CURS 14

Def: Fie (X, τ) un spațiu topologic. O mulțime A se numește neconexă dacă $\exists$ A_1, A_2 a.î.

1) $A_1 \cup A_2 = A$
2) $A_1 \neq \emptyset$ și $A_2 \neq \emptyset$
3) $\overline{A_1} \cap A_2 = A_1 \cap \overline{A_2} = \emptyset$

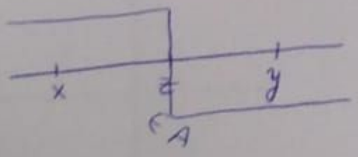
A se numește conexă dacă nu este neconexă

I. O mulțime $A \subset \mathbb{R}$ este conexă $\Leftrightarrow$ este un interval

A nu este interval $\Rightarrow$ A este neconexă

$\Downarrow$

$\exists$ x, y, z a.î. $x < y < z$ $\quad x, z \in A \quad y \notin A$



$A_1 = (-\infty; y) \cap A \qquad A_2 = (y; +\infty) \cap A$

$x \in A_1 \neq \emptyset \quad z \in A_2 \overset{A_2 \neq \emptyset}{\Rightarrow} \quad A_1 \cup A_2 = A \cap (\mathbb{R} \setminus z) = A \cap \mathbb{R} = A$

$\overline{A_1} \cap A_2 \subset (-\infty; y] \cap (y; +\infty) = \emptyset$

Prop
1) A_1 și A_2 conexe și $A_1 \cap A_2 \neq \emptyset \Rightarrow A_1 \cup A_2$ este conexă
2) A este conexă și $A \subset B \subset \overline{A} \Rightarrow B$ este conexă
3) $f: X \to Y$ cont. ; $A \subset X$ este conexă $\Rightarrow f(A)$ este conexă

1') $(A_i)_{i \in I}$ conexe și $\bigcap_{i \in I} A_i \neq \emptyset \Rightarrow \bigcup_{i \in I} A_i$ conexă

1) $\rho : [a;b] \to \mathbb{R}^n$ continuă $\Rightarrow$ $\rho([a;b])$ conexă

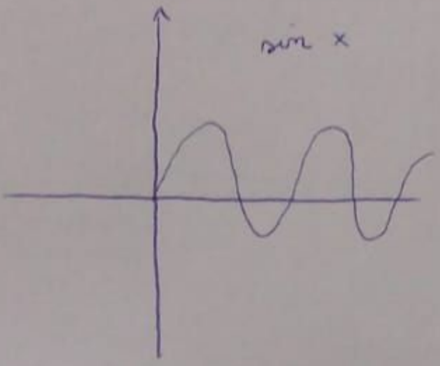
2) $A \subset \mathbb{R}^n$ s.n. stelată dacă $\forall x \in A \Rightarrow [a;x] \subset A$

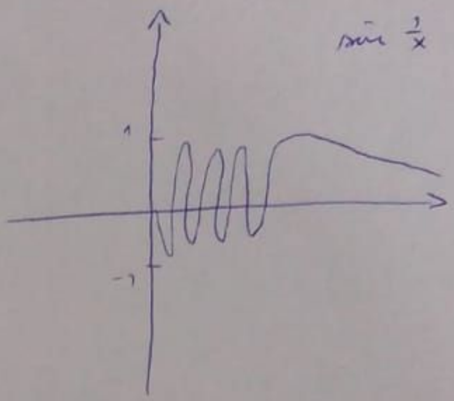
$([a;x])_{x \in A}$ - conexe

3) $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} \sin \frac{1}{x} & , x \neq 0 \\ \alpha & , x = 0 \end{cases}$$

f are prop. Darboux $\Leftrightarrow$ $\alpha \in [-1;1]$ $\Leftrightarrow$ G_f este conexă





$\sin \frac{1}{x}$ cont. pe $(0;+\infty) \cup (-\infty;0)$ (deci are P.D.)

$[-\varepsilon;\varepsilon]$ $\quad f([-\varepsilon;\varepsilon]) = [-1;1] \cup \{\alpha\}$

f are P.D $\Leftrightarrow$ $\alpha \in [-1;1]$

$$\underbrace{\overline{G_f|_{(0;+\infty)}}}_{\text{conexă}} = \underbrace{G_f|_{(0;+\infty)} \cup [-1;1] \times \{0\}}_{\text{conexă}}$$

$G_f|_{(0;\infty)}$ este conex $\Leftrightarrow$ $\alpha \in [-1;1]$

$G_f|_{(0;\infty)} \cap G_f|_{(-\infty;0)} = \{(0; \underbrace{\alpha}_{f(0)})\}$ $\Rightarrow$ G_f conexă

Exercitii recapitulative

$f: \mathbb{R}^2 \longrightarrow \mathbb{R}$

$$f(x,y) = \begin{cases} \dfrac{x^4 y^7}{x^{10}+y^{10}}, & x^2+y^2 \neq 0 \\ 0, & x=y=0 \end{cases}$$

1) cont. lui f. f este cont. pe $\mathbb{R}^2 \setminus \{(0,0)\}$

$$\lim_{\substack{x\to 0\\ y=ax}} \frac{x^4 a^7 x^7}{x^{10}(1+a^{10})} = \lim_{x\to 0} \frac{a^7}{1+a^{10}} = 0.$$

$$\frac{|x^4 y^7|}{x^{10}+y^{10}} \leq \left(\frac{x^{10}}{x^{10}+y^{10}}\right)^{\frac{4}{10}} \left(\frac{y^{10}}{x^{10}+y^{10}}\right)^{\frac{7}{10}}$$

$$(x^{10}+y^{10})^{\frac{4}{10}+\frac{7}{10}-1} \leq (x^{10}+y^{10})^{\frac{1}{10}} \underset{\substack{x\to 0\\ y\to 0}}{\longrightarrow} 0$$

b) $$\frac{\partial f}{\partial x} = \frac{4x^3y^7(x^{10}+y^{10}) - x^4 y^7 10x^9}{(x^{10}+y^{10})^2} = \frac{-6x^{13}y^7 + 4x^3 y^{17}}{(x^{10}+y^{10})^2}$$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{\substack{x\to 0\\ (y=0)}} \frac{\overset{=0}{f(x,0)} - \overset{=0}{f(0,0)}}{x-0} = 0$$

$$\lim_{\substack{x\to 0\\ y=ax}} \frac{\partial f}{\partial x}(x,ax) = \lim_{x\to 0} \frac{-6x^{13}a^7x^7 + 4x^3a^{17}x^{17}}{x^{20}(1+a^{10})^2} = \frac{-6a^7+4a^{17}}{(1+a^{10})^2}$$

depinde de $a \Rightarrow$

$\Rightarrow \frac{\partial f}{\partial x}$ disc. în $(0,0)$

Z

f este der în $(0,0) \iff \exists\, T \in L(\mathbb{R}^2, \mathbb{R})$ a. î.

$$\lim_{\substack{x\to 0\\ y\to 0}} \frac{f(x,y) - f(0,0) - T(x,y)}{|x|+|y|} = 0$$

$T(x,y) = ax + by$ $\qquad a = \frac{\partial f}{\partial x}(0,0) = 0$

$\qquad\qquad\qquad\qquad b = \frac{\partial f}{\partial y}(0,0) = 0$

$$\lim_{\substack{x\to 0\\ y=ax\\ x>0}} \frac{x^4 \cdot y^7}{(x^{10}+y^{10})\sqrt{x^2+y^2}} = \lim_{x\to 0} \frac{x^4 \cdot a^7 \cdot \cancel{x^7}}{x^{10}(1+a^{10})\cdot \cancel{x}\sqrt{1+a^2}} = \frac{a^7}{(1+a^{10})\sqrt{1+a^2}} \neq 0$$

$(a \neq 0)$

$\Rightarrow \nexists\, f($

-

$$\sum_{n\geq 1} x^n \cdot \frac{a(a+1)\dots(a+n)}{(n+10)!} \qquad x>0,\ a>0$$

$$\frac{a_{n+1}}{a_n} = x^{n+1} \cdot \frac{a(a+1)\dots(a+n)(a+n+1)}{(n+11)!} \cdot \frac{(n+10)!}{x^n \cdot a\cdot(a+1)\dots(a+n)} =$$

$$= \frac{a+n+1}{n+11} \longrightarrow x$$

$x>1 \Rightarrow$ div $(a_n \to \infty)$

$x<1 \Rightarrow$ conv

$x = 1$

Z

$\ldots m \quad = a_n$

$\boxed{x=1}$

$$\sum_{n=1}^{\infty} \frac{a(a+1)(\ldots)(a+n)}{(n+10)!}$$

R.D. $\lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}}-1\right) = \lim_{n\to\infty}\left(\frac{n+11}{a+n+1}-1\right)\cdot n =$

$= \lim_{n\to\infty} \frac{10-a}{a+n+1}\cdot n \longrightarrow 10-a$

$10-a>1 \Rightarrow$ conv. $\qquad 9>a$ conv.

$10-a<1 \Rightarrow$ div. $\qquad 9<a$ div

$$a=9 \Rightarrow \sum_{n\geq 1} \frac{9\cdot 10\cdot\ldots\cdot(9+n)}{(n+10)!} = \sum_{n\geq 1} \frac{1}{8!\cdot(n+10)} \underset{\sim}{} \sum \frac{1}{n} \text{ div.}$$

$$\sum_{n\geq 1} \frac{x^n}{\sqrt[n]{n!}}$$

$\lim_{n\to\infty} \sqrt[n]{x_n} = \lim_{n\to\infty} \frac{x_{n+1}}{x_n}$

$\lim_{n\to\infty} \sqrt[n]{n!} = \lim_{n\to\infty} \frac{(n+1)!}{n!} = \infty$

$\lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n} = \lim_{n\to\infty} \sqrt[n]{\frac{n!}{n}} = \lim_{n\to\infty} \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} =$

$= \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e}$

$$\sum_{n\geq 1} \frac{|x|^n}{\sqrt[n]{n!}} = a_n \qquad\qquad \sum_{n\geq 1} \frac{|x|^n}{n}$$

$x=0 \Rightarrow$ abs conv

$x \neq 0 \quad \frac{a_{n+1}}{a_n} = \frac{|x|^{n+1}}{\sqrt[n+1]{(n+1)!}} \cdot \frac{\sqrt[n]{n!}}{|x|^n} = |x| \cdot \underbrace{\frac{n+1}{\sqrt[n+1]{(n+1)!}}}_{\downarrow\ \frac{1}{e}} \cdot \underbrace{\frac{n}{n+1}}_{\downarrow\ 1} = |x|$

$|x|>1 \quad (a_n \to \infty) \Rightarrow$ div

$|x|<1 \quad \Rightarrow$ abs. conv

$|x|=1 \quad \sum_{n\geq 1} \frac{1}{\sqrt[n]{n!}} \sim \sum_{n\geq 1} \frac{1}{n} \quad (x=1 \text{ div.})$

$$\lim_{n\to\infty} \frac{\frac{1}{\sqrt[n]{n}}}{\frac{1}{n}} = (-1,\infty)\ ?$$

$x=-1 \quad \sum_{n\geq 1} (-1)^n \frac{1}{\sqrt[n]{n!}} = c_n$

$\frac{1}{\sqrt[n]{n!}} \to 0$

nu este abs. conv

$$\frac{c_{n+1}}{c_n} = \frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}} = \sqrt[n(n+1)]{\frac{(n!)^{n+1}}{(n+1)!^{\,n}}}$$

$$= \sqrt[n(n+1)]{\frac{(n!)^n \cdot n!}{(n!)^n \cdot (n+1)^n}} \leq 1$$

$a_n \searrow 0$ semi conv ([redacted] = conv, dar nu abs conv)

Nu o să avem ceva mai complicat de atât

7)

$A = \left\{ \frac{1}{n} \mid n \geq 1 \right\} \cup (3,4] \cup \left((7,8) \cap \mathbb{Q} \right)$

$\mathring{A} = \bigcup D = (3,4)$

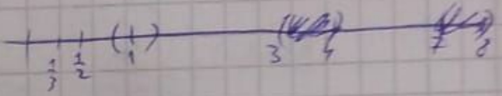
$D = \mathring{D}$
$D \subset A$

$A' = \{ a \in \mathbb{R} \mid \exists (x_n)_n \subset A \text{ a.î. } x_n \to a \text{ și } x_n \neq a \} =$

$\bar{A} = A \cup A' = [3,4] \cup [7,8] \cup \{0\} \cup \left\{ \frac{1}{n} \mid n \geq 1 \right\}$

$Fr(A) = \bar{A} \setminus \mathring{A} = [7,8] \cup \{0,3,4\} \cup \left\{ \frac{1}{n} \mid n \geq 1 \right\}$

$Iz(A) = A \setminus A' = \left\{ \frac{1}{n} \mid n \geq 1 \right\}$



$= [3,4] \cup [7,8] \cup \{0\}$

$(3,4) \subset A \Rightarrow (3,4) \subset \mathring{A}$

deschisă

$\mathring{A} \subset A \qquad \forall a \in A \setminus (3,4) \Rightarrow a \notin \mathring{A}$

$A \setminus (3,4) = \left\{ \frac{1}{n} \mid n \geq 1 \right\} \cup \left((7,8) \cap \mathbb{Q} \right)$

$a \in \mathring{A} \Leftrightarrow x_n \subset \mathbb{R} \setminus A$ a.î. $x_n \to a$

$x_n = a + \frac{\sqrt{2}}{n} \notin \mathbb{Q} \qquad x_n \to a$

$0 \in A' \qquad \frac{1}{n} \in A \qquad \frac{1}{n} \to 0$

$3 \in A' \qquad x_n = 3 + \frac{1}{n+1} \searrow 3$

$\in (3,4) \subset A$

$a \in (3,4) \qquad x_n = \underset{\downarrow a}{a} + \frac{1}{n+1} \cdot (4-a) > 3$

$x_n \in A$

$a \in [7;8] \Rightarrow \exists\, x_n \subset (7;8) \cap \mathbb{Q}$ a i $x_n \to a$ $x_n \neq a \Rightarrow$

$\Rightarrow [7;8] \subset A'$

$(A \cup B)' = A' \cup B'$

$A' = (\{\frac{1}{n}\})' \cup ((3;4])' \cup ((7;8) \cap \mathbb{Q})'$

(1) $a \in ((3;4])' \Rightarrow (x_n)_n \subset (3;4]$ a i $x_n \to a$ a i $x_n \to a$

$3 \leq a \leq 4$

(2) $a \in ((7;8) \cap \mathbb{Q})' \Rightarrow (x_n)_n \subset (7;8)$ a i $x_n \to a$

$\Rightarrow a \in [7;8]$

(3) $\{x_n\} \subset \{\frac{1}{n}\}$ $x_n \to a$ $\begin{cases} a = 0 \\ x_{n_0} = \frac{1}{n} \text{ si } x_n = \frac{1}{n} \; n \geq n \;??? \end{cases}$