CURS 13

10.01.2019

Def. Fie $D = \mathring{D} \subset \mathbb{R}^n$, $f : D \longrightarrow \mathbb{R}^m$, $a \in D$ și $u, v \in \mathbb{R}^n \setminus \{0\}$

$$\frac{\partial^2 f}{\partial u \partial v}(a) = \frac{\partial}{\partial v}\left(\frac{\partial f}{\partial u}(a)\right)$$

Ex: $f : \mathbb{R}^2 \to \mathbb{R}$, ~~$f(x)=$~~ $f(x,y,z) = x^3y^2 + y^3z^2$

$$\frac{\partial f}{\partial x} = 3x^2y^2$$

$$\frac{\partial f}{\partial y} = 2x^3y + 3y^2z^2$$

$$\frac{\partial f}{\partial z} = 2y^3z$$

Dacă $u = v$ $\quad \frac{\partial^2 f}{\partial u \partial v} = \frac{\partial^2 f}{\partial u^2}$

$$\frac{\partial^2 f}{\partial x^2} = 6xy^2$$

$$\frac{\partial^2 f}{\partial y \partial x} = 6x^2y$$

$$\frac{\partial^2 f}{\partial z \partial x} = 0$$

$$\frac{\partial^2 f}{\partial x \partial y} = 6x^2y$$

$$\frac{\partial^2 f}{\partial y^2} = 2x^3 + 6yz^2$$

$$\frac{\partial^2 f}{\partial z \partial x} = 6y^2z$$

$$\frac{\partial^2 f}{\partial x \partial z} = 0.$$

$$\frac{\partial^2 f}{\partial y \partial z} = 6y^2z$$

$$\frac{\partial^2 f}{\partial z^2} = 2y^3$$

$g: \mathbb{R}^3 \to \mathbb{R}^2$

$$f' = \left(\frac{\partial f}{\partial x} \quad \frac{\partial f}{\partial y} \quad \frac{\partial f}{\partial z}\right) = g = (g_1 \quad g_2 \quad g_3)$$

$$f'' = g' = \begin{pmatrix} \frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial y} & \frac{\partial g_1}{\partial z} \\ \frac{\partial g_2}{\partial x} & \frac{\partial g_2}{\partial y} & \frac{\partial g_2}{\partial z} \\ \frac{\partial g_3}{\partial x} & \frac{\partial g_3}{\partial y} & \frac{\partial g_3}{\partial z} \end{pmatrix} = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial z \partial y} \\ \frac{\partial^2 f}{xy} & \frac{\partial^2 f}{\partial y^2} & \frac{\partial^2 f}{\partial z \partial y} \\ \frac{\partial^2 f}{\partial x \partial z} & \frac{\partial^2 f}{\partial y \partial z} & \frac{\partial^2 f}{\partial z^2} \end{pmatrix}$$

$$f'' = \begin{pmatrix} 6xy^2 & 6x^2y & 0 \\ 6x^2y & 2x^3+6yz^2 & 6y^2z \\ 0 & 6y^2z & 2y^3 \end{pmatrix}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

<u>Teoremă</u> (Young)

Fie $D = \overset{\circ}{D} \subset \mathbb{R}^n$, $f: D \to \mathbb{R}^m$ a.î. $\exists f'$ pe D și $\exists f''(a)$

<u>Atunci</u> $\forall u, v \in \mathbb{R}^n \setminus \{0\} \Rightarrow \frac{\partial^2 f}{\partial v \partial u}(a) = \frac{\partial^2 f}{\partial u \partial v}(a) = f''(a)(u,v) =$

$= f''(a)(v,u)$

<u>Teoremă</u> (Schwartz)

Fie $D = \overset{\circ}{D} \subset \mathbb{R}^n$, $a \in D$, $f: D \to \mathbb{R}^m$ și $u, v \in \mathbb{R}^n \setminus \{0\}$.
Dacă $\exists \frac{\partial f}{\partial u}$ și $\frac{\partial f}{\partial v}$ pe D și $\exists \frac{\partial^2 f}{\partial u \partial v}$ pe D și este

cont. în $a \Rightarrow \exists \frac{\partial^2 f}{\partial v \partial u}(a) = \frac{\partial^2 f}{\partial u \partial v}(a)$

Ex: Fie $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$, $f(x,y) = e^{ax+by}$

$$\frac{\partial f}{\partial x} = a \cdot e^{ax+by}$$

$$\frac{\partial^2 f}{\partial x^2} = a^2 \cdot e^{ax+by}$$

$$\frac{\partial^n f}{\partial x^n} = a^n \cdot e^{ax+by}$$

$$\frac{\partial^{n+1} f}{\partial y \partial x^n} = a^n \cdot b \cdot e^{ax+by}$$

$$\frac{\partial^{n+m} f}{\partial^n y \partial^m x} = a^n \cdot b^m \cdot e^{ax+by} = \frac{\partial^{n+m} f}{\partial x^n \partial y^m}$$

$$\frac{\partial^8 f}{\partial x^3 \partial y^2 \partial x^2 \partial y} = a^5 b^3 e^{ax+by}$$

Def: $T: \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^m$ s.n. biliniară dacă

1) $T(x_1+x_2, y) = T(x_1, y) + T(x_2, y) \quad \forall x_1, x_2 \in \mathbb{R}^n$

2) $T(x, y_1+y_2) = T(x, y_1) + T(x y_2)$

3) $T(ax, y) = a \cdot T(x,y) = T(x, ay)$

T. s.n. asimetrică dacă $T(x,y) = T(y,x)$

Ex: $\langle , \rangle : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i = \langle y, x \rangle$$

$$x = y \quad \langle x, x \rangle = \sum_{i=1}^{n} x_i^2 = \|x\|_2^2 = d_2(x, 0_n)$$

$A \in \mathcal{M}_{m,n}(\mathbb{R}) \quad T_A : \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^m \quad T_A(x,y) = \langle Ax, y \rangle$ - biliniară

$e_1 = (1, 0, 0 \ldots 0)$
$e_2 = (0, 1, 0 \ldots 0)$

$T: \mathbb{R}^m \times \mathbb{R}^m \longrightarrow \mathbb{R}$ biliniară

$x, y \in \mathbb{R}^m \qquad x = (x_1 \ x_2 \ldots x_m) = \sum_{i=1}^{n} x_i e_i$

$\left(\underline{ex}: \ (a, b) = a \cdot \underset{e_1}{\underset{\shortparallel}{(1, 0)}} + b \cdot \underset{e_2}{\underset{\shortparallel}{(0, 1)}} \right)$

$$T(x, y) = T\left(\sum_{i=1}^{m} x_i e_i, \sum_{j=1}^{m} y_j e_j\right) = \sum_{i=1}^{m} x_i T(e_i, y) =$$

$$= \sum_{i=1}^{n} x_i y_j \cdot \underset{a_{ij}}{\underset{\shortparallel}{T(e_i, e_j)}} = \langle Ax, y \rangle$$

$$\langle x, y \rangle = x \ y^t = (x_1 \ldots x_m) \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}$$

$$\langle Ax, y \rangle = Ax \cdot y^t = \begin{pmatrix} \sum_{j=1}^{n} a_{1j} \cdot x_j \\ \vdots \\ \sum_{j=1}^{n} a_{mj} \cdot x_j \end{pmatrix}^t \cdot \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}$$

$$= \sum_{i=1}^{n} \left(\sum_{j=1}^{m} a_{ij} \cdot x_i \right) y_j$$

$L_2(\mathbb{R}^n, \mathbb{R}^n; \mathbb{R}^m) = \{ T: \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^m \mid T \text{ biliniară} \}$

$f: D = \mathring{D} \subset \mathbb{R}^n \qquad f': D \rightarrow L(\mathbb{R}^n; \mathbb{R}^m)$

$f'': D \rightarrow L(\mathbb{R}^n, L(\mathbb{R}^n; \mathbb{R}^m)) \approx L_2(\mathbb{R}^n, \mathbb{R}^n; \mathbb{R}^m)$

$T \in L_2 \rightarrow (x \rightarrow \overset{T_x}{T(x, \cdot)})$

$\underset{y}{\overset{\shortparallel}{L(\mathbb{R}^n; \mathbb{R}^m)}}$

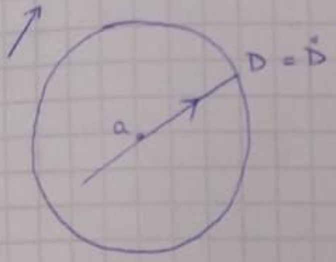
$T_x: \mathbb{R}^n \longrightarrow \mathbb{R}^m$

$T_x(y) = T(x, y)$

$T_{x_1 + x_2} = T_{x_1} + T_{x_2}$

T. Fermat

Fie $D = \mathring{D} \subset \mathbb{R}^m$, $f : D \to \mathbb{R}$ și $a \in D$ a.î. $\exists\, f'(a)$ și a să fie punct de extrem local. Atunci $f'(a) = 0$.



$v \in \mathbb{R}^m \setminus \{0\}$

$d_v\ (t \to a + tv)$

$f|d_v \quad \frac{\partial f}{\partial v}(a) = 0.$

a este un punct de extrem local pt f/d_v

Def: $A = A^t \in \mathcal{M}_{m,m}(\mathbb{R})$ – s.m. pozitivă dacă $\langle Ax, x \rangle \geq 0$ $\forall x \in \mathbb{R}^m$

– s.m. strict poz. dacă $\exists\ \varepsilon > 0$ a.î.

$\langle Ax, x \rangle \geq \varepsilon \cdot \langle x, x \rangle$

$A \leq 0 \iff -A \geq 0$

$$D = \begin{pmatrix} a_1 & & 0 \\ & a_2 & \\ 0 & & a_n \end{pmatrix} \qquad \langle Dx, x \rangle = \sum_{i=1}^{m} a_i \cdot x_i^2 \geq 0 \quad \forall x \iff$$

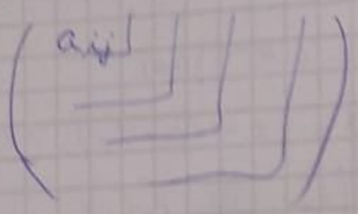
$\iff a_i \geq 0 \quad \forall i$

a_i – valorile proprii

$\det (A - \lambda I) = 0$

Pr. $A = (a_{ij})_{i,j=\overline{1,n}} = A^T$ este pozitivă $\iff$

$\det (a_{ij})_{i,j=\overline{1,k}} > 0 \quad \forall k=\overline{1,n}$



I. Fie $D = \mathring{D}$ ~~...~~ $f: D \to \mathbb{R}$, $a \in D$ a.î. $\exists f'$ pe D și $f''(a)$. Atunci

1) Dacă a este punct de minim local $\Rightarrow f'(a)=0$ și $f''(a) \geq$

2) Dacă $f'(a)=0$ și $f''(a)>0 \Rightarrow a$ este punct de minim local

3) Dacă a este punct de maxim local $\Rightarrow f'(a)=0$ și $f''(a) \leq$

4) Dacă $f'(a)=0$ și $f''(a)<0 \Rightarrow a$ este minim local

Dem când $n=1$

$f:(b,c) \to \mathbb{R}$ și $a \in (b,c)$

$\exists f''(a) \Rightarrow f(x) = f(a) + f'(a) + \frac{f''(a)}{2}(x-a)^2 + (x-a)^2\omega(x)$

$\lim\limits_{x\to a} \omega(x) = 0$

a minim local $\overset{\text{T. Fermat}}{\Longrightarrow} f'(a)=0$. $\underbrace{f(x)-f(a)}_{\geq 0} = (x-a)^2 \overbrace{\left(\frac{f''(a)}{2} + \omega(x)\right)}^{\geq 0}$

$\Rightarrow \frac{f''(a)}{2} + \omega(x) \geq 0 \quad \forall x, \ x \to a$

$\frac{f''(a)}{2} \geq 0$

2

Ex. $f: \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = x^2 + y^2$

~~$f(x) = x^2 + y^2$~~

$f(x,y) \geq 0 = f(0,0) \Rightarrow 0$ minim global

$f' = (2x\ 2y) = 0 \Rightarrow x = y = 0$

val. pr. 2 și 2

$f''_{(x,y)} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} > 0$

$\Delta_1 = 2 > 0$

$\Delta_2 = 4 > 0$

$+ + \Rightarrow$ punct de minim

Ex2 $f: \mathbb{R}^3 \to \mathbb{R}$ $f(x,y,z) = -x^2 + x \cdot y - y^2 - z^2$

$f' = (-2x + y,\ x - 2y,\ 2z) = 0$

$\begin{cases} -2x + y = 0 \\ x - 2y = 0 \\ z = 0 \end{cases}$

$\begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix} = 3 \neq 0 \Rightarrow x = y = z = 0$

$f'' = \begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix} -$

$\Delta_1 = -2$

$\Delta_2 = 3$

$\Delta_3 = -6$

$- + - \Rightarrow$ punct de maxim

<u>Ex 3</u> $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ $\quad f(x,y) = x^2 - y^2$

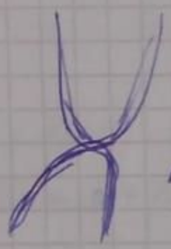
$$f' = \left(\frac{\partial f}{\partial x} \;\; \frac{\partial f}{\partial y} \right) = (2x \quad -2y) = 0 \Rightarrow x = y = 0$$

$$f'' = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$$

val. proprii $2 \quad -2$

$\Delta_1 = 2 > 0$

$\Delta_2 = -4 < 0$



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