CSE214 – Analysis of Algorithms

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https://github.com/FurkanGozukara/CSE214_2018

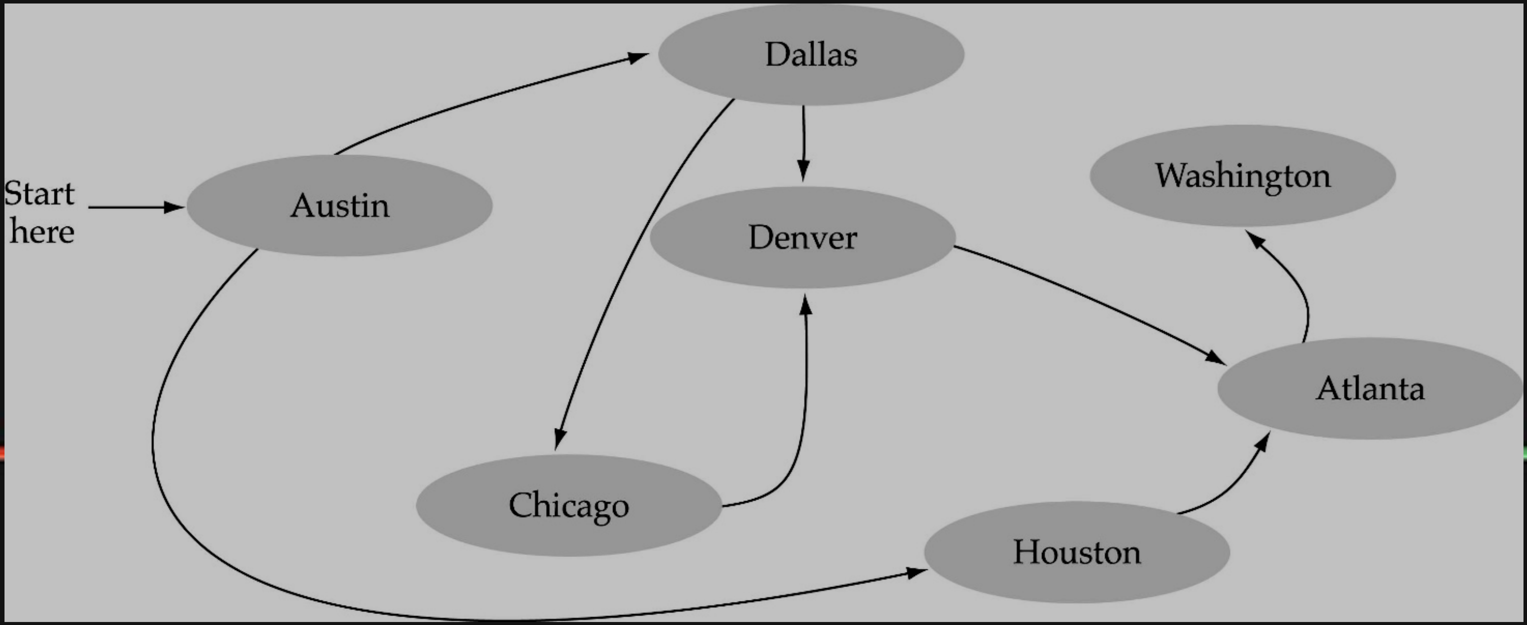
Lecture 10

Graphs

Based George Bebis Lecture Notes - Reno Logo University of Nevada

What is a graph?

- A data structure that consists of a set of nodes (*vertices*) and a set of edges that relate the nodes to each other
- The set of edges describes relationships among the vertices



Formal definition of graphs

- A graph G is defined as follows:

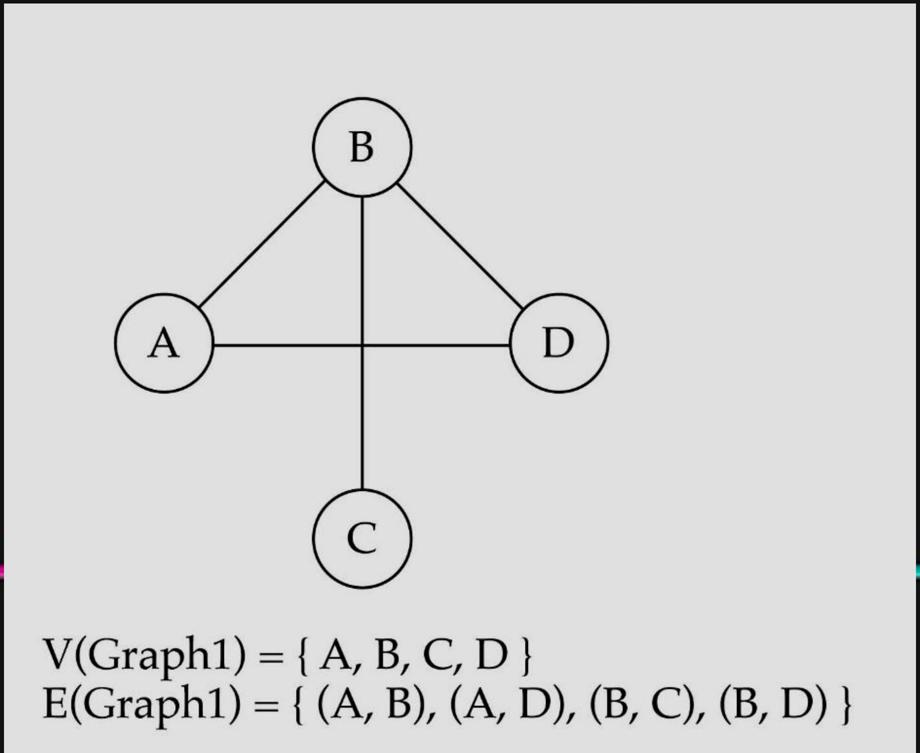
$$G=(V,E)$$

$V(G)$: a finite, nonempty set of vertices

$E(G)$: a set of edges (pairs of vertices)

Directed vs. undirected graphs

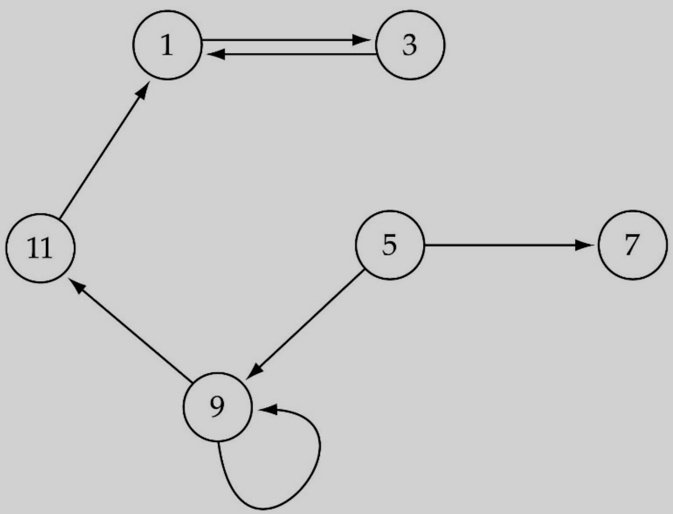
- When the edges in a graph have no direction, the graph is called *undirected*



Directed vs. undirected graphs (cont.)

- When the edges in a graph have a direction, the graph is called *directed* (or *digraph*)

(b) Graph2 is a directed graph.

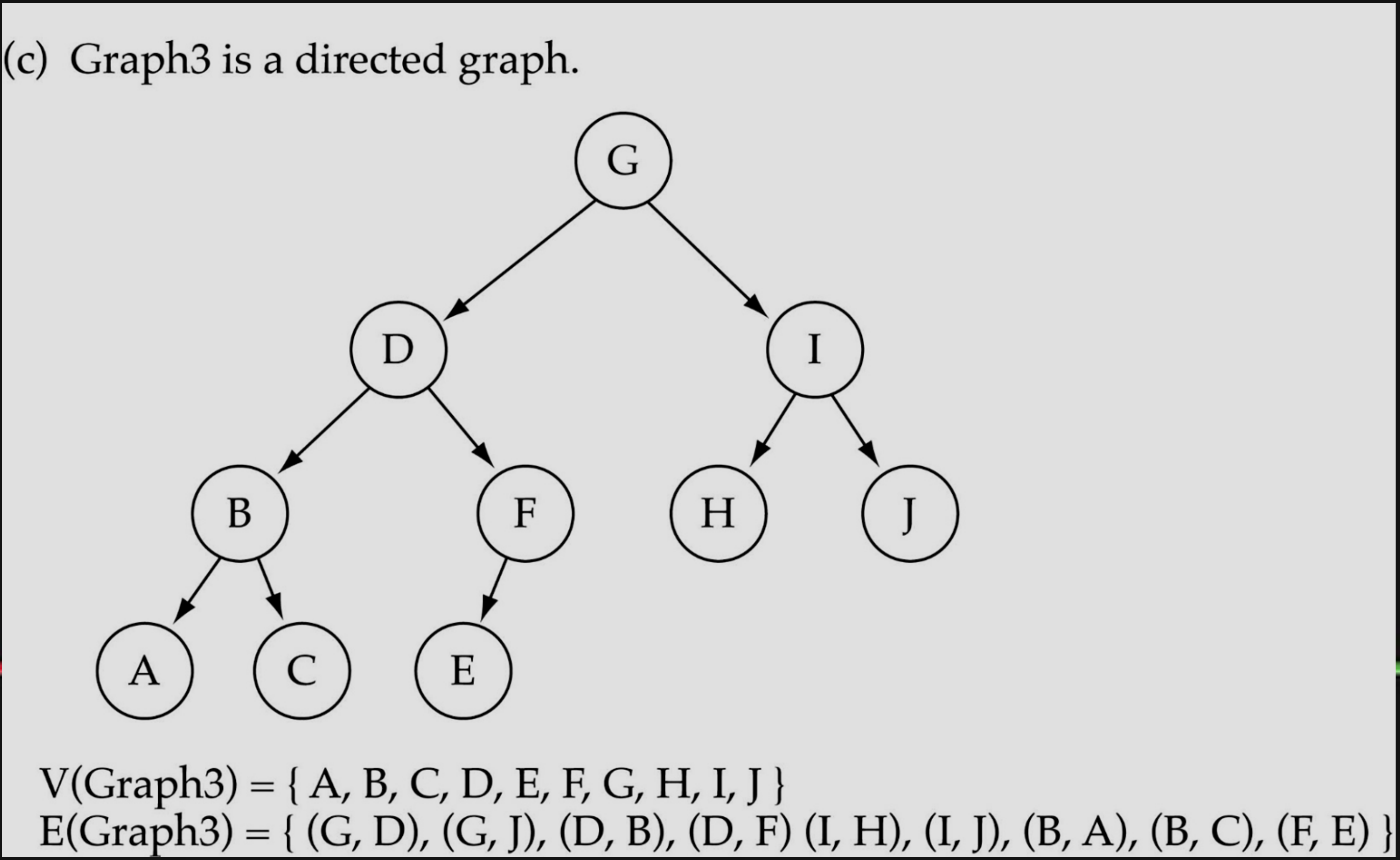


V(Graph2) = { 1, 3, 5, 7, 9, 11 }
E(Graph2) = {(1,3) (3,1) (5,9) (9,11) (5,7) , (9, 9), (11, 1) }

Warning: if the graph is directed, the order of the vertices in each edge is important !!

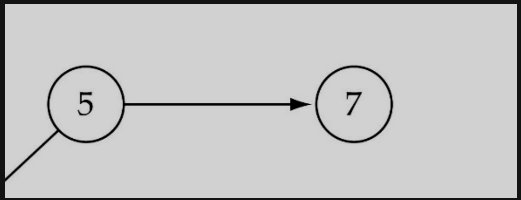
Trees vs graphs

- Trees are special cases of graphs!!



Graph terminology

- Adjacent nodes: two nodes are adjacent if they are connected by an edge



5 is adjacent to 7
7 is adjacent from 5

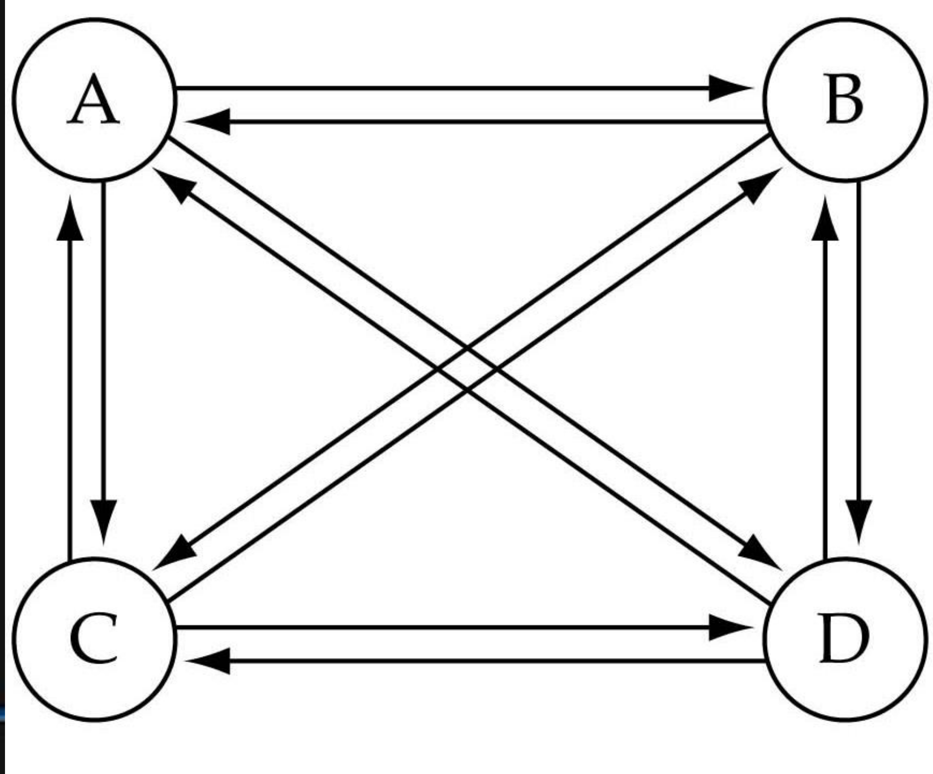
- Path: a sequence of vertices that connect two nodes in a graph
- Complete graph: a graph in which every vertex is directly connected to every other vertex

Graph terminology (cont.)

- What is the number of edges in a complete directed graph with N vertices?

*N * (N-1)*

$$O(N^2)$$



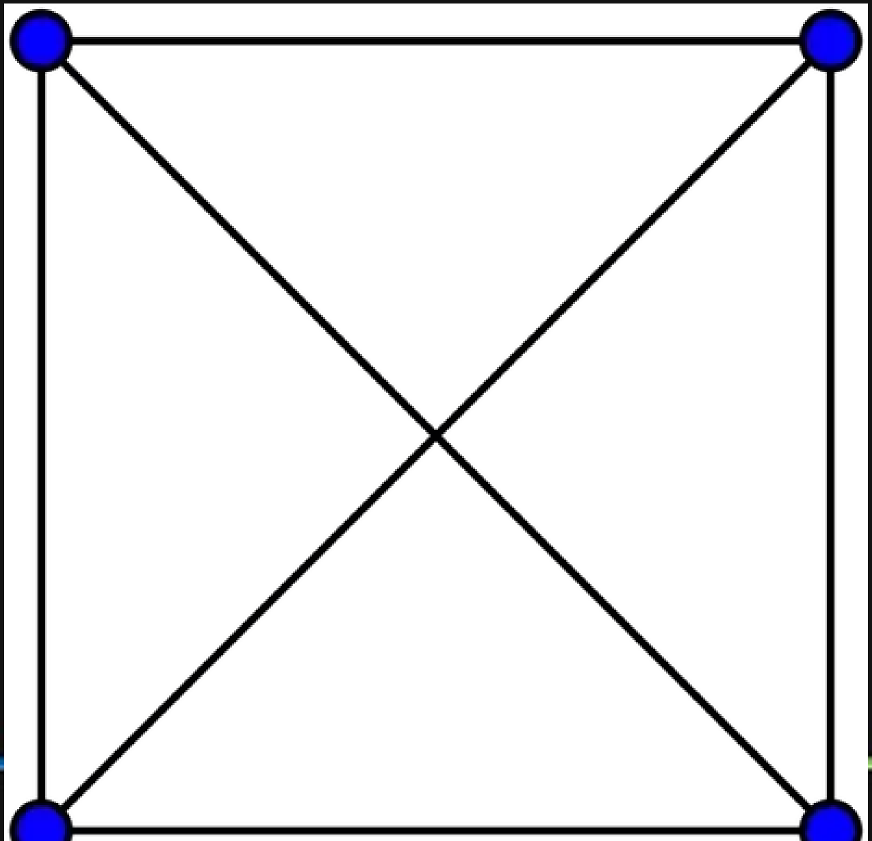
(a) Complete directed graph.

Graph terminology (cont.)

- What is the number of edges in a complete undirected graph with N vertices?

  $N * (N-1) / 2$

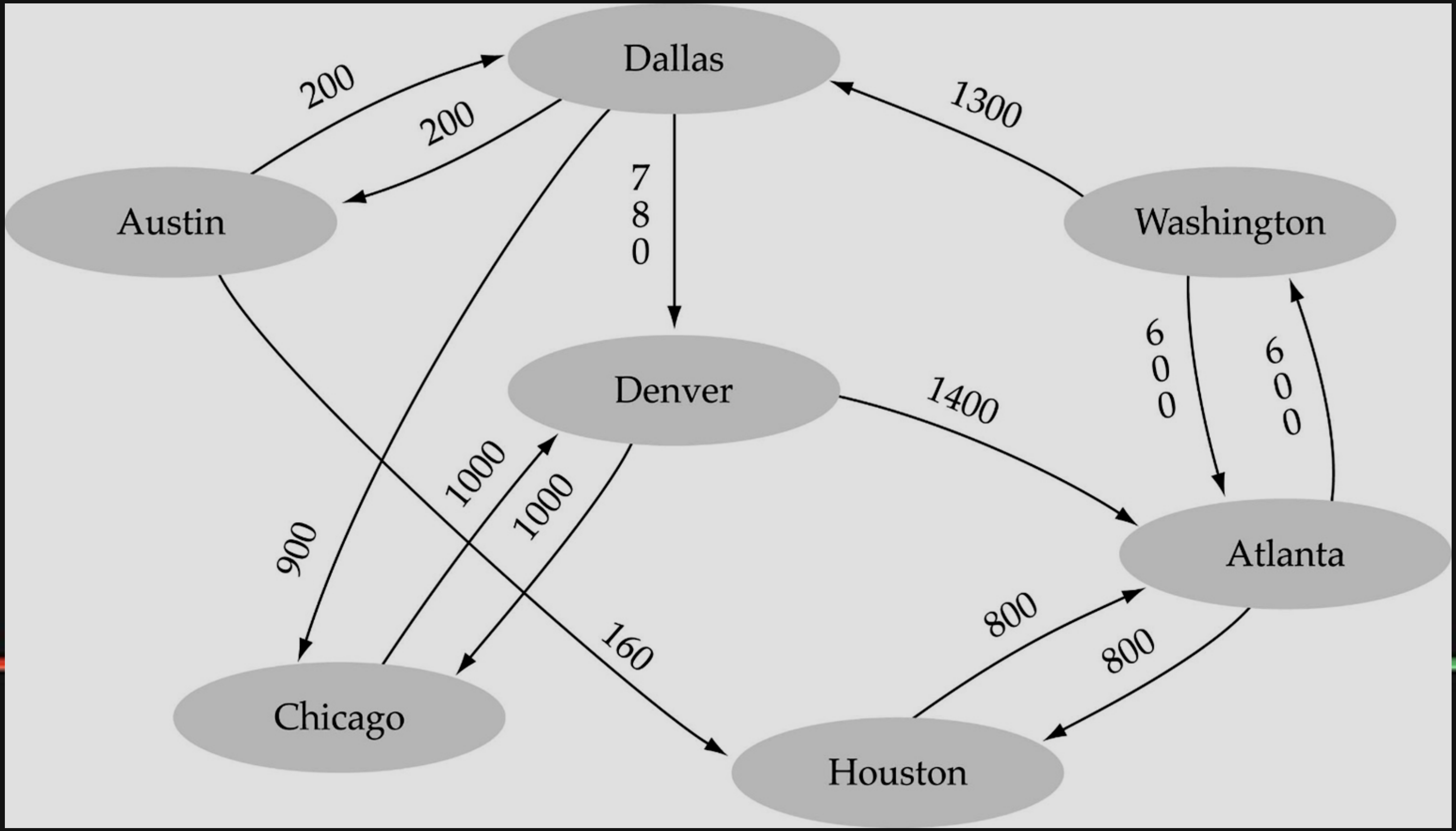
  $O(N^2)$



(b) Complete undirected graph

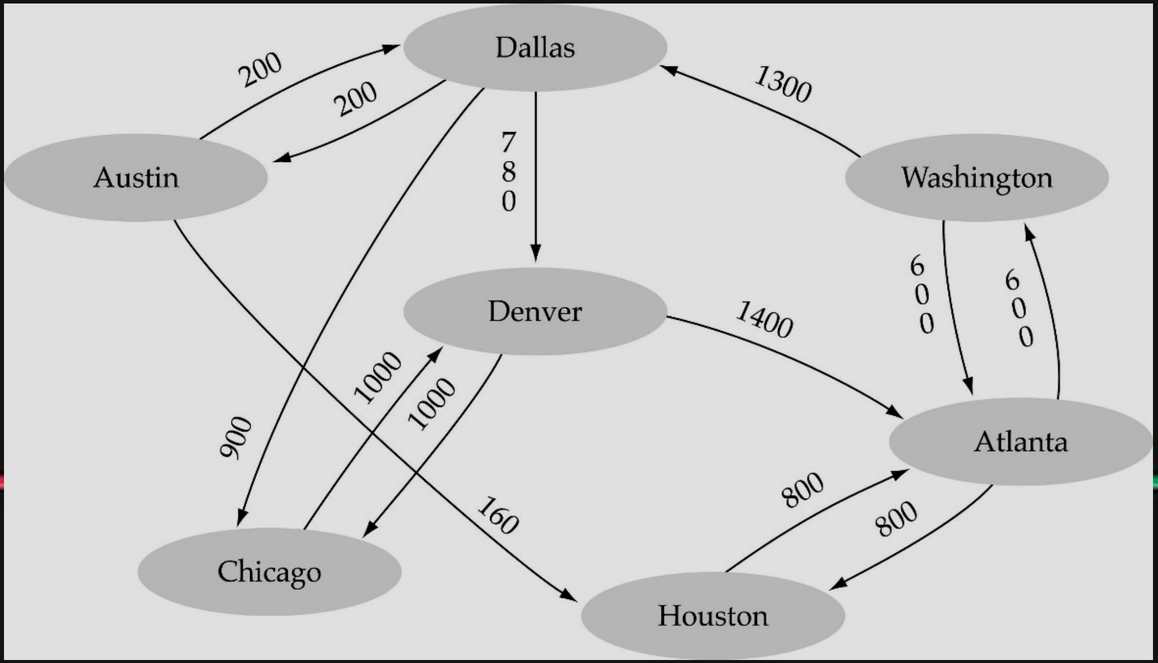
Graph terminology (cont.)

- Weighted graph: a graph in which each edge carries a value



Graph implementation

- Array-based implementation
  - A 1D array is used to represent the vertices
  - A 2D array (adjacency matrix) is used to represent the edges



Array-based implementation

graph

.numVertices 7

.vertices

[0]	"Atlanta "
[1]	"Austin "
[2]	"Chicago "
[3]	"Dallas "
[4]	"Denver "
[5]	"Houston "
[6]	"Washington"
[7]	
[8]	
[9]	

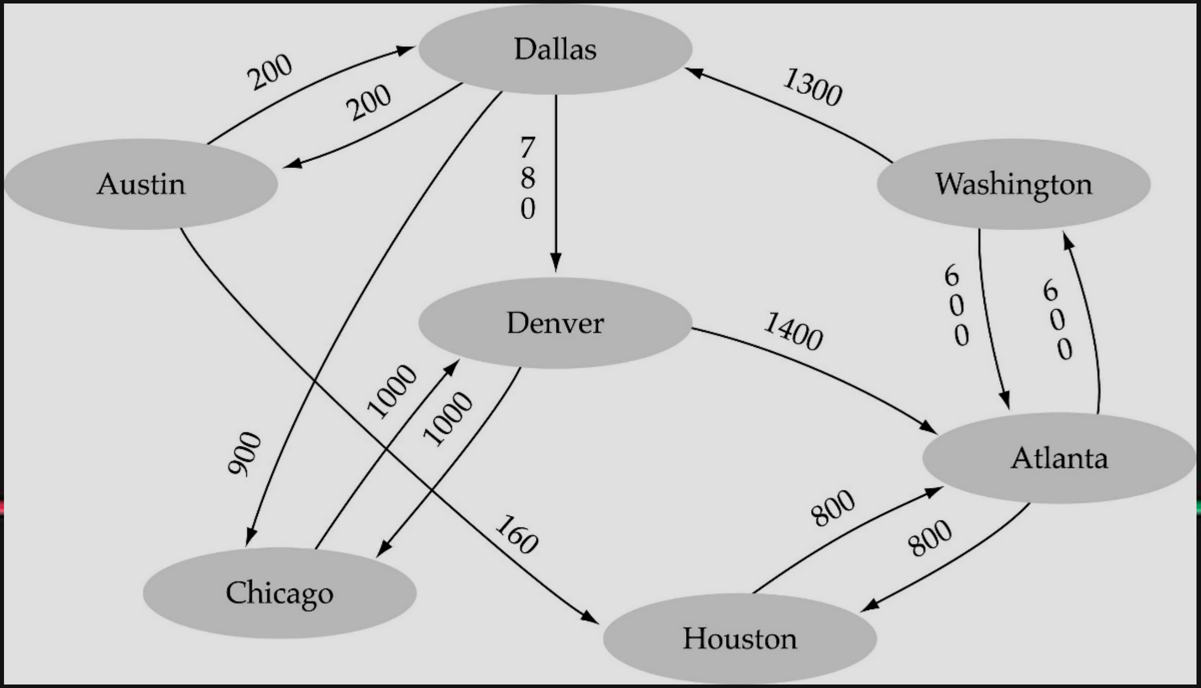
.edges

	[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]
[0]	0	0	0	0	0	800	600	•	•	•
[1]	0	0	0	200	0	160	0	•	•	•
[2]	0	0	0	0	1000	0	0	•	•	•
[3]	0	200	900	0	780	0	0	•	•	•
[4]	1400	0	1000	0	0	0	0	•	•	•
[5]	800	0	0	0	0	0	0	•	•	•
[6]	600	0	0	1300	0	0	0	•	•	•
[7]	•	•	•	•	•	•	•	•	•	•
[8]	•	•	•	•	•	•	•	•	•	•
[9]	•	•	•	•	•	•	•	•	•	•

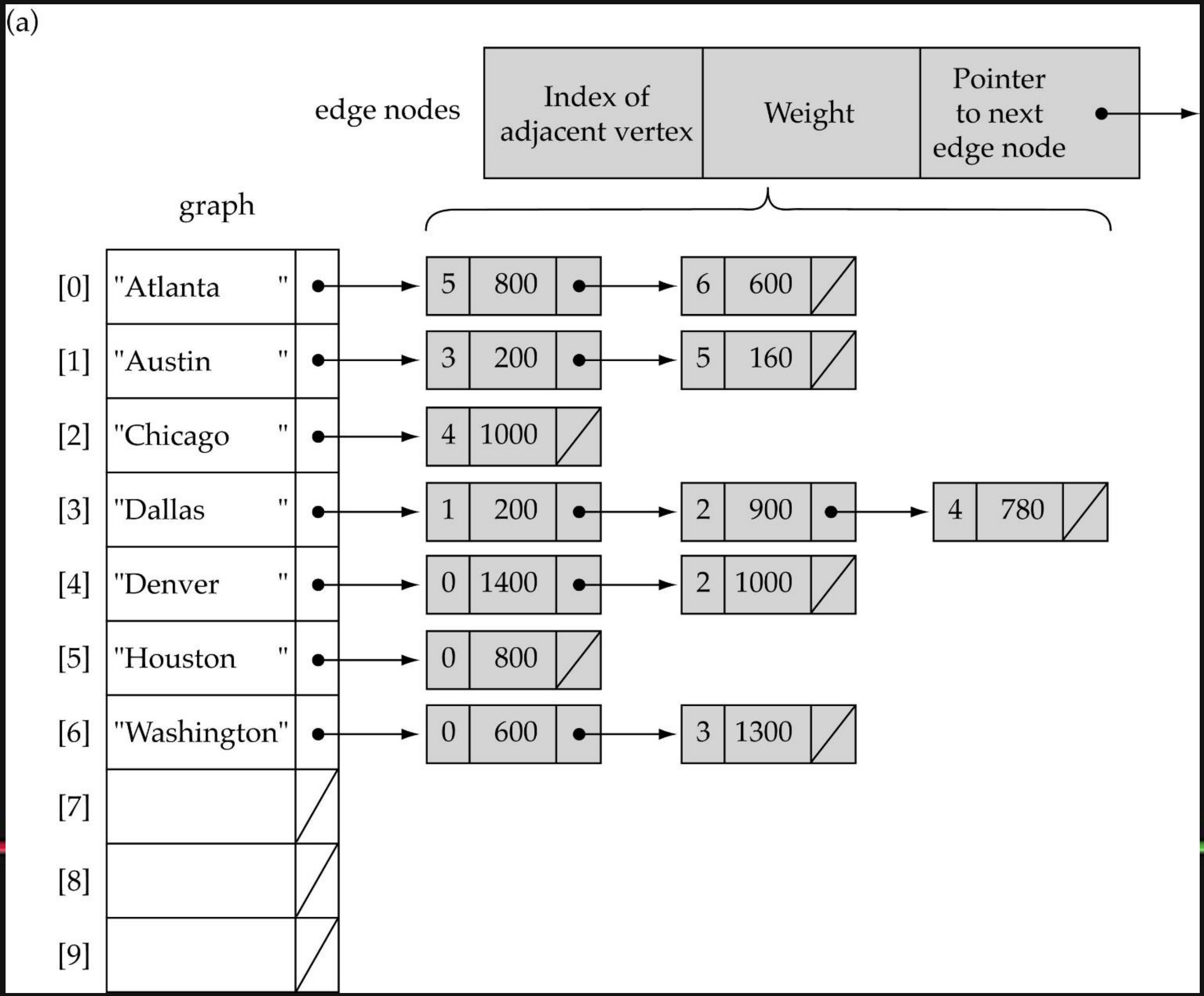
(Array positions marked '•' are undefined)

Graph implementation (cont.)

- Linked-list implementation
  - A 1D array is used to represent the vertices
  - A list is used for each vertex *v* which contains the vertices which are adjacent from *v* (adjacency list)



Linked-list implementation



Adjacency matrix vs. adjacency list representation

- **Adjacency matrix**
  - Good for dense graphs --$|E| \sim O(|V|^2)$
  - Memory requirements: $O(|V| + |E|) = O(|V|^2)$
  - Connectivity between two vertices can be tested quickly
- **Adjacency list**
  - Good for sparse graphs -- $|E| \sim O(|V|)$
  - Memory requirements: $O(|V| + |E|) = O(|V|)$
  - Vertices adjacent to another vertex can be found quickly

Graph searching

- *Problem*: find a path between two nodes of the graph (e.g., Austin and Washington)
- *Methods*: Depth-First-Search (DFS) or Breadth-First-Search (BFS)

Depth-First-Search (DFS)

- What is the idea behind DFS?
  - Travel as far as you can down a path
  - Back up *as little as possible* when you reach a "dead end" (i.e., next vertex has been "marked" or there is no next vertex)
- DFS can be implemented efficiently using a *stack*

What is a Stack?

- A *stack* is a list with the restriction that insertions and deletions can be performed in only one position, namely, the end of the list, called the *top*.
- The operations: push (insert) and pop (delete)

Breadth-First-Searching (BFS)

- What is the idea behind BFS?
  - Look at all possible paths at the same depth before you go at a deeper level
  - Back up *as far as possible* when you reach a "dead end" (i.e., next vertex has been "marked" or there is no next vertex)

Depth-First Graph Traversal Algorithm

Alyce Brady
Kalamazoo College

Search vs Traversal

- Tree Search: Look for a given node
  - stop when node found, even if not all nodes were visited
- Tree Traversal: Always visit all nodes

Depth-first Search

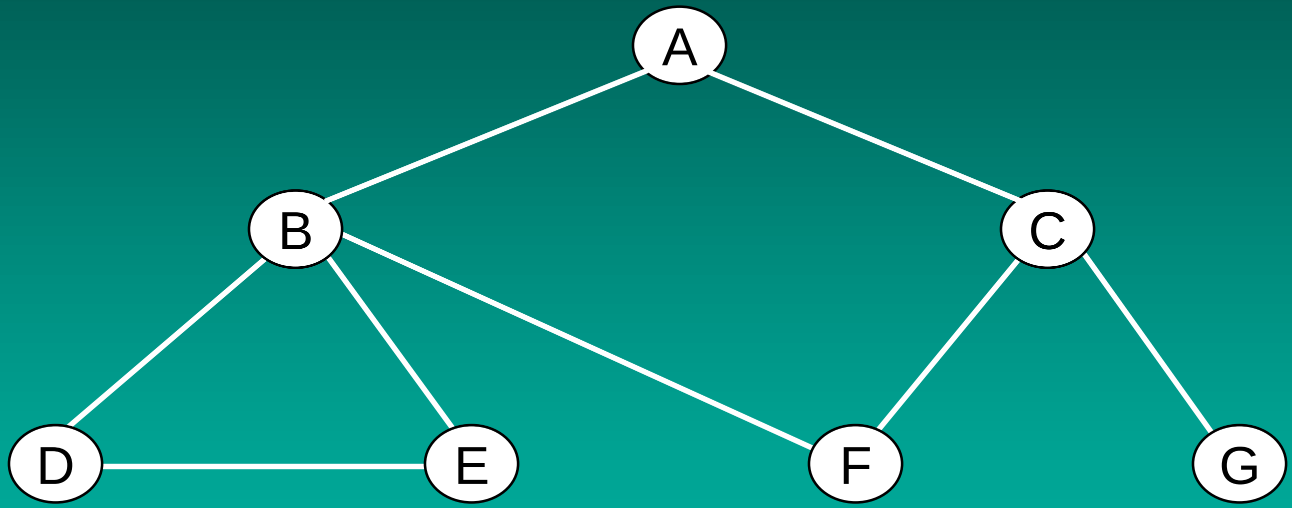
- Similar to Depth-first Traversal of a Binary Tree
- Choose a starting vertex
- Do a depth-first search on each adjacent vertex

Pseudo-Code for Depth-First Search

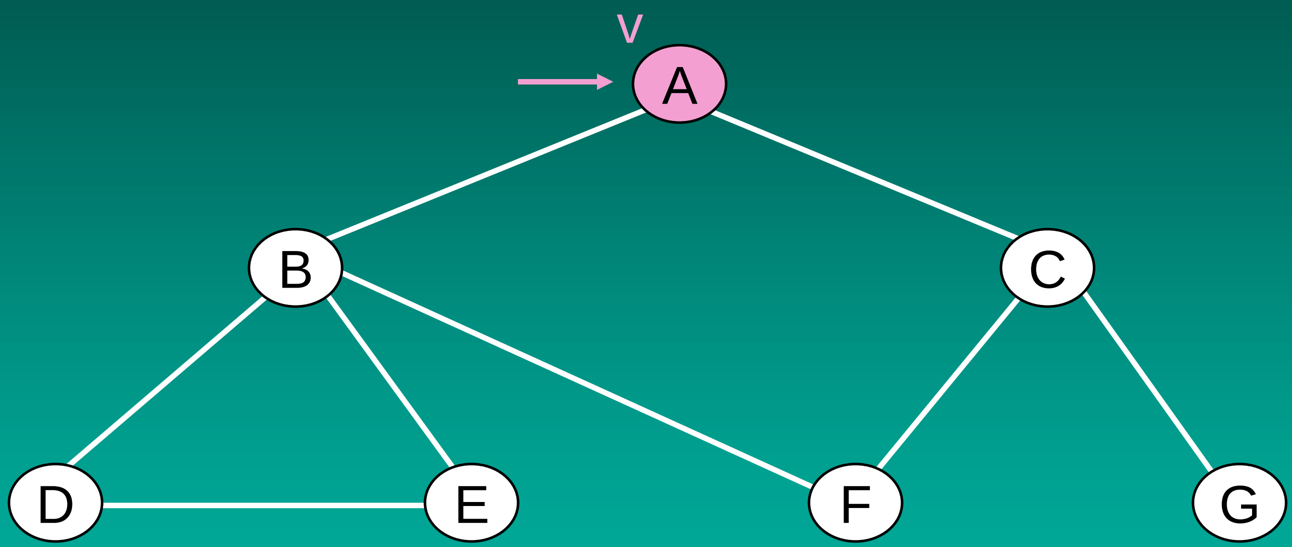
depth-first-search

    mark vertex as visited
    for each adjacent vertex
        if unvisited
            do a depth-first search on adjacent vertex

Depth-First Search

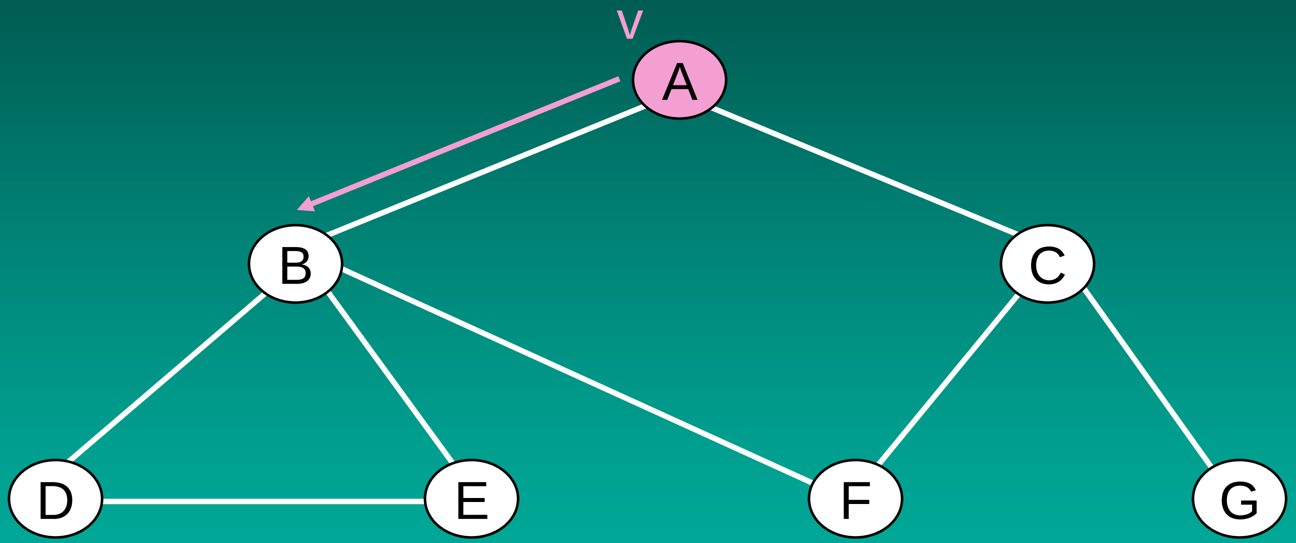


Depth-First Search



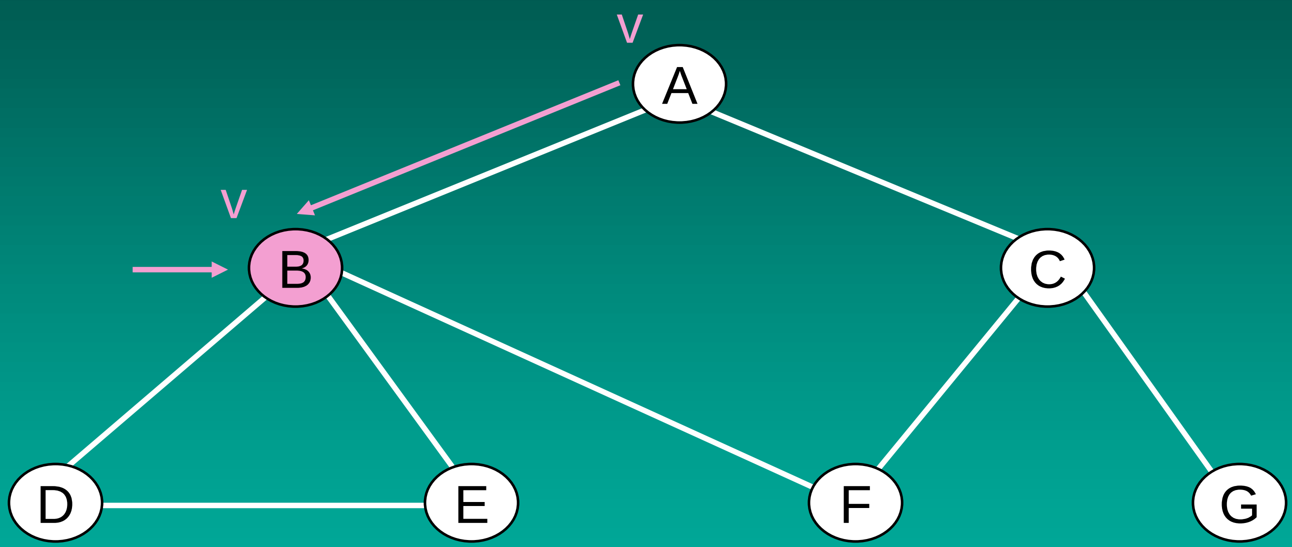
A

Depth-First Search



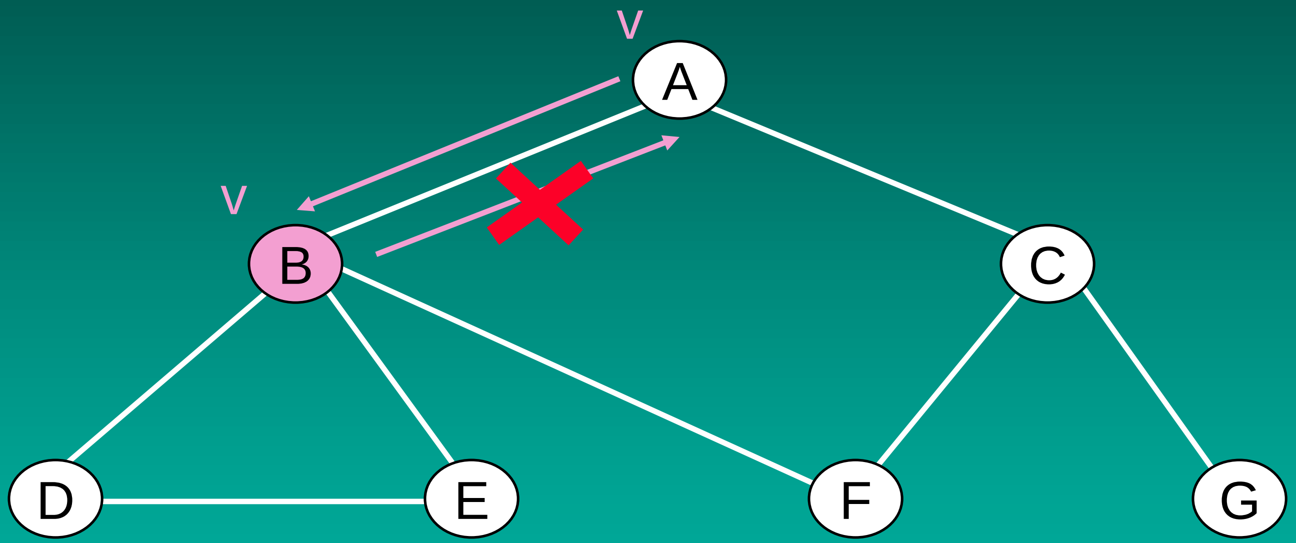
A

Depth-First Search



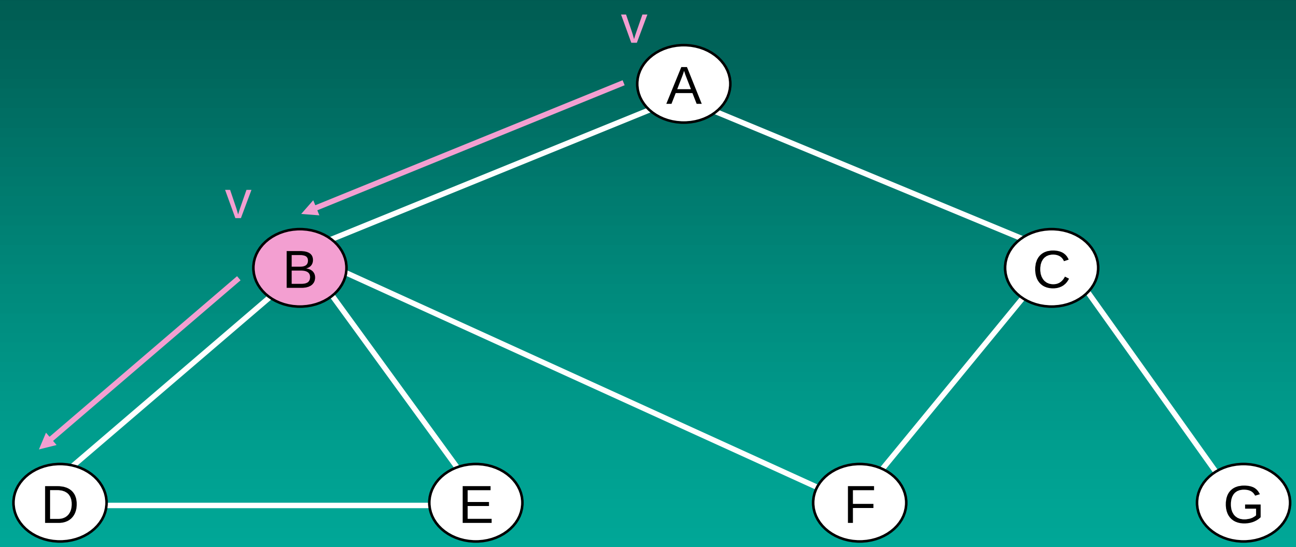
A B

Depth-First Search



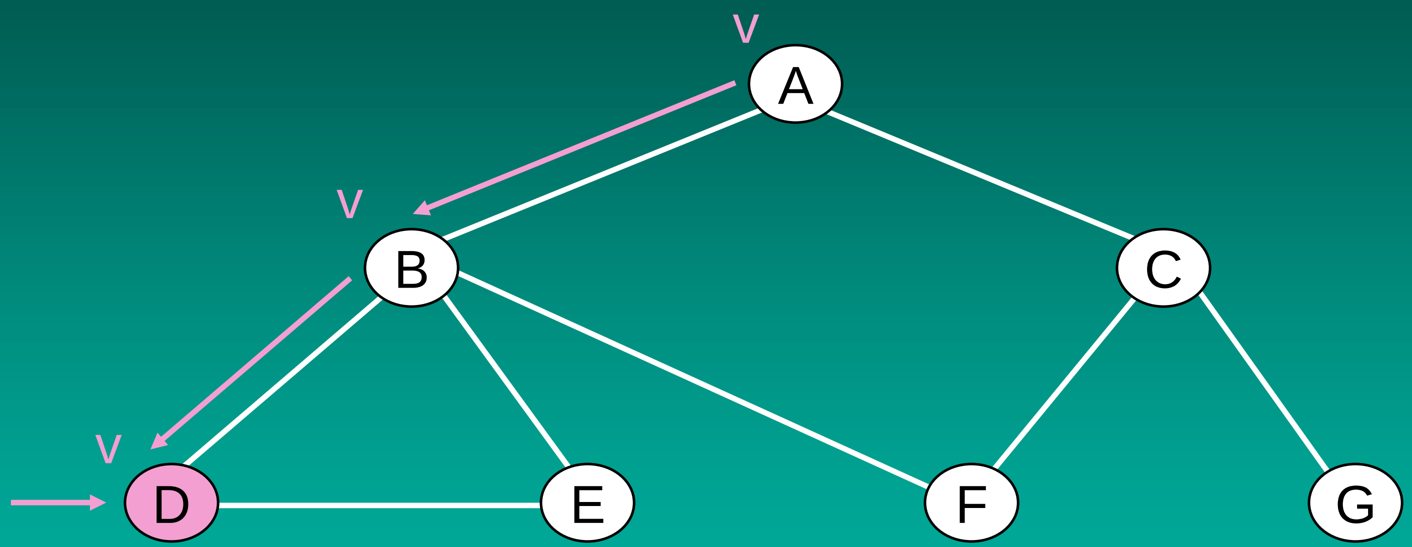
A B

Depth-First Search



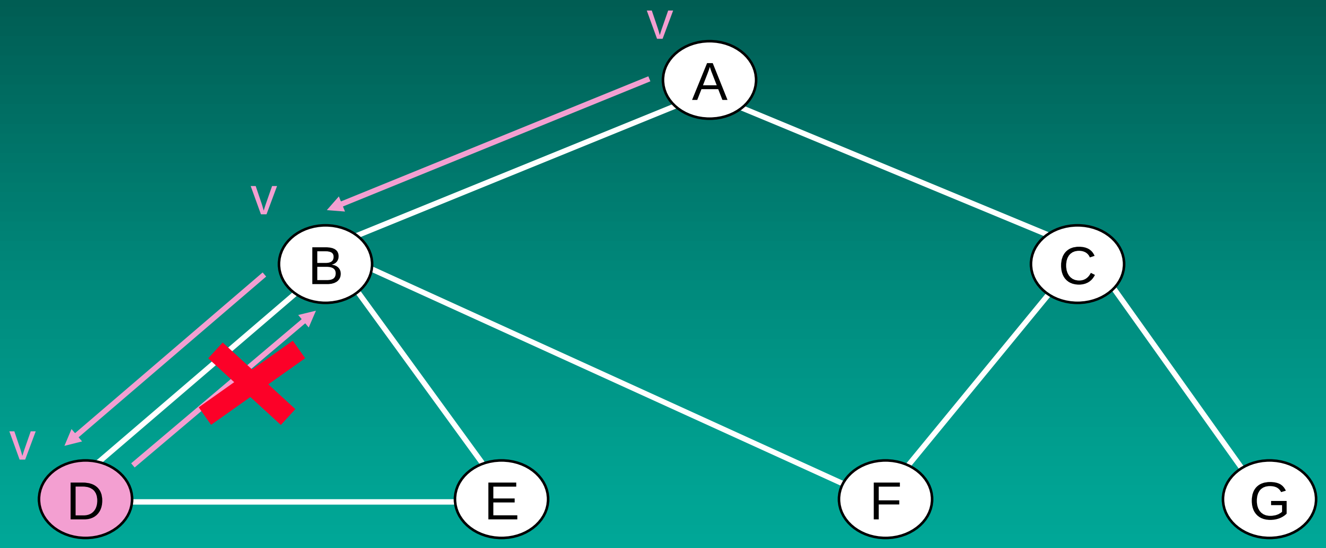
A B

Depth-First Search



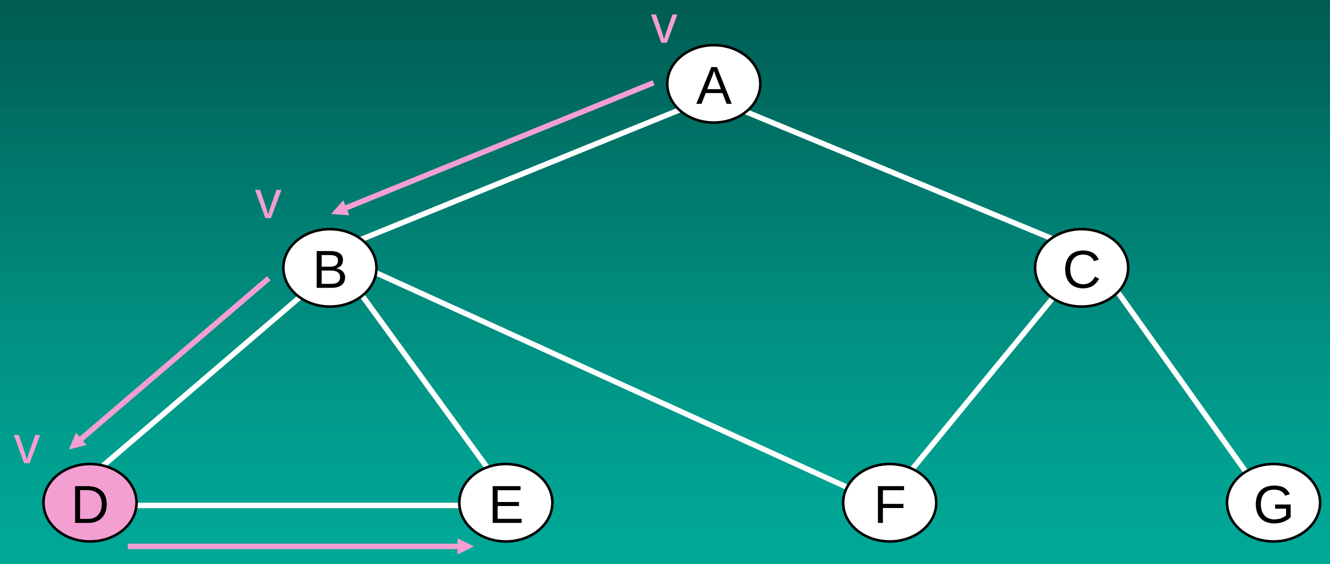
A B D

Depth-First Search



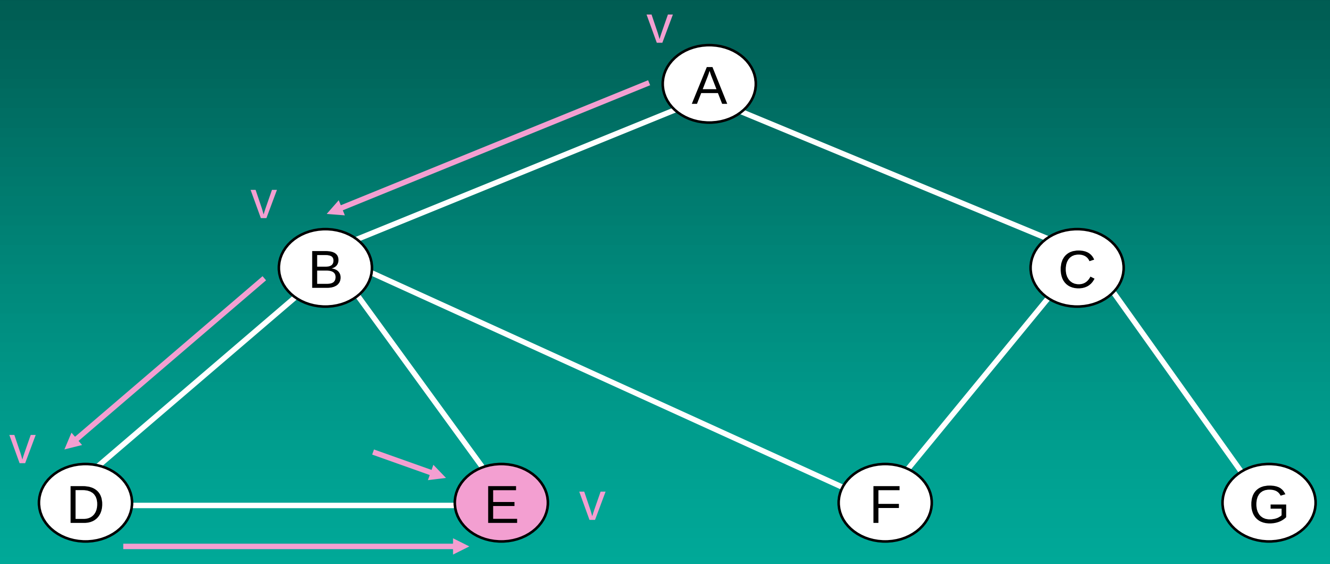
A B D

Depth-First Search



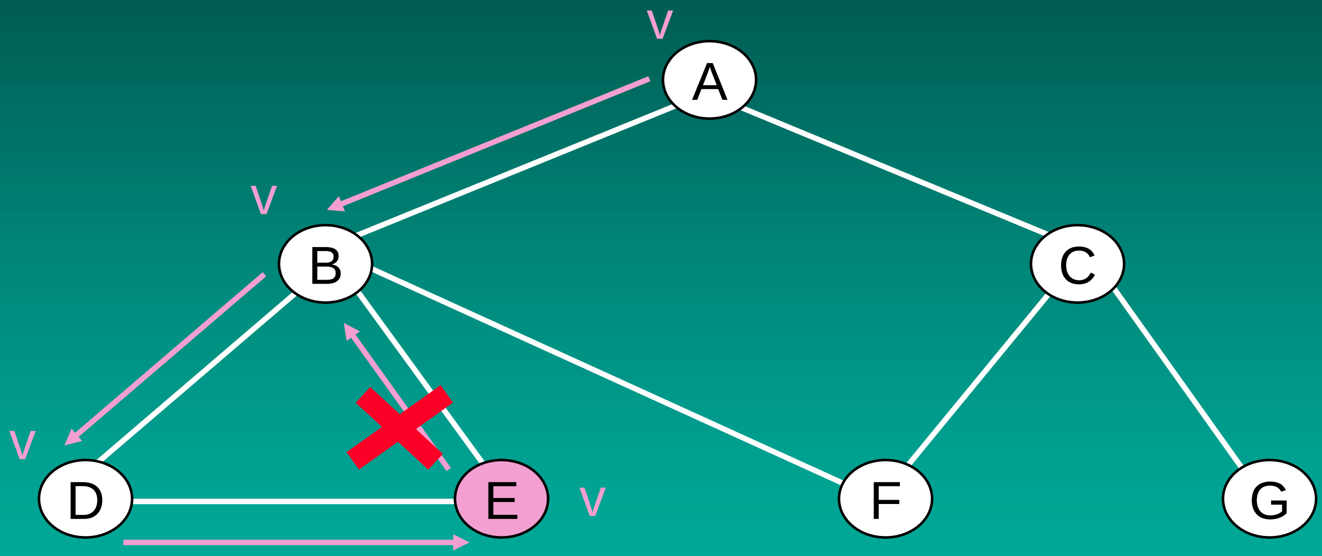
A B D

Depth-First Search



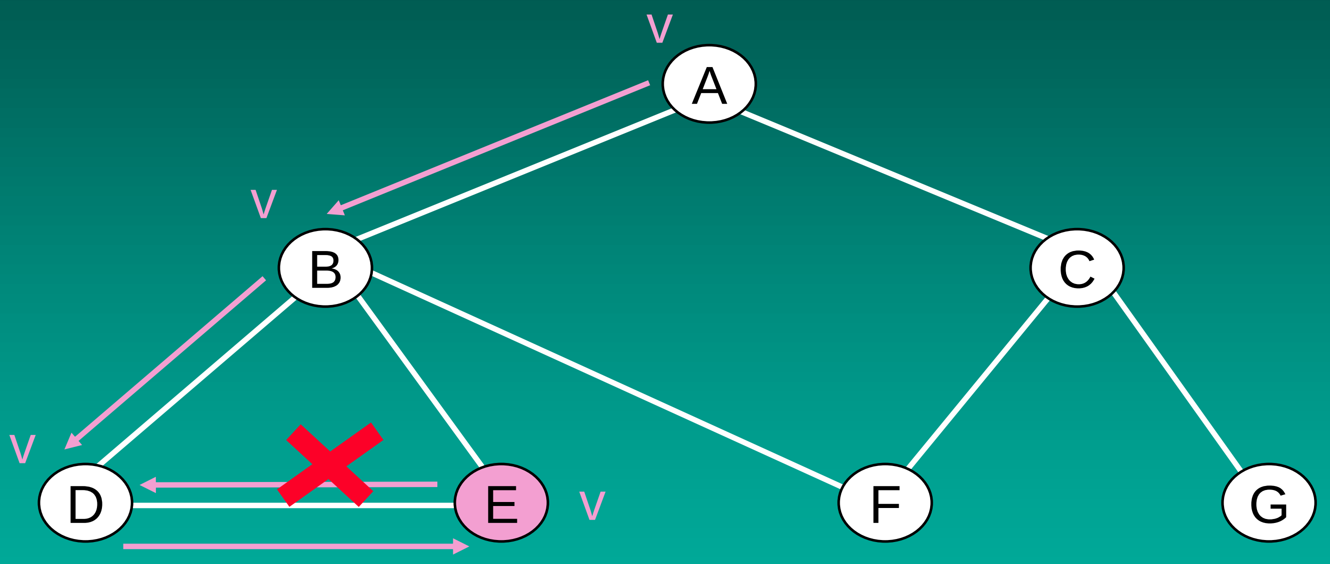
A B D E

Depth-First Search



A B D E

Depth-First Search



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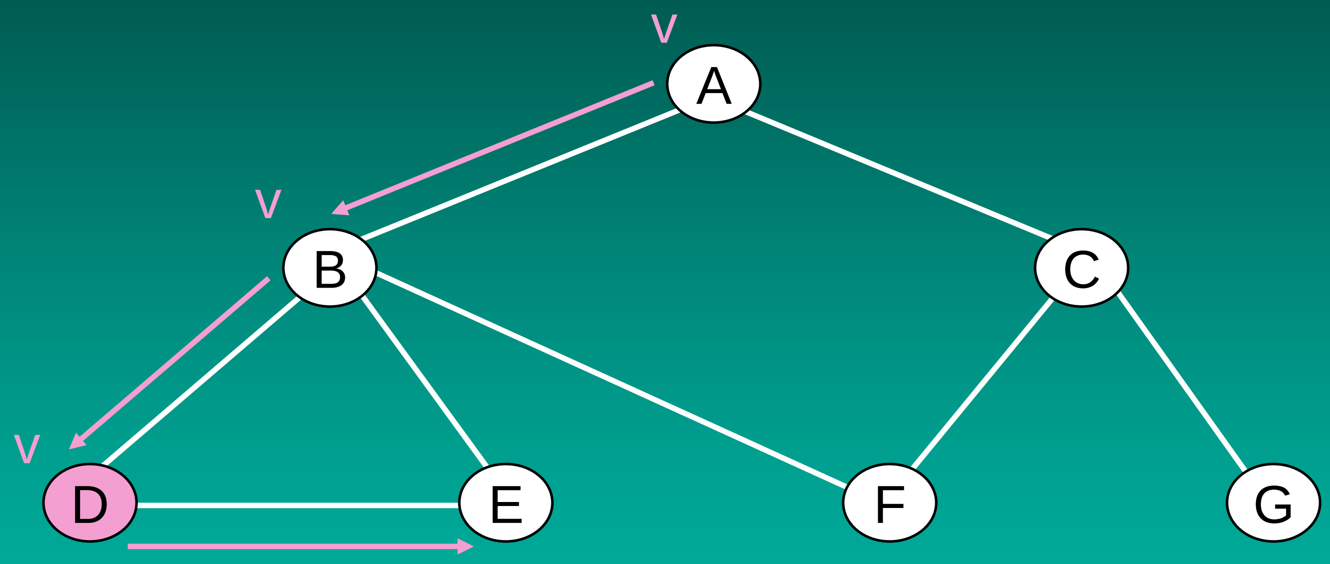
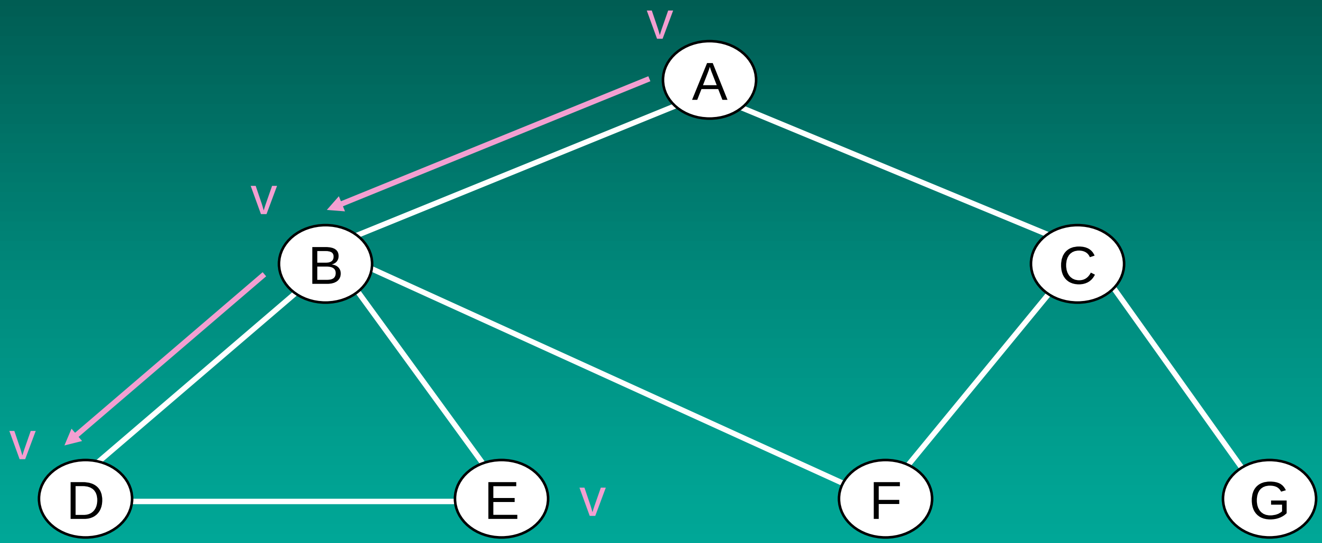
Depth-First Search



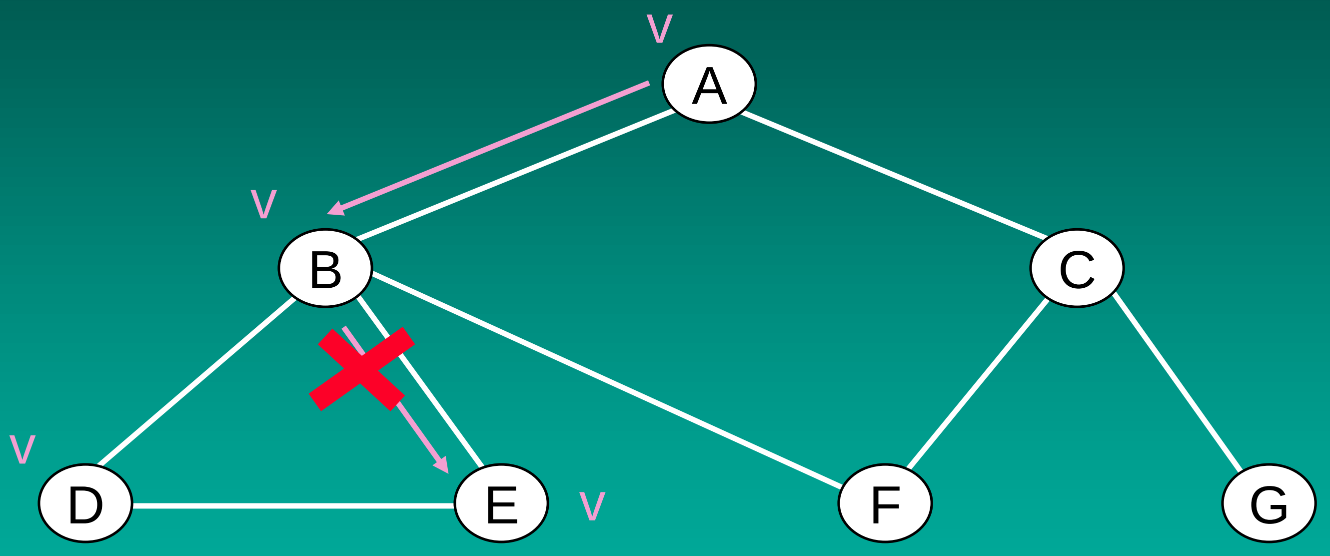
A B D E

Depth-First Search



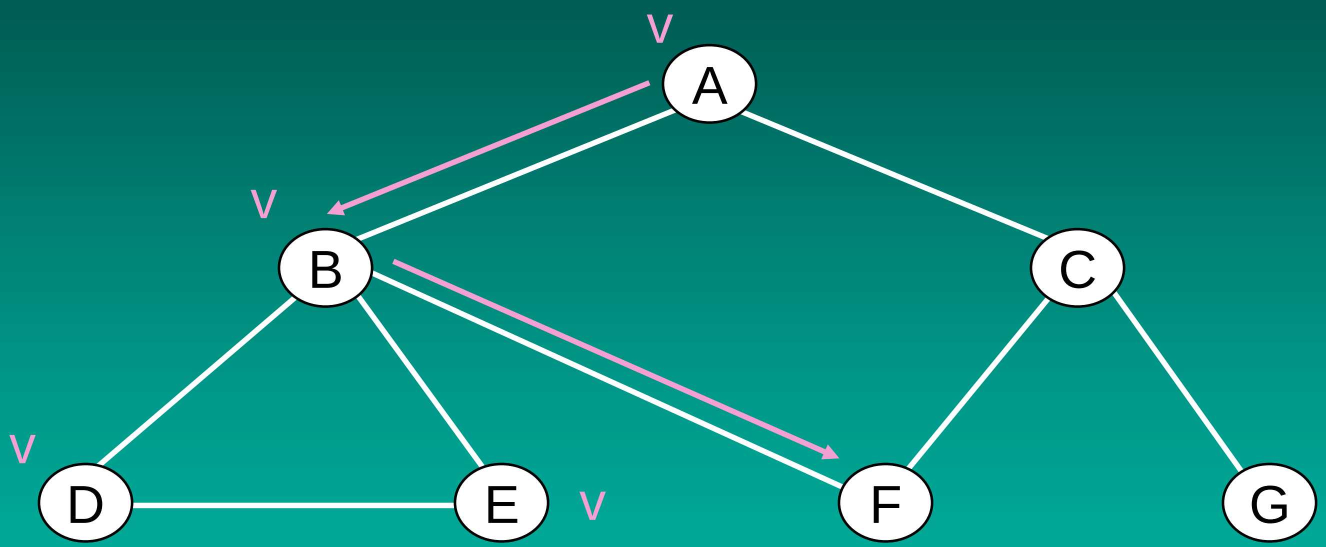
A B D E

Depth-First Search



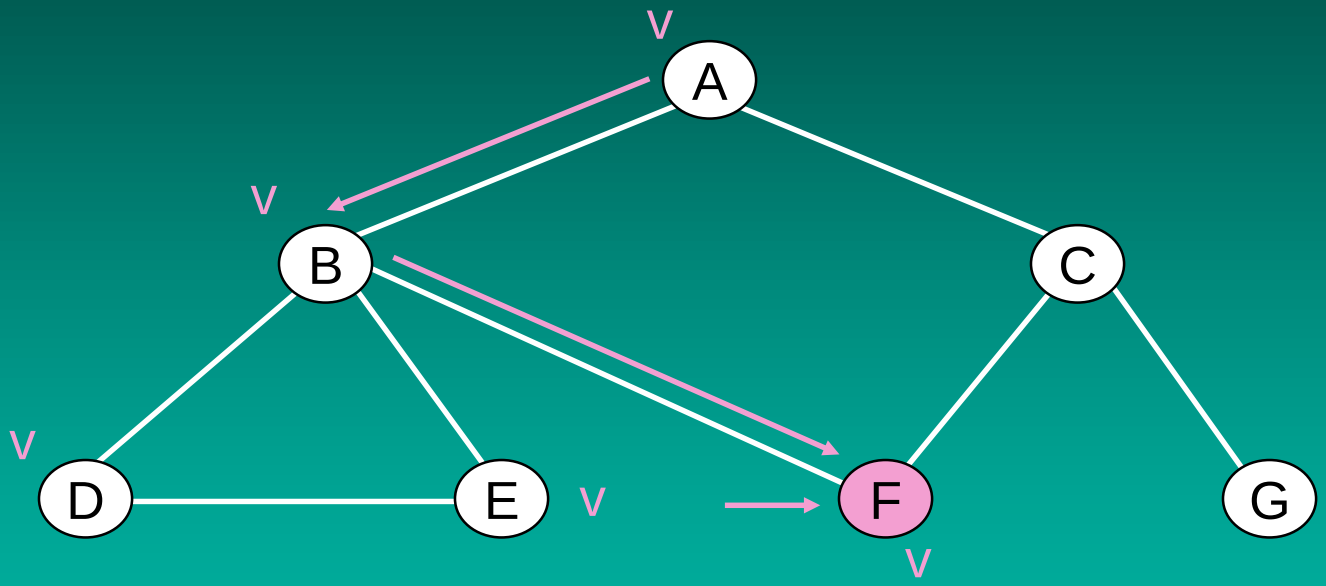
A B D E

Depth-First Search



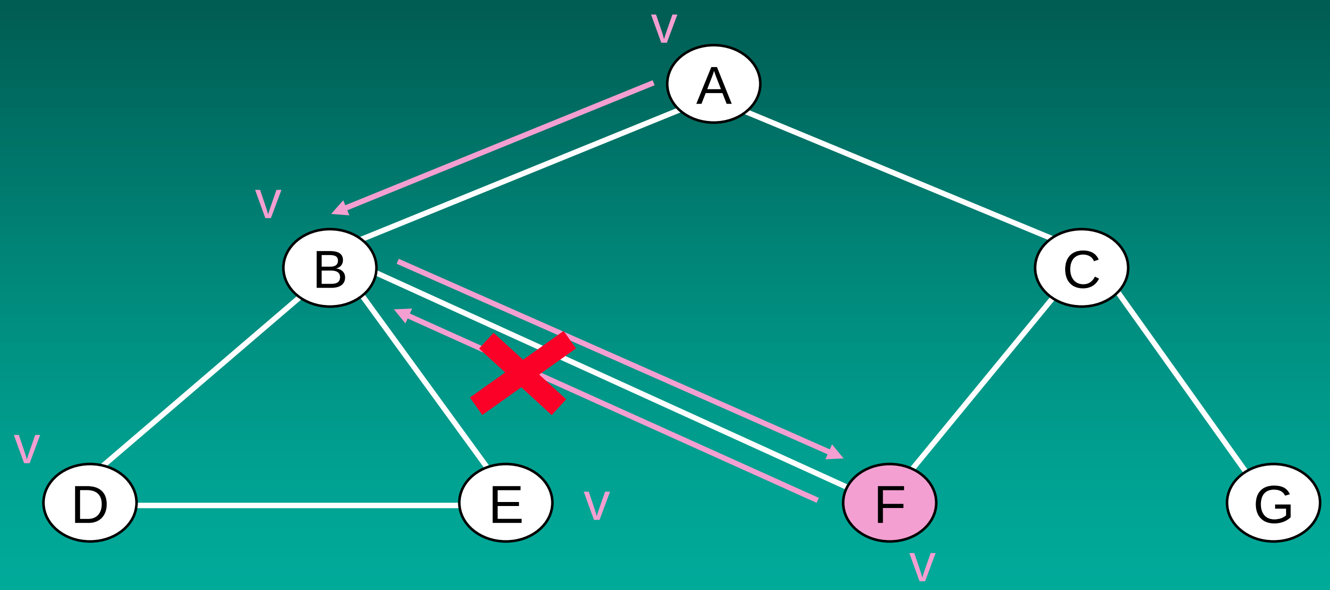
A B D E

Depth-First Search



A B D E F

Depth-First Search



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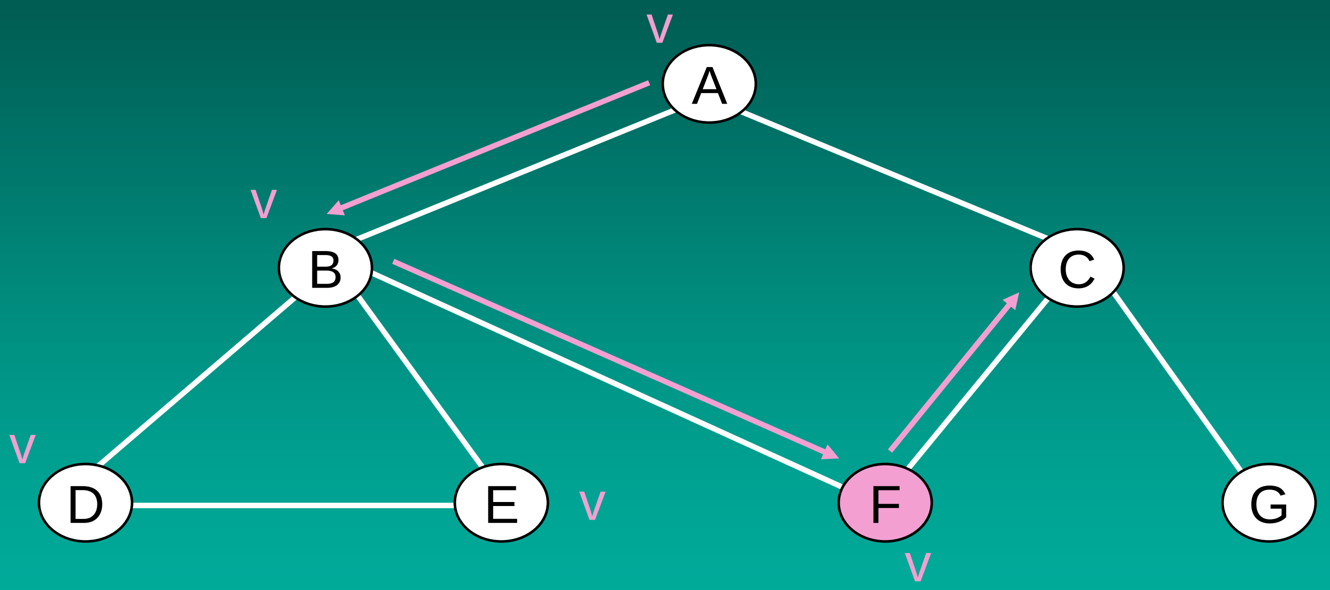
Depth-First Search



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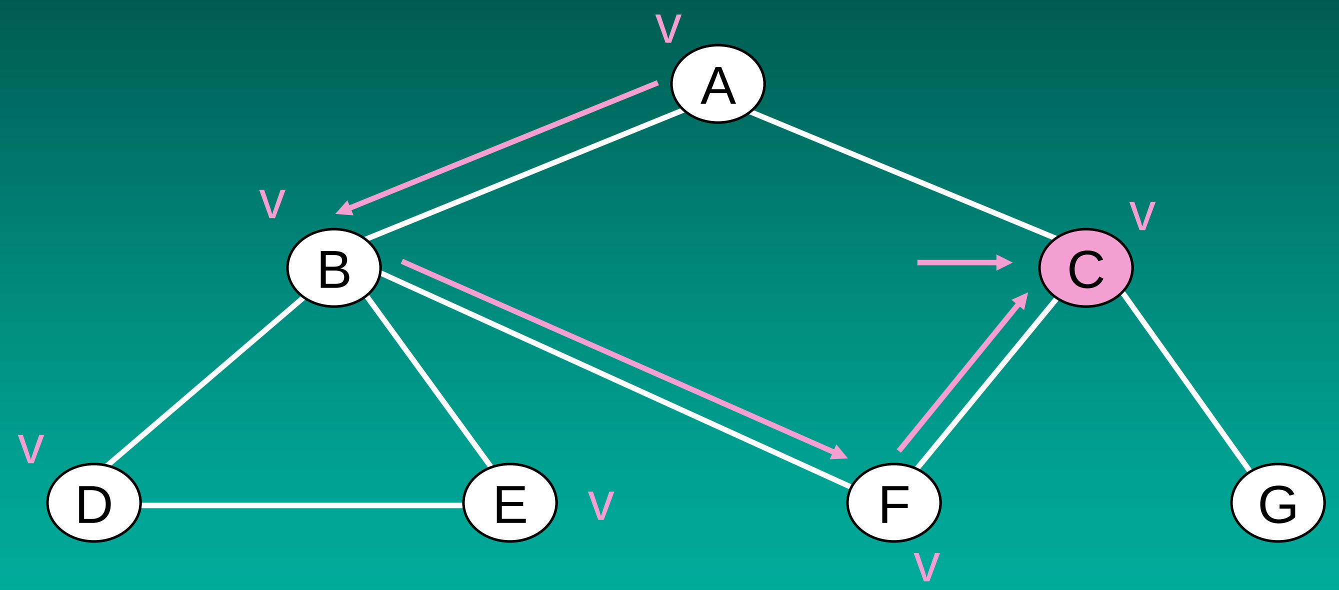
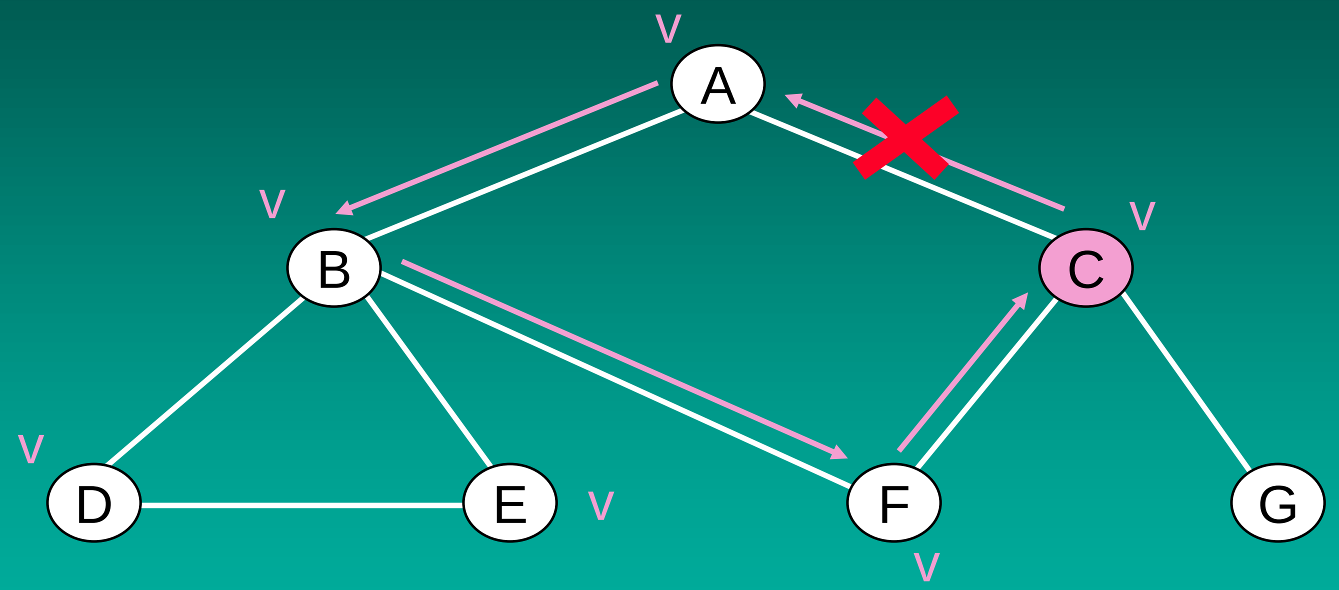
Depth-First Search



A B D E F C

Depth-First Search



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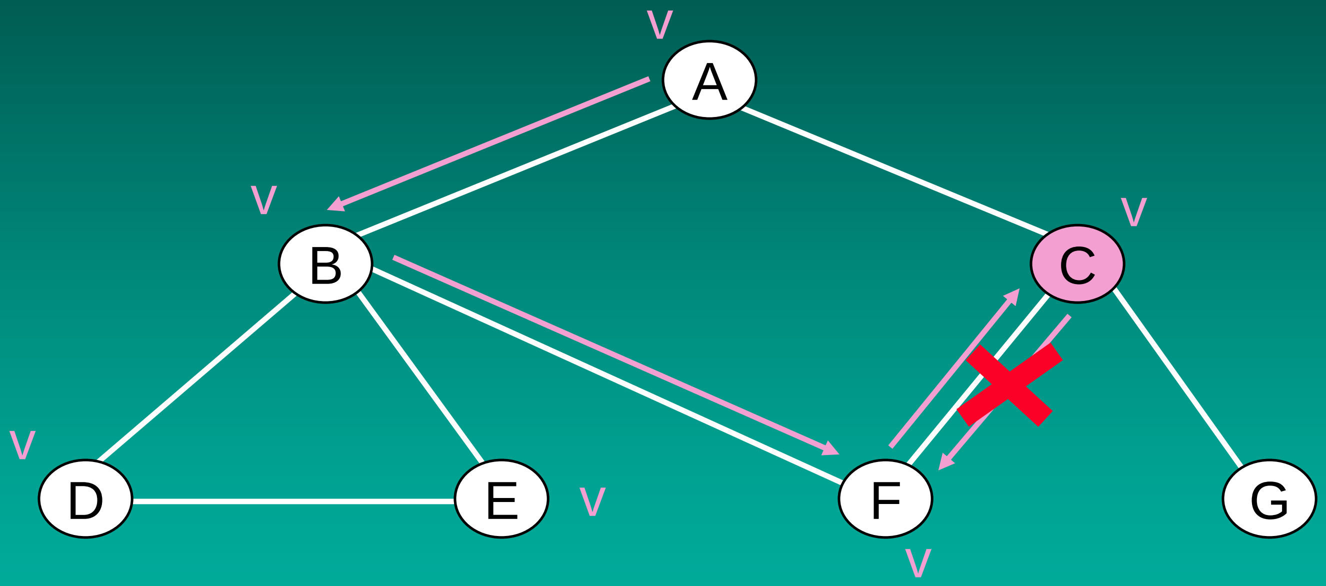

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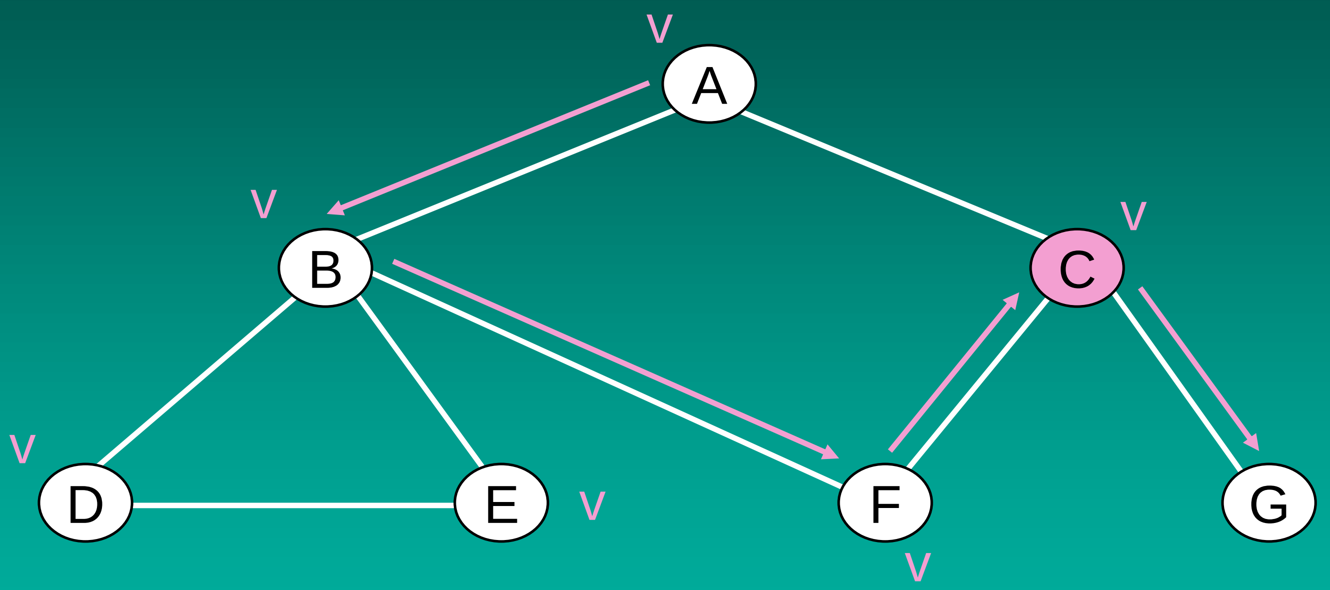

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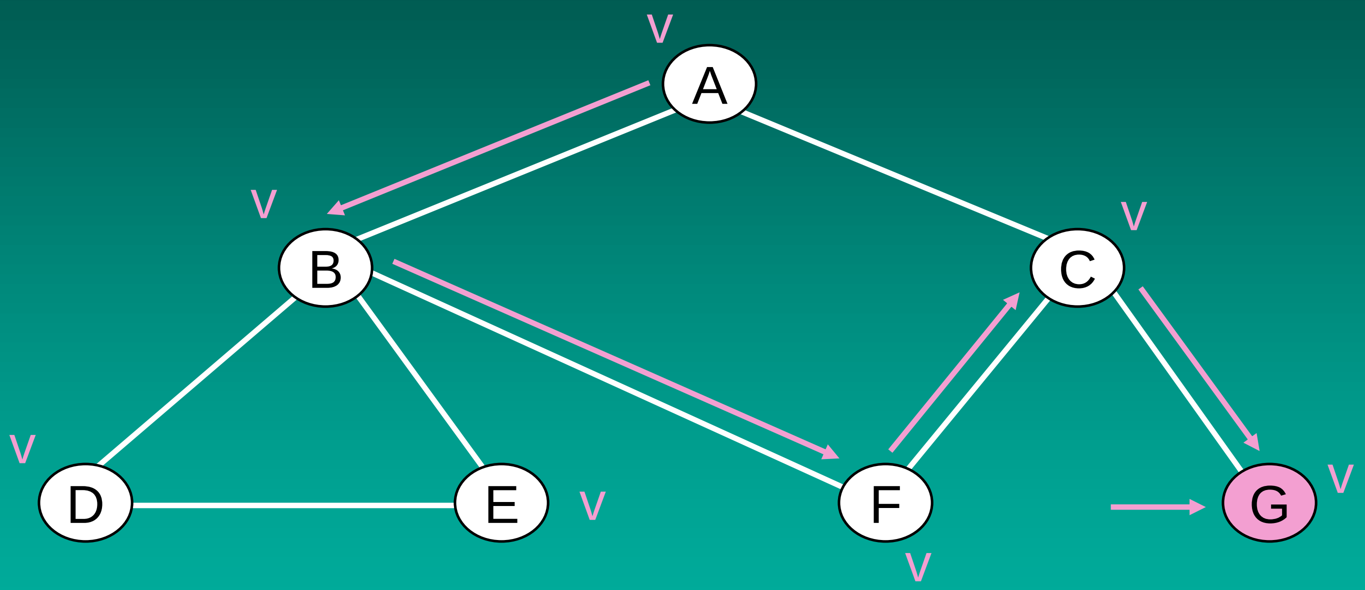
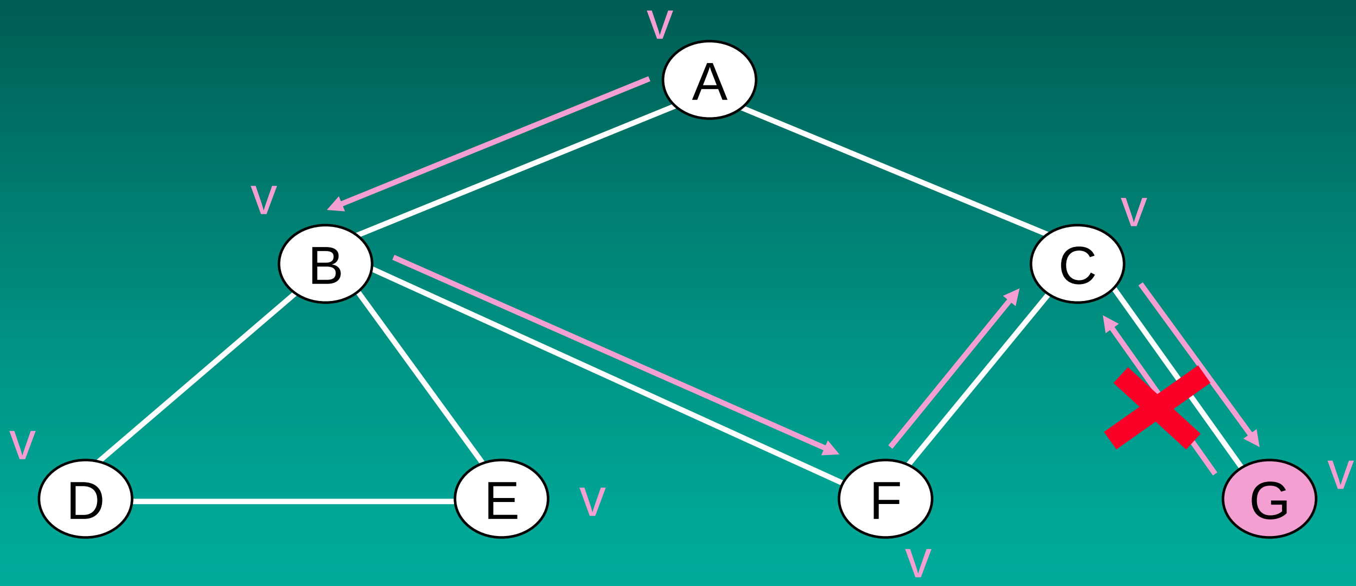


A B D E F C G

Depth-First Search



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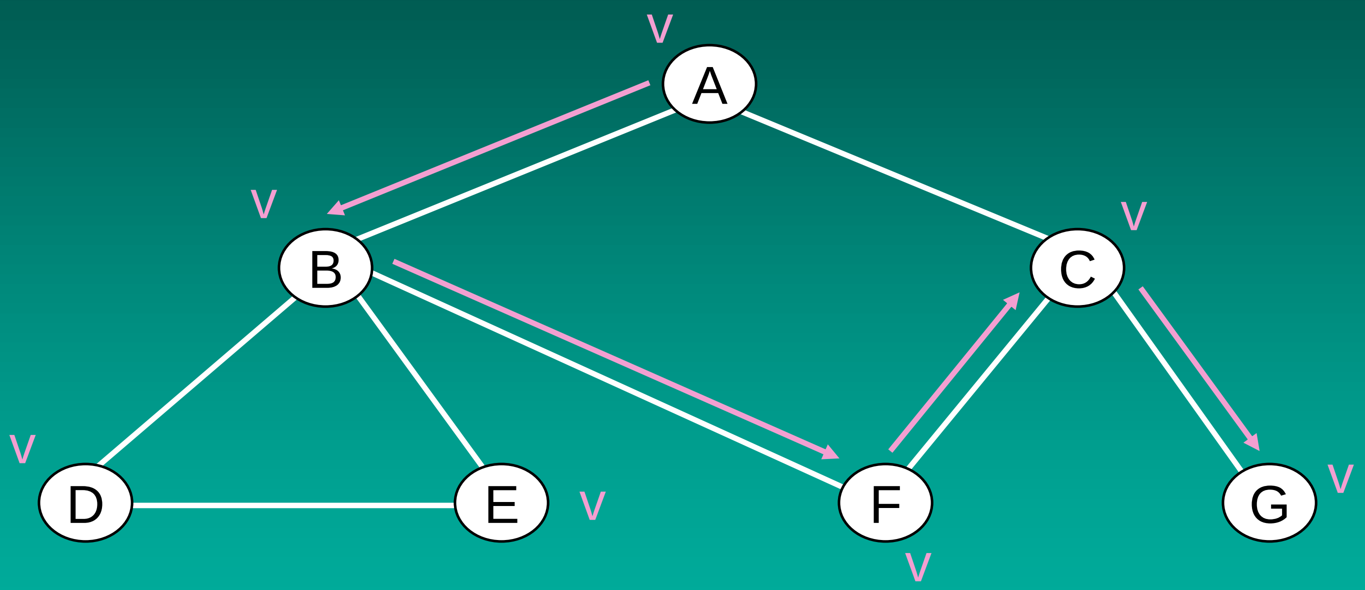

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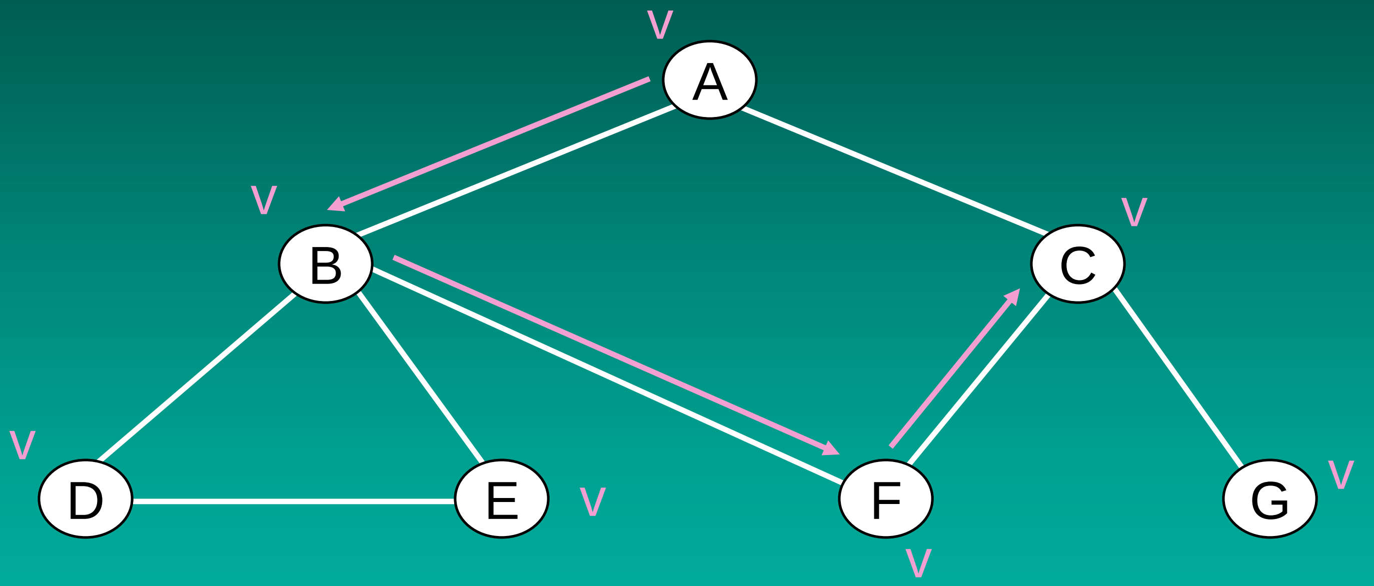

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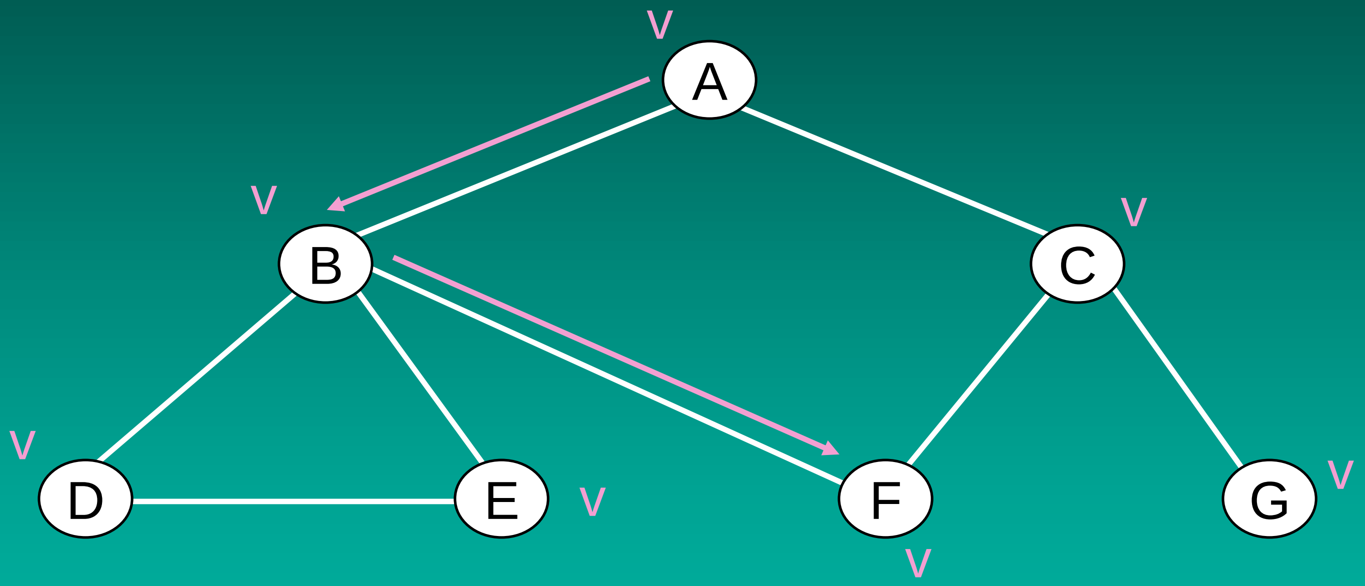
Depth-First Search



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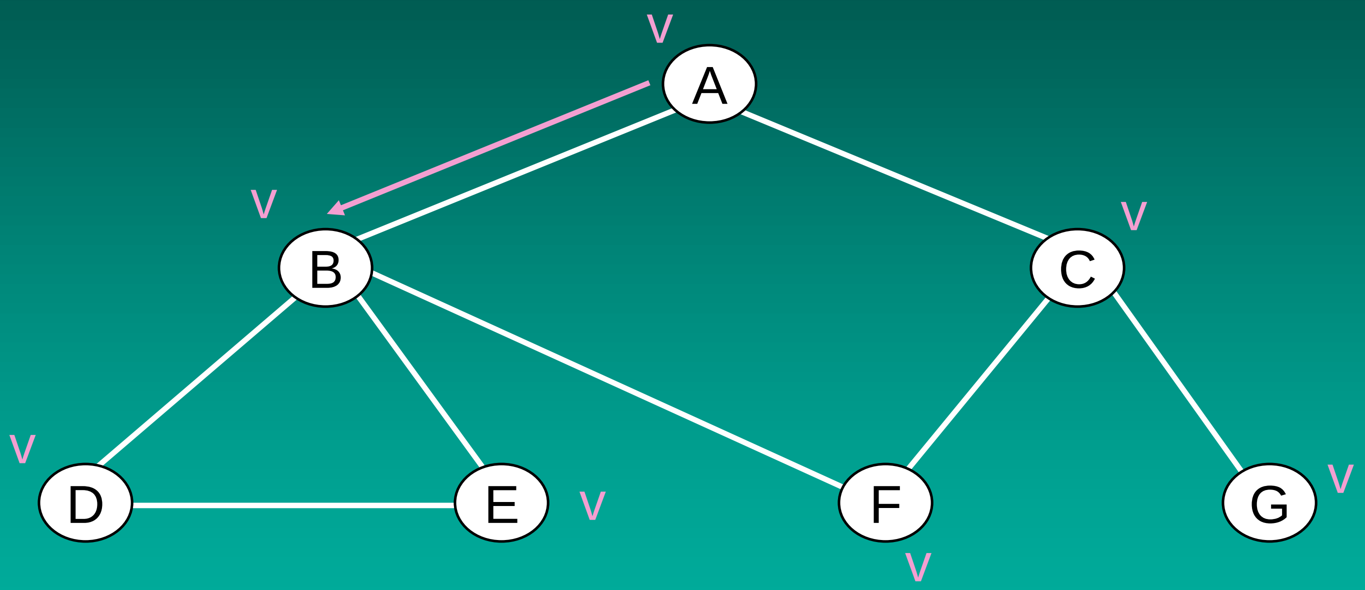
Depth-First Search



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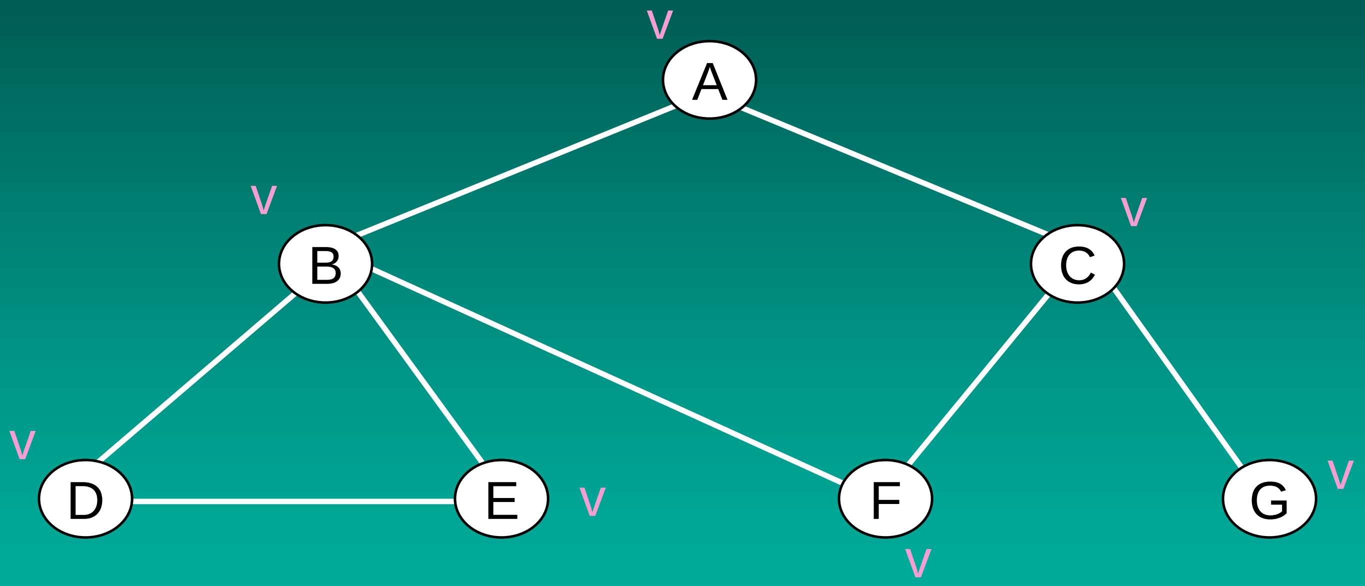
Depth-First Search



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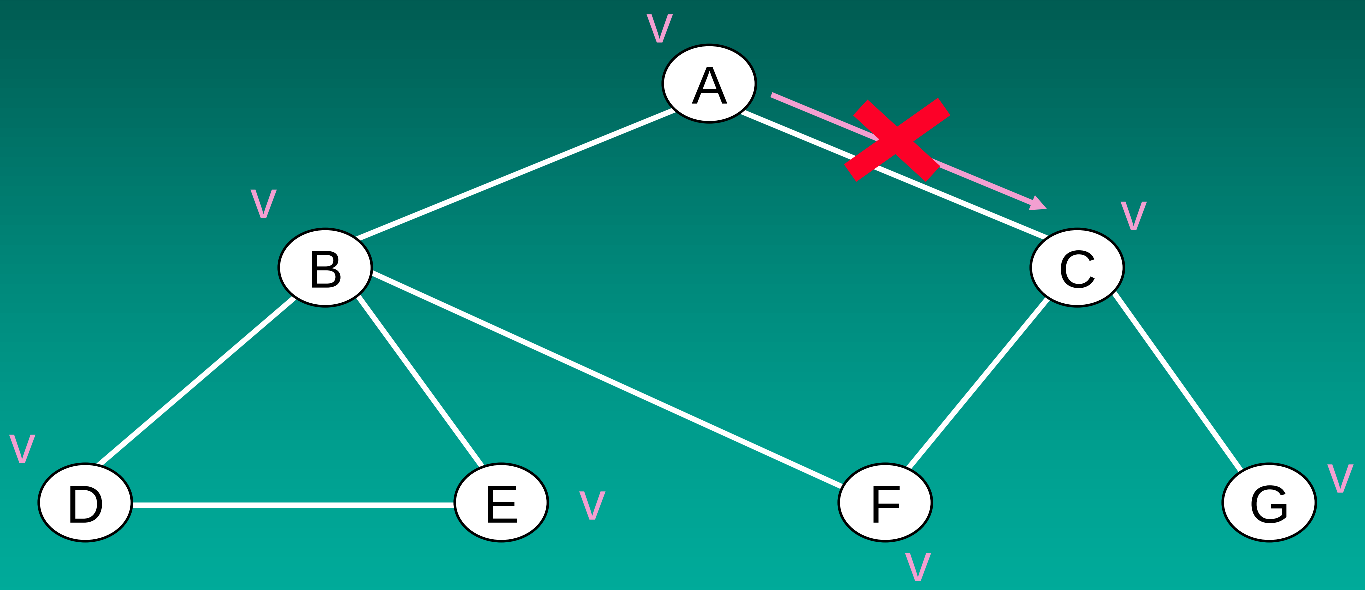

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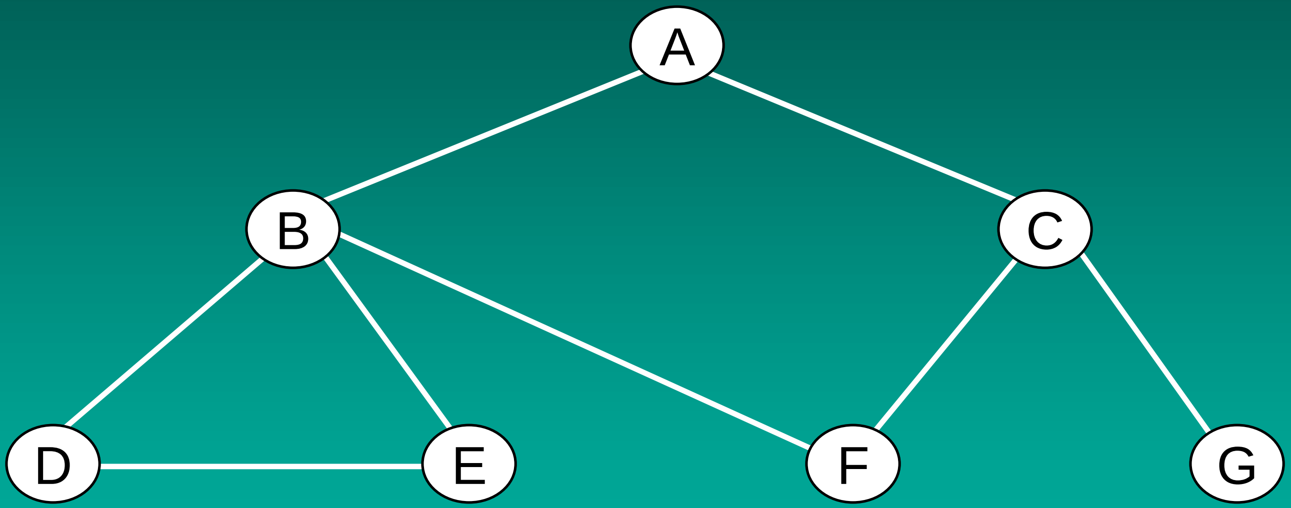
Depth-First Search



A B D E F C G

Time and Space Complexity for Depth-First Search

- Time Complexity
  - Adjacency Lists
    - Each node is marked visited once
    - Each node is checked for each incoming edge
    - O (v + e)
  - Adjacency Matrix
    - Have to check all entries in matrix: $O(n^2)$

Time and Space Complexity for Depth-First Search

- Space Complexity
  - – Stack to handle nodes as they are explored
    - Worst case: all nodes put on stack (if graph is linear)
    - O(n)

Breadth-First Graph Traversal Algorithm

Alyce Brady

Breadth-first Search

- Similar to Breadth-first Traversal of a Binary Tree

- Choose a starting vertex
- Search all adjacent vertices
- Return to each adjacent vertex in turn and visit all of its adjacent vertices

Pseudo-Code for Breadth-First Search

<u>breadth-first-search</u>

mark starting vertex as visited; put on queue

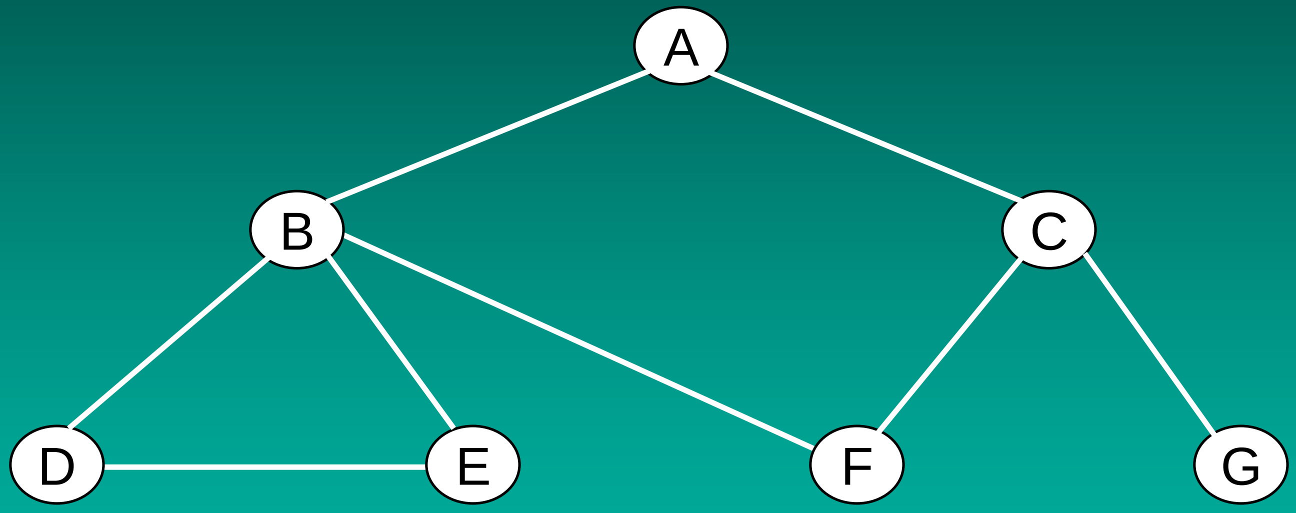
while the queue is not empty

dequeue the next node

for all unvisited vertices adjacent to this one

- mark vertex as visited
- add vertex to queue

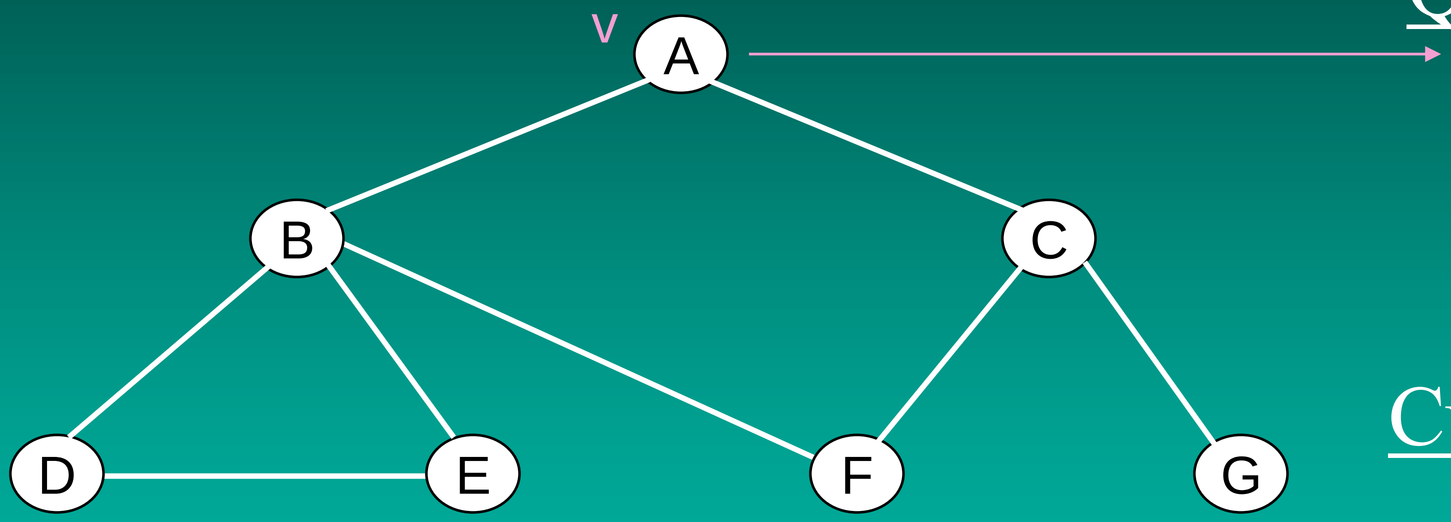
Breadth-First Search



Queue:

Current:

Breadth-First Search

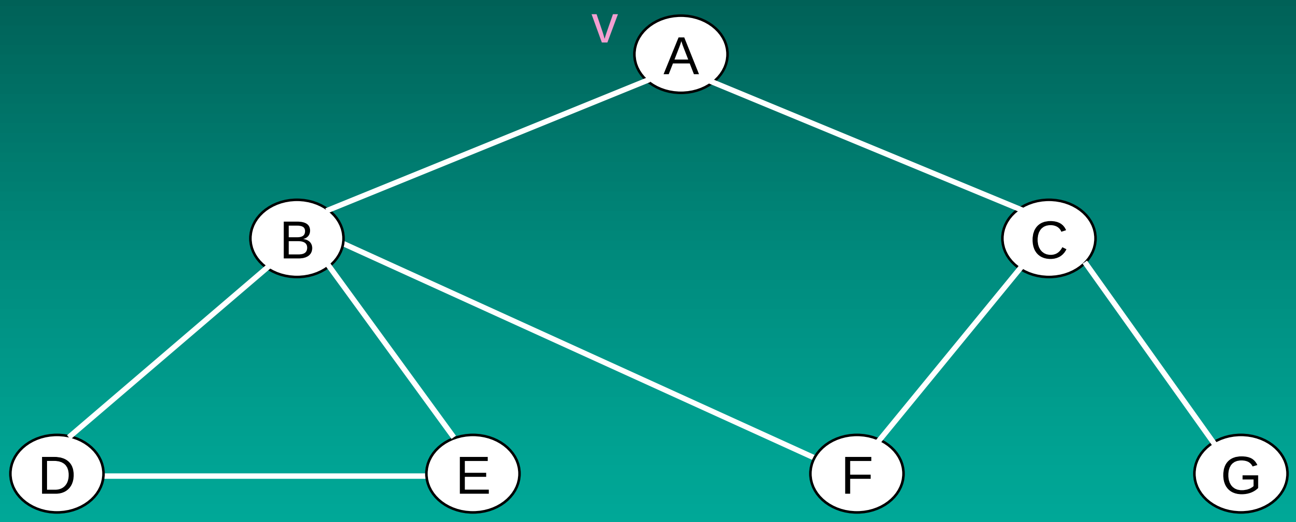


Queue:

A

Current:

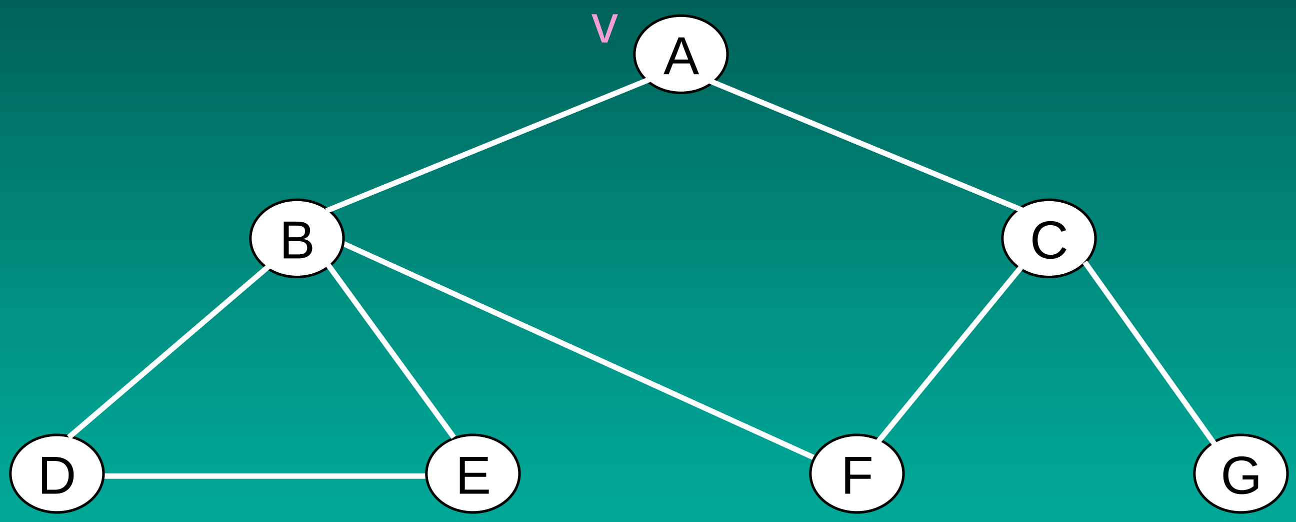
Breadth-First Search



Queue:
A

Current:
A

Breadth-First Search

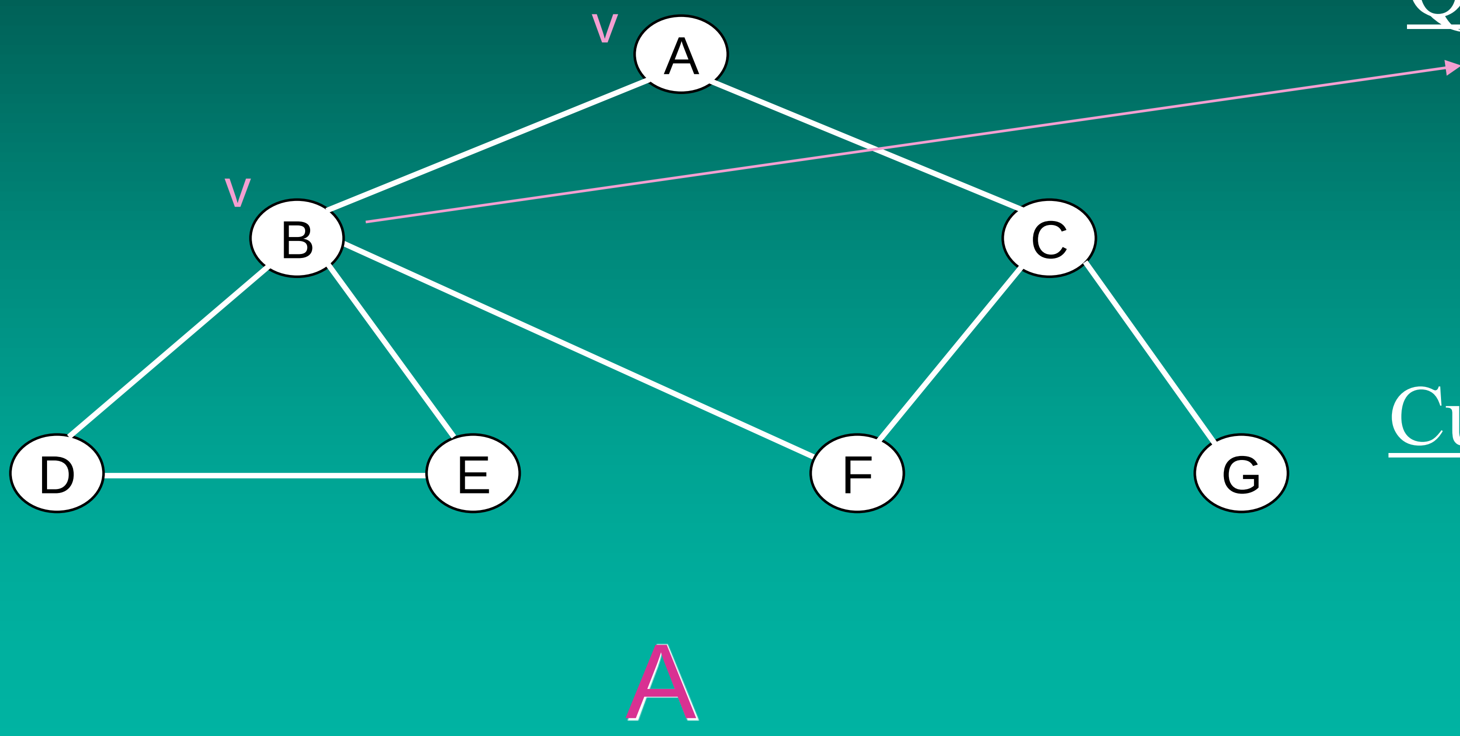


Queue:

Current:
A

A

Breadth-First Search



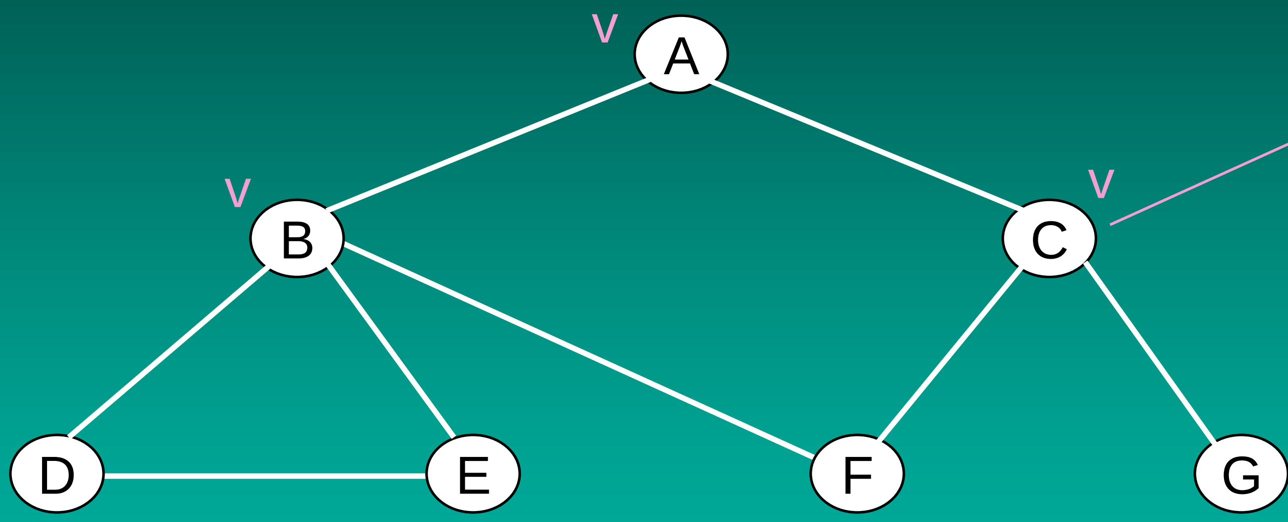
Queue:

B

Current:

A

Breadth-First Search

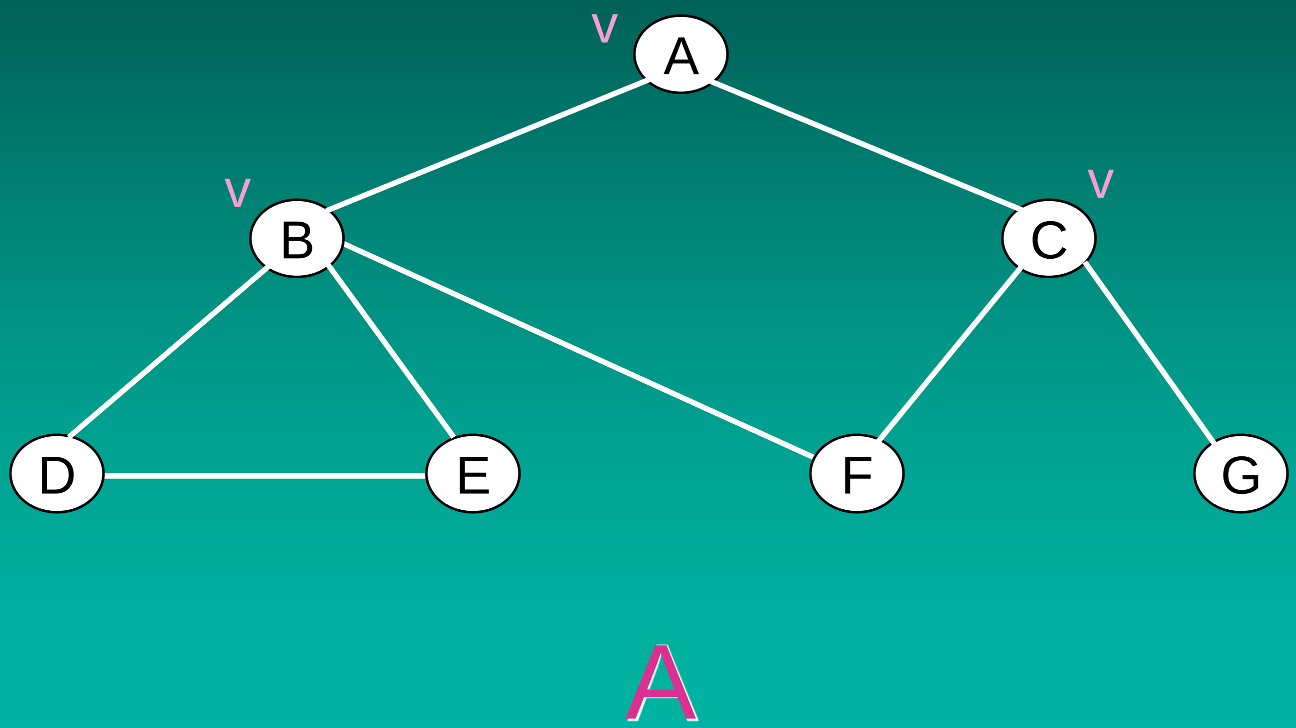


Queue:
C
B

Current:
A

A

Breadth-First Search



Queue:

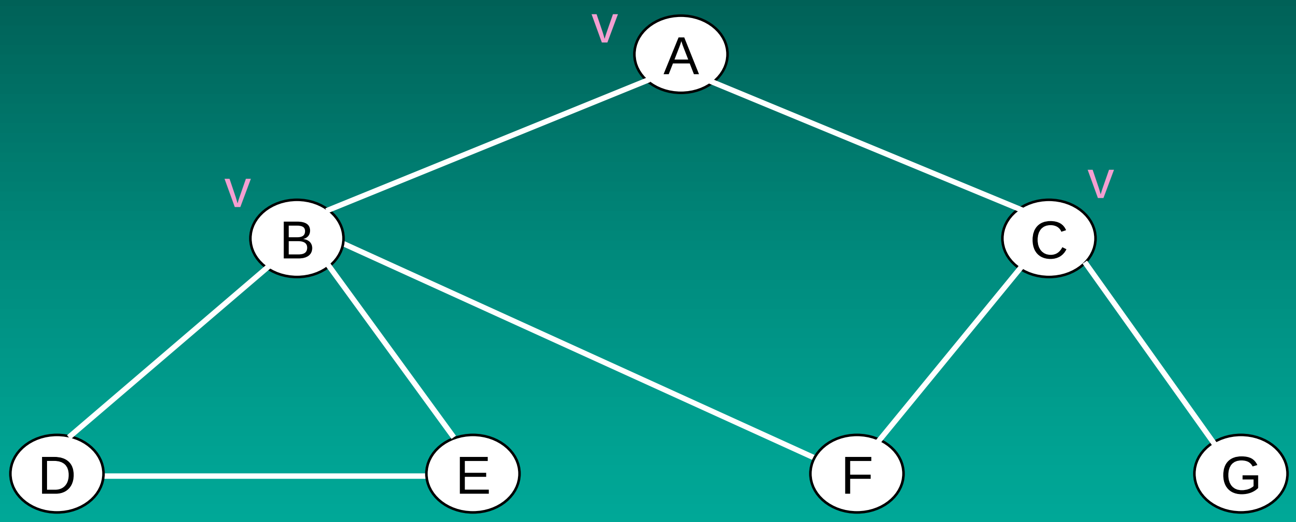
C

B

Current:

B

Breadth-First Search

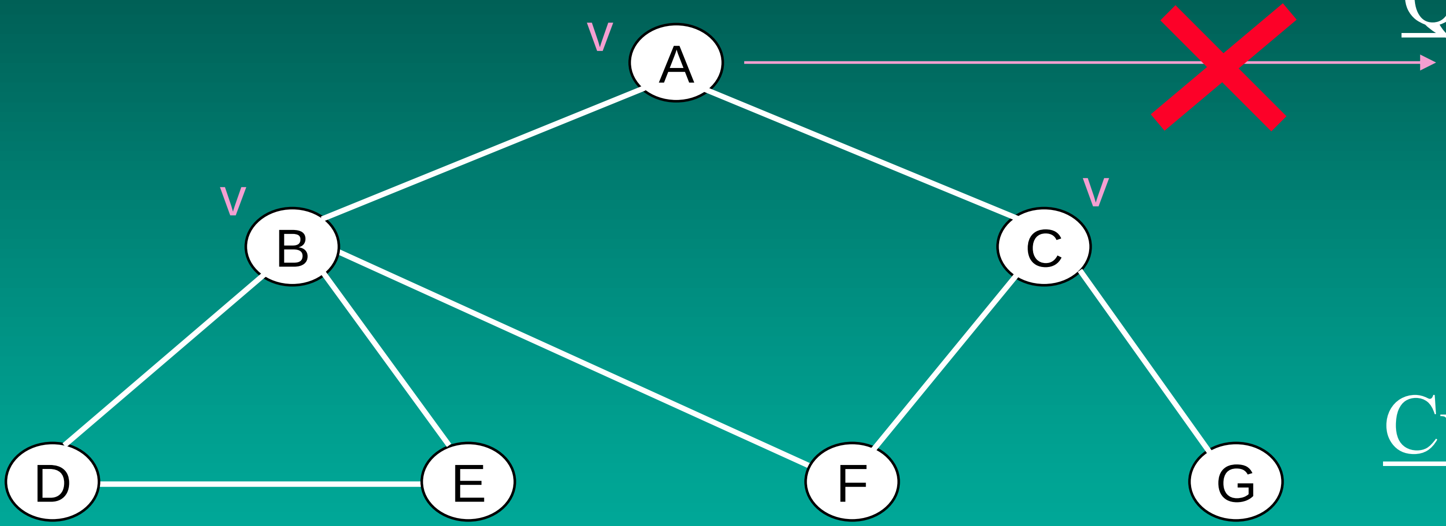


Queue:
C

Current:
B

A B

Breadth-First Search

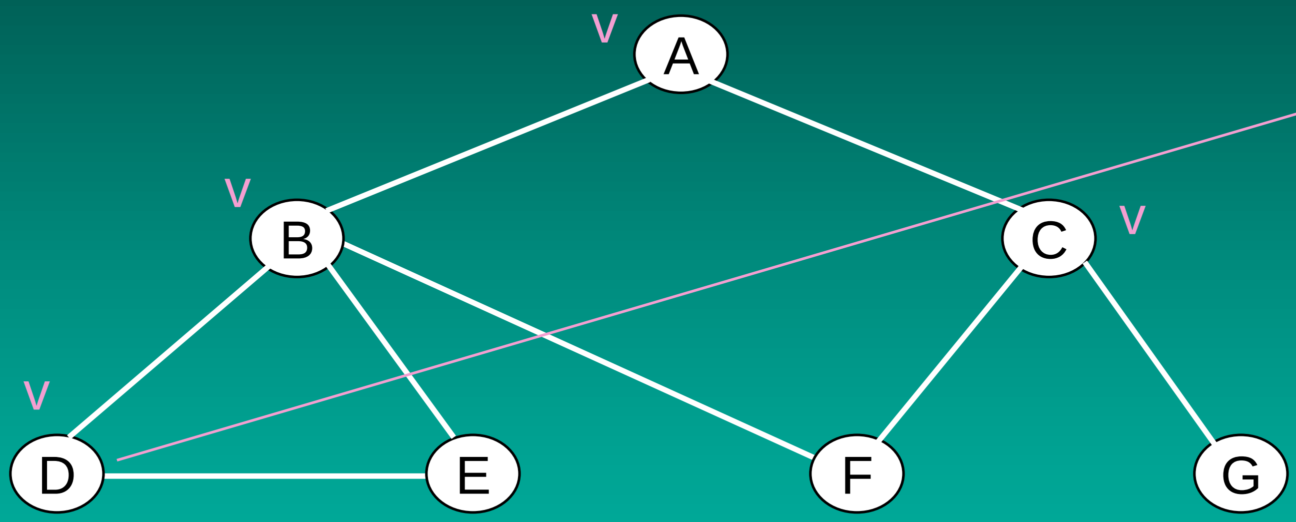


Queue:
C

Current:
B

A B

Breadth-First Search



Queue:

D

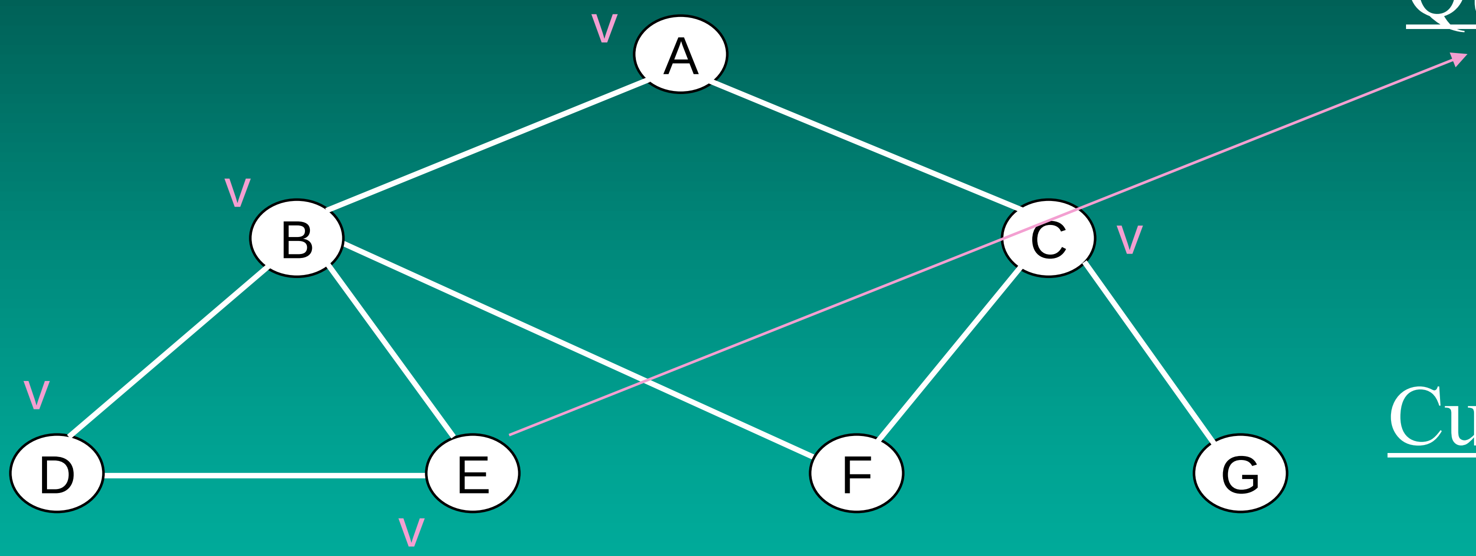
C

Current:

B

A B

Breadth-First Search

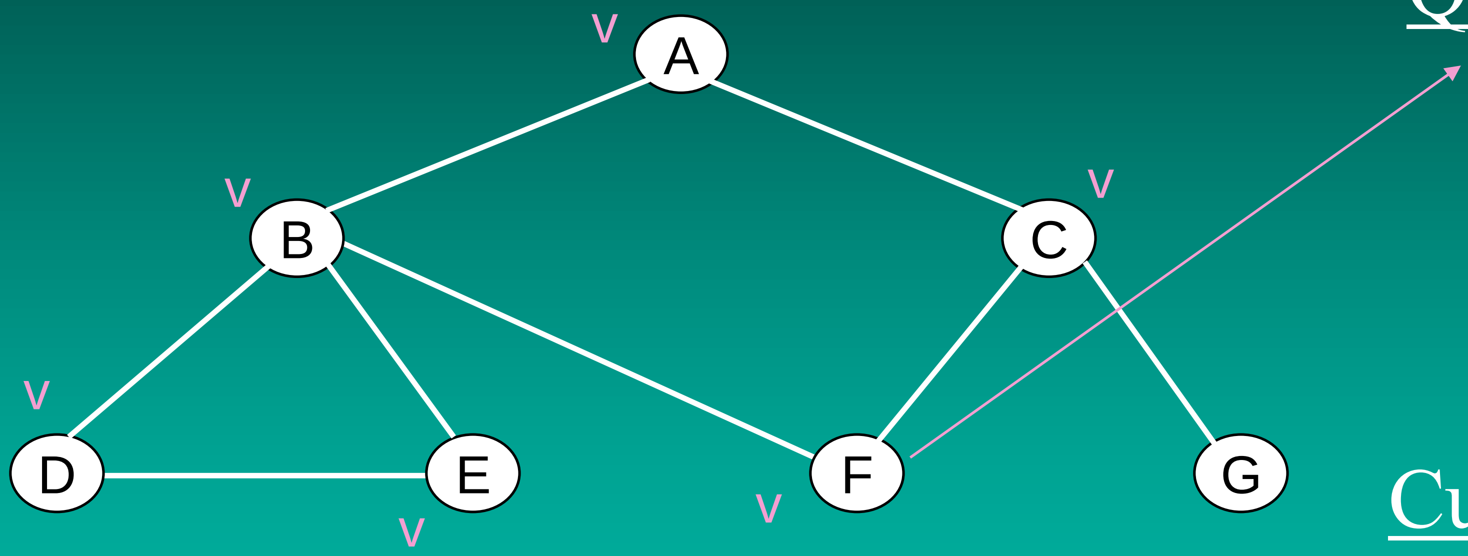


Queue:
E
D
C

Current:
B

A B

Breadth-First Search



Queue:

F

E

D

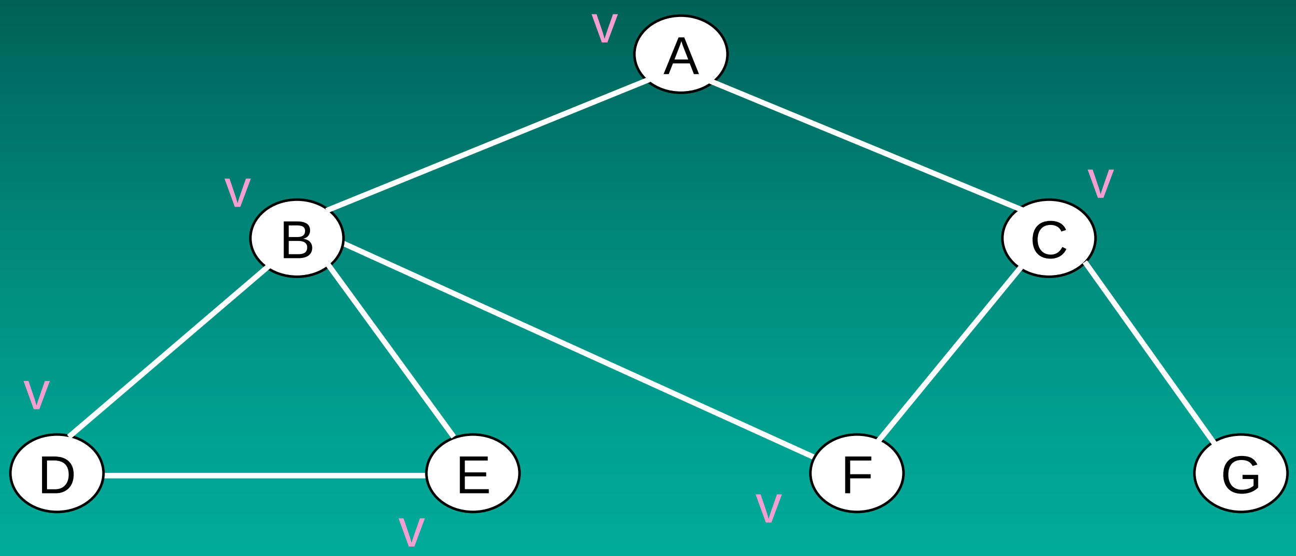
C

Current:

B

A B

Breadth-First Search

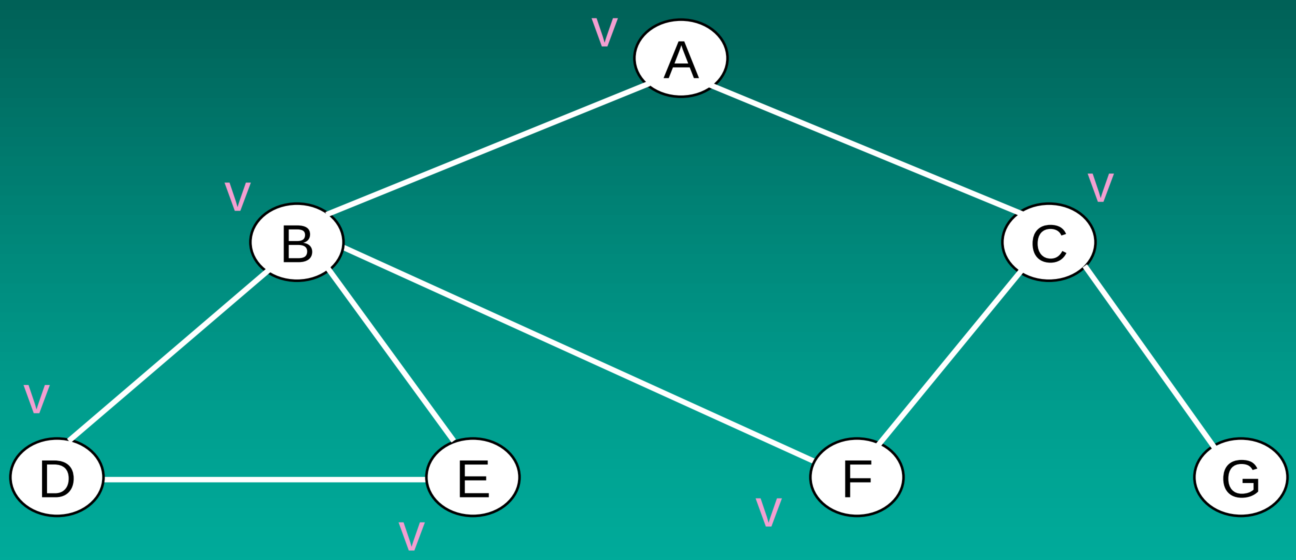


Queue:
F
E
D
C

Current:
C

A B

Breadth-First Search



<u>Queue:</u>

F

E

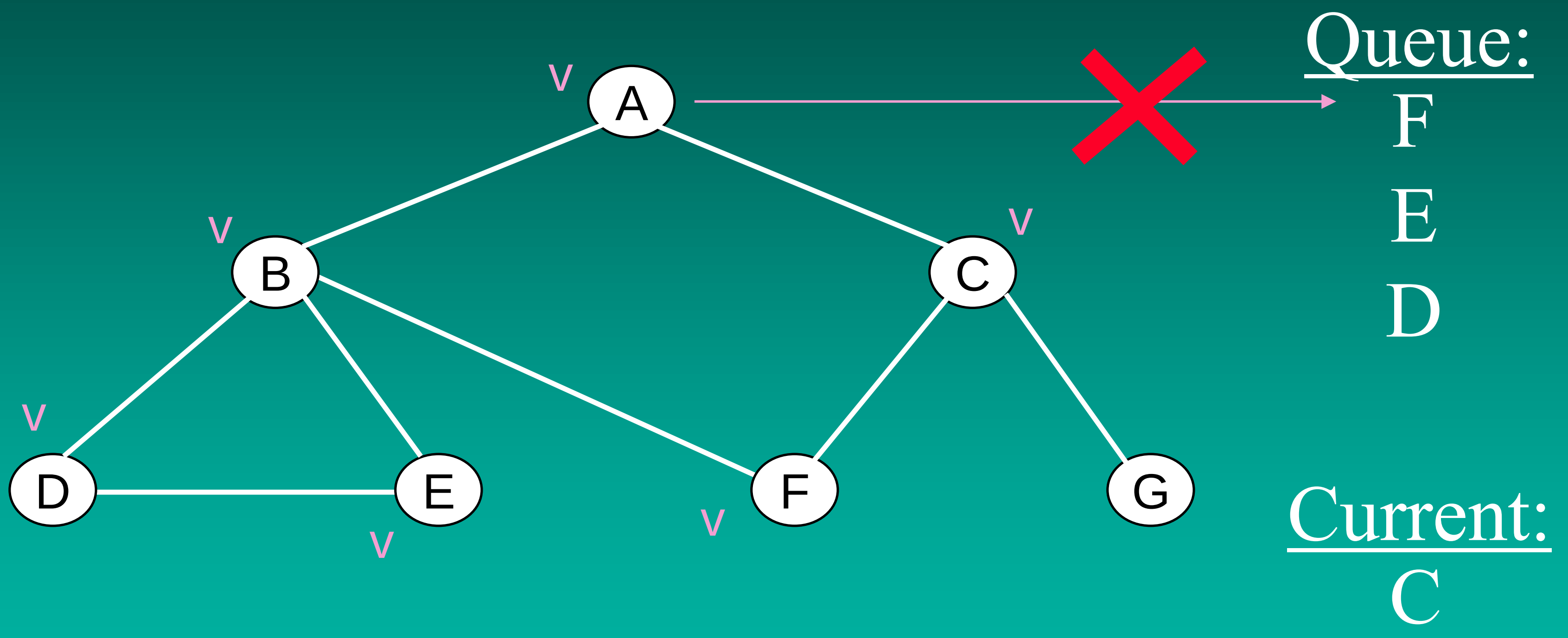
D

<u>Current:</u>

C

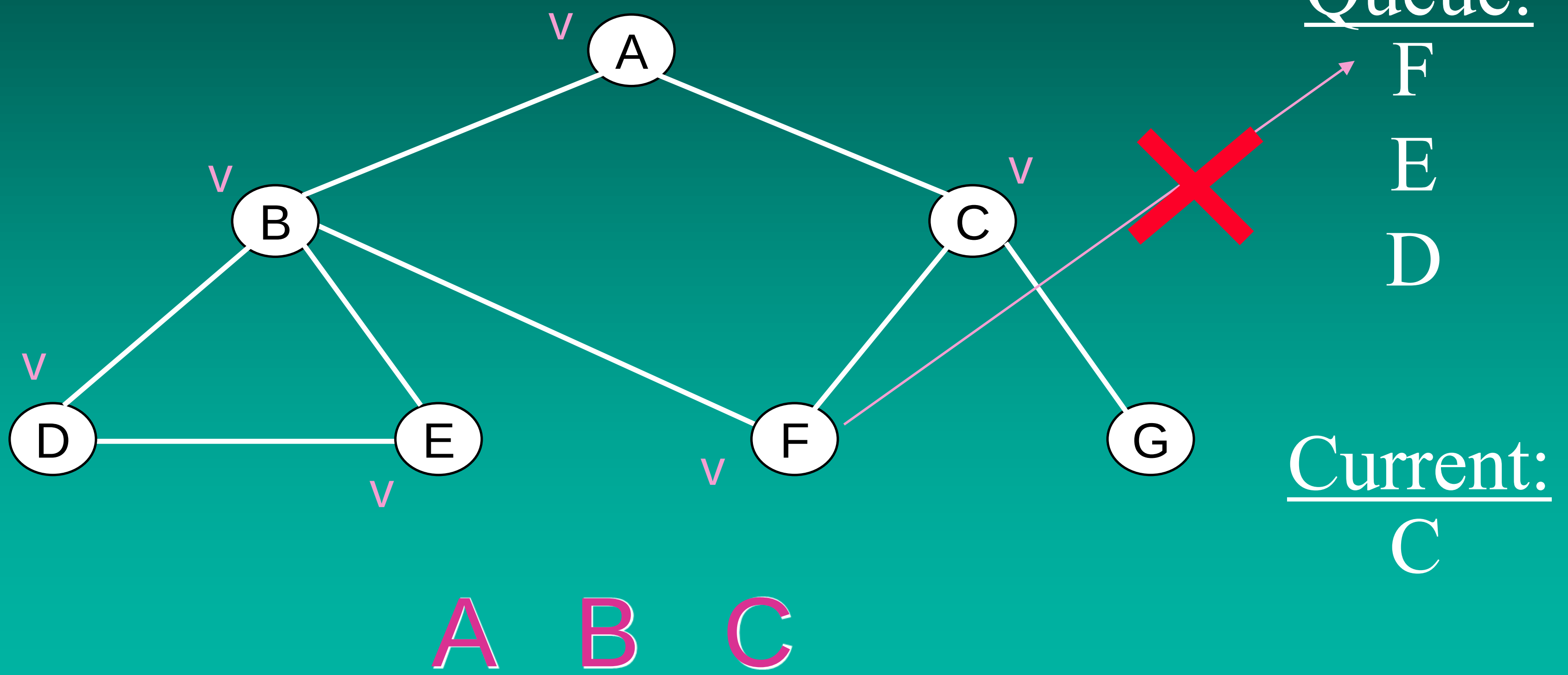
A B C

Breadth-First Search

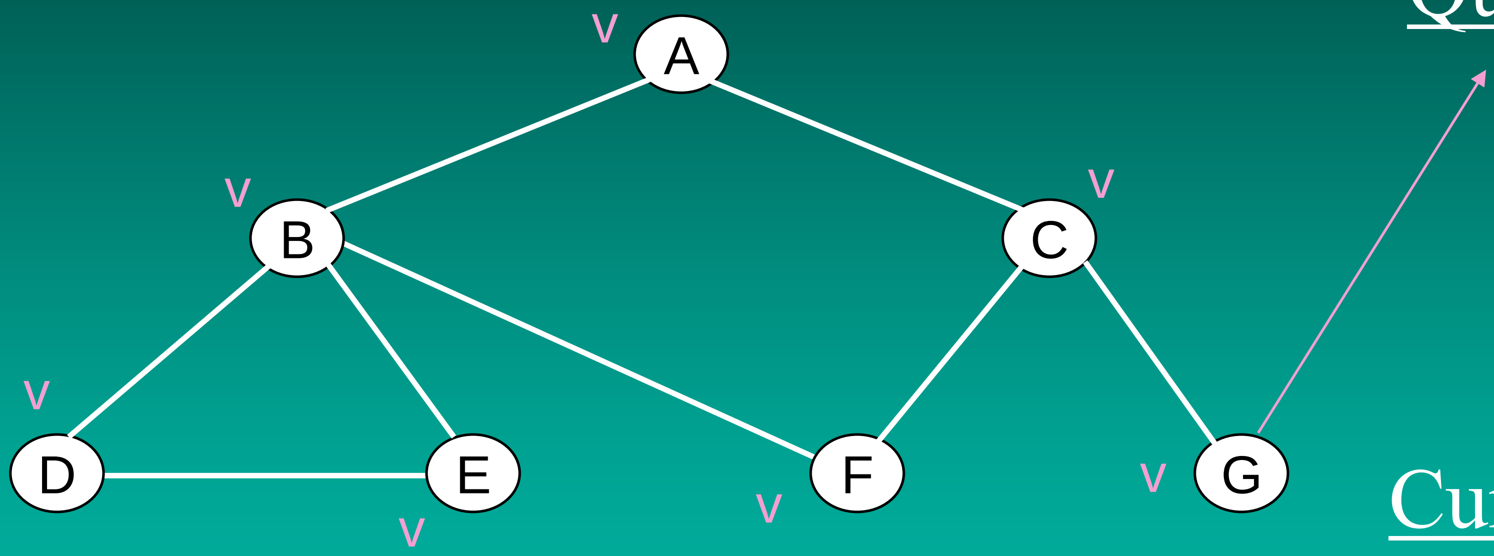


A B C

Breadth-First Search



Breadth-First Search

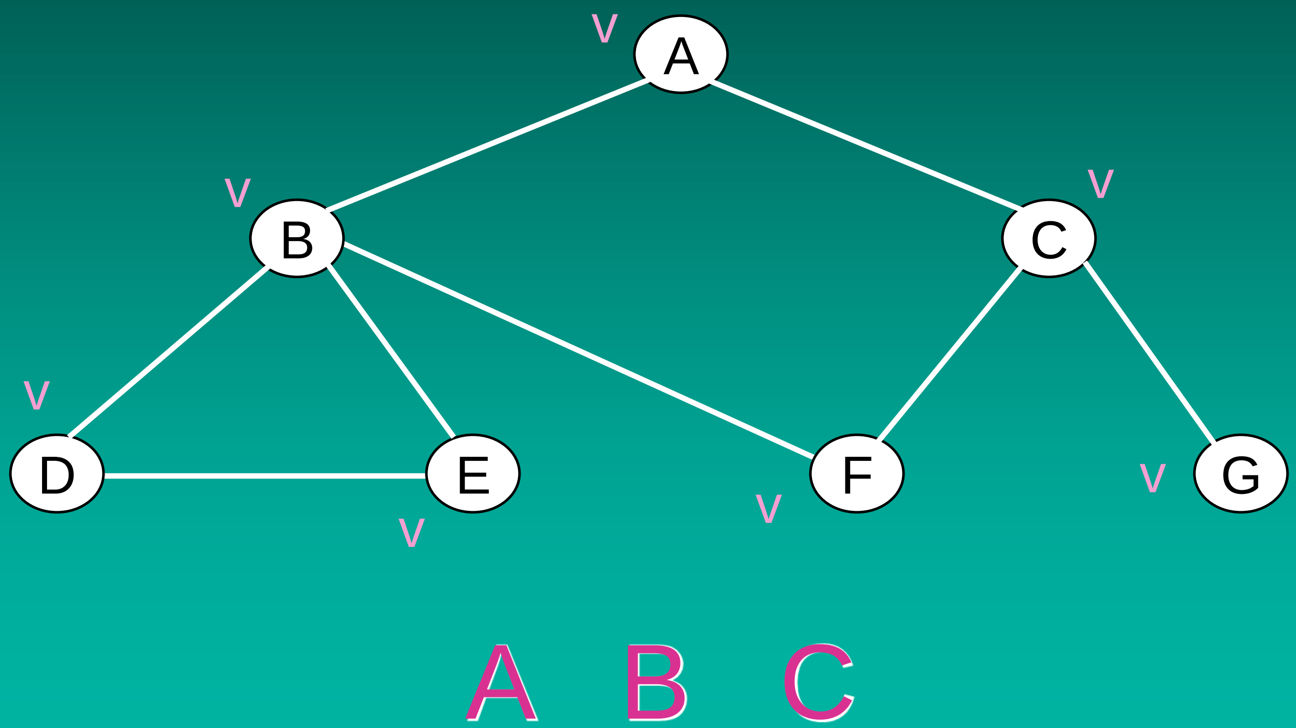


Queue:
G
F
E
D

Current:
C

A B C

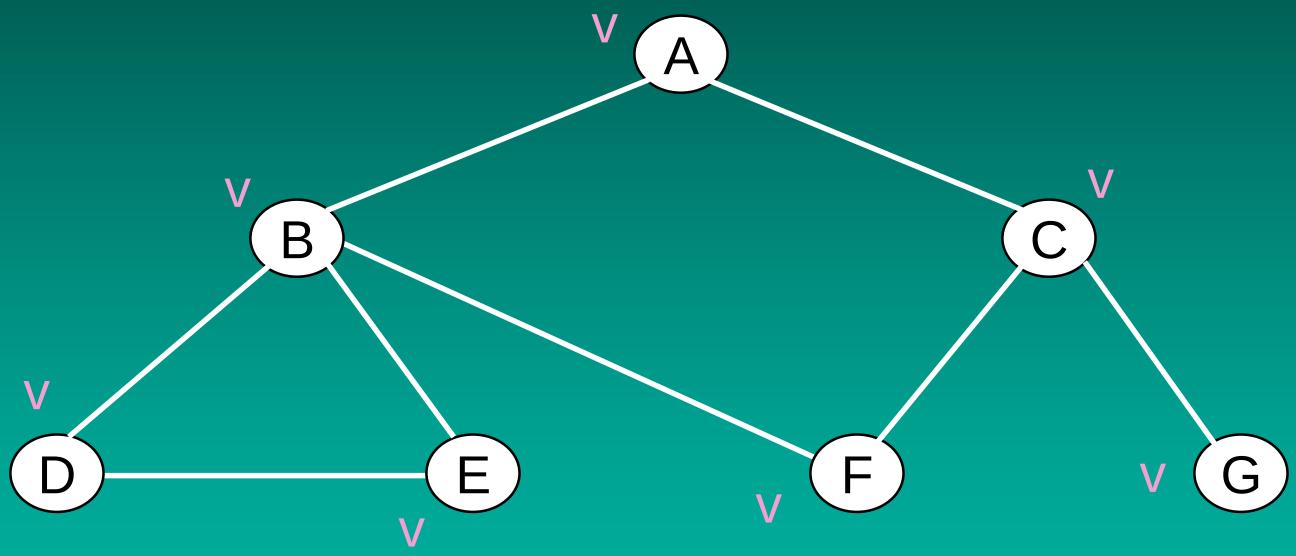
Breadth-First Search



Queue:
G
F
E
D

Current:
D

Breadth-First Search

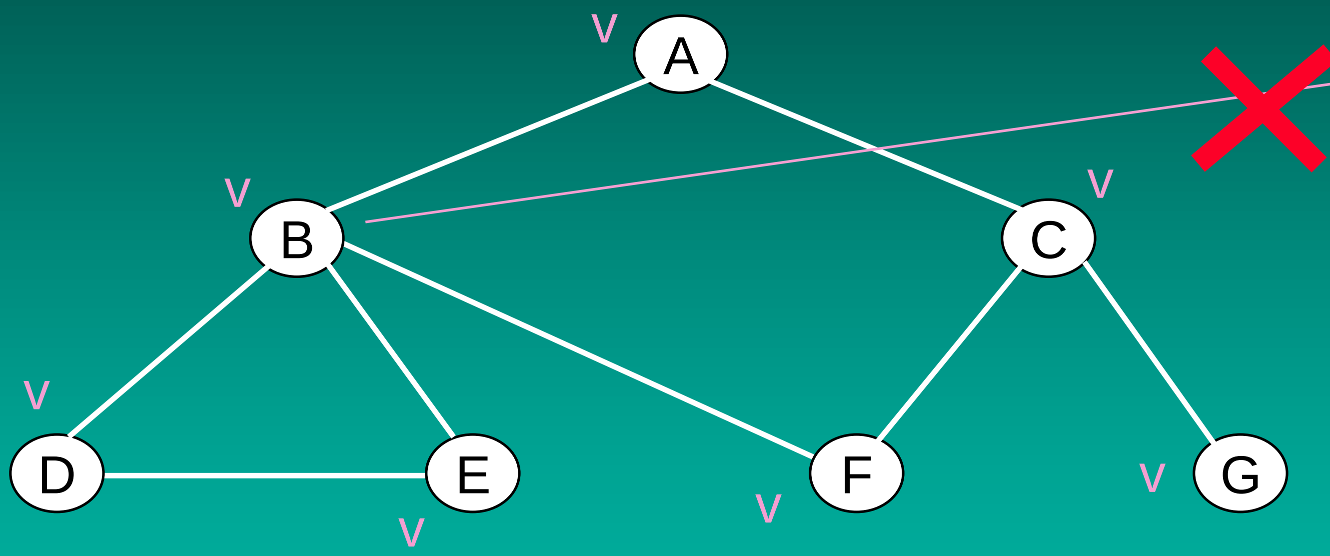


Queue:
G
F
E

Current:
D

A B C D

Breadth-First Search

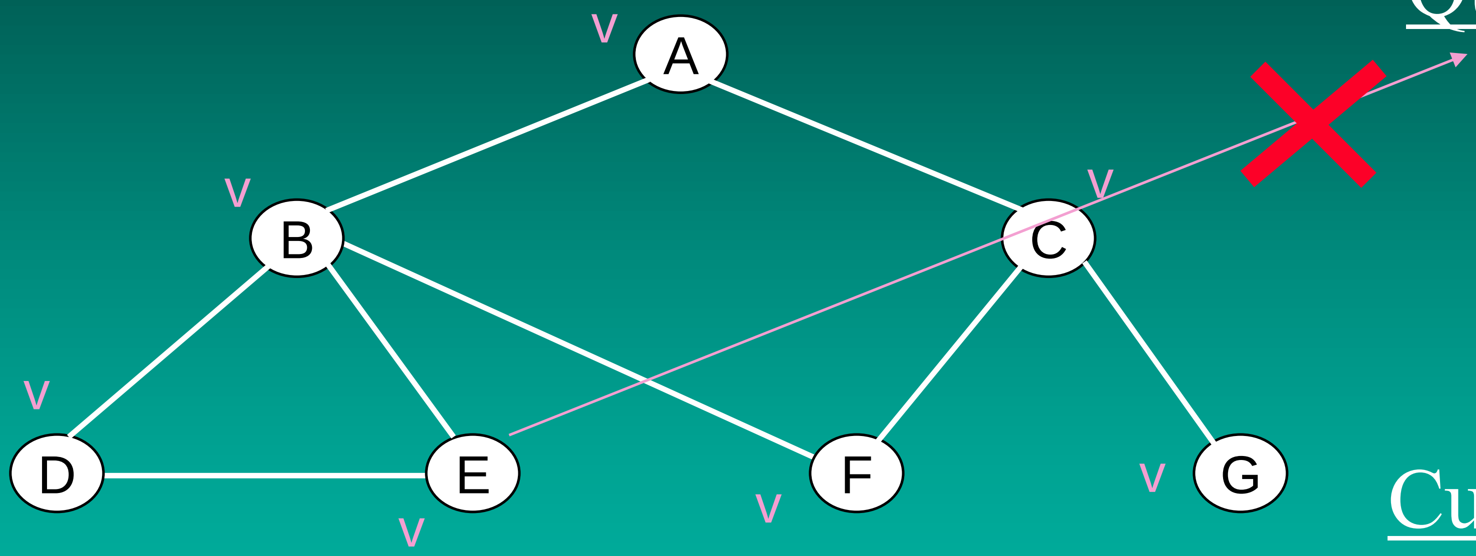


Queue:
G
F
E

Current:
D

A B C D

Breadth-First Search

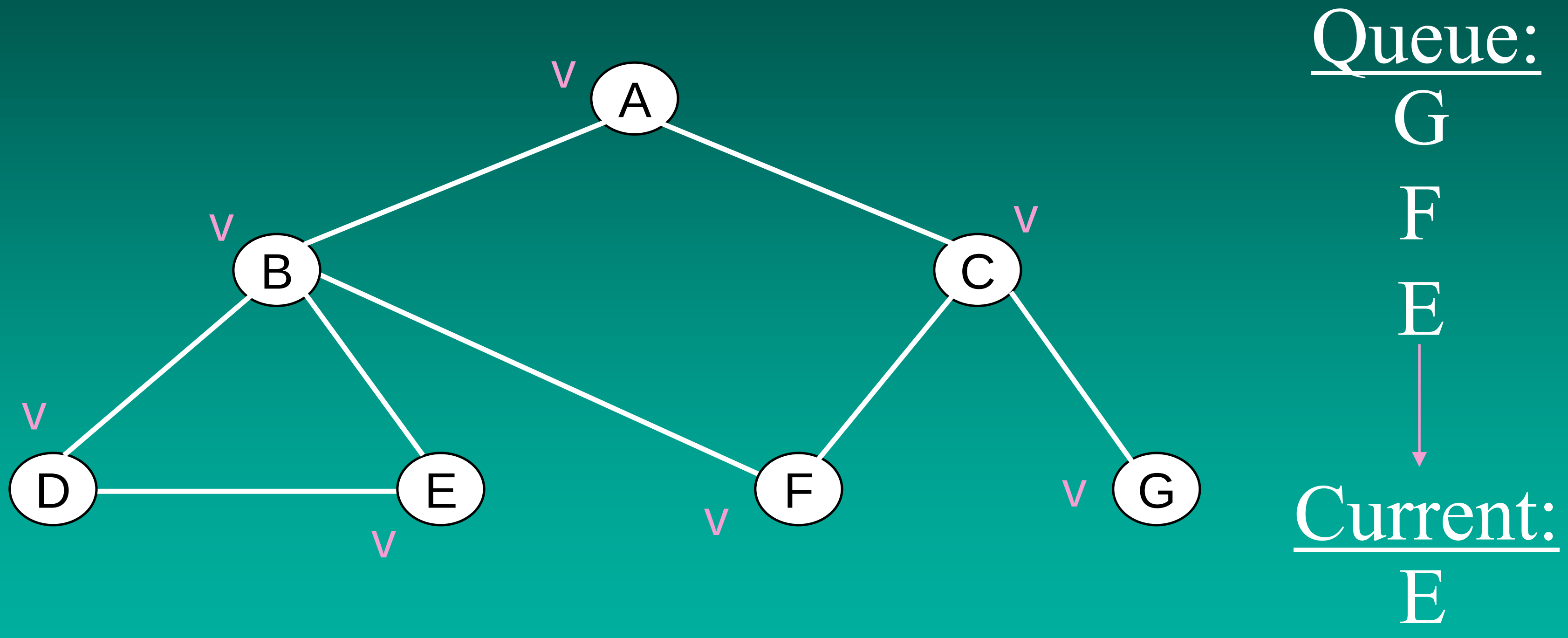


Queue:
G
F
E

Current:
D

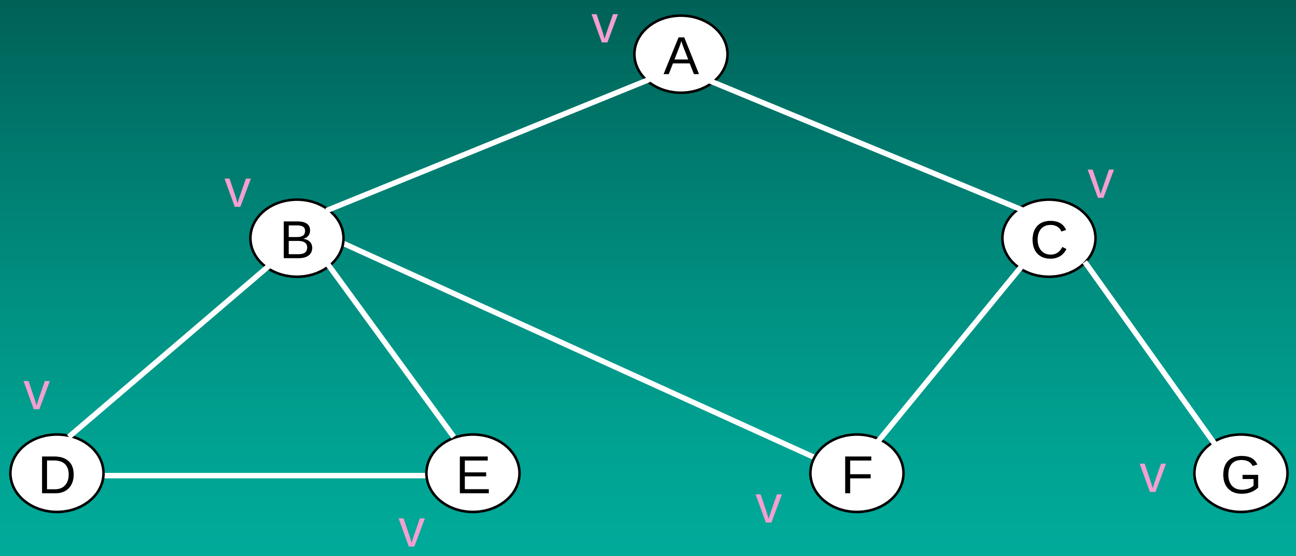
A B C D

Breadth-First Search



A B C D

Breadth-First Search

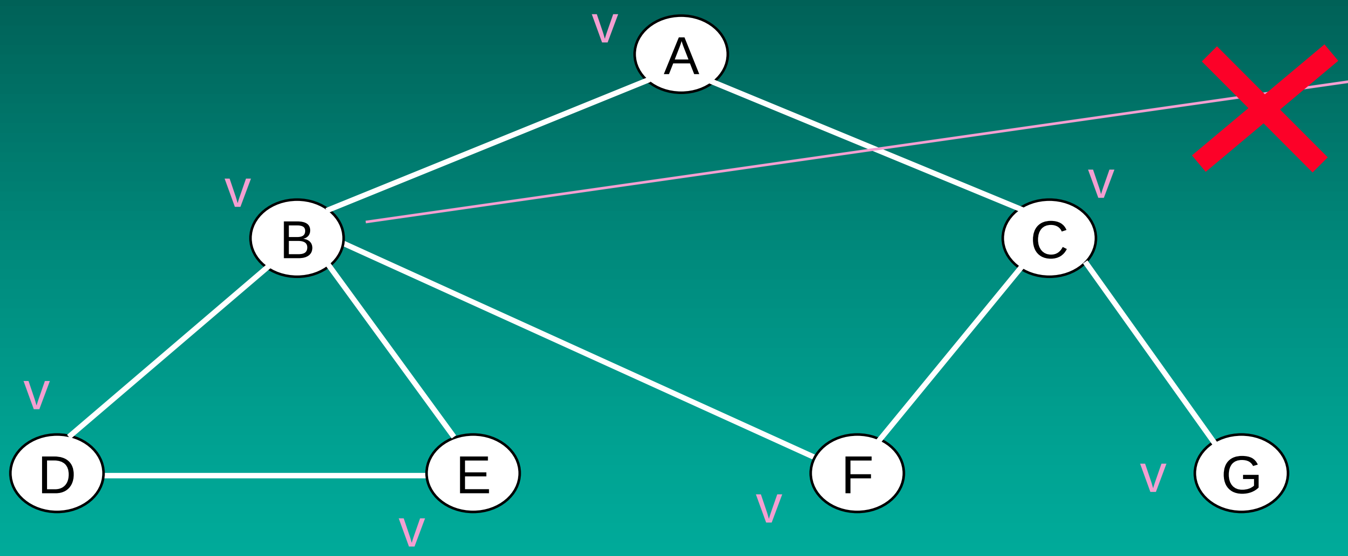


Queue:
G
F

Current:
E

A B C D E

Breadth-First Search

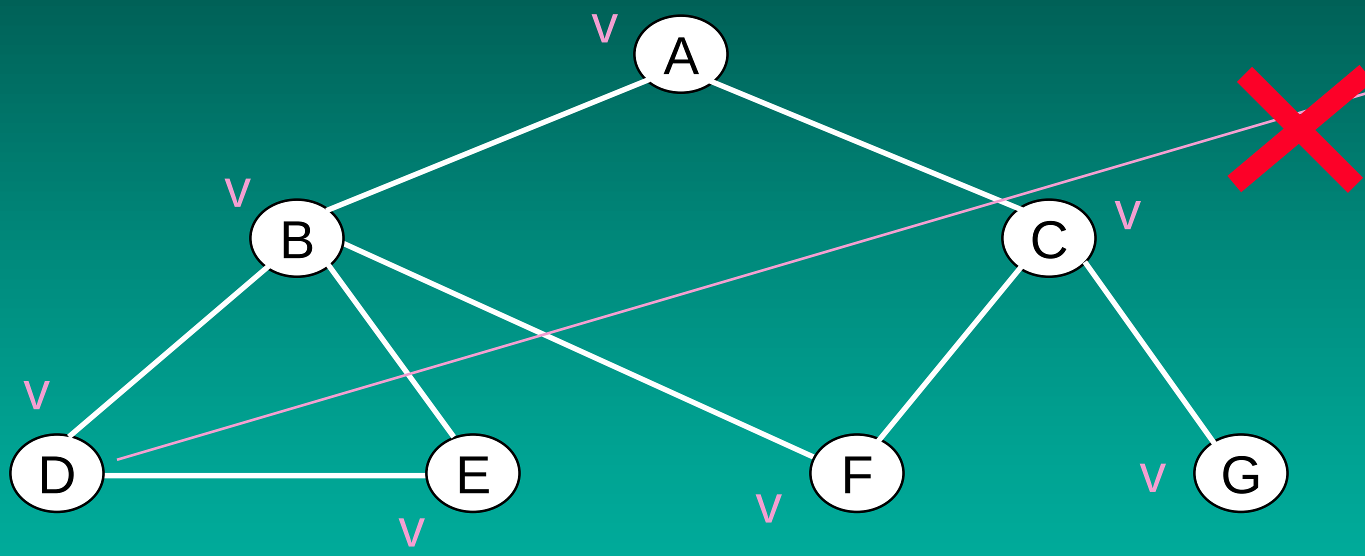


Queue:
G
F

Current:
E

A B C D E

Breadth-First Search

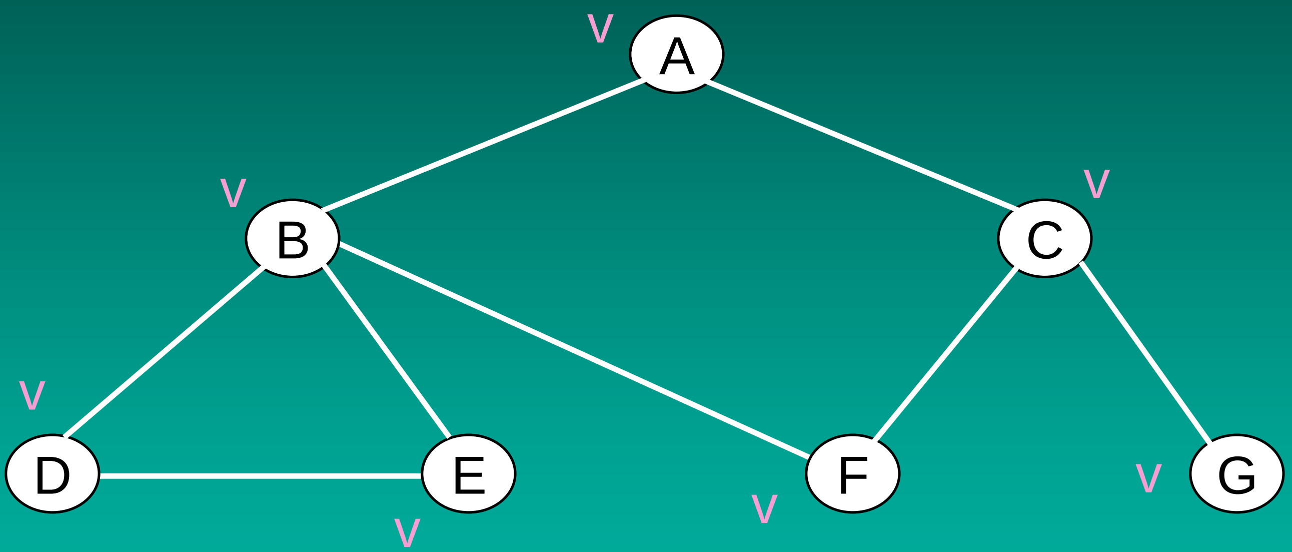


Queue:
G
F

Current:
E

A B C D E

Breadth-First Search

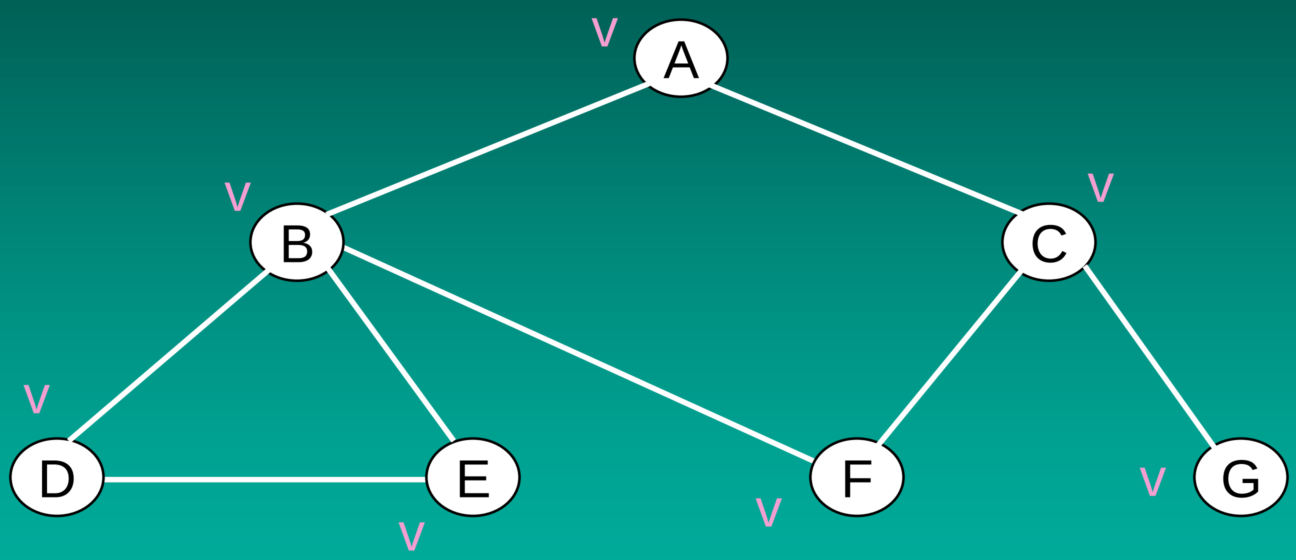


Queue:
G
F

Current:
F

A B C D E

Breadth-First Search

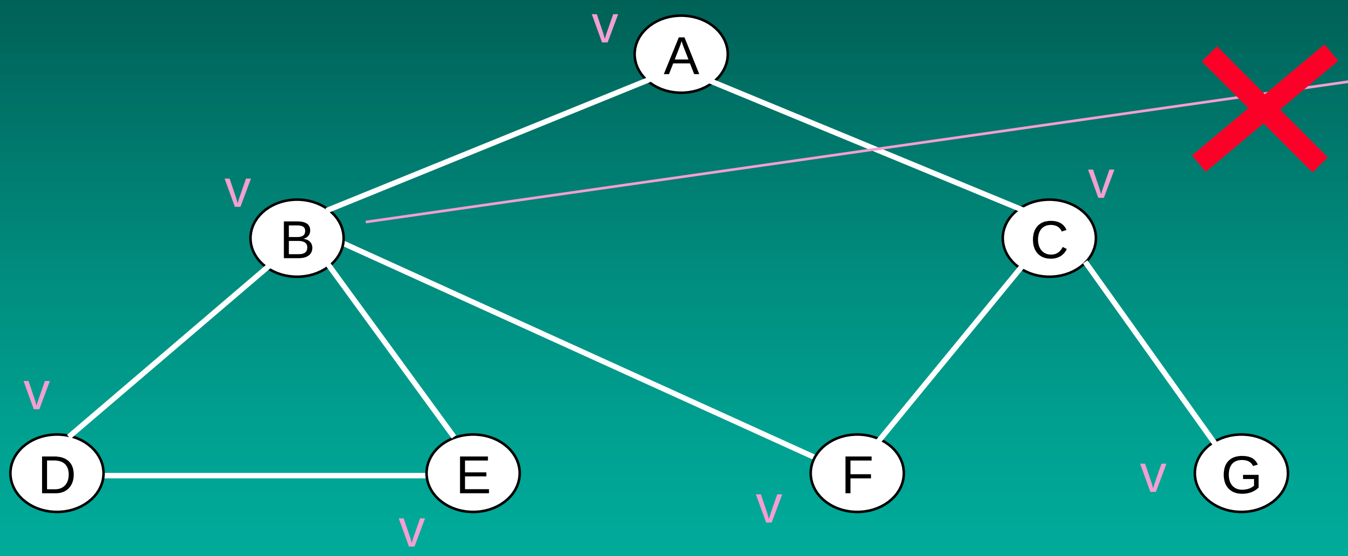


Queue:
G

Current:
F

A B C D E F

Breadth-First Search

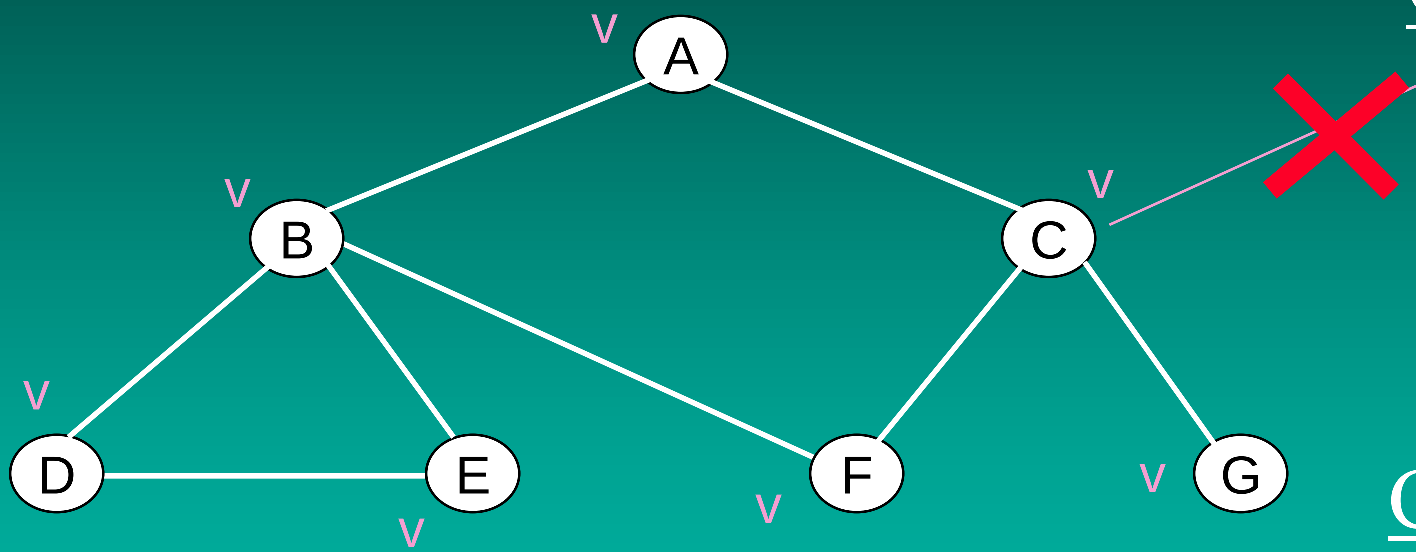


Queue:
G

Current:
F

A B C D E F

Breadth-First Search

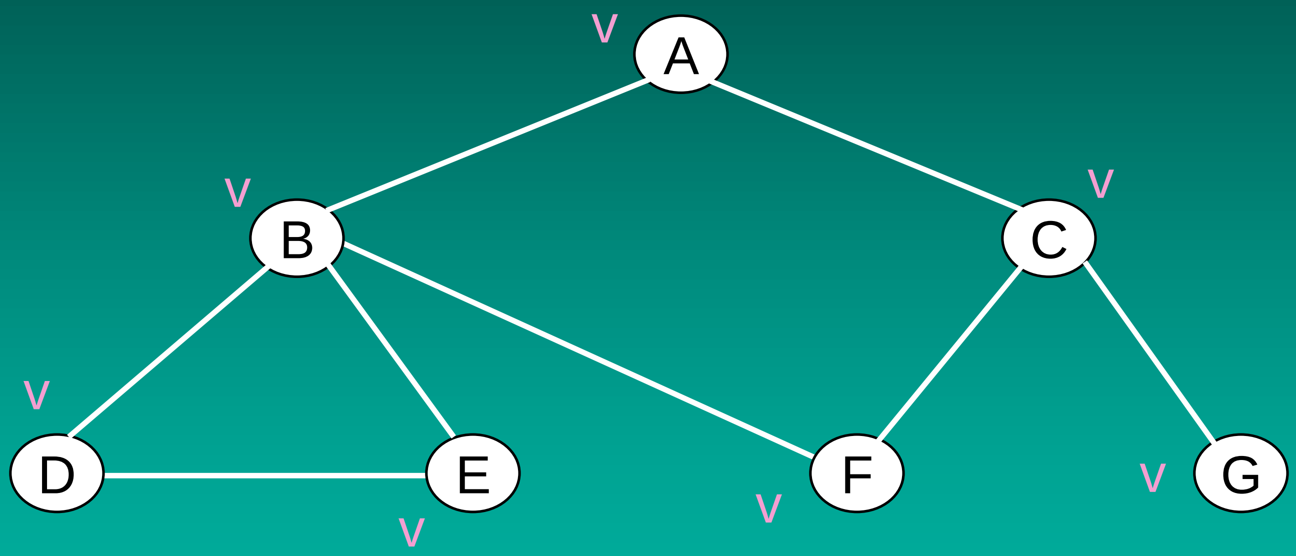


Queue:
G

Current:
F

A B C D E F

Breadth-First Search



Queue:

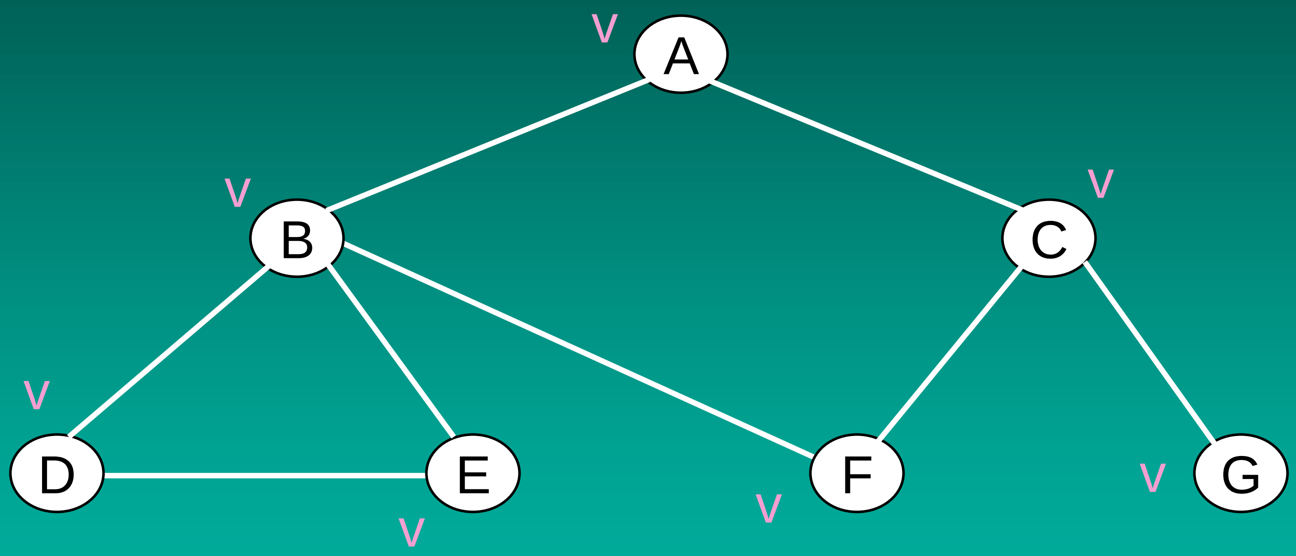
G

Current:

G

A B C D E F

Breadth-First Search



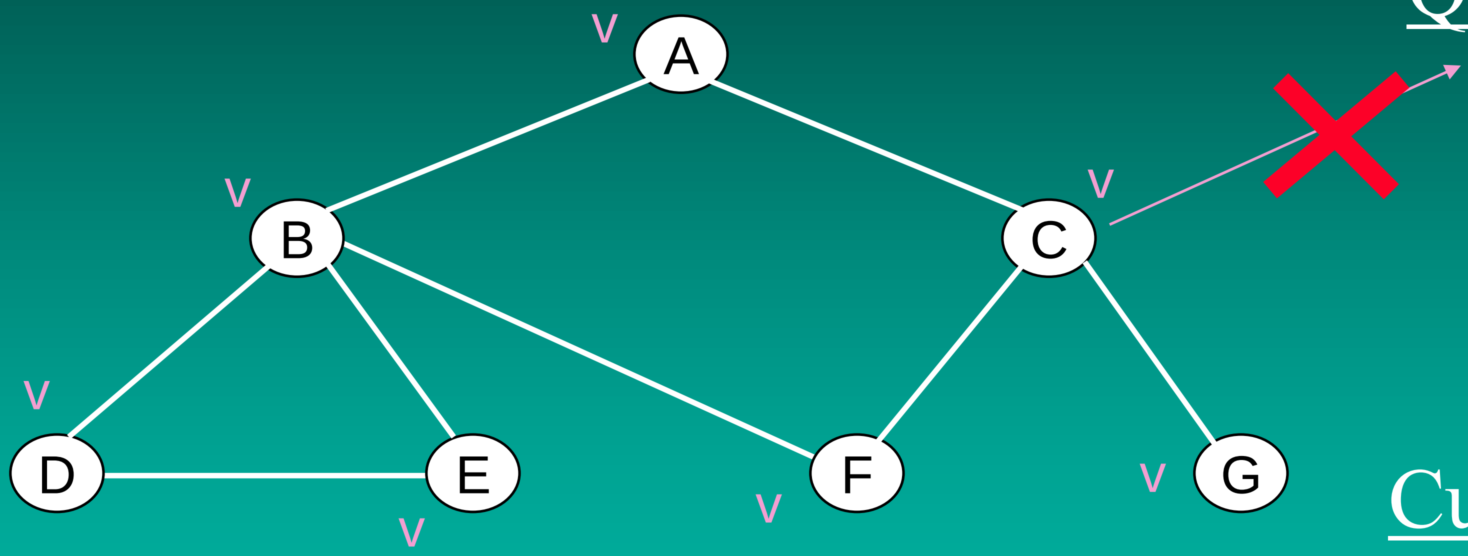
Queue:

Current:

G

A B C D E F G

Breadth-First Search

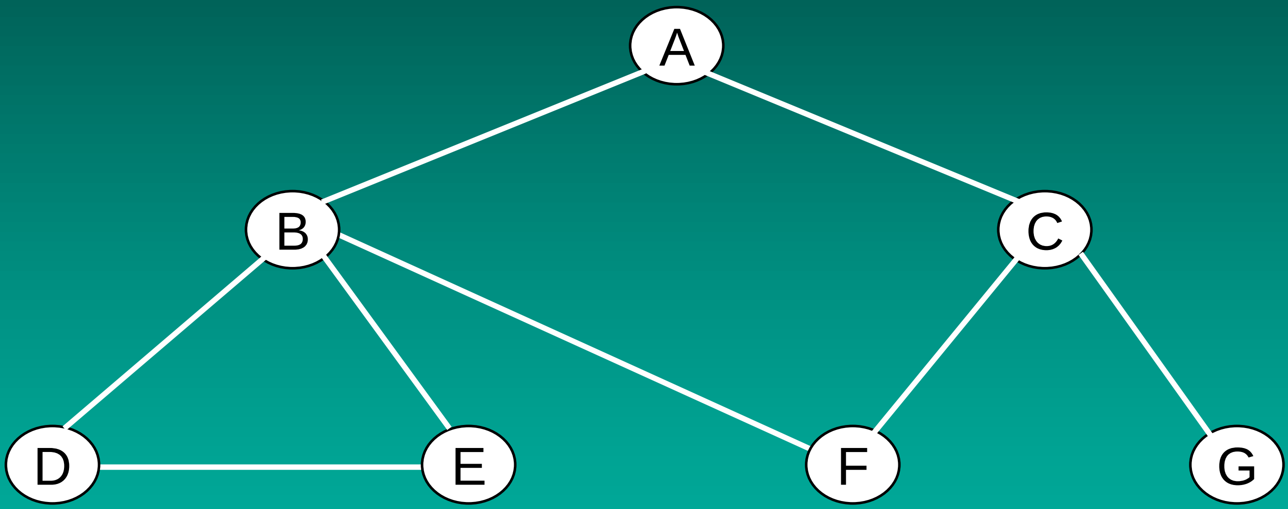


Queue:

Current:
G

A B C D E F G

Breadth-First Search



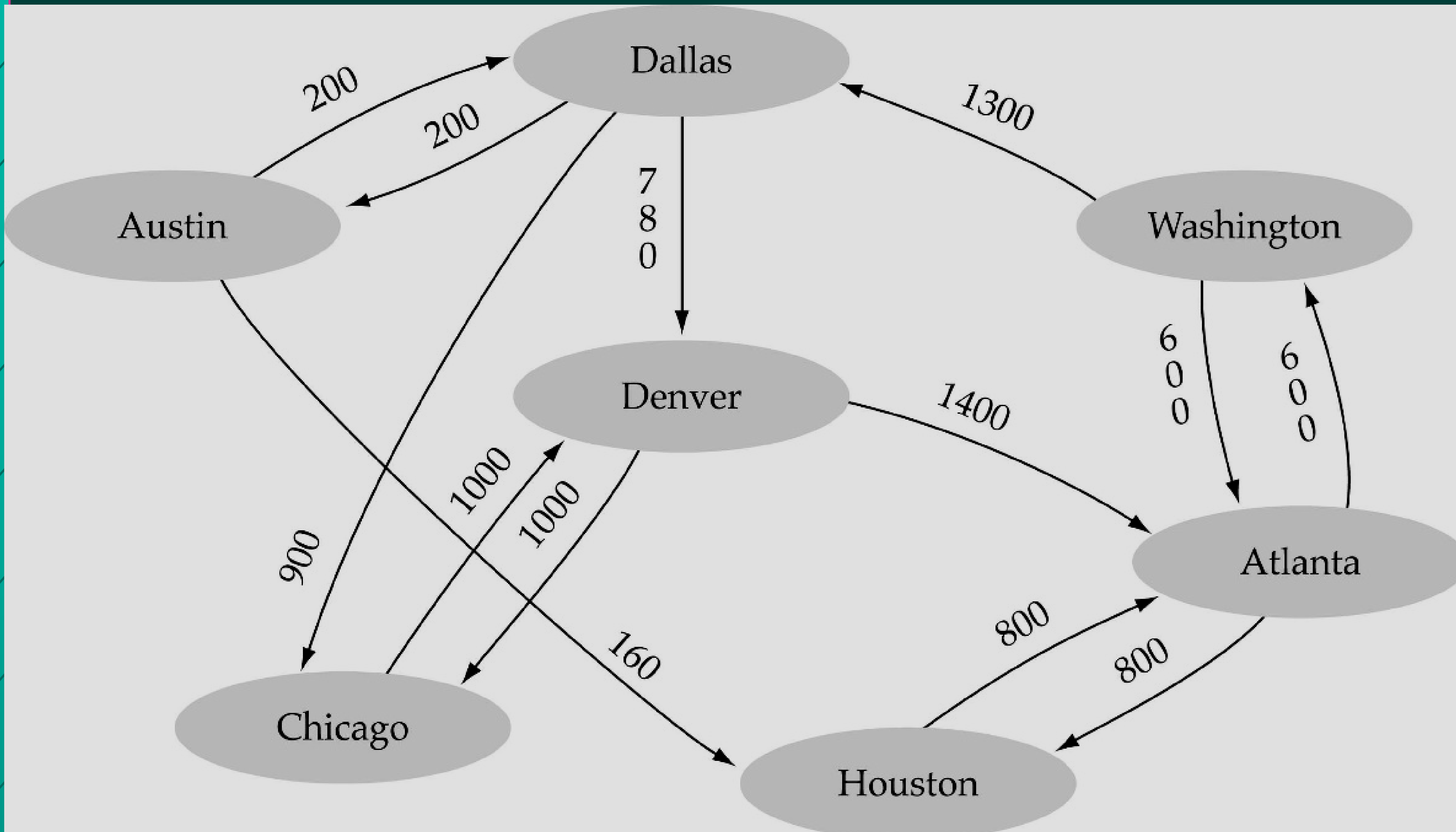
A B C D E F G

Time and Space Complexity for Breadth-First Search

- Time Complexity
  - Adjacency Lists
    - Each node is added to queue once
    - Each node is checked for each incoming edge
    - O (v + e)
  - Adjacency Matrix
    - Have to check all entries in matrix: $O(n^2)$

Time and Space Complexity for Breadth-First Search

- Space Complexity
  - – Queue to handle unexplored nodes
    - Worst case: all nodes put on queue (if all are adjacent to first node)
    - O(n)


Dallas
Austin
Washington
Denver
Atlanta
Chicago
Houston
200
200
1300
780
600
600
1400
1000
1000
900
160
800
800

Single-source shortest-path problem

- There are multiple paths from a source vertex to a destination vertex
- *Shortest path*: the path whose total weight (i.e., sum of edge weights) is minimum
- Examples:
  - Austin->Houston->Atlanta->Washington: 1560 miles
  - Austin->Dallas->Denver->Atlanta->Washington: 2980 miles

Single-source shortest-path problem (cont.)

- Common algorithms: *Dijkstra's* algorithm, *Bellman-Ford* algorithm
- BFS can be used to solve the shortest graph problem when the graph is <u>weightless</u> or all the weights are the same

(mark vertices before Enqueue)