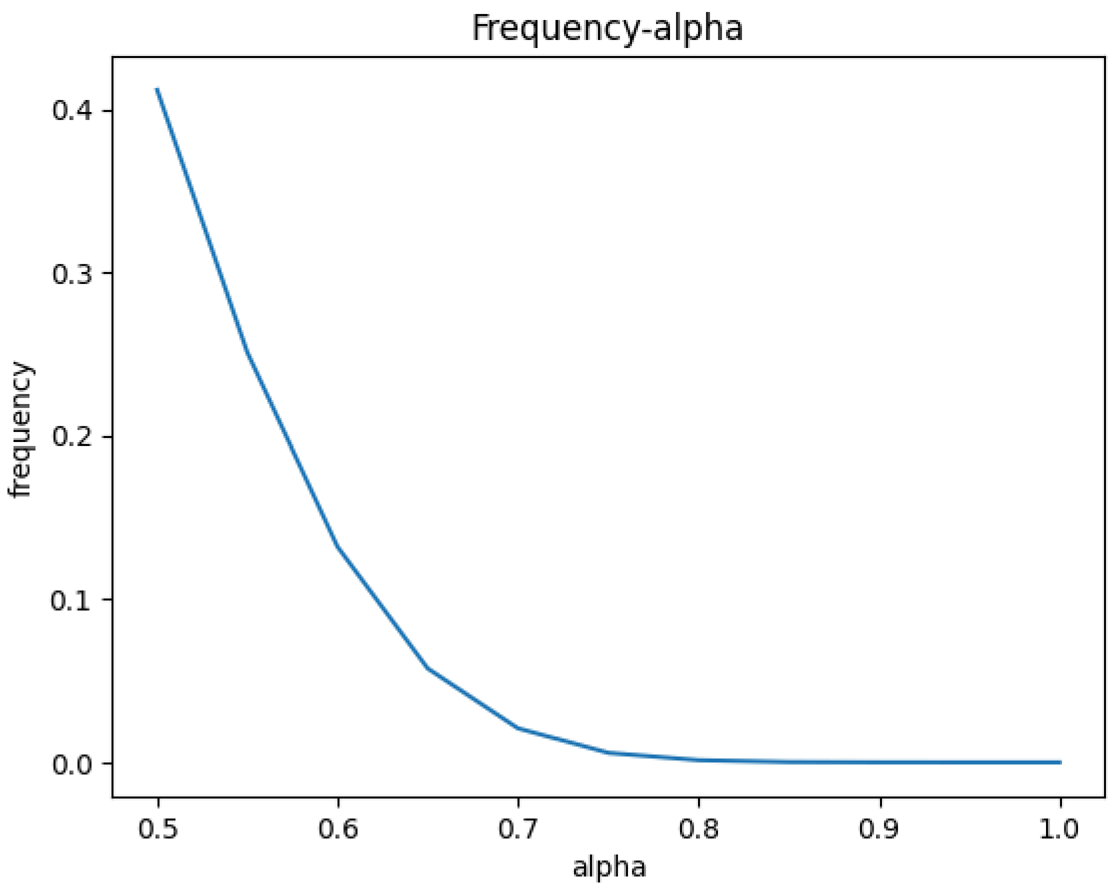
1.1

1.



2. The number of X is only 20. As a result, the number of possible values of $\sum_i^{20} X_i$ is 21 ranging from 0 to 1 with the step 0.05. Hence $P(\frac{1}{20}\sum_i^{20} X_i \geq 0.51) \Leftrightarrow P(\frac{1}{20}\sum_i^{20} X_i \geq 0.55)$

3,4,5:

Denote $Y = \frac{1}{20}\sum_{i=0}^{20} X_i$. Given $X_i \sim (0,1)$, $EX = \frac{1}{2}$, $DX = \frac{1}{4}$, $EY = \frac{1}{2}$, $DY = \frac{1}{40}$

- Markov's bound: $\frac{1}{2a}$
- Chebyshev's bound:

$$P(|Y - EY| \geq b) \leq \frac{DX}{b^2}$$
$$P(|Y - \frac{1}{2}| \geq b) \leq \frac{1}{40b^2}$$
$$P(Y \geq b + \frac{1}{2}) + P(Y \leq b - \frac{1}{2}) \leq \frac{1}{40b^2}$$

Let $0.5 \leq \alpha = b + \frac{1}{2} \leq 1, b - \frac{1}{2} \leq 0$, so $P(Y \leq b - \frac{1}{2}) = 0$,

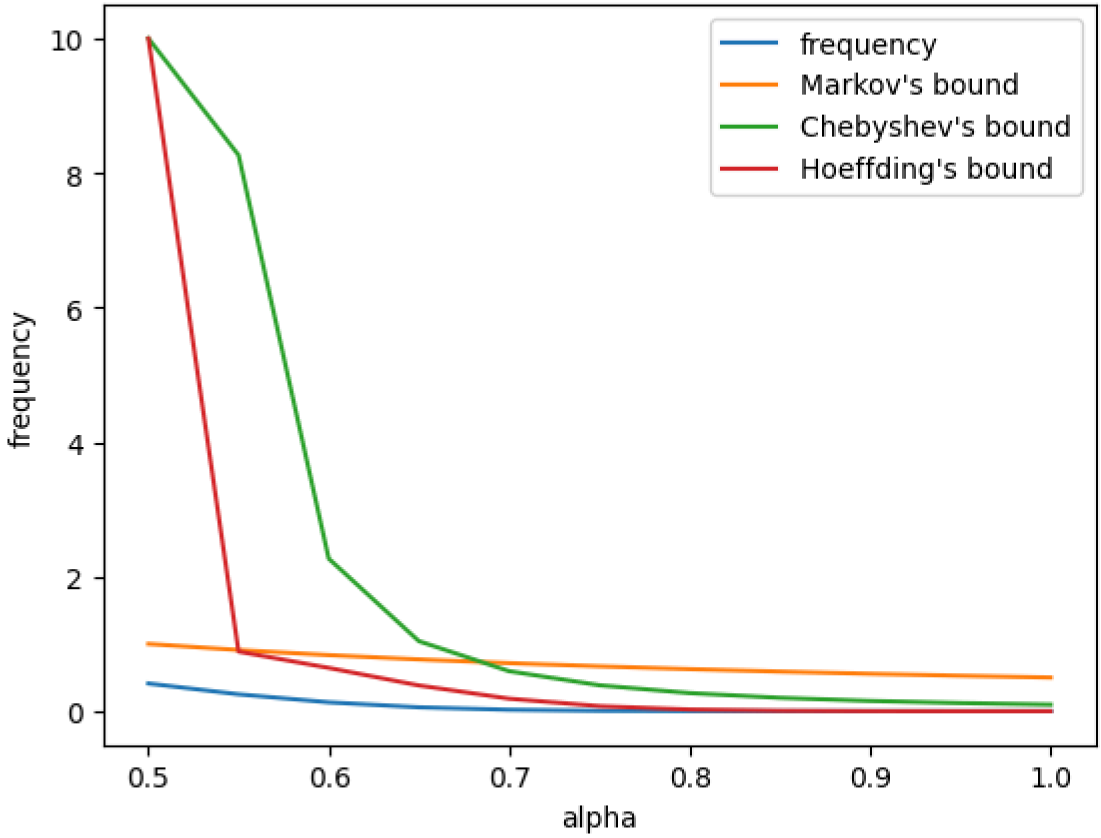
$$P(Y \geq \alpha) \leq \frac{1}{40(\alpha - \frac{1}{2})^2}$$

- Hoeffding's bound:

$$P(Y - EY \geq t) \leq exp(-40t^2)$$

let $\alpha = t + \frac{1}{2}$

$$P(Y \geq \alpha) \leq exp(-40(\alpha - \frac{1}{2})^2)$$



When $\alpha = 0.5$, the Chebyshev's bound and Hoeffding's bound is positive infinity. I just set them as 10.

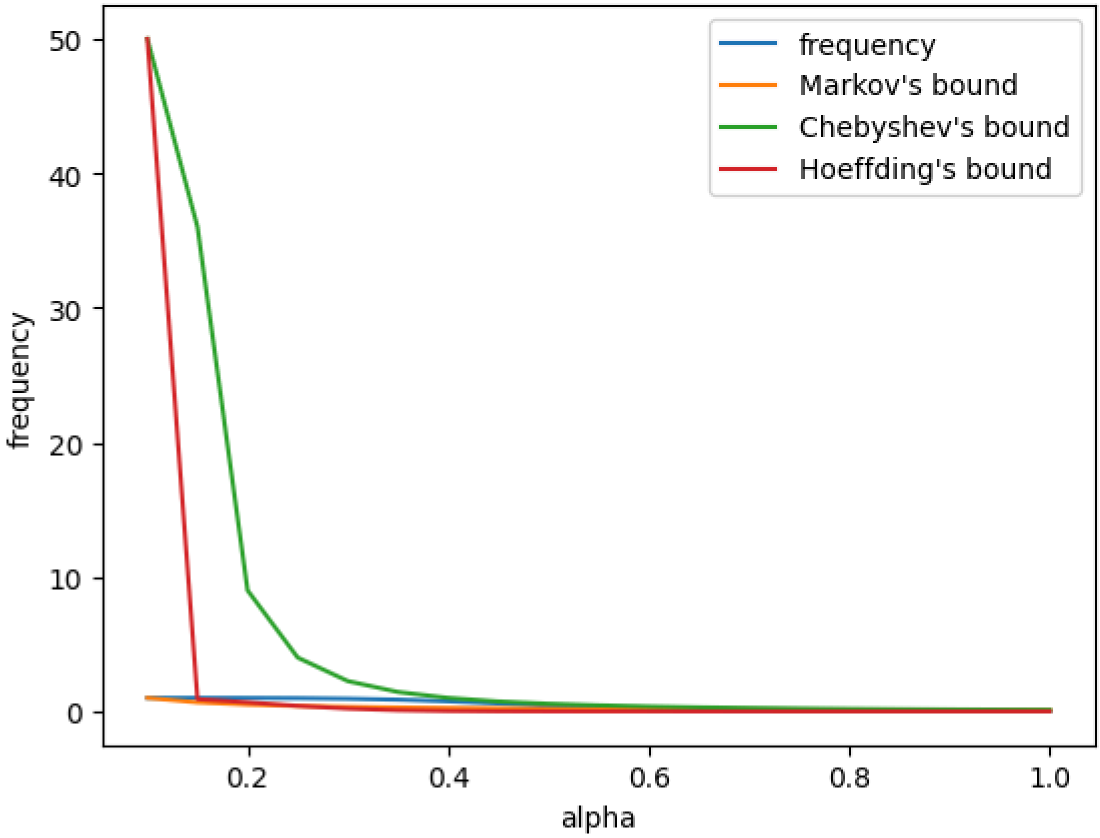
6. When α is close to 0.5, Markov's bound is more tight to the real frequency. When α gets close to 1, Chebyshev's bound and Hoeffding's bound become tighter and Hoeffding's bound is more accurate than others.
7. When $\alpha = 1$, there must 20 $X_i = 1$ and 0 $X_i = 0$, $P(\frac{1}{20}\sum_i^{20} X_i \geq 1) = (\frac{1}{2})^{20}$
   When $\alpha = 0.95$, there must more than 19 $X_i = 1$ $P(\frac{1}{20}\sum_i^{20} X_i \geq 1) = (\mathbb{C}_{20}^{19} + 1)(\frac{1}{2})^{20}$

For bias 0.1,

Markov's bound: $\frac{1}{10a}$

Chebyshev's bound: $\frac{9}{100(\alpha - 0.1)^2}$

Hoeffding's bound: $\exp(-40(\alpha - 0.1)^2)$



1.2

All X_i are the same

$X \sim (1, 0.5)$

2

$$P(\exists i : |S_i - p| \geq \epsilon) = |P(\cup_i^k |S_i - p| \geq \epsilon) \leq 2\sum_i^k \exp(2\epsilon^2 n) = 2k\exp(2n\epsilon^2)$$

3

1. $EL = 0, DL = EL^2$

$$\begin{aligned}\int_{-\infty}^{+\infty} x^2 \cdot \frac{1}{2b} \cdot \exp(-\frac{|x|}{b})dx &= 2\int_0^{+\infty} x^2 \cdot \frac{1}{2b} \cdot \exp(-\frac{x}{b})dx \\ &= -\int_0^{+\infty} x^2 d(\exp(-\frac{x}{b})) \\ &= -\int_0^{+\infty} 2x\exp(-\frac{x}{b})dx - x^2\exp(-\frac{x}{b})|_0^{+\infty} \\ &= 2b^2\end{aligned}$$

2. $\sigma^2 = 2b^2$

3. To prove,

$$\exp(-\frac{|X+1|+1}{b}) \le \exp(-\frac{|X|}{b}) \le \exp(-\frac{|X+1|-1}{b})$$

$\Leftrightarrow$

$$|X+1|+1 \ge |X| \ge |X+1|-1$$

$X > 0$

$$X+2 \ge X \ge X$$

$0 \ge X > -1$

$$X+2 \ge -X \ge X$$

$X \le -1$

$$-X \ge -X \ge -X-2$$

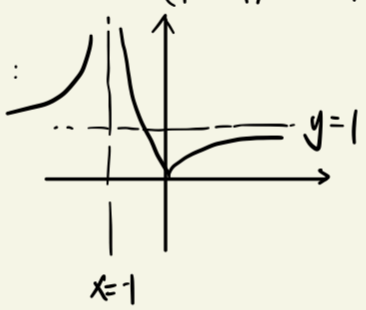
Try to find S and $f(a)$, satisfy

$$\frac{1}{\cancel{\sqrt{2\pi}\cdot a}} \cdot e^{\frac{x^2}{2a^2}} \le f(a) \cdot \frac{1}{\cancel{\sqrt{2\pi}\cdot a}} \cdot e^{\frac{(x+1)^2}{2a^2}}$$

$$\frac{x^2}{(x+1)^2} \le \ln f(a)$$

$$\left(1-\frac{1}{x+1}\right)^2 \le \ln f(a)$$

$y=\left(1-\frac{1}{x+1}\right)^2$:



$\Rightarrow$ $S \subseteq [0,+\infty)$

$f(a) \ge e$

4.