Chapter 2

1. From Example 2.2.1 we know that the general solution is given by

$$u(x,t) = \phi\left(x - \frac{5}{3}t\right).$$

Combined with the initial condition, we have $\phi(x) = \exp(-x^2)$. Therefore, the solution of the initial value problem is

$$u(x,t) = \exp\left(-\left(x - \frac{5}{3}t\right)^2\right).$$

2. The ODE for the characteristics is given by

$$\frac{dy}{dx} = x,$$

or by

$$y = \frac{1}{2}x^2 + C.$$

On a fixed characteristic curve, u satisfies

$$\frac{du}{dx} = u_x + u_y\frac{dy}{dx} = u,$$

from which it follows

$$u = Me^x.$$

The constant M depends on the characteristic curve and hence on C. Solving for C, we obtain the general solution

$$u(x,y) = f\left(y - \frac{1}{2}x^2\right)e^x.$$

3. The characteristics are given by

$$\frac{dx}{dt} = 1,$$

or by

$$x = t + C.$$

On a fixed characteristic curve, u satisfies

$$\frac{du}{dt} = u_t + u_x\frac{dx}{dt} = x = t + C,$$

from which it follows

$$u = \frac{1}{2}t^2 + Ct + M.$$

The constant M depends on the characteristic curve and hence on C. Solving for C, we obtain the general solution

$$u(x,t) = \frac{1}{2}t^2 + t(x-t) + f(x-t).$$

By the initial condition we see that

$$f(x) = \frac{1}{1+x^2}.$$

Therefore, the solution to the initial value problem is

$$u(x,t) = xt - \frac{1}{2}t^2 + \frac{1}{1+(x-t)^2}.$$

4. Note that $x = t$ is a characteristic curve, along which u is constant. But we are given $u(0,0) = 1 \neq 2 = u(1,1)$. This contradiction tells us that there is no smooth solution for this boundary value problem. Physically, if we consider a substance moving in $[0,1]$ with constant speed 1 to the right, it is clear that the behavior of the substance at $x = 1$ is completely determined by the behavior of it at $x = 0$.

5. (a). From Example 2.2.3 with $c(u) = u$, we know that the time t_s when shock first occurs is given by

$$t_s = \frac{-1}{c'(\phi(x_0))\phi'(x_0)} = \frac{-1}{-1} = 1,$$

since

$$\phi'(x_0) = \min_{x\in(-\infty,\infty)} \phi'(x) = -1 < 0.$$

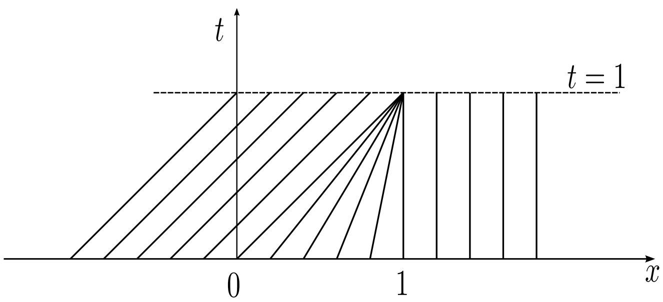

Figure 1 Characteristic curves for Burgers' equation with initial condition $\phi(x)$

(b). By Example 2.2.3, the solution is given by the implicit formula

$$u(x,t) = \phi(x-ut) = \begin{cases} 1, & x-ut \le 0, \\ 1-x+ut, & 0 < x-ut \le 1, \\ 0, & x-ut > 1, \end{cases}$$

from which it follows that

$$u(x,t) = \begin{cases} 1, & x \le t, \\ \dfrac{1-x}{1-t}, & t < x \le 1, \\ 0, & x > 1. \end{cases}$$

6. (a). For any fixed $t_0 \ge 0$, we think of $u(x,t_0)$ as the initial condition for Burger's equation. $\forall x_0 < y_0$, we consider the characteristic lines passing through (x_0,t_0) and (y_0,t_0), which are given by $x = u(x_0,t_0)(t-t_0)+x_0$ and $x = u(y_0,t_0)(t-t_0)+y_0$, respectively. If $u(x_0,t_0) > u(y_0,t_0)$, then the characteristic curves intersect at

$$t = \frac{y_0 - x_0}{u(x_0,t_0) - u(y_0,t_0)} + t_0 \ (> t_0),$$

as shown is the following graph. This contradicts the fact that u is C^1-smooth for $x \in (-\infty,\infty),\ t \ge 0$ and u is constant on each characteristic curve.

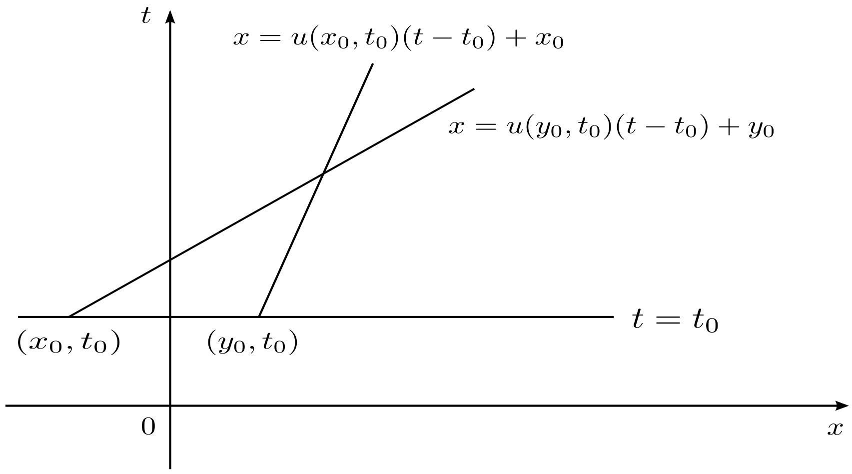

Figure 2 Characteristic curves for Burgers' equation with initial condition $u(x,t_0)$

(b). Note that $u > 0$, $u_x \ge 0$ and $u_t = -uu_x$. Obviously it follows $u_t \le 0$, which means u is non-increasing in $t \ge 0$ for each fixed x.