Exercise sheet 7

Linear Sigma model I.
Consider the Lagrangian

$$\mathscr{L} = \frac{1}{2}\left[(\partial_\mu \boldsymbol{\pi})^2 + (\partial_\mu \sigma)^2\right] - \frac{1}{2}m^2(\boldsymbol{\pi}^2 + \sigma^2) - \frac{\lambda}{4}(\boldsymbol{\pi}^2 + \sigma^2)^2 .$$

[You may interpret the fields $\boldsymbol{\pi} = (\pi_1, \pi_2, \pi_3)$ as pions with isospin 1, and σ as a sigma meson with isospin 0; the corresponding SU(2) symmetry has then the physical interpretation of isospin.]
a) Show that the Lagrangian $\mathscr{L}$ is invariant under the symmetry transformation

$$\Sigma \to \Sigma' = U\Sigma U^\dagger , \tag{1}$$

where

$$\Sigma \equiv \sigma + \mathrm{i}\boldsymbol{\tau}\cdot\boldsymbol{\pi} ,$$

$U = \exp(\mathrm{i}\boldsymbol{\alpha}\cdot\boldsymbol{\tau}/2)$ and $\boldsymbol{\tau} = (\tau^1, \tau^2, \tau^3)$ are the Pauli matrices. Find the corresponding conserved Noether currents. [Hint: Calculate first $\Sigma\Sigma^\dagger$.]
b.) Give the Feynman rules (i.e. specify propagators and vertices in momentum space) for this Lagrangian.

Projection operators.
Show that the inverse of the matrix $A = \sum_i a_i P_i$, where the P_i are projection operators (i.e. $\sum_i P_i = 1$ and $P_i P_j = \delta_{ij} P_j$), is given by $A^{-1} = \sum_i a_i^{-1} P_i$.

Dirac representation.
Show that the γ matrices in the Dirac representation,

$$\gamma^0 = 1 \otimes \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\gamma^i = \sigma^i \otimes \mathrm{i}\tau_2 = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix},$$

where σ_i and τ_i are the Pauli matrices, $\otimes$ denotes the tensor product, satisfy $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$.

Traces of gamma's.
Calculate

$$\mathrm{tr}[\not{a}\not{b}]$$

$$\mathrm{tr}[\not{a}\not{b}\not{c}\not{d}]$$

$$\gamma^\mu \not{a} \gamma_\mu$$

Solutions are discussed Monday, 04.03.19