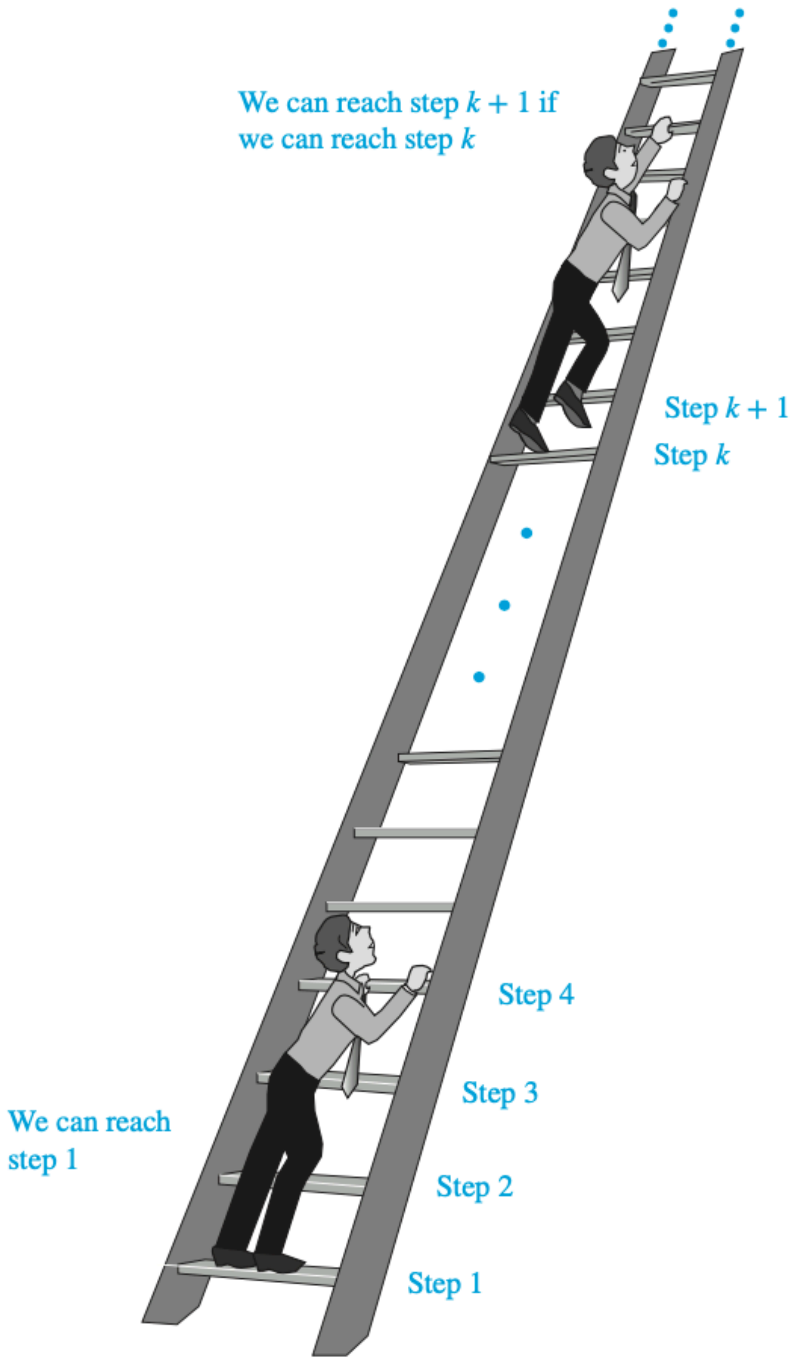
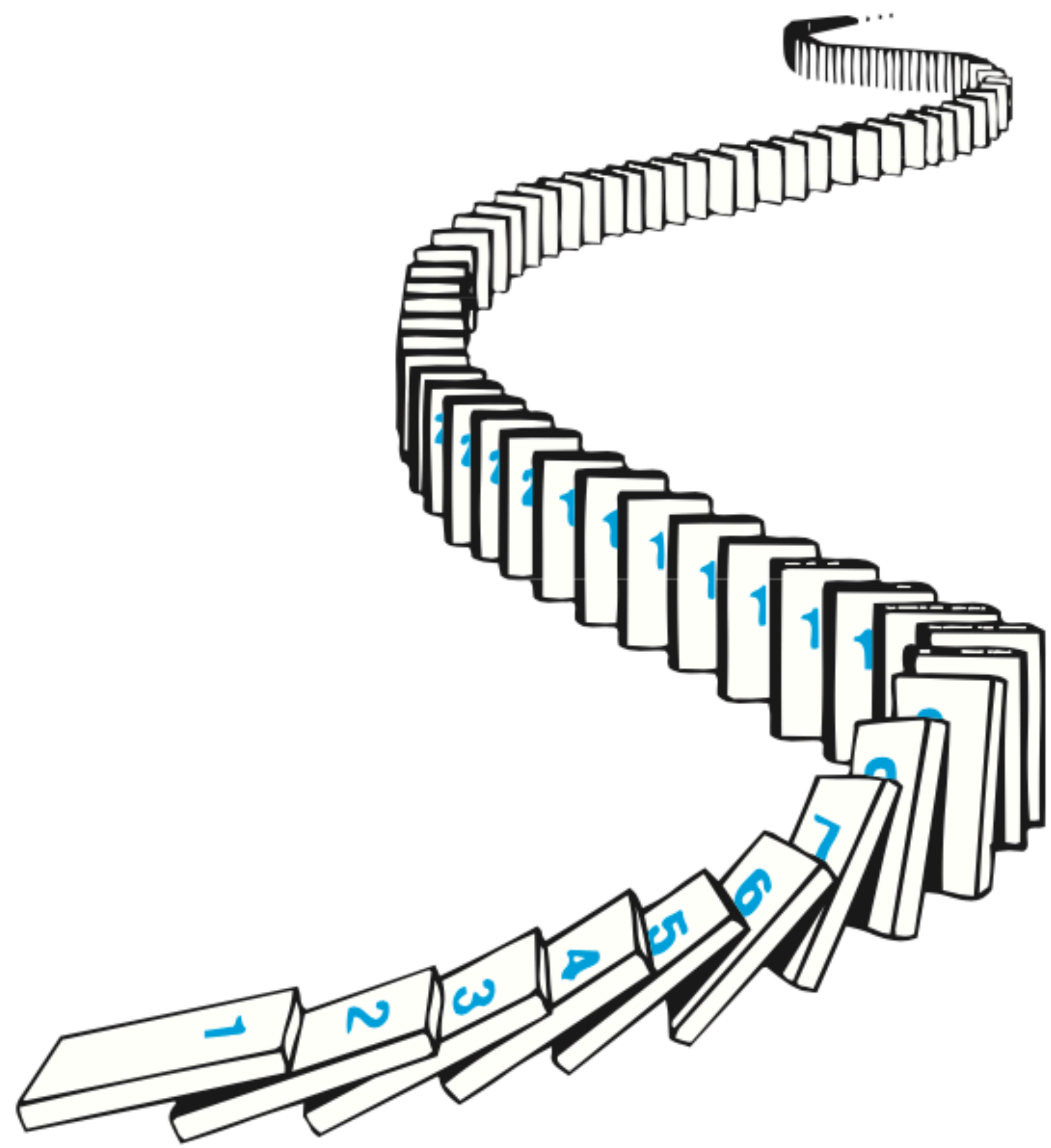
06 Induction and Recursion

CS201 Discrete Mathematics

Instructor: Shan Chen

Mathematical Induction





Mathematical Induction

Principle of Mathematical Induction

- Let $P(n)$ be a propositional function, i.e., $P(n)$ is either true or false when n is a specific number.

- **Principle of mathematical induction:** To prove that $P(n)$ is true for all positive integers n, we complete two steps:
  - **Basis step:** prove $P(1)$ is true
  - **Inductive step:** prove $\forall k \in \boldsymbol{Z}^+, P(k) \rightarrow P(k+1)$ is true

    ** the assumption "P(k) is true" is called the **inductive hypothesis***

- **Q:** Why is this principle valid?

- Proof by contradiction: Assume $P(n)$ is false for some integer $n \geq 1$, then the set S of all positive integer n such that $P(n)$ is false is non-empty. Let m be the smallest integer in S. ** why m exists?* We have $m \geq 2$ as $P(1)$ is true. However, since $P(m - 1)$ is true and $P(m-1) \rightarrow P(m)$ is true, $P(m)$ must be true, contradiction!

Principle of Mathematical Induction

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  - **Basis step:** prove $P(1)$ is true
  - **Inductive step:** prove $\forall k \in \mathbf{Z}^+, P(k) \to P(k + 1)$ is true

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- **Well-ordering principle** (axiom): every nonempty subset of the set of positive integers has a least/minimum element.
  - this principle is equivalent to principle of mathematical induction
    * *proof left as an exercise*

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Example

- Show that $1 + 2 + \cdots + n = n(n + 1)/2$ for any positive integer n.

- Proof by induction:
  - Let $P(n)$ be the proposition that the sum of the first n positive integers is equal to $n(n + 1)/2$.
  - **Basis step:** $P(1)$ is true, because $1 = 1(1 + 1)/2$.
  - **Inductive step:** From the inductive hypothesis, i.e., $P(k)$ is true for an arbitrary positive integer k, we need to show that $P(k + 1)$ is true, i.e., $1 + 2 + \cdots + k + 1 = (k + 1)((k + 1) + 1)/2$.

    $$1 + 2 + \cdots + k + (k + 1) = k(k + 1)/2 + k + 1$$
    $$= (k(k + 1) + 2(k + 1))/2 = (k + 1)(k + 2)/2 = (k + 1)((k + 1) + 1)/2$$
  - By mathematical induction, we know that $P(n)$ is true for all positive integers n. That is, we have proven that $1 + 2 + \cdots + n = n(n + 1)/2$ holds for all positive integers n.

Exercise (5 mins)

- For any positive integer n, $1 + 3 + 5 + \cdots + (2n - 1) = ?$ Prove it.

  - Show that $1 + 2 + \cdots + n = n(n + 1)/2$ for any positive integer n.

  - Proof by induction:
    - Let $P(n)$ be the proposition that the sum of the first n positive integers is equal to $n(n + 1)/2$.
    - **Basis step:** $P(1)$ is true, because $1 = 1(1 + 1)/2$.
    - **Inductive step:** From the inductive hypothesis, i.e., $P(k)$ is true for an arbitrary positive integer k, we need to show that $P(k + 1)$ is true, i.e., $1 + 2 + \cdots + k + 1 = (k + 1)((k + 1) + 1)/2$.

      $1 + 2 + \cdots + k + (k + 1) = k(k + 1)/2 + k + 1$
      $= (k(k + 1) + 2(k + 1))/2 = (k + 1)(k + 2)/2 = (k + 1)((k + 1) + 1)/2$
    - By mathematical induction, we know that $P(n)$ is true for all positive integers n. That is, we have proven that $1 + 2 + \cdots + n = n(n + 1)/2$ holds for all positive integers n.

Exercise *(5 mins)*

- For any positive integer n, $1 + 3 + 5 + \cdots + (2n - 1) = ?$ Prove it.
- $1 + 3 + 5 + \cdots + (2n - 1) = n^2$
  - $1 + 3 = 4$, $1 + 3 + 5 = 9$, …
- Proof by induction: (Let $P(n)$ be the above equation.)
  - **Basis step:** $P(1)$ is true, because $1 = 1^2$.
  - **Inductive step:** From the inductive hypothesis, i.e., $P(k)$ is true for an arbitrary positive integer k, we need to show that $P(k + 1)$ is true, i.e., $1 + 3 + \cdots + 2(k + 1) - 1 = (k + 1)^2$.

$$1 + 3 + \cdots + 2k - 1 + 2(k + 1) - 1 = k^2 + 2(k + 1) - 1$$
$$= k^2 + 2k + 2 - 1 = k^2 + 2k + 1 = (k + 1)^2$$

  - By mathematical induction, we know that $P(n)$ is true for all positive integers n. That is, we have proven that $1 + 3 + 5 + \cdots + (2n - 1) = n^2$ for all positive integers n.

Exercise *(5 mins)*

- Prove that for any integer $n \geq 2$, $2^{n+1} \geq n^2 + 3$

  - Show that $1 + 2 + \cdots + n = n(n + 1)/2$ for any positive integer n.

  - Proof by induction:
    - Let $P(n)$ be the proposition that the sum of the first n positive integers is equal to $n(n + 1)/2$.
    - **Basis step:** $P(1)$ is true, because $1 = 1(1 + 1)/2$.
    - **Inductive step:** From the inductive hypothesis, i.e., $P(k)$ is true for an arbitrary positive integer k, we need to show that $P(k + 1)$ is true, i.e., $1 + 2 + \cdots + k + 1 = (k + 1)((k + 1) + 1)/2$.

      $1 + 2 + \cdots + k + (k + 1) = k(k + 1)/2 + k + 1$
      $= (k(k + 1) + 2(k + 1))/2 = (k + 1)(k + 2)/2 = (k + 1)((k + 1) + 1)/2$
    - By mathematical induction, we know that $P(n)$ is true for all positive integers n. That is, we have proven that $1 + 2 + \cdots + n = n(n + 1)/2$ holds for all positive integers n.

Exercise *(5 mins)*

- Prove that for any integer $n \geq 2$, $2^{n+1} \geq n^2 + 3$
- Proof by induction:
  - Let $P(n)$ be $2^{n+1} \geq n^2 + 3$.
  - **Basis step:** $P(2)$ is true, because $2^{2+1} = 8 \geq 7 = 2^2 + 3$.
  - **Inductive step:** From the inductive hypothesis, i.e., $P(k)$ is true for an arbitrary integer $k \geq 2$, we need to show that $P(k + 1)$ is true:

$$2^{(k+1)+1} = 2 \cdot 2^{k+1} \geq 2(k^2 + 3) = 2k^2 + 6 = (k + 1)^2 - 2k - 1 + k^2 + 6$$
$$= (k + 1)^2 + (k - 1)^2 + 4 \geq (k + 1)^2 + 3$$

  - By mathematical induction, $P(n)$ is true for all integers $n \geq 2$.

Another Form of Induction

- We may have another form of mathematical induction as follows:
  - First suppose that we have a proof that $P(1)$ is true.
  - Next suppose that we have a proof that
    $\forall k \geq 1, P(1) \wedge P(2) \wedge \cdots \wedge P(k) \rightarrow P(k+1)$
  - Then,
    $P(1) \rightarrow P(2)$
    $P(1) \wedge P(2) \rightarrow P(3)$
    $P(1) \wedge P(2) \wedge P(3) \rightarrow P(4)$
    …
  - Iterating gives us a proof of $P(n)$ for all n

Strong Induction

- **Second principle of mathematical induction:** To prove that $P(n)$ is true for all positive integers n, we complete two steps:
  - **Basis step:** prove $P(1)$ is true
  - **Inductive step:** prove $\forall k \in \mathbf{Z}^+, P(1) \wedge \cdots \wedge P(k) \to P(k+1)$ is true
    ** the assumption "$P(1) \wedge P(2) \wedge \cdots \wedge P(k)$ is true" is called the* ***inductive hypothesis***

- This is called strong induction or complete induction, while the previous principle is called weak or incomplete induction.

- In practice, strong induction is often easier to apply than its weak form, because the inductive hypothesis is stronger.

- However, these two forms of induction are actually equivalent.
  - *proof left as an exercise*

Example

- Prove that every positive integer is a power of a prime or the product of powers of primes.

- Proof:
  - **Basis step:** *1* is a power of a prime number, $1 = 2^0$.
  - **Inductive step:**

    Inductive hypothesis: every positive integer that is $\leq k$ is a power of a prime or a product of powers of primes.

    If $k + 1$ is a prime power, $P(k + 1)$ is true. Otherwise, $k + 1$ must be a composite, i.e., a product of two smaller positive integers, each of which is, by the inductive hypothesis, a power of a prime or the product of powers of primes. Therefore, $P(k + 1)$ is true.
  - Finally, by strong induction, every positive integer is a power of a prime or a product of powers of primes.

Mathematical Induction Summary

- A typical proof by induction, showing that $P(n)$ is true for all integers $n \geq b$, consists of three steps:
  - **Basis step:** prove $P(b)$ is true
  - **Inductive step:** prove one of the following

    $\forall k \geq b, P(k) \rightarrow P(k+1)$ is true ***OR***

    $\forall k \geq b, P(b) \wedge \cdots \wedge P(k) \rightarrow P(k+1)$ is true
  - **Conclusion:** based on the (second) principle of mathematical induction, we conclude that $P(n)$ is true for all $n \geq b$.

- The assumption "$P(k)$ is true" ***OR*** "$P(1) \wedge P(2) \wedge \cdots \wedge P(k)$ *is true*" is called the inductive hypothesis (IH).
  - IH is used to prove "$P(k+1)$ *is true*".

Recursion

Recursion

- **Recursion:** a method of solving a computational problem where its solution depends on solutions to smaller instances of the same problem.
- Recursive computer programs or algorithms often lead to inductive analysis.
- A classical example of recursion is the Towers of Hanoi Problem.

Towers of Hanoi

- **Problem:** Find an efficient way to move all of the disks from one peg to another.
  - *3* pegs and *n* disks of different sizes
  - A legal move takes a disk from one peg and moves it onto another peg so that it is not on top of a smaller disk.

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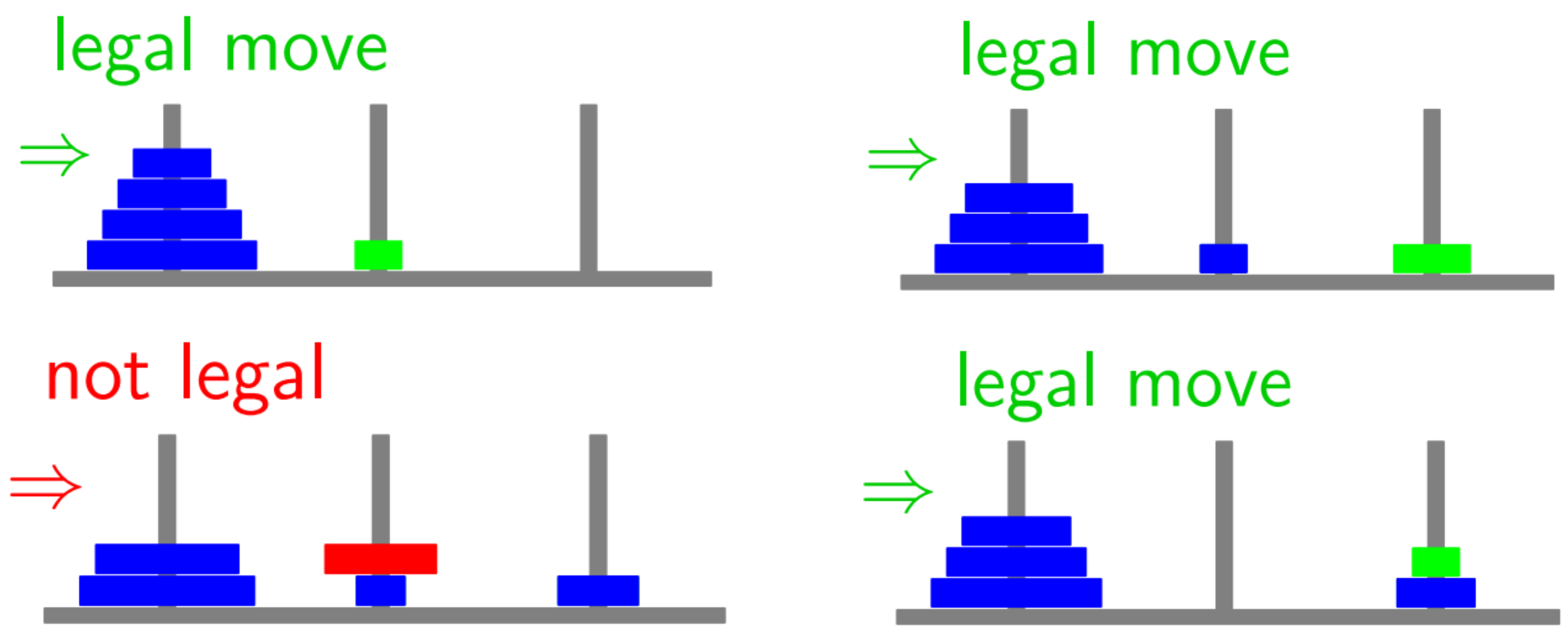

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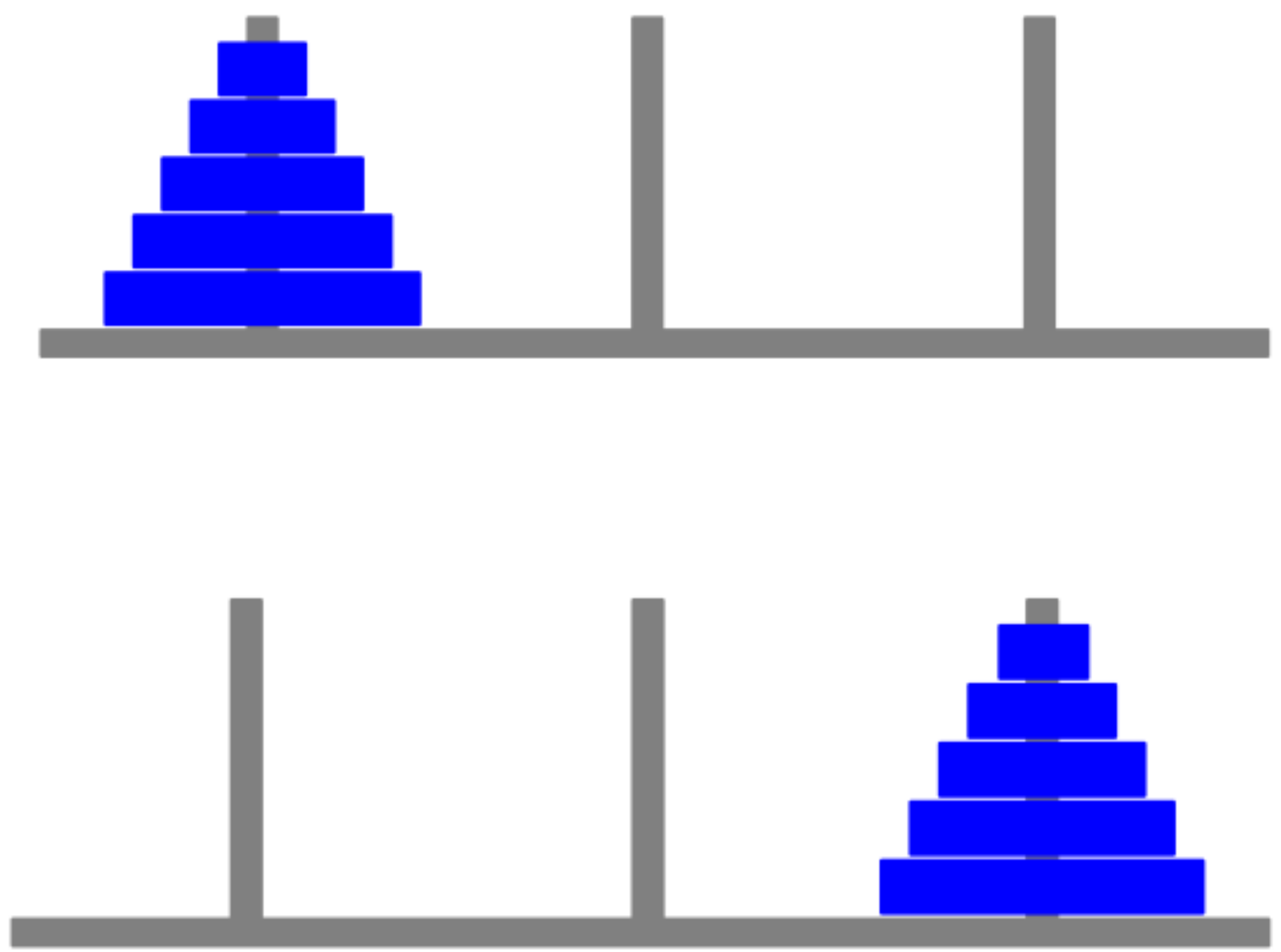
Towers of Hanoi

- **Problem:** Find an efficient way to move all of the disks from one peg to another, using only legal moves.

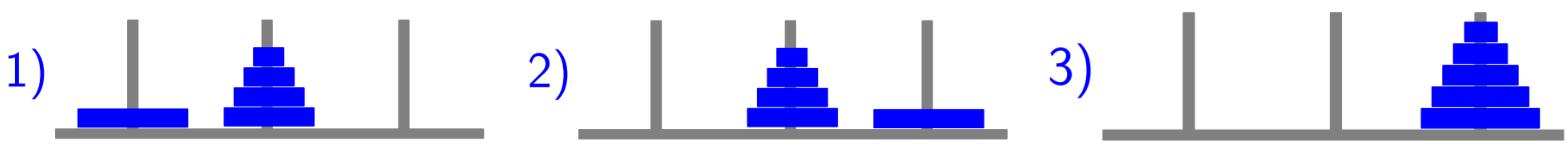


Towers of Hanoi

- **Problem:** Find an efficient way to move all of the disks from one peg to another, using only legal moves.

- **Solution** by recursion:
  - **Basis step:** If $n = 1$, moving one disk from one to another is easy.

  - **Recursive step:** If $n > 1$, we need three steps:



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Towers of Hanoi

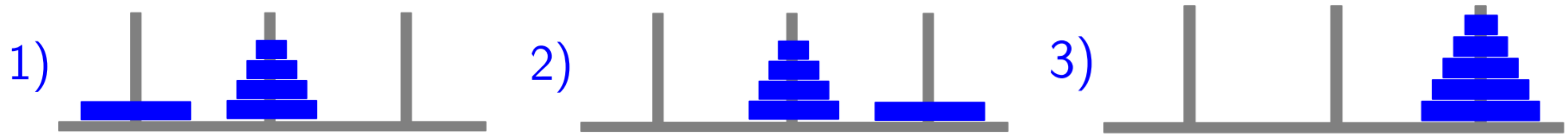
- **Problem:** Find an efficient way to move all of the disks from one peg to another, using only legal moves.
- **Solution** by recursion:

move n disks from peg a to peg c using peg b

```
public class Hanoi
{

    public void move(int n, char a, char b, char c)
    {
        if (n == 1)
            System.out.println("plate " + n + " from " + a + " to " + c);
        else
        {
            move(n-1,a,c,b);
            System.out.println("plate " + n + " from " + a + " to " + c);
            move(n-1,b,a,c);
        }

    }

```

Towers of Hanoi

- **Problem:** Find an efficient way to move all of the disks from one peg to another, using only legal moves.
- **Proof of correctness** by induction:
  - Let *P(n)* be the proposition that the solution is correct for *n*.
  - **Basis step:** *P(1)* is obviously true, i.e., the solution is correct when there is only one disk.
  - **Inductive step:** From the inductive hypothesis, i.e., *P(k)* is true for an arbitrary positive integer *k*, we need to show that *P(k + 1)* is true. That is, if our solution works for *k* disks, then we can build a correct solution for *k + 1* disks, which is true by the recursive step:



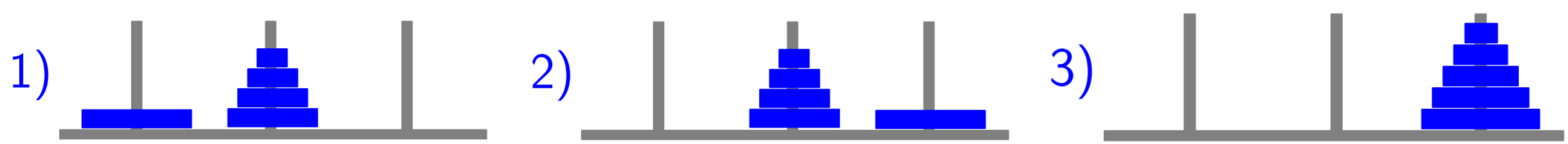
  - By mathematical induction, *P(n)* is true for all positive integer *n*.

Towers of Hanoi

- **Problem:** Find an efficient way to move all of the disks from one peg to another, using only legal moves.
- **Solution** by recursion: ***running time:*** *# disk moves $M(n)$ = ?*
  - **Basis step:** If $n = 1$, moving one disk from one to another is easy.

$M(1) = 1$

  - **Recursive step:** If $n > 1$, we need three steps:



$$M(n) = 2M(n-1) + 1 \text{ for } n > 1$$

Towers of Hanoi

- **Problem:** Find an efficient way to move all of the disks from one peg to another, using only legal moves.

- **Solving the running time:**

  $M(1) = 1$ $\qquad\qquad$ $M(n) = 2M(n - 1) + 1$ *for* $n > 1$

  - Iterating the above function gives:

    $M(1) = 1, M(2) = 3, M(3) = 7, M(4) = 15, M(5) = 31, \ldots$

  - We can guess that $M(n) = 2^n - 1$ and prove it by induction:

    Let $P(n)$ denote the above equation.

    Basis step: $P(1)$ is true, because $M(1) = 1 = 2^1 - 1$.

    Inductive step: Assume $P(k)$ is true for $k \geq 1$, i.e., $M(k) = 2^k - 1$.

    Then $P(k + 1)$ is true: $M(k + 1) = 2M(k) + 1 = 2(2^k - 1) + 1 = 2^{k+1} - 1$

    By mathematical induction, $P(n)$ is true for all positive n.

Towers of Hanoi

- **Problem:** Find an efficient way to move all of the disks from one peg to another, using only legal moves.
- Note that we applied induction twice:
  - first to prove **correctness** of the solution
  - second to derive the **closed-form running time** $M(n) = 2^n - 1$

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Recurrences

Recurrences

- A **recurrence equation** or **recurrence** for a function defined on the set of integers ≥ *b* tells us how to compute the *n*-th value from some or all the first *n - 1* values.

- To completely specify a function defined by a recurrence, we have to give the initial condition(s) (as known as the base case(s)) for the recurrence.

- Example: running time for Towers of Hanoi

$$M(n) = \begin{cases} 1 & \text{if } n = 1 \\ 2M(n-1) + 1 & \text{otherwise} \end{cases}$$

Example

- Let $S(n)$ be the number of subsets of a set of size n. We already learned that $S(n) = 2^n$, but now let us think about this recursively:
  - Consider the 8 subsets of $\{1, 2, 3\}$:

$\varnothing$	$\{1\}$	$\{2\}$	$\{1, 2\}$
$\{3\}$	$\{1, 3\}$	$\{2, 3\}$	$\{1, 2, 3\}$

  - The top 4 subsets are exactly the subsets of $\{1, 2\}$, while the bottom 4 subsets are the subsets of $\{1, 2\}$ with 3 added into each.
  - So, we get a subset of $\{1, 2, 3\}$ either by taking a subset of $\{1, 2\}$ or by adding 3 to a subset of $\{1, 2\}$.
  - This suggests that the recurrence for the number of subsets of an n-element set $\{1, 2, \ldots, n\}$ is

$$S(n) = \begin{cases} 1 & \text{if } n = 0 \\ 2S(n-1) & \text{if } n \geq 1 \end{cases}$$

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Example

- Let $S(n)$ be the number of subsets of a set of size n. We already learned that $S(n) = 2^n$, but now let us think about this recursively:

- Proof of correctness of the recurrence:

$$S(n) = \begin{cases} 1 & \text{if } n = 0 \\ 2S(n-1) & \text{if } n \geq 1 \end{cases}$$

  - The subsets of $\{1, 2, \ldots, n\}$ can be partitioned according to whether or not they contain the element n.

    Each subset S containing n can be constructed in a unique fashion by adding n to the subset $S - \{n\}$ that does not contain n.

    Each subset S not containing n can be constructed by removing n from the unique set $S \cup \{n\}$ that contains n.

    So, the number of subsets containing n is exactly the same as the number of subsets not containing n.

  - Therefore, if $n \geq 1$, then $S(n) = 2S(n - 1)$.

Example

- Let $S(n)$ be the number of subsets of a set of size n. We already learned that $S(n) = 2^n$, but now let us think about this recursively:

- Proof of the closed form $S(n) = 2^n$ for recurrence:

$$S(n) = \begin{cases} 1 & \text{if } n = 0 \\ 2S(n-1) & \text{if } n \geq 1 \end{cases}$$

  - *the proof (by induction) is left as an exercise*

Iterating a Recurrence

- Let $T(n) = rT(n-1) + a$, where r and a are constants.
- Find a recurrence that expresses

  $T(n)$ in terms of $T(n-2)$

  $T(n)$ in terms of $T(n-3)$

  $T(n)$ in terms of $T(n-4)$

  …
- Can we generalize this to find a closed-form solution?

Iterating a Recurrence

- Note that $T(n) = rT(n-1) + a$ implies that
  - $\forall i < n$, $T(n-i) = rT((n-i)-1)) + a$
- Then, we have

$$\begin{aligned} T(n) &= rT(n-1)+a \\ &= r(rT(n-2)+a)+a \\ &= r^2T(n-2)+ra+a \\ &= r^2(rT(n-3)+a)+ra+a \\ &= r^3T(n-3)+r^2a+ra+a \\ &= r^3(rT(n-4)+a)+r^2a+ra+a \\ &= r^4T(n-4)+r^3a+r^2a+ra+a. \end{aligned}$$

- Guess: $T(n) = r^nT(0) + a\sum_{i=0}^{n-1} r^i$

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Iterating a Recurrence

- The method we used to guess the solution is called iterating the recurrence, because we iteratively use the recurrence.

- Another approach is to iterate from the "bottom-up" instead of "top-down".
  - E.g., $T(n) = rT(n-1) + a$, where r and a are constants.

$$
\begin{aligned}
&T(0) = b \\
&T(1) = rT(0) + a = rb + a \\
&T(2) = rT(1) + a = r(rb + a) + a = r^2 b + ra + a \\
&T(3) = rT(2) + a = r^3 b + r^2 a + ra + a
\end{aligned}
$$

  - This would lead to the same guess: $T(n) = r^n T(0) + a \sum_{i=0}^{n-1} r^i$

Formula of Recurrences

- **Theorem:** If $T(n) = rT(n-1) + a$, $T(0) = b$, and $r \neq 1$, then for all non-negative integers n, we have:

$$T(n) = r^n b + a\frac{1-r^n}{1-r}$$

- Proof by induction:
  - **Basis step:** The formula is true for $n = 0$. *Why?*

$$T(0) = r^0 b + a\frac{1-r^0}{1-r} = b.$$

  - **Inductive step:** ?

Formula of Recurrences

- **Theorem:** If $T(n) = rT(n-1) + a$, $T(0) = b$, and $r \neq 1$, then for all non-negative integers n, we have:

$$T(n) = r^n b + a\frac{1-r^n}{1-r}$$

- Proof by induction:
  - **Basis step:** The formula is true for $n = 0$: $T(0) = r^0 b + a\frac{1-r^0}{1-r} = b.$
  - **Inductive step:**

$$\begin{aligned} T(n) &= rT(n-1) + a \\ &= r\left(r^{n-1}b + a\frac{1-r^{n-1}}{1-r}\right) + a \\ &= r^n b + \frac{ar - ar^n}{1-r} + a \\ &= r^n b + \frac{ar - ar^n + a - ar}{1-r} \\ &= r^n b + a\frac{1-r^n}{1-r}. \end{aligned}$$

Formula of Recurrences

- **Theorem:** If $T(n) = rT(n - 1) + a$, $T(0) = b$, and $r \neq 1$, then for all non-negative integers n, we have:

$$T(n) = r^n b + a\frac{1 - r^n}{1 - r}$$

- Example: $T(n) = 3T(n - 1) + 2$, $T(0) = 5$
  - Plugging $r = 3$, $a = 2$, $b = 5$ in the formula, we have:

$$T(n) = 3^n \cdot 5 + 2\frac{1 - 3^n}{1 - 3} = 3^n \cdot 6 - 1$$

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First-Order Linear Recurrences

- A recurrence of the form $T(n) = f(n)T(n-1) + g(n)$ is called a first-order linear recurrence.
  - First order: $T(n)$ depends upon going back one step, i.e., $T(n-1)$

    e.g., $T(n) = T(n-1) + 2T(n-2)$ is a second-order recurrence
  - Linear: the $T(n-1)$ only appears to the first power.

    e.g., $T(n) = (T(n-1))^2 + 3$ is a non-linear first-order recurrence

- When $f(n)$ is a constant, say r, the general solution is almost as easy as we derived before. Iterating the recurrence gives

$$
\begin{aligned}
T(n) &= rT(n-1) + g(n) \\
&= r(rT(n-2) + g(n-1)) + g(n) \\
&= r^2 T(n-2) + rg(n-1) + g(n) \\
&\vdots \\
&= r^n T(0) + \sum_{i=0}^{n-1} r^i g(n-i)
\end{aligned}
$$

First-Order Linear Recurrences

- **Theorem:** For any positive constants *a* and *r*, and any function *g* defined on nonnegative integers, the solution to the first-order linear recurrence

$$T(n) = \begin{cases} rT(n-1) + g(n) & \text{if } n > 0 \\ a & \text{if } n = 0 \end{cases}$$

is

$$T(n) = r^n a + \sum_{i=1}^{n} r^{n-i} g(i).$$

Exercise (5 mins)

- Solve $T(n) = 4T(n-1) + 2^n$ $(n > 0)$ with $T(0) = 6$
  Hint: express T(n) in terms of 4^n and 2^n

- **Theorem:** For any positive constants *a* and *r*, and any function *g* defined on nonnegative integers, the solution to the first-order linear recurrence

$$T(n) = \begin{cases} rT(n-1) + g(n) & \text{if } n > 0 \\ a & \text{if } n = 0 \end{cases}$$

is

$$T(n) = r^n a + \sum_{i=1}^{n} r^{n-i} g(i).$$

Divide-and-Conquer Recurrences

Divide and Conquer

- A divide-and-conquer algorithm recursively breaks down a problem into two or more sub-problems of the same or related type, until these become simple enough to be solved directly. The solutions to the sub-problems are then combined to give a solution to the original problem.

- Divide-and-conquer recurrences are usually of the form:

$$T(n) = \begin{cases} \text{something given} & \text{if } n \leq n_0 \\ r \cdot T(n/m) + g(n) & \text{if } n > n_0 \end{cases}$$

Binary Search

- Someone has chosen a number x between 1 and n. We need to discover x.
- We only need to ask two types of questions:
  - Is x greater than k?
  - Is x equal to k?
- Our strategy will be to always ask greater than questions, at each step halving our search range, until the range only contains one number, when we ask a final equal to question.

Binary Search

- Example: *n = 64, x = 35*

x > 32? — *Yes*

x > 48? — *No*

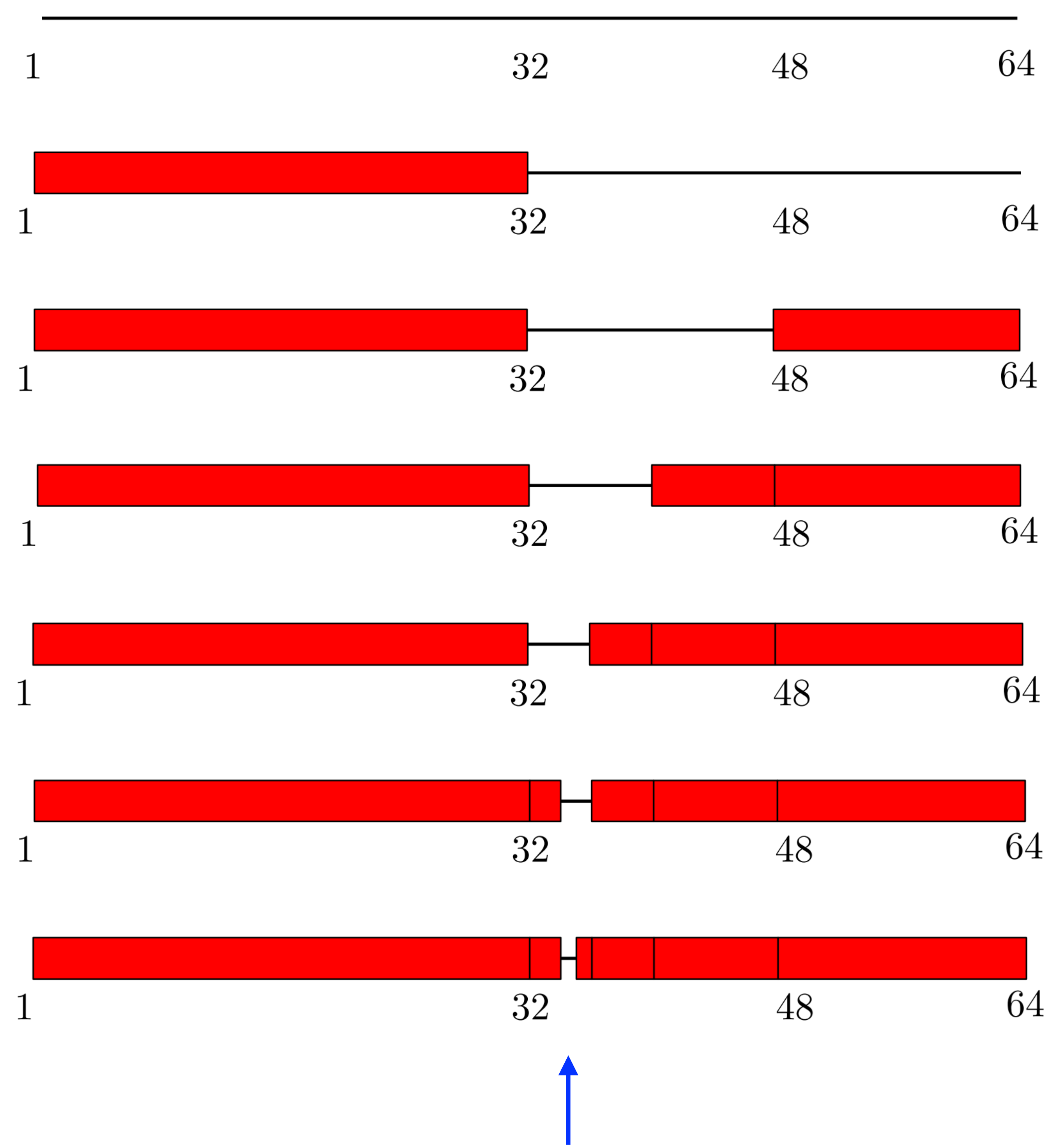
x > 40? — *No*

x > 36? — *No*

x > 34? — *Yes*

x > 35? — *No*

x = 35? — *Yes*



SUSTech

Binary Search

- Method: Each guess reduces the problem to one in which the range is only half as big.

- This divides the original problem into one that is only half as big; we can now (recursively) conquer this smaller problem.

- When n is a power of 2, $T(n)$, the number of comparisons in a binary search on $[1, n]$, satisfies

$$T(n) = \begin{cases} 1 & \text{if } n = 1 \\ T(n/2) + 1 & \text{if } n \geq 2 \end{cases}$$

- The inductive correctness proof is similar to the tower of Hanoi:
  - **Basis Step** ($n = 1$): only one "equal to" comparison is needed
  - **Inductive Step** ($n > 1$): one "great than" comparison + time to perform binary search on the remaining $n/2$ terms

SUSTech

Iterating Recurrences

- Example 1:

$$T(n) = \begin{cases} 1 & \text{if } n = 1 \\ T(n/2) + 1 & \text{if } n \geq 2 \end{cases}$$

- Solve it by algebraically iterating the recurrence:

$$T(n) = T\left(\frac{n}{2}\right) + 1 \qquad = \left(T\left(\frac{n}{2^2}\right) + 1\right) + 1$$

$$= T\left(\frac{n}{2^2}\right) + 2 \qquad = \left(T\left(\frac{n}{2^3}\right) + 1\right) + 2$$

$$= T\left(\frac{n}{2^3}\right) + 3$$

$$\vdots \qquad \vdots$$

$$= T\left(\frac{n}{2^i}\right) + i$$

End when $i = log_2 n$

$$\vdots \qquad \vdots$$

$$= T\left(\frac{n}{2^{\log_2 n}}\right) + \log_2 n \; = 1 + \log_2 n$$

SUSTech

Iterating Recurrences

- Example 1:

$$T(n) = \begin{cases} 1 & \text{if } n = 1 \\ T(n/2) + 1 & \text{if } n \geq 2 \end{cases}$$

- We just showed by algebraically iterating the recurrence that the solution is $T(n) = 1 + \log_2 n$.

- Note: Technically, we still need to use induction to prove that our solution is correct. Practically, we never explicitly perform this step, since it is obvious how the induction would work.

Iterating Recurrences

- Example 2:

$$T(n) = \begin{cases} T(1) & \text{if } n = 1 \\ 2T(n/2) + n & \text{if } n \geq 2 \end{cases}$$

- This corresponds to solving a problem of size *n*, by

  (i) solving *2* subproblems of size *n/2*

  (ii) doing *n* units of additional work

  or using *T(1)* work for "bottom" case of *n = 1*

- This is exactly how merge sort (from an algorithm course) works.

- How to solve it by algebraically iterating the recurrence?

Iterating Recurrences

- Example 2:

$$T(n) = \begin{cases} T(1) & \text{if } n = 1 \\ 2T(n/2) + n & \text{if } n \geq 2 \end{cases}$$

- Solve it by algebraically iterating the recurrence:

$$\begin{aligned}
T(n) = 2T\left(\tfrac{n}{2}\right) + n \quad & = 2\left(2T\left(\tfrac{n}{4}\right) + \tfrac{n}{2}\right) + n \\
= 4T\left(\tfrac{n}{4}\right) + 2n \quad & = 4\left(2T\left(\tfrac{n}{8}\right) + \tfrac{n}{4}\right) + 2n \\
= 8T\left(\tfrac{n}{8}\right) + 3n & \\
\vdots \qquad \vdots & \\
= 2^i T\left(\tfrac{n}{2^i}\right) + in & \\
\vdots \qquad \vdots & \\
= 2^{\log_2 n} T\left(\tfrac{n}{2^{\log_2 n}}\right) + (\log_2 n)n & = nT(1) + n\log_2 n
\end{aligned}$$

SUSTech

Iterating Recurrences

- Example 3:

$$T(n) = \begin{cases} 1 & \text{if } n = 1 \\ T(n/2) + n & \text{if } n \geq 2 \end{cases}$$

- Solve it by algebraically iterating the recurrence:

$$\begin{aligned}
T(n) &= T\left(\frac{n}{2}\right) + n \\
&= T\left(\frac{n}{2^2}\right) + \frac{n}{2} + n \\
&= T\left(\frac{n}{2^3}\right) + \frac{n}{2^2} + \frac{n}{2} + n \\
&\ \ \vdots \qquad \vdots \\
&= T\left(\frac{n}{2^i}\right) + \frac{n}{2^{i-1}} + \cdots + \frac{n}{2^2} + \frac{n}{2} + n \\
&\ \ \vdots \qquad \vdots \\
&= T\left(\frac{n}{2^{\log_2 n}}\right) + \frac{n}{2^{\log_2 n-1}} + \cdots + \frac{n}{2^2} + \frac{n}{2} + n \\
&= 1 + 2 + 2^2 + \cdots + \frac{n}{2^2} + \frac{n}{2} + n \ = 2n - 1
\end{aligned}$$

Exercise (5 mins)

- Solve this recurrence by algebraically iterating the recurrence:

$$T(n) = \begin{cases} 1 & \text{if } n < 3 \\ 3T(n/3) + n & \text{if } n \geq 3 \end{cases}$$

- Example 3:

$$T(n) = \begin{cases} 1 & \text{if } n = 1 \\ T(n/2) + n & \text{if } n \geq 2 \end{cases}$$

- Solve it by algebraically iterating the recurrence:

$$\begin{aligned} T(n) &= T\left(\frac{n}{2}\right) + n \\ &= T\left(\frac{n}{2^2}\right) + \frac{n}{2} + n \\ &= T\left(\frac{n}{2^3}\right) + \frac{n}{2^2} + \frac{n}{2} + n \\ &\vdots \quad\quad \vdots \\ &= T\left(\frac{n}{2^i}\right) + \frac{n}{2^{i-1}} + \cdots + \frac{n}{2^2} + \frac{n}{2} + n \\ &\vdots \quad\quad \vdots \\ &= T\left(\frac{n}{2^{\log_2 n}}\right) + \frac{n}{2^{\log_2 n - 1}} + \cdots + \frac{n}{2^2} + \frac{n}{2} + n \\ &= 1 + 2 + 2^2 + \cdots + \frac{n}{2^2} + \frac{n}{2} + n \end{aligned}$$

Iterating Recurrences

- Example 4:

$$T(n) = \begin{cases} 1 & \text{if } n = 1 \\ 4T(n/2) + n & \text{if } n \geq 2 \end{cases}$$

- Solve it by algebraically iterating the recurrence:

$$\begin{aligned}
T(n) = 4T\left(\tfrac{n}{2}\right) + n \qquad & = 4\left(4T\left(\tfrac{n}{2^2}\right) + \tfrac{n}{2}\right) + n \\
= 4^2 T\left(\tfrac{n}{2^2}\right) + \tfrac{4}{2}n + n \quad & = 4^2\left(4T\left(\tfrac{n}{2^3}\right) + \tfrac{n}{2^2}\right) + \tfrac{4}{2}n + n \\
= 4^3 T\left(\tfrac{n}{2^3}\right) + \tfrac{4^2}{2^2}n + \tfrac{4}{2}n + n & \\
\vdots \qquad \vdots & \\
= 4^i T\left(\tfrac{n}{2^i}\right) + \tfrac{4^{i-1}}{2^{i-1}}n + \cdots + \tfrac{4^2}{2^2}n + n & \\
\vdots \qquad \vdots & \\
= 4^{\log_2 n} T\left(\tfrac{n}{2^{\log_2 n}}\right) + \tfrac{4^{\log_2 n - 1}}{2^{\log_2 n - 1}}n + \cdots + \tfrac{4}{2}n + n = 2n^2 - n &
\end{aligned}$$

Three Different Behaviors

- Compare the iteration for the following recurrences ($T(1) = 1$):
  - $T(n) = T(n/2) + n$ $\quad$ $T(n) = 2n - 1 = \Theta(n)$
  - $T(n) = 2T(n/2) + n$ $\quad$ $T(n) = n + n \log_2 n = \Theta(n \log n)$
  - $T(n) = 4T(n/2) + n$ $\quad$ $T(n) = 2n^2 - n = \Theta(n^2)$
  - All three recurrences iterate $\log_2 n$ times. The size of subproblem in next iteration is half the size in the preceding iteration level.

- **Theorem:** Suppose that we have a recurrence $T(n) = aT(n/2) + n$, where $a \geq 1$ and $T(1) = \Theta(1)$. Then we have the following big Θ bounds on the solution:
  - If $a < 2$, then $T(n) = \Theta(n)$. ** the proof is left as an exercise*
  - If $a = 2$, then $T(n) = \Theta(n \log n)$. ** already proved in Example 2*
  - If $a > 2$, then $T(n) = \Theta(n^{\log_2 a})$. ** now let us prove this*

Three Different Behaviors

- Let $n = 2^i$. Prove that if $a > 2$, then $T(n) = \Theta(n^{\log_2 a})$.
- Iterating $T(n) = aT(n/2) + n$ as in Example *4* gives:

$$T(n) = a^i\, T\left(\frac{n}{2^i}\right) + \left(\frac{a^{i-1}}{2^{i-1}} + \frac{a^{i-2}}{2^{i-2}} + \cdots \frac{a}{2} + 1\right) n$$

$$T(n) = \underbrace{a^{\log_2 n}\, T(1)}_{\text{Work at ``bottom''}} + \underbrace{n \sum_{i=0}^{\log_2 n - 1} \left(\frac{a}{2}\right)^i}_{\text{Iterated Work}}$$

- Since $a > 2$, the geometric series is Θ of the largest term:

$$n \sum_{i=0}^{\log_2 n - 1} \left(\frac{a}{2}\right)^i = n \frac{1-(a/2)^{\log_2 n}}{1-a/2} = n\Theta((a/2)^{\log_2 n - 1})$$

Three Different Behaviors

- Let $n = 2^i$. Prove the third case: If $a > 2$, then $T(n) = \Theta(n^{\log_2 a})$.
- Iterating $T(n) = aT(n/2) + n$ as in Example *4* gives:

$$T(n) = a^{\log_2 n} T(1) + n \sum_{i=0}^{\log_2 n - 1} \left(\frac{a}{2}\right)^i$$

- Since $a > 2$, the geometric series is Θ of the largest term:

$$n \sum_{i=0}^{\log_2 n - 1} \left(\frac{a}{2}\right)^i = n \frac{1 - (a/2)^{\log_2 n}}{1 - a/2} = n\Theta\left((a/2)^{\log_2 n - 1}\right)$$

$$n \left(\frac{a}{2}\right)^{\log_2 n - 1} = \frac{2}{a} \cdot \frac{n \cdot a^{\log_2 n}}{2^{\log_2 n}} = \frac{2}{a} \cdot \frac{n \cdot a^{\log_2 n}}{n} = \frac{2}{a} \cdot a^{\log_2 n}$$

$$a^{\log_2 n} = \left(2^{\log_2 a}\right)^{\log_2 n} = \left(2^{\log_2 n}\right)^{\log_2 a} = n^{\log_2 a}$$

$$a^{\log_2 n} T(1) + n \sum_{i=0}^{\log_2 n - 1} \left(\frac{a}{2}\right)^i = \Theta\left(n^{\log_2 a}\right)$$

Three Different Behaviors

- **Theorem:** Suppose that we have a recurrence $T(n) = aT(n/2) + n$, where $a \geq 1$ and $T(1) = \Theta(1)$. Then we have the following big Θ bounds on the solution:
  - If $a < 2$, then $T(n) = \Theta(n)$. ** proof is left as an exercise*
  - If $a = 2$, then $T(n) = \Theta(n \log n)$. ** already proved in Example 2*
  - If $a > 2$, then $T(n) = \Theta(n^{\log_2 a})$. ** just proved*
- **Master Theorem:** Consider a recurrence $T(n) = aT(n/b) + cn^d$, where $a \geq 1$, $c > 0$, $d \geq 0$, integer $b \geq 2$, and $T(1) = \Theta(1)$. Then we have the following big Θ bounds on the solution:
  - If $a < b^d$, then $T(n) = \Theta(n^d)$.
  - If $a = b^d$, then $T(n) = \Theta(n^d \log n)$.
  - If $a > b^d$, then $T(n) = \Theta(n^{\log_b a})$.

07 Counting

To be continued…