Algorithms for Convex Optimization
Assignment 1

Note: Given a subset $A \subset \mathbb{R}^n$ and a vector $b \in \mathbb{R}^n$, the sum of A and b is defined as

$$A + b = b + A := A + \{b\}.$$

1. Given $a, b \in \mathbb{R}^n$ with $a \neq b$. Find the values of μ such that

$$\Omega_\mu = \{x \in \mathbb{R}^n : \|x - a\| \leq \mu \|x - b\|\}$$

   is convex.

2. Show the polar cone of

$$\Omega = \{(x, t) \in \mathbb{R}^n \times \mathbb{R} : \|x\|_1 \leq t\}$$

   is

$$\Omega^o = \{(x, t) \in \mathbb{R}^n \times \mathbb{R} : \|x\|_\infty \leq -t\}.$$

3. Let $\Omega \subset \mathbb{R}^n$ with $w^* \in \Omega$. Show that

$$C = \left\{x \in \mathbb{R}^n : x^T w^* \leq x^T w \;\text{ for all } w \in \Omega\right\}$$

   is a convex cone.

4. Let $f_1, \cdots, f_m : \mathbb{R}^n \to \overline{\mathbb{R}}$ be convex functions. Show that

$$\Omega = \{y := (y_1, \cdots, y_m) \in \mathbb{R}^m : \exists x \text{ such that } f_1(x) \leq y_1, \cdots, f_m(x) \leq y_m\}$$

   is convex.

5. Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ be closed convex. Show that for any $\alpha > 0$, the function $f(x) + \alpha \|x\|^2$ is coercive.

6. Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ be convex and $x^* \in \text{int}(\text{dom} f)$. If there exists a mapping $g : \text{int}(\text{dom} f) \to \mathbb{R}^n$ satisfying

   (a) $g(x) \in \partial f(x)$ for any $x \in \text{int}(\text{dom} f)$;

   (b) g is continuous at x^*.

   Then $\partial f(x^*) = \{\nabla f(x^*)\}$.

7. Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ be convex and let $\Omega \subset \text{dom}(\partial f)$ be a closed convex set. If $\cap_{x \in \Omega} \partial f \neq \varnothing$. Then f is affine over Ω.

8. Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ be convex and $x^* \in \text{int}(\text{dom} f)$. Recall that $f'(x^*; \cdot) : \mathbb{R}^n \to \mathbb{R}$ is well-defined and convex over $\mathbb{R}^n$. Show that $\partial f'(x^*; \cdot)(0) = \partial f(x^*)$.

9. Let $\Omega \subset \mathbb{R}^m_+$ and $f_1, \cdots, f_m : \mathbb{R}^n \to \overline{\mathbb{R}}$ are closed convex. Show that

$$f(x) = \sup_{y=(y_1,\cdots,y_m)\in\Omega} \sum_{i=1}^{m} y_i f_i(x)$$

is closed and convex.

10. Let $f_1, f_2 : \mathbb{R}^n \to \overline{\mathbb{R}}$ be closed convex and $\mathrm{dom} f_1 \cap \mathrm{dom}(f_2) \neq \varnothing$ is bounded. Define

$$f_{\max}(x) = \max\left\{f_1(x), f_2(x)\right\};$$
$$\varphi(x, \lambda) = \lambda f_1(x) + (1-\lambda) f_2(x).$$

Show that $\exists \lambda^* \in [0,1]$ such that

$$\min\ f_{\max} = \min\ \varphi(\cdot, \lambda^*)$$

11. The recession cone of Ω, a nonempty closed convex subset of $\mathbb{R}^n$, is defined by

$$R(\Omega) = \left\{d \in \mathbb{R}^n \middle| \forall \alpha \geq 0, \Omega + \alpha d \subset \Omega\right\}.$$

Show

(a) $R(\Omega)$ is a closed convex cone.

(b) $d \in R(\Omega)$ iff there exists $w \in \Omega$ such that $w + \alpha d \subset \Omega$ holds for $\forall \alpha \geq 0$.

(c) For a family of closed convex sets $\{\Omega_i\}_{i\in\mathbb{I}}$ satisfying $\cap_{i\in\mathbb{I}}\Omega_i \neq \varnothing$, show that

$$R\left(\cap_{i\in\mathbb{I}}\Omega_i\right) = \cap_{i\in\mathbb{I}} R\left(\Omega_i\right).$$

(d) If $d_u \in \mathbb{R}^n$ is a unit vector, show that $d_u \in R(\Omega)$ iff there exists unbounded sequence $\left\{w^k\right\}_{k=1}^{\infty} \subset \Omega$ satisfying

$$\frac{w^k}{\|w^k\|} \to d_u \text{ as } k \to \infty.$$

12. Let $f_1, f_2, \cdots, f_m : \mathbb{R}^n \to \mathbb{R}$ be real-valued convex with $\cap_{i=1}^m \mathrm{dom}\left(f_i\right) \neq \varnothing$.
Define the mapping $F : \mathbb{R}^n \to \mathbb{R}^m$ with the values

$$F(x) = \left(f_1(x), f_2(x), \cdots, f_m(x)\right)^T, \qquad \forall x \in \mathbb{R}^n.$$

Let $g : \mathbb{R}^m \to \overline{\mathbb{R}}$ satisfy:

(a) g is convex;

(b) For any $u, v \in \mathbb{R}^m$, if $u \leq v$, then $g(u) \leq g(v)$.

Suppose further that

$$F\left(\bigcap_{i=1}^{m} \mathrm{dom}\left(f_i\right)\right) := \left\{y \in \mathbb{R}^m \middle| \exists x \in \cap_{i=1}^m \mathrm{dom}\left(f_i\right) \text{ such that } y = F(x)\right\}.$$

is convex.
Then show the composite $g \circ F$ is convex.

13. Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ be proper convex with $\bar{x} \in \mathrm{int}\left(\mathrm{dom}(f)\right)$. For any $d \in \mathbb{R}^n$, define $\varphi_d : \mathbb{R} \to \overline{\mathbb{R}}$ with the values:

$$\varphi_d\left(\alpha\right) = f\left(\bar{x} + \alpha d\right).$$

Show that

(a) φ_d is convex for any $d \in \mathbb{R}^n$.

(b) $\bar{x}$ is a minimum of f iff for all $d \in \mathbb{R}^n$, $\bar{\alpha} = 0$ is a minimum of φ_d.

14. Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ be Lipschitz continuous with constant $L_f > 0$, i.e.,

$$|f(x) - f(y)| \leq L_f \|x - y\|, \qquad \forall x, y \in \mathrm{dom}(f).$$

Let $\Omega \subset \mathbb{R}^n$ be nonempty compact and $L > L_f$. Then show

(a) if $\bar{x}$ is a local minimizer of f over Ω, then $\bar{x}$ is a local minimizer of

$$f_L(x) = f(x) + L\mathrm{dist}\,(x; \Omega)$$

over $\mathbb{R}^n$.

(b) If $\bar{x}$ is a global minimizer of f_L over $\mathbb{R}^n$, then $\bar{x}$ is a global minimizer of f over Ω.

15. Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ be proper. Define

$$E_f = \left\{(x, \alpha) \in \mathbb{R}^n \times \mathbb{R} \middle| f(x) < \alpha\right\}.$$

Show that

$$E_f \subset \mathrm{epi}(f) \subset \mathrm{cl}\,(E_f).$$

16. Show

(a) If $\{\Omega_i\}_{i=1}^m \in \mathbb{R}^n$ be the family nonempty cones, then

$$\left(\prod_{i=1}^m \Omega_i\right)^\circ = \prod_{i=1}^m \Omega_i^\circ.$$

(b) If $\{\Omega_i\}_{i\in\mathbb{I}}$ be the family of nonempty cones, then

$$\left(\bigcup_{i\in\mathbb{I}} \Omega_i\right)^\circ = \bigcap_{i\in\mathbb{I}} \Omega_i^\circ.$$

(c) If $\Omega_1, \Omega_2 \subset \mathbb{R}^n$ be nonempty cones, then

$$(\Omega_1 + \Omega_2)^\circ = \Omega_1^\circ \cap \Omega_2^\circ.$$

17. Let $\Omega \subset \mathbb{R}^n$ be a closed convex cone. If $\mathrm{int}\,(\Omega^\circ) \neq \varnothing$, show that there exists $0 \neq p \in \mathbb{R}^n$ and $\delta > 0$ such that

$$\langle p, x\rangle \geq \delta \|x\|, \qquad \forall x \in \Omega.$$

18. Let $f : \mathbb{R} \to \mathbb{R}$ be scalar, convex and real-valued. For any $x \in \mathbb{R}$, show

(a) $\partial f\,(x)$ is a compact interval.

(b) $\partial f(x) = \{g \in \mathbb{R} | -f'(x; -1) \leq g \leq f'(x; 1)\}$.

19. Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ be convex and $x \in \mathrm{dom}(f)$. For any $\epsilon > 0$, define

$$\partial_\varepsilon f(x) := \left\{g \in \mathbb{R}^n \middle| f(y) \geq f(x) + \langle g, y - x\rangle - \varepsilon \quad \forall y \in \mathbb{R}^n\right\}.$$

Then show

$$\partial f(x) = \bigcap_{\varepsilon > 0} \partial_\varepsilon f(x).$$

20. We say $F : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is a set-valued mapping which maps $y \in \mathbb{R}^m$ to a subset of $\mathbb{R}^n$, that is to say, $F(y) \subset \mathbb{R}^n$. For simplicity, we assume that $F(y) \neq \varnothing$ for any $y \in \mathbb{R}^m$.

    We say the outer limit of a set-valued mapping F at $\bar{y}$ is defined as

    $$\operatorname*{Limsup}_{y \to \bar{y}} F(y) := \left\{ x \in \mathbb{R}^n : \exists \left\{ \left(y^k, x^k\right) \right\}_{k \geq 0} \subset \mathbb{R}^m \times \mathbb{R}^n \text{ satisfying } y^k \to \bar{y}, x^k \to x, x^k \in F\left(y^k\right) \right\}.$$

    We say F is outer semi-continuous at $\bar{y}$ if

    $$\operatorname*{Limsup}_{y \to \bar{y}} F(y) \subseteq F(\bar{y}).$$

    Let $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ be real-valued, lower semi-continuous and $F : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ be outer semi-continuous and compact-valued (i.e., $F(y)$ is compact for any $y \in \mathbb{R}^m$). Show that

    $$f^*(y) = \inf \{ f(x, y) : x \in F(y) \}$$

    is lower semi-continuous.