[MA303] Partial Differential Equations
2023 Fall Semester Midterm

Name ______________________ **Student ID** ______________________

1. (8 points) For each equation below, find its order, linearity and homogeneity.

(a) $u_{tx} + \sin u + u_{xxx} = x$

(b) $u_{tt} - (x^2 \cdot u_x)_x = 0$

(c) $u_x + 2023u_y = xu + e^y$

(d) $u_{xx}u_{yy} - u_{xy}^2 = x + y$

2. (8 points) Classify each of the following PDE as hyperbolic, elliptic, or parabolic. If the type changes in the xy-plane, find the region for each type.

(a) $u_{xx} + xu_{yy} - u_x = 0$

(b) $x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} = 0,\ (x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}$.

3. (a) (8 points) Use method of characteristics to find the solution $u(x, t)$ of

$$\begin{cases} u_t - u_x = u, & x > 0, t > 0, \\ u(x, 0) = \frac{1}{1+x^2}, & x > 0. \end{cases}$$

(b) (4 points) Please draw several characteristic curves and explain why we do not need the boundary condition on the half line $\{x = 0, t > 0\}$.

4. Consider the following initial value problem for Burger's equation

$$\begin{cases} u_t + uu_x = 0, \\ u(x, 0) = \phi(x) = \begin{cases} 1, & x \le 0, \\ 1 - x, & 0 < x \le 1, \\ 0, & x > 1. \end{cases} \end{cases}$$

(a) (4 points) Use the method of characteristics to find an implicit formula for u with **general** initial data $\phi(x)$ in details.

(b) (4 points) Derive the breakdown time t_s in details for the above **special** $\phi(x)$. Hint: Consider the point x_0 such that $\phi'(x_0) < 0$ and see what happens for u_x when t becomes large along the characteristic line issued from x_0 on the x-axis.

(c) (6 points) Solve the above problem (for the above **special** $\phi(x)$) before time t_s.

5. Solve the following two eigenvalue problems in details:

(a) (6 points)

$$\begin{cases} X''(x) + \lambda X(x) = 0, & x \in (0, l), \\ X(0) = 0, \quad X(l) = 0. \end{cases}$$

(b) (8 points)

$$\begin{cases} X''(x) + \lambda X(x) = 0, & x \in (0, l), \\ X(0) = 0, \quad X'(l) + hX(l) = 0. \end{cases}$$

Here h is a positive constant. You may write down your solution with the help of graph.

6. (16 points) Solve the following boundary-initial value problem using the method of separation of variables:

$$\begin{cases} u_t - a^2u_{xx} = -u, & x \in (0, \pi), \quad t > 0, \\ u(x, 0) = x + \pi, & x \in (0, \pi), \\ u(0, t) = 0, \quad u(\pi, t) = \pi e^{-t}, & t > 0. \end{cases}$$

7. Recall the fundamental solution of heat equation (heat kernel):

$$G(x,t;\xi) = \frac{1}{2a\sqrt{\pi t}} \exp\left(-\frac{(x-\xi)^2}{4a^2 t}\right), \quad t > 0.$$

(a) (4 points) Show in details that for $t > 0, x \in (-\infty, +\infty)$,

$$\int_{-\infty}^{+\infty} G(x,t;\xi)\,d\xi = 1.$$

(b) (2 points) Check in details that for any fixed ξ, G satisfies the homogeneous heat equation, i.e.

$$G_t = a^2 G_{xx}.$$

(c) (2 points) Write down a solution of the following Cauchy problem for heat equation without proof:

$$\begin{cases} u_t - a^2 u_{xx} = 0, & x \in (-\infty, +\infty), \quad t > 0, \\ u(x,0) = \phi(x), & x \in (-\infty, +\infty). \end{cases}$$

Here $\phi(x)$ is a bounded and continuous function on $\mathbb{R}$.

(d) (3 points) Show that if ϕ is odd and u is bounded in (c), then u is also odd in x for all $t > 0$.

(e) (3 points) If $\phi(x)$ in (c) is nonnegative and positive somewhere, what can you conclude for the bounded solution u for $t > 0$? Use the conclusion to explain that the heat equation has infinite propagation speed.

8. (a)(8 points) Prove the following statement using weak maximum (minimum) principle: (You should state this principle clearly first.)

Let Ω be a bounded region in $\mathbb{R}^n$ and $T > 0$. Let u be the smooth solution (and continuous up to parabolic boundary) of the following boundary-initial value problem:

$$\begin{cases} u_t - a^2 \Delta u = f(x,t), & (x,t) \in \Omega \times (0,T] \\ u(x,0) = \phi(x), & x \in \Omega, \\ u(x,t) = g(x,t), & (x,t) \in \partial\Omega \times [0,T]. \end{cases}$$

Then

$$\max_{\bar{\Omega} \times [0,T]} |u(x,t)| \le T \cdot \max_{\bar{\Omega} \times [0,T]} |f| + \max_{\bar{\Omega}} |\phi| + \max_{\partial\Omega \times [0,T]} |g|.$$

(b)(6 points) Is the following statement true? If yes, prove it. If no, find a counter-example.

Suppose on $(0,l) \times (0,T]$, $u(x,t)$ is smooth and continuous up to the parabolic boundary of $[0,l] \times [0,T]$. Moreover, u satisfies

$$u_t - a^2 u_{xx} - u_x \le 0.$$

Then the maximum of u must be achieved on the parabolic boundary of $[0,l] \times [0,T]$.