23-24 MA327 Differential Geometry Midterm Examination

12th April, 2024, 10:20am-12:10pm

Questions

1. (20 marks) Let $\alpha : I \to \mathbb{R}^3$ be a regular parametrized curve.
   (a) Write down the definition of the curvature κ and the torsion τ of α (in the latter case you may assume that κ is everywhere non-zero).
   (b) Show that if the curvature κ is everywhere zero, then the trace of α is inside a straight line.
   (c) Show that if the torsion τ is everywhere zero, then the trace of α lies inside a plane.

2. (15 marks) Let $\alpha : I \to \mathbb{R}^3$ be a regular parametrized curve in $\mathbb{R}^3$ (not necessarily by arc length) and let $\beta : J \to \mathbb{R}^3$ be an arc length reparametrization of $\alpha(I)$ by the arc length

$$s = s(t) = \int_{t_0}^{t} |\alpha(u)| du.$$

   Let $t = t(s)$ be the inverse function of s and set $d\alpha/dt = \alpha', d^2\alpha/dt^2 = \alpha''$. Prove that
   (a) $dt/ds = 1/|\alpha'|$, $d^2t/ds^2 = -(\alpha' \cdot \alpha''/|\alpha'|^4)$.
   (b) The curvature κ at $t \in I$ is

$$\kappa(t) = \frac{|\alpha' \wedge \alpha''|}{|\alpha'|^3}.$$

3. (15 marks) State the definition of a regular surface in $\mathbb{R}^3$. Show that

$$W = \{(x, y, z) : z = |x|\}$$

   is not a regular surface in $\mathbb{R}^3$.

4. (10 marks) Let S be a regular surface in $\mathbb{R}^3$ and $p = (0,0,0)$ is in S. Assume that L is a plane which intersects S only at one point p. Show that $L = T_pS$.

5. (20 marks) Let $C = \{(x,y,z) : x^2 + y^2 = 1\}$ be the cylinder in $\mathbb{R}^3$.
   (a) Show that C is a regular surface.
   (b) Find a parametrization $\mathcal{X}$ of C and calculate E, F, G (that is, the first fundamental form).
   (c) Find the regular parametrized curve $\alpha : [0,1] \to C$ such that $\alpha(0) = (0,1,0)$, $\alpha(1) = (1,0,1)$ and α is the shortest curve among all regular parametrized curves in C joining $(0,1,0)$ to $(1,0,1)$.

6. (20 marks) Let S be a connected regular surface, such that it is covered by three connected open sets V_i, where $i = 1,2,3$ and for each i we have $V_i = X_i(U_i)$ and

$$X_i : U_i \to S, \quad (u_i, v_i) \mapsto X_i(u_i, v_i),$$

is a parametrization of S. Assume also that

$$V_{12} = V_1 \cap V_2, \quad V_{23} = V_2 \cap V_3, \quad V_{31} = V_3 \cap V_1$$

are nonempty and connected.

   (a) Explain why the jacobian of the change of parameter

$$X_j^{-1} \circ X_i : X_i^{-1}(V_{ij}) \to X_j^{-1}(V_{ij}),$$
$$(u_i, v_i) \mapsto (u_j(u_i, v_i), v_j(u_i, v_i))$$

   has constant sign.

   (b) Let s_{ij} be the sign of the jacobian J_{ij} of $X_j^{-1} \circ X_i$. That is,

$$s_{ij} = \begin{cases} 1 & \text{if } J_{ij} > 0, \\ -1 & \text{if } J_{ij} < 0 \end{cases}.$$

   Show that S is orientable if and only if $s_{12}s_{23}s_{31} = 1$.