Home exam

In

FY3464

Quantum Field Theory I

By

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Executive summary:

Before beginning with the actual summary, I should point out that the text is based on self-help notes that I made for understanding the text. So, they are less formal than expected perhaps.[1]

The notes will start of with the crucial aspect of quantizing classical physics. Basically, turning classical physics to quantum mechanics (QM) sort of speak. We start with Newtonian mechanics, suitable to describe simple movement. And where you have to find the acceleration in 3 dims. Based on Fermat's principle (saying that a particle will use the shortest possible path to reach its destination) and that in classical physics at a given time t, we have a set of generalized coordinated q_i where i = 1, 2…N. These have the velocity, $q_i' = \frac{dq_i}{dt}$. These factors gives us the action:

$$S_{[q]} = \int_{t1}^{t2} L(q_i, q_i')\, dt \quad (1)$$

Where L is the Lagrangian of the system. The Lagrange makes it possible to reduce the problem of 3-dimension and many equations to a characteristic length. By putting $\delta\, S = 0$, we get the Euler-Lagrange equation:

$$\frac{dL}{dq_i} - \frac{d}{dt}\frac{\partial L}{\partial q_i'} = 0 \quad (2)$$

Giving us differential equations that describe the movement of the system. When quantizing we want to go from classical action S to a path and over to the transition amplitude, <q', t' | q, t>.

A Legendre transformation from an original function f(x, y) to a new function g(x, w) is often done by changing variable y to its conjugate variable w. Legendre-transformation is done to go from Lagrange, L(q, q') to Hamilton, H(q, p), where p is the momentum. This is useful due to time not being varied in Hamiltonian mechanics. The Hamiltonian gives the total energy of a system.

Einstein proposed that particles showed a particle-wave duality. When we are to find the QM amplitude for a particle at a position q_i at time t_i to reach position q_f at time t_f, one integrates over all possible path connecting the points with a weight factor given by the classical action for each path. It is stated as:

$$< q_f, t_f \mid q_i, t_i > = \int D\, q(t) \exp\left(\frac{iS_{[q(t)]}}{\hbar}\right) \quad (3)$$

Knowing the path integral is enough to solve scattering problems in QM. In (3) the Lagrangian is involved in the action. Diff. equations are very difficult to solve in Quantum Field Theory (QFT). However, there is a method called Green functions to remedy that. The Green function is basically the inverse of the L(q,q'), L^-1.

The initial and final states in QFT are generally free particles being described as harmonic oscillators. When the harmonic oscillator model is use, we rescale the L, before using annihilation (a) and creation (a+) operators to solve it. The resulting Hamiltonian is then:

$$H = \frac{w}{2}(aa^{+} + a^{+}a) \quad (4)$$

By using harmonic oscillators as model, we are able to reconstruct all exited states |n> from the ground-state,

$$|n > = \frac{1}{\sqrt{n!}}\, (a^{+})^{n}\, |0 > \quad (5)$$

An integration of interval given as $t = \pm \infty$ makes sure that the functional generated is Lorentz invariant (meaning that the variables are not changed during the Lorentz transformation). With the Green function many mathematical operations are possible; like adding time-ordered products of operators and path integrals, external sources and removing I and f states to obtain ground state/vacuum. The latter is done in order to create a probability amplitude with |0> at $t_i \rightarrow -\infty$ and |0> at $t_f \rightarrow \infty$ despite the external source J(t). By adding a coupling to J(t) and a damping factor $+i\epsilon q^2$ to the Lagrangian, we get the vacuum persistence amplitude:

$$Z[J] \equiv < 0, \infty\, |0, -\infty >_J = \int D\, q(t) \exp\left(i \int_{-\infty}^{\infty} dt\, (L + i\epsilon q^2)\right) \quad (6)$$

Z[J] is the probability amplitude that the fields will remain in the state they start out with after interactions.

With some of the background done with, we look at relativistic fields described by certain equations. The simplest field is the Klein-Gordon field for elementary particles with spin = 0. The process is so that one begins with the classical field theories, re-interpret the different variables like operators that follow canonical commutative relations, before finally solving it by finding eigenvalues and eigenstates to the Hamiltonian. By using the harmonic oscillator model, commutation rules, we get an equation for the field acting on vacuum and creating a particle at position x through the equation:

$$ø(x)|0> = \int \frac{d^3p}{(2\pi)^3} \frac{1}{E_p} \exp(ipx)\, |p> \quad (7)$$

with E_p being the energy = the same as w_p from the Hamiltonian.

The other field mentioned in this executive summary is the Dirac-field for particles that follow Fermi-Dirac statistics and has a spin of s = (1/2. 3/2,….etc). Later in the course, we are going to use QED and it should therefore be mentioned that in QFT anti-particles are created, in this regard positrons since we are dealing with electrons. Again, we quantize classical Dirac field equation. This is done partially below in order to introduce Clifford algebra. Dirac-equation for resting mass, m, is:

$$i\hbar\gamma^{\mu} \frac{\partial\Psi}{\partial x^{\mu}} - m\Psi(x) = 0 \ \ (8)$$

Where $\gamma^0 = \beta, \gamma^{i} = \beta\alpha i,\ i = 1,2,3$ are Dirac 4*4 matrices that satisfy anticommutation relations. $[\gamma^{\mu}, \gamma^{\nu}] = 2g^{\mu\nu}$. The accompanying field $\bar{\Psi}\,(x)$ is defined by:

$$\bar{\Psi}\,(x) = \Psi^{+}\,(x)\gamma^0 \ \ (9)$$

We can also write the Dirac matrices for 4D Minkowski space by something called Weyl representation.

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \ (10)$$

Fields that interact have a Hamiltonian given by:

$$H = H_0 + H_1 \ \ (11)$$

Where H_0 = free field and H_1 = interaction. We write S-matrix based on (11) and we get a covariant result:

$$S = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int \dots \int d^4x_1\, d^4x_2\, T\{H_1(x_1)\, H_1(x_2) \dots H_1\,(x_n)\} \ \ (12)$$

We want to find a bridge between S-matrix elements and the formalism derived up to now. And that can be done by finding the connection between Green functions derived from Z[J] and S-matrix elements. That can be done by the LSZ reduction formula. Due to space constraints it is not included in this text. But about it must be said that for each extra particle, a corresponding plan-wave component and Klein-Gordon operator is created.

Noether's theorem says that when a continuous symmetry of the Lagrange is obtained, then a corresponding conservation law exists. With symmetry one means that by transformation the variables q, q' and time are invariant. The transformation on the field ø can be written like:

$$ø\,(x) \rightarrow ø'(x) = ø(x) + \alpha\,\Delta ø(x) \quad (13)$$

where $\alpha =$ infinitesimal parameter and Δø is a deformation. We separate the symmetries into two classes: space-time symmetry and internal symmetry to a group of fields. The conserved Noether current is given by:

$$j^{\mu} = \frac{\delta L}{\delta \partial_{\mu} ø_{a}}\,\delta ø_{a} \quad (14)$$

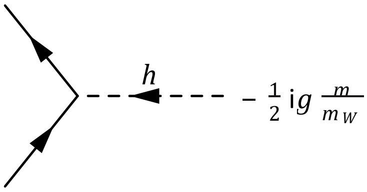
The very difficult diff. equations and particle reactions like 2 -> 2 are evaluated through the use of either perturbation theory or Feynman diagram. Perturbation theory is numerically based and the idea is to split the Hamiltonian into several parts which we can solve. One part is unsolvable (the perturbed part) and if it is small compared to the Hamilton, we can approximate a solution to the Hamilton. For many students however, the method of Feynman-diagrams is preferable due to its simplicity. Here one uses a set of pre-decided rules to create diagrams that you later quantize into a numerical solution to for example find the Feynman amplitude.

One important effect of perturbation theory in QED is that if it does not converge an emission of a soft photon or collinear photon is seen from an external line. With the propagator being:

$$\sim \frac{1}{(2\,E\,w\,(1-\cos(\theta)))} \quad (15)$$

w -> 0 = soft photon emission and θ -> 0 = collinear emission of photons.

Higgs decay into fermions and the optical theorem:



In the Standard Model, the Higgs particle h is a scalar particle that interacts with all fermions via a Yukawa coupling y proportional to the fermion mass m,

$$y = \frac{1}{2}\, g \frac{m}{m_W} \quad (16)$$

Figure 1: Schematic shows the Feynman-diagrams for the process of the Higgs particle interacting with fermions via a Yukawa coupling

a)

The question posed wants us to calculate the decay width $\Gamma\left(h \rightarrow \bar{f} f\right)$ of a Higgs particle with mass M into an antifermion-fermion pair. Decay width $\Gamma\ \left(\propto \frac{1}{\tau}\right)$ where τ = lifetime is a measure of the probability of a specific decay process occurring within a given amount of time in the parent particle's rest frame. The calculation here involves two steps: finding the amplitude, M_{fi}, of the process and then to integrate the amplitude over the phase space to get the $\sigma\ or\ \tau$. We want the latter obviously.

We have a figure to relate in figure 1. By using the Feynman rules for such diagrams we can create an amplitude, M_{fi}, for the diagram. I reference here to Peskin and Schroders book pages 800-802.[2] Like usual we begin from final to initial and use standard formulations of particle and anti-particle. We also have a vertex to consider (already given on the figure). This leads us the following:

$$iM = \bar{u}\left(-\frac{1}{2} ig \frac{m}{m_W}\right) v \quad (17)$$

Now we need to square the matrix element, sum over the fermion polarization and integrate over the phase space to find the decay width. We find the squared matrix by:

$$\left|M_{fi}\right|^2 = \frac{1}{4} \sum |M|^2,$$

Where the sum-sign represents summing over polarizations. And where $|M|^2 = M * M$.

Through a pregiven identity we also get:

$$\left[\bar{u}\left(-\frac{1}{2}ig\frac{m}{m_W}\right)v\right]^+ = \left[\bar{v}\left(-\frac{1}{2}ig\frac{m}{m_W}\right)u\right] \quad (18)$$

This all leads us to (with the momentums inserted whereas they were not before due to space and time constraints)

$$= \left(\bar{u}\,(p_2)\,v(p_3)\,\left(\frac{1}{4}\frac{g^2m^2}{m_W^2}\right)\left(\bar{v}\,(p_3)\,u(p_2)\right)\right.$$

By using the trace-identity called completeness relation, we get:

$$\sum u\,(p)\,\bar{u}\,(p) = (\not{p} + \mathrm{m}) \quad (19)$$

This is the completeness relation. Using this expression and taking the Trace we get:

$$= \frac{1}{4}\frac{m^2}{m_W^2}\,Tr\left(\left(\not{p}_2 + \frac{\mathrm{m}}{m_W}\right)\left(\not{p}_3 - \frac{m}{m_W}\right)\right)$$

Doing the multiplication and the trace relation given as $Tr(\not{a}\not{b}) = 4(ab)$ gives us:

$$= \frac{m^2}{m_W^2}\left(p_2p_3 - \frac{m^2}{m_W^2}\right)$$

From the handouts in the course FYS4555: Particle Physics at University of Oslo I have that when you work in the C.o.M frame, that one has the following relations given in equation (20) for e+e- colliders.[3] But I will use it as a blueprint for our own equation.

$$p_1 = (E,0,0,p), p_2 = (E,0,0,-p), p_3 = \left(E,\overrightarrow{p_f}\right), p_4 = \left(E,-\overrightarrow{p_f}\right) \quad (20)$$

We then get with mass M for the Higgs, the following:

$$p_1 = (M,0), p_2 = \left(E_f,\vec{p}\right), p_3 = \left(E_f,-\vec{p}\right) \quad (21)$$

The law of conservation of momentum is:

$$p(before) = p(after) \quad (22)$$

Becoming $p_1 = p_2 + p_3$ in our case and that gives us:

$$M = m(v_1 + v_2) \rightarrow M = m(2v)$$

When the velocity and mass are the same, as they are in this case. Transferring this to our case again, we then get something along the lines of:

$$M = 2E_f, \;\; where\; E_f^2 = p^2 + \frac{m^2}{m_W^2}$$

From this we get:

$$M = 2\sqrt{p^2 + \frac{m^2}{m_W^2}} \rightarrow p = \sqrt{\frac{M^2}{4} - \frac{m^2}{m_W^2}} \;\; (23)$$

We of course have, as defined:

$$p_2 p_3 - \frac{m^2}{m_W^2} = p$$

And from this we get an amplitude:

$$\sum \left|M_{fi}^2\right| = \frac{2m^2}{m_W^2}\left(\frac{M^2}{4} - \frac{m^2}{m_W^2}\right) N \;\; (24)$$

Here we have added an additional colour factor because it is a QCD Matrix rather than QED. This is the N in the equation.[4] Then we are only left with actually finding the decay width Γ. The equation for this I found in Mark Thomson's work, *Modern Particle Physics*.[5]

$$\Gamma\left(H \rightarrow f\bar{f}\right) = \frac{p}{8\pi M^2} < \left|M_{fi}^2\right| > (25)$$

We must add here that since initial fermions are not included then p/2. We have already found p and $|M^2|$, so we just do the calculation in equation (25).

$$\Gamma = \left(\frac{\left(\sqrt{\frac{M^2}{4} - \frac{m^2}{m_W^2}}\; N\left(\frac{2m^2}{m_W^2}\right)\left(\frac{M^2}{4} - \frac{m^2}{m_W^2}\right)\right)}{16\pi M^2}\right) \;\; (26)$$

Equation becomes after some more calculations:

$$\Gamma = \frac{2N}{16\pi M^2}\frac{m^2}{m_W^2}\frac{M^3}{6}\left(1 - \frac{4m^2}{M^2 m_W^2}\right)^{1.5} \;\; (27)$$

(27) is the partial decay width. Since we have three generations of fermions and particle and anti-particle, 3*2 = 6, we get the final expression for the total decay width given as:

$$\Gamma = \frac{N}{8\pi}\frac{m^2}{m_w^2}\ M\ \left(1 - \frac{4m^2}{M^2 m_W^2}\right)^{1.5} \quad (28)$$

b)

This problem wants us to show that a fermion loop leads to an additional minus sign in the Feynman amplitude. The basis for this question I take from Kachelreiss on pages 120 and 137.[1] It is there clearly defined that a closed loop with n propagators correspond to:

$$\bar{\psi}\ (x_1)\ \psi\ (x_1\ )\ \bar{\psi}\ (x_2)\psi\ (x_2)\ \dots\dots \bar{\psi}\ (x_n)\psi\ (x_n)$$

So for us to combine $\psi\ (x_1)\ and\ \psi\ (x_n)$ into T $\{\psi\ (x_n)\ \bar{\psi}\ (x_1)\}$, we must anticommute $\bar{\psi}\ (x_1)$ with the 2n-1 fields $\psi\ (x_1)\ \dots\dots \psi\ (x_n)$ and thereby generating a minus sign.

This is the formal definition, however, I feel that it does not help us commoners all that much in reaching our above-stated goal. We therefore take a closer look at a related theme known as crossing symmetry. This is relatable to our figure. It is basically explained that the spin sums of fermions and anti-fermions are related by the following identity.

$$\sum_s u\ (p,s)\ \bar{u}\ \ (p,s) = (\not{p} + m) = \ -(\not{p} + m) = -\sum_s v\ (p',s)\bar{v}\ \ (p',s)$$

The -1 factor for each exchanged external fermion pair is further explained by an exchange particle and anti-particle wavefunction for matrix elements where an in-going particle is replaced by an out-going particle for example.

Because of these aspects we must add an additional minus sign in the Feynman amplitude due to fermion loop.

c)

References:

[1] Michael Kachelreiss. *Lecture Notes for FY3464 & FY3466. Quantum Field Theory I and Quantum Field Theory II.* 2016. Trondheim

[2] Michael E. Peskin and Daniel V. Schroeder. *An introduction to Quantum Field Theory*. 1995. Reading, Massachusetts..

[3] Handout 4 in FYS4555: Particle Physics. Retrieved from: https://www.uio.no/studier/emner/matnat/fys/FYS4555/h18/timeplan/outlines/handout_4_2018.pdf. Last accessed 26.03.2020

[4] Handout 8 in FYS4555: Particle Physics. Retrieved from: https://www.uio.no/studier/emner/matnat/fys/FYS4555/h18/timeplan/outlines/handout_8_2018.pdf. Last accessed 26.03.2020.

{5] Mark Thomson, *Modern Particle Physics*, 2015. Cambridge.

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Appendix:

I am adding some stuff that was kind of irrelevant in the exam here. First I will add the figure including the p1, p2 and p3 momentum used in question a).

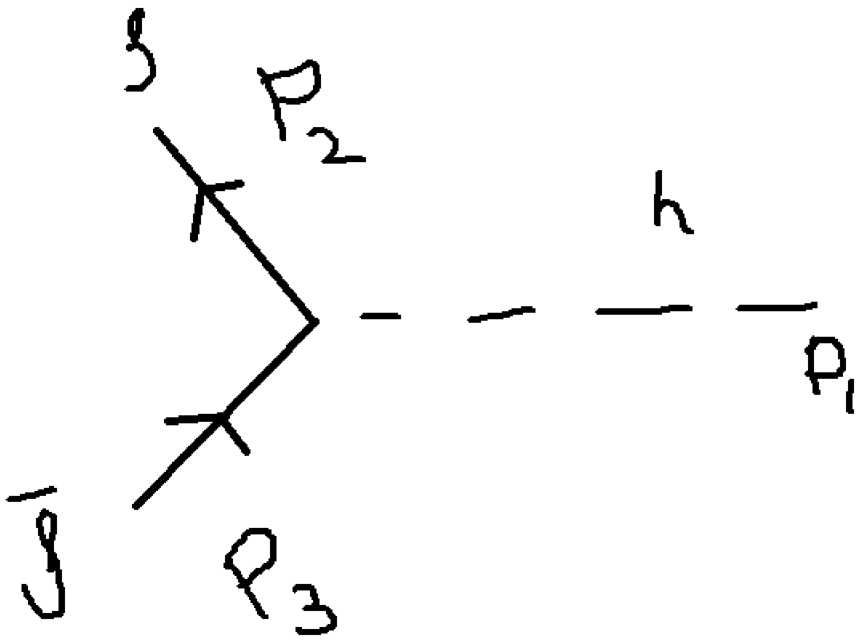

Figure 2: Schematic shows the same as in figure 1, but now with the momentum p1, p2 and p3.