CURS 12

20.12.2018

Def: Fie $f:(a,b) \to \mathbb{R}^n$ și $c \in (a,b)$.

Spunem că f e derivabilă în c dacă $\exists \lim_{x \to c} \frac{f(x)-f(c)}{x-c} \iff$

$\iff \exists f_c'(c), \forall c = \overline{1,n}$

$f = (f_1, \dots f_n)$

$f'(c) = (f'(c), f_c'(c) \dots f_n'(c))$

ex: $f:\mathbb{R} \to \mathbb{R}^3$, $f(x) = (x^5 + x, e^x, \sin x)$

$f'(x) = (5x^4+1, 2e^{2x}, \cos x)$

Def. Fie $D = \mathring{D} \subset \mathbb{R}^n$, $a \in D$, $f : D \to \mathbb{R}^m$

$v \in \mathbb{R}^n$, $v \neq 0$

$$\frac{\partial f}{\partial v}(a) = \lim_{t \to 0} \frac{f(a+tv) - f(a)}{t}$$

$\mathbb{R}^n$ $e_1 = (1, 0, 0, \dots, 0)$

$e_2 = (0, 1, \dots, 0)$

$\vdots$

$e_n = (0, 0, \dots, 1)$

$$x = (x_1, \dots, x_n) = \sum_{i=1}^{n} x_i \cdot e_i$$

$$\frac{\partial f}{\partial e_1} = \lim_{t \to 0} \frac{f(a_1 + t, a_2, \dots a_n) - f(a_1, \dots a_n)}{t}$$

$$= \lim_{t \to a_1} \frac{f(x, a_2, \dots a_n) - f(a_1, \dots a_n)}{x - a_1}$$

$$= \frac{\partial f}{\partial x_1}$$

$f : \mathbb{R}^2 \to \mathbb{R}^2$

$f(x, y) = x^3 y^2 + y^4 x$

$$\frac{\partial f}{\partial x} = 3x^2 y^2 + y^4$$

$$\frac{\partial f}{\partial y} = 2x^3 y + 4y^3 x$$

1. Fie $D=\mathring{D}$, $a\in D$ și $f:D\to\mathbb{R}^m$. Spunem că f este derivabilă dacă $\exists\ T\in L(\mathbb{R}^n,\mathbb{R}^m)$ a. î.

$$\lim_{x\to a}\frac{f(x)-f(a)-T(x-a)}{d_2(x,a)=\|x-a\|_2}=0$$

$T=f'(a)$.

:1 derivata este unică

pp $\exists\ T_1,T_2\in L(\mathbb{R}^n,\mathbb{R}^m)$ a. î. $\lim\limits_{x\to a}\frac{f(x)-f(a)-T_1(x-a)}{d_2(x,a)}=0$

$$\lim_{x\to a}\frac{f(x)-f(a)-T_2(x-a)}{d_2(x-a)}=0$$

(Vrem să arătăm că $T_1=T_2$)

le scădem

$$\lim_{x\to a}\frac{T_1(x-a)-T_2(x-a)}{d_2(x-a)}=0$$

$T_1-T_2=T$

$v\in\mathbb{R}^n\ v\neq0$

$x=a+tv$

$t\in\mathbb{R}$

$\Downarrow$

$$\lim_{t\to0}\frac{T(tv)}{d_2(a+tv,a)}=0\iff$$

$$\iff\lim_{t\to0}\frac{t\cdot T(v)}{d_2(tv,0)}$$

$$\iff\lim_{t\to0}\frac{\cancel{t}\cdot T(v)}{\cancel{t}\cdot d_2(v,0)}=0$$

$\Rightarrow T(v)=0\quad\forall v\neq0$

$T(0)=0$

$\Rightarrow T=0$

Def. Fie $D = \overset{\circ}{D} \subset \mathbb{R}^n$, $a \in D$ și $f: D \to \mathbb{R}^m$. f este derivabilă în $a \iff \exists\, T \in L(\mathbb{R}^n, \mathbb{R}^m)$ a. î.

$$\lim_{x \to a} \underbrace{\frac{f(x) - f(a) - T(x-a)}{d_2(x-a)}}_{\omega(x)} = 0$$

f este deriv. în $a \iff \exists\, T \in L(\mathbb{R}^n, \mathbb{R}^m)$ și $\omega: D \to \mathbb{R}^m$ a. î.

1) $f(x) = f(a) + T(x-a) + d_2(x-a)\,\omega(x)$

2) $\lim\limits_{x \to a} \omega(x) = 0$

Obs. 2 Dacă $\exists\, f'(a) \Rightarrow f$ este cont. în a

$$\lim_{x \to a} f(x) = \lim_{x \to a} \underbrace{f(a) + f'(a)\underset{\downarrow 0}{(x-a)}}_{\downarrow 0} + \underset{\downarrow 0}{d_2(x,a)}\,\underset{\downarrow 0}{\omega(x)} = f(a)$$

Obs. 3 Dacă $\exists\, f'(a) \Rightarrow \forall\, v \in \mathbb{R}^n$, $v \neq 0$

$\exists\, \frac{\partial f}{\partial v}(a) = f'(a)(v)$

$$\exists\, f'(a) \Rightarrow \lim_{x \to a} \frac{f(x) - f(a) - T(x-a)}{d_2(x,a)} = 0$$

$\Downarrow$

$x = a + t \cdot v$

$$\lim_{t \to 0} \frac{f(a+tv) - f(a) - f'(a)(tv)}{t \cdot d_2(\cancel{x}, a)}$$

$$\lim_{t \to 0} \frac{f(a+tv) - f(a)}{t} = f'(a)(v) = \frac{\partial f}{\partial v}$$

$T \in L(\mathbb{R}^n, \mathbb{R}^m)$, $x \in \mathbb{R}^n$, $x = (x_1, \dots x_n) = \sum_{i=1}^{n} x_i e_i$

$$T\left(\sum_{i=1}^{n} x_i e_i\right) = \sum_{i=1}^{n} T(x_i, e_i) = \sum_{i=1}^{n} x_i T(e_i)$$

$$T(e_i) \in \mathbb{R}^m = \begin{pmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{mi} \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{i=1}^{n} x_i a_{1i} \\ \vdots \\ \sum_{i=1}^{n} x_i a_{mi} \end{pmatrix} = A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad A = (a_{ij})$$

$$f'(a) = \left(\frac{\partial f}{\partial x_1}, \dots \frac{\partial f}{\partial x_n}\right) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}$$

$$f = \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix} \in \mathbb{R}^m$$

$$f'(a) = \left(\frac{\partial f_i}{\partial x_j}\right) \qquad \begin{matrix} i = \overline{1,m} - \text{col.} \\ j = \overline{1,n} - \text{lin.} \end{matrix}$$

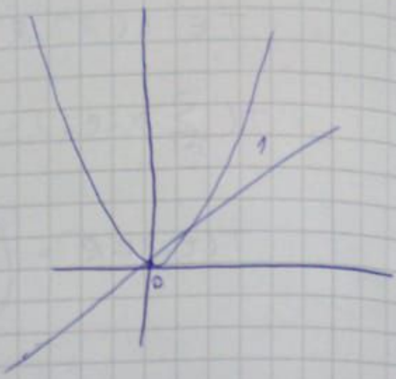
ex: $f: \mathbb{R}^2 \to \mathbb{R}^3$

$$f(x,y) = \begin{pmatrix} x^3 y^2 \\ e^{2x} y^2 \\ xy^3 + y^4 x^2 \end{pmatrix}$$

$$f'(x,y) = \begin{pmatrix} 3x^2y^2 & 2x^3y \\ 2e^{2x} \cdot y^2 & e^{2x} \cdot 2y \\ y^3 + 2xy^4 & 3xy^2 + 4y^3x^2 \end{pmatrix}$$

ex: $f: \mathbb{R}^2 \to \mathbb{R}$

$$f(x,y) = \begin{cases} 1 & y = x^2 \quad x \neq 0 \\ 0 & \text{rest} \end{cases}$$



$\frac{\partial f}{\partial v}(0,0) = 0 \quad \forall v$

f nu este cont. în 0

$f\left(\frac{1}{n}, \frac{1}{n^2}\right) = 1 \not\to 0$

Obs: Fie $f \quad B(a,r) \to \mathbb{R}^n$

1) Dacă $\exists \frac{\partial f}{\partial x_i} \; \forall i = \overline{1,n}$ sunt mărginite atunci f este continuă

2) Dacă $\exists \frac{\partial f}{\partial x_i} \; \forall i = \overline{1,n}$ pe $B(a,r)$ și sunt cont. în $a \Rightarrow \exists f'(a)$

Prop: Fie $D = \overset{\circ}{D} \in \mathbb{R}^n$, $f_1, f_2 : D \to \mathbb{R}^m$ și $a \in D$. Dacă $\exists f_1'(a)$ și $\exists f_2'(a)$, atunci $\Rightarrow$
$\exists (f_1 + f_2)'(a) = f_1'(a) + f_2'(a)$. $(\exists (\alpha f_1)'(a) = \alpha f_1'(a)$ $\forall \alpha \in \mathbb{R})$

$\exists f_1'(a) \Rightarrow \lim\limits_{x \to a} \frac{f_1(x) - f_1(a) - f_1'(a)(x-a)}{d_2(x-a)} = 0$

$\exists f_2'(a) \Rightarrow \lim\limits_{x \to a} \frac{f_2(x) - f_2(a) - f_2'(a)(x-a)}{d_2(x-a)} = 0$

+

$$\lim_{x \to a} \frac{(f_1+f_2)(x) - (f_1+f_2)(a) - \overbrace{(f_1'(a)+f_2'(a))}^{(f_1+f_2)'(a)}(x-a)}{d_1(x,a)} = 0$$

Prop. Fie $D = \overset{\circ}{D} \subset \mathbb{R}^n$, $G = \overset{\circ}{G} \subset \mathbb{R}^m$, $f: D \to G$, $g: G \to \mathbb{R}^p$ şi $a \in D$

$$\begin{array}{ccccc} & & \mathbb{R}^n & & \mathbb{R}^m & & \\ & & \cup & & \cup & & \\ D & \xrightarrow{f} & G & \xrightarrow{g} & \mathbb{R}^p \\ \cup & & & & \\ a & \longrightarrow & f(a) & \longrightarrow & g\,f(a) \end{array}$$

Dacă $\exists\, f'(a)$ şi $\exists\, g'(f(a)) \Rightarrow \exists\, (g \circ f)'(a) =$

$= g'(f(a) = f'(a)$

Prop. Fie $D = \overset{\circ}{D}$, $G = \overset{\circ}{G} \subset \mathbb{R}^n$, $f: D \to G$ bijectivă şi $a \in D$.

Dacă $\exists\, f'(a)$ şi $\exists\, (f'(a))^{-1}$ şi f^{-1} este cont. în $f(a) \Rightarrow$

$\Rightarrow \exists\, (f^{-1})'(f(a)) = (f'(a))^{-1}$