CURS 11

$f^{(n)} = (f^{(n-1)})' = (f')^{(n-1)}$ $\quad (f+g)^n = f^{(n)} + g^{(n)}$

$(\alpha f)^n = \alpha \cdot f^{(n)}$

$(fg)^{(n)} = \sum_{k=0} C_n^k \cdot f^{(k)} \cdot g^{(n+k)}$

$\exists f^n$ și g^n, $\exists (g \circ f)^{(n)}$

__Def__: $s(x) = \sum a_n (x-a)^n$ $\quad D = \{ x \mid s(x)$ este conv.$\}$

$\rho' = \frac{1}{\overline{\lim} \sqrt[n]{|a_n|}}$

__T. Cauchy - Hadamard__ $(a=0)$

1) a) $\rho = 0$ $\quad D = \{0\}$

b) $\rho = \infty$ $\quad D = \mathbb{R}$

c) $\rho \in (0; +\infty) \Rightarrow (-\rho; \rho) \subset D \subset [-\rho; \rho]$

2) $R < \rho$ $(\rho > 0) \Rightarrow s$ este uniform convergentă pe $[-R; R]$

3) $s_1(x) = \sum_{n \geq 1} n \cdot a_n - x^{n-1} \Rightarrow \rho_1 = \rho$

4) $s' = s_1$, $\quad s \in C^\infty$ pe $(-\rho; \rho) = \mathring{D}$

$s^{(m)} = s_m \quad s_m(x) = \sum_{k \geq m} k(k-1) \dots (k-m+1)\, a_k x^{k-m}$

$\qquad \qquad \| $

$\qquad (a_k x^k)^{(m)}$

1) $\sum_{n \geq 0} |a_n x^n|$

crit. $\sqrt{\ }$

$\overline{\lim_{n \to \infty}} \sqrt[n]{|a_n x|^n} = |x| \overline{\lim_{n \to \infty}} \sqrt[n]{|a_n|} = \frac{|x|}{\rho}$ $\quad$ <1 s. absolut conv. $\quad >1$ s. div.

$\frac{|x|}{\rho} < 1 \Rightarrow x \in (-\rho; \rho)$ abs. conv.

$\frac{|x|}{\rho} > 1 \Rightarrow x \in (-\infty; -\rho) \cup (\rho; +\infty)$ div.

2) Fie x a.î. $|x| < R < \rho$

$\overline{\lim} \sqrt[n]{|a_n x^n|} = \frac{|x|}{\rho} \leq \frac{R}{\rho} < 1$

$\alpha \in (\frac{R}{\rho}; 1)$

$\exists\, n_\alpha$ a.î. $\forall n \geq n_\alpha$ $\sqrt[n]{|a_n x^n|} \leq \alpha \Leftrightarrow |a_n x^n| \leq \alpha^n$

pe $[-R; R]$ $\frac{a_n x^n}{\alpha^n}$ este mărginită $\Rightarrow \exists\, c$ a.î.

$|a_n x^n| \leq c \cdot \alpha^n \quad \forall n$

$\sum_{n\geq 1} \alpha^n$ este conv. $\Rightarrow \sum_{n\geq 1} a_n \cdot x^n$ este uniform conv. pe $[-R; R]$

T. Fie $f_n, g \; (a, b) \to \mathbb{R}$ a.î. 1) $f_n' \xrightarrow[\text{și}]{u} g$ a.î.
2) $\exists\, c \in (a; b)$

$(f_n(c))_n$ să fie conv. $\Rightarrow \exists\, f: (a; b) \to \mathbb{R}$ a.î.

1) $f' = g$
și
2) $f_n \xrightarrow{u} f$ $\quad (a, b \in \mathbb{R})$

T. Fie $f_n: (a, b) \to \mathbb{R}$ derivabile a.î. $s = \sum_{n\geq 1} f_n$ să fie conv.

și $s_1 = \sum_{n\geq 1} f_n'$ să fie u.c. $\Rightarrow s' = s_1$

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3) $\rho_1 = \rho$

$$\rho_1 = \frac{1}{\overline{\lim} \sqrt[n]{n|a_n|}} = \frac{1}{\underbrace{\lim_{n\to\infty} \sqrt[n]{n}}_{=1} \ \overline{\lim} \sqrt[n]{|a_n|}} = \rho$$

$$\left(s_1 = \sum_{n\geq 1} a_n \cdot n x^{n-1}\right)$$

4) $\forall R<\rho$ pe $(-R;R)$

s și s_1 sunt u.c. $\Rightarrow s' = s_1$

Analog $s_1' = s_2$

$$s^{(m)} = \sum_{k\geq m} (a_k x^k)^{(m)} = \sum_{k\geq m} a_k (k-1) \dots (k-m+1) x^{k-m}$$

$$s^{(n)}(0) = n! \cdot a_n$$

<u>Exemplu</u> $s(x) = \sum_{n\geq 0} \frac{x^n}{n!}$ $\quad \rho = \infty \quad \lim_{n\to\infty} \sqrt[n]{n!} = \lim_{n\to\infty} \frac{(n-1)!}{n!} = \infty$

$\Rightarrow D = \mathbb{R}$

$$s_1(x) = \sum_{n\geq 1} \frac{n \cdot x^{n-1}}{n\cdot(n-1)!} = \sum_{n\geq 0} \frac{x^n}{n!} = s(x) \Rightarrow \boxed{s' = s} \Rightarrow$$

$\Rightarrow s = c\, e^x$

$g = s\cdot e^{-x} \quad g' = \underbrace{s'}_{=s} \cdot e^{-x} = s\cdot e^{-x} = 0 \Rightarrow g = c \Rightarrow$

$\Rightarrow s = c\cdot e^x$

$f:(a,b) \to \mathbb{R}$ deriv. în $c \Rightarrow f(x) = f(c) + f'(c)(x-c) +$
$+ (x-c)\,\omega(x)$

$\lim\limits_{x \to 0} \omega(x) = 0$

$\exists f''(c)$

$$f(x) = \underbrace{f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2}_{\text{polinomul Taylor asociat } f_{,2,c}(x)} + (x-c)^2\omega(x)$$

$\lim\limits_{x \to c} \omega(x) = 0$

Def. Fie $f:(a,b) \to \mathbb{R}$ derivabilă de $(n-1)$ ori pe $(a;b)$ și de n ori în c

S.n. polinomul Taylor de ordin n asociat funcției f în c

$$T_{f,n,c}(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k =$$

$$= f(c) + f'(c)(x-c) + \frac{f'(c)}{2!}(x-c)^2 + \dots + \frac{f^{(n)}(c)}{n!}(x-c)^n$$

T. (Taylor 1). Fie $f:(a,b) \to \mathbb{R}$ derivabilă de $(n-1)$ ori pe $(a;b)$ și de n ori în punctul c. Atunci $\exists\, \omega:(a,b) \to \mathbb{R}$ a.î. $f(x) = T_{f,n,c}(x) +$
$+ (x-c)^n\omega(x)$ și $\lim\limits_{x \to c} \omega(x) = 0$.

$$\lim_{x \to 0} \omega(x) = \lim_{x \to 0} \frac{f(x) - \sum\limits_{k=0}^{n} \frac{f^{(k)}(c)}{k!} \cdot (x-c)^k}{(x-c)^n} \overset{LH}{=}$$

$$= \lim_{x \to c} \frac{f'(x) - \sum\limits_{k=1}^{n} \frac{f^{(k-1)}(c)}{(k-1)!} \cdot (x-c)^{(k-1)}}{n \cdot (x-c)^{n-1}}$$

T. Taylor — Fie $f:(a,b)\to\mathbb{R}$ derivabilă de $(n+1)$ ori pe (a,b) și $c\in(a,b)$ $\forall x\in(a,b)$ $\exists\alpha$ între c și x a.î.

$$f(x) = T_{f,n,c}(x) + \frac{f^{(n+1)}(\alpha)}{(n+1)!}(x-c)^{n+1}$$

$(n=0 \quad f(x)=f(c)+f'(\alpha)(x-c) \quad$ T.L.)

$f(x)=e^x \cdot c=0 \quad f:\mathbb{R}\to\mathbb{R}$

$f^{(n)}=f \quad f^{(n)}(0)=e^0=1$

$$T_{f,n,c}(x)=\sum_{k=0}^{n}\frac{f^{(k)}(0)}{k!}x^k=\sum_{k=0}^{n}\frac{x^k}{k!}$$

$$e^x=f(x)=\sum_{k=0}^{n}\frac{x^k}{k!}+\frac{e^\alpha}{(n+1)!}x^{n+1}$$

$$e^x=\sum\frac{x^n}{n!} \qquad R_{f,n,c}(x) \qquad |x|\le M$$

$$\left|\frac{e^\alpha\cdot x^{n+1}}{(n+1)!}\right|\le\frac{e^M\cdot M^{n+1}}{(n+1)!}=a_n$$

$$\frac{a_{n+1}}{a_n}\to 0$$

Def: $f:(a,b)\to\mathbb{R}$ $\quad f=(f_1,\dots,f_n)$

f este der în c $\Rightarrow \lim\limits_{x\to c}\dfrac{f(x)-f(c)}{x-c} \Leftrightarrow \exists\lim\limits_{x\to c}\dfrac{f_i(x)-f_i(c)}{x-c}\in\mathbb{R}$, $\forall i=\overline{1,n}$

$\Leftrightarrow$

$\Leftrightarrow \quad \exists f_i'(c)$

Exemplu. $f:\mathbb{R}\to\mathbb{R}^3$

$$f(x)=(e^{3x}, x^4, \sin x)$$

$$f'(x)=(3e^{3x}, 4x^3, \cos x)$$

$f:\mathbb{R}^2\to\mathbb{R}^2$

$$f(x,y)=e^{2x}\cdot y^3$$

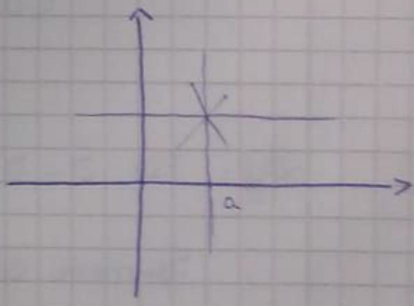
$$\frac{\partial f}{\partial x}(x,y)=2e^{2x}\cdot y^3$$

$$\frac{\partial f}{\partial y}(x,y)=e^{2x}\cdot 3y^2$$

$\cancel{\partial f}$

$$\frac{\partial f}{\partial x}(a,b)=\lim_{x\to a}\frac{f(x,b)-f(a,b)}{x-a}$$

$$\frac{\partial f}{\partial y}(a,b)=\lim_{y\to b}\frac{f(a,y)-f(a,b)}{y-b}$$



$0\neq v=(m,n)\in\mathbb{R}^2$

$$\frac{\partial f}{\partial v}(c)=\lim_{t\to 0}\frac{f(c+t\cdot v)-f(c)}{t}, \quad c\in(a;b)$$

$t\to c+tv$

Caz particular

$v = e_1 = (0,1) \qquad \longleftarrow \longrightarrow tv = (a+t, b)$

$$\frac{\partial f}{\partial e_1}(a,b) = \lim_{t\to 0} \frac{f(a+t,b) - f(a,b)}{t} = \lim_{x\to a} \frac{f(x,b) - f(a,b)}{x-a} = \frac{\partial f}{\partial x}(a,b)$$

$$\frac{\partial f}{\partial v}(a,b) = \lim_{t\to 0} \frac{f(a+tv, b+tv) - f(a,b)}{t} =$$

$f(x,y) = e^{2x} \cdot y^3$

$$= \lim_{t\to 0} \frac{e^{2(a+tv)}(b+tv)^3 + e^{2a}b^3}{t} =$$

$$= \lim_{t\to 0} \cancel{2m} \cdot e^{2(a+tm)}(b+tn)^3 + e^{2(a+tv)} + 3n \cdot (b+tn)^2$$

$$= m \cdot 2e^{2a} b^3 + n \cdot 3e^{2a} \cdot b^2 = m \cdot \frac{\partial f}{\partial x}(a,b) =$$

$$= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)\begin{pmatrix} m \\ n \end{pmatrix}$$

Def: Fie $\mathring{D} = D \subset \mathbb{R}^m$, $f : D \to \mathbb{R}^n$ şi $a \in D$

Spunem că f este derivabilă în a dacă $\exists\, T \in L(\mathbb{R}^m, \mathbb{R}^n)$

a. î. $\displaystyle\lim_{x\to 0} \frac{f(x) \cdot f(a) - T(x-a)}{d_2(x,a)} = 0 \iff$

$\iff f(x) = f(a) + T(x-a) + d_2(x,L)\, w(x)$

$$\lim_{x\to a} w(x) = 0$$

$$f'(c) = \lim_{x\to c} \frac{f(a) - f(c)}{a-c}$$

$$0 = \lim_{x\to c} \frac{f(x) - f(c) - f'(c)(x-c)}{|x-c| = d(x,c)}$$

$$f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$f(x,y) = \begin{pmatrix} x^3 y \\ e^{2x} \cdot y^2 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

$$f' = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} = \cancel{\begin{pmatrix} 3x^2 \cdot y^3 & 2e^{2x} y^2 \end{pmatrix}}$$

$$= \begin{pmatrix} 3x^2 y & x^3 \\ 2e^{2x} \cdot y^2 & 2e^{2x} y \end{pmatrix}$$