Final exam in FYS3120: Classical Mechanics and Electrodynamics

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Problem 1:

a)

We are tasked to draw a sketch of the set-up of the system suggested in the assignment of the text, including the reference frames involved. This was done in the program Pain and the figure follows in figure 1 below.

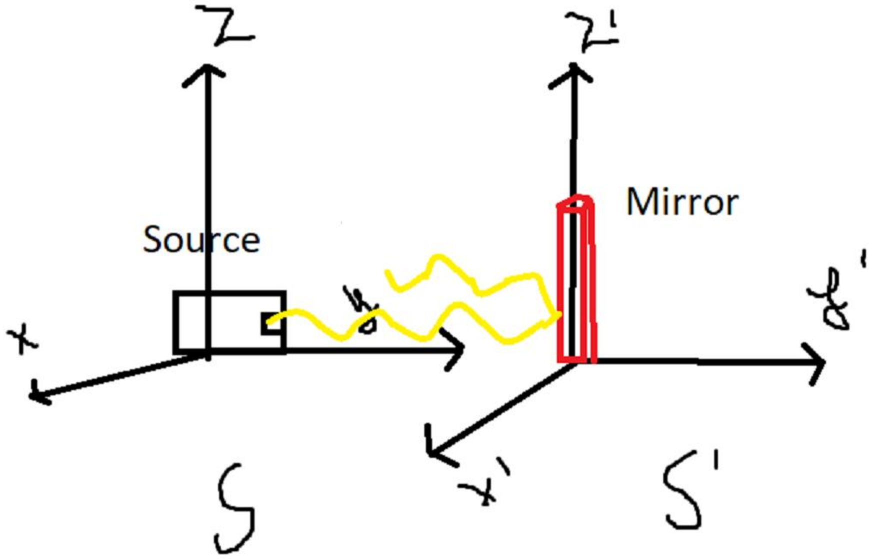

Figure 1: Schematic shows the set-up with the reference frames included

b)

We find the relationship between the frequency of the emitted and reflected light observed in S. We set the frequency of the photons emitted from the source as v_0 and the frequency of the reflected light as v. We use the transformation formula for four-momentum p^{μ} = (E/ c, $\vec{p}$) and the Planck relation E = hv to find the necessary relation.

The momentum four-vector has the following components: $P^0 = \frac{E}{c}$ and $P^i = p^i$. P^i are the three components of the particle momentum $\vec{p}$. The energy-momentum relation for particles without mass becomes E = $|\vec{p}|$c. So, by inserting it into the Planck relation we get:

$$hv = \vec{p}\text{c} \Rightarrow |\vec{p}| = \left(\frac{hv}{c}\right)$$

And this then becomes E = (hv/c)*c and this inserted into the four-momentum we get:

$$p^{\mu} = \left(\frac{hv_0}{c}, \frac{hv_0}{c}, 0, 0\right)$$

For emitted photons and remembering the notation from earlier. So, we want to find the momentum in the mirror rest frame. And in order to do so we do a Lorentz transformation:

$$P_E'^{\mu} = L_p^{\mu} \, P_E^p$$

We have that the mirror system is just boosted along the x-axis relative to the S frame. Therefore, we use the simple expression for the Lorentz transformation:

$$L = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Where $\beta = \frac{v}{c}$. The energy of the emitted photon in the mirror rest frame is given by, when divided by c:

$$P_E'^0 = L_0^0 \, P_E^0 + L_1^0 \, P_E' = \gamma \frac{hv_0}{c} - \beta\gamma \frac{hv_0}{c} = \frac{hv_0}{c} \, \gamma(1-\beta)$$

The transformed momentum is equal to this,

$$P_E'^1 = hv_0\gamma \, (1-\beta)$$

By putting $\gamma = \frac{h}{p}$, we can see that the frequency as seen by the mirror is:

$$v_0' = v_0 \, \gamma \, (1-\beta)$$

So, the frequencies are the same, but the directions are opposite. That means that the frequency of the reflected photon in the mirror rest frame is:

$$v_R' = v_0' = v_0 \, \gamma(1-\beta)$$

c)

We write down the four-vectors for an emitted photon, p_g^u, a reflected photon, $p_g'^u$, and the mirror, p_m^u, in terms of their respective relativistic momenta, $\overrightarrow{p_g^u}, \overrightarrow{p_g'}, \overrightarrow{p_m}$. This follows below, where I have omitted the arrow sign above the terms because it makes my text look uglier (hopefully it is ok):

$$p_g^u = (p_g, 0, 0, p_g)$$

$$p_g'^u = \left(p_g'^u, 0, 0, -p_g'^u\right)$$

$$p_m = (\gamma mc, 0, 0, p_m)$$

d)

In this particular question we are to do two calculations. They follow below.

$$p_g^u + p_m^u = \left(p_g + \gamma mc, 0, 0, p_g + p_m\right)$$

$$p_g p_m = \gamma mc\; p_{g_t} - p_{m_z} p_{g_z}$$

They don’t specify what g they want, but the point is that we must do an standard dot product. And I have denoted either t-component or z-component.

e)

We want to show that a relation holds. This relation is given by:

$$\left(p_g + p_m\right) p_g' = p_g p_m$$

What I will do here is to solve the left side of the equation and the right side of the equation individually and then compare them. We already solved the right side in the previous equation but write it down again.

$$p_g p_m = (\gamma mc) p_g - p_m p_g$$

Having done that, we go on to the left side.

$$\left(\left(p_g + \gamma mc, 0, 0, p_g + p_m\right) * \left({p'}_g, 0, 0, -p_g'\right)\right) = \left(p_g + \gamma mc\right)\left(p_g'\right) - \left(p_g + p_m\right)\left(p_g'\right)$$
$$= p_g' p_g + p_g'\,\gamma mc - \left(p_g' p_g + p_m p_g'\right) = p_g' \gamma mc - p_m p_g'$$

And since we have that the mass (knowing that momentum is given by p = mv) and velocity is the same, we conclude that the relation holds. Should also add that both left side and right side occurred in t-component and z-component respectively.

f)

Here we must find the momentum of the reflected photon in terms of the momentum of the emitted photon. We use the energy-momentum relation.

$$\left(p_g c\right)^2 + (mc^2)^2 = \left(p'_g c\right)^2 + (mc^2)^2$$

And we use the relation from the previous assignment:

$$p'_g = \frac{p_g p_m}{p_g + p_m}$$

Inserting this expression and doing some calculations lead us to the following final expression:

$$p'_g = \frac{p_g\sqrt{p_m^2 + m^2c^2} - p_g p_m}{p_m + 2p_g + \sqrt{p_m^2 + m^2c^2}}$$

g)

It is specified that in this problem the momentum of the photon is very small compared to the momentum of the mirror. We have that due to the reflection, the momentum of a photon changes, $p'_{ph} = -p_{ph}$, while total momentum of the system is conserved. Because of this the mirror must change its momentum accordingly. Having a photon with smaller momentum would lead to a smaller change of the momentum of the mirror. So, the change here in the reference frames will be smaller compared to a).

Problem 2:

a)

The task is to find the energy current density of the plane wave in terms of the electric field. The equation for the plane electromagnetic wave is given by:

$$\mathrm{E}\,(\vec{\mathrm{r}},\mathrm{t}) = \mathrm{E}_0\cos\left(\vec{\mathrm{k}}\vec{\mathrm{r}} - \mathrm{wt}\right) \quad (1)$$

Here we can use the equation for the energy current density, u, given by:

$$\mathrm{u} = \frac{1}{2}\left(\epsilon_0\,\vec{\mathrm{E}}^2\right) \quad (2)$$

Inserting (1) into (2), we get:

$$\mathrm{u} = \frac{1}{2}\left(\epsilon_0\left(\mathrm{E}_0^2\cos^2(\vec{\mathrm{k}}\,\vec{\mathrm{r}} - \mathrm{wt}\right)\right)$$

b)

Here we are to find the motion of the electron represented by an equation, $\vec{\mathrm{r}}\,(\mathrm{t})$. The limit here is non-relativistic because it is specified that $\mathrm{v} \ll \mathrm{c}$. The basic equation of motion for a charged particle in an electromagnetic field is:

$$\mathrm{ma} = \mathrm{qE} + \mathrm{q}(\mathrm{v\ x\ B})$$

We only consider the electrical field and get

$$\mathrm{ma} = \mathrm{q}\left(\mathrm{E}_0\cos(\vec{\mathrm{k}}\vec{\mathrm{r}} - \mathrm{wt})\right)$$

$$\mathrm{a} = \frac{\mathrm{q}}{\mathrm{m}}\left(\mathrm{E}_0\cos(\vec{\mathrm{k}}\vec{\mathrm{r}} - \mathrm{wt})\right)$$

For plane waves we have a constant phase surface for such waves, meaning that $\vec{k}\vec{r}$ = constant. I refer to this link for this particular relation:

http://www.iitg.ac.in/engfac/krs/public_html/lectures/ee340/2014/5_slides.pdf

(slide 5/55).

This then gives us:

$$a = \frac{q}{m}\left(E_0\cos(wt)\right)$$

We take the integral of this:

$$\int a = \int \frac{q}{m}\left(E_0\cos(wt)\right)$$

This then becomes:

$$v(t) = \frac{q}{mw}\,(E_0 \sin(wt)) + C_1$$

Integrating this again gives:

$$r(t) = -\frac{q}{mw^2}\,(E_0 \cos(wt)) + C_1 t + C_2$$

Where C_1 and C_2 are constants. We assumed that there were no collisions, it should be emphasized.

c)

In assignment 2 b) we found the velocity and we just set it as $v \ll c$ to get the expression we need. We then get:

$$\frac{q}{wm}\,E_0 \sin(wt) \ll c$$

Here we consider the velocity to be highest in either -1 or 1 and therefore set sin (wt) = 1. The final expression therefore becomes:

$$\frac{E_0}{w} \ll \frac{cm}{q}$$

Where m = electron mass.

d)

According to the text of the assignment we have to find the power P radiated by the electron. And it is also specified that the answer must be expressed in terms of the classical electron radius.

I will use the Larmor formula because it is the formula used in electrodynamics to calculate the total power by a non-relativistic point charge as it accelerates. The equation is given by:

$$P = \frac{2}{3}\,\frac{q^2 a^2}{4\pi\epsilon_0 c^3}$$

We have an equation for the acceleration given as, found earlier of course:

$$a = \frac{q}{m}\ (E_0 \cos(wt))$$

Inserting this into the equation for power gives us:

$$P = \frac{2}{3}\frac{q^4}{4\pi\epsilon_0 c^3 m^2}\ E_0^2 \cos(wt)$$

But here we have to take into account the relation for the classical electron radius, r_0.

$$r_0 = \frac{q^2}{4\pi\epsilon_0 mc^2}$$

Giving us the final expression of:

$$P = \frac{2}{3}\frac{q^2}{mc}\ r_0 E_0^2 \cos^2(wt)$$

e)

We are going to find the unit of the scattering cross section below. We've been given an equation.

$$< S > \sigma = < P >$$

$$\sigma = \frac{< P >}{< S >}$$

We have the individual units given as $S = \frac{W}{m^2}$ and $P = W$.

$$\sigma = \frac{W}{\frac{W}{m^2}} => m^2$$

m^2 is the unit for the cross section. However, it should be added that in particle physics, the conventional unit is the barn, b, with 1 b = $10^{-28}\ m^2$.

f)

The problem wants us to find the scattering cross section of a free electron. This is also called the Thomson scattering cross section. Furthermore, we must also evaluate it numerically.

So, we have the relation <P>/<S> to use here. <S> is taken from Leinaas book used in the course and <P> is found previously.

$$\sigma = \frac{\frac{2}{3}\frac{q^4}{4\pi\epsilon_0 c^3 m^2}\ E_0^2\cos^2(wt)}{\frac{1}{2}c\epsilon_0 E_0^2\cos^2(wt)}$$

This then becomes:

$$\sigma = \frac{4}{3}\frac{q^4}{4\pi\epsilon_0 c^4 m^2}$$

That further becomes:

$$\sigma = \frac{8\pi}{3}\frac{q^4}{16\pi^2\epsilon_0 c^4 m^2} => \frac{8\pi}{3}r_0^2$$

For the numerical solution, we use the value from the text of the assignment.

$$\sigma = \frac{8\pi}{3} * (2.8179 * 10^{-15}\ m)^2$$

$$\sigma = 6.65 * 10^{-29}\ m^2$$

g)

2 g) requires us to find the cross section for a bound electron. The result is pregiven in the text, and we are given a model where binding by a harmonic oscillator restoring force plus a dampening force to represent heat loss in matter:

$$F = -mw_0^2 r - m\Gamma r', F = mr''$$

We set:

$$F(t) = qE_0\cos(wt) \quad (*)$$

So, this gives us, with me substituting r(t) = x(t).

$$x''(t) + \Gamma x'(t) + w_0^2 x(t) = \frac{q}{m}E_0\cos(wt)$$

This equation has a transitory and a steady-state solution. We can easiest get the steady-state solution in the complex plane. We therefore write equation (*) as a real plus imaginary part.

$$F(t) = \bar{F}\,(\cos(wt) + i\sin(wt))$$

$$\bar{F} = F_0 \exp(i\phi)$$

And $F_0 = \frac{qE_0}{m}$, and phase modulation $\exp(i\phi) = \cos\phi + i\sin\phi$.

We do the same with the position:

$$x(t) = \bar{x}\,(\cos(wt) + i\sin(wt))$$

$$\bar{x} = x_0 \exp(i\phi)$$

The full expression for the complex solution for the electron motion is then:

$$x(t) = x_0 \exp\big(i\,(wt + \phi)\big) \;\; (**)$$

We take the real part of the equation (**) as the solution to equation (*) to get:

$$x(t) = x_0 \exp\,(wt + \phi)$$

We then proceed to determine the amplitude x_0 and phase angle ø. This is done by finding the derivatives of x(t):

$$x'(t) = iw\bar{x}\exp(iwt)$$

$$x''(t) = (iw)^2\bar{x}\exp(iwt)$$

This we substitute into equation (*) in order to get:

$$(iw)^2\bar{x} + (iw)\Gamma\,\bar{x} + w_0^2\bar{x} = \bar{F}$$

We then get the complex coefficient of x(t) and x_0 below:

$$\bar{x} = \frac{\bar{F}}{w_0^2 - w^2 + i\Gamma w}$$

$$x_0 = \frac{\left(\frac{q}{m}\right)E_0}{((w^2 - w_0^2)^2 + \Gamma^2 w^2)^{\frac{1}{2}}}$$

We insert the value for x_0 into equation (*):

$$x(t) = \left(\frac{q}{m}\right)E_0 \cos\frac{wt + \phi}{\big((w^2 - w_0^2) + \Gamma^2 w^2\big)^{\frac{1}{2}}}$$

Once steady-state is achieved, the electron oscillates at the frequency of the electric field of the incident beam. And the oscillation is out of phase with the electric field by phase angle ø.

We obtain <x''(t)> = $\frac{w^2 x_0^2}{2}$ and this gives us:

$$P = \frac{q^4 w^4}{3m^2 c^3} \frac{E_0^2}{((w^2 - w_0^2)^2 + \Gamma^2 w^2)}$$

So, we use this to find the cross section like we have done in the previous assignments.

$$\sigma(w) = \frac{\frac{q^4 w^4}{3m^2 c^3} \frac{E_0^2}{(w^2 - w_0^2)^2 + \Gamma^2 w^2}}{\frac{cE_0^2}{8\pi}}$$

This then becomes the final expression desired in the problem:

$$\sigma(w) = \frac{8\pi}{3} r_0^2 \frac{w^4}{(w^2 - w_0^2)^2 + w^2\Gamma^2}$$

h)

Previously we've found the cross sections of both free and bound electron of a function of the frequency of the incoming wave. With the given constants we draw the plots. Bot the plot and MATLAB code follows below.

```
w0 = 2e16;
d = 6e15;
r0 = 2.8179e-15;
k = ((8*pi)./3);

w = linspace(0,0.5e17);
f = (w.^4)./(((w.^2 - w0.^2).^2) + w.^2 + d.^2);

sigmafree = (k*(r0).^2);
sigmabound = (k*(r0).^2)*f;
plot(w,sigmabound,'b');
hold on
yline(sigmafree,'r');
xlabel('w');
ylabel('sigma');
title('sigma vs w');
```

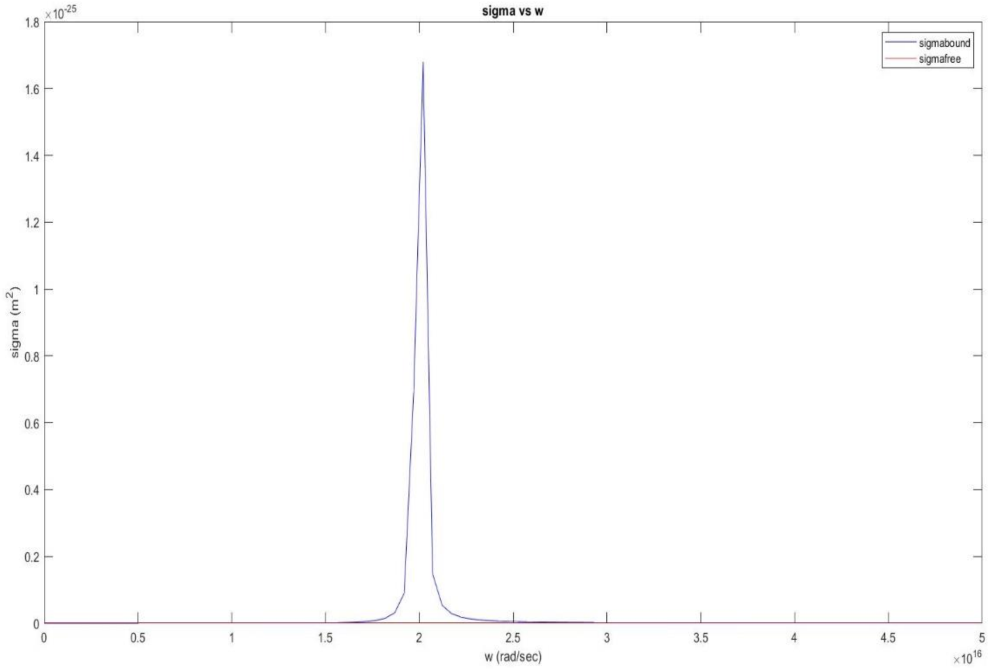

Figure 2: Schematic shows the plots of the two cross sections identified in the plots as sigmas

i)

In the following paragraphs follows an explanation of why the sky is blue.

We have previously seen the different cross sections. At very low energy $w \ll w_0$, the cross section becomes:

$$\sigma = \frac{8\pi}{3}\, r_0^2 \left(\frac{w}{w_0}\right)^4$$

This limit is called the Rayleigh scattering. The blue colour of the sky is because of Rayleigh scattering. During the light propagation through the atmosphere, the longer wavelengths pass through. We have in the visible spectra red, orange and yellow light. This passes through. Blue has light at 450 nm and has the shortest wavelength of the visible spectra. This light is absorbed by the gas molecules. The absorbed light is then radiated in different directions. We get scattering all over the sky and that is why the sky is blue.

In the cross section above shorter wavelengths are scattered the most, and longer wavelengths travel in straight lines.