MAT2400

Real Analysis

By

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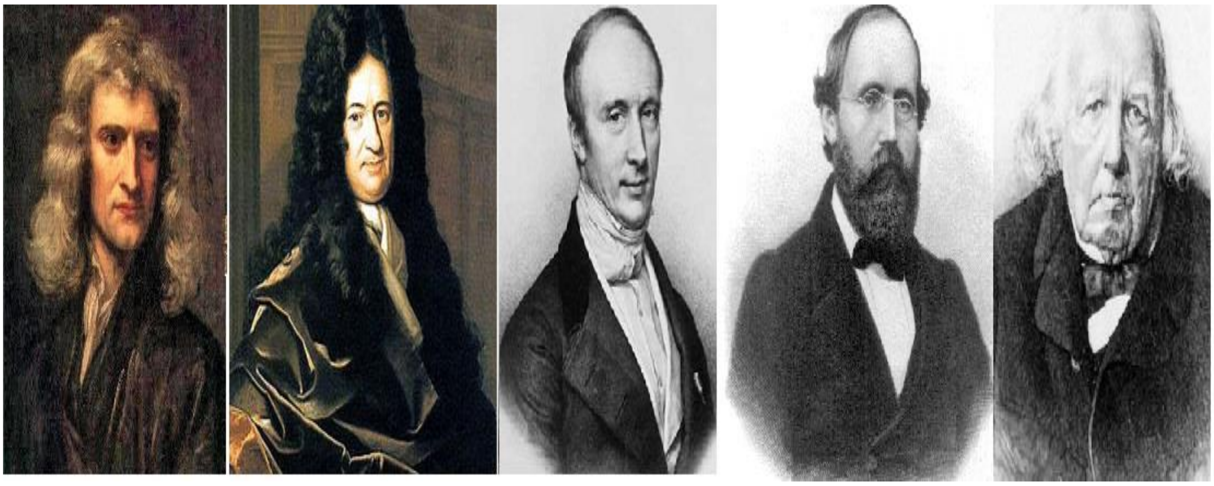


Figure 1: shows important mathematicians within the field of Real Analysis. From left to right: Newton, Leibniz, Cauchy, Riemann and Weierstrass

Problem 1:

a)

We are supposed to make a drawing of the Ball, B (x; r) with the given metrics:

$$d_1\ (x, y) = |x_1 - y_1| + |x_2 - y_2|$$

This is a standard ball in R^2. The drawing can be seen in figure 2.

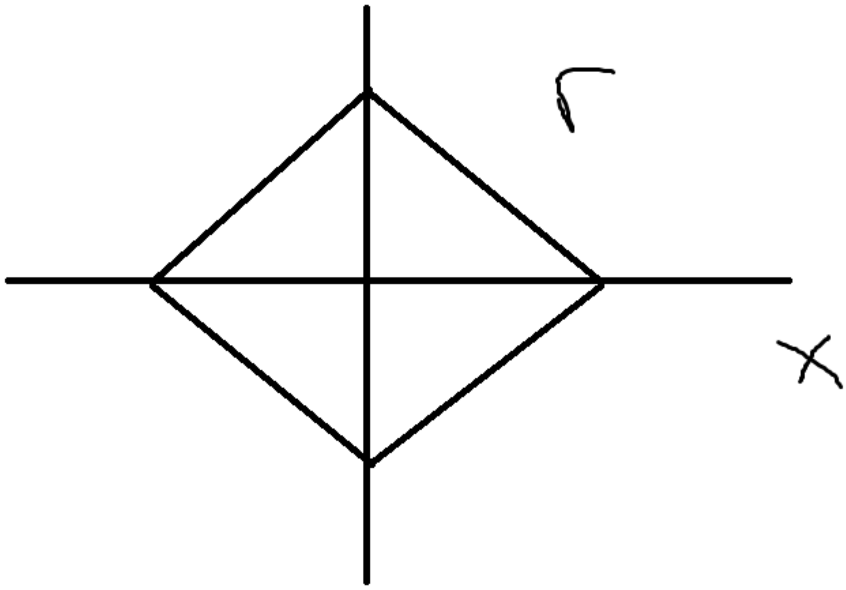


Figure 2: shows the metrics for assignment a)

b)

Same procedure as in the previous assignment. The metrics is:

$$d_2\ (x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$$

Drawing is seen in figure 3.

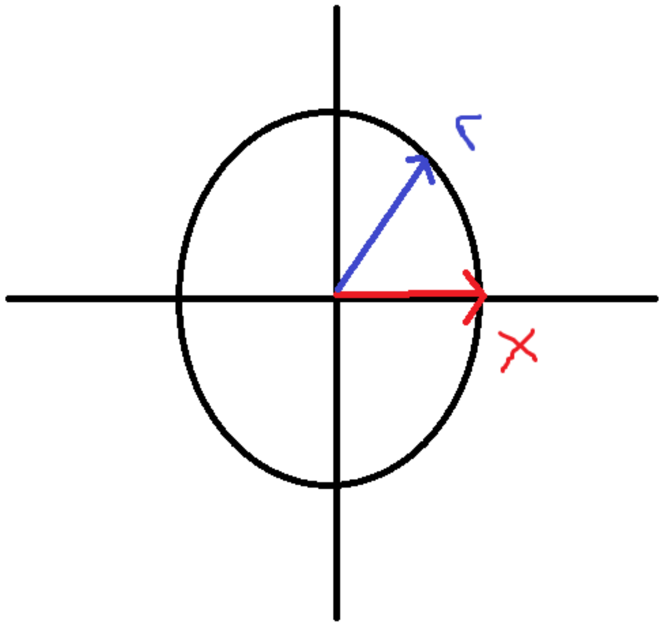


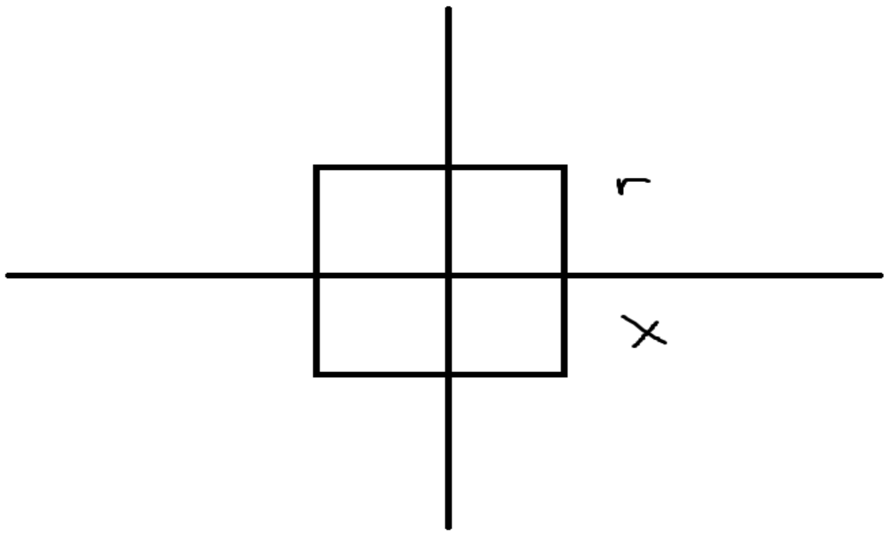
Figure 3: shows the parameters for the metrics in assignment b)

A basic circle is seen on figure 3.

c)

The third assignment has a new metrics given by:

$$d_\infty(x, y) = \max(|x_1 - y_1|, |x_2 - y_2|)$$



Figur 4: drawing shows the parameters from assignment c)

Problem 2:

a)

We are to show that d_1 and d_2 are equivalent metrics. And from the first Problem we know that the respective d' are given as:

$$d_1\,(x,y) = |x_1 - y_1| + |x_2 - y_2|$$

$$d_2\,(x,y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$$

In attempting to do this I intend to show that they are equivalent metrics by showing that they have the same convergent sequences. As mentioned in Introduction to Topology by Gamelin. And we know from Lindstrom that a geometrical definition of convergent sequences is that we can have a ball B around a with radius r and that means that the convergent sequence converges to a if the elements in the set are inside B.

What do we do from here? We draw both metrics in the same plot on MATLAB with the same a. So, if all the elements in the two metrics are inside the same ball, we have a convergence and an equivalence between the two metrics. The code is:

```
syms d1(x1,x2,y1,y2);
syms d2(x1,x2,y1,y2);
a = x1 - y1;
b = x2 - y2;

d1 = @(a,b) abs(a) + abs(b);
d2 = @(a,b) sqrt(a.^2 + b.^2);


gopster=fcontour(d1,'r');
hold on
gopster2=fcontour(d2,'b');
hold off


gopster.LevelList = 1;
gopster2.LevelList = 1;
xlabel x
ylabel y
grid on
gopster.YRange = [-1,1];
gopster.XRange = [-1,1];
axis equal
gopster2.YRange = [-1,1];
gopster2.XRange = [-1,1];
axis equal
```

Followed by a plot.

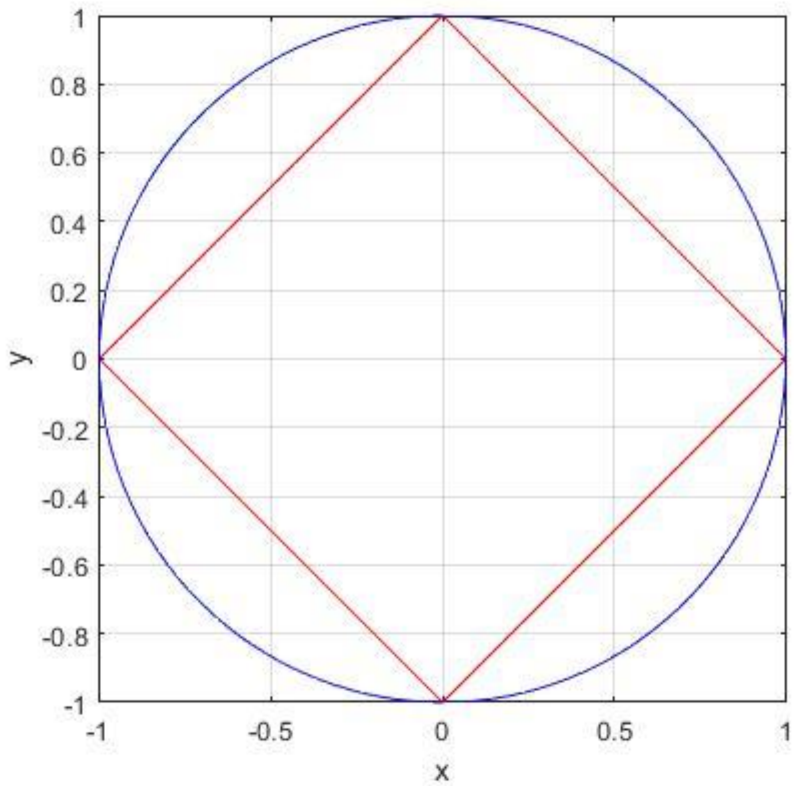


Figure 5: Schematic shows the d1 and d2 Balls. We see that the d1 Ball is completely inside the d2 and thereby suggesting same converging sequence and thereby equivalence.

Figure 5 shows the plot. We can see that the d1, given in red, has its values completely inside the d2, given in blue. This suggest that they converge toward the same number, a = 0, and therefore an equivalence is seen.

b)

We want to show that one of the d_1 or d_2 is not equivalent to the discrete metric given by:

$$d(x, y) = 0 \; if \; x = y, \qquad or \quad d(x, y) = 1 \; if \; x \neq y$$

It is suggested in the remarks that equivalent metrics share properties like continuity. In this assignment, this remark will be used to show that they are not equivalent. Definition 3.2.4 in Lindstrom's book says the following: "Assume that (X, d_X), (Y, d_Y) are two metric spaces. A function f: X → Y is continuous at a point a € X if for every $\epsilon > 0$ there is a $\delta > 0$ such that d_Y (f(x), f(a)) < ϵ whenever d_X (x,a) < δ.

It is inferred then that the definition for metric spaces is exactly the same as the usual definitions for functions of one or several variables. The only difference is that one now has metric instead to measure the distance. Therefore, it is valid to suggest that a continuous function is a function that does not have any abrupt changes in value, known as discontinuities. While a discrete metric space, as suggested by the equation, is a topological space where the points form a discontinuous sequence.

We therefore have that the discrete space is proven. So, we need to show that d1 or d2 is throughout the interval. Due that we refer to either figure 5 or figures 1 and 2. They both show continuity throughout the given domain and thereby that they are not equivalent to the discrete metric since they are continuous rather than discontinuous.

c)

The question posed wants us to prove that if ρ and γ are equivalent metrics on X, then (X, ρ) and (X, γ) have the same and closed sets. I will here rely on Propositions 3.3.3 and 3.3.4 from Lindstroms book. Proposition 3.3.3 states that a subset of a metric space X is open only if it consists of interior points (a bit rephrased). Proposition 3.3.4 states that a subset A of a metric room X is open only if its complement A^C is closed.

We have that the interior points of a set of one metrics will be the interior points of any other equivalent metrics. That means that all the interior points of one metric are also the interior points of all equivalent metrics. Thereby suggesting that based on Proposition 3.3.3 both metrics have the same open sets. With regards to closed sets and equivalent metrics we rely on Proposition 3.3.4. A set is closed if its complement is open. That means then that if a set is closed, then all its equivalent metric are also closed.

d)

The assignment tasks us to prove that if ρ and γ are equivalent metrics, then we can say the following: If $\{x_n\}_n$ is a sequence in X, then $\{x_n\}_n$ converges in (X, ρ) if and only if it converges in (X, γ).

We know from Problem 2 c) that when ρ *and* γ are equivalent metrics, the interior points of one metric are also the interior points of another metric. And we remember from Proposition 3.2.3 that a sequence in one metric cannot converge to more than one point. Those two aspects suggest that since ρ *and* γ have the same interior points and they converge to one point, we have that they must converge in both metrics in order to converge in one. And thereby inferring that for a sequence in X to converge in (X, ρ) it must converge in (X, γ).

e)

The intention here is to prove that if ρ and γ are equivalent metrics then (X, ρ) and (X, γ) have the same compact subsets. We again point to the remark suggesting that equivalent metrics share the continuity of function. Furthermore, Proposition 3.5.9 suggests that the (direct) image of a compact set under a continuous function is always compact. And the term direct image suggests that the "same" can also be used.

Problem 3:

a)

First assignment in Problem 3 requires us to compute and draw the graphs of f_k^- and f_k^+ for the function given within the domain. And we have also been given two identity functions in the text. I interpret them as below:

$$1 \; when \; x \in (-\infty, 0]$$

$$0 \quad when\ x \in [0, \infty)$$

That was the first one. The second is (hopefully I make myself understood with my notation).

$$1\ when\ x \in [0,2]$$

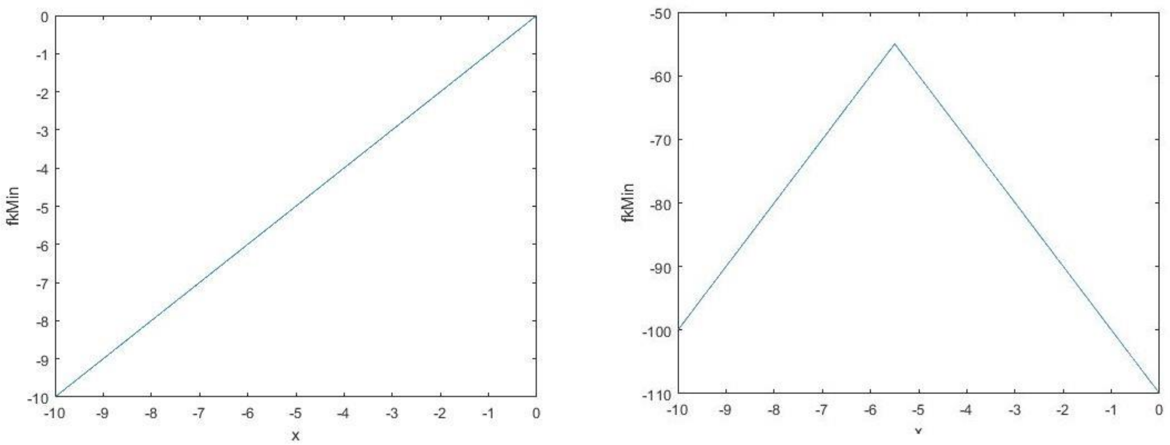
$$0\ when\ x \in [0,2]$$

```
x = linspace(-10,0,1000);
k = 10;
feval = x;
fkMin = zeros(1,1000);

for i = 1:1000
    fkMin(i) = min(feval + k*abs(x(i)-x));
end

plot(x,fkMin);
xlabel('x');
ylabel('fkMin');
```

The plots for the first function follows below. Meaning that lower bound comes first, then the second.



(I) (2)

Figure 6: Schematic of the first function mentioned. (1) is the lower bound and (2) is the upper bound.

For the second function, we get an identical shape, but with a different interval. The interval is now between [0,2]. This plot was not included here.

b)

We want to show that

$$f_k^-\ (x) \leq f(x) \leq f_k^+\ (x)\ for\ every\ x \epsilon X$$

In explaining this I will make use of Proposition 3.4.4. This proposition says that: Assume that (X, d) is a complete metric space. If A is a subset of X, (A, d_A) is complete if and only if A is closed.

We assume that the space is bounded and closed. We also assume that it is a convergent sequence. One reason for that is that the boundary limits are clearly defined in the identity function. We basically have that the sequence converges to a in (A, d_A). Then we know that the value of the function must within the space (subset) suggested for it. Here we have negative space for f_k^-, the boundary of f(x) gives a domain of space smaller than f_k^+ and bigger than f_k^-. Meaning that we have show what we wanted to show.

c)

In this particular assignment, I want to refer to the plots that I made on figure 6. They clearly indicate an approximation dominated by f(x). In fact for the lower end, they are identical. This proves that the statement is correct for the lower bound approximation.

For the upper bound this is less clear from the plots. The Moreau-Yosida approximations are traditionally convex shaped from research I’ve seen. I do not have a valid answer here sadly.