MA303 偏微分方程 第五次作业

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Chapter 4

Problem 1: (5’，任选一种方法即可)

Solution:

Similar to the 2-D case, we present both formal and rigorous treatments.

<u>**Formal treatment-3D case**</u>

Notice that $\delta(\boldsymbol{x})$ is radially symmetric about the origin. Then it is reasonable to seek radially symmetric solution:

$$G_0(\boldsymbol{x}) = F(r), \quad r = |\boldsymbol{x}|.$$

Then we have

$$\Delta G_0(\boldsymbol{x}) = \frac{d^2F}{dr^2} + \frac{2}{r}\frac{dF}{dr} = \frac{1}{r^2}\frac{d}{dr}\Big(r^2\frac{dF}{dr}\Big).$$

Since the fundamental solution $G_0(\boldsymbol{x})$ satisfies $-\Delta G_0(\boldsymbol{x}) = \delta(\boldsymbol{x})$, then we must have

$$\frac{d}{dr}\Big(r^2\frac{dF}{dr}\Big) = -r^2\delta(\boldsymbol{x}),$$

$$\begin{aligned}
r^2\frac{dF}{dr} &= -\int_0^r r^2\delta(\boldsymbol{x})dr \\
&= -\frac{1}{4\pi}\int_0^\pi\int_0^{2\pi}\int_0^r r^2\delta(\boldsymbol{x})\sin\theta\ drd\phi d\theta \\
&= -\frac{1}{4\pi}\int_{|\boldsymbol{x}|<r}\delta(\boldsymbol{x})d\boldsymbol{x} \\
&= -\frac{1}{4\pi}\int_{\mathbb{R}^3}\delta(\boldsymbol{x})d\boldsymbol{x} \\
&= -\frac{1}{4\pi}.
\end{aligned}$$

Therefore we have the ODE $\frac{dF}{dr} = -\frac{1}{4\pi r^2}$ and its solution $F(r) = \frac{1}{4\pi r} + C$, where C is a constant. Since constant C is a trivial solution of Laplace equation $\Delta u = 0$, we simply take $C = 0$. Finally we get that

$$G_0(\boldsymbol{x}) = \frac{1}{4\pi|\boldsymbol{x}|}.$$

<u>**Rigorous treatment-3D case**</u>

Radially symmetric solutions of Laplace equation are of the form $F(r) = \frac{C}{r}$ (except at the origin), where C is any constant. We need to find an appropriate C so that it is the fundamental solution interpreted as (4.3.4) on Page 54. We smooth out the singularity of F at 0 by defining

$$G_\epsilon(r) = \frac{C}{\sqrt{r^2 + \epsilon^2}},$$

where ϵ is a positive constant. We compute

$$-\Delta G_\epsilon = -\Big(\frac{\partial^2 G_\epsilon}{\partial r^2} + \frac{2}{r}\frac{\partial G_\epsilon}{\partial r}\Big) = \frac{3C\epsilon^2}{(r^2+\epsilon^2)^{\frac{5}{2}}}.$$

We want to choose C so that the righthand side converges to $\delta(\boldsymbol{x})$ weakly and (4.3.4) on Page 54 holds. The integral of the righthand side on $\mathbb{R}^3$ is equal to

$$\begin{aligned}
\int_{\mathbb{R}^3} \frac{3C\epsilon^2}{(r^2+\epsilon^2)^{\frac{5}{2}}} d\boldsymbol{x} &= \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \frac{3C\epsilon^2}{(r^2+\epsilon^2)^{\frac{5}{2}}} r^2 \delta(\boldsymbol{x}) \sin\theta \ dr d\phi d\theta \\
&= 12\pi C\epsilon^2 \int_0^{\infty} \frac{r^2}{(r^2+\epsilon^2)^{\frac{5}{2}}} dr \qquad\qquad (r = \epsilon\tan\theta) \\
&= 12\pi C\epsilon^2 \int_0^{\frac{\pi}{2}} \frac{\epsilon^2\tan^2\theta}{\epsilon^5\sec^5\theta}\epsilon\sec^2\theta d\theta \\
&= 12\pi C \int_0^{\frac{\pi}{2}} \sin^2\theta\cos\theta \ d\theta \\
&= 4\pi C,
\end{aligned}$$

so conditions (i) and (iii) in Theorem 4.3.1 on Page 53 are satisfied with $A = 4\pi C$. To verify condition (ii), take a fixed $r_0 > 0$, we estimate

$$\begin{aligned}
\int_{|\boldsymbol{x}|>r_0} \frac{|3C\epsilon^2|}{(r^2+\epsilon^2)^{\frac{5}{2}}} d\boldsymbol{x} &= 12\pi|C|\epsilon^2 \int_{r_0}^{\infty} \frac{r^2}{(r^2+\epsilon^2)^{\frac{5}{2}}} dr \\
&\leq 12\pi|C|\epsilon^2 \int_{r_0}^{\infty} \frac{r^2}{r^5} dr,
\end{aligned}$$

which clearly converges to 0 as $\epsilon \to 0^+$. Now according to Theorem 4.3.1 on Page 53, in the 3D case, the fundamental solution is given by

$$G_0(\boldsymbol{x}) = \frac{1}{4\pi|\boldsymbol{x}|}$$

by choosing $C = \frac{1}{4\pi}$. □

Problem 2: (5’，注意审题 spherical 是球体，属于 3D 情形)

注：书上的公式（4.2.4）是基于（4.2.1）推导得到的，考虑的是 2D 情形，使用时应注意。

Solutions:

(a) Since the shell is spherical and the boundary condition is symmetric, the temperature distribution will be radially symmetric. Then we can write $u = u(r)$ and obtain

$$u^{''}(r) + \frac{2}{r}u^{'}(r) = 0 \Rightarrow u(r) = \frac{a}{r} + b.$$

From the boundary condition, $u(1) = 100$ and $u^{'}(2) = -\gamma$, we finally obtain

$$u(r) = \frac{4\gamma}{r} + 100 - 4\gamma,\ 1 \le r \le 2$$

(b) By (a), if $\gamma \ge 0$, the hottest temperature will be $100°C$ and the coldest temperature will be $100 - 2\gamma°C$; If $\gamma < 0$, the hottest temperature will be $100 - 2\gamma°C$ and the coldest temperature will be $100°C$.

(c) By (a), let $r = 2$. Then $\gamma = 40$. □

Problem 5: (5'× 5)

Solutions:

(1) Since u should be bounded in $\overline{\Omega}$, by (4.2.4) of the textbook, we have

$$u(r,\theta) = C_0 + \sum_{n=1}^{\infty} r^n (A_n \cos(n\theta) + B_n \sin(n\theta)).$$

By the boundary condition, $u(R,\theta) = A\cos\theta$. Hence, we have

$$u(r,\theta) = \frac{A}{R} r \cos\theta.$$

□

(2) Since Ω and the boundary condition is symmetric, we can write $u = u(r)$ and then

$$u^{''}(r) + \frac{1}{r}u^{'}(r) = 1 \Rightarrow (ru^{'})^{'} = r.$$

Hence, we have

$$ru^{'} = \frac{r^2}{2} + C \Rightarrow u = \frac{r^2}{4} + C\ln r + D$$

As u should be bounded, $C = 0$. Meanwhile, by the boundary condition, $u(R) = 0$. Finally we obtain

$$u = \frac{r^2 - R^2}{4}.$$

□

(3) By hint, we set

$$u(r,\theta) = A_0(r) + \sum_{n=1}^{\infty} (A_n(r)\cos(n\theta) + B_n(r)\sin(n\theta)).$$

Since $\Delta u = \frac{A}{2} r^2 \sin(2\theta)$, we have

$$\begin{cases} A_n^{''}(r) + \frac{1}{r}A_n^{'}(r) - \frac{n^2}{r^2}A_n(r) = 0, \ n = 0,1,2,... \\ B_n^{''}(r) + \frac{1}{r}B_n^{'}(r) - \frac{n^2}{r^2}B_n(r) = 0, \ n \neq 2, \end{cases}$$

and

$$B_2^{''}(r) + \frac{1}{r}B_2^{'}(r) - \frac{4}{r^2}B_2(r) = \frac{A}{2}r^2.$$

First we can obtain that

$$A_n(r) = \begin{cases} c_0 \ln r + d_0, \ n = 0 \\ c_n r^n + d_n r^{-n}, \ n = 1,2,... \end{cases}$$

and $B_n(r) = \hat{c}_n r^n + \hat{d}_n r^{-n}$ for $n \neq 2$. To find $B_2(r)$, we try to substitute $B_2(r) = Cr^\alpha$ into the corresponding ODE and get

$$C(\alpha^2 - 4)r^{\alpha-2} = \frac{A}{2}r^2.$$

Hence $\alpha = 4$ and $C = \frac{A}{24}$, meaning that

$$B_2(r) = \hat{c}_2 r^2 + \hat{d}_2 r^{-2} + \frac{A}{24}r^4.$$

Now since u is bounded and by the boundary condition $u(R,\theta) = 0$, we have

$$u(r,\theta) = (\hat{c}_2 r^2 + \frac{A}{24}r^4)\sin(2\theta) = 0, \ \hat{c}_2 R^2 + \frac{A}{24}R^4 = 0$$

$$\Rightarrow u(r,\theta) = \frac{A}{24}r^2(r^2 - R^2)\sin(2\theta).$$

□

(4) Let $u(x,y) = X(x)Y(y)$. Since $\Delta u = 0$, we have

$$-\frac{X''}{X} = \frac{Y''}{Y} = -\lambda.$$

By the boundary condition, we have

$$\begin{cases} Y''(y) + \lambda Y(y) = 0, \ 0 < y < \pi \\ Y'(0) = 0, \ Y'(\pi) = 0 \end{cases} \Rightarrow \begin{cases} \lambda_n = n^2 \\ Y_n(y) = \cos(ny) \end{cases}, \ n = 0, 1, 2, ...$$

Then we have $u(x, y) = \sum_{n=0}^{\infty} X_n(x) Y_n(y)$ with

$$X_n''(x) - n^2 X_n(x) = 0 \quad \Rightarrow \quad X_n(x) = \begin{cases} a_0 x + b_0, & n = 0 \\ a_n e^{nx} + b_n e^{-nx}, & n = 1, 2, ... \end{cases}$$

Since $u(0, y) = 0$, $X_n(0) = 0$ for every n. Hence we have

$$u(x, y) = a_0 x + \sum_{n=1}^{\infty} a_n (e^{nx} - e^{-nx}) \cos(ny).$$

Since

$$u(\pi, y) = \cos^2 y = a_0 \pi + \sum_{n=1}^{\infty} a_n (e^{nx} - e^{-nx}) \cos(ny)$$

we can calculate each coefficient a_n:

$$\int_0^{\pi} a_0 \pi dy = \int_0^{\pi} \cos^2 y dy = \int_0^{\pi} \frac{1 + \cos 2y}{2} dy = \frac{\pi}{2} \quad \Rightarrow \quad a_0 = \frac{1}{2\pi},$$

$$\int_0^{\pi} a_n (e^{nx} - e^{-nx}) \cos^2 ny dy = \int_0^{\pi} \cos^2 y \cos ny dy = \int_0^{\pi} \frac{1 + \cos 2y}{2} \cos ny dy = \begin{cases} \frac{\pi}{4} & \text{for } n = 2, \\ 0 & \text{for } n \neq 2. \end{cases}$$

$$\Rightarrow \quad a_n = \begin{cases} \frac{1}{2(e^{2\pi} - e^{-2\pi})} & \text{for } n = 2, \\ 0 & \text{for } n \neq 2. \end{cases}$$

Hence, we have

$$u(x, y) = \frac{1}{2\pi} x + \frac{e^{2x} - e^{-2x}}{2(e^{2\pi} - e^{-2\pi})} \cos 2y.$$

□

(5) Similar to (4), let $u(x, y) = X(x) Y(y)$. Since $\Delta u = 0$, we have

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda.$$

By the boundary condition, we have

$$\begin{cases} X''(x) + \lambda X(y) = 0, \ 0 < y < \pi \\ X(0) = 0, \ X(a) = 0 \end{cases} \Rightarrow \begin{cases} \lambda_n = \left(\frac{n\pi}{a}\right)^2 \\ X_n(x) = \sin(\frac{n\pi x}{a}) \end{cases}, \ n = 1, 2, ...$$

Then we have $u(x,y) = \sum_{n=1}^{\infty} X_n(x) Y_n(y)$ with

$$Y_n''(y) - \left(\frac{n\pi}{a}\right)^2 Y_n(y) = 0 \quad \Rightarrow \quad Y_n(y) = c_n e^{\frac{n\pi}{a}y} + d_n e^{-\frac{n\pi}{a}y}, \ n = 1, 2, \cdots$$

Hence, we have

$$u(x,y) = \sum_{n=1}^{\infty} \sin(\frac{n\pi}{a}x)(c_n e^{\frac{n\pi}{a}y} + d_n e^{-\frac{n\pi}{a}y}). \tag{1}$$

Now we have to determine the coefficients c_n and d_n. By direct calculations,

$$\left(\frac{\partial u}{\partial y} + u\right)\bigg|_{y=0} = \sum_{n=1}^{\infty} \sin(\frac{n\pi}{a}x)\left(\frac{n\pi}{a}(c_n - d_n) + c_n + d_n\right)$$

Therefore, we must have

$$\frac{n\pi}{a}(c_n - d_n) + c_n + d_n = 0. \tag{2}$$

On the other hand,

$$u(x,b) = \sum_{n=1}^{\infty} \sin(\frac{n\pi}{a}x)(c_n e^{\frac{n\pi}{a}b} + d_n e^{-\frac{n\pi}{a}b}) = g(x),$$

from which we derive

$$\frac{a}{2}(c_n e^{\frac{n\pi}{a}b} + d_n e^{-\frac{n\pi}{a}b}) = \int_0^a g(x) \sin\left(\frac{n\pi x}{a}\right) dx. \tag{3}$$

Combining (2) with (3), we have

$$\begin{cases} c_n = \dfrac{2(n\pi - a)}{a\left[(n\pi - a)e^{\frac{n\pi}{a}b} + (n\pi + a)e^{-\frac{n\pi}{a}b}\right]} \displaystyle\int_0^a g(x)\sin\left(\frac{n\pi x}{a}\right) dx. \\ d_n = \dfrac{2(n\pi + a)}{a\left[(n\pi - a)e^{\frac{n\pi}{a}b} + (n\pi + a)e^{-\frac{n\pi}{a}b}\right]} \displaystyle\int_0^a g(x)\sin\left(\frac{n\pi x}{a}\right) dx. \end{cases} \tag{4}$$

Thus, the solution of our original problem is given by (1), where the coefficients c_n and d_n are determined by (4). $\square$

Problem 7: (5')

Proof: Suppose u is a solution of the BVP. Then by the divergence theorem, we have

$$\int_\Omega f \mathrm{dx} = \int_\Omega \Delta u \mathrm{dx} = \int_\Omega \nabla \cdot \nabla u \mathrm{dx} = \oint_{\partial\Omega} \frac{\partial u}{\partial \mathbf{n}} \mathrm{d}S = \oint_{\partial\Omega} g \mathrm{d}S.$$

$\square$

Problem 8: (5'+5')

(a) By Problem 7, in order that the BVP has a solution, one must have

$$C\,|\partial\Omega| = \int_{\Omega} -\delta(\mathrm{x}-\mathrm{x}_0)\mathrm{d}\mathrm{x} = -1,$$

where $|\partial\Omega|$ denotes the surface area of $\partial\Omega$. Hence $C = -\frac{1}{|\partial\Omega|}$. $\square$

(b) Notice that

$$\begin{cases} -\displaystyle\int_{\Omega} u(\mathrm{x})\Delta G_N(\mathrm{x};\mathrm{x}_0)\mathrm{d}\mathrm{x} = u(\mathrm{x}_0) \\ -\displaystyle\int_{\Omega} G_N(\mathrm{x};\mathrm{x}_0)\Delta u(\mathrm{x})\mathrm{d}\mathrm{x} = \int_{\Omega} G_N(\mathrm{x};\mathrm{x}_0) f(\mathrm{x})\mathrm{d}\mathrm{x}. \end{cases}$$

By Green's second identity, we have

$$u(\mathrm{x}_0) - \int_{\Omega} G_N(\mathrm{x};\mathrm{x}_0) f(\mathrm{x})\mathrm{d}\mathrm{x} = \oint_{\partial\Omega} \left[G_N(\mathrm{x};\mathrm{x}_0)\frac{\partial u}{\partial \mathbf{n}}(\mathrm{x}) - u(\mathrm{x})\frac{\partial G_N}{\partial \mathbf{n}}(\mathrm{x};\mathrm{x}_0)\right] \mathrm{d}S$$

$$= \oint_{\partial\Omega} G_N(\mathrm{x};\mathrm{x}_0) g(\mathrm{x})\mathrm{d}S + \frac{1}{|\partial\Omega|}\oint_{\partial\Omega} h(\mathrm{x})\mathrm{d}S$$

$$\Rightarrow u(\mathrm{x}_0) = \int_{\Omega} G_N(\mathrm{x};\mathrm{x}_0) f(\mathrm{x})\mathrm{d}\mathrm{x} + \oint_{\partial\Omega} G_N(\mathrm{x};\mathrm{x}_0) g(\mathrm{x})\mathrm{d}S + \frac{1}{|\partial\Omega|}\oint_{\partial\Omega} h(\mathrm{x})\mathrm{d}S.$$

$\square$