MA302 Functional Analysis, Midterm Exam
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Probelm 1. [25 pts] Let $C[0,1]$ be the space of real-valued continuous functions on the unit interval [0,1]

(i) For $f, g \in C[0,1]$, put $d(f,g) = \max_{x\in[0,1]} |f(x) - g(x)|$. Show that $d(\cdot,\cdot)$ is a metric on $C[0,1]$;

(ii) Show that $C[0,1]$ is complete with respect to d.

Problem 2. [25pts] Let $\mathcal{X}, \mathcal{Y}$ be two Banach spaces and $\mathcal{L} \subseteq \mathcal{L}(\mathcal{X}, \mathcal{Y})$ be a family of bounded linear operators from $\mathcal{X}$ to $\mathcal{Y}$. Suppose that there exists a subset $E \subseteq \mathcal{X}$ of the second category in $\mathcal{X}$ such that

$$\sup_{T\in\mathcal{L}} \|Tx\|_{\mathcal{Y}} < \infty, \ \forall x \in E$$

Show that

$$\sup_{T\in\mathcal{L}} \|T\| < \infty.$$

Problem 3. [30pts] Consider the space l^1 and l^∞ over $\mathbb{R}$,

$$l^1 := \{x = (\xi_i) : \sum_{i=1}^{\infty} |\xi_i| < \infty, \, \xi_i \in \mathbb{R}\}, \ \ \|x\|_1 = \sum_i |\xi_i|$$

$$l^\infty := \{x = (\xi_i) : \sup_{1\le i<\infty} |\xi_i| < \infty, \, \xi_i \in \mathbb{R}\}, \ \ \|x\|_\infty = \sup_i |\xi_i|$$

(i) For each $y = (\eta_i) \in l^1$, one can associate a bounded linear functional

$$l_y : l^\infty \to \mathbb{R},$$
$$x = (\xi_i) \to \sum_{i=1}^{\infty} \xi_i \eta_i.$$

Show that the map $y \to l_y$ is an isometry from l^1 into $(l^\infty)^*$.

(ii) Show that the above map $l^1 \to (l^\infty)^*$ is not surjective.

Problem 4. [20pts] Set $X = [0,1]$. For each $p \in (1, \infty]$, one can regard $L^p(X)$ as a subspace of $L^1(X)$ via the natural embedding $L^p(X) \hookrightarrow L^1(X), f \to f$.

(i) Is $\bigcup_{p\in(1,\infty]} L^p(X)$ a subset of first category in $L^1(X)$?

(ii) Construct an infinite dimensional closed subspace of $L^1(X)$ that is contained in $L^p(X)$, for all $p \in [1, \infty)$.