南方科技大学
SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY

考试科目：线性代数精讲　　开课单位：数学系
考试时长：120 分钟　　命题教师：线性代数精讲教师团队

题 号	1	2	3	4	5	6	7
分 值	30 分	10 分	20 分	10 分	20 分	10 分	10 分

本试卷共（7）大题，满分（110）分．(考试结束后请将试卷、答题本、草稿纸一起交给监考老师)

This test includes 7 questions. Write ***all your answers*** on the examination book.

Please put away all books, calculators, cell phones and other devices. Please write carefully and clearly in complete sentences. Your explanations are your only representative when your work is being graded.

Unless otherwise noted, vector spaces are over $\mathbb{F}$ and with finite dimensions, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$.

1. (30 points, 6 points each) Label the following statements as True or False. **Along with your answer, provide an informal proof, counterexample, or other explanation.**

(1,1,0)
(0,0,1)

(a) $\{(x_1, x_2, x_3) \in \mathbb{F}^3 : x_1x_2x_3 = 0\}$ is a subspace of $\mathbb{F}^3$.

(b) Define an operator $T \in \mathcal{L}(\mathbb{F}^2)$ by $T(x, y) = (0, x)$. Let $U = \{(0, z) : z \in \mathbb{F}\}$. There exists a subspace W of $\mathbb{F}^2$ that is invariant under T and such that $\mathbb{F}^2 = U \oplus W$.

(c) There exists a linear operator defined on $\mathbb{R}^4$ such that $T^4 = -I$.

(d) Suppose V is finite dimensional and $T \in \mathcal{L}(V)$ and $\lambda \in \mathbb{R}$. Then

$$\dim \operatorname{range}\,(T - \lambda I) < \dim V.$$

(e) In $M_{3\times2}(\mathbb{R})$, the following list

$$\overset{k_1}{\begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}}, \overset{k_2}{\begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}}, \overset{k_3}{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}}, \overset{k_4}{\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}}, \overset{k_5}{\begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}}$$

is linearly dependent. Where $M_{3\times2}(\mathbb{R})$ denotes the vector space consisting of all 3×2 real matrices.

2. (10 points) Let U_1, U_2 and U_3 be finite-dimensional real vector spaces. Let

$$T_1 : U_1 \to U_2 \text{ and } T_2 : U_2 \to U_3$$

be linear maps. Show that $T_2 \circ T_1$ is linear.

3. (20 points) Let $V = \mathcal{P}_3(\mathbb{R})$. Define $T \in \mathcal{L}(V)$ by

$$(Tp)(x) = xp'(x) + p(x) \text{ for all } p \in \mathcal{P}_3(\mathbb{R}).$$

   (a) Find the matrix representation of T with respect to the basis $1, x, x^2, x^3$ of $\mathcal{P}_3(\mathbb{R})$.

   (b) Find all the eigenvalues and eigenvectors of T.

   (c) Is T diagonalizable? If yes, please find a basis with respect to which T has a diagonal matrix. Otherwise, give an explanation.

4. (10 points) Let $V = M_{2\times 2}(\mathbb{R})$ be the vector space consisting of all 2×2 real matrices. Find a basis A_1, A_2, A_3, A_4 of V such that $A_j^2 = A_j, j = 1, 2, 3, 4$.

5. (20 points) Define $T : \mathbb{R}^3 \to \mathbb{R}^2$ by

$$T(x, y, z) = (4x + 5y + 6z, 7x + 8y + 9z).$$

Suppose φ_1, φ_2 denotes the dual basis of the standard basis of $\mathbb{R}^2$ and ϕ_1, ϕ_2, ϕ_3 denotes the dual basis of the standard basis of $\mathbb{R}^3$.

   (a) Describe the linear functionals $T'(\varphi_1)$ and $T'(\varphi_2)$.

   (b) Write $T'(\varphi_1)$ and $T'(\varphi_2)$ as linear combinations of ϕ_1, ϕ_2, ϕ_3.

6. (10 points) Let S, T be two linear operators defined on V satisfying

$$ST = S + T.$$

   (a) Suppose λ_1 is an eigenvalue of S, and λ_2 is an eigenvalue of T.

      i. Show that $\lambda_1 \neq 1$.

      ii. Show that $\lambda_2 \neq 1$.

   (b) Suppose u is an eigenvector of S with eigenvalue λ_1. Is u also an eigenvector of T? Justify your answer.

7. (10 points) Suppose $p, q \in \mathcal{P}(\mathbb{C})$ are nonconstant polynomials with no zeros in common. Let $m = \deg p$ and $n = \deg q$. Use linear algebra as outlined below in (a)-(c) to prove that there exist $r \in \mathcal{P}_{n-1}(\mathbb{C})$ and $s \in \mathcal{P}_{m-1}(\mathbb{C})$ such that

$$rp + sq = 1.$$

   (a) Define

$$T : \mathcal{P}_{n-1}(\mathbb{C}) \times \mathcal{P}_{m-1}(\mathbb{C}) \to \mathcal{P}_{m+n-1}(\mathbb{C})$$

   by

$$T(r, s)$$
$$\cancel{T(s, r)} = rp + sq.$$

   Show that the linear map T is injective.

   (b) Show that the linear map T in (a) is surjective.

   (c) Use (b) to conclude that there exist $r \in \mathcal{P}_{n-1}(\mathbb{C})$ and $s \in \mathcal{P}_{m-1}(\mathbb{C})$ such that $rp + sq = 1$.