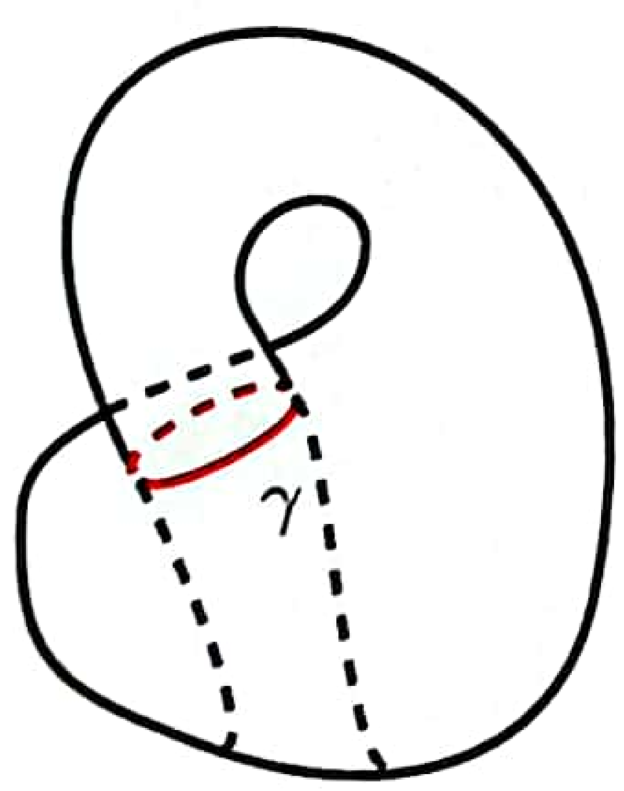
MA323 Topology Final Exam

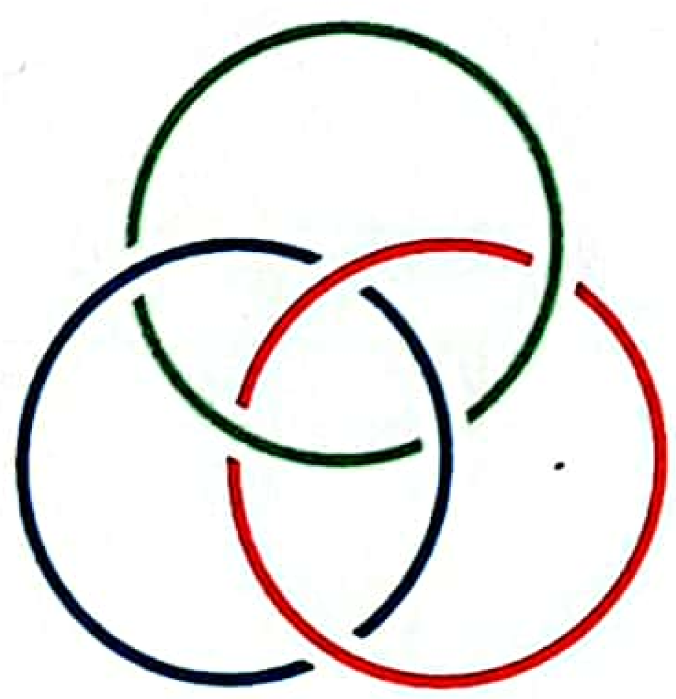
2:00–4:00 pm

January 16, 2024

1. (20 points)
   (a) Give an example of a topological space that is connected but not path connected. Remember to describe its topology. No justification is needed.
   (b) Show that a path-connected space is necessarily connected.

2. (40 points)
   (a) State the topological classification for compact, connected 2-manifolds without boundary.
   (b) Identify the class of the Klein bottle, with justification.
   (c) The Klein bottle cannot embed into $\mathbb{E}^3$. Show that the subspace $X \subset \mathbb{E}^3$ formed by a Klein bottle intersecting itself in a circle γ, as shown in the picture on the left below, is homotopy equivalent to $S^1 \vee S^1 \vee S^2$. (Hint: γ bounds a disc on the surface, which is homotopy equivalent to a point.)
   (d) Give a presentation for the fundamental group of X and draw the loops representing its generators in the picture (make a copy on your answer sheet first). Remember to mark their common base point.





3. (20 points) Given continuous maps $f, g\colon (S^1, 1) \to (Y, y_0)$, on the fundamental group, they induce homomorphisms $f_\pi, g_\pi\colon \pi_1(S^1, 1) \to \pi_1(Y, y_0)$. Show that $f \simeq g \operatorname{rel}\{1\}$ if and only if $f_\pi = g_\pi$.

4. (10 points) A map $f\colon X \to Y$ is *locally constant* if for each $x \in X$ there is an open set U with $x \in U$ and $f|_U$ constant. Prove or disprove: If X is connected and Y is any space, then every locally constant map is constant.

5. (10 points) Let $X = S^3 \setminus B$ be the link complement of the *Borromean rings* B as shown in the picture on the right above. Then $\pi_1(X)$ has a presentation of the form

$$\langle a, b, c \,|\, [a, [b^{-1}, c]],\ [b, [c^{-1}, a]],\ [c, [a^{-1}, b]] \rangle$$

where $[x, y] = xyx^{-1}y^{-1}$ and any one of the relations is redundant. Removing any component of the link results in a *trivial link* of two components. Describe the corresponding effect on the fundamental group.