南方科技大学
SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY

Course Name: Complex analysis **Dept.:** Mathematics
Exam Duration: 110 mins

Question No.	1	2	3	4	5	6	7	8	9	10
Score										

This exam paper contains 7 questions and the score is 100 in total.
(Please hand in your exam paper, answer sheet, and your scrap paper to the proctor when the exam ends.)

1. (15 points) Let $f : \mathbb{C} \to \mathbb{C}$ be a complex valued function which is written as

$$f(x, y) = u(x, y) + iv(x, y),$$

where u, v are continuously differentiable.

(a) Let $g(z) = \overline{f(\bar{z})}$, show that g is holomorphic $\boxed{\text{if and only if}}$ $\boxed{f}$ is holomorphic.

(b) Suppose $u(x, y) = xy - x + y$, find all possible v such that f is holomorphic. $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ ∂u

2. (10 points) Let $f(z)$ be a continuous function defined in the unit disk $D_1 = \{z \mid |z| < 1\}$. Assuming $f(z)^5$ and $f(z)^7$ are holomorphic, show that $f(z)$ itself is holomorphic.

3. (15 points) Consider the meromorphic function $f(z) = \frac{1}{e^{z^2}+1}$.

(a) Find all the poles of $f(z)$.

(b) Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series expression of $f(z)$ centered at 0, find the radius of convergence and explain your answer.

4. (15 points) Consider the strip $S = \{z \mid 0 < \mathrm{Im}(z) < 1\}$. Let $f : S \to \mathbb{C}$ be a holomorphic function on S such that it extends continuously to the closure $\bar{S}$ and has real values on the boundary.

(a) Show that there is an entire function F whose restriction to S is f. (Hints: use Schwarz reflection principle)

(b) Assuming $f : \bar{S} \to \mathbb{C}$ is bounded, show that f is a constant function. bounded + entire = cor

5. (15 points) For each of the following functions $f(z)$, determine the type of singularity, and compute the residue at the point z if it is a pole.

(a)

$$f(z) = \frac{e^z - e^{-z}}{z^3(e^z + e^{-z})}, \quad \text{at } z = -\frac{\pi i}{2}.$$

$-i-i = \frac{-2i}{z^3}$

(b)

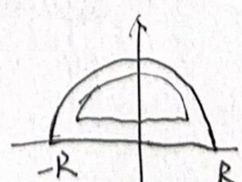
$$f(z) = \sin(\frac{1}{z}), \quad \text{at } z = 0.$$

6. (10 points) Calculate the following integral:

$\cos\theta = \sqrt{2}$?

$x+iy$

$$\int_0^{2\pi} \frac{1}{\sqrt{2} - \cos(\theta)} d\theta.$$



$e^{i\theta}$

7. (20 points) Consider a function $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ defined by

$$f(z) = \frac{\sin(z)}{z}.$$

(a) Show that f has a removable singularity at 0 and find the value that f can be extended continuously to 0.

(b) Show that $f(z) \neq 0$ for $|z| \leq 3$.

(c) Show that for any holomorphic function g defined in a neighborhood of the closed unit disk $\{z \mid |z| \leq 1\}$, we have the integral formula

$$g(z) = \frac{1}{2\pi i} \int_{C_1} \frac{g(w)}{\sin(w - z)} dw$$

for $|z| < 1$. Here C_1 is the unit circle oriented in counterclockwise direction.