Seminar 10

Transformări ortogonale. Endomorfisme simetrice

Ex1 Fie $(\mathbb{R}^3, g_0)$ s.v.e.r, cu str. canonică

$f \in End(\mathbb{R}^3)$, $A = [f]_{R_0,R_0} = \frac{1}{9}\begin{pmatrix} 8 & 1 & -4 \\ 1 & 8 & 4 \\ -4 & 4 & -7 \end{pmatrix}$

R_0 = reperul canonic.

a) Să se arate că $f \in O(\mathbb{R}^3)$ de speța 2.

(i.e. $f = s \circ R_\varphi$, R_φ = este rotație de unghi orientat φ și axă $<\{e_1\}>$, iar s = simetrie ortogonală față de $<\{e_1\}>^\perp$)

b) Să se afle unghiul de rotație și axa de rotație

c) Să se determine un reper ortonormat $R = \{e_1, e_2, e_3\}$ a.î. $[f]_{R,R} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix}$.

SOL

a) Arătăm că $A \in O(3)$ și $\det A = -1$.

$A \in O(3) \iff A \cdot A^T = I_3 \iff$

$\{f_1 = \frac{1}{9}(8,1,-4),\ f_2 = \frac{1}{9}(1,8,4),\ f_3 = \frac{1}{9}(-4,4,7)\}$ reper ortonormat $\iff \begin{cases} \|f_1\| = \|f_2\| = \|f_3\| = 1 \\ <f_1,f_2> = <f_1,f_3> = <f_2,f_3> = 0 \end{cases}$

$$A \cdot A^T = \frac{1}{81}\begin{pmatrix} 8 & 1 & -4 \\ 1 & 8 & 4 \\ -4 & 4 & -7 \end{pmatrix}\begin{pmatrix} 8 & 1 & -4 \\ 1 & 8 & 4 \\ -4 & 4 & -7 \end{pmatrix} = I_3.$$

$$\det A = \frac{1}{9^3}\begin{vmatrix} 8 & 1 & -4 \\ 1 & 8 & 4 \\ -4 & 4 & -7 \end{vmatrix} = -1.$$

Deci $f \in O(\mathbb{R}^3)$ de speța 2

b). $Tr A = \frac{1}{9}(8+8-7) = \frac{9}{9} = 1 = -1 + 2\cos\varphi \Rightarrow$

$2\cos\varphi = 2 \Rightarrow \cos\varphi = 1 \Rightarrow \varphi = 0$

Determinăm axa de rotație.

$$f(x) = -x \Rightarrow \begin{cases} 8x_1 + x_2 - 4x_3 = -9x_1 \\ x_1 + 8x_2 + 4x_3 = -9x_2 \\ -4x_1 + 4x_2 - 7x_3 = -9x_3 \end{cases} \Rightarrow \begin{cases} 17x_1 + x_2 - 4x_3 = 0 \\ x_1 + 17x_2 + 4x_3 = 0 \\ -4x_1 + 4x_2 + 2x_3 = 0 \end{cases}$$

$$\det\begin{pmatrix} 17 & 1 & -4 \\ 1 & 17 & 4 \\ -4 & 4 & 2 \end{pmatrix} = 0.$$

$$\begin{cases} 17x_1 + x_2 = 4x_3 & \Big| \cdot (-17) \\ x_1 + 17x_2 = -4x_3 \end{cases} \quad (+)$$

$$x_1(1-17^2) \quad = -4\cdot 18 x_3$$

$$x_1 \cdot 18 \cdot (-16) = -4 \cdot 18\, x_3 \Rightarrow 4x_1 = x_3 \Rightarrow x_1 = \frac{x_3}{4}$$

$$x_2 = 4x_3 - 17 \cdot \frac{x_3}{4} = \frac{(16-17)x_3}{4} \Rightarrow x_2 = -\frac{x_3}{4}.$$

$$(x_1, x_2, x_3) = \left(\frac{x_3}{4}, -\frac{x_3}{4}, x_3\right) = \frac{x_3}{4}\underbrace{(1,-1,4)}_{u}.$$

$\langle\{(1,-1,4)\}\rangle$ axa de rotație.

c) $\langle\{u\}\rangle^{\perp} = \{x \in \mathbb{R} \mid x_1 - x_2 + 4x_3 = 0\}$
$= \{(x_2 - 4x_3, x_2, x_3) \mid x_2, x_3 \in \mathbb{R}\} =$
$= \langle\{\underset{f_2}{(1,1,0)}, \underset{f_3}{(-4,0,1)}\}\rangle$

Aplicăm Gram-Schmidt pt reperul $\{f_2, f_3\}$.

$\bar{e}_2 = f_2 = (1,1,0)$

$\bar{e}_3 = f_3 - \dfrac{\langle f_3, \bar{e}_2\rangle}{\langle \bar{e}_2, \bar{e}_2\rangle} \cdot \bar{e}_2 = (-4,0,1) - \dfrac{-4}{2}(1,1,0) =$
$= (-4,0,1) + (2,2,0) = (-2,2,1)$

$\left\{e_2 = \frac{1}{\sqrt{2}}(1,1,0),\ e_3 = \frac{1}{3}(-2,2,1)\right\}$ reper ortonormat în $\langle\{u\}\rangle^{\perp}$

OBS $\bar{e}_2 \times \bar{e}_3 = \begin{vmatrix} i & j & k \\ 1 & 1 & 0 \\ -2 & 2 & 1 \end{vmatrix} = i\begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} - j\begin{vmatrix} 1 & 0 \\ -2 & 1 \end{vmatrix} + k\begin{vmatrix} 1 & 1 \\ -2 & 2 \end{vmatrix} = (1,-1,4)$

$e_1 = \frac{1}{3\sqrt{2}}(1,-1,4)$ versorul axei

$\mathcal{R} = \{e_1, e_2, e_3\}$ reper ortonormat, pozitiv orientat, în $\mathbb{R}^3$.

$$[f]_{\mathcal{R},\mathcal{R}} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad f = \text{simetrie față de } \langle\{e_1\}\rangle^{\perp}.$$

Ex2) $(\mathbb{R}^3, g_0)$, $u = (1,1,0)$

a) $<\{u\}>^{\perp} = ?$. Să se precizeze un reper ortonormat în $<\{u\}>^{\perp}$

b) Să se determine ecuația rotației de $\measuredangle \varphi = \frac{\pi}{2}$ în planul $<\{u\}>^{\perp}$, de axă $<\{u\}>$.

Sol.

a) $<\{u\}>^{\perp} = \{x \in \mathbb{R}^3 \mid x_1 + x_2 = 0\} = \{(-x_2, x_2, x_3) \mid x_2, x_3 \in \mathbb{R}\}$

$= <\{\underbrace{(-1,1,0)}_{f_2}, \underbrace{(0,0,1)}_{f_3}\}>.$

$\{f_2, f_3\}$ reper ortogonal în $<\{u\}>^{\perp}$

$\{e_2 = \frac{1}{\sqrt{2}}(-1,1,0),\ e_3 = (0,0,1)\}$ reper ortonormat în $<\{u\}>^{\perp}$.

$$f_2 \times f_3 = \begin{vmatrix} i & j & k \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = (1,1,0) = u.$$

$e_1 = \frac{1}{\sqrt{2}}(1,1,0)$ versorul axei de rotație.

$\mathcal{R} = \{e_1, e_2, e_3\}$ reper ortonormat, pozitiv orientat, în $\mathbb{R}^3$,

$$[f]_{\mathcal{R},\mathcal{R}} = A' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & +1 & 0 \end{pmatrix}$$

$\mathcal{R}_0 = \{e_1^0, e_2^0, e_3^0\}$ reperul canonic, $A = [f]_{\mathcal{R}_0,\mathcal{R}_0}$.

$\mathcal{R}_0 \xrightarrow{C} \mathcal{R}$, $C \in O(n)$

$A' = C^{-1}AC \Rightarrow A = CA'C^{-1} = CA'C^{T}$

$$C = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}$$

$$A = \frac{1}{2} \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & +1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & \sqrt{2} & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 & \sqrt{2} \\ 1 & 1 & -\sqrt{2} \\ -\sqrt{2} & \sqrt{2} & 0 \end{pmatrix}$$

$f: \mathbb{R}^2 \to \mathbb{R}^2$, $f(x) = \frac{1}{2}(x_1 + x_2 + \sqrt{2}x_3,\ x_1 + x_2 - \sqrt{2}x_3,\ -\sqrt{2}x_1 + \sqrt{2}x_2)$

OBS $(E, \langle \cdot, \cdot \rangle)$, $u \in E \setminus \{0_E\}$.

- $s \in End(E)$ simetria ortogonală față de hiperplanul $\langle\{u\}\rangle^{\perp}$ se scrie: $s(x) = x - 2\dfrac{\langle x, u\rangle}{\langle u, u\rangle} \cdot u$.
- $p \in End(E)$ proiecția ortogonală pe hiperplanul $\langle\{u\}\rangle^{\perp}$ se scrie: $p(x) = x - \dfrac{\langle x, u\rangle}{\langle u, u\rangle} \cdot u$.

$E = \langle u\rangle \oplus \langle u\rangle^{\perp}$

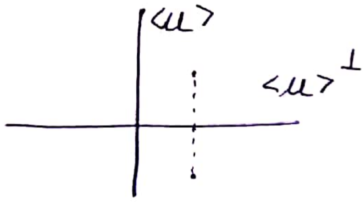
$\left\{\dfrac{u}{\|u\|}\right\}$ reper ortonormat în $\langle\{u\}\rangle$.

Fie $x' = x - \left\langle x, \dfrac{u}{\|u\|}\right\rangle \cdot \dfrac{u}{\|u\|} = x - \dfrac{\langle x, u\rangle}{\langle u, u\rangle} \cdot u$

Arătăm că $x' \in \langle u\rangle^{\perp}$

$$\langle x', u\rangle = \langle x, u\rangle - \frac{\langle x, u\rangle}{\cancel{\langle u, u\rangle}} \cancel{\langle u, u\rangle} = 0 \Rightarrow x' \in \langle u\rangle^{\perp}$$

$$x = \underbrace{\frac{\langle x, u\rangle}{\langle u, u\rangle} u}_{\in \langle u\rangle} + \underbrace{x'}_{\in \langle u\rangle^{\perp}}$$



$$\Rightarrow s(x) = -\frac{\langle x, u\rangle}{\langle u, u\rangle} u + x' =$$

$$s(x) = x - 2\frac{\langle x, u\rangle}{\langle u, u\rangle} \cdot u$$

$$s(x) = 2p(x) - x \Rightarrow p(x) = \frac{1}{2}(s(x) + x) =$$

$$p(x) = x - \frac{\langle x, u\rangle}{\langle u, u\rangle} u.$$

Ex3 $(\mathbb{R}^3, g_0)$, $u = (1, -1, 0)$

Să se scrie ec. simetriei ortogonale față de $\langle u\rangle^{\perp}$.

SOL

$$s(x) = x - 2\frac{\langle x, u\rangle}{\langle u, u\rangle} \cdot u =$$

$$= (x_1, x_2, x_3) - \cancel{2}\,\frac{x_1 - x_2}{\cancel{2}}(1, -1, 0) =$$

$$= (x_1, x_2, x_3) + (x_2 - x_1, x_1 - x_2, 0) =$$

$$= (x_2, x_1, x_3)$$

OBS $\langle u\rangle^{\perp} = \{x\in\mathbb{R}^3 \mid x_1 - x_2 = 0\} = \{(x_1, x_1, x_3) \mid x_1, x_3 \in \mathbb{R}\}$

$= \langle\{\underbrace{(1,1,0)}_{f_2}, \underbrace{(0,0,1)}_{f_3}\}\rangle$

$R = \{u, f_2, f_3\}$ reper ortogonal. $s(u) = -u$, $s(f_2) = f_2$, $s(f_3) = f_3$

$$[s]_{R,R} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \left(\begin{array}{c|cc} -1 & 0 & 0 \\ \hline 0 & \multicolumn{2}{c}{\multirow{2}{*}{I_2}} \\ 0 & & \end{array}\right)$$

OBS $(E, \langle\cdot,\cdot\rangle)$, $f \in \mathrm{Sim}(E) \Leftrightarrow \langle x, f(y)\rangle = \langle f(x), y\rangle, \forall x, y \in E$

Ex4 Fie $(E, \langle\cdot,\cdot\rangle)$ s.v.e.r, $\dim E = 2$.

Fie $Q_k : E \longrightarrow \mathbb{R}$, $k = \overline{1,3}$ definite prin

$Q_1(x) = \langle x, x\rangle$, $Q_2(x) = \langle f(x), x\rangle$, $Q_3(x) = \langle f(x), f(x)\rangle$,

$\forall x \in E$, unde $f \in \mathrm{Sim}(E)$.

Să se arate că: $Q_3(x) - \mathrm{Tr}(A_f) Q_2(x) + \det(A_f) Q_1(x) = 0$,

$\forall x \in E$, unde $A_f = [f]_{R,R}$, $R = \{e_1, e_2\}$ reper ortonormat.

(Q_k, $k = \overline{1,3}$ s.n. forme fundamentale)

SOL

$f \in \mathrm{Sim}(E) \Rightarrow A_f = A_f^T = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$

$\mathrm{Tr}(A_f) = a + c$, $\det(A_f) = ac - b^2$.

$f : E \to E$, $f(x) = (ax_1 + bx_2, bx_1 + cx_2)$

$f(e_1) = ae_1 + be_2$; $f(e_2) = be_1 + ce_2$.

- $Q_3(e_1) = \langle f(e_1), f(e_1)\rangle = \langle ae_1 + be_2, ae_1 + be_2\rangle = a^2 + b^2$

  $-\mathrm{Tr}(A_f) Q_2(e_1) = -(a+c)\langle f(e_1), e_1\rangle = -(a+c)\langle ae_1 + be_2, e_1\rangle =$

  $= -(a+c)a$

  $\det(A_f) Q_1(e_1) = (ac - b^2)\langle e_1, e_1\rangle = ac - b^2$

  $a^2 + b^2 - (a+c)a + ac - b^2 = 0$

- $Q_3(e_2) = \langle f(e_2), f(e_2)\rangle = \langle be_1 + ce_2, be_1 + ce_2\rangle = b^2 + c^2$

  $-\mathrm{Tr}(A_f) Q_2(e_2) = -(a+c)\langle f(e_2), e_2\rangle = -(a+c)\langle be_1 + ce_2, e_2\rangle$

  $= -(a+c)\cdot c$

  $\det(A_f) Q_1(e_2) = (ac - b^2)\langle e_2, e_2\rangle = ac - b^2$.

$b^2 + c^2 - (a+c)c + ac - b^2 = 0$. ; $x = x_1 e_1 + x_2 e_2$

$Q_3(x) - Tr(A_f)Q_2(x) + \det(A_f)\, Q_1(x) =$

$= x_1\left(Q_3(e_1) - Tr(A_f)Q_2(e_1) + \det(A_f)\, Q_1(e_1)\right) +$

$+ x_2\left(Q_3(e_2) - Tr(A_f)\, Q_2(e_2) + \det(A_f)\, Q_1(e_2)\right) = 0,$

$\forall x \in E$ (Q_1, Q_2, Q_3 sunt liniare).

<u>OBS</u>. $A = A^T$ 1) $f \in Sim(E)$

2) $Q : E \to \mathbb{R}$ formă pătratică

$$Q(x) = X^T A X = \sum_{i,j=1}^{n} a_{ij}\, x_i x_j$$

$$\langle x, f(x)\rangle = \langle \sum_{i=1}^{n} x_i e_i, f\left(\sum_{j=1}^{n} x_j e_j\right)\rangle =$$

$$= \sum_{i,j,k=1}^{n} \langle x_i e_i, a_{kj} x_j e_k\rangle = \sum_{i,j,k=1}^{n} a_{kj} x_i x_j \underbrace{\langle e_i, e_k\rangle}_{\delta_{ik}}$$

$$= \sum_{i,j=1}^{n} a_{ij} x_i x_j \Rightarrow \underline{\langle x, f(x)\rangle = Q(x), \forall x \in E}$$

Ex 5 Fie $f \in End(\mathbb{R}^3)$, $A = [f]_{R_0,R_0} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

a) Să se arate că $f \in Sim(\mathbb{R}^3)$; $f = ?$

b) Să se determine $Q : \mathbb{R}^3 \to \mathbb{R}$ forma pătratică asociată. Să se aducă Q la o formă canonică, efectuând o transformare ortogonală h; $h = ?$.
(schimbare de repere ortonormate)

<u>SOL</u>. a) $A = A^T \Rightarrow f \in Sim(\mathbb{R}^3)$

$f : \mathbb{R}^3 \to \mathbb{R}^3$, $f(x) = (x_1 + x_3,\ x_2,\ x_1 + x_3)$

b) $Q : \mathbb{R}^3 \to \mathbb{R}$, $Q(x) = x_1^2 + x_2^2 + x_3^2 + 2x_1 x_3$.

Aplicăm metoda valorilor proprii

$$P(\lambda) = \det(A - \lambda I_3) = 0 \Rightarrow \begin{vmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)\left[(1-\lambda)^2 - 1\right] = 0 \Rightarrow (1-\lambda)(-\lambda)(2-\lambda) = 0$$

$\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = 2$.

$V_0 = \{x \in \mathbb{R}^3 \mid A \cdot X = 0\} = \{(x_1, 0, -x_1) \mid x_1 \in \mathbb{R}\} = \langle\{(1,0,-1)\}\rangle.$

$\begin{cases} x_1 + x_3 = 0 \\ x_2 = 0 \end{cases}$

$V_1 = \{x \in \mathbb{R}^3 / AX = X\} = \{(0, x_2, 0) \mid x_2 \in \mathbb{R}\} = \langle\{(0,1,0)\}\rangle.$

$(A - J_3) X = 0 \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

$V_2 = \{x \in \mathbb{R}^3 \mid AX = 2X\} = \{(x_1, 0, x_1) \mid x_1 \in \mathbb{R}\} = \langle\{(1,0,1)\}\rangle$

$(A - 2J_3) X = 0 \; ; \; \begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$

$\mathcal{R} = \{e_1 = \frac{1}{\sqrt{2}}(1,0,-1),\ e_2 = (0,1,0),\ e_3 = \frac{1}{\sqrt{2}}(1,0,1)\}$ reper ortonormat

$[f]_{\mathcal{R},\mathcal{R}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

$x = \sum_{i=1}^{3} x_i' e_i' \implies Q(x) = x_2'^2 + 2x_3'^2$

$\mathcal{R}_0 = \{e_1^0, e_2^0, e_3^0\} \xrightarrow{C} \mathcal{R} \quad , \quad C \in O(3)$

reperul canonic

$C = [h]_{\mathcal{R},\mathcal{R}}.$

$h : \mathbb{R}^3 \to \mathbb{R}^3, \quad h(e_i^0) = e_i, \ \forall i = \overline{1,3}$

$C = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ -1 & 0 & 1 \end{pmatrix}$

$h(x) = \frac{1}{\sqrt{2}}(x_1 + x_3 \,;\, \sqrt{2}x_2 \,,\, -x_1 + x_3), \quad h \in O(\mathbb{R}^3).$

$\det C = \frac{1}{2\sqrt{2}} \sqrt{2} \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = \frac{\sqrt{2} \cdot 2}{2\sqrt{2}} = 1 \implies h \in SO(\mathbb{R}^3)$

OBS. $Q : \mathbb{R}^3 \to \mathbb{R}, \quad Q(x) = x_2'^2 + 2x_3'^2$

Fie $g : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ forma polară asociată

Este $(\mathbb{R}^3, g)$ un spațiu vectorial euclidian? NU.

- Q are signatura $(2,0) \implies Q$ nu este poz def.

Ex 6 $(\mathbb{R}^3, g_0)$

Fie $f:\mathbb{R}^3 \to \mathbb{R}^3$, $f(x) = \left(\frac{x_1}{\sqrt{2}} + \frac{x_2}{\sqrt{3}} + cx_3,\ ax_1 + \frac{x_2}{\sqrt{3}} + dx_3,\ \frac{x_1}{\sqrt{2}} + bx_2 + ex_3\right)$

a) Să se determine $a, b, c, d, e \in \mathbb{R}$ ai $f \in O(\mathbb{R}^3)$

b) —//— ai $f \in SO(\mathbb{R}^3)$

SOL

a) R_0 reper canonic

$$A = [f]_{R_0, R_0} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & c \\ a & \frac{1}{\sqrt{3}} & d \\ \frac{1}{\sqrt{2}} & b & e \end{pmatrix}$$

$A \in O(3) \iff \left\{ e_1 = \left(\frac{1}{\sqrt{2}}, a, \frac{1}{\sqrt{2}}\right), e_2 = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, b\right), e_3 = (c, d, e) \right\}$ reper ortonormat.

$\|e_1\| = 1 \Rightarrow \frac{1}{2} + a^2 + \frac{1}{2} = 1 \Rightarrow a^2 + 1 = 1 \Rightarrow \boxed{a = 0}$

$\|e_2\| = 1 \Rightarrow \frac{1}{3} + \frac{1}{3} + b^2 = 1 \Rightarrow b^2 = \frac{1}{3} \Rightarrow b = \pm\frac{1}{\sqrt{3}}$

$\|e_3\| = 1 \Rightarrow c^2 + d^2 + e^2 = 1 \ (*)$

$\langle e_1, e_2 \rangle = 0 \Rightarrow \frac{1}{\sqrt{6}} + \frac{b}{\sqrt{2}} = 0 \Rightarrow \boxed{b = -\frac{1}{\sqrt{3}}}$

$\langle e_1, e_3 \rangle = 0 \Rightarrow \frac{c}{\sqrt{2}} + \frac{e}{\sqrt{2}} = 0 \Rightarrow c = -e.$

$\langle e_2, e_3 \rangle = 0 \Rightarrow \frac{c}{\sqrt{3}} + \frac{d}{\sqrt{3}} + \frac{1}{\sqrt{3}} e = 0 \Rightarrow d = 2e.$

$(*) \Rightarrow e^2 + 4e^2 + e^2 = 1 \Rightarrow e^2 = \frac{1}{6} \Rightarrow e = \pm\frac{1}{\sqrt{6}}.$

1) $a = 0,\ b = -\frac{1}{\sqrt{3}},\ c = -\frac{1}{\sqrt{6}},\ d = \frac{2}{\sqrt{6}},\ e = \frac{1}{\sqrt{6}}$

2) $a = 0,\ b = -\frac{1}{\sqrt{3}},\ c = \frac{1}{\sqrt{6}},\ d = \frac{-2}{\sqrt{6}},\ e = -\frac{1}{\sqrt{6}}$

1) $A = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & +\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{pmatrix}$; 2) $A = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{pmatrix}$

b) 1) $\det A = \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{6}} \begin{vmatrix} 1 & 1 & -1 \\ 0 & 1 & 2 \\ 1 & -1 & 1 \end{vmatrix} = \frac{1}{6} \begin{vmatrix} 1 & 1 & -1 \\ 0 & 1 & 2 \\ 0 & -2 & 2 \end{vmatrix} = \frac{1}{6}(2+4) = 1$

$A \in SO(3) \Rightarrow f \in SO(\mathbb{R}^3)$

2) $\det A = \frac{1}{6} \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ 1 & -1 & -1 \end{vmatrix} = \frac{1}{6} \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & -2 & -2 \end{vmatrix} = \frac{1}{6}(-6) = -1$

f este de speța 2.

Temă (S_{10})

Ex1. $(\mathbb{R}^3, g_0)$, $x = (0,1,-1)$

a) Să se scrie ec. rotației de $\measuredangle \varphi = \pi$ și axă $<\{x\}>$.

b) Să se determine ec. simetriei ortogonale față de $<\{x\}>^{\perp}$.

Ex2 $(\mathbb{R}^3, g_0)$, $f \in End(\mathbb{R}^3)$

$f(x) = (x_1 + x_2 - x_3, x_1 + x_2 - x_3, -x_1 - x_2 + x_3)$

a) Să se arate că $f \in Sim(\mathbb{R}^3)$

b) Să se scrie forma pătratică Q asociată. Să se aducă Q la o formă canonică, efectuând o transformare ortogonală h. Să se precizeze h.

c) Fie $g: \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ forma polară asociată lui Q. Este $(\mathbb{R}^3, g)$ un spațiu vectorial euclidian?