Homework 1.

1, 2, 3, 5, 6, 7, 8, 9, 10, 11*, 12, 13, 16(1)&(4), 17*, 18*

24, 25, 28, 29, 32, 33.

1. 假设 $b\neq 0$, $c>0$，死亡力函数满足的条件是：

$$\begin{cases} \mu(x) \geq 0, & \forall x>0, \\ \int_0^{\infty} \mu(x)\,dx = \infty, \end{cases}$$

故应有：① $b>0$, $c>1$ 且 $a\geq -b$

or ② $0<c<1$, $a\geq -b$, $a>0$

or ③ $c=1$, $a>-b$

生存函数为 $s(x) = e^{-\int_0^x \mu(t)\,dt} = e^{-\int_0^x (a+bc^t)\,dt}$

$$= \begin{cases} e^{-ax-\frac{b}{\ln c}(c^x-1)}, & c\neq 1 \\ e^{-(a+b)x}, & c=1 \end{cases} \quad (x\geq 0).$$

密度函数为 $f(x) = s(x)\cdot\mu(x)$.

2. $s(x) = e^{-2x^4}$ $(x\geq 0)$ 满足：$\begin{cases} s(0)=1 \\ s(x) \text{ 单调下降，右连续} \\ s(x)\to 0,\ x\to\infty \text{时}. \end{cases}$

故可作为生存函数，且对 $x\geq 0$,

$$\mu(x) = \frac{f(x)}{s(x)} = -\frac{s'(x)}{s(x)} = 8x^3$$

$f(x) = s(x)\cdot\mu(x) = 8x^3e^{-2x^4}$, $F(x) = 1-s(x) = 1-e^{-2x^4}$

3. $s(x) = e^{-\int_0^x \mu(t)dt} = \frac{1}{(x+1)^5}$

$\Rightarrow \mathring{e}_0 = \int_0^\infty s(x)dx = \int_0^\infty \frac{1}{(x+1)^5}dx = \frac{1}{4}$

$e_0 = \sum_{n\geq 1} s(n) = \sum_{n\geq 1} \frac{1}{(n+1)^5}$

& $E(K(0)^2) = \sum_{n\geq 1}(2n-1)s(n) = \sum_{n\geq 1}\frac{2n-1}{(n+1)^5}$

$\Rightarrow Var(K(0)) = E(K(0)^2) - [E(K(0))]^2 = \sum_{n\geq 1}\frac{2n-1}{(n+1)^5} - e_0^2$

5. $\mathring{e}_x = \int_0^\infty t \cdot \lambda e^{-\lambda t}dt = \frac{1}{\lambda}$, & $P(T(x)\leq t) = \int_0^t f(s)ds = 1-e^{-\lambda t}$

中位数：令 $1-e^{-\lambda t} = \frac{1}{2} \Rightarrow t = \frac{1}{\lambda}\ln 2$，中位数 $\frac{1}{\lambda}\ln 2$

众数：因为 $f(t)$ 单调下降趋于 0，所以众数为 0.

6. $\mu(x) = \frac{5}{x+1}, x>0 \Rightarrow {}_xP_{10} = e^{-\int_0^x \mu(10+t)dt} = \left(\frac{11}{11+x}\right)^5$

$\Rightarrow \mathring{e}_{10} = \int_0^\infty {}_xP_{10}dx = \frac{11}{4}$. & $e_{10} = \sum_{x\geq 1} {}_xP_{10} = \sum_{x\geq 1}\left(\frac{11}{11+x}\right)^5$

& $E(K^2(10)) = \sum_{x\geq 1}(2x-1){}_xP_{10} = \sum_{x\geq 1}\left(\frac{11}{11+x}\right)^5(2x-1)$

$\Rightarrow Var(K(10)) = E[K^2(10)] - (E[K(10)])^2$

$= \sum_{x\geq 1}\left(\frac{11}{11+x}\right)^5(2x-1) - \left(\sum_{x\geq 1}\left(\frac{11}{11+x}\right)^5\right)^2$

7. ${}_{2|2}q_{20} = {}_2P_{20}\,{}_2q_{20} = e^{-0.02}(1-e^{-0.02}) \approx 0.0194$

${}_5P_{20} = e^{-\int_0^5 \mu(20+t)dt} = e^{-0.05} \approx 0.9512$.

8. 因为对 $t>0$，$_tp_x = e^{-\int_0^t \mu(x+s)ds} = e^{-\frac{1}{3}t^3}$

所以 $f_{T(x)}(t) = -\frac{d\,{}_tp_x}{dt} = t^2 e^{-\frac{1}{3}t^3}$

9. 证明: $\frac{d\mathring{e}_x}{dx} = \frac{d}{dx}\left(\int_0^\infty e^{-\int_x^{x+t}\mu(s)ds}\,dt\right) = \int_0^\infty\left(\frac{d}{dx}e^{-\int_x^{x+t}\mu(s)ds}\right)dt$

$= \int_0^\infty e^{-\int_x^{x+t}\mu(s)ds}\left(\mu(x)-\mu(x+t)\right)dt$

$= \mu(x)\int_0^\infty e^{-\int_x^{x+t}\mu(s)ds}\,dt - \int_0^\infty {}_tp_x\,\mu(x+t)\,dt$

$= \mu(x)\mathring{e}_x - 1$.

10. ${}_{17}p_{19} = \frac{s(36)}{s(19)} = \frac{8}{9}$

${}_{15}q_{36} = 1 - {}_{15}p_{36} = 1 - \frac{s(51)}{s(36)} = \frac{1}{8}$

$\mu(36) = \frac{-s'(36)}{s(36)} = \frac{1}{128}$.

11*. 令 $k_1+k_2+k_3+k_4 = l_0$

因为 $P({}_3\mathcal{D}_0 = k_1,\ {}_3\mathcal{D}_3 = k_2,\ {}_3\mathcal{D}_6 = k_3,\ {}_3\mathcal{D}_9 = k_4)$

$= C_{l_0}^{k_1}\,{}_3q_0^{k_1}\cdot C_{l_0-k_1}^{k_2}\,{}_{3|3}q_0^{k_2}\cdot C_{l_0-k_1-k_2}^{k_3}\,{}_{6|3}q_0^{k_3}\cdot C_{l_0-k_1-k_2-k_3}^{k_4}\,{}_{9|3}q_0^{k_4}$

$= \frac{l_0!}{k_1!\,k_2!\,k_3!\,k_4!}\,{}_3q_0^{k_1}\,{}_{3|3}q_0^{k_2}\,{}_{6|3}q_0^{k_3}\,{}_{9|3}q_0^{k_4}$

所以 $({}_3\mathcal{D}_0, {}_3\mathcal{D}_3, {}_3\mathcal{D}_6, {}_3\mathcal{D}_9)$ 服从多项分布.

2

(1) 令 T_i 表示第 i 个个体的寿命，则

$$E({}_3D_0) = E\left[\sum_{i=1}^{l_0} I_{\{0\le T_i<3\}}\right] = l_0 \cdot {}_3q_0 = \frac{1}{4} l_0 .$$

$$E({}_3D_3) = E\left[\sum_{i=1}^{l_0} I_{\{3\le T_i<6\}}\right] = l_0 \cdot {}_{3|3}q_0 = \frac{1}{4} l_0 .$$

同理 $E({}_3D_6) = E({}_3D_9) = \frac{1}{4} l_0$

(2). $$Var({}_3D_0) = l_0 \cdot {}_3q_0 \cdot {}_3p_0 = \frac{3}{16} l_0$$

$$Var({}_3D_3) = l_0 \cdot {}_{3|3}q_0 \cdot (1 - {}_{3|3}q_0) = \frac{3}{16} l_0$$

同理 ~~Var~~ $var({}_3D_6) = var({}_3D_9) = \frac{3}{16} l_0 .$

(3) 因为 $cov({}_3D_0, {}_3D_3) = E({}_3D_0\,{}_3D_3) - E({}_3D_0)E({}_3D_3)$

$$= E\left[\sum_{i=1}^{l_0} I_{\{0\le T_i<3\}} \sum_{j=1}^{l_0} I_{\{3\le T_j<6\}}\right] - \left(\frac{1}{4} l_0\right)^2$$

$$= E\left[\sum_{i=1}^{l_0}\sum_{j\ne i} I_{\{0\le T_i<3\}} I_{\{3\le T_j<6\}}\right] - \frac{l_0^2}{16}$$

$$= l_0(l_0-1)E(I_{\{0\le T_1<3\}})E(I_{\{3\le T_2<6\}}) - \frac{l_0^2}{16}$$

$$= l_0(l_0-1)\frac{1}{4}\cdot\frac{1}{4} - \frac{1}{16}l_0^2 = -\frac{1}{16} l_0$$

所以 ${}_3D_0, {}_3D_3$ 的相关系数为 $\frac{cov({}_3D_0, {}_3D_3)}{\sqrt{var({}_3D_0)var({}_3D_3)}} = \frac{1}{3}$

同理每两个随机变量的相关系数均为 $\frac{1}{3}$.

12. 证明: (1) $q_x = 1 - \frac{l_{x+1}}{l_x} = \frac{l_x - l_{x+1}}{l_x}$,

$$\mu(x) = -\frac{l_x'}{l_x} = \lim_{\Delta t \to 0} \frac{1}{l_x} \frac{l_x - l_{x+\Delta t}}{\Delta t}$$

当 l_{x+t} 为 t 的凸函数时.

$$l_{x+\Delta t} \le \Delta t\, l_{x+1} + (1-\Delta t) l_x = l_x - \Delta t (l_x - l_{x+1})$$

$$\Rightarrow \mu(x) \ge \lim_{\Delta t \to 0} \frac{1}{l_x} \cdot \frac{\Delta t (l_x - l_{x+1})}{\Delta t} = q_x$$

(2) 当 l_{x+t} 为 t 的凹函数时, $l_{x+\Delta t} \ge l_x - \Delta t (l_x - l_{x+1})$,

$$\Rightarrow \mu(x) \le \lim_{\Delta t \to 0} \frac{1}{l_x} \cdot \frac{\Delta t (l_x - l_{x+1})}{\Delta t} = q_x$$

13.

$$ {}_{15}p_{35} = \frac{l_{50}}{l_{35}} = \frac{928133}{966400} \approx 0.96$$

$$ {}_{25}q_{65} = 1 - {}_{35}p_{65} = 1 - \frac{l_{100}}{l_{65}} = 1 - \frac{3911}{785240} \approx 0.995.$$

16 (1) 证明 UDD 假设下 ${}_tq_x = t \cdot q_x$, 所以

$$m_x = \frac{q_x}{\int_0^1 {}_tp_x \, dx} = \frac{q_x}{\int_0^1 (1 - t q_x)\, dt} = \frac{q_x}{1 - \frac{1}{2} q_x} \Rightarrow \text{结论}.$$

(2) 解: 因为 $l_x = 100 - x$

$$\Rightarrow {}_{10}q_{50} = \frac{l_{50} - l_{60}}{l_{50}} = \frac{1}{5}, \quad {}_tp_{50} = \frac{l_{50+t}}{l_{50}} = 1 - \frac{t}{50}.$$

$$\Rightarrow {}_{10}m_{50} = \frac{{}_{10}q_{50}}{\int_0^{10} {}_tp_{50}\, dt} = \frac{\frac{1}{5}}{\int_0^{10} (1 - \frac{t}{50})\, dt} = \frac{1}{45}.$$

17.* 解：设原有死亡力为 $\mu_{50}(t)$，新的死亡力为 $\mu_{50}'(t)$，有

$$\mu_{50}'(t) = \mu_{50}(t) + 0.03(1-t),\quad 0<t<1$$

$$p_{50} = e^{-\int_0^1 \mu_{50}'(t)dt} = e^{-\int_0^1 [\mu_{50}(t) + 0.03(1-t)]dt}$$

$$= e^{-\int_0^1 \mu_{50}(t)dt} \cdot e^{-\int_0^1 0.03(1-t)dt}$$

$$= (1-0.006)\cdot e^{-0.015} \approx 0.98.$$

18* 证明：

① $$a(x) = \frac{\int_0^1 t\cdot {}_tp_x \mu_x(t)dt}{q_x} = \frac{\int_0^1 t \frac{l_{x+t}}{l_x}\mu_x(t)dt}{d_x/l_x}$$

$$= \frac{\int_0^1 t\, l_{x+t}\mu_x(t)dt}{d_x} = -\frac{\int_0^1 t\, d\, l_{x+t}}{d_x}$$

$$= \frac{-l_{x+1} + \int_0^1 l_{x+t}dt}{d_x} = \frac{-l_{x+1} + L_x}{d_x}$$

$$\Rightarrow a(x)d_x = L_x - l_{x+1}$$

② $$T_x = \int_0^\infty l_{x+s}ds = \int_0^1 l_{x+s}ds + \int_1^2 l_{x+s}ds + \cdots$$

$$= \int_0^1 l_{x+s}ds + \int_0^1 l_{x+1+s}ds + \cdots$$

$$= \sum_{k=0}^{\infty} L_{x+k}.$$

24. D.　25. A　28. B　29. B　32. A　33. A.

24. D $\int_0^{1.5} {}_tp_{60}\,dt = \int_0^1 {}_tp_{60}\,dt + \int_0^{0.5} {}_{1+t}p_{60}\,dt = \int_0^1 0.98\,dt + \int_0^{0.5} \frac{0.978}{0.98}\,dt = 1.478$

25. A $\int_0^{28} t\,{}_tp_{36}\,\mu_{36}(t)\,dt = \int_0^{28} t\,\frac{l_{36+t}}{l_{36}}\,\mu(36+t)\,dt = \int_0^{28} \frac{t}{8}\left(-\frac{d\,l_{36+t}}{d_{36+t}}\right)dt$

$= -\frac{1}{8}\int_0^{28} t\,d\,l_{36+t} = -\frac{1}{8}\left(t\,l_{36+t}\Big|_0^{28} - \int_0^{28} l_{36+t}\,dt\right)$

$= -\frac{1}{8}\left(28\times 6 - \int_0^{28}\sqrt{64-t}\,dt\right) \approx 3.67$

28 B. $\mu(25) = -\frac{s'(25)}{s(25)} = -\frac{-\frac{1}{100}}{1-\frac{25}{100}} = \frac{1}{75} \approx 0.0133$

29 B. $E(K(60)) = \sum_{n\geq 1} {}_np_{60} = \sum_{n\geq 1}(1+c)^{-n} = \frac{1}{c}$

$E(K^2(60)) = \sum_{n\geq 1}(2n-1)\,{}_np_{60} = \sum_{n\geq 1}(2n-1)(1+c)^{-n} = \frac{c+2}{c^2}$

$\mathrm{var}(K(60)) = E(K^2(60)) - \cancel{E(K(60))}\ E(K(60))^2 = \frac{c+2}{c^2} - \frac{1}{c^2} = \frac{c+1}{c^2}$

32. A $\mathring{e}_y = \int_0^{\infty} {}_tp_y\,dt = \frac{s(y+t)}{s(y)} = \frac{y+1}{y+1+t} = \frac{1}{2} \Rightarrow t = y+1$

33 A. $\begin{cases} {}_{1|1}q_{x+1} = p_{x+1} - {}_2p_{x+1} = p_{x+1} - p_{x+1}p_{x+2} = 0.095 \\ {}_{2|1}q_{x+1} = {}_2p_{x+1} - {}_3p_{x+1} = p_{x+1}p_{x+2}q_{x+3} = 0.171 \end{cases} \Rightarrow \begin{cases} p_{x+1} = \frac{19}{20} \\ p_x = \frac{9}{10} \end{cases}$

$\Rightarrow q_{x+1} + q_{x+2} = 2 - \frac{19}{20} - \frac{9}{10} = 0.15$