Problem 1. Let X be the set of all symmetric $n \times n$ matrices over $\mathbb{R}$, and $Y \subseteq X$ the subset of all positive semidefinite matrices.

(i) For two matrices $A, B \in X$, define $A \geq B$ if $A - B \in Y$. Prove that $\geq$ is a *partial order*, i.e. (1) $A \geq A$, (2) $A \geq B$ and $B \geq A$ implies $A = B$, (3) $A \geq B$ and $B \geq C$ implies $A \geq C$.

(ii) Prove that Y is *convex*, i.e. for $A, B \in Y$, we have $tA + (1-t)B \in Y$ for all real $t \in [0, 1]$.

(iii) Let $A, B \in Y$. Suppose that $AB = BA$. Prove that $AB \in Y$.

(iv) Let
$$Z := \{A \in X : \operatorname{tr}(AB) \geq 0 \text{ for all } B \in Y\},$$
where tr means trace. It is a known fact that $\operatorname{tr}(AB) = \operatorname{tr}(BA)$ for matrices A, B with compatible sizes. Prove that $Z = Y$.

Problem 2.

(i) Let $X := (-0.01, 2\pi) \times (-0.01, 2\pi)$. Consider $f : X \to \mathbb{R}$ given by $f(x, y) = (\cos x)(\cos y)$. Find all its local maximals (both points and values) and local minimals (both points and values).

(ii) Let X be the set of real 2×2 orthogonal matrices. Consider $f : X \to \mathbb{R}$ given by $A \mapsto \operatorname{tr} A^2$ where tr means trace. Find all its global maximum and minimum (both points and values).

Problem 3. We define an equivalence relation $\sim$ on $M_n(k)$ by
$$A \sim B \quad \text{if } B = S^\top AS \text{ for some invertible } S \in M_n(k) \text{ .}$$

(i) For $k = \mathbb{C}$ and $n = 2$, find a complete set of representatives for $M_n(k)/\sim$.

(ii) For $k = \mathbb{R}$ and $n = 2$, find a complete set of representatives for $M_n(k)/\sim$.

Problem 4. Let M be a finite inner product space over $\mathbb{C}$. For an ordered basis B, let G_B be its Gram matrix. A pair of *dual bases* consists of an ordered basis $B = (b_1, \ldots, b_m)$ of M and another ordered basis $C = (c_1, \ldots, c_m)$ of M such that
$$\langle b_i, c_j \rangle_M = \delta_{ij},$$
where $\delta_{ij} := 1$ if $i = j$ and $\delta_{ij} := 0$ if $i \neq j$.

(i) Let $b_1, \ldots, b_m$ and $c_1, \ldots, c_m$ be dual bases. Give a simple formula for an element $v \in M$ as a linear combination of $b_1, \ldots, b_m$.

(ii) Let $b_1, \ldots, b_m$ be an ordered basis of M. Prove that there exist $c_1, \ldots, c_m \in M$ such that $b_1, \ldots, b_m$ and $c_1, \ldots, c_m$ are dual bases.

(iii) Prove that, if B and C are dual bases, then G_B and G_C are inverses to each other.

(iv) Disprove that, if G_B and G_C are inverses to each other, then B and C are dual bases.

Problem 5. Consider square matrices over $\mathbb{C}$. It is a known fact that every matrix is upper triangularizable. Prove the following statements.

(i) Every matrix is unitarily triangularizable.

(ii) Eigenspaces with distinct eigenvalues of a normal matrix are orthogonal.

(iii) Every normal matrix is unitarily diagonalizable.

(iv) A matrix is unitarily diagonalizable if and only if it is normal.

Problem 6. Let $k := \mathbb{C}$. For a $A \in M_n(k)$, let

$$e^A := \sum_{i=0}^{\infty} \frac{A^i}{i!},$$

where we adopt the convention that $A^0 = I$. It is known facts that e^A always exists (the series converges) and $e^{A+B} = e^A e^B$ for $[A, B] = 0$. A *logarithm* of A is a matrix B such that $e^B = A$.

(i) Find all logarithms of the 2×2 identity matrix.

(ii) Let $\theta \in \mathbb{R}$ such that $\sin\theta \neq 0$. Let

$$A_\theta := \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Find all logarithms of A_θ.

(iii) Prove that, for a unitary matrix A, there exists a skew-Hermitian ($B^\dagger = -B$) logarithm B.

(iv) Prove that a logarithm of A exist if and only if A is invertible.

Problem 7. Let M be a finite k-module over an algebraically closed field k. Let $\mathfrak{g} := \mathrm{End}(M)$. For each $x \in \mathfrak{g}$, let $x = x_s + x_n$ be its (unique) Jordan-Chevalley decomposition in $\mathfrak{g}$. For each $x \in \mathfrak{g}$, let $\mathrm{ad}\ x : \mathfrak{g} \to \mathfrak{g}$ be the (a priori not necessarily linear) map given by $y \mapsto [x, y]$, where $[x, y] := xy - yx$.

(i) Prove that the map $\mathrm{ad}\ x$ is an element in $\mathrm{End}(\mathfrak{g})$, i.e. $\mathrm{ad}\ x$ is a linear transformation.

(ii) Prove that the map $\mathrm{ad}\ x_s$ is semisimple in $\mathrm{End}(\mathfrak{g})$.

(iii) Prove that the map $\mathrm{ad}\ x_n$ is nilpotent in $\mathrm{End}(\mathfrak{g})$.

(iv) Prove that $[\mathrm{ad}\ x_s, \mathrm{ad}\ x_n] = 0$ in $\mathrm{End}(\mathfrak{g})$.

Problem 8. Let $k := \mathbb{C}$, $d \geq 1$, and $c_0, \ldots, c_{d-1} \in k$. Let $p := x^d + c_{d-1}x^{d-1} + \cdots + c_1 x + c_0$ and suppose that $p = (x - \lambda_1)^{d_1} \cdots (x - \lambda_t)^{d_t}$ for distinct $\lambda_1, \ldots, \lambda_t \in k$. The companion matrix of p is

$$C := \begin{bmatrix} 0 & 0 & \ldots & 0 & -c_0 \\ 1 & 0 & \ldots & 0 & -c_1 \\ 0 & 1 & \ldots & 0 & -c_2 \\ \vdots & \vdots & \ddots & 0 & \vdots \\ 0 & 0 & \ldots & 1 & -c_{d-1} \end{bmatrix}_{d \times d}$$

(i) For $d = 2$, find an explicit similarity from C to its Jordan normal form.

(ii) Consider a recurrence $f_0 = 0$, $f_1 = 1$ and $f_{n+2} = af_{n+1} + bf_n$, where $a, b, f_i \in \mathbb{C}$. Give an explicit formula (without using matrices) for f_n in terms of a, b, n.

(iii) Prove that the only annihilating polynomial of C of degree at most $d - 1$ is 0.

(iv) Calculate the Jordan normal form of C.