Exercise sheet 2

Propagator as Green function.
Show that the Green function or propagator $K(x',t';x,t) = \langle x'|\exp[-\mathrm{i}H(t'-t)]|x\rangle$ of the Schrödinger equation is the inverse of the differential operator $(\mathrm{i}\partial_t - H)$. Hint: It is simpler to consider the advanced or retarded Green functions,

$$G^{\pm}(x',t';x,t) = G(x',t';x,t)\vartheta(\pm\tau) \tag{1}$$

where $\tau = t' - t$ and ϑ is the step function.

Lorentz invariant integration measure.
Show that $\mathrm{d}^3k/(2\omega_k)$ is a Lorentz invariant integration measure by a) calculating the Jacobian of a Lorentz transformation, b) showing that

$$\begin{aligned}\int \mathrm{d}^4k\ \delta(k^2-m^2)\vartheta(k^0)f(k^0,\boldsymbol{k}) &= \\ = \int \frac{\mathrm{d}^3k}{2\omega_k}\, f(\omega_k,\boldsymbol{k})\end{aligned} \tag{2}$$

holds for any function f.

Yukawa potential.
Show that the Yukawa potential $V(r) = \mathrm{e}^{-mr}/(4\pi r)$ is the Fourier transform of $(\boldsymbol{k}^2+m^2)^{-1}$, cf. Eq. (3.37).

Dimension of ϕ.
a.) Determine the mass dimension of a scalar field ϕ in d space-time dimensions.
b.) For which d has $\mathscr{L}_{\text{int}} = \lambda\phi^3$ ($\mathscr{L}_{\text{int}} = \lambda\phi^4$) a dimensionless coupling constant?
c.) Has the numerical value of λ for a $\lambda\phi^4$ theory classically a physical meaning? In quantum theory? [Hint: Can you rescale fields such that $\mathscr{L}(\lambda) = \lambda\mathscr{L}(1)$?]

Propagator at large $|\boldsymbol{x}|$.
Find the asymptotic behavior of the propagator $D_F(0,r)$ outside the light-cone.

Statistical mechanics.
Derive the connection between the partition function of statistical mechanics and the path integral in quantum mechanics,

$$Z = \mathrm{Tr}\,\mathrm{e}^{-\beta H} = \sum_n \langle n|\,\mathrm{e}^{-\beta H}\,|n\rangle \Leftrightarrow \int \mathcal{D}q(t)\exp\left(\mathrm{i}\int \mathrm{d}tL\right).$$

considering the latter in Euclidean time $\tau = t' - t = -\mathrm{i}\beta$.

Solutions are discussed Monday, 28.01.19