P120-175

HW3. 2, 4, 5, 6, 7, 8, 9, 10*

11, 12, 13, 15, 16, 18, 19, 20, 21.

2. 证明: $Var(\bar{a}_{\overline{T}|}) = \dfrac{{}^2\bar{A}_x - (\bar{A}_x)^2}{\delta^2} = \dfrac{[1-2\delta({}^2\bar{a}_x)] - (1-\delta\bar{a}_x)^2}{\delta^2}$

$= \dfrac{2(\bar{a}_x - {}^2\bar{a}_x)}{\delta} - (\bar{a}_x)^2$

4. 解: n年期生存年金 $Z = \bar{a}_{\overline{T(x)\wedge n}|} = \dfrac{1-e^{-\delta(T(x)\wedge n)}}{\delta}$, $P(Z \le z) = P\left(\dfrac{1-e^{-\delta(T(x)\wedge n)}}{\delta} \le z\right)$

$= P(\delta(T(x)\wedge n)) \le -\ln(1-\delta z))$

终身生存年金 $Z = \bar{a}_{\overline{T(x)}|} = \dfrac{1-e^{-\delta T(x)}}{\delta}$, $P(Z \le z) = P\left(\dfrac{1-e^{-\delta T(x)}}{\delta} \le z\right)$

$= P(\delta T(x) \le -\ln(1-\delta z))$

n年确定期生存年金 $Z = \bar{a}_{\overline{T(x)}|} - \bar{a}_{\overline{T(x)\wedge n}|}$

延期n年的终身生存年金 $Z = \bar{a}_{\overline{T(x)\vee n}|}$ $P(Z \le z) = P(\delta(T(x)\vee n) \le -\ln(1-\delta z))$

5. 证明: $a_{x:\overline{n}|} = \sum_{k=1}^{n} v^k {}_kp_x = v p_x \sum_{k=0}^{n-1} v^k {}_kp_{x+1} = {}_1E_x \ddot{a}_{x+1:\overline{n}|}$

${}_{n|}a_x = a_x - a_{x:\overline{n}|} = (\ddot{a}_x - 1) - (\ddot{a}_{x:\overline{n}|} - 1 + {}_np_x v^n)$

$= \ddot{a}_x - \ddot{a}_{x:\overline{n}|} - {}_nE_x = \dfrac{A_{x:\overline{n}|} - A_x}{d} - {}_nE_x$

$v\ddot{a}_{x:\overline{n}|} - a_{x:\overline{n-1}|} = v\ddot{a}_{x:\overline{n}|} - \ddot{a}_{x:\overline{n}|} + 1 = (v-1)\dfrac{1-A_{x:\overline{n}|}}{d} + 1$

$= \dfrac{(v+d-1) - (v-1)A_{x:\overline{n}|}}{d} = A_{x:\overline{n}|}$

6. 证明：记 $d_2$ 是2倍利息力对应的贴现率，则有 ${}^2\ddot{a}_x = \frac{1-{}^2A_x}{d_2}$.

因为 $1-d_2=(1-d)^2$，知 $d_2=2d-d^2 \Rightarrow {}^2A_x = 1-(2d-d^2)\,{}^2\ddot{a}_x$

故 $Var(v^{K+1}) = {}^2A_x - (A_x)^2$

$= 1-(2d-d^2)\,{}^2\ddot{a}_x - (1-d\ddot{a}_x)^2$

$= 2d(\ddot{a}_x - {}^2\ddot{a}_x) - d^2((\ddot{a}_x)^2 - {}^2\ddot{a}_x)$

因此 $Var(\ddot{a}_{\overline{K+1|}}) = \frac{1}{d^2}Var(v^{K+1}) = \frac{2}{d}(\ddot{a}_x - {}^2\ddot{a}_x) - ((\ddot{a}_x)^2 - {}^2\ddot{a}_x)$

7. 解：① $\bar{A}_x = \frac{i}{\delta}A_x = \frac{i}{\delta}(1-d\ddot{a}_x) = \frac{i}{\delta} - \frac{id}{\delta}\ddot{a}_x = \frac{i}{\delta} - \frac{i-d}{\delta}\ddot{a}_x$ ✓

② $\bar{A}_{x:\overline{n|}} = \bar{A}^1_{x:\overline{n|}} + A_{x:\overline{n|}} = \frac{i}{\delta}A^1_{x:\overline{n|}} + A_{x:\overline{n|}} \neq \frac{i}{\delta}A_{x:\overline{n|}}$ ×

③ $(\bar{I}\bar{A})_x = \frac{i}{\delta}\left((IA)_x - \left(\frac{1}{d}-\frac{1}{\delta}\right)A_x\right) \neq \frac{i}{\delta}(IA)_x$ ×

8. 解 对男性 $\bar{a}_x = \frac{1-\bar{A}_x}{\delta} = \frac{1-0.15}{0.1} = 8.5$

对女性 $\bar{a}_x = \frac{1-\bar{A}_x}{\delta} = 9.1$

对随机个体 $\bar{a}_x = \frac{1}{2}\times 8.5 + \frac{1}{2}\times 9.1 = \cancel{8.8}\ 8.8$

对男性 ${}^2\bar{A}_x = \delta^2 Var(\bar{a}_{\overline{T|}}) + (\bar{A}_x)^2 = 0.0725$

对女性 ${}^2\bar{A}_x = 0.0481$

对随机 ${}^2\bar{A}_x = \frac{1}{2}\times 0.0725 + \frac{1}{2}\times 0.0481 = \cancel{0.0603}\ 0.0603$

$\Rightarrow Var(\bar{a}_{\overline{T|}}) = \frac{{}^2\bar{A}_x - (\bar{A}_x)^2}{\delta^2} = 4.59$

9. 解: $P(\bar{a}_{\overline{T}|} > 2) = P\left(\frac{1-e^{-\delta T}}{\delta} > 2\right) = P\left(T > -\frac{\ln(1-2\delta)}{\delta}\right)$

$= P(T > 2.06) = {}_{2.06}p_x = e^{-\int_0^{2.06}\mu(x+t)dt}$

$= e^{-2.06(-0.24/\ln 0.4)} \approx 0.583.$

10* 解: $E(Y) = \ddot{a}_{\overline{1}|}P(K=0) + \ddot{a}_{\overline{2}|}P(K=1) + \ddot{a}_{\overline{3}|}P(K \geq 2)$

$= 1 \times \frac{1}{80} + (1+\frac{1}{1.05}) \times \frac{1}{80} + (1+\frac{1}{1.05}+\frac{1}{1.05^2}) \times \frac{78}{80}$

$\approx 2.825$

$E(Y^2) = (\ddot{a}_{\overline{1}|})^2 P(K=0) + (\ddot{a}_{\overline{2}|})^2 P(K=1) + (\ddot{a}_{\overline{3}|})^2 P(K \geq 2) \approx 8.032$

$Var(Y) = E(Y^2) - (E(Y))^2 = 0.051$

11. A. $cov(\bar{a}_{\overline{T}|}, v^T) = E(\bar{a}_{\overline{T}|}v^T) - E(\bar{a}_{\overline{T}|})E(v^T)$

$= Var(\bar{a}_{\overline{T}|} + v^T) - var(\bar{a}_{\overline{T}|}) - Var(v^T)$

$Var(\bar{a}_{\overline{T}|}) = \frac{{}^2\bar{A}_x - (\bar{A}_x)^2}{\delta^2}$ $\quad v^T = e^{-\delta T}$

$cov(\bar{a}_{\overline{T}|}, v^T) = E\left(\frac{v^T - v^{2T}}{\delta}\right) - \bar{a}_x \bar{A}_x = \frac{\bar{A}_x - {}^2\bar{A}_x}{\delta} - \left(\frac{1-\bar{A}_x}{\delta}\right)\bar{A}_x = \frac{\bar{A}_x^2 - {}^2\bar{A}_x}{\delta}$

12. D. $\ddot{a}_{x:\overline{3}|} = \sum_{j=0}^{2} v^j {}_jp_x = 1 + v\,{}_1p_x + v^2\,{}_2p_x = 1 + 0.9 \times 0.8 \times 2 + 3 \times 0.9^2 \times 0.8 \times 0.75 = 3.898$

$P(x > K) = P(x > 3.898)$ . 活到3年时 ${}_2p_x = 0.8 \times 0.75 = 0.6$

13. E. $E(Z) = 2 \times ({}_0p_x - {}_1p_x) + (2+3v) \times ({}_1p_x - {}_2p_x) + (2+3v+4v)\,{}_2p_x = 6.104$

$E(Z^2) = 2^2 \times ({}_0p_x - {}_1p_x) + (2+3v)^2 \times ({}_1p_x - {}_2p_x) + (2+3v+4v)^2\,{}_2p_x = 43.044$

$Var(Z) = E(Z^2) - (E(Z))^2 \approx 5.8$

15. D. $Var(\bar{a}_{\overline{T}|}) = \frac{100}{9}$ $\mu(x+t) = k$, $t>0$. $\delta = 4k$

$\bar{A}_x = \int_0^{\infty} v^t \, {}_tp_x \, \mu_x(t) \, dt = \int_0^{\infty} e^{-8kt} \cdot k \, dt = \frac{1}{8}$

${}^2\bar{A}_x = \int_0^{\infty} e^{-12kt} \cdot k \, dt = \frac{1}{12}$

$Var(\bar{a}_{\overline{T}|}) = \frac{{}^2\bar{A}_x - (\bar{A}_x)^2}{\delta^2} = \frac{\frac{1}{12} - \frac{1}{8^2}}{16k^2} = \frac{100}{9} \Rightarrow k \approx 0.020$.

16. D. $\ddot{a}_{x:\overline{4}|} = \sum_{j=0}^{3} v^j \, {}_jp_x = 1 + v \, {}_1p_x + v^2 \, {}_2p_x + v^3 \, {}_3p_x$

$= E(\ddot{a}_{\overline{K+1}|}) = \ddot{a}_{\overline{1}|} \, {}_{0|}q_x + \ddot{a}_{\overline{2}|} \, {}_{1|}q_x + \ddot{a}_{\overline{3}|} \, {}_{2|}q_x + \ddot{a}_{\overline{4}|} (1 - {}_{0|}q_x - {}_{1|}q_x - {}_{2|}q_x)$

$= 1 \times 0.33 + 1.93 \times 0.24 + 2.8 \times 0.16 + 3.62 \times 0.27$

$\approx 2.2$

18. A ${}_tp_x = e^{-\int_0^t \mu(x+s) ds} = e^{-0.01t}$ $\ddot{a}_x = \sum_{j=0}^{\infty} v^j \, {}_jp_x = \sum_{j=0}^{\infty} \left(\frac{1}{1+i}\right)^j e^{-0.01j}$

${}_{5|}\ddot{a}_x = \ddot{a}_x - \ddot{a}_{x:\overline{5}|} \approx 16.28 \Rightarrow P(S > {}_{5|}\ddot{a}_x) = 1 - P(S \le {}_{5|}\ddot{a}_x) = 1 - P(K(x) \le 21)$

$\approx 0.81$

19. E. $Y = \ddot{a}_{\overline{K(40)+1} - \overline{(K(40)+1) \wedge 10}|}$

$Z_1 = v^{K(40)+1}$ $Z_2 = v^{(K(40)+1) \wedge 10}$

$Y = 1 + v + \cdots + v^{K(40)+1} - (1 + v + \cdots v^{(K(40)+1) \wedge 10}) = 21(Z_2 - Z_1)$

20 A, $E_{Z_1 Z_2} = E(v^{K(40)+1 + (K(40)+1) \wedge 10}) = {}^2A'_{40:\overline{10}|} + v^{10} \, {}_{10|}A_{40} = {}^2A'_{40:\overline{10}|} + v^{10} \, {}_{10}E_{40} A_{50}$

$\approx 0.13$

21 E, $E(Y^2) = E(21^2 (Z_1^2 + Z_2^2 - 2Z_1 Z_2)) = 441 \times \Big(0.06741 - \left(\frac{1}{1+0.05}\right)^{10} {}_{10}E_{40} \cdot 0.12446$

$+ 0.590041 \cdot v^{10} + 0.06741 - 2 \times 0.13\Big)$

$\approx 83$.

24. D $\int_0^{1.5} {}_tp_{60}\,dt = \int_0^{1} {}_tp_{60}\,dt + \int_0^{0.5} {}_{1+t}p_{60}\,dt = \int_0^1 0.98\,dt + \int_0^{0.5} \frac{0.978}{0.98}\,dt = 1.478$

25. A $\int_0^{28} t\ {}_tp_{36}\,\mu_{36}(t)\,dt = \int_0^{28} t\,\frac{l_{36+t}}{l_{36}}\,\mu(36+t)\,dt = \int_0^{28} \frac{t}{8}\left(-\frac{d\,l_{36+t}}{d_{36+t}}\right)dt$

$= -\frac{1}{8}\int_0^{28} t\,d\,l_{36+t} = -\frac{1}{8}\left(t\,l_{36+t}\Big|_0^{28} - \int_0^{28} l_{36+t}\,dt\right)$

$= -\frac{1}{8}\left(28\times 6 - \int_0^{28}\sqrt{64-t}\,dt\right) \approx 3.67$

28 B. $\mu(25) = -\frac{S'(25)}{S(25)} = -\frac{-\frac{1}{100}}{1-\frac{25}{100}} = \frac{1}{75} \approx 0.0133$

29 B. $E(K(60)) = \sum_{n\geq 1} {}_np_{60} = \sum_{n\geq 1}(1+c)^{-n} = \frac{1}{c}$

$E(K^2(60)) = \sum_{n\geq 1}(2n-1)\,{}_np_{60} = \sum_{n\geq 1}(2n-1)(1+c)^{-n} = \frac{c+2}{c^2}$

$var(K(60)) = E(K^2(60)) - \cancel{E(K60)}\ E(K(60))^2 = \frac{c+2}{c^2} - \frac{1}{c^2} = \frac{c+1}{c^2}$

32. A $\mathring{e}_y = \int_0^{\infty} {}_tp_y\,dt = \frac{S(y+t)}{S(y)} = \frac{y+1}{y+1+t} = \frac{1}{2} \Rightarrow t = y+1$

33 A. $\begin{cases} {}_{1|1}q_{x+1} = p_{x+1} - {}_2p_{x+1} = p_{x+1} - p_{x+1}p_{x+2} = 0.095 \\ {}_{2|1}q_{x+1} = {}_2p_{x+1} - {}_3p_{x+1} = p_{x+1}p_{x+2}q_{x+3} = 0.171 \end{cases} \Rightarrow \begin{cases} p_{x+1} = \frac{19}{20} \\ p_{x+2} = \frac{9}{10} \end{cases}$

$\Rightarrow q_{x+1} + q_{x+2} = 2 - \frac{19}{20} - \frac{9}{10} = 0.15$