Algorithms for Convex Optimization
Assignment 3 Part I

1. Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ be proper closed and convex. Suppose that f^* is proper and σ-strongly convex ($\infty > \sigma > 0$). Show that
   (a) f is $\frac{1}{\sigma}$-smooth.
   (b) there exists a convex function g such that
   $$f = g \square \left(\frac{1}{2\sigma} \|\cdot\|^2 \right).$$

2. Define the inverse mapping of set-valued mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ by $F^{-1} : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ with the values
   $$F^{-1}(x) := \{y \in \mathbb{R}^n \mid x \in F(y)\} \text{ for any } x \in \mathbb{R}^m.$$
   Moreover the sum of F and $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is denoted as the set-valued mapping $F + G : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ with the values
   $$(F + G)(x) := \{y \in \mathbb{R}^m \mid \exists y_1 \in F(x), y_2 \in G(x) \text{ s.t. } y = y_1 + y_2\}.$$
   Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ be proper closed and convex. Show that
   (a)
   $$\operatorname{prox}_f = (I + \partial f)^{-1},$$
   where I is the identity operator on $\mathbb{R}^n$.
   (b) For any $\mu > 0$,
   $$\nabla M_f^\mu = \frac{1}{\mu} \left(I - (I + \mu \partial f)^{-1} \right).$$

3. Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ be proper closed and convex. Applying T2(a) and the firm nonexpansivity of the prox operator to show the monotonicity of ∂f, which was defined in Assignment 2, T9.

4. Let $f : \mathbb{R}^n \to \mathbb{R}$ be
   $$f(x) = \frac{1}{2} x^T A x + b^T x + c,$$
   where $A \in \mathbb{R}^{n \times n}$, $b \in \mathbb{R}^n$ and $c \in \mathbb{R}$. Show that
   (a) If A is symmetric and positive semidefinite, then
   $$\operatorname{prox}_f(x) = (A + I)^{-1}(x - b).$$
   (b) If A is symmetric (not necessarily positive semidefinite), then find the values of $\lambda \in \mathbb{R}_+$ when $\operatorname{prox}_{\lambda f}(x)$ is a singleton for any $x \in \mathbb{R}^n$, and give the expression of $\operatorname{prox}_{\lambda f}$ in this case.

5. Let $g : \mathbb{R}^n \to \overline{\mathbb{R}}$ be proper and σ-strongly convex. Let

$$f(x) = g(x) - \frac{c}{2} \|x\|^2 + \langle e, x \rangle,$$

where $c < \sigma$. Then give the expression of prox_f.

6. Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ be proper, closed (not necessarily convex). If f is bounded below, show that
   (a) $-\infty < M_f^\mu < \infty$ for any $\mu > 0$.
   (b) $M_f^\mu \le f$ for any $\mu > 0$.

7. Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ be proper, closed (not necessarily convex). Show the following properties are equivalent:
   (a) there exists $\mu > 0$ such that $M_f^\mu(x) > -\infty$ for some $x \in \mathbb{R}^n$.
   (b) there exists a polynomial function $q : \mathbb{R}^n \to \mathbb{R}$ with degree two or less such that $f \ge q$.
   (c) there exists $\mu > 0$ such that $f + \frac{1}{2\mu} \|\cdot\|^2$ is bounded from below.
   (d)

$$\liminf_{\|x\| \to \infty} \frac{f(x)}{\|x\|^2} > -\infty.$$

8. Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ be proper, closed and convex. Recalling the first prox theorem, we define $\varphi : \mathbb{R}_{++} \times \mathbb{R}^n \to \mathbb{R}^n$ by

$$\varphi(\mu, x) := \operatorname{prox}_{\mu f}(x).$$

Check the continuity of φ.

9. Consider the model

$$\min_{x \in \mathbb{R}^n} f(x) + g(x)$$

where f, g satisfy the standing assumption. If $g = 0$, the model reduces to the unconstrained smooth minimization problem

$$\min_{x \in \mathbb{R}^n} f(x),$$

while the proximal gradient method reduces to classical gradient method. Please consider the convergence analysis for the gradient method under both constant and backtracking stepsize rules.

10. Let $C \subset \mathbb{R}^n$ be closed convex. For any $x \in \mathbb{R}^n$ consider the normal cone $N_C(x)$ and the projection $P_C(x)$. Given $\bar{x} \in C$, we have
   (a) discuss the relation of $N_C(\bar{x}) + \bar{x}$ and $\left\{x \in \mathbb{R}^n \middle| P_C(x) = \bar{x}\right\}$.
   (b) Define the proximal normal cone $N_C^{\text{prox}}(\bar{x})$ by

$$N_C^{\text{prox}}(\bar{x}) := \left\{v \in \mathbb{R}^n \middle| \operatorname{dist}(\bar{x} + \alpha v, C) = \alpha \|v\| \text{ for some } \alpha > 0\right\},$$

   discuss the relation of $N_C(\bar{x})$ and $N_C^{\text{prox}}(\bar{x})$.