CURS 6

08.11.2018

$g: R \to R$

$g(x) = 0$

$\underbrace{x + g(x)}_{\text{notăm } f(x)} = x$

$\Downarrow$

$f(x) = x$

Def. Fie (X, d) un sp metric. O fct $f: X \to X$ s.n contracție dacă $\exists\ c \in (0,1)$ a.î. $d(f(x); f(y)) \leq c\, d(x,y)$

Ex. $f(a,b) \to R$ derivabilă a.î $|f'| < c < 1$

$a < x < y < b$

$|f(y) - f(x)| = |f'(z)(y-x)| = c|y-x|$

$\exists\ z \in (x;y)$

Teoremă (Principiul contracției al lui Banach)

Fie (X, d) un sp metric complet și $f: X \to X$ a.î. $d(f(x), f(y)) \leq c\, d(x,y)$ $\forall x,y \in X$ unde $c \in (0;1)$. Atunci $\exists!\ \alpha \in X$ a.î. a.î. $f(\alpha) = \alpha$, $\forall x \in X$, $\lim\limits_{n \to \infty} f^{[n]}(x) = \alpha$ n

$d(\alpha, f^{[n]}(x)) \leq \frac{c^n}{1-c} \cdot d(x, f(x))$

$\left(f^{[n]} = \underbrace{f \circ f \circ f \circ \dots \circ f}_{\text{de } n \text{ ori}} \right)$

Dem: (Pornim de la un x_0)

$x_0 \in X$

$x_n = f^{[n]}(x_0)$

$d(x_{n+1}, x_n) = d(f(x_n), f(x_{n-1})) \leq c\, d(x_n, x_{n-1}) \leq$

$\leq c^2 d(x_{n-1}, x_{n-2}) \leq \dots \leq c^n d(x_1, x_0)$

$d(x_{n+p}, x_n) \leq d(x_{n+p}, x_{n+p-1}) + d(x_{n+p}, x_{n+p-2}) + \dots +$
$+ d(x_{n+1}, x_n) \leq c^{n+p-1} d(x_1, x_0) + c^{n+p-2} d(x_1, x_0) + \dots +$
$+ c^n d(x_1, x_0) = c^n d(x_1, x_0)(c^{p-1} + c^{p-2} + \dots + 1) =$
$= c^n d(x_1, x_0) \left(\frac{1-c^p}{1-c} \right) \leq \frac{c^n}{1-c} d(x_1, x_0)$

$\Rightarrow (x_n)_n$ şir Cauchy $\Rightarrow x_n \to \alpha$

f cont. $\Big/ \Rightarrow x_n \to \alpha$

$f(x_n) \to f(\alpha)$

$\Rightarrow$

$\Rightarrow \alpha = f(\alpha)$

pp că $\exists\, \alpha$ şi β a.î. $f(\alpha) = \alpha$

$f(\beta) = \beta$ şi $\alpha \neq \beta$

$d(\alpha, \beta) = d(f(\alpha), f(\beta)) \leq c\, d(\alpha, \beta)$

$0 < (1-c)\, d(\alpha, \beta) \leq 0$

Contrad. $d(x_{n+p}, x_n) \leq \frac{c^n}{1-c} d(x_1, x_0)$ $\quad x_n \to \alpha$

$\downarrow_{p \to \infty}$

$d(\alpha, x_n) \leq \frac{c^n}{1-c} d(x_1, x_0)$

Def. O mulțime $V \subset \mathbb{R}$ s.n. vecinătate a lui $a \in \mathbb{R}$ dacă $\exists\, \varepsilon > 0$ a.î. $(a-\varepsilon, a+\varepsilon) \subset V$ $\quad \underset{\text{mult. vec. lui a}}{\mathcal{V}_a}$

Def. O mulțime $V \subset (X, d)$ s.n. vecinătate a lui $a \in X$ dacă $\varepsilon > 0$ a.î. $\underset{\text{bilă}}{B}(\underset{\text{centru}}{a}, \underset{\text{rază}}{\varepsilon}) \subset V$

$$\left(B(a, \varepsilon) = \{x \in X \mid d(a, x) < \varepsilon\}\right)$$

$\mathcal{V}_a = \{V \subset X \mid V \text{ vecinătate pt. } a\}$

Prop

1) Dacă $V_1, V_2 \in \mathcal{V}_a \Rightarrow V_1 \cap V_2 \in \mathcal{V}_a$
2) $V_1 \in \mathcal{V}_a \quad V_1 \subset V \Rightarrow V \in \mathcal{V}_a$
3) $a \in V \quad \forall V \in \mathcal{V}_a$
4) pt. $\forall\, \mathcal{V}_a \Rightarrow \exists\, W \in \mathcal{V}_x$ a.î. $W \in \mathcal{V}_x \quad \forall x \in W$

$r_1 < r_2 \quad B(a, r_1) \subset B(a, r_2)$

1) Dacă $V_1 \in \mathcal{V}_a \Rightarrow \exists\, r_1$ a.î. $B(a, r_1) \subset V_1$

Dacă $V_2 \in \mathcal{V}_a \Rightarrow \exists\, r_2$ a.î. $B(a, r_2) \subset V_2$

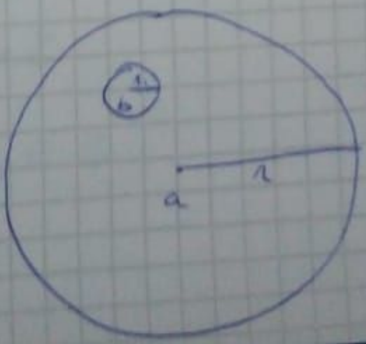
$r = \min(r_1, r_2) > 0$

$\Rightarrow B(a, r) \subset V_1 \cap V_2 \Rightarrow V_1 \cap V_2 \in \mathcal{V}_a$

2) Dacă $V_1 \in \mathcal{V}_a$, $V_1 \subset V$ $\Rightarrow \exists\, r > 0$ a.î. $B(a, r) \subset V_1 \subset V \Rightarrow V \in \mathcal{V}_a$

3) $a \in B(a, r) \subset V$

4) $B(a, r) \quad \in\ \in B(a, r)$



? $B(b, r_1) \subset B(a, r_1)$

$y \in B(b, r_1)$

$d(y,a) \leq d(y,b) + d(b,a) \leq r_1 + d(b,a) = r - d(a,b) + d(b,a) \Leftrightarrow$

$\Leftrightarrow y \in B(a, r)$

$(X, d) \quad (x_n)_n \subset X$ și $a \in X$

Def$_1$: $x_n \to a \quad \forall \varepsilon > 0 \Rightarrow \exists n_\varepsilon$ a.î. $n \geq n_\varepsilon \Rightarrow$

$\Updownarrow \quad \Rightarrow d(x_n, a) < \varepsilon \Leftrightarrow x_n \in B(a, \varepsilon)$

Def$_2$: $x_n \to a \quad \forall V \in \mathcal{V}_a \Rightarrow \exists n_V$ a.î. $\forall n \geq n_V \Rightarrow$

$\Rightarrow x_n \in V$

$D_1 \xrightarrow{\text{implică}} D_2 \quad V \in \mathcal{V}_a \Rightarrow \varepsilon_V > 0$ a.î. $B(a, \varepsilon_V) \subset V$

$\forall n \geq n_{\varepsilon_V} \Rightarrow x_n \in (a, \varepsilon_V) \subset V$

$D_2 \longrightarrow D_1 \quad \varepsilon > 0$

$B(a, \varepsilon) \in \mathcal{V}_a$

$\exists n_{B(a,\varepsilon)}$ a.î. $\forall n \geq n_{B(a,\varepsilon)} \Rightarrow x_n \in B(a, \varepsilon)$

$\Leftrightarrow d(x_n, a) < \varepsilon$

Def. Fie (X,d) un sp. metric. O mulțime D s.n. deschisă dacă $\forall a \in D \Rightarrow D \in \mathcal{V}_a \Leftrightarrow \forall a \in D \Rightarrow$

$\Rightarrow r_a > 0$ a. î. $B(a, r_a) \subset D \Leftrightarrow \forall a \in D \Rightarrow$

$\Rightarrow r_a > 0$ a. î. $D = \bigcup_{a \in D} B(a, r_a) \Leftrightarrow D = \overbrace{\bigcup_{a \in I} B(a,d)}^{\text{reuniune}}$

$\Rightarrow \mathcal{T} = \{ D \subset X \mid D \text{ deschisă} \}$ $(\mathcal{T}_d)$ topologia lui X

Obs. 1) $\emptyset, X \in \mathcal{T}_d$

2) $D_1, D_2 \in \mathcal{T}_d \Rightarrow D_1 \cap D_2 \in \mathcal{T}_d$

3) $(D_i)_{i \in I} \subset \mathcal{T}_d \Rightarrow \bigcup_{i \in I} D_i \in \mathcal{T}_d$

1) $X = \bigcup_{d \in X} B(0,1)$

2) $a \in D_1 \cap D_2 \Rightarrow r_1 > 0$ și $r_2 > 0$ a. î. $B(a, r_1) \subset D_1$ și

$B(a, r_2) \subset D$

$r = \min(r_1, r_2) > 0$

$B(a, r) \subset D_1 \cap D_2$

3) $a \in \bigcup_{i \in I} D_i \Rightarrow \exists j \in J$ $a \in D_j \in \mathcal{T}_d$

$\Rightarrow \exists r > 0$ a. î. $B(a, r) \in D_j \subset \bigcup_{i \in I} D_i$

Def. Fie X o mulțime. O mulțime $\mathcal{T} \subset \mathcal{P}(X)$ s.n. topologie dacă

1) $\emptyset, X \in \mathcal{T}$

2) $D_1, D_2 \in \mathcal{T} \Rightarrow D_1 \cap D_2 \in \mathcal{T}$

3) $(D_i)_i \subset \mathcal{T} \Rightarrow \bigcup_{i \in I} D_i \in \mathcal{T}$

O mulțime $D \in \mathcal{T}$ s.n. mulțime deschisă

O mulțime $F \subset X$ s.n. închisă dacă $X' \in \mathcal{T}$.

$\mathcal{F} = \{F \subset X / F$ închisă$\}$

O mulțime $V \subset X$ s.n. vecinătate a lui a dacă $\exists D \in \mathcal{T}_a$ a.î. $a \in D \subset V$

Propriet. vecin. ->

1) $V_1, V_2 \in \mathcal{V}_a \Rightarrow V_1 \cap V_2 \in \mathcal{V}_a$

2) $V_1 \in \mathcal{V}_a$, $V_1 \subset V \Rightarrow V \in \mathcal{V}_a$

3) $a \in V \ \forall V \in \mathcal{V}_a$

4) $\forall V \in \mathcal{V}_a \Rightarrow \exists W \subset V$ a.î. $a \in W$ și $W \in \mathcal{V}_x \ \forall x \in W$

$\mathcal{F}$: 1) $\emptyset, X \in \mathcal{F}$

2) $F_1, F_2 \in \mathcal{F} \Rightarrow F_1 \cap F_2 = \mathcal{F}$

3) $(F_i)_{i \in I} \in \mathcal{F} \Rightarrow \bigcap_{i \in I} F_i \subset \mathcal{F}$

$F_1, F_2 \in \mathcal{F} \Rightarrow X \setminus F_1, X \setminus F_2 \in \mathcal{T} \Rightarrow (X \setminus F_1) \cap (X \setminus F_2) \in \mathcal{T}$

$\Downarrow$

$X \setminus (F_1 \cup F_2) \Rightarrow$

~~$x_n \Rightarrow a$~~

$\Rightarrow F_1 \cup F_2 = \mathcal{F}$

$x_n \to a \Leftrightarrow \forall V \in \mathcal{V}_a \Rightarrow \exists n_V$ a.î. $\forall n \geq n_V \Rightarrow x_n \in V$

exemple

$(\mathbb{R}, d)$ $\quad (a, b)$

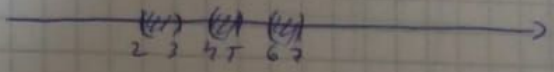
$B\left(\frac{a+b}{2} ; \frac{b-a}{2}\right)$ mulț. deschisă

$D \subset \mathbb{R}$

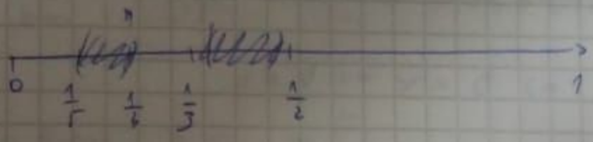
$D = \bigcup_{i \in I} (a, b)$

$\overbrace{(1, 2) \cup (3, 4)}^{desch} \in \tau$

$\bigcup_{n \geq 1} (n, n+1)$



$\bigcup_{n \geq 1} \left(\frac{1}{2n+1} ; \frac{1}{2n}\right)$



mulțimea închisă

$[1, 2] = \mathbb{R} \setminus (-\infty; 1) \cup (2; +\infty))$

$\{1\} = \mathbb{R} \setminus ((-\infty; 1) \cup (1; +\infty))$ - mulțime închisă

$\mathbb{Z} \quad \mathbb{R} \setminus \mathbb{Z} = \bigcup_{n \in \mathbb{Z}} (n, n+1) \in \tau$

Teoremă O mulțime din $\mathbb{R}$ este deschisă => este reuniunea unei familii cel mult numărabile de intervale deschise și disjuncte

$(X, \tau = \{\emptyset, X\})$

τ - topologie

$\cup$	$\emptyset$	X
$\emptyset$	$\emptyset$	X
X	X	X

$V \in \mathcal{V}_a$ dacă $\exists D \in \tau_a$ a.î. $a \in D \subset V$

$\Rightarrow D = X \Rightarrow V = X$

$\Rightarrow \mathcal{V}_a = \{X\}$

$x_n \longrightarrow a$

$x_n \in X \quad \forall n \geq 1$

Topologia indusă (X, τ)

$A \subset X$

$\tau_A = \{A \cap D \mid D \in \tau\}$

1) $\emptyset = A \cap \emptyset \quad (\emptyset \subset \tau) \quad A = A \cap X$

2) $G_1, G_2 \in \tau_A \Rightarrow D_1, D_2 = \tau_A$ a.î. $G_1 = A \cap D_1$ și $G_2 = A \cap D_2$

$G_1 \cap G_2 = A \cap \underbrace{(D_1 \cap D_2)}_{\in \tau} \in \tau_A$