UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in MAT2200 — Groups, rings, and fields. Re-exam

Day of examination: take-home exam August 2020

Examination hours: 9:00 (exam start) – 9:00 (7 days later)

This problem set consists of 3 pages.

Appendices: All.

Permitted aids: All.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Important: This is a take-home exam. You must provide justification for all your answers. You may deliver the solution to the exam written in Norwegian or English. For a short dictionary: field is "kropp", splitting field is "rotkropp", degree of an extension is "tallgrad", homomorphism is "homomorfi".

There are 15 subproblems distributed over 5 main problems. The 15 subproblems have equal weight. To pass the exam, you need to solve correctly 40% of the total of 15 subproblems.

Problem 1

(The questions (a) and (b) in this problem are not related to each other.)

1a

Consider the permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 3 & 10 & 4 & 7 & 8 & 5 & 1 & 6 & 2 & 9 \end{pmatrix}$$

in S_{10}. Write σ as a product of disjoint cycles. Determine if σ is an even or an odd permutation.

1b

Show that the following eight matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$
$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(Continued on page 2.)

form a group under matrix multiplication. Show that this group is isomorphic to the dihedral group D_4.

Problem 2

Let $(G, +)$ and $(G', +')$ be abelian groups. Let $\mathrm{Hom}(G, G')$ denote the set of all group homomorphisms from G to G'. Define an operation $*$ on $\mathrm{Hom}(G, G')$ as follows: for $\phi, \psi \in \mathrm{Hom}(G, G')$ let $\phi * \psi : G \to G'$ be given by

$$(\phi * \psi)(x) = \phi(x) +' \psi(x) \text{ for all } x \in G.$$

2a

Show that $\phi * \psi$ belongs to $\mathrm{Hom}(G, G')$ for all $\phi, \psi \in \mathrm{Hom}(G, G')$. Show that $(\mathrm{Hom}(G, G'), *)$ is an abelian group.

2b

Suppose that $G = \mathbb{Z}$. Show that $\mathrm{Hom}(\mathbb{Z}, G')$ is isomorphic to G'. Hint: consider the map $F : \mathrm{Hom}(\mathbb{Z}, G') \to G'$ given by $F(\phi) = \phi(1)$ for $\phi \in \mathrm{Hom}(\mathbb{Z}, G')$.

2c

Let $G = \mathbb{Z}$ and $G' = \mathbb{Z}_6$. List all the possible ring homomorphisms from $\mathbb{Z}$ to $\mathbb{Z}_6$. Hint: find $\phi \in \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}_6)$ such that $\phi(1)^2 = \phi(1)$.

Problem 3

Let G be a group of order $|G| = 225$.

3a

Show that G has a unique Sylow p-subgroup for each prime p that divides $|G|$. Explain why G is not a simple group.

3b

Show that G is the direct product of its Sylow p-subgroups. Show that G is abelian.

Problem 4

Let F be a field and let $F[x]$ be the ring of polynomials in one indeterminate x with coefficients in F.

(Continued on page 3.)

4a

Explain what it means for a polynomial $g(x)$ in $F[x]$ to be irreducible over F.

Let $f(x) = x^3 + cx + 3$ in $\mathbb{Z}_5[x]$, where c is an element in the field $\mathbb{Z}_5$.

4b

Find all values of c in $\mathbb{Z}_5$ such that $f(x)$ is irreducible over $\mathbb{Z}_5$.

4c

Find all values of c in $\mathbb{Z}_5$ such that the quotient ring $\mathbb{Z}_5[x]/\langle f(x)\rangle$ is a field.

4d

For the values of c found in part 4c give a basis for the field $\mathbb{Z}_5[x]/\langle f(x)\rangle$ seen as a vector space over $\mathbb{Z}_5$, and find the number of elements in this field.

Problem 5

Let $f(x) = x^3 - 5$ in $\mathbb{Q}[x]$.

5a

Show that $f(x)$ is irreducible over $\mathbb{Q}$.

5b

Find the splitting field K of $f(x)$ over $\mathbb{Q}$ and the degree $[K : \mathbb{Q}]$.

5c

Show that the Galois group $G(K/\mathbb{Q})$ is isomorphic to S_3.

5d

Set up the Galois correspondence between subgroups of $G(K/\mathbb{Q})$ and subfields $\mathbb{Q} \leq E \leq K$. List all the normal extensions E of $\mathbb{Q}$.

END.