1.

(a)

According to the description,

$$M'_{means} = (f_l(x) + z_l, g_x(x) + z'_l), l \in [t]$$

And

$$M_{means} = c^l = (c^l_1, c^l_2, \ldots, c^l_k), l \in [t]$$

Given

$$c^l_i = \frac{1}{\max\{1, n^{l-1}_i\}}(z_{l,i} + \sum_{j \in S^{(l-1)}_i} x_j)$$

and

$$f_l(x)_i = \sum_{j \in S^{(l-1)}_i} x_j$$

Then

$$\begin{aligned} c^l &= w^T(f_l(x) + z_l), \\ w_i &= \frac{1}{\max\{1, n^{l-1}_i\}}, i \in [k] \end{aligned}$$

Let Y and Y' be the output spaces of M_{means} and M'_{means}. So there is a post processing function $\Gamma : Y' \times Y' \rightarrow Y$ and let it do such mapping:

$$\begin{aligned} \Gamma(M'_{means}) &= \Gamma(f_l(x) + z_l, g_x(x) + z'_l) \\ &= w^T(f_l(x) + z_l) \\ &= c^l \\ &= M_{means}, l \in [t] \end{aligned}$$

If M'_{means} satisfies ρ-zCDP, it means M'_{means} satisfies $\sqrt{2\rho}$-differential privacy. Let $\epsilon = \sqrt{2\rho}$ and T' is the a set of possible outputs, $T' \subseteq Y'$. S and S' are neighboring datasets, we have

$$Pr(M'_{means}(S) \in T') \leq e^\epsilon Pr(M'_{means}(S') \in T')$$

And for M_{means}, $T \subseteq Y$, and Γ^{-1} is the inverse function of Γ. We have

$$\begin{aligned} Pr(M_{means}(S) \in T) = Pr(\Gamma^{-1}(M_{means}(S)) \in \Gamma^{-1}(T)) = Pr(M'_{means}(S) \in T') \\ Pr(M_{means}(S') \in T) = Pr(\Gamma^{-1}(M_{means}(S')) \in \Gamma^{-1}(T)) = Pr(M'_{means}(S') \in T') \end{aligned}$$

Then

$$Pr(M_{means}(S) \in T') \leq e^\epsilon Pr(M_{means}(S') \in T')$$

So M_{means} satisfies ϵ-privacy and the satisfies ρ-zCDP

(b)

Given x and x' are neighbors, there is only one sample different. These pair of samples ars denoted as x_p, $x'_p \in R^d$ and $x_p, x'_p \in S_\alpha^{l-1}$, $\alpha \in \{1, 2.., , k\}$.

So

$$f_l(x)_i - f_l(x')_i = \begin{cases} 0, & i \neq \alpha \\ x_p - x'_p, & i = \alpha \end{cases}$$

Then according to the Norm triangle inequality and it is known that for $\forall\, i : ||x_i||_2 \leq 1$, we have

$$\begin{aligned} ||f_l(x) - f_l(x')||_2 &= ||x_p - x'_p||_2 \\ &\leq ||x_p||_2 + ||x'_p||_2 \quad \text{(Norm triangle inequality)} \\ &\leq 2 \end{aligned}$$

(c)

We define two mechanisms $M_1^l(x) = f_l(x) + z_l$, $M_2^l(x, m) = (g_l(x) + z'_l, m), m \in R^k$, for $l \in [t]$. For M_2^l, denote the sample in the h-postion in neighboring datasets x and x' as x_h and $x'_h, h \in \{1, 2... d\}$. To maximize the distance, let $x_h \in S_j, j \in \{1, 2... k\}$ and $x'_h \notin S_j$. Then $||g_l(x) - g_l(x')||_2 = \sqrt{2}$, so its l_2 sensitivity, $\Delta_2 = \sqrt{2}$. As dicussed in (b), l_2 sensitivity of M_1^l, $\Delta_1 = 2$.

According to proposition 3, M_1^l satisfies $\frac{2^2}{2\sigma^2} = \frac{2}{\sigma^2}$-zCDP and M_2^l satisfies $\frac{\sqrt{2}^2}{2\sigma'^2} = \frac{1}{\sigma'^2}$-zCDP.

According to lemma 4, $M_2^l(x, M_1^l)$ satisfies $(\frac{2}{\sigma^2} + \frac{1}{\sigma'^2})$-zCDP.

According to lemma 4

$$M'_{means} = (M_2^1(x, M_2^1(x)), M_2^2(x, M_2^2(x)) \ldots M_2^t(x, M_2^t(x)))$$

satisfies $t \cdot (\frac{2}{\sigma^2} + \frac{1}{\sigma'^2}) = \frac{2t}{\sigma^2} + \frac{t}{\sigma'^2}$-zCDP

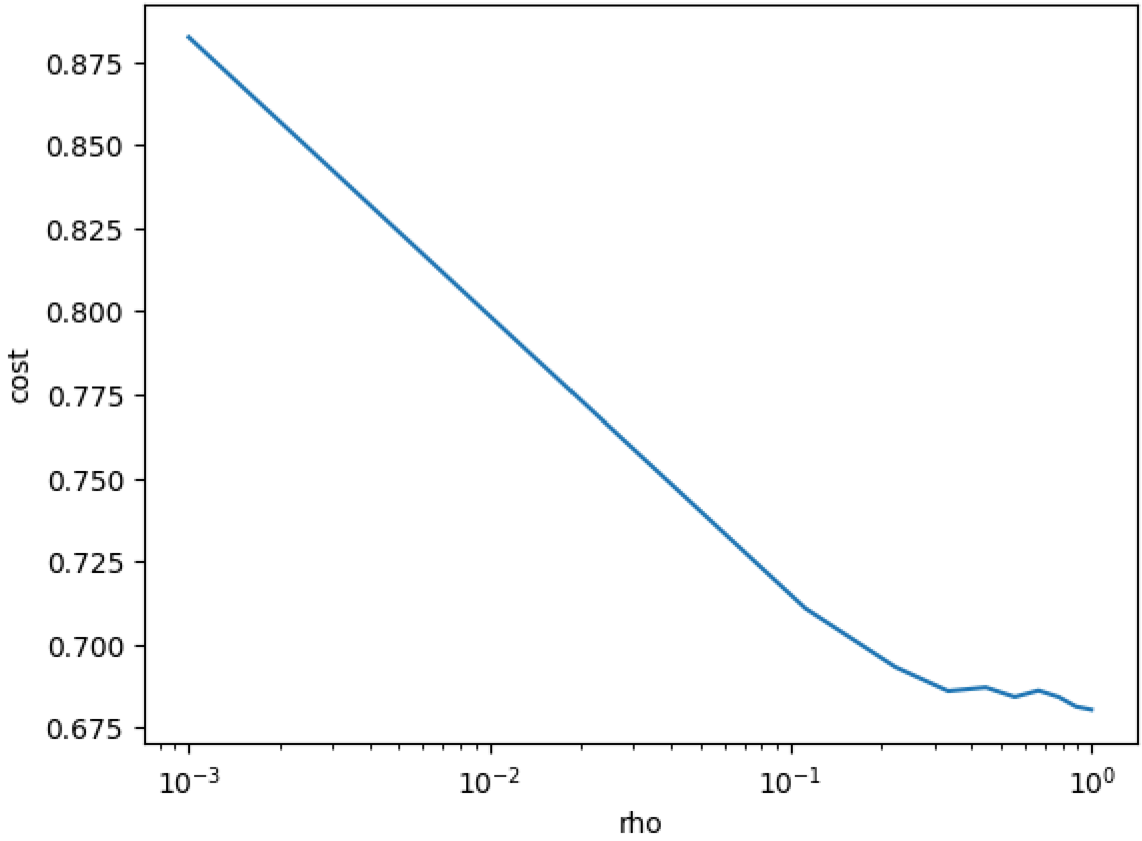
2.

(a)

To calculate σ and σ' from ρ, we set $\sigma = \sigma'$. Then according to the discussion in 1.(c), we have

$$\frac{3t}{\sigma^2} = \rho \Rightarrow \sigma = \sqrt{\frac{3t}{\rho}}$$

(b)



The core code is as following:

```
def compute_cost(points, centers):
    distances_squared = np.sum((points - centers[:,np.newaxis])**2, axis=-1)
    return np.mean(np.min(distances_squared, axis=0))

def add_gaussian_noise(x, sigma):
    return x + np.random.normal(0, sigma, x.shape)

def privacy_k_means(points, k, t, rho):
    sigma = np.sqrt(3*t/rho)

    initial_assignment = np.random.choice(range(k), n)
    cluster_indexes = [ (initial_assignment == i) for i in range(k) ]
    cluster_sizes = [ cluster_indexes[i].sum() for i in range(k) ]

    for l in range(t):
        cluster_sums = [ add_gaussian_noise(np.sum(points[cluster_indexes[i]],
axis=0), sigma) for i in range(k) ]
        centers = np.array([ cluster_sums[i] / max(1, cluster_sizes[i]) for i in
range(k) ])

        distances_squared = np.sum((points - centers[:,np.newaxis])**2, axis=-1)
        assignment = np.argmin(distances_squared, axis=0)

        cluster_indexes = [ (assignment == i) for i in range(k) ]
        cluster_sizes = [ add_gaussian_noise(cluster_indexes[i].sum(),sigma) for
i in range(k) ]
```

```python
    return centers


k = 5 # Number of clusters

rho_range = np.linspace(0.001, 1 ,10)
costs = []
for rho in rho_range: # number of iterations
    t = 5
    centers = privacy_k_means(points, k, t, rho=rho)
    costs.append(compute_cost(points, centers))

fig, ax = plt.subplots()
ax.set_xlabel('rho')
ax.set_ylabel('cost')
ax.plot(rho_range, costs)
plt.xscale('log')
plt.show()
```

Detailed code is published on [github: A2>A2.ipynb]()