(3.4) $S \subseteq \mathbb{R}$ and $X \sim \mathcal{N}(0, \sigma^2)$

$$\mathbb{P}[X \in S] = \int_S f_X(x)\,dx$$

$$= \int_S \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}}\,dx$$

$$X \sim \mathcal{N}(0, \sigma^2) \implies X+1 \sim \mathcal{N}(1, \sigma^2)$$

$$\implies \mathbb{P}[X+1 \in S] = \int_S \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-1)^2}{2\sigma^2}}\,dx$$

So $$\frac{f_X}{f_{X+1}} = \exp\left(\frac{-x^2 + (x-1)^2}{2\sigma^2}\right)$$

$$= \exp\left(\frac{-x^2 + x^2 - 2x - 1}{2\sigma^2}\right) = \exp\left(-\frac{(2x+1)}{2\sigma^2}\right)$$

this grows exponentially fast for $x \to -\infty$

$$\implies f_X = \exp\left(-\frac{(2x+1)}{2\sigma^2}\right) f_{X+1}$$

meaning for $x \ll 0$ $\quad f_X \gg f_{X+1}$

So we can never upper bound f_X with a function $f(\sigma) \cdot f_{X+1}$ because $f(\sigma)$ is constant (in x)

So no matter how big we choose $f(\sigma)$ we can always find b so that for all $x \leq b$ $\exp\left(-\frac{(2x+1)}{2\sigma^2}\right) > f(\sigma)$

so by choosing ~~only~~ $S = [a,b]$ we have

$$f(\sigma)\,\mathbb{P}[X+1 \in S] = \int_a^b \underbrace{f(\sigma)}_{\substack{< \exp\left(-\frac{(2x+1)}{2\sigma^2}\right) \\ \text{for all } x \in [a,b]}} f_{X+1}(x)\,dx < \int_a^b \exp\left(-\frac{2x+1}{2\sigma^2}\right) f_{X+1}(x)\,dx$$

$$= \mathbb{P}[X \in S]$$