# MA303 偏微分方程 第四次作业

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## Chapter 3

**Problem 5:** (5'+5'，(3) 已评讲不计入总分)

注：用分离变量法求解非齐次性问题的一般步骤请参考课本 P27-P28 的 Summary 部分。

**Solutions:**

(1) Let $u(x,t) = U(x,t) + w(x,t)$. From the boundary conditions $U(0,t) = U(l,t) = 0$, we have $w(0,t) = u_1, w(l,t) = u_2$. Then we consider

$$w(x,t) = u_1 + \frac{x}{l}(u_2 - u_1).$$

Therefore we have the new PDE:

$$\begin{cases} U_t = a^2 U_{xx}, & 0 < x < l,\ t > 0 \\ U(0,t) = 0, U(l,t) = 0, & t > 0 \\ U(x,0) = u_0 - u_1 - \dfrac{x}{l}(u_2 - u_1), & 0 < x < l. \end{cases}$$

Let $U(x,t) = X(x)T(t)$. Then combining with the above new PDE, we obtain the following eigenvalue problem:

$$\begin{cases} X''(x) + \lambda X(x) = 0,\ 0 < x < l, \\ X(0) = 0, X(l) = 0, \end{cases}$$

whose solution is given by

$$\begin{cases} \lambda_n = \left(\frac{n\pi}{l}\right)^2, \quad n = 1, 2, \cdots \\ X_n(x) = \sin\left(\frac{n\pi x}{l}\right). \end{cases}$$

For $T(t)$, we have

$$T'(t) + a^2 \lambda T(t) = 0,$$

whose solution is given by

$$T_n(t) = \phi_n e^{-\left(\frac{n\pi a}{l}\right)^2 t}, \quad n = 1, 2, \cdots$$

Hence, we have

$$U_n(x,t) = X_n(x)T_n(t) = \phi_n e^{-\left(\frac{n\pi a}{l}\right)^2 t} \sin\left(\frac{n\pi x}{l}\right), \quad n = 1, 2, \cdots$$

Let $U(x,t) = \sum\limits_{n=1}^{\infty} U_n(x,t)$. From the initial condition, we have

$$U(x,0) = u_0 - u_1 - \frac{u_2 - u_1}{l}x = \sum_{n=1}^{\infty} \phi_n \sin\left(\frac{n\pi x}{l}\right),$$

where the Fourier coefficients is determined by

$$\begin{aligned}\phi_n &= \frac{\int_0^l \left(u_0 - u_1 - \frac{u_2-u_1}{l}x\right)\sin\left(\frac{n\pi x}{l}\right)dx}{\int_0^l \sin^2\left(\frac{n\pi x}{l}\right)dx} \\ &= \frac{2(u_0-u_1)}{n\pi}[1-(-1)^n] - \frac{2(u_1-u_2)}{n\pi}(-1)^n, \quad n = 1, 2, \cdots\end{aligned} \tag{1}$$

Therefore, we have

$$u(x,t) = u_1 + \frac{u_2 - u_1}{l}x + \sum_{n=1}^{\infty} \phi_n e^{-\left(\frac{n\pi a}{l}\right)^2 t} \sin\left(\frac{n\pi x}{l}\right),$$

where $\phi_n$ is defined by (1).

From (1), we can see that $|\phi_n| \leq \frac{C}{n}$ for some constant C, and moreover that

$$\sum_{n=1}^{\infty} \left|\phi_n e^{-\left(\frac{n\pi a}{l}\right)^2 t} \sin\left(\frac{n\pi x}{l}\right)\right| \leq C \sum_{n=1}^{\infty} \frac{1}{n} e^{-\left(\frac{n\pi a}{l}\right)^2 t} \to 0 \ as\ t \to \infty.$$

Therefore, we have

$$\lim_{t\to\infty} u(x,t) = u_1 + \frac{u_2 - u_1}{l}x,$$

which is clearly a steady state. $\square$

(2) Let $u(x,t) = V(x,t)e^{-ht}$. Then the given problem becomes

$$\begin{cases} V_t = a^2 V_{xx} + ge^{ht}, & 0 < x < l,\ t > 0 \\ V(0,t) = 0,\ V(l,t) = 0, & t > 0 \\ V(x,0) = 0, & 0 < x < l. \end{cases}$$

Let $U(x,t) = X(x)T(t)$. Then combining with the above new PDE, we obtain the following eigenvalue problem:

$$\begin{cases} X''(x) + \lambda X(x) = 0,\ 0 < x < l, \\ X(0) = 0, X(l) = 0, \end{cases}$$

whose solution is given by

$$\begin{cases} \lambda_n = \left(\frac{n\pi}{l}\right)^2, \quad n = 1, 2, \cdots \\ X_n(x) = \sin\left(\frac{n\pi x}{l}\right). \end{cases}$$

We now expand $ge^{ht}$ in $\{X_n(x), n = 1, 2, \cdots\}$:

$$ge^{ht} = \sum_{n=1}^{\infty} \phi_n(t) \sin\left(\frac{n\pi x}{l}\right),$$

where the Fourier coefficients are given by

$$\phi_n(t) = \frac{2}{l} \int_0^l ge^{ht} \sin\left(\frac{n\pi x}{l}\right) dx = \frac{2g}{n\pi} [1 - (-1)^n] e^{ht}.$$

For $T(t)$, we have

$$\begin{cases} T'(t) + a^2 \lambda T(t) = \phi_n(t), \\ T_n(0) = 0. \end{cases}$$

whose solution is given by

$$T_n(t) = \frac{2g}{n\pi} \, \frac{[1 - (-1)^n]}{h + \left(\frac{n\pi a}{l}\right)^2} \left[ e^{ht} - e^{-\left(\frac{n\pi a}{l}\right)^2 t} \right], \quad n = 1, 2, \cdots$$

Let $V(x,t) = \sum\limits_{n=1}^{\infty} X_n(x) T_n(t)$. Then we have

$$u(x,t) = V(x,t) e^{-ht} = \sum_{n=1}^{\infty} \frac{2g}{n\pi} \, \frac{[1 - (-1)^n]}{h + \left(\frac{n\pi a}{l}\right)^2} \left( 1 - e^{-\left[h + \left(\frac{n\pi a}{l}\right)^2\right] t} \right) \sin\left(\frac{n\pi x}{l}\right).$$

□

(3) Assume that $u(x,y,t) = X(x)Y(y)T(t)$. Plugging this into the given PDE, we obtain the following:

$$\frac{T'(t)}{k^2 T(t)} = \frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = -\mu$$

and

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} - \mu = \lambda,$$

where $\lambda$ and $\mu$ are constants.

These and the boundary conditions lead to the following eigenvalue problems

$$\begin{cases} X''(x) + \lambda X(x) = 0, \quad 0 < x < a, \\ X(0) = 0, \quad X(a) = 0. \end{cases} \quad \textit{and} \quad \begin{cases} Y''(y) + (\mu - \lambda) Y(y) = 0, \quad 0 < y < b, \\ Y(0) = 0, \quad Y(b) = 0. \end{cases}$$

The eigenvalues and eigenfunctions for $(\lambda_n, X_n)$ are

$$\lambda_n = \big(\frac{n\pi}{a}\big)^2, \quad X_n(x) = \sin\big(\frac{n\pi x}{a}\big), \quad n = 1, 2, \cdots$$

The eigenvalues and eigenfunctions for $(\mu_{m,n}, Y_m)$ are

$$\mu_{m,n} = \big(\frac{n^2}{a^2} + \frac{m^2}{b^2}\big)\pi^2, \quad Y_m(y) = \sin\big(\frac{m\pi y}{b}\big), \quad m = 1, 2, \cdots$$

Let

$$u(x, y, t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} X_n(x)Y_m(y)T_{m,n}(t).$$

Then from the initial condition, we have

$$xy = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} c_{m,n} \sin\big(\frac{n\pi x}{a}\big) \sin\big(\frac{m\pi y}{b}\big),$$

where the double Fourier coefficients are determined by

$$c_{m,n} = \frac{4}{l^2}\int_0^l x \sin\big(\frac{n\pi x}{a}\big)dx \int_0^l y \sin\big(\frac{m\pi y}{b}\big)dy = (-1)^{(m+n)}\frac{4ab}{mn\pi^2}.$$

A substitution of these into the given PDE yields the following ODE

$$\begin{cases} T'_{m,n}(t) + \mu_{m,n}k^2 T_{m,n}(t) = 0, \quad m, n = 1, 2, \cdots \\ T_{m,n}(0) = c_{m,n}. \end{cases}$$

The solution of this ODE is given by

$$T_{m,n}(t) = c_{m,n}e^{-\mu_{m,n}k^2 t} = (-1)^{(m+n)}\frac{4ab}{mn\pi^2}e^{\left(-\left(\frac{n^2}{a^2}+\frac{m^2}{b^2}\right)k^2\pi^2 t\right)}.$$

We thus conclude finally that

$$u(x, y, t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(-1)^{(m+n)}\frac{4ab}{mn\pi^2}e^{\left(-\left(\frac{n^2}{a^2}+\frac{m^2}{b^2}\right)k^2\pi^2 t\right)} \sin\big(\frac{n\pi x}{a}\big) \sin\big(\frac{m\pi y}{b}\big).$$

□

## Problem 7: (5')

**Solution:**

By Theorem 3.7.1 and the comment following it, the bounded solution to the Cauchy problem for the heat equation is given by

$$u(x,t) = \int_{-\infty}^{\infty} \phi(\xi) G(x,t;\xi) d\xi = \frac{1}{2a\sqrt{\pi t}} \int_{-1}^{1} \phi(\xi) e^{-\frac{(x-\xi)^2}{4a^2 t}} d\xi.$$

Now, for any $t > 0$ we have $e^{-\frac{(x-\xi)^2}{4a^2 t}} \leq 1$. Thus

$$0 < u(x,t) \leq \frac{1}{2a\sqrt{\pi t}} \int_{-1}^{1} 1 dx = \frac{1}{a\sqrt{\pi t}}.$$

One then sees easily that $\lim_{t\to\infty} u(x,t) = 0$, meaning that $u$ decays as $t \to \infty$ with the rate $\frac{1}{\sqrt{t}}$. Physically, since there is no heat external source applied to the system, the initial energy in the interval (-1,1) is diffused to $\pm\infty$, resulting in the decay of temperature as $t \to \infty$. $\square$

## Problem 8: (5')

**Solution:**

In light of Theorem 3.7.1 and the comment following it, the bounded solution to the Cauchy problem for the heat equation is given by

$$u(x,t) = \int_{-\infty}^{\infty} \phi(\xi) G(x,t;\xi) d\xi = \frac{1}{2a\sqrt{\pi t}} \int_{-1}^{1} \phi(\xi) e^{-\frac{(x-\xi)^2}{4a^2 t}} d\xi.$$

Now, if $\phi$ is odd, then a change of variable of $\eta = -\xi$ yields

$$\begin{aligned} u(-x,t) &= \frac{1}{2a\sqrt{\pi t}} \int_{-\infty}^{\infty} \phi(\xi) e^{-\frac{(-x-\xi)^2}{4a^2 t}} d\xi \\ &= \frac{1}{2a\sqrt{\pi t}} \int_{-\infty}^{\infty} \phi(-\eta) e^{-\frac{(-x+\eta)^2}{4a^2 t}} d\eta \\ &= \frac{1}{2a\sqrt{\pi t}} \int_{-\infty}^{\infty} -\phi(\eta) e^{-\frac{(x-\eta)^2}{4a^2 t}} d\eta \\ &= -u(x,t). \end{aligned}$$

Similarly, if $\phi$ is even, then $u(-x,t) = u(x,t)$. $\square$

## Problem 9: (2'+2'+1')

**Solutions:**

(a) Let $S = Ke^x, t = T - \frac{\tau}{\sigma^2/2}$ and $v(x,\tau) = V(S,t)$. Then by the chain rule, we have

$$\begin{aligned} \frac{\partial v}{\partial \tau} &= \frac{\partial V}{\partial t}\frac{\partial t}{\partial \tau} = (rV - rSV_S - \frac{\sigma^2}{2} S^2 V_{SS}) \frac{-2}{\sigma^2}, \\ \frac{\partial V}{\partial S} &= \frac{\partial v}{\partial x}\frac{\partial x}{\partial S} = \frac{\partial v}{\partial x}\frac{e^{-x}}{K}, \end{aligned}$$

$$V_{SS} = \frac{\partial V_S}{\partial S} = \frac{\partial V_S}{\partial x}\frac{\partial x}{\partial S} = (v_{xx}\frac{e^{-x}}{K} - \frac{e^{-x}}{K}v_x)\frac{e^{-x}}{K}.$$

Thus

$$\begin{aligned}\frac{\partial v}{\partial \tau} &= \frac{-2}{\sigma^2}(rv - rKe^x\frac{\partial v}{\partial x}\frac{e^{-x}}{K} - \frac{\sigma^2}{2}K^2e^{2x}\frac{e^{-2x}}{K}(v_{xx}-v_x)) \\ &= v_{xx} - v_x - \frac{2r}{\sigma^2}v + \frac{2r}{\sigma^2}v_x \\ &= v_{xx} + (\frac{2r}{\sigma^2}-1)v_x - \frac{2r}{\sigma^2}v.\end{aligned}$$

(b) Let $u(x,\tau) = e^{\alpha x+\beta\tau}v(x,\tau)$. Then

$$\begin{aligned}\frac{\partial u}{\partial \tau} &= \beta e^{\alpha x+\beta\tau}v(x,\tau) + e^{\alpha x+\beta\tau}v_\tau(x,\tau), \\ \frac{\partial u}{\partial x} &= \alpha e^{\alpha x+\beta\tau}v(x,\tau) + e^{\alpha x+\beta\tau}v_x(x,\tau), \\ \frac{\partial^2 u}{\partial x^2} &= \alpha^2 e^{\alpha x+\beta\tau}v(x,\tau) + 2\alpha e^{\alpha x+\beta\tau}v_x(x,\tau) + e^{\alpha x+\beta\tau}v_{xx}.\end{aligned}$$

Combining the results in (a) with the condition $\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}$, we have

$$(2\alpha + 1 - \frac{2r}{\sigma^2})v_x + (\alpha^2 + \frac{2r}{\sigma^2} - \beta)v = 0,$$

implying that

$$\begin{cases} 2\alpha + 1 = \frac{2r}{\sigma^2} \\ \beta = \alpha^2 + \frac{2r}{\sigma^2}. \end{cases}$$

Hence, we have

$$\begin{cases} \alpha = \frac{\frac{2r}{\sigma^2} - 1}{2} \\ \beta = \frac{2r}{\sigma^2} + (\frac{r}{\sigma^2} - \frac{1}{2})^2. \end{cases}$$

(c) Now if $\phi(S) = \max(S-K, 0)$, then

$$u(x,0) = e^{\alpha x}\max(Ke^x - K, 0) = Ke^{\alpha x}\begin{cases} e^x - 1, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

The solution of the Cauchy problem is

$$u(x,\tau) = \frac{1}{2\sqrt{\pi\tau}}\int_{-\infty}^{\infty} e^{\alpha\xi}\phi(Ke^\xi)e^{-\frac{(x-\xi)^2}{4\tau}}d\xi.$$

Let $z = \frac{x-\xi}{\sqrt{2\tau}}$. Then

$$u(x,\tau) = I_1 - I_2,$$

where

$$I_1 = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{x}{\sqrt{2\tau}}} Ke^{(\alpha+1)(x-\sqrt{2\tau}z)} e^{-\frac{z^2}{2}} dz,$$

$$I_2 = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{x}{\sqrt{2\tau}}} Ke^{\alpha(x-\sqrt{2\tau}z)} e^{-\frac{z^2}{2}} dz.$$

Then

$$I_1 = \frac{1}{\sqrt{2\pi}} e^{\frac{\left(\frac{2r}{\sigma^2}+1\right)x}{2} + \frac{\tau\left(\frac{2r}{\sigma^2}+1\right)^2}{4}} \int_{-\infty}^{\frac{x}{\sqrt{2\tau}}} Ke^{-\frac{1}{2}\left[z+\sqrt{\frac{\tau}{2}}\left(\frac{2r}{\sigma^2}+1\right)\right]^2} dz.$$

Let $y = z + \sqrt{\frac{\tau}{2}}\left(\frac{2r}{\sigma^2}+1\right)$. Then

$$\begin{aligned} I_1 &= \frac{e^{\frac{\left(\frac{2r}{\sigma^2}+1\right)x}{2} + \frac{\tau\left(\frac{2r}{\sigma^2}+1\right)^2}{4}}}{\sqrt{2\pi}} \int_{-\infty}^{\frac{x}{\sqrt{2\tau}}+\sqrt{\frac{\tau}{2}}\left(\frac{2r}{\sigma^2}+1\right)} Ke^{-\frac{y^2}{2}} dy \\ &= Ke^{\frac{\left(\frac{2r}{\sigma^2}+1\right)x}{2} + \frac{\tau\left(\frac{2r}{\sigma^2}+1\right)^2}{4}} \Phi\Big(\frac{x}{\sqrt{2\tau}} + \sqrt{\frac{\tau}{2}}\big(\frac{2r}{\sigma^2}+1\big)\Big). \end{aligned}$$

Similarly, we can obtain that

$$I_2 = Ke^{\frac{\left(\frac{2r}{\sigma^2}-1\right)x}{2} + \frac{\tau\left(\frac{2r}{\sigma^2}-1\right)^2}{4}} \Phi\Big(\frac{x}{\sqrt{2\tau}} + \sqrt{\frac{\tau}{2}}\big(\frac{2r}{\sigma^2}-1\big)\Big).$$

Based on the above results, we finally have

$$\begin{aligned} u(x,\tau) &= I_1 - I_2 \\ &= e^{\frac{\left(\frac{2r}{\sigma^2}-1\right)x}{2} + \frac{\tau\left(\frac{2r}{\sigma^2}+1\right)^2}{4}} v(x,\tau), \end{aligned}$$

which implies that

$$v(x,\tau) = e^{-\frac{\left(\frac{2r}{\sigma^2}-1\right)x}{2} - \frac{\tau\left(\frac{2r}{\sigma^2}+1\right)^2}{4}} u(x,\tau).$$

Using the transformations in (a), we have

$$\begin{aligned} I_1' &= S\Phi\Bigg(\frac{\ln\left(\frac{S}{K}\right)}{\sqrt{\sigma^2(T-t)}} + \sqrt{\frac{\sigma^2(T-t)}{4}}\big(\frac{2r}{\sigma^2}+1\big)\Bigg) \\ &= S\Phi\Bigg(\frac{\ln\left(\frac{S}{K}\right) + (r+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\Bigg), \\ I_2' &= Ke^{-r(T-t)}\Phi\Bigg(\frac{\ln\left(\frac{S}{K}\right) + (r-\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\Bigg). \end{aligned} \tag{2}$$

Hence, we have

$$V(S,t) = v(x,\tau) = e^{-\frac{\left(\frac{2r}{\sigma^2}-1\right)x}{2} - \frac{\tau\left(\frac{2r}{\sigma^2}+1\right)^2}{4}} (I_1' + I_2'),$$

where $I_1'$ and $I_2'$ are defined in (2). □

**Problem 10:** (4'+4'+2')

**Solutions:**

(a) **Proof:** For any $T > 0$, applying the weak maximum principle to $u$ on $[0, l] \times [0, T]$, we have

$$u(x, t) \geq 0 \quad \forall (x, t) \in [0, l] \times [0, T].$$

Since $T > 0$ is arbitrary it follows that

$$u(x, t) \geq 0 \quad \forall (x, t) \in [0, l] \times [0, \infty).$$

If there exists $(x_0, t_0) \in (0, l) \times (0, \infty)$ such that $u(x_0, t_0) = 0$, then the strong minimum principle says that

$$u(x, t) \equiv 0 \quad \forall (x, t) \in [0, l] \times [0, t_0],$$

and so $\phi(x) = u(x, 0) \equiv 0$, a contradiction. Hence, we must have

$$u(x, t) > 0 \quad \forall (x, t) \in (0, l) \times (0, \infty).$$

□

(b) **Proof:** Let $w = u_t$. Then $w$ solves

$$\begin{cases} w_t - a^2 w_{xx} = 0, & 0 < x < l, \ t > 0, \\ w(0, t) = 0, \ w(l, t) = 0, & t > 0, \\ w(x, 0) = a^2 u_{xx}(x, 0) = a^2 \phi''(x) < 0, & 0 \leq x \leq l. \end{cases}$$

Essentially the same argument as that in (a) yields $w < 0$ ($u_t < 0$) for $(x, t) \in (0, l) \times (0, \infty)$. □

(c) Since $u_t = a^2 u_{xx}$ and at time $t = 0$, $u_{xx} < 0$ in $(0, l)$, we have $u_t < 0$ at $t = 0$ and in $(0, l)$. So $u$ is decreasing as t increases, at least when $t$ is small and $x \in (0, l)$. This behaviour continuous because $u_t < 0$ implies $u_{xx} < 0$, and vice versa.

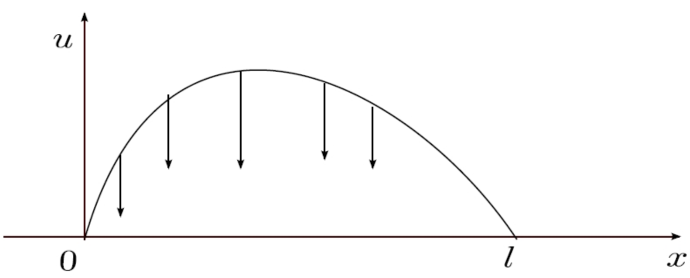


图 1: The behaviour of $u$ as $t$ increases if the initial value is concave down

If the initial value changes its concavity, then as illustrated in the following figure, in the intervals where $u$ is concave down, $u$ is decreasing in $t$; in the intervals where $u$ is concave up, $u$ is increasing in $t$. Recall because of the Dirichlet boundary condition, $\lim_{t\to\infty} u(x,t) = 0$. So the graph of $u$ becomes more and more concave down and at the mean time approaches the axis as $t$ increases.

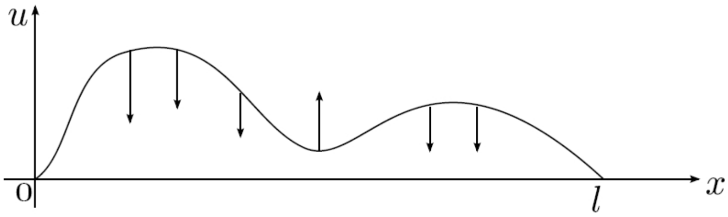


图 2: The behaviour of $u$ as $t$ increases if the initial value changes its concavity

□

## Problem 11:

(可参考课本 P38-P39 附录 3.2 的证明，考虑到有非数学系的同学选课，此题不计入总分。)

**Proof:** Notice that

$$\begin{aligned} u(x,t) &= \int_{-\infty}^{\infty} \phi(\xi)\frac{1}{2a\sqrt{\pi t}}e^{-\frac{(x-\xi)^2}{4a^2t}}d\xi \\ &= \Big(\int_{-\infty}^{x} + \int_{x}^{+\infty}\Big)\phi(\xi)\frac{1}{2a\sqrt{\pi t}}e^{-\frac{(x-\xi)^2}{4a^2t}}d\xi \\ &=: I_1 + I_2. \end{aligned}$$

Introducing a new variable

$$\eta = \frac{\xi - x}{2a\sqrt{t}},$$

we rewrite $I_2$ as

$$I_2 = \frac{1}{\sqrt{\pi}}\int_0^{+\infty} e^{-\eta^2}\phi(x+2a\sqrt{t}\eta)d\eta.$$

Now

$$I_2 - \frac{1}{2}\phi(x+0) = \frac{1}{\sqrt{\pi}}\int_0^{+\infty} e^{-\eta^2}\Big(\phi(x+2a\sqrt{t}\eta) - \phi(x+0)\Big)d\eta. \tag{3}$$

Since $\phi$ is bounded, we can find a constant $M$ such that $|\phi(y)| \le M$ for any y. Note that $e^{-\eta^2}$ is integrable on $(0,+\infty)$. Then for any $\epsilon > 0$, there exists a large $L > 0$ such that

$$\frac{1}{\sqrt{\pi}}\int_L^{+\infty} e^{-\eta^2}d\eta \le \frac{\epsilon}{2M}.$$

It follows that

$$\frac{1}{\sqrt{\pi}}\Big|\int_L^{+\infty} e^{-\eta^2}\Big(\phi(x+2a\sqrt{t}\eta) - \phi(x+0)\Big)d\eta\Big| \le 2M\cdot\frac{\epsilon}{2M} = \epsilon.$$

By definition, we have $\phi(x+2a\sqrt{t}\eta) \to \phi(x+0)$ as $t \to 0^+$. Therefore, there exists a small $\tau > 0$ such that

$$\Big|\phi(x+2a\sqrt{t}\eta)-\phi(x+0)\Big|\le\epsilon\quad\forall\, 0<t<\tau,\ 0<\eta\le L,$$

and hence

$$\begin{aligned}\frac{1}{\sqrt{\pi}}\Big|\int_0^L e^{-\eta^2}\Big(\phi(x+2a\sqrt{t}\eta)-\phi(x+0)\Big)d\eta\Big| &\le \frac{1}{\sqrt{\pi}}\int_0^L e^{-\eta^2}\epsilon d\eta\\ &\le \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} e^{-\eta^2}d\eta\epsilon=\epsilon,\end{aligned}$$

provided $0<t<\tau$. Then it follows from (3) that $I_2\to\frac{1}{2}\phi(x-0)$ as $t\to 0^+$. Similarly, we can show that $I_1\to\frac{1}{2}\phi(x-0)$ as $t\to 0^+$. □

## Problem 12: (3'× 5)

**Solutions:**

(a) **Proof:** We have by definition

$$E'(t)=2\int_0^l uu_t dx=2a^2\int_0^l uu_{xx}dx=2a^2\Big(uu_x\Big|_0^l-\int_0^l u_x^2dx\Big)=-2a^2\int_0^l u_x^2dx.$$

Then

$$E''(t)=-4a^2\int_0^l u_xu_{xt}dx=-4a^4\int_0^l u_xu_{xxx}dx=-4a^4\Big(u_xu_{xx}\Big|_0^l-\int_0^l u_{xx}^2dx\Big)=4a^4\int_0^l u_{xx}^2dx.$$

□

(b) **Proof:** Define a quadratic function of $r$ by

$$\xi(r)=\int_0^l\Big(f(x)+rg(x)\Big)^2dx=\int_0^l f^2(x)dx+2r\int_0^l f(x)g(x)dx+r^2\int_0^l g^2(x)dx.$$

Since $\xi(r)\ge 0$ for all $r\in\mathbb{R}$, we know its discriminant

$$\Delta=4\Big(\int_0^l f(x)g(x)dx\Big)^2-4\int_0^l f^2(x)dx\int_0^l g^2(x)dx\le 0,$$

from which follows the Cauchy-Schwarz inequality. □

(c) **Proof:** We have by definition and (b) that

$$(E')^2=4a^4\Big(\int_0^l uu_{xx}\Big)^2\le 4a^4\int_0^l u^2dx\int_0^l u_{xx}^2dx=EE'',\quad t\in[t_0,0).$$

□

(d) **Proof:** A direct calculation shows

$$(\ln E(t))'' = \frac{EE'' - (E')^2}{E^2} \geq 0, \quad t \in [t_0, 0).$$

□

(e) **Proof:** Let $g(t) = \ln E(t),\ t \in [t_0, 0)$. Then $g''(t) \geq 0$ on $[t_0, 0)$, meaning that $g$ is a convex function. Hence, by properties of convex functions, we have

$$g(t) \geq g'(t_0)(t - t_0) + g(t_0), \quad \forall\, t \in [t_0, 0).$$

Thus

$$-\infty = \lim_{t \to 0^-} g(t) \geq -t_0 g'(t_0) + g(t_0).$$

A contradiction. □