1. Let $A = \{1, 2, \cdots, 19\}$.
(a). (3 points) Find the number α of different binary operations on A.
(b). (2 points) How many nonisomorphic groups can you define on A?

2. (a). (5 points) Show that the following abelian groups are isomorphic:

$$\mathbb{Z}_{21} \times \mathbb{Z}_{90} \simeq \mathbb{Z}_{15} \times \mathbb{Z}_{126}$$

(b). (5 points) Describe all possible homomorphisms of $\mathbb{Z}_5 \times \mathbb{Z}_4$ to S_5.

3. (5 points) Let $(G, \cdot, e)$ be a group, $H, K, L \leq G$ with $K \subseteq L$. Show that $L \cap (HK) = (L \cap H)K$.

4. Let G be a group of order 42 .
(a). (3 points) Prove that G has a normal subgroup N of order 7 (hence G is not simple).
(b). (3 points) Let $g \in G$ of order 6 . Consider the action of g on the set of left N-cosets: $g \cdot xN := (gx)N$. Suppose there is an orbit of this action of length 6 . Describe the quotient group G/N.
(c). (4 points) Suppose $G = D_{21}$, will the fact in (b). still hold? If yes, prove it. If not, describe G/N under this condition.

5. (5 points) Let P be a Sylow p-subgroup of G. Prove that

$$|G : N_G(P)| \equiv 1 \bmod p$$