Algorithms for Convex Optimization
Assignment 3 Part II

11. Let $f : \mathbb{R}^n \to \mathbb{R}$ be convex and L-smooth, $g : \mathbb{R}^n \to \mathbb{R}$ be closed convex. Let $X^* = \arg\min \{F(x) := f(x) + g(x) : x \in \mathbb{R}^n\}$ and $F_{\text{opt}} = \min \{F(x) : x \in \mathbb{R}^n\}$. Suppose we are familiar with the notations $T_L(x)$ and $G_L(x)$. We say

    (a) the α-error bound property holds if for any $x \in \mathbb{R}^n$

    $$\|G_L(x)\| \geq \alpha \text{dist}(x, X^*).$$

    (b) the β-strong convexity holds if for any $x \in \mathbb{R}^n$

    $$\left\langle G_L(x), x - \text{proj}(x; X^*) \right\rangle \geq \beta \text{dist}^2(x, X^*).$$

    Show that

    (1). For any $x, y \in \mathbb{R}^n$,

    $$F(y) \geq F\left(T_L(x)\right) + \left\langle G_L(x), y - x \right\rangle + \frac{1}{2L} \left\| G_L(x) \right\|^2.$$

    (2). $(a) \to (b)$;

    (3). $(b) \to (a)$;

12. Let both $h, f : \mathbb{R}^n \to \mathbb{R}$ be convex and L-smooth. The sequence $\left\{x^k\right\}_{k \geq 0}$ is denoted by:

    $$x^{k+1} = x^k - \frac{1}{L}\left[\mu_k \nabla h(x_k) + (1 - \mu_k) \nabla f(x^k)\right].$$

    where $\mu^k = \frac{1}{k^{1-\alpha}}$ with $0 < \alpha < 1$. Let x^* minimizes h and f over $\mathbb{R}^n$. Show that

    (a)

    $$\min_{i=0,1,\cdots,k-1} \left\{f(x^k)\right\} - f(x^*) \leq O\left(\frac{1}{k}\right),$$

    (b)

    $$\min_{i=0,1,\cdots,k-1} \left\{F(x^k)\right\} - F(x^*) \leq O\left(\frac{1}{k^\alpha}\right).$$

    Hint: fundamental prox-grad inequality.