CURS 4

25.10.2018

Limita superioară și limita inferioară

Def: Un $a \in (X,d)$ este punct limită pentru un șir $(X_n)_n \subset X$ dacă $\exists\ X_{n_k} \longrightarrow a$.

ex: $X_n = \sin n \cdot \frac{\pi}{2} + \frac{1}{2n+1}$

$X_{2n} = \sin n\pi + \frac{1}{4n+1} \longrightarrow 0$

$X_{4n+1} = \sin\left(2n\pi + \frac{\pi}{2}\right) + \frac{1}{8n+3} \longrightarrow 1$

$X_{4n-1} \longrightarrow -1$

$0, 1, -1$, puncte limită

$\lim\limits_{n\to\infty} \sup X_n = 1 \qquad \underline{\lim\limits_{n\to\infty}}\ X_n = -1$

$\overline{\lim}\ x_n = \underline{\lim}\ x_n$

Propoziție Fie $(x_n)_n$ ...

$\exists\ X_{n_k} \mid k$ ...

Dem: $n_k < n$ ...

X_{n_k} ...

Pas $k=1$ $\quad \bar{\ell} = \lim\limits_{n\to}$ ...

$\ell \leq u_n$ ...

$u_{n_1} = $ ...

a.î. ...

$\Rightarrow$ ...

$X_{n_1} \in ($ ...

$0 \quad 1 \quad 0 \quad \frac{1}{2} \quad 1 \quad 0 \quad \frac{1}{3} \quad \frac{2}{3} \quad 1$

$[0,1]$ - puncte limită $\quad \overline{\lim_{n\to\infty}} x_n = 1 \quad \underline{\lim_{n\to\infty}} x_n = 0$

Def. Fie $(x_n)_n \subset \mathbb{R}$. Considerăm $u_n = \sup_{k\geq n} x_k$ și $v_n = \inf_{k\geq n} x_k$,

$$u_n \geq u_{n+1} \geq v_{n+1} \geq v_n \Rightarrow u_n \searrow \bar{\ell} \geq \underline{\ell} \nearrow v_n$$

S.n. lim sup. a lui $(x_n)_n$ $\lim_{n\to\infty} u_n = \inf_{n\geq 1} u_n$ și se

not. $\limsup_{n\to\infty} x_n = \overline{\lim_{n\to\infty}} x_n$

$\overline{\lim} x_n = \underline{\lim} x_n$

Propoziție Fie $(x_n)_n$ un șir mărginit de nr. reale. Atunci

$\exists$ x_{n_k}/k (subșir) convergent a.î. $x_{n_k} \to \overline{\lim} x_n = \bar{\ell}$

Dem $n_k < n_{k+1}$

x_{n_k} a.î. $|\bar{\ell} - x_{n_k}| < \frac{1}{k}$

Pas $k=1$ $\bar{\ell} = \lim_{n\to\infty} u_n \Rightarrow \exists m_1$ a.î. $\forall n \geq m_1 \Rightarrow$

$$\bar{\ell} \leq u_n < \bar{\ell} + 1$$

$u_{m_1} = \sup_{k\geq m_1} x_k \Rightarrow \exists n_1 \geq m_1$ a.î.

a.î. $u_{m_1} - 1 < x_{n_1} \leq u_{m_1}$ $\Rightarrow$

$\Rightarrow$ $\bar{\ell} - 1 \leq u_{m_1} - 1 < x_{n_1} \leq u_{m_1} < \bar{\ell} + 1$

$x_{n_1} \in (\bar{\ell} - 1; \bar{\ell} + 1)$ $\Longrightarrow |x_{n_1} - \bar{\ell}| < 1$

Pas k=2 $u_n \searrow \bar{\ell} \Rightarrow \exists\, m_2 > n_1$ a.î. $\forall n \geq m_2 \Rightarrow$

$\Rightarrow \bar{\ell} \leq u_n < \bar{\ell} + \frac{1}{2}$

$u_{m_2} = \sup_{k \geq m_2} x_k \Rightarrow \exists\, n_2 \geq m_2 > n_1$ a.î.

a.î. $u_{m_2} - \frac{1}{2} < x_{n_2} \leq u_{m_2}$

$\bar{\ell} - \frac{1}{2} < u_{m_2} - \frac{1}{2} < x_{n_2} \leq u_{m_2} < \bar{\ell} + \frac{1}{2} \Rightarrow |x_{n_2} - \bar{\ell}| < \frac{1}{2}$

Presupunem că am găsit x_{n_ℓ} a.î. $|x_{n_\ell} - \bar{\ell}| < \frac{1}{\ell}$

$\forall \ell = \overline{1,k}$

$u_n \searrow \bar{\ell} \Rightarrow \exists\, m_{k+1} > m_k$ a.î. $\forall n \geq m_k \Rightarrow$

$\Rightarrow \bar{\ell} \leq u_n < \bar{\ell} + \frac{1}{k+1}$

$u_{m_{k+1}} = \sup_{\ell \geq m_{k+1}} x_\ell \Rightarrow \exists\, n_{k+1} \geq m_k$ a.î.

a.î. $u_{m_{k+1}} - \frac{1}{k+1} < x_{n_{k+1}} \leq u_{m_{k+1}}$

$\Rightarrow \bar{\ell} - \frac{1}{k+1} < u_{m_{k+1}} - \frac{1}{k+1} < x_{n_{k+1}} \leq u_{m_{k+1}} < \bar{\ell} + \frac{1}{k+1}$

$\Rightarrow |\ell - x_{n_{k+1}}| < \frac{1}{k+1}$

Teoremă În R orice şir mărg are un subşir convergent

Teoremă Sp metric (R, d) este complet (orice şir Cauchy este convergent)

Teoremă În (R^n, d_2) orice şir mărginit are un subşir convergent

Dem: $n=2$ $z_n = (x_n, y_n) \in R^2$

$(z_n)_n$ este mărginit $\Rightarrow \exists M>0$ a.î. $d(z_n, 0) < M \Leftrightarrow$

$\Leftrightarrow \sqrt{x_n^2 + y_n^2} < M \; \forall n \in N$

$\Rightarrow |x_n| \le M$

$(x_n)_n$ este mărginit $\Rightarrow \exists$ un subşir (x_{n_k}) $x_{n_k} \to x$

~~(z_n)~~ $(y_n)_n$ este mărg $\Rightarrow (y_{n_k})$ este mărg. $\Rightarrow \exists (y_{n_{k_l}})_{l \ge 1}$

$y_{n_{k_l}} \to y$

$z_{n_{k_l}} = (x_{n_{k_l}}, y_{n_{k_l}}) \to (x, y)$

Teoremă (R^n, d_2) este un sp metric complet.

$$x_n \to a \quad \overset{a\in\mathbb{R}}{} \quad \forall \varepsilon>0 \ \exists n_\varepsilon \text{ a.î. } \forall n \geq n_\varepsilon \Rightarrow$$

$$a-\varepsilon < \varepsilon < a+\varepsilon$$

<u>Prop</u>: Fie $(x_n)_n$ un șir mărg. de nr. reale. Atunci

$\overline{\lim}\, x_n = a \Leftrightarrow$

1) $\forall \varepsilon>0 \ \exists n_\varepsilon$ a.î. $\forall n \geq n_\varepsilon \Rightarrow x_n < a+\varepsilon$

2) $\exists x_{n_k} \to a$

<u>Prop</u>: ~~Fie~~ Un șir mărg. este conv. $\Leftrightarrow \overline{\lim}\, x_n = \underline{\lim}\, x_n$

<u>Prop</u>: 1) Dacă $x_n \leq y_n \Rightarrow \overline{\lim}\, x_n \leq \overline{\lim}\, y_n$
și
$\underline{\lim}\, x_n \leq \underline{\lim}\, y_n$

2) $\overline{\lim}\,(-x_n) = -\underline{\lim}\, x_n$

3) $\overline{\lim}(x_n+y_n) \leq \overline{\lim}\, x_n + \overline{\lim}\, y_n$

4) $\overline{\lim}(x_n+y_n) \geq \underline{\lim}\, x_n + \overline{\lim}\, y_n$

5) Dacă $\exists \lim x_n \Rightarrow \overline{\lim}\,(x_n+y_n) = \lim x_n + \overline{\lim}\, y_n$

6) $\underline{\lim}\,(x_n+y_n) \geq \underline{\lim}\, x_n + \underline{\lim}\, y_n$

7) $\underline{\lim}\,(x_n+y_n) \leq \overline{\lim}\, x_n + \underline{\lim}\, y_n$

8) $x_n > 0 \quad y_n > 0$

$$\overline{\lim}\, \frac{1}{x_n} = \frac{1}{\underline{\lim}\, x_n}$$

9) $\overline{\lim}(x_n \cdot y_n) \leq (\overline{\lim}\, x_n) \cdot (\overline{\lim}\, y_n)$

<u>Prop.</u> Fie $(x_n)_n$ a.î. $x_n > 0$

$$\underline{\lim} \frac{x_{n+1}}{x_n} \leq \underline{\lim} \sqrt[n]{x_n} \leq \overline{\lim} \sqrt[n]{x_n} \leq \overline{\lim} \frac{x_{n+1}}{x_n},$$

În particular dacă $\exists \lim_{n \to \infty} \frac{x_{n+1}}{x_n} \Rightarrow \exists \lim_{n \to \infty} \sqrt[n]{x_n} =$

$= \lim_{n \to \infty} \frac{x_{n+1}}{x_n}$

ex. $\lim_{n \to \infty} \sqrt[n]{n!} = \lim_{n \to \infty} \frac{(n+1)!}{n!} = n+1 = \infty$

<u>Dem.</u> Vrem să arătăm că $\overline{\lim} \sqrt[n]{x_n} \leq \overline{\lim} \frac{x_{n+1}}{x_n}$

Dacă $\overline{\lim} \frac{x_{n+1}}{x_n} = \infty \Rightarrow$ ① este evidentă

Putem pp. că $\overline{\lim} \frac{x_{n+1}}{x_n} = a \in \mathbb{R}$

$\forall \varepsilon > 0 \ \exists n_\varepsilon$ a.î. $\forall n \geq n_\varepsilon \Rightarrow \frac{x_{n+1}}{x_n} < a + \varepsilon$

(înm. cu x_{n+1})

$n \geq n_\varepsilon \quad \frac{x_{n+2}}{x_n} = \frac{x_{n+2}}{x_{n+1}} \cdot \frac{x_{n+1}}{x_n} < (a+\varepsilon)^2$

$n \geq n_\varepsilon \Rightarrow \frac{x_{n+p}}{x_n} = \frac{x_{n+p}}{x_{n+p-1}} \cdot \frac{x_{n+p-1}}{x_{n+p-2}} \cdot \ldots$

$\cdot \frac{x_{n+1}}{x_n} \leq (a+\varepsilon)^p$

$\frac{\sqrt[n+p]{x_{n+p}}}{\sqrt[n+p]{x_n}} < (a+\varepsilon)^p$

$$\overline{\lim_{n\to\infty}} \sqrt[n]{X_n} = \overline{\lim_{p\to\infty}} \sqrt[n+p]{X_{n+p}} \le \lim_{p\to\infty} \underbrace{(a+\varepsilon)^{\frac{p}{n+p}}}_{a+\varepsilon} \cdot \lim_{p\to\infty} \underbrace{\sqrt[n+p]{X_n}}_{1}$$

$$= a+\varepsilon \text{ pt. } \forall \varepsilon>0 \Rightarrow \overline{\lim_{n\to\infty}} \sqrt[n]{X_n} \le a+\varepsilon \ \forall \varepsilon>0 \Rightarrow$$

$$\Rightarrow \overline{\lim_{n\to\infty}} \sqrt[n]{X_n} \le a = \overline{\lim_{n\to\infty}} \frac{X_{n+1}}{X_n}$$

<u>Teoremă</u> Dacă $b_n \nearrow \infty$ și $\exists \lim_{n\to\infty} \frac{a_{n+1}-a_n}{b_{n+1}-b_n} \Rightarrow$

$$\Rightarrow \exists \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{a_{n+1}-a_n}{b_{n+1}-b_n}$$

<u>ex.</u> $\lim_{n\to\infty} \frac{1+\frac{1}{2}+\dots+\frac{1}{n}}{\ln n}$

! $a_n = 1+\frac{1}{2}+\dots+\frac{1}{n}$

! $b_n = \ln n \nearrow \infty$

$$\lim_{n\to\infty} \frac{a_{n+1}-a_n}{b_{n+1}-b_n} = \lim_{n\to\infty} \frac{\cancel{1}+\cancel{\frac{1}{2}}+\dots+\cancel{\frac{1}{n}}+\frac{1}{n+1}-\cancel{1}-\cancel{\frac{1}{2}}-\dots-\cancel{\frac{1}{n}}}{\ln(n+1)-\ln n} =$$

$$= \lim_{n\to\infty} \frac{1}{n+1 \cdot \ln\left(\frac{n+1}{n}\right)} = \lim_{n\to\infty} \frac{1}{\ln\left(1+\frac{1}{n}\right)^{n+1}} = 1$$

$\Rightarrow a_n \to \infty$

=

Dem: $\lim\limits_{n\to\infty} \dfrac{a_{n+1}-a_n}{b_{n+1}-b_n} = a \in R$

$\forall \varepsilon > 0 \quad \exists n_\varepsilon$ a. î. $\forall n \geq n_\varepsilon \Rightarrow \left| \dfrac{a_{n+1}-a_n}{b_{n+1}-b_n} - a \right| < \varepsilon$

$\forall n \geq n_\varepsilon \quad (a-\varepsilon)(b_{n+1}-b_n) \leq \underbrace{a_{n+1}-a_n \leq (a+\varepsilon)(b_{n+1}-b_n)}$

ne ocupăm doar de partea asta

$\forall n \geq n_\varepsilon \Rightarrow a_{n+p} - a_n = \sum\limits_{k=0}^{p-1} (a_{n+k+1} - a_{n+k}) \leq$

$\leq \sum\limits_{k=0}^{p-1} (b_{n+k+1} - b_{n+k}) =$

se simplifică termen cu termen

$= (a+\varepsilon)(b_{n+p} - b_n) \quad \Big| : b_{n+p}$

$\dfrac{a_{n+p}}{b_{n+p}} - \overset{\to 0 \text{ când } p\to\infty}{\dfrac{a_n}{b_{n+p}}} \leq (a+\varepsilon)\left(1 - \dfrac{b_n}{b_{n+p}}\right)$

$\overline{\lim} \dfrac{a_{n+p}}{b_{n+p}} = \overline{\lim} \dfrac{a_{n+p}}{b_{n+p}} - \dfrac{a_n}{b_{n+p}} \leq (a-\varepsilon) \cdot \lim\limits_{p\to\infty}$

$\left(1 - \underbrace{\dfrac{b_n}{b_{n+p}}}_{0}\right) = a + \varepsilon$

arătat că

$\overline{\lim} \dfrac{a_{n+p}}{b_{n+p}} \leq a = \lim\limits_{n\to\infty} \dfrac{a_{n+1}-a_n}{b_{n+1}-b_n}$

Serii

→ o să avem și la examen studierea conv. unei serii

Def. O serie $\sum_{n\geq p} x_n = \sum_{n=p}^{\infty} x_n$ este o pereche de

șiruri $((x_n)_{n\geq p}, (y_n)_{n\geq p})$

→ sume parțiale ale lui x_n

$$y_n = \sum_{k=p}^{n} x_k$$

Spunem că seria $\sum_{n\geq p} x_n$ are limită (este conv)

dacă șirul $(y_n)_{n\geq p}$ are limită (este conv.)

$$\sum_{n\geq p} x_n = \lim_{n\to\infty} y_n$$

ex. ① $\sum_{n\geq 1} \frac{1}{n(n+1)}$

$$x_n = \frac{1}{n(n+1)}, \quad y_n = \sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \frac{1}{k} - \frac{1}{k+1}$$

se simplif termenii

$$= 1 - \frac{1}{n+1} \longrightarrow 1$$

ex ② $\sum_{n=0} a^n$

Caz I $|a|<1$, $x_n = a^n$

$$y_n = \sum_{k=0}^{n} a^k = 1+a+a^2+a^n = \frac{1-a^{n+1}}{1-a} \longrightarrow \frac{1}{1-a}$$

Caz 2 $a=1$ $y_n = \sum_{k=1}^{n} 1 = n \longrightarrow \infty$

ex. 3 $\sum_{n\geq 1} \frac{1}{(n+1)^2}$

$$y_n = \sum_{k=1}^{n} \frac{1}{(k+1)^2}$$

$$y_{n+1} - y_n = x_{n+1} = \frac{1}{(n+2)^2} \geq 0 \Rightarrow \text{sirul } y_n \nearrow$$

$$\frac{1}{(n+1)^2} \leq \frac{1}{n(n+1)} \Rightarrow y_n \leq \sum_{k=1}^{n} \frac{1}{k(k+1)} = 1 - \underbrace{\frac{1}{n+1}}_{\downarrow 0} \longrightarrow 1$$

sir mon. si mărg. $\Longrightarrow$ seria este convergentă

! $e^x = \sum_{n\geq 0} \frac{x^n}{n!}$

Obs: Dacă seriile $\sum_{n\geq p} x_n$ și $\sum_{n\geq p} y_n$ sunt conv. atunci seriile $\sum_{n\geq p} (x_n + y_n)$ și $\sum_{n\geq p} a \cdot x_n$ sunt conv. și

$$\sum_{n\geq p} (x_n + y_n) = \sum_{n\geq p} x_n + \sum_{n\geq p} y_n$$

$$\sum_{n\geq p} a \cdot x_n = a \cdot \sum_{n\geq p} x_p$$

Obs: Dacă seria $\sum_{n\geq 1} x_n$ este convergentă atunci șirul

$x_n \longrightarrow 0$

$x_{n+1} = y_{n+1} - y_n = a - a = 0 \quad (y_n \longrightarrow a)$

$\sum_{n\geq 1} (1 + (-1)^n)$ $\quad x_{2n} = 2 \not\longrightarrow 0$

$\downarrow$ $\quad x_{2n+1} = 0$

divergent

$\sum_{n\geq 1} \frac{1}{n} = \infty$

30.10.20

2. Fie $a >$

Ar. că se

Sol

$a_n = \sqrt[n]{}$

Vom arăt

Pentru

pt. și

crit. rad

$\lim_{n\to\infty}$

$= \lim_{n\to\infty}$