Operations Research Assignment 3

1. Calculate the dual of the quadratic program:

$$\min \frac{1}{2}x^T Qx - b^T x, \quad \text{s.t. } Ax \leq c,$$

   where $b \in \mathbb{R}^n$, $Q \in \mathbb{R}^{n\times n}$ is positive definite symmetric, $A \in \mathbb{R}^{m\times n}$ and $c \in \mathbb{R}^m$.

2. Solve the problem

$$\begin{aligned} \min\ & x_1 + \frac{2}{x_2} \\ \text{s.t.}\ & -x_2 + \frac{1}{2} \leq 0, \\ & -x_1 + x_2^2 \leq 0. \end{aligned}$$

3. Consider the problem

$$\begin{aligned} \min\ & f(x) = x_1 - x_2 \\ \text{s.t.}\ & g(x) = x_1 + x_2 - 1 \leq 0, \\ & x \in \mathbb{R}^2_+. \end{aligned}$$

   (a) Find the dual function $h(\lambda)$.

   (b) Formulate the dual problem.

   (c) Show that there is no duality gap and the set of Lagrange multipliers is equal to the set of dual optimal solutions.

4. Consider the problem (P)

$$\min\ f(x) \quad \text{s.t. } g_i(x) \leq 0,\ \forall i = 1, 2, \cdots, p.$$

   If all f, g_i are convex and differentiable, prove the equivalence of the following two claims:

   (a) the Slater condition holds;

   (b) the MFCQ holds at some feasible point.

5. Consider the problem

$$\min\ f(x) \quad \text{s.t. } g_i(x) \leq 0,\ \forall i = 1, 2, \cdots, p.$$

If $\bar{x}$ is feasible and there is a multiplier $\lambda \in \mathbb{R}_+^p$ with

$$
\begin{aligned}
&\bar{x} \text{ minmizes the unconstrained problem } \min_{x\in\mathbb{R}^n} L(x,\lambda) := f(x) + \lambda g_i(x) \\
0 =& \lambda_i g_i(\bar{x}), \ \forall i = 1,2,\cdots,p.
\end{aligned}
$$

Then $\bar{x}$ is optimal to (P).

6. Consider the nonlinear programming:

$$
\begin{aligned}
&\min\ f(x) \\
&\text{s.t. } h_i(x) = 0,\ i = 1,2,\cdots,p \\
&\qquad g_j(x) \le 0,\ j = 1,2,\cdots,q
\end{aligned}
$$

where all functions $f:\mathbb{R}^n \to \mathbb{R}$, $g_i:\mathbb{R}^n \to \mathbb{R}^p$ and $h_j:\mathbb{R}^n \to \mathbb{R}^q$ are continuously differentiable. Let x^* be a local minimizer. Denote the feasible region by

$$
\mathcal{F} := \left\{x \in \mathbb{R}^n \middle| h_i(x) = 0, g_j(x) \le 0 \text{ for any } i = 1,2,\cdots,p \text{ and } j = 1,2,\cdots q\right\}.
$$

(a) We denote the positive part of a scalar α by $\alpha^+ = \max\{0,\alpha\}$.
Prove that for any $k = 1,2,\cdots$, there exists x^k satisfying

i. x^k is a global minimizer of the auxiliary problem

$$
\min\ f(x) + \frac{k}{2}\left[\sum_{i=1}^{p}[h_i(x)]^2 + \sum_{j=1}^{q}\left[g_j^+(x)\right]^2\right] + \frac{1}{2}\|x - x^*\|^2, \qquad \text{subject to } x \in \mathbb{B}(x^*,\varepsilon).
$$

ii. $\left\{x^k\right\}_{k\ge 0} \to x^*$,

where $\varepsilon > 0$ is such that $f(x^*) \le f(x)$ for all $x \in \mathcal{F} \cap \mathbb{B}(x^*,\varepsilon)$.
Hint: firstly check all limit points of $\left\{x^k\right\}_{k\ge 0}$ are feasible.

(b) Check

$$
0 = \nabla f\left(x^k\right) + k\sum_{i=1}^{p} h_i\left(x^k\right)\nabla h_i\left(x^k\right) + k\sum_{j=1}^{q} g_j^+\left(x^k\right)\nabla g_j\left(x^k\right) + \left(x^k - x^*\right).
$$

(c) Denote

$$
\begin{aligned}
s_k &:= \sqrt{1 + k^2\left[\sum_{i=1}^{p}\left[h_i\left(x^k\right)\right]^2 + \sum_{j=1}^{q}\left[g_j^+\left(x^k\right)\right]^2\right]}, \\
\lambda_{k,0} &:= \frac{1}{s_k}, \quad \lambda_{k,i} := \frac{k h_i\left(x^k\right)}{s_k}, \quad \mu_{k,j} := \frac{k g_j^+\left(x^k\right)}{s_k}.
\end{aligned}
$$

Then the sequence $\{\lambda_{k,0}, \lambda_{k,1}, \cdots, \lambda_{k,p}, \mu_{k,1}, \cdots, \mu_{k,q}\}_{k\geq 1} \subseteq \mathbb{R}^{1+p+q}$ has at least one nonzero limit point.

(d) Assume $\{\lambda_0, \lambda_1, \cdots, \lambda_p, \mu_1, \cdots, \mu_q\}$ be a limit point. Then we have

$$0 = \lambda_0 \nabla f(x^*) + \sum_{i=1}^{p} \lambda_i \nabla h(x^*) + \sum_{j=1}^{q} \mu_j \nabla g(x^*),$$
$$\mu_j \geq 0, \ \mu_j g_j(x^*) = 0 \text{ for } j = 1, 2, \cdots, q.$$

(e) Assume the MFCQ holds at x^*, derive KKT condition from (d).

7. Let $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, $q \in \mathbb{R}^n$, $r \in \mathbb{R}$ and $P \in \mathbb{R}^{n\times n}$ be positive defined. Write down the KKT conditions and derive expressions of the primal solution x^* and the dual solutions v^* for the following problems.

(1).

$$\begin{aligned} \min \quad & \frac{1}{2}x^T x \\ s.t. \quad & Ax = b. \end{aligned}$$

(2).

$$\begin{aligned} \min \quad & \frac{1}{2}x^T P x + q^T x + r \\ s.t. \quad & Ax = b. \end{aligned}$$

8. Consider the l_1-regularized problem

$$\min_{x\in\mathbb{R}^n} \frac{1}{2}\|Ax - b\|_2^2 + \mu \|x\|_1,$$

where $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$ and μ is a scalar. Rewrite the above problem as a equality constrained ones by setting $r = Ax - b$ and derive the dual problem.

9. Consider the inequality

$$x^T Ax + b^T x + c \leq 0,$$

where A is a symmetric matrix. Suppose that C is the solutions set of the above inequality.

(i). Prove: C is convex if A is positive definite.

(ii). Let $\bar{C} := C \cap \{x \,|\, g^T x + \beta = 0\}$ and there exists $\lambda \in \mathbb{R}$ such that $A + \lambda g g^T$ is positive semidefinite. Show that $\bar{C}$ is convex.

10. Consider the nonlinear programming:

$$\begin{aligned}
\min \quad & f(x) \\
\text{s.t.} \quad & h_i(x) = 0 \quad i = 1, \cdots, p \\
& g_j(x) \le 0 \quad j = 1, \cdots, q \\
& x \in X,
\end{aligned}$$

where f, h_i, g_j are continuous and X are compact. Let f^* be the optimal value. Define

$$\theta(\mu) = \min_{x \in X} f(x) + \mu \left(\sum_{i=1}^{p} h_i^2(x) + \sum_{j=1}^{q} \max\{0, g_j(x)\}^2 \right).$$

Show that

(i). $\sup_{\mu \ge 0} \theta(\mu) = \lim_{\mu \to \infty} \theta(\mu)$.

(ii). $f^* \ge \sup_{\mu \ge 0} \theta(\mu)$.

(iii). Let x_μ solve the unconstrained problem

$$\min_{x \in X} f(x) + \mu \left(\sum_{i=1}^{p} h_i^2(x) + \sum_{j=1}^{q} \max\{0, g_j(x)\}^2 \right).$$

Then $f(x_\mu)$ is nondecreasing w.r.t. $\mu \ge 0$.

11. **Barrier functions**

If $g_i : \mathbb{R}^n \to \mathbb{R}$ is convex for $i = 1, \cdots, m$, define

$$S := \{x \in \mathbb{R}^n \mid g_i(x) < 0 \text{ for } i = 1, \cdots, m\}.$$

Show that

(i.) S is convex.

(ii). the inverse barrier

$$B_1(x) = -\sum_{i=1}^{m} \frac{1}{g_i(x)}$$

is convex on S.

(iii). the logarithmic

$$B_2(x) = -\sum_{i=1}^{m} \log[-g_i(x)]$$

is convex on S.