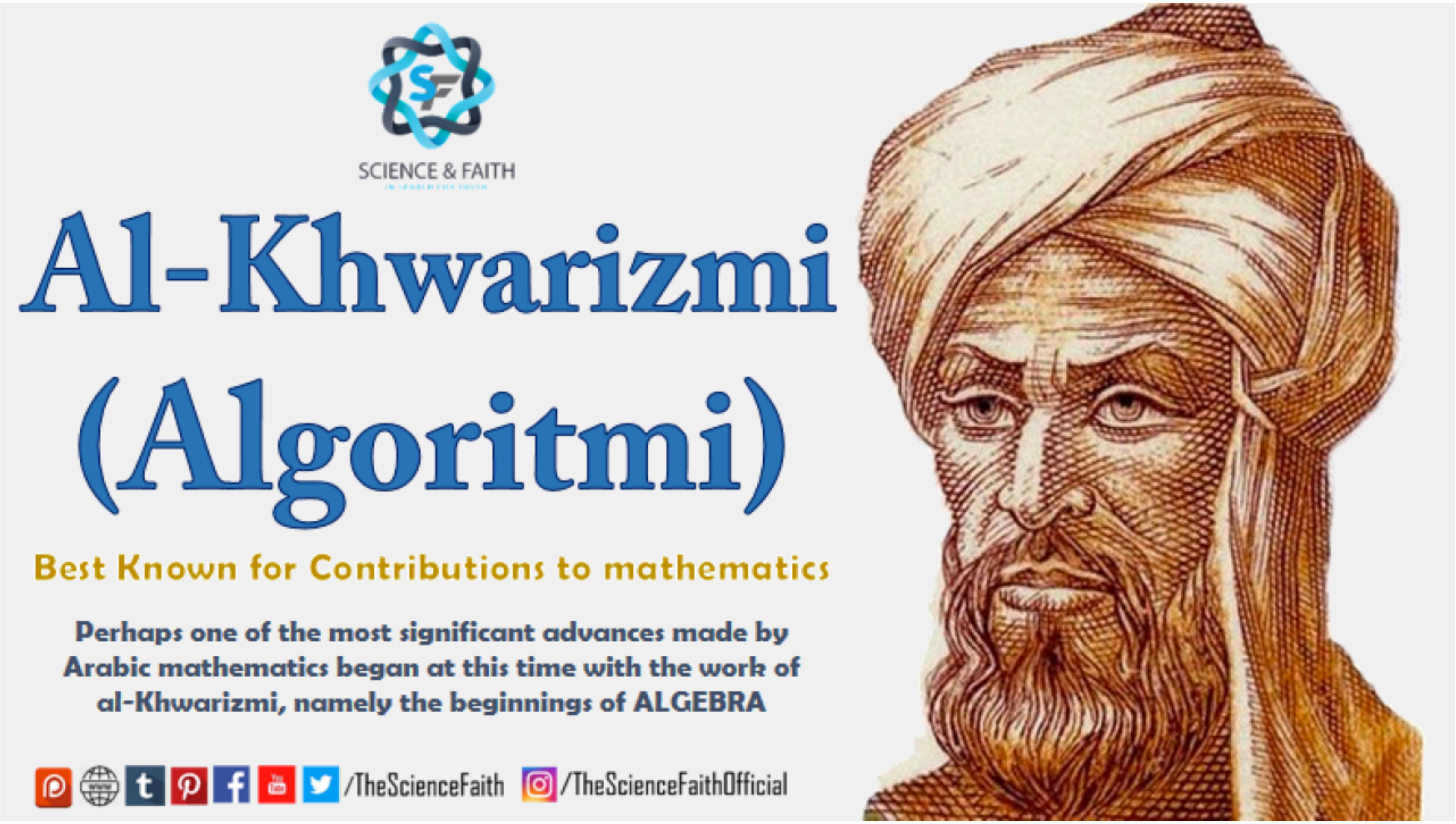
04 Complexity of Algorithms

CS201 Discrete Mathematics

Instructor: Shan Chen

Algorithms

- An algorithm is a finite sequence of precise instructions for performing a computation or for solving a problem.

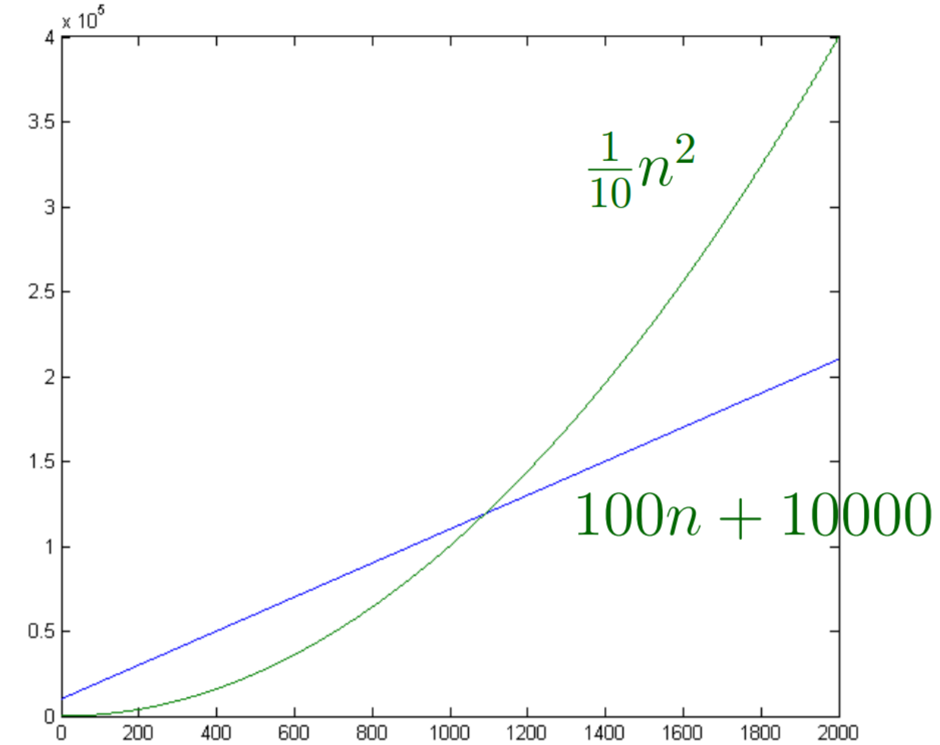


Al-Khwarizmi, Persian polymath

The Growth of Functions

Which Function is Larger?

- **Q:** Which function is "larger"? $n^2/10$ vs $100n + 10000$
- **A:** It depends on the value of n.
- In computer science, usually we are interested in what happens when the problem input size gets big.
- Note that when n is "large enough", $n^2/10$ gets bigger than $100n + 10000$ and stays bigger for larger n.



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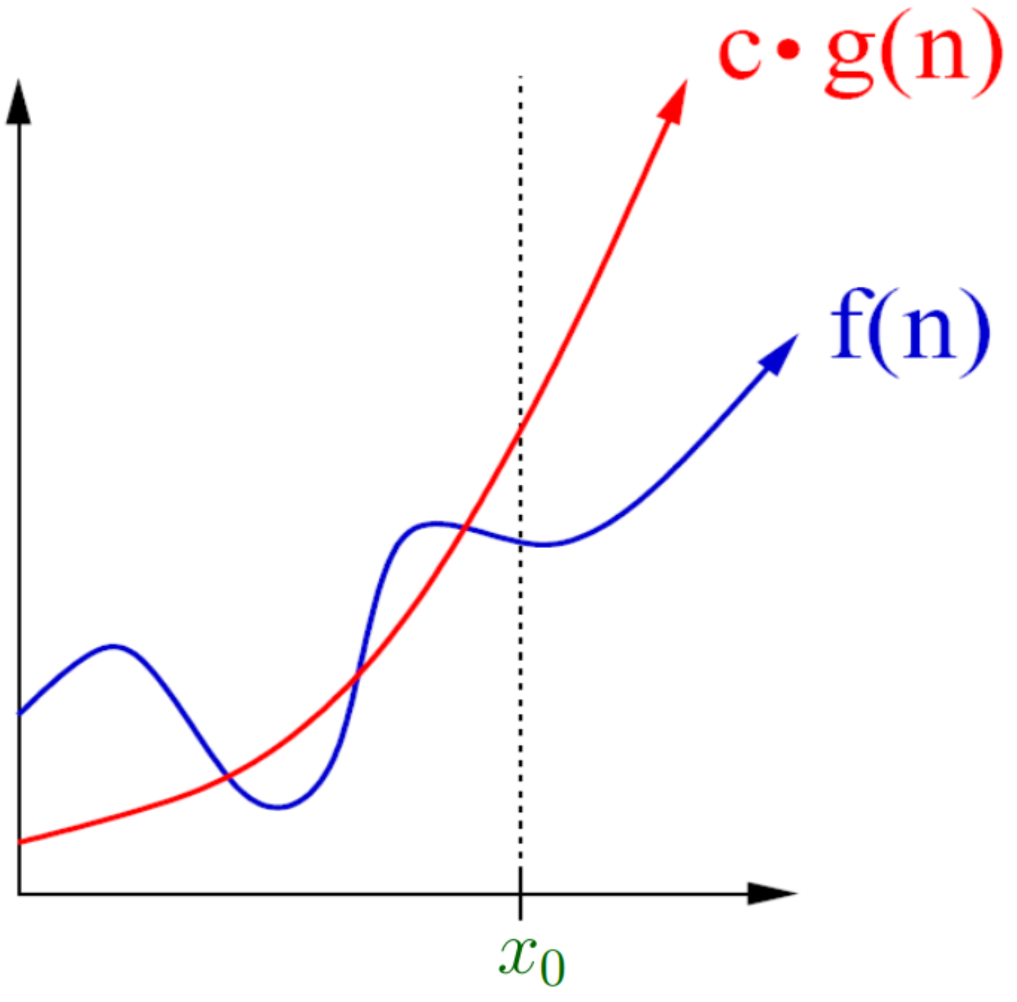
Big-*O* Notation

- **Definition:** Let f and g be functions from $\boldsymbol{Z}$ (or $\boldsymbol{R}$) to $\boldsymbol{R}$. We say that $f(x) = O(g(x))$ (read as *f(x) is big-oh of g(x)*), if there exist some positive constants c and x_0 such that

$$|f(x)| \leq c|g(x)|, \text{ whenever } x > x_0.$$



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Big-O Notation

- Example: $100n + 10000 = O(n^2/10)$
- Let $k = 2000$, we can verify that $\forall n > k,\ 100n + 10000 \leq n^2/10$
- Note that the opposite is not true, i.e., $n^2/10 \neq O(100n + 10000)$
  - proof by contradiction
- Some other functions with $O(n^2)$
  - $4n^2$
  - $8n^2 + 2n - 3$
  - $n^2/5 + n^{1/2} - 10 \log n$
  - $n(n - 3)$

Big-O Estimates for Polynomials

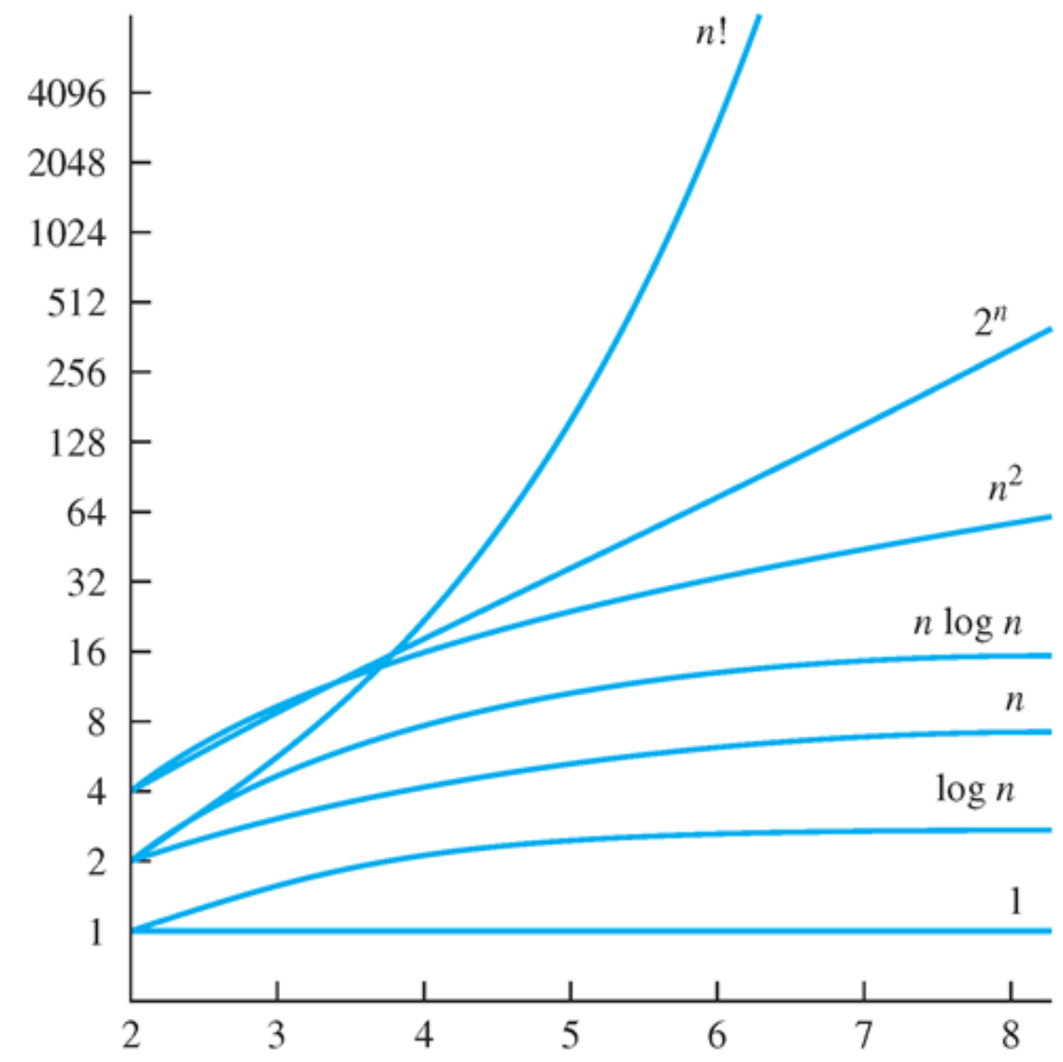
- **Theorem:** Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdot\cdot\cdot + a_1 x + a_0$, where a_0, a_1, …, a_n are real numbers. Then, $f(x) = O(x^n)$.
  - The leading term $a_n x^n$ of a polynomial dominates its growth.
- Proof:
  - Assuming $x > 1$, we have

$$\begin{aligned}
|f(x)| &= |a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0| \\
&\leq |a_n| x^n + |a_{n-1}| x^{n-1} + \cdots + |a_1| x + |a_0| \\
&= x^n (|a_n| + |a_{n-1}|/x + \cdots + |a_1|/x^{n-1} + |a_0|/x^n) \\
&\leq x^n (|a_n| + |a_{n-1}| + \cdots + |a_1| + |a_0|)
\end{aligned}$$

  - Choose $x_0 = 1$ and $c = |a_n| + |a_{n-1}| + \cdots + |a_1| + |a_0|$, then $|f(x)| \leq cx^n$ whenever $x > x_0$.

Some Big-O Estimates

- $1 + 2 + \cdots + n = O(n^2)$
- $n! = O(n^n)$
- $\log n! = O(n \log n)$
- $\log_a n = O(n)$ for $a > 0$
- $n^a = O(n^b)$ for $0 \leq a \leq b$
- $n^a = O(2^n)$ for $a \geq 0$



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Combination of Functions

- **Theorem:** If $f_1(x)$ is $O(g_1(x))$ and $f_2(x)$ is $O(g_2(x))$, then

  $$(f_1 + f_2)(x) = O(\max(|g_1(x)|, |g_2(x)|))$$

- Proof:
  - By definition, there exist constants C_1, C_2, k_1, k_2 such that

    $|f_1(x)| \leq C_1|g_1(x)|$ *when* $x > k_1$

    $|f_2(x)| \leq C_2|g_2(x)|$ *when* $x > k_2$

  - Let $g(x) = \max(|g_1(x)|, |g_2(x)|)$, when $x > \max(k_1, k_2)$ we have

    $$\begin{aligned} |(f_1 + f_2)(x)| &= |f_1(x) + f_2(x)| \leq |f_1(x)| + |f_2(x)| \\ &\leq C_1|g_1(x)| + C_2|g_2(x)| \leq C_1|g(x)| + C_2|g(x)| \\ &= (C_1 + C_2)|g(x)| \end{aligned}$$

  - The proof is concluded with $C = C_1 + C_2$ and $k = \max(k_1, k_2)$.

Combination of Functions

- **Theorem:** If $f_1(x)$ is $O(g_1(x))$ and $f_2(x)$ is $O(g_2(x))$, then
  $$(f_1f_2)(x) = O(g_1g_2(x))$$

- Proof: *very similar to the previous theorem*
  - By definition, there exist constants C_1, C_2, k_1, k_2 such that
    $$|f_1(x)| \le C_1|g_1(x)| \text{ when } x > k_1$$
    $$|f_2(x)| \le C_2|g_2(x)| \text{ when } x > k_2$$
  - Let $g(x) = g_1g_2(x)$, when $x > \max(k_1, k_2)$ we have
    $$\begin{aligned} |(f_1f_2)(x)| &= |f_1(x)f_2(x)| = |f_1(x)||f_2(x)| \\ &\le C_1|g_1(x)|C_2|g_2(x)| = C_1C_2|g_1(x)g_2(x)| \\ &= C_1C_2|g(x)| \end{aligned}$$
  - The proof is concluded with $C = C_1C_2$ and $k = \max(k_1, k_2)$.

Exercise *(3 mins)*

- Order the following functions by order of growth:
  - $f_1(n) = (1.5)^n$
  - $f_2(n) = 8n^3 + 17n^2 + 111$
  - $f_3(n) = (\log n)^2$
  - $f_4(n) = 2^n$
  - $f_5(n) = \log(\log n)$
  - $f_6(n) = n^2(\log n)^3$
  - $f_7(n) = 2^n(n^2 + 1)$
  - $f_8(n) = 8n^3 + n(\log n)^2$
  - $f_9(n) = 100000$
  - $f_{10}(n) = n!$

Big-Ω Notation

- **Definition:** Let f and g be functions from $\boldsymbol{Z}$ (or $\boldsymbol{R}$) to $\boldsymbol{R}$. We say that $f(x) = \Omega(g(x))$ (read as *f(x) is big-omega of g(x)*), if there exist some positive constants c and x_0 such that

$$|f(x)| \geq c|g(x)|, \text{ whenever } x > x_0.$$

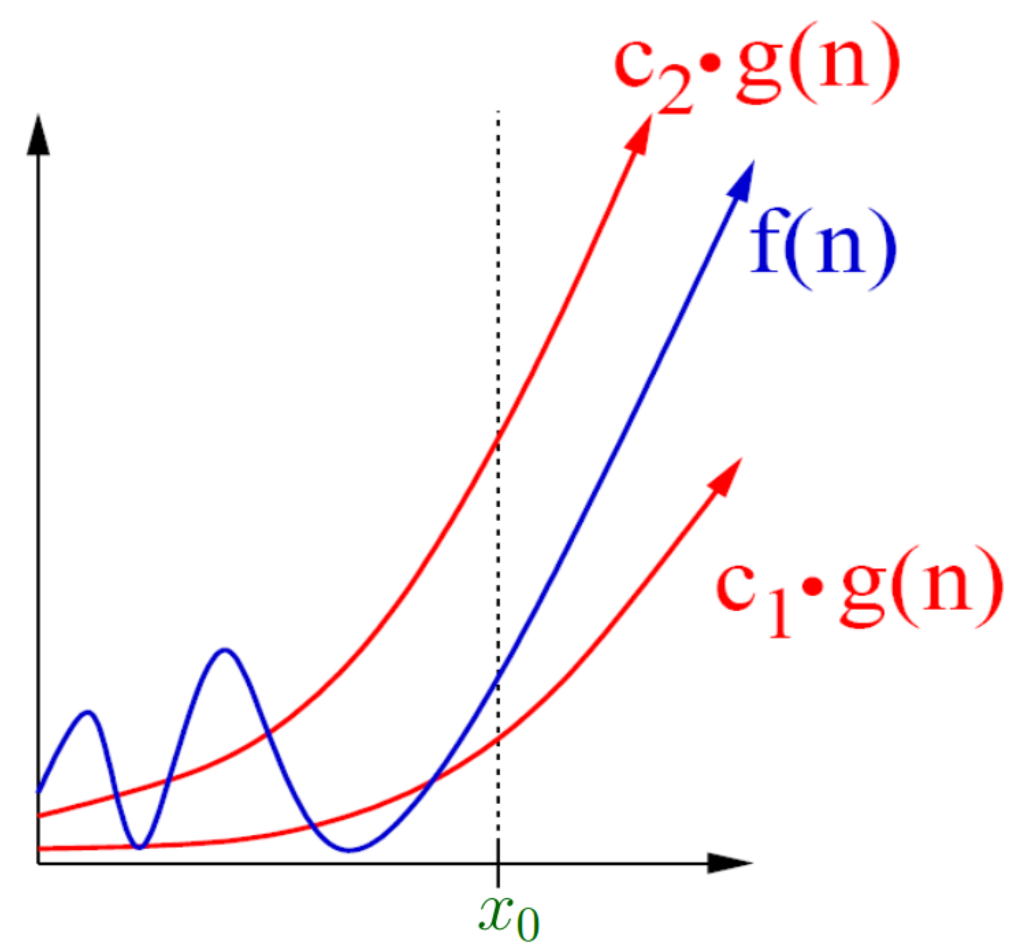
- Big-O gives an upper bound on the growth of a function, while Big-Ω gives a lower bound. Big-Ω tells us that a function grows at least as fast as the other function.

- Note: $f(x) = \Omega(g(x))$ if and only if $g(x) = O(f(x))$

Big-Θ Notation

- **Definition:** Let *f* and *g* be functions from ***Z*** (or ***R***) to ***R***. We say that *f(x) = Θ(g(x))* (read as *f(x) is big-theta of g(x)*), if they have the same order of growth: *f(x) = O(g(x))* and *f(x) = Ω(g(x))*.

- Note: *f(x) = Θ(g(x))* is equal to *g(x) = Θ(f(x))*



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Exercise (3 mins)

- True or false?
  - $3n^2 + 4n = \Theta(n)$?
  - $3n^2 + 4n = \Theta(n^2)$?
  - $3n^2 + 4n = \Theta(n^3)$?
  - $n/5 + 10n \log n = \Theta(n^2)$?
  - $n^2/5 + 10n \log n = \Theta(n \log n)$?
  - $n^2/5 + 10n \log n = \Theta(n^2)$?

Complexity of Algorithms

Computational Problems and Algorithms

- **Computational problem:** a task solved by a computer
  - a collection of instances (i.e., problem input, with size n) together with a (perhaps empty) set of solutions (output) for every instance
  - an instance is just a specific problem input, not the problem itself
- **Algorithm:** a finite sequence of precise instructions for performing a computation or for solving a problem
- We say an algorithm solves the problem if it halts with the correct output for every input instance

Computational Problems and Algorithms

- **Computational problem:** a collection of instances together with a (perhaps empty) set of solutions (output) for every instance
- **Algorithm:** a finite sequence of precise instructions for performing a computation or for solving a problem
- We say an algorithm solves the problem if it halts with the correct output for every input instance
- Example: algorithm for calculating the sum of $a_1, a_2, \ldots, a_n$
  - Step 1: set $S = 0$
  - Step 2: for $i = 1$ to n, $S := S + a_i$ (i.e., assign S the value $S + a_i$)
  - Step 3: output S
  - *example of a problem instance:* $< 8, 3, 6, 7, 1, 2, 9 >$ *(here* $n = 7$*)*

Time and Space Complexity

- The number of machine operations (addition, multiplication, comparison, assignment, etc.) needed in an algorithm is the time complexity of the algorithm, and the amount of memory needed is the space complexity of the algorithm.

- Example: algorithm for calculating the sum of $a_1, a_2, \ldots, a_n$
  - Step 1: set $S = 0$
  - Step 2: for $i = 1$ to n, $S := S + a_i$ (i.e., assign S the value $S + a_i$)
  - Step 3: output S
  - time complexity: $O(n)$ ** usually we ignore operations on iterator i*

    Step 2 takes n operations (in-place additions). Step 1 and 3 each take 1 operation. Altogether this algorithm takes $n + 2$ operations.
  - space complexity: $O(n)$

    The input numbers take $O(n)$ memory and S, i take $O(1)$ memory.

Example: Horner's Method

- Consider the evaluation of $f(x) = 1 + 2x + 3x^2 + 4x^3$
  - direct computation: 3 additions and 6 multiplications
  - better solution: evaluate $f(x) = 1 + x(2 + x(3 + 4x))$ instead, which takes 3 additions and 3 multiplications
- Problem: evaluation of $f(x) = a_0 + a_1x + \cdots + a_nx^n$
- **Horner's method:** $f(x) = a_0 + x(a_1 + x(a_2 + \cdots + x(a_{n-1} + xa_n) \cdots))$
  - Step 1: set $S = a_n$
  - Step 2: for $i = 1$ to n, $S := a_{n-i} + xS$
  - Step 3: output S
  - time complexity: $O(n)$

    Step 1 and 3 each take one operation. Step 2 takes $3n$ operations: n multiplications, n additions, n assignments.

Another Example

- Determine the time complexity of the following algorithm:

  for $i := 1$ to n

  for $j := 1$ to n

  $a := 2 * n + i * j;$

  end for

  end for

- Computing the value of a in each iteration takes 4 operations (two multiplications, one addition and one assignment). There are n^2 iterations in two loops. So it takes $n^2 \times 4 = 4n^2$ operations. The time complexity of this algorithm is $O(n^2)$.
  - Note that we can compute $2 * n$ only once but still $O(n^2)$ complexity.

Exercise (3 mins)

- Determine the time complexity of the following algorithm:

$S := 0$

for $i := 1$ to n

  for $j := 1$ to i

    $S := S + i * j$;

  end for

end for

Types of Complexity Analysis

- Example: (Insertion Sort)

Input: $A[1 \ldots n]$ is an array of numbers

for $j := 2$ to n

  $key = A[j]$;

  $i = j - 1$;

  while $i \geq 1$ and $A[i] > key$ do

    $A[i+1] = A[i]$;

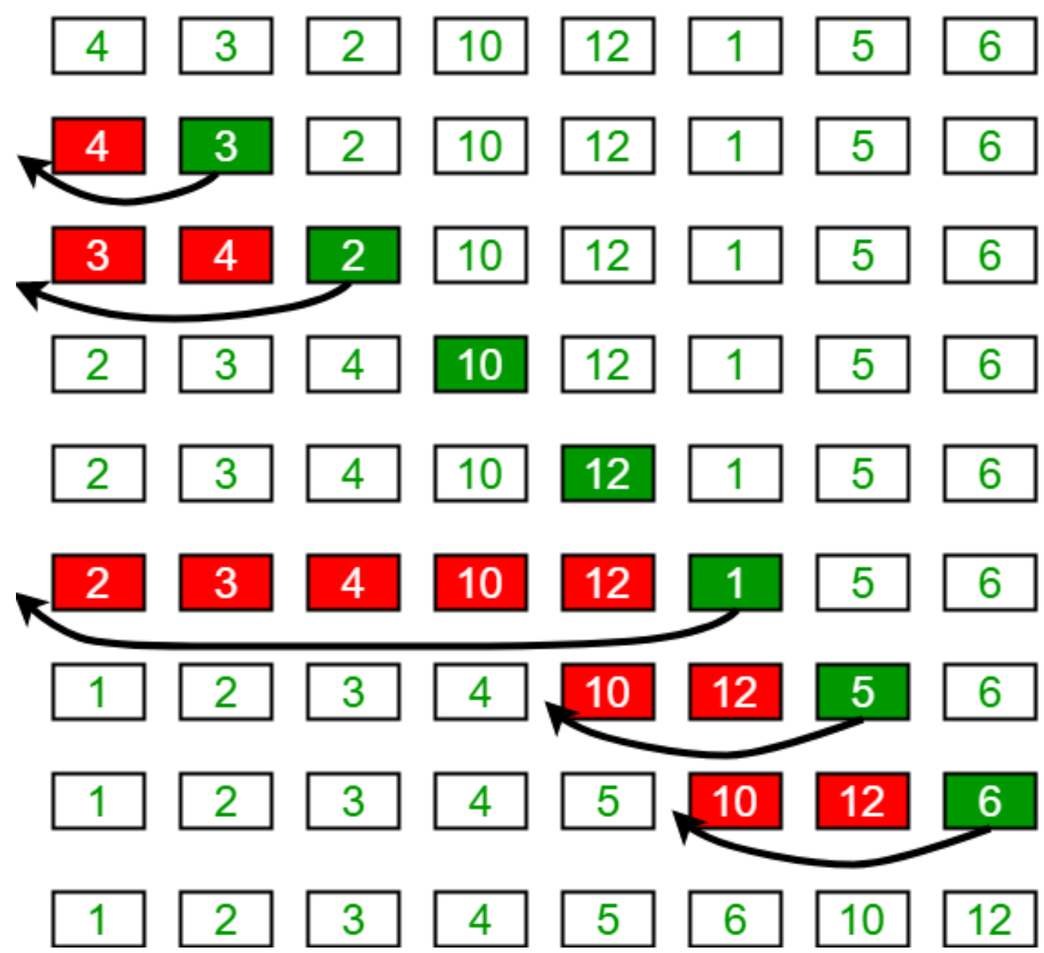
    $i - -$;

  end while

  $A[i+1] = key$;

end for

Insertion Sort Execution Example



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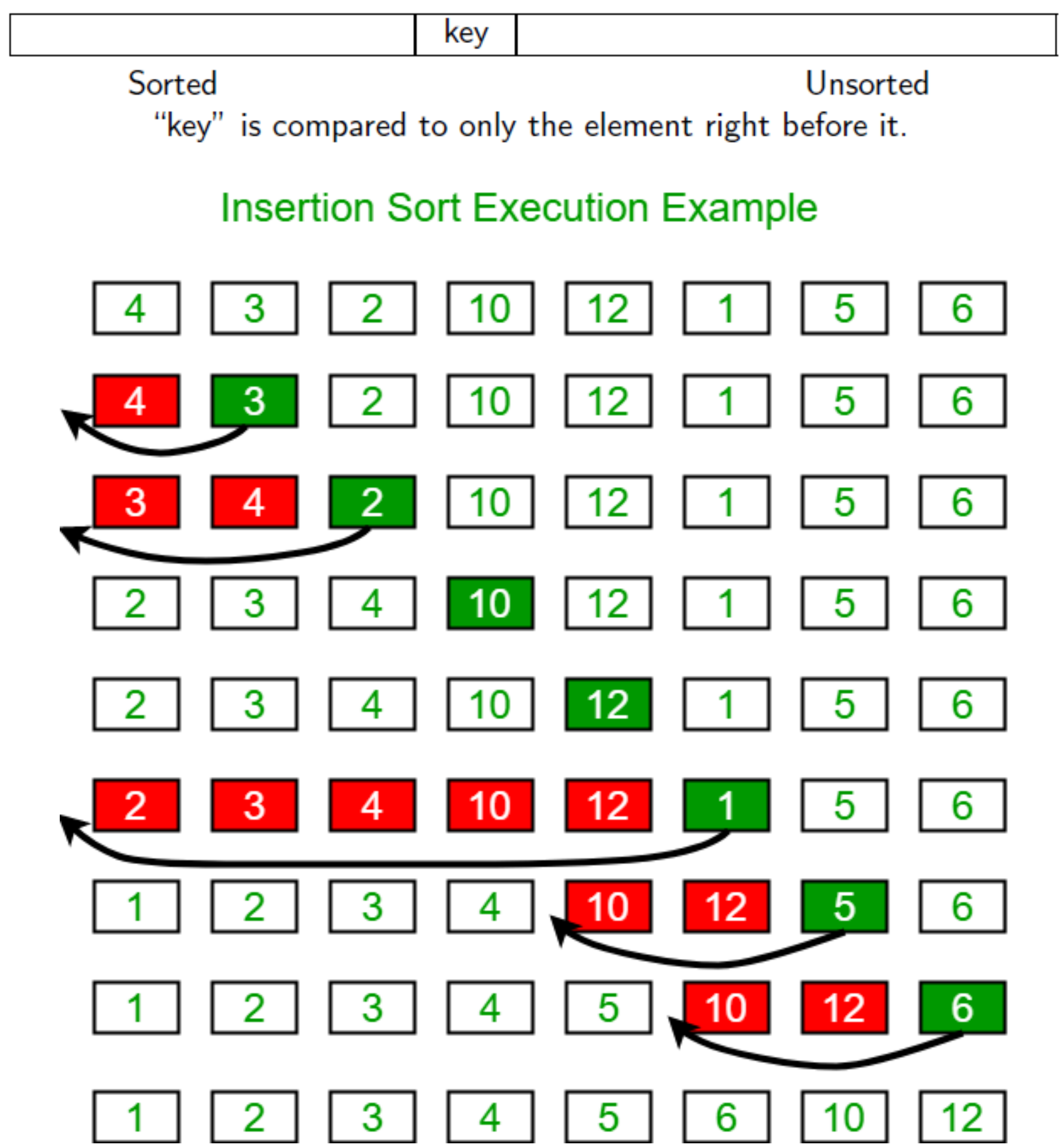
Complexity Analysis: Type I

- **Best-case complexity:** constraints on the input, other than size, resulting in the fastest possible running time for the given size.

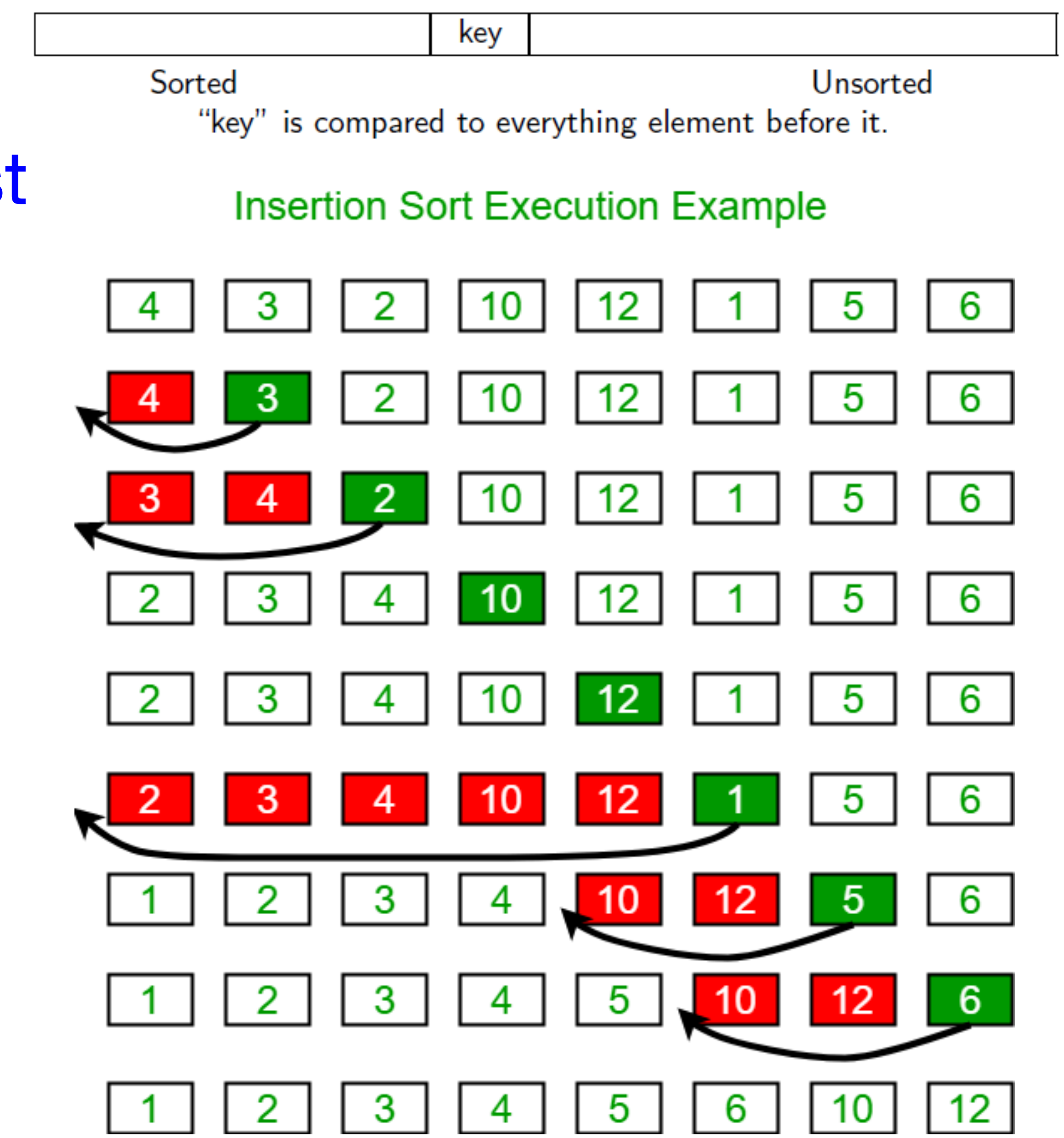
- Example: (Insertion Sort)
  - A[1] ≤ A[2] ≤ A[3] ≤ · · · ≤ A[n]
  - time complexity: *Θ(n)*

    n - 1 comparisons



Complexity Analysis: Type II

- **Worst-case complexity:** constraints on the input, other than size, resulting in the slowest possible running time for the given size.



- Example: (Insertion Sort)
  - $A[1] \geq A[2] \geq A[3] \geq \cdots \geq A[n]$
  - time complexity: $\Theta(n^2)$

$$\sum_{j=2}^{n} j - 1 = \frac{n(n-1)}{2}$$

comparisons & swaps

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Complexity Analysis: Type III

- **Average-case complexity:** average running time over all possible inputs for the given size (usually involving probability distribution on input instances)

- Example: (Insertion Sort)
  - if *n!* instances are equally likely
  - time complexity: $\Theta(n^2)$

$$\sum_{j=2}^{n} \frac{j-1}{2} = \frac{n(n-1)}{4}$$

comparisons & swaps

Some Thoughts on Algorithm Design

- Algorithm design is mainly about designing algorithms that have small Big-O running time.
- Being able to design good algorithms lets you identify the hard parts of your problem and handle them effectively.
- Too often, programmers try to solve problems using brute force techniques and end up with slow and complicated code!
- A few hours of abstract thought devoted to algorithm design could have speeded up and simplified the solution substantially!

Complexity of Problems

Dealing with Hard Problems

- What would you do if you cannot find an efficient algorithm for a given problem?

Blame yourself

Prove that no such algorithm exists

Dealing with Hard Problems

- Showing that a problem has efficient algorithms is relatively easy:
  - All that is needed is to demonstrate an algorithm.
- Proving that no efficient algorithm exists for a particular problem is difficult:
  - How can we prove the non-existence of something?
  - We will now learn about ***NP***-complete problems, which provide us with a way to approach this question.

Introduction to *NP*-Complete

- ***NP*-complete:** a very large class of problems (> 3000 are known) which is not known to have "efficient" solutions.
- It is known that if any one of the ***NP***-complete problems has an efficient solution then all of the ***NP***-complete problems have efficient solutions.
- Researchers have spent innumerable man-years trying to find efficient solutions to these problems but failed.
- So, ***NP***-complete problems are very likely to be hard.
- What we can do: prove that a hard problem is ***NP***-complete.
    - This shows no one can find an efficient solution so far.
- Next, we show how to define complexity classes formally.

Example Problem: COMPOSITE

- **COMPOSITE:** given a positive integer n, are there integers $d, k > 1$ such that $n = dk$?

- The naive algorithm for determining whether n is composite enumerates d from 2 to $n - 1$ to see if any of them divides n.
    - This takes $\Theta(n)$ division operations, which might seem linear time and very efficient. However, it is problematic to treat the value of n as the input size of the algorithm, because integer n is usually processed as a binary string of length $\Theta(\log_2 n)$ rather than of length $\Theta(n)$. An efficient algorithm should have time complexity "close" to its input size $\log_2 n$ rather than the input value n, e.g., integer multiplication $n \times n$ (show later) requires only $O((\log_2 n)^2)$ bit operations.
    - Therefore, we measure the input size $L = \log_2 n$ and the time complexity is $\Theta(n) = \Theta(2^L)$, which is exponential in L and hence very impractical. (Note that each integer division operation n/d also takes $O((\log_2 n)^2)$ bit operations but here we ignore it for simplicity.)

- To summarize, the input size is important for measuring complexity.

The Input Size of Problems

- Complexity of a problem is measured in terms of its input size.
- The input size of a problem is the number of bits needed to encode the input of the problem.
  - The optimal input size, determined by an optimal encoding method, is hard to compute in most cases.
  - For most problems, it is sufficient to choose some natural, and (usually) simple, encoding method and use its encoded input size.
- Example 1: COMPOSITE
  - What is the input size of this problem?

    Any integer $n > 0$ can be represented as a binary string $a_0 a_1 \cdots a_k$ of length $\lceil \log_2 (n+1) \rceil$. Therefore, a natural measure of the input size is $\lceil \log_2 (n+1) \rceil$ (or $\Theta(\log_2 n)$ for simplicity)

The Input Size of Problems

- Complexity of a problem is measured in terms of its input size.
- The input size of a problem is the number of bits needed to encode the input of the problem.
  - The optimal input size, determined by an optimal encoding method, is hard to compute in most cases.
  - For most problems, it is sufficient to choose some natural, and (usually) simple, encoding method and use its encoded input size.
- Example 2: Sort n integers $a_1, \ldots, a_n$
  - What is the input size of this problem?

    Using fixed-length encoding (i.e., all input numbers share the same length), we write every a_i as a binary string of the same length $m = \lceil \log_2 \max(|a_i| + 1) \rceil + 1$ (one extra bit for the +/- sign).

    This natural encoding gives an input size nm.

Decision and Optimization Problems

- **Decision problem:** a problem that has a yes or no answer.
  - E.g., "Given $n > 0$, is integer m such that $m^m < n$?"

- **Optimization problem:** a problem that asks for some answer that maximizes or minimizes a particular objective function.
  - E.g., "Given $n > 0$, what is the largest integer m such that $m^m < n$?"

- Given an algorithm for solving the optimization problem, solving the corresponding decision problem is usually trivial.
  - Contrapositive: if we prove that a given decision problem is hard to solve efficiently, then the corresponding optimization problem must be (at least as) hard.

- The other direction (decision → optimization) also often works.
  - E.g., use binary search to find m in the above examples.

Complexity Classes

- Computational Complexity Theory is a field that deals with:
  - classification of certain "decision problems" into several classes:
    the class of "easy" problems
    the class of "hard" problems
    the class of "hardest" problems
  - relations among the above classes
  - properties of problems in the above classes
- How to classify decision problems?
  - use polynomial-time algorithms (often called efficient algorithms)

Polynomial-Time Algorithms

- **Polynomial-time algorithm:** an algorithm that runs in time $O(n^c)$, where $c > 0$ is a constant independent of n, and n is the input size of the problem that the algorithm solves.
  - E.g., sorting algorithms
- When the input size of the algorithm is n^a (for any constant $a > 0$), the algorithm is still polynomial-time.
- Also, an algorithm that is composed by several polynomial-time algorithms is still polynomial-time.
- The above somehow shows why people choose polynomial-time to define efficient algorithms, because the common operations (e.g., addition, subtraction, multiplication, composition, etc.) are closed for polynomials.

Non-Polynomial-Time Algorithms

- **Non-polynomial-time algorithm:** an algorithm of which the running time is not $O(n^c)$ for any constant $c > 0$.
    - E.g., naive algorithm for solving the composite number problem
- Non-polynomial-time algorithms are usually impractical.
    - E.g., exponential-time 2^n for $n = 100$ takes billions of years!!!
- In reality, even polynomial-time algorithms could be impractical.
    - E.g., a $\Theta(n^{20})$ algorithm may not be very practical for $n = 100$.

Tractable Problems and Class *P*

- **Tractable problem:** a problem that is solvable in polynomial time (or the problem is in polynomial time). That is, there exists a polynomial-time algorithm that solves the problem.

- Class ***P*****:** consists of all decision problems that are solvable in polynomial time. That is, there exists a polynomial-time algorithm that decides if any given input is a yes-input or a no-input.
  - E.g., PRIMES (determining whether a number is prime) is in ***P***.

- How to prove that a decision problem is in ***P***?
  - find a polynomial-time algorithm (relatively easy)

- How to prove that a decision problem is not in ***P***?
  - prove that there is no polynomial-time algorithm for solving this problem (much much harder)

Certificates and Class *NP*

- A decision problem is usually formulated as: "Is there an object satisfying some conditions?"

- A certificate/witness/hint for a yes-input is a specific object that is used to verify/prove/show that this input is indeed a yes-input.
    - E.g., the COMPOSITE problem can be formulated as: "Is there an integer d $(1 < d < n)$ such that d divides n?". So, a certificate for a composite number n (i.e., n is a yes-input of COMPOSITE) can be one of such integer factors d.

- Class ***NP*** (nondeterministic polynomial-time): consists of all decision problems such that, for each yes-input, there exists a certificate such that a (universal) polynomial-time algorithm can use it to verify the input is indeed a yes-input.
    - E.g., COMPOSITE is in ***NP*** because the certificate can be verified in polynomial time (in the input size): the input size is $\Theta(\log_2 n)$ and checking if d divides n takes $O((\log_2 n)^2)$ bit operations.

P = ***NP*?**

- Whether $\boldsymbol{P} = \boldsymbol{NP}$ is one of the most important problems in CS.
- It is not hard to see that $\boldsymbol{P} \subseteq \boldsymbol{NP}$.
- Intuitively, $\boldsymbol{NP} \subseteq \boldsymbol{P}$ is doubtful.
  - Just being able to verify a certificate in polynomial time does not necessarily mean we can tell whether an input is a yes-input or a no-input in polynomial time, e.g., certificates may be hard to find.
  - So far, we are still far from solving it and do not know the answer. However, the search for such a solution has provided us with deep insights into what distinguishes “easy” problems from “hard” ones.

NP-Complete and *NP*-Hard

- ***NP*-complete:** consists of the hardest problems in ***NP***.
  - ***NP***-complete problems are reducible to each other, i.e., they are equivalently hard

    If solving problem A can be transformed into solving problem B, we say A reduces to B. This also means B is at least as hard as A.

- ***NP*-hard:** consists of problems at least as hard as ***NP***-complete.
  - some ***NP***-hard problems may not belong to ***NP***

05 Number Theory and Cryptography

To be continued…

Announcements

- Please submit your Undergraduate Students Declaration Form with your handwritten signature if you have not done so.
- Assignment 2 will be released today:
  - the deadline is before class on Oct 31
  - must be written in English
  - 100 points maximum but 105 in total
  - DO NOT CHEAT!