Linear programming

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History

- The word "programming" used traditionally by planners to describe the process of operations planning and resource allocation.
- In 1930s – 40s, this process could often be aided by solving LPs.
  Kantorovich: solutions to problems in production and transportation.
- The initial impetus came in the aftermath of World War II.
- In 1947, George Dantzig proposed the Simplex Method (poorly named great method1). Made the solution of LPs practical. But, it has exponential worst-case complexity.
- Advance in computer technology expand the applications of LP. Bringing people to study and apply LP extensively.

The Best of the 20th Century: Top 10 Algorithms by Barry A. Cipra

- 1946, von Neumann, Ulam, and Metropolis: Monte Carlo method
- 1947, Dantzig: the Simplex method
- 1950, Hestenes, Stiefel, and Lanczoz: Krylov subspace iteration methods
- 1951, Householder: decompositional approach to matrix computations
- 1957, Backus: Fortran optimizing compiler
- 1959 – 61: J.G.F. Francis of Ferranti Ltd.: QR algorithm
- 1962: Hoare: Quicksort
- 1965: Cooley and Tukey: the fast Fourier transform
- 1977, Ferguson and Forcade: integer relation detection algorithm
- 1987, Greengard and Rokhlin: fast multipole algorithm

Modern period

- 1950s – , Applications
- 1960s, Large-scale optimization
- 1970s, Complexity theory
- Khachyan, 1979, the ellipsoid algorithm, first polynomial-time algorithm, but impractical
- Karmakar, 1984, interior-point algorithms, lead to later interior-point methods.
- CPLEX 1.0, 1988, research shifts to commercial
- CPLEX acquired by ILOG, which was later acquired by IBM
- Gurobi, 2008
- Today: huge-scale, distributed, streaming LP

Example: the diet problem

- n different foods, jth food sells at price c_j per unit
- m basic nutrients; for balanced diet, receive at least b_i units of ith nutrient
- each unit of food j contains a_{ij} units of ith nutrient
- variable x_j : # units of food j in diet
- total cost: $c_1 x_1 + \cdots + c_n x_n$
- nutritional constraints: $a_{i1} x_1 + \cdots + a_{in} x_n \geq b_i, i = 1, \cdots, m.$

$$\begin{aligned} \min \ & c^T x \\ \text{subject to } & Ax = b \\ & x \geq 0. \end{aligned}$$

Graphic LP in 2D

In the problem, a company manufactures two iPod player models, both with 3.5-inch LCD but have different memory capacities:

- 16GB – two 8GB chips
- 8GB – one 8GB chip

Weekly resources are limited to

- 800 units of 3.5-inch antiglare LCDs
- 1000 units of 8GB memory chips
- 50 hours of total labor time. It takes 3 minutes of labor for each 16GB player, and 4 minutes of labor for each 8GB player.

For marketing reasons,

- Total production cannot exceed 700
- # 16GB players cannot exceed # 8GB players by more than 350

Profit, while remaining within the marketing guidelines, can be computed as

- $16 each 16GB player
- $10 each 8GB player

The current weekly production plan consists of 450 16GB players and 100 8GB players, make a profit of $16*450+$10*100=$8200. Management is seeking a new production plan that will increase the profit.

Variables:

- x_1: weekly produced units of 16GB players
- x_2: weekly produced units of 8GB players

Objective: to maximize the weekly profit $16x_1 + 10x_2$
Constraints:

- $x_1, x_2 \geq 0$
- LCD: $x_1 + x_2 \leq 800$
- Memory: $2x_1 + x_2 \leq 1000$
- Labor: $3x_1 + 4x_2 \leq 3000$
- Marketing total: $x_1 + x_2 \leq 700$
- Marketing mix: $x_1 - x_2 \leq 350$

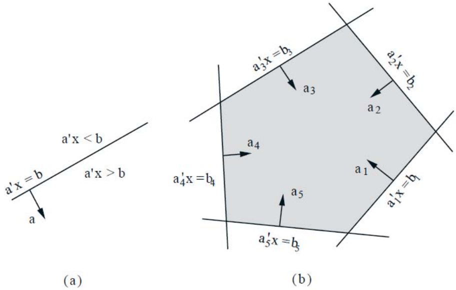
Graphical optimization

- 2D plot of the variables, constraints, and level curves of the objective
- feasible region is a polyhedron, possibly empty or unbounded
- three types of feasible points: interior, boundary, and extreme points
- level curves of the objective are parallel lines
- if there is a solution, there is an extreme point solution
- it is possible that the problem is feasible but has an unbounded $-\infty$ optimal objective
- they can be infinitely many solutions

Geometric concepts

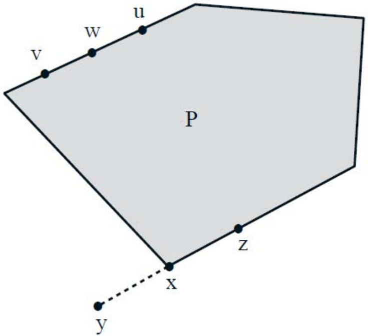
- A set S is convex if any $x, y \in S, \alpha x + (1-\alpha)y \in S$, $\forall \alpha \in [0,1]$
- Let $S := \{x_1, \cdots, x_K\}$.
  - $\text{span}(S) = \{\sum_{k=1}^{K} \lambda_k x^k : \lambda_k \in \mathbb{R},\ \forall k\}$
  - $\text{aff}(S) = \{\sum_{k=1}^{K} \lambda_k x^k : \sum_{k=1}^{K} \lambda_k = 1,\ \lambda_k \in \mathbb{R},\ \forall k\}$, called affine hull
  - $\text{cone}(S) = \{\sum_{k=1}^{K} \lambda_k x^k : \lambda_k \in \mathbb{R}_+,\ \forall k\}$, called convex cone
  - $\text{convex}(S) = \{\sum_{k=1}^{K} \lambda_k x^k : \sum_{k=1}^{K} \lambda_k = 1,\ \lambda_k \in \mathbb{R}_+,\ \forall k\}$, called convex hull
- The intersection of convex sets is a convex set

- consider $\mathbb{R}^n$
- $\{x : a^T x = b\}$ is called a hyperplane
- $\{x : a^T x \geq b\}$ is called a halfspace
- The intersection of finitely many halfspaces is called a polyhedron

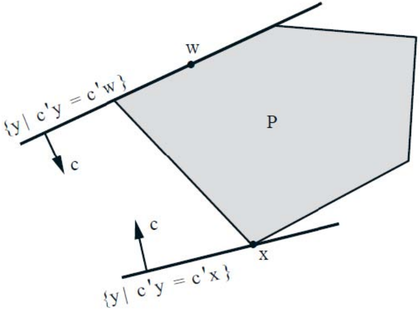


Consider the polyhedron $P = \{x : Ax \geq b\} \subseteq \mathbb{R}^n$

- $x \in P$ is an extreme point of P if $\nexists y, z \in P, y \neq x, z \neq x, 0 < \lambda < 1$, such that $x = \lambda y + (1 - \lambda)z$.
- an extreme point is not strictly within the line segment connecting two other points in P



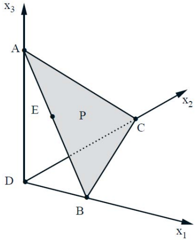
- $x \in P$ is a vertex of P if $\exists c, \ni c^T x < c^T z, \forall z \in P \backslash \{x\}$.
- a vertex is the unique minimizer of some linear function over P.



The standard simplex in $\mathbb{R}^3$:

$$P := \{(x_1, x_2, x_3) : x_1 + x_2 + x_3 = 1, x_1, x_2, x_3 \geq 0\}.$$

- Points A, B, C: each has 3 active (i.e., "$=$") constraints
- Point E: 2 active constraints. If add a constraint: $2x_1 + 2x_2 + 2x_3 = 2$. Then, 3 constraints are active at E, but they are not linearly independent.



A vertex or extreme point has n linearly independent active constraints

Standard form

- variable $x \in \mathbb{R}^n$
- cost vector $c \in \mathbb{R}^n$
- right-hand side vector $b \in \mathbb{R}^m$
- coefficient matrix $A \in \mathbb{R}^{m \times n}$
- standard form

$$
\begin{aligned}
\min \;\; & c^T x \\
\text{subject to } & Ax = b \\
& x \geq 0.
\end{aligned}
$$

Any non-standard form LP can be reformulated to the standard form. The standard form simplifies algorithms and unifies analysis.

(LP) $\min\ c^T x$ s.t. $\underline{Ax = b}$, $\underline{x \geq 0}$

$x \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, $A \in \mathbb{R}^{m \times n}$

$P = \{ x \in \mathbb{R}^n \mid Ax = b, x \geq 0 \}$

Assumption: $\underline{\text{rank}(A) = m}$

$Ax = b$ (m constraints)

$\hookrightarrow \underline{A'}x' = b'$

$\hookrightarrow$ full row rank!

general form

$\min\ c^T x$

s.t. $\underline{A_1 x \geq b_1}$

$A_2 x = b_2$

$\left\{ \begin{array}{l} A_1 x - \underline{y} = b_1 \\ y \geq 0 \end{array} \right.$

$\min\ c^T(u - v)$

s.t. $\left\{ \begin{array}{l} A_1(u-v) - y = b_1 \\ A_2(u-v) = b_2 \\ u \geq 0,\ v \geq 0,\ y \geq 0 \end{array} \right.$

$$\underbrace{\begin{bmatrix} A_1 & -A_1 & -I \\ A_2 & -A_2 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} u \\ v \\ y \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}}_{b}$$

free x

$\hookrightarrow \left\{ \begin{array}{l} x = \underline{u} - \underline{v}, \\ u \geq 0,\ v \geq 0 \end{array} \right.$

Conversion to the standard form

Consider

$$\begin{aligned} \min\ & c^T x \\ \text{subject to } & Ax \geq b \\ & x \geq 0. \end{aligned}$$

Introduce surplus or dummy variables s_i.

$$a_{i1}x_1 + \cdots + a_{in}x_n \geq b_i \Leftrightarrow a_{i1}x_1 + \cdots + a_{in}x_n - s_i = b_i, s_i \geq 0$$

New form

$$\begin{aligned} \min\ & c^T x \\ \text{subject to } & [A, -I_m] \left[\begin{array}{c} x \\ s \end{array} \right] = b \\ & x \geq 0, s \geq 0. \end{aligned}$$

General methods:

- “maximize” objective: minimize its negative
- $\leq$ constraint: add nonnegative slack variable
- $\geq$ constraint: subtract nonnegative slack variable
- $x_i \leq 0$: substitute x_i by $-x_i$ throughout
- free x_i : introduce $u_i, v_i \geq 0$ and substitute xi by $u_i - v_i$ throughout
- constraint $|x_i| \leq b_i$: replace by $x_i \leq b_i$ and $-x_i \leq b_i$
- objective $|x_i|$: introduce $u_i, v_i \geq 0$ and substitute
    - x_i by $u_i - v_i$
    - $|x_i|$ by $u_i + v_i$

Example

$$\begin{aligned} \max \;\; & x_2 - x_1 \\ \text{subject to } & 3x_1 = x_2 - 5 \\ & |x_2| \leq 2 \\ & x_1 \leq 0. \end{aligned}$$

Steps:

1. change to minimize $x_1 - x_2$
2. substitute $x1$ by $-x_1$
3. write $|x_2| \leq 2$ by $x_2 \leq 2$ and $-x_2 \leq 2$
4. introduce s_1 and s_2 and rewrite $x_2 + s_1 = 2$ and $-x_2 + s_2 = 2$

We obtain:

$$
\begin{aligned}
\text{minimize } & x_1 - x_2 \\
\text{subject to } & 3x_1 + u_2 - v_2 = 5 \\
& u_2 - v_2 + s_1 = 2 \\
& v_2 - u_2 + s_2 = 2 \\
& x_1, u_2, v_2, s_1, s_2 \geq 0.
\end{aligned}
$$