DM Assignment1 Report

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1. Concept Questions

1.1. Question 1

Note that every node in the clique links to every other node in the clique and the additional node, while the additional node does not link to any node (dead end).

So we have the Stochastic Web Matrix

$$\mathbf{M} = \begin{pmatrix} 0 & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{1}{n+1} \\ \frac{1}{n} & 0 & \cdots & \frac{1}{n} & \frac{1}{n+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{1}{n+1} \end{pmatrix}$$

and the Google Matrix

$$\mathbf{A} = \beta\mathbf{M} + (1-\beta)[\frac{\mathbf{1}}{\mathbf{n+1}}]_{n+1\times n+1}$$

Due to the exactly same property of nodes in the clique, we suppose the stationary distribution of all nodes are

$$\mathbf{r} = \begin{pmatrix} p_c \\ p_c \\ \vdots \\ p_c \\ p_s \end{pmatrix}$$

Plus, we have that $r = A \cdot r$, so we can get the equation that

$$p_c = \frac{\beta(n-1)}{n}p_c + \frac{(1-\beta)np_c}{n+1} + \frac{p_s}{n+1}$$
$$np_c + p_s = 1$$

So, we can get that $p_c = \frac{n}{n^2+n+\beta}$ and $p_s = \frac{n+\beta}{n^2+n+\beta}$.

1.2. Question 2

According to the graph, we have the Stochastic Web Matrix

$$\mathbf{M} = \begin{pmatrix} 0 & \frac{1}{2} & 1 & 0 \\ \frac{1}{3} & 0 & 0 & \frac{1}{2} \\ \frac{1}{3} & 0 & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} & 0 & 0 \end{pmatrix}$$

In case (a), we have the Teleport Matrix

$$\mathbf{T} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Because $\mathbf{r} = (\beta\mathbf{M} + (1-\beta)\mathbf{T})\mathbf{r}$, we can get that normalized $\mathbf{r} = \begin{pmatrix} \frac{3}{7} \\ \frac{4}{21} \\ \frac{4}{21} \\ \frac{4}{21} \end{pmatrix}$

1.3. Question 3

According to the description of the graph, we have that the adjacency matrix of the graph is

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

Thus we know that

$$\mathbf{A^T A} = \begin{pmatrix} 2 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

and that

$$\mathbf{A A^T} = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

Therefore, by repeating $h = \lambda\mu\mathbf{AA^T}h$ and $a = \lambda\mu\mathbf{A^T A}a$, where $\lambda = \frac{1}{\sum h_i}$ and $\mu = \frac{1}{\sum a_i}$, we can get the converged $h^* = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}$ and $a^* = \begin{pmatrix} \frac{1}{\sqrt{n}} \\ \frac{1}{\sqrt{n}} \\ \vdots \\ \frac{1}{\sqrt{n}} \\ \frac{1}{\sqrt{n}} \end{pmatrix}$

1.4. Question 4

Assume the eigenvectors of $\mathbf{M}$ is $\mathbf{x_1}, \mathbf{x_2}, \cdots, \mathbf{x_n}$ and their corresponding eigenvalues are $\lambda_1, \lambda_2, \cdots, \lambda_n$, with $|\lambda_1| > |\lambda_2| > \cdots > |\lambda_n|$. Because the column stochastic matrix $\mathbf{M}$ is usually dense, all eigenvectors of $\mathbf{M}$ are linearly independent.

Therefore, we can represent $\mathbf{r^{(0)}}$ as $c_1\mathbf{x_1} + c_2\mathbf{x_2} + \cdots + c_n\mathbf{x_n}$. So $\mathbf{Mr^0} = \mathbf{M}(c_1\mathbf{x_1} + c_2\mathbf{x_2} + \cdots + c_n\mathbf{x_n}) = c_1\lambda_1\mathbf{x_1} + c_2\lambda_2\mathbf{x_2} + \cdots + c_n\lambda_n\mathbf{x_n}$.

Repeating the operation above, we can have that $\mathbf{M}^k\mathbf{r^{(0)}} = c_1\lambda_1^k\mathbf{x_1} + c_2\lambda_2^k\mathbf{x_2} + \cdots + c_n\lambda_n^k\mathbf{x_n} = \lambda_1^k(c_1\mathbf{x_1} + c_2(\frac{\lambda_2}{\lambda_1})^k)\mathbf{x_2} + \cdots + c_n(\frac{\lambda_n}{\lambda_1})^k\mathbf{x_n})$. Because $\frac{\lambda_i}{\lambda_1} < 1$ when $i > 1$, when k approaches infinity, $(\frac{\lambda_i}{\lambda_1})^k$ approaches zero. So $\lim\limits_{k\to\infty} \mathbf{M}^k\mathbf{r^{(0)}} = \lambda_1^k c_1\mathbf{x}_1$.

So, as k increases, $\mathbf{M}^k\mathbf{r^{(0)}}$ approaches to $\mathbf{x_1}$, i.e. the principle eigenvector of $\mathbf{M}$,.