



南方科技大学
SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY

考试科目: 线性代数
考试时长: 120 分钟

开课单位: 数学系
命题教师: 线性代数教学团队

题号	1	2	3	4	5	6	7
分值	15 分	20 分	10 分	24 分	20 分	5 分	6 分

本试卷共 (7) 大题, 满分 (100) 分. (考试结束后请将试卷、答题本、草稿纸一起交给监考老师)
This exam paper contains 7 questions and the score is 100 in total. (Please hand in your exam paper, answer sheet, and your scrap paper to the proctor when the exam ends.)

1. (15 points, 3 points each) Multiple Choice. Only one choice is correct.
(共 15 分, 每小题 3 分) 选择题, 只有一个选项是正确的.

(1) Let

$$\alpha_1 = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \alpha_2 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \alpha_3 = \begin{bmatrix} 7 \\ 3 \\ c \end{bmatrix}.$$

If $\alpha_1, \alpha_2, \alpha_3$ are linearly dependent, then c equals

- (A) 5.
(B) 6.
(C) 7.
(D) 8.

假定

$$\alpha_1 = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \alpha_2 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \alpha_3 = \begin{bmatrix} 7 \\ 3 \\ c \end{bmatrix}.$$

若 $\alpha_1, \alpha_2, \alpha_3$ 线性相关, 则 c 的取值为

- (A) 5.
(B) 6.
(C) 7.
(D) 8.

- (2) Let A be an $m \times n$ real matrix and b be an $m \times 1$ real column vector. Which of the following statements is correct?

(A) If $Ax = b$ does not have any solution, then $Ax = 0$ has only the zero solution.

- (B) If $Ax = 0$ has infinitely many solutions, then $Ax = b$ has infinitely many solutions.
 (C) If $m < n$, both $Ax = b$ and $Ax = 0$ have infinitely many solutions.
 (D) If the rank of A is n , then $Ax = 0$ has only the zero solution.

设 A 为一个 $m \times n$ 实矩阵, b 为一个 m 维实列向量. 以下说法一定是正确的是?

- (A) 若 $Ax = b$ 无解, 则 $Ax = 0$ 只有零解.
 (B) 若 $Ax = 0$ 有无穷多解, 则 $Ax = b$ 有无穷多解.
 (C) 若 $m < n$, 则 $Ax = b$ 和 $Ax = 0$ 都有无穷多解.
 (D) 若 A 的秩为 n , 则 $Ax = 0$ 只有零解.
- (3) For which value of k does the system

$$\begin{cases} x_1 + 2x_2 - 4x_3 + 3x_4 = 0, \\ x_1 + 3x_2 - 2x_3 - 2x_4 = 0, \\ x_1 + 5x_2 + (5 - k)x_3 - 12x_4 = 0, \end{cases}$$

have exactly two free variables?

- (A) 5.
 (B) 4.
 (C) 3.
 (D) 2.

如果以下线性方程组有两个自由变量

$$\begin{cases} x_1 + 2x_2 - 4x_3 + 3x_4 = 0, \\ x_1 + 3x_2 - 2x_3 - 2x_4 = 0, \\ x_1 + 5x_2 + (5 - k)x_3 - 12x_4 = 0, \end{cases}$$

k 的取值为

- (A) 5.
 (B) 4.
 (C) 3.
 (D) 2.

- (4) Let $u, v \in \mathbb{R}^3$ and $\lambda \in \mathbb{R}$. Which of the following statements is false?

- (A) If u and v are nonzero vectors satisfying $u^T v = 0$, then u and v are linearly independent.
 (B) If $u + v$ is orthogonal to $u - v$, then $\|u\| = \|v\|$.
 (C) $u^T v = 0$ if and only if $u = 0$ or $v = 0$.
 (D) $\lambda v = 0$ if and only if $v = 0$ or $\lambda = 0$.

设 $u, v \in \mathbb{R}^3$, $\lambda \in \mathbb{R}$. 以下说法错误的是?

- (A) 如果 u 和 v 为满足 $u^T v = 0$ 的非零向量, 则 u 和 v 线性无关.
 (B) 如果 $u + v$ 和 $u - v$ 正交, 则 $\|u\| = \|v\|$.

(C) $u^T v = 0$ 当且仅当 $u = 0$ or $v = 0$.

(D) $\lambda v = 0$ 当且仅当 $v = 0$ or $\lambda = 0$.

(5) Let A and B be two $n \times n$ matrices. Which of the following assertions is false?

(A) If A, B are symmetric matrices, then AB is a symmetric matrix.

(B) If A, B are invertible matrices, then AB is an invertible matrix.

(C) If A, B are permutation matrices, then AB is a permutation matrix.

(D) If A, B are upper triangular matrices, then AB is an upper triangular matrix.

设 A 和 B 都为 n 阶矩阵. 以下说法错误的是?

(A) 如果 A, B 为对称矩阵, 则 AB 也为一个对称矩阵.

(B) 如果 A, B 为可逆矩阵, 则 AB 也为一个可逆矩阵.

(C) 如果 A, B 为置换矩阵, 则 AB 也为一个置换矩阵.

(D) 如果 A, B 为上三角矩阵, 则 AB 也为上三角矩阵.

2. (20 points, 5 points each) Fill in the blanks.

(共 20 分, 每小题 5 分) 填空题.

(1) Let

$$A = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & 3 & 2 \end{bmatrix}, a, b \in \mathbb{R}.$$

Then $A^{-1} =$ _____.

设

$$A = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & 3 & 2 \end{bmatrix}, a, b \in \mathbb{R},$$

则 $A^{-1} =$ _____.

(2) Let A be a 4×3 real matrix with rank 2 and $B = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ -1 & 0 & 3 \end{bmatrix}$. Then the rank AB is _____.

设 A 为一个 4×3 的实矩阵, B 为 $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ -1 & 0 & 3 \end{bmatrix}$. 如果矩阵 A 的秩为 2, 则 AB 的秩为 _____.

(3) Let $A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 2 & -2 & 2 \end{bmatrix}$. Then $A^{2024} =$ _____.

设 $A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 2 & -2 & 2 \end{bmatrix}$, 则 $A^{2024} =$ _____.

(4) Consider the system of linear equations:

$$Ax = b : \begin{cases} x = 2 \\ y = 3 \\ x + y = 6 \end{cases}$$

The least-squares solution for the system is _____.

考虑以下线性方程组:

$$Ax = b : \begin{cases} x = 2 \\ y = 3 \\ x + y = 6 \end{cases}$$

该线性方程组的最小二乘解为 _____.

3. (10 points) Let

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & -5 & 1 \\ 1 & -4 & -7 \end{bmatrix}.$$

Find an LU factorization of A .

设

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & -5 & 1 \\ 1 & -4 & -7 \end{bmatrix}.$$

求矩阵 A 的一个 LU 分解.

4. (24 points) Consider the following 4×5 matrix A and 4-dimensional column vector b :

$$A = \begin{bmatrix} 0 & 2 & 4 & 1 & 6 \\ 0 & 1 & 1 & 1 & 3 \\ 0 & 4 & 10 & 1 & 2 \\ 0 & -1 & -5 & 1 & 7 \end{bmatrix}, \quad b = \begin{bmatrix} 3 \\ 2 \\ -5 \\ 10 \end{bmatrix}$$

(a) Find a basis for each of the four fundamental subspaces of A .

(b) Find the complete solution to $Ax = b$.

考虑以下 4×5 矩阵 A 以及 4 维列向量 b :

$$A = \begin{bmatrix} 0 & 2 & 4 & 1 & 6 \\ 0 & 1 & 1 & 1 & 3 \\ 0 & 4 & 10 & 1 & 2 \\ 0 & -1 & -5 & 1 & 7 \end{bmatrix}, \quad b = \begin{bmatrix} 3 \\ 2 \\ -5 \\ 10 \end{bmatrix}$$

(a) 分别求矩阵 A 的四个基本子空间的一组基向量.

(b) 求 $Ax = b$ 的所有解.

5. (20 points) Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ and T be the linear transformation from $\mathbb{R}^{2 \times 2}$ to $\mathbb{R}^{2 \times 2}$ defined by

$$T(X) = XA + AX, X \in \mathbb{R}^{2 \times 2}.$$

Where $\mathbb{R}^{2 \times 2}$ denotes the vector space consisting of all 2×2 real matrices.

- (a) Find the matrix representation of T with respect to the following ordered basis

$$v_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, v_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, v_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

- (b) Find a matrix B such that

$$T(B) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

- (c) Find a matrix C such that

$$T(C) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

设 Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$, T 为按照以下方式定义的从 $\mathbb{R}^{2 \times 2}$ 到 $\mathbb{R}^{2 \times 2}$ 线性变换:

$$T(X) = XA + AX, X \in \mathbb{R}^{2 \times 2}.$$

其中 $\mathbb{R}^{2 \times 2}$ 表示所有 2×2 实矩阵构成的向量空间.

- (a) 求 T 在以下有序基

$$v_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, v_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, v_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

下的矩阵表示.

- (b) 求一个矩阵 B 使得

$$T(B) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

- (c) 求一个矩阵 C 使得

$$T(C) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

6. (5 points) Let A, B be two $n \times n$ real matrices satisfying $A^2 = A$ and $B^2 = B$. Show that if $(A+B)^2 = A+B$, then $AB = O$. Where O denotes the $n \times n$ zero matrix.

设 A, B 为满足 $A^2 = A$ 和 $B^2 = B$ 的 n 阶实矩阵. 证明: 如果 $(A+B)^2 = A+B$, 则 $AB = O$. 其中 O 表示 n 阶零矩阵.

7. (6 points) Let A be a 3×2 matrix, B be a 2×3 matrix such that

$$AB = \begin{bmatrix} 8 & 0 & -4 \\ -\frac{3}{2} & 9 & -6 \\ -2 & 0 & 1 \end{bmatrix}.$$

(a) Compute $(AB)^2$.

(b) Find BA .

设 A 为 3×2 矩阵, B 为 2×3 矩阵, 并且

$$AB = \begin{bmatrix} 8 & 0 & -4 \\ -\frac{3}{2} & 9 & -6 \\ -2 & 0 & 1 \end{bmatrix}.$$

(a) 计算 $(AB)^2$.

(b) 求 BA .