

Introduction to Computer Graphics

GAMES101, Lingqi Yan, UC Santa Barbara

Lecture 2: Review of Linear Algebra



Announcements

- Slides and recordings of Lecture 1 now available
- (Pre)-reading materials will be out before lectures

日期	主题
第1周	Feb 11 计算机图形学概述 [课件][录像]
	Feb 14 向量与线性代数 阅读材料：第 2 章 (Miscellaneous Math) , 第 5 章 (Linear Algebra)

- Happy Valentine's Day!

Last Lecture

- What is Computer Graphics?
- Why study Computer Graphics?
- Course Topics
- Course Logistics

A **Swift** and **Brutal** Introduction to Linear Algebra!

(in fact it's relatively easy...)

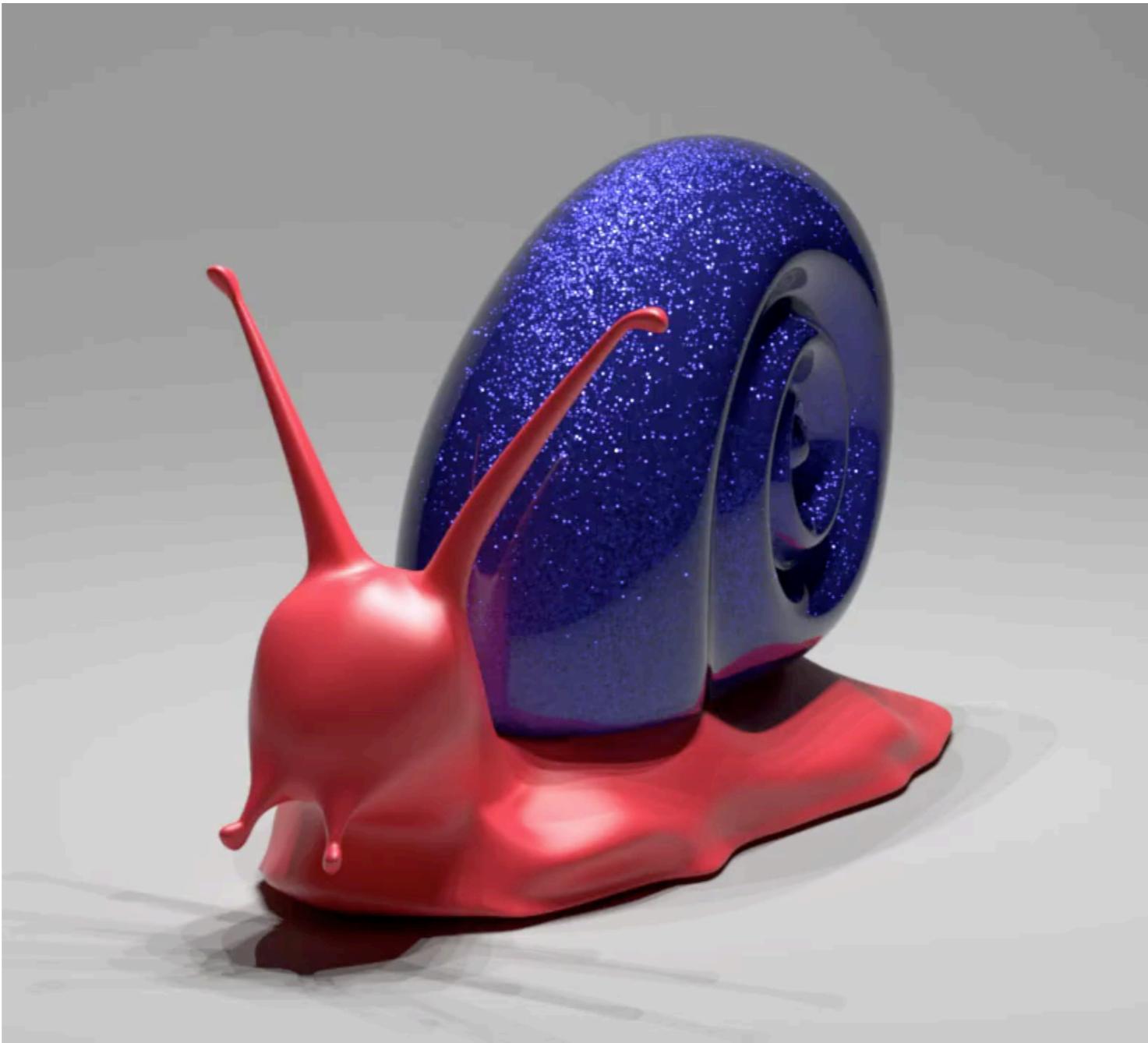
Graphics' Dependencies

- Basic mathematics
 - Linear algebra, calculus, statistics
- Basic physics
 - Optics, Mechanics
- Misc
 - Signal processing
 - Numerical analysis
- And a bit of aesthetics

This Course

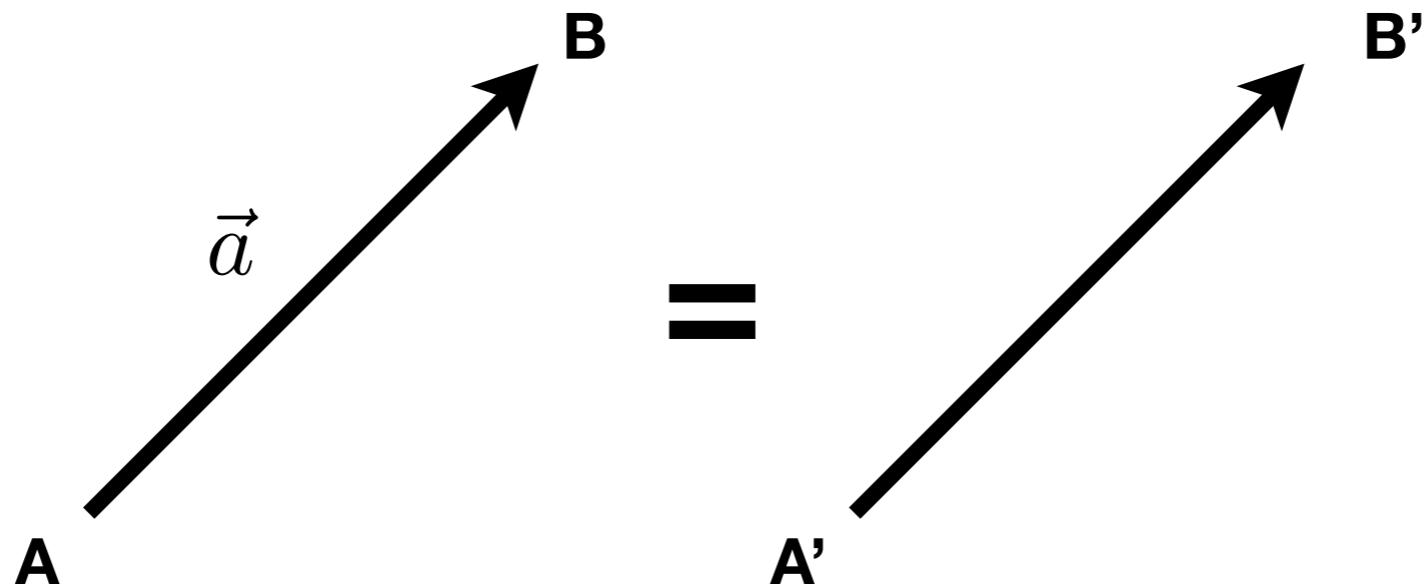
- More dependent on Linear Algebra
 - Vectors (dot products, cross products, ...)
 - Matrices (matrix-matrix, matrix-vector mult., ...)
- For example,
 - A point is a vector (?)
 - An operation like translating or rotating objects can be matrix-vector multiplication

An Example of Rotation



Rendering Glints on High-Resolution Normal-Mapped Specular Surfaces, Lingqi Yan, 2014

Vectors

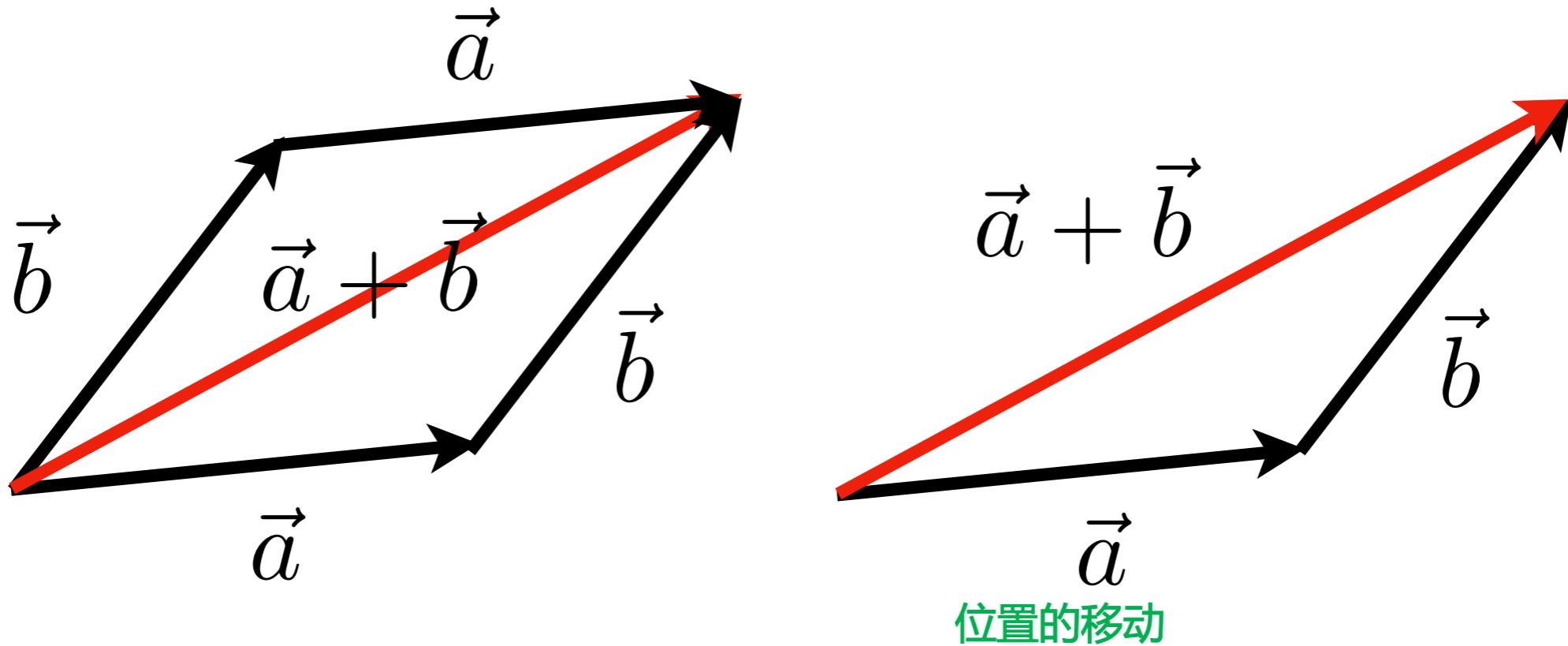


- Usually written as \vec{a} or in bold a
- Or using start and end points $\overrightarrow{AB} = B - A$
- Direction and length
- No absolute starting position

Vector Normalization

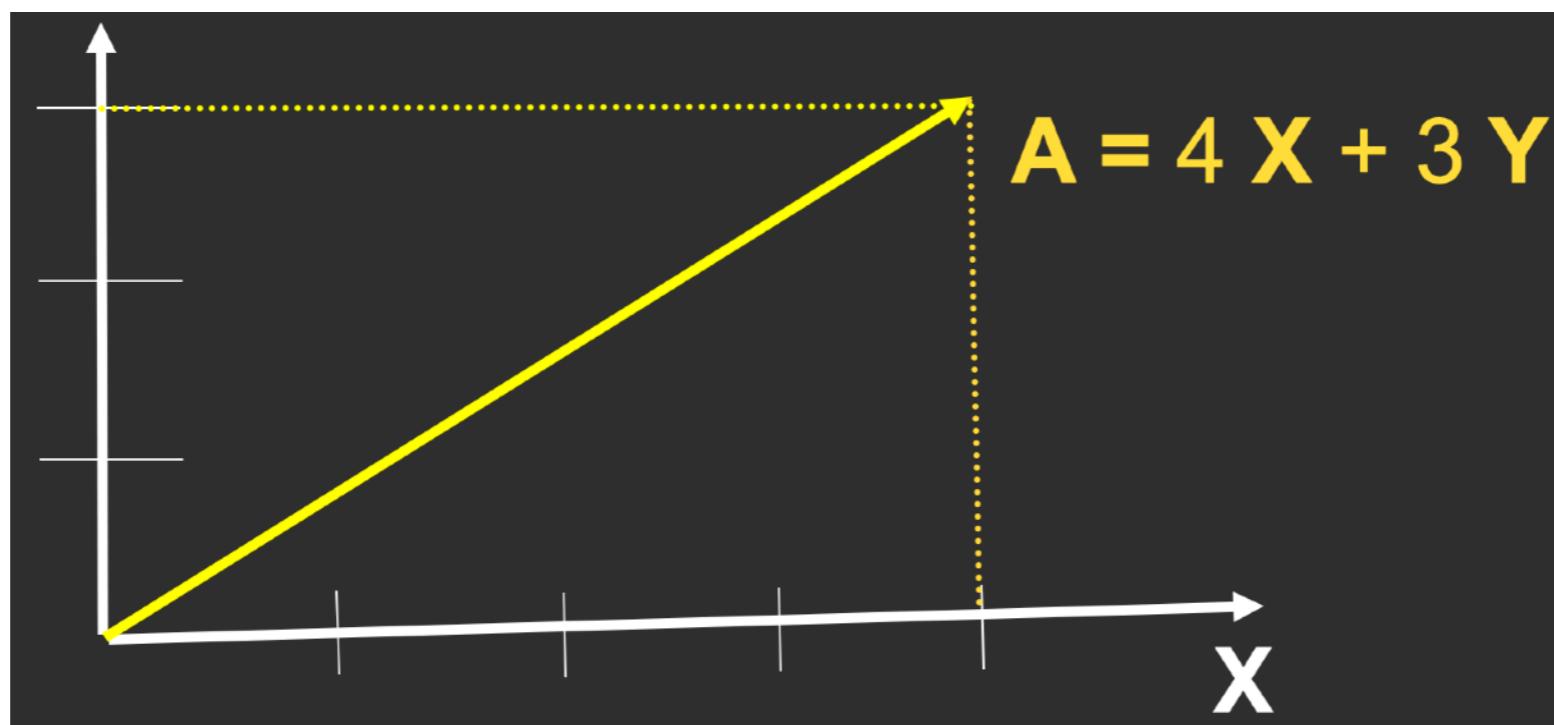
- Magnitude (length) of a vector written as $\|\vec{a}\|$
- Unit vector
 - A vector with magnitude of 1
 - Finding the unit vector of a vector (normalization): $\hat{a} = \vec{a}/\|\vec{a}\|$
 - Used to represent directions

Vector Addition



- Geometrically: Parallelogram law & Triangle law
- Algebraically: Simply add coordinates

Cartesian Coordinates



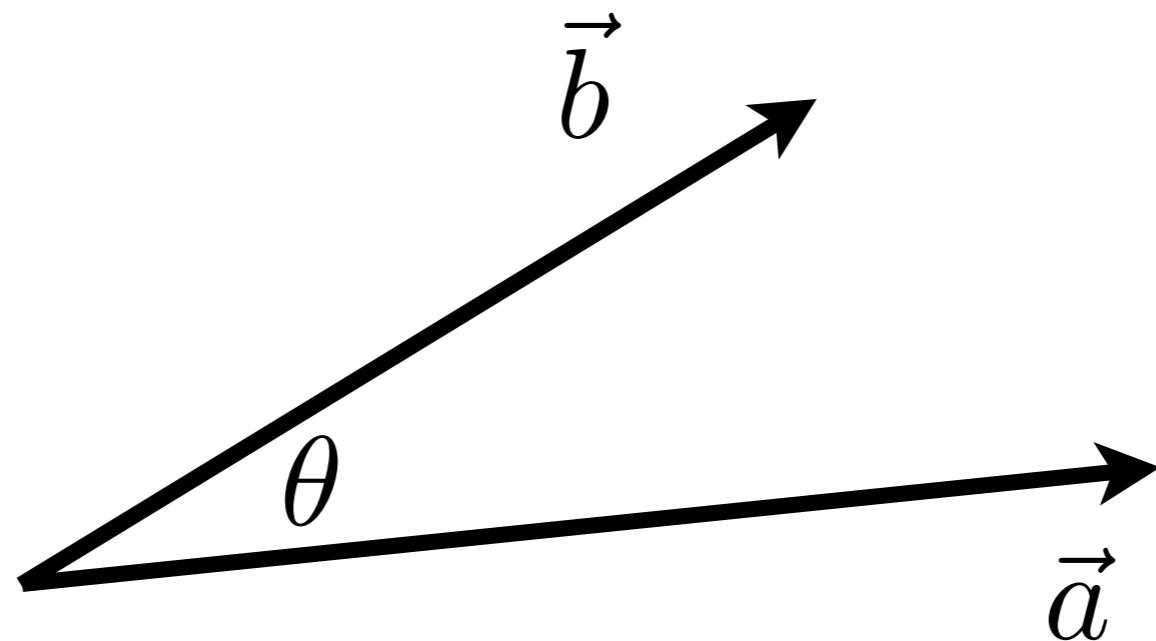
- X and Y can be any (usually **orthogonal unit**) vectors

$$\mathbf{A} = \begin{pmatrix} x \\ y \end{pmatrix} \quad \mathbf{A}^T = (x, y) \quad \|\mathbf{A}\| = \sqrt{x^2 + y^2}$$

Vector Multiplication

- Dot product
- Cross product
- Orthonormal bases and coordinate frames

Dot (scalar) Product



$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos \theta$$

- For unit vectors

$$\cos \theta = \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\| \|\vec{b}\|}$$

$$\cos \theta = \hat{a} \cdot \hat{b}$$

计算两个向量之间的夹角
按段两个向量之间的位置关系
投影

Dot (scalar) Product

- Properties

$$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$$

$$\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$$

$$(k\vec{a}) \cdot \vec{b} = \vec{a} \cdot (k\vec{b}) = k(\vec{a} \cdot \vec{b})$$

Dot Product in Cartesian Coordinates

- Component-wise multiplication, then adding up

- In 2D

$$\vec{a} \cdot \vec{b} = \begin{pmatrix} x_a \\ y_a \end{pmatrix} \cdot \begin{pmatrix} x_b \\ y_b \end{pmatrix} = x_a x_b + y_a y_b$$

- In 3D

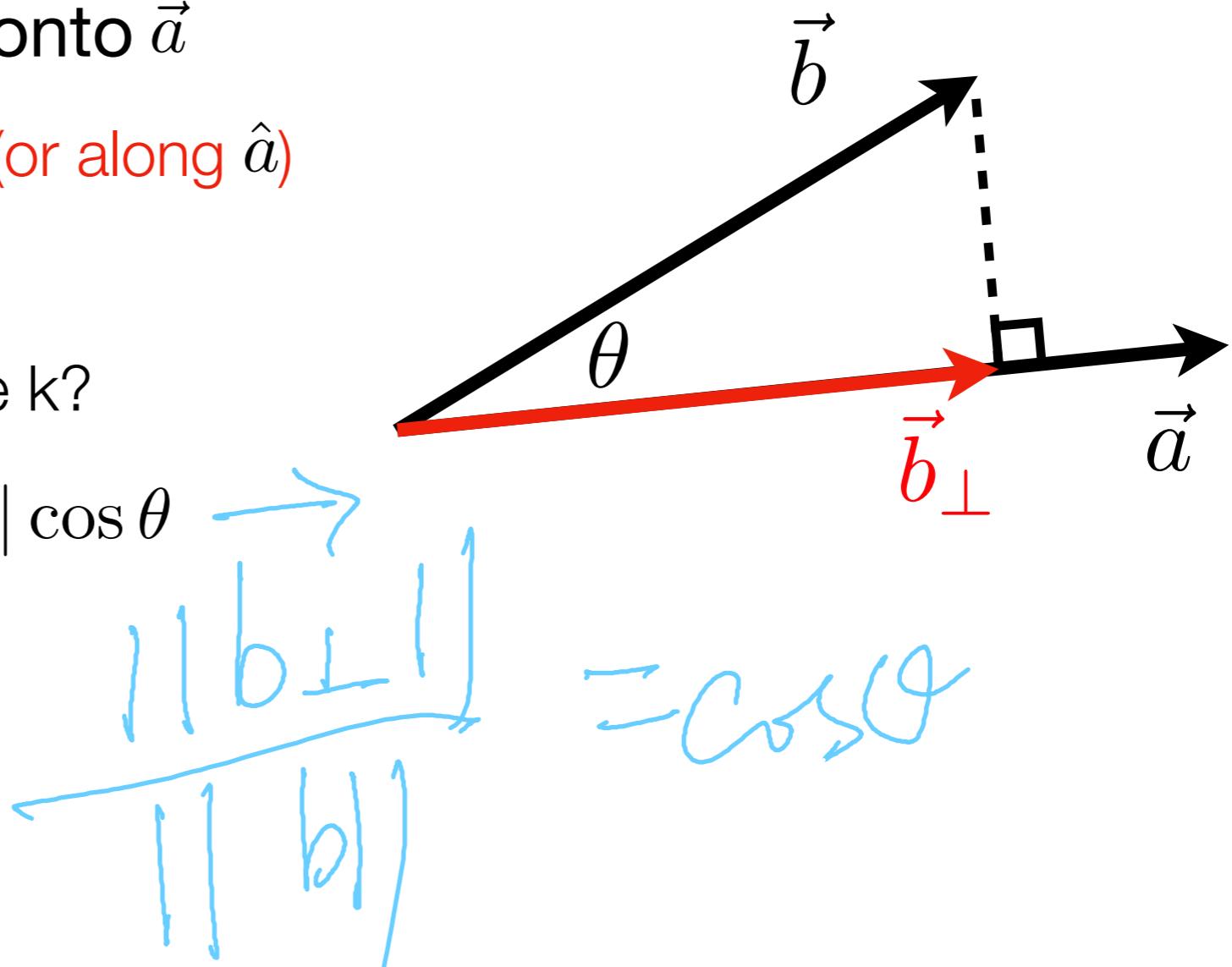
$$\vec{a} \cdot \vec{b} = \begin{pmatrix} x_a \\ y_a \\ z_a \end{pmatrix} \cdot \begin{pmatrix} x_b \\ y_b \\ z_b \end{pmatrix} = x_a x_b + y_a y_b + z_a z_b$$

Dot Product in Graphics

- Find angle between two vectors
(e.g. cosine of angle between light source and surface)
- Finding **projection** of one vector on another

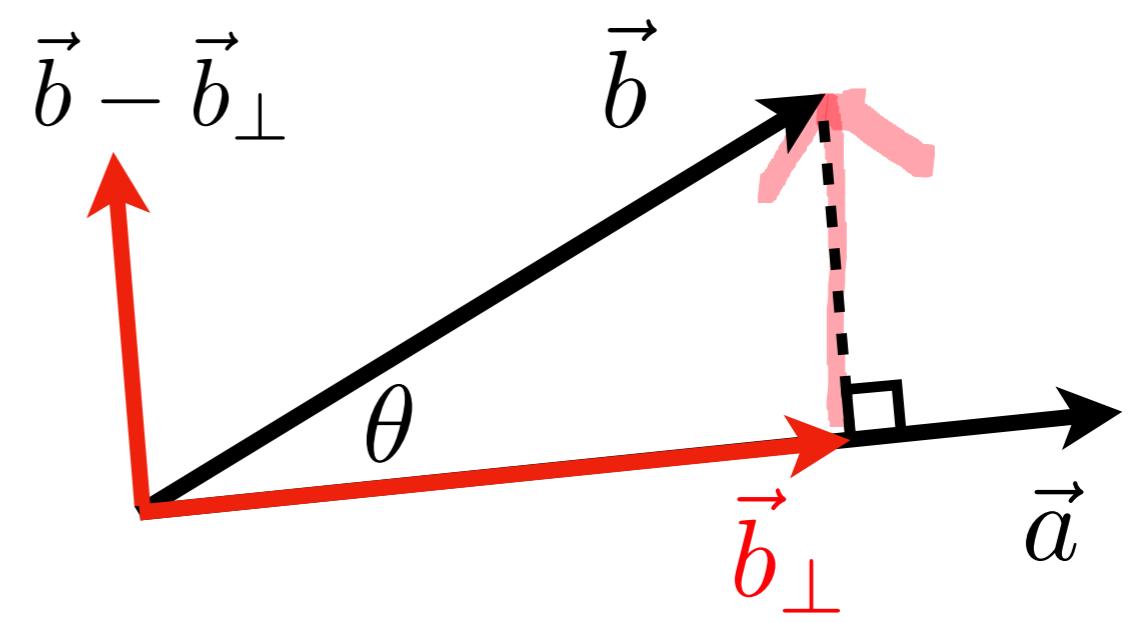
Dot Product for Projection

- \vec{b}_\perp : projection of \vec{b} onto \vec{a}
 - \vec{b}_\perp must be along \vec{a} (or along \hat{a})
 - $\vec{b}_\perp = k\hat{a}$
 - What's its magnitude k ?
 - $k = \|\vec{b}_\perp\| = \|\vec{b}\| \cos \theta$



Dot Product in Graphics

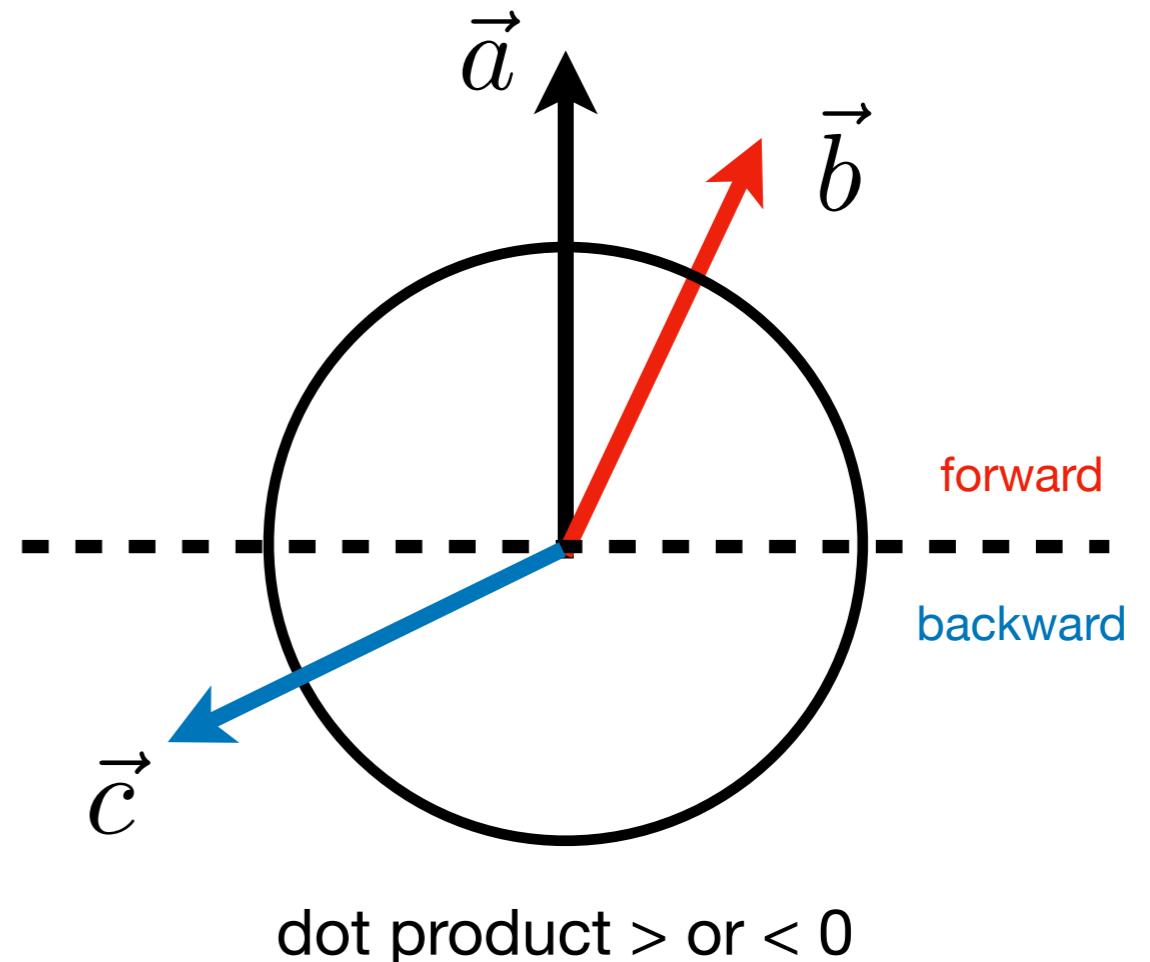
- Measure how close two directions are
- Decompose a vector
- Determine forward / backward



Dot Product in Graphics

- Measure how close two directions are
- Decompose a vector
- Determine forward / backward

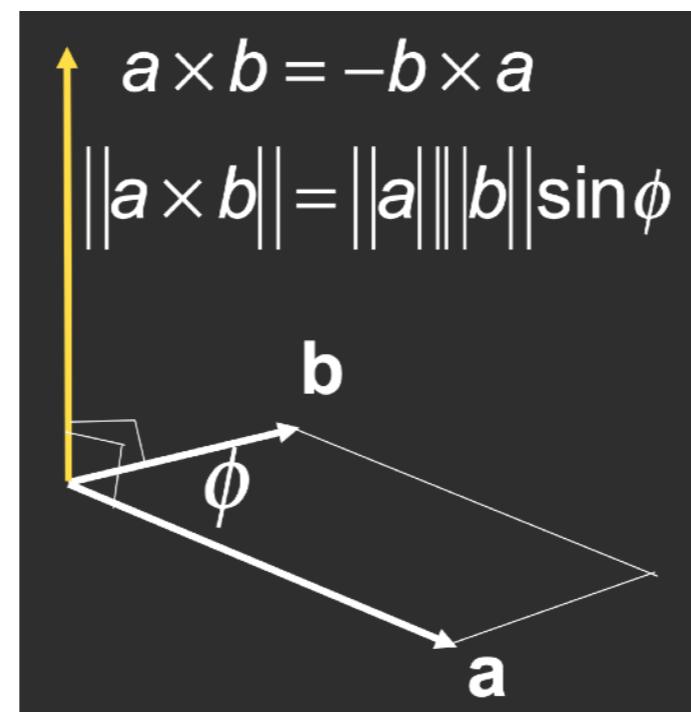
方向一致 点积为1
方向垂直 点积为0
方向相反 点积为-1



Vector Multiplication

- Dot product
- Cross product
- Orthonormal bases and coordinate frames

Cross (vector) Product



通过右手定则判断，
如 \mathbf{a} 叉乘 \mathbf{b} ，四指与 \mathbf{a} 向量平行，四指朝 \mathbf{b} 方向弯曲，大拇指的朝向就是叉积结果的方向

- Cross product is orthogonal to two initial vectors
- Direction determined by right-hand rule
- Useful in constructing coordinate systems (later)

Cross product: Properties

通过两个坐标轴的叉积获得其他坐标轴

$$\vec{x} \times \vec{y} = +\vec{z}$$

$$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$$

$$\vec{y} \times \vec{x} = -\vec{z}$$

$$\vec{y} \times \vec{z} = +\vec{x}$$

$$\vec{a} \times \vec{a} = \vec{0}$$

$\sin 0=0$
两向量夹角为0

$$\vec{z} \times \vec{y} = -\vec{x}$$

$$\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$$

$$\vec{z} \times \vec{x} = +\vec{y}$$

$$\vec{a} \times (k\vec{b}) = k(\vec{a} \times \vec{b})$$

$$\vec{x} \times \vec{z} = -\vec{y}$$

Cross Product: Cartesian Formula?

$$\vec{a} \times \vec{b} = \begin{pmatrix} y_a z_b - y_b z_a \\ z_a x_b - x_a z_b \\ x_a y_b - y_a x_b \end{pmatrix}$$

- Later in this lecture

$$\vec{a} \times \vec{b} = A^* b = \begin{pmatrix} 0 & -z_a & y_a \\ z_a & 0 & -x_a \\ -y_a & x_a & 0 \end{pmatrix} \begin{pmatrix} x_b \\ y_b \\ z_b \end{pmatrix}$$

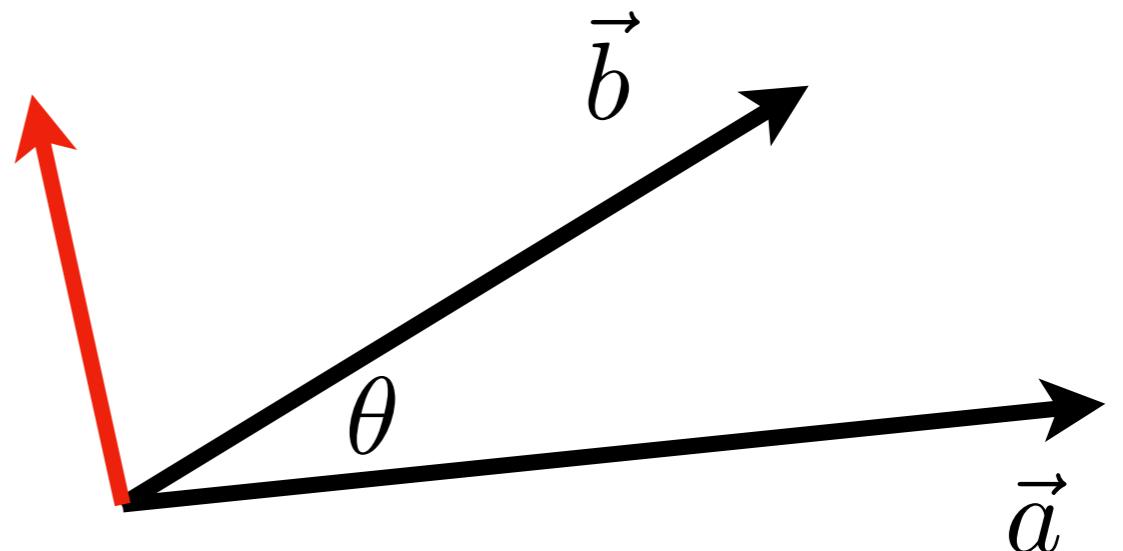
叉积矩阵形式

dual matrix of vector a

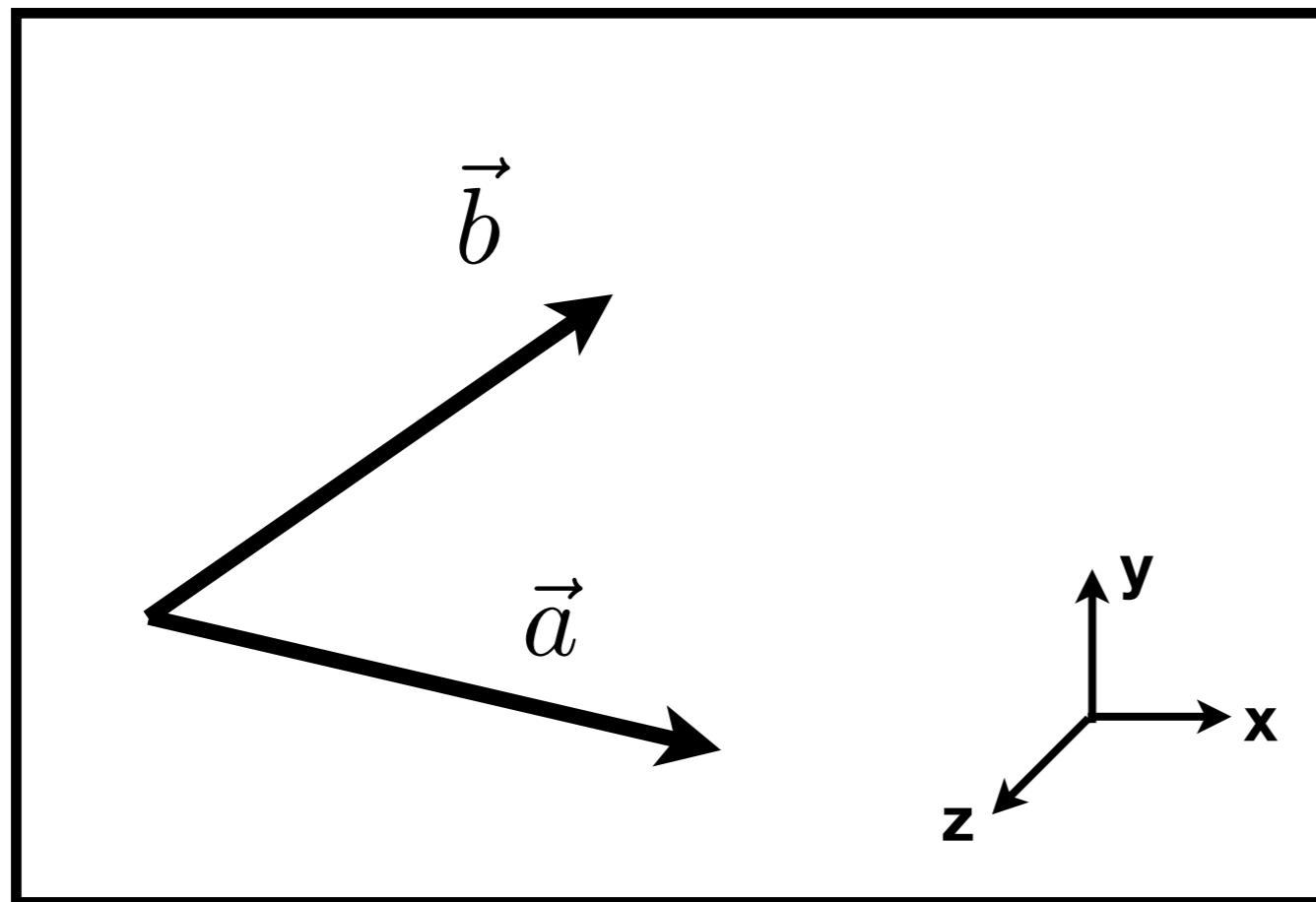
Cross Product in Graphics

- Determine left / right
- Determine **inside / outside**

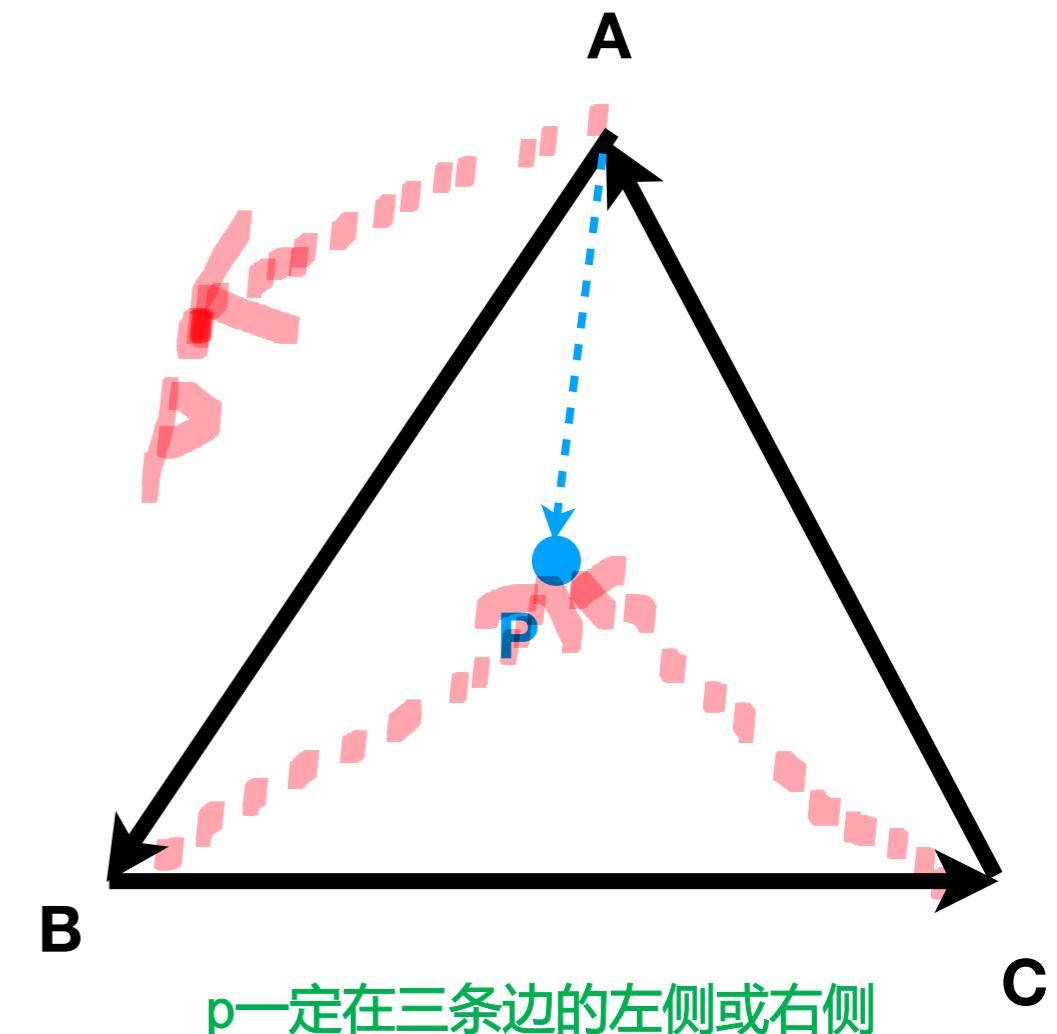
判断左右和内外



Cross Product in Graphics



$\vec{a} \times \vec{b}$ 叉积结果为正， \vec{a} 在 \vec{b} 的右侧
 $\vec{b} \times \vec{a}$ 叉积结果为负， \vec{b} 在 \vec{a} 的左侧



p一定在三条边的左侧或右侧

Vector Multiplication

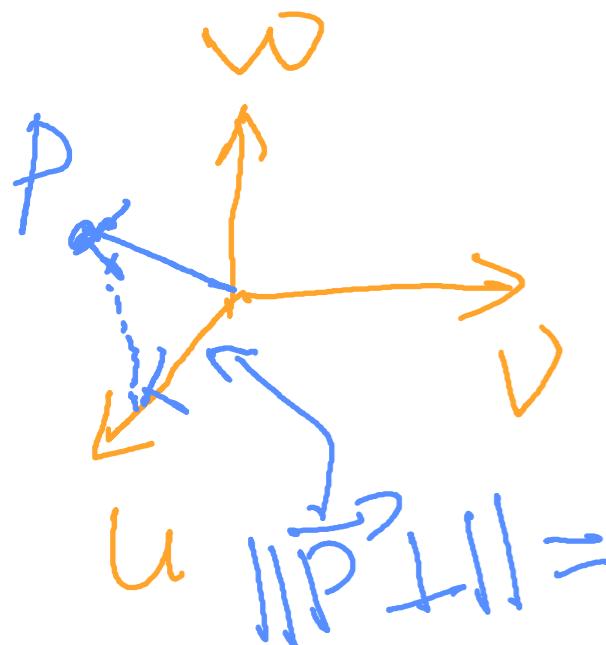
- Dot product
- Cross product
- Orthonormal bases and coordinate frames

Orthonormal Bases / Coordinate Frames

- Important for representing points, positions, locations
- Often, many sets of coordinate systems
 - Global, local, world, model, parts of model (head, hands, ...)
- Critical issue is transforming between these systems/bases
 - A topic for next week

Orthonormal Coordinate Frames

- Any set of 3 vectors (in 3D) that



$$\|\vec{u}\| = \|\vec{v}\| = \|\vec{w}\| = 1$$

$$\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{w} = \vec{u} \cdot \vec{w} = 0$$

$$u \quad \|\vec{P}\| =$$

$$\vec{w} = \vec{u} \times \vec{v} \quad (\text{right-handed})$$

$$\|\vec{P}\| \cdot \cos\theta$$

$$\vec{p} = (\vec{p} \cdot \vec{u})\vec{u} + (\vec{p} \cdot \vec{v})\vec{v} + (\vec{p} \cdot \vec{w})\vec{w}$$

$$\vec{p} \cdot \vec{u} = \|\vec{P}\| \cdot \|\vec{u}\| \cdot \cos\theta = \|\vec{P}\| \cdot \cos\theta = \|\vec{P}_{\perp}\|$$

(projection)

Questions?

Matrices

- Magical 2D arrays that haunt in every CS course
- In Graphics, pervasively used to represent **transformations**
 - Translation, rotation, shear, scale
(more details in the next lecture)



What is a matrix

- Array of numbers ($m \times n = m$ rows, n columns)

$$\begin{pmatrix} 1 & 3 \\ 5 & 2 \\ 0 & 4 \end{pmatrix}$$

- Addition and multiplication by a scalar are trivial:
element by element

Matrix-Matrix Multiplication

- # (number of) columns in A must = # rows in B
 $(M \times N) (N \times P) = (M \times P)$

$$\begin{pmatrix} 1 & 3 \\ 5 & 2 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} 3 & 6 & 9 & 4 \\ 2 & 7 & 8 & 3 \end{pmatrix}$$

Matrix-Matrix Multiplication

- # (number of) columns in A must = # rows in B
 $(M \times N) (N \times P) = (M \times P)$

$$\begin{pmatrix} 1 & 3 \\ 5 & 2 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} 3 & 6 & 9 & 4 \\ 2 & 7 & 8 & 3 \end{pmatrix} = \begin{pmatrix} 9 & ? & 33 & 13 \\ 19 & 44 & 61 & 26 \\ 8 & 28 & 32 & ? \end{pmatrix}$$

- Element (i, j) in the product is
the dot product of **row i from A** and **column j from B**

Matrix-Matrix Multiplication

- Properties
 - **Non-commutative**
(AB and BA are different in general)
 - Associative and distributive
 - $(AB)C = A(BC)$
 - $A(B+C) = AB + AC$
 - $(A+B)C = AC + BC$

Matrix-Vector Multiplication

- Treat vector as a column matrix ($m \times 1$)
- Key for transforming points (next lecture)
- Official spoiler: 2D reflection about y-axis

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ y \end{pmatrix}$$

Transpose of a Matrix

- Switch rows and columns ($ij \rightarrow ji$)

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}^T = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}$$

- Property

$$(AB)^T = B^T A^T$$

Identity Matrix and Inverses

$$I_{3 \times 3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$AA^{-1} = A^{-1}A = I$$

$$(AB)^{-1} = B^{-1}A^{-1}$$

Vector multiplication in Matrix form

- Dot product?

$$\vec{a} \cdot \vec{b} = \vec{a}^T \vec{b}$$
$$= (x_a \quad y_a \quad z_a) \begin{pmatrix} x_b \\ y_b \\ z_b \end{pmatrix} = (x_a x_b + y_a y_b + z_a z_b)$$

- Cross product?

$$\vec{a} \times \vec{b} = A^* b = \begin{pmatrix} 0 & -z_a & y_a \\ z_a & 0 & -x_a \\ -y_a & x_a & 0 \end{pmatrix} \begin{pmatrix} x_b \\ y_b \\ z_b \end{pmatrix}$$

dual matrix of vector a

An Example of General Transformation



The Sponza Scene, rendered by Lingqi Yan using Real-time Ray Tracing (RRTT)

Questions?

Next

- Transform!



Transformers: The Last Knight, 2017 movie

Thank you!

(And thank Prof. Ravi Ramamoorthi and Prof. Ren Ng for many of the slides!)