

THE NUMBER OF TWO-TERM TILTING COMPLEXES OVER SYMMETRIC ALGEBRAS WITH RADICAL CUBE ZERO

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ABSTRACT. In this paper, we compute the number of two-term tilting complexes for an arbitrary symmetric algebra with radical cube zero over an algebraically closed field. Firstly, we give a complete list of symmetric algebras with radical cube zero having only finitely many isomorphism classes of two-term tilting complexes in terms of their associated graphs. Secondly, we enumerate the number of two-term tilting complexes for each case in the list.

1. INTRODUCTION

Tilting theory plays an important role in the study of many areas of mathematics. A central notion of tilting theory is a tilting complex which is a generalization of a progenerator in Morita theory. Indeed, its endomorphism algebra is derived equivalent to the original algebra [16]. Hence it is a natural problem to give a classification of tilting complexes for a given algebra.

In this paper, we study a classification of two-term tilting complexes for an arbitrary symmetric algebra with radical cube zero over an algebraically closed field \mathbf{k} . Symmetric algebras with radical cube zero have been studied by Okuyama [15], Benson [7] and Erdmann–Solberg [11], and also appear in several areas such as [8, 13, 17]. Recently, Green–Schroll [12] showed that this class is precisely the Brauer configuration algebras with radical cube zero.

The study of symmetric algebras A with radical cube zero can be reduced to that of algebras with radical square zero. For example, as an application of τ -tilting theory ([3]), we find in Proposition 3.1 that the functor $-\otimes_A A/\text{soc } A$ gives a bijection

$$\text{2-tilt } A \longrightarrow \text{2-silt } (A/\text{soc } A).$$

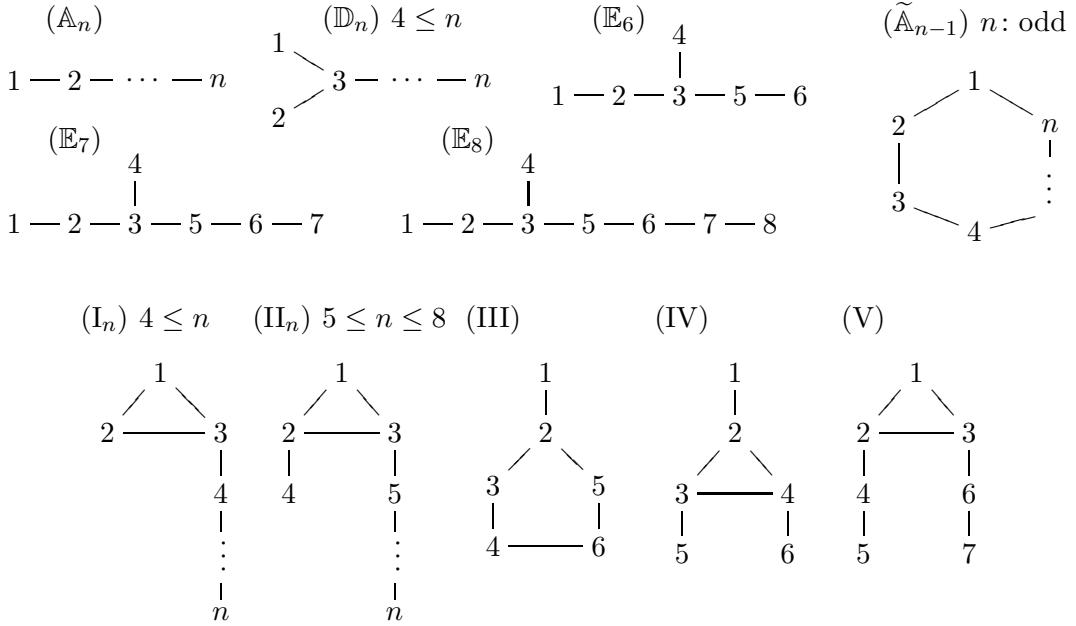
Here, we denote by $\text{2-tilt } A$ (respectively, $\text{2-silt } A$) the set of isomorphism classes of basic two-term tilting (respectively two-term silting) complexes for A . Notice that tilting complexes coincide with silting complexes for a symmetric algebra A ([4, Example 2.8]).

In [2, 5, 18], they study two-term silting theory (or equivalently τ -tilting theory) for algebras with radical square zero. The first author ([2]) gives a characterization of algebras with radical square zero which are τ -tilting finite (i.e., having only finitely many isomorphism classes of basic two-term silting complexes) by using the notion of single quivers, see Proposition 2.3(2). Using this result, we give a complete list of τ -tilting finite symmetric algebras with radical cube zero as follows.

Now, let A be a basic connected finite dimensional symmetric \mathbf{k} -algebra with radical cube zero. Let Q be the Gabriel quiver of A and Q° the quiver obtained from Q by deleting all loops. We show in Definition–Proposition 3.3 that Q° is the double quiver Q_G (see Definition 3.2) of a finite connected (undirected) graph G with no loops, i.e., $Q^\circ = Q_G$. We call G the graph of A .

Theorem 1.1. *Let A be a basic connected finite dimensional symmetric \mathbf{k} -algebra with radical cube zero. Then the following conditions are equivalent.*

- (1) *A is τ -tilting finite (or equivalently, $\text{2-tilt } A$ is finite).*
- (2) *The graph of A is one of graphs in the following list.*



The second author ([5]) classifies two-term silting complexes for an arbitrary algebra with radical square zero by using tilting modules over a path algebra (see Proposition 2.3(1)). Since the cardinality of the set of isomorphism classes of tilting modules over a path algebra is well known, this provides us an explicit way to compute the number of them. We use this result to determine the number $\# \text{2-tilt } A$ for each graph G in the list of Theorem 1.1.

Theorem 1.2. *In Theorem 1.1, the number $\# \text{2-tilt } A$ depends only on the graph G of A and is given as follows.*

G	\mathbb{A}_n	\mathbb{D}_n	\mathbb{E}_6	\mathbb{E}_7	\mathbb{E}_8	$\tilde{\mathbb{A}}_{n-1}$	I_n	II_5	II_6	II_7	II_8	III	IV	V
$\# \text{2-tilt } A$	$\binom{2n}{n}$	a_n	1700	8872	54066	2^{2n-1}	b_n	632	2936	11306	75240	3108	4056	17328

Here, for any $n \geq 4$, let $a_n := 6 \cdot 4^{n-2} - 2 \binom{2(n-2)}{n-2}$ and $b_n := 6 \cdot 4^{n-2} + 2 \binom{2n}{n} - 4 \binom{2(n-1)}{n-1} - 4 \binom{2(n-2)}{n-2}$.

We remark that the numbers for Dynkin graphs of type \mathbb{A} , \mathbb{D} and \mathbb{E} in the list are precisely the biCatalan numbers introduced by [6] in the context of Coxeter-Catalan combinatorics. Our results for Dynkin graphs are independently obtained by [9] in the study of biCambrian lattices for preprojective algebras.

We also remark that we can generalize our results for Brauer configuration algebras in terms of multiplicities. A Brauer configuration algebra is defined by a configuration and a multiplicity function. The configuration of a Brauer configuration algebra with radical cube zero corresponds to a graph [12]. By [10], one can show that the number of two-term tilting complexes over Brauer configuration algebras is independent of the multiplicity. Therefore, we can also apply our results to any Brauer configuration algebra obtained by replacing the multiplicity of a Brauer configuration associated with a graph in the list of Theorem 1.1.

This paper is organized as follows. In Section 2, we recall the definition of algebras with radical square zero and their two-term silting theory which are needed in this paper. In Section 3, we study symmetric algebras with radical cube zero together with the correspondence algebra with radical square zero. Our main results are Theorem 3.4 and Corollary 3.5 which provide us an explicit way to compute the number of two-term tilting complexes for a given symmetric

algebra with radical cube zero. In Section 4, we prove Theorems 1.1 and 1.2 by using results shown in the previous section.

2. PRELIMINARIES

Throughout this paper, \mathbf{k} is an algebraically closed field. We recall that any basic connected finite dimensional \mathbf{k} -algebra A is isomorphic to a bound quiver algebra $A \cong \mathbf{k}Q/I$, where Q is a finite connected quiver and I is an admissible ideal in the path algebra $\mathbf{k}Q$ of the quiver Q . We call $Q_A := Q$ the *Gabriel quiver* of A .

2.1 Silting complexes. Let A be a basic (not necessary connected) finite dimensional \mathbf{k} -algebra. We denote by $\text{mod } A$ the category of finitely generated right A -modules and by $\text{proj } A$ the category of finitely generated projective right A -modules. Let $\mathbf{K}^b(\text{proj } A)$ denote the homotopy category of bounded complexes of objects of $\text{proj } A$. For a complex $X \in \mathbf{K}^b(\text{proj } A)$, we say that X is *basic* if it is a direct sum of pairwise non-isomorphic indecomposable objects.

Definition 2.1. A complex T in $\mathbf{K}^b(\text{proj } A)$ is said to be *presilting* if it satisfies

$$\text{Hom}_{\mathbf{K}^b(\text{proj } A)}(T, T[i]) = 0$$

for all positive integers i . A presilting complex T is called a *silting complex* if it satisfies $\text{thick } T = \mathbf{K}^b(\text{proj } A)$, where $\text{thick } T$ is the smallest triangulated full subcategory which contains T and is closed under taking direct summands. In addition, a silting complex T is called a *tilting complex* if $\text{Hom}_{\mathbf{K}^b(\text{proj } A)}(T, T[i]) = 0$ for all non-zero integers i .

We restrict our interest to the set of two-term silting complexes. Here, a complex $T = (T^i, d^i)$ in $\mathbf{K}^b(\text{proj } A)$ is said to be *two-term* if it is isomorphic to a complex concentrated only in degree 0 and -1 , i.e.,

$$(T^{-1} \xrightarrow{d^{-1}} T^0) = \cdots \rightarrow 0 \rightarrow T^{-1} \xrightarrow{d^{-1}} T^0 \rightarrow 0 \rightarrow \cdots$$

We denote by $2\text{-silt } A$ (respectively, $2\text{-tilt } A$) the set of isomorphic classes of basic two-term silting (respectively, two-term tilting) complexes for A .

Now, we call $M \in \text{mod } A$ a *tilting module* if all the following conditions are satisfied: (i) the projective dimension of M is at most 1, (ii) $\text{Ext}_A^1(M, M) = 0$, and (iii) $|M| = |A|$, where $|M|$ denotes the number of pairwise non-isomorphic indecomposable direct summands of M . We denote by $\text{tilt } A$ the set of isomorphism classes of basic tilting A -modules. By definition, we can naturally regard a tilting A -module M as a tilting complex. More precisely, by taking a minimal projective presentation $P_1 \xrightarrow{f} P_0 \rightarrow M \rightarrow 0$ of M in $\text{mod } A$, the two-term complex $(P_1 \xrightarrow{f} P_0)$ provides a tilting complex in $\mathbf{K}^b(\text{proj } A)$.

The number of tilting modules over a path algebra of a Dynkin quiver is well known.

Proposition 2.2. (see [14] for example) *Let Q be a quiver whose underlying graph Δ is one of Dynkin graphs of type \mathbb{A} , \mathbb{D} and \mathbb{E} . Then the number $\#\text{tilt } \mathbf{k}Q$ is given by the following table and does not depend on the orientation of Q .*

Δ	\mathbb{A}_n ($n \geq 1$)	\mathbb{D}_n ($n \geq 4$)	\mathbb{E}_6	\mathbb{E}_7	\mathbb{E}_8
$\#\text{tilt } \mathbf{k}Q$	$\frac{1}{n+1} \binom{2n}{n}$	$\frac{3n-4}{2n} \binom{2(n-1)}{n-1}$	418	2431	17342

More generally, if Q is a disjoint union of Dynkin quivers Q_λ ($\lambda \in \Lambda$), then we have

$$(2.1) \quad \#\text{tilt } \mathbf{k}Q = \prod_{\lambda \in \Lambda} \#\text{tilt } \mathbf{k}Q_\lambda$$

and this number is completely determined by a collection of the underlying graphs Δ_λ of Q_λ for all $\lambda \in \Lambda$ as in Proposition 2.2.

2.2 Algebras with radical square zero. Let A be a basic connected finite dimensional \mathbf{k} -algebra. We say that A is an algebra with *radical square zero* (respectively, *radical cube zero*) if $J^2 = 0$ but $J \neq 0$ (respectively, $J^3 = 0$ but $J^2 \neq 0$), where J is the Jacobson radical of A . For simplicity, we abbreviate an algebra with radical square zero (respectively, radical cube zero) by a RSZ (respectively, RCZ) algebra.

We first recall that any basic connected finite dimensional RSZ \mathbf{k} -algebra A is isomorphic to a bound quiver algebra $\mathbf{k}Q/I$, where $Q := Q_A$ is the Gabriel quiver of A and I is the two-sided ideal in $\mathbf{k}Q$ generated by all paths of length 2.

Next, let $Q = (Q_0, Q_1)$ be a finite connected quiver, where Q_0 is the vertex set and Q_1 is the arrow set. We denote by Q^{op} the opposite quiver of Q . For a map $\epsilon: Q_0 \rightarrow \{\pm 1\}$, we define a quiver Q_ϵ , called a *single quiver* of Q , as follows:

- The set of vertices is Q_0 .
- We draw an arrow $a: i \rightarrow j$ in Q_ϵ whenever there exists an arrow $a: i \rightarrow j$ with $\epsilon(i) = +1$ and $\epsilon(j) = -1$.

Note that Q_ϵ is bipartite (i.e., each vertex is either a sink or a source), but not connected in general. Since it has no loops by definition, we have $Q_\epsilon = (Q^\circ)_\epsilon$, where Q° denotes the quiver obtained from Q by deleting all loops.

We give a connection between two-term silting complexes for a RSZ algebra and tilting modules over path algebras.

Proposition 2.3. *Let A be a basic connected finite dimensional RSZ \mathbf{k} -algebra and Q_A the Gabriel quiver of A . Let $Q := (Q_A)^\circ$ be the quiver obtained from Q_A by deleting all loops. Then the following statements hold.*

(1) ([5, Theorem 1.1]) *There is a bijection*

$$\text{2-silt } A \longrightarrow \bigsqcup_{\epsilon: Q_0 \rightarrow \{\pm 1\}} \text{tilt } \mathbf{k}(Q_\epsilon)^{\text{op}}.$$

(2) ([2, 5]) *The following conditions are equivalent.*

- (a) $\text{2-silt } A$ is finite.
- (b) *For every map $\epsilon: Q_0 \rightarrow \{\pm 1\}$, the underlying graph of the single quiver Q_ϵ is a disjoint union of Dynkin graphs of type \mathbb{A} , \mathbb{D} and \mathbb{E} .*

(3) *If one of equivalent conditions of (2) holds, we have*

$$\# \text{2-silt } A = \sum_{\epsilon: Q_0 \rightarrow \{\pm 1\}} \# \text{tilt } \mathbf{k}(Q_\epsilon)^{\text{op}}.$$

We remark that we can replace the quiver Q with the Gabriel quiver Q_A of A in Proposition 2.3 since we have $(Q_A)_\epsilon = Q_\epsilon$ for any map $\epsilon: Q_0 \rightarrow \{\pm 1\}$.

3. TWO-TERM TILTING COMPLEXES OVER SYMMETRIC RCZ ALGEBRAS

Let A be a basic connected finite dimensional symmetric RCZ \mathbf{k} -algebra. Then $\overline{A} := A/\text{soc } A$ is a RSZ algebra by definition. Moreover, the Gabriel quiver of \overline{A} coincides with the Gabriel quiver of A since $\text{soc } A$ is contained in the square of the Jacobson radical of A .

The following is basic. Here, we remember that silting complexes coincide with tilting complexes for a symmetric algebra A ([4, Example 2.8]). In particular, $\text{2-tilt } A = \text{2-silt } A$.

Proposition 3.1. [1, Theorem 3.3] *Let A be a basic connected finite dimensional symmetric RCZ \mathbf{k} -algebra and $\overline{A} := A/\text{soc } A$. Then the functor $-\otimes_A \overline{A}$ gives a bijection*

$$\text{2-tilt } A \longrightarrow \text{2-silt } \overline{A}.$$

Next, the following observations provide us a combinatorial framework of studying two-term tilting complexes over symmetric RCZ algebras.

Definition 3.2. For a finite connected graph G with no loops, we define a quiver Q_G as follows.

- The set of vertices of Q_G is the set of vertices of G .
- We draw two arrows $a^*: i \rightarrow j$ and $a^{**}: j \rightarrow i$ whenever there exists an edge a of G connecting i and j .

We call Q_G the *double quiver* of G . Notice that Q_G has no loops since so does G .

Definition-Proposition 3.3. Let A be a basic connected finite dimensional symmetric RCZ \mathbf{k} -algebra. Let Q_A be the Gabriel quiver of A and $Q := (Q_A)^\circ$ the quiver obtained from Q_A by deleting all loops. Then Q is the double quiver Q_G of a finite connected (undirected) graph G with no loops. We call G the graph of A .

Proof. For the Gabriel quiver Q_A of A , let $\pi: \mathbf{k}Q_A \rightarrow A$ be a canonical surjection. For any vertex i of Q_A , let P_i be the indecomposable projective A -module corresponding to i . By definition, P_i has Loewy length 3 and its simple socle is isomorphic to the simple top $S_i := P_i/P_iJ$.

We recall from [12, Proposition 5.6] that our algebra A is special multiserial (we refer to [12, Definition 2.2] for the definition of special multiserial algebras). Then each arrow $a: i \rightarrow j$ of Q_A determines the unique arrow $\sigma(a)$ such that $\pi(a\sigma(a)) \neq 0$, and the correspondence σ gives a permutation of the set of arrows of Q_A , see [12, Definition 4.8]. In addition, the element $\pi(a\sigma(a)\sigma^2(a) \cdots \sigma^{m-1}(a))$ lies in the socle of P_i , where m is the length of the σ -orbit containing the arrow a . Since P_i has Loewy length 3, $m = 2$ must hold. In particular, $\sigma(a)$ is the unique arrow $\sigma(a): j \rightarrow i$ such that $\pi(\sigma(a)a) \neq 0$.

Now, we can restrict the permutation σ to the subset consisting of all arrows which are not loops. Then we define a finite undirected graph G as follows: The set of vertices of G bijectively corresponds to the set of vertices of Q_A , and the set of edges of G is naturally given by the set of unordered pairs $\{a, \sigma(a)\}$ for all arrows a of Q_A which are not loops. Then G is the desired one as $(Q_A)^\circ = Q_G$ from our construction. \square

As we mentioned before, the algebras A and $\overline{A} := A/\text{soc } A$ have the same Gabriel quiver $Q_A = Q_{\overline{A}}$. Therefore, $(Q_A)^\circ = (Q_{\overline{A}})^\circ$ is the double quiver Q_G of a common finite connected graph G with no loops by Definition-Proposition 3.3.

Theorem 3.4. Let A be a basic connected finite dimensional symmetric RCZ \mathbf{k} -algebra and $\overline{A} := A/\text{soc } A$. Let Q_A be the Gabriel quiver of A and $Q := (Q_A)^\circ$ the quiver obtained from Q_A by deleting all loops.

- (1) The following conditions are equivalent.
 - (a) $2\text{-tilt } A$ is finite.
 - (b) $2\text{-silt } \overline{A}$ is finite.
 - (c) For every map $\epsilon: Q_0 \rightarrow \{\pm 1\}$, the underlying graph of the single quiver Q_ϵ is a disjoint union of Dynkin graphs of type \mathbb{A} , \mathbb{D} and \mathbb{E} .
- (2) Fix any vertex $v \in Q_0$. If one of the equivalent conditions in (1) is satisfied, then the following equalities hold.

$$\# 2\text{-tilt } A = \# 2\text{-silt } \overline{A} = 2 \cdot \sum_{\substack{\epsilon: Q_0 \rightarrow \{\pm 1\} \\ \epsilon(v)=+1}} \# \text{tilt } \mathbf{k}Q_\epsilon.$$

Proof. (1) It follows from Propositions 2.3(2) and 3.1.

(2) By Proposition 3.1, we have $\# 2\text{-tilt } A = \# 2\text{-silt } \overline{A}$. We show the second equality. Let v be a vertex in Q . By Proposition 2.3(1), we have

$$\# 2\text{-silt } \overline{A} = \sum_{\epsilon: Q_0 \rightarrow \{\pm 1\}} \# \text{tilt } \mathbf{k}(Q_\epsilon)^{\text{op}} = \sum_{\substack{\epsilon: Q_0 \rightarrow \{\pm 1\} \\ \epsilon(v)=+1}} \# \text{tilt } \mathbf{k}(Q_\epsilon)^{\text{op}} + \sum_{\substack{\epsilon: Q_0 \rightarrow \{\pm 1\} \\ \epsilon(v)=-1}} \# \text{tilt } \mathbf{k}(Q_\epsilon)^{\text{op}}.$$

For a map $\epsilon: Q_0 \rightarrow \{\pm 1\}$, we define a map $-\epsilon: Q_0 \rightarrow \{\pm 1\}$ by $(-\epsilon)(i) := -\epsilon(i)$ for all $i \in Q_0$. Since Q is the double quiver of the graph G of A , we have $Q_{-\epsilon} = (Q_\epsilon)^{\text{op}}$. This implies that Q_ϵ and $Q_{-\epsilon}$ have the same underlying graph Δ . By our assumption, Δ is a disjoint union of Dynkin graphs. Thus we obtain $\#\text{tilt } \mathbf{k}Q_\epsilon = \#\text{tilt } \mathbf{k}Q_{-\epsilon}$ because the number of non-isomorphic tilting modules over a path algebra of Dynkin type does not depend on orientation, see Proposition 2.2. Hence we have

$$\sum_{\substack{\epsilon: Q_0 \rightarrow \{\pm 1\} \\ \epsilon(v)=+1}} \#\text{tilt } \mathbf{k}Q_\epsilon = \sum_{\substack{\epsilon: Q_0 \rightarrow \{\pm 1\} \\ \epsilon(v)=+1}} \#\text{tilt } \mathbf{k}(Q_\epsilon)^{\text{op}} = \sum_{\substack{\epsilon: Q_0 \rightarrow \{\pm 1\} \\ \epsilon(v)=-1}} \#\text{tilt } \mathbf{k}(Q_\epsilon)^{\text{op}}.$$

This finishes the proof. \square

For our convenience, we restate Theorem 3.4 in terms of undirected graphs. Let $G = (G_0, G_1)$ be a finite connected graph with no loops, where G_0 is the set of vertices and G_1 is the set of edges. For each map $\epsilon: G_0 \rightarrow \{\pm 1\}$, let G_ϵ be the graph obtained from G by removing all edges between vertices i, j with $\epsilon(i) = \epsilon(j)$. From our construction, G_ϵ is precisely the underlying graph of the quiver Q_ϵ , where $Q := Q_G$ is the double quiver of G with vertex set $Q_0 = G_0$. In particular, Q_ϵ is a disjoint union of Dynkin quivers if and only if G_ϵ is a disjoint union of Dynkin graphs.

Now, we recall that, for a quiver Q whose underlying graph Δ is a disjoint union of Dynkin graphs, the number $\#\text{tilt } \mathbf{k}Q$ does not depend on orientation of Q and given by (2.1). Then, we set $|\Delta| := \#\text{tilt } \mathbf{k}Q$.

Corollary 3.5. *Let A be a basic connected finite dimensional symmetric RCZ \mathbf{k} -algebra and G the graph of A .*

(1) *The following conditions are equivalent.*

- (a) *$2\text{-tilt } A$ is finite.*
- (b) *For every map $\epsilon: G_0 \rightarrow \{\pm 1\}$, the graph G_ϵ is a disjoint union of Dynkin graphs of type \mathbb{A}, \mathbb{D} and \mathbb{E} .*

(2) *Assume that, for any $\epsilon: G_0 \rightarrow \{\pm 1\}$, the graph G_ϵ is a disjoint union of Dynkin graphs $\Delta_{\epsilon, \lambda}$ ($\lambda \in \Lambda_\epsilon$). Then for a fixed vertex v of G , the number $\#2\text{-tilt } A$ is equal to*

$$(3.1) \quad 2 \cdot \sum_{\substack{\epsilon: G_0 \rightarrow \{\pm 1\} \\ \epsilon(v)=+1}} |G_\epsilon| = 2 \cdot \sum_{\substack{\epsilon: G_0 \rightarrow \{\pm 1\} \\ \epsilon(v)=+1}} \prod_{\lambda \in \Lambda_\epsilon} |\Delta_{\epsilon, \lambda}|.$$

Proof. Let $Q := (Q_A)^\circ$, where Q_A is the Gabriel quiver of A . Then $Q = Q_G$ holds by Definition-Proposition 3.3. Then the assertion follows from Theorem 3.4 since $G_\epsilon = Q_\epsilon$ for any map $\epsilon: G_0 \rightarrow \{\pm 1\}$. \square

Definition 3.6. Keeping the notations in Corollary 3.5(2), we write $\|G\|$ for the number given by the left hand side of (3.1).

Example 3.7. (1) Let Q be a quiver whose underlying graph Δ is one of Dynkin graphs of type \mathbb{A}, \mathbb{D} and \mathbb{E} . Let A be the trivial extension of the path algebra $\mathbf{k}Q$ of Q by a minimal co-generator. It is easy to see that A is a symmetric RCZ algebra if Q is bipartite. In this case, the Gabriel quiver of A is precisely the double quiver Q_Δ of Δ , in other words, the graph of A is Δ . On the other hand, Q^{op} also determines the symmetric RCZ algebra, which is naturally isomorphic to A .

(2) Let $\Delta = \mathbb{E}_6$ and let A be the symmetric RCZ algebra obtained as in (1). In Figure 1, we describe single quivers of $Q := Q_{\mathbb{E}_6}$ associated to maps ϵ with $\epsilon(6) = +1$. Here, the notation i^σ denotes the vertex i with $\epsilon(i) = \sigma \in \{\pm 1\}$. Using the Corollary 3.5, we find that there are 1700 isomorphism classes of basic two-term tilting complexes over A as in the list of Theorem 1.2.

$Q := Q_{\mathbb{E}_6}$:	$1 \xleftrightarrow{\epsilon} 2 \xleftrightarrow{\epsilon} 3 \xleftrightarrow{\epsilon} 5 \xleftrightarrow{\epsilon} 6$	4	
Q_ϵ	4^+ $1^+ 2^+ 3^+ 5^+ 6^+$ ($\mathbb{A}_1, \mathbb{A}_1, \mathbb{A}_1, \mathbb{A}_1, \mathbb{A}_1, \mathbb{A}_1$)	4^+ $1^- \xrightarrow{\epsilon} 2^+ 3^+ 5^+ 6^+$ ($\mathbb{A}_2, \mathbb{A}_1, \mathbb{A}_1, \mathbb{A}_1, \mathbb{A}_1$)	4^+ $1^+ \xrightarrow{\epsilon} 2^- \xleftarrow{\epsilon} 3^+ 5^+ 6^+$ ($\mathbb{A}_3, \mathbb{A}_1, \mathbb{A}_1, \mathbb{A}_1$)
$\#\text{tilt k}Q_\epsilon$	1	2	5
	4^+ $1^+ 2^+ \xrightarrow{\epsilon} 3^- \xleftarrow{\epsilon} 5^+ 6^+$ ($\mathbb{A}_1, \mathbb{D}_4, \mathbb{A}_1$)	4^+ $1^- \xleftarrow{\epsilon} 2^+ \xrightarrow{\epsilon} 3^- \xleftarrow{\epsilon} 5^+ 6^+$ ($\mathbb{D}_5, \mathbb{A}_1$)	4^+ $1^+ \xrightarrow{\epsilon} 2^- \xleftarrow{\epsilon} 3^- \xleftarrow{\epsilon} 5^+ 6^+$ ($\mathbb{A}_2, \mathbb{A}_3, \mathbb{A}_1$)
	20	77	10
	4^- $1^+ 2^+ \xleftarrow{\epsilon} 3^+ 5^+ 6^+$ ($\mathbb{A}_1, \mathbb{A}_1, \mathbb{A}_2, \mathbb{A}_1, \mathbb{A}_1$)	4^- $1^- \xrightarrow{\epsilon} 2^+ \xleftarrow{\epsilon} 3^+ 5^+ 6^+$ ($\mathbb{A}_2, \mathbb{A}_2, \mathbb{A}_1, \mathbb{A}_1$)	4^- $1^+ \xrightarrow{\epsilon} 2^- \xleftarrow{\epsilon} 3^+ 5^+ 6^+$ ($\mathbb{A}_4, \mathbb{A}_1, \mathbb{A}_1$)
	2	4	14
	4^- $1^+ 2^+ \xrightarrow{\epsilon} 3^- \xleftarrow{\epsilon} 5^+ 6^+$ ($\mathbb{A}_1, \mathbb{A}_3, \mathbb{A}_1, \mathbb{A}_1$)	4^- $1^- \xrightarrow{\epsilon} 2^+ \xrightarrow{\epsilon} 3^- \xleftarrow{\epsilon} 5^+ 6^+$ ($\mathbb{A}_4, \mathbb{A}_1, \mathbb{A}_1$)	4^- $1^+ \xrightarrow{\epsilon} 2^- \xleftarrow{\epsilon} 3^- \xleftarrow{\epsilon} 5^+ 6^+$ ($\mathbb{A}_2, \mathbb{A}_2, \mathbb{A}_1, \mathbb{A}_1$)
	5	14	4
	4^+ $1^+ 2^+ 3^+ \xrightarrow{\epsilon} 5^- \xleftarrow{\epsilon} 6^+$ ($\mathbb{A}_1, \mathbb{A}_1, \mathbb{A}_1, \mathbb{A}_3$)	4^+ $1^- \xrightarrow{\epsilon} 2^+ \xrightarrow{\epsilon} 3^+ \xrightarrow{\epsilon} 5^- \xleftarrow{\epsilon} 6^+$ ($\mathbb{A}_2, \mathbb{A}_1, \mathbb{A}_3$)	4^+ $1^+ \xrightarrow{\epsilon} 2^- \xleftarrow{\epsilon} 3^+ \xrightarrow{\epsilon} 5^- \xleftarrow{\epsilon} 6^+$ ($\mathbb{A}_5, \mathbb{A}_1$)
	5	10	42
	4^+ $1^+ 2^+ \xrightarrow{\epsilon} 3^- \xleftarrow{\epsilon} 5^- \xleftarrow{\epsilon} 6^+$ ($\mathbb{A}_1, \mathbb{A}_3, \mathbb{A}_2$)	4^+ $1^- \xrightarrow{\epsilon} 2^+ \xrightarrow{\epsilon} 3^- \xleftarrow{\epsilon} 5^- \xleftarrow{\epsilon} 6^+$ ($\mathbb{A}_4, \mathbb{A}_2$)	4^+ $1^+ \xrightarrow{\epsilon} 2^- \xleftarrow{\epsilon} 3^- \xleftarrow{\epsilon} 5^- \xleftarrow{\epsilon} 6^+$ ($\mathbb{A}_2, \mathbb{A}_2, \mathbb{A}_2$)
	10	28	8
	4^- $1^+ 2^+ 3^+ \xrightarrow{\epsilon} 5^- \xleftarrow{\epsilon} 6^+$ ($\mathbb{A}_1, \mathbb{A}_1, \mathbb{A}_4$)	4^- $1^- \xrightarrow{\epsilon} 2^+ 3^+ \xrightarrow{\epsilon} 5^- \xleftarrow{\epsilon} 6^+$ ($\mathbb{A}_2, \mathbb{A}_4$)	4^- $1^+ \xrightarrow{\epsilon} 2^- \xleftarrow{\epsilon} 3^+ \xrightarrow{\epsilon} 5^- \xleftarrow{\epsilon} 6^+$ (\mathbb{E}_6)
	14	28	418
	4^- $1^+ 2^+ \xrightarrow{\epsilon} 3^- 5^- \xleftarrow{\epsilon} 6^+$ ($\mathbb{A}_1, \mathbb{A}_2, \mathbb{A}_1, \mathbb{A}_2$)	4^- $1^- \xrightarrow{\epsilon} 2^+ \xrightarrow{\epsilon} 3^- 5^- \xleftarrow{\epsilon} 6^+$ ($\mathbb{A}_3, \mathbb{A}_1, \mathbb{A}_2$)	4^- $1^+ \xrightarrow{\epsilon} 2^- \xleftarrow{\epsilon} 3^- 5^- \xleftarrow{\epsilon} 6^+$ ($\mathbb{A}_2, \mathbb{A}_1, \mathbb{A}_1, \mathbb{A}_2$)
	4	10	4
			2

FIGURE 1. A half of single quivers of the double quiver of \mathbb{E}_6 .

4. PROOF OF MAIN THEOREM

In this section, we prove Theorems 1.1 and 1.2. Throughout this section, G is a finite connected graph with no loops.

4.1 Proof of Theorem 1.1. By Corollary 3.5(1), the proof is completed with the following proposition.

Proposition 4.1. *Let G be a connected finite graph with no loops. Then the graph G_ϵ is a disjoint union of Dynkin graphs of type \mathbb{A} , \mathbb{D} and \mathbb{E} for every map $\epsilon: G_0 \rightarrow \{\pm 1\}$ if and only if G is one of the list in Theorem 1.1.*

In the following, we give a proof of Proposition 4.1 by removing extended Dynkin graphs from the collection G_ϵ of subgraphs of G . We start with removing extended Dynkin graphs of type $\tilde{\mathbb{A}}$. A graph is called an *n-cycle* if it is a cycle with exactly n vertices. In particular, it is called an *odd-cycle* if n is odd, and an *even-cycle* if n even.

Lemma 4.2. *The following statements are equivalent:*

- (1) *There exists a map $\epsilon: G_0 \rightarrow \{\pm 1\}$ such that G_ϵ contains an extended Dynkin graph of type $\tilde{\mathbb{A}}$ as a subgraph.*
- (2) *G contains an even-cycle as a subgraph.*

Proof. (2) \Rightarrow (1): Let G' be a subgraph of G which is an even-cycle. Since an even-cycle is a bipartite graph, there exists a map $\epsilon: G_0 \rightarrow \{\pm 1\}$ such that the underlying graph of G_ϵ contains G' as a subgraph. Hence the assertion follows.

(1) \Rightarrow (2): Assume that for some map $\epsilon: G_0 \rightarrow \{\pm 1\}$, the graph G_ϵ contains an extended Dynkin graph G' of type $\tilde{\mathbb{A}}$. Since G_ϵ is bipartite, so is G' . Hence G' is an even-cycle and a subgraph of G . This finishes the proof. \square

By Lemma 4.2, we may assume that G contains no even-cycle as a subgraph. In particular, G has no multiple edges. We give a connection between our graphs G_ϵ and subtrees of G . Recall that a *subtree* of G is a connected subgraph of G without cycles.

Proposition 4.3. *Assume that G contains no even-cycle as a subgraph. Let G' be a connected graph. Then the following statements are equivalent.*

- (1) *There exists a map $\epsilon: G_0 \rightarrow \{\pm 1\}$ such that G_ϵ contains G' as a subgraph.*
- (2) *G' is a subtree of G .*

In particular, there exists a naturally two-to-one correspondence between the set of connected graphs of the form G_ϵ and the set of subtrees of G .

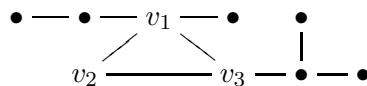
Proof. (2) \Rightarrow (1) is clear. We show (1) \Rightarrow (2). Since G has no even-cycle as a subgraph, then G_ϵ is tree by Lemma 4.2. Since G' is a subgraph of G , any subgraph of G' is a subtree of G . \square

For a tree, we have the following result.

Corollary 4.4. *Assume G is a tree. Then the graph G_ϵ is a disjoint union of Dynkin graphs of type \mathbb{A} , \mathbb{D} and \mathbb{E} for each map $\epsilon: G_0 \rightarrow \{\pm 1\}$ if and only if G is a Dynkin graph of type \mathbb{A} , \mathbb{D} and \mathbb{E} .*

Proof. It is well known that G is a Dynkin graph if and only if all subtrees of G are Dynkin graphs. The assertion follows from Proposition 4.3. \square

We remove extended Dynkin graphs of type $\tilde{\mathbb{D}}$. Assume that G contains at least two odd-cycles. Then there exists a subtree G' of G such that G' is an extended Dynkin graph of type $\tilde{\mathbb{D}}$. Moreover, by Proposition 4.3, there exists a map $\epsilon: G_0 \rightarrow \{\pm 1\}$ such that G_ϵ contains an extended Dynkin graph of type $\tilde{\mathbb{D}}$ as a subgraph. Hence we may assume that G contains at most one odd-cycle. By Corollary 4.4, it is enough to consider the case where G contains exactly one odd-cycle. Namely, G consists of an odd-cycle such that each vertex v in the odd-cycle is attached to a tree T_v .

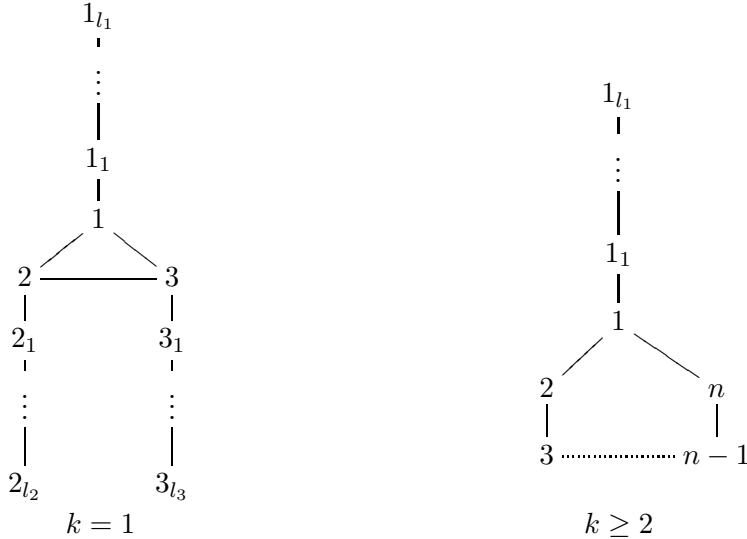


Lemma 4.5. *Fix an integer $k \geq 1$ and $n := 2k + 1$. Assume that G consists of an *n-cycle* such that each vertex v in the *n-cycle* is attached to a tree T_v . Then the following statements are equivalent:*

- (1) There exists a map $\epsilon: G_0 \rightarrow \{\pm 1\}$ such that G_ϵ contains an extended Dynkin graph of type $\tilde{\mathbb{D}}$ as a subgraph.
- (2) G contains an extended Dynkin graph of type $\tilde{\mathbb{D}}$ as a subgraph.
- (3) G satisfies one of the following conditions.
 - (a) There is a vertex v in the n -cycle such that the degree is at least four.
 - (b) There is a vertex v in the n -cycle such that the degree is exactly three and T_v is not a Dynkin graph of type \mathbb{A} .
 - (c) $k \geq 2$ and there are at least two vertices in the n -cycle such that the degrees are at least three.

Proof. (1) \Leftrightarrow (2) follows from Proposition 4.3. Moreover, we can easily check (2) \Leftrightarrow (3) because $\tilde{\mathbb{D}}_4$ has exactly one vertex whose degree is exactly four and $\tilde{\mathbb{D}}_l$ ($l \geq 5$) has exactly two vertices whose degree are exactly three. \square

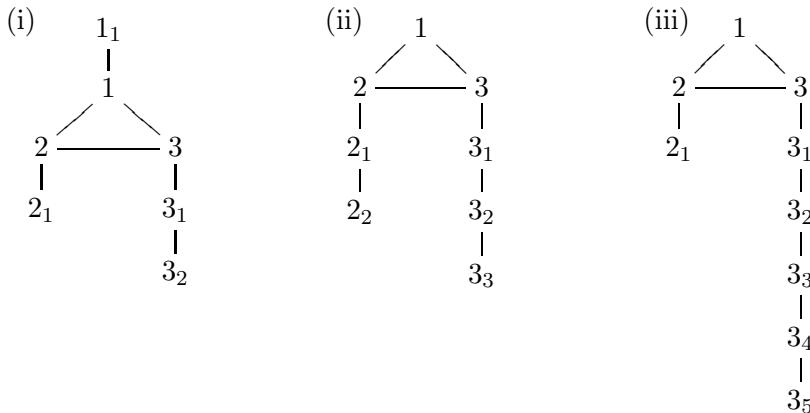
Fix an integer $k \geq 1$ and $n := 2k + 1$. By Lemma 4.5, we may assume that G is one of the following graphs:



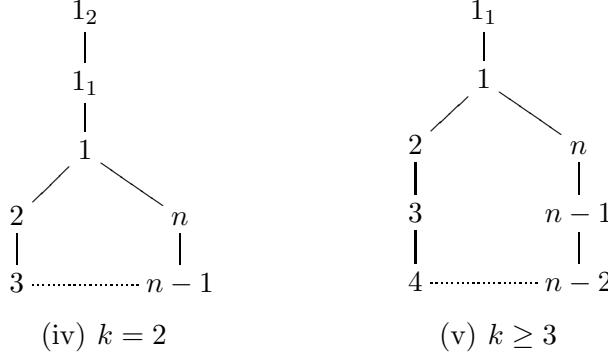
Finally, we remove extended Dynkin graphs of type $\tilde{\mathbb{E}}$.

Lemma 4.6. Fix an integer $k \geq 1$ and $n := 2k + 1$.

- (1) Assume that $k = 1$. The following graphs (i), (ii) and (iii) are the minimal graphs containing an extended Dynkin graph of type $\tilde{\mathbb{E}}$.



- (2) Assume that $k \geq 2$. The following graphs (iv) and (v) are the minimal graphs containing an extended Dynkin graph of type $\tilde{\mathbb{E}}$.



Proof. We can easily find extended Dynkin graphs $\tilde{\mathbb{E}}_6$, $\tilde{\mathbb{E}}_7$ and $\tilde{\mathbb{E}}_8$ in the graphs above. \square

Now we are ready to prove Proposition 4.1.

Proof of Proposition 4.1. If G is a tree, then the assertion follows from Corollary 4.4. We assume that G is not a tree. By Lemma 4.2, we may assume that G does not contain even-cycles as subgraphs. Then G does not contain extended Dynkin graphs as subgraphs if and only if G is one of the following classes:

- $(\text{I}_n)_{n \geq 4}$ in Theorem 1.1(2),
- proper connected non-tree subgraphs appearing in Lemma 4.6(i)–(v).

The second class coincides with the graphs $(\tilde{\mathbb{A}}_{n-1})_{n: \text{odd}}$, $(\text{I}_n)_{4 \leq n \leq 8}$, $(\text{II}_n)_{5 \leq n \leq 8}$, (III), (IV) and (V) in Theorem 1.1(2). Hence the assertion follows from Proposition 4.3. \square

We finish this subsection with proof of Theorem 1.1.

Proof of Theorem 1.1. The result follows from Corollary 3.5(1) and Proposition 4.1. \square

4.2 Proof of Theorem 1.2. We just compute the number of two-term tilting complexes for each graph in the list of Theorem 1.1. Our calculation is based on Theorem 3.4 and Corollary 3.5. For our purpose, we assume that G is a graph appearing in the list of Theorem 1.1 and let A be a basic connected finite dimensional symmetric RCZ algebra whose graph is G .

Keeping above notations, we determine the number $\# \text{2-tilt } A$, or equivalently, $\|G\|$ in Definition 3.6. First, for types \mathbb{A} and $\tilde{\mathbb{A}}$, the number is already computed by [5]:

Proposition 4.7. [5, Theorem 1.2] *The following equality holds.*

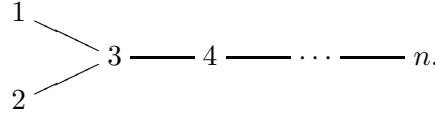
$$\# \text{2-tilt } A = \begin{cases} \binom{2n}{n} & \text{if } G = \mathbb{A}_n, \\ 2^{2n-1} & \text{if } G = \tilde{\mathbb{A}}_{n-1} \text{ for odd } n. \end{cases}$$

Secondly, we consider the case where G is a Dynkin graph of type \mathbb{D} . For simplicity, let $c_0 = 1$, $c_l := \binom{2l}{l}$ for each $l \geq 1$. Then we have $\|\mathbb{A}_l\| = c_l$ for all $l \geq 1$ by Proposition 4.7. In addition, let $\|\mathbb{A}_0\| := 2$.

Proposition 4.8. *Let $n \geq 4$ and $G = \mathbb{D}_n$. Then we have*

$$\# \text{2-tilt } A = 6 \cdot 4^{n-2} - 2c_{n-2}.$$

Proof. Let G be a graph as follows.



By Corollary 3.5, we have

$$(4.1) \quad \# \text{2-tilt } A = 2 \cdot \sum_{\substack{\epsilon: Q_0 \rightarrow \{\pm 1\} \\ \epsilon(3)=+1}} |G_\epsilon|.$$

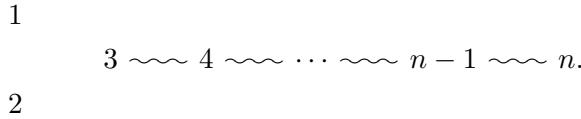
We study the right hand side of (4.1). Let M be the set of maps $\epsilon: G_0 \rightarrow \{\pm 1\}$ such that $\epsilon(3) = +1$. Clearly, M is a disjoint union of the following subsets:

- $M_1 := \{\epsilon \in M \mid \epsilon(1) = \epsilon(2) = \epsilon(3)\}$.
- $M_2 := \{\epsilon \in M \mid \epsilon(1) = -\epsilon(2) = \epsilon(3)\}$.
- $M_3 := \{\epsilon \in M \mid -\epsilon(1) = \epsilon(2) = \epsilon(3)\}$.
- $M_4 := \{\epsilon \in M \mid -\epsilon(1) = -\epsilon(2) = \epsilon(3) = \epsilon(4)\}$.
- $M_5 := \{\epsilon \in M \mid -\epsilon(1) = -\epsilon(2) = \epsilon(3) = -\epsilon(4)\} = \bigsqcup_{t=4}^n M_5(t)$, where

$$M_5(t) := \left\{ \epsilon \in M_5 \mid t = \min\{4 \leq j \leq n \mid \epsilon(j) = \epsilon(j+1)\} \right\}.$$

From now, we compute $n(i) := \sum_{\epsilon \in M_i} |G_\epsilon|$ for each $i \in \{1, \dots, 5\}$. In the following, the notation $i \sim j$ is replaced by an edge connecting i and j if $\epsilon(i) \neq \epsilon(j)$, otherwise nothing between them.

(i) Let $\epsilon \in M_1$. Then G_ϵ is given by



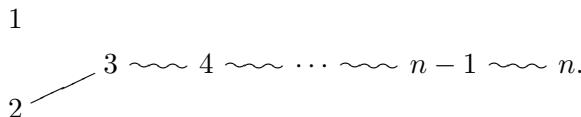
Let G' be the subgraph of G obtained by removing the vertices $\{1, 2\}$. Then we have $|G_\epsilon| = |G'_{\epsilon|_{\{3, \dots, n\}}} \cap \{3, \dots, n\}|$. Since G' is a Dynkin graph of type \mathbb{A}_{n-2} , we obtain

$$2n(1) = 2 \cdot \sum_{\substack{\epsilon: G'_0 \rightarrow \{\pm 1\} \\ \epsilon(3)=+1}} |G'_\epsilon| = ||\mathbb{A}_{n-2}|| = c_{n-2}$$

where the last equality follows from Proposition 4.7.

By an argument similar to (1), we can calculate other cases.

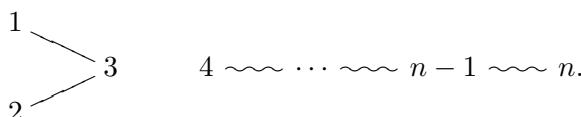
(ii) For each $\epsilon \in M_2$, the graph G_ϵ is given by



Then we can check $2n(2) = ||\mathbb{A}_{n-1}|| - ||\mathbb{A}_{n-2}|| = c_{n-1} - c_{n-2}$.

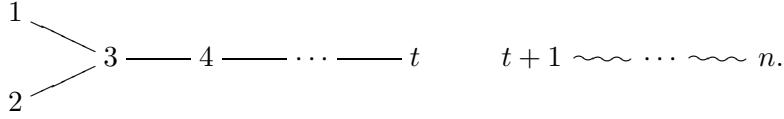
(iii) By the symmetry of G , we have $n(3) = n(2)$.

(iv) Let $\epsilon \in M_4$. Then G_ϵ is described as



Thus we find that $2n(4) = |\mathbb{A}_3| \cdot ||\mathbb{A}_{n-3}|| = 5c_{n-3}$.

(v) For $\epsilon \in M_5(t)$, the graph G_ϵ is given by



Then we obtain

$$\begin{aligned} 2n(5) &= \sum_{t=4}^n |\mathbb{D}_t| \cdot ||\mathbb{A}_{n-t}|| = \frac{3n-4}{2n} c_{n-1} \cdot 2c_0 + \sum_{t=4}^{n-1} \frac{3t-4}{2t} c_{t-1} c_{n-t} \\ &= \frac{3n-4}{2n} c_{n-1} + \sum_{t=4}^n \frac{3t-4}{2t} c_{t-1} c_{n-t}. \end{aligned}$$

To finish the proof, we need the following lemma.

Lemma 4.9. *For any positive integer n , the following equalities hold:*

- (1) $\sum_{t=1}^n c_{t-1} c_{n-t} = 4^{n-1}$.
- (2) $\sum_{t=1}^n \frac{1}{t} c_{t-1} c_{n-t} = \frac{1}{2} c_n$.

Proof. The equality (1) is well-known. The equality (2) is obtained by

$$\sum_{t=1}^n \frac{1}{t} c_{t-1} c_{n-t} = \frac{n+1}{2} \sum_{t=1}^n C_{t-1} C_{n-t} = \frac{n+1}{2} C_n = \frac{1}{2} c_n,$$

where $C_n := \frac{1}{n+1} c_n$ is the n -th Catalan number. \square

By Lemma 4.9, we obtain the equality

$$\begin{aligned} \sum_{t=4}^n \frac{3t-4}{2t} c_{t-1} c_{n-t} &= \frac{3}{2} \sum_{t=4}^n c_{t-1} c_{n-t} - 2 \sum_{t=4}^n \frac{1}{t} c_{t-1} c_{n-t} \\ &= \frac{3}{2} (4^{n-1} - c_{n-1} - 2c_{n-2} - 6c_{n-3}) - 2 \left(\frac{1}{2} c_n - c_{n-1} - c_{n-2} - 2c_{n-3} \right) \\ &= 6 \cdot 4^{n-2} - c_n + \frac{1}{2} c_{n-1} - c_{n-2} - 5c_{n-3}. \end{aligned}$$

By (i)–(v), we have

$$\begin{aligned} \# \text{2-tilt } A &= c_{n-2} + 2(c_{n-1} - c_{n-2}) + 5c_{n-3} + 6 \cdot 4^{n-2} - c_n + \frac{2n-2}{n} c_{n-1} - c_{n-2} - 5c_{n-3} \\ &= 6 \cdot 4^{n-2} - c_n + \frac{4n-2}{n} c_{n-1} - 2c_{n-2} \\ &= 6 \cdot 4^{n-2} - 2c_{n-2}, \end{aligned}$$

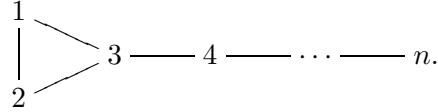
where the last equality follows from $c_n = \frac{2(2n-1)}{n} c_{n-1}$. \square

Thirdly, we give an enumeration for type (I). The number is obtained by using the result on type \mathbb{D} .

Proposition 4.10. *If $G = \mathbf{I}_n$, then we have*

$$\# \text{2-tilt } A = 6 \cdot 4^{n-2} + 2c_n - 4c_{n-1} - 4c_{n-2}.$$

Proof. Let G be a graph as follows.



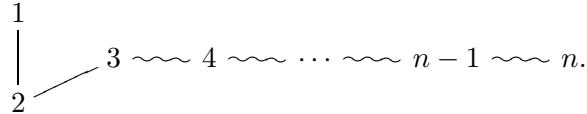
By using a similar method of the proof of proposition 4.8, we calculate the right-hand side of

$$\# \text{2-tilt } A = 2 \sum_{\substack{\epsilon: G_0 \rightarrow \{\pm 1\} \\ \epsilon(3) = +1}} |G_\epsilon|.$$

Let M and M_i ($1 \leq i \leq 5$) be sets of maps given in the proof of Proposition 4.8 and $\mathbf{m}(i) := \sum_{\epsilon \in M_i} |G_\epsilon|$. For each map $\epsilon \in M_1 \sqcup M_4 \sqcup M_5$, we have $G_\epsilon = (\mathbb{D}_n)_\epsilon$. Hence for each $i \in \{1, 4, 5\}$, we have

$$\mathbf{m}(i) = \sum_{\epsilon \in M_i} |G_\epsilon| = \sum_{\epsilon \in M_i} |(\mathbb{D}_n)_\epsilon| = \mathbf{n}(i).$$

Since $\mathbf{m}(2) = \mathbf{m}(3)$ holds by the symmetry of G , we have only to calculate $\mathbf{m}(2)$. For each map $\epsilon \in M_2$, the graph G_ϵ is given by



Then the calculation of $\mathbf{m}(2)$ is reduced to that of Dynkin graphs of type \mathbb{A} . In fact, let G' be the Dynkin graph \mathbb{A}_n . Then we have

$$\begin{aligned} \mathbf{m}(2) &= \sum_{\substack{\epsilon: G'_0 \rightarrow \{\pm 1\} \\ \epsilon(3) = +1}} |G'_\epsilon| - \sum_{\substack{\epsilon: G'_0 \rightarrow \{\pm 1\} \\ \epsilon(2) = \epsilon(3) = +1}} |G'_\epsilon| - \sum_{\substack{\epsilon: G'_0 \rightarrow \{\pm 1\} \\ -\epsilon(1) = -\epsilon(2) = \epsilon(3) = +1}} |G'_\epsilon| \\ &= \frac{1}{2} ||\mathbb{A}_n|| - \frac{1}{4} ||\mathbb{A}_2|| \cdot ||\mathbb{A}_{n-2}|| - \mathbf{n}(2). \\ &= \frac{1}{2} c_n - \frac{3}{2} c_{n-2} - \mathbf{n}(2). \end{aligned}$$

Therefore we obtain

$$\begin{aligned} \# \text{2-tilt } A &= 2(\mathbf{m}(1) + \mathbf{m}(2) + \mathbf{m}(3) + \mathbf{m}(4) + \mathbf{m}(5)) \\ &= 2(\mathbf{n}(1) + 2\mathbf{n}(2) + \mathbf{n}(4) + \mathbf{n}(5)) - 4\mathbf{n}(2) + 4\mathbf{m}(2) \\ &= ||\mathbb{D}_n|| - 4\mathbf{n}(2) + 4\mathbf{m}(2) \\ &= 6 \cdot 4^{n-2} - 2c_{n-2} - 4\mathbf{n}(2) + 2c_n - 6c_{n-2} - 4\mathbf{n}(2) \\ &= 6 \cdot 4^{n-2} + 2c_n - 4c_{n-1} - 4c_{n-2}. \end{aligned}$$

This finishes the proof. \square

For the remained finite series \mathbb{E} , (II), (III), (IV) and (V), we just compute the number by using the formula (3.1) in Corollary 3.5(2).

Proposition 4.11. *For each case \mathbb{E} , (II), (III), (IV) and (V), the number 2-tilt A_G is given by the table of Theorem 1.2.*

Proof. The number for \mathbb{E}_6 is shown in Example 3.7(2) and the others are similar. The detail is left to the reader. \square

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REFERENCES

- [1] T. Adachi, *The classification of τ -tilting modules over Nakayama algebras*, J. Algebra **452** (2016), 227–262.
- [2] T. Adachi, *Characterizing τ -tilting finite algebras with radical square zero*, Proc. Amer. Math. Soc. **144** (2016), no. 11, 4673–4685.
- [3] T. Adachi, O. Iyama, I. Reiten, *τ -tilting theory*, Compos. Math. **150** (2014), no. 3, 415–452.
- [4] T. Aihara, O. Iyama, *Silting mutation in triangulated categories*, J. Lond. Math. Soc. (2) **85** (2012), no. 3, 633–668.
- [5] T. Aoki, *Classifying torsion classes for algebras with radical square zero via sign decomposition*, arXiv:1803.03795v2.
- [6] E. Barnard, N. Reading, *Coxeter-biCatalan combinatorics*, J. Algebraic Combin. **47** (2018), no. 2, 241–300.
- [7] D.J. Benson, *Resolutions over symmetric algebras with radical cube zero*, J. Algebra **320** (2008), no. 1, 48–56.
- [8] S. Cautis, A. Licata, *Heisenberg categorification and Hilbert schemes*, Duke Math. J. **161** (2012), no. 13, 2469–2547.
- [9] L. Demonet, O. Iyama, N. Reading, I. Reiten, H. Thomas, *Lattice theory of torsion classes*, arXiv:1711.01785.
- [10] F. Eisele, G. Janssens, T. Raedschelders, *A reduction theorem for τ -rigid modules*, Math. Z. **290** (2018), no. 3–4, 1377–1413.
- [11] K. Erdmann, Ø. Solberg, *Radical cube zero weakly symmetric algebras and support varieties*, J. Pure Appl. Algebra **215** (2011), no. 2, 185–200.
- [12] E.L. Green, S. Schroll, *Multiserial and special multiserial algebras and their representations*, Adv. Math. **302** (2016), 1111–1136.
- [13] R.S. Huerfano, M. Khovanov, *A category for the adjoint representation*, J. Algebra **246** (2001), no. 2, 514–542.
- [14] M.A.A. Obaid, S.K. Nauman, W.M. Fakieh, C.M. Ringel, *The number of support-tilting modules for a Dynkin algebra*, J. Integer Seq. **18** (2015), no. 10, Article 15.10.6.
- [15] T. Okuyama, *On blocks of finite groups with radical cube zero*, Osaka J. Math. **23** (1986), no. 2, 461–465.
- [16] J. Rickard, *Morita theory for derived categories*, J. London Math. Soc. (2) **39** (1989), no. 3, 436–456.
- [17] P. Seidel, *Fukaya categories and Picard-Lefschetz theory*, Zurich Lectures in Advanced Mathematics. European Mathematical Society (EMS), Zürich, 2008.
- [18] X. Zhang, *τ -rigid modules for algebras with radical square zero*, arXiv:1211.5622v5.

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