Exercise sheet 4

Bernoulli, Euler, Riemann and Casimir.

a.) The function $f(t) = t/(e^t - 1)$ is the generating function for the Bernoulli numbers B_n, i.e.

$$f(t) = \frac{t}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} t^n \,. \tag{1}$$

Calculate the Bernoulli numbers B_n for $n \leq 3$.

b.) The Riemann ζ function can be defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{2}$$

for $s > 1$ and then analytically continued into the complex s plane. The Bernoulli numbers are connected to the Riemann ζ function with negative odd argument as

$$\zeta(1 - 2n) = -\frac{B_{2n}}{2n} \,. \tag{3}$$

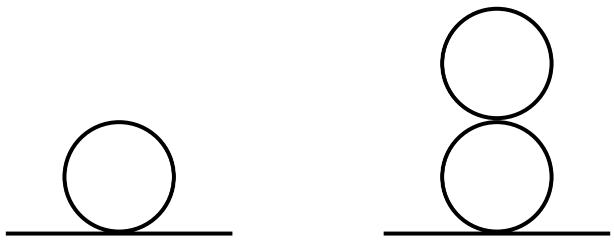
This allows you to find with magical ease the sum needed in the Casimir energy (in 1+1 dimensions), $\sum_{n=1}^{\infty} n = \zeta(-1) = -B_2/2 = -1/12$. Less magically, show using

$$\frac{1}{a} \frac{a}{e^a - 1} = \frac{1}{a} \sum_{n=0}^{\infty} \frac{B_n}{n!} a^n \tag{4}$$

that you can split the sum into a divergent and a finite part. The divergent term will cancel in the difference of the vacuum energy with and without plates, and the remaining finite term is determined by $B_2/2$.

Symmetry factors.

Find the symmetry factors for the following diagrams



Vacuum energy density.

Go through the calculation of the vacuum energy density in the script, section 4.3.2. Collect unclear points and errors.

Solutions are discussed Tuesday, 12.02.18

Expansion of the Gamma function.
Show that the Gamma function is given for $\varepsilon \to 0$ by

$$\Gamma(\varepsilon) = \frac{1}{\varepsilon} - \gamma + \mathcal{O}(\varepsilon)\,, \tag{5}$$

where the Euler-Mascheroni constant $\gamma = 0.5772\ldots$ is defined by $\gamma = -\psi(1)$ and ψ is the logarithmic derivative of the Gamma function,

$$\psi(z) = \frac{\mathrm{d}\ln\Gamma(z)}{\mathrm{d}z} = \frac{\Gamma'(z)}{\Gamma(z)}\,. \tag{6}$$

Show that ψ satisfies for integers

$$\psi(n+1) = 1 + \frac{1}{2} + \ldots + \frac{1}{n} - \gamma\,. \tag{7}$$

Solutions are discussed Tuesday, 12.02.18