Home Exam 2020 - FYS3120: Classical Mechanics and Electrodynamics

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1 Introduction

This is the home exam in Classical Mechanics and Electrodynamics course lectured at the University of Oslo done by Furkan Kaya. Its subject code is FYS3120. Due to the covid-19 virus pandemic all across the world, the work was done without any cooperation from classmates or supervisors. However, resources from Internet was used. And also Krita was used since I am sort of quarantined and I do not have the possibility to scan pages at my current location in a student dormitory.

2 Question 1: Coneheads

2.1 a)

The first assignment in this home exam has several question posed that needs to be answered. We must draw a figure of the set-up specified in the assignment, find the number of degrees of freedom, identify the appropriate generalised coordinated and relate these to the Cartesian coordinates.

Lets start with the drawing of the figure. This was done in the program called Krita and follows below in figure 1. I should also provide an explanation to figure 1. m represents the mass going in the path inside the hollow cone, α is the angle shown in the figure and required in the assignment. The particle's position is given by the polar coordinates (r, θ) of the projection of the position vector inside the x,y plane. The acceleration due to gravity is g in the negative z-direction.

Second task was to find the number of degrees of freedom. We will use the following equation in order to do so:

$$d = 3N - M \tag{1}$$

where d = degrees of freedom, 3N = number of variables and M = number of constraints. For more background, I refer to the syllabus by Leinaas. The particle can move in the x, y, z plane (standard Cartesian), but its movement is constrained by the cone. That gives us (by inserting into equation (1), the following:

$$d = 3 * 1 - 1 = 2 \tag{2}$$

2 degrees of freedom. This means that we can describe the system by standard polar coordinates, r and θ. In the drawing we had several parameters, they are as following. θ = around the cone, α = half-angle at the tip, r = the distance from the particle to the axis.

We have that if the particle's distance from the axis is r, then its height is r/(tan θ) and its distance along the cone is r/(sin θ). Having found the generalised coordinates above in the polar coordinates, we will therefore conclude this assignment by relating them to the Cartesian coordinates.

$$x = \frac{r}{sin(\theta)} \tag{3}$$

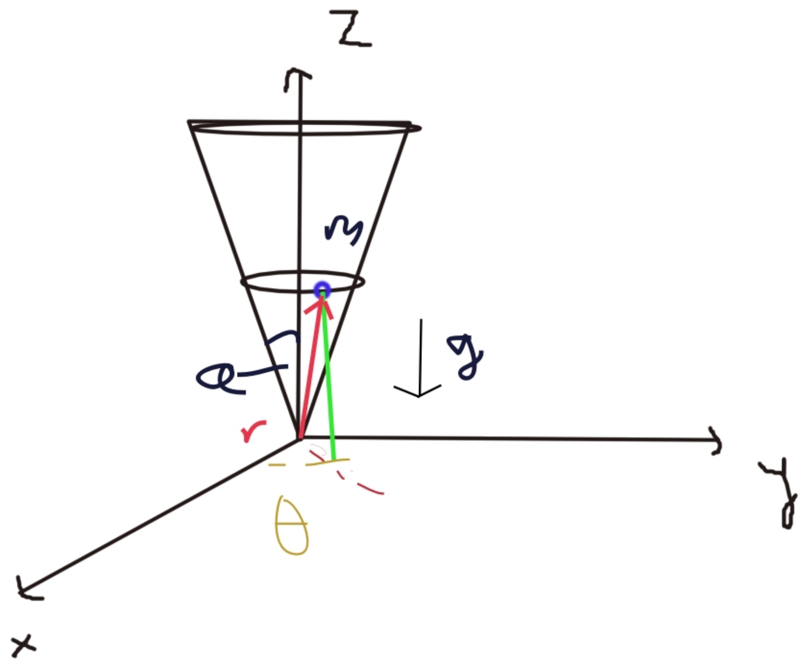

Figure 1: Schematic shows the figure of a hollow cone pointing downwards with all the parameters suggested in the text of the assignment. It includes the 3 axis, Polar coordinates, mass m, angle α and the gravity.

$$y = \frac{r}{cos\,(\theta)} \tag{4}$$

$$z = \frac{r}{tan\,(\theta)} \tag{5}$$

2.2 b)

Assignment b) wants us to find the Lagrangian. From Leinaas, we know that this is given by the relation:

$$L = T - V \tag{6}$$

representing that the Lagrangian (L) is given by the difference between kinetic (T) and potential (V) energy.

We know that the kinetic energy is given by $\frac{1}{2}mv^2$ and the potential energy in similar fashion is given by mgh. Breaking the velocity into components up long the cone and around the cone, we get for the velocity the following:

$$v^2 = \frac{\dot{r}^2}{sin^2\,(\alpha)} + r^2\dot{\theta}^2 \tag{7}$$

We have also found parts of V in the previous assignment by finding the relation between z and the polar coordinates, so this then leads us to the Lagrangian below:

$$L = \frac{1}{2}m(\frac{\dot{r^2}}{sin^2(\alpha)} + (\dot{r}\theta)^2) - mgr\frac{1}{tan\,(\alpha)} \tag{8}$$

Knowing from elementary mathematics that $cot\ x = \frac{1}{tan\,x}$, we find that the final version of the equation is written as:

$$L = \frac{1}{2}m(\dot{r^2}\,(1 + cot^2\,(\alpha)) + (r\dot{\theta})^2) - mgr\ cot\,(\alpha) \tag{9}$$

2.3 c)

The question says to find two conserved quantities in the system. Furthermore, we should show that they are conserved and what physical properties they represent. The two conserved quantities we find in the system are the angular momentum and forces. Angular momentum is the rotational equivalent of linear momentum. It is an important quantity physics because it is a conserved quantity, meaning that the total angular momentum of a closed system remains constant. We must then compare this notion with the figure we have drawn in figure 1. On that figure we can clearly see that the system is closed because we have constraints in the x,y,z-plane due to the cone. And the gravity pulls the particle down, so that it won't leave its path upwards and eject out of the system. The angular momentum is represented by θ in the equations above.

We also have conservation of forces in the system. This phenomena occurs when the mechanical system is in static equilibrium. Meaning that there is a balance between the forces that act on each part of the system, so that it has no movement. One particular aspect associated with this phenomena and relevant for our system, is that if we have a particle travelling in a closed loop, then the total work done by a conservative force is zero. As we can see on figure 1, the system is both closed and the movement is in a closed loop. And the forces are thereby conserved. Gravitation is an another example of a conservative force seen in the system.

After we've found the equations of motion in the next assignment, I will relate the conserved quantities quantitatively as well.

2.4 d)

The equation of motions are found for r and θ respectively. They follow Lagrange's equation given according to:

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{\theta}}) - \frac{\partial L}{\partial \theta} = 0 \tag{10}$$

Equation (10) is obviously only for θ, but r follows the same procedure.

We begin with θ and find the different parts in equation (10) separately.

$$\frac{\partial L}{\partial \theta} = 0 \tag{11}$$

$$\frac{\partial L}{\partial \dot{\theta}} = \frac{1}{2} m(r\dot{\theta})^2 = mr^2\dot{\theta} \tag{12}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \theta} = mr^2\ddot{\theta} \tag{13}$$

that finally becomes in Lagrange equation setup from equation (10):

$$mr^2\ddot{\theta} = 0 \tag{14}$$

This equation expresses the conservation of angular momentum from the previous assignment.

Second equation of motion considers the r parameter. We follow the same procedure as above. All variables are included below.

$$\frac{\partial L}{\partial r} = mr\dot{\theta}^2 - mg\,cot\,(\alpha) \tag{15}$$

$$\frac{\partial L}{\partial \dot{r}} = m\dot{r}\,(1 + cot\,(\alpha)) \tag{16}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = m\ddot{r}\,(1 + cot\,(\alpha)) \tag{17}$$

leading to when inserted into the Lagrange equations from (10).

$$m\ddot{r}\,(1 + cot\,(\alpha)) - mr\dot{\theta}^2 - mg\,cot\,(\alpha) = 0 \tag{18}$$

This equation represents the F = ma statement for the x-direction. We basically divide the expression by sin α and get something along the lines of:

$$\ddot{x} = (r\dot{\theta}^2)\,sin\,(\alpha) - g\,cos\,(\alpha) \tag{19}$$

Which is the same as Newtons second law for the system.

2.5 e)

It is here demanded that we must plot the phase space of the system. This is done in MATLAB and both code and plots follow below. However, before venturing on to the plots, I should emphasize that I was a bit confused by the term phase space. However, it should be clarified that it is in this course defined as: "Phase space is the 2d-dimensional space of generalized coordinates and their velocities (q, qdot), or the space of generalized coordinates and generalized momenta (q,p)." Taken verbatim from course's lecture notes. With this in mind I will continue on to plot the generalized coordinates and their velocities.

The equations are as in equations (14) and (18). Except for a small change in the expression in (14), Instead we write it as:

$$\frac{d}{dt}(mr^2\dot{\theta}) = 0 \tag{20}$$

We do this in order to integrate the expression one time. Then we get the following:

$$mr^2\dot{\theta} = C \tag{21}$$

with C being a constant. We put this into the equation and get the following expression that we use in the MATLAB-code.

$$\ddot{r} = \frac{sin^2(\alpha)}{m^2 r^3} - g\,cos\,(\alpha)\,sin\,(\alpha) \tag{22}$$

I made two programs and used the ode45-function from MATLAB to get the plot below. Code can be found in my own Github-account under the name Furkan Yekmal. The link for that is `https://github.com/FurkanYekmal/FYS3120`. A small, but relevant digression there. In our phase space there is a stable equi-

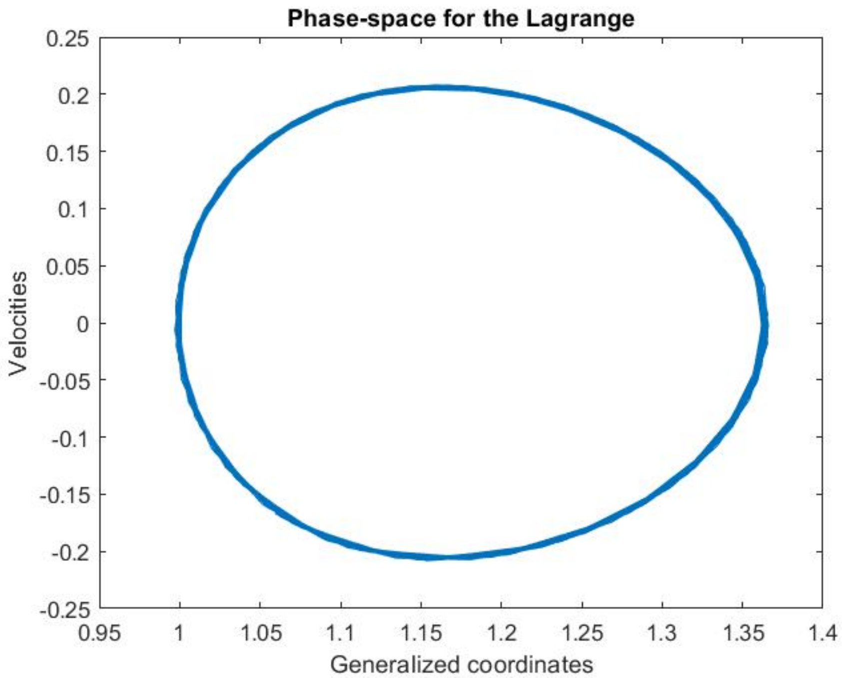

Figure 2: Schematic shows the phase space of the Lagrange. Generalized coordinates = r is on the x-axis and their velocities = r' are on the y-axis

librium when r is constant. Normally it would be both r and θ required to be constant, but we removed the latter from the equation earlier. This means that we can find the stable equilibrium by putt $\dot{r} = 0$. Here we integrate by time, because earlier we differentiated with the same parameter. We then get:

$$\dot{r} = (\frac{sin^2\,(\alpha)}{m^2 r^3} - g\,cos\,(\alpha)\,sin\,(\alpha))\,t \tag{23}$$

This expression is set as 0. And after some intermediate steps we get the following final expression for the stable equilibrium:

$$r_0 = \sqrt[3]{\frac{sin^2\,(\alpha)}{m^2(g\,cos\,(\alpha)\,sin\,(\alpha))\,t}} \tag{24}$$

Obviously I decided integrate based on t instead of θ. This was done because it double differentiation is done with t. The corresponding motion is a particle that is fixed at the r, while it moves with θ inside the hollow cone.

2.6 f)

Now we are asked to describe the motion in the rest of the phase space and find the equation of motion for a small perturbation around the stable equilibrium.

We see that the endpoints of the circle-like phase space is at r = 1 and roughly r = 1,35. We get a circular-like movement of the particle inside the hollow cone. The velocity also shows this because it varies between 0,2 and -0,2. This means that it gets to one end and back, while varying in between because of its path.

The small perturbation is given by r = $r_0 + \epsilon$. We already have $\ddot{r}$, and will below only give the new expression after expanding it about the equilibrium.

$$\ddot{r} = (\frac{sin^2(\alpha)}{m^2 r^3} - g\,cos\,(\alpha)\,sin\,(\alpha)) - (\ddot{r} = \frac{sin^2(\alpha)}{m^2 r^3} - g\,cos\,(\alpha)\,sin\,(\alpha)) * \epsilon \tag{25}$$

3 Question 2: The pi-factory

3.1 a)

Draw a figure of the setup from the assignment and answer the question about the number of collisions when M = m. The figure is seen on figure 3. With the drawing done, we look at the question associated with it. Meaning

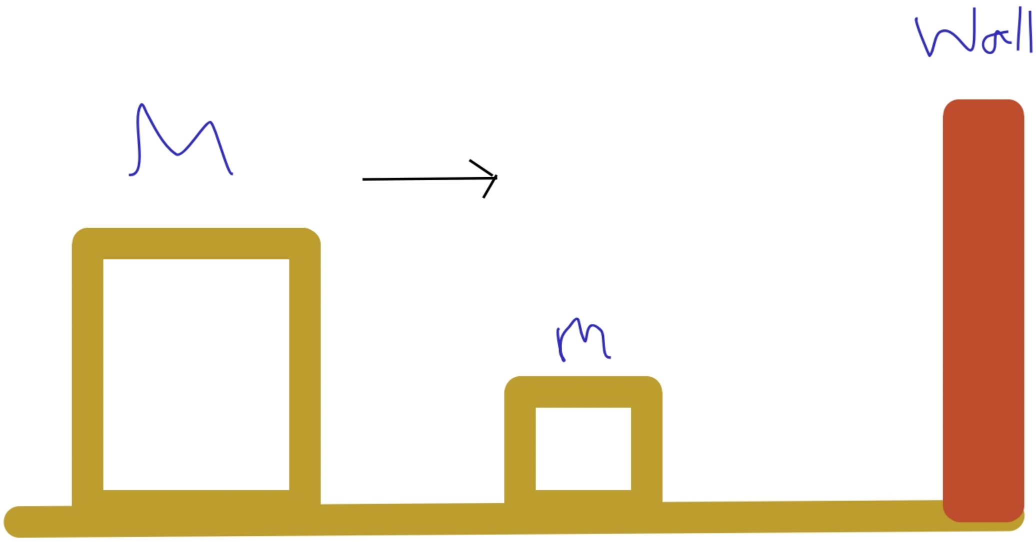

Figure 3: Schematic shows the setup as required in the text. We can see two components with masses M and m respectively. On the far-right there is a wall with infinite mass. Also worth noticing is the momentum for the component to the far-left.

that I will try to explain how this process is when M = m. First the block with mass M has a momentum and moves forward while the block with mass m stands still initially.

The block M hits the block m and transfers all its momentum to it. This means that the block M stops and the block m moves "forward" to hit the wall. The block m proceeds to collide with the wall and returns to collide with the block M and thereby transferring all its momentum to it. Meaning that block M stops and block M moves back toward the position it initially came from. Counting the collisions we get 3 collisions combined. This is a qualitative explanation.

3.2 b)

We want here to plot the phase space of the system and draw a sketch of the projection onto the 2D space (P, p), with P and p being being the large and small mass respectively. From the introductory physics-course , we know that P = mv. Important equations are the conservation of energy given as:

$$\frac{1}{2}Mv_M^2 + \frac{1}{2}mv_m^2 = constant \tag{26}$$

and the conservation of momentum given as:

$$Mv_M + mv_m = constant \tag{27}$$

They are both equal to constant because they are conserved quantities.

In our sketch we draw a two-axis diagram with P on one axis and p on an another axis. The drawing is done in Krita and it will be based on the 3 collision system from the previous assignment a) where M = m. This is done partly to simplify our task (by not having too many lines to draw) and partly for the betterment of the layout of the text. We know that the equation for conservation of energy is the equation for a circle. For reference it should be pointed out that P and p in the sketch (figure 4) are comparable to x and y in the circle equation. We can for example put M = m = $\sqrt{10}$. We also want an initial condition. Lets say that the wall is on the positive axis for x and y = 0. This is not something just made up, but based on 3 where the wall is to the right. So, we want to make a set of discrete lines within the circle given by the conservation of energy. We use the line equation from the conservation of momentum to decide where the next point of the line is going to be. But one constraint is that it has to be within the circle. We already know that M = m = $\sqrt{10}$, leading us to a slope:

$$slope = -\frac{M}{m} = -\frac{\sqrt{10}}{\sqrt{10}} = -1 \tag{28}$$

With an initial point, we get more point from the slope, as seen on figure 4. We continue with making more lines until there are no more collisions. The circle is easier to draw with M = m because we get the value for slope as 1, and by having r = 1 for circle, we get the sketch see on figure 4.

3.3 c)

Assignment says to re-scale the momenta to:

$$\hat{P} = \frac{P}{\sqrt{2M}} \ and \ \hat{p} = \frac{p}{\sqrt{2m}} \tag{29}$$

and find the angle between successive points in $(\hat{P},\ \hat{p})$.

I should mention that in the sketch from the previous assignment, the momenta had already been re-scaled, as clearly said in the text of it, so that a circle is sketched. If we had not re-scaled the figure would have been an ellipse. For the latter I politely refer to the Github-page.

Measuring an angle can be done in primarily two ways. One is manually by using a protractor or numerically. We already have a figure to relate to and we will therefore do it numerically before doing it manually for comparison. We can first refer to the wikipedia article for slope for the equation for finding the angles of slopes.

$$tan\,(\theta) = slope = \frac{rise}{run} \tag{30}$$

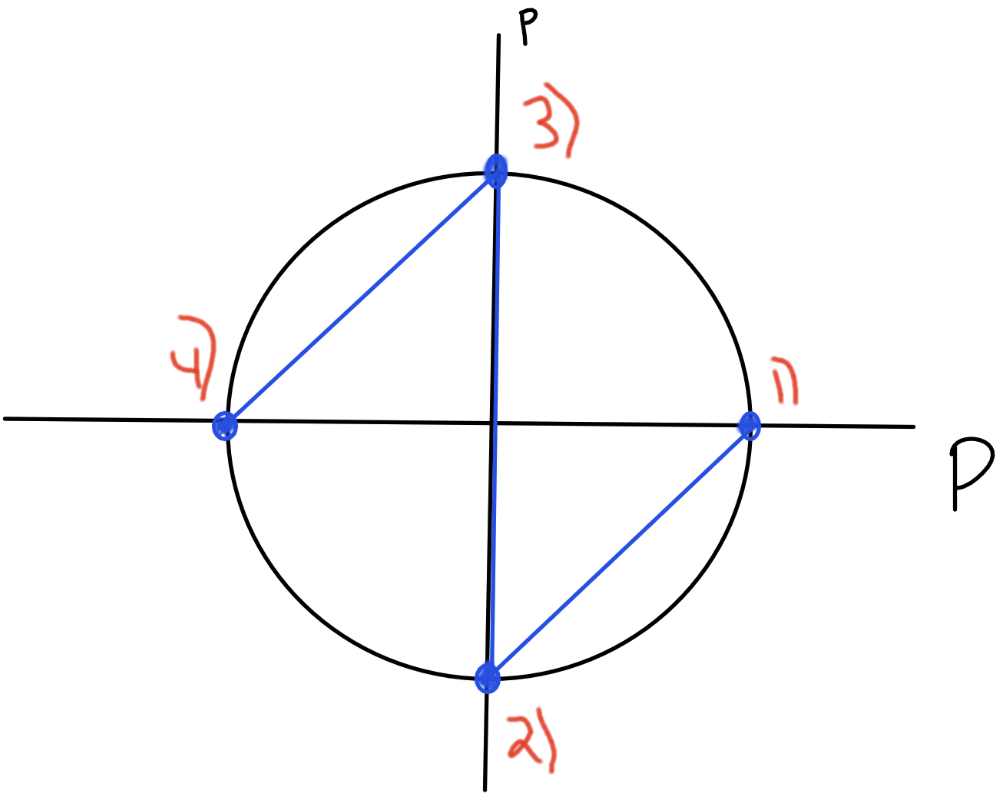

Figure 4: Schematic shows the sketch for this particular assignment (2b). We see the circle from the equation for conservation of energy, and the 3 lines for the 3 collisions. The lines come from the equation for conservation of momentum. Since we chose a slope = -1 and r = 1 for the circle, we see that the four number on the circle each represent the four points of the lines given in chronological order with 1) being the first and 4) the last.

giving us

$$tan\,(\theta) = -\frac{\sqrt{M}}{\sqrt{m}} \;=\; -1 \tag{31}$$

$$\theta = arctan\,(\frac{run}{-rise}) = arctan\,(\frac{\sqrt{m}}{\sqrt{M}}) \tag{32}$$

To get a positive tangent (for angles between 0 and 90 degrees) we mathematically use the negative of rise. This can be further confirmed by the graph. We should also say that the change in y-axis is rise and the run is the change in the x-axis. But obviously we could also write it as:

$$\theta = arctan((\frac{\frac{m}{\sqrt{2m}}}{\frac{M}{\sqrt{2M}}})) \tag{33}$$

with taking the re-scaling into consideration in an originally elliptical form. With regards to the last aspect of finding the angle, we can use the 3 collision model from above. Then we get:

$$\theta = arctan\,(1) \;=\; 45 \;\; degrees \tag{34}$$

We compare this with a manual measurement and it is pretty accurate.

3.4 d)

This is the final assignment in this midterm. It asks us to find the number of collisions for M » m. Some values for M is suggested and we are advised to understand the inscribed angle theorem.

We have seen that the angle θ is given by finding the arctan of the ratios of the two masses. Then by increasing the mass M comparable to mass m, we get a smaller angle θ. And likewise by increasing the mass m, we get a larger angle θ. A smaller angle gives us more points on the circle and therefore more collisions (I think it is safe to conclude this based on what we've done so far).

So, how is the inscribed angle theorem related to this. The theorem, as the assignment text implies, is kind of difficult to comprehend. But basically the theorem says that an angle θ inscribed in a circle is half the central angle 2θ that subtends the same arc on the circle. From this we get the figure 5 below. This figure explains that since the momentum lines have the same slope for all discrete lines, we can infer from this that by adding all 2θ (meaning all the circle arcs) we get to cover the entire circle with the given angle in radians as 2π. From this we can also calculate the amount of θ needed to reach this, where the amount of this given as N:

$$N * 2\theta \;=\; 2\pi \tag{35}$$

$$N * \theta \;=\; \pi \tag{36}$$

We then attempt to find the N with the suggested values for slope from the assignment text. The table follows below: When M is larger than then number of collisions increase proportionally to that.

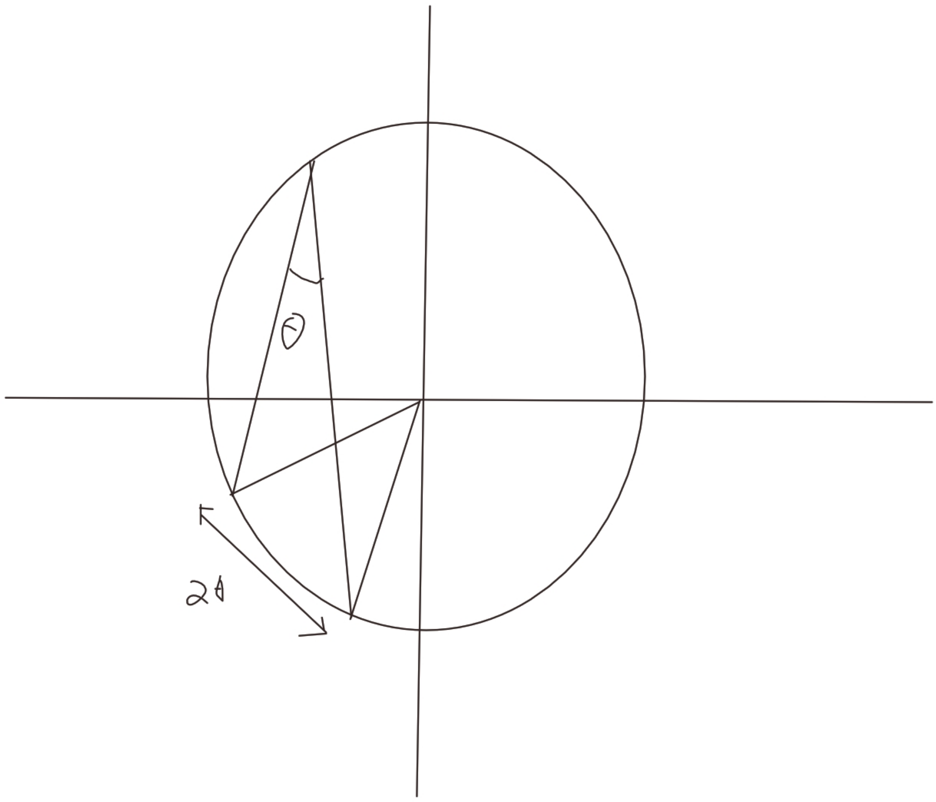

Figure 5: Schematic shows the angle θ, two discrete lines representing collisions, the central angle 2θ

Mass ratio	θ value	N
M:m	$\arctan \frac{\sqrt{m}}{\sqrt{M}}$	
100:1	0.0996687	31.52
10e4:1	9.99966669e-3	314.17
10e8:1	9.999999967e-5	31415.93

Table 1: Table shows Mass ratio in one column, θ value in the second and N in the third column