**MAT302 Functional Analysis, Final Exam**
Time: 14:00-16:00, June 18, 2024
Name:
ID:

Problem 1. [10 pts]
(1a) [5 pts] What is the definition of a Hilbert space?
(1b) [5 pts] State the uniform boundedness principle.

Problem 2. [20 pts] Equip the vector space $X = \{(\xi_i) : \xi_i \in \mathbb{R} \text{ for } i \in \mathbb{N}\}$ with the topology induced by the following metric

$$d(x, y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \cdot \frac{|\xi_i - \eta_i|}{1 + |\xi_i - \eta_i|}, \quad x = (\xi_i), y = (\eta_i).$$

Show that $X$ is complete.

Problem 3. [20 pts] Show that a Banach space $\mathcal{X}$ is finite-dimensional if and only if the unit sphere $S^1_{\mathscr{X}} := \{x \in X : \|x\| = 1\}$ in $\mathscr{X}$ is compact.

Problem 4. [20 pts] Find $a, b \in \mathbb{R}$ to minimize the expression

$$\int_0^1 |e^t - a - bt^2|^2 \mathrm{d}t.$$

Problem 5. [15 pts] Let $\mathscr{X}$ be a Banach space and $\{x_n\}$ be a sequence in $\mathscr{X}$ that converges weakly to $x$. Show that if $\|x_n\| \leqslant 1$ for all $n \in \mathbb{N}$, then $\|x\| \leqslant 1$.

Problem 6. [15 pts] Suppose $p, q \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$ and consider the space

$$\ell^p = \{x = (\xi_i) : \xi_i \in \mathbb{R} \text{ for } i \in \mathbb{N}, \sum_{i=1}^{\infty} |\xi_i|^p < \infty\}$$

Let $\{a_n\}$ be a sequence of real numbers such that for any element $x = (\xi_n) \in \ell^p$, the infinite sum $\sum_{n=1}^{\infty} a_n \xi_n$ is convergent. Show that $(a_n) \in \ell^q$, that is, $\sum_{n=1}^{\infty} |a_n|^q < \infty$.