$\boxed{\text{SEM 5}}$ Aplicatii liniare. Ker f, Im f.
Proiectii. simetrii

Ex. $f:\mathbb{R}^2 \to \mathbb{R}^2$, $f(x) = (2x_1 - x_2, x_1 + x_2)$.

a) $A = [f]_{R_0, R_0}$. $\qquad R_0 = \{e_1 = (1,0), e_2 = (0,1)\}$

b) $A' = [f]_{R', R'}$ $\qquad R' = \{e_1' = e_1 + e_2, e_2' = e_1 - e_2\}$.

c) $V = \{x \in \mathbb{R}^2 \mid x_1 - x_2 = 0\}$

$f(V) = ?$

SOL

a) $A = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$ $\qquad f(x) = y \Leftrightarrow Y = AX$.

$$\begin{pmatrix} 2x_1 - x_2 \\ x_1 + x_2 \end{pmatrix} = A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

sau $f(e_1) = a_{11} e_1 + a_{21} e_2 \Rightarrow f(1,0) = (2,1) = 2e_1 + 1 \cdot e_2$

$f(e_2) = a_{12} e_1 + a_{22} e_2 \Rightarrow f(0,1) = (-1,1) = -e_1 + e_2$.

b) (M1)

$$\begin{tikzcd} R_0 \arrow[r, "A_f"] \arrow[d, "C"'] & R_0 \arrow[d, "C"] \\ R_1 \arrow[r, "A'_f"] & R_1 \end{tikzcd}$$

$A' = C^{-1} A C$ $\qquad ; C = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$\begin{cases} e_1' = e_1 + e_2 \\ e_2' = e_1 - e_2. \end{cases}$ $\qquad e_i' = \sum_{j=1}^{2} c_{ji} e_j, i = \overline{1,2}$

$C^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$; $\det C = -2$; $C^* = \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}$

$$A' = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} & \frac{3}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{pmatrix}$$

(M2) $f(e_1') = f(1,1) = (1,2) = a e_1' + b e_2' = a(1,1) + b(1,-1) = (a+b, a-b)$

$\begin{cases} a + b = 1 \\ a - b = 2 \end{cases} \Rightarrow \begin{cases} a = \frac{3}{2} \\ b = -\frac{1}{2} \end{cases}$

$f(e_2') = f(1,-1) = (3,0) = c e_1' + d e_2' = (c+d, c-d)$

$\begin{cases} c + d = 1 \\ c - d = 0 \end{cases}$ $\qquad \begin{cases} c = \frac{3}{2} \\ d = \frac{3}{2} \end{cases}$

$$A' = \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} \frac{3}{2} & \frac{3}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{pmatrix}$$

c) $V = \{(x_1, x_1) \mid x_1 \in \mathbb{R}\} = \langle \{(1,1)\} \rangle$

$f(1,1) = (1,2)$

$f(V) = \langle \{(1,2)\} \rangle$.

Ex. $f: \mathbb{R}^3 \to \mathbb{R}^3$, $f(x) = (2x_1 + 2x_2, x_1 + x_3, x_1 + 3x_2 - 2x_3)$

a) f nu e izomorfism.

b) $V' = \{x \in \mathbb{R}^3 \mid x_1 + x_2 - x_3 = 0\}$

$V'' = \{x \in \mathbb{R}^3 \mid 3x_1 - 4x_2 - 2x_3 = 0\}$

$f/_{V'} : V' \to V''$ este izomorfism.

c) $f(V' \cap V'')$

SOL

a) $A = [f]_{\mathcal{R}_0, \mathcal{R}_0} = \begin{pmatrix} 2 & 2 & 0 \\ 1 & 0 & 1 \\ 1 & 3 & -2 \end{pmatrix}$

$\det A = \begin{vmatrix} 2 & 0 & 0 \\ 1 & -1 & 1 \\ 1 & 2 & -2 \end{vmatrix} = 0 \Rightarrow f$ nu e bijecție

b) $\dim V' = \dim V'' = 2$.

$V' = \{(x_1, x_2, x_1 + x_2) \mid x_1, x_2 \in \mathbb{R}\} = \langle\{(1,0,1),(0,1,1)\}\rangle$.

$\mathcal{R}' = \{(1,0,1),(0,1,1)\}$ reper în V'

($\dim V' = |\mathcal{R}'| = 2$ și $\mathcal{R}'$ este SG al lui V')

$f(\mathcal{R}') = \{\underbrace{f(1,0,1)}_{(2,2,-1)}, \underbrace{f(0,1,1)}_{(2,1,1)}\}$

$(2,2,-1) \in V''$: $3 \cdot 2 - 4 \cdot 2 - 2(-1) = 6 - 8 + 2 = 0$

$(2,1,1) \in V''$: $3 \cdot 2 - 4 \cdot 1 - 2 \cdot 1 = 6 - 4 - 2 = 0$.

$\operatorname{rg}\begin{pmatrix} 2 & 2 \\ 2 & 1 \\ -1 & 1 \end{pmatrix} = 2 = \max \xRightarrow{\text{crit LI}} f(\mathcal{R}') = \{(2,2,-1),(2,1,1)\}$ SLI în V''

$\dim V'' = |f(\mathcal{R}')| = 2$

$\Rightarrow f(\mathcal{R}')$ = reper în V''

$f/_{V'}$ duce reper în reper $\Rightarrow f/_{V'} : V' \to V'$ izomorfism.

c) $V' \cap V'' = \left\{x \in \mathbb{R}^3 \;\middle|\; \begin{cases} x_1 + x_2 - x_3 = 0 \\ 3x_1 - 4x_2 - 2x_3 = 0 \end{cases}\right.$ $\quad \begin{cases} x_1 + x_2 = x_3 \\ 3x_1 - 4x_2 = 2x_3 \end{cases} \Big| 4$

$7x_1 = 6x_3 \Rightarrow x_1 = \frac{6}{7}x_3$; $x_2 = \frac{1}{7}x_3$.

$V' \cap V'' = \langle\{(6,1,7)\}\rangle$; $f(6,1,7) = (14,13,-5)$

$f(V' \cap V'') = \langle\{(14,13,-5)\}\rangle$.

c) $V' = \{x \in \mathbb{R}^3 \mid \begin{cases} x_1 - x_2 + x_3 = 0 \\ x_1 + 2x_2 - x_3 = 0 \end{cases}\}$.

$rg\begin{pmatrix} 1 & -1 & 1 \\ 1 & 2 & -1 \end{pmatrix} = 2$

$\dim V' = 3 - 2 = 1.$

$\begin{cases} x_1 - x_2 = -x_3 \\ x_1 + 2x_2 = x_3 \end{cases} \Big| \cdot (-1) \quad \oplus$

$/ \quad 3x_2 = 2x_3 \Rightarrow x_2 = \frac{2}{3}x_3$

$x_1 = \frac{2}{3}x_3 - x_3 = -\frac{1}{3}x_3$.

$V' = \{(-\frac{1}{3}x_3, \frac{2}{3}x_3, x_3) \mid x_3 \in \mathbb{R}\} = <\{(-1,2,3)\}>$.

$f(-1,2,3) = (-1+4+3, -6, -1+2+3) = (6,-6,4)$

$f(V') = <\{(6,-6,4)\}>$.

Ex. $(\mathbb{R}^3, +, \cdot)/\mathbb{R}$, $V' = <\{(-1,2,3)\}>$.

$\mathbb{R}^3 = V' \oplus V''$

p = proiecția pe V' de-a lungul lui V''

s = simetria față de V'

Să se determine a) $p(0,1,1)$

b) $s(0,1,1)$

SOL

$p: V' \oplus V'' \longrightarrow V' \oplus V''$, $p(v' + v'') = v'$

$s \in End(\mathbb{R}^3)$ $\quad s = 2p - id_{\mathbb{R}^3}$.

$V'' = <\{(1,0,0),(0,0,1)\}>$. $\det\begin{pmatrix} -1 & 1 & 0 \\ 2 & 0 & 0 \\ 3 & 0 & 1 \end{pmatrix} \neq 0$.

$v = (0,1,1) = a(-1,2,3) + b(1,0,0) + c(0,0,1)$

$= (-a+b, 2a, 3a+c) \Rightarrow \begin{cases} -a+b = 0 \\ 2a = 1 \\ 3a + c = 1 \end{cases}$

$a = \frac{1}{2}, b = \frac{1}{2}, c = 1 - \frac{3}{2} = -\frac{1}{2}$.

$v = (0,1,1) = \underbrace{\frac{1}{2}(-1,2,3)}_{v'} + \underbrace{\frac{1}{2}(1,0,0) - \frac{1}{2}(0,0,1)}_{v''}$

$p(0,1,1) = (-\frac{1}{2}, 1, \frac{3}{2})$ $\quad s(0,1,1) = (-1,2,3) - (0,1,1)$

$= (-1,1,2)$

Ex $f:\mathbb{R}^3\to\mathbb{R}^3$, $f(x)=(x_1+2x_2+x_3,\ -x_1-2x_2-x_3,\ x_1+x_2+x_3)$

a) $A=[f]_{R_0,R_0}$.

b) Ker f, Im f (reper si sist. de ecuatii)

c) $V'=\left\{x\in\mathbb{R}^3 \;\middle|\; \begin{cases} x_1-2x_2+x_3=0\\ x_1+2x_2-x_3=0\end{cases}\right.$

$f(V')=?$

SOL

a) $A=\begin{pmatrix}1 & 2 & 1\\ -1 & -2 & -1\\ 1 & 1 & 1\end{pmatrix}$

b) $\operatorname{Ker} f=\{x\in\mathbb{R}^3 \mid AX=0\}$.

$\dim \operatorname{Ker} f = 3-\operatorname{rg}(A)=3-2=1$.

$$\begin{cases}-x_1-2x_2=x_3\\ x_1+x_2=-x_3 \quad \mid 2\end{cases} \quad \oplus$$

$$x_1 \quad / \quad = -x_3 \implies x_2=0$$

$\operatorname{Ker} f=\{(-x_3,0,x_3)\mid x_3\in\mathbb{R}\}=\langle\{(-1,0,1)\}\rangle$

M1 Completăm la un reper în $\mathbb{R}^3$

$\det\begin{pmatrix}-1 & 0 & 0\\ 0 & 0 & 1\\ 1 & 1 & 0\end{pmatrix}\neq 0$ $\{(-1,0,1),(0,0,1),(0,1,0)\}$ reper în $\mathbb{R}^3$

$\{f(0,0,1), f(0,1,0)\}$ reper în Im f (Th dim. pt. apl. liniare C5)

$f(0,0,1)=(1,-1,1)$; $f(0,1,0)=(2,-2,1)$.

$\operatorname{Im} f=\{a(1,-1,1)+b(2,-2,1)\mid a,b\in\mathbb{R}\}=$

$=\{(a+2b,-a-2b,a+b)\mid a,b\in\mathbb{R}\}$

$\begin{cases}a+2b=y_1\\ -a-2b=y_2\\ a+b=y_3\end{cases}$ $\left(\begin{array}{cc|c}1 & 2 & y_1\\ -1 & -2 & y_2\\ 1 & 1 & y_3\end{array}\right)$ $\Delta_c=\begin{vmatrix}1 & 2 & y_1\\ -1 & -2 & y_2\\ 1 & 1 & y_3\end{vmatrix}=0$

$\operatorname{Im} f=\{y\in\mathbb{R}^3\mid y_1+y_2=0\}$.

M2 $\operatorname{Im} f=\{y\in\mathbb{R}^3\mid \exists x\in\mathbb{R}^3 \text{ a.î. } f(x)=y\}$ $\begin{cases}x_1+2x_2+x_3=y_1\\ -x_1-2x_2-x_3=y_2\\ x_1+x_2+x_3=y_3\end{cases}$

$=\{y\in\mathbb{R}^3\mid y_1+y_2=0\}$

$=\{(y_1,-y_1,y_3)\mid y_1,y_3\in\mathbb{R}\}=\langle\{(1,-1,0),(0,0,1)\}\rangle$

$\dim \operatorname{Im} f=3-\dim\operatorname{Ker} f=2$ $\{(1,-1,0),(0,0,1)\}$ reper în Im f