Chapter 1

1. (1) 1st order, nonlinear.

(2) 2nd order, linear, nonhomogeneous.

(3) 2nd order, nonlinear.

(4) 2nd order, nonlinear.

2. (1) This is an elliptic equation since $a_{11}a_{22} - a_{12}^2 = 5 - 4 = 1 > 0$.

(2) This is a parabolic equation since $a_{11}a_{22} - a_{12}^2 = 4 - 4 = 0$.

(3) This is a hyperbolic equation since $a_{11}a_{22} - a_{12}^2 = -3 - 1 = -4 < 0$.

3. We rewrite the PDE as

$$a_{11}u_{x_1x_1} + 2a_{12}u_{x_1x_2} + a_{22}u_{x_2x_2} + b_1u_{x_1} + b_2u_{x_2} + cu = 0, \tag{0.1}$$

and define $a_{12} = a_{21}$. Change independent variables by

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \tag{0.2}$$

where

$$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix},$$

is a "rotation matrix", *i.e.*, orthogonal matrix ($B^TB = I = BB^T$). In terms of the new independent variables y_1 and y_2, the second order terms in (0.1) are changed according to

$$a_{11}u_{x_1x_1} + 2a_{12}u_{x_1x_2} + a_{22}u_{x_2x_2} = \sum_{k,l=1}^{2} a_{kl}u_{x_kx_l} = \sum_{i,j=1}^{2} \left(\sum_{k,l=1}^{2} a_{kl}b_{ik}b_{jl} \right) u_{y_iy_j}.$$

Note that the coefficient $\sum\limits_{k,l=1}^{2} a_{kl}b_{ik}b_{jl}$ of $u_{y_iy_j}$ is just the ij-th element of the matrix BAB^T, where $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$.

If (0.1) is invariant under the change of variables (0.2), the coefficient of $u_{y_iy_j}$ should be equal to the coefficient of $u_{x_ix_j}$, or in matrix form, we have

$$BAB^T = A, \qquad \text{for any orthogonal matrix } B. \tag{0.3}$$

On the other hand, from Linear Algebra, we know that there exists an orthogonal matrix B, such that

$$BAB^T = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

Thus, $A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, *i.e*, $a_{12} = a_{21} = 0$.

Next we show $\lambda_1 = \lambda_2$. By (0.3),

$$BAB^T B = AB, \text{ or, } BA = AB, \qquad \text{for any orthogonal matrix } B,$$

from which it follows

$$\begin{pmatrix} \lambda_1 b_{11} & \lambda_2 b_{12} \\ \lambda_1 b_{21} & \lambda_2 b_{22} \end{pmatrix} = \begin{pmatrix} \lambda_1 b_{11} & \lambda_1 b_{12} \\ \lambda_2 b_{21} & \lambda_2 b_{22} \end{pmatrix}.$$

Therefore, we have $\lambda_2 b_{12} = \lambda_1 b_{12}$. Obviously, there exists an orthogonal matrix B such that $b_{12} \neq 0$, from which we can conclude that $\lambda_2 = \lambda_1$. We have shown that $a_{12} = 0,\ a_{11} = a_{22} = \lambda_1 = \lambda_2$.

By Chain Rule, the first order terms $b_1 u_{x_1} + b_2 u_{x_2}$ are changed to

$$\begin{aligned} b_1 u_{x_1} + b_2 u_{x_2} &= b_1 \left(\frac{\partial u}{\partial y_1} \frac{\partial y_1}{\partial x_1} + \frac{\partial u}{\partial y_2} \frac{\partial y_2}{\partial x_1} \right) + b_2 \left(\frac{\partial u}{\partial y_1} \frac{\partial y_1}{\partial x_2} + \frac{\partial u}{\partial y_2} \frac{\partial y_2}{\partial x_2} \right) \\ &= b_1 \left(\frac{\partial u}{\partial y_1} b_{11} + \frac{\partial u}{\partial y_2} b_{21} \right) + b_2 \left(\frac{\partial u}{\partial y_1} b_{12} + \frac{\partial u}{\partial y_2} b_{22} \right) \\ &= (b_1 b_{11} + b_2 b_{12}) \frac{\partial u}{\partial y_1} + (b_1 b_{21} + b_2 b_{22}) \frac{\partial u}{\partial y_2}. \end{aligned}$$

If (0.1) is invariant under the change of variables (0.2), then

$$\begin{cases} b_1 = b_1 b_{11} + b_2 b_{12}, \\ b_2 = b_1 b_{21} + b_2 b_{22}, \end{cases}$$

or, in matrix form,

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.$$

This means the vector $\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ is invariant under any rotation, thus we must have $\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \mathbf{0}$, *i.e*, $b_1 = b_2 = 0$. Now the proof is complete.

4. (1) Elliptic since $a_{11}a_{22} - a_{12}^2 = (1 + x^2)(1 + y^2) > 0$.

(2) $a_{11}a_{22} - a_{12}^2 = (1 + y)^2 \geq 0$. Parabolic on $y = -1$; elliptic otherwise.

(3) Parabolic since $a_{11}a_{22} - a_{12}^2 = e^{2x}e^{2y} - e^{2(x+y)} = 0$.

(4) $a_{11}a_{22} - a_{12}^2 = y$. Elliptic on $y > 0$; parabolic on $y = 0$ and hyperbolic otherwise.

(5) $a_{11}a_{22} - a_{12}^2 = xy$. Elliptic on $xy > 0$; parabolic on $xy = 0$ and hyperbolic otherwise.

(6) $a_{11}a_{22} - a_{12}^2 = -xy$. Elliptic on $xy < 0$; parabolic on $xy = 0$ and hyperbolic otherwise.

5. We have

$$\begin{aligned} u_t &= 2f'(x+2t) - 2g'(x-2t), \\ u_x &= f'(x+2t) + g'(x-2t), \\ u_{tt} &= 4f''(x+2t) + 4g''(x-2t), \\ u_{xx} &= f''(x+2t) + g''(x-2t). \end{aligned}$$

It is obvious that $u_{tt} - 4u_{xx} = 0$.

6. (1) Case 1: $A = 0$

Since the equation is hyperbolic, we have $B^2 - 4AC > 0$, then $B \neq 0$. (*) becomes

$$\left(u_x + \frac{C}{B}u_y\right)_y = 0.$$

It follows that

$$u_x + \frac{C}{B}u_y = f(x).$$

The characteristics of this equation are given by

$$\frac{dy}{dx} = \frac{C}{B},$$

hence

$$y = \frac{C}{B}x + M.$$

Now on a fixed characteristic curve (so constant M is fixed), we have

$$\frac{du}{dx} = u_x + \frac{C}{B}u_y = f(x),$$

from which we derive

$$u = F(x) + \tilde{M},$$

where F is antiderivative of f. The constant $\tilde{M}$ depends on the characteristic curve, and hence on M. Let $\tilde{M} = G(M)$ and solve for M, then we obtain that, in this case, the solutions to (*) are given by

$$u(x,y) = F(x) + G\left(y - \frac{C}{B}x\right).$$

Case 2: $A \neq 0$

Note that the equation $A\lambda^2 + B\lambda + C = 0$ has two different solutions, denoted by λ_1 and λ_2. We make the following change of variables,

$$\begin{cases} \xi = \lambda_1 x + y, \\ \eta = \lambda_2 x + y. \end{cases}$$

Direct calculations show that (*) now becomes

$$u_{\xi\eta} = 0,$$

which clearly has general solution $u = f(\xi) + g(\eta)$. Therefore, in this case, we obtain the general solutions of (*),

$$\begin{aligned} u(x,y) &= f(\lambda_1 x + y) + g(\lambda_2 x + y) \\ &= f\left(\frac{-B + \sqrt{B^2 - 4AC}}{2A} x + y\right) + g\left(\frac{-B - \sqrt{B^2 - 4AC}}{2A} x + y\right). \end{aligned}$$

(2) Case 1: $A = 0$

Note that we have $B^2 = 4AC = 0$, then in this case equation (*) reduces to

$$u_{yy} = 0,$$

which obviously has general solution

$$u(x,y) = f(x)y + g(x).$$

Case 2: $A \neq 0$

We make the following change of variables,

$$\begin{cases} \xi = -\frac{B}{2A} x + y, \\ \eta = -\frac{B}{2A} x - y. \end{cases}$$

It can be easily checked that under the new variables (*) becomes

$$u_{\eta\eta} = 0,$$

from which it follows

$$u(\xi, \eta) = \eta f(\xi) + g(\xi).$$

Therefore, the general solution to (*) is given by

$$u(x,y) = -\left(\frac{B}{2A} x + y\right) f\left(-\frac{B}{2A} x + y\right) + g\left(-\frac{B}{2A} x + y\right).$$