midterm exam

1. True or False. You are not required to give any justification of your answer. (2*5=10)

T **(1)** Let $\Omega \subseteq \mathbb{R}^n$ and $\bar{x}$ is an interior point of Ω. Then any vector $d \in \mathbb{R}^n \backslash \{0\}$ is a feasible direction at $\bar{x} \in \Omega$.

(2) The sequence $\left\{\log\left(1+\frac{1}{k}\right)\right\}_{k\geq 1}$ exhibits a linear rate of convergence.

T **(3)** Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth function with $\bar{x} \in \mathbb{R}^n$. Then $-\nabla f(\bar{x})$ is the max-rate descending direction of f at $\bar{x}$ if $\bar{x}$ is not stationary.

T **(4)** The quasi-Newton methods avoid calculation of second derivatives of the objective function.

T **(5)** If a standard-form linear programming problem has a feasible solution, then there exists an extreme point solution.

2. (5+5=10). Answer True or False for each of the following statements and justify your answer.

   (a) We say a set $C \subseteq \mathbb{R}^n$ is locally closed around $\bar{x} \in C$ if there is $r > 0$ such that the set $C \cap \mathbb{B}_r(\bar{x})$ is closed, where $\mathbb{B}_r(\bar{x})$ stands for the closed ball with center $\bar{x}$ and radius $r > 0$. Then $\Omega \subset \mathbb{R}^n$ is closed if it is locally closed around any $x \in \Omega$.

   (b) Applying the Newton's method to solve $\min\limits_{x\in\mathbb{R}^n} f(x) = \frac{1}{2}x^TQx - b^Tx$, where Q is symmetric and invertible. It takes at least two iterations to obtain the optimal solution.

3. (16) linear programming.

$$\begin{aligned} \max \quad & 4x_1 + 3x_2 + 5x_3 \\ s.t. \quad & 3x_1 + x_2 + 3x_3 \leq 30 \\ & 2x_1 + 2x_2 + 3x_3 \leq 40 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

4. (6+10=16).

   (a) The table below shows the amount of materials A and materials B needed to make each product. The right column shows that the total amount of A and B available to make these productions. Establish integer linear programming model to maximize the total profit

	Production 1	Production 2	Total available
material A	3	5	300
material B	7	5	350
Profit of each production	7.5	10	

(b) $0-1$ integer programming.

$$\begin{aligned}\max \quad & z = 26x_1 + 28x_2 + 9x_3 + 10x_4 \\ s.t. \quad & 14x_1{+}17x_2 + 6x_3 + 7x_4 \leq 24 \\ & 11x_1{+}11x_2 + 3x_3 + 4x_4 \leq 14 \\ & 8x_1{+}7x_2 + 2x_3 \leq 8 \\ & x_1{+}x_2 \geq 1 \\ & x_1, x_2, x_3, x_4 \subset \{0,1\}\end{aligned}$$

5. (6+6+6=18). Consider the optimality conditions on minimizing a differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ over a closed set Ω. For any $x \in \Omega$, the tangent direction of Ω at x is given by

$$T(x) := \left\{ d \in \mathbb{R}^n \middle| \exists \left\{t_k, d^k\right\}_{k\geq 0} \subset \mathbb{R}_{++} \times \mathbb{R}^n \text{ with } t_k \downarrow 0, d^k \to d \text{ as } k \to \infty \text{ and } x + t_k d^k \in \Omega \text{ for all } k \geq 0 \right\}.$$

(a) If $\bar{x}$ is a local minimizer, then $d^T \nabla f(\bar{x}) \geq 0$ for all $d \in T(\bar{x})$.

(b) If $\bar{x}$ is a local minimizer, then $\bar{d} = 0$ is a minimizer of the problem

$$\min_d \ d^T \nabla f(\bar{x}), \quad \text{s.t. } d \in T(\bar{x}).$$

(c) Let x^* be a feasible point. If there exists $\eta > 0$ such that

$$d^T \nabla f(x^*) \geq \eta \|d\|, \ \forall d \in T(x^*).$$

Then x^* is a strict local minimizer.

6. (4+4+5+4+5+5+3=30). Let $f : \mathbb{R}^n \to \mathbb{R}$ be μ-strongly convex and L-smooth ($0 < \mu < L$). Consider the following algorithm:

Initialize $y^0 = x^0$.

for $k = 0, 1, 2, \cdots$ **do**

$$\begin{aligned} x^{k+1} &= y^k - \frac{1}{L} \nabla f\left(y^k\right) \\ y^{k+1} &= x^{k+1} + \frac{1-\theta}{1+\theta}\left(x^{k+1} - x^k\right), \end{aligned}$$

where $\theta=\sqrt{\frac{\mu}{L}}$.

Moreover define an estimate function sequence $\{\varphi_k:\mathbb{R}^n\to\mathbb{R}\}_{k\geq 0}$ as follows:

$$\begin{aligned}\varphi_0(x)&=f\left(x^0\right)+\frac{\mu}{2}\left\|x-x^0\right\|^2;\\ \varphi_{k+1}(x)&=(1-\theta)\,\varphi_k(x)+\theta\left[f\left(y^k\right)+\left\langle\nabla f\left(y^k\right),x-y^k\right\rangle+\frac{\mu}{2}\left\|x-y^k\right\|^2\right]\text{ for }k=0,1,2,\cdots\end{aligned}$$

Show that

(a) Use the strong convexity of f, we have

$$\varphi_{k+1}(x)\leq(1-\theta)\,\varphi_k(x)+\theta f(x).$$

(b) Use (a) by induction, we have

$$\varphi_k(x)\leq\left[1-(1-\theta)^k\right]f(x)+(1-\theta)^k\,\varphi_0(x).$$

(c) Observe that φ_k has the quadratic form for any $k=0,1,2,\cdots$. Define $\varphi_k^*=\min_x\varphi_k(x)$. Then

$$\begin{aligned}\varphi_0^*&=f\left(x^0\right),\\ \varphi_{k+1}^*&=(1-\theta)\,\varphi_k^*+\theta f\left(y^k\right)-\frac{1}{2L}\left\|\nabla f\left(y^k\right)\right\|^2+\theta\,(1-\theta)\left[\left\langle\nabla f\left(y^k\right),z^k-y^k\right\rangle+\frac{\mu}{2}\left\|z^k-y^k\right\|^2\right],\end{aligned}$$

where

$$\begin{aligned}z^0&=x^0;\\ z^{k+1}&=(1-\theta)\,z^k+\theta\left(y^k-\frac{1}{\mu}\nabla f\left(y^k\right)\right).\end{aligned}$$

Hint: Suppose φ_k has the form $\varphi_k(x)=\varphi_k^*+\frac{\mu}{2}\left\|x-z^k\right\|^2$ and by induction.

(d) By the L-smoothness of f, we have

$$f\left(x^{k+1}\right)\leq f\left(y^k\right)-\frac{1}{2L}\left\|\nabla f\left(y^k\right)\right\|^2.$$

(e) Use the recursions of z^k in (c), we have

$$z^k=\frac{1+\theta}{\theta}y^k-\frac{1}{\theta}x^k.$$

(f) Use (c), (d) and (e) by induction, we have $\varphi_k^*\geq f\left(x^k\right)$.

(g) Use (b) and (f) to show that

$$f\left(x^k\right)-f\left(x^*\right)\leq(1-\theta)^k\left[\varphi_0\left(x^*\right)-f\left(x^*\right)\right].$$