Suggested Solutions to A.1

1. *Proof.* Notice that we are already familiar with how to justify Part I of this theorem, that is to say, we have a procedure which can update some feasible solution x to another feasible solution $x - \epsilon y$. The only issue is how to show $x - \epsilon y$ is optimal if x is optimal feasible.

   Just show $c^T(x - \epsilon y) \le c^T z$ does hold for any feasible solution z. Notice that $c^T x \le c^T z$ holds for any feasible solution z, so just show $c^T(x - \epsilon y) \le c^T x$ or equivalently, $c^T y \ge 0$.
   Consider the following two cases.

   Case I: $y \ge 0$.

   Then $x + y$ is still feasible, so $c^T(x + y) \ge c^T x$, ie. $c^T y \ge 0$.

   Case II: $\exists\, y_1, \cdots, y_p$ such that some $y_i < 0$.

   Then let $\delta = min\{-\frac{x_i}{y_i} | i = 1, \cdots, p, y_i < 0\}$. Obviously, $x + \delta y$ is feasible, so $c^T(x + \delta y) \ge c^T x$, ie. $c^T y \ge 0$.
   Everything is done now. □

2. *Proof.* Suppose the LP under consideration has the form

$$\text{minimize } c^T x \text{ subject to } Ax = b,\ x \ge 0.$$

   Then we derive that $\bar{x} + \text{span}\left(\left\{\delta^{(j)}\right\}_{j\in\mathbb{J}}\right)$ is the solution set of the linear system $Ax = b$ by classical linear algebra. Hence for any $y \in F$, there exists $\{\alpha_j\}_{j\in\mathbb{J}}$ such that

$$y = \bar{x} + \sum_{j\in\mathbb{J}} \alpha_j \delta^{(j)}.$$

   noticing $Ay = b$ and meanwhile, $y \ge 0$ implies that $\alpha_j \ge 0$ for $j \in \mathbb{J}$.

   Hence $F \subseteq \bar{x} + \text{cone}\left(\left\{\delta^{(j)}\right\}_{j\in\mathbb{J}}\right)$. □

3. *Proof.* (i). Notice the fact that for any $x \in \mathbb{R}^n$, we have $x = \sum_{i=1}^n \left(e_i^T x\right) e_i$ and

$$e^T x = e^T \sum_{i=1}^n \left(e_i^T x\right) e_i = \sum_{i=1}^n \left(e_i^T x\right)\left(e^T e_i\right) = \sum_{i=1}^n \left(e_i^T x\right).$$

   Hence,

$$x \in \Delta_n \iff \begin{cases} e_i^T x \ge 0 \text{ for } i = 1, \cdots, n \\ \sum_{i=1}^n e_i^T x = 1 \end{cases} \iff x = \sum_{i=1}^n \left(e_i^T x\right) e_i \in \text{convex}\left(\{e_1, e_2, \cdots, e_n\}\right).$$

   (ii). WLOG, assume that $\alpha^2 = \max\left\{\alpha^1, \alpha^2, \cdots, \alpha^n\right\}$.
   On one hand, $\left(\alpha^1, \alpha^2, \cdots, \alpha^n\right) e_2 = \alpha^2$ implies that LHS $\le$ RHS .
   On the other hand, for any $x \in \Delta_n$ and so $\sum_{i=1}^n \left(e_i^T x\right) = 1$, we have

$$\begin{aligned}\left(\alpha^1, \alpha^2, \cdots, \alpha^n\right) x &= \sum_{i=1}^n \alpha^i \left(e_i^T x\right) \\ &= \sum_{i=1}^n \left(\alpha^i - \alpha^2\right)\left(e_i^T x\right) + \alpha^2 \sum_{i=1}^n \left(e_i^T x\right) \\ &\le \alpha^2,\end{aligned}$$

where the last inequality holds since $x \in \Delta_n$, or equivalently, $e_i^T x \geq 0$ for $i = 1, \cdots, n$ and $\sum_{i=1}^n e_i^T x = 1$. Hence, RHS $\leq$ LHS .

$\square$

4. *Proof.* x^* globally minimizes f over $\Omega \Leftrightarrow f(x^*) \leq f(x),\ \forall x \in \Omega \Rightarrow f(x^*) \leq f(x),\ \forall x \in \Omega \Leftrightarrow x^*$ globally minimizes f over Ω. $\square$

5. *Proof.* It's sufficient to show $\exists \varepsilon > 0$ such that $f(x^*) < f(x)$ for all $x \in \mathbb{B}(x^*, \varepsilon) \subset \Omega'$. $\square$

6. *Proof.* Let $\bar{d} \in \mathbb{R}^n$ be a feasible direction of Ω at x^*. Set $x = (x_1, \cdots, x_n)^T$ and $d = (d_1, \cdots, d_n)$. Recall the 3rd-order Taylor expansion,

$$f\left(x^* + \alpha \bar{d}\right) = f(x^*) + \alpha D_f(x^*)(\bar{d}) + \frac{\alpha^2}{2} D_f^2(x^*)(\bar{d}, \bar{d}) + \frac{\alpha^3}{6} D_f^3(x^*)\left(\bar{d}, \bar{d}, \bar{d}\right) + o\left(\alpha^3\right),$$

$\square$

where $D_f(x^*) : \mathbb{R}^n \to \mathbb{R}$ has the form:

$$D_f(x^*)(d) = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(x^*) d_i = d^T \nabla f(x^*), \quad \forall d \in \mathbb{R}^n.$$

$D_f^2(x^*) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ has the form:

$$D_f^2(x^*)(d^1, d^2) = \sum_{i,j=1}^n \frac{\partial^2 f}{\partial x_i x_j}(x^*) d_i^1 d_j^2 = \left(d^1\right)^T \nabla^2 f(x^*) d^2, \quad \forall d^1, d^2 \in \mathbb{R}^n.$$

$D_f^3(x^*) : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ has the form:

$$D_f^3(x^*)(d^1, d^2, d^3) = \sum_{i,j,k=1}^n \frac{\partial^3 f}{\partial x_i x_j x_k}(x^*) d_i^1 d_j^2 d_k^3, \quad \forall d^1, d^2, d^3 \in \mathbb{R}^n.$$

Then we state the third-order necessary condition: Let x^* be a local minimizer of f over Ω and d is a feasible direction at x^*. If $d^T \nabla f(x^*) = 0$ and $d^T \nabla^2 f(x^*) d = 0$, then

$$D_f^3(x^*)(d, d, d) \geq 0,$$

where $D_f^3(x^*) : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is the third-order derivative of f at x^*.

7. *Proof.* If $x^* \in \operatorname{int}(\Omega)$ is a local minimizer, then

$$0 = \nabla f(x^*) = -\left(\frac{1}{x_1}, \cdots, \frac{1}{x_n}\right)^T.$$

$\square$

8. *Proof.* Let $D(x)$ be the feasible directions of Ω at x. Then

$$D(x) \bigcup \left\{(0,0)^T\right\} = \begin{cases} \mathbb{R}^2 & \text{if } x_1 + x_2 < 1, x_1 > 0, x_2 > 0 \\ \mathbb{R} \times \mathbb{R}^+ & \text{if } x_1 + x_2 < 1, x_1 > 0, x_2 = 0 \\ \mathbb{R}_+ \times \mathbb{R} & \text{if } x_1 + x_2 < 1, x_1 = 0, x_2 > 0 \\ \mathbb{R}_+^2 & \text{if } x_1 + x_2 < 1, x_1 = 0, x_2 = 0 \\ \left\{(d_1, d_2)^T \middle| d_1 + d_2 \leq 0\right\} & \text{if } x_1 + x_2 = 1, x_1 > 0, x_2 > 0 \\ \left\{(d_1, d_2)^T \middle| d_1 + d_2 \leq 0, d_1 \leq 0, d_2 \geq 0\right\} & \text{if } x_1 + x_2 = 1, x_1 > 0, x_2 = 0 \\ \left\{(d_1, d_2)^T \middle| d_1 + d_2 \leq 0, d_1 \geq 0, d_2 \leq 0\right\} & \text{if } x_1 + x_2 = 1, x_1 = 0, x_2 > 0 \\ \left\{(0,0)^T\right\} & \text{if } x_1 + x_2 = 1, x_1 = 0, x_2 = 0 \end{cases}$$

We want to find a feasible point x such that $d^T \nabla f(x) = -3d_1 - 2d_2 \geq 0$ for all $d \in D(x)$.

(a) If $x_1 + x_2 < 1, x_1 > 0, x_2 > 0$, negative. (pick $d_1 = 1, d_2 = 0$).

(b) If $x_1 + x_2 < 1, x_1 > 0, x_2 = 0$, negative. (pick $d_1 = 1, d_2 = 0$).

(c) If $x_1 + x_2 < 1, x_1 = 0, x_2 > 0$, negative. (pick $d_1 = 1, d_2 = 0$).

(d) If $x_1 + x_2 < 1, x_1 = 0, x_2 = 0$, negative. (pick $d_1 = 1, d_2 = 0$).

(e) If $x_1 + x_2 = 1, x_1 > 0, x_2 > 0$, negative. (pick $d_1 = -1, d_2 = 2$).

(f) If $x_1 + x_2 = 1, x_1 > 0, x_2 = 0$, bingo. Notice that $0 \leq d_2 \leq -d_1$ and so $0 \geq -2d_2 \geq 2d_1$. Then $-3d_1 - 2d_2 \geq -d_1 \geq 0$.

(g) If $x_1 + x_2 = 1, x_1 > 0, x_2 = 0$, negative. (pick $d_1 = 1, d_2 = -1$).

(h) If $x_1 + x_2 = 1, x_1 = 0, x_2 = 0$, negative.

In conclusion, $(1, 0)^T$ is the unique optimal solution. □

9. *Proof.* (d). For any $\{d^k\}_{k\geq 0} \subseteq T(x)$ with $d^k \to d$ as $k \to \infty$, just show that $d \in T(x)$.
Notice that for any $k \geq 0$,
there exist $\{d^{k,n}\}_{n\geq 0} \to d^k$ and a positive sequence $\{t_{k,n}\}_{n\geq 0} \downarrow 0$ satisfying $x + t_{k,n}d^{k,n} \in \Omega$ for any $n \geq 0$.
Setting $\epsilon_0 = 1$, there exists k_0 such that

$$\left\|d - d^{k_0}\right\| < 1.$$

moreover, there exists n_0 such that

$$\left\|d^{k_0} - d^{k_0,n_0}\right\| < 1, \quad |t_{k_0,n_0}| < 1 \text{ and } x + t_{k_0,n_0}d^{k_0,n_0} \in \Omega.$$

Setting $\epsilon_1 = \frac{1}{2}$, there exists $k_1 > k_0$ such that

$$\left\|d - d^{k_1}\right\| < \frac{1}{2}.$$

moreover, there exists n_1 such that

$$\left\|d^{k_1} - d^{k_1,n_1}\right\| < \frac{1}{2} \quad |t_{k_1,n_1}| < \frac{1}{2} \text{ and } x + t_{k_1,n_1}d^{k_1,n_1} \in \Omega.$$

...

Setting $\epsilon_i = \frac{1}{i}$, there exists $k_i > k_{i-1}$ such that

$$\left\|d - d^{k_i}\right\| < \frac{1}{i}.$$

moreover, there exists n_i such that

$$\left\|d^{k_i} - d^{k_i,n_i}\right\| < \frac{1}{i} \quad |t_{k_i,n_i}| < \frac{1}{i} \text{ and } x + t_{k_i,n_i}d^{k_i,n_i} \in \Omega.$$

That is to say, there exist $\{d^{k_i,n_i}\}_{i\geq 0} \to d$ and $\{t_{k_i,n_i}\}_{i\geq 0} \to 0^+$ with $x + t_{k_i,n_i}d^{k_i,n_i} \in \Omega$ for any $i \geq 0$. WLOG, we assume that $\{t_{k_i,n_i}\}_{i\geq 0} \downarrow 0$ and let $\bar{d^i} = d^{k_i,n_i}$, $t_{k_i,n_i} = \bar{t}_i$ for any $i \geq 0$. Then we have $\{\bar{d^i}\}_{i\geq 0} \to d$, $0 < \{\bar{t}_i\}_{i\geq 0} \downarrow 0$ with $x + \bar{t}_i\bar{d^i} \in \Omega$ for any $i \geq 0$.

□

10. *Proof.* (a). Let $d \in G_0(\bar{x})$, then
on one hand, for each $i \in I(\bar{x})$, $d^T\nabla g_i(\bar{x}) < 0$, then there exists $\varepsilon_i > 0$ such that $g_i(\bar{x} + \alpha d) < g(\bar{x})$ for any $0 < \alpha < \varepsilon_i$.
on the other hand, for each $i \notin I(\bar{x})$, $g_i(\bar{x}) < 0$, then there exists $\varepsilon_i > 0$ such that $g_i(\bar{x} + \alpha d) < g(\bar{x})$ for any $0 < \alpha < \varepsilon_i$.
Then let $\varepsilon = \min\{\varepsilon_i : i = 1, 2, \cdots, m\}$, we have $g_i(\bar{x} + \alpha d) < g_i(\bar{x})$ for any $0 < \alpha < \varepsilon$ and $i = 1, 2, \cdots, m$.
That is to say, $d \in D(x)$.

□