Algorithms for Convex Optimization
Suggested Solutions to A.1

1. (a) If $\mu = 1$, then

$$\begin{aligned}
&x \in \Omega_1 \\
\Leftrightarrow & \|x-a\| \leq \|x-b\| \\
\Leftrightarrow & \|x-a\|^2 \leq \|x-b\|^2 \\
\Leftrightarrow & 0 \geq \|x-a\|^2 - \|x-b\|^2 = -2(a-b)^T x + \|a\|^2 - \|b\|^2 .
\end{aligned}$$

   Then Ω_1 is a half-space.

   (b) If $0 \leq \mu < 1$, then

$$\begin{aligned}
&x \in \Omega_1 \\
\Leftrightarrow & 0 \geq \|x-a\|^2 - \mu^2 \|x-b\|^2 = (1-\mu^2)\left[\left\|x - \frac{a-\mu^2 b}{1-\mu^2}\right\|^2 - \frac{\mu^2}{(1-\mu^2)^2}\|a-b\|^2\right] \\
\Leftrightarrow & \left\|x - \frac{a-\mu^2 b}{1-\mu^2}\right\|^2 \leq \frac{\mu^2}{(1-\mu^2)^2}\|a-b\|^2 .
\end{aligned}$$

   Then Ω_μ is the closed ball with center $\frac{a-\mu^2 b}{1-\mu^2}$ and radius $\frac{\mu}{1-\mu^2}\|a-b\|$.

   (c) If $\mu > 1$, then Ω_μ is the complement of the open ball with center $\frac{a-\mu^2 b}{1-\mu^2}$ and radius $\frac{\mu}{1-\mu^2}\|a-b\|$.

   Quick Fact: Recall the definition of the so-called Apollonius circle.

5. Recall that $\mathrm{dom}(\partial f) \neq \varnothing$, then choose $\tilde{x} \in \mathrm{dom}(\partial f)$ and $g \in \partial f(\tilde{x})$, we have

$$f(x) + \alpha\|x\|^2 \geq g^T(x-\tilde{x}) + \alpha\|x\|^2 \geq \alpha\|x\|^2 - \|g\| \cdot \|x\| - \|g\| \cdot \|\tilde{x}\| .$$

6. For any $x \in \mathrm{int}(\mathrm{dom}\ f)$, we have

$$0 \leq \frac{f(x) - f(x^*) - g(x^*)^T(x-x^*)}{\|x-x^*\|} \leq \frac{(g(x)-g(x^*))^T(x-x^*)}{\|x-x^*\|} \leq \|g(x)-g(x^*)\| ,$$

   taking a limit as $x \to x^*$, we get f is differential at x^* with $\nabla f(x^*) = g(x^*)$. Then $\partial f(x^*) = \{\nabla f(x^*)\} = \{g(x^*)\}$.

7. Choose $g \in \cap_{x\in\Omega}\partial f$ and fixed $\bar{x} \in \Omega$, we have for any $x \in \Omega$.

$$\begin{aligned}
f(x) &\geq f(\bar{x}) + g^T(x-\bar{x}) \\
f(\bar{x}) &\geq f(x) + g^T(\bar{x}-x).
\end{aligned}$$

   Then $f(x) = f(\bar{x}) + g^T(x-\bar{x})$

8. Recall that both $\partial f'(x^*;\cdot)(0)$ and $\partial f(x^*)$ are closed and convex sets. Just show for any $d \in \mathbb{R}^n$,

$$\sigma_{\partial f'(x^*;\cdot)(0)}(d) = \sigma_{\partial f(x^*)}(d),$$

Claim 1: we get $\partial f(x^*) \subset \partial f'(x^*;\cdot)(0)$, and so $\sigma_{\partial f'(x^*;\cdot)(0)}(d) \geq \sigma_{\partial f(x^*)}(d)$.

Claim 2: we get $f'(x^*;d) \geq \sigma_{\partial f'(x^*;\cdot)(0)}(d)$.

Then Claim 1, Claim 2 and the max formula $f'(x;d) = \sigma_{\partial f(x^*)}(d)$ imply the desired conclusion.

9. For any $y \in \Omega$, we define $F_y : \mathbb{R}^n \to \overline{\mathbb{R}}$ with the form $F_y = \sum_{i=1}^m y_i f_i$. Then F_y is closed convex (think of why). Moreover, the sup-function $f = \sup_{y\in\Omega} F_y$ is closed convex (think of why).

10. It's easy to show that $\min f_{\max} \geq \min \varphi(\cdot,\lambda)$ for any $\lambda \in [0,1]$. Next we show that

$$\min f_{\max} = \sup_{0\leq\lambda\leq 1} \left(\min \varphi(\cdot,\lambda)\right).$$

Define $\varphi_{\min} : \mathbb{R} \to (-\infty,\infty]$ with the values $\varphi_{\min}(\lambda) = \min\varphi(\cdot,\lambda)$ if $0 \leq \lambda \leq 1$, while $\varphi_{\min}(\lambda) = \infty$ otherwise.

Define x_λ as the point with $\varphi(x_\lambda,\lambda) = \min\varphi(\cdot,\lambda)$ for any $0 \leq \lambda \leq 1$.

Claim 1: $\varphi_{\min}$ is univariate concave and continuous on $[0,1]$, and $\exists\lambda^* \in [0,1]$ such that

$$\varphi_{\min}(\lambda^*) = \varphi(x_{\lambda^*},\lambda^*) = \sup_{0\leq\lambda\leq 1} \varphi_{\min}(\lambda) = \sup_{0\leq\lambda\leq 1} \left(\min\varphi(\cdot,\lambda)\right).$$

Claim 2: Let $g_\lambda = f_1(x_\lambda) - f_2(x_\lambda)$ for any $0 \leq \lambda \leq 1$. Then we get

$$\varphi_{\min}(\lambda') \leq \varphi_{\min}(\lambda) + g_\lambda(\lambda' - \lambda)$$

and g_λ is decreasing on λ.

Claim 3: We have

$$g^+_{\lambda^*} \geq \max\{g_{\lambda^*},0\} \geq g_{\lambda^*} \geq \min\{g_{\lambda^*},0\} \geq g^-_{\lambda^*},$$

where $g^+_{\lambda^*} = \lim\limits_{\lambda\to\lambda^*+} g_\lambda = \inf\limits_{0\leq\lambda\leq\lambda^*} g_\lambda$ and $g^-_{\lambda^*} = \lim\limits_{\lambda\to\lambda^*-} g_\lambda = \sup\limits_{\lambda^*\leq\lambda\leq 1} g_\lambda$.

For any $\eta \in [0,1]$,

$$\begin{aligned}
\varphi_{\min}(\lambda^*) =&\eta \lim_{\lambda\to\lambda^*+} \varphi_{\min}(\lambda) + (1-\eta)\lim_{\lambda\to\lambda^*-} \varphi_{\min}(\lambda)\\
=&\eta \lim_{\lambda\to\lambda^*+} [\lambda f_1(x_\lambda) + (1-\lambda)f_2(x_\lambda)] + (1-\eta)\lim_{\lambda\to\lambda^*-} [\lambda f_1(x_\lambda) + (1-\lambda)f_2(x_\lambda)]\\
=&\eta \lim_{\lambda\to\lambda^*+} [f_2(x_\lambda) + \lambda g_\lambda] + (1-\eta)\lim_{\lambda\to\lambda^*-} [f_2(x_\lambda) + \lambda g_\lambda]\\
=&\lambda^*\left[\eta g^+_{\lambda^*} + (1-\eta)g^-_{\lambda^*}\right] + \eta\lim_{\lambda\to\lambda^*+} f_2(x_\lambda) + (1-\eta)\lim_{\lambda\to\lambda^*-} f_2(x_\lambda)\\
\geq&\lambda^*\left[\eta g^+_{\lambda^*} + (1-\eta)g^-_{\lambda^*}\right] + f_2(x_{\lambda^*}).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\varphi_{\min}(\lambda^*) =&\eta \lim_{\lambda\to\lambda^*+} \varphi_{\min}(\lambda) + (1-\eta)\lim_{\lambda\to\lambda^*-} \varphi_{\min}(\lambda)\\
=&\eta \lim_{\lambda\to\lambda^*+} [\lambda f_1(x_\lambda) + (1-\lambda)f_2(x_\lambda)] + (1-\eta)\lim_{\lambda\to\lambda^*-} [\lambda f_1(x_\lambda) + (1-\lambda)f_2(x_\lambda)]\\
=&\eta \lim_{\lambda\to\lambda^*+} [f_1(x_\lambda) - (1-\lambda)g_\lambda] + (1-\eta)\lim_{\lambda\to\lambda^*-} [f_1(x_\lambda) - (1-\lambda)g_\lambda]\\
=&-(1-\lambda^*)\left[\eta g^+_{\lambda^*} + (1-\eta)g^-_{\lambda^*}\right] + \eta\lim_{\lambda\to\lambda^*+} f_1(x_\lambda) + (1-\eta)\lim_{\lambda\to\lambda^*-} f_1(x_\lambda)\\
\geq&-(1-\lambda^*)\left[\eta g^+_{\lambda^*} + (1-\eta)g^-_{\lambda^*}\right] + f_1(x_{\lambda^*}).
\end{aligned}$$

(a) Case I: If $g_{\lambda^*}^- = g_{\lambda^*}^+$.
Then $g_{\lambda^*}^- = g_{\lambda^*}^+ = 0$ and so

$$\varphi_{\min}(\lambda^*) \geq \max\{f_1(x_{\lambda^*}), f_2(x_{\lambda^*})\} = f(x_{\lambda^*}) \geq \min f_{\max}.$$

(b) Case II: If $g_{\lambda^*}^- < g_{\lambda^*}^+$.
Then pick $\eta = \frac{-g_{\lambda^*}^-}{g_{\lambda^*}^+ - g_{\lambda^*}^-}$ (check $0 \leq \eta \leq 1$) and so

$$\varphi_{\min}(\lambda^*) \geq \max\{f_1(x_{\lambda^*}), f_2(x_{\lambda^*})\} = f(x_{\lambda^*}) \geq \min f_{\max}.$$

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11. (a) i. $R(\Omega)$ is a cone. If $d \in R(\Omega)$, then for any $\beta \geq 0$ we have

$$\Omega + \alpha(\beta d) = \Omega + (\alpha\beta) d \subset \Omega, \quad \alpha \geq 0,$$

where the last equality holds since $d \in R(\Omega)$.

ii. $R(\Omega)$ is convex. For any $d_1, d_2 \in R(\Omega)$ and $0 \leq \lambda \leq 1$, just show

$$w + \alpha[\lambda d_1 + (1-\lambda) d_2] \in \Omega, \quad \forall w \in \Omega, \alpha \geq 0. \tag{*}$$

Notice that $w + \alpha[\lambda d_1 + (1-\lambda) d_2] = \lambda(w + \alpha d_1) + (1-\lambda)(w + \alpha d_2)$, $w + \alpha d_1, w + \alpha d_2 \in \Omega$ and Ω is convex, then $(*)$ holds.

iii. $R(\Omega)$ is closed. If $d \leftarrow \{d^k\} \subset R(\Omega)$, then

$$w + \alpha d^k \in \Omega, \quad \forall w \in \Omega, \alpha \geq 0, k = 1, 2, \cdots.$$

Observing that $w + \alpha d^k \to w + \alpha d$ as $k \to \infty$ and Ω is closed, $w + \alpha d \in \Omega$ holds. Hence $d \in R(\Omega)$.

(b) i. Sufficiency. For any $\bar{w} \in \Omega$, we show that $\bar{w} + \alpha d \in \Omega$ holds for any $\alpha \geq 0$. Fix $\bar{w}$ and α, we show that

$$\bar{w} + \alpha d + \frac{w - \bar{w}}{k} \in \Omega \quad \forall k.$$

Actually,

$$\bar{w} + \alpha d + \frac{w - \bar{w}}{k} = \left(1 - \frac{1}{k}\right)\bar{w} + \frac{1}{k}(w + k\alpha d),$$

and $w + k\alpha d \in \Omega$. Then $\bar{w} + \alpha d \in \Omega$ since Ω is closed. Convex

ii. Necessity.

(c) $d \in R(\cap_i \Omega_i) \Leftrightarrow$ for some $w^* \in \cap_i \Omega_i$, $w^* + \alpha d \in \cap_i \Omega_i \Rightarrow$ for any i, we have $w^* \in \Omega_i$ and $w + \alpha d \in \Omega_i$.
$d \in \cap_i R(\Omega_i) \Rightarrow$ choose $w^* \in \cap_i \Omega_i$, for any i we have $w^* + \alpha d \in \Omega_i \Rightarrow w^* + \alpha d \in \cap_i \Omega_i \Rightarrow d \in R(\cap_i \Omega_i)$.

(d) i. Necessity. Pick $w \in \Omega$ and set $w^k = w + k d_u$, then for k sufficiently large,

$$\left\| \frac{w + k d_u}{\|w^k\|} - d_u \right\| \leq \frac{\|w\|}{\|w^k\|} + \left| \frac{k}{\|w^k\|} - 1 \right|. \to 0$$

Then we show $\frac{\|w^k\|}{k} \to 1$ as $k \to \infty$. Actually,

$$\frac{\|w^k\|^2}{k^2} = \frac{\langle w + k d_u, w + k d_u \rangle}{k^2} = \frac{w^T w + 2k w^T d_u + k^2}{k^2} \to 1.$$

ii. Sufficiency. Fix $w \in \Omega$ and $\alpha \geq 0$, just show that $w + \alpha d_u \in \Omega$. Actually,

$$w + \alpha d_u \leftarrow \left(1 - \frac{\alpha}{\|w^k\|}\right) w + \frac{\alpha}{\|w^k\|} w^k \in \Omega.$$

13. (a)
    (b) i. Necessity.
        ii. Sufficiency. If there exists x^* such that $f(x^*) < f(\bar{x})$, consider the function $\varphi_{x^*-\bar{x}}$, let $0 < \alpha < 1$,

$$\begin{aligned}\varphi_{x^*-\bar{x}}(\alpha) =& f(\bar{x} + \alpha(x^* - \bar{x})) = f((1-\alpha)\bar{x} + \alpha x^*) \\ \leq& (1-\alpha) f(\bar{x}) + \alpha f(x^*) \\ <& f(\bar{x}) = \varphi_{x^*-\bar{x}}(0).\end{aligned}$$

Then 0 is not a local minimizer of $\varphi_{x^*-\bar{x}}$.

14. (a) There exists $\delta > 0$ such that for any $x \in \mathbb{B}_\delta(\bar{x}) \cap \Omega$,

$$f(x) \geq f(\bar{x}).$$

Next we show for any $z \in \mathbb{B}_{\frac{\delta}{2}}(\bar{x})$, $f_L(z) \geq f_L(\bar{x}) = f(\bar{x})$. Actually, choose $z_\Omega \in P_\Omega(z)$ (think of why $P_\Omega(z) \neq \varnothing$), we have

$$\begin{aligned}f_L(z) - f_L(\bar{x}) =& f(z) - f(\bar{x}) + L\|z - z_\Omega\| \\ =& f(z) - f(z_\Omega) + L\|z - z_\Omega\| + f(z_\Omega) - f(\bar{x}) \\ \geq& -L_f\|z - z_\Omega\| + L\|z - z_\Omega\| + f(z_\Omega) - f(\bar{x}) \\ \geq& (L - L_f)\|z - z_\Omega\| + f(z_\Omega) - f(\bar{x}) \\ >& 0,\end{aligned}$$

where the last inequality holds since $z_\Omega \in \mathbb{B}_\delta(\bar{x}) \cap \Omega$, noticing that

$$\|z_\Omega - \bar{x}\| \leq \|z_\Omega - z\| + \|z - \bar{x}\| \leq 2\|z - \bar{x}\| \leq \delta.$$

(b) It's sufficient to show $\bar{x} \in \Omega$. If not, choose $\bar{x}_\Omega \in P_\Omega(\bar{x})$, then

$$\begin{aligned}f_L(\bar{x}) - f_L(\bar{x}_\Omega) =& f(\bar{x}) - f(\bar{x}_\Omega) + L\|\bar{x} - \bar{x}_\Omega\| \\ \geq& -L_f\|\bar{x} - \bar{x}_\Omega\| + L\|\bar{x} - \bar{x}_\Omega\| \\ >& 0.\end{aligned}$$

15. Let $(x, \alpha) \in \mathrm{epi}(f)$, then $(x, \alpha) \leftarrow \left\{\left(x, \alpha + \frac{1}{k}\right)\right\} \subset E_f$. So $(x, \alpha) \in \mathrm{cl}(E_f)$.

17. Choose $y \in \mathrm{int}(\Omega^\circ)$,

Case I: $y = 0$, then $\Omega = \{0\}$. Trivial.

Case II: $y \neq 0$, then there exists $\delta > 0$ such that $\mathbb{B}_\delta(y) \subset \mathrm{int}(\Omega^\circ)$. For any $0 \neq x \in \Omega$, notice that $y + \delta\frac{x}{\|x\|} \in \Omega^\circ$, i.e,

$$\langle y + \delta\frac{x}{\|x\|}, x\rangle \leq 0.$$

i.e,

$$\langle -y, x\rangle \geq \delta\|x\|.$$

Obviously, $\langle -y, 0\rangle \geq \delta\|0\|$. So $p = -y$.

18.
- (heuristic)
  Recall that $\partial f(x)$ is nonempty, closed, convex and bounded. Need to show a closed convex and bounded subset of $\mathbb{R}$ is a compact interval.
  Quick Facts:
  - Every closed set in $\mathbb{R}$ is a countable union of disjoint closed intervals.
  - Every convex set in $\mathbb{R}$ is a connected set.

  $f'(x;1) = \max\{g : g \in \partial f(x)\}$; while $f'(x;-1) = \max\{-g : g \in \partial f(x)\} = -\min\{g : g \in \partial f(x)\}$, so $\partial f(x) = [-f'(x;-1), f'(x;1)]$.
- (formal)

$$\begin{aligned} g \in \partial f(x) &\Leftrightarrow f(y) - f(x) \geq g(y-x), \quad \forall y \neq x \\ &\Leftrightarrow g \leq \varphi(y) \quad \forall y > x \text{ and } g \geq \varphi(y) \quad \forall y < x \\ &\Leftrightarrow \sup_{(-\infty,x)} \varphi(y) \leq g \leq \inf_{(x,\infty)} \varphi(y). \end{aligned}$$

where

$$\varphi(y) = \frac{f(y) - f(x)}{y - x}.$$

Notice that φ is increasing in (x, ∞) and decreasing in $(-\infty, x)$. So

$$\begin{aligned} \sup_{(-\infty,x)} \varphi(y) = \lim_{y \to x^-} \varphi(y) &= \lim_{y \to x^-} \frac{f(y) - f(x)}{y - x} \quad \text{let } y = x - \alpha \\ &= \lim_{\alpha \to 0+} \frac{f(x-\alpha) - f(x)}{-\alpha} = -f'(x;-1). \end{aligned}$$

Similarly,

$$\inf_{(x,\infty)} \varphi(y) = f'(x;1).$$