MA303 偏微分方程 第六次作业

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Chapter 4

Problem 3: $5' + 5'$

Solutions:

3.(a) By the maximum principle, we have

$$\max_{\mathrm{x}\in\overline{D}} u(\mathrm{x}) = \max_{\mathrm{x}\in\partial D} u(\mathrm{x}) = \max\{3\sin 2\theta + 1\} = 4.$$

(b) Since u is a harmonic function, by the mean value property, we have

$$u(0,0) = \frac{1}{4\pi}\int_{|\mathrm{x}|=2} u(\mathrm{x})\mathrm{dx} = \frac{1}{4\pi}\int_{r=2}(3\sin 2\theta + 1)\cdot r\mathrm{d}r\mathrm{d}\theta$$

$$= \frac{1}{4\pi}\int_0^{2\pi} 2\cdot(3\sin 2\theta + 1)\mathrm{d}\theta = 1.$$

□

Problem 4: $5'$

Solution:

4. Take $M = (x, y)$ and $M_0 = (x_0, y_0)$. Let

$$M_0^x = (x_0, -y_0),\ M_0^y = (-x_0, y_0),\ M_0^0 = (-x_0, -y_0)$$

Then by the method of reflection, we have

$$G(M; M_0) = \frac{1}{2\pi}\ln\frac{|M - M_0^x|}{|M - M_0|} + \frac{1}{2\pi}\ln\frac{|M - M_0^y|}{|M - M_0^0|}$$

$$= \frac{1}{4\pi}\ln\frac{(x - x_0)^2 + (y + y_0)^2}{(x - x_0)^2 + (y - y_0)^2} + \frac{1}{4\pi}\ln\frac{(x + x_0)^2 + (y - y_0)^2}{(x + x_0)^2 + (y + y_0)^2}$$

$\square$

Problem 9: $5 \times 3'$

Solutions:

(i) By (4.4.13) of the textbook, the solution to the given equations is

$$u(\mathrm{x}) = \int_{\mathbb{R}^3} G_0(\mathrm{x} - \mathrm{x}_0) f(\mathrm{x}_0) \mathrm{d}\mathrm{x}_0 = \int_{|\mathrm{x}_0| \le 1} G_0(\mathrm{x} - \mathrm{x}_0) \mathrm{d}\mathrm{x}_0$$

$$= \frac{1}{4\pi} \int_{|\mathrm{x}_0| \le 1} \frac{1}{|\mathrm{x} - \mathrm{x}_0|} \mathrm{d}\mathrm{x}_0.$$

(ii) By (i), we have

$$|\mathrm{x}|\, u(\mathrm{x}) = \frac{1}{4\pi} \int_{|\mathrm{x}_0| \le 1} \frac{|\mathrm{x}|}{|\mathrm{x} - \mathrm{x}_0|} \mathrm{d}\mathrm{x}_0.$$

Since when $|\mathrm{x}|$ is large, then

$$|\mathrm{x}| - 1 \le |\mathrm{x}| - |\mathrm{x}_0| \le |\mathrm{x} - \mathrm{x}_0| \le |\mathrm{x}| + |\mathrm{x}_0| \le |\mathrm{x}| + 1.$$

Therefore we have

$$\frac{|\mathrm{x}|}{|\mathrm{x}| + 1} \le \frac{|\mathrm{x}|}{|\mathrm{x} - \mathrm{x}_0|} \le \frac{|\mathrm{x}|}{|\mathrm{x}| - 1}.$$

This indicates that as a function of x_0, when $|\mathrm{x}| \to \infty$, $\frac{|\mathrm{x}|}{|\mathrm{x} - \mathrm{x}_0|} \to 1$ uniformly. Hence

$$\lim_{|\mathrm{x}| \to \infty} |\mathrm{x}|\, u(\mathrm{x}) = \frac{1}{4\pi} \int_{|\mathrm{x}_0| \le 1} \lim_{|\mathrm{x}| \to \infty} \frac{|\mathrm{x}|}{|\mathrm{x} - \mathrm{x}_0|} \mathrm{d}\mathrm{x}_0$$

$$= \frac{1}{4\pi} \int_{|\mathrm{x}_0| \le 1} 1 \mathrm{d}\mathrm{x}_0 = \frac{1}{4\pi} \cdot \frac{4\pi}{3} = \frac{1}{3}.$$

(iii) By (ii), as $|\mathrm{x}|$ is large:

$$u(\mathrm{x}) \approx \frac{1}{3\,|\mathrm{x}|} = \frac{\frac{4\pi}{3}}{4\pi\,|\mathrm{x}|}.$$

This is because $u(\mathrm{x})$ is the electric potential at point x induced by the electric charge distributed with density function f. From f, we learn that the charges distribute in the unit ball centered at 0 uniformly and the total amount of the charges is $\frac{4\pi}{3}$. When $|\mathrm{x}|$ is large, the ball can be seen as a point, hence the eletric potential at $\mathrm{x} \approx \frac{\frac{4\pi}{3}}{4\pi|\mathrm{x}|}$. $\square$

Problem 10: $5'$

Proof: For an arbitrary pair of points x and y, let $r = |x - y|$. Then we have:

$$B_r(\mathrm{x}) \subset B_{2r}(\mathrm{y}).$$

Since u is harmonic, by the mean value property, we have

$$u(\mathrm{x}) = \frac{1}{|B_r(\mathrm{x})|} \int_{B_r(\mathrm{x})} u(\mathrm{x}_0)\mathrm{dx}_0.$$

Since u is non-negative and $B_r(\mathrm{x}) \subset B_{2r}(\mathrm{y})$, we have

$$\frac{1}{|B_r(\mathrm{x})|} \int_{B_r(\mathrm{x})} u(\mathrm{x}_0)\mathrm{dx}_0 \leq \frac{1}{|B_r(\mathrm{x})|} \int_{B_{2r}(\mathrm{y})} u(\mathrm{x}_0)\mathrm{dx}_0$$

$$= \frac{2^n}{|B_{2r}(\mathrm{y})|} \int_{B_{2r}(\mathrm{y})} u(\mathrm{x}_0)\mathrm{dx}_0.$$

Also by the mean value property, we have

$$\frac{1}{|B_{2r}(\mathrm{y})|} \int_{B_{2r}(\mathrm{y})} u(\mathrm{x}_0)\mathrm{dx}_0 = u(\mathrm{y}).$$

Hence

$$u(\mathrm{x}) \leq 2^n u(\mathrm{y}),\ \forall \mathrm{x} \in \mathbb{R}^n \Rightarrow \sup u \leq 2^n u(\mathrm{y}).$$

Since y is arbitrary, we eventually get the Harnack inequality:

$$\sup u \leq 2^n \inf u.$$

$\square$

Problem 11: $5'$

Proof: If u is bounded below, $\inf u$ is a well defined real number. Then by the Harnack inequality we have proved in Problem 10, we have

$$\sup u - \inf u = \sup(u - \inf u) \leq 2^n \inf(u - \inf u).$$

Since $\inf(u - \inf u) = \inf u - \inf u = 0$, we have

$$0 \le \sup u - \inf u \le 0 \Rightarrow \sup u = \inf u,$$

which indicates u is a constant. For the case that u is bounded above, we can prove that u is a constant by applying the similar discussion to $\sup u - u$. □

Problem 12: $5 \times 2'$

Solutions:

(i) Let A be the exterior of $B_R(0)$. Suppose $\exists \mathrm{x}_0 \in A$ such that $v(x_0) \le 0$. Since u is harmonic in A, we have

$$-\Delta v(\mathrm{x}) \ge 0,\ \mathrm{x} \in \Omega = A \cap B_{|\mathrm{x}_0|+1}(0).$$

Clearly, Ω is bounded. Since $\partial B_R(0) \subset \partial\Omega$ and v is positive on $\partial B_R(0)$, in $\bar{\Omega}$, v takes its minimum in the interior. By the strong minimum principle, v is constant in $\bar{\Omega}$. Hence

$$v(\mathrm{x}) = v(\mathrm{x}_0) \le 0,\ \mathrm{x} \in \partial B_R(0),$$

which is a contradiction. □

(ii) In 3D, we have

$$G_0(\mathrm{x}_0) = \frac{1}{4\pi\,|\mathrm{x}|} > 0.$$

By the result of (i), $\forall \mathrm{x}$ such that $|\mathrm{x}| > R$:

$$\frac{v(\mathrm{x})}{G_0(\mathrm{x})} = M - \frac{u(\mathrm{x})}{G_0(\mathrm{x})} > 0 \Rightarrow \frac{u(\mathrm{x})}{G_0(\mathrm{x})} < M.$$

Since $\partial B_R(0)$ is compact, $\min_{\mathrm{x}\in\partial B_R(0)} u(\mathrm{x})$ is a well defined real number. Hence $\exists N > 0$ such that

$$\hat{v}(\mathrm{x}) = NG_0(\mathrm{x}) + u(\mathrm{x}) > 0,\ \forall \mathrm{x} \in \partial B_R(0).$$

Applying the same method of (i) to $\hat{v}$, we obtain that $\hat{v}$ is positive in the exterior of $B_R(0)$. Similar as above, we have

$$\frac{\hat{v}(\mathrm{x})}{G_0(\mathrm{x})} = N + \frac{u(\mathrm{x})}{G_0(\mathrm{x})} > 0 \Rightarrow \frac{u(\mathrm{x})}{G_0(\mathrm{x})} > -N,$$

$\forall \mathrm{x}$ such that $|\mathrm{x}| > R$. Then as $|\mathrm{x}| \to \infty$:

$$\frac{|u(\mathrm{x})|}{G_0(\mathrm{x})} \leq \max\{M,\ N\}.$$

This shows that u decays at infinity at least as fast as the fundamental solution G_0. $\square$