

Introduction to Computer Graphics

AMES101, Lingqi Yan, UC Santa Barbara

Lecture 3: Transformation



Announcements

- Assignment 0 will be released soon
- Do not register the homework submission system with QQ email

Last Lecture

- Vectors
 - Basic operations: addition, multiplication
- Dot Product
 - Forward / backward (dot product positive / negative)
- Cross Product
 - Left / right (cross product outward / inward)
- Matrices

This Week

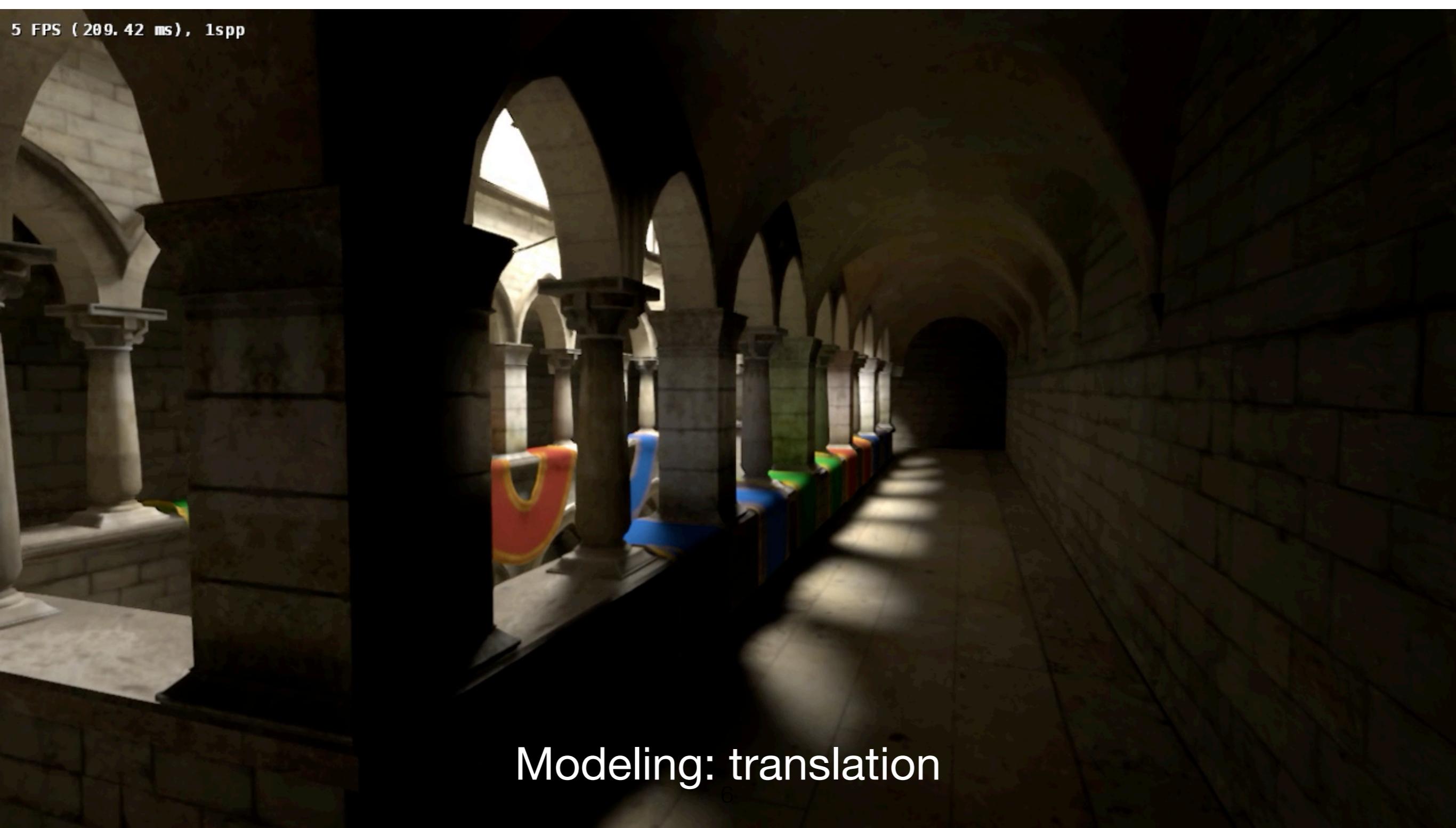
- Transformation!
- Today
 - Why study transformation
 - 2D transformations: rotation, scale, shear
 - Homogeneous coordinates
 - Composing transforms
 - 3D transformations

Today

- Why study transformation
 - Modeling
 - Viewing
- 2D transformations
- Homogeneous coordinates

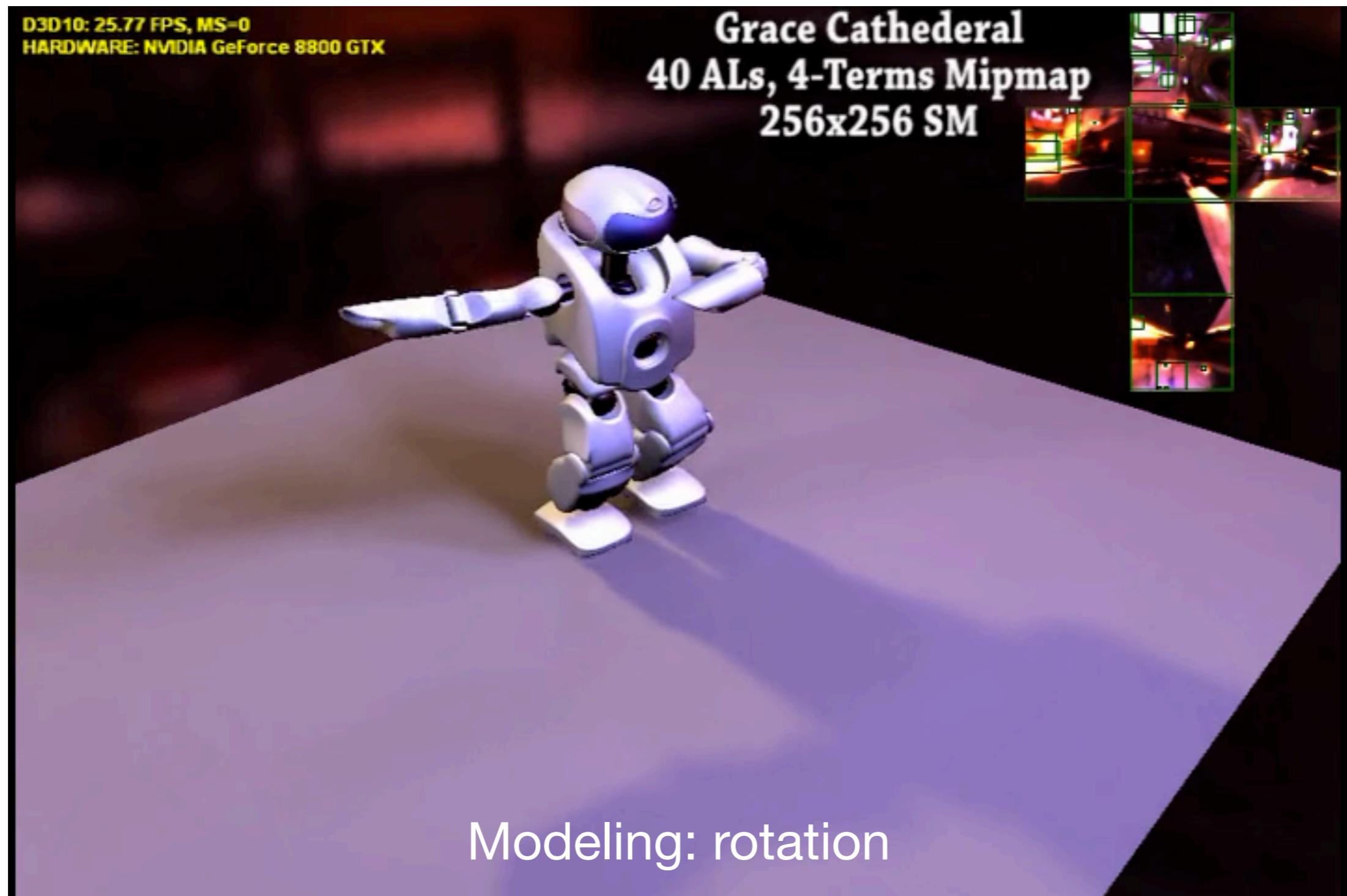
Why Transformation?

5 FPS (209.42 ms), 1spp



Modeling: translation

Why Transformation?



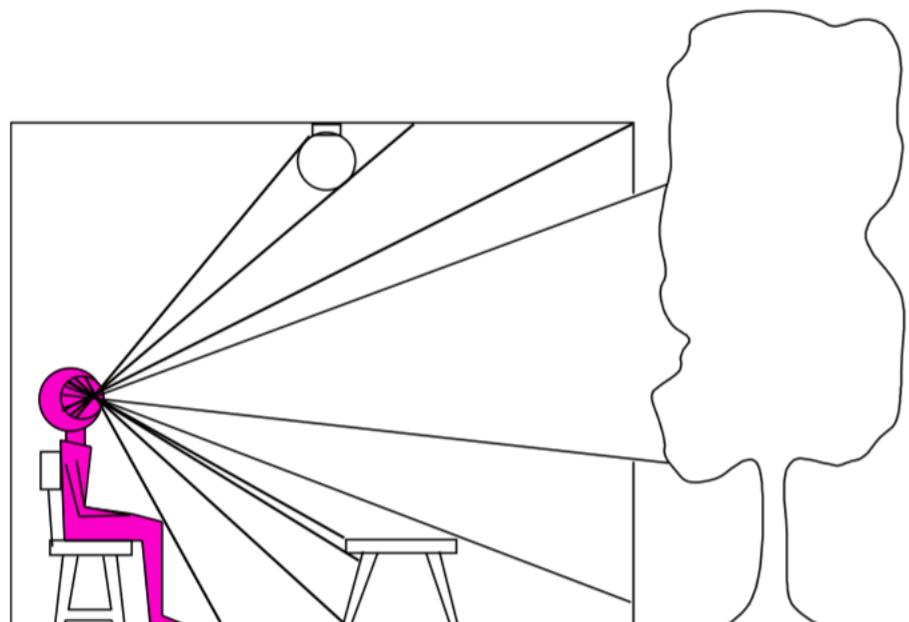
Why Transformation?

P I X A R

Modeling: scaling

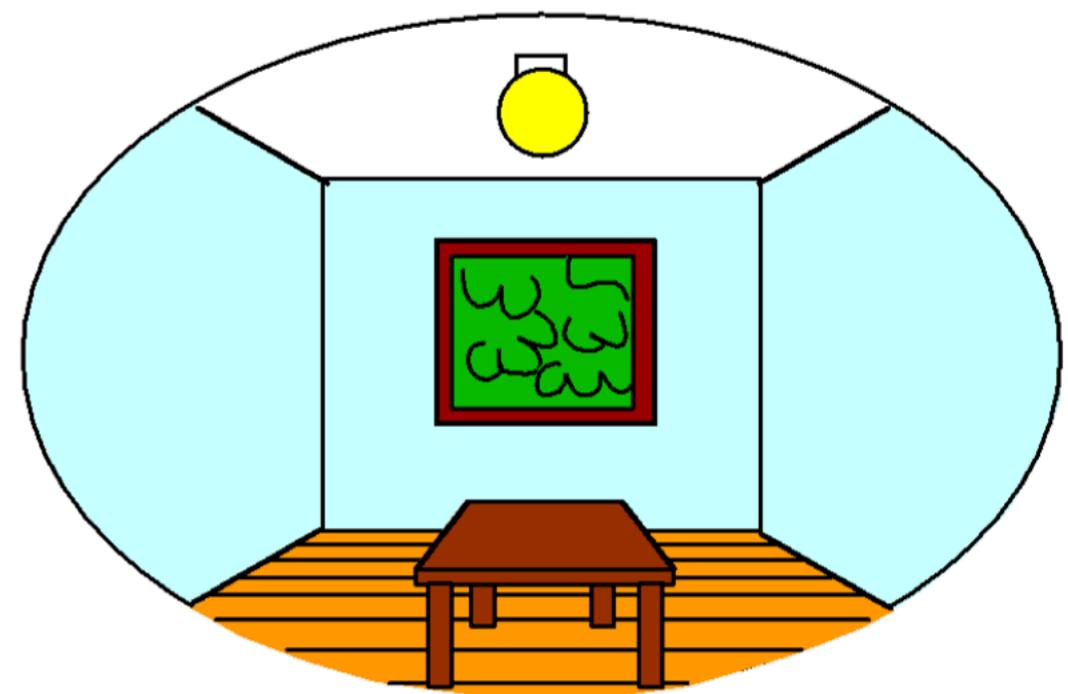
Why Transformation?

3D world



Point of observation

2D image



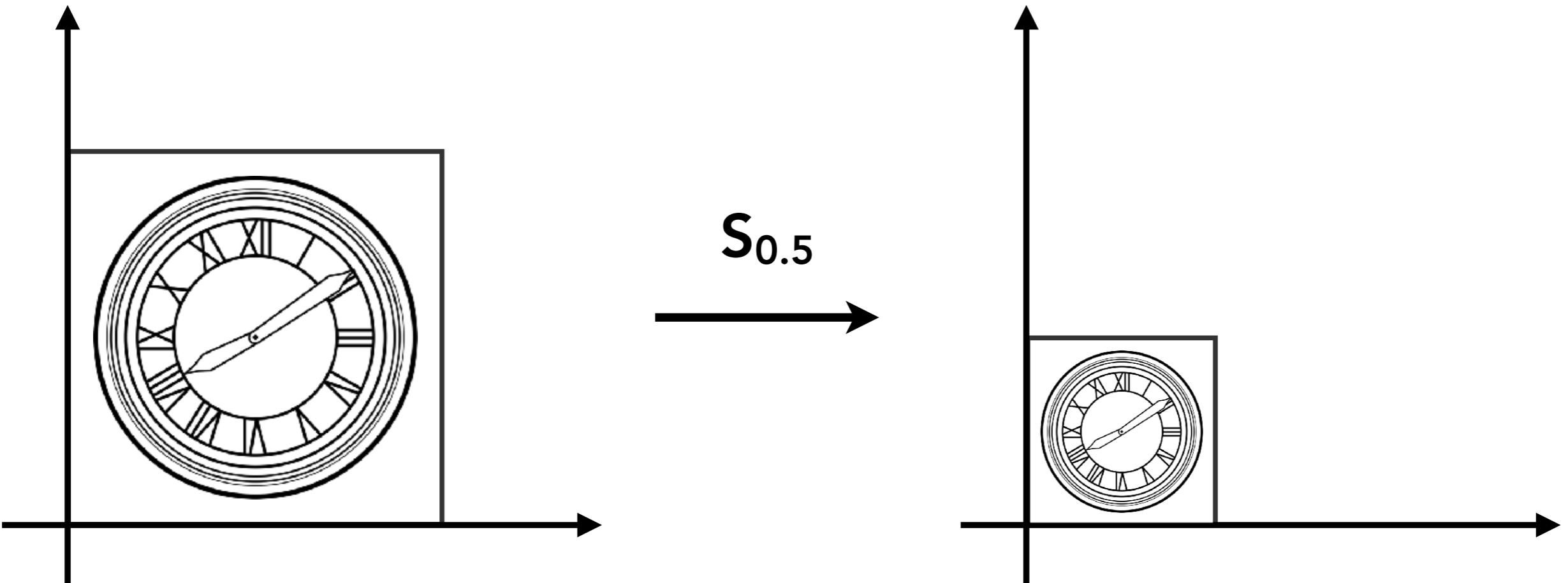
Figures © Stephen E. Palmer, 2002

Viewing: (3D to 2D) projection

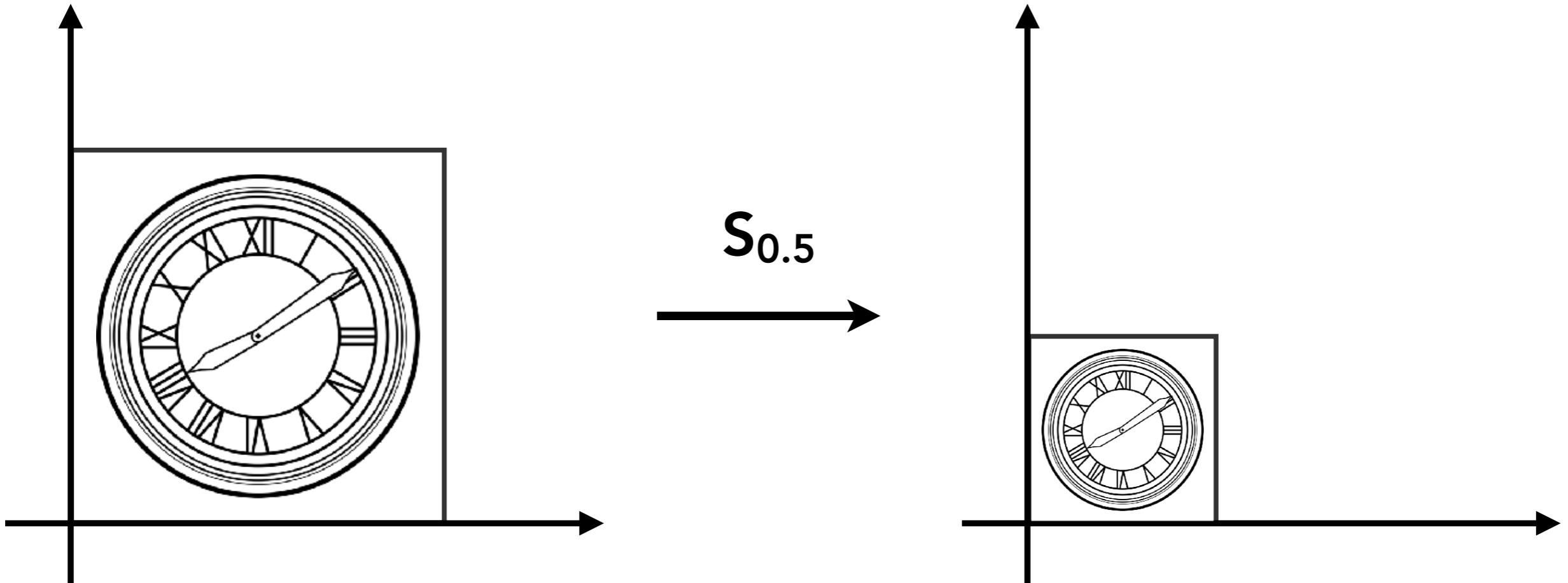
Today

- Why study transformation
- 2D transformations
 - Representing transformations using matrices
 - Rotation, scale, shear
- Homogeneous coordinates

Scale



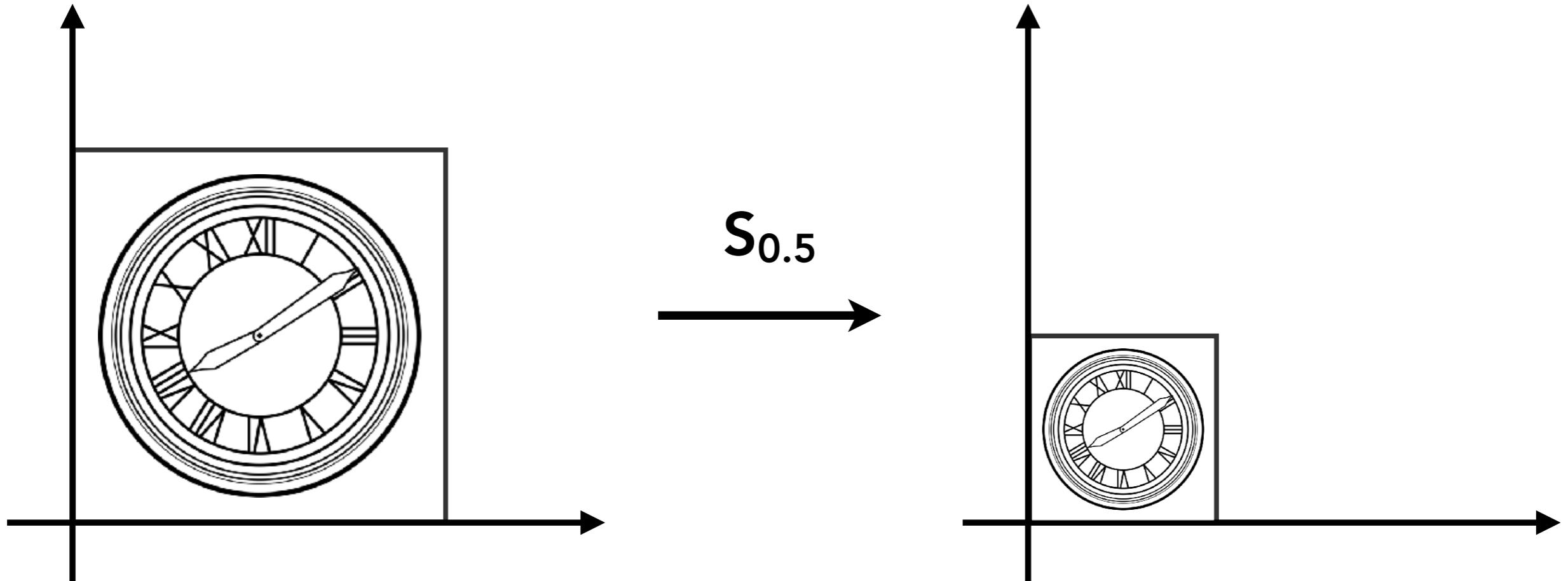
Scale Transform



$$x' = sx$$

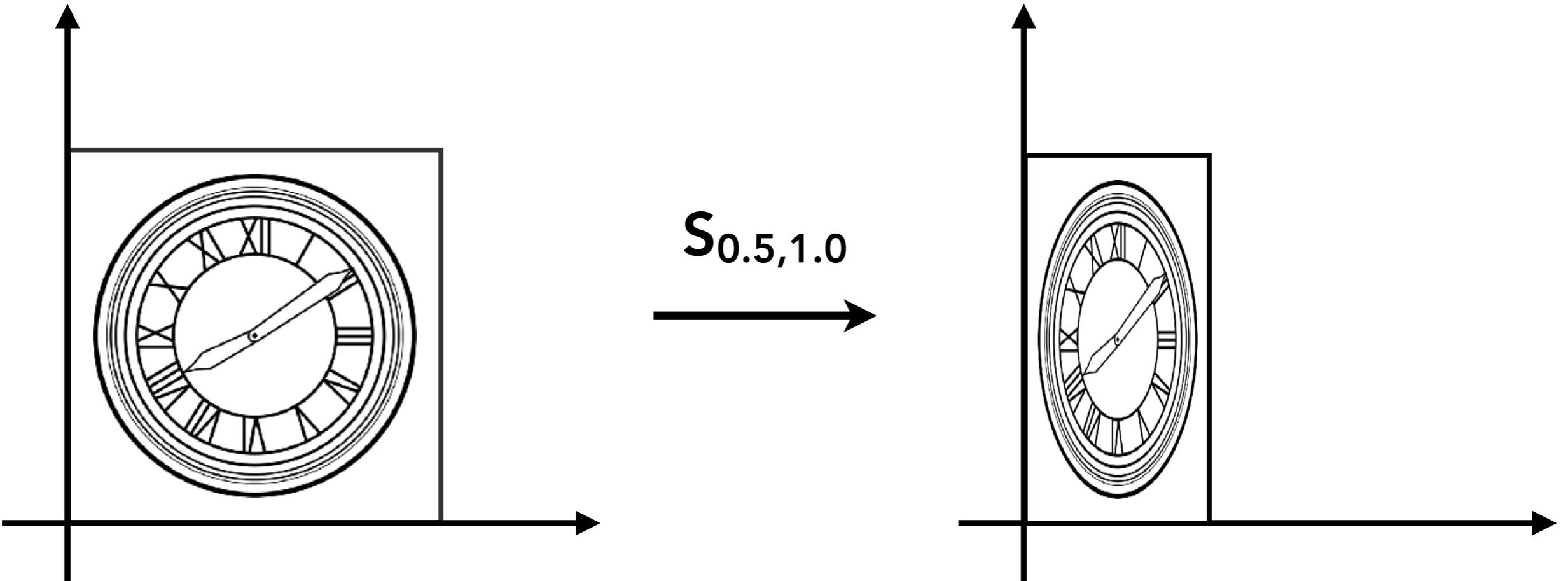
$$y' = sy$$

Scale Matrix



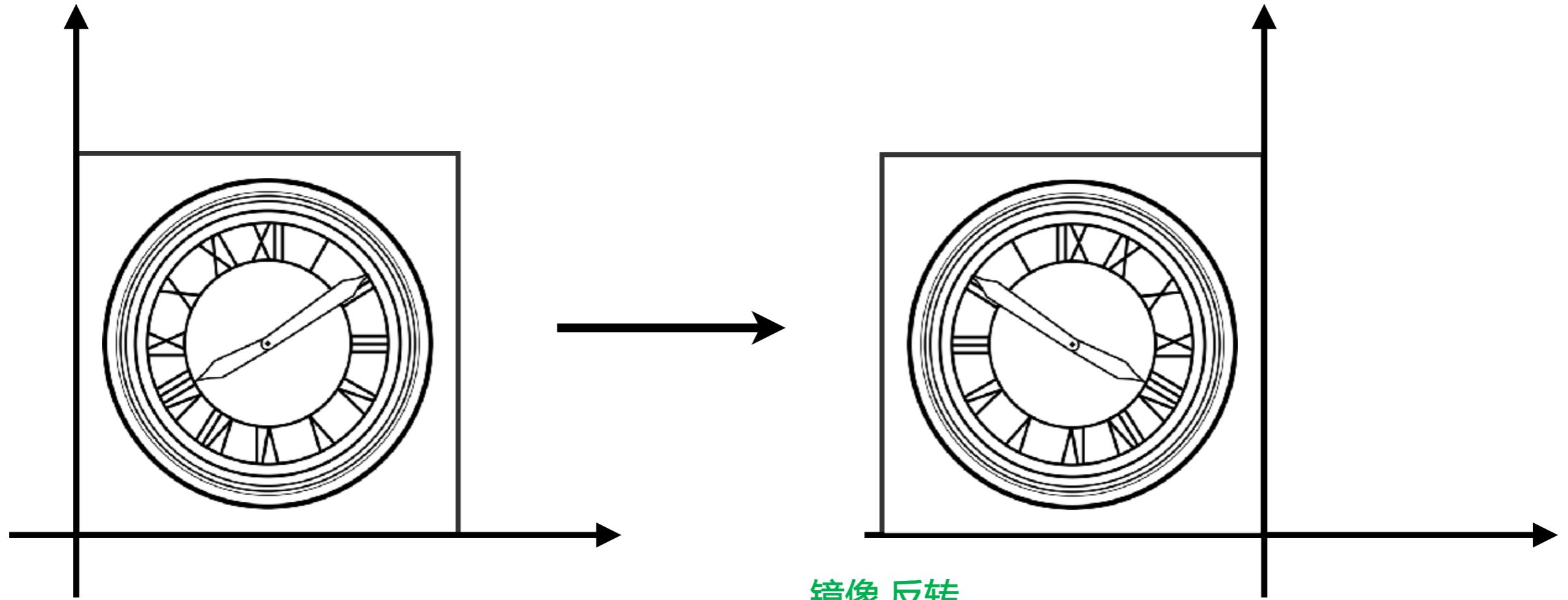
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Scale (Non-Uniform)



$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Reflection Matrix



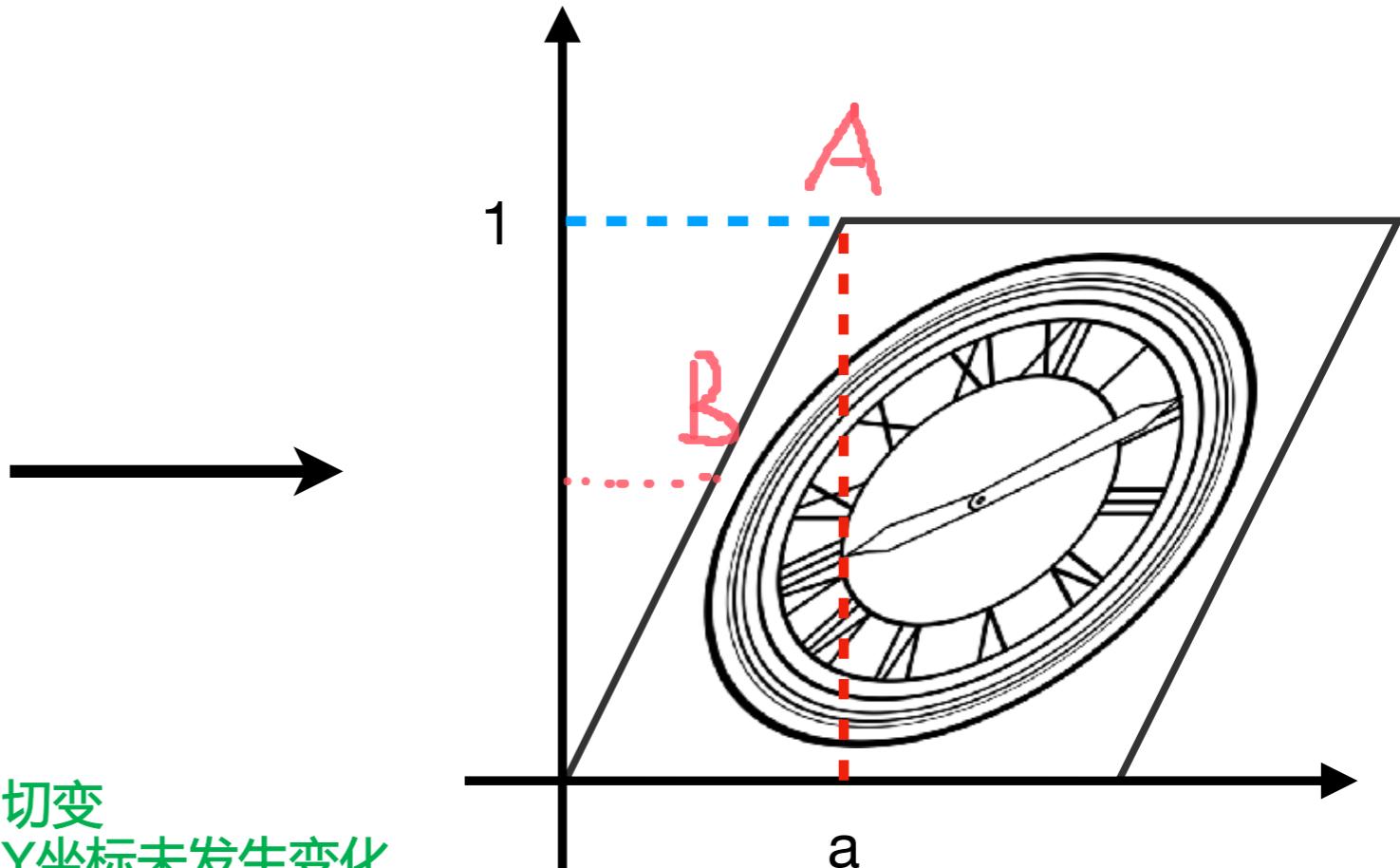
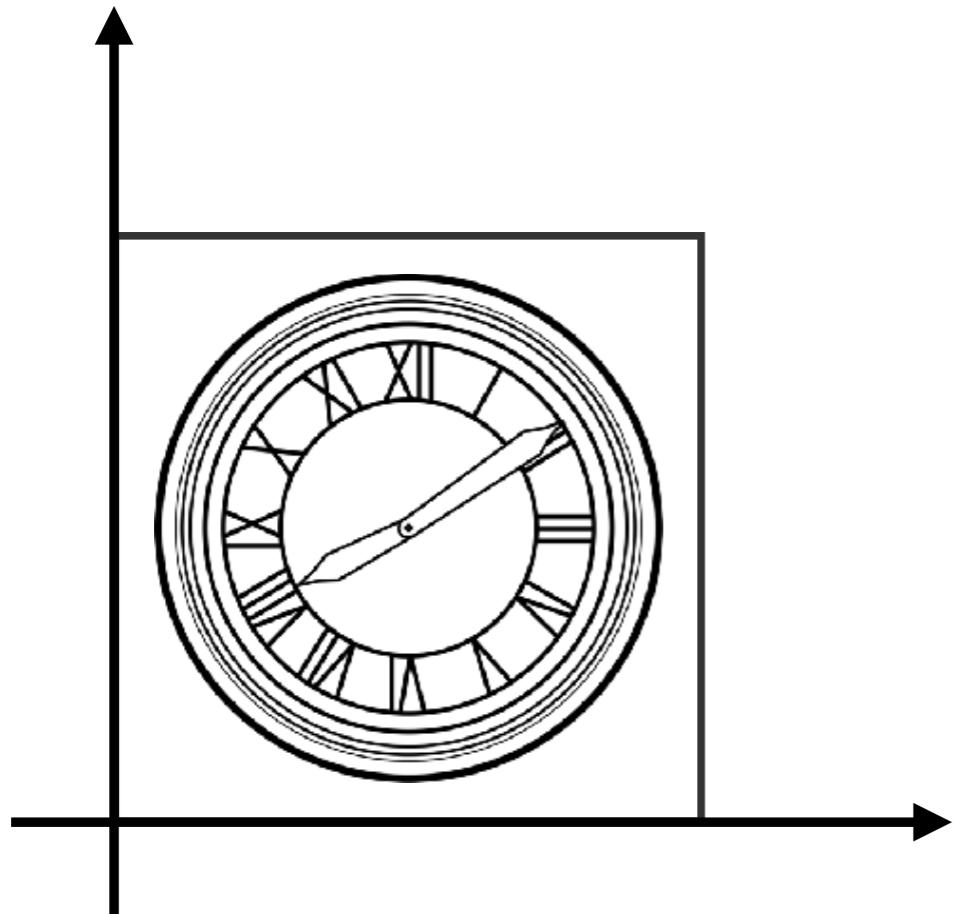
Horizontal reflection:

$$x' = -x$$

$$y' = y$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Shear Matrix



Hints:

Horizontal shift is 0 at $y=0$

Horizontal shift is a at $y=1$

Vertical shift is always 0

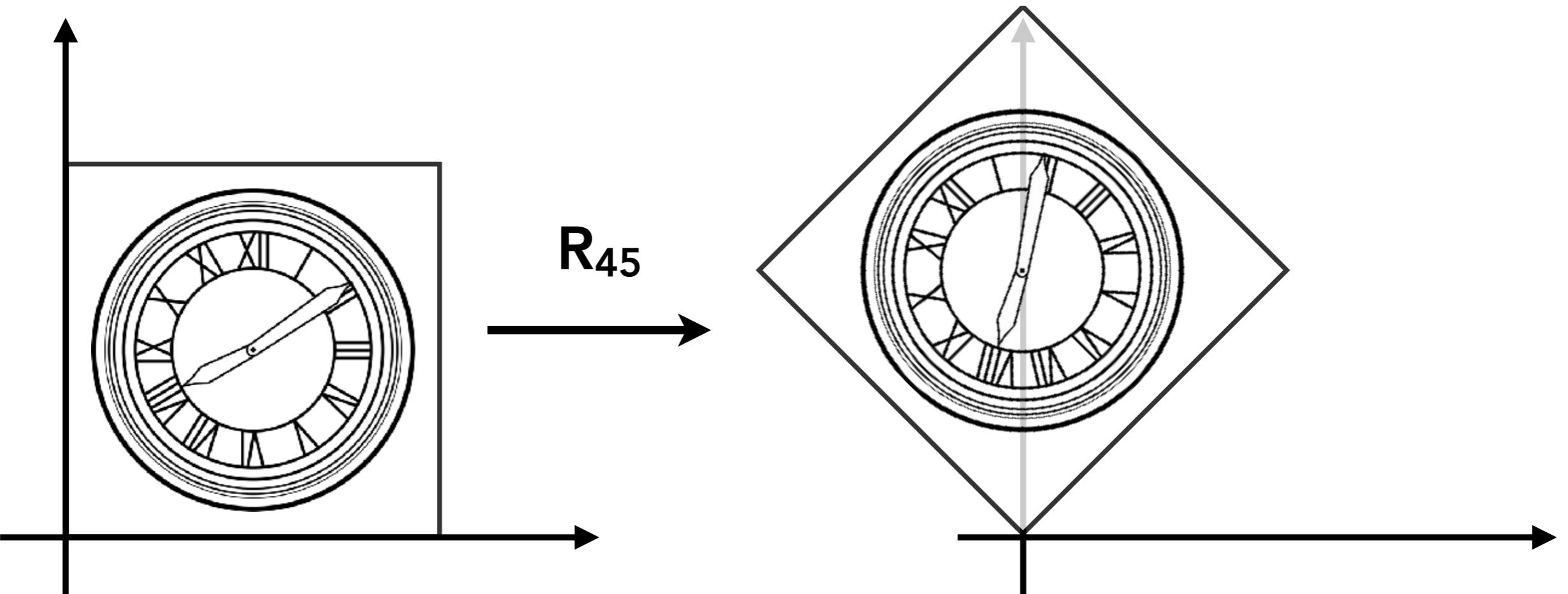
切变
Y坐标未发生变化
X坐标与Y坐标存在比例关系
如点A, X坐标为 $a \cdot 1$
点B, X坐标为 $a \cdot 0.5$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

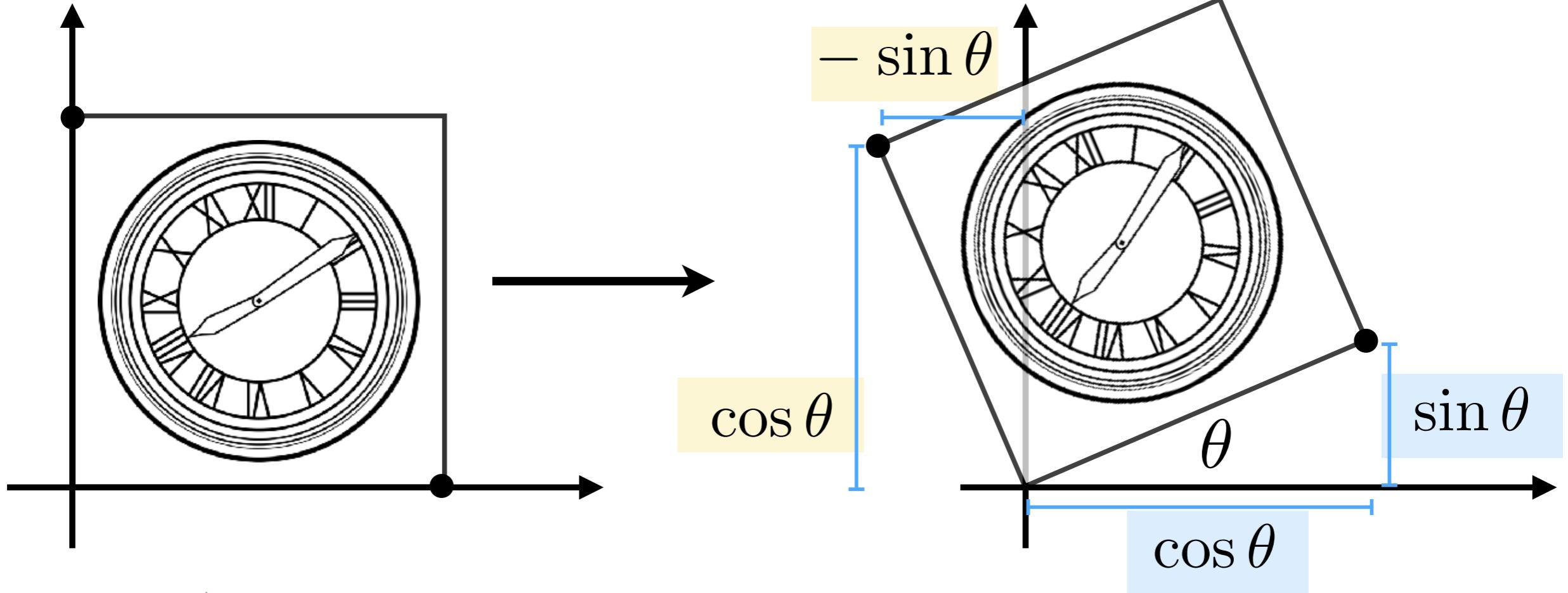
$$\begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

垂直方向切变矩阵

Rotate (about the origin (0, 0), CCW by default)



Rotation Matrix



Handwritten notes illustrating the derivation of the rotation matrix:

- Diagram showing a point (x, y) being transformed into (x', y') .
- Diagram showing a vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ being rotated by θ into $\begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}$.
- Equation: $(x, y) \approx (x', y')$.
- Equation: $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$
- Equation: $\begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$
- Equation: $\cos \theta = A \cdot 1 + B \cdot 0 = A$
- Equation: $\sin \theta = C \cdot 1 + D \cdot 0 = C$

$$\mathbf{R}_\theta = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

Linear Transforms = Matrices

(of the same dimension)

$$x' = a x + b y$$

$$y' = c x + d y$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

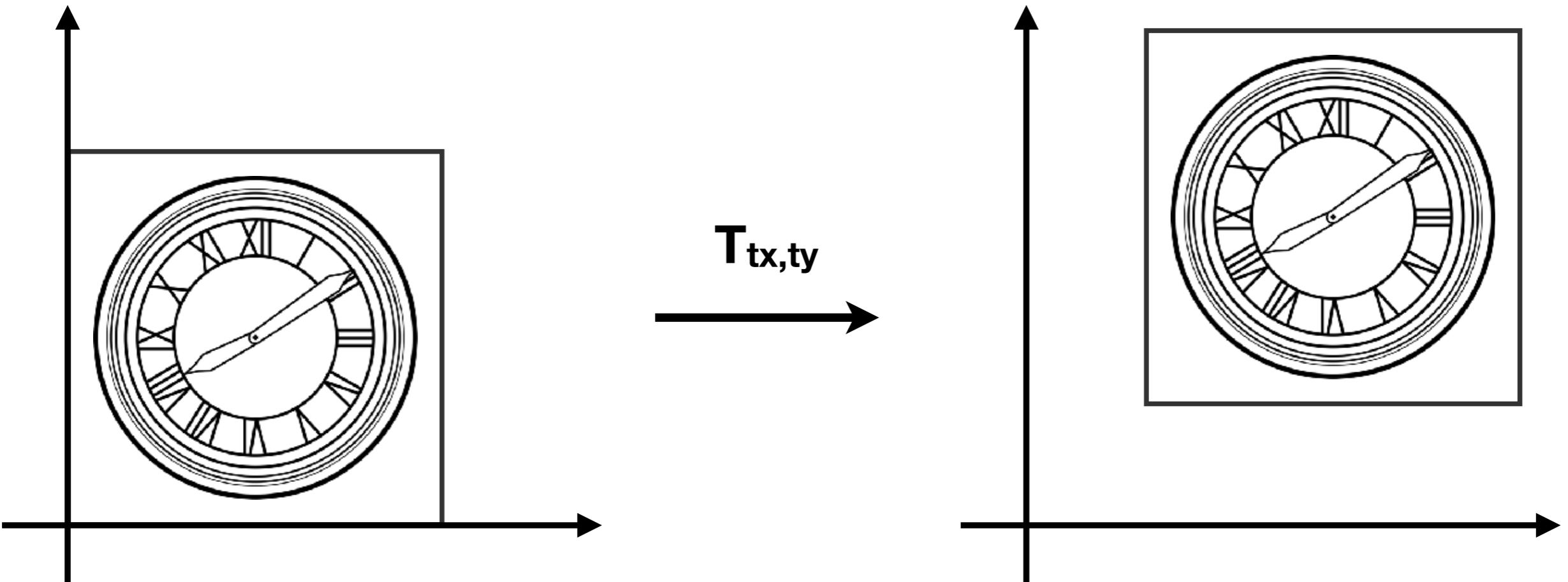
$$\mathbf{x}' = \mathbf{M} \mathbf{x}$$

Questions?

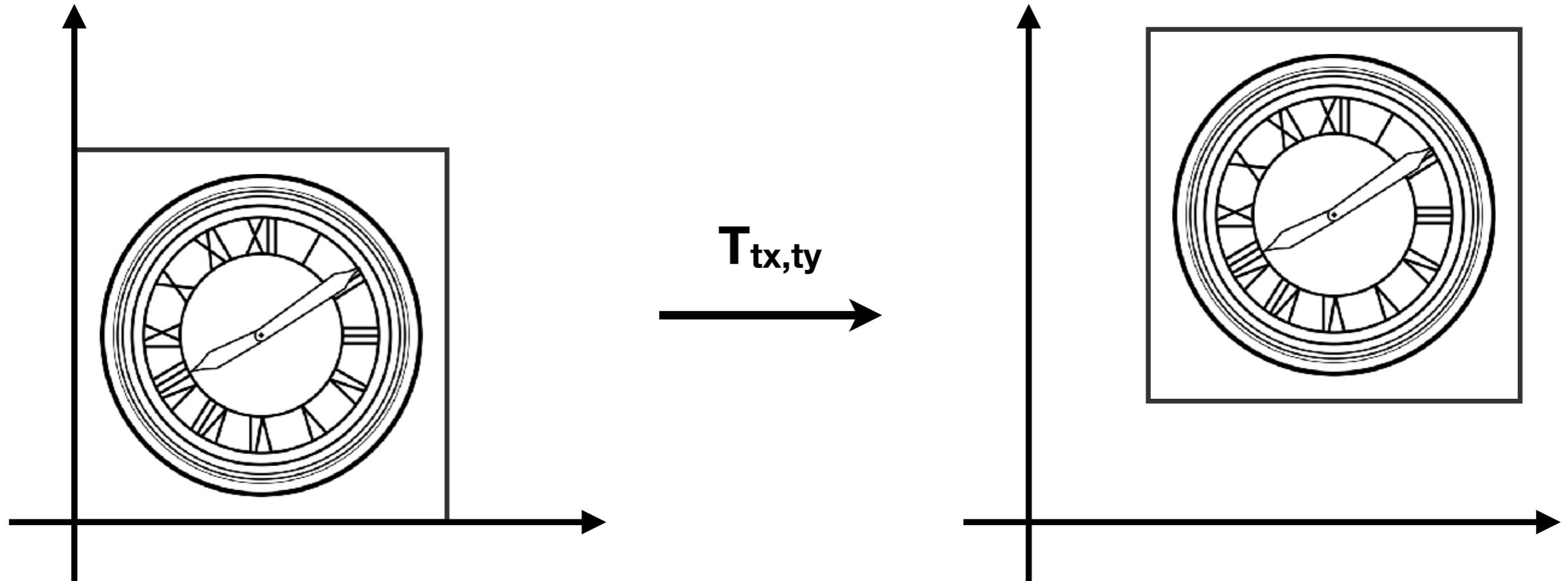
Today

- Why study transformation
- 2D transformations
- **Homogeneous coordinates**
 - Why homogeneous coordinates
 - Affine transformation

Translation



Translation??



$$x' = x + t_x$$

$$y' = y + t_y$$

Why Homogeneous Coordinates

- Translation cannot be represented in matrix form

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$

(So, translation is NOT linear transform!)

- But we don't want translation to be a special case
- Is there a unified way to represent all transformations?
(and what's the cost?)

Solution: Homogenous Coordinates

Add a third coordinate (w-coordinate)

- 2D point = $(x, y, 1)^T$
- 2D vector = $(x, y, 0)^T$

Matrix representation of translations

$$\begin{pmatrix} x' \\ y' \\ w' \end{pmatrix} = \begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} x + t_x \\ y + t_y \\ 1 \end{pmatrix}$$

对向量平移不改变向量

What if you translate a vector?

Homogenous Coordinates

Valid operation if w-coordinate of result is 1 or 0

- vector + vector = vector

- point - point = vector

- point + vector = point

- point + point = ??

In homogeneous coordinates,

$$\begin{pmatrix} x \\ y \\ w \end{pmatrix} \text{ is the 2D point } \begin{pmatrix} x/w \\ y/w \\ 1 \end{pmatrix}, w \neq 0$$

按照上述表达方式，齐次坐标中，两个点相加，表示这个两点的中点
w为2

Affine Transformations

Affine map = linear map + translation 仿射变换

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} t_x \\ t_y \end{pmatrix}$$

Using homogenous coordinates: 齐次坐标

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} a & b & t_x \\ c & d & t_y \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

2D Transformations

Scale

$$\mathbf{S}(s_x, s_y) = \begin{pmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Rotation

$$\mathbf{R}(\alpha) = \begin{pmatrix} \cos \alpha & -\sin \alpha & 0 \\ \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

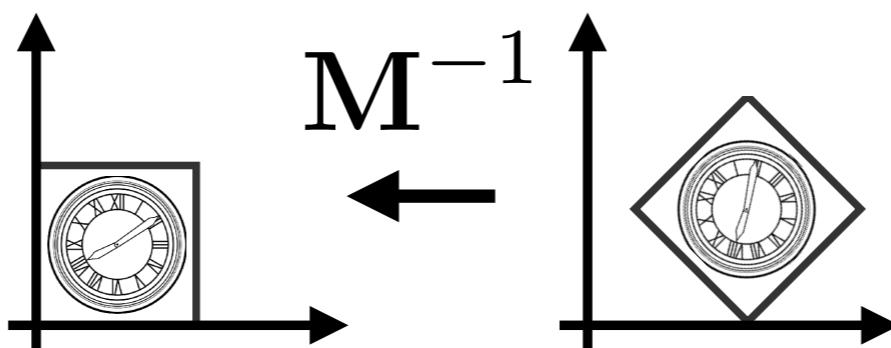
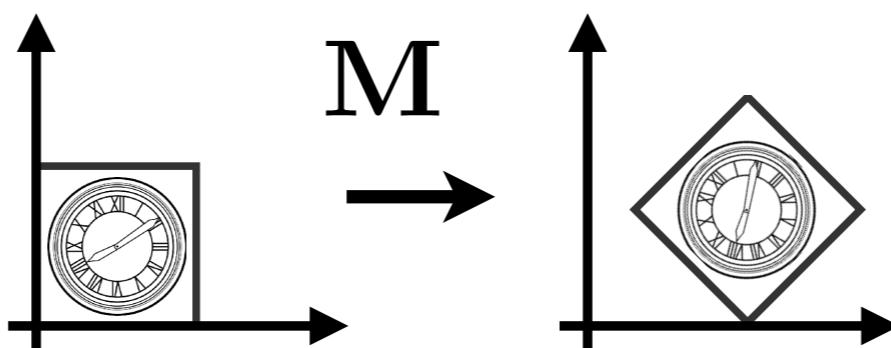
Translation

$$\mathbf{T}(t_x, t_y) = \begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{pmatrix}$$

Inverse Transform

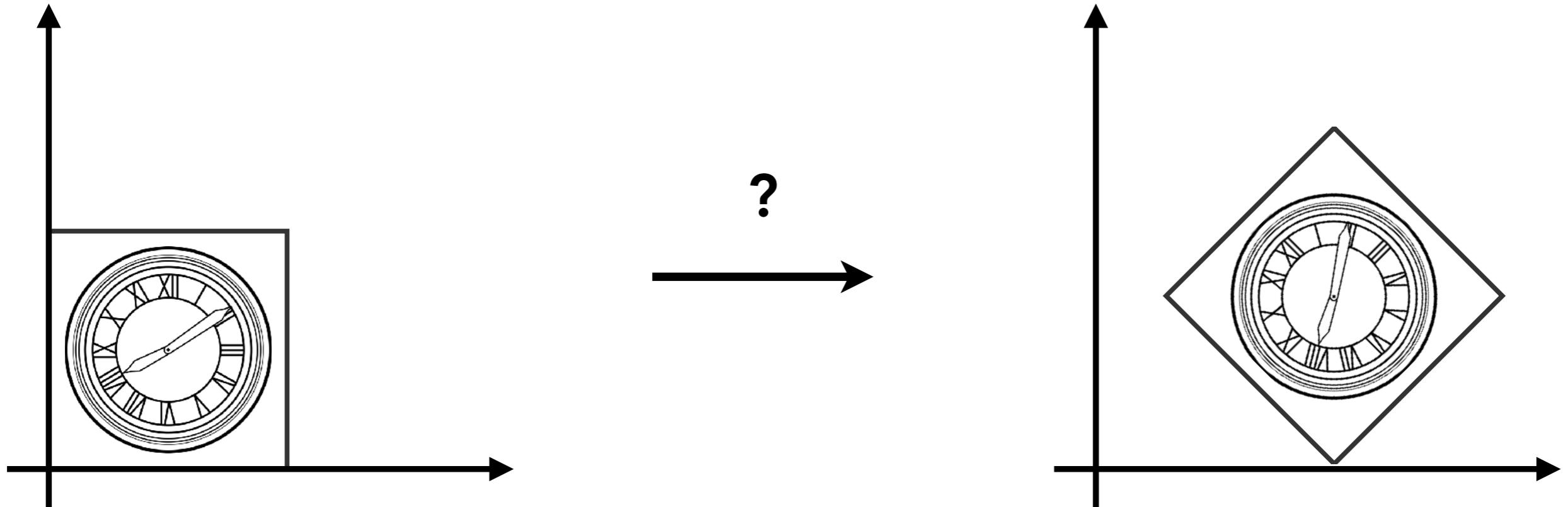
$$M^{-1}$$

M^{-1} is the inverse of transform M in both a matrix and geometric sense

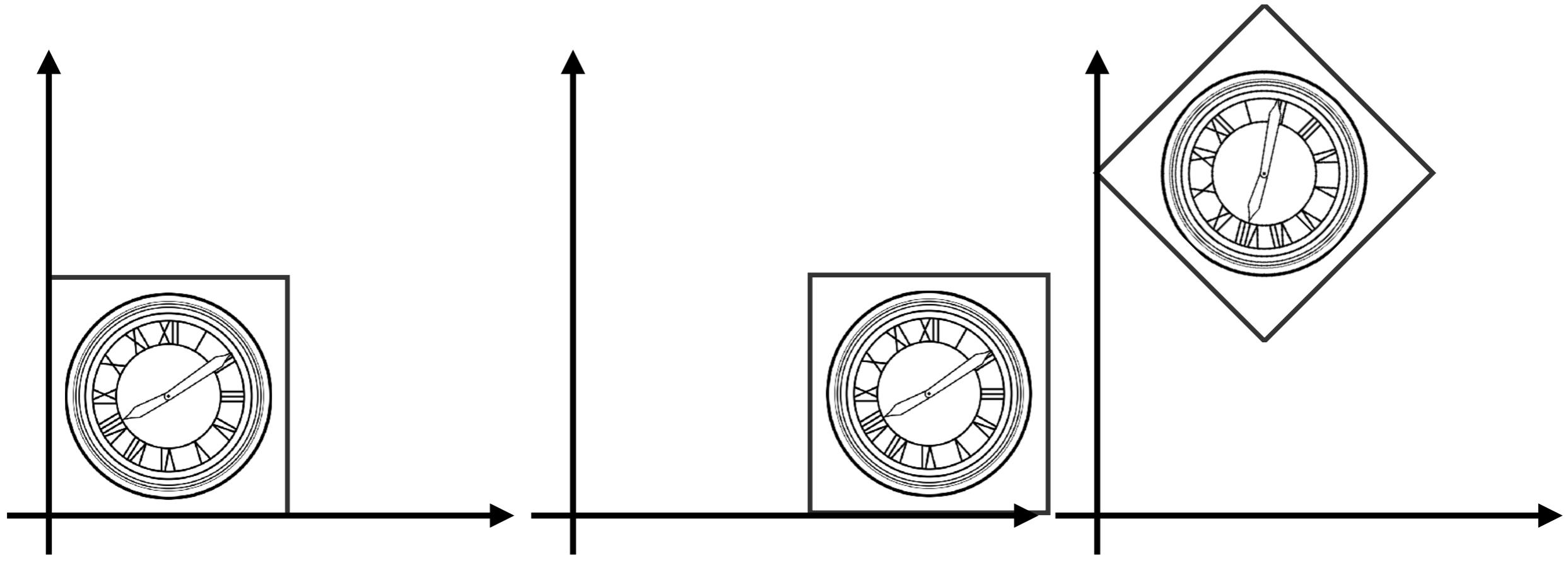


Composing Transforms

Composite Transform



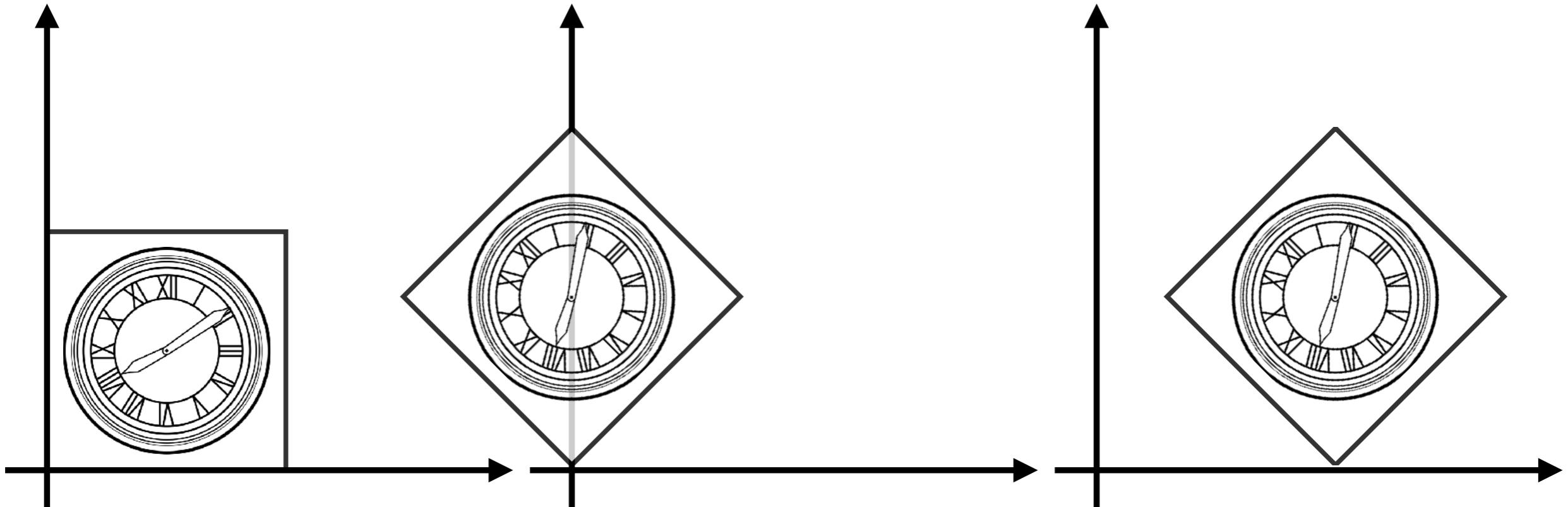
Translate Then Rotate?



$$T_{(1,0)}$$

$$R_{45} \cdot T_{(1,0)}$$

Rotate Then Translate



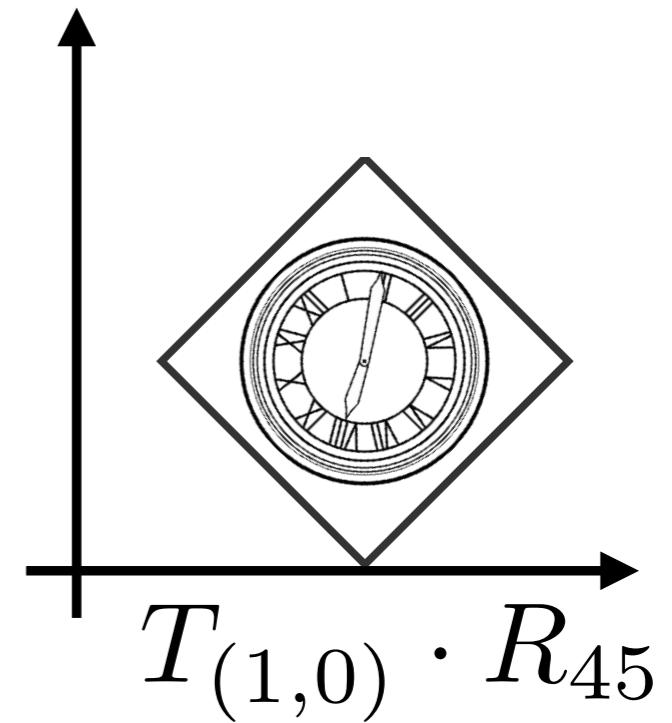
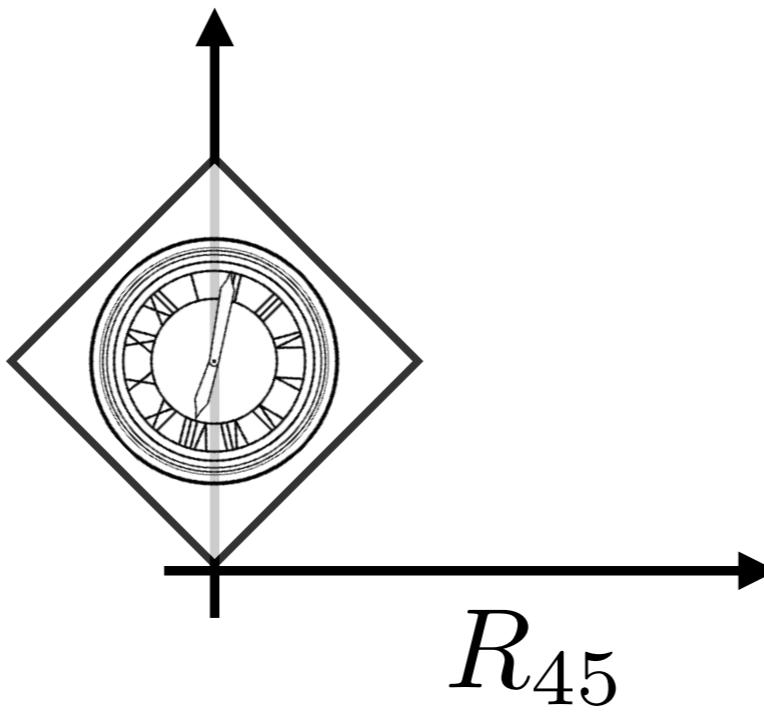
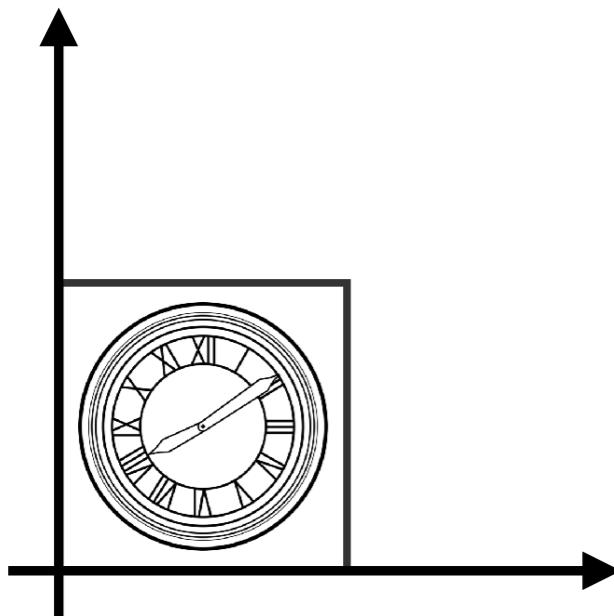
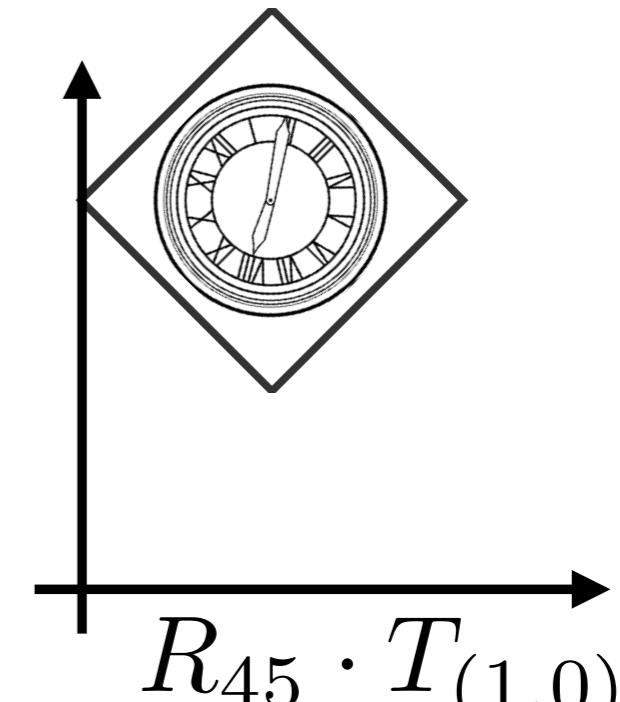
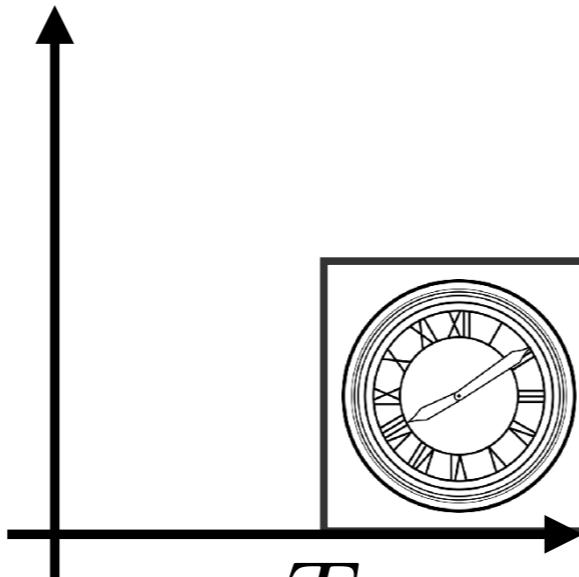
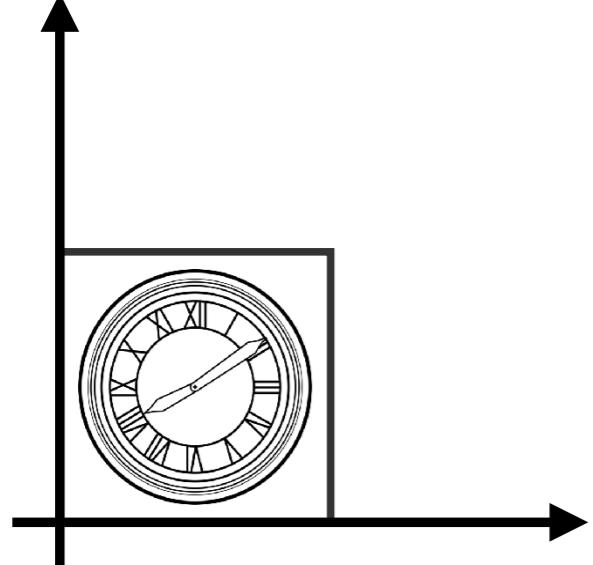
矩阵没有交换律，乘法的顺序影响大

先旋转后平移

$$R_{45}$$

$$T_{(1,0)} \cdot R_{45}$$

Transform Ordering Matters!



Transform Ordering Matters!

Matrix multiplication is not commutative

$$R_{45} \cdot T_{(1,0)} \neq T_{(1,0)} \cdot R_{45}$$

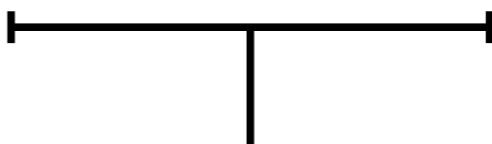
Note that matrices are applied right to left:

$$T_{(1,0)} \cdot R_{45} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos 45^\circ & -\sin 45^\circ & 0 \\ \sin 45^\circ & \cos 45^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Composing Transforms

Sequence of affine transforms A_1, A_2, A_3, \dots

- Compose by matrix multiplication
 - Very important for performance!

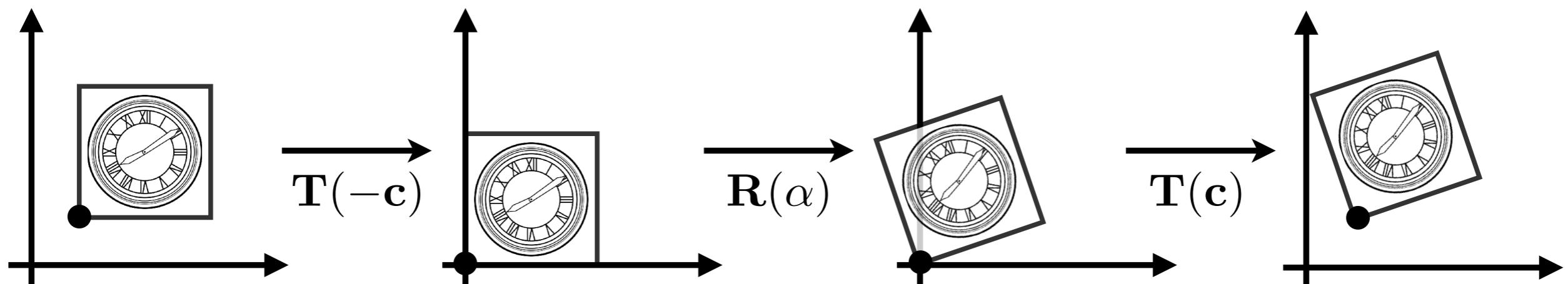
$$A_n(\dots A_2(A_1(\mathbf{x}))) = \mathbf{A}_n \cdots \mathbf{A}_2 \cdot \mathbf{A}_1 \cdot \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$


Pre-multiply n matrices to obtain a single matrix representing combined transform

Decomposing Complex Transforms

How to rotate around a given point c ?

1. Translate center to origin
2. Rotate
3. Translate back



Matrix representation?

以任意点为旋转中心
移动至原点后旋转后平移

$$T(c) \cdot R(\alpha) \cdot T(-c)$$

3D Transforms

3D Transformations

Use homogeneous coordinates again:

- 3D point = $(x, y, z, 1)^T$
- 3D vector = $(x, y, z, 0)^T$

In general, (x, y, z, w) ($w \neq 0$) is the 3D point:

$$(x/w, y/w, z/w)$$

3D Transformations

Use 4×4 matrices for affine transformations

$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{pmatrix} a & b & c & t_x \\ d & e & f & t_y \\ g & h & i & t_z \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

Affine map = linear map + translation

What's the order?

Linear Transform first or Translation first?

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} t_x \\ t_y \end{pmatrix}$$

Using homogenous coordinates:

先线性变换再平移

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} a & b & t_x \\ c & d & t_y \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

Thank you!

(And thank Prof. Ravi Ramamoorthi and Prof. Ren Ng for many of the slides!)