

# On Stability and Convergence of a Three-layer Semi-discrete Scheme for an Abstract Analogue of the Ball Integro-differential Equation

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## Abstract

We consider the Cauchy problem for a second-order nonlinear evolution equation in a Hilbert space. This equation represents the abstract generalization of the Ball integro-differential equation. The general nonlinear case with respect to terms of the equation which include a square of a norm of a gradient is considered. A three-layer semi-discrete scheme is proposed in order to find an approximate solution. In this scheme, the approximation of nonlinear terms that are dependent on the gradient is carried out by using an integral mean. We show that the solution of the nonlinear discrete problem and its corresponding difference analogue of a first-order derivative is uniformly bounded. For the solution of the corresponding linear discrete problem, it is obtained high-order *a priori* estimates by using two-variable Chebyshev polynomials. Based on these estimates we prove the stability of the nonlinear discrete problem. For smooth solutions, we provide error estimates for the approximate solution. An iteration method is applied in order to find an approximate solution for each temporal step. The convergence of the iteration process is proved.

**Keywords and phrases:** Cauchy problem, Three-layer semi-discrete scheme, Nonlinear integro-differential equation, Stability and convergence, Abstract analogue of beam equation, Chebyshev polynomials.

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## Introduction

In the present work, we consider the Cauchy problem in the Hilbert space for a nonlinear second-order abstract differential equation. Coefficients in the main part of the equation are self-adjoint positively defined, in general, unbounded operators. Our goal is to find an approximate solution to this problem. To do so, we apply a three-layer symmetrical semi-discrete scheme. In this scheme, nonlinear terms are approximated by using integral mean.

The considered equation represents an abstract generalization of J. M. Ball beam equation (see [6]). J. M. Ball has generalized Kirchhoff type nonlinear equation for beam, that was obtained by S. Woinowsky-Krieger (see [37]), by introducing damping terms, in order to account for the effect of external and internal damping.

Investigation of the topics related to the classic Kirchhoff equation started with Bernstein's well-known paper (see [7]). In this paper existence and uniqueness issues for local as well as global solutions of initial-boundary value problem for the Kirchhoff string equation is studied. The issues of solvability of the classical and generalized Kirchhoff equations were later considered by many authors: A. Arosio, S. Panizzi [1], L. Berselli, R. Manfrin [8], P. D'Ancona, S. Spagnolo [12], [13], R. Manfrin [22], L. A. Medeiros [24], M. Matos [23], K. Nishihara [25], S. Panizzi [26]. In the works [1], [12], [13], [22] and [25] issues of well-posedness and global solvability

are thoroughly studied for a generalized Kirchhoff equation. In [26] the existence of a global solution with low regularity is studied for Kirchhoff-type equations. An abstract analogue of the Kirchhoff-type beam equation is considered in the work by L. A. Medeiros [24], where the existence and uniqueness theorem for the regular solution of the Cauchy problem is proved. The same abstract nonlinear equation, strengthened by the first derivative with respect to time, is discussed in the work by P. Biler and E. H. de Brito (see [9], [14]), where most attention is paid to study of the behaviour of Cauchy problem. We should note that participation of the square of the main operator in the linear part of this equation essentially helps to obtain the necessary *a priori* estimates.

The following works are dedicated to approximate solutions of initial-boundary value problems for classical and generalized Kirchhoff equations: A. I. Christie, J. Sanz-Serna [11], T. Geveci, I. Christie [15], I.-S. Liu, M.A. Rincon [21], J. Peradze [27], J. Rogava, M. Tsiklauri [33], [32] and in [36]. An algorithm of approximate solution for the dynamic beam equation is studied in [15]. This algorithm represents a combination of the Galerkin method for spatial coordinates and the finite difference method for the time coordinate. The same combination of the methods is investigated for the classic Kirchhoff equation in [27]. Design of algorithms for finding numerical solutions and their investigations for initial-boundary value problems of some classes integro-differential equations are considered in the book of T. Jangveladze, Z. Kiguradze and B. Neta [16].

As it was mentioned before J. M. Ball - generalized the Kirchhoff beam equation by introducing damping terms, to account for the effect of external and internal damping. For an approximate solution of the initial-boundary problem of this equation, S. M. Choo and S. K. Chung proposed the finite difference method (see [10]). In this work stability and convergence of the approximate solution is investigated.

As far as we know, issues of approximate solution of abstract analogue of Kirchhoff-type equation for a beam are less studied. In the present paper, investigations of stability and convergence of the designed semi-discrete scheme for second-order (complete kind) nonlinear operator differential equation that represents the abstract analogue of a model of J. M. Ball for the beam is based on two facts: (a)  $(u_k - u_{k-1})/\tau$  and  $B^{1/2}u_k$  are uniformly bounded ( $u_k$  is an approximate solution, and  $\tau$  is time step; linear operator  $B$  is included in the main part of the equation); (b) For the solution of the corresponding linear problem an *a priori* estimation is obtained where on the left-hand side power  $s$  and on the right-hand side power  $s - 1$  of the operator  $B$  is included. These facts give the possibility to weaken the nonlinear terms in the given nonlinear equation so much that, to make it possible to apply Grönwall's lemma. Besides, it is not required to impose any essential restriction for the temporal step  $\tau$ .

## 1 Statement of the problem and semi-discrete scheme

Let us consider the following Cauchy problem in Hilbert space  $H$ :

$$\begin{aligned} & \frac{d^2u}{dt^2} + a_1 B \frac{du}{dt} + a_2 Bu + \psi_1 \left( \|A^{1/2}u\|^2 \right) Au \\ & + \frac{d}{dt} \left( \psi_2 \left( \|A^{1/2}u\|^2 \right) \right) Au + \psi_3 \left( \|u\|^2 \right) u \\ & + Cu + N \frac{du}{dt} + M(u) = f(t), \quad t \in ]0, \bar{t}], \end{aligned} \quad (1.1)$$

$$u(0) = \varphi_0, \quad u'(t)|_{t=0} = \varphi_1 \quad (1.2)$$

where  $A$  and  $B$  are self-adjoint, positively defined (generally unbounded) operators with the domains  $D(A)$  and  $D(B)$  which are everywhere dense in  $H$ , besides, the following conditions are fulfilled

$$\|Au\|^2 \leq b_0^2 (Bu, u), \quad \forall u \in D(B) \subset D(A), \quad b_0 = \text{const} > 0, \quad (1.3)$$

where by  $\|\cdot\|$  and  $(\cdot, \cdot)$  are defined correspondingly the norm and scalar product in  $H$ ;  $\psi_1(s)$ ,  $\psi_2(s)$  and  $\psi_3(s)$ ,  $s \in [0, +\infty[$  are twice continuously differentiable nonnegative functions, besides

$\psi_2(s)$  is increasing function;  $C$  is linear operator, which satisfies the following condition

$$\|Cu\| \leq a_0 \|Au\| , \quad \forall u \in D(A) \subset D(C), \quad a_0 = \text{const} > 0; \quad (1.4)$$

$N$  is linear bounded operator; nonlinear operator  $M(\cdot)$  satisfies Lipschitz condition;  $a_1$  and  $a_2$  are positive constants  $\varphi_0$  and  $\varphi_1$  are given vectors from  $H$ ;  $u(t)$  is a twice continuously differentiable, unknown function with values in  $H$  and  $f(t)$  is given continuous function with values in  $H$ .

As in the linear case (see S. G. Krein [19]) vector function  $u(t)$  with values in  $H$ , defined on the interval  $[0, \bar{t}]$  is called a solution of the problem (1.1)-(1.2) if it satisfies the following conditions: (a)  $u(t)$  is twice continuously differentiable in the interval  $[0, \bar{t}]$ ; (b)  $u(t), u'(t) \in D(B)$  for any  $t$  from  $[0, \bar{t}]$  and  $Bu(t)$  and  $Bu'(t)$  are continuous functions; (c)  $u(t)$  satisfies equation (1.1) on the  $[0, \bar{t}]$  interval and the initial condition (1.2). Here continuity and differentiability is meant by metric  $H$ .

Equation (1.1) is an abstract analogue of the following equation

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} + a_1 \frac{\partial^4}{\partial x^4} \left( \frac{\partial u}{\partial t} \right) + a_2 \frac{\partial^4 u}{\partial x^4} - \left( \alpha + \beta \int_0^l [\partial_\xi u(\xi, t)]^2 d\xi \right) \frac{\partial^2 u}{\partial x^2} \\ - \gamma \frac{\partial}{\partial t} \left( \int_0^l [\partial_\xi u(\xi, t)]^2 d\xi \right) \frac{\partial^2 u}{\partial x^2} + \delta \frac{\partial u}{\partial t} = f(t), \quad (x, t) \in ]0, l[ \times ]0, \bar{t}], \end{aligned} \quad (1.5)$$

where  $a_1, a_2, \beta$  and  $\gamma$  are positive and  $\alpha, \delta$  any constants.

At the first time, equation (1.5) was considered by J. M. Ball in [6]. In this paper, J. M. Ball investigated the existence, uniqueness and asymptotic behaviour of the solution of equation (1.5) using the topological method.

We look for an approximate solution of the problem (1.1)-(1.2) using the following semi-discrete scheme

$$\begin{aligned} \frac{u_{k+1} - 2u_k + u_{k-1}}{\tau^2} + a_1 B \frac{u_{k+1} - u_{k-1}}{2\tau} + a_2 B \frac{u_{k+1} + u_{k-1}}{2} \\ + a_{1,k} A \frac{u_{k+1} + u_{k-1}}{2} + d_k A \frac{u_{k+1} + u_{k-1}}{2} + a_{3,k} \frac{u_{k+1} + u_{k-1}}{2} \\ + Cu_k + N \frac{u_{k+1} - u_{k-1}}{2\tau} + M(u_k) = f_k, \end{aligned} \quad (1.6)$$

where  $f_k = f(t_k)$ ,  $k = 1, \dots, n-1$ ,  $t_k = k\tau$ ,  $\tau = \bar{t}/n$  ( $n > 1$ ),

$$\begin{aligned} a_{1,k} &= \tilde{\psi}_1(\gamma_{k-1}, \gamma_{k+1}), \quad \gamma_k = \|A^{1/2}u_k\|^2, \\ d_k &= \frac{\psi_2(\gamma_{k+1}) - \psi_2(\gamma_{k-1})}{2\tau}, \quad a_{3,k} = \tilde{\psi}_3(\|u_{k-1}\|^2, \|u_{k+1}\|^2), \end{aligned}$$

and where function  $\tilde{\psi}_1(a, b)$  (analogously of  $\tilde{\psi}_3(a, b)$ ) is defined using the following formula

$$\tilde{\psi}_1(a, b) = \frac{1}{b-a} \int_a^b \psi_1(s) ds. \quad (1.7)$$

It is clear that if interval  $b-a$  is small enough, then (1.7) formula gives good approximation of  $\psi_1(s)$  function at  $s = (a+b)/2$ .

In (1.6), nonlinear terms are approximated using integral mean. This approach first was used in [27] and [32].

As an approximate solution  $u(t)$  of problem (1.1)-(1.2) at point  $t_k = k\tau$  we declare  $u_k$ ,  $u(t_k) \approx u_k$ .

**Remark 1.1.** From (1.3) condition it follows that

$$\|Au\| \leq b_0 \|B^{1/2}u\|, \quad \forall u \in D(B) \subset D(A). \quad (1.8)$$

It is known, that  $D(B)$  is a core of  $B^{1/2}$  (see [17], p. 354). It means that, for every  $u \in D(B^{1/2})$  there exists sequence  $u_n \in D(B)$  such that,  $u_n \rightarrow u$  and  $B^{1/2}u_n \rightarrow B^{1/2}u$ . From here, according to (1.8) it follows that  $Au_n$  is Cauchy sequence and it is clear that, since  $H$  is complete, this sequence is convergent.  $u \in D(A)$  and  $Au_n \rightarrow Au$  as  $A$  is closed operator. From here and (1.8) it follows that:

$$\|Au\| \leq b_0 \|B^{1/2}u\|, \quad \forall u \in D(B^{1/2}) \subset D(A). \quad (1.9)$$

## 2 Uniform boundedness of solution of discrete problem and difference analogue of the first-order derivative

The following theorem takes place (below everywhere  $c$  denotes positive constant).

**Theorem 2.1.** For discrete problem (1.6) the vectors  $(u_k - u_{k-1})/\tau$  and  $B^{1/2}u_k$  are uniformly bounded, i.e. there exist constants  $c_1$  and  $c_2$  (independent of  $n$ ) such that

$$\left\| \frac{u_k - u_{k-1}}{\tau} \right\| \leq c_1, \quad \|B^{1/2}u_k\| \leq c_2, \quad k = 1, \dots, n.$$

**Proof.** If we multiply both sides of the equality (1.6) on vector  $u_{k+1} - u_{k-1} = (u_{k+1} - u_k) + (u_k - u_{k-1})$ , we obtain

$$\begin{aligned} & \alpha_{k+1} - \alpha_k + 2a_1\tau \|B^{1/2}\delta u_k\|^2 + \frac{1}{2}\beta_{k+1} - \frac{1}{2}\beta_{k-1} + \frac{1}{2}a_{1,k}(\gamma_{k+1} - \gamma_{k-1}) \\ & + \frac{1}{2}d_k(\gamma_{k+1} - \gamma_{k-1}) + \frac{1}{2}a_{3,k}(\vartheta_{k+1} - \vartheta_{k-1}) = (g_k, u_{k+1} - u_{k-1}), \end{aligned} \quad (2.1)$$

where

$$\begin{aligned} \alpha_k &= \left\| \frac{u_k - u_{k-1}}{\tau} \right\|^2, \quad \beta_k = \|B^{1/2}u_k\|^2, \quad \gamma_k = \|A^{1/2}u_k\|^2, \quad \vartheta_k = \|u_k\|^2, \\ g_k &= f_k - Cu_k - N\delta u_k - M(u_k), \quad \delta u_k = \frac{u_{k+1} - u_{k-1}}{2\tau}. \end{aligned}$$

According to (1.7) we get:

$$\begin{aligned} a_{1,k}(\gamma_{k+1} - \gamma_{k-1}) &= \tilde{\psi}_1(\gamma_{k-1}, \gamma_{k+1})(\gamma_{k+1} - \gamma_{k-1}) \\ &= \int_0^{\gamma_{k+1}} \psi_1(s)ds - \int_0^{\gamma_{k-1}} \psi_1(s)ds, \\ a_{3,k}(\vartheta_{k+1} - \vartheta_{k-1}) &= \int_0^{\vartheta_{k+1}} \psi_3(s)ds - \int_0^{\vartheta_{k-1}} \psi_3(s)ds. \end{aligned}$$

Besides according to the monotonicity of function  $\psi_2(s)$ , the following estimation is valid

$$d_k(\gamma_{k+1} - \gamma_{k-1}) = \frac{1}{2\tau}(\psi_2(\gamma_{k+1}) - \psi_2(\gamma_{k-1}))(\gamma_{k+1} - \gamma_{k-1}) \geq 0.$$

Then from (2.1) we get

$$\lambda_{k+1} \leq \lambda_k + |(g_k, u_{k+1} - u_{k-1})|, \quad (2.2)$$

where  $\lambda_k = \alpha_k + \frac{1}{2}(\beta_k + \beta_{k-1} + \mu_k + \mu_{k-1} + \nu_k + \nu_{k-1})$ ,

$$\mu_k = \int_0^{\gamma_k} \psi_1(s) ds, \quad \nu_k = \int_0^{\vartheta_k} \psi_3(s) ds.$$

If we use Schwarz inequality, condition (1.4) and Remark 1.1 we obtain

$$\begin{aligned} & |(g_k, u_{k+1} - u_{k-1})| \\ & \leq \|u_{k+1} - u_{k-1}\| \left( \|f_k\| + a_0 b_0 \sqrt{\beta_k} + \|M(u_k)\| + \|N\delta u_k\| \right). \end{aligned}$$

From here follows

$$\begin{aligned} |(g_k, u_{k+1} - u_{k-1})| & \leq \tau (\sqrt{\alpha_{k+1}} + \sqrt{\alpha_k}) (\|f_k\| + \|M(u_k)\| \\ & \quad + c_1 \sqrt{\beta_k} + c_0 (\sqrt{\alpha_{k+1}} + \sqrt{\alpha_k})) \end{aligned} \quad (2.3)$$

where  $c_0 = \frac{1}{2} \|N\|$ ,  $c_1 = a_0 b_0$ .

For nonlinear operator  $M(\cdot)$  due to Lipschitz condition we have

$$\begin{aligned} \|M(u_k)\| & \leq c (\|u_k - u_{k-1}\| + \|u_{k-1} - u_{k-2}\| + \dots + \|u_1 - u_0\|) + \|M(u_0)\| \\ & = c\tau (\sqrt{\alpha_k} + \sqrt{\alpha_{k-1}} + \dots + \sqrt{\alpha_1}) + \|M(u_0)\|. \end{aligned} \quad (2.4)$$

If we insert inequality (2.4) into (2.3) we get

$$|(g_k, u_{k+1} - u_{k-1})| \leq \tau (\sqrt{\alpha_{k+1}} + \sqrt{\alpha_k}) \sigma_k, \quad (2.5)$$

where

$$\sigma_k = c\tau \sum_{i=1}^k \sqrt{\alpha_i} + c_1 \sqrt{\beta_k} + c_0 (\sqrt{\alpha_{k+1}} + \sqrt{\alpha_k}) + \|f_k\| + \|M(u_0)\|.$$

From (2.2) according to (2.5) it follows

$$\lambda_{k+1} \leq \lambda_k + \varepsilon_k, \quad (2.6)$$

where  $\varepsilon_k = \tau (\sqrt{\alpha_{k+1}} + \sqrt{\alpha_k}) \sigma_k$ .

Obviously from (2.6) we obtain

$$\lambda_{k+1} \leq \lambda_1 + \tau \sum_{i=1}^k (\sqrt{\alpha_i} + \sqrt{\alpha_{i+1}}) \sigma_i,$$

from here we have

$$\delta_{k+1}^2 \leq \delta_1^2 + \tau \sum_{i=1}^k (\delta_i + \delta_{i+1}) \sigma_i, \quad \delta_k = \sqrt{\lambda_k}.$$

from here follows the following inequality

$$\delta_{k+1} \leq \delta_1 + 2\tau \sum_{i=1}^k \sigma_i \leq c\tau \sum_{i=1}^k \delta_i + 2c_0 \tau \delta_{k+1} + \eta_k, \quad (2.7)$$

where

$$\eta_k = \delta_1 + c \|M(u_0)\| + 2\tau \sum_{i=1}^k \|f_i\|.$$

If we assume that  $1 - 2\tau c_0 = 1 - \tau \|N\| > 0$ , then from (2.7) we have

$$\delta_{k+1} \leq c\tau \sum_{i=1}^k \delta_i + c\eta_k.$$

From here, according to discrete analogue of Grönwall's lemma, we have

$$\delta_{k+1} \leq c \exp(c\tau) (\eta_k + \tau\delta_1).$$

From here it follows that  $\alpha_k$  and  $\beta_k$  are uniformly bounded.  $\square$

**Remark 2.1.** From uniform boundedness of vectors  $B^{1/2}u_k$  follows uniform boundedness of  $Au_k$  vectors (see inequality (1.9)). From this fact and the following inequality

$$\|A^{1/2}u\| = \|A^{-1/2}(Au)\| \leq \frac{1}{\sqrt{m_A}} \|Au\|, \quad \forall u \in D(A), \quad (2.8)$$

where  $m_A > 0$  is lower bound of operator  $A$  ( $(Au, u) \geq m_A(u, u)$ ), follows that  $A^{1/2}u_k$  is uniformly bounded.

**Remark 2.2.** According to the triangle inequality, from uniform boundedness of  $(u_{k+1} - u_k)/\tau$  vectors follows uniform boundedness of  $(u_{k+1} - u_{k-1})/2\tau$  vectors.

**Remark 2.3.** From uniform boundedness of  $\|Au_k\|$  and  $\|(u_{k+1} - u_k)/\tau\|$  follows uniform boundedness of

$$|(\gamma_k - \gamma_{k-1})/\tau|.$$

### 3 Estimations for two-variable Chebyshev polynomials

To obtain *a priori* estimations for the main linear part of the difference equation (1.6) we require estimations for a specific class of polynomials that we call two-variable Chebyshev polynomials. These polynomials are defined using the following recurrence relation (see [31]):

$$\begin{aligned} U_{k+1}(x, y) &= xU_k(x, y) - yU_{k-1}(x, y), \quad k = 1, 2, \dots, \\ U_1(x, y) &= x, \quad U_0(x, y) \equiv 1. \end{aligned} \quad (3.1)$$

$U_k(x, y)$  we call two-variable Chebyshev polynomials, as  $U_k(2x, 1)$  represents Chebyshev polynomials of the second kind (see, e.g. [35]).

From recurrence relation (3.1) using induction, we get the following formula

$$U_k(x, y) = \sqrt{y^k} U_k(\xi, 1), \quad \xi = \frac{x}{\sqrt{y}}, \quad y > 0. \quad (3.2)$$

The formula (3.2) is important as it relates  $U_k(x, y)$  polynomials with classic Chebyshev polynomials (we assume that in classic Chebyshev polynomials  $x$  variable is replaced by  $x/2$ ). Let us introduce the following domains:

$$\begin{aligned} \Delta &= \{(x, y) : |y| < 1 \text{ and } |x| < y + 1\}, \\ \Omega^+ &= \{(x, y) : 4y - x^2 > 0\}, \quad \Omega^- = \{(x, y) : 4y - x^2 < 0\}, \\ \Delta^+ &= \{(x, y) \in \Delta : x \geq 0\}, \quad \Omega_1 = \Omega^+ \cap \Delta^+, \quad \Omega_2 = \Omega^- \cap \Delta^+. \end{aligned}$$

It is well-known that roots of the classic Chebyshev polynomials are in  $] -1, 1[$  (see, e.g., [35]). From here, according to formula (3.2), it follows that, for any fixed positive  $y$  roots of the polynomial  $U_k(x, y)$  are inside  $] -2\sqrt{y}, 2\sqrt{y}[$ . Besides if we take into consideration, that

$U_k(\pm 2, 1) = (-1)^k(k+1)$  and  $|U_k(2\xi, 1)|$  reaches its maximum on boundary (see, e.g., [35]), then from formula (3.2) follows the following estimation

$$|U_k(x, y)| \leq U_k(2\sqrt{y}, y) = (k+1)\sqrt{y^k}, \quad (x, y) \in \Omega^+. \quad (3.3)$$

From above discussion, we conclude that for any positive  $y$ ,  $U_k(x, y)$  is increasing function regarding  $x$  variable, when  $x \geq 2\sqrt{y}$ . Besides from recurrence relation (3.1) it follows that, for any fixed  $y \leq 0$ ,  $U_k(x, y)$  is increasing function regarding to  $x$  variable, when  $x \geq 0$ . From here we obtain

$$|U_k(x, y)| \leq U_k(1+y, y) = 1 + y + \dots + y^k, \quad (3.4)$$

where  $y \geq -1$  and  $|x| \leq 1+y$ .

From (3.4) follows the following estimation

$$|U_k(x, y)(1-y)| \leq 1, \quad (x, y) \in \Delta. \quad (3.5)$$

We also need estimation for  $U_k(x, y) - y^m U_{k-1}(x, y)$ ,  $m = 0, 1$ , polynomials, where  $(x, y) \in \Delta^+$ .

The following inequality is simply obtained

$$|U_k(x, 1) - U_{k-1}(x, 1)| \leq \frac{2}{\sqrt{2+x}}, \quad x \in ]-2, 2]. \quad (3.6)$$

According to formula (3.2) and inequality (3.6) the following estimation is valid

$$|U_k(x, y) - \sqrt{y}U_{k-1}(x, y)| = \sqrt{y^k} |U_k(\xi, 1) - U_{k-1}(\xi, 1)| \leq \sqrt{2y^k}, \quad (3.7)$$

where  $\xi = x/\sqrt{y}$ ,  $(x, y) \in \Omega_1$ .

Let us estimate the following difference  $U_k(x, y) - yU_{k-1}(x, y)$ , when  $(x, y) \in \Omega_1$ . According to inequalities (3.7) and (3.3) we have

$$|U_k(x, y) - yU_{k-1}(x, y)| \leq \sqrt{2}, \quad (x, y) \in \Omega_1. \quad (3.8)$$

Let us estimate the following difference  $U_k(x, y) - yU_{k-1}(x, y)$ , when  $(x, y) \in \Omega_2$  and  $y > 0$ . For that we require the following formulas:

$$U_k(x, y) = \sqrt{y^k} \sum_{i=0}^k C_{k+i+1}^{2i+1} (\xi - 2)^i, \quad (3.9)$$

$$U_k(x, y) - \sqrt{y}U_{k-1}(x, y) = \sqrt{y^k} \sum_{i=0}^k C_{k+i}^{2i} (\xi - 2)^i, \quad (3.10)$$

where  $\xi = x/\sqrt{y}$ ,  $C_k^i$  are binomial coefficients ( $C_k^0 = 1$ ).

Using simple transformation from (3.9) we obtain formula (3.10). The formula (3.9) can be obtained by using Taylor expansion of  $U_k(\xi, 1)$  at  $\xi = 2$ . We need to consider that  $U_k^{(i)}(2, 1) = i!C_{k+i+1}^{2i+1}$ .

From the following equality

$$U_k(x, y) - yU_{k-1}(x, y) = (U_k(x, y) - \sqrt{y}U_{k-1}(x, y)) + (1 - \sqrt{y}) \sqrt{y}U_{k-1}(x, y),$$

according to formulas (3.9) and (3.10), it follows that, for any fixed  $y$  from  $]0, 1]$  interval  $U_k(x, y) - yU_{k-1}(x, y)$  is increasing function regarding  $x$ , when  $x \geq 2\sqrt{y}$ . From here follows

$$U_k(2\sqrt{y}, y) - yU_{k-1}(2\sqrt{y}, y) \leq U_k - yU_{k-1} \leq U_k(1+y, y) - yU_{k-1}(1+y, y), \quad (3.11)$$

where  $y > 0$  and  $(x, y) \in \Omega_2$ .

If we insert  $U_k(2\sqrt{y}, y) = (k+1)\sqrt{y^k}$  and (3.4) in the relation (3.11) then we obtain the following estimation

$$\sqrt{y^k} ((k+1)(1 - \sqrt{y}) + \sqrt{y}) \leq U_k(x, y) - yU_{k-1}(x, y) \leq 1, \quad (3.12)$$

where  $y > 0$  and  $(x, y) \in \Omega_2$ .

We straightforwardly obtain the following inequality

$$0 \leq U_k(x, y) - yU_{k-1}(x, y) \leq 1, \quad (3.13)$$

where  $y \leq 0$  and  $(x, y) \in \Omega_2$ .

From estimations (3.8), (3.12) and (3.13) we have

$$|U_k(x, y) - yU_{k-1}(x, y)| \leq \sqrt{2}, \quad (x, y) \in \Delta^+. \quad (3.14)$$

Analogously to (3.14) we obtain

$$|U_k(x, y) - U_{k-1}(x, y)| \leq \sqrt{2}, \quad (x, y) \in \Delta^+. \quad (3.15)$$

## 4 High Order *a priori* estimations for three-layer semi-discrete scheme corresponding to the second-order evolution equation

The three-layer scheme is natural for the second-order evolution equation and the two-layer scheme is natural for the first-order evolution equation. Investigation of a three-layer scheme is more difficult than an investigation of a two-layer scheme. This difficulty can be somehow simplified if we reduce the second-order evolution equation to the first-order evolution equation by introducing additional unknowns. In this case, a self-adjoint operator is replaced by an operator matrix that is not self-adjoint anymore. This makes it complicated to investigate the corresponding discrete problem.

Obtaining such *a priori* estimations from where follows stability and convergence of the nonlinear semi-discrete scheme (1.6), is based on high order accuracy *a priori* estimations for corresponding linearized semi-discrete scheme. In this section, we obtain *a priori* estimation for a three-layer semi-discrete scheme for the second-order evolution equation. In this estimation on the left-hand side, we have positive  $s$  power for the main operator, while on the right-hand side we have  $(s - 1)$ . This allows us to make weaken nonlinear part of the equation in such a way that, using the results obtained in the previous section we are able to use Grönwall's lemma and obtain the final estimation. To obtain these estimations we require to construct the exact representation of three-layer recurrence relations with operator coefficients by using two-variable Chebyshev polynomials. These kinds of estimates were obtained before by one of the authors of the presented papers (see [31], [30]).

Important results for constructing and investigating approximate schemes for the Cauchy problem for second-order evolution equations were obtained by the following authors: G. A. Baker [2], G. A. Baker, J. H. Bramble [3], G. A. Baker, V. A. Dougalis, S. M. Serbin [4], L. A. Bales [5], J. Kačur [18], O. Ladyzhenskaya [20], M. Pultar [28], P. E. Sobolevskij, L. M. Chebotarova [34].

Let us consider in Hilbert space  $H$  the following linear difference equation

$$\frac{u_{k+1} - 2u_k + u_{k-1}}{\tau^2} + a_1 B \frac{u_{k+1} - u_{k-1}}{2\tau} + a_2 B \frac{u_{k+1} + u_{k-1}}{2} = f_k, \quad (4.1)$$

where  $k = 1, \dots, n - 1$ ,  $u_0$ ,  $u_1$  and  $f_k$  are the given vectors from  $H$ .

Difference equation (4.1) represents main part of the nonlinear equation (1.6), and obviously corresponds main part of the equation (1.1).

The following lemma takes place.

**Lemma 4.1.** Let  $B$  be self-adjoint positively defined operator and  $1 - \tau a > 0$ ,  $a = a_2/a_1$ . Then for scheme (4.1) the following a priori estimation is valid:

$$\begin{aligned} \|B^s u_{k+1}\| &\leq \sqrt{2} \|B^s u_0\| + \frac{1}{a_1} \left\| B^{s-1} \frac{\Delta u_0}{\tau} \right\| + \frac{\tau}{2} (1 + \tau a) \left\| B^s \frac{\Delta u_0}{\tau} \right\| \\ &+ \frac{\tau}{a_1} \sum_{i=1}^k \left\| B^{s-1} f_i \right\|, \quad u_0, u_1 \in D(B^s), \quad f_i \in D(B^{s-1}), \end{aligned} \quad (4.2)$$

$$\left\| \frac{\Delta u_k}{\tau} \right\| \leq a \|u_0\| + \sqrt{2} \left\| \frac{\Delta u_0}{\tau} \right\| + \sqrt{2} \tau \sum_{i=1}^k \|f_i\|, \quad (4.3)$$

where  $s \geq 0$ ,  $k = 1, \dots, n-1$ ,  $\Delta u_k = u_{k+1} - u_k$  ( $B^0 = I$ ).

**Remark 4.1.** In section 2 (see Theorem 2.1 and Remark 2.1, Remark 2.2 and Remark 2.3) there was shown that  $\|(u_{k+1} - u_k)/\tau\|$ ,  $\|A^{1/2} u_k\|$ ,  $\|A u_k\|$ ,  $\|B^{1/2} u_k\|$  and  $|(\gamma_k - \gamma_{k-1})/\tau|$  are uniformly bounded. These results along with (4.2) and (4.3) allows to obtain such a priori estimations for semi-discrete scheme (1.6) from where follows stability and convergence of the presented method.

**Remark 4.2.** A priori estimations (4.2) and (4.3) have independent meaning, as constants on the right-hand side are absolute constants (does not depend on interval length). Besides these constants cannot be improved. To obtain these estimations were possible by constructing exact representations for the solution of difference equation (4.1).

**Proof of Lemma 4.1.** Let us rewrite difference equation (4.1) in the following form to prove inequality (4.2)

$$B_0 u_{k+1} - 2I u_k + B_1 u_{k-1} = \tau^2 f_k, \quad (4.4)$$

where

$$B_0 = I + \frac{\tau}{2} a_1 B + \frac{\tau^2}{2} a_2 B, \quad B_1 = B_0 - \tau a_1 B.$$

From (4.4) we get

$$u_{k+1} = L u_k - S u_{k-1} + \frac{\tau^2}{2} L f_k, \quad (4.5)$$

where  $L = 2B_0^{-1}$ ,  $S = B_1 B_0^{-1}$ .

Let us note that  $L$  and  $S$  are self-adjoint, bounded linear operators in a Hilbert space  $H$ . Besides it is obvious that  $L$  and  $S$  are commutative.

Using mathematical induction for (4.5) recurrence relation we get

$$u_{k+1} = U_k(L, S) u_1 - S U_{k-1}(L, S) u_0 + \frac{\tau^2}{2} \sum_{i=1}^k U_{k-i}(L, S) L f_i, \quad (4.6)$$

where operator polynomial  $U_k(L, S)$  satisfies the following recurrence relation:

$$\begin{aligned} U_k(L, S) &= L U_{k-1}(L, S) - S U_{k-2}(L, S), \quad k = 1, 2, \dots, \\ U_0(L, S) &= I, \quad U_{-1}(L, S) = 0. \end{aligned} \quad (4.7)$$

Scalar polynomials  $U_k(x, y)$  corresponding to  $U_k(L, S)$  satisfy (3.1) recurrence relation.

From (4.6) using simple transformation we have

$$u_{k+1} = \tau U_k(L, S) \frac{\Delta u_0}{\tau} + (U_k(L, S) - S U_{k-1}(L, S)) u_0 + \frac{\tau^2}{2} \sum_{i=1}^k U_{k-i}(L, S) L f_i. \quad (4.8)$$

If we apply operator  $B^s$  ( $s \geq 0$ ) on both sides of equality (4.8) and move on to the norm we obtain

$$\begin{aligned} \|B^s u_{k+1}\| &\leq \tau \left\| B^s U_k(L, S) \frac{\Delta u_0}{\tau} \right\| + \|U_k(L, S) - SU_{k-1}(L, S)\| \|B^s u_0\| \\ &+ \frac{\tau^2}{2} \sum_{i=1}^k \|B^s U_{k-i}(L, S) L f_i\|. \end{aligned} \quad (4.9)$$

Using simple transformations we get:

$$S = \frac{1}{1 + \tau a} L - \frac{1 - \tau a}{1 + \tau a} I, \quad (4.10)$$

$$I - S = \frac{\tau a_1}{2} BL. \quad (4.11)$$

From here we have

$$\begin{aligned} \tau \|B^s U_k(L, S) L f\| &= \tau \|U_k(L, S) BL(B^{s-1} f)\| \\ &= \frac{2}{a_1} \|U_k(L, S)(I - S)(B^{s-1} f)\| \leq \frac{2}{a_1} \|U_k(L, S)(I - S)\| \|B^{s-1} f\|, \end{aligned} \quad (4.12)$$

where  $f \in D(B^{s-1})$ .

As it is known the norm of an operator function, when the argument of the function represents self-adjoint bounded operator, is equal to  $C$ -norm of the corresponded scalar function (see, e.g., [29], Chapter VII). Using this and representation (4.10) we have

$$\|U_k(L, S)(I - S)\| \leq \max_{y \in \sigma(S)} |U_k(\eta(y), y)(1 - y)|, \quad (4.13)$$

where  $\eta(y) = (1 + \tau a)y + (1 - \tau a)$ .

Let us estimate the spectrum of operator  $L$ . As, according to the condition,  $B$  is a self-adjoint and positively defined operator, therefore, we have  $\sigma(L) \subset [0, 2]$ . According to this relation, from the representation (4.10) we have

$$\sigma(S) \subset [-1, 1]. \quad (4.14)$$

If we consider relation (4.14) and estimation (3.5) we get

$$\begin{aligned} \max_{y \in \sigma(S)} |U_k(\eta(y), y)(1 - y)| &\leq \max_{y \in [-1, 1]} |U_k(\eta(y), y)(1 - y)| \\ &\leq \max_{(x, y) \in \Delta^+} |U_k(x, y)(1 - y)| \leq 1. \end{aligned}$$

From here and (4.13) it follows

$$\|U_k(L, S)(I - S)\| \leq 1. \quad (4.15)$$

It is obvious, from (4.12) using (4.15) we have

$$\tau \|B^s U_k(L, S) L f\| \leq \frac{2}{a_1} \|B^{s-1} f\|, \quad f \in D(B^{s-1}). \quad (4.16)$$

Also from (4.16) follows the inequality

$$\tau \|B^s U_k(L, S) f\| \leq \frac{1}{a_1} \|B^{s-1} B_0 f\| \leq \frac{1}{a_1} \|B^{s-1} f\| + \frac{\tau}{2}(1 + \tau a) \|B^s f\|, \quad (4.17)$$

where  $f \in D(B^s)$ .

Let us estimate operator  $U_k - SU_{k-1}$ . Analogously, we have

$$\|U_k(L, S) - SU_{k-1}(L, S)\| \leq \max_{y \in \sigma(S)} |U_k(\eta(y), y) - y U_{k-1}(\eta(y), y)|.$$

If we consider relation (4.14) and estimation (3.14) we get

$$\max_{y \in \sigma(S)} |U_k(\eta(y), y) - yU_{k-1}(\eta(y), y)| \leq \max_{(x,y) \in \Delta^+} |U_k(x, y) - yU_{k-1}(x, y)| \leq \sqrt{2}.$$

So, we have

$$\|U_k(L, S) - SU_{k-1}(L, S)\| \leq \sqrt{2}. \quad (4.18)$$

If we insert estimations (4.16), (4.17) and (4.18) into inequality (4.9) we obtain estimation (4.2).

Let us prove estimation (4.3). From the formula (4.8) according to the recurrence relation (4.7) we get

$$\begin{aligned} \left\| \frac{\Delta u_k}{\tau} \right\| &\leq \tau^{-1} \|(L - S - I)U_{k-1}(L, S)\| \|u_0\| \\ &+ \|U_k - U_{k-1}\| \left\| \frac{\Delta u_0}{\tau} \right\| + \frac{\tau}{2} \sum_{i=1}^k \|U_{k-i} - U_{k-i-1}\| \|L\| \|f_i\|. \end{aligned} \quad (4.19)$$

According to equality (4.11) we have

$$L - S - I = I - S - (2I - L) = I - S - (1 + \tau a)(I - S) = -\tau a(I - S).$$

If we consider this representation and estimation (4.15), we get

$$\tau^{-1} \|(L - S - I)U_{k-1}(L, S)\| = a \|U_k(L, S)(I - S)\| \leq a. \quad (4.20)$$

Analogously to (4.18), according to estimation (3.15) we obtain

$$\|U_k(L, S) - U_{k-1}(L, S)\| \leq \sqrt{2}. \quad (4.21)$$

From inequality (4.19) using estimations (4.20), (4.21) and  $\|L\| \leq 2$  *a priori* estimation (4.3) follows.  $\square$

**Remark 4.3.** Analogously to (4.3) the following estimation can be obtained

$$\left\| B^s \frac{\Delta u_k}{\tau} \right\| \leq a \|B^s u_0\| + \sqrt{2} \left\| B^s \frac{\Delta u_0}{\tau} \right\| + \sqrt{2} \tau \sum_{i=1}^k \|B^s f_i\|, \quad (4.22)$$

where  $u_0, u_1, f_i \in D(B^s)$ ,  $s \geq 0$ .

## 5 The *a priori* estimates for perturbation of the solution of the discrete problem

The goal of the section is to show the stability of the scheme (1.6). As additivity does not take place for nonlinear cases, therefore it is natural that we try to obtain the *a priori* estimates exactly for the solution perturbation. From here (analogously to a linear problem) automatically follows the stability and convergence of the nonlinear scheme.

In this section, based on the results of the previous sections, we obtain the *a priori* estimates for the solution of the semi-discrete scheme (1.6) and perturbation of the corresponding first-order difference.

The following theorem takes place (below everywhere  $c$  denotes positive constant).

**Theorem 5.1.** Let  $u_k$  and  $\bar{u}_k$  be solutions of difference equation (1.6) corresponding to initial vectors  $(u_0, u_1, f_k)$  and  $(\bar{u}_0, \bar{u}_1, \bar{f}_k)$ , components of which are sufficiently smooth. Then for  $z_k = u_k - \bar{u}_k$  the following estimates are true

$$\begin{aligned} & \|B^{1/2}z_{k+1}\| + \left\| \frac{\Delta z_k}{\tau} \right\| \\ & \leq c \left( \|B^{1/2}z_0\| + \left\| \frac{\Delta z_0}{\tau} \right\| + \tau \left\| B^{1/2} \frac{\Delta z_0}{\tau} \right\| + \tau \sum_{i=1}^k \|f_i - \bar{f}_i\| \right), \end{aligned} \quad (5.1)$$

where  $k = 1, \dots, n-1$ ,  $\Delta z_k = z_{k+1} - z_k$ .

In this section and the following ones the notation  $\delta z_k = \frac{z_{k+1} - z_{k-1}}{2\tau}$  is applied.

Let us prove the corresponding auxiliary lemma.

**Lemma 5.1.** The following inequality is true

$$|d_k - \bar{d}_k| \leq c (\|Az_{k+1}\| + \|Az_{k-1}\| + \|\delta z_k\|), \quad (5.2)$$

where  $z_k = u_k - \bar{u}_k$ ,

$$d_k = \delta\psi_2(\gamma_k), \quad \bar{d}_k = \delta\psi_2(\bar{\gamma}_k), \quad \gamma_k = \|A^{1/2}u_k\|^2, \quad \bar{\gamma}_k = \|A^{1/2}\bar{u}_k\|^2.$$

**Proof.** The following representation is valid

$$d_k - \bar{d}_k = (\delta\gamma_k - \delta\bar{\gamma}_k) I_{1,k} + \delta\bar{\gamma}_k I_{2,k}, \quad (5.3)$$

where

$$I_{1,k} = \int_0^1 \psi'_2(l_k(\xi)) d\xi, \quad I_{2,k} = \int_0^1 [\psi'_2(l_k(\xi)) - \psi'_2(\bar{l}_k(\xi))] d\xi,$$

and where

$$l_k(\xi) = \gamma_{k-1} + (\gamma_{k+1} - \gamma_{k-1})\xi, \quad \bar{l}_k(\xi) = \bar{\gamma}_{k-1} + (\bar{\gamma}_{k+1} - \bar{\gamma}_{k-1})\xi.$$

Using standard transformations we get

$$\delta\gamma_k - \delta\bar{\gamma}_k = (Az_{k+1}, \delta u_k + \delta\bar{u}_k) + (\delta z_k, A(u_{k-1} + \bar{u}_{k-1})).$$

From here using the Schwarz inequality follows

$$|\delta\gamma_k - \delta\bar{\gamma}_k| \leq \|Az_{k+1}\| (\|\delta u_k\| + \|\delta\bar{u}_k\|) + \|\delta z_k\| (\|Au_{k-1}\| + \|A\bar{u}_{k-1}\|). \quad (5.4)$$

If we consider that all terms on the right-hand side of the inequality (5.4) are bounded (see Remark 2.1 and Remark 2.2), then we obtain

$$|\delta\gamma_k - \delta\bar{\gamma}_k| \leq c (\|Az_{k+1}\| + \|\delta z_k\|). \quad (5.5)$$

Let us estimate integrals  $I_{1,k}$  and  $I_{2,k}$ . If we use the change of variables for the integral  $I_{2,k}$  we obtain

$$I_{2,k} = \int_0^1 \int_{\bar{l}_k(\xi)}^{l_k(\xi)} \psi''_2(\eta) d\eta d\xi. \quad (5.6)$$

From (5.6) we have

$$|I_{2,k}| \leq c \int_0^1 |\chi(\xi)| d\xi, \quad (5.7)$$

where

$$\begin{aligned}\chi(\xi) &= (\gamma_{k-1} - \bar{\gamma}_{k-1}) + ((\gamma_{k+1} - \bar{\gamma}_{k+1}) - (\gamma_{k-1} - \bar{\gamma}_{k-1})) \xi, \\ c &= \max |\psi_2''(\eta)| < +\infty, \quad 0 \leq \eta \leq \max_k (\gamma_k, \bar{\gamma}_k) < +\infty.\end{aligned}$$

Regarding inequality (5.7) it should be noted that since  $\gamma_k$  and  $\bar{\gamma}_k$  are uniformly bounded (see Remark 2.1) therefore  $\max_k (\gamma_k, \bar{\gamma}_k) < +\infty$ , i.e. the interval where we have to find a maximum value of  $|\psi_2''(\eta)|$  is finite. From here follows that  $\max |\psi_2''(\eta)| < +\infty$  (according to the condition  $\psi_2''(\eta)$  is continuous). So, from (5.7) we have

$$|I_{2,k}| \leq c \int_0^1 |\chi(\xi)| d\xi \leq c (|\gamma_{k-1} - \bar{\gamma}_{k-1}| + |\gamma_{k+1} - \bar{\gamma}_{k+1}|). \quad (5.8)$$

As vectors  $A^{1/2}u_k$  are uniformly bounded, for the difference  $\gamma_k - \bar{\gamma}_k$  the following estimation is valid

$$|\gamma_k - \bar{\gamma}_k| = \left( \sqrt{\gamma_k} + \sqrt{\bar{\gamma}_k} \right) \left| \sqrt{\gamma_k} - \sqrt{\bar{\gamma}_k} \right| \leq c \|A^{1/2}z_k\| \leq c \|Az_k\|. \quad (5.9)$$

From (5.8) according to (5.9) follows

$$|I_{2,k}| \leq c (\|Az_{k+1}\| + \|Az_{k-1}\|). \quad (5.10)$$

For the integrals  $I_{1,k}$  the following estimation is valid

$$|I_{1,k}| \leq c, \quad c = \max |\psi_2'(s)| < +\infty, \quad 0 \leq s \leq \max_k (\gamma_k, \bar{\gamma}_k) < +\infty. \quad (5.11)$$

From (5.3) according to inequalities (5.5), (5.10), (5.11) and Remark 2.3 follows (5.2).  $\square$

Let us return to proof of **Theorem 5.1**.

**Proof.** According to (1.6) difference  $z_k = u_k - \bar{u}_k$  satisfies the following equation

$$\frac{z_{k+1} - 2z_k + z_{k-1}}{\tau^2} + a_1 B \frac{z_{k+1} + z_{k-1}}{2} + a_2 B \frac{z_{k+1} + z_{k-1}}{2} = -\frac{1}{2} g_k, \quad (5.12)$$

where  $k = 1, \dots, n-1$ ,

$$\begin{aligned}g_k &= g_{1,k} + g_{2,k} + g_{3,k} + g_{4,k} + g_{5,k}, \\ g_{1,k} &= a_{1,k} A(u_{k+1} + u_{k-1}) - \bar{a}_{1,k} A(\bar{u}_{k+1} + \bar{u}_{k-1}), \\ g_{2,k} &= d_k A(u_{k+1} + u_{k-1}) - \bar{d}_k A(\bar{u}_{k+1} + \bar{u}_{k-1}), \\ g_{3,k} &= a_{3,k} A(u_{k+1} + u_{k-1}) - \bar{a}_{3,k} A(\bar{u}_{k+1} + \bar{u}_{k-1}), \\ g_{4,k} &= 2(Cz_k + N\delta z_k), \quad g_{5,k} = 2((M(u_k) - M(\bar{u}_k)) - (f_k - \bar{f}_k)),\end{aligned}$$

and where  $\bar{a}_{1,k} = \tilde{\psi}_1(\bar{\gamma}_{k+1}, \bar{\gamma}_{k-1})$ ,  $\bar{\gamma}_k = \|A^{1/2}\bar{u}_k\|^2$  (analogously are defined  $\bar{a}_{3,k}$  and  $\bar{d}_k$ ).

For scheme (5.12), according to Lemma 4.1 the following *a priori* estimations are valid (see (4.2) and (4.3)):

$$\|B^{1/2}z_{k+1}\| \leq c \left( \|B^{1/2}z_0\| + \left\| \frac{\Delta z_0}{\tau} \right\| + \tau \left\| B^{1/2} \frac{\Delta z_0}{\tau} \right\| + \tau \sum_{i=1}^k \|g_i\| \right), \quad (5.13)$$

$$\left\| \frac{\Delta z_k}{\tau} \right\| \leq c \left( \|z_0\| + \left\| \frac{\Delta z_0}{\tau} \right\| + \tau \sum_{i=1}^k \|g_i\| \right). \quad (5.14)$$

Let us estimate each  $g_{.,k}$  separately. For  $g_{1,k}$  we have

$$g_{1,k} = (a_{1,k} - \bar{a}_{1,k}) (Au_{k+1} + Au_{k-1}) + \bar{a}_{1,k} (Az_{k+1} + Az_{k-1}). \quad (5.15)$$

Using simple transformations for difference  $a_{1,k} - \bar{a}_{1,k}$  we get

$$a_{1,k} - \bar{a}_{1,k} = \int_0^1 \int_{\bar{l}_k(\xi)}^{l_k(\xi)} \psi'_1(\eta) d\eta d\xi, \quad (5.16)$$

From (5.16) according to Remark 2.1 we have

$$|a_{1,k} - \bar{a}_{1,k}| \leq c \int_0^1 |\chi(\xi)| d\xi \leq c (\|\gamma_{k-1} - \bar{\gamma}_{k-1}\| + \|\gamma_{k+1} - \bar{\gamma}_{k+1}\|), \quad (5.17)$$

where  $c = \max |\psi'_1(\eta)| < +\infty$ ,  $0 \leq \eta \leq \max_k (\gamma_k, \bar{\gamma}_k) < +\infty$ .

From (5.15) using (5.17), we get

$$\begin{aligned} \|g_{1,k}\| &\leq c (\|\gamma_{k-1} - \bar{\gamma}_{k-1}\| + \|\gamma_{k+1} - \bar{\gamma}_{k+1}\|) (\|Au_{k+1}\| + \|Au_{k-1}\|) \\ &\quad + |\bar{a}_{1,k}| (\|Az_{k+1}\| + \|Az_{k-1}\|). \end{aligned} \quad (5.18)$$

According to Remark 2.1,  $\bar{a}_{1,k}$  is uniformly bounded. Indeed we have

$$|\bar{a}_{1,k}| = \left| \int_0^1 \psi_1(\bar{l}_k(\xi)) d\xi \right| \leq c, \quad (5.19)$$

where  $c = \max |\psi_1(s)| < +\infty$ ,  $0 \leq s \leq \max_k (\gamma_k, \bar{\gamma}_k)$ .

If we insert estimations (5.9) and (5.19) in (5.18) and take into account, that  $\|Au_k\|$  is uniformly bounded, we obtain

$$\|g_{1,k}\| \leq c (\|Az_{k+1}\| + \|Az_{k-1}\|). \quad (5.20)$$

Let us estimate vector  $g_{2,k}$ . We have

$$g_{2,k} = (d_k - \bar{d}_k) (Au_{k+1} + Au_{k-1}) + \bar{d}_k (Az_{k+1} + Az_{k-1}). \quad (5.21)$$

According to Remark 2.1, for  $\bar{d}_k$  the following estimation is valid

$$|\bar{d}_k| = |\delta \bar{\gamma}_k| \left| \int_0^1 \psi'_2(\bar{l}_k(\xi)) d\xi \right| \leq c |\delta \bar{\gamma}_k|, \quad (5.22)$$

where  $c = \max |\psi'_2(s)| < +\infty$ ,  $0 \leq s \leq \max_k (\gamma_k, \bar{\gamma}_k)$ .

According to Remark 2.2, from (5.22) follows that  $\bar{d}_k$  is uniformly bounded. Vector  $\|Au_k\|$  is also uniformly bounded (see Remark 2.1). Using this and inequality (5.2) from (5.21) follows

$$|g_{2,k}| \leq c (\|Az_{k+1}\| + \|Az_{k-1}\| + \|\delta z_k\|). \quad (5.23)$$

Let us estimate the vector  $g_{3,k}$ . If in representation of  $g_{1,k}$  operator  $A$  is replaced by identity operator, then we get  $g_{3,k}$  (we assume that  $\psi_1$  respectively is replaced by  $\psi_3$ ). According to this for  $g_{1,k}$  analogously as for  $g_{3,k}$  the following estimation is true

$$\|g_{3,k}\| \leq c (\|z_{k+1}\| + \|z_{k-1}\|) \leq c (\|Az_{k+1}\| + \|Az_{k-1}\|). \quad (5.24)$$

If we consider that operator  $C$  satisfies condition (1.4), and  $N$  is bounded, then for  $g_{4,k}$  we get

$$|g_{4,k}| \leq c (\|Az_k\| + \|\delta z_k\|). \quad (5.25)$$

If we take into account that operator  $M(\cdot)$  satisfies Lipschitz condition, then for  $g_{5,k}$  we obtain

$$\|g_{5,k}\| \leq c \|z_k\| + \|f_k - \bar{f}_k\| \leq c \|Az_k\| + \|f_k - \bar{f}_k\|. \quad (5.26)$$

Finally, according to inequalities (5.20), (5.23), (5.24), (5.25) and (5.26) for the vector  $g_k$  the following estimation is valid

$$\|g_k\| \leq \|f_k - \bar{f}_k\| + c \left( \|Az_{k+1}\| + \|Az_k\| + \|Az_{k-1}\| + \left\| \frac{\Delta z_k}{\tau} \right\| + \left\| \frac{\Delta z_{k-1}}{\tau} \right\| \right). \quad (5.27)$$

Let us introduce the following denotations

$$\begin{aligned} \delta_k &= \|B^{1/2}z_0\| + \left\| \frac{\Delta z_0}{\tau} \right\| + \tau \left\| B^{1/2} \frac{\Delta z_0}{\tau} \right\| + \tau \sum_{i=1}^k \|f_i - \bar{f}_i\|, \\ \varepsilon_k &= \|B^{1/2}z_k\| + \left\| \frac{\Delta z_{k-1}}{\tau} \right\|. \end{aligned}$$

From (5.27) according to (1.9) we have

$$\|g_k\| \leq \|f_k - \bar{f}_k\| + c (\varepsilon_{k+1} + \varepsilon_k + \|B^{1/2}z_{k-1}\|). \quad (5.28)$$

Using this inequality, from (5.13) we obtain

$$\|B^{1/2}z_{k+1}\| \leq c\delta_k + c\tau \sum_{i=1}^k (\varepsilon_{i+1} + \varepsilon_i + \|B^{1/2}z_{i-1}\|). \quad (5.29)$$

Using simple transformation from (5.29) we get

$$\|B^{1/2}z_{k+1}\| \leq c\delta_k + c\tau \sum_{i=1}^{k+1} \varepsilon_i. \quad (5.30)$$

Analogously, according to (5.28) and  $\|z_0\| \leq \|B^{1/2}z_0\|$ , from (5.14) follows

$$\left\| \frac{\Delta z_k}{\tau} \right\| \leq c\delta_k + c\tau \sum_{i=1}^{k+1} \varepsilon_i. \quad (5.31)$$

If we add inequalities (5.30) and (5.31), we get

$$\varepsilon_{k+1} \leq c\delta_k + c\tau \sum_{i=1}^{k+1} \varepsilon_i. \quad (5.32)$$

If we request that  $\tau$  satisfies the condition  $\tau \leq q/c$  ( $0 < q < 1$ ), then from (5.30) we get

$$\varepsilon_{k+1} \leq c\delta_k + c\tau \sum_{i=1}^k \varepsilon_i.$$

From here by the induction can be obtained (discrete analogue of Grönwall's lemma)

$$\varepsilon_{k+1} \leq c (1 + c\tau)^{k-1} (\delta_k + \tau\varepsilon_1). \quad (5.33)$$

From (5.33), using inequality

$$\|B^{1/2}z_1\| \leq \|B^{1/2}z_0\| + \tau \left\| B^{1/2} \frac{\Delta z_0}{\tau} \right\|,$$

follows estimation (5.1).  $\square$

## 6 Estimate of the error of approximate solution

In this section, using the results of the previous sections, we prove the theorem, which considers the error estimate for the approximate solution. This theorem represents an almost trivial result of the theorem proved in the previous section. However, estimation of the approximation error for scheme (1.6) because of nonlinear terms, requires additional calculations.

Before formulating the theorem regarding the convergence (error estimate of the approximate solution) of the scheme (1.6), we would like to make a note about the well-posedness of the problem (1.1), (1.2). We mean from the beginning that the original continuous problem is well-posed and the solution is sufficiently smooth. Obviously, we require the smoothness of the solution to find a convergence order (rate). If we demand the minimal smoothness which is necessary for the well-posedness of the problem, then the convergence is guaranteed, but we are not able to establish an order. If we increase the smoothness order by one unit, then the convergence rate is equal to one (in this, as well as in the previous case, it is sufficient to take  $u_1 = \varphi_0 + \tau\varphi_1$ ). However, a convergence rate becomes two if we level up smoothness by two and define starting vector  $u_1$  using the following formula

$$u_1 = \varphi_0 + \tau\varphi_1 + \frac{\tau^2}{2}\varphi_2, \quad (6.1)$$

where  $\varphi_2 = u''(0)$ ,  $u''(0)$  is defined from the equation (1.1) via  $\varphi_0$  and  $\varphi_1$  (we assume that  $\varphi_0, \varphi_1 \in D(B)$ ).

The further increase of smoothness of the solution does not make sense, as the approximation order of the scheme (1.6) is not more than two (obviously, the convergence order generally does not exceed the approximation order).

Let us formulate above stated as a theorem (below everywhere  $c$  denotes positive constant).

**Theorem 6.1.** *Let the problem (1.1), (1.2) be well-posed. Besides, the following conditions are fulfilled:*

- (a)  $\varphi_0, \varphi_1, \varphi_2 \in D(B)$ ;
- (b) *solution  $u(t)$  of problem (1.1), (1.2) is continuously differentiable to third order including and  $u'''(t)$  satisfies Lipschitz condition;*
- (c)  *$u''(t) \in D(B)$  for every  $t$  from  $[0, \bar{t}]$  and function  $Bu''(t)$  satisfy Lipschitz condition.*

*Then for scheme (1.6), (6.1) the following estimates are true*

$$\max_{1 \leq k \leq n-1} \left( \|B^{1/2}\tilde{z}_{k+1}\| + \left\| \frac{\Delta\tilde{z}_k}{\tau} \right\| \right) \leq c\tau^2, \quad (6.2)$$

where  $\tilde{z}_k = u(t_k) - u_k$  is an error of approximate solution,  $\Delta\tilde{z}_k = \tilde{z}_{k+1} - \tilde{z}_k$ .

**Proof.** Let us introduce the following notations:

$$\delta u(t_k) = \frac{u(t_{k+1}) - u(t_{k-1})}{2\tau}, \quad \hat{u}(t_k) = \frac{u(t_{k+1}) + u(t_{k-1})}{2}.$$

Let us write down the equation (1.1) at point  $t = t_k$  in the following form

$$\begin{aligned} & \frac{\Delta^2 u(t_{k-1})}{\tau^2} + a_1 B \delta u(t_k) + a_2 B \hat{u}(t_k) + \tilde{\psi}_1(\zeta_{k-1}, \zeta_{k+1}) A \hat{u}(t_k) \\ & + \delta \psi_2(\zeta_k) A \hat{u}(t_k) + \tilde{\psi}_3(\|u(t_{k-1})\|^2, \|u(t_{k+1})\|^2) \hat{u}(t_k) \\ & + C u(t_k) + N \delta u(t_k) + M(u(t_k)) = f(t_k) + r_\tau(t_k), \end{aligned} \quad (6.3)$$

where  $\zeta_k = \|A^{1/2}u(t_k)\|^2$  ( $\zeta(t) = \|A^{1/2}u(t)\|^2$ ),

$$r_\tau(t_k) = \sum_{j=0}^5 r_{j,\tau}(t_k), \quad (6.4)$$

and where

$$\begin{aligned}
r_{0,\tau}(t_k) &= \frac{\Delta^2 u(t_{k-1})}{\tau^2} - u''(t_k), \quad r_{6,\tau}(t_k) = N(\delta u(t_k) - u'(t_k)), \\
r_{2,\tau}(t_k) &= \frac{1}{2}\tilde{\psi}_1(\zeta_{k-1}, \zeta_{k+1})A(\Delta^2 u(t_{k-1})) + (\tilde{\psi}_1(\zeta_{k-1}, \zeta_{k+1}) - \psi_1(\zeta_k))Au(t_k), \\
r_{3,\tau}(t_k) &= a_1B(\delta u(t_k) - u'(t_k)), \quad r_{1,\tau}(t_k) = \frac{1}{2}a_2B(\Delta^2 u(t_{k-1})), \\
r_{4,\tau}(t_k) &= \delta\psi_2(\zeta_k)\frac{1}{2}A(\Delta^2 u(t_{k-1})) + (\delta\psi_2(\zeta_k) - (\psi_2(\zeta(t_k))'_t))Au(t_k), \\
r_{5,\tau}(t_k) &= \tilde{\psi}_3(\|u(t_{k-1})\|^2, \|u(t_{k+1})\|^2)\Delta^2 u(t_{k-1}) \\
&\quad + (\tilde{\psi}_3(\|u(t_{k-1})\|^2, \|u(t_{k+1})\|^2) - \psi_3(\|u(t_k)\|^2))u(t_k).
\end{aligned}$$

From (6.3) and (1.6) according to Theorem 5.1 we obtain

$$\begin{aligned}
&\|B^{1/2}\tilde{z}_{k+1}\| + \left\|\frac{\Delta\tilde{z}_k}{\tau}\right\| \\
&\leq c \left( \|B^{1/2}\tilde{z}_0\| + \left\|\frac{\Delta\tilde{z}_0}{\tau}\right\| + \tau \left\|B^{1/2}\frac{\Delta\tilde{z}_0}{\tau}\right\| + \tau \sum_{i=1}^k \|r_\tau(t_k)\| \right). \tag{6.5}
\end{aligned}$$

If we carry out the routine calculations we obtain

$$\|r_\tau(t_k)\| \leq c\tau^2. \tag{6.6}$$

It is obvious that according to the conditions (a), (b) and (c) of the Theorem 6.1, and the equality (6.1) the following inequalities are true

$$\|B^{1/2}(\Delta\tilde{z}_0)\| + \left\|\frac{\Delta\tilde{z}_0}{\tau}\right\| \leq c\tau^2, \tag{6.7}$$

From (6.5), taking into account (6.6) and (6.7), follows (6.2).  $\square$

## 7 Iterative method for discrete problem

Let us rewrite equation (1.6) in the following form

$$T_k v_{k+1} = \frac{1}{2}\tau\psi_2(\gamma_{k-1})Av_{k+1} - \tau Nv_{k+1} + \tilde{f}_k, \tag{7.1}$$

where  $v_{k+1} = (u_{k+1} + u_{k-1})/2$ ,

$$\begin{aligned}
T_k &= (2 + \tau^2 a_{3,k})I + \tau(a_1 + \tau a_2)B + \tau b_k A, \quad b_k = \tau a_{1,k} + \frac{1}{2}\psi_2(\gamma_{k+1}), \\
\tilde{f}_k &= \tau\tilde{g}_k + \tau a_1 B u_{k-1} + 2u_k, \quad \tilde{g}_k = \tau f_k - \tau M(u_k) + N u_{k-1} - \tau C u_k.
\end{aligned}$$

Equation (7.1) is solved by using the following iteration

$$T_{k,m} v_{k+1}^{(m+1)} = \frac{1}{2}\tau\psi_2(\gamma_{k-1})Av_{k+1}^{(m)} - \tau Nv_{k+1}^{(m)} + \tilde{f}_k, \tag{7.2}$$

where  $m = 0, 1, \dots$ ,

$$\begin{aligned}
T_{k,m} &= (2 + \tau^2 a_{3,k}^{(m)})I + \tau(a_1 + \tau a_2)B + \tau b_k^{(m)}A, \quad b_k^{(m)} = \tau a_{1,k}^{(m)} + \frac{1}{2}\psi_2(\gamma_{k+1}^{(m)}), \\
a_{1,k}^{(m)} &= \tilde{\psi}_1(\gamma_{k-1}, \gamma_{k+1}^{(m)}), \quad \gamma_{k+1}^{(m)} = \left\|A^{1/2}u_{k+1}^{(m)}\right\|^2, \quad u_{k+1}^{(m)} = 2v_{k+1}^{(m)} - u_{k-1}, \\
a_{3,k}^{(m)} &= \tilde{\psi}_3\left(\|u_{k-1}\|^2, \left\|u_{k+1}^{(m)}\right\|^2\right), \quad v_{k+1}^{(0)} = (u_k + u_{k-1})/2.
\end{aligned}$$

**Remark 7.1.** We already required that  $A$  and  $B$  to be self-adjoint positively defined operators. Besides condition (1.3) is valid. From here follows that  $D(A) \subset D(B^{1/2})$  (see Remark 1.1). In order to show convergence of iterative method (7.2) we require the following condition to be fulfilled  $D(A) = D(B^{1/2})$ . From here follows that operator  $B^{1/2}A^{-1}$  (as it is closed operator defined in whole Hilbert space  $H$ ) is bounded according to the closed graph theorem. Let us denote norm of this operator by  $c_1 = \|B^{1/2}A^{-1}\|$ .

**Remark 7.2.** Condition  $D(A) = D(B^{1/2})$  is automatically fulfilled for equation (1.5), if unknown function and its derivatives satisfy homogeneous boundary conditions. In this case  $B^{1/2} = A$ , where  $A$  is an expansion of symmetrical operator  $(-\partial_{xx}^2)$  (with homogeneous boundary conditions) till a self-adjoint operator.

**Remark 7.3.** Regarding iteration (7.2) it is important that  $T_{k,m}$  is self-adjoint positively defined operator ( $a_{3,k}^{(m)}$ ,  $a_{1,k}^{(m)}$  and  $\psi_2(\gamma_{k+1}^{(m)})$  are nonnegative, as according to the condition  $\psi_1(s)$ ,  $\psi_2(s)$  and  $\psi_3(s)$ ,  $s \in [0, +\infty[$ , are nonnegative functions). It is well-known that from here follows that  $R(T_{k,m}) = H$ , i.e. equation  $T_{k,m}u = g$ , for any vector  $g$  from  $H$  has unique solution  $u \in D(B)$  and it continuously depends on  $g$ .

Let us prove convergence of iteration (7.2). The prove depends on many standard transformations. The first step is to prove uniform boundedness of vectors  $w_{k+1}^{(m)} = B^{1/2}v_{k+1}^{(m)}$  obtained by iteration (7.2). This fact is important, as it gives a certain opportunity that iteration (7.2) might be converged.

**Step 1.** Prove uniform boundedness of vector sequence  $w_{k+1}^{(m)}$  obtained by using iteration (7.2).

**Proof.** Let us introduce the following notations:

$$\begin{aligned} S_{k,m} &= \left(2 + \tau^2 a_{3,k}^{(m)}\right) I + a_\tau B, \quad S_k = \left(2 + \tau^2 a_{3,k}\right) I + a_\tau B, \quad a_\tau = \tau(a_1 + \tau a_2), \\ P_{k,m} &= B^{1/2}T_{k,m}^{-1}, \quad Q_{k,m} = B^{1/2}S_{k,m}^{-1/2}, \quad P_k = B^{1/2}T_k^{-1}, \quad Q_k = B^{1/2}S_k^{-1/2}. \end{aligned}$$

If we define the vector  $v_{k+1}^{(m+1)}$  from the equation (7.2), after applying the operator  $B^{1/2}$  and move on to the norm, we get

$$\begin{aligned} \|w_{k+1}^{(m+1)}\| &\leq \frac{1}{2}\tau\psi_2(\gamma_{k-1})\|P_{k,m}\|\|AB^{-1/2}\|\|w_{k+1}^{(m)}\| \\ &\quad + \tau\|P_{k,m}\|\|N\|\|B^{-1/2}\|\|w_{k+1}^{(m)}\| + \|P_{k,m}\tilde{f}_k\|. \end{aligned} \quad (7.3)$$

Operator  $T_{k,m}$  should be written in the following form

$$T_{k,m} = S_{k,m}^{1/2}G_{k,m}S_{k,m}^{1/2}, \quad G_{k,m} = I + \tau b_k^{(m)}S_{k,m}^{-1/2}AS_{k,m}^{-1/2}. \quad (7.4)$$

According to (7.4) we have

$$\|P_{k,m}\| \leq \|Q_{k,m}\|\|G_{k,m}^{-1}\|\|S_{k,m}^{-1/2}\|. \quad (7.5)$$

The following estimations can be obtained easily:

$$\|Q_{k,m}\| \leq \frac{1}{\sqrt{\tau a_1}}, \quad \|S_{k,m}^{-1/2}\| \leq \frac{1}{\sqrt{2}}, \quad \|G_{k,m}^{-1}\| \leq 1. \quad (7.6)$$

If we insert inequalities (7.6) into (7.5), we get

$$\|P_{k,m}\| = \|B^{1/2}T_{k,m}^{-1}\| \leq \frac{1}{\sqrt{2\tau a_1}}. \quad (7.7)$$

As  $\gamma_k$  is uniformly bounded (see Remark 2.1), we have

$$\psi_2(\gamma_k) \leq M_2, \quad M_2 = \max \psi_2(s) < +\infty, \quad 0 \leq s \leq \max_k \gamma_k < +\infty. \quad (7.8)$$

From (7.3) according to estimations (7.7), (7.8),  $\|AB^{-1/2}\| \leq b_0$  (see (1.9)) and  $\|B^{-1/2}\| \leq 1/\sqrt{m_B}$  ( $m_B$  is a lower bound of the operator  $B$ ) follows

$$\left\| w_{k+1}^{(m+1)} \right\| \leq \sqrt{\tau} M_3 \left\| w_{k+1}^{(m)} \right\| + \left\| P_{k,m} \tilde{f}_k \right\|, \quad (7.9)$$

where  $M_3$  is a positive constant (independent of  $m$  and  $n$ ).

Let us estimate norm of the vector

$$P_{k,m} \tilde{f}_k = \tau P_{k,m} \tilde{g}_k + \tau a_1 P_{k,m} B u_{k-1} + 2 P_{k,m} u_k. \quad (7.10)$$

According to Theorem 2.1,  $Au_k$ ,  $M(u_k)$  and  $u_k$  vectors are uniformly bounded, (uniform boundedness of vectors  $M(u_k)$  follows from (2.4)), also according to (1.4) we have  $\|Cu_k\| \leq a_0 \|Au_k\|$ . From here follows that vectors  $\tilde{g}_k$  are uniformly bounded. According to this from (7.7) follows that there exists such constant  $M_0$  (independent of  $n$  and  $m$ ), that

$$\sqrt{\tau} \|P_{k,m} \tilde{g}_k\| \leq \frac{1}{\sqrt{2a_1}} \|\tilde{g}_k\| \leq M_0. \quad (7.11)$$

Let us estimate second summand on the right-hand side of equality (7.10). Using formula (7.4) and inequalities (7.6) we get the following estimation

$$\tau \|P_{k,m} B u_{k-1}\| = \tau \|Q_{k,m} G_{k,m}^{-1} Q_{k,m} B^{1/2} u_{k-1}\| \leq \frac{1}{a_1} \|B^{1/2} u_{k-1}\|. \quad (7.12)$$

From here according to Theorem 2.1 follows

$$\tau \|P_{k,m} B u_{k-1}\| \leq M_1, \quad (7.13)$$

where  $M_1$  is positive constant (independent of  $n$  and  $m$ ).

Eventually, we need to estimate third summand in equality (7.10). Let us rewrite it in the following form

$$P_{k,m} u_k = B^{1/2} (T_{k,m}^{-1} - C_{k,m}^{-1}) u_k + B^{1/2} C_{k,m}^{-1} u_k, \quad (7.14)$$

where  $C_{k,m} = \left(2 + \tau^2 a_{3,k}^{(m)}\right) I + \tau b_k^{(m)} A$ .

Obviously, we have

$$T_{k,m}^{-1} - C_{k,m}^{-1} = T_{k,m}^{-1} (C_{k,m} - T_{k,m}) C_{k,m}^{-1} = -a_\tau T_{k,m}^{-1} B C_{k,m}^{-1}. \quad (7.15)$$

If we insert (7.15) into (7.14) and consider (7.4), we get

$$\begin{aligned} P_{k,m} u_k &= \left(B^{1/2} A^{-1}\right) C_{k,m}^{-1} (Au_k) \\ &\quad - a_\tau Q_{k,m} G_{k,m}^{-1} Q_{k,m} \left(B^{1/2} A^{-1}\right) C_{k,m}^{-1} (Au_k). \end{aligned} \quad (7.16)$$

From (7.16) according to (7.6),  $\|C_{k,m}^{-1}\| \leq 1/2$ , Remark 7.1 and also Theorem 2.1 we have

$$\|P_{k,m} u_k\| \leq M_4 \|Au_k\| \leq M_5. \quad (7.17)$$

where  $M_5$  is positive constant (independent of  $n$  and  $m$ ),  $M_4 = c_1 (2 + \tau a_2/a_1)$ .

Let us insert inequalities (7.11), (7.13) and (7.17) into (7.10), we get

$$\left\| P_{k,m} \tilde{f}_k \right\| \leq M_6, \quad M_6 = \sqrt{\tau} M_0 + a_1 M_1 + 2 M_5. \quad (7.18)$$

From (7.9) according to (7.18) we have

$$\left\| w_{k+1}^{(m+1)} \right\| \leq \sqrt{\tau} M_3 \left\| w_{k+1}^{(m)} \right\| + M_6. \quad (7.19)$$

Let  $\tau$  satisfies condition  $\sqrt{\tau}M_3 \leq q < 1$ . Then from (7.19) follows

$$\|w_{k+1}^{(m)}\| \leq q^m \|w_{k+1}^{(0)}\| + \frac{M_6}{1-q}. \quad (7.20)$$

Inequality (7.20) shows, that vector sequence  $w_{k+1}^{(m)} = B^{1/2}v_{k+1}^{(m)}$  ( $v_{k+1}^{(0)} = (u_{k+1}^{(0)} + u_{k-1})/2$ ,  $u_{k+1}^{(0)} = u_k$ ) obtained by iterative method is uniformly bounded, i.e. there exists such constant  $M_7$  (independent of  $n$  and  $m$ ), that  $\|w_{k+1}^{(m)}\| \leq M_7$  (here we again used Theorem 2.1 for vector  $w_{k+1}^{(0)}$ ).  $\square$

**Step 2.** Let us estimate norm of error  $Z_{k+1}^{(m)} = B^{1/2}(v_{k+1} - v_{k+1}^{(m+1)})$ . Let us define  $v_{k+1}$  and  $v_{k+1}^{(m+1)}$  from equalities (7.1) and (7.2), respectively. If we apply  $B^{1/2}$  to the difference of these vectors and perform corresponding transformation, we get

$$\begin{aligned} Z_{k+1}^{(m+1)} &= \frac{1}{2}\tau\psi_2(\gamma_{k-1})B^{1/2}L_{k,m}Lw_{k+1} + \frac{1}{2}\tau\psi_2(\gamma_{k-1})P_{k,m}LZ_{k+1}^{(m)} \\ &\quad - \tau B^{1/2}L_{k,m}Sw_{k+1} - \tau P_{k,m}SZ_{k+1}^{(m)} + B^{1/2}L_{k,m}\tilde{f}_k, \end{aligned} \quad (7.21)$$

where  $L = AB^{-1/2}$ ,  $S = NB^{-1/2}$  and  $L_{k,m} = T_k^{-1} - T_{k,m}^{-1}$ .

Let us note, that in representation of (7.21) the most complicated term is  $B^{1/2}L_{k,m}\tilde{f}_k$ , as it does not have a small parameter  $\tau$  as a multiplier, that provides to obtain an estimation from where follows convergence of the iteration method (7.2). For this term, using simple transformation we get

$$\begin{aligned} B^{1/2}L_{k,m}\tilde{f}_k &= -P_k(T_k - T_{k,m})T_{k,m}^{-1}\tilde{f}_k = -\tau^2\zeta_{3,k}^{(m)}P_kT_{k,m}^{-1}\tilde{f}_k \\ &\quad - \tau^2\zeta_{1,k}^{(m)}P_kAT_{k,m}^{-1}\tilde{f}_k - \frac{1}{2}\tau\zeta_{2,k}^{(m)}P_kAT_{k,m}^{-1}\tilde{f}_k, \end{aligned} \quad (7.22)$$

where  $\zeta_{2,k}^{(m)} = \psi_2(\gamma_{k+1}) - \psi_2(\gamma_{k+1}^{(m)})$ ,  $\zeta_{j,k}^{(m)} = a_{j,k} - a_{j,k}^{(m)}$ ,  $j = 1, 3$ .

Analogously to (7.4) we can represent operator  $T_k$  in the following form:

$$T_k = S_k^{1/2}G_kS_k^{1/2}, \quad G_k = I + \tau b_kS_k^{-1/2}AS_k^{-1/2}. \quad (7.23)$$

Analogously to inequalities (7.6) we have:

$$\|Q_k\| \leq \frac{1}{\sqrt{\tau a_1}}, \quad \|S_k^{-1/2}\| \leq \frac{1}{\sqrt{2}}, \quad \|G_k^{-1}\| \leq 1. \quad (7.24)$$

From (7.23) using inequalities (7.24) follows

$$\|P_k\| = \|B^{1/2}T_k^{-1}\| \leq \frac{1}{\sqrt{2\tau a_1}}. \quad (7.25)$$

Using estimations (7.24), (7.25), (7.6), and (7.7), analogously to inequalities (7.11), (7.12) and (7.17), we can obtain:

$$\tau \|P_kAT_{k,m}^{-1}\| = \tau \|P_kLP_{k,m}\| \leq \frac{b_0}{2a_1}, \quad (7.26)$$

$$\tau\sqrt{\tau} \|P_kAT_{k,m}^{-1}Bu_{k-1}\| = \tau\nu_0 \|LB^{1/2}T_{k,m}^{-1}Bu_{k-1}\| \leq b_0\nu_0 a_1^{-1} \|B^{1/2}u_{k-1}\|, \quad (7.27)$$

$$\begin{aligned} \sqrt{\tau} \|P_{k,m}AT_{k,m}^{-1}u_k\| &\leq \nu_0 \|C_{k,m}^{-1}Au_k - a_\tau(LB^{1/2}T_{k,m}^{-1}BA^{-1}C_{k,m}^{-1})Au_k\| \\ &\leq \nu_0 (1 + (a_1 + \tau a_2)b_0c_1a_1^{-1}) \|Au_k\|, \quad \nu_0 = 1/\sqrt{2a_1}. \end{aligned} \quad (7.28)$$

To obtain inequality (7.28) the following representations  $T_{k,m}^{-1} = C_{k,m}^{-1} - a_\tau T_{k,m}^{-1} BC_{k,m}^{-1}$  and  $B^{1/2}T_{k,m}^{-1}B^{1/2} = Q_{k,m}G_{k,m}Q_{k,m}$  are used.

From inequalities (7.26), (7.27) and (7.28), according to Theorem 2.1 it follows that there exists such positive constant  $M_8$  (independent of  $n$  and  $m$ ) that

$$\sqrt{\tau} \left\| P_k A T_{k,m}^{-1} \tilde{f}_k \right\| \leq M_8 . \quad (7.29)$$

According to the representation (7.4) and the estimation (7.7) we have:

$$\sqrt{\tau} \left\| P_k T_{k,m}^{-1} \right\| \leq \frac{1}{2} \nu_0 , \quad (7.30)$$

$$\tau \left\| P_k T_{k,m}^{-1} B u \right\| \leq \frac{1}{2a_1} \left\| B^{1/2} u \right\| , \quad u \in D(B) . \quad (7.31)$$

From the inequalities (7.30) and (7.31) analogously to (7.29) we have

$$\sqrt{\tau} \left\| P_k T_{k,m}^{-1} \tilde{f}_k \right\| \leq M_9 , \quad (7.32)$$

where  $M_9$  is a positive constant (independent of  $n$  and  $m$ ).

If we apply norms in (7.22) and take into consideration the estimations (7.29) and (7.32), we get

$$\left\| B^{1/2} L_{k,m} \tilde{f}_k \right\| \leq \sqrt{\tau} \left( \tau M_9 \left| \zeta_{3,k}^{(m)} \right| + \tau M_8 \left| \zeta_{1,k}^{(m)} \right| + M_8 \left| \zeta_{2,k}^{(m)} \right| \right) . \quad (7.33)$$

Now, let us estimate norm of the operator  $B^{1/2} L_{k,m}$ . From (7.22) we have

$$\left\| B^{1/2} L_{k,m} \right\| \leq \tau^2 \left| \zeta_{3,k}^{(m)} \right| \left\| P_k T_{k,m}^{-1} \right\| + \tau \left\| P_k A T_{k,m}^{-1} \right\| \left( \tau \left| \zeta_{1,k}^{(m)} \right| + \left| \zeta_{2,k}^{(m)} \right| \right) .$$

From here according to inequalities (7.26) and (7.30) we have

$$\left\| B^{1/2} L_{k,m} \right\| \leq \frac{1}{2} \tau^{3/2} \nu_0 \left| \zeta_{3,k}^{(m)} \right| + b_0 \nu_0^2 \left( \tau \left| \zeta_{1,k}^{(m)} \right| + \left| \zeta_{2,k}^{(m)} \right| \right) . \quad (7.34)$$

Let us note that, from uniform boundedness of vectors  $w_{k+1}^{(m)}$  follows uniform boundedness of vectors  $A v_{k+1}^{(m)}$  (see inequality (1.9)). From here according to the inequality (2.8) follows uniform boundedness of vectors  $A^{1/2} v_{k+1}^{(m)}$ .

If we insert the terms  $\gamma_{k+1}^{(m)}$  and  $\gamma_{k-1}$  into the inequality (5.17) instead of the following ones  $\gamma_{k+1}$  and  $\bar{\gamma}_{k-1}$ , respectively, we obtain

$$\begin{aligned} \left| \zeta_{1,k}^{(m)} \right| &\leq K_1 \left| \gamma_{k+1} - \gamma_{k+1}^{(m)} \right| \leq 2K_1 K_2 \left\| A^{1/2} u_{k+1} - A^{1/2} u_{k+1}^{(m)} \right\| \\ &= 4K_1 K_2 \left\| A^{1/2} v_{k+1} - A^{1/2} v_{k+1}^{(m)} \right\| , \end{aligned} \quad (7.35)$$

where  $K_1 = \max | \psi'_1(s) | < +\infty$ ,  $0 \leq s \leq K_2 = \max_{(k,m)} (\gamma_k, \gamma_k^{(m)}) < +\infty$ .

From (2.8) using inequality (1.9) it follows that

$$\left\| A^{1/2} u \right\| \leq \frac{b_0}{\sqrt{m_A}} \left\| B^{1/2} u \right\| , \quad \forall u \in D(B^{1/2}).$$

According to this inequality, from (7.35) we have

$$\left| \zeta_{1,k}^{(m)} \right| \leq K_3 \left\| w_{k+1} - w_{k+1}^{(m)} \right\| . \quad (7.36)$$

where  $K_3$  is a positive constant (independent of  $n$  and  $m$ ).

Analogously we get

$$\left| \zeta_{j,k}^{(m)} \right| \leq K_4 \left\| w_{k+1} - w_{k+1}^{(m)} \right\|, \quad j = 2, 3, \quad K_4 = \text{const} > 0. \quad (7.37)$$

If we insert (7.36) and (7.37) inequalities into (7.33) and (7.34) we get:

$$\left\| B^{1/2} L_{k,m} \tilde{f}_k \right\| \leq \sqrt{\tau} K_5 \left\| w_{k+1} - w_{k+1}^{(m)} \right\|, \quad K_5 = \text{const} > 0, \quad (7.38)$$

$$\left\| B^{1/2} L_{k,m} \right\| \leq K_6 \left\| w_{k+1} - w_{k+1}^{(m)} \right\|, \quad K_6 = \text{const} > 0. \quad (7.39)$$

If we move on to norms in the equality (7.21) and consider the inequalities (7.38), (7.39), (7.7),  $\|L\| \leq b_0$  (see (1.9)) and Theorem 2.1 we get

$$\left\| Z_{k+1}^{(m+1)} \right\| \leq \sqrt{\tau} K_0 \left\| Z_{k+1}^{(m)} \right\|, \quad (7.40)$$

where  $K_0$  is a positive constant (independent of  $n$  and  $m$ ).

Let  $\tau$  satisfies condition  $\sqrt{\tau} K_0 \leq q < 1$ . Then from (7.40) follows that  $\left\| Z_{k+1}^{(m)} \right\| \leq q^m \left\| Z_{k+1}^{(0)} \right\|$ , which means that (7.2) iteration is convergent.

So, we proved the following theorem

**Theorem 7.1.** *If operators  $A$  and  $B$  satisfy conditions from section 1 and  $D(A) = D(B^{1/2})$ , then (7.2) iteration converges with geometric progression speed.*

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