Operations Research Assignment 1

Linear programming

1. Show if an linear programming in standard form has an optimal feasible solution, then there exists an optimal BFS.

2. Consider the LP in standard form, let F be the feasible set and $\left\{\delta^{(j)}\right\}_{j\in\mathbb{J}}$ be the set of edge directions of an BFS $\bar{x}$. Show

$$F \subseteq \bar{x} + \text{cone}\left(\left\{\delta^{(j)}\right\}_{j\in\mathbb{J}}\right).$$

3. The unit simplex over $\mathbb{R}^n$ is denoted by Δ_n and is comprised by all nonnegative vectors whose components sum up to one:

$$\Delta_n := \left\{x \in \mathbb{R}^n \,\middle|\, x \geq 0,\, e^T x = 1\right\},$$

where e stands for the n-length column vector of all ones. Show

(i). $\Delta_n = \text{convex}\left(\{e_1, e_2, \cdots, e_n\}\right)$, where e_i is the n-length column vector whose ith component is one while all the others are zeros.

(ii). For any $\{\alpha_i\}_{i=1}^n \subset \mathbb{R}$, we have

$$\max\{\alpha_1, \alpha_2, \cdots, \alpha_n\} = \max_{x\in\Delta_n}\left(\alpha_1, \alpha_2, \cdots, \alpha_n\right)x.$$

Optimality Conditions

4. Show that if x^* is a global minimizer of f over Ω, and $x^* \in \Omega' \subset \Omega$. Then x^* is a global minimizer of f over Ω'.

5. If x^* is a local minimizer of f over Ω and is a interior point of Ω, $\Omega \subset \Omega'$. Then x^* is a local minimizer of f over Ω'.

6. Let $f : \mathbb{R}^n \to \mathbb{R}$ be of class C^3 and x^* is a local minimizer of f over a closed set $\Omega \subset \mathbb{R}^n$. Review the first-order necessary condition and the second-order necessary condition. Similarly, state and prove the third-order necessary condition.

7. Consider minimizing $f : \mathbb{R}^n \to \mathbb{R}$ over a closed set $\Omega \subset \mathbb{R}^n_{++}$ and

$$f(x) = -\sum_{i=1}^n \log\left(x_i\right) \text{ for } x = \left(x_1, x_2, \cdots, x_n\right)^T \in \Omega.$$

Then all optimal solutions lie on the boundary of Ω.

8. Consider the following linear programming:

$$\min -3x_1 - 2x_2 \quad \text{s.t. } x_1 + x_2 \leq 1, x_1 \geq 0, x_2 \geq 0.$$

Use the first-order necessary condition to find the unique optimality feasible solution.

9. Consider minimizing the differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ over a closed set Ω. For any $x \in \Omega$, the feasible directions of Ω at x is given by

$$D(x) := \left\{ d \in \mathbb{R}^n \middle| d \neq 0, \exists \varepsilon > 0 \text{ such that } x + \alpha d \in \Omega \text{ for all } 0 < \alpha < \varepsilon \right\},$$

the tangent direction of Ω at x is given by

$$T(x) := \left\{ d \in \mathbb{R}^n \middle| \exists \left\{ t_k, d^k \right\}_{k \geq 0} \subset \mathbb{R}_{++} \times \mathbb{R}^n \text{ with } t_k \downarrow 0, d^k \to d \text{ as } k \to \infty \text{ and } x + t_k d^k \in \Omega \text{ for all } k \geq 0 \right\}.$$

the descent directions of f at x is given by

$$F(x) = \left\{ d \in \mathbb{R}^n \middle| \exists \varepsilon > 0 \text{ such that } f(x + \alpha d) < f(x) \text{ for all } 0 < \alpha < \varepsilon \right\}.$$

Then

(a) $\bar{x} \in \Omega$ is a local minimizer if and only if $D(\bar{x}) \cap F(\bar{x}) = \varnothing$.

(b) For any $x \in \Omega$, $F_0(x) \subset F(x)$, where

$$F_0(x) := \left\{ d \in \mathbb{R}^n \middle| \nabla f(x)^T d < 0 \right\}.$$

(c) $D(\bar{x}) \cap F_0(\bar{x}) = \varnothing$ if $\bar{x} \in \Omega$ is a local minimizer.

(d) For any $x \in \Omega$, $T(x)$ is closed.

(e) For any $x \in \Omega$, $\mathrm{cl}(D(x)) \subset T(x)$.

(f) $T(\bar{x}) \cap F_0(\bar{x}) = \varnothing$ if $\bar{x} \in \Omega$ is a local minimizer.

10. Continue to the above question: If $\Omega := \left\{ x \in \mathbb{R}^n \middle| g_i(x) \leq 0 \text{ for } i = 1, 2, \cdots, m \right\}$, where $g_i : \mathbb{R}^n \to \mathbb{R}$ is differentiable for $i = 1, 2, \cdots, m$. For any $x \in \Omega$, the active index set at x is given by

$$I(x) := \left\{ i \middle| i \in \{1, 2, \cdots, m\}, g_i(x) = 0 \right\}.$$

Then

(a) For any $x \in \Omega$, $G_0(x) \subset D(x)$, where

$$G_0(x) := \left\{ d \in \mathbb{R}^n \middle| \nabla g_i(x)^T d < 0 \text{ for each } i \in I(x) \right\}.$$

(b) $G_0(\bar{x}) \cap F_0(\bar{x}) = \varnothing$ if $\bar{x} \in \Omega$ is a local minimizer.