The Proximal Gradient Method

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1. The Composite Model

The Composite Model

$$\min_{x \in \mathbb{R}^n} \{F(x) \equiv f(x) + g(x)\} \tag{1}$$

Standing Assumption (SA):

(A). $g : \mathbb{R}^n \to \overline{\mathbb{R}}$ is proper closed and convex.

(B). $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is proper and closed, $\mathrm{dom}(f)$ is convex, $\mathrm{dom}(g) \subseteq \mathrm{int}(\mathrm{dom}(f))$, and f is L_f-smooth over $\mathrm{int}(\mathrm{dom}(f))$.

(C). The optimal set of problem (1) is nonempty and denoted by X^*. The optimal value of the problem is denoted by F_{opt}.

Three special cases of the general model (1)

- **Smooth unconstrained minimization.** If $g \equiv 0$ and dom$(f) = \mathbb{R}^n$, then (1) reduces to

$$\min_{x \in \mathbb{R}^n} f(x),$$

  where $f : \mathbb{R}^n \to \mathbb{R}$ is L_f-smooth over $\mathbb{R}^n$.

- **Convex constrained smooth minimization.** If $g = \delta_C$, where $C \subset \mathbb{R}^n$ is a nonempty closed and convex set, then (1) amounts to

$$\min_{x \in \mathbb{R}^n} f(x), \quad \text{s.t. } x \in C \qquad \text{or } \min_{x \in C} f(x),$$

  where f is L_f-smooth over int(dom(f)) and $C \subset$ int(dom(f)).

- **l_1-regularized minimization.** If $g(x) = \lambda \|x\|_1$ for some $\lambda > 0$ and dom$(f) = \mathbb{R}^n$, then (1) amounts to

$$\min_{x \in \mathbb{R}^n} \left\{ f(x) + \lambda \|x\|_1 \right\},$$

  where $f : \mathbb{R}^n \to \mathbb{R}$ is L_f-smooth over $\mathbb{R}^n$.

Motivation:

- Solve the Smooth unconstrained model by the gradient method:

$$
\begin{aligned}
x^{k+1} =& x^k - t_k \nabla f\left(x^k\right) \\
=& \underset{x \in \mathbb{R}^n}{\arg\min} \left\{ f\left(x^k\right) + \langle \nabla f\left(x^k\right), x - x^k \rangle + 0 + \frac{1}{2t_k} \left\| x - x^k \right\|^2 \right\}
\end{aligned}
$$

- Solve the Convex constrained smooth model by the projected gradient method:

$$
\begin{aligned}
x^{k+1} =& P_C\left(x^k - t_k \nabla f\left(x^k\right)\right) \\
=& \underset{x \in C}{\arg\min} \left\{ f\left(x^k\right) + \langle \nabla f\left(x^k\right), x - x^k \rangle + \frac{1}{2t_k} \left\| x - x^k \right\|^2 \right\} \\
=& \underset{x \in \mathbb{R}^n}{\arg\min} \left\{ f\left(x^k\right) + \langle \nabla f\left(x^k\right), x - x^k \rangle + \delta_C(x) + \frac{1}{2t_k} \left\| x - x^k \right\|^2 \right\}
\end{aligned}
$$

It's natural to generalize the above idea to the more general model (1):

$$
\begin{aligned}
x^{k+1} &= \underset{x \in \mathbb{R}^n}{\arg\min} \left\{ f\left(x^k\right) + \langle \nabla f\left(x^k\right), x - x^k \rangle + g(x) + \frac{1}{2t_k} \left\| x - x^k \right\|^2 \right\} \\
&= \underset{x \in \mathbb{R}^n}{\arg\min} \left\{ t_k g(x) + \frac{1}{2} \left\| x - \left(x^k - t_k \nabla f\left(x^k\right) \right) \right\|^2 \right\} \\
&= \mathsf{prox}_{t_k g}\left(x^k - t_k \nabla f\left(x^k\right) \right).
\end{aligned}
$$

From now on, we will take the stepsize as $t_k = \frac{1}{L_k}$, leading to the following description.

The Proximal Gradient Method

- **Initialization:** pick $x^0 \in \mathrm{int}(\mathrm{dom} f)$.
- **General Step:** for any $k = 0, 1, 2, \cdots$ execute the following steps:
  (a). pick $L_k > 0$;
  (b). set $x^{k+1} = \mathrm{prox}_{\frac{1}{L_k} g}\left(x^k - \frac{1}{L_k}\nabla f(x^k)\right)$.

Definitions:

Prox-grad Operator: Suppose that f and g satisfy (A) and (B) of SA and let $L > 0$. Then $T_L^{f,g}$: int(dom f) $\to \mathbb{R}^n$ is the prox-grad operator associated with f, g, L defined by

$$T_L^{f,g}(x) = \text{prox}_{\frac{1}{L}g}\left(x - \frac{1}{L}\nabla f(x)\right) \text{ for any } x \in \text{int}(\text{dom} f).$$

Gradient Mapping: Suppose that f and g satisfy (A) and (B) of SA and let $L > 0$. Then $G_L^{f,g}$: int(dom f) $\to \mathbb{R}^n$ is the gradient mapping associated with f, g, L defined by

$$G_L^{f,g}(x) = L\left(x - T_L^{f,g}(x)\right) \text{ for any } x \in \text{int}(\text{dom} f).$$

3. Analysis of the PGM—The Nonconvex Case
3.1 Sufficient Decrease

Notations

We set $T_L \equiv T_L^{f,g}$ and $G_L \equiv G_L^{f,g}$ when there's no ambiguity.

Lemma: (sufficient decrease lemma).

Suppose that f and g satisfy properties (A) and (B) of SA. Let $F = f + g$. Then for any $x \in \operatorname{int}(\operatorname{dom} f)$ and $L \in (\frac{L_f}{2}, \infty)$ the following inequality holds:

$$F(x) - F\left(T_L(x)\right) \geq \frac{L - \frac{L_f}{2}}{L^2} \left\|G_L(x)\right\|^2 .$$

Especially,

$$F(x) - F\left(T_{L_f}(x)\right) \geq \frac{1}{2L_f} \left\|G_{L_f}(x)\right\|^2 .$$

The gradient mapping G_L “measures” the optimality.

Theorem

Let f and g satisfy properties (A) and (B) of SA and let $L > 0$. Then

(a). $G_L^{f,g_0}(x) = \nabla f(x)$ *for any* $x \in$ *int*$(\mathrm{dom} f)$, *where* $g_0(x) \equiv 0$;

(b). *for* $x^* \in$ *int*$(\mathrm{dom} f)$, *it holds that* $G_L^{f,g}(x^*) = 0$ *if and only if* x^* *is a stationary point of problem (1).*

Corollary (necessary and sufficient optimality condition under convexity).

Let f and g satisfy properties (A) and (B) of SA and let $L > 0$. Suppose that in addition f is convex. Then for $x^* \in \text{dom}(g)$, it holds that $G_L^{f,g}(x^*) = 0$ if and only if x^* is an optimal solution of problem (1).

The next result establishes monotonicity properties of $\|G_L(x)\|$ w.r.t. the parameter L.

Theorem (monotonicity of the gradient mapping)

Suppose that f and g satisfy properties (A) and (B) of SA. Suppose that $L_1 \geq L_2 > 0$. Then for any $x \in$ int(dom$f$),

$$\|G_{L_1}(x)\| \geq \|G_{L_2}(x)\|, \quad \frac{\|G_{L_1}(x)\|}{L_1} \leq \frac{\|G_{L_2}(x)\|}{L_2}.$$

Lemma: Lipschitz continuity of the gradient mapping

Suppose that f and g satisfy properties (A) and (B) of SA. Then

(a). $\|G_L(x) - G_L(y)\| \leq (2L + L_f)\|x - y\|$ for any $x, y \in \text{int}(\text{dom} f)$.

(b). $\left\|G_{L_f}(x) - G_{L_f}(y)\right\| \leq 3L_f\|x - y\|$ for any $x, y \in \text{int}(\text{dom} f)$.

Lemma: firm nonexpansivity of $\frac{3}{4L_f}G_{L_f}$

Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is convex and L_f-smooth $(L_f > 0)$ over $\mathbb{R}^n$, and let $g : \mathbb{R}^n \to \overline{\mathbb{R}}$ be proper closed and convex. Then

(a). the gradient mapping G_{L_f} satisfies the relation

$$\langle G_{L_f}(x) - G_{L_f}(y), x - y\rangle \geq \frac{3}{4L_f}\left\|G_{L_f}(x) - G_{L_f}(y)\right\|^2$$

for any $x, y \in \mathbb{R}^n$.

(b). $\left\|G_{L_f}(x) - G_{L_f}(y)\right\| \leq \frac{4L_f}{3}\|x - y\|$ for any $x, y \in \mathbb{R}^n$.

Lemma: monotonicity of the norm of the gradient mapping with respect to the pro-grad operator

Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is convex and L_f-smooth $(L_f > 0)$ over $\mathbb{R}^n$, and let $g : \mathbb{R}^n \to \overline{\mathbb{R}}$ be proper closed and convex. Then

$$\left\| G_{L_f}\left(T_{L_f}(x)\right) \right\| \leq \left\| G_{L_f}(x) \right\|.$$

Stepsize Strategies

- **Constant.** $L_k = \bar{L} \in \left(\frac{L_f}{2}, \infty\right)$ for all k.
- **Backtracking procedure B1.**
  The procedure requires three parameters (s, γ, η), where $s > 0$, $\gamma \in (0, 1)$ and $\eta > 1$.
  The choice of L_k is done as follows.
  First, L_k is set to be equal to the initial guess s.
  Then, while

$$F(x^k) - F\left(T_{L_k}(x^k)\right) < \frac{\gamma}{L_k} \left\| G_{L_k}(x^k) \right\|^2,$$

  we set $L_k := \eta L_k$.

- Backtracking procedure B1 (to be continued).
  In other words, L_k is chosen as $L_k = s\eta^{i_k}$, where i_k is the smallest nonnegative integer for which the condition

$$F(x^k) - F\left(T_{s\eta^{i_k}}(x^k)\right) \geq \frac{\gamma}{s\eta^{i_k}} \left\|G_{s\eta^{i_k}}\right\|^2.$$

  is satisfied.

Remark

Under SA,

1. the backtracking procedure is finite when $L_k \geq \frac{L_f}{2(1-\gamma)}$.
2. Compute the finite upper bound on L_k:

$$L_k \leq \max\left\{s, \frac{\eta L_f}{2(1-\gamma)}\right\}.$$

Lemma (sufficient decrease of the PGM).

Suppose that SA holds. Let $\{x^k\}_{k\geq 0}$ be the sequence generated by the proximal gradient method for solving problem (1) with either a constant stepsize defined by $L_k = \bar{L} \in (\frac{L_f}{2}, \infty)$ or with a stepsize chosen by the backtracking procedure B1 with parameter (s, γ, η), where $s > 0$, $\gamma \in (0,1)$, $\eta > 1$. Then for any $k \geq 0$,

$$F(x^k) - F(x^{k+1}) \geq M \left\| G_d(x^k) \right\|^2,$$

where

$$M = \begin{cases} \frac{\bar{L} - \frac{L_f}{2}}{\left(\bar{L}\right)^2}, & \text{constant stepsize,} \\ \frac{\gamma}{\max\left\{s, \frac{\eta L_f}{2(1-\gamma)}\right\}}, & \text{backtracking,} \end{cases}$$

and

$$d = \begin{cases} \bar{L}, & \text{constant stepsize,} \\ s, & \text{backtracking.} \end{cases}$$

Theorem (convergence of the PGM-nonconvex case.)

Suppose that SA holds and let $\{x^k\}_{k\geq 0}$ be the sequence generated by the proximal gradient method for solving problem (1) either with a constant stepsize defined by $L_k = \bar{L} \in (\frac{L_f}{2}, \infty)$ or with a stepsize chosen by the backtracking procedure B1 with parameters (s, γ, η), where $s > 0$, $\gamma \in (0, 1)$, $\eta > 1$. Then

(a). the sequence $\{F(x^k)\}_{k\geq 0}$ is nonincreasing. In addition, $F(x^{k+1}) < F(x^k)$ if and only if x^k is not a stationary of problem (1);

(b). $G_d(x^k) \to 0$ as $k \to \infty$, where

$$d = \begin{cases} \bar{L}, & \text{constant stepsize,} \\ s, & \text{backtracking.} \end{cases}$$

(c).

$$\min_{n=0,\cdots,k} \left\| G_d(x^k) \right\| \leq \frac{\sqrt{F(x^0) - F_{\text{opt}}}}{\sqrt{M(k+1)}},$$

where

$$M = \begin{cases} \dfrac{\bar{L} - \frac{L_f}{2}}{\left(\bar{L}\right)^2}, & \text{constant stepsize,} \\ \dfrac{\gamma}{\max\left\{s, \frac{\eta L_f}{2(1-\gamma)}\right\}}, & \text{backtracking,} \end{cases}$$

(d). all limit points of the sequence $\{x^k\}_{k \geq 0}$ are stationary points of problem (1).

Theorem (fundamental prox-grad inequality).

Suppose that f and g satisfy properties (A) and (B) of SA. For any $x \in \mathbb{R}^n$, $y \in \operatorname{int}(\operatorname{dom} f)$ and $L > 0$ satisfying

$$f\left(T_L(y)\right) \leq f(y) + \langle \nabla f(y), T_L(y) - y \rangle + \frac{L}{2}\left\|T_L(y) - y\right\|^2,$$

it holds that

$$F(x) - F\left(T_L(y)\right) \geq \frac{L}{2}\left\|x - T_L(y)\right\|^2 - \frac{L}{2}\left\|x - y\right\|^2 + l_f(x, y),$$

where

$$l_f(x, y) = f(x) - f(y) - \langle \nabla f(y), x - y \rangle.$$

Remark

Suppose that f and g satisfy properties (A) and (B) of SA. For any $x \in \mathbb{R}^n$, $y \in \mathrm{int}(\mathrm{dom} f)$, the inequality

$$F(x) - F\left(T_{L_f}(y)\right) \geq \frac{L_f}{2}\left\|x - T_{L_f}(y)\right\|^2 - \frac{L_f}{2}\|x - y\|^2 + l_f(x, y)$$

holds.

Corollary (sufficient decrease lemma-second version).

Suppose that f and g satisfy properties (A) and (B) of SA. For any $x \in \mathrm{int}(\mathrm{dom} f)$ for which

$$f\left(T_L(x)\right) \leq f(x) + \langle \nabla f(x), T_L(x) - x \rangle + \frac{L}{2} \left\| T_L(x) - x \right\|^2,$$

it holds that

$$F(x) - F\left(T_L(x)\right) \geq \frac{1}{2L} \left\| G_L(x) \right\|^2.$$

Stepsize Strategies

- **Constant.** $L_k = L_f$ for all k.
- **Backtracking procedure B2.**
  The procedure requires two parameters (s, η), where $s > 0$, and $\eta > 1$.
  Define $L_{-1} = s$.
  At iteration $k (k \geq 0)$ the choice of L_k is done as follows.
  First, L_k is set to be equal to L_{k-1}.
  Then, while

$$f\left(T_{L_k}(x^k)\right) > f(x^k) + \langle \nabla f(x^k), T_{L_k}(x^k) - x^k \rangle + \frac{L_k}{2} \left\| T_{L_k}(x^k) - x^k \right\|^2,$$

  we set $L_k := \eta L_k$.

- **Backtracking procedure B2 (to be continued).**
  In other words, L_k is chosen as $L_k = L_{k-1}\eta^{i_k}$, where i_k is the smallest nonnegative integer for which the condition

$$f\left(T_{L_{k-1}\eta^{i_k}}(x^k)\right) \leq f(x^k) + \langle \nabla f(x^k), T_{L_{k-1}\eta^{i_k}}(x^k) - x^k \rangle + \frac{L_k}{2}\left\|T_{L_{k-1}\eta^{i_k}}(x^k) - x^k\right\|^2$$

  is satisfied.

Remark (upper and lower bounds on L_k)

Under SA, the constants L_k that the backtracking procedure B2 produces satisfy the following bounds for all $k \geq 0$:

$$s \leq L_k \leq \max\{\eta L_f, s\},$$

which can be rewritten as $\beta L_f \leq L_k \leq \alpha L_f$, where

$$\alpha = \begin{cases} 1, & \text{constant,} \\ \max\left\{\eta, \frac{s}{L_f}\right\}, & \text{backtracking,} \end{cases} \qquad \beta = \begin{cases} 1, & \text{constant,} \\ \frac{s}{L_f}, & \text{backtracking.} \end{cases}$$

Remark (monotonicity of the proximal gradient method)

Under SA and either of two stepsize rules, for any $k \geq 0$, we obtain the inequality

$$F(x^k) - F\left(x^{k+1}\right) \geq \frac{L_k}{2}\left\|x^k - x^{k+1}\right\|^2.$$

Theorem ($O(\frac{1}{k})$ rate of convergence of proximal gradient).

Suppose that SA holds and that in addition f is convex. Let $\{x^k\}_{k\geq 0}$ be the sequence generated by the proximal gradient method for solving problem (1) with either a constant stepsize rule in which $L_k \equiv L_f$ for all $k \geq 0$ or the backtracking procedure B2. Then for any $x^* \in X^*$ and $k \geq 0$,

$$F(x^k) - F_{\text{opt}} \leq \frac{\alpha L_f \left\|x^0 - x^*\right\|^2}{2k},$$

where $\alpha = 1$ in the constant stepsize setting and $\alpha = \max\left\{\eta, \frac{s}{L_f}\right\}$ if the backtracking rule is employed.

Definition (Fejér monotonicity)

A sequence $\{x^k\}_{k\geq 0} \subseteq \mathbb{R}^n$ is called Fejér monotone with respect to a set $S \subseteq \mathbb{R}^n$ if

$$\left\|x^{k+1} - y\right\| \leq \left\|x^k - y\right\| \text{ for all } k \geq 0 \text{ and } y \in S.$$

Theorem (convergence under Fejér monotonicity)

Let $\{x^k\}_{k\geq 0} \subseteq \mathbb{R}^n$ be a sequence, and let S be a set satisfying $D \subseteq S$, where D is the set comprising all the limit points of $\{x^k\}_{k\geq 0}$. If $\{x^k\}_{k\geq 0}$ is Fejér monotone with respect to S, then it converges to a point in D.

Theorem (Fejér monotonicity of the sequence generated by the proximal gradient method).

Suppose that SA holds and that in addition f is convex. Let $\{x^k\}_{k\geq 0}$ be the sequence generated by the proximal gradient method for solving problem (1) with either a constant stepsize rule in which $L_k \equiv L_f$ for all $k \geq 0$ or the backtracking procedure B2. Then for any $x^* \in X^*$ and $k \geq 0$,

$$\left\| x^{k+1} - x^* \right\| \leq \left\| x^k - x^* \right\|.$$

Theorem (convergence of the sequence generated by the proximal gradient method).

Suppose that SA holds and that in addition f is convex. Let $\{x^k\}_{k\geq 0}$ be the sequence generated by the proximal gradient method for solving problem (1) with either a constant stepsize rule in which $L_k \equiv L_f$ for all $k \geq 0$ or the backtracking procedure B2. Then the sequence $\{x^k\}_{k\geq 0}$ converges to an optimal solution of problem (1).

Theorem ($O(\frac{1}{k})$ rate of convergence of the minimal norm of the gradient mapping).

Suppose that SA holds and that in addition f is convex. Let $\{x^k\}_{k\geq 0}$ be the sequence generated by the proximal gradient method for solving problem (1) with either a constant stepsize rule in which $L_k \equiv L_f$ for all $k \geq 0$ or the backtracking procedure B2. Then for any $x^* \in X^*$ and $k \geq 1$,

$$\min_{n=0,1,\cdots,k} \left\| G_{\alpha L_f}(x^n) \right\| \leq \frac{2\alpha^{1.5} L_f \left\| x^0 - x^* \right\|}{\sqrt{\beta} k},$$

where $\alpha = \beta = 1$ in the constant stepsize setting and $\alpha = \max\left\{\eta, \frac{s}{L_f}\right\}, \beta = \frac{s}{L_f}$ if the backtracking rule is employed.

Theorem ($O(\frac{1}{k})$ rate of convergence of the norm of the gradient mapping under the constant stepsize rule).

Suppose that SA holds and that in addition f is convex and L_f-smooth over $\mathbb{R}^n$. Let $\{x^k\}_{k\geq 0}$ be the sequence generated by the proximal gradient method for solving problem (1) with a constant stepsize rule in which $L_k \equiv L_f$ for all $k \geq 0$. Then for any $x^* \in X^*$ and $k \geq 0$,

(a).

$$\left\|G_{L_f}(x^{k+1})\right\| \leq \left\|G_{L_f}(x^k)\right\|;$$

(b).

$$\left\|G_{L_f}(x^k)\right\| \leq \frac{2L_f\left\|x^0 - x^*\right\|}{k+1}.$$

5. Analysis of the PGM—The Strongly Convex Case

Theorem (linear rate of convergence of the proximal gradient method—strongly convex case).

Suppose that SA holds and that in addition f is σ-convex. Let $\{x^k\}_{k\geq 0}$ be the sequence generated by the proximal gradient method for solving problem (1) with either a constant stepsize rule in which $L_k \equiv L_f$ for all $k \geq 0$ or the backtracking procedure B2, and let x^* be the unique minimum of problem (1). Then for any $k \geq 0$,

(a). $\left\|x^{k+1} - x^*\right\|^2 \leq \left(1 - \frac{\sigma}{\alpha L_f}\right) \left\|x^k - x^*\right\|^2$;

(b). $\left\|x^{k+1} - x^*\right\|^2 \leq \left(1 - \frac{\sigma}{\alpha L_f}\right)^k \left\|x^0 - x^*\right\|^2$;

(c). $F(x^{k+1}) - F_{\text{opt}} \leq \frac{\alpha L_f}{2} \left(1 - \frac{\sigma}{\alpha L_f}\right)^{k+1} \left\|x^0 - x^*\right\|^2$,

where $\alpha = 1$ in the constant stepsize setting and $\alpha = \max\left\{\eta, \frac{s}{L_f}\right\}$ if the backtracking rule is employed.

no strong:just Fejer monotone

The Composite Model

$$\min_{x \in \mathbb{R}^n} \{F(x) \equiv f(x) + g(x)\} \tag{2}$$

Assumption 2:

(A). $g : \mathbb{R}^n \to \overline{\mathbb{R}}$ is proper closed and convex.

(B). $f : \mathbb{R}^n \to \mathbb{R}$ is L_f-smooth and convex.

(C). The optimal set of problem (1) is nonempty and denoted by X^*. The optimal value of the problem is denoted by F_{opt}.

Fast proximal gradient method
Fast iterative shrinkage-thresholding algorithm (FISTA)

FISTA

- **Input:** (f, g, x^0), where f and g satisfy properties (A) and (B) in Assumption 2 and $x^0 \in \mathbb{R}^n$.
- **Initialization:** set $y^0 = x^0$ and $t_0 = 1$.
- **General Step:** for any $k = 0, 1, 2, \cdots$ execute the following steps:
  (a). pick $L_k > 0$;
  (b). set $x^{k+1} = \mathrm{prox}_{\frac{1}{L_k} g}\left(y^k - \frac{1}{L_k}\nabla f(y^k)\right)$;
  (c). set $t_{k+1} = \frac{1+\sqrt{1+4t_k^2}}{2}$;
  (d). compute $y^{k+1} = x^{k+1} + \frac{t_k - 1}{t_{k+1}}\left(x^{k+1} - x^k\right)$.

Stepsize Strategies

- **Constant.** $L_k = L_f$ for all k.
- **Backtracking procedure B3.**
  The procedure requires two parameters (s, η), where $s > 0$ and $\eta > 1$. Define $L_{-1} = s$. At iteration $k (k \geq 0)$ the choice of L_k is done as follows:
  First, L_k is set to be equal to L_{k-1}.
  Then, while
  $$f\left(T_{L_k}(y^k)\right) > f(y^k) + \langle \nabla f(y^k), T_{L_k}(y^k) - y^k \rangle + \frac{L_k}{2} \left\| T_{L_k}(y^k) - y^k \right\|^2,$$
  we set $L_k := \eta L_k$.

- Backtracking procedure B3 (to be continued).
In other words, L_k is chosen as $L_k = L_{k-1}\eta^{i_k}$, where i_k is the smallest nonnegative integer for which the condition

$$f\left(T_{L_{k-1}\eta^{i_k}}(y^k)\right) \leq f(y^k) + \langle \nabla f(y^k), T_{L_{k-1}\eta^{i_k}}(y^k) - y^k \rangle + \frac{L_k}{2} \left\| T_{L_{k-1}\eta^{i_k}}(y^k) - y^k \right\|^2$$

is satisfied.

Remark (upper and lower bounds on L_k)

Under Assumption 2, the constants L_k that the backtracking procedure B3 produces satisfy the following bounds for all $k \geq 0$:

$$s \leq L_k \leq \max\{\eta L_f, s\},$$

which can be rewritten as $\beta L_f \leq L_k \leq \alpha L_f$, where

$$\alpha = \begin{cases} 1, & \text{constant,} \\ \max\left\{\eta, \frac{s}{L_f}\right\}, & \text{backtracking,} \end{cases} \qquad \beta = \begin{cases} 1, & \text{constant,} \\ \frac{s}{L_f}, & \text{backtracking.} \end{cases}$$

Lemma

Let $\{t_k\}_{k\geq 0}$ be the sequence defined by

$$t_0 = 1,\, t_{k+1} = \frac{1+\sqrt{1+4t_k^2}}{2}, \quad k \geq 0.$$

Then $t_k \geq \frac{k+2}{2}$ for all $k \geq 0$.

Theorem ($O(\frac{1}{k^2})$ rate of convergence of FISTA).

Suppose that Assumption 2 holds. Let $\{x^k\}_{k\geq 0}$ be the sequence generated by FISTA for solving problem (1) with either a constant stepsize rule in which $L_k \equiv L_f$ for all $k \geq 0$ or the backtracking procedure B3. Then for any $x^* \in X^*$ and $k \geq 0$,

$$F(x^k) - F_{\text{opt}} \leq \frac{2\alpha L_f \left\| x^0 - x^* \right\|^2}{(k+1)^2},$$

where $\alpha = 1$ in the constant stepsize setting and $\alpha = \max\left\{\eta, \frac{s}{L_f}\right\}$ if the backtracking rule is employed.

Remark (alternative choice for t_k).

A close inspection of the proof of Theorem ($O(\frac{1}{k^2})$ rate of convergence of FISTA) reveals that the result is correct if $\{t_k\}_{k\geq 0}$ is any sequence satisfying the following two properties for any $k \geq 0$:

(a). $t_k \geq \frac{k+2}{2}$;

(b). $t_{k+1}^2 - t_{k+1} \leq t_k^2$.

The choice $t_k = \frac{k+2}{2}$ also satisfies these two properties. The validity of (a) is obvious; to show (b), note that

$$
\begin{aligned}
t_{k+1}^2 - t_{k+1} = t_{k+1}(t_{k+1} - 1) = \frac{k+3}{2} \cdot \frac{k+1}{2} &= \frac{k^2+4k+3}{4} \\
\leq \frac{k^2+4k+4}{4} = \frac{(k+2)^2}{4} &= t_k^2.
\end{aligned}
$$

Remark

Note that FISTA has an $O(\frac{1}{k^2})$ rate of convergence in function values, while the proximal gradient method has an $O(\frac{1}{k})$ rate of convergence. This improvement was achieved despite the fact that the dominant computational steps at each iteration of both methods are essentially the same: one gradient evaluation and one prox computation.

Example (l_1-regularized minimization)

Consider the following model:

$$\min_{x\in\mathbb{R}^n} f(x) + \lambda \|x\|_1 ,$$

where $\lambda > 0$ and $f : \mathbb{R}^n \to \mathbb{R}$ is assumed to be convex and L_f-smooth.

The proximal gradient method (or iterative shrinkage-thresholding algorithm (ISTA)) with constant stepsize $\frac{1}{L_f}$:

$$x^{k+1} = \mathcal{T}_{\frac{\lambda}{L_f}}\left(x^k - \frac{1}{L_f}\nabla f(x^k)\right).$$

Recall that $\mathcal{T}_\alpha : \mathbb{R}^n \to \mathbb{R}^n_+$ is the soft thresholding operator associated with $\alpha > 0$ defined by

$$\mathcal{T}_\alpha(x) \equiv ([|x_i| - \alpha]_+ \mathrm{sgn}(x))_{i=1}^n .$$

The fast proximal gradient method (or fast iterative shrinkage-thresholding algorithm (FISTA)) with constant stepsize $\frac{1}{L_f}$:

(a). set $x^{k+1} = \mathcal{T}_{\frac{\lambda}{L_f}}\left(y^k - \frac{1}{L_f}\nabla f(y^k)\right)$;

(b). set $t_{k+1} = \frac{1+\sqrt{1+4t_k^2}}{2}$;

(c). compute $y^{k+1} = x^{k+1} + \frac{t_k - 1}{t_{k+1}}\left(x^{k+1} - x^k\right)$.

Example (l_1-regularized least squares).

Consider

$$\min_{x \in \mathbb{R}^n} \frac{1}{2} \|Ax - b\|_2^2 + \lambda \|x\|_1 ,$$

where $\lambda > 0$ and $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$. Note that the function $\frac{1}{2} \|Ax - b\|_{2,2}$ is L-smooth with

$$L = \left\|A^T A\right\|_2^2 = \lambda_{\max}(A^T A).$$

The update step of ISTA:

$$x^{k+1} = \mathcal{T}_{\frac{\lambda}{L_k}} \left(x^k - \frac{1}{L_k} A^T (Ax^k - b) \right)$$

The update step of FISTA:

(a). set $x^{k+1} = \mathcal{T}_{\frac{\lambda}{L_k}} \left(y^k - \frac{1}{L_k} A^T (Ay^k - b) \right)$;

(b). set $t_{k+1} = \frac{1+\sqrt{1+4t_k^2}}{2}$;

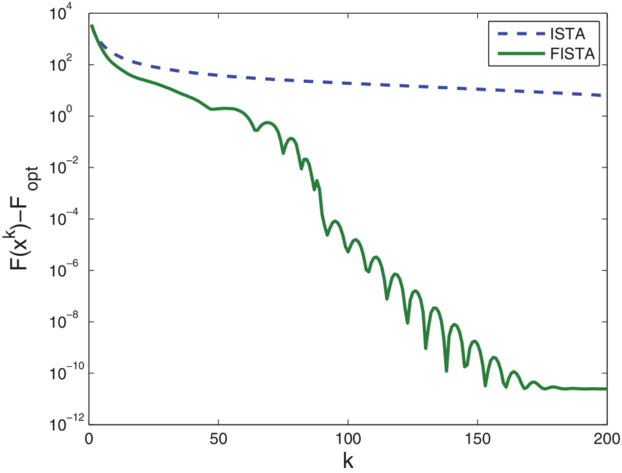
(c). compute $y^{k+1} = x^{k+1} + \frac{t_k - 1}{t_{k+1}} \left(x^{k+1} - x^k \right)$.

Instance performance:

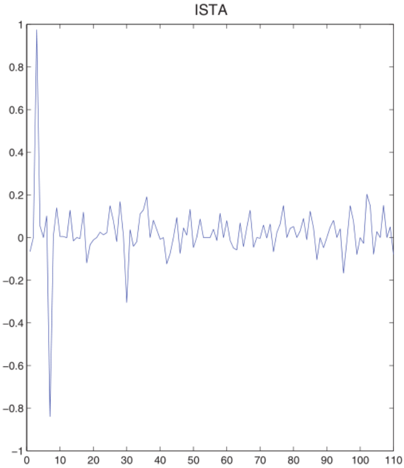
Let

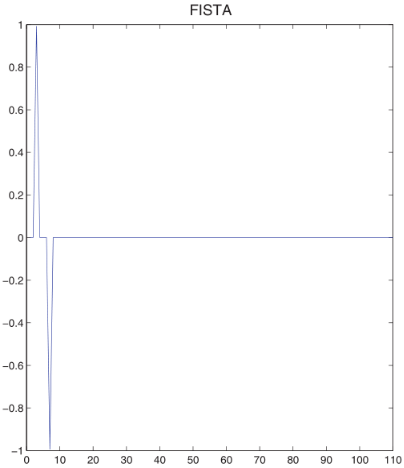
- $\lambda = 1$,
- $A \in \mathbb{R}^{100\times 110}$ where the components of A were independently generated using a standard normal distribution.
- The "true" vector is $x_{\text{true}} = e_3 - e_7$.
- $b = Ax_{\text{true}}$.

Let the initial vector $x = e$. The distances to optimality in terms of function values of the sequences generated by the two methods as a function of the iteration index are plotted:



the vectors that were obtained by 200 iterations of ISTA and FISTA:





- **Input:** (f, g, x^0), where f and g satisfy properties (A) and (B) in Assumption 2 and $x^0 \in \mathbb{R}^n$.
- **Initialization:** set $y^0 = x^0$ and $t_0 = 1$.
- **General Step:** for any $k = 0, 1, 2, \cdots$ execute the following steps:
  (a). pick $L_k > 0$;
  (b). set $z^k = \text{prox}_{\frac{1}{L_k} g}\left(y^k - \frac{1}{L_k}\nabla f(y^k)\right)$;
  (c). choose $x^{k+1} \in \mathbb{R}^n$ such that $F(x^{k+1}) \leq \min\{F(z^k), F(x^k)\}$
  (d). set $t_{k+1} = \frac{1+\sqrt{1+4t_k^2}}{2}$;
  (e). compute $y^{k+1} = x^{k+1} + \frac{t_k}{t_{k+1}}\left(z^k - x^{k+1}\right) + \frac{t_k - 1}{t_{k+1}}\left(x^{k+1} - x^k\right)$.

Theorem ($O(\frac{1}{k^2})$ rate of convergence of MFISTA).

Suppose that Assumption 2 holds. Let $\{x^k\}_{k\geq 0}$ be the sequence generated by MFISTA for solving problem (1) with either a constant stepsize rule in which $L_k \equiv L_f$ for all $k \geq 0$ or the backtracking procedure B3. Then for any $x^* \in X^*$ and $k \geq 0$,

$$F(x^k) - F_{\text{opt}} \leq \frac{2\alpha L_f \left\|x^0 - x^*\right\|^2}{(k+1)^2},$$

where $\alpha = 1$ in the constant stepsize setting and $\alpha = \max\left\{\eta, \frac{s}{L_f}\right\}$ if the backtracking rule is employed.