MA303 偏微分方程 第三次作业

12232859 徐临风

Chapter 3

Problem 1: (3’)

Proof: Take any arbitrary patch ΔS on S and let Φ be the total rate of thermal energy crossing ΔS in the direction of $\vec{n}$. Then since on S, $u_1 = u_2$, by Fourier's Law, we have

$$\int_{\Delta S} -k_1 \frac{\partial u_1}{\partial \vec{n}} \mathrm{dS} = \Phi = \int_{\Delta S} -k_2 \frac{\partial u_2}{\partial \vec{n}} \mathrm{dS}.$$

As ΔS is arbitrary, we can conclude that $-k_1 \frac{\partial u_1}{\partial \vec{n}} = -k_2 \frac{\partial u_2}{\partial \vec{n}}$ on S. □

Problem 2: (4’+1’)

注：(b) scaling relationship 言之有理即可。

Proof:

(a) Fix $x \in \partial\Omega_1$. Consider $f(\tau) = u_2(x + \tau\vec{n})$. By Taylor's expansion, we have

$$f(\tau) \approx f(0) + f^{'}(0)\tau = u_2(x) + \frac{\partial u_2}{\partial \vec{n}}\tau.$$

From the text, $x + \delta\tau \in \partial\Omega_2$, hence we have the following:

$$H = f(\delta) \approx u_2(x) + \frac{\partial u_2}{\partial \vec{n}}\delta = u_1(x) + \frac{\partial u_2}{\partial \vec{n}}\delta.$$

$$\Rightarrow \frac{\partial u_2}{\partial \vec{n}} \approx -\frac{1}{\delta}(u_1 - H).$$

Since x is arbitrary, by Exercise 1 in Chapter 3, on $\partial\Omega_1$, we have

$$k_1 \frac{\partial u_1}{\partial \vec{n}} + \frac{k_2}{\delta}(u_1 - H) = 0.$$

(b) From (a), we have on $\partial\Omega_1$ that

$$\frac{k_2}{\delta} = -\frac{k_1}{u_1 - H}\frac{\partial u_1}{\partial \vec{n}}.$$

To insulating the body well, the level of thermal isolation should remain the same as the thickness of coating δ changes, meaning that

$$\frac{k_2}{\delta} = Const.$$

$\square$

Problem 3: (5'+5')

注：对于此类专门考查求解特征值问题的题目，需要像课本 P18-P19 的 Eigenvalue problem 例子一样给出**详细的求解过程**；下面关于 (a) 的参考解答只给出了结果，具体过程类似于课本 P18-P19 的 Eigenvalue problem 例子。另外建议**写出特征对中下标 n 的取值范围**。

Solutions:

(a) Let $\tau = x + l$. Then for $Y(\tau) = X(\tau - l)$, we have

$$\begin{cases} Y^{''}(\tau) + \lambda Y(\tau) = 0, \ 0 < \tau < 2l \\ Y^{'}(0) = 0, \ Y(2l) = 0. \end{cases}$$

Solve this eigenvalue problem, we have

$$\begin{cases} \lambda_n = \left(\frac{(\frac{n}{2} - \frac{1}{4})\pi}{l} \right)^2 \\ Y_n(\tau) = \cos \frac{(\frac{n}{2} - \frac{1}{4})\pi}{l} \tau \end{cases}, \ n = 1, 2, \ldots$$

Hence, the solution to the original eigenvalue problem is the following:

$$\begin{cases} \lambda_n = \left(\frac{(\frac{n}{2} - \frac{1}{4})\pi}{l} \right)^2 \\ X_n(x) = \cos \frac{(\frac{n}{2} - \frac{1}{4})(x+l)\pi}{l} \end{cases}, \ n = 1, 2, \ldots$$

(b) Since $X(x)$ is a periodic function with period l, we have

$$X(x + l) = X(x) \quad \forall x. \tag{1}$$

The general solution to the ODE $X''(x) + \lambda X(x) = 0$ for the case $\lambda \neq 0$ can be written as

$$X(x) = C_1 e^{-\sqrt{-\lambda}x} + C_2 e^{\sqrt{-\lambda}x}. \tag{2}$$

The periodic condition (1), together with (2), leads to

$$e^{\sqrt{-\lambda}l} = 1,$$

meaning that

$$\sqrt{-\lambda_n} l = i(2\pi n) \Rightarrow \lambda_n = \frac{4\pi^2 n^2}{l^2},\ n = 1, 2, ...$$

We should also consider the case $\lambda = 0$, in which case the general solution to the ODE $X''(x) + \lambda X(x) = 0$ can be written as

$$X(x) = ax + b.$$

The periodic condition (1) forces $X(x)$ to be a constant, and $X(x)$ is an eigenfunction if the constant $b \neq 0$, which is possible.

We summarize the result as follows.

The eigenvalues should satisfy

$$\lambda_n = \frac{4\pi^2 n^2}{l^2},\ n = 0, 1, 2, ...$$

The corresponding eigenfunction is

$$X_n(x) = a_n \cos \frac{2\pi n}{l} x + b_n \sin \frac{2\pi n}{l} x,\ n = 0, 1, 2, ...$$

□

Problem 4: (5'+2'+5')

评分标准：$\lambda < 0$ 的情形写出(3)式 (3')，分类讨论特征对的存在性 (2');
$\lambda = 0$ 的情形写出对于特征对结果完整的讨论 (2')
$\lambda > 0$ 的情形写出(4)式 (3')，完整表示出 $h > 0$(课本例题) 和 $h < 0$ 的特征对结果 (2')

Solution:

Plugging the boundary conditions $X(0) = 0$ and $X'(l) + hX(l) = 0$ into (**??**), we consider the following cases:

Case 1: $\lambda < 0$.

In order that $\lambda < 0$ is an eigenvalue we must have

$$\frac{\mu}{hl} = -\frac{e^{\mu} - e^{-\mu}}{e^{\mu} + e^{-\mu}} = -\tanh(\mu), \tag{3}$$

where $\mu = \sqrt{-\lambda} l$.

It follows that there exists a unique negative eigenvalue $\lambda = -\left(\frac{\mu}{l}\right)^2$ if the slope of the straight line $y = \frac{\mu}{hl}$ is larger than the slope of $y = -\tanh(\mu)$ at $\mu = 0$, which is equivalent to

$$\frac{1}{hl} > -1.$$

We refer readers to see Figure 1.

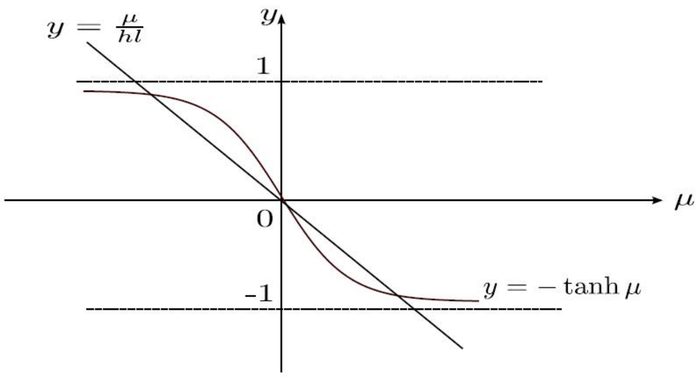

图 1: Case 1: $\lambda < 0$

Case 2: $\lambda = 0$.

In this case, if $h \neq \frac{-1}{l}$, then $\lambda = 0$ is not an eigenvalue. Otherwise, $\lambda = 0$ is an eigenvalue, and its eigenfunction is x.

Case 3: $\lambda > 0$.

In order that $\lambda > 0$ is an eigenvalue we must have

$$\frac{\mu}{hl} = -\tan(\mu), \tag{4}$$

where $\mu = \sqrt{\lambda}l$. (see Example 3.5.2 in textbook ($h > 0$) for details, $h < 0$ is similar)

Then there are infinitely many eigenvalues and eigenfunctions:

$$\begin{cases} \lambda_n = \left(\frac{\mu_n}{l}\right)^2 & n = 1, 2, 3, \ldots \\ X_n(x) = \sin\frac{\mu_n x}{l} & n = 1, 2, 3, \ldots. \end{cases}$$

Note that in Case 1, the solutions of the corresponding IBVP for the homogeneous heat equation will grow exponentially, because

$$u(x,t) = \phi_1 e^{-\lambda_1 a^2 t} X_1(x) + \phi_2 e^{-\lambda_2 a^2 t} X_2(x) + \cdots$$

and $\lambda_1 < 0, \lambda_1 < \lambda_2 < \lambda_3 < \cdots$ □

Problem 6: (10')

评分标准：正确求解特征值问题 (3')；求出 $T_n(t)$ 的完整表达式 (3')；写出 $u(x,t)$ 的完整表达式 (1')；求出 $\lim\limits_{t\to\infty} u(x,t)$(1')；物理解释 (1')；将结论扩展到高维 (1')

注：关于分离变量法的求解步骤请参考课本 P21 的 Summary 部分，对于特征值问题的求解如果熟练可以直接用课本 P23-P24 中一些特殊情况的结论，但要注意 x 的取值范围。求解特征值问题建议写出特征

对中下标 n 的取值范围，尤其是对于需要考虑 $n=0$ 的情形，避免漏掉第一项。

Solution:

(a) Let $u(x,t)=X(x)T(t)$. Then the corresponding eigenvalue problem for the given differential equation would be

$$\begin{cases} X^{''}(x)+\lambda X(x)=0,\ 0<x<l \\ X^{'}(0)=0,\ X^{'}(l)=0 \end{cases}$$

The solution to this eigenvalue problem is

$$\begin{cases} \lambda_n=(\frac{n\pi}{l})^2 \\ X_n(x)=\cos\frac{n\pi}{l}x \end{cases},\ n=0,1,2,...$$

Set $u(x,t)=\sum_{n=0}^{\infty}X_n(x)T_n(t)$. Then by the initial condition, we have

$$x=u(x,0)=\sum_{n=0}^{\infty}T_n(0)\cos\frac{n\pi}{l}x.$$

$$\Rightarrow T_n(0)=\begin{cases} \frac{1}{l}\int_0^l x\mathrm{dx}=\frac{l}{2},\ n=0 \\ \frac{2}{l}\int_0^l x\cos\frac{n\pi}{l}x\mathrm{dx}=\frac{2l}{n^2\pi^2}((-1)^n-1),\ n=1,2,3,... \end{cases}$$

Since $u_t=a^2u_{xx}$, we also have

$$T_n^{'}(t)-(\frac{n\pi a}{l})^2T_n(t)=0.$$

Hence $T_n(t)=T_n(0)\ e^{-(\frac{n\pi a}{l})^2t}$ and the solution of the original problem is

$$u(x,t)=\sum_{n=0}^{\infty}X_n(x)T_n(t)=\frac{l}{2}+\sum_{n=1}^{\infty}\frac{2l((-1)^n-1)}{n^2\pi^2}e^{-(\frac{n\pi a}{l})^2t}\cos\frac{n\pi}{l}x.$$

$$\Rightarrow\lim_{t\to\infty}u(x,t)=\frac{l}{2}=\frac{1}{l}\int_0^l u(\hat{x},0)\mathrm{d}\hat{\mathrm{x}}.$$

(b) Physically, this means that the Neumann boundary condition implies there is no energy exchange on the boundary, and hence the temperature of the system would tend to the average of the initial temperature. Generally, in higher spatial dimension, we have

$$\lim_{t\to\infty}u(x,t)=\frac{1}{|\Omega|}\int_\Omega u(\hat{x},0)\mathrm{d}\hat{\mathrm{x}}.$$

□