Seminar 9

22.04.2020.

Spații vectoriale euclidiene
Produs scalar. Produs vectorial. Produs mixt

Ex1 $(\mathbb{R}^2,+,\cdot)/\mathbb{R}$, $g:\mathbb{R}^2\times\mathbb{R}^2\to\mathbb{R}$, $g(x,y)=ax_1y_1+bx_1y_2+bx_2y_1+cx_2y_2$.

a) g formă biliniară, simetrică

b) g produs scalar $\Leftrightarrow \begin{cases} a>0 \\ ac-b^2>0. \end{cases}$

Sol

a) $G=\begin{pmatrix} a & b \\ b & c \end{pmatrix}$ matricea asociată lui g în raport cu reperul canonic.

$g(x,y)=X^TGY$, $X=\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $Y=\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$; $G=G^T$

$\Rightarrow g$ formă biliniară, simetrică

b) $Q:\mathbb{R}^2\to\mathbb{R}$, $Q(x)=g(x,x)=ax_1^2+2bx_1x_2+cx_2^2$

• Criteriul Jacobi $\quad Q$ poz def $\Leftrightarrow \begin{cases} \Delta_1=a>0 \\ \Delta_2=\det G=ac-b^2>0 \end{cases}$

$\exists$ un reper ai $Q(x)=\frac{1}{a}x_1'^2+\frac{a}{ac-b^2}x_2'^2$.

• Metoda Gauss $\quad Q$ poz def $\Leftrightarrow (2,0)$ signatura

$Q(x)=\frac{1}{a}(a^2x_1^2+2abx_1x_2)+cx_2^2=\frac{1}{a}(ax_1+bx_2)^2+x_2^2\left(c-\frac{b^2}{a}\right)$

Fie schimbarea de reper:

$\begin{cases} x_1'=ax_1+bx_2 \\ x_2'=x_2 \end{cases} \Rightarrow Q(x)=\frac{1}{a}x_1'^2+\frac{ac-b^2}{a}x_2'^2$

Signatura este $(2,0) \Leftrightarrow \begin{cases} \frac{1}{a}>0 \\ \frac{ac-b^2}{a}>0 \end{cases} \Leftrightarrow \begin{cases} a>0 \\ ac-b^2>0 \end{cases}$

Ex $(\mathbb{R}^3, +, \cdot)/\mathbb{R}$, $g: \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ formă biliniară și
$G = \begin{pmatrix} 3 & 2 & 0 \\ 2 & 2 & 2 \\ 0 & 2 & 1 \end{pmatrix}$ matricea asociată în raport cu reperul canonic. Este $(\mathbb{R}^3, g)$ spațiu vectorial euclidian real?

SOL
$G = G^T \Rightarrow g$ formă biliniară, simetrică.
Este g poz. definită?
$\Delta_1 = 3 > 0$; $\Delta_2 = \begin{vmatrix} 3 & 2 \\ 2 & 2 \end{vmatrix} = 6 - 4 > 0$
$\Delta_3 = \det G = 3(2-4) - 2(2-0) = -6-4 = -10 < 0$
Criteriul Jacobi $\Rightarrow Q$ nu e poz. def.
$\Rightarrow g$ nu este produs scalar.

Ex $(\mathbb{R}^3, g_0)$, $g_0: \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$, $g_0(x,y) = x_1 y_1 + x_2 y_2 + x_3 y_3$
g_0 = produs scalar canonic.
Fie $U = \{x \in \mathbb{R}^3 \mid x_1 + x_2 - x_3 = 0\}$.
a) $U^\perp = ?$
b) Să se afle un reper ortonormat $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$ în $\mathbb{R}^3$, unde $\mathcal{R}_1$, respectiv $\mathcal{R}_2$ — // — în U, respectiv $U^\perp$.

Sol.
a) $U = \{x \in \mathbb{R}^3 \mid g_0((x_1, x_2, x_3), (1, 1, -1)) = 0\}$
$\Rightarrow U^\perp = \langle \{(1,1,-1)\} \rangle$.
$\mathbb{R}^3 = U \oplus U^\perp$, $\dim U = 3 - 1 = 2$, $\dim U^\perp = 1$.

b) $U = \{(x_1, x_2, x_1 + x_2) \mid x_1, x_2 \in \mathbb{R}\}$
$$(x_1, x_2, x_1+x_2) = x_1 \underbrace{(1,0,1)}_{f_1} + x_2 \underbrace{(0,1,1)}_{f_2}$$
$\left.\begin{array}{l}\{f_1, f_2\} \text{ este SG pt } U \\ \dim U = 2\end{array}\right\} \Rightarrow \{f_1, f_2\}$ reper V în U

Aplicăm procedeul Gram-Schmidt.
$e_1 = f_1 = (1, 0, 1)$
$e_2 = f_2 - \frac{\langle f_2, e_1 \rangle}{\langle e_1, e_1 \rangle} e_1 = (0,1,1) - \frac{1}{2}(1,0,1) = \left(-\frac{1}{2}, 1, \frac{1}{2}\right) =$

$$\begin{cases} e_1 = (1,0,1) \\ e_2 = \frac{1}{2}(-1,2,1) \end{cases} \Rightarrow \{e_1, e_2\} \text{ reper ortogonal în } U.$$

$R_1 = \left\{\frac{1}{\sqrt{2}}(1,0,1), \frac{1}{\sqrt{6}}(-1,2,1)\right\}$ reper ortonormat în U

$e_3 = (1,1,-1)$

$R_2 = \left\{\frac{1}{\sqrt{3}}(1,1,-1)\right\}$ reper ortonormat în $U^\perp$

$R = R_1 \cup R_2$ reper ortonormat în $\mathbb{R}^3$.

Ex. $(\mathbb{C}, +, \cdot)_{\mathbb{R}}$, $g: \mathbb{C} \times \mathbb{C} \to \mathbb{R}$ formă biliniară
$(\mathbb{C} \simeq \mathbb{R}^2)$ $G = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}$ matricea asociată în raport cu reperul canonic $R = \{\underset{(1,0)}{1}, \underset{(0,1)}{i}\}$

a) $(\mathbb{C}, g)$ este spațiu vectorial euclidian real.

b) $u = 2 - i$ versor în raport cu g

c) $\langle\{u\}\rangle^\perp$

d) Să se ortonormeze R în raport cu g.

e) Intersecția dintre cercul unitar în $(\mathbb{C}, g_0)$ și în $(\mathbb{C}, g)$?

SOL

a) $g: \mathbb{C} \times \mathbb{C} \to \mathbb{R}$, $g(z, z') = x_1 y_1 + 2x_1 y_2 + 2x_2 y_1 + 5x_2 y_2$
$z = x_1 + i x_2$; $z' = y_1 + i y_2$.

$g(z, z') = X^T G Y$, $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, $G = G^T \Rightarrow$

g este formă biliniară, simetrică

g $G = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}$ $\quad \begin{matrix} \Delta_1 = 1 > 0 \\ \Delta_2 = 5 - 4 > 0 \end{matrix} \Rightarrow Q: \mathbb{C} \to \mathbb{R}$

$Q(z) = x_1^2 + 4x_1 x_2 + 5x_2^2$ e poz. def.

$\exists$ un reper a.î. $Q(z) = x_1'^2 + x_2'^2$.

SAU $Q(z) = (x_1 + 2x_2)^2 + x_2^2$

$$\begin{cases} x_1' = x_1 + 2x_2 \\ x_2' = x_2 \end{cases} \Rightarrow Q(z) = x_1'^2 + x_2'^2$$

g este produs scalar.

b) $g(u,u) = Q(u) = (2-2)^2 + (-1)^2 = 1 \Rightarrow u$ versor.

$Q(z) = (x_1 + 2x_2)^2 + x_2^2$

$u = 2 - i$, $x_1 = 2, x_2 = -1$

c) $<\{u\}>^{\perp} = \{z \in \mathbb{C} \mid g(z,u) = 0\} = \{z = x_1 + i x_2 \mid x_2 = 0\} = \mathbb{R}$

$g(z,z') = x_1 y_1 + 2x_1 y_2 + 2x_2 y_1 + 5x_2 y_2$.

$z = x_1 + i x_2$, $u = 2 - i$

$y_1 = 2, y_2 = -1$.

$g(z,u) = 2x_1 + 2(-1)x_1 + 2(2)x_2 + 5x_2(-1) =$

$= 2x_1 - 2x_1 + 4x_2 - 5x_2 = -x_2$.

d) $\mathcal{R} = \{\underset{\substack{\parallel \\ (1,0)}}{\overset{= f_1}{1}}, \underset{\substack{\parallel \\ (0,1)}}{\overset{= f_2}{i}}\}$

Aplicăm Gram-Schmidt

$e_1 = f_1 = 1$.

$e_2 = f_2 - \dfrac{g(f_2,e_1)}{g(e_1,e_1)} \cdot e_1 = i - \dfrac{2}{1} \cdot 1 = -2 + i$

$g(f_2,e_1) = g(f_1,f_2) = g(1,i) = 0 + 2 + 0 + 0 = 2$

$x_1 = 1, x_2 = 0$, $y_1 = 0, y_2 = 1$.

$g(e_1,e_1) = Q(1) = (1+0)^2 + 0^2 = 1$.

$x_1 = 1, x_2 = 0$.

$\{e_1, e_2\}$ reper ortogonal în $\mathbb{C}$.

$g(e_1,e_1) = 1 \Rightarrow e_1 =$ versor.

$g(e_2,e_2) = g(-2+i, -2+i) = g(2-i, 2-i) = 1 \Rightarrow e_2 =$ versor

$\{1, -2+i\}$ reper ortonormat în $\mathbb{C}$

e) $(\mathbb{C}, g_0)$ $\quad S^1_{g_0} = \{z \in \mathbb{C} \mid \underset{\substack{\parallel \\ Q_0(z)}}{\|z\|} = 1\} = \{z \in \mathbb{C} \mid x_1^2 + x_2^2 = 1\}$

$x_1 = \cos t, x_2 = \sin t$

$(\mathbb{C}, g)$ $\quad S^1_g = \{z \in \mathbb{C} \mid \underset{\substack{\parallel \\ (x_1+2x_2)^2 + x_2^2}}{Q(z)} = 1\}$.

$S^1_{g_0} \cap S^1_g$: $\cos^2 t + 4 \sin t \cos t + 5 \sin^2 t = 1$

$4 \sin t \cos t + 4 \sin^2 t = 0$.

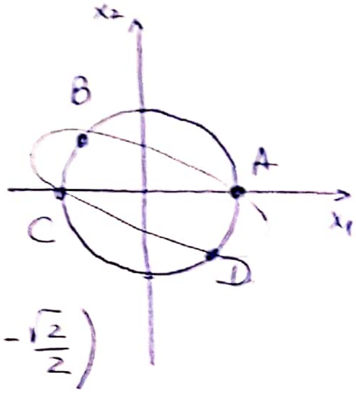
$4\sin t(\cos t+\sin t)=0$

a) $\sin t=0 \Rightarrow t=k\pi,\ k\in\mathbb{Z} \Rightarrow t\in\{0,\pi\}$

b) $\cos t+\sin t=0 \Rightarrow tg\, t=-1 \Rightarrow t=-\frac{\pi}{4}+k\pi,\ k\in\mathbb{Z}$,

$t\in[0,2\pi)$ $\Rightarrow t\in\left\{\frac{3\pi}{4},\frac{5\pi}{4}\right\}$

$A(1,0),\ B\left(-\frac{\sqrt2}{2},\frac{\sqrt2}{2}\right),\ C(-1,0),\ D\left(\frac{\sqrt2}{2},-\frac{\sqrt2}{2}\right)$



Ex $(E,\langle\cdot,\cdot\rangle)$ s.v.e.r., $u,v\in E$. UAE

1) $u\perp v$

2) $\|u+v\|=\|u-v\|$

3) $\|u+v\|^2=\|u\|^2+\|v\|^2$.

SOL $1\Leftrightarrow 2$

$u\perp v \Leftrightarrow \langle u,v\rangle=0$

(*) $\|u+v\|^2=\langle u+v,u+v\rangle=\|u\|^2+\|v\|^2+2\langle u,v\rangle$

$\|u-v\|^2=\langle u-v,u-v\rangle=\|u\|^2+\|v\|^2-2\langle u,v\rangle$

$\|u+v\|=\|u-v\| \Leftrightarrow \langle u,v\rangle=0$

$1\Leftrightarrow 3$.

(*): $\|u+v\|^2=\|u\|^2+\|v\|^2 \Leftrightarrow \langle u,v\rangle=0$.

Ex $(\mathbb{R}^3,g_0)$, $R=\{f_1=(1,2,3), f_2=(0,1,1), f_3=(1,2,5)\}$

a) R este reper. Să se ortonormeze, utilizând Gram-Schmidt.

b) $f_1\times f_2$

c) $f_1\wedge f_2\wedge f_3$. Care este volumul paralelipipedului construit pe f_1,f_2,f_3?

SOL

a) $\det\begin{pmatrix}1&0&1\\2&1&2\\3&1&5\end{pmatrix}=\begin{vmatrix}1&0&0\\2&1&0\\3&1&2\end{vmatrix}=2\neq0 \Rightarrow R$ este SLI, dar $\dim\mathbb{R}^3=3$ $\Rightarrow$ R reper.

$c_3'=c_3-c_1$

Aplicăm Gram-Schmidt

$e_1=f_1=(1,2,3)$

$e_2=f_2-\frac{\langle f_2,e_1\rangle}{\langle e_1,e_1\rangle}e_1=(0,1,1)-\frac{5}{14}(1,2,3)=\left(-\frac{5}{14},\frac{4}{14},\frac{-1}{14}\right)$

$e_2 = \frac{1}{14}(-5,4,-1)$

$e_3 = f_3 - \frac{\langle f_3,e_1\rangle}{\langle e_1,e_1\rangle}\cdot e_1 - \frac{\langle f_3,e_2\rangle}{\langle e_2,e_2\rangle}\cdot e_2.$

• $\langle f_3,e_1\rangle = \langle f_3,f_1\rangle = 20$; $\langle e_1,e_1\rangle = 14$

$f_1=(1,2,3),\ f_3=(1,2,5)$

• $\langle f_3,e_2\rangle = \frac{1}{14}(-5+8-5) = -\frac{2}{14}$

$\langle e_2,e_2\rangle = \frac{1}{14^2}(25+16+1) = \frac{42}{14^2}$

$$e_3 = (1,2,5) - \frac{20}{14}\cdot(1,2,3) - \frac{-\frac{2}{14}}{\frac{42}{14^2}}\cdot\frac{1}{14}(-5,4,-1)$$

$$= (1,2,5) - \frac{10}{7}(1,2,3) + \frac{1}{21}(-5,4,-1)$$

$$= \frac{1}{21}(21-30-5,\ 42-60+4,\ 105-90-1)$$

$$= \frac{1}{21}(-14,-14,14) = \frac{14}{21}(-1,-1,1) = \frac{2}{3}(-1,-1,1)$$

$\left\{e_1=(1,2,3),\ e_2=\frac{1}{14}(-5,4,-1),\ e_3=\frac{2}{3}(-1,-1,1)\right\}$ reper ortogonal în $\mathbb{R}^3$

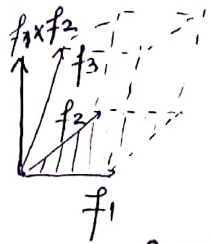
$\left\{\frac{1}{\sqrt{14}}(1,2,3),\ \frac{1}{\sqrt{42}}(-5,4,-1),\ \frac{1}{\sqrt{3}}(-1,-1,1)\right\}$ reper ortonormat în $\mathbb{R}^3$.

b) $f_1\times f_2 = \begin{vmatrix} i & j & k \\ 1 & 2 & 3 \\ 0 & 1 & 1 \end{vmatrix} = i\begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} - j\begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix} + k\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix}$

$= (-1,-1,1)$

<u>OBS</u> $Sp\{e_1,e_2\} = Sp\{f_1,f_2\}$

$e_3 = \alpha\cdot f_1\times f_2$



c) $V_{paralelipiped} = |f_1\wedge f_2\wedge f_3|$

$= |f_3\wedge f_1\wedge f_2| = |\langle f_3, f_1\times f_2\rangle|$

$= |-1-2+5| = 2$

$f_1\times f_2 = (-1,-1,1)$

$f_3 = (1,2,5)$

OBS $(\mathbb{R}^3, g_0)$, $\{e_1, e_2, e_3\}$ reperul canonic.

$$\begin{cases} e_1 \times e_2 = e_3 \\ e_2 \times e_3 = e_1 \\ e_3 \times e_1 = e_2 \end{cases} ; \quad e_1 \times e_2 = \begin{vmatrix} e_1 & e_2 & e_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = e_3.$$

Ex $(\mathbb{R}^3, g_0)$, $U = \{x \in \mathbb{R}^3 \mid x_1 - 2x_2 + 2x_3 = 0\}$,

$u = \frac{1}{3}(1,-2,2)$, $v = \frac{1}{\sqrt{2}}(0,1,1)$

a) $u \perp v$, $\|u\| = \|v\| = 1$.

b) $w = ?$ a.î. $\{u, v, w\}$ reper ortonormat, pozitiv orientat

c) $U^\perp$

d) Precizați un reper ortonormat în U.

SOL

a) $\langle u, v \rangle = \frac{1}{3\sqrt{2}} \langle (1,-2,2), (0,1,1) \rangle = 0$

$\|u\| = \frac{1}{3}\sqrt{1+4+4} = \frac{3}{3} = 1$

$\|v\| = \frac{1}{\sqrt{2}}\sqrt{1+1} = \frac{\sqrt{2}}{2} = 1$.

b) $u \times v = \frac{1}{3\sqrt{2}} \begin{vmatrix} i & j & k \\ 1 & -2 & 2 \\ 0 & 1 & 1 \end{vmatrix} = \frac{1}{3\sqrt{2}}(-4, -1, 1)$

$\|u \times v\| = \frac{1}{3\sqrt{2}}\sqrt{16+1+1} = \frac{3\sqrt{2}}{3\sqrt{2}} = 1$.

$w = u \times v = \frac{1}{3\sqrt{2}}(-4,-1,1)$

$\{u, v, w\}$ reper ortonormat, pozitiv orientat

c) $U^\perp = \{x \in \mathbb{R}^3 \mid g_0((x_1,x_2,x_3),(1,-2,2)) = 0\} = \langle\{(1,-2,2)\}\rangle$

OBS $U = \{x \in \mathbb{R}^3 \mid x_1 - 2x_2 + 2x_3 = 0\}$.

$= \{(2x_2 - 2x_3, x_2, x_3) \mid x_2, x_3 \in \mathbb{R}\}$

$\quad \overset{\|}{} x_2(2,1,0) + x_3(-2,0,1)$

$\qquad\qquad \underset{f_1}{\|} \qquad \underset{f_2}{\|}$

$\{f_1, f_2\}$ reper în U

$x \in U^{\perp} \Rightarrow \begin{cases} g_0(x, f_1) = 0 \\ g_0(x, f_2) = 0 \end{cases} \Rightarrow \begin{cases} 2x_1 + x_2 = 0 \\ -2x_1 + x_3 = 0 \end{cases}$

$f_1 = (2,1,0)$
$f_2 = (-2,0,1)$

$x_2 = -2x_1$
$x_3 = 2x_1$.

$U^{\perp} = \{ (x_1, -2x_1, 2x_1) \mid x_1 \in \mathbb{R}$

$\quad = x_1 (1,-2,2)$.

d) $\mathcal{R} = \{f_1, f_2\}$ reper $\forall$ în U

$e_1 = f_1 = (2,1,0)$

$e_2 = f_2 - \frac{\langle f_2, e_1 \rangle}{\langle e_1, e_1 \rangle} \cdot e_1 = (-2,0,1) - \frac{-4}{5}(2,1,0) =$

$= (-2,0,1) + \left(\frac{8}{5}, \frac{4}{5}, 0\right) = \left(-2 + \frac{8}{5}, \frac{4}{5}, 0+1\right) =$

$= \left(-\frac{2}{5}, \frac{4}{5}, 1\right) = \frac{1}{5}(-2,4,5)$

$\mathcal{R}' = \{e_1, e_2\}$ reper ortogonal în U

$\mathcal{R}'' = \left\{ \frac{1}{\sqrt{5}}(2,1,0), \frac{1}{3\sqrt{5}}(-2,4,5) \right\}$ reper ortonormat în U

$\mathcal{R} \longrightarrow \mathcal{R}' \longrightarrow \mathcal{R}''$

$\mathcal{R}, \mathcal{R}', \mathcal{R}''$ sunt la fel orientate.