# CSE214 – Analysis of Algorithms

PhD Furkan Gözükara, Toros University

https://github.com/FurkanGozukara/CSE214_2018

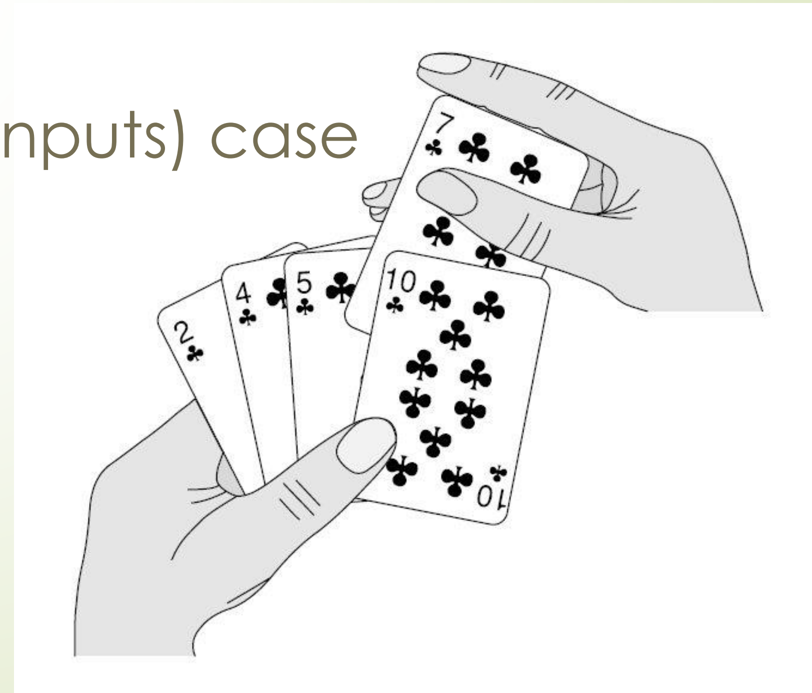
## *Lecture 6*

## *Linear Sorting*

*Based on Andrew Davison's Lecture Notes*

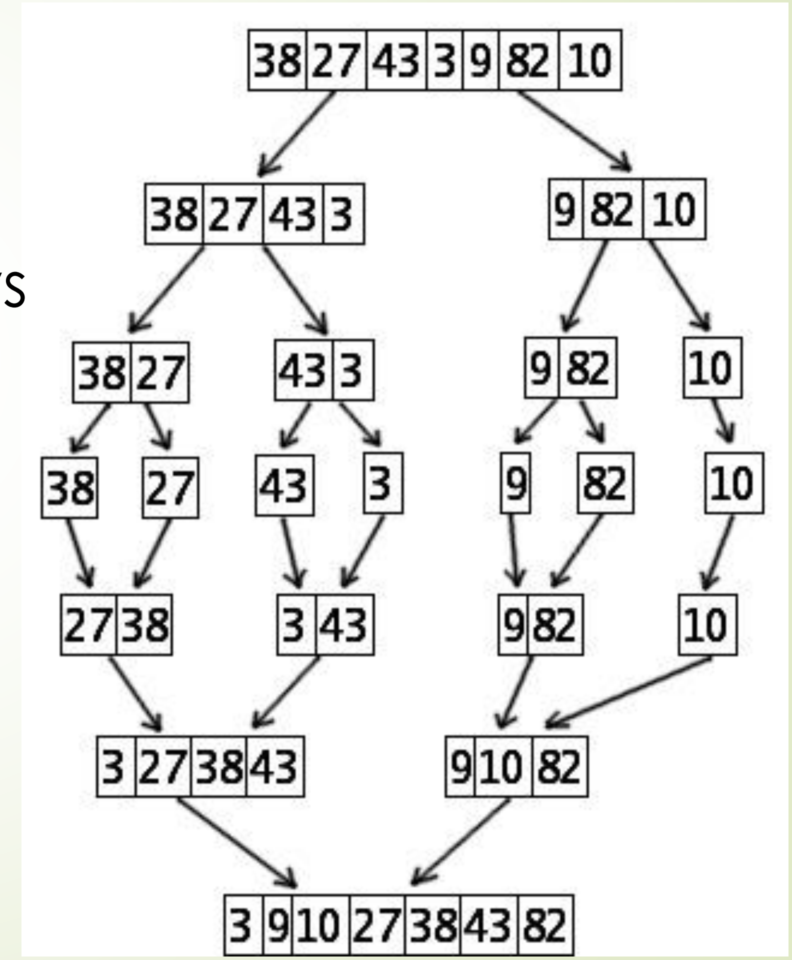
# Sorting So Far

- **Insertion sort**:
  - Easy to code
  - Fast on small inputs (less than ~50 elements)
  - Fast on nearly-sorted inputs
  - $O(n^2)$ worst case
  - $O(n^2)$ average (equally-likely inputs) case
  - $O(n^2)$ reverse-sorted case



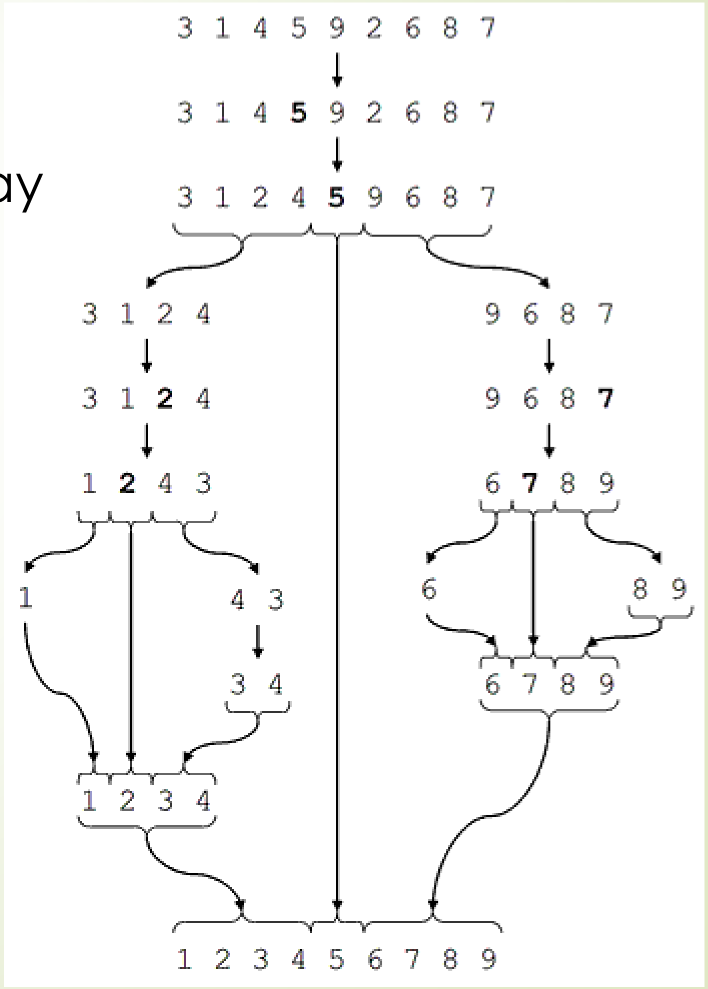
# Sorting So Far

- **Merge sort**:
  - Divide-and-conquer:
    - Split array in half
    - Recursively sort subarrays
    - Linear-time merge step
  - O(n lg n) worst case
  - Doesn't sort in place



# Sorting So Far

- **Quicksort**:
  - Divide-and-conquer:
    - Partition array into two subarrays, recursively sort
    - All of 1st subarray < all of 2nd subarray
    - No merge step needed!
  - O(n lg n) average case
  - Fast in practice
  - O($n^2$) worst case
    - for sorted input
    - this is avoided by using a randomized pivot



# How Fast Can We Sort?

- Selection Sort, Bubble Sort, Insertion Sort: *$O(n^2)$*
- Heap Sort, Merge sort: *$O(nlgn)$*
- Quicksort: *$O(nlgn)$ - average*
- What is common to all these algorithms?
  - Make **comparisons** between input elements

$a_i < a_j$, $a_i \leq a_j$, $a_i = a_j$, $a_i \geq a_j$, or $a_i > a_j$

# How Fast Can We Sort?

- All these sorting algorithms are *comparison sorts*
  - gain ordering information about a sequence using the comparison of two elements (=, <, >)
  - Theorem: all **comparison sorts** are **O(n lg n)** or slower
  - The **best speed for sorting is O(n)**
    - we must look at all the data
    - for that we must use sorting algorithms which don't require comparisons of all the data

## Can we do better?

- Linear sorting algorithms
  - Counting Sort
  - Radix Sort
  - Bucket sort
- Make certain assumptions about the data
- Linear sorts are NOT “comparison sorts”

# Non-Comparison Based Sorting

- Many times we have restrictions on our keys
  - Deck of cards: Ace->King and four suites
  - Social Security Numbers
  - Employee ID’s
- We will examine three algorithms which under certain conditions can run in **O(*n*)** time.
  - Counting sort
  - Radix sort
  - Bucket sort

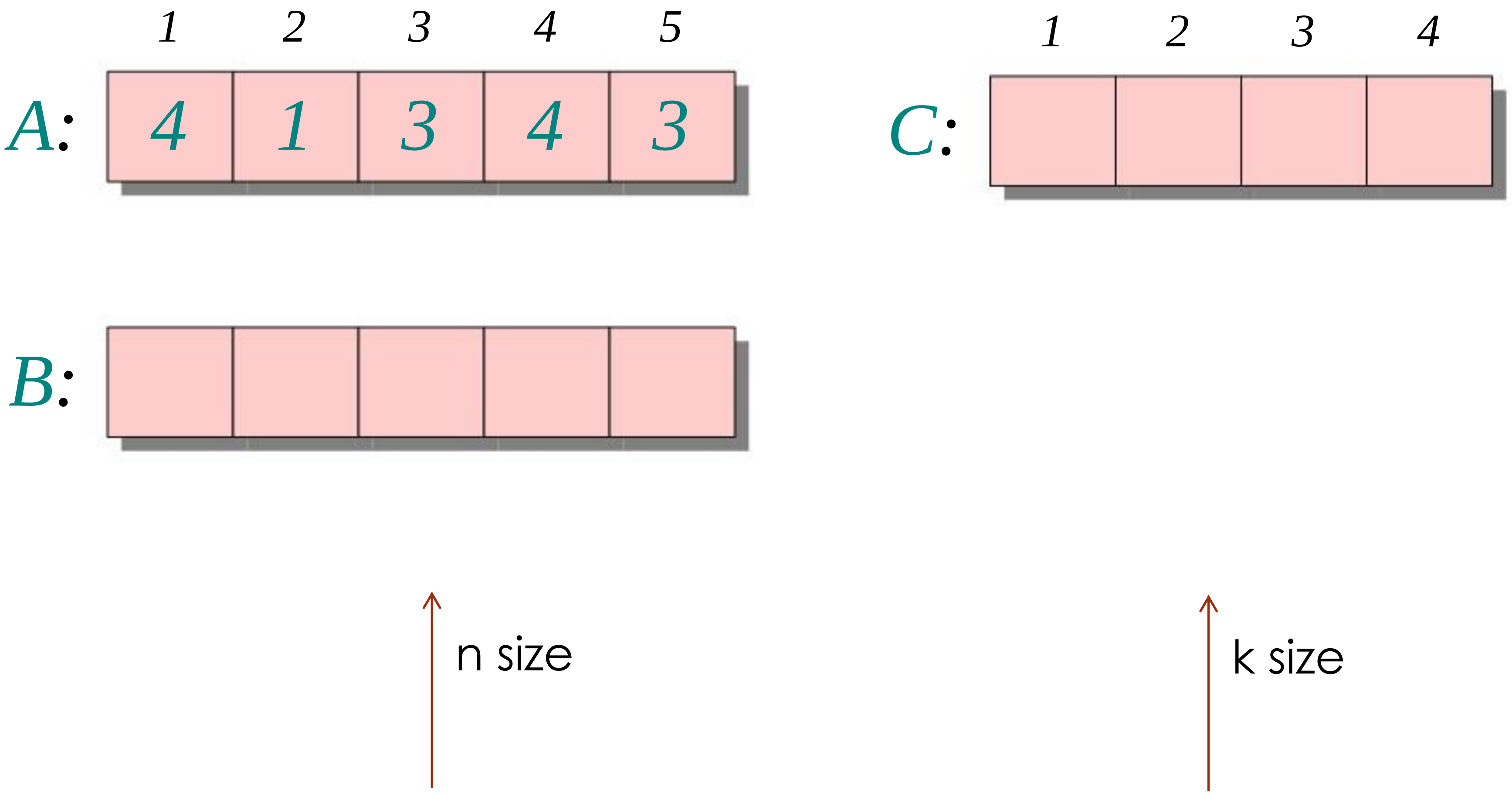
# Counting Sort

- Does no comparisons between the array elements!
- This depends on an assumption about the numbers being sorted
  - We assume numbers are integers in the range *1.. k*
  - *running time depends on k, so might be slower than comparison sorts*
- The algorithm:
  - Input: A[1 .. *n*], where A[j] ∈ {1, 2, 3, …, *k*}
  - Output: B[1 .. *n*], sorted (notice: not sorting in place)
  - Also: Array C[1 .. ***k***] for auxiliary storage

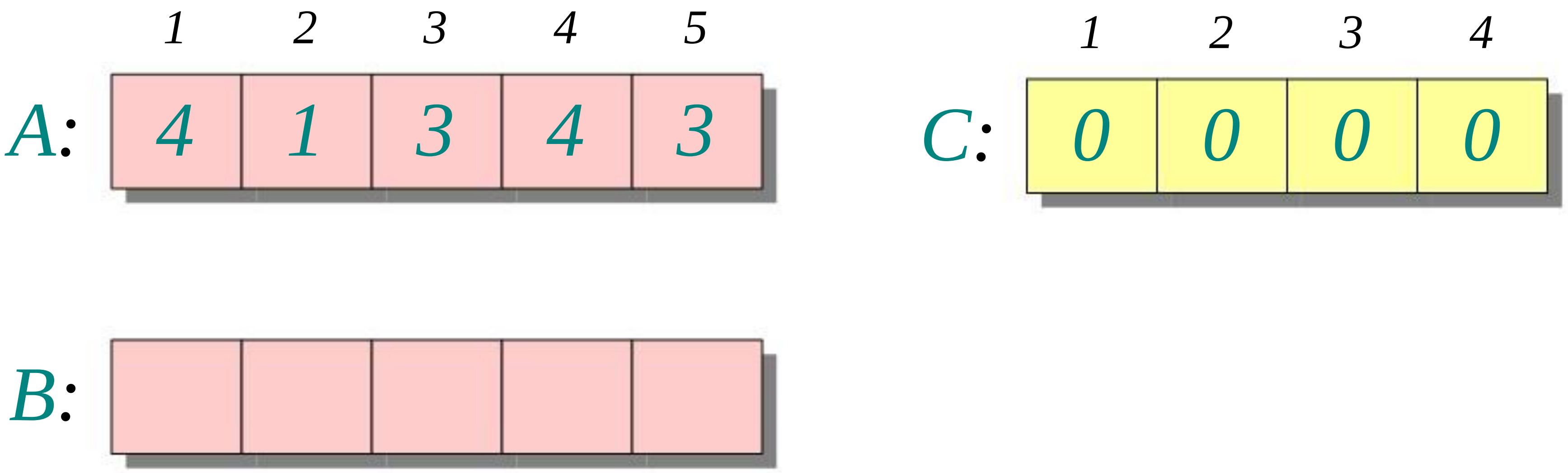
# Pseudocode

```
CountingSort(int[] A, int[] B, int k)    // sort A into B; elems range: 1..k
for i ← 1 to k    // initialization count occurences array
    do C[i] ←0

for j ← 1 to n     // counting
    do C[A[j]] ← C[A[j]] + 1    //  C[i] = |{key == i}|

for i ← 2 to k    // summing
    do C[i] ←C[i] + C[i–1]       //  C[i] = |{key ≤ i}|

for j ← n down to 1     // create output array (distribution)
    do   B[ C[ A[j] ] ]  ← A[j]
         C[ A[j] ]  ←  C[ A[j] ] –1
```

# *Counting-sort example*

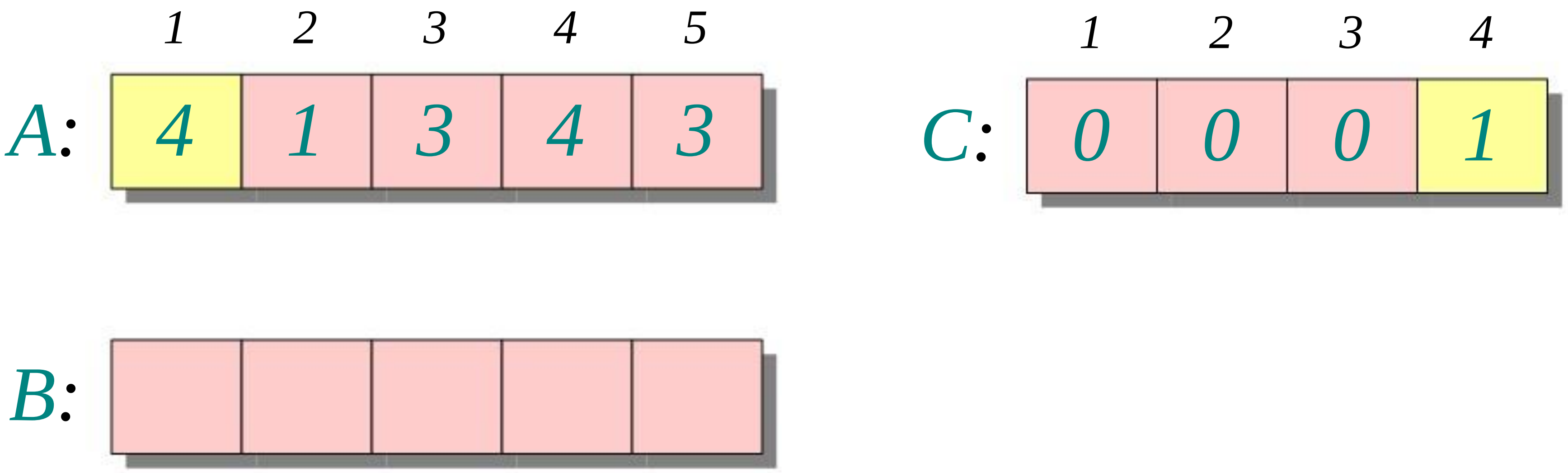


## Loop 1



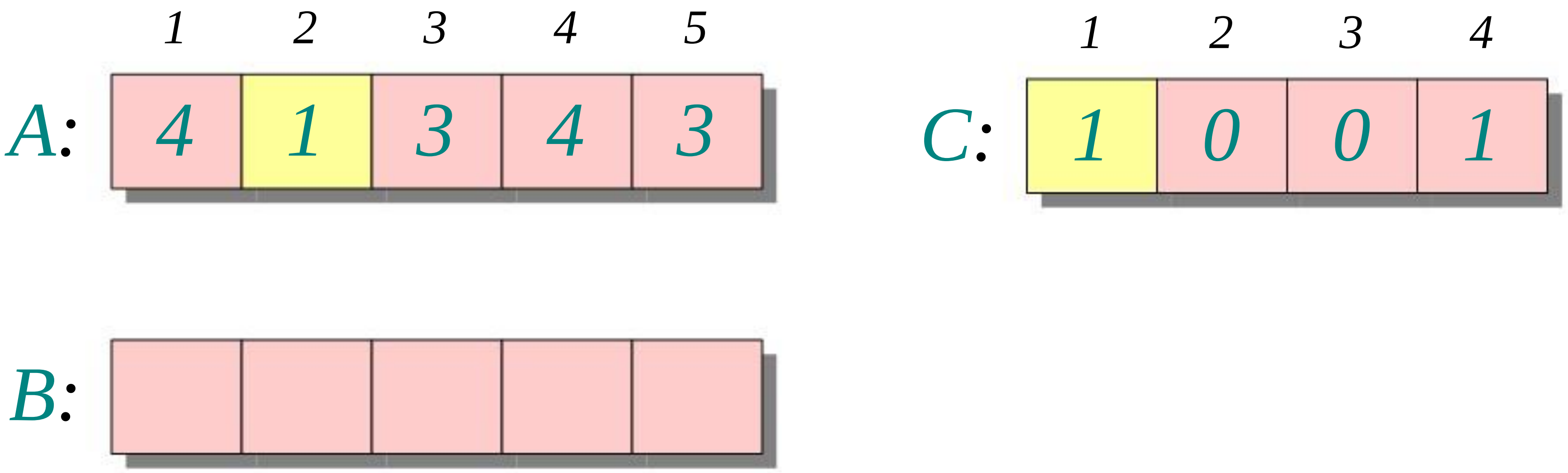
```
for $i \leftarrow 1$ to $k$
    do $C[i] \leftarrow 0$
```

## Loop 2



```
for $j \leftarrow 1$ to $n$
    do $C[A[j]] \leftarrow C[A[j]] + 1$
                              // $C[i] == |\{key = i\}|$
```

## Loop 2



```
for $j \leftarrow 1$ to $n$
    do $C[A[j]] \leftarrow C[A[j]] + 1$
```

// $C[i] = |\{key == i\}|$

# Loop 2

| | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| A: | 4 | 1 | 3 | 4 | 3 |

| | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| C: | 1 | 0 | 1 | 1 |

| | | | | | |
|---|---|---|---|---|---|
| B: | | | | | |

```
for $j \leftarrow 1$ to $n$
    do $C[A[j]] \leftarrow C[A[j]] + 1$
                                        // $C[i] = |\{key == i\}|$
```

## Loop 2

| | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| A: | 4 | 1 | 3 | 4 | 3 |

| | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| C: | 1 | 0 | 1 | 2 |

| | | | | | |
|---|---|---|---|---|---|
| B: | | | | | |

```
for $j \leftarrow 1$ to $n$
    do $C[A[j]] \leftarrow C[A[j]] + 1$
```

// $C[i] = |\{key == i\}|$

## Loop 2

C is the number of data occurences

| | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| A: | 4 | 1 | 3 | 4 | 3 |

| | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| C: | 1 | 0 | 2 | 2 |

| | | | | | |
|---|---|---|---|---|---|
| B: | | | | | |

```
for $j \leftarrow 1$ to $n$
    do $C[A[j]] \leftarrow C[A[j]] + 1$
```

// $C[i] = |\{key == i\}|$

# Loop 3

C is changed to hold prefix sums

|  | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| A: | 4 | 1 | 3 | 4 | 3 |
| B: |  |  |  |  |  |

|  | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| C: | 1 | 0 | 2 | 2 |
| C': | 1 | 1 | 2 | 2 |

```
for $i \leftarrow 2$ to $k$
    do $C[i] \leftarrow C[i] + C[i-1]$
```

// $C[i] = |\{\text{key} \leq i\}|$

## Loop 3

A:

| 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|
| 4 | 1 | 3 | 4 | 3 |

C:

| 1 | 2 | 3 | 4 |
|---|---|---|---|
| 1 | 0 | 2 | 2 |

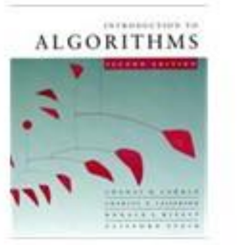
B:

| | | | | |
|---|---|---|---|---|
| | | | | |

C':

| | | | |
|---|---|---|---|
| 1 | 1 | 3 | 2 |

```
for $i \leftarrow 2$ to $k$
    do $C[i] \leftarrow C[i] + C[i-1]$        // $C[i] = |\{key \le i\}|$
```



## Loop 3

A:

| 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|
| 4 | 1 | 3 | 4 | 3 |

C:

| 1 | 2 | 3 | 4 |
|---|---|---|---|
| 1 | 0 | 2 | 2 |

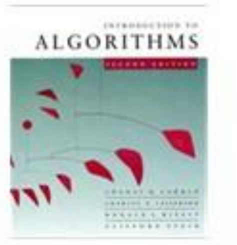
B:

| | | | | |
|---|---|---|---|---|
| | | | | |

C':

| | | | |
|---|---|---|---|
| 1 | 1 | 3 | 5 |

```
for $i \leftarrow 2$ to $k$
    do $C[i] \leftarrow C[i] + C[i-1]$        // $C[i] = |\{key \leq i\}|$
```



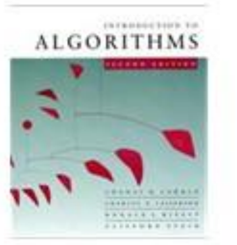
# Loop 4

B is will store the A data in its correct position (distribution) based on C

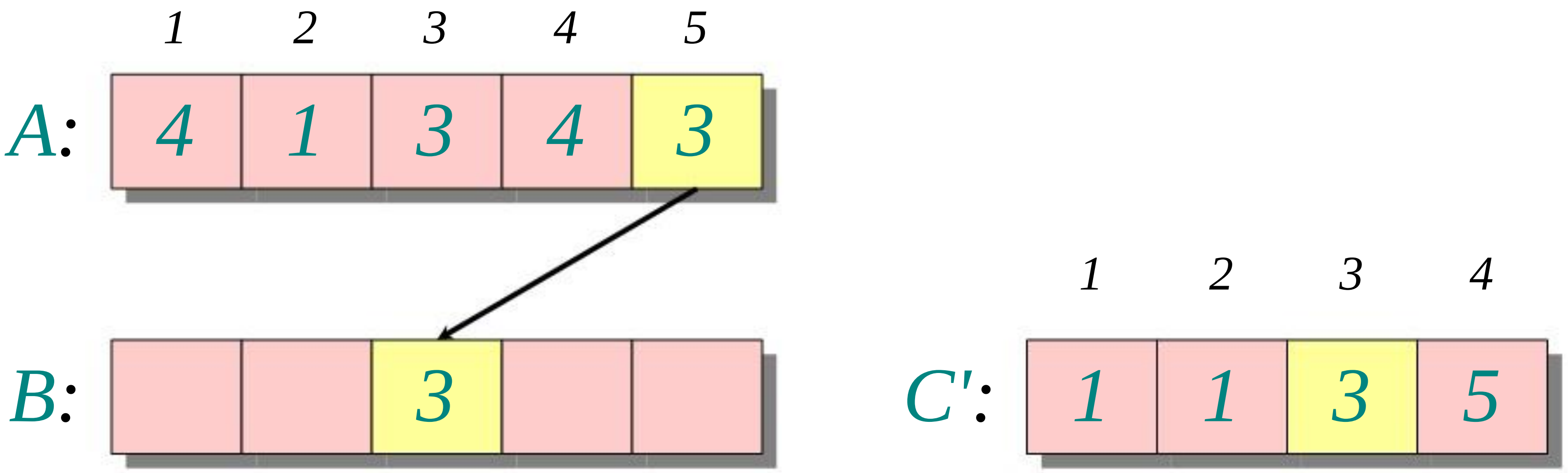
| | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| A: | 4 | 1 | 3 | 4 | 3 |
| B: | | | | | |

| | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| C': | 1 | 1 | 3 | 5 |

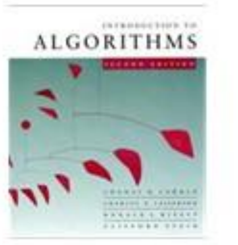
```
for $j \leftarrow n$ downto $1$
    do $B[C[A[j]]] \leftarrow A[j]$
       $C[A[j]] \leftarrow C[A[j]] - 1$
```



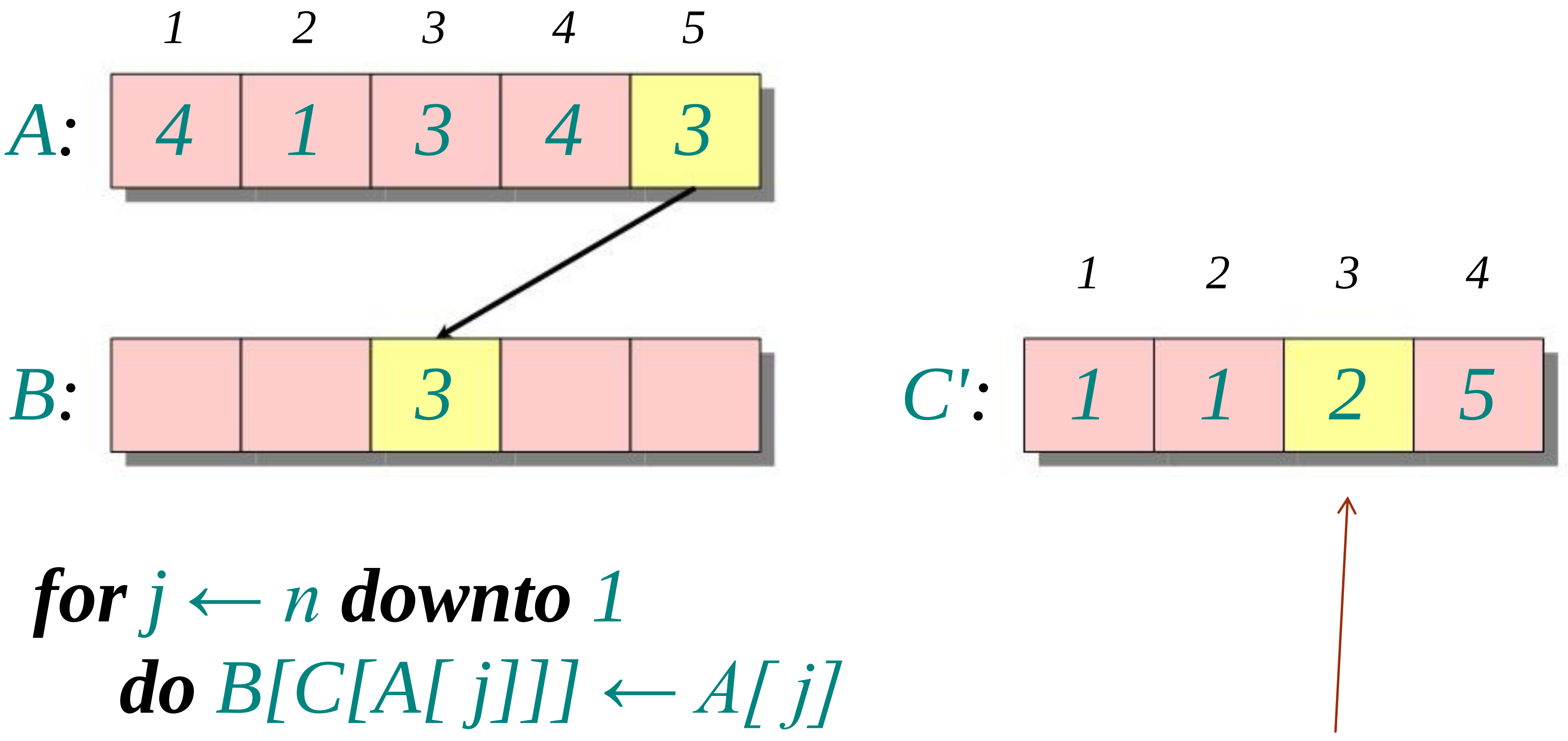
## Loop 4



```
for $j \leftarrow n$ downto $1$
    do $B[C[A[j]]] \leftarrow A[j]$
       $C[A[j]] \leftarrow C[A[j]] - 1$
```

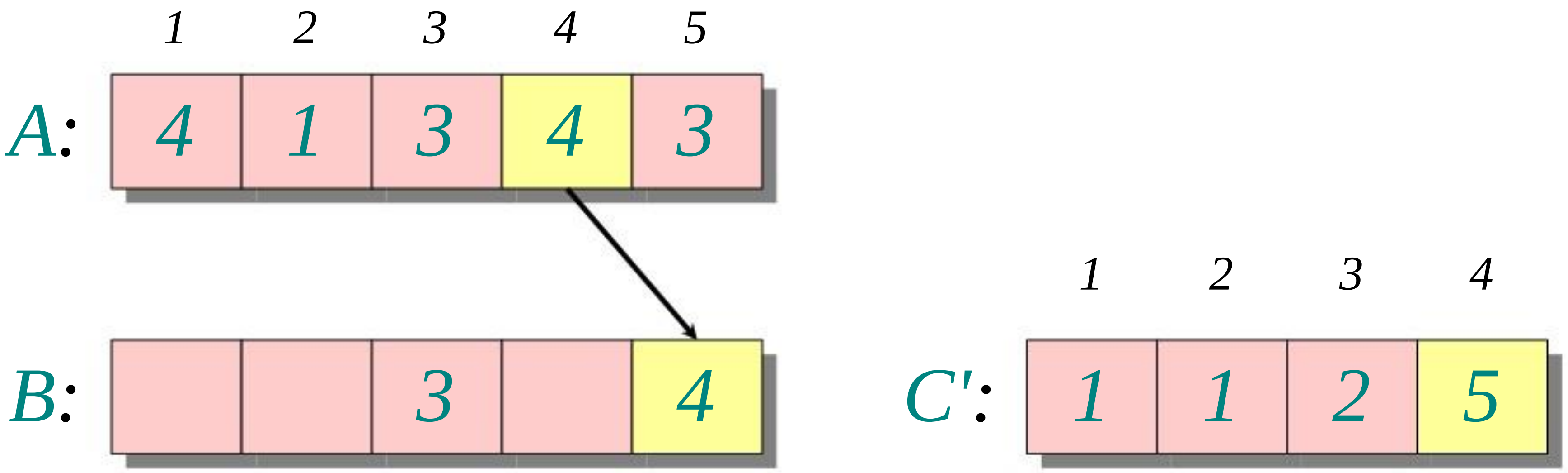


# Loop 4

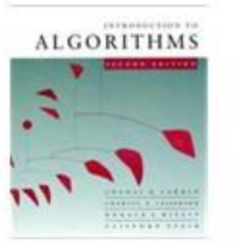


```
for $j \leftarrow n$ downto $1$
    do $B[C[A[j]]] \leftarrow A[j]$
       $C[A[j]] \leftarrow C[A[j]] - 1$        // decr counter
```

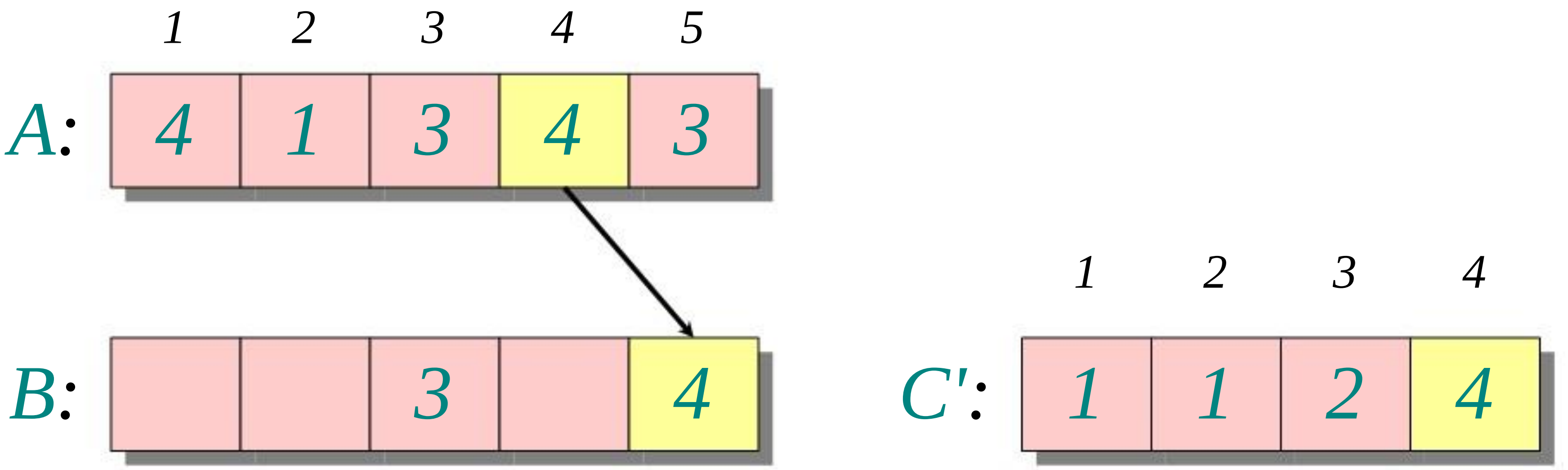
## Loop 4



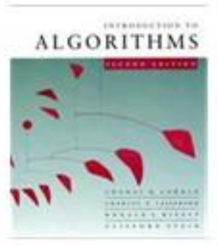
```
for $j \leftarrow n$ downto $1$
    do $B[C[A[j]]] \leftarrow A[j]$
       $C[A[j]] \leftarrow C[A[j]] - 1$
```



## Loop 4

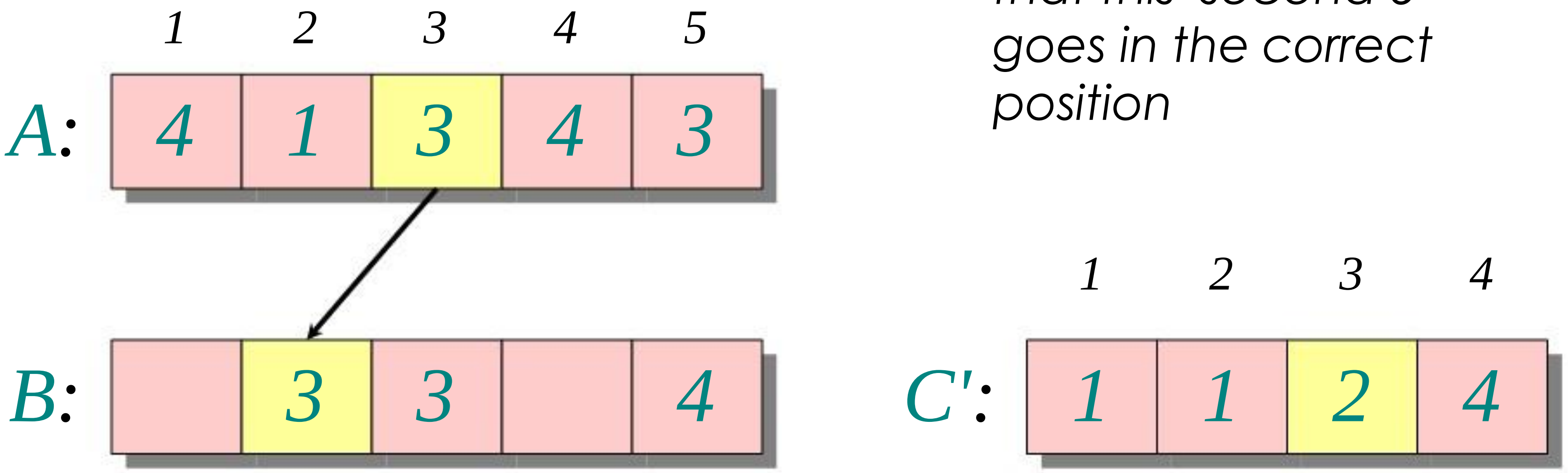


```
for $j \leftarrow n$ downto $1$
    do $B[C[A[j]]] \leftarrow A[j]$
       $C[A[j]] \leftarrow C[A[j]] - 1$        // decr counter
```



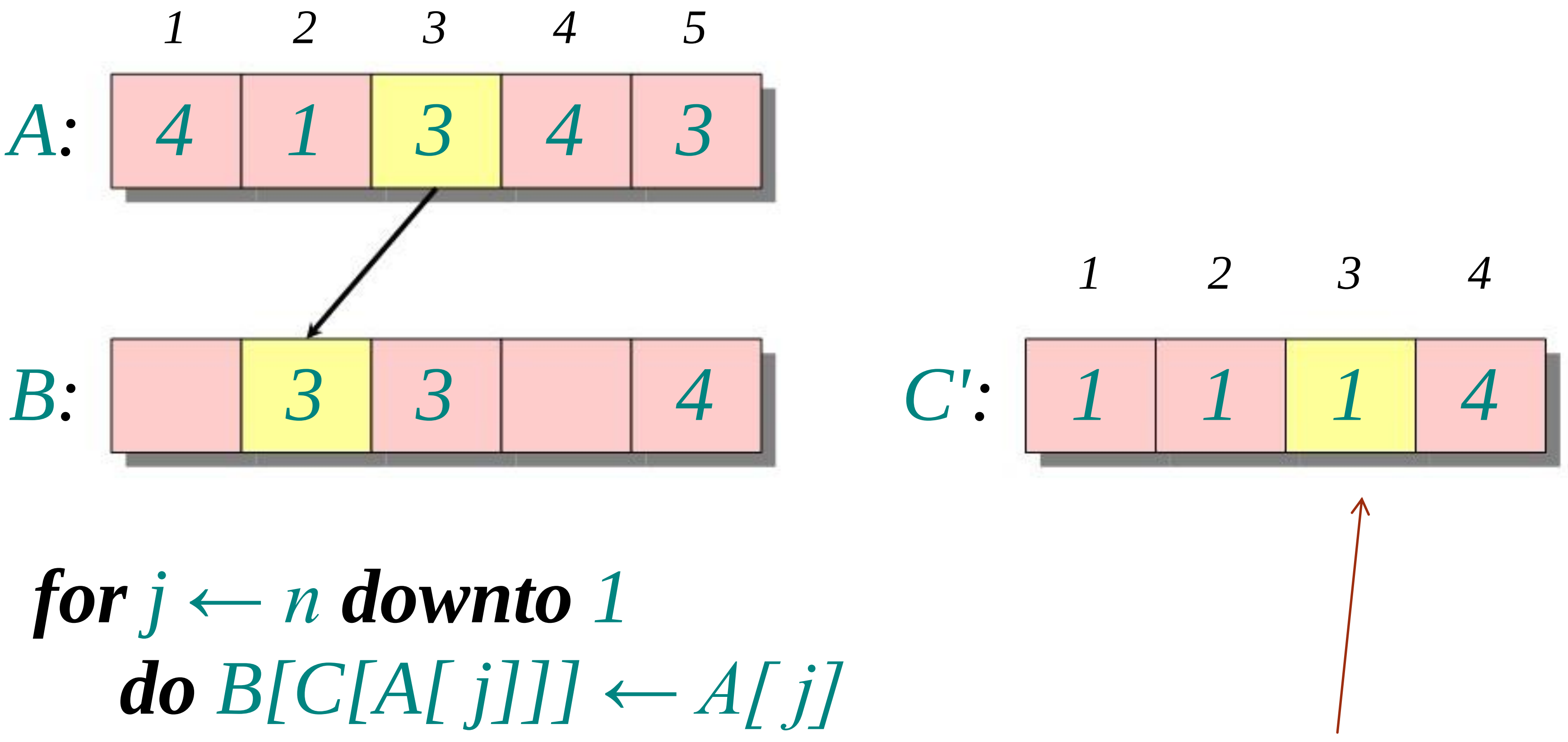
## Loop 4

the decrement means that this second 3 goes in the correct position



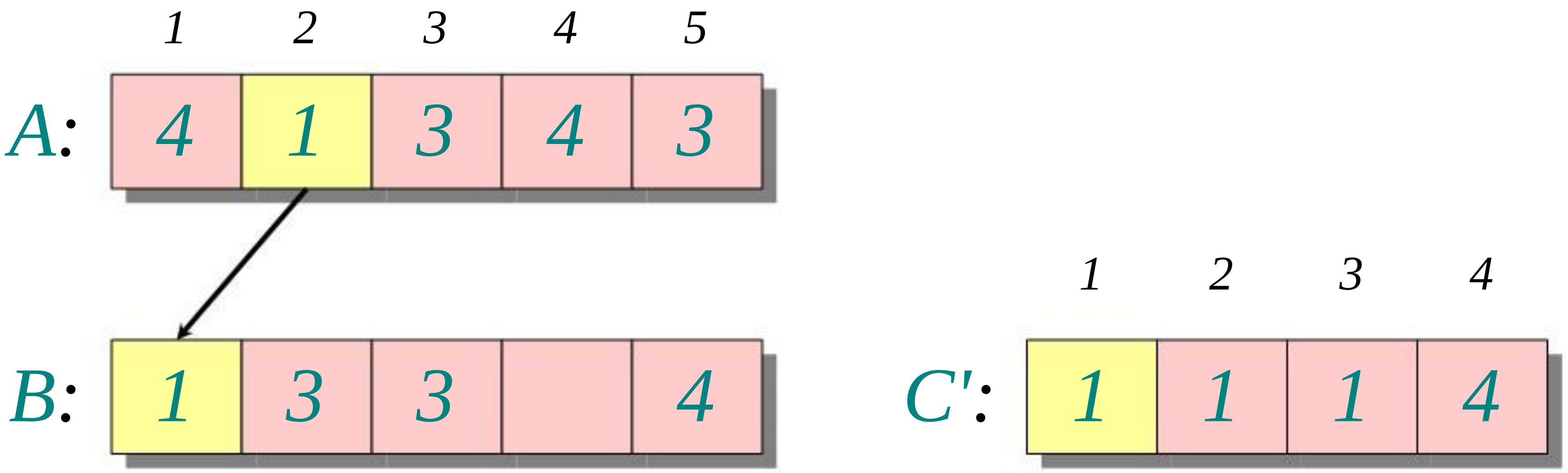
```
for $j \leftarrow n$ downto $1$
    do $B[C[A[j]]] \leftarrow A[j]$
       $C[A[j]] \leftarrow C[A[j]] - 1$
```

## Loop 4

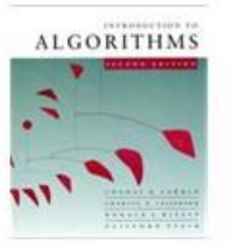


```
for $j \leftarrow n$ downto $1$
    do $B[C[A[j]]] \leftarrow A[j]$
       $C[A[j]] \leftarrow C[A[j]] - 1$        // decr counter
```

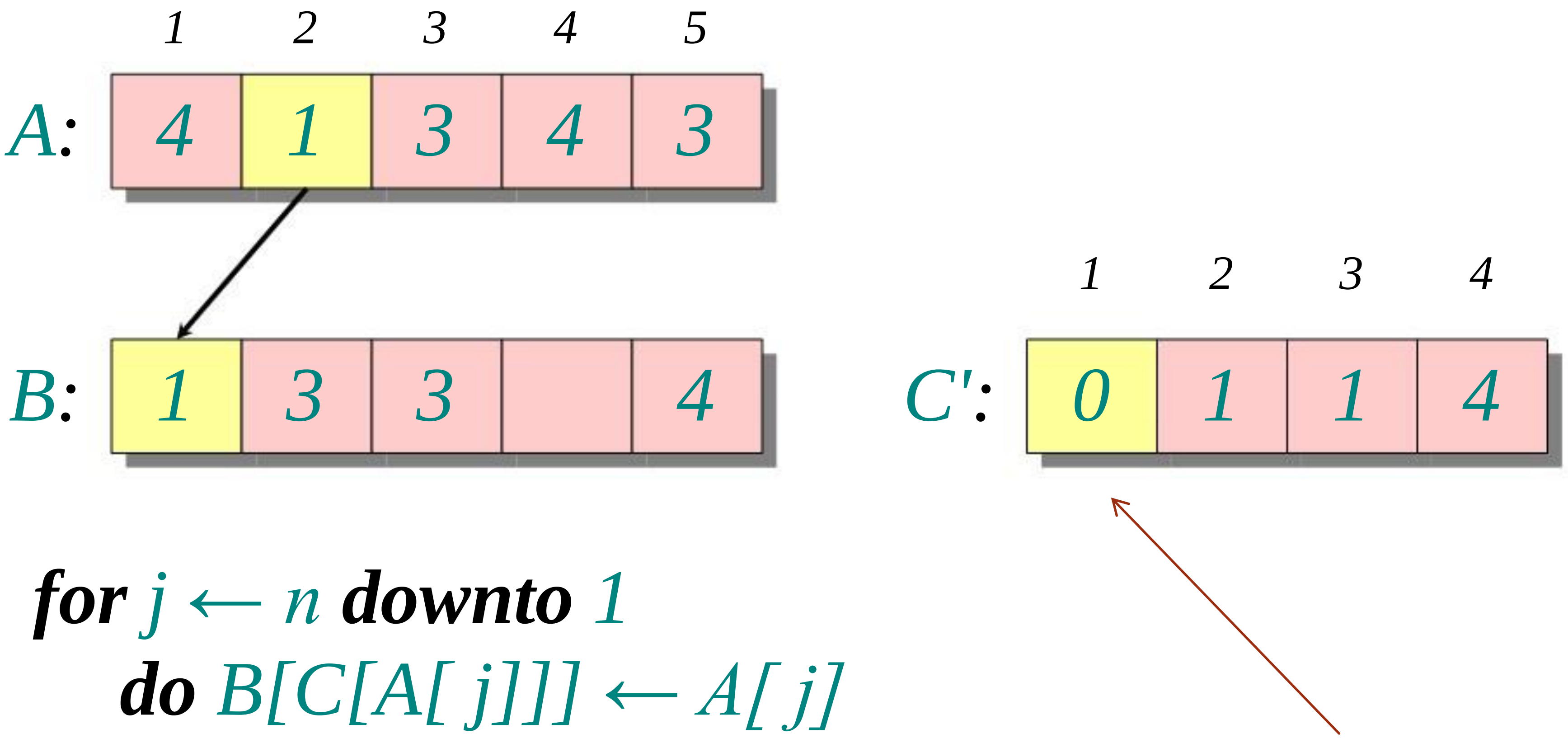
## Loop 4



```
for $j \leftarrow n$ downto $1$
    do $B[C[A[j]]] \leftarrow A[j]$
       $C[A[j]] \leftarrow C[A[j]] - 1$
```



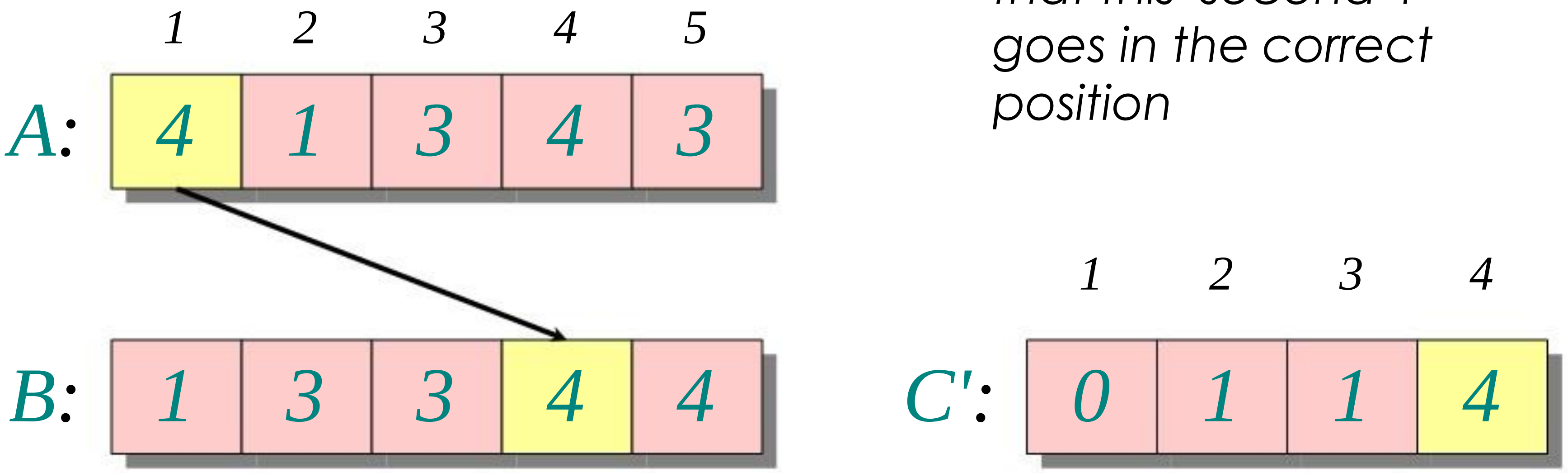
# Loop 4



```
for $j \leftarrow n$ downto $1$
    do $B[C[A[j]]] \leftarrow A[j]$
       $C[A[j]] \leftarrow C[A[j]] - 1$        // decr counter
```

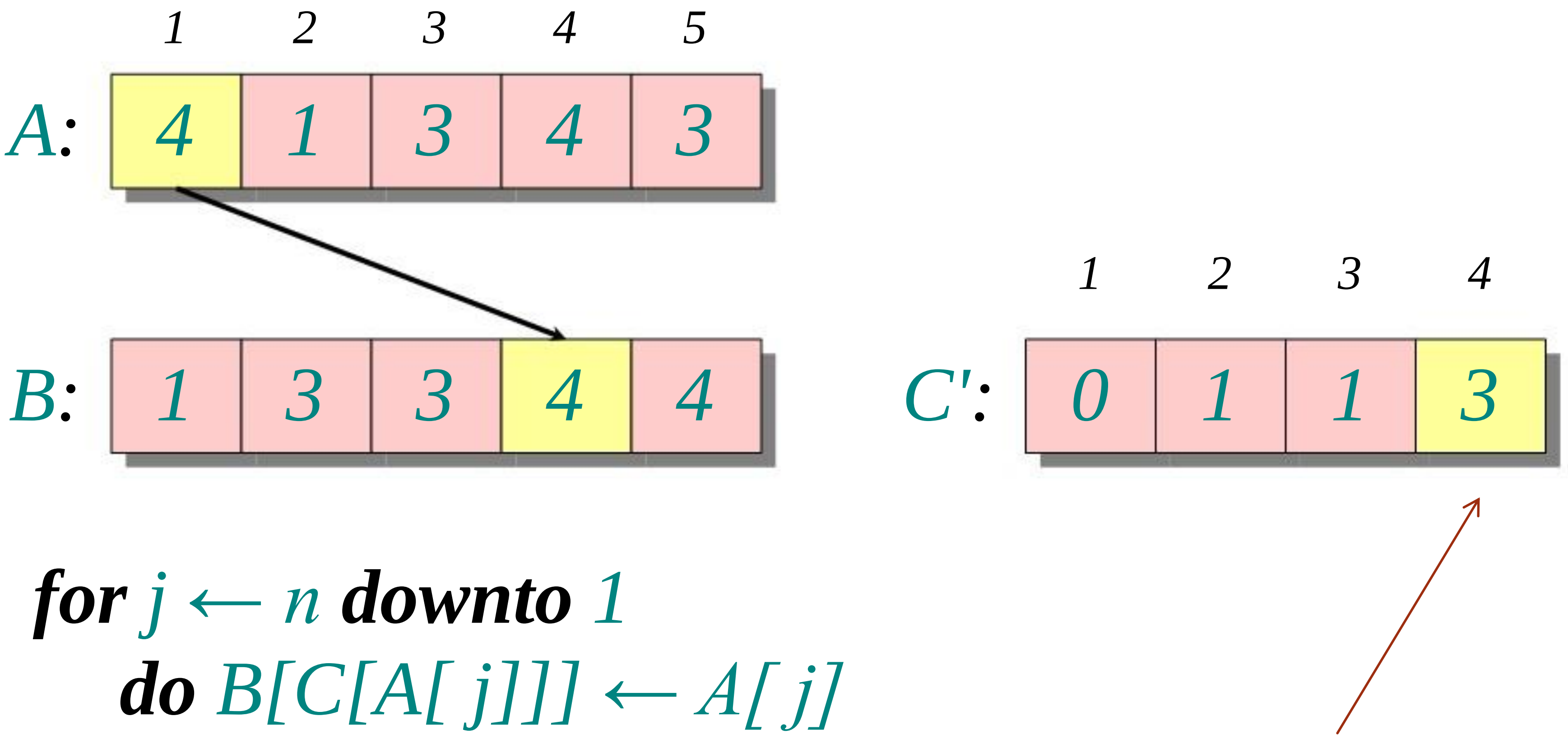
## Loop 4

the decrement means that this second 4 goes in the correct position

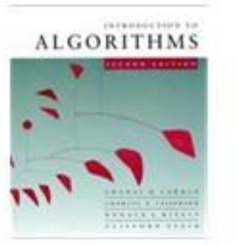


```
for $j \leftarrow n$ downto $1$
    do $B[C[A[j]]] \leftarrow A[j]$
       $C[A[j]] \leftarrow C[A[j]] - 1$
```

## Loop 4



```
for $j \leftarrow n$ downto $1$
    do $B[C[A[j]]] \leftarrow A[j]$
       $C[A[j]] \leftarrow C[A[j]] - 1$        // decr counter
```



## B vs C

| | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| B: | 1 | 3 | 3 | 4 | 4 |

| | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| C': | 1 | 1 | 3 | 5 |

*In the end, each element i occupies the range*
*B[C[i-1]+1 … C[i]]*

# Analysis

```
O(k)      { for $i \leftarrow 1$ to $k$
          {     do $C[i] \leftarrow 0$
O(n)      { for $j \leftarrow 1$ to $n$
          {     do $C[A[j]] \leftarrow C[A[j]] + 1$
O(k)      { for $i \leftarrow 2$ to $k$
          {     do $C[i] \leftarrow C[i] + C[i-1]$
          { for $j \leftarrow n$ downto $1$
O(n)      {     do $B[C[A[j]]] \leftarrow A[j]$
          {        $C[A[j]] \leftarrow C[A[j]] - 1$
__________
O(n + k)
```

## Running Time

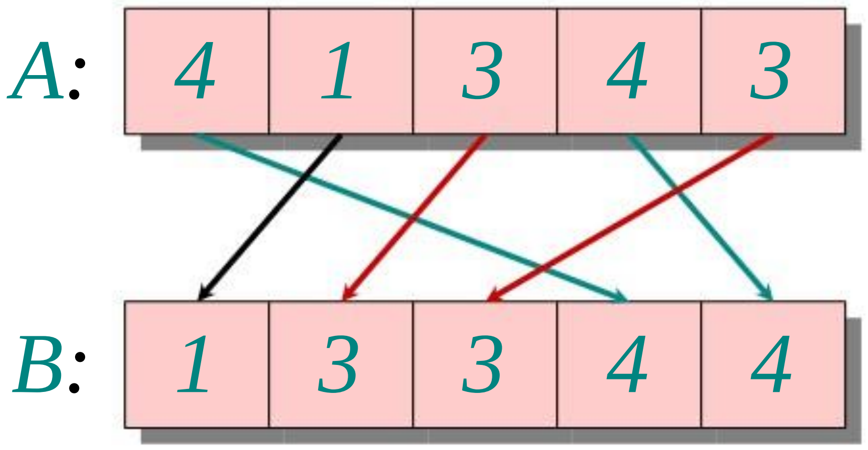
- Total time: **O($n$ + $k$)**
  - Usually, $k$ = O($n$)
  - Thus counting sort runs in **O(n)** time

- But sorting is $\Omega$($n$ lg $n$)!
  - No contradiction -- this is not a comparison sort (in fact, there are *no* comparisons at all!)

  - Notice that this algorithm is *stable*
    - A sorting algorithm is said to be stable if two objects with equal keys appear in the same order in sorted output as they appear in the input array to be sorted

## *Stable Sorting*

*Counting sort is a* ***stable*** *sort: it preserves the input order among equal elements.*



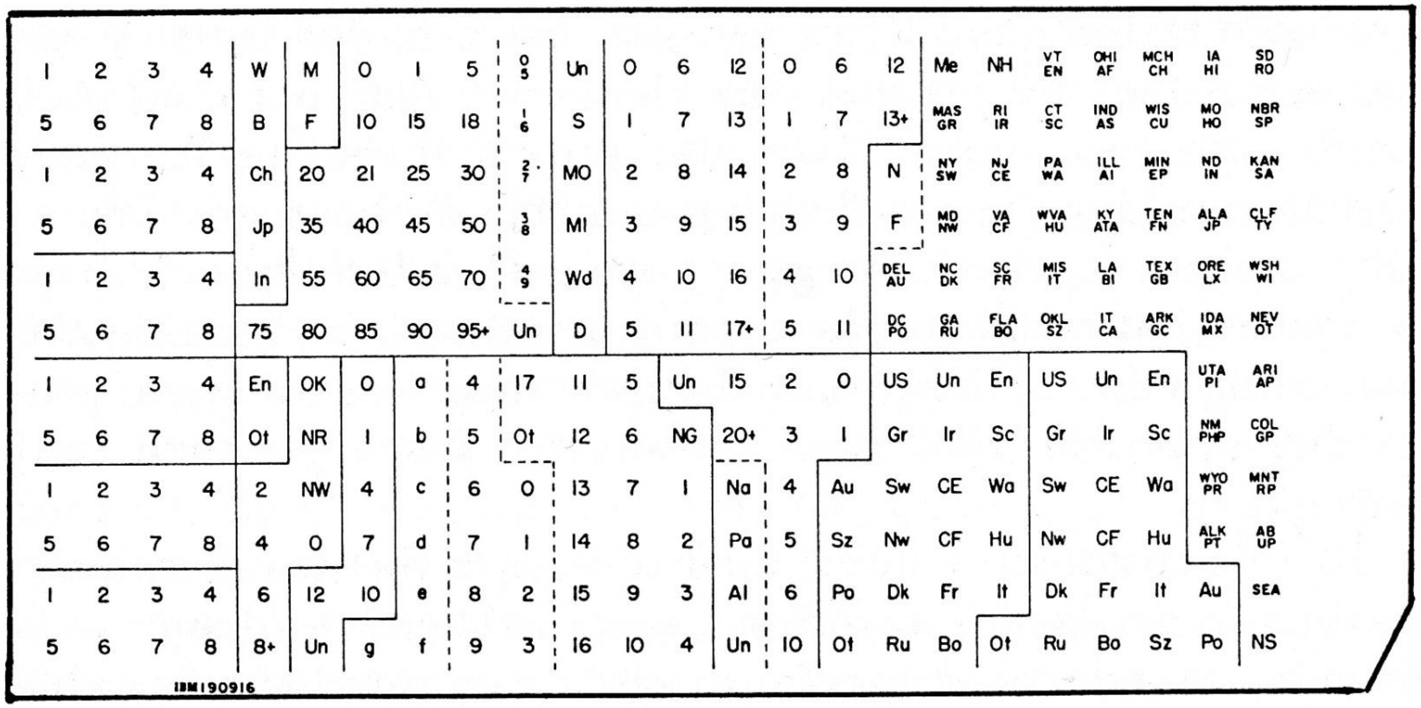
# Drawbacks of Counting Sort

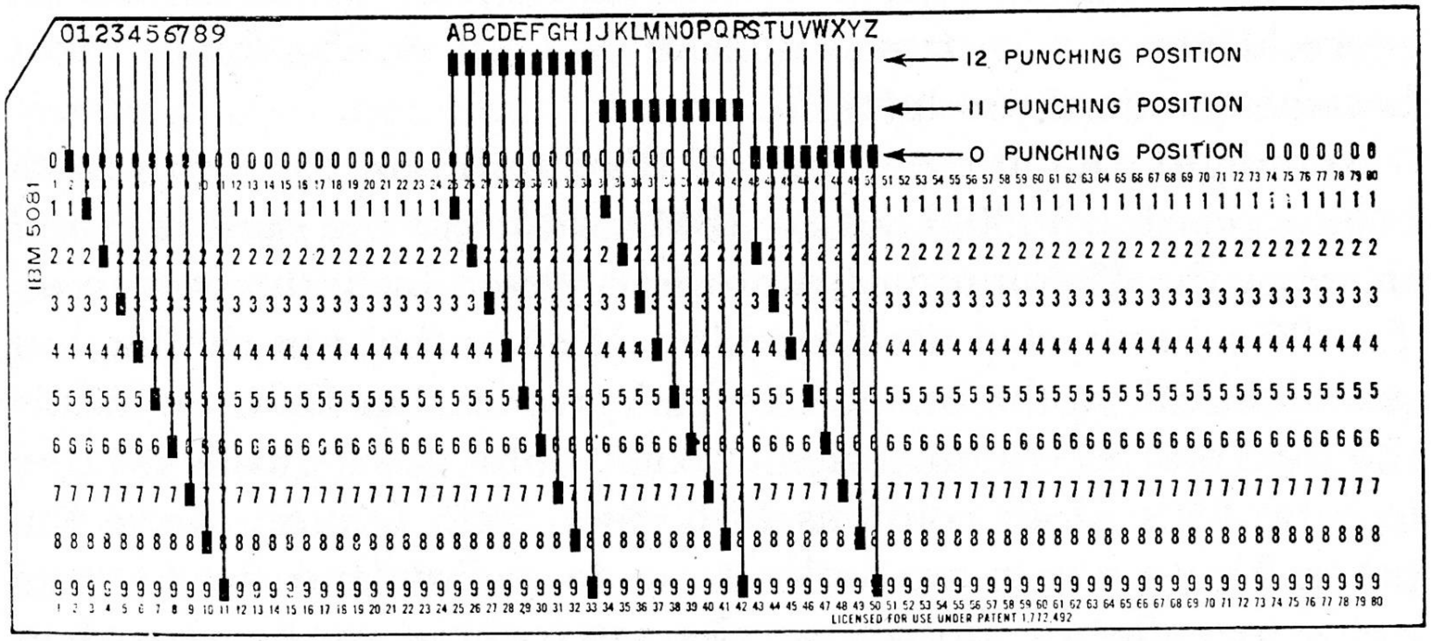
- *Why don't we always use counting sort?*
- Because it depends on the range *k* of the elements

- *Could we use counting sort to sort 32 bit integers?*
- Yes, on x64 and enough ram memory having computers ($2^{32}$ = 4,294,967,296)
  - k is used for the size of the C array = 4 GB

# Radix Sort

- **Origin**: Herman Hollerith's card-sorting machine for the 1890 U.S. census
  - probably the oldest implemented sorted algorithm
  - uses punch cards
- a digit-by-digit sort
- Hollerith's original (**bad**) idea: sort on the **most-significant digit** first
  - Problem: lots of intermediate piles of cards (i.e. extra temporary arrays) to keep track of the calculations


IBM190916


0123456789
ABCDEFGHIJKLMNOPQRSTUVWXYZ
12 PUNCHING POSITION
11 PUNCHING POSITION
0 PUNCHING POSITION
IBM 5081
LICENSED FOR USE UNDER PATENT 1,772,492

- **Good** idea: sort on the **least-significant digit** first with a *stable* sort
  - preserves the relative order of equal elements
  - this is a property of counting sorts, as seen earlier

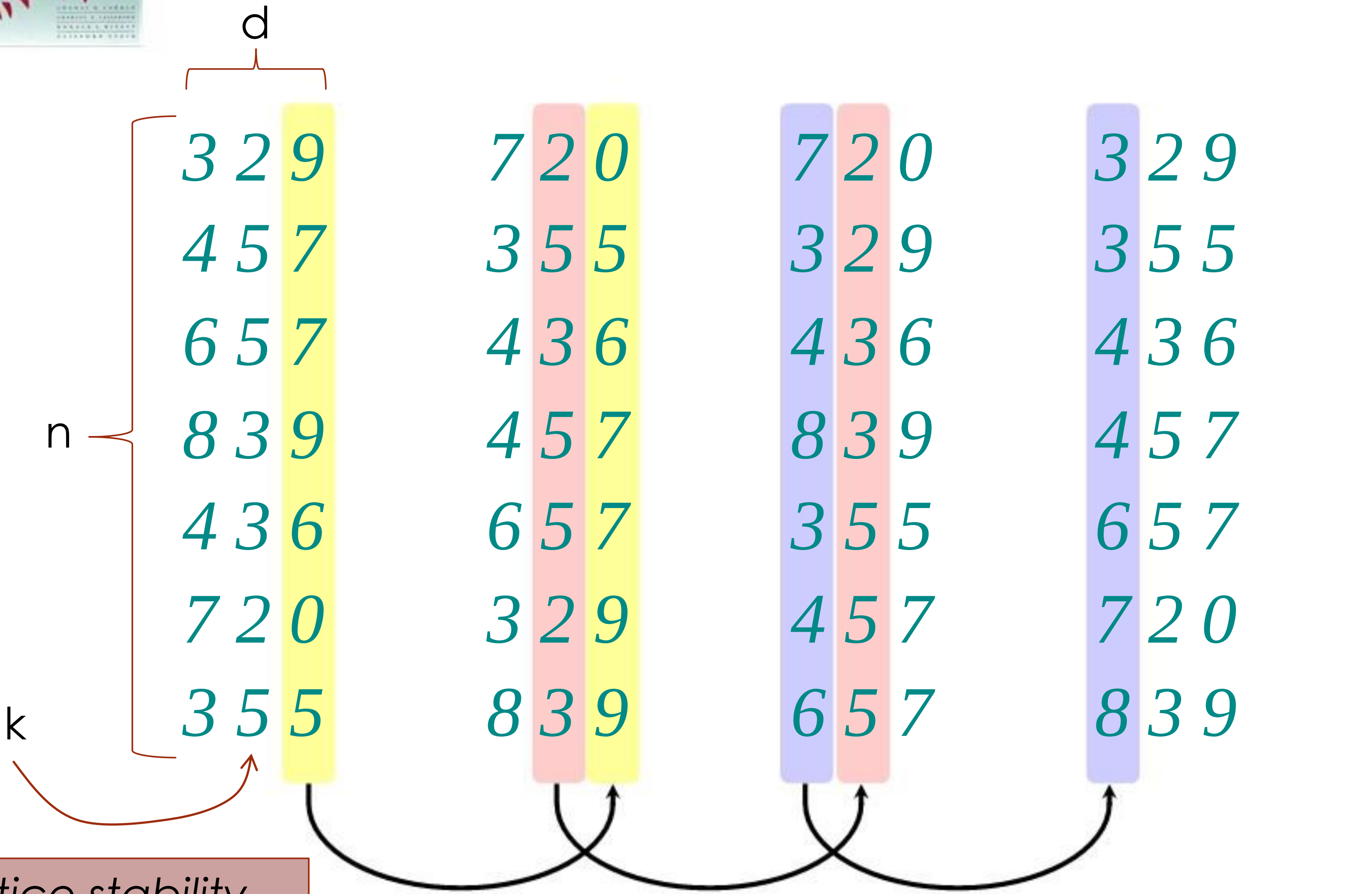
- Code:

```
RadixSort(A, d){
    for i = 1 to d
        StableSort(A) on digit i
}
```

*this can be any sorting algorithm, but must be stable*

# *Operation of Radix Sort*

n = size of array = 7
k = range = 0..9 = 10
d = no. digits = 3



| | | | |
|---|---|---|---|
| 3 2 9 | 7 2 0 | 7 2 0 | 3 2 9 |
| 4 5 7 | 3 5 5 | 3 2 9 | 3 5 5 |
| 6 5 7 | 4 3 6 | 4 3 6 | 4 3 6 |
| 8 3 9 | 4 5 7 | 8 3 9 | 4 5 7 |
| 4 3 6 | 6 5 7 | 3 5 5 | 6 5 7 |
| 7 2 0 | 3 2 9 | 4 5 7 | 7 2 0 |
| 3 5 5 | 8 3 9 | 6 5 7 | 8 3 9 |

*notice stability of 7's and 9's*

## Running Time

- *What StableSort() will we use to sort on digits?*
- Counting sort is a good choice:
  - Sort $n$ numbers on digits that range from 0.. $k$
  - Time: $O(n + k)$
- Each pass over ***n*** **numbers** with ***d*** **digits** takes time $O(n+k)$, so total time $O(dn + dk)$
  - When $d$ is constant and $k = O(n)$, takes **O(*n*)** time

## Radix Sort Speed

- How do we sort 1 million 64-bit numbers?
  - Treat each number as four 16-bit 'digit's
    - n = 1 million; d = 4
  - Can sort in just four passes
- Compares well with typical O(*n* log *n*) comparison sort
  - Requires approximately log *n* = 20 operations per number being sorted
- Downside: unlike quicksort, radix sort has poor locality of reference
  - harder to cache data in RAM to increase memory access speed

- In general, radix sort based on counting sort is
  - asymptotically fast (i.e., O($n$))
  - simple to code
  - a good choice

- To think about: *can radix sort be used on floating-point numbers?*

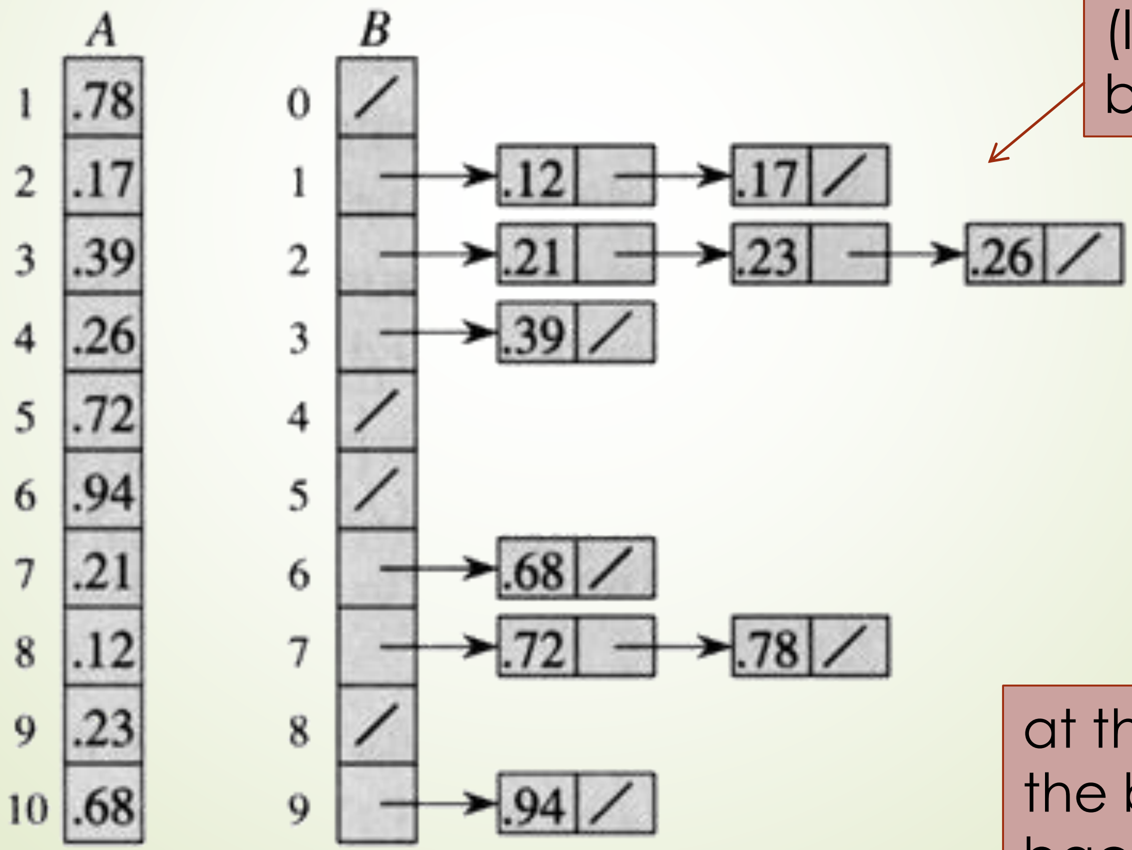
## Bucket Sort

```
Bucket-Sort(int[] A, int x, int y)
1. divide interval [x, y) into n equal-sized subintervals (buckets)
2. distribute the n input keys into the buckets
3. sort the numbers in each bucket (e.g., with
      insertion sort)
4. scan the (sorted) buckets in order and produce
      the output array (usually A)
```

**Assumption**: input elements are distributed uniformly over some known range, e.g., [0,1), so all the elements in A are greater than or equal to 0 but less than 1 .

# Bucket Sort Example

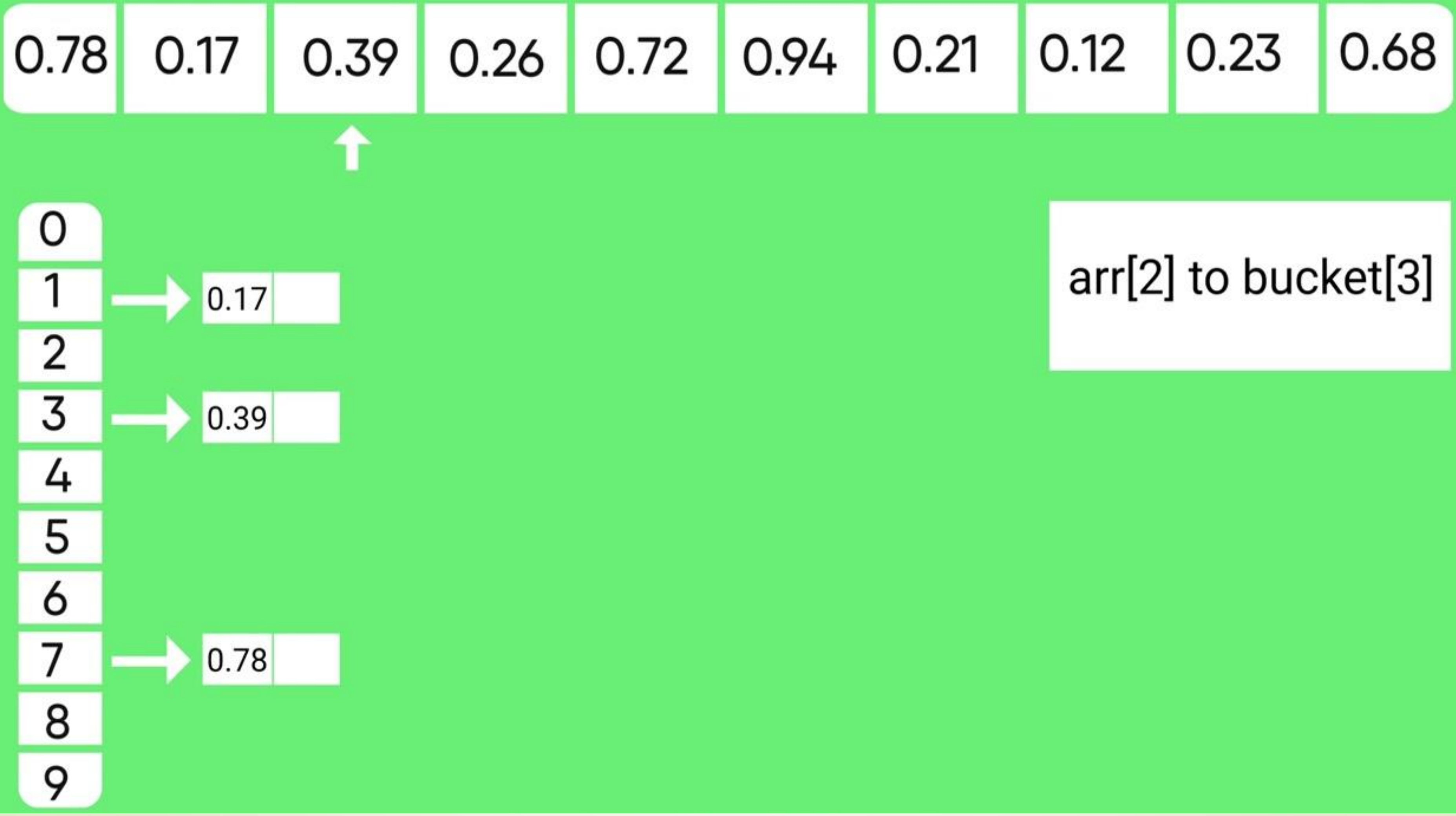
- Sort A[] into the ten 'buckets' in B[], assuming the data in A is in [0..1)



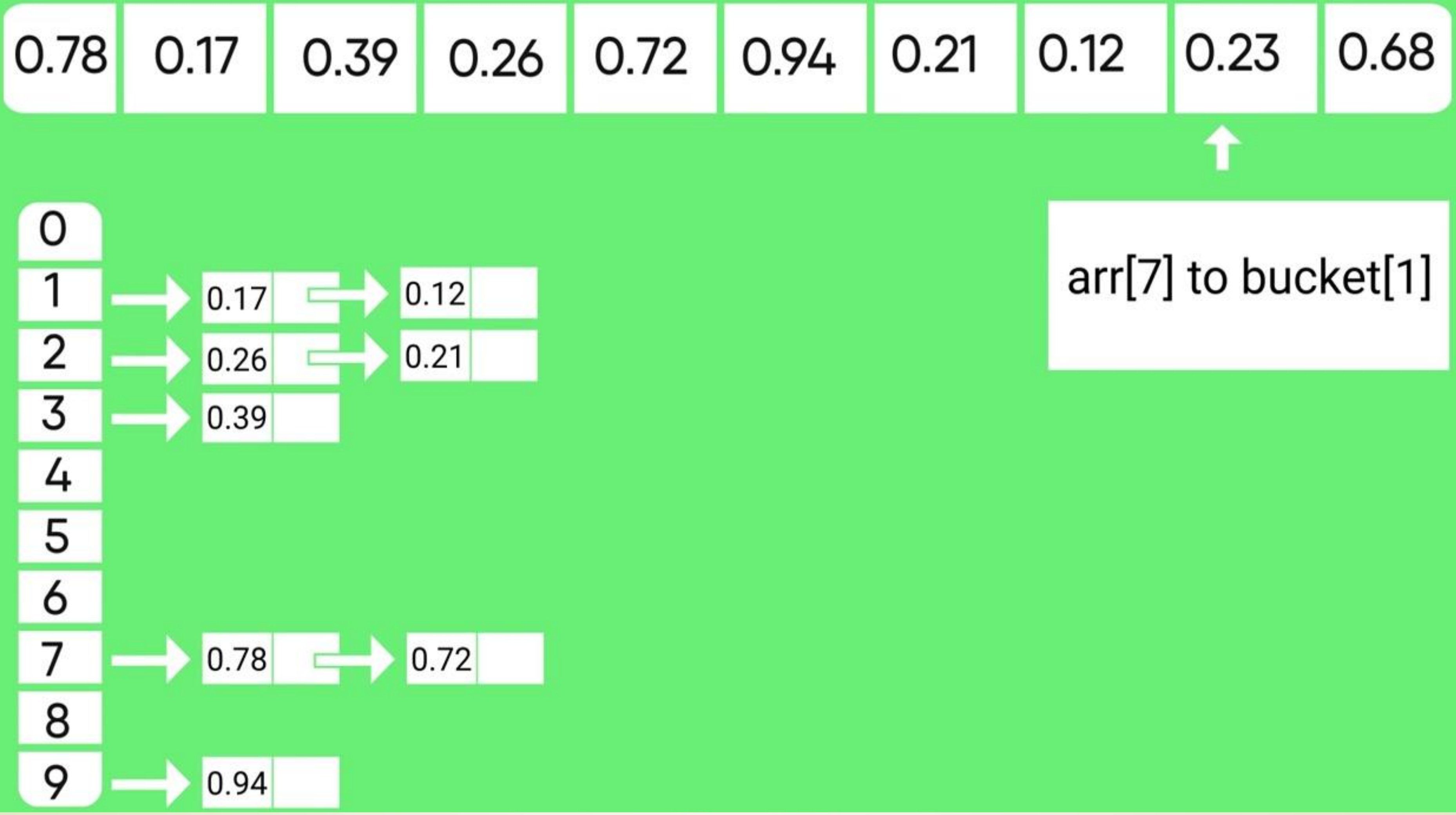
each bucket (list) must be sorted

at the end, copy the buckets (lists) back into A[]

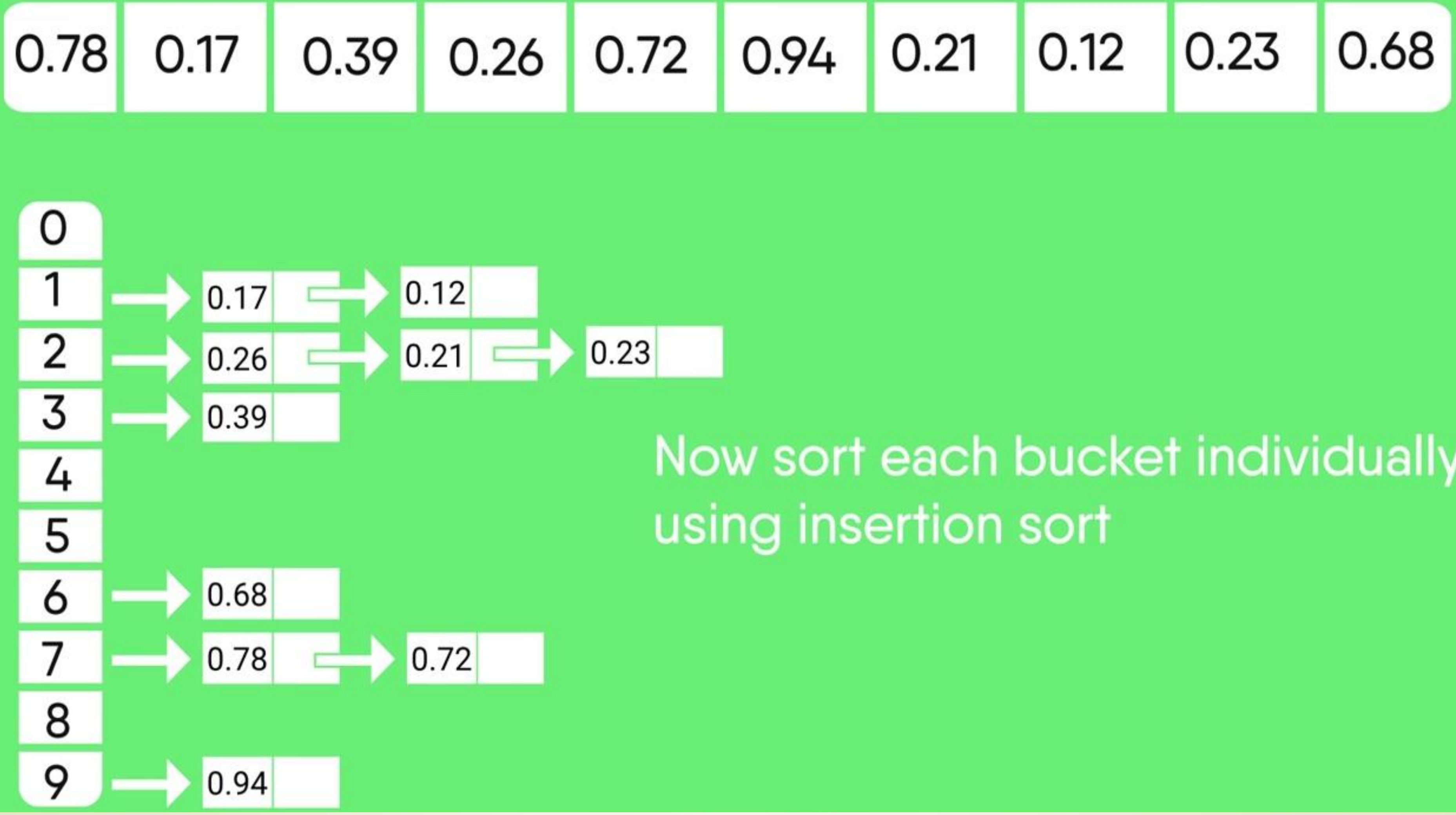
| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 0.78 | 0.17 | 0.39 | 0.26 | 0.72 | 0.94 | 0.21 | 0.12 | 0.23 | 0.68 |



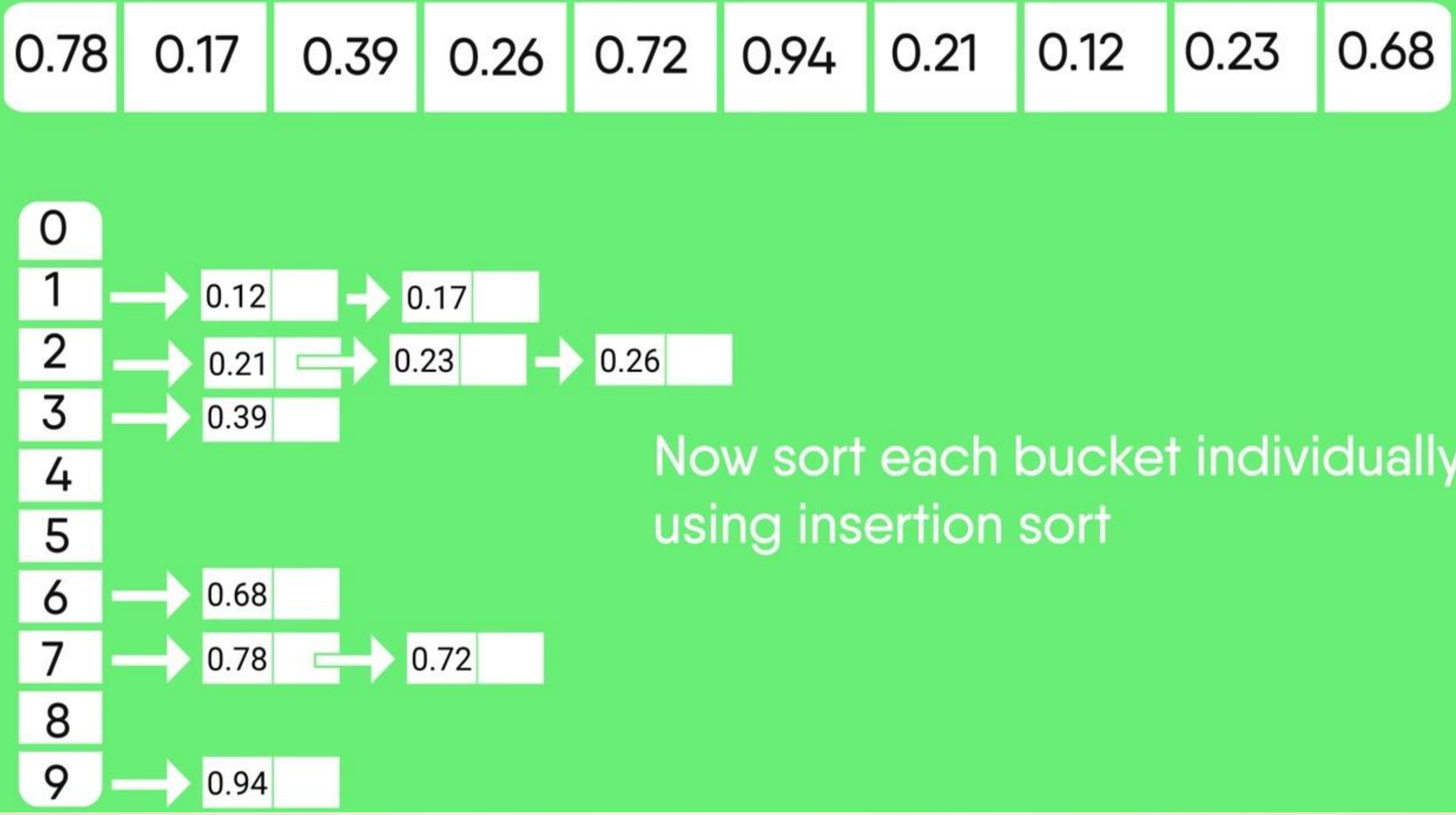
arr[2] to bucket[3]



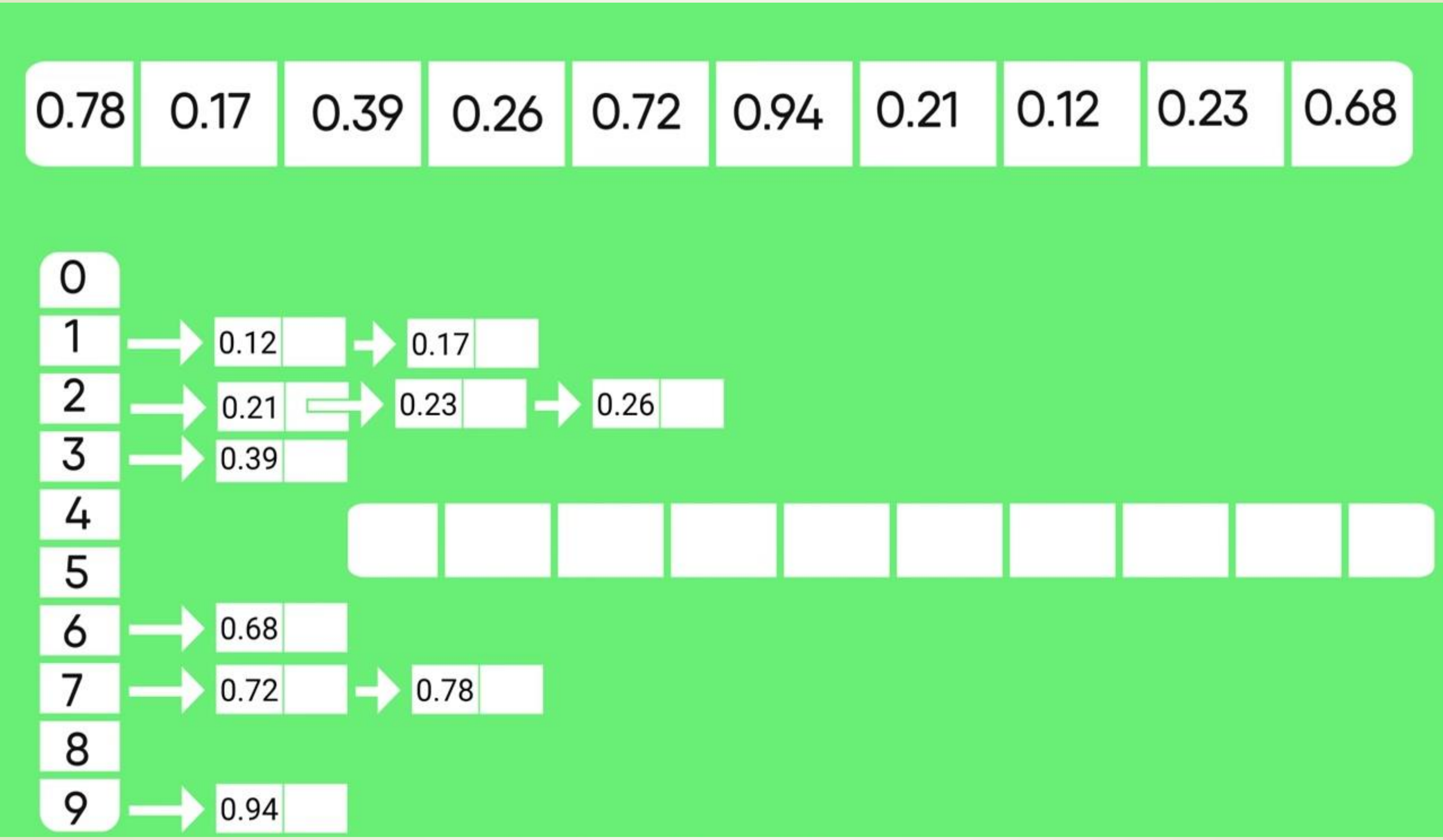
arr[7] to bucket[1]

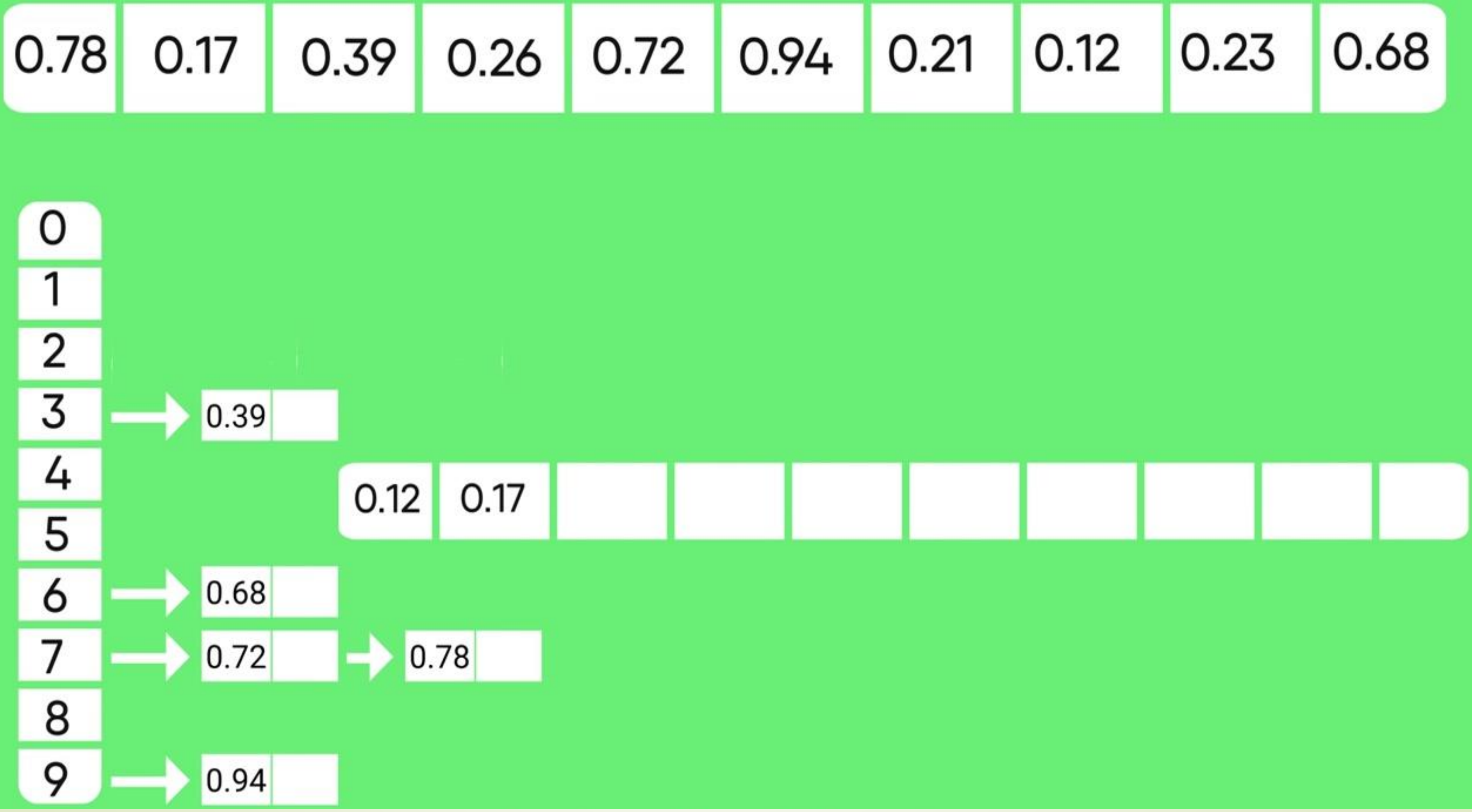


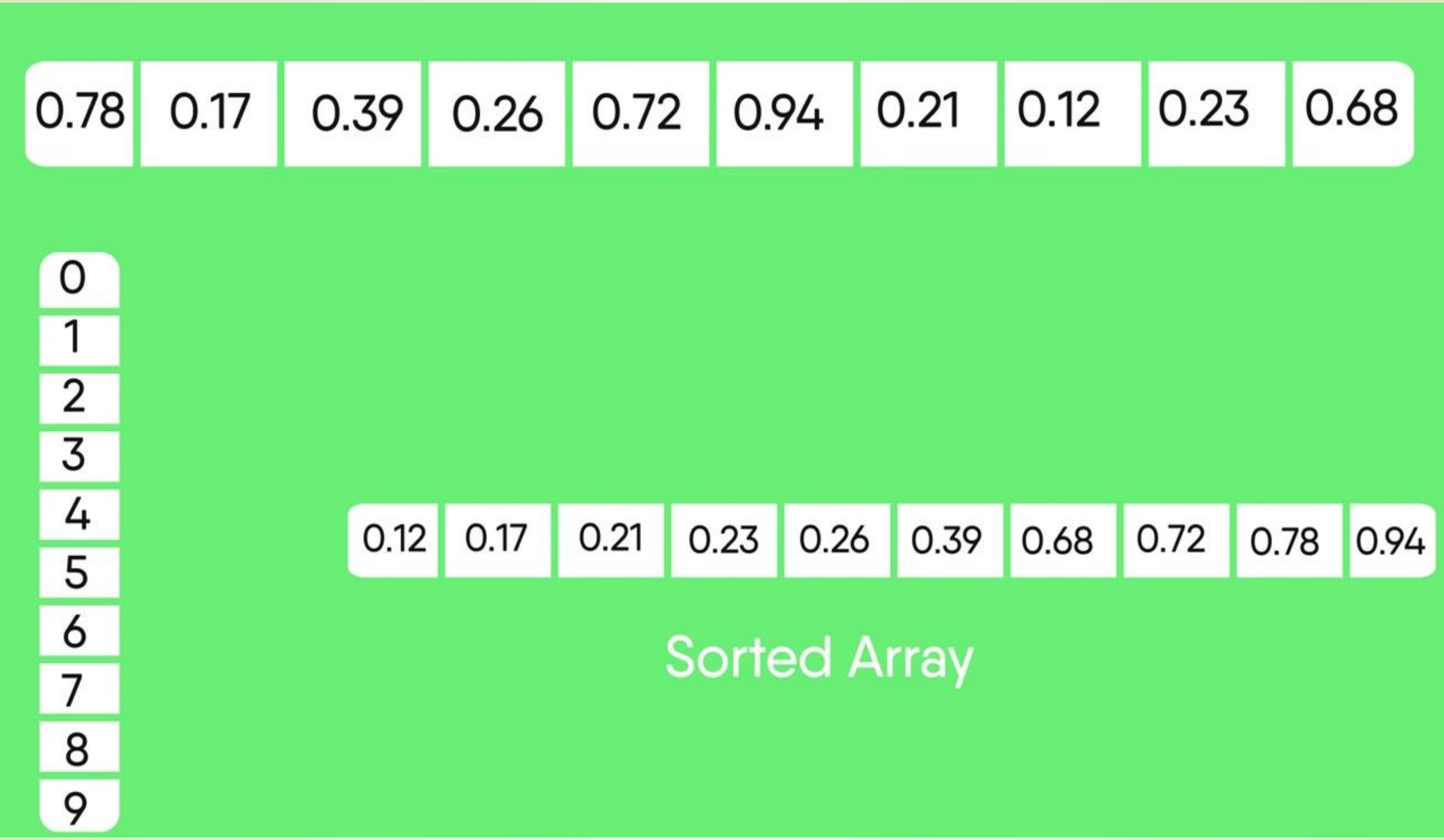
Now sort each bucket individually using insertion sort



Now sort each bucket individually using insertion sort


0.78 0.17 0.39 0.26 0.72 0.94 0.21 0.12 0.23 0.68
0
1 → 0.12 → 0.17
2 → 0.21 → 0.23 → 0.26
3 → 0.39
4
5
6 → 0.68
7 → 0.72 → 0.78
8
9 → 0.94


0.78
0.17
0.39
0.26
0.72
0.94
0.21
0.12
0.23
0.68
0
1
2
3
4
5
6
7
8
9
0.39
0.12
0.17
0.68
0.72
0.78
0.94


0.78 0.17 0.39 0.26 0.72 0.94 0.21 0.12 0.23 0.68
0
1
2
3
4
5
6
7
8
9
0.12 0.17 0.21 0.23 0.26 0.39 0.68 0.72 0.78 0.94
Sorted Array

## Running Time

- O(n) expected time
- Step 1: O(1) for each interval = O(n) time total
- Step 2: O(n) time
- Step 3: The expected number of elements in each bucket is O(1), so total is O(n)
- Step 4: O(n) time to scan the n buckets containing a total of n input elements

## Bucket Sort Review

- Pro's:
  - Fast
  - asymptotically fast (i.e., **O(*n*)** when distribution is uniform)
  - simple to code
  - good for a rough sort
- Con's:
  - doesn't sort in place

# Why use Bucketsort and not Quicksort?

- If we know **k** up front, and it is small (k << n) then bucket sort can efficiently run faster than Quicksort, since n*log(n), the average for Quicksort, will be more than (n + k), which is the average for bucket sort.

- I.e., **sortedList = (n*log(n) > n + k) ? bucketSort(list) : quicksort(list);**

- For example, bucketsort may be preferred for streams over quicksort because the max char to be ordered compared to the minimum value can be as small as the ASCII value of 0 to the ASCII value of z, hence reducing K to a number around 60

# Summary of Linear Sorts

## Non-Comparison Based Sorts

|  | Running Time |  |  |  |
|---|---|---|---|---|
|  | worst-case | average-case | best-case | in place |
| Counting Sort | O(n + k) | O(n + k) | O(n + k) | no |
| Radix Sort | O(d(n + k')) | O(d(n + k')) | O(d(n + k')) | no |
| Bucket Sort |  | O(n) |  | no |

n = size of array
k = range
d = digits
k' = a subrange of k