Exercise sheet 3

Scalar fields.
The most general expression for the Lagrange density $\mathscr{L}$ of N scalar fields ϕ_i which is Lorentz invariant and at most quadratic in the fields is

$$\mathscr{L} = \frac{1}{2}A_{ij}\partial^\mu\phi_i\partial_\mu\phi_j - \frac{1}{2}B_{ij}\phi_i\phi_j - C\,.$$

What constraints have the coefficients A_{ij}, B_{ij}, C to satisfy? Show that $\mathscr{L}$ can be recast into "canonical form"

$$\mathscr{L} = \frac{1}{2}\partial^\mu\phi_i\partial_\mu\phi_i - \frac{1}{2}b_i\phi_i\phi_i - C$$

by linear field redefinitions.

Maxwell Lagrangian.
a.) Derive the Lagrangian for the photon field A_μ from the source-free Maxwell equation $\partial_\mu F^{\mu\nu} = 0$. (Compare to the procedure for a scalar field in the notes.) b.) What is the meaning of the unused set of Maxwell equations?

Green functions.
Show that the connected and the unconnected n-point Green functions are identical for $n = 2$,

$$G(x_1, x_2) = \mathcal{G}(x_1, x_2)\,,$$

while they differ in general for $n \geq 3$. [Recall that $\langle 0|0\rangle = 1$ and $\langle 0|\phi(x)|0\rangle = 0$.]

Volume of a sphere in arbitrary dimensions.
a.) Calculate the volume of the unit sphere S^{n-1} defined by $x_1^2 + \ldots + x_n^2 = 1$ in $\mathbb{R}^n$.
b.) Generalize the result to arbitrary (not necessarily integer) dimensions and show that it agrees with the familiar results for $n = 1, 2$ and 3.
(Recall that mathematicians distinguish between a sphere S^n and a ball B^n.)

Solutions are discussed Monday, 04.02.19