COMPLEX ANALYSIS (H) MIDTERM EXAM

Instructions

- Allotted time: 4:20-6:10pm
- Partial marks will be awarded for correct reasoning

(1) True or False? No need to justify your answer.

i An entire function that does not take on any real values is constant.

ii The Bessel function, defined by the power series

$$J_r(z) = \left(\frac{z}{2}\right)^r \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+r)!} \left(\frac{z}{2}\right)^{2n}$$

where r is a positive integer, is entire.

iii All values of r^i, for $r \in \mathbb{R} \setminus \{0\}$, lie on the unit circle.

iv A continuous function defined on the closed unit disk can be uniformly approximated on the closed unit disk by a sequence of holomorphic functions.

(30 marks)

(2) Evaluate, where C is the positively oriented circle centered at the origin with radius 1

$$\int_C \frac{z^4 + 3z^2 + 1}{z^{16}} dz$$

(10 marks)

(3) Evaluate

$$\int_0^{+\infty} \frac{\cos(x)}{x^2 + b^2} dz$$

when $b > 0$.

(20 marks)

(4) Let γ be the closed curve parameterised by $e^{\pi i t}$, $t \in [0, 4]$. What is the winding number of γ around the origin?

(5 marks)

(5) Suppose $c \in \mathbb{C}$ satisfies $|c| > e$. Calculate the number (with multiplicities) of solutions of the equation $e^z = cz^n$ for $|z| < 1$.

(10 marks)

(6) Give an example of a subset of $\mathbb{C}$ on which a branch of the multivalued function $\log(1-z^2)$ can be defined. It is enough to draw the subset, no need to give a formula.
For $z = e^{i\frac{\pi}{4}}$, list all values of $\log(1-z^2)$.

(15 marks)

(7) Let $f(z)$ be a function holomorphic on Ω, where Ω is open and contains the closed unit disk $\mathbb{D}$ centered on the origin. Show that if $|f(z^2)| \geq |f(z)|$ for all z in the interior of $\mathbb{D}$, then f is constant.

(10 marks)

$z^{\frac{1}{z^2}}$