64 digit

1. floating - point number: $(-1)^{s} \cdot 2^{c-1023} (1+f)$ 尾数 (mantissa)

s: 1, c: 11, f: 52

· range: $-1.79 \times 10^{-308} \sim 1.79 \times 10^{308}$

2. operation: ① chopping: $fl(y) = 0.d_1 \cdots d_k \times 10^n$, $d_{k+1} \cdots$ cut

② rounding: $fl(y) = 0.\delta_1 \cdots \delta_k \times 10^n$, add $5 \times 10^{n-(k+1)}$ then chop

i) $d_{k+1} \geq 5$, $\tilde{d}_k = d_k + 1$

ii) $d_{k+1} < 5$, $\tilde{d}_k = d_k$

3. Significant digits: $\frac{|p - p^*|}{|p|} \leq 5 \times 10^{-t}$

p^* approximate to p to t - significant digits.

for chopping: $\frac{0.d_{k+1} \cdots \times 10^{n-k}}{0.d_1 \cdots \times 10^{n}} \leq \frac{1}{0.1} \times 10^{-k} < 5 \times 10^{-(k-1)}$

at least $(k-1)$ significant digits.

If $d_{k+1} < 5$, $\frac{d_{k+1}}{d_1} < 5 \Rightarrow \boxed{\frac{|y - fl(y)|}{|y|}} \leq 5 \times 10^{-k}$

k significant digits

4. reduce roundoff - error:

① $\frac{a}{b}$, $a \gg b$ is bad

② $a \approx b$, $a-b$ doing is bad (easy to chop)

③ $x+y$, $x \gg y$ is bad

④ $f(x) = a_n x^n + \cdots + a_0$

$= \{\cdots\{(a_n x + a_{n-1})x + a_{n-2}\}x \cdots\}x + a_0$

Horner's algorithm.

2° approaching method:

①° Bisection

Th1: $|p_n - p^*| \le \frac{b-a}{2^n}$ for $b_n - a_n = \frac{b_1 - a_1}{2^{n-1}} = \frac{b-a}{2^{n-1}}$

$p^* \in [a_n, b_n]$, $|p_n - p^*| \le \frac{1}{2}(b_n - a_n) = \frac{b-a}{2^n}$

②° fixed-point: $g(p) - p = 0 \Rightarrow g(p) = p$

∃ Th1: $g \in C[a,b]$, $g([a,b]) \subseteq [a,b]$

g has at least 1 fix point:

proof: 1° $g(a) = a$ or $g(b) = b$

2° $h(x) = g(x) - x \Rightarrow h(x) \in C[a,b]$

$h(a) > 0$, $h(b) < 0 \Rightarrow \exists\, h(p) = 0, p \in [a,b]$

$\Rightarrow g(p) = p$

! Th2: ① $g \in C[a,b]$, $g[a,b] \subseteq [a,b]$

② $\exists\, 0 < k < 1$ s.t. $|g'(x)| \le k, \forall x \in (a,b)$

proof ① in Th1; ②: suppose $\exists\, p \ne q$, $p, q \in [a,b]$

s.t. $p = g(p)$, $q = g(q) \Rightarrow \exists\, \xi \in (p,q)$, s.t.

$g'(\xi) = \frac{g(p) - g(q)}{p - q} = 1$, $|g'(\xi)| < 1$ is not hold.

iteration: $x^2 = x + 1 \Rightarrow x = \sqrt{x+1}$

$g(x) = \sqrt{x+1} \Rightarrow p_n = \sqrt{p_{n-1} + 1}$

converge speed:

Th: ① $g[a,b] \subseteq [a,b]$, $g \in C[a,b]$

② $|g'(x)| \le k < 1 \quad \forall x \in (a,b)$

③ $\forall p_0 \in [a,b]$, $p_n = g(p_{n-1}) \quad \forall n \ge 1$

$|p_n - p| \le k^n \cdot \max\{p_0 - a, b - p_0\}$

Thm: ① $p_n \to p$ to unique p ② $|p_n - p| \le \frac{k^n}{1-k}|p_1 - p_0|, \forall n \ge 1$

proof: ①, ② $\exists!\ p \in [a,b],\ g(p) = p.$

$|p_n - p| = |g(p_{n-1}) - g(p)| = |g'(\xi)||p_{n-1} - p| \le k|p_{n-1} - p|$

$\le k^n |p_0 - p| \le k^n \max\{p_0 - a, b - p_0\}$

$[a \quad p \quad b]$

$|p_{n+1} - p_n| \le k^n |p_1 - p_0| \Rightarrow \forall m > n, |p_m - p_n| \le \sum_{k=n}^{m-1} |p_{k+1} - p_k|$

$\le k^n |p_1 - p_0|(1 + \cdots + k^{m-n-1}) \le \frac{k^n}{1-k}|p_1 - p_0|$

3° Newton-method, $p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)}$

eg: $f(x) = \cos x - x$,

$f'(x) = -\sin x - 1 \quad g(x) = x - \frac{\cos x - x}{-\sin x - 1}$

By analizing convergence. $\exists \lambda, \alpha$ s.t. $\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^\alpha} = \lambda$

$\{p_n\} \to p$ of order α with asymptotic error λ.

Th1: $f \in C^2[a,b]$, $p \in [a,b]$ s.t. $f(p) = 0$, $f'(p) \ne 0$ → Convergence

$\Rightarrow \exists\ \delta$ s.t. Newton-method converge to p for any $p_0 \in [p-\delta, p+\delta]$

Th2: ① $g \in C[a,b]$, $g[a,b] \subseteq [a,b]$, $|g'(x)| \le k < 1\ \forall x \in (a,b)$, g' continuous on $[a,b]$

(by fix point) ② $g'(p) \ne 0$, p is a fixed point,

p_n converge linearly: $\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|} < 1$

Th3: ① $p = g(p)$, $g'(p) = 0$

② g'' continuous, $|g''(x)| < M$ on I (open)

$\exists\ \delta > 0$ s.t. $p_0 \in [p-\delta, p+\delta]$, $p_n = g(p_{n-1})\ \forall n \ge 1$ converge

$g'(p)$ comes from $x = x - \phi(x)f(x) = g(x)$
$g'(p)=0$

at least quadratically.

proof: $g(x) = g(p) + g'(p)(x-p) + \frac{g(\xi)}{2}(x-p)^2$
$= p + \frac{1}{2}g''(\xi)(x-p)^2$

$\Rightarrow p_{n+1} = p + \frac{1}{2}g''(\xi_n)(p_n - p)^2$

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^2} \le \frac{M}{2}$$

4° Secant method: $f'(p_{n+1}) \approx \frac{f(p_{n+1}) - f(p_n)}{p_{n+1} - p_n}$

$$\Rightarrow p_n = p_{n-1} - \frac{p_{n-1} - p_{n-2}}{f(p_{n-1}) - f(p_{n-2})} f(p_{n-1})$$

3° interpolation: by weierstrass thm, using $\{(x_k, f(x_k))\}_{k=0}^n$
interpolation $\exists!$ /.

1° Lagrange interpolation: $\{x_k\}_{k=0}^n$

$$L_k(x) = \frac{(x-x_0)\cdots(x-x_n)}{(x_k-x_0)\cdots(x_k-x_n)} = \prod_{\substack{i=0\\ i\neq k}}^{n} \frac{x-x_i}{x_k-x_i}$$

$\Rightarrow L_k(x_j) = \delta_{kj}$, Rk.: i) $p(x)$ degree at most n
ii) $p(x_j) = f(x_j)$

Th1: $\{x_k\}_{k=0}^n \Rightarrow p(x) = \sum_{k=0}^{n} f(x_k)L_k(x)$
(set $R(x) = p(x) - \sum_{k=0}^{n} f(x_k)L_k(x)$, $R(x)=0$

Th: $f\in C^{n+1}[a,b]$, $x_0 < \cdots < x_n$, $\forall x \in [a,b]$
$\exists \xi$ s.t. $f(x) - P_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)\cdots(x-x_n)$

proof:..

$f(x_0) = \cdots = f(x_n) \Rightarrow \exists \xi\in(a,b)$, s.t. $f^{(n)}(\xi) = 0$

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eg: $f(x)=\frac{1}{x}$, $x_0=2$, $x_1=2.75$, $x_2=4$

$f(x_0)=\frac{1}{2}$ $f(x_1)=\frac{4}{11}$, $f(x_2)=\frac{1}{4}$

$l_0(x)=\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}$ . . . $R(x)=\frac{f^{(3)}(\xi)}{3!}(x-2)(x-2.75)(x-4)$

$|R(x)|\leq\frac{1}{16}|(x-2)(x-2.75)(x-4)|\leq\frac{9}{256}$

2° Neville's method: Set $P_{m_1\cdots m_k}(x)$ using $\{x_{m_j}\}_{j=1}^{k}$

eg: $x_0=1$, $x_1=2$, $x_2=3$, $x_3=4$, $x_4=6$, $f(x)=e^x$

$$P_{1,2,4}=\frac{(x-3)(x-6)}{(2-3)(2-6)}e^2+\frac{(x-2)(x-6)}{(3-2)(3-6)}e^3+\frac{(x-2)(x-3)}{(6-2)(6-3)}e^6$$

Th:

$$P_n(x)=\frac{(x-x_j)P_{0,\ldots,j-1,j+1,\ldots,k}(x)-(x-x_i)P_{0,1,\ldots,i-1,i+1,\ldots,k}(x)}{x_i-x_j}$$

find $P_n(x)=f(x)$ on $\{x_i\}_{i=1}^{k}$

3° Newton method: using $1, x-x_0, (x-x_0)(x-x_1), \ldots, (x-x_0)\cdots(x-x_{n-1})$

$p(x)=a_0+a_1(x-x_0)+\cdots+a_n(x-x_0)\cdots(x-x_{n-1})$

find that $a_0=f(x_0)$, $a_1=\frac{f(x_1)-f(x_0)}{x_1-x_0}$, $f[x_i,\ldots,x_{i+k}]=\frac{f[x_{i+1},\ldots,x_{i+k}]-f[x_i,\ldots,x_{i+k-1}]}{x_{i+k}-x_i}$

$p(x_1)=a_0+a_1(x_1-x_0)$

$$p(x)=f[x_0]+\sum_{k=1}^{n}f[x_0,\ldots,x_k](x-x_0)(x-x_1)\cdots(x-x_{k-1})$$

OBS: $f(x)=p(x)+f[x,x_0,\ldots,x_n](x-x_0)\cdots(x-x_n)$

P_1: $f[x_0,\ldots,x_n]=\sum_{k=0}^{n}\frac{f(x_k)}{(x_k-x_0)\cdots(x_k-x_{k-1})(x_k-x_{k+1})\cdots(x_k-x_n)}$

P_2: $f\in C^n[a,b]$, $x_0,\ldots,x_n\in[a,b]$ (distinct) $\Rightarrow\exists\,\xi\in(a,b)$

s.t. $f[x_0,\ldots,x_n]=\frac{f^{(n)}(\xi)}{n!}$

$$f[x_0,\dots,x_k] = \frac{\Delta^k f(x_0)}{k!\,h^k}$$

when equal step: $p(x) = f(x_0) + \sum_{k=1}^{n} \binom{s}{k} \Delta^k f(x_0)$

$$p(x) = \sum_{k=0}^{n} (-1)^k \binom{-s}{k} \nabla^k f(x_n)$$

$$\begin{cases} \nabla p_n = p_n - p_{n-1} \\ \nabla^k p_n = \nabla^{k-1} p_n - \nabla^{k-1} p_{n-1} \end{cases}$$

4° Hermite polynomial:

at most $2n+1$ degree $\leftarrow$ $H(x) = \sum_{j=0}^{n} f(x_j) H_j(x) + \sum_{j=0}^{n} f'(x_j) \hat{H}_j(x)$

where $H_j(x) = [1 - 2(x - x_j) L_j'(x_j)] L_j^2(x)$

$\hat{H}_j(x) = (x - x_j) L_j^2(x)$

$H_j(x_i) = \delta_{ij}$, $\hat{H}_j'(x_i) = \delta_{ij}$

$\hat{H}_j(x_i) = 0$, $H_j'(x_i) = 0$

$$f(x) = H(x) + \frac{(x-x_0)^2 \cdots (x-x_n)^2}{(2n+2)!} f^{(2n+2)}(\xi)$$

5° spline. $\begin{cases} \varphi_j(x_j) = \delta_{ij}, & 0 \le i,j \le n \\ \varphi_j |_{[x_j, x_{j+1}]} \text{ is linear.} \end{cases}$

error bound: $\max\limits_{x \in [x_0, x_n]} |f(x) - I_n[f(x)]| \le \frac{1}{8} h^2 \max\limits_{x \in [x_0, x_n]} |f''(x)|$

$$h = \max_{0 \le j \le n} |x_{j+1} - x_j|$$

Cubic spline: set $S_j(x) = a_j + b_j x + c_j x^2 + d_j x^3$.

For $4n$ co-efficients: $\qquad S_j = a_j + b_j(x - x_j) + c_j(x - x_j)^2 + d_j(x - x_j)^3$

a) $S_{j+1}(x_{j+1}) = S_j(x_{j+1})$ $\quad 2n$

b) $S'_{j+1}(x_{j+1}) = S'_j(x_{j+1})$ $\quad n-1$

c) $S''_{j+1}(x_{j+1}) = S''_j(x_{j+1})$, $n-1$

d) i) $S''(x_0) = S''(x_n) = 0$ (natural bound)

or: ii) $S'(x_0) = f'(x_0)$, $S'(x_n) = f'(x_n)$ (clamped bound)

Date.

4th numerical integral and differentiation.

4.1 Nu-differentiation:

$$f'(x_k) \approx \sum_{j=0}^{n} a_{kj} f(x_j)$$

$$f'(x_k) = \sum_{j=0}^{n} L_j'(x_k) f(x_j) + \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{\substack{j=0 \\ j\neq k}}^{n} (x_k - x_j)$$

eg : 2-points:

$$f'(x_0) = \frac{f(x_1) - f(x_0)}{x_1 - x_0} + \frac{1}{2}(x_0 - x_1) f''(\xi) \quad \text{forward}$$

$$f'(x_1) = \frac{f(x_1) - f(x_0)}{x_1 - x_0} + \frac{1}{2}(x_1 - x_0) f''(\xi)$$

back-ward

3-points:

forward

$$f'(x_0) = \frac{-3f(x_0) + 4f(x_0+h) - f(x_0+2h)}{2h} + \frac{h^2}{3} f^{(3)}(\xi)$$

backward

$$f'(x_0) = \frac{f(x_0-2h) - 4f(x_0-h) + 3f(x_0)}{2h} + \frac{h^2}{3} f^{(3)}(\xi)$$

central

$$f'(x_0) = \frac{f(x_0+h) - f(x_0-h)}{2h} - \frac{h^2}{6} f^{(3)}(\xi)$$

Round-error: $f(x_0 \pm h) = \hat{f}(x_0 \pm h) + e(x_0 \pm h)$

$(e = \frac{\varepsilon}{h})$

$$\left| f'(x_0) - \frac{\hat{f}(x_0+h) - \hat{f}(x_0-h)}{2h} \right| \leq \frac{h^2}{6} M + \frac{\varepsilon}{h}$$

4.7 Numerical - integral

1°: $\int_a^b f(x)dx = \sum_{k=0}^{n} a_k f(x_k) + E(f)$

$a_k = \int_a^b L_k(x)dx$, $E(f) = \frac{1}{(n+1)!}\int_a^b f^{(n+1)}(\xi)\prod_{j=0}^{n}(x-x_j)dx$

make equal-spacing:

① $h=1$, 2 points: $\int_a^b f(x)dx = \frac{b-a}{2}\left(f(b)+f(a)\right) - \frac{(b-a)^3}{12}f''(\xi)$

$\int_a^b \frac{f^{(2)}(\xi)}{6}(x-a)(x-b)dx$

② $h=2$, 3 points:

$\int_a^b f(x)dx = \frac{(b-a)}{6}\left[f(a)+4f\left(\frac{b+a}{2}\right)+f(b)\right] - \frac{(b-a)^5}{2^5\times 90}f^{(4)}(\xi)$

2°: Degree of accuracy: largest positive $n\in\mathbb{Z}$ s.t. formula exact for x^k $(k\in[1,n])$

For ①: $f''=0$ is $x^0, x^1 \Rightarrow$ degree 1

②: $f^{(4)}=0 \Rightarrow$ degree 3.

3° general: Newton-Cotes:

i) n even, $f\in C^{n+2}[a,b]$, $\int_a^b f(x)dx = \sum_{k=0}^{n} a_k f(x_k) + \frac{h^{n+3}f^{(n+2)}(\xi)}{(n+2)!}\int_0^n t^2(t-1)\cdots(t-n)dt$

ii) n is odd, $f\in C^{n+1}[a,b]$, $\int_a^b f(x)dx = \sum_{k=0}^{n} a_k f(x_k) + \frac{h^{n+2}f^{(n+1)}(\xi)}{(n+1)!}\int_0^n t\cdots(t-n)dt$

open-Newton-Cotes:

$n=0$, $\int_{x_{-1}}^{x_1} f(x)dx = (x_1-x_{-1})f\left(\frac{x_1+x_{-1}}{2}\right) + \frac{h^3}{3}f''(\xi)$.

$n=1$, $\int_{x_{-1}}^{x_2} f(x)dx = \frac{(x_2-x_{-1})}{2}\left(f(x_0)+f(x_1)\right] + \frac{3h^3}{4}f''(\xi)$

$n=2$: $\int_{x_{-1}}^{x_3} f(x)dx = \frac{x_3-x_{-1}}{3}\left[2f(x_0)-f(x_1)+2f(x_2)\right] + \frac{14h^5}{45}f^{(4)}(\xi)$

round-off: $e = \left| \sum_{k=0}^{n} a_k [f(x_k) - \hat{f}(x_k)] \right| \le (b-a)\varepsilon$

$\le \varepsilon \sum_{k=0}^{n} |a_k| \le (b-a)\varepsilon,$

Since $\sum_{k=0}^{n} a_k = \int_a^b \sum_{k=0}^{n} 1 \cdot l_k(x)\,dx = b-a$

This kind quadrature formula is stable (doesn't change by #notes)

4° composite: $\int_a^b f(x)dx = \frac{h}{2}\left[f(x_0) + 2[f(x_1) + \cdots + f(x_{n-1})] + f(x_n)\right] - \frac{b-a}{12} h^2 f''(\xi)$

Composite Simpson's:

$$\int_a^b f(x)dx = \frac{h}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + f(x_n)\right] - \frac{b-a}{180} h^4 f^{(4)}(\xi)$$

Lemma: $f \in C[a,b]$, $\forall x_j$ $(j \in [1,n])$ $\exists \xi \in (a,b)$ s.t. $f(\xi) = \frac{1}{n}\sum_{k=1}^{n} f(x_k)$

proof:

$$n \cdot \min_{x \in [a,b]} f(x) \le \frac{1}{n}\sum_{k=1}^{n} f(x_k) \le n \cdot \max_{x \in [a,b]} f(x)$$

by intermediate: $\exists \xi \in (a,b)$, $f(\xi) = \frac{1}{n}\sum_{k=1}^{n} f(x_k)$

by pre-analysis, round-off also $\varepsilon(b-a)$

5° Gaussian quadrature:

$$S(f) = \sum_{k=0}^{n} a_k f(x_k), \quad a_k = \int_a^b L_k(x)\,dx$$

has $(2n+1)$ degree of precise.

$O((b-a)^5)$

$\int_{-1}^{1} f(x)\,dx = f\left(-\frac{\sqrt{3}}{3}\right) + f\left(\frac{\sqrt{3}}{3}\right)$ $\rightarrow$ degree of precision 3 since $1 \sim x^3$ use for calculate co-efficient

$$\Rightarrow \int_a^b f(x)\,dx = \frac{b-a}{2}\int_{-1}^{1} f\left[\frac{b-a}{2}(t+1)+a\right] dt$$

$\int_{-1}^{1} f(x)\,dx \approx \frac{5}{9} f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{\frac{3}{5}}\right)$ $\rightarrow$ degree of precision : 5

5.2 get generally formula:

Gaussian quadrature $\Leftrightarrow \forall p(x) \in P_n$, $\int_a^b p(x) W_n(x)\,dx = 0$

$P_0(x) = 1$, $P_2(x) = x^2 - \frac{1}{3}$

$P_1(x) = x$, $P_3(x) = x^3 - \frac{3}{5}x$

$P_{n+1}(x) = xP_n(x) - \frac{n(n+1)}{2n+1} P_{n-1}(x)$

Th. $x_1, \dots, x_n$ are roots of n-th Legendre poly-nomial

$$C_i = \int_{-1}^{1} \prod_{\substack{j=1 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}, \quad i \in [1, n], \Rightarrow \int_{-1}^{1} f(x)\,dx = \sum_{i=1}^{n} C_i f(x_i)$$

eg: $n = 2$, $P_2(x) = x^2 - \frac{1}{3}$, $x_1 = \frac{\sqrt{3}}{3}$, $x_2 = -\frac{\sqrt{3}}{3}$

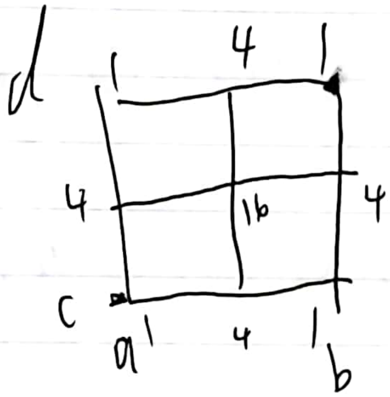
$$C_1 = \int_{-1}^{1} \frac{x - \frac{\sqrt{3}}{3}}{-\frac{\sqrt{3}}{3} - \frac{\sqrt{3}}{3}}\,dx = 1, \quad C_2 = \int_{-1}^{1} \frac{x + \frac{\sqrt{3}}{3}}{\frac{\sqrt{3}}{3} + \frac{\sqrt{3}}{3}}\,dx = 1$$

generally: $\int_a^b f(x)dx \approx \frac{b-a}{2}\sum_{k=1}^{n} c_k f[\frac{b-a}{2}(x_k+1)+a]$

6° mutiple integral:

like $\int_a^b\int_c^d f(x,y)dydx = \int_a^b \frac{d-c}{6}[f(x,c)+4f(x,\frac{c+d}{2})+f(x,d)]$

$= \frac{(b-a)(d-c)}{36}[(f(a,c)+ \ldots$



7° improper integral:

$g \in C^5$. $S = \int_a^b \frac{g(x)}{(x-a)^p}dx = \underbrace{\int_a^b \frac{P_4(x)}{(x-a)^p}dx}_{\text{integrable}} + \underbrace{\int_a^b \frac{g(x)-P_4(x)}{(x-a)^p}dx}_{\text{Simpson's method}}$

5^{th} initial value problem (IVPs)

1° $\begin{cases} y'(t) = f(t, y(t)) \\ y(a) = \alpha \end{cases}$ $t \in (a, b)$

Theoretically: continuous: $|f(u) - f(u^*)| \to 0$ as $u \to u^*$

Lipschitz: $|f(u) - f(u^*)| \sim |u - u^*|$

differentiable: $|f(u) - f(u^*)| \le |u - u^*| \max_y |f(y)|$

def: $T^* = \min\{t_1, t_0 + \frac{a}{S}\}$, $S = \max_{(t,u) \in D} |f(t,u)|$

$D = \{(t, y) \mid t \in [t_0, t_1], |y - a| \le a\}$.

Euler method: $y(t_{n+1}) = y(t_n + h) = y(t_n) + h y'(t_n) + \frac{h^2}{2} y''(\xi_n)$

$y_{n+1} = y_n + f(t_n, y_n) h$

For differential: $\frac{y_{n+1} - y_n}{h} = f(t_n, y_n)$

For integral: $y_{n+1} - y_n \approx h f(t_n, y_n)$

eg1: $\begin{cases} y' = y - t^2 + 1, \ t \in [0, 2] \\ y(0) = 0.5 \end{cases}$ choose $h = 0.5$

$y_1 = y_0 + h(y_0 - t_0^2 + 1) = 1.25$

$y_2 = y_1 + h(y_1 - t_1^2 + 1)$ $|\tilde{y}_n - y_n| \le e^{LT} |\delta_0|$ stable.

2° measure error. consistent $|e_n| \le \frac{Mh}{2L}(e^{Lt_n} - 1)$

Def: local truncation error: $\Leftarrow \lim_{h \to 0} \max_{1 \le i \le N} \left|\frac{T_i}{h}\right| = \lim_{h \to 0} \max_{1 \le i \le N} \frac{h |y''(\xi_i)|}{2} = 0$

$y_{n+1} = y_n + h f(t_n, y_n) + T_{n+1}$ convergent $\Leftarrow \lim_{h \to 0} \max_{1 \le i \le N} |e_i| = \lim_{h \to 0} \max_{1 \le i \le N} \frac{Mh(e^{Lt_n} - 1)}{2L}$

$= 0$

$T_{n+1} = y(t_{n+1}) - [y(t_n) + h f(t_n, y(t_n))]$

$= y_n + h y'_n(t_n) + \frac{h^2}{2} \cdot y''(\xi) - y(t_n) - h f(t_n, y_n)$

$= \frac{h^2}{2} y''(\xi)$

$y(t_{n+1}) - y_{n+1} = e_{n+1}$

$y(t_n) - y_n = e_n$

$M = \max_{t\in[0,T]} |y''(t)|$, $|T_n| \le \frac{1}{2}Mh^2$

$e_{n+1} = e_n + h[f(t_n, y(t_n)) - f(t_n, y_n)] + T_{n+1}$

from $\begin{cases} y_{n+1} = y_n + hf(t_n, y_n) \\ y(t_{n+1}) = y(t_n) + hf(t_n, y(t_n)) + T_{n+1} \end{cases}$

$|e_{n+1}| \le |e_n| + \frac{Mh^2}{2} + h|f(t_n, y(t_n)) - f(t_n, y_n)|$

lipstz

$\le (1+hL)|e_n| + \frac{1}{2}Mh^2$ - - - - ①

Too find more precise error bound:

1. $x \ge -1$, $m \in \mathbb{Z}^+$, $0 \le (1+x)^m \le e^{mx}$

2. $s, t \in \mathbb{R}^+$, $\{a_i\}_{i=0}^{k}$ s.t. $a_0 \ge -\frac{t}{s}$
and $a_{i+1} \le (1+s)a_i + t$ for each $i \in [0, k-1]$,
$\Rightarrow a_{i+1} \le e^{(i+1)s}(a_0 + \frac{t}{s}) - \frac{t}{s}$

proof: $a_{i+1} \le (1+s)a_i + t \le (1+s)^{i+1}a_0 + \frac{(1+s)^{i+1}-1}{s}t$

$\le e^{(i+1)s}(a_0 + \frac{t}{s}) - \frac{t}{s}$

Finally find Th about error.

Th: $\begin{cases} y' = f(t,y) \\ y(0) = y_0 \end{cases}$, $D \in [0,T] \times \mathbb{R}$

f is Lipschitz-continuous in y
$M = \max_D |y''|$, $\{y_k\}_{k=0}^{N}$ generate
by $y_n = y_{n-1} + hf(t_{n-1}, y_{n-1})$

$\Rightarrow |y(t_n) - y_n| \le \frac{M}{2L}(e^{Lt_n} - 1)h$, proof: ①

$$|y_n - y(t_n)| = |e_n| \le |1+hL||e_n| + \frac{1}{2}Mh^2 \le \cdots \le |1+hL|^n |e_0| + \frac{Mh^2}{2}\frac{(1+hL)^n - 1}{(1+hL)-1}$$

$$\le e^{nhL}|e_0| + \frac{(e^{Lnh}-1)M}{2L}h$$

$$nh = t_n \quad = \cancel{\tfrac{1}{2L}} e^{Lt_n}|e_0| + \frac{(e^{Lt_n}-1)M}{2L}h \to \text{if } t_n L >> 1$$

then we need h small.

Γ $I \subseteq (a,b)$ open, u, β ; if $u \in C^1(I) \Rightarrow u'(t) \le \beta(t)u(t)$,

$\Rightarrow u(t) \le u(a)\exp\left[\int_a^t \beta(s)ds\right] \quad \forall t \in I.$

proof: set $v(t) = \exp\left(\int_a^t \beta(s)ds\right) \Rightarrow v' = \beta v$

with $v(a) = 1$, $v(t) > 0 \quad \forall t \in I$.

$$\frac{d}{dt}\left(\frac{u(t)}{v(t)}\right) = \frac{u'(t)v(t) - \beta(t)u(t)v(t)}{v^2(t)} \le 0$$

$\Rightarrow \frac{u(t)}{v(t)} \le \frac{u(a)}{v(a)} = u(a) \Rightarrow u(t) \le u(a)\cdot\exp\left(\int_a^t \beta(s)ds\right)$ 」

3° other methods: $y(t_k) = y(t_{k-1}) + hf(t_{k-1}, y(t_{k-1})) + T_{k-1}$

① forward Euler: $y_k = y_{k-1} + hf(t_{k-1}, y_{k-1})$

$T_k^{(1)} \Rightarrow O(h^2)$

② $y_k = y_{k-1} + hf(t_k, y_k) \to O(h^2)$, implicit

Date /

③ Trapezoial rule:

$$y_k = y_{k-1} + \frac{h}{2}\left[f(t_{k-1}, y_{k-1}) + f(t_k, y_k)\right]$$

$$T_k = y'_{k-1} h + \frac{h^2}{2} y''_{k-1} + \frac{h^3}{3!} y'''(\xi) - \frac{h}{2} y'_{k-1} - \frac{h}{2}\left(f(t_{k-1}, y_{k-1}) + f_t h + f_y y'_{k-1} + O(h^2)\right)$$

④ modified Euler:

$O(h^3)$

$$y_k = y_{k-1} + \frac{h}{2}\left[f(t_{k-1}, y_{k-1}) + f\left(t_k, y_{k-1} + h f(t_{k-1}, y_{k-1})\right)\right]$$

⑤ $y_k = y_{k-1} + h f\left(t_{k-1} + \frac{h}{2}, y_{k-1} + \frac{h}{2} f(t_{k-1}, y_{k-1})\right)$

$T = O(h^3)$

4° analysis error: $y_{n+1} = y_n + h\varphi(t_n, y_n, h)$

① L.T.E: $T_{n+1} = y(t_{n+1}) - [y(t_n) + h\varphi(t_n, y(t_n); h)]$

② consistent: $\lim\limits_{h\to 0} \max\limits_{1\le i\le N} \left|\frac{T_i}{h}\right| = 0$

③ convergence: $\lim\limits_{h\to 0} \max\limits_{1\le i\le N} |y(t_i) - y_i| = 0$

order of accuracy: $|y(t_i) - y_i| = O(h^p)$

Thm: $\begin{cases} y' = f(t,y), \ 0 \le t \le T \\ y(0) = y_0, \end{cases}$, $y_{n+1} = y_n + h\varphi(t_n, y_n, h)$

Suppose $\exists h_0 > 0$ s.t. $\varphi(t,y,h)$ satisfies Lipschitz continuous in y over $D = [0,T] \times \mathbb{R} \times [0,h_0]$

i) method stable

proof: set $\tilde{y}_0 = y_0 + \delta$, $\tilde{y}_{n+1} = \tilde{y}_n + h\varphi(t_n, \tilde{y}_n, h)$. def: $w_n = \tilde{y}_n - y_n$

$\Rightarrow w_{n+1} = w_n + h[\underbrace{\varphi(t_n, \tilde{y}_n, h) - \varphi(t_n, y_n, h)}]$. Lipschitz.

$|w_{n+1}| \le (1+Lh)w_n \le \cdots \le (1+Lh)^{n+1}|w_0|$

$\Rightarrow |w_n| \le e^{t_n L}\delta$, stable at small δ.

ii) consistent $\Leftrightarrow$ convergence

proof: $\frac{T_{n+1}}{h} = \frac{y(t_{n+1}) - y(t_n)}{h} - \varphi(t_n, y(t_n), h)$

$h \to 0$, RHS: $y'(t_n) - f(t_n, y(t_n)) = 0$

[convergent then $h \to 0$, $\varphi(t_n, y(t_n), 0) = f(t_n, y(t_n))$]

$\Leftrightarrow \frac{T_{n+1}}{h} = 0$ as $h \to 0$:

iii) If LTE $|T_n| \le ch^{p+1}$ then

$|y(t_n) - y_n| \le \frac{c}{L}(e^{t_n L} - 1)h^p \ \forall nh \le T$

how to choose step h.

④ Strong stable: $|y_{n+1}| \le |y_n|$

use $\begin{cases} y' = \lambda y \quad (Re(\lambda) < 0) \\ y(0) = a \end{cases} \Rightarrow y(t) = a e^{\lambda t}$ ⓪ ↓

① $t \to +\infty$, $y(t) \to 0$.

1° forward Euler: $y_{n+1} = y_n + \lambda h y_n = (1+\lambda h) y_n$

let $z = \lambda h \Rightarrow |1+z| \le 1$ $(-1, 0)$

$\hat{\lambda} = Re(\lambda) \Rightarrow 0 \le h \le -\frac{2}{\hat{\lambda}}$

2° backward Euler: $|1-z| \ge 1 \Rightarrow$ use

3° trapezoidal rule: $y_{n+1} = y_n + \frac{h}{2}(y_n + y_{n+1})$

$\Rightarrow |z-2| \ge |z+2| \Rightarrow -4z \ge 4z$

In General: $y_{n+1} = u(\lambda h) y_n$, $|u(\lambda h)| \le 1$

4° modified Euler: $y_{n+1} = y_n + \frac{h}{2}[\lambda y_n + \lambda(y_n + h\lambda y_n)]$
$= \left(1 + \lambda h + \frac{(\lambda h)^2}{2}\right) y_n$

$\boxed{b_m = 0}$ explicit

$b_m \ne 0$ implicit. $\Leftarrow$

5° multi-step methods

generally: $y_{i+1} = a_{m-1} y_i + \cdots + a_0 y_{i+1-m} +$
$h[b_m f(t_{i+1}, y_{i+1}) + \cdots + b_0 f(t_{i+1-m}, y_{i+1-m})]$

eg1: approach $y'(t_{i+1})$

$$\frac{3y(t_{i+1}) - 4y(t_i) + y(t_{i-1})}{2h} = y'(t_{i+1}) + \frac{h^2}{3} y^{(3)}(\xi) = f(t_{i+1}, y(t_{i+1}))$$

$$\Downarrow$$

$$\frac{3y_{i+1} - 4y_i + y_{i-1}}{2h} = f(t_{i+1}, y_{i+1})$$

$\tau_{i+1} = O(h^3)$ $(|y'''| \le M)$

eg2: Adams-Bashforth:

~~P4t~~ $m=2$, $y_{i+1} = y_i + h\left(\frac{3}{2} f(t_i, y_i) - \frac{1}{2} f(t_{i-1}, y_{i-1})\right)$, $\tau = O(h^3)$

$m=3$, $\tau_{i+1} = O(h^3)$