Deci, $\sum_{n=1}$ ...

01.1...

CURS 5

SERII DE NR REALE

Def. O pereche de șiruri $((x_n)_{n\geq p}, (y_n)_{n\geq p})$ unde $y_n = \sum_{k=p}^{n} x_k$ s.n. serie $\sum_{n\geq p} x_n = \sum_{n=p}^{\infty} x_n$

x_n - termenii seriei

y_n - sumele parțiale

$\lim_{n\to\infty} y_n = \sum_{n\geq p} x_n$

Obs. $\sum_{n\geq 1} x_n$ ...

$\Rightarrow$ serie

y_{n+1} - ...

Obs: Dacă $\sum_{n\geq 1} x_n$ și $\sum_{n\geq 1} y_n$ sunt serii convergente $\Rightarrow$

$\Rightarrow \sum_{n\geq 1} (x_n + y_n)$ și $\sum_{n\geq 1} a \cdot x_n$ sunt convergente

și $\sum_{n\geq 1} (x_n + y_n) = \sum_{n\geq 1} x_n + \sum_{n\geq 1} y_n$, $\sum_{n\geq 1} a \cdot x_n =$

$= a \cdot \sum_{n\geq 1} x_n$

Obs.: Dacă $\sum_{n\geq 1} x_n \overset{not}{=} a \in \mathbb{R}$ (este conv.) $\Rightarrow x_n \longrightarrow 0$

$x_n = y_n - y_{n-1} \longrightarrow a - a = 0$

ex $\sum_{n\geq 1} \frac{1}{n} = \infty$ $\qquad \frac{1}{n} \to \infty$

$$\lim_{n\to\infty} \frac{1 + \frac{1}{2} + \dots + \frac{1}{n}}{\ln n} = 1$$

Obs. $\sum_{n\geq 1} x_n$ unde $x_n \geq 0$ atunci

$\Rightarrow$ seria este conv. $\Leftrightarrow$ este mărg.

$y_{n+1} - y_n = x_{n+1} \geq 0 \Rightarrow \cancel{\#}\ y_n \nearrow$

Un şir $(x_n)_n \in \mathbb{R}$ s.n. Cauchy dacă $\forall \varepsilon > 0$ $\exists n_\varepsilon$ a.î. $\forall n \geq n_\varepsilon$ $\forall p \in \mathbb{N} \Rightarrow$

$$\Rightarrow |x_{n+p} - x_n| < \varepsilon$$

<u>CRITERIUL 1</u> Cauchy O serie $\sum_{n\geq 1} x_n$ este conv $\Leftrightarrow$ $\forall \varepsilon > 0$

$$\exists n_\varepsilon \text{ a.î. } n \geq n_\varepsilon \Rightarrow \left|\sum_{k=n}^{n+p} x_k\right| < \varepsilon$$

$$\sum_{k=n}^{n+p} x_k = y_{n+p} - y_{n-1}$$

<u>Def.</u> O serie $\sum_{n\geq 1} x_n$ s.n. absolut conv. dacă seria

$\sum_{n\geq 1} |x_n|$ este conv

<u>CRITERIUL 2</u> O serie abs conv. este conv.

<u>Dem</u>: $\sum_{n\geq 1} |x_n|$ este abs. conv.

Asta înseamnă că pt. $\forall$ $\varepsilon > 0$ $\exists n_\varepsilon$ a.i. $\forall n \geq n_\varepsilon$ şi $\forall p \in \mathbb{N}$

$$\sum_{k=n}^{n+p} |x_k| < \varepsilon$$

$$\Downarrow$$

$$\geq$$

$$\left|\sum_{k=n}^{n+p} x_n\right| \xRightarrow{\text{Crit Cauchy}} \sum_{n\geq 1} x_n \text{ este conv.}$$

CRITERIUL 3 (Abel)

Dacă $a_n \searrow 0$ și $\exists M$ a.î. $\left|\sum_{k=1}^{n} x_k\right| \le M \Rightarrow$

$\Rightarrow \sum_{n\ge 1} a_n \cdot x_n$ este conv.

CRITERIUL 4 (Leibnitz - crit. serii alternante)

$x_n = (-1)^n \quad a_n \searrow 0 \qquad \sum (-1)^n \cdot a_n$ este conv.

$a_n = \frac{1}{n} \rightarrow \sum (-1)^n \cdot \frac{1}{n}$ - conv.

dar nu este abs. conv.

CRITERIUL 3 => CRITERIUL 4

$x_n = (-1)^n$

$\sum_{k=1}^{n} x_n = \sum_{k=1}^{n} (-1)^n = -1 + 1 - 1 + 1 \ldots \in \{-1; 0\}$

ex:

$\sum_{n\ge 1} \frac{1}{n} \sin nx$ conv.

$\sum_{k=1}^{n} \sin k \cdot x = \ldots$ convergentă

CRITERIUL CONDENSĂRII

$a_n \searrow 0 \Rightarrow \sum_{n\ge 1} a_n \sim \sum_{n\ge 1} 2^n a_{2^n}$

$\sum_{n\ge 1} \frac{1}{x} \sim \sum_{n\ge 1} 2^n \frac{1}{2^x}$

$$\sum_{n\geq 1} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{2} + \dots + \frac{1}{2^n+1} + \dots + \frac{1}{2^{n+1}}$$

. . .

$$A = \sum 2^n a_{2^n} \geq \sum_{n\geq 1} a_n = \sum_{n\geq 0} \left(\sum_{k=2^n+1} a_k \right)$$

$$a_2 \geq a_k \geq a_{2^{n+1}}$$

$$\cancel{2^n \geq k \geq}$$

$$2^n \leq k \leq 2^{n+1}$$

$$\sum 2^n a_{2^n} \text{ e conv.} \Rightarrow \sum 2^n a_{2^n} < \infty \Rightarrow \sum_{n\geq} a_n < \infty$$

$$\sum_{n\geq 1} \frac{1}{n^\alpha}$$

$$\lim_{n\to\infty} \frac{1}{n^\alpha} = \begin{cases} 0, & \alpha > 0 \\ 1, & \alpha \leq 0 \\ \infty, & \alpha < 0 \end{cases} \Big/ \Rightarrow \text{div}$$

$$\alpha > 0 \quad \frac{1}{n^\alpha} \downarrow 0$$

$$\sim \sum_{n\geq 1} 2^n \frac{1}{(2^n)^\alpha} = \sum_{n\geq 1} 2^{n-n\alpha} = \sum_{n\geq 1} (2^{1-\alpha})^n$$

$$\Downarrow$$

conv.

$$\sum_{n\geq 0} a^n = \begin{cases} \frac{1}{1-a}, & a < 1 \\ \infty, & a \geq 1 \end{cases} \quad a \geq 0$$

ne ajutăm de

$$\Updownarrow$$

$$2^{1-\alpha} < 1 = 2^0$$

$$\Updownarrow$$

$$\boxed{\alpha > 1}$$

$$\sum_{n\geq 2} \frac{1}{n(\ln n)^\alpha}$$

$\Downarrow$

pt. că avem log.

$$\frac{1}{n(\ln n)^\alpha} \searrow 0$$

$f(x) = x \cdot \ln x^\alpha$

pt. a studia monot

$$\Rightarrow \sum_{n\geq 2} \cancel{2^n} \cdot \frac{1}{\cancel{2^n}(\ln 2^n)^\alpha} =$$

$$= \sum_{n\geq 2} \frac{1}{n^\alpha (\ln 2)^\alpha} \sim \sum_{n\geq 2} \frac{1}{n^\alpha}$$ (pt. că $(\ln 2)^\alpha$ nu modifică)

$\Downarrow$

care e conv. $\Leftrightarrow \boxed{\alpha > 1}$

CRITERIUL 6 (comparației)

Fie $\sum_{n\geq 1} a_n$ și $\sum_{n\geq 1} b_n$ unde $a_n, b_n \geq 0$

Dacă $\exists M > 0$ și $\exists n_0$ a.î. $\forall n \geq n_0 \Rightarrow a_n < M b_n$ atunci

($b_n > 0 \Leftrightarrow \lim \frac{a_n}{b_n} < \infty$)

1) Dacă $\sum_{n\geq 1} b_n < \infty \Rightarrow \sum_{n\geq 1} a_n < \infty$

2) Dacă $\sum_{n\geq 1} a_n = \infty \Rightarrow \sum b_n = \infty$

$\underline{V2}$ $\quad 0 < \underline{\lim} \frac{a_n}{b_n} \leq \overline{\lim} \frac{a_n}{b_n} < \infty \Rightarrow \sum_{n\geq 1} a_n \sim \sum_{n\geq 1} b_n$

$\underline{V3}$ $\quad \lim_{n\to\infty} \frac{a_n}{b_n} \in (0, \infty) \to \sum_{n\geq 1} a_n \sim \sum_{n\geq 1} b_n$

$\underline{ex}$. $\sum_{n\geq 1} \underbrace{\frac{\sqrt{n}}{3n^2+1}}_{b_n} \sim \sum_{n\geq 1} \underbrace{\frac{1}{n^{\frac{3}{2}}}}_{a_n}$ conv.

$$\frac{a_n}{b_n} \to 3$$

$$\sum_{n\geq 1} \sqrt{n+1} - \sqrt{n-1} = \sum_{n\geq 1} \underbrace{\frac{2}{\sqrt{n+1} + \sqrt{n-1}}}_{a_n} \sim \sum \underbrace{\frac{1}{\sqrt{n}}}_{b_n}$$

$$\frac{a_n}{b_n} \longrightarrow \underline{1}$$

$\underline{Dem}$ $\quad \sum_{n\geq 1} a_n = \underbrace{\sum_{k=1}^{n_0-1} a_k}_{= c} + \sum_{k\geq n_0} a_k \leq -c + \sum_{k\geq n_0} M b_k \leq$

$$\leq c + M \sum_{k\geq 1} b_k$$

CRITERIUL 7 (RAPORTULUI)

Fie $\sum_{n\geq 1} a_n$ cu $a_n > 0$

① Dacă $\exists\, n_0$ și $\alpha < 1$ a.î. $\frac{a_{n+1}}{a_n} < \alpha$ pt. $\forall n \geq n_0 \Leftrightarrow$

$\left(\Leftrightarrow \overline{\lim} \frac{a_{n+1}}{a_n} < 1\right)$

$\Rightarrow \sum_{n\geq 1} a_n$ este conv. ~~$(a_n \to 0)$~~

② Dacă $\exists\, n_0$ și $\alpha > 1$ a.î. $\frac{a_{n+1}}{a_n} > \alpha$ pt. $\forall n \geq n_0 \Leftrightarrow$

$\left(\Leftrightarrow \overline{\lim} \frac{a_{n+1}}{a_n} > 1\right)$

$\Rightarrow a_n \to \infty$

$\Rightarrow \sum_{n\geq 1} a_n$ este div.

ex. $\sum_{n\geq 1} \underbrace{\frac{x^n}{n^2+1}}_{a_n}$ $x > 0$

$$\frac{a_{n+1}}{a_n} = \frac{x^{\cancel{n+1}}}{\cancel{x^n}} \cdot \underbrace{\frac{n^2+1}{(n+1)^2+1}}_{\to 1} = x$$

DACĂ

$x > 1 \Rightarrow$ serie div

$x < 1 \Rightarrow$ serie conv.

$x = 1 \Rightarrow$ crit. nu decide

$\Rightarrow \sum_{n\geq 1} \frac{1}{n^2+1} \sim \sum_{n\geq 1} \frac{1}{n^2}$ conv.

Conv: $\overline{\lim} \sqrt[n]{|a_n x|^n} = |x| \, \overline{\lim} \sqrt[n]{|a_n|} = \frac{|x|}{\rho}$

[redacted]

$\frac{|x|}{\rho} < 1 \Rightarrow$ s. conv

$\frac{|x|}{\rho} > 1$ s. div

$$\sum_{n \geq 1} \frac{x^n}{n!}$$

$$\lim_{n\to\infty} \sqrt[n]{n!} = \lim_{n\to\infty} \left(\frac{(n+1)!}{n!}\right)^{-1} = \lim_{n\to\infty} (n+1)^{-1} = 0 \Rightarrow$$

$$\Rightarrow \rho = \frac{1}{0} = \infty$$

$\sum_{n \geq 1} \frac{x^n}{n^\alpha}$ $\quad \lim_{n\to\infty} \sqrt[n]{\frac{1}{n^\alpha}} = \lim_{n\to\infty} \frac{n^\alpha}{(n+1)^\alpha} = 1 \quad \rho = 1$

$(-1;1) \subset D \subset [-1;1]$

$x = 1 \quad \sum_{n \geq 1} \frac{1}{n^\alpha}$ conv. $\Leftrightarrow \alpha > 1$

$\alpha > 1 \quad D = [-1;1]$

$$\lim_{n\to\infty} \frac{1}{n^\alpha} = \begin{cases} 0, & \alpha > 0 \\ 1 & \alpha > 0 \\ \infty & \alpha < 0 \end{cases}$$

$\alpha \leq 0 \quad \frac{1}{n^\alpha} \to 0 \quad D = (-1;1)$

$0 < \alpha \le 1$

$x = 1$ $\quad \sum \frac{1}{n^\alpha} = \infty$

$x = -1$ $\quad \sum (-1)^n \frac{1}{n^\alpha}$ conv. $\quad \frac{1}{n^\alpha} \searrow 0$

$D = [-1, 1)$

CRITERIUL 9 (Raabe - Duhamel)

$\sum_{n \ge 1} a_n > 0 \quad \ell = \lim_{n \to \infty} n \cdot \left(\frac{a_n}{a_{n+1}} - 1 \right)$

1) $\ell > 1 \Rightarrow$ s conv.

2) $\ell < 1 \Rightarrow$ s div.

$\frac{a_{n+1}}{a_n} \to 2$

$n \cdot \left(\frac{1}{2} - 1 \right) \to -\infty$

SERII CU TERMENI POZ.

PAS 1 crit rap
v radic

PAS 2 crit. comp sau crit. R-D

Serii cu termeni oarecare $\sum_{n\geq 1} x_n$

PAS 1 Studiem absolut conv.

$$\sum_{n\geq 1} |x_n|$$

PAS 2 $\lim_{n\to\infty} x_n$

Dacă $\lim_{n\to\infty} x_n \neq 0 \Rightarrow$ s. div.

PAS 3 $x_n \to 0$

~~ex~~

$\sum x_n$ este abs. conv.