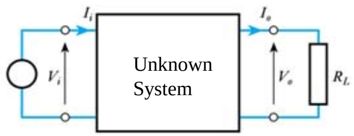
Frequency Characteristics of AC Circuits

Chapter 8

- We then define voltages and currents at the input and output



(b) A typical arrangement

- Then

$$\text{voltage gain } (A_v) = \frac{V_0}{V_i}$$

$$\text{current gain } (A_i) = \frac{I_0}{I_i}$$

$$\text{power gain } (A_p) = \frac{P_0}{P_i}$$

The Decibel (dB)

8.3

- The power gain of amplifiers is often given in **decibels**, that allows gains of individual stages to be added
- The decibel is a dimensionless figure for power gain

$$\text{Power gain (dB)} = 10 \log_{10} \frac{P_{out}}{P_{input}}$$

- In terms of voltage gain, the power gain is given as:

$$\text{Power gain (dB)} = 10 \log_{10}\left(\frac{V_{out}^{\ 2}}{V_{in}^{\ 2}} \cdot \frac{R_{in}}{R_{out}}\right) \approx 20 \log_{10} \frac{V_2}{V_1}$$

$$\text{Power gain (dB)} = 20 \log_{10} (\text{Voltage gain})$$

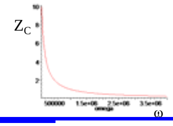
- Generally used even when $R_{out} \neq R_{in}$

Frequency response

8.4

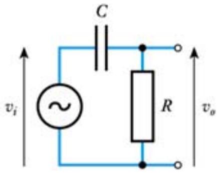
- The way in which the gain of a circuit changes with frequency is termed its **frequency response** or **transfer function**
- These variations take the form of variations in the *magnitude* of the gain and in the *phase response*

A High-Pass *RC* Network



- The transfer function is

$$\frac{v_o}{v_i}=\frac{\mathbf{Z_R}}{\mathbf{Z_R}+\mathbf{Z_C}}=\frac{R}{R-j\frac{1}{\omega C}}=\frac{1}{1-j\frac{1}{\omega CR}}$$



- the *magnitude* of the voltage gain is

$$|\text{Voltage gain}|=\frac{1}{\sqrt{1^2+\left(\frac{1}{\omega CR}\right)^2}}$$

- When $1/(\omega CR)$ = 1

$$|\text{Voltage gain}|=\frac{1}{\sqrt{1+1}}=\frac{1}{\sqrt{2}}=0.707 \qquad \text{Power gain}=\frac{1}{2}=-3dB$$

8.5

- The half power point is the **cut-off frequency**
  - the angular frequency ω_C at which this occurs is

$$\frac{1}{\omega_C CR}=1$$

$$\omega_C=\frac{1}{CR}=\frac{1}{\mathrm{T}}\ \text{rad/s}$$

  - where T is the time constant of the *CR* network. Also

$$f_c=\frac{\omega_c}{2\pi}=\frac{1}{2\pi CR}\ \text{Hz} \quad \text{and} \quad \frac{v_o}{v_i}=\frac{1}{1-j\frac{1}{\omega CR}}=\frac{1}{1-j\frac{1}{(2\pi f)\left(\frac{1}{2\pi f_c}\right)}}=\frac{1}{1-j\frac{f_c}{f}}$$

8.6

$$\frac{v_o}{v_i}=\frac{1}{1-j\frac{f_c}{f}}$$

- Consider the behaviour of the circuit at different frequencies:
- **When $f \gg f_c$**
  - $f_c/f \ll 1$, the voltage gain ≈ 1
- **When $f = f_c$**

$$\frac{v_o}{v_i}=\frac{1}{1-j\frac{f_c}{f}}=\frac{1}{1-j}=\frac{1\times(1+j)}{(1-j)\times(1+j)}=\frac{(1+j)}{2}=0.5+0.5j$$

Output leads input by 45°

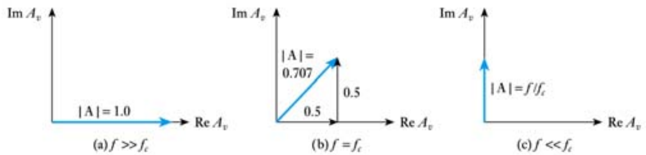
- **When $f \ll f_c$**

$$\frac{v_o}{v_i}=\frac{1}{1-j\frac{f_c}{f}}\approx\frac{1}{-j\frac{f_c}{f}}=j\frac{f}{f_c}$$

Output leads input by 90°

8.7

- The behaviour in these three regions can be illustrated using phasor diagrams

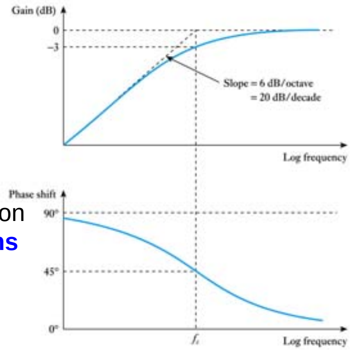


- At *low* frequencies the gain is linearly related to frequency. ½ the frequency ⇒ ½ the voltage gain
  ½ the frequency ⇒ ¼ the power gain
  Power gain falls at -6dB/octave (-20dB/decade)

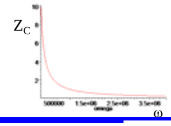
8.8

A High-Pass *RC* Network (high frequencies passed to the output)

- Frequency response of the high-pass network
  - the gain response has two *asymptotes* that meet at the cut-off frequency
  - figures the transfer function are called **Bode diagrams**



Phase shifts important in control systems

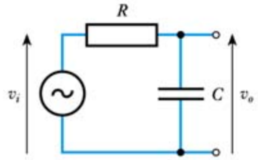


8.6

A Low-Pass *RC* Network

- Transposing the *C* and *R* gives

$$\frac{v_o}{v_i} = \frac{\mathbf{Z_C}}{\mathbf{Z_R}+\mathbf{Z_C}} = \frac{-j\frac{1}{\omega C}}{R-j\frac{1}{\omega C}} = \frac{1}{1+j\omega CR}$$



- At high frequencies
  - ω is large, voltage gain $\rightarrow 0$

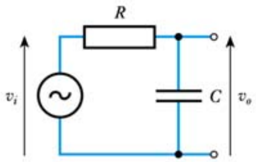
(capacitor looks like a wire⇒nearly all the voltage drops across resistor)

- At low frequencies
  - ω is small, voltage gain ≈ 1

(capacitor looks like a huge resistor⇒hardly any voltage drop across resistor)

- A similar analysis to before gives

$$|\text{Voltage gain}| = \frac{1}{\sqrt{1+(\omega CR)^2}}$$



- Therefore when, when $\omega CR = 1$

$$|\text{Voltage gain}| = \frac{1}{\sqrt{1+1}} = \frac{1}{\sqrt{2}} = 0.707$$

- Which is the cut-off frequency

- Therefore
  - the angular frequency ω_C at which this occurs is given by

$$\omega_C CR = 1$$

$$\omega_C = \frac{1}{CR} = \frac{1}{T}\text{ rad/s}$$

  - where T is the time constant of the *CR* network, and as before

$$f_C = \frac{\omega_C}{2\pi} = \frac{1}{2\pi CR}\text{ Hz}$$

- Substituting $\omega = 2\pi f$ and $CR = 1/2\pi f_C$ in the earlier equation gives

$$\frac{v_o}{v_i} = \frac{1}{1 + j\omega CR} = \frac{1}{1 + j\dfrac{\omega}{\omega_C}} = \frac{1}{1 + j\dfrac{f}{f_C}}$$

- This is similar, but not the same, as the transfer function for the high-pass network (+j instead of –j)

8.13

- Consider the behaviour of this circuit at different frequencies:
- **When $f \ll f_c$**
  – $f/f_c \ll 1$, the voltage gain ≈ 1
- **When $f = f_c$**

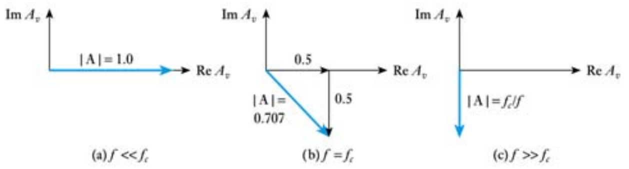
$$\frac{v_o}{v_i} = \frac{1}{1 + j\dfrac{f}{f_c}} = \frac{(1-j)(1+j)}{(1+j)} = \frac{(1-j)}{2} = 0.5 - 0.5j$$

- **When $f \gg f_c$**

$$\frac{v_o}{v_i} = \frac{1}{1 + j\dfrac{f}{f_C}} \approx \frac{1}{j\dfrac{f}{f_C}} = -j\frac{f_C}{f}$$

8.14

- The output voltage lags the input voltage

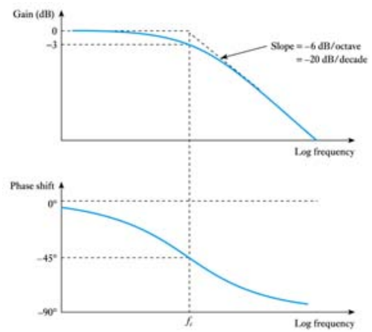


- At *high* frequencies the gain is linearly related to frequency.
- Power gain falls at 6dB/octave (20dB/decade)

8.15

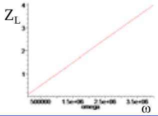
A Low-Pass *RC* Network (low frequencies passed to the output)

- Frequency response of the low-pass network
  – the gain response has two *asymptotes* that meet at the cut-off frequency
  – you might like to compare this with the Bode Diagram for a high-pass network



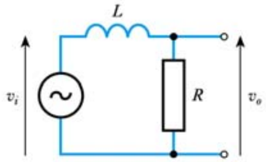
8.16

A Low-Pass *RL* Network



8.7

- Low-pass networks can also be produced using *RL* circuits
  - these behave similarly to the corresponding *CR* circuit
  - the voltage gain is



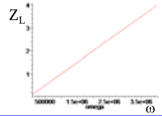
$$\frac{v_o}{v_i} = \frac{\mathbf{Z_R}}{\mathbf{Z_R}+\mathbf{Z_L}} = \frac{R}{R+j\omega L} = \frac{1}{1+j\omega\frac{L}{R}}$$

At Low frequencies
L looks like a wire
At High frequencies
L looks like a big resistor

  - the cut-off frequency is

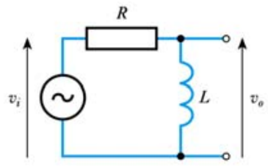
$$\omega_c = \frac{R}{L} = \frac{1}{T}\text{rad/s} \qquad f_c = \frac{\omega_c}{2\pi} = \frac{R}{2\pi L}\text{Hz}$$

A High-Pass *RL* Network



8.8

- High-pass networks can also be produced using *RL* circuits
  - these behave similarly to the corresponding *CR* circuit
  - the voltage gain is



$$\frac{v_o}{v_i} = \frac{\mathbf{Z_L}}{\mathbf{Z_R}+\mathbf{Z_L}} = \frac{j\omega L}{R+j\omega L} = \frac{1}{1+\frac{R}{j\omega L}} = \frac{1}{1-j\frac{R}{\omega L}}$$

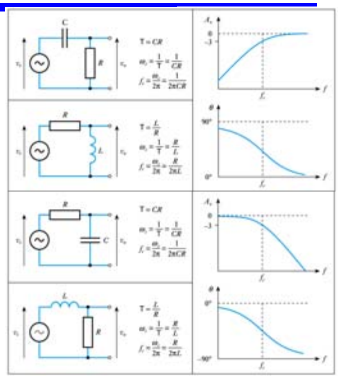
  - the cut-off frequency is

$$\omega_c = \frac{R}{L} = \frac{1}{T}\text{rad/s} \qquad f_c = \frac{\omega_c}{2\pi} = \frac{R}{2\pi L}\text{Hz}$$

A Comparison of *RC* and *RL* Networks

8.9

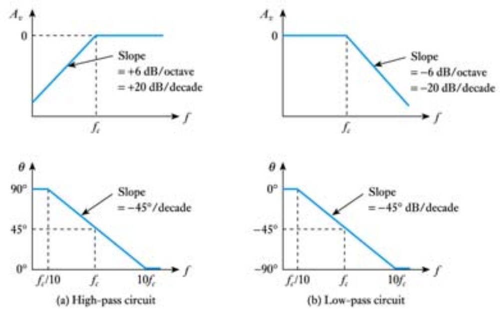
- Circuits using RC and RL techniques have similar characteristics
  - see **Figure 8.12** in the course text



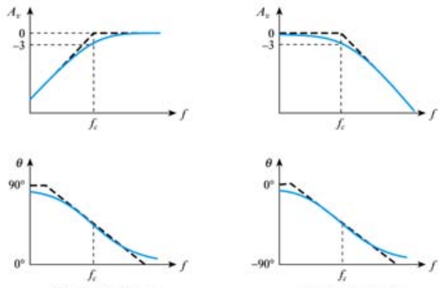
Bode Diagrams

Video 8A
8.10

- Straight-line approximations



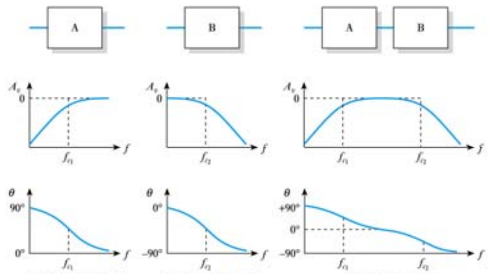
- Creating more detailed Bode diagrams



(a) High-pass circuit (b) Low-pass circuit

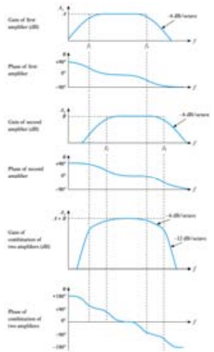
Combining the Effects of Several Stages

- The effects of several stages 'add' in bode diagrams

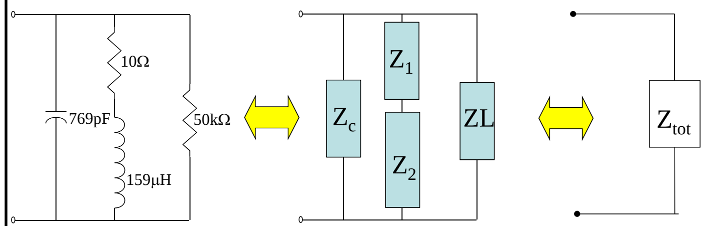


(a) High-pass circuit (b) Low-pass circuit (c) Band-pass circuit

- Multiple high- and low-pass elements may also be combined
  - this is illustrated in **Figure 8.16** in the course text



Why did we say complex impedances made things easier?



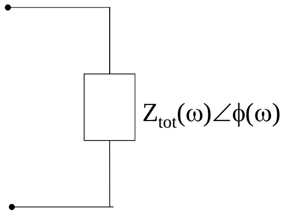
$$\frac{1}{Ztot} = \frac{1}{Z_c} + \frac{1}{(Z_1 + Z_2)} + \frac{1}{ZL} \quad \Leftrightarrow \text{Simple Arithmetic} \Rightarrow \quad Ztot = \frac{ZL\,(Z1 + Z2)\,Zc}{Zc\,Z1 + Zc\,Z2 + ZL\,Z1 + ZL\,Z2 + ZL\,Zc}$$

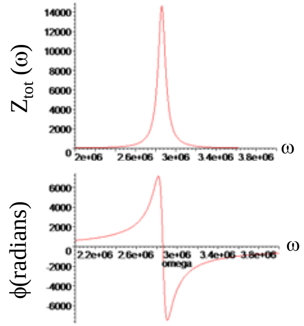
Substitute in complex, frequency dependent values, and let Maple solve it to get

$Ztot(\omega) = Z_{re} + j \cdot Z_{im}$

Plot

How does the total impedance change with frequency?





RLC Circuits and Resonance

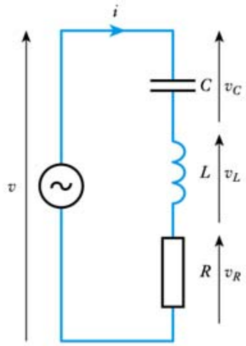
8.12

- **Series *RLC* circuits**
  - the impedance is given by

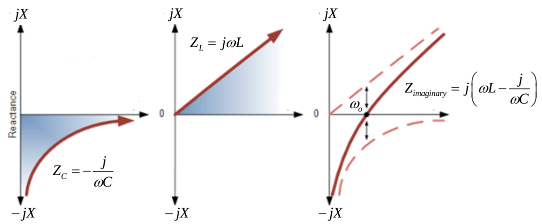
$$\mathbf{Z} = R + j\omega L + \frac{1}{j\omega C} = R + j(\omega L - \frac{1}{\omega C})$$

  - if the magnitude of the **reactance** of the inductor and capacitor are equal, the imaginary part is zero, and **the impedance is simply *R***
  - this occurs at frequency ω_o when

$$\omega_o L = \frac{1}{\omega_o C} \qquad \omega_o^2 = \frac{1}{LC} \qquad \omega_o = \frac{1}{\sqrt{LC}}$$



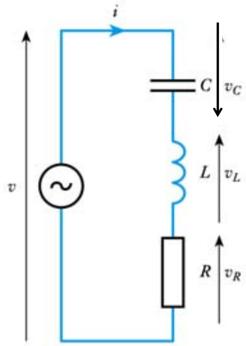
How do the impedances change with frequency



- At one value of ω the total reactive impedance is 0
  - As if the capacitor and inductor were replaced by a wire
- The impedance is a minimum value of R

At resonance, voltages in C and L 180° out of phase

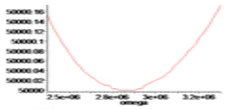
- Resonance hits a sweet spot where the phasing between the voltage in the capacitor and the inductor are out of phase and cancel each other
- As a result, no net voltage drop across that part of the circuit (it looks like a wire), and the total resistance reduces to R, where v_R=V



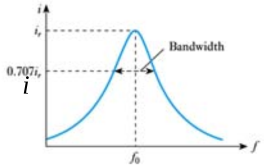
For R=50kΩ, C=769 pF, L =159µH,
$\omega_o = 2.86\times10^6$ rad/s ⇒ f_o = 455 kHz

- This situation is referred to as **resonance**
  - the frequency at which is occurs is the **resonant frequency**

$$\omega_0 = \frac{1}{\sqrt{LC}} \qquad f_o = \frac{1}{2\pi\sqrt{LC}}$$

Z


  - in the **series resonant circuit**, the *impedance* is at a ***minimum*** at resonance
  - the *current* is at a maximum at resonance



- The resonant effect can be quantified by the **quality factor, Q**
  - this is the ratio of the power stored (in L or C) to the power dissipated (in R) in each cycle
  - it can be shown that

$$\text{Quality factor } Q = \frac{X_L}{R} = \frac{X_C}{R}$$

  - and

$$Q = \frac{1}{R}\sqrt{\left(\frac{L}{C}\right)}$$

- The series *RLC* circuit is an **acceptor circuit**
  - the narrowness of bandwidth is determined by the Q

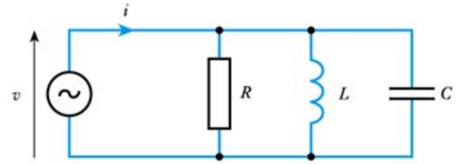
Not Proven
Stole på oss!

$$\text{Quality factor } Q = \frac{\text{Resonant frequency}}{\text{Bandwidth}} = \frac{f_o}{B}$$

  - combining this equation with the earlier one gives

$$B = \frac{R}{2\pi L}\ \text{Hz}$$

- **Parallel *RLC* circuits**



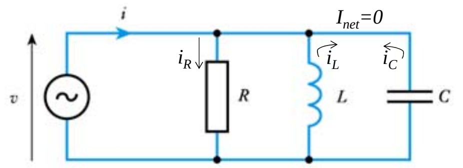
  - as before

$$\frac{1}{Ztot} = \frac{1}{R} + j\left(-\frac{1}{\omega L} + \omega C\right) \qquad \omega_o = \frac{1}{\sqrt{LC}} \qquad f_o = \frac{1}{2\pi\sqrt{LC}}$$

Separate into real and imaginary parts

- $Z_{tot} = \frac{1}{\omega^2 L^2 + R^2(\omega^2 CL - 1)} \left\{ R\omega^2 L^2 - jR^2 \omega L(\omega^2 CL - 1) \right\}$
- and when $(\omega_o)^2 \cdot CL = 1$ we see
  - Reactive (Imaginary) part vanishes
  - Real part = R
- At resonance, impedance reaches a ***maximum*** and
  $Z_{tot} = R$

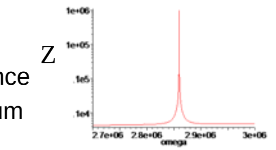
At resonance the currents in the capacitor and inductor are 180° out of phase and cancel

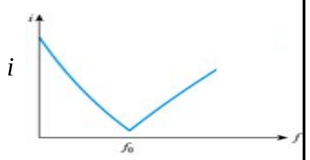


- The reactive currents are flowing back and forth between the L and the C with a phase (tank circuit)
- At resonance, they are 180° out of phase and $I_{net}=0$
- Since there is no current flowing there, the L-C part of the circuit looks like an open circuit and $i = i_R$

For R=50kΩ, C=769 pF, L =159μH,
$\omega_o=2.86\text{x}10^6$ rad/s ⇒ f_o = 455 kHz

- The parallel arrangement is a **rejector circuit**
  - in the **parallel resonant circuit**, the *impedance* is at a maximum at resonance
  - the *current* is at a minimum at resonance
  - in this circuit





$$Q = R\sqrt{\left(\frac{C}{L}\right)}$$

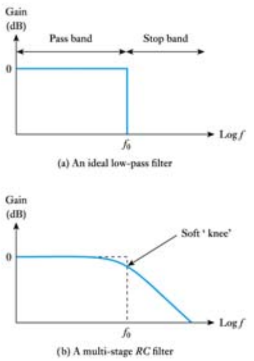
$$B = \frac{1}{2\pi RC}\text{Hz}$$

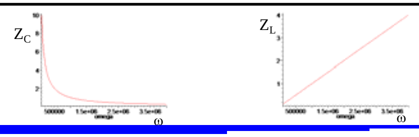
Filters


8.13

- ***RC* and *RL* Filters**
- The *RC* and *RL* networks considered earlier are **first-order** or **single-pole** filters
  - these have a maximum roll-off of 6 dB/octave
  - they also produce a maximum of 90° phase shift
- Combining multiple stages can produce filters with a greater ultimate roll-off rates (12 dB, 18 dB, etc.) but such filters have a very soft 'knee'

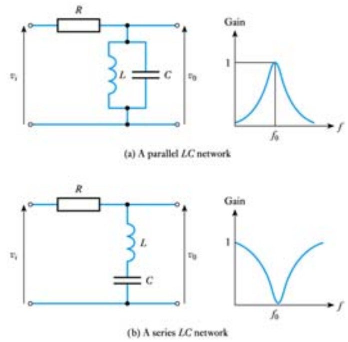
- An ideal filter would have constant gain and zero phase shift for frequencies within its **pass band**, and zero gain for frequencies outside this range (its **stop band**)
- Real filters do not have these idealised characteristics



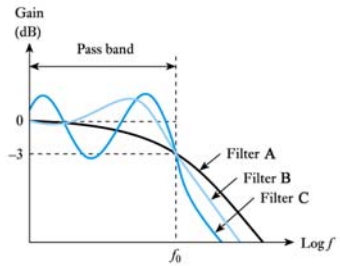


- ***LC* Filters**
- Simple LC filters can be produced using series or parallel tuned circuits
  - these produce narrow-band filters with a centre frequency f_o

$$f_o = \frac{1}{2\pi\sqrt{LC}}$$

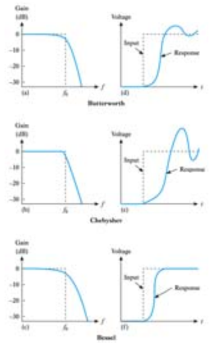


- **Active filters**
  - combining an op-amp with suitable resistors and capacitors can produce a range of filter characteristics
  - these are termed **active filters**



- Common forms include:
- **Butterworth**
  - optimised for a flat response
- **Chebyshev**
  - optimised for a sharp 'knee'
- **Bessel**
  - optimised for its phase response

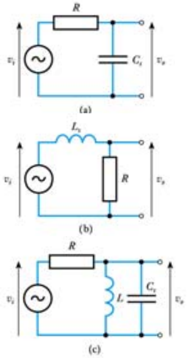
  see **Section 8.13.3** of the course text for more information on these



Stray Capacitance and Inductance

8.14

- All circuits have stray capacitance and stray inductance
  - these unintended elements can dramatically affect circuit operation
  - for example:
    - (a) C_s adds an unintended low-pass filter
    - (b) L_s adds an unintended low-pass filter
    - (c) C_s produces an unintended resonant circuit and can produce instability



Further Study

Video 8B Further Study

- The Further Study section at the end of Chapter 8 looks at the filter needed to select a single radio station from the multitude being transmitted.
- You are required to design a simply RLC filter to select a given station.
- When you have finished your design watch the video to see why this is not as simple as it perhaps appears.

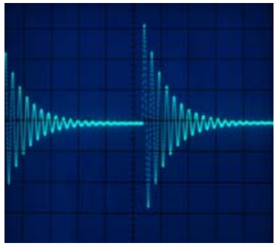
Key Points

- The reactance of capacitors and inductors is dependent on frequency
- Single *RC* or *RL* networks can produce an arrangement with a single upper or lower cut-off frequency
- In each case the angular cut-off frequency ω_o is given by the reciprocal of the time constant T
- For an *RC* circuit $\mathrm{T} = CR$, for an *RL* circuit $\mathrm{T} = L/R$
- Resonance occurs when the reactance of the capacitive element cancels that of the inductive element
- Simple *RC* or *RL* networks represent single-pole filters
- Active filters produce high performance without inductors
- Stray capacitance and inductance are found in all circuits

Transient Behaviour

Chapter 9

- Introduction
- Charging Capacitors and Energising Inductors
- Discharging Capacitors and De-energising Inductors
- Response of First-Order Systems
- Second-Order Systems
- Higher-Order Systems



Introduction

9.1

- So far we have looked at the behaviour of systems in response to:
  - fixed DC signals
  - constant AC signals
- We now turn our attention to the operation of circuits before they reach steady-state conditions
  - this is referred to as the **transient response**
- We will begin by looking at simple *RC* and *RL* circuits

Charging Capacitors and Energising Inductors

9.2

Capacitor Charging

- Consider the circuit shown here
  - Applying Kirchhoff's voltage law

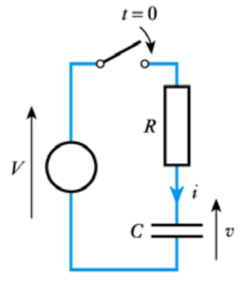
$$iR + v = V$$

  - Now, in a capacitor

$$i = C\frac{dv}{dt}$$

  - which substituting gives

$$CR\frac{dv}{dt} + v = V$$



- The above is a first-order differential equation with constant coefficients
- Assuming $V_C = 0$ at $t = 0$, this can be solved to give

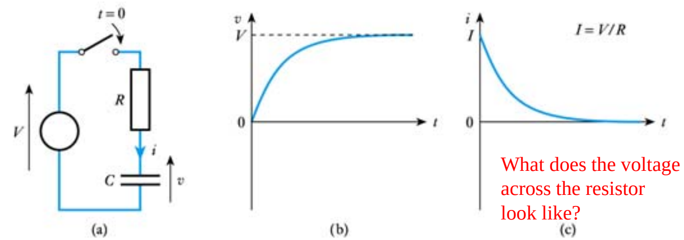
$$v = V(1 - e^{-\frac{t}{CR}}) = V(1 - e^{-\frac{t}{T}})$$

  - see **Section 9.2.1** of the course text for this analysis
- Since $i = Cdv/dt$ this gives (assuming $V_C = 0$ at $t = 0$)

$$i = Ie^{-\frac{t}{CR}} = Ie^{-\frac{t}{T}}$$

  - where $I = V/R$

- Thus both the voltage and current have an exponential form



What does the voltage across the resistor look like?

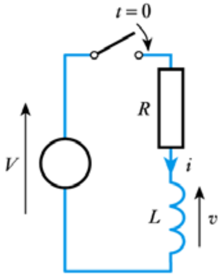
Inductor energising

KVL ⇒ iR + v = V, and v=L(di/dt)

- A similar analysis of this circuit gives

$$v = Ve^{-\frac{Rt}{L}} = Ve^{-\frac{t}{T}}$$

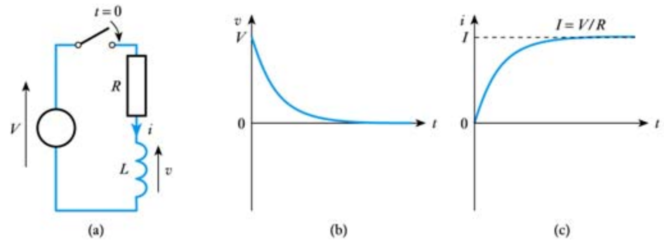
$$i = I(1-e^{-\frac{Rt}{L}}) = I(1-e^{-\frac{t}{T}})$$



where $I = V/R$

– see **Section 9.2.2** for this analysis

- Thus, again, both the voltage and current have an exponential form



Discharging Capacitors and De-energising Inductors

9.3

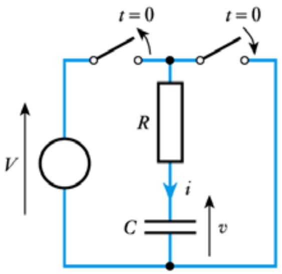
Capacitor discharging

- Consider this circuit for discharging a capacitor
  - At $t = 0$, $V_C = V$
  - From Kirchhoff's voltage law

$$iR + v = 0$$

  - giving

$$CR\frac{dv}{dt} + v = 0$$



- Solving this as before gives

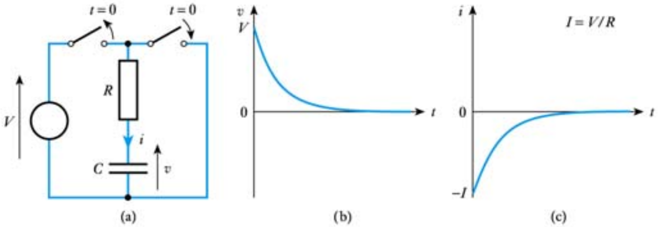
$$v = Ve^{-\frac{t}{CR}} = Ve^{-\frac{t}{T}}$$

$$i = -Ie^{-\frac{t}{CR}} = -Ie^{-\frac{t}{T}}$$

  – where $I = V/R$

  – see **Section 9.3.1** for this analysis

- In this case, both the voltage and the current take the form of decaying exponentials



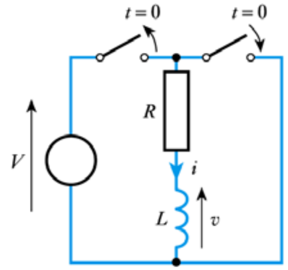
Inductor de-energising

- A similar analysis of this circuit gives

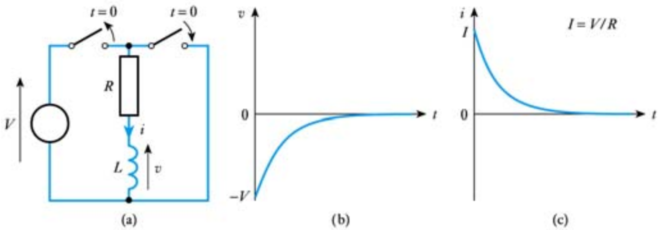
$$v = -Ve^{-\frac{Rt}{L}} = -Ve^{-\frac{t}{T}}$$

$$i = Ie^{-\frac{Rt}{L}} = Ie^{-\frac{t}{T}}$$

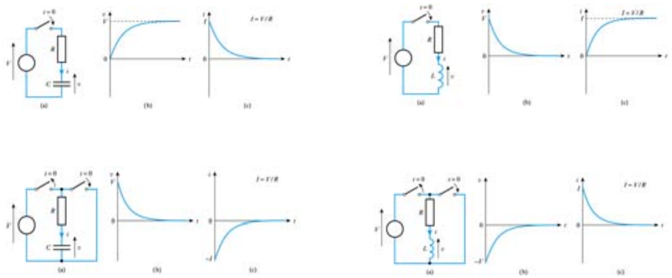
  - where $I = V/R$
  - see **Section 9.3.1** for this analysis



- And once again, both the voltage and the current take the form of decaying exponentials



- A comparison of the four circuits



9.4

Response of First-Order Systems

- **Initial and final value formulae**
  - increasing or decreasing exponential waveforms (for either voltage or current) are given by:

$$v = V_f + (V_i - V_f)e^{-t/T}$$

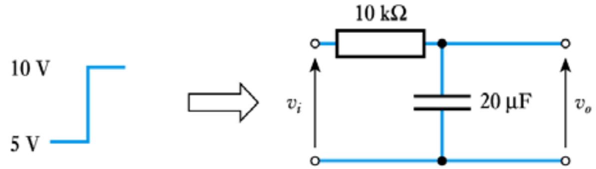
$$i = I_f + (I_i - I_f)e^{-t/T}$$

  - where V_i and I_i are the *initial* values of the voltage and current
  - where V_f and I_f are the *final* values of the voltage and current
  - the first term in each case is the **steady-state response**
  - the second term represents the **transient response**
  - the combination gives the **total response** of the arrangement

8.57

Example – see **Example 9.3** from course text

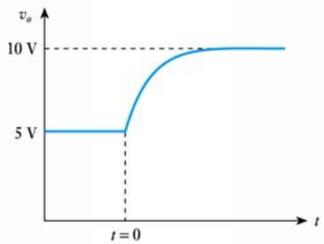
The input voltage to the following *CR* network undergoes a step change from 5 V to 10 V at time t = 0. Derive an expression for the resulting output voltage



8.58

Here the initial value is 5 V and the final value is 10 V. The time constant of the circuit equals $CR = 10 \times 10^3 \times 20 \times 10^{-6} = 0.2$s. Therefore, from above, for $t \geq 0$

$$\begin{aligned} v &= V_f + (V_i - V_f)e^{-t/T} \\ &= 10 + (5 - 10)e^{-t/0.2} \\ &= 10 - 5e^{-t/0.2} \text{ volts} \end{aligned}$$



And the current?

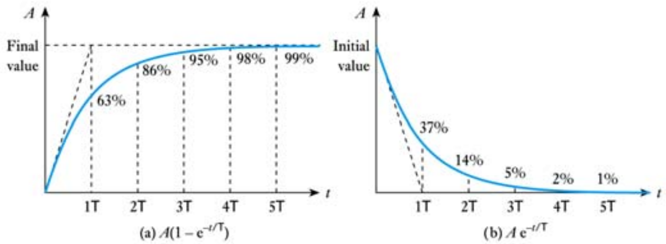
$i = C(dv/dt)$

$i = (V_f - V_i)/R \cdot e^{-t/T}$

$I_i = (V_f - V_i)/R$, $I_f = 0$

8.59

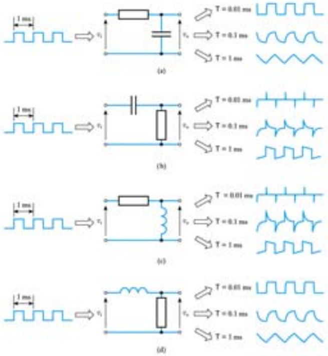
The nature of exponential curves



8.60

Response of first-order systems to a square waveform

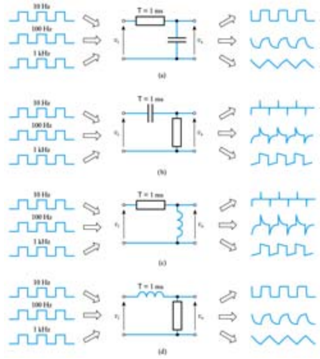
– see Section 9.4.3



8.61

Response of first-order systems to a square waveform of different frequencies

– see Section 9.4.3



8.62

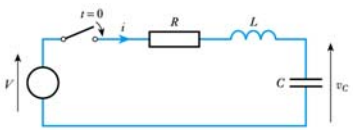
Second-Order Systems

9.5

- Circuits containing both capacitance and inductance are normally described by second-order differential equations. These are termed **second-order systems**
  – for example, this circuit is described by the equation

$v_R+v_L+v_C$ = V and v_L=L(di/dt), and v_R=iR but i = C(dv_C/dt). So

$$LC\frac{d^2v_C}{dt^2}+RC\frac{dv_C}{dt}+v_C=V$$



8.63

- When a step input is applied to a second-order system, the form of the resultant transient depends on the relative magnitudes of the coefficients of its differential equation. The general form of the response is

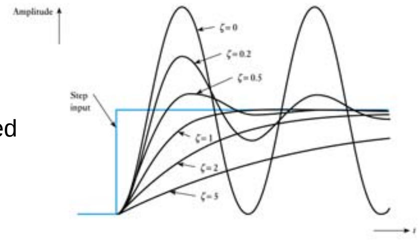
$$\frac{1}{\omega_n^2}\frac{d^2y}{dt^2}+\frac{2\zeta}{\omega_n}\frac{dy}{dt}+y=x$$

  – where ω_n is the **undamped natural frequency** in rad/s and ζ (Greek Zeta) is the **damping factor**

8.64

Response of second-order systems

ζ =0 undamped
ζ <1 under damped
ζ =1 critically damped
ζ >1 over damped



Higher-Order Systems


9.6

- Higher-order systems are those that are described by third-order or higher-order equations
- These often have a transient response similar to that of the second-order systems described earlier
- Because of the complexity of the mathematics involved, they will not be discussed further here

Further Study


Video 9B Further Study

- The Further Study section at the end of Chapter 9 considers the problem of determining the time constant of a circuit, so that the initial and final value theorems can be applied.

- Two sample circuits are given so that you can test your understanding.
- Calculate the time constants of the circuits and then check your results by looking at the video.

Key Points

- The charging or discharging of a capacitor, and the energising and de-energising of an inductor, are each associated with exponential voltage and current waveforms
- Circuits that contain resistance, and either capacitance or inductance, are termed first-order systems
- The increasing or decreasing exponential waveforms of first-order systems can be described by the initial and final value formulae
- Circuits that contain both capacitance and inductance are usually second-order systems. These are characterised by their undamped natural frequency and their damping factor