HW2. 1, 2, 3, 4, 6, 7*, 8

11, 12, 16, 17, 20, 21.

1. 解. $\bar{A}_x = \int_0^{\infty} e^{-\delta t} {}_tp_x \mu_x(t)\, dt = \sum_{k=0}^{\infty} \int_k^{k+1} e^{-\delta t} {}_tp_x \mu_x(t)\, dt$

$\geq \sum_{k=0}^{\infty} e^{-\delta(k+1)} \int_k^{k+1} {}_tp_x \mu_x(t)\, dt = \sum_{k=0}^{\infty} e^{-\delta(k+1)} {}_kp_x q_{x+k} = A_x$

2. 证明: $\bar{A}_x = \int_0^{\infty} e^{-\delta t} {}_tp_x \mu_x(t)\, dt = \int_0^{\infty} e^{-\delta t} \cdot e^{-\mu t} \cdot \mu\, dt = \frac{\mu}{\mu+\delta}$

3. 解. ${}_tp_x = e^{-\int_x^{x+t} \mu(s) ds} = \frac{1+x}{1+x+t}$

$\Rightarrow \bar{A}_x = \int_0^{\infty} e^{-\delta t} \frac{1+x}{1+x+t} \cdot \frac{1}{1+x+t}\, dt = 1 - \delta \int_0^{\infty} \frac{1+x}{1+x+t} e^{-\delta t}\, dt$

$\Rightarrow \frac{d\bar{A}_x}{dx} = -\delta \int_0^{\infty} \left[d\left(\frac{1+x}{1+x+t} e^{-\delta t}\right) \Big/ dx \right] dt = -\delta \int_0^{\infty} e^{-\delta t} \frac{t}{(1+x+t)^2}\, dt.$

因为 $e^{-\delta t} \frac{t}{(1+x+t)^2}$ 在 $t \in (0, +\infty)$ 上恒大于0 $(\delta > 0)$ $\Rightarrow \frac{d\bar{A}_x}{dx} < 0.$

4. 解 ${}_tp_x = \frac{100-x-t}{100-x}$, $\mu_x(t) = \frac{1}{100-x-t}$

$\bar{A}^1_{40:\overline{25|}} = \int_0^{25} e^{-\delta t} {}_tp_{40} \mu_{40}(t)\, dt = \int_0^{25} e^{-0.05t} \cdot \frac{1}{60}\, dt$

$= \frac{1}{3}(1 - e^{-1.25}) \approx 0.238.$

6. 证明: $\bar{A}^1_{x:\overline{n|}} = \int_0^n e^{-\delta t} {}_tp_x \mu_x(t)\, dt = -\int_0^n e^{-\delta t} d\,{}_tp_x$

$= 1 - e^{-\delta n} {}_np_x - \delta \int_0^n {}_tp_x e^{-\delta t}\, dt.$

$$\Rightarrow \frac{d\bar{A}^1_{x:\overline{n}|}}{dx} = -e^{-\delta n}\frac{d\,{}_np_x}{dx} - \delta\int_0^n e^{-\delta t}\frac{d\,{}_tp_x}{dx}dt$$

$$= -e^{-\delta n}{}_np_x(\mu(x)-\mu(x+n)) - \delta\int_0^n e^{-\delta t}{}_tp_x(\mu(x)-\mu(x+t))dt.$$

$$= -\mu(x)\left(e^{-\delta t}{}_np_x + \delta\int_0^n e^{-\delta t}{}_tp_x dt\right) + \mu(x+n)e^{-\delta t}{}_np_x + \delta\int_0^n {}_tp_x\mu(x+t)e^{-\delta t}dt$$

$$= -\mu(x)A_{x:\overline{n}|} + \mu(x)\int_0^n {}_tp_x\, de^{-\delta t} + \mu(x+n)A_{x:\overline{n}|}^{\;\,1} + \delta\bar{A}^1_{x:\overline{n}|}$$

$$= -\mu(x)A_{x:\overline{n}|} + \left[\mu(x)\,{}_np_xe^{-\delta n} - \mu(x) + \mu(x)\int_0^n e^{-\delta t}{}_tp_x\,\mu(x+t)dt\right] + \mu(x+n)A_{x:\overline{n}|}^{\;\,1} + \delta\bar{A}^1_{x:\overline{n}|}$$

$$= (\mu(x)+\delta)\cancel{A_{x:\overline{n}|}}\ \bar{A}^1_{x:\overline{n}|} + \mu(x+n)A_{x:\overline{n}|}^{\;\,1} - \mu(x)$$

7.* 证明：因为 $\dfrac{d\bar{A}_x}{dx} = \delta\bar{A}_x + \mu(x)(\bar{A}_x - 1)$

所以 $$\int_y^\infty \mu(x)v^{x-y}(1-\bar{A}_x)dx = \int_y^\infty v^{x-y}\left(\delta\bar{A}_x - \frac{d\bar{A}_x}{dx}\right)dx$$

$$= \int_y^\infty e^{-\delta(x-y)}\delta\bar{A}_x dx - \int_y^\infty e^{-\delta(x-y)}d\bar{A}_x$$

$$= -\int_y^\infty \bar{A}_x\, de^{-\delta(x-y)} - \int_y^\infty e^{-\delta(x-y)}d\bar{A}_x$$

$$= -\bar{A}_x e^{-\delta(x-y)}\Big|_y^\infty = \bar{A}_y$$

8. 解 $$\bar{A}_x = \int_0^\infty e^{-\delta t}\cdot\frac{2}{10\sqrt{2\pi}}e^{-\frac{t^2}{200}}dt.$$

$$= 2e^{\frac{1}{8}}\int_0^\infty \frac{1}{10\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{t+5}{10}\right)^2}dt \quad \left(x = \frac{t+5}{10}\right)$$

$$= 2e^{\frac{1}{8}}\int_{\frac{1}{2}}^\infty \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}dx = 2e^{\frac{1}{8}}\left(1-\Phi\left(\tfrac{1}{2}\right)\right) \approx 0.70.$$

$${}^2\bar{A}_x = \int_0^\infty e^{-2\delta t}\cdot\frac{2}{10\sqrt{2\pi}}e^{-\frac{t^2}{200}}dt = 2e^{\frac{1}{2}}\int_1^\infty\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}dy = 2e^{\frac{1}{2}}(1-\Phi(1)) \approx 0.52$$

11, C. 12, E. 16, B. 17, B. 20, C. 21, A..

11, C, 未来生存时间 $T_1(x), \cdots, T_{100}(x) \Rightarrow Z = 1000\sum_{j=1}^{100} v^{T_j(x)}$

$\bar{A}_x = E(Z) = 0.06 \qquad Var(Z) = {}^2\bar{A}_x - (E(Z))^2 = 0.01 - 0.06^2 = 0.0064$

$P\left\{1000\sum_{j=1}^{100} v^{T_j(x)} \le 100\nu\right\} = P\left\{\frac{\sum_{j=1}^{100} v^{T_j(x)} - 6}{8} < \frac{\frac{\nu}{10} - 6}{8}\right\} = 0.95$

$\Rightarrow \nu \approx 73.$

12, E. $l_x = 110 - x, \quad 0 \le x \le 110. \quad \delta = 0.05$

$v = e^{-0.05}, \quad Z = v^{T(x)} = e^{-0.05T(x)}$

$F_Z(z) = P(Z < z) = P(v^{T(x)} \le z) = P\left(T(x) \ge -\frac{\ln z}{\delta}\right) = 1 - \frac{\ln z}{2.5}$

$f_Z(0.8) = \frac{1}{2.5 \times 0.8} \approx 0.36.$

16. B. $E(Z_3) = v^n {}_nP_x = 0.1, \quad E(Z_2) = E(Z_1) + E(Z_3) = 0.37, \quad E(Z_3)^2 = 0.02$

$Var(Z_3) = 0.01 \qquad Var(Z_2) = Var(Z_3) + Var(Z_1) - 2E(Z_1)E(Z_3) = 0.049$

17 B. $Var(X+Y) = Var(X) + Var(Y) + 2(EXY - EXEY) = 5.188$

20 C, ${}^2\bar{A}^1_{x:\overline{2}|} = \int_0^2 v^{2t}{}_tP_x \mu_x(t)\,dt = \int_0^1 v^{2t} q_x\,dt + P_x \int_0^1 v^{2t+2} q_{x+1}\,dt$

$= (0.1 + 0.9 \times 0.2v^2)\int_0^1 v^{2t}\,dt = (0.1 + 0.18v^2)\frac{1}{2\delta}(1 - v^2) = 0.218$

21, A. $A_x = A^1_{x:\overline{n}|} + {}_nE_x \cdot A_{x+n} = y + A_{x:\overline{n}|}^{\ \ 1} \cdot A_{x+n}$

$= y + (A_{x:\overline{n}|} - A^1_{x:\overline{n}|})A_{x+n} = y + (u - y)z = y(1-z) + uz.$