Problem set 12 FYS3120: Classical Mechanics and Electrodynamics
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Problem 1 a) We have to show that the charge density and current satisfy the relation

$$\frac{\partial \lambda}{\partial t} + \frac{\partial I}{\partial z} = 0 \qquad (1)$$

and later to find λ as a function of z and t.

The current is, as specified in the text of the assignment,

$$(2) \quad I(z,t) = I_0 \cos\left(\frac{\pi z}{a}\right) \cos(wt), \quad [-a/2 < z < a/2]$$

we do a partial derivation:

$$\frac{\partial I}{\partial z} = -\frac{I_0 \pi}{a} \sin\left(\frac{\pi z}{a}\right) \cos(wt)$$

$$\frac{\partial \lambda}{\partial t} = -I_0 w \cos\left(\frac{\pi z}{a}\right) \sin(wt)$$

Inserted in (2) we get

$$-\frac{I_0 \pi}{a} \sin\left(\frac{\pi z}{a}\right) \cos(wt) - I_0 w \cos\left(\frac{\pi z}{a}\right) \cdot \sin(wt)$$

$$-I_0\left(\frac{\pi}{a}\sin\left(\frac{\pi z}{a}\right)\cos(wt) + w\cos\left(\frac{\pi z}{a}\right)\cdot\sin(wt)\right) = 0$$

$$\frac{\pi}{a}\sin\left(\frac{\pi z}{a}\right)\cos(wt) = -w\cos\left(\frac{\pi z}{a}\right)\cdot\sin(wt)$$

(3)

We set this as (3). And we know that

$\frac{\partial \lambda}{\partial t} = -\frac{\partial I}{\partial z}$. so inserted into (1)

we get $-\frac{\partial I}{\partial z} + \frac{\partial I}{\partial z} = 0$

And thereby showing that it is true.

The second part of the assignment was to find λ as a function of z and t.

Da har vi

$$\frac{\partial \lambda}{\partial t} = -w\cos\left(\frac{\pi z}{a}\right)\sin wt$$

By integration of t on both sides we get the following expression

$$\lambda(t) = -w \cos\left(\frac{\pi z}{a}\right) \cos wt + C$$

Ved $t = 0 \Rightarrow \lambda = 0$, så da har vi

$$0 = \cos\left(\frac{\pi z}{a}\right) + C$$

$$C = -\cos\left(\frac{\pi z}{a}\right)$$

That in the end becomes:

$$\lambda(t) = \cos\left(\frac{\pi z}{a}\right) \cos wt - \cos\left(\frac{\pi z}{a}\right)$$

$$\underline{\underline{\lambda(t) = \cos\left(\frac{\pi z}{a}\right)\left(\cos wt - 1\right)}}$$

b) We want to show that the electric dipole moment of the antenna has the form

$$\vec{p}(t) = p_0 \sin(wt)\, \hat{e}_z,$$

with $\hat{e}_z$ as the unit vector along the z-axis. And we will determine the constant

p_0. The electric dipole moment is given by

$$p(t) = \int_{-a/2}^{a/2} \lambda(z,t)\, z\, dz$$

For the time derivative we find

$$\dot{p} = \int_{-a/2}^{a/2} \frac{\partial I}{\partial z}\, z\, dz$$

$$= -\int_{-a/2}^{a/2} \frac{\partial}{\partial z}(zI)\, dz + \int_{-a/2}^{a/2} I\, dz$$

$$= \int_{-a/2}^{a/2} I\, dz \Rightarrow \frac{2a}{\pi} I_0 \cos(\omega t)$$

This then makes $p(t) = p_0 \sin(\omega t)\,\hat{e}_z$

with $\underline{\underline{p_0 = 2aI_0/\pi\omega}}$

c) We are tasked to use the expressions for electric dipole radiation to determine the elect and magnetic fields, and so find the type of polarization of the radiation.

The general expressions for the radiation fields are:

$$B(\vec{r}, t) = -\frac{\mu_0}{4\pi c}\frac{\hat{n}}{r} \times \ddot{\vec{p}}_{ret}$$

$$E(\vec{r}, t) = c\vec{B}(\vec{r}, t) \times \hat{n}$$

with $\vec{p}_{ret} = \vec{p}(t - r/c)$ and $\hat{n} = \vec{r}/r$

So, we find $\vec{B}$ first,

$$p_{ret} = p_0 \sin(w(t - \frac{r}{c}))$$

$$\dot{p}_{ret} = w p_0 \vec{k} \cos(w(t - \frac{r}{c}))$$

$$\ddot{p}_{ret} = -w^2 p_0 \vec{k} \sin(w(t - \frac{r}{c}))$$

This then becomes on the x-axis: $\vec{r} = r\vec{i}$

Giving us:

$$\vec{B}(\vec{r}, t) = -\frac{\mu_0}{4\pi c r}(-w^2 p_0) \sin(w(t - \frac{r}{c}))\vec{i} \times \vec{k}$$

Giving us:

$$\vec{B}(\vec{r},t) = -\frac{\mu_0 w^2 p_0}{4\pi c r} \sin\left(w\left(t-\frac{r}{c}\right)\right) \vec{i} \times \vec{k}$$

$$E(\vec{r},t) = -\frac{\mu_0 w^2 p_0}{4\pi c r} \sin\left(w\left(t-\frac{r}{c}\right)\right) \vec{k}$$

The radiation is linearly polarized in the direction of the z-axis.

Problem 2 a) We do the same as above, same procedure.

$$\vec{p} = ql(\cos wt\ \hat{e}_x + \sin wt\ \hat{e}_y)$$

$$\vec{p}' = wql(-\sin wt\ \hat{e}_x + \cos wt\ \hat{e}_y)$$

$$\vec{p}'' = w^2 ql(-\cos wt\ \hat{e}_x - \sin wt\ \hat{e}_y)$$

$$\vec{n} = (\sin\theta\cos\phi)\hat{e}_x + (\sin\theta\sin\phi)\hat{e}_y + (\cos\theta)\hat{e}_z$$

With $t = t - \frac{r}{c}$ (as before) we get:

$$\vec{B}(\vec{r},t) = -\frac{\mu_0}{4\pi c r}\left(\hat{n} \times \ddot{\vec{p}}_{ret}\right)$$

$$\hat{n} \times \ddot{\vec{p}}_{ret} = \begin{array}{ccc} \hat{e}_x & \hat{e}_y & \hat{e}_z \end{array} \left(\begin{array}{ccc} \sin\theta \cdot \cos\phi & \sin\theta \cdot \sin\phi & \cos\theta \\ \cos\omega t & \sin\omega t & 0 \end{array} \right)$$

(cross product done in matrix form)

$$\left(-\sin\omega t \cdot \cos\theta\right)\hat{e}_x - \left(\sin\omega t \sin\theta \cos\phi - \cos\omega t \sin\theta \sin\phi\right)\hat{e}_y$$

$$+ \left(\sin\omega t \sin\theta \cos\phi - \cos\omega t \sin\theta \sin\phi\right)\hat{e}_z$$

$$\vec{B} = -\frac{\mu_0 \omega^2 q l}{4\pi c r} \times \text{cross product}$$

$$B_0 = -\frac{\mu_0 \omega^2 q l}{4\pi c r}$$

$\vec{E}$ then becomes:

$$\vec{E}(\vec{r},t) = c\,\vec{B}(\vec{r},t) \times \vec{n}$$

b) The assignment requires to show that the x-direction is linearly polarized. And to find the polarization in the z-direction as well.

We take a look at the equation we found in a). In that one $\hat{e}_x$ shows one variable and that it goes in one direction. This says a linear polarization is what we get.

For the z-direction, we see that there are both x and y-plane movement. In mathematics we get the equation for circle. So circular polarization is what we see.

c) We want to find the time-averaged expression for the energy-density of the radiation.

The energy density is given by the relation

$$u = \frac{1}{2}\left(\varepsilon_0 \vec{E}^2 + \frac{1}{\mu_0}\vec{B}^2\right)$$

$$u = \frac{1}{\mu_0} \vec{B}^2 = \frac{1}{\mu_0} B_0^2 \Big(\big(\sin^2 \omega t \cos^2\theta \big) - \big(\sin \omega t \sin\theta \cos\phi - \cos\omega t \sin\theta \sin\phi \big)^2 + \big(\sin \omega t \sin\theta \cos\phi - \cos\omega t \sin\theta \sin\phi \big)^2$$

becoming

$$u = \frac{1}{\mu_0} B_0^2 \left(\cos^2\theta + \sin^2\theta \sin\left(\omega t_{ret} - \varphi\right)\right)$$

time-averaged $\overline{\sin^2 \omega t} = \frac{1}{2}$

becoming the final expression

$$\overline{u} = \frac{1}{2\mu_0} B_0^2 \left(1 + \cos^2\theta\right)$$

It is maximum along the z-direction.