Monoids explained to an imperative programmer

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Structure of this presentation

- Monoid definition and simple example

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    - more friendly notation and function composition

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- Why would we ever want to abstract to a monoid?

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  - Monoids where $x \cdot y = y \cdot x$ are called commutative monoids

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We can also very easily show that $\mathbb{Z}/\{0\}$ under multiplication is also a monoid: we use 1 as the identity element.

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In general, for all types τ, $v : \tau$ is 'v is a member of type τ'.

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Note how $x : int$ and $f : int \rightarrow int$ use the same notation. We also define $a \rightarrow b \rightarrow c \equiv (a \rightarrow b) \rightarrow c$, i.e. the type signature is left associative.

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We know that $f : int \to int$. Now consider any $v : int$. In Java, we might use the function like a = f(v);. What is the type of a? Well, we take a $f : int \to int$ and feed an int to it so we get an int back.

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Consider the following.

```
public class TwoBasicFunc {
    public static int f(int x) {
        return x + 1;
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    public static int g(int x) {
        return x * 2;
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We have $f : int \to int$ and $g : int \to int$.
Let's say we want to apply g first and then f to the result. In Java we'd right `f(g(v));`. With our new notation we write $f\ g\ v$. Note that application is left associative here.

Let's not stop on *int*s though and go generic.

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In our notation, this is simply $id : A \rightarrow A$: nothing special about it!

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We can very easily see that $g\ x : b$. So what's the type of $f\ (g\ x)$?
Well, it's just c! $g\ x$ provides us a value of type b and we apply $f : b \rightarrow c$ to it.

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Well, it's just c! $g\ x$ provides us a value of type b and we apply $f : b \to c$ to it.
Mind that we need the parenthesis in $f\ (g\ x)$ as we have defined function application to be left associative.
Without parenthesis, we'd get $f\ g\ x \equiv (f\ g)\ x$. But we can't have $f\ g$ because f takes a value of type b and g is of type $a \to b$.

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Simple stuff! It turns out that h is just function application that we've been doing all the time!

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The idea is that we just wrote a function h that takes any function of type $a \rightarrow b$ and a value of type a. As we saw with id, a and b can be anything: the types are polymorphic.

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That's it! It's that simple. This operation is so common that we define $h\ f\ g\ x = f\ (g\ x)$ to be $(f \circ g)\ x$. So just $(f \circ g)$ is of type $a \to c$ and $(f \circ g)\ x$ is of type c.
We can give this new function $(f \circ g)$ a name: $f' = f \circ g$.

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- Identity
  - $\forall f \in F : id \circ f = f \circ id = f$
  - Here $id\ x = x$ is our identity element. id is polymorphic so it gets the type $(\tau \to \tau) \to (\tau \to \tau)$

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  - $\forall f, g \in F : f \circ g \in F$
  - $((f : \tau \to \tau) \circ (g : \tau \to \tau)) : \tau \to \tau$
- Associativity
  - $\forall f, g, h \in F : (f \circ g) \circ h = f \circ (g \circ h)$
  - $\forall (x : \tau), \forall f, g, h \in F : (f \circ g) \circ h\, x = f\, (g \circ h)\, x = f\, (g\, (h\, x))$
  - $\forall (x : \tau), \forall f, g, h \in F : f \circ (g \circ h)\, x = f\, ((g \circ h)\, x) = f\, (g\, (h\, x))$
- Identity
  - $\forall f \in F : id \circ f = f \circ id = f$
  - Here $id\ x = x$ is our identity element. id is polymorphic so it gets the type $(\tau \to \tau) \to (\tau \to \tau)$
  - $\forall (x : \tau), \forall f \in F : (f \circ id)\, x = f\, (id\ x) = f\ x$

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  - Mind that by definition of id, $id\ f\ x = f\ x$ and $f\ (id\ x) = f\ x$. Not applying to x, we just get f in both cases.

It's important to note that for functions to form a monoid under composition, the functions must be of a uniform type: that is, for any function f^n where $f^0 : a$, $f^1 : a \to a$, f^k takes k arguments of type a.

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  - Python's *reduce* with the unit used as the initial value
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  - Gives us assurance about what we can do
  - Composing small parts is the way to control complexity
- Only a step away from the ever powerful monads!

Here's output from a proof of concept monoid code written in Java.

```
Folding left: [2, 3, 4]
9
Folding right: [2, 3, 4]
9
Folding left [Why, , hello , world!]
Why, hello world!
Folding right [Why, , hello , world!]
Why, hello world!
Folding left [f x = x * x, f x = x + 5, f x = x * 3]
Applying the result of the fold to 7: 676
Folding right [f x = x * x, f x = x + 5, f x = x * 3]
Applying the result of the fold to 7: 676
Folding left [[1, 3, 4], [25, 7, 16], [735, 17, 8]]
[1, 3, 4, 25, 7, 16, 735, 17, 8]
Folding right [[1, 3, 4], [25, 7, 16], [735, 17, 8]]
[1, 3, 4, 25, 7, 16, 735, 17, 8]
```

Questions

Hopefully I miraculously managed to fit in my assigned time.
Feel free to ask any questions or point out any mistakes.

Contact

mk440@bath.ac.uk; fuuzetsu@fuuzetsu.co.uk
Get these slides and all code at
https://github.com/ShanaTsunTsunLove/foundations-talk

A quote from the #haskell IRC channel

* roconnor: where are all the category theoriest? why don't they already have all the answers for us?
* edwardk: roconnor: this is the point in your career where you look around for the cavalry and realize that you're it ;)