ECE685D HW5

Zanwen Fu

November 2025

1 Solution to 2.1

The joint distribution of a Gaussian–Bernoulli RBM is

$$p(v,h) = \frac{1}{Z}\exp\left(\sum_i\sum_j W_{ij}h_j\frac{v_i}{\sigma_i} - \sum_i \frac{(v_i-b_i)^2}{2\sigma_i^2} + \sum_j \alpha_j h_j\right),$$

derived from the energy

$$E(v,h;\theta) = -\left(\sum_i\sum_j W_{ij}h_j\frac{v_i}{\sigma_i} - \sum_i \frac{(v_i-b_i)^2}{2\sigma_i^2} + \sum_j \alpha_j h_j\right).$$

(a) Conditional distribution $p(h_j = 1 \mid v)$

Starting from

$$p(h \mid v) \propto p(v,h),$$

we keep only the terms depending on h:

$$p(h \mid v) \propto \exp\left(\sum_j h_j\left(\alpha_j + \sum_i W_{ij}\frac{v_i}{\sigma_i}\right)\right).$$

Because $h_j \in \{0,1\}$ and hidden units factorize,

$$p(h_j = 1 \mid v) = \frac{\exp\left(\alpha_j + \sum_i W_{ij}\frac{v_i}{\sigma_i}\right)}{1+\exp\left(\alpha_j + \sum_i W_{ij}\frac{v_i}{\sigma_i}\right)}.$$

Thus,

$$p(h_j = 1 \mid v) = \sigma\left(\alpha_j + \sum_i W_{ij}\frac{v_i}{\sigma_i}\right).$$

(b) Conditional distribution $p(v_i \mid h)$

Using Bayes' rule,

$$p(v_i \mid h) \propto p(v_i, h),$$

and keeping only terms involving v_i,

$$p(v_i \mid h) \propto \exp\left(\sum_j W_{ij} h_j \frac{v_i}{\sigma_i} - \frac{(v_i - b_i)^2}{2\sigma_i^2}\right).$$

Let

$$A_i = \sum_j W_{ij} h_j / \sigma_i.$$

Then the exponent becomes

$$\sum_j W_{ij} h_j \frac{v_i}{\sigma_i} - \frac{(v_i - b_i)^2}{2\sigma_i^2} = -\frac{1}{2\sigma_i^2}\Big[(v_i - b_i)^2 - 2\sigma_i^2 A_i v_i\Big].$$

Completing the square yields

$$p(v_i \mid h) \propto \exp\left(-\frac{(v_i - \mu_i)^2}{2\sigma_i^2}\right), \qquad \mu_i = b_i + \sigma_i \sum_j W_{ij} h_j.$$

Therefore,

$$p(v_i \mid h) = \mathcal{N}\left(v_i;\, b_i + \sigma_i \sum_j W_{ij} h_j,\ \sigma_i^2\right).$$

Integral form (optional, as allowed in the question):

$$p(v_i = x \mid h) = \frac{1}{C(h)} \exp\left(\sum_j W_{ij} h_j \frac{x}{\sigma_i} - \frac{(x - b_i)^2}{2\sigma_i^2}\right),$$

where $C(h)$ is the normalization constant obtained by integrating over x.