

# CURS 5

## SERII DE AIR REALE

Def. O pereche de serii  $((X_n)_{n \geq p}, (y_n)_{n \geq p})$  unde  $\sum_{k=p}^{\infty} X_{n+k}$  n.m. serie  $\sum_{n \geq p} X_n = \sum_{n=p}^{\infty} X_n$

$X_n$  - termenii seriei  
 $y_n$  - sumele parțiale

$$\lim_{n \rightarrow \infty} y_n = \sum_{n \geq p} X_n$$



Obs: Dacă  $\sum_{n=1}^{\infty} x_n$  și  $\sum_{n=1}^{\infty} y_n$  sunt serii convergente  $\Rightarrow$

$\Rightarrow \sum_{n=1}^{\infty} (x_n + y_n)$  și  $\sum_{n=1}^{\infty} a \cdot x_n$  sunt convergente

$$\text{și } \sum_{n=1}^{\infty} (x_n + y_n) = \sum_{n=1}^{\infty} x_n + \sum_{n=1}^{\infty} y_n, \quad \sum_{n=1}^{\infty} a \cdot x_n = a \cdot \sum_{n=1}^{\infty} x_n$$

Obs: Dacă  $\sum_{n=1}^{\infty} x_n \in \mathbb{R}$  (este conv.)  $\Rightarrow x_n \rightarrow 0$

$$x_n = y_n - y_{n-1} \rightarrow a - a = 0$$

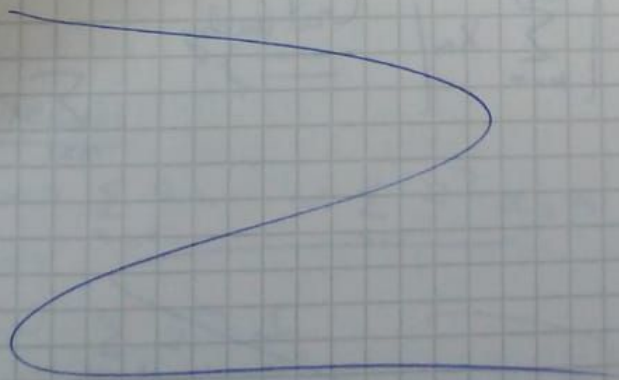
ex  $\sum_{n=1}^{\infty} \frac{1}{n} = \infty \quad \frac{1}{n} \rightarrow 0$

$$\lim_{n \rightarrow \infty} \frac{1 + \frac{1}{2} + \dots + \frac{1}{n}}{\ln n} = 1$$

Obs:  $\sum_{n=1}^{\infty} x_n$  unde  $x_n \geq 0$  atunci

$\Rightarrow$  serie este conv.  $\Leftrightarrow$  este mărg.

$$y_{n+1} - y_n = x_{n+1} \geq 0 \Rightarrow y_n \nearrow$$





Un sir  $(x_n)_n \in \mathbb{R}$  s.m. Cauchy dacă  $\forall \varepsilon > 0 \exists m_\varepsilon$  a.i.  
 $\forall n \geq m_\varepsilon \quad \forall p \in \mathbb{N} \Rightarrow$

$$\Rightarrow |x_{n+p} - x_n| < \varepsilon$$

CRITERIUL 1 Cauchy O serie  $\sum_{n=1}^{\infty} x_n$  este conv  $\Leftrightarrow \forall \varepsilon > 0$

$$\exists m_\varepsilon \text{ a.i. } n \geq m_\varepsilon \Rightarrow \left| \sum_{k=n}^{n+p} x_k \right| < \varepsilon$$

$$\sum_{k=n}^{n+p} x_k = y_{n+p} - y_n$$

Def. O serie  $\sum_{n=1}^{\infty} x_n$  s.m. absolut conv. dacă serie

$$\sum_{n=1}^{\infty} |x_n| \text{ este conv}$$

CRITERIUL 2 O serie abs conv. este conv.

Dem:  $\sum_{n=1}^{\infty} |x_n|$  este abs conv.

Asta înseamnă  $\forall \varepsilon > 0 \exists m_\varepsilon$  a.i.  $\forall n \geq m_\varepsilon$  și  $\forall p \in \mathbb{N}$   
 că pt.

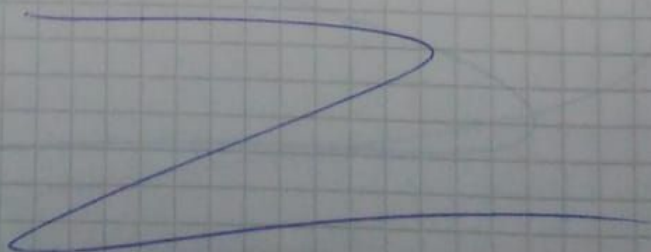
$$\sum_{k=n}^{n+p} |x_k| < \varepsilon$$



$\Rightarrow$

$$\left| \sum_{k=n}^{n+p} x_k \right| \quad \text{Crit Cauchy}$$

$$\sum_{n=1}^{\infty} x_n \text{ este conv.}$$



### CRITERIUL 3 (Abel)

Dacă  $a_n \searrow 0$  și  $\exists M$  a.i.  $\left| \sum_{k=1}^n x_k \right| \leq M \Rightarrow$

$\Rightarrow \sum_{n \geq 1} a_n \cdot x_n$  este conv.

### CRITERIUL 4 (Leibniz - crit. sîn alternant)

$x_n = (-1)^n$   $a_n \searrow 0$   $\sum (-1)^n \cdot a_n$  este conv.

$a_n = \frac{1}{n} \rightarrow \sum (-1)^n \cdot \frac{1}{n}$  conv.

dar nu este abs. conv.

### CRITERIUL 3 $\Rightarrow$ CRITERIUL 4

$x_n = (-1)^n$

$$\sum_{k=1}^n x_k = \sum_{k=1}^n (-1)^k = -1 + 1 - 1 + 1 - \dots \in \{-1, 0\}$$

ex:

$\sum_{n \geq 1} \frac{1}{n} \sin nx$  conv.

$\sum_{k=1}^n \sin k \cdot x = \dots$  convergență

### CRITERIUL CONDENSĂRII

$$a_n \searrow 0 \Rightarrow \sum_{n \geq 1} a_n \sim \sum_{n \geq 1} 2^n a_{2^n}$$

$$\sum_{n \geq 1} \frac{1}{n} \sim \sum_{n \geq 1} 2^n \frac{1}{2^n}$$



$$\sum_{n \geq 1} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{2^m} + \dots + \frac{1}{2^{m+1}} + \dots + \frac{1}{2^{n+1}}$$

$$n = \sum_{k=2^m+1}^{2^{m+1}} a_k = \sum_{m \geq 0} \left( \sum_{k=2^m+1}^{2^{m+1}} a_k \right)$$

$$a_1 \geq a_k \geq a_{2^{m+1}}$$

$$\frac{2^m}{2^m} \leq k \leq \frac{2^{m+1}}{2^m}$$

$$\sum 2^m a_{2^m} \text{ e conv. } \Rightarrow \sum 2^m a_{2^m} < A \Rightarrow \sum a_n < A$$

$$\sum_{n \geq 1} \frac{1}{n^\alpha}$$

$$\lim_{n \rightarrow \infty} \frac{1}{n^\alpha} = \begin{cases} 0 & , \alpha > 0 \\ 1 & , \alpha = 0 \\ \infty & , \alpha < 0 \end{cases} \quad \text{div}$$

$$\alpha > 0 \quad \frac{1}{n^\alpha} \downarrow 0$$

$$\sim \sum_{n \geq 1} 2^n \frac{1}{(2^n)^\alpha} = \sum_{n \geq 1} 2^{n-n\alpha} = \sum_{n \geq 1} (2^{1-\alpha})^n$$

$$\sum_{n \geq 0} a^n = \begin{cases} \frac{1}{1-a} & , a < 1 \\ \infty & , a \geq 1 \end{cases}$$

↓  
conv.

me gusta de  $2^{1-\alpha} < 1 = 2^0$

↓

$\alpha > 1$

$$\sum_{n=2}^{\infty} \frac{1}{n (\ln n)^{\alpha}}$$

pt. c. a.   
 even   
 log

$$\frac{1}{n (\ln n)^{\alpha}} \searrow 0$$

$$f(x) = x \cdot \ln x^{\alpha}$$

pt. a studia monot

$$\Rightarrow \sum_{n=2}^{\infty} \frac{1}{2^n (\ln 2^n)^{\alpha}}$$

$$= \sum_{n=2}^{\infty} \frac{1}{n^{\alpha} (\ln 2)^{\alpha}} \sim \sum_{n=2}^{\infty} \frac{1}{n^{\alpha}} \quad (\text{pt. c. a. } (\ln 2)^{\alpha} \text{ nu modifica})$$



Care e conv.  $\Leftrightarrow \boxed{\alpha > 1}$

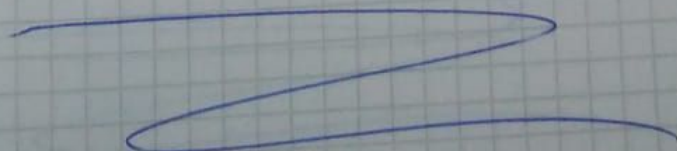
### CRITERIUL 6 (comparatiei)

Fie  $\sum_{n=1}^{\infty} a_n$  si  $\sum_{n=1}^{\infty} b_n$  unde  $a_n, b_n \geq 0$

Daca  $\exists M > 0$  si  $\exists m_0$  a.i.  $\forall n \geq m_0 \Rightarrow a_n < M b_n$  atunci:  
 $(b_n > 0 \Leftrightarrow \lim_{n \rightarrow \infty} \frac{a_n}{b_n} < \infty)$

1) Daca  $\sum_{n=1}^{\infty} b_n < \infty \Rightarrow \sum_{n=1}^{\infty} a_n < \infty$

2) Daca  $\sum_{n=1}^{\infty} a_n = \infty \Rightarrow \sum_{n=1}^{\infty} b_n = \infty$





V2  $0 < \underline{\lim} \frac{a_n}{b_n} \leq \overline{\lim} \frac{a_n}{b_n} < \infty \Rightarrow \sum_{n=1}^{\infty} a_n \sim \sum_{n=1}^{\infty} b_n$

V3  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} \in (0, \infty) \Rightarrow \sum_{n=1}^{\infty} a_n \sim \sum_{n=1}^{\infty} b_n$

ex.  $\sum_{n=1}^{\infty} \underbrace{\frac{\sqrt{n}}{3n^2+1}}_{a_n} \sim \sum_{n=1}^{\infty} \underbrace{\frac{1}{n^2}}_{b_n} \text{ conv.}$

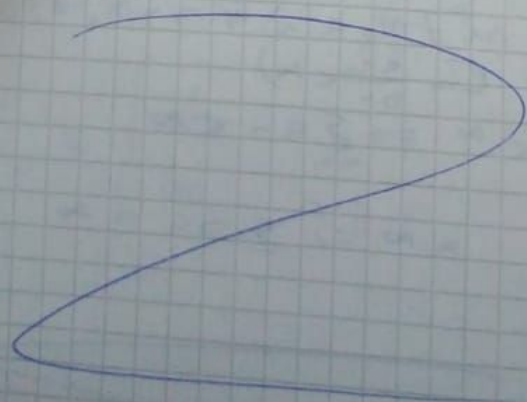
$\frac{a_n}{b_n} \rightarrow 3$

$\sum_{n=1}^{\infty} \underbrace{\sqrt{n+1} - \sqrt{n-1}}_{a_n} \sim \sum_{n=1}^{\infty} \underbrace{\frac{1}{\sqrt{n}}}_{b_n}$

$\frac{a_n}{b_n} \rightarrow 1$

Dem  $\sum_{n=1}^{\infty} a_n = \underbrace{\sum_{k=1}^{n_0-1} a_k}_C + \sum_{k \geq n_0} a_k \leq -C + \sum_{k \geq n_0} M b_k \leq$

$\leq C + M \sum_{k \geq 1} b_k$



## CRITERIUL 7 (RAPORTULUI)

Fie  $\sum_{n=1}^{\infty} a_n$  cu  $a_n > 0$

① Dacă  $\exists n_0$  și  $x < 1$  a.i.  $\frac{a_{n+1}}{a_n} < x$  <sup>pt</sup>  $\forall n \geq n_0$  sau  
( $\Leftrightarrow \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} < 1$ )

$\Rightarrow \sum_{n=1}^{\infty} a_n$  este conv. ( ~~$q_n > 0$~~ )

② Dacă  $\exists n_0$  și  $x > 1$  a.i.  $\frac{a_{n+1}}{a_n} > x$  <sup>pt</sup>  $\forall n \geq n_0$  sau

( $\Leftrightarrow \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} > 1$ )

$\Rightarrow a_n \rightarrow \infty$

$\Rightarrow \sum_{n=1}^{\infty} a_n$  este div.

ex.  $\sum_{n=1}^{\infty} \underbrace{\frac{x^n}{n^2+1}}_{a_n} \quad x > 0 \quad a$

$$\frac{a_{n+1}}{a_n} = \frac{x^{n+1}}{x^n} \cdot \frac{n^2+1}{(n+1)^2+1} = x$$

$x > 1 \Rightarrow$  serie div

$x < 1 \Rightarrow$  serie conv.

$x = 1 \Rightarrow$  crit. nu decide

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2+1} \sim \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ conv.}$$



$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n x|^n} = |x| \lim_{n \rightarrow \infty} \sqrt[n]{n!} = \frac{|x|}{e}$$

$$\frac{|x|}{e} < 1 \Rightarrow \text{conv.}$$

$$\frac{|x|}{e} > 1 \Rightarrow \text{div.}$$

$$\sum_{n=1}^{\infty} \frac{x^n}{n!}$$

$$\lim_{n \rightarrow \infty} \sqrt[n]{n!} = \lim_{n \rightarrow \infty} \left( \frac{(n+1)}{n!} \right)^{-1} = \lim_{n \rightarrow \infty} (n+1)^{-1} = 0 \Rightarrow$$

$$\Rightarrow f = \frac{1}{0} = \infty$$

$$\sum_{n=1}^{\infty} \frac{x^n}{n^\alpha} \quad \lim_{n \rightarrow \infty} \sqrt[n]{\frac{1}{n^\alpha}} = \lim_{n \rightarrow \infty} \frac{n^\alpha}{(n+1)^\alpha} = 1 \quad f = 1$$

$$(-1; 1) \subset D \subset [-1; 1]$$

$$x = 1 \quad \sum_{n=1}^{\infty} \frac{1}{n^\alpha} \quad \text{conv.} \Leftrightarrow \alpha > 1$$

$$\alpha > 1 \quad D = [-1; 1]$$

$$\lim_{n \rightarrow \infty} \frac{1}{n^\alpha} = \begin{cases} 0 & \alpha > 0 \\ 1 & \alpha = 0 \\ \infty & \alpha < 0 \end{cases}$$

$$\alpha \leq 0 \quad \frac{1}{n^\alpha} \rightarrow 0 \quad D = (-1; 1)$$

$$0 < x < 1$$

$$x=1 \quad \sum \frac{1}{n^x} = \infty$$

$$x=-1 \quad \sum (-1)^n \frac{1}{n^x} \quad \text{conv.} \quad \frac{1}{n^x} \searrow 0$$

$$D = [-1, 1)$$

CRITERIUL 9 (Raabe - Duhamel)

$$\sum_{n=1}^{\infty} a_n > 0 \quad \ell = \lim_{n \rightarrow \infty} n \cdot \left( \frac{a_n}{a_{n+1}} - 1 \right)$$

- 1)  $\ell > 1 \Rightarrow \Delta$  conv.
- 2)  $\ell < 1 \Rightarrow \Delta$  div.

$$\frac{a_{n+1}}{a_n} \rightarrow 2$$

$$n \cdot \left( \frac{1}{2} - 1 \right) \rightarrow -\infty$$





## SERII CU TERMENI POZ.

PAS 1 crit prop  
radic

PAS 2 crit comp sau crit. R-D

Serii cu termeni oarecare

$$\sum_{n=1}^{\infty} x_n$$

PAS 1 Studiem absolut conv.

$$\sum_{n=1}^{\infty} |x_n|$$

PAS 2  $\lim_{n \rightarrow \infty} x_n$

Dacă  $\lim_{n \rightarrow \infty} x_n \neq 0 \Rightarrow \text{ser. div.}$

PAS 3  $x_n \rightarrow 0$

~~ex~~

$\sum x_n$  este abs conv.

