2024 Spring Semester Midterm Exam Paper

Please note that the underlying spaces are finite-dimensional Euclidean spaces endowed with an inner product $\langle \cdot, \cdot \rangle$ and the Euclidean norm $\|\cdot\| := \sqrt{\langle \cdot, \cdot \rangle}$.

1. True or False. You are not required to give any justification of your answer. (2*10=20)

   (a) A set $\Omega \subseteq \mathbb{R}^n$ is closed if it is locally closed around any $x \in \Omega$.

   (b) **Remark:** A set $C \subseteq \mathbb{R}^n$ is locally closed around $\bar{x} \in C$ if there is $r > 0$ such that the set $C \cap \mathbb{B}_r(\bar{x})$ is closed, where $\mathbb{B}_r(\bar{x})$ stands for the closed ball with center $\bar{x}$ and radius $r > 0$.

   (c) Let $x^* \in \Omega \subseteq \mathbb{R}^n$, if $f(x) > f(x^*)$ for all $x \in \Omega \backslash \{x^*\}$, then x^* is a strict global minimizer of $f : \mathbb{R}^n \to \mathbb{R}$ over Ω.

   (d) A set $\Omega \subseteq \mathbb{R}^n$ is convex if for any $x, y \in \Omega$ and $\lambda \in \mathbb{R}$ it holds that $\lambda x + (1-\lambda)y \in \Omega$.

   (e) $f : \mathbb{R}^n \to \mathbb{R}$ is σ-strongly convex function if and only if $f(\cdot) - \frac{\sigma}{2}\|\cdot\|^2$ is convex.

   (f) Let $\Omega \subseteq \mathbb{R}^n$, $\bar{x}$ is an interior point of Ω. Then any vector $d \in \mathbb{R}^n \backslash \{0\}$ is a feasible direction at $\bar{x} \in \Omega$.

   (g) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth function with $\bar{x} \in \mathbb{R}^n$. Then the gradient vector $\nabla f(\bar{x})$ is max-rate ascending direction of f at $\bar{x}$.

   (h) Consider the fixed-step-size gradient algorithm for the objective function $f : \mathbb{R}^n \to \mathbb{R}$ with L_f−Lipschitz continuous gradients over $\mathbb{R}^n$. If the initial point x^0 is chosen arbitrarily and constant stepsize $\alpha_k = \bar{\alpha}$ for $k := 0, 1, \cdots$. Then the iteration $\{f(x^k)\}_{k \geq 0}$ is decreasing for any constant stepsize $\bar{\alpha} > 0$.

   (i) The sequence $\{x^k\}_{k \geq 0}$ converges linearly to $\bar{x}$ if $x^k \to \bar{x}$ as $k \to \infty$ and
   $$\frac{\|x^{k+1} - \bar{x}\|}{\|x^k - \bar{x}\|} \to 1 \text{ as } k \to \infty.$$

   (j) If a standard-form linear programming has a solution, then there exists an extreme point solution.

   (k) If linear programming in the standard form has a feasible solution, then there exists a basic feasible solution.

2. (5+5=10). Answer True or False for each of the following statements and justify your answer.

   (a) If x^* is a global minimizer of f over Ω, and $x^* \in \Omega' \subset \Omega$. Then x^* is a local minimizer of f over Ω'.

   (b) Applying the Newton's method to solve
   $$\min_{x \in \mathbb{R}^n} \frac{1}{2}\|Ax - b\|^2,$$
   where A is a $m \times n$ real matrix with full column rank and $b \in \mathbb{R}^m$, it takes finite iterations to obtain the solution.

3. (15*2=30)

   (a) Apply simplex method to solve the following linear programming problem.
   $$\begin{aligned} \max \quad & 4x_1 + 3x_2 + 5x_3 \\ s.t. \quad & 3x_1 + x_2 + 3x_3 \leq 30 \\ & 2x_1 + 2x_2 + 3x_3 \leq 40 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

(b) Apply the KKT conditions to find all optimal solutions of the following convex programming:

$$\begin{aligned} \min\ & x^2 + y^2 - \ln\left(x^2 y^2\right) \\ \text{s.t. } & x \le \ln y \\ & x \ge 1, y \ge 1. \end{aligned}$$

4. (6*4=24) Consider the convex problem

$$\min f(x) \text{ s.t. } x \in \Omega.$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is convex and continuously differentiable, while Ω is a nonempty convex subset in $\mathbb{R}^n$. Let $\bar{x} \in \Omega$ and denote the collection of all feasible directions of Ω at $\bar{x}$ by $F(\bar{x})$, i.e.,

$$F(\bar{x}) := \{0 \neq d \in \mathbb{R}^n \mid \exists \bar{\alpha} > 0 \text{ such that } \bar{x} + \alpha d \in \Omega \text{ for any } \alpha \in (0, \bar{\alpha}]\}.$$

In addition, set

$$\begin{aligned} D(\bar{x}) &= \left\{d \in \mathbb{R}^n \mid \nabla f(\bar{x})^T d < 0\right\}, \\ \text{cone}F(\bar{x}) &= \left\{\sum_{i=1}^n \lambda_i d_i \,\middle|\, \lambda_i \ge 0, d_i \in F(\bar{x}) \text{ where } i := 1, 2, \cdots, n \text{ for all } n \in \mathbb{N}\right\}. \end{aligned}$$

Show the following assertions:

(i) $\{x - \bar{x} \mid \text{ for any } x \in \Omega \backslash \{\bar{x}\}\} \subseteq F(\bar{x})$.

(ii) $F(\bar{x}) \cup \{0\}$ is a **convex cone**.

Hint: A set $C \subseteq \mathbb{R}^n$ is a cone if $\lambda C \subseteq C$ for any $\lambda \ge 0$.

(iii) If $F(\bar{x}) \cap D(\bar{x}) = \varnothing$, then $\bar{x}$ is a global minimizer of f over Ω.

Hint: Use the first-order optimality condition for constrained optimization under convex setting.

(iv) If $\bar{x}$ is a global minimizer, then $\text{cone}F(\bar{x}) \cap D(\bar{x}) = \varnothing$.

Hint: For a set $C \subseteq \mathbb{R}^n$, we have $C = \text{cone}C$ if C is a convex cone.

5. (4+6+6=16) Consider the unconstrained optimization problem

$$\min f(x) \quad \text{s.t.} \quad x \in \mathbb{R}^n, \tag{P}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is L_f-smooth. Assume $X^* \subset \mathbb{R}^n$, the optimal set of (P), is nonempty. Let f^* be the optimal value. Recall the gradient descent (GD) method to find the minimal value of f. You can use the following inequality:

$$f(y) \le f(x) + \nabla f(x)^T (y - x) + \frac{L_f}{2} \|y - x\|^2 \text{ for any } x, y$$

(a) Write down the iterative process of GD method with the constant step size L_f.

(b) Show the sufficiently descending property of $\left\{f\left(x^k\right)\right\}_{k \ge 0}$, i.e,

$$f\left(x^k\right) - f\left(x^{k+1}\right) \ge \frac{1}{2L_f} \left\|\nabla f\left(x^k\right)\right\|^2.$$

(c) show that

$$\min_{n=0,1,\cdots,k} \|\nabla f(x^n)\| \le \sqrt{\frac{2L_f\left(f\left(x^0\right) - f^*\right)}{k+1}}.$$