MID-SEMESTER TEST: ABSTRACT ALGEBRA

(Duration: 120mins)

(1) (15 pts) Give a ring R and a subset I of R, where

$$R = \left\{ \begin{bmatrix} a_1 & a_2 \\ 0 & a_3 \end{bmatrix} \mid a_i \in \mathbb{Z} \right\}, \qquad I = \left\{ \begin{bmatrix} b_1 & b_2 \\ 0 & b_3 \end{bmatrix} \mid b_i \in 3\mathbb{Z} \right\}.$$

(i) (6 pts) Characterize invertible elements and zero-divisors of R.
(ii) (3 pts) Prove that I is an ideal o f R.
(iii) (3 pts) Describe the elements of the factor ring R/I.
(iv) (3 pts) How many elements are there in R/I?

(2) (15 pts) Let $R = \{a_0, a_1, \dots, a_n\}$ be an integral domain. Prove the following statements:
(i) (5 pts) if $0 \neq a \in R$, then $aR = R$, and a is invertible;
(ii) (5 pts) for any integer $m \mid n$ and any element $a \in R$, we have $ma = 0$;
(iii) (5 pts) $n = p^e$ for some prime p and some positive integer e.

(3) (i) (15 pts) List all subgroups and normal subgroups of
(a) (5 pts) the dihedral group $D_{2n} = \langle r, s | r^n = s^2 = 1, r^s = r^{-1} \rangle$.
(b) (5 pts) the *quaternion group* $Q_8 = \langle a, b | a^4 = 1, b^2 = a^2, a^b = a^{-1} \rangle$.
(c) (3 pts) $Q_8 \times \mathbb{Z}_2$.
(ii) (2 pts) Find a subgroup of $Q_8 \times \mathbb{Z}_4$ which is not normal.

(4) (20 pts) Let R be a commutative ring with identity 1.
(i) (5 pts) Show that, for any positive integer n and any elements $a, b \in R$, there exists an element c such that $(a - b)c = a^n - b^n$.
(ii) (3 pts) Let $a \in R$ be a nilpotent element (namely, $a^n = 0$ for some positive integer n). Prove that, if j is a positive integer, then $1 - a^j$ is invertible, and find its inverse.
(iii) (5 pts) Prove that, if a, b are nilpotent elements of R, then $a + b$ is nilpotent.
(iv) (3 pts) Let I be the set of all nilpotent elements of R. Prove that I is an ideal.
(v) (4 pts) Is there any nilpotent element in the factor ring R/I ? why ?

(5) (20 pts) Let $J = \mathbb{Z}[\sqrt{-5}] = \{x + y\sqrt{-5} \mid x, y \in \mathbb{Z}\}$. Prove the following statements:
(i) (4 pts) J has exactly two invertible elements: $1, -1$;
(ii) (4 pts) 2 divides $(1 + \sqrt{-5})(1 - \sqrt{-5})$, but 2 divides none of $(1 + \sqrt{-5})$ and $(1 - \sqrt{-5})$;
(iii) (4 pts) if $2 = ab$ with $a, b \in J$, then one of a, b is invertible, namely, 2 is an irreducible;
(iv) (4 pts) J is an integral domain, but J is not a unique factorization domain;
(v) (4 pts) Is J a principal ideal domain ? why ?

(6) (15 pts) Prove the following statements.
(i) (5 pts) Prove that a finite extension of $\mathbb{F}_p$ is a simple extension.
(ii) (7 pts) Find a number α such that $\mathbb{Q}(\sqrt{2}, \sqrt{3}) = \mathbb{Q}(\alpha)$.
(iii) (3 pts) Prove that a finite extension of $\mathbb{Q}$ is a simple extension.