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Quantization Conditions and the Double Copy

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Abstract: We formulate Wilson loop observables as products of eikonal Wilson lines given in terms of on-shell scattering amplitudes. Using these, we derive the Dirac-Schwinger-Zwanziger quantization condition and its gravitational (Taub-NUT) double copy, where we find a relativistic generalisation of the usual non-relativistic gravitational quantization condition. We also compute the relativistic Wilson loop for an anyon-anyon system, obtaining a similar relativistic generalisation of the Aharonov-Bohm phase for gravitational anyons.

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1 Introduction

Long ago, Dirac showed that if a single magnetic monopole with charge g exists in the universe, then electric charge e must be quantized in terms of g via [1]

$$eg = 2\pi\hbar n \ , \tag{1.1}$$

with n an integer[1]. This was further generalised independently by both Schwinger [2] and Zwanzinger [3, 4] to include particles with both electric *and* magnetic charge - dyons - which leads to the constraint

$$e_1 g_2 - e_2 g_1 = 2\pi\hbar n \ , \tag{1.2}$$

known as the generalised Dirac-Zwanziger-Schwinger quantization condition. This was also extended to the gravitational case by Dowker and Roche [5] (see also [6, 7]), where in the non-relativistic limit it was found that gravitational mass m is quantized in terms of the NUT charge ℓ as

$$m\ell = \frac{1}{4}\hbar n \ . \tag{1.3}$$

It is natural to wonder whether or not the gauge and gravitational quantization conditions are related by the double copy, especially since the double copy relationship has been established

[1] Note that we have restored $\hbar$ in this expression, and do so where necessary throughout this paper, such that the quantum and classical results extracted from the eikonal phases and Wilson loops are clearly distinguished.

between magnetic monopoles and Taub-NUT spacetimes [8–12]. Recently, there have been several developments in the areas of both dualities [9, 10, 13–20] and eikonal physics [21–29] as examined through the lens of on-shell scattering amplitudes.

In this paper we will examine this question by combining these lines of enquiry. The quantization conditions can be derived by considering the Aharonov-Bohm phase associated to a Dirac string, which we demand is undetectable. Typically, this involves demanding the single-valuedness of the wave function or by demanding that two potentials defined on different regions of space have a well-defined overlap. In this paper, we will take a different approach, and demand that observables built out of on-shell scattering amplitudes be gauge invariant. This has often led to interesting insights, for example much of the physics of black holes is very simply captured by some relatively simple on-shell amplitudes [30–36]. The observable of interest will be the phase of the Wilson loop, a central observable in gauge theories and the relativistic cousin of the Aharonov-Bohm phase. There have been some very interesting recent papers on Wilson loops (and Wilson lines) and the double copy [11, 37], typically focussing on the double copy at the level of the fields. We shall take a different approach here and formulate the Wilson loop entirely in terms of on-shall amplitudes by way of the eikonal phase (itself a Wilson line) and find that this leads us neatly to the Dirac-Zwanziger-Schwinger quantization condition. We then consider the double copy of the Wilson loop phase and discover a relativistic generalisation of the gravitational quantization conditions, where the relativistic four momentum p^μ is quantized in terms of a dual 'NUT' momentum k^μ. Finally, we we derive the famous Aharonov-Bohm phase associated to gauge and gravitational anyons and show that in the gravitational case a similar relativistic generalisation is found.

2 Scattering Amplitudes and Eikonal Physics

For $2 \to 2$ scattering, the relativistic S-matrix can be expanded in terms of the eikonal phase [38–43], which captures much of the classical physics. The eikonal phase is currently undergoing somewhat of a renaissance in the on-shell scattering amplitudes program, especially where gravity is concerned. Although not itself an observable, it provides an important connection between amplitudes and observables, encoding a number of useful quantities of a physical system. For example, changes in spin, classical impulses, and (as we will show) certain Wilson loops can be derived from the relevant eikonal phase.

We start by defining the eikonal phase for a four-particle amplitude $\mathcal{A}_4[p_1, p_2 \to p_1', p_2']$. In momentum space, such an amplitude can be expressed purely in terms of masses M_i and the Mandelstam invariants s and t, given by

$$s = (p_1 + p_2)^2, \qquad t = (p_1 - p_1')^2 \ . \tag{2.1}$$

When required, we will assume the momentum is related via

$$p_1' = p_1 + \frac{q}{2}, \qquad p_2' = p_2 - \frac{q}{2}\,, \tag{2.2}$$

such that $t = q^2/4$.

By now, it has become standard to express the eikonal phase in terms of the transverse Fourier transform

$$e^{\frac{i}{\hbar}\chi} - 1 = i \int \hat{\mathrm{d}}^D q\, \hat{\delta}(2p_1 \cdot q)\, \hat{\delta}(2p_2 \cdot q)\, e^{iq\cdot b} \mathcal{A}_4[p_1, p_2 \to p_1', p_2']\big|_{q^2 \to 0}\,. \tag{2.3}$$

The eikonal phase χ can be expanded in powers of coupling constant, encompassing various loop contributions from the amplitude, albeit with a number of constraints from unitarity. In general one has to be careful in applying the eikonal phase at higher order, however in this paper we will only be interested in leading order effects, i.e.

$$\chi = \frac{\hbar}{4M_1 M_2} \int \hat{\mathrm{d}}^D q\, \hat{\delta}(u_1 \cdot q) \hat{\delta}(u_1 \cdot q)\, e^{iq\cdot b} \mathcal{A}_4[p_1, p_2 \to p_1', p_2']\big|_{q^2 \to 0}\,. \tag{2.4}$$

As claimed, one can readily extract the leading-order impulse of one of the particles, say particle 1, taken to be a probe in the presence of a much heavier particle (in this case, particle 2), from the eikonal phase χ. This is achieved by taking the derivative of χ with respect to the projected impact parameter: $\Pi^\mu \frac{\partial}{\partial b_\nu}$. The projector is a dimensionally-dependent quantity, and in four dimensions is given by

$$\Pi^\mu{}_\nu = (\gamma^2 - 1)^{-1} \epsilon^{\mu\rho\alpha\beta} \epsilon_{\nu\rho\gamma\delta} u_{1\alpha} u_{2\beta}\, u_1^\gamma u_2^\delta\,, \tag{2.5}$$

whereas in three dimensions by

$$\Pi^\mu{}_\nu = (\gamma^2 - 1)^{-1} \epsilon^{\mu\rho\alpha} \epsilon_{\nu\rho\gamma} u_{1\alpha} u_2^\gamma\,. \tag{2.6}$$

In both cases, the projector ensures that the resulting expression stays transverse to both the incoming velocities u_1^μ and u_2^μ meaning its components lie in the impact parameter plane. As such, we can express the leading-order impulse as

$$\Delta p_1^\mu = \Pi^\mu{}_\nu \frac{\partial \chi}{\partial b_\nu} = \frac{\hbar}{4M_1 M_2} \Pi^\mu{}_\nu \frac{\partial}{\partial b_\nu} \int \hat{\mathrm{d}}^4 q\, \hat{\delta}(u_1 \cdot q) \hat{\delta}(u_1 \cdot q)\, e^{iq\cdot b} \mathcal{A}_4[p_1, p_2 \to p_1', p_2']\big|_{q^2 \to 0}\,. \tag{2.7}$$

Having given a brief overview on relating scattering amplitudes to the eikonal phasee, and how to extract the classical leading-order impulse from it, let us proceed by considering examples in both electromagnetic and gravitational settings.

2.1 Dyon-Dyon Eikonal Phase and the Electromagnetic Impulse

Here, we wish to determine the eikonal phase associated with a pair of spinning dyons with masses M_i and momentum $p_i^\mu = M_i u_i^\mu$ with $i = 1, 2$. To do so, we first need to determine the relevant four-particle scattering amplitude. We will suppress factors of $\hbar$ for much of the calculations, restoring it when useful.

The three-particle amplitude of an electromagnetically charged spin-s particle with mass M_1 emmiting massless particle of helicity ± 1 is given by [44]

$$\mathcal{A}_3\left[1^s, 1^{s\prime}, q^{\pm 1}\right] = \sqrt{2} e M_1 x_1^{\pm} \frac{\langle \mathbf{11'} \rangle^{2s}}{M_1^{2s}}. \tag{2.8}$$

To construct *dyon* amplitudes, we duality-rotate the above three-particle amplitudes and take the infinite spin limit, finding [10]

$$\mathcal{A}_3\left[1, 1', q^{\pm 2}\right] = \sqrt{2} e M_1 x_1^{\pm} e^{\pm(i\theta_1 + q\cdot a_1)}, \quad \mathcal{A}_3\left[2, 2, q^{\pm 2}\right] = \sqrt{2} e M_2 x_2^{\pm} e^{\pm(i\theta_2 + q\cdot a_2)}\,, \tag{2.9}$$

where a_1^μ and a_2^μ parametrise the (classical) spins of particles 1 and 2, respectively.

The four-particle scattering amplitude is then given by

$$\begin{aligned}
\mathcal{A}_4 &= \frac{2M_1 M_2 e^2}{q^2}\left(\frac{x_1}{x_2} e^{-i(\theta_1-\theta_2)-q(a_1+a_2)} + \frac{x_2}{x_1} e^{i(\theta_1-\theta_2)+q(a_1+a_2)}\right) \\
&= \frac{2M_1 M_2}{q^2}\Big[(e_1 - i g_1)(e_2 + i g_2)\left(u_1\cdot u_2 + i\frac{\epsilon(\eta, u_1, q, u_2)}{q\cdot\eta}\right) e^{-iq\cdot a} \\
&\qquad + (e_1 + i g_1)(e_2 - i g_2)\left(u_1\cdot u_2 - i\frac{\epsilon(\eta, u_1, q, u_2)}{q\cdot\eta}\right) e^{iq\cdot a}\Big]\,,
\end{aligned} \tag{2.10}$$

where we have defined $a \equiv -i(a_1 + a_2)$, along with the duality-rotated couplings

$$e_i \equiv e\cos\theta_i, \qquad g_i \equiv e\sin\theta_i. \tag{2.11}$$

The dyon-dyon eikonal phase is then given by

$$\begin{aligned}
\chi_{\text{dyon}} &= \frac{1}{2}\int \hat{\mathrm{d}}^4 q\, \hat{\delta}(u_1\cdot q)\hat{\delta}(u_1\cdot q)\frac{e^{iq\cdot b}}{q^2}\Big[(e_1 - i g_1)(e_2 + i g_2)\left(u_1\cdot u_2 + i\frac{\epsilon(\eta, u_1, q, u_2)}{q\cdot\eta}\right) e^{-iq\cdot a} \\
&\qquad + (e_1 + i g_1)(e_2 - i g_2)\left(u_1\cdot u_2 - i\frac{\epsilon(\eta, u_1, q, u_2)}{q\cdot\eta}\right) e^{iq\cdot a}\Big] \\
&= \mathrm{Re}\,(e_1 - i g_1)(e_2 + i g_2)\int \hat{\mathrm{d}}^4 q\, \hat{\delta}(u_1\cdot q)\hat{\delta}(u_1\cdot q)\frac{e^{iq\cdot(b-a)}}{q^2}\left(u_1\cdot u_2 + i\frac{\epsilon(\eta, u_1, q, u_2)}{q\cdot\eta}\right)\,.
\end{aligned} \tag{2.12}$$

To extract the leading-order probe particle impulse from the eikonal phase, we need simply

refer back to eq. (2.7), from which determine the following result:

$$\begin{aligned}\Delta p_1^\mu &= \Pi^\mu{}_\nu \frac{\partial}{\partial b_\nu}\chi_{\text{dyon}} \\ &= \text{Re}\, i(e_1 - ig_1)(e_2 + ig_2)\int \hat{\text{d}}^4 q\, \hat{\delta}(u_1\cdot q)\hat{\delta}(u_1\cdot q)\,\frac{e^{iq\cdot(b-a)}}{q^2}\left(q^\mu\, u_1\cdot u_2 - i\epsilon^\mu(u_1,q,u_2)\right)\ ,\end{aligned} \tag{2.13}$$

where we have used that $q^\mu\epsilon(\eta,u_1,q,u_2) = -(q\cdot\eta)\epsilon^\mu(u_1,q,u_2) + \mathcal{O}(q^2)$. Eq. (2.13) exactly corresponds to the dyon-dyon generalisation of the result found in [10], firmly establishing the applicability of the eikonal phase, and its efficiency at extracting classical observables.

2.2 The Gravitational Eikonal Phase and Impulse via the Double Copy

We can of course carry out the same procedure in a gravitational setting. From an amplitudes perspective, one can simply apply the double copy to the electromagnetic case, mapping the spinning dyon scattering amplitude to that of two interacting duality-rotated Kerr black holes (Kerr-Taub-NUT's, cf. [10]). Thus, the corresponding set-up in gravity, is to consider two duality-rotated Kerr black holes: a probe of mass M_1 and momentum p_1^μ, interacting with a heavy black hole of mass $M_2(\gg M_1)$ and momentum p_2^μ. Up to a constant of proportionality, the relevant three-point gravitational amplitude $\mathcal{M}_3$ is simply the square of $\mathcal{A}_3$, and thus upon a duality rotation, and taking the infinite spin limit, we find

$$\mathcal{M}_3\left[1,1',q^{\pm 2}\right] = \frac{\kappa}{2}M_1^2 x_1^{\pm 2}e^{\pm(i\theta_1+q\cdot a_1)},\quad \mathcal{M}_3\left[2,2,q^{\pm 2}\right] = \frac{\kappa}{2}M_2^2 x_2^{\pm 2}e^{\pm(i\theta_2+q\cdot a_2)}, \tag{2.14}$$

where $\kappa^2 = 32\pi G$. The long-range amplitude that we are interested in is therefore given by

$$\begin{aligned}\mathcal{M}_4 &= \left(\frac{\kappa}{2}\right)^2\frac{M_1^2M_2^2}{q^2}\left(\left(\frac{x_1}{x_2}\right)^2 e^{i(\theta_1-\theta_2)+q\cdot(a_1+a_2)} + \left(\frac{x_2}{x_1}\right)^2 e^{-i(\theta_1-\theta_2)-q\cdot(a_1+a_2)}\right) \\ &= \frac{2M_1M_2}{q^2}\Big[(Q_1+i\tilde{Q}_1)(Q_2-i\tilde{Q}_2)\Big(u_1\cdot u_2 + i\frac{\epsilon(\eta,u_1,q,u_2)}{q\cdot\eta}\Big)^2 e^{iq\cdot a} \\ &\qquad + (Q_1-i\tilde{Q}_1)(Q_2+i\tilde{Q}_2)\Big(u_1\cdot u_2 - i\frac{\epsilon(\eta,u_1,q,u_2)}{q\cdot\eta}\Big)^2 e^{-iq\cdot a}\Big]\ ,\end{aligned} \tag{2.15}$$

As an interesting aside, note that amplitude (2.15) is invariant under *double* duality, that is to say, if we scatter two Kerr-Taub-NUT objects ($Q_1 = Q_2 = 0$) or two Kerr black holes ($\tilde{Q}_1 = \tilde{Q}_2 = 0$), the scattering amplitude is identical, up to coupling labels. This reflects the fact that Hodge duality acting twice is trivial. The interesting physics arises when we consider the scattering of a Kerr black hole off a Kerr-Taub-NUT, i.e. $\tilde{Q}_1 = Q_2 = 0$, or two gravitational dyons.

Returning to the problem at hand, using (2.15) we determine the gravitational eikonal phase

to be

$$\chi_{\text{KTN}} = \frac{1}{2}\int \hat{\mathrm{d}}^4 q\, \hat{\delta}(u_1\cdot q)\hat{\delta}(u_1\cdot q)\frac{e^{iq\cdot b}}{q^2}\Big[(Q_1+i\tilde{Q}_1)(Q_2-i\tilde{Q}_2)\Big(u_1\cdot u_2 + i\frac{\epsilon(\eta,u_1,q,u_2)}{q\cdot\eta}\Big)^2 e^{iq\cdot a} \\ + (Q_1-i\tilde{Q}_1)(Q_2+i\tilde{Q}_2)\Big(u_1\cdot u_2 - i\frac{\epsilon(\eta,u_1,q,u_2)}{q\cdot\eta}\Big)^2 e^{-iq\cdot a}\Big]$$

$$
\begin{aligned}
&= \mathrm{Re}\,(Q_1 + i\tilde{Q}_1)(Q_2 - i\tilde{Q}_2) \int \hat{\mathrm{d}}^4 q\, \hat{\delta}(u_1 \cdot q)\hat{\delta}(u_1 \cdot q) \frac{e^{iq\cdot(b+a)}}{q^2} \Big(u_1 \cdot u_2 + i \frac{\epsilon(\eta, u_1, q, u_2)}{q \cdot \eta} \Big)^2 \\
&= \mathrm{Re}\,(Q_1 + i\tilde{Q}_1)(Q_2 - i\tilde{Q}_2) \int \hat{\mathrm{d}}^4 q\, \hat{\delta}(u_1 \cdot q)\hat{\delta}(u_1 \cdot q) \frac{e^{iq\cdot(b+a)}}{q^2} \\
&\qquad \times \Big[\cosh(2w) + 2i \frac{(u_1 \cdot u_2)\epsilon(\eta, u_1, q, u_2)}{q \cdot \eta} \Big] \,, \qquad (2.16)
\end{aligned}
$$

where in going from the second to the third equality, we have expanded the square, and noted that

$$
\epsilon^2(\eta, u_1, q, u_2) = -(q \cdot \eta)((u_1 \cdot u_2)^2 - 1) + \mathcal{O}(u_i \cdot q) + \mathcal{O}(q^2) \,, \qquad (2.17)
$$

in which we can ignore terms proportional to $q \cdot u_1$ or $q \cdot u_2$, noting that these do not contribute to the Wilson loop due to the presence of the delta functions. Moreover, we neglect contact terms $\mathcal{O}(q^2)$ since this are set to zero in the definition of the eikonal phase. Finally, we have used that $u_i^2 = -1$ $(i = 1, 2)$, and $u_1 \cdot u_2 = -\gamma = -\cosh(w)$, such that $2(u_1 \cdot u_2)^2 - 1 = \cosh(2w)$, where w is the rapidity.

As in the electromagnetic case, it is then a simple procedure to extract the leading-order impulse of the the probe Kerr-Taub-NUT from χ_{KTN}. Indeed, using eq. (2.7), the result is

$$
\begin{aligned}
\Delta p_1^\mu &= \Pi^\mu{}_\nu \frac{\partial}{\partial b_\nu} \chi_{\mathrm{KTN}} \\
&= \mathrm{Re}\,(Q_1 + i\tilde{Q}_1)(Q_2 - i\tilde{Q}_2) \int \hat{\mathrm{d}}^4 q\, \hat{\delta}(u_1 \cdot q)\hat{\delta}(u_1 \cdot q)\, \frac{e^{iq\cdot(b+a)}}{q^2} \\
&\qquad \times \left(i q^\mu \cosh(2w) - 2\cosh(w) \epsilon^\mu(u_1, q, u_2) \right) \,, \qquad (2.18)
\end{aligned}
$$

where we have used that $q^\mu \epsilon(\eta, u_1, q, u_2) = (q \cdot \eta)\epsilon^\mu(u_1, q, u_2) + \mathcal{O}(q^2)$. This result is again in exact agreement with that found in [10]. Of course, this is to be expected, but is nevertheless an elegant and efficient way to derive the classical impulse, bearing in mind that the eikonal phase encodes a number of physical observables. Indeed, we shall now move on to discuss how (in certain cases) Wilson loops can be derived, given knowledge of the appropriate eikonal phases. This is the subject of the next section.

2.3 Anyon-Anyon Eikonal Phase and the Electromagnetic and Gravitational Impulse

We can repeat the analysis conducted in the dyon case for anyons in 2+1 dimensions, and in doing so, making contact with some of the results derived in [45]. The efficiency the eikonal phase approach affords us, means that the results can be obtained in a few lines.

The four particle amplitude in the small m limit is given by

$$
\mathcal{A}_4[1, 2, 1', 2'] = \frac{2e_1 e_2 M_1 M_2}{q^2 + m^2} \left(u_1 \cdot u_2 + i \frac{m\epsilon(u_1, u_2, q)}{q^2} + \frac{m^2}{4m_1 m_2} \right) \,, \qquad (2.19)
$$

such that the three-dimensional electromagnetic eikonal phase is then

$$\chi_{\text{anyon}} = \frac{e_1 e_2}{2} \int \hat{\mathrm{d}}^3 q \, \hat{\delta}(u_1 \cdot q) \hat{\delta}(u_1 \cdot q) \, \frac{e^{iq\cdot b}}{q^2 + m^2} \left(u_1 \cdot u_2 + i \frac{m\epsilon(u_1, u_2, q)}{q^2} \right) . \tag{2.20}$$

For gravitational anyons, the four particle amplitude given by [45]

$$\mathcal{M}_4[1, 2, 1', 2'] = 2\kappa^2 \frac{m_1^2 m_2^2}{q^2 + m^2} \left(\cosh 2w + 2i u_1 \cdot u_2 \frac{m\epsilon(u_1, u_2, q)}{q^2} \right) - 2\kappa^2 \frac{m_1^2 m_2^2 \sinh^2 w}{q^2} , \tag{2.21}$$

giving an eikonal phase

$$\begin{aligned} \chi_{\text{grav anyon}} = & \frac{\kappa^2 m_1 m_2}{2} \int \hat{\mathrm{d}}^3 q \, \hat{\delta}(u_1 \cdot q) \hat{\delta}(u_1 \cdot q) \, \frac{e^{iq\cdot b}}{q^2 + m^2} \left(\cosh 2w + 2i u_1 \cdot u_2 \frac{m\epsilon(u_1, u_2, q)}{q^2} \right) \\ & - \frac{\kappa^2 m_1 m_2}{2} \int \hat{\mathrm{d}}^3 q \, \hat{\delta}(u_1 \cdot q) \hat{\delta}(u_1 \cdot q) \, \frac{e^{iq\cdot b}}{q^2} \sinh^2 w . \end{aligned} \tag{2.22}$$

As a check, we can derive the impulses in both cases, as the eikonal phases will be used later on. The electromagnetic impulse is given by

$$\begin{aligned} \Delta p_1^\mu &= \Pi^\mu{}_\nu \frac{\partial}{\partial b_\nu} \chi_{\text{anyon}} \\ &= \frac{i e_1 e_2}{2} \int \hat{\mathrm{d}}^3 q \, \hat{\delta}(u_1 \cdot q) \hat{\delta}(u_1 \cdot q) \, \frac{e^{iq\cdot b}}{q^2 + m^2} \left(q^\mu u_1 \cdot u_2 - i m \epsilon^\mu(u_1, u_2) \right) , \end{aligned} \tag{2.23}$$

where we have used

$$q^\mu \epsilon(q, u_1, u_2) = -m^2 \epsilon^\mu(u_1, u_2) + \mathcal{O}(m^2/m_1) . \tag{2.24}$$

In the gravity case, we find that the impulse is

$$\begin{aligned} \Delta p_1^\mu &= \Pi^\mu{}_\nu \frac{\partial}{\partial b_\nu} \chi_{\text{grav anyon}} \\ &= \frac{\kappa^2}{2} m_1 m_2 \int \hat{\mathrm{d}}^3 q \; \hat{\delta}(u_1 \cdot q) \hat{\delta}(u_2 \cdot q) \, e^{-iq\cdot b} \\ &\qquad \times \left(\frac{q^\mu \cosh 2w}{q^2 + m^2} - \frac{2 i m \epsilon^\mu(u_1, u_2) \cosh w}{q^2 + m^2} - \frac{q^\mu \sinh^2 w}{q^2} \right) . \end{aligned} \tag{2.25}$$

Upon inspection, it is found that both the electromagnetic and gravitational results are again in exact agreement with those determined previously in [45], affirming the validity of the eikonal phase in 2+1 dimensions.

3 Wilson Loops and Scattering Amplitudes

The Wilson loop is a gauge invariant quantity in gauge theory, derived from the holonomy of the gauge connection around some closed contour $\mathcal{C}$. It essentially corresponds to a phase factor, and is observable via the Aharonov-Bohm effect. Generically, it is defined as the trace of the path-ordered exponential of a gauge field A_μ integrated around a closed spacetime contour $\mathcal{C}$

$$W[A_\mu] = \mathrm{Tr}\left(\mathcal{P}\exp\left(\frac{ie}{\hbar}\oint_{\mathcal{C}} A_\mu(x)\mathrm{d}x^\mu\right)\right) , \tag{3.1}$$

where $\mathcal{P}$ is the path-ordering operator.

In the case of electromagnetism, the gauge field is $U(1)$ and thus Abelian, meaning that eq. (3.1) reduces to

$$W[A_\mu] = \exp\left(\frac{ie}{\hbar}\oint_{\mathcal{C}} A_\mu(x)\mathrm{d}x^\mu\right) = \exp\left(\frac{ie}{\hbar}\oint_{\mathcal{C}} A_\mu(x(\tau))\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\mathrm{d}\tau\right), \tag{3.2}$$

where τ is an affine parameter such that contour is given by $\mathcal{C} = \{x^\mu = x^\mu(\tau), \tau \in [0,1]\}$.

As mentioned, the Wilson loop is an observable, and in this analysis we wish to connect it to an on-shell scattering amplitude, from which we can readily calculate it. To do so, we will first break our contour $\mathcal{C}$ into two paths, $\mathcal{C} = \gamma_+ + \gamma_-$, where each path corresponds to evaluating x^μ as a straight line. This breaks the Wilson loop into a product of Wilson lines, however we need to be careful to ensure that our two chosen paths do indeed produce a spacetime loop.

We will choose the two paths to be determined by $x_1^\mu = u_1^\mu\tau + b^\mu$ and $x_2^\mu = -u_1^\mu\tau - b^\mu$ where $u_1^\mu \equiv \frac{p_1^\mu}{M_1}$ is the proper velocity of a particle probing the potential generated by a particle with momentum p_2^μ, which sources the electromagnetic (or gravitational) potential.

At $\tau \to \infty$, the proper velocity picks out points on the celestial sphere, since

$$\left.\frac{x_1^\mu}{\sqrt{-x_1^2}}\right|_{\tau\to\pm\infty} = \pm u_1^\mu = -\left.\frac{x_2^\mu}{\sqrt{-x_2^2}}\right|_{\tau\to\pm\infty} . \tag{3.3}$$

In Fig. 3 we present a diagram of how this closed contour is constructed from the above argument.

As such, the Wilson loop can be expressed as

$$\begin{aligned} W[A_\mu] &= \exp\left(\frac{ie}{\hbar}\oint_{\gamma_1+\gamma_2} A_\mu(x(\tau))\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\mathrm{d}\tau\right) \\ &= \exp\left(\frac{ie}{\hbar}\int_{-\infty}^{\infty}\mathrm{d}\tau\ \left[A_\mu(u_1\tau + b)u_1^\mu - A_\mu(u_1\tau - b)u_1^\mu\right]\right) . \end{aligned} \tag{3.4}$$

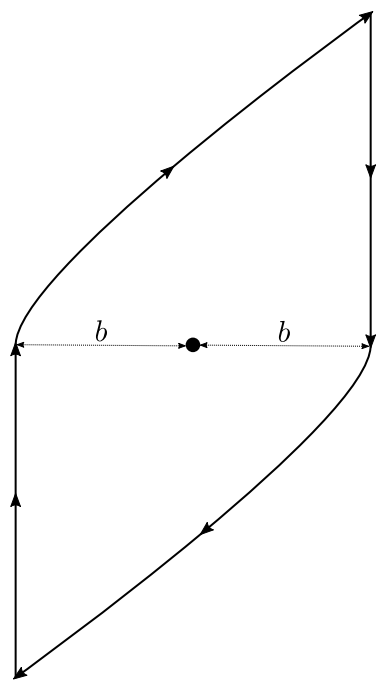

Figure 1. Closed contour as a product of Wilson Lines.

We note, however, that this is only *strictly* valid for attractive potentials, since this always ensures that the paths will have crossed by the time they reach asymptotic infinity. In principle, for a potential that vanishes as $r \to \infty$, any two paths could be connected by a translation (or rotation) at asymptotic infinity, where the gauge field is zero. However, this has the potential to introduce effects arising from large gauge transformations and care is likely needed. In this paper, we will only consider attractive potentials and so we will not comment on this further.

To recast this in terms of an on-shell scattering amplitude, we first recall a well-known quantity from relativistic electromagnetism, namely the relation between the gauge field A^μ and the electric potential V. Indeed, for a relativistic charged point particle subject to an external electromagnetic field, the potential is given by

$$V(p_1, q) = e\, u_1 \cdot A(q) = \frac{e}{M_1} p_1 \cdot A(q) \,. \tag{3.5}$$

With this expression in hand, we can simply employ the (relativistic) Born approximation[2] to relate the potential V to a corresponding 4-point scattering amplitude $\mathcal{A}_4$. At leading order,

[2]Such a potential can be derived using old-fashioned perturbation theory via the Lippmann-Schwinger equations. Interestingly, it is possible to derive the relativistic eikonal approximation from such a potential [46].

it can be shown that they are related by

$$\langle p|V|p+q\rangle = \frac{\mathcal{A}_4(p,p+q)}{4M_1M_2} \ , \tag{3.6}$$

where $\mathcal{A}_4(p,p+q)$ is the $2\to 2$ scattering amplitude.

We can use this relation to write the Wilson loop, at leading order, as

$$\begin{aligned} W[A_\mu] &= \exp\left(\frac{1}{4M_1M_2\hbar}\int_{-\infty}^{\infty} d\tau \int \hat{\mathrm{d}}^D q\, \hat{\delta}(u_2\cdot q)\, \mathcal{A}_4[p_1,p_2\to p_1',p_2']e^{iq\cdot u_1\tau}\left(e^{-iq\cdot b}-e^{iq\cdot b}\right)\right) \\ &= \exp\left(-\frac{i}{2M_1M_2\hbar}\int \hat{\mathrm{d}}^D q\, \hat{\delta}(u_1\cdot q)\, \hat{\delta}(u_2\cdot q)\, \mathcal{A}_4[p_1,p_2\to p_1',p_2']\sin(q\cdot b)\right) . \end{aligned} \tag{3.7}$$

We recognise the phase as a sum of two eikonal phases defined in eq. (2.4), each evaluated at different values of the impact parameter b.

It is useful to note that any part of the amplitude which is an even function of q^2 will not contribute to the Wilson loop, since the integral will vanish. This is because only the spatial part of q is non-zero (due to the delta functions) and is therefore an even function such that

$$\int \hat{\mathrm{d}}^4 q\, \hat{\delta}(u_1\cdot q)\, \hat{\delta}(u_2\cdot q)\, f(q^2)\sin(q\cdot b) = 0 \ . \tag{3.8}$$

3.1 Quantization of Electric Charge from Amplitudes

Now that we have established the connection between Wilson loops and scattering amplitudes, we can compute the Wilson loop generated by a dyon-dyon amplitude. Let us do so in the spinless case, i.e. $a^\mu = 0$, as including spin does not add anything of any particular relevance[3].

Starting from the dyon-dyon eikonal phase (2.10) (with $a^\mu = 0$), and massaging it into a form that exposes the different charge sectors, we can determine the corresponding Wilson loop, by plugging it into (3.7). As per eq. (3.8) and the explanation preceding it, we can neglect any parts of the amplitude that are an even function of q^2, as their integrals vanish. As such, we can consider the reduced amplitude,

$$\tilde{\mathcal{A}}_4 = 4M_1M_2(e_1g_2 - e_2g_1)\frac{\epsilon(\eta,u_1,q,u_2)}{q^2(q\cdot\eta)} \ , \tag{3.9}$$

[3]For non-zero a^μ the Wilson loop has a more complicated structure, however, it turns out that the spin contribution is already gauge invariant, and therefore provides no further constraints. As it only serves to complicate the expression, distracting from the main result, we work in the spinless case in the analysis of this section. The argument is the same in the gravitational case.

which readily gives us the following Wilson loop:

$$W[A_\mu] \;=\; \exp\left(\frac{1}{\hbar}(e_1 g_2 - e_2 g_1)\epsilon(\eta, u_1, u_2)_\mu J^\mu\right), \tag{3.10}$$

where the J^μ is given by (for further details of this result see appendix A)

$$J^\mu = -2i \int \frac{\mathrm{d}^4 q}{(2\pi)^2}\delta(u_1\cdot q)\delta(u_2\cdot q)\frac{q^\mu}{q^2(q\cdot\eta)}\sin(q\cdot b) = \frac{ib_\perp^\mu}{|\epsilon(\eta_\perp, u_1, u_2, b_\perp)|}\ , \tag{3.11}$$

and we have noted that for $\eta_\perp\cdot u_i = 0$ and $b_\perp\cdot u_i = 0$ $(i = 1, 2)$, one finds that $\epsilon^2(\eta_\perp, u_1, u_2, b_\perp) = -|\beta\gamma|^2\left(b_\perp^2\eta_\perp^2 - (b_\perp\cdot\eta_\perp)^2\right)$.

Using this result in eq. (3.10), the Wilson loop can be expressed succinctly, as

$$W[A_\mu] = \exp\left(\frac{i}{\hbar}\,(e_1 g_2 - e_2 g_1)\mathrm{sgn}(\epsilon(\eta_\perp, u_1, u_2, b_\perp))\right)\ , \tag{3.12}$$

where $\mathrm{sgn}(x) = \frac{x}{|x|}$ is the standard signum function. Now, in its current form, eq. (3.12) clearly depends on the gauge vector $\eta_\perp^\mu$ (or at least its projection onto the impact parameter plane). However, the Wilson loop is an observable quantity and should therefore be gauge invariant. That is, it should be independent of whatever gauge vector we choose. Notice, though, that the gauge vector only appears in the signum function, and therefore, if we choose some other vector $\tilde{\eta}^\mu$, the function can differ by at most a sign from the initial gauge. Thus, the gauge invariance of $W[A_\mu]$ can be ensured by requiring that is insensitive to the change $\eta^\mu \to -\eta^\mu$. This leads us to enforce the condition

$$W[A_\mu] - W[A_\mu]\Big|_{\eta\to-\eta} = 2i\,\mathrm{sgn}(\epsilon(\eta_\perp, u_1, u_2, b_\perp))\sin\left(\frac{1}{\hbar}(e_1 g_2 - e_2 g_1)\right) \stackrel{!}{=} 0\ . \tag{3.13}$$

We see that this is readily satisfied if the following constraint on the dyonic charges holds:

$$e_1 g_2 - e_2 g_1 = n\hbar\pi\ , \tag{3.14}$$

which is precisely the celebrated charge quantisation condition. The important point to note here, is that this result was arrived at via a fully on-shell amplitudes approach, with no mention of any geometry (or wavefunctions), which is typically relied upon in the standard field theory calculation. We have shown here, that simply by demanding the gauge invariance of the Wilson loop, a necessary requirement of it being an observable, one can derive such results purely from the (leading-order) difference of two eikonal phases of the relevant four-point scattering amplitude.

3.2 Quantization of Momentum from Amplitudes

In this section we will turn to quantization conditions in gravity. To do so, we consider the generalisation of a Wilson loop from gauge theory to the case of perturbative gravity. The

gravitational Wilson loop then corresponds to the phase experienced by a scalar test particle in a gravitational field. This can be formulated in terms of the proper length of a closed curve $\mathcal{C}$ traversed by a probe particle of mass m, and to leading-order in the gravitational coupling κ, is given by [11, 47, 48]

$$W[h_{\mu\nu}] = \exp\left(-\frac{i\kappa m}{2\hbar}\oint_{\mathcal{C}} h_{\mu\nu}(x(\tau))\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\mathrm{d}\tau\right) . \tag{3.15}$$

To see this, we note that in the full covariant theory, the Wilson loop must take the form (in the proper time parametrisation)

$$\Phi[h_{\mu\nu}] = \exp\left(\frac{im}{\hbar}\oint_{\mathcal{C}}\left(-g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\right)^{1/2}\mathrm{d}\tau\right) . \tag{3.16}$$

Expanding the metric as $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$ and expand the square-root, we find that to leading-order in κ, the integrand becomes

$$\left(-g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\right)^{1/2} = 1 - \frac{\kappa}{2}h_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} + \mathcal{O}(\kappa^2) . \tag{3.17}$$

Inserting this expansion back into eq. (3.16) and absorbing the constant term (independent of $h_{\mu\nu}$) into an overall normalisation, we immediately recover eq. (3.15).

From a scattering amplitudes perspective, however, the setup is identical to the electromagnetic case, where now we simply replace the photon exchange amplitude with a graviton exchange amplitude, i.e.

$$\begin{aligned} W[h_{\mu\nu}] &= \exp\left(\frac{1}{4M_1M_2\hbar}\int \hat{\mathrm{d}}^4q\,\hat{\delta}(u_1\cdot q)\,\hat{\delta}(u_2\cdot q)\,\mathcal{M}_4[p_1,p_2\to p_1',p_2']\left(e^{-iq\cdot b} - e^{iq\cdot b}\right)\right) \\ &= \exp\left(-\frac{i}{2M_1M_2\hbar}\int \hat{\mathrm{d}}^4q\,\hat{\delta}(u_1\cdot q)\,\hat{\delta}(u_2\cdot q)\,\mathcal{M}_4[p_1,p_2\to p_1',p_2']\sin(q\cdot b)\right) . \end{aligned} \tag{3.18}$$

To arrive at the above expression, we have used the fact that the perturbative expansion of the relativistic Lagrangian for a point particle, subject to an external gravitational field, is proportional to eq. (3.17) (with the proportionality constant being the mass of the probe particle). Furthermore, as in the electromagnetic case, to leading order in the Born approximation we can relate the relativistic momentum space potential (experienced by the test particle) with the four-point graviton exchange amplitude (in the probe particle limit). Thus, for a probe particle of mass m_1 and momentum p_1, we have

$$V(p_1,q) = \frac{\kappa}{2M_1}h_{\mu\nu}(q)p_1^\mu p_1^\nu = \frac{\mathcal{M}_4[p_1,p_2\to p_1',p_2']}{4M_1M_2} . \tag{3.19}$$

We will consider the scattering of two duality-rotated Kerr black holes (Kerr-Taub-NUT's),

the established double copy of the spinning dyon [10]. The relevant long-range four-point scattering amplitude in this case is the same as that used in section 2.2, i.e., eq. (2.15). As in the electromagnetic case, adding the classical spin does not change the story at all in terms of the quantization condition, and we therefore set $a^\mu = 0$ in the following analysis. This effectively reduces the problem to the case of two Taub-NUT's (previously explored in the non-relativistic setting [5–7]).

To simplify things, as in the discussion on eikonal phases, we can make use of eq. (2.17), as well as making the different charge sectors apparent. This enables us to instead consider a simplified amplitude of the form

$$\mathcal{M}_4 = \frac{2M_1M_2}{q^2}\Big[(Q_1Q_2 + \tilde{Q}_1\tilde{Q}_2)\cosh(2w) + 2(Q_1\tilde{Q}_2 - Q_2\tilde{Q}_1)\frac{(u_1\cdot u_2)\epsilon(\eta, u_1, q, u_2)}{q\cdot\eta}\Big] \,. \tag{3.20}$$

It is interesting to note that the Schwarzschild part of the amplitude is an even function of q^2, and therefore, from eq. (3.8), it cannot contribute to the Wilson loop. Consequently, we can completely neglect this term, and focus our attention on the parity odd term, proportional to the Levi-Civita tensor. The Wilson loop under consideration then becomes

$$W[h_{\mu\nu}] = \exp\left(\frac{1}{\hbar}(Q_1\tilde{Q}_2 - Q_2\tilde{Q}_1)(u_1\cdot u_2)\epsilon_\mu(\eta, u_1, u_2)\, J^\mu\right) \,, \tag{3.21}$$

where J^μ is given by eq. (3.11). Accordingly, the gravitational Wilson loop is given by

$$\begin{aligned} W[h_{\mu\nu}] &= \exp\left(\frac{i}{\hbar}\,(Q_1\tilde{Q}_2 - Q_2\tilde{Q}_1)(u_1\cdot u_2)\mathrm{sgn}\big(\epsilon(\eta_\perp, u_1, u_2, b_\perp)\big)\right) \\ &= \exp\left(\frac{4\pi i}{\hbar}\,\gamma(m_1\ell_2 - m_2\ell_1)\mathrm{sgn}\big(\epsilon(\eta_\perp, u_1, u_2, b_\perp)\big)\right) \,, \end{aligned} \tag{3.22}$$

where we have made the identification $Q_i\tilde{Q}_j = 4\pi m_i\ell_j$, where m_i is the mass and ℓ_j the NUT parameter. In order for this to be gauge invariant, as it should be as an observable, it should be independent of the gauge vector η^μ. Following the same approach as the electromagnetic case, this gives the generalized quantization condition as

$$\gamma(m_1\ell_2 - m_2\ell_1) = \frac{1}{4}\hbar n \,. \tag{3.23}$$

We can go further, however, and note that we can write this as

$$(p_1\cdot k_2 - p_2\cdot k_1) = \frac{1}{4}\hbar n \,, \tag{3.24}$$

where $p_i^\mu = m_i u_i^\mu$ and $k_i^\mu = \ell_i u_i^\mu$ are the momentum and dual 'magnetic' momentum respectively. The momenta as defined agree explicitly with the momentum and dual momentum discussed in [49] in the context of boosted Taub-NUT solutions. Interestingly, in Ref. [50]

the authors discovered that such a magnetic momentum must be present in the superalgebra if supersymmetric systems are to respect gravitational duality.

The appearance of *momentum* in the quantization condition is not entirely surprising from the perspective of the double copy, which relates colour degrees of freedom (in this case U(1)) with kinematic degrees of freedom. Here we see exactly this enacted, with products of charge being replaced by dot products of momentum.

3.3 Electromagnetic & Gravitational Phases for Anyons

We now return to the study of anyons in electromagnetism and gravity, this time to compute the associated Wilson loops. In the electromagnetic case, we plug in the eikonal phase (2.20) into the Wilson loop formula for $D = 3$ to find

$$\begin{aligned} W[A_\mu] &= \exp\left(2ie_1e_2 \int \hat{\mathrm{d}}^3q\, \hat{\delta}(u_1\cdot q)\hat{\delta}(u_1\cdot q)\, \frac{\sin(q\cdot b)}{q^2+m^2}\left(\frac{m\epsilon(u_1,u_2,q)}{q^2}\right)\right) \\ &= \exp\left(i\frac{e_1e_2}{m}\left(1-e^{-m|b_\perp|}\right)\mathrm{sgn}(\epsilon(u_1,u_2,b_\perp))\right)\ , \end{aligned} \tag{3.25}$$

where we have used

$$\begin{aligned} \int \hat{d}^3x\ \hat{\delta}(q\cdot u_1)\hat{\delta}(q\cdot u_2)\frac{q^\mu}{q^2(q^2+m^2)}\sin(q\cdot b) &= -\frac{i}{2|\beta\gamma|m^2}(1-e^{m|b_\perp|})\frac{b_\perp^\mu}{|b_\perp|} \\ &= \frac{(1-e^{m|b_\perp|})b_\perp^\mu}{2m^2\sqrt{\epsilon(u_1,u_2,b_\perp)^2}}\ . \end{aligned} \tag{3.26}$$

In the gravitational case, we plug in eq. (2.22) to find

$$\begin{aligned} W[h_{\mu\nu}] &= \exp\left(2i\kappa^2m_1m_2 \int \hat{\mathrm{d}}^3q\, \hat{\delta}(u_1\cdot q)\hat{\delta}(u_1\cdot q)\, \frac{\sin(q\cdot b)}{q^2+m^2}\left(u_1\cdot u_2\frac{m\epsilon(u_1,u_2,q)}{q^2}\right)\right) \\ &= \exp\left(\frac{\kappa^2m_1m_2(u_1\cdot u_2)}{m}\left(1-e^{-m|b_\perp|}\right)\mathrm{sgn}(\epsilon(u_1,u_2,b_\perp))\right)\ . \end{aligned} \tag{3.27}$$

Unlike in the four-dimensional case, there is no gauge dependence here - the sign of the phase is entirely determined by physical quantities, and so this phase should be entirely observable. Furthermore, we find that in the non-relativistic large m Chern-Simons limit (keeping e/m fixed), we recover the well known Aharonov-Bohm phase for both the anyon [51, 52] and gravitational anyon [52–54], thus providing a good consistency check.

4 Discussion

In this paper we have explored the nature of charge quantization in both electromagnetism and gravity in the context of on-shell scattering amplitudes. In particular, we worked within the framework of on-shell eikonal physics, from which we determined the relevant

leading-order contribution for a dyon-dyon and a two Kerr-Taub-NUT scattering process in electromagnetism and GR, respectively. In the gravitational case we employed the double copy to map the spinning dyon amplitude to that of two interacting duality rotated Kerr black holes. We further gave an example of how the eikonal phase contains information about physical observables in straightforwardly deriving the associated particle impulses in each case. These results are in exact agreement with those previously derived, where the impulse was computed directly.

Having established the form of the leading order eikonal phase in the context of scattering amplitudes, we went on to demonstrate how one can construct certain Wilson loops (i.e. those for which the potential is attractive) from a sum of eikonal phases, each evaluated at different values of the associated impact parameter of the scattering processes considered. Working in the spinless case[4], we were led very nicely to the key result of this paper: that charge quantization can be seen directly from on-shell scattering amplitudes, with no mention of wavefunctions, or an underlying geometry. Indeed, by simply imposing the gauge invariance of the Wilson loop one naturally arrives at the known charge quantization condition for dyons. We extended this analysis to the case of gravity, in which we considered the scattering of two Taub-NUT's, i.e. the spinless limit of duality rotated Kerr black holes (so-called Kerr-Taub-NUT's). In doing so we derived an analogous quantization condition from the associated Wilson loop, in this case telling us that relativistic four-momentum is quantized in terms of the dual 'NUT' momentum. This result is particularly nice as it generalises what had previously been found to a fully relativistic setting. This is explicitly seen through the appearance of a Lorentz factor in the quantization condition at the level of the mass and dual mass aspects of the Taub-NUT's.

Finally, we utilised the scattering amplitudes construction of the eikonal phase and Wilson loops to explore the analogous set-up in three spacetime dimensions. In doing so, we made contact with some of the results derived in [45], for both topologically massive gauge theory and its gravitational double copy. This served to highlight the validity and efficacy of the approach adopted here. In 2+1 dimensions, the analogous set-ups to consider are the scattering of anyons and gravitational anyons. Through computing the relevant eikonal phases in both cases, we were readily able to derive the associated probe particle impulses, which agree exactly with the results found previously in [45]. We then went a step further to consider the Wilson loops corresponding to the electromagnetic and gravitational set-ups in 2+1 dimensions. As a consistency check, it can be shown that one recovers the well known Aharonov-Bohm phase in both cases. Interestingly, in opposition to the 4D case, the Wilson loop has no gauge dependence, with the sign of the phase being determined entirely by physical quantities. As such, there are no corresponding quantization conditions placed on the anyon and gravitational anyon charges.

Given the fruitful nature of eikonal physics and Wilson loops, there are a number of possible

[4] Adding spin does not provide further constraints and serves only to complicate the expression

directions to explore in the future. For example, it would be interesting to examine the double copy properties of boosted Taub-NUT solutions and their relation to gravitational shockwaves, which were recently investigated in [55]. Another intriguing extension, would be to consider the non-Abelian case in which colour is added in the gauge theory sector. Indeed, there has already been some interesting research into classical Yang-Mills observables from amplitudes, for example, the colour impulse (as an analogue to momentum impulse) [56]. It would be instructive to explore these sorts of structures further in the context of eikonal phases and Wilson loops. This would of course require more care in constructing the closed contour in the Wilson loop, since in this case path ordering would matter. Finally, one might wonder whether or not the Wilson loop could be defined via the analytic continuation of the eikonal (itself a Wilson line), much in the same way that scattering amplitudes have been used recently to describe bound orbits [57–59].

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A Fourier Transforms

In the analysis of this paper, we have made use of a specialised Fourier transform, in which we project the Fourier transformed function onto the impact parameter plane. Here we shall give a detailed derivation of the most general inverse Fourier transform, of which we utilise in section 3 on Wilson loops (albeit specialised to $a^\mu = 0$).

Let us begin by defining the inverse eikonal Fourier transform as

$$\mathcal{F}_\pm[f(q)] \equiv \int \hat{\mathrm{d}}^4 q\, \hat{\delta}(u_1 \cdot q)\hat{\delta}(u_2 \cdot q)\, e^{\pm iq\cdot b}\, f(q). \tag{A.1}$$

The function of interest, of which we shall determine the inverse eikonal Fourier transform of, is

$$f^\mu(q, \pm a, \eta) = \frac{q^\mu}{q^2(q\cdot\eta)}\, e^{\pm iq\cdot a}\ . \tag{A.2}$$

It helps to choose a frame here, and so we pick

$$u_1^\mu = (\gamma, \gamma\beta, 0, 0), \qquad u_2 = (1, 0, 0, 0). \tag{A.3}$$

For simplicity, we also define $\tilde{b}^\mu_\pm = b^\mu \pm a^\mu$. such that $\tilde{b}^\mu_\pm|_{a^\mu=0} = b^\mu$. Then, using a Schwinger parametrization, along with the useful result

$$\int \hat{\mathrm{d}}^2 q_\perp\, \frac{q^\mu_\perp}{q^2_\perp} e^{\pm iq_\perp\cdot b_\perp} = \pm\frac{ib^\mu_\perp}{2\pi|b_\perp|^2}\ , \tag{A.4}$$

we can determine the eikonal Fourier transform, as follows:

$$\begin{aligned}
\mathcal{F}_\pm\left[f^\mu(q,\pm a,\eta)\right] &= \int \hat{\mathrm{d}}^4 q\, \hat{\delta}(u_1\cdot q)\hat{\delta}(u_2\cdot q)\; e^{\pm i q\cdot \tilde{b}^\mu_\pm}\, \frac{q^\mu}{q^2(q\cdot\eta)} \\
&= -i \lim_{\epsilon\to 0}\int_0^\infty \mathrm{d}\lambda\, e^{-\epsilon\lambda} \int \hat{\mathrm{d}}^4 q\, \hat{\delta}(u_1\cdot q)\hat{\delta}(u_2\cdot q)\; \frac{q^\mu}{q^2} e^{\pm i q\cdot \tilde{b}^\mu_\pm} e^{i\lambda q\cdot\eta} \\
&= -i \frac{1}{|\beta\gamma|} \lim_{\epsilon\to 0}\int_0^\infty \mathrm{d}\lambda\, e^{-\epsilon\lambda} \int\; \hat{\mathrm{d}}^2 q_\perp\; \frac{q^\mu_\perp}{q^2_\perp} e^{\pm i q_\perp\cdot(\tilde{b}^\mu_{\pm,\perp}\pm\lambda\eta_\perp)} \\
&= \mp \frac{1}{2\pi|\beta\gamma|} \lim_{\epsilon\to 0}\int_0^\infty \mathrm{d}\lambda\, e^{-\epsilon\lambda}\, \frac{\tilde{b}^\mu_{\pm,\perp}\pm\lambda\eta^\mu_\perp}{|\tilde{b}_{\pm,\perp}\pm\lambda\eta_\perp|^2} \\
&= \mp\alpha_\pm(\tilde{b}_{\pm,\perp},\eta_\perp)\,\tilde{b}^\mu_{\pm,\perp} + \beta_\pm(\tilde{b}_{\pm,\perp},\eta_\perp)\,\eta^\mu_\perp\,, \qquad \text{(A.5)}
\end{aligned}$$

where we have defined the coefficients of $\tilde{b}_{\pm,\perp}$ and $\eta^\mu_\perp$, as

$$\alpha_\pm(\tilde{b}_{\pm,\perp},\eta_\perp) = \frac{1}{2\pi|\beta\gamma|}\lim_{\epsilon\to 0}\int_0^\infty \mathrm{d}\lambda\, \frac{e^{-\epsilon\lambda}}{|\tilde{b}_{\pm,\perp}\pm\lambda\eta_\perp|^2} = \frac{\left(\frac{\pi}{2}\mp 2\arctan\left(z(\tilde{b}_{\pm,\perp},\eta_\perp)\right)\right)}{2\pi|\epsilon(\eta_\perp,u_1,\tilde{b}_{\pm,\perp},u_2)|}\,, \qquad \text{(A.6)}$$

$$\beta_\pm(\tilde{b}_{\pm,\perp},\eta_\perp) = -\frac{1}{2\pi|\beta\gamma|}\lim_{\epsilon\to 0}\int_0^\infty \mathrm{d}\lambda\, e^{-\epsilon\lambda}\, \frac{\lambda}{|\tilde{b}_{\pm,\perp}\pm\lambda\eta_\perp|^2}\,, \qquad \text{(A.7)}$$

in which we have introduced the compact notation $z(\tilde{b}_{\pm,\perp},\eta_\perp) = (\tilde{b}^2_{\pm,\perp}\eta^2_\perp - (\tilde{b}_{\pm,\perp}\cdot\eta_\perp)^2)^{-1/2}\tilde{b}_{\pm,\perp}\cdot\eta_\perp$, and noted that $|\epsilon(\eta_\perp,u_1,\tilde{b}_{\pm,\perp},u_2)| = |\beta\gamma|(\tilde{b}^2_{\pm,\perp}\eta^2_\perp - (\tilde{b}_{\pm,\perp}\cdot\eta_\perp)^2)^{1/2}$.

We have deliberately left $\beta_\pm(\tilde{b}_{\pm,\perp},\eta_\perp)$ unevaluated, as it technically contains a divergent part, and therefore would need to be further regulated. However, in this paper, we such quantities are ultimately contacted with a Levi-Civita tensor of the form $\epsilon_\mu(\eta_\perp,u_1,u_2)$, and so the $\beta(\tilde{b}_{\pm,\perp},\eta_\perp)\eta^\mu_\perp$ is completely irrelevant, as it automatically vanishes. Moreover, any observables should be independent of the gauge vector $\eta^\mu_\perp$, and as a result of these facts, we therefore do not concern ourselves further with the exact details of computing $\beta_\pm(\tilde{b}_\perp,\eta_\perp)$.

Returning to eq. (A.5), we see that in the case where $a^\mu = 0$, the result reduces to

$$\mathcal{F}_\pm\left[\frac{q^\mu}{q^2(q\cdot\eta)}\right] = \mp\frac{\left(\frac{\pi}{2}\mp 2\arctan\left(z(\eta_\perp,b_\perp)\right)\right)}{2\pi|\epsilon(\eta_\perp,u_1,b_\perp,u_2)|}\, b^\mu_\perp\,. \qquad \text{(A.8)}$$

Using this result, one can readily derive the expression in (3.11) in the main body:

$$J^\mu = -2i\left(\mathcal{F}_+\left[\frac{q^\mu}{q^2(q\cdot\eta)}\right] - \mathcal{F}_-\left[\frac{q^\mu}{q^2(q\cdot\eta)}\right]\right) = \frac{i b^\mu_\perp}{|\epsilon(\eta_\perp,u_1,b_\perp,u_2)|}\,. \qquad \text{(A.9)}$$

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