Problem set 5

In

FYS 3120

Classical Mechanics and Electrodynamics

By

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Problem 1

a)

We are asked to write down the Lagrangian L in terms of coordinates $\overrightarrow{r_1}\ and\ \overrightarrow{r_2}$ of a two-body system with objects with mass $m_1 and\ m_2$. From previous assignments we remember that the Lagrange is given by the relation: L = T – V, where T = kinetic energy and V = potential energy. From this and the parameters above we get:

$$L = \frac{1}{2}m_1\overrightarrow{r_1}'^2 + \frac{1}{2}\ m_2\vec{r}'^2 - V(\overrightarrow{r_1} - \overrightarrow{r_2})$$

It should be pointed out that the r are r’ because they are derived and represent the speed.

b)

In this assignment we have been given the following changes:

$$\vec{r} = \overrightarrow{r_1} - \overrightarrow{r_2}$$

$$\vec{R} = \frac{m_1}{m_1 + m_2}\ \overrightarrow{r_1} + \frac{m_2}{m_1 + m_2}\ \overrightarrow{r_2}$$

We will then do R to $\overrightarrow{r_1}\ og\ \overrightarrow{r_2}$ that further becomes (after some more arithmetic.):

$$\overrightarrow{r_1} = \frac{m_1 + m_2}{m_1}\ \vec{R} - \frac{m_2}{m_1}\ \overrightarrow{r_2}$$

$$\overrightarrow{r_1} = \vec{R} + \frac{m_2}{m_1}\ (\vec{R} - \overrightarrow{r_2})$$

$$\overrightarrow{r_1} = \frac{(m_1 + m_2)}{m_1}\ \vec{R} + \frac{m_2}{m_1}\ (\overrightarrow{r_1} - \vec{r})$$

$$\overrightarrow{r_1} = \vec{R} - \frac{m_2}{m_1 + m_2}\ \vec{r}$$

And we get likewise for $\overrightarrow{r_2}$:

$$\overrightarrow{r_2} = \vec{R} - \frac{m_1}{m_1 + m_2}\ \vec{r}$$

This is then inserted into the Lagrange we found in assignment a). This becomes:

$$\frac{1}{2}m_1\left(\vec{R}^2+\left(\frac{m_2}{m_1+m_2}\vec{r}\right)^2+2\vec{R}\vec{r}\frac{m_2}{m_1+m_2}\right)$$
$$+\frac{1}{2}m_2\left(\vec{R}^2+\left(\frac{m_1}{m_1+m_2}\vec{r}\right)^2-2\vec{R}\vec{r}\frac{m_1}{m_1+m_2}\right)-V(r)$$

From this we get the expression we are supposed to, meaning this:

$$\frac{(m_1+m_2)\vec{R}^2}{2}+\frac{1}{2}\,\mu\vec{r}^2-V(r)=0$$

Where R and r are the derivative, although the sign lacks above.

c)

We are solving the Lagrange equation for the new coordinates. First, we investigate the R coordinate:

$$\frac{\partial L}{\partial R'}=(m_1+m_2)\,R'$$

$$\frac{d}{dt}\frac{\partial L}{\partial R''}=0$$

The two equations above inserted into $\frac{\partial L}{\partial R'}-\frac{d}{dt}\frac{\partial L}{\partial R''}=\ 0$ gives us;

$$(m_1+m_2)\,R'=0$$

We will also look at the r coordinate as well.

$$\frac{\partial L}{\partial r'}=\mu r''$$

$$\frac{\partial L}{\partial r}=\ -V'(r)$$

$$\frac{d}{dt}\frac{\partial L}{\partial r'}=\mu r''$$

Which becomes in the Lagrange equation set up.

$$\mu r''+V'(r)=0$$

An equation for R is given as:

$$R = R_0 + Vt$$

With R_0 with as an initial condition.

d)

Before elaborating on the question posed, I refer to the following link:

https://www.google.com/url?sa=t&rct=j&q=&esrc=s&source=web&cd=18&ved=2ahUKEwiphpe3xPLnAhVlk4sKHcCJB7UQFjARegQICRAB&url=http%3A%2F%2Fwww.physics.usu.edu%2Ftorre%2F3550_Fall_2013%2FLectures%2F08.pdf&usg=AOvVaw156xbhsLxQogIkmS_CHPUr

Rather than having the Lagrange based on r_1 and r_2 for the respective bodies 1 and 2 with the masses m_1 and m_2, we have a Lagrange with the R as the center of mass position and r as the relative position. I interpret this as having L for a combined system rather than a two-part system.

Problem 2

a)

Problem 2 concerns the brachistochrone problem postulated by Bernoulli. We are supposed to find the solution to the brachistochrone problem by using the correspondence between the variational problem and Lagrange equation. Assuming an initial velocity of zero, we get:

$$\frac{1}{2}mv^2 + mgy = 0$$

$$v = \sqrt{-2gy}$$

We have the distance segment given as:

$$ds = \sqrt{1 + (y')^2}$$

The magnitude of the velocity becomes:

$$v = \frac{ds}{dt} => dt = \frac{ds}{v} = \sqrt{\frac{1 + (y')^2}{-2gy}}$$

The time T is $\int dt$. And from this we get what we are supposed to in the assignment.

$$T = \int_{x_A}^{xB} \sqrt{\frac{1+(y')^2}{-2gy}}$$

b)

It is suggested in the text of the assignment that x plays the role of t in the usual formulation. Since there is no explicit x dependency, so it indicates H as a constant motion.

$$H = \frac{\partial L}{\partial y'} - L = \frac{1}{2}\left(\frac{1+y'^2}{-2gy}\right)^{-0.5}\left(\frac{2y'}{(-2gy)}\right)y' - L = E$$

This becomes:

$$\frac{y'^2}{-2gy} - \left(\frac{1+y'^2}{-2gy}\right) = E\left(\frac{1+y'^2}{-2gy}\right)^{0.5}$$

And further:

$$\left(\frac{1}{2gy}\right)^2 = \left(E\left(\frac{1+y'^2}{-2gy}\right)\right)^2$$

$$\frac{1}{4g^2y^2} = E^2\left(\frac{1+y'^2}{-2gy}\right)$$

And putting:

$$(1+y'^2)y = \frac{1}{E^2 gy}$$

Gives us the relation we want by putting k^2 like the stuff on the right hand side.

c)

I insert (8) and (9) into (7) to see what I get. But first we will find y'.

$$y' = -\frac{1}{2}k^2\sin(\theta)$$

Inserting gives us:

$$(1 + \left(-\frac{1}{2}k^2 \sin(\theta)\right)^2 * \left(\frac{1}{2}k^2(\cos(\theta) - 1)\right) = -k^2$$

That finally becomes:

$$\frac{1}{8}\, k^2 \sin^2\theta \cos\theta = -\frac{k^2}{2} - 1$$

This is my final answer on this assignment since I could not go any further. It is 99 percent to be wrong, but I hope you will accept my total answer.

d)

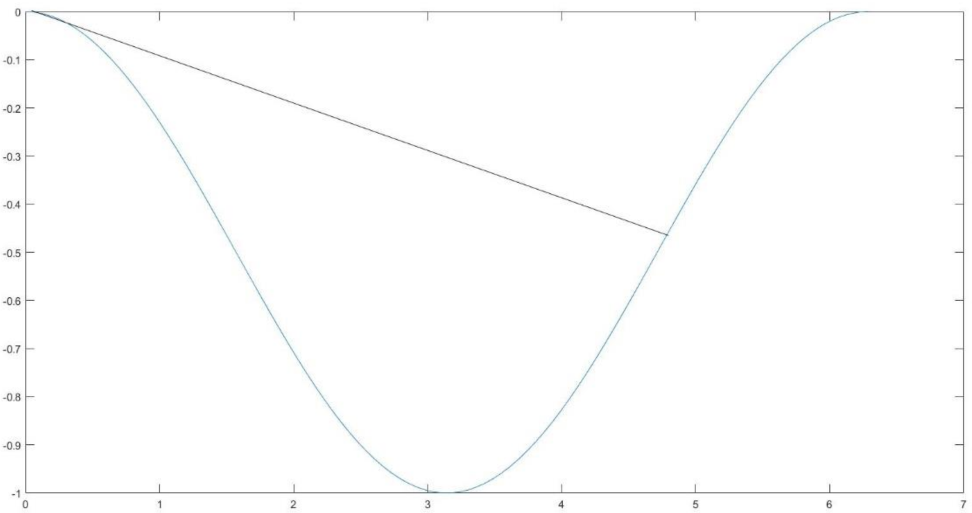
We are going to plot the curve y(θ) of the brachistochrone problem. In the assignment above we have already replaced x in y(x) with theta. I made a script in MATLAB for this particular task. The code is as following:

```
k = 1;
theta = linspace(0,2*pi);

y1 = (1./2) * k.^2 * (cos(theta) - 1);

plot(theta,y1);
```

And the plot follows below as well:



Figur 1: Schematic shows the brachistochrone plot

e)

We have the following relations:

$$\frac{dy}{d\theta} = -\frac{1}{2}k^2 \sin(\theta) = 0 => \theta = \pi$$

$$y = -k^2$$

$$x = \frac{1}{2}k^2\pi$$

$$\frac{y}{x} = -\frac{k^2}{\frac{1}{2}k^2\pi} = -\frac{2}{pi}\ x$$