

$P_6 = (a_{12} - a_{22}) \times (b_{21} + b_{22})$

$P_7 = (a_{11} - a_{21}) \times (b_{11} + b_{12})$

$c_{11} = P_5 + P_4 - P_2 + P_6$

$c_{12} = P_1 + P_2$

$c_{21} = P_3 + P_4$

$c_{22} = P_5 + P_1 - P_3 - P_7$

7 recursive multiply calls

18 add/sub operations

Does not rely on commutativity of multiplication

Matrix Multiplication: Strassen's Idea

$$P_1 = a_{11} \mathbf{x} (b_{12} - b_{22})$$

$$P_2 = (a_{11} + a_{12}) \mathbf{x} b_{22}$$

$$P_3 = (a_{21} + a_{22}) \mathbf{x} b_{11}$$

$$P_4 = a_{22} \mathbf{x} (b_{21} - b_{11})$$

$$P_5 = (a_{11} + a_{22}) \mathbf{x} (b_{11} + b_{22})$$

$$P_6 = (a_{12} - a_{22}) \mathbf{x} (b_{21} + b_{22})$$

$$P_7 = (a_{11} - a_{21}) \mathbf{x} (b_{11} + b_{12})$$

e.g. Show that $c_{12} = P_1 + P_2$

$$\begin{aligned} c_{12} &= P_1 + P_2 \\ &= a_{11}(b_{12} - b_{22}) + (a_{11} + a_{12})b_{22} \\ &= a_{11}b_{12} - a_{11}b_{22} + a_{11}b_{22} + a_{12}b_{22} \\ &= a_{11}b_{12} + a_{12}b_{22} \end{aligned}$$

Strassen's Algorithm

1.**Divide**: Partition A and B into $(n/2) \times (n/2)$ submatrices. Form terms to be multiplied using **+** and **–**.

2.**Conquer**: Perform **7** multiplications of $(n/2) \times (n/2)$ submatrices recursively.

3.**Combine**: Form C using **+** and **–** on $(n/2) \times (n/2)$ submatrices.

Recurrence: $T(n) = 7\,T(n/2) + \Theta(n^2)$

Strassen's Algorithm: Solving the Recurrence

$$T(n) = 7\,T(n/2) + \Theta(n^2)$$

$a = 7, \quad b = 2, \quad f(n) = \Theta(n^2), \quad n^{\log_b a} = n^{\lg 7}$

Case 1: $\dfrac{n^{\log_b a}}{f(n)} = \Omega(n^{\varepsilon})$ $\Rightarrow$ $T(n) = \Theta(n^{\log_b a})$

$T(n) = \Theta(n^{\lg 7})$

Note: $\lg 7 \approx 2.81$

Strassen's Algorithm

- The number 2.81 may not seem much smaller than 3

- But, it is significant because the difference is in the exponent.

- For example: $3000^{2.81} = 5{,}898{,}080{,}907$, $3000^3 = 27{,}000{,}000{,}000$

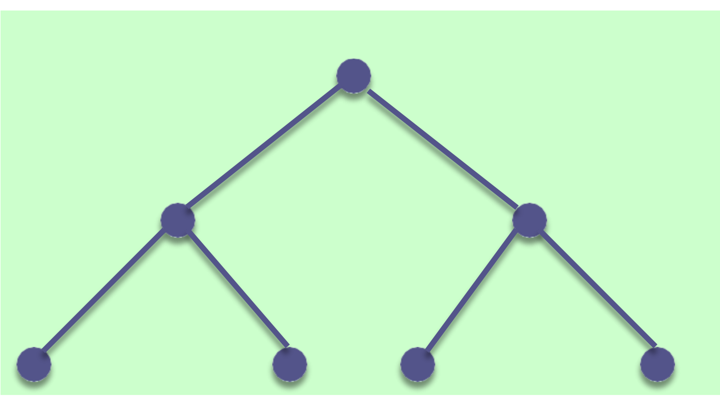
- Strassen's algorithm beats the ordinary algorithm on today's machines for $n \geq 30$ or so.

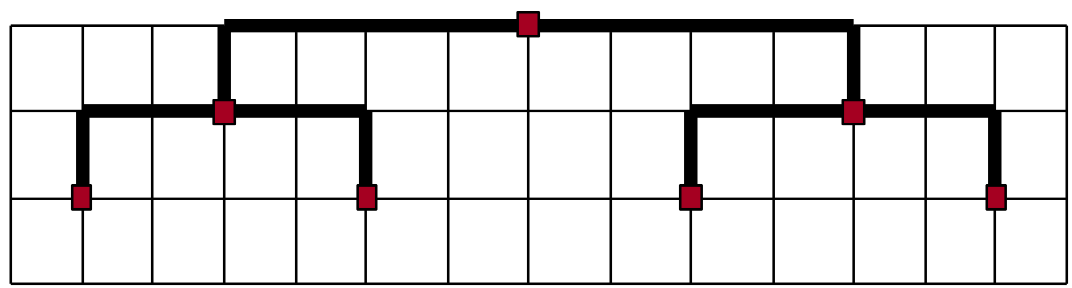
- Best to date: $\Theta(n^{2.376\ldots})$ *(of theoretical interest only)*

VLSI (Very-large-scale integration) Layout: Binary Tree Embedding

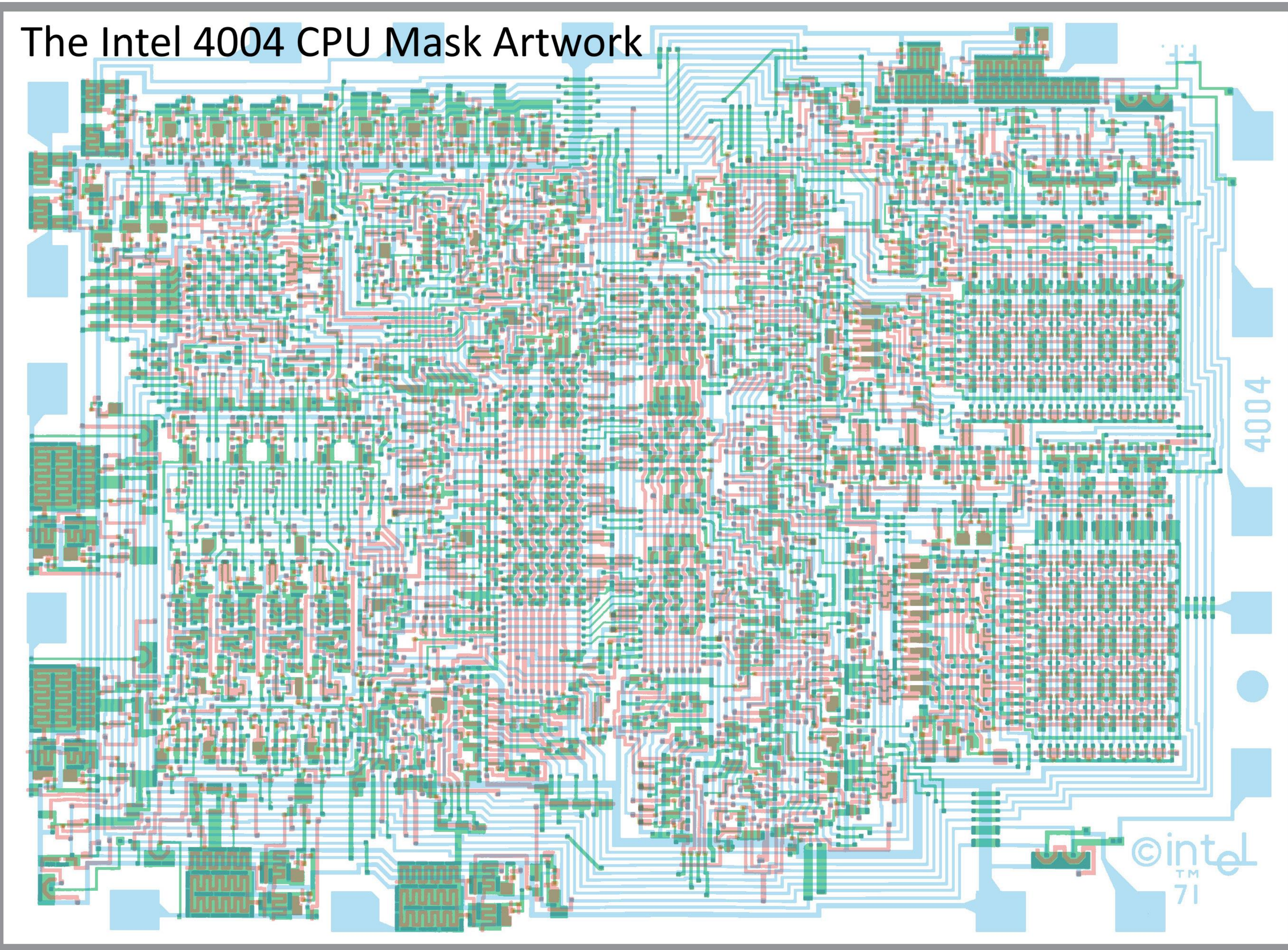
- Problem: Embed a complete binary tree with n leaves into a 2D grid with minimum area.

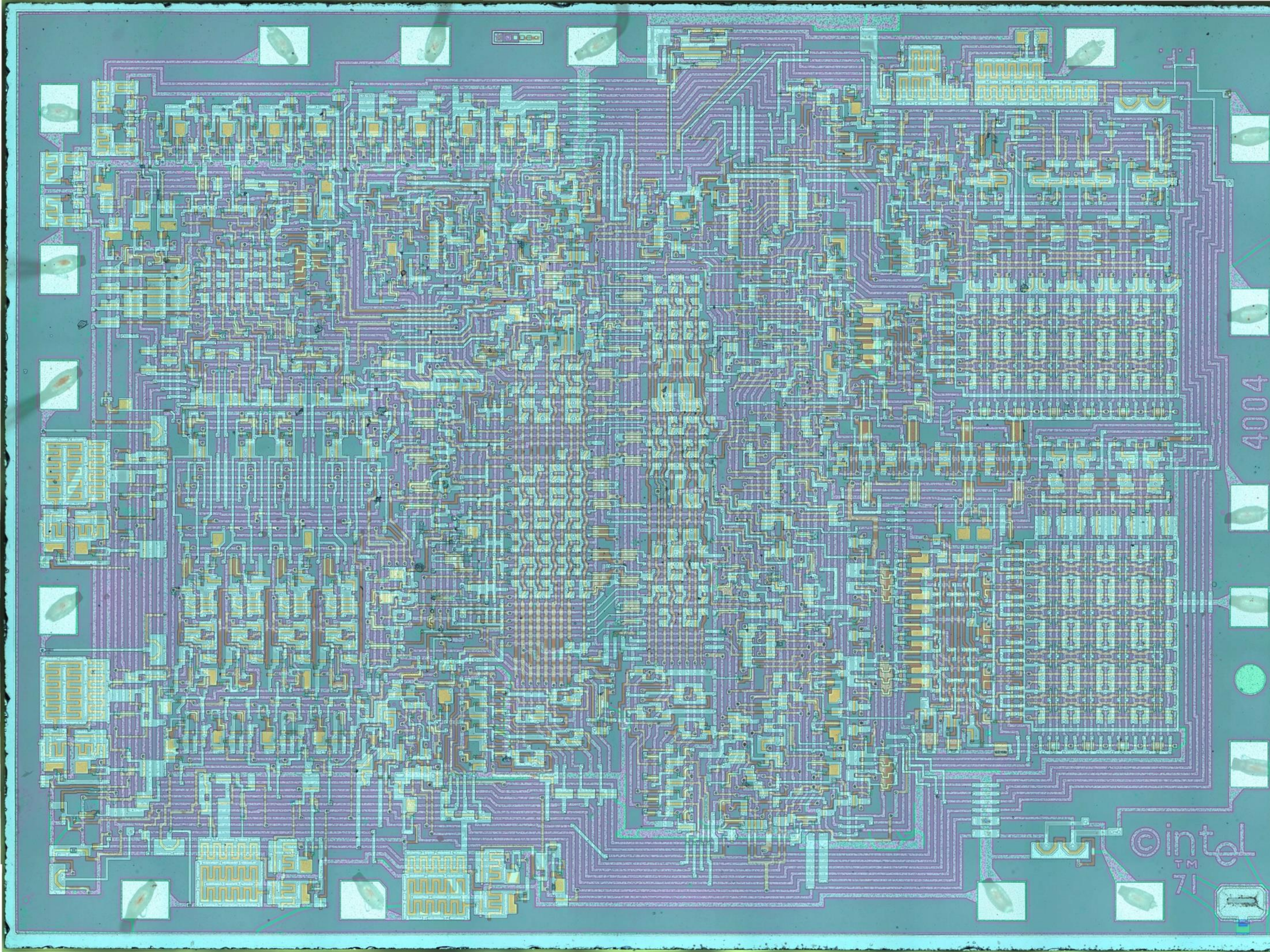
- Example:





The Intel 4004 CPU Mask Artwork

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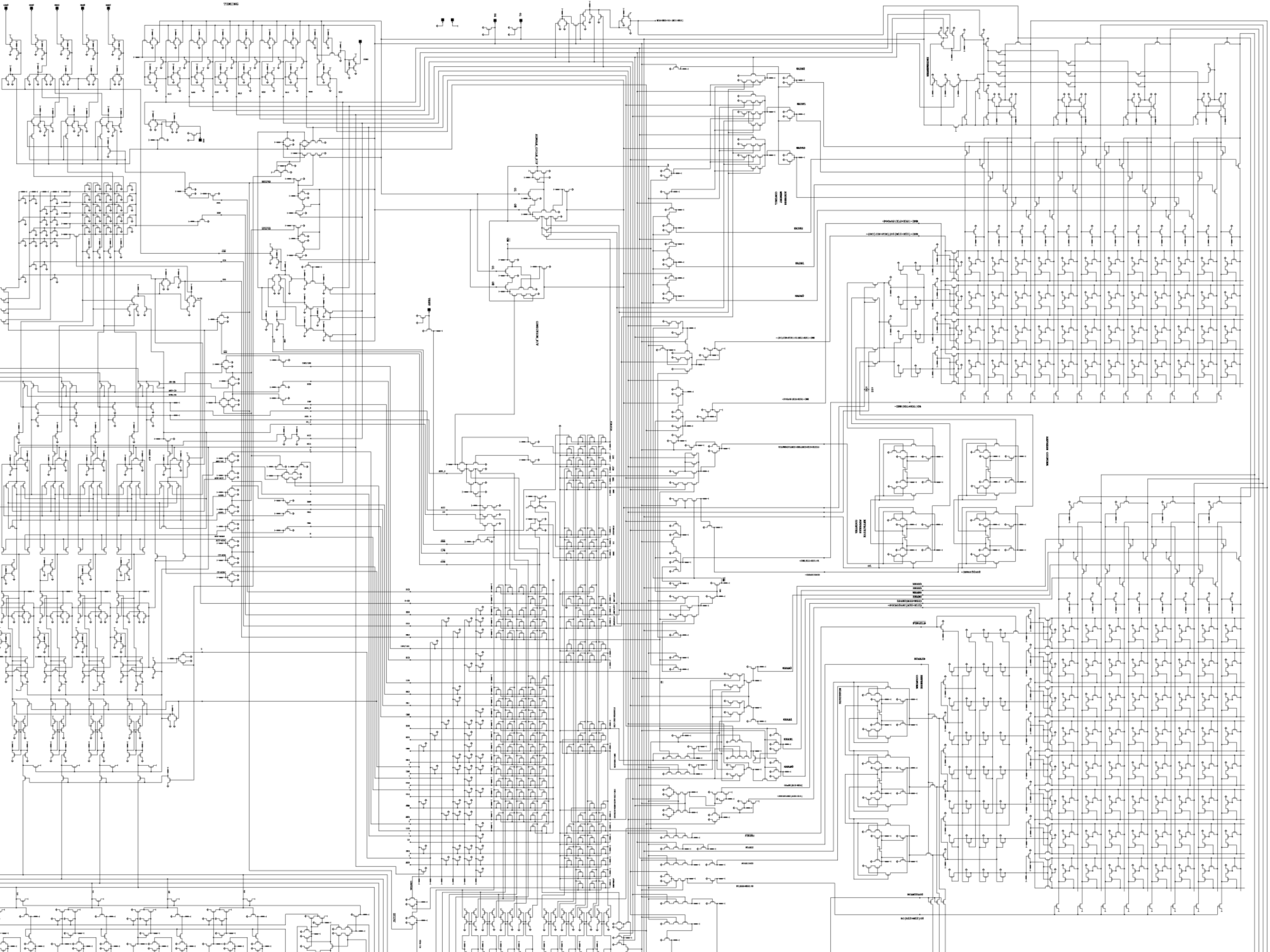
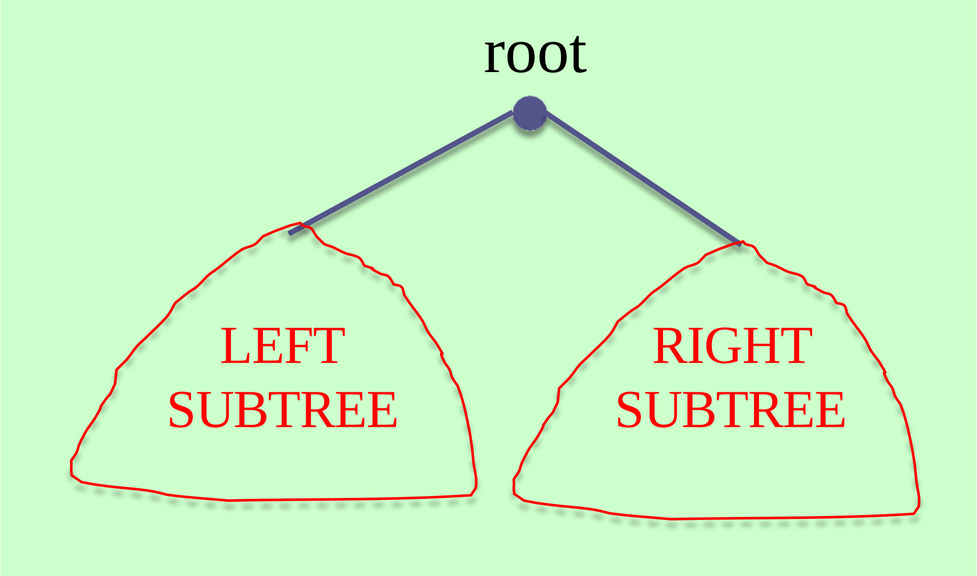

TIMING

Binary Tree Embedding

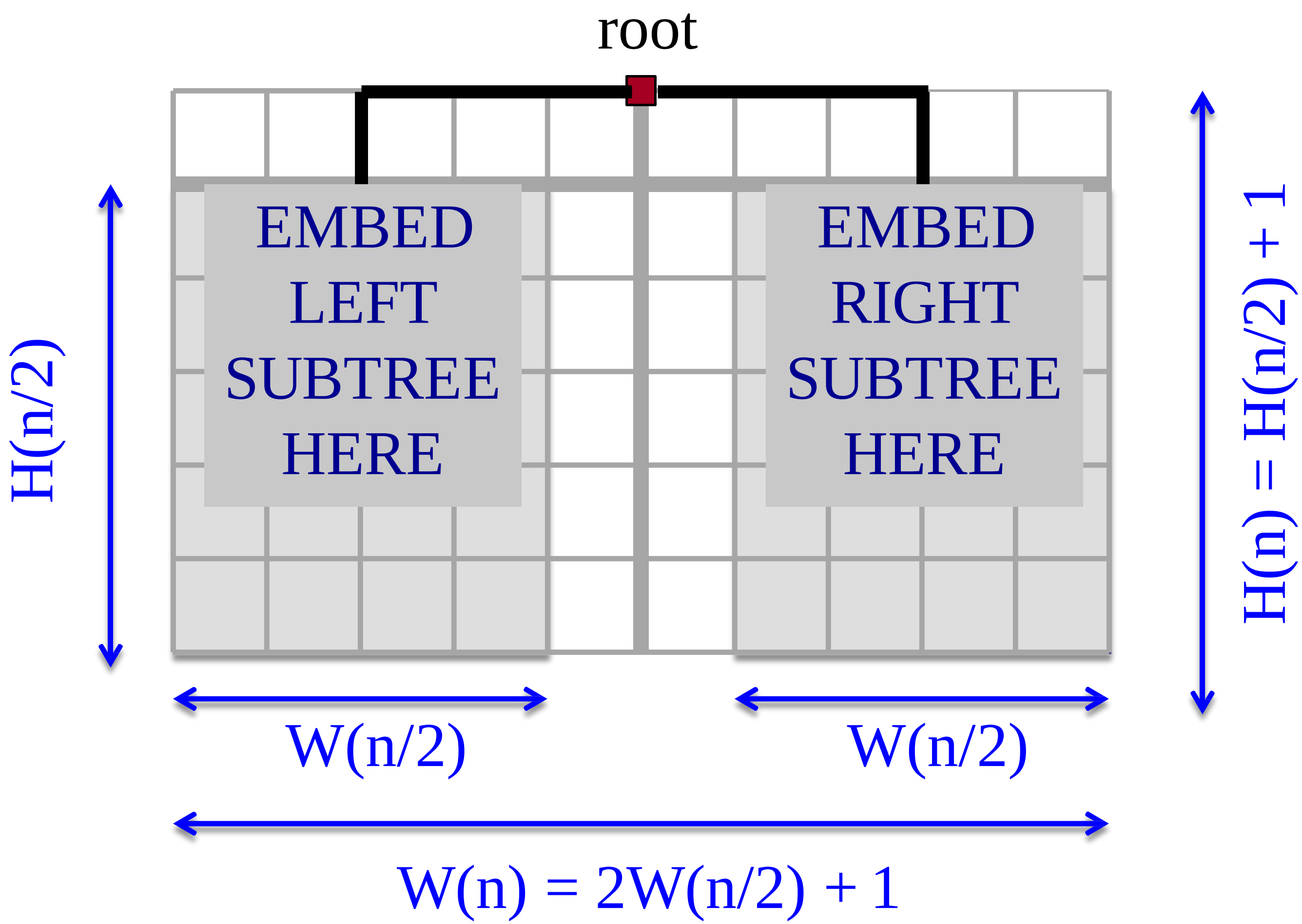
- Use divide and conquer



1. Embed the root node
2. Embed the left subtree
3. Embed the right subtree

What is the min-area required for n leaves?

Binary Tree Embedding



Master Theorem: Reminder

T(n) = aT(n/b) + f(n)

Case 1: $\dfrac{n^{\log_b a}}{f(n)} = \Omega(n^{\varepsilon})$ → $T(n) = \Theta(n^{\log_b a})$

Case 2: $\dfrac{f(n)}{n^{\log_b a}} = \Theta(\lg^k n)$ → $T(n) = \Theta(n^{\log_b a} \lg^{k+1} n)$

Case 3: $\dfrac{f(n)}{n^{\log_b a}} = \Omega(n^{\varepsilon})$

and $a\,f(n/b) \le c\,f(n)$ for $c < 1$ → $T(n) = \Theta(f(n))$

Binary Tree Embedding

- Solve the recurrences:

$W(n) = 2W(n/2) + 1$

$H(n) = H(n/2) + 1$

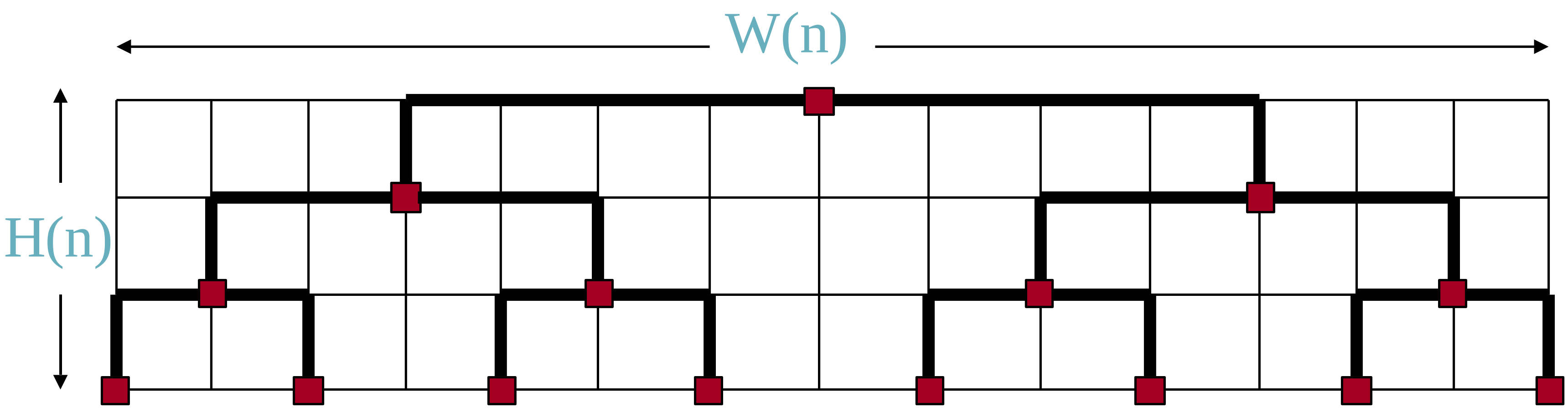
$\Rightarrow W(n) = \Theta(n)$

$\Rightarrow H(n) = \Theta(\lg n)$

- $Area(n) = \Theta(n\lg n)$

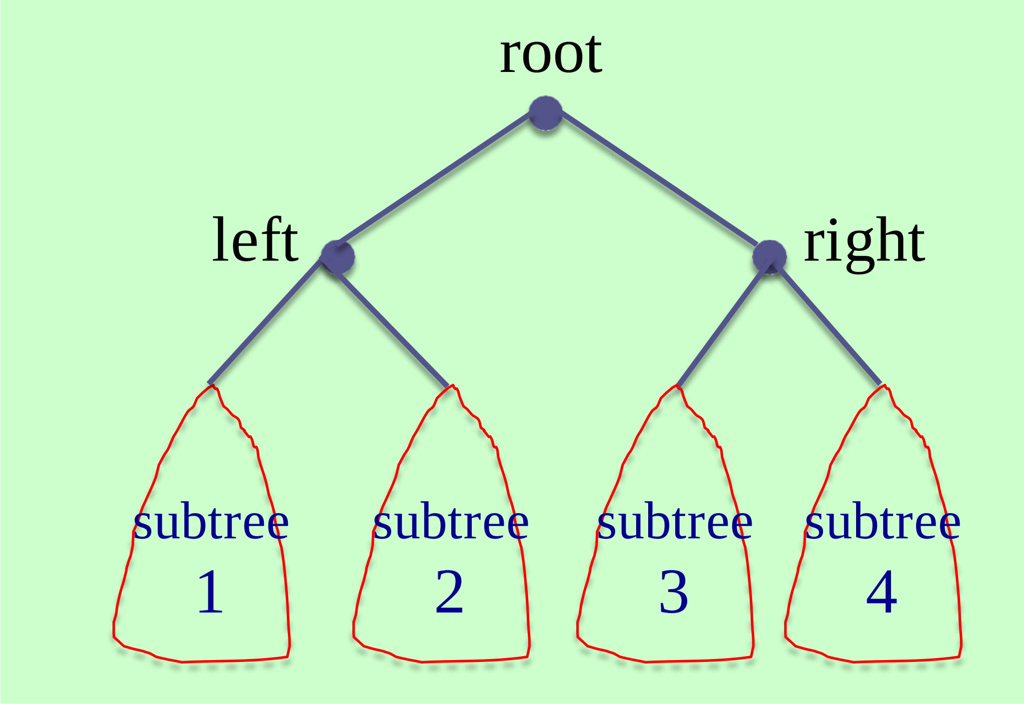
Binary Tree Embedding

Example:



Binary Tree Embedding: H-Tree

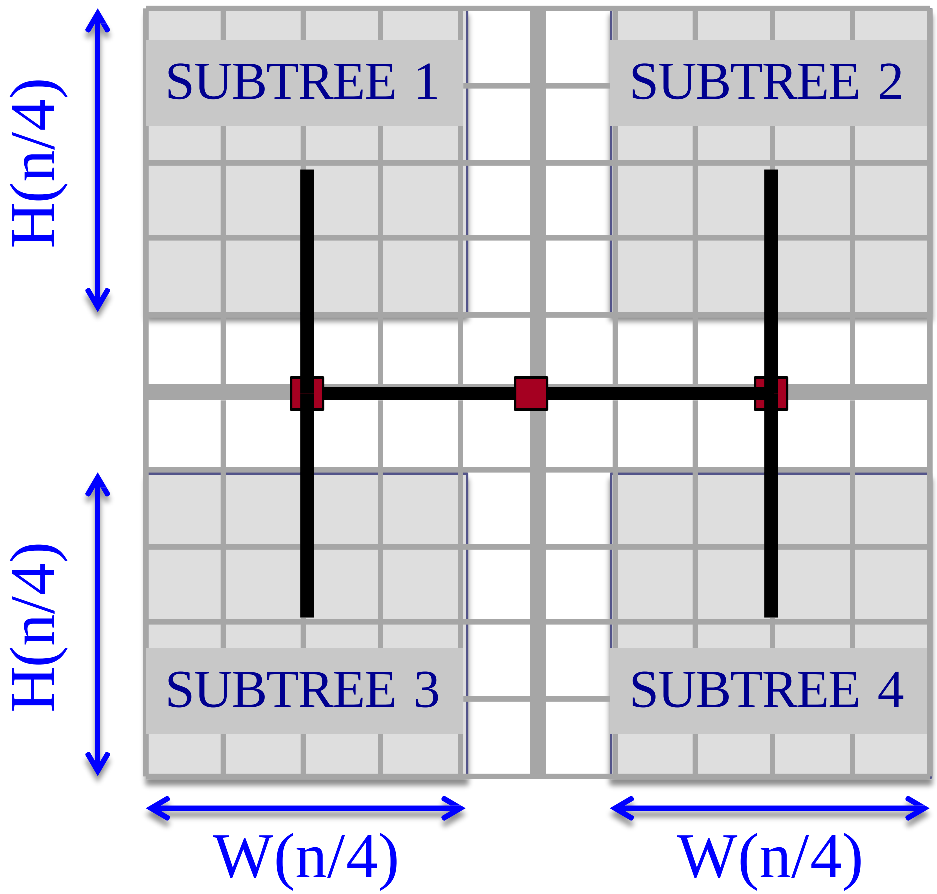
- Use a different divide and conquer method



1. Embed root, left, right nodes
2. Embed subtree 1
3. Embed subtree 2
4. Embed subtree 3
5. Embed subtree 4

What is the min-area required for n leaves?

Binary Tree Embedding: H-Tree



$$W(n) = 2W(n/4) + 1$$

$$H(n) = 2H(n/4) + 1$$

Master Theorem: Reminder

T(n) = aT(n/b) + f(n)

Case 1: $\dfrac{n^{\log_b a}}{f(n)} = \Omega(n^{\varepsilon})$ → $T(n) = \Theta(n^{\log_b a})$

Case 2: $\dfrac{f(n)}{n^{\log_b a}} = \Theta(\lg^k n)$ → $T(n) = \Theta(n^{\log_b a} \lg^{k+1} n)$

Case 3: $\dfrac{f(n)}{n^{\log_b a}} = \Omega(n^{\varepsilon})$

and $a\, f(n/b) \le c\, f(n)$ for $c < 1$ → $T(n) = \Theta(f(n))$

Binary Tree Embedding: H-Tree

- Solve the recurrences:

  $W(n) = 2W(n/4) + 1$

  $H(n) = 2H(n/4) + 1$

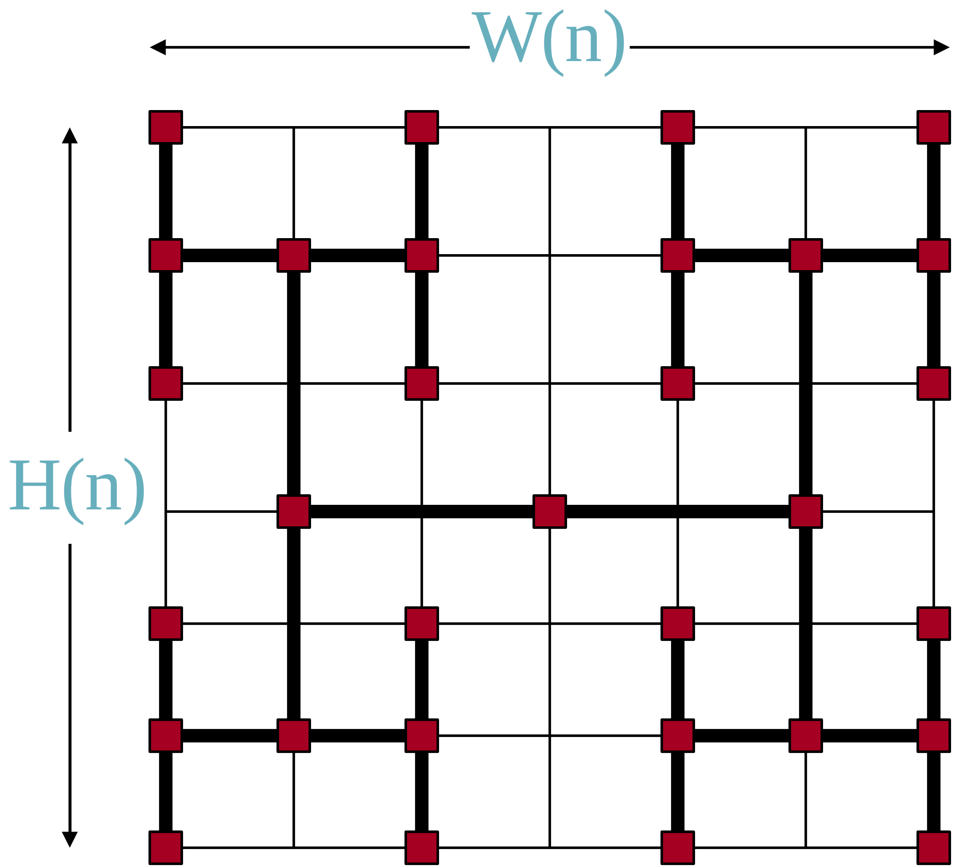
  ➔ $W(n) = \Theta(\sqrt{n})$

  ➔ $H(n) = \Theta(\sqrt{n})$

- $Area(n) = \Theta(n)$

Binary Tree Embedding: H-Tree

Example:



Maximum Subarray Problem

- ***Input***: An array of values
- ***Output***: The contiguous subarray that has the largest sum of elements

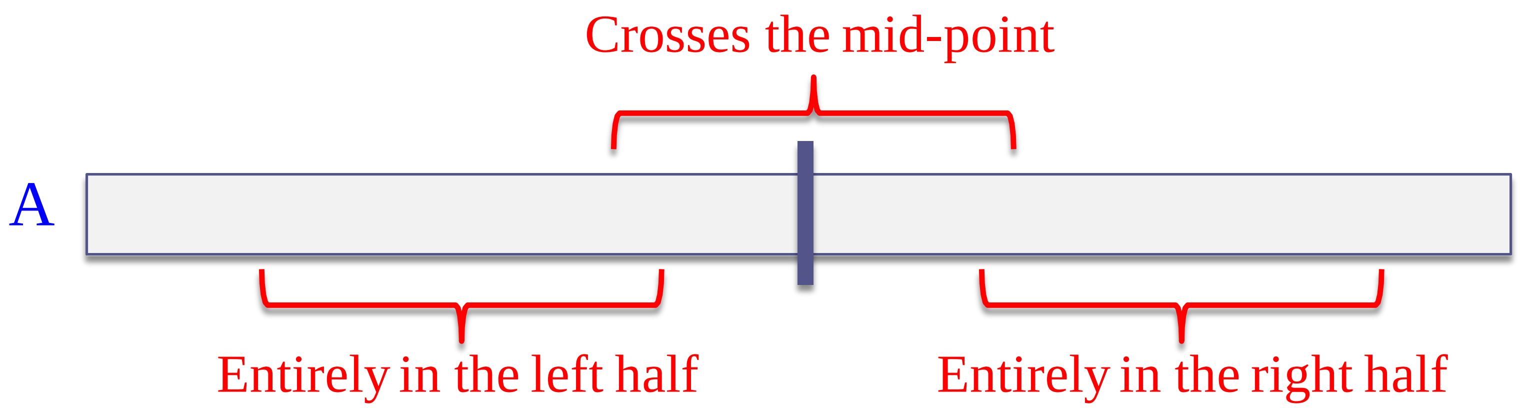
Input array:

13	-3	-25	20	-3	-16	-23	18	20	-7	12	-22	-4	7

the maximum contiguous subarray

Maximum Subarray Problem: Divide & Conquer

- *Basic idea*:
  - Divide the input array into 2 from the middle
  - Pick the best solution among the following:
    1. The max subarray of the left half
    2. The max subarray of the right half
    3. The max subarray crossing the mid-point



Maximum Subarray Problem: Divide & Conquer

- ***Divide***: Trivial (divide the array from the middle)
- ***Conquer***: Recursively compute the max subarrays of the left and right halves
- ***Combine***: Compute the max-subarray crossing the mid-point *(can be done in $\Theta(n)$ time)*. Return the max among the following:
  1. the max subarray of the left subarray
  2. the max subarray of the right subarray
  3. the max subarray crossing the mid-point

Conclusion

- Divide and conquer is just one of several powerful techniques for algorithm design.
- Divide-and-conquer algorithms can be analyzed using recurrences and the master method (so practice this math).
- Can lead to more efficient algorithms