<u>Mandatory assignment in MAT2200</u>

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Problem 1: How many non-isomorphic abelian groups of order $7^2 * 5^3$ are there? How many of these are cyclic and what is the order?

Answer:

We use theorem 11.12 to get the possibilities:

$$7 * 7 * 5 * 5 * 5 = 6125$$

$$49 * 5 * 5 * 5 = 6125$$

$$49 * 25 * 5 = 6125$$

$$7 * 7 * 25 * 5 = 6125$$

$$7 * 7 * 125 = 6125$$

$$49 * 125 = 6125$$

Thus there are 6 different abelian groups. We then use the corollary at 11.6 that says that the gcd must be 1 to find that there is 1 cyclic group.

The order is:

$$7^2 * 5^3 = 6125$$

Problem 2: Let R be the group of real numbers under addition and C ∗ the group of non-zero complex numbers under multiplication. Define φ : R → C ∗ by φ(r) = cos(2πr) + i sin(2πr) for r ∈ R

a) Show that φ is a homomorphism of groups. Find Ker(φ) and φ[R].

Answer:

So, we have that a function $f: G \to H$ between two groups is a homomorphism when

$$f(xy) = f(x)\, f(y) \quad (1)$$

For all x and y in G.

Here the multiplication in xy is in G and the multiplication in f(x) f(y) is in H, meaning that a homomorphism from G to H is a function that transforms the operation in G to the operation in H.

In order to be able to continue with our derivation from equation (1), we set $r = x + y$ and inserted into the main equation from the text we get:

$$f(x,y) = \cos((2\pi(x+y)) + i\sin(2\pi(x+y))$$

From Rottman it is given that the two addition laws for sine and cosine-functions are:

$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$

$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$

Further leading us to:

$$f(x,y) = \cos 2\pi x \cos\ 2\pi y - \sin 2\pi x \sin 2\pi y + i\ (\sin 2\pi x \cos 2\pi y + \cos 2\pi x \sin 2\pi y)$$

$$f(x,y) = (\cos 2\pi x + i \sin 2\pi x) \cos 2\pi y + (-\sin 2\pi x + i \cos 2\pi x) \sin 2\pi y$$

Which is, as we see, the same as f(x) f(y), where the x- function is multiplied by the y-function variable. This then gives us that the equation (2) is the same as equation (1) and that f is a homomorphism.

The second part wants us to find Ker(ø). We want to find the:

$$ø(r) = e \quad (3)$$

$$ø\ (r) = 1$$

The Kernel of ø is y = $2\pi n$, n = 0,1,2…. Meaning that it is an integer n = Z.

Final question to answer: the transfinite cardinal number is n.

b) **<u>Show that the quotient group R/Z is isomorphic to the group U = {z ∈ C | |z| = 1} endowed with multiplication as operation</u>**.

<u>Answer:</u>

The group U is the circle group. By using Euler's formula on the function given in the text of the assignment we get $ø = e^{i2\pi r}$. We have also shown in the previous assignment that R -> C is a homomorphism. Also shown was that Ker ø = Z.

With the first isomorphism theorem we get that the image of ø is isomorphic to R/Z since in the theorem it is stated that the image of ø is isomorphic to R/Ker (ø) and we have already shown that Ker (ø) = Z.

Problem 3:

a) **Show that if {Hi}i∈I is a family of normal subgroups of a group G, then the intersection T i∈I Hi is also a normal subgroup of G.**

Answer:

We take that $x \epsilon \cap_{i \epsilon I} \; H_i$ then for any g in G:

$$\forall i \epsilon I, x^g := gxg^{-1} \, \epsilon N_i \rightarrow x^g \; \epsilon \; \cap_{i \epsilon I} N_i \; \ldots ..$$

Let G = D4, the dihedral group with elements {ρ0, ρ1, ρ2, ρ3, μ1, μ2, δ1, δ2} as defined in Section 8 in Fraleigh's book

b) **Show that the elements of G can be listed as {ι, σ, σ2 , σ3 , ρ, σρ, σ2 ρ, σ3 ρ}, where σ is a rotation and ρ is a symmetry of the regular 4-gon**

Answer:

I should say that to begin with we have that the elements in G = D_4 mentioned above are those on page 80 in Fraleighs book. Our task is to show that they can be listed as in the text of the assignment b).

I will use Cayles theorem on the assignment. That means we will first make the table and then matrices. Should also add that I will use ϵ as the identity.

$$G = \{\epsilon, \sigma, \sigma^2, \sigma^3, \rho, \sigma\rho, \sigma^2\rho, \sigma^3\rho)$$

Then follows the table:

V	ϵ	σ	σ^2	σ^3	ρ	$\sigma\rho$	$\sigma^2\rho$	$\sigma^3\rho$
ϵ	ϵ	σ	σ^2	σ^3	ρ	$\sigma\rho$	$\sigma^2\rho$	$\sigma^3\rho$
σ	σ	σ^2	σ^3	ϵ	$\sigma^2\rho$	$\sigma^3\rho$	$\sigma\rho$	ρ
σ^2	σ^2	σ^3	ϵ	σ	$\sigma\rho$	ρ	$\sigma^3\rho$	$\sigma^2\rho$
σ^3	σ^3	ϵ	σ	σ^2	$\sigma^3\rho$	$\sigma^2\rho$	ρ	$\sigma\rho$
ρ	ρ	$\sigma^3\rho$	$\sigma\rho$	$\sigma^2\rho$	ϵ	σ^2	σ^3	σ
$\sigma\rho$	$\sigma\rho$	$\sigma^2\rho$	ρ	$\sigma^3\rho$	σ^2	ϵ	σ	σ^3
$\sigma^2\rho$	$\sigma^2\rho$	ρ	$\sigma^3\rho$	$\sigma\rho$	σ	σ^3	ϵ	σ^2
$\sigma^3\rho$	$\sigma^3\rho$	$\sigma\rho$	$\sigma^2\rho$	ρ	σ^3	σ	σ^2	ϵ

From this we get:

$$\epsilon = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$$

$$\sigma^2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$$

$$\sigma^3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}$$

$$\rho = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$$

$$\sigma\rho = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix}$$

$$\sigma^2\rho = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}$$

$$\sigma^3\rho = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 3 & 1 \end{pmatrix}$$

This shows that they can be listed in G.

c) Find all normal subgroups of G and indicate the subgroup diagram consisting of normal subgroups (you can draw a diagram similar to Figure 8.13 in Fraleigh or simply refer to that diagram). Give an example of subgroups K and H of G such that K E H E G but K is not normal in G.

Answer:

This question can be found on the last page. This because they were done by handwriting and required a figure as well. This figure was scanned on a printer.

Problem 4: The goal of this problem is to show that there are only two groups, up to isomorphism, of order 6, namely Z6 and S3.

a) Show first that the elements {ρ1, ρ1, ρ2, μ1, μ2, μ3} of S3 used in Fraleigh's book can be written in the form {ι, ρ, ρ2 , τ, ρτ, ρ2 τ}, where ι is the identity permutation on {1, 2, 3}, ρ = ρ1 and τ = μ1.

Answer:

We just go straight to the permutations here.

$$l = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$$

$$\rho = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

$$\rho^2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

$$\tau = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

$$\rho\,\tau = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

$$\rho^2\,\tau = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

These are the only symmetries of S_3 and show that they can be found on the desired form.

b) **Show that there is x ∈ G with x 2 =6 e. Hint: Argue by contrapositive. If x 2 = e for all x ∈ G, then G is abelian (Exercise 32, section 4). For distinct elements e, x, y in G, consider the subgroup {e, x, y, xy} and find a contradiction using Lagrange's theorem.**

Answer:

So, first we take into account that we denote by e the identity element. We know that the group G har order 6 and the elements $H = \{e, x, y, xy\}$. H is a subgroup of G. The order of a group is the number of elements in that group. We then see that H has 4 elements and therefore an order of 4. From this we see that G has an order of 6 and H has an order of 4.

A set is closed under a scalar multiplication if one can multiply any two sets and the result is still a number in the set. This is done by using $x^2 = y^2 = (xy)^2 = e$. We also have that xy = yx.

The Lagrange theorem says that the order of a subgroup H of a group G divides the order of G. It also states that the number of elements in any subgroup of a finite group must divide evenly into the number of elements in this group. From the above we have that 6/4 = 3/2, this number is obviously not an integer. This means that we have a x in G, but where $x^2 \neq e$. This is for this subgroup. If subgroup had three elements, then we would have had an order of 2.

c) **Assume x 3 6= e. Conclude, again by Lagrange's theorem, that G = H. Therefore in this case G is cyclic of order 6. Thus G ∼= Z6.**

Answer:

We have a x $\epsilon\ G$ and $x^2 \neq\ e\ let\ H\ =\ \{x\}$. We also have that x^3 ≠ e. I want to prove that G is cyclic of order 6.

We have a generator x. We assume $that\ x^2 \neq\ e$ and $x^3 \neq\ e$, meaning that they are different elements in the set. By using having x^2 ≠ e, then x^4 is also ≠ e. From this analogy we can also set x^5 in the set. This then gives us a set H as:

$$H = \{e, x, x^2, x^3, x^4, x^5\}$$

This gives H = 6. And thereby H = G. And that G is cyclic of order 6.

d) **<u>Assume now that *G* is not cyclic. Then *x*3 = *e* and *H* = h*x*i ={*e*, *x*, *x*2} is a cyclic subgroup of *G* of order 3. Choose *y* 2 *G* with *y* /2 *H*. Since (*G* : *H*) = 2, we have that *G* = *H* [*Hy* = {*e*, *x*, *x*2, *y*, *xy*, *x*2*y*}. (1) Note that all elements listed in (1) are pairwise distinct.</u>**

<u>Answer:</u>

Before venturing on to answering the assignment, I will list the parameters and constraints of the assignment given in the text of it.

- G is not cyclic
- G har order of 6
- $x^3 = e$
- H = $< x > = \{e, x, x^2\}$ is a cyclic group of order 3
- This gives us G/H = 6/3 = 2

First we are tasked to show that $y^2 = e$. Since G is finite, then y^2 must be one of the "values" in the set:

$$G = H\ \cup Hy = \{e, x, x^2, y, xy, x^2y\} \quad (4)$$

What we want to do here is to show that $y^2 = e$ by showing that y^2 ≠ the other values in the group G. We first notice that $G = H\ \cup Hy$. This means the group G consists of the values in the subgroup H in union with the subgroup H multiplied with y.

This fact help us show that if $y^2 = x$ then the generator in the group must be G = $< y >$. This is not true because it has already been stated that G $< x >$. Since y ≠ x and cannot be the generator, then it must be $y^2 \neq x^2$ also.

For y^2 to be equal to y then y must be e. Because only y = 1 multiplied with itself gives $y^2 = 1$. Since e and y are two distinct values in the set, then we know that $y^2 \neq y$. Also it must be said that $\sqrt{y^2} = \pm y$. Furthering strengthening our proposition. And all values in the group are pairwise distinct.

We are left with xy and x^2y. And we begin with xy. x^2y. We know that x is the generator of the subgroup H (which is cyclic). This means that x ≠ y because that would mean that G is cyclic. So, from that we see that $y^2 \neq xy$ since we know that y^2 is the multiplication of two components that are the same.

On x^2y I again use the same argument as for xy. We also know that $x^3 = e$. So that would mean that if $x^2y = e$, then it would mean x = y, that the group is cyclic and the everything up to now has been wrongfully stated. These factors leave us with just for option for $y^2, y^2 = e$.

Next step is to compute the multiplication table for G with the elements in (4). For comparison we compare with the results obtained in a). The multiplication table is:

	e	x	x^2	y	xy	x^2y
e	e	x	x^2	y	xy	x^2y
x	x	x^2	e	x^2y	y	xy
x^2	x^2	e	x	xy	x^2y	y
y	y	xy	x^2y	e	x	x^2
xy	xy	x^2y	y	x^2	e	x
x^2y	x^2y	y	xy	x	x^2	e

This then becomes the matrices we see on the next page.

$$e = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$$

$$x = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

$$x^2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

$$y = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

$$xy = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

$$x^2y = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

It is the same values and table, but not entirely the same in y, xy and x^2y compared to $\rho, \rho\tau, \rho^2\tau$.

e) Show that *yx* = *x*2*y* (which matches __ = _2_ in *S*3) as follows: *yx* must be one of the elements listed in (1), and all possibilities except *yx* = *x*2*y* lead to a contradiction.

Answer:

Assignment 4 e) requires that we show that yx = x^2y. And like in assignment 4 d) we will do that by showing that all other possibilities in the set are not viable. Here, we've already been given on beforehand that $yx \neq e$. That leaves us with x, x^2 and y. We start with x. Obviously we have if yx = x, then y = e because yx = x → $y = \frac{x}{x} \rightarrow y = 1$. But this is not possible because $y^2 = e \neq y$ from the previous assignment. We then go on to yx = x^2. If y = x, then we must also have G = $< y >$. This was also shown not to be true because G is not cyclic while H is.

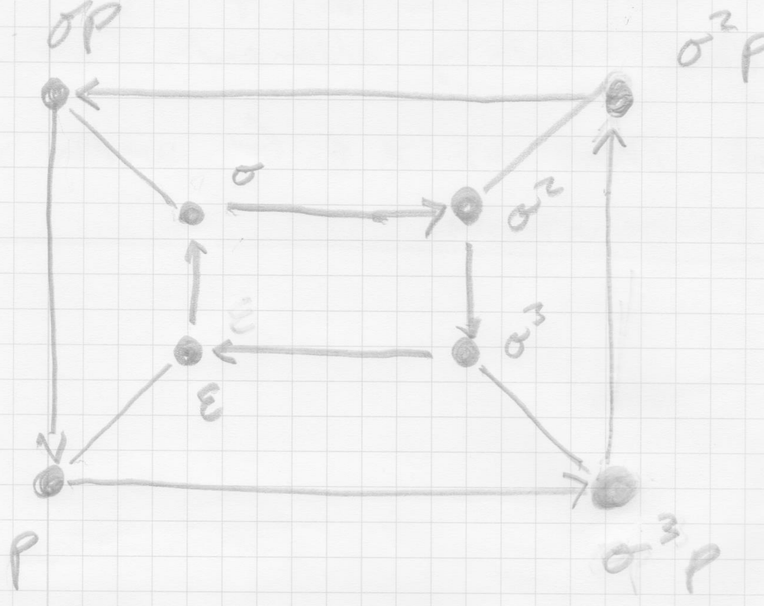
The last "value" is y. If xy = y → x =1, but x is a multiplication of e. Meaning that we are left with $yx = x^2y$.

Second part is to show that if yx = xy, then G = $< xy >$. We show that xy = yx by the law of associativity for groups. This means that if x*y, then it is equal to y*x. We have previously seen that the generator is x. If yx = xy → $x = x \ and \ y = y$, and of them must be e. Therefore, we can conclude that G = $< xy >$.

The group is then:

$$G = \{e, xy, x^2y^2, x^3y^3, x^4y^4, x^5y^5\}$$

Obviously G is now cyclic.



$\varepsilon, \sigma, \sigma^2, \sigma^3, \rho,$
$\sigma\rho, \sigma^2\rho, \sigma^3\rho$

The figure above represents a Cayley-diagram for the D_4 from the previous assignment.

We observe that ε and D_4 from the figure in the syllabus are normal subgroups. We let N be a normal subgroup of D_4. And note that:

$$\rho = x\sigma\rho x^{-1}, \quad \sigma^2\rho = x\sigma^3\rho x^{-1},$$
$$\rho\sigma\rho = \sigma^2\rho\sigma^3\rho = \sigma^2$$

Which gives us that if $\rho \in N$, then $\sigma\rho \in N$, and so on. And $\rho\sigma^2\rho = \sigma$.
Thus if N contains $\sigma^2\rho$ and $\sigma^3\rho$ and $N \neq G$, then N either contains $\rho, \sigma\rho$ or $\sigma^2\rho, \sigma^3\rho$. That then gives us the following sets.

$N = \{\varepsilon, \rho, \sigma\rho, \sigma^2\}$, $N = \{\varepsilon, \sigma^2\rho, \sigma^3\rho,$

$N = \{\varepsilon, \sigma, \sigma^2, \sigma^3\}$ and $N = \{\varepsilon, \sigma^2\}$

This leads us to have one 2-element normal subgroup, meaning

$N = \{\varepsilon, \sigma^2\}$. In addition to $\{\varepsilon\}$ and D_4.

We then want the diagram consisting of normal subgroups. From the diagram in Fraleigh, we know that it is:

$$\begin{array}{c} D_4 \ \text{(same)} \\ | \\ \{\varepsilon, \sigma^2\} \ (\rho_0, \rho_2 \text{ in Fraleigh}) \\ | \\ \varepsilon \ (\rho_0 \text{ i Fraleigh}) \end{array}$$

The final part of the assignment was to find subgroups K and H of G.