CSE214 – Analysis of Algorithms

PhD Furkan Gözükara, Toros University

https://github.com/FurkanGozukara/CSE214_2018

Lecture 3

Solving Recurrences

Based on Cevdet Aykanat's and Mustafa Ozdal's Lecture Notes - Bilkent

Solving Recurrences

- Reminder: Runtime (T(n)) of *MergeSort* was expressed as a recurrence

$$T(n) = \begin{cases} \Theta(1) & \text{if } n=1 \\ 2T(n/2) + \Theta(n) & \text{otherwise} \end{cases}$$

Kod Satırı	Maliyet
`MergeSort(A, left, right) {`	$T(n)$
`  if (left < right) {`	$\Theta(1)$
`    mid = floor((left + right) / 2);`	$\Theta(1)$
`    MergeSort(A, left, mid);`	$T(n/2)$
`    MergeSort(A, mid+1, right);`	$T(n/2)$
`    Merge(A, left, mid, right);`	$\Theta(n)$
`  }`	
`}`	

- $T(n) =$ $\Theta(1)$, $n = 1$ için

  $T(n) = 2T(n/2) + f(n)$,$n > 1$için

- Özyineli bir denklemdir

Solving Recurrences

- We will focus on 3 techniques in this lecture:

1. Substitution method

2. Recursion tree approach

3. Master method

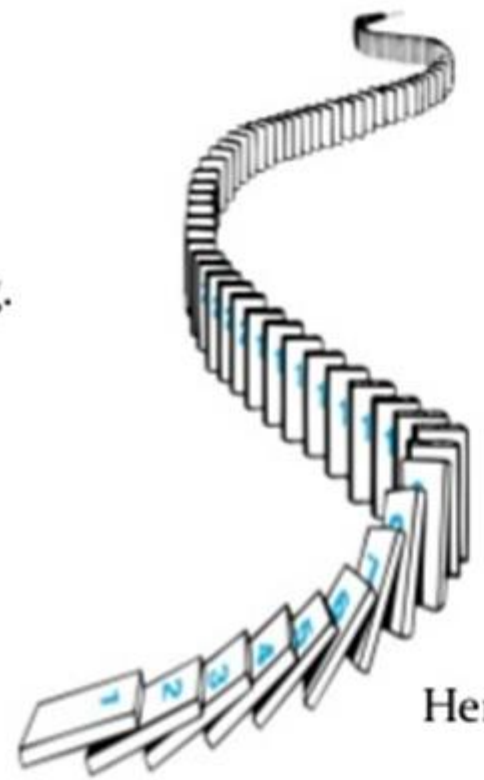
Substitution Method

- The most general method:
  1. Guess
  2. Prove by induction
  3. Solve for constants

Remembering How Mathematical Induction Works

Consider an infinite sequence of dominoes, labeled 1,2,3, …, where each domino is standing.

Let $P(n)$ be the proposition that the nth domino is knocked over.



We know that the first domino is knocked down, i.e., $P(1)$ is true .

We also know that if whenever the kth domino is knocked over, it knocks over the $(k + 1)$st domino, i.e, $P(k) \rightarrow P(k + 1)$ is true for all positive integers k.

Hence, all dominos are knocked over.

$P(n)$ is true for all positive integers n.

Substitution Method: Example

Solve $T(n) = 4T(n/2) + n$ *(assume $T(1) = \Theta(1)$)*

1. Guess $T(n) = O(n^3)$ (need to prove O and Ω separately)

2. Prove by induction that $T(n) \leq cn^3$ for large n (i.e. $n \geq n_0$) Inductive hypothesis: $T(k) \leq ck^3$ for any $k < n$

   Assuming ind. hyp. holds, prove $T(n) \leq cn^3$

Substitution Method: Example – cont'd

Original recurrence: $T(n) = 4T(n/2) + n$

From inductive hypothesis: $T(n/2) \leq c(n/2)^3$

Substitute this into the original recurrence:

$$T(n) \leq 4c\,(n/2)^3 + n$$

$$= (c/2)\,n^3 + n$$

$$(c/2)\,n^3 + n \leq cn^3 \text{ ?}$$

Substitution Method: Example – cont'd

- To prove

  $T(n) \le cn^3$

- We can choose $c \ge 2$ and $n_0 \ge 1$
- But, the proof is not complete yet.
- Reminder: Proof by induction:

1. Prove the base cases → *haven't proved the base cases yet*
2. Inductive hypothesis for smaller sizes
3. Prove the general case

Substitution Method: Example – cont'd

- We need to prove the base cases

Base: $T(n) = \Theta(1)$ for small n (e.g. for $n = n_0$)

- We should show that:

  $$\text{“}\Theta(1)\text{”} \le cn^3 \quad \text{for } n = n_0$$

  This holds if we pick c big enough

- So, the proof of $T(n) = O(n^3)$ is complete.
- But, is this a tight bound?

Example: A tighter upper bound?

- Original recurrence: $T(n) = 4T(n/2) + n$
- Try to prove that $T(n) = O(n^2)$,

  i.e. $T(n) \leq cn^2$ for all $n \geq n_0$

- Ind. hyp: Assume that $T(k) \leq ck^2$ for $k < n$
- Prove the general case: $T(n) \leq cn^2$

Example (cont'd)

- Original recurrence: $T(n) = 4T(n/2) + n$
- Ind. hyp: Assume that $T(k) \leq ck^2$ for $k < n$
- Prove the general case: $T(n) \leq cn^2$

$$
\begin{aligned}
T(n) &= 4T(n/2) + n \\
&\leq 4c(n/2)^2 + n \\
&cn^2 + n \leq cn^2 \\
&\cancel{= O(n^2)}
\end{aligned}
$$

Wrong! We must prove exactly

Example (cont'd)

- Original recurrence: $T(n) = 4T(n/2) + n$
- Ind. hyp: Assume that $T(k) \leq ck^2$ for $k < n$
- Prove the general case: $T(n) \leq cn^2$

- So far, we have:

$$T(n) \leq cn^2 + n$$

No matter which positive c value we choose,

this does not show that $T(n) \leq cn^2$

Proof failed?

Example (cont'd)

- What was the problem?
  - *The inductive hypothesis was not strong enough*
- Idea: Start with a stronger inductive hypothesis
  - *Subtract* a low-order term
- Inductive hypothesis: $T(k) \le c_1 k^2 - c_2 k$ for $k < n$
- Prove the general case: $T(n) \le c_1 n^2 - c_2 n$

Example (cont'd)

- Original recurrence: $T(n) = 4T(n/2) + n$
- Ind. hyp: Assume that $T(k) \leq c_1k^2 - c_2k$ for $k < n$
- Prove the general case: $T(n) \leq c_1n^2 - c_2n$

$$
\begin{aligned}
T(n) &= 4T(n/2) + n \\
&\leq 4\,(c_1(n/2)^2 - c_2(n/2)) + n \\
&= c_1n^2 - 2c_2n + n
\end{aligned}
$$

$$c_1n^2 - 2c_2n + n \leq c_1n^2 - c_2n$$

$$n - c_2n \leq 0 \qquad \text{for } n(c_2 - 1) \geq 0$$

$$n(1 - c_2) \leq 0 \qquad \text{choose } c_2 \geq 1$$

Example (cont'd)

- We now need to prove

$$T(n) \le c_1 n^2 - c_2 n$$

for the base cases.

$T(n) = \Theta(1)$ for $1 \le n \le n_0$ (implicit assumption)

"$\Theta(1)$" $\le c_1 n^2 - c_2 n$ for n small enough (i.e. $n = n_0$)

We can choose c_1 large enough to make this hold

e.g. $c_1 = 2$, $c_2 = 1$, $n = 1$

- We have proved that $T(n) = O(n^2)$

Substitution Method: Example 2

- For the recurrence $T(n) = 4T(n/2) + n$, prove that $T(n) = \Omega(n^2)$

  i.e. $T(n) \geq cn^2$ for any $n \geq n_0$
- Ind. hyp: $T(k) \geq ck^2$ for any $k < n$
- Prove general case: $T(n) \geq cn^2$

$$\begin{aligned} T(n) &= 4T(n/2) + n \\ &\geq 4c\,(n/2)^2 + n \\ &= cn^2 + n \geq cn^2 \end{aligned}$$

since $n > 0$

Proof succeeded – no need to strengthen the ind. hyp as in the last example

Example 2 (cont'd)

- We now need to prove that $T(n) \geq cn^2$ for the base cases

$T(n) = \Theta(1)$ for $1 \leq n \leq n_0$ (implicit assumption)

"$\Theta(1)$" $\geq cn^2$ for $n = n_0$

n_0 is sufficiently small (i.e. c=1, n=1)

We can choose c small enough for this to hold

- We have proved that $T(n) = \Omega(n^2)$

Substitution Method - Summary

1. Guess the asymptotic complexity

1. Prove your guess using induction
    1. Assume inductive hypothesis holds for $k < n$
    2. Try to prove the general case for n

        Note: MUST prove the EXACT inequality

        CANNOT ignore lower order terms

        If the proof fails, strengthen the ind. hyp. and try again
    3. Prove the base cases (usually straightforward)

$T(1) = 4 \quad T(\boxed{n}) = 2T(\boxed{n}/2) + 4\boxed{n}$

Build Solution

$$T(n) = 2\underbrace{T(n/2)} + 4n$$
$$= 2\left[2T\left(\frac{n}{2^2}\right) + 4\frac{n}{2}\right] + 4n$$
$$= 2^2 \underbrace{T\left(\frac{n}{2^2}\right)} + 4n + 4n$$
$$= 2^2\left[2T\left(\frac{n}{2^3}\right) + 4\frac{n}{2^2}\right] + 4n + 4n$$
$$= 2^3 T\left(\frac{n}{2^3}\right) + 4n + 4n + 4n$$
$$= 2^3\left[2T\left(\frac{n}{2^4}\right) + 4\frac{n}{2^3}\right] + 4n + 4n + 4n$$
$$= 2^4 T\left(\frac{n}{2^4}\right) + \underbrace{4n + 4n + 4n + 4n}_{4}$$
$$= 2^i T\left(\frac{n}{2^i}\right) + i(4n)$$
$$\frac{n}{2^i} = 1 \Rightarrow n = 2^i \Rightarrow \log_2 n = i$$

Expand Scratch

$$T(n/2) = \underbrace{2T\left(\frac{n}{2^2}\right) + 4\left(\frac{n}{2}\right)}$$
$$T\left(\frac{n}{2^2}\right) = 2T\left(\frac{n}{2^3}\right) + 4\left(\frac{n}{2^2}\right)$$
$$T\left(\frac{n}{2^3}\right) = \underbrace{2T\left(\frac{n}{2^4}\right) + 4\frac{n}{2^3}}$$

$$2^{\log_2 n} = n^{\log_2 2} = n$$

$$\Rightarrow 2^{\log_2 n}\,\underline{T(1)} + (\log_2 n)(4n)$$
$$= n(4) + 4n\log_2 n$$
$$= 4n + 4n\log_2 n =$$
$$O(n \log n)$$

Recursion Tree Method

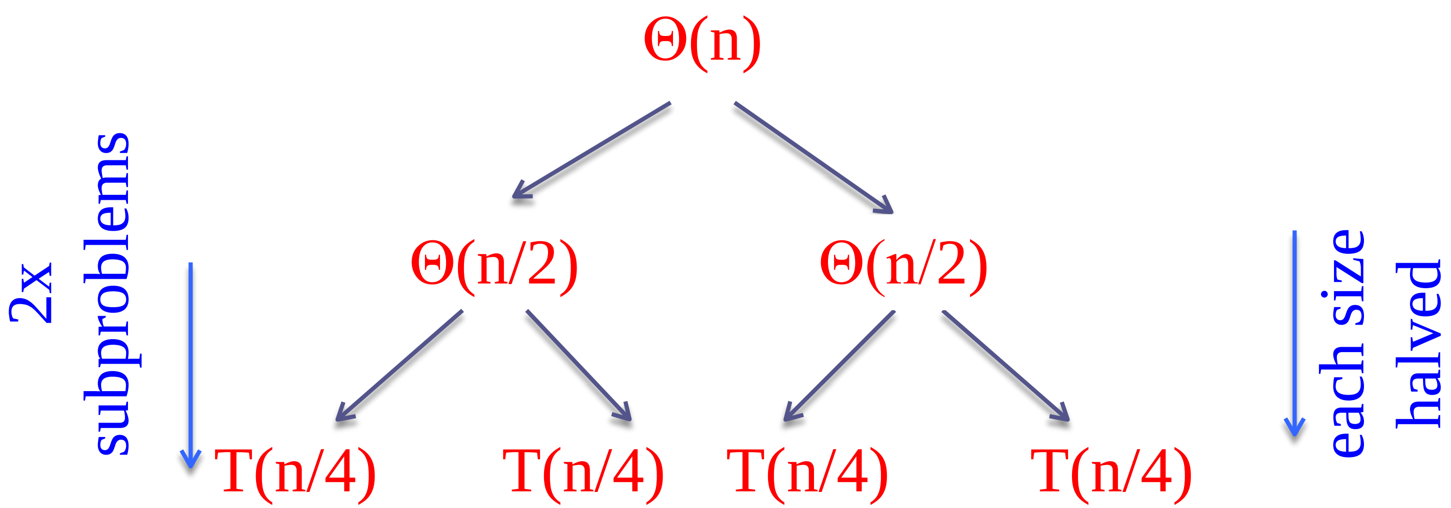
- A recursion tree models the runtime costs of a recursive execution of an algorithm.
- The recursion tree method is good for generating guesses for the substitution method.
- The recursion-tree method can be unreliable.
  - Not suitable for formal proofs
- The recursion-tree method promotes intuition, however.

Solve Recurrence: $T(n) = 2T(n/2) + \Theta(n)$

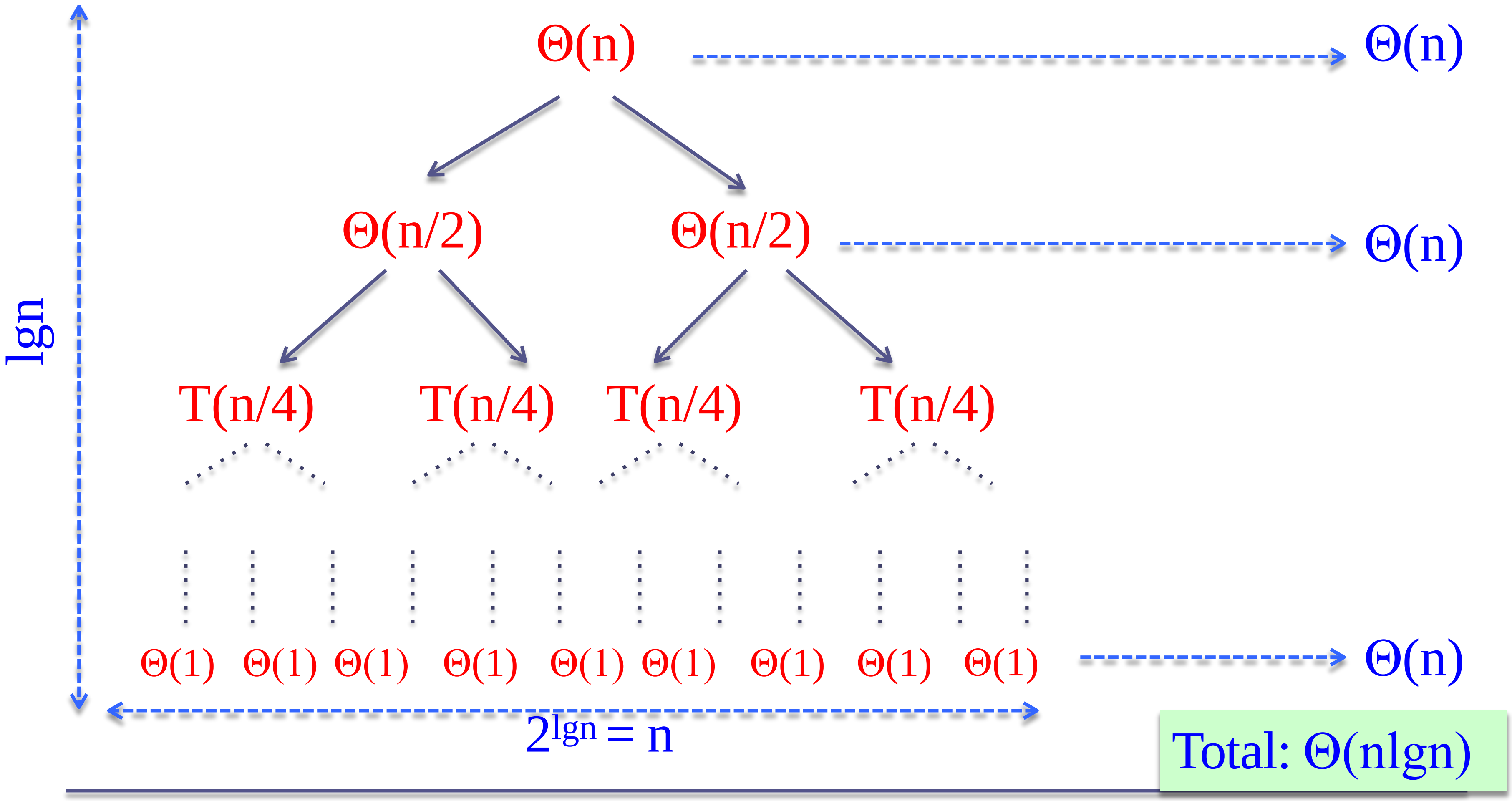
$\Theta(n)$

$T(n/2)$ $\quad\quad$ $T(n/2)$

Solve Recurrence: $T(n) = 2T\,(n/2) + \Theta(n)$



Solve Recurrence: $T(n) = 2T(n/2) + \Theta(n)$



Example of Recursion Tree

Solve $T(n) = T(n/4) + T(n/2) + n^2$:

Example of Recursion Tree

Solve $T(n) = T(n/4) + T(n/2) + n^2$:

$$T(n)$$

Example of Recursion Tree

Solve $T(n) = T(n/4) + T(n/2) + n^2$:

$$
\begin{tikzcd}
 & n^2 \arrow[dl, dash] \arrow[dr, dash] & \\
T(n/4) & & T(n/2)
\end{tikzcd}
$$

Example of Recursion Tree

Solve $T(n) = T(n/4) + T(n/2) + n^2$:

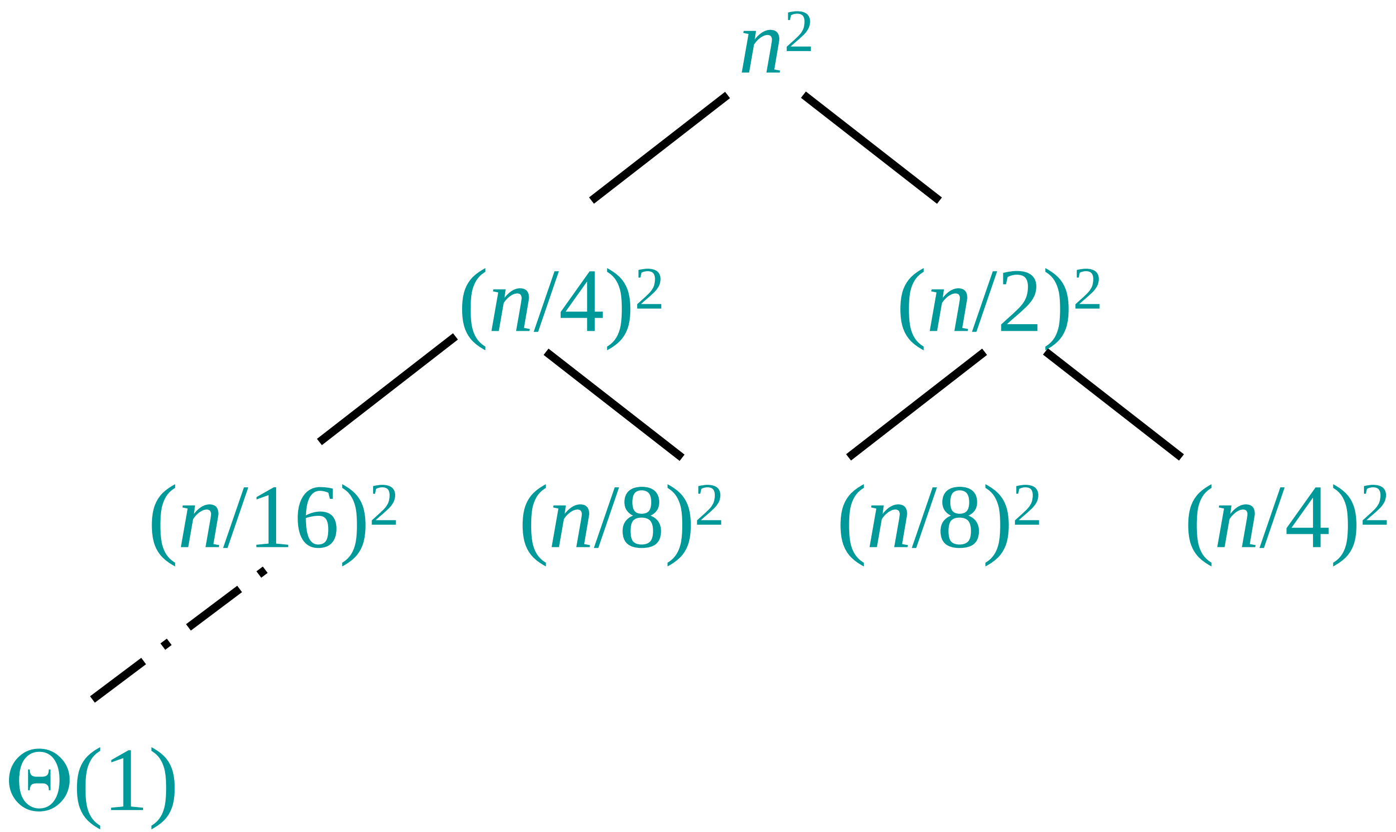
$$n^2$$

$$(n/4)^2 \qquad (n/2)^2$$

$$\mathrm{T}(n/16) \qquad \mathrm{T}(n/8) \qquad \mathrm{T}(n/8) \qquad \mathrm{T}(n/4)$$

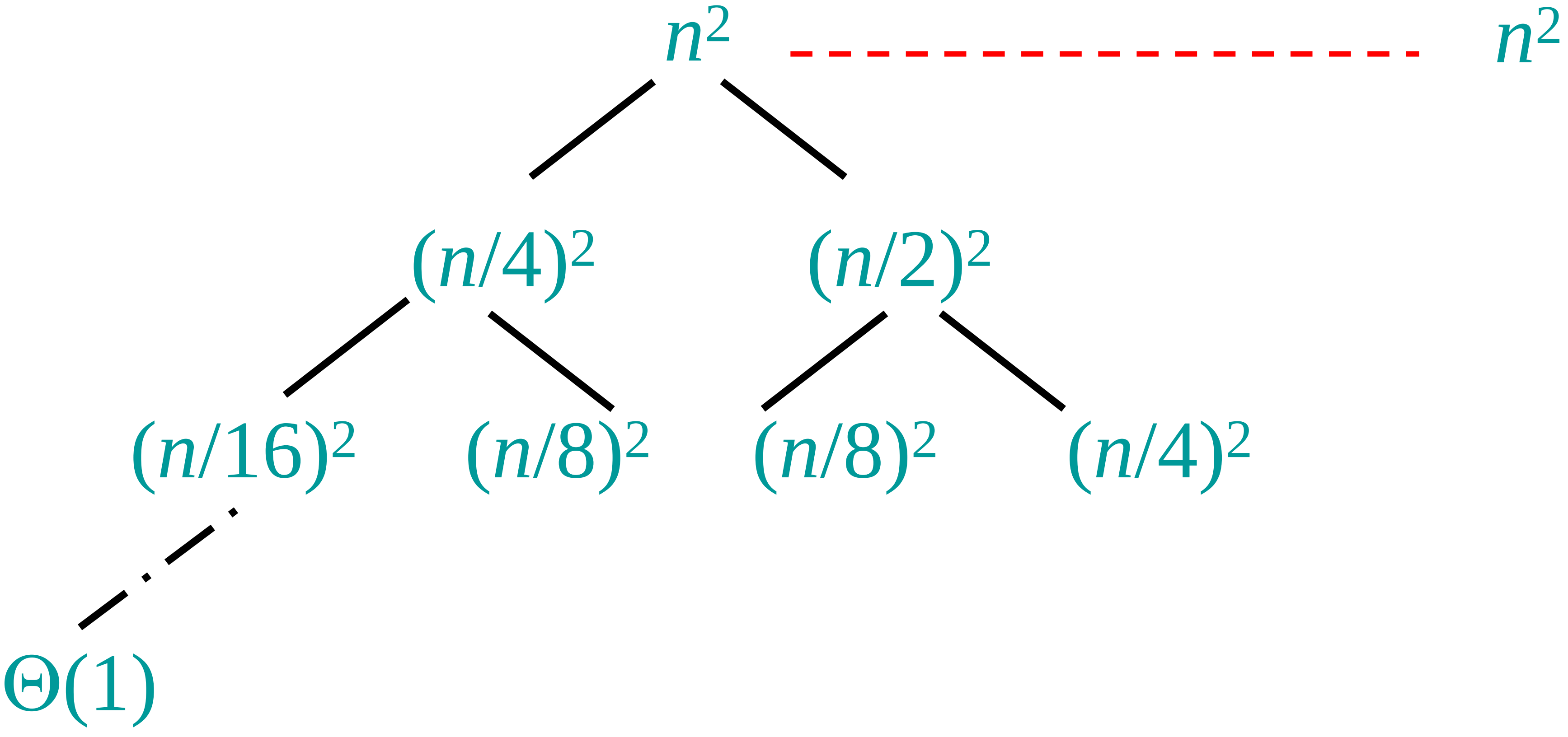
Example of Recursion Tree

Solve $T(n) = T(n/4) + T(n/2) + n^2$:



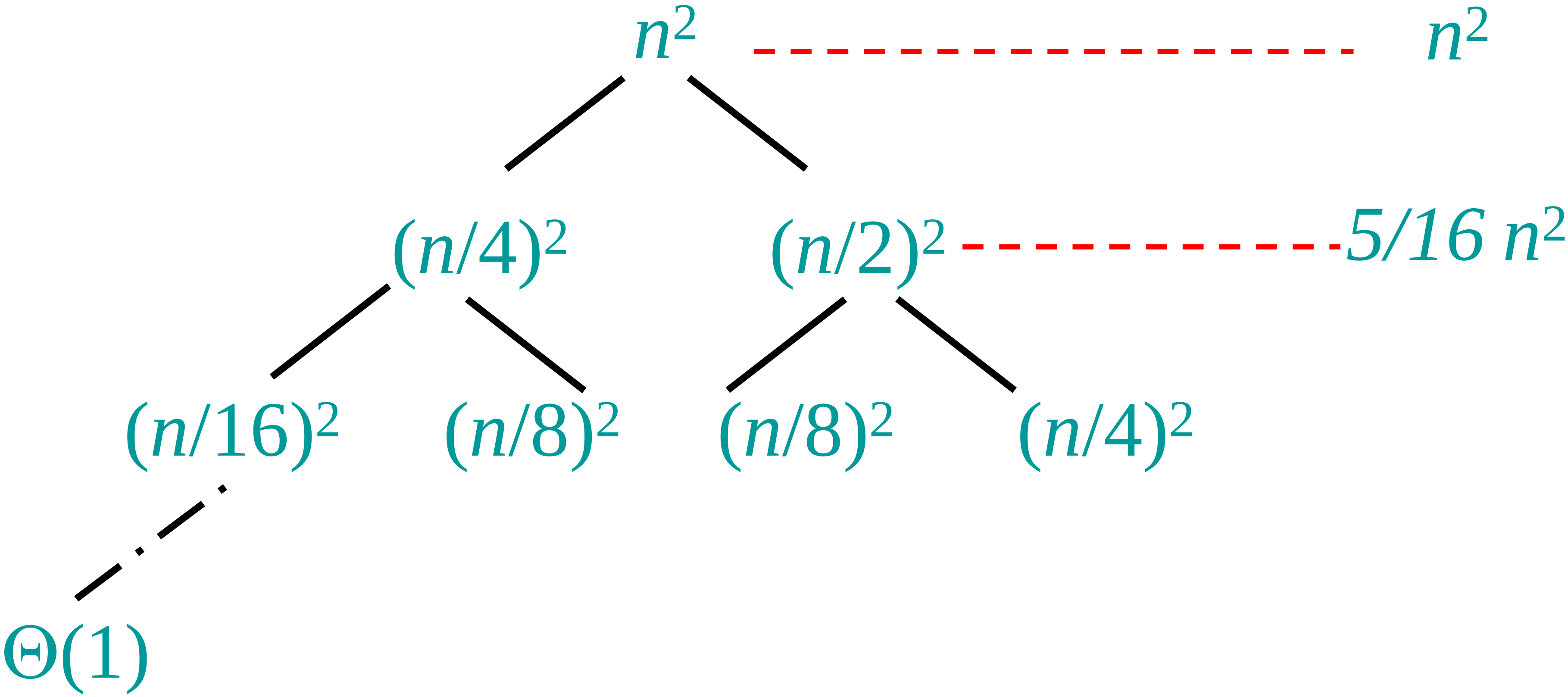
Example of Recursion Tree

Solve $T(n) = T(n/4) + T(n/2) + n^2$:



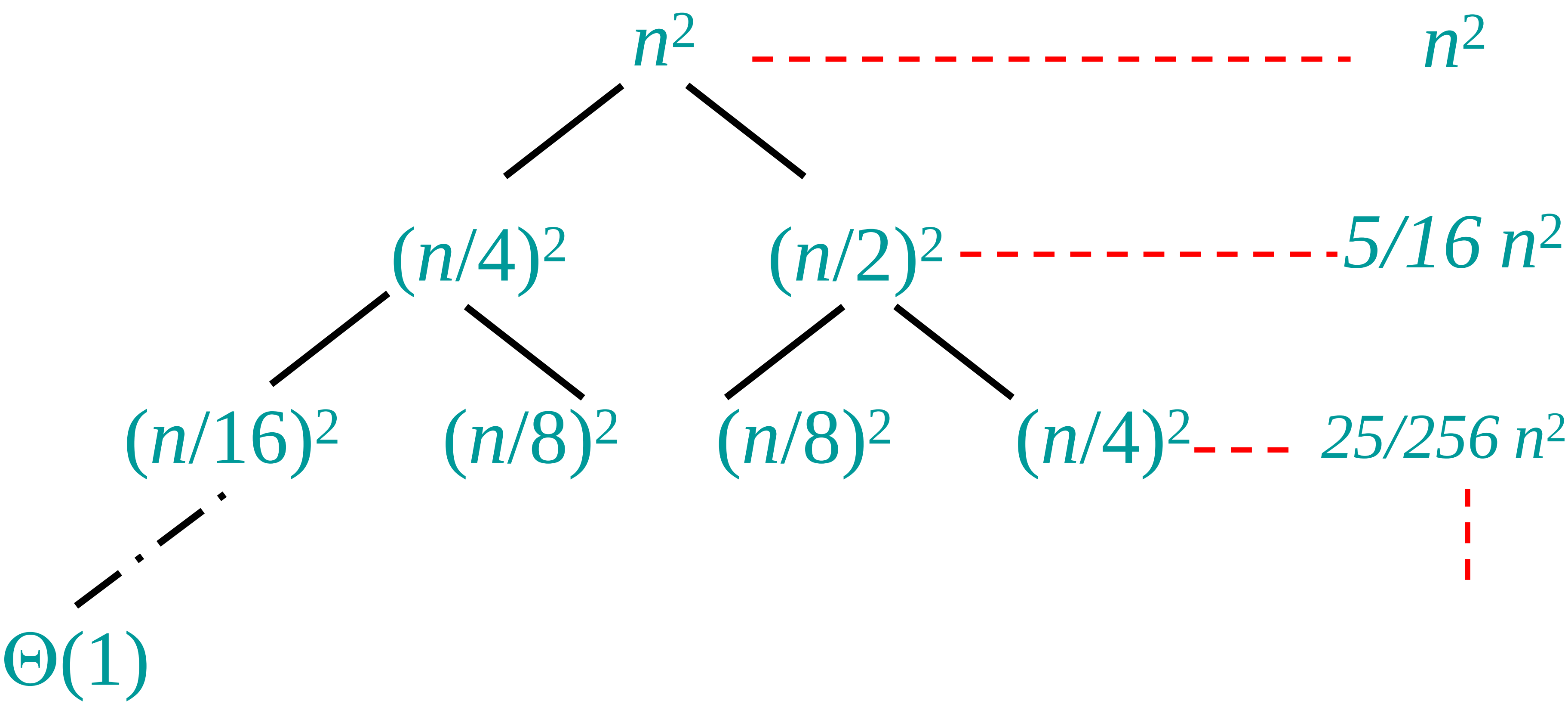
Example of Recursion Tree

Solve $T(n) = T(n/4) + T(n/2) + n^2$:



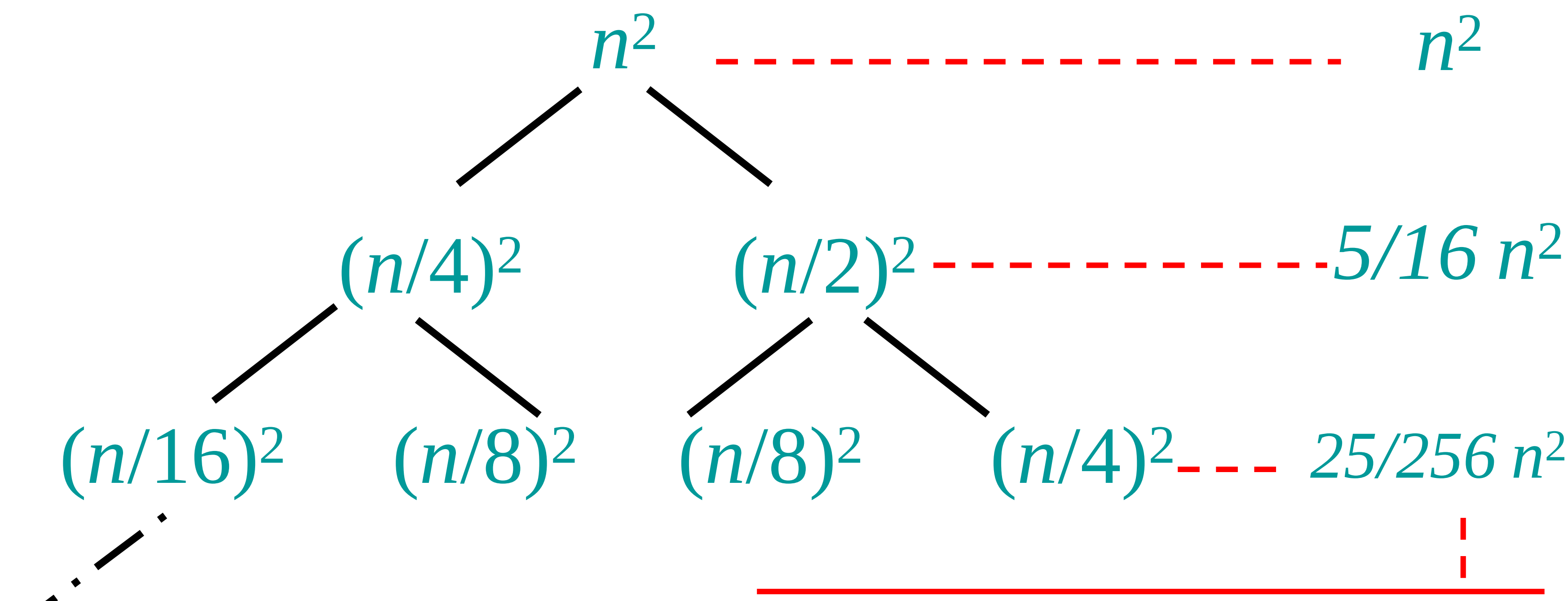
Example of Recursion Tree

Solve $T(n) = T(n/4) + T(n/2) + n^2$:



Example of Recursion Tree

Solve $T(n) = T(n/4) + T(n/2) + n^2$:



$\Theta(1)$

Total $= n^2 (1 + 5/16 + (5/16)^2 + (5/16)^3 + \ldots)$

$= \Theta(n^2)$ geometric series

Geometric Series Reminder

$$1 + x + x^2 + \cdots = \frac{1}{1-x} \quad \text{for } |x| < 1$$

$$1 + x + x^2 + \cdots + x^n = \frac{1-x^{n+1}}{1-x} \quad \text{for } x \neq 1$$

The Master Method

- A powerful black-box method to solve recurrences (solving divide and conquer type problems).

- The master method is utilized for algorithms that divides problem into a times n/b size pieces

$$T(n) = aT(n/b) + f(n)$$

where $a \geq 1$, $b > 1$, and f is asymptotically positive.

The Master Method

- The cost of every step is calculated (e.g. dividing cost, merge cost of the pieces, etc.) and sum of these costs are shown as $f(n)$

- Then the master method formula is used to have guess of the run time cost of the algorithm

Logaritma Nedir? Logaritma Formülleri Özellikleri

Logaritma Tanımı: a, b ∈ R^+ ve a≠1 olmak üzere a^x= b denklemini sağlayan x sayısına $\log_a b$ denir ve b'nin a tabanında logaritması diye okunur.

1) $\log_a x = b$ ise $x = a^b$ $\quad log_2 8 = 3 \mid 8 = 2^3$

2) $\log_a(A.B) = \log_a A + \log_a B$ $\quad log_2(4*8) = log_2(32) = 5 = log_2(4) + log_2(8) = 2+3$

3) $\log_a(A/B) = \log_a A - \log_a B$ $\quad log_2(16/4) = log_2(4) = 2 = log_2(16) - log_2(4) = 4-2 = 2$

4) $\log_a A^n = n.\log_a A$ $\quad log_2 8^2 = log_2 64 = 6 = 2 * log_2 8 = 2*3 = 6$

5) $\log_{a^m} A^n = \frac{n}{m} \log_a A$ $\quad log_{2^3} 8^2 = log_8 64 = 2 = (2/3) * log_2 8 = 2/3 * 3 = 2$

6) $\log_{(a^n)} x = \frac{1}{n} .\log_a x$ $\quad log_{2^3} 8 = log_8 8 = 1 = (1/3) * log_2 8 = 1/3 * 3 = 1$

7) $\log_a x = (\log_b x)/(\log_b a)$ [taban değiştirme] $\quad log_4 16 = 2 = log_2 16 - log_2 4 = 4-2 = 2$

8) $a^{\log_a x} =$ $\quad 2^{log_2 8} = 2^3 = 8 = 8$

9) $\log_a \sqrt[n]{A} = \frac{1}{n} \log_a A$ $\quad log_2 \sqrt[3]{8} = log_2 2 = 1 = \left(\frac{1}{3}\right) * log_2 8 = \frac{1}{3} * 3 = 1$

10) $\log_{1/a} x = -\log_a x$ $\quad log_{1/2} 8 = -log_2 8 = -3 \mid 8 = \left(\frac{1}{2}\right)^{-3}$

11) $\log_a b.\log_b c.\log_c d = \log_a d$ $\quad log_2 4 * log_4 16 * log_{16} 256 = 2*2*2 = 8 = log_2 256$

12) $\log_a b = 1/\log_b a$ veya $\log_a b.\log_b a = 1$

The Master Method: 3 Cases

- Recurrence: $T(n) = aT(n/b) + f(n)$
- Compare $f(n)$ with $n^{\log_b a}$
- Intuitively:

Case 1: $f(n)$ grows *polynomially slower* than $n^{\log_b a}$

Case 2: $f(n)$ grows *at the same rate* as $n^{\log_b a}$

Case 3: $f(n)$ grows *polynomially faster* than $n^{\log_b a}$

The Master Method: Case 1

- Recurrence: $T(n) = aT(n/b) + f(n)$

Case 1: $\dfrac{n^{\log_b a}}{f(n)} = \Omega(n^{\varepsilon})$ for some constant $\varepsilon > 0$

i.e., $f(n)$ grows polynomialy slower than $n^{\log_b a}$ (by an n^{ε} factor).

Solution: $T(n) = \Theta(n^{\log_b a})$

The Master Method: Case 2 (simple version)

- Recurrence: $T(n) = aT(n/b) + f(n)$

<u>Case 2</u>: $\dfrac{f(n)}{n^{\log_b a}} = \Theta(1)$

i.e., f (n) and $n^{\log_b a}$ *grow at similar rates*

<u>Solution</u>: $T(n) = \Theta(n^{\log_b a} \lg n)$

The Master Method: Case 3

Case 3: $\dfrac{f(n)}{n^{\log_b a}} = \Omega(n^{\varepsilon})$ for some constant $\varepsilon > 0$

i.e., $f(n)$ grows polynomialy faster than $n^{\log_b a}$ (by an n^{ε} factor).

and the following regularity condition holds:

$a\,f(n/b) \le c\,f(n)$ for some constant $c < 1$

Solution: $T(n) = \Theta(f(n))$

Example: T(n) = 4T(n/2) + n

a = 4

b = 2

f(n) = n

$$n^{\log_b a} = n^2$$

f(n) grows *polynomially* slower than $n^{\log_b a}$

$$\frac{n^{\log_b a}}{f(n)} = \frac{n^2}{n} = n = \Omega(n^{\varepsilon})$$

for ε = 1

CASE 1

T(n) = Θ($n^{\log_b a}$)

T(n) = Θ(n^2)

Example: $T(n) = 4T(n/2) + n^2$

$a = 4$

$b = 2$

$f(n) = n^2$

$n^{\log_b a} = n^2$

$f(n)$ grows at similar rate as $n^{\log_b a}$

$f(n) = \Theta(n^{\log_b a}) = n^2$

CASE 2

$T(n) = \Theta(n^{\log_b a} \lg n)$

$\boxed{T(n) = \Theta(n^2 \lg n)}$

Example: $T(n) = 4T(n/2) + n^3$

a = 4

b = 2

$f(n) = n^3$

$n^{\log_b a} = n^2$

f(n) grows *polynomially* faster than $n^{\log_b a}$

$$\frac{f(n)}{n^{\log_b a}} = \frac{n^3}{n^2} = n = \Omega(n^{\varepsilon})$$

for $\varepsilon = 1$

seems like CASE 3, but need to check the regularity condition

Regularity condition: $a\,f(n/b) \le c\,f(n)$ for some constant $c < 1$

$4\,(n/2)^3 \le cn^3$ for $c = 1/2$

CASE 3 → $T(n) = \Theta(f(n))$ → $T(n) = \Theta(n^3)$

Example: T(n) = 4T(n/2) + n²/lgn

a = 4

b = 2

f(n) = n²/lgn

$$n^{\log_b a} = n^2$$

f(n) grows slower than $n^{\log_b a}$

but is it polynomially slower?

$$\frac{n^{\log_b a}}{f(n)} = \frac{n^2}{\frac{n^2}{\lg n}} = \lg n \neq \Omega(n^{\varepsilon})$$

for any ε > 0

is **not** CASE 1

Master method does not apply!

The Master Method: Case 2 (general version)

- Recurrence: $T(n) = aT(n/b) + f(n)$

Case 2: $\dfrac{f(n)}{n^{\log_b a}} = \Theta(\lg^k n)$ for some constant $k \geq 0$

Solution: $T(n) = \Theta(n^{\log_b a} \lg^{k+1} n)$

Example: $T(n) = 4T(n/2) + n^2/\lg n$

a = 4

b = 2

$f(n) = n^2/\lg n$

Solution:

$$\frac{\frac{n^2}{\log n}}{n^{\log_2 4}} = \frac{\frac{n^2}{\log n}}{n^2} = \frac{1}{\log n} = \log_n 2 = log^{-1} n$$

$$T(n) = \Theta(n^{\log_2 4} \lg^{1+1} n)$$

$$= \Theta(n^2 \lg\lg n)$$

General Method (Akra-Bazzi)

$$T(n) = \sum_{i=1}^{k} a_i T(n/b_i) + f(n)$$

Let p be the unique solution to

$$\sum_{i=1}^{k} (a_i / b^p_i) = 1$$

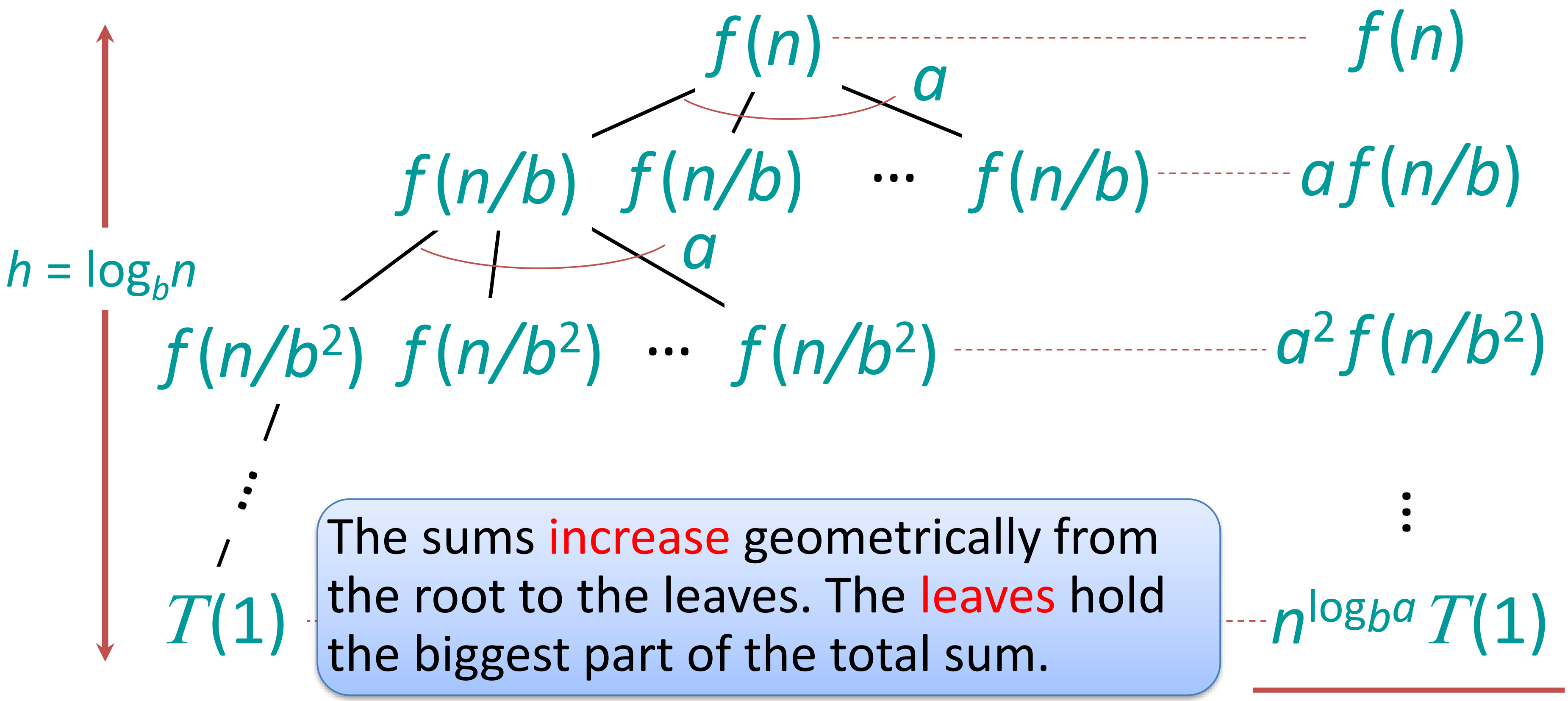
Then, the answers are the same as for the master method, but with n^p instead of $n^{\log_b a}$

(*Akra and Bazzi also prove an even more general result*.)

Case 1 Explained

f(n) is smaller than leaf sum.

$f(n) < O(n^{\log_b a})$



$\Theta(n^{\log_b a})$

Case 2 Explained

f(n) is roughly equal to the leaf sum.

f(n) = O($n^{\log_b a}$)

$h = \log_b n$

$f(n)$ .... $f(n)$

a

$f(n/b)$ $f(n/b)$ … $f(n/b)$ .... $af(n/b)$

a

$f(n/b^2)$ $f(n/b^2)$ … $f(n/b^2)$ .... $a^2 f(n/b^2)$

⋮

$T(1)$ .... $n^{\log_b a}\, T(1)$

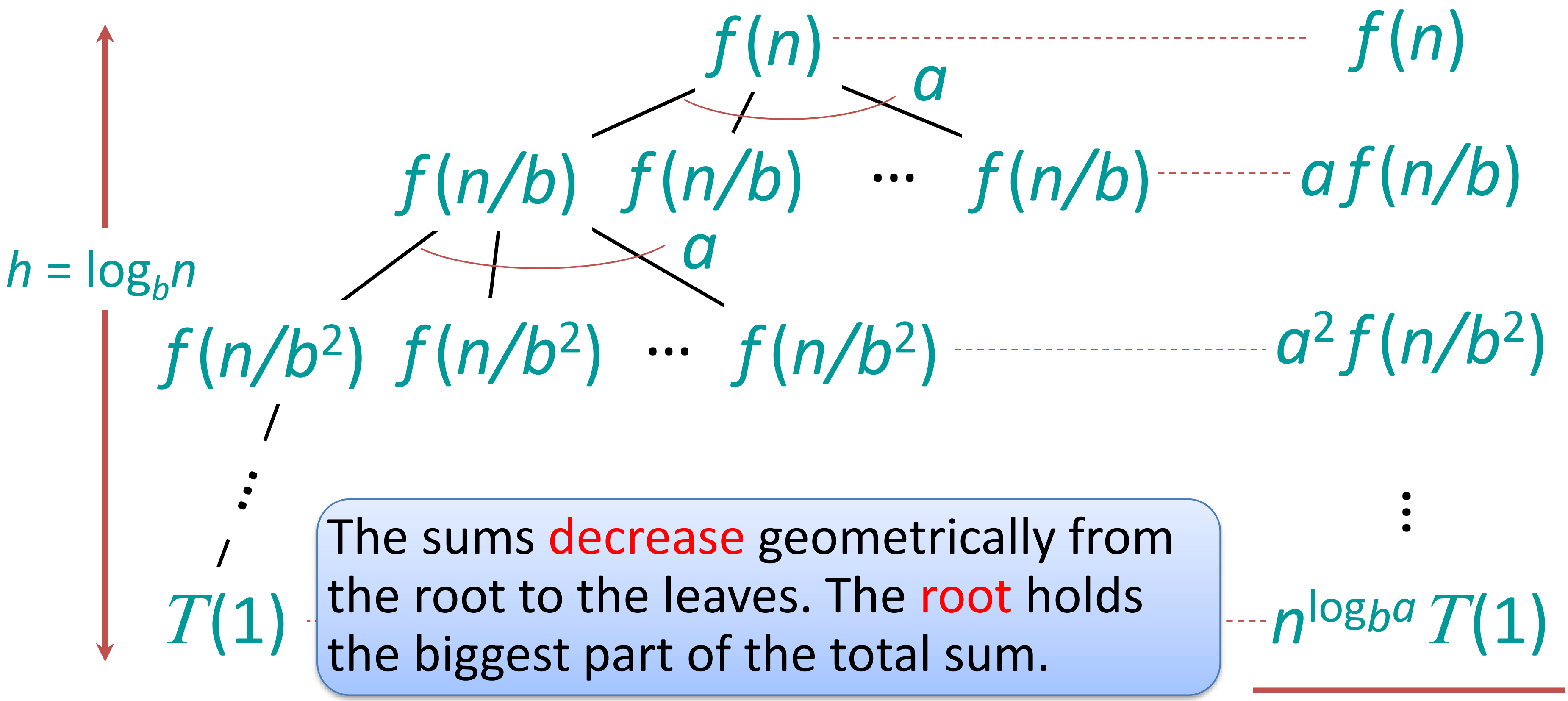
The sums are approximately the same on each of the levels (Θ(total of all sums)).

$\Theta(n^{\log_b a} * \log n)$

Case 3 Explained

a f(n/b) is getting smaller at lower levels (see def^n)

f(n) > O($n^{\log_b a}$)



$\Theta(f(n))$

Proof of Master Theorem: Case 1 and Case 2

- Recall from the recursion tree (note $h = \lg_b n$=tree height)

$$T(n) = \underbrace{\Theta(n^{\log_b a})}_{\text{Leaf cost}} + \underbrace{\sum_{i=0}^{h-1} a^i f(n/b^i)}_{\text{Non-leaf cost} = g(n)}$$

Proof of Case 1

- $\dfrac{n^{\log_b a}}{f(n)} = \Omega(n^{\varepsilon})$ for some $\varepsilon > 0$
- $\dfrac{n^{\log_b a}}{f(n)} = \Omega(n^{\varepsilon}) \Rightarrow \dfrac{f(n)}{n^{\log_b a}} = O(n^{-\varepsilon}) \Rightarrow f(n) = O(n^{\log_b a - \varepsilon})$
- $g(n) = \sum_{i=0}^{h-1} a^i O\left((n/b^i)^{\log_b a-\varepsilon}\right) = O\left(\sum_{i=0}^{h-1} a^i (n/b^i)^{\log_b a-\varepsilon}\right)$
- $= O\left(n^{\log_b a-\varepsilon} \sum_{i=0}^{h-1} a^i b^{i\varepsilon} / b^{i \log_b a}\right)$

Case 1 (cont')

$$\sum_{i=0}^{h-1} \frac{a^i b^{i\varepsilon}}{b^{i \log_b a}} = \sum_{i=0}^{h-1} a^i \frac{(b^\varepsilon)^i}{(b^{\log_b a})^i} = \sum a^i \frac{b^{\varepsilon i}}{a^i} = \sum_{i=0}^{h-1} (b^\varepsilon)^i$$

= An increasing geometric series since b > 1

$$= \frac{b^{\varepsilon h} - 1}{b^\varepsilon - 1} = \frac{(b^h)^\varepsilon - 1}{b^\varepsilon - 1} = \frac{(b^{\log_b n})^\varepsilon - 1}{b^\varepsilon - 1} = \frac{n^\varepsilon - 1}{b^\varepsilon - 1} = O(n^\varepsilon)$$

Case 1 (cont')

- $$g(n) = O\left(n^{\log_b a - \varepsilon} O(n^{\varepsilon})\right) = O\left(\frac{n^{\log_b a}}{n^{\varepsilon}} O(n^{\varepsilon}) \right)$$
  $$= O(n^{\log_b a})$$

- $$T(n) = \Theta(n^{\log_b a}) + g(n) = \Theta(n^{\log_b a}) + O(n^{\log_b a})$$
  $$= \Theta(n^{\log_b a})$$

Q.E.D.

Proof of Case 2 (limited to *k=0*)

- $$\frac{f(n)}{n^{\log_b a}} = \Theta(\lg^0 n) = \Theta(1) \Rightarrow f(n) = \Theta(n^{\log_b a}) \Rightarrow f(n/b^i) = \Theta\left((\frac{n}{b^i})^{\log_b a} \right)$$
- $$\therefore g(n) = \sum_{i=0}^{h-1} a^i \Theta\left((n/b^i)^{\log_b a} \right)$$

$$= \Theta\left(\sum_{i=0}^{h-1} a^i \frac{n^{\log_b a}}{b^{i \log_b a}} \right) = \Theta\left(n^{\log_b a} \sum_{i=0}^{h-1} a^i \frac{1}{(b^{\log_b a})^i} \right) = \Theta\left(n^{\log_b a} \sum_{i=0}^{h-1} a^i \frac{1}{a^i} \right)$$

$$= \Theta\left(n^{\log_b a} \sum_{i=0}^{\log_b n - 1} 1 \right) = \Theta\left(n^{\log_b a} \log_b n \right) = \Theta\left(n^{\log_b a} \lg n \right)$$

- $$T(n) = n^{\log_b a} + \Theta(n^{\log_b a} \lg n)$$
$$= \Theta\left(n^{\log_b a} \lg n \right)$$

Q.E.D.