4th March, 2020

MAT2200

Mandatory assignment 1 of 1

Submission deadline

Thursday 19th MARCH 2020, 14:30 in Canvas (canvas.uio.no).

Instructions

You can choose between scanning handwritten notes or typing the solution directly on a computer (for instance with LaTeX). The assignment must be submitted as a single PDF file. Scanned pages must be clearly legible. The submission must contain your name, course and assignment number.

It is expected that you give a clear presentation with all necessary explanations. Remember to include all relevant plots and figures. Students who fail the assignment, but have made a genuine effort at solving the exercises, are given a second attempt at revising their answers. All aids, including collaboration, are allowed, but the submission must be written by you and reflect your understanding of the subject. If we doubt that you have understood the content you have handed in, we may request that you give an oral account.

In exercises where you are asked to write a computer program, you need to hand in the code along with the rest of the assignment. It is important that the submitted program contains a trial run, so that it is easy to see the result of the code.

Application for postponed delivery

If you need to apply for a postponement of the submission deadline due to illness or other reasons, you have to contact the Student Administration at the Department of Mathematics (e-mail: studieinfo@math.uio.no) well before the deadline.

All mandatory assignments in this course must be approved in the same semester, before you are allowed to take the final examination.

Complete guidelines about delivery of mandatory assignments:

uio.no/english/studies/admin/compulsory-activities/mn-math-mandatory.html

GOOD LUCK!

To pass the assignment, you need a score of 60 %.

Problem 1. (10 points)

How many non-isomorphic abelian groups of order $7^2 \cdot 5^3$ are there? How many of these are cyclic and what is their order?

Problem 2. (10 points) Let $\mathbb{R}$ be the group of real numbers under addition and $\mathbb{C}^*$ the group of non-zero complex numbers under multiplication. Define $\phi : \mathbb{R} \to \mathbb{C}^*$ by $\phi(r) = \cos(2\pi r) + i\sin(2\pi r)$ for $r \in \mathbb{R}$.

(a) Show that ϕ is a homomorphism of groups. Find $\mathrm{Ker}(\phi)$ and $\phi[\mathbb{R}]$.

(b) Show that the quotient group $\mathbb{R}/\mathbb{Z}$ is isomorphic to the group $U = \{z \in \mathbb{C} \mid |z| = 1\}$ endowed with multiplication as operation.

Problem 3. (20 points)

(a) Show that if $\{H_i\}_{i\in I}$ is a family of normal subgroups of a group G, then the intersection $\bigcap_{i\in I} H_i$ is also a normal subgroup of G.

Let $G = D_4$, the dihedral group with elements $\{\rho_0, \rho_1, \rho_2, \rho_3, \mu_1, \mu_2, \delta_1, \delta_2\}$ as defined in Section 8 in Fraleigh's book.

(b) Show that the elements of G can be listed as

$$\{\iota, \sigma, \sigma^2, \sigma^3, \rho, \sigma\rho, \sigma^2\rho, \sigma^3\rho\},$$

where σ is a rotation and ρ is a symmetry of the regular 4-gon.

(c) Find all normal subgroups of G and indicate the subgroup diagram consisting of normal subgroups (you can draw a diagram similar to Figure 8.13 in Fraleigh or simply refer to that diagram). Give an example of subgroups K and H of G such that $K \trianglelefteq H \trianglelefteq G$ but K is not normal in G.

Problem 4. (60 points)

The goal of this problem is to show that there are only two groups, up to isomorphism, of order 6, namely $\mathbb{Z}_6$ and S_3.

(a) Show first that the elements $\{\rho_1, \rho_1, \rho_2, \mu_1, \mu_2, \mu_3\}$ of S_3 used in Fraleigh's book can be written in the form $\{\iota, \rho, \rho^2, \tau, \rho\tau, \rho^2\tau\}$, where ι is the identity permutation on $\{1, 2, 3\}$, $\rho = \rho_1$ and $\tau = \mu_1$.

Let G be a group of order 6. Denote by e the identity element.

(b) Show that there is $x \in G$ with $x^2 \neq e$. Hint: Argue by contrapositive. If $x^2 = e$ for all $x \in G$, then G is abelian (Exercise 32, section 4). For distinct elements e, x, y in G, consider the subgroup $\{e, x, y, xy\}$ and find a contradiction using Lagrange's theorem.

Now let $x \in G$ be such that $x^2 \neq e$ and let $H = \langle x \rangle$.

(c) Assume $x^3 \neq e$. Conclude, again by Lagrange's theorem, that $G = H$. Therefore in this case G is cyclic of order 6. Thus $G \cong \mathbb{Z}_6$.

(d) Assume now that G is not cyclic. Then $x^3 = e$ and $H = \langle x \rangle = \{e, x, x^2\}$ is a cyclic subgroup of G of order 3. Choose $y \in G$ with $y \notin H$. Since $(G : H) = 2$, we have that

$$G = H \cup Hy = \{e, x, x^2, y, xy, x^2y\}. \tag{1}$$

Note that all elements listed in (1) are pairwise distinct.

Show that $y^2 = e$ as follows: since G is finite, the element y^2 must be one of e, x, x^2, y, xy, x^2y, and only the first possibility, $y^2 = e$, does not lead to a contradiction. For example, if $y^2 = x$, show that $G = \langle y \rangle$. This contradicts our assumption that G is not cyclic. If $y^2 = xy$, show that $y \in H$, contradicting the choice of y. The remaining possibilities are similar.

Assume now we have proved $y^2 = e$. The next step is to compute the multiplication table for G with the elements as listed in (1) and show that it is the same as the multiplication table for S_3 as expressed in part (a) of this problem. Therefore an isomorphism $G \cong S_3$ is obtained by mapping e to ι, x to ρ, y to τ, and similarly for the remaining three elements.

(e) Show that $yx = x^2y$ (which matches $\tau\rho = \rho^2\tau$ in S_3) as follows: yx must be one of the elements listed in (1), and all possibilities except $yx = x^2y$ lead to a contradiction. For example, if $yx = e$ then $y = x^{-1} \in H$, a contradiction. Prove in a similar way that yx cannot be x, x^2 or y. Show also that if $yx = xy$ then $G = \langle xy \rangle$, which contradicts our assumption that G is not cyclic. Finally, compute the remaining products of elements in G.

Problem 5. (Optional. Does not count towards the total score.) Consider the symmetric group S_n on n letters $\{1, 2, \ldots, n\}$ for $n \geq 2$. Given $\sigma \in S_n$, an ordered pair (k, l) with $1 \leq k < l \leq n$ is called an inversion if $\sigma(k) > \sigma(l)$. Let $\mathrm{inv}(\sigma)$ denote the number of inversions associated to σ. Then σ is called even (or odd) if $\mathrm{inv}(\sigma)$ is even (or odd, respectively). Prove that a transposition $\sigma = (i, j)$, for $i, j \in \{1, 2, \ldots, n\}$, $i \neq j$, is an odd permutation.

Hint: Without loss of generality assume that $i < j$. Then, for a given ordered pair (k, l) with $1 \leq k < l \leq n$, determine if it is an inversion by comparing k, l with i, j. For example, if $k, l \notin \{i, j\}$ then $\sigma(k) = k < l = \sigma(l)$, so such pairs (k, l) are not inversions, and similarly for the case $k < l = i < j$. Consider the remaining cases, and count the number of inversions.