Suggested Solutions to Midterm Exam

1. Solutions:
   (a) True.
   Check f is coercive and so f attains its minimal value over $\mathbb{R}^n$.
   (b) False.
   Let $f(x) = \|x\|$ if $x \neq 0$ and $f(0) = 1$.
   (c) True.
   Notice that $\mathrm{epi} g = \cap_{y\in\Omega} \mathrm{epi} f(\cdot, y)$.
   (d) True.
   By the sum rule, just check $x \in \mathrm{int}\,(\mathrm{dom}(f_1)) \cap \mathrm{int}\,(\mathrm{dom}(f_2)) = \mathrm{int}\,(\mathrm{dom}(f_1)) \cap \mathbb{R}^n = \mathrm{int}\,(\mathrm{dom}(f_1))$.
2. Choose $g \in \cap_{x\in\Omega}\partial f$ and fixed $\bar{x} \in \Omega$, we have for any $x \in \Omega$,
$$
\begin{aligned}
f(x) &\geq f(\bar{x}) + g^T(x - \bar{x}),\\
f(\bar{x}) &\geq f(x) + g^T(\bar{x} - x).
\end{aligned}
$$
   Then $f(x) = f(\bar{x}) + g^T(x - \bar{x})$.
3.
$$
\begin{aligned}
f(y) - f(x) - (y-x)^T\nabla f(x) &= \int_0^1 (y-x)^T\left[\nabla f\left(x + t(y-x)\right) - \nabla f(x)\right] dt\\
&\leq \int_0^1 \left|(y-x)^T\left[\nabla f\left(x + t(y-x)\right) - \nabla f(x)\right]\right| dt\\
&\leq \int_0^1 \|y-x\| \cdot \left\|\nabla f\left(x + t(y-x)\right) - \nabla f(x)\right\| dt\\
&\leq \int_0^1 \|y-x\| \cdot L\left\|t(y-x)\right\|^p dt\\
&= L\|y-x\|^{p+1}\int_0^1 t^p dt = \frac{L}{p+1}\|y-x\|^{p+1}.
\end{aligned}
$$
4. (a)
   (b) At least the conjugate correspondence theorem should be pointed out.
   (c) Let
$$
\phi(h) = \left(f_1\Box f_2\right)(x+h) - \left(f_1\Box f_2\right)(x) - h^T\nabla f_2\left(x - u(x)\right).
$$
   We want to show $\phi(h) = o(\|h\|)$.
   Notice that
$$
\left(f_1\Box f_2\right)(x+h) = f_1(u(x+h)) + f_2(x+h-u(x+h)) \leq f_1(u(x)) + f_2(x+h-u(x)).
$$
   Then
$$
\begin{aligned}
\phi(h) =& f_1(u(x+h)) + f_2(x+h-u(x+h)) - f_1(u(x)) - f_2(x-u(x)) - h^T\nabla f_2(x-u(x))\\
\leq& f_1(u(x)) + f_2(x+h-u(x)) - f_1(u(x)) - f_2(x-u(x)) - h^T\nabla f_2(x-u(x))\\
=& f_2(x+h-u(x)) - f_2(x-u(x)) - h^T\nabla f_2(x-u(x))\\
\leq& L\|h\|^2. \quad \text{(descent lemma)}
\end{aligned}
$$

On the other hand, notice that $0 = \phi(0) = \phi\left(\frac{1}{2}(-h) + \frac{1}{2}h\right) \leq \frac{1}{2}\phi(-h) + \frac{1}{2}\phi(h)$, we have $-\phi(h) \leq \phi(-h) \leq L\left\|-h\right\|^2 = L\left\|h\right\|^2$, i.e., $\phi(h) \geq -L\left\|h\right\|^2$. Hence $|\phi(h)|/\left\|h\right\| \leq L\left\|h\right\| \to 0$ as $h \to 0$.

5. (a) Let $g_1, g_2 \in \partial_\epsilon f(\bar{x})$ and $0 \leq \lambda \leq 1$. Then

$$
\begin{aligned}
&\frac{f(x) - f(\bar{x}) - (\lambda g_1 + (1-\lambda)g_2)^T (x - \bar{x})}{\|x - \bar{x}\|} \\
=&\lambda \frac{f(x) - f(\bar{x}) - g_1^T (x - \bar{x})}{\|x - \bar{x}\|} + (1-\lambda) \frac{f(x) - f(\bar{x}) - g_2^T (x - \bar{x})}{\|x - \bar{x}\|}
\end{aligned}
$$

Then

$$
\begin{aligned}
&\liminf_{x \to \bar{x}} \frac{f(x) - f(\bar{x}) - (\lambda g_1 + (1-\lambda)g_2)^T (x - \bar{x})}{\|x - \bar{x}\|} \\
=&\liminf_{x \to \bar{x}} \left[\lambda \frac{f(x) - f(\bar{x}) - g_1^T (x - \bar{x})}{\|x - \bar{x}\|} + (1-\lambda) \frac{f(x) - f(\bar{x}) - g_2^T (x - \bar{x})}{\|x - \bar{x}\|}\right] \\
\geq&\lambda \liminf_{x \to \bar{x}} \frac{f(x) - f(\bar{x}) - g_1^T (x - \bar{x})}{\|x - \bar{x}\|} + (1-\lambda) \liminf_{x \to \bar{x}} \frac{f(x) - f(\bar{x}) - g_2^T (x - \bar{x})}{\|x - \bar{x}\|} \\
=&-\lambda\epsilon - (1-\lambda)\epsilon = -\epsilon.
\end{aligned}
$$

(b) $g \in \partial_\epsilon f(\bar{x}) \Leftrightarrow \forall \eta > 0$, there exists $\delta_\eta > 0$ such that

$$
\frac{f(x) - f(\bar{x}) - g^T (x - \bar{x})}{\|x - \bar{x}\|} \geq -\epsilon - \eta
$$

holds for any $x \in \mathbb{B}(\bar{x}, \delta_\eta)$, or equivalently,

$$
f_{g,\eta}(x) = f(x) - f(\bar{x}) - g^T (x - \bar{x}) + (\epsilon + \eta)\left\|x - \bar{x}\right\| \geq 0 = f_{g,\eta}(\bar{x})
$$

holds for any $x \in \mathbb{B}(\bar{x}, \delta_\eta)$.

(c) i. $LHS \supseteq RHS$. Trivial.

ii. $LHS \subseteq RHS$. For any $\bar{x} \neq x \in \mathbb{R}^n$, let $x_t = \bar{x} + t(x - \bar{x})$. Then

$$
\begin{aligned}
-\epsilon \leq& \liminf_{x_t \to \bar{x}} \frac{f(x_t) - f(\bar{x}) - g^T (x_t - \bar{x})}{\|x_t - \bar{x}\|} \\
=& \liminf_{t \to 0} \frac{f((1-t)\bar{x} + tx) - f(\bar{x}) - tg^T (x - \bar{x})}{t\left\|x - \bar{x}\right\|} \\
\leq& \liminf_{t \to 0} \frac{(1-t)f(\bar{x}) + tf(x) - f(\bar{x}) - tg^T (x - \bar{x})}{t\left\|x - \bar{x}\right\|} \\
=& \frac{f(x) - f(\bar{x}) - g^T (x - \bar{x})}{\|x - \bar{x}\|},
\end{aligned}
$$

i.e.,

$$
g^T (x - \bar{x}) \leq f(x) - f(\bar{x}) + \epsilon\left\|x - \bar{x}\right\|.
$$

6. (a) Slater's condition: $\exists \tilde{x}$ such that $g_i(\tilde{x}) < 0$ for $i = 1, 2, \cdots, m$.
KKT condition: there exists $\lambda_1, \cdots, \lambda_m \geq 0$ for which

$$
\begin{aligned}
&0 \in \partial f(x^*) + \sum_{i=1}^m \lambda_i \partial g_i(x_i), \\
&\lambda_i g_i(x^*) = 0, \quad i = 1, 2, \cdots, m
\end{aligned}
$$

(b) Define $g := \max\{g_1, g_2, \cdots, g_m\}$ and recall that $g(x^*) = 0$ and $g'(x^*;\cdot) = \max\{g_i'(x^*;\cdot) : i \in I(x^*)\}$. Notice that the MFCQ holds at x^* $\Leftrightarrow$ there exists d such that $g'(x^*;d) < 0$, while the Slater's condition holds $\Leftrightarrow$ there exists $\tilde{x}$ such that $g(\tilde{x}) < 0$.

If the MFCQ holds at x^* w.r.t some direction d, notice that

$$g'(x^*;d) = \lim_{\alpha \to 0^+} \frac{g(x^* + \alpha d) - g(x^*)}{\alpha}.$$

Then there exists $\bar{\alpha} > 0$ such that

$$\frac{g(x^* + \bar{\alpha} d) - g(x^*)}{\bar{\alpha}} < -\left|\frac{g'(x^*;d)}{2}\right|,$$

i.e.,

$$g(x^* + \bar{\alpha} d) < g(x^*) - \bar{\alpha}\left|\frac{g'(x^*;d)}{2}\right| < 0,$$

i.e., $x^* + \bar{\alpha} d$ is a strictly feasible point, which means the Slater's condition holds.

If the Slater's condition holds, we pick $g(\tilde{x}) < 0$, then

$$g'(x^*;\tilde{x} - x^*) \le g(\tilde{x}) - g(x^*) \le g(\tilde{x}) < 0,$$

i.e., the MFCQ holds at x^* w.r.t the direction $\tilde{x} - x^*$.