LØSNINGSFORSLAG

UNIK4540 Matematisk modellering av dynamiske systemer

Oddvar Hallingstad

20040302

OPPGAVESETT 1

Oppgave 1

a) *Det er gitt et sett med basisvektorer* $\{\vec{p}_1, \vec{p}_2\}$ *som utspenner planet. Vinkelen fra* $\vec{p}_1$ *til* $\vec{p}_2$ *er* 30° *og deres lengder er henholdsvis* 1 *og* 2*. Finn den duale basis og benytt den til å dekomponere en vektor* $\vec{r}$ *med lengden 1 i basisen* $\{\vec{p}_1, \vec{p}_2\}$*. Vinkelen fra* $\vec{p}_2$ *til* $\vec{r}$ *er* 30°*.*

Løsning: **Teori**:

Duale basis:

$$\langle \vec{p}_i, \vec{p}_j \rangle = \delta_{ij} \tag{1}$$

hvor det indre produktet i $\overrightarrow{R}^2$:

$$\begin{aligned}
\left\langle \vec{p}_i, \vec{p}_j^* \right\rangle &= |\vec{p}_i||\vec{p}_j^*| \cos(\angle \vec{p}_i, \vec{p}_j^*) && (2)\\
\langle \vec{p}_1, \vec{p}_1^* \rangle &= 1 = |\vec{p}_1||\vec{p}_1^*| \cos(\angle \vec{p}_1, \vec{p}_1^*) && (3)\\
\langle \vec{p}_1, \vec{p}_2^* \rangle &= 0 = |\vec{p}_1||\vec{p}_2^*| \cos(\angle \vec{p}_1, \vec{p}_2^*) \Longrightarrow \angle \vec{p}_1, \vec{p}_2^* = \pm\frac{\pi}{2} && (4)\\
\langle \vec{p}_2, \vec{p}_1^* \rangle &= 0 = |\vec{p}_2||\vec{p}_1^*| \cos(\angle \vec{p}_2, \vec{p}_1^*) \Longrightarrow \angle \vec{p}_2, \vec{p}_1^* = \pm\frac{\pi}{2} && (5)\\
\langle \vec{p}_2, \vec{p}_2^* \rangle &= 1 = |\vec{p}_2||\vec{p}_2^*| \cos(\angle \vec{p}_2, \vec{p}_2^*) && (6)
\end{aligned}$$

Utregning:

$$\cos(\angle \vec{p}_1, \vec{p}_1^*) > 0 \Longrightarrow \angle \vec{p}_1, \vec{p}_1^* \in \left\langle -\frac{\pi}{2}, \frac{\pi}{2} \right\rangle \tag{7}$$

$\Longrightarrow$retningen på $\vec{p}_1^*$ er bestemt. Se i Figure: 1.

$$\angle \vec{p}_1^*, \vec{p}_1 = \angle \vec{p}_1^*, \vec{p}_2 - \angle \vec{p}_1, \vec{p}_2 = \frac{\pi}{2} - \phi \tag{8}$$

$$\cos(\angle \vec{p}_2, \vec{p}_2^*) > 0 \Longrightarrow \angle \vec{p}_2, \vec{p}_2^* \in \left\langle -\frac{\pi}{2}, \frac{\pi}{2} \right\rangle \tag{9}$$

$\Longrightarrow$retningen på $\vec{p}_2^*$ er bestemt. Se i Figure: 1.

$$\angle \vec{p}_2, \vec{p}_2^* = \angle \vec{p}_1, \vec{p}_2^* - \angle \vec{p}_1, \vec{p}_2 = \frac{\pi}{2} - \phi \tag{10}$$

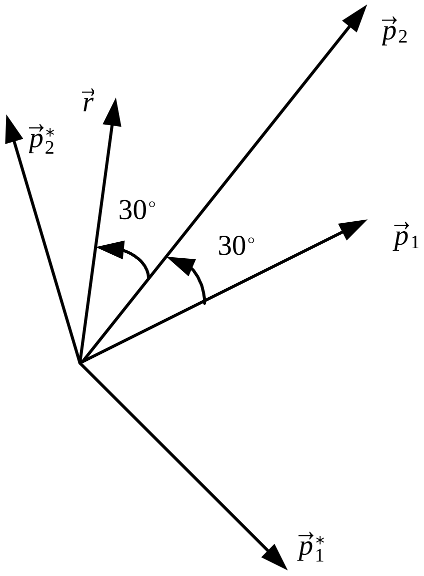

Figure 1:

$$|\vec{p}_1||\vec{p}_1^*|\cos(\frac{\pi}{2}-\phi) = 1 \Longrightarrow |\vec{p}_1^*| = \frac{1}{|\vec{p}_1|\cos(\frac{\pi}{2}-\phi)} = \frac{1}{|\vec{p}_1|\sin(\phi)} \tag{11}$$

$$|\vec{p}_2||\vec{p}_2^*|\cos(\frac{\pi}{2}-\phi) = 1 \Longrightarrow |\vec{p}_2^*| = \frac{1}{|\vec{p}_2|\cos(\frac{\pi}{2}-\phi)} = \frac{1}{|\vec{p}_2|\sin(\phi)} \tag{12}$$

hvor $\phi = \frac{\pi}{6} \Longrightarrow$

$$|\vec{p}_1^*| = 2,\ |\vec{p}_2^*| = 1 \tag{13}$$

Dekomponering av $\vec{r}$:

$$r_1^p = \langle \vec{r}, \vec{p}_1^* \rangle = |\vec{r}||\vec{p}_1^*|\cos(\angle \vec{r}, \vec{p}_1^*) = 2\cos(-\frac{\pi}{2}-\frac{\pi}{6}) = -1 \tag{14}$$

$$r_2^p = \langle \vec{r}, \vec{p}_2^* \rangle = |\vec{r}||\vec{p}_2^*|\cos(\angle \vec{r}, \vec{p}_2^*) = 1\cos(\frac{\pi}{6}) = \frac{\sqrt{3}}{2} \tag{15}$$

$$\vec{r} = -\vec{p}_1 + \frac{\sqrt{3}}{2}\vec{p}_2 \tag{16}$$

b) *Løs det samme problemet i de ortonormalebasisen* $\{\vec{q}_1, \vec{q}_2\}$.

Løsning: Ortogonale system$\Longrightarrow q_1 = q_1^*, q_2 = q_2^*$

Dekomponering av $\vec{r}$:

$$r_1^q = \langle \vec{r}, \vec{q}_1^* \rangle = |\vec{r}||\vec{q}_1^*|\cos(\angle \vec{r}, \vec{q}_1^*) = 1\cos(-\frac{\pi}{2}-\frac{\pi}{6}) = -\frac{1}{2} \tag{17}$$

$$r_2^q = \langle \vec{r}, \vec{q}_2^* \rangle = |\vec{r}||\vec{q}_2^*|\cos(\angle \vec{r}, \vec{q}_2^*) = 1\cos(-\frac{\pi}{6}) = \frac{\sqrt{3}}{2} \tag{18}$$

$$\vec{r} = -\frac{1}{2}\vec{q}_1 + \frac{\sqrt{3}}{2}\vec{q}_2 \tag{19}$$

Oppgave 2

Projeksjons operatoren P defineres som $P(\vec{r})\vec{x} = \langle \vec{x}, \vec{r} \rangle \vec{r}$, hvor $\vec{r}$ er enhetsvektor i retningen vektoren $\vec{x}$ skal projiseres ned på.

a) *Vis at operatoren er lineær, det vil si at $P(\vec{r})(a\vec{x} + b\vec{y}) = aP(\vec{r})\vec{x} + bP(\vec{r})\vec{y}$.*

Løsning:

$$\begin{aligned} P(a\vec{x} + b\vec{y}) &= \langle a\vec{x} + b\vec{y}, \vec{r} \rangle \vec{r} && (20)\\ &= (\langle a\vec{x}, \vec{r} \rangle + \langle b\vec{y}, \vec{r} \rangle)\vec{r} && (21)\\ &= (a \langle \vec{x}, \vec{r} \rangle + b \langle \vec{y}, \vec{r} \rangle)\vec{r} && (22)\\ &= a \langle \vec{x}, \vec{r} \rangle \vec{r} + b \langle \vec{y}, \vec{r} \rangle \vec{r} && (23)\\ &= aP(\vec{r})\vec{x} + bP(\vec{r})\vec{y} && (24) \end{aligned}$$

b) *Finn matriserepresentasjonen i den generelle basisen $\{\vec{p}_1, \vec{p}_2\}$. Hva blir matriserepresentasjonen når basisvektoren $\vec{p}_1, \vec{p}_2$ og vektoren $\vec{r}$ er gitt som i oppgave 1a)?*

Løsning: Matriserepresentasjon i p-basis ved $P(r)^p = [p_y^p]$.

$$\vec{y} = P(\vec{r})\vec{x} \Leftrightarrow \underline{y}^p = P(\vec{r})^p \underline{x}^p \qquad (25)$$

hvor

$$p_{ij}^p = \langle P(\vec{r})\vec{p}_j, \vec{p}_i^* \rangle$$

$$\begin{aligned} p_{11}^p &= \langle P(\vec{r})\vec{p}_1, \vec{p}_1^* \rangle = \langle \langle \vec{p}_1, \vec{r} \rangle \vec{r}, \vec{p}_1^* \rangle = \langle \vec{p}_1, \vec{r} \rangle \langle \vec{r}, \vec{p}_1^* \rangle && (26)\\ p_{12}^p &= \langle P(\vec{r})\vec{p}_2, \vec{p}_1^* \rangle = \langle \langle \vec{p}_2, \vec{r} \rangle \vec{r}, \vec{p}_1^* \rangle = \langle \vec{p}_2, \vec{r} \rangle \langle \vec{r}, \vec{p}_1^* \rangle && (27)\\ p_{21}^p &= \langle P(\vec{r})\vec{p}_1, \vec{p}_2^* \rangle = \langle \langle \vec{p}_1, \vec{r} \rangle \vec{r}, \vec{p}_2^* \rangle = \langle \vec{p}_1, \vec{r} \rangle \langle \vec{r}, \vec{p}_2^* \rangle && (28)\\ p_{22}^p &= \langle P(\vec{r})\vec{p}_2, \vec{p}_2^* \rangle = \langle \langle \vec{p}_2, \vec{r} \rangle \vec{r}, \vec{p}_2^* \rangle = \langle \vec{p}_2, \vec{r} \rangle \langle \vec{r}, \vec{p}_2^* \rangle && (29)\\ P(\vec{r})^p &= \begin{bmatrix} \langle \vec{p}_1, \vec{r} \rangle \langle \vec{r}, \vec{p}_1^* \rangle & \langle \vec{p}_2, \vec{r} \rangle \langle \vec{r}, \vec{p}_1^* \rangle \\ \langle \vec{p}_1, \vec{r} \rangle \langle \vec{r}, \vec{p}_2^* \rangle & \langle \vec{p}_2, \vec{r} \rangle \langle \vec{r}, \vec{p}_2^* \rangle \end{bmatrix} && (30) \end{aligned}$$

fra a) har vi $\langle \vec{r}, \vec{p}_1^* \rangle = -1$ og $\langle \vec{r}, \vec{p}_2^* \rangle = \frac{\sqrt{3}}{2}$.

Regner ut:

$$\begin{aligned} \langle \vec{p}_1, \vec{r} \rangle &= 1 \cos(\frac{\pi}{3}) = \frac{1}{2} && (31)\\ \langle \vec{p}_2, \vec{r} \rangle &= 2 \cos(\frac{\pi}{6}) = \sqrt{3} && (32)\\ p_{11}^p &= \frac{1}{2}(-1) = -\frac{1}{2} && (33)\\ p_{12}^p &= \sqrt{3}(-1) = -\sqrt{3} && (34)\\ p_{21}^p &= \frac{1}{2}\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{4} && (35)\\ p_{22}^p &= \sqrt{3}\frac{\sqrt{3}}{2} = \frac{3}{2} && (36)\\ P(\vec{r})^p &= \begin{bmatrix} -\frac{1}{2} & -\sqrt{3} \\ \frac{\sqrt{3}}{4} & \frac{3}{2} \end{bmatrix} && (37) \end{aligned}$$

$$\vec{y}^p = P(\vec{r})^p \vec{x}^p \Leftrightarrow \begin{bmatrix} y_1^p \\ y_2^p \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & -\sqrt{3} \\ \frac{\sqrt{3}}{4} & \frac{3}{2} \end{bmatrix} \begin{bmatrix} x_1^p \\ x_2^p \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}x_1^p - \sqrt{3}x_2^p \\ \frac{\sqrt{3}}{4}x_1^p + \frac{3}{2}x_2^p \end{bmatrix} \tag{38}$$

$$\vec{y} = P(\vec{r})\vec{x} = (-\frac{1}{2}x_1^p - \sqrt{3}x_2^p)\vec{p}_1 + (\frac{\sqrt{3}}{4}x_1^p + \frac{3}{2}x_2^p)\vec{p}_2 \tag{39}$$

c) *Finn matriserepresentasjonen i den ortonormale basisen* $\{\vec{q}_1, \vec{q}_2\}$.

Løsning:

$$\langle \vec{r}, \vec{q}_1^* \rangle = \frac{1}{2} \tag{40}$$

$$\langle \vec{r}, \vec{q}_2^* \rangle = \frac{\sqrt{3}}{2} \tag{41}$$

$$\langle \vec{q}_1, \vec{r} \rangle = \frac{1}{2} \tag{42}$$

$$\langle \vec{q}_2, \vec{r} \rangle = \frac{\sqrt{3}}{2} \tag{43}$$

$$p_{11}^p = \langle P(\vec{r})\vec{q}_1, \vec{q}_1^* \rangle = \langle \vec{q}_1, \vec{r} \rangle \langle \vec{r}, \vec{q}_1^* \rangle = (-\frac{1}{2})(-\frac{1}{2}) = \frac{1}{4} \tag{44}$$

$$p_{12}^p = \langle P(\vec{r})\vec{q}_2, \vec{q}_1^* \rangle = \langle \vec{q}_2, \vec{r} \rangle \langle \vec{r}, \vec{q}_1^* \rangle = \frac{\sqrt{3}}{2}(-\frac{1}{2}) = -\frac{\sqrt{3}}{4} \tag{45}$$

$$p_{21}^p = \langle P(\vec{r})\vec{q}_1, \vec{q}_2^* \rangle = \langle \vec{q}_1, \vec{r} \rangle \langle \vec{r}, \vec{q}_2^* \rangle = -\frac{1}{2}\frac{\sqrt{3}}{2} = -\frac{\sqrt{3}}{4} \tag{46}$$

$$p_{22}^p = \langle P(\vec{r})\vec{q}_2, \vec{q}_2^* \rangle = \langle \vec{q}_2, \vec{r} \rangle \langle \vec{r}, \vec{q}_2^* \rangle = \frac{\sqrt{3}}{2}\frac{\sqrt{3}}{2} = \frac{3}{4} \tag{47}$$

$$P(\vec{r})^q = \begin{bmatrix} \frac{1}{4} & -\frac{\sqrt{3}}{4} \\ -\frac{\sqrt{3}}{4} & \frac{3}{4} \end{bmatrix} \tag{48}$$

$$\vec{y}^q = P(\vec{r})^q \vec{x}^q \Leftrightarrow \begin{bmatrix} y_1^q \\ y_2^q \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & -\frac{\sqrt{3}}{4} \\ -\frac{\sqrt{3}}{4} & \frac{3}{4} \end{bmatrix} \begin{bmatrix} x_1^q \\ x_2^q \end{bmatrix} = \begin{bmatrix} \frac{1}{4}x_1^q - \frac{\sqrt{3}}{4}x_2^q \\ -\frac{\sqrt{3}}{4}x_1^q + \frac{3}{4}x_2^q \end{bmatrix} \tag{49}$$

$$\vec{y} = P(\vec{r})\vec{x} = (\frac{1}{4}x_1^q - \frac{\sqrt{3}}{4}x_2^q)\vec{q}_1 + (-\frac{\sqrt{3}}{4}x_1^q + \frac{3}{4}x_2^q)\vec{q}_2 \tag{50}$$

d) *Velg* $\vec{r} = \vec{q}_1$ *og finn matriserepresentasjonen av operatoren i den ortonormale basisen* $\{\vec{q}_1, \vec{q}_2\}$.

Løsning:

$$\vec{r} = \vec{q}_1 \tag{51}$$

$$\langle \vec{r}, \vec{q}_1^* \rangle = 1 \tag{52}$$

$$\langle \vec{r}, \vec{q}_2^* \rangle = 0 \tag{53}$$

$$\langle \vec{q}_1, \vec{r} \rangle = 1 \tag{54}$$

$$\langle \vec{q}_2, \vec{r} \rangle = 0 \tag{55}$$

$$p_{11}^p = \langle P(\vec{r})\vec{q}_1, \vec{q}_1^* \rangle = \langle \vec{q}_1, \vec{r} \rangle \langle \vec{r}, \vec{q}_1^* \rangle = 1 \tag{56}$$

$$p_{12}^p = \langle P(\vec{r})\vec{q}_2, \vec{q}_1^* \rangle = \langle \vec{q}_2, \vec{r} \rangle \langle \vec{r}, \vec{q}_1^* \rangle = 0 \tag{57}$$

$$p_{21}^p = \langle P(\vec{r})\vec{q}_1, \vec{q}_2^* \rangle = \langle \vec{q}_1, \vec{r} \rangle \langle \vec{r}, \vec{q}_2^* \rangle = 0 \tag{58}$$

$$p_{22}^p = \langle P(\vec{r})\vec{q}_2, \vec{q}_2^* \rangle = \langle \vec{q}_2, \vec{r} \rangle \langle \vec{r}, \vec{q}_2^* \rangle = 0 \tag{59}$$

$$P(\vec{r})^q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \tag{60}$$

$$\vec{y} = P(\vec{r})\vec{x} = x_1^q\vec{q}_1$$