Multiple Objective Programming

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Outline

1. What is Multiple Objective Programming (MOP)?
2. Solution set of MOP
3. Image set of MOP
4. Scalarization Methods
5. Approaches Based on Non-Pareto Optimality
6. Hierarchical sequence method

What is Multiple Objective Programming?

In many practical problems, there are often more than one standard for measuring a design. For example:

- Designing a missile requires the longest range, the most fuel-efficient, the lightest weight, and the highest precision.
- In the fastest and most designed design of the aircraft that includes the aircraft's shape design, not only is the aircraft's total weight required to be the lightest, but it also requires the farthest range in the case of a certain fuel consumption.
- For determining a rubber formula, it is often necessary to examine multiple indicators such as strength, hardness, deformation, and elongation.

This type of problem has become a multiple objective programming, in many areas such as early source economics and environmental protection issues.

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Solution set of multiple objective programming (MOP)

Consider the following multiple objective programming:

$$\text{(MOP)} \qquad V_{-\min} \quad F(x) = (f_1(x), \cdots, f_p(x))^T$$
$$\text{s.t.} \quad g_i(x) \leq 0, i = 1, 2, \cdots, m,$$

where $x \in \mathbf{R}^n, p \geq 2$. Let $\mathbb{R} = \{x \in \mathbf{R}^n | g_i(x) \leq 0, i = 1, 2, \cdots, m\}$.

Definition

Assume $x^* \in \mathbb{R}$. If $f_i(x^*) \leq f_i(x)$ for all $x \in \mathbb{R}$ and $i = 1, \cdots, p$, then x^* is the absolute optimal solution of MOP, and $F(x^*)$ is the absolute optimal value of MOP.

Example

Let $f_1(x) = x^2 + 1$, $f_2(x) = x^2$, consider the following problem:

$$\begin{aligned} V_{-\min} \quad & F(x) = (f_1(x), f_2(x))^T \\ \text{s.t.} \quad & -1 \le x \le 1. \end{aligned}$$

The optimal solution is $x^* = 0$, and the absolute optimal value is $F(x^*) = (1, 0)^T$.

If $f_1(x) = (x-1)^2, f_2(x) = x^2$, then there is no absolute optimal solution for the above problem. Therefore, we must find solutions in another sense.

Give some notations as follows:

$$F\left(x^1\right) = \left(f_1\left(x^1\right), f_2\left(x^1\right), \cdots, f_p\left(x^1\right)\right)^T$$
$$F\left(x^2\right) = \left(f_1\left(x^2\right), f_2\left(x^2\right), \cdots, f_p\left(x^2\right)\right)^T$$

($<$) $F(x^1) < F(x^2)$ is equivalent to $f_j(x^1) < f_j(x^2),\ j = 1, \cdots, p$.

($\leq$) $F(x^1) \leq F(x^2)$ is equivalent to $f_j(x^1) < f_j(x^2),\ j = 1, \cdots, p$, and at least there exist some j_0 such that $f_{j_0}(x^1) < f_{j_0}(x^2)$.

($\leqq$) $F(x^1) \leqq F(x^2)$ is equivalent to $f_j(x^1) \leq f_j(x^2),\ j = 1, \cdots, p$.

Definition

Let $x^* \in \mathbb{R}$, if there are not $x \in \mathbb{R}$ satisfies that $F(x) \leq F(x^*)$, then x^* is the **effective solution** (or **Pareto solution**), it is also called **non-inferior solution**.

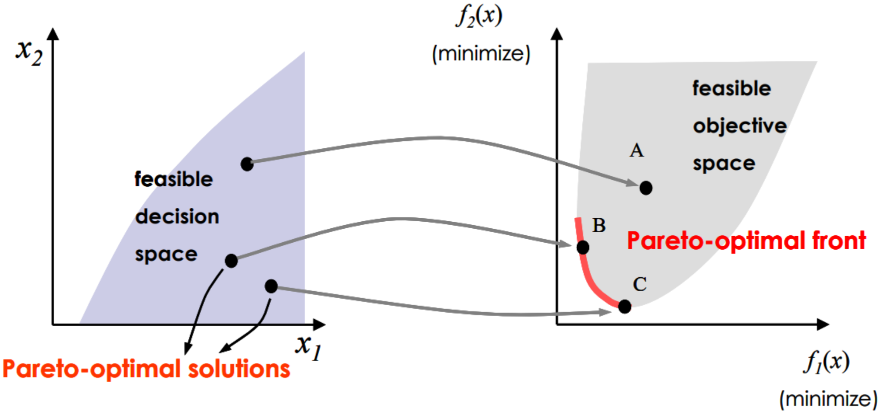
Definition

Let $x^* \in \mathbb{R}$, if there are not $x \in \mathbb{R}$ satisfies that $F(x) < F(x^*)$, then x^* is the **weak effective solution** (or **weak Pareto solution**), it is also called **weak non-inferior solution**.

Definition

Let $x^* \in \mathbb{R}$, if there is no $x' \in \mathbb{R}, x' \neq x^*$, such that $F(x') \leqq F(x^*)$, then x^* is the **strictly efficient solution**.

Graphical Depiction of Pareto Optimal Solution



$\mathbb{R}_s^*$: the strictly efficient solution set of MOP.
$\mathbb{R}^*$: the absolute optimal solution set of MOP.
$\mathbb{R}_p^*$: the effective solution set of MOP.
$\mathbb{R}_{wp}^*$: the weak effective solution set of MOP.
Consider the single objective programming:

$$\begin{cases} \min & f_j(x) \\ \text{s.t.} & g_i(x) \leq 0, i = 1, 2, \cdots, m, \end{cases}$$

where $j = 1, \cdots$. Let $\mathbb{R}_j^*$ be the optimal solution set of the above problem, then

$$\mathbb{R}^* = \cap_{j=1}^p \mathbb{R}_j^*.$$

Theorem

$\mathbb{R}_p^* \subset \mathbb{R}_{wp}^* \subset \mathbb{R}.$

Theorem

$\mathbb{R}_j^* \subset \mathbb{R}_{wp}^*,\ j = 1, \cdots, p.$

Theorem

$\mathbb{R}^* \subset \mathbb{R}_p^*.$

From the above theorem, we have

$$\mathbb{R}^* \subset \mathbb{R}_p^* \subset \mathbb{R}_{wp}^* \subset \mathbb{R}$$

$$\mathbb{R}_j^* \subset \mathbb{R}_{wp}^*, \quad j = 1, 2, \cdots, p.$$

Definition

If $f_1(x), \cdots, f_p(x)$ and $g_1(x), \cdots, g_p(x)$ in MOP are convex function, then MOP is convex multiple objective programming.

In general, for convex multiple objective programming, $\mathbb{R}^*_{wp}$ and $\mathbb{R}^*_p$ not necessarily convex.

Definition

A point $\hat{x} \in X$ is called a **properly efficient solution** $(\mathbb{R}^*_{pe})$ of the MOP in the sense of Geoffrion if $\hat{x} \in \mathbb{R}^*_p$ and if there exists $M > 0$ such that for each $k = 1, \ldots, p$ and each $x \in X$ satisfying $f_k(x) < f_k(\hat{x})$ there exists an $l \neq k$ with $f_l(x) > f_l(\hat{x})$ and $(f_k(\hat{x}) - f_k(x)) / (f_l(x) - f_l(\hat{x})) \leq M$.

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Image set of multiple objective programming (MOP)

For MOP, taking a feasible solution $x^0 \in \mathbb{R}$, the $F(x^0) = (f_1(x^0), \cdots, f_p(x^0))^T$ can be regarded as a point in Euclidean space R^p. Generally, for any $x \in \mathbb{R}$, it is always obtain a point $F(x)$ in $\mathbb{R}^p$. Hence, we can define a mapping:

$$x \to F(x).$$

Let $F(\mathbb{R}) := \{F(x)|x \in \mathbb{R}\}$, then $F(\mathbb{R})$ is called the image set of the feasible set $\mathbb{R}$ under the mapping F.

- For any $x^0 \in \mathbb{R}$, then $F(x^0) \in F(\mathbb{R})$ is call the image at x^0 under the mapping F.
- If for some $F^0 \in F(\mathbb{R}$, there are at least one point $x^0 \in \mathbb{R}$ such that $F(x^0) = F^0$, then x^0 is the original image.
- In general, the image set $F(\mathbb{R})$ of convex multiple objective programming not necessarily convex.

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Methods for dealing with MOP–Scalarization Methods

Scalarization Methods

1. The Weighted-Sum Approach
2. The Weighted t-th Power Approach
3. The Weighted Quadratic Approach
4. The Guddat et al. Approach
5. The ε-Constraint Approach
6. The Improved ε-Constraint Approach
7. The Penalty Function Approach
8. The Benson Approach
9. Reference Point Approaches
10. Direction-Based Approaches
11. Ideal point method
12. Square sum weighting method

The traditional approach to solving MOPs is by scalarization which involves formulating an MOP-related SOP by means of a real-valued scalarizing function typically being a function of the objective functions of the MOP, auxiliary scalar or vector variables, and/or scalar or vector parameters.

The Weighted-Sum Approach

In the weighted-sum approach a weighted sum of the objective functions is minimized:

$$\begin{aligned} \min \sum_{j=1}^{p} \quad & \lambda_j f_j(x) \\ \text{s.t.} \quad & x \in \mathbb{R}, \end{aligned}$$

where $\lambda_i > 0, i = 1, 2, \cdots, p$.

Theorem

1. *Let $\lambda \geq 0$. If $\hat{x} \in \mathbb{R}$ is an optimal solution of problem in the weighted-sum approach, then $\hat{x} \in \mathbb{R}^*_{wp}$. If $\hat{x} \in \mathbb{R}$ is a unique optimal solution of problem in the weighted-sum approach, then $\hat{x} \in \mathbb{R}^*_{p}$.*
2. *Let $\lambda > 0$. If $\hat{x} \in \mathbb{R}$ is an optimal solution of problem in the weighted-sum approach, then $\hat{x} \in \mathbb{R}^*_{pe}$.*
3. *Let the MOP be convex. A point $\hat{x} \in \mathbb{R}$ is an optimal solution of problem in the weighted-sum approach for some $\lambda > 0$ if and only if $\hat{x} \in \mathbb{R}^*_{pe}$.*

The Weighted t-th Power Approach

In the weighted t -th power approach a weighted sum of the objective functions taken to the power of t is minimized:

$$\min \sum_{j=1}^{p} \lambda_j \left(f_j(x)\right)^t$$
$$\text{s.t.} \quad x \in \mathbb{R}$$

where $\lambda_i > 0, i = 1, 2, \cdots, p.$ and $t > 0.$

Theorem

1. *For all $t > 0$, if $\hat{x} \in \mathbb{R}$ is an optimal solution of problem in the weighted t-th power approach, then $\hat{x} \in \mathbb{R}_p^*$.*
2. *If a point $\hat{x} \in \mathbb{R}$ is efficient then there exists a $\hat{t} > 0$ such that for every $t \geq \hat{t}$, the point $\hat{x}$ an optimal solution of problem in the weighted t-th power approach.*

The Weighted Quadratic Approach

In the weighted quadratic approach a quadratic function of the objective functions is minimized:

$$\begin{aligned} \min \quad & F(x)^T Q F(x) + q^T F(x) \\ \text{s.t.} \quad & x \in \mathbb{R} \end{aligned}$$

where Q is a $p \times p$ matrix and q is a vector in R^p.

Theorem

Under conditions of quadratic Lagrangian duality, if $\hat{x} \in \mathbb{R}$ is efficient then there exist a symmetric $p \times p$ matrix Q and a vector $q \in \mathbb{R}^p$ such that $\hat{x}$ is an optimal solution of problem in the weighted quadratic approach.

The Guddat et al. Approach

Let x^0 be an arbitrary feasible point for the MOP. Consider the following problem:

$$\min \sum_{j=1}^{p} \lambda_j f_j(x)$$
$$\text{S.T.} f_j(x) \leq f_j\left(x^0\right), j = 1, \ldots, p$$
$$x \in \mathbb{R}$$

where $\lambda_i > 0, i = 1, 2, \cdots, p$.

Theorem

Let $\lambda > 0$. A point $x^0 \in \mathbb{R}$ is an optimal solution of problem in the Guddat et al. approach if and only if $x^0 \in \mathbb{R}_p^*$.

The ε-Constraint Approach

In the ε-constraint method one objective function is retained as a scalar-valued objective while all the other objective functions generate new constraints. The j-th ε-constraint problem is formulated as:

$$\begin{aligned} \min \quad & f_j(x) \\ \text{s.t.} \quad & f_i(x) \leq \varepsilon_i, i = 1, \ldots, p; i \neq j \\ & x \in \mathbb{R}. \end{aligned}$$

Theorem

1. *If, for some* $j, j \in \{1, \ldots, p\}$*, there exists* $\varepsilon_{-j} \in \mathbb{R}^{p-1}$ *such that* $\hat{x}$ *is an optimal solution of problem in the* ε*-constraint approach, then* $\hat{x} \in \mathbb{R}^{*}_{wp}$.
2. *If, for some* $j, j \in \{1, \ldots, p\}$*, there exists* $\varepsilon_{-j} \in \mathbb{R}^{p-1}$ *such that* $\hat{x}$ *is a unique optimal solution of problem in the* ε*-constraint approach, then* $\hat{x} \in \mathbb{R}^{*}_{p}$
3. *A point* $\hat{x} \in X$ *is efficient if and only if there exists* $\varepsilon \in \Psi$ *such that* $\hat{x}$ *is an optimal solution of problem in the* ε*-constraint approach for every* $j = 1, \ldots, p$ *and with* $f_i(\hat{x}) = \varepsilon_i, i = 1, \ldots, p, i \neq j$.

The Improved ε-Constraint Approach

The improved constraint approach tries to overcome those difficulties using the following two scalarizations:

$$\begin{aligned} \min \quad & f_j(x) - \sum_{i \neq j} \lambda_i l_i \\ \text{s.t.} \quad & f_i(x) + l_i \leq \varepsilon_i, i = 1, \ldots, p; i \neq j \\ & l_i \geq 0, i = 1, \ldots, p; i \neq j \\ & x \in \mathbb{R}. \end{aligned}$$

$$\begin{aligned} \min \quad & f_j(x) + \sum_{i \neq j} \lambda_i l_i \\ \text{s.t.} \quad & f_i(x) - l_i \leq \varepsilon_i, i = 1, \ldots, p; i \neq j \\ & l_i \geq 0, i = 1, \ldots, p; i \neq j \\ & x \in \mathbb{R}, \end{aligned}$$

where weights $\lambda_i \geq 0, i \neq j$.

Theorem

1. *Let* $\lambda > 0$. *If* $(\hat{x}, \hat{l})$ *is an optimal solution of problem in the improved* ε*-constraint approach, then* $\hat{x} \in \mathbb{R}^*_p$.
2. *Let* $\lambda > 0$. *If* $(\hat{x}, \hat{l})$ *is an optimal solution of problem in the improved* ε*-constraint approach with* $\hat{l} > 0$ *then* $\hat{x} \in \mathbb{R}^*_{pe}$
3. *If* $\hat{x} \in \mathbb{R}^*_{pe}$ *then, for every* $j \in \{1, \ldots, p\}$, *there exist* $\varepsilon, \hat{l}$, *and* $\lambda \in \mathbb{R}^p_>$ *such that* $(\hat{x}, \hat{l})$ *is an optimal solution of problem in the improved* ε*-constraint approach.*

The Penalty Function Approach

The scalarized MOP assumes the form:

$$\begin{aligned} \min \quad & \sum_{j=1}^{p} \lambda_j \max\left\{f_j(x) - M, 0\right\}^2 \\ \text{s.t.} \quad & x \in X, \end{aligned}$$

where $\lambda_i \geq 0, M$ is a scalar such that $M < f_j\left(x^0\right), j = 1, \ldots, p,$ and $x^0 \in \mathbb{R}$. The scalarized MOP' becomes:

$$\begin{aligned} \min \quad & \sum_{j=1}^{p} \lambda_j \max\left\{f_j(x) - M, 0\right\}^2 + M^2 \sum_{j=1}^{p} \max\left\{g_j(x), 0\right\} \\ \text{s.t.} \quad & x \in \mathbb{R}, \end{aligned}$$

where $\lambda_j \geq 0, M < 0$ and such that $M < f_j(\hat{x}), j = 1, \ldots, p,$ and $\hat{x} \in \mathbb{R}$.

Theorem

1. *If x^0 is an optimal solution of problem in the penalty function approach, then $x^0 \in \mathbb{R}_p^*$ for the MOP.*
2. *Let $\hat{x}$ be an optimal solution of problem in the penalty function approach. If $\hat{x}$ is a feasible solution of the MOP,then $\hat{x}^0 \in \mathbb{R}_p^*$ for MOP.*

The Benson Approach

Benson introduces an auxiliary vector variable $l \in R^p$ and uses a known feasible point x^0 in the following scalarization:

$$\begin{aligned} \max \quad & \sum_{j=1}^{p} l_j \\ \text{s.t.} \quad & f_j(x) + l_j = f_j\left(x^0\right), j = 1, \ldots, p \\ & l \geq 0 \\ & x \in \mathbb{R} \end{aligned}$$

Theorem

1. *The point* $x^0 \in \mathbb{R}$ *is efficient if and only if the optimal objective value of problem in the Benson approach is equal to zero.*
2. *If* $(\hat{x}, \hat{l})$ *is an optimal solution of problem in the Benson approach with a positive optimal objective value then* $\hat{x} \in \mathbb{R}^*_p$.
3. *Let the MOP be convex. If no finite optimal objective value of problem in the Benson approach exists then* $\mathbb{R}^*_{pe} = \emptyset$.

Reference Point Approaches

- **Distance-Function-Based Approaches**: These methods employ a distance function, typically based on a norm, to measure the distance between a utopia (or ideal) point and the points in the Pareto set. Let $d : R^p \times R^p \to R$ denote a distance function. The generic problem is formulated as:

$$\begin{aligned} \min \quad & d(F(x), r) \\ \text{s.t.} \quad & x \in \mathbb{R}, \end{aligned}$$

  where $r \in R^p$ is a reference point.

- **The Achievement Function Approach**: A certain class of real-valued functions $s_r : R^p \to R$, referred to as achievement functions, is used to scalarize the MOP. The scalarized problem is given by

$$\begin{aligned} \min \quad & s_r(F(x)) \\ \text{s.t.} \quad & x \in \mathbb{R}. \end{aligned}$$

- **The Weighted Geometric Mean Approach**: Consider the weighted geometric mean of the differences between the nadir point y^N and the objective functions with the weights in the exponents

$$
\begin{aligned}
\max \quad & \Pi_{j=1}^{p} \left(y_j^N - f_j(x)\right)^{\lambda_j} \\
\text{s.t.} \quad & f_j(x) \leq y_j^N, j = 1, \ldots, p, \\
& x \in \mathbb{R},
\end{aligned}
$$

where $\lambda_j \geq 0$.

- **Goal Programming**: The vector of these goals produces a reference point in the objective space and therefore goal programming can be viewed as a variation of the reference point approaches. Let $r \in R^p$ be a goal. The general formulation of GP is

$$\begin{aligned} \min \quad & a\left(\delta^-, \delta^+\right) \\ \text{s.t.} \quad & f_k(x) + \delta_k^- - \delta_k^+ = r_k, k = 1, \ldots, p, \\ & x \in \mathbb{R}, \end{aligned}$$

where $\delta^-, \delta^+ \in R^p$ are variables representing negative and positive deviations from the goal r, and $a\left(\delta^-, \delta^+\right)$ is an achievement function.

Direction-Based Approaches

- **The Roy Approach**:

$$
\begin{aligned}
\max \quad & \alpha \\
\text{s.t.} \quad & f_j(x) + \alpha e \leq r_j, j = 1, \ldots, p, \\
& x \in \mathbb{R}
\end{aligned}
$$

where $e \in R^p$ is a vector of ones and determines the fixed direction of search. Depending on the choice of the reference point r the approach finds a (weakly) efficient solution.

- **The Goal-Attainment Approach**: Given a (feasible or infeasible) goal vector r and a direction $d \leq 0$ along which the search is performed the goal-attainment approach is formulated as

$$\begin{aligned} \max \quad & \alpha \\ \text{s.t.} \quad & f_j(x) - \alpha d_j \leq r_j, j = 1, \ldots, p, \\ & x \in \mathbb{R}. \end{aligned}$$

- **The Reference Direction Approach**: This is a more general approach with an unrestricted search direction $d \in R^p$ and an auxiliary vector variable $l \in R^p$:

$$\begin{aligned} \max \quad & \alpha \\ \text{s.t.} \quad & f_j(x) - \alpha d_j + l_j = r_j, j = 1, \ldots, p, \\ & l \geq 0, \\ & x \in \mathbb{R}. \end{aligned}$$

- **The Pascoletti and Serafini Approach**: With the inclusion of non-negative slack variables they arrive at above method with a feasible reference point $r \in Y$ and a search direction $d \leq 0$. Additionally, to move on the Pareto set they parametrize the reference point r or the search direction d, and obtain:

$$
\begin{aligned}
\max\quad & \alpha \\
\text{s.t.}\quad & f_j(x) - \alpha(d_j + \alpha_1 \Delta d_j) + l_j = (r_j + \alpha_2 \Delta r_j), k = 1, \dots, p, \\
& l \geq 0, \\
& x \in \mathbb{R}.
\end{aligned}
$$

- **The Modified Pascoletti and Serafini Approach**: Since a solution to problem in the reference direction approach may not be finite, the following modification has been developed:

$$\begin{aligned} \max \quad & \{\alpha,\ \|l\|_p\} \\ \text{s.t.} \quad & f_j(x) - \alpha d_j + l_j = r_j, j = 1, \ldots, p, \\ & l \geq 0, \\ & x \in \mathbb{R}. \end{aligned}$$

Ideal point method

Firstly, one should solve p single objective programming problem:

$$\begin{aligned} \min \quad & f_i(x), i = 1, 2, \cdots, p \\ \text{s.t.} \quad & g_i(x) \le 0, i = 1, 2, \cdots, m \end{aligned}$$

Let $f_i^* = \min_{x \in R} f_i(x)$, then $F^* = \left(f_1^*, f_2^*, \cdots, f_p^*\right)^T$ is called the ideal point method. Generally, it can not be obtained. Hence, we hope to find a point and try to get close to it. Then, solving the following problem:

$$\min \ h(x) \triangleq h(F(x)) = \sqrt{\sum_{i=1}^{p} \left(f_i(x) - f_i^*\right)^2} = \|F(x) - F^*\|_2^2$$

$$\text{s.t.} \quad x \in \mathbb{R}.$$

Square sum weighting method

Firstly, finding a best possible lower bound $f_1^0, \cdots, f_p^0$ of each single objective programming problem, that is,

$$\min_{x \in \mathbb{R}} f_i(x) \leq f_i^0, i = 1, 2, \cdots, p.$$

Then, establishing evaluation function as follows:

$$h(x) = h(F(x)) = \sqrt{\sum_{i=1}^{p} \lambda_i \left(f_i(x) - f_i^0\right)^2},$$

where $\lambda_1, \cdots, \lambda_p$ a given set of weight coefficients, and they satisfies

$$\sum_{i=1}^{p} \lambda_i = 1, \lambda_i > 0, i = 1, 2, \cdots, p.$$

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The Lexicographic Approach

The lexicographic approach makes use of the lexicographic relation and assumes a ranking of the objective functions according to their importance. Let π be a permutation of $\{1, \ldots, p\}$ and assume that $f_{\pi(k)}$ is more important than $f_{\pi(k+1)}$ $k = 1, \ldots, p-1$. Let $f^{\pi} : \mathbb{R}^n \to \mathbb{R}^p$ be $\left(f_{\pi(1)}, \ldots, f_{\pi(p)}\right)$. The lexicographic problem is formulated as

$$\begin{aligned} \min \quad & f^{\pi}(x) \\ \text{s.t.} \quad & x \in \mathbb{R}. \end{aligned}$$

The Max-Ordering Approach

The max-ordering approach makes use of the max-ordering relation and does only consider the objective function f_k which has the highest (worst) value. The max- ordering problem is formulated as

$$\begin{aligned} \min \max_{k=1,\cdots,p} \quad & f_k(x) \\ \text{s.t.} \quad & x \in \mathbb{R}. \end{aligned}$$

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Methods for dealing with MOP–Hierarchical sequence method

Hierarchical sequence method for MOP is that sorting the objections by importance, for example, $f_1(x)$ is the most important, $f_2(x)$ times, ..., and the last objection is $f_p(x)$. Firstly, solving the optimal solution x^1 and optimal value f_1^* of problem:

$$(\mathrm{P}_1) \qquad \begin{aligned} \min \ & f_1(x) \\ \text{s.t.} \ & x \in \mathbb{R}. \end{aligned}$$

Then, solving the optimal solution x^2 and optimal value f_2^* of problem:

$$(\mathrm{P}_2) \qquad \begin{aligned} \min \ & f_2(x) \\ \text{s.t.} \ & x \in \mathbb{R}_1, \end{aligned}$$

where $\mathbb{R}_1 := \mathbb{R} \cap \{x | f_1(x) \leq f_1^*\}$.

Continue to solve the optimal solution x^3 and optimal value f_3^* of problem:

$$(\mathrm{P})_3 \qquad \min \ f_3(x) \quad \text{s.t.} \ x \in \mathbb{R}_2,$$

where $\mathbb{R}_2 := \mathbb{R}_1 \cap \{x | f_2(x) \leq f_2^*\}$. Continue this way until solve the optimal solution x^p and optimal value f_p^* of the pth problem:

$$(\mathrm{P})_p \qquad \min \ f_p(x) \quad \text{s.t.} \ x \in \mathbb{R}_{p-1},$$

where $\mathbb{R}_{p-1} := \mathbb{R}_{p-2} \cap \{x | f_{p-2}(x) \leq f_{p-2}^*\}$. Hence, we can obtain the optimal solution of MOP in the sense of a hierarchical sequence, that is, $x^* = x^p$, and $F^* = (f_1(x^*), \cdots, f_p(x^*))^T$ is the optimal value of MOP.

Other methods

- Descent Methods
- Set-Oriented Methods
  - The Balance and Level Set Approaches
  - The ϵ-Efficiency Approach
  - Continuation Methods
  - Covering Methods